Chapter 1 – Functions and relations

Solutions to Exercise 1A

1 a $\{8, 11\}$

b $\{8, 11\}$

c $\{1, 3, 8, 11, 18, 22, 23, 24, 25, 30\}$

d $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$

e $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$

f $\{1, 8, 11, 25, 30\}$

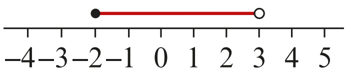
2 a $\{3, 18, 22, 23, 24\}$

b $\{25, 30, 32\}$

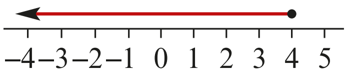
c $\{3, 18, 22, 23, 24\}$

d $\{1, 25, 30\}$

3 a



b



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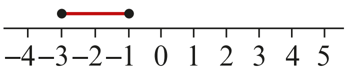

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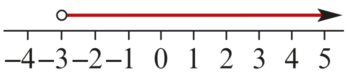
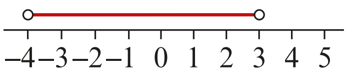
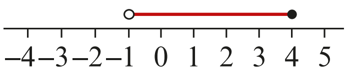


e



f



4 a $X \cap Y = \{7, 9\}$

b $X \cap Y \cap Z = \{7, 9\}$

c $X \cup Y = \{2, 3, 5, 7, 9, 11, 15, 19, 23\}$

d $X \backslash Y = \{2, 3, 5, 11\}$

e $Z \backslash Y = \{2\}$

f $X \cap Z = \{2, 7, 9\}$

g $[-2, 8] \cap X = \{2, 3, 5, 7\}$

h $(-3, 8] \cap Y = \{7\}$

i $(2, \infty) \cap Y = \{7, 9, 15, 19, 23\}$

j $(3, \infty) \cup Y = (3, \infty)$

5 a $X \cap Y = \{a, e\}$

b $X \cup Y = \{a, b, c, d, e, i, o, u\}$

c $X \backslash Y = \{b, c, d\}$

d $Y \backslash X = \{i, o, u\}$

6 a $B \cap C = \{6\}$

b $B \backslash C = \{2, 4, 8, 10\}$

c $A \backslash B = \{1, 3, 5, 7, 9\}$

d $A \backslash B = \{1, 3, 5, 7, 9\}$
$A \backslash C = \{2, 4, 5, 7, 8, 10\}$
$(A \backslash B) \cup (A \backslash C) = \{1, 2,$
$3, 4, 5, 7, 8, 9, 10\}$

e $B \cap C = \{6\}$
$A \backslash (B \cap C) = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$

f $A \backslash B = \{1, 3, 5, 7, 9\}$
$A \backslash C = \{2, 4, 5, 7, 8, 10\}$

$(A \backslash B) \cap (A \backslash C) = \{5, 7\}$

g $B \cup C = \{1, 2, 3, 4, 6, 8, 9, 10\}$
$A \backslash (B \cup C) = \{5, 7\}$

h $A \cap B \cap C = \{6\}$

7 **a** $[-3, 1)$

b $(-4, 5]$

c $(-\sqrt{2}, 0)$

d $(-\frac{1}{\sqrt{2}}, \sqrt{3})$

e $(-\infty, -3)$

f $(0, \infty)$

g $(-\infty, 0)$

h $[-2, \infty)$

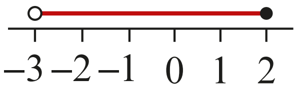
8 **a** $(-2, 3)$

b $[-4, 1)$

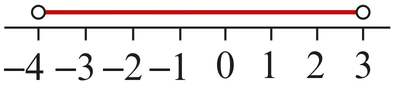
c $[-1, 5]$

d $(-3, 2]$

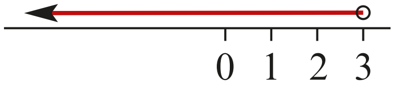
9 **a**



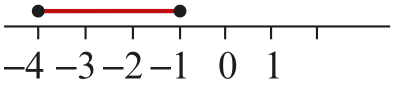
b



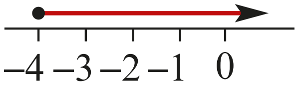
c



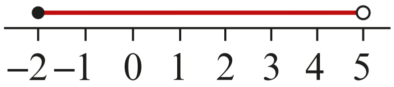
d



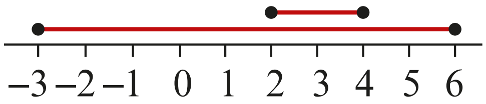
e



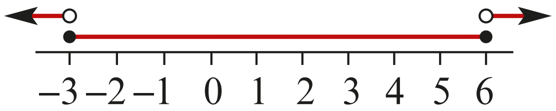
f



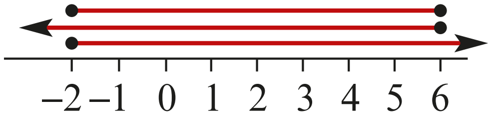
10 **a**



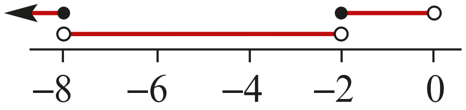
b



c



d



Solutions to Exercise 1B

1 **a** Domain = $\mathbb{R}$

range = $[-2, \infty)$

b Domain = $(-\infty, 2]$

range = $\mathbb{R}$

c Domain = $(-2, 3)$

range = $[0, 9)$

d Domain = $(-3, 1)$

range = $(-6, 2)$

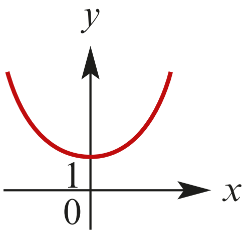
e Domain = $[-4, 0]$

range = $[0, 4]$

f Domain = $\mathbb{R}$

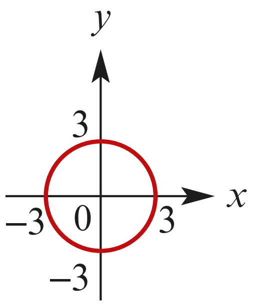
range = $(-\infty, 2)$

2 **a**



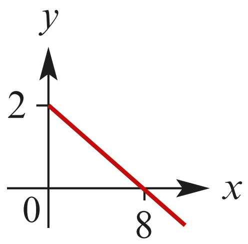
Domain = $\mathbb{R}$
range = $[1, \infty)$

b



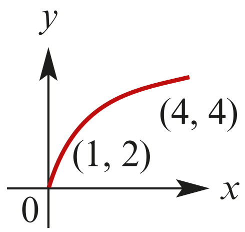
Domain = $[-3, 3]$
range = $[-3, 3]$

c



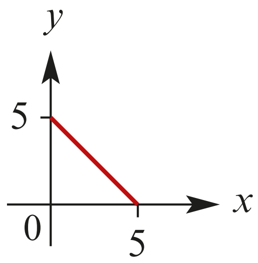
Domain = $\mathbb{R}^+ \cup \{0\}$
range = $(-\infty, 2]$

d



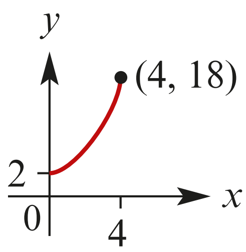
Domain = $[0, \infty)$
range = $[0, \infty)$

e



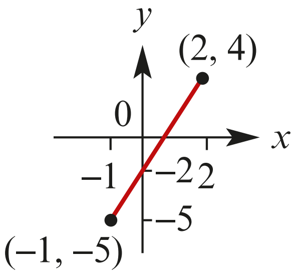
Domain = $[0, 5]$
range = $[0, 5]$

f



Domain = $[0, 4]$
range = $[2, 18]$

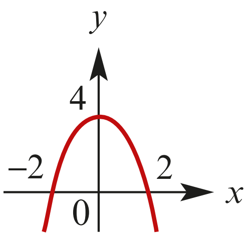
g



Domain = $[-1, 2]$

range = $[-5, 4]$

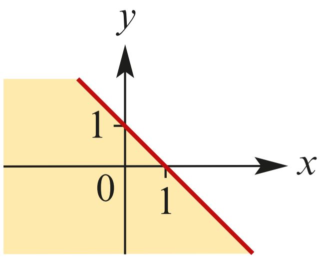
h



Domain = $\mathbb{R}$

range = $(-\infty, 4]$

i



Domain = $\mathbb{R}$

Range = $\mathbb{R}$

3 **a** not a function

Domain = $\{-1, 1, 2, 3\}$

range = $\{1, 2, 3, 4\}$

b function

Domain = $\{-2, -1, 0, 1, 2\}$

range = $\{-4, -1, 0, 3, 5\}$

c not a function

Domain = $\{-2, -1, 2, 4\}$

range = $\{-2, 1, 2, 4, 6\}$

d function

Domain = $\{-1, 0, 1, 2, 3\}$

range = $\{4\}$

4 **a** function Domain = $\mathbb{R}$

range = $\{4\}$

b not a function

Domain = $\{2\}$

range = $\mathbb{Z}$

c function

Domain = $\mathbb{R}$

range = $\mathbb{R}$

d not a function

Domain = $\mathbb{R}$

range = $\mathbb{R}$

e not a function

Domain = $[-4, 4]$

range = $[-4, 4]$

5 $f(x) = 2x^2 + 4x;$

$g(x) = 2x^3 + 2x - 6$

a
$$f(-1) = 2(-1)^2 + 4(-1) = -2$$
$$f(2) = 2(2)^2 + 4(2) = 16$$
$$f(-3) = 2(-3)^2 + 4(-3) = 6$$
$$f(2a) = 2(2a)^2 + 4(2a) = 8a^2 + 8a$$

b
$$g(-1) = 2(-1)^3 + 2(-1) - 6 = -10$$
$$g(2) = 2(2)^3 + 2(2) - 6 = 14$$
$$g(3) = 2(3)^3 + 2(3) - 6 = 54$$
$$\begin{aligned} g(a-1) &= 2(a-1)^3 + 2(a-1) - 6 \\ &= 2(a^3 - 3a^2 + 3a - 1) + 2a - 8 \\ &= 2a^3 - 6a^2 + 8a - 10 \end{aligned}$$

6 $g(x) = 3x^2 - 2$

a $g(-2) = 3(-2)^2 - 2 = 10$

$g(4) = 3(4)^2 - 2 = 46$

b **i** $g(-2) = 3(-2)^2 - 2 = 12x^2 - 2$

ii $g(x-2)^2 = 3(x-2)^2 - 2 =$
$3x^2 - 12x + 10$

iii $g(x+2)^2 = 3(x+2)^2 - 2 =$
$3x^2 + 12x + 10$

iv $g(x^2) = 3(x^2)^2 - 2 = 3x^4 - 2$

7 $f(x) = 2x - 3$

a $f(3) = 2(3) - 3 = 3$

b $f(x) = 11$

$11 = 2x - 3x = 7$

c $f(x) = 4x$

$$4x = 2x - 3$$
$$2x = -3$$
$$x = \frac{-3}{2}$$

d $f(x) > x$

$$2x - 3 > x$$
$$x > 3$$

8 $g(x) = 6x + 7 \quad h(x) = 3x - 2$

a $6x + 7 = 3x - 2$

$$3x = -9$$
$$x = -3$$

b $6x + 7 > 3x - 2$

$$3x > -9$$
$$x > -3$$

c $3x - 2 = 0$

$$x = \frac{2}{3}$$

9 **a** $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 3$

b $3y + 4x = 12$

$$3y = 12 - 4x$$
$$y = 4 - \frac{4x}{3}$$

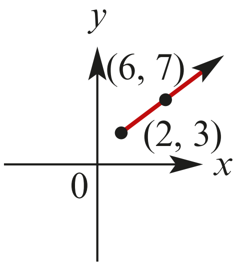
$f: \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{-4x}{3} + 4$

c $f: [0, \infty) \to \mathbb{R}$ where $f(x) = 2x - 3$

d $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2 - 9$

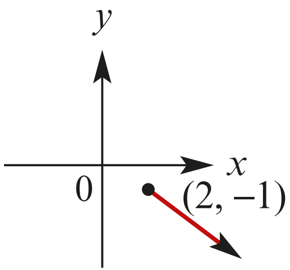
e $f: [0, 2] \to \mathbb{R}$ where $f(x) = 5x - 3$

10 **a**



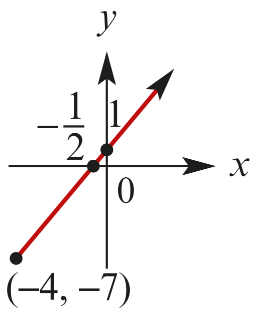
range = $[3, \infty)$

b



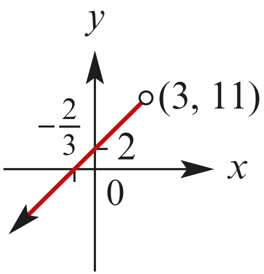
range = $(-\infty, -1]$

c



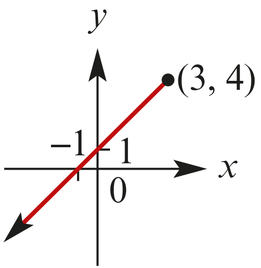
range = $[-7, \infty)$

d



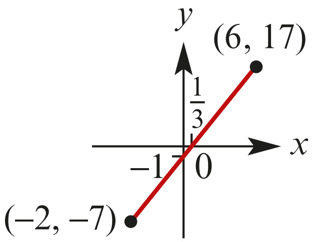
range = $(-\infty, 11)$

e



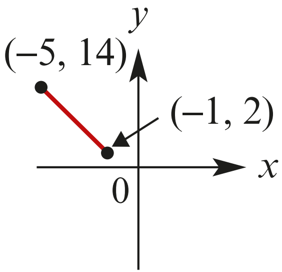
range = $(-\infty, 4]$

f



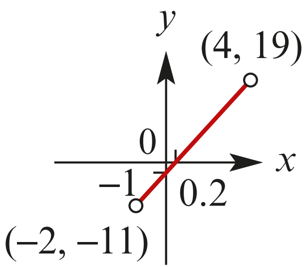
range = $[-7, 17]$

g



range = $[2, 14]$

h



range = $(-11, 19)$

11 $f(x) = 2x^2 - 6x + 1; \quad g(x) = 3 - 2x$

a $f(2) = 2(2)^2 - 6(2) + 1 = -3$

$f(-3) = 2(-3)^2 - 6(-3) + 1 = 37$

$f(-2) = 2(-2)^2 - 6(-2) + 1 = 21$

b $g(-2) = 3 - 2(-2) = 7$

$g(1) = 3 - 2(1) = 1$

$g(-3) = 3 - 2(-3) = 9$

c **i** $f(a) = 2a^2 - 6a + 1$

ii $$\begin{aligned} f(a+2) &= 2(a+2)^2 \\ &\quad - 6(a+2) + 1 \\ &= 2a^2 + 2a - 3 \end{aligned}$$

iii $g(-a) = 3 + 2a$

iv $g(2a) = 3 - 4a$

v $$\begin{aligned} f(5-a) &= 2(5-a)^2 \\ &\quad - 6(5-a) + 1 \\ &= 2a^2 - 14a + 21 \end{aligned}$$

vi $f(2a) = 8a^2 - 12a + 1$

vii $$\begin{aligned} g(a) + f(a) &= (2a^2 - 6a + 1) \\ &\quad + (3 - 2a) \\ &= 2a^2 - 8a + 4 \end{aligned}$$

viii $g(a) - f(a) = (3 - 2a)$

$$- (2a^2 - 6a + 1)$$

$$= -2a^2 + 4a + 2$$

12 $f(x) = 3x^2 + x - 2$

a $f(x) = 0$

$3x^2 + x - 2 = 0$

using the quadratic formula

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(3)(-2)}}{2(3)}$$

$$x = \frac{-1 \pm \sqrt{25}}{6}$$

$$x = -1, \frac{2}{3}$$

in set notation

$$\left\{-1, \frac{2}{3}\right\}$$

b $f(x) = x$

$3x^2 + x - 2 = x$

$3x^2 = 2$

$$x^2 = \frac{2}{3}$$

$$x = \pm\sqrt{2/3}$$

in set notation

$$\left\{-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right\}$$

c $f(x) = -2$

$3x^2 + x - 2 = -2$

$3x^2 + x = 0$

$x(3x + 1) = 0$

$\therefore$ *either* $x = 0$ *or* $3x + 1 = 0$

$$x = 0, \frac{-1}{3}$$

in set notation

$$\left\{0, \frac{-1}{3}\right\}$$

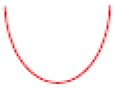
d $f(x) > 0$

$3x^2 + x - 2 > x$

from (a), the x-intercepts are $-1, \frac{2}{3}$

as the coefficient of $x^2 > 0$

the shape of the graph $y = f(x)$ is



$\therefore$ $f(x) > 0$ for

$x \in (-\infty, -1) \cup (\frac{2}{3}, \infty)$

e $f(x) > x$

$3x^2 + x - 2 > x$

$3x^2 - 2 > 0$

$$x^2 > \frac{2}{3}$$

$$x \in \left(-\infty, \frac{-\sqrt{2}}{\sqrt{3}}\right) \cup \left(\frac{\sqrt{2}}{\sqrt{3}}, \infty\right)$$

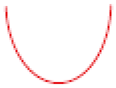
f $f(x) \le -2$

$3x^2 + x - 2 \le -2$

from (c), the x-intercepts are $-\frac{1}{3}, 0$

as as the coefficient of $x^2 > 0$

the shape of the graph $y = f(x)$ is



$$\therefore \quad f(x) \le -2 \text{ for } x \in \left[\frac{-1}{3}, 0\right]$$

13 $f(x) = x^2 + x$

a $f(-2) = (-2)^2 + (-2) = 2$

b $f(2) = (2)^2 + (2) = 6$

c $f(-a) = (-a)^2 + (-a) = a^2 - a$

d $f(a) + f(-a) = (a^2 + a) + (a^2 - a)$

$$= 2a^2$$

e $f(a) - f(-a) = (a^2 + a) - (a^2 - a)$

$= 2a$

f $f(a^2) = (a^2)^2 + (a^2) = a^4 + a^2$

14 $g(x) = 3x - 2$

a $g(x) = 4$

$3x - 2 = 4$

$x = 2$

b $g(x) > 4$

$3x - 2 > 4$

$x > 2$

in set notation

$\{x : x > 2\}$

c $g(x) = a$

$3x - 2 = a$

$x = \dfrac{a+2}{3}$

d $g(-x) = 6$

$-3x - 2 = 6$

$x = \dfrac{-8}{3}$

e $g(2x) = 4$

$6x - 2 = 4$

$x = 1$

f $\dfrac{1}{g(x)} = 6$

$1 = 6g(x)$

$1 = 6(3x - 2)$

$1 = 18x - 12$

$18x = 13$

$x = \dfrac{13}{18}$

15 **a** $f(x) = kx - 1$

$3 = 3k - 1$

$k = \dfrac{4}{3}$

b $f(x) = x^2 - k$

$3 = 9 - k$

$k = 6$

c $f(x) = x^2 + kx + 1$

$3 = 9 + 3k + 1$

$k = \dfrac{-7}{3}$

d $f(x) = \dfrac{k}{x}$

$3 = \dfrac{k}{3}$

$k = 9$

e $f(x) = kx^2$

$3 = 9k$

$k = \dfrac{1}{3}$

f $f(x) = 1 - kx^2$

$$3 = 1 - 9k$$

$$9k = -2$$

$$k = \frac{-2}{9}$$

16 **a** $5x - 4 = 2$

$$x = \frac{6}{5}$$

b $\frac{1}{x} = 5$

$$x = \frac{1}{5}$$

c $\frac{1}{x^2} = 9$

$$x = \pm\frac{1}{3}$$

d $x = \frac{1}{x} = 2$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0$$

$$x = 1$$

e $(x + 1)(x - 2) = 2$

$\therefore$ either $x + 1 = 0$ or $x - 2 = 0$

$x = -1 \quad x = 2$

$\therefore \quad x = -1, 2$

Solutions to Exercise 1C

1 **a** The functions which are one - to - one are **b** and **c**

2 **a** The functions which are one - to - one are **b**,**d** and **f**

3 **a** The graphs of functions are **i**, **iii**, **iv**, **vi**, **vii**, and **viii**.

b The graphs of one - to - one functions are **iii**, and **vii**.

4 $y^2 = x + 2, x \geq -2$

$y = \pm\sqrt{x+2}$

two possible functions f and g are

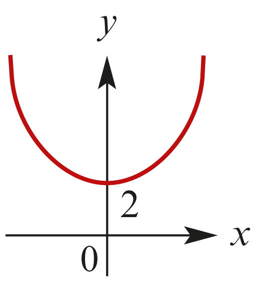
$f : [-2, \infty) \to \mathbb{R} \quad f(x) = \sqrt{x+2}$

range of $f : [0, \infty) = \mathbb{R}^+ \cup \{0\}$

$g : [-2, \infty) \to \mathbb{R}\ g(x) = -\sqrt{x-2}$

range of $g : (-\infty, 0] = \mathbb{R}^- \cup \{0\}$

5 **a**



b two possible functions are the right half

$g_1 : [0, \infty) \to \mathbb{R} \quad g_1(x) = x^2 + 2$

and the left half

$g_2 : (-\infty, 0) \to \mathbb{R} \quad g_2(x) = x^2 + 2$

6 **a** Domain: $\mathbb{R}$ range: $\mathbb{R}$

b Domain: $\mathbb{R}^+ \cup \{0\}$ range: $\mathbb{R}^+ \cup \{0\}$

c Domain: $\mathbb{R}$ range: $[-2, \infty)$

d Domain: $[-4, 4]$ range: $[0, 4]$

e Domain: $\mathbb{R}\backslash\{0\}$ range: $\mathbb{R}\backslash\{0\}$

f Domain: $\mathbb{R}$ range: $(-\infty, 4]$

g Domain $[3, \infty)$ range: $[0, \infty)$

7 **a** Domain: $\mathbb{R}$ range: $\mathbb{R}$

b Domain: $\mathbb{R}$ range: $[-2, \infty)$

c Domain [-3, 3] range: $[0, 3]$

d Domain: $\mathbb{R}\backslash\{1\}$ range: $\mathbb{R}\backslash\{0\}$

8 **a** $\mathbb{R}\backslash\{3\}$

b $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

c $\mathbb{R}$

d $[4, 11]$

e $\mathbb{R}\backslash\{-1\}$

f $h(x) = \sqrt{(x+1)(x-2)}$
Domain : $(-\infty, -1] \cup [2, \infty)$

g $\mathbb{R}\backslash\{-1, 2\}$

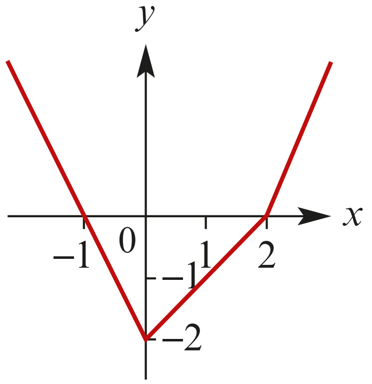
h Domain: $(-\infty, -2) \cup [1, \infty)$

i $f(x) = \sqrt{x(1-3x)}$ Domain : $\left[0, \frac{1}{3}\right]$

j $[-5, 5]$

k $[3, 12]$

9 **a**



b $[-2, \infty)$

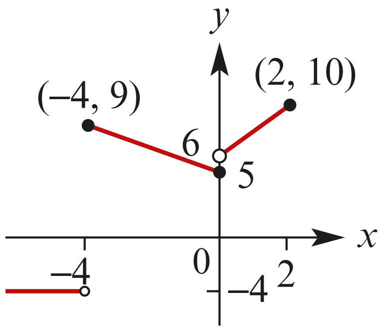
10 Domain: $(-3, 0] \cup [1, 3)$

range: $[-2, 3)$

11 Domain: $[-5, 4]$

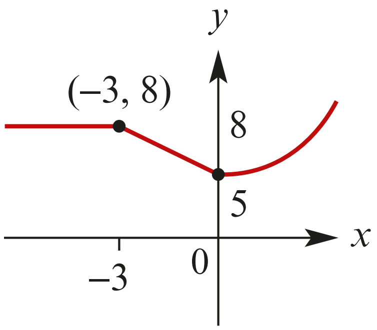
range: $[-4, 0) \cup [2, 5]$

12 **a**



b Domain $= (-\infty, 2]$
range $= [5, 10] \cup \{-4\}$

13 **a**



b range $= [5, \infty\}$

14 $f(x) = \begin{cases} \frac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$

a $f(-4) = 2(-4) = -8$

b $f(0) = 2(0) = 0$

c $f(4) = \frac{1}{(4)} = \frac{1}{4}$

d $f(a+3) = \begin{cases} \frac{1}{a+3}, & a > 0 \\ 2a + 6, & a \le 0 \end{cases}$

e $f(2a) = \begin{cases} \frac{1}{2a}, & a > \frac{3}{2} \\ 4a, & a \le \frac{3}{2} \end{cases}$

f $f(a-3) = \begin{cases} \frac{1}{a-3}, & a > 6 \\ 2a - 6, & a \le 6 \end{cases}$

15 **a** $f(0) = 4$

b $f(3) = \sqrt{(3) - 1} = \sqrt{2}$

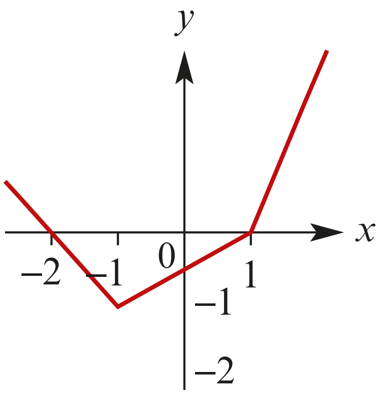
c $f(8) = \sqrt{(8) - 1} = \sqrt{7}$

d $f(a+1) = \begin{cases} \sqrt{a}, & a \ge 0 \\ 4, & a < 0 \end{cases}$

e $f(a-1) =$

$$\begin{cases} \sqrt{a-2}, & a-1>1 \Rightarrow a \geq 2 \\ 4, & a-0<1 \Rightarrow a<2 \end{cases}$$

16 **a**



range = $[-1, \infty)$

17 $$f(x) = \begin{cases} ax+b, & x<-2 \\ cx+d, & -2 \leq x \leq 3 \\ ex+f, & x>3 \end{cases}$$

using the points given

$$f(x) = \begin{cases} -x-4, & x<-2 \\ \frac{1}{2}x-1, & -2 \leq x \leq 3 \\ -\frac{1}{2}x+2, & x>3 \end{cases}$$

18 **a** Even

b Odd

c Neither

d Even

e Odd

f Neither

19 **a** Even

b Even

c Odd

d Odd

e Neither

f Even

g Neither

h Neither

i Even

Solutions to Exercise 1D

1 **a** $(f+g)(x) = 3x + x + 2$

$= 4x + 2$

Domain: $\mathbb{R}$

$(fg)(x) = 3x(x+2)$

$= 3x^2 + 6x$

Domain: $\mathbb{R}$

b $(f+g)(x) = 1 - x^2 + x^2 = 1$

Domain: $(0, 2]$

(from Domain $(g) \cap$ Domain (f))

$(fg)(x) = (1 - x^2)x^2$

$= x^2 - x^4$

Domain: $(0, 2]$

(from Domain $(g) \cap$ Domain(f))

c $(f+g)(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}} = \dfrac{x+1}{\sqrt{x}}$

Domain: $[1, \infty)$ (from g)

$(fg)(x) = \dfrac{\sqrt{x}}{\sqrt{x}}$

$= 1$

Domain: $[1,\infty)$ (from g)

d $(f+g)(x) = x^2 + \sqrt{4-x}$

Domain: $[0, 4]$ (from g)

$(fg)(x) = x^2\sqrt{4-x}$

Domain: $[0, 4]$ (from g)

2 **a** functions f and h are even, g and k are odd

b $(f+h)(x) = x^2 + 1 + \dfrac{1}{x^2}, x \in \mathbb{R}\backslash\{0\}$

it is even

$(fh)(x) = 1 + \dfrac{1}{x^2}, x \in \mathbb{R}\backslash\{0\}$

it is even

$(g+k)(x) = x + \dfrac{1}{x}, x \in \mathbb{R}\backslash\{0\}$

it is odd

$(gk)(x) = 1, x \in \mathbb{R}\backslash\{0\}$

it is even

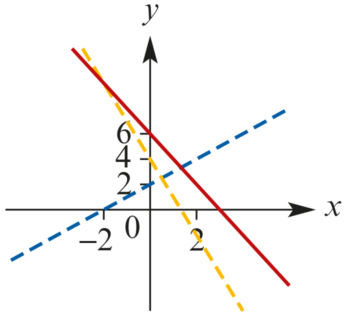
$(f+g)(x) = x^2 + x + 1, x \in \mathbb{R}$

it is neither odd nor even

$(fg)(x) = x^3 + x, x \in \mathbb{R}$

it is odd

3



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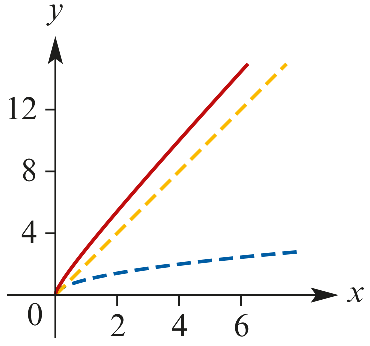

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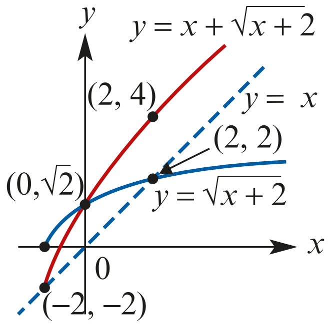

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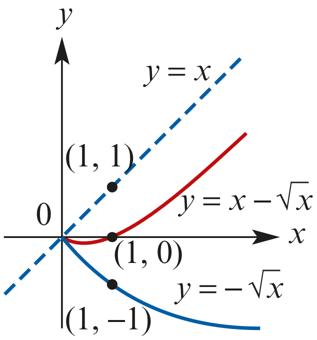

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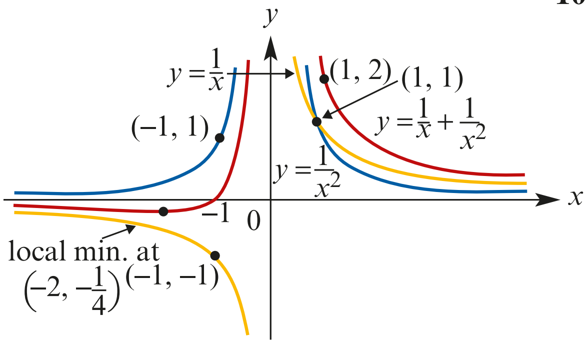
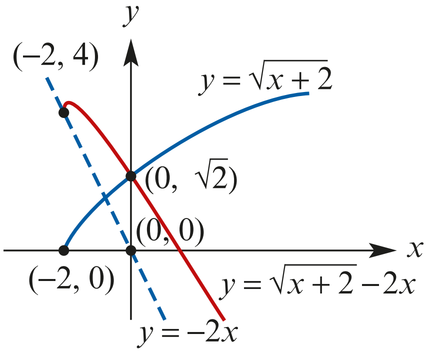
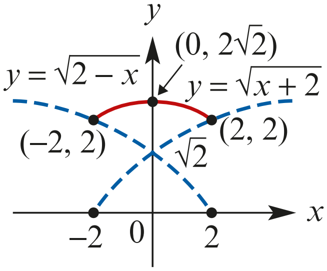


8 **a**



b



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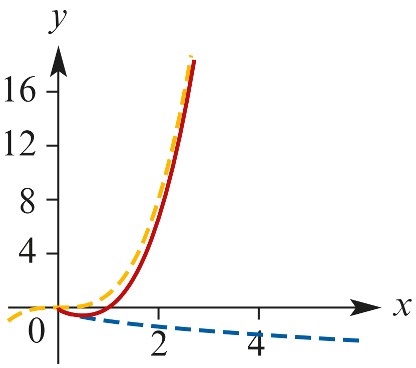

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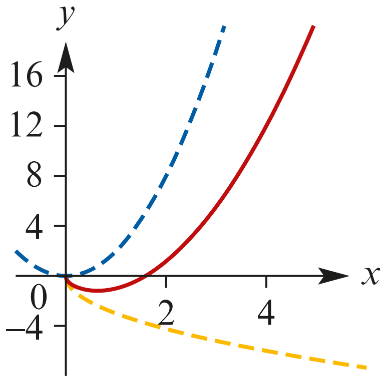

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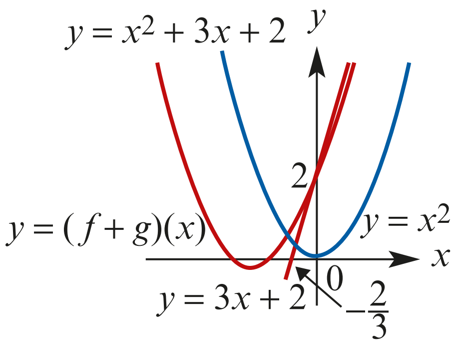
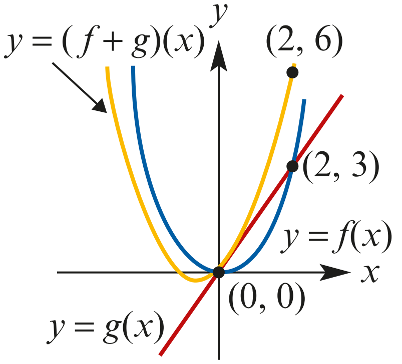
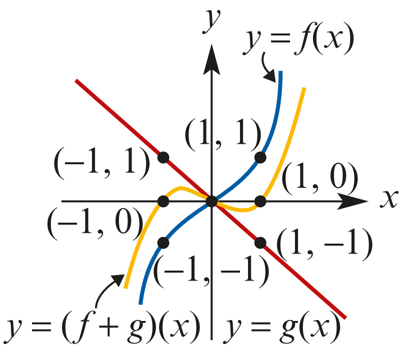


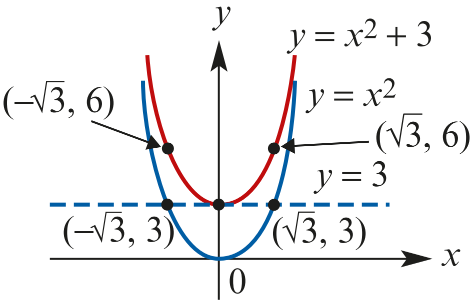
12 **a**



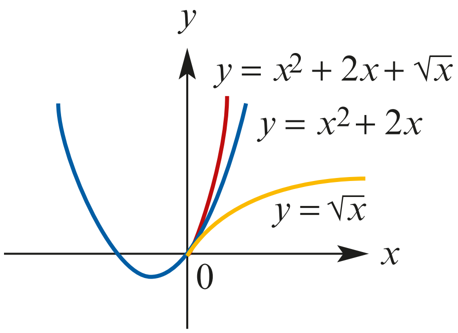
b


y
y = f(x)
(1, 1)
(−1, 1)
(1, 0)
x
(−1, 0)
(1, −1)
(−1, −1)
y = (f + g)(x)
y = g(x)

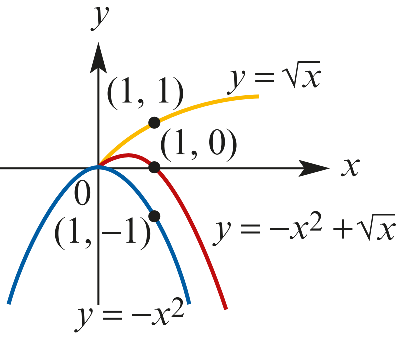
13 a


y
y = x² + 3
y = x²
(−√3, 6)
(√3, 6)
y = 3
(−√3, 3)
(√3, 3)
x
0

b


y
y = x² + 2x + √x
y = x² + 2x
y = √x
x
0

c


y
(1, 1)
y = √x
(1, 0)
x
0
(1, −1)
y = −x² + √x
y = −x²

Solutions to Exercise 1E

1 **a** $f(g(x)) = 2(2x) - 1 = 4x - 1$

$g(f(x)) = 2(2x - 1) = 4x - 2$

b $f(g(x)) = 4(2x + 1) + 1 = 8x + 5$

$g(f(x)) = 2(4x + 1) + 1 = 8x + 3$

c $f(g(x)) = 2(2x - 3) - 1 = 4x - 7$

$g(f(x)) = 2(2x - 1) - 3 = 4x - 5$

d $f(g(x)) = 2(x^2) - 1 = 2x^2 - 1$

$g(f(x)) = (2x - 1)^2 = 4x^2 - 4x + 1$

e $f(g(x)) = 2(x - 5)^2 + 1$

$= 2x^2 - 20x + 51$

$g(f(x)) = (2x^2 + 1) - 5$

$= 2x^2 - 4$

f $f(g(x)) = 2(x^2) + 1 = 2x^2 + 1$

$g(f(x)) = (2x + 1)^2$

2 **a** $f \circ h(x) = 2(3x + 2) - 1 = 6x + 3$

b $h(f(x)) = 3(2x - 1) + 2 = 6x - 1$

c $f \circ h(2) = 6(2) + 3 = 15$

d $h \circ f(2) = 6(2) - 1 = 11$

e $f(h(3)) = 6(3) + 3 = 21$

f $h(f(-1)) = 6(-1) - 1 = -7$

g $f \circ h(0) = 6(0) + 3 = 3$

3 **a** $f \circ h(x) = (3x + 1)^2 + 2(3x + 1)$

$= 9x^2 + 12x + 3$

b $h \circ f(x) = 3(x^2 + 2x) + 1 = 3x^2 + 6x + 1$

c $f \circ h(3) = 9(3)^2 + 12(3) + 3 = 120$

d $h \circ f(3) = 3(3)^2 + 6(3) + 1 = 46$

e $f \circ h(0) = 9(0)^2 + 12(0) + 3 = 3$

f $h \circ f(0) = 3(0)^2 + 6(0) + 1 = 1$

4 **a** $h \circ g : \mathbb{R}^+ \to \mathbb{R}, h \circ g(x) = \dfrac{1}{(3x + 2)^2}$

b $g \circ h : \mathbb{R}\backslash\{0\}, g \circ h(x) = \dfrac{3}{x^2} + 2$

c $h \circ g(1) = \dfrac{1}{(3(1) + 2)^2} = \dfrac{1}{25}$

d $g \circ h(1) = \dfrac{3}{(1)^2} + 2 = 5$

5 **a** range$(f) = [-4, \infty)$

range$(g) = \mathbb{R}^+ \cup \{0\}$

b $f \circ g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, f \circ g(x) = x - 4$

range(f ∘ g) = $[-4, \infty)$

c $g \circ f$ does not exist because the range of f is not a subset of the Domain of g

6 **a** $f(g(x)) = \dfrac{1}{2}(2x) = x$

$f \circ g : \mathbb{R}\backslash\left\{\dfrac{1}{2}\right\} \to \mathbb{R}, f \circ g(x) = x$

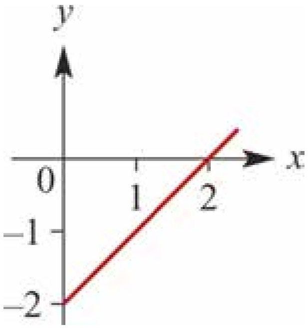
Range: $\mathbb{R}\backslash\left\{\dfrac{1}{2}\right\}$

b $g \circ f : \mathbb{R}\backslash\{0\} \to R, g \circ f(x) = x$

Range : $\mathbb{R}\backslash\{0\}$

7 **a** the range of is $[-2, \infty)$, which is not a subset of the Domain of g, $\therefore g \circ f$ does not exist.

b $f \circ g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, f \circ g(x) = x - 2$



8 **a** the range of g is $[-1, \infty)$, which is not a subset of the Domain of $f((-\infty, 3])$,
$\therefore \quad f \circ g$ does not exist.

b the range of g^* needs to be $[-1, 3]$ at most.
$g^* : [-2, 2] \to \mathbb{R}, g^*(x) = x^2 - 1$
$f \circ g^* : [-2, 2] \to \mathbb{R}, f \circ g^*(x) = 4 - x^2$

9 **a** The range of g is $\mathbb{R}$, which is not a subset of the Domain of f,
$f \circ g$ does not exist.

b the range of g needs to be $\mathbb{R}^+$ at most.
$\therefore$ let $g_1 : \{x : x < 3\} \to \mathbb{R}$,
$g_1(x) = 3 - x$
then $f \circ g_1 : \{x : x < 3\} \to \mathbb{R}$,
$$f \circ g_1(x) = \frac{1}{\sqrt{3 - x}}$$

10 **a** the Domain of f is R, the range of g is $\mathbb{R}^+ \cup \{0\}$
$\therefore \quad f \circ g$ exists.

b Range of f is $\mathbb{R}^+ \cup \{0\}$
Domain of g is $(-\infty, 3]$
The range of f is not a subset of the Domain of g
$\therefore \quad g \circ f$ does not exist.

11 **a** S is the maximal Domain of f,
$\therefore S = [-2, 2]$

b Range of $f = [0, 2]$
range of $g = [1, \infty)$

c $f \circ g$ is not defined as the range of g is not a subset of the Domain of f.
$g \circ f$ is defined as the range of f is a subset of the Domain of g.

12 For both $f \circ g$ and $g \circ f$ to exist, the range of g must be a subset of the Domain of $f$and the range must be a subset of the Domain of g.

Domain of$f : [2, \infty]$; Range of $f : (-\infty, a - 2]$

Domain of $g : (-\infty, 1]$; range of $g : [a, \infty)$

So $a \geq 2$ from $f \circ g$

& $a - 2 \leq 1$ from $g \circ f$

$\therefore \quad 2 \leq a \leq 3$

Solutions to Exercise 1F

1 **a** Let $y = f^{-1}(x)$ then

$$x = 2y + 3$$
$$y = \frac{x-3}{2}$$
$$f^{-1}(x) = \frac{x-3}{2}$$

b Let $y = f^{-1}(x)$ then

$$x = 4 - 3y$$
$$y = \frac{4-x}{3}$$
$$f^{-1}(x) = \frac{4-x}{3}$$

c Let $y = f^{-1}(x)$ then

$$x = 4y + 3$$
$$y = \frac{x-3}{4}$$
$$f^{-1}(x) = \frac{x-3}{4}$$

2 **a** Let $y = f^{-1}(x)$ then

$$x = y - 4$$
$$f^{-1}(x) = y = x + 4$$

b Let $y = f^{-1}(x)$ then

$$x = 2y$$
$$f^{-1}(x) = y = \frac{x}{2}$$

c Let $y = f^{-1}(x)$ then

$$x = \frac{3}{4}y$$
$$f^{-1}(x) = y = \frac{4}{3}x$$

d Let $y = f^{-1}(x)$ then

$$x = \frac{3y-2}{4}$$
$$3y = 4x + 2$$
$$f^{-1}(x) = y = \frac{4x+2}{3}$$

3 **a** Let $y = f^{-1}(x)$ then

$$x = 2y - 4$$
$$f^{-1}(x) = y = \frac{x+4}{2}$$

Domain (f^{-1}) = range $(f) = [-8, 8]$

$\therefore$ $f^{-1} : [-8, 8] \to R, f^{-1}(x) = \dfrac{x+4}{2}$

range (f^{-1}) = Domain $(f) = [-2, 6]$

b let $g^{-1}(x) = y$ then

$$x = \frac{1}{9-y}$$
$$9 - y = \frac{1}{x}$$
$$g^{-1}(x) = y = 9 - \frac{1}{x}$$

Domain (g^{-1}) = range $(g) = \mathbb{R}^-$

$\therefore$ $g^{-1} : R^- \to \mathbb{R}, g^{-1}(x) = 9 - \dfrac{1}{x}$

range (g^{-1}) = Domain $(g) = (9, \infty)$

c Let $h^{-1}(x) = y$. Then

$$x = y^2 + 2$$
$$y^2 = x - 2$$
$$y = \pm\sqrt{x-2}$$

but range (h^{-1}) = Domain (h)

$= \mathbb{R}^+ \cup (0\}$

$\therefore$ $h^{-1}(x) = y = \sqrt{x-2}$

Domain (h^{-1}) = range $(h) = [2, \infty)$

$\therefore$ $h^{-1}[2, \infty) \to \mathbb{R}, h^{-1}(x) = \sqrt{x-2}$

range $(h^{-1}) = [0, \infty)$

d Let $f^{-1}(x) = y$. Then

$$x = 5y - 2$$

$$f^{-1}(x) = y = \frac{x+2}{5}$$

Domain $(f^{-1}) =$ range $(f) =$ $[-17, 28]$

$\therefore$ $f^{-1}[-17, 28] \to R, f^{-1}(x) = \dfrac{x+2}{5}$

range $(f^{-1}) =$ Domain $(f) = [-3, 6]$

e Let $g^{-1}(x) = y$. Then $x = y^2 - 1$

$$y^2 = x + 1$$

$$y = \pm\sqrt{x+1}$$

but range $(g^{-1}) =$ Domain $(g) =$ $(1, \infty)$

$\therefore$ $g^{-1}(x) = \sqrt{x+1}$

Domain $(g^{-1}) =$ range $(g) = (0, \infty)$

$\therefore$ $g^{-1}(0, \infty) \to R, g^{-1}(x) = \sqrt{x+1}$

range$(g^{-1}) = (1, \infty)$

f Let $h^{-1}(x) = y$. Then $x = \sqrt{y}$

$$h^{-1}(x) = y = x^2$$

Domain $(h^{-1}) =$ range$(h) = R^+$

$\therefore$ $h^{-1}\colon R^+ \to R, h^{-1}(x) = x^2$

range $(h^{-1}) =$ Domain $(h) = R^+$

4 **a** Interchange x and y

$x = y^2 + 2y$

Completing the square:

$(y+1)^2 - x - 1 = 0$

$$y + 1 = \pm\sqrt{1+x}$$

$$y = -1 \pm \sqrt{1+x}$$

but range (g^{-1})=Domain $(g) =$ $[-1, \infty)$

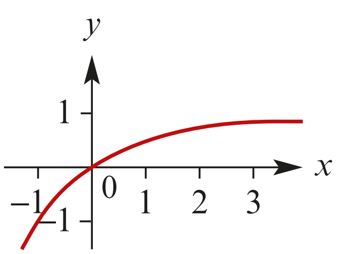
$\therefore$ $g^{-1}(x) = y = \sqrt{1+x} - 1$

Domain (g^{-1}) =range $(g) = [-1, \infty)$

$g^{-1}[-1, \infty) \to \mathbb{R}, g^{-1}(x) = \sqrt{1+x} - 1$

range $(g^{-1}) = [-1, \infty)$

b



5 Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{1}{x} - 3$

Let $y = f^{-1}(x)$. Then we can write

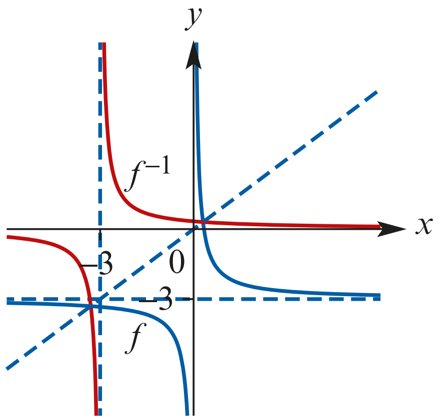
$x = \dfrac{1}{y} - 3$

Hence $y = \dfrac{1}{x+3}$.

That is $f^{-1}(x) = \dfrac{1}{x+3}$.

The Domain of f^{-1} is $\mathbb{R}\backslash\{-3\}$

$f^{-1}\colon \mathbb{R} \setminus \{-3\} \to \mathbb{R}, f^{-1}(x) = \dfrac{1}{x+3}$



6 **a** to find f^{-1} (2), use $f(x) = 2$

$$2 = 3 - 2x$$

$$f^{-1}(2) = x = \frac{1}{2}$$

Domain $f^{-1} =$ range$(f) = [-3, 3]$

7 **a** Let $f^{-1}(x) = y$

$$x = 2y$$

$$f^{-1}(x) = y = \frac{x}{2}$$

Domain $f^{-1} =$ range$(f) = [-2, 6]$

range f^{-1} = Domain(f) = $[-1, 3]$

∴ $f^{-1}[-2, 6] \to R, f^{-1}(x) = \frac{x}{2}$

b Let $f^{-1}(x) = y$

$$x = 2y^2 - 4$$

$$y^2 = \frac{(x+4)}{2}$$

$$y = \pm\sqrt{\frac{(x+4)}{2}}$$

but range f^{-1} = Domain(f) = $[-0, \infty)$

∴ $f^{-1}(x) = y = \sqrt{\frac{(x+4)}{2}}$

Domain f^{-1} = range(f) = $[-4, \infty)$

∴ $f^{-1}[-4, \infty) \to R, f^{-1}(x) = \sqrt{\frac{(x+4)}{2}}$

range $f^{-1} = [0, \infty)$

c $\{(4, 2), (6, 1), (8, 3), (11, 5)\}$

Domain = $\{4, 6, 8, 11\}$

range = $\{1, 2, 3, 5\}$

d Let $h^{-1}(x) = y$.Then

$$x = \sqrt{-y}$$

$$h^{-1}(x) = y = -x^2$$

Domain h^{-1} = range(h) = $\mathbb{R}^+$

∴ $h^{-1}: \mathbb{R}^+ \to \mathbb{R}, h^{-1}(x) = -x^2$

range $(h^{-1}) = \mathbb{R}^-$

e Let $f^{-1}(x) = y$. Then

$$x = y^3 + 1$$

$$f^{-1}(x) = y = (x-1)^{\frac{1}{3}}$$

Domain (f^{-1}) = range(f) = $\mathbb{R}$

$f^{-1} : R \to \mathbb{R}, f^{-1}(x) = (x-1)^3$

range $(f^{-1}) = \mathbb{R}$

f Let $g^{-1}(x) = y$. Then

$$x = (y+1)^2$$

$$y = \pm\sqrt{x} - 1$$

but range (g^{-1}) = Domain(g)

= $(-1, 3)$

∴ $g^{-1}(x) = y = \sqrt{x} - 1$

Domain g^{-1} = range(g) = $(0, 16)$

∴ $g^{-1} : (0, 16) \to \mathbb{R}, g^{-1}(x) = \sqrt{x} - 1$

range $(g^{-1}) = (-1, 3)$

g Let $g^{-1}(x) = y$. Then

$$x = \sqrt{y-1}$$

$$g^{-1}(x) = y = x^2 + 1$$

Domain g^{-1} = range(g) = $[0, \infty)$

∴ $g^{-1} : [0, \infty) \to \mathbb{R}, g^{-1}(x) = x^2 + 1$

range $g^{-1} = [1, \infty)$

h Let $h^{-1}(x) = y$. Then

$$x = \sqrt{4 - y^2}$$

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2}$$

but range (h^{-1}) = Domain (h) = $[0, 2]$

∴ $h^{-1}(x) = y = \sqrt{4 - x^2}$

Domain (h^{-1}) = range (h) = $[0, 2]$

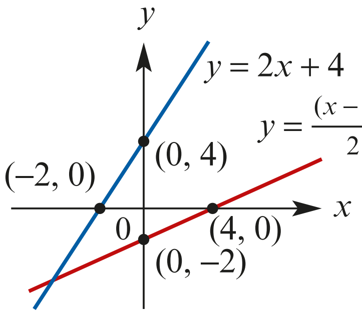
∴ h^{-1}: $[0, 2] \to \mathbb{R}, h^{-1}(x) = \sqrt{4 - x^2}$

range $(h^{-1}) = [0, 2]$

8 **a** $x = 2y + 4$

$$y = \frac{x-4}{2}$$

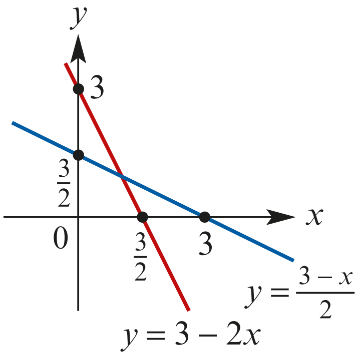
implied Domain: $\mathbb{R}$ and range: $\mathbb{R}$



b $$x = \frac{3 - f^{-1}(x)}{2}$$

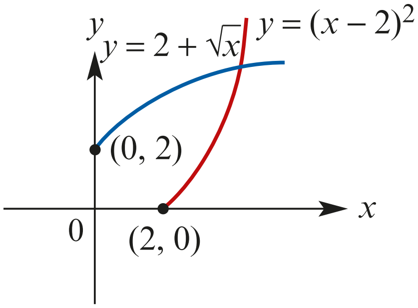
$f^{-1}(x) = 3 - 2x$

implied Domain: $\mathbb{R}$
and range: $\mathbb{R}$



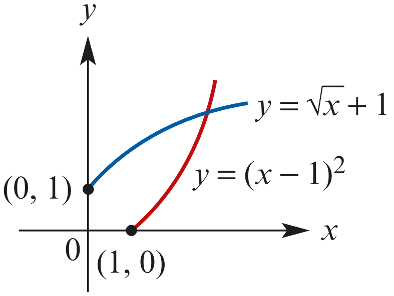
c $$x = (f^{-1}(x) - 2)^2$$

$\pm\sqrt{x} + 2 = f^{-1}(x)$
but range (f^{-1}) =dom$(f) = [2, \infty)$
$\therefore f^{-1}(x) = \sqrt{x} + 2$
Domain: $[0, \infty)$
range: $[2, \infty)$



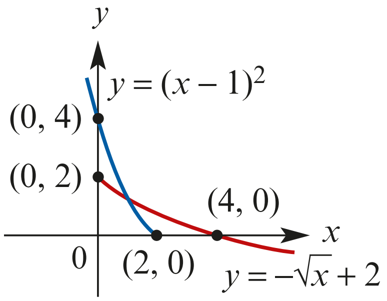
d $$x = (f^{-1}(x) - 1)^2$$

$f^{-1}(x) = \sqrt{x} + 1$
Domain: $[0, \infty)$
range: $[1, \infty)$

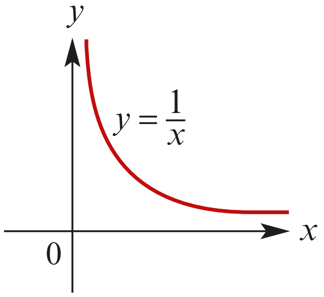


e similar to (c)
but $f^{-1}(x) = -\sqrt{x} + 2$
Domain: $[0, \infty)$

range: $(-\infty, 2]$

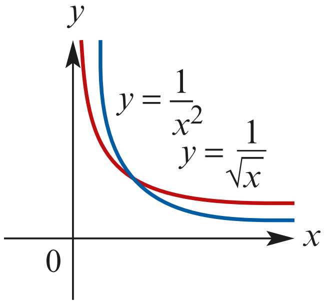


f $f^{-1}(x) = \frac{1}{x}$
Domain: R^+
range : R^+



g $$x = \frac{1}{(f^{-1}(x))^2}$$

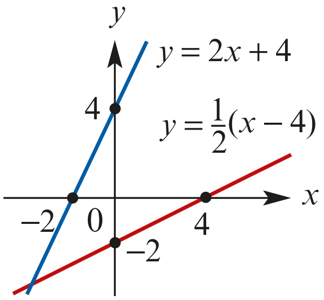
$f^{-1}(x) = \pm\frac{1}{\sqrt{x}}$
but range $f^{-1}(x)$ =Domain$(f) = \mathbb{R}^+$
$\therefore f^{-1}(x) = \frac{1}{\sqrt{x}}$
Domain: $\mathbb{R}^+$
range: $\mathbb{R}^+$



h $$x = \frac{1}{2}(h^{-1}(x) - 4)$$

$h^{-1}(x) = 2x + 4$

implied Domain: $\mathbb{R}$
and range: $\mathbb{R}$



9 **a** $x = \sqrt{f^{-1}(x)} + 2$

$(x - 2)^2 = f^{-1}(x)$

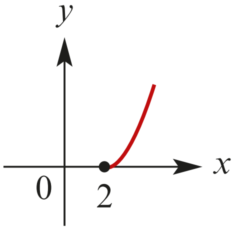
$f^{-1}(x) = x^2 - 4x + 4$

$f^{-1}(x) = (x - 2)^2$

Therefore,

$f^{-1}: [2, \infty) \rightarrow \mathbb{R}$,

$f^{-1}(x) = (x - 2)^2$



b $x = \dfrac{1}{f^{-1}(x) - 3}$

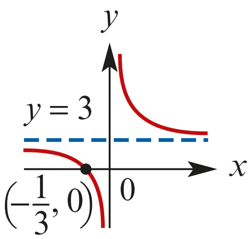
$f^{-1}(x) - 3 = \dfrac{1}{x}$

$f^{-1}(x) = \dfrac{1}{x} + 3$

Therefore,

$f^{-1}: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$,

$f^{-1}(x) = \dfrac{1}{x} + 3$



c $x = \sqrt{f^{-1}(x) - 2} + 4$

$f^{-1}(x) - 2 = (x - 4)^2$

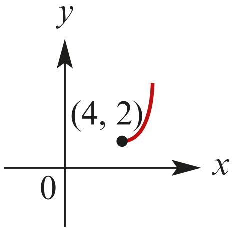
$f^{-1}(x) = x^2 - 8x + 18$

$f^{-1}(x) = (x - 4)^2 + 2$

Therefore,

$f^{-1}: [4, \infty) \rightarrow \mathbb{R}$,

$f^{-1}(x) = (x - 4)^2 + 2$



d $x = \dfrac{3}{f^{-1}(x) - 2} + 1$

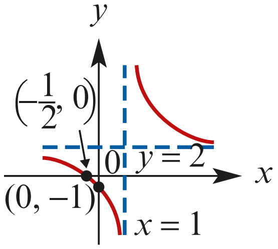
$f^{-1}(x) - 2 = \dfrac{3}{x - 1}$

$f^{-1}(x) = \dfrac{3}{x - 1} + 2$

Therefore,

$f^{-1}: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$,

$f^{-1}(x) = \dfrac{3}{x - 1} + 2$



e $x = \dfrac{5}{f^{-1}(x) - 1} - 1$

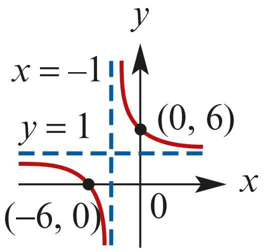
$f^{-1}(x) - 1 = \dfrac{5}{x + 1}$

$f^{-1}(x) = \dfrac{5}{x + 1} + 1$

Therefore,

$f^{-1}: \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}$,

$f^{-1}(x) = \dfrac{5}{x + 1} + 1$



f

$$x = \sqrt{2 - f^{-1}(x)} + 1$$

$$(x-1)^2 = 2 - f^{-1}(x)$$

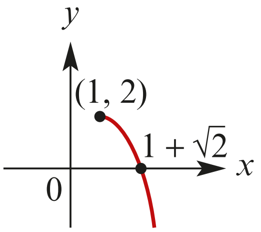
$$f^{-1}(x) = 2 - (x-1)^2$$

$$f^{-1}(x) = -x^2 + 2x + 1$$

Therefore,

$f^{-1}\colon [1, \infty) \to \mathbb{R}$,

$f^{-1}(x) = 2 - (x-1)^2$



10 a

$$f(x) = 1 + \frac{2}{x-1}$$

$$x = 1 + \frac{2}{f^{-1}(x) - 1}$$

$$x - 1 = \frac{2}{f^{-1}(x) - 1}$$

$$f^{-1}(x) - 1 = \frac{2}{x-1}$$

$$f^{-1}(x) = 1 + \frac{2}{x-1}$$

$$f^{-1}(x) = \frac{x+1}{x-1}$$

b $f(x) = \sqrt{x-2}$

$$x = \sqrt{f^{-1}(x) - 2}$$

$$x^2 = f^{-1}(x) - 2$$

$$f^{-1}(x) = x^2 + 2$$

c

$$f(x) = \frac{2x+3}{3x-2}$$

$$= \frac{\frac{2}{3}(3x-2) + \frac{4}{3} + 3}{3x-2}$$

$$= \frac{2}{3} + \frac{\frac{13}{3}}{3x-2}$$

$$= \frac{2}{3} + \frac{13}{9x-6}$$

$$x = \frac{2}{3} + \frac{13}{9f^{-1}(x) - 6}$$

$$x - \frac{2}{3} = \frac{13}{9f^{-1}(x) - 6}$$

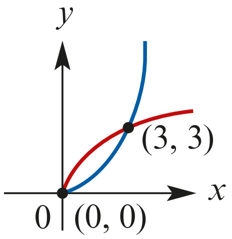
$$9f^{-1}(x) - 6 = \frac{13}{x - \frac{2}{3}}$$

$$3f^{-1}(x) - 2 = \frac{13}{3x-2}$$

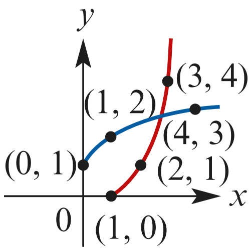
$$3f^{-1}(x) = \frac{13 + 6x - 4}{3x-2}$$

$$f^{-1}(x) = \frac{2x+3}{3x-2}$$

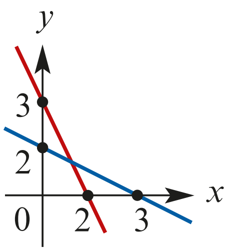
11 a



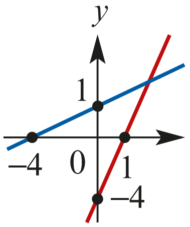
b



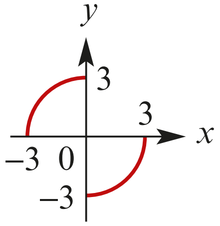
c



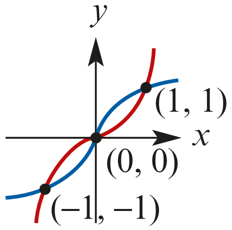
d



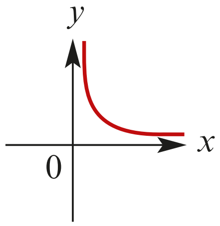
e



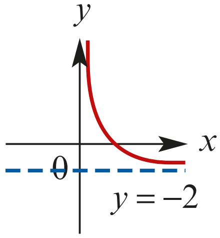
f



g



h



12 **a** C

b B

c D

d A

13 **a** $3 - x \geq 0$

$x \leq 3$

$\therefore \quad A = (-\infty, 3]$

b minimum b is at the turning point

i.e. $b = 0$

let $g^{-1}(x) = y$

$x = 1 - y^2$

$y = \pm\sqrt{1 - x}$

,but range (g^{-1}) =Domain $(g) = [0, 2]$

$\therefore y = \sqrt{1 - x}$

Domain (g^{-1}) =range$(g) = [-3, 1]$

$\therefore \; g^{-1} : [-3, 1] \to R, g^{-1}(x) = \sqrt{1 - x}$

14 $b = -2, \quad g^{-1}(x) = -2 + \sqrt{x + 4}$

15 $a = 3, \quad f^{-1}(x) = 3 - \sqrt{x + 9}$

16 **a**

$$x = \frac{3}{g^{-1}(x)}$$

$$g^{-1}(x) = \frac{3}{x}$$

Domain = $\mathbb{R}\backslash\{0\}$

b

$$x = \sqrt[3]{g^{-1}(x) + 2} - 4$$

$$(x + 4)^3 = g^{-1}(x) + 2$$

$$g^{-1}(x) = (x + 4)^3 - 2$$

Domain = $\mathbb{R}$

$$x = 2 - \sqrt{h^{-1}(x)}$$

c $$\sqrt{h^{-1}(x)} = 2 - x$$

$$h^{-1}(x) = (x-2)^2$$

Domain (h^{-1}) = range $(h) = (-\infty,\ 2]$

$$x = \frac{3}{f^{-1}(x)} + 1$$

d $$f^{-1}(x) = \frac{3}{x-1}$$

Domain = $\mathbb{R}\backslash\{1\}$

e $$x = 5 - \frac{2}{(h^{-1}(x) - 6)^3}$$

$$\frac{2}{5-x} = (h^{-1}(x) - 6)^3$$

$$h^{-1}(x) = \sqrt[3]{\frac{2}{5-x}} + 6$$

Domain = $\mathbb{R}\backslash\{5\}$

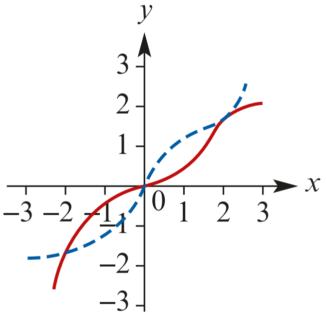
f $$x = \frac{1}{(g^{-1}(x) - 1)^{\frac{3}{4}}} + 2$$

$$(g^{-1}(x) - 1)^{3/4} = \frac{1}{x-2}$$

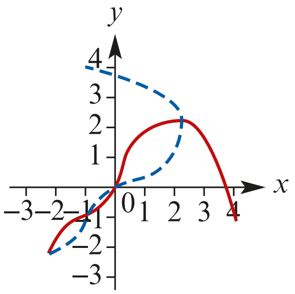
$$g^{-1}(x) = \frac{1}{(x-2)^{\frac{4}{3}}} + 1$$

Domain = $(2, \infty)$

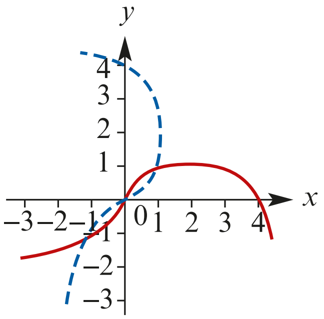
17 **a** Inverse is a function



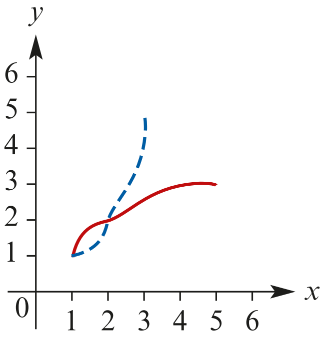
b Inverse is not a function



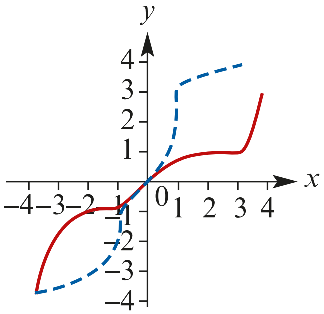
c Inverse is not a function



d Inverse is a function



e Inverse is not a function



18 a $$f(x) = \frac{x+3}{2x-1}$$

$$\text{Domain} = \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$$

$$f(x) = \frac{\frac{1}{2}(2x-1) + \frac{7}{2}}{2x-1}$$

$$= \frac{1}{2} + \frac{7}{2(2x-1)}$$

$$\text{range} = \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$$

Since range(f) =Domain(f)

$f \circ f$ is defined.

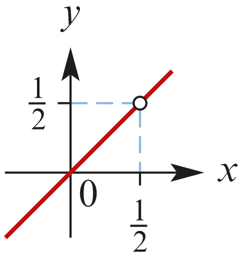
b $$f \circ f(x) = \frac{1}{2} + \frac{\frac{7}{2}}{2\frac{(2x+3)}{(2x-1)} - 1}$$

$$= \frac{1}{2} + \frac{7}{2(\frac{2x+6-2x+1}{2x-1})}$$

$$= \frac{1}{2} + \frac{7(2x-1)}{14}$$

$$= \frac{1}{2} + x - \frac{1}{2}$$

$$f \circ f(x) = x, \ x \in \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$$



c Since $f \circ f(x) = x$ and $f^{-1} \circ f(x) = x$

$$f^{-1} = f = \frac{x+3}{2x-1}, \ x \in \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$$

Solutions to Exercise 1G

1 a Maximal Domain = $\mathbb{R} \setminus \{0\}$;
Range = $\mathbb{R}^+$

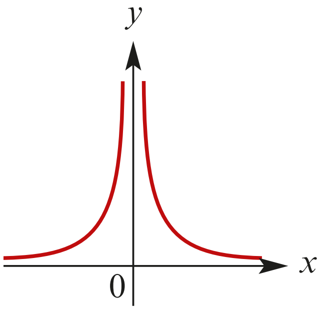
b **i** $\frac{1}{16}$

ii $\frac{1}{16}$

iii 16

iv 16

c



2 a Odd

b Even

c Odd

d Odd

e Even

f Odd

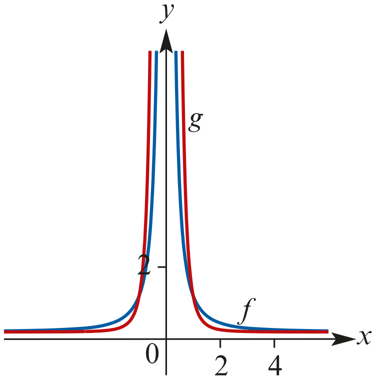
3 a $f(x) = g(x)$

$$x^{-2} = x^{-4}$$

$$x^2 = 1$$

$x = 1$ or $x = -1$

b



4 a $f(x) = g(x)$

$x = 0$ is one solution.

Now assume$x \neq 0$

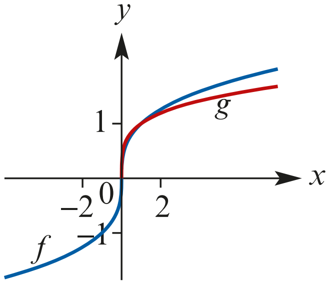
$$x^{\frac{1}{3}} = x^{\frac{1}{4}}$$

$$x^{\frac{1}{3}-\frac{1}{4}} = 1$$

$$x^{\frac{1}{12}} = 1$$

$$\therefore x = 1 \text{ or } x = 0$$

b



5 a $x = \left(f^{-1}(x)\right)^7$

$f^{-1}(x) = x^{\frac{1}{7}}$

Domain of f^{-1} = range of $f = \mathbb{R}$

$f^{-1} : \mathbb{R} \to \mathbb{R},\ f^{-1}(x) = x^{\frac{1}{7}}$

b $x = \left(f^{-1}(x)\right)^6$

$f^{-1}(x) = x^{\frac{1}{6}}$

Domain of f^{-1} = range of $f = [0, \infty)$

$f^{-1}: [0, \infty) \to \mathbb{R}, f^{-1}(x) = -x^{\frac{1}{6}}$

c
$$x = 27\left(f^{-1}(x)\right)^3$$
$$\frac{x}{27} = \left(f^{-1}(x)\right)^3$$
$$f^{-1}(x) = \left(\frac{x}{27}\right)^{\frac{1}{3}} = \frac{1}{3}x^{\frac{1}{3}}$$

Domain of f^{-1} = range of $f = \mathbb{R}$

$f^{-1}: \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \frac{1}{3}x^{\frac{1}{3}}$

d
$$x = 16\left(f^{-1}(x)\right)^4$$
$$\frac{x}{16} = \left(f^{-1}(x)\right)^4$$
$$f^{-1}(x) = \left(\frac{x}{16}\right)^{\frac{1}{4}} = \frac{1}{2}x^{\frac{1}{4}}$$

Domain of f^{-1} = range of $f = (16, \infty)$

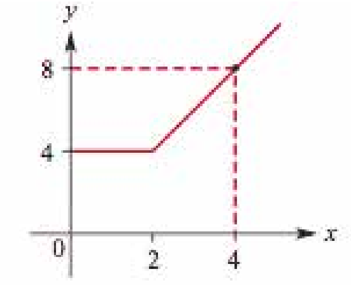
$f^{-1}: (16, \infty) \to \mathbb{R}, f^{-1}(x) = \frac{1}{2}x^{\frac{1}{4}}$

Solutions to Exercise 1H

1 For $0 \leq x \leq 2$, the cost is \$ 4. For $x > 2$, the cost is \$4 + \$2 for each extra km over 2 km, i.e \$2(x – 2). Hence:

$$f(x) = \begin{cases} 4 & \text{if } 0 \leq x \leq 2 \\ 4 + 2(x - 2) & \text{if } x > 2 \end{cases}$$

$$= \begin{cases} 4 & \text{if } 0 \leq x \leq 2 \\ 2x & \text{if } x > 2 \end{cases}$$



2 The box has length $(36 - 2x)$ cm, width $(20 - 2x)$ cm and height x cm. So the volume V cm^3 is given by

$$V = x(20 - 2x)(36 - 2x)$$
$$= 4x(10 - x)(18 - x)$$

where $x > 0$ and $x < 10$ for a box to exist.
The Domain is $[0, 10]$.

3 a Perimeter $= 2x + 2y = 160$, so $y = 80 - x$. The area can be found by subtracting a rectangle of dimensions 12 by $(y - 20)$ from a rectangle of dimensions x by y:

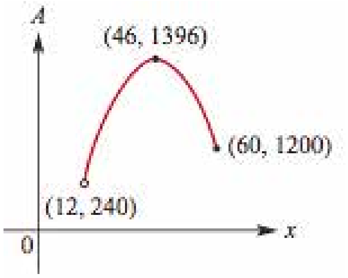
$$A = xy - 12(y - 20)$$
$$= x(80 - x) - 12(60 - x)$$
$$= -x^2 + 80x + 12x - 720$$
$$= -x^2 + 92x - 720$$

b $x > 12$; also $y > 20$ implies $80 - x > 20$ so that $x < 60$.
The Domain is $[12, 60]$.

c The function is a quadratic with (non-included) endpoints where $x = 12, 60$. When $x = 12, A = 240$; when $x = 60, A = 1200$.
Endpoints are $(12, 240)$ and $(60, 1200)$.
There is a turning point where

$$x = -\frac{b}{2a}$$
$$= -\frac{92}{-2} = 46$$

Then $A = 1396$.
The graph is shown here.



d The maximum area is 1396 m^2 and it occurs for $x = 46$ and $y = 80 - 46 = 34$.

4 a i $S = 2x^2 + 2 \leftrightarrow 2x \leftrightarrow h + 2 \leftrightarrow x \leftrightarrow h$
$= 2x^2 + 6xh$

ii $V = 2x^2h$ where $h = \dfrac{V}{2x^2}$

$$S = 2x^2 + 6x \leftrightarrow \frac{V}{2x^2}$$
$$= 2x^2 + \frac{3V}{x}$$

b $x > 0$, so maximal Domain is $(0, \infty)$.

c $V = 1000$ so $S = 2x^2 + \dfrac{3000}{x}$.
A sketch using a CAS calculator shows that there is an endpoint maximum where $x = 2$. Then $S = 1508$ m^2.

5 Let x be the with of the rectangle and y be the length of the rectangle.
The diagonal has length $2a$.
$\therefore x^2 + y^2 = 4a^2$
$\therefore y^2 = 4a^2 - x^2$
$\therefore y = \sqrt{4a^2 - x^2}$
$\therefore$ Area $= xy = x(\sqrt{4a^2 - x^2})$
The Domain is clearly [0,2a].

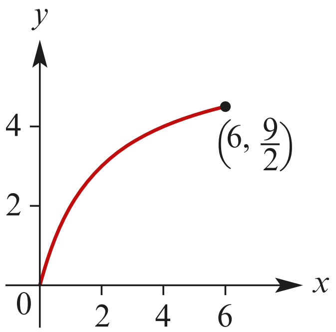
6 The coordinates of C are $\left(a, \dfrac{6}{a+2}\right)$

a Area $= a \times \dfrac{6}{a+2} = \dfrac{6a}{a+2}$

b Domain $= [0, 6]$; Range $= \left[0, \dfrac{9}{2}\right]$

c Maximum value$= \left[0, \dfrac{9}{2}\right]$

d



7 **a** Distance is speed by time, so during the first 45 minutes, the man runs a distance of $\dfrac{2}{60}t = \dfrac{1}{30}t$ km; after 45 minutes, he has run $\dfrac{3}{2}$ km and thereafter adds a distance of $\dfrac{4}{60}t = \dfrac{1}{15}t$ during the next 30 minutes. Hence:

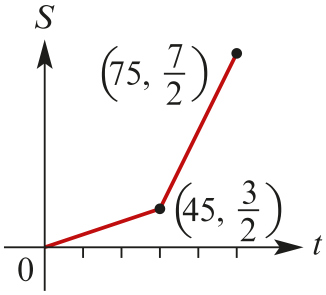
$$S(t) = \begin{cases} \dfrac{1}{30}t & \text{if } 0 \le t \le 45 \\ \dfrac{3}{2} + \dfrac{1}{15}(t - 45) & \text{if } 45 < t \le 75 \end{cases}$$

$$= \begin{cases} \dfrac{1}{30}t & \text{if } 0 \le t \le 45 \\ \dfrac{1}{15}t - \dfrac{3}{2} & \text{if } 45 < t \le 75 \end{cases}$$

$a = \dfrac{1}{30}, \quad b = \dfrac{1}{15}, \quad c = 45,$

$d = -\dfrac{3}{2}, \quad e = 75$

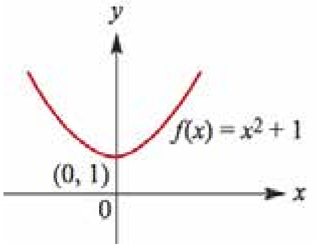
b The graph comprises two line segments as shown here.



c The range is $\left[0, \dfrac{7}{2}\right]$.

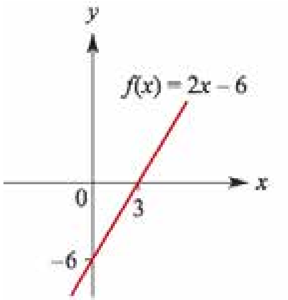
Solutions to technology-free questions

1 a



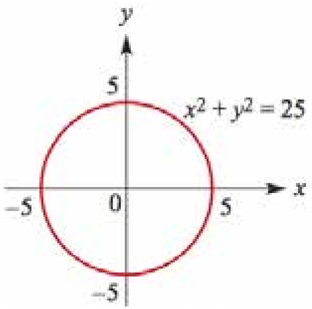
Domain = $\mathbb{R}$, range = $[1, \infty)$

b



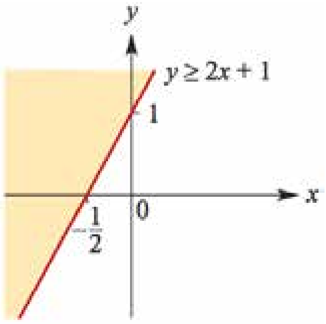
Domain = $\mathbb{R}$, range = $\mathbb{R}$

c



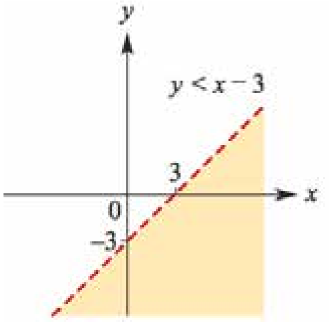
Domain = $[-5, 5]$, range = $[-5, 5]$

d



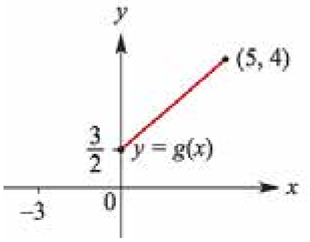
Domain = $\mathbb{R}$, range = $\mathbb{R}$

e



Domain = $\mathbb{R}$, range = $\mathbb{R}$

2 a



b range = $[1.5, 4]$

c Interchange x and y and solve for y:

$$x = \frac{y+3}{2}$$

$$y + 3 = 2x$$

$$y = 2x - 3$$

$$g^{-1} : [1.5, 4] \to \mathbb{R}, g^{-1}(x) = 2x - 3$$

Domain = $[1.5, 4]$, range = $[0, 5]$

d

$$g(x) = 4$$

$$\frac{x+3}{2} = 4$$

$$x + 3 = 8$$

$$x = 5$$

$$\{x : g(x) = 4\} = \{5\}$$

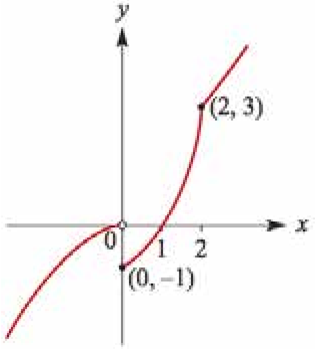
e If $g^{-1}(x) = 4$, then $x = g(4) = 3.5$.
$\{x : g^{-1}(x) = 4\} = \{3.5\}$
(Alternatively, solve the equation $2x - 3 = 4$ for x.)

3 **a**
$$5x + 1 = 2$$
$$5x = 1$$
$$x = \frac{1}{5}$$
$$\{x : g(x) = 2\} = \left\{\frac{1}{5}\right\}$$

b If $g^{-1}(x) = 2$, then $x = g(2) = 11$.
$\{x : g(x) = 2\} = \{11\}$

c
$$\frac{1}{5x + 1} = 2$$
$$5x + 1 = \frac{1}{2}$$
$$5x = -\frac{1}{2}$$
$$x = -\frac{1}{10}$$
$$\left\{x : \frac{1}{g(x)} = 2\right\} = \left\{-\frac{1}{10}\right\}$$

4



5 **a** $2x - 6 \neq 0$, so $x \neq 3$
Domain = $R \backslash \{3\}$

b
$$x^2 - 5 > 0$$
$$\left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right) > 0$$
$$x < -\sqrt{5} \text{ or } x > \sqrt{5}$$
Domain = $R \backslash \left[-\sqrt{5}, \sqrt{5}\right]$

c $(x - 1)(x + 2) \neq 0$, so $x \neq 1, -2$
Domain = $\mathbb{R} \backslash \{1, -2\}$

d
$$25 - x^2 \geq 0$$
$$(5 - x)(5 + x) \geq 0$$
$$-5 \leq x \leq 5$$
Domain = $[-5, 5]$

e $x - 5 \geq 0$ and $15 - x \geq 0$
$5 \leq x \leq 15$
Domain = $[5, 15]$

f $3x - 6 \neq 0$, so $x \neq 2$
Domain = $\mathbb{R} \backslash \{2\}$

6
$$\begin{aligned}(f + g)(x) &= (x + 2)^2 + x - 3 \\ &= x^2 + 4x + 4 + x - 3 \\ &= x^2 + 5x + 1\end{aligned}$$
$$(fg)(x) = (x - 3)(x + 2)^2$$

7
$$\begin{aligned}(f + g)(x) &= (x - 1)^2 + 2x \\ &= x^2 + 1\end{aligned}$$
$$(f + g) : [1, 5] \to \mathbb{R}, (f + g)(x) = x^2 + 1$$
$$(fg)(x) = 2x(x - 1)^2$$
$$(fg) : [1, 5] \to \mathbb{R}, (fg)(x) = 2x(x - 1)^2$$

8 $f(3) = 8$, so range of f is $[8, \infty)$ (the graph of $y = f(x)$ is increasing for $x \geq 3$).
Hence Domain of f^{-1} is $[8, \infty)$ and the range is $[3, \infty)$.

Interchange x and y and solve for y:

$$x = y^2 - 1$$

$$y^2 = x + 1$$

$$y = \sqrt{x+1} \text{ (as } y > 0)$$

$$f^{-1} : [8, \infty) \to \mathbb{R}, f^{-1}(x) = \sqrt{x+1}$$

9 **a** $(f + g)(x) = -x^2 + 2x + 3$

b $(fg)(x) = -x^2(2x + 3)$

c

$$(f + g)(x) = 0$$

$$\{x\colon (f + g)(x) = 0\}$$

$$= \{-1, 3\}$$

$$-x^2 + 2x + 3 = 0$$

$$-(x^2 - 2x - 3) = 0$$

$$-(x + 1)(x - 3) = 0$$

$$x = -1, 3$$

10 $f(2) = 2$, so range of f is $(-\infty, 2]$ (the graph of $y = f(x)$ is a straight line with endpoint at (2, 2)).

Interchange x and y and solve for y:

$$x = 3y - 4$$

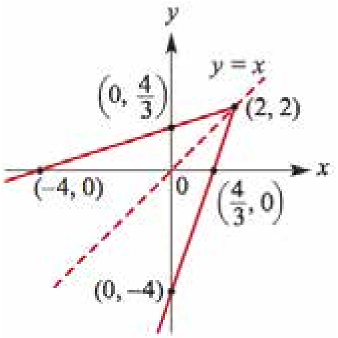
$$3y = x + 4$$

$$y = \frac{x+4}{3}$$

$$f^{-1} : (-\infty, 2] \to R, f^{-1}(x) = \frac{x+4}{3}$$

The graphs are straight lines, reflections of each other in the line $y = x$, each with endpoint $(2, 2)$.

The graph of $y = f(x)$ has axes intercepts $\left(\frac{4}{3}, 0\right)$, $(0, -4)$. The graph of $y = f^{-1}(x)$ has axes intercepts $(-4, 0)$, $\left(0, \frac{4}{3}\right)$.



11 **a**

$$x = 8\left(f^{-1}(x)\right)^3$$

$$\frac{x}{8} = \left(f^{-1}(x)\right)^3$$

$$f^{-1}(x) = \left(\frac{x}{8}\right)^{\frac{1}{3}} = \frac{1}{2}x^{\frac{1}{3}}$$

Domain of f^{-1} = range of $f = \mathbb{R}$

b

$$x = 32\left(f^{-1}(x)\right)^5$$

$$\frac{x}{32} = \left(f^{-1}(x)\right)^5$$

$$f^{-1}(x) = \left(\frac{x}{32}\right)^{\frac{1}{5}} = \frac{1}{2}x^{\frac{1}{5}}$$

Domain of f^{-1} = range of $f = (-\infty, 0]$

c

$$x = 64\left(f^{-1}(x)\right)^6$$

$$\frac{x}{64} = \left(f^{-1}(x)\right)^6$$

$$f^{-1}(x) = \left(\frac{x}{64}\right)^{\frac{1}{6}} = \frac{1}{2}x^{\frac{1}{6}}$$

Domain of f^{-1} = range of $f = [0, \infty]$

d

$$x = 10\,000\left(f^{-1}(x)\right)^4$$

$$\frac{x}{10\,000} = \left(f^{-1}(x)\right)^4$$

$$f^{-1}(x) = \left(\frac{x}{10\,000}\right)^{\frac{1}{4}} = \frac{1}{10}x^{\frac{1}{4}}$$

Domain of f^{-1} = range of $f = (10\,000, \infty)$

12 **a** $f \circ g(x) = f(-x^3)$

$= -2x^3 + 3$

b $g \circ f(x) = g(2x + 3)$

$= -(2x + 3)^3$

c $g \circ g(x) = g(-x^3)\text{i}$

$= (-x^3)^3$

$= -x^9$

d $f \circ f(x) = f(2x + 3)$

$= 2(2x + 3) + 3$

$= 4x + 9$

e $f \circ (f + g)(x) = f(f + g(x))$

$= f(-x^3 + 2x + 3)$

$= 2(-x^3 + 2x + 3) + 3$

$= -2x^3 + 4x + 9$

f $f \circ (f - g)(x) = f(f - g(x))$

$= f(2x + 3 + x^3)$

$= 2(2x + 3 + x^3) + 3$

$= 2x^3 + 4x + 9$

g $f \circ (f \cdot g)(x) = f(f \cdot g(x))$

$= f(-2x^4 - 3x^3)$

$= 2(-2x^4 - 3x^3) + 3$

$= -4x^4 - 6x^3 + 3$

13 $x \geq -1$ or $x \leq -9$

14 $h^{-1}(x) = \left(\dfrac{x - 64}{2}\right)^{\frac{1}{5}}$

Solutions to multiple-choice questions

1 E $6 - 2x \geq 0$

$6 \geq 2x$

$3 \geq x$

$\therefore\ (-\infty, 3]$

2 B $f : [-1, 3) \to R, f(x) = -x^2$

$f(3) = -9$; maximum 0 at $x = 0$

$\therefore (-9, 0]$.

3 E $f(x) = 3x^2 + 2x$

$f(2a) = 3(2a)^2 + 2(2a)$

$f(2a) = 12a^2 + 4a$

4 C $f(x) = 2x - 3$

letf(x) $= 2(f^{-1}(x)) - 3$

$f(x) + 3 = 2(f^{-1}(x))$

$f^{-1}(x) = \dfrac{f(x) + 3}{2}$

$f^{-1}(x) = \dfrac{x + 3}{2}$

$f^{-1}(x) = \dfrac{1}{2}x + \dfrac{3}{2}$

5 E $f : (a, b] \to R, f(x) = 10 - x, a < b$

The minimum is:

$f(b) = 10 - b$

The maximum is:

$f(a) = 10 - \text{a}$

$\therefore\ [10 - b, 10 - a)$

6 C As a is a negative real number:

$f(a + 3) = -(a + 3) + 6$

$f(a + 3) = -a + 3$

7 D $f(x) = (x + 3)^2 - 6$ Graph must be one to one to have an inverse function.

Turning point of function is at $(-3, -6)$

Domain must be a sub set of either: $(-\infty, -3]$ or $[-3, \infty)$

$\therefore [6, \infty)$

8 B An inverse only exists if the function is one to one.

$g : [-4, 4] \to R, g(x) = \sqrt{16 - x^2}$

Is not one to one for the specified Domain.

9 B The asymptote is at $x = -2$ therefore the asymptote of the inverse is at $y = -2$.

10 C $f(x) = \dfrac{2x + 1}{x - 1} = 2 + \dfrac{3}{x - 1}$

Therefore asymptotes $x = 1$ and $y = 2$.

11 B $f(x) = 3x^2$ and $g(x) = 2x + 1$

$\therefore f(g(x)) = 3(2x + 1)^2$

$f(g(x)) = 12x^2 + 12x + 3$

$\therefore f(g(a)) = 12a^2 + 12a + 3$

12 E $f(x) = x^2 + 2x - 6 = (x + 1)^2 - 7$

$\therefore$ vertex has coordinates $(-1, -7)$

$f(-2) = (-2)^2 + 2(-2) - 6 =$

$4 - 4 - 6 = -6$

$f(4) = (4)^2 + 2(4) - 6 = 18$

$\therefore$ range $= [-7, 18)$

13 C If $a > b$ then $a^{\frac{1}{5}} > b^{\frac{1}{5}}$

14 C Maximal Domain

$= (-1, \infty) \cap (-\infty, 4] = (-1, 4]$

15 A Domain of f^{-1} = Range of

$f = (\sqrt{7}, \infty)$

$x = \sqrt{2f^{-1}(x) + 3}$

$\therefore f^{-1}(x) = \frac{1}{2}(x^2 - 3)$

16 B $5 - x = -2 \Rightarrow x = 7$
$5 - x = 3 \Rightarrow x = 2$ ∴ Domain of $f = (2, 7]$

17 A $g : R\{3\} \to \mathbb{R}, g(x) = \frac{1}{x-3} + 2$

Let $x = \frac{1}{g^{-1}(x) - 3} + 2$

$g^{-1}(x) - 3 = \frac{1}{x-2}$

$g^{-1}(x) = \frac{1}{x-2} + 3$

$x \neq 2$

$\therefore$ dom $g^{-1}(x) = R\backslash\{2\}$

18 C $f(g(x)) = g(f(x))$

$$\frac{6}{3x-2} = \frac{18}{x-2}$$

$$6x - 2 = 54x - 36$$

$$x = \frac{1}{2}$$

$$f(g(\frac{1}{2})) = \frac{6}{3 \times \frac{1}{2} - 2} = -12$$

$$a + b = -12 + \frac{1}{2} = -\frac{23}{2}$$

19 B

20 B Asymptotes of $y(x)$ occur at
$x + 1 = 0$
$\therefore x = -1$
And at $y = -2$
∴ Asymptotes of $y^{-1}(x)$ occur at:
$y = -1$ and $x = -2$

21 C Asymptotes of $\frac{-2}{(x+3)^4} - 5$ occur when $x + 3 = 0$
$\therefore x = -3$
And when $y = -5$

22 A $f : [0, \infty) \to R,\ f(x) = (x-2)^2\ f(x)$ does not have an inverse function as it is not a one to one function.

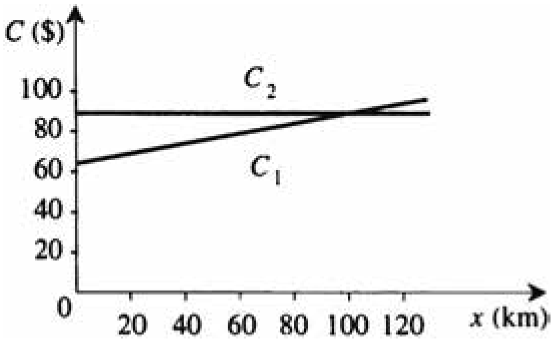
23 D Note that the graph of $y = \frac{1}{x^4}$ will be like that of $y = \frac{1}{x^2}$, but 'steeper'. Looking at the alternatives, D stands out: its Domain runs from negative to positive numbers with 0 removed. for numbers close to 0, the value of y will be very large. As $x \to 0, f(x) \to \infty$. Its range is actually $[1, \infty)$. (Checking each of the remaining alternatives shows that the range is correct in each case.)

Solutions to extended-response questions

1 **a** $C_1(x) = 0.25x + 64$

$C_2(x) = 89$

b



c From the graph or using the inequality

$0.25x + 64 > 89$

$0.25x > 25$

$x > 100$

Method 2 is cheaper than Method 1 if the distance travelled is greater than 100 km.

From this it can be seen that Method 2 is cheaper than Method 1 if the distance travelled is more than 100 km.

2 **a** Area of each face $= x^2$

$\therefore$ the total surface area, $S = 6x^2$

b The volume, $V = x^3$

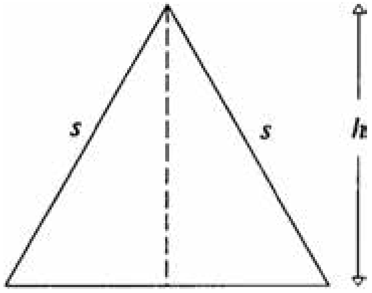
$$\therefore \quad x = V^{\frac{1}{3}}$$

and $S = 6V^{\frac{2}{3}}$

3 **a** The triangle is equilateral.

Area $A = \frac{1}{2}s^2 \sin 60°$ (Area of triangle $= \frac{1}{2}bc \sin A$)

$= \frac{\sqrt{3}}{4}s^2 \ldots \langle 1 \rangle$



b By Pythagoras' Theorem, $h^2 = s^2 - \frac{s^2}{4}$

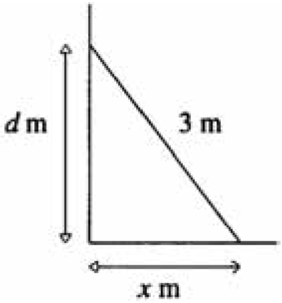
$$= \frac{3s^2}{4}$$

$$\therefore \quad h = \frac{\sqrt{3}s}{2} \quad \text{and} \quad s = \frac{2h}{\sqrt{3}}$$

by $\langle 1 \rangle A = \frac{\sqrt{3}}{4}\left(\frac{2h}{\sqrt{3}}\right)^2 = \frac{\sqrt{3}}{4} \times \frac{4h^2}{3} = \frac{\sqrt{3}h^2}{3}$

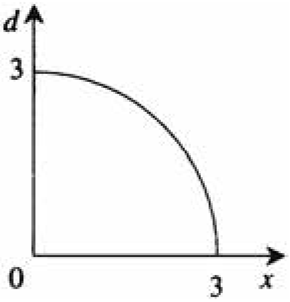
4 a By Pythagoras' Theorem $d^2 = 9 - x^2$

$$\therefore \quad d = \sqrt{9 - x^2}$$



b maximal Domain = $[0, 3]$

The range of the function is $[0, 3]$



5 Let d km be the distance travelled.

The time taken for journey travelling at 80 km per hour

$= \frac{d}{2} \div 80 = \frac{d}{160}$

The time taken for journey travelling at x km per hour

$= \frac{d}{2} \div x = \frac{d}{2x}$

$\therefore$ Total time taken $= \frac{d}{160} + \frac{d}{2x} = \frac{d}{2}\left(\frac{1}{80} + \frac{1}{x}\right) = \frac{d}{2}\left(\frac{x+80}{80x}\right)$

Average speed $= \frac{\text{distance travelled}}{\text{total time taken}}$

$$\therefore \quad S(x) = d \div \frac{d}{2}\left(\frac{x+80}{80x}\right)$$

$$= d \times \frac{2}{d} \times \frac{80x}{x+80}$$

$$= \frac{160x}{x+80}$$

Domain of S is $[0, \infty)$

6 Volume of cylinder = $\pi r^2 h$

a The diameter has length 12 cm.

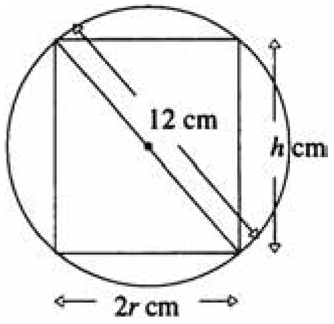
By Pythagoras' Theorem

$$12^2 = h^2 + 4r^2 \ldots \langle 1 \rangle$$

$$\therefore \quad r^2 = \frac{12^2 - h^2}{4}$$

$$V_1(h) = \frac{\pi}{4}\left(144 - h^2\right)h$$

$$= \pi\left(36 - \frac{h^2}{4}\right)h$$



As $V_1 > 0, h > 0$ and $r > 0$ Domain of $V_1 = (0, 12)$

b by $\langle 1 \rangle$

$$h^2 = 144 - 4r^2$$

$$\therefore \quad h = \sqrt{144 - 4r^2} = 2\sqrt{36 - r^2}$$

$$\therefore \quad V_2(r) = \pi r^2 \times 2\sqrt{36 - r^2}$$

$$= 2\pi r^2 \sqrt{36 - r^2}$$

Domain of $V_2 = (0, 6)$

7 **a**

	Domain	range
f	R	R
g	R	R

ran f = dom g g of exists,

$g \text{ o } f\ (x) = g(x + 1) = 2 + (1 + x)^3$

b $g \text{ o} f$ is a one-to-one function

$\therefore$ $(g \text{ o} f)^{-1}$ is defined,

Solve the equation $g \text{ o } f(x) = 10$

$$2 + (1 + x)^3 = 10$$

$$\therefore \quad (1 + x)^3 = 8$$

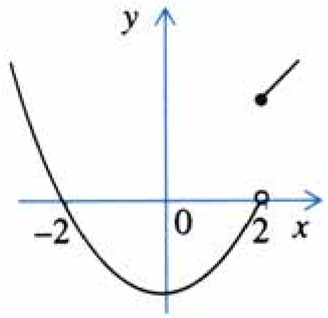
$$\therefore \quad 1 + x = 2$$

$$\therefore \quad x = 1$$

$\therefore$ $(g \text{ o} f)^{-1}$ is defined, $(g \text{ o} f)^{-1}(10) = 1$

8 $f(x) = \begin{cases} x^2 - 4 & \text{if } x \in (-\infty, 2) \\ x & \text{if } x \in [2, \infty) \end{cases}$

a



b **i** $f(-1) = 1 - 4 = -3$ as $-1 \in (-\infty, 2)$

ii $f(3) = 3$ as $3 \in [2, \infty)$

c $S = (-\infty, 0]$ as f is one to one for this interval. and $-1 \in S$.

d $h(x) = 2x$, then $f(h(x)) = f(2x)$

$$f(2x) = \begin{cases} (2x)^2 - 4 & \text{if } 2x \in (-\infty, 2) \\ 2x & \text{if } 2x \in [2, \infty) \end{cases}$$

Therefore $f \circ h(x) = \begin{cases} 4x^2 - 4 & \text{if } x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$

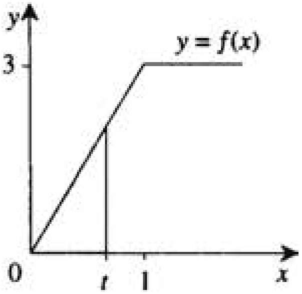
Now $h \circ f(x) = h\left(\begin{cases} x^2 - 4 & \text{if } x \in (-\infty, 2) \\ x & \text{if } x \in [2, \infty) \end{cases}\right)$

$$h \circ f(x) = \begin{cases} 2x^2 - 8 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$$

9 For $0 \leq t \leq 1$

Area $= \frac{1}{2} \times t \times 3t$

$= \frac{3}{2}t^2$



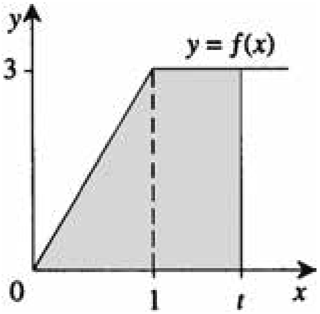
For $t > 1$

Area = area of triangle△ + area of rectangle□

$= \frac{1}{2} \times 1 \times 3 + 3(t - 1)$

$= \frac{3}{2} + 3t - 3$

$= 3t - \frac{3}{2}$



$$A(t) = \begin{cases} \frac{3}{2}t^2 & \text{for } 0 \le t \le 1 \\ 3t - \frac{3}{2} & \text{for } t > 1 \end{cases}$$

Domain of $A = [0, \infty)$

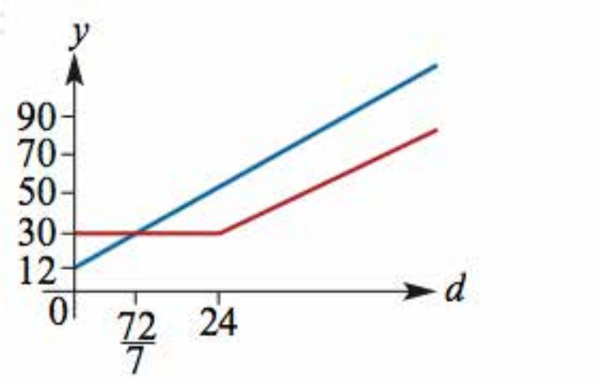
Range of $A = [0, \infty)$

10 **a** Charge is 0.35 per 200 metres. That is $5 \times 0.35 = \$1.75$ per km.

Hence $S(d) = 1.75d + 12$

b \$30 for distance under 24 km, Then $1.5(d - 24) + 30 = 1.5d - 36 + 30 = 1.5d - 6$ for distances over 24 km. $T(d) = \begin{cases} 30 & 0 \le d \le 24 \\ 1.5d - 6 & d > 24 \end{cases}$

c



d **i** $S(17) = 1.75 \times 17 + 12 = \$41.75$

ii $T(15) = \$30$

e $S(45) = 1.75 \times 45 + 12 = \$90.75$

$T(45) = 1.5 \times 45 - 6 = \$61.50$

Thrifty is cheaper.

f The graphs cross before $d = 24$

$$30 < 1.75d + 12$$

$$18 < 1.75d$$

$$\frac{72}{7} < d$$

11 **a** Let $x = \dfrac{ay + b}{cy + d}$

$\therefore x(cy + d) = ay + b$

and $xcy - ay = b - xd$

$y(xc - a) = b - xd$

$\therefore y = \dfrac{b - xd}{xc - a}$

Hence $f^{-1} : \mathbb{R}\backslash\left\{\dfrac{a}{c}\right\} \to \mathbb{R},\ f^{-1}(x) = \dfrac{b - xd}{xc - a}$

For the range of f note: $f(x) = \dfrac{ax + b}{cx + d} = \dfrac{a}{c} + \dfrac{cb - da}{c(cx + d)}$ (by division)

$\therefore$ range of $f = \mathbb{R}\backslash\left\{\dfrac{a}{c}\right\}$

and Domain of f^{-1} = range of $f = \mathbb{R}\backslash\left\{\dfrac{a}{c}\right\}$

range of f^{-1} = Domain of $f = \mathbb{R}\backslash\left\{\dfrac{-d}{c}\right\}$

b **i** For $f(x) = \dfrac{3x + 2}{3x + 1}$

$a = 3, b = 2, c = 3, d = 1$

and $f^{-1}(x) = \dfrac{2 - x}{3x - 3}$; Domain of $f^{-1} = \mathbb{R}\backslash\{1\}$

ii For $f(x) = \dfrac{3x + 2}{2x - 3}$

$a = 3, b = 2, c = 2, d = -3$

and $f^{-1}(x) = \dfrac{3x + 2}{2x - 3}$; Domain of $f^{-1} = \mathbb{R}\backslash\left\{\dfrac{3}{2}\right\}$

iii For $f(x) = \dfrac{x - 1}{-x - 1}$

$f^{-1}(x) = \dfrac{x - 1}{-x - 1} = \dfrac{1 - x}{x + 1}$; Domain of $f^{-1} = \mathbb{R}\backslash\{-1\}$

iv For $f(x) = \dfrac{-x + 1}{x + 1}$

$f^{-1}(x) = \dfrac{1 - x}{x + 1}$; Domain of $f^{-1} = \mathbb{R}\backslash\{-1\}$

c If $f^{-1} = f$ then Domain of f^{-1} = Domain of f

$\therefore \dfrac{a}{c} = \dfrac{-d}{c}$ (we will assume $c \neq 0$)

$\therefore a = -d$

As $f(x) = \dfrac{ax + b}{cx + d}$

and $f^{-1}(x) = \dfrac{b - xd}{xc - a}$

If $a = -d$ $f^{-1}(x) = \dfrac{ax + b}{cx + d} = f(x)$

$\therefore$ For $c \neq 0$ $f^{-1} = f \Leftrightarrow a = -d$

12 **a** **i** $YB = r$ cm (sides of square)

ii $ZB = r$ cm (sides of square)

iii $AZ = (x - r)$ cm

iv $CY = (3 - r)$ cm

b $CY = CX = 3 - r$ (tangents from a point)
$AX = AZ = x - r$ (tangents from a point)
Therefore $AC = AX + XC = x - r + 3 - r = x + 3 - 2r$
Using Pythagoras' Theorem for triangle ABC

$$x^2 + 9 = (x + 3 - 2r)^2$$

i.e. $x^2 + 9 = (x + 3)^2 - 4r(x + 3) + 4r^2$

$$\therefore \quad x^2 + 9 = x^2 + 6x + 9 - 4rx - 12r + 4r^2$$

$$\therefore \quad 0 = 6x - 4rx - 12r + 4r^2$$

$$\therefore \quad 0 = 2r^2 - 2r(x + 3) + 3x$$

$$\therefore \quad r = \frac{2(x + 3) \pm \sqrt{4(x + 3)^2 - 24x}}{4}$$

$$= \frac{2x + 6 \pm \sqrt{4(x^2 + 6x + 9) - 24x}}{4}$$

$$= \frac{2x + 6 \pm \sqrt{4x^2 + 36}}{4}$$

$$= \frac{x + 3 \pm \sqrt{x^2 + 9}}{2}$$

But $r < \dfrac{x + 3}{2}$

$$\therefore r = \frac{x + 3 - \sqrt{x^2 + 9}}{2}$$

When $x = 4$,

c **i**

$$r = \frac{7 - \sqrt{25}}{2}$$

i.e. $r = 1$

ii When $r = \dfrac{1}{2}$

$$\frac{1}{2} = \frac{(x + 3) - \sqrt{x^2 + 9}}{2}$$

$$\therefore \quad -2 - x = -\sqrt{x^2 + 9}$$

$$\therefore 4 + 4x + x^2 = x^2 + 9$$

$$\therefore \quad 4x = 5$$

$x = \dfrac{5}{4}$ (Note this must be tested because of squaring)

13 $f(x) = \dfrac{px+q}{x+r}$ $x \in R\backslash\{-r, r\}$ for $x \in R\backslash\{-r, r\}$

a $f(x) = f(-x)$

implies

$$\frac{px+q}{x+r} = \frac{-px+q}{-x+r}$$

$$\therefore \quad (-x+r)(px+q) = (-px+q)(x+r)$$

$$\therefore -px^2 - qx + pxr + qr = -px^2 - pxr + qx + qr$$

$$\therefore \quad 2pxr = 2qx$$

$$\therefore \quad pr = q$$

$$\therefore \quad f(x) = \frac{px+pr}{x+r}$$

$$\therefore \quad f(x) = p$$

b $f(-x) = -f(x)$

implies

$$\frac{-px+q}{-x+r} = \frac{-px-q}{x+r}$$

$$\therefore -px^2 + qx - prx + qr = px^2 + qx - pxr - qr$$

$$\therefore \quad 2px^2 - 2qr = 0$$

$$\text{i.e. } px^2 = qr$$

$\therefore \quad p = \dfrac{qr}{x^2}$ since $x \neq 0$.

Substitute for p in $f(x) = \dfrac{Px+q}{x+r}$:

$$f(x) = \frac{\frac{qr}{x}+q}{x+r}$$

$$= \frac{qr+qx}{x(x+r)}$$

$$= \frac{q(x+r}{x(x+r)}$$

$= \dfrac{q}{x}$ (make that $x \pm -r$)

c **i** If $p = 3$, $q = 8$ and $r = -3$

$$f(x) = \frac{3x+8}{x-3}$$

Consider $x = \dfrac{3y+8}{y-3}$

$$yx - 3x = 3y + 8$$

$$\therefore yx - 3y = 3x + 8$$

$$\therefore y(x - 3) = 3x + 8$$

$$\therefore \quad y = \frac{3x + 8}{x - 3}$$

$f(x)$ Hence $f^{-1}(x) = \dfrac{3x + 8}{x - 3}$

Domain of $f^{-1} = R\backslash\{3\}$

ii $$x = \frac{3x + 8}{x - 3}$$

$$3x + 8 = x^2 - 3x$$

$$0 = x^2 - 6x - 8$$

$$\therefore \quad x = \frac{6 \pm \sqrt{36 + 32}}{2}$$

$$= \frac{6 \pm 2\sqrt{9 + 8}}{2}$$

$$= 3 \pm \sqrt{17}$$

14 a $f : R\backslash\{1\} \to R, f(x) = \dfrac{x + 1}{x - 1}$

Note: For this function $f = f^{-1}$ from Question 10.

i $$f(2) = \frac{2 + 1}{2 - 1} = 3$$

$$f(f(2)) = f(3) = \frac{3 + 1}{3 - 1} = 2$$

$$f(f(f(2))) = f(2) = 3$$

ii $f(f(x)) = x$ for all x

b $f : R\backslash\{-1\} \rightarrow R,$

$$\begin{aligned}
f(x) &= \frac{x-3}{x+1} \\
f(f(x)) &= f\left(\frac{x-3}{x+1}\right) \\
&= \frac{\frac{x-3}{x+1} - 3}{\frac{x-3}{x+1} + 1} \\
&= \frac{x - 3 - 3x - 3}{x - 3 + x + 1} \\
&= \frac{-x-3}{x-1} \\
f(f(f(x))) &= f\left(\frac{-x-3}{x-1}\right) \\
&= \frac{\frac{-x-3}{x-1} - 3}{\frac{-x-3}{x-1} + 1} \\
&= \frac{-x - 3 - 3x + 3}{-x - 3 + x - 1} \\
&= \frac{-4x}{-4} \\
&= x \\
i.e.:\ f(fx) &= f^{-1}(x)
\end{aligned}$$

Chapter 2 – Coordinate geometry

Solutions to Exercise 2A

1 **a** $3x - 4 = 2x + 6$

$x = 10$

b $8x - 4 = 3x + 1$

$5x = 5$

$x = 1$

c $3(2 - x) - 4(3 - 2x) = 14$

$6 - 3x - 12 + 8x = 14$

$5x - 20 = 0$

$x = 4$

d $\frac{3x}{4} - 4 = 17$

$\frac{x}{4} = 7$

$x = 28$

e $6 - 3y = 5y - 62$

$8y = 68$

$y = \frac{17}{2}$

f $\frac{2}{3x - 1} = \frac{3}{7}$

$14 = 9x - 3$

$x = \frac{17}{9}$

g $\frac{2x - 1}{3} = \frac{x + 1}{4}$

$8x - 4 = 3x + 3$

$5x = 7$

$x = \frac{7}{5}$

h $\frac{2(x - 1)}{3} - \frac{(x + 4)}{2} = \frac{5}{6}$

$4x - 4 - 3x - 12 = 5$

$x = 21$

i $4y - \frac{3y + 4}{2} + \frac{1}{3} = \frac{5(4 - y)}{3}$

$24y - 9y - 12 + 2 = 40 - 10y$

$25y = 50$

$y = 2$

j $\frac{x + 1}{2x - 1} = \frac{3}{4}$

$4x + 4 = 6x - 3$

$2x = 7$

$x = \frac{7}{2}$

2 **a** $x - 4 = y \ldots (1)$

$4y - 2x = 8 \ldots (2)$

$(2) + 2 \times (1) \Rightarrow 4y - 8 = 8 + 2y$

$2y = 16$

$y = 8$

$\Rightarrow x = 12$

b $9x + 4y = 13 \ldots (1)$

$2x + y = 2 \ldots (2)$

$(1) - 4 \times (2) \Rightarrow x = 5$

$\Rightarrow 10 + y = 2$

$y = -8$

c $7x - 3y = 18 \ldots (1)$

$22x + 5y = 11 \ldots (2)$

$5 \times (1) + 3 \times (2) \Rightarrow 41x = 123$

$x = 3$

$\Rightarrow 6 + 5y = 11$

$y = 1$

d $5x + 3y = 13 \ldots (1)$

$7x + 2y = 16 \ldots (2)$

$3 \times (2) - 2 \times (1) \Rightarrow 11x = 22$

$x = 2$

$\Rightarrow 10 + 3y = 13$

$y = 1$

e $19x + 17y = 0 \ldots (1)$

$2x - y = 53 \ldots (2)$

From $(1) y = \dfrac{-19}{17}x$

$\Rightarrow 2x + \dfrac{19}{17}x = 53$

$53x = 17 * 53$

$x = 17$

$\Rightarrow 34 - y = 53$

$y = -19$

f $\dfrac{x}{5} + \dfrac{y}{2} = 5 \ldots (1)$

$x - y = 4 \ldots (2)$

$(2) + 2 \times (1) \Rightarrow \dfrac{7x}{5} = 14$

$x = 10$

$\Rightarrow 10 - y = 4$

$y = 6$

3 $l = w + 4 \ldots (1)$

$2(l - 5) + 2(w - 2) = 18 \ldots (2)$

Substitute from (1) into (2)

$w - 1 + w - 2 = 9$

$w = 6$ cm

$\Rightarrow l = 10$ cm

4 Let g represent the number of goals scored, and $t$the number of throws.

$t_0 = t_j \ldots (1)$

$g_0 = 2g_j \ldots (2)$

$t_j + 2g_j = 11 \ldots (3)$

$t_j + 2g_0 = 19 \ldots (4)$

Subtract Equation(3) from Equation(4)

$\Rightarrow 2g_0 - 2g_j = 8$

Substitute from Equation (2)

$4g_j - 2g_j = 8$

$g_j = 4$

$g_0 = 8$

John scored 4 goals and David scored 8.

5 **a** $w = 800 + 20n$

b $w = 800 + 20(30)$

$\$w = \$1400$

c $1620 = 800 + 20n$

$20n = 820$

$n = 41$ units

6 **a** $V = 250 + 15t$

b $V = 250 + 15(60)$

$V = 1150\text{L}$

c $5000 = 250 + 15t$

$t = \dfrac{4750}{15} = \dfrac{950}{3}$ min

$t = 5\text{h } 16 \text{ min } 40\text{s}$

7 **a** $V = 10000 - 10t$

b $V = 10000 - 10(60)$

$= 9400$ L

c $0 = 10000 - 10t$

$t = 1000$ min

$t = 16$ h 40 min

8 $\dfrac{x}{240} + \dfrac{x}{320} = \dfrac{35}{60}$

$\dfrac{7x}{12} = 80 \times \dfrac{7}{12}$

$x = 80$ km

9 $\dfrac{x}{48} + \dfrac{x}{4.8} = 24 - 2$

$11x = 22 \times 48$

$x = 96$ km

10 **a** $C = 100 + 25t$

b **i** $C = 100 + 252$

$\$C = \$150$

ii $C = 100 + 25(2.5)$

$\$C = \$162.50$

c **i** $375 = 100 + 25t$

$t = 11$ h

ii $400 = 100 + 25t$

$t = 12$ h

Solutions to Exercise 2B

1 **a** $ax + n = m$

$$x = \frac{m - n}{a}$$

b $ax + b = bx$

$$x = \frac{b}{b - a}$$

c $\dfrac{ax}{b} + c = 0$

$$x = \frac{-bc}{a}$$

d $px = qx + 5$

$$x = \frac{5}{p - q}$$

e $mx + n = nx - m$

$$(m - n)x = -(n + m)$$

$$x = \frac{n + m}{n - m}$$

f $\dfrac{1}{x + a} = \dfrac{b}{x}$

$$x = b(x + a)$$

$$(1 - b)x = ba$$

$$x = \frac{ba}{1 - b}$$

g $\dfrac{b}{x - a} = \dfrac{2b}{x + a}$

$$bx + ab = 2bx - 2ab$$

$$bx = 3ab$$

$$x = 3a,\ b \neq 0$$

h $\dfrac{x}{m} + n = \dfrac{x}{n} + m$

$$nx + n^2m = mx + m^2n$$

$$(n - m)x = nm(m - n)$$

$$x = -mn$$

i $-b(ax + b) = a(bx - a)$

$$-bax - b^2 = abx - a^2$$

$$2abx = a^2 - b^2$$

$$x = \frac{a^2 - b^2}{2ab}$$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

$$p^2 - (p^2 + 2pq)x = q^2 + q^2x$$

$$p^2 - q^2 = (p + q)^2x$$

$$x = \frac{p - q}{p + q}$$

k $bx - ab = ax + 2b$

$$(b - a)x = 3ab$$

$$x = \frac{3ab}{b - a}$$

l $\dfrac{x}{a - b} + \dfrac{2x}{a + b} = \dfrac{1}{a^2 - b^2}$

$$x(a + b) + 2x(a - b) = 1$$

$$x(a + b + 2a - 2b) = 1$$

$$x = \frac{1}{3a - b}$$

m $\dfrac{p - qx}{t} + p = \dfrac{qx - t}{p}$

$$p^2 - qpx + p^2t = qtx - t^2$$

$$qtx + qpx = p^2 + p^2t + t^2$$

$$x = \frac{p^2 + p^2t + t^2}{q(t + p)}$$

n $\dfrac{1}{x + a} + \dfrac{1}{x + 2a} = \dfrac{2}{x + 3a}$

$$(x + 2a)(x + 3a) + (x + a)(x + 3a)$$
$$= 2(x + a)(x + 2a)$$

$$2x^2 + 9ax + 9a^2 = 2x^2 + 6ax + 4a^2$$

$$3ax = -5a^2$$

$$x = \frac{-5a}{3}$$

2 a $ax + y = c \ldots (1)$

$x + by = d \ldots (2)$

$(1) - a \times (2)$

$\Rightarrow y(1 - ab) = c - ad$

$$y = \frac{c - ad}{1 - ab}$$

Equation (2)$-b\times$ Equation (1)

$\Rightarrow x(1 - ab) = d - bc$

$$x = \frac{d - bc}{1 - ab}$$

b $ax - by = a^2 \ldots (1)$

$bx - ay = b^2 \ldots (2)$

$b\times$ Equation(1)$-a\times$ Equation (2)

$\Rightarrow (-b^2 + a^2)y = a^2b - b^2a$

$$y = \frac{ab(a - b)}{a^2 - b^2}$$

$$y = \frac{ab}{a + b}$$

$a\times$ Equation(1) $-b\times$ Equation (2)

$\Rightarrow (a^2 - b^2)x = a^3 - b^3$

$$x = \frac{a^3 - b^3}{a^2 - b^2}$$

$$x = \frac{a^2 + ab + b^2}{a + b}$$

c $ax + by = t \ldots (1)$

$ax - by = s \ldots (2)$

$(1) + (2) \Rightarrow 2ax = t + s$

$$x = \frac{t + s}{2a}$$

Equation (1) - Equation (2)

$\Rightarrow 2by = t - s$

$$y = \frac{t - s}{2b}$$

d $ax + by = a^2 + 2ab - b^2 \ldots (1)$

$bx + ay = a^2 + b^2 \ldots (2)$

$a \times (1) - b \times (2)$

$\Rightarrow (a^2 - b^2)x$

$= a^3 + 2a^2b - ab^2 - a^2b - b^3$

$$x = \frac{a^3 + a^2b - ab^2 - b^3}{a^2 - b^2}$$

$$x = \frac{(a + b)(a^2 - b^2)}{a^2 - b^2}$$

$x = a + b$

Substitute into (2)

$\Rightarrow b(a + b) + ay = a^2 + b^2$

$ay = a^2 + b^2 - ba - b^2$

$y = a - b$

e $(a + b)x + cy = bc \ldots (1)$

$(b + c)y + ax = -ab \ldots (2)$

$a \times (1) - (a + b) \times (2)$

$\Rightarrow (ac - ab - b^2 - ac - bc)y$

$= abc + a^2b + b^2 \; y = \dfrac{ab(c + a + b)}{-b(a + b + c)}$

$$y = -a$$

Substitute into (2)

$\Rightarrow (-ab - ac) + ax = -ab$

$x = -b + b + c$

$x = c$

f $3(x - a) - 2(y + a) = 5 - 4a \ldots (1)$

$\Rightarrow 3x - 2y = 5 + a$

$2(x + a) + 3(y - a) = 4a - 1 \ldots (2)$

$\Rightarrow 2x + 3y = 5a - 1$

$3 \times (1) + 2 \times (2)$

$\Rightarrow 13x = 15 + 3a + 10a - 2$

$$13x = 13 + 13a$$

$$x = 1 + a$$

Substitute into (1)

$$\Rightarrow 3 + 3a - 2yt = 5 + a$$

$$y = -1 + a$$

$$y = a - 1$$

3 **a** $s = a(2a + 1)$

$s = 2a^2 + a$

b $h = a(2 + h)$

$h = 2a + ah$

$(1 - a)h = 2a$

$$h = \frac{2a}{1 - a}$$

$$s = a\left(\frac{2a}{1 - a}\right)$$

$$s = \frac{2a^2}{1 - a}$$

$$h = \frac{1}{1 + a}$$

c $as = a + \dfrac{1}{1 + a}$

$$s = 1 + \frac{1}{a + a^2}$$

$$s = 1 + \frac{1}{a + a^2}$$

$$s = \frac{a^2 + a + 1}{a^2 + a}$$

d $ah = a + h$

$(a - 1)h = a$

$$h = \frac{a}{a - 1}$$

$$as = s + \frac{a}{a - 1}$$

$$(a - 1)s = \frac{a}{a - 1}$$

$$s = \frac{a}{(a - 1)^2}$$

e $s = (3a^2)^2 + a(3a^2)$

$s = 9a^4 + 3a^3$

$s = 3a^3(3a + 1)$

f $as = a + 2(a - s)$

$as = a + 2a - 2s$

$(a + 2)s = 3a$

$$s = \frac{3a}{a + 2}$$

g $s = 2 + a\left(a - \dfrac{1}{a}\right) + \left(a - \dfrac{1}{a}\right)^2$

$$s = 2 + a^2 - 1 + a^2 - 2 + \frac{1}{a^2}$$

$$s = 2a^2 - 1 + \frac{1}{a^2}$$

h $3s - ah = a^2 \ldots (1)$

$as + 2h = 3a \ldots (2)$

$2\times$ Eq(1)$+a\times$ Eq(2)$\Rightarrow (6 + a^2)s = 5a^2$

$$s = \frac{5a^2}{6 + a^2}$$

4 $ax + by = p \ldots (1)$

$bx - ay = q \ldots (2)$

$a \times (1) + b \times (2) \Rightarrow (a^2 + b^2)x = pa + bq$

$$x = \frac{ap + bq}{a^2 + b^2}$$

$b \times (1) - a \times (2) \Rightarrow (b^2 + a^2)y = bp - aq$

$$y = \frac{bp - aq}{a^2 + b^2}$$

5 $bx + ay = ab \ldots (1)$

$ax + by = ab \ldots (2)$

$a \times (1) - b \times (2)$

$\Rightarrow (a^2 - b^2)y = ab(a - b)$

$$y = \frac{ab}{a + b}$$

$b \times (1) - a \times (2)$

$\Rightarrow (b^2 - a^2)x = ab(b - a)$

$$x = \frac{ab}{a + b}$$

Solutions to Exercise 2C

1 **a** $\sqrt{205}$

b $\left(1, -\frac{1}{2}\right)$

c $-\frac{13}{6}$

d $13x + 6y = 10$

e $13x + 6y = 43$

f $13y - 6x = -\frac{25}{2}$

2 **a** $\left(3, \frac{15}{2}\right)$

b $\left(\frac{-5}{2}, -2\right)$

c $\left(\frac{3}{2}, \frac{1}{2}\right)$

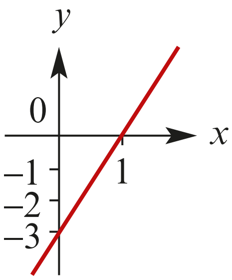
3 **a** $(4, 7)$

b $(5, -2)$

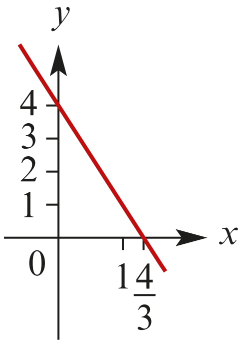
c $(2, 19)$

d $(-2, -9)$

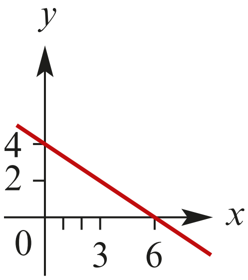
4 **a**



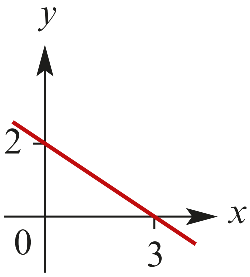
b



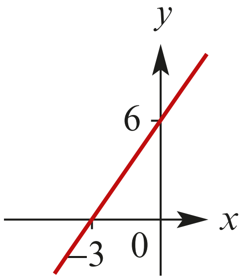
c



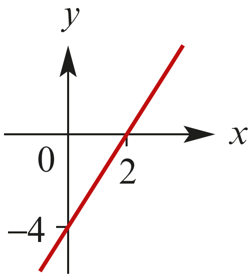
d



e



f



5 **a** $y - 2 = 2(x - 4)$

$y = 2x - 6$

b $y - 4 = -3(x + 3)$

$y = -3x - 5$

$m = \frac{4}{3}$

c $y - 3 = \frac{4}{3}(x - 1)$

$y = \frac{4}{3}x + \frac{5}{3}$ or $3y - 4x = 5$

d $m = 2$

$y - 5 = 2(x - 2)$

$y = 2x + 1$

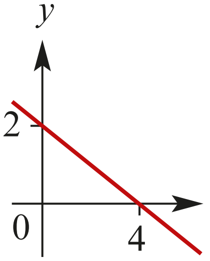
6 **a** $\dfrac{\frac{x}{-3} + \frac{y}{2}}{\frac{y}{2} - \frac{x}{3}} = 1$

b $\frac{x}{4} + \frac{y}{6} = 1$

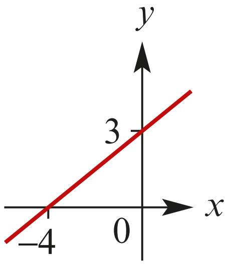
c $\dfrac{\frac{x}{-4} + \frac{y}{-3} = 1}{\left(-\frac{x}{4} - \frac{y}{3} = 1\right)}$

d $\dfrac{\frac{x}{6} + \frac{y}{-2} = 1}{\frac{x}{6} - \frac{y}{2} = 1}$

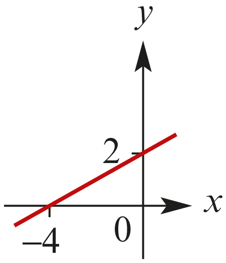
7 **a** $\frac{x}{4} + \frac{y}{2} = 1$



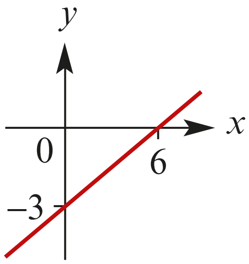
b $\frac{y}{3} - \frac{x}{4} = 1$



c $\frac{y}{2} - \frac{x}{4} = 1$



d $\frac{x}{6} - \frac{y}{3} = 1$



8 (600,35) (800,46)

$m = \frac{11}{200}$

$C - 35 = \frac{11}{200}(n - 600)$

$C = \frac{11}{200}n + 2$

$C(1000) = \frac{11}{200}(1000) + 2$

$\$C = \$57$

9 **a** (120,775) (160,975)

$$m = 5$$
$$C - 775 = 5(n - 120)$$
$$C = 5n + 175$$

b *yes*

c \$175

10 **a** $d = \sqrt{1^2 + 2^2}$

$= \sqrt{5} \approx 2.236$

b $d = \sqrt{1^2 + 1^2}$

$= \sqrt{2} \approx 1.414$

c $d = \sqrt{5^2 + 2^2}$

$= \sqrt{29} \approx 5.385$

d $d = \sqrt{2^2 + 18^2}$

$= \sqrt{326}$

$= 2\sqrt{82} \approx 18.111$

e $d = \sqrt{4^2 + 2^2}$

$= \sqrt{20}$

$= 2\sqrt{5} \approx 4.472$

f $d = \sqrt{3^2 + 4^2}$

$= 5$

11 **a** **i** $y - 6 = 2(x + 1)$

$y = 2x + 4$

ii $y - 6 = \frac{-1}{2}(x - 1)$

$y = \frac{-1}{2}x + \frac{13}{2}$ or $2y + x = 13$

b **i** $y - 3 = -2(x - 2)$

$y = -2x + 7$

ii $y - 3 = \frac{1}{2}(x - 2)$

$y = \frac{1}{2}x + 2$

or $2y - x = 4$

12 $(3, 3)$, $m = \frac{-1}{-3/6} = 2$

$y - 3 = 2(x - 3)$

$y = 2x - 3$

13 $5 = \sqrt{3^2 + (y + 1)^2}$

$25 = 9 + (y + 1)^2$

$y + 1 = \pm 4$

$y = -1 \pm 4$

$y = -5, 3$

14 $10 = \sqrt{8^2 + (y - 6)^2}$

$100 = 64 + (y - 6)^2$

$y - 6 = \pm 6$

$y = 6 \pm 6$

$y = 0, 12$

15 $26 = \sqrt{10^2 + (y - 8)^2}$

$676 = 100 + (y - 6)^2$

$y - 8 = \pm 576$

$y = 8 \pm 24$

$y = -16, 32$

16 **a** **i** $y - 3 = \frac{-2}{5}(x + 1)$

$$y = \frac{-2}{5}x + \frac{13}{5} \text{ or } 5y + 2x = 13$$

ii $y - 3 = \frac{-4}{5}(x + 1)$

$$y = \frac{-4}{5}x + \frac{11}{5} \text{ or } 5y + 4x = 11$$

b **i** $y - 3 = \frac{5}{2}(x + 1)$

$$y = \frac{5}{2}x + \frac{11}{2} \text{ or } 2y - 5x = 11$$

ii $y - 3 = \frac{5}{4}(x + 1)$

$$y = \frac{5}{4}x + \frac{17}{4} \text{ or } 4y - 5x = 17$$

17 **a** $m = \frac{6 - 1}{4 + 4} = \frac{5}{8}$

$$\theta = \tan^{-1}\left(\frac{5}{8}\right) = 32.01°$$

b $m = \frac{-1}{2}$

$$\theta = \tan^{-1}\left(\frac{-1}{2}\right) = 153.43°$$

c $m = \frac{3}{2}$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.31°$$

d $m = \frac{-10}{6}$

$$\theta = \tan^{-1}\left(\frac{-5}{3}\right) = 120.96°$$

18 $m_1 = 2$

$m_2 = -3$

$\theta_1 = 63.43°$

$\theta_2 = 108.43°$

$\alpha = \theta_2 - \theta_1 = 45°$

19 $\sqrt{(-2 - a)^2 + (-2)^2} = 2\sqrt{5^2 + 1^2}$

square both sides

$4 + 4a + a^2 + 4 = 4(26)$

$a^2 + 4a - 96 = 0$

$(a + 2)^2 - 100 = 0$

$a + 2 = \pm 10$

$a = -12, 8$

20 **a** $\frac{5 - 7}{7 - 1} = \frac{-1}{3}$

$m = 3$

midpoint $= (4, 6)$

$y - 6 = 3(x - 4)$

$y = 3x - 6$

b $\overrightarrow{BC}$ *has* $m = 1$

$1y = x - 2$

$2y = 3x - 6$

$x = 2, y = 0$

point of intersection: $(2,0)$

21 $k = h + 1 \ldots (1)$

$\sqrt{h^2 + (2 - k)^2} = 5 \ldots (2)$

$\Rightarrow h^2 + (k - 2)^2 = 25$

Substitute in (1)

$\Rightarrow h^2 + (h - 1)^2 = 25$

$2h^2 - 2h + 1 = 25$

$2h^2 - 2h - 24 = 0$

$h^2 - h - 12 = 0$

$(h + 3)(h - 4) = 0$

$h = -3, 4$

Substitute in (1)

$\Rightarrow k = -2, 5$

$(h,\ k) = (-3,\ -2)$ or(4,5)

22 $P = (3, 0),\ Q = (0, 2)$

a $\overrightarrow{QR} : y - 2 = \frac{1}{2}x$

$y = \frac{1}{2}x + 2$

if $x = 2a,$

$y = a + 2$

$R = (2a, a + 2)$

b $\overrightarrow{PR}$: *has* $m = \frac{a + 2}{2a - 3}$

but $m = -2$

$\therefore \quad -2 = \frac{a + 2}{2a - 3}$

$6 - 4a = a + 2$

$4 = 5a$

$a = \frac{4}{5}$

23 **a** AB has gradient $-3m$

$-3m = \frac{1 - 4}{1 + 1} = \frac{-3}{2}$

$m = \frac{1}{2}$

b $AC : 1y = \frac{3}{2}x - \frac{1}{2}$

$BC : y - 4 = \frac{1}{2}x + \frac{1}{2}$

$2y = \frac{1}{2}x + \frac{9}{2}$

$1 - 20 = x - 5$

$x = 5$

$\Rightarrow 2y = \frac{5}{2} + \frac{9}{2} = 7$

$C = (5, 7)$

c $AC = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$

$AB = \sqrt{2^2 + 3^2} = \sqrt{13}$

$AC = 2AB$

QED

24 **a** BC has gradient $= \frac{-1}{\text{grad}(AB)} = \frac{1}{3}$

$y - 8 = \frac{1}{3}(x - 2)$

$BC : y = \frac{1}{3}x + \frac{22}{3}$ or $3y - x = 22$

b $1y = \frac{1}{3}x + \frac{22}{3}$

$2y = x - 2$

$2 - 10 = \frac{2x}{3} + \frac{28}{3}$

$x = 14$

$\Rightarrow 2y = 12$

$C = (14, 12)$

c *because it is a rectangle*

$D = C - (B - A)$

$= (14, 12) - ((2, 8) - (4, 2))$

$= (14, 12) - (-2, 6)$

$= (16, 6)$

d Area $= AB \times BC$

$= \sqrt{2^2 + 6^2} \times \sqrt{12^2 + 4^2}$

$= \sqrt{40} \times \sqrt{160}$

$= 2\sqrt{10} \times 4\sqrt{10}$

$= 80$ square units

25 **a** (2,3)

b BD has$m = -5$
$y - 3 = -5(x - 2)$
$y = -5x + 13$

c **i** BC has $m = \dfrac{1}{\text{grad}(AC)} = \dfrac{3}{2}$
$y - 1 = \dfrac{3}{2}(x - 5)$
$y = \dfrac{3}{2}x - \dfrac{13}{2}$ or $2y = 3x - 13$

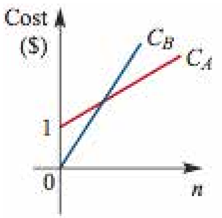
ii $1y = \dfrac{3}{2}x - \dfrac{13}{2}$
$2y = -5x + 13$
$1 - 20 = \dfrac{13x}{2} - \dfrac{39}{2}$
$x = 3$
$\Rightarrow 2y = -2$
$B = (3, -2)$

iii $D = A + (C - B)$
$D = (-1, 5) + (5, 1) - (3, \ -2)$
$D = (1, 8)$

Solutions to Exercise 2D

1 **a** $C_A = 0.4n + 1; C_B = 0.6n$

b The graphs are straight lines as shown here.



c $C_A = C_B$

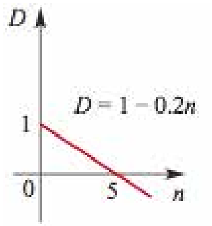
$0.4n + 1 = 0.6n$

$0.2n = 1$

$n = 5$

The charge is the same for 5 km.

d $D = C_A\ C_B = 1\ 0.2n$, so the graph is a straight line with intercept (0, 1) and gradient -0.2.



D gives the difference in charges of the two firms in terms of the distance travelled.

2 **a** Since the journey lasts 4 hours, $4 - T$ hours are spent on country roads.

b **i** 90 km/h for T hours gives a distance of $90T$ km.

ii 70 km/h for $4 - T$ hours gives a distance of $70(4 - T)$ km.

c **i** $90T + 70(4 - T) = 300$

$20T = 20$

$T = 1$

ii 90 km on the freeway, $70 \times 3 = 210$ km.

3 Let $L = at + b$, since a constant rate of decrease means the relation is linear.

a $t = 20,\ L = 3000$: $20a + b = 3000 \quad \ldots 1$

$t = 35,\ L = 1200$: $35a + b = 1200 \quad \ldots 2$

$2 - 1$: $15a = -1800$, so $a = -120$

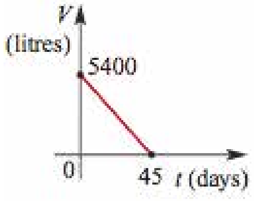
Sub in 1: $b = 3000 - 20(-120)$

$= 5400$

$L = -120t + 5400$

b 5400 litres (at $t = 0$ when it was filled)

c The tank will be empty when $-120t + 5400 = 0 \Rightarrow t = 45$



d From **c**, the domain is [0,45]

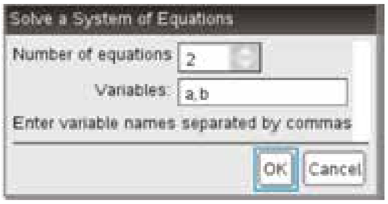
e 45 days

f The coefficient of t, i.e. -120, in the linear relation represents the rate of ‘increase’. So the water is decreasing, or leaving the tank, at 120 litres/day

Graphic calculator techniques for

question

In a **Calculator** page insert the linear equation template (b>**Algebra**>**Solve System of Equations**>**Solve System of Equations**) and complete the dialogue box as shown.



Solve a System of Equations

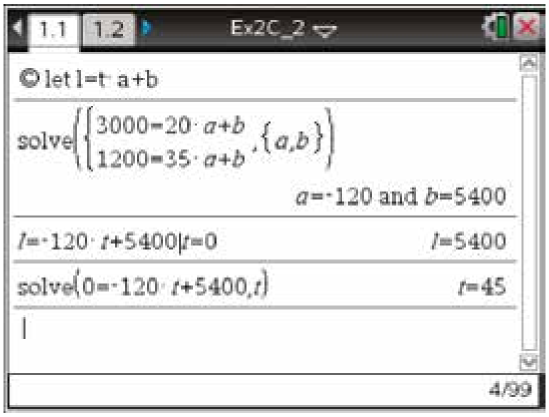
Number of equations: 2

Variables: a,b

Enter variable names separated by commas

Defining the rule(b>**Actions**>**Define**) allows you to use the rule elsewhere by just typing in *l*.

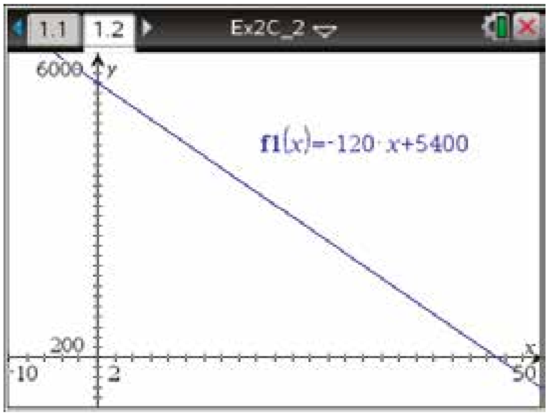
To find when the tank will be empty use the solve command (b>**Algebra**>**Solve**) with the equation equal to zero.



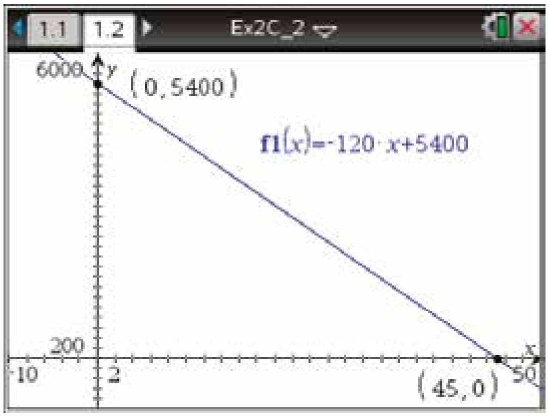
```
© let l=t· a+b
solve({3000=20· a+b ,{a,b})
      {1200=35· a+b
                                  a=-120 and b=5400
l=-120· t+5400|t=0                           l=5400
solve(0=-120· t+5400,t)                        t=45
                                               4/99
```

Insert a **Graphs** page (/ + **I**) and type in the rule $f1(x) = -120x + 5400$

Use b>**Window/Zoom**>**Window Settings** to set an appropriate window.

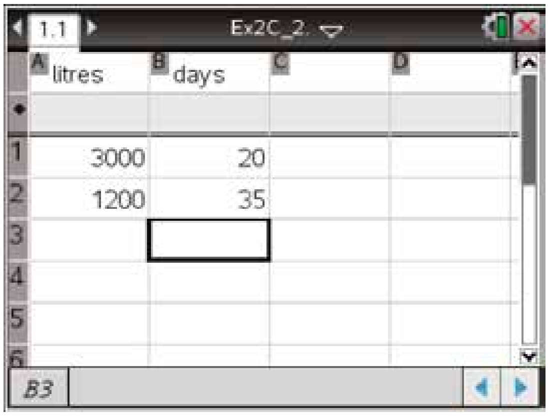


Note: the default digit display for **Graphs** is float3. You can increase this either in the settings or by placing the cursor over the designated value and pressing the + key. Hence 5.4E+3 now displays as 5400.



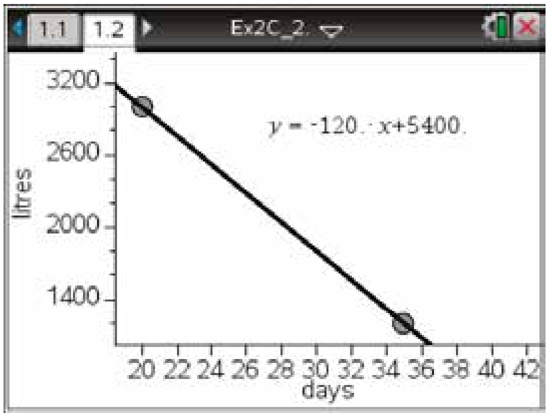
Alternative method as is current example

In a Lists & Spreadsheet page enter the list data as shown.



	A litres	B days	C	D
◆				
1	3000	20		
2	1200	35		
3				
4				
5				

Insert a **Data & Statistics** page(/ + I) and plot the data as shown.
A linear regression can be obtained using b>**Analyze>Regression>Show Linear (mx+b)**



4 **a** $y = \dfrac{22.5}{10}x$

$= \dfrac{9}{4}x$

b $OA = \sqrt{10^2 + 22.5^2}$

$= 24.622$ km(24622 m)

c $y - 9 = \dfrac{9 - 22.5}{23 - 10}(x - 23)$

$y = -\dfrac{27}{26}x + \dfrac{855}{26}$

d The midpoint of AB has coordinates $\left(\dfrac{33}{2}, \dfrac{61}{4}\right)$ and the perpendicular has gradient $\dfrac{26}{27}$, so its equation is:

$y - \dfrac{61}{4} = \dfrac{26}{27}\left(x - \dfrac{33}{2}\right)$

As the port has x-coordinate 52, substitute to find the y-coordinate:

$y - \dfrac{61}{4} = \dfrac{26}{27}\left(52 - \dfrac{33}{2}\right)$

$y = \dfrac{5339}{108}$

5 **a** **i** grad $AB = \dfrac{2 - 4}{2\ 1.5} = -4$

ii grad $AD =$ grad $BC = \dfrac{6 - 4}{6 - 1.5}$

$= \dfrac{4}{9}$

b **i** $y - 4 = \dfrac{4}{9}(x - 1.5)$

$y = \dfrac{4}{9}x + \dfrac{10}{3}$

ii $y - 6 =$ d$(x - 6)$

$y = -4x + 30$

c Notice that A is '0.5 across and 2 down' from B, so D is '0.5 across and 2 down' from C, so its coordinates are (6.5, 4).
As B and D have the same y-coordinate, the equation of BD is $y = 4$.
The equation of AC is given by:

$y - 6 = \dfrac{6 - 2}{6 - 2}(x - 6)$

$y = x$

d The diagonals intersect at (4, 4).

6 **a** M is the midpoint of AC

Coordinates of $M\left(\dfrac{5 + 9}{2}, \dfrac{0 + 10}{2}\right) =$ $M(7, 5)$

Coordinates of $N\left(\dfrac{13 + 9}{2}, \dfrac{0 + 10}{2}\right) =$ $M(11, 5)$

b **i** Gradient of $AC = \dfrac{10}{4} = \dfrac{5}{2}$

$\therefore$ Equation of lineAC is

$y - 0 = \dfrac{5}{2}(x - 5)$

ii Gradient of $BC = \dfrac{10}{-4} = -\dfrac{5}{2}$

$\therefore$ Equation of lineBC is

$y - 0 = -\frac{5}{2}(x - 13)$

iii Gradient of $MN = 0$

$\therefore$ Equation of line MN is

$y = 5$

c Line perpendicular to AC has gradient$-\frac{2}{5}$

$\therefore$ equation of line passing through M perpendicular to AC has equation

$y - 5 = -\frac{2}{5}(x - 7)$

Line perpendicular to BC has gradient$\frac{2}{5}$

$\therefore$ equation of line passing through N perpendicular to BC has equation

$y - 5 = \frac{2}{5}(x - 11)$

Intersect when

$-\frac{2}{5}(x - 7) = \frac{2}{5}(x - 11)$

$$7 - x = x - 11$$

$$18 = 2x$$

$$x = 9$$

When $x = 9, y = \frac{21}{5}$

Solutions to Exercise 2E

1 **a** $3x + 2y = 6 \ldots (1)$

$x - y = 7 \ldots (2)$

Multiply (2) by 2 and add to (1)

$5x = 20$

$x = 4$

$\therefore y = -3$

b $2x + 6y = 0 \ldots (1)$

$y - x = 2 \ldots (2)$

Multiply (2) by 2 and add to (1)

$8y = 4$

$$y = \frac{1}{2}$$

$$\therefore x = -\frac{3}{2}$$

c $4x - 2y = 7 \ldots (1)$

$5x + 7y = 1 \ldots (2)$

Multiply (2) by 4, (1)by 5 and subtract

$38y = -31$

$$y = -\frac{31}{38}$$

$$\therefore x = \frac{51}{38}$$

d $2x - y = 6 \ldots (1)$

$4x - 7y = 5 \ldots (2)$

Multiply (1) by 2,and subtract

$-5y = -7$

$$y = \frac{7}{5}$$

$$\therefore x = \frac{37}{10}$$

2 **a** one solution

b infinitely many solutions

c no solutions

3 , The two corresponding lines are parallel but not equal, and have no intersection.

4 $x - y = 6$

$y = x - 6$

Let$x = \lambda, y = \lambda - 6$

$y = \lambda - 6$

5 Lines are parallel. The gradients are the same and the lines have a common point.

$$3x + my = 5 \ldots (1)$$

$$(m + 2)x + 5y = m \ldots (2)$$

$$\text{Gradient of (1)} = -\frac{3}{m}$$

$$\text{Gradient of (2)} = -\frac{m + 2}{5}$$

For the lines to coincide:

$$\frac{3}{m} = \frac{m + 2}{5}$$

$$m^2 + 2m - 15 = 0$$

$$(m - 3)(m + 5) = 0$$

$$m = 3 \text{ or } m = -5$$

a If $m = 3$ the equations are

$3x + 3y = 5 \ldots (1)$

$5x + 5y = 5 \ldots (2)$

The lines don't coincide.

If $m = -5$ the equations are

$3x - 5y = 5 \ldots (1)$

$-3x + 5y = -5 \ldots (2)$

The lines coincide.

b A unique solution if $m \in \mathbb{R}\backslash\{3, -5\}$

No solution if $m = 3$

Alternative method

$$\begin{bmatrix} 3 & m \\ m+2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ m \end{bmatrix}$$

$$\begin{bmatrix} 3 & m \\ m+2 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & m \\ m+2 & 5 \end{bmatrix} = 15 - m^2 - 2m$$

$$m^2 + 2m - 15 = 0$$

$$(m + 5)(m - 3) = 0$$

$$m = -5, 3$$

substitute $m = -5$

$$3x - 5y = 5$$

$$-3x + 5y = -5$$

infinite solutions

Substitute $m = 3$

$3x + 3y = 5$

$5x + 5y = 3$

No solutions

$\therefore$ **a** $m = -5$ **b** $m = 3$

6 $m = 9$

7 **a** $mx + 2y = 8 \ldots (1)$

$4x - (2 - m)y = 2m \ldots (2)$

$4 \times (1) - m \times (2)$

$\Rightarrow (8 + m(m - 2))y = 32 - 2m^2$

$$y = \frac{32 - 2m^2}{m^2 - 2m + 8}$$

$$y = \frac{2(m + 4)(m - 4)}{(m - 4)(m + 2)}$$

$$y = \frac{2(m + 4)}{(m + 2)}$$

$m \neq 4, -2$

Substitute in (1)

$$\Rightarrow mx + 4\frac{(m - 4)(m + 2)}{(m - 4)(m + 2)} = 8$$

$$x = \frac{8(m - 4)(m + 2) - 4(m + 4)(m - 4)}{m(m - 4)(m + 2)}$$

$$x = \frac{4(m - 4)(2m + 4 - m - 4)}{m(m - 4)(m + 2)}$$

$$x = \frac{4(m - 4)m}{m(m - 4)(m + 2)}$$

$$x = \frac{4}{m + 2}, \quad m \neq -2, 0, 4,$$

b Values to test: $m = -2, 0, 4$

$m - 2$

$-2x + 2y = 8 \ldots (1)$

$4x - 4y = -4 \ldots (2)$

No solutions

$m = 0$

$10 + 2y = 8$

$y = 4$

$2 - 4x - 2y = 0$

$4x - 8 = 0$

$x = 2$ *unique solution*

(*Note*: *this value for m would not have appeared if equation* 2 *had been used to find x*.)

$m = 4$

$4x + 2y = 8 \ldots (1)$

$4x + 2y = 8 \ldots (2)$

infinite solutions

$\therefore$

i $m = -2$

ii $m = 4$

8 **a** $2x - 3y = 4 \ldots (1)$

$x + ky = 2 \ldots (2)$

$(1) - 2 \times (2)$

$\Rightarrow (3 - 2k)y = 0$

$y = 0,\ k \neq \dfrac{-3}{2}$

$\Rightarrow\ 2x = 4$

$x = 2$

b $k = \frac{-3}{2}$

9 $x + 5y = 4 \ldots (1)$

$2x + by = c \ldots (2)$

Gradient of (1) $= -\frac{1}{5}$

Gradient of (2) $= -\frac{2}{b}$

For the lines to be parallel or coincide:

$\frac{1}{5} = \frac{2}{b}$

$b = 10$

a A unique solution for $b \in \mathbb{R}\setminus\{10\}$

b If $b = 10$ and $c = 8$ the corresponding lines coincide and there are infinitely many solutions.

c If $b = 10$ and $c \neq 8$ the corresponding lines are parallel and there are no solutions

Solutions to Exercise 2F

1 **a**
$$2x + 3y - z = 12 \dots (1)$$
$$2y + z = 7 \dots (2)$$
$$2y - z = 5 \dots (3)$$

Add (2) and (3)

$$4y = 12$$
$$y = 3$$
$$\therefore z = 1$$

Substitute in (1) to find x

$x = 2$

b
$$x + 2y + 3z = 13 \dots (1)$$
$$-x - y + 2z = 2 \dots (2)$$
$$-x + 3y + 4z = 26 \dots (3)$$

Add (1) and (2)

$$y + 5z = 15 \dots (4)$$

Subtract (2) from(3)

$$4y + 2z = 24$$
$$2y + z = 12 \dots (5)$$

Multiply (4) by 2 and

subtract from (4)

$$-9z = -18$$
$$z = 2$$
$$\therefore y = 5$$
$$\therefore x = -3$$

c
$$x + y = 5 \dots (1)$$
$$y + z = 7 \dots (2)$$
$$z + x = 12 \dots (3)$$

Subtract (2) from (3)

$$x - y = 5 \dots (4)$$

Add (1) and(4)

$$\therefore x = 5$$
$$\therefore y = 0$$
$$\therefore z = 7$$

d
$$x - y - z = 0 \dots (1)$$
$$5x + 20z = 50 \dots (2)$$
$$10y - 20z = 30 \dots (3)$$

Simplify (2) and (3)

$$x + 4z = 10 \dots (4)$$
$$y - 2z = 3 \dots (5)$$

Subtract (5) from (4)

$$(x - y) + 6z = 7 \dots (6)$$

Subtract (1) from (6)

$$7z = 7$$
$$\therefore z = 1$$
$$\therefore x = 6$$
$$\therefore y = 5$$

2 **a** $y - 4z = -2$

$y = 4z - 2$

b $z = \lambda$

$y = 4\lambda - 2$

$\therefore x + 8\lambda - 4 - 3\lambda = 4$

$x = 8 - 5\lambda$

3 **a** $-y + 5z = 15 \quad (2) + (1)$

$-y + 5z = 15 \quad (3) - (2)$

b They are the same

c $z = \lambda$

$-y + 5\lambda = 15$

$y = 5\lambda - 15$

d $x + 10\lambda - 30 + 3\lambda = 13$

$x = 43 - 13\lambda$

4 **a** (1) +(2)

$\Rightarrow 2z = 10$

$z = 5$

Substitute into (1)

$\Rightarrow x - y + 5 = 4$

let $y = \lambda$

$x = \lambda - 1$

b Let $z = \lambda$

Substitute in (2)

$\Rightarrow x = 3 + \lambda$

Substitute into (1)

$\Rightarrow 6 + 2\lambda - y + \lambda = 6$

$y = 3\lambda$

c $(1) + 2 \times (2)$

$6x + 3z = 14$

Let $z = \lambda$

$6x = 14 - 3\lambda$

$x = \dfrac{14 - 3\lambda}{6}$

Substitute into (2)

$\dfrac{14 - 3\lambda}{6} + y + \lambda = 4$

$y = 4 - \dfrac{14 + 3\lambda}{6}$

$y = \dfrac{10 - 3\lambda}{6}$

5 $x + y + z + w = 4 \ldots (1)$

$x + 3y + 3z = 2 \ldots (2)$

$x + y + 2z - w = 6 \ldots (3)$

$(3) - (1)$

$\Rightarrow z - 2w = 2$

Let $z = t,\ t \in \mathbb{R}$

$w = \dfrac{t - 2}{2} = \dfrac{1}{2}t - 1$

$(2) - (3)$ gives

$2y + z + w = -4$

$2y + t + \dfrac{1}{2}t - 1 = -4$

$2y = -3 - \dfrac{3}{2}t$

$y = -\dfrac{3}{2} - \dfrac{3}{4}t$

Substitute into (1)

$x - \dfrac{3}{2} - \dfrac{3}{4}t + t + \dfrac{1}{2}t - 1 = 4$

$x = 6\dfrac{1}{2} - \dfrac{3}{4}t = \dfrac{26 - 3t}{4}$

when $w = 6$, $\dfrac{t - 2}{2} = 6$ so $t = z = 14$

$y = \dfrac{-3(14 + 2)}{4} = -12$

$x = \dfrac{26 - 42}{4} = -4$

6 **a** $3x - y + z = 4 \ldots (1)$

$x + 2y - z = 2 \ldots (2)$

$-x + y - z = -2 \ldots (3)$

$(2) - (3) \Rightarrow 2x + y = 4$

$(3) + (1) \Rightarrow 2x = 2$

$x = 1$

$y = 2$

Substitute into (3)

$\Rightarrow -1 + 2 - z = -2$

$z = 3$

b $x - y - z = 0 \ldots (1)$

$3y + 3z = -5 \ldots (2)$

$3 \times (1) + (2)$

$\Rightarrow 3x = -5$

$x = \dfrac{-5}{3}$

$3y = -5 - 3z$

Let $z = \lambda$

$y = \dfrac{-5 - 3\lambda}{3}$

c $12x - y + z = 0$

$2y + 2z = 2$

Let $z = \lambda$

$y = 2 - 2\lambda$

$2x - 2 + 2\lambda + \lambda = 0$

$2x = 2 - 3\lambda$

$x = \dfrac{2 - 3\lambda}{2}$

Solutions to technology-free questions

1 **a** $3x - 2 = 4x + 6$

$4x - 3x = -2 - 6$

$x = -8$

b $\dfrac{x+1}{2x-1} = \dfrac{4}{3}$

$3(x + 1) = 4(2x - 1)$

$3x + 3 = 8x - 4$

$8x - 3x = 3 + 4$

$5x = 7$

$x = \dfrac{7}{5}$

c $\dfrac{3x}{5} - 7 = 11$

$\dfrac{3x}{5} = 18$

$3x = 90$

$x = 30$

d $\dfrac{2x+1}{5} = \dfrac{x-1}{2}$

$2(2x + 1) = 5(x - 1)$

$4x + 2 = 5x - 5$

$x = 7$

2 **a** $y = x + 4 \quad \ldots 1$

$5y + 2x = 6 \quad \ldots 2$

Substitute 1 into 2:

$5(x + 4) + 2x = 6$

$5x + 20 + 2x = 6$

$7x = -14$

$x = -2$

Substitute into 1:

$y = -2 + 4 = 2$

b $\dfrac{x}{4} - \dfrac{y}{3} = 2 \quad \ldots 1$

$y - x = 5 \quad \ldots 2$

Multiply 1 by 12:

$3x - 4y = 24 \quad \ldots 3$

Multiply 2 by 3:

$3y - 3x = 15 \quad \ldots 4$

3 + 4 gives $-y = 39$, so $y = -39$

Substitute into 2:

$-39 - x = 5$, so $x = -44$

3 **a** $\dfrac{n+m}{b}$

b $\dfrac{b}{c+b}$

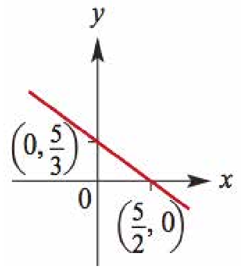
c d

d $\dfrac{6}{q-p}$

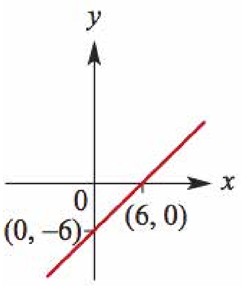
e $\dfrac{m+n}{m-n}$

f $\dfrac{a^2}{a-1}$

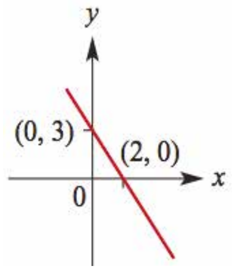
4 **a** intercepts $\left(\dfrac{5}{2}, 0\right), \left(0, \dfrac{5}{3}\right)$



b intercepts (6, 0), (0, −6)



c intercepts $(2, 0)$, $(0, 3)$



5 a $y - 3 = -2(x - 1)$

$y = -2x + 5$

b $$m = \frac{8-4}{3-1} = 2$$

$y - 4 = 2(x - 1)$

$y = 2x + 2$

c $y = -2x + 6$ has gradient $m_1 = -2$; for the gradient m_2 of a perpendicular line: $m_1 m_2 = -1$

$$m_2 = \frac{-1}{-2} = \frac{1}{2}$$

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

d $y = 6 - 2x$ has gradient -2; a parallel line has the same gradient:

$y - 1 = -2(x - 1)$

$y = -2x + 3$

6 $$\begin{aligned}\text{distance} &= \sqrt{(2-(-1))^2 + (4-6)^2}\\ &= \sqrt{3^2 + (-2)^2}\\ &= \sqrt{9+4}\\ &= \sqrt{13}\end{aligned}$$

7 $$\text{Midpoint} = \left(\frac{4+(-2)}{2}, \frac{6+8}{2}\right) = (1, 7)$$

8 a Let (x, y) be the coordinates of Y.

$$\left(\frac{x+(-6)}{2}, \frac{y+2}{2}\right) = (8, 3)$$

$$\therefore \frac{x-6)}{2} = 8 \text{ and } \frac{y+2}{2} = 3$$

$\therefore x = 22$ and $y = 4$

b Let (x, y) be the coordinates of Y.

$$\left(\frac{x+(-1)}{2}, \frac{y+(-4)}{2}\right) = (2, -8)$$

$$\therefore \frac{x-1)}{2} = 2 \text{ and } \frac{y-4}{2} = -8$$

$\therefore x = 5$ and $y = -12$

9 $$\begin{aligned}\sqrt{(10-5)^2 + (y-12)^2} &= 13\\ 25 + (y-12)^2 &= 169\\ (y-12)^2 &= 144\\ y - 12 &= \pm 12\\ y = 0 \text{ or } y &= 24\end{aligned}$$

10 $$mx - 4y = m + 3 \ldots (1)$$

$$4x + (m+10)y = -2 \ldots (2)$$

$$\text{Gradient of (1)} = \frac{m}{4}$$

$$\text{Gradient of (2)} = -\frac{4}{m+10}$$

Infinitely many or no solutions when the gradients are the same.

$$-\frac{m}{4} = \frac{4}{m+10}$$

$m^2 + 10m + 16 = 0$

$(m + 8)(m + 2) = 0$

$$m = -8 \text{ or } m = -2$$

a Checking back in the equations there are infinitely many solutions when $m = -2$.
Equation (1) becomes $-2x - 4y = 1$
Equation (2) becomes $4x + 8y = -2$

b There is a unique solution for $m \in \mathbb{R}\backslash\{-2, -8\}$

11 **a**

$$2x - 3y + z = 6 \ldots (1)$$

$$-2x + 3y + z = 8 \ldots (2)$$

Add (1) and (2)

$$2z = 14$$

$$z = 7$$

Substitute in (1)

$2x - 3y = -1$

$$\therefore y = \frac{2x+1}{3}$$

Let $x = \lambda$

The solution is $(\lambda, \frac{2\lambda+1}{3}, 7)$ where $\lambda \in \mathbb{R}$

b

$$x - z + y = 6 \ldots (1)$$

$$2x + z = 4 \ldots (2)$$

Let $z = \lambda$

Then $x = \frac{4-\lambda}{2}$

Substitute in (1)

$$\frac{4-\lambda}{2} - \lambda + y = 6$$

$$\therefore y = \frac{8+3\lambda}{2}$$

The solution is $(\frac{4-\lambda}{2}, \frac{3\lambda+8}{2}, \lambda)$ where $\lambda \in \mathbb{R}$

Solutions to multiple-choice questions

1 E $y = mx + c$

The m (gradient) value is $-\frac{1}{2}$,

It passes through the point (1, 4)

$4 = -\frac{1}{2}1 + c$

$\therefore\ c = \frac{9}{2}$

$\therefore\ y = -\frac{1}{2}x + \frac{9}{2}$

2 E $y = -2x + 4$

Point $(a, 3)$

$3 = -2a + 4$

$a = \frac{1}{2}$

3 D Line passes through the points $(-2, 0)$ and $(0, -1)$

$\therefore\ \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-1 - 0}{0 - -2}$

m (gradient) $= -\frac{1}{2}$

Perpendicular line $= -\frac{1}{m}$

$\therefore\ -\frac{1}{-\frac{1}{2}} = 2$

4 C Midpoint at $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$

$= \left(-\frac{2}{2}, \frac{17}{2}\right)$

$= (-1, 8.5)$

5 B $2ax - 10by = 22$

$+ \quad + \quad +$

$4ax + 10by = 2$

$\therefore\ 6ax = 24$

$\therefore\ x = \frac{4}{a}$

Do not need to solve for y as there is only one possible option.

6 A Line passes through the points $(3, -2)$ and $(-1, 10)$

$\therefore\ \frac{y_2 - y_1,}{x_2 - x_1}$

$= \frac{10 - -2}{-1 - 3}$

$= \frac{12}{-4}$

$= -3$

7 A Eqn 1: $y = 2x + 3$

Eqn 2: $y = \frac{ax}{3} + \frac{4}{3}$

To be parallel gradients must be the same.

$\therefore\ \frac{a}{3} = 2$

$\therefore\ a = 6$

8 C $y = mx + c$

$m = \frac{10 - -2}{3 - -1}$

$m = 3$

Passes through the point (3, 10)

$\therefore\ 10 = 9 + c$

$\therefore\ c = 1$

$\therefore\ y = 3x + 1$

9 B Distance between x points

$= |x_2 - x_1|$

$= |5 - 1|$

$= 4$

Distance between y points

$= y_2 - y_1$

$= |-2 - 4|$

$= 6$

Using Pythagoras

$\sqrt{4^2 + 6^2}$

$= \sqrt{52}$

$= 2\sqrt{13}$

10 C $y = mx + c$

Passes through points (4, 0) and (0, −3)

$$m = \frac{-3-0}{0-4}$$

$$m = \frac{3}{4}$$

Y intercept = −3

$\therefore$ c = −3

$$\therefore\ f(x) = \frac{3}{4} - 3$$

11 D

$$bx + 3y = 0 \ldots (1)$$

$$4x + (b+1)y = 0 \ldots (2)$$

$$\text{Gradient of } (1) = -\frac{b}{3}$$

$$\text{Gradient of } (2) = -\frac{4}{b+1}$$

Infinitely many solutions when the gradients are the same.

$$\frac{b}{3} = \frac{4}{b+1}$$

$$b^2 + b - 12 = 0$$

$$(b+4)(b-3) = 0$$

$$b = -4 \text{ or } b = 3$$

12 A

$$(a-1)x + 5y = 7 \ldots (1)$$

$$3x + (a-3)y = 0 \ldots (2)$$

$$\text{Gradient of } (1) = -\frac{a-1}{5}$$

$$\text{Gradient of } (2) = -\frac{3}{a-3}$$

Infinitely many or no solutions when the gradients are the same.

$$\frac{a-1}{5} = \frac{3}{a-3}$$

$$a^2 - 4a - 12 = 0$$

$$(a-6)(a+2) = 0$$

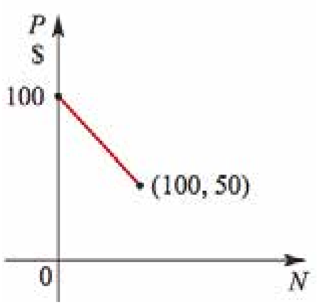
$$a = -2 \text{ or } a = 6$$

13 D $\left(\frac{0+4}{2}, \frac{d-6}{2}\right) = \left(2, \frac{d-6}{2}\right)$

14 C Gradient of line segment joining (3, 0) and (0, −6) is $\frac{6}{3} = 2$ Gradient of line perpendivular to this is -$\frac{1}{2}$

Solutions to extended-response questions

1 a Graph is a straight line passing through (100, 50) and (50, 75).



Note that extending it back to the *P* axis shows that the intercept is (0, 100); this is confirmed in part **b** below.

b Relationship is linear: $P = aN + b$

$$P = 50, N = 100: \quad 50 = 100a + b \quad \ldots 1$$

$$P = 75, N = 50: \quad 75 = 50a + b \quad \ldots 3$$

$$1 - 2: \quad 50a = -25$$

$$a = -\frac{1}{2}$$

which implies $b = 100$

Hence $P = -\frac{1}{2}N + 100$.

c **i** $N = 88$: $\quad P = -\frac{1}{2} \times 88 + 100$

$$= 56$$

So the price would be \$56.

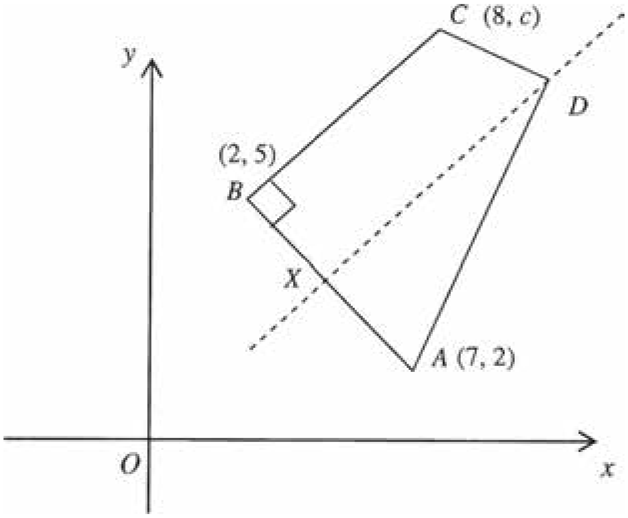
ii $P = 60$: $\quad 60 = -\frac{1}{2}N + 100$

$$\frac{1}{2}N = 40$$

$$N = 80$$

So the number of jackets would be 80.

2



a Midpoint of $AB = \left(\frac{7+2}{2}, \frac{5+2}{2}\right) = \left(\frac{9}{2}, \frac{7}{2}\right)$

Gradient of $AB = \frac{5-2}{2-7} = -\frac{3}{5}$

Therefore equation of perpendicular bisector of AB is

$y - \frac{7}{2} = \frac{5}{3}\left(x - \frac{9}{2}\right)$

Therefore $y = \frac{5}{3}x - 4$

b Solving the equations $y = 4x - 26$ and $y = \frac{5}{3}x - 4$ simultaneously for x and y will give the coordinates of D

Consider $4x - 26 = \frac{5}{3}x - 4$

$\frac{7x}{3} = 22$

$x = \frac{66}{7}$

Substitute $x = \frac{66}{7}$ in the equation $y = 4x - 26$ to give $y = \frac{82}{7}$

Coordinates of D are $\left(\frac{66}{7}, \frac{82}{7}\right)$

c Line BC is perpendicular to line AB. Therefore gradient of BC is $\frac{5}{3}$

d $B(2, 5)$ and $C(8, c)$. The gradient of BC can also be written as $\frac{5-c}{-6}$

Therefore $\frac{5-c}{-6} = \frac{5}{3}$

Solving for c gives $c = 15$

e The area will be found by calculating the area of triangle DXA and trapezium

BCDX. Let X be the midpoint of AB. From the above the coordinates of X are $\left(\frac{9}{2}, \frac{7}{2}\right)$

$$\text{Distance } XD = \sqrt{\left(\frac{66}{7} - \frac{9}{2}\right)^2 + \left(\frac{82}{7} - \frac{7}{2}\right)^2}$$

$$= \sqrt{\frac{8993}{98}}$$

$$= \frac{23\sqrt{34}}{14}$$

$$\text{Distance } XA = \text{distance } XB = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$

$$= \sqrt{\frac{17}{2}}$$

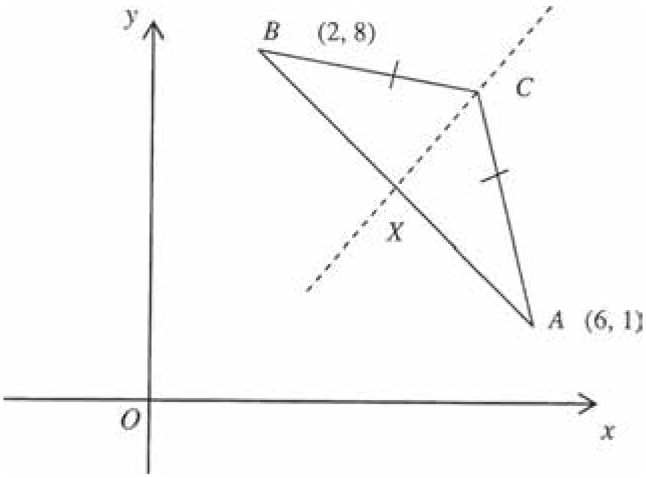
Area = area of triangle DXA + area of trapezium $BCDX$.

$$= \frac{1}{2}XA \times XB + \frac{1}{2}BX(BC + XD)$$

$$= \frac{1}{2}AX(BX + 2XD)$$

$$= \frac{629}{14}$$

3



a Midpoint of $BC = \left(4, \frac{9}{2}\right)$

Gradient of $AB = \frac{8-1}{2-6} = -\frac{7}{4}$

Therefore gradient of perpendicular bisector $= \frac{4}{7}$

The equation of the perpendicular bisector is

$$y - \frac{9}{2} = \frac{4}{7}(x - 4)$$

Therefore $y = \frac{4}{7}x + \frac{31}{14}$

b The perpendicular bisector passes through C as the triangle is isosceles.

When $x = 3.5, y = \frac{4}{7} \times 3.5 + \frac{31}{14} = \frac{59}{14}$

The coordinates of C are $\left(\frac{7}{2}, \frac{59}{14}\right)$

c The length of $AB = \sqrt{(6 - 2)^2 + (1 - 8)^2} = \sqrt{65}$

d The area of the triangle $= \frac{1}{2} \times \sqrt{65} \times XC$ where X is the midpoint of AB

$XC = \sqrt{\left(\frac{7}{2} - 4\right)^2 + \left(\frac{59}{14} - \frac{9}{2}\right)^2} = \frac{\sqrt{65}}{14}$

Therefore area $= \frac{1}{2} \times \sqrt{65} \times \frac{\sqrt{65}}{14} = \frac{65}{28}$ square units.

4 $A(-4, 6)$ and $B(6, -7)$

a Midpoint $= \left(\frac{-4 + 6}{2}, \frac{6 + -2}{2}\right) = \left(1, -\frac{1}{2}\right)$

b/c The length $AB = \sqrt{(-7 - 6)^2 + (6 - -4)^2} = \sqrt{269}$ = the distance between A and B

d gradient of $AB = \frac{6 - -7}{-4 - 6}$

$= -\frac{13}{10}$

The equation of AB is $y - 6 = -\frac{13}{10}(x + 4)$

Rearranging gives

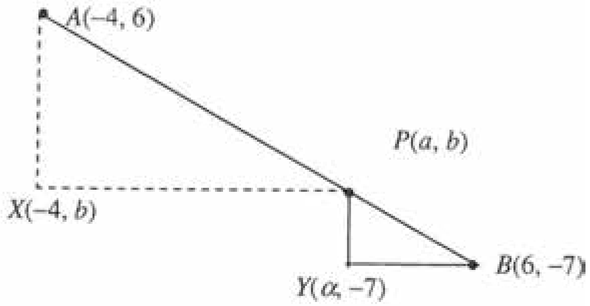
$y = -\frac{13}{10}x + \frac{4}{5}$

e The perpendicular bisector has gradient $\frac{10}{13}$

The equation is $y + \frac{1}{2} = \frac{10}{13}(x - 1)$

Therefore $y = \frac{10}{13}x - \frac{33}{26}$

f



Triangles AXP and PYB are similar with scale factor 3.

$AX : PY = 3 : 1$

Therefore $\dfrac{6-b}{b+7} = \dfrac{3}{1}$

Therefore $b = -\dfrac{15}{4}$

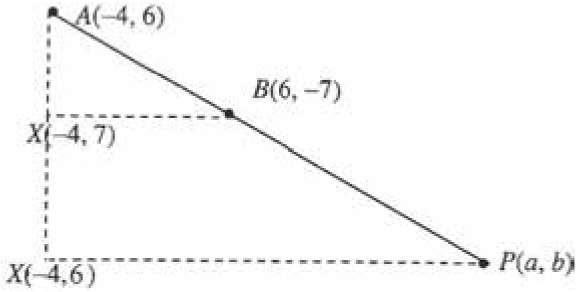
Also $XP : YB = 3$

$\dfrac{a+4}{6-a} = 3$

$a = \dfrac{7}{2}$

coordinates of P are $\left(\dfrac{7}{2}, -\dfrac{15}{4}\right)$

g



Triangles AXB and AYP are similar with scale factor 3.

Therefore

$\dfrac{a+4}{10} = \dfrac{3}{1}$

$a = 26$

Also $\dfrac{b-6}{-7-6} = 3$

$b = -33$

The coordinates of P are $(26, -33)$

5 a 25% of 500 = 125

125 litres of acid is required to produce 500 litres of a 25% acid solution.

b Let x denote the amount of 30% solution.

Let y denote the amount of 18% solution.

$\therefore \quad x + y = 500 \quad 1$

$0.3x + 0.18y = 125 \quad 3$

From 1 $y = 500 - x$. Substitute in 2

$\therefore \quad 0.3x + 0.18(500 - x) = 125$

$\therefore \quad (0.3 - 0.18)x + 90 = 125$

$\therefore \quad 0.12x = 35$

$\therefore \quad x = \dfrac{875}{3}$

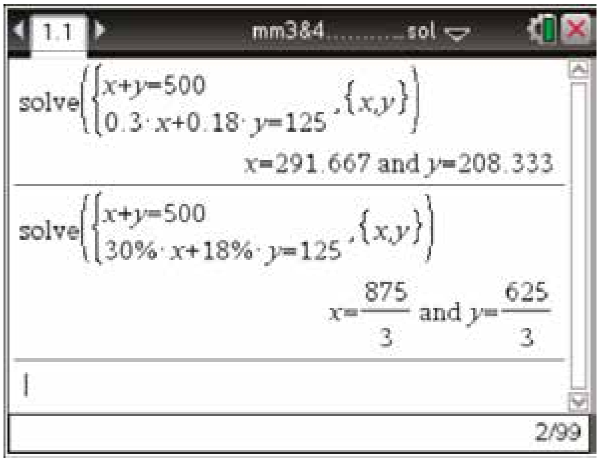
Substitute in 1 $y = 500 - \dfrac{875}{3} = \dfrac{625}{3}$

$\frac{875}{3}$ litres of the 30% solution and $\frac{625}{3}$ litres of the 18% solution are required.

Graphical Calculator techniques for Question 6.

In a **Calculator** page use **Algebra>Solve System of Equations>Solve System of Equations**.

For exact answers enter decimal inputs as fractions such as 30/100 or 30 as shown.



$\text{solve}\left(\begin{cases} x+y=500 \\ 0.3\cdot x+0.18\cdot y=125 \end{cases},\{x,y\}\right)$

$x=291.667$ and $y=208.333$

$\text{solve}\left(\begin{cases} x+y=500 \\ 30\%\cdot x+18\%\cdot y=125 \end{cases},\{x,y\}\right)$

$x=\frac{875}{3}$ and $y=\frac{625}{3}$

Chapter 3 – Transformations

Solutions to Exercise 3A

1 **a** $(-2, 5) \to (-2 + 1, 5 - 2) = (-1, 3)$

b $(-2, 5) \to (-2 - 3, 5 + 5) = (-5, 10)$

c $(-2, 5) \to (-2 - 1, 5 - 6) = (-3, -1)$

d $(-2, 5) \to (-2 - 3, 5 + 2) = (-5, 7)$

e $(-2, 5) \to (-2 - 1, 5 + 1) = (-3, 6)$

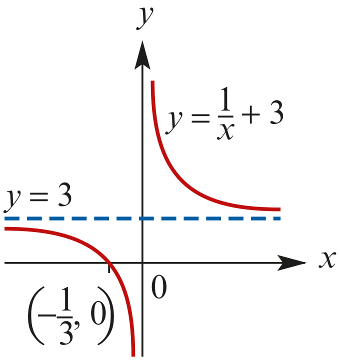
2 **a** $y = \dfrac{1}{x - 2} - 3$

b $y = \dfrac{1}{x + 2} + 3$

c $y = \dfrac{1}{x - \frac{1}{2}} + 4 = \dfrac{2}{2x - 1} + 4$

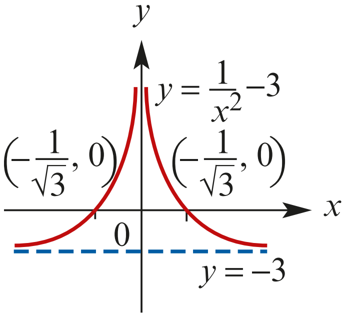
3 **a** Domain $= \mathbb{R} \setminus \{0\}$

Range $= \mathbb{R} \setminus \{3\}$



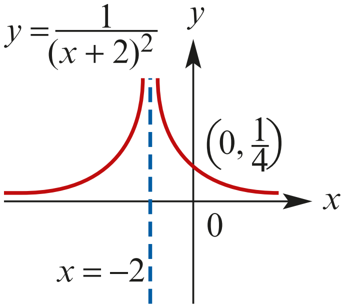
b Domain $= \mathbb{R} \setminus \{0\}$

Range $= (-3, \infty)$



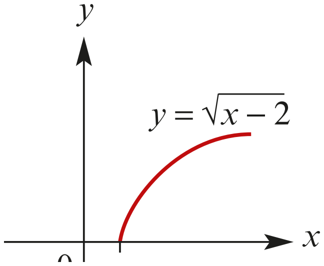
c Domain $= \mathbb{R} \setminus \{-2\}$

Range $= \mathbb{R}^+$



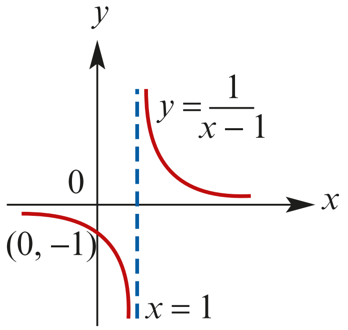
d Domain $= [2, \infty)$

Range $= \mathbb{R}^+ \cup \{0\}$



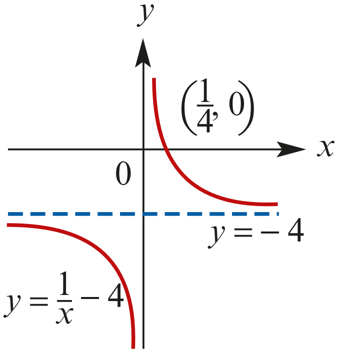
e Domain $= \mathbb{R} \setminus \{1\}$

Range $= \mathbb{R} \setminus \{0\}$



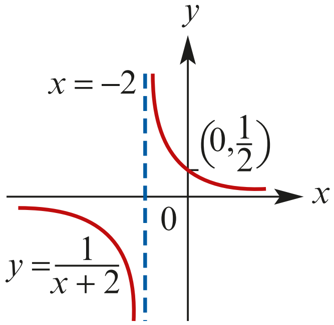
f Domain = $\mathbb{R} \setminus \{0\}$

Range = $\mathbb{R} \setminus \{-4\}$



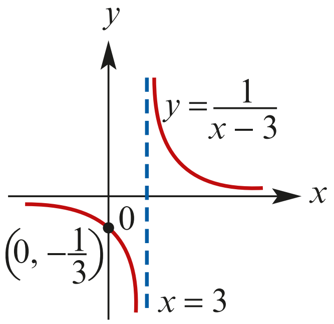
g Domain = $\mathbb{R} \setminus \{-2\}$

Range = $\mathbb{R} \setminus \{0\}$



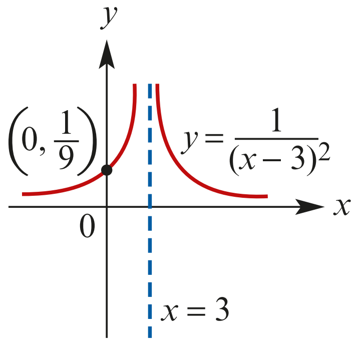
h Domain = $\mathbb{R} \setminus \{3\}$

Range = $\mathbb{R} \setminus \{0\}$



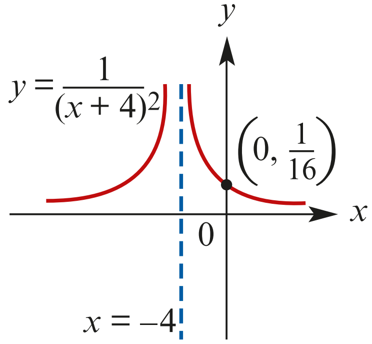
i Domain = $\mathbb{R} \setminus \{3\}$

Range = $\mathbb{R}^+$



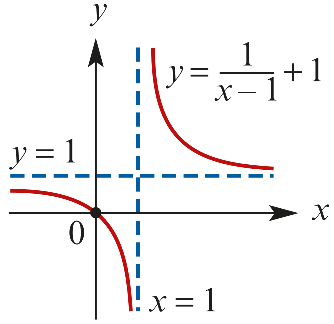
j Domain = $\mathbb{R} \setminus \{-4\}$

Range = $\mathbb{R}^+$



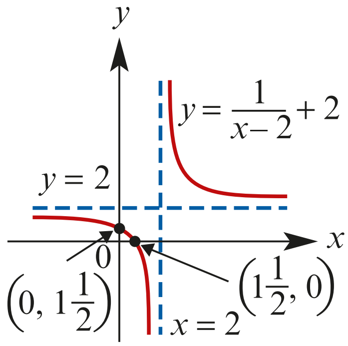
k Domain = $\mathbb{R} \setminus \{1\}$

Range = $\mathbb{R} \setminus \{1\}$

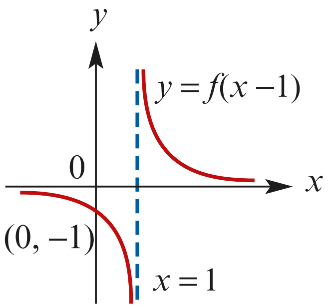


l Domain = $\mathbb{R} \setminus \{2\}$

Range = $\mathbb{R} \setminus \{2\}$



4 a



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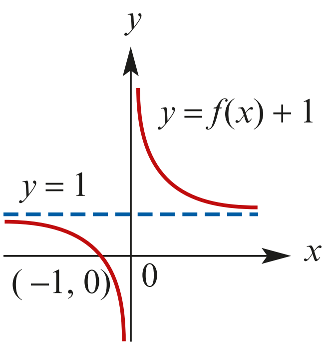

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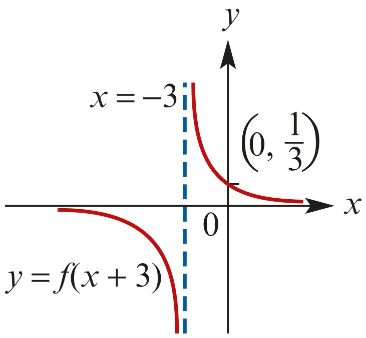

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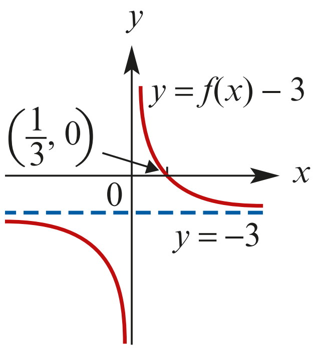

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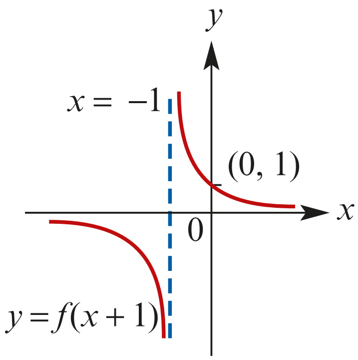

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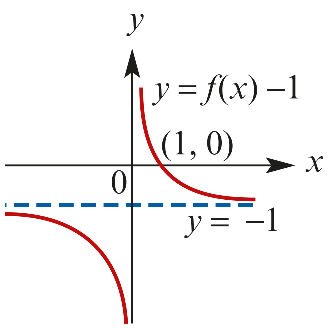


5 a Translation of 5 to the left;
$(x, y) \rightarrow (x - 5, y)$

b Translation of 2 up; $(x, y) \rightarrow (x, y + 2)$

c Translation of 4 up; $(x, y) \rightarrow (x, y + 4)$

d Translation $(x, y) \rightarrow (x, y + 3)$

e Translation $(x, y) \rightarrow (x - 3, y)$

6 a i $y - 1 = (x - 7)^{\frac{1}{4}}$; $y = (x - 7)^{\frac{1}{4}} + 1$

ii $y + 6 = (x + 2)^{\frac{1}{4}}$; $y = (x + 2)^{\frac{1}{4}} - 6$

iii $y + 3 = (x - 2)^{\frac{1}{4}}$; $y = (x - 2)^{\frac{1}{4}} - 3$

iv $y - 4 = (x + 1)^{\frac{1}{4}}$; $y = (x + 1)^{\frac{1}{4}} + 4$

b i $y = \sqrt[3]{(x - 7)} + 1$

ii $y = \sqrt[3]{(x + 2)} - 6$

iii $y = \sqrt[3]{(x - 2)} - 3$

iv $y = \sqrt[3]{(x + 1)} + 4$

c i $y = \dfrac{1}{(x - 7)^3} + 1$

ii $y = \dfrac{1}{(x + 2)^3} - 6$

iii $y = \dfrac{1}{(x - 2)^3} - 3$

iv $y = \dfrac{1}{(x + 1)^3} + 4$

d i $y = \dfrac{1}{(x - 7)^4} + 1$

ii $y = \dfrac{1}{(x + 2)^4} - 6$

iii $y = \dfrac{1}{(x - 2)^4} - 3$

iv $y = \dfrac{1}{(x + 1)^4} + 4$

7 a $x' = x - 3$ and $y' = y + 2$
$\therefore x = x' + 3$ and $y = y' - 2$
$\therefore y = (x - 2)^2 + 3$ maps to
$y' - 2 = (x' + 3 - 2)^2 + 3$
The image is $y = (x + 1)^2 + 5$

b $x' = x + 3$ and $y' = y - 3$
$\therefore x = x' - 3$ and $y = y' + 3$
$\therefore y = 2(x + 3)^2 + 3$ maps to
$y' + 3 = 2(x' - 3 + 3)^2 + 3$
The image is $y = 2x^2$

c $x' = x + 4$ and $y' = y - 2$
$\therefore x = x' - 4$ and $y = y' + 2$
$\therefore y = \dfrac{1}{(x - 2)^2} + 3$ maps to
$y' + 2 = \dfrac{1}{(x' - 4 - 2)^2} + 3$
The image is $y = \dfrac{1}{(x - 6)^2} + 1$

d $x' = x - 1$ and $y' = y + 1$
$\therefore x = x' + 1$ and $y = y' - 1$
$\therefore y = (x + 2)^3 + 1$ maps to
$y' - 1 = (x' + 1 + 2)^3 + 1$
The image is $y = (x + 3)^3 + 2$

e $x' = x - 1$ and $y' = y + 1$
$\therefore x = x' + 1$ and $y = y' - 1$
$\therefore y = \sqrt[3]{x - 3} + 2$ maps to
$y' - 1 = \sqrt[3]{x' + 1 - 3} + 2$
The image is $y = \sqrt[3]{x - 2} + 3$

8 a Write
$y = \dfrac{1}{x^2}$ and $y' = \dfrac{1}{(x' - 2)^2} + 3$
Therefore, choose:
$y = y' - 3$ and $x = x' - 2$
$\therefore y' = y + 3$ and $x' = x + 2$
That is, $(x, y) \rightarrow (x + 2, y + 3)$

b Write
$y = \dfrac{1}{x}$ and $y' = \dfrac{1}{(x' + 2)} - 3$

Therefore, choose:

$y = y' + 3$ and $x = x' + 2$

$\therefore y' = y - 3$ and $x' = x - 2$

That is, $(x, y) \rightarrow (x - 2, y - 3)$

c Write

$y = \sqrt{x}$ and $y' = \sqrt{x' + 4} + 2$

Therefore, choose:

$y = y' - 2$ and $x = x' + 4$

$\therefore y' = y + 2$ and $x' = x - 4$

That is, $(x, y) \rightarrow (x - 4, y + 2)$

Solutions to Exercise 3B

1 **a** $x' = x, y' = 3y$

$\therefore y = \frac{1}{x}$ maps to $\frac{y'}{3} = \frac{1}{x'}$

The image is $y = \frac{3}{x}$

b

c $x' = 3x, y' = y$

$\therefore y = \frac{1}{x}$ maps to $y' = \frac{1}{\frac{x'}{3}}$

The image is $y = \frac{3}{x}$

2 **a** $x' = x, y' = 2y$

$\therefore y = \frac{1}{x^2}$ maps to $\frac{y'}{2} = \frac{1}{(x')^2}$

The image is $y = \frac{2}{x^2}$

b

c $x' = 2x, y' = y$

$\therefore y = \frac{1}{x^2}$ maps to $y' = \frac{1}{\left(\frac{x'}{2}\right)^2}$

The image is $y = \frac{4}{x^2}$

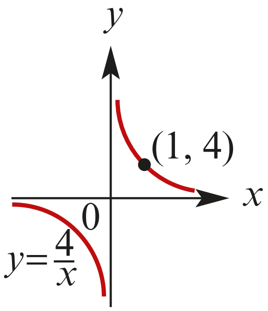
3 **a** $y = 2\sqrt{x}$

b $y = \sqrt{\frac{x}{2}}$

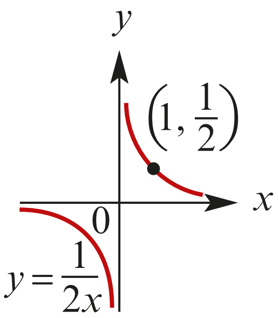
4 **a** $y = 2x^3$

b $y = \frac{x^3}{8}$

5 **a**



b



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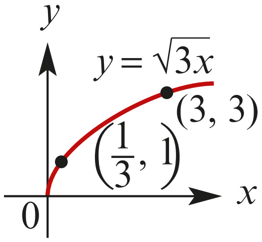

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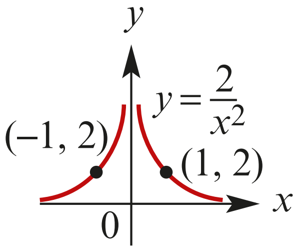
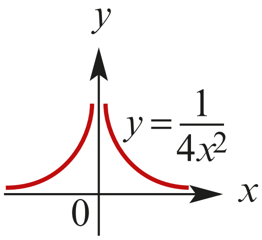


6 **a**



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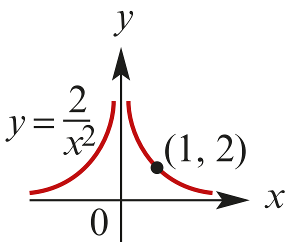

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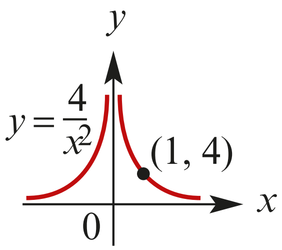

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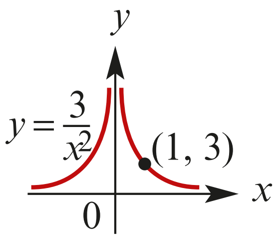

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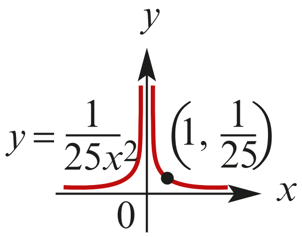

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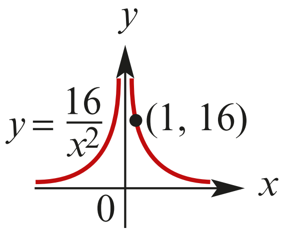
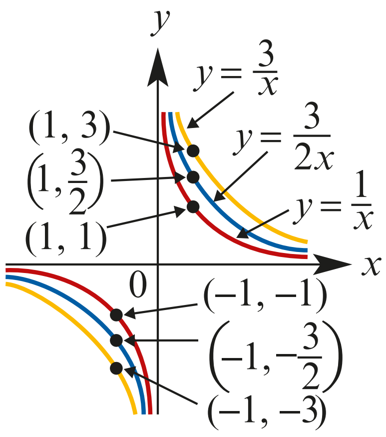


7



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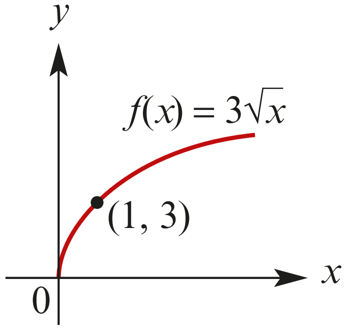


9 **a** Dilation factor $\frac{1}{5}$ from the y-axis

b Dilation factor $\sqrt{5}$ from the x-axis

10 **a** Let $y = f(x) = \frac{1}{x^2}$ and

$y' = f_1(x') = \frac{5}{(x')^2}$

Then rewrite as $y = \frac{1}{x^2}$ and

$\frac{y'}{5} = \frac{1}{(x')^2}$.

Choose $\frac{y'}{5} = y$ and $x = x'$.

One transformation is $y' = 5y$ and

$x' = x$

A dilation of factor 5 from the x-axis.

b Let $y = f(x) = \sqrt{x}$ and

$y' = f_1(x') = 4\sqrt{x'}$

Then rewrite as $y = \sqrt{x}$ and
$\frac{y'}{4} = \sqrt{x'}$.
Choose $\frac{y'}{4} = y$ and $x = x'$.
One transformation is $y' = 4y$ and
$x' = x$
A dilation of factor 4 from the x-axis.

c Let $y = f(x) = \sqrt{x}$ and
$y' = f_1(x') = \sqrt{5x'}$
Then rewrite as $y = \sqrt{x}$ and
$y' = \sqrt{5x'}$.
Choose $y' = y$ and $x = 5x'$.
One transformation is $y' = y$ and
$x' = \frac{1}{5}x$
A dilation of factor $\frac{1}{5}$ from the y-axis.

d Let $y = f(x) = \sqrt{\frac{x}{3}}$ and
$y' = f_1(x') = \sqrt{x'}$
Then rewrite as $y = \sqrt{\frac{x}{3}}$ and
$y' = \sqrt{x'}$.
Choose $y' = y$ and $\frac{x}{3} = x'$.
One transformation is $y' = y$ and
$x' = \frac{1}{3}x$
A dilation of factor $\frac{1}{3}$ from the y-axis.

e Let $y = f(x) = \frac{1}{4x^2}$ and
$y' = f_1(x') = \frac{1}{(x')^2}$
Then rewrite as $y = \frac{1}{(2x)^2}$ and
$y' = \frac{1}{(x')^2}$.
Choose $y' = y$ and $2x = x'$.
One transformation is $y' = 5y$ and
$x' = 2x$
A dilation of factor 2 from the y-axis.

11 **a** **i** $y = 4x^2$

ii $y = \frac{2}{3}x^2$

iii $y = (2x)^2 = 4x^2$

iv $y = \left(\frac{x}{5}\right)^2 = \frac{1}{25}x^2$

b **i** $y = \frac{4}{x^2}$

ii $y = \frac{2}{3x^2}$

iii $y = \frac{1}{4x^2}$

iv $y = \frac{25}{x^2}$

c **i** $y = 4\sqrt[3]{x}$

ii $y = \frac{2}{3} \times \sqrt[3]{x}$

iii $y = \sqrt[3]{2x}$

iv $y = \sqrt[3]{\frac{x}{5}}$

d **i** $y = \frac{4}{x^3}$

ii $y = \frac{2}{3x^3}$

iii $y = \frac{1}{8x^3}$

iv $y = \frac{125}{x^3}$

e **i** $y = \frac{4}{x^4}$

ii $y = \dfrac{2}{3x^4}$

iii $y = \dfrac{1}{16x^4}$

iv $y = \dfrac{625}{x^4}$

f **i** $y = 4\sqrt[4]{x}$

ii $y = \dfrac{2}{3} \times \sqrt[4]{x}$

iii $y = \sqrt[4]{2x}$

iv $y = \sqrt[4]{\dfrac{x}{5}}$

g **i** $y = 4x^{\frac{1}{5}}$

ii $y = \dfrac{2}{3}x^{\frac{1}{5}}$

iii $y = (2x)^{\frac{1}{5}}$

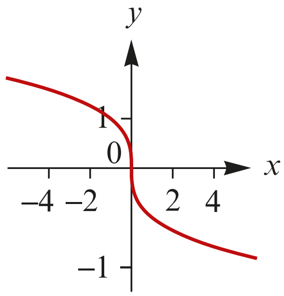
iv $y = \left(\dfrac{x}{5}\right)^{\frac{1}{5}}$

Solutions to Exercise 3C

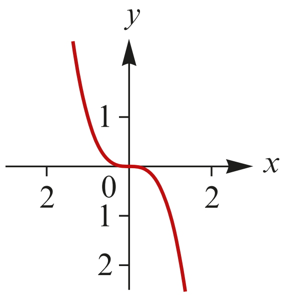
1 **a** $y = -(x - 1)^2$

b $y = (x + 1)^2$

2 **a** Domain = $\mathbb{R}$



b Domain = $\mathbb{R}$



3 Reflection in the *y*-axis

4 **a** **i** $y = -x^3$

ii $y = -x^3$

b **i** $y = -\sqrt[3]{x}$

ii $y = -\sqrt[3]{x}$

c **i** $y = \dfrac{-1}{x^3}$

ii $y = \dfrac{-1}{x^3}$

d **i** $y = \dfrac{-1}{x^4}$

ii $y = \dfrac{1}{x^4}$

e **i** $y = -x^{\frac{1}{3}}$

ii $y = -x^{\frac{1}{3}}$

f **i** $y = -x^{\frac{1}{5}}$

ii $y = -x^{\frac{1}{5}}$

g **i** $y = -x^{\frac{1}{4}}$

ii $y = (-x)^{\frac{1}{4}}$

Solutions to Exercise 3D

1 Part a will be done with the method.

a **i** The mapping is

$(x, y) \to (x + 2, 2y - 3) = (x', y')$

Hence $x' = x + 2$ and $y' = 2y - 3$

This implies $x = x' - 2$ and

$y = \dfrac{y' + 3}{2}$

$\therefore y = x^2$ maps to

$\dfrac{y' + 3}{2} = (x' - 2)^2$

That is, $y = 2(x - 2)^2 - 3$

ii The mapping is

$(x, y) \to (3x - 2, y - 4) = (x', y')$

Hence $x' = 3x - 2$ and $y' = y - 4$

This implies $x = \dfrac{x' + 2}{3}$ and

$y = y' + 4$

$\therefore y = x^2$ maps to

$y' + 4 = \left(\dfrac{x' + 2}{3}\right)^2$

That is, $y = \left(\dfrac{x + 2}{3}\right)^2 - 4$

iii The mapping is

$(x, y) \to (3x - 2, y - 4) = (x', y')$

Hence $x' = -x$ and $y' = 2y$

This implies $x = -x'$ and $y = \dfrac{y'}{2}$

$\therefore y = x^2$ maps to $y = 2x^2$

b **i** $y = 2\sqrt[3]{x - 2} - 3$

ii $y = \sqrt[3]{\dfrac{x + 2}{3}} - 4$

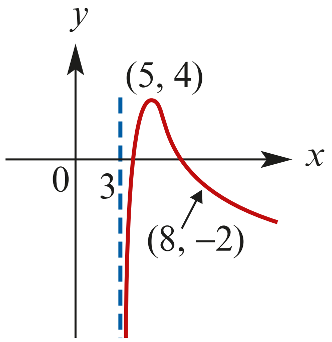
iii $y = -2\sqrt[3]{x}$

c **i** $y = \dfrac{2}{(x - 2)^2} - 3$

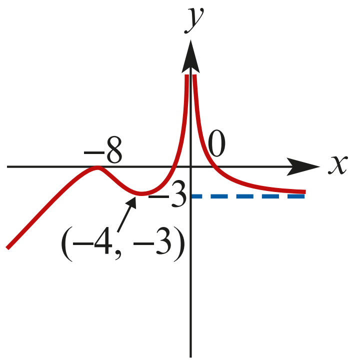
ii $y = \dfrac{9}{(x + 2)^2} - 4$

iii $y = \dfrac{2}{x^2}$

2



3



4 **a** **i** $y = -2(x - 3)^2 - 4$

ii $y = -(2(x - 3)^2) - 4$

$y = -2(x - 3)^2 + 4$

iii $y = 2(-(x - 3)^2) - 4$

$y = -2(x - 3)^2 - 4$

iv $y = 2(-(x - 3)^2 - 4)$

$y = -2(x - 3)^2 - 8$

v $y = -2((x - 3)^2 - 4)$

$y = -2(x - 3)^2 + 8$

vi $y = 2(-((x - 3)^2 - 4))$

$y = -2(x - 3)^2 + 8$

b **i** $y = -2\sqrt[3]{x - 3} - 4$

ii $y = -2\sqrt[3]{x-3} + 4$

iii $y = -2\sqrt[3]{x-3} - 4$

iv $y = -2\sqrt[3]{x-3} - 8$

v $y = -2\sqrt[3]{x-3} + 8$

vi $y = -2\sqrt[3]{x-3} + 8$

c **i** $y = \dfrac{-2}{(x-3)^2} - 4$

ii $y = \dfrac{-2}{(x-3)^2} + 4$

iii $y = \dfrac{-2}{(x-3)^2} - 4$

iv $y = \dfrac{-2}{(x-3)^2} - 8$

v $y = \dfrac{-2}{(x-3)^2} + 8$

vi $y = \dfrac{-2}{(x-3)^2} + 8$

d **i** $y = -2(x-3)^4 - 4$

ii $y = -2(x-3)^4 + 4$

iii $y = -2(x-3)^4 - 4$

iv $y = -2(x-3)^4 - 8$

v $y = -2(x-3)^4 + 8$

vi $y = -2(x-3)^4 + 8$

e **i** $y = \dfrac{-2}{(x-3)^3} - 4$

ii $y = \dfrac{-2}{(x-3)^3} + 4$

iii $y = \dfrac{-2}{(x-3)^3} - 4$

iv $y = \dfrac{-2}{(x-3)^3} - 8$

v $y = \dfrac{-2}{(x-3)^3} + 8$

vi $y = \dfrac{-2}{(x-3)^3} + 8$

f **i** $y = \dfrac{-2}{(x-3)^4} - 4$

ii $y = \dfrac{-2}{(x-3)^4} + 4$

iii $y = \dfrac{-2}{(x-3)^4} - 4$

iv $y = \dfrac{-2}{(x-3)^4} - 8$

v $y = \dfrac{-2}{(x-3)^4} + 8$

vi $y = \dfrac{-2}{(x-3)^4} + 8$

g **i** $y = \dfrac{-2}{(x-3)^2} - 4$

ii $y = \dfrac{-2}{(x-3)^2} + 4$

iii $y = \dfrac{-2}{(x-3)^2} - 4$

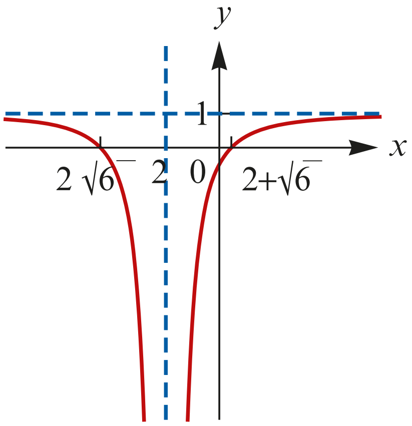
iv $y = \dfrac{-2}{(x-3)^2} - 8$

v $y = \dfrac{-2}{(x-3)^2} + 8$

vi $y = \dfrac{-2}{(x-3)^2} + 8$

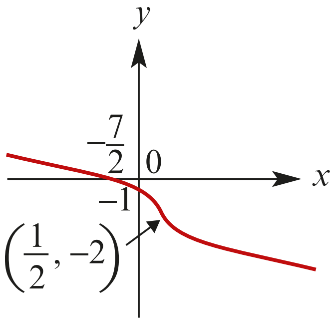
5 $y = -\sqrt{\dfrac{x+12}{3}}$

6 **a**



b $y = 1 - \frac{6}{(x+2)^2}$

7 **a**



b $y = (1 - 2x)^{\frac{1}{3}} - 2$

Solutions to Exercise 3E

1 **a** **i** Dilation of factor 2 from the x-axis, then translation 1 unit to the right and 3 units up

ii Reflection in the x-axis, then translation 1 unit to the left and 2 units up

iii Dilation of factor $\frac{1}{2}$ from the y-axis, then translation $\frac{1}{2}$ unit to the left and 2 units down

b **i** Dilation of factor 2 from the x-axis, then translation 3 units to the left

ii Translation 3 units to the left and 2 units up

iii Translation 3 units to the right and 2 units down

c **i** Translation 3 units to the left and 2 units up

ii Dilation of factor $\frac{1}{3}$ from the y-axis and dilation of factor 2 from the x-axis

iii Reflection in the x-axis, then translation 2 units up

2 **a** Translation 1 unit to the left and 6 units down

b Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{3}{2}$ units up and 1 unit to the left

c Translation 1 unit to the left and 6 units up

d Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{5}{2}$ units up and 1 unit to the left

e Dilation of factor 2 from the y-axis, then translation of 1 unit to the left and 6 units down

3 **a** Dilation of factor $\frac{1}{5}$ from the x-axis, then translation $\frac{7}{5}$ units up and 3 units to the left

b Dilation of factor 3 from the y-axis, then translation 2 units to the right and 5 units down

c Reflection in the x-axis, dilation of factor $\frac{1}{3}$ from the x-axis, translation $\frac{7}{3}$ units up, dilation of factor 3 from the y-axis, translation 1 unit to the right

d Reflection in the y-axis, translation 4 units to the right, dilation of factor $\frac{1}{2}$ from the x-axis

e Reflection in the y-axis, translation 4 units to the right, reflection in the x-axis, dilation of factor $\frac{1}{2}$ from the x-axis, translation $\frac{15}{2}$ units up

4 a Dilation of factor 2 from the x-axis, then translation 1 unit to the right and 3 units up

b Dilation of factor 2 from the x-axis, then translation 4 units to the left and 7 units down

c Reflection in the y-axis and dilation of factor 4 from the x-axis (in either order), then translation 1 unit to the right and 5 units down

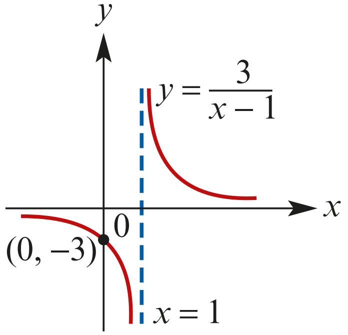
d Reflection in the x-axis, then translation 1 unit to the left and 2 units up

e Reflection in the y-axis and dilation of factor 2 from the x-axis (in either order), then translation 3 units up

f Translation 3 units to the left and 4 units down, then reflection in either axis and dilation of factor $\frac{1}{2}$ from the x-axis (in either order)

Solutions to Exercise 3F

1 **a** $f(x) = \dfrac{3}{x-1}$



Asymptotes

$y = 0$

$x - 1 = 0$

$x = 1$

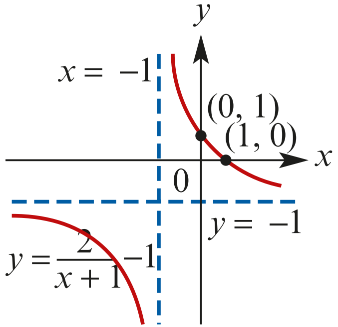
Axis intercepts

$y = \dfrac{3}{0-1} = -3$

$0 = \dfrac{3}{x-1}$ ∴ no x- axis intercept.

Range $\mathbb{R}\backslash\{0\}$

b $y = \dfrac{2}{x-1} - 1$



Asymptotes

$y = 0 - 1 = -1$

$x + 1 = 0$

$x = -1$

Axis intercepts

$0 = \dfrac{2}{x+1} - 1$

$x + 1 = 2$

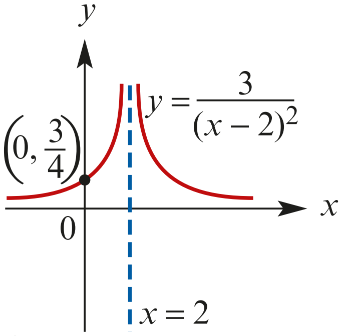
$x = 1$

$y = \dfrac{2}{0+1} - 1$

$y = 1$

Range $\mathbb{R}\backslash\{-1\}$

c $y = \dfrac{3}{(x-2)^2}$



Asymptotes

$y = 0$

$x - 2 = 0$

$x = 2$

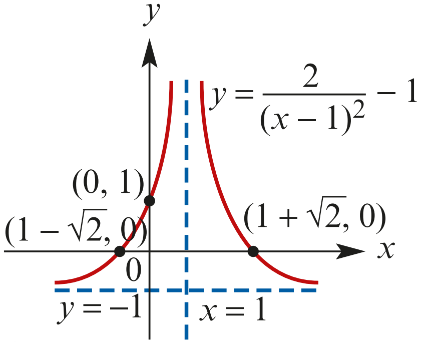
Axis intercepts

$0 = \dfrac{3}{(x-2)^2}$ ∴ no x- axis intercept.

$y = \dfrac{3}{(-2)^2} = \dfrac{3}{4}$

Range = $\mathbb{R}^+$

d $y = \dfrac{2}{(x-1)^2} - 1$



Asymptotes

$y = 0 - 1 = -1$
$x - 1 = 0$
$x = 1$

Axis intercepts

$$0 = \frac{2}{(x-1)^2} - 1$$

$x - 1 = \pm\sqrt{2}$

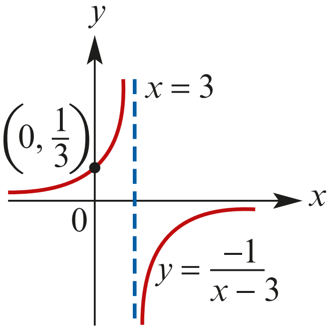
$$x = 1 \pm \sqrt{2}$$

$$y = \frac{2}{(0-1)^2} - 1$$

$$y = 1$$

Range = $(-1, \infty)$

e $y = \dfrac{-1}{x-3}$



Asymptotes

$y = 0$
$x - 3 = 0$
$x = 3$

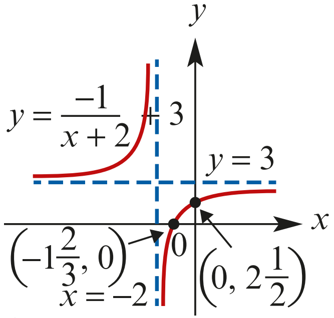
Axis intercepts

$0 = \dfrac{-1}{x-3}$ ∴ no x- axis intercept.

$y = \dfrac{-1}{0-3} = \dfrac{1}{3}$

Range = $\mathbb{R}\backslash\{0\}$

f $y = \dfrac{-1}{x+2} + 3$



Asymptotes

$y = 0 + 3 = 3$
$x + 2 = 0$
$x = -2$

Axis intercepts

$y = \dfrac{-1}{0+2} = 3 = \dfrac{5}{2}$

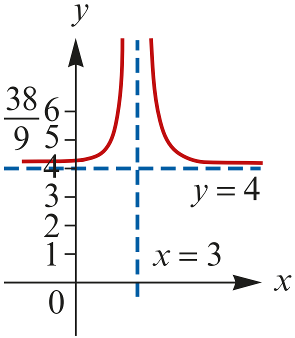
$0 = \dfrac{-1}{x+2} + 3$

$3x + 6 = 1$

$x = \dfrac{-5}{3}$

Range = $\mathbb{R}\backslash\{3\}$

g $y = \dfrac{2}{(x-3)^2} + 4$

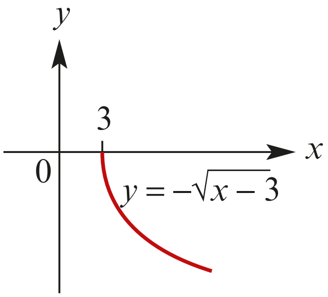


Asymptotes

$x - 3 = 0$
$x = 3$
$y = 0 + 4 = 4$

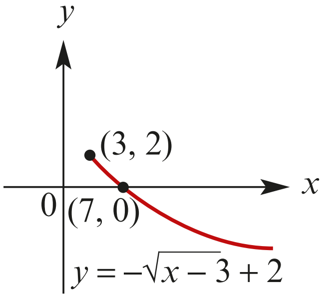
Range = $(4, \infty)$

2 a



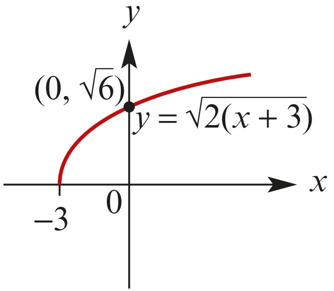
Range = $\mathbb{R}^- \cup \{0\}$

b



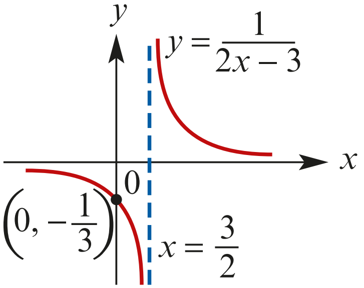
Range = $(-\infty, 2]$

c



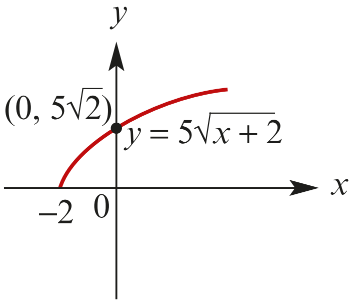
Range = $\mathbb{R}^+ \cup \{0\}$

d



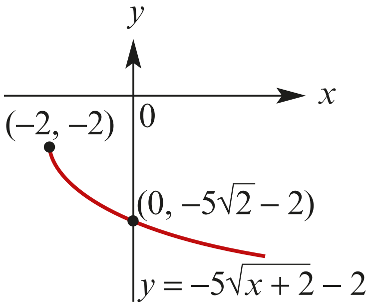
Range = $\mathbb{R} \setminus \{0\}$

e



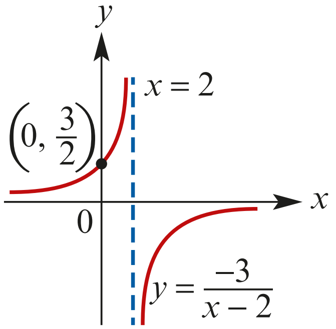
Range = $\mathbb{R}^+ \cup \{0\}$

f



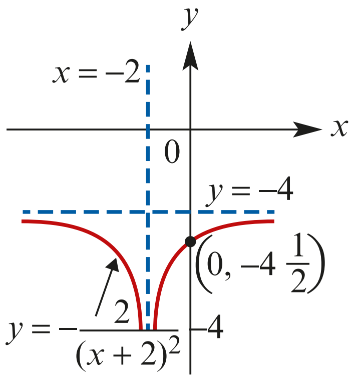
Range = $(-\infty, -2]$

g



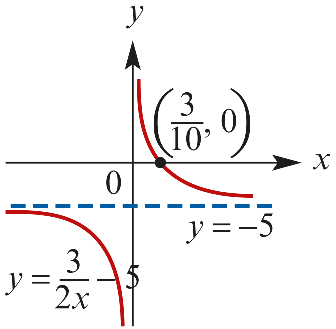
Range = $\mathbb{R} \setminus \{0\}$

h



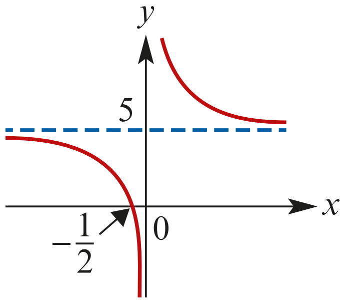
Range = $(-\infty, -4)$

i



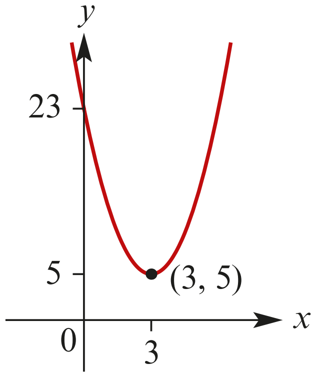
Range = $\mathbb{R} \setminus \{-5\}$

j



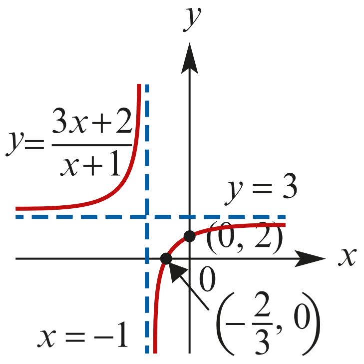
Range = $\mathbb{R} \setminus \{5\}$

k



Range = $[5, \infty)$

3

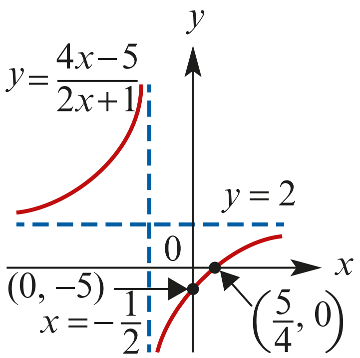


$$\frac{3x+2}{x+1} = \frac{3x+3-1}{x+1}$$

$$= 3(x+1) - \frac{1}{x+1}$$

$$= 3 - \frac{1}{x+1}$$

4

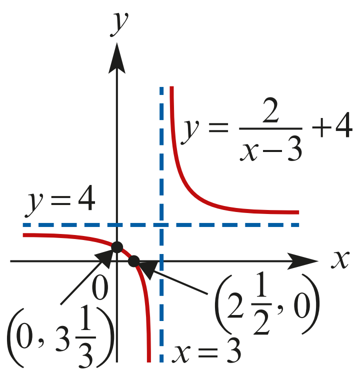


$$\frac{4x-5}{2x+1} = \frac{4x+2-7}{2x+1}$$

$$= \frac{2(2x+1)}{2x+1} - \frac{7}{2x+1}$$

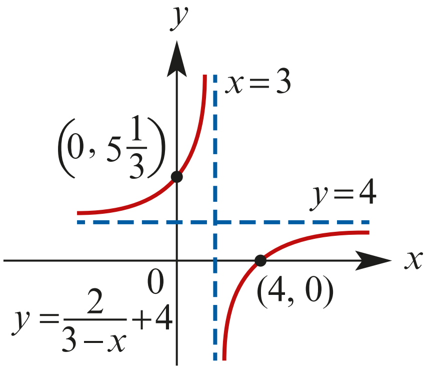
$$= 2 - \frac{7}{2x+1}$$

5 **a**



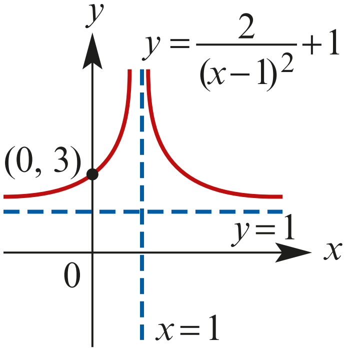
Range = ℝ\{4}

b



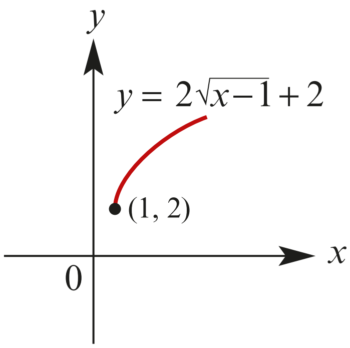
Range = R\{4}

c



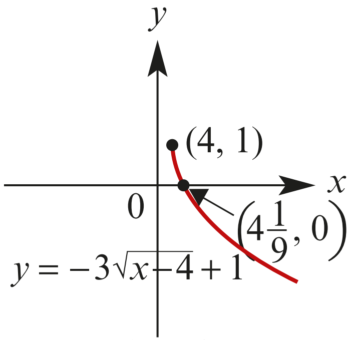
Range = (1, ∞)

d



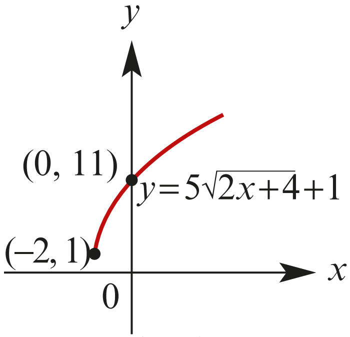
Range = [2, ∞)

e



Range = (−∞, 1]

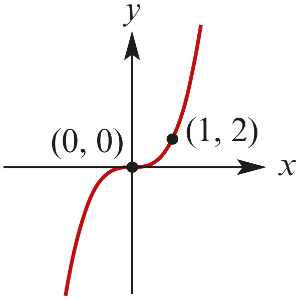
f



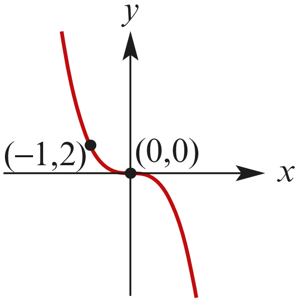
Range = [1, ∞)

Solutions to Exercise 3G

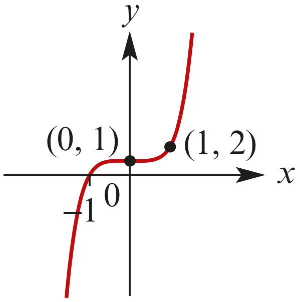
1 a



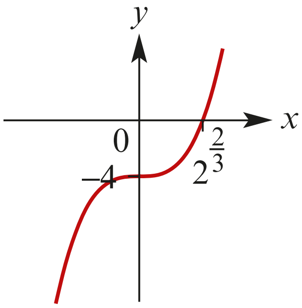
b



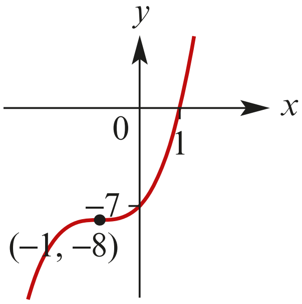
c



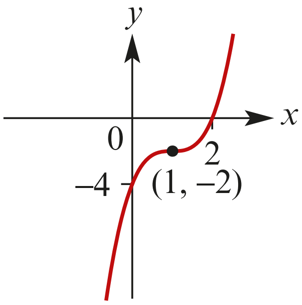
d



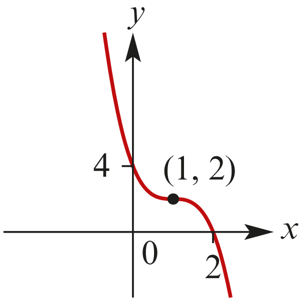
e



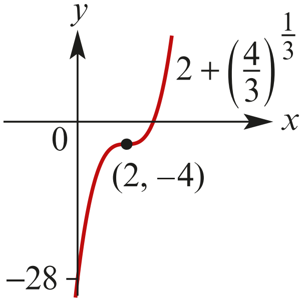
f



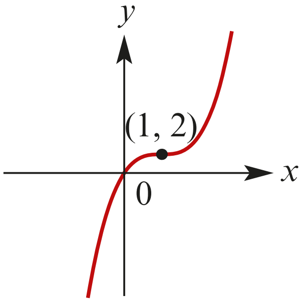
g



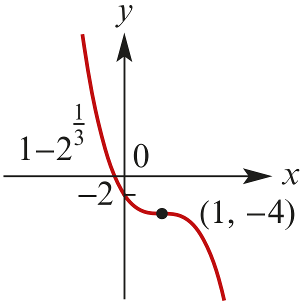
h



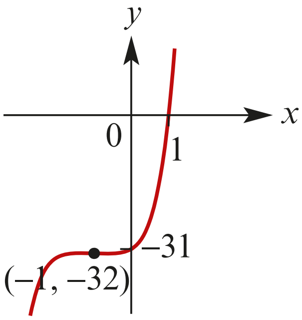
i



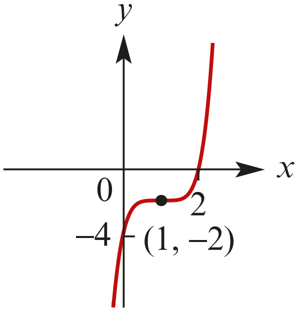
j



k



l



2 $h = 0$ and $k = 4$

$y = ax^3 + 4$

When $x = 1, y = 1$

$\therefore 1 = a + 4$

$\therefore a = -3$

$\therefore y = -3x^3 + 4$

3 **a** $y = 3x^3$

b $y = (x + 1)^3 + 1$

c $y = -(x - 2)^3 - 3$

d $y = 2(x + 1)^3 - 2$

e $y = \dfrac{x^3}{27}$

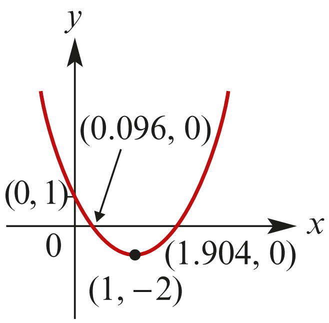
4 **a** $y = \dfrac{(3 - x)^3}{27} + 1$

b Dilation of factor 3 from the x-axis, reflection in the x-axis, then translation 1 unit to the left and 4 units up

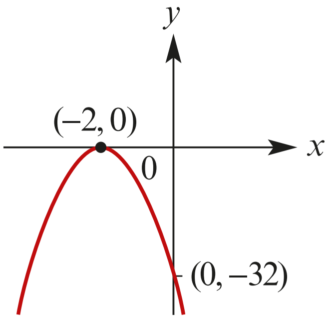
5 $y = \dfrac{(x + 2)^4}{16} - 1$

6 Dilation of factor 3 from the x-axis, reflection in the x-axis, then translation 1 unit to the right and 5 units up

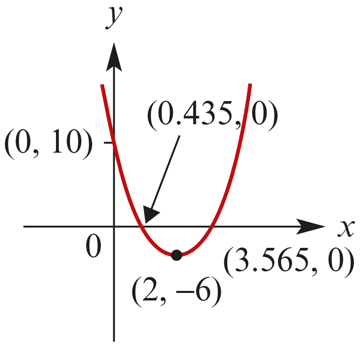
7 **a**



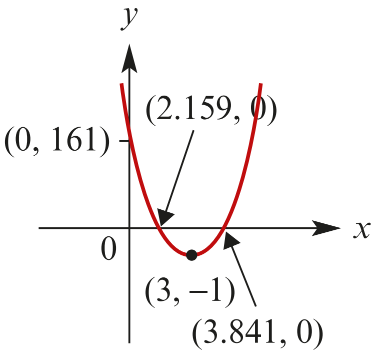
b



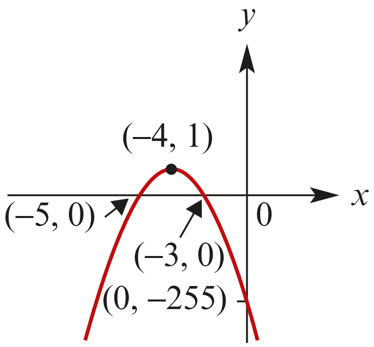
c



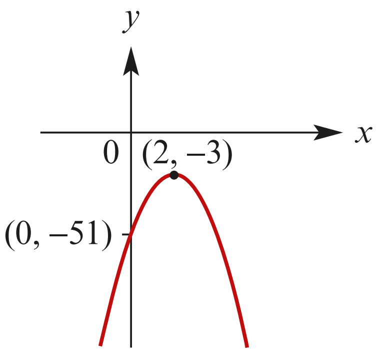
d



e



f



8 $h = -2$ and $k = 3$

Passes through $(0, -6)$

$y = a(x + 2)^4 + 3$

$-6 = 16a + 3$

$a = -\frac{9}{16}$

$y = -\frac{9}{16}(x + 2)^4 + 3$

9 $h = 1$ and $k = 7$

Passes through $(0, 23)$

$y = a(x - 1)^4 + 7$

$23 = a + 7$

$a = 16$

$y = 16(x - 1)^4 + 7$

Solutions to Exercise 3H

1

$$4 = a + b \ldots (1)$$

$$1 = \frac{a}{3} + b \ldots (2)$$

Equation (1) − Equation (2)

$$3 = \frac{2a}{3}$$

$$a = \frac{9}{2}$$

From (1)

$$b = -\frac{1}{2}$$

2 **Asymptotes**: $x = 1,\ y = 2$

$x + b = 0$ $\quad$ $y = B$

$1 + b = 0$ $\quad$ $B = 2$

$b = -1$

Point: (0, 1)

$$1 = \frac{A}{-1} + 2$$

$$A = 1$$

3

$$1 = A + B \ldots (1)$$

$$6 = 3A + B \ldots (2)$$

Equation (2) − Equation (1)

$$5 = 2A$$

$$A = \frac{5}{2}$$

From (1)

$$B = -\frac{3}{2}$$

4

$$y = A\sqrt{x} + B$$

$$5 = A\sqrt{1} + \mathrm{B} \ldots (1)$$

$$= A + B$$

$$11 = A\sqrt{16} + \mathrm{B} \ldots (2)$$

$$= 4\mathrm{A} + \mathrm{B}$$

Equation (2) − Equation (1)

$$\Rightarrow 3A = 6$$

$$A = 2$$

From (1)

$$\Rightarrow 5 = 2 + B$$

$$B = 3$$

5

$$y = \frac{A}{x^2} + B$$

$$1 = \frac{A}{1^2} + B \ldots (1)$$

$$= A + B$$

$$7 = \frac{A}{0.5^2} + B \ldots (2)$$

$$= 4A + B$$

Equation (2) − Equation (1)

$$\Rightarrow 3A = 6$$

$$A = 2$$

From (1)

$$1 = 2 + B$$

$$B = -1$$

6 $y = \dfrac{A}{(x + b)^2} + B$

Asymptotes

$x = -2 \quad y = -3$

$x + b = 0$ $\quad$ $y = 0 + B$

$-2 + b = 0 \qquad B = -3$

$b = 2$

Point: $(0, -1)$

$-1 = \frac{A}{2^2} - 3$

$A = 8$

7

$$y = \frac{a}{x^3} + b$$

$$-1 = \frac{a}{1^3} + b = a + b \ldots (1)$$

$$\frac{3}{4} = \frac{a}{2^3} + b = \frac{1}{8}a + b \ldots (2)$$

Equation (2) − Equation (1)

$$\Rightarrow \quad \frac{-7}{8}a = \frac{7}{4}$$

$$a = -2$$

From (1)

$$\Rightarrow -1 = -2 + b$$

$$b = 1$$

8

$$y = ax^{\frac{1}{3}} + b$$

$$-8 = a + b \ldots (1)$$

$$4 = -a + b \ldots (2)$$

Equation (2) + Equation (1)

$$2b = -4$$

$$b = -2$$

From (2)

$$a = -6$$

Solutions to Exercise 3I

1 $T:\mathbb{R}\to\mathbb{R}, T(x,y)=(x-2,2y+3)$

a **i** $T(-2,5)=(-2-2,2\times5+3)$
$=(-4,13)$

ii $T(4,2)=(4-2,2\times2+3)$
$=(2,7)$

b $x'=x-2, y'=2y+3$

Therefore:

$x=x'+2$ and $y=\dfrac{y'-3}{2}$

$y=2^x$ is mapped to $\dfrac{y'-3}{2}=2^{x'+2}$

Hence $y'=2\times2^{x'+2}+3$

which can be wriiten as $y'=8\times2^x+3$

2 Let $T(x,y)=(ax+h,by+k)$

Given: $T(-1,7)=(-7,-3)$ and $T(-2,-3)=(4,6)$

Hence:

$-a+h=-7\ldots(3)$

$-2a+h=4\ldots(4)$

and

$7b+k=-3\ldots(1)$

$-3b+k=6\ldots(2)$

From (1) and (2):

$a=-11$ and $h=-18$

From (3) and (4):

$b=-\dfrac{9}{10}$ and $k=\dfrac{33}{10}$

3 **a** $T_2(T_1(x,y))=T_2(2x,2y-3)$

$=(-2x+2,-2y+3-3)$

$=(-2x+2,-2y)$

b $T_1(T_2(x,y))=T_1(-x+2,y-3)$

$=(-2x+4,2(y-3)-3)$

$=(-2x+2,2y-9)$

4 Let $T^{-1}(x,y)=(a,b)$.

Then,

$T(T^{-1}(x,y))=T(a,b)$

$=(-2a+2,b-3)$

Also $T(T^{-1}(x,y))=(x,y)$

Hence

$x=-2a+2$ and $y=b-3$

$a=\dfrac{x-2}{-2}$ and $b=y+3$

Hence $T^{-1}(x,y)=\left(\dfrac{2-x}{2},y+3\right)$

5 $[2,5]\to[2\times2+6,2\times5+6]=[10,16]$

$[-3,7]\to[-3-3,7-3]=[-6,4]$

6 **a** $f(0)=0, f(4)=16$. Function is strictly increasing. Therefore range $=[0,16]$

b $T(x,y)=(-2x,2y+4)$

$x'=-2x$ and $y'=2y+4$

$x=-\dfrac{x'}{2}$ and $y=\dfrac{y'-4}{2}$

Therefore $y=x^2$ is mapped to

$\dfrac{y'-4}{2}=\left(-\dfrac{x'}{2}\right)^2$

Simplifyng.

$y'=\dfrac{(x')^2}{2}+4$

Domain $=[-8,0]$, Range $[4,36]$

7 $T_1(x,y)=\left(\dfrac{1}{2}x,y-3\right)$,$T_2(x,y)=(-x,y+3)$ and $T_3(x,y)=(-2x,y-3)$.

a $T_2(T_1(x,y))=T_2\left(\dfrac{1}{2}x,y-3\right)$

$=\left(-\dfrac{1}{2},y-3+3\right)$

$=\left(-\dfrac{1}{2}x,y\right)$

b $$\begin{aligned} T_1(T_2(x,y)) &= T_1(-x, y+3) \\ &= \left(-\frac{1}{2}x, y+3-3\right) \\ &= \left(-\frac{1}{2}x, y\right) \end{aligned}$$

c $$\begin{aligned} T_3(T_1(x,y)) &= T_3\left(\frac{1}{2}x, y-3\right) \\ &= \left(-2\times\frac{1}{2}x, y-3-3\right) \\ &= (-x, y-6) \end{aligned}$$

d $$\begin{aligned} T_1(T_3(x,y)) &= T_1(-2x, y-3) \\ &= \left(\frac{1}{2}\times -2x, y-3-3\right) \\ &= (-x, y-6) \end{aligned}$$

e $$\begin{aligned} T_2(T_3(x,y)) &= T_2(-2x, y-3) \\ &= \left(2x, y-3+3\right) \\ &= (2x, y) \end{aligned}$$

f $$\begin{aligned} T_3(T_2(x,y)) &= T_3(-x, y+3) \\ &= (2x, y+3-3) \\ &= (2x, y) \end{aligned}$$

8 a Let $T^{-1}(x,y) = (a,b)$.
Then,
$$\begin{aligned} T(T^{-1}(x,y)) &= T(a,b) \\ &= (-a+2, -b-3) \end{aligned}$$
Also $T(T^{-1}(x,y)) = (x,y)$
Hence
$x = -a+2$ and $y = -b-3$
$a = -x+2$ and $b = -y-3$
Hence $T^{-1}(x,y) = (2-x, -y-3)$

b Let $T^{-1}(x,y) = (a,b)$.
Then,
$$\begin{aligned} T(T^{-1}(x,y)) &= T(a,b) \\ &= (-a+2, -b-3) \end{aligned}$$
Also $T(T^{-1}(x,y)) = (x,y)$
Hence
$x = a+2$ and $y = b-3$
$a = x-2$ and $b = y+3$
Hence $T^{-1}(x,y) = (x-2, y+3)$

c Let $T^{-1}(x,y) = (a,b)$.
Then,
$$\begin{aligned} T(T^{-1}(x,y)) &= T(a,b) \\ &= (-3a-2, 6-b) \end{aligned}$$
Also $T(T^{-1}(x,y)) = (x,y)$
Hence
$x = -3a-2$ and $y = 6-b$
$a = \dfrac{x+2}{-3}$ and $b = 6-y$
Hence $T^{-1}(x,y) = \left(-\dfrac{x+2}{3}, 6-y\right)$

d Let $T^{-1}(x,y) = (a,b)$.
Then,
$$\begin{aligned} T(T^{-1}(x,y)) &= T(a,b) \\ &= (-2a+3, 4-b) \end{aligned}$$
Also $T(T^{-1}(x,y)) = (x,y)$
Hence
$x = -2a+3$ and $y = 4-b$
$a = \dfrac{x-3}{-2}$ and $b = 4-y$
Hence $T^{-1}(x,y) = \left(\dfrac{3-x}{2}, 4-y\right)$

9 a $f(x)$ is a strictly increasing function.
Range = $[-1, 8]$.

b Let $x' = -x+3$ and $y' = -2y+4$.
This implies $x = -x'+3$ and
$y = \dfrac{4-y'}{2}$
Hence $y = f(x) = x^3$ is mapped to
$\dfrac{4-y'}{2} = (-x'+3)^3$
That is,

$$4 - y' = 2(3 - x')^3$$
$$-y'2(3 - x')^3 - 4$$
$$y' = 2(x' - 3)^3 + 4$$

For the domain. The transformation has a reflection in the y-axis.
$[-1, 2] \rightarrow [-2 + 3, 1 + 3] = [1, 4]$
For the range.The transformation has a reflection in the x-axis.
$[-1, 8] \rightarrow [-16 + 4, 2 + 4] = [-12, 6]$

10 **a** $T_1(x, y) = (x - 5, y + 2)$ and $T_2(x, y) = \left(-x, \frac{1}{2}y\right)$.

i $T_1(T_2(x, y)) = T_1\left(-x, \frac{1}{2}y\right)$
$= \left(-x - 5, \frac{1}{2}y + 2\right)$

ii $T_2(T_1(x, y)) = T_1(x - 5, y + 2)$
$= \left(5 - x, \frac{1}{2}(y + 2)\right)$

iii $T_1(T_1(x, y)) = T_1(x - 5, y + 2)$
$= (x - 10, y + 4)$

b **i**
- a reflection in the y-axis.
- a dilation of factor $\frac{1}{2}$ from the x-axis.
- a translation of 5 units in the negative direction of the x-axis.
- a translation of 2 units in the positive direction of the y-axis.

ii
- a reflection in the y-axis.
- a dilation of factor $\frac{1}{2}$ from the x-axis.
- a translation of 5 units in the positive direction of the x-axis.
- a translation of 1 units in the positive direction of the y-axis.

iii
- a translation of 10 units in the negative direction of the x-axis.
- a translation of 4 units in the positive direction of the y-axis.

11 $T_1(x, y) = (3x, 2y)$,
$T_2(x, y) = (x + 3, y - 2)$
and $T_3(x, y) = (-x, y)$

a **i**
$T_1(T_2(T_3(x, y))) = T_1(T_2(-x, y))$
$= T_1(-x + 3, y - 2)$
$= (-3x + 9, 2y - 4)$

ii
$T_2(T_1(T_3(x, y))) = T_2(T_1(-x, y))$
$= T_1(-3x, 2y)$
$= (-3x + 3, 2y - 2)$

iii
$T_3(T_1(T_2(x, y))) = T_3(T_1(x + 3, y - 2))$
$= T_1(3x + 9, 2y - 4)$
$= (-3x - 9, 2y - 4)$

b **i**
- a reflection in the y-axis.
- a dilation of factor 3 from the y-axis.
- a dilation of factor 2 from the x-axis.
- a translation of 9 units in the positive direction of the x-axis.

- a translation of 4 units in the negative direction of the y-axis.

ii
- a reflection in the y-axis.
- a dilation of factor 3 from the y-axis.
- a dilation of factor 2 from the x-axis.
- a translation of 3 units in the positive direction of the x-axis.
- a translation of 4 units in the negative direction of the y-axis.

iii
- a reflection in the y-axis.
- a dilation of factor 3 from the y-axis.
- a dilation of factor 2 from the x-axis.
- a translation of 9 units in the ngative direction of the x-axis.
- a translation of 4 units in the negative direction of the y-axis.

12 Let $y = \sqrt{x}$ and $y' = -3\sqrt{2x' - 5} + 6$

Rearrange the second equation to:

$$\frac{y' - 6}{-3} = \sqrt{2x' - 5}$$

Let $y = \dfrac{y' - 6}{-3}$ and $x = 2x' - 5$

Then,

$$y' = -3y + 6 \text{ and } x' = \frac{x+5}{2} = \frac{x}{2} + \frac{5}{2}$$

Hence choose: $a = \dfrac{1}{2}, h = \dfrac{5}{2}$ and $b = -3, k = 6$

13 a Let $x = \dfrac{1}{5y + 2}$

then $5y + 2 = \dfrac{1}{x}$

$$5y = \frac{1}{x} - 2$$

$$y = \frac{1}{5x} - \frac{2}{5}$$

Therefore,

$$f^{-1} : \mathbb{R}\backslash\{0\}, f^{-1}(x) = \frac{1}{5x} - \frac{2}{5}$$

b Let $y = \dfrac{1}{5x - 2}$ and $y' = \dfrac{1}{5x'} - \dfrac{2}{5}$

Rearrange the second equation:

$$y' = \frac{1}{5x'} - \frac{2}{5}$$

$$y' + \frac{2}{5} = \frac{1}{5x'}$$

Let $y' + \dfrac{2}{5} = y$ and $5x' = 5x - 2$

Therefore $y' = y - \dfrac{2}{5}$ and $x' = x - \dfrac{2}{5}$

14 Let $T_1 = (a_1x + h_1, b_1y + k_1)$ and $T_2 = (a_2x + h_2, b_2y + k_2)$

a

$$\begin{aligned} T_1(T_2(x,y)) &= T_1(a_2x + h_2, b_2y + k_2) \\ &= (a_1(a_2x + h_2) + h_2, b_1(b_2y + k_2) + k_1) \\ &= (a_1a_2x + a_1h_2 + h_1, b_1b_2y + b_1k_2 + k_1) \end{aligned}$$

Let $S = T_1(T_2(x,y))$

Let $S^{-1}(x,y) = (r,s)$

Then

$$\begin{aligned} S(S^{-1}(x,y)) &= S(r,s) \\ &= (a_1a_2r + a_1h_2 + h_1, b_1b_2s + b_1k_2 + k_1) \end{aligned}$$

Also $S(S^{-1}(x,y)) = (x,y)$

Solve $a_1a_2r + a_1h_2 + h_1 = x$ for r and $b_1b_2s + b_1k_2 + k_1 = y$ for s

We have:

$$r = \frac{1}{a_1a_2}(x - (a_1h_2 + h_1))$$

and $s = \dfrac{1}{b_1b_2}(y - (b_1k_2 + k_1))$

Therefore$(T_1 \circ T_2)^{-1}(x, y) =$
$\left(\frac{1}{a_1 a_2}\left(x - (a_1 h_2 + h_1)\right), \frac{1}{b_1 b_2}\left(y - (b_1 k_2 + k_1)\right)\right)$

b $T_1^{-1}(x, y) = \left(\frac{1}{a_1}(x - h_1), \frac{1}{b_1}(y - k_1)\right)$

$T_2^{-1}(x, y) = \left(\frac{1}{a_2}(x - h_2), \frac{1}{b_2}(y - k_2)\right)$

$$T_2^{-1}(T_1^{-1}(x, y)) = T_2^{-1}\left(\frac{1}{a_1}(x - h_1), \frac{1}{b_1}(y - k_1)\right)$$

$$= \left(\frac{1}{a_2}\left(\frac{1}{a_1}(x - h_1)\right) - h_2, \frac{1}{b_2}\left(\frac{1}{b_1}(y - k_1)\right) - k2\right)$$

$$= \left(\frac{1}{a_1 a_2}(x - h_1 - a_2 h_2), \frac{1}{b_1 b_2}(x - k_1 - a_2 k_2)\right)$$

c From **a & b** $(T_1 \circ T_2)^{-1} = T_2^{-1} \circ T_1^{-1}$

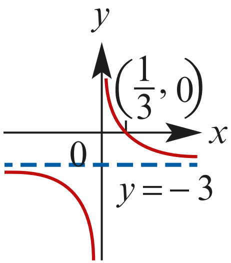
Solutions to technology-free questions

1 **a** $y = \frac{1}{x} - 3,\ x \neq 0$

no y intercept

$y = 0$: $x = \frac{1}{3}$

asymptotes: $x = 0$ & $y = -3$

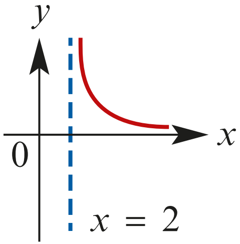


range = $\mathbb{R}\backslash\{-3\}$

b $y = \frac{1}{x-2},\ x > 2$

no intercepts

asymptotes: $x = 2$ & $y = 0$



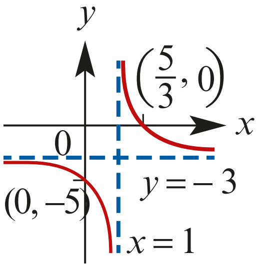
range = $(0, \infty)$

c $y = \frac{2}{x-1} - 3,\ x \neq 1$

$x = 0$: $y = -5$

$y = 0$: $\frac{2}{x-1} = 3 \Rightarrow x = \frac{5}{3}$

asymptotes: $x = 1$ & $y = -3$

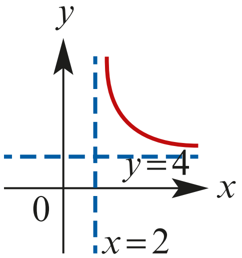


range = $\mathbb{R}\backslash\{-3\}$

d $y = -\frac{3}{2-x} + 4,\ x > 2$

no intercepts

asymptotes: $x = 2$ & $y = 4$



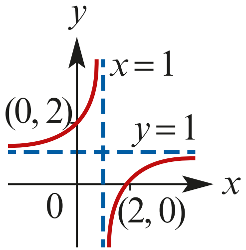
range = $(4, \infty)$

e $y = 1 - \frac{1}{x-1},\ x \neq 1$

$x = 0$: $y = 2$

$y = 0$: $\frac{1}{x-1} = 1 \Rightarrow x = 2$

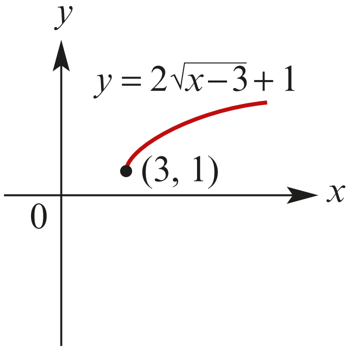
asymptotes: $x = 1$ & $y = 1$



range = $R\backslash\{1\}$

2 **a** $y = 2\sqrt{x-3} + 1$

$x \geq 3$; $y \geq 1$; endpoint $(3, 1)$



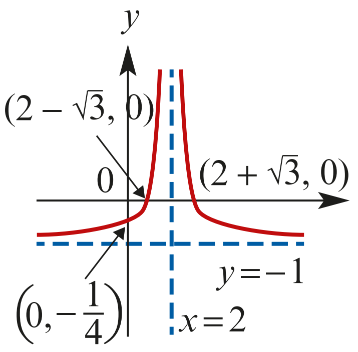
b $y = \frac{3}{(x-2)^2} - 1$

$x = 0$: $y = -\frac{1}{4}$

$y = 0$: $\frac{3}{(x-2)^2} = 1$

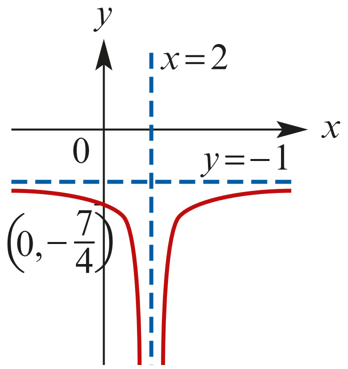
$(x-2)^2 = 3 \Rightarrow \ x = 2 \pm \sqrt{3}$

asymptotes: $x = 2$ & $y = -1$

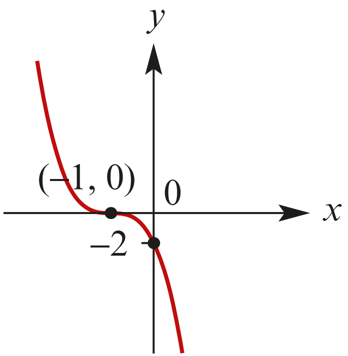


c $y = \dfrac{-3}{(x-2)^2} - 1$

This is a reflection in the line $y = -1$ of the graph in part **b** above. There are no x intercepts, the y intercept is at $y = -\dfrac{7}{4}$ and the asymptotes are the same.



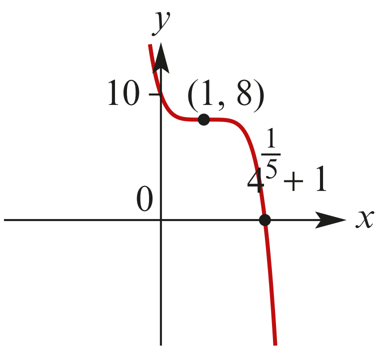
3 a



Point of zero gradient $(-1, 0)$;

Axis intercepts $(-1, 0)$, $(0, -2)$

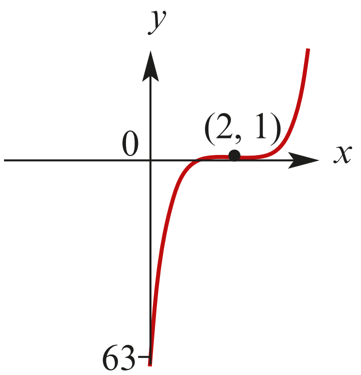
b



Point of zero gradient $(1, 8)$;

Axis intercepts $(4^{\frac{1}{5}}, 0)$, $(0, 10)$

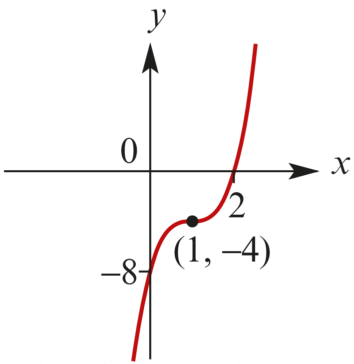
c



Point of zero gradient $(2, 1)$;

Axis intercepts $(-\tfrac{1}{2})^{\frac{1}{5}} + 2, 0)$, $(0, -63)$

d



Point of zero gradient $(1, -4)$;

Axis intercepts $(2, 0)$, $(0, -8)$

4 $y = a\sqrt{x} + b$

$(1, 6)$ and $(16, 12)$ lie on the curve

$6 = a + b \ldots (1)$

$12 = 4a + b \ldots (2)$

Subtract (1) from (2)

$6 = 3a$

$a = 2$

$\therefore b = 4$

5 $x' = x - 4$ and $y' = -2y - 1$

$\therefore x = x' + 4$ and $y = -\dfrac{y' + 1}{2}$

$\therefore$ the image of $y = \sqrt{x}$ under this transformation is $-\dfrac{y' + 1}{2} = \sqrt{x' + 4}$

The image is $y = -2\sqrt{x + 4} - 1$

6 $x' = 3x - 4$ and $y' = -y - \dfrac{1}{2}$

$\therefore x = \dfrac{x' + 4}{3}$ and $y = -y' - \dfrac{1}{2}$

$\therefore$ the image of $y = 2\sqrt{x - 4}+3$ under this transformation is

$$-y' - \frac{1}{2} = 2\sqrt{\frac{x' + 4}{3} - 4} + 3$$

The image is $y = -2\sqrt{\dfrac{x - 8}{3}} - \dfrac{7}{2}$

7 $(1, 3)$: $3 = a + b \quad \ldots (1)$

$(3,7)$: $7 = \dfrac{a}{3} + b \quad \ldots (2)$

Subtract (1) from (2):

$\dfrac{a}{3} - a = 4$

$-\dfrac{2a}{3} = 4$

$a = -6$

Substitute into (1): $b = 9$

8 **a** $(x, y) \to (-x, y) \to (-2x, y) \to (-2x + 4, y + 6) = (x', y')$

$\therefore x' = -2x + 4$ and $y' = y + 6$

$\therefore x = \dfrac{4 - x'}{2}$ and $y = y' - 6.$

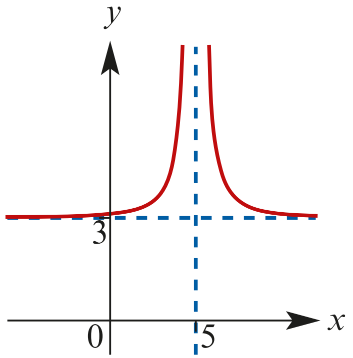
$\therefore y = -x^2$ maps to

$y' - 6 = -\left(\dfrac{4 - x'}{2}\right)^2$ That is to

$y = -\left(\dfrac{x - 4}{2}\right)^2 + 6$

b Reflection in the x-axis, dilation of factor 4 from the x-axis, then translate1 unit to the left and 6 units up

9 Dilation of factor 3 from the x-axis, then translation 5 units to the right and 3 units up



Asymptotes $x = 5$, $y = 3$; Intercept $\left(0, \dfrac{78}{25}\right)$

10 Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{3}{2}$ units up

11 Dilation of factor $\frac{1}{2}$ from the x-axis, then translation 3 units to the left and 2 units down

Solutions to multiple-choice questions

1 B $(3, -4) \to (3, -1) \to (3, 1)$

2 B $y = x^3 + 4 \to y = x^3 + 1 \to y = (x - 2)^3 + 1$

3 B

4 E $y = x^2 \to y = -x^2 \to y = -(x + 4)^2 - 3$

5 D Asymptotes at $x = -3$ and $y = -2$

$\therefore b = 3$ and $c = -2$

6 A Let $y = x^{\frac{1}{3}}$ Reflection in the y-axis: $y = -x^{\frac{1}{3}}$ Dilation by a factor of 5 units from the x-axis: $y = -5x^{\frac{1}{3}}$

7 D $x' = 3x - 2$ and $y' = -y - 1$

$\therefore x = \dfrac{x' + 2}{3}$ and $y = -y' - 1$

The image is $-y' - 2 = \sqrt[3]{\dfrac{x' + 2}{3}}$

$\therefore y = -\sqrt[3]{\dfrac{x' + 2}{3}} - 2$

8 A Rearranging $\dfrac{y + 4}{3} = \dfrac{1}{2x + 1}$

Choose $x = 2x' + 1$ and $y = \dfrac{y' + 4}{3}$

$\therefore x' = \dfrac{1}{2}x - \dfrac{1}{2}$ and $y' = 3y - 4$

9 A Rearrange $-\dfrac{y - 3}{5} = \dfrac{1}{2x - 1}$

Choose $x' = 2x - 1$ and

$y' = -\dfrac{y - 3}{5} = -\dfrac{y}{5} + \dfrac{3}{5}$

10 A $g(f(x) = (3x - 2)^2 - 4(3x - 2) + 2.$

Therefore $x = 3x' - 2$

$\therefore x' = \dfrac{x + 2}{3} = \dfrac{x}{3} - \dfrac{2}{3}$

.

Solutions to extended-response questions

1 **a** $\mathbb{R}\backslash\{-2\}$

b
- dilation of factor 24 from the x-axis
- translation of 2 in the negative direction of the x-axis
- translation of 6 in the negative direction of the y-axis

c $f(0) = \frac{24}{2} - 6 = 12 - 6 = 6$

$\therefore$ y axis intercept is 6: $(0, 6)$

$f(x) = 0$ implies

$$\frac{24}{x+2} - 6 = 0$$

$$\therefore 24 = 6(x+2)$$

$$\therefore 24 = 6x + 12$$

$$\therefore x = 2$$

$y = f(x)$ intercepts with the x axis at $(2, 0)$

d $g{:}(-2, \infty) \to R,\ g(x) = \frac{24}{x+2} - 6$

Consider $x = \frac{24}{y+2} - 6$

i.e. $(y+2)x = 24 - 6(y+2)$

$\therefore yx + 6y = 24 - 12 - 2x$

$y(x+6) = 12 - 2x$

$\therefore y = \frac{12 - 2x}{x+6}$

$\therefore g^{-1}(x) = \frac{12-2x}{x+6} = -2 + \frac{24}{x+6}$

e $\therefore$ domain of g^{-1} = range of $g = (-6, \infty)$

Forg$(x) = x$

$$\frac{24}{x+2} - 6 = x$$

f,g i.e. $\dfrac{12-2x}{x+6} = x$

$\therefore 12 - 2x = x^2 + 6x$

$\therefore x^2 + 8x - 12 = 0$

$\therefore x = \dfrac{-8 \pm \sqrt{64+48}}{2}$

$= \dfrac{-8 \pm \sqrt{112}}{2}$

$= \dfrac{-8 \pm 4\sqrt{7}}{2}$

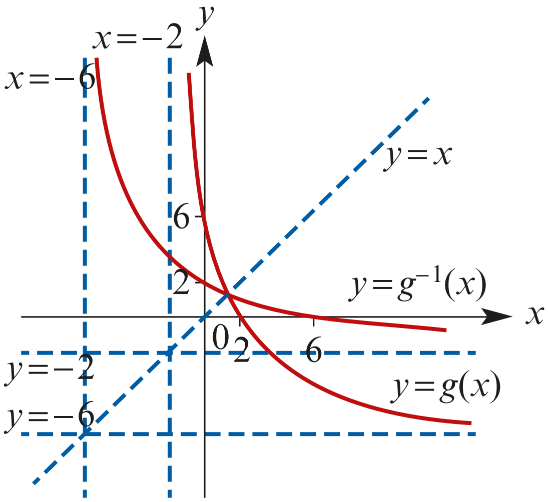
$= -4 \pm 2\sqrt{7}$

But $x \in (-6, \infty)$

Therefore $x = -4 + 2\sqrt{7}$

The graphs of $y = g(x)$ and $y = g^{-1}(x)$ intersect where $y = x$

$\therefore$ they intersect at $x = -4 + 2\sqrt{7}$



2 $f : D \to R,\ f(x) = 4 - 2\sqrt{2x+6}$

a $2x + 6 \geq 0$

i.e. $x \geq -3$

$\therefore$ domain is $[-3, \infty)$

b
- dilation of factor $\dfrac{1}{2}$ from the y axis
- dilation of factor 2 from the x axis
- reflection in the x axis
- translation 3 units in the negative direction of the x axis
- translation 4 units in the positive direction of the y axis

c $f(0) = 4 - 2\sqrt{6}$

$\therefore y = f(x)$ cuts the y axis at $(0, 4 - 2\sqrt{6})$

When $4 - 2\sqrt{2x+6} = 0$

$4 = 2\sqrt{2x+6}$

i.e. $2 = \sqrt{2x+6}$

$\therefore 4 = 2x + 6$

$\therefore x = -1$

$\therefore y = f(x)$ cuts the x axis at $(-1, 0)$

d Consider $x = 4 - 2\sqrt{2y+6}$

Then $2\sqrt{2y+6} = 4 - x$

Squaring both sides yields

$$4(2y+6) = (4-x)^2$$

$$\therefore 8y + 24 = 16 - 8x + x^2$$

$$\begin{aligned}\therefore y &= \frac{1}{8}(x^2 - 8x - 8)\\ &= \frac{1}{8}(x^2 - 8x + 16 - 24)\\ &= \frac{1}{8}(x-4)^2 - 3\end{aligned}$$

i.e. $f^{-1}(x) = \frac{1}{8}(x-4)^2 - 3$

e The domain of f^{-1} = range of $f = (-\infty, 4]$

f,g

$$f(x) = x$$

$$4-2\sqrt{2x+6} = x$$

implies $2\sqrt{2x+6} = 4 - x$

$$\therefore 4(2x+6) = 16 - 8x + x^2$$

$$\therefore 8x + 24 = 16 - 8x + x^2$$

$$\therefore x^2 - 16x - 8 = 0$$

$$\therefore x = \frac{16 \pm \sqrt{256+32}}{2} = \frac{16 \pm \sqrt{288}}{2} = \frac{16 \pm 12\sqrt{2}}{2}$$

$x = 8 \pm 6\sqrt{2}$

and the required solution is $x = 8 - 6\sqrt{2}$

The curves intersect at two other points

Consider $4 - 2\sqrt{2x+6} = \frac{1}{8}(x^2 - 8x - 8)$

Use a CAS calculator to find the other solutions.

It can be shown that they intersect on the line

$y = -x$

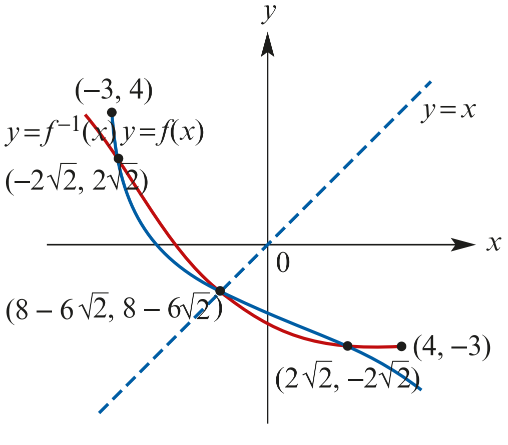
$$4 - 2\sqrt{2x+6} = -x$$

$$-2\sqrt{2x+6} = -x - 4$$

$$4(2x+6) = 16 + 8x + x^2$$

$$\therefore x^2 = 8$$

$$x = \pm 2\sqrt{2}$$



3 a i $(x, y) \to (x,\ ky)$, so $(25, 625) \to (25, 15)$

$\therefore k = \dfrac{15}{625} = \dfrac{3}{125}$

Dilation of factor $\dfrac{3}{125}$ from the x axis

ii $(x, y) \to (x, -y)$

iii $(x, y) \to (x + 25, y + 15)$

iv $(x, y) \to \left(x + 25, \dfrac{-3}{125}y + 15\right)$

b i $y = \dfrac{-3}{125}(x - 25)^2 + 15$

ii $(x, y) \to (x + 50, y)$

iii $y = \dfrac{-3}{125}(x - 75)^2 + 15$

c i Dilation factor from the x axis

$(x, y) \to (x,\ ky)$

$\left(\dfrac{m}{2}, \dfrac{m^2}{4}\right) \to \left(\dfrac{m}{2}, n\right)$

$$\therefore k = \frac{n}{\frac{m^2}{4}} = \frac{4n}{m^2}$$

reflection in x axis $(x, y) \to (x, -y)$

translation $(x, y) \to \left(x + \dfrac{m}{2}, y + n\right)$

overall $(x, y) \to \left(x + \dfrac{m}{2}, \dfrac{-4n}{m^2}y + n\right)$

ii $y = \dfrac{-4n}{m^2}\left(x - \dfrac{m}{2}\right)^2 + n$

iii $y = \dfrac{-4n}{m^2}\left(x - \dfrac{3m}{2}\right)^2 + n$

4 a $\mathbb{R}\backslash\left\{\dfrac{4}{3}\right\}$

b $a = \dfrac{4}{3}$

c Consider

$$x = \frac{3}{(3y-4)^2} + 6$$

$$x - 6 = \frac{3}{(3y-4)^2}$$

$$\frac{x-6}{3} = \frac{1}{(3y-4)^2}$$

$$\therefore (3y-4)^2 = \frac{3}{x-6}$$

$$\therefore y = \frac{1}{3}\sqrt{\frac{3}{x-6}} + \frac{4}{3} \quad \text{as range of } f^{-1} = \text{domain of } f = \left(\frac{4}{3}, \infty\right)$$

d Consider $\dfrac{3}{(3x-4)^2} + 6 = x$ as $y = f(x)$ and $y = f^{-1}(x)$ intersect on the line $y = x$

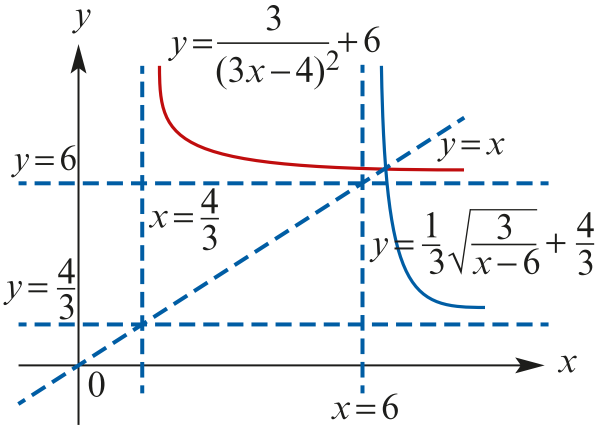
$\dfrac{3}{(3x-4)^2} = x - 6$

$3 = (x-6)(3x-4)^2$

$x = 6.015$

(Solve the equation with the 'solve' command of a CAS calculator.)

e



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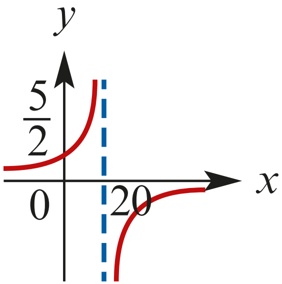


$g(x) = \dfrac{50x}{20-x}$

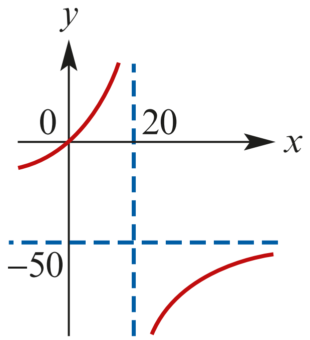
ȧ $\dfrac{1000}{20-x}-50$

$$= \frac{1000-50(20-x)}{20-x}$$

$$= \frac{1000-1000+50x}{20-x}$$

$$= \frac{50x}{20-x} = g(x)$$

ii



iii $20f(x)-50 = 20\left(\dfrac{50}{20-x}\right)-50$

$$= \frac{1000-50(20-x)}{20-x}$$

$$= \frac{1000-1000+50x}{20-x}$$

$$= \frac{50x}{20-x} = g(x)$$

b Consider $x = \dfrac{50y}{20-y}$

$(20-y)x = 50y$

$\therefore 20x = 50y + yx$

$\therefore 20x = y(50+x)$

$y = \dfrac{20x}{50+x}$

$\therefore g{-}1(x) = \dfrac{20x}{50+x}$

6 **a** **i** $(x, y) \to (x+3, y+5) \to (y+5, x+3)$

(x, y) maps to a unique point (x', y')

Hence $x' = y+5$ and $y' = x+3$

Hence $y = x'-5$ and $x = y'-3$

Therefore the graph of $y = f(x)$ maps to the graph of $x' \; 5 = f(y'-3)$

The inverse function exists and therefore

$y' = f^{-1}(x' - 5) + 3$

ii $(x, y) \to (y, x) \to (y + 3, x + 5)$

(x, y) maps to a unique point (x', y')

Hence $x' = y + 3$ and $y' = x + 5$

Hence $y = x' - 3$ and $x = y' - 5$

Therefore the graph of $y = f(x)$ maps to the graph of $x' - 3 = f(y' - 5)$

The inverse function exists and therefore

$y' = f^{-1}(x' - 3) + 5$

iii $(x, y) \to (5x, 3y) \to (3y, 5x)$

(x, y) maps to a unique point (x', y')

Hence $x' = 3y$ and $y' = 5x$

Hence $y = \dfrac{x'}{3}$ and $x = \dfrac{y'}{5}$

Therefore the graph of $y = f(x)$ maps to the graph of $\dfrac{x'}{3} = f\left(\dfrac{y'}{5}\right)$

The inverse function exists and therefore

$y' = 5f^{-1}\left(\dfrac{x'}{3}\right)$

iv $(x, y) \to (y, x) \to (5y, 3x)$

(x, y) maps to a unique point (x', y')

Hence $x' = 5y$ and $y' = 3x$

Hence $y = \dfrac{x'}{5}$ and $x = \dfrac{y'}{3}$

Therefore the graph of $y = f(x)$ maps to the graph of $\dfrac{x'}{5} = f\left(\dfrac{y'}{3}\right)$

The inverse function exists and therefore

$y' = 3f^{-1}\left(\dfrac{x'}{5}\right)$

b $x' = ay + b$ and $y' = cx + d$

Therefore

Therefore $y = \dfrac{x' - b}{a}$ and $x = \dfrac{y' - d}{c}$

The graph of $y = f(x)$ maps to the graph of $\dfrac{x' - b}{a} = f\left(\dfrac{y' - d}{c}\right)$

Therefore as the inverse function exists $y' = cf^{-1}\left(\dfrac{x' - b}{a}\right) + d$

From

$x' = ay + b$ and $y' = cx + d$:

the graph of $y = f(x)$ has undergone the following sequence of transformations:

A reflection in the line $y = x$, then a dilation of factor c from the x axis and factor a from the y axis, and a translation of b units in the positive direction of the x axis and

d units in the positive direction of the y axis.

7 **a** Range of $g = [-9, 6]$

b **i** $k > 6$ or $k < -9$

ii $k = 6$ or $k = 9$

iii $\frac{7}{2} < k < 6$ or $-9 < k < -\frac{14}{3}$

iv $0 < k < \frac{7}{2}$ or $-\frac{14}{3} < k < -\frac{9}{2}$

v $k = 0$ or $k = -\frac{9}{2}$

vi $-\frac{9}{2} < k < 0$

c
$$\begin{aligned}\frac{1}{2}f(x+6) &= \frac{1}{2}[(x+6)^2 - 9] \\ &= \frac{1}{2}[x^2 + 12x + 27] \\ -\frac{2}{3}f(x-6) &= -\frac{2}{3}[(x-6)^2 - 9] \\ &= -\frac{2}{3}[x^2 - 12x + 27]\end{aligned}$$

$$\begin{aligned}g(x) &= \begin{cases}\frac{1}{2}f(x+6) & -10 \le x < -3 \\ f(x) & -3 \le x < 3 \\ -\frac{2}{3}f(x-6) & 3 \le x \le 10\end{cases} \\ &= \begin{cases}\frac{1}{2}[x^2 + 12x + 27] & -10 \le x < -3 \\ x^2 - 9 & -3 \le x < 3 \\ -\frac{2}{3}[x^2 - 12x + 27] & 3 \le x \le 10\end{cases}\end{aligned}$$

d The transformation is the sequence

- dilation of factor $\frac{1}{3}$ from the y-axis
- dilation of factor 2 from the x-axis
- reflection in the x-axis

The range of g is $[-9, 6]$. The dilation from the $x-$axis takes this to $[-18, 12]$ and the reflection in the x-axis takes it to $[-12, 18]$

e The domain of g is $[-10, 10]$. This maps to $[-8, 12]$
The range of g is $[-9, 6]$. This maps to $[-8, 22]$

f **i** $y = x^2 - 9$ maps to $y = -2x'^2 + 12x' + 2$
Completing the square for the image rule:

$$\begin{aligned}-2x^2 + 12x + 2 &= -2[x^2 - 6x - 1]\\ &= -2[x^2 - 6x + 9 - 10]\\ &= -2(x - 3)^2 + 20\end{aligned}$$

Therefore:

$$\frac{y' - 20}{-2} = (x' - 3)^2$$

We write: $\frac{y' - 20}{-2} = y + 9$ and $x' = x + 3$
Hence $y' = -2y + 2$ and $x' = x + 3$
The rule is $(x, y) \to (x + 3, -2y + 2)$

ii We look at domains of the piecewise defined functions first:
$[-10, -3) \to [-7, 0)$
$[-3, 3) \to [0, 6)$
$[3, 10] \to [6, 13]$

The rules

Rule 1 $y = \frac{1}{2}f(x + 6)$ maps to:

$$\frac{y' - 2}{-2} = \frac{1}{2}f(x' - 3 + 6) = \frac{1}{2}f(x' + 3)$$

That is

$$\begin{aligned}y' &= -f(x' + 3) + 2\\ &= -(x' + 3)^2 + 9 + 2\\ &= -x'^2 - 6x' + 2\end{aligned}$$

Rule 2

Maps to $y = -2x^2 + 12x + 2$

Rule 3

$y = -\frac{2}{3}f(x - 6)$ maps to:

$$\frac{y' - 2}{-2} = -\frac{2}{3}f(x' - 3 - 6) = -\frac{2}{3}f(x' - 9)$$

That is.

$$y' = \frac{4}{3}f(x' - 9) + 2$$

$$y' = \frac{4}{3}x^2 + 24x + 98$$

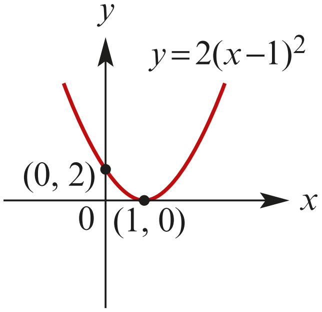
Image

$$\begin{cases} -x^2 - 6x + 2 & \text{if } -7 \leq x < 0 \\ -2x^2 + 12x + 2 & \text{if } 0 \leq x < 6 \\ \dfrac{4}{3}x^2 + 24x + 98 & \text{if } 6 \leq x \leq 13 \end{cases}$$

Chapter 4 – Polynomial functions

Solutions to Exercise 4A

1 a



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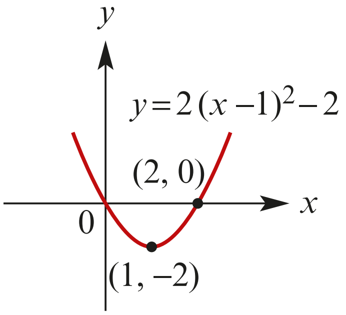

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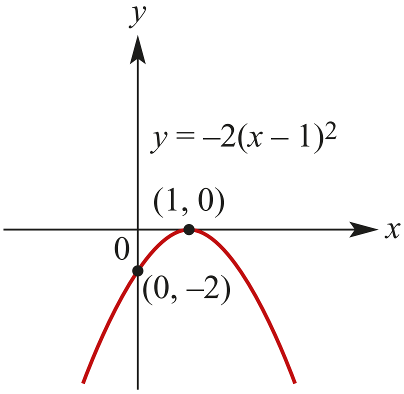

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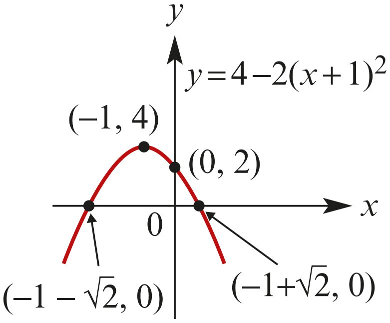

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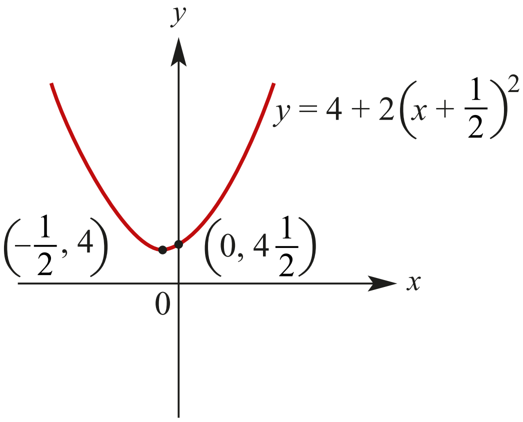

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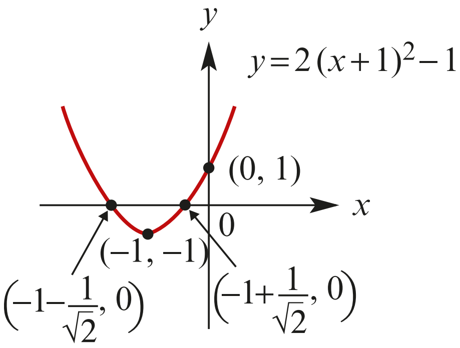

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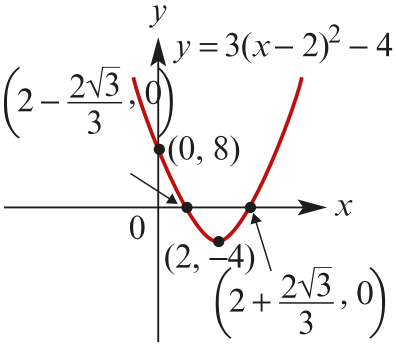

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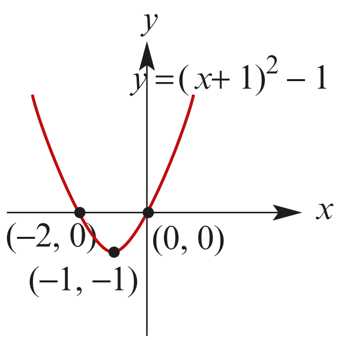

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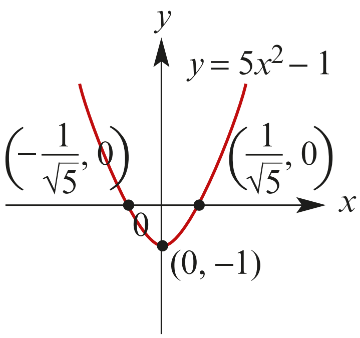

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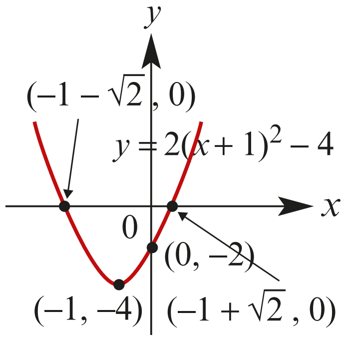


2 **a** $f(x) = x^2 + 3x - 2$

$= x^2 + 3x + 2.25 - 2.25 - 2$

$= (x + 1.5)^2 - 4.25$

Minimum= −4.25 and the range is $[-4.25, \infty)$

b $f(x) = x^2 - 6x + 8$

$= x^2 - 6x + 9 - 9 + 8$

$= (x - 3)^2 - 1$

Minimum= −1 and the range is $[-1, \infty)$

c $f(x) = 2x^2 + 8x - 6$

$= 2(x^2 + 4x - 3)$

$= 2(x^2 + 4x + 4) - 14$

$= 2(x + 2)^2 - 14$

Minimum= −14 and the range is $[-14, \infty)$

d $f(x) = 4x^2 + 8x - 7$

$= 4(x^2 + 2x) - 7$

$= 4(x^2 + 2x + 1) - 4 - 7$

$= 4(x + 1)^2 - 11$

Minimum= −11 and the range is $[-11, \infty)$

e $f(x) = 2x^2 - 5x$

$= 2\left(x^2 - \frac{5}{2}x\right)$

$= 2\left(x^2 - \frac{5}{2}x + \frac{25}{16}\right) - \frac{25}{8}$

$= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8}$

Minimum= $\frac{-25}{8}$ and the range is $\left[\frac{-25}{8}, \infty\right)$

f $f(x) = -3x^2 - 2x + 7$

$= -3\left(x^2 - \frac{2}{3}x\right) + 7$

$= -3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + \frac{1}{3} + 7$

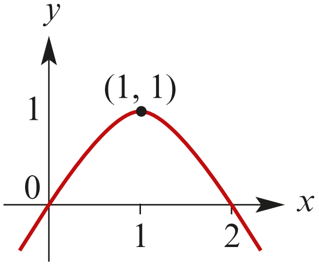
$= -3\left(x + \frac{1}{3}\right)^2 + \frac{22}{3}$

maximum = $\frac{22}{3}$ and the range is $\left(-\infty, \frac{22}{3}\right]$

g $$\begin{aligned} f(x) &= -2x^2 + 9x + 11 \\ &= -2\left(x^2 - \frac{9}{2}x\right) + 11 \\ &= -2\left(x^2 - \frac{9}{2}x + \frac{81}{16}\right) + \frac{81}{8} + 11 \\ &= -2\left(x - \frac{9}{4}\right)^2 + \frac{169}{8} \end{aligned}$$

maximum = $\frac{169}{8}$ and the range is $\left(-\infty, \frac{169}{8}\right]$

3 a



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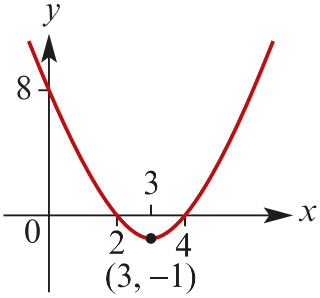

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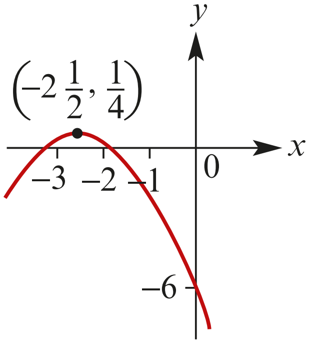

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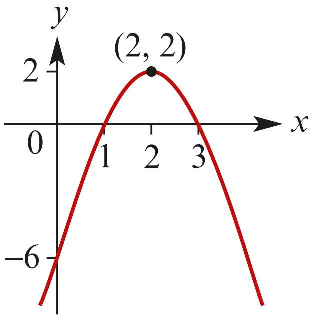

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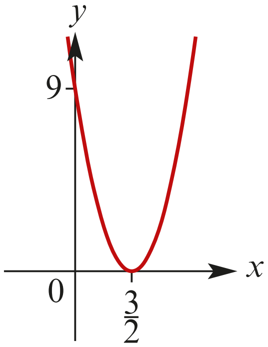

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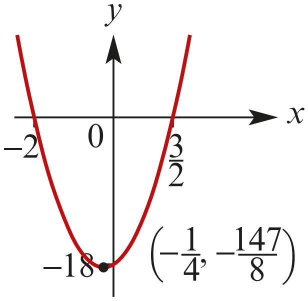
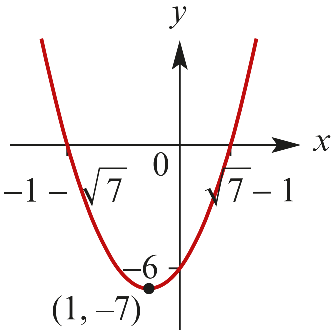
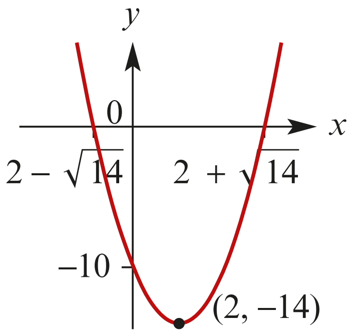


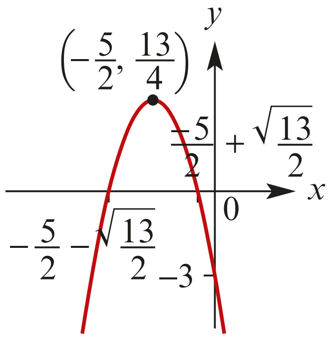
4 a $y = (x + 1)^2 - 7$



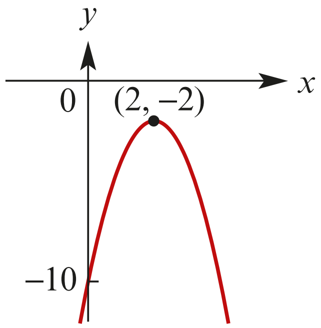
b $y = (x - 2)^2 - 14$



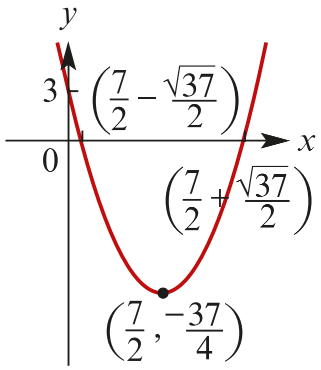
c $y = \frac{13}{4} - \left(x + \frac{5}{2}\right)^2$



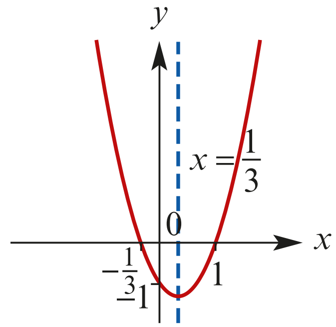
d $y = -2(x - 2)^2 - 2$



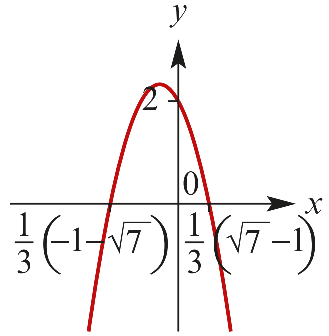
e $y = \left(x - \frac{7}{2}\right)^2 - \frac{37}{4}$



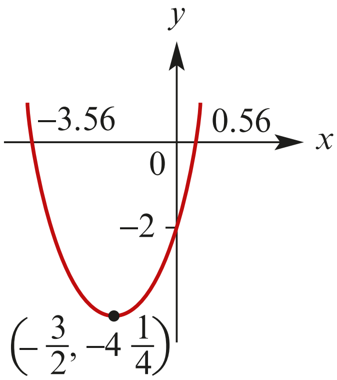
5



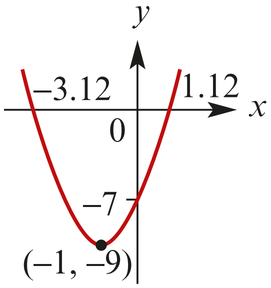
6



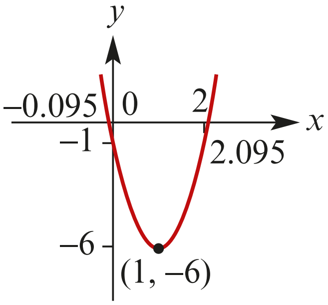
7 **a**



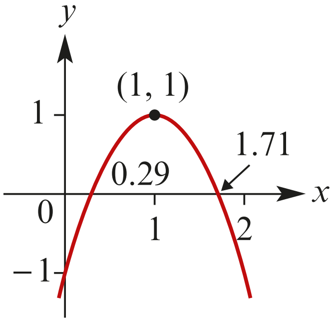
b



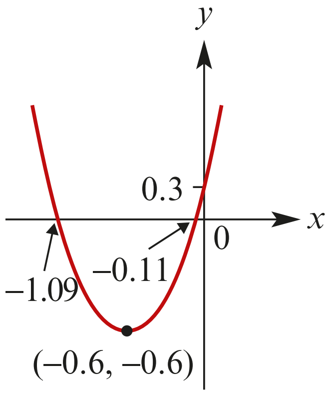
c



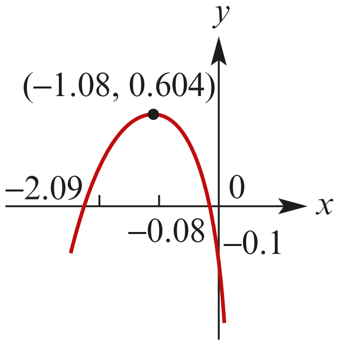
d



e



f



8 **a** *B*

b *D*
by looking at turning point coordinates

9 **a** *C* *x*-axis intercepts

b *B* turning point *x*-value

c *D* turning point coordinates

d *A* turning point *x*-value

10 **a** $b^2 - 4ac = 25 - 8$
> 0
$\therefore$ it crosses the *x*-axis

b $b^2 - 4ac = 4 - 4 \times -4 - 1$
$= 4 - 16$
< 0
$\therefore$ it does not intersect the *x*-axis

c $b^2 - 4ac = 36 - 36 = 0$
$\therefore$ it touches the *x*-axis

d $b^2 - 4ac = 9 - 4 \times 8 \times -2$
$= 9 + 64$
> 0
$\therefore$ it crosses the *x*-axis

e $b^2 - 4ac = 4 - 60 < 0$
$\therefore$ it does not intersect the *x*-axis

f $b^2 - 4ac = 1 - 4$
< 0
$\therefore$ it does not intersect the *x*-axis

11 $mx^2 - 2mx + 3 = 0$
$b^2 - 4ac = 4m^2 - 12m$
$= 4m(m - 3)$

a $4m(m - 3) > 0$

$m < 0$ or $m > 3$

b $4m(m - 3) = 0$

$m = 3$

($m = 0$, is not a solution as it gives $3 = 0$)

12 $\Delta = 36m^2 - 16(4m + 1)$

$= 36m^2 - 64m - 16$

$= 4(9m^2 - 16m - 4)$

$= 4(9m + 2)(m - 2)$

Perfect square if $\Delta = 0$

$\therefore m = -\frac{2}{9}$ or $m = 2$

13 $\Delta = 4a^2 - 4(a + 2)(a - 3)$

$= 4a^2 - 4(a^2 - a - 6)$

$= 4a + 24$

No solutions if $\Delta < 0$

$\therefore a < -6$

14 $\Delta = (a + 1)^2 - 4(a - 2)$

$= a^2 + 2a + 1 - 4a + 8$

$= a^2 - 2a + 9$

$= (a - 1)^2 + 8$

$\therefore \Delta > 0$ for all values of a

15 **a** $(k + 1)x^2 - 2x - k = 0$

$b^2 - 4ac = 4 + k(k + 1)$

need to show

$4 + k(k + 1) > 0$

i.e. $k(k + 1) > -4$

$LHS = k^2 + k$

$= k^2 + k + \frac{1}{4} - \frac{1}{4}$

$= (k + \frac{1}{2})^2 - \frac{1}{4}$

> -4

which is what is required

16 $\Delta = 4k^2 + 20k$

$= 4k(k + 5)$

a $\Delta > 0 \Leftrightarrow k \in (-\infty, -5) \cup (0, \infty)$.

b $\Delta = 0 \Leftrightarrow k = 0$ or $k = -5$

17 $\Delta = 4k^2 - 4(k + 2)(k - 3)$

$= 4k^2 - 4(k^2 - k - 6)$

$= 4(k + 6)$

a Two solutions if $k > -6$

b One solution if $k = -6$

18 **a** $ax^2 - (a + b)x + b = 0$

$(a + b)^2 - 4ab = a^2 + 2ab + b^2 - 4ab$

$= (a - b)^2 \geq 0$

$\therefore$ the equation always has at least one solution

Solutions to Exercise 4B

1 $y = k(x + 3)(x + 2)$

When $x = 1, y = -24$

$\therefore -24 = k(4)(3)$

$\therefore k = -2$

$\therefore y = -2(x + 3)(x + 2)$

2 $y = k(x + 3)(2x + 3)$

When $x = 1, y = 20$

$\therefore 20 = k(4)(5)$

$\therefore k = 1$

$\therefore y = (x + 3)(2x + 3)$

3 $y = a(x + 2)^2 + 4$

When $x = 4, y = 58$

$\therefore 58 = 36a + 4$

$\therefore a = \frac{54}{36} = \frac{3}{2}$

$\therefore y = \frac{3}{2}(x + 2)^2 + 4$

4 $y = a(x + 2)^2 - 3$

When $x = -3, y = -5$

$\therefore -5 = a - 3$

$\therefore a = -2$

$\therefore y = -2(x + 2)^2 - 3$

5 Passes through $(1, 19)$, $(0, 18)$ and $(-1, 7)$

The equation has form $y = ax^2 + bx + 18$

$$19 = a + b + 18 \ldots (1)$$

$$7 = a - b + 18 \ldots (2)$$

Equation (1) – Equation (2)

$$12 = 2b$$

$$\therefore b = 6$$

$$\therefore a = -5$$

$$\therefore y = -5x^2 + 6x + 18$$

6 Passes through $(2, -14)$, $(0, 10)$ and $(-4, 10)$

The equation has form $y = ax^2 + bx + 10$

$$-14 = 4a + 2b + 10 \ldots (1)$$

$$10 = 16a - 4b + 10 \ldots (2)$$

$2 \times$ Equation (1) + Equation (2)

$$-18 = 24a + 30$$

$$\therefore a = -2$$

$$\therefore b = -8$$

$$\therefore y = -2x^2 - 8x + 10$$

7 **a** $y = ax^2 + bx + c$

$c = 4$(y-intercept)

$b = 0$(x-value at turning point)

$y = ax^2 + 4$

$x = 5, y = 0$

$0 = 25a + 4$

$a = \frac{-4}{25}$

$y = \frac{-4}{25}x^2 + 4$

b $y = a(x + h)^2 + k$

$y = ax_2$

$x = 2, y = -4$

$-4 = 4a$

$a = -1$

$y = -x^2$

c $y = a(x + b)(x + c)$

$y = a(x + 2)(x + 0)$

$y = ax^2 + 2ax$

$x = 1, y = 3$

$3 = a + 2a$

$a = 1$

$y = x^2 + 2x$

d $y = a(x + b)(x + c)$

$y = a(x + 0)(x - 2)$

$y = ax^2 - 2ax$

$x = -1, y = -3$

$-3 = a + 2a$

$a = -1$

$y = -x^2 + 2x$

e $y = a(x + b)(x + c)$

$y = a(x - 1)(x - 4)$

$y = ax^2 - 5ax + 4a$

$4a = 4(y\text{-}intercept)$

$a = 1$

$y = x^2 - 5x + 4$

f $y = a(x + b)(x + c)$
$y = a(x + 1)(x - 5)$
$y = ax^2 - 4ax - 5a$
$-5a = -5$ (y-intercept)
$a = 1$
$y = x^2 - 4x - 5$

g $y = a(x + h)^2 + k$
$y = a(x - 1)^2 - 2$
$x = -1, y = 2$
$2 = 4a - 2$
$a = 1$
$y = (x - 1)^2 - 2 = x^2 - 2x - 1$

h $y = a(x + h)^2 + k$
$y = a(x - 2)^2 + 2$
$x = 0, y = 6$
$6 = 4a + 2$
$a = 1$
$y = (x - 2)^2 + 2 = x^2 - 4x + 6$

8 left hand curve
$-y = ax^2 + x + c$

$c = -5$ C

$x = 4, y = 1$ B

$1 = 16a + 4 - 5$

$a = \frac{1}{8}$

$y = \frac{1}{8}x^2 + x - 5$

right hand curve

$y = ax^2 + x + c$

$c = 1$ D

$y = ax^2 + x + 1$

$x = 4, y = 3$ A

$3 = 16a + 4 + 1$

$16a = -2$

$a = -\frac{1}{8}$

$y = -\frac{1}{8}x^2 + x + 1$

9 $f(x) = A(x + b)^2 + B$

$= A(x + 2)^2 + 4$(vertex)

$f(0) = 8$

$8 = 4a + 4$

$A = 1, b = 2, B = 4$

$f(x) = (x + 2)^2 + 4$

Solutions to Exercise 4C

1 **a** $P(1) = 3$

b $P(-1) = -5$

c $P(2) = 7$

d $P(-2) = -21$

e $P(\frac{1}{2}) = \frac{17}{8}$

f $P(-\frac{1}{2}) = -\frac{9}{8}$

2 **a** $P(0) = 6$

b $P(1) = 6$

c $P(2) = 18$

d $P(-1) = 12$

e $P(a) = a^3 + 3a^2 - 4a + 6$

f $P(2a) = 8a^3 + 12a^2 - 8a + 6$

3 **a**
$$\begin{aligned} P(2) &= 0 \\ 8 + 12 - 2a - 30 &= 0 \\ -2a &= 10 \\ a &= -5 \end{aligned}$$

b
$$\begin{aligned} P(3) &= 68 \\ 27 + 9a + 15 - 14 &= 68 \\ 9a &= 40 \\ a &= \frac{40}{9} \end{aligned}$$

c
$$\begin{aligned} P(1) &= 6 \\ 1 - 1 - 2 + c &= 6 \\ c &= 8 \end{aligned}$$

d
$$\begin{aligned} P(-1) = P(2) &= 0 \\ 2 + 5 + a - b + 12 &= 0 \\ a - b &= -19 \ldots (1) \\ 128 - 40 + 4a + 2b + 12 &= 0 \\ 4a + 2b &= -100 \\ 2a + b &= -50 \ldots (2) \end{aligned}$$

Equation (1) + Equation(2)

$$\begin{aligned} 3a &= -69 \\ a &= -23 \\ \therefore b &= -4 \end{aligned}$$

e
$$\begin{aligned} P(3) = P(1) &= 0 \\ 3^5 - 2 \times 3^4 + 27a + 9b + 36 - 36 &= 0 \\ 81 + 27a + 9b &= 0 \\ 3a + b &= -9 \ldots (1) \\ 1 - 2 + a + b + 12 - 36 &= 0 \\ a + b &= 25 \ldots (2) \end{aligned}$$

Equation (1) − Equation(2)

$$\begin{aligned} 2a &= -34 \\ a &= -17 \\ \therefore b &= 42 \end{aligned}$$

4 **a** $2x^3 - x^2 + 2x + 2$

b $2x^3 + 5x$

c $2x^3 - x^2 + 4x - 2$

d $6x^3 - 3x^2 + 9x$

e $-2x^4 + 5x^3 - 5x^2 + 6x$

f $4x - x^3$

g $2x^3 + 4x + 2$

h $2x^5 + 3x^4 + x^3 + 6x^2$

5 **a** $x^3 - 5x^2 + 10x - 8$

b $x^3 - 7x^2 + 13x - 15$

c $2x^3 - x^2 - 7x - 4$

d $x^2 + (b + 2)x^2 + (2b + c)x + 2c$

e $2x^3 - 9x^2 - 2x + 3$

6 **a** $(x + 1)(x^2 + bx + c) =$
$x^3 + (b + 1)x^2 + (c + b)x + c$

b $x^3 - x^2 - 6x - 4 =$
$x^3 + (b + 1)x^2 + (c + b)x + c$
for all x. $\therefore (b + 1) = -1, c = -4$ and
$c + b = -6$
$\therefore b = -2$ and $c = -4$

c $x^3 - x^2 - 6x - 4 = (x + 1)(x^2 - 2x - 4)$
$\therefore x^3 - x^2 - 6x - 4 =$
$(x + 1)(x + \sqrt{5} - 1)(x - \sqrt{5} - 1)$

7 **a** $2x^3 - 18x^2 + 54x - 49 =$
$a(x^3 - 9x^2 + 27x - 27) + b$
Equating coefficients
$a = 2$ and $-27a + b = -49$
$\therefore a = 2$ and $b = 5$

b $-2x^3 + 18x^2 - 54x + 52 =$
$a(x^3 + 3cx^2 + 3c^2x + c^3) + b$
Equating coefficients
$a = -2$ and $3ca = 18$ and $52 = ac^3 + b$
$\therefore a = -2, c = -3$ and $b = -2$

c $x^3 - 5x^2 - 2x + 24 =$
$a(x^3 + 3cx^2 + 3c^2x + c^3) + b$
Equating coefficients: For x^3: $a = 1$
For x^2: $-5 = 3c$
For x: $-2 = 3c^2$ which is impossible

8 $A(x + 3) + B(x + 2) = 4x + 9$

$(A + B)x + (3A + 2B) = 4x + 9$
by equating coefficients
(1) $A + B = 4$
(2) $3A + 2B = 9$
(2) + 2(1) ⇒ $A = 1$
(1) ⇒ $B = 3$

9 **a** $x^2 - 4x + 10 = Ax^2 + 2ABx + AB^2 + C$
by equating coefficients
(1) $A = 1$
$2AB = 4$
(2) $AB^2 + C = 10$
(1) ⇒ $= 2B = 4$
$B = -2$
⇒ 2 ⇒ $4 + C = 10$
$C = 6$

b $4x^2 - 12x + 14 = Ax^2 + 2AB + C$
by equating coefficients
$A = 4$
(1) $2AB = -12$
(2) $AB^2 + C = 14$
⇒ (1) ⇒ $8B = -12$
$B = \frac{-3}{2}$
(2) ⇒ $4 \times \frac{9}{4} + C = 14$
$C = 5$

c $x^3 - 9x^2 + 27x - 22 = A(x + B)^3 + C$ $\quad B = -3$

$(x - 3)^3 + 5 = A(x + B)^3 + C$ $\quad C = 5$

$A = 1$

Solutions to Exercise 4D

1 a

$$\begin{array}{r} x^2-5x+6 \\ x+4\overline{)x^3-x^2-14x+24} \\ \underline{x^3+4x^2} \\ -5x^2-14x \\ \underline{-5x^2-20x} \\ 6x+24 \\ \underline{6x+24} \\ 0 \end{array}$$

b

$$\begin{array}{r} 2x^2+7x-4 \\ x-3\overline{)2x^3+x^2-25x+12} \\ \underline{2x^3-6x^2} \\ 7x^2-25x \\ \underline{7x^2-21x} \\ -4x+12 \\ \underline{-4x+12} \\ 0 \end{array}$$

2 a

$$\begin{array}{r} x^2-4x-3+\dfrac{34}{x+3} \\ x+3\overline{)x^3-x^2-15x+25} \\ \underline{x^3+3x^2} \\ -4x^2-15x \\ \underline{-4x^2-12} \\ -3x+25 \\ \underline{-3x-9} \\ 34 \end{array}$$

b

$$\begin{array}{r} 2x^2+6x+14+\dfrac{54}{x-3} \\ x-3\overline{)2x^3+0x^2-4x+12} \\ \underline{2x^3-6x^2} \\ 6x^2-4x \\ \underline{6x^2-18x} \\ 14x+12 \\ \underline{14x-42} \\ 54 \end{array}$$

3 a

$$\begin{array}{r} x^2-\dfrac{5}{2}x-\dfrac{15}{4}+\dfrac{145}{4(2x+3)} \\ 2x+3\overline{)2x^3-2x^2-15x+25} \\ \underline{2x^3+3x^2} \\ -5x^2-15x \\ \underline{-5x^2-\dfrac{15}{2}x} \\ -\dfrac{15}{2}x+25 \\ \underline{-\dfrac{15}{2}x-\dfrac{45}{4}} \\ \dfrac{145}{4} \end{array}$$

b

$$\begin{array}{r} 2x^2+6x+7+\dfrac{33}{2x-3} \\ 2x-3\overline{)4x^3+6x^2-4x+12} \\ \underline{4x^3-6x^2} \\ 12x^2-4x \\ \underline{12x^2-18x} \\ 14x+12 \\ \underline{14x-21} \\ 33 \end{array}$$

4 **a**
$$\begin{array}{r} 2x^2 - x + 12 \\ x-3\overline{)2x^3 - 7x^2 + 15x - 3} \\ \underline{2x^3 - 6x^2 \phantom{{}+15x-3}} \\ -x^2 + 15x \phantom{{}-3} \\ \underline{-x^2 + 3x \phantom{{}-3}} \\ 12x - 3 \\ \underline{12x - 36} \\ 33 \end{array}$$

$$\frac{2x^3 - 7x^2 + 15x - 3}{x-3} = 2x^2 - x + 12 + \frac{33}{x-3}$$

b
$$\begin{array}{r} 5x^4 + 8x^3 - 8x^2 + 6x - 6 \\ x+1\overline{)5x^5 + 13x^4 - 2x^2 - 6} \\ \underline{5x^5 + 5x^4 \phantom{{}-2x^2-6}} \\ 8x^4 + 0x^3 \\ \underline{8x^4 + 8x^3} \\ -8x^3 - 2x^2 \\ \underline{-8x^3 - 8x^2} \\ 6x^2 + 0x \\ \underline{6x^2 + 6x} \\ -6x - 6 \end{array}$$

$$\frac{5x^5 + 13x^4 - 2x^2 - 6}{x+1} = 5x^4 + 8x^3 - 8x^2 + 6x - 6$$

5 **a**
$$\begin{array}{r} x^2 - 9x + 27 \\ x^2-2\overline{)x^4 - 9x^3 + 25x^2 - 8x - 2} \\ \underline{x^4 - 2x^2 \phantom{{}-8x-2}} \\ -9x^3 - 8x \\ \underline{-9x^3 + 18x} \\ 27x^2 - 2 \\ \underline{27x^2 - 54} \\ -26x + 52 \end{array}$$

$$\frac{x^4 - 9x^3 + 25x^2 - 8x - 2}{x^2-2} = x^2 - 9x + 27 - 26\left(\frac{x-2}{x^2-2}\right)$$

b

c
$$\begin{array}{r} x^2 + x + 2 \\ x^2-1\overline{)x^4 + x^3 + x^2 - x - 2} \\ \underline{x^4 + 0x^3 - x^2 \phantom{{}-x-2}} \\ x^3 + 2x^2 - x \\ \underline{x^3 + 0x^2 - x} \\ 2x^2 + 0x - 2 \\ \underline{2x^2 - 2} \\ 0 \end{array}$$

6 **a** remainder $= P(-2)$

$= (-2)^3 + 3(-2) - 2 = -16$

b $P(x) = (1 - 2a)x^2 + 5ax$

$+ (a - 1)(a - 8)$

$P(2) = 0$

$P(1) \neq 0$

$P(2) = 4 - 8a + 10a + a^2 - 9a + 8$

$= a^2 - 7a + 12$

$(a - 3)(a - 4) = 0$

$a = 3, 4$

$P(1) = 1 - 2a + 5a + a^2 - 9a + 8$

$= a^2 - 6a + 9$

$= (a - 3)^2$

$P(1) \neq 0, \quad \therefore a \neq 3, \quad \therefore a = 4$

7 **a** $f(x) = 6x^3 + 5x^2 - 17x - 6$

$f(2) = 6 \times 8 + 5 \times 4 - 17 \times 2 - 6$

$= 48 + 20 - 34 - 6$

$= 28$

b $f(-2) = (6 \times -8) + (5 \times 4)$

$- (17 \times -2) - 6$

$= -48 + 20 + 34 - 6$

$= 0$

c $f(x) = (x + 2)(6x^2 - 7x - 3)$

$= (x + 2)(3x + 1)(2x - 3)$

8 **a** $P(-1) = -1 + (k - 1) - (k - 9) - 7$

$= -1 + k - 1 - k + 9 - 7$

$= 0$

$\therefore$ for any value of k, $P(x)$ is divisible by $x + 1$

b $P(2) = 8 + 4(k - 1) + 2(k - 9) - 7$

$P(2) = 12$

$1 + 4k - 4 + 2k - 18 = 12$

$6k - 21 = 12$

$6k = 33$

$k = \frac{11}{2}$

9 $f(x) = 2x^3 + ax^2 - bx + 3$

a $f(-3) = 0 = -54 + 9a + 3b + 3$

$9a + 3b = 51$

$3a + b = 17 \ldots (1)$

$f(2) = 15 = 16 + 4a - 2b + 3$

$4a - 2b = -4$

$2a - b = -2 \ldots (2)$

$(1) + (2)$

$\Rightarrow\ 5a = 15$

$a = 3$

Sub in (1) $\Rightarrow$ $b = 8$

b $f(x) = (x + 3)(2x^2 - 3x + 1)$

$= (x + 3)(2x - 1)(x - 1)$

$\therefore$ the other two factors are $(2x - 1)$ & $(x - 1)$

10 **a** $f(x) = 4x^3 + ax^2 - 5x + b$

$$f\left(\frac{3}{2}\right) = -8 = 4 \times \left(\frac{27}{8}\right) + a \times \left(\frac{9}{4}\right) - 5 \times \left(\frac{3}{2}\right) + b$$

$$-8 = \frac{27}{2} + \frac{9}{4}a - \frac{15}{2} + b$$

$-32 = 54 + 9a - 30 + 4b$

$9a + 4b = -56 \ldots (1)$

$f(3) = 10 = 4 \times 27 + a \times 9 - 5 \times 3 + b$

$10 = 108 + 9a - 15 + b$

2 $\quad 9a + b = -83$

$1 - 2 \Rightarrow \quad 3b = 27$

$b = 9$

Sub in 2 $\Rightarrow$ $9a + 9 = -83$

$a = \frac{-92}{9}$

11 $P(2) = (3)^4$

$= 81$

12 $P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$

a $P1 = -1 - 3 + 2 - 2 + 3 + 1$

$= 2 \neq 0$

$\therefore$ $(x - 1)$ is not a factor

$P(-1) = -1 - 3 - 2 - 2 - 3 + 1$

$= -10 \neq 0$

$\therefore$ $(x + 1)$ is not a factor

b $P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$

$$\begin{array}{r@{\,}l}
 & x^3 - 3x^2 + 3x - 5 \\ \cline{2-2}
x^2 - 1 \big) & x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1 \\
 & x^5 - x^3 \\ \cline{2-2}
 & -3x^4 + 3x^3 - 2x^2 \\
 & -3x^4 + 3x^2 \\ \cline{2-2}
 & \qquad 3x^3 - 5x^2 + 3x \\
 & \qquad 3x^3 - 3x \\ \cline{2-2}
 & \qquad\qquad -5x^2 + 6x + 1 \\
 & \qquad\qquad -5x^2 + 5 \\ \cline{2-2}
 & \qquad\qquad\qquad 6x - 4
\end{array}$$

$P(x) = (x^3 - 3x^2 + 3x - 5)(x^2 - 1) + 6x - 4$

$\therefore$ the remainder when $(x^3 - 3x^2 + 3x - 5)$ is divided by $(x^2 - 1)$ is $6x - 4$

13 $P(-1) = -2 - 5 + 4 + 3 =$

$\therefore (x + 1)$ is factor

$2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x^2 - 7x + 3)$

$= (x + 1)(2x - 1)(x - 3)$

14 **a** $P(x) = x^4 + x^3 - x^2 - 3x - 6$

$P(\sqrt{3}) = 9 + 3\sqrt{3} - 3 - 3\sqrt{3} - 6$

$= 0$

$P(\sqrt{-3}) = 9 - 3\sqrt{3} - 3 + 3\sqrt{3} - 6$

$= 0$

b the quadratic factor is

$(x + \sqrt{3})(x - \sqrt{3})$

$= (x^2 - 3)$

$\therefore$ $P(x) = (x^2 - 3)(x^2 + x + 2)$

$\therefore$ an other factor is $(x^2 + x + 2)$

15 **a** $(2a + 3b)(4a^2 - 6ab + 9b^2$

b $(4 - a)(a^2 + 4a + 16)$

c $(5x + 4y)(25x^2 - 20xy + 16y^2)$

d $2a(a^2 + 3b^2)$

16 **a** $(2x - 1)(2x + 3)(3x + 2)$

b $(2x - 1)(2x^2 + 3)$

17 **a** $(2x - 3)(2x^2 + 3x + 6)$

b $(2x - 3)(2x - 1)(2x + 1)$

18 **a** $x = -4, 2, 3$

b $x = 0, 2$

c $x = \frac{1}{2}, 2$

d $x = -2, 2$

e $x = 0, -2, 2$

f $x = 0, -3, 3$

g $x = 1, -2, \frac{-1}{4}, \frac{1}{3}$

h $x = 1, -2$

i $x = 1, -2, \frac{1}{3}, \frac{3}{2}$

19 Use a CAS calculator to solve $y = 0$ in each case to obtain the x-exis intercepts.

a $(-1, 0), (0, 0), (2, 0)$

b $(-2, 0), (0, 6), (1, 0), (3, 0)$

c $(-1, 0), (0, 6), (2, 0), (3, 0)$

d $\left(\frac{-1}{2}, 0\right), (0, 2), (1, 0), (2, 0)$

e $(-2, 0), (-1, 0), (0, -2), (1, 0)$

f $(-1, 0), \left(\frac{-2}{3}, 0\right), (0, -6), (3, 0)$

g $(-4, 0), (0, -16), \left(-\frac{2}{5}, 0\right), (2, 0)$

h $\left(\frac{-1}{2}, 0\right), (0, 1), \left(\frac{1}{3}, 0\right), (1, 0)$

i $(-2, 0), \left(-\frac{-3}{2}, 0\right), (0, -30), (5, 0)$

20 $16p - 10 + q = 0 \ldots (1)$

$16 - 16 - 4p - 2q - 8 = 0 \ldots (2)$

$\Rightarrow \quad 2p + q + 4 = 0$

$(1) - (2) \Rightarrow \quad 14p - 14 = 0$

$p = 1$

Sub in 1 $\Rightarrow \quad 16 - 10 + q = 0$

$q = -6$

21 $f(x) = x^4 - x^3 + 5x^2 + 4x - 36$

$f(-1) = 1 + 1 + 5 - 4 - 36$

$= -33$

22 **a** $(x - 9)(x - 13)(x + 11)$

b $(x + 11)(x - 9)(x - 11)$

c $(x + 11)(2x - 9)(x - 11)$

d $(x + 11)(2x - 13)(2x - 9)$

23 **a** $(x - 1)(x + 1)(x - 7)(x + 6)$

b $(x - 3)(x + 4)(x^2 + 3x + 9)$

24 **a** $(x - 9)(x - 5)(2x^2 + 3x + 9)$

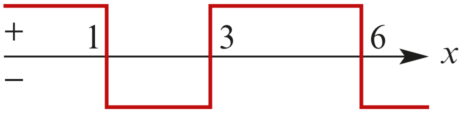
b $(x + 5)(x + 9)(x^2 - x + 9)$

c $(x - 3)(x + 5)(x^2 + x + 9)$

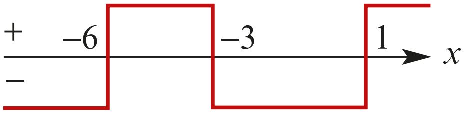
d $(x - 4)(x - 3)(x + 5)(x + 6)$

Solutions to Exercise 4E

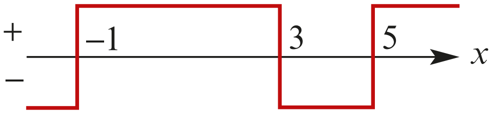
1 a



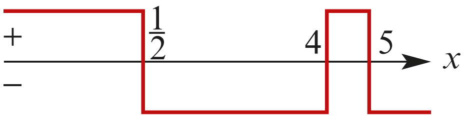
b



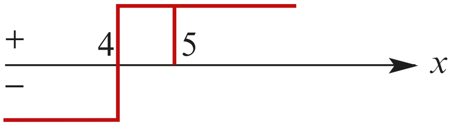
c



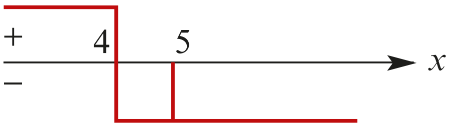
d



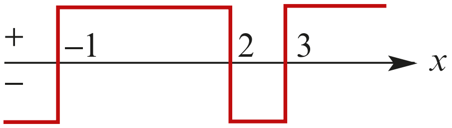
e



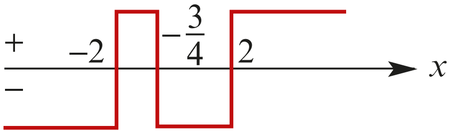
f



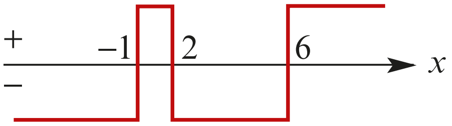
2 a



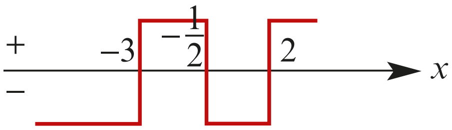
b



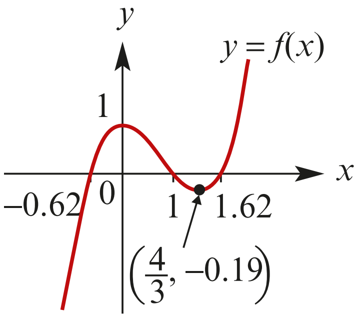
c



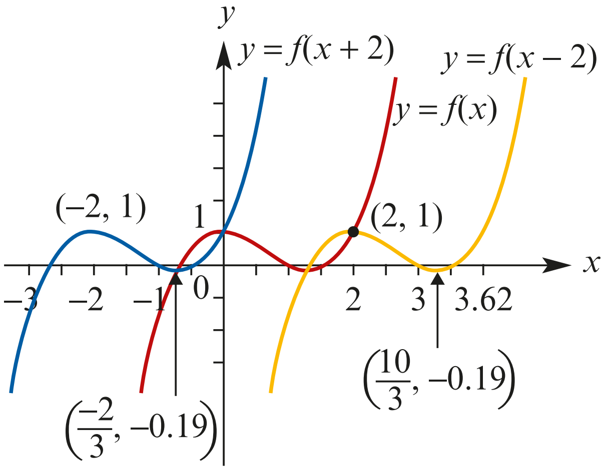
d



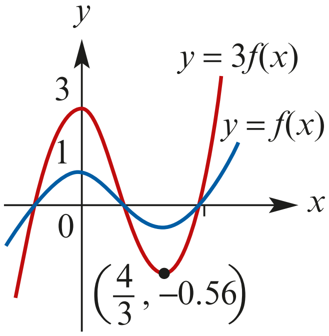
3 a



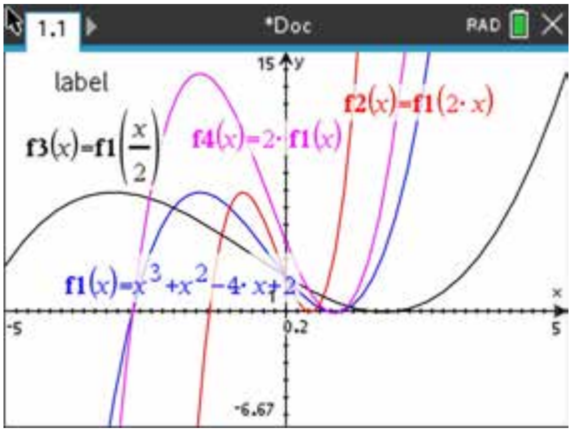
b



For clarity the graph of $y = 3f(x)$ is shown on separate axes:

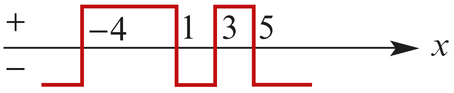


4

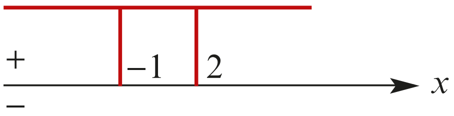

1.1
*Doc
RAD
label
$f3(x)=f1\left(\frac{x}{2}\right)$
$f4(x)=2\cdot f1(x)$
$f2(x)=f1(2\cdot x)$
$f1(x)=x^3+x^2-4\cdot x+2$
15
y
x
-5
0.2
5
-6.67

Solutions to Exercise 4F

1 **a**



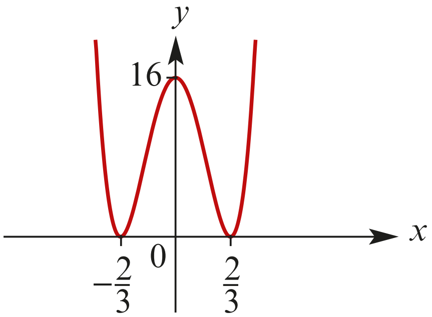
b



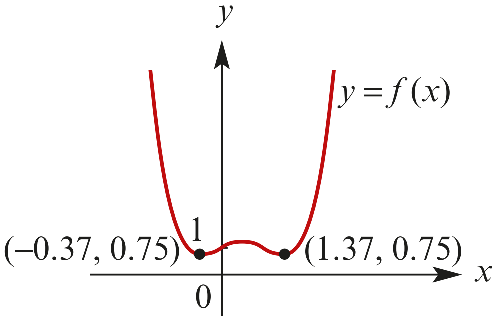
2 (0, 16)

$\left(\frac{2}{3}, 0\right)$

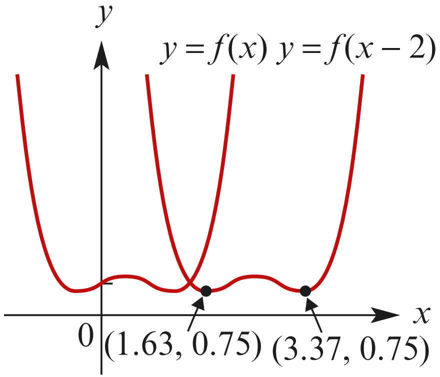
$\left(-\frac{2}{3}, 0\right)$



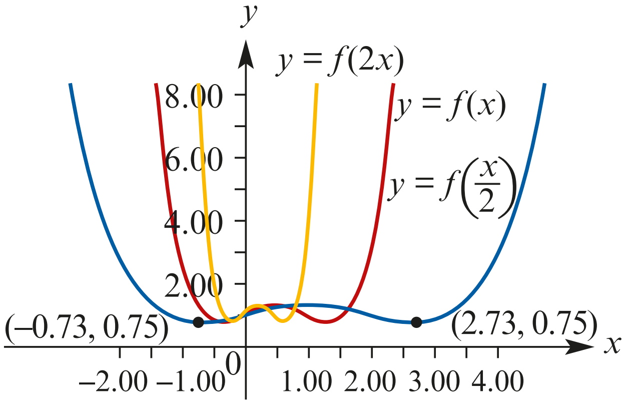
3 **a**



b

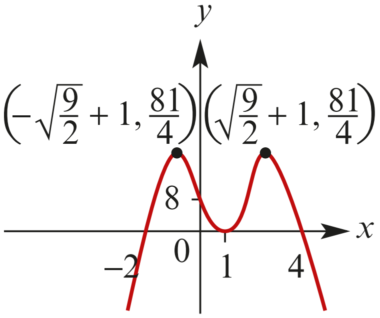


Graphs of dilations shown on separate axes for clarity:

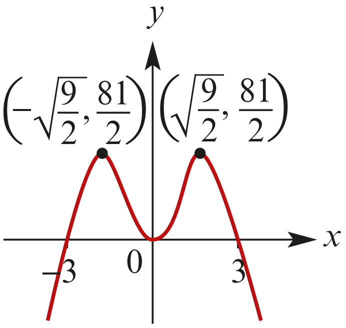


Turning points for $y = f(2x)$ are at (−0.18, 0.75) and (0.68, 0.75)

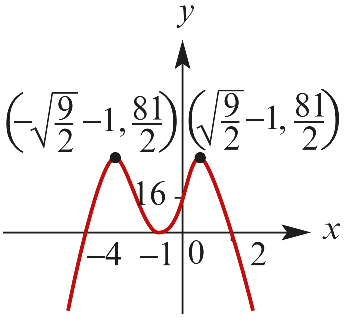
4 **a**



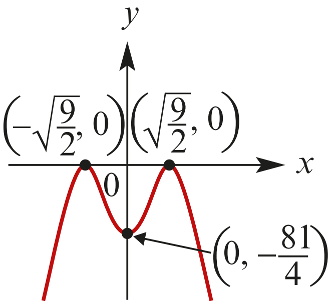
b


y
$\left(-\sqrt{\frac{9}{2}}, \frac{81}{2}\right)$
$\left(\sqrt{\frac{9}{2}}, \frac{81}{2}\right)$
x
-3
0
3

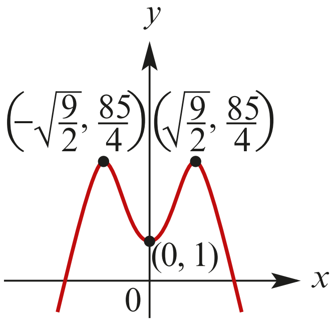
c


y
$\left(-\sqrt{\frac{9}{2}}-1, \frac{81}{2}\right)$
$\left(\sqrt{\frac{9}{2}}-1, \frac{81}{2}\right)$
16
x
-4
-1
0
2

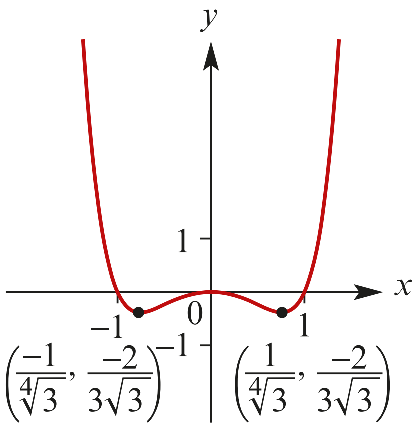
d


y
$\left(-\sqrt{\frac{9}{2}}, 0\right)$
$\left(\sqrt{\frac{9}{2}}, 0\right)$
x
0
$\left(0, -\frac{81}{4}\right)$

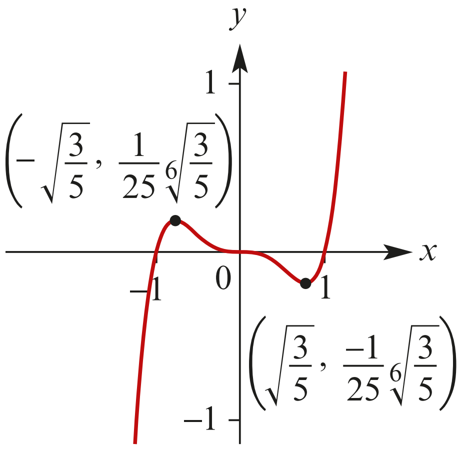
e


y
$\left(-\sqrt{\frac{9}{2}}, \frac{85}{4}\right)$
$\left(\sqrt{\frac{9}{2}}, \frac{85}{4}\right)$
(0, 1)
x
0

5


y
1
x
-1
0
1
-1
$\left(\frac{-1}{\sqrt[4]{3}}, \frac{-2}{3\sqrt{3}}\right)$
$\left(\frac{1}{\sqrt[4]{3}}, \frac{-2}{3\sqrt{3}}\right)$

6


y
1
$\left(-\sqrt{\frac{3}{5}}, \frac{1}{25}\sqrt[6]{\frac{3}{5}}\right)$
x
-1
0
1
$\left(\sqrt{\frac{3}{5}}, \frac{-1}{25}\sqrt[6]{\frac{3}{5}}\right)$
-1

Solutions to Exercise 4G

1 **a** $y = a(x-5)^3 - 2$

When $x = 4, y = 0$

$0 = -a - 2$

$a = -2$

b $y = a(x-1)(x+1(x+2)$

When $x = 3, y = 120$

$120 = a(2)(4)(5)$

$a = 3$

c $y = ax^3 + bx$

$(2, -20)$ and $(-1, 20)$ lie on the graph

$$-20 = 8a + 2b$$

$$-10 = 4a + b \ldots (1)$$

$$20 = -a - b \ldots (2)$$

Add (1)(2)

$$10 = 3a$$

$$a = \frac{10}{3}$$

$$b = -\frac{70}{3}$$

2 We know that the y-intercept is 5.

Consider $f(x) = ax^3 + bx^2 + cx + 5$

$f(-1) = 14, \therefore -a + b - c + 5 = 14$

$-a + b - c = 9 \ldots (1)$

$f(1) = 0, \therefore a + b + c + 5 = 0$

$a + b + c = -5 \ldots (2)$

$f(2) = -19, \therefore 8a + 4b + c + 5 = -19$

$8a + 4b + 2c = -24$

Add (1) and (2)

$$2b = 4$$

$$b = 2$$

$$\therefore a + c = -7$$

$$\text{and } 4a + c = -16$$

$$\therefore 3a = -9$$

$$\therefore a = -3 \text{ and } c = -4$$

3 Note: A CAS calculator can be used for all questions in this exercise, but should be used for questions 5 and 6.

$$y = a(x-b)(x-c)(x-d)$$

$$b = -5, c = -2, d = 6$$

$$y = a(x+5)(x+2)(x-6)$$

$$x = 0, y = -11$$

$$-11 = -60a$$

$$a = \frac{11}{60}$$

$$y = \frac{11}{60}(x+5)(x+2)(x-6)$$

4 $y = a(x-b)(x-c)^2$

$y = a(x+1)(x-3)^2$

$x = 0, y = 5$

$5 = 9a$

$a = \frac{5}{9}$

$y = \frac{5}{9}(x+1)(x-3)^2$

5 a $y = ax^3 + bx^2 + cx + d$

$(0, 1) \Rightarrow d = 1$

$y = ax^3 + bx^2 + cx + 1$

$(1, 3) \Rightarrow 3 = a + b + c + 1$

$a + b + c = 2 \ldots (1)$

$(-1, -1) \Rightarrow -1 = -a + b - c + 1$

$-a + b - c = 2 \quad \ldots (2)$

$(1) + (2) \Rightarrow 2b = 0$

$b = 0$

$(2, 11) \Rightarrow 11 = 8a + 2c + 1$

$4a + c = 5 \ldots (3)$

$(3) + (2) \Rightarrow 3a = 3$

$a = 1, c = 1$

$y = x^3 + x + 1$

b $y = ax^3 + bx^2 + cx + d$

$(0, 1) = d = 1$

$(1, 1) = 1 = a + b + c + 1$

$a + b + c = 0 \ldots (1)$

$(-1, 1) = 1 = -a + b - c + 1$

$-a + b - c = 0 \ldots (2)$

$(1) + (2) \Rightarrow b = 0$

$(2, 7) \Rightarrow 7 = 8a + 2c + 1$

$4a + c = 3 \ldots (3)$

$3 + 2 \Rightarrow 3a = 3$

$a = 1$

Sub in $\Rightarrow (1) \Rightarrow c = -1$

$y = x^3 - x + 1$

c $y = ax^3 + bx^2 + cx + d$

$(0, -2) \Rightarrow d = -2$

$(1, 0) \Rightarrow 0 = a + b + c - 2$

$a + b + c = 2 \quad \ldots (1)$

$(-1, -6) \Rightarrow -6 = -a + b - c - 2$

$-a + b - c = -4 \ldots (2)$

$(2, 12) \Rightarrow 12 = 8a + 4b + 2c - 2$

$4a + 2b + c = 7 \ldots (3)$

$(1) + (2) : 2b = -2$

$b = -1$

Sub in $\Rightarrow (3) \Rightarrow 4a + c = 9$

Sub in $\Rightarrow (2) \Rightarrow -a - c = -3$

$(3) + (2) \Rightarrow 3a = 6$

$a = 2$

Sub in $\Rightarrow (3) \Rightarrow c = 1$

$y = 2x^3 - x^2 + x - 2$

6 a $y = a(x - b)(x - c)(x - d)$

$y = a(2x + 1)(x - 1)(x - 2)$

$2 = 2a$

$a = 1$

$y = (2x + 1)(x - 1)(x - 2)$

b $y = ax^3 + bx^2 + cx$

$(1, 0.75) \Rightarrow \frac{3}{4} = a + b + c \ldots (1)$

$(2, 3) \Rightarrow 3 = 8a + 4b + 2c \ldots (2)$

$(-2, -3) \Rightarrow -3 = -8a + 4b - 2c \ldots (3)$

$(2) + (3) \Rightarrow 8b = 0$

$b = 0$

$(2) - 2(1) \Rightarrow \frac{6}{4} = 6a$

$a = \frac{1}{4}$

Sub in $(2) \Rightarrow 3 = 2 + 2c$

$c = \frac{1}{2}$

$y = \frac{1}{4}x^3 + \frac{1}{2}x = \frac{1}{4}x(x^2 + 2)$

c $y = a(x - b)(x - c)^2$
$y = a(x + 1)x^2$
$2 = a(2)1^2$
$a = 1$
$y = x^2(x + 1)$

d $y = a(x - b)(x - c)(x - d)$
$y = a(x + 2)(x + 1)(x - 1)$
$x = 0, y = -2$
$y = (x + 2)(x + 1)(x - 1)$

e $y = a(x - b)(x - c)^2$
$y = a(x + 2)(x - 3)^2$
$x = 0, y = 18$
$y = (x + 2)(x - 3)^2$

7 **a** $y = -2x^3 - 25x^2 + 48x + 135$

b $y = 2x^3 - 30x^2 + 40x + 13$

8 **a** $y = -2x^4 + 22x^3 - 10x^2 - 37x + 40$

b $y = x^4 - x^3 + x^2 + 2x + 8$

c $y = \frac{31}{36}x^4 + \frac{5}{4}x^3 - \frac{157}{36}x^2 - \frac{5}{4}x + \frac{11}{2}$

Solutions to Exercise 4H

1 **a** $kx^2 + x + k = 0$

$$x = \frac{-1 \pm \sqrt{1 - 4k^2}}{2k},$$

$k \in \left[, -\frac{1}{2}, \frac{1}{2}\right] \backslash \{0\}$ since $k^2 \leq \frac{1}{4}$.

(Note: If $k = 0, x = 0$)

b $x^3 - 7ax^2 + 12a^2x = 0$

$\Rightarrow x(x^2 - 7ax + 12a^2) = 0$

$x(x - 3a)(x - 4a) = 0$

$x = 0, 3a, 4a$

c $x(x^3 - a) = 0$

$x = 0, (a)^{\frac{1}{3}}$

d $x^2 - kx + k = 0$

$x = \dfrac{k \pm \sqrt{k^2 - 4k}}{2}, k \leq 0$ a $k \geq, 4,$

since $k^2 - 4k \geq 0$

e $x(x^2 - a) = 0$

$x = 0, \pm\sqrt{a}, a \geq 0$

f $x^4 - a^4 = 0$

$(x^2 + a^2)(x^2 - a^2) = 0$

$(x^2 + a^2)(x - a)(x + a) = 0$

$x = -a, a$

g $(x - a)^2(x - b) = 0$

$x = a, b$

h $(x - a)^4(a - x^3)(x^2 - a) = 0$

$x = a, (a)^{\frac{1}{3}}, \pm\sqrt{a}\ if a \geq 0$

2 **a** $ax^3 + b = 2c$

$$x^3 = \frac{2c - b}{a}$$

$$x = \left(\frac{2c - b}{a}\right)^{\frac{1}{3}}$$

b $ax^3 - b = c$

$$x^3 = \frac{b + c}{a}$$

$$x = \left(\frac{b + c}{a}\right)^{\frac{1}{3}}$$

c $a - bx^2 = c$

$$x^2 = \frac{a - c}{b}$$

$$x = \left(\frac{a - c}{b}\right)^{\frac{1}{2}}$$

d $x^{\frac{1}{3}} = a$

$x = a^3$

e $(x)^{\frac{1}{n}} + c = a$

$$(x)^{\frac{1}{n}} = a - c$$

$$x = (a - c)^n$$

f $a(x - 2b)^3 = c$

$$(x - 2b)^3 = \frac{c}{a}$$

$$x - 2b = \left(\frac{c}{a}\right)^{\frac{1}{3}}$$

$$x = 2b + \left(\frac{c}{a}\right)^{\frac{1}{3}}$$

g $ax^{\frac{1}{3}} = b$

$$x = \left(\frac{b}{a}\right)^3$$

h $x^3 = c + d$

$$x = (c + d)^{\frac{1}{3}}$$

3 a

$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x - 1) = 0$$

$$x = 0, 1$$

$$y = 0, 1$$

Pts. (0, 0) & (1, 1)

b

$$2x^2 = x$$

$$2x^2 - x = 0$$

$$x(2x - 1) = 0$$

$$x = 0, \frac{1}{2}$$

Pts. (0, 0) $\left(\frac{1}{2}, \frac{1}{2}\right)$

c

$$y = x^2 - x,$$

$$y = 2x + 1$$

$$\Rightarrow x^2 - x = 2x + 1$$

$$x^2 - 3x - 1 = 0$$

$$x = \frac{3 \pm \sqrt{9 + 4}}{2}$$

$$x = \frac{3 \pm \sqrt{13}}{2}$$

$$y = 2x + 1 = 4 \pm \text{V}13$$

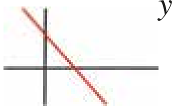
$$co - ords = \left(\frac{3 - \sqrt{13}}{2}, 4 - \sqrt{13}\right),$$

$$\left(\frac{3 + \sqrt{13}}{2}, 4 + \sqrt{13}\right)$$

4 a

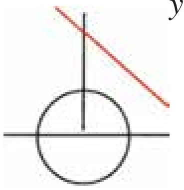
$$y = 16 - x$$

$$y^2 = (16 - x)^2$$



$$y^2 + x^2 = 178$$

$$y^2 = 178 - x^2$$



$$178 - x^2 = (16 - x)^2$$

$$178 - x^2 = 256 - 32x + x^2$$

$$2x^2 - 32x + 78 = 0$$

$$x^2 - 16x + 39 = 0$$

$$(x - 3)(x - 13) = 0$$

$$x = 3, 13$$

Pts. (3, 13), (13, 3)

b

$$\begin{aligned} y^2 &= 125 - x^2 \\ y &= 15 - x \\ \Rightarrow \qquad y^2 &= 225 - 30x + x^2 \\ x^2 - 30x + 225 &= 125 - x^2 \\ 2x^2 - 30x + 100 &= 0 \\ x^2 - 15x + 50 &= 0 \\ (x-5)(x-10) &= 0 \\ x &= 5, 10 \end{aligned}$$

Pts. (5, 10), (10, 5)

c

$$\begin{aligned} y^2 &= 185 - x^2 \\ y &= x - 3 \\ y^2 &= x^2 - 6x + 9 \\ x^2 - 6x + 9 &= 185 - x^2 \\ 2x^2 - 6x - 176 &= 0 \\ x^2 - 3x - 88 &= 0 \\ (x+8)(x-11) &= 0 \\ x &= -8, 11 \end{aligned}$$

Pts. (−8, −11), (11, 8)

d

$$\begin{aligned} y^2 &= 97 - x^2 \\ y &= 13 - x \\ y^2 &= 169 - 26x + x^2 \\ x^2 - 26x + 169 &= 97 - x^2 \\ 2x^2 - 26 + 72 &= 0 \\ x^2 - 13x + 36 &= 0 \\ (x-9)(x-4) &= 0 \\ x &= 4, 9 \end{aligned}$$

Pts. (4, 9), (9, 4)

e

$$\begin{aligned} y^2 &= 106 - x^2 \\ y &= x - 4 \\ y^2 &= x^2 - 8x + 16 \\ x^2 - 8x + 16 &= 106 - x^2 \\ 2x^2 - 8x - 90 &= 0 \\ x^2 - 4x - 45 &= 0 \\ (x+5)(x-9) &= 0 \\ x &= -5, 9 \end{aligned}$$

Pts. (−5, −9), (9, 5)

5 a

$$\begin{aligned} y &= 28 - x \ldots (1) \\ xy &= 187 \ldots (2) \\ \Rightarrow \qquad x(28 - x) &= 187 \\ -x^2 + 28x &= 187 \\ x^2 - 28x + 187 &= 0 \\ x &= \frac{28 \pm \sqrt{784 - 748}}{2} \\ x &= \frac{28 \pm 6}{2} \\ x &= 11, 17 \\ \Rightarrow \qquad pts &= (11, 17), (17, 11) \end{aligned}$$

b

$$\begin{aligned} y &= 51 - x \\ x(51 - x) &= 518 \\ x^2 - 51x + 518 &= 0 \\ x &= \frac{51 \pm \sqrt{2601 - 2072}}{2} \\ x &= \frac{51 \pm \sqrt{529}}{2} \\ x &= \frac{51 \pm 23}{2} \\ x &= 14, 37 \\ \Rightarrow pts &= (14, 37), (37, 14) \end{aligned}$$

c

$$y = x - 5$$

$$xy = 126$$

$$x^2 - 5x = 126$$

$$x^2 - 5x - 126 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 504}}{2}$$

$$x = \frac{5 \pm 23}{2}$$

$$x = -9, 14$$

$$\Rightarrow \quad pts = (-9, -14), (14, 9)$$

6

$$y^2 = 25 - (x - 5)^2$$

$$= 25 - x^2 + 10x - 25$$

$$y^2 = -x^2 + 10x \ldots (1)$$

$$y = 2x$$

$$y^2 = 4x^2 \ldots (2)$$

$$4x^2 = -x^2 + 10x$$

$$x^2 - 2x = 0$$

$$x = 0, 2$$

$$pts = (0, 0), (2, 4)$$

7 $x = \dfrac{1}{x - 2} + 3$

$$x(x - 2) = 1 + 3(x - 2)$$

$$x^2 - 2x = 1 + 3x - 6$$

$$x^2 - 5x + 5 = 1$$

$$x = \frac{5 \pm \sqrt{25 - 20}}{2}$$

$$x = \frac{5 \pm \sqrt{5}}{2}$$

$$pts = \left(\frac{5 + \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}\right),$$

$$\left(\frac{5 - \sqrt{5}}{2}, \frac{5 - \sqrt{5}}{2}\right)$$

8 a

$$\frac{y}{4} - \frac{x}{5} = 1 \ldots (1)$$

$$\Rightarrow \quad y = \frac{4}{5}x + 4$$

$$x^2 + 4x + y^2 = 12 \ldots (2)$$

$$\Rightarrow \quad y^2 = 12 - 4x - x^2$$

$$\Rightarrow \quad y^2 = \frac{16}{25}x^2 + \frac{32}{5}x + 16$$

$$16\left(\frac{1}{25}x^2 + \frac{2}{5}x + 1\right) = 12 - 4x - x^2$$

$$16x^2 + 160x + 400 = 300 - 100x - 25x^2$$

$$41x^2 + 260x + 100 = 0$$

$$x = \frac{-260 \pm \sqrt{67600 - 16400}}{82}$$

$$x = \frac{-130 \pm 80\sqrt{2}}{41}$$

Sub in (1)

$$\left(\frac{-130 - 80\sqrt{2}}{41}, \frac{60 - 64\sqrt{2}}{41}\right),$$

$$\left(\frac{-130 + 80\sqrt{2}}{41}, \frac{60 + 64\sqrt{2}}{41}\right)$$

9

$$-x = \frac{1}{x+2} - 3$$

$$-x^2 - 2x = 1 - 3x - 6$$

$$x^2 - x - 5 = 0$$

$$x = \frac{1 \pm \sqrt{1+20}}{2}$$

$$x = \frac{1 \pm \sqrt{21}}{2}$$

$$pts = \left(\frac{1+\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right),$$

$$\left(\frac{1-\sqrt{21}}{2}, \frac{\sqrt{21}-1}{2}\right)$$

10

$$y = \frac{9}{4}x + 1$$

$$y^2 = \frac{81}{16}x^2 + \frac{9}{2}x + 1$$

$$9x = \frac{81}{16}x^2 + \frac{9}{2}x + 1$$

$$\frac{81}{16}x^2 - \frac{9}{2}x + 1 = 0$$

$$x = \frac{\frac{9}{2} \pm \sqrt{\frac{81}{4} - \frac{81}{4}}}{\frac{81}{8}}$$

$$x = \frac{9}{2} \times \frac{8}{81}$$

$$x = \frac{4}{9}$$

$$\text{co ords} = \left(\frac{4}{9}, 2\right)$$

11

$$y^2 = 9 - x^2$$

$$y = 2x + 3\sqrt{5}$$

$$y^2 = 4x^2 + 12\sqrt{5x} + 45$$

$$9 - x^2 = 4x^2 + 12\sqrt{5x} + 45$$

$$5x^2 + 12\sqrt{5x} + 36 = 0$$

$$x = \frac{-12\sqrt{5} \pm \sqrt{144 \times 5 - 20 \times 36}}{10}$$

$$x = \frac{-12\sqrt{5}}{10}$$

$$x = \frac{-6\sqrt{5}}{5}$$

$$y = \frac{-12\sqrt{5}}{10} + 3\sqrt{5}$$

$$y = \frac{3\sqrt{5}}{5}$$

$$pts = \left(\frac{-6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}\right)$$

12

$$\frac{1}{4}x + 1 = -\frac{1}{x}$$

$$\frac{1}{4}x^2 + x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1-1}}{\frac{1}{2}}$$

$$x = -2$$

$$pt = \left(-2, \frac{1}{2}\right)$$

13

$$x - 1 = \frac{2}{x-2}$$

$$(x-1)(x-2) = 2$$

$$x^2 - 3x + 2 = 2$$

$$x(x-3) = 0$$

$$x = 0, 3$$

$$pts = (0, -1), (3, 2)$$

14 **a**

$$5x - 4y = 7$$
$$4y = 5x - 7$$
$$y = \frac{5x - 7}{4}$$
$$xy = 6$$
$$x\left(\frac{5x - 7}{4}\right) = 6$$
$$5x^2 - 7x - 24 = 0$$
$$x = \frac{7 \pm \sqrt{49 + 480}}{10}$$
$$x = \frac{7 \pm \sqrt{529}}{10}$$
$$x = \frac{-8}{5}, 3$$
$$pts = \left(\frac{-8}{5}, \frac{-15}{4}\right), (3, 2)$$

b

$$y = \frac{37 - 2x}{3}$$
$$xy = 45$$
$$37x - 2x^2 = 135$$
$$2x^2 - 37x + 135 = 0$$
$$x = \frac{37 \pm \sqrt{1369 - 1080}}{4}$$
$$x = 5, 13.5$$
$$pts = (5, 9), \left(13.5, \frac{10}{3}\right)$$

c

$$5x - 3y = 18$$
$$y = \frac{5x - 18}{3}$$
$$xy = 24$$
$$5x^2 - 18x = 72$$
$$x = \frac{18 \pm \sqrt{324 + 1440}}{10}$$
$$x = \frac{18 \pm 42}{10}$$
$$x = -\frac{12}{5}, 6$$
$$pts = \left(-\frac{12}{5}, -10\right), (6, 4)$$

15 $x^2 + ax + b$ *div* by $x + c$

$$(-c)^2 + a(-c) + b = 0$$
$$c^2 - ac + b = 0$$

16

$$x + 2 = \frac{160}{x}$$
$$x^2 + 2x - 160 = 0$$
$$x = \frac{-2 \pm \sqrt{4 + 640}}{2}$$
$$x = -1 \pm \sqrt{161}$$
$$pts = \left(-1 - \sqrt{161}, 1 - \sqrt{161}\right),$$
$$\left(\sqrt{161} - 1, \sqrt{161} + 1\right)$$

17 $y = -7x + 14, \ y = 5x + 12$

18 $m < -7$ or $m > 1$

19 $c = -8$ or $c = 4$

20 **a** $mx = \frac{1}{x} + 5$

$mx^2 - 5x - 1 = 0$

$x = \frac{5 \pm \sqrt{25 + 4m}}{2m}, m \pm 0$

Note that if $m = 0, x = -\frac{1}{5}$.

b $25 + 4m = 0$

$$m = \frac{-25}{4}$$

$$x = \frac{5}{\frac{-25}{2}}$$

$$x = \frac{-2}{5}$$

$pt\left(\frac{-2}{5}, \frac{5}{2}\right)$

c $25 + m < 0$

$$m < \frac{-25}{4}$$

21 $x^2 + x + 4 = kx + b$

$x^2 + (1 - k)x + (4 - b) = 0$

$\Delta = (1 - k)^2 - 4(4 - b)$

$= 1 - 2k + k^2 - 16 + 4b$

$\Delta = 0 \Rightarrow k^2 - 2k + 4b - 15 = 0$

If $b = 3$ then:

$k^2 - 2k - 3 = 0 \Rightarrow k = 3$ or $k = -1$

$y = 3x + 3, \ y = -x + 3$

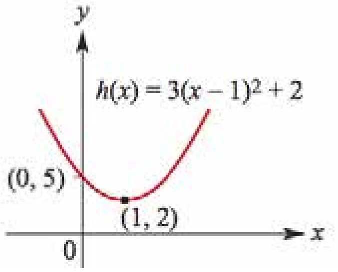
Solutions to technology-free questions

1 **a** $h(x) = 3(x-1)^2 + 2$

$x = 0$: $y = 3(-1)^2 + 2 = 5$

$y = 0$: no solutions

TP (1, 2); no x int; y int (0, 5)



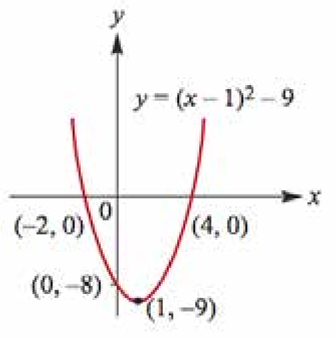
b $h(x) = (x-1)^2 - 9$

$x = 0$: $y = (-1)^2 - 9 = -8$

$y = 0$: $(x-1)^2 - 9 = 0$

$x - 1 = \pm 3$, so $x = -2, 4$

TP(1, −9); x int (−2, 0), (4, 0);
y int (0, −8)



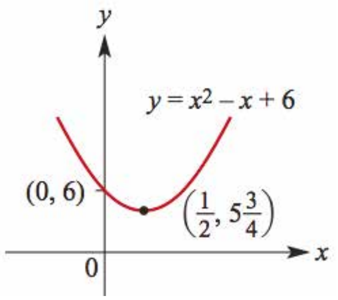
c $f(x) = x^2 - x + 6$

$x = 0$: $y = 6$

$y = 0$: no solutions ($b^2 - 4ac < 0$)

$$x^2 - x + 6 = \left(x - \frac{1}{2}\right)^2 + 5\frac{3}{4}$$

TP$\left(\frac{1}{2}, 5\frac{3}{4}\right)$; no x int; y int (0, 6)



d $f(x) = x^2 - x - 6$

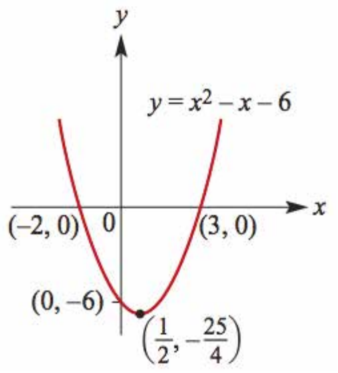
$x = 0$: $y = -6$

$y = 0$: $x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$, so $x = -2, 3$

$$x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - 6\frac{1}{4}$$

TP$\left(\frac{1}{2}, -6\frac{1}{4}\right)$; x int (−2, 0), (3, 0);
y int (0, −6)



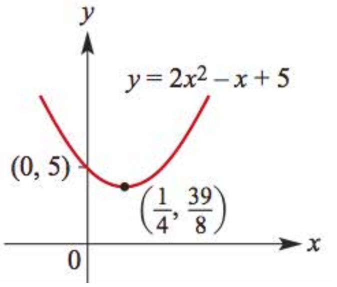
e $f(x) = 2x^2 - x + 5$

$x = 0$: $y = 5$

$y = 0$: no solutions ($b^2 - 4ac < 0$)

$$2x^2 - x + 5 = 2\left(x - \frac{1}{4}\right)^2 + 4\frac{7}{8}$$

TP$\left(\frac{1}{4}, 4\frac{7}{8}\right)$; no x int; y int (0, 5)



f $h(x) = 2x^2 - x - 1$

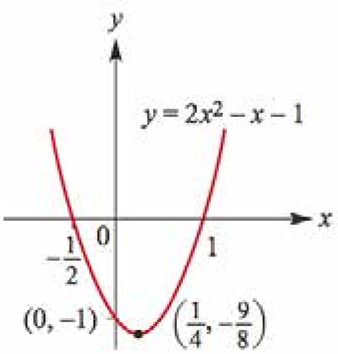
$x = 0$: $y = -1$

$y = 0$: $2x^2 - x - 1 = 0$

$(2x + 1)(x - 1) = 0$, so $x = -\frac{1}{2}, 1$

$2x^2 - x - 1 = 2\left(x - \frac{1}{4}\right)^2 - 1\frac{1}{8}$

$\text{TP}\left(\frac{1}{4}, -1\frac{1}{8}\right)$; x int $\left(-\frac{1}{2}, 0\right)$, $(1, 0)$;
y int $(0, -1)$



2 $(1, 1): 1 = a + b \quad 1$

$(2, 5): 5 = 4a + b \quad 2$

Subtract 2 from 1:

$3a = 4$

$a = \frac{4}{3}$

Substitute into 1: $b = -\frac{1}{3}$

3
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{2 \pm \sqrt{4 - 4(3)(-10)}}{6}$$
$$= \frac{2 \pm \sqrt{124}}{6} = \frac{1}{3}\left(1 \pm \sqrt{31}\right)$$

4 **a** $f(x) = 2(x - 1)^3 - 16$

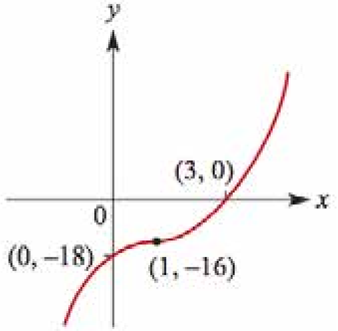
$x = 0$: $y = 2(-1)^3 - 16 = -18$

$y = 0$: $2(x - 1)^3 - 16 = 0$

$(x - 1)^3 = 8$, $x - 1 = 2$, so $x = 3$

zero gradient: $(1, -16)$

x int $(3, 0)$; y int $(0, -18)$



b $g(x) = -(x + 1)^3 + 8$

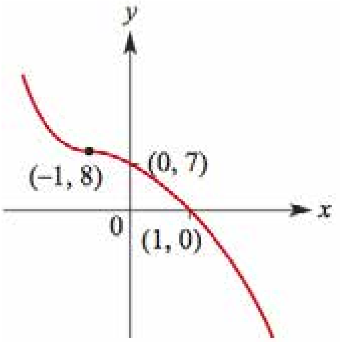
$x = 0$: $y = -(1)^3 + 8 = 7$

$y = 0$: $-(x + 1)^3 + 8 = 0$

$(x + 1)^3 = 8$, $x + 1 = 2$, so $x = 1$

zero gradient: $(-1, 8)$

x int $(1, 0)$; y int $(0, 7)$



c $h(x) = -(x + 2)^3 - 1$

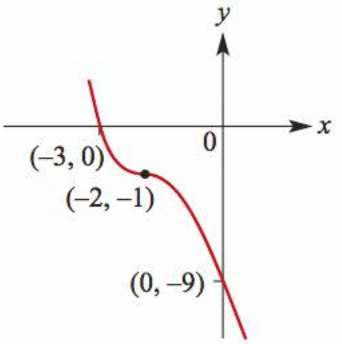
$x = 0$: $y = -(2)^3 - 1 = -9$

$y = 0$: $-(x + 2)^3 - 1 = 0$

$(x + 2)^3 = -1$, $x + 2 = -1$, so $x = -3$

zero gradient: $(-2, -1)$

x int $(-3, 0)$; y int $(0, -9)$



d $f(x) = (x+3)^3 - 1$

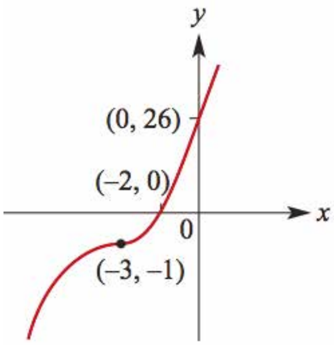
$x = 0$: $y = (3)^3 - 1 = 26$

$y = 0$: $(x+3)^3 - 1 = 0$

$(x+3)^3 = 1, x + 3 = 1$, so $x = -2$

zero gradient: $(-3, -1)$

x int $(-2, 0)$; y int $(0, 26)$



e $f(x) = 1 - (2x-1)^3$

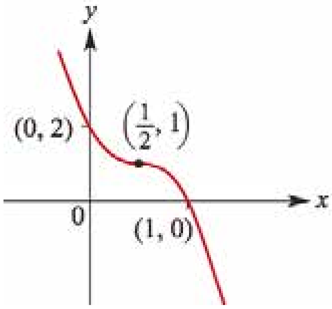
$x = 0$: $y = 1 - (-1)^3 = 2$

$y = 0$: $1 - (2x-1)^3 = 0$

$(2x-1)^3 = 1, 2x - 1 = 1$, so $x = 1$

zero gradient: $\left(\frac{1}{2}, 1\right)$

x int $(1, 0)$; y int $(0, 2)$



5 **a** $(x+2)^2 - 4$

b $3(x+1)^2 - 3$

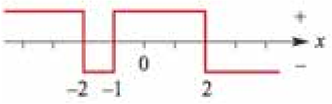
c $(x-2)^2 + 2$

d $2\left(x - \frac{3}{2}\right)^2 - \frac{17}{2}$

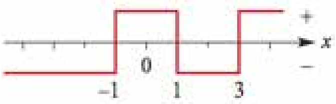
e $2\left(x - \frac{7}{4}\right)^2 - \frac{81}{8}$

f $-\left(x - \frac{3}{2}\right)^2 - \frac{7}{4}$

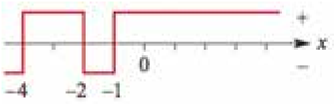
6 **a**



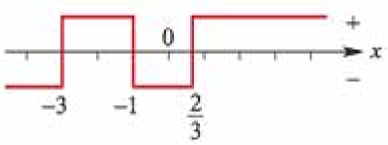
b



c



d



7 **a** $P(x) = x^3 + 3x^2 - 4x + 2$

$P(-1) = (-1)^3 + 3(-1)^2 - 4(-1) + 2$

$= 8$

b $P(x) = x^3 - 3x^2 - x + 6$

$P(2) = 2^3 - 3 \times 2^2 - 2 + 6$

$= 0$

c $P(x) = 2x^3 + 3x^2 - 3x - 2$

$P(-2) = 2(-2)^3 + 3(-2)^2 - 3(-2) - 2$

$= 0$

8 From the x intercepts, the rule must be $y = a(x+3)(x+2)(x-7)$

$x = 0$: $y = a(3)(2)(-7) = -42a$

But the y intercept is $(0, -42)$ and hence $-42a = -42$, so $a = 1$.

Thus $y = (x+3)(x+2)(x-7)$.

9 **a** $(x-2)(x+1)(x+3)$

b $(x-1)(x+1)(x-3)$

c $(x-1)(x+1)(x-3)(x+2)$

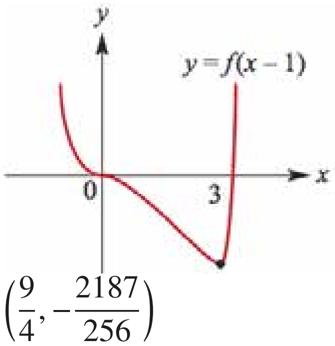
d $\frac{1}{4}(x-1)(2x+3+\sqrt{13})(2x+3-\sqrt{13})$

10 $x^2+4 = 1\times(x^2-2x+2)+2x+2$

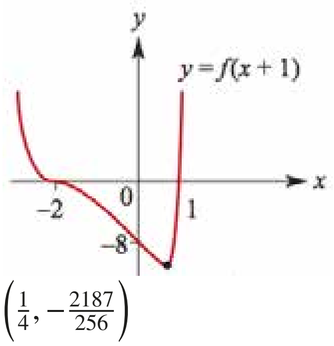
11 $a=-6$

12 $f(x)=(x+1)^3(x-2)$ Note: The tp on the diagrams are incorrect

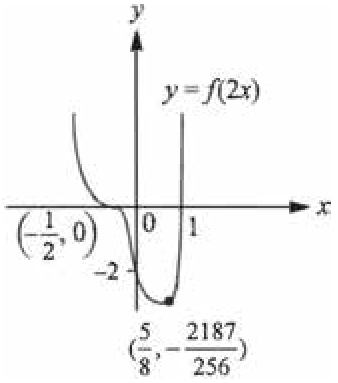
a $y=f(x-1)$
Translate the given graph 1 unit right.
The new intercepts are $(0,0)$, $(3,0)$.
The new minimum is at $(\frac{9}{4},-\frac{2187}{256})$
since the y value does not change.



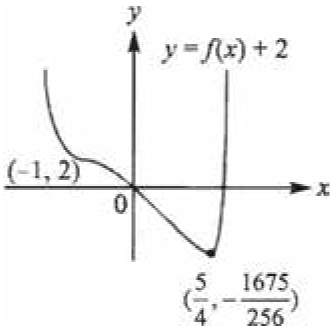
b $y=f(x+1)$
Translate the given graph 1 unit left.
The new x intercepts are $(-2,0)$, $(1,0)$.
$x=0$: $y=f(1)=2^3(-2)=-16$, so the new y intercept is $(0,-16)$.
The new minimum is at $\left(\frac{1}{4},-\frac{2187}{256}\right)$
since the y value does not change.



c $y=f(2x)$
Dilate the given graph $\frac{1}{2}$ unit from the y axis.
The new x intercepts are $(1,0)$, $(-\frac{1}{2},0)$.
The new y intercept stays at $(0,-2)$.
The new minimum is at $(\frac{5}{8},-\frac{2187}{256})$
since the y value does not change.



d $y=f(x)+2$
Translate the given graph 2 units up.
This makes the origin an intercept.
A second x intercept is between $\frac{5}{4}$ and 2.
The minimum has the same x value of $\frac{5}{4}$ and y value of $-\frac{2187}{256}+2=-\frac{1675}{256}$.
The new minimum is at $\left(\frac{5}{4},-\frac{1675}{256}\right)$.



13 $k = \pm 8$

14 $(4, -5),\ (3, 9)$

15 $a = 3,\ b = \frac{5}{6},\ c = -\frac{13}{12}$

16 $64x^3 + 144x^2 + 108x + 27$

17 $a = 1,\ b = -1,\ c = 4$

18 If $4x^2 - 2px + p + 3 = 0$ has no solution:

$$4p^2 - 16(p + 3) < 0$$
$$p^2 - 4p - 12 < 0$$
$$(p - 6)(p + 2) < 0$$

Hence no real solutions for $-2 < p < 6$

19 The rule of the cubic function is of the form $y = ax^3 + bx^2 + cx + d$. Since its graph passes through $(0, 6)$, $d = 6$. Write the equation as $y - 6 = ax^3 + bx^2 + cx$. Use the remaining points to form three simultaneous equations in a, b, and c.

$(1, 1)$: $-5 = a + b + c$ 1

$(2, 4)$: $-2 = 8a + 4b + 2c$ 2

$(3, 9)$: $3 = 27a + 9b + 3c$ 3

$2{-}21$: $6a + 2b = 8$ or equivalently
$3a + b = 4$ 4
$2 - 31$: $24a + 6b = 18$ or equivalently
$4a + b = 3$ 5
$5 - 4$ gives $a = -1$.
Substitution into 4 gives $b = 7$.
Substitution into 1 gives $c = -11$.
Hence $a = -1, b = 7, c = -11, d = 6$ and so $y = -x^3 + 7x^2 - 11x + 6$.

Solutions to multiple-choice questions

1 **E** $= 5x^2 - 10x - 2$
$= 5x^2 - 2x - 2$
$= 5x - 2x + 1 - 1 - 2$
$= 5(x - 2)^2 - 1 - 2$
$= 5(x - 2)^2 - 5 - 2$
$= 5(x - 2)^2 - 7$

2 **D** There are 2 real roots when the determinant > 0
$b^2 - 4ac > 0$
$36 + 12m > 0$
$12m > -36$
$m > -3$

3 **E** $x^3 + 27$
$= x^3 + 3^3$
$a^3 + b^3 = (ax + b)(ax^2 - abx + b^2)$
Where $a = 1$ and $b = 3$
$(x + 3)(x^2 - 3x + 9)$

4 **C** The equation is a cubic.
From null factor theorem:
The only possible options are B and C
Sub in an x value to determine if the graph has a positive or negative y value:
When $x = 2$
Option C: $y = 16 \times -6$
Option D: $y = 4 \times 4$
Therefore it must be option C

5 **E** $x - 1$ is a factor
$\therefore$ $1^3 + 3(1)^2 - 2a + 1 = 0$
$-2a = -5$
$a = \frac{5}{2}$

6 **A** Check by expanding:
For option A,
$(3x + 2y)(2x - 4y)$
$= 6x^2 - 12xy + 4xy - 8y^2$
$= 6x^2 - 8xy - 8y^2$

7 **C** Looking at the part of the graph shown, we can see that at $x = 1$, the graph is also showing a turning point. Therefore we can see that the answer must be either D or C, as the x-intercept points in the other graphs either show points of inflection (i.e. $f(x) = (x - 1)^3$), or an intercept where the graph doesn't change direction (i.e. $f(x) = x^2(x - 1)$. Then substitute values into the equations to check which one of C or D it is. Looking at C, you can see that for all values of x greater than zero other than 1, the function will be equal to a number less than zero. Looking at D, you can see that for all values of x greater than zero other than 1, the function will be equal to a number greater than zero.

8 **E** Expand the outer set of brackets to get the function into turning point for m. So $p(x) = 3((x - 2)^2 + 4)$ becomes $p(x) = 3(x - 2)^2 + 12$. Therefore the graph is shifted right 2 and up 12 from the origin. The answer is

9 C From the graph there is a intercept at $x = c$ and turning point at $(b, 0)$. So the polynomial must have functions $(x - c)$ and $(x - b)^2$

Now $(x - b)^2$ is the same as $(b - x)^2$.

$y = (x - c)(b - x)^2$ fits.

(Note: that option *D* gives a reflection in the *x*-axis of the graph given.)

10 C We can see immediately by looking at the equation that the function will touch the *x*-axis when $x = b$, and when $x = -c$. The remaining factor of the function is $(x^2 + a)$ and we know that is *a* positive real number. When we attempt to solve for *x*, we get the following: $x^2 = -a$.

Knowing that *a* is a positive real number, we realise that the solutions are not real numbers and hence are not roots.

11 C

$$-x^2 + 2x - 12 = kx - 3$$

$$-x^2 + (2 - k)x - 9 = 0$$

$$\Delta = (2 - k)^2 - 36$$

$$\Delta > 0 \Rightarrow (2 - k)^2 > 36$$

$$\therefore k < -4 \text{ or } k > 8$$

12 B

Solutions to extended-response questions

1 **a** The graph passes through the point (15, 20)

$\therefore \quad 20 = k \times 15^3 \times (20 - 15)$

$$\therefore \quad k = \frac{4}{15^3}$$

$$\therefore \quad k = \frac{4}{3375} \approx 0.0019$$

$$\therefore \quad R = \frac{4t^3}{3375}(20 - t)$$

b When $t = 10 \quad R = \dfrac{4 \times 10^3}{3375} \times 10$

$$= \frac{4 \times 10^4}{3375}$$

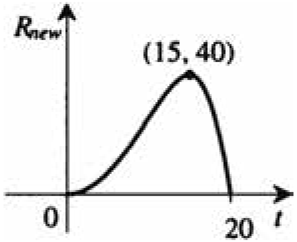
$$= \frac{4 \times 80}{27}$$

$$= \frac{320}{27}$$

The rate of flow $= \dfrac{320}{27}$mL/ min when $t = 10$

$\left(\dfrac{320}{27} \approx 11.852\right)$

c **i**



Note: This graph is given by a dilation of factor 2 from the t-axis

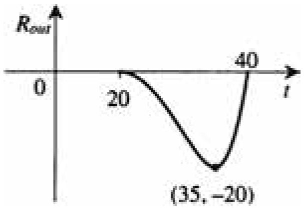
ii When $t = 10$

$$R_{new} = 2 \times \frac{4}{3375} \times 10^3 \times 10$$

$$= \frac{640}{27} - \text{mL/min}$$

The rate of flow $= \dfrac{640}{27}$ mL/min when $t = 10\left(\dfrac{640}{27} \approx 23.704\right)$

d **i** The hint gives that R_{out} is obtained by a translation of 20 units to the right.



$\therefore\ (t, R) \rightarrow (t + 20, R)$

$\therefore\ t' = t + 20$ and $R' = R$

$\therefore\ R = kt^3(20 - t)$ is transformed to

$$R' = k(t' + 20)^3(20 - (t' - 20))$$

$$= k(t' - 20)^3(40 - t')$$

A reflection in the x-axis give

$R_{out} = -k(t - 20)^3(40 - t)$

ii When $t = 30$, $R_{out} = \dfrac{-320}{27}$ mL/ min$\left(-\dfrac{320}{27} \approx -11.852\right)$

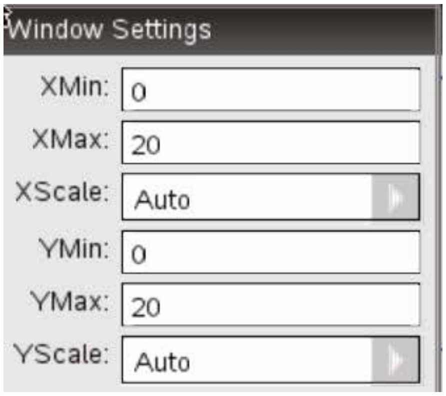
Note: the simplest way to obtain this is to move $\left(10, \dfrac{320}{7}\right) \rightarrow \left(30, \dfrac{-320}{7}\right)$ with this transformation

The rate of flow out is $\dfrac{320}{27}$ mL/ min

Calculator technique for question:

a In a Graphs page enter the rule: $f1(x) = 4/3375x^3(20 - x)$.

Suitable window settings are:



Window Settings

XMin: 0

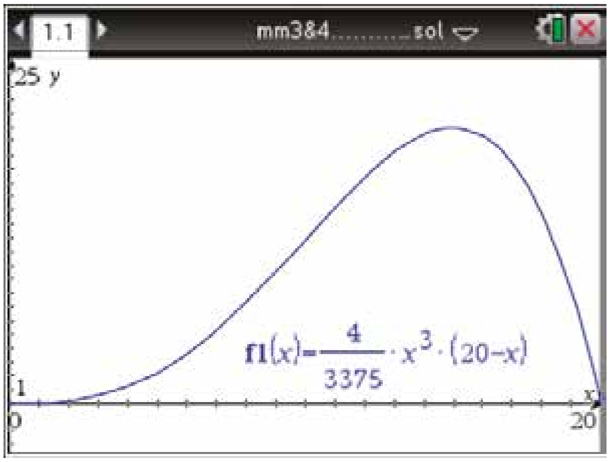
XMax: 20

XScale: Auto

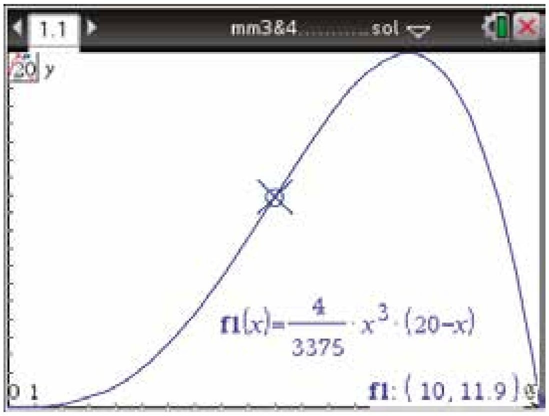
YMin: 0

YMax: 20

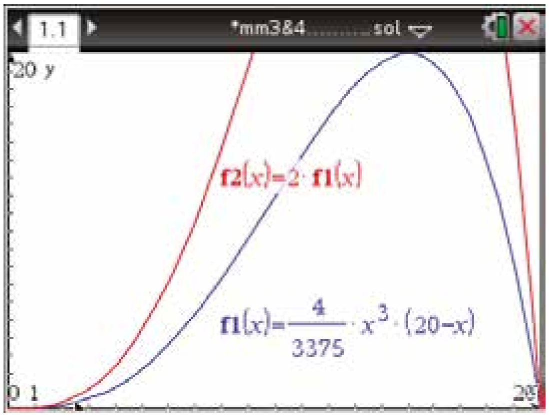
YScale: Auto



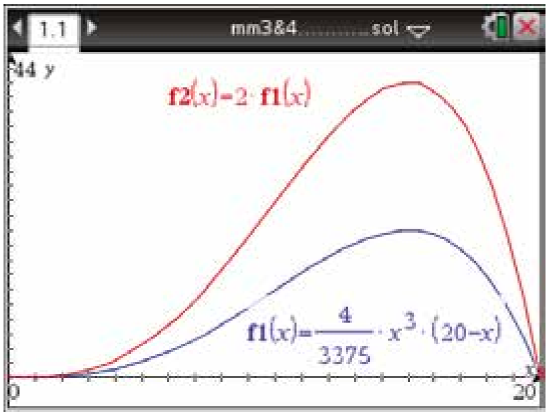
b The rate of flow when t = 10 is obtained by using Graph Trace from the Trace menu and typing in 10. Press.
Hint: press d to exit the Graph Trace tool.



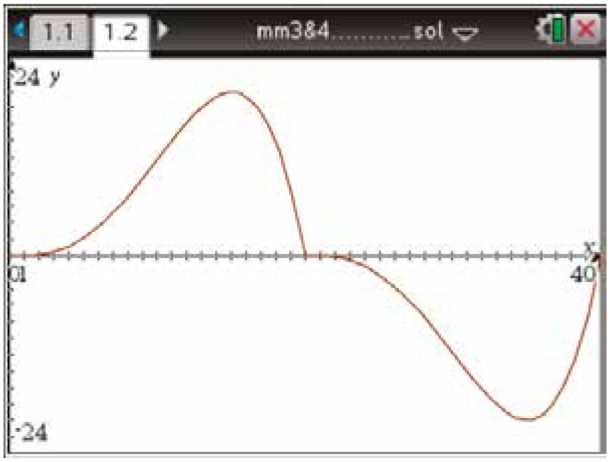
c The new function is obtained by entering $f2(x) = 2f1(x)$ in the function entry line (press e or /+G to show the function entry line if required). Press to plot the new graph.



Change the window settings to show key points of both graphs. Hint: use b>Window/Zoom>ZoomFit

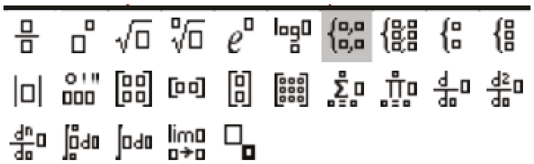


In order to see the graphs by R against t and Rout against t a hybrid function must be entered as shown,



$$f3(x) = \begin{cases} \left(\dfrac{4}{3375}\right)x^3(20 - x) & \text{for } 0 \geq x \geq 20 \\ -\left(\dfrac{4}{3375}\right)(x - 20)^3(40 - x) & \text{for } 20 \geq x \geq 40 \end{cases}$$

Insert a new Graphs page (/ + I) From the math templates palette (t)select the piecewise template.



The graph is as shown. For this choose Xmin = 0 and Xmax = 40. Adjust values.

2 **a** **i** When $t = 0$, $V = 4 \times 9^3 = 2916$
The volume is 2916 m^3

ii When t = 9, V = 0

b The volume is 0 m^3

c

$$512 = 4(9 - t)^3$$

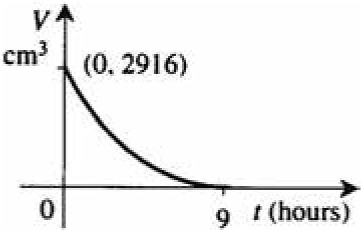
$$128 = (9 - t)^3$$

$$128^{\frac{1}{3}} = 9 - t$$

$$t = 9 - 128^3$$

$$= 9 - 4 \times 2^{\frac{1}{3}} \approx 3.9603$$

After 3.96 hr the volume is 512m^3



3 a i $V = \frac{1}{3}\pi \times 4 \times (18 - 2)$

$= \frac{\pi}{3} \times 4 \times 16$

$= \frac{64\pi}{3}$

Volume is $= \frac{64\pi}{3}$ cm^3 when $x = 2$

ii $V = \frac{1}{3}\pi \times 3^2 \times (18 - 3)$

$= \pi \times 45$

$= 45\pi$

Volume is 45πcm^3 when$x = 3$.

iii $V = \frac{1}{3}\pi \times 4^2 \times (18 - 4)$

$= \frac{\pi}{3} \times 16 \times 14$

$= \frac{224\pi}{3}$

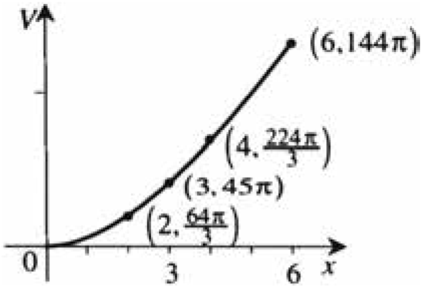
Volume is $= \frac{224\pi}{3}$ cm^3 when $x = 4$.

b When the bowl is full, depth is 6 cm.

When $x = 6$, $V = \frac{1}{3}\pi \times 36 \times 12$

$= 144\pi$

The volume of water is 144 πcm^3 when the bowl is full.



c If $V = \frac{325\pi}{3}$, $\frac{325\pi}{3} = \frac{1}{3}\pi x^2(18 - x)$

which implies $325 = x^2(18 - x)$

and $\therefore\ x^3 - 18x^2 + 325 = 0$

Let $P(x) = x^3 - 18x^2 + 325$

$P(5) = 5^3 - 18 \times 5^2 + 325 = 0$

which, by the Factor Theorem, implies that $x - 5$ is a factor.

$\therefore$ $P(x) = (x - 5)(x^2 - 13x + 65)$

$x^2 - 13x - 65 = 0$ implies $x = \frac{13 \pm \sqrt{169 + 4 \times 65}}{2}$

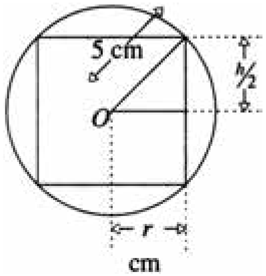
$= \dfrac{13 \pm \sqrt{429}}{2}$

but these two values of x lie outside the domain of $V = (0, 6)$

$\therefore$ $x = 5$ is the only solution.

i.e. the depth of the water when $V = \dfrac{325\pi}{3}$ is 5 cm.

4 a



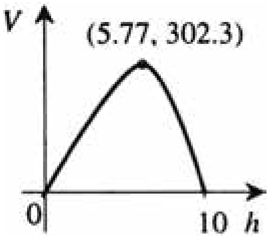
By Pythagoras' Theorem

$$r^2 + \left(\frac{h}{2}\right)^2 = 25$$

$$\therefore\ r^2 = 25 - \left(\frac{h}{2}\right)^2$$

i.e. $r = \dfrac{1}{2}\sqrt{100 - h^2}$

b



$$\begin{aligned}\therefore\ \text{Volume of cylinder} &= \frac{1}{2}\pi r^2 h \\ &= \pi \times \frac{1}{4}(100 - h^2)h \\ &= \frac{1}{4}\pi h(100 - h^2)\end{aligned}$$

c $V = \frac{1}{4}\pi h(100 - h^2)$

When $h = 6$

$V = \frac{1}{4} \times \pi \times 6(100 - 36)$

$= 96\pi$

The volume of the cylinder is $96\pi\text{cm}^3$

d When $V = 48\pi$

$$48\pi = \frac{1}{4}\pi h(100 - h^2)$$

$$\therefore 192 = 100h - h^3$$

$$\therefore h^3 - 100h + 192 = 0$$

$$\text{Let } P(h) = h^3 - 100h + 192$$

$$P(2) = 2^3 - 100 \times 2 + 192$$

$$= 0$$

$\therefore h - 2$is a factor

$\therefore P(h) = (h - 2)(h^2 + 2h - 96)$

$P(h) = 0$ implies $h = 2$ or $h^2 + 2h - 96 = 0$

$$\therefore h = \frac{-2 \pm \sqrt{4 + 4 \times 96}}{2}$$

$$= \frac{-2 \pm \sqrt{388}}{2}$$

$$= -1 \pm \sqrt{97}$$

But $h > 0, \therefore$ the only solutions are $h = 2$ and $h = -1 + \sqrt{97}$

When $h = 2$

$r = \frac{1}{2}\sqrt{100 - 4}$

$= \frac{1}{2}\sqrt{96}$

$= 2\sqrt{6}$

When $h = -1 + \sqrt{97} \approx 8.849$

$r \approx \frac{1}{2}\sqrt{100 - 78.30}$

≈ 2.33

When the volume of the cylinder is 48π cm^3 the height is 2 cm and the radius $2\sqrt{6} \approx 4.899$ cm.

OR the height is $(-1 + \sqrt{97}) \approx 8.849$ and the radius is ≈ 2.33 cm.

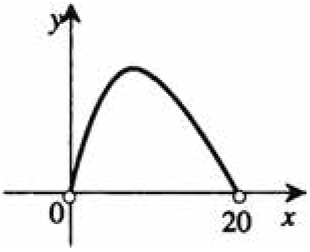
5 **a** $V = (84 - 2x)(40 - 2x)x$

b $84 - 2x > 0$ and $40 - 2x > 0$ and $x > 0$

$\therefore$ $x < 42$ and $x < 20$ and $x < 0$

$\therefore$ maximal domain = $(0, 20)$

c



d **i** 5 760 cm^3

ii 12 096 cm^3

iii 13 056 cm^3

iv 12 800 cm^3

e Use **Intersection** from the **Analyze Graph** menu
$x = 13.50$ or $x = 4.18$ (answers given correct to two decimal places)

f 13 098.71 cm^3 (use **Maximum** from the **Analyze Graph** menu)

6 **a** **i** $A = 2x(16 - x^2)$

ii $0 < x < 4$

b **i** $A = 6(16 - 9)$
$= 42$

ii $x = 0.82$ or $x = 3.53$ (use **Intersection** from the **Analyze Graph** menu)

c **i** $V = xA$
$= 2x^2(16 - x^2)$

ii $x = 2.06$ or $x = 3.43$ (use **Intersection** from the **Analyze Graph** menu)

7 **a** $A = yx + \frac{\pi}{2}x^2$

b **i** $100 = y + \pi x$

$\therefore$ $y = 100 - \pi x$

ii $A = (100 - \pi x)x + \frac{\pi}{2}x^2$

$= 100x - \pi x^2 + \frac{\pi}{2}x^2$

$= 100x - \frac{\pi}{2}x^2$

iii $\left(0, \frac{100}{\pi}\right)$ as $x > 0$ and $y > 0$ which implies $100 - \pi x > 0$

c $x = 12.425$

Intersection from the Analyze Graph menu has been used.

d **i** $V = \frac{x}{50}\left(\frac{\pi}{2}x^2 + yx\right)$

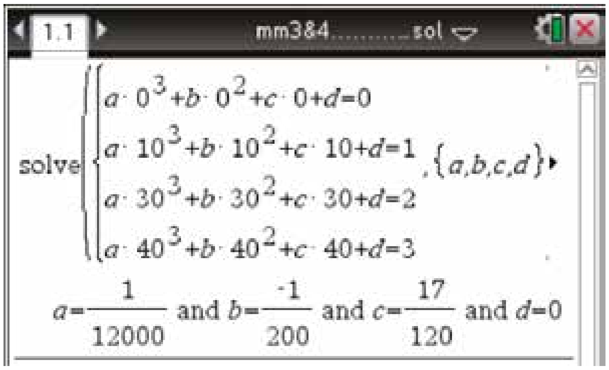
$= \frac{x}{50}\left(100x - \frac{\pi}{2}x^2\right)$

$= \frac{x^2}{50}\left(100 - \frac{\pi}{2}x\right) \quad x \in \left(0, \frac{100}{\pi}\right)$

ii $V = 248.5$ m^3 using $x = 12.425$ when $A = 1000$

iii Using Intersection from the Analyze Graph menu gives $x = 18.84$

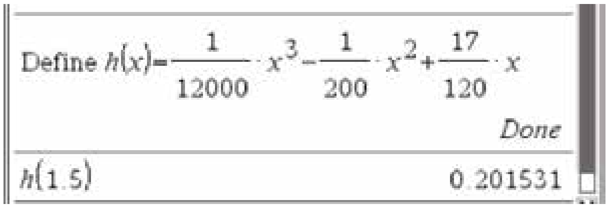
8a In a **Calculator** page solve the system of equations using b**>Algebra>Solve System of Equations>Solve System of Equations**.



$$\text{solve}\left(\begin{cases} a\cdot 0^3+b\cdot 0^2+c\cdot 0+d=0 \\ a\cdot 10^3+b\cdot 10^2+c\cdot 10+d=1 \\ a\cdot 30^3+b\cdot 30^2+c\cdot 30+d=2 \\ a\cdot 40^3+b\cdot 40^2+c\cdot 40+d=3 \end{cases},\{a,b,c,d\}\right)$$

$$a=\frac{1}{12000} \text{ and } b=\frac{^-1}{200} \text{ and } c=\frac{17}{120} \text{ and } d=0$$

b Define the function $h(x)$
Find the height when $x = 1.5$ m



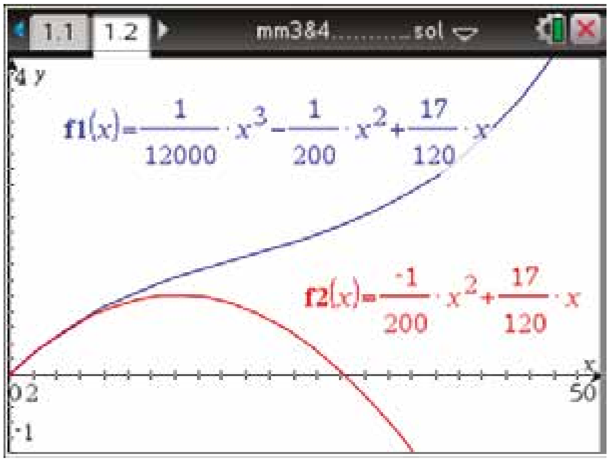
$$\text{Define } h(x)=\frac{1}{12000}\cdot x^3-\frac{1}{200}\cdot x^2+\frac{17}{120}\cdot x$$

Done

$h(1.5)$ 0.201531

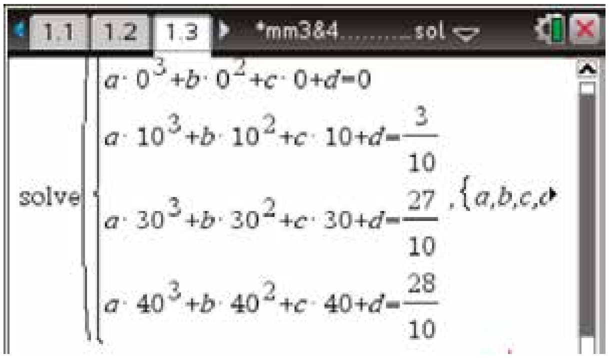
c In a **Graphs** page, enter the two functions

The coefficient of x^3, although small, is clearly influential.



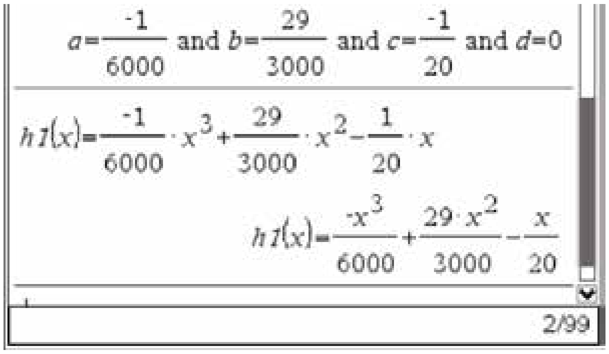
d Solve the system of equations.

Hint: to obtain exact (fraction) answers the decimal values in the system of equations can be written as fractions as shown.



$$\text{solve}\left(\begin{cases} a\cdot 0^3+b\cdot 0^2+c\cdot 0+d=0 \\ a\cdot 10^3+b\cdot 10^2+c\cdot 10+d=\frac{3}{10} \\ a\cdot 30^3+b\cdot 30^2+c\cdot 30+d=\frac{27}{10} \\ a\cdot 40^3+b\cdot 40^2+c\cdot 40+d=\frac{28}{10} \end{cases},\{a,b,c,d\right.$$

Alternatively use **b>Number>Approximate to Fraction** and edit the tolerance to 5.E-5)

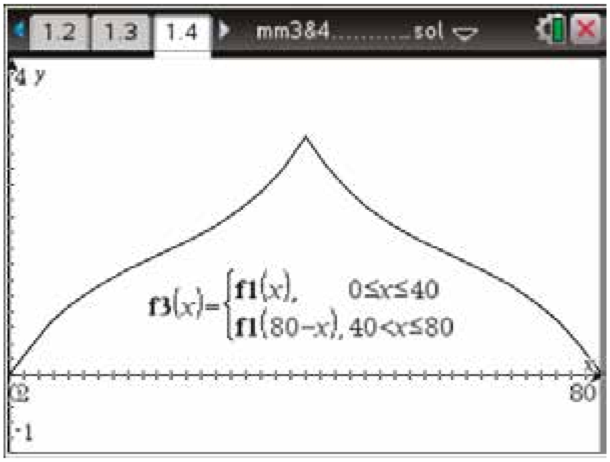


$a=\frac{-1}{6000}$ and $b=\frac{29}{3000}$ and $c=\frac{-1}{20}$ and $d=0$

$h1(x)=\frac{-1}{6000}\cdot x^3+\frac{29}{3000}\cdot x^2-\frac{1}{20}\cdot x$

$$h1(x)=\frac{-x^3}{6000}+\frac{29\cdot x^2}{3000}-\frac{x}{20}$$

e (i) in a **Graphs** page enter the hybrid function using the piecewise template from the **Math Template** palette (t)



$$\mathbf{f3}(x)=\begin{cases}\mathbf{f1}(x), & 0\le x\le 40\\ \mathbf{f1}(80-x), & 40<x\le 80\end{cases}$$

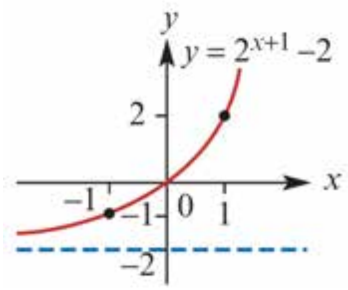
The result is as shown.

e (ii) The second section of the graph is formed by a reflection of the graph of $y = f1(\mathrm{x}), \mathrm{x} \in (0, 40)$ in the line $x = 40$

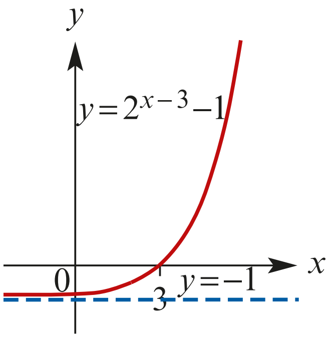
Chapter 5 – Exponential and logarithmic functions

Solutions to Exercise 5A

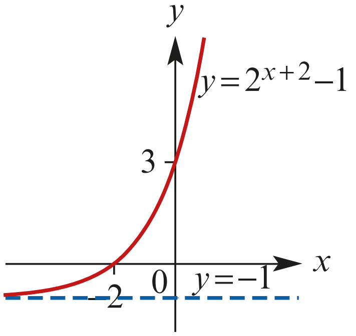
1 **a** Range = $(-2, \infty)$



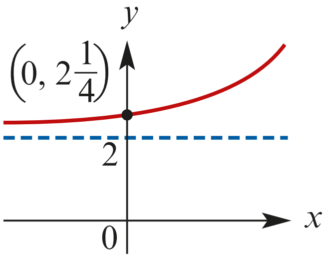
b Range = $(-1, \infty)$



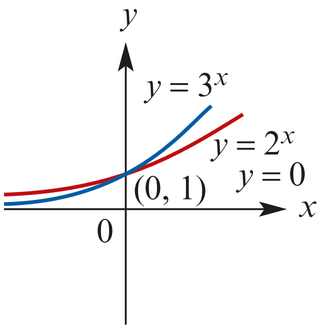
c Range = $(-1, \infty)$



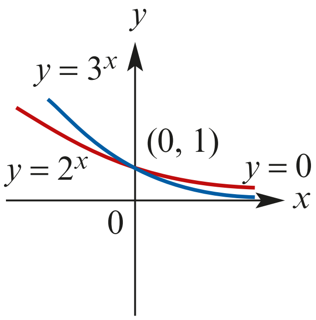
d Range = $(2, \infty)$



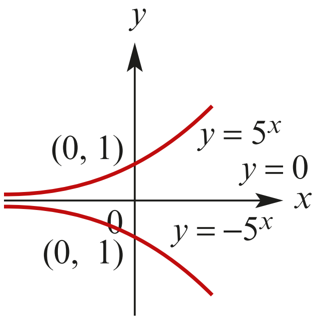
2 **a**



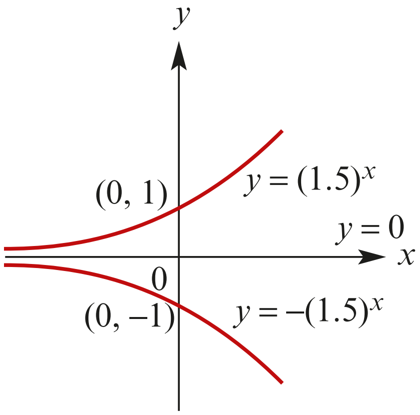
b



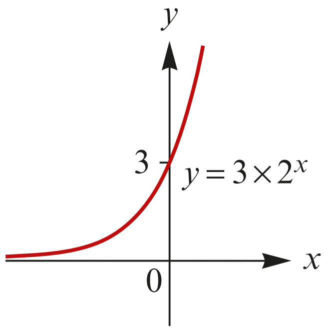
c



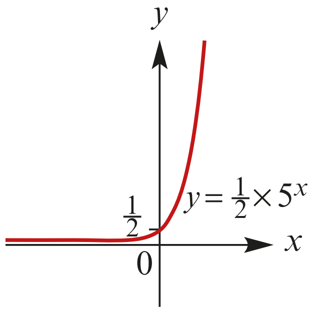
d



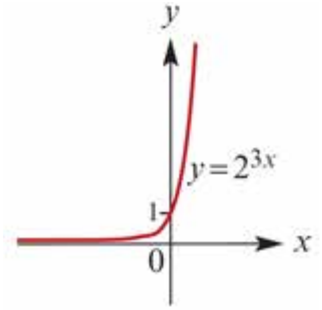
3 **a** Range = $(0, \infty)$



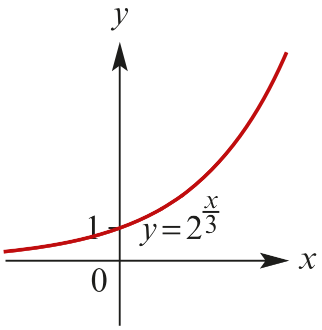
b Range = $(0, \infty)$



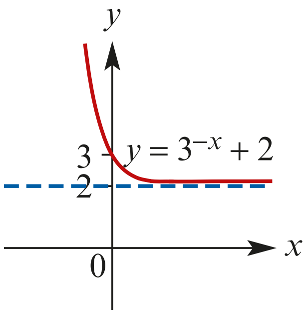
c Range = $(0, \infty)$



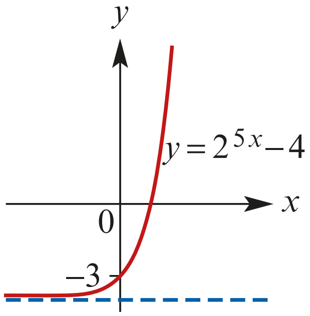
d Range = $(0, \infty)$



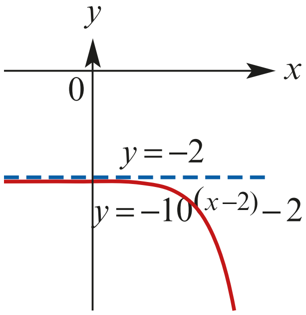
4 **a** Range = $(2, \infty)$



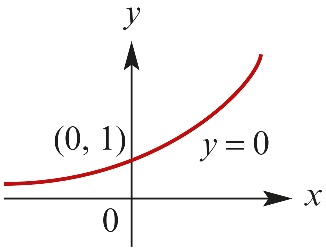
b Range = $(-4, \infty)$



c Range = $(-\infty, 2)$

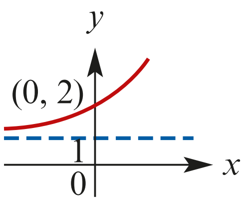


5 **a**



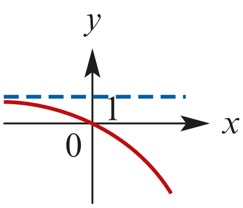
Range = $\mathbb{R}^+$

b

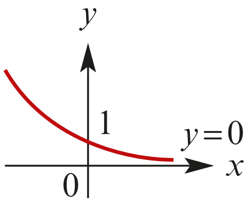


Range = $(1, \infty)$

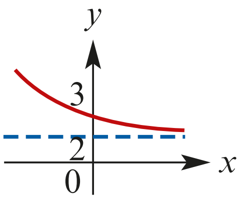
c Range = $(-\infty, 1)$



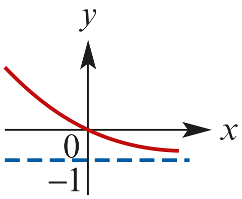
d Range = $\mathbb{R}^+$



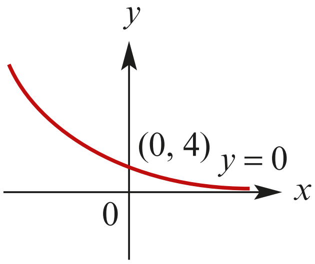
e Range = $(2, \infty)$



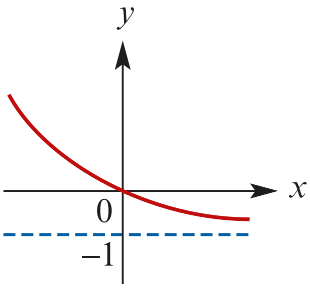
f Range = $(-1, \infty)$



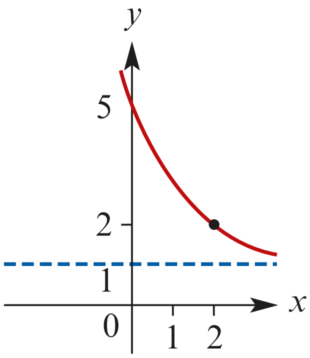
6 **a** Range = $\mathbb{R}^+$



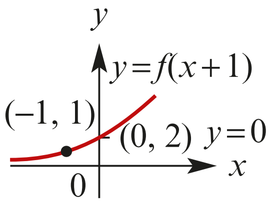
b Range = $(-1, \infty)$



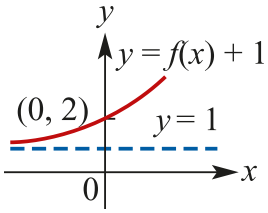
c Range = $(1, \infty)$



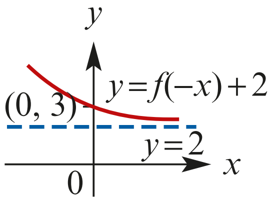
7 **a**



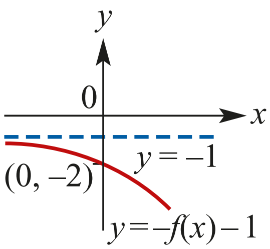
b



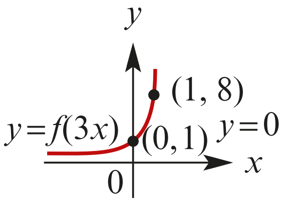
c



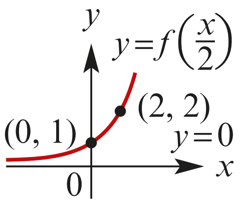
d



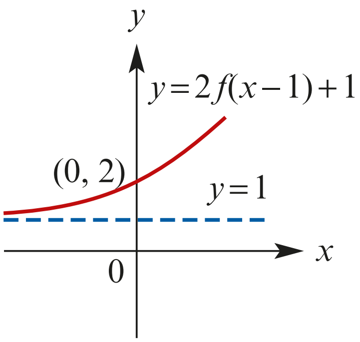
e



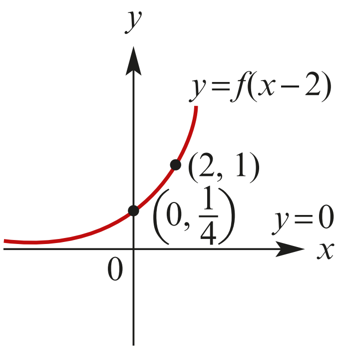
f



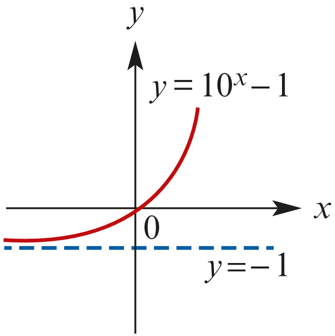
g



h

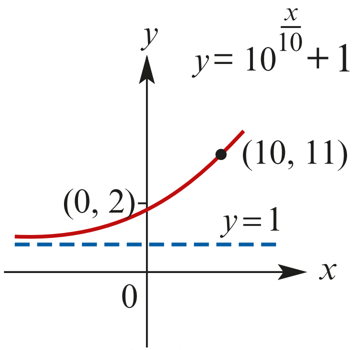


8 a



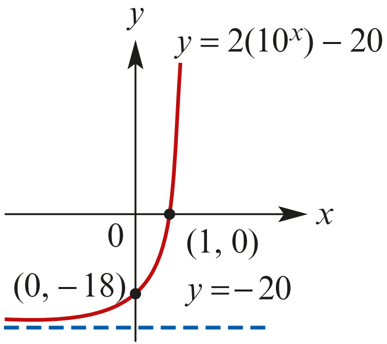
range = $(-1, \infty)$

b



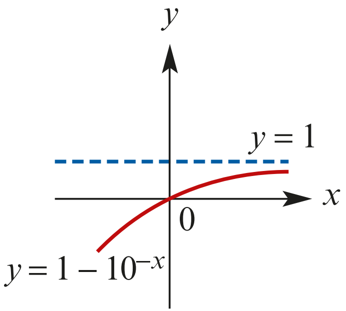
range = $(1, \infty)$

c



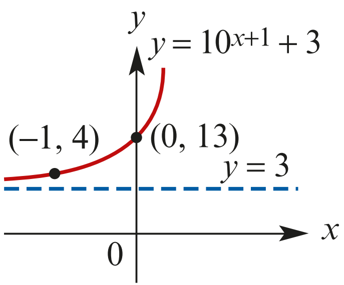
range = $(-20, \infty)$

d



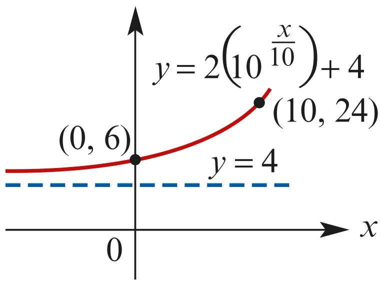
range = $(-\infty, 1)$

e



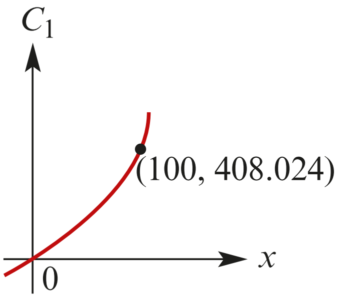
range = $(3, \infty)$

f



range = $(4, \infty)$

9 a



b $C_1 = 10000((1.0004)^x - 1)$

i $C_1 = 10000((1.0004)^{100} - 1)$

$= 10000(1.040802 - 1)$

$= \$408.02$

ii $C_1 = 10000((1.0004)^{300} - 1)$

$= 10000(1.127470 - 1)$

$= \$1,274.70$

c $1000 = 10000((1.0004)^x - 1)$

$(1.0004)^x = 1.1$
Use the 'solve' command of a CAS calculator to solve for x. This gives $x = 238.32\ldots\ x = 239$ days
(you must round up in this case)

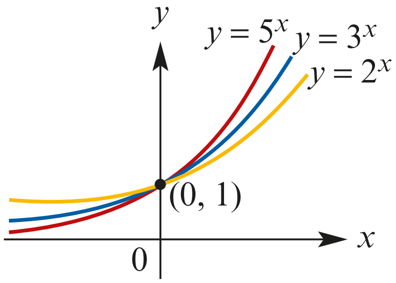
d i

ii to find $C_2 < C_1$
find $C_2 = C_1$ then round up using the CAS calculator at

$C_2 = C_1, x = 301.16$
$\therefore$ for $C_2 < C_1$
minimum $x = 302$ days

10 $y = 100(1.02)^x$
what is x when $y = 200$?
$2 = (1.02)^x$
Use the 'solve' command of a CAS calculator to solve for x. This gives $x = 35.003$. So your money has not quite doubled after 35 days; it will take 36 days.

11 a i

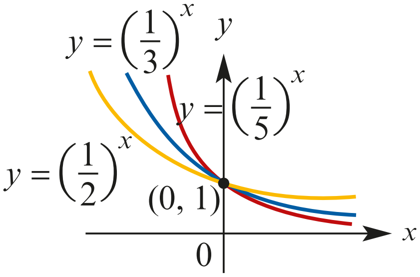


ii $x < 0$

iii $x > 0$

iv $x = 0$ (read off graph)

b i

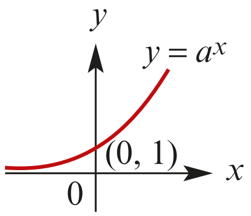


ii $x > 0$

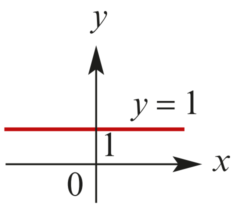
iii $x < 0$

iv $x = 0$ (read off graph)

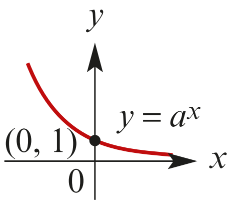
c **i**



ii

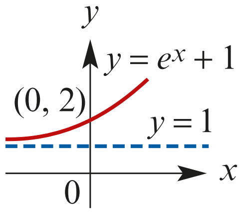


iii



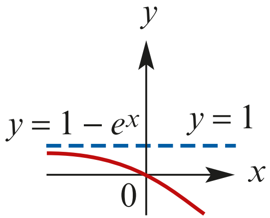
Solutions to Exercise 5B

1 **a**



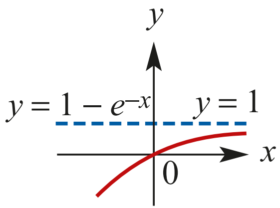
range = $(1, \infty)$

b



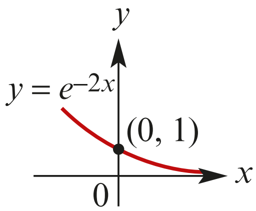
range = $(-\infty, 1)$

c



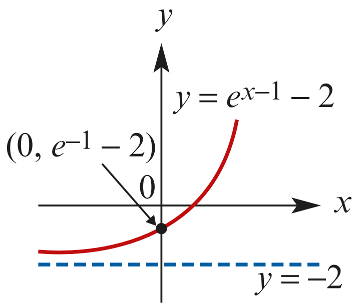
range = $(-\infty, 1)$

d



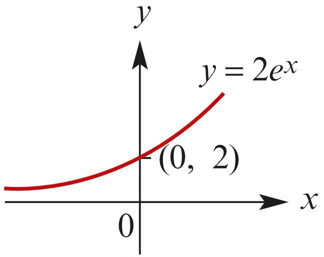
range = $(0, \infty)$

e



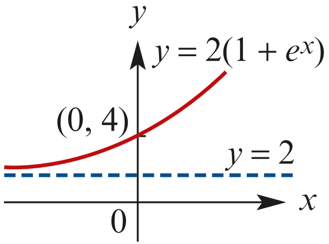
range = $(-2, \infty)$

f



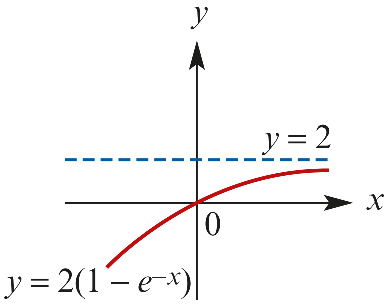
range = $(0, \infty)$

g



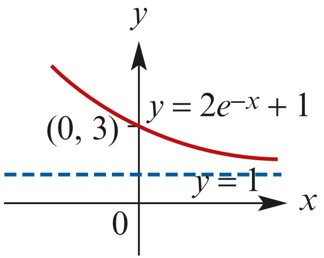
range = $(2, \infty)$

h



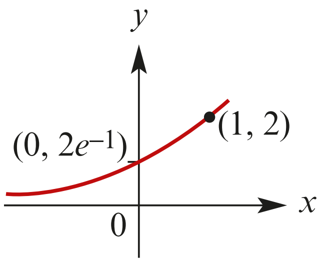
range = $(-\infty, 2)$

i



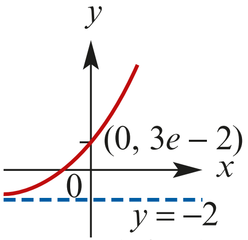
range = $(1, \infty)$

j



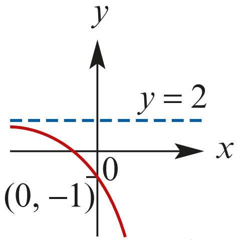
range = $(0, \infty)$

k



range = $(-2, \infty)$

l



range = $(-\infty, 2)$

2 **a** Translation 2 units to the left and 3 units down

b Dilation of factor 3 from the x-axis, then translation 1 unit to the left and 4 units down

c Dilation of factor 5 from the x-axis and factor $\frac{1}{2}$ from the y-axis, then translation $\frac{1}{2}$ unit to the left

d Reflection in the x-axis, then translation 1 unit to the right and 2 units up

e Dilation of factor 2 from the x-axis, reflection in the x-axis, then translation 2 units to the left and 3 units up

f Dilation of factor 4 from the x-axis and factor $\frac{1}{2}$ from the y-axis, then translation 1 unit down

3 **a** $y = -2e^{x-3} - 4$

b $y = 4 - e^{2x-3}$

c $y = -2e^{x-3} - 4$

d $y = -2e^{x-3} - 8$

e $y = -2e^{x-3} + 8$

f $y = -2e^{x-3} + 8$

4 **a** Translation 2 units to the right and 3 units up

b Translation 1 unit to the right and 4 units up, then dilation of factor $\frac{1}{3}$ from the x-axis

c Translation $\frac{1}{2}$ unit to the right, then dilation of factor $\frac{1}{5}$ from the x-axis and factor 2 from the y-axis

d Translation 1 unit to the left and 2 units down, then reflection in the x-axis

e Translation 2 units to the right and 3 units down, then dilation of factor $\frac{1}{2}$ from the x-axis and reflection in the x-axis

f Translation 1 unit up, then dilation of factor $\frac{1}{4}$ from the x-axis and factor 2 from the y-axis

5 **a** $x = 1.146$ or $x = -1.841$

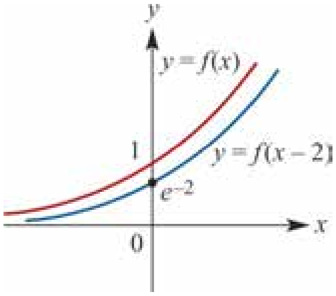
b $x = -0.443$

c $x = -0.703$

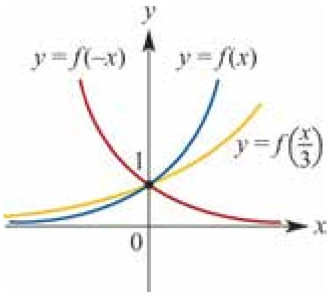
d $x = 1.857$ or $x = 4.536$

6

a b i



ii iii



Solutions to Exercise 5C

1 **a** $3x^2y^2 + 2x^4y^6 = 6x^6y^9$

b $\dfrac{12x^8}{4x^2} = 3x^6$

c $\dfrac{18x^2y^3}{3x^4y} = 6x^{-2}y^2$

$= \dfrac{6y^2}{x^2}$

d $(4x^4y^2)^2 \div (2(x^2y)^4)$

$= 16x^8y^4 \div (2x^8y^4)$

$= 8$

e $(4x^0)^2$

$= 4^2$

$= 16$

f $\dfrac{15(x^5y^{-2})^4}{3(x^4y)^{-2}} = 5x^{20}y^{-8}x^8y^2$

$= 5x^{28}y^{-6}$

$= \dfrac{5x^{28}}{y^6}$

g $\dfrac{3(2x^2y^3)^4}{2x^3y^2} = \dfrac{3 * 16x^8y^{12}}{2x^3y^2}$

$= 24x^5y^{10}$

h $(8x^3y^6)^{\frac{1}{3}} = 2xy^2$

i $\dfrac{x^2 + y^2}{x^{-2} + y^{-2}} = \dfrac{x^2 + y^2}{\dfrac{1}{x^2} + \dfrac{1}{y^2}}$

$= \dfrac{x^2 + y^2}{\dfrac{y^2 + x^2}{x^2y^2}}$

$= x^2y^2$

2 **a** $3^x = 81$

$3^x = 3^4$

$x = 4$

b $81^x = 9$

$81^x = 81^{\frac{1}{2}}$

$x = \dfrac{1}{2}$

c $2^x = 256$

$2^x = 2^8$

$x = 8$

d $625^x = 5$

$625^x = 625^{\frac{1}{4}}$

$x = \dfrac{1}{4}$

e $32^x = 8$

$2^{5x} = 2^3$

$5x = 3$

$x = \dfrac{3}{5}$

f $5^x = 125$

$5^x = 5^3$

$x = 3$

g $16^x = 1024$

$2^{4x} = 2^{10}$

$x = \dfrac{5}{2}$

h $2^{-x} = \frac{1}{64}$

$2^{-x} = 2^{-6}$

$x = 6$

i $5^{-x} = \frac{1}{625}$

$5^{-x} = 5^{-4}$

$x = 4$

3 a $5^{2n} \times 25^{2n-1} = 625$

$5^{2n} \times 5^{4n-2} = 5^4$

$5^{6n-2} = 5^4$

$6n - 2 = 4$

$n = 1$

b $4^{2n-2} = 1$

$4^{2n-2} = 4^0$

$2n - 1 = 0$

$n = 1$

c $4^{2n-1} = \frac{1}{256}$

$4^{2n-1} = 4^{-4}$

$2n - 2 = -4$

$n = \frac{-3}{2}$

d $\frac{3^{n-2}}{9^{2-n}} = 27$

$3^{n-2} \times 3^{-2n-4} = 3^3$

$3^{3n-6} = 3^3$

$3n - 6 = 3$

$n = 3$

e $2^{2n-2} \times 4^{-3n} = 64$

$2^{2n-2} \times 2^{-6n} = 2^6$

$2^{-4n-2} = 2^6$

$-4n - 2 = 6$

$-4n = 8$

$n = -2$

f $2^{n-4} = 8^{4-n}$

$2^{n-4} = 2^{12-3n}$

$n - 4 = 12 - 3n$

$n = 4$

g $27^{n-2} = 9^{3n+2}$

$3^{3n-6} = 32^{6n+4}$

$3n - 6 = 6n + 4$

$3n = -10$

$n = \frac{-10}{3}$

h $8^{6n+2} = 8^{4n-1}$

$6n + 2 = 4n - 1$

$2n = -3$

$n = \frac{-3}{2}$

i $125^{4-n} = 5^{6-2n}$

$5^{12-3n} = 5^{6-2n}$

$12 - 3n = 6 - 2n$

$n = 6$

j $2^{n-1} \times 4^{2n+1} = 16$

$$\begin{aligned} 2^{n-1} \times 2^{4n+2} &= 2^4 \\ 2^{5n+1} &= 2^4 \\ 5n + 1 &= 4 \\ 5n &= 3 \\ n &= \frac{3}{5} \end{aligned}$$

k $(27 \times 3^n)^n = 27^n \times 3^{\frac{1}{4}}$

$$\begin{aligned} (3^3 \times 3^n)^n &= 3^{3n} \times 3^{\frac{1}{4}} \\ (3^{3+n})^n &= 3^{3n+\frac{1}{4}} \\ 3^{3n+n^2} &= 3^{3n+\frac{1}{4}} \\ 3n + n^2 &= 3n + \frac{1}{4} \\ n^2 &= \frac{1}{4} \\ n^2 &= \pm\frac{1}{2} \end{aligned}$$

4 **a** $3^{2x} - 2(3^x) - 3 = 0$

$$\begin{aligned} \Rightarrow (3^x - 3)(3^x + 1) &= 0 \\ 3^x &= 3; \quad -1 \\ \therefore x &= 1; \\ x &= 1 \end{aligned}$$

b $5^{2x} - 23(5^x) - 50 = 0$

$$\begin{aligned} \Rightarrow (5^x - 25)(5^x + 2) &= 0 \\ 5^x &= 25; \quad -2 \\ \therefore x &= 2; \\ x &= 2 \end{aligned}$$

c $5^{2x} - 10(5^x) + 25 = 0$

$$\begin{aligned} (5^x - 5)^2 &= 0 \\ 5^x &= 5 \\ x &= 1 \end{aligned}$$

d $2^{2x} - 6(2^x) + 8 = 0$

$$\begin{aligned} \Rightarrow (2^x - 2)(2^x - 4) &= 0 \\ 2^x &= 2, 4 \\ x &= 1, 2 \end{aligned}$$

e $8(3^x) - 6 = 2(3^{2x})$

$$\begin{aligned} 3^{2x} - 4(3^x) + 3 &= 0 \\ (3^x - 3)(x - 1) &= 0 \\ 3^x &= 3, 1 \\ x &= 1, 0 \end{aligned}$$

f $2^{2x} - 20(2^x) + 64 = 0$

$$\begin{aligned} (2^x - 16)(2^x - 4) &= 0 \\ 2^x &= 16, 4 \\ x &= 4, 2 \end{aligned}$$

g $4^{2x} - 5(4^x) + 4 = 0$

$$\begin{aligned} (4^x - 4)(4^x - 1) &= 0 \\ 4^x &= 4, 1 \\ x &= 1, 0 \end{aligned}$$

h $3(3^{2x}) - 28(3^x) + 9 = 0$

$$\begin{aligned} (3(3^x) - 1)(3^x - 9) &= 0 \\ 3^x &= \frac{1}{3}, 9 \\ x &= -1, 2 \end{aligned}$$

i $$7(7^{2x}) - 8(7^x) + 1 = 0$$

$$(7(7^x) - 1)(7^x - 1) = 0$$

$$7^x = \frac{1}{7}, 1$$

$$x = -1, 0$$

Solutions to Exercise 5D

1 **a** 3

b −4

c −3

d 6

e 6

f −7

2 Note: the natural logarithm function $\log_e x$ is often written $\ln x$; this notation is used here.

a $\ln 6$

b $\ln 4$

c $\ln 10^6 = 6 \ln 10$

d $\ln 7$

e $\ln \dfrac{1}{3 \times 4 \times 5} = \ln \dfrac{1}{60} = -\ln 60$

f $\ln(uv \times uv^2 \times uv^3) = \ln u^3v^6$

$= 3 \ln uv^2$

g $7 \ln x = \ln x^7$

h $\ln\left(\dfrac{(x+y)(x-y)}{(x^2-y^2)}\right)$

$= \ln 1$

$= 0$

3 **a** $x = 10^2 = 100$

b $\log_2 x = 4$

$x = 2^4 = 16$

c $x - 5 = e^0 = 1$

$x = 6$

d $x = 2^6 = 64$

e $\ln(x + 5) = 3$

$x + 5 = e^3$

$x = e^3 - 5 \approx 15.086$

f $2x = e^0 = 1$

$x = \dfrac{1}{2}$

g $2x + 3 = e^0 = 1$

$2x = -2$

$x = -1$

h $x = 10^{-3}$

$= \dfrac{1}{1000}$

i $\log_2(x - 4) = 5$

$x - 4 = 2^5 - 32$

$x = 36$

4 **a** $\log_{10} x = \log_{10} 15$

$x = 15$

b $\ln x = \ln 5$

$x = 5$

c $\ln x = \ln\left(8^{\frac{2}{3}}\right)$

$= \ln 4$

$x = 4$

d $\ln(2x^2 - x) = 0, x > 0$

$$2x^2 - x = 1$$

$$2x^2 - x - 1 = 0$$

$$(2x + 1)(x - 1) = 0$$

$$x = -\frac{1}{2}, 1$$

since $x > 0, x = 1$

e $\ln x^2 - \ln(x - 1) = \ln(x + 3)$

$$\ln \frac{x^2}{x - 1} = \ln(x + 3)$$

$$x^2 = (x + 3)(x - 1)$$

$$x^2 = x^2 + 2x - 3$$

$$2x = 3$$

$$x = \frac{3}{2}$$

5 **a** $\log_{10}(3 \times 9) = \log_{10} 27$

b $\log_2\left(\frac{24}{6}\right) = \log_2 4 = 2$

c $\frac{1}{2}(\log_{10} a - \log_{10} b)$

$$= \frac{1}{2}\left(\log_{10} \frac{a}{b}\right)$$

$$= \log_{10} \sqrt{\frac{a}{b}}$$

d $1 + \log_{10} a - \log_{10}(b)^{\frac{1}{3}}$

$$= \log_{10} 10 + \log_{10}\left(\frac{a}{b^{\frac{1}{3}}}\right)$$

$$= \log_{10}\left(\frac{10a}{b^{\frac{1}{3}}}\right)$$

e $\log_{10\sqrt{36}} - \log_{10}(27)^{\frac{1}{3}} - \log_{10}(64)^{\frac{2}{3}}$

$$= \log_{10} 6 - \log_{10} 3 - \log_{10} 16$$

$$= \log_{10}\left(\frac{6}{3 \times 16}\right)$$

$$= \log_{10}\left(\frac{1}{8}\right)$$

6 **a** $\log_{10} 10 = 1$

b $\log_{10} 5 + \log_{10} 8 - \log_{10} 4$

$$= \log_{10} 10$$

$$= 1$$

c $\log_2 \sqrt{2} + \log_2 1 + \log_2 4$

$$= \log_2 4\sqrt{2}$$

$$= 2\frac{1}{2}$$

$$= \frac{5}{2}$$

d $\log_{10} 25 + \log_{10} 4 + \log_{10} 10$

$$= \log_{10} 1000$$

$$= 3$$

e $\log_{10} 16 - \log_{10} 16$

$$= 0$$

7 **a** $\log_3\left(\frac{1}{3^x}\right) = \log_3(3^{-x})$

$$= -x \log_3 3$$

$$= -x$$

b $\log_2 x - \log_2 y^2 + \log_2(xy^2)$

$$= \log_2(x^2)$$

$$= 2 \log_2 x$$

c $\ln(x^2 - y^2) - \ln(x - y) - \ln(x + y)$

$$= \ln\left(\frac{x^2 - y^2}{(x - y)(x + y)}\right)$$

$$= \ln 1 = 0$$

8 a $\ln(x^2 - 2x + 8) = \ln x^2$

$$x^2 - 2x + 8 = x^2$$

$$2x = 8$$

$$x = 4$$

b

$$\ln(5x) - \ln(3 - 2x) = \ln e$$

$$\ln(5x) = \ln(e(3 - 2x))$$

$$3e - 2ex = 5x$$

$$(5 + 2e)x = 3e$$

$$x = \frac{3e}{5 + 2e} \approx 0.7814$$

9 a $\ln x + \ln(3x + 1) = \ln e$

$$\ln(3x^2 + x) = \ln e$$

$$3x^2 + x - e = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 12e}}{6}$$

$$but\ x > \frac{-1}{3}$$

$$x = \frac{-1 + \sqrt{1 + 12e}}{6}$$

$$\approx 0.7997$$

b $8e^{-x} - e^x - 2 = 0$

$$8 - e^{2x} - 2e^x = 0$$

$$(e^x)^2 + 2e^x - 8 = 0$$

$$(e^x + 4)(e^x - 2) = 0$$

$$e^x = -4, 2$$

But $e^x > 0$, so:

$$e^x = 2$$

$$x = \ln 2 \approx 0.6931$$

10 a $\log_x 81 = 4$

$$x^4 = 81$$

$$x = 3$$

b $\log_x \frac{1}{32} = 5$

$$x^5 = \frac{1}{32}$$

$$x = \frac{1}{2}$$

11 $\ln x^2 + \ln 4 = \ln(9x - 2)$

$$4x^2 = 9x - 2$$

$$4x^2 - 9x + 2 = 0$$

$$(4x - 1)(x - 2) = 0$$

$$x = \frac{1}{4}, 2$$

12

$$\log_a N = \frac{1}{2}(\log_a 24 - \log_a 0.375 - \log_a 729)$$

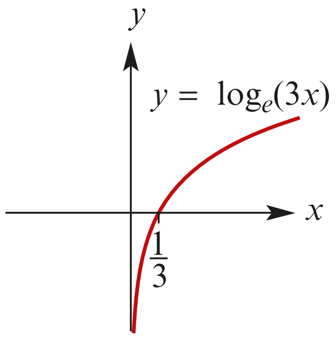
$$= \frac{1}{2}\left(\log_a \frac{64}{729}\right)$$

$$= \log_a \frac{8}{27}$$

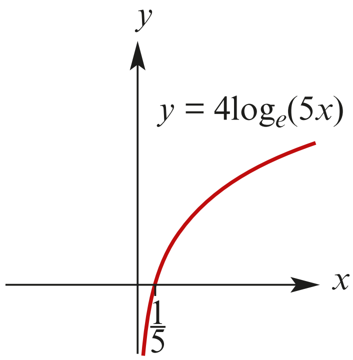
$$N = \frac{8}{27}$$

Solutions to Exercise 5E

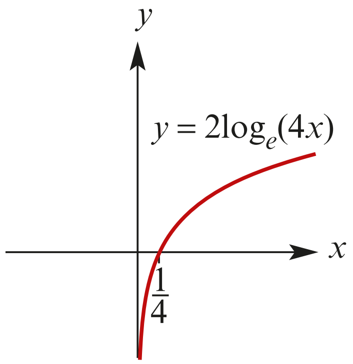
1 **a**



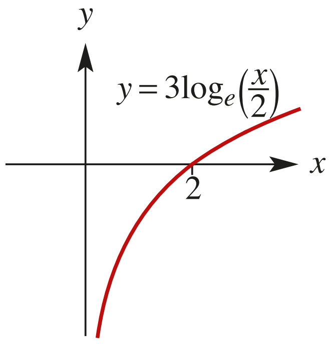
b



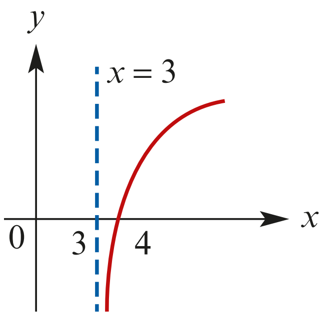
c



d

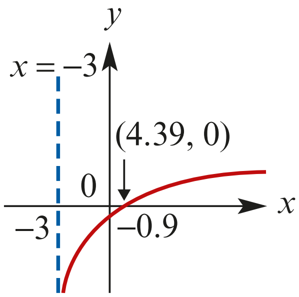


2 **a**



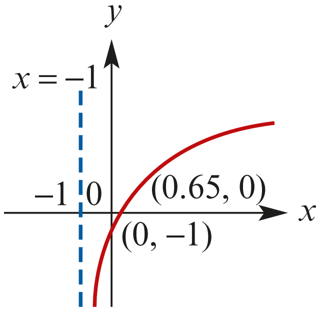
domain = $(3, \infty)$, range = $\mathbb{R}$

b



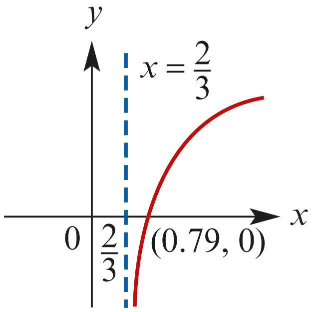
domain = $(-3, \infty)$, range = $\mathbb{R}$

c



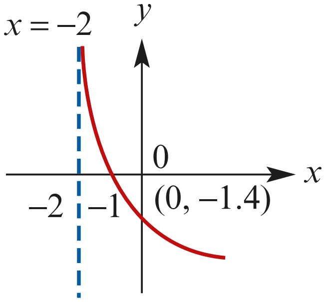
domain = $(-1, \infty)$, range = $\mathbb{R}$

d



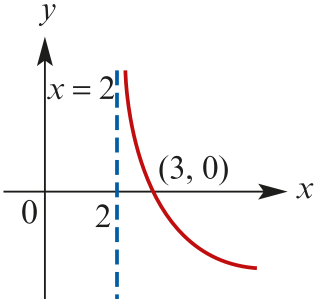
domain = $\left(\frac{2}{3}, \infty\right)$, range = $\mathbb{R}$

e



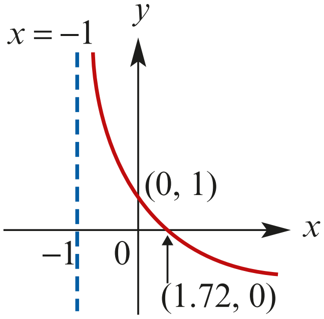
domain = $(-2, \infty)$, range = $\mathbb{R}$

f



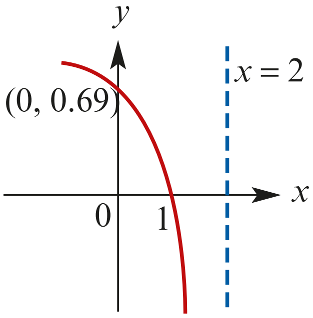
domain = $(2, \infty)$, range = $\mathbb{R}$

g



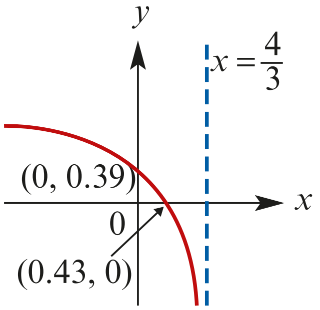
domain = $(-1, \infty)$, range = $\mathbb{R}$

h



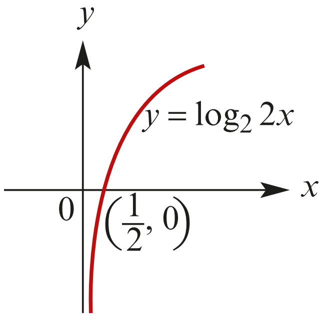
domain = $(-\infty, 2)$, range = $\mathbb{R}$

i



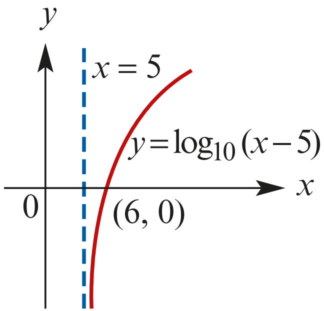
domain = $\left(-\infty, \frac{4}{3}\right)$, range = R

3 a



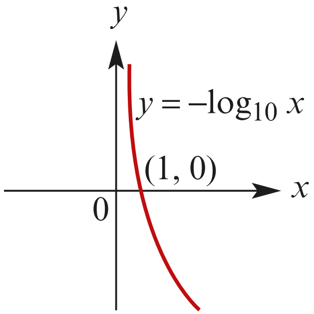
domain = $\mathbb{R}^+$

b



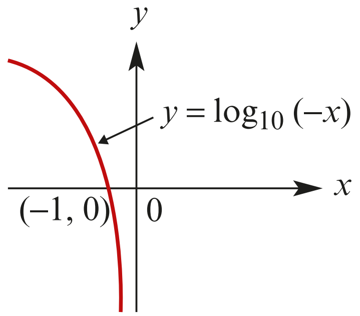
domain = $(5, \infty)$

c



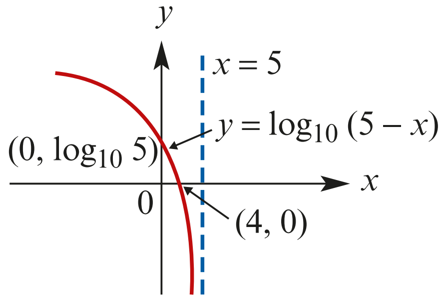
domain = $\mathbb{R}^+$

d



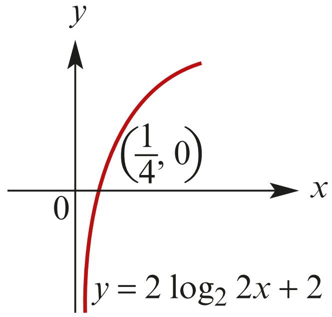
domain = $\mathbb{R}^-$

e



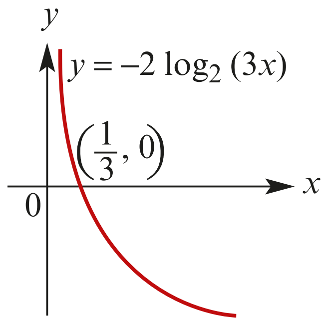
domain = $(-\infty, 5)$

f



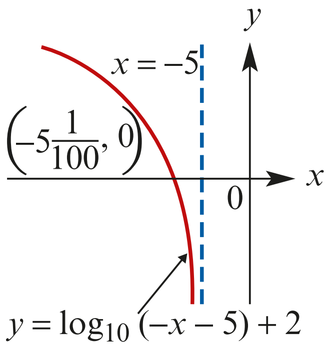
domain = R^+

g



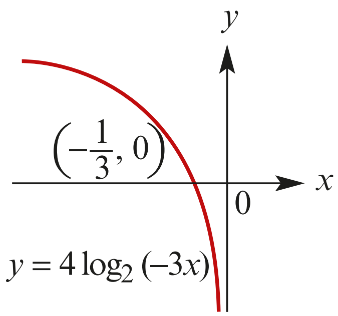
domain = $\mathbb{R}^+$

h



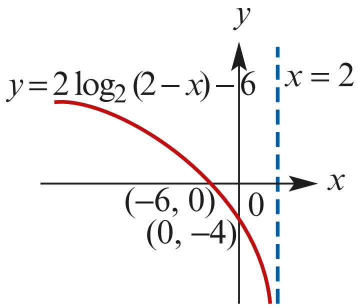
domain = $(-\infty, -5)$

i



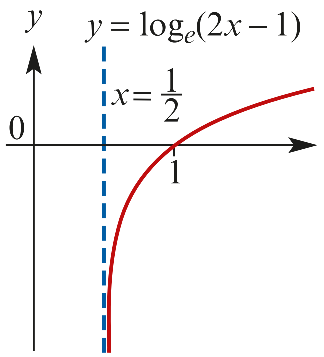
domain = $\mathbb{R}^-$

j



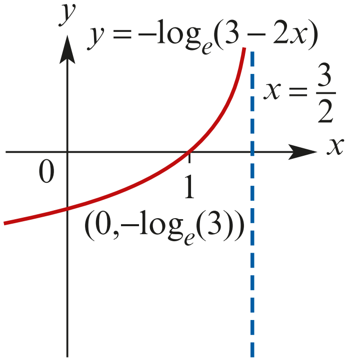
domain = $(-\infty, 2)$

k



Domain = $(\frac{1}{2}, \infty)$

l



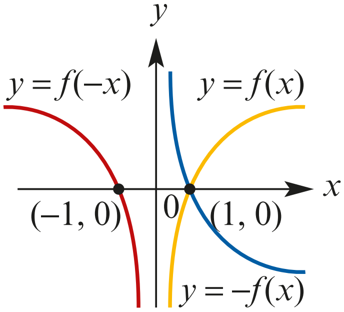
Domain = $(-\infty, \frac{3}{2})$

4 **a** $x = 1.557$

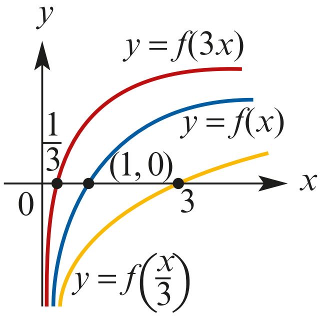
b $x = 1.189$

5 **b**

i ii



iii iv



6 A dilation of factor $\log_e 3$ from the y-axis

7 A dilation of factor $\frac{1}{\log_e 2}$ from the y-axis

Solutions to Exercise 5F

1 $a + b = 5 \ldots (1)$

$ae^4 + b = 11 \ldots (2)$

$(2) - (1)$

$a(e^4 - 1) = 6$

$$a = \frac{6}{e^4 - 1}$$

$$\therefore b = 5 - \frac{6}{e^4 - 1}$$

$$b = \frac{5e^4 - 11}{e^4 - 1}$$

2 $a \log_e(5 + b) = 0 \ldots (1)$

$a \log_e(10 + b) = 2 \ldots (2)$

From (1)

$\log_e(5 + b) = 0$

$5 + b = e^0$

$b = -4$

From (2)

$\therefore a \log_e 6 = 2$

$$\therefore a = \frac{2}{\log_e 6}$$

3 $y = ae^x + b$

$x \to -\infty, y \to 4$

$4 = b$

$x = 0, y = 6$

$6 = a + b$

$= a + 4$

$a = 2$

4 $y = ae^x + b$

$x = 0, y = 0$

$0 = a + b$

$a = -b$

$x = 1, y = 14$

$14 = ae + b$

$= (e - 1)a$

$$a = \frac{14}{e - 1} \approx 8.148$$

$$b = \frac{-14}{e - 1} \approx -8.148$$

5 $y = ae^{-bx}$

$x = 3, y = 50$

$50 = ae^{-3b} \ldots (1)$

$x = 6, y = 10$

$10 = ae^{-6b} \ldots (2)$

$$\frac{1}{2} \Rightarrow 5 = e^{3b}$$

$3b = \ln 5$

$$b = \frac{1}{3} \ln 5$$

$$b = \ln(5)^{\frac{1}{3}}$$

$$y = a \times 5^{\frac{-x}{3}}$$

$\therefore a = 250$

6 $f(x) = ae^{-x} + b$

$x \to \infty, f(x) \to 500$

$500 = b$

$x = 0, f(x) = 700$

$700 = a + 500$

$a = 200$

Sub in equation $\Rightarrow 50 = a \times \frac{1}{5}$

$a = 250$

7 $y = a\log_2 x + b$

$x = 8, y = 10$

$10 = 3a + b \ldots (1)$

$x = 32, y = 14$

$14 = 5a + b \ldots (2)$

$(2) - (1) \Rightarrow a = 2$

Sub in (1) $\Rightarrow 10 = 6 + b$

$b = 4$

8 $y = a\log_2(x - b)$

$x \to 5, y \to -\infty$

$b = 5$

$x = 7,\ y = 3$

$3 = a\log_2(7 - 5)$

$a = 3$

9 $y = ae^{bx}$

$x = 3,\ y = 10$

$x = 6,\ y = 50$

$10 = ae^{3b} \ldots (1)$

$50 = ae^{6b} \ldots (2)$

$\frac{(2)}{(1)} \Rightarrow 5 = e^{3b}$

$b = \ln(5)^{\frac{1}{3}}$

$y = a \times 5^{\frac{x}{3}}$

Sub in (1) $\Rightarrow 10 = a \times 5$

$a = 2$

10 $y = a\log_2(x - b)$

$x = 5, y = 2$

$2 = a\log_2(5 - b) \ldots (1)$

$x = 7, y = 4$

$4 = a\log_2(7 - b) \ldots (2)$

$(2) \div (1) \Rightarrow$

$2\log_2(5 - b) = \log_2(7 - b)$

$(5 - b)^2 = 7 - b$

$b^2 - 10b + 25 = -b + 7$

$b^2 - 9b + 18 = 0$

$(b - 6)(b - 3) = 0$

$b = 3$ *or* 6

since $\log_2(x)$ is only defined for $x > 0$ and $\log_2(5 - b)$ is one of the points, $b = 6$ is impossible

$\therefore\ b = 3$

Sub in (1) $\Rightarrow 2 = a\log_2(5 - 3)$

$a = 2$

11 $y = a\ln(x - b) + c$

vertical asymptote $x = 1, \therefore b = 1$

$$y = a\ln(x-1) + c$$

$$x = 3, y = 10$$

$$10 = a\ln 2 + c \ldots (1)$$

$$x = 5, y = 12$$

$$12 = a\ln 4 + c \ldots (2)$$

$$= 2a\ln 2 + c$$

$(2) - (1) \Rightarrow a\ln 2 = 2$

$$a = \frac{2}{\ln 2} \approx 2.885$$

$$y = 2\log_2(x-1) + c$$

Sub in $(1) \Rightarrow 10 = 2\log_2 2 + c$

$$c = 8$$

12

$$f(x) = a\ln(-x) + b$$

$$x = -2, \quad f(-2) = 6$$

$$6 = a\ln 2 + b \ldots (1)$$

$$x = -4, \quad f(-4) = 8$$

$$8 = a\ln(4) + b \ldots (2)$$

$$8 = 2a\ln 2 + b$$

$(2) - (1) \Rightarrow 2 = a\ln 2$

$$a = \frac{2}{\ln 2} \approx 2.885$$

Sub in $1 \Rightarrow 6 = 2 + b$

$$b = 4$$

Solutions to Exercise 5G

1 **a** $$\begin{aligned} \log_2 8 &= k\log_2 7 + 2 \\ 3 - 2 &= k\log_2 7 \\ 1 &= k\log 27 \\ k &= \frac{1}{\log_2 7} \end{aligned}$$

b $$\begin{aligned} \log_2 7 - x\log_2 7 &= 4 \\ (1 - x)\log 27 &= 4 \\ 1 - x &= \frac{4}{\log_2 7} \\ x &= 1 - \frac{4}{\log_2 7} \\ x &= \frac{\log_2(7) - 4}{\log_2 7} \end{aligned}$$

c $$\begin{aligned} \log_e 7 - x\log_e 14 &= 1 \\ \log_e 7 - 1 &= x\log_e 14 \\ x &= \frac{\log_e 7 - 1}{\log_e 14} \end{aligned}$$

2 **a** 2.58

b −0.32

c 2.18

d 1.16

e −2.32

f −0.68

g −2.15

h −1.38

i 2.89

j −1.7

k −4.42

l 5.76

m −6.21

n 2.38

o 2.80

3 **a** $x < 2.81$

b $x > 1.63$

c $x < -0.68$

d $x \leq 3.89$

e $x \geq 0.57$

4 **a** $x = \log_2 5$

b $$\begin{aligned} 2x - 1 &= \log_3 8 \\ 2x &= \log_3(8) + 1 \\ x &= \frac{\log_3(8) + 1}{2} \end{aligned}$$

c $$\begin{aligned} 3x + 1 &= \log_7 20 \\ 3x &= \log_7(20) - 1 \\ x &= \frac{\log_7(20) - 1}{3} \end{aligned}$$

d $x = \log_3 7$

e $x = \log_3 6$

f $x = \log_5 6$

g Let $a = 3^x$

$a^2 - 9a + 8 = 0$

$(a - 8)(a - 1) = 0$

$\therefore a = 8 \text{ or } a = 1$

$\therefore 3^x = 8 \text{ or } 3^x = 1$

$\therefore x = \log_3 8 \text{ or } x = 0$

h Let $a = 5^x$

$a^2 - 4a - 5 = 0$

$(a - 5)(a + 1) = 0$

$\therefore a = 5 \text{ or } a = -1$

$\therefore 5^x = 5 \text{ or } 5^x = -1$

$\therefore x = 1$

5 **a** $7^x > 52 \Leftrightarrow x > \log_7 52$

b $3^{2x-1} < 40 \Leftrightarrow 2x - 1 < \log_3 40$

$\Leftrightarrow 2x < \log_3(40) + 1$

$\Leftrightarrow x < \frac{1}{2}(\log_3(40) + 1)$

$= \frac{1}{2}(\log_3(120))$

c $4^{3x+1} \geq 5 \Leftrightarrow 3x + 1 \geq \log_4 5$

$\Leftrightarrow 3x \geq \log_4(5) - 1$

$\Leftrightarrow x \geq \frac{1}{3}\log_4(\frac{5}{4})$

$= \frac{1}{6}\log_2(\frac{5}{4})$

d $3^{x-5} \leq 30 \Leftrightarrow x - 5 \leq \log_3 30$

$\Leftrightarrow 3x \leq \log_3(30) + 5$

$\Leftrightarrow x < \frac{1}{3}(\log_3(30) + 5)$

$= \log_3(7290)$

e $3^x < 106 \Leftrightarrow x < \log_3 106$

f $5^x < 0.6 \Leftrightarrow x \leq \log_5 0.6$

6 **a** $a\log_2 7 = 3 - \log_6 14$

$a\log_2 7 = \log_6 216 - \log_6 14$

$a\log_2 7 = \log_6\left(\frac{108}{7}\right)$

$$a = \frac{\log_6\left(\frac{108}{7}\right)}{\log_2 7}$$

$$a = \frac{\ln\left(\frac{108}{7}\right)}{\ln 6} \times \frac{\ln 2}{\ln 7}$$

$$a = \frac{2.73622}{1.791759} \times \frac{0.69314}{194591}$$

$a = 1.5271138 \times 0.356207$

$a = 0.544$

b $\log_3 18 = \log_{11} k$

$$\log_{11} k = \frac{\ln 18}{\ln 3}$$

$= 2.6309$

$k = 11^{2.6309}$

$k = 549.3$

7 $\log_r p = q \Rightarrow p = r^q \quad (1)$

$\log_q(r) = p \Rightarrow r = q^p \quad (2)$

Raise both sides of (2) to the power q:

$r^q = (q^p)^q$

$p = q^{pq}$ (from (1))

Change to logarithm form:

$\log_q p = pq$

8 $u = \log_9 x$

a $x = 9^u$

b $\log_9(3x) = \log_9(3 \times 9^u)$

$= \log_9 9^u + \log_9 3$

$= u + \frac{1}{2}$

c $x = 9^u$

$$\Rightarrow \log_x x = \log_x 9^u$$

$$\Rightarrow 1 = \mathrm{u} \log_x 9$$

$$\Rightarrow \frac{1}{\mathrm{u}} = \frac{1}{2} \log_x 81$$

$$\log_x 81 = \frac{2}{u}$$

9 $\log_5 x = 16 \log_x 5$

$$\Rightarrow \frac{\ln x}{\ln 5} = \frac{16 \ln 5}{\ln x}$$

$$(\ln\ x)^2 = 16(\ln 5)^2$$

$$\ln x = \pm 4 \ln 5$$

$$x = e^{\pm \ln 625}$$

$$x = 625, \frac{1}{625}$$

10 $q^p = 25 \Rightarrow p = \log_q 25$

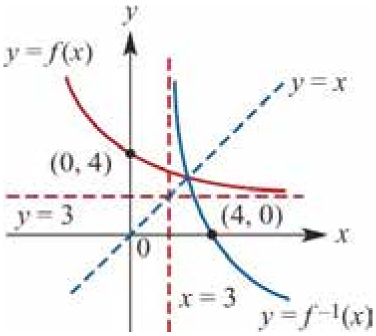
$$\log_5 q = \frac{\log_q q}{\log_q 5} = \frac{1}{\log_q 5}$$

$$= \frac{2}{\log_q 25} = \frac{2}{p}$$

Solutions to Exercise 5H

1 $f^{-1}: (-2, \infty) \to \mathbb{R},\ f^{-1}(x) = \log_e(x + 2)$

2



to find $f^{-1}(x)$,

$$f(x) = e^{-x} + 3$$

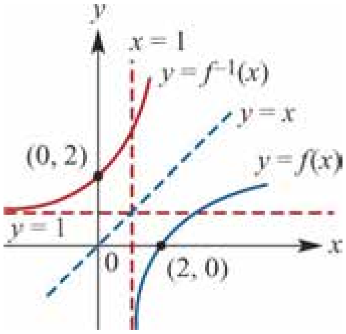
$$\therefore \quad x = e^{-f^{-1}(x)} + 3$$

$$x - 3 = e^{-f^{-1}(x)}$$

$$-f^{-1}(x) = \ln(x - 3)$$

$$f^{-1}(x) = -\ln(x - 3)$$

3



to find $f^{-1}(x)$,

$$f(x) = \ln(x - 1)$$

$$x = \ln(f^{-1}(x) - 1)$$

$$e = f^{-1}(x) - 1$$

$$f^{-1}(x) = e^x + 1$$

4 $x = e^{\frac{y+4}{3}}$

5 **a**

$$\text{domain}(f) = \mathbb{R}^+$$

$$\text{range}(f) = \mathbb{R}$$

$$\therefore \quad \text{domain}(f^{-1}) = \mathbb{R}$$

$$\text{range}(f^{-1}) = \mathbb{R}^+$$

$$f(x) = \ln 2x$$

$$x = \ln(2f^{-1}(x))$$

$$2f^{-1}(x) = \mathrm{e}^x$$

$$f^{-1}(x) = \frac{1}{2}e^x$$

b

$$\text{domain} f = \mathrm{R}^+$$

$$\text{range} f = \mathrm{R}$$

$$\therefore \quad \text{domain} f^{-1} = \mathrm{R}$$

$$\text{range} f^{-1} = \mathrm{R}^+$$

$$f(x) = 3\ln(2x) + 1$$

$$x = 3\ln(2f^{-1}(x)) + 1$$

$$\frac{x-1}{3} = \ln(2f^{-1}(x))$$

$$2f^{-1}(x) = e^{\frac{x-1}{3}}$$

$$f^{-1}(x) = \frac{1}{2}e^{\frac{x-1}{3}}$$

c

$$\text{domain}(f) = \mathrm{R}, \text{range(f)} = (2, \infty)$$

$$\therefore \quad \text{domain}(f^{-1}) = (2, \infty), \text{range}(\mathrm{f}^{-1}) = \mathrm{R}$$

$$f(x) = e^x + 2$$

$$\therefore \quad x = e^{f^{-1}(x)} + 2$$

$$x - 2 = e^{f^{-1}(x)}$$

$$f^{-1}(x) = \ln(x - 2)$$

d domain$(f) = \mathbb{R}$, range(f) $= \mathbb{R}^+$

$\therefore$ domain$(\mathrm{f}^{-1}) = \mathrm{R}^+$, range$(\mathrm{f}^{-1}) = \mathbb{R}$

$$f(x) = e^{x+2}$$
$$\therefore \quad x = e^{f^{-1}(x)+2}$$
$$\ln x = f^{-1}(x) + 2$$
$$f^{-1}(x) = \ln x - 2$$

e domain$(f) = \left(-\frac{1}{2}, \infty\right)$,

range$(f) = \mathbb{R}$

$\therefore$ domain$(\mathrm{f}^{-1}) = \mathbb{R}$,

range$(f^{-1}) = \left(-\frac{1}{2}, \infty\right)$

$$f(x) = \ln(2x + 1)$$
$$x = \ln(2f^{-1}(x) + 1)$$
$$e^x = 2f^{-1}(x) + 1$$
$$f^{-1}(x) = \frac{e^x - 1}{2}$$

f domain(f) $= \left(-\frac{2}{3}, \infty\right)$,

range(f) = R

$\therefore$ domain$(\mathrm{f}^{-1}) = \mathbb{R}$,

range$(f^{-1}) = \left(-\frac{2}{3}, \infty\right)$

$$f(x) = 4\ln(3x + 2)$$
$$x = 4\ln(3f^{-1}(x) + 2)$$
$$e^{\frac{x}{4}} = 3f^{-1}(x) + 2$$
$$f^{-1}(x) = \frac{e^{\frac{x}{4}} - 2}{3}$$

g

domain$(f) = (-1, \infty)$, range(f) = R

$\therefore$ domain$(\mathrm{f}^{-1}) = \mathbb{R}$, range$(\mathrm{f}^{-1}) = (-1, \infty)$

$$f(x) = \log_{10}(x + 1)$$
$$f(x) = \log_{10}(f^{-1}(x) + 1)$$
$$f^{-1}(x) + 1 = 10^x$$
$$f^{-1}(x) = 10^x - 1$$

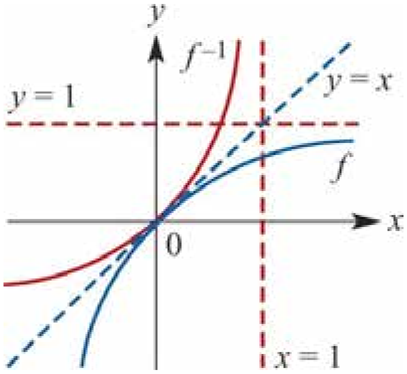
h domain(f) = R, range(f) $= \mathbb{R}^+$

$\therefore$ domain$(\mathrm{f}^{-1}) = \mathbb{R}^+$, range$(\mathrm{f}^{-1}) = \mathbb{R}$

$$f(x) = 2e^{x-1}$$
$$x = 2e^{(f^{-1}(x)-1)}$$
$$\frac{x}{2} = e^{(f^{-1}(x)-1)}$$
$$f^{-1}(x) - 1 = \ln\left(\frac{x}{2}\right)$$
$$f^{-1}(x) = \ln\left(\frac{x}{2}\right) + 1$$

6

a c



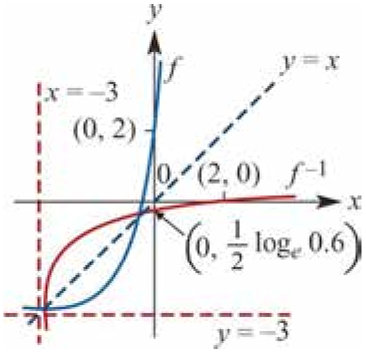
b range(f) $= (-\infty, 1)$

$\therefore$ domain$(\mathrm{f}^{-1}) = (-\infty, 1)$

$$f(x) = 1 - e^{-x}$$
$$x = 1 - e^{-f^{-1}(x)}$$
$$-f^{-1}(x) = \ln(1 - x)$$
$$f^{-1}(x) = -\ln(1 - x)$$

7

a c



b

$$\begin{aligned} f(x) &= 5e^{2x} - 3 \\ x &= 5e^{2f^{-1}(x)} - 3 \\ \frac{x+3}{5} &= e^{2f^{-1}(x)} \\ 2f^{-1}(x) &= \ln\left(\frac{x+3}{5}\right) \\ f^{-1}(x) &= \frac{1}{2}\ln\left(\frac{x+3}{5}\right) \end{aligned}$$

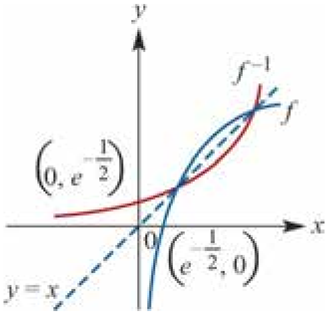
domain $= (-3, \infty)$

$\therefore \quad f^{-1} : (-3, \infty) \to \mathbb{R},$

$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+3}{5}\right)$

8

a c



b

$$\begin{aligned} f(x) &= 2\ln x + 1 \\ x &= 2\ln f^{-1}(x) + 1 \\ \frac{x-1}{2} &= \ln f^{-1}(x) \\ f^{-1}(x) &= e^{(\frac{x-1}{2})} \end{aligned}$$

range $(f^{-1}) = \text{domain}(f) = \mathbb{R}^+$

9 $t = \frac{-1}{k}\log_e\left(\frac{P-b}{A}\right)$

10 **a** $\frac{y-5}{2} = \ln x$

$$x = e^{\left(\frac{y-5}{2}\right)}$$

b $\frac{P}{\text{A}} = e^{-6x}$

$$\begin{aligned} -6x &= \ln\left(\frac{P}{A}\right) \\ x &= -\frac{1}{6}\ln\left(\frac{P}{A}\right) \end{aligned}$$

c $\frac{y}{a} = x^n$

$$n = \log_x\left(\frac{y}{a}\right) = \frac{\ln\left(\frac{y}{a}\right)}{\ln x}$$

d $10^x = \frac{y}{5}$

$$x = \log_{10}\left(\frac{y}{5}\right)$$

e $\ln(2x) = \frac{5-y}{3}$

$$\begin{aligned} 2x &= e^{\left(\frac{5-y}{3}\right)} \\ x &= \frac{1}{2}e^{\left(\frac{5-y}{3}\right)} \end{aligned}$$

f $\frac{y}{6} = x^{2n}$

$$2n = \log_x\left(\frac{y}{6}\right)$$

$$n = \frac{1}{2}\log_x\left(\frac{y}{6}\right) = \frac{1}{2}\left(\frac{\ln\left(\frac{y}{6}\right)}{\ln x}\right)$$

$$y = \ln(2x - 1)$$

g $2x - 1 = e^y$

$$x = \frac{e^y + 1}{2}$$

h $y = 5(1 - e^{-x})$

$$e^{-x} = 1 - \frac{y}{5}$$

$$-x = \ln\left(\frac{5 - y}{5}\right)$$

$$x = -\ln\left(\frac{5 - y}{5}\right) = \ln\left(\frac{5}{5 - y}\right)$$

11 **a** $f(x) = 2e^x - 4$

$$\frac{x + 4}{2} = e^{f^{-1}(x)}$$

$$f^{-1}(x) = \ln\left(\frac{x + 4}{2}\right)$$

b using the CAS calculator (0.895, 0.895), (−3.962, −3.962)

12 **a** $f(x) = 2\ln(x + 3) + 4$

$$x = 2\ln(f^{-1}(x) + 3) + 4$$

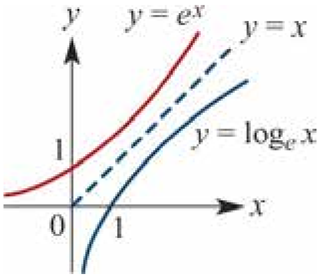
$$\left(\frac{x - 4}{2}\right) = \ln(f^{-1}(x) + 3)$$

$$f^{-1}(x) + 3 = e^{\left(\frac{x-4}{2}\right)}$$

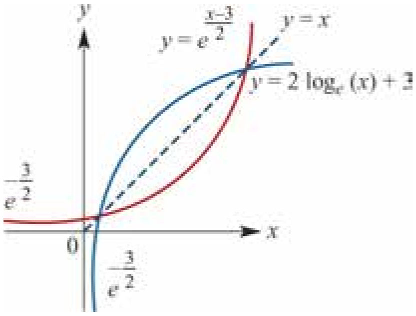
$$f^{-1}(x) = e^{\left(\frac{x-4}{2}\right)} - 3$$

b using the CAS calculator (8.964, 8.964), (−2.969, −2.969)

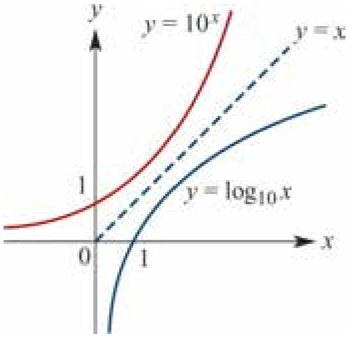
13 **a** **i**



ii



iii



b $f(x)$ and $g(x)$ are inverse functions

Solutions to Exercise 5I

1 a $N = 1000 \times 2^{\frac{t}{15}}$

b 50 minutes

2 $d = d_o 10^{mt}$

When $t = 1$, $d = 52$ cm

When $t = 3$, $d = 80$

Consider the equations

$52 = d_o 10^m \qquad \ldots(1)$

$80 = d_o 10^{3m} \ldots(2)$

Divide (2) by (1)

$$\frac{80}{52} = 10^{2m}$$

$$2m = \log_{10}\left(\frac{20}{13}\right)$$

$$m = \frac{1}{2}\log_{10}\left(\frac{20}{13}\right) \approx 0.094$$

Substitute in (1)

$$52 = d_0 10^{\frac{1}{2}\log_{10}\left(\frac{20}{13}\right)}$$

$$\therefore \quad 52 = d_0 10^{\log_{10}\left(\frac{20}{13}\right)^{\frac{1}{2}}}$$

Hence $52 = \left(\frac{20}{13}\right)^2 1/2d_0$

and $d_0 = \left(\frac{13}{20}\right)^{\frac{1}{2}} \times 52$

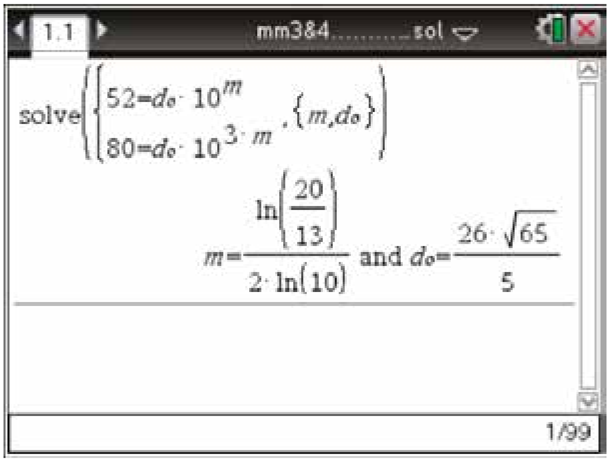
$\therefore \quad m \approx 0.094$ and $d_0 \approx 41.9237$

Graphic calculator techniques for question

In a **Calculator** page use:

b>**Algebra>Solve System of Equations>Solve System of Equations** and enter as shown opposite.

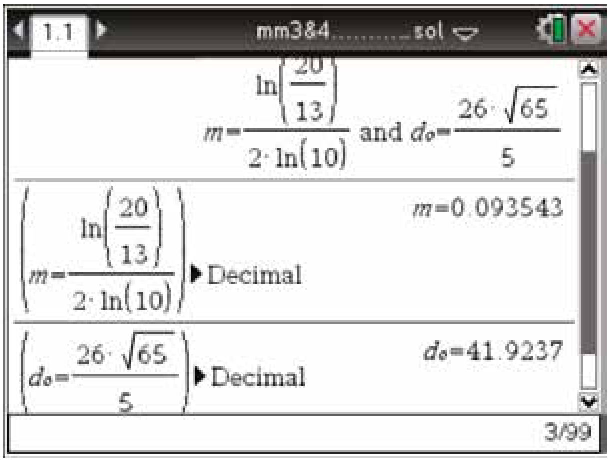
Hint: do can be entered using a template from t, otherwise just use d0.



$\text{solve}\left(\begin{cases}52=d_o\cdot 10^m\\80=d_o\cdot 10^{3\cdot m}\end{cases},\{m,d_o\}\right)$

$m=\frac{\ln\left(\frac{20}{13}\right)}{2\cdot\ln(10)}$ and $d_o=\frac{26\cdot\sqrt{65}}{5}$

Approximate the solutions using b>**Number>Convert to Decimal.**



$m=\frac{\ln\left(\frac{20}{13}\right)}{2\cdot\ln(10)}$ and $d_o=\frac{26\cdot\sqrt{65}}{5}$

$\left(m=\frac{\ln\left(\frac{20}{13}\right)}{2\cdot\ln(10)}\right)$ ▸Decimal $\quad m=0.093543$

$\left(d_o=\frac{26\cdot\sqrt{65}}{5}\right)$ ▸Decimal $\quad d_o=41.9237$

3 a $N = N_0 e^{kt}$

i When $t = 0$, $N = 20\,000$

$\therefore \quad 20\,000 = N_0 e^0$

i.e. $\quad N_0 = 20\,000$

ii $N = 20\,000e^{kt}$

When $t = 1$, $N = 20\,000$and20% of $20\,000 = 16\,000$

$\therefore \quad 16\,000 = 20\,000 \times e^k$

$\therefore \quad e^k = 0.8$

$\therefore \quad k = \log_e(0.8) \approx -0.223$

b When $N = 5000$

$$5000 = 20\,000e^{\log_e(08)t}$$

$$\therefore \quad 0.25 = 0.8^t$$

$$\text{and } t = \frac{\log_e(0.25)}{\log_e(0.8)}$$

$$\approx 6.2126$$

It takes about 6.2 years for there to be 5000 people infected.

4 $M = M_0e^{-kt}$

When $t = 0, M = 10$

When $t = 140, M = 5$

a

$$10 = M_0e^0$$

$$\therefore \quad M_0 = 10$$

Also $5 = 10e^{-140k}$

$$0.5 = e^{-140k}$$

$$k = \frac{-1}{140}\log_e(0.5)$$

$$= \frac{1}{140}\log_e(2) \approx 0.00495$$

$$= 4.95 \times 10^{-3}$$

b When $t = 70$ $M = 10e^{\frac{-1}{140}\log_e(2)\times 70}$

$$= 10e^{-05\log_e 2}$$

$$= 10 \times 2^{-0.5}$$

$$\approx 7.0711$$

The mass is 7.07 g after 70 days.

c When $M = 2$ $\quad 2 = 10e^{\frac{-1}{140}(\log_e 2)t}$

$$\therefore \quad 0.2 = 2^{\frac{t}{140}}$$

$$\therefore \quad t = -140\frac{\log_e(0.2)}{\log_e(2)} \approx 325.07$$

After 325 days the mass remaining is 2g.

5 **a** $A(t = 1690) = \frac{1}{2}A_0$

$$A_0e^{-1690k} = \frac{1}{2}A_0$$

$$2 = e^{1690k}$$

$$\log_e 2 = 1690k$$

$$k = \frac{\log_e 2}{1690}$$

b $A = 0.2A_0$

$$A_0e^{-\frac{\log_e 2}{1690}t} = \frac{1}{5}A_0$$

$$5 = e^{\frac{\log_e 2}{1690}t}$$

$$\log_e 5 = \frac{\log_e 2}{1690}t$$

$$t = 1690\frac{\log_e 5}{\log_e 2}$$

$$= 3924$$

6 $A = A_0e^{kt}$

When $t = 0, A = 20$

$\therefore A_0 = 20$

Half life is 24 000 years.

$\therefore 10 = 20e^{24000k}$

$\therefore k = \frac{1}{24000}\log_e(\frac{1}{2})$

When does 20% remain?

$20e^{kt} = 4$

$e^{kt} = \frac{1}{5}$

$t = \frac{1}{k}\log_e(\frac{1}{5})$

$t \approx 55726$ years

7 $A = A_0e^{kt}$

$A = \frac{1}{2}A_0$ when $t = 5730$

$\therefore \frac{1}{2} = e^{5730k}$

$\therefore k = \frac{1}{5730}\log_e(\frac{1}{2})$

When does 40% remain?

$e^{kt} = 0.4$

$e^{kt} = \frac{2}{5}$

$t = \frac{1}{k}\log_e(\frac{2}{5})$

$t \approx 7575$ years

8 $P = P_0e^{kt}$

When $t = 0, P = 10000$

$\therefore P_0 = 10000$

$A = 15000$ when $t = 13$

$\therefore \frac{3}{2} = e^{13k}$

$\therefore k = \frac{1}{13}\log_e(\frac{3}{2})$

$P = 10000e^{kt}$

a When $t = 16$

$P = 10000e^{16k}$

$\therefore P = 16471$

b $30000 = 10000e^{kt}$

$\log_e 3 = kt$

$t = \frac{1}{k}\log_e(3)$

$t \approx 35$

9 $C = C_0(1.12)^n$

$M = M_0(0.94)^n$

$M_0 = 5C_0$

$\therefore M = 5C_0(0.94)^n$

$C > 5M \Leftrightarrow (1.12)^n > 25(0.94)^n$

This happens after approximately 18.4 years.

10 $P(h) = 1000 \times 10^{-0.05428h}$

a 607 millibars

b 6.389 km

11 $P = 500000(1.1)^n$

$4000000 = 500000(1.1)^n$

$8 = (1.1)^n$

$n \approx 21.82$

12 $T = T_0e^{-kt}$

When $t = 0, T = 100$

$\therefore T_0 = 100$

$T = 40$ when $t = 5$

$\therefore \frac{2}{5} = e^{-5k}$

$\therefore k = -\frac{1}{5}\log_e(\frac{2}{5})$

$T = 100e^{-kt}$

When $t = 15$

$T = 100e^{-15k}$

$\therefore T = 6.4$

13 $N = N_0e^{kt}$

$101 = N_0e^{2k} \ldots (1)$

$203 = N_0e^{4k} \ldots (2)$

Divide(2) by (1)

$$\frac{203}{101} = e^{2k}$$

$$k = \frac{1}{2}\log_e\left(\frac{203}{101}\right)$$

$k \approx 0.349, \ N_0 \approx 50.25$

14 **a** $k = \log_e\left(\frac{5}{4}\right)$

b 7.21 hours

15 **a** $N = a \times b^t$

$1000 = a$

$15\,000 = 1000 \times b^5$

$15 = b^5$

$a = 1000, \; b = 15^{\frac{1}{5}}$

b $N > 5000$ 3 hours

$1000b^t > 5000$

$b^t > 5$

$b > 2.97\ldots$

c $N > 1\,000\,000$

$1000b^t > 1\,000\,000$

$b^t > 1000$

$t > 12.75\ldots$

13 hours

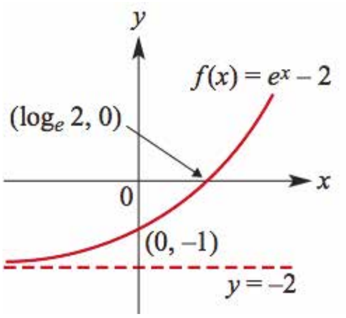
d 664 690

Solutions to Technology-free questions

1 a $y = e^x - 2$

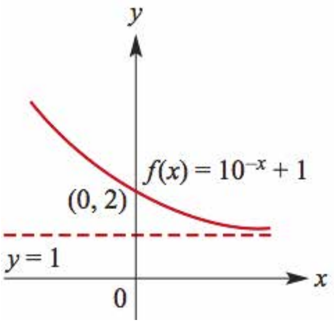
$x = 0$: $y = -1$

$y = 0$: $e^x = 2 \Rightarrow x = \log_e 2$
asymptote: $y = -2$



b $y = 10^{-x} + 1$

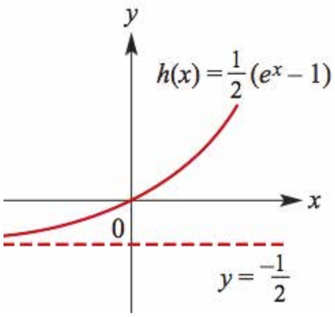
$x = 0$: $y = 2$
no x intercepts as $y > 1$
asymptote: $y = 1$



c $y = \frac{1}{2}(e^x - 1)$

$x = 0$: $y = 0$

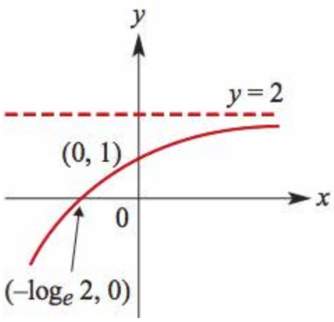
asymptote: $y = -\frac{1}{2}$



d $y = 2 - e^{-x}$

$x = 0$: $y = 1$

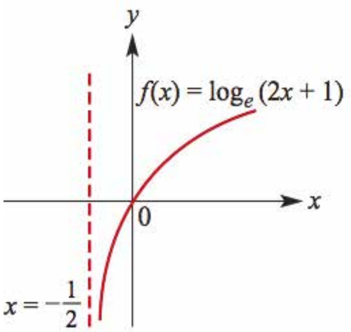
$y = 0$: $e^{-x} = 2 \Rightarrow x = -\log_e 2$
asymptote: $y = 2$



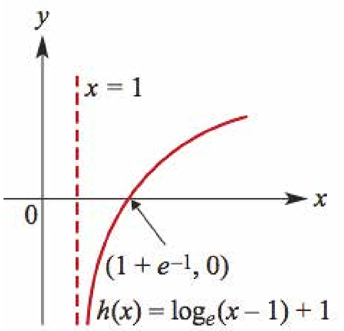
e $y = \log_e(2x + 1)$

$x = 0$: $y = 0$

asymptote: $x = -\frac{1}{2}$



f $y = \log_e(x - 1) + 1$
no y intercepts as $x > 1$
$y = 0$: $\log_e(x - 1) = -1 \Rightarrow x = 1 + e^{-1}$
asymptote: $x = 1$

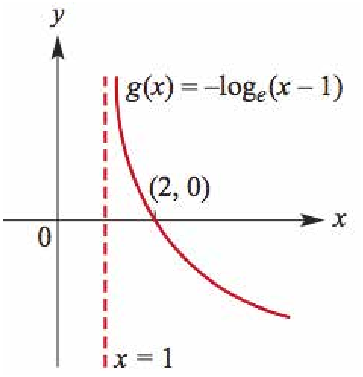


g $y = -\log_e(x-1)$

no y intercepts as $x > 1$

$y = 0$: $-\log_e(x-1) = 0 \Rightarrow x = 2$

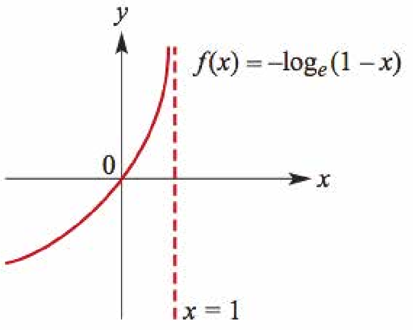
asymptote: $x = 1$



h $y = -\log_e(1-x)$

$x = 0$: $y = 0$

asymptote: $x = 1$



2 **a** $f(x) = e^{2x} - 1$

domain = $\mathbb{R}$, range = $(-1, \infty)$

The domain of f^{-1} is $(-1, \infty)$.

Interchange x and y and solve for y:

$$x = e^{2y} - 1$$

$$e^{2y} = x + 1$$

$$2y = \log_e(x+1)$$

$$y = \frac{1}{2}\log_e(x+1)$$

$f^{-1}: (-1, \infty) \to \mathbb{R}, f^{-1}(x) = \frac{1}{2}\log_e(x+1)$

b $f(x) = 3\log_e(x-2)$

domain = $(2, \infty)$, range = $\mathbb{R}$

The domain of f^{-1} is $\mathbb{R}$.

Interchange x and y and solve for y:

$$x = 3\log_e(y-2)$$

$$\log_e(y-2) = \frac{x}{3}$$

$$y = e^{\frac{x}{3}} + 2$$

$f^{-1}: \mathbb{R} \to \mathbb{R},\ f^{-1}(x) = e^{\frac{x}{3}} + 2$

c $f(x) = \log_{10}(x+1)$

domain = $(-1, \infty)$, range = $\mathbb{R}$

The domain of f^{-1} is $\mathbb{R}$.

Interchange x and y and solve for y:

$$x = \log_{10}(y+1)$$

$$y = 10^x - 1$$

$f^{-1}: \mathbb{R} \to \mathbb{R}, f^{-1}(x) = 10^x - 1$

d $f(x) = 2^x + 1$

domain = $\mathbb{R}^+$, range = $(2, \infty)$

The domain of f^{-1} is $(2, \infty)$.

Interchange x and y and solve for y:

$$x = 2^y + 1$$

$$2^y = x - 1$$

$$y = \log_2(x-1)$$

$f^{-1}: (2, \infty) \to \mathbb{R}, f^{-1}(x) = \log_2(x-1)$

3 **a**
$$\begin{aligned}\log_e y &= \log_e(x) + 2\\ &= \log_e(x) + \log_e(e^2)\\ &= \log_e(e^2 x)\\ y &= e^2 x\end{aligned}$$

b
$$\begin{aligned}\log_{10} y &= \log_{10} x + 1\\ &= \log_{10} x + \log_{10} 10\\ &= \log_{10} 10x\\ y &= 10x\end{aligned}$$

c $\log_2 y = 3\log_2 x + 4$

$$= \log_2 x^3 + \log_2 2^4$$

$$= \log_2 16x^3$$

$$y = 16x^3$$

d $\log_{10} y = -1 + 5\log_{10} x$

$$= -\log_{10} 10 + \log_{10} x^5$$

$$= \log_{10} \frac{x^5}{10}$$

$$y = \frac{x^5}{10}$$

e $\log_e y = 3 - \log_e x$

$$= \log_e e^3 - \log_e x$$

$$= \log_e \frac{e^3}{x}$$

$$y = \frac{e^3}{x}$$

f $\log_e y = 2x - 3$

$$y = e^{2x-3}$$

4 **a** $3^x = 11$

$$x = \log_3 11$$

$x = \dfrac{\log_e 11}{\log_e 3}$ by change of base

(Alternatively, take logarithms to base *e* of both sides and simplify, as in part **c** below.)

b $2^x = 0.8$

$$x = \log_2(0.8)$$

$= \dfrac{\log_e(0.8)}{\log_e 2}$ by change of base

c

$$2^x = 3^{x+1}$$

$$\log_e 2^x = \log_e 3^{x+1}$$

$$x\log_e 2 = (x+1)\log_e 3$$

$$x\log_e 2 - x\log_e 3 = \log_e 3$$

$$x(\log_e 2 - \log_e 3) = \log_e 3$$

$$x = \frac{\log_e 3}{\log_e 2 - \log_e 3}$$

$$= \frac{\log_e 3}{\log_e\left(\frac{2}{3}\right)}$$

5 **a**

$$2^{2x} - 2^x - 2 = 0$$

$$(2^x)^2 - 2^x - 2 = 0$$

$$(2^x - 2)(2^x + 1) = 0$$

$$2^x = 2, -1$$

But $2^x > 0$ for all real x, so the only solution is given by $2^x = 2$, i.e. $x = 1$.

b $\log_e(3x - 1) = 0$

$$3x - 1 = 1$$

$$3x = 2$$

$$x = \frac{2}{3}$$

c $\log_{10}(2x) + 1 = 0$

$$\log_{10}(2x) = -1$$

$$2x = 10^{-1}$$

$$= \frac{1}{10}$$

$$x = \frac{1}{20}$$

d

$$\begin{aligned}10^{2x} - 7 \times 10^x + 12 &= 0\\ (10^x)^2 - 7 \times 10^x + 12 &= 0\\ (10^x - 3)(10^x - 4) &= 0\\ 10^x &= 3, 4\\ x &= \log_{10} 3, \log_{10} 4\end{aligned}$$

6 $y = 3\log_2(x + 1) + 2$

$x = 0$: $y = 3\log_2 1 + 2 = 2$
y intercept: $(0, 2)$, so $b = 2$.

$y = 0$: $3\log_2(x + 1) + 2 = 0$

Solving for x:

$3\log_2(x + 1) = -2$

$$\begin{aligned}\log_2(x + 1) &= -\frac{2}{3}\\ x + 1 &= 2^{-\frac{2}{3}}\\ x &= 2^{-\frac{2}{3}} - 1\end{aligned}$$

x intercept: $\left(2^{-\frac{2}{3}} - 1, 0\right)$ so $a = 2^{-\frac{2}{3}} - 1$.

7 $f(k) = 5\log_{10}(k + 1) = 6$, so solving for k:

$5 \log 10(k + 1) = 6$

$$\begin{aligned}\log 10(k + 1) &= \frac{6}{5}\\ k + 1 &= 10^{\frac{6}{5}}\\ k &= 10^{\frac{6}{5}} - 1\end{aligned}$$

8 $4e^{3x} = 287$

$$\begin{aligned}e^{3x} &= \frac{287}{4}\\ 3x &= \log_e\left(\frac{287}{4}\right)\\ x &= \frac{1}{3}\log_e\left(\frac{287}{4}\right)\end{aligned}$$

9 $3\log_a x = 3 + \log_a 8$

$$\begin{aligned}&= 3 + \log_a 2^3\\ &= 3 + 3\log_a 2\\ &= 3(1 + \log_a 2)\\ \log_a x &= 1 + \log_a 2\\ &= \log_a a + \log_a 2\\ &= \log_a 2a\\ x &= 2a\end{aligned}$$

10 Given $3^x = 4^y = 12^z$

$x\log_e(3) = y\log_e(4) = z\log_e(12)$

$x\log_e(3) = y\log_e(4) =$

$z(\log_e(3) + \log_e(4))$

$$\frac{xy}{x + y}$$

$$= \frac{\dfrac{z(\log_e(3) + \log_e(4))}{\log_e(3)} \times \dfrac{z(\log_e(3) + \log_e(4))}{\log_e(4)}}{\dfrac{z(\log_e(3) + \log_e(4))}{\log_e(3)} + \dfrac{z(\log_e(3) + \log_e(4))}{\log_e(4)}}$$

$$= \frac{\dfrac{z^2(\log_e(3) + \log_e(4))}{\log_e(3)} \times \dfrac{(\log_e(3) + \log_e(4))}{\log_e(4)}}{z\dfrac{(\log_e(3) + \log_e(4))}{\log_e(3)} + \dfrac{(\log_e(3) + \log_e(4))}{\log_e(4)}}$$

$$= \frac{z^2(\log_e(3) + \log_e(4))^2}{z(\log_e(3)\log_e(4) + \log_e(4)\log_e(4)) + (\log_e(3)\log_e(3) + \log_e(4)\log_e(3))}$$

$$= \frac{z^2(\log_e(3) + \log_e(4))^2}{z(\log_e(3) + \log_e(4))^2}$$

$= z$

OR

$x = \dfrac{z}{\log_{12} 3}$ and $y = \dfrac{z}{\log_{12} 4}$

$$xy = \frac{z^2}{\log_{12} 3 \log_{12} 4}$$

$$x + y = \frac{z}{\log_{12} 3} + \frac{z}{\log_{12} 4}$$

$$= \frac{z(\log_{12} 3 + \log_{12} 4)}{\log_{12} 3 \log_{12} 4}$$

Therefore $\dfrac{xy}{x + y} = z$

11 $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$

$$\begin{aligned}&= \log_2(12^2 \times 5^3) - \log_2(15 \times 150)\\ &= \log_2 \frac{2^4 \times 3^2 \times 5^3}{3^2 \times 5^3 \times 2}\\ &= \log_2 8\\ &= 3\end{aligned}$$

12 **a** $\log_p 7 + \log_p k = 0$

$$\begin{aligned}\log_p 7k &= 0\\ 7k &= 1\\ k &= \frac{1}{7}\end{aligned}$$

b $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$

$$\begin{aligned}\log_q \frac{3^4 \times 2^2}{144} &= 2\\ \frac{3^4 \times 2^2}{144} &= q^2\\ q &= \frac{3}{2}\end{aligned}$$

13

$$\begin{aligned}\ln y &= a + b\ln x\\ \ln y - b\ln x &= a\\ \ln \frac{y}{x^b} &= a\\ \frac{y}{x^b} &= e^a\\ y &= e^a x^b\end{aligned}$$

14 The range of f is the range of a complete log function, which is $\mathbb{R}$. So the domain of f^{-1} is $\mathbb{R}$.

15 $y = f(x) = e^{2x} - 3ke^x + 5$
$(0, 0)$: $1 - 3k + 5 = 0$, so $k = 2$
Hence $y = e^{2x} - 6e^x + 5$.
$x \to -\infty, e^{2x} - 6e^x + 5 \to 0 + 0 + 5 = 5$, so the horizontal asymptote is $y = 5$ and therefore $b = 5$.
Now find when $y = 0$, i.e.

$$\begin{aligned}e^{2x} - 6e^x + 5 &= 0\\ (e^x - 1)(e^x - 5) &= 0\\ e^x &= 1, 5\\ x &= 0, \log_e 5\end{aligned}$$

$x = 0$ corresponds to the intercept $(0, 0)$, so $x = \log_e 5$ corresponds to the intercept $(a, 0)$. Thus $a = \log_e 5$.

16

$$\begin{aligned}3^x &= e^{kx}\\ x\log_e(3) &= kx \text{ for all } x\\ k &= \log_e(3)\end{aligned}$$

17 **a** $f^{-1}(x) = \frac{1}{3}\log_e(x + 4)$,
dom $f^{-1} = (-4, \infty)$

b $\frac{1}{3x + 4} - 4$

18

$$\begin{aligned}f(27) &= 27\\ k\log_3(27) &= 27\\ 3k &= 27\\ k &= 9\end{aligned}$$

19 **a**

$$\begin{aligned}x^3 - 3x^2 - 6x + 8 &= 0\\ (x - 1)(x^2 - 2x - 8) &= 0\\ (x - 1)(x - 4)(x + 2) &= 0\\ x = 1 \text{ or } x = 4 &\text{ or } x = -2\end{aligned}$$

b $e^x = 1$ or $e^x = 4$ or $e^x = -2$
Hence $x = 0$ or $x = \log_e(4)$

20 **a** Domain of $f \circ g$ = domain of $g = \mathbb{R}$

$f \circ g(x) = \log_e(2x^2 + 4)$

Range = $[\log_e(4), \infty)$

b Domain of $h^{-1} = [\log_e(4), \infty)$

Consider

$$\begin{aligned} x &= \log_e(2y^2 + 4) \\ e^x &= 2y^2 + 4 \\ 2y^2 &= e^x - 4 \\ y^2 &= \frac{e^x - 4}{2} \\ y &= -\sqrt{\frac{e^x - 4}{2}} \end{aligned}$$

$h^{-1}(x) = -\sqrt{\dfrac{e^x - 4}{2}}$

Range of $h^{-1} = \mathbb{R}^-$

21 Let $g(x) = 2^x$ and $f(x) = x^2 - 12x + 32$

a

$$\begin{aligned} f(g(x)) &= 0 \\ 2^{2x} - 12 \times 2^x + 32 &= 0 \\ (2^x - 8)(2^x - 4) &= 0 \\ x = 3 &\text{ or } x = 2 \end{aligned}$$

b

$$\begin{aligned} g(f(x)) &= 1 \\ 2^{x^2-12x+32} &= 1 \\ x^2 - 12x + 32 &= 0 \\ x = 4 &\text{ or } x = 8 \end{aligned}$$

c

$$\begin{aligned} f(g^{-1}(x)) &= 0 \\ (\log_2 x)^2 - 12\log_2(x) + 32 &= 0 \\ \log_2(x) = 4 &\text{ or } \log_2(x) = 8 \\ x = 2^4 &\text{ or } x = 2^8 \\ x = 16 &\text{ or } x = 256 \end{aligned}$$

22 $e^{\log_e \frac{a}{2}} - e^{-\log_e \frac{a}{2}} + 1 = 0$

$$\begin{aligned} \frac{a}{2} - \frac{2}{a} + 1 &= 0 \\ a^2 - 4 + 2a &= 0 \\ a^2 + 2a - 4 &= 0 \\ a &= \frac{-2 \pm \sqrt{4 + 16}}{2} \\ &= -1 \pm \sqrt{5} \\ \therefore a &= -1 + \sqrt{5} \end{aligned}$$

23 $\log_{ab} x = \dfrac{\log_a(x)}{\log_a(ab)}$

$$\begin{aligned} &= \frac{\log_a(x)}{\log_a(a) + \log_a(b)} \\ &= \frac{\log_a(x)}{1 + \log_a b)} \end{aligned}$$

$$\begin{aligned} \frac{1 - \log_{14}(2)}{\log_1 42} &= \frac{\log_{14} 14 - \log_{14}(2)}{\log_{14} 2} \\ &= \frac{\log_{14} 7}{\log_{14} 2} \\ &= \log_2 7 \end{aligned}$$

24

$$\begin{aligned} [f(x)]^2 + [g(x)]^2 &= 5 \\ (e^x + e^{-x})^2 + (e^x - e^{-x})^2 &= 5 \\ e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x} &= 5 \\ 2e^{2x} + 2e^{-2x} &= 5 \\ 2e^{4x} + 2 &= 5e^{2x} \\ 2e^{4x} - 5e^{2x} + 2 &= 0 \\ (2e^{2x} - 1)(e^{2x} - 2) &= 0 \\ x = -\frac{1}{2}\log_e(2) &\text{ or } x = \frac{1}{2}\log_e(2) \end{aligned}$$

25 **a** Let $y = 3^x$ and $y' = 3^{x'+2} - 2$

Rearranging the second equation

$y' + 2 = 3^{x'+2}$

Therefore we can write:

$x = x' + 2$ and $y = y' + 2$

Hence $x' = x - 2$ and $y' = y - 2$

$c = d = -2$

b
$$\begin{aligned} 3^{x+2} - 2 &= 3^x \\ 9 \times 3^x - 3^x &= 2 \\ 8 \times 3^x &= 2 \\ 3^x &= \frac{1}{4} \\ x &= -\log_3(4) \end{aligned}$$

When $x = -\log_3 4$,

$y = 3^{-\log_3 4} = \dfrac{1}{4}$

Intersect at $\left(-\log_3 4, \dfrac{1}{4}\right)$

c Let $x = 3^{y+2} - 2$

$y = \log_3(x + 2) - 2$

$f^{-1} : (-2, \infty) \to \mathbb{R}, f^{-1}(x) = \log_3(x + 2) - 2$

26 **a** $f(-x) = f(x)$

b $2(e^u + e^{-u})$

c 0

d $e^{2u} + e^{-2u}$

e $g(-x) = -g(x)$

f $2e^x, 2e^{-x}, e^{2x} - e^{-2x}$

Solutions to multiple-choice questions

1 C $$\begin{aligned} 4\log_b x^2 &= \log_b 16 + 8 \\ &= \log_b 2^4 + 8 \\ &= 4\log_b 2 + 8 \end{aligned}$$

$$4\log_b \frac{(x)^2}{2} = 8$$

$$\log_b \frac{x^2}{2} = 2$$

$$\frac{x^2}{2} = b^2$$

$$x^2 = 2b^2$$

$$x = \pm\sqrt{2}b$$

2 D $\log_e 4e^{3x}$

$$= \log_e 4 + \log_e e^{3x}$$

$$= \log_e 4 + 3x$$

3 B $3\log_3(x-4)$

$$= x - 4$$

4 E The Functions g and h here the same domain of $R\backslash\{-1\}$, so $B = C$. It follows that either option **D** or **E** Must be true.
Now range $(g) = R\backslash\{0\}$.
Using a CAS calculator to plot the graph of h shows that range $(h) \neq R\backslash\{0\}$.

5 A As $x = 5$
$\log_{10}(5k-3) = 2$

$$5k - 3 = 10^2$$

$$5k = 103$$

$$k = \frac{103}{5}$$

6 C $3^{4\log_3 x + \log_3 4x}$

$$= 3^{\log_3 x^4 + \log_3 4x}$$

$$= 3^{\log_3 4x^5}$$

$$= 4x^5$$

7 B Using the 'solve' command CAS calculator gives $x = 0.2755\ldots$, so $x \approx 0.28$.

8 A The graph is translated 3 units in the negative direction of the y axis
$\therefore$ $b = -3$
When $x = 0, y = 0$
$\therefore$ $0 = ae^0 - 3$

$$0 = a - 3$$

$$a = 3$$

9 C $f : R^+ \to R, f(x) = \log_5 x$

$(5, 0)$

$0 \neq \log_5 5$

$0 \neq 1$
The graph does not pass through the point (5,0).

10 D $3\log_2 x - 7\log_2(x-1) = 2 + \log_2 y$

$$\log_2 \frac{x^3}{(x-1)^7} = 2 + \log_2 y$$

$$\log_2 \frac{x^3}{(x-1)^7} - \log_2 y = 2$$

$$\log_2 \frac{x^3}{y(x-1)^7} = 2$$

$$\frac{x^3}{y(x-1)^7} = 2^2$$

$$y = \frac{x^3}{4(x-1)^7}$$

11 A
$$\begin{aligned} e^{2x} - 12 &= -e^x \\ e^{2x} + e^x - 12 &= 0 \\ (e^x + 4)(e^x - 3) &= 0 \\ \therefore e^x &= 3 \\ x &= \log_e(3) \end{aligned}$$

12 C

13 C

14 D

15 B Consider
$$\begin{aligned} x &= e^{3y+4} \\ 3y + 4 &= \log_e(x) \\ 3y &= \log_e(x) - 4 \\ y &= \frac{1}{3}(\log_e(x) - 4) \\ f^{-1}(x) &= \frac{1}{3}(\log_e(x) - 4) \end{aligned}$$
Domain of f^{-1} = range of $f = (e^4, \infty)$

16 D $f(6x) = 2\log_e(18x) = \log_e(324x^2)$
$f(6x) = f(y) \Rightarrow y = 324x^2$

Solutions to extended-response questions

1 The temperature, T°C, of a liquid x minutes after it begins to cool is given by $T = 90(0.98)^x$

a When $x = 10$

$$T = 90(0.98)^{10}$$
$$= 73.5366$$

b When $T = 27$

$$27 = 90(0.98)^x$$
$$\frac{27}{90} = 10.98^x$$
$$0.3 = 0.98^x$$
$$\therefore \quad \log_e(0.3) = x\log_e(0.98)$$
$$\therefore \quad x = \frac{\log_e(0.3)}{\log_e(0.98)}$$
$$= 59.5946$$

2 Let P denote the population of the village in years after 1800.

$P = 240(1.06)^n$

When $n = 0$, $P = 240$

a When $n = 20$

$P = 240(1.06)^{20} = 769.71$

At the beginning of 1820 the population is approximately 770.

b If $P = 2500$

$$2500 = 240(1.06)^n$$
$$\frac{2500}{240} = (1.06)^n$$
$$\text{i.e. } \frac{125}{12} = (1.06)^n$$

Taking logarithms of both sides

$$\log_e\left(\frac{125}{12}\right) = n\log(1.06)$$
$$\therefore \quad n = \frac{\log_e\left(\frac{125}{12}\right)}{\log_e(1.06)}$$
$$= 40.217$$

The population will reach 2500 in the year 1840.

3 $V = ke^{-\lambda t}$

a as $V = 22\,497$ when $t = 0$

$k = 22\,497$

After one year the value of the car is \$18 000

$\therefore$ Take logarithms, base e of both sides.

$\log_e 18\,000 = \log_e(22\,497)\lambda$

$$\therefore \quad \lambda = \log_e\left(\frac{22\,497}{18\,000}\right)$$

$$\approx 0.223$$

$$\approx 0.22 \text{ (correct to two decimal places)}$$

b $V = 22\,497e^{-0.22\times 3}$

when $t = 3$

$V = 22\,497e^{-0.22\times 3}$

$= 11\,627.60$

The value is \$11 627.6 after 3 years. (This is obtained by taking $\lambda = 0.22$)

4 $\$M$ is the value of a particular house in a certain area t years after January lst 1988.

a It is given that $M = Ae^{-pt}$

and when $t = 0$, $M = \$65\,000$

$\therefore$ $A = 65\,000$

Furthermore when $t = 1$, $M = 61\,000$

$\therefore$ $61\,000 = 65\,000e^{-p}$

$$\therefore \quad \frac{61}{65} = e^{-p}$$

$$\text{and} \quad -p = \log_e\left(\frac{61}{65}\right)$$

$$\text{i.e.} \quad p = \log_e\left(\frac{65}{61}\right)$$

$$p = 0.635$$

$\therefore$ $A = 65\,000$ and $p = 0.064$ to two significant figures.

b $M = 65\,000e^{-pt}$

When $t = 5$

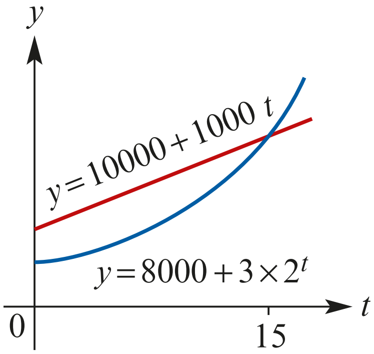
$M = 65\,000e^{-5p}$

$= 47\,199.687$

To the nearest hundred the value is \$47 200

5 **a** $N_A(t) = 10\,000 + 1\,000t$

$N_C(t) = 8\,000 + 3 \times 2^t$



b **i** Using **intersect** from the **CALC** menu the point of intersection of the two graphs has coordinates (12.21, 22209.62)

ii $t = 12.21$ i.e. on Jan 13

iii 22 210

c **i**

$$10\,000 + 100t = 8000 + 3 \times 2^t$$

$$\therefore \quad 2000 + 1000t = 3 \times 2^t$$

$$\therefore \quad \frac{2000 + 1000t}{3} = 2^t$$

$$\therefore \quad \log_{10}\left(\frac{2000 + 1000t}{3}\right) = t\log_{10} 2$$

$$\therefore \quad \log_{10} 1000 + \log_{10}\left(\frac{2 + t}{3}\right) = t\log_{10} 2$$

$$\therefore \quad t = \frac{1}{\log_{10} 2}\left(3 + \log_{10}\left(\frac{2 + t}{3}\right)\right)$$

ii (12.21, 12.21) is found by

d

$$N_c(15) = N_A(15)$$

$$\therefore \quad 8000 + c \times 2^{15} = 10\,000 + 1000 \times 15$$

$$\therefore \quad c \times 2^{15} = 17\,000$$

$$\therefore \quad c = 0.52$$

6 $n = A(1 - e^{-Bt})$

a **i** When $t = 2, n = 10\,000$ and when $t = 4, n = 15\,000$

$10\,000 = A(1 - e^{-2B}) \qquad (1)$

and $15\,000 = A(1 - e^{-4B}) \qquad (2)$

Divide (2) by (1)

$$\frac{3}{2} = \frac{A(1 - e^{-4B})}{A(1 - e^{-2B})}$$

$$\therefore \quad 3(1 - e^{-2B}) = 2(1 - e^{-4B})$$

$$\therefore \quad 3 - 3e^{-2B} = 2 - 2e^{-4B}$$

$$\therefore \quad 1 + 2e^{-4B} - 3e^{-2B} = 0$$

ii Let $a = e^{-2B}$

Then $1 + 2a^2 - 3a = 0$

i.e. $2a^2 - 3a + 1 = 0$

iii $\therefore (2a - 1)(a - 1) = 0$

$\therefore a = \frac{1}{2}$ or $a = 1$

iv $\therefore e^{-2B} = \frac{1}{2}$ or $e^{-2B} = 1$

$\therefore -2B = \log_e\left(\frac{1}{2}\right)$ or $-2B = 0$

$\therefore B = \frac{1}{2}\log_e 2$ or $B = 0$, and then $A \in R^+$ and $n = 0$ for any A.

v Substitute in (1)

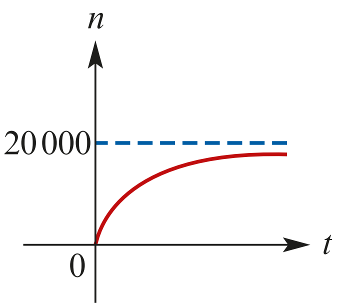
$$10\,000 = A\left(1 - e^{-\log_e 2}\right)$$

$$10\,000 = A\left(1 - e^{\log_e \frac{1}{2}}\right)$$

$$10\,000 = A\left(\frac{1}{2}\right)$$

$$\therefore \quad A = 20\,000$$

b



c $18\,000 = 20\,000\left(1 - e^{\left(-\frac{1}{2}\log_e 2\right)}\right)$

$18\,000 = 20\,000\left(1 - 2^{-\frac{1}{2}}\right)$

$$\therefore \quad \frac{9}{10} = 1 - 2^{-\frac{1}{2}}$$

$$\therefore \quad 2^{-\frac{1}{2}} = 0.1$$

$$-\frac{t}{2}\log_e 2 = \log_e 0.1$$

$$\therefore \quad t = \frac{2\log_e 10}{\log_e 2} \approx 6.644$$

After 6.65 hours the population is 18 000

7 $P = 75(10^{-0.15h})$

a When $h = 0, P = 75$

The barometric pressure is 75 cm of mercury when $h = 0$.

b When $h = 10, P = 75 \times 10^{-1.5} = 2.3717$

The barometric pressure is 2.37 cm when $h = 10$.

c When $P = 60$

$$60 = 75 \times 10^{-0.15h}$$

$$\therefore \quad 0.8 = 10^{-0.15h}$$

$$\therefore \quad \log_{10}(0.8) = -0.15h$$

$$\therefore \quad h = \frac{-1}{0.15}\log_{10}(0.8)$$

$$= 0.646 \text{ km}$$

The barometric pressure is 60 cm of mercury then $h = 0.646$.

8 $A = A_0e^{kt}$

When $t = 1$, $a = 60.7$

When $t = 6$, $a = 5$

Consider the equations

$60.7 = A_0e^{kt}$ (1)

$5 = A_0e^{kt}$ (2)

Divide 2 by 1

$$\frac{50}{607} = e^{5k}$$

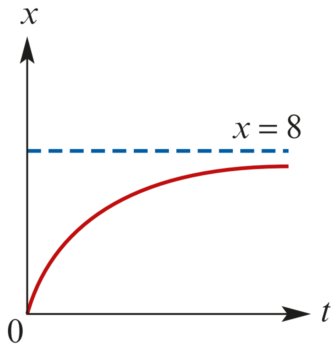
$$\therefore \quad k = \frac{1}{5}\log_e\left(\frac{50}{607}\right) \approx -0.4993 \approx -0.5$$

Substitute in (1)

$$60.7 = A_0\left(\frac{50}{607}\right)^{\frac{1}{5}}$$

$$\therefore \quad A_0 = 60.7 \times \left(\frac{607}{50}\right)^{\frac{1}{5}} \approx 100.007 \approx 100$$

9 **a**



Note: When $t = 0, x = 8(1 - 1) = 0$
As $t \to \infty, e^{-0.2t} \to 0 \therefore x \to 8$

b **i** When $t = 0, x = 8(1 - 1) = 0$ Amount reacted after 0 min is 0 gram

ii When $t = 2, x = 8(1 - e^{-04}) \approx 2.64$ Amount reacted after 2 min is ≈ 2.64 gram

iii When $t = 10, x = 8(1 - e^{-2}) \approx 6.92$ Amount reacted after 10 min is ≈ 6.92 gram

c When $x = 7, 7 = 8(1 - e^{-02\times t})$

$$0.875 = 1 - e^{-0.2t}$$

$$e^{-0.2t} = 0.125$$

$$-0.2t = \log_e(0.125)$$

$$t = -5\log_e(0.125)$$

$$= 5\log_e 8$$

$$\approx 10.397$$

After 10.4 minutes there is 7 g of the substance which has reacted.

10 $T - T_s = (T_0 - T_s)e^{-kt}$

$$T_s = 15^\circ$$

$$T_0 = 96^\circ$$

a When $t = 5, T = 40$

$$\therefore \quad 40 - 15 = (96 - 15)e^{-5k}$$

$$25 = 81e^{-5k}$$

$$e^{-5k} = \frac{25}{81}$$

$$-5k = \log_e \frac{25}{81}$$

$$k = -\frac{1}{5}\log_e \frac{25}{81}$$

$$\approx 0.235$$

b When $t = 10$

$$T - 15 = (96 - 15)_{e}^{\frac{1}{5}\left(\log_e \frac{25}{81}\right)\times 10}$$

$$\text{i.e. } T - 15 = 81 \times \left(\frac{25}{81}\right)^2$$

$$T = 22.716$$

The temperature of the egg is 22.7°C when $t = 10$.

c When $T = 30$

$$30 - 15 = (96 - 15)_{e}^{\frac{1}{5}\left(\log_e \frac{25}{81}\right)}$$

$$\frac{15}{81} = \left(\frac{25}{81}\right)^{\frac{t}{5}}$$

$$\text{i.e. } \frac{5}{27} = \left(\frac{25}{81}\right)^{\frac{t}{5}}$$

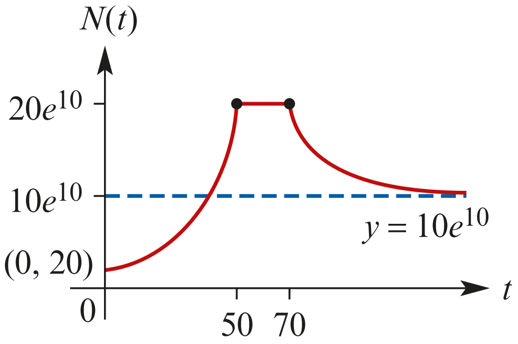
$$\therefore \quad \frac{\log_e\left(\frac{5}{27}\right)}{\log_e\left(\frac{25}{81}\right)} = \frac{t}{5}$$

$$\therefore \quad t \approx 7.17$$

The egg reaches a temperature of 30°C after 7.17 minutes.

11 $$N(t) = \begin{cases} 20e^{0.2t} & 0 \le t \le 50 \\ 20e^{10} & 50 < t \le 70 \\ 10e^{10}(e^{70-t} + 1) & t > 70 \end{cases}$$

a



b **i** $N(10) = 20e^{0.2\times10} \quad (0 \le t \le 50)$

$= 20e^2$

≈ 147.78

ii $N(40) = 20e^{0.2\times40} \quad (0 \le t \le 50)$

$= 20e^8$

$\approx 59\,619.16$

iii $N(60) = 20e^{10} \quad (50 < t \le 70)$

$\approx 440\,529.32$

iv $N(80) = 10e^{10}(e^{70-80} + 1)(t > 70)$

$= 10\text{e}^{10}(e^{-10} + 1)$

$= 10(1 + e^{10})$

$\approx 220\,274.66$

c **i** Considering the graph

$N = 2968$ for $0 \le t \le 50$

$\therefore \quad 2968 = 20.e^{0.2t}$

$148.4 = e^{0.2t}$

$\therefore \quad t = 5\log_e(148.4)$

$= 24.99955$

After 25 days the population is 2968.

ii For $N = 21\,932$, $0 \le t \le 50$. This can be seen from the graph above.

$21\,932 = 20.e^{0.2\mathrm{t}}$

$1096.6 = e^{0.2t}$

$$\therefore \quad t = 5\log_e(1096.6)$$
$$\approx 34.9998$$

After 35 days the population is 21932.

Chapter 6 – Circular functions

Solutions to Exercise 6A

1 a $50° = \frac{50}{180}\pi$

$= \frac{5\pi}{18}$

b $136° = \frac{136}{180}\pi$

$= \frac{34\pi}{45}$

c $250° = \frac{250}{180}\pi$

$= \frac{25\pi}{18}$

d $340° = \frac{340}{180}\pi$

$= \frac{17\pi}{9}$

e $420° = \frac{420}{180}\pi$

$= \frac{7\pi}{3}$

f $490° = \frac{490}{180}\pi$

$= \frac{49\pi}{18}$

2 a $\frac{\pi}{3} = \frac{180°}{3} = 60°$

b $\frac{5\pi}{6} = 180° \times \frac{5}{6} = 150°$

c $\frac{4\pi}{3} = 180° \times \frac{4}{3} = 240°$

d $\frac{7\pi}{9} = 180° \times \frac{7}{9} = 140°$

e $3.5\pi = \frac{7\pi}{2} = \frac{7}{2} \times 180° = 630°$

f $\frac{7\pi}{5} = \frac{7}{5} \times 180° = 252°$

3 a $0.8 = \frac{180°}{\pi} \times 0.8 = 45.84°$

b $1.64 = \frac{180°}{\pi} \times 1.64 = 93.97°$

c $2.5 = \frac{180°}{\pi} \times 2.5 = 143.24°$

d $3.96 = \frac{180°}{\pi} \times 3.96 = 226.89°$

e $4.18 = \frac{180°}{\pi} \times 4.18 = 239.50°$

f $5.95 = \frac{180°}{\pi} \times 5.95 = 340.91°$

4 a $37° = \frac{\pi}{180°} \times 37° = 0.65$

b $74° = \frac{\pi}{180°} \times 74° = 1.29$

c $115° = \frac{\pi}{180°} \times 115° = 2.01$

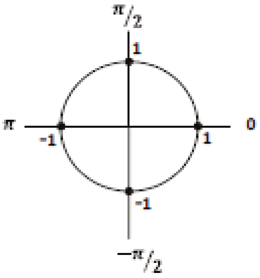
d $122.25° = \frac{\pi}{180°} \times 122.25° = 2.13$

e $340° = \frac{\pi}{180°} \times 340° = 5.93$

f $132.5° = \frac{\pi}{180°} \times 132.5° = 2.31$

Solutions to Exercise 6B

1



a $\sin 3\pi = 0$

b $\cos\left(-\frac{5\pi}{2}\right) = 0$

c $\sin\left(\frac{7\pi}{2}\right) = -1$

d $\cos 3\pi = -1$

e $\sin(-4\pi) = 0$

f $\tan -\pi = 0$

g $\tan 2\pi = 0$

h $\tan -2\pi = 0$

i $\cos(23\pi) = \cos\pi = -1$

j $\cos\left(\frac{49\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

k $\cos(35\pi) = \cos\pi = -1$

l $\cos\left(\frac{-45\pi}{2}\right) = \cos\left(\frac{-\pi}{2}\right) = 0$

m $\tan(24\pi) = \tan(0) = 0$

n $\cos(20\pi) = \cos(0) = 1$

2 **a** 0.99

b 0.52

c −0.87

d 0.92

e −0.67

f −0.23

g −0.99

h 0.44

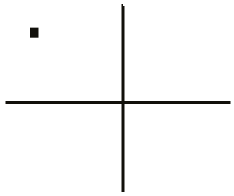
i −34.23

j −2.57

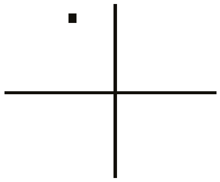
k 0.95

l 0.75

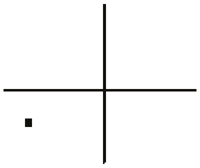
3 **a** $\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$



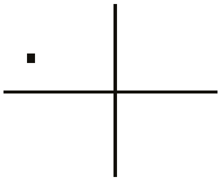
b $\cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$



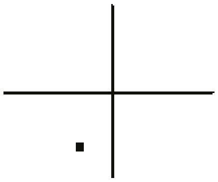
c $\cos\left(\frac{7\pi}{6}\right) = \frac{-\sqrt{3}}{2}$



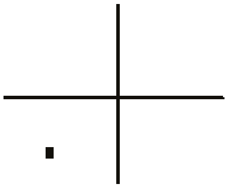
d $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$



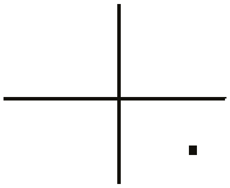
e $\cos\left(\frac{4\pi}{3}\right) = \frac{-1}{2}$



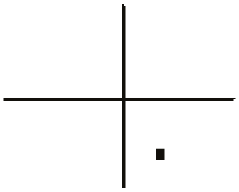
f $\sin\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}}$



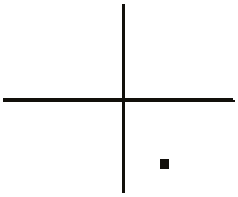
g $\sin\left(\frac{7\pi}{4}\right) = \frac{-1}{\sqrt{2}}$



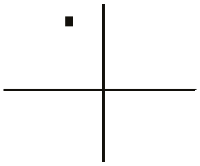
h $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$



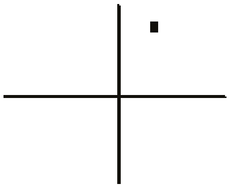
i $\cos\left(\frac{11\pi}{3}\right) = \frac{1}{2}$



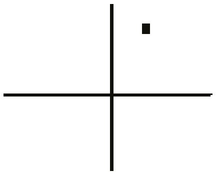
j $\sin\left(\frac{200\pi}{3}\right) = \frac{\sqrt{3}}{2}$



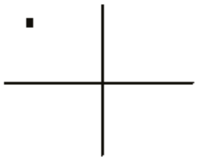
k $\cos\left(\frac{-11\pi}{3}\right) = \frac{1}{2}$



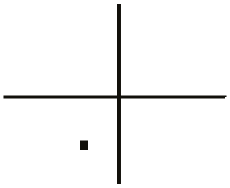
l $\sin\left(\frac{25\pi}{3}\right) = \frac{\sqrt{3}}{2}$



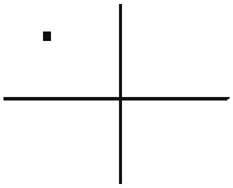
m $\sin\left(\frac{-13\pi}{4}\right) = \frac{1}{\sqrt{2}}$



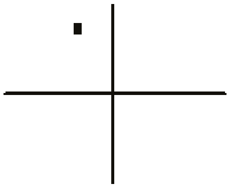
n $\cos\left(\frac{-20\pi}{3}\right) = \frac{-1}{2}$



o $\sin\left(\frac{67\pi}{4}\right) = \frac{1}{\sqrt{2}}$



p $\cos\left(\frac{68\pi}{3}\right) = \frac{-1}{2}$



q $\tan\left(\frac{11\pi}{3}\right) = -\sqrt{3}$

r $\tan\left(\frac{200\pi}{3}\right) = -\sqrt{3}$

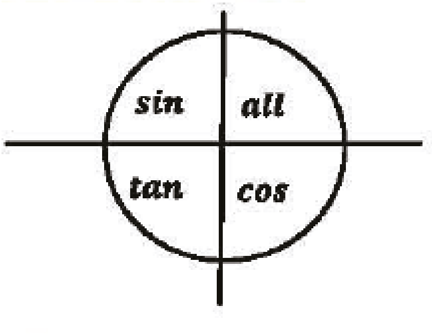
s $\tan\left(\frac{-11\pi}{6}\right) = \frac{1}{\sqrt{3}}$

t $\tan\left(\frac{25\pi}{3}\right) = \sqrt{3}$

u $\tan\left(-\frac{13\pi}{4}\right) = -1$

v $\tan\left(-\frac{25\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

4



a $\sin(\pi - \theta) = \sin\theta = 0.52$

b $\cos(\pi + x) = -\cos x = -0.68$

c $\sin(2\pi + \theta) = \sin\theta = 0.52$

d $\tan(\pi + \alpha) = \tan(\alpha) = 0.4$

e $\sin(\pi + \theta) = -\sin\theta = -0.52$

f $\cos(2\pi - x) = \cos(-x) = \cos x = 0.68$

g $\tan(2\pi - \alpha) = \tan(-\alpha)$

$= -\tan\alpha$

$= -0.4$

h $\cos(\pi - x) = -\cos x = -0.68$

i $\sin(-\theta) = -\sin\theta = -0.52$

j $\cos(-x) = \cos x = 0.68$

k $\tan(-\alpha) = -\tan(\alpha) = -0.4$

5 **a** 0.4

b −0.7

c 0.4

d 1.2

e −0.4

f 0.7

g −1.2

h −0.7

i −0.4

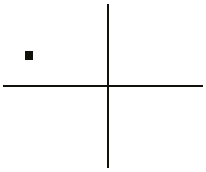
j 0.7

k −1.2

6 **a** $\sin(150°) = \frac{1}{2}$

$\cos(150°) = \frac{-\sqrt{3}}{2}$

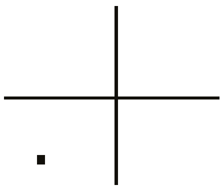
$\tan(150°) = \frac{-1}{\sqrt{3}}$



b $\sin(225^\circ) = \frac{-1}{\sqrt{2}}$

$\cos(225^\circ) = \frac{-1}{\sqrt{2}}$

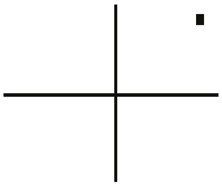
$\tan(225^\circ) = 1$



c $\sin(405^\circ) = \frac{1}{\sqrt{2}}$

$\cos(405^\circ) = \frac{1}{\sqrt{2}}$

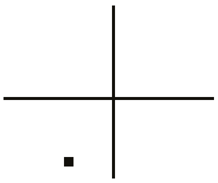
$\tan(405^\circ) = 1$



d $\sin(-120^\circ) = \frac{-\sqrt{3}}{2}$

$\cos(-120^\circ) = \frac{-1}{2}$

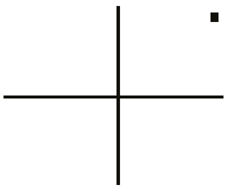
$\tan(-120^\circ) = \sqrt{3}$



e $\sin(-315^\circ) = \frac{1}{\sqrt{2}}$

$\cos(-315^\circ) = \frac{1}{\sqrt{2}}$

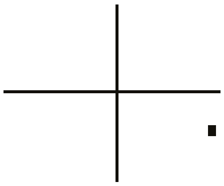
$\tan(-315^\circ) = 1$



f $\sin(-30^\circ) = \frac{-1}{2}$

$\cos(-30^\circ) = \frac{\sqrt{3}}{2}$

$\tan(-30^\circ) = \frac{-1}{\sqrt{3}}$



(ensure calculator is in radians not degrees)

Solutions to Exercise 6C

1 $\sin x = 0.3, \cos x = 0.6, \tan x = 0.7$

a $\cos(-\alpha) = \cos\alpha = 0.6$

b $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha = 0.6$

c $\tan(-\theta) = -\tan\theta = -0.7$

d $\cos\left(\frac{\pi}{2} - x\right) = \sin x = 0.3$

e $\sin(-x) = -\sin x = -0.3$

f $\tan\left(\frac{\pi}{2} - \theta\right) = \text{cotan}(\theta) = \frac{10}{7}$

g $\cos\left(\frac{\pi}{2} + x\right) = -\sin x = -0.3$

h $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha = 0.6$

i
$$\begin{aligned}\sin\left(\frac{3\pi}{2} + \alpha\right) &= \cos(\pi + \alpha)\\ &= -\cos\alpha = -0.6\end{aligned}$$

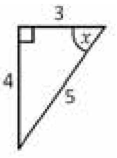
j
$$\begin{aligned}\cos\left(\frac{3\pi}{2} - x\right) &= \cos\left(\frac{-\pi}{2} - x\right)\\ &= \cos\left(\frac{\pi}{2} + x\right)\\ &= -\sin x\\ &= -0.3\end{aligned}$$

k $\tan\left(\frac{3\pi}{2} - \theta\right) = \frac{1}{\tan\theta} = \frac{10}{7}$

l $\cos\left(\frac{5\pi}{2} - \theta\right) = \sin x = 0.3$

2 **a** $\cos x = \frac{3}{5}$

$\frac{3\pi}{2} \le x \le 2\pi$

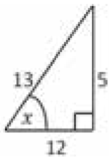


Method 1

$$\begin{aligned}\cos^2 x + \sin^2 x &= 1\\ \frac{9}{25} + \sin^2 x &= 1\\ \sin^2 x &= \frac{16}{25}\\ \sin x &= \frac{\pm 4}{5}\\ \frac{3\pi}{2} \le x \le 2\pi, \sin x &= \frac{-4}{5}\\ \tan x &= \frac{\sin x}{\cos x} = \frac{-4}{3}\end{aligned}$$

b $\sin x = \frac{5}{13}$

$\frac{\pi}{2} < x < \pi$



Method 2

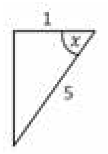
from the triangle,

$\cos x = \frac{-12}{13}$ SOH **CAH** TOA

$\tan x = \frac{\sin x}{\cos x} = \frac{-5}{12}$

c $\cos x = \frac{1}{5}$

$\frac{3\pi}{2} < x < 2\pi$



$\cos^2 x = \frac{1}{25}$

$\therefore \sin^2 x = 1 - \cos^2 x$

$\sin^2 x = \frac{24}{25}$

$\sin x = \frac{\pm 2\sqrt{6}}{5}$

since $\frac{3\pi}{2} < x < 2\pi,\ \sin x = \frac{-2\sqrt{6}}{5}$

$\tan x = \frac{\sin x}{\cos x} = -2\sqrt{6}$

d $\sin x = -\frac{12}{13}$

$\therefore \cos^2 x = 1 - \sin^2 x$

$= \frac{25}{169}$

Since $\pi < x < \frac{3\pi}{2}$

$\cos x = -\frac{5}{13}$

$\tan x = \frac{\sin x}{\cos x}$

$= \frac{12}{5}$

e $\cos x = \frac{4}{5}$

$\therefore \cos^2 x = 1 - \sin^2 x$

$= \frac{9}{25}$

Since $\frac{3\pi}{2} < x < 2\pi$

$\sin x = -\frac{3}{5}$

$\tan x = \frac{\sin x}{\cos x}$

$= -\frac{3}{4}$

f $\sin x = -\frac{5}{13}$

$\therefore \cos^2 x = 1 - \sin^2 x$

$= 1 - \frac{25}{169}$

$= \frac{144}{169}$

Since $\pi < x < \frac{3\pi}{2}$

$\cos x = -\frac{12}{13}$

$\tan x = \frac{\sin x}{\cos x}$

$= \frac{5}{12}$

g $\cos x = \frac{8}{10}$

$\therefore \cos^2 x = 1 - \sin^2 x$

$= \frac{36}{100}$

Since $\frac{3\pi}{2} < x < 2\pi$

$\sin x = -\frac{3}{5}$

$\tan x = \frac{\sin x}{\cos x}$

$= -\frac{3}{4}$

Solutions to Exercise 6D

1 **a** $2\pi,\ 3$

b $\frac{2\pi}{3},\ 5$

c $\pi,\ \frac{1}{2}$

d $6\pi,\ 2$

e $\frac{\pi}{2},\ 3$

f $2\pi,\ \frac{1}{2}$

g $4\pi,\ 3$

h $3\pi,\ 2$

2 **a** Dilation of factor 4 from the x-axis, dilation of factor $\frac{1}{3}$ from the y-axis; Amplitude = 4; Period = $\frac{2\pi}{3}$

b Dilation of factor 5 from the x-axis, dilation of factor 3 from the y-axis; Amplitude = 5; Period = 6π

c Dilation of factor 6 from the x-axis, dilation of factor 2 from the y-axis; Amplitude = 6; Period = 4π

d Dilation of factor 4 from the x-axis, dilation of factor $\frac{1}{5}$ from the y-axis; Amplitude = 4; Period = $\frac{2\pi}{5}$

3 **a** Dilation of factor 2 from the x-axis, dilation of factor $\frac{1}{3}$ from the y-axis; Amplitude = 2; Period = $\frac{2\pi}{3}$

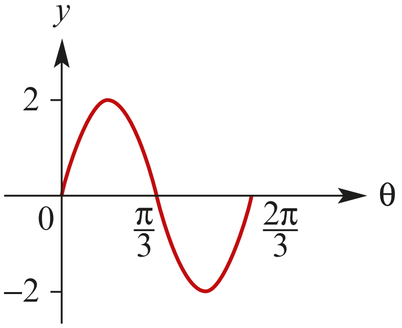
b Dilation of factor 3 from the x-axis, dilation of factor 4 from the y-axis; Amplitude = 3; Period = 8π

c Dilation of factor 6 from the x-axis, dilation of factor 5 from the y-axis; Amplitude = 6; Period = 10π

d Dilation of factor 3 from the x-axis, dilation of factor $\frac{1}{7}$ from the y-axis; Amplitude = 3; Period = $\frac{2\pi}{7}$

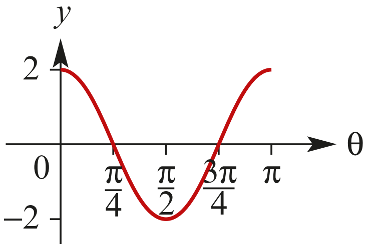
4 **a** Amplitude = 2

Period = $\frac{2\pi}{3}$



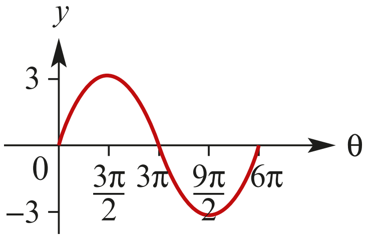
b Amplitude = 2

Period = π



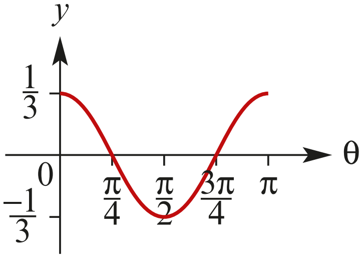
c Amplitude = 3

Period = 6π



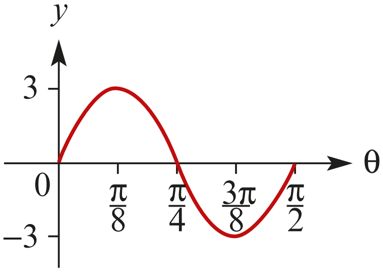
d Period = π

Amplitude = $\frac{1}{3}$



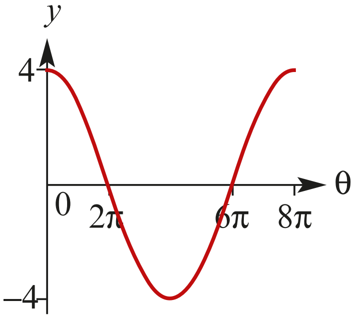
e Amplitude = 3

Period = $\frac{\pi}{2}$



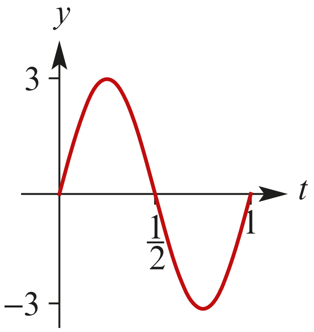
f Amplitude = 4

Period = 8π



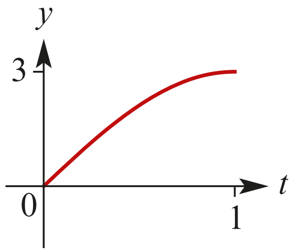
5 Period = $\frac{2\pi}{2\pi}$ = 1

Amplitude = 3



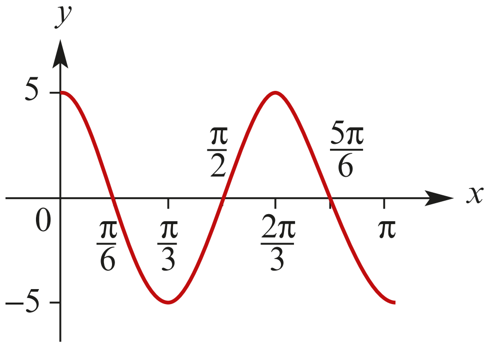
6 Period = = $\frac{2\pi}{\frac{\pi}{2}}$ = 4

Amplitude = 3



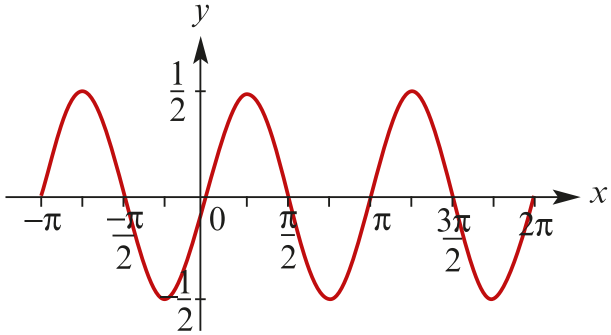
7 Period = = $\frac{2\pi}{3}$

Amplitude = 5



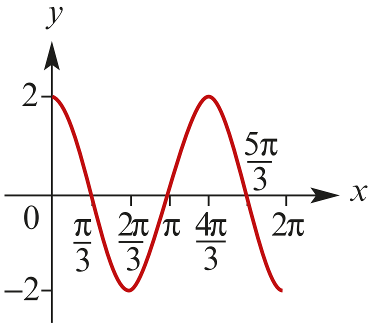
8 Period = = $\frac{2\pi}{2}$ = π

Amplitude = $\frac{1}{2}$



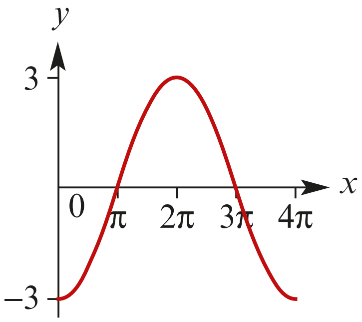
9 Period = = $\frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$

Amplitude = 2



10 Period = = $\frac{2\pi}{\frac{1}{2}} = 4\pi$

Amplitude = 3



11 $y = \sin x$

Dilation of factor 2 from the x-axis

$\Rightarrow\ y = 2\sin x$

Dilation of factor 3 from the y-axis

$\Rightarrow\ y = 2\sin\left(\frac{x}{3}\right)$

12 $y = \cos x$

Dilation of factor $\frac{1}{2}$ from the x-axis

$\Rightarrow\ y = \frac{1}{2}\cos x$

Dilation of factor 3 from the y-axis

$\Rightarrow\ y = \frac{1}{2}\cos\left(\frac{x}{3}\right)$

13 $y = \sin x$

Dilation of factor $\frac{1}{2}$ from the x-axis

$\Rightarrow\ y = \frac{1}{2}\sin x$

Dilation of factor 2 from the y-axis

$\Rightarrow\ y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

Solutions to Exercise 6E

1 **a** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

b $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

c $\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$

d $\frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

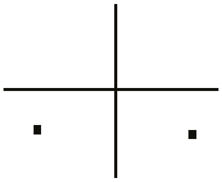
e $\frac{\pi}{2}, \frac{5\pi}{2}$

f $\pi, 3\pi$

2 **a** $\sin x = \frac{-1}{2}$

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

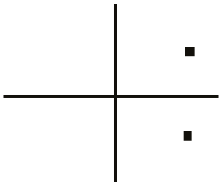
$$x = \frac{-5\pi}{6}, \frac{-\pi}{6}$$



b $\cos x = \frac{\sqrt{3}}{2}$

$$x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

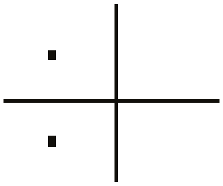
$$x = \frac{-\pi}{6}, \frac{\pi}{6}$$



c $\cos x = \frac{-\sqrt{3}}{2}$

$$x = \cos^{-1} \frac{-\sqrt{3}}{2}$$

$$x = \frac{-5\pi}{6}, \frac{5\pi}{6}$$



3 **a** $\sqrt{2}\sin x = 1$

$$\sin x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

b $\sqrt{2}\cos x = -1$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

c $2\cos x = -\sqrt{3}$

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

d $2\sin x + 1 = 0$

$$\sin x = -\frac{1}{2}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

e $\sqrt{2}\cos x = 1$

$$\cos x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{4}$$

f $4\cos x = -2$

$$\cos x = -\frac{1}{2}$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

4 **a** $\sin x = 0.6$

calculator gives $x_1 \cong 0.6435$

second answer is $\pi - x_1 \cong 2.498$

b $\cos x = 0.8$

calculator gives $x_1 \cong 0.6435$

second answer is $2\pi - x_1 \cong 5.640$

c $\sin x = -0.45$

calculator gives $x_1 \cong 5.816$

second answer is $\pi - x_1 \cong 3.608$

d $\cos x = -0.2$

calculator gives $x_1 \cong 1.772$

second answer is $2\pi - x_1 \cong 4.511$

5 **a** $\sin\theta^\circ = 0.3$

calculator gives $\theta_1 = 17.46$

second answer is $180^\circ - \theta_1 = 162.54$

b $\cos\theta^\circ = 0.4$

calculator gives $\theta_1 = 66.42$

second answer is $360^\circ - \theta_1 = 293.58$

c $\sin\theta^\circ = -0.8$

calculator gives $\theta_1 = 306.87$

second answer is $180^\circ - \theta_1 = 233.13$

d $\cos\theta^\circ = -0.5$

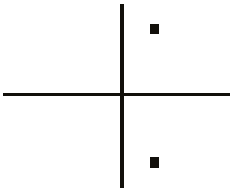
$\theta_1 = 120$

second answer is $\theta_2 - 360^\circ - \theta_1 = 240$

6 **a** $\cos(\theta^\circ) = \frac{1}{2}$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

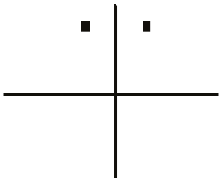
$$\theta = 60, 300$$



b $\sin(\theta^\circ) = \frac{\sqrt{3}}{2}$

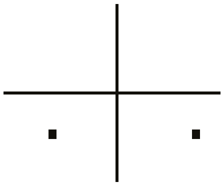
$$\theta^\circ = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\theta = 60, 120$$



c $\sin(\theta^\circ) = \frac{-1}{\sqrt{2}}$

$$\theta = 225, 315$$

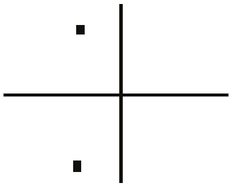


d $2\cos\theta^\circ + 1 = 0$

$$\cos\theta^\circ = \frac{-1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

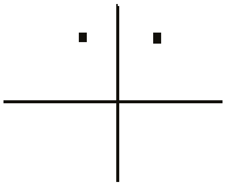
$$\theta = 120, 240$$



e $\sin(\theta^\circ) = \frac{\sqrt{3}}{2}$

$$\theta^\circ = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

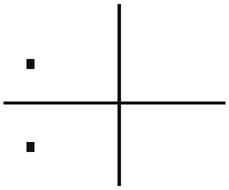
$$\theta = 60, 120$$



f $\cos(\theta^\circ) = \frac{-\sqrt{3}}{2}$

$$\theta^\circ = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

$$\theta = 150, 210$$



7 **a** $\sin 2\theta = -\frac{1}{2}$

$$2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

b $\cos 2\theta = \frac{\sqrt{3}}{2}$

$$2\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

c $\sin 2\theta = \frac{1}{2}$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

d $\sin 3\theta = -\frac{1}{\sqrt{2}}$

$$3\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{23\pi}{4}$$

$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}\left(= \frac{5\pi}{4}\right),$$

$$\frac{21\pi}{12}\left(= \frac{7\pi}{4}\right), \frac{23\pi}{12}$$

e $\cos 2\theta = -\frac{\sqrt{3}}{2}$

$$2\theta = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$$

$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$$

f $\sin 2\theta = -\frac{1}{\sqrt{2}}$

$$2\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$$

$$\theta = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

8 **a** $\cos 3x = -\dfrac{\sqrt{3}}{2}$

$$3x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6}$$

$$x = \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$$

b $\sin 2x = \dfrac{1}{2}$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

c $\cos 3x = \dfrac{1}{\sqrt{2}}$

$$3x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}$$

$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

d $\sin 3x = \dfrac{1}{2}$

$$3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$$

$$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$$

e $\sin 2x = \dfrac{1}{\sqrt{2}}$

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

f $\cos 3x = -\dfrac{\sqrt{3}}{2}$

$$3x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6}$$

$$x = \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$$

g $x = \dfrac{5\pi}{12}, \dfrac{7\pi}{12}, \dfrac{13\pi}{12}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \dfrac{23\pi}{12}$

h $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

i $x = \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{11\pi}{8}, \dfrac{13\pi}{8}$

9 **a** 2.03444, 2.67795, 5.17604, 5.81954

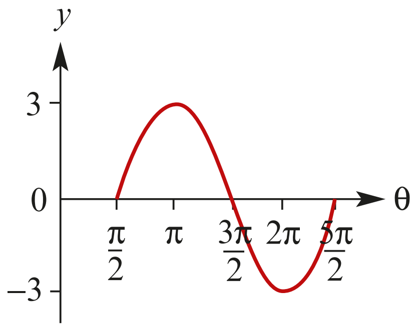
b 1.89255, 2.81984, 5.03414, 5.96143

c 0.57964, 2.56195, 3.72123, 5.70355

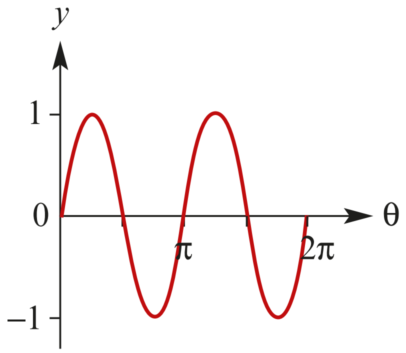
d 0.309098, 1.7853, 2.40349, 3.87969, 4.49789, 5.97409

Solutions to Exercise 6F

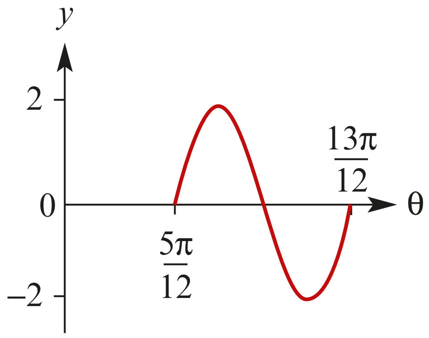
1 **a** Period = 2π; Amplitude = 3; $y = \pm 3$



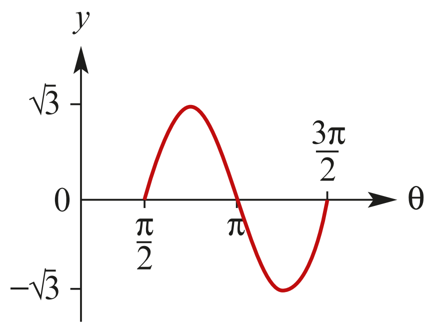
b Period = π; Amplitude = 1; $y = \pm 1$



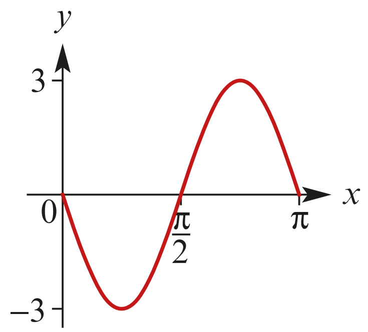
c Period = $\frac{2\pi}{3}$; Amplitude = 2; $y = \pm 2$



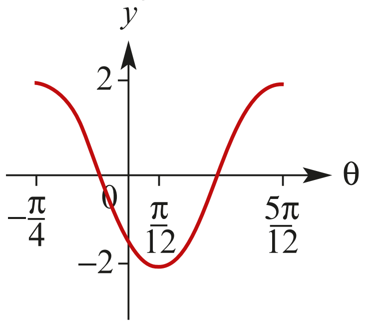
d Period = π; Amplitude = $\sqrt{3}$;
$y = \pm\sqrt{3}$



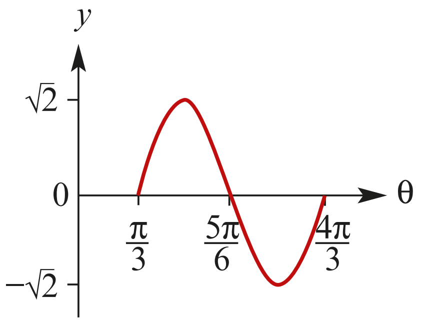
e Period = π; Amplitude = 3; $y = \pm 3$



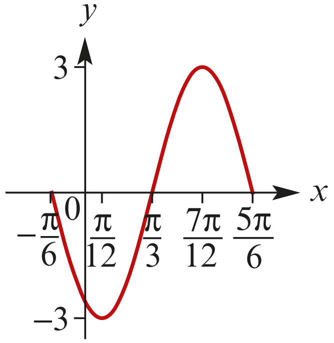
f Period = $\frac{2\pi}{3}$; Amplitude = 2; $y = \pm 2$



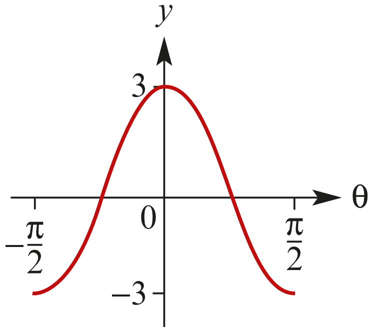
g Period = π; Amplitude = $\sqrt{2}$;
$y = \pm\sqrt{2}$



h Period = π; Amplitude = 3; $y = \pm 3$

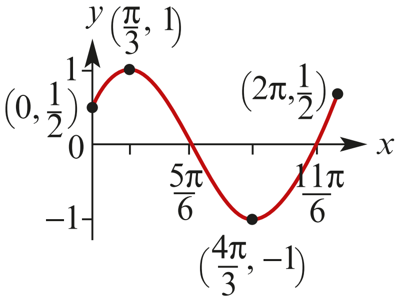


i Period = π; Amplitude = 3; $y = \pm 3$



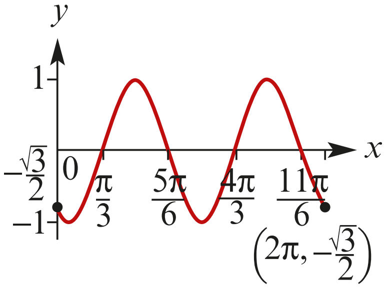
2 **a** $f(0) = \frac{1}{2},\ f(2\pi) = \frac{1}{2}$

b



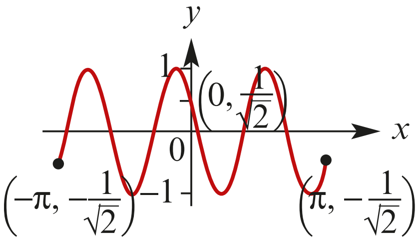
3 **a** $f(0) = -\frac{\sqrt{3}}{2},\ f(2\pi) = -\frac{\sqrt{3}}{2}$

b



4 **a** $f(-\pi) = -\frac{1}{\sqrt{2}},\ f(\pi) = -\frac{1}{\sqrt{2}}$

b



5 **a** $y = 3\sin\left(\frac{x}{2}\right)$

b $y = 3\sin(2x)$

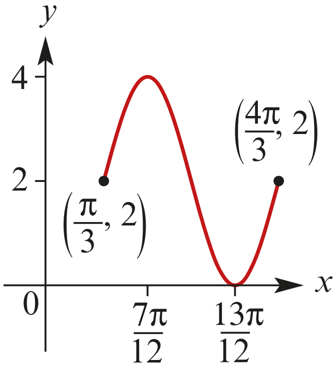
c $y = 2\sin\left(\frac{x}{3}\right)$

d $y = \sin 2\left(x - \frac{\pi}{3}\right)$

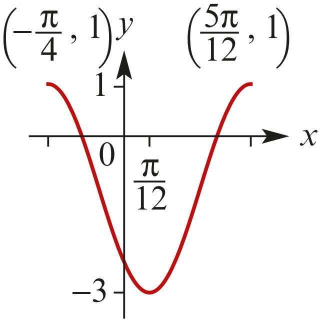
e $y = \sin \frac{1}{2}\left(x + \frac{\pi}{3}\right)$

Solutions to Exercise 6G

1 **a**



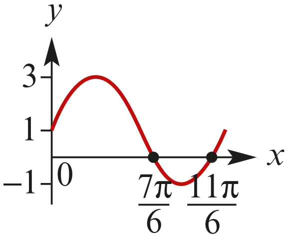
b



2 **a** $2 \sin x + 1 = 0$

$$\sin x = -\frac{1}{2}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

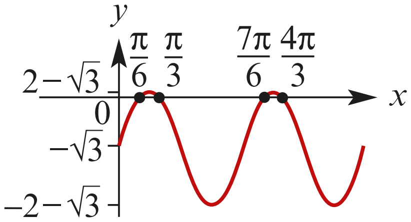


b $2 \sin 2x - \sqrt{3} = 0$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

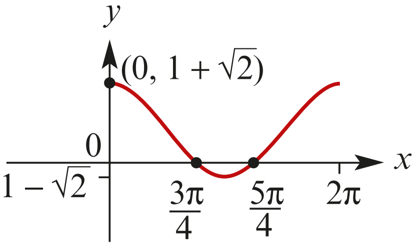
$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$



c $\sqrt{2} \cos x = -1$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

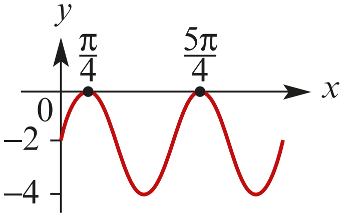


d $2 \sin 2x - 2 = 0$

$$\sin 2x = 1$$

$$2x = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

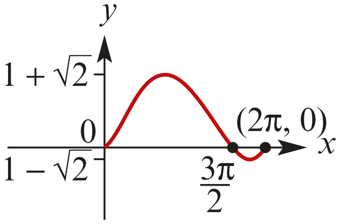


e $\sqrt{2}\sin(x - \frac{\pi}{4}) = -1$

$$\sin(x - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$$

$$x - \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4}, -\frac{\pi}{4}$$

$$x = 0, \frac{3\pi}{2}, 2\pi$$



3 a ***y*-axis intercept**

$y = -2$

***x*-axis intercepts**

$2\sin(3x) = 2$

$$\sin(3x) = 1$$

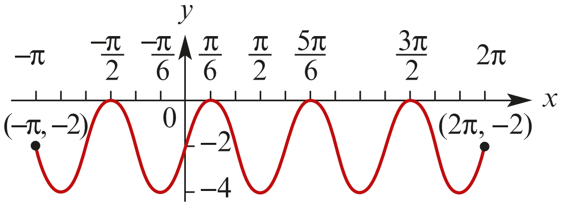
$$3x = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

Endpoints

When $x = -\pi, y = -2$

When $x = 2\pi, y = -2$



b ***y*-axis intercept**

$y = -\sqrt{2}$

***x*-axis intercepts**

$\cos(3(x - \frac{\pi}{4}) = 0$

$$3(x - \frac{\pi}{4}) = -\frac{7\pi}{2}, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{6}, \frac{9\pi}{2}$$

$$(x - \frac{\pi}{4}) = -\frac{7\pi}{6}, -\frac{5\pi}{6}, -\frac{3\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}$$

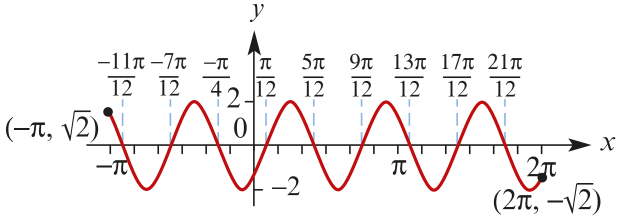
$$(x - \frac{3\pi}{12}) = -\frac{14\pi}{12}, -\frac{10\pi}{12}, -\frac{6\pi}{12}, -\frac{2\pi}{12}, \frac{2\pi}{12}, \frac{6\pi}{12}, \frac{10\pi}{12}, \frac{14\pi}{12}, \frac{18\pi}{12}$$

$$x = -\frac{11\pi}{12}, -\frac{7\pi}{12} - \frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$$

Endpoints

When $x = -\pi, y = \sqrt{2}$

When $x = 2\pi, y = -\sqrt{2}$



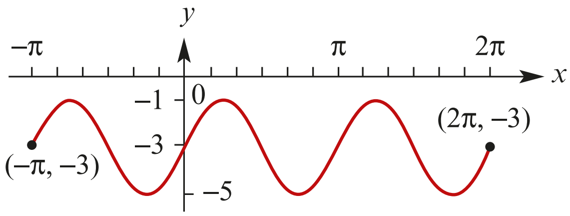
c ***y*-axis intercept**

$y = -3$

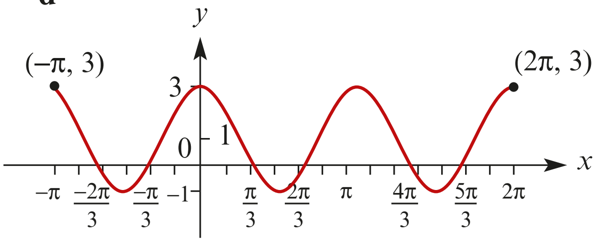
Endpoints

When $x = -\pi, y = -3$

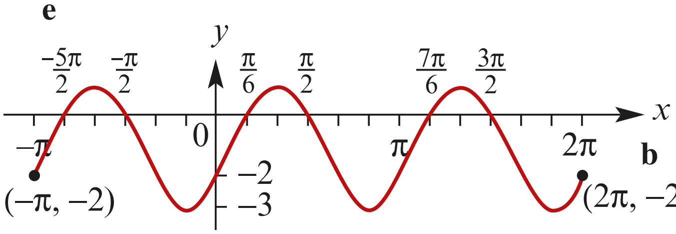
When $x = 2\pi, y = -3$



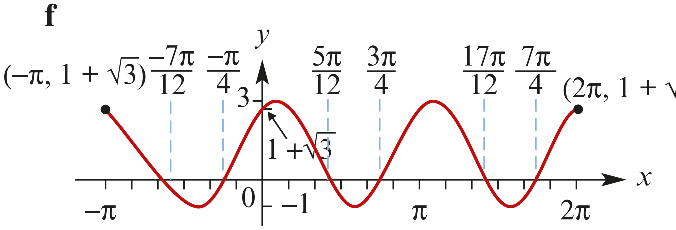
d



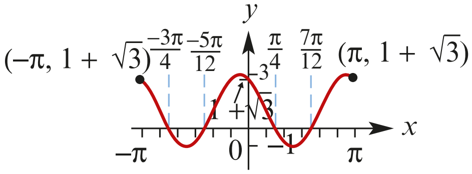
e



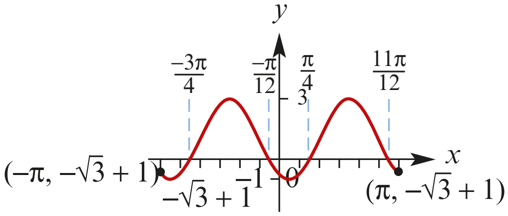
f



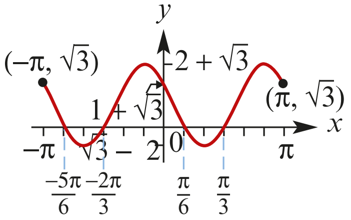
4 a



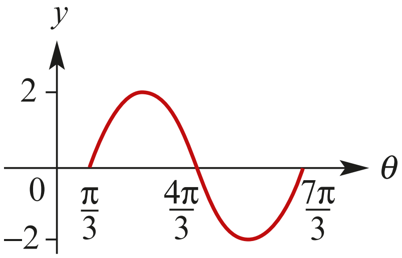
b



c



5 a

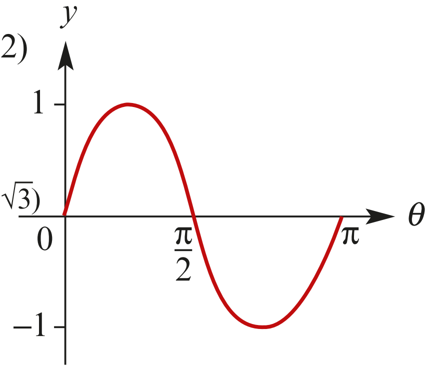


Period = 2π

Amplitude = 2

Range = $[-2, 2]$

b

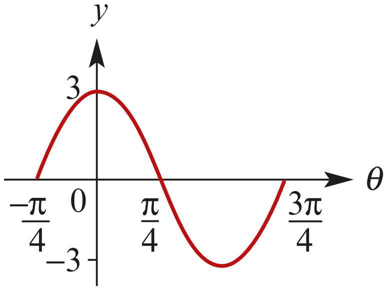


Period = π

Amplitude = 1

Range = $[-1, 1]$

c

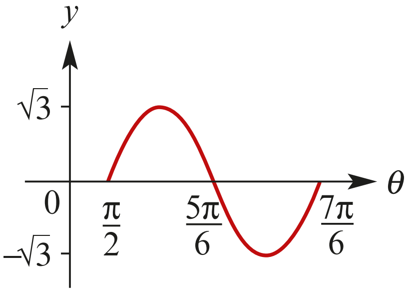


Period = π

Amplitude = 3

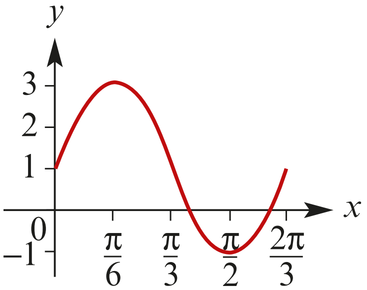
Range = $[-3, 3]$

d



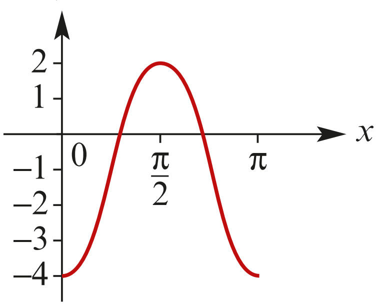
Period = $\frac{2\pi}{3}$
Amplitude = $\sqrt{3}$
Range = $[-\sqrt{3}, \sqrt{3}]$

e



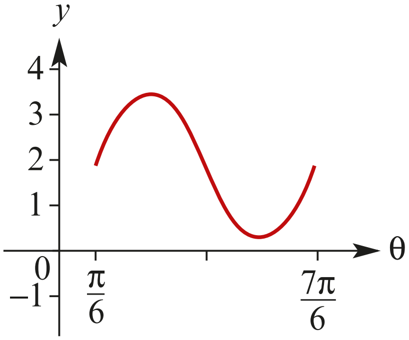
Period = $\frac{2\pi}{3}$
Amplitude = 2
Range = $[-1, 3]$

f Period = π



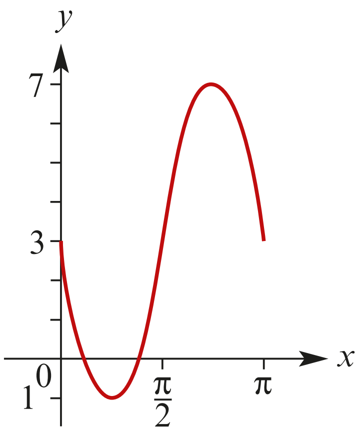
Amplitude = 3
Range = $[-4, 2]$

g



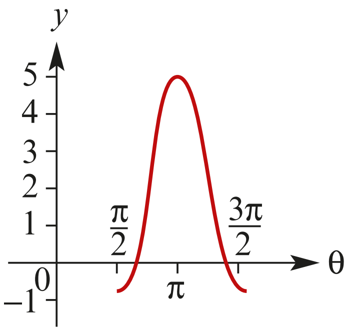
Period = π
Amplitude = $\sqrt{2}$
Range = $[-\sqrt{2}+2, 2+\sqrt{2}]$

h



Period = π
Amplitude = 4
Range = $[-1, 7]$

i



Period = π
Amplitude = 3
Range = $[-1, 5]$

6 **a** $y = \frac{1}{2}\cos\left(\frac{1}{3}\left(x - \frac{\pi}{4}\right)\right)$

b $y = 2\cos\left(x - \frac{\pi}{4}\right)$

c $y = -\frac{1}{3}\cos\left(x - \frac{\pi}{3}\right)$

7 **a**
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis

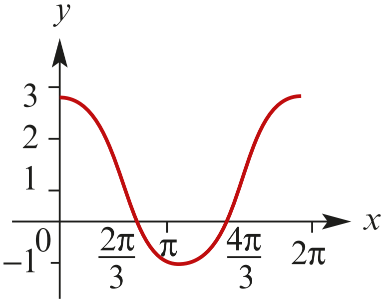
b
- Dilation of factor 3 from the x-axis

- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation $\dfrac{\pi}{3}$ units to the right

c
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation $\dfrac{\pi}{3}$ units to the right and 2 units up

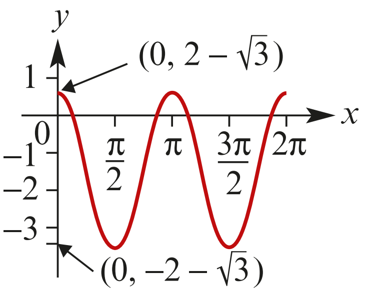
d
- Dilation of factor 2 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation $\dfrac{\pi}{3}$ units to the right and 5 units up

8 a



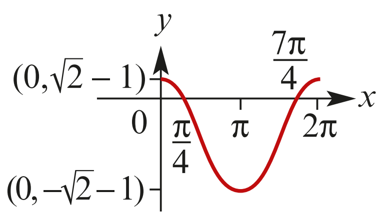
Intercepts: $\left(\dfrac{2\pi}{3}, 0\right), \left(\dfrac{4\pi}{3}, 0\right)$

b



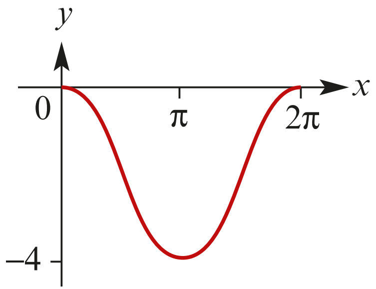
Intercepts: $\left(\dfrac{\pi}{12}, 0\right), \left(\dfrac{11\pi}{12}, 0\right), \left(\dfrac{13\pi}{12}, 0\right), \left(\dfrac{23\pi}{12}, 0\right)$

c



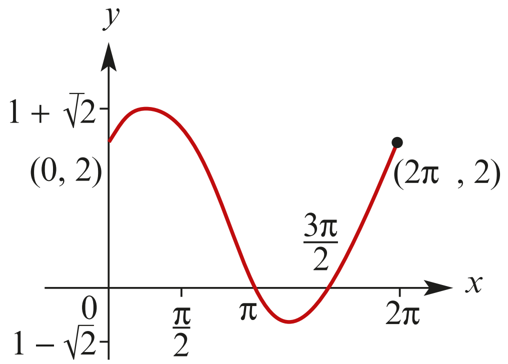
Intercepts: $\left(\dfrac{\pi}{4}, 0\right), \left(\dfrac{7\pi}{4}, 0\right)$t

d



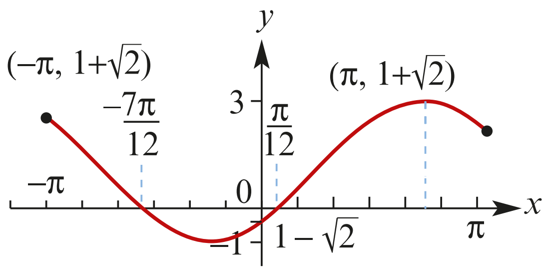
Intercepts: $(0, 0), (2\pi, 0)$

e

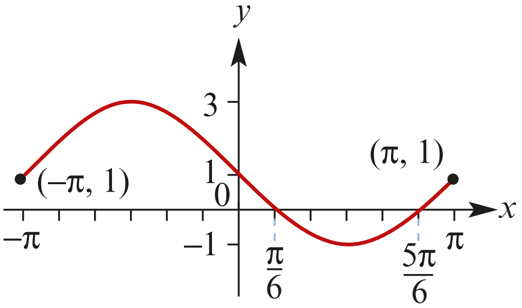


Intercepts: $(\pi, 0), \left(\dfrac{3\pi}{2}, 0\right)$

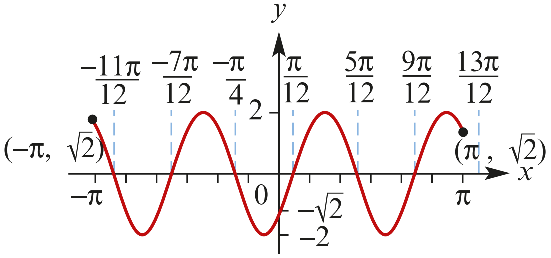
9 a



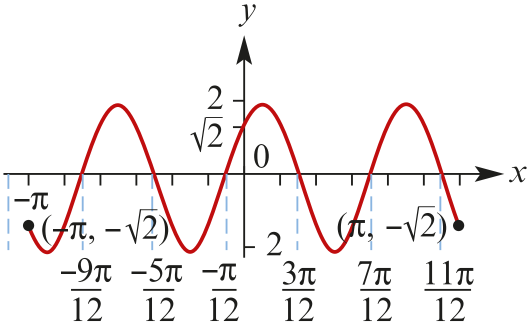
b


y
3
1
0
−1
(−π, 1)
(π, 1)
−π
π
x
$\frac{\pi}{6}$
$\frac{5\pi}{6}$

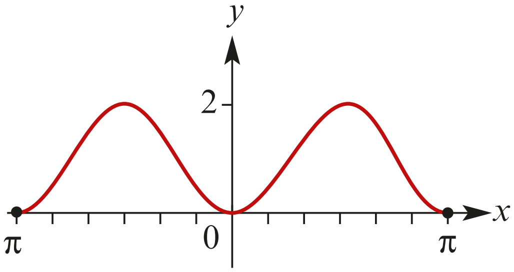
c


y
$\frac{-11\pi}{12}$
$\frac{-7\pi}{12}$
$\frac{-\pi}{4}$
$\frac{\pi}{12}$
$\frac{5\pi}{12}$
$\frac{9\pi}{12}$
$\frac{13\pi}{12}$
2
(−π, √2)
(π , √2)
x
−π
0
π
−√2
−2

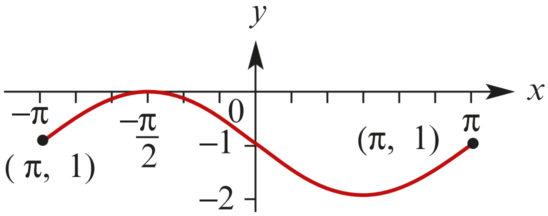
d


y
2
√2
0
x
−π
(−π, −√2)
(π, −√2)
2
$\frac{-9\pi}{12}$
$\frac{-5\pi}{12}$
$\frac{-\pi}{12}$
$\frac{3\pi}{12}$
$\frac{7\pi}{12}$
$\frac{11\pi}{12}$

e

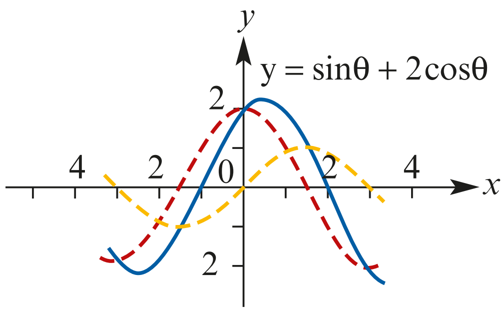

y
2
x
π
0
π

f

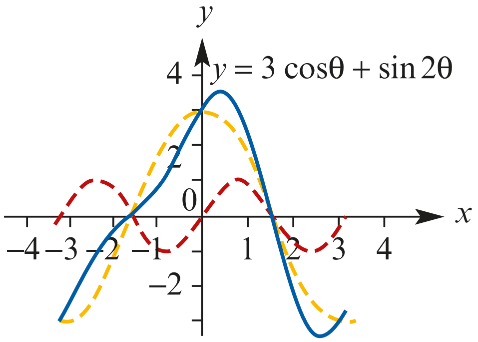

y
x
−π
$\frac{-\pi}{2}$
0
π
−1
−2
(π, 1)
(π, 1)

Solutions to Exercise 6H

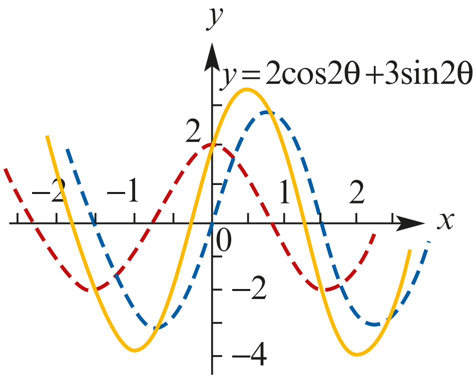
1 a



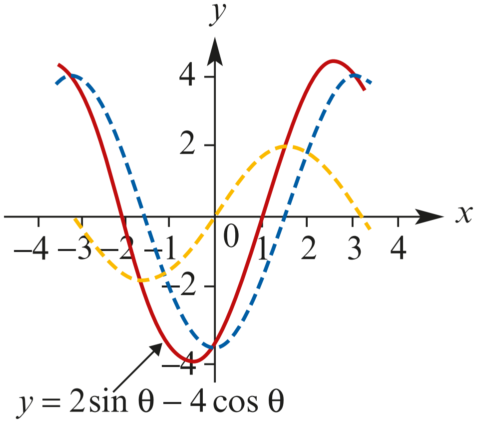
d



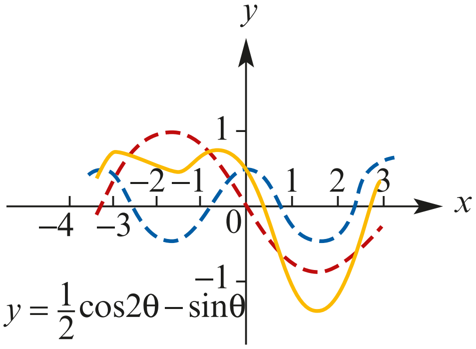
b



e



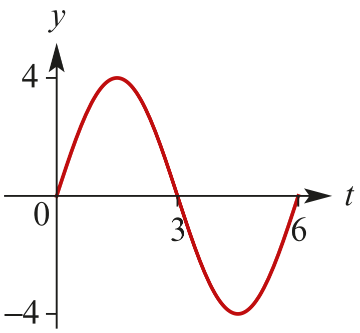
c



Solutions to Exercise 6I

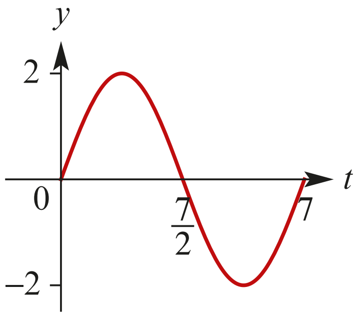
1 **a** $n = \frac{\pi}{3}$

$A = 4$



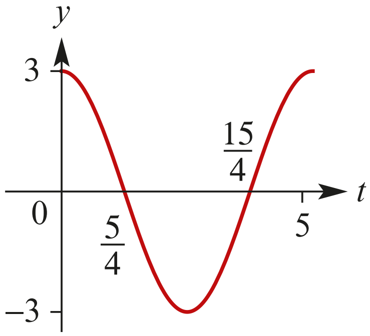
b $n = \frac{2\pi}{7}$

$A = 2$



c $n = \frac{2\pi}{5}$

$A = 3$



2 $A = 3,\ n = \frac{\pi}{4}$

3 $A = -4,\ n = \frac{\pi}{6}$

4 $A = 0.5,\ \varepsilon = \frac{-\pi}{3}$

5 $A = 3,\ n = 3,\ b = 5$

6

$$A = 4$$

$$\frac{2\pi}{n} = 8$$

$$n = \frac{\pi}{4}$$

$$t = 2,\ y = 0$$

$$0 = 4\sin\left(\frac{\pi}{4} \times 2 + \varepsilon\right)$$

$$\sin\left(\frac{\pi}{2} + \varepsilon\right) = 0$$

$$\varepsilon = -\frac{\pi}{2} + x\pi, x \in \mathbb{Z}$$

(*i.e.* ε can be an infinity of no.s, separated by π)

7

$$A = 2$$

$$\frac{2\pi}{n} = 6$$

$$n = \frac{\pi}{3}$$

$$t = 1,\ y = 1$$

$$1 = 2\sin\left(\frac{\pi}{3} + \varepsilon\right)$$

$$\sin\left(\frac{\pi}{3} + \varepsilon\right) = \frac{1}{2}$$

$$\frac{\pi}{3} + \varepsilon = \left(\frac{\pi}{6}, \frac{5\pi}{6}\right) + 2x\pi$$

$$\varepsilon = \left(\frac{-\pi}{6}, \frac{\pi}{2}\right) + 2x\pi,\ x \in \mathbb{Z}$$

8

$$A = \frac{6 - (-2)}{2}$$

$$A = 4$$

$$d = 6 - A = 6 - 4$$

$$d = 2$$

$$\frac{2\pi}{n} = 8$$

$$n = \frac{\pi}{4}$$

$$t = 2,\ y = 2$$

$$2 = 4\sin\left(\frac{\pi}{4} \times 2 + \varepsilon\right) + 2$$

$$\sin\left(\frac{\pi}{2} + \varepsilon\right) = 0$$

$$\frac{\pi}{2} + \varepsilon = 0 + x\pi$$

$$\varepsilon = -\frac{\pi}{2} + x\pi, x \in \mathbb{Z}$$

9

$$A = \frac{4 - 0}{2} = 2$$

$$d = 4 - A = 2$$

$$\frac{2\pi}{n} = 6$$

$$n = \frac{\pi}{3}$$

$$t = 1,\ y = 3$$

$$3 = 2\sin\left(\frac{\pi}{3} \times 1 + \varepsilon\right) + 2$$

$$\sin\left(\frac{\pi}{3} + \varepsilon\right) = \frac{1}{2}$$

$$\frac{\pi}{3} + \varepsilon = \left(\frac{\pi}{6}, \frac{7\pi}{6}\right) + 2x\pi$$

$$\varepsilon = \left(\frac{-\pi}{6}, \frac{5\pi}{6}\right) + 2x\pi, x \in \mathbb{Z}$$

note: for $Q1$, $Q2$, $Q5$, $Q6$, $Q7$ & $Q8$ A could take the negative of the value given if ε is changed to $\varepsilon + \pi$

Solutions to Exercise 6J

1 **a** $T = \frac{\pi}{3}$

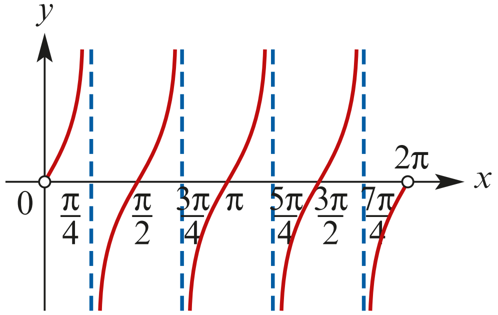
b $T = \frac{\pi}{\frac{1}{2}} = 2\pi$

c $T = \frac{\pi}{\frac{3}{2}} = \frac{2\pi}{3}$

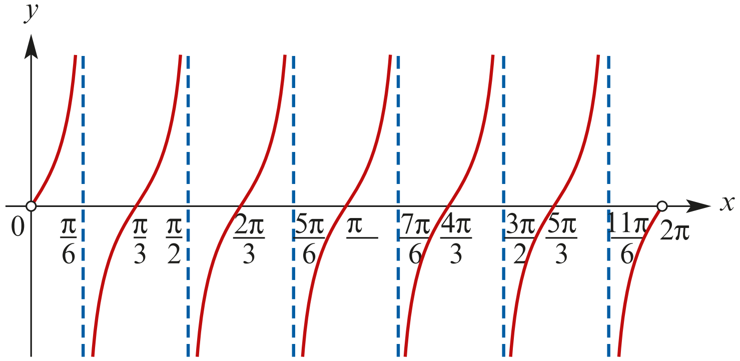
d $T = \frac{\pi}{\pi} = 1$

e $T = \frac{\pi}{\frac{\pi}{2}} = 2$

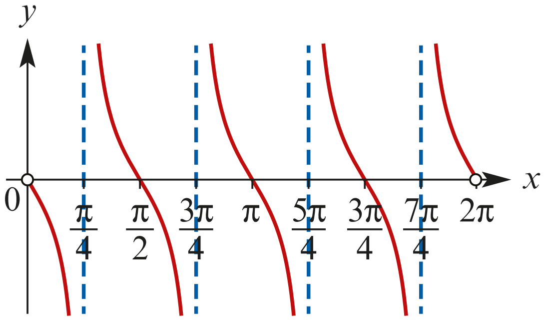
2 **a**



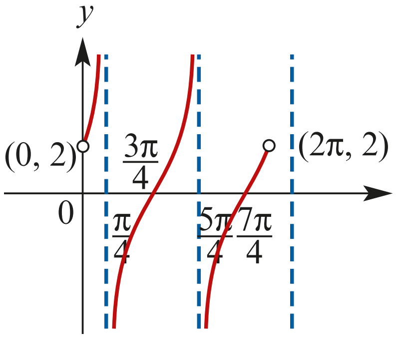
b



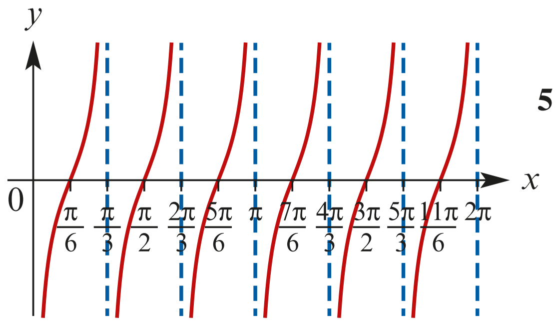
c



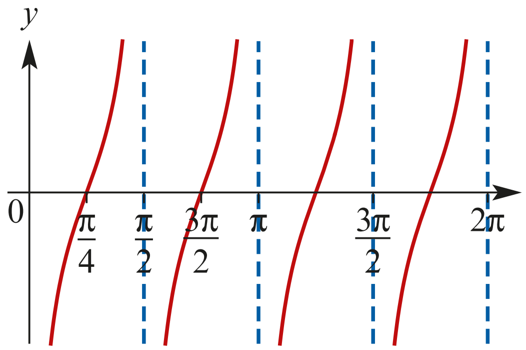
3 **a**



b



c



4 **a** $\tan(2x) = 1$

$$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

b $\tan(2x) = -1$

$$2x = -\frac{\pi}{4}, -\frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4},$$

$$x = -\frac{\pi}{8}, -\frac{5\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8},$$

c $\tan(2x) = -\sqrt{3}$

$$2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3},$$

$$x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6},$$

d $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

e $-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

5 $\tan(2(x - \frac{\pi}{3})) = 1$

$$(2(x - \frac{\pi}{3})) = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$x - \frac{\pi}{3} = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$$

$$x - \frac{8\pi}{24} = \frac{3\pi}{24}, \frac{15\pi}{24}, \frac{27\pi}{24}, \frac{39\pi}{44}$$

$$x = \frac{11\pi}{24}, \frac{23\pi}{24}, \frac{35\pi}{24}, \frac{47\pi}{24}$$

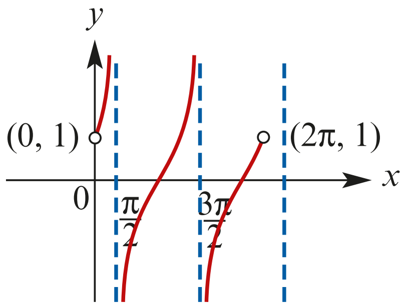
6 $\tan((x - \frac{\pi}{4})) = \sqrt{3}$

$$(x - \frac{\pi}{4}) = \frac{\pi}{3}, \frac{4\pi}{3}$$

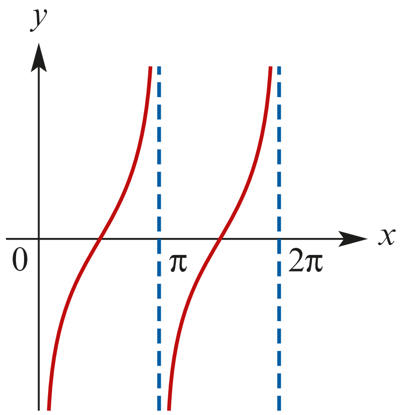
$$x - \frac{3\pi}{12} = \frac{4\pi}{12}, \frac{16\pi}{12}$$

$$x = \frac{7\pi}{12}, \frac{19\pi}{12}$$

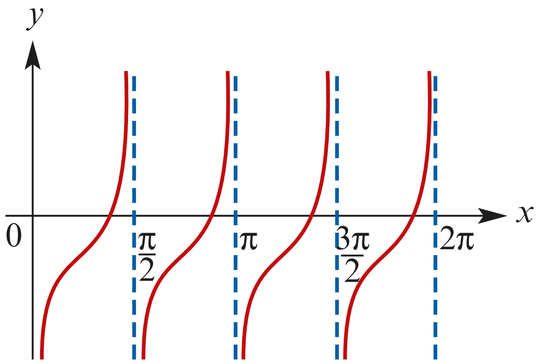
7 **a**



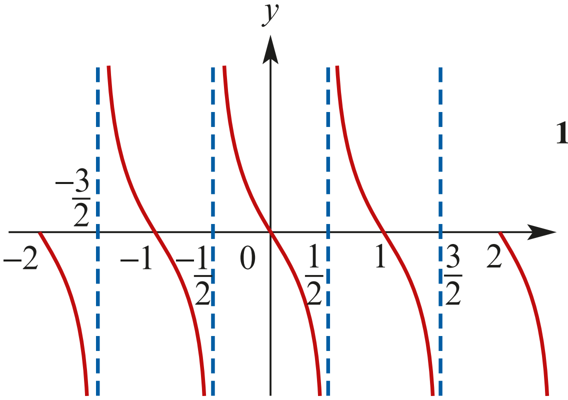
b



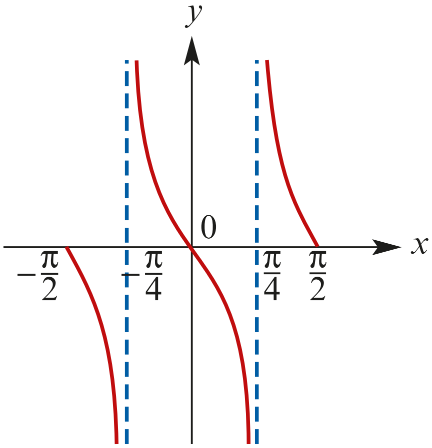
c



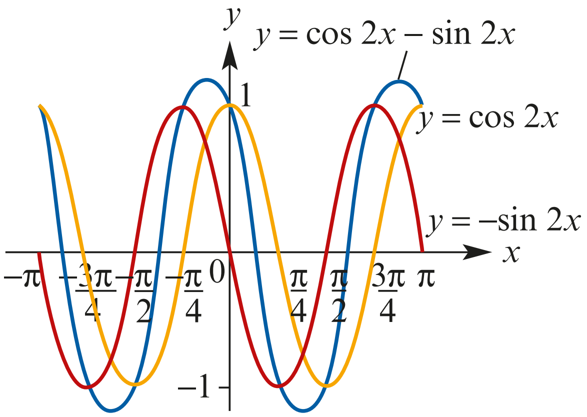
8



9



10 **a** **c**



b $\left(\frac{-5\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-\pi}{8}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{7\pi}{8}, \frac{1}{\sqrt{2}}\right)$

11 **a** $\sqrt{3}\sin x = \cos x$

$$\sqrt{3}\tan x = 1$$

$$\tan x = \frac{1}{\sqrt{3}}$$

$$x = \frac{\pi}{6}, \frac{7\pi}{6}$$

b $\sin(4x) = \cos(4x)$

$$\tan(4x) = 1$$

$$4x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}, \frac{25\pi}{4}, \frac{29\pi}{4}$$

$$x = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$$

c $\sqrt{3}\sin(2x) = \cos(2x)$

$$\tan(2x) = \frac{1}{\sqrt{3}}$$

$$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$$

d $\tan(2x) = \dfrac{-1}{\sqrt{3}}$

$$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$

$$x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

e

$\sin(3x) = -\cos(3x)$

$\tan(3x) = -1$

$$3x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

f $\tan x = \dfrac{1}{2}$

using the CAS calculator $x = 0.4636$, 3.6052

g $\tan x = 2$

using the CAS calculator x = 1.1071, 4.2487

h $\tan(2x) = -1$

$$2x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

$$x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$$

i $\sqrt{3}\sin(3x) = \cos(3x)$

$$\tan(3x) = \frac{1}{\sqrt{3}}$$

$$3x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \frac{25\pi}{6}, \frac{31\pi}{6}$$

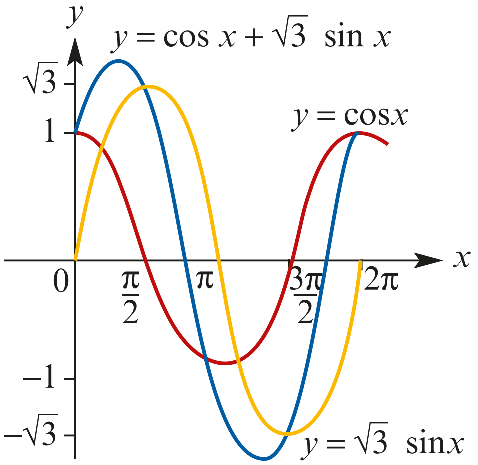
$$x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$$

j $\tan(3x) = \sqrt{3}$

$$3x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}, \frac{16\pi}{3}$$

$$x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$$

12a,c



b $\left(\dfrac{\pi}{6}, -\dfrac{\sqrt{3}}{2}\right), \left(\dfrac{7\pi}{6}, -\dfrac{\sqrt{3}}{2}\right)$

13 **a** $\tan\left(2x - \dfrac{\pi}{4}\right) = \sqrt{3}$

$$2x - \frac{\pi}{4} = \cdots \frac{-2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \cdots$$

$$2x = \cdots \frac{-5\pi}{12}, \frac{7\pi}{12}, \frac{19\pi}{12}, \frac{31\pi}{12}, \frac{43\pi}{12}, \cdots$$

$$x = \cdots \frac{-5\pi}{24}, \frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}, \cdots$$

but $0 \le x \le 2\pi$

$$\therefore \quad x = \frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}$$

b $\tan(2x) = \dfrac{-1}{\sqrt{3}}$

$$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$

$$x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

c $\tan\left(3x - \frac{\pi}{6}\right) = -1$

$$3x - \frac{\pi}{6} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$$

$$3x = \frac{11\pi}{12}, \frac{23\pi}{12}, \frac{35\pi}{12}, \frac{47\pi}{12}, \frac{59\pi}{12}, \frac{71\pi}{12}$$

$$x = \frac{11\pi}{36}, \frac{23\pi}{36}, \frac{35\pi}{36}, \frac{47\pi}{36}, \frac{59\pi}{36}, \frac{71\pi}{36}$$

14 asymptotes at $t = (2k + 1)\frac{\pi}{6}$

period $= T = \frac{\pi}{3}$

$T = \frac{\pi}{n},$

$\therefore \quad n = 3$

$t = \frac{\pi}{12},\ y = 5$

$5 = A\tan\left(\frac{\pi}{4}\right) = A \times 1$

$A = 5$

15 $T = \frac{\pi}{n}$

$T = 2$

$\therefore\ \mathrm{n} = \frac{\pi}{2}$

$t = \frac{1}{2},\ y = 6$

$6 = A\tan\left(\frac{\pi}{4}\right)$

$A = 6$

Solutions to Exercise 6K

1 a $\cos^{-1}(1) = 0$

i $2\pi \pm 0 = 2\pi$

ii $4\pi \pm 0 = 4\pi$

iii $-4\pi \pm 0 = -4\pi$

b $\cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$

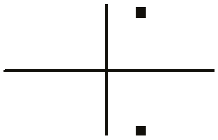
i $2\pi \pm \frac{2\pi}{3} = \frac{4\pi}{3}, \frac{8\pi}{3}$

ii $4\pi \pm \frac{2\pi}{3} = \frac{10\pi}{3}, \frac{14\pi}{3}$

iii $-4\pi \pm \frac{2\pi}{3} = \frac{-14\pi}{3}, \frac{-10\pi}{3}$

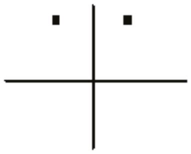
2 a $\cos x = \frac{\sqrt{3}}{2}$

$x = \left(\frac{\pi}{6}, \frac{-\pi}{6}\right) + 2n\pi \quad n \in \mathbb{Z}$



b $2\sin 3x = \sqrt{3}$

$\sin 3x = \frac{\sqrt{3}}{2}$

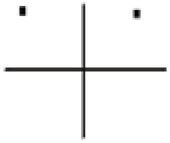


$3x = \left(\frac{\pi}{3}, \frac{2\pi}{3}\right) + 2n\pi \quad n \in \mathbb{Z}$

$x = \left(\frac{\pi}{9}, \frac{2\pi}{9}\right) + \frac{2n\pi}{3} \quad n \in \mathbb{Z}$

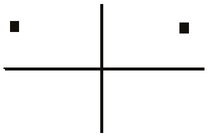
c $\tan x = \sqrt{3}$

$x = \frac{\pi}{3} + n\pi, \quad n \in \mathbb{Z}$



3 a $\sin x = \frac{1}{2}$

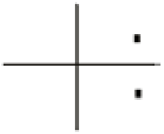
$x = \frac{\pi}{6}, \frac{5\pi}{6}$



b $\cos 2x = \frac{\sqrt{3}}{2}$

$2x = \frac{\pi}{6}, \frac{11\pi}{6}$

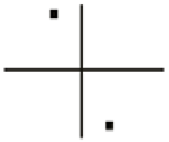
$x = \frac{\pi}{12}, \frac{11\pi}{12}$



c $\tan 2x = -\sqrt{3}$

$2x = \frac{2\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{3}, \frac{5\pi}{6}$



4 $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$

$= n\pi + (-1)^n\left(\frac{\pi}{6}\right)$

$n = -2, \ x = -2\pi + \frac{\pi}{6} = -\frac{11\pi}{6}$

$n = -1, \ x = -\pi - \frac{\pi}{6} = -\frac{7\pi}{6}$

$n = 0, \ x = \frac{\pi}{6}$

$n = 1, \ x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

. . .

$x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

5 $x = 2n\pi \pm (\pi = 2x\pi \pm)\cos^{-1}\left(\frac{1}{2}\right)$

$= 2n\pi \pm \frac{\pi}{3}$

$n = 0, 1: \ x = \frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

6 **a** $2\left(x + \frac{\pi}{3}\right) = \pm\frac{\pi}{3} + 2n\pi, \quad n \in \mathbb{Z}$

$x + \frac{\pi}{3} = n\pi \pm \frac{\pi}{6}$

$x = \left(\frac{-\pi}{2}, \frac{-\pi}{6}\right) + n\pi$

b $\tan\left(2\left(x + \frac{\pi}{4}\right)\right) = \sqrt{3}$

$2\left(x + \frac{\pi}{4}\right) = \frac{\pi}{3} + n\pi, n \in \mathbb{Z}$

$x + \frac{\pi}{4} = \frac{\pi}{6} + \frac{n\pi}{2}$

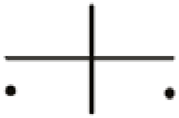
$x = \frac{-\pi}{12} + \frac{n\pi}{2}$

c $\sin\left(x + \frac{\pi}{3}\right) = \frac{-1}{2}$

$x + \frac{\pi}{3} = \frac{7\pi}{6}, \frac{11\pi}{6} + 2n\pi, n \in \mathbb{Z}$

$x = \left(\frac{5\pi}{6}, \frac{9\pi}{6}\right) + 2n\pi$

$= \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right) + 2x\pi$



7 $\cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$2x + \frac{\pi}{4} = \pm\frac{\pi}{4} + 2n\pi n \in \mathbb{Z}$

$2x = \left(0, -\frac{\pi}{2}\right) + 2n\pi$

$x = \left(0, -\frac{\pi}{4}\right) + n\pi$

$x = -2\pi + \frac{3\pi}{4}, -\pi, -\pi + \frac{3\pi}{4},$

$0, \frac{3\pi}{4}, \pi, \pi + \frac{3\pi}{4}$

$x = \frac{-5\pi}{4}, -\pi, \frac{-\pi}{4}, 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$

8 $\tan\left(\frac{\pi}{6} - 3x\right) = \frac{1}{\sqrt{3}}$

$\frac{\pi}{6} - 3x = \frac{\pi}{6} + n\pi \quad n \in \mathbb{Z}$

$3x - \frac{\pi}{6} = n\pi - \frac{\pi}{6} \quad n \in \mathbb{Z}$

The−vebecomes part of n

$3x = n\pi$

$x = \frac{n\pi}{3} \quad n \in \mathbb{Z}$

$x = -\pi, \frac{-2\pi}{3}, \frac{-\pi}{3}, 0$

9

$\sin(4\pi x) = \frac{-\sqrt{3}}{2}$

$4\pi x = \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right) + 2n\pi \quad n \in \mathbb{Z}$

$x = \left(\frac{1}{3}, \frac{5}{12}\right) + \frac{n}{2} \quad n \in \mathbb{Z}$

$x = -1 + \frac{1}{3}, -1 + \frac{5}{12}, \frac{-1}{2} + \frac{1}{3},$

$\frac{-1}{2} + \frac{5}{12}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2} + \frac{1}{3}, \frac{1}{2} + \frac{5}{12}$

$x = \frac{-2}{3}, \frac{-7}{12}, \frac{-1}{6}, \frac{-1}{12},$

$\frac{1}{3}, \frac{5}{12}, \frac{5}{6}, \frac{11}{12}$

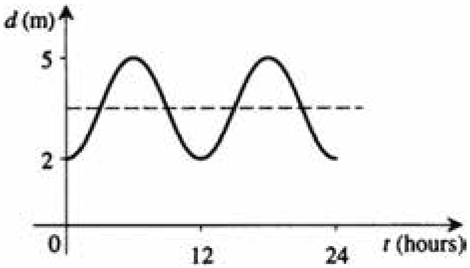
Solutions to Exercise 6L

1 **a** From the graph

i The range of the function is [2, 5] and the amplitude is 1.5.

ii The period is 12.

iii The function is of the form
$d = a\sin(nt + \varepsilon) + b$



The amplitude is 1.5.
Therefore $a = 1.5$
The period is 12.
Therefore $\frac{2p}{n} = 12$ and $n = \frac{\pi}{6}$
The centre of motion is at $d = 3\frac{1}{2}$.
Therefore $b = 3\frac{1}{2}$

$$\therefore \quad d = 1.5\sin\left(\frac{\pi t}{6} + \varepsilon\right) + 3.5$$

When $\quad t = 0, \ d = 2$

$$\therefore \quad 2 = 1.5\sin(\varepsilon) + 3.5$$

$$-1.5 = 1.5\sin(\varepsilon)$$

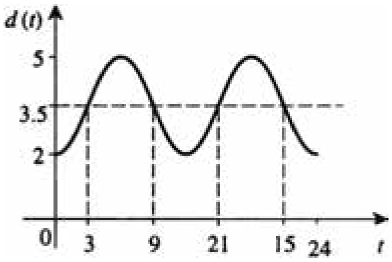
$\sin(\varepsilon) = -1$ and $\varepsilon = \frac{3\pi}{2}$

$$\therefore \quad d = 1.5\sin\left(\frac{\pi t}{6} + \frac{3\pi}{2}\right) + 3.5$$

But $\sin\left(\theta + \frac{3\pi}{2}\right) = \cos\theta$

$$\therefore \quad d = 3.5 - 1.5\cos\left(\frac{\pi t}{6}\right)$$

iv The length of the hour hand is 1.5 m. This is given by the amplitude.



b When is $d(t) \le 3.5$?
Consider $d(t) = 3.5$

$$3.5 = 3.5 - 1.5\cos\left(\frac{\pi t}{6}\right)$$

$$\therefore \quad \cos\left(\frac{\pi t}{6}\right) = 0$$

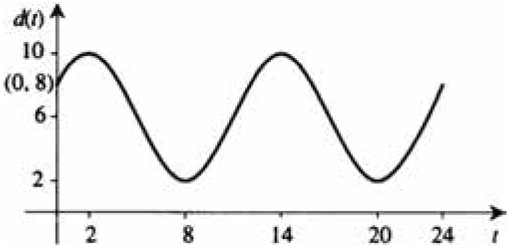
And $\frac{\pi t}{6} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $\frac{5\pi}{2}$ or $\frac{7\pi}{2}$ or $\cdots$

$\therefore \quad t = 3$ or 9 or 15 or 21 or $\cdots$

$\therefore \quad$ From the graph

$d(t) < 3.5$ for $t \in [0, 3) \cup (9, 15) \cup (21, 24]$

2 **a** $d(t) = 6 + 4\cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$



Centre: $d = 6$
Range: $[6 - 4, \ 6 + 4] = [2, 10]$
Period: $2\pi \div \frac{\pi}{6} = 12$

When $t = 0$, $d(0) = 6 + 4\cos\left(\frac{-\pi}{3}\right)$

$$= 6 + 4\frac{1}{2} = 8$$

When $t = 24$,
$d(24) = 6 + 4\cos\left(4\pi - \frac{\pi}{3}\right)$

$$= 6 + 4 \times \frac{1}{2} = 8$$

b Highest level is 10 m.
Consider $10 = 6 + 4\cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$

$$1 = \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$$

$\therefore \ \frac{\pi t}{6} = \frac{\pi}{3}$ (No need to consider other solution as question asks for earliest time.)

$\therefore t = 2$

The water is first at its highest at 2:00 a.m.

c When $d(t) = 2$

$$2 = 6 + 4\cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$$

$$-1 = \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$$

$$\therefore \ \frac{\pi t}{6} = \frac{\pi}{3} = \pi \text{ or } 3\pi \text{ or } 5\pi \text{ or } \cdots$$

$$\therefore \ \frac{\pi t}{6} = \frac{4\pi}{3} \text{ or } \frac{10\pi}{3} \text{ or } \frac{16\pi}{3} \text{ or } \cdots$$

$$\therefore t = 8 \text{ or } 20 \text{ or } 32 \text{ or } \cdots$$

Only 8 and 20 are in the required domain.

$\therefore$ The water is 2 m up the wall at 8:00 a.m. and 8:00 p.m.

3 a The time between high tides is 12 hours, so the period = 12

$\frac{2\pi}{n} = 12 \Rightarrow n = \frac{\pi}{6}$

The average depth is 5 metres. Therefore $b = 5$.

The high tide is 8 m. Therefore amplitude = 8 − 5 = 3 and $A = 3$

$$\therefore \ h(t) = 3\sin\left(\frac{\pi t}{6} + \varepsilon\right) + 5$$

When $t = 0$, $h = 8$ (t is the number of hours after 12:00 noon.)

$$\therefore \ 8 = 3\sin(\varepsilon) + 5$$

$$\therefore \ \sin(\epsilon) = 1 \text{ and } \varepsilon = \frac{\pi}{2}$$

$$\therefore \ h(t) = 3\sin\left(\frac{\pi t}{6} + \frac{\pi}{2}\right) + 5$$

$$= 3\cos\left(\frac{\pi t}{6}\right) + 5$$

b When $h = 6$

$$1 = 3\cos\left(\frac{\pi t}{6}\right)$$

$$\frac{1}{3} = \cos\left(\frac{\pi t}{6}\right)$$

$$\therefore \ \frac{\pi t}{6} = \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 2\pi - \cos^{-1}$$

$$\times\left(\frac{1}{3}\right) \text{ or } 2\pi + \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 4\pi - \cos^{-1}\left(\frac{1}{3}\right)$$

$$\therefore \ t = \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right) \text{ or } 12 - \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right) \text{ or } 12$$

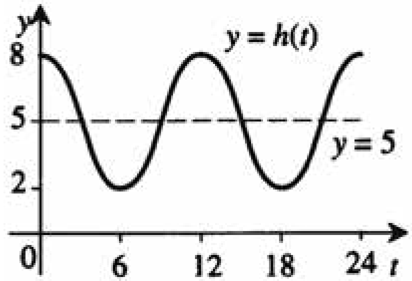
$$+ \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right) \text{ or } 24 - \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right)$$

$$\approx 2.351 \text{ or } 9.649 \text{ or } 14.351 \text{ or } 21.649$$

Depth of the water is 6 metres at the following times (times measured from 12 noon).

2:21 p.m. 9:39 p.m. 2:21 a.m. 9:39 a.m.

c



4 a Greatest distance occurs when $\sin 3t = 1$

$\therefore$ greatest distance = 3 + 2 = 5m.

b Least distance occurs when $\sin 3t = -1$

$\therefore$ least distance = 3 − 2 = 1m.

c When $x = 5$

$$5 = 3 + 2\sin 3t$$

$$2 = 2\sin 3t$$

$$1 = \sin 3t$$

$$3t = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \cdots$$

$$\therefore \quad t = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{9\pi}{6} \text{ or } \cdots$$

For $0 \le t \le 5$ the times are: 0.524 sec, 2.618 sec, 4.712 sec

d When $x = 3$

$$3 = 3 + 2\sin 3t$$

$$0 = \sin 3t$$

$$3t = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } \cdots$$

$$t = 0 \text{ or } \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ or } \pi \text{ or } \cdots$$

For $0 \le t \le 3$ the times are: 0 sec, 1.047 sec, 2.094 sec

e The particle oscillates about the point $x = 3$ from $x = 1$ to $x = 5$.

5 $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$ or $0 \le t \le 24$ gives the temperature inside the house and $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$ for $0 \le t \le 24$ gives the temperature outside the house.

a When $t = 4$ (time measured from 4:00 a.m.)

$$A = 21 - 3\cos\frac{4\pi}{12}$$

$$= 21 - 3\cos\frac{\pi}{3}$$

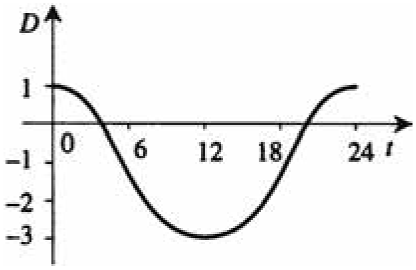
$$= 21 - 1.5$$

$$= 19.5$$

i.e. the temperature outside the house is 19.5°C at 8:00 a.m.

b $D = A - B = 2\cos\left(\frac{\pi t}{12}\right) - 1$

c



d The inside temperature is less than the outside temperature.

This occurs when

$A < B$

$\Leftrightarrow A - B < 0$

$\Leftrightarrow D < 0$

Consider $\quad D = 0$

$$0 = 2\cos\left(\frac{\pi t}{12}\right) - 1$$

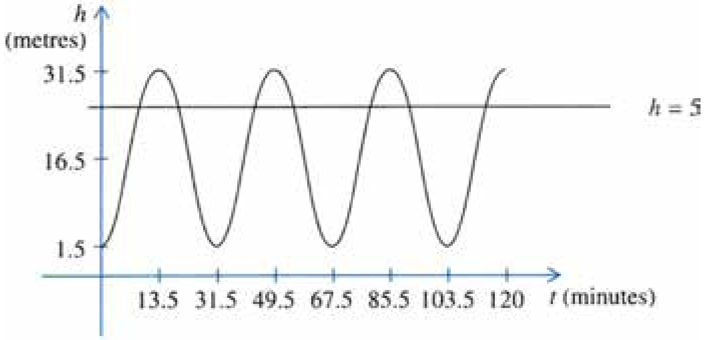
Implies $\quad \frac{1}{2} = \cos\left(\frac{\pi t}{12}\right)$

$$\therefore \quad \frac{\pi t}{12} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ or } \cdots$$

$$\therefore \quad t = 4 \text{ or } 20 \text{ or } \cdots$$

For $0 \le t \le 24$, $D < 0$ for $t \in (4, 20)$, i.e. $4 < t < 20$.

6 a



b When $t = 0$, $h = 15\sin(-45)^\circ + 16.5$ $= 5.89$ m (correct to two decimal places)

c Solving the equation $h(t) = 5$

$$\begin{aligned} 5 &= 15\sin(10t - 45)^\circ + 16.5 \\ -11.5 &= 15\sin(10t - 45)^\circ \\ -\frac{23}{30} &= \sin(10t - 45)^\circ \end{aligned}$$

The first positive solution is $t = 27.51$ seconds correct to two decimal places.

d There are 6 points of intersection with the graph of $h = 5$

e 20 times

f

$$\begin{aligned} t = 100,\ h(100) &= 15\sin(100 - 45) + 16.5 \\ &= 15\sin 955^\circ + 16.5 \\ &\approx 4.21 \text{ metres} \end{aligned}$$

g The phase shift will be different for Hamish; the range and the period will be the same. Consider $k(t) = 15\sin(10t + c)^\circ + 16.5$

$$\begin{aligned} \text{When } t = 0,\ k(0) &= 1.5 \\ 15\sin(+c)^\circ + 16.5 &= 1.5 \\ \sin c^\circ &= -1 \\ c^\circ &= 270^\circ \\ k(t) &= 15\sin(10t + 270)^\circ + 16.5 \\ t &= 100, \\ k(100) &= 15\sin(1270^\circ) + 16.5 \\ &\approx 13.9 \text{ metres} \end{aligned}$$

Solutions to technology-free questions

1 Note that $x \in [-\pi,\ 2\pi]$ throughout.

a $\sin x = \dfrac{1}{2}$

x is in the first or second quadrant.

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

b $2\cos x = -1$, so $\cos x = -\dfrac{1}{2}$

x is in the second or third quadrant.

$x = -\dfrac{2\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

c $2\cos x = \sqrt{3}$, so $\cos x = \dfrac{\sqrt{3}}{2}$

x is in the first or fourth quadrant.

$x = -\dfrac{\pi}{6}, \dfrac{\pi}{6}, \dfrac{11\pi}{6}$

d $\sqrt{2}\sin x + 1 = 0$, so $\sin x = -\dfrac{1}{\sqrt{2}}$

x is in the third and fourth quadrants.

$x = -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

e $4\sin x + 2 = 0$, so $\sin x = -\dfrac{1}{2}$

x is in the third or fourth quadrant.

$x = -\dfrac{5\pi}{6}, -\dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

f $\sin 2x + 1 = 0$, so $\sin 2x = -1$

$$2x = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$$

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

g $\cos 2x = -\dfrac{1}{\sqrt{2}}$

$2x$ is in the second or third quadrant.

$$2x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$$

$$x = -\frac{5\pi}{8}, -\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$$

h $2\sin 3x - 1 = 0$, so $\sin 3x = \dfrac{1}{2}$

3x is in the first or second quadrant.

$$3x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

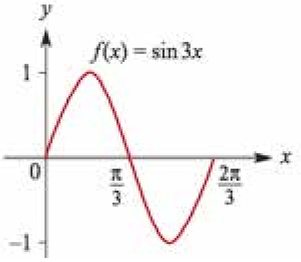
$$\frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$$

$$x = -\frac{11\pi}{18}, -\frac{7\pi}{18}, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}$$

$$\frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$$

2 **a** $y = \sin 3x$

$y = 0: x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ for one cycle



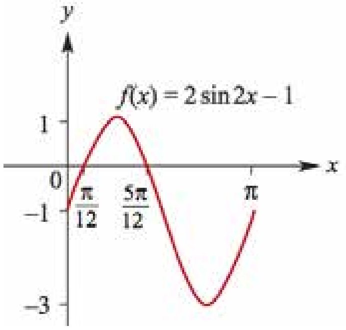
b $y = 2\sin 2x - 1$

$y = 0: \sin 2x = \frac{1}{2}$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}$ for one cycle

$x = \frac{\pi}{12}, \frac{5\pi}{12}$

$x = 0: y = -1$



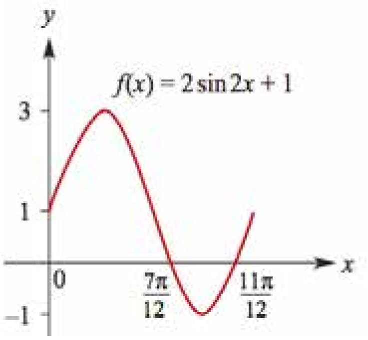
c $y = 2\sin 2x + 1$

$y = 0$: $\sin 2x = -\frac{1}{2}$

$2x = \frac{7\pi}{6}, \frac{11\pi}{6}$ for one cycle

$x = \frac{7\pi}{12}, \frac{11\pi}{12}$

$x = 0$: $y = 1$

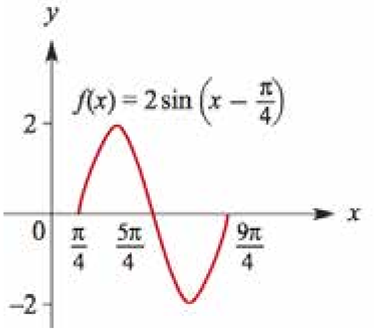


d $y = 2\sin\left(x - \frac{\pi}{4}\right)$ so translate the graph of $y = 2\sin x$ by $\frac{\pi}{4}$ to the right.

$y = 0$: $\sin\left(x - \frac{\pi}{4}\right) = 0$

$x - \frac{\pi}{4} = 0, \pi, 2\pi$ for one cycle

$x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

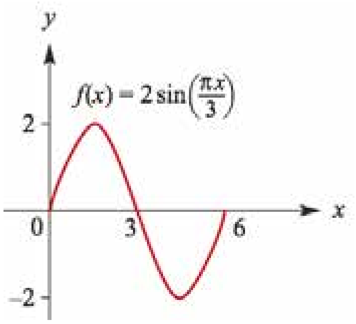


e $y = 2\sin\frac{\pi x}{3}$

$y = 0$: $\sin\frac{\pi x}{3} = 0$

$\frac{\pi x}{3} = 0, \pi, 2\pi$ for one cycle

$x = 0, 3, 6$



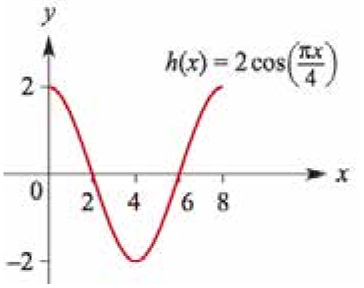
f $y = 2\cos\frac{\pi x}{4}$

$y = 0\colon \cos\frac{\pi x}{4} = 0$

$\frac{\pi x}{4} = \frac{\pi}{2}, \frac{3\pi}{2}$ for one cycle

$x = 2, 6$

$x = 0\colon y = 2$



3 Note that $x \in [0, 360]$ throughout.

a $\sin x^\circ = 0.5$

x° is in the first or second quadrant.

$x = 30, 150$

b $\cos(2x)^\circ = 0$

$2x = 90, 270, 450, 630$

$x = 45, 135, 225, 315$

c $2\sin x^\circ = -\sqrt{3}$, so $\sin x^\circ = -\frac{\sqrt{3}}{2}$

x° is in the third or fourth quadrant.

$x = 240, 300$

d $\sin(2x + 60)^\circ = -\frac{\sqrt{3}}{2}$

$(2x + 60)^\circ$ is in the third or fourth quadrant.

$$2x + 60 = 240, 300, 600, 660$$
$$2x = 180, 240, 540, 600$$
$$x = 90, 120, 270, 300$$

e $2\sin\left(\frac{1}{2}x\right)^\circ = \sqrt{3}$, so $\sin\left(\frac{1}{2}x\right)^\circ = \frac{\sqrt{3}}{2}$

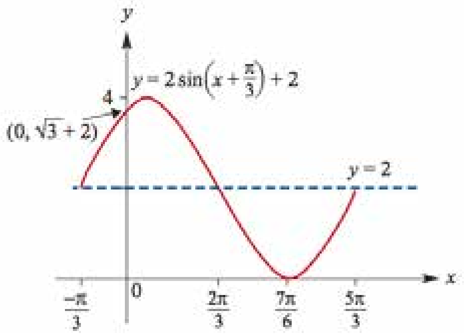
$\left(\frac{1}{2}x\right)^\circ$ is in the first or second quadrant.

$$\frac{1}{2}x = 60, 120$$
$$x = 120, 240$$

4 **a** $y = 2\sin\left(x + \frac{\pi}{3}\right) + 2$

The graph is that of $y = 2\sin x$ translated $\frac{\pi}{3}$ units to the left and 2 units up.

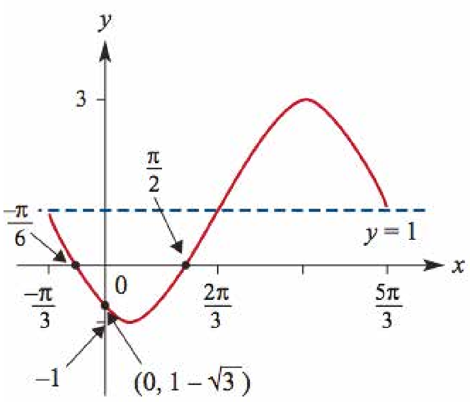
$$y = 0: \sin\left(x + \frac{\pi}{3}\right) = -1$$
$$x + \frac{\pi}{3} = \frac{3\pi}{2} \text{ for one cycle}$$
$$x = \frac{7\pi}{6}$$
$$x = 0: y = \sqrt{3} + 2$$



b $y = -2\sin\left(x + \frac{\pi}{3}\right) + 1$

The graph is that of $y = -2\sin x$ translated $\frac{\pi}{3}$ units to the left and 1 unit up.

$$y = 0: \sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$
$$x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6} \text{ for one cycle}$$
$$x = -\frac{\pi}{6}, \frac{\pi}{2}$$
$$x = 0: y = 1 - \sqrt{3}$$



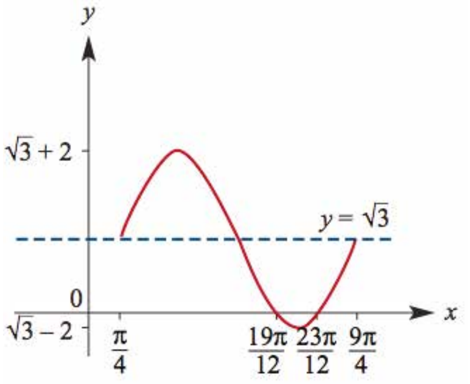
c $y = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}$

The graph is that of $y = 2\sin x$ translated $\frac{\pi}{4}$ units to the right and $\sqrt{3}$ units up.

$$y = 0: \sin\left(x - \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$$

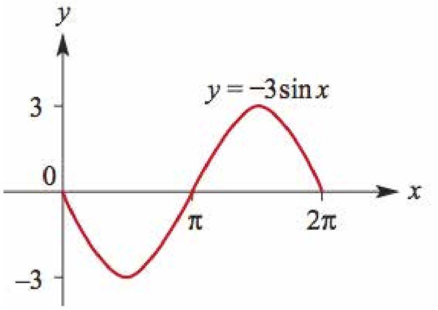
$$x - \frac{\pi}{4} = \frac{4\pi}{3}, \frac{5\pi}{3} \text{ for one cycle}$$

$$x = \frac{19\pi}{12}, \frac{23\pi}{12}$$



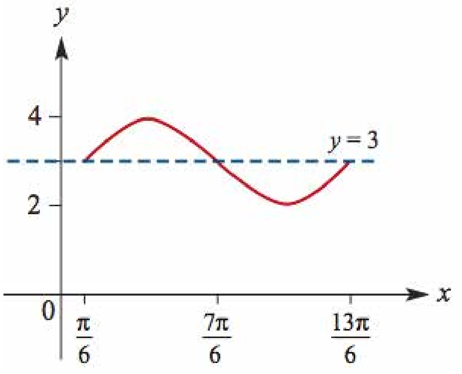
d $y = -3\sin x$

$y = 0: x = 0, \pi, 2\pi$ for one cycle



e $y = \sin\left(x - \frac{\pi}{6}\right) + 3$

The graph is that of $y = \sin x$ translated $\frac{\pi}{6}$ units to the right and 3 units up, so there are no x intercepts.



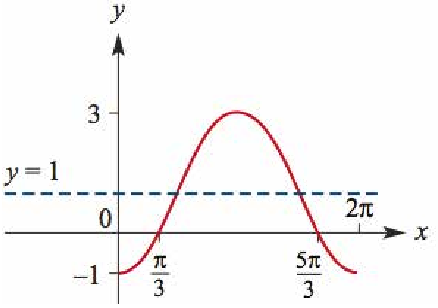
f $y = 2\sin\left(x - \frac{\pi}{2}\right) + 1$

Now $\sin\left(x - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - x\right) = -\cos x$, so $y = -2\cos x + 1$ is an equivalent form.
The graph is that of $y = -2\cos x$ translated 1 unit up.

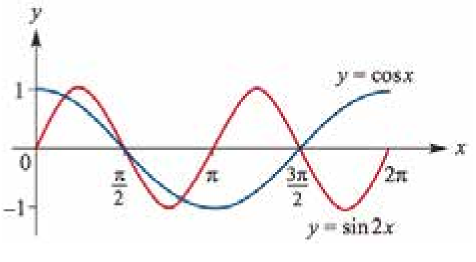
$y = 0$: $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ for one cycle

$x = 0$: $y = -1$



5 The graphs are shown below.



a The line with equation $y = 0.6$ cuts the curve with equation $y = \sin 2x$ four times. The equation has 4 solutions.

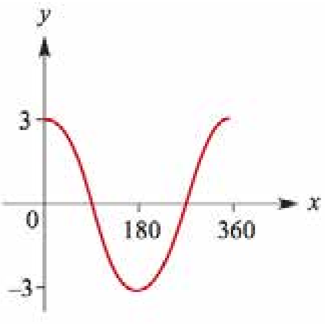
b The curve with equation $y = \sin 2x$ cuts the curve with equation $y = \cos x$ four times. The equation has 4 solutions.

c Rewrite the equation in the form:

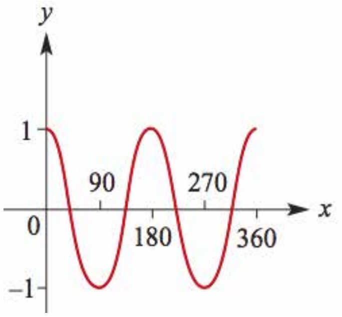
$\sin 2x - 1 = \cos x$

The curve with equation $y = \sin 2x - 1$ is that of $y = \sin 2x$ translated 1 unit down. Looking at the graphs above, it is clear that translating the sine graph 1 unit down means that two intersections with the cosine graph are lost and only two remain. The equation has 2 solutions.

6 **a** $y = 3\cos x°$



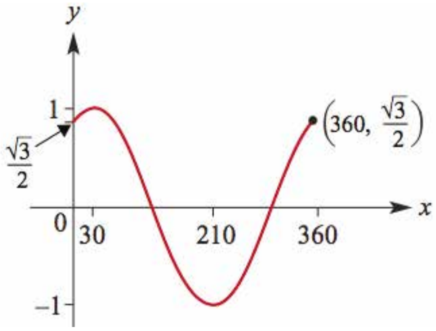
b $y = \cos 2x°$



c $y = \cos(x - 30)°$

The graph is that of $y = \cos x°$ translated 30° to the right.

$x = 0, 360 : y = \frac{\sqrt{3}}{2}$



7 Note that $x \in [-\pi, \pi]$ throughout.

a $\tan x = \sqrt{3}$

x is in the first or third quadrant.

$X = -\frac{2\pi}{3}, \frac{\pi}{3}$

b $\tan x = -1$

x is in the second or fourth quadrant.

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$

c $\tan 2x = -1$

$2x$ is in the second or fourth quadrant.

$$2x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$x = -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$$

d $\tan(2x) + \sqrt{3} = 0$, so $\tan 2x = -\sqrt{3}$

$2x$ is in the second or fourth quadrant.

$$2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$$

$$x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$$

8 $\tan(x) = \sqrt{3}$

$$x = -\frac{2\pi}{3}, \frac{\pi}{3}$$

9 **a** $a \cos \frac{\pi}{6} = \sin \frac{\pi}{6}$

$$\frac{\sqrt{3}a}{2} = \frac{1}{2}$$

$$a = \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

b $\tan x = \frac{\sqrt{3}}{3}$

$x = \frac{\pi}{6}, -\frac{5\pi}{6}$

10 **a** $\sin 2x = -1$

$$2x = -\frac{\pi}{2} + 2n\pi$$

$$x = -\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$$

b $\cos 2x = 1$

$$3x = 2n\pi$$

$$x = -\frac{2n\pi}{3}, n \in \mathbb{Z}$$

c $\tan x = -1$

$$x = n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$$

Solutions to multiple-choice questions

1 C $3\sin\left(\frac{1}{2}x - \pi\right) + 4$

$$\text{Period} = \frac{2\pi}{n},\ n = \frac{1}{2}$$

$$= \frac{2\pi}{\frac{1}{2}}$$

Period $= 4\pi$

2 A $f(x) = 5\cos\left(2x - \frac{\pi}{3}\right) - 7$

Range $= [-7 + 5,\ -7 - 5]$

Range $= [-2,\ -12]$

3 E $y = \sin x$

A dilation of factor $\frac{1}{2}$ from the y-axis:

$y = \sin(2x)$

A translation of $\frac{\pi}{4}$ in the positive direction of the x axis:

$y = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

4 D $f : R \to R, f(x) = a\sin(bx) + c$

Period $= \frac{2\pi}{b}$

5 A $3\sin(x) - 1 = b$

It is only possible for the equation to have one positive real number solution at the turning point:

Max value $= 2$

6 C

7 C $f(x) = p\cos 5x + q,\ p > 0$

$f(x) \leq 0$

$0 \geq p\cos 5x + q$

Maximum y value must be negative, this value occurs at $x = 0$

$\therefore\ 0 \geq p\cos 0 + q$

$\therefore\ p \leq -q$

8 B One rotation = period

$\frac{2\pi}{6\pi} = \frac{1}{3}$

9 C

10 E $y = \cos x$

A dilation of factor 2 from the x-axis:

$y = 2\cos x$

A translation of $\frac{\pi}{4}$ in the positive direction of the x-axis:

$y = 2\cos\left(x - \frac{\pi}{4}\right)$

11 C

12 B Period of graph shown:

$\frac{2\pi}{n} = 8$

$n = \frac{\pi}{4}$

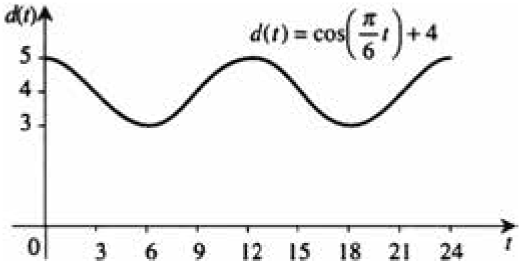
Graph is translated 3 units in the positive direction of the y-axis. As the graph is initially positive it must be a sine function.

$\therefore\ y = 3 + 3\sin\left(\frac{\pi x}{4}\right)$

Solutions to extended-response questions

1 The time between high tide and low tide is 6 hours.
Assume the function modelling the river is sinusoidal.
i.e. $d(t) = a\sin(nt + \varepsilon) + b$
where $d(t)$ is the depth at time t (measured from 0)
Period $= 12 \therefore \dfrac{2\pi}{n} = 12$ and $n = \dfrac{\pi}{6}$
Average depth $= 4$ m i.e. $d = 4$ is the centre $\therefore\ b = 4$
Highest value $= 5 \therefore$ amplitude $= 5 - 4 = 1$
Range $= [3, 5]$ and $a = 1$

a $d(t) = \sin\left(\dfrac{\pi t}{6} + \varepsilon\right) + 4$



Also there is a high tide at 12:00.

$\therefore$ When $t = 12, d = 5$

$$\therefore \quad 5 = \sin(2\pi + \varepsilon) + 4$$

i.e. $\sin(\varepsilon) = 1$

and $\therefore$ $\varepsilon = \dfrac{\pi}{2}$

$$\therefore \quad d(t) = \sin\left(\frac{\pi t}{6} + \frac{\pi}{2}\right) + 4$$

But $\sin\left(\theta + \dfrac{\pi}{2}\right) = \cos\theta$

$$\therefore \quad d(t) = \cos\left(\frac{\pi t}{6}\right) + 4$$

b For $d(t) \geq 4$ consider first $d = 4$

$$4 = \cos\left(\frac{\pi t}{6}\right) + 4$$

$$\cos\left(\frac{\pi t}{6}\right) = 0$$

$$\therefore \quad \frac{\pi t}{6} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \cdots$$

$\therefore \quad t = 3$ or 9 or 15 or $\cdots$

$\therefore \quad d \geq 4$ for $t \in [0, 3] \cup [9, 15] \cup \ldots$

The boat may enter the harbour after 9:00 a.m. but it must leave by 3:00 p.m.

c For $d \geq 3.5$

First consider $d = 3.5$

$$3.5 = \cos\left(\frac{\pi t}{6}\right) + 4$$

$$\therefore \quad -0.5 = \cos\left(\frac{\pi t}{6}\right)$$

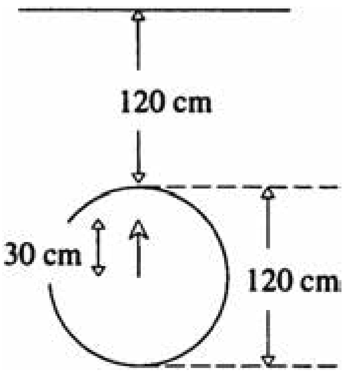
$$\therefore \quad \frac{\pi t}{6} = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } \frac{8\pi}{3} \text{ or } \cdots$$

$$\therefore \quad t = 4 \text{ or } 8 \text{ or } 16 \text{ or } \cdots$$

$\therefore$ $d \geq 3.5$ for $t \in [0, 4] \cup [8, 16] \cup \ldots$ (See graph above)

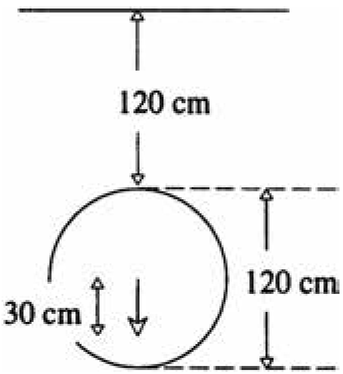
A boat can enter the river after 8:00 a.m. but must leave before 4:00 p.m.

2 a



The minimum distance $= 120 + 30$

$= 150$ cm



The maximum distance $= 120 + 60 + 30$

$= 210$ cm

The mean distance = 180 cm

b $y = A\sin(nt + \varepsilon) + b$

From the above:

Mean distance is 180 cm $\therefore b = 180$

Range $= [150, 210]$

$\therefore$ amplitude $= 30, A = 30$

Period $= 12$ $\therefore \frac{2\pi}{n} = 12$

i.e. $n = \frac{\pi}{6}$

$$\therefore \quad y = 30\sin\left(\frac{\pi t}{6} + \varepsilon\right) + 180$$

When $t = 0$, distance is minimum

$$\therefore \quad y = 150$$

$$\therefore \quad 150 = 30\sin(\varepsilon) + 180$$

$$\therefore \quad \sin(\varepsilon) = -1$$

$$\varepsilon = -\frac{\pi}{2}$$

$$\therefore \quad y = 30\sin\left(\frac{\pi t}{6} - \frac{\pi}{2}\right) + 180$$

$$= 180 - 30\cos\left(\frac{\pi t}{6}\right) \text{ Since } \sin\left(\theta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -\cos\theta.$$

c **i** When $t = 2$

$$y = 180 - 30\cos\left(\frac{\pi \times 2}{6}\right)$$

$$= 180 - 30\cos\left(\frac{\pi}{3}\right)$$

$$= 180 - 15$$

$$= 165$$

The distance from the ceiling to the tip of the hour hand is 165 cm at 2:00.

ii When $t = 23$

$$y = 180 - 30\cos\left(\frac{\pi \times 23}{6}\right)$$

$$= 180 - 30\cos\left(\frac{-\pi}{6}\right)$$

$$= 180 - 30 \times \frac{\sqrt{3}}{2}$$

$$= 180 - 15\sqrt{3}$$

$$\approx 154\,\text{cm}$$

The distance from the ceiling to the top of the hour hand is approx. 154 cm at 23:00.

d When $y = 200$

$$200 = 180 - 30\cos\left(\frac{\pi t}{6}\right)$$

$$\frac{-2}{3} = \cos\left(\frac{\pi t}{6}\right)$$

$$\therefore \quad \frac{\pi t}{6} = \pi - \cos^{-1}\left(\frac{2}{3}\right) \text{ or } \pi + \cos^{-1}\left(\frac{2}{3}\right) \text{ or } \cdots$$

$$\therefore \quad t = 6 - \frac{6}{\pi}\cos^{-1}\left(\frac{2}{3}\right) \text{ or } 6 + \frac{6}{\pi}\cos^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \quad t \approx 4.39 \text{ or } 7.61$$

The tip of the hour hand is 200 cm below the ceiling at 7:36 and 4:24.

3 **a** Amplitude = 3 When $t = 0$, $y = -3$ and therefore $a = -3$

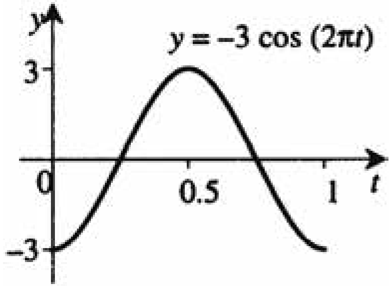
Period = 1 $\therefore \frac{2\pi}{n} = 1$

$$\therefore \quad n = 2\pi$$

$$\therefore \quad a = -3 \text{ and } n = 2\pi$$

$$y = -3\cos(2\pi t)$$

b



c **i** When $y = 1.5$

$$\begin{aligned} 1.5 &= -3\cos(2\pi t) \\ -\frac{1}{2} &= \cos(2\pi t) \\ \frac{2\pi}{3} &= 2\pi t \\ \therefore \quad t &= \frac{1}{3} \end{aligned}$$

The centre of the weight is 1.5 cm above 0 after $\frac{1}{3}$ second.

ii When $y = -1.5$

$$\begin{aligned} -1.5 &= -3\cos(2\pi t) \\ \frac{1}{2} &= \cos(2\pi t) \\ \frac{\pi}{3} &= 2\pi t \\ \frac{1}{6} &= t \end{aligned}$$

The centre of the weight is 1.5 cm below 0 after $\frac{1}{6}$ second.

d When $y = -1$

$$\begin{aligned} -1 &= -3\cos(2\pi t) \\ \frac{1}{3} &= \cos(2\pi t) \\ \cos^{-1}\left(\frac{1}{3}\right) &= 2\pi t \\ \therefore t &= \frac{1}{2\pi}\cos^{-1}\left(\frac{1}{3}\right) \\ &\approx 0.196 \end{aligned}$$

It reaches a point 1 cm below 0 after 0.196 seconds.

4 **a** $y = a\sin(nt + \varepsilon) + b$

The average inflow is $100\,000\text{ m}^3/\text{day}$ $\therefore$ $b = 100\,000$

Minimum flow is 80 000 and maximum 120 000.

$\therefore$ range = [80 000, 120 000]

Amplitude is 20 000 $\therefore$ $a = 20\,000$

The period is 365 days $\therefore$ $\dfrac{2\pi}{n} = 365$

and therefore $n = \dfrac{2\pi}{365}$

$$\therefore\ y = 20\,000\sin\left(\frac{2\pi t}{365} + \epsilon\right) + 100\,000$$

When $t = 121, y = 120\,000$

$$\therefore\ 120\,000 = 20\,000\sin\left(\frac{2\pi \times 121}{365} + \varepsilon\right) + 100\,000$$

$$\therefore\ \sin\left(\frac{2\pi \times 121}{365} + \varepsilon\right) = 1$$

$$\frac{2\pi \times 121}{365} + \varepsilon = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \cdots$$

$$\therefore\ \varepsilon = \frac{\pi}{2} - \frac{2\pi \times 121}{365} \text{ or } \frac{5\pi}{2} - \frac{2\pi \times 121}{365} \text{ or } \frac{9\pi}{2} - \frac{2\pi \times 121}{365} \text{ or } \ldots$$

$$\approx -0.512 \text{ or } 5.77 \text{ or } \ldots$$

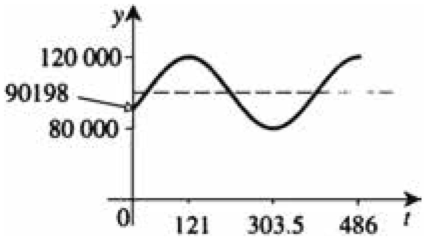
Choose $\varepsilon = 5.77$

$$\therefore\ y = 20\,000\sin\left(\frac{2\pi t}{365} + 5.77\right) + 100\,000$$

b When $t = 0$,

$$y = 20\,000\sin(5.77) + 100\,000$$

$$\approx 90\,198.33$$



c **i** When $y = 90\,000$

$$90\,000 = 20\,000\sin\left(\frac{2\pi t}{365} + 5.77\right) + 100\,000$$

$$-\frac{1}{2} = \sin\left(\frac{2\pi t}{365} + 5.77\right)$$

$$\therefore\ \frac{2\pi t}{365} + 5.77 = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{23\pi}{6} \text{ or } \cdots$$

$$\therefore\ \frac{2\pi t}{365} = \frac{7\pi}{6} - 5.77 \text{ or } \frac{11\pi}{6} - 5.77 \text{ or } \frac{19\pi}{6} - 5.77 \text{ or } \frac{23\pi}{6} - 5.77 \text{ or } \cdots$$

$$\frac{2\pi t}{365} \approx -2.1058 \text{ or } -0.01141 \text{ or } 4.1773 \text{ or } 6.2717$$

(Negative values are not considered.)

$\therefore$ $t \approx 242.7$ or $t \approx 364.3$

i.e. when $t = 242.7$ and $t = 364.3$ the inflow per day is $90\,000\text{ m}^3/\text{day}$.

ii When $y = 110\,000$

$$110\,000 = 20\,000\sin\left(\frac{2\pi t}{365} + 5.77\right) + 100\,000$$

$$\frac{1}{2} = \sin\left(\frac{2\pi t}{365} + 5.77\right)$$

$$\frac{2\pi t}{365} + 5.77 = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6}$$

$$t = 60.2 \text{ or } t = 181.8$$

(Negative values not considered.)

i.e. when $t = 60.2$ and $t = 181.8$ the inflow is 110 000 m^3/day.

d When $t = 152$

$$y = 20\,000\sin\left(\frac{2\pi \times 152}{365} + 5.77\right) + 100\,000$$

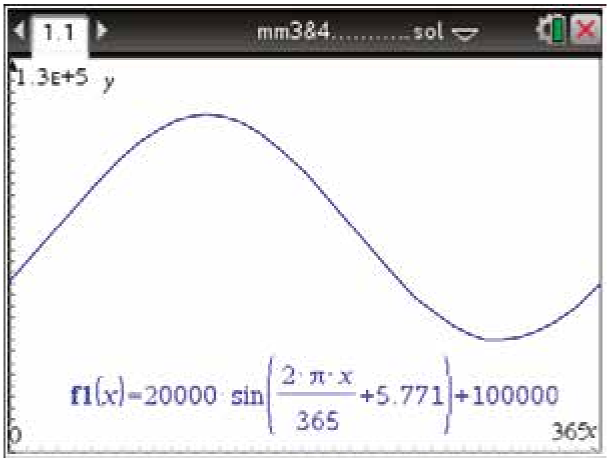
$$= 117219$$

The inflow rate is 117 219 m^3/day on 1 June.

Graphic calculator techniques for question 7

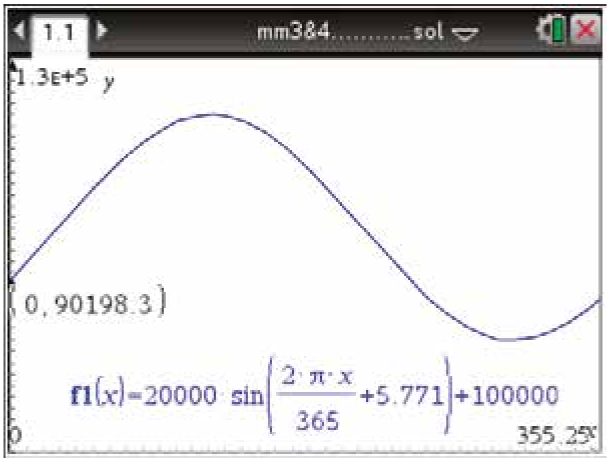
In a **Graphs** page enter the rule (note that *x* must be used here instead of *t*) in the **function entry line.**

Because of the magnitude of the numbers in this problem it is useful to increase the number of display digits using b>**Settings** and change the **Display Digits** to **Auto** Set the **WINDOW** (b**>Window/Zoom>Window Settings)** at Xmin = 0, Xmax = 365; Ymin= 60000, Ymax = 130000. The graph appears as shown.

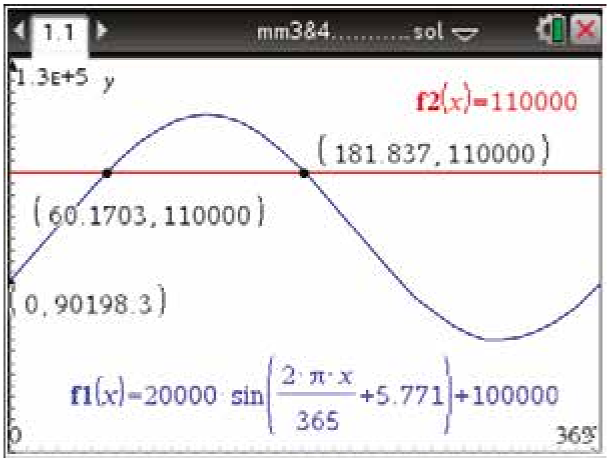


$f1(x)=20000\cdot\sin\left(\frac{2\cdot\pi\cdot x}{365}+5.771\right)+100000$

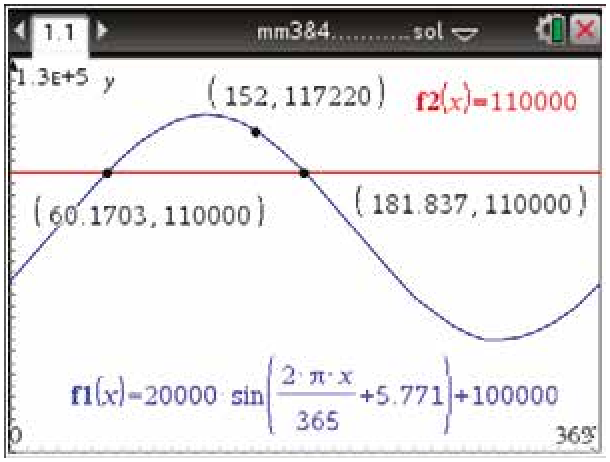
The value of Y when $x = 0$ can be found several ways. Use b**>Geometry>Points & Lines>Point On** to place a point on the graph. Press d to exit the **Point On** tool. By double clicking on the x-coordinate you can edit this to 0.
Alternatively, use b**>Trace>Graph Trace** and type 0. (An "x =" box will appear as soon as you start typing a value).
Hence y = 90198.3 when x = 0.



In order to find the t values for which y = 110 000, press e or /+G) to show the function entry line and enter **f2(x)** = 110000 (use **Intersection** from the **Analyze Graph** menu to find each of the required values.
Hint: using b**>Geometry>Points & Lines>Intersection Point/s** will find all intersections at once.



The value when $x = 152$ can be found by double clicking on the x-coordinate of the point found earlier and changing to 152 or using the b**>Trace>Graph Trace** and editing the *x*-coordinate to 152.
Alternatively, insert (/+I)a **Calculator** page and type in f1(152)



5 $d = 12 + 12\cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$

a **i** When $t = 5.7$

$$\begin{aligned} d &= 12 + 12\cos\frac{\pi}{6}\left(5.7 + \frac{1}{3}\right) \\ &= 1.8276 \times 10^{-3} \approx 1.83 \times 10^{-3} \text{ hours} \end{aligned}$$

ii When $t = 2.7$

$$\begin{aligned} d &= 12 + 12\cos\frac{\pi}{6}\left(2.7 + \frac{1}{3}\right) \\ &= 11.79 \text{ hours} \end{aligned}$$

b When $d = 5$

$$\begin{aligned} 5 &= 12 + 12\cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right) \\ \frac{-7}{12} &= \cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right) \end{aligned}$$

$$\therefore \quad \frac{\pi}{6}\left(t + \frac{1}{3}\right) = 2.1936 \text{ or } 4.089$$

$$\therefore \quad t = 3.856 \text{ or } 7.477$$

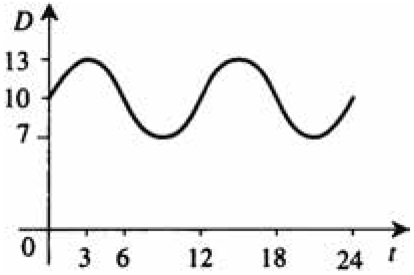
There will be 5 hours of daylight on 25th April and 14th August.

6 **a** Period $= 2\pi \div \frac{\pi}{6}$

$= 12$

For $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$

$$\begin{aligned} D(0) &= 10 \\ D(24) &= 10 + 3\sin\left(\frac{\pi \times 24}{6}\right) \\ &= 10 \end{aligned}$$



b For $D(t) \geq 8.5$, first consider

$$\begin{aligned} 8.5 &= 10 + 3\sin\left(\frac{\pi t}{6}\right) \\ -\frac{1}{2} &= \sin\left(\frac{\pi t}{6}\right) \end{aligned}$$

$$\therefore \quad \frac{\pi t}{6} = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{23\pi}{6} \text{ or } \ldots$$

$$\therefore \quad t = 7 \text{ or } 11 \text{ or } 19 \text{ or } 23 \text{ or } \ldots$$

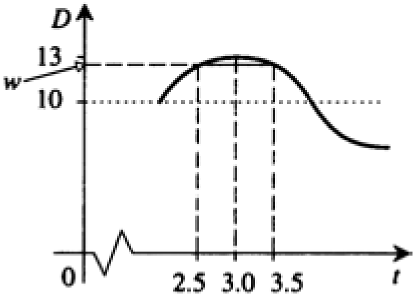
From the graph is can be seen that $D(t) \geq 8.5$ for $t \in [0, 7] \cup [11, 19] \cup [23, 24]$

c The maximum depth is 13 m.
From the graph the required period of time is [2.5, 3.5]
The largest value of w occurs for $t = 2.5$

$$w = 10 + 3\sin\left(\frac{2.5\pi}{6}\right)$$

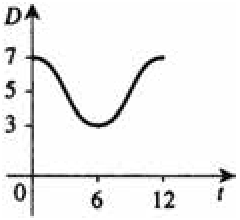
$$= 12.898$$

The largest value of w is 12.898.



7 **a** $D = p + q\cos(rt)°$
High tide is 7 m.
Low tide is 3 m.
Low tide occurs 6 hours after high tide.
High tide occurs when $\cos(rt)° = 1$ and low tide occurs when $\cos(rt)° = -1$.
$\therefore\ D = p + q\cos(rt)°$
gives $7 = p + q$ ①
and $3 = p - q$ ②
Adding ① and ② gives $2p = 10$
$p = 5$
Therefore from ① $\quad q = 2$
Hence $D = 5 + 2\cos(rt)°$
The period is 12 $\therefore\ \dfrac{360}{r} = 12$
and $r = 30$
$\therefore\ D = 5 + 2\cos(30t)°$

b



c Low tide occurs when $t = 6$. The depth at low tide is 3 m.
$5 + 2\cos(30t)° = 4$

$$\cos(30t)° = -\frac{1}{2}$$

$$\therefore\ 30t = 120 \text{ or } 240 \text{ or } \ldots$$

$$\therefore\ t = 4 \text{ or } 8 \text{ or } \ldots$$

The ship may enter the harbour 2 hours after low tide.

8 **a** $a = b = 10,\ \theta = \dfrac{\pi}{3}$

i $A = \dfrac{1}{2} \times 100 \times \sin\dfrac{\pi}{3}$

$= 50 \times \dfrac{\sqrt{3}}{2}$

$= 25\sqrt{3}$ square units

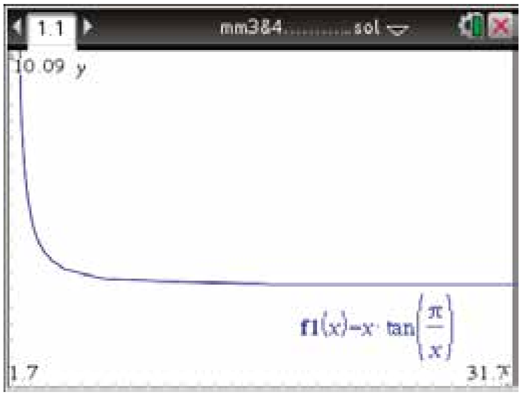
ii $P = 10 + 10 + \sqrt{100 + 100 - 200\cos\dfrac{\pi}{3}}$

$= 20 + \sqrt{200 - 100}$

$= 30$

b $P = A$ implies
$20 + 10\sqrt{2 - 2\cos\theta} = 50\sin\theta$
$\Leftrightarrow 2 + \sqrt{2 - 2\cos\theta} = 5\sin\theta$
Plot the graphs of $y = 2 + \sqrt{2 - 2\cos\theta}$ and $y = 5\sin\theta$ to find the point of intersection.
Intersection occurs where $\theta = 0.53$ or $\theta = 2.27$

c If $a = b = 6$
$A = 18\sin\theta$ and
$P = 12 + \sqrt{72 - 72\cos\theta}$
$\quad = 12 + 6\sqrt{2 - 2\cos\theta}$
Graph $y = P - A$ for $\theta \in (0, \pi)$ and note that the minimum > 0, so $P - A > 0 = P > A$.



d If $\theta = \dfrac{\pi}{2}$ and $a = 6$
$A = P$ implies

$$3b = 6 + b + \sqrt{36 + b^2}$$

i.e. $\quad 2b - 6 = \sqrt{36 + b^2}$

$\therefore \quad 4b^2 - 24b + 36 = 36 + b^2$

$\therefore \quad 3b^2 - 24b = 0$

$\therefore \quad 3b(b - 8) = 0$

$\therefore \quad b = 0$ or $b = 8$

$b = 0$ does not satisfy the original equation. Therefore $b = 8$

e If $a = 10$ and $b = 6$

$30 \sin\theta = 16 + \sqrt{136 - 120\cos\theta}$

$15 \sin\theta = 8 + \sqrt{34 - 30\cos\theta}$

$\theta = 0.927$ or $\theta = 1.837$ (from a cas calculator using the 'solve' command)

f If $a = b$ and $\theta = \dfrac{\pi}{3}$

$A = P$

implies

$$\frac{a^2}{2} \times \frac{\sqrt{3}}{2} = 2a + \sqrt{2a^2 - a^2}$$

$$\therefore \quad \frac{\sqrt{3a^2}}{4} = 2a + \sqrt{a^2}$$

$$\frac{\sqrt{3a^2}}{4} = 3a$$

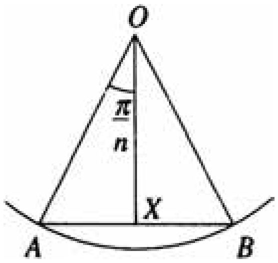
$$\therefore \quad \frac{\sqrt{3a^2}}{4} - 3a = 0$$

$$\therefore \quad a\left(\frac{\sqrt{3}}{4}a - 3\right) = 0$$

$$\therefore a = \frac{12}{\sqrt{3}} = 4\sqrt{3} \text{ since } a > 0.$$

9 **a** The n sided polygon consists of n isosceles triangles.

The angle for each triangle at the centre of the circle is $\dfrac{2\pi}{n}$



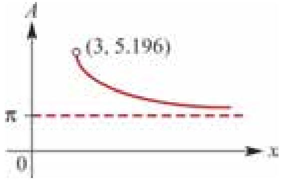
Length of $OX = 1$

Length of $AB = 2\tan\left(\dfrac{\pi}{n}\right)$

$\therefore$ Area of triangle $= \dfrac{1}{2} \times 2\tan\left(\dfrac{\pi}{n}\right) \times 1 = \tan\left(\dfrac{\pi}{n}\right)$

b Area of the polygon is $n\tan\left(\dfrac{\pi}{n}\right)$ (n triangles).

c The horizontal asympote is $y = \pi$



d **i** $n = 3$

Area of polygon $= 3\tan\left(\dfrac{\pi}{3}\right) = 3\sqrt{3}$ difference $= 3\sqrt{3} - \pi \approx 2.055$

ii $n = 4$

Area of polygon $= 4\tan\left(\frac{\pi}{4}\right) = 4$ difference $= 4 - \pi \approx 0.858$

iii $n = 12$

Area of polygon $= 12\tan\left(\frac{\pi}{12}\right) \approx 3.215$ difference ≈ 0.0738

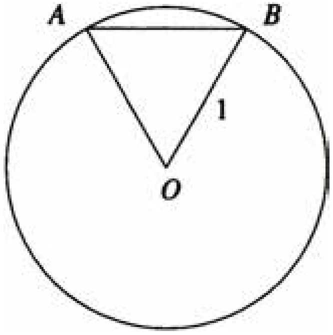
iv $n = 50$

Area of polygon $= 50\tan\left(\frac{\pi}{50}\right) =\approx 3.1457$difference ≈ 0.0041

e The circles are similar

$\therefore$ the area $= nr\tan\left(\frac{\pi}{n}\right)$

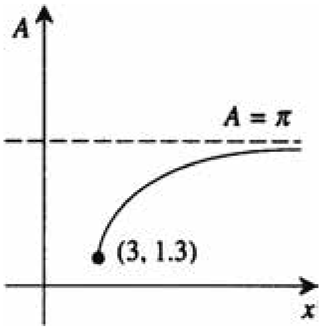
f **i** So the area is $n\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$



ii The polygon consists of n isosceles triangles.

The area of each triangle

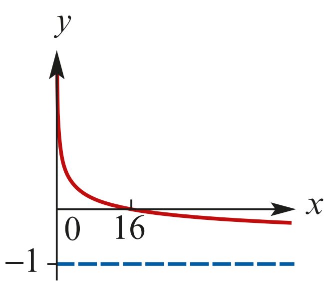
is $\frac{1}{2}\sin\left(\frac{2\pi}{n}\right) = \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$



Chapter 7 – Functions revisited

Solutions to Exercise 7A

1 a

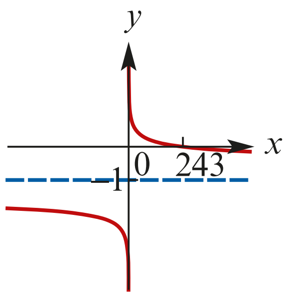


Domain = $\mathbb{R}^+$

Range = $(-1, \infty)$

Neither odd nor even

b



Domain = $\mathbb{R} \setminus \{0\}$

Range = $\mathbb{R} \setminus \{-1\}$

Neither odd nor even

2 a $32^{\frac{2}{5}} = (32^{\frac{1}{5}})^2 = 4$

b $(-32)^{\frac{2}{5}} = (-32)^{\frac{1}{5}})^2 = 4$

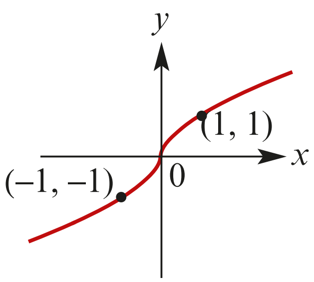
c $32^{\frac{3}{5}} = (32^{\frac{1}{5}})^3 = 8$

d $(-32)^{\frac{3}{5}} = (-32)^{\frac{1}{5}})^3 = -8$

e $(-8)^{\frac{5}{3}} = ((-8)^{\frac{1}{3}})^5 = -32$

f $(-27)^{\frac{4}{3}} = ((-27)^{\frac{1}{3}})^4 = 81$

3 a

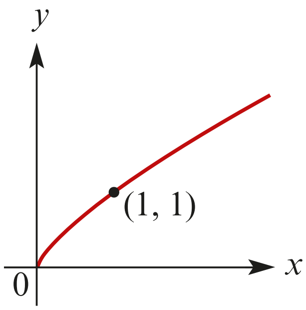


Domain = $\mathbb{R}$

Range = $\mathbb{R}$

Odd

b



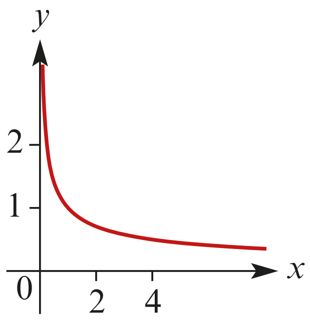
Domain = $\mathbb{R}^+ \cup \{0\}$

Range = $\mathbb{R}^+ \cup \{0\}$

Neither

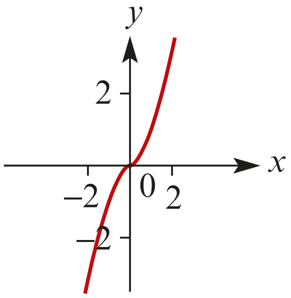
4 a i Domain = $\mathbb{R}^+$; Range = $\mathbb{R}^+$; Asymptotes: $x = 0$, $y = 0$

ii



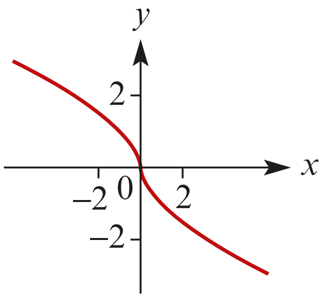
b i Domain = $\mathbb{R}$; Range = $\mathbb{R}$

ii



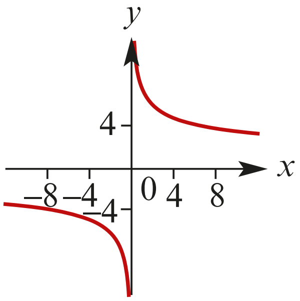
c **i** Domain = $\mathbb{R}$; Range = $\mathbb{R}$

ii



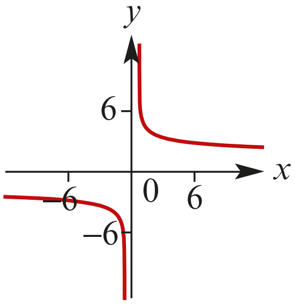
d **i** Domain = $\mathbb{R} \setminus \{0\}$;
Range = $\mathbb{R} \setminus \{0\}$;
Asymptotes: $x = 0$, $y = 0$

ii



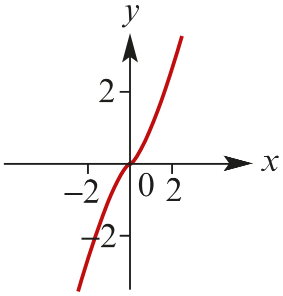
e **i** Domain = $\mathbb{R} \setminus \{0\}$;
Range = $\mathbb{R} \setminus \{0\}$;
Asymptotes: $x = 0$, $y = 0$

ii



f **i** Domain = $\mathbb{R}$; Range = $\mathbb{R}$

ii



5 We can assume $x \neq 0$ in the this question witout effecting the result.

a $x^{\frac{3}{2}} > x^2$

$1 > x^{2-\frac{3}{2}}$

$1 > x^{\frac{1}{2}}$

$x < 1$

$\therefore x \in (0, 1)$

b $x^{\frac{3}{2}} < x^{-2}$

$x^{2+\frac{3}{2}} < 1$

$x^{\frac{5}{2}} < 1$

$x < 1$

$\therefore x \in (0, 1)$

6 **a** Odd

b Even

c Odd

d Odd

e Even

f Odd

Solutions to Exercise 7B

1 **a** $h(x) = f \circ g(x),\ f(x) = e^x,\ g(x) = x^3$

b $h(x) = f \circ g(x),\ f(x) = \sin x,$
$g(x) = 2x^2$

c $h(x) = f \circ g(x),\ f(x) = x^n,$
$g(x) = x^2 - 2x$

d $h(x) = f \circ g(x),\ f(x) = \cos x,$
$g(x) = x^2$

e $h(x) = f \circ g(x),\ f(x) = x^2,$
$g(x) = \cos x$

f $h(x) = f \circ g(x),\ f(x) = x^4,$
$g(x) = x^2 - 1$

g $h(x) = f \circ g(x),\ f(x) = x^2,$
$g(x) = \cos(2x)$

h $h(x) = f \circ g(x),\ f(x) = x^3 - 2x,$
$g(x) = x^2 - 2x$

2 **a** $f \circ f^{-1}(x) = x$

$$\Leftrightarrow 4e^{f^{-1}(x)} = x$$

$$\Leftrightarrow e^{f^{-1}(x)} = \frac{x}{4}$$

$$\Leftrightarrow f^{-1}(x) = \log_e\left(\frac{x}{4}\right)$$

Therefore,
$f^{-1}\colon (0, \infty) \to \mathbb{R},\ f^{-1}(x) = \dfrac{1}{3}\log_e\left(\dfrac{x}{4}\right)$

b $f \circ g^{-1}(x) = x$

$$\Leftrightarrow \frac{2}{\sqrt[3]{g^{-1}(x)}} = x$$

$$\Leftrightarrow 2 = x\sqrt[3]{g^{-1}(x)}$$

$$\Leftrightarrow g^{-1}(x) = \frac{8}{x^3}$$

Therefore,
$g^{-1}\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ g^{-1}(x) = \dfrac{8}{x^3}$

c $f \circ g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f \circ g(x) = 4e^{\frac{6}{\sqrt[3]{x}}}$

d $g \circ f\colon \mathbb{R} \to \mathbb{R},\ g \circ f(x) = \dfrac{2}{\sqrt[3]{4e^{3x}}}$

e Let $h(x) = f \circ g(x)$

$h(x) = 4e^{\frac{6}{\sqrt[3]{x}}}$

$h \circ h^{-1}(x) = x$

$$\Leftrightarrow 4e^{\frac{6}{\sqrt[3]{h^{-1}(x)}}} = x$$

$$\Leftrightarrow e^{\frac{6}{\sqrt[3]{h^{-1}(x)}}} = \frac{x}{4}$$

$$\Leftrightarrow \frac{6}{\sqrt[3]{h^{-1}(x)}} = \log_e\left(\tfrac{x}{4}\right)$$

$$\Leftrightarrow h^{-1}(x) = \left(\frac{6}{\log_e(\frac{x}{4})}\right)^3$$

Therefore,
$(f \circ g)^{-1}\colon \mathbb{R}^+ \to \mathbb{R},$
$(f \circ g)^{-1}(x) = \left(\dfrac{6}{\log_e(\frac{x}{4})}\right)^3$

f $(g \circ f)^{-1}\colon \mathbb{R}^+ \to \mathbb{R},\ (g \circ f)^{-1}(x) = \dfrac{1}{3}\log_e\left(\dfrac{2}{x^3}\right)$

3 **a** $f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = x^{\frac{5}{2}}$
Both f and f^{-1} are strictly increasing

b $f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = -x^{\frac{5}{2}}$

Both f and f^{-1} are strictly decreasing

c $f^{-1}: \mathbb{R}^+ \to \mathbb{R}, f^{-1}(x) = x^{\frac{2}{5}}$
Both f and f^{-1} are strictly increasing

4 **a** **i** $f \circ g(x) = 3\sin(2x^2)$,
$g \circ f(x) = g(3\sin 2x)^2 = 9\sin^2(2x)$

ii $\mathrm{ran}(f \circ g) = [-3, 3]$,
$\mathrm{dom}(f \circ g) = \mathbb{R}$,
$\mathrm{ran}(g \circ f) = [0, 9]$, $\mathrm{dom}(g \circ f) = \mathbb{R}$

b **i** $f \circ g(x) = -2\cos(2x^2)$,
$g \circ f(x) = g(-2\cos 2x) =$
$4\cos^2(2x)$

ii $\mathrm{ran}(f \circ g) = [-2, 2]$,
$\mathrm{dom}(f \circ g) = \mathbb{R}$,
$\mathrm{ran}(g \circ f) = [0, 4]$, $\mathrm{dom}(g \circ f) = \mathbb{R}$

c **i** $f \circ g(x) = e^{x^2}$, $g \circ f(x) = e^{2x}$

ii $\mathrm{ran}(f \circ g) = (1, \infty)$,
$\mathrm{dom}(f \circ g) = \mathbb{R}$,
$\mathrm{ran}(g \circ f) = (0, \infty)$,
$\mathrm{dom}(g \circ f) = \mathbb{R}$

d **i** $f \circ g(x) = e^{2x^2} - 1$,
$g \circ f(x) = (e^{2x} - 1)^2$

ii $\mathrm{ran}(f \circ g) = [0, \infty)$,
$\mathrm{dom}(f \circ g) = \mathbb{R}$,
$\mathrm{ran}(g \circ f) = [0, \infty)$,
$\mathrm{dom}(g \circ f) = \mathbb{R}$

e **i** $f \circ g(x) = -2e^{x^2} - 1$,
$g \circ f(x) = (2e^x + 1)^2$

ii $\mathrm{ran}(f \circ g) = (-\infty, -3]$,
$\mathrm{dom}(f \circ g) = \mathbb{R}$,
$\mathrm{ran}(g \circ f) = (1, \infty)$,
$\mathrm{dom}(g \circ f) = \mathbb{R}$

f **i** $f \circ g(x) = \log_e(2x^2)$,
$g \circ f(x) = (\log_e(2x))^2$

ii $\mathrm{ran}(f \circ g) = \mathbb{R}$,
$\mathrm{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$,
$\mathrm{ran}(g \circ f) = [0, \infty)$,
$\mathrm{dom}(g \circ f) = \mathbb{R}^+$

g **i** $f \circ g(x) = \log_e(x^2 - 1)$,
$g \circ f(x) = (\log_e(x - 1))^2$

ii $\mathrm{ran}(f \circ g) = \mathbb{R}$,
$\mathrm{dom}(f \circ g) = \mathbb{R} \setminus [-1, 1]$,
$\mathrm{ran}(g \circ f) = [0, \infty)$,
$\mathrm{dom}(g \circ f) = (1, \infty)$

h **i** $f \circ g(x) = -\log_e(x^2)$,
$g \circ f(x) = (\log_e x)^2$

ii $\mathrm{ran}(f \circ g) = \mathbb{R}$,
$\mathrm{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$,
$\mathrm{ran}(g \circ f) = [0, \infty)$,
$\mathrm{dom}(g \circ f) = \mathbb{R}^+$

5 **a** $g \circ f(x) = g\left(2x - \dfrac{\pi}{3}\right) = \sin\left(2x - \dfrac{\pi}{3}\right)$

b $(x, y) \to (2x' - \dfrac{\pi}{3}, y')$
$\therefore x' = \dfrac{x + \frac{\pi}{3}}{2} = \dfrac{1}{2}x + \dfrac{\pi}{6}$ and $y' = y$
Dilation of factor $\frac{1}{2}$ from the y-axis,
then translation $\frac{\pi}{6}$ units to the right

6 **a** $g \circ f: (\frac{1}{3}, \infty) \to \mathbb{R}$,

$$\begin{aligned} g \circ f(x) &= g(3x - 2) \\ &= \log_e(3x - 2 + 1) \\ &= \log_e(3x - 1) \end{aligned}$$

b Write $y' = \log_e(3x' - 1)$ and
$y = \log_e(x + 1)$
Then choose,
$y' = y$ and $3x' - 1 = x + 1$
That is $y' = y$ and $x' = \dfrac{x + 2}{3}$

7 **a** $[g(x)]^2 - 7g(x) + 12 = 0$

$(g(x) - 3)(g(x) - 4) = 0$

$\therefore g(x) = 3 \text{ or } g(x) = 4$

b $[g(x)]^2 - 7xg(x) + 12x^2 = 0$

$(g(x) - 3x)(g(x) - 4x) = 0$

$\therefore g(x) = 3x \text{ or } g(x) = 4x$

8 $e^{g(x)} = 2x - 1$

$g(x) = \log_e(2x - 1)$

$\therefore g(x) = 3x \text{ or } g(x) = 4x$

9 $f(x) = e^{4x}, \qquad g(x) = 2\sqrt{x}$

a $g(f(x)) = 2\sqrt{e^{4x}} = 2(e^{4x})^{\frac{1}{2}}$

$= 2e^{2x}$

b $x = 2e^{2(g\circ f)^{-1}(x)}$

$\ln\dfrac{x}{2} = 2(g\circ f)^{-1}(x)$

$(g\circ f)^{-1}(x) = \dfrac{1}{2}\ln\dfrac{x}{2}$

c $x = 2\sqrt{g^{-1}(x)}$

$\dfrac{x}{2} = \sqrt{g^{-1}(x)}$

$g^{-1}(x) = \dfrac{x^2}{4}$

$(f\circ g^{-1})(x) = e^{4\left(\frac{x^2}{4}\right)}$

$= e^{x^2}$

10 $f(x) = e^{-2x}, g(x) = x^3 + 1$

a $x = e^{-2f^{-1}(x)}$

$\ln x = -2f^{-1}(x)$

$f^{-1}(x) = \dfrac{-1}{2}\ln x$

$x = (g^{-1}(x))^3 + 1$

$x - 1 = (g^{-1}(x))^3$

$g^{-1}(x) = (x - 1)^{\frac{1}{3}}$

b $f\circ g(x) = \mathrm{e}^{-2(x^3+1)}$

$= \mathrm{e}^{-2x^3-2}$

range $(f\circ g) = \mathbb{R}^+$

since range $(-2x^3 - 2) = \mathbb{R}$

and range $(e^x) = \mathbb{R}^+$

$g\circ f(x) = (e^{-2x})^3 + 1$

$= e^{-6x} + 1$

range $(g\circ f) = (1, \infty)$

11 **a** $f\colon (-1, \infty) \to \mathbb{R}, \quad f(x) = \dfrac{1}{x+1}$

domain (f) = $(-1, \infty)$,

$\therefore$ range $(f) = \mathbb{R}^+$

$x = \dfrac{1}{f^{-1}(x) + 1}$

$f^{-1}(x) + 1 = \dfrac{1}{x}$

$f^{-1}(x) = \dfrac{1}{x} - 1$

range $(f^{-1}) = (-1, \infty)$,

domain $(f^{-1}) = \mathbb{R}^+$

$\therefore f^{-1}\colon \mathbb{R}^+ \to \mathbb{R}, \quad f^{-1}(x) = \dfrac{1}{x} - 1$

b $f(x) = f^1(x) = x$

$f(x) = x$

$\dfrac{1}{x+1} = x$

$(x+1)x = 1$

$x^2 + x - 1 = 0$

$x = \dfrac{-1 \pm \sqrt{1+4}}{2}$

but $x > 0$, (domain(f^{-1}))

$\therefore \quad x = \dfrac{-1 + \sqrt{5}}{2}$

$x = \dfrac{\sqrt{5} - 1}{2}$

12 a $f(x) = \ln(x+1)$

$x = \ln(f^{-1}(x) + 1)$

$e^x = f^{-1}(x) + 1$

$f^{-1}(x) = e^x - 1$

$f^{-1}: R \to R, f^{-1}(x) = e^x - 1$

$g(x) = x^2 + 2x$, domain(g) $= (-1, \infty)$

range $(g) = (g(-1), \infty)$

$= (-1, \infty)$

$x = (g^{-1}(x))^2 + 2g^{-1}(x)$

$(g^{-1}(x))^2 + 2(g^{-1}(x)) - x = 0$

$(g^{-1}(x)^2 + 1)^2 - x - 1 = 0$

$g^{-1}(x) + 1 = \pm\sqrt{(x+1)}$

$g^{-1}(x) + 1 = \pm\sqrt{x+1}$

$g^{-1}(x) = -1 \pm \sqrt{x+1}$

but $g^{-1}(x) > -1$

$\therefore \quad g^{-1}(x) = -1 + \sqrt{x+1}$

$g^{-1}: (-1, \infty) \to R, g^{-1}(x) =$
$\sqrt{(x+1)} - 1$

b $f \circ g(x) = \ln(x^2 + 2x + 1)$

$= \ln((x+1)^2)$

$= 2\ln(x+1)$

($Since$ $domain$ $g = (-1, \infty)$)

13 $f \circ g(x) = \text{In}\left(\dfrac{1}{x}\right)$

$= -\ln(x)$

$f(x) + f \circ g(x) = \ln x - \ln x = 0$

14 $h(g(x)) = \sqrt{\dfrac{(5x^2+3)-3}{5}}$

$= \sqrt{\dfrac{5x^2}{5}}$

$= \sqrt{x^2}$

$= |x|$

15 a $f(g(x)) = (x^2 - 4 - 4)(x^2 - 4 - 6)$

$= (x^2 - 8)(x^2 - 10)$

$f(g(x)) = x^4 - 18x^2 + 80$

$g(f(x)) = ((x-4)(x-6))^2 - 4$

$= (x^2 - 10x + 24)^2 - 4$

$= x^4 - 20x^3 + 48x^2 + 100x^2$

$- 480x + 576 - 4$

$g(f(x)) = x^4 - 20x^3 + 148x^2$

$- 480x + 572$

b $g(f(x)) - f(g(x)) = 158$

$x^4 - 20x^3 + 148x^2 - 480x + 572$

$-x^4 + 18x^2 - 80 = 158$

$-20x^3 + 166x^2 - 480x + 334 = 0$

$10x^3 - 88x^2 + 240x - 167 = 0$

CAS calculator gives $x = 1$ as a solution

$\Rightarrow (x-1)(10x^2 - 73x + 167) = 0$

$x = 1, x = \dfrac{73 \pm \sqrt{5329 - 6680}}{20}$

$x = 1, \dfrac{73 \pm \sqrt{-1351}}{20}$

$\Downarrow$

no real solutions

$\therefore \quad x = 1$

16 **a**

$$f(x) = 4 - x^2$$

$$f(f(x)) = 4 - (4 - x^2)^2$$

$$= 4 - (16 - 8x^2 + x^4)$$

$$= -12 + 8x^2 - x^4$$

$$f(f(x)) = 0$$

$$\Rightarrow x^4 - 8x^2 + 12 = 0$$

$$(x^2)^2 - 8x^2 + 12 = 0$$

$$(x^2 - 6)(x^2 - 2) = 0$$

$$x^2 = 2, 6$$

$$x = \pm\sqrt{2}, \pm\sqrt{6}$$

17 $f(x) = e^x - e^{-x}$

a

$$LHS = e^{(-x)} - e^{-(-x)}$$

$$= e^{-x} - e^x$$

$$RHS = -e^x + e^{-x}$$

$$= e^{-x} - e^x$$

$$= LHS \quad QED$$

b

$$RHS = e^{3x} - e^{-3x} - 3e^x + 3e^{-x}$$

$$= e^{3x} - 3e^x + 3e^{-x} - e^{-3x}$$

$$LHS = (e^x - e^{-x})^3$$

$$= e^{3x} - 3e^x + 3e^{-x} - e^{-3x}$$

$$= RHS \quad QED$$

18 Consider,

$af^{-1}(x) + b = x$

$\therefore f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$

If$f^{-1}(x) = 6x + 3$

$a = \frac{1}{6}$ and $-\frac{b}{a} = 3$

$\therefore a = \frac{1}{6}$ and $b = -\frac{1}{2}$

19

$$\frac{f^{-1}(x) + 2}{f^{-1}(x) - 1} = x$$

$$f^{-1}(x) + 2 = x(f^{-1}(x) - 1)$$

$$f^{-1}(x)(1 - x) = -2 - x$$

$$f^{-1}(x) = \frac{x + 2}{x - 1}$$

20 $\ln(g(x)) = ax + b$

$$g(x) = e^{ax+b}$$

$$g(0) = 1$$

$$\therefore \quad 1 = e^b$$

$$\therefore \quad b = 0$$

$$g(x) = e^{ax}$$

$$g(1) = e^6$$

$$e^6 = e^a$$

$$a = 6$$

$$g(x) = e^{6x}$$

21 Let $y = f^{-1}(x)$

a

$$\frac{e^y + e^{-y}}{2} = x$$

$$e^y + e^{-y} = 2x$$

$$e^{2y} + 1 = 2xe^y$$

$$e^2y - 2xe^y + 1 = 0$$

$$e^y = \frac{1}{2}(2x \pm \sqrt{4x^2 - 4})$$

$$y = \log_e(x \pm \sqrt{x^2 - 1})$$

But Range of f^{-1}= Domain of $f = [0, \infty)$ and Domain of f^{-1}= Range of $f = [1, \infty) \therefore f^{-1} : [1, \infty) \to \mathbb{R},$

$f^{-1}(x) = \log_e(x + \sqrt{x^2 - 1})$

b $g^{-1}: \mathbb{R} \to \mathbb{R}, g^{-1}(x) = \log_e(x + \sqrt{x^2 + 1})$

c Yes

d Yes

22 a If $x > y$ then $f(x) > f(y)$ and
If $x > y$ then $g(x) > g(y)$
Hence $x > y \Rightarrow f(x) > f(y) \Rightarrow g(f(x)) > g(f(y))$.

b If $x > y$ then $f(x) < f(y)$ and
If $x > y$ then $g(x) < g(y)$
Hence if $x > y \Rightarrow f(x) < f(y) \Rightarrow g(f(x)) > g(f(y))$.

c The composite function will be strictly decreasing.

Solutions to Exercise 7C

1 $f(x) = e^{-2x}, g(x) = -2x$

a **i** $(f + g)(x) = e^{-2x} - 2x$

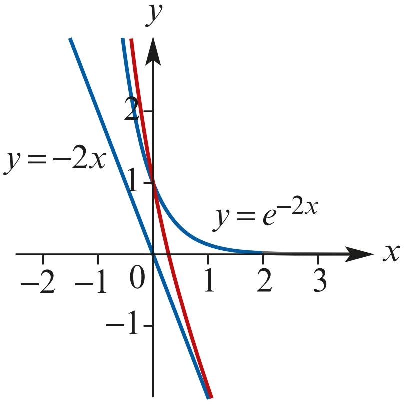
ii $(fg)(x) = -2xe^{-2x}$

b **i** $(f + g)\left(\frac{-1}{2}\right) = e^{-1} + 1$

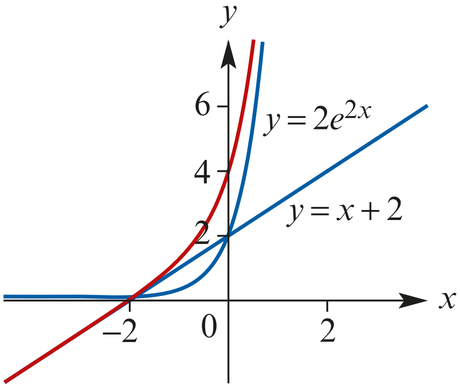
ii $(fg)\left(\frac{-1}{2}\right) = 1 \times e^{-1}$

$= e^{-1}$

2



3



4 $f(x) = \sin\left(\frac{\pi x}{2}\right), g(x) = -2x$

a **i** $(f + g)(x) = \sin\left(\frac{\pi x}{2}\right) - 2x$

ii $(fg)(x) = -2x\sin\left(\frac{\pi x}{2}\right)$

b **i** $(f + g)(1) = \sin\left(\frac{\pi}{2}\right) - 2$

$= 1 - 2$

$= -1$

ii $(fg)(1) = -2\sin\left(\frac{\pi}{2}\right)$

$= -2$

5 $f(x) = \cos\left(\frac{\pi x}{2}\right), g(x) = e^x$

a **i** $(f + g)(x) = \cos\left(\frac{\pi x}{2}\right) + e^x$

ii $(fg)(x) = e^x\cos\left(\frac{\pi x}{2}\right)$

b **i** $(f + g)(0) = \cos(0) + e^0$

$= 1 + 1$

$= 2$

ii $(fg)(0) = 1 \times 1$

$= 1$

6 Let $g(x) = \frac{f(x) + f(-x)}{2}$ and $h(x) = \frac{f(x) - f(-x)}{2}$

We have, $g(-x) = g(x)$. That is $g(x)$ is even.

We have, $h(-x) = -h(x)$. That is $h(x)$ is odd.

$f(x) = h(x) + g(x)$

Solutions to Exercise 7D

1 **a** $f(x-y) = 2(x-y)$

$$= 2x - 2y$$

$$= f(x) - f(y)$$

b $f(x-y) = (x-y-3)$

$$\neq f(x) - f(y)$$

2 $f(x-y) = k(x-y)$

$$= kx - ky$$

$$= f(x) - f(y)$$

3 $f(x+y) = 2(x+y) + 3$

$$= 2x + 2y + 3$$

$$= 2x + 3 + 2y + 3 - 3$$

$$= f(x) + f(y) - 3$$

$$a = -3$$

4 $f(x) + f(y) = \dfrac{3}{x} + \dfrac{3}{y}$

$$= \frac{3(x+y)}{xy}$$

$$= (x+y)f(xy)$$

5 $(g(x))^2 = g(x)$

$$(g(x))^2 - g(x) = 0$$

$$g(x)(g(x)-1) = 0$$

$$g(x) = 0, 1$$

6 $\dfrac{1}{g(x)} = g(x)$

$$(g(x))^2 = 1$$

$$g(x) = \pm 1$$

7 $f(x) = x^3$

$$f(x+y) = (x+y)^3$$

$$f(x) + f(y) = x^3 + y^3$$

Let $x = 1, y = 1$

$$f(x+y) = 8$$

$$f(x) + f(y) = 2$$

8 $f(x) = \sin x$

$$LHS = f(x+y) = \sin(x+y)$$

$$RHS = f(x) + f(y) = \sin x + \sin y$$

let x $= \dfrac{\pi}{2}, y = \dfrac{\pi}{2}$

$$LHS = \sin(\pi) = 0$$

$$RHS = \sin\frac{\pi}{2} + \sin\frac{\pi}{2} = 2 \neq LHS\ QED$$

(any non zero numbers would work)

9 $f(x) = \dfrac{1}{x^2}$

$$LHS = f(x) + f(y)$$

$$= \frac{1}{x^2} + \frac{1}{y^2}$$

$$= \frac{y^2}{x^2y^2} + \frac{x^2}{x^2y^2}$$

$$= (x^2 + y^2)\frac{1}{x^2y^2}$$

$$= (x^2 + y^2)\frac{1}{(xy)^2}$$

$$= (x^2 + y^2)f(xy)$$

$$= RHS\ \ QED$$

10 $h(x) = x^2$

a Let $x = 1, y = 1$

$LHS = (1 + 1)^2 = 4$

$RHS = (1)^2 + (1)^2 = 2 \neq LHS\, QED$
(any non zero numbers would work)

b
$$\begin{aligned} LHS &= (x + y)^2 \\ &= x^2 + 2xy + y^2 \\ RHS &= (x)^2 + (y)^2 \\ &= x^2 + y^2 \\ LHS &= RHS + 2xy \end{aligned}$$

$\therefore$ given LHS $= RHS$

$$2xy = 0$$
i.e., $x = 0$ *or* $y = 0$ *QED*

11 $g(x) = 2^{3x}$
$$\begin{aligned} LHS &= 2^{3(x+y)} \\ &= 2^{3x+3y} \\ &= 2^{3x} \times 2^{3y} \\ &= g(x) \times g(y) \\ &= RHS\, QED \end{aligned}$$

12 $f(x) = x^n$
$f(xy) = (xy)^n = x^n y^n$
(by the indices laws)
$= f(x)f(y)$ *QED*
$f\left(\frac{x}{y}\right) = \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
(by the indices laws)
$= \frac{f(x)}{f(y)}$ *QED*

13 $f(x) = ax, \quad a \in R\backslash\{0, 1\}$
$$\begin{aligned} f(xy) &= axy \\ &= f(x)f(y) = ax \times ay \\ &= a^2xy \end{aligned}$$

let $x = 1, \quad y = 1$
(any non zero numbers would work)
$$\begin{aligned} f(xy) &= axy = a \\ f(x)f(y) &= a^2xy = a^2 \\ if\ f(x)f(y) &= f(xy) \\ a^2 &= a \\ a^2 - a &= 0 \\ a &= 0, 1 \end{aligned}$$
but $a \neq 0, 1$
$\therefore \quad f(x)f(y) \neq f(xy)$
for the case shown

14 $f : \mathbb{R}\backslash\{-1\} \to \mathbb{R}, f(x) = \frac{1}{x+1}.$
$f(f(x)) = f(\frac{1}{x+1}) = \frac{1}{\frac{1}{x+1} + 1} =$
$\frac{x+1}{x+2}$
$f(x + 1) = \frac{1}{x+2}$
$$\begin{aligned} \therefore f(f(x)) + f(x + 1) &= \frac{x+1}{x+2} + \frac{1}{x+2} \\ &= \frac{x+2}{x+2} \\ &= 1 \end{aligned}$$

15 Let $f(x) = x^2, g(x) = 2, h(x) = 3$
$f \circ (g + h)(x) = f(5) = 25$
$f \circ g(x) + f \circ h(x) = f(2) + f(3) = 13$

16
$$\begin{aligned} g + h \circ f(x) &= (g + h)f(x) \\ &= g(f(x) + h(f(x)) \\ &= g \circ f(x) + h \circ f(x) \end{aligned}$$

17 For $x > 0$

$$\begin{aligned} f(g(x)) - f(x) &= f(xe^x) - \log_e x \\ &= \log_e(xe^x) - \log_e x \\ &= \log_e x + \log_e(e^x) - \log_e x \\ &= \log_e(e^x) \\ &= x \end{aligned}$$

$$\begin{aligned} \frac{g(f(x)}{f(x)} &= \frac{g(\log_e x)}{\log_e x} \\ &= \frac{\log_e x \times e^{\log_e x}}{\log_e x} \\ &= \frac{x \log_e x}{\log_e x} \\ &= x \end{aligned}$$

Solutions to Exercise 7E

1 $f(x) = mx - 4, \quad m \in R\backslash\{0\}$

a $0 = mx - 4$

$mx = 4$

$x = \dfrac{4}{m}$

b $\dfrac{4}{m} \leq 1$

$\therefore \quad 4 \leq m, \quad m < 0$

c $x = mf^{-1}(x) - 4$

$x + 4 = mf^{-1}(x)$

$f^{-1}(x) = \dfrac{x+4}{m}, \quad \text{domain} = \mathbb{R}$

d $x = mx - 4$

$(m-1)x - 4 = 0$

$x = \dfrac{4}{m-1}$

check $\text{f}\left(\dfrac{4}{\text{m}-1}\right) = \dfrac{4m}{m-1} - 4$

$= \dfrac{4m - 4m + 4}{m-1}$

$= \dfrac{4}{m-1}$

co-ordinates $\left(\dfrac{4}{m-1}, \dfrac{4}{m-1}\right)$,

$m \in \mathbb{R}\backslash\{0, 1\}$

e $y = ax + b$

$a = \dfrac{-1}{m}$ (normalline)

$y = \dfrac{-x}{m} + b$

$(0, -4)$

$\Rightarrow \quad -4 = b$

$y = \dfrac{-x}{m} - 4$

2 $f(x) = -2x + c$

a $0 = -2x + c$

$-c = -2x$

$x = \dfrac{c}{2}$

b $\dfrac{c}{2} \leq 1$

$c \leq 2$

c $x = -2f^{-1}(x) + c$

$x - c = -2f^{-1}(x)$

$f^{-1}(x) = \dfrac{c-x}{2}, \quad \text{domain} = \mathbb{R}$

d $x = -2x + c$

$3x = c$

$x = \dfrac{c}{3}$

$y = x$

$co - ords = \left(\dfrac{c}{3}, \dfrac{c}{3}\right)$

e $y = ax + b$

$(0, c)$

$\Rightarrow \quad b = c$

$y = ax + c$

normal line

$\Rightarrow \quad a = \dfrac{-1}{-2}$

$a = \dfrac{1}{2}$

$y = \dfrac{x}{2} + c$

3 $y = x^2 - bx$

a $x^2 - bx = 0$

$$x(x - b) = 0$$

$$x = 0, b$$

b $x^2 - bx + \frac{b^2}{4} - \frac{b^2}{4} = 0$

$$\left(x - \frac{b}{2}\right)^2 - \frac{b^2}{4} = 0$$

$$co - ords: \left(\frac{b}{2}, \frac{-b^2}{4}\right)$$

c **i** $-x = x^2 - bx$

$$x^2 - (b - 1)x = 0$$

$$x(x - (b - 1)) = 0$$

$$x = 0, b - 1$$

$$y = -x = 0, 1 - b$$

$$co - ords: (0, 0), (b - 1, 1 - b)$$

ii $b - 1 = 0,$

$$b = 1$$

iii $b - 1 \neq 0,$

$$b \neq 1$$

$$b \in \mathbb{R}\backslash\{1\}$$

4 $y = ax^2 + bx + c$

When $x = -1, y = 6$

When $x = 1, y = 4$

$$6 = a - b + c \ldots (1)$$

$$4 = a + b + c \ldots (2)$$

Equation (1) – Equation (2)

$$2 = -2b$$

$$b = -1$$

Substitute in (1)$6 = a + 1 + c$

$$a = 5 - c$$

$$y = (5 - c)x^2 - x + c$$

5 **a** $(1 + h)^2 = 8$

$$1 + h = \pm 2\sqrt{2}$$

$$h = -1 \pm 2\sqrt{2}$$

b $(a + 1)^2 = 8$

$$a^2 = 8$$

$$a = \pm 2\sqrt{2}$$

c $a = ax^2 + bx$

$$y = a\left(x^2 + \frac{b}{a}x\right)$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$$

vertex (1,8)

$$(\mathbf{1}) \quad \frac{-b}{2a} = 1, \quad (\mathbf{2}) \quad -\frac{b^2}{4a} = 8$$

$$\frac{(\mathbf{2})}{(\mathbf{1})} \Rightarrow \quad \frac{b}{2} = 8$$

$$b = 16$$

$$\text{Sub in } (\mathbf{1}) \Rightarrow \quad \frac{-16}{2a} = 1$$

$$\frac{-8}{a} = 1$$

$$a = -8$$

check

$$\Rightarrow (\mathbf{2}) \Rightarrow \frac{-(16)^2}{4^* - 8} = 8$$

$$\frac{256}{32} = 8$$

$$\frac{2}{2^5} = 2^3 \quad \text{correct}$$

$$a = -8, b = 16$$

6 $f(x) = \sqrt{2a - x}$

a $2a - x \geq 0$

$2a \geq x$, so domain $= (-\infty, 2a]$

b
$$x = \sqrt{2a - x}$$
$$x^2 = 2a - x$$
$$x^2 + x - 2a = 0$$
$$x = \frac{-1 \pm \sqrt{1 + 8a}}{2}$$
$but\ y > 0, \quad \therefore x > 0$
$$\therefore \quad x = \frac{-1 \pm \sqrt{1 + 8a}}{2}$$
$y = x,$
$\therefore \quad co - ords =$
$$\left(\frac{-1 + \sqrt{1 + 8a}}{2}, \frac{-1 + \sqrt{1 + 8a}}{2}\right)$$

c
$$\frac{-1 + \sqrt{1 + 8a}}{2} = 1$$
$$-1 + \sqrt{1 + 8a} = 2$$
$$\sqrt{1 + 8a} = 3$$
$$1 + 8a = 9$$
$$8a = 8$$
$$a = 1$$

d
$$\frac{-1 + \sqrt{1 + 8a}}{2} = 2$$
$$\sqrt{1 + 8a} = 5$$
$$1 + 8a = 25$$
$$8a = 24$$
$$a = 3$$

e
$$\frac{-1 + \sqrt{1 + 8a}}{2} = c$$
$$\sqrt{1 + 8a} = 2c + 1$$
$$1 + 8a = 4c^2 + 4c + 1$$
$$8a = 4c^2 + 4c$$
$$a = \frac{c^2 + c}{2}$$

7 $f(x) = (x^2 - ax)^2$

a
$$0 = (x^2 - ax)^2$$
$$0 = x(x - a)$$
$$x = 0, a$$
$co - ords = (0, 0), (a, 0)$

b
$$f(0) = (0 - 0)^2$$
$$= 0$$
$co - ords = (0, 0)$

c
$$x \in [0, a]$$
$$f(x) = (ax - x^2)^2$$
$$= \left(-\left(x^2 - ax + \frac{a^2}{4}\right) + \frac{a^2}{4}\right)^2$$
$$= \left(-\left(x - \frac{a}{2}\right)^2 + \frac{a^2}{4}\right)^2$$
maximum value is $\frac{a^4}{16}$

d
$$f(-1) = 16$$
$$16 = ((-1)^2 - a(-1))^2$$
$$\pm 4 = 1 + a$$
$$1 + a = \pm 4$$
$$a = -1 \pm 4$$
$$a = -5, 3$$

8 **a**
$$-ae^{bx} + c = 0$$
$$e^{bx} = \frac{c}{a}$$
$$bx = \ln\frac{c}{a}$$
$$x = \frac{1}{b}\ln\frac{c}{a}$$

b $c\ln(x+a) = b$

$$\ln(x+a) = \frac{b}{c}$$

$$x + a = e^{\frac{b}{c}}$$

$$x = e^{\frac{b}{c}} - a$$

c $\ln(cx - a) = 0$

$$\ln(cx - a) = 1$$

$$cx = 1 + a$$

$$x = \frac{1+a}{c}$$

d $e^{ax+b} = c$

$$ax + b = \ln c$$

$$x = \frac{\ln c - b}{a}$$

9 $f(x) = c\ln(x - a)$

a $x - a = 0$

$$x = a$$

b $x - a = 1$

$$x = 1 + a$$

$$co - ords = (1 + a, 0)$$

c $1 = c\ln(x - a)$

$$x - a = e^{\frac{1}{c}}$$

$$x = a + e^{\frac{1}{c}}$$

$$co - ords = (a + e^{\frac{1}{e}}, 0)$$

d $1 = c\ln(2 - a)$

$$c = \frac{1}{\ln(2-a)}$$

10 $f(x) = e^{x-1} - b$

a $y = 0 - b$

$y = -b$

b $0 = e^{x-1} - b$

$x - 1 = \ln b$

$x = \ln b + 1$

$co - ords = (\ln b + 1, 0)$

c **i** $\ln b + 1 = 0$

$$\ln\ b = -1$$

$$b = e^{-1}$$

$$b = \frac{1}{e}.$$

ii $\ln b + 1 < 0$

$$\ln b < -1$$

$$b < e^{-1}$$

$$b < \frac{1}{e}$$

but $b > 0$ as given, else there is no intercept

$$\therefore\quad 0 < b < \frac{1}{e}$$

11 $y = ax^3 + bx^2 + cx + d$

$(-1, 6)$

$\Rightarrow$ ① $6 = -a + b - c + d$

$(1, -2)$

$\Rightarrow$ ② $-2 = a + b + c + d$

② + ① $4 = 2b + 2d$

$$b = 2 - d$$

$(2,4)$

$\Rightarrow$ ③ $4 = 8a + 4b + 2c + d$

② − ① $\Rightarrow$ $2a + 2c = -8$

$$a + c = -4$$

$$a = -4 - c$$

Sub in ③ ⇒

$$4 = 8(-4 - c) + 4(2 - d) + 2c + d$$

$$4 = -32 - 8c + 8 - 4d + 2c + d$$

$$4 = -24 - 6c - 3d$$

$$6c + 3d = -28$$

$$6c = -28 - 3d$$

$$c = \frac{-28 - 3d}{6}$$

$$a = -4 - \mathrm{c}$$

$$a = 4 - \mathrm{c}$$

$$a = \frac{-24 + 28 + 3d}{6}$$

$$a = \frac{3d + 4}{6}$$

12 $y = \left(\frac{c - 8}{2}\right)x^2 + \left(\frac{20 - 3c}{2}\right)x + c$

$$b^2 - 4ac$$

$$= \left(\frac{20 - 3c}{2}\right)^2 - 4\left(\frac{c - 8}{2}\right)c$$

$$= \frac{400 - 120c + 9c^2}{4} - 2c^2 + 16c$$

$$= 100 - 30c + \frac{9}{4}c^2 - 2c^2 + 4c$$

$$= \frac{1}{4}c^2 - 14c + 100$$

a $b^2 - 4ac = 0$

$$c^2 - 56c + 400 = 0$$

$$c = \frac{56 \pm \sqrt{1536}}{2}$$

$$c = 28 \pm 8\sqrt{6}$$

b $b^2 - 4ac > 0$

$c^2 - 56c + 400 > 0$

$c < 28 - 8\sqrt{6}$ or $c > 28 + 8\sqrt{6}$

but $c \neq 8$ (since *if* $c = 8$, the function becomes linear

∴ $c < 8$, or

$8 < c < 28 - 8\sqrt{6}$ or $c > 28 + 8\sqrt{6}$

13 a $y = ax^3 + bx^2 + cx + d$

(−2, 8)

⇒ ① $8 = -8a + 4b - 2c + d$

(1,1)

⇒ ② $1 = a + b + c + d$

(3,4)

⇒③ $4 = 27a + 9b + 3c + d$

③ − 3② ⇒ ④ $1 = 24a + 6b - 2d$

2③ + 3① ⇒ ⑤ $32 = 30a + 30b + 5d$

⑤ ⇒ $\frac{32}{5} = 6a + 6b + d$

④ − ⑤ ⇒ $1 - \frac{32}{5} = 18a - 3d$

$$-\frac{9}{5} = 6a - d$$

$$a = \frac{5d - 9}{30}$$

Sub in ⑤ ⇒ $\frac{32}{5} = \frac{5d - 9}{5} + 6b + d$

$$32 = 5d - 9 + 30b + 5d$$

$$30b = 41 - 10d$$

$$b = \frac{41 - 10d}{30}$$

Sub in ② ⇒ $1 = \frac{5d - 9}{30} + \frac{41 - 10d}{30} + c + d$

$$30d = 5d - 9 + 41 - 10d + 30c + 30d$$

$$-2 = 30c + 25d$$

$$c = \frac{-2 - 25d}{30}$$

14 a

b $y = \frac{1}{x}$

$$\frac{y' - 2}{k} = \frac{1}{\left(\frac{x' - 3}{-4}\right)} \quad \text{(from (a))}$$

$$y' - 2 = \frac{-4k}{x' - 3}$$

$$y' = \frac{-4k}{x' - 3} + 2$$

c $0 = \frac{-4k}{0 - 3} + 2$

$$\frac{4}{3}k = -2$$

$$k = \frac{-6}{4} = \frac{-3}{2}$$

15 a

b

$$y = 2^x$$

$$\frac{y' + 2}{2} = 2^{\left(\frac{x'-a}{4}\right)}$$

$$y' + 2 = 2^{\left(\frac{x'-a}{4}+1\right)}$$

$$y' = 2^{\left(\frac{x'+4-a}{4}\right)} - 2 = 2 \times 2^{\frac{(x'-a)}{4}} - 2$$

c $0 = 2^{\left(\frac{0+4-a}{4}\right)-2}$

$$2 = 2^{\left(\frac{4-a}{4}\right)}$$

$$\frac{4 - a}{4} = 1$$

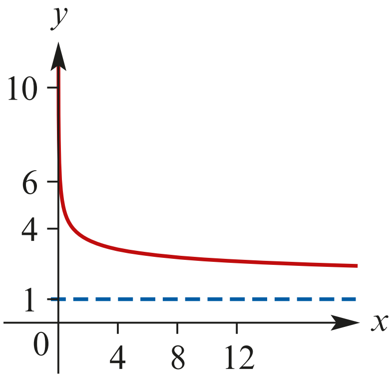
$$4 - a = 4$$

$$-a = 0$$

$$a = 0$$

Solutions to Technology-free questions

1 a

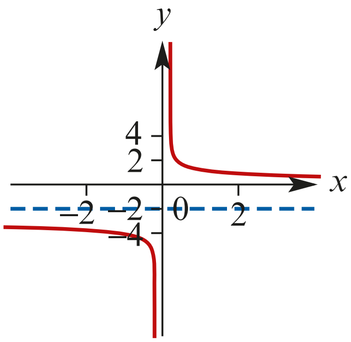


Domain = $\mathbb{R}^+$

Range = $(1, \infty)$

Neither

b



Domain = $\mathbb{R} \setminus \{0\}$

Range = $\mathbb{R} \setminus \{-2\}$

Neither

2 a $243^{\frac{2}{5}} = 3^2 = 9$

b $(-243)^{\frac{2}{5}} = (-3)^2 = 9$

c $243^{\frac{3}{5}} = 3^3 = 27$

d $(-243)^{\frac{2}{5}} = (-3)^2 = 9$

e $(-27)^{\frac{5}{3}} = (-3)^5 = -243$

f $(-125)^{\frac{4}{3}} = (-5)^4 = 625$

3 a i $f \circ g(x) = 3\cos(2x^2)$, $g \circ f(x) = 9\cos^2(2x)$

ii $\text{dom}(f \circ g) = \mathbb{R}$, $\text{ran}(f \circ g) = [-3, 3]$, $\text{dom}(g \circ f) = \mathbb{R}$, $\text{ran}(g \circ f) = [0, 9]$

b **i** $f \circ g(x) = \log_e(3x^2)$, $g \circ f(x) = (\log_e(3x))^2$

ii $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$, $\text{ran}(f \circ g) = \mathbb{R}$, $\text{dom}(g \circ f) = \mathbb{R}^+$, $\text{ran}(g \circ f) = [0, \infty)$

c **i** $f \circ g(x) = \log_e(2 - x^2)$, $g \circ f(x) = (\log_e(2 - x))^2$

ii $\text{dom}(f \circ g) = (-\sqrt{2}, \sqrt{2})$, $\text{ran}(f \circ g) = (-\infty, \log_e 2)$, $\text{dom}(g \circ f) = (-\infty, 2)$, $\text{ran}(g \circ f) = [0, \infty)$

d **i** $f \circ g(x) = -\log_e(2x^2)$, $g \circ f(x) = (\log_e(2x))^2$

ii $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$, $\text{ran}(f \circ g) = \mathbb{R}$, $\text{dom}(g \circ f) = (0, \infty)$, $\text{ran}(g \circ f) = [0, \infty)$

4 **a** $h(x) = f \circ g(x)$, $g(x) = x^2$, $f(x) = \cos x$ (*Note:* answer not unique)

b $h(x) = f \circ g(x)$, $g(x) = x^2 - x$, $f(x) = x^n$ (*Note:* answer not unique)

c $h(x) = f \circ g(x)$, $g(x) = \sin x$, $f(x) = \log_e x$ (*Note:* answer not unique)

d $h(x) = f \circ g(x)$, $g(x) = \sin(2x)$, $f(x) = -2x^2$ (*Note:* answer not unique)

e $h(x) = f \circ g(x)$, $g(x) = x^2 - 3x$, $f(x) = x^4 - 2x^2$ (*Note:* answer not unique)

5 **a** **i** $(f + g)(x) = 2\cos\left(\frac{\pi x}{2}\right) + e^{-x}$

ii $(fg)(x) = 2e^{-x}\cos\left(\frac{\pi x}{2}\right)$

b **i** $(f + g)(0) = 3$

ii $(fg)(0) = 2$

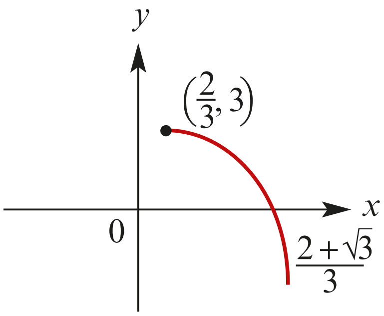
6 $f : [a, \infty) \to \mathbb{R}, f(x) = -(3x - 2)^2 + 3$

Turning point at $(\frac{2}{3}, 3)$

a $\frac{2}{3}$

b $(-\infty, 3]$

c



d

$$f(f^{-1}(x)) = x$$

$$-3(f^{-1}(x) - 2)^2 + 3 = x$$

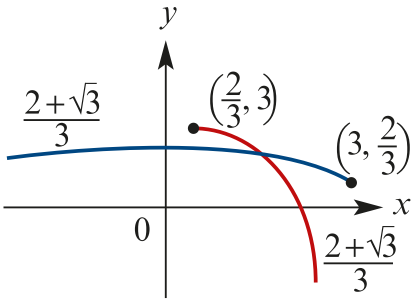
$$(f^{-1}(x) - 2)^2 = -\frac{x-3}{3}$$

$$f^{-1}(x) = 2 \pm \sqrt{-\frac{x-3}{3}}$$

Because of domain and range of f

$f^{-1}(x) = \dfrac{2 + \sqrt{3-x}}{3}$, ran $= [\frac{2}{3}, \infty)$, dom $= (-\infty, 3]$

e



7 $f(x) = c\log_e(x - a)$

a $x = a$

b $c\log_e(x - a) = 0 \Rightarrow x - a = e^0 \Rightarrow x = a + 1$

Therefore cooordinates of the x-axis intercept are $(a + 1, 0)$

c $c\log_e(x - a) = c \Rightarrow \log_e(x - a) = 1 \Rightarrow x = a + e^1$

Therefore cooordinates of the point where the curve crosses the line $y = c$ is $(e + a, c)$

d $$f(f^{-1}(x)) = x$$

$$c\log_e(f^{-1}(x) - a) = x$$

$$\log_e(f^{-1}(x) - a) = \frac{x}{c}$$

$$f^{-1}(x) = e^{\frac{x}{c}} + a$$

e (a, ∞)

f $f^{-1}(1) = 2 \Rightarrow f(2) = 1$

$f^{-1}(2) = 4 \Rightarrow f(4) = 2$

$$c\log_e(2 - a) = 1 \ldots (1)$$

$$c\log_e(4 - a) = 2 \ldots (2)$$

Equation (2) ÷ Equation (2)

$$\frac{\log_e(4 - a)}{\log_e(2 - a)} = 2$$

$$\log_e(4 - a) = 2\log_e(2 - a)$$

$$4 - a = (2 - a)^2$$

$$4 - a = 4 - 4a + a^2$$

$$a^2 - 3a = 0$$

$$a(a - 3) = 0$$

$$a = 0 \text{ or } a = 3$$

But $a = 3$ does not satisfy our equations

$\therefore c = \dfrac{1}{\log_e 2}, a = 0$

8 Consider,

$af^{-1}(x) + b = x$

$\therefore f^{-1}(x) = \dfrac{x}{a} - \dfrac{b}{a}$

If $f^{-1}(x) = 4x - 6$

$a = \dfrac{1}{4}$ and $-\dfrac{b}{a} = -6$

$\therefore a = \dfrac{1}{4}$ and $b = -\dfrac{3}{2}$

9 **a** $f^{-1}(x) = \left(\dfrac{x - 1}{3}\right)^3$

b $f^{-1}(x) = \left(\frac{x+2}{4}\right)^3$

c $f^{-1}(x) = \frac{1}{3}\left((x-4)^{\frac{1}{3}} + 2\right)$

d $f^{-1}(x) = \left(\frac{3-x}{2}\right)^{\frac{1}{3}}$

10 $$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(a \sin x) \\ &= \sqrt{a^2 - a^2 \sin^2 x} \\ &= a\sqrt{1 - \sin^2(x)} \\ &= a\sqrt{\cos^2(x)} \\ &= a \cos x \end{aligned}$$

Note that $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and therefore $\cos x > 0$.

Solutions to multiple-choice questions

1 B

$h(x) = \dfrac{x^4 + 2}{x^2}$

Split $h(x)$ into two separate fractions:

$$h(x) = x^2 + \frac{2}{x^2}$$

$$\therefore \quad f(x) = x^2, g(x) = \frac{2}{x^2}$$

2 E The graph of

$f: R \to R, f(x) = \cos(x)$

Is not a one to one function.

3 E

A: $e^{x+y} = e^x \times e^y$, so A is true.

B: $\log_e xy = \log_e x + \log_e^y$, so B is true.

C: $\log_e x^y = y \log_e x$, so C is true.

D: $f^{-1}(1) = \log_e 1 = 0$, true for any x, y.

4 D

$$\begin{aligned} f(x) &= \cos x \\ g(x) &= 3x^2 \\ g(f(x)) &= 3(\cos x)^2 \\ g\left(f\left(\frac{\pi}{3}\right)\right) &= 3\left(\cos\frac{\pi}{3}\right)^2 \\ g\left(f\left(\frac{\pi}{3}\right)\right) &= 3\left(\frac{1}{4}\right) \\ g\left(f\left(\frac{\pi}{3}\right)\right) &= \frac{3}{4} \end{aligned}$$

5 E

$f: R \to R, f(x) = (x - 2)^2$ is not an even function as it is a parabola that has been translated 2 units right, so it is not symmetrical about the y-axis.

6 E

$$y = 2ax + \cos 2x$$

When $x = \pi, y = 0$

$$\begin{aligned} 0 &= 2\pi a + \cos 2\pi \\ -1 &= 2\pi a \\ a &= -\frac{1}{2\pi} \end{aligned}$$

7 B

$$\begin{aligned} x > 5, g(x) &= \log_e(x-5) \\ 2[g(x)] &= g(f(x)) \\ 2[g(x)] &= \log_e(x-5)^2 \\ \log_e(x-5)^2 &= \log_e(f(x)-5) \\ (x-5)^2 &= f(x)-5 \\ x^2-10x+25 &= f(x)-5 \\ f(x) &= x^2-10x+30 \end{aligned}$$

8 C

9 D

10 C Domain of $(f+g) = (-\infty, 3) \cap [2, \infty) = [2, 3)$

11 B $\dfrac{y'-2}{-4} = \sin\left(3x' - \dfrac{\pi}{3}\right)$

$\therefore$ choose $y' = -4y + 2$ and $x' = \dfrac{1}{3}x + \dfrac{\pi}{9}$

12 D $\dfrac{y+3}{2} = \sin\left(2x - \dfrac{\pi}{4}\right)$ to $y' = \sin x'$ $\therefore$ choose $y' = \dfrac{y+3}{2} = \dfrac{1}{2}y + \dfrac{3}{2}$ and $x' = 2x - \dfrac{\pi}{4}$

Solutions to extended-response questions

1 **a** The range of $f: \mathbb{R}^+ \to \mathbb{R}, f(x) = e^{-x}$ is (0,1)

The range of $g: (-\infty, 0) \to R, g(x) = \dfrac{1}{x-1}$ is $\mathbb{R}^-$

b domain of f^{-1} = range of f = (0, 1)

domain of g^{-1} = range of $g = R^+$

To determine the rule for f^{-1} consider

$$x = e^{-y}$$

$$\log_e x = -y$$

$$-\log_e x = y$$

Therefore $f^{-1}(x) = -\log_e x$

To determine the rule for g^{-1} consider

$$x = \frac{1}{y-1}$$

Taking the reciprocal of both sides

$$y - 1 = \frac{1}{x}$$

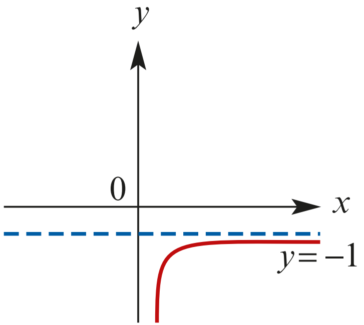
$$\text{and } y = \frac{1}{x} + 1$$

$$g^{-1}(x) = \frac{1}{x} + 1$$

c **i** $g \circ f(x)$ is defined as range of $f \subseteq$ domain of g

$$g \circ f(x) = g(f(x)) = g(e^{-x}) = \frac{1}{e^{-x} - 1} = \frac{e^x}{1 - e^x}$$

ii



$$g \circ f(x) = \frac{e^x}{1 - e^x} = -1 + \frac{1}{1 - e^x}$$

d **i** $g \circ f(x) = \dfrac{e^x}{1 - e^x}$ with domain = R^+ For the inverse consider

$$\frac{e^y}{1 - e^y} = x$$

Solve for y

$x(1-e^y) = e^y$

$x - xe^y = e^y$

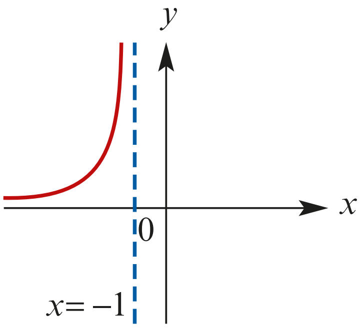
Therefore $\quad e^y(1+x) = x$

and $\quad e^y = \dfrac{x}{1+x}$

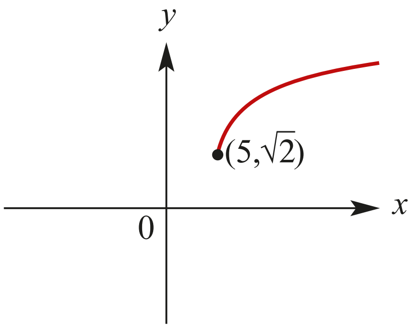
Therefore $\quad y = \log_e\left(\dfrac{x}{1+x}\right)$

$(g \circ f)^{-1}(x) = \log_e\left(\dfrac{x}{1+x}\right)$ and domain of $(g \circ f)^{-1}$ = range of $g \circ f = (-\infty, -1)$

ii



2 a i $f : [5, \infty) \to \mathbb{R}, f(x) = \sqrt{x-3}$



ii range of = $[\sqrt{2}, \infty)$

iii For the inverse rule consider $x = \sqrt{y-3}$

Square both sides and make y the subject.

$y = x^2 + 3$

and $f^{-1}(x) = x^2 + 3$. The domain of the inverse function is $[\sqrt{2}, \infty)$

b i h: $[4, \infty) \to \mathbb{R},\ h(x) = \sqrt{x-p}$

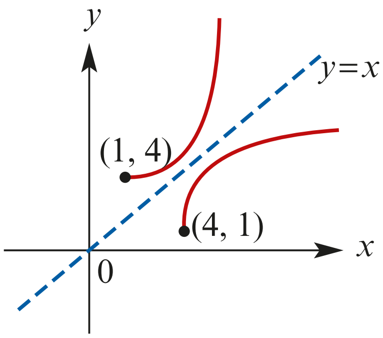
The inverse function has domain $[1, \infty)$.

The function h is increasing and therefore $\sqrt{4-p} = 1$

Therefore $p = 3$.

ii Proceeding as above the rule is $h^{-1}(x) = x^2 + 3$.

iii



3 $f\colon (0, \pi) \to \mathbb{R}, f(x) = \sin x$

$g\colon [1, \infty) \to \mathbb{R}, g(x) = \frac{1}{x}$

a range of $f = (0, 1)$

b range of $g = (0, 1]$

c $f \circ g$ is defined as the range of g $\subseteq$ domain of f as $1 < \pi$

$f \circ g(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x}\right)$

d $g \circ f$ is not defined as the range of $f \not\subset$ domain of g

e Consider

$x = \frac{1}{y}$ and solve for y.

Therefore $g^{-1}(x) = \frac{1}{x}$

The domain of g^{-1} = range of g = $(0, 1]$

The range of g^{-1} = domain of $g = [1, \infty)$

f Range of $f = (0, 1) \subseteq (0, 1]$ = domain of g^{-1}.

Therefore g^{-1} of is defined and

$g^{-1} \circ f(x) = g^{-1}(f(x)) = g^{-1}(\sin x) = \frac{1}{\sin\ x}$

The domain of $g^{-1} \circ f$ = domain of $f = (0,\ \pi)$

The range of $g^{-1} \circ f = [1, \infty)$

4 **a** $a = 2$

b $c = 2 - k\log_e(2)$

c $k = \frac{10}{\log_e(\frac{d+2}{2})}$

d $k = 10$

5 **a** Require that the range of $g \subseteq$ domain of f.

That is $(-\infty, b] \subseteq [0, \infty)$

Consider,

$$x^2 + 6x - 4 > 0$$

$\Leftrightarrow x < -3 - \sqrt{13}$ or $x > -3 + \sqrt{13}$

Therefore choose $b = -3 - \sqrt{13}$

b Domain of $f(g(x))$ is $(-\infty, -3 - \sqrt{13}]$.

The rule is $f(g(x)) = \sqrt{x^2 + 6x + 5}$

Range $= [3, \infty)$

c Consider the domain of $y = f(g(x)$. It is $(\infty, -3 - \sqrt{13}]$

Consider

$$x = \sqrt{y^2 + 6y + 5}$$

$$x^2 = y^2 + 6y + 5$$

$$x^2 = (y + 3)^2 - 4$$

$$\pm\sqrt{x^2 + 4} = (y + 3)$$

$$y = -3 \pm \sqrt{x^2 + 4}$$

The inverse will have rulet $h(x) = -3 - \sqrt{x^2 + 4}$

The domain $= [3, \infty)$ and the range $(\infty, -3 - \sqrt{13}]$

Chapter 8 – Revision of Chapters 1–7

Solutions to technology-free questions

1 **a** Domain = $\mathbb{R} \setminus \{0\}$; Range = $\mathbb{R} \setminus \{2\}$

b $3x - 2 \geq 0 \Rightarrow x \geq \frac{2}{3}$

The endpoint is $\left(\frac{2}{3}, 3\right)$

Domain = $\left[\frac{2}{3}, \infty\right)$; Range = $(-\infty, 3]$

c Domain = $\mathbb{R} \setminus \{2\}$; Range = $(3, \infty)$

d Domain = $\mathbb{R} \setminus \{2\}$; Range = $\mathbb{R} \setminus \{4\}$

e $x - 2 \geq 0 \Rightarrow x \geq 0$

The endpoint is $(2, -5)$

Domain = $[2, \infty)$; Range = $(-5, \infty]$

2 $\sqrt{f^{-1}(x) - 2} + 4 = x$

$$\therefore \sqrt{f^{-1}(x) - 2} = x - 4$$

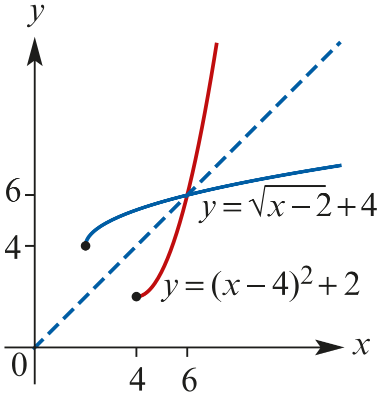
$$\therefore f^{-1}(x) - 2 = (x - 4)^2$$

$$\therefore f^{-1}(x) = (x - 4)^2 + 2$$

The domain of f^{-1} = range of f = $[4, \infty)$

Hence,

$f^{-1}\colon [4, \infty) \to \mathbb{R}$, $f^{-1}(x) = (x - 4)^2 + 2$



3 $\dfrac{f^{-1}(x) - 2}{f^{-1}(x) + 1} = x$

$$\therefore f^{-1}(x) - 2 = x(f^{-1}(x) + 1)$$

$$\therefore f^{-1}(x) - xf^{-1}(x) = x + 2$$

$$\therefore f^{-1}(x)(1 - x) = x + 2$$

$$\therefore f^{-1}(x) = \frac{x + 2}{1 - x}$$

The domain of f^{-1} = range of f = $\mathbb{R} \setminus \{1\}$

Hence,

$f^{-1}\colon \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x + 2}{1 - x}$

4 **a** $2e^{f^{-1}(x)} - 1 = x$

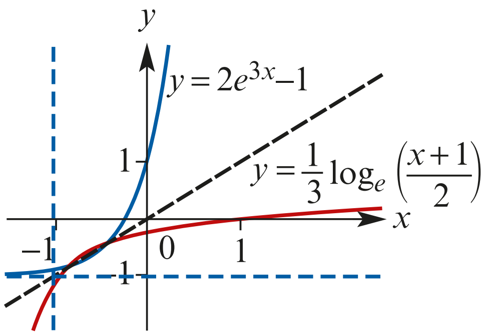
$$\therefore e^{f^{-1}(x)} = \frac{x + 1}{2}$$

$$\therefore f^{-1}(x) = \log_e\left(\frac{x + 1}{2}\right)$$

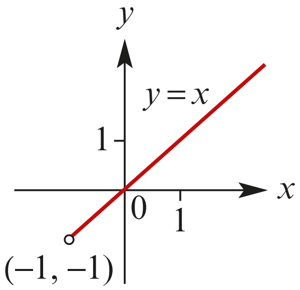
$f^{-1}(x) = \frac{1}{3}\log_e\left(\frac{x + 1}{2}\right)$,

dom f^{-1} = $(-1, \infty)$

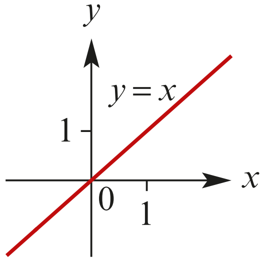
b



c



d



e $y = 2x$

5 $2\log_{10} 5 + 3\log_{10} 2 - \log_{10} 20$

$$= \log_{10} 25 + \log_{10} 8 - \log_{10} 20$$

$$= \log_{10} \frac{200}{20}$$

$$= \log_{10} 10$$

$$= 1$$

6

$$3\log_a x = 3 + \log_a 12$$

$$\log_a x^3 - \log_a 12 = 3$$

$$\log_a\left(\frac{x^3}{12}\right) = 3$$

$$\frac{x^3}{12} = a^3$$

$$x^3 = 12a^3$$

$$x = \sqrt[3]{12}a$$

7 $2^{-x} = 2^9$

$x = -9$

8 $\log_e(x + 12) = 1 + \log_e(2 - x)$

$$\log_e\left(\frac{x + 12}{2 - x}\right) = 1$$

$$\frac{x + 12}{2 - x} = e^1$$

$$x + 12 = 2e - ex$$

$$(e + 1)x = 2e - 12$$

$$x = \frac{2e - 12}{e + 1}$$

9 $\log_a 4 \times \log_{16} a = \log_a 4 \times \times \dfrac{\log_a a}{\log_a 16}$

$$= 2\log_a 2 \times \frac{\log_a a}{4\log_a 2}$$

$$= \frac{1}{2}$$

10 $4e^{2x} = 9$

$$e^{2x} = \frac{9}{4}$$

$$2x = \log_e\left(\frac{9}{4}\right)$$

$$x = \frac{1}{2}\log_e\left(\frac{9}{4}\right) = \log_e\left(\frac{3}{2}\right)$$

11 **a** When $x = 0$, $f(0) = 2\log_e 2$

When $f(x) = 0$

$$2\log_e(x + 2) = 0$$

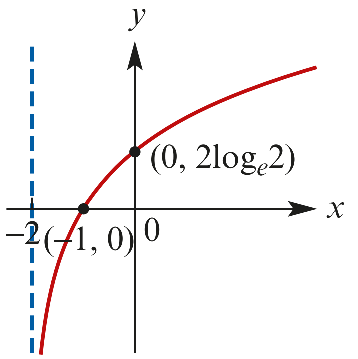
$$\log_e(x + 2)t = 0$$

$$x + 2 = e^0$$

$$x = -1$$

$\therefore a = -1$ and $b = 2\log_e 2$

b



12 $2^{4x} - 5 \times 2^{2x} + 4 = 0$

$$\begin{aligned} \text{Let } a &= 2^{2x} \\ a^2 - 5a + 4 &= 0 \\ (a-4)(a-1) &= 0 \\ a = 4 \text{ or } a &= 1 \\ \therefore 2^{2x} = 4 \text{ or } 2^{2x} &= 1 \\ \therefore x = 0 \text{ or } x &= 1 \end{aligned}$$

13 $\sin\left(\dfrac{3x}{2}\right) = \dfrac{1}{2}$

$$\frac{3x}{2} = -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = -\frac{7\pi}{9} \text{ or } x = \frac{\pi}{9} \text{ or } x = \frac{5\pi}{9}$$

14 **a** Range $= [2, 8]$; Period $= 6$

b $\cos\left(2x + \dfrac{\pi}{6}\right) = \dfrac{1}{2}$

$$\begin{aligned} 2x + \frac{\pi}{6} &= \frac{\pi}{3}, \frac{5\pi}{3} \\ 2x + \frac{\pi}{6} &= \frac{2\pi}{6}, \frac{10\pi}{6} \\ 2x &= \frac{\pi}{6}, \frac{9\pi}{6} \end{aligned}$$

$$x = \frac{\pi}{12} \text{ or } x = \frac{3\pi}{4}$$

15 Consider the gradients of the two lines

Gradient $\ell_1 = -m$

and Gradient $\ell_2 = -\dfrac{2}{m-1}$

If the gradients are equal

$$\begin{aligned} -m &= -\frac{2}{m-1} \\ m &= \frac{2}{m-1} \end{aligned}$$

$$m^2 - m - 20$$

$$(m-2)(m+1) = 0$$

$$m = 2 \text{ or } m = -1$$

a Therefore a unique soluton when the lines are not parallel, $m \in \mathbb{R} \setminus \{-1, 2\}$

b If $m = 2$

$2x + y = 2$

$2x + y = -4$

The lines are parallel but do not coincide.

There is no solution.

$m = -1$ is checked in the next part.

c If $m = -1$

$-x + y = 2$

$2x - 2y = -4$

The lines coincide and there are infinitely many solutions.

16 $y = \dfrac{a}{x^2} + b$

When $x = 1, y = -1$

When $x = -2, y = \dfrac{1}{2}$

$$a + b = -1 \ldots (1)$$

$$\frac{a}{4} + b = \frac{1}{2} \ldots (2)$$

Equation (1) − Equation (2)

$$\frac{3a}{4} = -\frac{3}{2}$$

$\therefore a = -2$ and $b = 1$

17 $\Delta = m^2 - 8$

a $\Delta = 0 \Rightarrow m = \pm 2\sqrt{2}$

b $\Delta > 0 \Rightarrow m > 2\sqrt{2}$ or $m < -2\sqrt{2}$

c $\Delta < 0 \Rightarrow -2\sqrt{2} < m < 2\sqrt{2}$

18
$$\begin{aligned}\Delta &= 4(a^2 + a)^2 - 24a(a + 1)\\ &= 4(a^4 + 2a^3 + a^2) - 24a^2 - 24a\\ &= 4a^4 + 8a^3 + 4a^2 - 24a^2 - 24a\\ &= 4a(a^3 + 2a^2 - 5a - 6)\\ &= 4a(a - 2)(a + 1)(a + 3)\end{aligned}$$
For one solution $\Delta = 0$. Therefore,$a = 2$ or $a = -1$ or $a = -3$

19 **a** **i** $\frac{a + 3}{2} = 0, \therefore a = -3$

ii $\sqrt{(a - 3)^2 + (-2 - 1)^2} = \sqrt{13}$
$$\begin{aligned}a^2 - 6a + 9 + 9 &= 13\\ a^2 - 6a + 5 &= 0\\ (a - 5)(a - 1) &= 0\end{aligned}$$
$a = 5$ or $a = 1$

iii $\frac{3}{3 - a} = \frac{1}{2}$
$$\begin{aligned}6 &= 3 - a\\ a &= -3\end{aligned}$$

b If $a = -2$ the gradient of the line is $\frac{3}{5}$
The equation of the line is
$y - 1 = \frac{3}{5}(x - 3)$
or $5y - 3x + 4 = 0$
The angle the line makes with the positive direction of the x-axis is $\tan^{-1}\left(\frac{3}{5}\right)$.

20 **a** Odd

b $f^{-1}(x) = \sqrt[3]{\frac{x}{2}}$

c **i** 2

ii -1

iii $f^{-1}(x) = f(x)$
$$\begin{aligned}\sqrt[3]{\frac{x}{2}} &= 2x^3\\ \frac{x}{2} &= 8x^9\\ x - 16x^9 &= 0\\ x(1 - 16x^8) &= 0\\ x = 0 \text{ or } x &= \left(\frac{1}{16}\right)^{\frac{1}{8}}\\ x = 0 \text{ or } x = 2^{-\frac{1}{2}} &\text{ or } x = -2^{-\frac{1}{2}}\end{aligned}$$

21 **a** 4

b $\sqrt{5}$

c $2 - 2a$

d $\sqrt{2a - 5}$

e $x = -8$

f $x = \frac{103}{2}$

g $x < 1$

22 **a** **i** $f \circ g(x) = 4x^2 + 8x - 3$

ii $g \circ f(x) = 16x^2 - 16x + 3$

iii $g \circ f^{-1}(x) = \frac{1}{16}(x^2 + 14x + 33)$

b Dilation of factor $\frac{1}{4}$ from the y-axis, then translation $\frac{3}{4}$ units to the right

c Translation 1 unit to the left and 1 unit down

23
$$1 - \sin\left(\frac{x}{4}\right) = \sin\left(\frac{x}{4}\right)$$
$$2\sin\left(\frac{x}{4}\right) = 1$$
$$\sin\left(\frac{x}{4}\right) = \frac{1}{2}$$
$$\frac{x}{4} = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{11\pi}{6}, -\frac{7\pi}{6}$$
$$x = \frac{2\pi}{3}$$
The only solution

24 $x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

25
$$Ae^k = 4 \ldots (1) \qquad A = \frac{8}{5} \text{ and}$$
$$Ae^{2k} = 10 \ldots (2)$$
Equation (2) ÷ Equation (1)
$$e^k = \frac{5}{2}$$
$$k = \log_e\left(\frac{5}{2}\right)$$
$$k = \log_e\left(\frac{5}{2}\right)$$

26 **a** Period= $\frac{2\pi}{5}$

b Amplitude = 8

c **i** Consider $y = \sin x$ and $y' = 8\sin 5x'$

Reorganise the second equation to $\frac{y'}{8} = \sin 5x'$

Then we can write, $\frac{y'}{8} = y$ and $x = 5x'$

Hence $y' = 8y$ and $x' = \frac{x}{5}$ Hence a sequence of transformations is:

- A dilation of 8 from the x-axis.
- A dilation of $\frac{1}{5}$ from the y-axis.

ii Consider $y = \cos x = \sin\left(x - \frac{\pi}{2}\right)$ and $y' = 8\sin 5x'$

Reorganise the second equation to
$\frac{y'}{8} = \sin 5x'$
Then we can write,
$\frac{y'}{8} = y$ and $x - \frac{\pi}{2} = 5x'$
and therefore,
$y' = 8y$ and $x' = \frac{x}{5} - \frac{\pi}{10}$
Hence a sequence of transformations is:

- A dilation of 8 from the x-axis.
- A dilation of $\frac{1}{5}$ from the y-axis.
- A translation of $\frac{\pi}{10}$ in the negative direction of the x-axis.

27 **a** $P(1) = 0 \Rightarrow 1 + a + b - 12 + 4 = 0$
$\Rightarrow a + b = 7 \ldots (1)$
$P(2) = 0 \Rightarrow 16 + 8a + 4b - 24 + 4 = 0$
$\Rightarrow 8a + 4b = 4$
$\Rightarrow 2a + b = 1 \ldots (2)$
From (1) and (2)
$a = -6$ and $b = 13$

b

$$P(x) = (x-1)(x-2)(x^2+cx+d)$$
$$= x^4 - 6x^3 + 13x^2 - 12x + 4$$
$$\Rightarrow (x^2-3x+2)(x^2+cx+d)$$
$$= x^4 - 6x^3 + 13x^2 - 12x + 4$$
$$\Rightarrow 2d = 4 \text{ and } 2c - 3d = -12$$
$$\Rightarrow d = 2 \text{ and } 2c = -6$$
$$\Rightarrow P(x) = (x-1)(x-2)(x^2-3x+2)$$
$$\Rightarrow P(x) = (x-1)^2(x-2)^2$$

28 $g(x) = \frac{1}{2}f(x+4)$ and
$h(x) = 2g(5x-11) + 3$
$g(5x-11) = \frac{1}{2}f(5x-11+4)$
$= \frac{1}{2}f(5x-7)$
Therefore,
$h(x) = f(5x-7) + 3$

29 Let $f(x) = h(x) - x$
$= x^3 + (a-1)x + b$
We know,
$f(2) = 0$ and $f(3) = 0$
Therefore,
$8 + 2(a-1) + b = 0 \ldots (1)$
$27 + 3(a-1) + b = 0 \ldots (2)$
which become,
$2a + b = -6 \ldots (1')$
$3a + b = -24 \ldots (2')$
Hence, $a = -18$ and $b = 30$

30 $2^{2n} - 8 \times 2^n + 10 = 10$
$2^{2n} - 8 \times 2^n = 0$
$2^n(2^n - 8) = 0$
Therefore, $n = 3$

31 $f(g(x)) = 0$
$e^{2x} - 7 \times e^x + 6 = 0$
$(e^x - 6)(e^x - 1) = 0$
$x = \log_e 6$ or $x = 0$

32 Write,
$y = 2x\cos x$ and $y' = 2(x' - 5\pi)\cos x'$
Rewrite the second equation as,
$y' = -2(x' - 5\pi)\cos(x' - 5\pi)$
Then we can write,
$y' = -y$ and $x' - 5\pi = x$
Hence we can write,
$(x, y) \rightarrow (x + 5\pi, -y)$

33 **a** $2x^3 - 3x^2 - 11x + 6 \geq 0$
$\Leftrightarrow (2x-1)(x-3)(x+2) \geq 0$
$\Leftrightarrow -2 \leq x \leq \frac{1}{2}$ or $x \geq 3$

b $-x^3 + x^2 - 4x > 0$
$\Leftrightarrow x(x^2 - x + 4) < 0$
$\Leftrightarrow x < 0$

34 **a** $f(g(x) = f(\log_e(x))$
$= e^{3\log_e(x)+2}$
$= e^2 e^{\log_e(x^3)}$
$= e^2 x^3$

b $f(g(2) = 8e^2$
$k = 8$

Solutions to multiple-choice questions

1 D Domain = $[-1, 3)$ since -1 is included and 3 is excluded.

2 A For each value of $x > 0$, the rule $x = 2y^2$, $x \geq 0$ gives two value for y, so is not a function.

3 B Require $2 - x > 0$, i.e. $x < 2$. Implied domain = $(-\infty 2)$

4 E
$$f\left(-\frac{1}{a}\right) = \frac{-\frac{1}{a}}{-\frac{1}{a} - 1} = \frac{-1}{-1 - a} = \frac{1}{a + 1}$$

5 E
$$(f + g)\left(\frac{3\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) + g\left(\frac{3\pi}{2}\right) = \sin(3\pi) + 2\sin\left(\frac{3\pi}{2}\right) = -2$$

6 C
$$f(g(3)) = f(18) = 56$$

7 A
$$\begin{aligned} \text{dom } f &= [0, 6];\ \text{dom } g = (-\infty, 2] \\ \text{dom } (f + g) &= \text{dom } f \cap \text{dom } g = [0, 2] \end{aligned}$$

8 B
$$fg(x) = (2x^2 + 1)(3x + 2) = 6x^3 + 4x^2 + 3x + 2$$

9 C Require $4 - x^2 \geq 0$, i.e.
$(2 - x)(2 + x) \geq 0$
$-2 \leq x \leq 2$
Implied domain = $[-2, 2]$

10 A For $x \leq 0$, the gradient is -2 and the y intercept is $(0, -2)$; the equation is $y = -2x - 2$ for $x \leq 0$. For $x > 0$, the gradient is 1 and the y intercept is $(0, -2)$; the equation is $y = x - 2$ for $x > 0$.

11 C Reflect the graph of $y = f(x)$ in the line $y = x$.
Then the endpoint $(4, -2)$ reflects to $(-2, 4)$.
Only the third graph fits.

12 C $a = 1$

13 B

14 A For $x < 2$, the straight line has gradient | and the y-intercept is $(0, -3)$; the equations is $y = x - 3$ for $x < 2$. For $x \geq 2$, the curve has equation $y = (x - 2)^2$.

15 E $f(2) = 0$, $f(3) = 2$ so dom f^{-1} = ram $f = [0, 2]$. For f, $y = 2x - 4$. For f^{-1}, interchange x and y and solve for y.

$$\begin{aligned} x &= 2y - 4 \\ x + 4 &= 2y \\ y &= \frac{x + 4}{2} \end{aligned}$$

Hence:
$f^{-1} : [0, 2] \to R$, $f^{-1}(x) = \dfrac{x + 4}{2}$

16 D The only one-to-one graph.

17 E For f, $y = 3x - 2$
For f^{-1}, interchange x and y and solve for y.

$$x = 3y - 2$$
$$x + 2 = 3y$$
$$y = \frac{1}{3}(x + 2)$$

18

C $y = \frac{4}{5}x - 4 = \frac{4}{5}(x - 5)$

x − axis intcept is (5, 0)

y − as is intercept is (0, −4).

$$\text{area } OAB = \frac{1}{2}(OA)(OB)$$
$$= \frac{1}{2}(5)(4)$$
$$= 10 \text{ square units}$$

19 D $2x - 3y = 12$ ①

$3x - 2y = 13$ ②

② − ① gives:

$x + y = 1$

(Note: in this case, you do not need to solve for x and y explicitly, although it is not wrong to do so.)

20 D $7x - 6y = 20$ ①

$3x + 4y = 2$ ②

3 × ② + 2 × (*i*) gives :

$$9x + 14x = 6 + 40$$
$$23x = 46$$
$$x = 2$$

21 B $0 = -\frac{\pi a}{2} + \sin\left(-\frac{\pi}{2}\right)$

$$1 = -\frac{\pi a}{2}$$
$$a = -\frac{2}{\pi}$$

22 C

23 A $kx - 1 = x^2 + 3x$

$$x^2 + (3 - k)x + 1 = 0$$
$$\Delta = (3 - k)^2 - 4$$
$$= 9 - 6k + k^2 - 4$$
$$= k^2 - 6k + 5$$
$$= (k - 1)(k - 5)$$
$$\Delta < 0 \Rightarrow 1 < k < 5$$

24 B $f(g(7)) = f(5) = 7$

25 E

$$3\sin(2x) + \sqrt{3}\cos(2x) = 0$$
$$3\sin(2x) = -\sqrt{3}\cos(x)$$
$$\tan(2x) = -\frac{1}{\sqrt{3}}$$
$$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}$$
$$x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}$$

26 E The graph is that of a hyperbola with asymptotes $x = 1$ and $y = -3$.

The equation is of the form

$$y = \frac{a}{x - 1} - 3$$

$x = 0, y = -4$: $\quad -4 = \frac{a}{-1} - 3$

$$-4 = -a - 3$$
$$a = 1$$
$$y = \frac{1}{x - 1} - 3$$

(Check: $x = \frac{4}{3}$, $y = 0$ as expected.)

27 A As $x \to \pm\infty$, $y = f(x) \to -2$

$y = -2$ is an asymptote So the range is $R\backslash\{-2\}$

28 D Vertex at (2, 3) means

$y = a(x - 2)^2 + 3$.

Only the fourth option in which

$a = 1$ fits.

29 D Require $f(-x) = f(x)$ for any value of x.
If $f(x) = -x^2$, then

$$\begin{aligned} f(-x) &= -(-x)^2 \\ &= -x^2 \\ &= f(x) \end{aligned}$$

So $f(x) = -x^2$ is an even function of x.
(A quick check reveals that none of the other functions is even.)

30 A The factor '$x + 2$' indicates a translation of 2 units to the left.
So $(x, y) \rightarrow (x - 2, y)$.
The factor '3' indicates a dilation of factor 3 from the x-axis.

31 E Require $x - 2 \geq 0$, i.e. $x \geq 2$.
Maximal domain = $[2, \infty)$

32 D The graph has endpoint (3, 1) so its equation must be of the form
$y = a\sqrt{x - 3} + 1$
$x = 4,\ y = 0{:}0 = a\sqrt{1} + 1$

$$a = -1$$

$y = -\sqrt{x - 3} + 1$

33 E $\dfrac{3}{(x-2)^2} > 0$ for any $x \neq 2$.
So $\dfrac{3}{(x-2)^2} + 4 > 4$ for any $x \neq 2$.
The range is $(4, \infty)$.

34 A

$$\begin{aligned} 3(1)^2 + k(1) + 1 &= k + 4 \\ &= 0 \\ k &= -4 \end{aligned}$$

35 E Let $P(x) = x^3 - 5x^2 + x + k$.
$P(x)$ is divisible by $x + 1$, so

$$\begin{aligned} P(-1) &= 0. \\ (-1)^3 - 5(-1)^2 + (-1) + k &= 0 \\ -1 - 5 - 1 + k &= 0 \\ k &= 7 \end{aligned}$$

36 C The graph could be a cubic with minimum turning point at $(-2, 0)$ and another x - intercept at $(2, 0)$.
Equation is $y = a(x + 2)^2(x - 2)$
$x = 0,\ y > 0{:} = 8a > 0$

$$a < 0$$

Only the third option fits.

37 D The graph could be a cubic with a Stationary points of inflexion as $(-1, 2)$.
Equation is $y = a(x + 1)^3 + 2$.
Only the fourth option fits.
$y = -\frac{1}{2}(x + 1)^3 + 2$, then when $x = 0$
(Check: If

$$\begin{aligned} y &= -\frac{1}{2} + 2 \\ &= 1\frac{1}{2} \end{aligned}$$

which is consistent with the graph.)

38 C $P(-1) = -1 + 2 + 5 - 6 = 0$
So $(x + 1)$ is a factor.
Option B expanded has a constant term of +6.
Option C expanded has constant term of −6.
Option D expanded has constant term of +6
Only option C fits.
(Alternatively, divide the cubic by $(x + 1)$ and factorise the resulting quadratic.)

39 E For $f.y = mx + 3$

For f^{-1}, interchange x and y and solve for y.

$$\begin{aligned} x &= my + 3 \\ x - 3 &= my \\ y &= \frac{1}{m}(x-3) \\ &= \frac{1}{m}x - \frac{3}{m} \end{aligned}$$

Hence $a = \frac{1}{m}, b = -\frac{3}{m}$

40 B Remainder is given by $P(2)$

$$\begin{aligned} P(2) &= 2(2)^3 - 2(2)^2 + 3(2) + 1 \\ &= 16 - 8 + 6 + 1 \\ &= 15 \end{aligned}$$

41 B Let $P(x) = x^3 + 2x^2 + ax - 4$

Given $P(-1) = 1$

$$\begin{aligned} (-1)^3 + 2(-1)^2 + a(-1) - 4 &= 1 \\ -1 + 2 - a - 4 &= 1 \\ -a - 3 &= 1 \\ a &= -4 \end{aligned}$$

42 C The graph could be a quartic with minimum tuning point at (–2, 0) and (2, 0).

$$\begin{aligned} \text{Equation is } y &= a(x+2)^2(x-2)^2 \\ &= a((x+2)(x-2))^2 \\ &= a(x^2-4)^2 \end{aligned}$$

When $x = 0$, $y = a \times (-4)^2 = 16a$

For the graph the y intercept is posture so $a > 0$

Only the third alternative fits.

43 D As $x \to \infty$, $f(x) \to 1$,

Since $e^{-x} > 0$ for all x, $f(x) > 1$ for all x.

Hence the range of f is $(1, \infty)$ and this is the domain of f^{-1}

44 B For f, $y = 2\log_e x + 1$

for f^{-1}, interchange x and y and solve for y.

$$\begin{aligned} x &= 2\log_e y + 1 \\ x - 1 &= \log_e y \\ \log_e y &= \frac{1}{2}(x-1) \\ y &= e^{\frac{1}{2}(x-1)} \end{aligned}$$

So $f^{-1}(x) = e^{\frac{1}{2}(x-1)}$

45 B $\log_e(-1+2) = \log_e 1 = 0$, so range of $g = (0, \infty)R^+$

$e^{-0} = 1$ and as $x \to \infty$, $e^{-x} \to 0$

Hence the range of the function with rule $y = f(g(x))$ is $(0, 1)$.

46 E For f, $y = e^x - 1$

For f^{-1}, interchange x and y and solve for y.

$$\begin{aligned} x &= e^y - 1 \\ x + 1 &= e^g \\ y &= \log_e(x+1) \end{aligned}$$

So $f^{-1}(x) = \log_e(x+1)$.

47 A $f(4) = \log_e(4-3) = \log_e 1 = 0$, so f has range $[0, \infty)$

and this is the domain of the inverse.

48 C For f, $y = e^{x-1}$

For f^{-1}, interchange x and y and solve for y.

$$\begin{aligned} x &= e^{y-1} \\ \log_e x &= y - 1 \\ y &= 1 + \log_e x \end{aligned}$$

So $f^{-1}(x) = 1 + \log_e x$

49 D For f, $y = \log_e \frac{x}{2}$

For f^{-1}, interchange x and y and solve for y.

$$x = \log_e \frac{y}{2}$$

$$e^x = \frac{y}{2}$$

$$y = 2e^x$$

So $f^{-1}(x) = 2e^x$.

50 B Require $3x - 2 > 0$, i.e.

$$3x > 2$$

$$x > \frac{2}{3}$$

So f is defined for $x \in \left(\frac{2}{3}, \infty\right)$.

51 C The Graph of f has asymptote $x = -2$.
Reflecting it in the line $y = x$ means its inverse has asymptote $y = -2$.
Only the third option fits.

52 C Method 1

$$\log_2 8x = \log_2 8 + \log_2 x$$

$$= \log_2 2^3 + \log_2 x$$

$$= 3\log_2 2 + \log_2 x$$

$$= \log_2 x + 3$$

$$\log_2 2x = \log_2 2 + \log_2 x$$

$$= \log_2 x + 1$$

So the equation becomes

$$\log_2 x + 3 + \log_2 x + 1 = 6$$

$$2\log_2 x = 2$$

$$\log_2 x = 1$$

$$x = 2$$

Method 2

$$\log_2 8x - \log_2 2x = 6$$

$$\log_2(8x \times 2x) = 6$$

$$\log_2(16x^2) = 6$$

$$16x^2 = 2^6$$

$$= 64$$

$$x^2 = 4$$

$$x = \pm 2$$

But $x > 0$, so $x = 2$.

53 A $\log_{10} x = y(\log_{10} 3) + 1$

$$= \log_{10} 3^y + \log_{10} 10$$

$$= \log_{10}(10(3^y))$$

$$x = 10(3^y)$$

54 B Graph has gradient -2 and y intercept $(0, 2)$.
Equation is $\log_e N = -2t + 2$

$$N = e^{2-2t}$$

55 A As $x \to -\infty$, $y \to 1$, so the rule must involve $e^{-x.}$
When $x = 0$, $y = 0$.
Only the first option fits both of these.

56 B $x = -2$ is a vertical asymptote and the domain is $(-2, \infty)$, so only the second and fourth options are possible.
The graph through $(0, 0)$.
B: $\log_e \frac{1}{2}(0 + 2) = \log_e 1 = 0$
E: $\frac{1}{2}\log_e(0 + 2) = \frac{1}{2}\log_e(2) \neq 0$
So the second option fits.

57 D Period $= \frac{5\pi}{12} - \left(-\frac{\pi}{4}\right) = \frac{8\pi}{12} = \frac{2\pi}{3}$
Range $= [-4, 0]$ so amplitude $= 2$ and there is a vertical translation of 2 units down.

these rule out options A and B.
When $\theta = \frac{\pi}{4}$, $y = 0$.
In the third and fifth options, when $\theta = -\frac{\pi}{4}$, $y = -4$; in the fourth option:

$$y = 2\cos 3\left(-\frac{\pi}{4} + \frac{\pi}{4}\right) - 2$$

$$2\cos 0 - 2$$

$$= 2 - 2$$

$$= 0$$

So the fourth option fits.

58 D The minimum value of f is $2 - 3 = -1$
The maximum value of f is $2 + 3 = 5$.
The range of f is $[-1, 5]$

59 A When $x = \frac{\pi}{6}$, $y = 0$. Only options A and C satisfy this.
When $\frac{\pi}{6} < x < \frac{7\pi}{6}$, $y > 0$.
This is true for option A but false for option C.

60 A Amplitude 3, period $= \frac{2\pi}{2} = \pi$

61 D The minimum value of f is -3.
The maximum value of f is 3.
The range of f is $[-3, 3]$

62 B $P(x) = 0 \Rightarrow x - 2a = 0$ or

$x + a = 0$ or $x^2a = 0$.

So $x = 2a$ or $x = a$ or $x^2 = -a$.
But $\alpha > 0$ so $x^2 = -a$ has no solutions.
The equation has 2 decimal real solutions.

63 D The gradient of the given straight live is -2.
For perpendicular lines, $m_1m_2 = -1$.
So $-2m_2 = -1$, going $m_2 = \frac{1}{2}$.

64 B Since x and y are interchanged for the inverse, there must be an asymptote will equation $x = 6$ for the inverse function.
So the inverse has vertical asymptote with equation $x = 6$.

65 E Period $= \frac{2\pi}{a}$

66 C $f(18) = 32 = 2^5, f(34) = 64 = 2^6$.

$g(2^5) = \log_2 2^5 = 5$

$g(2^6) = \log_2 2^6 = 6$

The range of $g \circ f$ is $[5, 6]$

67 C Interchange x & y: $x = y^2 - 4y + 5$

Solve for y: $x = (y - 2)^2 + 1$

$(y - 2)^2 = x - 1$

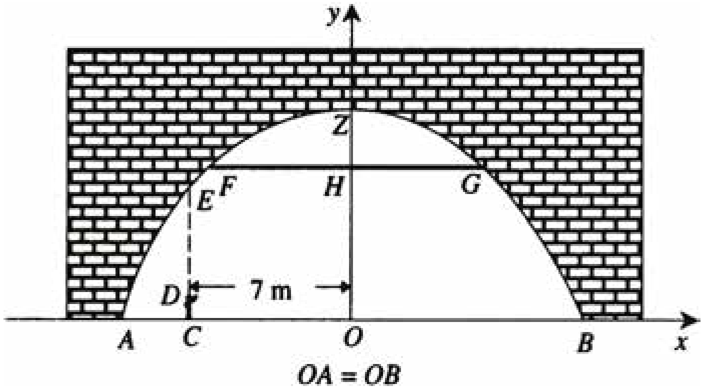
$y = 2 \pm \sqrt{x - 1}$

Because of domain restriction, the rule of the inverse is$y = 2 - \sqrt{x - 1}$

68 B The vertex occurs when $x = 2$. The only one-to -one function is that with domain $(2, \infty)$

69 C The domain of $f + g$ is $[-5, -3)$.
$h(x) = (f + g)(x) = x^2 - 5x + 6$. The vertex occurs when $x = \frac{5}{2}$ which is outside the domain. $h(-5) = 56$ and $h(-3) = 30$. Therefore domain of $h^{-1} = rangeof$h$= (30, 56)$

Solutions to extended-response questions



1 **a** The graph is of the form $y = ax^2 + b$

The vertex is at $(0, 9)$.

Therefore $b = 9$

The width of the arch is 20m.

Therefore the x-axis intercepts are at $(10, 0)$ and $(-10, 0)$

When $x = 10,\ y = 0$.

Hence $b = 9$

$$\text{and} \quad 0 = a \times 100 + 9$$

$$\therefore \quad a = \frac{-9}{100} = -0.09$$

b The equation of the curve is $y = \dfrac{-9}{100}x^2 + 9$

When $x = -7$ $y = \dfrac{-9}{100} \times 49 + 9 = 4.59$

The man is 1.8 m high.

$\therefore$ E is $(4.56 - 1.8)$ m $= 2.79$ m above the man's head.

c OH is 6.3 m

$\therefore$ Consider $y = 6.3$

$$6.3 = \frac{-9}{100}x^2 + 9$$

$$\frac{-2.7}{-9} \times 100 = x^2$$

$$\therefore \quad 30 = x^2$$

$$\therefore \quad x = \pm\sqrt{30}$$

The length of the bar is $2\sqrt{30}$ m ≈ 10.95 m.

2 **a** Let $P(x) = 2x^3 + ax^2 - 72x - 18$

By the Remainder Theorem

$$P(-5) = 17$$

$$\text{i.e. } 2 \times (-5)^3 + a(-5)^2 - (72 \times -5) - 18 = 17$$

$$250 + 25a + 360 - 18 = 17$$

$$\therefore \ 25a = -75$$

$$a = -3$$

b $2x^3 = x^2 + 5x + 2$

Let $P(x) = 2x^3 - x^2 - 5x - 2$

$P(-1) = -2 - 1 + 5 - 2 = 0$

$\therefore$ By the Factor Theorem $x + 1$ is a factor.

Dividing $P(x)$ by $x + 1$

$P(x) = (x + 1)(2x^2 - 3x - 2)$

For $P(x) = 0$

$x = -1$ or $2x^2 - 3x - 2 = 0$

and $2x^2 - 3x - 2 = 0$

implies $(2x + 1)(x - 2) = 0$

$\therefore$ $x = -\frac{1}{2}$ or $x = 2$

i.e. $x = -\frac{1}{2}$, $x = 2$ and $x = -1$ are solutions to the equation $2x^3 = x^2 + 5x + 2$

c $x^2 - 5x + 7$ leaves the same remainder when divided by $x - b$ or $x - c$

By the Remainder Theorem

$$b^2 - 5b + 7 = c^2 - 5c + 7$$

$$\Leftrightarrow \quad b^2 - c^2 = 5(b - c)$$

$$\Leftrightarrow \quad (b - c)(b + c) = 5(b - c)$$

$$\Leftrightarrow \quad b + c = 5 \text{ as } b \neq c$$

$$\therefore \quad b = 5 - c$$

and if $4bc = 21$

$$4(5 - c)c = 21$$

$$20c - 4c^2 - 21 = 0$$

$$\therefore \quad 4c^2 - 20c + 21 = 0$$

$$(2c - 3)(2c - 7) = 0$$

which implies

$$c = \frac{3}{2} \text{ or } c = \frac{7}{2}$$

If $c = \frac{3}{2}$, $4 \times b \times \frac{3}{2} = 21$

$\therefore$ $b = \frac{7}{2}$

If $c = \frac{7}{2}$, $b = \frac{3}{2}$

As $b > c$ the required values are $b = \frac{7}{2}$, $c = \frac{3}{2}$

3 **a** If $ax^2 + 7x + 3$is positive for all x then $a > 0$ and $ax^2 + 7x + 3 = 0$ has no solutions.

$\Delta = 49 - 12a$

$\Delta < 0 \Leftrightarrow 49 - 12a < 0$

$\Leftrightarrow a > \dfrac{49}{12}$

$\therefore$ minimum integer value $= 5$

b $-3x^2 + bx - 4 < 0$ for all x then $-3x^2 + bx - 4 = 0$ has no solutions.

$\Delta = b^2 - 48$

$\Delta < 0 \Leftrightarrow b^2 - 48 < 0$

$\Leftrightarrow -4\sqrt{3} < b < 4\sqrt{3}$

$\therefore$ minimum integer value $= -6$

c **i** Assume $a + b + c = 0$

Then $b = -(a + c)$ $b^2 - 4ac = (a + c)^2 - 4ac$

$$= a^2 - 2ac + c^2$$

$$= (a - c)^2$$

ii Assume $b - a - c = 0$

Then $b = (a + c)$ $b^2 - 4ac = (a + c)^2 - 4ac$

$$= a^2 - 2ac + c^2$$

$$= (a - c)^2$$

iii If $\Delta = b^2 - 4ac$ is a perfect square the quadratic has rational solutions.

iv For example if $a = 4, b = -2$ and $c = -2$ we have

$4x^2 - 2x - 2 = 2(2x^2 - x - 1) = 2(2x + 1)(2x - 1)$

If we have $b = 6, a = 1, c = 5$ we have

$x^2 + 6x + 5 = (x + 5)(x + 1)$ and so on.

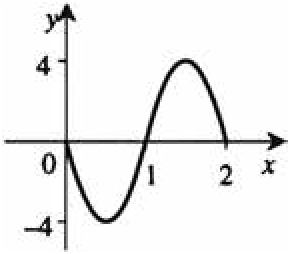
v $x^2 - 3x - 54 = (x - 9)(x + 6)$

4 **a** $x = -4\sin \pi t$

b **i** When $t = 0$, $x = -4\sin 0 = 0$

ii When $t = \dfrac{1}{2}$, $x = -4\sin\dfrac{\pi}{2} = -4$

iii When $t = 1$, $x = -4\sin\pi = 0$



c When $x = 2$

$$2 = -4\sin \pi t$$

$$\therefore\ -\frac{1}{2} = \sin(\pi t)$$

$$\text{i.e. } \frac{7\pi}{6} = \pi t$$

$$\therefore\ t = \frac{7}{6}$$

d Period $= \frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$ seconds

5 **a** $h = ax - x^2 - x = x(a - 1) - x^2$

b Maximum occurs when $x = \frac{a - 1}{2}$

c Maximum

$$= \frac{a-1}{2} \times (a - 1) - \left(\frac{a-1}{2}\right)^2$$

$$= \left(\frac{a-1}{2}\right)^2$$

d **i** $\left(\frac{a-1}{2}\right)^2 = \frac{1}{4}$

$\frac{a-1}{2} = \pm\frac{1}{2}$

$a = 2$ since $a > 1$

ii $\left(\frac{a-1}{2}\right)^2 = 1$

$\frac{a-1}{2} = \pm 1$

$a = 3$ since $a > 1$

iii $\left(\frac{a-1}{2}\right)^2 = 5$

$\frac{a-1}{2} = \pm\sqrt{5}$

$a = 1 + 2\sqrt{5}$ since $a > 1$

iv $\left(\frac{a-1}{2}\right)^2 = 9$

$\frac{a-1}{2} = \pm 3$

$a = 7$ since $a > 1$

v $\left(\dfrac{a-1}{2}\right)^2 = 10$

$\dfrac{a-1}{2} = \pm\sqrt{10}$

$a = 1 + 2\sqrt{10}$ since $a > 1$

6 **a** $x^2 - ax = bx - x^2$

$2x^2 - (a+b)x = 0$

$x(2x - (a+b)) = 0$

$x = 0$ and $x = \dfrac{a+b}{2}$

When $x = \dfrac{a+b}{2}$

$$f\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^2 - a\left(\frac{a+b}{2}\right)$$

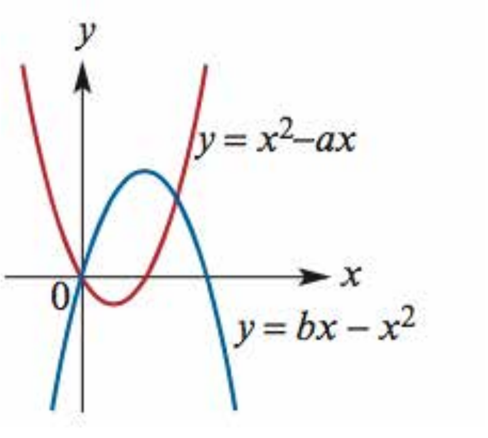
$$= \frac{a^2+2ab+b^2}{4} - \left(\frac{a^2+ba}{2}\right)$$

$$= \frac{a^2+2ab+b^2-2a^2-2ba}{4}$$

$$= \frac{b^2-a^2}{4}$$

Therefore points of intersection $(0, 0), \left(\dfrac{a+b}{2}, \dfrac{b^2-a^2}{4}\right)$

b



c $PQ = bx - x^2 - x^2 + ax$

$= (b+a)x - 2x^2$

d Maximum occurs for $x = \dfrac{b+a}{4}$

∴ maximum

$$= (b+a) \times \frac{b+a}{4} - 2\left(\frac{b+a}{4}\right)^2$$
$$= \frac{(b+a)^2}{4} - \frac{(b+a)^2}{8}$$
$$= \frac{(b+a)^2}{8}$$

7 a $y = -1.25\cos(2\pi t) + 1.25$

i When $t = 0$

$y = -1.25\cos(0) + 1.25$

$= 0$

ii When $t = \frac{1}{2}$

$y = -1.25\cos(\pi) + 1.25$

$= 1.25 + 1.25$

$= 2.5$

iii When $t = 1$

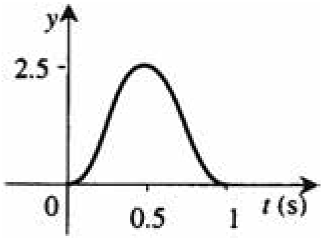
$y = -1.25\cos 2\pi + 1.25$

$= 0$

b Period $= \frac{2\pi}{n} = \frac{2\pi}{2\pi} = 1$

One revolution of the rope takes 1 second.

c



d $2 = -1.25\cos(2\pi t) + 1.25$

$\frac{0.75}{-1.25} = \cos(2\pi t)$

$-0.6 = \cos(2\pi t)$

$\therefore\ 2\pi t = \cos^{-1}(-0.6)$ (only first solution required)

$t = \frac{1}{2\pi}\cos^{-1}(-0.6)$

≈ 0.3524

It is 2 metres above the ground after 0.35 seconds.

8 $P(t) = 150 \times 10^6\, e^{kt}$

a From section 5.8, chapter 5 of *EMM Units 3 & 4*, $k = 0.0296$ (i.e. 2.96% as a decimal).

b $P(0) = 150 \times 10^6 e^0$

$\therefore$ Population on lst Jan 1950 is 150×10^6

c
$$\begin{aligned} P(50) &= 150 \times 10^6 \times e^{0.0296 \times 50} \\ &= 150 \times 10^6 \times 4.3929 \\ &= 658941852.1 \\ &\approx 6.589418521 \times 10^8 \end{aligned}$$

Population is approximately 6.589×10^8 on January 1st 2000.

When $P = 300 \times 10^6$

d $300 \times 10^6 = 150 \times 10^6 e^{0.0296t}$

$\therefore \quad 2 = e^{0.0296t}$

Taking logarithms of both sides of the equation

$$\frac{1}{0.0296} \log_e 2 = t$$

$\therefore\ t \approx 23.417$ years

The population is 300×10^6 after 23.417 years.

9 **a** $T = Ae^{-kt} + 15$, there $0 \le t \le 10$

When $t = 0, T = 95$

$$95 = A + 15$$

$$\therefore \quad A = 80$$

When $t = 2,\ T = 55$

$$55 = 80e^{-2k} + 15$$

$$\therefore \quad 0.5 = e^{-2k}$$

Taking logarithms both sides

$$-\frac{1}{2} \log_e 5 = k$$

$$\therefore \quad k = \log e(2^{\frac{1}{2}})$$

$k \approx 0.3466$

b At midnight $t = 0$

$$\therefore \quad T = 80e^{(-\log_e(2^{\frac{1}{2}}))\times 10}$$

$$= 80e^{\log_e 2^{-5}} + 15$$

$$= 80 \times \frac{1}{32} + 15$$

$$= 17.5$$

The temperature is 17.5°C at midnight.

c Graph is decreasing

When $T = 24°$

$$24 = 80e^{-\log_e(2^{\frac{1}{2}})t} + 15$$

$$\frac{9}{80} = e^{-\log_e(2^{\frac{1}{2}})t}$$

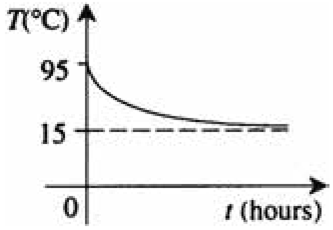
$$\log_e\left(\frac{90}{80}\right) = -\log_e(2^{\frac{1}{2}})t$$

$$\frac{\log_e\left(\frac{9}{80}\right)}{-\log_e(2^{\frac{1}{2}})} = t$$

$\therefore \quad t = 6.304$

This is 6 hours 18 minutes and 14 seconds after 2:00 pm, i.e. 8:18:14 pm.

Jenny first recorded a temperature less than 24° at 9:00 p.m. (Note: temperature is recorded on the hour)



d $T = 80e^{-\log_e(2^{\frac{1}{2}})t} + 15$

10 a If $V = 25$ and $\alpha = 45°$

$$x = \frac{25^2 \sin 90}{10}$$

$$= \frac{625}{10}$$

$$= 62.5$$

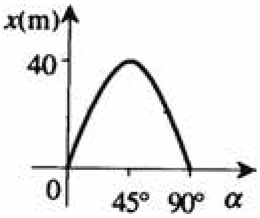
The distance the ball is kicked is 62.5 m.

b For $V = 20$

$$x = \frac{400 \sin 2\alpha}{10}$$

$$= 40 \sin 2\alpha$$

Period = 180°; amplitude = 40



c If $x = 30$ and $V = 20$

$$30 = \frac{400 \sin 2\alpha}{10}$$

$$\frac{3}{4} = \sin 2\alpha$$

$$\therefore \quad 2\alpha = \text{Sin}^{-1}\left(\frac{3}{4}\right) \text{ or } 180° - \sin^{-1}\left(\frac{3}{4}\right)$$

$$\alpha = \frac{1}{2}\text{Sin}^{-1}\left(\frac{3}{4}\right) \text{ or } 90° - \frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)$$

$$\approx 24.3° \text{ or } 65.7°$$

The angle projection is 24.3° or 65.7°

11 **a** Area $= 0.02\left(0.92^{\frac{1}{10}}\right)^x$

$$= 0.02(0.92)^{\frac{x}{10}}$$

b When $x = \frac{5}{3}$

$$\text{Area} = (0.02)\left((0.92)^{\frac{5}{30}}\right)$$

$$= (0.02)\left((0.92)^{\frac{1}{6}}\right)$$

$$= 0.0197$$

Area is 0.0197 mm^2 when $x = \frac{5}{3}$

c load = strength × cross-sectional area

$$= (0.92)^{10-3x} \times (0.02) \times (0.92)^{\frac{x}{10}}$$

$$= (0.92)^{10-3x+\frac{x}{10}} \times 0.02$$

$$= (0.92)^{\frac{100-29x}{10}} \times 0.02 = 0.02(0.92)^{10-2.9x}$$

d If load $= 0.02 \times (0.92)^{2.5}$

$$0.02 \times (0.92)^{2.5} = (0.92)^{\frac{100-29x}{10}} \times 0.02$$

$$(0.92)^{2.5} = (0.92)^{\frac{100-29x}{10}}$$

$$\therefore \quad \frac{100 - 29x}{10} = 2.5$$

i.e. $\quad 100 - 29x = 25$

$\therefore \quad 75 = 29x$

and $\quad x \approx 2.59$

Therefore the cable cannot exceed 2.59 m in length.

12 **a** The period of the function

$= \frac{2\pi}{b} = 2\pi \div = \frac{\pi}{6} = 12$

i Therefore length of OR is 12 units

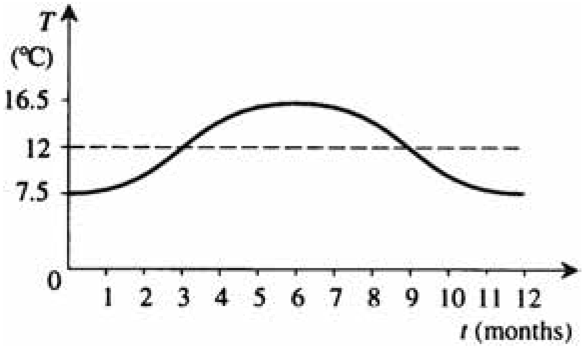
ii Q is at the minimum value

$\therefore \quad OQ = h - k$

R is at the maximum value

$\therefore \quad OR = h + k$

b



c From a $h + k = 16.5$ and $h - k = 7.5$

Consider as simultaneous equations

$h + k = 16.5$ ①

$h - k = 7.5$ ②

Add ① and ②

$2h = 24$

$h = 12$

and from $\langle 1 \rangle k = 4.5$

13 **a** *For Carriage A*

Stop 1 Illumination = $0.83I$

Stop 2 Illumination = $(0.83)^2 I$

Stop n Illumination = $(0.83)^n I$

For Carriage B

Stop 1 Illumination = $0.89 \times 0.66I$

Stop 2 Illumination = $(0.89)^2 \times 0.66I$

Stop n Illumination = $(0.89)^n \times 0.66I$

b *Illuminations equal implies*

$(0.83)^n I = (0.89)^n \times 0.66I$

$\therefore \quad \left(\frac{0.83}{0.89}\right)^n = 0.66$

Taking logarithms of both sides

$$n\log_e\left(\frac{0.83}{0.89}\right) = \log_e(0.66)$$

$$n = \frac{\log_e(0.66)}{\log_e\left(\frac{0.83}{0.89}\right)}$$

$$\approx 5.95$$

The illumination is approximately equal after the sixth stop.

14 a i $y = 1 - a(x-3)^2$

When $y = 0$

$1 - a(x-3)^2 = 0$

$\therefore\ 1 = a(x-3)^2$

$\therefore\ (x-3)^2 = \frac{1}{a}$

$\therefore\ x = 3 \pm \sqrt{\frac{1}{a}}$

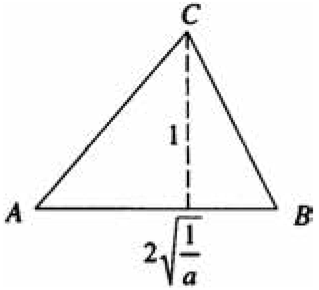
$\left(3 + \sqrt{\frac{1}{a}}, 0\right)$ and $\left(3 - \sqrt{\frac{1}{a}}, 0\right)$

ii AB has length $2\sqrt{\frac{1}{a}}$

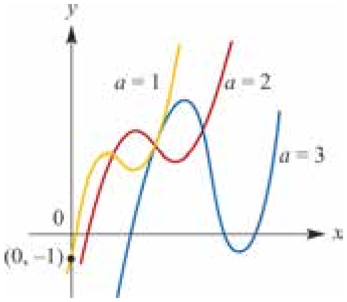
C has coordinates $(3, 1)$

Therefore the area $= \frac{1}{2} \times 2\sqrt{\frac{1}{a}} \times 1$

$= \frac{1}{\sqrt{a}}$ square units



b i



ii $-\frac{4}{27}a^3 + a = 0$

$\therefore\ a\left(-\frac{4}{27}a^2 + 1\right) = 0$

$\therefore\ a = 0$ or $a = \pm\sqrt{\frac{27}{4}}$

$= \pm\frac{3\sqrt{3}}{2}$

But $a > 0$. Therefore $a = \dfrac{3\sqrt{3}}{2}$

iii $-\dfrac{4}{27}a^3 + a < 0$

$\Leftrightarrow -\dfrac{4}{27}a^2 + 1 < 0$ (as $a > 0$)

$\Leftrightarrow a^2 > \dfrac{27}{4}$

$\Leftrightarrow a > \dfrac{3\sqrt{3}}{2}$ (as $a > 0$)

iv $-\dfrac{4}{27}a^3 + a = -1$

$-4a^3 + 27a + 27 = 0$

Using a CAS calculator yields $a = 3$ is a solution.

Consider

$(a - 3)(-4a^2 - 12a - 9) = 0$

i.e. $(a - 3)(4a^2 + 12a + 9) = 0$

and $4a^2 + 12a + 9 > 0$ for all a

$\therefore$ $a = 3$ is the only solution.

v $-\dfrac{4}{27}a^3 + a = 1$

$-4a^3 + 27a - 27 = 0$

Using a graphics calculator yields $a = \dfrac{3}{2}$ is a solution.

$\therefore$ $-4a^3 + 27a - 27 = (2a - 3)(-2a^2 - 3a + 9)$

and $-2a^2 - 3a + 9 = 0$

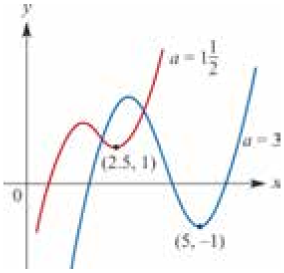
implies $2a^2 + 3a - 9 = 0$

$\therefore$ $(2a - 3)(a + 3) = 0$

$\therefore$ $a = \dfrac{3}{2}$ or $a = -3$

$a = \dfrac{3}{2}$ is the only solution.

vi



Graphic calculator techniques for 12b

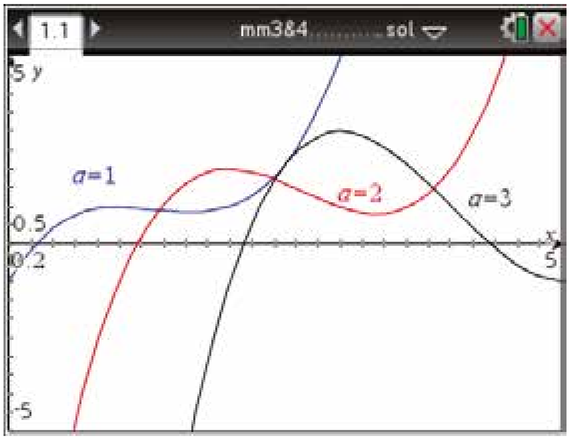
b i In a Graphs page enter

$f1(x) = (x - 1)^2(x - 2) + 1,$

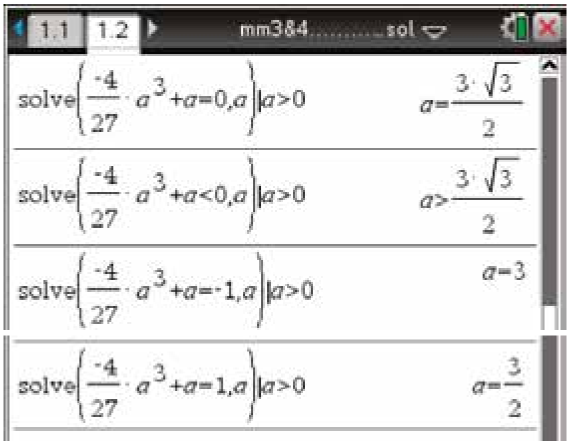
$f2(x) = (x - 2)^2(x - 4) + 2$ an

d$f3(x) = (x - 3)^2(x - 6) + 3$

Set an appropriate window to show key points.



b ii – v In **a Calculator** page, use the **Solve** command. Note the domain restrictions.



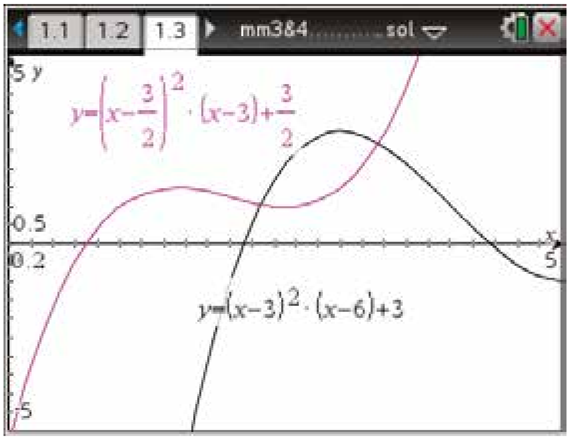
$\text{solve}\left(\frac{-4}{27}\cdot a^3+a=0,a\right)|a>0$ $\quad a=\frac{3\cdot\sqrt{3}}{2}$

$\text{solve}\left(\frac{-4}{27}\cdot a^3+a<0,a\right)|a>0$ $\quad a>\frac{3\cdot\sqrt{3}}{2}$

$\text{solve}\left(\frac{-4}{27}\cdot a^3+a=-1,a\right)|a>0$ $\quad a=3$

$\text{solve}\left(\frac{-4}{27}\cdot a^3+a=1,a\right)|a>0$ $\quad a=\frac{3}{2}$

b vi Plot in a **Graphs** page.



c **i** $\left(a, -\frac{4}{27}a^3 + a\right)$

ii $PS = a - \left(-\frac{4}{27}a^3 + a\right)$

$= \frac{4}{27}a^3$

$SQ = \frac{5a}{3} - a$

$= \frac{2a}{3}$

iii Area $= \frac{1}{2} \times SQ \times PS$

$= \frac{1}{2} \times \frac{2a}{3} \times \frac{4}{27}a^3$

$= \frac{4a^4}{81}$

iv $\frac{4a^4}{81} = 4$

$\therefore\ a^4 = 81$

$\therefore\ a = 3$ (since $a > 0$)

v $\frac{4a^4}{81} = 1500$

$\therefore\ a^4 = \frac{81 \times 1500}{4}$

$\therefore\ a^4 = 81 \times 375$

$a = 3\sqrt[4]{375}$ (since $a > 0$)

15 a $D = at^2 + bt + c$

When $t = 0, D = 1.8$

Therefore $c = 1.8$

When $t = 1, D = 1.6$

Therefore

$1.6 = a + b + 1.8$ and rearranging gives,

$-0.2 = a + b$ ①

When $t = 3, D = 1.5$

Therefore

$1.5 = 9a + 3b + 1.8$ and rearranging gives,

$-0.3 = 9a + 3b$

Dividing both sides of the equation by 3 gives

$-0.1 = 3a + b$ ②

Subtract ① from ②

$0.1 = 2a$

Therefore $a = 0.05$. Substituting in (1) gives that $b - 0.25$

$D = 0.05t^2 - 0.25t + 1.8$

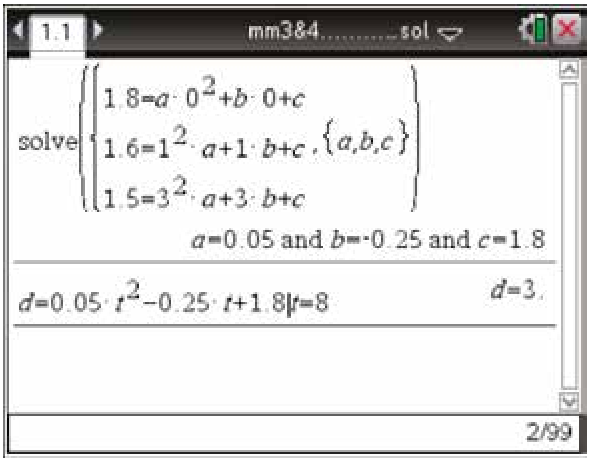
b When $t = 8, D = 0.05 \times 64 - 0.25 \times 8 + 1.8 = 3$

The deficit is 3 000 000 Ningteak dollars

Graphic calculator techniques for question

In a **Calculator** page use: b>**Algebra**>**Solve System of Equations**>**Solve System of Equations** and enter as shown opposite.

Substitute $t = 8$ into equation.



$\text{solve}\left(\begin{cases}1.8=a\cdot 0^2+b\cdot 0+c\\ 1.6=1^2\cdot a+1\cdot b+c\\ 1.5=3^2\cdot a+3\cdot b+c\end{cases},\{a,b,c\}\right)$

$a=0.05$ and $b=-0.25$ and $c=1.8$

$d=0.05\cdot t^2-0.25\cdot t+1.8|t=8$ $\quad d=3.$

16 $R = at^2 + bt + c$

When $t = 0, R = 7.5$

Therefore $c = 7.5$

When $t = 4, R = 9$

Therefore $9 = 16a + 4b + 7.5$

and $1.5 = 16a + 4b$

Diving both sides by 4 gives

$\frac{3}{8} = 4a + b$ ①

When $t = 6, R = 8$

$8 = 36a + 6b + 7.5$

$0.5 = 36a + 6b$

Divide both sides by 6

$\frac{1}{12} = 6a + b$ ②

Subtract ② from ①

$\frac{3}{8} - \frac{1}{12} = -2a$

Therefore $a = -\frac{7}{48}$ and substituting in ① gives $b = \frac{23}{24}$

Thus $R = -\frac{7}{48}t^2 + \frac{23}{24}t + \frac{15}{2}$

When $t = 8, R = \frac{35}{6}$. The rate is $\frac{35}{6}$mm/h at 12:00 noon.

The rainfall is greatest when $t = -\frac{-b}{2a} = -\frac{23}{24} \div -\frac{7}{24} = \frac{23}{7}$

The rainfall was heaviest at 7: 17 am.

17 **a**

$$N = a\log_{10}(bP)$$

$$45 = a\log_{10}(b) \qquad \ldots(1)$$

$$90 = a\log_{10}(10b) \qquad \ldots(2)$$

Subtract (1) from (2)

$$45 = a\log_{10}(10b) - a\log_{10}(b) = a\log_{10}(10) + a\log_{10}(b) - a\log_{10}(b)$$

$$\therefore a = 45$$

Substitute in (1)

$$45 = 45\log_{10} b$$

$$b = 10$$

b N is an increasing function of P.

Therefore maximum when $P = 20$

Therefore maximum,

$N \approx 104$ dB

c $45\log_{10} 10P = 75$

$$\log_{10} 10P = \frac{5}{3}$$

$$10P = 10^{\frac{5}{3}}$$

$$P = 10^{\frac{2}{3}} \approx 4.65$$

Maximum power setting is 4.

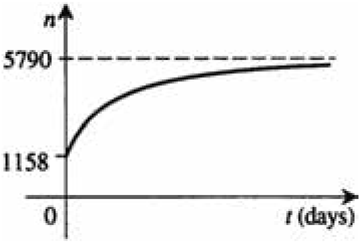
18 $n = \frac{c}{1 + ae^{-bt}} \quad t \geq 0$

$n = \frac{5790}{1 + 4e^{-0.03t}}$ for $c = 5790,\ a = 4$ and $b = 0.03$

a **i** $n = 5790$ is the horizontal asymptote

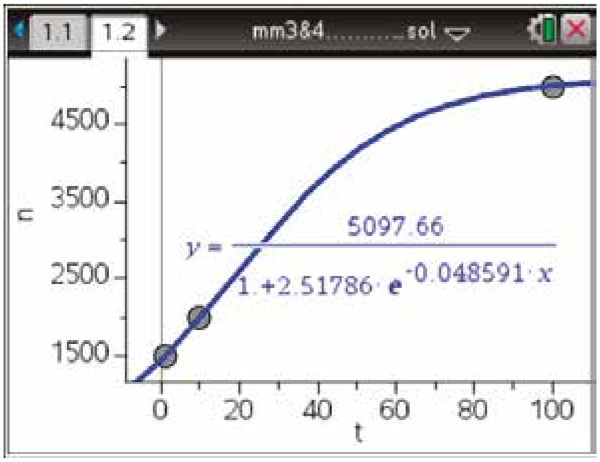
ii when $t = 0, n = \frac{5790}{5} = 1158$

iii



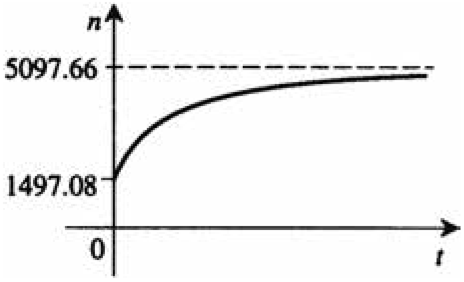
iv
$$\begin{aligned} 4000 &= \frac{5790}{1 + 4e^{-0.03t}} \\ \therefore \quad 1 + 4e^{-0.03t} &= \frac{579}{400} \\ \therefore \quad 4e^{-0.03t} &= \frac{179}{400} \\ \therefore \quad e^{-0.03t} &= \frac{179}{1600} \\ \therefore \quad t &= -\frac{100}{3}\log_e\left(\frac{179}{1600}\right) \\ &= \frac{100}{3}\log_e\left(\frac{1600}{179}\right) \end{aligned}$$

b **i** Enter the data in a **Lists & Spreadsheet** page.
Plot the data in a **Data & Statistics** page.
Determine the logistic regression using b>**Analyze**>**Regression**>**Show Logistic** (**d**=0)
The result is as shown.



$$y = \frac{5097.66}{1.+2.51786 \cdot e^{-0.048591 \cdot x}}$$

ii

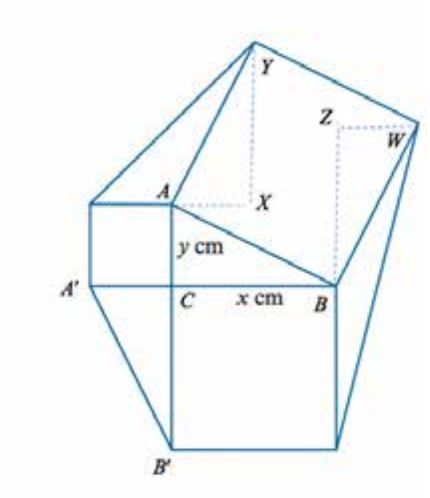


19 **a** The three squares have areas x^2, y^2 and $x^2 + y^2$. (The length of each the edges of the largest squareis $\sqrt{x^2 + y^2}$ by Pythagoras's theorem).

The two right-angled triangles each have area $\frac{1}{2}xy$. As do the other two triangles.

See diagram to prove triangles XYA and ZBW are congruent to triangle CBA.

Hence $A = 2(x^2 + xy + y^2)$



b Substituting $y = 7 - x$into the expression for A

$$A = 2(x^2 + xy + y^2)$$
$$= 2((7 - x)^2 + (7 - x)x + x^2)$$
$$= 2(49 - 14x + x^2 + 7x - x^2 + x^2)$$
$$= 2(49 - 7x + x^2)$$

$A = 2(49 - 7x + x^2)$ is a quadratic with turning point minimum when $x = \frac{7}{2}$.

When $x = \frac{7}{2}, A = \frac{147}{2}$. The corresponding value of y is $x = \frac{7}{2}$.

c Substituting $y = a - x$into the expression for A

$$A = 2(x^2 + xy + y^2)$$
$$= 2((a - x)^2 + (a - x)x + x^2)$$
$$= 2(a^2 - 2ax + x^2 + ax - x^2 + x^2)$$
$$= 2(a^2 - ax + x^2)$$

Again a quadratic with minimum turning point when $x = y = \frac{a}{2}$

$A = 2(a^2 - \frac{a^2}{2} + \frac{a^2}{4}$

$A = \frac{3}{4}a^2$

d Minimum occurs when triangle ABC is isosceles.The hexagon has an area $\frac{3}{4}$ of the area of a square of side length a.

20 a $f(g(x) = f(8x - 4)$

$= (8x - 4)^2 + 12a(8x - 4) + 6a^2$

$= 64x^2 - 64x + 16 + 96ax - 48a + 6a^2$

$= 64x^2 + 32(3a - 2)x + 6a^2 - 48a + 16$

b $64x^2 + 32(3a - 2)x + 6a^2 - 48a + 16$

$= 64[x^2 + \frac{1}{2}(3a - 2) + \frac{1}{64}(6a^2 - 48a + 16)]$

$= 64[x^2 + \frac{1}{2}(3a - 2) + +\frac{1}{16}(3a - 2)^2) - \frac{1}{16}(9a^2 - 12a + 4) + \frac{1}{64}(6a^2 - 48a + 16)]$

$= 64[(x + \frac{1}{4}(3a - 2))^2] - 30a^2$

c Range of $f(g(x)) = [-30a^2, \infty)$

d $g^{-1}(x) = \dfrac{x + 4}{8}$

$f(g^{-1}(x)) = f\left(\dfrac{x + 4}{8}\right) = \dfrac{1}{64}x^2 + \dfrac{12a + 1}{8}x + 6a^2 + 6a + \dfrac{1}{4}$

Completing the square

$f(g^{-1}(x)) = \frac{1}{64}(x + 4(12a + 1))^2 - 30a^2$

Range = $[-30a^2, \infty)$

e

$$f(g(2) = 630$$

$$f(12) = 630$$

$$6a^2 + 144a + 144 = 630$$

$$6a^2 + 144a - 486 = 0$$

$$a^2 + 24a - 81 = 0$$

$$(a + 27)(a - 3) = 0$$

$$\therefore a = 3$$

21 **a** Range of $g = [a, \infty)$ and range of $f = (-\infty, a + 1]$
Domain of $f = [-1, \infty)$ and domain of $g = (-\infty, 2]$
$f \circ g$ is defined if range of g is contained in domain of f.
therefore $[a, \infty) \subseteq [-1, \infty)$. Therefore $a \geq -1$.
$g \circ f$ is defined if range of f is contained in domain of g.
therefore $(-\infty, a + 1] \subseteq (-\infty, 2]$. Therefore $a \leq 1$. Largest set $S = [-1, 1]$

b $g(f(x)) = (a - x)^2 + a = (x - a)^2 + a$. Domain of $g \circ f$ =domain of $f = (-\infty, a + 1]$
The vertex of this parabola= (a, a)

Solutions for algorithms and pseudocode

You are advised to look at the Pseudocode appendix to this book and the appropriate programming appendix.

1 a

```
define $f(n)$:
    $sum \leftarrow 0$
    for $i$ from 1 to $n$
        $sum \leftarrow sum + (-1)^i \times i$
    end for
    return $sum$
```

S_i	i	sum
S_1	1	−1
S_2	2	1
S_3	3	−2
S_4	4	2
S_5	5	−3
S_6	6	3

b

```
define $g(n)$:
    $sum \leftarrow 0$
    for $i$ from 1 to $n$
        $sum \leftarrow sum + 3 \times i$
    end for
    return $sum$
```

c

```
define $p(n)$:
    $product \leftarrow 1$
    for $i$ from 1 to $n$
        $product \leftarrow product \times i$
    end for
    return $ptoduct$
```

2 a i $n \leq 20$; So $f(11) = 11^2 + 1 = 122$. Note withuse of else..else if statement with this program the condition $n \leq 20$ implies $10 < n \leq 20$.

ii $n \leq 40$; So $f(40) = 40^2 + 2 = 1602$.

iii $n \leq 40$; So $f(34) = 40^2 + 2 = 1158$.

b First redefine $f(n)$ to print out $(n, f(n))$ and then use a `for` loop.

```
define $f(n)$:
  if $n \leq 10$ then
    $T \leftarrow n^2$
  else if $n \leq 20$ then
    $T \leftarrow n^2 + 1$
  else if $n \leq 40$ then
    $T \leftarrow n^2 + 2$
  else if $n \leq 50$
    $T \leftarrow n^2 + 3$
  else:
    $T \leftarrow$ "value out of domain"
  end if
  return $n, T$
```

The for loop

```
for $i$ from 1 to 50
  print($n, f(n)$
end for
```

3 **a** The volume is 32 m^3.

Therefore $xyz = 32 \ldots (1)$.

Let x m and y m be the length and width of the base.

$S = xy + 2xz + 2yz$

From (1), $z = \dfrac{32}{xy}$.

Therefore, $S = xy + \dfrac{64}{y} + \dfrac{64}{x}$

b $S(1,1) = 129$; $S(2,2) = 68$; $S(3,3) = \dfrac{155}{3} \approx 51.667$; $S(4,4) = 48$;

$S(5,5) = \dfrac{208}{5} = 50.6$

c From our calculations in **b** you can see that the minimum surface area (denoted by *min* in the program) is less than or equal to 48. We choose a the larger value of 100 for the starting value of *min*.

We note that with the following program we are of course working with integer values. Hence lookinga tvalues of x and y from 1 to 32 will be sufficient.

```
$min \leftarrow 100$
$xmin \leftarrow 1$
$ymin \leftarrow 1$
for $x$ from 1 to 32
  for $y$ from 1 to 32
    print $min, xmin, ymin$
    $S \leftarrow x \times y + \frac{64}{x} + \frac{64}{y}$
      if $S \le min$ then
        $min \leftarrow S$
        $xmin \leftarrow x$
        $ymin \leftarrow y$
      end if
  end for
end for
```

Note: It is necessary to assign values to *xmin* and *ymin* from the beginning as well as changing the position of the print statement.

d Minimum surface area is 48 m^2 when $x = y = 4$. This is the the actual minimum. It does occur for these integer values.

e
```
$min \leftarrow 100$
for $x$ from 1 to 64
  for $y$ from 1 to 64
    $S \leftarrow x \times y + \frac{128}{x} + \frac{128}{y}$
      if $S \le min$ then
        $min \leftarrow S$
        $xmin \leftarrow x$
        $ymin \leftarrow y$
      end if
  end for
end for
print $min, xmin, ymin$
```

The algorithm returns 76.2, 5, 5. In this case it is not the true minimum. The actual minimum does not occur for these values but it is close.The actual minimum is approximately 76.1953 when $(x, y) \approx (5.03968, 5.03968)$.

f

```
$min \leftarrow 100$
for $x$ from $1$ to $24$
  for $y$ from $1$ to $24$
    $S \leftarrow 2 \times x \times y + \frac{48}{x} + \frac{48}{y}$
      if $S \leq min$ then
        $min \leftarrow S$
        $xmin \leftarrow x$
        $ymin \leftarrow y$
      end if
  end for
end for
print $min, xmin, ymin$
```

The algoritm returns $50, 3, 3$. In this case it is not the true minimum. The actual minimum does not occur for these values but it is close.The actual minimum is approximately 49.922 when $(x, y) \approx (2.8845, 2.8845)$.

4 We note in this case we are only interested in integer values and so the algorithm will return the ‘real world values’

```
$min \leftarrow 2010000$
for $x$ from 1 to 100
  $y \leftarrow 100 - x$
  $C \leftarrow x^3 + 100x^2 + y^3 + y^2 + 10\,000$
  if $C \le min$ then
    $min \leftarrow S$
    $xmin \leftarrow x$
    $ymin \leftarrow y$
  end if
end for
print $min, xmin, ymin$
```

Note that $C(100) = 2010000$. The minimum cost is \$451 444 when $x = 38$ and $y = 62$. There are no other values of x and y that give this minimum. In a later example a method for making sure of this is undertaken.

5 **a** $Profit = Selling\ price - Cost\ price$

So margin per kilogram is given by $a - 70$ per kilogram for product A and $b - 80$ per kilogram for product B. Let P denote the profit.

$$P = (a - 70)x + (b - 80)y$$

Using the rules $x = 240(b - a)$ and $y = 240(150 + a - b)$ we have

$$P = 240(b - a)(a - 70) + 240(150 + a - b)(b - 80)$$

b We assume that $b > a$ and $240(150 + a - b) > 0$. The upper limits are chosen as 1000 cents.

```
$max \leftarrow 1$
for $a$ from 70 to 1000
  for $y$ from 80 to 1000
    $P \leftarrow 240(a - 70)(b - a) + 240(b - 80)(150 + a - 2b)$
    if $(P \ge max$ and $150 + a - 2 * b \ge 0$ and $b > a)$ then
      $max \leftarrow P$
      $amax \leftarrow a$
      $bmax \leftarrow b$
    end if
  end for
end for
print $(max, amax, bmax)$
```

The maximum profit is \$300 000 when $a = 110$ and $b = 115$. The program gives the correct result without the conditions $150 + a - 2 * b \ge 0$ and $b > a$

6 **a**

```
$max \leftarrow 1$
for $x$ from 1 to 48
  for $y$ from 1 to 48
    $P \leftarrow xy(48 - x - y)$
      if $P > max$ and $(x + y) < 48$ then
        $max \leftarrow P$
        $xmax \leftarrow x$
        $ymax \leftarrow y$
      end if
  end for
end for
for $x$ from 1 to 48
  for $y$ from 1 to 48
    $P \leftarrow xy(48 - x - y)$
    if $(P = max)$ then
      print $(max, x, y)$
    end if
  end for
end for
```

b

```
$max \leftarrow 1$
for $x$ from 1 to 64
  for $y$ from 1 to 64
    $P \leftarrow xy(64 - x - y)$
      if $P > max$ and $(x + y) < 64$ then
        $max \leftarrow P$
        $xmax \leftarrow x$
        $ymax \leftarrow y$
      end if
  end for
end for
for $x$ from 1 to 64
  for $y$ from 1 to 64
    $P \leftarrow xy(64 - x - y)$
    if $(P = max)$ then
      print $(max, x, y)$
    end if
  end for
end for
```

Three solutions: $x = 21, y = 21, z = 22$; $x = 21, y = 22, z = 21$; $x = 22, y = 22, z = 21$

c

```
$min \leftarrow 10\,000$
for $x$ from 1 to 27
  for $y$ from 1 to 27
    $S \leftarrow x^2 + y^2 + (27 - x - y)^2$
      if $S < min$ and $(x + y) < 27$ then
        $min \leftarrow S$
        $xmin \leftarrow x$
        $ymin \leftarrow y$
      end if
  end for
end for
for $x$ from 1 to 27
  for $y$ from 1 to 27
    $S \leftarrow x^2 + y^2 + (27 - x - y)^2$
    if $(S = min)$ then
      print $(min, x, y)$
     end if
  end for
end for
```

7 **a** $C = 3000x + 4000y$

b It is evident that the more trips that are taken tthe greater the cost. You don't need to check very large values of x and y. $min \leftarrow 10\,000$

```
for $x$ from 0 to 50
  for $y$ from 0 to 50
    $C \leftarrow 3000x + 4000y$
      if $C \leq min$ and $20x + 40y \geq 200$ and $6x + 4y \geq 36$ then
        $min \leftarrow C$
        $xmin \leftarrow x$
        $ymin \leftarrow y$
      end if
  end for
end for
print $(min, xmin, ymin)$
```

The result is $C = 24\,000$, $x = 4$, $y = 3$

8 **a**

a	b	m	$count$
1	1.5	1.25	1
1.25	1.5	1.375	2
1.375	1.5	1.4375	3
1.375	1.4375	1.40625	4

b

```
define $f(x)$:
  return $2^x - 7$

$a \leftarrow 2$
$b \leftarrow 3$
$m \leftarrow 2.5$
$count = 0$
while $b - a > 2 \times 0.0001$
    if $f(a) \times f(m) < 0$ then
        $b \leftarrow m$
    else
        $a \leftarrow m$
    end if
    $m \leftarrow \frac{a+b}{2}$
    $count \leftarrow count + 1$
end while
print $m, count$
```

a	b	m	$count$
2.5	3	2.75	1
2.75	3	2.875	2
2.75	2.875	2.8125	3
2.75	2.8125	2.78125	4

c

```
define $f(x)$:
  return $\sin x - 0.7$

$a \leftarrow \frac{\pi}{6}$
$b \leftarrow \frac{\pi}{4}$
$m \leftarrow \frac{5\pi}{24}$
$count = 0$
while $b - a > 2 \times 0.0001$
    if $f(a) \times f(m) < 0$ then
        $b \leftarrow m$
```

```
        else
            $a \leftarrow m$
        end if
        $m \leftarrow \frac{a+b}{2}$
        $count \leftarrow count + 1$
    end while
    print $m, count$
```

a	b	m	$count$
0.654498...	0.785398...	0.719948...	1
0.719948...	0.785398...	0.752673...	2
0.7526732...	0.785398...	0.769035...	3
0.769035...	0.785398...	0.777216...	4

Chapter 9 – Differentiation of polynomials, power functions and rational functions

Solutions to Exercise 9A

1 $f(x) = -x^2 + 2x + 1$

$f(-1) = -(-1)^2 + 2(-1) + 1 = -2$

$f(4) = -(4)^2 + 2 \times 4 + 1 = -7$

$$\text{Average rate of change} = \frac{f(4) - f(-1)}{4 - (-1)} = \frac{-7 - (-2)}{5} = -1$$

2 $f(x) = 6 - x^3$

$f(-1) = 6 + 1 = 7$

$f(1) = 6 - 1 = 5$

$$\text{Average rate of change} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2}{2} = -1$$

3 $f(x) = x^2 + 5x$

a Gradient

$$= \frac{(2 + h)^2 + 5(2 + h) - 14}{2 + h - 2}$$

$$= \frac{4 + 4h + h^2 + 10 + 5h - 14}{h}$$

$$= \frac{9h + h^2}{h}$$

$= 9 + h$

b $\lim\limits_{h \to 0} 9 + h = 9$

4 **a** $\lim\limits_{h \to 0} \dfrac{4x^2h^2 + xh + h}{h}$

$= \lim\limits_{h \to 0}(4x^2h + x + 1)$

$= x + 1$

b $\lim\limits_{h \to 0} \dfrac{2x^3h - 2xh^2 + h}{h}$

$= \lim\limits_{h \to 0}(2x^3 - 2xh + 1)$

$= 2x^3 + 1$

c $\lim\limits_{h \to 0}(40 - 50h)$

$= 40$

d $\lim\limits_{h \to 0} 5h$

$= 0$

e $\lim\limits_{h \to 0} 5$

$= 5$

f $\lim\limits_{h \to 0} \dfrac{30h^2x^2 + 20h^2x + h}{h}$

$= \lim\limits_{h \to 0}(30hx^2 + 20hx + 1)$

$= 1$

g $\lim\limits_{h \to 0} \dfrac{3h^2x^3 + 2hx + h}{h}$

$= \lim\limits_{h \to 0}(3hx^3 + 2x + 1)$

$= 2x + 1$

h $\lim\limits_{h \to 0} 3x$

$= 3x$

i $\lim\limits_{h \to 0} \dfrac{3x^3h - 5x^2h^2 + hx}{h}$

$= \lim\limits_{h \to 0}(3x^3 + 5x^2h + x)$

$= 3x^3 + x$

j $\lim\limits_{h \to 0}(6x - 7h)$

$= 6x$

5 $y = x^3 - x$

a $$grad = \frac{rise}{run}$$
$$= \frac{(1+h)^3 - (1+h) - 0}{(1+h) - 1}$$
$$= \frac{1 + 3h + 3h^2 + h^3 - 1 - h}{h}$$
$$= \frac{h^3 + 3h^2 + 2h}{h}$$
$$= h^2 + 3h + 2$$

b $$grad = \lim_{h \to 0} grad(PQ)$$
$$= \lim_{h \to 0}(h^2 + 3h + 2)$$
$$= 2$$

6 $f(x) = x^2 - 2$
$$\frac{f(x+h) - f(x)}{h} = \frac{((x+h)^2 - 2) - (x^2 - 2)}{h}$$
$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= 2x + h$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0}(2x + h)$$
$$f'(x) = 2x$$

7
$$y = x^2 + 2x + 5$$
$$grad(PQ) = \frac{rise}{run}$$
$$= \frac{((2+h)^2 + 2(a+h) + 5) - ((2)^2 + 2(2) + 5)}{(2+h) - (2)}$$
$$= \frac{4 + 4h + h^2 + 4 + 2h + 5 - 4 - 4 - 5}{h}$$
$$= \frac{h^2 + 6h}{h}$$
$$= h + 6$$
$$grad(P) = \lim_{h \to 0}(grad(PQ))$$
$$= \lim_{h \to 0}(h + 6)$$
$$= 6$$

8 **a** $f(x) = 5x^2$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{5(x+h)^2 - 5x^2)}{h}$$
$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h}$$
$$= \lim_{h \to 0} 10x + 5h$$
$$= 10x$$

b $f(x) = 3x + 2$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{3(x+h) + 2 - 3x - 2}{h}$$
$$= \lim_{h \to 0} \frac{3h}{h}$$
$$= \lim_{h \to 0} 3$$
$$= 3$$

c $f(x) = 5$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5 - 5}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

d

$$f(x) = 3x^2 + 4x + 3$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 + 4(x+h) + 3 - 3x^2 - 4x - 3}{h}$$

$$= \lim_{h\to 0}$$

$$\frac{3x^2 - 3x^2 + 6xh + 3h^2 + 4x - 4x + 4h + 3 - 3}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 + 4h}{h}$$

$$= \lim_{h\to 0}(6x + 3h + 4)$$

$$= 6x + 4$$

e

$$f(x) = 5x^3 - 5$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5(x+h)^3 - 5 - 5(x)^3 + 5}{h}$$

$$= \lim_{h\to 0} \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 - 5x^3}{h}$$

$$= \lim_{h\to 0}(15x^2 + 15xh + 5h^2)$$

$$= 15x^2$$

f

$$f(x) = 5x^2 - 6x$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5(x+h)^2 + 6(x+h) - 5x^2 + 6x}{h}$$

$$= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - 6x - 6h - 5x^2 + 6x}{h}$$

$$= \lim_{h\to 0}(10x + 5h - 6)$$

$$= 10x - 6$$

9 a i $$\frac{f(a+h) - f(a)}{h} = \frac{12h}{2h} = 6$$

ii $$\frac{f(a+h) - f(a-h)}{2h} = \frac{h^2 + 6h}{h} = h + 6$$

iii $$\frac{f(a+h) - f(a)}{h} = \frac{12h}{2h} = 6$$

b i $$\frac{f(a+h) - f(a)}{h} = \frac{h^2 + 10h}{h} = h + 10$$

ii $$\frac{f(a+h) - f(a-h)}{2h} = \frac{20h}{2h} = 10$$

c i $$\frac{f(a+h) - f(a)}{h} = \frac{h^3 + 9h^2 + 24h}{h} = h^2 + 9h + 24$$

ii $$\frac{f(a+h) - f(a-h)}{2h} = \frac{2h^3 + 48h}{2h} = h^2 + 24$$

d i $$\frac{f(a+h) - f(a)}{h} = \frac{h^3 + 6h^2 + 14h}{h} = h^2 + 6h + 14$$

ii $$\frac{f(a+h)-f(a-h)}{2h} = \frac{2h^3 + 28h}{2h}$$
$$= h^2 + 14$$

Solutions to Exercise 9B

1 **a** $f(x) = x^5$

$f'(x) = 5x^4$

b $f(x) = 4x^7$

$f'(x) = 7 \times 4x^6$

$= 28x^6$

c $f(x) = 6x$

$f'(x) = 6$

d $f(x) = 5x^2 - 4x + 3$

$f'(x) = 2 \times 5x - 4$

$= 10x - 4$

e $f(x) = 4x^3 + 6x^2 + 2x - 4$

$f'(x) = 3 \times 4x^2 + 2 \times 6x + 2$

$= 12x^2 + 12x + 2$

f $f(x) = 5x^4 + 3x^3$

$f'(x) = 4 \times 5x^3 + 3 \times 3x^2$

$= 20x^3 + 9x^2$

g $f(x) = -2x^2 + 4x + 6$

$f'(x) = -4x + 4$

h $f(x) = 6x^3 - 2x^2 + 4x - 6$

$f'(x) = 18x^2 - 4x + 4$

2 **a** $f(x) = 2x^3 - 5x^2 + 1$

$f'(x) = 6x^2 - 10x$

$f'(1) = -4$

b $f(x) = -2x^3 - x^2 - 1$

$f'(x) = -6x^2 - 2x$

$f'(1) = -8$

c $f(x) = x^4 - 2x^3 + 1$

$f'(x) = 4x^3 - 6x^2$

$f'(1) = -2$

d $f(x) = x^5 - 3x^3 + 2$

$f'(x) = 5x^4 - 9x^2$

$f'(1) = -4$

3 **a** $f(x) = 2x^3 - 5x^2 + 2$

$f'(x) = 6x^2 - 10x$

$f'(1) = -4$

b $f(x) = -2x^3 - 3x^2 + 2$

$f'(x) = -6x^2 - 6x$

$f'(2) = -36$

4 **a** $\dfrac{dy}{dt} = 3t^2$

b $\dfrac{dx}{dt} = 3t^2 - 2t$

c $\dfrac{dz}{dx} = x^3 + 9x^2$

5 **a** $y = -2x$

$\dfrac{dy}{dx} = -2$

b $y = 7$

$\dfrac{dy}{dx} = 0$

c $y = 5x^3 - 3x^2 + 2x + 1$

$\frac{dy}{dx} = 15x^2 - 6x + 2$

d $y = \frac{2}{5}x^3 - \frac{8}{5}x + \frac{12}{5}$

$\frac{dy}{dx} = \frac{6}{5}x^2 - \frac{8}{5}$

e $y = (2x + 1)(x - 3)$

$= 2x^2 - 5x - 3$

$\frac{dy}{dx} = 4x - 5$

f $y = 3x(2x - 4)$

$= 6x^2 - 12x$

$\frac{dy}{dx} = 12x - 12$

g $y = \frac{10x^7 + 2x^2}{x^2}$

$= 10x^5 + 2$

$\frac{dy}{dx} = 50x^4$

h $y = \frac{9x^4 + 3x^2}{x}$

$= 9x^3 + 3x$

$\frac{dy}{dx} = 27x^2 + 3$

6 a $\frac{d}{dx}(2x^2 - 5x^3) = 4x - 15x^2$

b $\frac{d}{dz}(-2z^2 - 6z) = -4z - 6$

c $\frac{d}{dz}(6z^3 - 4z^2 + 3) = 18z^2 - 8z$

d $\frac{d}{dx}(-2x - 5x^3) = -2 - 15x^2$

e $\frac{d}{dz}(-2z^2 - 6z + 7) = -4z - 6$

f $\frac{d}{dz}(-z^3 - 4z^2 + 3) = -3z^2 - 8z$

7 a $y = 2x^2 - 4x + 1, \ \frac{dy}{dx} = -6$

$\frac{dy}{dx} = 4x - 4$

$-6 = 4x - 4$

$4x = -2$

$x = \frac{-1}{2}$

$y = \frac{1}{2} + 2 + 1 = \frac{7}{2}$

co-ords $= \left(\frac{-1}{2}, \frac{7}{2}\right)$

b $y = 4x^3, \ \frac{dy}{dx} = 48$

$\frac{dy}{dx} = 12x^2$

$48 = 12x^2$

$x^2 = 4$

$x = \pm 2$

$y = \pm 32$

co-ords = (−2, −32) *and* (2, 32)

c $y = x(5 - x), \ \frac{dy}{dx} = 1$

$y = 5x - x^2$

$\frac{dy}{dx} = 5 - 2x$

$1 = 5 - 2x$

$-2x = -4$

$x = 2$

$y = 2(3) = 6$

co-ords = (2, 6)

d $y = x^3 - 3x^2, \; \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 3x^2 - 6x$$

$$0 = 3x^2 - 6x$$

$$0 = 3x(x - 2)$$

$$x = 0, 2$$

$$y = 0 - 0 = 0, \; y = 8 - 12 = -4$$

co-ords = (0, 0) *and* (2, −4)

8 **a** $\tan 45° = 1$

$\therefore$ gradient = 1

$\frac{dy}{dx} = 4x - 3$ Therefore

When gradient = 1

$$4x - 3 = 1$$

$$\therefore x = 1$$

$$f(1) = 7$$

the tangent line at the point (1, 7) makes an angle of tan 45° with the positive direction of the x-axis.

b Gradient = 2

$$\frac{dy}{dx} = 4x - 3$$

When gradient = 2

$$4x - 3 = 2$$

$$\therefore x = \frac{5}{4}$$

$$f(\frac{5}{4}) = \frac{59}{8}$$

Therefore the tangent line at the point $\left(\frac{5}{4}, \frac{59}{8}\right)$ is parallel to the line $y = 2x + 8$

9 $\frac{dy}{dx} = 2x - 1$

a $2x - 1 = 1$

$$x = 1$$

b $2x - 1 = -1$

$$x = 0$$

c $2x - 1 = \sqrt{3}$

$$x = \frac{1}{2}\left(1 + \sqrt{3}\right)$$

$$= \frac{1 + \sqrt{3}}{2}$$

10 **a** $y = x^2 + 3x, \; (1, 4)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = 2x + 3$$

When $x = 1, \frac{dy}{dx} = 5$

$$\therefore \tan\theta = 5$$

$$\therefore \theta \approx 78.69°$$

b $y = -x^2 + 2x, \; (1, 1)$ Let θ be the angle between the tangent line and the x-axis. $\frac{dy}{dx} = -2x + 2$

When $x = 1, \frac{dy}{dx} = 0$

$$\therefore \tan\theta = 0$$

$$\therefore \theta = 0°$$

c $y = x^3 + x, \; (0, 0)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = 3x^2 + 1$$

When $x = 0, \frac{dy}{dx} = 1$

$$\therefore \tan\theta = 1$$

$$\therefore \theta = 45°$$

d $y = -x^3 - x,\ (0,0)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = -3x^2 - 1$$

When $x = 0, \dfrac{dy}{dx} = -1$

$$\therefore \tan\theta = -1$$

$$\therefore \theta = 135°$$

e $y = x^4 - x^2,\ (1,0)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = 4x^3 - 2x$$

When $x = 1, \dfrac{dy}{dx} = 2$

$$\therefore \tan\theta = 2$$

$$\therefore \theta \approx 63.43°$$

f $y = x^4 - x^2,\ (-1,0)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = 4x^3 - 2x$$

When $x = -1, \dfrac{dy}{dx} = -2$

$$\therefore \tan\theta = -2$$

$$\therefore \theta \approx 116.57°$$

11 a $y = (2x-1)^2$

$$= 4x^2 - 4x + 1$$

$$\frac{dy}{dx} = 8x - 4$$

b $y = \dfrac{x^3 + 2x^2}{x}$

$$= x^2 + 2x$$

$$\frac{dy}{dx} = 2x + 2$$

c $y = 2x^3 - 6x^2 + 18x$

$$\frac{dy}{dx} = 6x^2 - 12x + 18$$

$$= 6(x^2 - 2x + 3)$$

$b^2 - 4ac = 6(4 - 12) < 0$

$\therefore$ $\dfrac{dy}{dx}$ *does not intersect the x-axis and since x = 0 gives* $\dfrac{dy}{dx} = 3, \dfrac{dy}{dx} > 0$ *for all x (as opposed to* $\dfrac{dy}{dx} < 0$ *for all x)*

d $y = \dfrac{x^3}{3} - x^2 + x$

$$\frac{dy}{dx} = x^2 - 2x + 1$$

$$= (x-1)^2$$

$\therefore$ $\dfrac{dy}{dx} \geq 0,$

since any number squared is non-negative

12 a $y = x^2 + 2x + 1,\ x = 3$

$$y = 3^2 + 2(3) + 1$$

$$= 9 + 6 + 1$$

$$y = 16$$

$$\frac{dy}{dx} = 2x + 2$$

$$\frac{dy}{dx} = 2(3) + 2$$

$$= 8$$

b $y = x^2 - x - 1,\ x = 0$

$$y = -1$$

$$\frac{dy}{dx} = 2x - 1$$

$$\frac{dy}{dx} = -1$$

c $y = 2x^2 - 4x,\ x = -1$

$y = 2(-1)^2 - 4(-1)$

$= 2 + 4$

$y = 6$

$\dfrac{dy}{dx} = 4x - 4$

$\dfrac{dy}{dx} = 4(-1) - 4$

$= -8$

d $y = (2x + 1)(3x - 1)(x + 2),\ x = 4$

$y = 6x^3 + 13x^2 + x - 2$

$= 6(4)^3 + 13(4)^2 + (4) - 2$

$y = 6 \times 64 + 13 \times 16 + 4 - 2$

$y = 384 + 208 + 2$

$y = 594$

$y = 6x^3 + 13x^2 + x - 2$

$\dfrac{dy}{dx} = 18x^2 + 26x + 1$

$x = 4,\quad \dfrac{dy}{dx} = 18 \times 16 + 26 \times 4 + 1$

$= 393$

e $y = (2x + 5)(3 - 5x)(x + 1), x = +1$

$y = -10x^3 - 25x^2 + 6x^2 - 10x + 6x$

$- 25x + 15x + 15$

$y = -10x^3 - 29x^2 - 4x + 15$

$x = +1,\ y = -10 - 29 - 4 + 15$

$y = -28$

$\dfrac{dy}{dx} = -30x^2 - 58x - 4$

$x = +1,\ \dfrac{dy}{dx} = -30 - 58 - 4$

$= -92$

f $y = (2x\ - 5)^2, x = 2\frac{1}{2}$

$x = 2\frac{1}{2},\ y = (5 - 5)^2$

$y = 0$

$y = 4x^2 - 20x + 25$

$\dfrac{dy}{dx} = 8x - 20$

$x = 2\frac{1}{2},\ \dfrac{dy}{dx} = 4 \times 5 - 20$

$= 0$

13 $f(x) = 3(x - 1)^2$

a $0 = 3(x - 1)^2$

$x = 1$

b $f'(x) = 3(x^2 - 2x + 1)$

$f'(x) = 3(2x - 2)$

$= 6(x - 1)$

$0 = 6(x - 1)$

$x = 1$

c $0 < 6(x - 1)$

$x - 1 > 0$

$x > 1;\text{i.e.}(1, \infty)$

d $0 > 6(x - 1)$

$x - 1 < 0$

$x < 1;\text{i.e.}(-\infty, 1)$

e $10 = 6(x - 1)$

$x - 1 = \dfrac{5}{3}$

$x = \dfrac{8}{3}$

f $27 = 3(x-1)^2$

$9 = (x-1)^2$

$x - 1 = \pm 3$

$x = -2, 4$

14 a $x < -1,\ x > 1$
i.e. $x \in R\backslash[-1, 1]$

b $-1 < x < 1$
i.e. $x \in (-1, 1)$

c $x = -1, 1$

15 a $-1 < x < 0.5,\ x > 2$
i.e. $x \in \left(-1, \frac{1}{2}\right) \cup (2, \infty)$

b $x < -1,\ \frac{1}{2} < x < 2$
i.e. $x \in (-\infty, -1) \cup \left(\frac{1}{2}, 2\right)$

c $x = -1,\ \frac{1}{2}, 2$

16 a $x > -1,\ x \neq 2$
i.e. $x \in \left(\frac{-1}{4}, 2\right) \cup (2, \infty)$

b $x < \frac{-1}{4}$
i.e. $x \in \left(-\infty, \frac{-1}{4}\right)$

c $x = \frac{-1}{4}, 2$

17 $y = x^2 - 4x - 8$

a $\frac{dy}{dx} = 2x - 4$

$\frac{dy}{dx} = 0,$

$0 = 2x - 4$

$2x = 4$

$x = 2$

$y = 4 - 8 - 8 = -12$
co-ords = (2, −12)

b $\frac{dy}{dx} = 2$

$2 = 2x - 4$

$2x = 6$

$x = 3$

$y = -11$
co-ords= (3, −11)

c $3x + 2y = 8$

$\Rightarrow y = \frac{8}{2} - \frac{3}{2}x$

$= 4 - \frac{3}{2}x$

$\frac{dy}{dx} = \frac{-3}{2}$

$\frac{-3}{2} = 2x - 4$

$x = \frac{5}{4}$

$y = -\frac{183}{16}$

co-ords = $\left(\frac{5}{4}, -\frac{183}{16}\right)$

18 a $f'(x) = x^2 > 0$ for all $x \neq 0$.
Therefore strictly increasing for $\mathbb{R}\backslash\{0\}$.
Also $f(0) = 0$ and $f(b) > 0$ for all $b > 0$ and $f(b) < 0$ for all $b < 0$.
Therefore strictly increasing for all

$x \in \mathbb{R}$.

b $f'(x) = -x^2 < 0$ for all $x \neq 0$.
Therefore strictly decreasing for $\mathbb{R}\backslash\{0\}$.
Also $f(0) = 0$ and $f(b) < 0$ for all $b > 0$ and $f(b) > 0$ for all $b < 0$.
Therefore strictly decreasing for all $x \in \mathbb{R}$.

19 **a** Assume $x > y$ and $x \geq 0$ and $y \geq 0$.
Then

$$\begin{aligned} x &> y \\ \Leftrightarrow x - y &> 0 \\ \Leftrightarrow (x-y)(x+y) &> 0 \\ \Leftrightarrow x^2 - y^2 &> 0 \\ \Leftrightarrow x^2 &> y^2 \end{aligned}$$

b Assume $x > y$ and $x \leq 0$ and $y \leq 0$.
Then

$$\begin{aligned} x &> y \\ \Leftrightarrow x - y &> 0 \\ \Leftrightarrow (x-y)(x+y) &< 0 \\ \Leftrightarrow x^2 - y^2 &< 0 \\ \Leftrightarrow x^2 &< y^2 \end{aligned}$$

20 $f'(x) = 2x - 1$
$2x - 1 > 0 \Leftrightarrow x > \frac{1}{2}$
$\therefore$ strictly increasing for $x > \frac{1}{2}$ We also know that $f(x) > f(\frac{1}{2})$ for all $x \in \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$
$\therefore$ strictly increasing for $\left[\frac{1}{2}, \infty\right)$
If $x < \frac{1}{2}$ then $f(x) > \frac{1}{2}$
Hence $\left[\frac{1}{2}, \infty\right)$ is the largest interval for which f is strictly increasing.

21 **a** $(\infty, -1]$

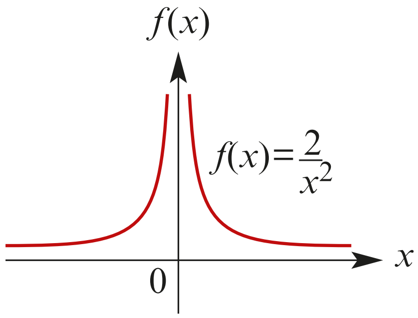
b $[2, \infty)$

c $[-\infty, 0]$

d $[\frac{3}{2}, \infty)$

Solutions to Exercise 9C

1 a



b $$grad(PQ) = \frac{rise}{run}$$
$$= \frac{f(1+h) - f(1)}{(1+h) - 1}$$
$$= \frac{\frac{2}{(1+h)^2}\ \frac{2}{1^2}}{h}$$
$$= \frac{1}{h}\left(\frac{2}{(1+h)^2} - 2\frac{(1+h)^2}{(1+h)^2}\right)$$
$$= \frac{1}{h}\left(\frac{2 - 2(1+2h+h^2)}{1+2h+h^2}\right)$$
$$= \frac{1}{h}\left(\frac{-4h - 2h^2}{1+2h+h^2}\right)$$
$$grad(PQ) = \frac{-4 - 2h}{1 + 2h + h^2}$$

c $$grad(P) = \lim_{h\to 0} grad(PQ)$$
$$= \lim_{h\to 0} \frac{-4 - 2h}{1 + 2h + h^2}$$
$$= \frac{-4}{1}$$
$$grad(P) = -4$$

2 a $$\frac{f(x+h) - f(x)}{h}$$
$$= \left(\frac{1}{x+h-3} - \frac{1}{x-3}\right) \times \frac{1}{h}$$
$$= \left(\frac{x-3-(x+h-3)}{(x+h-3)(x-3)}\right) \times \frac{1}{h}$$
$$= \left(\frac{-h}{(x+h-3)(x-3)}\right) \times \frac{1}{h}$$
$$= \left(\frac{-1}{(x+h-3)(x-3)}\right)$$

Hence

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{-1}{(x+h-3)(x-3)}$$
$$= -\frac{1}{(x-3)^2}$$

b $$\frac{f(x+h) - f(x)}{h}$$
$$= \left(\frac{1}{x+h+2} - \frac{1}{x+2}\right) \times \frac{1}{h}$$
$$= \left(\frac{x+2-(x+h+2)}{(x+h+2)(x+2)}\right) \times \frac{1}{h}$$
$$= \left(\frac{-h}{(x+h+2)(x+2)}\right) \times \frac{1}{h}$$
$$= \left(\frac{-1}{(x+h+2)(x+2)}\right)$$

Hence

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{-1}{(x+h+2)(x+2)}$$
$$= -\frac{-1}{(x+2)^2}$$

3

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{(x+h)^{-4}-x^{-4}}{h}$$
$$= \left(\frac{1}{(x+h)^4}-\frac{1}{x^4}\right)\times\frac{1}{h}$$
$$= \left(\frac{x^4-(x+h)^4}{x^4(x+h)^4}\right)\times\frac{1}{h}$$
$$= \left(\frac{x^4-(x^4+4x^3h+6x^2h^2+4xh^3+h^4)}{x^4(x+h)^4}\right)\times\frac{1}{h}$$
$$= \left(\frac{-(4x^3h+6x^2h^2+4xh^3+h^4)}{x^4(x+h)^4}\right)\times\frac{1}{h}$$
$$= \frac{-(4x^3+6x^2h+4xh^2+h^3)}{x^4(x+h)^4}$$

Hence

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\left(\frac{-(4x^3+6x^2h+4xh^2+h^3)}{x^4(x+h)^4}\right)$$
$$= -\frac{4x^3}{x^8}$$
$$= -\frac{4}{x^5}$$

4 a $y = 3x^{-2}+5x^{-1}+6$

$$\frac{dy}{dx} = -6x^{-3}-5x^{-2}$$

b $y = 5x^{-3}+6x^2$

$$\frac{dy}{dx} = -15x^{-4}+12x$$

c $f(x) = -5x^{-3}+4x^{-2}+1$

$f'(x) = 15x^{-4}-8x^{-3}$

d $f(x) = 6x^{-3}+3x^{-2}$

$f'(x) = -18x^{-4}-6x^{-3}$

e $y = 4+2x^{-1}$

$$\frac{dy}{dx} = -2x^{-2}$$

5 a $y = 2-4z^{-1}$

$$\frac{dy}{dz} = 4z^{-2}$$

b $y = 6z^{-3}+z^{-2}$

$$\frac{dy}{dz} = -18z^{-4}-2z^{-3}$$

c $y = 16-z^{-3}$

$$\frac{dy}{dz} = 3z^{-4}$$

d $f(z) = 4z^{-1}+z-z^2$

$f'(z) = -4z^{-2}+1-2z$

e $f(z) = 6z^{-2}-2z^{-3}$

$f'(z) = -12z^{-3}+6z^{-4}$

f $f(x) = 6x^{-1}-3x^2$

$f'(x) = -6x^{-2}-6x$

6 a $y = x^{-2}+x^3$

$$\frac{dy}{dx} = -2x^{-3}+3x^2$$

$x = 2,$

$$\frac{dy}{dx} = \frac{-2}{8}+3\times 4$$
$$= \frac{-1}{4}+12$$
$$\frac{dy}{dx} = \frac{47}{4} = 11\frac{3}{4}$$

b $y = x^{-2} - x^{-1}$

$\frac{dy}{dx} = -2x^{-3} + x^{-2}$

$x = 4,$

$\frac{dy}{dx} = \frac{-2}{64} + \frac{1}{16}$

$= \frac{-1}{32} + \frac{1}{16}$

$\frac{dy}{dx} = \frac{1}{32}$

c $y = x^{-2} - x^{-1}$

$\frac{dy}{dx} = -2x^{-3} + x^{-2}$

$x = 1,$

$\frac{dy}{dx} = -2 + 1$

$\frac{dy}{dx} = -1$

d $y = 1 + x^3 - x^{-2}$

$\frac{dy}{dx} = 3x^2 + 2x^{-3}$

$x = 1,$

$\frac{dy}{dx} = 3 + 2$

$\frac{dy}{dx} = 5$

7 $f'(x) = 10x^{-4} > 0$ for all $x \neq 0$

8 $y = \frac{x^2 - 1}{x} = x - \frac{1}{x} = x - x^{-1}$

$\frac{dy}{dx} = 1 + x^{-2}$

$\frac{dy}{dx} = 5$

$5 = 1 + \frac{1}{x^2}$

$\frac{1}{x^2} = 4$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

9 $y = ax^2 + bx^{-1}$

$x = 2,\ y = -2$

$1 - 2 = 4a + \frac{b}{2}$

$\frac{dy}{dx} = 2ax - bx^{-2}$

$x = 2,\ \frac{dy}{dx} = -5$

$2 - 5 = 4a - \frac{b}{4}$

$1 - 2 \Rightarrow 3 = \frac{3b}{4}$

$b = 4$

Sub in 1 $\Rightarrow -2 = 4a + 2$

$4a = -4$

$a = -1$

$y = -x^2 + \frac{4}{x}$

10 $y = 2x^{-1} - 4x^{-2}$

$y = 0,$

$0 = 2x^{-1} - 4x^{-2}$

$0 = 2x - 4$

$x = 2$

$$\frac{dy}{dx} = -2x^{-2} + 8x^{-3}$$

$x = 2,$

$$\frac{dy}{dx} = \frac{-2}{4} + \frac{8}{8}$$

$$= \frac{-1}{2} + 1$$

$$= \frac{1}{2}$$

11 $y = \frac{9}{x} + bx^2$

$= ax^{-1} + bx^2$

$x = 3,\ y = 6$

$$1\ 6 = \frac{9}{3} + 9b$$

$$\frac{dy}{dx} = ax^{-2} + 2bx$$

$$x = 3,\ \frac{dy}{dx} = 7$$

$$2\ 7 = \frac{-a}{9} + 6b$$

$31 \Rightarrow 18 = a + 27b \quad b = 1$

$92 \Rightarrow 63 = -a + 54b$

$31 + 92 \Rightarrow 81 = 81b$

$$\text{Sub in } 2 \Rightarrow 7 = \frac{-a}{9} + 6$$

$$\frac{-a}{9} = 1$$

$$a = -9$$

$$y = \frac{-9}{x} + x^2$$

12 $y = \frac{5}{3}x + kx^2 - \frac{8}{9}x^3$

$$\frac{dy}{dx} = \frac{5}{3} + 2kx - \frac{8}{3}x^2$$

$$at\ x = \frac{-1}{2}$$

$$\frac{dy}{dx} = \frac{5}{3} - k - \frac{2}{3}$$

$$\frac{dy}{dx} = 1 - k$$

$at\ x = 1,$

$$\frac{dy}{dx} = \frac{5}{3} + 2k - \frac{8}{3}$$

$$\frac{dy}{dx} = 2k - 1$$

$$2k - 1 = \frac{-1}{1-k}(perpendicular)$$

$(2k - 1)(k - 1) = 1$

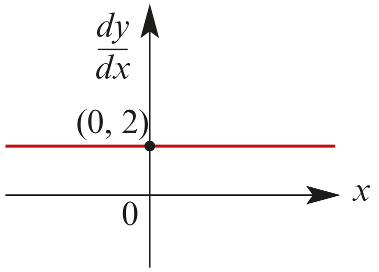
$2k^2 - 3k + 1 = 1$

$2k^2 - 3k = 0$

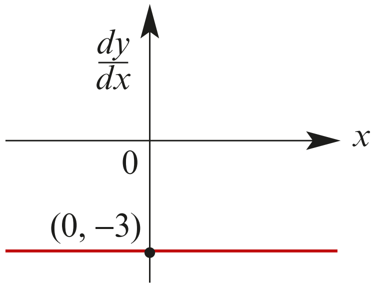
$$k(2k - 3) = 0 \Rightarrow k = 0,\ \frac{3}{2}$$

Solutions to Exercise 9D

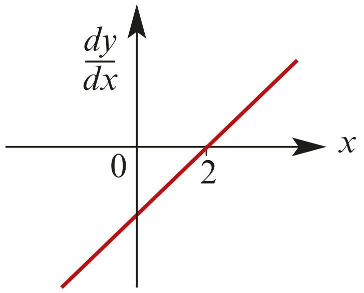
1 a



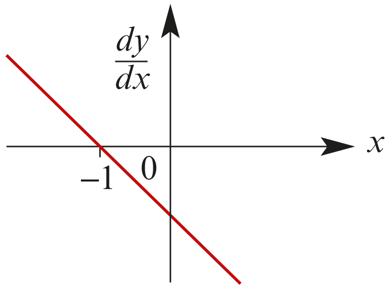
b



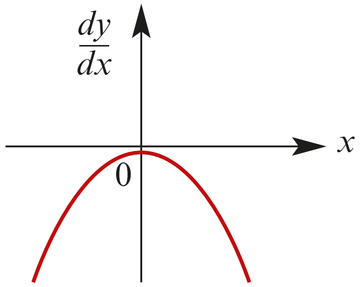
c



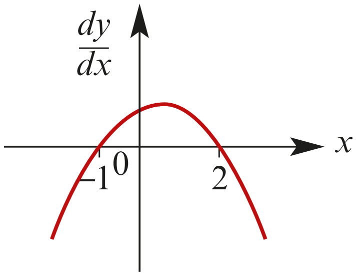
d



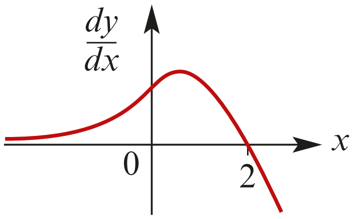
e



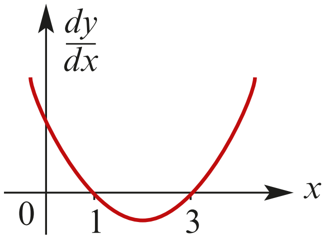
f



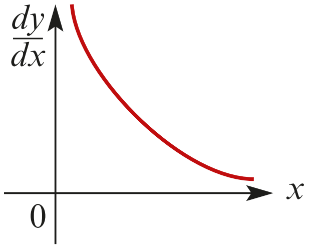
g



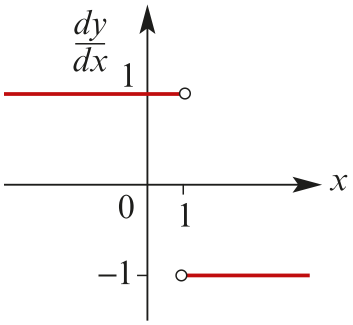
h



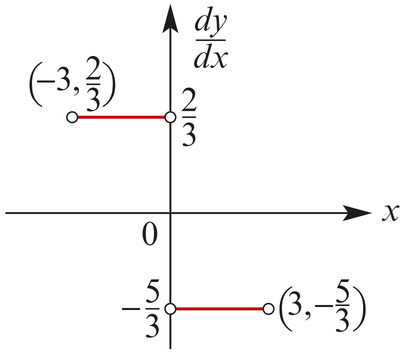
i



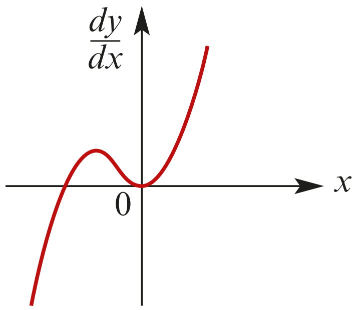
2 a



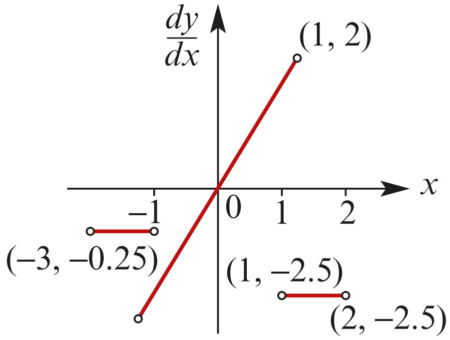
b


$\frac{dy}{dx}$
$\left(-3, \frac{2}{3}\right)$
$\frac{2}{3}$
x
0
$-\frac{5}{3}$
$\left(3, -\frac{5}{3}\right)$

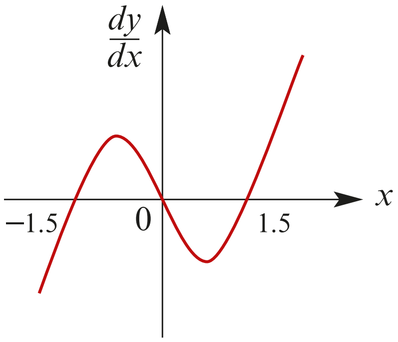
c


$\frac{dy}{dx}$
x
0

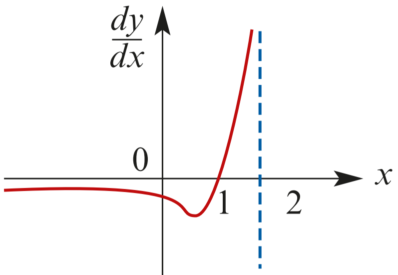
d


$\frac{dy}{dx}$
(1, 2)
x
–1
0
1
2
(–3, –0.25)
(1, –2.5)
(2, –2.5)

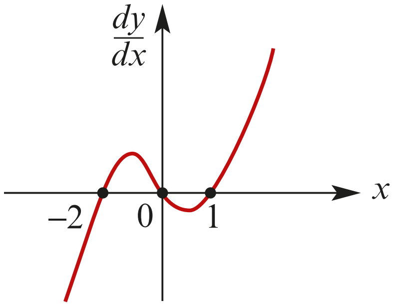
e


$\frac{dy}{dx}$
x
–1.5
0
1.5

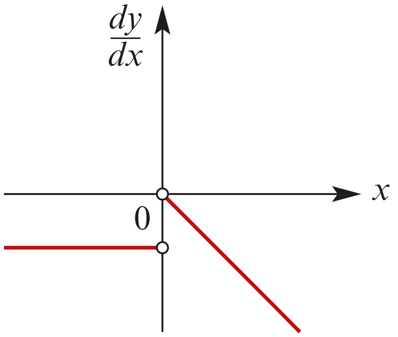
f


$\frac{dy}{dx}$
0
x
1
2

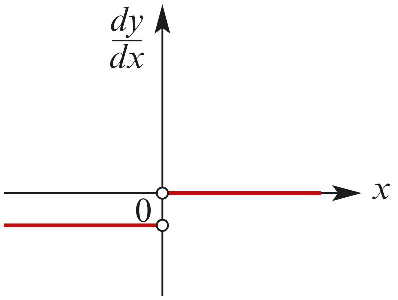
g


$\frac{dy}{dx}$
x
–2
0
1

h


$\frac{dy}{dx}$
x
0

i


$\frac{dy}{dx}$
x
0

3 **a** D

b F

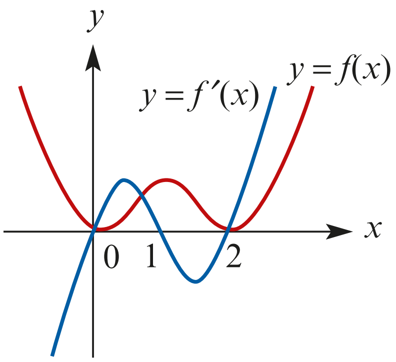
c B

d C

e A

f E

4 **a**



b **i** 0

ii 0

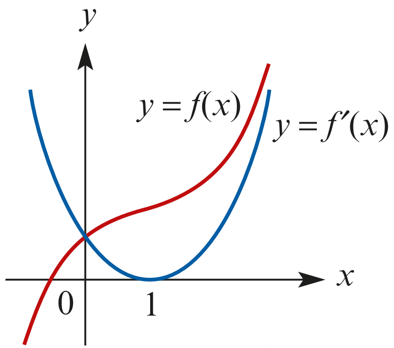
iii 0

iv 96

c **i** 1

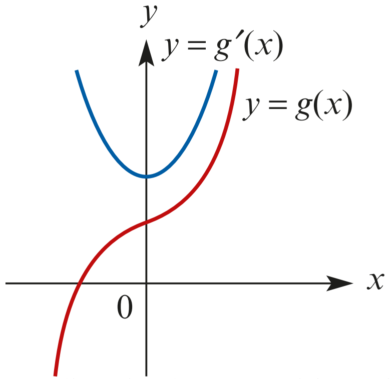
ii 0.423

5



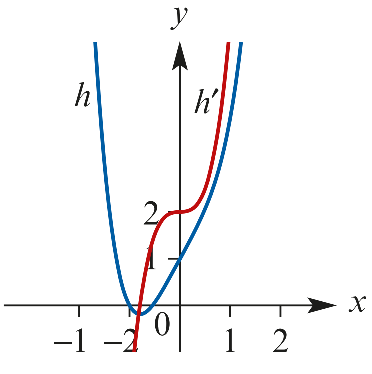
Gradient is 0 at $\left(1, \frac{4}{3}\right)$
Gradient is positive for $R \backslash \{1\}$

6



Gradient is always positive,minimum gradient where $x = 0$

7 **a**



b **i** $x = -1.495$ *or* $x = 0.798$

ii $x = 0.630$

Solutions to Exercise 9E

1 a $y = (x^2 + 1)^4$

Let $u = x^2 + 1,\ y = u^4$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4u^3 \times 2x \\ &= 4(x^2 + 1)^3 \times 2x \\ &= 8x(x^2 + 1)^3 \end{aligned}$$

b $y = (2x^2 - 3)^5$

Let $u = 2x^2 - 3,\ y = u^5$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4x \times 5u^4 \\ &= 4x \times 5(2x^2 - 3)^4 \\ &= 20x(2x^2 - 3)^4 \end{aligned}$$

c $y = (6x + 1)^4$

Let $u = 6x + 1,\ y = u^4$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 6 \times 4u^3 \\ &= 24(6x + 1)^3 \end{aligned}$$

d $y = (ax + b)^n$

Let $u = ax + b,\ y = u^n$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= a \times nu^{(n-1)} \\ &= an(ax + b)^{n-1} \end{aligned}$$

e $y = (ax^2 + b)^n$

Let $u = ax^2 + b,\ y = u^n$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2ax \times nu^{(n-1)} \\ &= 2anx(ax^2 + b)^{n-1} \end{aligned}$$

f $y = (1 - x^2)^{-3}$

Let $u = 1 - x^2, y = u^{-3}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -2x - 3u^{-4} \\ &= 6x(1 - x^2)^{-4} \end{aligned}$$

g $y = (x^2 - x^{-2})^{-3}$

Let $u = x^2 - x^{-2}, y = u^{-3}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= (2x + 2x^{-3}) \times -3u^{-4} \\ &= -6(x + x^{-3})(x^2 - x^{-2})^{-4} \end{aligned}$$

h $y = (1 - x)^{-1}$

Let $u = 1 - x,\ y = u^{-1}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -1 \times -u^{-2} \\ &= (1 - x)^{-2} \end{aligned}$$

2 a

$$\begin{aligned}
y &= (x^2 + 2x + 1)^3\\
y &= ((x + 1)^2)^3\\
y &= (x + 1)^6\\
\textit{Let } u &= x + 1,\ y = u^6\\
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\
&= 1 \times 6u^5\\
&= 6(x + 1)^5
\end{aligned}$$

b

$$\begin{aligned}
y &= (x^3 + 2x^2 + x)^4\\
\textit{Let } u &= x^3 + 2x^2 + x,\ y = u^4\\
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\
&= (3x^2 + 4x + 1) \times 4u^3\\
&= 4(3x + 1)(x + 1)(x^3 + 2x^2 + x)^3\\
&= 4(3x + 1)(x + 1)(x(x + 1)^2)^3\\
&= 4x^3(3x + 1)(x + 1)(x + 1)^6\\
&= 4x^3(3x + 1)(x + 1)^7
\end{aligned}$$

c

$$\begin{aligned}
y &= (6x^3 + 2x^{-1})^4\\
\textit{Let } u &= 6x^3 + 2x^{-1},\ y = u^4\\
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\
&= (18x^2 - 2x^{-2}) \times 4u^3\\
&= 8(9x^2 - x^{-2})(6x^3 + 2x^{-1})^3
\end{aligned}$$

d

$$\begin{aligned}
y &= (x^2 + 2x + 1)^{-2}\\
&= ((x + 1)^2)^{-2}\\
&= (x + 1)^{-4}\\
\textit{Let } u &= x + 1,\ y = u^{-4}\\
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\
&= 1 \times -4u^{-5}\\
&= -4(x + 1)^{-5}
\end{aligned}$$

3 Let $y = \dfrac{16}{3x^3 + x} = 16(3x^3 + x)^{-1}$

Let $u = 3x^3 + x$

Then $y = 16u^{-1}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\
&= -16u^{-2} \times (9x^2 + 1)\\
&= \frac{-16(9x^2 + 1)}{(3x^3 + x)^2}
\end{aligned}$$

When $x = 1$

$$\frac{dy}{dx} = -10$$

4 Let $y = \dfrac{1}{x^2 + 1} = (x^2 + 1)^{-1}$

Let $u = x^2 + 1$

Then $y = u^{-1}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\
&= -u^{-2} \times (2x)\\
&= \frac{-2x}{(x^2 + 1)^2}
\end{aligned}$$

When $x = 1$

$$\frac{dy}{dx} = -\frac{1}{2}$$

When $x = -1$

$$\frac{dy}{dx} = \frac{1}{2}$$

5

$$\begin{aligned}
F(x) &= f(g(x))\\
F'(x) &= g'(x)f'(g(x))\\
&= 2x\sqrt{3g(x) + 4}\\
&= 2x\sqrt{3x^2 + 1}
\end{aligned}$$

6 **a** *Let* $h(x) = [f(x)]^n$

Let $g(x) = x^n$

then $h(x) = g(f(x))$

$h'(x) = g'(f(x)) \times f'(x)$

$= n(f(x))^{n-1} \times f'(x)$

b *Let* $h(x) = (f(x))^{-1}$

Let $g(x) = x^{-1}$

then $h(x) = g(f(x))$

$h'(x) = g'(f(x)) \times f'(x)$

$= -(f(x))^{-2} \times f'(x)$

7 $\dfrac{dy}{dx} = \dfrac{2x - 3}{x^2(x - 3)^2}$

$\dfrac{dy}{dx} = 0 \Rightarrow x = \dfrac{3}{2}$

8 $h(x) = f(g(x))$

$h'(x) = g'(x)f'(g(x))$

Therefore,

$h'(3) = g'(3)f'(g(3))$

$= 6f'(4)$

$= 6 \times 8$

$= 48$

Solutions to Exercise 9F

1

$$\frac{f(x+h)-f(x)}{h}$$

$$= (2\sqrt{x+h} - 2\sqrt{x}) \times \frac{1}{h}$$

$$= (2\sqrt{x+h} - 2\sqrt{x}) \times \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \times \frac{1}{h}$$

$$= \frac{2(x+h-x)}{\sqrt{x+h} + \sqrt{x}} \times \frac{1}{h}$$

$$= \frac{2}{\sqrt{x+h} + \sqrt{x}}$$

Hence

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{x+h} + \sqrt{x}}$$

$$= -\frac{1}{\sqrt{x}}$$

2 **a** $$\frac{d\left(x^{\frac{1}{5}}\right)}{dx} = \frac{1}{5}x^{-\frac{4}{5}}$$

b $$\frac{d\left(x^{\frac{5}{2}}\right)}{dx} = \frac{5}{2}x^{\frac{3}{2}}$$

c $$\frac{d\left(x^{\frac{5}{2}} - x^{\frac{3}{2}}\right)}{dx} = \frac{5}{2}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$

d $$\frac{d\left(3x^{\frac{1}{2}} - 4x^{\frac{5}{3}}\right)}{dx} = \frac{3}{2}x^{-\frac{1}{2}} - 4 \times \frac{5}{3}x^{\frac{2}{3}} = \frac{3}{2}x^{-\frac{1}{2}} - \frac{20}{3}x^{\frac{2}{3}}$$

e $$\frac{d\left(x^{-\frac{6}{7}}\right)}{dx} = \frac{-6}{7}x^{-\frac{13}{7}}$$

f $$\frac{d\left(x^{\frac{1}{4}} + 4x^{\frac{1}{2}}\right)}{dx} = \frac{-1}{4}x^{-\frac{5}{4}} + 2x^{-\frac{1}{2}}$$

3 **a**

$$f(x) = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(27) = \frac{1}{3} \times \frac{1}{(27^{\frac{1}{3}})^2}$$

$$= \frac{1}{3} \times \frac{1}{3^2} = \frac{1}{27}$$

b

$$f(x) = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(-8) = \frac{1}{3} \times (-8)^{-\frac{2}{3}}$$

$$= \frac{1}{3}(-2)^{-2}$$

$$= \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

c

$$f(x) = x^{\frac{2}{3}}$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$f'(27) = \frac{2}{3} \times (27)^{-\frac{1}{3}}$$

$$= \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$$

d

$$f(x) = x^{\frac{5}{4}}$$

$$f'(x) = \frac{5}{4}x^{\frac{1}{4}}$$

$$f'(16) = \frac{5}{4} \times (16)^{\frac{1}{4}}$$

$$= \frac{5}{4} \times 2 = \frac{5}{2}$$

4 **a** $\frac{d}{dx}(\sqrt{2x+1})$

$$= 2 \times \frac{1}{2\sqrt{2x+1}}$$

$$= \frac{1}{\sqrt{2x+1}}$$

b $\frac{d}{ax}(\sqrt{4-3x})$

$$= -3 \times \frac{1}{2\sqrt{4-3x}}$$

$$= \frac{-3}{2\sqrt{4-3x}}$$

c $\frac{d}{dx}(\sqrt{x^2+2})$

$$= 2x \times \frac{1}{2\sqrt{x^2+2}}$$

$$= \frac{x}{\sqrt{x^2+2}}$$

d $\frac{d}{dx}(4-3x)^{\frac{1}{3}}$

$$= -3 \times \frac{1}{3(4-3x)^{\frac{2}{3}}}$$

$$= -(4-3x)^{-2/3}$$

e $\frac{d}{dx}\left(\frac{x^2+2}{\sqrt{x}}\right)$

$$= \frac{d}{dx}\left(x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}\right)$$

$$= \frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{3}{2}}$$

f $\frac{d}{dx}(3\sqrt{x}(x^2+2x))$

$$= \frac{d}{dx}(3x^{\frac{5}{2}} + 6x^{\frac{3}{2}})$$

$$= \frac{15}{2}x^{\frac{3}{2}} + \frac{18}{2}x^{\frac{1}{2}}$$

$$= \frac{15}{2}x^{\frac{3}{2}} + 9\sqrt{x}$$

5 **a** *Let* $u = x^2 \pm a^2$

$$LHS = \frac{d}{dx}(\sqrt{x^2 \pm a^2}) = \frac{d}{dx}(u^{\frac{1}{2}})$$

$$= \frac{d}{dx}(u^{\frac{1}{2}}) \times \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \times 2x$$

$$= \frac{x}{\sqrt{x^2 \pm a^2}} = RHS \quad QED$$

b *Let* $u = a^2 - x^2$

$$LHS = \frac{d}{dx}(\sqrt{a^2 - x^2}) = \frac{d}{dx}(\sqrt{u})$$

$$= \frac{d}{ax}(\sqrt{u}) \times \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \times -2x$$

$$= \frac{x}{\sqrt{a^2 - x^2}}$$

6 $y = (x + \sqrt{x^2 + 1})^2$

Let $u = x + \sqrt{x^2 + 1},\ y = u^2$

$$LHS = \frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= \left(\frac{d(x)}{dx} + \frac{d(\sqrt{x^2 + 1})}{dx}\right) \times \frac{dy}{du}$$

Let $w = x^2 + 1$

$$\frac{dy}{dx} = \left(\frac{d(x)}{dx} + \frac{d(\sqrt{w})}{dw} \times \frac{dw}{dx}\right) \times \frac{dy}{du}$$

$$= \left(1 + \frac{1}{2\sqrt{w}} \times 2x\right) \times 2\mathrm{u}$$

$$\frac{dy}{dx} = \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \times 2(x + \sqrt{x^2 + 1})$$

$$= \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right) \times 2(x + \sqrt{x^2 + 1})$$

$$= \frac{2\left(x + \sqrt{x^2 + 1}\right)^2}{\sqrt{x^2 + 1}}$$

$$= \frac{2y}{\sqrt{x^2 + 1}} = RHS\ \ QED$$

7 **a** *Let* $u = x^2 + 2$

$$\frac{d(\sqrt{x^2 + 2})}{dx} = \frac{d(\sqrt{u})}{du} \times \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \times 2x$$

$$= \frac{x}{\sqrt{x^2 + 2}}$$

b *Let* $u = x^3 - 5x$

$$\frac{d\left((x^3 - 5x)^{\frac{1}{3}}\right)}{dx} = \frac{d\left(u^{\frac{1}{3}}\right)}{du} \times \frac{du}{dx}$$

$$= \frac{1}{3}u^{-\frac{2}{3}} \times (3x^2 - 5)$$

$$= \frac{1}{3}(3x^2 - 5)(x^3 - 5x)^{-\frac{2}{3}}$$

$$= \frac{3x^2 - 5}{3\sqrt[3]{(x^3 - 5x)^2}}$$

c *Let* $u = x^2 + 2x$

$$\frac{d\left((x^2 + 2x)^{\frac{1}{5}}\right)}{dx} = \frac{d\left(u^{\frac{1}{5}}\right)}{du} \times \frac{du}{dx}$$

$$= \frac{1}{5}u^{-\frac{4}{5}} \times 2x + 2$$

$$= \frac{2x + 2}{5(x^2 + 2x)^{\frac{4}{5}}}$$

Solutions to Exercise 9G

1 **a** $f(x) = e^{5x}$

$f' x = 5e^{5x}$

b $f(x) = 7e^{-3x}$

$f'(x) = -21e^{-3x}$

c $f(x) = 3e^{-4x} + e^{x} - x^2$

$f'(x) = -12e^{-4x} + e^{x} - 2x$

d $f(x) = e^{x} - 1 + e^{-x}$

$f'(x) = e^{x} - e^{-x}$

e $f(x) = \dfrac{4e^{2x} - 2e^{x} + 1}{2e^{2x}}$

$= -e^{-x} + \dfrac{1}{2}e^{-2x}$

$f'(x) = e^{-x} - e^{-2x}$

$= e^{-2x}(e^{x} - 1)$

f $f(x) = e^{2x} + e^{4} + e^{-2x}$

$f'(x) = 2e^{2x} - 2e^{-2x}$

2 **a** $-6x^2e^{-2x^3}$

b $2xe^{x^2} + 3$

c $(2x - 4)e^{x^2-4x} + 3$

d $(2x - 2)e^{x^2-2x+3} - 1$

e $-\dfrac{1}{x^2}e^{\frac{1}{x}}$

f $\dfrac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

3 Let $y = e^{\frac{x}{2}} + 4x$

Then $\dfrac{dy}{dx} = \dfrac{1}{2}e^{\frac{x}{2}} + 4$

a When$x = 0, y = \dfrac{9}{2}$

b When$x = 1, \dfrac{1}{2}e^{\frac{1}{2}} + 4$

4 Let $y = e^{x^2+3x} + 2x$

Then $\dfrac{dy}{dx} = (2x + 3)e^{x^2+3x} + 2$

a When$x = 0, y = 5$

b When$x = 1, 5e^{4} + 2$

5 **a** $2f'(x)e^{2f(x)}$

b $2e^{2x}f'(e^{2x})$

6 **a** $y = (e^{2x} - 1)^4$

Let $u = e^{2x} - 1, y = u^4$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= (2e^{2x}) \times 4u^3\\ &= 8e^{2x}(e^{2x} - 1)^3\end{aligned}$$

b $y = e^{\sqrt{x}}$

Let $u = \sqrt{x}, y = e^{u}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= e^{u} \times \frac{1}{2\sqrt{x}}\\ &= e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}}\\ &= \frac{1}{2\sqrt{x}}e^{\sqrt{x}}\end{aligned}$$

c $y = (e^x - 1)^{\frac{1}{2}}$

Let $u = e^x - 1, y = u^{\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{2}u^{-\frac{1}{2}} \times e^x\\ &= \frac{1}{2}e^x(e^x - 1)^{-\frac{1}{2}}\end{aligned}$$

d $y = e^{x^{\frac{2}{3}}}$

Let$u = x^{\frac{2}{3}}, y = e^u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= e^u \times \frac{2}{3}x^{-\frac{1}{3}}\\ &= \frac{2}{3}x^{-\frac{1}{3}}e^{x^{\frac{2}{3}}}\end{aligned}$$

e $(2x - 3)e^{(x-1)(x-2)}$

f $e^{e^x + x}$

Solutions to Exercise 9H

1 a $\dfrac{dy}{dx} = \dfrac{2}{x}$

b $\dfrac{dy}{dx} = \dfrac{4}{2x} = \dfrac{2}{x}$

c $\dfrac{dy}{dx} = 2x + \dfrac{3}{x}$

d $\dfrac{dy}{dx} = \dfrac{3}{x} - \dfrac{1}{x^2} = \dfrac{3x - 1}{x^2}$

e $\dfrac{dy}{dx} = \dfrac{3}{x} + 1 = \dfrac{3 + x}{x}$

f $\dfrac{dy}{dx} = \dfrac{1}{x + 1}$

g $\dfrac{dy}{dx} = \dfrac{2}{2x + 4} = \dfrac{1}{x + 2}$

h $\dfrac{dy}{dx} = \dfrac{3}{3x - 1}$

i $\dfrac{dy}{dx} = \dfrac{6}{6x - 1}$

2 a $\dfrac{dy}{dx} = \dfrac{3}{x}$

b $\dfrac{dy}{dx} = \dfrac{3(\log_e x)^2}{x}$

c $\dfrac{dy}{dx} = \dfrac{2x + 1}{x^2 + x - 1}$

d $\dfrac{dy}{dx} = \dfrac{3x^2 + 2x}{x^3 + x^2}$

e $\dfrac{dy}{dx} = \dfrac{4}{2x + 3}$

f $\dfrac{dy}{dx} = \dfrac{4}{2x - 3}$

3 a $f(x) = \log_e(x^2 + 1)$

$$f'(x) = 2x \times \frac{1}{x^2 + 1}$$

$$= \frac{2x}{x^2 + 1}$$

b $f(x) = \log_e(e^x)$

$$f'(x) = e^x \times \frac{1}{e^x}$$

$$= 1$$

4 a $y = \log_e x$

$x = e,$

$y = \ln e = 1$

$$\frac{dy}{dx} = \frac{1}{x}$$

$x = e,$

$$\frac{dy}{dx} = \frac{1}{e} = e^{-1}$$

b $y = \ln(x^2 + 1)$

$x = e,$

$y = \ln(e^2 + 1)$

$$\frac{dy}{dx} = 2x \times \frac{1}{x^2 + 1}$$

$$= \frac{2x}{x^2 + 1}$$

$x = e,$

$$\frac{dy}{dx} = \frac{2e}{e^2 + 1}$$

c $y = \ln(-x)$

$x = -e,$

$y = \log_e e = 1$

$$\frac{dy}{dx} = \frac{-1}{-x} = \frac{1}{x}$$

$x = -e,$

$$\frac{dy}{dx} = \frac{-1}{e} = -e^{-1}$$

d $y = x + \log_e x$

$x = 1,$

$y = 1 + \log_e 1 = 1$

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

$x = 1,$

$$\frac{dy}{dx} = 1 + 1 = 2$$

e $y = \log_e(x^2 - 2x + 2)$

$x = 1,$

$y = \log_e 1 = 0$

$$\frac{dy}{dx} = (2x - 2)\frac{1}{x^2 - 2x + 2}$$

$x = 1,$

$$\frac{dy}{dx} = 0$$

f $y = \log_e(2x - 1)$

$x = \frac{3}{2},$

$y = \log_e 2$

$$\frac{dy}{dx} = \frac{2}{2x - 1}$$

$x = \frac{3}{2},$

$$\frac{dy}{dx} = \frac{2}{2} = 1$$

5 $f(x) = \ln(\sqrt{x^2 + 1})$

$$f'(x) = 2x \times \frac{1}{2\sqrt{x^2 + 1}} \times \frac{1}{\sqrt{x^2 + 1}}$$

$$= \frac{x}{x^2 + 1}$$

alternatively,

$$f(x) = \ln\left((x^2 + 1)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\ln(x^2 + 1)$$

$$f'(x) = \frac{1}{2} \times 2x \times \frac{1}{x^2 + 1}$$

$$= \frac{x}{x^2 + 1}$$

$$f'(1) = \frac{1}{1 + 1} = \frac{1}{2}$$

6 $\frac{d}{dx}(\ln(x^2 + x + 1))$

$$= (2x + 1) \times \frac{1}{x^2 + x + 1}$$

$$= \frac{2x + 1}{x^2 + x + 1}$$

7 $f(x) = \ln(x^2 + 1)$

$$f'(x) = 2x \times \frac{1}{x^2 + 1}$$

$$= \frac{2x}{x^2 + 1}$$

$$f'(3) = \frac{6}{9 + 1} = \frac{3}{5}$$

8 $\frac{d}{dx}(\ln(f(x))) = f'(x) \times \frac{1}{f(x)}$

$$= \frac{f'(x)}{f(x)}$$

$$\begin{aligned} x &= 0, \\ \frac{d}{dx}(\ln(f(x))) &= \frac{f'(0)}{f(0)} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

Solutions to Exercise 9I

1 a $5\cos 5x$

b $-5\sin 5x$

c $5\sec^2 5x$

d $\cos x \times 2\sin x = \sin 2x$

e $3\sec^2(3x + 1)$

f $-2x\sin(x^2 + 1)$

g $2\sin\left(x - \frac{\pi}{4}\right)\cos\left(x - \frac{\pi}{4}\right)$

h $-2\cos\left(x - \frac{\pi}{3}\right)\sin\left(x - \frac{\pi}{3}\right)$

i $6\sin^2\left(2x + \frac{\pi}{6}\right)\cos\left(2x + \frac{\pi}{6}\right)$

j $6\cos\left(2x + \frac{\pi}{4}\right)\sin^2\left(2x + \frac{\pi}{4}\right)$

2 a $y = \sin 2x$

$$\frac{dy}{dx} = 2\cos\ 2x$$

$x = \frac{\pi}{8}$,

$$y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\frac{dy}{dx} = 2\cos\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

b $y = \sin 3x$

$$\frac{dy}{dx} = 3\cos 3x$$

$x = \frac{\pi}{6}$,

$$y = \sin\frac{\pi}{2} = 1$$

$$\frac{dy}{dx} = 3\cos\frac{\pi}{2} = 0$$

c $y = 1 + \sin 3x$

$$\frac{dy}{dx} = 3\ \cos 3x$$

$x = \frac{\pi}{6}$,

$$y = 1 + \sin\frac{\pi}{2} = 1 + 1 = 2$$

$$\frac{dy}{dx} = 3\cos\frac{\pi}{2} = 0$$

d $y = \cos^2 2x$

$$\begin{aligned}\frac{dy}{dx} &= -2\sin 2x \times 2\cos 2x\\ &= -4\sin 2x\cos 2x\\ &= -2\sin 4x\end{aligned}$$

$x = \frac{\pi}{4}$,

$$y = \cos^2\frac{\pi}{2} = 0$$

$$\frac{dy}{dx} = -2\sin\pi = 0$$

e $y = \sin^2 2x$

$$\begin{aligned}\frac{dy}{dx} &= -2\cos 2x \times 2\sin 2x\\ &= 4\cos 2x\sin 2x\\ &= 2\sin 4x\end{aligned}$$

$x = \frac{\pi}{4}$,

$$y = \sin^2\frac{\pi}{2} = 1$$

$$\frac{dy}{dx} = 2\sin\pi = 0$$

f $y = \tan 2x$

$$\frac{dy}{dx} = 2\sec^2 2x$$

$$x = \frac{\pi}{8},$$

$$y = \tan\frac{\pi}{4} = 1$$

$$\frac{dy}{dx} = 2\sec^2\frac{\pi}{4}$$

$$= 2 \times (\sqrt{2})^2$$

$$= 4$$

3 a $f(x) = 5\cos x - 2\sin 3x$

$$f'(x) = -5\sin x - 6\cos 3x$$

b $f(x) = \cos x + \sin x$

$$f'(x) = -\sin x + \cos x$$

$$= \cos x - \sin x$$

c $f(x) = \sin x + \tan x$

$$f'(x) = \cos x + \sec^2 x$$

d $f(x) = \tan^2 x$

$$f'(x) = \sec^2 x \times 2\tan x$$

$$= 2\tan x\sec^2 x$$

4 a $y = 2\cos\left(\frac{\pi x}{180}\right)$

$$\frac{dy}{dx} = \frac{-2\pi}{180}\sin\left(\frac{\pi x}{180}\right)$$

$$= \frac{-\pi}{90}\sin(x°)$$

b $y = 3\sin\left(\frac{\pi x}{180}\right)$

$$\frac{dy}{dx} = \frac{3\pi}{180}\cos\left(\frac{\pi x}{180}\right)$$

$$= \frac{\pi}{60}\cos(x°)$$

c $y = \tan\left(\frac{3\pi x}{180}\right)$

$$y = \tan\left(\frac{\pi x}{60}\right)$$

$$\frac{dy}{dx} = \frac{\pi}{60}\sec^2\left(\frac{\pi x}{60}\right)$$

$$= \frac{\pi}{60}\sec^2(3x°)$$

5 a $y = -\ln(\cos x)$

$$\frac{dy}{dx} = -\sin x \times -1 \times \frac{1}{\cos x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

b $y = -\log_e(\tan x)$

$$\frac{dy}{dx} = -\sec^2 x \times \frac{1}{\tan x}$$

$$= -\frac{1}{\cos x\sin x}$$

6 a $2\cos xe^{2\sin x}$

b $-2\sin(2x)e^{\cos 2x}$

Solutions to Exercise 9J

1 a

$$y = (2x^2 + 6)(2x^3 + 1)$$

$$\frac{dy}{dx} = (2x^2 + 6)\frac{d}{dx}(2x^3 + 1)$$

$$+ (2x^3 + 1)\frac{d}{dx}(2x^2 + 6)$$

$$= (2x^2 + 6)(6x^2) + (2x^3 + 1)(4x)$$

$$= 12x^4 + 36x^2 + 8x^4 + 4x$$

$$= 20x^4 + 36x^2 + 4x$$

b

$$y = 3x^{\frac{1}{2}}(2x + 1)$$

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}\frac{d}{dx}(2x + 1)$$

$$+ 3(2x + 1)\frac{d}{dx}x^{\frac{1}{2}}$$

$$= 3x^{\frac{1}{2}} \times 2 + 3(2x + 1) \times \frac{1}{2x^{\frac{1}{2}}}$$

$$= 6x^{\frac{1}{2}} + 3x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$$

$$= 9x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$$

c

$$y = 3x(2x - 1)^3$$

$$\frac{dy}{dx} = 3x\frac{d}{dx}((2x - 1)^3)$$

$$+ 3(2x - 1)^3\frac{d}{dx}(x)$$

Let $u = 2x - 1$

$$\frac{dy}{dx} = 3x\frac{du}{dx} \times \frac{d(u)^3}{dx} + 3(2x - 1)^3$$

$$= 3x(2 \times 3u^2) + 3(2x - 1)^3$$

$$= 18x(2x - 1)^2 + 3(2x - 1)^3$$

$$= 3(2x - 1)^2(6x + (2x - 1))$$

$$= 3(2x - 1)^2(8x - 1)$$

d

$$y = 4x^2(2x^2 + 1)^2$$

$$\frac{dy}{dx} = 4x^2\frac{d}{dx}(2x^2 + 1)^2 + 4(2x^2 + 1)^2\frac{dy}{dx}(x^2)$$

$$= 4x^2\Big((2x^2 + 1)\frac{d}{dx}(2x^2 + 1)$$

$$+ (2x^2 + 1)\frac{d}{dx}(2x^2 + 1)\Big)$$

$$+ 4(2x^2 + 1)^2 \times 2x$$

$$= 8x^2(2x^2 + 1)(4x) + 8x(2x^2 + 1)^2$$

$$= 32x^3(2x^2 + 1) + 8x(2x^2 + 1)^2$$

$$= 8x(2x^2 + 1)(4x^2 + 2x^2 + 1)$$

$$= 8x(2x^2 + 1)(6x^2 + 1)$$

e

$$y = (3x + 1)^{\frac{3}{2}}(2x + 4)$$

$$\frac{dy}{dx} = (3x + 1)^{\frac{3}{2}}\frac{d}{dx}(2x + 4)$$

$$+ (2x + 4)\frac{d}{dx}(3x + 1)^{\frac{3}{2}}$$

Let $u = 3x + 1$

$$\frac{dy}{dx} = (3x + 1)^{\frac{3}{2}}(2) + (2x + 4)\left(\frac{d\left(u^{\frac{3}{2}}\right)}{dx} \times \frac{du}{dx}\right)$$

$$= 2(3x + 1)^{\frac{3}{2}} + (2x + 4)\left(\frac{3}{2}u^{\frac{1}{2}} \times 3\right)$$

$$= 2(3x + 1)^{\frac{3}{2}} + \frac{9}{2}(2x + 4)(3x + 1)^{\frac{1}{2}}$$

$$= 2(3x + 1)^{\frac{3}{2}} + 9(x + 2)(3x + 1)^{\frac{1}{2}}$$

$$= (2(3x + 1) + 9(x + 2))(3x + 1)^{\frac{1}{2}}$$

$$= (6x + 2 + 9x + 18)(3x + 1)^{\frac{1}{2}}$$

$$= (15x + 20)(3x + 1)^{\frac{1}{2}}$$

$$= 5(3x + 4)(3x + 1)^{\frac{1}{2}}$$

f

$$\begin{aligned}
y &= (x^2 + 1)(2x - 4)^{\frac{1}{2}} \\
\frac{dy}{dx} &= (x^2 + 1)\frac{d}{dx}(2x - 4)^{\frac{1}{2}} \\
&\quad + (2x - 4)^{\frac{1}{2}}\frac{d}{dx}(x^2 + 1) \\
\textit{Let } u &= 2x - 4 \\
\frac{dy}{dx} &= (x^2 + 1)\left(\frac{d\left(u^{\frac{1}{2}}\right)}{dx} \times \frac{du}{dx}\right) \\
&\quad + (2x - 4)^{\frac{1}{2}} + 2x \\
&= (x^2 + 1)(\frac{1}{2u^{\frac{1}{2}}} \times 2) + 2x(2x - 4)^{\frac{1}{2}} \\
&= (x^2 + 1)(2x - 4)^{-\frac{1}{2}} + 2x(2x - 4)^{\frac{1}{2}} \\
&= \frac{(x^2 + 1) + 2x(2x - 4)}{\sqrt{2x - 4}} \\
&= \frac{(x^2 + 1) + 4x^2 - 8x}{\sqrt{2x - 4}} \\
&= \frac{5x^2 - 8x + 1}{\sqrt{2x - 4}}
\end{aligned}$$

g

$$\begin{aligned}
y &= x^3(3x^2 + 2x + 1)^{-1} \\
\frac{dy}{dx} &= x^3\frac{d}{dx}(3x^2 + 2x + 1)^{-1} \\
&\quad + (3x^2 + 2x + 1)^{-1}\frac{d}{dx}x^3 \\
\textit{Let } u &= 3x^2 + 2x + 1 \\
\frac{dy}{dx} &= x^3\left(\frac{d(u^{-1})}{dx} \times \frac{du}{dx}\right) + (3x^2 + 2x + 1)^{-1} + 3x^2 \\
&= x^3(-u^{-2} \times (6x + 2)) + 3x^2(3x^2 + 2x + 1)^{-1} \\
&= -x^3(6x + 2)(3x^2 + 2x + 1)^{-2} \\
&\quad + 3x^2(3x^2 + 2x + 1)^{-1} \\
&= \frac{-x^3(6x + 2) + 3x^2(3x^2 + 2x + 1)}{(3x^2 + 2x + 1)^2} \\
&= \frac{-6x^4 - 2x^3 + 9x^4 + 6x^3 + 3x^2}{(3x^2 + 2x + 1)^2} \\
&= \frac{3x^4 + 4x^3 + 3x^2}{(3x^2 + 2x + 1)^2} \\
&= \frac{x^2(3x^2 + 4x + 3)}{(3x^2 + 2x + 1)^2}
\end{aligned}$$

h

$$\begin{aligned}
y &= x^4(2x^2 - 1)^{\frac{1}{2}} \\
\frac{dy}{dx} &= x^4\frac{d}{dx}(2x^2 - 1)^{\frac{1}{2}} + (2x^2 - 1)^{\frac{1}{2}}\frac{d}{dx}x^4 \\
\textit{Let } u &= 2x^2 - 1 \\
\frac{dy}{dx} &= x^4\left(\frac{d\left(u^{\frac{1}{2}}\right)}{dx} \times \frac{du}{dx}\right) + (2x^2 - 1)^{\frac{1}{2}} + 4x^3 \\
&= x^4\left(\frac{1}{2}u^{-\frac{1}{2}} \times 4x\right) + 4x^3(2x^2 - 1)^{\frac{1}{2}} \\
&= 2x^5(2x^2 - 1)^{-\frac{1}{2}} + 4x^3(2x^2 - 1)^{\frac{1}{2}} \\
&= (2x^5 + 4x^3(2x^2 - 1))(2x^2 - 1)^{-\frac{1}{2}} \\
&= (2x^5 + 8x^5 - 4x^3)(2x^2 - 1)^{-\frac{1}{2}} \\
&= (10x^5 - 4x^3)(2x^2 - 1)^{-\frac{1}{2}} \\
&= 2x^3(5x^2 - 2)(2x^2 - 1)^{-\frac{1}{2}}
\end{aligned}$$

i

$$
\begin{aligned}
y &= x^2(x^2 + 2x)^{\frac{1}{3}}\\
\frac{dy}{dx} &= x^2\frac{d}{dx}(x^2 + 2x)^{\frac{1}{3}} + (x^2 + 2x)^{\frac{1}{3}}\frac{d}{dx}(x^2)\\
Let\ u &= x^2 + 2x\\
\frac{dy}{dx} &= x^2\left(\frac{d\left(u^{\frac{1}{3}}\right)}{dx} \times \frac{du}{dx}\right) + (x^2 + 2x)^{\frac{1}{3}} \times 2x\\
&= x^2\left(\frac{1}{3}u^{-\frac{2}{3}} \times (2x + 2)\right) + 2x(x^2 + 2x)^{\frac{1}{3}}\\
&= \frac{2}{3}x^2(x + 1)(x^2 + 2x)^{-\frac{2}{3}} + 2x(x^2 + 2x)^{\frac{1}{3}}\\
&= \left(\frac{2}{3}x^3 + \frac{2}{3}x^2 + 2x(x^2 + 2x)\right)(x^2 - 2x)^{-\frac{2}{3}}\\
&= \left(\frac{2}{3}x^3 + \frac{2}{3}x^2 + 2x^3 + 4x^2\right)(x^2 - 2x)^{-\frac{2}{3}}\\
&= x^2\left(\frac{8}{3}x + \frac{14}{3}\right)(x^2 + 2x)^{-\frac{2}{3}}\\
&= \frac{2}{3}x^2(4x + 7)(x^2 + 2x)^{-\frac{2}{3}}
\end{aligned}
$$

j $\dfrac{4(5x^2 - 4)^2(5x^2 + 2)}{x^3}$

k $\dfrac{3(x^6 - 16)}{x^4}$

l $\dfrac{2x^3(9x^2 - 8)}{5(x(x^2 - 1))^{4/5}}$

2 a

$$
\begin{aligned}
f(x) &= e^x(x^2 + 1)\\
f'(x) &= e^x + 2x + (x^2 + 1) \times e^x\\
&= e^x(x^2 + 2x + 1)\\
&= ex(x + 1)^2
\end{aligned}
$$

b

$$
\begin{aligned}
f(x) &= e^{2x}(x^3 + 3x + 1)\\
f'(x) &= e^{2x}(3x^2 + 3) + (x^3 + 3x + 1) \times 2e^{2x}\\
&= e^{2x}(3x^2 + 3 + 2x^3 + 6x + 2)\\
&= e^{2x}(2x^3 + 3x^2 + 6x + 5)
\end{aligned}
$$

c

$$\begin{aligned} f(x) &= e^{4x+1}(x+1)^2 \\ f'(x) &= e^{4x+1} \times 2(x+1) + (x+1)^2 \times 4e^{4x+1} \\ &= e^{4x+1}(4(x+1)^2 + 2(x+1)) \\ &= e^{4x+1}(4x^2 + 8x + 4 + 2x + 2) \\ &= e^{4x+1}(4x^2 + 10x + 6) \\ &= e^{4x+1}(2x+2)(2x+3) \end{aligned}$$

d

$$\begin{aligned} f(x) &= e^{-4x}(x+1)^{\frac{1}{2}} \\ f'(x) &= e^{-4x} \times \frac{1}{2}(x+1)^{\frac{-1}{2}} + (x+1)^{\frac{1}{2}} \times -4e^{-4x} \\ &= e^{-4x}\left(\frac{1}{2}(x+1)^{\frac{-1}{2}} - 4(x+1)^{\frac{-1}{2}}\right) \\ &= e^{-4x}(x+1)^{\frac{-1}{2}}\left(\frac{1}{2} - 4(x+1)\right) \\ &= e^{-4x}(x+1)^{\frac{-1}{2}}\left(-4x - \frac{7}{2}\right) \\ &= \frac{-8x-7}{2e^{4x}\sqrt{x+1}} \end{aligned}$$

3 **a** $f'(x) = \ln x \times 1 + x + \dfrac{1}{x}$

$= \ln x + 1$

b $f'(x) = \ln x \times 4x + 2x^2 \times \dfrac{1}{x}$

$= 2x(1 + 2\ln x)$

c $f'(x) = e^x \times \dfrac{1}{x} + \ln x \times e^x$

$= e^x\left(\dfrac{1}{x} + \ln x\right)$

d $f'(x) = \ln(-x) \times 1 + x + \dfrac{-1}{-x}$

$= \ln(-x) + 1$

4 **a** $f'(x) = 4x^3e^{-2x} - 2x^4e^{-2x}$

$= 2x^3e^{-2x}(2 - x)$

b $f'(x) = 2e^{2x+3}$

c Let $y = (e^{2x} + x)^{\frac{3}{2}}$

Let $u = e^{2x} + x$

Then $y = u^{\frac{3}{2}}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{3}{2}u^{\frac{1}{2}} \times (2e^{2x} + 1) \\ &= \frac{3}{2}(e^{2x} + x)^{\frac{1}{2}} \times (2e^{2x} + 1) \end{aligned}$$

d Let $y = \dfrac{1}{x}e^x$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{x^2} \times e^x + \frac{1}{x} \times e^x \\ &= \frac{e^x(x-1)}{x^2} \end{aligned}$$

e Let $y = e^{\frac{1}{2}x^2}$

$$\frac{dy}{dx} = xe^{\frac{1}{2}x^2}$$

f Let $y = (x^2 + 2x + 2)e^{-x}$

$$\begin{aligned} \frac{dy}{dx} &= (2x+2)e^{-x} - (x^2 + 2x + 2)e^{-x} \\ &= e^{-x}(2x + 2 - x^2 - 2x - 2) \\ &= -x^2e^{-x} \end{aligned}$$

5 **a** $\dfrac{d}{dx}\left(e^x f(x)\right) = e^x f(x) + e^x f'(x)$

$= e^x(f(x) + f'(x))$

b $\dfrac{d}{dx}\left(\dfrac{e^x}{f(x)}\right) = \dfrac{e^x f(x) - e^x f'(x)}{(f(x))^2}$

c $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$

d

$$\frac{d}{dx}\left(e^x(f(x))^2\right) = e^x(f(x))^2 + 2e^x f(x)f'(x)$$

6 a $\frac{d}{dx}(x^3 \cos x) = 3x^2 + (-\sin x)x^3$

$$= x^2(3\cos x - x\sin x)$$

b $2x\cos x - (1 + x^2)\sin x$

c $\frac{d}{dx}(e^{-x}\sin x)$

$$= e^{-x}\sin x + e^{-x}\cos x$$

$$= e^{-x}(\cos x - \sin x)$$

d $6\cos x - 6x\sin x$

e $3\cos(3x)\cos(4x) - 4\sin(4x)\sin(3x)$

f $2\sin(2x) + 2\tan(2x)\sec(2x)$

g $12\sin x + 12x\cos x$

h $\frac{d}{dx}(x^2e^{\sin x})$

$$= 2xe^{\sin x} + x^2\cos xe^{\sin x}$$

$$= xe^{\sin x}(2 + x\cos x)$$

i $\frac{d}{dx}(x^2\cos^2 x)$

$$= 2x\cos^2 x - 2\sin x\cos x \times x^2$$

$$= 2x\cos^2 x - x^2\sin 2x$$

j $\frac{d}{dx}(e^x\tan x)$

$$= e^x\tan x + e^x\sec^2 x$$

$$= e^x(\tan x + \sec^2 x)$$

7 a $f(x) = e^x\sin x$

$$f'(x) = e^x\sin x + e^x\cos x$$

$$= e^x(\sin x + \cos x)$$

$$f'(\pi) = e^\pi(\sin\pi + \cos\pi)$$

$$= -e^\pi$$

b $f(x) = \cos^2 2x$

$$f'(x) = -2\sin 2x \times 2\cos 2x$$

$$= -4(\sin 2x\cos 2x)$$

$$= -2\sin 4x$$

$$f'(\pi) = -2\sin 4\pi$$

$$= -2\sin 0$$

$$= 0$$

8 Let $y = \frac{d}{dx}(f(x)\log_e x)$

$= f'(x)\log_e x + \frac{1}{x}f(x)$

When $x = 1$

$y = 4\log_e 1 + 2 = 2$

Solutions to Exercise 9K

1 a

$$y = \frac{x}{x+4}$$

$$\frac{dy}{dx} = \frac{(x+4)\frac{dx}{dx} - x\frac{d(x+4)}{dx}}{(x+4)^2}$$

$$= \frac{(x+4) - x}{(x+4)^2}$$

$$= \frac{4}{(x+4)^2}$$

b

$$y = \frac{x^2 - 1}{x^2 + 1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d(x^2-1)}{dx}(x^2-1)\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{(x^2+1) \times 2x + (1 - x^2) \times 2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

c

$$y = \frac{x^{\frac{1}{2}}}{1+x}$$

$$\frac{dy}{dx} = \frac{(1+x)\frac{d\left(x^{\frac{1}{2}}\right)}{dx}x^{\frac{1}{2}}\frac{d(1+x)}{dx}}{(1+x)^2}$$

$$= \frac{(1+x)\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{(1+x)^2}$$

$$= \frac{\frac{1}{2}(1+x) - x}{x^{\frac{1}{2}}(1+x)^2}$$

$$= \frac{\frac{1}{2} - \frac{1}{2}x}{x^{\frac{1}{2}}(1+x)^2}$$

$$= \frac{x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{2(1+x)^2}$$

d

$$y = \frac{(x+2)^3}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d(x+2)^3}{dx} - (x+2)^3\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{(x^2+1) \times 3(x+2)^2 - (x+2)^3 \times 2x}{(x^2+1)^2}$$

$$= \frac{(3(x^2+1) - 2x(x+2))(x+2)^2}{(x^2+1)^2}$$

$$= \frac{(3x^2 + 3 - 2x^2 - 4x)(x+2)^2}{(x^2+1)^2}$$

$$= \frac{(x^2 - 4x + 3)(x+2)^2}{(x^2+1)^2}$$

$$= \frac{(x-3)(x-1)(x+2)^2}{(x^2+1)^2}$$

e

$$y = \frac{x-1}{x^2+2}$$

$$\frac{dy}{dx} = \frac{(x^2+2)\frac{d(x-1)}{dx} - (x-1)\frac{d(x^2+2)}{dx}}{(x^2+2)^2}$$

$$= \frac{(x^2+2) - (x-1) \times 2x}{(x^2+2)^2}$$

$$= \frac{x^2 + 2 - 2x^2 + 2x}{(x^2+2)^2}$$

$$= \frac{-x^2 + 2x + 2}{(x^2+2)^2}$$

f

$$y = \frac{x^2+1}{x^2-1}$$

$$\frac{dy}{dx} = \frac{(x^2-1)\frac{d(x^2+1)}{dx}(x^2+1)\frac{d(x^2-1)}{dx}}{(x^2-1)^2}$$

$$= \frac{(x^2-1) \times 2x - (x^2+1) \times 2x}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

g

$$y = \frac{3x^2 + 2x + 1}{x^2 + x + 1}$$

$$\frac{dy}{dx} = \frac{(x^2 + x + 1)\frac{d}{dx}(3x^2 + 2x + 1) - (3x^2 + 2x + 1)\frac{d}{dx}(x^2 + x + 1)}{(x^2 + x + 1)^2}$$

$$= \frac{(x^2 + x + 1)(6x + 2) - (3x^2 + 2x + 1) \times (2x + 1)}{(x^2 + x + 1)^2}$$

$$= \frac{6x^3 + 8x^2 + 8x + 2 - 6x^3 - 7x^2 - 4x - 1}{(x^2 + x + 1)^2}$$

$$= \frac{x^2 + 4x + 1}{(x^2 + x + 1)^2}$$

h

$$y = \frac{2x + 1}{2x^3 + 2x}$$

$$\frac{dy}{dx} = \frac{(2x^3 + 2x)\frac{d}{dx}(2x + 1) - (2x + 1)\frac{d}{dx}(2x^3 + 2x)}{(2x^3 + 2x)^2}$$

$$= \frac{(2x^3 + 2x) \times 2 - (2x + 1)(6x^2 + 2)}{(2x^3 + 2x)^2}$$

$$= \frac{4x^3 + 4x - 12x^3 - 6x^2 - 4x - 2}{(2x^3 + 2x)^2}$$

$$= \frac{-8x^3 - 6x^2 - 2}{(2x^3 + 2x)^2}$$

$$= \frac{-(4x^3 + 3x^2 + 1)}{2(x^3 + x)^2}$$

2 a

$$y = (2x + 1)^4 x^2$$

$$x = 1,\ y = (2(1) + 1)^4 \times (1)^2$$

$$y = 3^4$$

$$y = 81$$

$$\frac{dy}{dx} = x^2\frac{d((2x + 1)^4)}{dx} + (2x + 1)^4\frac{d(x^2)}{dx}$$

$$= x^2 \times 2 \times 4(2x + 1)^3 + (2x + 1)^4 \times 2x$$

$$= 8x^2(2x + 1)^3 + 2x(2x + 1)^4$$

$$= (8x^2 + 4x^2 + 2x)(2x + 1)^3$$

$$= (8x^2 + 2x(2x + 1))(2x + 1)^3$$

$$= 2x(6x + 1)(2x + 1)^3$$

$$x = 1,$$

$$\frac{dy}{dx} = 2(1)(6(1) + 1)(2(1) + 1)^3$$

$$= 2(7)(3)^3$$

$$= 14 \times 27$$

$$\frac{dy}{dx} = 378$$

b $y = x^2(x+1)^{\frac{1}{2}}$

$x = 0,$

$y = (0)^2(0+1)^{\frac{1}{2}}$

$y = 0$

$$\frac{dy}{dx} = x^2\frac{d\left((x+1)^{\frac{1}{2}}\right)}{dx} + (x+1)^{\frac{1}{2}}\frac{d(x^2)}{dx}$$

$$= x^2 \times \frac{1}{2\sqrt{x+1}} + \sqrt{x+1} \times 2x$$

$$= \frac{x^2 + 2x(x+1) \times 2x}{2\sqrt{x+1}}$$

$$= \frac{x^2 + 4x^2 + 4x}{2\sqrt{x+1}}$$

$$= \frac{5x^2 + 4x}{2\sqrt{x+1}}$$

$x = 0,$

$$\frac{dy}{dx} = \frac{5(0)^2 + 4(0)}{2\sqrt{0+1}}$$

$$\frac{dy}{dx} = 0$$

c $y = x^2(2x+1)^{\frac{1}{2}}$

$x = 0,$

$y = 0$

$$\frac{dy}{dx} = x^2\frac{d(2x+1)^{\frac{1}{2}}}{dx} + (2x+1)\frac{dx^2}{dx}$$

$$= x^2\frac{d(2x+1)^{\frac{1}{2}}}{dx} + 2x(2x+1)$$

$x = 0,$

$$\frac{dy}{dx} = 0 + 0$$

$$\frac{dy}{dx} = 0$$

d $y = \frac{x}{x^2+1}$

$x = 1,$

$y = \frac{1}{1+1}$

$y = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{dx}{dx} - x\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{x^2 + 1 - x \times 2x}{(x^2+1)^2}$$

$$= \frac{1 - x^2}{(x^2+1)^2}$$

$x = 1,$

$$\frac{dy}{dx} = \frac{1-1}{(1+1)^2}$$

$$\frac{dy}{dx} = 0$$

e

$y = \frac{2x+1}{x^2+1}$

$x = 1,$

$y = \frac{2+1}{1+1}$

$y = \frac{3}{2}$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d(2x+1)}{dx} - (2x+1)\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{(x^2+1) \times 2 - (2x+1) \times 2x}{(x^2+1)^2}$$

$$= \frac{2x^2 + 2 - 4x^2 - 2x}{(x^2+1)^2}$$

$$= \frac{2(-x^2 - x + 1)}{(x^2+1)^2}$$

$x = 1,\ \frac{dy}{dx} = \frac{2(-1-1+1)}{(2)^2} = -\frac{1}{2}$

3 a

$$
\begin{aligned}
f(x) &= (x+1)(x^2+1)^{\frac{1}{2}} \\
f'(x) &= (x+1)\frac{d(x^2+1)^{\frac{1}{2}}}{dx} \\
&\quad + (x^2+1)^{\frac{1}{2}}\frac{d(x+1)}{dx} \\
&= (x+1)\left(\frac{d(x^2+1)^{\frac{1}{2}}}{d(x^2+1)} \times \frac{d(x^2+1)}{dx}\right) \\
&\quad + (x^2+1)^{\frac{1}{2}} \\
&= (x+1)\left(\frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x\right) + (x^2+1)^{\frac{1}{2}} \\
&= (x^2+1)^{-\frac{1}{2}}(x^2+x+x^2+1) \\
&= (x^2+1)^{-\frac{1}{2}}(2x^2+x+1)
\end{aligned}
$$

b

$$
\begin{aligned}
f(x) &= (x^2+1)(x^3+1)^{\frac{1}{2}} \\
f'(x) &= (x^2+1)\frac{d(x^3+1)^{\frac{1}{2}}}{dx} \\
&\quad + (x^3+1)^{\frac{1}{2}}\frac{d(x^2+1)}{dx} \\
&= (x^2+1)\left(\frac{d(x^3+1)^{\frac{1}{2}}}{d(x^3+1)} \times \frac{d(x^3+1)}{dx}\right) \\
&\quad + (x^3+1)^{\frac{1}{2}} \times 2x \\
&= (x^2+1)\left(\frac{1}{2}(x^3+1)^{-\frac{1}{2}} \times 3x^2\right) + 2x(x^3+1)^{\frac{1}{2}} \\
&= \frac{3}{2}x^2(x^2+1)(x^3+1)^{-\frac{1}{2}} + 2x(x^3+1)^{\frac{1}{2}} \\
&= (x^3+1)^{-\frac{1}{2}}\left(\frac{3}{2}x^4 + \frac{3}{2}x^2 + 2x^4 + 2x\right) \\
&= (x^3+1)^{-\frac{1}{2}}\left(7x^4 + \frac{3}{2}x^2 + 2x\right) \\
&= x\left(\frac{7}{2}x^3 + \frac{3}{2}x + 2\right)(x^3+1)^{-\frac{1}{2}}
\end{aligned}
$$

c

$$
\begin{aligned}
f(x) &= \frac{2x+1}{x+3} \\
f^1(x) &= \frac{(x+3)\frac{d(2x+1)}{dx} - (2x+1)\frac{d(x+3)}{dx}}{(x+3)^2} \\
&= \frac{(x+3) \times 2 - (2x+1)}{(x+3)^2} \\
&= \frac{2x+6-2x-1}{(x+3)^2} \\
&= \frac{5}{(x+3)^2}
\end{aligned}
$$

4 a
$$
\begin{aligned}
f'(x) &= \frac{e^x(e^{3x}+3) - 3e^{3x}e^x}{(e^{3x}+3)^2} \\
&= \frac{3e^x - 2e^{4x}}{(e^{3x}+3)^2}
\end{aligned}
$$

b
$$
\begin{aligned}
f'(x) &= \frac{-\sin x(x+1) - \cos x}{(x+1)^2} \\
&= -\frac{\sin x(x+1) + \cos x}{(x+1)^2}
\end{aligned}
$$

c
$$
\begin{aligned}
f'(x) &= \frac{\frac{1}{x} \times (x+1) - \log_e x}{(x+1)^2} \\
&= \frac{(x+1) - x\log_e x}{x(x+1)^2}
\end{aligned}
$$

5 a
$$
\begin{aligned}
f'(x) &= \frac{\frac{1}{x} \times x - \log_e x}{(x^2} \\
&= \frac{1 - \log_e x}{x^2}
\end{aligned}
$$

b
$$
\begin{aligned}
f'(x) &= \frac{\frac{1}{x}(x^2+1) - 2x\log_e x}{(x^2+1)^2} \\
&= \frac{x^2 + 1 - 2x^2\log_e x}{x(x^2+1)^2}
\end{aligned}
$$

6 **a**

$$\begin{aligned} f(x) &= \frac{e^{3x}}{e^{3x}+3} \\ f'(x) &= \frac{(e^{3x}+3)\frac{d}{dx}e^{3x} - e^{3x}\frac{d}{dx}(e^{3x}+3)}{(e^{3x}+3)^2} \\ &= \frac{3e^{3x}(e^{3x}+3) - 3e^{3x}(e^{3x})}{(e^{3x}+3)^2} \\ &= \frac{9e^{3x}}{(e^{3x}+3)^2} \end{aligned}$$

b

$$\begin{aligned} f(x) &= \frac{e^{x}+1}{e^{x}-1} \\ f'(x) &= \frac{(e^x-1)\frac{d}{dx}(e^x+1) - (e^X+1)\frac{d}{dx}(e^x-1)}{(e^x-1)^2} \\ &= \frac{(e^x-1)e^x + (-e^x-1)(e^x)}{(e^x-1)^2} \\ &= \frac{-2e^x}{(e^x-1)^2} \end{aligned}$$

c

$$\begin{aligned} f(x) &= \frac{e^{2x}+2}{e^{2x}-2} \\ f'(x) &= \frac{(e^x-2)\frac{d}{dx}(e^{2x}+2) - (e^{2x}+2)\frac{d}{dx}(e^{2x}-2)}{(e^{2x}-2)^2} \\ &= \frac{(e^{2x}-2)2e^{2x} - (e^{2x}+2)2e^{2x}}{(e^{2x}-2)^2} \\ &= \frac{-8e^{2x}}{(e^{2x}-2)^2} \end{aligned}$$

7 **a**

$$\begin{aligned} f(x) &= \frac{2x}{\cos x} \\ f'(x) &= \frac{\cos x \times 2 - 2x(-\sin x)}{\cos^2 x} \\ &= \frac{2\cos x + 2x\sin x}{\cos^2 x} \\ f'(\pi) &= \frac{2\cos(\pi) + 2\pi\sin(\pi)}{(\cos(\pi))^2} \\ &= \frac{-2}{1} \\ &= -2 \end{aligned}$$

b

$$\begin{aligned} f(x) &= \frac{3x^2+1}{\cos x} \\ f'(x) &= \frac{\cos x(6x) - (3x^2+1)(-\sin x)}{\cos^2 x} \\ &= \frac{6x\cos x + (3x^2+1)\sin x}{\cos^2 x} \\ f'(\pi) &= \frac{6\pi\cos(\pi) + (3\pi^2+1)\sin(\pi)}{(\cos(\pi))^2} \\ &= \frac{-6\pi}{1} \\ &= -6\pi \end{aligned}$$

c

$$\begin{aligned} f(x) &= \frac{e^x}{\cos x} \\ f'(x) &= \frac{\cos x e^x + \sin x e^x}{\cos^2 x} \\ f'(\pi) &= \frac{(\cos\pi + \sin\pi)e^{\pi}}{\cos^2\pi} \\ &= -e^{\pi} \end{aligned}$$

d

$$\begin{aligned} f(x) &= \frac{\sin x}{x} \\ f'(x) &= \frac{x\cos x - \sin x}{x^2} \\ f'(\pi) &= \frac{\pi\cos\pi - \sin\pi}{\pi^2} \\ &= \frac{-\pi}{\pi^2} \\ &= \frac{-1}{\pi} \end{aligned}$$

Solutions to Exercise 9L

1 **a** $\lim\limits_{x\to2}(17) = 17$

b $\lim\limits_{x\to6}(x-3) = 6-3 = 3$

c $\lim\limits_{x\to\frac{1}{2}}(2x-5) = 1-5 = -4$

d $\lim\limits_{t\to-3}\left(\dfrac{(t+2)}{(t-5)}\right) = \dfrac{-3+2}{-3-5} = \dfrac{-1}{-8} = \dfrac{1}{8}$

e $\lim\limits_{t\to2} = \left(\dfrac{t^2+2t+1}{t+1}\right)$

$= \lim\limits_{t\to2}\left(\dfrac{(t+1)^2}{t+1}\right) = \lim\limits_{t\to2}(t+1) = 3$

f $\lim\limits_{x\to0}\left(\dfrac{(x+2)^2-4}{x}\right)$

$= \lim\limits_{x\to0}\left(\dfrac{x^2+4x+4-4}{x}\right)$

$= \lim\limits_{x\to0}\left(\dfrac{x^2+4x}{x}\right) = \lim\limits_{x\to0}(x+4) = 4$

g $\lim\limits_{t\to1} = \left(\dfrac{t^2-1}{t-1}\right)$

$= \lim\limits_{t\to1}\left(\dfrac{(t+1)(t-1)}{\mathrm{t}-1}\right)$

$= \lim\limits_{t\to1}(t+1) = 2$

h $\lim\limits_{x\to9}(\sqrt{x+3}) = \sqrt{9+3}$

$= \sqrt{12} = 2\sqrt{3}$

i $\lim\limits_{x\to0}\left(\dfrac{x^2-2x}{x}\right)$

$= \lim\limits_{x\to0}(x-2) = -2$

j $\lim\limits_{x\to2} = \left(\dfrac{x^3-8}{x-2}\right)$

$= \lim\limits_{x\to2}\left(\dfrac{(x-2)(x^2+2x+4)}{(x-2)}\right)$

$= \lim\limits_{x\to2}(x^2+2x+4) = 12$

k $\lim\limits_{x\to2} = \left(\dfrac{3x^2-x-10}{x^2+5x-14}\right)$

$= \lim\limits_{x\to2}\left(\dfrac{(3x+5)(x-2)}{(x+7)(x-2)}\right)$

$= \lim\limits_{x\to2}\dfrac{3x+5}{x+7} = \dfrac{11}{9}$

l $\lim\limits_{x\to1} = \left(\dfrac{x^2-3x+2}{x^2-6x+5}\right)$

$= \lim\limits_{x\to1}\left(\dfrac{(x-1)(x-2)}{(x-1)(x-4)}\right)$

$= \lim\limits_{x\to1}\dfrac{x-2}{x-5} = \dfrac{1}{4}$

2 **a** $x = 3$, *since* $f(3) \neq \lim\limits_{x\to3}(f(x))$, $x = 4$, *since* $\lim\limits_{x\to4^+}(f(x)) \neq \lim\limits_{x\to4^-}(f(x))$

b $x = 7$, *since* $\lim\limits_{x\to7^+}(f(x)) \neq \lim\limits_{x\to7^-}(f(x))$

3 **a** *value to test*: $x = 0$

$\lim\limits_{x\to0^-}(f(x)) = \lim\limits_{x\to0}(f(-2x+2)) = 2$

$\lim\limits_{x\to0^+}(f(x)) = \lim\limits_{x\to0}(3x) = 0 \neq$

$\lim\limits_{x\to0^-}(f(x))$

$\therefore$ *there is a discontinuity at* $x = 0$

b *value to test*: $x = 1$

$\lim\limits_{x\to1^-}(f(x)) = \lim\limits_{x\to1}(f(-2x+1)) =$

$-2(1)+1 = -1$

$\lim\limits_{x\to1^+}(f(x)) = \lim\limits_{x\to1}(x^2+2) = 1^2+2$

$= 3 \neq \lim\limits_{x\to1^-}(f(x))$

$\therefore$ *there is a discontinuity at* $x = 1$

c *value to test*: $x = -1,\ 0$

$\lim\limits_{x \to 1^-} (f(x)) = \lim\limits_{x \to -1} (-x) = -(-1) = 1$

$\lim\limits_{x \to -1^+} (f(x)) = \lim\limits_{x \to -1} (x^2) = (-1)^2 = 1$

$= \lim\limits_{x \to -1^-} (f(x))$

$f(-1) = -(-1) = 1 = \lim\limits_{x \to -1} (f(x))$

$\therefore$ *$f(x)$ is continuous at $x = -1$*

$\lim\limits_{x \to 0} -(f(x)) = \lim\limits_{x \to 0}(x^2) = (0)^2 = 0$

$\lim\limits_{x \to 0^+} (f(x)) = \lim\limits_{x \to 0}(-3x + 1) = -3(0) + 1$

$= 1 \neq \lim\limits_{x \to 0^-} (f(x))$

$\therefore$ *there is one discontinuity at $x = 0$*

4 **a** *value to test*: $x = 1, 7$

$\lim\limits_{x \to 1^-} (f(x)) = \lim\limits_{x \to -1} (2) = 2$

$\lim\limits_{x \to -1^+} (f(x))$

$= \lim\limits_{x \to -1} ((x - 4)^2 - 9)$

$= (1 - 4)^2 - 9$

$= 0 \neq \lim\limits_{x \to -1^-} (f(x))$

$\therefore$ *there is a discontinuity at $x = 1$*

$\lim\limits_{x \to 7^-} (f(x)) = (\lim\limits_{x \to 7}(x - 4)^2 - 9)$

$= (7 - 4)^2 - 9 = 0$

$\lim\limits_{x \to 7^-} (f(x)) = \lim\limits_{x \to 7}(x - 7) = 7 - 7 =$

$\lim\limits_{x \to 7^-} (f(x))$

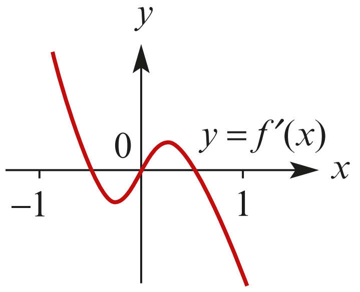
$f(7) = 7 - 7 = 0 = \lim\limits_{x \to 7} f(x)$

$f(x)$ is continuous at $x = 7$

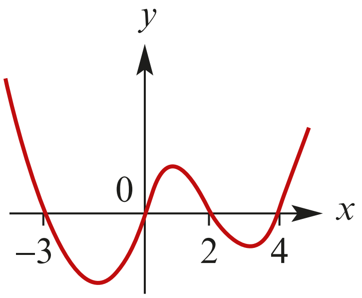
$\therefore$ *$f(x)$ is continuous for all $x \in R\backslash\{1\}$*

Solutions to Exercise 9M

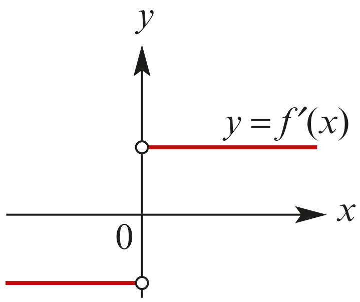
1 a



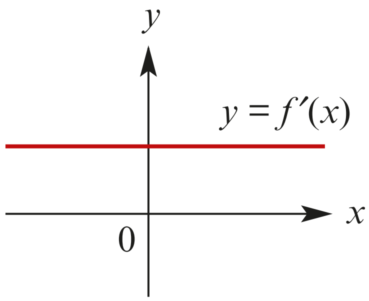
b



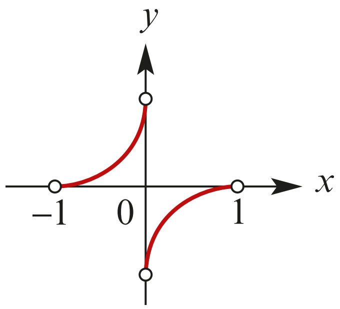
c



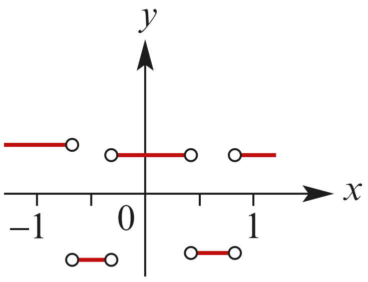
d



e



f



2 $x > 0,\ f'(x) = \dfrac{d}{dx}(-x^2 + 3)$

$= -2x + 3$

$x < 0,\ f'(x) = \dfrac{d}{dx}(3x + 1)$

$= 3$

test $x = 0$

$\lim\limits_{x\to 0^-}(f(x)) = \lim\limits_{x\to 0}(3x + 1) = 1$

$\lim\limits_{x\to 0^+}(f(x)) = \lim\limits_{x\to 0}(-x^2 + 3x + 1) = 1 = \lim\limits_{x\to 0^-}(f(x))$

$f(0) = -(0)^2 + 3(0) + 1 = 1 = \lim\limits_{x\to 0}(f(x))$

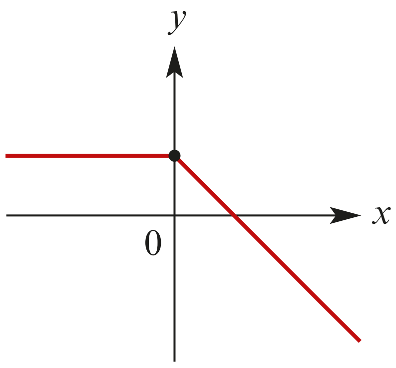
$\therefore$ $f(x)$ is continous at $x = 0$

$\lim\limits_{x\to 0^-}(f'(x)) = \lim\limits_{x\to 0}(3) = 3$

$\lim\limits_{x\to 0^+}(f'(x)) = \lim\limits_{x\to 0}(-2x + 3) = 3 = \lim\limits_{x\to 0^-}(f'(x))$

$f(x)$ is differentiable at $x = 0$

$$f'(x) = \begin{cases} -2x + 3 & if\ x \geq 0 \\ 3 & if\ x < 0 \end{cases}$$



3

$x > 1,\ f'(x) = \frac{d}{dx}(x^2 + 2x + 1)$

$= 2x + 2$

$x < 1,\ f'(x) = \frac{d}{dx}(-2x + 3)$

$= -2$

test $x = 1$

$\lim\limits_{x \to 1^-}(f(x)) = \lim\limits_{x \to 1}(-2x + 3) = -2 + 3 = 1$

$\lim\limits_{x \to 1^+}(f(x)) = \lim\limits_{x \to 1}(x^2 + 2x + 1) = 1 + 2 + 1$

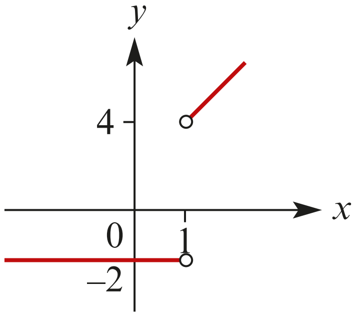
$= 4 \neq \lim\limits_{x \to 1^-}(f(x))$

$\therefore$ $f(x)$is discontinous &

$\therefore$ not differentiable at $x = 1$

$\therefore$ $f'(x)$ is defined for $x \in R\backslash\{1\}$

$$f'(x) = \begin{cases} 2x + 2 & if\ x > 1 \\ -2 & if\ x < 0 \end{cases}$$



4 $x > -1,\ f'(x) = \frac{d}{dx}(-x^2 - 2x + 1)$

$= -2x - 2$

$x < -1,\ f'(x) = \frac{d}{dx}(-2x + 3)$

$= -2$

test $x = -1$

$\lim\limits_{x \to -1^-}(f(x)) = \lim\limits_{x \to -1}(-2x + 3) = 2 + 3 = 5$

$\lim\limits_{x \to -1^+}(f(x)) = \lim\limits_{x \to -1}(-x^2 - 2x + 1) = -1 + 2 + 1$

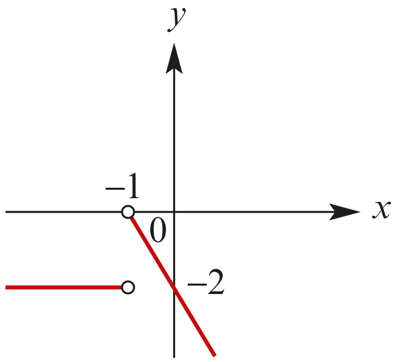
$= 2 \neq \lim\limits_{x \to -1^-}(f(x))$

$\therefore$ $f(x)$ is not continuous &

$\therefore$ not differentiable at$x = -1$

$\therefore$ $f'(x)$ is defined for$x \in R\backslash\{-1\}$

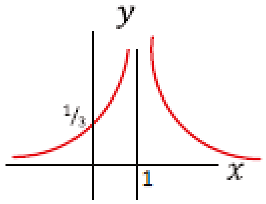
$$f'(x) = \begin{cases} -2x - 2 & if\ x > -1 \\ -2 & if\ x < -1 \end{cases}.$$



5 **a** $f'(x) = \frac{1}{3}(x - 1)^{-\frac{2}{3}}$

$f'(x)$ is defined for $x \in \mathbb{R}\backslash\{1\}$ (since

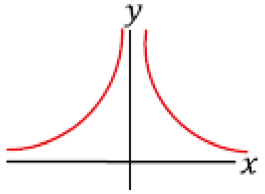
$x = 1$ gives $f'(x) = \frac{1}{0}$)



b $f'(x) = \frac{1}{5}x^{\frac{-4}{5}}$

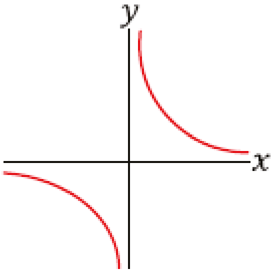
$f'(x)$ is defined for $x \in \mathbb{R}\backslash\{0\}$ (*since*

$x = 0$ gives $f'(x) = \frac{1}{0}$)



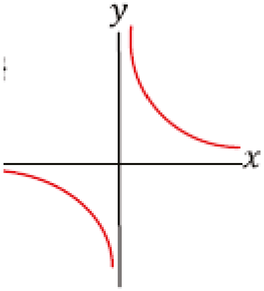
c $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

$f'(x)$ is defined for $x \in \mathbb{R}\backslash\{0\}$



d $f'(x) = \frac{2}{5}(x + 2)^{-\frac{3}{5}}$

$f'(x)$ is defined for $x \in \mathbb{R}\backslash\{-2\}$



Solutions to Technology-free questions

1 a Average rate of change $= \dfrac{26-10}{2}$

$= 8$

b $\dfrac{dy}{dx} = 2x$

When $x = -4$, $\dfrac{dy}{dx} = -8$

2 a

$$y = x + \sqrt{1-x^2} = x + (1-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 1 + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)$$

$$= 1 - \frac{x}{\sqrt{1-x^2}}$$

b

$$y = \frac{4x+1}{x^2+3}$$

$$\frac{dy}{dx} = \frac{(x^2+3)(4)-(4x+1)(2x)}{(x^2+3)^2}$$

$$= \frac{12-2x-4x^2}{(x^2+3)^2}$$

c

$$y = \sqrt{1+3x} = (1+3x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(1+3x)^{-\frac{1}{2}} \times 3$$

$$= \frac{3}{2\sqrt{1+3x}}$$

d

$$y = \frac{2+\sqrt{x}}{x} = 2x^{-1} + x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -2x^{-2} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$= -\frac{2}{x^2} - \frac{1}{2x^{\frac{3}{2}}}$$

e

$$y = (x-9)\sqrt{x-3} = (x-9)(x-3)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = (1)(x-3)^{\frac{1}{2}} + (x-9) \times \frac{1}{2}(x-3)^{-\frac{1}{2}}$$

$$= \frac{2(x-3)+(x-9)}{2\sqrt{x-3}}$$

$$= \frac{3x-15}{2\sqrt{x-3}}$$

f

$$y = x\sqrt{1+x^2} = x(1+x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = (1)(1+x^2)^{\frac{1}{2}} + x \times \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)$$

$$= \frac{1+2x^2}{\sqrt{1+x^2}}$$

g

$$y = \frac{x^2-1}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

h

$$y = \frac{x}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)(1)-(x)(2x)}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

i

$$y = (2+5x^2)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3}(2+5x^2)^{-\frac{2}{3}} \times 10x$$

$$= \frac{10x}{3}(2+5x^2)^{-\frac{2}{3}}$$

j $y = \dfrac{2x+1}{x^2+2}$

$$\frac{dy}{dx} = \frac{(x^2+2)(2)-(2x+1)(2x)}{(x^2+2)^2}$$
$$= \frac{4-2x-2x^2}{(x^2+2)^2}$$

k $y = (3x^2+2)^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{2}{3}(3x+2)^{-\frac{1}{3}} \times 6x$$
$$= 4x(3x^2+2)^{-\frac{1}{3}}$$

3 **a** $y = 3x^2 - 4$

$$\frac{dy}{dx} = 6x$$
$$= -6(\text{ at } x = -1)$$

b $y = \dfrac{x-1}{x^2+1}$

$$\frac{dy}{dx} = \frac{(x^2+1)(1)-(x-1)(2x)}{(x^2+1)^2}$$
$$= \frac{1+2x-x^2}{(x^2+1)^2}$$
$$= 1(\text{at } x = 0)$$

c $y = (x-2)^5$

$$\frac{dy}{dx} = 5(x-2)^4$$
$$= 5 \text{ (at } x = 1)$$

d $y = (2x+2)^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}(2x+2)^{-\frac{1}{3}} \times 2$$
$$= \frac{2}{3}(2x+2)^{-\frac{2}{3}}$$
$$= \frac{2}{3}(8)^{-\frac{2}{3}} \text{ (at } x = 3)$$
$$= \frac{2}{3}(2^3)^{-\frac{2}{3}}$$
$$= \frac{2}{3} \times 2^{-2}$$
$$= \frac{1}{6}$$

4 **a** $y = \log_e(x+2)$

$$\frac{dy}{dx} = \frac{1}{x+2}$$

b $y = \sin(3x+2)$

$$\frac{dy}{dx} = 3\cos(3x+2)$$

c $y = \cos\left(\dfrac{x}{2}\right)$

$$\frac{dy}{dx} = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$$

d $y = e^{x^2-2x}$

$$\frac{dy}{dx} = (2x-2)e^{x^2-2x}$$

e $y = \log_e(3-x)$

$$\frac{dy}{dx} = -\frac{1}{3-x} = \frac{1}{x-3}$$

f $y = \sin(2\pi x)$

$$\frac{dy}{dx} = 2\pi\cos(2\pi x)$$

g $y = \sin^2(3x+1)$

$$\frac{dy}{dx} = 2\sin(3x+1) \times 3\cos(3x+1)$$

$$= 6\sin(3x+1)\cos(3x+1)$$

$$= 3\sin(6x+2)$$

as $\sin(2a) = 2\sin(a)\cos(a)$

h $y = \sqrt{\log_e x} = (\log_e x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(\log_e x)^{\frac{1}{2}} \times \frac{1}{x} = \frac{1}{2x\sqrt{\log_e x}}$$

i

$$y = \frac{2\log_e 2x}{x} = 2x^{-1}\log_e 2x$$

$$\frac{dy}{dx} = -2x^{-2}\log_e 2x + 2x^{-1} \times \frac{2}{2x}$$

$$= -\frac{2\log_e 2x}{x^2} + \frac{2}{x^2} = \frac{2 - 2\log_e 2x}{x^2}$$

j $y = x^2\sin(2\pi x)$

$$\frac{dy}{dx} = 2x\sin(2\pi x) + 2\pi x^2\cos(2\pi x)$$

5 a $y = e^x \sin 2x$

$$\frac{dy}{dx} = e^x\sin 2x + 2e^x\cos 2x$$

b $y = 2x^2\log_e x$

$$\frac{dy}{dx} = 4x\log_e x + 2x^2 \times \frac{1}{x}$$

$$= 4x\log_e x + 2x$$

c $y = \frac{\log_e x}{x^3} = x^{-3}\log_e x$

$$\frac{dy}{dx} = -3x^{-4}\log_e x + x^{-3} \times \frac{1}{x}$$

$$= \frac{1 - 3\log_e x}{x^4}$$

d

$$y = \sin 2x \cos 3x$$

$$\frac{dy}{dx} = (2\cos 2x)\cos 3x + \sin 2x(-3\sin 3x)$$

$$= 2\cos 2x\cos 3x - 3\sin 2x\sin 3x$$

e $y = \frac{\sin 2x}{\cos 2x} = \tan 2x$

$$\frac{dy}{dx} = 2\sec^2 2x$$

(Alternatively, use the quotient rule.)

f $y = \cos^3(3x+2)$

$$\frac{dy}{dx} = 3\cos^2(3x+2) \times -3\sin(3x+2)$$

$$= -9\cos^2(3x+2)\sin(3x+2)$$

g

$$y = x^2\sin^2(3x)$$

$$\frac{dy}{dx} = 2x\sin^2(3x)$$

$$+ x^2(2\sin(3x) \times 3\cos(3x))$$

$$= 2x\sin^2(3x) + 6x^2\sin(3x)\cos(3x)$$

$$= 2x\sin^2(3x) + 3x^2\sin(6x)$$

as $\sin(2a) = 2\sin(a)\cos(a)$

6 a $y = e^{2x} + 1$

$$\frac{dy}{dx} = 2e^{2x}$$

$$= 2e^2 \text{ (at } x = 1)$$

b $y = e^{x^2+1}$

$$\frac{dy}{dx} = 2xe^{x^2+1}$$

$$= 0 \text{ (at } x = 0)$$

c $y = 5e^{3x} + x^2$

$$\frac{dy}{dx} = 15e^{3x} + 2x$$

$$= 15e^3 + 2 \text{ (at } x = 1)$$

d $y = 5 - e^{-x}$

$\dfrac{dy}{dx} = e^{-x}$

$= 1$ (at $x = 0$)

7 a $y = e^{ax}$

$\dfrac{dy}{dx} = ae^{ax}$

b $y = e^{ax+b}$

$\dfrac{dy}{dx} = ae^{ax+b}$

c $y = e^{a-bx}$

$\dfrac{dy}{dx} = -be^{a-bx}$

d $y = be^{ax} - ae^{bx}$

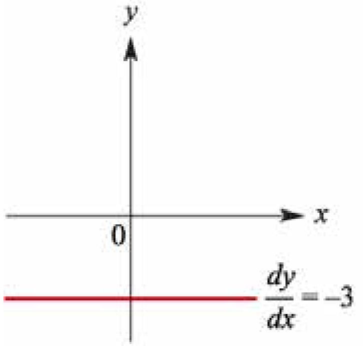
$\dfrac{dy}{dx} = abe^{ax} - abe^{bx}$

$= ab(e^{ax} - e^{bx})$

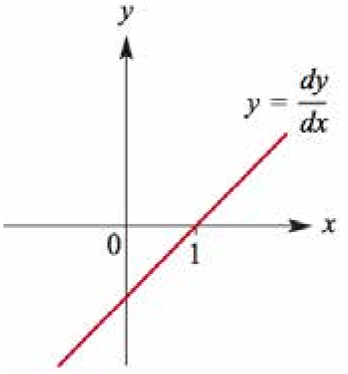
e $y = \dfrac{e^{ax}}{e^{bx}} = e^{ax-bx} = e^{(a-b)x}$

$\dfrac{dy}{dx} = (a-b)e^{(a-b)x}$

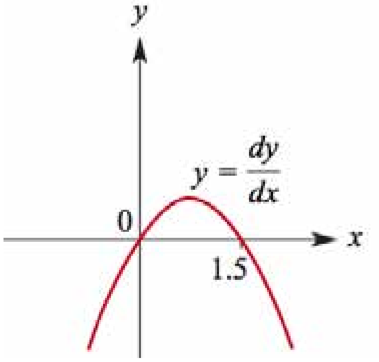
8 a $y = 3 - 3x$ so $\dfrac{dy}{dx} = -3$



b Graph looks parabolic, so derivative graph will be linear. Also, there is a turning point where $x = 1$, so the derivative function will be zero at $x = 1$.



c Graph looks cubic, so derivative graph will be quadratic. Also, there are turning points where $x = 0$ and $x = 1.5$, so derivative function will be zero at $x = 0, 1.5$. Finally, the gradient goes from negative to positive to negative, so the gradient graph will be an inverted parabola.



9 $y = \left(4x + \dfrac{9}{x}\right)^2$

$\dfrac{dy}{dx} = 2\left(4x + \dfrac{9}{x}\right)\left(4 - \dfrac{9}{x^2}\right)$

$= \dfrac{2(4x^2 + 9)(4x^2 - 9)}{x^3}$

Then $\dfrac{dy}{dx} = 0$ provided $4x^2 - 9 = 0$ (since $4x^2 + 9 > 0$ for all values of x).

Hence $x = \pm\dfrac{3}{2}$.

10 a $$y = \frac{2x-3}{x^2+4}$$

$$\frac{dy}{dx} = \frac{(x^2+4)(2)-(2x-3)(2x)}{(x^2+4)^2}$$
$$= \frac{2x^2+8-4x^2+6x}{(x^2+4)^2}$$
$$= \frac{8+6x-2x^2}{(x^2+4)^2}$$

b Note that $x^2 + 4 > 0$ for all values of x. So only check the numerators.

$y > 0$ provided $2x - 3 > 0$, i.e. $x > \frac{3}{2}$.

$\frac{dy}{dx} = 0$ provided $8 + 6x - 2x^2 > 0$, which is equivalent to $4 + 3x - x^2 > 0$.

$4 + 3x - x^2 = (4 - x)(1 + x) > 0$ provided $-1 < x < 4$ (since the corresponding quadratic graph is an inverted parabola with x- axis intercepts of -1 and 4).

So y and $\frac{dy}{dx}$ are both positive provided $x \in \left(\frac{3}{2}, \infty\right) \cap (-1, 4)$, i.e. $\left(\frac{3}{2}, 4\right)$.

11 a $$y = xf(x)$$

$$\frac{dy}{dx} = (x)(f'(x)) + (1)(f(x))$$
$$= xf'(x) + f(x)$$

b $$y = \frac{1}{f(x)}$$

$$\frac{dy}{dx} = \frac{-f'(x)}{[f(x)]^2}$$

c $$y = \frac{x}{f(x)}$$

$$\frac{dy}{dx} = \frac{f(x) + xf'(x)}{[f(x)]^2}$$

d $$y = \frac{x^2}{[f(x)]^2}$$

$$\frac{dy}{dx} = \frac{[f(x)]^2(2x) - (x^2)(2f(x)f'(x))}{[f(x)]^2}$$
$$= \frac{[f(x)](2xf(x) - 2x^2f'(x))}{[f(x)]^4}$$
$$= \frac{2xf(x) - 2x^2f'(x)}{[f(x)]^3}$$

12 a $f \circ g(x) = 2\cos^3 x - 1$

b $g \circ f(x) = \cos(2x^3 - 1)$

c $g' \circ f(x) = -\sin(2x^3 - 1)$

d $(g \circ f)'(x) = -(6x^2)\sin(2x^3 - 1)$

e $\frac{3}{2}$

f $-\frac{3\sqrt{3}}{4}$

13 $f(x) = 3 + 6x^2 - 2x^3$

$f'(x) = 12x - 6x^2$

$f'(x) > 0 \Rightarrow 6x(2 - x) > 0$

Therefore positive gradient for $(0, 2)$

14 For $y = x^3$, $\frac{dy}{dx} = 3x^2$

For $y = x^3 + x^2 + x - 2$, $\frac{dy}{dx} = 3x^2 + 2x + 1$

$3x^2 = 3x^2 + 2x + 1$

$\Leftrightarrow x = -\frac{1}{2}$

15 $y = bx^2 - cx$

Therefore,

$0 = 4b - c \ldots (1)$

$\frac{dy}{dx} = 2bx - c$

$\frac{dy}{dx} = 1$ when $x = 4 \Rightarrow 8b - c = 1 \ldots (2)$

$\therefore b = \frac{1}{4}, c = 1$

16 **a** $e^{2x} - 16e^x - 36 = 0$

$(e^x - 18)(e^x + 2) = 0$

$x = \log_e(18)$

b $f'(x) = 2e^2x - 16e^x$

$f'(x) = 0 \Rightarrow 2e^x(e^x - 8) = 0$

$\therefore x = \log_e 8 = 3\log_e 2$

Coordinates

$f(3\log_e 2) = e^{6\log_e 2} - 16^{3\log_e 2} - 36$

$= 2^6 - 16 \times 2^3 - 36$

$= 64 - 16 \times 8 - 36 = -100$

$(3\log_e 2, -100)$

c $2e^x(e^x - 8) > 0 \Leftrightarrow x > 3\log_e(2)$

d $\dfrac{f(\log_e(18)) - f(\log_e(8))}{\log_e(18) - \log_e(8)} = \dfrac{50}{\log_e \frac{3}{2}}$

Solutions to multiple-choice questions

1 A

$$\text{Average rate of change} = \frac{e + 1 - (1)}{1}$$
$$= e$$

2 C $f : R\backslash\{7\} \rightarrow R, f(x) = 5 + \frac{5}{(7 - x)^2}$

$$f(x) = 5 + 5(7 - x)^{-2}$$
$$f'(x) = 10(7 - x)^{-3}$$
$$f'(x) = \frac{10}{(7 - x)^3}$$
$$f'(x) > 0$$
$$\therefore \quad (7 - x)^3 > 0$$
$$x < 7$$

3 A $y = f(g(x))$

$$g(x) = 2x^4$$
$$\therefore \quad y = f(2x^4)$$
$$\frac{dy}{dx} = 8x^3 f'(2x^4)$$

4 A $f(x) = x^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

As $3x^{\frac{2}{3}} \neq 0$, the gradient is undefined at this point.

5 B $y = \frac{k}{2(x^3 + 1)}$

$$y' = \frac{-3kx^2}{2(x^3 + 1)^2}$$
$$\therefore \quad 1 = \frac{-3kx^2}{2(x^3 + 1)^2}$$
$$1 = \frac{-3k}{8}$$
$$k = \frac{-8}{3}$$

6 C The gradient is positive when: $x < -3$ or $x > 2$

7 D $f(x) = 4x(2 - 3x)$

$$f(x) = 8x - 12x^2$$
$$f'(x) = 8 - 24x$$
$$f'(x) < 0$$
$$8 - 24x < 0$$
$$x > \frac{8}{24}$$
$$x > \frac{1}{3}$$

8 D $\frac{f(4) - f(2)}{2} = \frac{4\log_e(4) - 2\log_2}{2}$

$$= 2\log_e 4 - \log_e 2$$
$$= 4\log_e 2 - \log_e 2$$
$$= 3\log_e(2)$$

9 A $y = (x + 3)(x - 2)$

$y = x^2 + x - 6$

$\frac{dy}{dx} = 2x + 1$

When $\frac{dy}{dx} = -7$

$-8 = 2x$

$x = -4$

When $x = -4$

$y = (-4)^2 - 4 - 6$

$y = 6$

Coordinates $= (-4, 6)$

10 B $y = ax^2 - bx$

$\frac{dy}{dx} = 2ax - b$

When $\frac{dy}{dx} = 0, x = 2$

$0 = 4a - b$

$4a = b$

Sub into: $y = ax^2 - bx$

$y = ax^2 - 4ax$

$y = ax(x - 4)$

Using null factor theorem:

$x = 0, \ x = 4$

11 E $f(x) = \frac{4x^4 - 12x^2}{3x - k}$

$f'(k) = 2 \Rightarrow 5k^2 - 3 = 2$

$\Leftrightarrow k = 1$ or $k = -1$

Test values with calculator. $k = 1$

12 C $h(x) = \frac{f(x)}{g(x)}$

$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$

$h'(2) = \frac{-4 \times 6 - 8 \times 3}{36}$

$= -\frac{4}{3}$

13 A

$$\begin{aligned} f(g(x)) &= x \\ g'(x)f'(g(x)) &= 1 \\ g'(6) \times f'(g(6)) &= 1 \\ g'(6) \times f'(8) &= 1 \\ g'(6) \times 4 &= 1 \\ g'(6) &= \frac{1}{4} \end{aligned}$$

Solutions to extended-response questions

1 $f(1) = 6,\ g(1) = -1,\ g(6) = 7$ and $f(-1) = 8$

$f'(1) = 6,\ g'(1) = -2, f'(-1) = 2$ and $g'(6) = -1$

a **i** $f \circ g)'(1) = g'(1)f'(g(1)) = -2 \times f'(-1) = -2 \times 2 = -4$

ii $(g \circ f)'(1) = f'(1)g'(f(1)) = 6 \times g'(6) = 6 \times (-1) = -6$

iii $(fg)'(1) = f'(1)g(1) + g'(1)f(1) = 6 \times (-1) + (-2) \times 6 = -18$

iv $(gf)'(1) = f'(1)g(1) + g'(1)f(1) = 6 \times (-1) + (-2) \times 6 = -18$

v $\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - g'(1)f(1)}{[g(1)]^2} = \frac{6 \times -1 - (-2 \times 6)}{[-1]^2} = 6$

vi $\left(\frac{g}{f}\right)'(1) = \frac{g'(1)f(1) - f'(1)g(1)}{[f(1)]^2} = \frac{-2 \times 6 - (6 \times -1))}{[6]^2} = -\frac{1}{6}$

b For $f(x) = ax^3 + bx^2 + cx + d.$, $f(1) = 6$ and $f(-1) = 8$

Therefore

$a + b + c + d = 6 \qquad 1$

$-a + b - c + d = 8 \qquad 2$

Also $f'(x) = 3ax^2 + 2bx + c, f'(1) = 6$ and $f'(-1) = 2$

Therefore

$3a + 2b + c = 6 \qquad 3$

$3a - 2b + c = 2 \qquad 4$

Subtract 4 from 3 to give $4b = 4$ and $b = 1$

Add 1 and 2

$2b + 2d = 14$ and as $b = 1,\ d = 6$

From 1

$a + c = -1 \qquad 5$

and from 4

$3a + c = 4 \qquad 6$

Subtract 5 from 6

$2a = 5$ and therefore $a = \frac{5}{2}$ and $c = -\frac{7}{2}$

2 $f'(x) = 0$ for $x = 1$ and $x = 5$

$f'(x) > 0$ for $x > 5$ and $x < 1$

$f'(x) < 0$ for $1 < x < 5$

$f(1) = 6$ and $f(5) = 1$

a The graph of $y = f(x+2)$ is obtained from the graph of $y = f(x)$ by a translation of 2 units in the negative direction of the x-axis.

i Therefore $\frac{dy}{dx} = 0$ for $x = -1$ and $x = 3$

ii $\frac{dy}{dx} > 0$ for $x > 3$ and $x < -1$

b The graph of $y = f(x-2)$ is obtained from the graph of $y = f(x)$ by a translation of 2 units in the positive direction of the x-axis.

i Therefore $\frac{dy}{dx} = 0$ for $x = 3$ and $x = 7$.

ii The coordinates at which the gradient is zero are $(3, 6)$ and $(7, 1)$

c The graph of $y = f(2x)$ is obtained from the graph of $y = f(x)$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

i Therefore $\frac{dy}{dx} = 0$ for $x = \frac{1}{2}$ and $\frac{5}{2}$

ii The coordinates at which the gradient is zero are $\left(\frac{1}{2}, 6\right)$ and $\left(\frac{5}{2}, 1\right)$

d The graph of $y = f\left(\frac{x}{2}\right)$ is obtained from the graph of $y = f(x)$ by a dilation of factor 2 from the y-axis.

i Therefore $\frac{dy}{dx} = 0$ for $x = 2$ and $x = 10$

ii The coordinates at which the gradient is zero are $(2, 6)$ and $(10, 1)$

e The graph of $y = 3f\left(\frac{x}{2}\right)$ is obtained from the graph of $y = f(x)$ by a dilation of factor 2 from the y-axis and factor 3 from the x-axis.

i Therefore $\frac{dy}{dx} = 0$ for $x = 2$ and $x = 10$

ii The coordinates at which the gradient is zero are $(2, 18)$ and $(10, 3)$

3 $f(x) = (x-\alpha)^n(x-\beta)^m$ where m and n are positive integers with $m > n$ and $\beta > \alpha$

a $f(x) = 0$ implies $x = \alpha$ or $x = \beta$

b Using the product rule

$$\begin{aligned} f'(x) &= n(x-\alpha)^{n-1}(x-\beta)^m + m(x-\alpha)^n(x-\beta)^{m-1} \\ &= (x-\alpha)^{n-1}(x-\beta)^{m-1}[n(x-\beta)+m(x-\alpha)] \\ &= (x-\alpha)^{n-1}(x-\beta)^{m-1}[x(n+m)-(n\beta+m\alpha)] \end{aligned}$$

c $f'(x) = 0$ implies $x = \alpha$ or $x = \beta$ or $x = \dfrac{n\beta + m\alpha}{n+m}$

d **i** If m and n are odd then $m-1$ and $n-1$ are even.
Therefore $(x-\alpha)^{n-1}(x-\beta)^{m-1} \geq 0$ for all x
and $f'(x) > 0$ for $x > \dfrac{n\beta + m\alpha}{n+m}$ and $x \neq \beta$

ii If m is odd then $m-1$ is even and $(x-\beta)^{m-1} \geq 0$ for all x
Therefore $f'(x) > 0$ if and only if $(x-\alpha)^{n-1}[x(n+m)-(n\beta+m\alpha)] > 0$
If n is even then $(x-\alpha)^{n-1} > 0$ if and only if $x - \alpha > 0$.
Together gives
$(x-\alpha)^{n-1}[x(n+m)-(n\beta+m\alpha)] > 0$ is equivalent to both factors positive or both factors negative.

If both are positive:
$x > \alpha$ and $x > \dfrac{n\beta + m\alpha}{n+m}$
and as $\beta > \alpha$, $\dfrac{n\beta + m\alpha}{n+m} > \alpha +$ and thus $x > \dfrac{n\beta + m\alpha}{n+m}$

If both are negative
$x < \alpha$ and $x < \dfrac{n\beta + m\alpha}{n+m}$ and hence $x < \alpha$

4 $f(x) = \dfrac{x^n}{1+x^n}$ where n is an even integer.

a $1 - \dfrac{1}{x^n+1} = \dfrac{x^n+1-1}{x^n+1} = \dfrac{x^n}{1+x^n}$

b $f(x) = \dfrac{nx^{n-1}}{(x^n+1)^2}$

c $0 < \dfrac{1}{x^n+1} \leq 1$ as n is even. Therefore $-1 \leq -\dfrac{1}{x^n+1} < 0$ and $0 \leq 1 - \dfrac{1}{x^n+1} < 1$

d $f'(x) = 0$ implies $\dfrac{n^{xn-1}}{(x^n+1)^2} = 0$ implies $x = 0$

e $f'(x) > 0$ for $\dfrac{nx^{n-1}}{(x^n+1)^2} > 0$ which implies $x > 0$

Chapter 10 – Applications of differentiation

Solutions to Exercise 10A

1

$$
\begin{aligned}
y &= x^2 - 1\\
\frac{dy}{dx} &= 2x\\
x &= 2,\\
\frac{dy}{dx} &= 4\\
\text{tangent}: y &= 4x + c\\
x &= 2, y = 3\\
3 &= 8 + c\\
c &= -5\\
y &= 4x - 5
\end{aligned}
$$

2

$$
\begin{aligned}
y &= x^2 + 3x - 1\\
x &= 0, y = -1\\
\frac{dy}{dx} &= 2x + 3\\
x &= 0,\\
\frac{dy}{dx} &= 3
\end{aligned}
$$

normal:

$$
\begin{aligned}
grad &= -\frac{-1}{4}\\
y &= \frac{-1}{3}x + c\\
x &= 0, y = -1\\
-1 &= c\\
y &= \frac{-x}{3} - 1
\end{aligned}
$$

3

$$
\begin{aligned}
y &= x^2 - 5x + 6\\
&= (x - 3)(x - 2)\\
y &= 0, x = 2, 3\\
\frac{dy}{dx} &= 2x - 5
\end{aligned}
$$

When $x = 2$,

$$
\begin{aligned}
\frac{dy}{dx} &= 4 - 5\\
&= -1
\end{aligned}
$$

Gradient of normal = 1

$$
\begin{aligned}
y &= x + c\\
&= x + c\\
x &= 2, y = 0\\
0 &= 2 + c\\
c &= -2\\
y &= x - 2
\end{aligned}
$$

When $x = 3$,

$$
\begin{aligned}
\frac{dy}{dx} &= 6 - 5\\
&= 1
\end{aligned}
$$

Gradient of normal = -1

$$
\begin{aligned}
y &= -x + c\\
&= -x + c\\
x &= 3, y = 0\\
0 &= -3 + c\\
c &= 3\\
y &= 3 - x
\end{aligned}
$$

4 $y = (2x + 1)^9$

$\frac{dy}{dx} = 2 \times 9(2x + 1)^8$

$= 18(2x + 1)^8$

$x = 0,$

$\frac{dy}{dx} = 18(1)^8$

$= 18$

tangent:

$y = 18x + c$

$x = 0, y = 1$

$1 = c$

$y = 18x + 1$

normal:

$y = \frac{-1}{18}x + c$

$x = 0, y = 1$

$1 = c$

$y = \frac{-1}{18}x + 1$

5 $y = x^2 - 5$

$\frac{dy}{dx} = 2x$

$\frac{dy}{dx} = 3$

$3 = 2x$

$x = \frac{3}{2}$

$x = \frac{3}{2},$

$y = \left(\frac{3}{2}\right)^2 - 5$

$= \frac{9}{4} - 5$

$y = \frac{-11}{4}$

co-ords $= \left(\frac{3}{2}, \frac{-11}{4}\right)$

$y = 3x + c$

$x = \frac{3}{2}, y = \frac{-11}{4}$

$\frac{-11}{4} = 3 \times \frac{3}{2} + c$

$c = \frac{-11}{4} - \frac{9}{2}$

$= \frac{-11}{4} - \frac{18}{4}$

$c = \frac{-29}{4}$

6 a $y = x^2 - 2$

$x = 1, y = 1 - 2 = -1$

$\frac{dy}{dx} = 2x$

$x = 1, \frac{dy}{dx} = 2$

i tangent:

$$y = 2x + c$$

$$x = 1, y = -1$$

$$-1 = 2 + c$$

$$c = -3$$

$$y = 2x - 3$$

ii normal:

$$y = \frac{-1}{2}x + c$$

$$x = 1, y = -1$$

$$-1 = \frac{-1}{2} + c$$

$$c = \frac{-1}{2}$$

$$y = \frac{-1}{2}x - \frac{1}{2}$$

b $y = x^2 - 3x - 1$

$$x = 0, y = -1$$

$$\frac{dy}{dx} = 2x - 3$$

$$x = 0, \frac{dy}{dx} = -3$$

i tangent:

$$y = -3x + c$$

$$x = 0, y = -1$$

$$-1 = c$$

$$y = -3x - 1$$

ii normal:

$$y = \frac{-1}{-3}x + c$$

$$= \frac{1}{3}x + c$$

$$x = 0, y = -1$$

$$-1 = c$$

$$y = \frac{1}{3}x - 1$$

c $y = \frac{1}{x}$

$$x = -1, y = -1$$

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

$$x = -1, \frac{dy}{dx} = -1$$

i tangent:

$$y = -x + c$$

$$x = -1, y = -1$$

$$-1 = 1 + c$$

$$c = -2$$

$$y = -x - 2$$

ii normal:

$$y = \frac{-1}{-1}x + c$$

$$= x + c$$

$$x = -1, y = -1$$

$$-1 = -1 + c$$

$$c = 0$$

$$y = x$$

d $y = (x - 2)(x^2 + 1)$

$= x^3 - 2x^2 + x - 2$

$x = -1,$

$y = -1 - 2 - 1 - 2$

$= -6$

$\frac{dy}{dx} = 3x^2 - 4x + 1$

$x = -1,$

$\frac{dy}{dx} = 3 + 4 + 1$

$= 8$

i tangent:

$y = 8x + c$

$x = -1, y = -6$

$-6 = -8 + c$

$c = 2$

$y = 8x + 2$

ii normal:

$y = \frac{-1}{8}x + c$

$x = -1$

$y = -6$

$-6 = \frac{1}{8} + c$

$c = -6\frac{1}{8} = -\frac{-49}{8}$

$y = \frac{-1}{8}x - \frac{49}{8}$

e $y = \sqrt{3x + 1}$

$x = 0, \quad y = \sqrt{1} = 1$

$\frac{dy}{dx} = 3 \times \frac{1}{2\sqrt{3x + 1}}$

$= \frac{3}{2\sqrt{3x + 1}}$

$x = 0,$

$\frac{dy}{dx} = \frac{3}{2\sqrt{1}}$

$\frac{dy}{dx} = \frac{3}{2}$

i tangent:

$y = \frac{3}{2}x + c$

$x = 0, y = 1$

$1 = c$

$y = \frac{3}{2}x + 1$

ii normal:

$y = \frac{-2}{3}x + c$

$x = 0, y = 1$

$1 = c$

$y = \frac{-2}{3}x + 1$

f $y = \sqrt{x}$

$x = 1, y = 1$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$x = 1,$

$\frac{dy}{dx} = \frac{1}{2}$

i tangent:

$$y = \frac{1}{2}x + c$$

$$x = 1, y = 1$$

$$1 = \frac{1}{2} + c$$

$$c = \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

ii normal:

$$y = -2x + c$$

$$x = 1, y = 1$$

$$1 = -2 + c$$

$$c = 3$$

$$y = -2x + 3$$

g $y = x^{\frac{2}{3}} + 1$

$$x = 1, y = 2$$

$$\frac{dy}{dx} = \frac{2}{3}x^{\frac{-1}{3}}$$

$$x = 1,$$

$$\frac{dy}{dx} = \frac{2}{3}$$

i tangent:

$$y = \frac{2}{3}x + c$$

$$x = 1, y = 2$$

$$2 = \frac{2}{3} + c$$

$$c = \frac{4}{3}$$

$$y = \frac{2x + 4}{3}$$

ii normal:

$$y = \frac{-3}{2}x + c$$

$$x = 1, y = 2$$

$$2 = \frac{-3}{2} + c$$

$$c = \frac{7}{2}$$

$$y = \frac{7 - 3x}{2}$$

h $y = x^3 - 8x$

$$x = 2,$$

$$y = 8 - 16$$

$$= -8$$

$$\frac{dy}{dx} = 3x^2 - 8$$

$$x = 2,$$

$$\frac{dy}{dx} = 12 - 8$$

$$= 4$$

i tangent:

$$y = 4x + c$$

$$x = 2, y = -8$$

$$-8 = 8 + c$$

$$c = -16$$

$$y = 4x - 16$$

ii normal:

$$y = \frac{-1}{4}x + c$$

$$x = 2, y = -8$$

$$-8 = \frac{-1}{2} + c$$

$$c = -7\frac{1}{2}$$

$$= \frac{-15}{2}$$

$$y = \frac{-x}{4} - \frac{15}{2}$$

i $y = x^3 - 3x^2 + 2$

$$x = 2,$$

$$y = 8 - 3 \times 4 + 2$$

$$y = -2$$

$$\frac{dy}{dx} = 3x^2 - 6x$$

$$x = 2,$$

$$\frac{dy}{dx} = 3 \times 4 - 6 \times 2$$

$$= 0$$

i tangent:

$$y = c$$

$$x = 2, y = -2$$

$$c = -2,$$

$$y = -2$$

ii normal:

$$x = 2$$

j $y = 2x^3 + x^2 - 4x + 1$

$$x = 1,$$

$$y = 2 + 1 - 4 + 1$$

$$y = 0$$

$$\frac{dy}{dx} = 6x^2 + 2x - 4$$

$$x = 1,$$

$$\frac{dy}{dx} = 6 + 2 - 4$$

$$= 4$$

i tangent:

$$y = 4x + c$$

$$x = 1, y = 0$$

$$0 = 4 + c$$

$$c = -4$$

$$y = 4x - 4$$

ii normal:

$$y = \frac{-1}{4}x + c$$

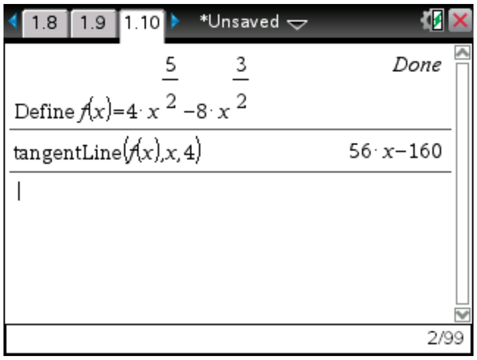
$$x = 1, y = 0$$

$$0 = \frac{-1}{4} + c$$

$$c = \frac{1}{4}$$

$$y = \frac{-1}{4}x + \frac{1}{4}$$

7 $y = 56x - 160$



Define $f(x)=4\cdot x^{\frac{5}{2}}-8\cdot x^{\frac{3}{2}}$ Done

tangentLine$(f(x),x,4)$ $56\cdot x-160$

8 **a** $y = \frac{x^2 - 1}{x^2 + 1}$

$x = 0, y = \frac{-1}{1} = -1$

$y = \frac{x^2 + 1 - 2}{x^2 + 1}$

$y = 1 - \frac{2}{x^2 + 1}$

$y = 1 - 2(x^2 + 1)^{-1}$

$\frac{dy}{dx} = 2x \times -1 \times -2(x^2 + 1)^{-2}$

$= \frac{4x}{(x^2 + 1)^2}$

$x = 0, \frac{dy}{dx} = 0$

tangent:

$y = 0 \times x + c$

$x = 0, y = -1$

$-1 = c$

$y = -1$

b $y = \sqrt{3x^2 + 1}$

$x = 1,$

$y = \sqrt{3 + 1}$

$= 2$

$\frac{dy}{dx} = 6x \times \frac{1}{2\sqrt{3x^2 + 1}}$

$= \frac{3x}{\sqrt{3x^2 + 1}}$

$x = 1,$

$\frac{dy}{dx} = \frac{3}{\sqrt{3 + 1}}$

$= \frac{3}{2}$

tangent:

$y = \frac{3}{2}x + c$

$x = 1, y = 2$

$2 = \frac{3}{2} + c$

$c = \frac{1}{2}$

$y = \frac{3}{2}x + \frac{1}{2}$

c $y = \frac{1}{2x - 1}$

$x = 0,$

$y = \frac{1}{-1}$

$y = -1$

$\frac{dy}{dx} = 2 \times -1 \times \frac{1}{(2x - 1)^2}$

$= \frac{-2}{(2x - 1)^2}$

$x = 0,$

$\frac{dy}{dx} = \frac{-2}{(-1)^2}$

$= -2$

tangent:

$y = -2x + c$

$x = 0, y = -1$

$c = -1$

$y = -2x - 1$

d $y = \dfrac{1}{(2x-1)^2}$

$x = 1,$

$y = \dfrac{1}{(1)^2}$

$y = 1$

$\dfrac{dy}{dx} = -2 \times 2 \times \dfrac{1}{(2x-1)^3}$

$= \dfrac{-4}{(2x-1)^3}$

$x = 1,$

$\dfrac{dy}{dx} = \dfrac{-4}{(1)^3}$

$= -4$

tangent:

$y = -4x + c$

$x = 1, y = 1$

$1 = -4 + c$

$c = 5$

$y = -4x + 5$

9 a $y = \sin 2x$

$\dfrac{dy}{dx} = 2\cos 2x$

$x = 0,$

$y = \sin 0 = 0$

$\dfrac{dy}{dx} = 2\cos 0 = 2$

$y = 2x + c$

$x = 0,\ y = 0$

$c = 0$

$y = 2x$

b $y = \cos 2x$

$\dfrac{dy}{dx} = -2\sin 2x$

$x = \dfrac{\pi}{2},$

$y = \cos \pi = -1$

$\dfrac{dy}{dx} = -\sin \pi = 0$

$y = -1$

c $y = \tan x$

$\dfrac{dy}{dx} = \sec^2 x$

$x = \dfrac{\pi}{4},$

$y = \tan \dfrac{\pi}{4} = 1$

$\dfrac{dy}{dx} = \sec^2 \dfrac{\pi}{4} = \sqrt{2}^2 = 2$

$y = 2x + \mathrm{c}$

$x = \dfrac{\pi}{4},\ y = 1$

$1 = \dfrac{\pi}{2} + c$

$c = 1 - \dfrac{\pi}{2}$

$y = 2x + 1 - \dfrac{\pi}{2}$

d $y = \tan 2x$

$\dfrac{dy}{dx} = 2\sec^2 2x$

$x = 0,$

$y = \tan 0 = 0$

$\dfrac{dy}{dx} = 2\sec^2 0 = 2$

$y = 2x + c$

$x = 0,\ y = 0$

$0 = c$

$y = 2x$

e $y = \sin x + x\sin 2x$

$$\frac{dy}{dx} = \cos x + \sin 2x + 2x\cos 2x$$

$x = 0,$

$y = \sin 0 + 0\sin 0 = 0$

$$\frac{dy}{dx} = \cos 0 + 2\sin 0 + 0 = 1$$

$y = x + c$

$x = 0, \quad y = 0$

$c = 0$

$y = x$

f $y = x - \tan x$

$$\frac{dy}{dx} = 1 - \sec^2 x$$

$x = \frac{\pi}{4},$

$$y = \frac{\pi}{4} - \tan\frac{\pi}{4} = \frac{\pi}{4} - 1$$

$$\frac{dy}{dx} = 1 - \sec^2\frac{\pi}{4} = 1 - 2 = -1$$

$y = -x + c$

$x = \frac{\pi}{4},\ y = \frac{\pi}{4} - 1$

$$\frac{\pi}{4} - 1 = \frac{-\pi}{4} + c$$

$$c = \frac{\pi}{2} - 1$$

$$y = -x + \frac{\pi}{2} - 1$$

10 **a** $f(x) = e^x + e^{-x}$

$f'(x) = e^x - e^{-x}$

$f'(0) = 1 - 1 = 0$

$y = c$

$f(0) = 1 + 1 = 2$

$2 = c$

$y = 2$

b $f(x) = \dfrac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x + e^{-x}}{2}$$

$$f'(0) = \frac{1+1}{2} = 1$$

$y = x + c$

$$f(0) = \frac{1-1}{2} = 0$$

$0 = c$

$y = x$

c $f(x) = x^2e^{2x}$

$f'(x) = 2xe^{2x} + 2x^2e^{2x}$

$= 2xe^{2x}(x^2 + x)$

$f'(1) = 2e^2(1 + 1)$

$= 4e^2$

$y = 4e^2x + c$

$f(1) = 1 \times e^2 = e^2$

$e^2 = 4e^2 + c$

$c = -3e^2$

$y = 4e^2x - 3e^2$

d $f(x) = e^{\sqrt{x}}$

$$f'(x) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$$

$$f'(1) = \frac{e}{2}$$

$$y = \frac{e}{2}x + c$$

$f(1) = e^1 = e$

$$e = \frac{e}{2} + c$$

$$c = \frac{e}{2}$$

$$y = \frac{e}{2}(x + 1)$$

e $f(x) = xe^{x^2}$

$$f'(x) = e^{x^2} + 2x^2e^{x^2}$$
$$= e^{x^2}(2x^2 + 1)$$
$$f'(1) = e^1(2 + 1)$$
$$= 3e$$
$$y = 3ex + c$$
$$f'(1) = 1 \times e^1 = e$$
$$e = 3e + c$$
$$c = -2e$$
$$y = 3ex - 2e$$

f $f(x) = x^2e^{-x}$

$$f'(x) = 2xe^{-x} - x^2e^{-x}$$
$$= e^{-x}(2x - x^2)$$
$$f'(2) = e^{-2}(4 - 4) = 0$$
$$y = c$$
$$f(2) = 2^2e^{-2} = \frac{4}{e^2}$$
$$\frac{4}{e^2} = c$$
$$y = \frac{4}{e^2}$$

11 a $f(x) = \ln x$

$$f'(x) = \frac{1}{x}$$
$$f'(1) = 1$$
$$y = x + c$$
$$\left(1, 0\right) \Rightarrow 0 = 1 + c$$
$$c = -1$$
$$y = x - 1$$

For the normal the gradient is −1

The equation of the normal is

$y = x + 1$

b $f(x) = \ln(2x)$

$$f'(x) = \frac{2}{2x} = \frac{1}{x}$$
$$f'\left(\frac{1}{2}\right) = 2$$
$$y = 2x + c$$
$$\left(\frac{1}{2}, 0\right) \Rightarrow 0 = 1 + c$$
$$c = -1$$
$$y = 2x - 1$$

c $f(x) = \ln(kx)$

$$f'(x) = \frac{k}{kx} = \frac{1}{x}$$
$$f\left(\frac{1}{k}\right) = k$$
$$y = kx + c$$
$$\left(\frac{1}{k}, 0\right) \Rightarrow 0 = 1 + c$$
$$c = -1$$
$$y = kx - 1$$

12 a $y = x^{\frac{1}{5}}$

$$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}}$$

When $x = 0, y = 0, \frac{dy}{dx}$ not defined.

Therefore equation of tangent

$$x = 0$$

b $y = x^{\frac{3}{5}}$

$$\frac{dy}{dx} = \frac{3}{5}x^{-\frac{2}{5}}$$

When $x = 0, y = 0, \frac{dy}{dx}$ not defined.

Therefore equation of tangent

$$x = 0$$

c $y = (x - 4)^{\frac{1}{3}}$

$y = 0 \Rightarrow x = 4$

$\dfrac{dy}{dx} = \dfrac{1}{3}(x - 4)^{-\dfrac{2}{3}}$

$x = 4,$

$\dfrac{dy}{dx}$ is undefined

$\therefore$ tangent is $x = 4$

d $y = (x + 5)^{\frac{2}{3}}$

$y = 0 \Rightarrow x = -5$

$\dfrac{dy}{dx} = \dfrac{2}{3}(x + 5)^{-\dfrac{1}{3}}$

$x = -5,$

$\dfrac{dy}{dx}$ is undefined

$\therefore$ tangent is $x = -5$

e $y = (2x + 1)^{\frac{1}{3}}$

$y = 0 \Rightarrow x = -\dfrac{1}{2}$

$\dfrac{dy}{dx} = \dfrac{2}{3}(2x + 1)^{-\frac{2}{3}}$

$x = -\dfrac{-1}{2},$

$\dfrac{dy}{dx}$ is undefined

$\therefore$ tangent is $x = -\dfrac{1}{2}$

f $y = (x + 5)^{\frac{4}{5}}$

$y = 0 \Rightarrow x = -5$

$\dfrac{dy}{dx} = \dfrac{4}{5}(x + 5)^{-\frac{1}{5}}$

$x = -5,$

$\dfrac{dy}{dx}$ is undefined

$\therefore$ tangent is $x = -5$

13 $y = \tan 2x$

$\dfrac{dy}{dx} = 2\sec^2 2x$

$x = \dfrac{\pi}{8},$

$y = \tan \dfrac{\pi}{4} = 1$

$\dfrac{dy}{dx} = 2\sec^2 \dfrac{\pi}{4} = 4$

$y = 4x + c$

$x = \dfrac{\pi}{8}, \ y = 1$

$1 = \dfrac{\pi}{2} + c$

$c = 1 - \dfrac{\pi}{2}$

$y = 4x + 1 - \dfrac{\pi}{2}$

$x = 0,$

$y = 1 - \dfrac{\pi}{2}$

$A = \left(0, 1 - \dfrac{\pi}{2}\right)$

$OA = \dfrac{\pi}{2} - 1$

14 $y = 2e^x$

$\dfrac{dy}{dx} = 2e^x$

$\therefore \dfrac{dy}{dx} = 2e^a$ when $x = a$

Gradient of the line segment joining $(a, 2e^a)$ and the origin is $\dfrac{2e^a}{a}$

$\therefore \dfrac{2e^a}{a} = 2e^a$

$\therefore a = 1$

15 $y = \log_e x$

$\frac{dy}{dx} = \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{1}{a}$ when $x = a$

Gradient of the line segment joining $(a, \log_e a)$ and the origin is $\frac{\log_e a}{a}$

$\therefore \frac{\log_e a}{a} = \frac{1}{a}$

$\therefore \log_e a = 1 \therefore a = e$

16 $y = x^2 + 2x$

$\frac{dy}{dx} = 2x + 2$

$\therefore \frac{dy}{dx} = 2a + 2$ when $x = a$

Gradient of the line segment joining $(a, a^2 + 2a)$ and the origin is $\frac{a^2 + 2a}{a}$

$\therefore \frac{a^2 + 2a}{a} = 2a + 2$

$\therefore a^2 + 2a = 2a^2 + 2a$

$\therefore a = 0$

17 $y = x^3 + x$

$\frac{dy}{dx} = 3x^2 + 1$

$\therefore \frac{dy}{dx} = 3a^2 + 1$ when $x = a$

Gradient of the line segment joining $(a, a^3 + a)$ and the point $(1, 1)$ is

$\frac{a^3 + a - 1}{a - 1}$

$\therefore 3a^2 + 1 = \frac{a^3 + a - 1}{a - 1}$

$\therefore (3a^2 + 1)(a - 1) = a^3 + a - 1$

$\therefore 3a^3 + a - 3a^2 - 1 = a^3 + a - 1$

$\therefore 2a^3 - 3a^2 = 0$

$\therefore a^2(2a - 3) = 0$

$\therefore a = 0$ or $a = \frac{3}{2}$

Solutions to Exercise 10B

1 **a** Average rate of change $= \dfrac{f(3) - f(2)}{3 - 2}$

$$= \frac{45 - 24}{1}$$

$$= 21$$

b Average rate of change

$$= \frac{f(2+h) - f(2)}{2 + h - 2}$$

$$= \frac{3(2+h)^2 + 6(2+h) - 24}{h}$$

$$= \frac{3(4 + 4h + h^2) + 6(2+h) - 24}{h}$$

$$= \frac{18h + 3h^2)}{h}$$

$$= 18 + 3h$$

c $f'(x) = 6x + 6$

$f'(2) = 18$

2 **a** $\dfrac{dV}{dt}$

b $\dfrac{dS}{dr}$

c $\dfrac{dV}{dx}$

d $\dfrac{dA}{dt}$

e $\dfrac{dV}{dh}$

3 **a** $I = \dfrac{4}{(t+1)^2}$

$$\frac{dI}{dt} = \frac{-8}{(t+1)^3}$$

$t = 10$,

$$\frac{dI}{dt} = \frac{-8}{11^3}$$

$$= \frac{-8}{1331}$$

i.e. I wanes *by* ≈ 0.006 units/day

4 $V(t) = 1000(90 - t)^3$

a $V'(t) = -3000(90 - t)^2$

it empties at $3000(90 - t)^2$ m^3/day

b $V(t) = 0$,

$$1000(90 - t)^3 = 0$$

$t = 90$ days

c $V(0) = 1000(90)^3$

$= 729\,000\,000$ m^3

d $V'(t) = -300\,000$

$$-300\,000 = -3000(90 - t)^2$$

$$(90 - t)^2 = 100$$

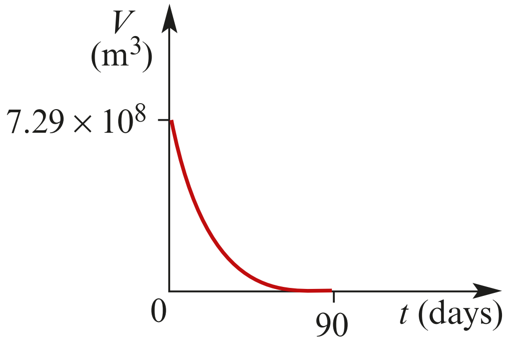
$$90 - t = \pm 10$$

$$t = 90 \pm 10$$

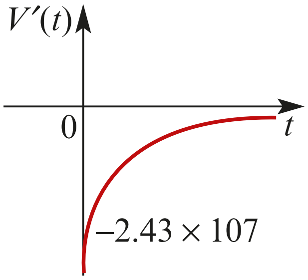
since $t \in [0, 90]$

$t = 80$th day

e



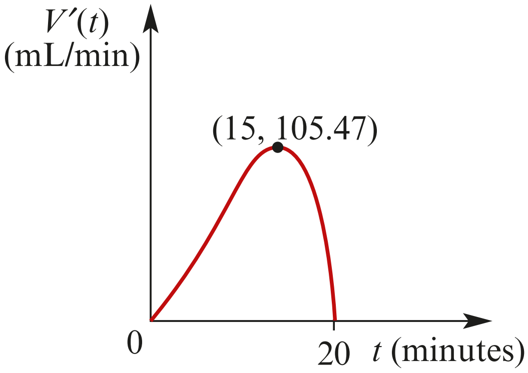
f



5 $V(t) = \frac{1}{160}\left(5t^4 - \frac{t^5}{5}\right), 0 \leq t \leq 20$

a $V'(t) = \frac{1}{160}(20t^3 - t^4)$ ml/min

b



c

$$\frac{d}{dt} = \frac{1}{160}(60t^2 - 4t^3)$$

$$= \frac{1}{40}(15t^2 - t^3)$$

$$\frac{d}{dt} = 0,$$

$$15t^2 - t^3 = 0$$

$$t^2(15 - t) = 0$$

$$t = 0, 15$$

max flow occurs at

$t = 15$

(using graph to determine max. or min. status)

6 **a** $t \approx 100, 250, 500$

(read off graph-turning points)

b draw tangent at $t = 200$,

use $\frac{rise}{run}$ to find gradient

$\frac{dV}{dt} \cong 430\,000\ \text{m}^3/\text{day}$

(be careful *re*: vertical scale)

c $t = 100, V \cong 4 \times 10^7$

$t = 250, V \cong 8 \times 10^7$

$\frac{rise}{run} \approx \frac{4 \times 10^7}{150} \approx 270\,000\ \text{m}^3/\text{day}$

d $100 < t < 250$ or $t > 500$

7 a

$$P = P_0e^{-kt}$$

When $t = 0, P = 30$

$\therefore \quad 30 = P_0e^0$

$\therefore \quad P_0 = 30$

When $t = 8, P = 10$

$\therefore \quad 10 = 30e^{-8k}$

$\therefore \quad \frac{1}{3} = e^{-8k}$

$\therefore \quad \log_e\left(\frac{1}{3}\right) = -8k$

and
$$k = -\frac{1}{8}\log_e\left(\frac{1}{3}\right)$$
$$= \frac{1}{8}\log_e(3) \approx 0.1373$$

b When $P = 8, 8 = 30\,e^{\left(-\frac{1}{8}\log_e(3)\right)t}$

$$\therefore \quad \frac{4}{15} = e^{\left(\log_e(3)^{\frac{1}{8}}\right)t}$$
$$\therefore \quad \frac{4}{15} = 3^{-\frac{t}{8}}$$
$$\therefore \quad \frac{15}{4} = 3^{\frac{t}{8}}$$
$$\therefore \quad t = \frac{8\log_e\left(\frac{15}{4}\right)}{\log_e(3)} \approx 9.625$$

The pressure would be 8 units after approximately 9.625 hours.

c **i**
$$\frac{dP}{dt} = -30ke^{-kt}$$

where $k = \frac{1}{8}\log_e 3$

When $t = 0, \frac{dP}{dt} = -30ke^0$
$$= -30k$$
$$= -\frac{30}{8}\log_e 3$$
$$= -\frac{15}{4}\log_e 3$$

The rate of loss is $\frac{15}{4}\log_e 3 \approx 4.120$ units per hour when $t = 0$.

ii When $t = 8, \frac{dP}{dt} = -30ke^{-8k}$
$$= -\frac{15}{4} \times \log_e 3 \times e^{-\log_e 3}$$
$$= -\frac{15}{4} \times \log_e 3 \times \frac{1}{3}$$
$$= -\frac{5}{4}\log_e 3$$

This rate of loss is $\frac{5}{4}\log_e 3 \approx 1.373$ units per hour when $t = 8$.

8 a
$$\frac{dT}{dt} = -\frac{45}{2}e^{-0.3t}$$

Also, $e^{-0.3t} = \frac{1}{75}(T - 15)$

$$\therefore \quad \frac{dT}{dt} = -\frac{45}{150}(T - 15)$$
$$= -0.3(T - 15)$$

b **i** When $T = 90, t = 0$

$$\therefore \quad \frac{dT}{dt} = -\frac{45}{2}$$

ii When $T = 60$

$$\frac{dT}{dt} = -0.3(60 - 15) = -13.5$$

iii When $T = 30$

$$\frac{dT}{dt} = -0.3(30 - 15) = -4.5$$

9

$$\begin{aligned}
y &= 3x + 2\cos x\\
\frac{dy}{dx} &= 3 - 2\sin x
\end{aligned}$$

$$-1 \leq \sin 2x \leq 1$$

$$-2 \leq -2\sin 2x \leq 2$$

$$-1 \leq 3 - 2\sin 2x \leq 5 \therefore \frac{dy}{dx} > 0 \quad QED$$

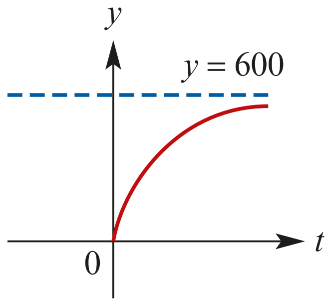
10 $V(t) = 3 + 2\sin\frac{t}{4}$

a $$\begin{aligned}
V(10) &= 3 + 2\sin\left(\frac{5}{2}\right)\\
&\approx 4.197
\end{aligned}$$

b $$\begin{aligned}
V'(t) &= \frac{1}{2}\cos\frac{t}{4}\\
V'(10) &= \frac{1}{2}\cos\left(\frac{5}{2}\right)\\
&\approx -0.4
\end{aligned}$$

11 $y = 600(1 - e^{-0.5t})$

a



b $$\begin{aligned}
\frac{dy}{dx} &= 600(0.5e^{-0.5t})\\
&= 300e^{-0.5t}
\end{aligned}$$

$$t = 9,$$

$$\frac{dy}{dx} = 300e^{-4.5} \approx 3.33$$

12 **a** $$\begin{aligned}
y &= e^{-2x}\\
-2x &= \ln y\\
x &= \frac{-1}{2}\ln y\\
\frac{dy}{dx} &= \frac{-1}{2y}\\
\frac{dy}{dx} &= -2y
\end{aligned}$$

b $$\begin{aligned}
y &= Ae^{kx}\\
\frac{dy}{dx} &= Ake^{kx}\\
&= k(Ae^{kx})\\
&= ky
\end{aligned}$$

13 $m = 2e^{-0.2t}$

a $$\begin{aligned}
t &= 12,\\
m &= 2e^{-2.4}\\
&\approx 0.18\ kg
\end{aligned}$$

b $$\begin{aligned}
t &= 0,\\
m &= 2\\
m &- 1,\\
1 &= 2e^{-0.2t}\\
-0.2t &= \ln\frac{1}{2}\\
0.2t &= \ln 2\\
t &= 5\ln 2 \approx 3.47\ hours
\end{aligned}$$

c **i** $$\begin{aligned}
e^{-0.2t} &= \frac{1}{4}\\
0.2t &= \ln 4\\
t &= 10\ln 2 \approx 6.93\ hours
\end{aligned}$$

ii $e^{-0.2t} = \frac{1}{8}$

$$0.2t = \ln 8$$

$$t = 15 \ln 2 \approx 10.4 \textit{ hours}$$

d

$$\begin{aligned} \frac{dm}{dt} &= -\frac{2}{5}e^{\frac{-t}{5}} \\ &= -\frac{1}{5}(2e^{\frac{-t}{5}}) \\ &= -\frac{1}{5}\text{ m/hr} \\ \text{Rate of decay} &= \frac{1}{5}\text{ m} \end{aligned}$$

Solutions to Exercise 10C

1 a

$$\begin{aligned} f(x) &= x^3 - 12x \\ f'(x) &= 3x^2 - 12 \\ f'(x) &= 0 \\ 3x^2 - 12 &= 0 \\ x^2 - 4 &= 0 \\ x &= \pm 2 \\ f(\pm 2) &= \pm 8 \mp 24 \\ &= \mp 16 \\ \text{co-ords} &= (-2, 16), (2, -16) \end{aligned}$$

b

$$\begin{aligned} g(x) &= 2x^2 - 4x \\ g'(x) &= 4x - 4 \\ g'(x) &= 0, \\ 4x - 4 &= 0 \\ x &= 1, \\ g(1) &= 2 - 4 \\ &= -2 \\ \text{co-ords} &= (1, -2) \end{aligned}$$

c

$$\begin{aligned} h(x) &= 5x^4 - 4x^5 \\ h'(x) &= 20x^3 - 20x^4 \\ h'(x) &= 0, \\ 20x^3 - 20x^4 &= 0 \\ x^3(1 - x) &= 0 \\ x &= 0, 1 \\ h(0) &= 0, \\ h(1) &= 1 \\ \text{co-ords} &= (0, 0), (1, 1) \end{aligned}$$

d

$$\begin{aligned} f(t) &= 8t + 5t^2 - t^3, t > 0 \\ f'(t) &= 8 + 10t - 3t^2 \\ f'(t) &= 0, \\ 3t^2 + 10t + 8 &= 0 \\ t &= \frac{10 \pm \sqrt{100 + 16}}{6} \\ &= \frac{10 \pm 14}{6} \\ &= \frac{-2}{3}, 4 \\ t > 0, \therefore\ t &= 4 \\ f(4) &= 32 + 80 - 64 = 48 \\ \text{co-ords} &= (4, 48) \end{aligned}$$

e

$$\begin{aligned} g(z) &= 8z^2 - 3z^4 \\ g'(z) &= 16z - 12z^3 \\ g'(z) &= 0, \\ 16z - 12z^3 &= 0 \\ (3z^2 - 4)z &= 0 \\ z &= 0, \frac{\pm 2}{\sqrt{3}} \\ g(0) &= 0, \\ g\left(\frac{\pm 2}{\sqrt{3}}\right) &= 8 \times \frac{4}{3} - 3 \times \frac{16}{9} \\ &= \frac{32}{3} - \frac{16}{3} \\ &= \frac{16}{3} \\ \text{co-ords} &= \left(\frac{\pm 2}{\sqrt{3}}, \frac{16}{3}\right), (0, 0) \end{aligned}$$

f $f(x) = 5 - 2x + 3x^2$

$f'(x) = -2 + 6x$

$f'(x) = 0,$

$6x - 2$

$x = \frac{1}{3}$

$f\left(\frac{1}{3}\right) = 5 - \frac{2}{3} + \frac{1}{3}$

$= 4\frac{2}{3} = \frac{14}{3}$

co-ords $= \left(\frac{1}{3}, \frac{14}{3}\right)$

g $h(x) = x^3 - 4x^2 - 3x + 20,$

$x > 0$

$h'(x) = 3x^2 - 8x - 3$

$h'(x) = 0$

$3x^2 - 8x - 3 = 0$

$x = \dfrac{8 \pm \sqrt{64 + 36}}{6}$

$= \frac{-2}{6}, \frac{18}{6}$

$x > 0, \therefore x = \frac{18}{6} = 3$

$h(3) = 27 - 36 - 9 + 20$

$= 2$

co-ords $= (3, 2)$

h

$f(x) = 3x^4 - 16x^3 + 24x^2 - 10$

$f'(x) = 12x^3 - 48x^2 + 48x$

$f'(x) = 0,$

$x(x^2 - 4x + 4) = 0$

$x(x - 2)^2 = 0$

$x = 0, 2$

$f(0) = -10$

$f(2) = 3 \times 16 - 16 \times 8 + 24 \times 4 - 10$

$= 48 - 128 + 96 - 10$

$= -80 + 86$

$= 6$

co-ords $= (0, -10), (2, 6)$

2 a $f(x) = e^{2x} - 2x$

$f'(x) = 2e^{2x} - 2$

$f'(x) = 0 \Rightarrow e^{2x} = 1$

$\Rightarrow x = 0$

Coordinates of stationary point: (0,1)

b $f(x) = x\log_e(3x)$

$f'(x) = \log_e(3x) + 1$

$f'(x) = 0 \Rightarrow \log_e(3x) = -1$

$\Rightarrow x = \frac{1}{3e}$

Coordinates of stationary point: $(\frac{1}{3e}, -\frac{1}{3e})$

c $f(x) = \cos(2x), x \in [-\pi, \pi]$

$f'(x) = -2\sin(2x)$

$f'(x) = 0 \Rightarrow \sin(2x) = 0$

$\Rightarrow 2x = -2\pi, -\pi, 0, \pi, 2\pi$

$\Rightarrow x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

Coordinates of stationary point :

$(-\pi, 1), (-\frac{\pi}{2}, -1), (0, 1), (\frac{\pi}{2}, -1), (\pi, 1)$

d $f(x) = xe^x$

$f'(x) = e^x + xe^x = e^x(1 + x)$

$f'(x) = 0 \Rightarrow x = -1$

Coordinates of stationary point: $(-1, -\frac{1}{e})$

e

$$f(x) = x^2e^x$$

$$f'(x) = x^2e^x + 2xe^x = xe^x(2 + x)$$

$f'(x) = 0 \Rightarrow x = -2, 0$
Coordinates of stationary point:

$(-2, \frac{4}{e^2}), (0, 0)$

f

$$f(x) = 2x\log_e(x)$$

$$f'(x) = 2\log_e(3x) + 2$$

$$f'(x) = 0 \Rightarrow \log_e(x) = -1$$

$$\Rightarrow x = \frac{1}{e}$$

Coordinates of stationary point:$(\frac{1}{e}, -\frac{2}{e})$

3 **a** $f(x) = x^2 - ax + 9$

$$f'(x) = 2x - a$$

$$f'(3) = 0,$$

$$6 - a = 0$$

$$a = 6$$

b

$$h(x) = x^3 - bx^2 - 9x + 7$$

$$h'(x) = 3x^2 - 2bx - 9$$

$$h'(-1) = 0,$$

$$3 + 2b - 9 = 0$$

$$2b = 6$$

$$b = 3$$

4

$$y = x^3 + bx^2 + cx + d$$

$$\frac{dy}{dx} = 3x^2 + 2bx + c$$

When $x = 0, y = 3$

$$\therefore d = 3$$

When $x = 1, y = 3$

$$\therefore 1 + b + c + 3 = 3$$

$$\therefore b + c = -1 \ldots (1)$$

When $x = 1, \frac{dy}{dx} = 0$

$$\therefore 2b + c = -3 \ldots (2)$$

Subtract (1) from (2)

$$b = -2$$

$$\therefore c = 1$$

5 $y = ax^2 + bx + c$

$x = 1, y = -3$

(1) $-3 = a + b + c$

$$\frac{dy}{dx} = 2ax + b$$

$$x = 2, \frac{dy}{dx} = 4$$

(2) $4 = 4a + b$

$$x = 1, \frac{dy}{dx} = 0$$

(3) $0 = 2a + b$

(2) − (3) ⇒ $4 = 2a$

$$a = 2$$

sub in (3) ⇒ $b + 4 = 0$

$$b = -4$$

sub in (1) ⇒ $-3 = 2 - 4 + c$

$$c = -1$$

$$y = 2x^2 - 4x - 1$$

6

$$y = ax^3 + bx^2 + cx + d$$

$$x = 0, y = 7\frac{1}{2}$$

$$d = \frac{15}{2}$$

$$x = 3, y = -6$$

$$-6 = 27a + 9b + 3c + \frac{15}{2}$$

$$-\frac{27}{2} = 27a + 9b + 3c$$

$$-\frac{9}{2} = 9a + 3b + c \ldots (1)$$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$x = 0, \frac{dy}{dx} = -3$$

$$-3 = c$$

$$\text{sub in (1)} \Rightarrow -\frac{9}{2} = 9a + 3b - 3$$

$$-\frac{3}{2} = 9a + 3b \ldots (2)$$

$$x = 3, \frac{dy}{dx} = 0$$

$$0 = 27a + 6b - 3$$

$$9a + 2b = 1 \ldots (3)$$

$$(2) - (3) \Rightarrow b = \frac{-5}{2}$$

$$\text{sub in (3)} \Rightarrow 9a - 5 = 1$$

$$9a = 6$$

$$a = \frac{2}{3}$$

$$y = \frac{2x^2}{3} - \frac{5x^2}{2} - 3x + \frac{15}{2}$$

7 $y = ax + b(2x - 1)^{-1}$

a $x = 2, y = 7$

$$1 \quad 7 = 2a + \frac{b}{3}$$

$$\frac{dy}{dx} = a - \frac{2b}{(2x - 1)^2}$$

$$x = 2, \frac{dy}{dx} = 0$$

$$0 = a - \frac{2b}{9}$$

$$2 \quad a = \frac{2b}{9}$$

$$\text{sub in 1} \Rightarrow 7 = \frac{4b}{9} + \frac{b}{3}$$

$$7 = \frac{7b}{9}$$

$$b = 9$$

$$\text{sub in 2} \Rightarrow a = 2$$

b

$$y = 2x + \frac{9}{(2x - 1)}$$

$$\frac{dy}{dx} = 2 - \frac{18}{(2x - 1)^2}$$

$$\frac{dy}{dx} = 0$$

$$\frac{18}{(2x - 1)^2} = 2$$

$$(2x - 1)^2 = 9$$

$$2x - 1 = \pm 3$$

$$2x = 1 \pm 3$$

$$x = -1, 2$$

$$x = -1,$$

$$y = -2 + \frac{9}{-3}$$

$$= -5$$

$$\text{co-ords} = (-1, -5)$$

8

$$y = (2x-1)^n(x+2)$$

$$\frac{dy}{dx} = \frac{d}{dx}(2x-1)^n \times (x+2)(2x-1)^n$$

$$\times \frac{d}{dx}(x+2)$$

$$= 2n(2x-1)^{n-1} \times (x+2)$$

$$+ (2x-1)^n$$

$$= (2x-1)^{n-1}(2n(x+2) + (2x-1))$$

$$= (2x-1)^{n-1}(2nx + 4n + 2x - 1)$$

$$= (2x-1)^{n-1}((2n+2)x + (4n-1))$$

$$\frac{dy}{dx} = 0,$$

$$0 = (2x-1)^{n-1}((2n+2)x + (4n-1))$$

$$2x - 1 = 0 \text{ or } (2n+2)x + (4n-1) = 0$$

$$x = \frac{1}{2} \text{ or } x = \frac{1-4n}{2n+2}$$

9 $y = (x^2-1)^n$

$$\frac{dy}{dx} = 2x \times n(x^2-1)^{n-1}$$

$$= 2nx(x^2-1)^{n-1}$$

$$= 2nx((x+1)(x-1))^{n-1}$$

$$\frac{dy}{dx} = 0,$$

$$2nx((x+1)(x-1))^{n-1} = 0$$

$$x = 0, -1, 1$$

10 $y = \frac{x}{x^2+1}$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$\frac{dy}{dx} = 0,$$

$$1 - x^2 = 0$$

$$x = \pm 1$$

$$y = \frac{\pm 1}{2}$$

$$\text{co-ords} = \left(1, \frac{1}{2}\right), \left(-1, \frac{-1}{2}\right)$$

Solutions to Exercise 10D

1 a $0 = 4x^2$

$x = 0$

	0	
+	0	+
/	—	/

inflexion

b $0 = (x - 2)(x + 5)$

$x = -5, 2$

	−5		2	
+	0	−	0	+
/	—	\	—	/
	max.		min.	

c $0 = (x + 1)(2x - 1)$

$x = -1, \frac{1}{2}$

	−1		$\frac{1}{2}$	
+	0	−	0	+
/	—	\	—	/
	max.		min.	

d $0 = -x^2 + x + 12$

$0 = -(x^2 - x - 12)$

$0 = -(x - 4)(x + 3)$

$x = -3, 4$

	−3		4	
−	0	+	0	−
\	—	/	—	\
	min.		max.	

e $0 = x^2 - x - 12$

$0 = -(x - 4)(x + 3)$

$x = -3, 4$

	−3		4	
+	0	−	0	+
/	—	\	—	/
	max.		min.	

f $0 = 5x^4 - 27x^3$

$0 = x^3(5x - 27)$

$x = 0, \frac{27}{5}$

	0		$\frac{27}{5}$	
+	0	−	0	+
/	—	\	—	/
	max.		min.	

g $0 = (x - 1)(x - 3)$

$x = 1, 3$

	1		3	
+	0	−	0	+
/	—	\	—	/
	max.		min.	

h $0 = -(x - 1)(x - 3)$

$x = 1, 3$

	1		3	
−	0	+	0	−
\	—	/	—	\
	min.		max.	

2 a $y = x^3 - 12x$

$\frac{dy}{dx} = 3x^2 - 12$

$\frac{dy}{dx} = 0$

$x^2 - 4 = 0$

$x = -2, +2$

$x = -2.5,$

$\frac{dy}{dx} > 0$ ╱

$x = -1.5,$

$\frac{dy}{dx} < 0$ ╲

$\therefore\ x = -2$ is a max

$x = 1.5,$

$\frac{dy}{dx} < 0$ ╲

$x = 2.5,$

$\frac{dy}{dx} > 0$ ╱

$\therefore\ x = 2$ is a min

b $y = 3x^2 - x^3$

$\frac{dy}{dx} = 6x - 3x^2$

$\frac{dy}{dx} = 0,$

$2x - x^2 = 0$

$x(x - 2) = 0$

$x = 0, 2$

$x = -0.5,$

$\frac{dy}{dx} < 0$ ╲

$x = 0,$

$\frac{dy}{dx} = 0$ —

$x = 1,$

$\frac{dy}{dx} > 0,$ ╱

$x = 2,$

$\frac{dy}{dx} = 0$

$x = 2.5,$

$\frac{dy}{dx} < 0$ ╲

$\therefore\ x = 0$ is a min.

$\therefore\ x = 2$ is a max.

c $y = x^3 - 5x^2 + 3x$

$\frac{dy}{dx} = 3x^2 - 10x + 3$

$\frac{dy}{dx} = 0,$

$(3x - 1)(x - 3) = 0$

$x = \frac{1}{3}, 3$

$x = 0,$

$\frac{dy}{dx} > 0$ ╱

$x = \frac{1}{3},$

$\frac{dy}{dx} = 0$ —

$x = 1,$

$\frac{dy}{dx} < 0$ ╲

$x = 3,$

$\frac{dy}{dx} = 0$ ─

$x = 4,$

$\frac{dy}{dx} > 0$ ╱

$\therefore\ x = \frac{1}{3}$ is a max.

$x = 3$ is a min.

d

$$y = 3 - x^3$$

$$\frac{dy}{dx} = -3x^2$$

$\frac{dy}{dx} = 0,$

$x = 0$

$x = -1,$

$\frac{dy}{dx} < 0$ ╲

$x = 0,$

$\frac{dy}{dx} = 0$ ─

$x = 1,$

$\frac{dy}{dx} < 0$ ╲

$x = 0$ is a stationary point of infection

e

$$y = 3x^4 + 16x^3 + 22x^2 + 3$$

$$\frac{dy}{dx} = 12x^3 + 48x^2 + 48x$$

$\frac{dy}{dx} = 0,$

$$x(x^2 + 4x + 4) = 0$$

$$x(x + 2)^2 = 0$$

$$x = -2, 0$$

$x = -3,$

$\frac{dy}{dx} = -27 \times 12$

$+ 48 \times 9 - 48 \times 3 < 0$ ╲

$x = -2,$

$\frac{dy}{dx} = 0$ ─

$x = -1,$

$\frac{dy}{dx} = -12 + 48 - 48 < 0$ ╲

$x = 0,$

$\frac{dy}{dx} = 0$ ─

$x = 1,$

$\frac{dy}{dx} = 12 + 48 + 48 > 0$ ╱

$\therefore\ x = -2$ is a stationary point of infection

$x = 0$ is a min.

f $y = x^3 - x$

$$\frac{dy}{dx} = 3x^2 - 1$$

$\frac{dy}{dx} = 0,$

$$x^2 = \frac{1}{3}$$

$x = \frac{\pm 1}{\sqrt{3}}$

$x = -1,$

$\frac{dy}{dx} > 0$ /

$x = \frac{-1}{\sqrt{3}},$

$\frac{dy}{dx} = 0$ —

$x = 0,$

$\frac{dy}{dx} < 0$ \

$x = \frac{+1}{\sqrt{3}},$

$\frac{dy}{dx} = 0$ —

$x = 1,$

$\frac{dy}{dx} > 0$ /

$\therefore\ x = \frac{-1}{\sqrt{3}}$ is a max

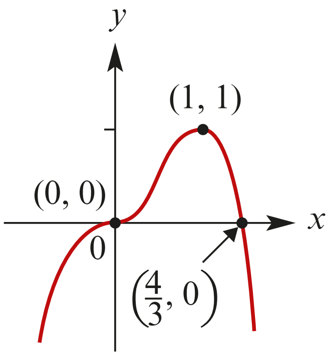
$x = \frac{1}{\sqrt{3}}$ is a min

3 a i $y = 0$

$4x^3 - 3x^4 = 0$

$x^3(4 - 3x) = 0$

$x = 0, \frac{4}{3}$



ii

$\frac{dy}{dx} = 12x^2 - 12x^3$

$\frac{dy}{dx} = 0,$

$12x^2 - 12x^3 = 0$

$x^2(1 - x) = 0$

$x = 0, 1$

$x = 0, y = 0$

(0, 0) is a stationary point of inflection

$x = 1, y = 4 - 3 = 1$

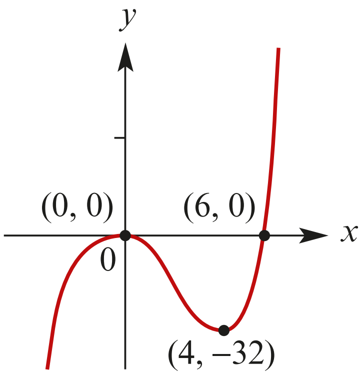
(1, 1) is a maximum turning point

b i $y = x^3 - 6x^2$

$y = 0$

$x^2(x - 6) = 0$

$x = 0, 6$



ii $\frac{dy}{dx} = 3x^2 - 12x$

$\frac{dy}{dx} = 0$

$3x(x - 4) = 0$

$x = 0, 4$

$x = 0, y = 0$

(0, 0) is a maximum turning point

$x = 4,$

$y = 64 - 96$

$= -32$

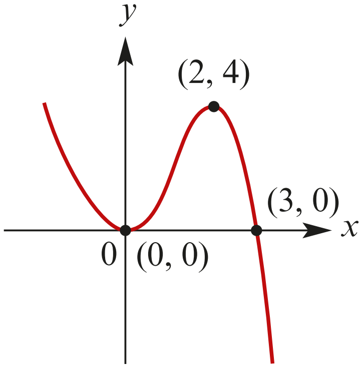
(4, −32) is a minimum turning point

c **i** $y = 3x^2 - x^3$

$y = 0$

$x^2(3 - x) = 0$

$x = 0, 3$



ii $\frac{dy}{dx} = 6x - 3x^2$

$\frac{dy}{dx} = 0$

$3x(2 - x) = 0$

$x = 0, 2$

$x = 0, y = 0$

(0,0) is a minimum turning point

$x = 2,$

$y = 3 \times 4 - 8$

$= 4$

(2, 4) is a maximum turning point

d **i**

$y = x^3 + 6x^2 + 9x + 4$

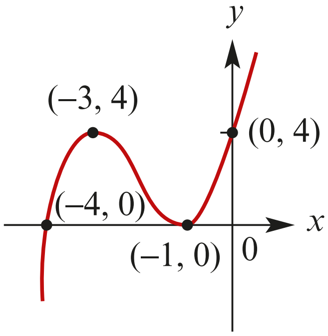
$y = 0$

$x^3 + 6x^2 + 9x + 4 = 0$

$(x + 4)(x + 1)^2 = 0$

$x - \text{ints} : x = -4, -1$

$y - \text{int} : y = 4$



ii

$\frac{dy}{dx} = 3x^2 + 12x + 9$

$\frac{dy}{dx} = 0$

$3x^2 + 12x + 9 = 0$

$x^2 + 4x + 3 = 0$

$(x + 3)(x + 1) = 0$

$x = -3, -1$

$x = -3,$

$y = -27 + 6 \times 9 - 9 \times 3 + 4$

$y = -27 + 54 - 27 + 4$

$y = 4$

$(-3, 4)$

is a maximum turning point

$x = -1,$

$y = 0$

$(-1, 0)$

is a minimum turning point

e **i** $y = (x^2 - 1)^5$

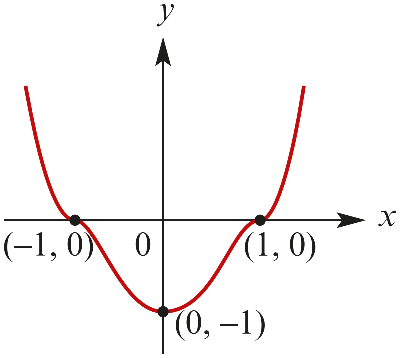
$y = 0$

$x^2 - 1 = 0$

$x - \text{ints} : x = \pm 1$

$x = 0, y = -1$

$y - \text{int} : y = -1$



ii $\frac{dy}{dx} = 2x \times 5(x^2 - 1)^4$

$= 10x(x^2 - 1)^4$

$\frac{dy}{dx} = 0$

$10x(x^2 - 1)^4 = 0$

$x = 0, \pm 1$

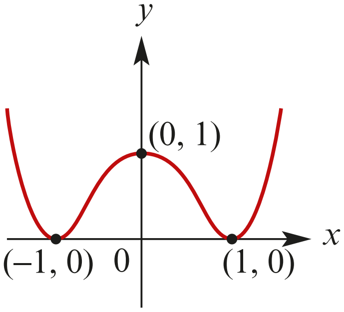
$x = 0,$

$y = (-1)^5 = -1$

$(0, -1)$
is a minimum turning point
$x = \pm 1,$

$y = 0$
$(\pm 1, 0)$
are stationary point of inflection

f **i** $y = (x^2 - 1)^4$
$x^2 = 1$
x-ints: $x = \pm 1$
$x = 0, y = 1$
y-int: $y = 1$



ii $\frac{dy}{dx} = 8x(x^2 - 1)^3$
$\frac{dy}{dx} = 0,$
$x(x^2 - 1)^3 = 0$
$x = 0, \pm 1$
$(-1, 0), (1, 0)$
are minimum turning points
$(0, 1)$ are maximum turning points

4 **a** $y = 2x^3 + 3x^2 - 12x + 7$

$\frac{dy}{dx} = 6x^2 + 6x - 12$

$\frac{dy}{dx} = 0,$

$0 = 6x^2 + 6x - 12$

$x^2 + x - 2 = 0$

$(x - 1)(x + 2) = 0$

$x = -2, 1$ both are turning points

$x = -3,$

$$\frac{dy}{dx} = 6(9 - 3 - 2)$$

> 0

$x = 0,$

$$\frac{dy}{dx} = -12 < 0$$

$\therefore\ x = -2$ is a max

$x = -2,$

$y = -2 \times 8 + 3 \times 4 + 12 \times 2 + 7$

$y = -16 + 12 + 24 + 7$

$= 27$

$(-2, 27)$ is a max

$x = 2$

$$\frac{dy}{dx} = 6(4 + 2 - 2)$$

>0

$\therefore\ x = 1$ is a min

$x = 1,$

$y = 2 + 3 - 12 + 7$

$x = 1, y = 0$

(1,0) is a *min*

b see above, (1,0)
is a point on the curve

c $y = 0$
$2x^3 + 3x^2 - 12x + 7 = 0$ (from **a**),
we know (1,0) is a turning point.

$\therefore (x - 1)^2$ is a factor,

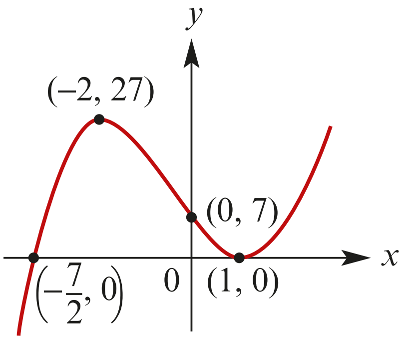
$\therefore\ y = (x - 1)^2(2x + 7)$

$(x - 1)^2 (2x + 7) = 0$

$x = ints : x = 1, \dfrac{-7}{2}$

$x = 0, y = 7, y - \text{int} : y = 7$

d



5 a $P(x) = x^3 + ax^2 + b$

$P'(x) = 3x^2 + 2ax$

$= x(3x + 2a)$

$P'(0) = 0$

$\therefore\ x = 0$

is a stationary point for all values of *a* and *b*

b

$$P'(-2) = 0$$

$$3(-2)^2 + 2a(-2) = 0$$

$$12 - 4a = 0$$

$$a = 3$$

$$P(-2) = 6$$

$$(-2)^3 + 3(-2)^2 + b = 6$$

$$-8 + 12 + b = 6$$

$$4 + b = 6$$

$$b = 2$$

$$P(x) = x^3 + 3x^2 + 6$$

$$P'(x) = 3x^2 + 6x$$

$$P'(-1) = 3 - 6$$

$$<0 \quad \diagdown$$

$$P'(1) = 3 + 6$$

$$>0 \quad \diagup$$

$x = 0$ is a min.

Local minimum at $(0, 2)$

$$P'(-3) = 27 - 18 > \quad 0 \diagup$$

$x = -2$ is a max.

Local maximum at $(-2, 6)$

6 a

$$f(x) = (2x - 1)^5(2x - 4)^4$$

$$f(0) = (-1)^5(-4)^4$$

y-intercept $= -256$

co-ords$(0, -256)$

$$f(x) = 0,$$

$$0 = (2x - 1)^5(2x - 4)^4$$

$$x = \frac{1}{2}, 2$$

x-intercepts $= \frac{1}{2}, 2$

co-ords $\left(\frac{1}{2}, 0\right), (2, 0)$

b

$$f'(x) = (2x - 1)^5$$

$$\left(\frac{d}{dx}((2x - 4)^4) + (2x - 4)^4 \frac{d}{dx}((2x - 1)^5)\right)$$

$$= (2x - 1)^5 \times 2 \times 4(2x - 4)^3$$

$$+ (2x - 4)^4 \times 2 \times 5(2x - 1)^4$$

$$= (2x - 4)^3(2x - 1)^4((2x - 1) \times 8$$

$$+ (2x - 4) \times 10)$$

$$= (2x - 4)^3(2x - 1)^4(16x - 8 + 20x - 40)$$

$$= (2x - 4)^3(2x - 1)^4(36x - 48)$$

$$= 12(3x - 4)(2x - 4)^3(2x - 1)^4$$

$$f'(x) = 0,$$

$$12(3x - 4)(2x - 4)^3(2x - 1)^4 = 0$$

$x = \frac{4}{3}$ (turning point)

or $x = 2$ (turning point)

or $x = \frac{1}{2}$(stationary point of inflection)

$$f'(1) = 12(3-4)(2-4)^3(2-1)^4$$

$$> 0$$

$$\diagup$$

$$f'(1.5) = 12(4.5-4)(3-4)^3(3-1)^4$$

$$\diagdown$$

$$< 0$$

$$f'(3) = 12(9-4)(6-4)^3(6-1)^4$$

$$\diagup$$

$$> 0$$

$x = \frac{4}{3}$ is a max.

$x = 2$ is a min.

$$f\left(\frac{4}{3}\right) = \left(\frac{8}{3}-1\right)^5\left(\frac{8}{3}-4\right)^4\left(\frac{5}{3}\right)^5\left(\frac{-4}{3}\right)^4$$

$$\approx 40.6$$

$\left(\frac{4}{3}, 40.6\right)$ is a max.

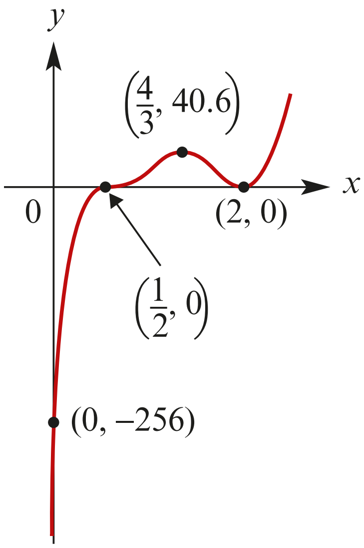
$$f(2) = (4-1)^5(4-4)^4$$

$$= 0$$

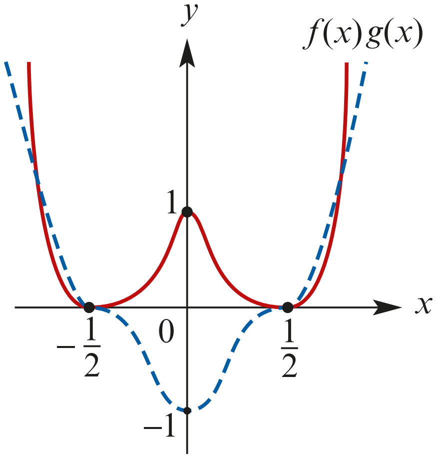
$(2, 0)$ is a min.

$$f\left(\frac{1}{2}\right) = 0$$

$\left(\frac{1}{2}, 0\right)$ is a stationary point of inflection



7 a



b **i** $(4x^2 - 1)^6 > (4x^2 - 1)^5$

$4x^2 - 1 > 1 \; if \; (4x^2 - 1)^5 > 0$

$4x^2 > 2 \; if \; (4x^2 - 1)^5 > 0$

$x^2 > \frac{1}{2} \; if \; (4x^2 - 1)^5 > 0$

$|x| > \frac{1}{\sqrt{2}} \; if \; (4x^2 - 1)^5 > 0$

$x^2 < \frac{1}{4} \; if \; (4x^2 - 1)^5 < 0$

$|x| < \frac{1}{2} \; if \; (4x^2 - 1)^5 < 0$

$\therefore \; |x| > \frac{1}{\sqrt{2}} \text{ or } |x| < \frac{1}{2}$

ii

$f'(x) = 8x \times 6(4x^2 - 1)^5$

$g'(x) = 8x \times 6(4x^2 - 1)^4$

$f'(x) > g'(x)$

$8x \times 6(4x^2 - 1)^5 > 8x \times 6(4x^2 - 1)^4$

$6x(4x^2 - 1) > 5x$

$if \; x > 0,$

$6(4x^2 - 1) > 5$

$(4x^2 - 1) > \frac{5}{6}$

$4x^2 > \frac{11}{6}$

$x^2 > \frac{11}{24}$

$x > \sqrt{\frac{11}{24}}$

$x > \frac{\sqrt{66}}{12}$

$if \; x < 0,$

$6(4x^2 - 1) < 5$

$(4x^2 - 1) < \frac{5}{6}$

$4x^2 < \frac{11}{6}$

$x^2 < \frac{11}{24}$

$x^2 < \frac{66}{144}$

$x > \frac{-\sqrt{66}}{12}, x < 0$

$x \neq \pm\frac{1}{2}$

$\therefore \; \frac{-\sqrt{66}}{12} < x < \frac{-1}{2},$

or $\frac{-1}{2} < x < 0,$

or $x > \frac{\sqrt{66}}{12}$

8 **a**

$y = x^3 + x^2 - 8x - 12$

$x = 0, y = -12$

$y\text{-intercept} = (0, -12)$

$y = 0$

$x^3 + x^2 - 8x - 12 = 0$

try $x = 3$ (a factor of -12)

$27 + 9 - 24 - 12$

$= 0$

$\therefore (x - 3)$ is a factor

$(x - 3)(x^2 + 4x + 4) = 0$

$(x - 3)(x + 2)^2 = 0$

$x = 3, -2$

$$x\text{-intercepts} = (-2, 0), (3, 0)$$

$$\frac{dy}{dx} = 3x^2 + 2x - 8$$

$$= (3x - 4)(x + 2)$$

$$\frac{dy}{dx} = 0,$$

$$(3x - 4)(x + 2) = 0$$

$$x = -2, \frac{4}{3}$$

$$x = -2, y = 0$$

$$x = \frac{4}{3}$$

$$y = \left(\frac{4}{3}\right)^3 + \left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) - 12$$

$$= \frac{64}{27} + \frac{16}{9} - \frac{32}{3} - 12$$

$$= \frac{64 + 48 - 288 - 324}{27}$$

$$= \frac{-500}{27}$$

$$= -18\frac{14}{27}$$

stationary points are $(-2, 0)$ max and $\left(\frac{4}{3}, \frac{-500}{27}\right)$ min

b $y = 4x - 18x^2 + 48x - 290$

$$= 2(2x^3 - 9x^2 + 24x - 145)$$

$$x = 0, y = -290$$

y-intercept $= (0, -290)$

$$y = 0$$

$$2x^3 - 9x^2 + 24x - 145 = 0$$

using CAS calculator

$$x = 5$$

$$y = 2(x - 5)(2x^2 + x + 29)$$

$$2x^2 + x + 29 = 0,$$

$$x = \frac{-1 \pm \sqrt{1 - 232}}{4}$$

no real solutions

$$y = 0, x = 5$$

x-intercept $= (5, 0)$

$$\frac{dy}{dx} = 12x^2 - 36x + 48$$

$$= 12(x^2 - 3x + 4)$$

$$\frac{dy}{dx} = 0,$$

$$x^2 - 3x + 4 = 0$$

$$x = \frac{3 \pm \sqrt{9 - 16}}{2}$$

no real solutions

$\therefore$ y has no stationary points

9 a $f(x) = 3x^4 + 4x^3$

$$f(x) = 12x^3 + 12x^2$$

$$f'(x) = 0,$$

$$12x^2(x + 1) = 0$$

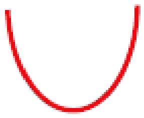
$$x = -1, 0$$

$$f'(0) = 0$$

$(0, 0)$, stationary point of inflection

$$f(-1) = 3 - 4$$
$$= -1$$

$(-1, -1)$ *min.*, since
$f(x)$ is shaped and $(0, 0)$
is a stationary point of inflection



b $$f(x) = x^4 + 2x^3 - 1$$
$$f'(x) = 4x^3 + 6x^2$$
$$f'(x) = 0,$$
$$2x^2(2x + 3) = 0$$

$x = 0$ (stationary point of inflection)
$x = \frac{-3}{2}$ (turning point)
$f(0) = -1$
$(0, -1)$
is a stationary point of inflection

$$f'(-2) = 4 \times -8 + 6 \times 4$$

$$< 0$$
$$f'(-1) = -4 + 6$$

$$> 0$$
$$f'\left(-\frac{3}{2}\right) = \frac{81}{16} + \frac{-27}{4} - 1$$
$$= \frac{-43}{16}$$

$\left(\frac{-3}{2}, \frac{-43}{16}\right) = (-1.5, -2.6875)$ is a min.

c $$f(x) = 3x^3 - 3x^2 + 12x + 9$$
$$= 3(x^3 - x^2 + 4x + 3)$$
$$f'(x) = 3(3x^2 - 2x + 4)$$
$$f'(x) = 0,$$
$$3x^2 - 2x + 4 = 0$$
$$x = \frac{2 \pm \sqrt{4 - 48}}{6}$$

no real solutions
$\therefore f(x)$ has no stationary points

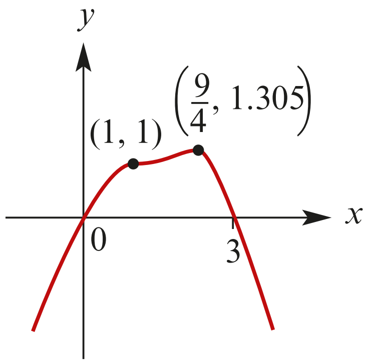
10 $f(x) = \frac{1}{8}(x - 1)^3(8 - 3x) + 1$

a $$f(0) = \frac{1}{8}(-1)^3(8) + 1$$
$$= 1 - 1$$
$$= 0 \; QED$$
$$f(3) = \frac{1}{8}(2)^3(-1) + 1$$
$$= -1 + 1$$
$$= 0 \quad QED$$

b $$f'(x) = \frac{1}{8}(x - 1)^3\frac{d}{dx}(8 - 3x)$$
$$+ \frac{1}{8}(8 - 3x)\frac{d}{dx}(x - 1)^3$$
$$= \frac{-3}{8}(x - 1)^3\frac{3}{8}(8 - 3x)(x - 1)^2$$
$$= \frac{3}{8}(x - 1)^2((8 - 3x) - (x - 1))$$
$$= \frac{3}{8}(x - 1)^2(9 - 4x) \quad QED$$

want x such that $f'(x) \geq 0$
$\frac{3}{8}(x - 1)^2(9 - 4x) \geq 0$
since $(x - 1)^2 \geq 0$,
$9 - 4x \geq 0$
$$x \leq \frac{9}{4}$$

c



11 $y = 3x^4 - 44x^3 + 144x^2$

$$\frac{dy}{dx} = 12x^3 - 132x^2 + 288x$$

$$= 12x(x^2 - 11x + 24)$$

$$= 12x(x - 8)(x - 3)$$

$$\frac{dy}{dx} = 0,$$

$x = 0, 3, 8$

$x = 0,$

$y = 0$

(0, 0) is a minimum turning point

$x = 3,$

$y = 3^5 - 44 \times 27 + 144 \times 9$

$= 243 - 1188 + 1296$

$= 351$

(3,351) is a maximum turning point

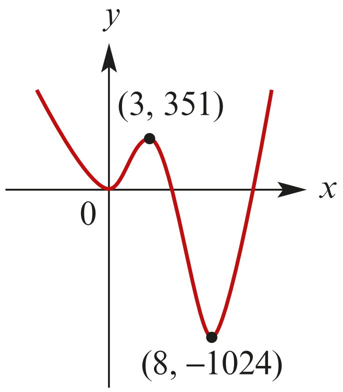
$x = 8,$

$y = 3 \times 8^4 - 44 \times 8^3 + 144 \times 64$

$= 12288 - 22528 + 9216$

$= -1024$

(8, −1024) is a minimum turning point



12 **a** $x = -1$ (stationary point of inflection)

$x = 1$ (*min*)

$x = 5$ (*max*)

b $x = 0$ (*max*)

$x = 2$ (*min*)

c $x = -4$ (*min*)

$x = 0$ (*max*)

d $x = -3$ (*min*)

$x = 2$ (stationary point of inflection)

13 **a** $y = x^4 - 16x^2$

$$\frac{dy}{dx} = 4x^3 - 32x$$

$$= 4x(x^2 - 8)$$

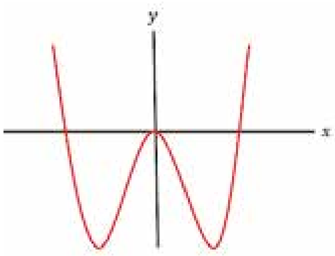
$$\frac{dy}{dx} = 0,$$

$x = 0, \pm 2\sqrt{2}$

$x = 0, x = \pm 2\sqrt{2},$

$y = 0\ y = -64$

Since the x-intercepts are ±4, 0 we can sketch the graph.



hence (0,0) is a maximum
($\pm 2\sqrt{2}, -64$) are minimums

b $y = x^{2m} - 16x^{2m-2}$

$$\frac{dy}{dx} = (x^{2m} - 16x^{2m-2})$$
$$= 2mx^{2m-1} - 16(2m-2)x^{2m-3}$$
$$= 2x^{2m-3}(mx^2 - 16(m-1))$$
$$\frac{dy}{dx} = 0$$
$$x^{2m-3}(mx^2 - 16(m-1)) = 0$$
$$x = 0, mx^2 - 16(m-1) = 0$$
$$x^2 = \frac{16(m-1)}{m}$$
$$x = \pm\sqrt{\frac{16(m-1)}{m}}$$
$$= \pm 4\sqrt{\frac{(m-1)}{m}}$$

If $x = 0$ then $y = 0$.

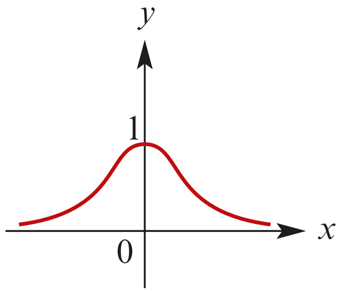
$$\text{If} x = 4\sqrt{\frac{(m-1)}{m}}$$
$$y =$$
$$\left(4\sqrt{\frac{(m-1)}{m}}\right)^{2m} - 16\left(4\sqrt{\frac{(m-1)}{m}}\right)^{2m-2}$$
$$= 16m\left(\frac{(m-1)}{m}\right)^{m} - 16 \times 16^{m-1}\left(\frac{(m-1)}{m}\right)^{m-1}$$
$$= 16m\left(\frac{m-1}{m}\right)^{m} - \left(\frac{(m-1)}{m}\right)^{m-1}$$
$$= 16^{m}\left(\frac{(m-1)}{m}\right)^{m-1}\left(1 - \frac{(m-1)}{m}\right)$$
$$= 16^{m}\left(\frac{(m-1)}{m}\right)^{m-1}\frac{1}{m}$$

stationary points are :

$(0, 0) min$

$\left(\pm 4\sqrt{\frac{(m-1)}{m}}, \frac{16^m(m-1)^{m-1}}{m^m}\right) max$

14



15

$$f(x) = x^2e^x \qquad in\ set$$
$$f'(x) = 2xe^x + x^2e^x$$
$$= e^x(x^2 + 2x)$$
$$f'(x) < 0,$$
$$e^x(x^2 + 2x) < 0$$
$$x^2 + 2x < 0$$
$$x(x + 2) < 0$$
$$x < 0 \ \& \ x > -2$$
$$\therefore \ -2 < x < 0$$
$$notation, \{x: \ -2 < x < 0\}$$

in set notation, $\{x: \ -2 < x < 0\}$

16

$$f(x) = 100e^{-x^2+2x-5}$$

$$f'(x) = 100(-2x+2)e^{-x^2+2x-5}$$

$$f'(x) > 0,$$

$$100(-2x+2)e^{-x^2+2x-5} > 0$$

$$-2x + 2 > 0$$

$$x < 1$$

hence maximum $f(x)$ *occurs at* $x = 1$.

$f'(1) = 100e^{-1+2-5}$

$= 100e^{-4}$

$\cong 1.83$

17 $f(x) = e^x - 1 - x$

a $f'(x) = e^x - 1$

$f'(x) = 0,$

$e^x = 1$

$x = 0$

$f(0) = e^0 - 1 - 0$

min $f(x) = 0$

b *min* $f(x) = 0$

$\therefore \ f(x) \geq 0$

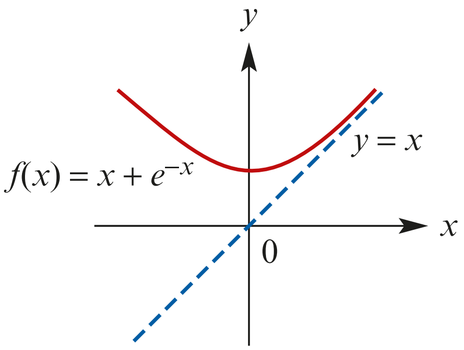
$e^x - 1 - x \geq 0$

$e^x \geq 1 + x$ *QED*

18 **a** (0, 1) min

b $y = x$

c



19 $y = e^x(px^2 + qx + r)$

$$\frac{dy}{dx} = e^x(px^2 + qx + r) + e^x(2px + q)$$

$$= e^x(px^2(q + 2p)x + (r + q))$$

$x = 0, y = 9$

$9 = e^0(0 + 0 + r)$

$9 = r$

$y = e^x(px^2 + qx + 9)$

$$\frac{dy}{dx} = e^x(px^2(q + 2p)x + (q + 9))$$

$x = 1, y = 0$

$0 = e^1(p + q + 2p + q + 9)$

$13p + 2q + 9 = 0$

$x = 3, y = 0$

$0 = e^3(9p + 3(q + 2p) + (q + 9))$

$9p + 3q + 6p + q + 9 = 0$

$215p + 4q + 9 = 0$

$2 - 21 \Rightarrow 9p - 9 = 0$

$p = 1$

sub in $1 \Rightarrow 3 + 2q + 9 = 0$

$2q = -12$

$q = -6$

$y = e^x(x^2 - 6x + 9)$

20 **a** $y = e^{4x^2-8x}$

$$\frac{dy}{dx} = (8x - 8)e^{4x^2-8x}$$

b $\frac{dy}{dx} = 0$

$8(x-1)e^{4x^2-8x} = 0$

$x - 1 = 0$

$x = 1$

$x < 1, f'(x) < 0$

$x > 1, f'(x) > 0$

$\left(\text{since } e^{4x^2-8x} > 0\right)$

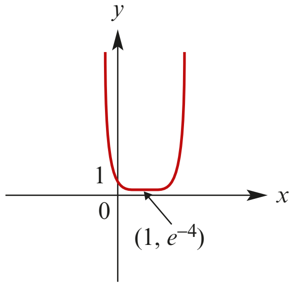
$\therefore x = 1$ *is a min.*

$x = 1,$

$y = e^{4-8} = e^{-4}$

$\therefore (1, e^{-4})$ *is a min.*

c



d $x = 2,$

$y = e^{16-16} = 1$

$\frac{dy}{dx} = 8(x-1)e^{16-16}$

$= 8$

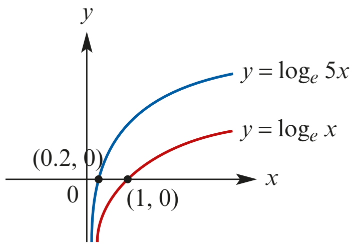
$x = 2, y = 1$

$1 = -\frac{1}{4} + c$

$c = \frac{5}{4}$

$y = -\frac{1}{8}x + \frac{5}{4}$

21



tangents are parallel for any given value of x

22 $f(x) = x^2 \ln x$

a $f'(x) = 2x \ln x + \frac{x^2}{x}$

$= x(2 \ln x + 1)$

b $f(x) = 0,$

$x^2 \ln x = 0$

$x^2 = 0, \quad \ln x = 0$

$x = 0, \quad x = 1$

$x = 0, 1$ *but* $x > 0, \quad \therefore x = 1$

c $f'(x) = 0$

$x(2 \ln x + 1) = 0$

$x = 0, 2 \ln x + 1 = 0$

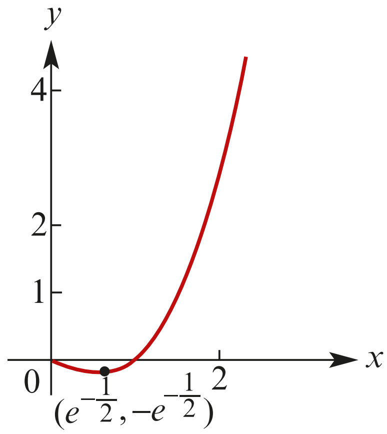
$\ln x = \frac{-1}{2}$

$x = \frac{1}{\sqrt{e}}$

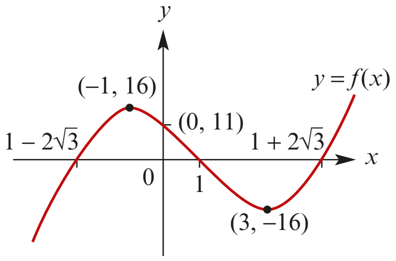
$x = 0, \frac{1}{\sqrt{e}}$ *but* $x > 0,$

$\therefore x = \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$

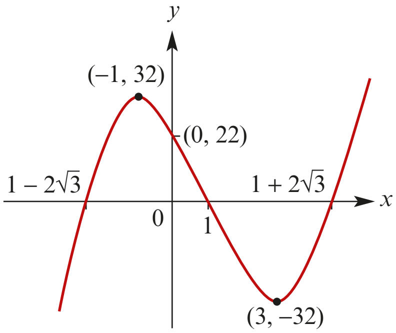
d $x = e^{-\frac{1}{2}}, y = \left(e^{-\frac{1}{2}}\right)^2 \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2}e^{-1}$



23 a

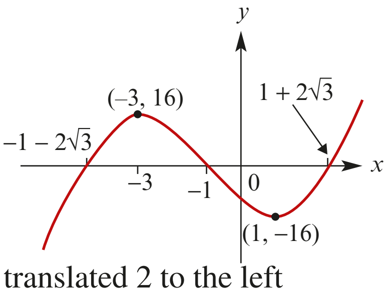


b



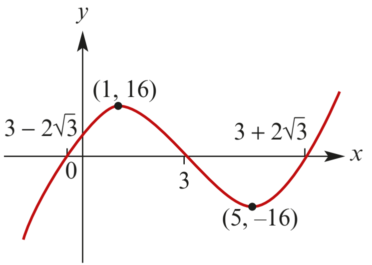
graph dilated by factor 2 from *x*-axis

c



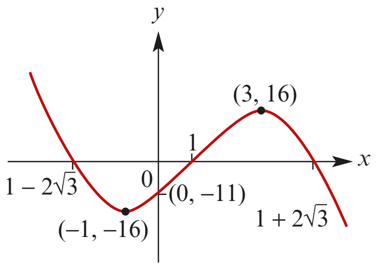
translated 2 to the left

d



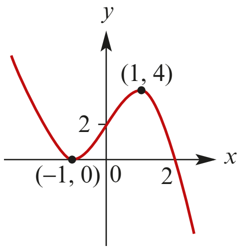
translated 2 to the right

e

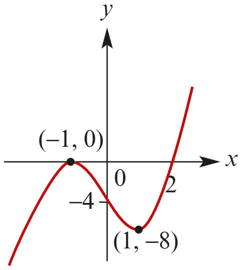


reflected in the *x*-axis

24 a



b



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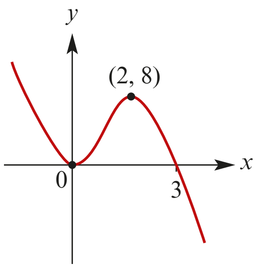
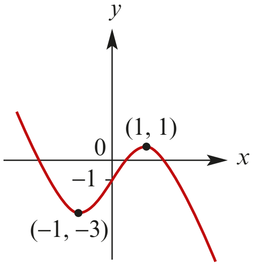
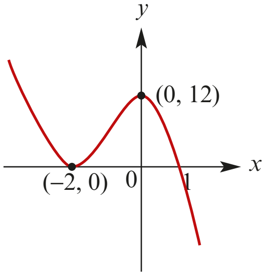


d



e



25 **a** $A' = (a + l, 0)$

$B' = (b + l, 0)$

b $P' = (h + l, kp)$

26 **a**

$$f(x) = 2\cos x - 2\cos^2 x + 1$$

$$f'(x) = -2\sin x + 4\sin x\cos x$$

$$f'(x) = 0 \Rightarrow 2\sin x(2\cos x - 1) = 0$$

$$\Rightarrow \sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$\Rightarrow x = 0, \pi, 2\pi \text{ or}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$(0, 1)$, $(\pi, -3)$, $(2\pi, 1)$ *are min.*

$\left(\frac{\pi}{3}, \frac{3}{2}\right), \left(\frac{5\pi}{3}, \frac{3}{2}\right)$ *are max.*

b

$$f(x) = 2\cos x + 2\sin xcox$$

$$f'(x) = -2\sin x + 2\cos x\cos x - 2\sin x\sin x$$

$$= -2\sin x + 2(1 - 2\sin^2 x)$$

$$= 2(-2\sin^2 x - \sin x + 1)$$

$$f'(x) = 0,$$

$$\sin x = \frac{1 \pm \sqrt{1 + 8}}{-4}$$

$$\sin x = \frac{-1 \pm 3}{4}$$

$$\sin x = \frac{1}{2}, -1$$

$$x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{5\pi}{6}$$

$$f\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)$$

$$= \sqrt{3} + \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}$$

$$f\left(\frac{3\pi}{2}\right) = 2\cos\left(\frac{3\pi}{2}\right) + \sin 3\pi$$

$$= 0$$

$$f\left(\frac{5\pi}{6}\right) = 2\cos\left(\frac{5\pi}{6}\right) + \sin(\frac{10\pi}{6})$$

$$= -\sqrt{3} - \frac{\sqrt{3}}{2}$$

$$= \frac{-3\sqrt{3}}{2}$$

$\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$ *max.*

$\left(\frac{3\pi}{2}, 0\right)$ *stationary point of inflection*

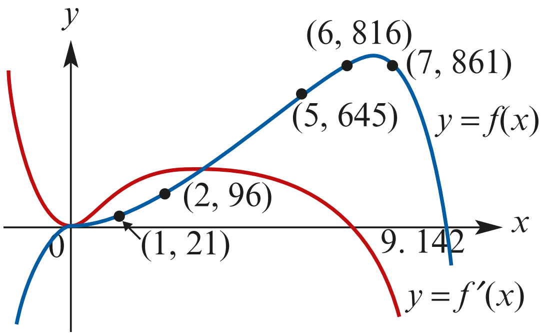
$\left(\frac{5\pi}{6}, \frac{-3\sqrt{3}}{2}\right)$ *min.*

c Max $x = \frac{\pi}{2}, \frac{3\pi}{2}$; Min $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

d Max $x = \frac{\pi}{3}$; Infl $x = \pi$; Min $x = \frac{5\pi}{3}$

27 **a** and

b



Using a CAS calculator:
$y = -x^4 + 8x^3 + 10x^2 + 4x$
loc max at (6.761,867.07)
no stationary point of inflexion, since at
$x = 0, \dfrac{dy}{dx} = 4.$

c −960

d Use the ‘solve’ command of a CAS calculator, giving:
$x = 4.317$ or $x = 8.404$

Solutions to Exercise 10E

1 $f : [-3, 3] \to \mathbb{R}, f(x) = 2 - 8x^2$

Local maximum at $(0, 2)$

$f(-3) = 2 - 8(-3)^2 = 2 - 72 = -70$

$f(3) = 2 - 2(3)^2 = 2 - 72 = -70$

Therefore absolute maximum of f is 2 and absolute minimum is -70

2

$$f : [-3, 2] \to R,$$
$$f(x) = x^3 + 2x + 3$$
$$f'(x) = 3x^2 + 2$$

$f'(x)$ has no real solution

$\therefore$ $f(x)$ has no stationary points

$\therefore$ $f(-3)$ is absolute minimum

$f(2)$ is absolute maximum

$f(-3) = -27 - 6 + 3$

$abs.\ min. = -30$

$f(2) = 8 + 4 + 3$

$abs.\ max. = 15$

3 $f : \left[-\frac{3}{2}, \frac{5}{2}\right] \to R,$

$$\begin{aligned}
f(x) &= 2x^3 - 6x^2 \\
f'(x) &= 6x^2 - 12x \\
f'(x) &= 0, \\
x(x - 2) &= 0 \\
x &= 0, 2 \\
f(0) &= 0 \\
f(2) &= 16 - 24 \\
&= -8 \\
f\left(\frac{-3}{2}\right) &= \frac{-27}{4} - \frac{27}{2} \\
&= \frac{-81}{4} = -20.25 \\
f\left(\frac{5}{2}\right) &= \frac{125}{4} - \frac{75}{2} \\
&= \frac{-25}{4}
\end{aligned}$$

absolute $min = \frac{-81}{4}$

absolute $max = 0$

4 $f : [-2, 6] \to R,\ f(x) = 2x^4 - 8x^2$

$$f'(x) = 8x^3 - 16x$$

$$f'(x) = 0,$$

$$8x(x^2 - 2) = 0$$

$$x = \pm\sqrt{2}, 0$$

$$f(\pm\sqrt{2}) = 8 - 16$$

$$= -8$$

$$f(0) = 0$$

$$f(-2) = 32 - 32 = 0$$

$$f(6) = 2 \times 6^4 - 8 \times 6^2$$

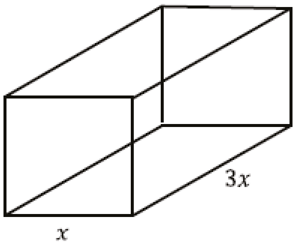
$$= 2592 - 288$$

$$= 2304$$

absolute *min* = −8

absolute *max* = 2304

5



$$4x + 4(3x) + 4y = 20$$

$$4x + y = 5$$

$$y = 5 - 4x$$

a $V = x(3x)y$

$$= 3x^2(5 - 4x)$$

$$V = 15x^2 - 12x^3$$

QED

b $\dfrac{dV}{dx} = 30x - 36x^2$

c $\dfrac{dV}{dx} = 0,$

$$30x - 36x^2 = 0$$

$$x(5 - 6x) = 0$$

$$x = 0, \frac{5}{6}$$

since $x = 0$ gives

$V = 0$, it is not the *max*,

$\therefore\ x = \dfrac{5}{6}$ is the max

co-ords $= \left(\dfrac{5}{6}, \dfrac{125}{36}\right)$

d

there are no turning points, so test the end points,

$x = 0,$

$V = 0$

$x = 0.8,$

$$V = 15 \times \frac{16}{25} - 12 \times \frac{64}{125}$$

$$= \frac{1200 - 768}{125}$$

absolute *max* : $V = \dfrac{432}{125} = 3.456\text{ cm}^3$

when $x = 0.8$

e turning point at $x = \dfrac{5}{6}$,

test the endpoints, $\dfrac{5}{6}$

$x = 0,$

$V = 0,$

$x = 1,$

$V = 15 - 12$

$= 3$

$x = \dfrac{5}{6},$

$$V = 15 \times \frac{16}{25} - 12 \times \frac{125}{216}$$

$$= \frac{375 - 250}{36}$$

absolute max : $V = \dfrac{125}{36} = 3.472\,\text{cm}^3$

when $x = \dfrac{5}{6}$

6 $x + y = 30, z = xy$

a $x \in [2, 5]$,

$y = 30 - x$

$y \in [25, 28]$, i.e. $25 \le y \le 28$

b $z = x(30 - x)$

$\text{z} = 30x - x^2$

$\dfrac{dz}{dx} = 30 - 2x$

$\dfrac{dz}{dx} = 0, x = 15$

this is outside the domain $x \in [2, 5]$

$\therefore$ values to test are

$x = 2, 5$

$x = 2,$

$z = 60 - 4 = 56,$

$x = 5,$

$z = 150 - 25 = 125$

absolute minimum = 56

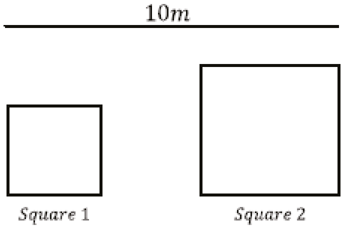
absolute maximum = 125

7 **a** $\dfrac{1}{(x-4)^2} - \dfrac{1}{(x-1)^2}$

b $\left(\dfrac{5}{2}, \dfrac{4}{3}\right)$

c Absolute max $= \dfrac{3}{2}$; Absolute min $= \dfrac{4}{3}$

8



a square 1 has perimeter x, i.e. side $\left(\dfrac{x}{4}\right)$

$A_1 = \dfrac{x^2}{16}$

square 2 has perimeter $(10 - 4)$, $i.e.$ side $\left(\dfrac{10 - x}{4}\right)$

$A_2 = \dfrac{(10 - x)^2}{16}$

$$\begin{aligned} A &= A_1 + A_2 \\ &= \frac{x^2 + (10 - x)^2}{16} \\ &= \frac{x^2 + 100 - 20x + x^2}{16} \\ &= \frac{2x^2 - 20x + 100}{16} \\ &= \frac{1}{8}(x^2 - 10x + 50)\ QED \end{aligned}$$

b $\dfrac{dA}{dx} = \dfrac{1}{8}(2x - 10)$

$= \dfrac{1}{4}(x - 5)$

c $\dfrac{dA}{dx} = 0,$

$x = 5$

d $A = \dfrac{1}{8}(x^2 - 10x + 50)$

$x \in [0, 1]$ $A(0) = \dfrac{25}{4}$ and $A(1) = \dfrac{41}{8}$

The maximum is $\dfrac{25}{4}\text{m}^2$ but only one square is formed

9 $g : [2.1, 8] \to R,\ g(x) = x + \dfrac{1}{x-2}$

$$g'(x) = 1 - \frac{1}{(x-2)^2}$$

$$g'(x) = 0,$$

$$\frac{1}{(x-2)^2} = 1$$

$$(x-2)^2 = 1$$

$$x - 2 = \pm 1$$

$$x = 2 \pm 1$$

$$x \in [2.1, 8]$$

$$x = 3,$$

values to test :

$f(2.1), f(3), f(8)$

$$f(2.1) = 2.1 + \frac{1}{0.1}$$

$$= 12.1$$

$$f(3) = 3 + \frac{1}{1}$$

$$= 4$$

$$f(8) = 8 + \frac{1}{6}$$

$$= 8\frac{1}{6}$$

absolute minimum = 4

absolute maximum = 12.1

10 $f : [0, 3] \to R,\ f(x) = \dfrac{1}{x+1} + \dfrac{1}{4-x}$

$$f(x) = \frac{1}{x+1} - \frac{1}{x-4}$$

a $f'(x) = \dfrac{-1}{(x+1)^2} - \dfrac{-1}{(x-4)^2}$

$$= \frac{1}{(x-4)^2} - \frac{1}{(x+1)^2}$$

b $f'(x) = 0,$

$$\frac{1}{(x-4)^2} = \frac{1}{(x+1)^2}$$

$$(x-4)^2 = (x+1)^2$$

$$(x-4) = \pm(x+1)$$

$$if\ x - 4 = x + 1$$

$$-4 = -1$$

does not work

$$\therefore\ x - 4 = -(x+1) = -x - 1$$

$$2x - 3 = 0$$

$$x = \frac{3}{2}$$

$$f\left(\frac{3}{2}\right) = \frac{1}{\left(\frac{5}{2}\right)} + \frac{1}{\left(\frac{5}{2}\right)}$$

$$= \frac{4}{5}$$

$$\text{co-ords} = \left(\frac{3}{2}, \frac{4}{5}\right)$$

c values to test:

$f(0), f(3), f\left(\dfrac{3}{2}\right)$

$$f(\frac{3}{2}) = \frac{4}{5}$$

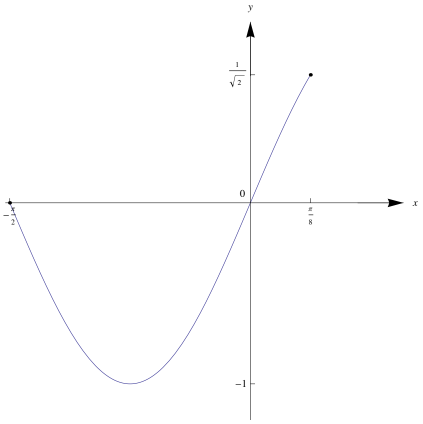
$$f(0) = \frac{1}{1} + \frac{1}{4} = \frac{5}{4}$$

$$f(3) = \frac{1}{4} + \frac{1}{1} = \frac{5}{4}$$

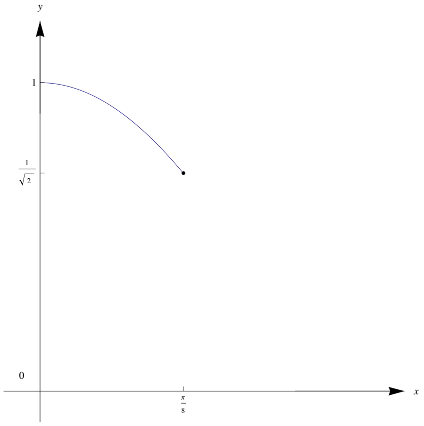
absolute minimum = $\dfrac{4}{5}$

absolute maximum = $\dfrac{5}{4}$

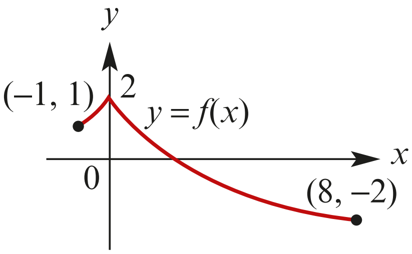
11 Absolute max = $\frac{\sqrt{2}}{2}$; Absolute min = -1



12 Absolute max = 1; Absolute min = $\frac{\sqrt{2}}{2}$

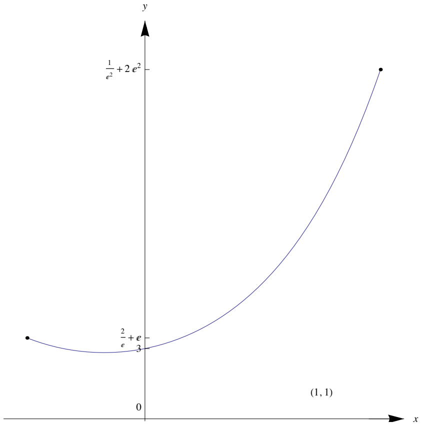


13



absolute maximum = 2
absolute minimum = −2

14 Absolute max = $\frac{1}{e^2} + 2e^2$;
Absolute min = $2\sqrt{2}$



15 $f(x) = 2e^{(x-1)^2}$
$f(-2) = 2e^9$ and $f(2) = 2e$
$f'(x) = 4(x-1)e^{(x-1)^2}$
$f'(x) = 0$ implies $x = 1$
$f(1) = 2$
Absolute max = $2e^9$; Absolute min = 2

16 Absolute max = $-\log_e 10$;
Absolute min = $-\frac{10}{e}$

Solutions to Exercise 10F

1 Let x m be the width of the rectangle

Let y m be the length of the rectangle

$2x + 2y = 100 \Rightarrow x + y = 50$

Area, $A = xy = x(50 - x) = 50x - x^2$

$$\frac{dA}{dx} = 50 - 2x$$

$$\frac{dA}{dx} = 0 \Rightarrow x = 25$$

$\therefore$ maximum area $= 25 \times 25 = 625$ m^2.

2

$$x = y = 4; x, y > 0;$$

$x^3 + y^2$ is a min.

$$\text{let } z = x^3 + y^2$$

$$y = 4 - x$$

$$z = x^3 + (4 - x)^2$$

$$= x^3 + 16 - 8x + x^2$$

$$= x^3 + x^2 - 8x + 16$$

$$\frac{dz}{dx} = 3x^2 + 2x - 8$$

$$\frac{dz}{dx} = 0,$$

$$3x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 96}}{6}$$

$$x = \frac{-2 \pm 10}{6}$$

$$x = -2, \frac{4}{3}$$

but $x > 0$,

$$\therefore \; x = \frac{4}{3}$$

$$y = 4 - x = \frac{8}{3}$$

3 $x + y = 100 \quad P = xy$

$$y = 100 - x$$

$$P = x(100 - x)$$

$$= 100x - x^2$$

$$\frac{dp}{dx} = 100 - 2x$$

$$\frac{dp}{dx} = 0,$$

$$x = 50$$

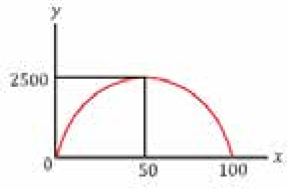
this gives max P

$$x = 50, y = 100 - 50 = 50 = x$$

QED

$$P = 50^2$$

$$P = 2500$$



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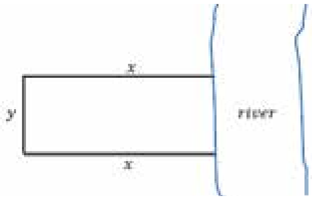


$$y + 2x = 4$$

$$y = 4 - 2x$$

$$A = xy$$

$$= x(4 - 2x)$$

$$= 4x - 2x^2$$

$$\frac{dA}{dx} = 4 - 4x$$

$$\frac{dA}{dx} = 0,$$

$x = 1$ km

the farmer should make one side 2 km long and the other two sides 1 km long, using 2 km of river.

5 $p, q > 0$

$$p^3 q = 9$$

$$\Rightarrow q = \frac{9}{p^3}$$

$$z = 16p + 3q$$

$$z = 16p + \frac{27}{p^3}$$

$$\frac{dz}{dx} = 16 - \frac{81}{p^4}$$

$$\frac{dz}{dx} = 0,$$

$$\frac{81}{p^4} = 16$$

$$\frac{3}{p} = 2 \textit{ since } p > 0$$

$$p = \frac{3}{2}$$

$$q = \frac{9}{p^3} = \frac{9}{\left(\frac{27}{8}\right)} = \frac{8}{3}$$

6 $SA = 150$, base has side $x (x >, not \geq 0)$

a $\text{SA} = 2x^2 + 4xh$

$$150 = 2x^2 + 4xh$$

$$h = \frac{75 - x^2}{2x} \quad QED$$

b $V = x^2 h$

$$= x^2 \left(\frac{75 - x^2}{2x}\right)$$

$$V = \frac{75x - x^3}{2}$$

c $\frac{dv}{dx} = \frac{75}{2} - \frac{3}{2}x^2$

$$\frac{dv}{dx} = 0,$$

$$x^2 = \frac{75}{2} \times \frac{2}{3}$$

$$= 25$$

$$x = 5\,\text{cm}$$

$$V = \frac{75 \times 5 - 5^2}{2} = \frac{375 - 125}{2} = 125\,\text{cm}^3$$

7 $P = 100n - 0.4n^2 - 160$

a **i** $\frac{dP}{dn} = 100 - 0.8n$

$\frac{dP}{dn} = 0$ implies $100 = 0.8n$

$\therefore n = 125$

A maximum occurs when $n = 125$ as P is a quadratic with negative coeffient of n^2.

ii When $n = 125$, $P = 100 \times 125 - 0.4 \times 125^2 - 160 = 6090$

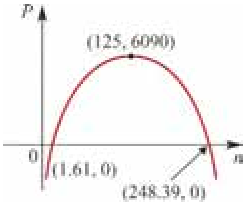
Maximum daily profit is \$ 6090.

b When $P = 0$,

$$n = -\frac{100 \pm \sqrt{100^2 - .4 \times 04 \times 160}}{-0.8}$$

$\therefore \; n \approx 1.6, 248.4$

In this problem a continuous model for a discrete situation has been used.



c $P > 0$ implies $2 \leq n \leq 248$ (Note: n can only take integer values)

d Let \$$P$ be the profit per article

$$\therefore\ p = \frac{P}{n}\left(= \frac{\text{total profit}}{\text{no. of articles}}\right)$$

$$= \frac{100n - 0.4n^2 - 160}{n}$$

$$= 100 - 0.4n - \frac{160}{n}$$

In order to find the maximum profit per article consider the derivative of p with resect to n.

$$\frac{dp}{dn} = -0.4 + \frac{160}{n^2}$$

$$\frac{dp}{dn} = 0 \text{ implies } 0.4 = \frac{160}{n^2}$$

$$\therefore\ n^2 = \frac{160}{0.4}$$

i.e. $n^2 = 400$

$$\therefore\ n = 20$$

The gradient chart indicates maximum:

	< 20	20	> 20
sign $f'(x)$	+ve	0	−ve
shape	/	−	\

i.e. selling 20 articles maximises the profit per article.

8

$$S(x) = -x^3 + 3x^2 + 360x + 5000,$$

$$x \in [6, 20]$$

$$S'(x) = -3x^2 + 6x + 360$$

$$S'(x) = 0,$$

$$x^2 - 2x - 120 = 0$$

$$(x + 10)(x - 12) = 0$$

$$x = -10, 12$$

but $x \in [6, 20]$

$$x = 12$$

values to test :

$$S(6), S(12), S(20)$$

$$S(6) = -216 + 108 + 2160 + 5000$$

$$= 2052 + 5000 = 7052$$

$$S(12) = -1728 + 432 + 4320 + 5000$$

$$= 3024 + 5000 = 8024$$

$$S(20) = -8000 + 1200 + 7200 + 5000$$

$$= 400 + 5000 = 5400$$

absolute maximum = 12°C

$S(12) = 8024$ salmon

9 $M(x) = \frac{-1}{30}(x^3 - 14x^2 + 32x - 50),$

$0 \le x \le 10$

$$M'(x) = \frac{-1}{30}(3x^2 - 28x + 32)$$

$$M'(x) = 0,$$

$$3x^2 - 28x + 32 = 0$$

$$x = \frac{28 \pm \sqrt{784 - 384}}{6}$$

$$x = \frac{28 \pm 20}{6}$$

$$x = \frac{4}{3}, 8$$

values to test:

$$x = 0, x = \frac{4}{3}, x = 8, x \to \infty$$

$$\begin{aligned}
\mathrm{M}(0) &= \frac{50}{30} = \frac{5}{3} \\
\mathrm{M}\left(\frac{4}{3}\right) &= \frac{1}{30}\left(50 - 30 \times \frac{4}{3} + 14 \times \frac{16}{9} - \frac{64}{27}\right) \\
&= \frac{1}{30}\left(\frac{1}{27}(1350 - 1152 + 672 - 64)\right) \\
&= \frac{806}{810} \\
\mathrm{M}\left(\frac{4}{3}\right) &= \frac{403}{405} \\
\mathrm{M}(8) &= \frac{1}{30}(50 - 32 \times 8 + 14 \\
&\quad \times 64 - 512) \\
&= \frac{178}{30} \\
&= \frac{89}{15}
\end{aligned}$$

.

Maximum M occurs at $x = 8$ mm

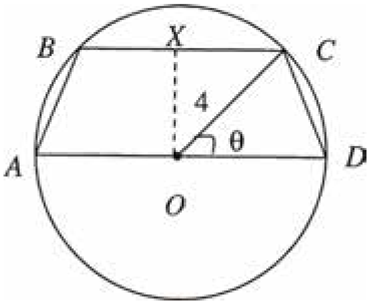
Minimum when $x = \frac{4}{3}$

10 **a** Let X be the midpoint of BC.

Angle $XCO = \theta$

Therefore $XC = 4\cos\theta$

$BC = 8\cos\theta$



b

$$\begin{aligned}
\text{Area} &= \frac{1}{2} \times (8 + 8\cos\theta) \times 4\sin\theta \\
&= 16\sin\theta(1 + \cos\theta) \\
A &= 16\sin\theta(1 + \cos\theta) \\
\frac{dA}{d\theta} &= 16[\cos\theta(1 + \cos\theta) - \sin^2\theta)] \\
&= 16[\cos^2\theta - \sin^2\theta + \cos\theta] \\
&= 16[\cos^2\theta - (1 - \cos^2\theta) + \cos\theta] \\
&= 16[2\cos^2\theta + \cos\theta - 1]
\end{aligned}$$

$\frac{dA}{d\theta} = 0$ implies

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

For the figure to exist $\cos\theta = \frac{1}{2}$

which implies $\theta = \frac{\pi}{3}$

Therefore maximum area

$$= 16\sin\frac{\pi}{3}\left(1 + \cos\frac{\pi}{3}\right)$$

$$= 16 \times \frac{\sqrt{3}}{2} \times \frac{3}{2} = 12\sqrt{3} \text{ square units}$$

11 $$\begin{aligned}
\textit{distance} &= \sqrt{(x-3)^2 + y^2} \\
&= \sqrt{x^2 - 6x + 9 + x^4}
\end{aligned}$$

want minimum distance

$$\begin{aligned}
\frac{d}{dx}(x^4 + x^2 - 6x + 9) &= 0 \\
4x^3 + 2x - 6 &= 0 \\
2x^3 + x - 3 &= 0
\end{aligned}$$

try $x = 1$

$2 + 1 - 3 = 0$ ✓

$$(x-1)(2x^2+2x+3)=0$$

$$x-1=0,\ 2x^2+2x+3=0$$

$$x=1 \quad x=\frac{-2\pm\sqrt{4-24}}{4}$$

no solution

$x = 1$ is the only solution

$y = 1$

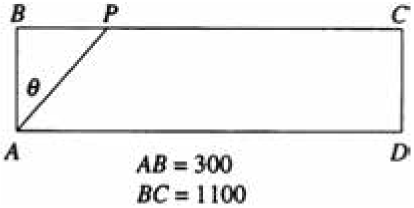
$\therefore$ (1, 1) is the point on $y = x^2$ closest to (3, 0)

12 **a** $\dfrac{AB}{AP} = \cos\theta$

$$\therefore AP = \frac{300}{\cos\theta}$$

$\therefore$ time taken to run from

$$A \text{ to } P = \frac{300}{\cos\theta}\times\frac{1}{4} = \frac{75}{\cos\theta}$$



b

$$PC = BC - BA\tan\theta$$

$$= 1100 - 300\tan\theta$$

$\therefore$ the time taken to run from P to C

$$= \frac{1100-300\tan\theta}{5}$$

$$= 220 - 60\tan\theta$$

c Let T denote the total time then T= time to run from A to P+ time taken to run from P to C

$$= \frac{75}{\cos\theta} + 220 - 60\tan\theta$$

$$= \frac{75}{\cos\theta} + 220 - 60\frac{\sin\theta}{\cos\theta}$$

$$= \frac{75-60\sin\theta}{\cos\theta} + 220$$

d The quotient rule gives

$$\frac{dT}{d\theta} = \frac{\cos\theta(-60\cos\theta)+\sin\theta(75-60\sin\theta)}{\cos^2\theta}$$

$$\left(\text{Note: } \frac{d}{d\theta}(220) = 0\right)$$

$$= \frac{-60\cos^2\theta + 75\sin\theta - 60\sin^2\theta}{\cos^2\theta}$$

$$= \frac{-60[\cos^2\theta+\sin^2\theta]+75\sin\theta}{\cos^2\theta}$$

$$= \frac{75\sin\theta - 60}{\cos^2\theta}$$

e

$$\frac{dT}{d\theta} = 0$$

implies $\dfrac{75\sin\theta-60}{\cos^2\theta} = 0$

$$\therefore \sin\theta = \frac{60}{75} = \frac{4}{5}$$

$$\therefore \theta = \sin^{-1}\left(\tfrac{4}{5}\right)$$

(Only the acute angle solution needs to be considered).

$\theta \approx 53.13°$

In order to confirm a minimum consider the following

When $\theta = 60°$,

$$\frac{dT}{d\theta} = \frac{75\sin 60° - 60}{\cos^2\theta} > 0$$

When $\theta = 50°$,

$$\frac{dT}{d\theta} = \frac{75\sin 50° - 60}{\cos^2\theta} < 0$$

$\therefore$ a minimum occurs when $\theta = \sin^{-1}\left(\dfrac{4}{5}\right)$

f

When $\sin\theta = \dfrac{4}{5}, T = \dfrac{75 - 60\times\frac{4}{5} + 220}{\frac{3}{5}}$

$$= 45 + 220$$

$$= 265$$

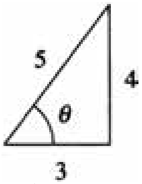
$\therefore$ minimum time taken is 265 seconds.

If $\sin\theta = \frac{4}{5}, \tan\theta = \frac{4}{5}$

$\therefore BP = BA\tan\theta$

$= 300 \times \frac{4}{5}$

$= 400$



P is 400 metres from *B* for a minimum time.

13

$N(t) = 50te^{-0.1t}$

$N'(t) = 50e^{-0.1t} - 5te^{-0.1t}$

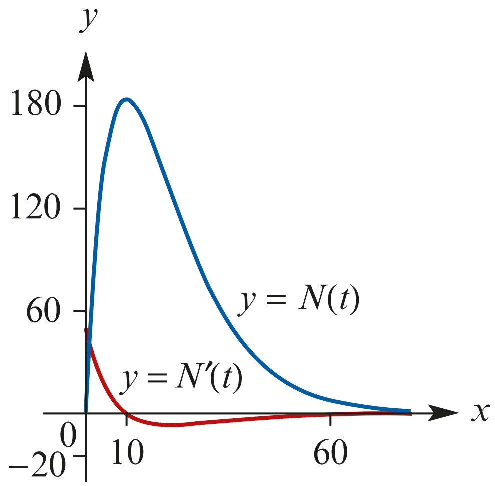
$= 5e^{-0.1t}(10 - t)$

$N'(t) = 0 \Rightarrow t = 10$

Therefore maximum population when $t = 10$

$N(10) = 500e^{-1}$

14 **a**



b Maximum rate of increase

$= N'(0) = 50$

$N''(t) = -5e^{-0.1x} - 0.5e^{-0.1x}(10 - x)$

$N''(t) = 0 \Rightarrow t = 20$

$N'(20) = -\frac{50}{e^2}$

Maximum rate of decrease

$= N'(20) = \frac{50}{e^2}$

15 $V(t) = \frac{3}{4}\left(10t^2 - \frac{t^3}{3}\right) 0 \le t \le 20$

a **i** $V(0) = 0$ The volume of water is 0 mL when $t = 0$

ii $V(20) = \frac{3}{4}\left(10 \times 20^2 - \frac{20^3}{3}\right)$

$= \frac{3 \times 400}{4}\left(10 - \frac{20}{3}\right)$

$= 3 \times 100\left(\frac{30 - 20}{3}\right)$

$= 1000$

The volume of water is 1000 mL when $t = 20$

b $V'(tf) = \frac{3}{4}\left(20t - \frac{3t^3}{3}\right)$

$= \frac{3}{4}(20t - t^2)$

c

Domain of $V(t) = [0, 20]$

$V(0) = 0$ and $V(20) = 1000$

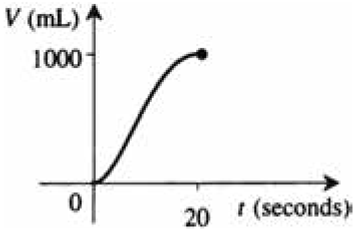
$V'(t) = 0$ implies $20t - t^2 = 0$

$\therefore\ t(20 - t) = 0$

$\therefore\ t = 0$ or $t = 20$

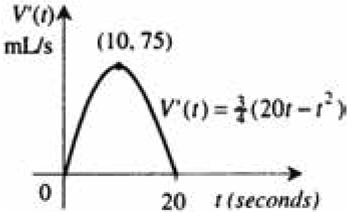
A gradient chart

t	<	0	<<	20	>
sign of $V'(t)$	$-ve$	0	$+ve$	0	$-ve$
shape	\	–	/	–	\



reveals a local minimum at (0,0) and a local minimum at (20,1000)

d The graph of $V'(t)$ against t is a parabola with t intercepts 0 and 20.



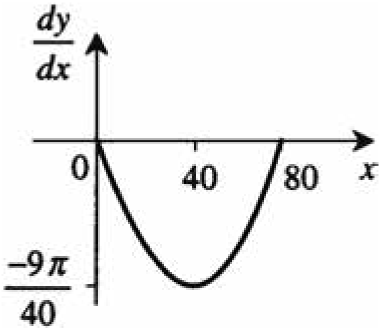
The maximum occurs when $t = 10$ and

$$V'(10) = \frac{3}{4}(200 - 100)$$
$$= 75$$

e The flow is greatest after 10 seconds and the flow is 75 mL/s.

16 a $$\frac{dy}{dx} = -\frac{18\pi}{80}\sin\left(\frac{\pi x}{80}\right) \quad x \in [0, 80]$$
$$= -\frac{9\pi}{40}\sin\left(\frac{\pi x}{80}\right)$$

b



For the graph of $\frac{dy}{dx}$ against x

amplitude $= \frac{18\pi}{80} = \frac{9\pi}{40}$

period $= 2\pi \div \frac{\pi}{80} = 2\pi \times \frac{80}{\pi} = 160$

When $x = 0$, $\frac{dy}{dx} = 0$

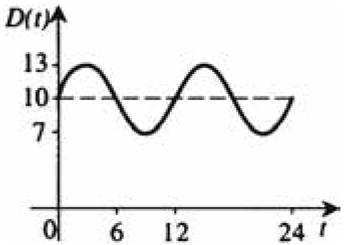
When $x = 80$, $\frac{dy}{dx} = \frac{18\pi}{80}\sin\left(\frac{\pi \times 80}{80}\right)$

$$= \frac{18\pi}{80}\sin\pi = 0$$

c Maximum gradient magnitude occurs where $\sin\left(\frac{\pi x}{80}\right) = \pm 1$

This occurs when $x = 40$ for $0 \le x \le 80$

17 a



The depth of the harbour at time t is given by

$D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right) 0 \le t \le 24$

amplitude = 3

period $= 2\pi \div \frac{\pi}{6} = 2\pi \times \frac{6}{\pi} = 12$

centre $D = 10$

range $= [10 - 3, 10 + 3] = [7, 13]$

b $D(t) \geq 8.5$

$\Leftrightarrow 10 + 3\sin\left(\frac{\pi t}{6}\right) \geq 8.5$

which is equivalent to

$$3\sin\left(\frac{\pi t}{6}\right) \geq -1.5$$

$$\sin\left(\frac{\pi t}{6}\right) \geq -\frac{1}{2}$$

Consider $\sin\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$

then $\frac{\pi t}{6} = \frac{7\pi}{6}$

or $\frac{11\pi}{6}$ or $\frac{19\pi}{6}$ or $\frac{23\pi}{6}$ or $\cdots$

$t = 7$ or 11 or 19 or 23 or . . .

From the graph and considering the domain $[0, 24]$

$\{t : D(t) \geq 8.5\} = [0, 7] \cup [11, 19] \cup [23, 24]$

c The rate of change of depth is given by the derivative function

$D'(t) = \frac{3\pi}{6}\cos\left(\frac{\pi t}{6}\right) = \frac{\pi}{2}\cos\left(\frac{\pi t}{6}\right)$

i $D'(t) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$

The rate at which the depth is changing when $t = 3$ is 0 metres/hour.

ii $D'(6) = \frac{\pi}{2}\cos\left(\frac{6\pi}{6}\right) = \frac{\pi}{2}\cos(\pi) = -\frac{\pi}{2}$

The rate at which the depth is changing when $t = 3$ is $-\frac{\pi}{2}$ metres/hours.

(This means that the depth is decreasing at a rate of $\frac{\pi}{2}$ metres/hour).

iii $D'(12) = \frac{\pi}{2}\cos(2\pi) = \frac{\pi}{2}$

The depth is increasing at a rate of $\frac{\pi}{2}$ metres/hours

d The function which describes the rate is

$D'(t) = \frac{\pi}{2}\cos\left(\frac{\pi t}{6}\right)$

i $D'(t)$ has a maximum when

$\cos\left(\frac{\pi t}{6}\right) = 1$

$\therefore \frac{\pi t}{6} = 0$ or 2π or 4π or $\cdots$

$t = 0$ or 12 or 24 or $\cdots$

For the required domain the depth is increasing most rapidly when $t = 0$ or $t = 12$ or $t = 24$

ii The depth is decreasing most rapidly when

$\cos\left(\frac{\pi t}{6}\right) = -1$

$\therefore$ when $\frac{\pi t}{6} = \pi$ or 3π or 5π or $\cdots$

$\therefore t = 6$ or 18 or 30 or$\cdots$

For the required domain the depth is decreasing most rapidly when $t = 6$ or 18

Solutions to Exercise 10G

1 $f(x) = (x-1)^2(x-b)\ ,\ b > 1$

a $$\begin{aligned} f(x) &= (x^2-2x+1)(x-b) \\ &= x^3-(2+b)x^2+(1+2b)x-b \\ f'(x) &= 3x^2-2(2+b)x+1+2b \\ &= (x-1)(3x-2b-1) \end{aligned}$$

b $$\begin{aligned} f'(x) &= 0, \\ (x-1)(3x-(1+2b)) &= 0 \\ x = 1, x &= \frac{1+2b}{3} \\ f(1) &= 1-2-b+1+2b-b = 0 \\ f\left(\frac{1+2b}{3}\right) &= \left(\frac{2b-2}{3}\right)^2\left(\frac{2b+1-3b}{3}\right) \\ &= \frac{4}{9}(b-1)^2\left(\frac{1-b}{3}\right) \\ &= \frac{-4}{27}(b-1)^3 \end{aligned}$$

$$\text{co-ords} = (1,0)\ \&\ \left(\frac{1+2b}{3}, \frac{-4(b-1)^3}{27}\right)$$

c $\frac{2b+1}{3} > 1$ as $b > 1$ so the other stationary point is a local minimum; hence the point (1,0) is always a local maximum.

d $$\begin{aligned} \frac{1+2b}{3} &= 4 \\ 1+2b &= 12 \\ b &= \frac{11}{2} \end{aligned}$$

2 $y = x^4 - 4x^2$

a $$\begin{aligned} \frac{dy}{dx} &= 4x^3-8x \\ \frac{dy}{dx} &= 0, \\ 4x(x^2-2) &= 0 \\ x &= 0, \pm\sqrt{2} \\ x &= 0, y = 0 \\ &(0,0) \\ x &= \pm\sqrt{2} \\ y &= 4-4(2) \\ &= -4 \\ &(\pm\sqrt{2}, -4) \end{aligned}$$

b $$\begin{aligned} &(x,y) \to (x+a, y+b) \\ &(0,0) \to (a,b) \\ &(\pm\sqrt{2}, -4) \to (a \pm \sqrt{2}, b-4) \end{aligned}$$

3 **a**

$$\begin{aligned} f(x) &= ax^3+bx^2+cx \\ f(1) = 10 &\Rightarrow a+b+c = 10\ldots(1) \\ f'(x) &= 3ax^2+2bx+c \\ f'(1) = 0 &\Rightarrow 3a+2b+c = 0\ldots(2) \end{aligned}$$

Multiply (1) by 2 and subtract from (2)

$a - c = -20 \Rightarrow a = c - 20$

Substitute for a in (1)

$c - 20 + b + c = 10$

$\therefore b = 30 - 2c$

b

$$f'(3) = 0 \Rightarrow 27a + 6b + c = 0$$

$$\therefore 27(c - 20) + 6(30 - 2c) + c = 0$$

$$27c - 540 + 180 - 12c + c = 0$$

$$16c - 360 = 0$$

$$c = \frac{360}{16} = \frac{45}{2}$$

4 $f : [0, \infty] \to R, f(x) = x^2 - ax^3$

$a > 0$

a $f'(x) = 2x - 3ax^2$

$= x(2 - 3ax)$

$f'(x) < 0,$

$x(2 - 3ax) < 0$

since $x \geq 0, 2 - 3ax < 0$

$\Rightarrow x > \frac{2}{3a}$

The end point is also to be included since $f(\frac{2}{3a}) > f(x)$ for any $x > \frac{2}{3a}$

$f(x)$ is strictly decreasing when $x \geq \frac{2}{3a}$

$f'(x) > 0,$

$x(2 - 3ax) > 0$

since $x \geq 0, 2 - 3ax > 0$

$\Rightarrow x < \frac{2}{3a}$

$0 < x < \frac{2}{3a}$

The end points are included since $f(0) < f(x) < \frac{2}{3a}$ for all $0 < x < \frac{2}{3a}$

$f(x)$ is strictly increasing when $0 \leq x \leq \frac{2}{3a}$

b $f'\left(\frac{1}{a}\right) = \frac{1}{a}\left(2 - 3\frac{a}{a}\right)$

$= \frac{-1}{a}$

$y = \frac{-x}{a} + c$

$x = \frac{1}{a}, y = 0$

$0 = \frac{-1}{a^2} + c$

$c = \frac{1}{a^2}$

$y = \frac{-x}{a} + \frac{1}{a^2}$

c $y = ax + c$

$x = \frac{1}{a}, y = 0$

$0 = 1 + c$

$c = -1$

$y = ax - 1$

d $max.$ at $x = \frac{2}{3a}$

$f\left(\frac{2}{3a}\right) = \frac{4}{9a^2} - \frac{a \times 8}{27a^3} \quad = \frac{1}{a^2}\left(\frac{4}{9} - \frac{8}{27}\right)$

$= \frac{4}{27a^2}$

range $= \left(-\infty, \frac{4}{27a^2}\right]$

5 a i $y = (x - 3)^2$

$\frac{dy}{dx} = 2(x - 3)$

$= 2x - 6$

$x = a,$

$\frac{dy}{dx} = 2a - 6$

ii $m = 2a - 6$

b $x = a$

$$\begin{aligned} y &= (a-3)^2 \\ &= a^2 - 6a + 9 \\ &\Rightarrow (a, a^2 - 6a + 9) \end{aligned}$$

c

$$\begin{aligned} y &= mx + c \\ &= (2a-6)x + c \\ x &= a, y = (a-3)^2 \\ (a-3)^2 &= 2a(a-3) + c \\ c &= (a-3)(a-3-2a) \\ c &= (a-3)(-a-3) \\ y &= (a-3)(2x-a-3) \\ &= 2(a-3)x - a^2 + 9 \end{aligned}$$

d

$$\begin{aligned} y &= 0, \\ 2x - a - 3 &= 0 \\ 2x &= a + 3 \\ x &= \frac{a+3}{2} \end{aligned}$$

6 a

$$\begin{aligned} f(x) &= x^4 \\ \Rightarrow f(x+h) &= (x+h)^4 \\ f(1+h) &= 16 \\ (1+h)^4 &= 16 \\ 1 + h &= \pm 2 \\ h &= -1 \pm 2 \\ h &= -3, 1 \end{aligned}$$

b

$$\begin{aligned} f(x) &= x^3 \\ \Rightarrow f(ax) &= (ax)^3 \\ f(a) &= 8 \\ a^3 &= 8 \\ a &= 2 \end{aligned}$$

c

$$\begin{aligned} y &= ax^4 - bx^3 \\ \frac{dy}{dx} &= 4ax^3 - 3bx^2 \\ x &= 1, \\ \frac{dy}{dx} &= 0 \\ 4a &= 3b \ldots (1) \\ x &= 1, y = 16 \\ 16 &= a - b \\ a &= 16 + b \ldots (2) \end{aligned}$$

sub in (1) $\Rightarrow 4(16 + b) = 3b$

$$\begin{aligned} 64 + 4b &= 3b \\ b &= -64 \end{aligned}$$

sub in (2) $\Rightarrow a = -48$

7 a

$$\begin{aligned} f(x) &= (x-a)^2(x-1) \\ &= (x^2 - 2ax + a^2)(x-1) \\ &= x^3 - (2a+1)x^2 + (a^2+2a)x - a^2 \\ f'(x) &= 3x^2 - (4a+2)x + (a^2+2a) \\ f'(x) &= 0, \\ 3x^2 - (4a+2)x + (a^2+2a) &= 0 \\ x &= \frac{4a + 2 \pm \sqrt{4(2a+1)^2 - 4(3a^2+6a)}}{6} \\ &= \frac{2a + 1 \pm \sqrt{4a^2 + 4a + 1 - 3a^2 - 6a}}{3} \\ &= \frac{2a + 1 \pm \sqrt{a^2 - 2a + 1}}{3} \\ &= \frac{(2a+1) \pm (a-1)}{3} \\ x &= a, \frac{a+2}{3} \\ f(a) &= 0, \end{aligned}$$

$$f\left(\frac{a+2}{3}\right) = \left(\frac{a+2}{3} - \frac{3a}{a}\right)^2\left(\frac{a+2}{3} - \frac{3}{3}\right)$$

$$= \left(\frac{2-2a}{3}\right)^2\left(\frac{a-1}{3}\right)$$

$$= \frac{4}{9}(1-a)^2\frac{-1}{3}(1-a)$$

$$= \frac{-4(1-a)^3}{27}$$

co-ords = $(a, 0), \left(\frac{a+2}{3}, \frac{-4(1-a)^3}{27}\right)$

b Since $a > 1, \frac{-4(1-a)^3}{27} > 0$,
Hence $(a, 0)$ is a local minimum and $\left(\frac{a+2}{3}, \frac{-4(1-a)^3}{27}\right)$ is a local maximum.

c **i** $f'(1) = 3 - 4a - 2 + a^2 + 2a$

$$= a^2 - 2a + 1$$

$$= (a-1)^2$$

$$y = (a-1)^2x + c$$

$f'(1) = 0$,

$$0 = (a-1)^2 + c$$

$$c = -(a-1)^2$$

$$y = (a-1)^2(x-1)$$

ii $f'(a) = 0$

$$y = c$$

$$f(a) = 0 = c$$

$$y = 0$$

iii $f'\left(\frac{a+1}{2}\right)$

$$= \frac{3(a+1)^2}{4} - (2a+1)(a+1) + (a^2 + 2a)$$

$$= \frac{\begin{gathered}3a^2 + 6a + 3 - 8a^2\\ -12a - 4 + 4a^2 + 8a\end{gathered}}{4}$$

$$= \frac{-a^2 + 2a - 1}{4}$$

$$= \frac{-(a-1)^2}{4}$$

$$f\left(\frac{a+1}{2}\right) = \left(\frac{a+1}{2} - a\right)^2\left(\frac{a+1}{2} - 1\right)$$

$$= \frac{1}{4}(1-a)^2\frac{1}{2}(a-1)$$

$$= \frac{1}{8}(a-1)^3$$

$$y = \frac{-(a-1)^2}{4}x + c$$

$$x = \frac{a+1}{2}, y = \frac{1}{8}(a-1)^3$$

$$\frac{1}{8}(a-1)^3 = \frac{-1}{4}(a-1)^2\left(\frac{a+1}{2}\right) + c$$
$$c = \frac{1}{4}(a-1)^2\left(\frac{a-1}{2} + \frac{a+1}{2}\right)$$
$$= \frac{1}{4}(a-1)^2$$
$$y = \frac{1}{4}(a-1)^2(-x+a)$$
$$= \frac{-1}{4}(a-1)^2(x-a)$$

8 **a** $f'(x) = (x-1)^2\frac{d}{dx}(x-b)^2 + (x-b)^2\frac{d}{dx}(x-1)^2$
$$= 2(x-1)(x-b)((x-1)+(x-b))$$
$$= 2(x-1)(x-b)(2x-b-1)$$

b $f'(x) = 0,$
$$x = 1, b, \frac{b+1}{2}$$
$$f(1) = 0, f(b) = 0,$$
$$f\left(\frac{b+1}{2}\right) = \left(\frac{b-1}{2}\right)^2\left(\frac{1-b}{2}\right)^2$$
$$= \left(\frac{b-1}{2}\right)^4$$
$$\text{co-ords} = (1,0)\,(b,0)\left(\frac{b+1}{2}, \frac{(b-1)^4}{16}\right)$$

c $\frac{b+1}{2} = 2$
$$b = 1 = 4$$
$$b = 3$$

9 $a = \frac{1}{486},\ b = 0,\ c = \frac{-1}{161},\ d = \frac{1459}{243}$

10 $f(x) = ax^4 + bx^3 + cx^2 + dx$

a $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f(1) = 1$

$1 = a + b + c + d \ldots (1)$

$f'(1) = 0$

$0 = 4a + 3b + 2c + d \ldots (2)$

$f'(-1) = 4$

$4 = a - b + c - d \ldots (3)$

$(1) - (3) \Rightarrow \quad -3 = 2b + 2d$

$$b = \frac{-3}{2} - d$$

$(2) - (1) - (3) \Rightarrow \ -5 = 2a + 3b + d$

$$-5 = 2a - \frac{9}{2} - 3d + d$$

$$-\frac{1}{2} = 2a - 2d$$

$$a = d - \frac{1}{4}$$

sub in (1) $\Rightarrow \quad 1 = \left(d - \frac{1}{4}\right) + \left(\frac{-3}{2} - d\right)c + d$

$$1 = -\frac{7}{4} + c + d$$

$$c = \frac{11}{4} - d$$

b $f'(4) = 0$

$$0 = 4a(64) + 3b(16) + 2c(4) + d$$

$$0 = 256a + 48b + 8c + d$$

$$0 = 256\left(\frac{4d - 1}{4}\right) + 48\left(\frac{-3 - 2d}{2}\right) + 8\left(\frac{11 - 4d}{4}\right) + d$$

$$0 = 256d - 64 - 72 - 48d + 22 - 8d + d$$

$$0 = 201d - 114$$

$$d = \frac{114}{201}$$

$$d = \frac{38}{67}$$

Solutions to Exercise 10H

1 **a** $x_0 = 2$
$x_1 = 2.166\ldots$
$x_2 = 2.1510\ldots$
$x3 = 2.1509\ldots$
$x_4 = 2.1509\ldots$
$x = 2.151$ correct to 3 decimal places.

b $x_0 = -2$
$x_1 = -1.8$
$x_2 = -1.75\ldots$
$x_3 = -1.747\ldots$
$x_4 = -1.74767\ldots$
$x = -1.75$ correct to 2 decimal places.

c $x_0 = -2$
$x_1 = -1.8$
$x_2 = -1.75\ldots$
$x_3 = -1.747\ldots$
$x_4 = -1.74767\ldots$
$x = -1.75$ correct to 2 decimal places.

d $x_0 = 2$
$x_1 = 2.642\ldots$
$x_2 = 2.555\ldots$
$x_3 = 2.55419\ldots$
$x_4 = 2.55419\ldots$
$x = 2.554$ correct to 3 decimal places.

e $x_0 = 2$
$x_1 = 1.6266\ldots$
$x_2 = 1.5662\ldots$
$x_3 = 1.5644\ldots$
$x_4 = 1.56446\ldots$
$x = 1.564$ correct to 3 decimal places.

2 $f(x) = x^3 - 3$

$f'(x) = 3x^2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$\therefore x_{n+1} = x_n - \frac{x_n^3 - 3}{3x_n^2}$$

$$= \frac{3x_n^3 - (x_n^3 - 3)}{3x_n^2}$$

$$= \frac{2x_n^3 + 3}{3x_n^2}$$

$x_0 = 2$
$x_1 = 1.5833\ldots$
$x_2 = 1.4544\ldots$
$x_3 = 1.4423\ldots$
$x_4 = 1.44224\ldots$
$x_5 = 1.442249\ldots$

3 $f(x) = x^3 - 2x - 1$

$f'(x) = 3x^2 - 2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$\therefore x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 1}{3x_n^2 - 2}$$

$$= \frac{3x_n^3 - 2x_n - (x_n^3 - 2x_n - 1)}{3x_n^2 - 2}$$

$$= \frac{2x_n^3 + 1}{3x_n^2 - 2}$$

$x_0 = 2$
$x_1 = 1.7$
$x_2 = 1.6230\ldots$
$x_3 = 1.61805\ldots$
$x_4 = 1.61803\ldots$
$x_5 = 1.61803\ldots$

4 $f(x) = x^4 - 2x^3 + 1$

$$f'(x) = 4x^3 - 6x^2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$\therefore x_{n+1} = x_n - \frac{x_n^4 - 2x_n^3 + 1}{4x_n^3 - 6x_n^2}$$

$$= \frac{4x_n^4 - 6x_n^3 - (x_n^4 - 2x_n^3 + 1))}{4x_n^3 - 6x_n^2}$$

$$= \frac{3x_n^4 - 4x_n^3 - 1}{4x_n^3 - 6x_n^2}$$

5 $f(x) = x^5 - 158$

$$f'(x) = 5x^4$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$\therefore x_{n+1} = x_n - \frac{x_n^5 - 158}{5x_n^4}$$

$$= \frac{5x_n^5 - (x_n^5 - 158))}{5x_n^4}$$

$$= \frac{4x_n^5 + 158}{5x_n^4}$$

$x_0 = 3$
$x_1 = 2.7901\ldots$
$x_2 = 2.75352\ldots$
$x_3 = 2.75252\ldots$
$x_4 = 2.752525920389\ldots$
$x_5 = 2.752525920388\ldots$

6 a $x_0 = 0.6$
$x_1 = 0.6355088\ldots$
$x_2 = 0.6412015\ldots$

b $f(x) = x^2 + e^{-\frac{1}{2}x} - 7$

$$f'(x) = 2x - \frac{1}{2}e^{-\frac{1}{2}x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$\therefore x_{n+1} = x_n - \frac{x_n^2 + e^{-\frac{1}{2}x_n} - 7}{2x - \frac{1}{2}e^{-\frac{1}{2}x_n}}$$

$$\therefore x_1 = x_0 - \frac{x_0^2 + e^{-\frac{1}{2}x_0} - 7}{2x_0 - \frac{1}{2}e^{-\frac{1}{2}x_0}}$$

$$= -2 - \frac{4 + e - 7}{-4 - \frac{1}{2}e}$$

$$= -\frac{22}{e + 8}$$

7 a $f(x) = \log_e x - \frac{x}{4}$

$$f'(x) = \frac{1}{x} - \frac{1}{4}$$

$$f'(x) \geq 0$$

$$\frac{1}{x} - \frac{1}{4} \geq 0$$

$$x \leq 4$$

Therefore $x \in (0, 4)$

b

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$\therefore x_{n+1} = x_n - \frac{\log_e x_n - \frac{x_n}{4}}{\frac{1}{x_n} - \frac{1}{4}}$$

$$= x_n - \frac{4x_n\left(\log_e x_n - \frac{x_n}{4}\right)}{4 - x_n}$$

$$= \frac{4x_n - x_n^2 - 4x_n\left(\log_e x_n - \frac{x_n}{4}\right)}{4 - x_n}$$

$$= \frac{4x_n(1 - \log_e x_n)}{4 - x_n}$$

c

$$f(e) = \log_e e - \frac{e}{4} \quad = 1 - \frac{e}{4}$$

$$f'(e) = \frac{1}{e} - \frac{1}{4}$$

$$y - \left(1 - \frac{e}{4}\right) = \left(\frac{1}{e} - \frac{1}{4}\right)(x - e)$$

$$y - \left(\frac{4-e}{4}\right) = \left(\frac{4-e}{4e}\right)(x - e)$$

$$y - \left(\frac{4-e}{4}\right) = \left(\frac{4-e}{4e}\right)x - \left(\frac{4-e}{4}\right)$$

$$y = \left(\frac{4-e}{4e}\right)x$$

d At $x = a$ the equation of the tangent is:

$$y - \left(\frac{4\log_e(a) - a}{4}\right) = \left(\frac{4-a}{4a}\right)(x - a)$$

When $y = 0$,

$$x - a = -\left(\frac{4\log_e(a) - a}{4}\right) \times \left(\frac{4a}{4-a}\right)$$

$$x = \left(\frac{a - 4\log_e(a)}{1}\right) \times \left(\frac{a}{4-a}\right) + a$$

$$x = \left(\frac{a^2 - 4a\log_e(a) + 4a - a^2}{4-a}\right)$$

$$x = \frac{4a(1 - \log_e a)}{4 - a}$$

If $a \in (0, 4)$ and $a < e$ then $\log_e a < 1$ and $x > 0$

e

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0}$$

$$= 1 + \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= 1 + \frac{1}{3}$$

$$= \frac{4}{3}$$

f

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0}$$

$$= 6 - \frac{\log_e 6 - \frac{3}{2}}{-\frac{1}{12}}$$

$$= 6 - 12(\log_e 6 - \frac{3}{2})$$

$$= 6 - 12\log_e 6 - 18$$

$$= 12\log_e 6 - 12$$

g **Solution 1** $x_0 = 1$

$x_1 = 1.33333\ldots$

$x_2 = 1.42463\ldots$

$x_3 = 1.42959\ldots$

$x_4 = 1.42961\ldots$

Solution 2

$x_0 = 6$

$x_1 = 9.5011\ldots$

$x_2 = 8.6453\ldots$

$x_3 = 8.6132\ldots$

$x_4 = 8.61316\ldots$

Solutions to technology-free questions

1 **a** $y = x^3 - 8x^2 + 15x$

$$\frac{dy}{dx} = 3x^2 - 16x + 15$$

$$= -1 \text{ (at } x = 4)$$

For the tangent:

$$y + 4 = -1(x - 4)$$

$$y = -x$$

b Tangent meets curve again when

$$x^3 - 8x^2 + 15x = -x$$

$$x^3 - 8x^2 + 16x = 0$$

$$x(x^2 - 8x + 16) = 0$$

$$x(x - 4)^2 = 0$$

Thus $x = 0$ and then $y = 0$ ($x = 4$ corresponds to the given point).

The tangent meets the curve again at the point $(0, 0)$.

2 At $x = a, y = 3a^2$

$$y = 3x^2$$

$$\frac{dy}{dx} = 6x$$

$$= 6a \text{ (at } x = a)$$

For the tangent:

$$y - 3a^2 = 6a(x - a)$$

$$y = 6ax - 3a^2$$

$x = 0 : y = -3a^2$, so the tangent meets the y axis where $y = -3a^2$.

3 $y = x^3 - 7x^2 + 14x - 8$

$$\frac{dy}{dx} = 3x^2 - 14x + 14$$

$$= 3 \text{ (at } x = 1)$$

For the tangent, $x = 1$ gives $y = 0$, so:

$$y - 0 = 3(x - 1)$$

$$y = 3x - 3$$

A parallel tangent has gradient 3, so:

$$\frac{dy}{dx} = 3$$

$$3x^2 - 14x + 14 = 3$$

$$3x^2 - 14x + 11 = 0$$

$$(3x - 11)(x - 1) = 0$$

$$x = 1, \frac{11}{3}$$

The x coordinate of a second point with the same gradient is $x = \frac{11}{3}$.

4 **a** Average rate is given by

$$\frac{A(3) - A(2)}{3 - 2} = \frac{9\pi - 4\pi}{1} = 5\pi$$

b $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$= 6\pi \text{ (at } r = 3)$$

Instantaneous rate is 6π

5 **a** $f(x) = 4x^3 - 3x^4$

$$f'(x) = 12x^2 - 12x^3$$

$$= 12x^2(1 - x)$$

$$= 0 \text{ if } x = 0, 1$$

$$x = 0, y = 0; x = 1, y = 1$$

The stationary points have coordinates $(0, 0)$ and $(1, 1)$.

$x < 0, f'(x) > 0; f'(0) = 0;$

$0 < x < 1, f'(x) > 0$; so $(0, 0)$ is a stationary point of inflexion.

$0 < x < 1, f'(x) > 0; f'(1) = 0;$

$x > 1, f'(x) < 0$; so $(1, 1)$ is a maximum.

b $g(x) = x^3 - 3x - 2$

$$\begin{aligned} g'(x) &= 3x^2 - 3 \\ &= 3(x+1)(x-1) \\ &= 0 \text{ if } x = -1, 1 \end{aligned}$$

$x = -1, y = 0; x = 1, y = -4$

The stationary points have coordinates $(-1, 0)$ and $(1, -4)$.

$x < -1, f'(x) > 0; f'(-1) = 0;$
$-1 < x < 1, f'(x) < 0;$ so $(-1, 0)$ is a maximum.

$-1 < x < 1, f'(x) < 0; f'(1) = 0;$
$x > 1, f'(x) > 0;$ so $(1, -4)$ is a minimum.

c $h(x) = x^3 - 9x + 1$

$$\begin{aligned} g'(x) &= 3x^2 - 9 \\ &= 3(x^2 - 3) \\ &= 3(x + \sqrt{3})(x - \sqrt{3}) \\ &= 0 \text{ if } x = -\sqrt{3}, \sqrt{3} \end{aligned}$$

$x = -\sqrt{3}, y = 6\sqrt{3} + 1$

$x = \sqrt{3}, y = -6\sqrt{3} + 1$

The stationary points have coordinates $(-\sqrt{3}, 6\sqrt{3} + 1)$ and $(\sqrt{3}, -6\sqrt{3} + 1)$.

$x < -\sqrt{3}, f'(x) > 0; f'(-\sqrt{3}) = 0;$

$-\sqrt{3} < x < \sqrt{3}, f'(x) < 0;$

so there is a maximum at

$(-\sqrt{3}, 6\sqrt{3} + 1)$.

$-\sqrt{3} < x < \sqrt{3}, f'(x) < 0;$

$f'(\sqrt{3}) = 0;$

$x > \sqrt{3}, f'(x) > 0;$

so there is a minimum at

$(\sqrt{3}, -6\sqrt{3} + 1)$.

6 $y = x^3 - 6x^2 + 9x = x(x-3)^2$

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \\ &= 3(x-1)(x-3) \\ &= 0 \text{ if } x = 1, 3 \end{aligned}$$

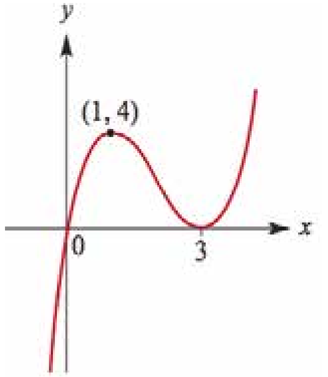
$x = 1, y = 4; x = 3, y = 0$

The stationary points have coordinates $(1, 4)$ and $(3, 0)$. Also it is evident from the factorised form that $(3, 0)$ is a stationary point of inflexion. Then $(1, 4)$ must be a maximum.

For the intercepts:

$y = 0, x = 0, 3$

The graph is shown below.



7 $\frac{dy}{dx} = (x-1)^2(x-2)$

$= 0 \text{ if } x = 1, 2$

There are stationary points where $x = 1$ and $x = 2$.

$x < 1, \frac{dy}{dx} < 0; \frac{dy}{dx} = 0$ at $x = 1;$

$1 < x < 2, \frac{dy}{dx} < 0;$

so there is a stationary point of inflexion at $x = 1$.

$1 < x < 2, \frac{dy}{dx} < 0; \frac{dy}{dx} = 0$ at $x = 2;$

$x > 2, \frac{dy}{dx} > 0;$ so there is a minimum at $x = 2$.

8 $y = x^3 - 3x^2 - 9x + 11$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

$$= -9 \text{ (at } x = 2)$$

Also when $x = 2, y = -11$

For the tangent:

$$y + 11 = -9(x - 2)$$

$$y = -9x + 7$$

9 $f(x) = (x - 1)^{\frac{4}{5}}$

a The function is differentiable for $R\backslash\{1\}$. $f'(x) = \frac{4}{5}(x - 1)^{-\frac{1}{5}}$

b $f'(0) = \frac{4}{5}, f'(2) = -\frac{4}{5}$

For the tangent at (2, 1):

$$y - 1 = \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x - \frac{3}{5}$$

For the tangent at (0, 1):

$$y - 1 = -\frac{4}{5}(x - 0)$$

$$y = -\frac{4}{5}x + 1$$

c $\frac{4}{5}x - \frac{3}{5} = -\frac{4}{5}x + 1$

$$\frac{8}{5}x = \frac{8}{5}$$

$$x = 1$$

When $x = 1, y = \frac{1}{5}$, so $\left(1, \frac{1}{5}\right)$ is the point of intersection of the tangents.

10 For a sphere of radius r and volume V, $V = \frac{4}{3}\pi r^3$.

a $\frac{dV}{dr} = 4\pi r^2$

$$= 64\pi \text{ if } r = 4$$

The rate of increase of volume with respect to the change in radius is 64π cm^3/cm when the radius is 4 cm.

b $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$$= 4\pi r^2 \times 1$$

$$= 4\pi r^2$$

$$= 64\pi \text{ if } r = 4$$

The rate of increase of volume with respect to time is 64π cm^3/s when the radius is 4 cm.

(An alterative is to use the initial conditions to express r in terms of t, i.e. $r = 1 + t$, and then V in terms of t; then differentiate to get the result directly.)

11 $\theta = \frac{1}{4}e^{100t}$

a $\frac{d\theta}{dt} = 25e^{100t}$ °C/s

b $\frac{d\theta}{dt} = 25e^5 \left(\text{at } t = \frac{1}{20}\right)$

So the rate of increase is $25e^5$ °C/s.

12 $y = e^x$

$$\frac{dy}{dx} = e^x$$

$$= e \text{ (at } x = 1) \; y - e \quad = e(x - 1)$$

$$y - e = ex - e$$

$$y = ex$$

13 $D = 50e^{kt}$

a $\dfrac{dD}{dt} = 50ke^{kt}$

$= k \times 50e^{kt} = kD$

Thus $\dfrac{dD}{dt} = cD$, where $c = k$.

b $\dfrac{dD}{dt} = kD = 0.2 \times 100 = 20$ cm/year

14 $y = e^{3x} + e^{-3x}$

$\dfrac{dy}{dx} = 3e^{3x} - 3e^{-3x}$

$= 0$ if

$3e^{3x} = 3e^{-3x}$

$e^{6x} = 1$

$x = 0$

When $x = 0, y = 2$.

Since $y \to \infty$ when $x \to \pm\infty$, it is evident that $y = 2$ is a minimum.

15 **a** Let the equation of the line of the third side be $y = mx + c$

The point (1, 1) is on the line.

Therefore,

$1 = m + c \Rightarrow c = 1 - m$

The equation of the line is,

$y = mx + (1 - m)$

When $x = 0, y = 1 - m$.

When $y = 0, x = \dfrac{m-1}{m}$

Intersection of $y = 3x$ **with** $y = mx + (1 - m)$**.**

$mx + (1 - m) = 3x$

$(m - 3)x = m - 1$

$x = \dfrac{m-1}{m-3}$ and $y = \dfrac{3m-3}{m-3}$

Area of triangle$= \dfrac{1}{2} \times \dfrac{m-1}{m} \times \dfrac{3m-3}{m-3}$

$= \dfrac{3(m-1)^2}{2m(m-3)}$

b Let $A = \dfrac{3(m-1)^2}{2m(m-3)}$

$\dfrac{dA}{dm} = \dfrac{-3(m-1)(m+3)}{2m^2(m-3)^2}$

If $m = 1$ the line is $y = mx$ and it intersects with $y = 3x$ at the origin. No triangle formed, If $m = -3$, $\dfrac{dA}{dm} < 0$ for $m < -3$ and $\dfrac{dA}{dm} > 0$ for $m > -3$.

Therefore local minimum.

16 **a** $f'(x) = 1 - \dfrac{4}{\sqrt{x}}$

$f'(x) = 0 \Rightarrow x = 16$

$f(16) = 16 - 32 = -16$

Local minimum at $X(16, -16)$

b $f(x) = 0 \Rightarrow \sqrt{x}(\sqrt{x} - 8) = 0$

Therefore, $x = 64$

c **i** $f'\left(\dfrac{64}{9}\right) = 1 - \dfrac{4}{\sqrt{\dfrac{64}{9}}}$

$= 1 - \dfrac{3}{2} = -\dfrac{1}{2}$

Equation of tangent

$y + \dfrac{128}{9} = -\dfrac{1}{2}(x - \dfrac{64}{9})$

$y = -\dfrac{1}{2}x - \dfrac{32}{3}$

ii $f'(64) = 1 - \dfrac{4}{\sqrt{64}}$

$= 1 - \dfrac{1}{2} = \dfrac{1}{2}$

Equation of tangent

$y = \dfrac{1}{2}(x - 64)$

$y = \dfrac{1}{2}x - 32$

d $\frac{1}{2}x - 32 = -\frac{1}{2}x - \frac{32}{3}$

$$x = \frac{64}{3}$$

$$\therefore y = \frac{1}{2} \times \frac{64}{3} - 32$$

$$= \frac{64}{3}$$

Coordinates $P\left(\frac{64}{3}, -\frac{64}{3}\right)$

e Three points lie on the line $y = -x$.Use similar triangles to see $OP = 4OX$. Therefore $OP : OX = 3 : 1$

17 a $y = \log_e x$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$= \frac{1}{e} \text{ (at } x = e)$$

$$y - 1 = \frac{1}{e}(x - e)$$

$$y - 1 = \frac{1}{e}x - 1$$

$$y = \frac{1}{e}x$$

b $y = 2\sin\left(\frac{x}{2}\right)$

$$\frac{dy}{dx} = \cos\left(\frac{x}{2}\right)$$

$$= \frac{1}{\sqrt{2}} \left(\text{at } x = \frac{\pi}{2}\right)$$

$$y - \sqrt{2} = \frac{1}{\sqrt{2}}(x - \frac{\pi}{2})$$

$$y = \frac{1}{\sqrt{2}}x - \frac{\pi}{2\sqrt{2}} + \sqrt{2}$$

c $y = \cos x$

$$\frac{dy}{dx} = -\sin x$$

$$= 1 \left(\text{at } x = \frac{3\pi}{2}\right)$$

$$y = (1)\left(x - \frac{3\pi}{2}\right)$$

$$y = x - \frac{3\pi}{2}$$

d $y = \log_e(x^2)$

$$\frac{dy}{dx} = \frac{2}{x}$$

$$= -\frac{2}{\sqrt{e}} \text{ (at } x = -\sqrt{e})$$

$$y - 1 = -\frac{2}{\sqrt{e}}(x + \sqrt{e})$$

$$y = -\frac{2}{\sqrt{e}}x - 1$$

Solutions to multiple-choice questions

1 A $y = x^2 - x - 5$

$\frac{dy}{dx} = 2x - 1$

Gradient of tangent equation=4

$\therefore \frac{dy}{dx} = 4$

$\therefore 4 = 2x - 1$

$x = \frac{5}{2}$

$y = \left(\frac{5}{2}\right)^2 - \frac{5}{2} - 5$

$y = -\frac{5}{4}$

Sub x and y values into $y = 4x + c$

$-\frac{5}{4} = 10 + c$

$c = -\frac{45}{4}$

2 E Since the gradient changes from negative to positive at point a, this is a local minimum.

Since the gradient remains the same at, before and after point b, this is a stationary point of inflection.

3 E The graph of the second function is obtained from the graph of the first function by this sequence of transformations:

(1) a reflection in the x-axis

(2) a dilation of factor 2 from the x-axis

(3) a dilation of factor 2 from y-axis

(4) a translation of k units vertically up

The point $(0, 0)$ transforms to $(0, k)$ and is now a maximum due to the reflection (the dilation leave no effect).

The maximum point $a, f(a)$ of the original graph transforms as follows:

(1) $(a, -f(a)$; local minimum

(2) $(a, -2f(a)$; local minimum

(3) $(2a, -2f(a)$; local minimum

(4) $(2a, -2f(a) + k)$ local minimum

4 B $f(x) = x^3 - x^2 - 1$

$f'(x) = 3x^2 - 2x$

Stationary points occur when

$f'(x) = 0$

$3x^2 - 2x = 0$

$x(3x - 2) = 0$

Using the null factor theorem:

$x = 0$ and $x = \frac{2}{3}$

5 C As it is a local minimum the gradient of the tangent is 0. Therefore it is a horizontal line which goes through the point (2, 4)

$y = mx + c$

$m = 0$

$\therefore\ y = 4$

6 B
$$V = -10x(2x^2 - 6)$$
$$V = -20x^3 + 60x$$
$$\frac{dV}{dx} = -60x^2 + 60$$
$$0 = -60x^2 + 60$$
$$60x^2 = 60$$
$$x = \pm 1$$

When $x = -1, V = -80$

$$V \neq -80$$

$\therefore$ Maximum volume occurs when $x = 1$

7 A
$$f(x) = ax^3 + bx^2 + cx + d$$
$$f'(x) = 3ax^2 + 2bx + c$$
$$\Delta = 4b^2 - 12ac$$
$$\Delta < 0 \Rightarrow 4b^2 - 12ac < 0$$
$$\Leftrightarrow b^2 - 3ac < 0$$
$$\Leftrightarrow a > \frac{b^2}{3c}$$

8 D $f : R \to R, f(x) = e^x - ex$
$$f'(x) = e^x - e$$
$$f'(x) = 0$$
$$e^x - e = 0$$
$$e^x = e$$
$$x = 1$$

Turning point occurs at $x = 1$

Sub into $f(x)$ to find y coordinate:

$y = e - e$

$y = 0$

$\therefore (1, 0)$

9 E $y = e^{ax}$

Tangent at point $\left(\frac{1}{a}, e\right)$

$$\frac{dy}{dy} = ae^{ax}$$

At $x = \dfrac{1}{a}$

$$\frac{dy}{dx} = ae^{\frac{a}{a}}$$
$$\frac{dy}{dx} = ae$$

Equation of tangent: $y = aex + c$

Sub in point $\left(\frac{1}{a}, e\right)$

$$e = ae\frac{1}{a} + c$$
$$e = e + c$$
$$c = 0$$

$\therefore$ equation of tangent:

$y = aex$

10 A
$$N = 4000e^{0.2t}$$
$$\frac{dN}{dt} = 800e^{0.2t}$$

When $t = 3$

$$\frac{dN}{dt} = 800e^{0.6} \approx 1458$$

11 E Stationary point pf inflection when $x = -8$

12 B $y = e^{-x} - 1$

Point where equation crosses the y-axis:

$x = 0, y = 0$ coordinate: $(0, 0)$

$$\frac{dy}{dx} = -e^{-x}$$

Gradient of tangent at $x = 0$:

$$\frac{dy}{dx} = -1$$

Equation of tangent:

$y = -x + c$

Sub in point $(0, 0)$:

$c = 0$

$\therefore$ Equation of tangent:

$y = -x$

13 D $f(x) = x^3 - 9x^2 + 24x + c$

$$f'(x) = 3x^2 - 18x + 24$$

$$f'(x) = 0 \Rightarrow x^2 - 6x + 8 = 0$$

$$\Leftrightarrow x = 4 \text{ or} x = 2$$

$$f(4) = c + 16 \text{ and } f(2) = c + 20$$

Local minimum at $(4, c + 16)$ and a local maximum at $(2, c + 20)$

The local minimum has to be below the x axis and the local maximum above. Therefore $c + 16 < 0$ and $c + 20 > 0$

That is, $c \in (-20, -16)$

14 C

$$f(x) = e^{ax} - \frac{ax}{e}$$

$$f'(x) = ae^{ax} - \frac{a}{e}$$

$$= 0 \text{ if}$$

$$ae^{ax} = \frac{a}{e}$$

$$e^{ax} = \frac{1}{e} = e^{-1}$$

$$ax = -1$$

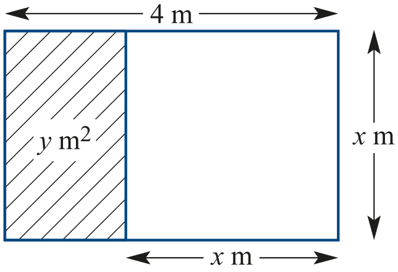
$$x = -\frac{1}{a}$$

$$f\left(-\frac{1}{a}\right) = e^{-1} - \frac{-1}{e} = \frac{2}{e}$$

The coordinates of the turning point are $\left(-\frac{1}{a}, \frac{2}{e}\right)$.

Solutions to extended-response questions

1

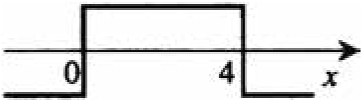


a Shaded area $= 4x - x^2$

i.e. $y = 4x - x^2$

b As $y > 0, 4x - x^2 > 0$

i.e $x(4 - x) > 0$



$\therefore\ y > 0$ for $0 < x < 4$

The possible values of x are $0 < x < 4$.

c $\dfrac{dy}{dx} = 4 - 2x$

$\dfrac{dy}{dx} = 0$ implies $x = 2$

Note: $y = 4x - x^2$ is a quadratic with negative coefficient of x^2.

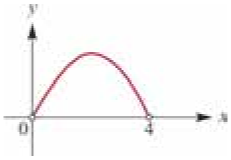
When $x = 2, y = 8 - 4 = 4$.

The maximum value of y is 4 and this occurs when $x = 2$.

d $y = 4x - x^2$ is a quadratic with negative coefficient of x^2

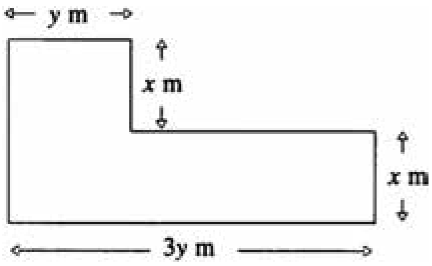
or gradient to the left of $x = 2$ is positive and to the right negative.

e



f From the graph the possible values for y are $0 < y < 4$.

2



a $A = xy + 3xy = 4xy$

b Perimeter $= 48$

$$\therefore\ 48 = 6y + 4x$$

$$y = \frac{48 - 4x}{6}$$

$$= 8 - \frac{2}{3}x$$

c $A = 4xy$

$$= 4x\left(8 - \frac{2}{3}x\right)$$

$$= 32x - \frac{8x^2}{3}$$

d $\dfrac{dA}{dx} = 32 - \dfrac{16x}{3}$

$\dfrac{dA}{dx} = 0$ implies $x = \dfrac{96}{16} = 6$. Maximum as quadratic with negative coefficient of x^2

When $x = 6, y = 8 - \dfrac{2}{3} \times 6 = 4$

e When $x = 6$

$$A = 32 \times 6 - \frac{8}{3} \times 36$$

$$= 96$$

The maximum area is 96 m^2.

3 **a** Cost is $(12 + 0.008x)$ dollars per kilometre plus \$14.40 per hour for the driver, where x is the speed of the truck in km/h

i Cost per kilometre for tmck travelling at 40 km/h

$$= (12 + 0.008 \times 40) + 14.40 \times \frac{1}{40}$$

$$= 12.68$$

i.e. the cost per kilometre is \$12.68.

ii Cost per kilometre for truck travelling at 64 km/h

$$= (12 + 0.008 \times 64) + \frac{1}{64} \times 14.40$$

$$= 12.737$$

i.e. the cost per kilometre is \$12.74.

b Let C be the cost per kilometre.

$$C = (12 + 0.008x) + \frac{14.40}{x}$$

$$= 12 + 0.008x + \frac{14.40}{x}$$

c To sketch the graph we first differentiate to determine tuming points.

For $C = 12 + 0.008x + \frac{14.40}{x}$

$$\frac{dc}{dx} = 0.008 - \frac{14.40}{x^2}$$

and stationary points occur for $\frac{dC}{dx} = 0$.

This implies

$$0.008x^2 = 14.40$$

$$x^2 = 1800$$

$$x = 30$$

$$\approx 42.426$$

A sign chart is used to determine the nature of the stationary point.

	$< 30\sqrt{2}$	$30\sqrt{2}$	$> 30\sqrt{2}$
sign $f'(x)$	-ve	0	+ve
shape	\	–	/

$\therefore$ A minimum occurs where $x = 30\sqrt{2}$.

When $x = 30\sqrt{2}$

$$C = 12 + 0.008 \times 30\sqrt{2} + \frac{14.40}{30\sqrt{2}}$$

$$= 12 + 0.24\sqrt{2} + \frac{0.48}{\sqrt{2}}$$

$$= 12 + 0.24\sqrt{2} + 0.24\sqrt{2}$$

$$= 12 + 0.48\sqrt{2}$$

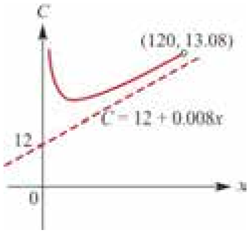
$$\approx 12.679$$

$\therefore$ minimum at $(30\sqrt{2}, 12 + 0.48\sqrt{2})$

When $x = 120$

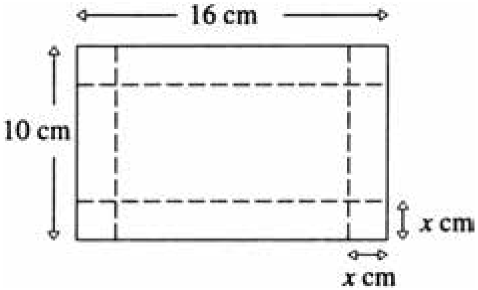
$$C = 12 + 0.008 \times 120 + \frac{14.40}{120}$$
$$= 12 + 0.96 + 0.12$$
$$= 13.08$$

It is also observed that as $x \to 0, C \to \infty$ and that as x large, the graph gets close to that of $c = 12 + 0.08x$.



d From the above, the truck should be driven at $30\sqrt{2} \approx 42.43$ km/hr.

4 **a**



$\therefore$ length of box = $(16 - 2x)$ cm
width of box = $(10 - 2x)$ cm
height of box = x cm
Volume = length × width × height

$$= (16 - 2x)(10 - 2x)x$$
$$= 4(8 - x)(5 - x)x$$
$$= 4(40 - 13x + x^2)x$$
$$= 4(x^3 - 13x^2 + 40x) \text{ cm}^3$$

b All dimensions are positive.
$10 - 2x > 0$ and $16 - 2x > 0$ and $x > 0$
$\therefore \quad x < 5$ and $x < 8$ and $x > 0$
$\therefore \quad 0 < x < 5$

c Let $V = 4(x^3 - 13x^2 + 40x)$

$$\frac{dV}{dx} = 4(3x^2 - 26 + 40)$$

$$\frac{dV}{dx} = 0 \text{ implies } 4(3x^2 - 26x + 40) = 0$$

$\therefore\ 3x^2 - 26x + 40 = 0$

$\therefore\ (3x - 20)(x - 2) = 0$

$\therefore\ x = \frac{20}{3}$ or $x = 2$

but $0 < x < 5 \therefore\ x = 2$

d A gradient chart reveals there is a maximum when $x = 2$:

	< 2	2	> 2
sign $\frac{dV}{dx}$	+ve	0	−ve
shape	/	−	\

When$x = 2$,

$16 - 2x = 12$

$10 - 2x = 6$

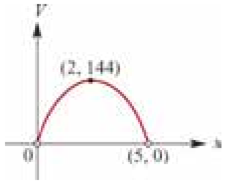
∴ The dimensions of the box for maximum volume are:
2 cm, 6 cm, 12 cm

e Maximum when $x = 2$

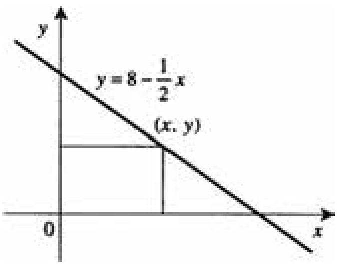
$$\begin{aligned}\therefore\ V_{\max} &= 4(5-2)(8-2)2\\ &= 4 \times 3 \times 6 \times 2\\ &= 144\end{aligned}$$

The maximum volume is 144 cm^3

f



5



Area of rectangle = length × width
Let A denote the area.

Let x denote the width.

Let y denote the length.

$$A = xy$$

$$= x\left(8 - \frac{x}{2}\right)$$

$$= 8x - \frac{x^2}{2}$$

Consider the derivative of A with respect to x.

$$\frac{dA}{dx} = 8 - x$$

$\frac{dA}{dx} = 0$ implies $x = 8$

As A is a quadratic function with negative coefficient of x^2, a maximum occurs where $x = 8$.

When $x = 8$, $A = 8 \times 8 - \frac{8^2}{2} = 32$

$\therefore$ Maximum area = 32 square units.

6 **a** $$T = k + 2w^2$$

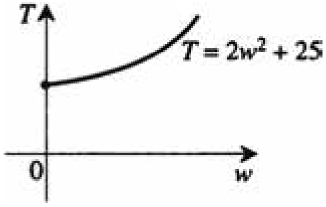
When $w = 5, T = 75$

$\therefore\ 75 = k + 50$

i.e. $k = 25$

So: $T = 2w^2 + 25$

b



c Average time in seconds per kg = $\frac{T}{w} = \frac{25}{w} + 2w$

d **i** Let A be the average time.

$$A = \frac{25}{w} + 2w$$

Minimum occurs when $\frac{dA}{dw} = 0$.

$$\frac{dA}{dw} = -25w^{-2} + 2 = 0$$

which implies $w^2 = \frac{25}{2}$

i.e. $w = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

A gradient chart confirms minimum:

w	$< \frac{5}{\sqrt{2}}$	$\frac{5}{\sqrt{2}}$	$> \frac{5}{\sqrt{2}}$
sign of $\frac{dA}{dw}$	−ve	0	+ve
shape	/	−	\

$\therefore \frac{5\sqrt{2}}{2}$kg $\approx$ 3.54 kg yields the minimum average machinery time.

ii When $w = \frac{5\sqrt{2}}{2}$

$$A = \frac{2 \times 5\sqrt{2}}{2} + \frac{25}{\left(\frac{5\sqrt{2}}{2}\right)}$$

$$= 5\sqrt{2} + 5\sqrt{2}$$

$$= 10\sqrt{2}$$

$\therefore$ minimum average machine time is $10\sqrt{2} \approx 14.14$ seconds.

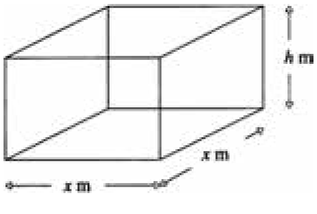
7 Let the base have dimension x m by x m and h m be the height of the tank.
The volume of a cuboid = length × width × height

$$= x \times x \times h$$

$$= x^2h$$

For this tank volume = 500 m^3

$$\therefore\ x^2h = 500\ 1$$



Let A m^2 be the area of sheet metal required.

$A = x^2 + 4xh$

(Note: The tank is open.)

From equation (1)

$$h = \frac{500}{x^2}$$

$$\therefore\ A = x^2 + 4x\left(\frac{500}{x^2}\right)$$

$$= x^2 + \frac{2000}{x}$$

Differentiating to find a minimum:

$$\frac{dA}{dx} = 2x - \frac{2000}{x^2}$$

$$\therefore\ \frac{dA}{dx} = 0 \text{ implies } x^3 = 1000$$

$$\therefore\quad x = 10$$

The gradient chart shows a minimum occurs when $x = 10$.

x	<	10	>
sign of $\frac{dA}{dx}$	−ve	0	+ve
shape	\	−	/

When $x = 10, h = 5$

Therefore the dimensions necessary for a minimum surface area are 10 m × 10 m × 5 m

8 **a** Area of bottom $= x^2 + x^2 = 2x^2$

Area of top $= x^2$

Area of sides $= xh + xh + xh + xh = 4xh$

$\therefore$ total area $= 4xh + 3x^2$

i.e. $C = 4xh + 3x^2$

b Volume $V = x^2h$

For Volume $= 12$ m^3

$$12 = x^2h$$

$$\text{i.e. } h = \frac{12}{x^2}$$

$$\text{and } C = 4x\left(\frac{12}{x^2}\right) + 3x^2$$

$$= \frac{48}{x} + 3x^2$$

c It is preferable to complete **d**

before sketching the graph.

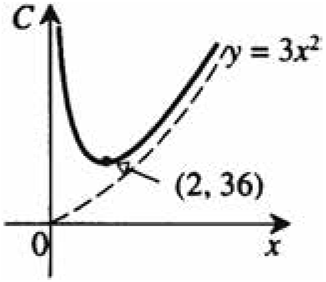
d **i**

$$\frac{dC}{dx} = -\frac{48}{x^2} + 6x$$

$$\frac{dC}{dx} = 0 \text{ implies } -\frac{48}{x^2} + 6x = 0$$

which implies $6x = \frac{48}{x^2}$

$\therefore\ x^3 = 8$ and $x = 2$



The gradient chart is as shown:

x	<2	2	>2
sign $\frac{dC}{dx}$	−ve	0	+ve
shape	\	−	/

$\therefore$ a minimum when $x = 2$ When $x = 2$, the dimensions are 2 m, 2 m, 3 m $\left(h = \frac{12}{2^2}\right)$

ii When $x = 2$

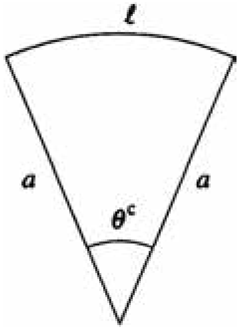
$$C = 12 + \frac{48}{2} = 12 + 24 = 36$$

$\therefore$ The minimum area is 36 m^2

9 **a** The area of a sector $A = \frac{1}{2}r^2\,\theta$

In this case $r = a$

$\therefore\; A = \frac{1}{2}a^2\theta$



b The length of the wire $= a + a + \ell$

$= 2a + \ell$

where $\ell = a\theta$

Therefore as the wire is $1\text{ m} = 100\text{ cm}$ in length

$$100 = 2a + a\theta$$

$$\therefore\; 100 = a(\theta + 2)$$

$$\text{i.e. } a = \frac{100}{\theta + 2}$$

$$\therefore\; A = \frac{1}{2}\left(\frac{100}{\theta + 2}\right)^2\theta$$

c Differentiating to find maximum

$$A = \frac{10^4}{2}\left(\frac{1}{\theta + 2}\right)^2\theta$$

Using the product rule

$$\frac{dA}{d\theta} = 5000\left[\frac{1}{(\theta + 2)^2} - \frac{2\theta}{(\theta + 2)^3}\right]$$

$$\frac{dA}{d\theta} = 0 \text{ implies } \frac{1}{(\theta + 2)^2} = \frac{2\theta}{(\theta + 2)^3}$$

$$\therefore (\theta + 2)^3 - 2\theta(\theta + 2)^2 = 0$$

$$\therefore (\theta + 2)^2[\theta + 2 - 2\theta] = 0$$

$$\therefore \theta = 2 \text{ or } \theta = -2$$

but $\theta > 0 \therefore\; \theta = 2$

The gradient chart will show a maximum i.e. A is maximum when $\theta = 2$

d When $\theta = 2, A = \frac{1}{2}\left(\frac{100}{2+2}\right)^2 \times 2$

$$= \frac{1}{2} \times 25^2 \times 2$$

$$= 625$$

The maximum area is 625 cm^2

10 **a** $L = 2r + r\theta$

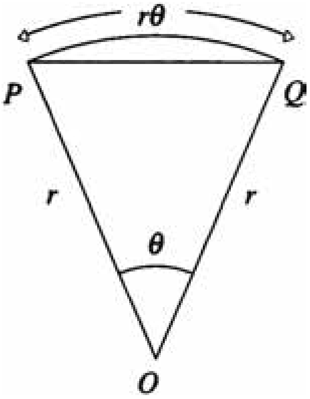
$\therefore\ \theta = \frac{1}{r}(L - 2r)$ 1

Area of sector $= \frac{1}{2}r^2\theta$

$$= \frac{1}{2}r^2\left(\frac{1}{r}(L - 2r)\right)$$

$$= \frac{1}{2}r(L - 2r)$$

$$= \frac{1}{2}rL - r^2$$



b **i** The area of the sector $A = \frac{1}{2}rL - r^2$

$\therefore\ \frac{dA}{dr} = \frac{1}{2}L - 2r$

and $\frac{dA}{dr} = 0$ implies $\frac{1}{2}L - 2r = 0$, so $r = \frac{L}{4}$.

ii Substituting $r = \frac{L}{4}$ in 1 gives

$$\theta = \frac{1}{\frac{L}{4}}\left(L - 2 \times \frac{L}{4}\right)$$

$$= \frac{4}{L}\left(L - \frac{L}{2}\right)$$

$$= \frac{4}{L} \times \frac{L}{2} = 2$$

iii A stationary point occurs when $r = \dfrac{L}{4}$.

If $r < \dfrac{L}{4}, \dfrac{dA}{dr} > 0$

$r < \dfrac{L}{4}, \dfrac{dA}{dr} < 0$ (gradients considered locally)

So the stationary point is a maximum.

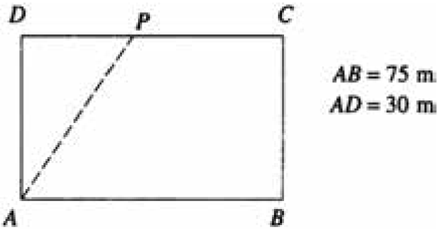
c Area of sector $= \dfrac{1}{2}r^2\theta$

When $\theta = 2, r = \dfrac{L}{4}$

$$\begin{aligned}
\therefore \text{Area of sector} &= \frac{1}{2} \times \frac{L^2}{16} \times 2 \\
&= \frac{L^2}{16} \\
\text{Area of triangle} &= \frac{1}{2}r^2 \sin 2 \\
&= \frac{1}{2} \times \frac{L^2}{16} \sin 2 \\
&= \frac{L^2 \sin 2}{32} \\
\frac{\text{Area of triangle}}{\text{Area of sector}} &= \frac{L^2 \sin 2}{32} \div \frac{L^2}{16} \\
&= \frac{L^2 \sin 2}{32} \times \frac{16}{L^2} \\
&= \frac{\sin 2}{2} = 0.4546\cdots
\end{aligned}$$

$\therefore$ Area of triangle $\approx 45.5\%$ area of sector

11



a Let T be the total time in seconds and $DP = x$ (m) (Note: the position of P varies)

T = time to swim to AP + time to run PC + time to get out

$$\text{time to swim } AP = \frac{AP}{\text{speed of swimming}} = \frac{\sqrt{900 + x^2}}{1} \text{ (Pythagoras' Theorem)}$$

$$\text{time to run } PC = \frac{75 - x}{1\frac{2}{3}} = \frac{3}{5}(75 - x)$$

$$\text{time to get out} = 2$$

$$T = \sqrt{900 + x^2} + \frac{3}{5}(75 - x) + 2$$

b $\frac{d}{dx}\left(\sqrt{900 + x^2}\right) = 2x \times \frac{1}{2}(900 + x^2)^{-\frac{1}{2}}$ (Chain rule)

$\therefore \frac{dT}{dx} = x(x^2 + 900)^{-\frac{1}{2}} - \frac{3}{5}$

c **i** Minimum occurs when $\frac{dT}{dx} = 0$

$\frac{dT}{dx} = 0$ implies $x(x^2 + 900)^{-\frac{1}{2}} = \frac{3}{5}$

$$\therefore \frac{x}{x(x^2 + 900)^{\frac{1}{2}}} = \frac{3}{5}$$

and $5x = 9(x^2 + 900)^{\frac{1}{2}}$

Squaring both sides yields

$$25x^2 = 9x^2 + 8100$$

$$\therefore 6x^2 = 8100$$

$$\therefore x^2 = \frac{8100}{16}$$

and $x = \frac{90}{4}$ (Note: $x \geq 0$ and so positive root is chosen)

$= 22\frac{1}{2}$

A gradient chart reveals a local minimum when $x = 22\frac{1}{2}$

ii The minimum time occurs when $x = \frac{90}{4}$

$$\text{When } x = \frac{90}{4}, T = \sqrt{900 + \left(\frac{90}{4}\right)^2} + \frac{3}{5}\left(75 - \frac{90}{4}\right) + 2$$

$$= \sqrt{\frac{22500}{16}} + 3\left(15 - \frac{18}{4}\right) + 2$$

$$= \frac{150}{4} + 3 \times \frac{42}{4} + 2$$

$$= 71$$

The minimum time is 71 seconds.

d If the boy runs from A to D and then from D to C

$$\text{time} = \frac{30}{1\frac{2}{3}} + \frac{75}{1\frac{2}{3}}$$

$$= \frac{30}{\frac{5}{3}} + \frac{75}{\frac{5}{3}}$$

$$= \frac{3}{5} \times \frac{105}{1} = 63$$

It takes 63 seconds to run from A to D and then from D to C.

12 **a** For $y = e^x$, $\dfrac{dy}{dx} = e^x$

When $x = 1$, $y = e$ and $\dfrac{dy}{dx} = e$

Therefore the equation of the tangent is given by

$$y - e = e(x - 1)$$

i.e. $y - e = ex - e$

$\therefore$ $y = ex$ is the equation of the tangent.

b For $y = e^{2x}$, $\dfrac{dy}{dx} = 2e^{2x}$

when $x = \dfrac{1}{2}$, $y = e + \dfrac{dy}{dx} = 2e$

The equation of the tangent at $\left(\dfrac{1}{2}, e\right)$ is given by

$$y - e = 2e\left(x - \frac{1}{2}\right)$$

i.e. $y - e = 2ex - e$

$\therefore$ $y = 2ex$ is the equation of the tangent at $\left(\dfrac{1}{2}, e\right)$

c For $y = e^{kx}$, $\dfrac{dy}{dx} = ke^{kx}$

when $x = \dfrac{1}{k}$, $y = e$ and $\dfrac{dy}{dx} = ke$

The equation of the tangent is

$$y - e = ke\left(x - \frac{1}{k}\right)$$

i.e. $y = kex$

d Consider the equation of the tangent at the point (a, e^{ka})

which passes through the origin for the curve with equation $y = e^{kx}$

$\dfrac{dy}{dx} = ke^{kx}$ and at (a, e^{ka}), $\dfrac{dy}{dx} = ke^{ka}$

$\therefore$ The equation of the tangent is

$y - 0 = ke^{ka}(x - 0)$

i.e. $y = ke^{ka}x$

Also the gradient of the tangent can be determined as the gradient of a straight line joining the point (a, e^{ka}) and (0,0)

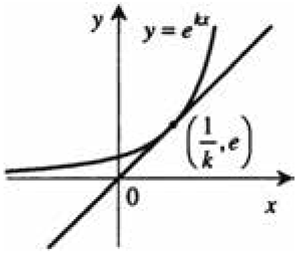
$$\text{Gradient} = \frac{e^{ka} - 0}{a - 0} = \frac{e^{ka}}{a}$$

$$\therefore \quad ke^{ka} = \frac{e^{ka}}{a}$$

$$\therefore \quad a = \frac{1}{k}$$

and $e^{ka} = e$

$\therefore$ Equation of tangent is $y = kex$

e 

Solving $e^{kx} = x$ is equivalent to solving the pair of equations

$y = e^{kx}$

$y = x$

simultaneously.

i There is a single solution to the equation $e^{kx} = x$ if $y = x$ is a tangent to the curve $y = e^{kx}$.

From (d) this occurs only if $ke = 1$ i.e. if $k = \frac{1}{e}$ for $k > 0$.

There is always a unique real root for $k \le 0$ (check the graph of $y = e^{kx}$ for $k \le 0$)

ii For no real roots, there are no solutions to the pair of equations

$y = e^{kx}$

and $y = x$

For $k = \frac{1}{e}$, $y = x$ is a tangent.

For $k > \frac{1}{e}$, the curve $y = e^{kx}$ does not meet the line$y = x$.

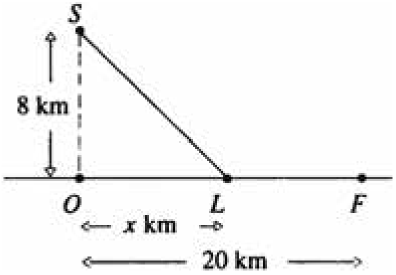
13 a Distance $SL = \sqrt{64 + x^2}$ (Pythagoras' Theorem)

Time taken = time taken for SL+ time taken for LF

$$= \frac{SL}{\text{rowing speed}} + \frac{LF}{\text{running speed}}$$

$$= \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$

$$\therefore T(x) = \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$



b Differentiating to find minimum

$$T'(x) = \frac{x}{5}(64 + x^2)^{-\frac{1}{2}} - \frac{1}{15}$$

$$T'(x) = 0$$

implies $\dfrac{x}{5(64 + x^2)^{\frac{1}{2}}} = \dfrac{1}{15}$

$\therefore\ 15x = 5(64 + x^2)^{\frac{1}{2}}$

Squaring both sides yields

$$225x^2 = 25(64 + x^2)$$

i.e. $200x^2 = 25 \times 64$

and $x = \dfrac{5 \times 8}{10\sqrt{2}}$ (Note: positive root is chosen as $x \geq 0$)

$$= \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

A gradient chart reveals a minimum.

When $x = 2\sqrt{2}$

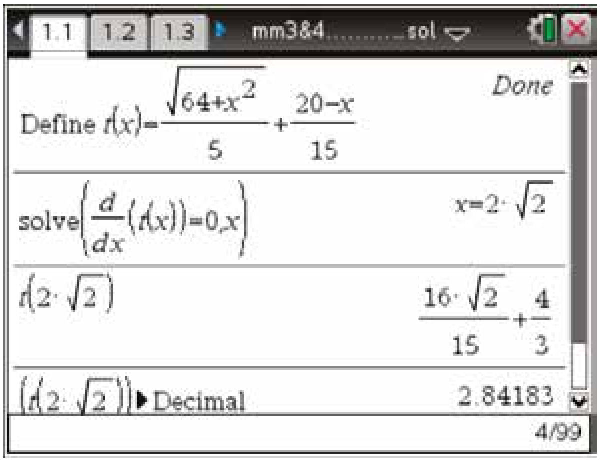
$$\begin{aligned} T &= \frac{\sqrt{64 + 8}}{5} + \frac{20 - 2\sqrt{2}}{15} \\ &= \frac{6\sqrt{2}}{5} + \frac{4}{3} - \frac{2\sqrt{2}}{15} \\ &= \frac{18\sqrt{2} - 2\sqrt{2} + 20}{15} \\ &= \frac{16\sqrt{2} + 20}{15} \\ &\approx 2.84 \end{aligned}$$

The minimum time is $\dfrac{16\sqrt{2} + 20}{15}$ hours ≈ 2.84 hours ≈ 2 hours 50 minutes 31 seconds

Graphic calculator techniques for question 13

In a **Calculator** page, define the function.
To find the *x*-value where the minimum occurs, solve the derivative equalling zero.

Hence minimum occurs at $\left(2\sqrt{2}, \frac{16\sqrt{2}}{15} + \frac{4}{3}\right)$



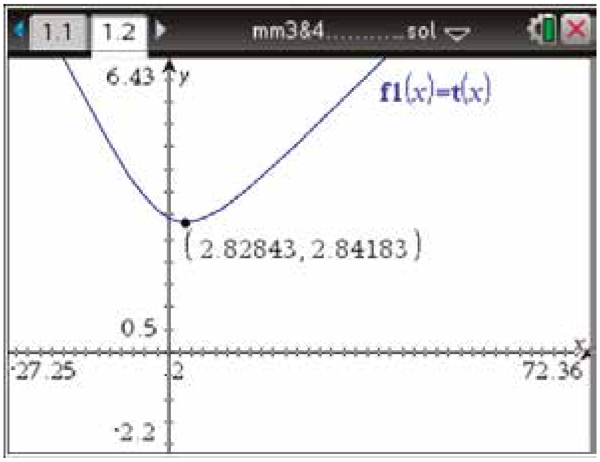
Define $t(x)=\frac{\sqrt{64+x^2}}{5}+\frac{20-x}{15}$ Done

$\text{solve}\left(\frac{d}{dx}(t(x))=0,x\right)$ $x=2\cdot\sqrt{2}$

$t(2\cdot\sqrt{2})$ $\frac{16\cdot\sqrt{2}}{15}+\frac{4}{3}$

$(t(2\cdot\sqrt{2}))$▸Decimal 2.84183

In a **Graphs** page enter the function t(x).
Find the minimum
using b>**Analyze Graph**>**Minimum**.



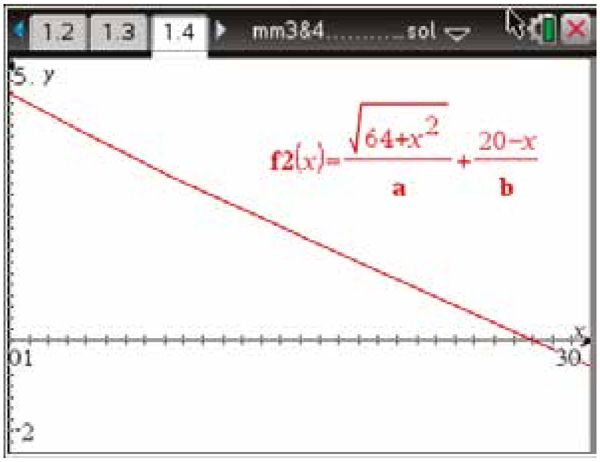
A further investigation can be made by considering

$$T(x) = \frac{\sqrt{64 + x^2}}{A} + \frac{20 - x}{B}$$

where A is the rowing speed and B is the running speed.
In the problem, store A = 20 and B = 5 in a **Calculator** page
Enter the formula in the **Function Entry Line** as shown by the graph label.
The result is as expected. It is best to row straight to F.

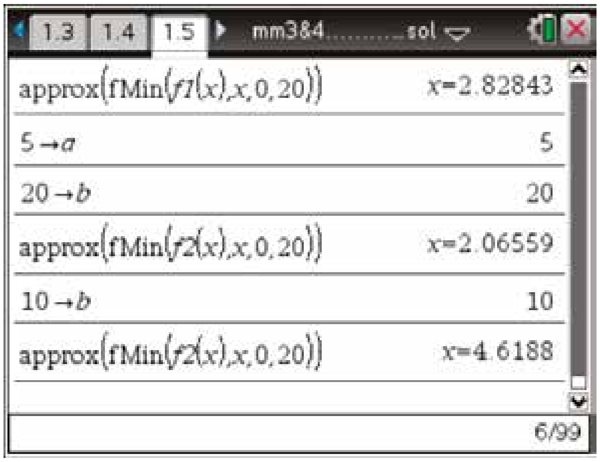
20 →a 20

5 →b 5



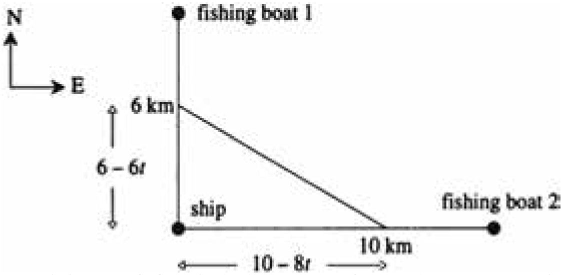
Investigate the minimum value, holding A = 5 and varying B. The use of **fMin(,** b>**Calculus**>**Function Minimum**, is a good way of doing this. The screen shows a possible investigation. The first value of **fMin(** was obtained with A = 5, B = 15. i.e. f1(x). The following values are obtained by altering the B value (A is constant) in function f2(x).

Note: **approx**(is used to give decimal answers for easier comparison. Other options to give decimal answers can also be used.



```
approx(fMin(f1(x),x,0,20))        x=2.82843
5→a                                       5
20→b                                     20
approx(fMin(f2(x),x,0,20))        x=2.06559
10→b                                     10
approx(fMin(f2(x),x,0,20))         x=4.6188
                                       6/99
```

14



Position of fishing boat 1 after t hours = $(6 - 6t)$ km North

Position of fishing boat 2 after t hours $(10 - 8t)$ km East

Distance apart after t hours

$= \sqrt{(6 - 6\text{t})^2 + (10 - 8t)^2}$ (Pythagoras' Theorem)

Let D km be the distance apart after t hours

$$D = \sqrt{(6 - 6t)^2 + (10 - 8t)^2}$$

and $D^2 = (6 - 6t)^2 + (10 - 8t)^2$

The minimum value of D will occur for the same value of t as the minimum of D^2.

$$\frac{d(D^2)}{dt} = -12(6 - 6t) - 16(10 - 8t)$$

$$= -72 + 72t - 160 + 128t$$

$$= 200t - 232$$

$\frac{d(D^2)}{dt} = 0$ implies

$$t = \frac{232}{200} = 1.16$$

This is a local minimum as D^2 vs t is a parabola with positive coefficient of t^2.

The boats are closest 1.16 hours after noon, i.e. after 1 hour 9 minutes and 36 seconds.

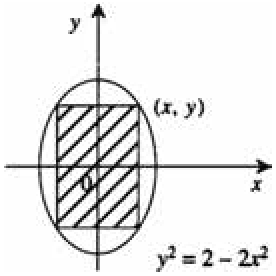
When $t = 1.16$

$$D = \sqrt{(6 - 6 \times 1.16)^2 + (10 - 8 \times 116)^2}$$

$= 1.2$

The least distance between the two fishing boats is 1.2 km.

15 a



$$\text{Area of rectangle, } A = 2x \times 2\text{y}$$
$$= 4x\sqrt{2-2x^2}$$
$$= 4x(2-2x^2)^{\frac{1}{2}}$$

b $2 - 2x^2 > 0$ for the relation to be defined

$\therefore\ 1 > x^2$

and $-1 < x < 1$

But $0 < x < 1$ (as x is the half–width of the beam)

$\therefore$ allowable values are $x \in (0, 1)$

c Using the product rule and chain rule

$$A = 4x(2-2x^2)^{\frac{1}{2}}$$
$$\frac{dA}{dx} = 4(2-2x^2)^{\frac{1}{2}} - 2x(2-2x^2)^{\frac{1}{2}} \times 4x$$
$$= 4(2-x^2)^{\frac{1}{2}} - \frac{8x^2}{(2-2x^2)^{\frac{1}{2}}}$$
$$= \frac{4(2-x^2)-8x^2}{(2-2x^2)^{\frac{1}{2}}}$$
$$= \frac{8-8x^2-8x^2}{(2-2x^2)^{\frac{1}{2}}} = \frac{8-16x^2}{(2-2x^2)^{\frac{1}{2}}}$$

Maximum will occur when $\frac{dA}{dx} = 0$

$$\text{When } \frac{dA}{dx} = 0$$
$$8 = 16x^2$$
$$x^2 = \frac{1}{2}$$
$$\therefore\ x = \pm\frac{1}{\sqrt{2}}$$

but $x \in (0, 1)$ $\therefore\ x = \frac{1}{\sqrt{2}}$

A gradient chart shows local maximum when $x = \frac{1}{\sqrt{2}}$. When $x = \frac{1}{\sqrt{2}}$, $y = \pm 1$

d When $x = \dfrac{1}{\sqrt{2}}$

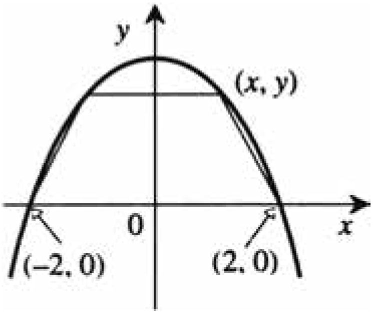
$$A = \frac{4}{\sqrt{2}} \times \left(2 - 2 \times \left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{1}{2}}$$

$$= \frac{4}{\sqrt{2}} \times (2 - 1)^{\frac{1}{2}}$$

$$= 2\sqrt{2}$$

$\therefore$ The maximum cross-sectional area of the beam is $2\sqrt{2}$ square units.

16



a Area of a trapezoid $= \dfrac{h}{2}(a + b)$

where h is the height of the trapezoid and a and b are the lengths of the opposite parallel sides.

$\therefore$ Area of the trapezoid $= \dfrac{y}{2}(4 + 2x)$

But $y = 4 - x^2$

$\therefore$ Area, $A = \dfrac{(4 - x^2)}{2}(4 + 2x)$

$= \dfrac{1}{2}(4 - x^2)(2x + 4)$

b Using the product rule

$$\frac{dA}{dx} = \frac{1}{2}[-2x(2x + 4) + 2(4 - x^2)]$$

$$= \frac{1}{2}[-4x^2 - 8x + 8 - 2x^2]$$

$$= \frac{1}{2}[-6x^2 - 8x + 8] = -3x^2 - 4x + 4$$

$\frac{dA}{dx} = 0$ implies $3x^2 + 4x - 4 = 0$

$\therefore\ (3x - 2)(x + 2) = 0$

$\therefore\ x = \frac{2}{3}$ or $x = -2$

$\frac{dA}{dx} = -(3x - 2)(x + 2)$

When $x > \frac{2}{3}, \frac{dA}{dx} < 0$ (locally)

When $x < \frac{2}{3}, \frac{dA}{dx} > 0$

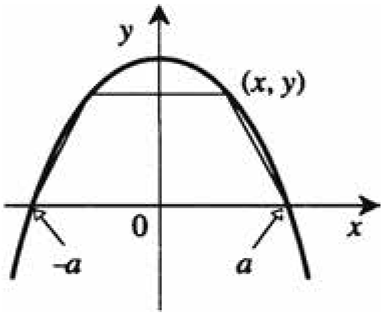
$\therefore$ local maximum when $x = \frac{2}{3}$

$\therefore$ The trapezoid has its greatest area when $x = \frac{2}{3}$

c **i** $A = \frac{1}{2} \times y(2a + 2x)$

$= y(a + x)$

$= (a^2 - x^2)(a + x)$



ii Using the product rule

$$\frac{dA}{dx} = a^2 - x^2 + (-2x)(a + x)$$

$$= a^2 - x^2 - 2xa - 2x^2$$

$$= a^2 - 2xa - 3x^2$$

$$= (a + x)(a - 3x)$$

iii $\frac{dA}{dx} = 0$ implies $x = \frac{a}{3}$ or $x = -a$

when $x > \frac{a}{3}, \frac{dA}{dx} < 0$ (locally)

When $x < \frac{a}{3}, \frac{dA}{dx} > 0$

$\therefore$ maximum when $x = \frac{a}{3}$

17

$$\begin{aligned} N(t) &= 24te^{-0.2t} \\ N'(t) &= 24e^{\frac{-t}{5}} - \frac{24}{5}te^{\frac{-t}{5}} \\ &= e^{\frac{-t}{5}}\left(24 - \frac{24t}{5}\right) \\ N'(t) &= 0, \\ 24 - \frac{24t}{5} &= 0 \\ \frac{t}{5} &= 1 \\ t &= 5 \\ N(5) &= 120e^{-1} \\ &= \frac{120}{e} = 44 \text{ bacteria} \end{aligned}$$

(*round because it is a discrete quantity not a continuous one*)

18 **a** $y = -t^3 + bt^2 + ct$

When $t = 1, y = 10$

When $t = 2, y = 24$

$\therefore\ 10 = -1 + b + c$

and $24 = -8 + 4b + 2c$

$\therefore\ 11 = b + c \quad 1$

and $32 = 4b + 2c \quad 2$

Subtract 2×1 from 2

$10 = 2b$

$\therefore\ b = 5$ and from 1, $c = 6$

$\therefore\ y = -t^3 + 5t^2 + 6t$

b $y = -t^3 + 5t^2 + 6t$

i y is the rate of increase.

$\therefore$ to determine when area covered by the plant is a maximum consider $y = 0$

i.e. $-t^3 + 5t^2 + 6t = 0$

$-t(t^2 - 5t - 6) = 0$

$\therefore\ t = 0$ or $(t - 6)(t + 1) = 0$

$\therefore\ t = 6$ or $t = -1$ or t (Note: $t \geq 0$ and when $t = 0, y = 0$)

$y = -t(t - 6)(t + 1)$

When $t > 6, y < 0$ (locally)

When $t < 6, y > 0$

$\therefore$ local maximum then $t = 6$

The area is a maximum 6 weeks after planting.

ii The rate of increase $y = -t^3 + 5t^2 + 6t$

To determine maximum rate consider

$$\frac{dy}{dt} = -3t^2 + 10t + 6$$

$$\frac{dy}{dt} = 0 \text{ implies } -3t^2 + 10t + 6 = 0$$

The quadratic formula gives

$$\begin{aligned} t &= \frac{-10 \pm \sqrt{100 + 72}}{-6} \\ &= \frac{-10 \pm \sqrt{172}}{-6} \\ &= \frac{-10 \pm 2\sqrt{43}}{-6} \\ &= \frac{10 \mp 2\sqrt{43}}{6} = \frac{-5 \mp \sqrt{43}}{3} \approx 3.852 \text{ or } -0.519 \end{aligned}$$

$t \geq 0$ in this example and a gradient chart reveals that a maximum rate of increase occurs when $t = 3.852$.

i.e. The rate of increase is a maximum after 3.852 weeks.

c This question requires antidifferentiation at year 11 MM 1 & 2 standard.

$y = -t^3 + 5t^2 + 6t$

$$\therefore \text{ Area } = -\frac{t^4}{4} + \frac{5t^3}{3} + \frac{6t^2}{2} + c$$

When $t = 0$, area $= 100\,\text{cm}^2\ \therefore\ c = 100$

$$\therefore \text{ Area } = \frac{-t^4}{4} + \frac{5t^3}{3} + 3t^2 + 100$$

When $t = 4$

$$\text{Area} = \frac{-4^4}{4} + \frac{5 \times 4^3}{3} + 3 \times 16 + 100$$

$$= -4^3 + \frac{5}{3} \times 4^3 + 3 \times 16 + 100$$

$$= \frac{2 \times 4^3}{+} 3 \times 16 + 100$$

$$= \frac{2 \times 64}{3} + 48 + 100$$

$$= 42\frac{2}{3} + 48 + 100$$

$$= 190\frac{2}{3}$$

The plant will cover $190\frac{2}{3}$ cm^2 after 4 weeks

d After 6 weeks the rate becomes negative which implies the plant begins to recede.

Area $= 244$ cm^2 after 6 weeks

$7 = 218$ cm^2 after 7 weeks

$= 121$ cm^2 after 8 weeks

The area becomes "negative" between 8 and 9 weeks. The model is not valid after this. Once the area begins to decrease the model is questionable.

19 $f(x) = x^3 - 3x^2 + 6x - 10$

a $f'(x) = 3x^2 - 6x + 6$

$f'(x) = 3$ implies $3x^2 - 6x + 6 = 3$

$$\therefore\ x^2 - 2x + 2 = 1$$

$$\therefore\ x^2 = 2x + 1 = 0$$

$$\therefore\ (x - 1)^2 = 0$$

$$x = 1$$

and $f(1) = 1 - 3 + 6 - 10 = -6$

The coordinates of the point where $f'(x) = 3$ are $(1, -6)$

b $f(x) = 3x^2 - 6x + 6$

$$= 3[x^2 - 2x + 2]$$

$$= 3[x^2 - 2x + 1 + 1]$$

$$= 3[(x - 1)^2 + 1]$$

$$= 3(x - 1)^2 + 3$$

c $(x - 1)^2 > 0$ for all $x \in R\backslash\{1\}$

$\therefore\ f(x) > 3$ for all $x \in R\backslash\{1\}$

20 a $y = ax^3 + bx^2 + cx + d$ passes through the x-axis at $(1, 0)$

$\therefore\ 0 = a + b + c + d \ldots\ldots (1)$

Gradient $= 0$ when $x = 1$ and $x = \dfrac{1}{3}$

gradient function, $\dfrac{dy}{dx} = 3ax^2 + 2abx + c$

$\therefore\ 0 = 3a + 2b + c \ldots\ldots (2)$

and $0 = \dfrac{a}{3} + \dfrac{2}{3}b + c \ldots\ldots (3)$

Finally it passes through the point $\left(\dfrac{1}{3}, \dfrac{4}{27}\right)$

$\therefore\ \dfrac{4}{27} = \dfrac{a}{27} + \dfrac{b}{9} + \dfrac{c}{3} + d \ldots\ldots (4)$

Subtract 3 and 2

$0 = \dfrac{8a}{3} + \dfrac{4b}{3}$

$\therefore\ 0 = 2a + b$

i.e. $b = -2a \ldots (5)$

Substitute in (1) for b

$0 = a - 2a + c + d$

i.e. $0 = -a + c + d \ldots (6)$

Substitute in (4) for b

$\dfrac{4}{27} = \dfrac{a}{27} - \dfrac{2a}{9} + \dfrac{c}{3} + d$

$\dfrac{4}{27} = -\dfrac{5a}{27} + \dfrac{c}{3} + d \ldots (7)$

Subtract (7) from (6)

$$-\frac{4}{27} = -\frac{22a}{27} + \frac{2c}{3}$$

$$\text{i.e } -4 = -22a + 18c$$

$$\text{and } -2 = -11a + 9c$$

$$\therefore\ c = \frac{-2 + 11a}{9}$$

Substitute in (2) for b and c

$$0 = 3a - 4a + 2 - \frac{2 + 11a}{9}$$

$$0 = \frac{9a + -2 + 11a}{9}$$

$$\therefore\ a = 1$$

and $b = -2a = -2$

and $c = -\frac{2 + 11}{9} = 1$

From (1)

$$0 = a + b + c + d$$

$$0 = 1 - 2 + 1 + d$$

$$\therefore\ d = 0$$

i.e. $a = 1, b = -2, c = 1, d = 0$

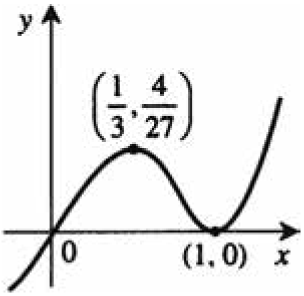
b $\dfrac{dy}{dx} = 3ax^2 + 2bx + c$

$= 3x^2 - 4x + 1$

$= (3x - 1)(x - 1)$

$\dfrac{dy}{dx} < 0$ for $\dfrac{1}{3} < x < 1$

c



$y = x^3 - 2x^2 + x$

$= x(x - 1)^2$

$\dfrac{dy}{dx} = 0$ when $x = \dfrac{1}{3}$

and $x = 1$

when $x = \dfrac{1}{3}, y = \dfrac{4}{27}$

and $\dfrac{dy}{dx} < 0$ for $x \in \left(\dfrac{1}{3}, 1\right)$

21 $V = \dfrac{\pi}{3}((y + 630)^3 - 630^3)$

a When $y = 40$

$V = \dfrac{\pi}{3}((40 + 630)^3 - 630^3)$

$= \dfrac{\pi}{3}(670^3 - 630^3)$

$= \dfrac{\pi}{3}(50\,716\,000)$

$\approx 53109\,671.0$

Volume of water in reservoir = 53 109 671.0 m^3

b $\dfrac{dV}{dy} = \dfrac{\pi}{3}(3(y + 630)^2)$

$= \pi(y + 630)^2$

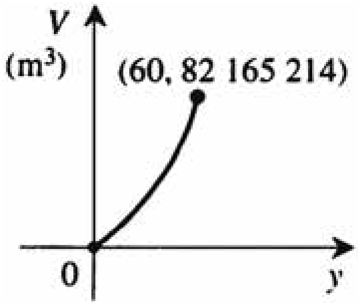
c $\frac{dV}{dy} > 0$ for all $v \in R$, gives that the function is increasing and the gradient increases as y increases.

d $V = \frac{\pi}{3}((y + 630)^3 - 630^3)$

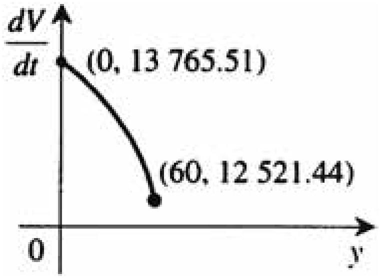
$= \frac{\pi}{3}((690)^3 + (630)^3)$ when $y = 60$

$= \frac{\pi}{3}(78\,462\,000)$

$= 82\,165\,214\,\text{m}^3$



e $\frac{dV}{dt} = 20\,000 - 0.005\pi(y + 630)^2$



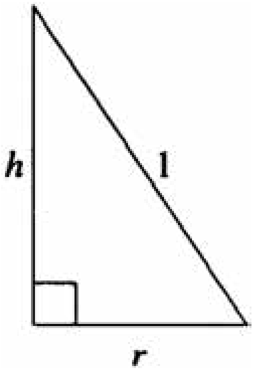
The graph of $\frac{dV}{dt}$ against y is a parabola.
It is the graph of $z = -x^2$ transformed by a dilation of 0.005π from the x-axis followed by a translation of 630 units "to the left" and 20 000 units "up".
The domain is $0 \le y \le 60$.
When $y = 60$, $\frac{dV}{dt} = 12521.44$
When $y = 0$, $\frac{dV}{dt} = 13765.51$

22 **a** **i** The circumference of the base of the cone is equal to the length of the sector formed. Hence $2\pi r = (2\pi - \theta)$ (radius of circle is one)

$\therefore\ 2\pi r = 2\pi - \theta$

$$r = \frac{2\pi - \theta}{2\pi}$$

ii



$$h^2 + r^2 = 1$$

$$\therefore\ h^2 = 1 - r^2$$

From (i), $r = \dfrac{2\pi - \theta}{2\pi}$

$$\therefore\ h^2 = 1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2$$

and $h = \sqrt{1 - \left(\dfrac{2\pi - \theta}{2\pi}\right)^2}$

iii $V = \dfrac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{2\pi - \theta}{2\pi}\right)^2 \sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$$

b When $\theta = \dfrac{\pi}{2}$

$$V = \frac{\pi}{3}\left(\frac{2\pi - \frac{\pi}{4}}{2\pi}\right)^2 \sqrt{\frac{4\pi^2 - (4\pi^2 - 4\pi\theta + \theta^2)}{4\pi^2}}$$

$$= \frac{\pi}{3}\left(\frac{7}{8}\right)^2 \sqrt{\frac{4\pi \times \frac{\pi}{4} - (\frac{\pi}{4})^2}{4\pi^2}}$$

$$= \frac{49\pi}{192}\sqrt{\frac{1 - \frac{1}{16}}{4}}$$

$$= \frac{49\pi}{384}\sqrt{\frac{15}{16}}$$

$$= \frac{49\pi}{1536}\sqrt{15}$$

c $0.3 = \frac{\pi}{3}\left(\frac{2\pi - \theta}{2\pi}\right)^2 \sqrt{\frac{4\pi\theta - \theta^2}{4\pi^2}}$ Solving using a CAS calculator $\theta = 0.3281$

$$= \frac{(2\pi - \theta)^2}{24\pi^2}\sqrt{4\pi\theta - \theta^2}$$

d (Note: $0 < \theta < \pi$)

i maximum occurs at $\theta \approx 1.153$

ii maximum volume is $V \approx 0.403$ cm^2

e $V = \frac{(2\pi - \theta)^2}{24\pi^2}(4\pi\theta - \theta^2)^{\frac{1}{2}}$

$$\frac{dV}{d\theta} = \frac{1}{24\pi^2}\left[-2(2\pi - \theta)(4\pi\theta - \theta^2)^{\frac{1}{2}} + \frac{1}{2}(2\pi - \theta)^2(4\pi - 2\theta)(4\pi\theta - \pi^2)^{-\frac{1}{2}}\right]$$

$$= \frac{(2\pi - \theta)}{24\pi^2}\left[\frac{-2(4\pi\theta - \theta^2) + (2\pi - \theta)^2}{(4\pi\theta - \theta^2)^{\frac{1}{2}}}\right]$$

$\frac{dV}{d\theta} = 0$ implies $\theta = 2\pi$

or $-8\pi\theta + 2\theta^2 + 4\pi^2 - 4\pi\theta + \theta^2 = 0$

i.e $3\theta^2 - 12\pi\theta + 4\pi^2 = 0$

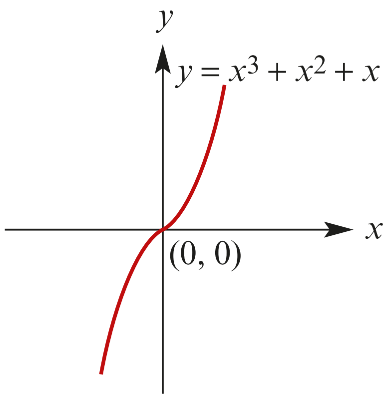
$$\therefore\ \theta = \frac{12\pi \pm \sqrt{144\pi^2 - 48\pi^2}}{6}$$

$$\theta = \frac{12\pi \pm 4\sqrt{6\pi^2}}{6}$$

$$\therefore\ \theta = \frac{6\pi - 2\pi\sqrt{6}}{3} \approx 1.153$$

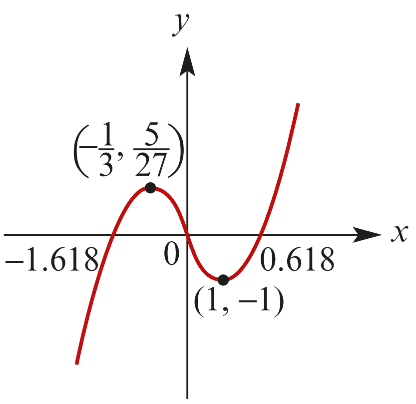
Maximum volume is 0.403 cm^2

23 a i

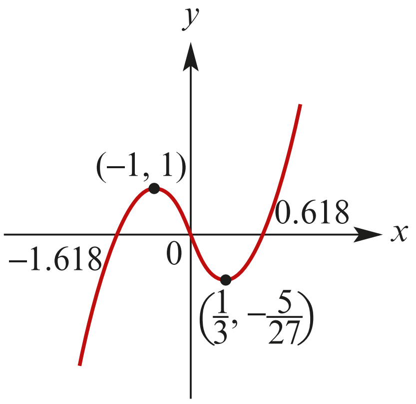


No stationary points

ii



iii



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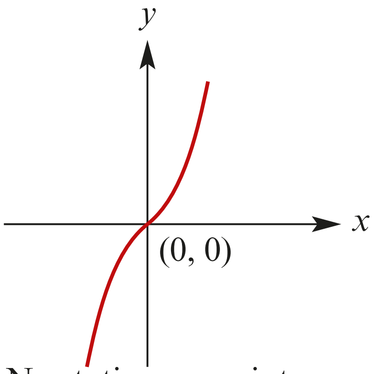


No stationary point

b **i** $f'(x) = 3x^2 + 2ax + b$

ii $f'(x) = 0$ implies $3x^2 + 2ax + b = 0$

$$\begin{aligned}\therefore \ x &= \frac{-2a \pm \sqrt{4a^2 - 4 \times 3 \times b}}{6} \\ &= \frac{-2a \pm \sqrt{4a^2 - 12b}}{6} \\ &= \frac{-a \pm \sqrt{a^2 - 3b}}{3}\end{aligned}$$

c **i** If $a^2 - 3b = 0$, the cubic has one stationary point given by $x = \frac{-9}{3} = -3$.

ii If $b = 3, a^2 = 9$ and $a = \pm 3$

$\therefore x = -1$ or $x = 1$

If $b = 3, a = 3$ and $x = -1$

$$\begin{aligned} y &= (-1)^3 + 3(-1)^2 + 3(-1) \\ &= -1 + 3 - 3 \\ &= -1 \end{aligned}$$

Coordinates $(-1, -1)$

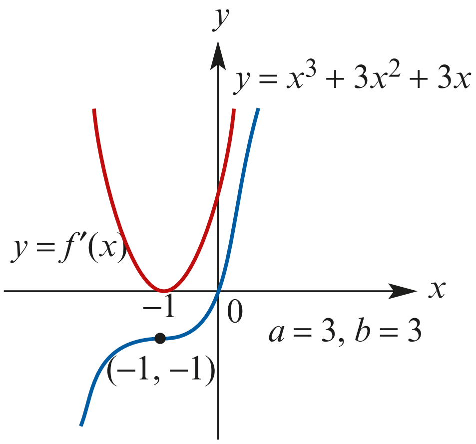
If $b = 3, a = -3$ and $x = 1$

$$\begin{aligned} y &= (1)^3 - 3(1)^2 + 3(1) \\ &= 1 - 3 + 3 \\ &= 1 \end{aligned}$$

Coordinates $(1, 1)$

Each is a stationary point of inflexion

iii



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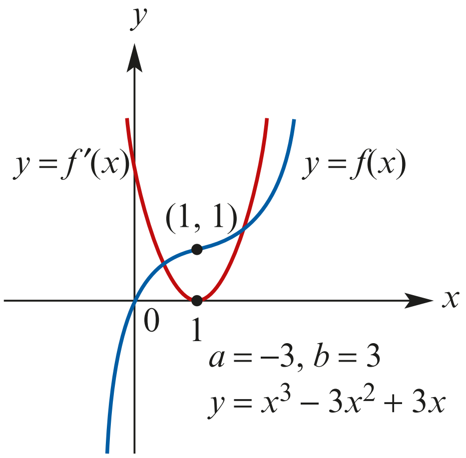


d No stationary points exist if $a^2 < 3b$

24 Let $y = \dfrac{\log_e x}{x}$

$$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \log_e x}{x^2}$$

$$= \frac{1 - \log_e x}{x^2}$$

$$\frac{dy}{dx} = 0 \text{ implies } \frac{1 - \log_e x}{x^2} = 0$$

$$\therefore x = e$$

$\dfrac{dy}{dx} < 0$ for $1 - \log_e x < 0 \Leftrightarrow \log_e x > 1 \Leftrightarrow x > e$

$\dfrac{dy}{dx} < 0$ for $1 - \log_e x > 0 \Leftrightarrow \log_e x < 1 \Leftrightarrow x < e$

$\therefore$ a maximum for $x = e$

When $x = e, y = \dfrac{\log_e}{e} = \dfrac{1}{e}$

i.e. The ratio of the logarithm of a number to the number is a maximum when $x = e$.

25 **a** **i** $f(x) = 6x^4 - x^3 + ax^2 - 6x + 8$

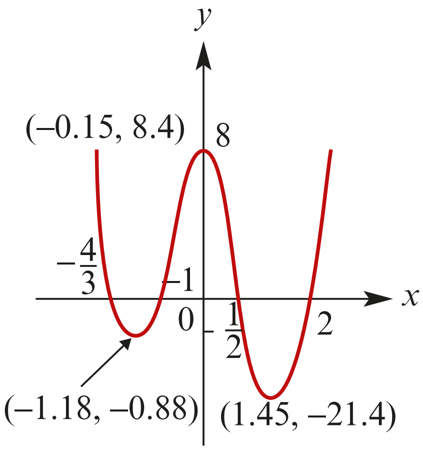
If $x + 1$ is a factor $f(-1) = 0$

i.e. $f(-1) = 6 + 1 + a + 6 + 8 = 0$

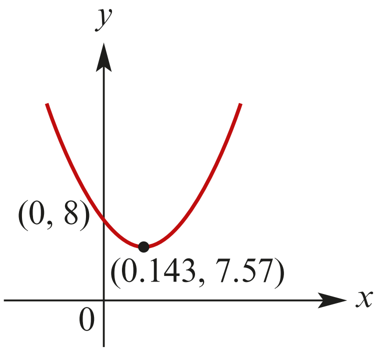
$\therefore\ a + 21 = 0$

$a = -21$

ii



b **i**



ii minimum = 7.57 when $x = 0.143$

iii $g'(x) = 24x^3 - 3x^2 + 42x - 6$

iv
$$g'(x) = 0$$
$$24x^3 - 3x^2 + 42x - 6 = 0$$
$$x = 0.1427$$

v $g'(\text{o}) = -6; g'(10) = 24\,114$

vi $\frac{d}{dx}(g(x))$ can be written as $g''(x)$, meaning the derivative of derivative.
$g''(x) = 72x^2 - 6x + 42$

vii $g''(x) = 0$ implies

$12x^2 - x + 7 = 0$

But $\Delta = 1 - 4 \times 12 \times 7 < 0$

$\therefore$ no stationary points

Hence the graph of

$y = g'(x)$ has positive gradient for all x. There is only one solution of $g'(x) = 0$.

26 **a** $f(x) = (x - a)^2(x - b)^2 a > 0\ b > 0$

$$\begin{aligned} f'(x) &= 2(x-a)(x-b)^2 + 2(x-b)(x-a)^2 \\ &= 2(x-a)(x-b)(x-b+x-a) \\ &= 2(x-a)(x-b)(2x-(b+a)) \end{aligned}$$

b **i** $f'(x) = 0$ implies $x = a$ or $x = b$ or $x = \frac{b+a}{2}$

ii $x = a$ or $x = b$

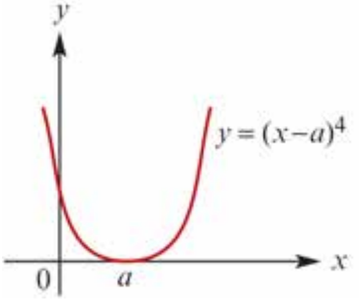
c Stationary points
$(a, 0)\ (b, 0)$

When $x = \dfrac{a+b}{2}$

$$f\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2} - a\right)^2\left(\frac{a+b}{2} - b\right)^2$$
$$= \left(\frac{a+b-2a}{2}\right)^2\left(\frac{a+b-2b}{2}\right)^2$$
$$= \left(\frac{b-a}{4}\right)^2\left(\frac{a-b}{4}\right)^2$$
$$= \frac{(a-b)^4}{16}$$

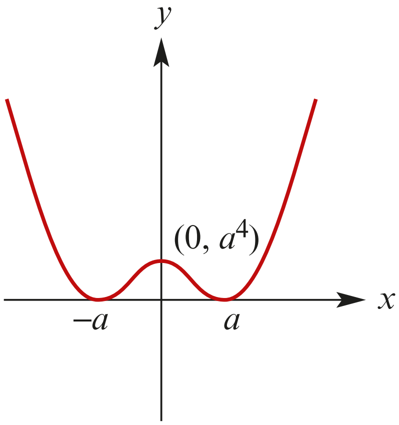
$\therefore$ coordinates $\left(\dfrac{(a+b)}{2}, \dfrac{(a-b)^4}{16}\right)$

e **i**



ii If $a = -b$ coordinates are $(a, 0)$ $(-a, 0)$ $(0, a^4)$

iii



27 **a** $f(x) = (x-a)^3(x-b)$

$$f'(x) = 3(x-a)^2(x-b) + (x-a)^3$$
$$= (x-a)^2[3(x-b) + (x-a)]$$
$$= (x-a)^2[4x - (3b+a)]$$

b **i** $f'(x) = 0$ implies $x = a$ or $x = \dfrac{3b+a}{4}$

ii $f(x) = 0$ implies $x = a$ or $x = b$

c $(a, 0)$ is a stationary point of inflection as $f'(a + h)$ and $f'(a - h)$ have the same sign where h is a small number.

If $x = \dfrac{3b + a}{4}$

$$\begin{aligned} f(x) &= \left(\frac{3b + a}{4} - a\right)^3 \left(\frac{3b + a}{4} - b\right) \\ &= \left(\frac{3b + a - 4a}{4}\right)^3 \left(\frac{3b + a - 4b}{4}\right) \\ &= \left(\frac{3b - 3a}{4}\right)^3 \left(\frac{a - b}{4}\right) \\ &= -\frac{27}{256}(b - a)^4 \end{aligned}$$

If $x > \dfrac{3b + a}{4}$ then $f'(x) > 0$

If $x < \dfrac{3b + a}{4}$ then $f'(x) < 0$

$\therefore$ local minimum at $\left(\dfrac{3b + a}{4}, -\dfrac{27}{256}(b - a)^4\right)$

d Calculator

e If $a = -b$

stationary points are $(a, 0)$ and $\left(-\dfrac{a}{2}, \dfrac{27a^4}{16}\right)$

f **i** If a local minimum at $x = 0$, $\dfrac{3b + a}{4} = 0$, i.e. $a = -3b$ or $b = -\dfrac{a}{3}$.

g If there is a turning point for $x = \dfrac{a + b}{2}$

then $\dfrac{a + b}{2} = \dfrac{3b + a}{4}$

$\therefore\ 2a + 2b = 3b + a$

$\therefore\ 0 = b - a$

$\therefore\ b = a$

If $b = a \quad f(x) = (x - a)^4$

28 $f\colon (0, 6] \to R, f(x) = x \log_e x + 1$

a $f'(x) = \log_e x + x \times \dfrac{1}{x} = \log_e x + 1$ (product rule)

b $f'(x) = 0$ implies $\log_e x = -1$

$\therefore x = e^{-1}$

When $x > e^{-1}$, $\log_e x + 1 > 0$

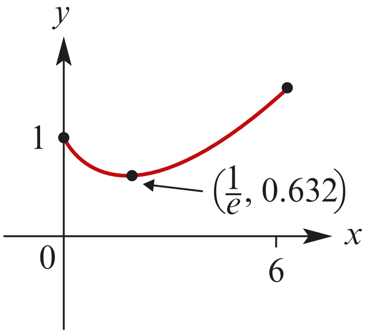
When $x < e^{-1}$, $\log_e x + 1 < 0$

$\therefore$ a minimum when $x = e^{-1} \approx 0.37$,

i.e. during the fourth month of its life.

$$\begin{aligned}\text{When } x = e^{-1}, f(x) &= \frac{1}{e}\log_e e^{-1} + 1\\ &= -\frac{1}{e} + 1\\ &\approx 0.632\end{aligned}$$

c

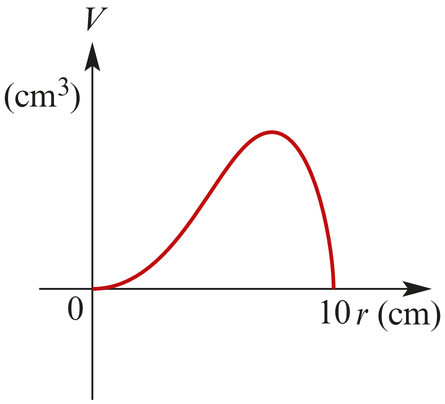


d The mouse's ability to memorise is a maximum after 6 years.

29 a i
$$y^2 + r^2 = 100$$
$$\therefore\ y = \sqrt{100 - r^2}$$
$$\therefore\ \text{height} = 2y = 2\sqrt{100 - r^2}$$

ii
$$\begin{aligned}V &= \pi r^2 h\\ &= \pi r^2\left(2\sqrt{100 - r^2}\right)\\ &= 2\pi r^2\sqrt{100 - r^2}\end{aligned}$$

b i



ii Maximum volume is 2418.4 cm^3

This occurs when $r = 8.165$ and $h = 11.55$

iii Use the 'solve' command of a CAS calculator.

$r = 6.456$ or $r = 9.297$

c **i** $V = 2\pi r^2(100 - r^2)^{\frac{1}{2}}$

$$\begin{aligned}\frac{dV}{dr} &= 4\pi r(100 - r^2)^{\frac{1}{2}} - \frac{2r}{2}(100 - r^2)^{-\frac{1}{2}} \times 2\pi r^2\\ &= 2\pi r\left[2(100 - r^2)^{\frac{1}{2}} - \frac{r^2}{(100 - r^2)^{\frac{1}{2}}}\right]\\ &= 2\pi r\frac{[2(100 - r^2) - r^2]}{(100 - r^2)^{\frac{1}{2}}}\\ &= 2\pi r\frac{[200 - 3r^2]}{(100 - r^2)^{\frac{1}{2}}}\end{aligned}$$

ii If $\frac{dV}{dr} = 0, 200 - 3r^2 = 0$

$\therefore\ 3r^2 = 200$

$$\therefore\ r^2 = \frac{200}{3}$$

$$r = \sqrt{\frac{200}{3}} = \sqrt{\frac{600}{3}}$$

$$= \frac{10\sqrt{6}}{3}$$

$\therefore$ maximum volume is given by

$$V\max = 2\pi \times \frac{200}{3}\left(100 - \frac{200}{3}\right)^{\frac{1}{2}}$$

$$= \frac{400\pi}{3}\left(\frac{100}{3}\right)^{\frac{1}{2}} = \frac{4000\pi}{3} \times \frac{\sqrt{3}}{3}$$

$$= \frac{4000\sqrt{3}\pi}{9}$$

d **i** Calculator

ii $\frac{dV}{dr} > 0$ for $r \in \left(0, \frac{20\sqrt{6}}{6}\right)$

iii $\frac{dV}{dr}$ is increasing for $r \in (0, 5.21)$

30 **a** Surface area = $\pi r^2 + 2\pi rh + 2\pi r^2$

$\therefore 3\pi r^2 + 2\pi rh = 100\pi$

$\therefore 3r^2 + 2rh = 100$

$\therefore h = \dfrac{100 - 3r^2}{2r}$

b $V = \pi r^2 h + \dfrac{2}{3}\pi r^3$

$$= \pi r^2 \left(\frac{100 - 3r^2}{2r}\right) + \frac{2}{3}\pi r^3$$

$$= \pi r \left(\frac{100 - 3r^2}{2}\right) + \frac{2}{3}\pi r^3$$

$$= \frac{\pi r}{6}(300 - 9r^2 + 4r^2)$$

$$= \frac{\pi r}{6}(300 - 5r^2)$$

c defined for $r > 0$ and $300 - 5r^2 > 0$

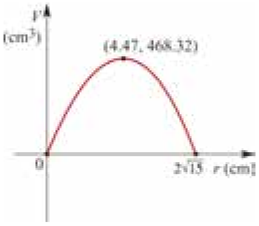
i.e. $r^2 < 60$

$r < 2\sqrt{15}$

d $V = \dfrac{\pi}{6}(300r - 5r^3)$

$\dfrac{dV}{dr} = \dfrac{\pi}{6}(300 - 15r^2)$

e



31 **a** **i** $30x^2y = 3000$

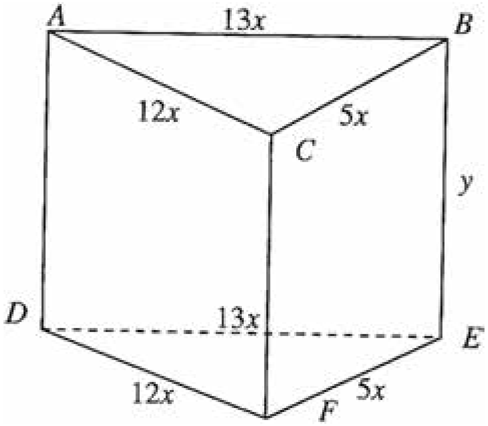
$$y = \frac{100}{x^2}$$

ii $S = 12xy + 5xy + 13xy + 60x^2$

$$= 30xy + 60x^2$$

$$= 30x\frac{100}{x^2} + 60x^2$$

$$= \frac{3000}{x} + 60x^2$$



b **i** $\frac{dS}{dx} = -\frac{3000}{x^2} + 120x$

ii $\frac{dS}{dx} = 0$ implies $3000 = 120x^3$

Therefore $x^3 = 250$

and hence $x = 5^{\frac{2}{3}}$

When $x = 5^{\frac{2}{3}}$, $S \approx 1539\,\text{cm}^2$

c $\frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt} = \left(-\frac{3000}{x2} + 120x\right) \times 0.5$

When $x = 10$, $\frac{dS}{dt} = \left(-\frac{3000}{10^2} + 1200\right) \times 0.5$

$= 585\,\text{cm}^2/\text{s}$

32 **a** $f(x) = \frac{100\,000}{1 + 100e^{-3x}}$

Using the Chain rule

$$f'(x) = -\frac{100\,000}{(1 + 100e^{-0.3x})^2} \times -30e^{-0.3x}$$

$$= \frac{3000\,000e^{-0.3x}}{(1 + 100e^{-0.3x})^2}$$

b **i** When $x = 0$, $f'(0) = \frac{3\,000\,000}{(1 + 100)^2} = 294.08$

The rate of growth is 294 kangaroos per year when $x = 0$

ii When $x = 4$, $f'(4) = \frac{3\,000\,000e^{-1.2}}{(1 + 100e^{-1.2})^2} = 933.0498$

The rate of growth is 933 kangaroos per year when $x = 4$

33 **a** f is defined for

$6 - 0.2x > 0$

$\Leftrightarrow 6 > 0.2x$

$\Leftrightarrow \frac{6}{0.2} > x$

$\Leftrightarrow 30 > x$

$\therefore a = 30$

b $f(0) = 8\log_e 6$

When $f(x) = 0$

$8\log_e(6 - 0.2x) = 0$

which implies

$6 - 0.2x = 1$

$5 = \frac{1}{5}x$

$25 = x$

$\therefore$ $(25, 0)$ and $(0, 8\log_e 6)$ are the coordinates of the axes intercepts

c
$$f(x) = 8\log_e(6 - 0.2x)$$
$$f'(x) = \frac{-8}{5(6 - 0.2x)}$$

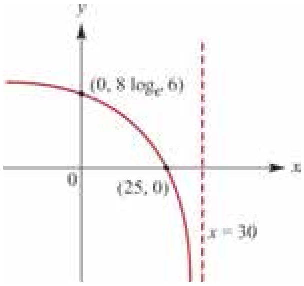
when $x = 20$

$$f'(20) = \frac{-8}{5(6 - 4)}$$
$$= \frac{-4}{5} = -0.8$$

d Consider
$$x = 8\log_e(6 - 0.2y)$$
$$e^{\frac{x}{8}} = 6 - \frac{y}{5}$$
$$\therefore\ y = 5\left(6 - e^{\frac{x}{8}}\right)$$
$$\therefore\ f^{-1}(x) = 5\left(6 - e^{\frac{x}{8}}\right)$$

e The domain of f^{-1} is R

f



34 **a** Calculator

b $g'(x) = \cos(x)e^{\sin x}$

$g'(x) = 0$ implies $\cos x = 0$ as$e^{\sin x} \neq 0$

$\therefore$ the stationary points occur at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

The coordinates of the stationary points are

$\left(\frac{\pi}{2}, e\right)$ and $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$

c range $= \left[\frac{1}{e}, e\right]$

d period $= 2\pi$ as $g(x + 2\pi) = g(x)$

35 **a**

$$y = e^x$$

$$\frac{dy}{dx} = e^x$$

When $x = 0, \frac{dy}{dx} = 1$

Therefore equation of tangent is $y = x + 1$

b Identical transformations applied to the curve and the tangent retain the relationship, i.e. the image of the tangent is tangent to the image of the curve.

c Consider the curve with equation $y = a\,f(bx)$

Then the gradient at $\left(\frac{x_1}{b}\right)$ is given by $\frac{dy}{dx} = ab\,f'(bx) = ab\,f'(x_1)$

But the gradient of $y = f(x)$ at x_1 is $f'(x_1) = m$

$\therefore$ gradient of $y = a\,f(bx)$ is abm

$\therefore$ equation of the tangent at $\left(\frac{x_1}{b}, y_1 a\right)$

is $y - y_1 a = abm\left(x - \frac{x_1}{b}\right)$

$\therefore y = bam\left(x - \frac{x_1}{b}\right) + y_1 a$

$$= bamx - amx_1 + y_1a$$

But $y_1 = mx_1 + c$ and

$$\therefore y = a(bmx - mx_1 + y_1)$$

$$y = a(bmx - mx_1 + mx_1 + c)$$

$$\therefore y = a(bmx + c)$$

36 **a** **i** When $t = 0, x = \frac{60}{5e^0 - 3} = \frac{60}{2} = 30$

When $t = 0$, there are 30 g not dissolved.

ii When $t = 5, x = \frac{60}{5e^{5\lambda} - 3}$ where $\lambda = \frac{1}{2}\log_e \frac{6}{5}$

$$= \frac{60}{5e^{\frac{5}{2}\log\left(\frac{6}{5}\right)} - 3}$$

$$= \frac{60}{5 \times \left(\frac{6}{5}\right)^{\frac{5}{2}} - 3}$$

$$\approx 12.2769$$

When $t = 5$ there are 12.28 g not dissolved.

b $\frac{dx}{dt} = 5\lambda e^{\lambda t} \times -\frac{60}{(5e^{\lambda t} - 3)^2}$ (Chain rule)

$$= -\frac{300\lambda e^{\lambda t}}{(5e^{\lambda t} - 3)^2}$$

c **i** $$x = \frac{60}{5e^{\lambda t} - 3}$$

$$\therefore \quad 5xe^{\lambda t} - 3x = 60$$

$$\therefore \quad e^{\lambda t} = \frac{3x + 60}{5x}$$

$$\therefore \quad \frac{dx}{dt} = -\frac{300\lambda e^{\lambda t}}{\left(5e^{\lambda t} - 3\right)^2}$$

$$= -300\lambda\left(\frac{3x + 60}{5x}\right) \div \left(\frac{5(3x + 60)}{5x} - 3\right)^2$$

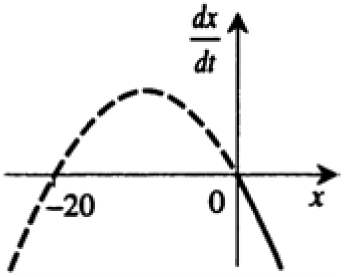
$$= -60\lambda\left(\frac{3x + 60}{x}\right) \div \left(\frac{3x + 60 - 3x}{x}\right)^2$$

$$= -60\lambda\left(\frac{3x + 60}{x}\right) \times \frac{x^2}{3600}$$

$$= -\lambda(x + 20) \times \frac{x}{20}$$

$$= -\frac{\lambda x^2}{20} - \lambda x$$

ii



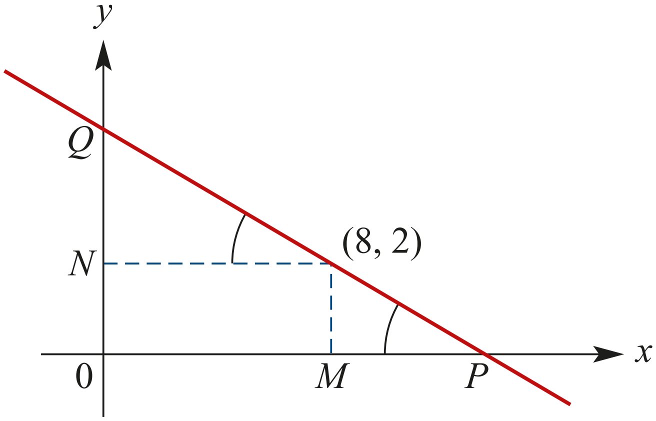
$$\frac{dx}{dt} = -\lambda\left(\frac{x^2}{20} + x\right)$$

$$= -\lambda x\left(\frac{x}{20} + 1\right)$$

x-axis intercepts $x = 0$ and $x = -20$ domain $= [0, \infty)$

iii Rate of dissolving increases where x is the amount of material not dissolved.

37



a Let $y = \frac{1}{\tan\theta}$

Let $u = \tan\theta$

Then $y = \dfrac{1}{u}$ and $\quad \dfrac{dy}{d\theta} = \dfrac{dy}{du} \times \dfrac{du}{d\theta}$

$$
\begin{aligned}
&= -\frac{1}{u^2} \times \sec^2\theta \\
&= -\frac{1}{\tan^2\theta} \times \sec^2\theta \\
&= -\frac{\cos^2\theta}{\sin^2\theta} \times \sec^2\theta \\
&= -\frac{1}{\sin^2\theta} \\
&= -\text{cosec}^2\theta
\end{aligned}
$$

b $\dfrac{2}{MP} = \tan\theta$

$\therefore MP = \dfrac{2}{\tan\theta}$

c $\dfrac{NQ}{8} = \tan\theta$

$\therefore NQ = 8\tan\theta$

d $OP + OQ = OM + MP + ON + NQ$

$$
\begin{aligned}
&= 8 + \frac{2}{\tan\theta} + 2 + 8\tan\theta \\
&= 10 + 8\tan\theta + \frac{2}{\tan\theta}
\end{aligned}
$$

e Let $x = OP + OQ$

i.e. $x = 10 + 8\tan\theta + \dfrac{2}{\tan\theta}$

$\dfrac{dx}{d\theta} = -2\,\text{cosec}^2\theta + 8\sec^2\theta$

f minimum occurs when $\dfrac{dx}{d\theta} = 0$

$$-2[\text{cosec}]^2\theta + 8\sec^2\theta = 0$$

$$\therefore\ -\frac{2}{\sin^2\theta} + \frac{8}{\cos^2\theta} = 0$$

$$\therefore\ \frac{2}{\sin^2\theta} = \frac{8}{\cos^2\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta} = \frac{2}{8} = \frac{1}{4}$$

$$\therefore \tan^2\theta = \frac{1}{4}$$

$$\text{and } \tan\theta = \pm\frac{1}{2}$$

$$\text{We know } 0 < \theta < \frac{\pi}{2}$$

$$\therefore \tan\theta = \frac{1}{2}$$

$$\therefore \theta > 26.6^\circ$$

If $\theta > 26.6^\circ, \dfrac{dx}{d\theta} < 0$ (locally)

If $\theta < 26.6^\circ, \dfrac{dx}{d\theta} > 0$

$\therefore$ minimum when $\theta = 26.6^\circ$

If $\tan\theta = \dfrac{1}{2}$

$$\begin{aligned} x &= \frac{2}{\tan\theta} + 8\tan\theta + 10 \\ &= \frac{2}{\frac{1}{2}} + 8 \times \frac{1}{2} + 10 \\ &= 4 + 4 + 10 = 18 \end{aligned}$$

The minimum value of x is 18 units.

38 Let $f : R \to R, f(x) = e^x - e^{-x}$

a $f'(x) = e^x + e^{-x}$

b $f(x) = 0$ implies $e^x - e^{-x} = 0$

i.e. $e^x - \dfrac{1}{e^x} = 0$

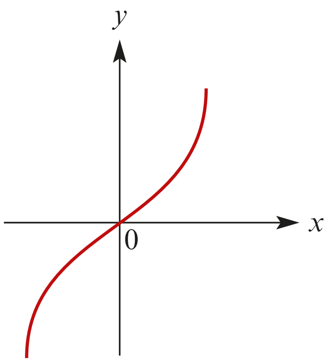
$\therefore \quad e^{2x} - 1 = 0$

$\therefore \quad 2x = 0$

which implies $x = 0$

c $f'(x) = e^x + e^{-x}$ and both e^x and e^{-x} are positive, so $f'(x) > 0$ for all x.

d



The result that $f'(x) > 0$ for all x is used

39 **a** $(\log_e x)^2 = 2\log_e x$

is equivalent to $(\log_e x)^2 - 2\log_e x = 0$

i.e. $\log_e x[\log_e x - 2] = 0$

which implies $\log_e x = 0$ or $\log_e x = 2$

$\therefore x = 1$ or $x = e^2$

b

For $y = 2\log_e x$

$$\frac{dy}{dx} = \frac{2}{x}$$

For $y = (\log_e x)^2$

Let $u = \log_{2^e} x$

Then $y = u^2$

and
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= 2u.\frac{1}{x}$$
$$= \frac{2\log_e x}{x}$$

The gradient of $y = 2\log_e x$ at $x = 1$ is 2

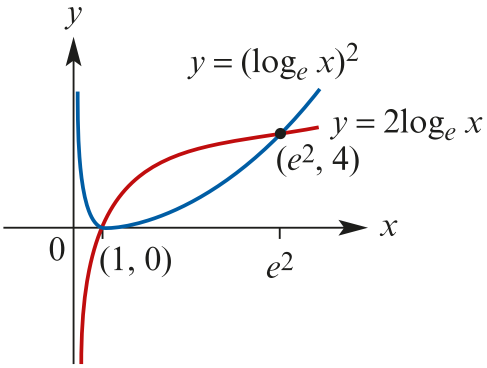
The gradient of $y = (\log_e x)^2$ at $x = 1$ is 0

This information is now used to sketch the graphs.

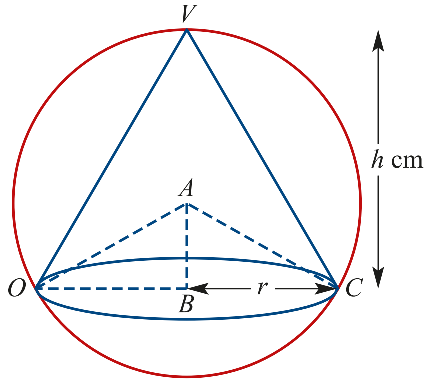
c Note: $y = (\log_e x)^2 \geq 0$ for all x and

$(\log_e x)^2 \to \infty$ as $x \to 0$

d $\therefore \left\{x\colon 2\log_e x > (\log_e x)^2\right\} = \left\{x\colon 1 < x < e^2\right\}$



40 A cross-section of the solids is as shown.



a $h = VA + AE$

$= a + a\cos\theta$

where a is the radius of the sphere

b $r = a\sin\ \theta$

c $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(a^2\sin^2\theta)(a + a\cos\theta)$

$= \frac{1}{3}\pi a^3\sin^2\theta(1 + \cos\theta)$

d Using the product rule

$$\frac{dV}{d\theta} = \frac{1}{3}\pi a^3[\sin^2\theta \times -\sin\theta + 2\sin\theta\cos\theta(1 + \cos\theta)]$$

$$= \frac{1}{3}\pi a^3[-\sin^3\theta + 2\sin\theta\cos\theta(1 + \cos\theta)]$$

$$\frac{dV}{d\theta} = 0 \text{ implies } \frac{1}{3}\pi a^3[-\sin^3\theta + 2\sin\theta\cos\theta(1 + \cos\theta)] = 0$$

$\therefore$ $\sin^3\theta = 2\sin\theta\cos\theta(1 + \cos\theta)$

For $\sin\theta \neq 0$

$$\sin^2\theta = 2\cos\theta + 2\cos^2\theta$$

Using $\sin^2\theta = 1 - \cos^2\theta$

$$1 - \cos^2\theta = 2\cos\theta + 2\cos^2\theta$$

which implies $3\cos^2\theta + 2\cos\theta - 1 = 0$

This is a quadratic equation in $\cos\theta$. It factorises to give the following:

$(3\cos\theta - 1)(\cos\theta + 1) = 0$

$$\therefore \cos\theta = \frac{1}{3} \text{ or } \cos\theta = -1$$

A gradient chart confirms a maximum volume occurs when

$\cos\theta = \frac{1}{3}\left(\text{when } \theta = \cos^{-1}\left(\frac{1}{3}\right)\right)$, i.e. $\theta \approx 70.53°$

e $$V = \frac{1}{3}\pi a^3 \sin^2 4(1 + \cos\theta)$$

Using $\sin^2\theta = 1 - \cos^2\theta$

$$V = \frac{1}{3}\pi a^3(1 - \cos^2\theta)(1 + \cos\theta)$$

When $\cos\theta = \frac{1}{3}$

$$\begin{aligned} V &= \frac{1}{3}\pi a^3\left(1 - \frac{1}{9}\right)\left(1 + \frac{1}{3}\right) \\ &= \frac{1}{3}\pi a^3 \times \frac{8}{9} \times \frac{4}{3} \\ &= \frac{32\pi a^3}{81} \end{aligned}$$

The maximum volume is $\frac{32\pi a^3}{81}$ cm^3

41 **a** $y = \dfrac{Ae^{bt}}{1 + Ae^{bt}}$

Dividing through by $1 + Ae^{bt}$ gives

$y = 1 - \dfrac{1}{1 + Ae^{bt}}$

and as $Ae^{bt} > 0$ for all t, $\dfrac{1}{1 + Ae^{bt}} < 1$

Hence $0 < y < 1$

b By using the quotient rule

$$\begin{aligned} \frac{dy}{dt} &= \frac{(1 + Ae^{bt})bAe^{bt} - bA^2e^{2bt}}{(1 + Ae^{bt})^2} \\ &= \frac{bAe^{bt}}{(1 + Ae^{bt})^2} \end{aligned}$$

c As $$y = \frac{Ae^{bt}}{1 + Ae^{bt}}$$

$$y(1 + Ae^{bt}) = Ae^{bt}$$

$\therefore$ $$y + yAe^{bt} = Ae^{bt}$$

and $$y = Ae^{bt}(1 - y)$$

$\therefore$ $$Ae^{bt} = \frac{y}{1 - y}$$

d **i** From the result of **b**

$$\frac{dy}{dt} = \frac{bAe^{bt}}{(1 + Ae^{bt})^2}$$

Substituting $Ae^{bt=}\frac{y}{1-y}$

$$\begin{aligned}\frac{dy}{dx} &= b\left(\frac{y}{1-y}\right) \div \left(1 + \frac{y}{1-y}\right)^2\\ &= b\left(\frac{y}{1-y}\right) \times (1-y)^2\\ &= by(1-y)\end{aligned}$$

ii $\frac{dy}{dx} = by(1 - y)$
is a quadratic expression in y with negative coefficient of y^2 (b is a positive constant)
$\therefore$ a maximum occurs when $y = 0.5$.

e From **c** $Ae^{bt} = \frac{y}{1-y}$

$\therefore$ when $A = 0.01, b = 0.7$ and $y = 0.5$

$$\begin{aligned}0.01e^{0.7t} &= 1\\ \therefore \quad e^{0.7t} &= 100\\ \therefore \quad 0.7t &= \log_e 100\\ \therefore \quad t &= \frac{10}{7}\log_e 100 \approx 6.578\end{aligned}$$

$\therefore$ The bacteria are increasing at the fastest rate when $t = 7$ (to the nearest hour).

42 Let $f(x) = \frac{e^x}{x}$

a $f(x) = e^x.x^{-1}$
The product rule gives

$$\begin{aligned}f'(x) &= e^x x^{-1} - e^x x^{-2}\\ &= \frac{e^x}{x} - \frac{e^x}{x^2}\\ &= \frac{xe^x - e^x}{x^2}\end{aligned}$$

b If

$$f'(x) = 0$$

$$\frac{xe^x - e^x}{x^2} = 0$$

which implies $e^x(x - 1) = 0$
and as $e^x \neq 0$ for all $x \in R^+$, $x = 1$

c There is a stationary point when $x = 1$ and $f(1) = e$.

Therefore there is a stationary point at $(1, e)$.

If $x > 1$ $f'(x) = \dfrac{e^x(x-1)}{x^2} > 0$ (Note: domain of f' is R^+)

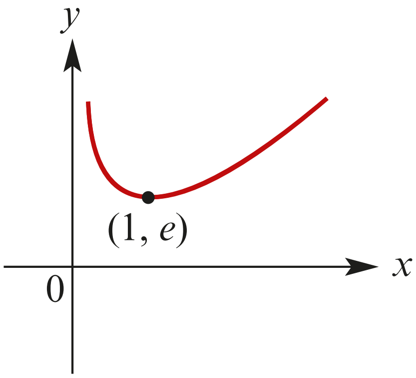
If $x < 1$ $f'(x) = \dfrac{e^x(x-1)}{x^2} < 0$

$\therefore$ there is a minimum at$(1, e)$

d **i** $$\frac{f'(x)}{f(x)} = \frac{xe^x - e^x}{x^2} \times \frac{x}{e^x}$$
$$= \frac{x-1}{x}$$

ii $\lim_{x\to\infty} \dfrac{f'(x)}{f(x)} = \lim\limits_{x\to\infty}\left(1 - \dfrac{1}{x}\right) = 1$

$\therefore$ $f'(x) \approx f(x)$ when x is very large



e $n = \dfrac{ae^{kt}}{t}$

When $t = 65, n = \dfrac{ae^{65k}}{65}$

When $t = 30, n = \dfrac{ae^{30k}}{30}$

The population of birds is the same for the years 1930 and 1965.

$$\therefore \quad \frac{ae^{65k}}{65} = \frac{ae^{30k}}{30}$$
$$\therefore \quad e^{35k} = \frac{65}{30} = \frac{13}{6}$$
$$\therefore \quad 35k = \log_e\left(\frac{13}{6}\right)$$
$$\therefore \quad k = \frac{1}{35}\log_e\left(\frac{13}{6}\right) \approx 0.0221$$

f Minimum occurs when $\dfrac{dn}{dt} = 0$

Using the quotient rule

$$\frac{dn}{dt} = \frac{a(kte^{kt} - e^{kt})}{t^2}$$
$$= \frac{ae^{kt}(kt-1)}{t^2}$$

$\dfrac{dn}{dt} = 0$ implies $t = \dfrac{1}{k}$

This is a local minimum

and $t = \dfrac{1}{\frac{1}{35}\log_e\left(\frac{13}{6}\right)} = \dfrac{35}{\log_e\left(\frac{13}{6}\right)} \approx 45.27$

The minimum population occurred in 1945.

43 **a** When $t = 0, N = 1000$

When $t = 5, N = 10\,000$

As $N = Ae^{kt}$

$1000 = Ae^0$

which implies $A = 1000$

Also $10\,000 = 1000e^{5k}$

which implies

$e^{5k} = 10$

$\therefore k = \frac{1}{5}\log_e 10$

i.e. $A = 1000$ and $k = \frac{1}{5}\log_e 10 \approx 0.46$

b $\dfrac{dN}{dt} = kAe^{kt}$, where $A = 1000$ and $k = \frac{1}{5}\log_e 10$

c $\dfrac{dN}{dt} = kN$ as $N = Ae^{kt}$

d **i** When $t = 4$

$$\frac{dN}{dt} = \frac{1}{5}\log_e 10 \times 1000e^{\frac{4}{5}\log_e 10}$$

$$= 200\log_e 10 \times 10^{\frac{4}{5}}$$

i.e. the rate of growth when $t = 4$ is 2905.7 bacteria/hour.

ii When $t = 50$

$$\frac{dN}{dt} = 200(\log_e 10)e^{\frac{50}{5}\log_e 10}$$

$$= 200\log_e 10 \times 10^{10}$$

$$= 2\log_e 10 \times 10^{12}$$

$$\approx 4.61 \times 10^{12}$$

i.e. the rate of growth when $t = 50$ is 4.61×10^{12} bacteria/hour.

44 $T(t) = p + q\cos(\pi r t)$ where p, q and r are constants

a From the graph:

i The period is 12 $\therefore \quad \frac{2\pi}{\pi r} = 12$

which implies $r = \frac{1}{6}$

ii The amplitude is $\frac{(20-4)}{2} = 8$ which implies $q = 8$

The centre is $T = \frac{20+4}{2} = 12$ which implies $p = 12$

b $T'(t) = -\pi r q \sin(\pi r t)$

$$\therefore T'(3) = -\frac{4\pi}{3}\sin\left(\frac{3\pi}{6}\right)$$

$$= -\frac{4\pi}{3}\sin\left(\frac{\pi}{2}\right) = -\frac{4\pi}{3}$$

The hours of night are decreasing at a rate of $\frac{4\pi}{3}$ hours/month when $t = 3$

$$T'(9) = -\frac{4\pi}{3}\sin\left(\frac{3\pi}{2}\right) = \frac{4\pi}{3}$$

The hours of night are increasing at a rate of $\frac{4\pi}{3}$ hours/month when $t = 9$

c Average rate of change from $t = 0$ to $t = 6$

$$= \frac{T(6) - T(0)}{6 - 0}$$

when $T(t) = 12 + 8\cos\left(\frac{\pi t}{6}\right)$

$$\therefore \text{ Average rate of change} = \frac{12 + 8\cos(\pi) - (12 + 8\cos(0))}{6}$$

$$= \frac{12 - 8 - 12 - 8}{6}$$

$$= -\frac{16}{6} = -\frac{8}{3}$$

i.e. the average rate of change for time interval $[0, 6]$ is $-\frac{8}{3}$ hours/month.

d $T'(t) = -\frac{4\pi}{3}\sin\left(\frac{\pi t}{6}\right)$

Rate of change of hours is maximum (in the sense of maximum increasing rate) when $\sin\left(\frac{\pi t}{6}\right) = -1$

This occurs when

$$\frac{\pi t}{6} = \frac{3\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \ldots$$

i.e. $t = 9$ or 21 or ...

The rate of change of hours of night is a miximum after 9 months.

45 **a** Area A = length × width

$$= x \times 2\cos(3x)$$

$$= 2x\ \cos(3x)$$

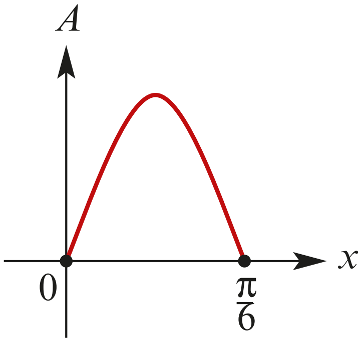
b **i** $\dfrac{dA}{dx} = 2\cos(3x) - 6x\sin(3x)$

ii When $x = 0, \dfrac{dA}{dx} = 2$

When $x = \dfrac{\pi}{6}, \dfrac{dA}{dx} = -6 \times \dfrac{\pi}{6}\sin\dfrac{\pi}{2}$

$$= -\pi$$

c **i**



ii Either use the 'Intersect' feature of a CAS calculator of the graph screen or use the 'solve' command at the calculator screen to solve the equation $2x\cos 3x = 0.2$.

$x = 0.105$ or $x = 0.449$

iii maximum area is 0.374

Use the 'max' feature of a CAS calculator of the graph screen or use the 'flex' command at the calculator screen with the instruction $0 < x < \dfrac{y}{6}$. when $x = 0.287$

d **i** $\dfrac{dA}{dx} = 2\cos(3x) - 6x\sin(3x)$

$\dfrac{dA}{dx} = 0$ implies $\tan(3x) = \dfrac{1}{3x}$

ii The co-ordinates of the points of intersection are (0.287, 1.16), founded as with **c ii**.

46 **a** **i** $N'(t) = -1 + \dfrac{1}{10}e^{\frac{t}{20}}$

ii $N'(t) = 0$ implies $10 = e^{\frac{t}{20}}$

$\therefore \quad t = 20\log_e 10 \approx 46.05$

When $t = 20\log_e 10$

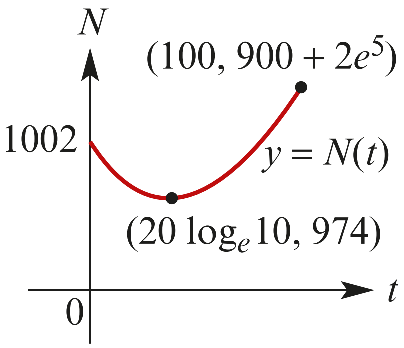
$$\begin{aligned} N(t) &= 1000 - 20\log_e 10 + 2e^{\log_e 10} \\ &= 1000 - 20\log_e 10 + 20 \\ &= 1020 - 20\log_e 10 \\ &\approx 973.95 \end{aligned}$$

Minimum population is 974

iii $N(0) = 1000 + 2 = 1002$

iv $$\begin{aligned} N(100) &= 1000 - 100 + 2e^5 \\ &= 900 + 2e^5 \\ &\approx 1196.826 \end{aligned}$$

v



b $N_2(t) = 1000 - t^{\frac{1}{2}} + 2e^{\frac{t^{\frac{1}{2}}}{20}}$

i $$\begin{aligned} N_2(0) &= 1000 - 0 + 2 \\ &= 1002 \end{aligned}$$

ii $$\begin{aligned} N_2(100) &= 1000 - 10 + 2e^{\frac{1}{2}} \\ &= 990 + 2e^{\frac{1}{2}} \end{aligned}$$

iii $$\begin{aligned} N_2'(t) &= -\frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{2}t^{-\frac{1}{2}} \times \frac{1}{20} \times 2e^{\frac{t^{\frac{1}{2}}}{20}} \\ &= \frac{1}{2}t^{\frac{1}{2}}\left(-1 + \frac{1}{10}e^{\frac{t^{\frac{1}{2}}}{20}}\right) \end{aligned}$$

$$N_2'(t) = 0 \text{ implies } e^{\frac{t^{\frac{1}{2}}}{20}} = 10$$

$$\therefore \quad \frac{t^{\frac{1}{2}}}{20} = \log_e 10$$

$$\therefore \quad t^{\frac{1}{2}} = 20\log_e 10$$

$$\therefore \quad t = (20\log_e 10)^2$$

When $t = (20\log_e 10)^2$

$$N_2(t) = 1000 - 20\log_e 10 + 2e^{\log_s 10}$$

$$= 1000 - 20\log_e 10 + 20$$

Minimum population is 974

c $N_3(t) = 1000 - t^{\frac{3}{2}} + 2e^{\frac{t}{20}}$

i Using a CAS calculator with the 'min' feature at the graph screen, the minimum population is
297 when $t = 100.24$

d **i** $N_3'(t) = -\frac{3}{2}t^{\frac{1}{2}} + \frac{1}{10}e^{\frac{t}{20}}$

ii $N_3'(t) = 0$

$$\frac{3}{2}t^{\frac{1}{2}} = \frac{1}{10}e^{\frac{t}{20}}$$

$$15t^{\frac{1}{2}} = e^{\frac{t}{20}}$$

$$t = 20\log_e(15\sqrt{t})$$

47 **a** $y = (2x^2 - 5x)e^{ax}$

$(3, 10):\ 10 = 3e^{3a}$

$$e^{3a} = \frac{10}{3}$$

$$a = \frac{1}{3}\log e\left(\frac{10}{3}\right)$$

b **i** $y = 0$: $2x^2 - 5x = 0$ (since $e^{ax} > 0$)

$x(2x - 5) = 0$

$$x = 0, \frac{5}{2}$$

ii $$\frac{dy}{dx} = (4x - 5)e^{ax} + (2x^2 - 5x) \times ae^{ax}$$

$$= (2ax^2(4 - 5a)x - 5)e^{ax}$$

$= 0$ if

$2ax^2 + (4 - 5a)x - 5 = 0$

$$x = \frac{-4 + 5a \pm \sqrt{16 - 40a + 25a^2 + 40a}}{4a}$$

$$= \frac{-4 + 5a \pm \sqrt{25a^2 + 16}}{4a}$$

48 **a** $f(x) = x^3 - x^2 - x + 12$

$f'(x) = 3x^2 - 2x - 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

$$\therefore x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - x_n + 12}{3x_n^2 - 2x_n - 1}$$

$$= \frac{x_n(3x_n^2 - 2x_n - 1) - (x_n^3 - x_n^2 - x_n + 12)}{3x_n^2 - 2x_n - 1}$$

$$= \frac{(3x_n^3 - 2x_n^2 - x_n) - (x_n^3 - x_n^2 - x_n + 12)}{3x_n^2 - 2x_n - 1}$$

$$= \frac{2x_n^3 - x_n^2 - 12}{3x_n^2 - 2x_n - 1}$$

b **i**

```
define $f(x)$:
  return $x^3 - x^2 - x + 12$

define $Df(x)$:
  return $3x^2 - 2x - 1$

$x \leftarrow -2$
while $f(x) > 10^{-6}$ or $f(x) < -10^{-6}$
    $x \leftarrow x - \frac{f(x)}{Df(x)}$
    print $x, f(x)$
end while
```

ii $x_0 = -2$

$x_1 = -2.1333333333333333$

$x_2 = -2.125838367918746$

$x_3 = -2.12581366303931$

$x_4 = -2.125813662771433$

It takes two iterations to reach this accuracy.

c $f'(1) = 0$. There is a local minimum. The tangent is parallel to the x-axis

d It take 35 iterations to reach the solution to 4 decimal places.

$f'(0) = -1$. The tangent has equation $y = -x + 12$. The first iteration gives $x_1 = 12$.

It has moved away. The turning points are at $x = 1$ and $x = -\frac{1}{3}$.

Chapter 11 – Integration

Solutions to Exercise 11A

1

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$f(x_0) = 0, f(x_1) = 5, f(x_2) = 14, f(x_3) = 27, f(x_4) = 44$

$$\frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{2}(0 + 10 + 28 + 54 + 44)$$

$$= \frac{1}{2} \times 136$$

$$= 68$$

2

$f(x_0) = 1, f(x_1) = \frac{1}{2}, f(x_2) = \frac{1}{3}, f(x_3) = \frac{1}{4}, f(x_4) = \frac{1}{5}$

$$\frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{2}(1 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{5})$$

$$= \frac{1}{2} \times \frac{101}{30}$$

$$= \frac{101}{60}$$

3

$f(x_0) = 5, f(x_1) = 3.5, f(x_2) = 2.5, f(x_3) = 2.2, f(x_4) = 2$

$$\frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= 0.5(5 + 7 + 5 + 4.4 + 2)$$

$$= 11.7$$

4 **a** $\frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+f(x_6)\right]$

$= \frac{1}{4}(0+2.5+4+4.5+4+2.5+0)$

$= \frac{35}{8}$

b 4.48. (15 trapeziums)

5 $\frac{1}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+\cdots+2f(9)+f(10)\right]$

$= 0.5(0+7+7.4+7.6+7.8+7.8+8+8+7.4+6.6+2.9)$

$= 36.75$

6

$$f(x_0)=1, f(x_1)=\frac{16}{17}, f(x_2)=\frac{4}{5}, f(x_3)=\frac{16}{25}, f(x_4)=\frac{1}{2}$$

$\frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]$

$= \frac{1}{8}(1+\frac{32}{17}+\frac{8}{5}+\frac{32}{25}+\frac{1}{2})$

$= \frac{1}{8}\times\frac{5323}{850}$

$= \frac{5323}{6800}$

$\approx 0.782794\ldots$

Therefore $\pi \approx 3.13$

7 **a** 4.371

b 1.128

8 109.5 m^2

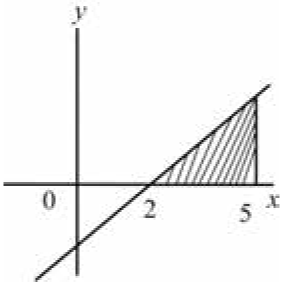
9 **a** The definite integral represents the trangular region shown.

The trangle has base $5-2=3$ units.

When $x=5$, $y=5-2=3$, so the triangle has height 3 units Area $= \frac{1}{2}\times 3\times 3$

$= \frac{9}{2}$ square units

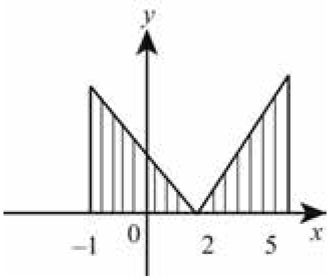
Hence $\int_2^5 (x-2)\,dx = \frac{9}{2}$.



b The definits integral represents two equal traingular regions show.

Formal, area $= 2 \times \frac{9}{2} = 9$ square units

Hence $\int_{-1}^{5} |x-2|\,dx = 9$.



c The definits integral represents the trapezium region shown.

The distance between the parallel sides is 1 unit.

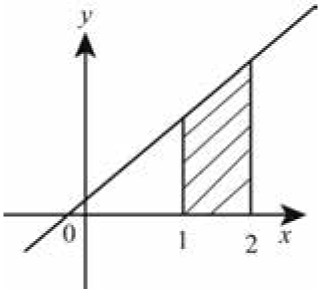
When $x = 1, y = 2 + 1 = 3$ units.

When $x = 2, y = 4 + 1 = 5$ units.

$$\text{Area} = \frac{1}{2} \times 1 \times (3+5)$$

$$= 4 \text{ square units}$$

Hence $\int_1^2 (2x+1)\,dx = 4$



Solutions to Exercise 11B

1 a $\int \frac{1}{2}x^3\,dx$

$= \frac{1}{2} \times \frac{1}{4}x^4 + c$

$= \frac{1}{8}x^4 + c$

b $\int 5x^3 - 2x\,dx$

$= 5 \times \frac{x^4}{4} - 2 \times \frac{x^2}{2} + c$

$= \frac{5}{4}x^4 - x^2 + c$

c $\int \frac{4}{5}x^3 - 3x^2\,dx$

$= \frac{4}{5} \times \frac{x^4}{4} - 3\frac{x^3}{3} \times + c$

$= \frac{1}{5}x^4 - x^3 + c$

d $\int 6z - 3z^2 - z + 2\,dz$

$= \int 3z^2 + 5z + 2\,dz$

$= -3\frac{z^3}{3} + 5\frac{z^2}{2} + 2z + c$

$= -z^3 + \frac{5}{2}z^2 + 2z + c$

2 a $\frac{dy}{dx} = x^{-3}$

$y = -\frac{1}{2}x^{-2} + c = -\frac{1}{2x^2} + c$

b $\frac{dy}{dx} = 4\sqrt[3]{x} = 4x^{\frac{1}{3}}$

$y = 4 \times \frac{3}{4}x^{\frac{4}{3}} + c = 3x^{\frac{4}{3}} + c$

c $\frac{dy}{dx} = x^{\frac{1}{4}} + x^{-\frac{3}{5}}$

$y = \frac{4}{5}x^{\frac{5}{4}} + \frac{5}{2}x^{\frac{2}{5}} + c$

3 a $\int 3x^{-2}\,dx$

$= 3\frac{x^{-1}}{-1} + c$

$= \frac{-3}{x} + c$

b $\int 2x^{-4} + 6x\,dx$

$= 2\frac{x^{-3}}{-3} + 6\frac{x^2}{2} + c$

$= \frac{-2}{3}x^{-3} + 3x^2 + c$

c $\int 2x^{-2} + 6x^{-3}\,dx$

$= -2x^{-1} - 3x^{-2} + c$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}}\,dx$

$= 3\frac{x^{\frac{4}{3}}}{\frac{4}{3}} - 5\frac{x^{\frac{9}{4}}}{\frac{9}{4}} + c$

$= \frac{9}{4}x^{\frac{4}{3}} - \frac{20}{9}x^{\frac{9}{4}} + c$

e $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}}\,dx$

$= \frac{12}{7}x^{\frac{7}{4}} - \frac{14}{3}x^{\frac{3}{2}} + c$

f $\int 4x^{\frac{3}{5}} + 12x^{\frac{5}{3}}\,dx$

$= \frac{5}{2}x^{\frac{8}{5}} + \frac{9}{2}x^{\frac{8}{3}} + c$

4 a $\dfrac{dy}{dx} = 2x - 3$

$$y = \frac{2x^2}{2} - 3x + c$$
$$= x^2 - 3x + c$$
$$x = 1,\ y = 1$$
$$1 = 1 - 3 + c$$
$$c = 3$$
$$y = x^2 - 3x + 3$$

b $\dfrac{dy}{dx} = x^3$

$$y = \frac{x^4}{4} + c$$
$$x = 0,\ y = 6$$
$$6 = c$$
$$y = \frac{x^4}{4} + 6$$

c $y = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{x^2}{2} + c$

$$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 + c$$
$$x = 4,\ y = 6$$
$$6 = \frac{2}{3} \times 8 + \frac{1}{2} \times 16 + c$$
$$6 = \frac{16}{3} + 8 + c$$
$$c = -2 - \frac{16}{3} = \frac{-22}{3}$$
$$y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 - \frac{22}{3}$$

5 a $\displaystyle\int \sqrt{x}(2 + x)\,dx = \int 2x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

$$= \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c$$

b $\displaystyle\int \frac{3z^4 + 2z}{z^3}\,dz = \int 3z + 2z^{-2}\,dz$

$$= \frac{3z^2}{2} - 2z^{-1} + c$$
$$= \frac{3z^2}{2} - \frac{2}{z} + c$$
$$= \frac{3z^3 - 4}{2z} + c$$

c $\displaystyle\int \frac{5x^3 + 2x^2}{x},dx = \int 5x^2 + 2x\,dx$

$$= \frac{5x^3}{3} + x^2 + c$$
$$= \frac{5x^3 + 3x^2}{3} + c$$

d $\displaystyle\int \sqrt{x}(2x + x^2)\,dx = \int 2x^{\frac{3}{2}} + x^{\frac{5}{2}}\,dx$

$$= \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + c$$

e $\displaystyle\int x^2(2 + 3x^2)\,dx = \int 2x^2 + 3x^4\,dx$

$$= \frac{2x^3}{3} + \frac{3x^5}{5} + c$$

f $\displaystyle\int \sqrt[3]{x}(x + x^4)\,dx = \int x^{\frac{4}{3}} + x^{\frac{13}{3}}\,dx$

$$= \frac{3}{7}x^{\frac{7}{3}} + \frac{3}{16}x^{\frac{16}{3}} + c$$

6 $f'(x) = 3x^2 - x^{-2}$

$$f(x) = 3\frac{x^3}{3} - \frac{x^{-1}}{-1} + c$$
$$= x^3 + \frac{1}{x} + c$$
$$f(2) = 0$$
$$0 = 8 + \frac{1}{2} + c$$
$$c = \frac{-17}{2}$$
$$f(x) = x^3 + \frac{1}{x} - \frac{17}{2}$$

7 $\dfrac{ds}{dt} = 3t - \dfrac{8}{t^2} = 3t - 8t^{-2}$

$$s = 3\frac{t^2}{2} - \frac{8t^{-1}}{-1} + c$$

$$= \frac{3}{2}t^2 + \frac{8}{t} + c$$

$$t = 1,\ s = \frac{3}{2}$$

$$\frac{3}{2} = \frac{3}{2} + 8 + c$$

$$c = -8$$

$$s = \frac{3}{2}t^2 + \frac{8}{t} - 8$$

8 **a** $f'(x) = 16x + k$

$f'(2) = 0$s

$$0 = 32 + k$$

$$k = -32$$

$$f'(x) = 16x - 32$$

b $f(x) = 16\dfrac{x^2}{2} - 32x + c$

$$= 8x^2 - 32x + c$$

$$f(2) = 1$$

$$1 = 32 - 64 + c$$

$$c = 33$$

$$f(x) = 8x^2 - 32x + 33$$

$$f(7) = 8 \times 49 - 32 \times 7 + 33$$

$$= 201$$

Solutions to Exercise 11C

1 **a** $$\int (2x-1)^2\,dx$$

$$= \frac{1}{2\times 3}(2x-1)^3 + c$$

$$= \frac{1}{6}(2x-1)^3 + c$$

b $$\int (2-t)^3\,dt$$

$$= \frac{1}{-1\times 4}(2-t)^4 + c$$

$$= \frac{-1}{4}(2-t)^4 + c$$

c $$\int (5x-2)^3\,dx$$

$$= \frac{1}{5\times 4}(5x-2)^4 + c$$

$$= \frac{1}{20}(5x-2)^4 + c$$

d $$\int (4x-6)^{-2}\,dx$$

$$= -\frac{1}{4}(4x-6)^{-1} + c$$

$$= -\frac{1}{4(4x-6)} + c$$

$$= \frac{1}{24-16x} + c$$

e $$\int (6-4x)^{-3}\,dx$$

$$= \frac{1}{8}(6-4x)^{-2} + c$$

$$= \frac{1}{8(6-4x)^2} + c$$

f $$\int (4x+3)^{-3}\,dx$$

$$= -\frac{1}{8}(4x+3)^{-2} + c$$

$$= -\frac{1}{8(4x+3)^2} + c$$

g $$\int (3x+6)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3\times\frac{3}{2}}(3x+6)^{\frac{3}{2}} + c$$

$$= \frac{2}{9}(3x+6)^{\frac{3}{2}} + c$$

h $$\int (3x+6)^{\frac{-1}{2}}\,dx$$

$$= \frac{1}{3\times\frac{1}{2}}(3x+6)^{\frac{1}{2}} + c$$

$$= \frac{2}{3}(3x+6)^{\frac{1}{2}} + c$$

i $$\int (2x-4)^{\frac{7}{2}}\,dx$$

$$= \frac{1}{2\times\frac{9}{2}}(2x-4)^{\frac{9}{2}} + c$$

$$= \frac{1}{9}(2x-4)^{\frac{9}{2}} + c$$

j $$\int (3x+11)^{\frac{4}{2}}\,dx$$

$$= \frac{1}{3\times\frac{7}{3}}(3x+11)^{\frac{7}{2}} + c$$

$$= \frac{1}{7}(3x+11)^{\frac{7}{2}} + c$$

k $\int (2-3x)^{\frac{1}{2}}\,dx$

$= \dfrac{1}{-3 \times \dfrac{3}{2}}(2-3x)^{\frac{3}{2}} + c$

$= \dfrac{-2}{9}(2-3x)^{\frac{3}{2}} + c$

l $\int (5-2x)^4\,dx$

$= \dfrac{1}{-2 \times 5}(5-2x)^5 + c$

$= \dfrac{-1}{10}(5-2x)^5 + c$

2 **a** $\int \frac{1}{2}x^{-1}\,dx$

$= \frac{1}{2}\log_e x + c$

b $\int \dfrac{1}{3x+2}\,dx$

$= \frac{1}{3}\log_e(3x+2) + c$

c $\int \dfrac{4}{1+4x}\,dx$

$= \log_e(1+4x) + c$

d $\int \dfrac{5}{3x-2}\,dx$

$= \frac{5}{3}\log_e(3x-2) + c$

e $\int \dfrac{3}{1-4x}\,dx$

$= -\frac{3}{4}\log_e(1-4x) + c$

f $\int \dfrac{3}{2-\frac{x}{2}}\,dx$

$= -6\log_e(\dfrac{4-x}{2}) + c$

$= -6\log_e(x-4) + c_2$

3 **a** $\int \frac{5}{x}\,dx$

$= 5\log_e|x| + c$

b $\int \dfrac{3}{x-4}\,dx$

$= 3\log_e|x-4| + c$

c $\int \dfrac{10}{2x+1}\,dx$

$= \dfrac{10}{2}\log_e|2x+1| + c$

$= 5\log_e|2x+1| + c$

d $\int \dfrac{6}{5-2x}\,dx$

$= \dfrac{6}{-2}\log_e|5-2x| + c$

$= -3\log_e|2x-5| + c$

e $\int 6(1-2x)^{-1}\,dx$

$= -3\log_e|1-2x| + c$

f $\int (4-3x)^{-1}\,dx$

$= \dfrac{1}{-3}\ln|4-3x| + c$

$= \dfrac{-1}{3}\ln|3x-4| + c$

4 **a** $3x + \log_e|x| + c$

b $x + \log_e|x| + c$

c $-\dfrac{1}{x+1} + c$

d $2x + \dfrac{x^2}{2} + \log_e|x| + c$

e $-\dfrac{3}{2(x-1)^2} + c$

f $-2x + \log_e |x| + c$

5 **a** $\dfrac{dy}{dx} = \dfrac{1}{2x}\ \ x > 0$

$y = \dfrac{1}{2}\log_e x + c$

$x = e^2,\ y = 2$

$2 = \dfrac{1}{2}\log_e e^2 + c$

$2 = \dfrac{1}{2} \times 2 + c$

$c = 1$

$y = \dfrac{1}{2}\log_e x + 1\ ,\ x > 0$

b $\dfrac{dy}{dx} = \dfrac{2}{5 - 2x}$

$y = \dfrac{2}{-2}\log_e |5 - 2x| + c$

$y = -\log_e |2x - 5| + c$

$x = 2, y = 10$

$10 = -\log_e 1 + c$

$c = 10$

$y = -\log_e 2x - 5 + 10\ ,\ x < \dfrac{5}{2}$

6 $f'(x) = \dfrac{10}{x - 5}$

$f(x) = 10\log_e |x - 5| + c$

$f(5 + e) = 10$

$10 = 10\log_e e + c$

$c = 0$

$f(x) = 10\log_e x - 5\ ,\ x > 5$

7 **a** $\displaystyle\int \frac{x}{x + 1}\,dx$

$= \displaystyle\int 1 - \frac{1}{x + 1}\,dx$

$= x - \log_e |x + 1| + c$

b $\displaystyle\int \frac{1 - 2x}{x + 1}\,dx$

$= \displaystyle\int -2 + \frac{3}{x + 1}\,dx$

$= -2x + 3\log_e |x + 1| + c$

c $\displaystyle\int \frac{2x + 1}{x + 1}\,dx$

$= \displaystyle\int 2 - \frac{1}{x + 1}\,dx$

$= 2x - \log_e |x + 1| + c$

8 $\dfrac{dy}{dx} = \dfrac{3}{x - 2}$

$y = 3\log_e |x - 2| + c$

$x = 0,\ y = 10$

$10 = 3\log_e |-2| + c$

$c = 10 - 3\log_e 2$

$y = 3\log_e |x - 2| + 10 - 3\log_e 2$

$y = 3\log_e\left(\dfrac{|x - 2|}{2}\right) + 10$

You can complete it without using the absolute value function.

$$\frac{dy}{dx} = \frac{3}{x-2}$$
$$= -\frac{3}{2-x}$$
$$y = 3\log_e(2-x) + c$$
$$x = 0,\ y = 10$$
$$10 = 3\log_e 2 + c$$
$$c = 10 - 3\log_e 2$$
$$y = 3\log_e(2-x) + 10 - 3\log_e 2$$
$$y = 3\log_e\left(\frac{2-x}{2}\right) + 10$$

9 $$\frac{dy}{dx} = \frac{5}{2-4x}$$
$$y = \frac{5}{-4}\log_e|2-4x| + c$$
$$y = \frac{-5}{4}\log_e|4x-2| + c$$
$$x = -2,\ y = 10$$
$$10 = \frac{-5}{4}\log_e|-8-4| + c$$
$$10 = \frac{-5}{4}\log_e 10 + c$$
$$c = 10 + \frac{5}{4}\log_e 10$$
$$y = \frac{5}{4}\log_e 10 - \frac{5}{4}\log_e|4x-2| + 10$$
$$y = \frac{5}{4}\log_e\left|\frac{10}{4x-2}\right| + 10$$
$$y = \frac{5}{4}\log_e\left|\frac{5}{2x-1}\right| + 10$$

To satisy the conditions you can write the rule as $y = \frac{5}{4}\log_e\frac{5}{1-2x} + 10$

10 $$\frac{dy}{dx} = \frac{5}{2-4x}$$
$$y = -\frac{5}{4}\log_e|2x-1| + c$$
$$x = 1,\ y = 10$$
$$10 = -\frac{5}{4}\log_e|1| + c$$
$$c = 10$$
$$y = -\frac{5}{4}\log_e|2x-1| + 10$$
$$y = \frac{5}{4}\log_e\left|\frac{1}{2x-1}\right| + 10$$

To satisy the conditions you can write the rule as $y = \frac{5}{4}\log_e\frac{1}{2x-1} + 10$

Solutions to Exercise 11D

1 **a** $\frac{1}{6}e^{6x} + c$

b $\frac{1}{2}e^{2x} + \frac{3}{2}x^2 + c$

c $-\frac{1}{3}e^{-3x} + x^2 + c$

d $-\frac{1}{2}e^{-2x} + \frac{1}{2}e^{2x} + c$

2 **a** $\int e^{2x} - e^{\frac{x}{2}}\,dx$

$= \frac{1}{2}e^{2x} - 2e^{\frac{x}{2}} + c$

b $\int e^x + e^{-x}\,dx$

$= e^x - e^{-x} + c$

c $\int 2e^{3x} - e^{-x}\,dx$

$= \frac{2}{3}e^{3x} + e^{-x} + c$

d $\int 5e^{\frac{x}{3}} - 3e^{\frac{x}{5}}\,dx$

$= 15e^{\frac{x}{3}} - 15e^{\frac{x}{5}} + c$

e $\frac{9}{2}e^{\frac{2x}{3}} - \frac{15}{7}e^{\frac{7x}{5}} + c$

f $\frac{15}{4}e^{\frac{4x}{3}} - \frac{9}{2}e^{\frac{2x}{3}} + c$

3 **a** $\frac{dy}{dx} = e^{2x} - x$

$y = \frac{1}{2}e^{2x} - \frac{x^2}{2} + c$

$x = 0, y = 5$

$5 = \frac{1}{2} - 0 + c$

$c = \frac{9}{2}$

$y = \frac{1}{2}\left(e^{2x} - x^2 + 9\right)$

b $\frac{dy}{dx} = 3e^{-x} - e^x$

$y = -3e^{-x} - e^x + c$

$x = 0, y = 4$

$4 = -3 - 1 + c$

$c = 8$

$y = -3e^{-x} - e^x + 8$

4 $\dfrac{dy}{dx} = ae^{-x} + 1$

$x = 0, \dfrac{dy}{dx} = 3$

$3 = a + 1$

$a = 2$

$\dfrac{dy}{dx} = 2e^{-x} + 1$

$y = -2e^{-x} + x + c$

$x = 0, y = 5$

$5 = -2 + 0 + c$

$c = 7$

$y = -2e^{-x} + x + 7$

$x = 2,$

$y = -2e^{-2} - 2 + 7$

$y = 9 - \dfrac{2}{e^2}$

5 $\dfrac{dy}{dx} = e^{kx}$

a $x = 1,$

$\dfrac{dy}{dx} = e^k$

Tangent

$y = e^k x + c$

$x = 0, y = 0$

$0 = c$

$y = e^k x$

$x = 1, y = e^2$

$e^2 = e^k$

$k = 2$

$\dfrac{dy}{dx} = e^{2x}$

b $y = \dfrac{1}{2}e^{2x} + c$

$x = 1, y = e^2$

$e^2 = \dfrac{1}{2}e^2 + c$

$c = \dfrac{1}{2}e^2$

$y = \dfrac{1}{2}e^{2x} + \dfrac{1}{2}e^2$

6 $\dfrac{dy}{dx} = -e^{kx}$

a $x = 1,$

$\dfrac{dy}{dx} = -e^k$

Tangent

$y = -e^k x + c$

$x = 0, y = 0$

$0 = c$

$y = -e^k x$

$x = 1, y = -e^3$

$-e^3 = -e^k$

$k = 3$

$\dfrac{dy}{dx} = -e^{3x}$

b $y = -\dfrac{1}{3}e^{3x} + c$

$x = 1, y = -e^3$

$-e^3 = -\dfrac{1}{3}e^3 + c$

$c = -\dfrac{2}{3}e^3$

$y = -\dfrac{1}{3}e^{3x} - \dfrac{2}{3}e^3$

Solutions to Exercise 11E

1 **a** $\int_1^2 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_1^2$

$= \frac{2^3}{3} - \frac{1^3}{3}$

$= \frac{7}{3}$

b $\int_{-1}^3 x^3\,dx$

$= \left[\frac{x^4}{4}\right]_{-1}^2$

$= \frac{81}{4} - \frac{1}{4}$

$= 20$

c $\int_0^1 x^3 - xdx$

$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1$

$= \left(\frac{1}{4} - \frac{1}{2}\right) - (0 - 0)$

$= \frac{-1}{4}$

d $\int_{-1}^2 (x + 1)^2\,dx$

$= \left[\frac{1}{3}(x + 1)^3\right]_{-1}^2$

$= \frac{1}{3}(3)^3 - \frac{1}{3}(0)^3$

$= 9$

e $\int_1^2 \frac{1}{x^2}\,dx$

$= \left[\frac{-1}{x}\right]_1^2$

$= \frac{-1}{2} - \frac{-1}{1}$

$= \frac{1}{2}$

f $\int_1^4 x^{\frac{1}{2}} + 2x^2\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}x^3\right]_1^4$

$= \frac{2}{3}\left(4^{\frac{3}{2}} + 4^3\right) - \frac{2}{3}\left(1^{\frac{3}{2}} + 1^3\right)$

$= \frac{2}{3}(8 + 64) - \frac{4}{3}$

$= \frac{140}{3}$

g $\int_0^2 x^3 + 2x^2 + x + 2\,dx$

$= \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + 2x\right]_0^2$

$= \frac{16}{4} + \frac{2}{3} \times 8 + \frac{4}{2} + 2 \times 2 - 0$

$= 4 + \frac{16}{3} + 2 + 4$

$= \frac{46}{3} = 15\frac{1}{3}$

h $\int_1^4 2x^{\frac{3}{2}} + 5x^3 \, dx$

$= \left[\frac{4}{5}x^{\frac{5}{2}} + \frac{5}{4}x^4\right]_1^4$

$= \left(\frac{4}{5} \times 32 + \frac{5}{4} \times 256\right) - \left(\frac{4}{5} + \frac{5}{4}\right)$

$= \frac{128}{5} + 320 - \frac{4}{5} - \frac{5}{4}$

$= \frac{6871}{20} = 343\frac{11}{20}$

2 **a** $\int_0^1 (2x + 1)^3 \, dx$

$= \left[\frac{1}{8}(2x + 1)^4\right]_0^1$

$= \frac{1}{8}(3)^4 - \frac{1}{8}(1)^4$

$= \frac{81}{8} - \frac{1}{8}$

$= 10$

b $\int_0^2 (4x + 1)^{\frac{-1}{2}} \, dx$

$= \left[\frac{2}{4}(4x + 1)^{\frac{1}{2}}\right]_0^2$

$= \frac{1}{2}(9)^{\frac{1}{2}} - \frac{1}{2}(1)^{\frac{1}{2}}$

$= \frac{3}{2} - \frac{1}{2}$

$= 1$

c $\int_1^2 (1 - 2x)^2 \, dx$

$= \left[\frac{1}{-6}(1 - 2x)^3\right]_1^2$

$= \frac{-1}{6}(-3)^3 + \frac{1}{6}(-1)^3$

$= \frac{27}{6} - \frac{1}{6}$

$= \frac{13}{3}$

d $\int_0^1 (3 - 2x)^{-2} \, dx$

$= \left[\frac{1}{2}(3 - 2x)^{-1}\right]_0^1$

$= \frac{1}{2}(1)^{-1} - \frac{1}{2}(3)^{-1}$

$= \frac{1}{2} - \frac{1}{6}$

$= \frac{1}{3}$

e $\int_0^2 (3 + 2x)^{-3} \, dx$

$= \left[\frac{1}{-4}(3 + 2x)^{-2}\right]_0^2$

$= \frac{-1}{4}(7)^{-2} + \frac{1}{4}(3)^{-2}$

$= \frac{1}{36} - \frac{1}{196}$

$= \frac{10}{441}$

f $\int_{-1}^1 (4x + 1)^3 \, dx$

$= \left[\frac{1}{16}(4x + 1)^4\right]_{-1}^1$

$= \frac{1}{16}(5)^4 - \frac{1}{16}(-3)^4$

$= \frac{625}{16} - \frac{81}{16}$

$= \frac{544}{16}$

$= 34$

g $\displaystyle\int_0^1 (2-x)^{\frac{1}{2}}\,dx$

$$= \left[\frac{2}{-3}(2-x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{-2}{3}(1)^{\frac{3}{2}} + \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2}{3}(2^{\frac{3}{2}} - 1)$$

$$\simeq 1.22$$

h $\displaystyle\int_3^4 (2x-4)^{\frac{-1}{2}}\,dx$

$$= \left[\frac{2}{2}(2x-4)^{\frac{1}{2}}\right]_3^4$$

$$= \sqrt{4} - \sqrt{2}$$

$$= 2 - \sqrt{2}$$

i $\displaystyle\int_0^1 (3+2x)^{-2}\,dx$

$$= \left[\frac{1}{-2}(3+2x)^{-1}\right]_0^1$$

$$= \frac{-1}{2}(5)^{-1} + \frac{1}{2}(3)^{-1}$$

$$= \frac{1}{6} - \frac{1}{10}$$

$$= \frac{1}{15}$$

3 a $\displaystyle\int_0^1 e^{2x}\,dx$

$$= \left[\frac{1}{2}e^{2x}\right]_0^1$$

$$= \frac{1}{2}e^2 - \frac{1}{2}$$

b $\displaystyle\int_0^1 e^{-2x} + 1\,dx$

$$= \left[\frac{-1}{2}e^{-2x} + x\right]_0^1$$

$$= \left(\frac{-1}{2}e^{-2} + 1\right) - \left(\frac{-1}{2}\right)$$

$$= \frac{3}{2} - \frac{1}{2e^2}$$

c $\displaystyle\int_0^1 2e^{\frac{x}{3}} + 2\,dx$

$$= \left[6e^{\frac{x}{3}} + 2x\right]_0^1$$

$$= (6e^{\frac{1}{3}} + 2) - (6)$$

$$= 6e^{\frac{1}{3}} - 4$$

d $\displaystyle\int_{-2}^2 \frac{e^x + e^{-x}}{2}\,dx$

$$= \left[\frac{e^x - e^{-x}}{2}\right]_{-2}^2$$

$$= \frac{e^2 - e^{-2}}{2} - \frac{e^{-2} - e^2}{2}$$

$$= e^2 - e^{-2}$$

4 a $\displaystyle\int_0^4 h(x)\,dx = 5$

$$\int_0^4 2h(x)\,dx$$

$$= 2\int_0^4 h(x)\,dx = 10$$

b $\displaystyle\int_0^4 (h(x) + 3)\,dx$

$$= \int_0^4 h(x)\,dx + \int_0^4 3\,dx$$

$$= 5 + 12$$

$$= 17$$

c $\int_4^0 h(x)\,dx$

$= -\int_0^4 h(x)\,dx$

$= -5$

d $\int_0^4 (h(x) + 1)\,dx$

$= \int_0^4 h(x)\,dx + \int_0^4 dx$

$= 5 + 4$

$= 9$

e $\int_0^4 (h(x) - x)\,dx$

$= \int_0^4 h(x)\,dx - \int_0^4 xdx$

$= 5 - \left[\frac{x^2}{2}\right]_0^4$

$= 5 - 8$

$= -3$

5 a $\int_0^4 \frac{1}{x-6}\,dx$

$= -\int_0^4 \frac{1}{6-x}\,dx$

$= \left[\log_e(6-x)\right]_0^4$

$= (\log_e(2) - \log_e(6))$

$= \log_e\left(\frac{1}{3}\right)$

b $\int_2^4 \frac{1}{2x-3}\,dx$

$= \left[\frac{1}{2}\log_e(2x-3)\right]_2^4$

$= \frac{1}{2}\log_e 5 - \frac{1}{2}\log_e 1$

$= \frac{1}{2}\log_e 5$

c $\int_5^6 \frac{3}{2x+7}\,dx$

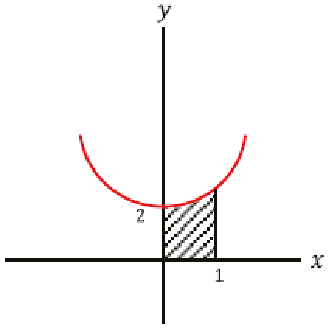
$= \left[\frac{3}{2}\log_e(2x+7)\right]_5^6$

$= \frac{3}{2}\log_e 19 - \frac{3}{2}\log_e 17$

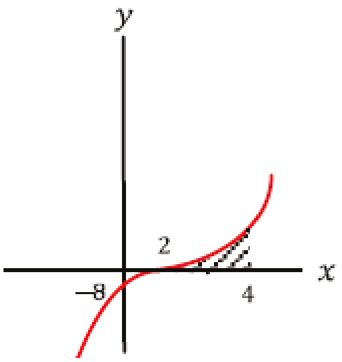
$= \frac{3}{2}\log_e\left(\frac{19}{17}\right)$

Solutions to Exercise 11F

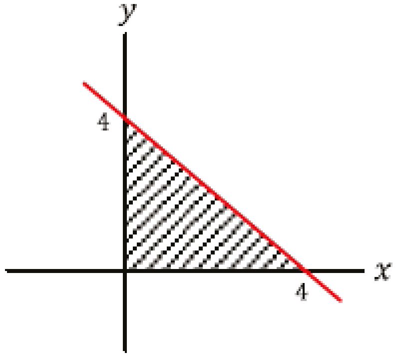
1 **a** $$A = \int_0^1 y\,dx$$
$$= \int_0^1 3x^2 + 2\,dx$$
$$= \Big[x^3 + 2x\Big]_0^1$$
$$= (1 + 2) - (0 + 0)$$
$$= 3$$



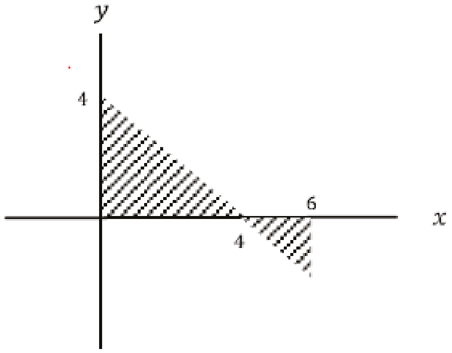
b $$A = \int_2^4 y\,dx$$
$$= \int_2^4 x^3 - 8\,dx$$
$$= \left[\frac{x^4}{4} - 8x\right]_2^4$$
$$= (64 - 32) - (4 - 16)$$
$$= 32 + 12$$
$$= 44$$



c **i** $$A = \int_0^4 y\,dx$$
$$= \int_0^4 4 - x\,dx$$
$$= \left[4x - \frac{x^2}{2}\right]_0^4$$
$$= (16 - 8) - 0$$
$$= 8$$



ii $$A = \int_0^4 y\,dx - \int_4^6 y\,dx$$
$$= 8 - \left[4x - \frac{x^2}{2}\right]_4^6 \qquad (\textit{from } (i))$$
$$= 8 - ((24 - 18) - (16 - 8))$$
$$= 8 - (6 - 8)$$
$$= 10$$



2 a $A = x^2 - 2x$

$$= x(x-2)$$

$$y = 0,\ x = 0, 2$$

$$A = -\int_0^2 y\,dx$$

$$= -\int_0^2 x^2 - 2x\,dx$$

$$= -\left[\frac{x^3}{3} - x^2\right]_0^2$$

$$= -\left(\frac{8}{3} - 4\right)$$

$$= \frac{4}{3}$$

b $y = (4-x)(3-x)$

$$y = 0,\ x = 3, 4$$

$$A = -\int_3^4 y\,dx$$

$$A = -\int_3^4 (x-4)(x-3)\,dx$$

$$A = -\int_3^4 -x^2 - 7x + 12\,dx$$

$$= -\left[\frac{x^3}{3} - \frac{7x^2}{2} + 12x\right]_3^4$$

$$= \left(\left(\frac{64}{3} - \frac{7\times 16}{2} + 48\right) - \left(\frac{27}{3} - \frac{7\times 9}{2} + 36\right)\right)$$

$$= -\left(\left(\frac{40}{3}\right) - \left(\frac{27}{2}\right)\right)$$

$$= \frac{1}{6}$$

c $y = (x+2)(x+7)$

$$y = 0,\ x = -2, 7$$

$$A = \int_{-2}^{7} y\,dx$$

$$= \int_{-2}^{7} -x^2 + 5x + 14\,dx$$

$$= \left[\frac{-x^3}{3} + \frac{5x^2}{2} + 14x\right]_{-2}^{7}$$

$$= \left(\frac{-343}{3} + \frac{5\times 49}{2} + 98\right) - \left(\frac{8}{3} + \frac{20}{2} - 28\right)$$

$$= \left(\frac{637}{6}\right) + \left(\frac{46}{3}\right)$$

$$= 121.5$$

d $y = x^2 - 5x + 6$

$$= (x-2)(x-3)$$

$$y = 0,\ x = 2, 3$$

$$A = -\int_2^3 y\,dx$$

$$= -\int_2^3 x^2 - 5x + 6\,dx$$

$$= -\left[\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right]_2^3$$

$$= -\left(\left(\frac{27}{3} - \frac{45}{2} + 18\right) - \left(\frac{8}{3} - \frac{20}{2} + 12\right)\right)$$

$$= \left(\frac{14}{3}\right) - \left(\frac{9}{2}\right)$$

$$= \frac{1}{6}$$

e $y = 3 - x^2$

$$= (\sqrt{3} + x)(\sqrt{3} + x)$$

$$y = 0, x = \pm\sqrt{3}$$

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} y\,dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 \, dx$$

$$= \left[3x - \frac{x^3}{3}\right]_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \left(3\sqrt{3} - \frac{3\sqrt{3}}{3}\right) - \left(-3\sqrt{3} + \frac{3\sqrt{3}}{3}\right)$$

$$= 2\sqrt{3} + 2\sqrt{3}$$

$$= 4\sqrt{3}$$

f $y = x^3 - 6x^2$

$$= x^2(x - 6)$$

$y = 0,\ x = 0, 6$

$$A = -\int_0^6 y dx$$

$$= -\int_0^6 x^3 - 6x^2 \, dx$$

$$= \int_0^6 6x^2 - x^3 \, dx$$

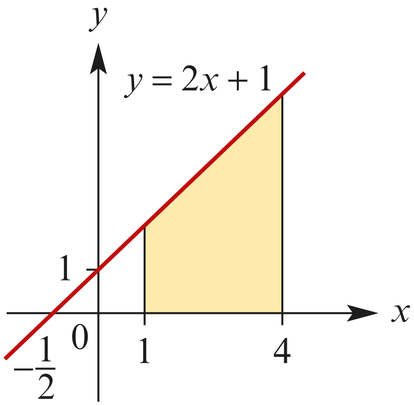
$$= \left[2x^3 - \frac{x^4}{4}\right]_0^6$$

$$= 2 \times (216) - \frac{1296}{4}$$

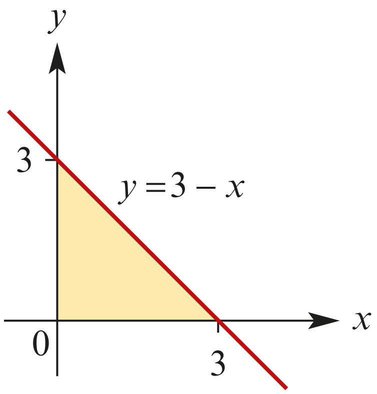
$$= 432 - 324$$

$$= 108$$

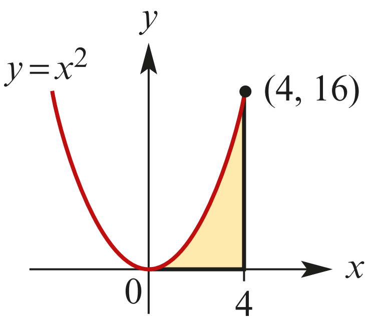
3 a



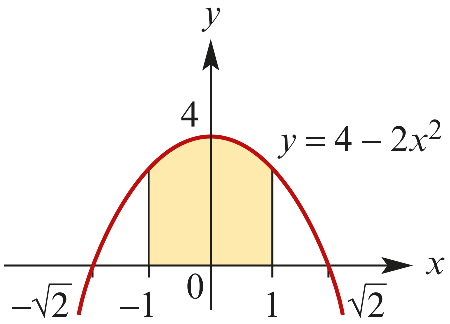
b



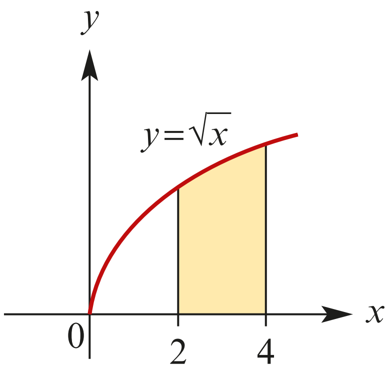
c



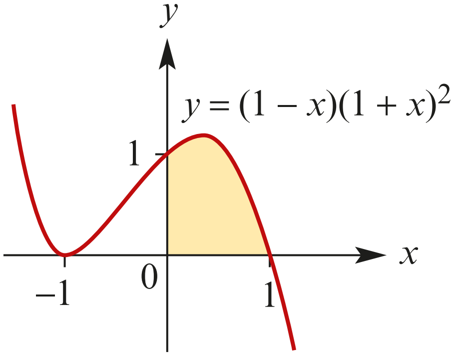
d



e



f



4 $y = 3x + 2x^{-2}$

$y = 0,$

$3x = \frac{-2}{x^2}$

$3x^3 = -2$

$x^3 = \frac{-2}{3}$

$x = \left(\frac{-2}{3}\right)^{\frac{1}{3}}$

which is not in the region under consideration

$$\therefore A = \int_2^5 3x + 2x^{-2}\,dx$$

$$= \left[\frac{3}{2}x^2 - 2x^{-1}\right]_2^5$$

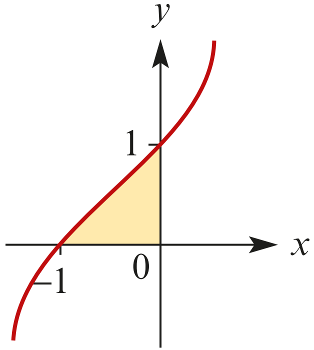
$$= \left(\frac{3}{2}\times 25 - \frac{2}{5}\right) - \left(\frac{3}{2}\times 4 - \frac{2}{2}\right)$$

$$= \frac{75}{2} - \frac{2}{5} - 6 + 1$$

$$= \frac{375 - 4 - 50}{10}$$

$$= \frac{321}{10} \text{ square units}$$

5



$$A = \int_{-1}^{0} f(x)\,dx \quad (\textit{from graph})$$

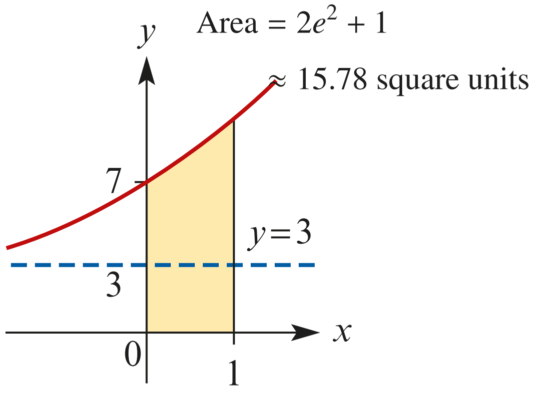
$$= \int_{-1}^{0} 1 + x^3\,dx$$

$$= \left[x + \frac{x^4}{4}\right]_{-1}^{0}$$

$$= 0 - \left(-1 + \frac{1}{4}\right)$$

$$= \frac{3}{4} \text{ square units}$$

6



$$A = \int_0^1 f(x)\,dx \quad (\textit{from graph})$$

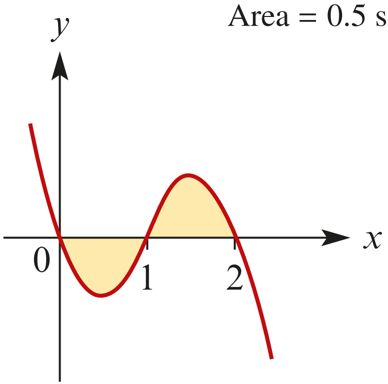
$$= \int_0^1 4e^{2x} + 3\,dx$$

$$= \left[2e^{2x} + 3x\right]_0^1$$

$$= (2e^2 + 3) - (2 + 0)$$

$$= 2e^2 + 1 \approx 15.78 \text{ square units}$$

7 Area = 0.5 square units



$$A = -\int_0^1 y\,dx + \int_1^2 y\,dx \quad (\textit{from graph})$$

$$= -\int_0^1 x(2-x)(x-1)\,dx$$

$$+ \int_1^2 x(2-x)(x-1)\,dx$$

$$= -\int_0^1 -x^3 + 3x^2 - 2x\,dx$$

$$+ \int_1^2 -x^3 + 3x^2 - 2x\,dx$$

$$= \left[\frac{-x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2}\right]_0^1$$

$$+ \left[\frac{-x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2}\right]_1^2$$

$$= \left[\frac{-x^4}{4} + x^3 - x^2\right]_0^1 + \left[\frac{-x^4}{4} + x^3 - x^2\right]_1^2$$

$$= 0 - \left(\frac{-1}{4} + 1 - 1\right) + \left(\frac{-16}{4} + 8 - 4\right)$$

$$- \left(\frac{-1}{4} + 1 - 1\right)$$

$$= \frac{1}{2} \text{ square units}$$

8 **a** $\int_{-1}^{4} x(3-x)\,dx$

$$= \int_{-1}^{4} 3x - x^2\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4}$$

$$= \left(\frac{3 * 16}{2} - \frac{64}{3}\right) - \left(\frac{3}{2} + \frac{1}{3}\right)$$

$$= 24 - \frac{64}{3} - \frac{3}{2} - \frac{1}{3}$$

$$= \frac{45}{2} - \frac{65}{3}$$

$$= \frac{5}{6} \text{ square units}$$

b assuming the graph shown is $y = x(3-x)$,

$$A = -\int_{-1}^{0} x(3-x)\,dx + \int_{0}^{3} x(3-x)\,dx$$

$$- \int_{3}^{4} x(3-x)\,dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 - \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_{-1}^0 - \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_3^4$$

from (a)

$$= \left(\frac{3 \times 9}{2} - \frac{27}{3}\right) - (0) - 0 + \left(\frac{3}{2} + \frac{1}{3}\right)$$

$$- \left(\frac{3}{2} \times 16 - \frac{64}{3}\right) + \left(\frac{3 \times 9}{2} - \frac{27}{3}\right)$$

$$= \frac{49}{6}$$

9 **a** $y^2 = 9(1-x)$

A: $x = 0$,

$y^2 = 9(1-0)$

$= 9$

$y = \pm 3$

but A is above the x-axis

$$so\ A = (0,3)$$

$$B{:}y = 0$$

$$0 = 9(1-x)$$

$$1-x = 0$$

$$x = 1$$

$$B = (1,0)$$

b $A = \int_0^3 \left(1 - \frac{y^2}{9}\right)dy$

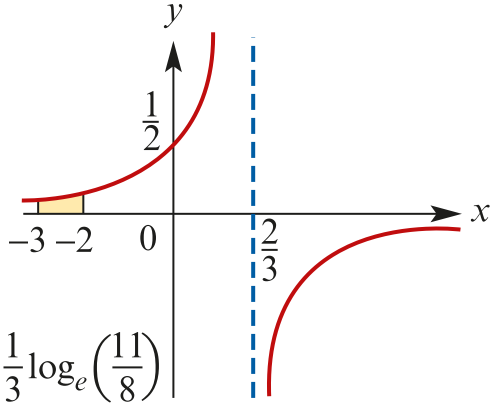
(0 *to* A, *where* 0 *is the origin*)

$$= \left[y - \frac{y^3}{27}\right]_0^3$$

$$= \left(3 - \frac{27}{27}\right) - 0$$

$= 2$ square units

10



$$A = -\int_{-3}^{-2} y\,dx \quad (\textit{from graph})$$

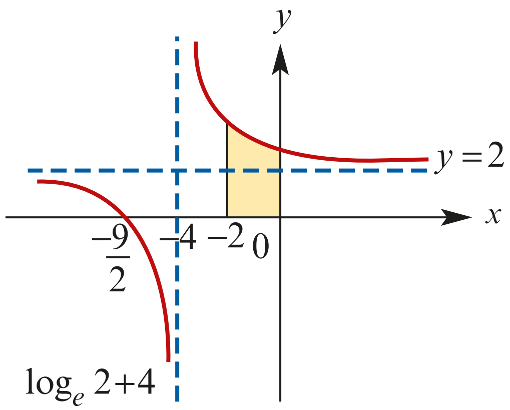
$$= -\int_{-3}^{-2} \frac{1}{2-3x}\,dx$$

$$= \left[\frac{-1}{3}\log_e|2-3x|\right]_{-3}^{-2}$$

$$= \frac{-1}{3}\log_e|8| + \frac{1}{3}\log_e|11|$$

$$= \frac{1}{3}\log_e\left(\frac{11}{8}\right)$$

11



$$A = -\int_{-2}^{0} y\,dx \quad (\textit{from graph})$$

$$= \int_{-2}^{0} 2 + \frac{1}{x+4}\,dx$$

$$= \left[2x + \log_e|x+4|\right]_{-2}^{0}$$

$$= (0 + \log_e 4) - (-4 + \log_e 2)$$

$$= 2\log_e 2 + 4 - \log_e 2$$

$$= \log_e 2 + 4$$

12 **a** RHS $= e^{x(\ln a)}$

$$= e^{(\ln a^x)}$$

$$= a^x$$

$$= LHS \quad QED$$

b $\frac{d}{dx}(a^x)$

$= \frac{d}{dx}(e^{x(\ln a)})$

$= \ln a e^{x \ln a}$

$= a^x \ln a$

$\int a^x \, dx$

$= \int e^{x \ln a} \, dx$

$= \frac{1}{\ln a} e^{x \ln a} + c$

$= \frac{a^x}{\ln a} + c$

c $\int_0^b a^x \, dx$

$= \left[\frac{a^x}{\ln a}\right]_0^b \quad \textit{from} \quad (b)$

$= \frac{a^b}{\ln a} - \frac{1}{\ln a}$

$= \frac{1}{\ln a}(a^b - 1) \quad QED$

Solutions to Exercise 11G

1 **a** $$\int \cos 3x\,dx$$
$$= \frac{1}{3}\sin 3x$$

b $$\int \sin \frac{1}{2}x\,dx$$
$$= -2\cos \frac{1}{2}x$$

c $$\int 3\cos 3x\ x\,dx$$
$$= \sin 3x$$

d $$\int 2\sin \frac{1}{2}x\,dx$$
$$= -4\cos \frac{1}{2}x$$

e $$\int \sin\left(2x - \frac{\pi}{3}\right)\,dx$$
$$= \frac{-1}{2}\cos\left(2x - \frac{\pi}{3}\right)$$

f $$\int \cos 3x + \sin 2x\,dx$$
$$= \frac{1}{3}\sin 3x - \frac{1}{2}\cos 2x$$

g $$\int \cos 4x - \sin 4x\,dx$$
$$= \frac{1}{4}\sin 4x + \frac{1}{4}\cos 4x$$

h $$\int \frac{-1}{2}\sin 2x + \cos 3x\,dx$$
$$= \frac{1}{4}\cos 2x + \frac{1}{3}\sin 3x$$

i $$\int \frac{-1}{2}\cos\left(2x + \frac{\pi}{3}\right)\,dx$$
$$= \frac{-1}{4}\sin\left(2x + \frac{\pi}{3}\right)$$

j $$\int \sin \pi x\,dx$$
$$= \frac{-1}{\pi}\cos \pi x$$

2 **a** $$\int_0^{\frac{\pi}{4}} \sin x\,dx$$
$$= [-\cos x]_0^{\frac{\pi}{4}}$$
$$= \frac{-1}{\sqrt{2}} + 1$$
$$= 1 - \frac{1}{\sqrt{2}}$$

b $$\int_0^{\frac{\pi}{4}} \cos 2x\,dx$$
$$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}(1) - 0$$
$$= \frac{1}{2}$$

c $$\int_{\frac{-\pi}{2}}^{\frac{\pi}{4}} \cos\theta\,d\theta$$
$$= [\sin\theta]_{\frac{-\pi}{2}}^{\frac{\pi}{4}}$$
$$= \frac{1}{\sqrt{2}} - -1$$
$$= 1 + \frac{1}{\sqrt{2}}$$

d $$\int_0^{\frac{\pi}{2}} \sin\theta + \cos\theta\,d\theta$$
$$= [-\cos\theta + \sin\theta]_0^{\frac{\pi}{2}}$$
$$= (0 + 1) - (-1 + 0)$$
$$= 2$$

e $\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$

$= \left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}(-1) + \frac{1}{2}(1)$

$= 1$

f $\int_0^{\frac{\pi}{3}} \cos 3\theta + \sin 3\theta \, d\theta$

$= \left[\frac{1}{3}\sin 3\theta - \frac{1}{3}\cos 3\theta\right]_0^{\frac{\pi}{3}}$

$= \left(0 - \frac{1}{3}(-1)\right) - \left(0 - \frac{1}{3}(1)\right)$

$= \frac{2}{3}$

g $\int_0^{\frac{\pi}{3}} \cos 3\theta + \sin\left(\theta - \frac{\pi}{3}\right) d\theta$

$= \left[\frac{1}{3}\sin 3\theta - \cos\left(\theta - \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}}$

$= (0 - 1) - \left(0 - \frac{1}{2}\right)$

$= \frac{-1}{2}$

h $\int_0^{\pi} \sin \frac{x}{4} + \cos \frac{x}{4} \, dx$

$= \left[-4cos\frac{x}{4} + 4\sin\frac{x}{4}\right]_0^{\pi}$

$= \left(-4\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right)\right) - (-4(1) + 0)$

$= 4$

i $\int_0^{\frac{\pi}{4}} \sin\left(2x - \frac{\pi}{3}\right) dx$

$= \left[\frac{-1}{2}\cos\left(2x - \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{4}}$

$= \frac{-1}{2}\cos\left(\frac{\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{-\pi}{3}\right)$

$= \frac{-\sqrt{3}}{4} + \frac{1}{4}$

$= \frac{1 - \sqrt{3}}{4}$

j

$\int_0^{\pi} \cos 2x - \sin \frac{x}{2} \, dx$

$= \left[\frac{1}{2}\sin\ 2x + 2\cos\frac{x}{2}\right]_0^{\pi}$

$= \left(\frac{1}{2}\sin\ 2\pi + 2\cos\frac{\pi}{2}\right) - \left(\frac{1}{2}\sin 0 + 2\cos 0\right)$

$= -2$

3 $A = \int_0^{\frac{\pi}{2}} y \, dx$

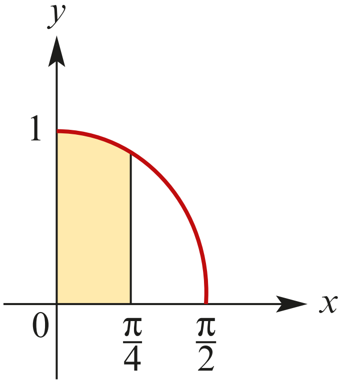
$= \int_0^{\frac{\pi}{2}} \sin \frac{1}{2}x \, dx$

$= \left[-2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$

$= -2\cos\frac{\pi}{4} + 2\cos 0$

$= 2 - \sqrt{2}$

4 a



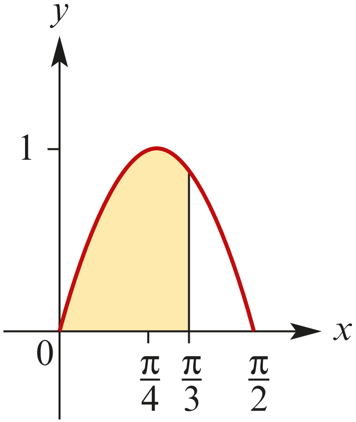
$$\int_0^{\frac{\pi}{4}} \cos x\, dx$$

$$= [\sin x]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} - 0$$

$$= \frac{1}{\sqrt{2}}$$

b



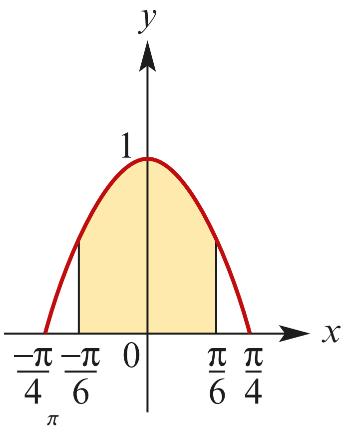
$$\int_0^{\frac{\pi}{3}} \sin 2x\, dx$$

$$= \left[\frac{-1}{2}\cos 2x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{-1}{2}\left(\frac{-1}{2}\right) + \frac{-1}{2}(1)$$

$$= \frac{3}{4}$$

c



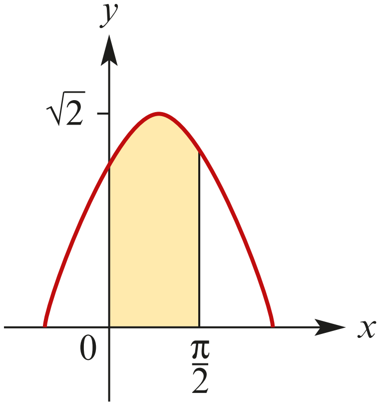
$$\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos 2x\, dx$$

$$= \left[\frac{1}{2}\sin 2x\right]_{\frac{-\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{-\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2}$$

d



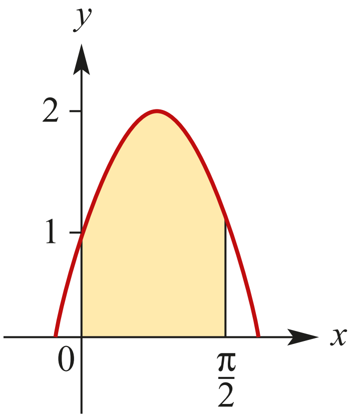
$$\int_0^{\frac{\pi}{2}} \cos\theta + \sin\ \theta\, d\theta$$

$$= [\sin\theta - \cos\theta]_0^{\frac{\pi}{2}}$$

$$= (1 - 0) - (0 - 1)$$

$$= 2$$

e



$$\int_0^{\frac{\pi}{2}} \sin 2\theta + 1\, d\theta$$

$$= \left[\frac{-1}{2}\cos 2\theta + \theta\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{-1}{2}(-1) + \frac{\pi}{2}\right) - \left(\frac{-1}{2}(-1) + 0\right)$$

$$= 1 + \frac{\pi}{2}$$

f $$\int_{\frac{-x}{4}}^{\frac{x}{4}} 1 - \cos 2\theta\, d\theta$$

$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - \frac{1}{2}(1) - (-\frac{\pi}{4} - \frac{1}{2}(-1)$$

$$= \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{4} - \frac{1}{2}$$

$$= \frac{\pi}{2} - 1$$

5 a $$\int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{\pi}{4}\right) dx$$

$$= \left[\frac{-1}{2}\cos\left(2x + \frac{\pi}{4}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{-1}{2}\cos\left(\frac{5\pi}{4}\right) + \frac{1}{2}\cos\left(\frac{\pi}{4}\right)$$

$$= \frac{-1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}$$

b $$\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{6}\right) dx$$

$$= \left[\frac{1}{3}\sin\left(3x + \frac{\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{3}\sin\left(\frac{7\pi}{6}\right) - \frac{1}{3}\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{3}\left(\frac{-1}{2}\right) - \frac{1}{3}\left(\frac{1}{2}\right)$$

$$= \frac{-1}{3}$$

c $$\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right) dx$$

$$= \left[\frac{1}{3}\sin\left(3x + \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{3}\sin\left(\frac{4\pi}{3}\right) - \frac{1}{3}\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{3}\left(\frac{-\sqrt{3}}{2}\right) - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-1}{\sqrt{3}}$$

d $\int_0^{\frac{\pi}{4}} \cos(3\pi - x)\,dx$

$$= \left[-\sin(3\pi - x)\right]_0^{\frac{\pi}{4}}$$

$$= [-\sin(x - 3\pi)]_0^{\frac{\pi}{4}}$$

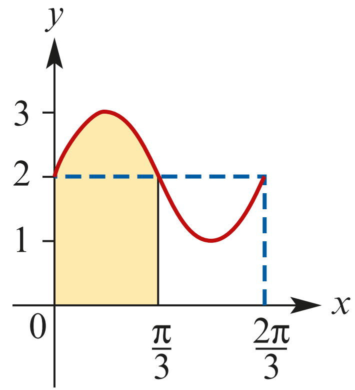
$$= \sin\left(-2\pi - \frac{3\pi}{4}\right) - \sin(-2\pi - \pi)$$

$$= \sin\left(-\frac{3\pi}{4}\right) - \sin(-\pi)$$

$$= \frac{-1}{\sqrt{2}} - 0$$

$$= \frac{-1}{\sqrt{2}}$$

6



$$A = \int_0^{\frac{\pi}{3}} 2 + \sin 3x\,dx$$

$$= \left[2x - \frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{2x}{3} - \frac{1}{3}(-1)\right) - \left(0 - \frac{1}{3}(1)\right)$$

$$= \frac{2}{3}(\pi + 1)$$

Solutions to Exercise 11H

1 a $\int_1^4 \sqrt{x}\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$

$= \frac{16}{3} - \frac{2}{3}$

$= \frac{14}{3}$

b $\int_{-1}^1 (1+x)^2\,dx$

$= \left[\frac{1}{3}(1+x)^3\right]_{-1}^1$

$= \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3$

$= \frac{8}{3}$

c $\int_0^8 x^{\frac{1}{3}}\,dx$

$= \left[\frac{3}{4}x^{\frac{4}{3}}\right]_0^8$

$= \frac{3}{4} \times 16 - 0$

$= 12$

d

$\int_0^{\frac{\pi}{3}} \cos 2x - \sin\frac{1}{2}x\,dx$

$= \left[\frac{1}{2}\sin 2x + 2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\sin\frac{2\pi}{3} + 2\cos\frac{\pi}{6} - \frac{1}{2}\sin 0 + 2\cos 0$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) - 0 - 2(1)$

$= \frac{5\sqrt{3}}{4} - 2$

e $\int_1^2 e^{2x} + \frac{4}{x}\,dx$

$= \left[\frac{1}{2}e^{2x} + 4\log_e|x|\right]_1^2$

$= \frac{1}{2}e^4 + 4\log_e 2 - \frac{1}{2}e^2 - 4\log_e 1$

$= \frac{1}{2}e^4 - \frac{1}{2}e^2 + 4\log_e 2$

f

$\int_0^{\frac{\pi}{2}} \sin 2x + \cos 3x\,dx$

$= \left[\frac{-1}{2}\cos 2x + \frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{-1}{2}(-1) + \frac{1}{3}(-1)\right) - \left(\frac{-1}{2}(1) + \frac{1}{3}(0)\right)$

$= \frac{1}{2} - \frac{1}{3} + \frac{1}{2}$

$= \frac{2}{3}$

g

$\int_0^{\pi} \sin\frac{x}{4} + \cos\frac{x}{4}\,dx$

$= \left[-4\cos\frac{x}{4} + 4\sin\frac{x}{4}\right]_0^{\pi}$

$= \left(-4\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right)\right) - (-4(1) + 4(0))$

$= 4$

h $\int_0^{\frac{\pi}{2}} 5x + \sin 2x \, dx$

$$= \left[\frac{5x^2}{2} - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{5\left(\frac{\pi}{2}\right)^2}{2} - \frac{1}{2}(-1)\right) - \left(0 - \frac{1}{2}(1)\right)$$

$$= \frac{5\pi^2}{8} + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{5\pi^2}{8}$$

i

$$\int_1^4 \left(2 + \frac{1}{x}\right)^2 dx$$

$$= \int_1^4 4 + \frac{4}{x} + \frac{1}{x^2} \, dx$$

$$= \left[4x + 4\log_e |x| - \frac{1}{x}\right]_1^4$$

$$= \left(16 + 4\log_e 4 - \frac{1}{4}\right) - (4 + 4\log_e 1 - 1)$$

$$= 16 + 8\log_e 2 - \frac{1}{4} - 4 + 1$$

$$= 12\frac{3}{4} + 8\log_e 2$$

j $\int_0^1 x^2 - x^3 \, dx$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4}$$

$$= \frac{1}{12}$$

2 $A = \int_0^{\frac{\pi}{3}} \sin x \, dx$

$$= \left[-\cos x\right]_0^{\frac{\pi}{3}}$$

$$= -\cos\frac{\pi}{3} + \cos 0$$

$$= \frac{-1}{2} + 1$$

$= \frac{1}{2}$ square units

3 a $\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

Therefore, $\int \frac{1}{\cos^2 x} dx$

$$= \frac{\sin x}{\cos x} + c$$

$$= \tan x + c$$

b

$$\frac{d}{dx}\left(\frac{\cos 2x}{\sin 2x}\right)$$

$$= \frac{2\sin 2x \frac{d}{dx}(\cos 2x) - 2\cos 2x \frac{d}{dx}(\sin 2x)}{\sin^2(2x)}$$

$$= \frac{2(-\sin^2 2x - \cos^2 2x)}{\sin^2 2x}$$

$$= -\frac{2}{\sin^2 2x}$$

Therefore, $\int \frac{1}{\sin^2 2x} dx$

$$= -\frac{\cos 2x}{2\sin 2x} + c$$

c $\dfrac{d}{dx}(\log_e(3x^2 + 7))$

$= 6x \times \dfrac{1}{3x^2 + 7}$

$= \dfrac{6x}{3x^2 + 7}$

$\displaystyle\int \frac{x}{3x^2 + 7}\, dx$

$= \dfrac{1}{6}\displaystyle\int \frac{6x}{3x^2 + 7}\, dx$

$= \dfrac{1}{6} \log_e [3x^2 + 7]_0^2$

$= \dfrac{1}{6} \log_e \left(\dfrac{19}{7}\right)$

d $\dfrac{d}{dx}(x \sin x))$

$= x \cos x + \sin x$

Therefore,

$\int x \cos x + \sin x\, dx = x \sin x + c$

$\int_0^{\frac{\pi}{4}} x \cos x + \sin x\, dx = \Big[x \sin x\Big]_0^{\frac{\pi}{4}}$

$\int_0^{\frac{\pi}{4}} x \cos x\, dx = \Big[\cos x + x \sin x\Big]_0^{\frac{\pi}{4}}$

$= \dfrac{\pi\sqrt{2}}{8} + \dfrac{\sqrt{2}}{2} - 1$

4 **a** $1 + \log_e(2x),\ -x + x\log_e(2x)$

b $x + 2x\log_e(2x),\ \dfrac{1}{2}x^2 \log_e(2x) - \dfrac{x^2}{4}$

c

$\dfrac{d}{dx}\left(x + \sqrt{1 + x^2}\right)$

$= 1 + \dfrac{2x}{2\sqrt{1 + x^2}}$

$= 1 + \dfrac{x}{\sqrt{1 + x^2}}$

$\dfrac{d}{dx}\left(\log_e (x + \sqrt{1 + x^2})\right)$

$= \left(1 + \dfrac{x}{\sqrt{1 + x^2}}\right) \times \dfrac{1}{x + \sqrt{1 + x^2}}$

$= \left(\dfrac{\sqrt{1 + x^2} + x}{\sqrt{1 + x^2}}\right) \times \dfrac{1}{x + \sqrt{1 + x^2}}$

$= \dfrac{1}{\sqrt{1 + x^2}}$

$\displaystyle\int_0^1 \frac{1}{\sqrt{1 + x^2}}\, dx$

$= \Big[\log_e(x + \sqrt{1 + x^2})\Big]_0^1$

$= \log_e(1 + \sqrt{1 + 1}) - \log_e(0 + \sqrt{1 + 0})$

$= \log_e(1 + \sqrt{2}) - \log_e 1$

$= \log_e(1 + \sqrt{2})$

5 $\dfrac{d}{dx}\left(e^{\sqrt{x}}\right)$

$= \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

Therefore,

$\int_1^2 \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}\, dx = \Big[e^{\sqrt{x}}\Big]_1^2 = 2e^{\sqrt{2}} - 2e$

6 $6\sin^2(2x)\cos(2x),\ \dfrac{1}{6}$

7 *using the CAS calculator's 'integral' command:*

a 139.68

b 18.50

c −0.66

d −23.76

e 2.06

f 0.43

8 a $LHS = \dfrac{2x+3}{x-1}$

$= \dfrac{2x-2+2+3}{x-1}$

$= \dfrac{2(x-1)}{x-1} + \dfrac{5}{x-1}$

$= 2 + \dfrac{5}{x-1}$

$= RHS\ \ QED$

b $\displaystyle\int_2^4 \frac{2x+3}{x-1}\,dx$

$\displaystyle = \int_2^4 2 + \frac{5}{x-1}\,dx$

$= \Big[2x + 5\log_e|x-1|\Big]_2^4$

$= (8 + 5\log_e 3) - (4 + 5\log_e 1)$

$= 4 + 5\log_e 3$

9 a $LHS = \dfrac{5x-4}{x-2}$

$= \dfrac{5x-4-6+6}{x-2}$

$= \dfrac{5(x-2)}{x-2} + \dfrac{6}{x-2}$

$= 5 + \dfrac{6}{x-2}$

$= RHS\ \ QED$

b $\displaystyle\int_3^4 \frac{5x-4}{x-2}\,dx$

$\displaystyle = \int_3^4 5 + \frac{6}{x-2}\,dx$

$= \Big[5x + 6\log_e|x-2|\Big]_3^4$

$= (20 + 6\log_e 2) - (15 + 6\log_e 1)$

$= 5 + 6\log_e 2$

10 a $y = \left(1 - \frac{1}{2}x\right)^8$

$\dfrac{dy}{dx} = \dfrac{-1}{2} \times 8\left(1 - \frac{1}{2}x\right)^7$

$= -4\left(1 - \frac{1}{2}x\right)^7$

$\displaystyle\int \left(1 - \frac{1}{2}x\right)^7 dx$

$\displaystyle = \frac{-1}{4}\int -4\left(1 - \frac{1}{2}x\right)^7 dx$

$= \dfrac{-1}{4}\left(1 - \frac{1}{2}x\right)^8 + c$

b $y = \log_e|\cos x|$

$\dfrac{dy}{dx} = -\sin x \times \dfrac{1}{\cos x}$

$= -\tan x$

$\displaystyle\int_0^{\frac{\pi}{3}} \tan x\,dx$

$\displaystyle = \int_0^{\frac{\pi}{3}} -\tan x\,dx$

$= -\Big[\log_e|\cos x|\Big]_0^{\frac{\pi}{3}}$

$= -\log_e|\cos\frac{\pi}{3}| + \log_e|\cos 0|$

$= -\log_e\dfrac{1}{2} + \log_e 1$

$= \log_e 2$

11 $f'(x) = \sin\left(\frac{1}{2}x\right)$

$$f(x) = -2\cos\left(\frac{1}{2}x\right) + c$$

$$f\left(\frac{4\pi}{3}\right) = 2,$$

$$2 = -2\cos\left(\frac{2\pi}{3}\right) + c$$

$$2 = -2\left(\frac{-1}{2}\right) + c$$

$$2 = 1 + c$$

$$c = 1$$

$$f(x) = -2\cos\left(\frac{1}{2}x\right) + 1$$

12 **a** $f'(x) = \cos 2x$

$$f(x) = \frac{1}{2}\sin 2x + c$$

$$f(\pi) = 1,$$

$$1 = \frac{1}{2}\sin 2\pi + c$$

$$c = 1$$

$$f(x) = \frac{1}{2}\sin 2x + 1$$

b $f'(x) = \frac{3}{x}$

$$f(x) = 3\log_e |x| + c$$

$$f(1) = 6,$$

$$6 = 3\log_e 1 + c$$

$$c = 6$$

$$f(x) = 3\log_e |x| + 6$$

c $f'(x) = e^{\frac{x}{2}}$

$$f(x) = 2e^{\frac{x}{2}} + c$$

$$f(0) = 1,$$

$$1 = 2 + c$$

$$c = -1$$

$$f(x) = 2e^{\frac{x}{2}} - 1$$

13 $\frac{d}{dx}(x\sin 3x) = \sin 3x + 3x\cos 3x$

$$\int_0^{\frac{\pi}{6}} x\cos 3x\, dx$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{6}} 3x\cos 3x + \sin 3x - \sin 3x\, dx$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{6}} 3x\cos 3x + \sin 3x - \frac{1}{3}\int_0^{\frac{\pi}{6}} \sin 3x\, dx$$

$$= \frac{1}{3}\Big[x\sin 3x\Big]_0^{\frac{\pi}{6}} - \frac{1}{3}\left[\frac{-1}{3}\cos 3x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{3}\left(\frac{\pi}{6}\sin\frac{\pi}{2} - 0\right) - \frac{1}{3}\left(\frac{-1}{3}\cos\frac{\pi}{2} + \frac{1}{3}\cos 0\right)$$

$$= \frac{\pi}{18} - \frac{1}{9}$$

14 $y = a + b\sin\left(\frac{\pi x}{2}\right)$

(0, 1)

$\Rightarrow 1 = a + b\sin 0$

$1 = a$

(3, 3)

$\Rightarrow 3 = 1 + b\sin\left(\frac{3\pi}{2}\right)$

$$3 = 1 - b$$

$$b = -2$$

$$y = 1 - 2\sin\left(\frac{\pi x}{2}\right)$$

$$x = 0, y = 1$$

$$x = 1, y = 1 - 2\sin\frac{\pi}{2}$$

$$= -1$$

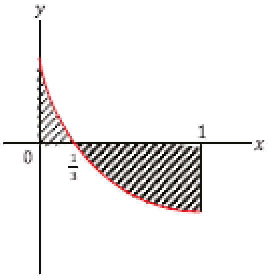
$\therefore$ *there is an* $x -$ *intercept*

$$0 = 1 - 2\sin\left(\frac{\pi}{2}x\right)$$

$$\sin\frac{\pi}{2}x = \frac{1}{2}$$

$$\frac{\pi}{2}x = \frac{\pi}{6}$$

$$x = \frac{1}{3}$$



$$\therefore A = \int_0^{\frac{1}{3}} y dx - \int_{\frac{1}{3}}^{1} y\,dx$$

$$= \int_0^{\frac{1}{3}} 1 - 2\sin\frac{\pi x}{2}\,dx$$

$$- \int_{\frac{1}{3}}^{1} 1 - 2\sin\frac{\pi x}{2}\,dx$$

$$= \left[x + \frac{4}{\pi}\cos\frac{\pi x}{2}\right]_0^{\frac{1}{3}} - \left[x + \frac{4}{\pi}\cos\frac{\pi x}{2}\right]_{\frac{1}{3}}^{1}$$

$$= \left(\frac{1}{3} + \frac{4}{\pi}\cos\frac{\pi}{6}\right) - \left(0 + \frac{4}{\pi}\cos 0\right)$$

$$- \left(1 + \frac{4}{\pi}\cos\frac{\pi}{2}\right) + \left(\frac{1}{3} + \frac{4}{\pi}\cos\frac{\pi}{6}\right)$$

$$= \frac{-1}{3} + \frac{4\sqrt{3}}{\pi} - \frac{4}{\pi} \approx 0.5987 \text{ square units}$$

15 *using the CAS calculator's 'integral' command:*

a 1.450 square units

b 1.716 square units

16 *using the CAS calculator's 'integral' command:*
0.1345

17 $f'(x) = x + \sin 2x$

$$f(x) = \frac{x^2}{2} - \frac{1}{2}\cos 2x + c$$

$$f(0) = 1,$$

$$1 = 0 - \frac{1}{2}\cos 0 + c$$

$$c = \frac{3}{2}$$

$$f(x) = \frac{x^2 - \cos 2x + 3}{2}$$

18 **a** $\int f(x)\,dx$

$$= \int g'(x)\,dx$$

$$= g(x) + c$$

$$= (x^2 + 1)^3 + c$$

b $\int h(x)\,dx$

$$= \int k'(x)\,dx$$

$$= k(x) + c$$

$$= \sin x^2 + c$$

c $\displaystyle\int f(x) + h(x)\,dx$

$$= \int g'(x) + k'(x)\,dx$$

$$= g(x) + k(x) + c$$

$$= (x^2 + 1)^3 + \sin x^2 + c$$

d $\displaystyle\int -f(x)\,dx$

$$= -\int g'(x)\,dx$$

$$= -g(x) + c$$

$$= -(x^2 + 1)^3 + c$$

e $\displaystyle\int f(x) - 4\,dx$

$$= \int g'(x)\,dx - \int 4\,dx$$

$$= g(x) - 4x + c$$

$$= (x^2 + 1)^3 - 4x + c$$

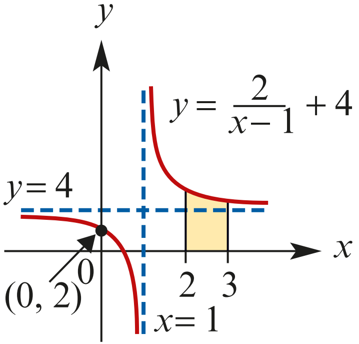
f $\displaystyle\int 3h(x)\,dx$

$$= 3\int k'(x)\,dx$$

$$= 3k(x) + c$$

$$= 3\sin x^2 + c$$

19



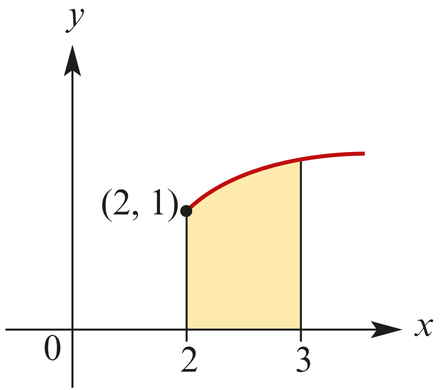
$$\int_2^3 \frac{2}{x-1} + 4\,dx$$

$$= [2\log_e|x-1| + 4x]_2^3$$

$$= (2\log_e 2 + 12) - (2\log_e 1 + 8)$$

$$= 4 + 2\log_e 2$$

20



$$\int_2^3 \sqrt{2x-4} + 1\,dx$$

$$= \left[\frac{2}{3\times 2}(2x-4)^{\frac{3}{2}} + x\right]_2^3$$

$$= \left(\frac{1}{3}(2)^{\frac{3}{2}} + 3\right) - \left(\frac{1}{3}(0)^{\frac{3}{2}} + 2\right)$$

$$= 1 + \frac{1}{3}(2)^{\frac{3}{2}}$$

21 **a** $\displaystyle\int_3^4 \sqrt{x-2}\,dx$

$$= \left[\frac{2}{3}(x-2)^{\frac{3}{2}}\right]_3^4$$

$$= \frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}$$

$$= \frac{2}{3}(2\sqrt{2} - 1)$$

b $\displaystyle\int_0^2 \sqrt{2-x}\,dx$

$$= \left[\frac{-2}{3}(2-x)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{-2}{3}(0)^{\frac{3}{2}} + \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2^{\frac{5}{2}}}{3}$$

c $\int_0^1 \frac{1}{3x+1}\,dx$

$= \left[\frac{1}{3}\log_e|3x+1|\right]_0^1$

$= \frac{1}{3}\log_e 4 + \frac{1}{3}\log_e 1$

$= \frac{2}{3}\log_e 2$

d $\int_1^2 \frac{1}{2x-1} + 3\,dx$

$= \left[\frac{1}{2}\log_e|2x-1| + 3x\right]_1^2$

$= \left(\frac{1}{2}\log_e|3| + 6\right) - \left(\frac{1}{2}\log_e 1 + 3\right)$

$= \frac{1}{2}\log_e 3 + 3$

e $\int_{2.5}^3 \sqrt{2x-5} - 6\,dx$

$= \left[\frac{2}{3} \times \frac{1}{2} \times (2x-5)^{\frac{3}{2}} - 6x\right]_{2.5}^3$

$= \left(\frac{1}{3}(1)^{\frac{3}{2}} - 18\right) - \left(\frac{1}{3}(0)^{\frac{3}{2}} - 15\right)$

$= \frac{1}{3} - 3$

$= \frac{-8}{3}$

f $\int_3^4 \frac{1}{\sqrt{x-2}}\,dx$

$= \left[2(x-2)^{\frac{1}{2}}\right]_3^4$

$= 2(2)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}}$

$= 2\sqrt{2} - 2$

Solutions to Exercise 11I

1 $y_1 = 12 - x - x^2, y_2 = x + 4$

$12 - x - x^2 = x + 4$

$x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x = 2, -4$

to test which graph is higher in this interval:

$x = 0, y_1 = 12, y_2 = 4$

$$A = \int_{-4}^{2} y_1 - y_2 \, dx$$

$$= \int_{-4}^{2} (12 - x - x^2) - (x + 4) \, dx$$

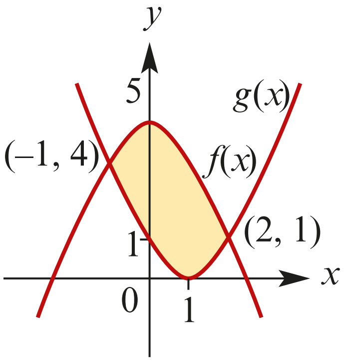
$$= \int_{-4}^{2} 8 - 2x - x^2 \, dx$$

$$= \left[8x - x^2 - \frac{x^3}{3}\right]_{-4}^{2}$$

$$= \left(16 - 4 - \frac{8}{3}\right) - \left(-32 - 16 + \frac{64}{3}\right)$$

$= 36$ units^2

2



$f(x) = 5 - x^2, g(x) = (x - 1)^2$

$5 - x^2 = (x - 1)^2$

$5 - x^2 = x^2 - 2x + 1$

$2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x - 1) = 0$

$x = -1, 2$

to test which graph is higher in this interval:

$f(0) = 5, \; g(0) = (-1)^2 = 1$

$$\therefore A = \int_{-1}^{2} f(x) - g(x) \, dx$$

$$= \int_{-1}^{2} 5 - x^2 - (x^2 - 2x + 1) \, dx$$

$$= \int_{-1}^{2} 4 + 2x - 2x^2 \, dx$$

$$= \left[4x + x^2 - \frac{2x^3}{3}\right]_{-1}^{2}$$

$$= \left(8 + 4 - \frac{16}{3}\right) - \left(-4 + 1 + \frac{2}{3}\right)$$

$= 16 - 6 - 1$

$= 9$ units^2

3 **a** $y_1 = x + 3, y_2 = 12 + x - x^2$

$x + 3 = 12 + x - x^2$

$x^2 - 9 = 0$

$(x + 3)(x - 3) = 0$

$x = -3, 3$

to test which graph is higher in this interval:

$x = 0, y_1 = 3, y_2 = 12$

$$A = \int_{-3}^{3} y_2 - y_1 \, dx$$

$$= \int_{-3}^{3} (12 + x - x^2) - (x + 3) \, dx$$

$$= \int_{-3}^{3} 9 - x^2 \, dx$$

$$= \left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= \left(27 - \frac{27}{3}\right) - \left(-27 + \frac{27}{3}\right)$$

$$= 54 - \frac{54}{3}$$

$$= 54 - 18$$

$$= 36 \text{ units}^2$$

b $y_1 = 3x + 5, y_2 = x^2 + 1$

$3x + 5 = x^2 + 1$

$x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

$x = -1, 4$

to test which graph is higher in this interval:

$x = 0, y_1 = 5, y_2 = 1$

$$A = \int_{-1}^{4} y_1 - y_2 \, dx$$

$$= \int_{-1}^{4} (3x + 5) - (1 + x^2) \, dx$$

$$= \int_{-1}^{4} 4 + 3x - x^2 \, dx$$

$$= \left[4x + \frac{3}{2}x^2 - \frac{x^3}{3}\right]_{-1}^{4}$$

$$= \left(16 - 24 - \frac{64}{3}\right) - \left(-4 + \frac{3}{2} + \frac{1}{3}\right)$$

$$= \frac{125}{6} \text{ units}^2$$

c $y_1 = 3 - x^2, y_2 = 2x^2$

$3 - x^2 = 2x^2$

$x^2 = 1$

$x = \pm 1$

to test which graph is higher in this interval:

$x = 0, y_1 = 3, y_2 = 0$

$$A = \int_{-1}^{1} y_1 - y_2 \, dx$$

$$= \int_{-1}^{1} (3 - x^2) - (2x^2) \, dx$$

$$= \int_{-1}^{1} 3 - 3x^2 \, dx$$

$$= [3x - x^3]_{-1}^{1}$$

$$= (3 - 1) - (-3 + 1)$$

$$= 4 \text{ units}^2$$

d $y_1 = x^2, y_2 = 3x$

$x^2 = 3x$

$x^2 - 3x = 0$

$x(x - 3) = 0$

$x = 0, 3$

to test which graph is higher in this interval:

$x = 1, y_1 = 1, y_2 = 3$

$$A = \int_{0}^{3} y_2 - y_1 \, dx$$

$$= \int_{0}^{3} 3x - x^2 \, dx$$

$$= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{0}^{3}$$

$$= \left(\frac{27}{2} - 9\right) - (0 - 0)$$

$$= \frac{9}{2} \text{ units}^2$$

e $y_1^2 = x,\ x - y_2 = 2$

$y_1 = \pm\sqrt{x},\ y_2 = x - 2$

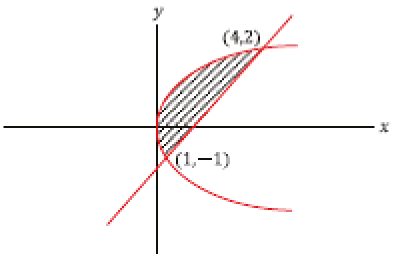
$\pm\sqrt{x} = x - 2$

$x = x^2 - 4x + 4$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 1, 4$



$$A = \int_0^1 \sqrt{x} - \sqrt{x}\,dx$$

$$+ \int_1^4 \sqrt{x} - (x-2)\,dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right]_1^4$$

$$= \frac{4}{3} + \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= \frac{20}{3} - \frac{2}{3} - \frac{3}{2}$$

$$= 6 - \frac{3}{2}$$

$$= \frac{9}{2} \text{ units}^2$$

4 a $P = \int_{-1}^{0} e - e^{-x}dx + \int_0^1 e - e^x\,dx$

$$= 2\int_0^1 e - e^x\,dx$$

$$= 2[ex - e^x]_0^1$$

$$= 2(e - e) - 2(0 - 1)$$

$$= 2 \text{ units}^2$$

b $Q = \int_0^1 e^x - e^{-x}\,dx$

$$= \left[e^x + e^{-x}\right]_0^1$$

$$= \left(e + \frac{1}{e}\right) - (1 + 1)$$

$$= e + \frac{1}{e} - 2 \approx 1.086 \text{ units}^2$$

5

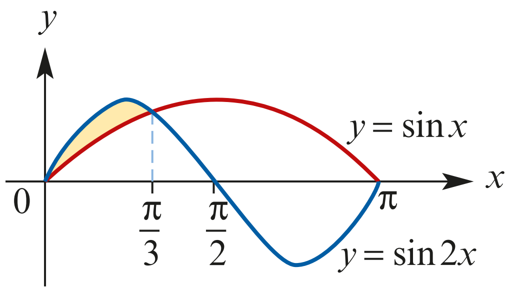
$$A = \int_0^{\frac{7\pi}{6}} (\sin x) - \left(\frac{-1}{2}\right)dx$$

$$= \left[-\cos x + \frac{1}{2}x\right]_0^{\frac{7\pi}{6}}$$

$$= \left(-\left(\frac{-\sqrt{3}}{2}\right) + \frac{1}{2} \times \frac{7\pi}{6}\right) - \left(-(1) + \frac{1}{2}(0)\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{12} + 1 \approx 3.699 \text{ units}^2$$

6



$$A = \int_0^{\frac{\pi}{3}} \sin 2x - \sin x\,dx \text{ (\textit{from the graph})}$$

$$= \left[\frac{-1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{-1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right)$$

$$- \left(\frac{-1}{2}\cos 0 + \cos 0\right)$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{2}$$

$$= \frac{1}{4} \text{ units}^2$$

7

$$A = \int_0^{\frac{\pi}{6}} \cos x - \sin 2x\,dx$$

$$+ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x - \cos x\,dx$$

$$= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}}$$

$$+ \left[\frac{-1}{2}\cos 2x - \cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(\sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3}\right) - \left(\sin 0 + \frac{1}{2}\cos 0\right)$$

$$+ \left(\frac{-1}{2}\cos\pi - \sin\frac{\pi}{2}\right) - \left(\frac{-1}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{6}\right)$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2} + \frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{4}$$

$$= \frac{1}{2}\text{ units}^2$$

8

$$e^x = 2 + 3e^{-x}$$

$$e^{2x} - 2e^x - 3 = 0$$

$$(e^x - 3)(e^x + 1) = 0$$

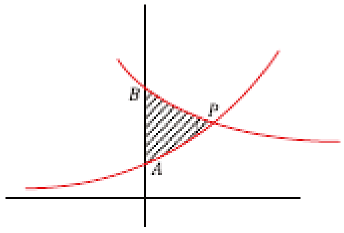
$$e^x = -1, 3$$

Since $e^x > 0$, $e^x = 3$

$$x = \log_e 3$$

$$y = e^x = 3$$

$$P = (\log_e 3, 3)$$



$$Area = \int_0^{\log_e 3} 2 + 3e^{-x} - e^x\,dx$$

$$= \left[2x - 3e^{-x} - e^x\right]_0^{\log_e 3}$$

$$= (2\log_e 3 - 3e^{\log_e \frac{1}{3}} - e^{\log_e 3}) - (0 - 3 - 1)$$

$$= 4 + 2\log_e 3 - 1 - 3$$

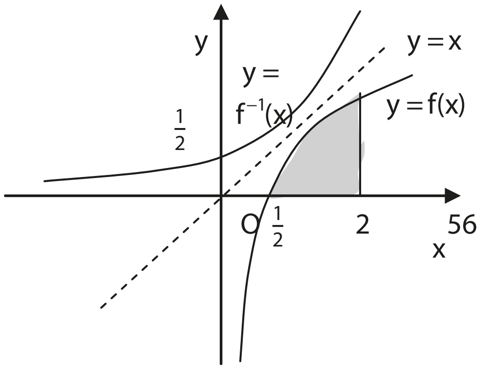
$$= 2\log_e 3$$

$$\approx 2.197\text{ units}^2$$

9 **a** $f : R^+ \to R, f(x) = \log_e(2x)$

Consider $x = \log_e(2y)$. Solving for y gives $y = \frac{1}{2}e^x$

$f^{-1}(x) = \frac{1}{2}e^x$ and the domain of $f^{-1} = R$



b $b\int_0^{\log_e 4} f^{-1}(x)\,dx$

$$= \int_0^{\log_e 4} \frac{1}{2}e^x\,dx$$

$$= \left[\frac{1}{2}e^x\right]_0^{\log_e 4} = \frac{3}{2}$$

c By symmetry $\int_{\frac{1}{2}}^{2} f(x)\,dx$

$$= 4\log_e(2) - \frac{3}{2}$$

Solutions to Exercise 11J

1 a
$$\text{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\text{av} = \frac{1}{2-0}\int_0^2 x(2-x)\,dx$$
$$= \frac{1}{2}\int_0^2 (2x - x^2)\,dx$$
$$= \frac{1}{2}\left[x^2 - \frac{1}{3}x^3\right]_0^2$$
$$= \frac{1}{2}\left(4 - \frac{8}{3}\right) = \frac{2}{3}$$

b
$$\text{av} = \frac{1}{\pi - 0}\int_0^\pi \sin(x)\,dx$$
$$= \frac{1}{\pi}\Big[-\cos(x)\Big]_0^\pi$$
$$= \frac{1}{\pi}((-(-1)) - (-1)) = \frac{2}{\pi}$$

c
$$\text{av} = \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \sin(x)\,dx$$
$$= \frac{2}{\pi}\Big[-\cos(x)\Big]_0^{\frac{\pi}{2}}$$
$$= \frac{2}{\pi}((-0) - (-1)) = \frac{2}{\pi}$$

d
$$\text{av} = \frac{1}{\frac{2\pi}{n} - 0}\int_0^{\frac{2\pi}{n}} \sin(nx)\,dx$$
$$= \frac{n}{2\pi}\left[-\frac{1}{n}cos(nx)\right]_0^{\frac{2x}{n}}$$
$$\frac{1}{2\pi}((-1) - (-1)) = 0$$
$$= \frac{1}{2\pi}((-1) - (-1)) = 0$$

e
$$\text{av} = \frac{1}{2-(-2)}\int_{-2}^2 (e^x + e^{-x})\,dx$$
$$= \frac{1}{4}\Big[e^x - e^{-x}\Big]_{-2}^2$$
$$= \frac{1}{4}((e^2 - e^{-2}) - (e^{-2} - e^2))$$
$$= \frac{1}{2}(e^2 - e^{-2})$$

2
$$\text{av temp} = \frac{1}{10-0}\int_0^{10} 50e^{-\frac{t}{2}}\,dt$$
$$= \frac{1}{10}\left[\frac{50}{\left(-\frac{t}{2}\right)}e^{-\frac{t}{2}}\right]_0^{10}$$
$$= -10(e^{-5} - e^{-0})$$
$$= 10(1 - e^{-5}) \approx 9.93°\text{C}$$

3
$$\text{mean value} = \frac{1}{a-0}\int_0^a x(a-x)\,dx$$
$$= \frac{1}{a}\int_0^a (ax - x^2)\,dx$$
$$= \frac{1}{a}\left[\frac{1}{2}ax^2 - \frac{1}{3}x^3\right]_0^a$$
$$= \frac{1}{a}\left(\frac{1}{2}a^3 - \frac{1}{3}a^3\right)$$
$$= \frac{a^2}{6}$$

4 a $pv^{0.9} = 300 \Rightarrow p = 300v^{-0.9}$

$$\text{av pressure } = \frac{1}{1-\frac{1}{2}}\int_{\frac{1}{2}}^{1} 300v^{-0.9}dv$$

$$2\left[\frac{300}{0.1}v^{0.1}\right]_{\frac{1}{2}}^{1}$$

$$= 6000\left(1-\left(\frac{1}{2}\right)^{0.1}\right)$$

$$= 3000(2-2^{0.9})$$

$$\approx 401.8 \text{ N/m}^2$$

b $v = 3t + 1$, so:

$t = 0,\ v = 1,\ t = 1,\ v = 4$

av pressure

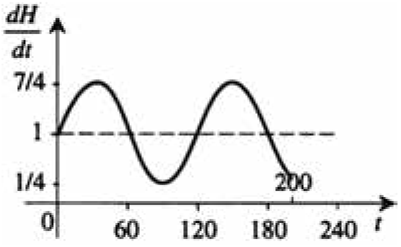
$$= \frac{1}{4-1}\int_{1}^{4} 300v^{-0.9}dv$$

$$= \frac{1}{3}\left[\frac{300}{0.1}v^{0.1}\right]_{1}^{4}$$

$$= 1000((4)^{0.1}-1)$$

$$\approx 148.7 \text{ N/m}^2$$

5 a $\frac{dH}{dt} = 1 + \frac{3}{4}\sin\left(\frac{\pi t}{60}\right)$ for $t \in [0,200]$



b $\frac{dH}{dt} > 1.375 \Leftrightarrow 1 + 4\frac{3}{4}\sin\left(\frac{\pi t}{60}\right) > \frac{11}{8}$

$$\Leftrightarrow \frac{3}{4}\sin\left(\frac{\pi t}{60}\right) > \frac{3}{8}$$

$$\Leftrightarrow \sin\left(\frac{\pi t}{60}\right) > \frac{1}{2}$$

Consider the equation $\sin\left(\frac{\pi t}{60}\right) = \frac{1}{2}$

This is equivalent to

$\frac{\pi t}{60} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{13\pi}{6}$ or $\frac{17\pi}{6}$ or . . .

i.e. $t = 10$ or 50 or 130 or 170 or $\cdots$

For the required domain and by observation from graph $\frac{dH}{dt} > 1.375$ for $t \in (10, 50) \cup (130, 170)$

c The rate of heat loss is greatest when $\sin\left(\frac{\pi t}{60}\right) = 1$

This occurs when

$\frac{\pi t}{60} = \frac{\pi}{2}$ or $\frac{5\pi}{2}$ or $\frac{9\pi}{2}$ or . . .

$\Rightarrow t = 30$ or 150 or 270 or$\cdots$

$\therefore$ rate of heat loss is greatest when $t = 30$ or 150 for t in $[0, 200]$

d **i** The total heat loss for $t \in [0, 120] = \int_0^{120} 1 + \frac{3}{4}\sin\left(\frac{\pi t}{60}\right)dt$

$$= \left[t - \frac{3}{4}\times\frac{60}{\pi}\cos\left(\frac{\pi t}{60}\right)\right]_0^{120}$$

$$= 120 - \frac{45}{\pi}\cos 2\pi$$

$$-\left(0 - \frac{45}{\pi}\cos 0\right)$$

$$= 120 - \frac{45}{\pi} + \frac{45}{\pi}$$

$$= 120$$

$\therefore$ 120 kilojoules lost over the 120 days.

ii Total heat lost for $t \in [0, 200] = \int_0^{200)} 1 + \frac{3}{4}\sin\left(\frac{\pi t}{60}\right)dt$

$$= \left[t - \frac{45}{\pi}\cos\left(\frac{\pi t}{60}\right)\right]_0^{200}$$

$$= 200 - \frac{45}{\pi}\cos\left(\frac{200\pi}{60}\right) - \left(0 - \frac{45}{\pi}\right)$$

$$= 200 - \frac{45}{\pi}\cos\left(\frac{10\pi}{3}\right) + \frac{45}{\pi}$$

$$= 200 - \frac{45}{\pi}\times -\frac{1}{2} + \frac{45}{\pi}$$

$$= 200 + \frac{45}{2\pi} + \frac{45}{\pi}$$

$$= 200 + \frac{135}{2\pi}$$

$$\approx 221.48 \text{ kilojules}$$

6 $\dfrac{dV}{dt} = 1000 - 30t^2 + 2t^3 \; 0 \le t \le 15$

a When $t = 0$, $\dfrac{dV}{dt} = 1000$

The rate of flow is 1000 million litres/hour= 10^9 litres/hour.

When $t = 2$,

$= 1000 - 30 \times 4 + 2 \times 2^3 =$

$1000 - 120 + 16$

$= 896$ million litres/hour $= 8.96 \times 10^8$ litres/hour

b **i** To find stationary points, let $R = \dfrac{dV}{dt} = 1000 - 30t^2 + 2t^3$

Stationary points occur when $\dfrac{dR}{dt} = 0$

$$\frac{dR}{dt} = -60t + 6t^2$$

$$= -6t(10 - t)$$

$\dfrac{dR}{dt} = 0$ implies $t = 0$ or $t = 10$

A gradient chart for $\dfrac{dR}{dt}$ is as shown:

t	< 0	0	<<	10	> 10
sing of $\frac{dR}{dt}$	+ ve	0	-ve	0	+ ve
shape	/	–	\	–	/

$\therefore$ a local maximum at (0,1000) and a local minimum when $t = 10$.

when $t = 10$

$R = \dfrac{dV}{dt} = 1000 - 30 \times 10^2 + 2 \times 10^3$

$= 1000 - 3000 + 2000$

$= 0$

$\therefore$ local minimum at $(10,\ 0)$

When $t = 15$, $\dfrac{dV}{dt} =$

$1000 - 30 \times 15^2 + 2 \times 15^3$

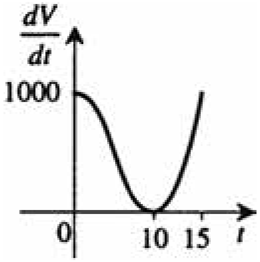
$= 1000 - 30 \times 225 + 2 \times 3375$

$= 1000$

$\therefore$ The maximum flow occurs when $t = 0$ and $t = 15$

ii The maximum flow is 1000 millon litres/hour.

c



d **i** Area beneath the graph between $t = 0$ and $t = 10$

$$= \int_0^{10} 1000 - 30t^2 + 2t^3 \, dt$$

$$= \left[1000t - \frac{30 \times t^3}{3} + \frac{2t^4}{4}\right]_0^{10}$$

$$= 5000$$

ii 5000 million litres flowed out in the first 10 hours.

7 **a** $R{:}[0,\ \infty) \to R,\ R(t) = 10\log_e(t + 1)$

When $t = 5,\ R(5) = 10\log_e(6) \approx 17.918$

When $t = 5$, the rate of growth is ≈ 17.918 penguins per year.

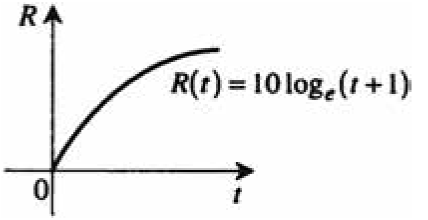
When $t = 10,\ R(10) = 10\log_e(10) \approx 23.978$

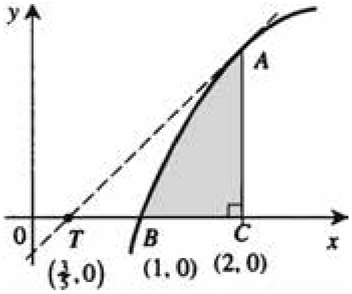
When $t = 10$, the rate of growth is 23.978 penguins per year.

When $t = 100,\ R(100) = 10\log_e(100) \approx 46.151$

When $t = 100$, the rate of growth is 46.151 penguins per year.

b





c For the inverse function consider

$$t = 10\log_e(y+1)$$

$$\therefore \frac{t}{10} = \log_e(y+1)$$

$$\therefore e^{\frac{t}{10}} = y + 1$$

$$\therefore y = e^{\frac{t}{10}} - 1$$

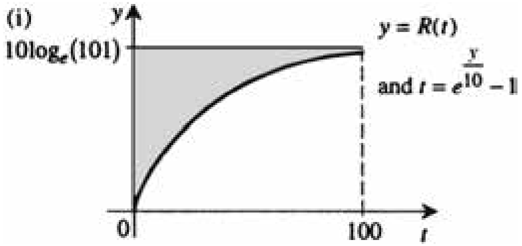
and the inverse function is

$R^{-1}(t) = e^{\frac{t}{10}} - 1$

The domain of R^{-1} = range of

$R = R^+ \cup \{0\}$

d **i**



$\therefore$ required area = area of rectangle − area

$$= 100 \times 10\log_e(101) - \int_0^{10\log_e(101)} (e^{\frac{y}{10}} - 1)\,dy$$

$$= 1000\log_e(101) - \left[10e^{\frac{y}{10}} - y\right]_0^{10\log_e 101}$$

$$= 1000\log_e(101) - \left[10e^{\log_e 101} - 10\log_e(101) - (10e^0 - 0)\right]$$

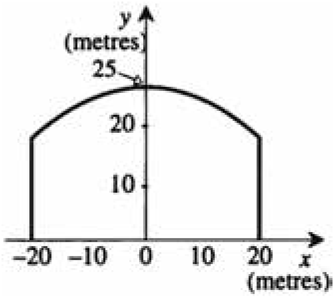
$$= 1000\log e(101) - [1010 - 10\log_e(101) - 10]$$

$$= 1000\log_e(101) - 1000$$

$$\approx 3661.27$$

ii The penguin population has increased by 3661 penguins over 100 years.

8



Area of cross section

$= \int_{-20}^{20}(25 - 0.02x^2)\,dx$

The symmetry of f gives that the area of cross section

$= 2\int_0^{20}(25 - 0.02x^2)\,dx$

$= 2\left[25x - \frac{x^3}{150}\right]_0^{20}$

$= 2\left(25 \times 20 - \frac{20^3}{150}\right)$

$= 2\left(500 - \frac{8000}{150}\right)$

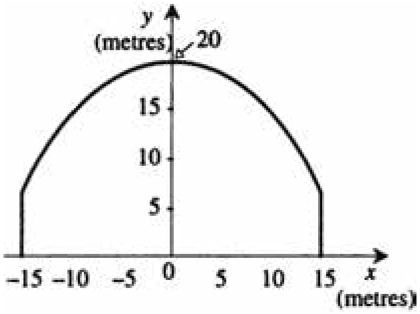
$= 893\frac{1}{3}$

The volume of the hall = area of cross section × length

$= 893\frac{1}{3} \times 80$

$= 71\#466\frac{2}{3}$ m^3

9 **a**



Area of cross section

$= \int_{-15}^{15}(20 - 0.06x^2)\,dx$

The symmetry of f gives that the area of cross section

$= 2\int_0^{15}(20 - 0.06x^2)\,dx$

$= 2\left[20x - \frac{x^3}{50}\right]_0^{15}$

$= 2\left(20 \times 15 - \frac{15^3}{30}\right)$

$= 465$

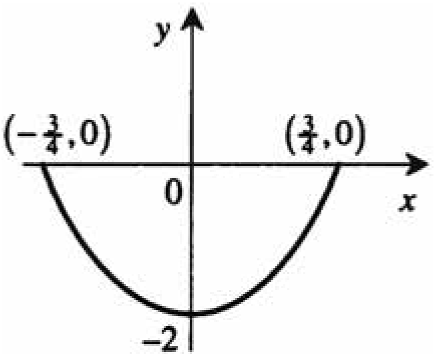
The area of the cross section is 465 m^2

b The volume of the hangar = area of cross section × length

$= 465 \times 100$

$= 46\,500$ m^3

10



The parabola is of the form $y = ax^2 + b$

When $x = 0,\ y = -2$

When $y = 0,\ x = \pm\frac{3}{4}$

$0 = a(\frac{3}{4})^2 - 2$

and $\frac{2 \times 16}{9} = a$

i.e. $a = \frac{32}{9}$

∴ The equation of the parabola is

$y = \frac{32}{9}x^2 - 2$

The total volume of the trough = area of cross section × length To determine the area of cross section

consider $2\int_0^{\frac{3}{4}} \frac{32x^2}{9} - 2\,dx$

$$= 2\left[\frac{32x^3}{27} - 2x\right]^{\frac{3}{4}}$$

$$= 2\left(\frac{32}{27} \times \frac{27}{64} - 2 \times \frac{3}{4}\right)$$

$$= 2\left(\frac{1}{2} - 1\frac{1}{2}\right)$$

$$= -2$$

∴ The cross sectional area is 2 m^2

The total volume = $2 \times l = 2l$ m^3, where ℓ is the length of the trough.

When the trough is half full the volume is l m^3 and the cross sectional area is 1 m^3 The shaded area = 1 m^2

Shaded area $= 2\int_0^a \left(\frac{32}{9}a^2 - 2\right) - \left(\frac{32}{9}a^2 - 2\right)dx$

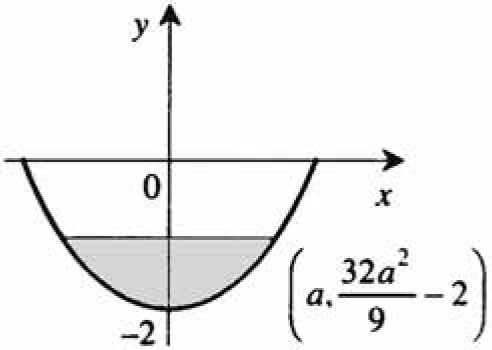
$$= 2\left[\frac{32}{9}a^2x - 2x - \frac{32}{27}x^3 + 2x\right]_0^a$$

$$= 2\left[\frac{32a^3}{9} - \frac{32a^3}{27}\right]$$

$$= 2 \times \left[\frac{64a^3}{27}\right]$$

$$= \frac{128a^3}{27}$$

$$\therefore \frac{128\,a^3}{27} = 1$$



which implies $a^3 = \frac{27}{128}$

$$\therefore a^3 = \frac{3^3}{2^7}$$

$$\therefore a = \frac{3}{2^{\frac{7}{3}}}$$

When $x = \frac{3}{2^{\frac{7}{3}}}$ $\quad y = \frac{32}{9} \times \left(\frac{3}{2^{\frac{7}{3}}}\right)^2 - 2$

$$= \frac{32}{9} \times -\frac{9}{2^{\frac{14}{3}}} - 2$$

$$= \frac{2^5}{2^{\frac{14}{3}}} - 2$$

$$= 2^{\frac{1}{3}} - 2$$

∴ the depth $= 2 - (2 - 2^{\frac{1}{3}})$

$= 2^{\frac{1}{3}}$ metres ≈ 1.26 metres

The depth of the water is 1.26 metres when it is half full.

11 **a** $y = 3 - 3\cos\left(\frac{x}{3}\right)$ for $x \in [-3\pi, 3\pi]$. The maximum value of the function is 6 and hence the maximum height of 6 metres

b The area $= 2\int_0^{3\pi}\left(3 - 3\cos\left(\frac{x}{3}\right)\right)dx$

$= 2[3x - 9\sin(\frac{x}{3})]_0^{3\pi} = 18\pi$

The area is 18π m^2

c **i** $\frac{dy}{dx} = \sin\left(\frac{x}{3}\right)$. When

$x = a$, $y = 3 - 3\cos\left(\frac{a}{3}\right)$ and $\frac{dy}{dx} = \sin\left(\frac{a}{3}\right)$.

Therefore the equation of the normal is

$$y - \left(3 - 3\cos\left(\frac{a}{3}\right)\right)$$

$$= -\frac{1}{\sin\left(\frac{a}{3}\right)}(x - a)$$

ii If it passes through (9,0),

$$0 - 3 + 3\cos\left(\frac{a}{3}\right) = -\frac{1}{\sin\left(\frac{a}{3}\right)}(9 - a)$$

Solving numerically gives $a = 5.409$

12 **a** $\frac{dV}{dt} = 3\left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2\right]$

i When $t = 0$, $\frac{dV}{dt} = 3[1 + 0 + 2] = 9$

ii When $t = 2$, $\frac{dV}{dt}$

$$= 3\left[-1 + \frac{1}{\sqrt{2}} + 2\right] = 3\frac{(\sqrt{2} + 2)}{2}$$

iii When $t = 4$, $\frac{dV}{dt} = 3[1 + 1 + 2]$
$= 12$

b From the graph maximum value is 12 and the minimum value is 0.834

c The volume through the pipe in the first 8 minutes

$$= 3\int_0^8 \left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2\right] dt$$

$$= 3\left[\frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right) - \frac{8}{\pi}\cos\left(\frac{\pi t}{8}\right) + 2t\right]_0^8$$

$$= 3\left(\frac{8}{\pi} + 16 - \left(-\frac{8}{\pi}\right)\right) = 48\left(1 + \frac{1}{\pi}\right)$$

$$= \frac{48(\pi + 1)}{\pi} \text{ litres}$$

Solutions to Technology-free questions

1 **a** $\displaystyle\int_2^3 x^3\,dx = \left[\frac{1}{4}x^4\right]_2^3 = \frac{65}{4}$

b Since $\sin x$ is an odd function, the integral is 0. (Alternatively work through the integral.)

c $\displaystyle\int_a^{4a} (a^{\frac{1}{2}} - x^{\frac{1}{2}})\,dx$

$$= \left[a^{\frac{1}{2}}x - \frac{2}{3}x^{\frac{3}{2}}\right]_a^{4a}$$

$$= \left(4a^{\frac{3}{2}} - \frac{2}{3} \times 8a^{\frac{3}{2}}\right) - \left(a^{\frac{3}{2}} - \frac{2}{3}a^{\frac{3}{2}}\right)$$

$$= -\frac{5a^{\frac{3}{2}}}{3}$$

d $\displaystyle\int_1^4 \frac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}}\,dx$

$$= \int_1^4 3x^{-\frac{1}{2}} - 5x^{\frac{1}{2}} - x^{-\frac{3}{2}}\,dx$$

$$= \left[6x^{\frac{1}{2}} - \frac{10}{3}x^{\frac{3}{2}} + 2x^{\frac{-1}{2}}\right]_1^4$$

$$= \left(12 - \frac{80}{3} + 1\right) - \left(6 - \frac{10}{3} + 2\right)$$

$$= -\frac{55}{3}$$

e $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2\,\theta d\theta = \left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$$= \left(\frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{1}{2}$$

f $\displaystyle\int_1^e \frac{1}{x}\,dx = [\log_e x]_1^e$

$$= \log_e e - \log_e 1 = 1$$

g $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2\left(\theta + \frac{\pi}{4}\right) d\theta$

$$= \left[-\frac{1}{2}\cos 2\left(\theta + \frac{\pi}{4}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left(-\frac{1}{2}\cos\frac{3\pi}{2}\right) - \left(-\frac{1}{2}\cos\frac{\pi}{2}\right)$$

$$= 0$$

h

$$\int_0^{\pi} \sin 4\theta\,d\theta = \left[-\frac{1}{4}\cos 4\theta\right]_0^{\pi}$$

$$= \left(-\frac{1}{4}\cos 4\pi\right) - \left(-\frac{1}{4}\cos 0\right)$$

$$= 0$$

2 $\displaystyle\int_{-1}^2 x + 2f(x)\,dx$

$$= \int_{-1}^2 x dx + 2\int_{-1}^2 f(x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^2 + 2 \times 5$$

$$= \frac{23}{2}$$

3 $\displaystyle\int_1^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^1 f(x)\,dx$

$$= 1 - (-2) = 3$$

4

$$\int_3^{-2} f(x)\,dx = -\int_{-2}^3 f(x)\,dx$$

$$= -\int_{-2}^1 f(x)\,dx - \int_1^3 f(x)\,dx$$

$$= -2 - (-6) = 4$$

5 $\int_0^2 (x+1)^7\,dx = \left[\frac{1}{8}(x+1)^8\right]_0^2 = 820$

6 $\int_0^1 (3x+1)^3\,dx = \left[\frac{1}{3\times 4}(3x+1)^4\right]_0^1$

$= \frac{85}{4}$

7 If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_0^9 f(x)\,dx = [F(x)]_0^9$$

$$= F(9) - F(0) = 5$$

Also by the chain rule

$\frac{d}{dx}(F(3x)) = 3f(3x)$, so:

$$\int_0^3 f(3x)\,dx = \frac{1}{3}[F(3x)]_0^3$$

$$= \frac{1}{3}(F(9) - F(0))$$

$$= \frac{5}{3}$$

8 If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_1^4 f(x)\,dx = [F(x)]_1^4$$

$$= F(4) - F(1) = 5$$

Also by the chain rule

$\frac{d}{dx}(F(3x+1)) = 3f(3x+1)$, so:

$$\int_0^1 f(3x+1)\,dx = \frac{1}{3}[F(3x+1)]_0^1$$

$$= \frac{1}{3}(F(4) - F(1))$$

$$= \frac{5}{3}$$

9 **a** Signed Area

$= -\frac{1}{2}(3+4) - \frac{1}{2}\times 3 + \frac{1}{2}\times 5 + \frac{1}{2}(5+12)$

$= -\frac{7}{2} - \frac{3}{2} + \frac{5}{2} + \frac{17}{2}$

$= 6$

b

$$\int_0^4 f(x)\,dx = -\int_0^4 x^2 - 4\,dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_0^4 \qquad = \frac{64}{3} - 16$$

$$= \frac{16}{3}$$

c Signed Area

$= -0.25 \times \left(8 + \frac{63}{8}\right) - 0.25 \times \left(\frac{63}{8} + 7\right) - 0.25 \times \left(7 + \frac{37}{8}\right) - 0.25 \times \frac{37}{8} + 0.25 \times \frac{61}{8} + 0.25 \times \left(\frac{61}{8} + 19\right)$

$= -3.1875$

10 The area of the shaded region from $x = a$ to $x = b$ is given by $\int_a^b f(x) - g(x)\,dx$.

The area of the shaded region from $x = b$ to $x = c$ is given by $\int_b^c g(x) - f(x)\,dx$.

The area of the shaded region from $x = c$ to $x = d$ is given by $\int_c^d f(x) - g(x)\,dx$.

The area of the shaded region is

$\int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx$
$+ \int_c^d f(x) - g(x)\,dx$

11 **a** The curves intersect where $2x + x^2 = 15$, i.e. $x^2 + 2x - 15 = 0$. Hence $(x-3)(x+5) = 0$, so P has coordinates $(3, 9)$.

Q has coordinates $(7.5,\ 0)$.

b The area of the shaded region is

$$\int_0^3 x^2\,dx + \int_3^{7.5} 15 - 2x\,dx$$

$$= \left[\frac{1}{3}x^3\right]_0^3 + [15x - x^2]_3^{7.5}$$

$$= 9 + (112.5 - 56.25) - (45 - 9)$$

$$= 29.25$$

12 a area $A = \int_1^2 10x^{-2}\,dx$

$$= [-10x^{-1}]_1^2$$

$$= -5 - (-10) = 5$$

b $\int_2^p 10x^{-2}\,dx = \int_p^5 10x^{-2}\,dx$

$$[-10x^{-1}]_2^p = [-10x^{-1}]_p^5$$

$$5 - \frac{10}{p} = -2 + \frac{10}{p}$$

$$\frac{20}{p} = 7$$

$$p = \frac{20}{7}$$

13 The area of the shaded region is

$$\int_2^4 16x^{-2} - 0.5x + 1\,dx$$

$$+ \int_4^5 0.5x - 1 - 16x^{-2}\,dx$$

$$= [-16x^{-1} - 0.25x^2 + x]_2^4$$

$$+ [0.25x^2 - x + 16x^{-1}]_4^5$$

$$= (-4 - 4 + 4) - (-8 - 1 + 2)$$

$$+ (6.25 - 5 + 3.2) - (4 - 4 + 4)$$

$$= 3.45$$

14 a When $x = 0, 6y - y^2 = 0$, so $y = 0, 6$. Thus A has coordinates $(0, 6)$. For point B, solve $y = 6y - y^2$, i.e. $y^2 - 5y = 0$, so $y = 0$ or $y = 5$. As $y = x$, then B has coordinates (5, 5).

b area

$$P = \int_0^5 y\,dy + \int_5^6 6y - y^2\,dy$$

$$= \left[\frac{1}{2}y^2\right]_0^5 + \left[3y^2 - \frac{1}{3}y^3\right]_5^6$$

$$= \frac{25}{2} + (108 - 72) - \left(75 - \frac{125}{3}\right)$$

$$= \frac{91}{6} = 15\frac{1}{6}$$

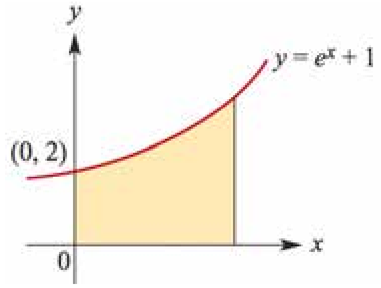
c Area bounded by the parabola and the y axis is given by

$$\int_0^6 6y - y^2\,dy = \left[3y^2 - \frac{1}{3}y^3\right]_0^6$$

$$= 36$$

So area $Q = 36 -$ area $P = 20\frac{5}{6} = \frac{125}{6}$.

15 a y intercept is $(0, 2)$

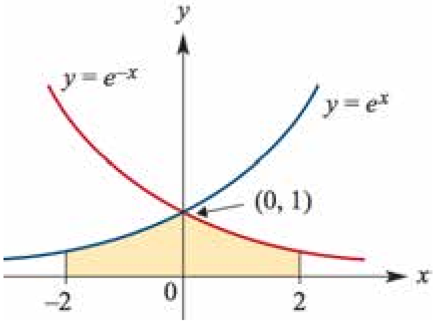


b $\int_0^2 e^x + 1\,dx = [e^x + x]_0^2$

$= (e^2 + 2) - (1 + 0)$

$= e^2 + 1$

16 **a** The graphs intersect at (0, 1) .



b $\int_0^2 e^{-x}\,dx + \int_{-2}^0 e^x\,dx = 2\int_{-2}^0 e^x\,dx$

$= 2[e^x]_{-2}^0$

$= 2 - 2e^{-2}$

17 **a** $\int_0^1 e^x\,dx = [e^x]_0^1$

$= e - 1$

b area $= 2(e - 1)$

18 $\frac{1}{3}\int_1^4 \sqrt{x}\,dx$

$= \frac{1}{3}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$

$= \frac{1}{3}\left(\frac{2}{3} \times 8 - \frac{2}{3}\right)$

$= \frac{14}{9}$

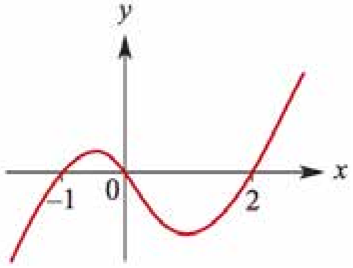
19 area $= \int_0^1 2e^{2x} + 3\,dx$

$= [e^{2x} + 3x]_0^1$

$= (e^2 + 3 + -(1 + 0)$

$= e^2 + 2$

20 The intercepts are $(-1, 0)$, $(0, 0)$, $(2, 0)$.



area $= \int_{-1}^0 x(x-2)(x+1)\,dx$

$- \int_0^2 x(x-2)(x+1)\,dx$

$= \int_{-1}^0 x^3 - x^2 - 2x\,dx$

$- \int_0^2 x^3 - x^2 - 2x\,dx$

$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_{-1}^0$

$- \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_0^2$

$= -\left(\frac{1}{4} + \frac{1}{3} - 1\right) - \left(4 - \frac{8}{3} - 4\right)$

$= \frac{37}{12} = 3\frac{1}{12}$

21 $\displaystyle\int_{\ln 2}^{\frac{5}{2}} e^x - 2 - (-e^x + 2)\ dx$

$\displaystyle = \int_{\ln 2}^{\frac{5}{2}} 2e^x - 4\ dx$

$\displaystyle = \left[2e^x - 4x\right]_{\ln 2}^{\frac{5}{2}}$

$\displaystyle = 2e^{\frac{5}{2}} - 4 \times \frac{5}{2} - 4 + 4\ln 2$

$\displaystyle = 2e^{\frac{5}{2}} + 4\ln 2 - 14$

22 **a** Equation of normal

$f(x) = 6 - e^{-2x}$

$f'(x) = 2e^{-2x}$

$f'(0) = 2 \Rightarrow$ Gradient of normal $= -\dfrac{1}{2}$

$y - 5 = -\dfrac{1}{2}x$

$y = -\dfrac{1}{2}x + 5$

Therefore D has coordinates $(10, 0)$

For C consider

$0 = 6 - e^{-2x}$

$-2x = \ln 6$

$x = -\dfrac{1}{2}\ln 6$

Therefore C has coordinates

$\left(-\dfrac{1}{2}\ln 6, 0\right)$

b Area of the triangle to the right

$= \dfrac{1}{2} \times 10 \times 5 = 25$

Area of the region to the left

$\displaystyle\int_{\frac{1}{2}\ln(\frac{1}{6})}^{0} 6 - e^{-2x}\ dx$

$\displaystyle = \left[6x + \frac{1}{2}e^{-2x}\right]_{\frac{1}{2}\ln(\frac{1}{6})}^{0}$

$\displaystyle = \frac{1}{2} - \left(3\ln\left(\frac{1}{6}\right) + \frac{1}{2} \times 6\right)$

$\displaystyle = -\frac{5}{2} - 3\ln\left(\frac{1}{6}\right)$

Therefore total area $= \dfrac{45}{2} + 3\ln 6$

23 **a** $2\sin(\pi x) + 1 = 0$

$\sin(\pi x) = -\dfrac{1}{2}$

$\pi x = \dfrac{7\pi}{6} . \dfrac{11\pi}{6}$

$x = \dfrac{7}{6}, \dfrac{11}{6}$

Coordinates $C\left(\dfrac{7}{6}, 0\right), D\left(\dfrac{11}{6}, 0\right)$

b $\displaystyle\int_{0}^{\frac{7}{6}} 2\sin(\pi x) + 1\ dx$

$\displaystyle = \left[-\frac{2}{\pi}\cos \pi x + x\right]_{0}^{\frac{7}{6}}$

$\displaystyle = -\frac{2}{\pi}\cos\left(\frac{7\pi}{6}\right) + \frac{7}{6} + \frac{2}{\pi}$

$\displaystyle = \frac{\sqrt{3} + 2}{\pi} + \frac{7}{6}$

$\displaystyle\int_{\frac{7}{6}}^{\frac{11}{6}} 2\sin(\pi x) + 1\ dx$

$\displaystyle = \left[-\frac{2}{\pi}\cos \pi x + x\right]_{\frac{7}{6}}^{\frac{11}{6}}$

$\displaystyle = -\frac{2\sqrt{3}}{\pi} + \frac{2}{3}$

Total area$= \dfrac{\sqrt{3} + 2}{\pi} + \dfrac{7}{6} + \dfrac{2\sqrt{3}}{\pi} - \dfrac{2}{3}$

$$= \frac{1}{2} + \frac{3\sqrt{3} + 2}{\pi}$$

24 a
$$\begin{aligned} 8 - x &= \frac{12}{x} \\ 8x - x^2 &= 12 \\ x^2 - 8x + 12 &= 0 \\ x &= 6 \text{ or } x = 2 \end{aligned}$$

Coordinates (6, 2), (2, 6)

b $\displaystyle\int_2^6 8 - x - \frac{12}{x}$

$$\begin{aligned} &= \left[8x - \frac{x^2}{2} - 12\ln x\right]_2^6 \\ &= 16 + 12\ln\left(\frac{1}{3}\right) \\ &= 16 - 12\ln 3 \end{aligned}$$

25 a
$$\begin{aligned} \int_0^2 e^{-x} + x\,dx &= \left[-e^{-x} + \frac{1}{2}x^2\right]_0^2 \\ &= (-e^{-2} + 2) - (-1) \\ &= 3 - e^{-2} \end{aligned}$$

b $\displaystyle\int_{-2}^{-1} x + \frac{1}{x - 1}\,dx$

$$\begin{aligned} &= \left[\frac{1}{2}x^2 + \log_e|x - 1|\right]_{-2}^{-1} \\ &= \left(\frac{1}{2} + \log_e|-2|\right) - (2 + \log_e|-3|) \\ &= \log_e\frac{2}{3} - \frac{3}{2} \end{aligned}$$

c
$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin x + x\,dx &= \left[-\cos x + \frac{1}{2}x^2\right]_0^{\frac{\pi}{2}} \\ &= \left(0 + \frac{\pi^2}{8}\right) - (-1) \\ &= \frac{\pi^2}{8} + 1 \end{aligned}$$

d $\displaystyle\int_{-4}^{-5} e^x + \frac{1}{2 - 2x}\,dx$

$$\begin{aligned} &= \int_{-4}^{-5} e^x + \frac{1}{2} \times \frac{1}{1 - x}\,dx \\ &= \left[e^x - \frac{1}{2}\log_e|1 - x|\right]_{-4}^{-5} \\ &= \left(e^{-5} - \frac{1}{2}\log_e|6|\right) - \left(e^{-4} - \frac{1}{2}\log_e|5|\right) \\ &= e^{-5} - e^{-4} + \frac{1}{2}\log_e\frac{5}{6} \end{aligned}$$

Solutions to multiple-choice questions

1 C $\int_0^2 3f(x) + 2\,dx$

$$= \int_0^2 3f(x)\,dx + \int_0^2 2\,dx$$

$$= 3\int_0^2 f(x)\,dx + [2x]_0^2$$

$$= 3\int_0^2 f(x)\,dx + 4$$

2 D $\int_3^5 f(x)\,dx = F(5) - F(3)$

$$\therefore F(5) = \int_3^5 f(x)\,dx + F(3)$$

$$= \int_3^5 f(x)\,dx + 4$$

3 C $\frac{1}{a}\int_0^a x^3 - 2x^2\,dx = \frac{9}{12}$

$$\frac{1}{a}\left(\frac{a^4}{4} - \frac{2a^3}{3}\right) = \frac{9}{12}$$

$$\frac{a^3}{4} - \frac{2a^2}{3} = \frac{9}{12}$$

$$3a^3 - 8a^2 - 9 = 0$$

$$(a-3)(3a^2 + a + 3) = 0$$

$$\therefore a = 3$$

4 B The area of the shaded region from $x = 0$ to $x = 2$ is given by $\int_0^2 f(x) - g(x)\,dx$.
The area of the shaded region from $x = 2$ to $x = 5$ is given by $\int_2^5 g(x) - f(x)\,dx$.
The area of the shaded region is $\int_0^2 f(x) - g(x)\,dx + \int_2^5 g(x) - f(x)\,dx$.

5 A $\frac{dy}{dx} = \frac{ax}{2} + 1$

$$y = \frac{ax^2}{4} + x + c$$

$$x = 0, y = 1 \text{ so } c = 1$$

$$y = \frac{ax^2}{4} + x + 1$$

6 D $f'(x) = -6\sin 3x$

$$f(x) = 2\cos 3x + c$$

$$f\left(\frac{2\pi}{3}\right) = 3$$

$$c = 3 - 2\cos 2\pi = 1$$

$$f(x) = 2\cos 3x + 1$$

7 C $\int_{-5}^4 f(x)\,dx = 2$ and $\int_{11}^4 f(x)\,dx = 6$

$\therefore \int_4^{11} f(x)\,dx = -6$ and

$\int_{-5}^{11} f(x)\,dx = -4$

8 C

$$\frac{dy}{dx} = ae^{-x} + 2$$

$$x = 0, \frac{dy}{dx} = 5, \text{ so } a + 2 = 5, \text{ i.e. } a = 3.$$

$$y = -3e^{-x} + 2x + c$$

$$x = 0, y = 1, \text{ so } -3 + c = 1, \text{ i.e. } c = 4.$$

$$y = -3e^{-x} + 2x + 4$$

If $x = 2,\ y = -3e^{-2} + 4 + 4 = -\dfrac{3}{e^2} + 8$.

9 C $R(t) = 5e^{-0.1t}$ litres/minute.
Since $R(t)$ is the **rate** of flow, it is equal to $\dfrac{dV}{dt}$ where V L is the volume of water at time t. Thus the outflow in the first 3 minutes is given by

$$\int_0^3 5e^{-0.1t}\,dt = -\frac{5}{0.1}\left[e^{-0.1t}\right]_0^3$$
$$= -50(e^{-0.3} - 1)$$
$$= 12.959\ldots$$

To the nearest litre, this is 13 1itres.

10 C $Area(A) = xy$

$2x + 2y = 200$

$x + y = 100$

$A = x(100 - x)$

$$Average = \frac{1}{100}\int_0^{100}(100 - x)x\,dx$$
$$= \frac{1}{100}\int_0^{100} 100x - x^2\,dx$$
$$= \frac{1}{100}\left[50x^2 - \frac{x^3}{3}\right]_0^{100}$$
$$= \frac{1}{100}(50 \times 100^2 - \frac{100^3}{3})$$
$$= 50 \times 100 - \frac{100^2}{3}$$
$$= 100(50 - \frac{100}{3})$$
$$= 100 \times \frac{50}{3}$$
$$= \frac{5000}{3} \approx 1667$$

11 D $A \approx \dfrac{1}{2}(f(2) + 2f(3) + 2f(4) + f(5))$

$= \dfrac{1213}{32}$

$\dfrac{3147}{80} - A = \dfrac{229}{160}$

12 D By symmetry, the shaded regions have equal area, so the total area is given by

$$2\int_{\pi-a}^{\pi} \sin x\,dx = 2[-\cos x]_{\pi-a}^{\pi}$$
$$= 2\,(-\cos\pi + \cos(\pi - a))$$
$$= 2(1 - \cos a)$$

Solutions to extended-response questions

1 **a** For $y = x - \frac{1}{x^2} = x - x^{-2}$

$\frac{dy}{dx} = 1 + 2x^{-3}$

When $x = 2$, $\frac{dy}{dx} = 1 + \frac{2}{23} = 1\frac{1}{4}$

When $x = 2, y = 2 - \frac{1}{4} = \frac{7}{4}$

The equation of the tangent is

$$y - \frac{7}{4} = \frac{5}{4}(x - 2)$$

$\therefore 4y - 7 = 5x - 10$

and $4y - 5x = -3$

b When $y = 0$, $x = \frac{3}{5}$ The coordinates are $\left(\frac{3}{5}, 0\right)$

c When $y = 0$, $x - \frac{1}{x^2} = 0$

implies $x^3 - 1 = 0$

i.e. $x = 1$ The coordinates are $(1, 0)$

d Required area = Area of triangle ATC – Area of should region

$$= \frac{1}{2}\left(2 - \frac{3}{5}\right) \times \frac{7}{4} - \int_1^2 x - x^{-2}\, dx$$

$$= \frac{1}{2} \times \frac{7}{5} \times \frac{7}{4} - \left[\frac{x^2}{2} + x^{-1}\right]_1^2$$

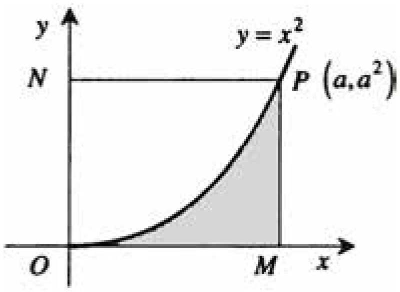
$$= \frac{49}{40} - \left(\left(2 + \frac{1}{2}\right) - \left(\frac{1}{2} + 1\right)\right)$$

$$= \frac{49}{40} - 1$$

$$= \frac{9}{40}$$

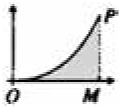
e The required ratio $= \frac{9}{40} : \frac{49}{40} = 9 : 94$

2 a



Let M, have coordinates $(a,\ 0)$

Area of OPM



$$= \int_0^a x^2\,dx$$

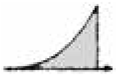
$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

The coordinates of P are $(a,\ a^2)$

Area of OPN

= area of rectangle $OMPN$ – area of OPM

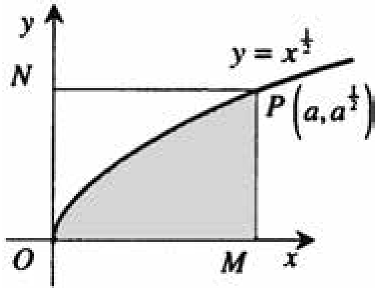


$$= a \times a^2 - \frac{a^3}{3}$$

$$= \frac{2a^3}{3}$$

$\therefore$ The ratio of the areas $= \dfrac{2a^3}{3} : \dfrac{a^3}{3} = 2 : 1$

b



Let M have coordinates $(a,\ 0)$.

$$\text{Area shaded} = \int_0^a x^{\frac{1}{2}}\,dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{2a^{\frac{3}{2}}}{3}$$

Area of rectangle $OMPN = a \times a^{\frac{1}{2}} = a^{\frac{3}{2}}$

$\therefore$ shaded area $= \dfrac{2}{3}$ of the area of rectangle $OMPN$.

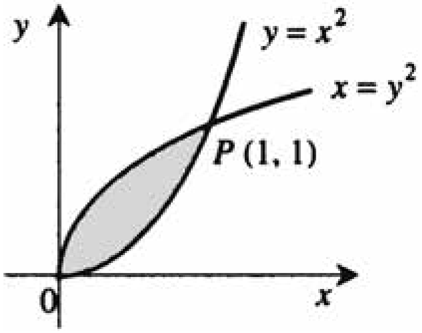
c Let M have coordinates $(a,\ 0)$. Then the coordinates of P are $(a,\ a^n)$ Area of region enclosed by PM, the x-axis and the curve

$$= \int_0^a x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^a = \frac{a^{n+1}}{n+1}$$

Area of rectangle $OMPN = a \times a^n = a^{n+1}$

$\therefore$ Area of described region $= \dfrac{1}{n+1}$ (area of rectangle)

3 a



The parabolas intersect at (1, 1)

$$\therefore \text{ the area } = \int_0^1 x^{\frac{1}{2}} - x^2\,dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \tfrac{2}{3} - \tfrac{1}{3}$$

$$= \tfrac{1}{3}$$

The area is $\frac{1}{3}$ square units.

b For $y = x^n$ and $y^n = x$

$x^{\frac{1}{n}} = x^n$

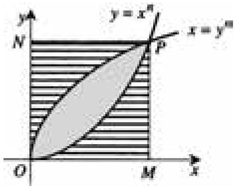
which implies $1 = x^{n-\frac{1}{n}}$

i.e. $1 = x^{\frac{n^2-1}{n}}$

$\therefore x = (1)^{\frac{n}{n^2-1}} = \pm 1$

If n is even, $x = 1$ is the only solution. All such pairs of curves intersect at (1, 1).

c The coordinates of P are (1, 1)



$$\text{Area } = \int_0^1 x^{\frac{1}{n}} - x^n\,dx$$

$$= \left[\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} - \frac{x^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{\frac{1}{n}+1} - \frac{1}{n+1}$$

$$= \frac{n}{n+1} - \frac{1}{n+1}$$

$$= \frac{n-1}{n+1} \text{ square units}$$

d Area with shading $= 1 - \left(\frac{n-1}{n+1}\right)$

$$= \frac{n+1-n+1}{n+1}$$

$$= \frac{2}{n+1} \text{ square units}$$

e For $n = 10$, Area $= \dfrac{10-1}{10+1} = \dfrac{9}{11}$ square units

For $n = 100$, Area $= \dfrac{100-1}{100+1} = \dfrac{99}{101}$ square units

For $n = 1000$, Area $= \dfrac{1000-1}{1000+1} = \dfrac{999}{1001}$ square units

f For $\dfrac{n-1}{n+1} = 1 + \dfrac{2}{n-1}$, as $n \to \infty, \dfrac{n-1}{n+1} \to 1$

4 a

$$\frac{d\theta}{dt} = e^{2.6t}$$

$$\therefore \theta = \frac{1}{2.6}e^{2.6t} + c$$

$$= \frac{5}{13}e^{2.6t} + c$$

when $t = 0,\ \theta = 30$

$$\therefore c = 30 - \frac{5}{13} = \frac{385}{13}$$

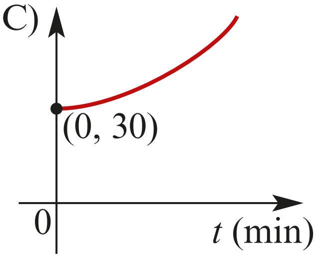
$$\text{and } \theta = \frac{5}{13}e^{2.6t} + \frac{385}{13}$$

when $t = 3$

$$\theta = \frac{5}{13}e^{2.6\times 3} + \frac{385}{13} = 968.3$$

The temperature is 968.3°C after 3 minutes.

b



c

When $\theta = 500$

$$500 = \frac{5}{13}e^{2.6t} + \frac{385}{13}$$

$$\therefore \quad \frac{6115}{13} \times \frac{13}{5} = e^{2.6t}$$

$$1223 = e^{2.6t}$$

$$\therefore \quad t = \frac{5}{13}\log_e(1223)$$

$$\approx 2.734$$

The temperature is 500° after 2.734 minutes.

d The average rate of change for interval [1, 2]

$$= \frac{\theta(2) - \theta(1)}{2 - 1}$$

$$= \frac{5}{13}e^{5.2} + \frac{385}{13} - \left(\frac{5}{13}e^{2.6} + \frac{385}{13}\right)$$

$$= \frac{5}{13}\left(e^{5.2} - e^{2.6}\right)$$

$$\approx 64.5$$

The average rate of change for the interval [1, 2] is 64.5° per minute.

5 $\frac{dx}{dt} = ve^{-t}$, where $v = 5 \times 10^4$ m/s

a When $t = 0$, $\frac{dx}{dt} = 5 \times 10^4$ m/s

b $\frac{dx}{dt} = \frac{v}{e^t} = \frac{5 \times 10^4}{e^t}$

as $t \to \infty$, $\frac{dx}{dt} \to 0$

c The distance travelled between $t = 0$ and $t = 20$

$$= \int_0^{20} 5 \times 10^4 e^{-t}\, dt$$

$$= \left[-5 \times 10^4 e^{-t}\right]_0^{20}$$

$$= -5 \times 10^4 \times e^{-20} + 5 \times 10^4$$

$$= 5 \times 10^4(1 - e^{-20})\text{metres}$$

d $\frac{dx}{dt} = ve^{-t}$

$\therefore x = -ve^{-t} + c$

When $t = 0$, $x = 0$

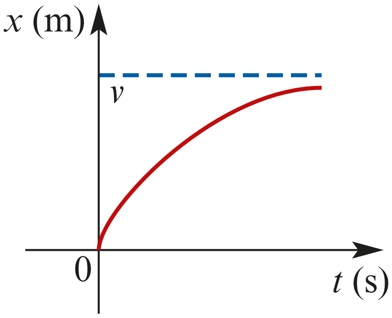
$\therefore = -v + c$

i.e. $c = v$

$\therefore x = v - ve^{-t}$

$= v(1 - e^{-t})$

e



6 **a** Let $y = e^{-3x}\sin 2x$

then, using the produce rule,

$$\frac{dy}{dx} = -3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x \text{ ①}$$

For $y = e^{-3x}\cos 2x$

$$\frac{dy}{dx} = 3e^{-3x}\cos 2x - 2e^{-3x}\sin 2x \text{ ②}$$

b From ①

$$\int\left(-3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x\right)dx = e^{-3x}\sin 2x + c_1$$

i.e. $-3\int e^{-3x}\sin 2x dx + 2\int e^{-3x}\cos 2x dx = e^{-3x}\sin 2x + c_1$③

From ②

$$\int(-3e^{-3x}\cos 2x - 2e^{-3x}\sin 2x)\,dx = e^{-3x}\cos 2x + c_2$$

i.e. $-3\int e^{-3x}\cos 2x - 2\int e^{-3x}\sin 2x dx = e^{-3x}\cos 2x + c_2$④

c Let $a = \int e^{-3x}\sin 2x dx$ and $b = \int e^{-3x}\cos 2x dx$

Then the equations can be rewritten as

$-3a + 2b = e^{-3x}\sin 2x + c_1$

$-3b - 2a = e^{-3x}\cos 2x + c_2$

Multiply ⑤ by 3 and ⑥ by 2 and add:

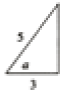
$-9a - 4a = 3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x + (3c_1 + 2c_2)$

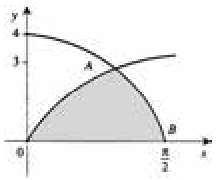
$\therefore -13\int e^{-3x}\sin 2x dx = 3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x + (3c_1 + 2c_2)$

i.e. $\int(e^{-3x}\sin 2x)\,dx = -\frac{1}{13}(3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x) + C$

where $C = \frac{3c_1 + 3c_2}{-13}$

7 a





i $3\sin a = 4\cos a$

$\therefore \frac{3\sin a}{\cos a} = 4$

$\tan a = \frac{4}{3}$

ii Consider the triangle

Then $\sin(a) = \frac{4}{5}$ and $\cos(a) = \frac{3}{5}$

b Area of the shaded region $= \int_0^a 3\sin\ dx + \int_a^{\frac{\pi}{4}} 4\cos x dx$

$= [-3\cos x]_0^a + [4\sin x]_a^{\frac{\pi}{4}}$

$= -3\cos a - (-3) + 4\sin\frac{\pi}{2} - 4\sin a$

$= -3 \times \frac{3}{5} + 3 + 4 - 4 \times \frac{4}{5}$

$= -\frac{9}{5} + 7 - \frac{16}{5}$

$= -5 + 7$

$= 2$

Area of the shaded region = 2 square units.

8 a $y = x\log_e x$

Using the product rule gives

$\frac{dy}{dx} = \log_e x + x \times \frac{1}{x}$

$= \log_e x + 1$

Also $\int(\log_e x + 1)\,dx = x\log_e x + c$

$\therefore \int \log_e x dx + x = x\log_e x + c$

and $\int_1^e \log e x dx = [x\log\ ex - x]_1^e$

$= e\log_e e - e - (e\log_e 1 - 1)$

$= e - e - 0 + 1$

$\therefore \int_1^e \log_e x dx = 1$

b $y = x(\log_e x)^n$

Using the product rule:

$\frac{dy}{dx} = (\log_e x)^n + x \times \frac{1}{x} \times n(\log_e x)^{n-1}$

$= (\log_e x)^n + n(\log_e x)^{n-1}$

c $I_n = \int_1^e (\log_e x)^n \, dx, \textit{ and } I_{n-1} = \int_1^e (\log_e x)^{n-1} \, dx$

From $\textcircled{\dagger}$

$\int (\log_e x)^n + n(\log_e x)^{n-1} \, dx = x(\log_e x)^n + c$

$\therefore I_n + nI_{n-1} = \Big[x(\log_e x)^n\Big]_1^e$

$= e$

d $I_3 = \int_1^3 (\log_e x)^3 \, dx$

From (c) $\quad I_3 = e - 3I_2$

$= e - 3[e - 2I_1]$

$= e - 3e + 6I_1$

$= -2e + 6 \quad$ by (a)

9 To find the point of intersection, consider $x^2 = by$ and $y^2 = ax$

as a simultaneous pair.

$\therefore y = \dfrac{x^2}{b}$ and $\left(\dfrac{x^2}{b}\right)^2 = ax$

which implies $x^3 = ab^2$

and $x = a^{\frac{1}{3}} b^{\frac{2}{3}}$ Substitute for x in

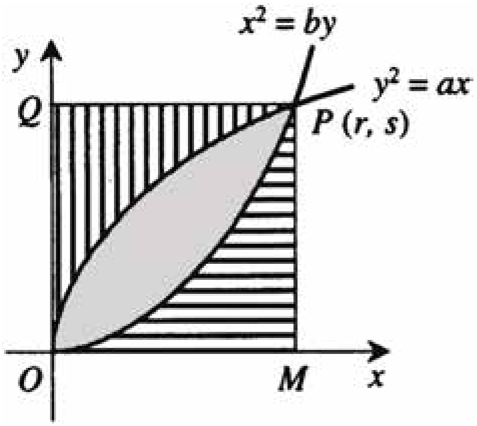
$y^2 = ax$

$\therefore y^2 = a\left(a^{\frac{1}{3}} b^{\frac{2}{3}}\right)$

$= a^{\frac{1}{3}} b^{\frac{2}{3}}$

$\therefore y = a^{\frac{2}{3}} b^{\frac{1}{3}}$

$\therefore r = a^{\frac{1}{3}} b^{\frac{2}{3}}$ and $s = a^{\frac{2}{3}} b^{\frac{1}{3}}$



The area of the region shaded horizontally ▤

$$= \int_0^{a^{\frac{1}{3}} b^{\frac{2}{3}}} \frac{x^2}{b} \, dx$$

$$= \frac{1}{b}\left[\frac{x^3}{3}\right]_0^{a^{\frac{1}{3}} b^{\frac{1}{3}}}$$

$$= \frac{1}{3b} \times a \times b^2$$

$$= \frac{ab}{3}$$

The area of the region shaded vertically ▥

$$= \int_0^{a^{\frac{2}{3}}b^{\frac{1}{3}}} \frac{y^2}{a}\,dy$$

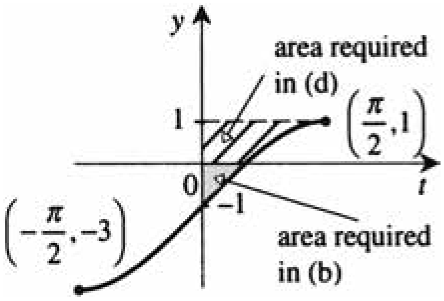
$$= \left[\frac{y^3}{3a}\right]_0^{a^{\frac{2}{3}}b^{\frac{1}{3}}}$$

$$= \frac{ab}{3}$$

The area of rectangle $OMPQ = a^{\frac{2}{3}}b^{\frac{1}{3}} \times a^{\frac{1}{3}}b^{\frac{2}{3}} = ab$

$\therefore$ All three regions have area $\frac{ab}{3}$

10 a



b $\int_0^{\frac{\pi}{6}} 2\sin x - 1\,dx = [-2\cos x - x]_0^{\frac{\pi}{6}}$

$= -2\cos\frac{\pi}{6} - \frac{\pi}{6} - (-2\cos 0 - 0)$

$= -2 \times \frac{\sqrt{3}}{2} - \frac{\pi}{6} + 2$

$= 2 - \sqrt{3} - \frac{\pi}{6}$

c For the inverse of $f(x) = 2\sin x - 1$

consider $x = 2\sin y - 1$

$\frac{(x+1)}{2} = \sin y$

and $y = \sin^{-1}\left(\frac{x+1}{2}\right)$

i.e. $f^{-1}(x) = \sin^{-1}\left(\frac{x+1}{2}\right)$

The domain of f^{-1} = range of $f = [-3, 1]$

d $\displaystyle\int_0^1 f^{-1}(x)\,dx = \text{ area of rectangle } - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x)\,dx$

$$\begin{aligned}
&= \frac{\pi}{2} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin x - 1)\,dx \\
&= \frac{\pi}{2} - [-2\cos x - x]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} + [2\cos x + x]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} + \left[0 + \frac{\pi}{2} - \left(2\cos\frac{\pi}{6} + \frac{\pi}{6}\right)\right] \\
&= \frac{\pi}{2} + \frac{\pi}{2} - \sqrt{3} - \frac{\pi}{6} \\
&= \frac{5\pi}{6} - \sqrt{3}
\end{aligned}$$

The ‘integral’ command of a CAS could be used in this question.

11 **a** For $y = e^{\frac{x}{10}}(10 - x)$

$$\begin{aligned}
\frac{dy}{dx} &= \tfrac{1}{10}e^{\frac{x}{10}}(10 - x) - e^{\frac{x}{10}} \\
&= e^{\frac{x}{10}}\left(1 - \frac{x}{10} - 1\right) \\
&= -\frac{x}{10}e^{\frac{x}{10}}
\end{aligned}$$

For $y = \sqrt{100 - x^2} = \left(100 - x^2\right)^{\frac{1}{2}}$

$$\begin{aligned}
\frac{dy}{dx} &= -x(100 - x^2)^{-\frac{1}{2}} \\
&= \frac{-x}{(100 - x^2)^{\frac{1}{2}}}
\end{aligned}$$

b When $x = 0$, $\dfrac{dy}{dx} = 0$ for both functions

c When $x = 10$, $\dfrac{dy}{dx} = -e$

d $\int_0^{10} e^{\frac{x}{10}}(10 - x)\,dx = 71.828\ldots$

$\int_0^{10} \sqrt{10^2 - x^2}\ dx = 78.5398\ldots$

$\therefore$ area between the curves = 6.7118 square units

e percentage error = $\dfrac{6.7118}{25\pi} \times 100 = 8.55\%$

f Equation of the chord is $y = 10 - x$

Area of the shaded region = $25\pi - 50 = 28.54$ square units

g **i**

$$\frac{d}{dx}\left(e^{\frac{x}{10}}(10-x)\right) = \tfrac{1}{10}e^{\frac{x}{10}}(10-x) - e^{\frac{x}{10}}$$

$$\therefore \tfrac{1}{10}\int_0^{10} e^{\frac{x}{10}}(10-x)\,dx = \left[e^{\frac{x}{10}}(10-x)\right]_0^{10} + \int_0^{10} e^{\frac{x}{10}}\,dx$$

$$= -(10) + \left[10e^{\frac{x}{10}}\right]_0^{10}$$

$$= -10 + 10e - 10$$

$$= 10e - 20$$

$$\therefore \int_0^{10} e^{\frac{x}{10}}(10-x)\,dx = 10(10e - 20)$$

ii $\therefore$ exact area of shaded region = $25\pi - 100e + 200$ square units

12 $R(t) = 10e^{-(\frac{t}{10})}\sin\left(\frac{\pi t}{3}\right)$

a **i** $R(0) = 0$

ii $R(3) = 10e^{-\frac{3}{10}}\sin\pi = 0$

b $R'(\text{t}) = -e^{-\frac{t}{10}}\sin\left(\frac{\pi t}{3}\right) + \frac{10\pi}{3}e^{-\frac{t}{10}}\cos\left(\frac{\pi t}{3}\right)$

$$= e^{-\frac{t}{10}}\left[\frac{10\pi}{3}\cos\left(\frac{\pi t}{3}\right) - \sin\left(\frac{\pi t}{3}\right)\right]$$

c **i** $R'(t) = 0$ implies

$\frac{10\pi}{3}\cos\left(\frac{\pi t}{3}\right) = \sin\left(\frac{\pi t}{3}\right)$ as $e^{-\frac{t}{10}} \neq 0$

$$\therefore \tan\left(\frac{\pi t}{3}\right) = \frac{10\pi}{3}$$

$$\therefore \frac{\pi t}{3} = \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } \pi + \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 2\pi + \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 3\pi + \tan^{-1}\left(\frac{10\pi}{3}\right)$$

$$\therefore t = \frac{3}{\pi}\tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 3 + \frac{3}{\pi}\tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 6 + \frac{3}{\pi}\tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 9 + \tan^{-1}\left(\frac{10\pi}{3}\right)$$

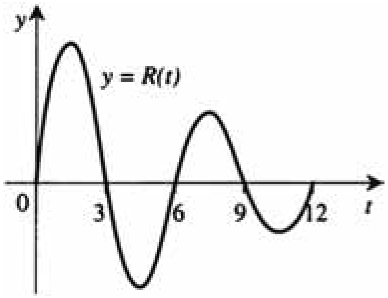
ii stationary points (1.409, 8.646) and (7.409, 4.745) loc max (4.409, −6.405) and (10.409, −3.515) loc min

d $R(t) = 0$ implies $\sin\left(\frac{\pi t}{3}\right) = 0$ as $10 \therefore e^{-\frac{t}{10}} \neq 0$

$\therefore \frac{\pi t}{3} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\,\pi \text{ or } 4\pi$

$t = 0$ or 3 or 6 or 9 or 12

e



f **i** Use a CAS calculator to find areas

$\int_0^3 R(t)\,dt \approx 16.47337 \quad \therefore \quad$ 16.47 litres flowed in

ii $\int_3^6 R(t)\,dt \approx -12.20377 \quad \therefore \quad$ 12.20 litres flowed out

iii Total amount of water in the device

$= 16.47337\ldots - 12.20377\ldots + 4$

$= 4.2695 + 4$

$= 8.2695$

There are approximately 8.27 litres in the device

g $\int_0^{30} R(t)\,dt \approx 8.9918\ldots$

$\therefore$ There are $4 + 8.9918 = 12.992$ litres in the device after 30 minutes. (Use a CAS calculator with this problem.)

13 **a** If $\cos 2x = 2\cos^2 x - 1$ and $\cos 2x = 1 - 2\sin^2 x$

$$\begin{aligned}\frac{1-\cos 2x}{1+\cos 2x} &= \frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}\\ &= \frac{2\sin^2 x}{2\cos^2 x}\\ &= \tan^2 x\\ &= \sec^2 x - 1\end{aligned}$$

b $$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{1+\cos 2x}\,dx &= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx\\ &= \Big[\tan x - x\Big]_0^{\frac{\pi}{4}}\\ &= \tan\frac{\pi}{4} - \frac{\pi}{4} = 1 - \frac{\pi}{4}\end{aligned}$$

Chapter 12 – Revision of Chapters 9–11

Solutions to technology-free questions

1 $y = \dfrac{x^2 - 1}{x^4 - 1}$

a $$\frac{dy}{dx} = \frac{(x^4 - 1)\dfrac{d(x^2 - 1)}{dx} - (x^2 - 1)\dfrac{d(x^4 - 1)}{dx}}{(x^4 - 1)^2}$$
$$= \frac{2x(x^4 - 1) - 4x^3(x^2 - 1)}{(x^4 - 1)^2}$$
$$= \frac{2x^5 - 2x - 4x^5 + 4x^3}{(x^4 - 1)^2}$$
$$= \frac{-2(x^5 - 2x^3 + x)}{(x^4 - 1)^2}$$
$$= \frac{-2x(x^2 - 1)^2}{(x^4 - 1)^2}$$

b $\dfrac{dy}{dx} = 0,$

$0 = \dfrac{-2x(x^2 - 1)^2}{(x^4 - 1)^2}$

looking at the numerator

$x = 0,\ x^2 = 1$

$x = -1, 0, 1$

looking at the denominator

$x^4 \neq 1$

$x \neq \pm 1$

$\therefore\ x = 0$

in set notation, $\{0\}$

2 $y = (3x^2 - 4x)^4$

Let $u = 3x^2 - 4x$.

Then $y = u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = 4u^3 \times (6x - 4)$$
$$= 8(3x - 2)(3x^2 - 4x)^3$$

3 $f(x) = x^2 \log_e(2x)$

$$f'(x) = 2x\log_e(2x) + x^2 \times \frac{1}{x}$$
$$= 2x\log_e(2x) + x$$

4 **a** $f(x) = e^{2x+1}$

$f'(x) = 2e^{2x+1}$

$f'(b) = 2e^{2b+1}$

The tangent is at the point (b, e^{2b+1})

Gradient of line from the point to the origin is $\dfrac{e^{2b+1}}{b}$

$\therefore \dfrac{e^{2b+1}}{b} = 2e^{2b+1}$

$\therefore b = \dfrac{1}{2}$

b $f(b) = e^{2b+1} + k$

$f'(b) = 2e^{2b+1}$

$\therefore \dfrac{e^{2b+1} + k}{b} = 2e^{2b+1}$

$\therefore e^{2b+1} + k = 2be^{2b+1}$

$\therefore k = (2b - 1)e^{2b+1}$

5 $$y = x^{\frac{1}{3}} + c$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

When $x = 8, y = a$

$$\therefore a = 2 + c \ldots (1)$$

When $x = 8, \dfrac{dy}{dx} = \dfrac{1}{12}$

$$\therefore m = \frac{1}{12}$$

When $x = 8, a = 8m - 8$

$$\therefore a = \frac{2}{3} - 8 = -\frac{22}{3}$$

Substitute in (1)

$$-\frac{22}{3} = 2 + c$$

$$c = -\frac{28}{3}$$

6 Average value$= \dfrac{1}{2}\displaystyle\int_0^2 \frac{1}{3x+1}, dx$

$$= \frac{1}{6}\Big[\log_e(3x+1)\Big]_0^2$$

$$= \frac{1}{6}\log_e 7$$

7 **a** $\displaystyle\int \frac{3}{5x-2}\,dx = \frac{3}{5}\log_e(5x-2) + c$

b $\displaystyle\int \frac{3}{(5x-2)^2}\,dx = \frac{3}{10-25x}$

8 **a** $g(x) = 3x^2 - 5f(x)$

$$g'(x) = 6x - 5f'(x)$$

$$g'(3) = 6 \times 3 - 5f'(3x)$$

$$= 18 - 5 \times 5$$

$$g'(3) = -7$$

b $g(x) = \dfrac{3x+1}{f(x)}$

$$g'(x) = \frac{f(x)\dfrac{d(3x+1)}{dx} - f(x)(3x+1)}{(f(x))^2}$$

$$= \frac{3f(x) - (3x+1)f'(x)}{(f(x))^2}$$

$$g'(3) = \frac{3f(3) - (9+1)f'(3)}{(f(3))^2}$$

$$= \frac{3 \times -2 - 10 \times 5}{(-2)^2}$$

$$= \frac{-6-50}{4}$$

$$g'(3) = -14$$

c $$g(x) = [f(x)]^2$$

$$g'(x) = 2f(x)f'(x)$$

$$\therefore g'(3) = 2 \times (-2) \times 5$$

$$= -20$$

9 **a** $g(x) = \sqrt{x}f(x)$

$$g'(x) = \sqrt{x}f'(x) + \frac{f(x)}{2f\sqrt{x}}$$

$$g'(4) = \sqrt{4}f'(4) + \frac{f(4)}{2\sqrt{4}}$$

$$= 2 \times 2 + \frac{6}{4}$$

$$g'(4) = \frac{11}{2} = 5\frac{1}{2}$$

b $g(x) = \frac{f(x)}{x}$

$$g'(x) = \frac{xf(x) - f(x)\frac{dx}{dx}}{x^2}$$

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

$$g'(4) = \frac{4f'(4) - f(4)}{4^2}$$

$$= \frac{4 \times 2 - 6}{16}$$

$$g'(4) = \frac{1}{8}$$

10 $f(x) = f(g(x))$

$f'(x) = \sqrt{3x + 4}$, $g(x) = x^2 - 1$

$\therefore$ $g'(x) = 2x$

$$f'(x) = f'(g(x)) \times g'(x)$$

$$= \sqrt{3(x^2 - 1) + 4} \times 2x$$

$$= \sqrt{3(x^2 - 3) + 4} \times 2x$$

$$f'(x) = 2x\sqrt{3x^2 + 1}$$

11 $f(x) = 2x^2 - 3x + 5$

a $f'(x) = 4x - 3$

b $f'(0) = -3$

c $f'(x) = 1$
$4x - 3 = 1$
$4x = 4$
$x = 1$
in set notation$\{1\}$

12 $\frac{d}{dx}\left(\log_e 3f(x)\right) = \frac{f'(x)}{f(x)}$

$$y = \sqrt{a - x} = (a - x)^{\frac{1}{2}}$$

Let $u = a - x$.

Then$y = u^{\frac{1}{2}}$

13 $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(-1)$$

$$= -\frac{1}{2\sqrt{a - x}}$$

When $x = 1$, $\frac{dy}{dx} = -6$

$$\therefore, -\frac{1}{2\sqrt{a - 1}} = -6$$

$$1 = 12\sqrt{a - 1}$$

$$\frac{1}{144} = a - 1$$

$$a = \frac{145}{144}$$

14 Area of region A $= \int_0^1 -x^2 - x + 2\,dx$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_0^1$$

$$= \frac{7}{6}$$

Area of region B $=-\int_1^m -x^2 - x + 2\,dx$

$$= -\left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_1^m$$

$$= \frac{m^3}{3} + \frac{m^2}{2} - 2m + \frac{7}{6}$$

$$\text{Area A} = \text{Area B}$$

$$\frac{m^3}{3} + \frac{m^2}{2} - 2m = 0$$

$$2m^3 + 3m^2 - 12m = 0$$

$$m(2m^2 + 3m - 12) = 0$$

$$m = 0 \text{ or } m = \frac{-3 \pm \sqrt{9 + 96}}{4}$$

$$m = 0 \text{ or } m = \frac{-3 \pm \sqrt{105}}{4}$$

$$\text{But } m > 1, \therefore m = \frac{-3 + \sqrt{105}}{4}$$

$$f(x) = x^3 + 3x^2 - 4$$

$$f'(x) = 3x^2 + 6x$$

15 a $f'(x) = 0 \Rightarrow 3x(x + 2) = 0$

$$\therefore x = 0 \text{ or } x = -2$$

$$f(0) = -4, f(-2) = 0$$

b $\int_{-2}^{2} f(x)\,dx = \left[\frac{x^4}{4} + x^3 - 4x\right]_{-2}^{2} = 0$

c $\int_{0}^{2} f(x)\,dx = \left[\frac{x^4}{4} + x^3 - 4x\right]_{0}^{2} = 4$

d Area $= -\int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$

$$= \frac{19}{2}$$

16 $f(x) = \frac{1}{3x - 1} = (3x - 1)^{-1}$

$$f'(x) = \frac{d(3x - 1)}{dx} \times -(3x - 1)^{-2}$$

$$= \frac{-3}{(3x - 1)^2}$$

$$f'(2) = \frac{-3}{(6 - 1)^2}$$

$$f'(2) = \frac{-3}{25}$$

17 $y = 1 - x^2$

$$\frac{dy}{dx} = -2x$$

$$LHS = x\frac{dy}{dx} + 2$$

$$= x \times -2x + 2$$

$$= 2 - 2x^2$$

$$= 2(1 - x^2)$$

$$= 2y = RHS \;\; QED$$

18 $A = 4\pi r^2$

$$\frac{dA}{dr} = 8\pi r$$

$$r = 3, \frac{dA}{dr} = 8\pi \times 3$$

$$= 24\pi$$

19 $y = 1.8x^2$

$$\frac{dy}{dx} = 3.6x$$

$$\frac{dy}{dx} = 1, \; x = \frac{1}{3.6}$$

$$x = \frac{10}{36}$$

$$x = \frac{10}{36}$$

$$y = \frac{18}{10} \times \left(\frac{10}{36}\right)^2$$

$$= \frac{18 \times 10}{(36)^2}$$

$$= \frac{10}{2 \times 36}$$

$$= \frac{10}{72}$$

$$\text{co-ords} = \left(\frac{10}{36}, \frac{10}{72}\right)$$

$$\approx (0.28, 0.14)$$

20 $y = 3x^2 - 4x + 7$

$$\frac{dy}{dx} = 6x - 4$$

$$\frac{dy}{dx} = 0,$$

$$6x - 4 = 0$$

$$x = \frac{2}{3}$$

21 $y = \dfrac{x^2 + 2}{x^2 - 2}$

$$y = \frac{x^2 - 2 + 4}{x^2 - 2}$$

$$y = 1 + \frac{4}{x^2 - 2}$$

$$y = 1 + 4(x^2 - 2)^{-1}$$

$$\frac{dy}{dx} = 2x \times -4(x^2 - 2)^{-2}$$

$$= \frac{-8x}{(x^2 - 2)^2}$$

22 $z = 3y + 4,$

$$y = 2x - 1$$

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$= 3 \times 2$$

$$= 6$$

23 $y = (5 - 7x)^9$

$$\frac{dy}{dx} = -7 \times 9(5 - 7x)^8$$

$$= -63(5 - 7x)^8$$

24 $y = 3x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{3}{3} = x^{-\frac{2}{3}}$$

$$= x^{-\frac{2}{3}}$$

$$x = 27,$$

$$\frac{dy}{dx} = (27)^{-\frac{2}{3}}$$

$$= (3)^{-2}$$

$$= \frac{1}{9}$$

25 $y = \sqrt{5 + x^2}$

$$\frac{dy}{dx} = 2x \times \frac{1}{2\sqrt{5 + x^2}}$$

$$= \frac{x}{\sqrt{5 + x^2}}$$

$$x = 2,$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{5 + 4}}$$

$$= \frac{2}{3}$$

26 $y = (x^2 + 3)(2 - 4x - 5x^2)$

$$\frac{dy}{dx} = (x^2 + 3)(-4 - 10x)$$

$$+ (2 - 4x - 5x^2)(2x)$$

$$x = 1,$$

$$\frac{dy}{dx} = (1 + 3)(-4 - 10) + (2 - 4 - 5)(2)$$

$$= 4 \times -14 + (-7) \times 2$$

$$= 5 \times -14$$

$$= -70$$

27 $y = \frac{x}{1+x^2}$

$$\frac{dy}{dx} = \frac{(1+x^2)\frac{dx}{dx} - x\frac{d(1+x^2)}{2dx}}{(1+x^2)^2}$$

$$= \frac{(1+x^2) - x \times 2x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

$x = 1,$

$$\frac{dy}{dx} = \frac{1-1}{(1+1)^2} = 0$$

28

$$y = \frac{2+x}{x^2+x+1}$$

$$\frac{dy}{dx} = \frac{x^2+x+1\frac{d(2+x)}{dx} - (2+x)\frac{d(x^2+x+1)}{dx}}{(x^2+x+1)^2}$$

$$= \frac{(x^2+x+1) - (2+x)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{x^2+x+1-2x^2-5x-2}{(x^2+x+1)^2}$$

$$= \frac{-x^2-4x-1}{(x^2+x+1)^2}$$

$x = 0,$

$$\frac{dy}{dx} = \frac{-1}{1}$$

$$= -1$$

29 $f(x) = \frac{1}{2x+1}$

a

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{1}{2x+2h+1} - \frac{1}{2x+1}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h} \times \left(\frac{2x+1-(2x+2h+1)}{(2x+1)(2x+2h+1)}\right)\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h} \times \left(\frac{-2h}{4x^2+4xh+2x+2x+2h+1}\right)\right)$$

$$= \lim_{h\to 0}\left(\frac{-2}{4x^2+4xh+4x+2h+1}\right)$$

$$= \frac{-2}{4x^2+4x+1}$$

$$= \frac{-2}{(2x+1)^2}$$

b $f'(0) = \frac{-2}{(1)^2}$

$= -2$

30 $f(x) = x^3 + 3x^2 - 1$

a $f'(x) = 3x^2 + 6x$

$f'(x) = 0,$

$3x^2 + 6x = 0$

$x(x+2) = 0$

$x = 0, -2$

in set notation, $\{-2, 0\}$

b $f'(x) > 0$

$3x^2 + 6x > 0$

$x(x+2) > 0$

$x < -2,\ x > 0$

$\textit{in set notation}, R\backslash[-2, 0]$

c $f'(x) < 0$

$3x^2 + 6x < 0$

$x(x+2) < 0$

$-2 < x < 0$

$\textit{in set notation}, (-2, 0)$

31 $y = \dfrac{x}{1-x}$

a

$$\begin{aligned} y &= \frac{x-1+1}{1-x} \\ y &= \frac{-(1-x)}{1-x} + \frac{1}{1-x} \\ &= -1 - \frac{1}{1-x} \\ &= -1 - (x-1)^{-1} \\ \frac{dy}{dx} &= -1 - (x-1)^{-2} \\ &= \frac{1}{(x-1)^2} \end{aligned}$$

b

$$\begin{aligned} y &= \frac{x}{1-x} \\ (1-x)y &= x \\ y - yx &= x \\ x(1+y) &= y \\ x &= \frac{y}{1+y} \\ x &= \frac{y+1}{1+y} - \frac{1}{1+y} \\ x &= 1 - \frac{1}{1+y} \\ &= 1 - (y+1)^{-1} \\ \frac{dx}{dy} &= -1 \times -(y+1)^{-2} \\ &= (y+1)^{-2} \\ \frac{dy}{dx} &= \left(\frac{dx}{dy}\right)^{-1} = ((y+1)^{-2})^{-1} \\ \frac{dy}{dx} &= (y+1)^2 \end{aligned}$$

32

$$\begin{aligned} y &= (x^2+1)^{-\frac{3}{2}} \\ \frac{dy}{dx} &= 2x \times \frac{-3}{2}(x^2+1)^{-\frac{5}{2}} \\ &= -3x(x^2+1)^{-\frac{5}{2}} \end{aligned}$$

33

$$\begin{aligned} y &= x^4 \\ \frac{dy}{dx} &= 4x^3 \\ LHS &= x \times \frac{dy}{dx} \\ &= x \times 4x^3 \\ &= 4x^4 \\ &= 4y \\ &= RHS\ QED \end{aligned}$$

34 $f'(x) = 10x^4 > 0$ for all$x \neq 0$
$f(b) = 2b^5 > f(0) = 0$ for all $b > 0$
$f(b) = 2b^5 < f(0) = 0$ for all $b < 0$

35 **a** $$\int_0^{\frac{\pi}{2}} 2\sin\left(\frac{x}{2}\right)dx$$
$$= \left[-4\cos\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$$
$$= 4 - 2\sqrt{2}$$

b $$\int_0^{\frac{3}{2}} e^{\frac{x}{2}}\,dx$$
$$= \left[2e^{\frac{x}{2}}\right]_0^{\frac{3}{2}}$$
$$= 2(e^{\frac{3}{4}} - 1)$$

c $$\int_{\frac{1}{2}}^{1} \frac{1}{2x}\,dx$$
$$= \left[\frac{1}{2}\log_e(x)\right]_{\frac{1}{2}}^{1}$$
$$= \frac{1}{2}\log_e 2$$

d $$\int_{-1}^{-\frac{1}{2}} \frac{1}{2x}\,dx$$
$$= -\int_{\frac{1}{2}}^{1} \frac{1}{2x}\,dx$$
$$= -\left[\frac{1}{2}\log_e(x)\right]_{\frac{1}{2}}^{1}$$
$$= -\frac{1}{2}\log_e 2$$

e $$\int_3^4 \frac{1}{2(x-2)^2}\,dx$$
$$= \frac{1}{4}$$

f $$\int_2^4 \frac{1}{(3x-2)^2}\,dx$$
$$= \frac{1}{20}$$

36 **a**
$$\begin{aligned}
f(x) &= a\sqrt{x+1} - x - 1\\
f'(x) &= \frac{a}{2\sqrt{x+1}} - 1\\
f'(x) &= 0\\
\Rightarrow \frac{a}{2\sqrt{x+1}} - 1 &= 0\\
2\sqrt{x+1} &= a\\
x + 1 &= \frac{a^2}{4}\\
x &= \frac{a^2}{4} - 1\\
f\left(\frac{a^2}{4} - 1\right) &= \frac{a^2}{4}
\end{aligned}$$

b **i**
$$\begin{aligned}
f(3) &= 16\\
a\sqrt{4} - 3 - 1 &= 16\\
2a &= 20\\
a &= 10
\end{aligned}$$

ii
$$\begin{aligned}
f(35) &= 24\\
f'(35) &= -\frac{1}{6}\\
\therefore y - 24 &= -\frac{1}{6}(x - 35)\\
y &= -\frac{1}{6}x + \frac{35}{6} + 24\\
y &= -\frac{1}{6}x + \frac{179}{6}
\end{aligned}$$

iii $\left(0, \frac{179}{6}\right), (179, 0)$

37 $f'(x) = -6x^2 < 0$for all$x \neq 0$
$f(b) = -6b^3 < f(0) = 0$ for all $b > 0$

$f(b) = -6b^3 > f(0) = 0$ for all $b < 0$

38 a $$f(x) = e^{-mx+2} + 4x$$
$$f'(x) = -me^{-mx+2} + 4$$
$$f'(x) = 0 \Rightarrow e^{-mx+2} = \frac{4}{m}$$
$$\Rightarrow -mx + 2 = \log_e \frac{4}{m}$$
$$\Rightarrow x = \frac{1}{m}\left(2 - \log_e \frac{4}{m}\right)$$

b $$\frac{1}{m}\left(2 - \log_e \frac{4}{m}\right) < 0$$
$$\Leftrightarrow 2 - \log_e \frac{4}{m} < 0$$
$$\Leftrightarrow \log_e \frac{4}{m} > 2$$
$$\Leftrightarrow \frac{4}{m} > e^2$$
$$\Leftrightarrow m < 4e^{-2}$$

39 a Let$f(x) = y = x \sin x$

$$f'(x) = \frac{dy}{dx} = \sin x + x \cos x$$

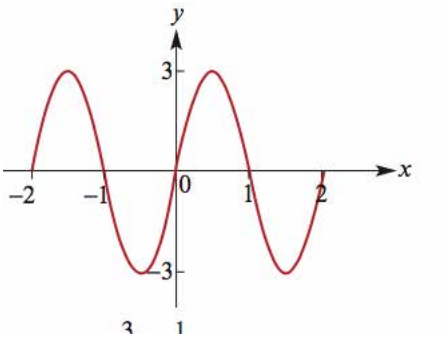
Let $A(a, f(a))$ be a point on the graph of $f(x) = x \sin x$ which has tangent that passes through the origin.
Then gradient of chord OA = gradient of the tangent at A. That is
$$\frac{a \sin a - 0}{a - 0} = \sin a + a \cos a$$
$$a \sin a = a \sin a + a^2 \cos a$$
$$\therefore a^2 \cos a = 0$$
This implies $a = 0$ or $\cos a = 0$, $-\pi \leq a \leq \pi$
$$\therefore a = 0 \text{ or } \frac{\pi}{2} \text{ or } -\frac{\pi}{2}.$$
The coordinates are
$$(0, 0), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

b Let$f(x) = y = x \cos 2x$

$$f'(x) = \frac{dy}{dx} = \cos 2x - 2x \sin 2x$$

Let $A(a, f(a))$ be a point on the graph of $f(x) = x \cos 2x$ which has tangent that passes through the origin.
Then gradient of chord OA = gradient of the tangent at A. That is
$$\frac{a \cos 2a - 0}{a - 0} = \cos 2a - 2a \sin 2a$$
$$a \cos 2a = a \cos 2a - 2a^2 \sin 2a$$
$$\therefore -a^2 \sin 2a = 0$$
This implies $a = 0$ or $\sin 2a = 0$, $-\pi \leq a \leq \pi$
$$\therefore a = 0 \text{ or } \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \text{ or } \pi \text{ or } -\pi.$$
The coordinates are
$$(0, 0), \left(\frac{\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), (\pi.\pi), (-\pi, -\pi),$$

40 a



b $f(x) = 3 \sin(\pi x)$

$f'(x) = 3\pi \cos(\pi x)$

Equation of tangent when $x = \frac{1}{2}$

$$f'\left(\frac{1}{2}\right) = 0$$
$y = 3$

c

$$\int_0^{\frac{1}{2}} f(x) - \frac{x}{4}\, dx = \int_0^{\frac{1}{2}} 3\sin(\pi x) - \frac{x}{4}\, dx$$

$$= \left[-\frac{3}{\pi}\cos(\pi x) - \frac{x^2}{8} \right]_0^{\frac{1}{2}}$$

$$= -\frac{1}{32} - (-\frac{3}{\pi})$$

$$= \frac{3}{\pi} - \frac{1}{32}$$

41 Where does the line $y = 5$ meet the hyperbola $y = 4 + \frac{2}{x}$

$$4 + \frac{2}{x} = 5$$

$$\frac{2}{x} = 1$$

$$x = 2$$

$$\int_{\frac{1}{2}}^{2} 4 + \frac{2}{x} - 5\, dx$$

$$= \left[4x + 2\ln x - 5x \right]_{\frac{1}{2}}^{2}$$

$$= (8 + 2\ln(2) - 10) - (2 - 2\ln(2) - \frac{5}{2})$$

$$= -\frac{3}{2} + 4\ln(2)$$

Area of the rectangle bounded by the line $x = \frac{1}{2}$, $y = 5$, $y = 8$ and the y-axis is $\frac{3}{2}$.

Therefore area of the shaded region is $4\log_e(2)$

42 $h(x) = (ax^2 + b)e^{cx}$

$h'(x) = 2axe^{cx} + c(ax^2 + b)e^{cx}$

$h'(x) = (2ax + cax^2 + cb)e^{cx}$

- $h(0) = -4$

  $\therefore be^0 = -4$

  $\therefore b = -4$

- $h'(0) = 8 \Rightarrow cbe^0 = 8$

  $\therefore -4c = 8$

  $\therefore c = -2$

- $h'(x) = 0$ when $x = -1$

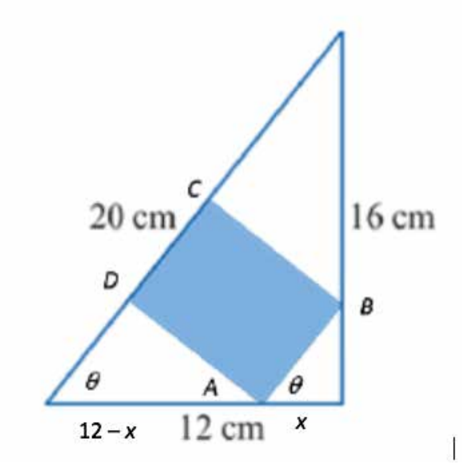
  $-2a + ca + cb = 0$

  $-2a - 2a + 8 = 0$

  $-4a + 8 = 0$

  $a = 2$

43



$$\frac{x}{AB} = \cos\theta$$

$$\frac{AD}{12 - x} = \sin\theta$$

$$\sin\theta = \frac{4}{5} \text{ and } \cos\theta = \frac{3}{5}$$

Area of rectangle $= AD \times AB$

$$= \frac{x}{\cos\theta} \times (12 - x)\sin\theta$$

$$= \frac{5x}{3}(12 - x) \times \frac{4}{5}$$

$$= \frac{4x}{3}(12 - x)$$

Therefore maximum when $x = 6$

Area = 48 cm^2

$AB = 10$ cm and $AD = \frac{24}{5}$ cm

Solutions to multiple-choice questions

1 B Stationary points when $x = 0, 4, -7$
Checking gradients

- $x > 4 \Rightarrow f'(x) > 0$
- $0 < x < 4 \Rightarrow f'(x) < 0$
- $-7 < x < 0 \Rightarrow f'(x) < 0$
- $x < -7 \Rightarrow f'(x) > 0$

On;y ome local maximum.

2 C

$f(-2) = -15$

$f(2) = 9$

$f'(x) = 6x^2 - 2x - 2 = 2(3x^2 - x - 1)$

$f'(x) = 0 \Rightarrow x = \frac{1}{6}(1 \pm \sqrt{13})$

$f(\frac{1}{6}(1 + \sqrt{13}) < 9$

and $f(\frac{1}{6}(1 - \sqrt{13}) < 9$

Therefore absolute maximum when $x = 2$

3 D $y = \sin 2x + 1$

$\frac{dy}{dx} = 2\cos 2x$

$x = 0: \frac{dy}{dx} = 2\cos 0 = 2$

4 C $f'(x) = \left(\frac{1}{\sqrt{3}}\cos x - \sin x\right)e^{\frac{x}{\sqrt{3}}}$

$f'(x) = 0 \Rightarrow \frac{1}{\sqrt{3}}\cos x - \sin x = 0$

$\Rightarrow \tan x = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$

The sum is $\frac{21\pi}{6} = \frac{7\pi}{2}$

5 B Do a rough sketch.

6 E

$$\text{grad } PQ = \frac{(2+h)^3 - 2^3}{(2+h) - 2} = \frac{(2+h)^3 - 8}{h} = 12 + 6h + h^2$$

7 A

$$\frac{[f(x+h) - f(x)]}{h} = \frac{\frac{3}{x+h} - \frac{3}{x}}{h} = \frac{3x - 3(x+h)}{hx(x+h)} = \frac{-3h}{hx(x+h)} = -\frac{3}{x(x+h)}$$

8 D $y = ce^{2x}$

$$\frac{dy}{dx} = 2ce^{2x}$$

$= 2c$ (when$x = 0$)

$= 11$

$\Rightarrow c = 5.5$

9 B $y = bx^2 - cx = x(bx - c)$

$y = 0$ if $x = 0$ or $x = \frac{c}{b}$.

Since the graph crosses the x-axis at $(4, 0), \frac{c}{b} = 4$, i.e. $c = 4b$ ①

$\frac{dy}{dx} = 2bx - c$

$= 1$ *at* $(4, 0)$

So $8b - c = 1$, *i.e.* $c = 8b - 1$ ②

② − ① gives $0 = 4b - 1$

$$b = \frac{1}{4}$$

$$c = 4 \times \frac{1}{4}.$$

10 A The graph shows two local maximum and local minimum; there are no stationary points of inflection. Here, $f'(x) = 0$ at 3 points

11 E $f(x) = 4 - e^{-2x}$

$f'(x) = 2e^{-2x}$

The graph is a decaying exponential with the x-axis as a horizontal asymptote.

Only the last graph fits.

12 B The gradient of the gives graph is zero at some negative value of x and again at $x = 0$.

The gradient goes from positive to negative to positive through these two stationary points.

Only the second graph fits.

13 C

$2\pi r + h = 30$

$\Rightarrow h = 30 - 2\pi r$

$\therefore V = \pi r^2(30 - 2\pi r)$

$= 30\pi r^2 - 2\pi^2 r^3$

$\frac{dV}{dr} = 60\pi r - 60\pi^2 r^2$

$\frac{dV}{dr} = 0 \Rightarrow 6\pi r(10 - \pi r) = 0$

For maximum, $r = \frac{10}{\pi}$

14 D Since the derivations are equal, the functions differ by at most of constant.

So $g(x) = f(x) + c$

$= 3x^2 + 2 + c$

$g(2) = 29$

$12 + 2 + c = 29$

$c = 15$

$g(x) = 3x^2 + 2 + 15$

$= 3x^2 + 17$

(Alternatively, $g(x) = 3x^2 + k$ for some constant k, so only options A, B and D are posible. Substitute x = 2 in each to see which gives 29.)

15 E $f'(x) = ke^{kx} - ke^{-kx}$

$= ke^{-kx}(e^{2kx} - 1)$

Case(1): $k > 0$

Then $f'(x) > 0$ provided $e^{2kx} - 1 > 0$

i.e $e^{2kx} - 1$

$2kx < 0$ (Since $e^0 = 1$)

$x > 0$ (Since $k > 0$)

Case(2): $k < 0$

Then $f(x) > 0$ provided $e^{2kx} - 1 < 0$

i.e $e^{2kx} < 1$

$2kx < 0$

$x > 0$ (Since $k > 0$)

In either case, $f'(x) > 0$ for $x > 0$

16 B $f'(x) = (x^2 - 9)g(x)$

Given that $g(x) < 0$ there are stationary points when $x = -3$ and $x = 3$.

$x < -3 \Rightarrow f'(x) < 0$

$-3 < x < 3 \Rightarrow f'(x) > 0$

$x > 3 \Rightarrow f'(x < 0$

f has a local minimum when $x = -3$ and a local maximum when $x = 3$.

17 **C** Average rate $= \dfrac{V(4) - V(2)}{4 - 2}$

$$= \frac{45 - 15}{2}$$

$$= 15 \text{ m}^3/\text{ min}$$

18

E Gradient $PQ = \dfrac{f(x+h) - f(x)}{(x+h) - (x)}$

$$= \frac{\begin{array}{c}[(x+h)^2 - 2(x+h) + 1] \\ -[x^2 - 2x + 1]\end{array}}{h}$$

$$= \frac{\begin{array}{l}x^2 + 2xh + h^2 - 2x \\ \quad -2h + 1 - x^2 \\ \quad +2x - 1\end{array}}{h}$$

$$= \frac{2xh + h^2 - 2h}{h}$$

$$= 2x - 2 + 2h$$

19 **D** The gradient of the gives graph is zero at a point in $(-3, -1)$ and again at a point in $(0, 2)$. The gradient goes from position to negative to positive through these two points.
Only the fourth graph fits.

20 **E** The gradient of a tangent to $y = f(x)$ at $x = a$ is given by
$\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$
If a=2, this becomes
$\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$

21 **A** From the graph, strictly increasing for $[-3, 2]$

22 **D** The graph of $y = -x^2 + 4x + 3$ is an inverted parabola will a maximum tuning point.

$$\frac{dy}{dx} = -2x + 4$$

$$= 0 \text{ if } x = 2$$

The $y = -4 + 8 + 3 = 7$, which is the required maximum value.

23 **E** $h'(x) = f'(x)g(x) + g'(x)f(x)$

$$\therefore h'(2) = f'(2)g(2) + g'(2)f(2)$$

$$= -6 \times -3 + 7 \times 4$$

$$= 18 + 28$$

$$= 46$$

24 **A** $y = x^2 - x^3$

$$\frac{dy}{dx} = 2x - 3x^2$$

$$= x(2 - 3x)$$

$$= 0 \text{ if } x = 0, \frac{2}{3}$$

There are stationary point where $x = 0$ and $x = \dfrac{2}{3}$

25 **D** $\dfrac{dy}{dx} = 2x + 3$

When $x = 2, y = 10$ and $\dfrac{dy}{dx} = 7$
Equation of tangent. $y - 10 = 7(x - 2)$
$\therefore y = 7x - 4$

26 **D** Let $F(t)$ be an an antiderivative of $\sqrt{t^3 + 4t}$

$$\int_0^x \sqrt{t^3 + 4t}\, dt = F(x) - F(0)$$

$$\therefore f(x) = F(x) - F(0)$$

$$\therefore f'(x) = F'(x) = \sqrt{t^3 + 4t}$$

$$\therefore f'(1) = \sqrt{1 + 4} = \sqrt{5}$$

27 **A** $f'(x) = x^2 + \dfrac{1}{x}$

$$f(x) = \frac{1}{3}x^3 + \log_e x + c$$

(Since the condition has $x > 0$)

$$f(1) = \frac{1}{3} + \log_e 1 + c$$
$$= \frac{1}{3} + c$$
$$= \frac{1}{3} \text{ if } c = 0$$
$$f(x) = \frac{1}{3}x^3 + \log_e x$$

28 C $\int_2^3 f(x)dx = [F(x)]_2^3$
$$= F(3) - F(2)$$

29 C Area $= \int_{\frac{\pi}{2}}^{\pi} \sin x\, dx - \int_{\pi}^{\frac{3\pi}{2}} \sin x\, dx$
$$= \int_{\frac{\pi}{2}}^{\pi} \sin x\, dx + \int_{\frac{3\pi}{2}}^{\pi} \sin x\, dx$$

30 D The Straight line crosses the x-axis at $x = -1$
Area $= \int_{-1}^{2} x + 1dx - \int_{-2}^{-1} x + 1dx$

31 B $\frac{dy}{dx} = \frac{1}{x^2}$
$$= x^{-2}$$
$$y = -x^{-1} + c$$
$$= -\frac{1}{x} + c$$
$y = 2$ where $x = 1$
$$2 = -1 + c$$
$$c = 3$$
$$y = -\frac{1}{x} + 3$$

32 B $\int_0^{16} \frac{dx}{2x+1} = \left[\frac{1}{2}\log_e(2x+1)\right]_0^{16}$
$$= \frac{1}{2}\log_e 33 - \frac{1}{2}\log_e 1$$
$$= \frac{1}{2}\log_e 33$$

So $k = \sqrt{33}$

33 B $f(x) = 8\sqrt{x+1} - (x+1)$
$$f'(x) = \frac{4}{\sqrt{x+1}} - 1$$
y-axis intercept is $(0, 7)$
When $x = 0$,
$8\sqrt{x+1} - (x+1) = 0$
$$8 = \sqrt{x+1}$$
$$x = 63$$
x-axis intercept is $(63, 0)$
Gradient of the line connecting the two intercepts is $-\frac{1}{9}$
$$f'(x) = -\frac{1}{9}$$
$$\frac{4}{\sqrt{x+1}} - 1 = -\frac{1}{9}$$
$$\frac{4}{\sqrt{x+1}} = \frac{8}{9}$$
$$\frac{\sqrt{x+1}}{4} = \frac{9}{8}$$
$$\sqrt{x+1} = \frac{9}{2}$$
$$x + 1 = \frac{81}{4}$$
$$x = \frac{77}{4}$$

34 E Area $= -\int_{-3}^{0} f(x)dx + \int_0^4 f(x)dx$
Which is not the same as any a of the first four options.

35 A Area $= \int_0^2 \frac{1}{3-x}dx$
$$= [-\log_e(3-x)]_0^2$$
$$= -\log_e 1 + \log_e 3$$
$$= \log_e 3$$

36

A $$\int_a^b \sin 2x\,dx = \left[-\frac{1}{2}\cos 2x\right]_a^b$$

$$= -\frac{1}{2}\cos 2b + -\frac{1}{2}\cos 2a$$

$$= \frac{1}{2}(\cos 2a - \cos 2b)$$

For A: Substituting $b = \frac{3\pi}{4}$ and $a = \frac{\pi}{4}$ give, $\frac{1}{2}\left(\cos\frac{\pi}{2} - \cos\frac{3\pi}{2}\right) = \frac{1}{2}(0 - 0) = 0$ as required.

(Checking each other option shows that none of these gives zero.)

37 **A** Since the shaded region is below the x-axis, its area is given by $\int_0^a -f(x)dx$.

38 **A**

$$A = x(8\sqrt{x+16} - (x+16)$$

$$= 8x\sqrt{x+16} - (x^2 + 16x)$$

$$\frac{dA}{dx} = 8\sqrt{x+16} + \frac{4x}{\sqrt{x+16}} - 2x - 16$$

$$\frac{dA}{dx} = 0 \Rightarrow x = 8(\sqrt{5} + 1)$$

$$A(8(\sqrt{5}+1)) = 256$$

39 **D** $f'(x) = 3x^2 + 6x - 9$

$$= 3(x^2 + 2x - 3)$$

$$= 3(x+3)(x-1)$$

$$> 0 \text{ if } x < -3 \text{ or } x > 1$$

So the function is stictly increasing if $x \le -3$ or $x \ge 1$

40 **B** $$\frac{1}{a-1}\int_1^a x^3 - 3x^2\,dx = 43$$

$$\frac{1}{a-1}\left[\frac{x^4}{4} - x^3\right]_1^a = 43$$

$$\frac{a^4}{4} - a^3 - \left(\frac{1}{4} - 1\right) = 43(a-1)$$

$$\frac{a^4}{4} - a^3 + \frac{3}{4} = 43(a-1)$$

$$a^4 - 4a^3 + 3 = 172(a-1)$$

$$a^4 - 4a^3 - 172a + 175 = 0$$

$$a = 7$$

41 **D** $$f(x) = 2x^3 + ax^2 + bx$$

$$f'(x) = 6x^2 + 2ax + b$$

$$f'(-1) = 0 \text{ and } f'(4) = 0$$

$$6 - 2a + b = 0\ldots(1)$$

$$96 + 8a + b = 0\ldots(2)$$

$$90 + 10a = 0$$

$$\therefore a = -9 \text{ and } b = -24$$

42 **C** $f'(x) = \sin 2x$

$$f(x) = -\frac{1}{2}\cos 2x + c$$

$f(0) = 3$, so

$$3 = -\frac{1}{2}\cos 0 + c$$

$$= -\frac{1}{2} + c$$

$$c = 3\frac{1}{2}$$

$$f(x) = -\frac{1}{2}\cos 2x + 3\frac{1}{2}$$

43 **B** $f'(x) = 3x^2 - 2x - 1$

$$= (3x+1)(x+1)$$

$$= 0 \text{ if } x = -\frac{1}{3}, 1$$

The gradient is positive if $x < -\frac{1}{3}$ and negative if $-\frac{1}{3} < x < 1$.
So there is a local maximum at $x = -\frac{1}{3}$.
The gradient is negative if $-\frac{1}{3} < x < 1$ and positive if $x > 1$.
So there is a local minimum at $x = 1$. $f(1) = 1 - 1 - 1 + 2 = 1$ there is a local minimum at $(1, 1)$.

44 A $$\text{Area} = \int_0^{\frac{\pi}{2}} \cos x\, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx$$
$$= [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$
$$= (1 + 0) - (-1 - 1)$$
$$= 3 \text{ square units}$$

45 C $x + y = 1$ so $y = 1 - x$
$$p = x^2 + xy - y^2$$
$$= x^2 + x(1 - x) - (1 - x)^2$$
$$= x^2 + x - x^2 - 1 + 2x - x^2$$
$$= -x^2 + 3x - 1$$
$$\frac{d\text{P}}{dx} = -2x + 3$$
$$= 0 \text{ if } x = \frac{3}{2}$$
and this corresponds to a maximum since the graph of P against x is an inverted parabola.

46 B $$y = e^{-\cos x}$$
$$\frac{dy}{dx} = e^{-\cos x} \times \sin x$$
$$= \sin x e^{-\cos x}$$
where $x = \frac{\pi}{3}, \frac{dy}{dx} = \sin\frac{\pi}{3} e^{-\cos\frac{\pi}{3}}$
$$= \frac{\sqrt{3}}{2} e^{-\frac{1}{2}}$$
and this is the gradient of the tangent at $x = \frac{\pi}{3}$.
Using $m_1 m_2 = -1$
$$m_2 = -1 \div \left(\frac{\sqrt{3}}{2} e^{-\frac{1}{2}}\right)$$
$$= -\frac{2}{\sqrt{3}} e^{\frac{1}{2}}$$
$$= \frac{-2e^{\frac{1}{2}}}{\sqrt{3}}$$

47 C Since $(1, 3)$ is a maximum point on the graph, $f'(1) = 0$.
The equation of the tangent is
$y - 3 = 0\,(x - 1)$
$$y = 3$$

48 D $$P = -x^2 + 6x + 4$$
$$\frac{dP}{dx} = -2x + 6$$
$$= 0 \text{ if } x = 3$$
and this corresponds to a maximum since the graph of P against x is an inverted parabola.
When $x = 3, P = -9 + 18 + 4$
$$= 13$$
The maximum value is 13.

49 B Stationary points
$f'(x) = x^3 - x^2 - 1$ occur when
$f'(x) = 0$.

$$f'(x) = 3x^3 - 2x$$
$$= x(3x - 2)$$
$$= 0 \text{ if } x = 0, \frac{2}{3}$$

50 D $f'(2) = \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$
$$= \lim_{h \to 0} h^2 + 6h + 12$$
$$= 12$$

51 B $y = x^2e^x$
$$\frac{dy}{dx} = x^2e^x + 2xe^x$$
$$= xe^x(x + 2)$$
$$= 0 \text{ if } x = 0, 2.$$
Note that $y \geq 0$ and when $x = 0$, $y = 0$, So the minimum value is 0.

52 E $f(x) = a\sin(3x)$
$$f'(x) = 3a\cos(3x)$$
$$f'(\pi)3a\cos 3\pi$$
$$= -3a$$
$$= 2 \text{ if } a = -\frac{2}{3}$$

53 Local maximum when $x = a$. Therefore $a = 2$

54 A Let $D(x)$ be the distance from $(4, 0)$ to $\sqrt{x}$
$$[D(x)]^2 = (x - 4)^2 + x$$
$$\frac{d}{dx}([D(x)]^2) = 2(x - 4) + 1$$
$$= 2x - 7$$
$$\frac{d}{dx}([D(x)]^2) = 0 \Rightarrow x = \frac{7}{2}$$
$$D\left(\frac{7}{2}\right) = \sqrt{\left(\frac{7}{2} - 4\right)^2 + \frac{7}{2}}$$

Therefore minimum distance is $\dfrac{\sqrt{15}}{2}$

55 B

$\sin(2x) = \cos x$

Using calculator $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

From symmetry we see that:
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x) - g(x)\,dx =$$
$$-\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} g(x) - f(x)\,dx$$
Therefore total area
$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x) - g(x)\,dx$$
$$+ \int_{0}^{\frac{\pi}{6}} g(x) - f(x)\,dx$$

56 C $y = 2e^{3x} - 1$
$$\frac{dy}{dx} = 6e^{3x}$$
$$6e^{3x} = 6$$
$$e^{3x} = 1$$
$$\therefore x = 0$$
Equation of tangent:

$y - 1 = 6x$

When $y = 0$, $x = -\dfrac{1}{6}$

57 C $\int_a^0 f(x)\,dx = n \Rightarrow \int_0^a f(x)\,dx = -n$

$\int_0^a 2f(x) - 1\,dx = 2\int_0^a f(x) - \int_0^a 1\,dx$

$= -2n - a$

58 A
$$f(x) = \frac{a}{x^2} + x - 2$$
$$f'(x) = -\frac{2a}{x^3} + 1$$
$$f'(x) = 0 \Rightarrow -2a + 1 = 0$$
$$\therefore a = \frac{1}{2}$$

59 D

$$y = 2x^3 + ax^2 + 1$$

$$\frac{dy}{dx} = 6x^2 + 2ax$$

When $x = -1, y = a - 1, \frac{dy}{dx} = 6 - 2a$

Equation of tangent at $x = -1$

$y - (a - 1) = (6 - 2a)(x + 1)$

When $x = 0, y = 0$

Therefore,

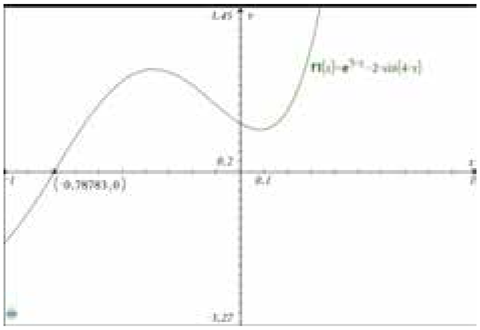
$1 - a = 6 - 2a \Rightarrow a = 5$

60 A

$$\begin{aligned} f(g(x)) &= x \\ (f \circ g)'(x) &= 1 \\ \therefore g'(x)f'(g(x)) &= 1 \\ g'(4)f'(g(4)) &= 1 \\ g'(4)f'(6) &= 1 \\ g'(4) &= \frac{1}{4} \end{aligned}$$

61 D From the graph, there is a stationary point of inflexion at x = −3 and a minimum stationary point at $x = \frac{9}{4}$. A quick check of each option shows that the fourth is not true, as there are two stationary point on the graph. (Checking the other options shows each is true.)

62 E Use a CAS calculator to plot the curve on the interval $(-1, 1)$. It shows one x intercept at $x = -0.78783$.



Hence the required area is given by

$$A = \int_{-1}^{-0.78783} 2\sin 4x - e^{5x}\,dx + \int_{-0.78783}^{1} e^{5x} - 2\sin 4x\,dx$$

$= 30.02$ to 2 dp

where the integrals have been evaluated using a CAS calculator.

Solutions to extended-response questions

1 **a** $S = 50 + 30e^{-\frac{1}{3}t}$

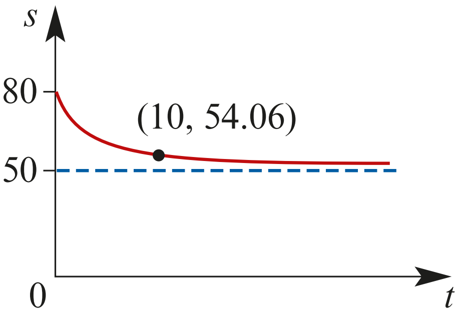
When $t = 10$

$s = 50 + 30e^{-2}$

≈ 54.06

There are 54.06 g of salt in the mixture after 10 minutes.

b



c $\dfrac{ds}{dt} = -6e^{\frac{1}{5}t}$

d Considering $s = 50 + 30e^{-\frac{1}{5}t}$

Solve for $e^{-\frac{1}{5}t}$

$\therefore \dfrac{s - 50}{30} = e^{-\frac{1}{5}t}$

Substitute in $\dfrac{ds}{dt} = -6e^{-\frac{1}{5}t}$

to yield $\dfrac{ds}{dt} = -6\left(\dfrac{S - 50}{30}\right)$

$= \dfrac{1}{5}(50 - s)$

e When $t = 0$

$s = 50 + 30e^{0}$

$= 80$

The volume of water is 100 litres.

$\therefore$ Concentration $= \dfrac{80}{100} = 0.8$g/litre

f Concentration $= \dfrac{s}{100}$

$= \dfrac{50 + 30e^{-\frac{1}{5}t}}{100}$

Concentration = 0.51 gram/litre

implies $0.51 = \dfrac{50 + 30e^{-\frac{1}{5}t}}{100}$

$\therefore\ 51 = 50 + 30e^{-\frac{1}{5}t}$

$\dfrac{1}{30} = e^{-\frac{1}{5}t}$

and therefore

$t = -5\log_e \dfrac{1}{30}$

$= 5\log_e 30$

≈ 17.006

The concentration first reaches 0.51 g/litre after about 17 seconds.

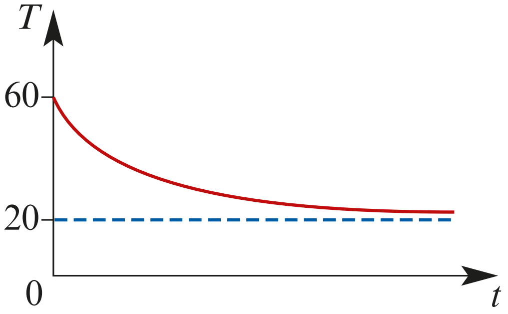
2 **a** $T = 40e^{-0.36t} + 20,\ t \geq 0$

When $t = 0$

$T = 60$

The initial temperature of the body was 60° C.

b



c $\dfrac{dT}{dt} = -14.4e^{-0.36t}$

d Since $T = 40e^{-0.36t} + 20$

$e^{-0.36t} = \dfrac{T - 20}{40}$

Hence $\dfrac{dT}{dt} = -14.4\left(\dfrac{T-20}{40}\right) = -0.36(T - 20)$

3 **a** $f(t) = 1000e^{-0.5t}$

$f(0) = 1000$

Initially there were 1000 F-type spores.

50% of the initial number is 500.

Consider

$500 = 1000e^{-0.5t}$

$0.5 = e^{-0.5t}$

$\therefore\ t = -2\log_e(0.5)$

$= 2\log_e 2$

≈ 1.386

It takes about 1.386 minutes to kill half of the F-type spores.

b $f(0) = 1000$ and $g(1000) = 1200$
Initially there are 1000 F-type spores and 1200 G-type spores,
so there are 2200 line spores of both types.
For $t = 5$

$f(5) = 1000e^{-0.5\times5}$ and $g(5) = 1200e^{-0.7\times5}$

$= 1000e^{-2.5}$ $= 1200e^{-3.5}$

Percentage of spores still alive after 5 minutes $= \dfrac{f(5) + g(5)}{f(0) + g(0)} \times \dfrac{100}{1}$

$= \dfrac{1000e^{-2.5} + 1200e^{-3.5}}{2200} \times \dfrac{100}{1}$

$= \dfrac{1000\, e^{-2.5} + 1200\, e^{-3.5}}{22}$

≈ 5.378

$\therefore$ Percentage of spores still alive after 5 minutes is 5.378. %

c Total no. of spores $= 1000e^{-0.5t} + 1200e^{-0.7t}$
i.e. $T = 1000e^{-0.5t} + 1200e^{-0.7t}$ where T is the total number of spores

$$\frac{dT}{dt} = -500e^{-0.5} - 840e^{-0.7t}$$

When $t = 5$

$$\frac{dT}{dt} = -500e^{-0.5\times5} - 840e^{0.7\times5}$$

$$= -500e^{-2.5} - 840e^{-3.5}$$

$$\approx -66.408$$

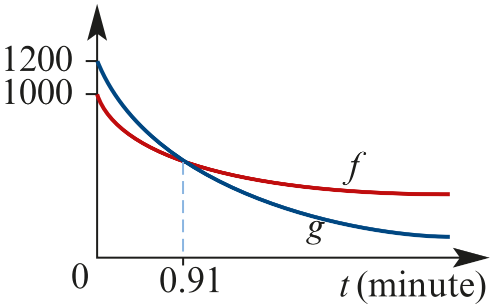
When $t = 5$, the rate at which the total number of spores is decreasing is 66.4 spores per minute.

d Live F-type spores = live G-type spores
When $f(t) = g(t)$
i.e. $1000e^{-0.5t} = 1200e^{-0.7t}$
which implies $e^{0.2t} = \dfrac{1200}{1000}$

$$\text{and } t = \frac{1}{0.2}\log_e(\frac{6}{5})$$
$$= 5\log_e$$
$$\approx 0.9116$$

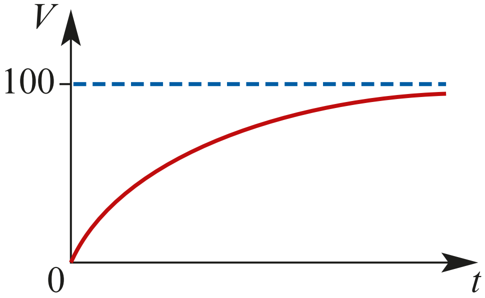
∴ the number of live F-type spores = the number of live G-type spores when $t = 0.9116$.

e number of spores



4 $V = 100(1 - e^{-0.2t})$

a



b **i** Acceleration $= \frac{dV}{dt} = 100 \times 0.2e^{-0.2t}$

$$= 20e^{-0.2t} m/s^2$$

ii From

$$\begin{aligned} V &= 100(1 - e^{-0.2t}) \\ \frac{V}{100} &= 1 - e^{-0.2t} \\ \text{and } e^{-0.2t} &= 1 - \frac{V}{100} \\ &= \frac{100 - V}{100} \\ \therefore \ \frac{dV}{dt} &= 20e^{-0.2t} \\ &= 20\left(\frac{100 - V}{100}\right) \\ &= \frac{1}{5}(100 - V)\text{m/s}^2 \end{aligned}$$

c when $V = 80$

$$\begin{aligned} 80 &= 100(1 - e^{-0.2t}) \\ 0.8 &= 1 - e^{-0.2t} \\ e^{-0.2t} &= 0.2 \\ \therefore \ -0.2t &= \log_e(0.2) \\ \text{i.e. } t &= -5\log_e(0.2) \\ &= 5\log_e 5 \end{aligned}$$

When the velocity of the body is 80 m/s, $t = 5\log_e 5 \approx 8.05$ seconds.

5 $C = 0.05x^2 + 5x + 500$

The average cost $A = \dfrac{C}{x}$

i.e. $A = 0.05x + 5 + \dfrac{500}{x}$

$$\frac{dA}{dx} = 0.05 - \frac{500}{x^2}$$

$\dfrac{dA}{dx} = 0$ implies

$$0.05 = \frac{500}{x^2}$$

i.e. $x^2 = 10\,000$

$x = 100$

$\dfrac{dA}{dx} > 0$ for $x > 100$

and $\dfrac{dA}{dx} < 0$ for $x < 0$

$\therefore$ a local minimum at $x = 100$

i.e. 100 units per annum minimises the average cost per unit.

6 $T = T_0e^{-kt}$

a $\dfrac{dT}{dt} = -kT_0e^{-kt}$

$= -kT$

$\therefore\ \dfrac{dT}{dt}$ is proportional to T

b When $t = 0,\ T = 100 - 30 = 70$

i.e. $T_0 = 70$

When $t = 20,\ T = 70 - 30 = 40$

$\therefore\ 40 = 70e^{-20k}$

$\dfrac{4}{7} = e^{-20k}$

and $-20k = \log_e\left(\dfrac{4}{7}\right)$

$k = -\dfrac{1}{20}\log_e\left(\dfrac{4}{7}\right)$

$= \dfrac{1}{20}\log_e\left(\dfrac{7}{4}\right)$

$= 0.028$ (correct to 3 decimal places)

c $\dfrac{dT}{dt} = -kT_0e^{-kt}$

$$= -\frac{70}{20}\log_e\left(\frac{7}{4}\right)e^{-\frac{30}{20}\log_e\left(\frac{7}{4}\right)}$$

$$= -\frac{70}{20}\log_e\left(\frac{7}{4}\right)e^{\log e\left(\frac{7}{4}\right)^{\frac{3}{2}}}$$

$$= -\frac{70}{20}\times\left(\frac{4}{7}\right)^{\frac{3}{2}}\log_e\left(\frac{7}{4}\right)$$

≈ -0.846

The temperature is decreasing at a rate of 0.846 degrees/minute.

7 **a** **i** $p(t) = 0.2 - 0.2e^{-\frac{t}{20}} + 0.1e^{-\frac{t}{10}} t \geq 0$

$\therefore\ p(10) = 0.2 - 0.2e^{-\frac{1}{2}} + 0.1e^{-1}$

≈ 0.1155 (correct to four decimal places)

ii As $t \to \infty$

$e^{-\frac{t}{20}} \to 0$ and $e^{-\frac{t}{10}} \to 0$

$\therefore\ p(t) \to 0.2$

The proportion approaches 0.2

b $p'(t) = 0.01e^{-\frac{t}{20}} - 0.01e^{-\frac{t}{10}}$

Let $N(t)$ be the number of new cases per day

$N(t) = kp'(t)$

$= k\left(0.01e^{-\frac{t}{20}} - 0.01e^{-\frac{t}{10}}\right)$

To find maximum, differentiate to find $N'(t)$ and solve the equation $N'(t) = 0$

$N'(t) = \left(-0.0005e^{-\frac{t}{20}} + 0.001e^{-\frac{t}{10}}\right)k$

$N'(t) = 0$ implies

$0.0005\, e^{-\frac{t}{20}} = 0.001e^{-\frac{t}{10}}$

$\therefore\ e^{\frac{t}{20}} = 2$

$\therefore\ t = 20\log_e 2 \approx 13.86$

$N'(t) < 0$ for $t > 20\log_e 2$ and $N'(t) > 0$ for $t < 20\log_e 2$

which implies a local maximum at $t = 20\log_e 2$

The number of new cases per day is a maximum when $t = 20\log_e 2$

i.e. after 13.86 days.

8 Let $\$x$ be the rent per month from each apartment.

The number of apartments occupied $= 70 - 2\frac{(x - 500)}{20}$

$$= \frac{700 - x + 500}{10}$$

$$= \frac{1200 - x}{10}$$

Let $\$R$ be the total revenue

then $R = \frac{x(1200 - x)}{10}$

$= \frac{1}{10}(1200x - x^2)$

$\frac{dR}{dx} = 0$ implies $x = 600$

this is a maximum as R is a quadratic function of x with negative coefficient of x^2 i.e. the price per apartment to maximise monthly revenue is \$600.

9 $V = \frac{5 \times 10^4}{(t + 1)^2}$

a When $t = 0,\ V = 5 \times 10^4$

i.e. the initial volume of liquid is 5×10^4 m^3

b $V = (5 \times 10^4)(t + 1)^{-2}$

$\therefore\ \frac{dV}{dt} = -10 \times 10^4(t + 1)^{-3}$ (chain rule)

$$= -\frac{10^5}{(t + 1)^3}$$

When $t = 1, \frac{dV}{dt} = -\frac{10^5}{2^3} = -12\ 500$

i.e. the rate of change of the volume of liquid with respect to time is $-12\ 500$ m^3/day.

c $V(4) = \dfrac{5 \times 10^4}{5^2} = \dfrac{10^4}{5} = 2000$

$V(1) = \dfrac{5 \times 10^4}{2^2} = 12\,500$

$\therefore$ the average rate of change of V with respect to t for

the interval $[1, 4] = \dfrac{2000 - 12\,500}{4 - 1}$

$= -3500$

The average rate of change for the interval [1,4] is-3500 m^3/day.

d $\dfrac{5 \times 10^4}{(t+1)^2} < 1$

$5 \times 10^4 < (t+1)^2$

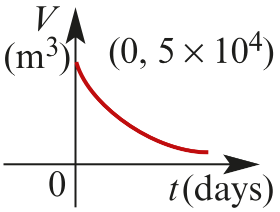
$\therefore\ t + 1 > \sqrt{5 \times 10^4};\ t > 0$

$\therefore\ t > 100\sqrt{5} - 1$

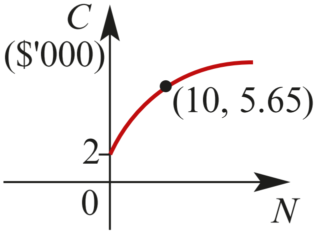
$100\ \sqrt{5} - 1 \approx 222.61$

$\therefore$ the amount of liquid in the pool is less than one cubic metre after $100\ \sqrt{5} - 1 \approx$ 222.61 days.

e



10 a



b $\dfrac{dC}{dN} = \dfrac{1}{4}.3N^2(N^3 + 16)^{-\frac{3}{4}}$ (chain rule)

$= \dfrac{3N^2}{4(N^3 + 16)^{\frac{3}{4}}}$

c The rate of change of cost in \$ 1000 s with respect to the number of bottle tops produced.

11 Profit = Selling Price − Cost Price

Selling Price $= \frac{800}{p^2} \times p = \frac{800}{p}$

Cost Price $= \frac{800}{p^2} \times 2 = \frac{1600}{p^2}$

$\therefore$ Profit $= \frac{800}{p} - \frac{1600}{p^2}$

Let R denote profit

then $R = \frac{800}{p} - \frac{1600}{p^2}$

$$\frac{dR}{dp} = -\frac{800}{p^2} + \frac{3200}{p^3}$$

For maximum profit consider $\frac{dR}{dp} = 0$

$$-\frac{800}{p^2} + \frac{3200}{p^3} = 0 \ (p \neq 0)$$

$$-800p + 3200 = 0$$

$$p = 4$$

A sign diagram shows a local maximum

p	<	4	>
$\frac{dR}{dp}$	+	0	
sign	/	−	\

When $p = 4, R = \frac{800}{4} - \frac{1600}{4^2}$

$= 200 - 100$

$= 100$

$\therefore$ The selling price is \$4 to maximise profit and the number of items sold is $\frac{800}{16} = 50$

12 $y = (ax + b)^{-2}$

When $x = 0, y = \frac{1}{4}$

$$\frac{1}{4} = b^{-2}$$

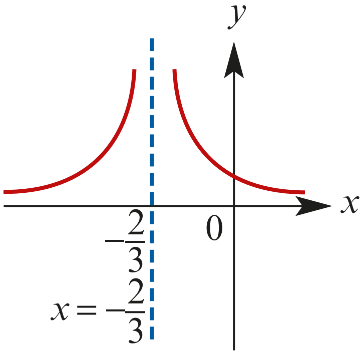
$\therefore\ b^2 = 4$

and $b = \pm 2$

$$\frac{dy}{dx} = -2a(ax + b)^{-3}$$

When $x = 0, \frac{dy}{dx} = -\frac{3}{4}$

$\therefore\ -\frac{3}{4} = -2a(b)^{-3}$



Substituting $b = 2$ gives

$-\frac{3}{4} = -2a \times \frac{1}{8}$

$\therefore \quad a = 3$

For $b = -2$

$a = 3$

$\therefore$ The possible pairs are (3, 2) and (−3, −2)

For $a = 3,\ b = 2$ (equivalently $a = -3, b = -2$)

$y = \frac{1}{(3x+2)^2}$

13 **a** Cost per hour $= 160 + \frac{1}{100}V^3$ dollars

A journey of 1000 km at 10 km/hr takes $\frac{1000}{10} = 100$ hours.

$$\therefore \text{ Cost of journey } = \left(160 + \frac{1}{100} \times 10^3\right)100 = (160 + 10)100 = 17\,000$$

The cost of the journey = \$17 000

b Time for a journey of 1000 km at Vkm/hr $= \frac{1000}{V}$ hours

$$\therefore \ C = \left(160 + \frac{1}{100}V^3\right)\frac{1000}{V} = \frac{160\,000}{V} + 10V^2$$

c In order to sketch the graph it is necessary to investigate stationary points.

$$\frac{dC}{dV} = -\frac{160\,000}{V^2} + 20V$$

$\frac{dC}{dV} = 0$ implies $20V = \frac{160\,000}{V^2}$

$\therefore \ V^3 = 8000$

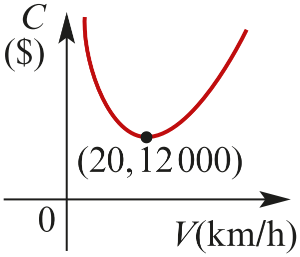
i.e. $V = 20$

When $V = 20,\ C = \frac{160\,000}{20} + 10 \times 20^2$

$= 12\ 000$

When $V > 20, \dfrac{dC}{dV} > 0$

When $0 < V < 20, \dfrac{dC}{dV} < 0$



$\therefore$ there is a local minimum at (20, 12 000)

For $C = \dfrac{160\ 000}{V} + 10V^2$

as $V \to \infty,\ C \to 10V^2$

as $V \to 0,\ C \to \infty$

$\therefore$ the graph is as shown here.

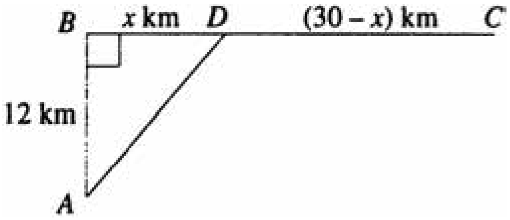
d From the above the most economical speed is 20 km/hr and the minimum cost is \$12 000.

e From the graph the minimum will occur when $V = 16$

$$C = \left(160 + \frac{1}{100}V^3\right)\frac{1000}{V}$$
$$= \frac{160\ 000}{V} + 10V^2$$
$$= \frac{160\ 000}{16} + 10 \times 16^2$$
$$= 10\ 000 + 2560$$
$$= 12\ 560$$

$\therefore$ the minimum cost is \$12 560 when the maximum speed is 16 km/hr.

14 a



Let $BD = x$, where D is the point where the camper reaches the shore.

Then $DC = 30 - x$

and by Pythagoras' Theorem

$AD = \sqrt{x^2 + 12^2}$

The camper rows at 5 km/hr. Therefore the time taken to row from

A to $D = \dfrac{\sqrt{x^2 + 144}}{5}$ hours.

The camper walks at 8 km/hr. Therefore the time taken to walk from

D to $C = \dfrac{(30 - x)}{8}$ hours.

The total time, T (hours), for the trip is given by

$$T = \frac{\sqrt{x^2 + 144}}{5} + \frac{30 - x}{8} = \frac{(x^2 + 144)^{\frac{1}{2}}}{5} + \frac{30 - x}{8}$$

To find the minimum time consider stationary point

$$\therefore \ \frac{dT}{dx} = \frac{2x \times \frac{1}{2}(x^2 + 144)^{-\frac{1}{2}}}{5} - \frac{1}{8}$$

$$= \frac{x}{5(x^2 + 144)^{\frac{1}{2}}} - \frac{1}{8}$$

$$\frac{dT}{dx} = 0 \text{ implies } \frac{x}{5(x^2 + 144)^{\frac{1}{2}}} = \frac{1}{8}$$

$$\therefore \ 8x = 5(x^2 + 144)^{\frac{1}{2}}$$

Squaring both sides

$$64x^2 = 25(x^2 + 144)$$

$$39x^2 = 25 \times 144$$

$$x^2 = \frac{25 \times 144}{39}$$

$$\therefore \ x = \frac{60}{\sqrt{39}} = \frac{60\sqrt{39}}{39} \approx 9.61$$

A gradient chart reveals a minimum

x	<	$\frac{60\sqrt{39}}{39}$	>
sign of $\frac{dT}{dx}$	−	0	+
shape	\	−	/

$\therefore$ the camper should land 9.61 km from B to minimise the time of the journey.

b If C is 24 km from B

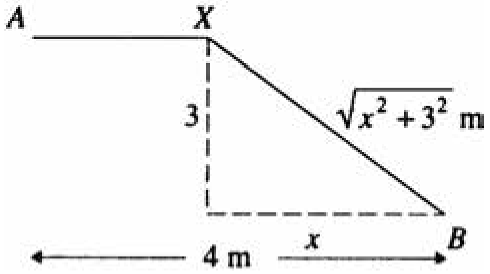
$$T = \frac{(x^2 + 144)^{\frac{1}{2}}}{5} + \frac{24 - x}{8}$$

$$\frac{dT}{dx} = \frac{x}{5(x^2 + 144)^{\frac{1}{2}}} - \frac{1}{8}$$

$\therefore$ local minimum is the same as **a**.

i.e. the camper should still row to a point 9.61 km from B.

15



Let C be the cost of laying the pipe

Distance $AX = 4 - x$

The cost of laying the pipe along $AX = 10(4 - x)$ dollars

The cost of laying section $XB = 25(x^2 + 9)^{\frac{1}{2}}$

$\therefore\ C = 10(4 - x) + 25(x^2 + 9)^{\frac{1}{2}}$

To find the minimum consider $\frac{dC}{dx}$

$$\frac{dC}{dx} = -10 + 25x(x^2 + 9)^{-\frac{1}{2}}$$

$$= -10 + \frac{25x}{(x^2 + 9)^{\frac{1}{2}}}$$

$\frac{dC}{dx} = 0$ implies $10 = \frac{25x}{(x^2 + 9)^{\frac{1}{2}}}$

$\therefore\ 10(x^2 + 9)^{\frac{1}{2}} = 25x$

$2\,(x^2 + 9)^{\frac{1}{2}} = 5x$

Squaring both sides gives

$4(x^2 + 9) = 25x^2$

$4x^2 + 36 = 25x^2$

$\therefore\ 36 = 21x^2$

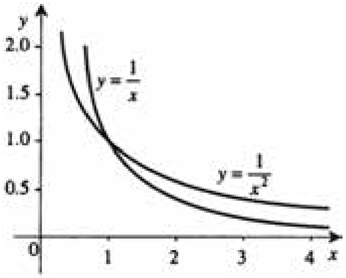
$\therefore\ x = \sqrt{\frac{12}{7}}$

x	$<$	$\sqrt{\frac{12}{7}}$	$>$
$\frac{dC}{dx}$	$-$	0	$+$
shape	\	$-$	/

A minimum occurs when $x = \sqrt{\frac{12}{7}}$

$\therefore$ Length of pipe on the surface should be $\left(4 - \sqrt{\frac{12}{7}}\right) \approx 2.7$ metres in order to minimise costs.

16 a



$g(x) > h(x)$

$\Leftrightarrow \frac{1}{x} > \frac{1}{x^2}$

$\Leftrightarrow x > 1$ (Multiply both sides by x^2. Note $x > 0$)

$\therefore \{x : g(x) > h(x)\} = \{x : x > 1\}$

b $g'(x) = -\frac{1}{x^2}$

$h'(x) = -\frac{2}{x^3}$

$g'(x) > h'(x)$

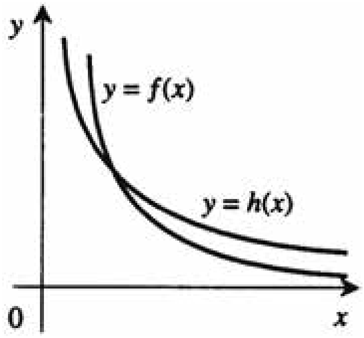
$\Leftrightarrow -\frac{1}{x^2} > -\frac{2}{x^3}$

$\Leftrightarrow -x > -2$ (Multiply both sides by x^3 : Note $x^3 > 0$)

$\therefore\ x < 2$

$\{x : g'(x) > h'(x)\} = \{x : 0 < x < 2\}$

c



$f(x) = \frac{1}{x^3}$

$h(x) = \frac{1}{x^2}$

$h(x) > f(x)$

$\Leftrightarrow \frac{1}{x^2} > \frac{1}{x^3}$

$\Leftrightarrow x > 1$ (Multiply both sides by
x^3 :
Note $x^3 > 0$)

$\therefore\ \{x: h(x) > f(x)\} = \{x: x > 1\}$

$f'(x) = -\frac{3}{x^4},\ h'(x) = -\frac{2}{x^3}$

$h'(x) > f'(x)$

$\Leftrightarrow -\frac{2}{x^3} > \frac{-3}{x^4}$

(Multiply both sides by x^4: Note $x > 0$)

$-2x > -3$

$\therefore\ x < \frac{3}{2}$

$\therefore\ \left\{x: h'(x) > f'(x)\right\} = \left\{(x: 0 < x < \frac{3}{2}\right\}$

d $f_1(x) = \frac{1}{x^n},\ f_2(x) = \frac{1}{x^{n+1}}$

$f_1(x) > f_2(x)$

$\Leftrightarrow \dfrac{1}{x^n} > \dfrac{1}{x^{n+1}}$ (Multiply both sides by x^{n+1}: Note $x^{n+1} > 0$)

$\Leftrightarrow x > 1$

$\therefore\ \{x: f_1(x) > f_2(x)\} = \{x: x > 1\}$

$f_1'(x) = -\dfrac{n}{x^{n+1}} \quad f_2'(x) = -\dfrac{-(n+1)}{x^{n+2}}$

$f_1'(x) > f_2'(x)$

$\Leftrightarrow -\dfrac{n}{x^{n+1}} > -\dfrac{(n+1)}{x^{n+2}}$

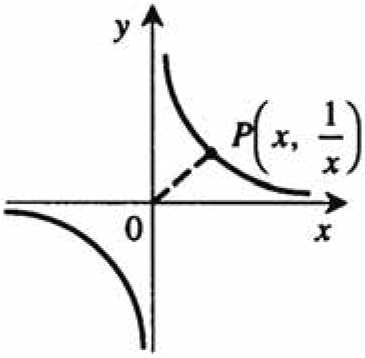
(Multiplying both sides by x^{n+2} : Note $x^{n+2} > 0$)

$\therefore\ -nx > -(n+1)$

and $x < \dfrac{(n+1)}{n}$

$\therefore\ \{x: f_1'(x) > f_2'(x)\} = \left\{x: 0 < x < \dfrac{n+1}{n}\right\}$

17 a



Let $D = OP$

$D^2 = x^2 + \dfrac{1}{x^2}$

It is sufficient to minimise D^2 to minimise D

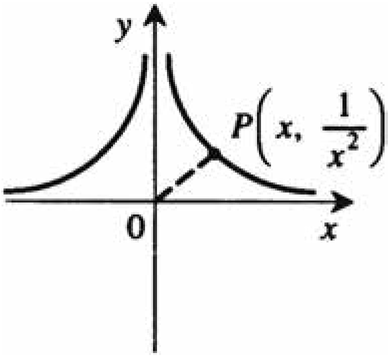
$\therefore\ \dfrac{d(D^2)}{dx} = 2x - \dfrac{2}{x^3}$

$\dfrac{d(D^2)}{dx} = 0$ implies $x^4 = 1$

$\therefore\ x = \pm 1$

A sign diagram confirms a minimum at the points (1, 1) and (−1, −1)

b



Let $D = OP$

then $D^2 = x^2 + \dfrac{1}{x^4}$

$\dfrac{d(D^2)}{dx} = 2x - \dfrac{4}{x^5}$

$\dfrac{d(D^2)}{dx} = x$ implies $x^6 = 2$

i.e. $x = \pm\sqrt[6]{2}$

When $x = \pm\sqrt[6]{2},\ y = \dfrac{1}{\sqrt[3]{2}}$

As before, a sign diagram reveals minimum distance for $P\left(\sqrt[6]{2}, \dfrac{1}{\sqrt[3]{2}}\right)$ and $P\left(-\sqrt[6]{2}, \dfrac{1}{\sqrt[3]{2}}\right)$

c Let $D = OP$

then $D^2 = x^2 + \frac{1}{x^{2n}}$

$$\frac{d(D^2)}{dx} = 2x - \frac{2n}{x^{2n+1}}$$

$$\frac{d(D^2)}{dx} = 0 \text{ implies}$$

$$2x = \frac{2n}{x^{2n+1}}$$

$\therefore \quad 2x^{2n+2} = 2n$

and $x = \pm n^{\frac{1}{2n+2}}$

$\therefore \quad y = n^{-\frac{n}{2n+2}}$

18 **a** Let y m be the width of each window

$\therefore \ 6xy = 36$

$$y = \frac{6}{x}$$

The width of the wall = $8 + 3y$

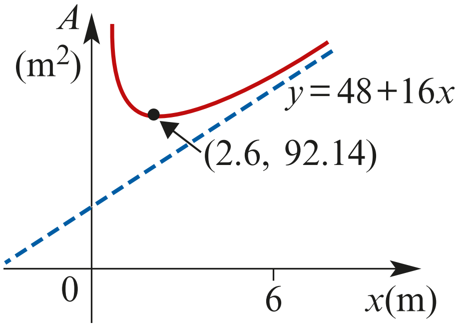
The height of the wall = $6 + 2x$

$\therefore$ Area of the brickwork $A = (8 + 3y)(6 + 2x) - 36$

$$= \left(8 + \frac{18}{x}\right)(6 + 2x) - 36$$

$$= 48 + \frac{108}{x} + 16x$$

b



c In order to find the value of x which will give the minimum amount of brickwork, consider:

$$\frac{dA}{dx} = -\frac{108}{x^2} + 16$$

$$\frac{dA}{dx} = 0 \text{ implies } x^2 = \frac{108}{16}$$

$$\therefore \ x = \pm\frac{3\sqrt{3}}{2}$$

But $x > 0$ $\quad \therefore\; x = \dfrac{3\sqrt{3}}{2} \approx 2.62$

A sign diagram shows local minimum

x	$<$	$\dfrac{3\sqrt{3}}{2}$	$>$
$\dfrac{dA}{dx}$	$-$	0	$+$
shape	\	–	/

When $x = \dfrac{3\sqrt{3}}{2}$, $y = 6 \div \dfrac{3\sqrt{3}}{2} = 6 \times \dfrac{2}{3\sqrt{3}}$

$= \dfrac{4\sqrt{3}}{3} \approx 2.3$

$\therefore$ The dimensions of each window are height $\dfrac{3\sqrt{3}}{2}$ metres and width $\dfrac{4\sqrt{3}}{3}$ metres; the minimum area of brickwork is $48 + 48\sqrt{3} \approx 131.14$ metres.

d $x \geq 1$ and $y \geq 1$

implies $x \geq 1$ and $\dfrac{6}{x} \geq 1$

implies $1 \leq x \leq 6$

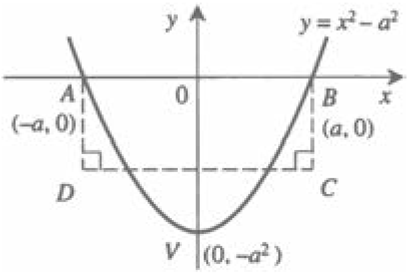
From the graph the maximum will occur at either $x = 1$ or $x = 6$

If $x = 1$, $A = 48 + 108 + 16 = 172$

If $x = 6$, $A = 48 + \dfrac{108}{6} + 16 \times 6 = 162$

$\therefore$ The maximum amount of brickwork which could be used is 172 m^2

19 a



b Area $= -\int_{-a}^{a} x^2 - a^2 dx$

$= -2\int_{0}^{a} x^2 - a^2 dx$

$= -2\left[\dfrac{x^3}{3} - a^2 x\right]_0^a$

$= \dfrac{4a^3}{3}$ square units

See graph above.

Length $AB = 2a$

If the area of rectangle $ABCD$ is $\dfrac{4a^3}{3}$ square units

b $BC = \dfrac{4a^3}{3} \div 2a$

$= \dfrac{4a^3}{3} \times \dfrac{1}{2a} \quad = \dfrac{2a^2}{3}$ units

c $\dfrac{\text{length of } BC}{\text{length of } OV} = \dfrac{2a^2}{3} \div a^2 = \dfrac{2}{3}$, a ratio of 2:3.

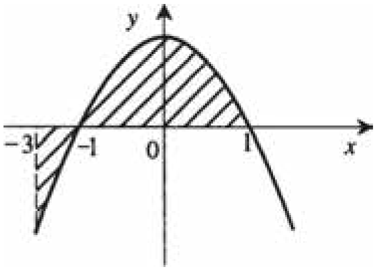
20 a $\int_{-3}^{1}(1-t^2)dt = \left[t - \frac{t^3}{3}\right]_{-3}^{1}$

$= 1 - \frac{1}{3} - \left(-3 - \frac{(-3)^3}{3}\right)$

$= \frac{2}{3} - (-3 + 9)$

$= \frac{2}{3} - (6)$

$= -5\frac{1}{3}$



b $\int_{a}^{1}(1-t^2)dt = 0$

implies $\left[t - \frac{t^3}{3}\right]_{a}^{1} = 0$

$\therefore\ 1 - \frac{1}{3} - \left(a - \frac{a^3}{3}\right) = 0$

$\frac{2}{3} - a + \frac{a^3}{3} = 0$

$\therefore\ a^3 - 3a + 2 = 0$

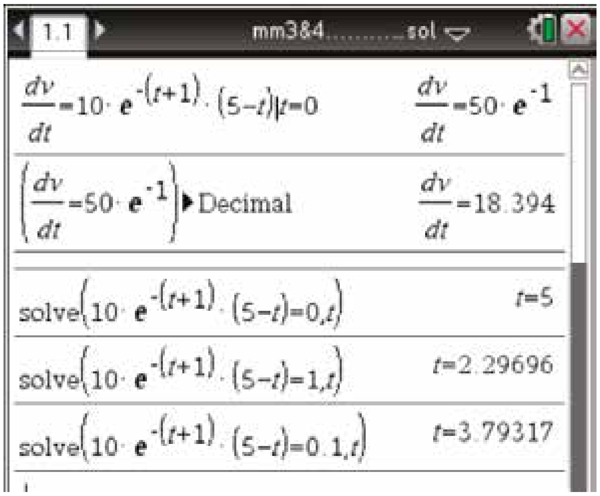
c From b $\int_{a}^{1}(1-t^2)\,dt = 0$ is equivalent to
$a^3 - 3a + 2 = 0$
By the factor theorem $(a-1)$ is a factor
and $a^3 - 3a + 2 = (a-1)(a^2 + a - 2) = (a-1)^2(a+2)$
$\therefore\ \int_{a}^{1}(1-t^2)\,dt = 0$ for $a = 1$ and $a = -2$

21 a **i** When t = 0, $\frac{dV}{dt} = 10e^{-1} \times 5 = 50e^{-1}$ litres/minute

ii $\frac{dV}{dt} = 0$ implies $10e^{-(t+1)}(5-t) = 0$ and therefore t = 5

iii The rate is 1 litre/minute when t = 2 minutes and 18 seconds (to the nearest second)

iv The rate is less than 0.1 litres/minute for the first time when t = 3 minutes and 48 seconds (to the nearest second)



$\frac{dv}{dt}=10\cdot e^{-(t+1)}\cdot(5-t)|t=0$ $\quad \frac{dv}{dt}=50\cdot e^{-1}$

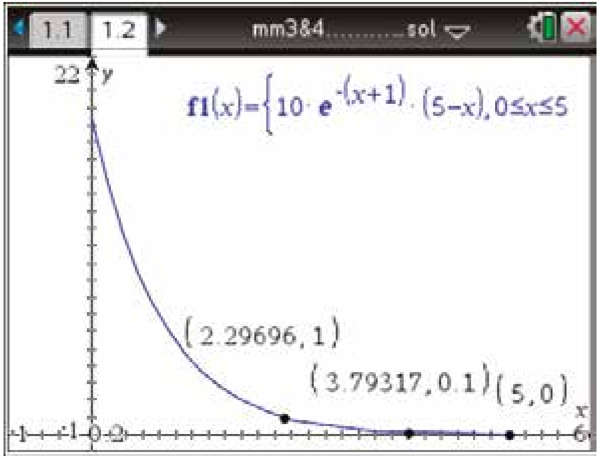
$\left(\frac{dv}{dt}=50\cdot e^{-1}\right)$▸Decimal $\quad \frac{dv}{dt}=18.394$

$\text{solve}\left(10\cdot e^{-(t+1)}\cdot(5-t)=0,t\right)$ $\quad t=5$

$\text{solve}\left(10\cdot e^{-(t+1)}\cdot(5-t)=1,t\right)$ $\quad t=2.29696$

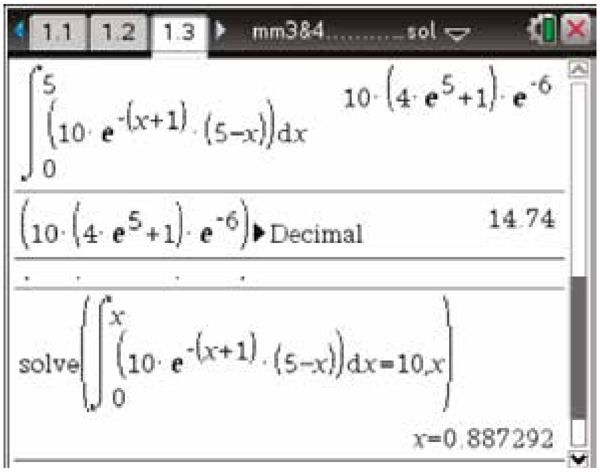
$\text{solve}\left(10\cdot e^{-(t+1)}\cdot(5-t)=0.1,t\right)$ $\quad t=3.79317$

Solving graphically. Enter $f1(x) = 10e^{-(x+1)}(5-1)|0 \le x \le 5$ Use the **Point On** tool (b>**Geometry**>**Points & Lines**) and edit the y-coordinate.



b Use **Integral** from the **Calculus** menu. There are 14.74 litres of water in the tank after 5 minutes.

c The time (to the nearest second) that there is 10 litres in the tank is 53 seconds.



$\int_0^5 \left(10\cdot e^{-(x+1)}\cdot(5-x)\right)dx$ $\quad 10\cdot\left(4\cdot e^5+1\right)\cdot e^{-6}$

$\left(10\cdot\left(4\cdot e^5+1\right)\cdot e^{-6}\right)$ ▶Decimal $\quad 14.74$

$\text{solve}\left(\int_0^x \left(10\cdot e^{-(x+1)}\cdot(5-x)\right)dx=10,x\right)$

$x=0.887292$

22 **a** $A_1 = 5$

b $E_1 = 5 - \dfrac{3}{\ln 2} = 0.6719\ldots$

1 **a** $A_2 = \dfrac{3}{2} + 3 = 4.5$

b $E_2 = 4.5 - \dfrac{3}{\ln 2} = 0.1719\ldots$

2 $A_4 = 4.37132$
$E_4 = 4.37132 - \dfrac{3}{\ln 2} = 0.0432\ldots$
$A_8 = 4.333891$
$E_4 = 4.33891 - \dfrac{3}{\ln 2} = 0.0108\ldots$

23 **a** $f'(x) = 1 - \dfrac{1}{x}, \qquad x > 0$

i $1 - \dfrac{1}{x} < 0$

$\Leftrightarrow \dfrac{1}{x} > 1$

$\Leftrightarrow x < 1$

ii $1 - \dfrac{1}{x} = 0$

$\Leftrightarrow \dfrac{1}{x} = 1$

$\Leftrightarrow x = 1$

iii $1 - \frac{1}{x} > 0$

$\Leftrightarrow \frac{1}{x} < 1$

$\Leftrightarrow x > 1$

Also $f'(x) = 1 - \frac{1}{x} < 1$ for $x > 1$

b Local minimum $(1, 1)$

c $1 - \frac{1}{x} = \frac{1}{n}$

$\Leftrightarrow 1 - \frac{1}{n} = \frac{1}{x}$

$\Leftrightarrow \frac{n-1}{n} = \frac{1}{x}$

$\Leftrightarrow x = \frac{n}{n-1}$

d If the tangent at $P(a, f(a))$ passes through the origin.

$\frac{a - \ln a - 0}{a - 0} = 1 - \frac{1}{a}$

$\Leftrightarrow \frac{a - \ln a}{a} = \frac{a-1}{a}$

$\Leftrightarrow \ln(a) = 1$

$\Leftrightarrow a = e$

e $f(e^{-1}) = e^{-1} + 1$

$f'(e^{-1}) = 1 - e$

Equation of tangent

$y - (e^{-1} + 1) = (1 - e)(x - e^{-1}$

$y - e^{-1} - 1 = (1 - e)x - e^{-1}(1 - e)$

$y - e^{-1} - 1 = (1 - e)x - e^{-1} + 1$

$y = (1 - e)x + 2$

f $f(e^n) = e^n - n$

$f'(e^n) = 1 - e^{-n}$

Equation of tangent

$y - (e^n - n) = (1 - e^{-n})(x - e^n)$

$y - e^n + n = (1 - e^{-n})x - e^n + 1$

$y = (1 - e^{-n})x + 1 - n$

When $x = 0$

$y = 1 - n$

g $y = x\ln x \Rightarrow \dfrac{dy}{dx} = \ln x + 1$

Hence $\displaystyle\int x - \ln x\, dx = \frac{x^2}{2} - x\ln x + x + c$

h $\displaystyle\int_1^e x - \ln x\, dx = \left[\frac{x^2}{2} - x\ln x + x\right]_1^e$

$= \dfrac{e^2}{2} - e + e - (\dfrac{1}{2} + 1)$

$= \dfrac{e^2 - 3}{2}$

24 **a** $f'(x) = 1 + \cos x$

$f''(x) = -\sin x$

b We know $-1 \leq \cos x \leq 1$

Therefore, $0 \leq 1 + \cos x \leq 2$

That is, $0 \leq f'(x) \leq 2$

c $f''(x) = 0 \Leftrightarrow \sin x = 0$

Therefore $x = -4\pi, -3\pi, 2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi$

d Stationary points of inflection occur when $f'(x) = 0$ and $f''(x) = 0$ $f'(x) = 0 \Leftrightarrow x = -3\pi, -\pi, \pi, 3\pi$

Coordinates of the stationary points of inflection are:

$(-3\pi, -3\pi), (-\pi, -\pi), (\pi, \pi)$ and $(3\pi, 3\pi)$

e $g'(x) = \dfrac{1}{2} + \cos x$

$g'(x) = 0 \Leftrightarrow x = -\dfrac{2\pi}{3}, -\dfrac{4\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

f Coordinates of stationary points are:

$\left(-\dfrac{4\pi}{3}, \dfrac{\sqrt{3}}{2} - \dfrac{2\pi}{3}\right)$

$\left(-\dfrac{2\pi}{3}, -\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{3}\right)$

$\left(\dfrac{2\pi}{3}, \dfrac{\sqrt{3}}{2} + \dfrac{\pi}{3}\right)$

$\left(\dfrac{4\pi}{3}, -\dfrac{\sqrt{3}}{2} + \dfrac{2\pi}{3}\right)$

Solutions for algorithms and pseudocode

1 **a**

```
define $f(x)$:
    return $\sin(x)$
$h \leftarrow 0.5$
$a \leftarrow \frac{\pi}{3}$
$m \leftarrow \cos\left(\frac{\pi}{3}\right)$
$i \leftarrow 0$
$f1 \leftarrow 1$
$f2 \leftarrow 1$
print("Known value",$m$)
while (abs($f1 - m$) > 0.001) or (abs($f2 - m$) > 0.001)
    $f1 \leftarrow \frac{f(a+h) - f(a-h)}{2h}$
    $f2 \leftarrow \frac{f(a+h) - f(a)}{h}$
    print($i, h, f1, f2$)
    $i \leftarrow i + 1$
    $h \leftarrow \frac{h}{2}$
end while
```

Note: abs($f1 - m$) > 0.001) means the same as $f1 - m > 0.001$ or $f1 - m < -0.001$

Known value 0.5

i	h	$f1$	$f2$
0	0.5	0.4794 . . .	0.2673 . . .
1	0.25	0.4948 . . .	0.38711 . . .
2	0.125	0.4986 . . .	0.4446. . .
3	0.0625	0.49967 . . .	0.47262 . . .
⋱	⋱	⋱	⋱
8	0.001953125	0.49999968 . . .	0.49915395 . . .

The first approximation reaches the tolerance level at $i = 3$

The second approximation reaches the tolerance level at $i = 8$

b

```
define $f(x)$
    return $\log_e(x)$
$h \leftarrow 0.5$
$m \leftarrow 0.4$
$a \leftarrow 2.5$
$i \leftarrow 0$
$f1 \leftarrow 1$
$f2 \leftarrow 1$
```

```
print("Known exact value",$m$)
while (abs($f1 - m$) > 0.001) or (abs($f2 - m$) > 0.001)
    $f1 \leftarrow \dfrac{f(a+h) - f(a-h)}{2h}$
    $f2 \leftarrow \dfrac{f(a+h) - f(a)}{h}$
    print($i, h, f1, f2$)
    $i \leftarrow i + 1$
    $h \leftarrow \dfrac{h}{2}$
end while
```

Known exact value 0.4

i	h	$f1$	$f2$
0	0.5	0.40546 . . .	0.36464 . . .
1	0.25	00.40134 . . .	0.381240 . . .
2	0.125	0.40033 . . .	0.39032. . .
$\ddots$	$\ddots$	$\ddots$	$\ddots$
6	0.0078125	0.40000 . . .	0.39937 . . .

The first approximation reaches the tolerance level at $i = 2$

The second approximation reaches the tolerance level at $i = 6$

c

```
define $f(x)$
    return $x^4 - \log_e(x)$
$h \leftarrow 0.5$
$m \leftarrow 3$
$a \leftarrow 1$
$i \leftarrow 0$
$f1 \leftarrow 1$
$f2 \leftarrow 1$
print("Known exact value",$m$)
while (abs($f1 - m$) > 0.001) or (abs($f2 - m$) > 0.001)
    $f1 \leftarrow \dfrac{f(a+h) - f(a-h)}{2h}$
    $f2 \leftarrow \dfrac{f(a+h) - f(a)}{h}$
    print($i, h, f1, f2$)
    $i \leftarrow i + 1$
    $h \leftarrow \dfrac{h}{2}$
end while
```

Known exact value 3

i	h	$f1$	$f2$
0	0.5	3.90138 . . .	7.31406 . . .
1	0.25	03.22834 . . .	4.8730 . . .
2	0.125	3.0572 . . .	3.8721. . .
3	0.0625	3.014319 . . .	3.4208 . . .
4	0.03125	3.00358 . . .	3.20674 . . .
5	0.015625	3.00089 . . .	3.10246 . . .
⋱	⋱	⋱	⋱
12	0.0001220703125	3.00000 . . .	3.00079 . . .

The first approximation reaches the tolerance level at $i = 5$

The second approximation reaches the tolerance level at $i = 12$

2 **a** We start by noting that you may prefer to start your output with the initial value. This can be done by bringing the print statement up to the top.

```
define $f(x)$
  return $-x^3 + 5x^2 - 3x + 4$
define $Df(x)$
  return $-3x^2 + 10x - 3$
$x \leftarrow 3.8$
$n = 0$
while $(f(x) > 10^{-10}$ or $f(x) < -10^{-10})$
    print $(n, x, f(x))$
    $n \leftarrow n + 1$
    $x = x - f(x)/(Df(x))$
end while
```

The code for **i -vi** is exactly the same with the function and its derivative being replaced.

b **i** $f(x) = \sin(2x) - x$

$f'(x) = 2\cos(2x) - 1$

$$\begin{aligned} x_{n+1} &= x_n - \frac{\sin(2x_n) - x_n}{2\cos(2x_n) - 1} \\ &= \frac{x_n(2\cos(2x_n) - 1) - (\sin(2x_n) - x_n)}{2\cos(2x_n) - 1} \\ &= \frac{2x_n\cos(2x_n) - \sin(2x_n)}{2\cos(2x_n) - 1} \end{aligned}$$

ii $f(x) = \cos(2x) - x$

$f'(x) = -2\sin(2x) - 1$

$$\begin{aligned} x_{n+1} &= x_n - \frac{\cos(2x_n) - x_n}{-2\sin(2x_n) - 1} \\ &= \frac{x_n(-2\sin(2x_n) - 1) - (\cos(2x_n) - x_n)}{-2\sin(2x_n) - 1} \\ &= \frac{-2x_n \sin(2x_n) - \cos(2x_n)}{-2\sin(2x_n) - 1} \\ &= \frac{2x_n \sin(2x_n) + \cos(2x_n)}{2\sin(2x_n) + 1} \end{aligned}$$

iii $f(x) = \log_e(x) - 0.25x$

$f'(x) = \frac{1}{x} - 0.25$

$$\begin{aligned} x_{n+1} &= x_n - \frac{\log_e(x_n) - 0.25x_n}{\frac{1}{x_n} - 0.25} \\ &= \frac{x_n\left(\frac{1}{x_n} - 0.25\right) - (\log_e(x_n) - 0.25x_n)}{\frac{1}{x_n} - 0.25} \\ &= \frac{1 - \log_e(x_n)}{\frac{1}{x_n} - 0.25} \\ &= \frac{x_n - x_n \log_e(x_n)}{1 - 0.25x_n} \end{aligned}$$

iv $f(x) = e^x - \log_e x - 3$

$f'(x) = e^x - \frac{1}{x}$

$$\begin{aligned} x_{n+1} &= x_n - \frac{e^{x_n} - \log_e x_n - 3}{e^{x_n} - \frac{1}{x_n}} \\ &= \frac{e^{x_n} x_n - 1 - (e^{x_n} - \log_e x_n - 3)}{e^{x_n} - \frac{1}{x_n}} \\ &= \frac{e^{x_n} x_n - e^{x_n} + \log_e x_n + 2}{e^{x_n} - \frac{1}{x_n}} \end{aligned}$$

v $f(x) = \sin x - \log_e x - 3$

$f'(x) = \cos x - \frac{1}{x}$

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{\sin x_n - \log_e x_n - 3}{\cos x_n - \dfrac{1}{x_n}} \\
&= \frac{x_n(\cos x_n - \dfrac{1}{x_n}) - (\sin x_n - \log_e x_n - 3)}{\cos x_n - \dfrac{1}{x_n}} \\
&= \frac{x_n \cos x_n - 1 - \sin x_n + \log_e x_n + 3}{\cos x_n - \dfrac{1}{x_n}} \\
&= \frac{x_n \cos x_n - \sin x_n + \log_e x_n + 2}{\cos x_n - \dfrac{1}{x_n}}
\end{aligned}
$$

vi $f(x) = (x-2)^2 - \log_e x$

$f'(x) = 2(x-2) - \dfrac{1}{x}$

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{(x_n - 2)^2 - \log_e x_n}{2(x_n - 2) - \dfrac{1}{x_n}} \\
&= \frac{x_n(2(x_n - 2) - \dfrac{1}{x_n}) - ((x_n - 2)^2 - \log_e x_n}{2(x_n - 2) - \dfrac{1}{x_n}} \\
&= \frac{2x_n^2 - 4x_n - 1 - (x_n - 2)^2 + \log_e x_n}{2(x_n - 2) - \dfrac{1}{x_n}} \\
&= \frac{2x_n^2 - 4x_n - 1 - (x_n^2 - 4x_n + 4) + \log_e x_n}{2(x_n - 2) - \dfrac{1}{x_n}} \\
&= \frac{x_n^2 - 5 + \log_e x_n}{2(x_n - 2) - \dfrac{1}{x_n}}
\end{aligned}
$$

3 a

$x_0 =$	1.5
$x_1 =$	1.8695652 . . .
$x_2 =$	1.7994524 . . .
$x_3 =$	1.7963279 . . .

b

$x_0 =$	2
$x_1 =$	1.967741935 . . .
$x_2 =$	1.966903756 . . .
$x_3 =$	1.9669032026 . . .

c

$x_0 =$	0.5
$x_1 =$	0.636363636 . . .
$x_2 =$	0.618381618 . . .
$x_3 =$	0.618034117 . . .

d

$x_0 =$	0.6
$x_1 =$	−0.61260621626 . . .
$x_2 =$	−1.21530959125 . . .
$x_3 =$	−1.2780177811 . . .

4

```
define $f(x)$:
  return $\sin x - \frac{x}{4}$
define $Df(x)$:
  return $\cos x - \frac{1}{4}$
define $Df(x)$:
  return $-\sin x$
$x \leftarrow 3$
while $f(x) > 10^{-6}$ or $f(x) < -10^{-6}$
    $x \leftarrow x - \frac{2f(x) \times Df(x)}{2(Df(x))^2 - 2f(x) \times DDf(x)}$
    print $x, f(x)$
end while
```

5 **a** 47.5425. The actual value is 47.25

b **i** Change the first line in the while loop to: $strip \leftarrow f(left) \times h$

ii Change the first line in the while loop to: $strip \leftarrow f(right) \times h$

c Define $f(x) = 2^x$
$a \leftarrow 0$
$b \leftarrow 2$
$n \leftarrow 100$
and then as in the given code.

Chapter 13 – Discrete random Variables and their probability distribution

Solutions to Exercise 13A

1

1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T

2 HH1, HH2, HH3, HH4, HH5, HH6,
HT1, HT2, HT3, HT4, HT5, HT6,
TH1, TH2, TH3, TH4, TH5, TH6,
TT1, TT2, TT3, TT4, TT5, TT6

3 **a** $\frac{4}{52} = \frac{1}{13}$

b $\frac{3}{4}$

c $\frac{16}{52} = \frac{4}{13}$

d $\frac{8}{52} = \frac{2}{13}$

4 **a** $\frac{3}{6} = \frac{1}{2}$

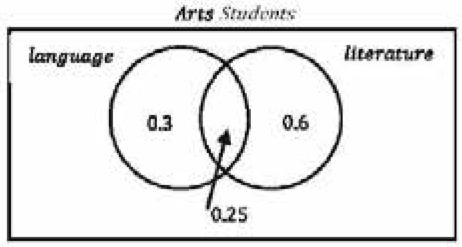
b $\frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}$

5 $\Pr(S \cup L) = \Pr(S) + \Pr(L) - \Pr(S \cap L)$

$= 0.7 + 0.6 - 0.5$

$= 0.8$

6



$\Pr(\text{Lit} \cup \text{Lan})$

$= \Pr(\text{Lit}) + \Pr(\text{Lan}) - \Pr(\text{Lan} \cap \text{Lit})$

$= 0.3 + 0.6 - 0.25$

$= 0.65$

7 **a** $0.05 + 0.02 - 0.003 = 0.067$

b $0.05 - 0.003 = 0.047$

8 $1 - 0.75 - 0.12 - 0.08 = 0.05 = 5\%$

9 let $\Pr(A)$ be the probability that an adult owns a car & $\Pr(B)$ be the probability that an adult is employed

$\Pr(A) = 0.7,\ \Pr(B) = 0.6$

$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{0.6}{0.7} = \frac{6}{7}$

10 **a** $\frac{17}{500}$

b $\frac{18}{500} = \frac{9}{250}$

c $\frac{30 + 45 + 33 + 39 + 17}{500}$

$= \frac{164}{500} = \frac{41}{125}$

d $\frac{10 + 17 + 2 + 1 + 11}{500} = \frac{41}{500}$

11 **a** $\Pr(\text{guns}) = \frac{130}{200} = \frac{13}{20}$

b $\Pr(\text{guns} \cap \text{male}) = \frac{70}{200} = \frac{7}{20}$

12 **a** Pr(head) $\approx \frac{114}{200} = \frac{57}{100}$

b Pr(ten) $\approx \frac{40}{380} = \frac{2}{19}$

c Pr(2 heads) $\approx \frac{54}{200} = \frac{27}{100}$

d Pr(3 sixes)$\approx \frac{2}{500} = \frac{1}{250}$

13 $$\begin{aligned}\text{Pr(White)} &= \frac{\text{Area of white}}{\text{Total area}}\\ &= \frac{30^2}{50^2}\\ &= \frac{900}{2500}\\ &= \frac{9}{25}\end{aligned}$$

14 **a** $$\begin{aligned}\text{Pr(Green)} &= \frac{\text{Area of green}}{\text{Total area}}\\ &= \frac{\frac{1}{2}\pi r^2}{\pi r^2}\\ &= \frac{1}{2}\end{aligned}$$

b $$\begin{aligned}\text{Pr(Yellow)} &= \frac{\text{Area of yellow}}{\text{Total area}}\\ &= \frac{\frac{1}{6}\pi r^2}{\pi r^2}\\ &= \frac{1}{6}\end{aligned}$$

c $$\begin{aligned}\text{Pr(Not Yellow)} &= 1 - \text{Pr}(Yellow)\\ &= \frac{5}{6}\end{aligned}$$

15

	C	C'	
T	0.32	0.13	0.45
T'	0.33	0.22	0.55
	0.65	0.35	

a $\Pr(T \cap C') = 0.13$

b $\Pr(T \cap C) = 0.32$

16

	S	S'	
D	0.25	0.15	0.40
D'	0.42	0.18	0.60
	0.67	0.33	

a $\Pr(D) = 0.4$

b $\Pr(S) = 0.67$

c $\Pr(D' \cap S') = 0.18$

17

	A	A'	
S	0.53	0.12	0.65
S'	0.18	0.17	0.35
	0.71	0.29	

a $\Pr(S') = 0.35$

b $\Pr(A \cap S'S) = 0.18$

c $\Pr(A' \cap S) = 0.12$

d $\Pr(A' \cap S') = 0.17$

18 **a** $\Pr(A) = 0.42$,
$\Pr(B) = 0.76$,

$\Pr(A \cup B) = 0.82$
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$0.82 = 0.42 + 0.76 - \Pr(A \cap B)$
$\Pr(A \cap B) = 0.36$

b $\Pr(A \cap B) + \Pr(A \cap B') = \Pr(A)$
$\Pr(A \cap B') = \Pr(A) - \Pr(A \cap B)$
$\Pr(A \cap B') = 0.42 - 0.36 = 0.06$

Solutions to Exercise 13B

1 **a** $\Pr(RR) = 0.25 \times 0.8 = 0.2$

b $\Pr(R'R') = 0.75 \times 0.9 = 0.675$

c $\Pr(R \text{ Sunday})$
$= \Pr(RR) + \Pr(R'R')$
$= 0.2 + 0.075 = 0.275$

2 **a** $\Pr(B|A) = \dfrac{\Pr(B \cap A)}{\Pr(A)}$
$= \dfrac{1}{6}$

b $\Pr(A|B) = \dfrac{\Pr(B \cap A)}{\Pr(B)}$
$= \dfrac{1}{3}$

3 **a** $\Pr(A \cap B) = \Pr(B|A)\Pr(A)$
$= 0.1 \times 0.6 = 0.06$

b $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{1}{5}$

4 $\Pr(C|F) = \dfrac{\Pr(C \cap F)}{Pr(F)} = \dfrac{0.3}{0.5} = \dfrac{3}{5}$

5 Let H be the event poor harvest.
Let D be the event disease.

$$\Pr(D|H) = \frac{\Pr(D \cap H)}{\Pr(H)}$$
$$= \frac{\Pr(H|D)\Pr(D)}{\Pr(H|D')\Pr(D') + \Pr(H|D)\Pr(D)}$$
$$= \frac{0.8 \times 0.3}{0.8 \times 0.3 + 0.5 \times 0.7}$$
$$= \frac{24}{59}$$

6 **a** $\dfrac{500}{1000} = \dfrac{1}{2}$

b $\dfrac{385}{1000} = \dfrac{77}{200}$

c $\dfrac{200}{385} = \dfrac{40}{77}$

d $\dfrac{200}{500} = \dfrac{2}{5}$

7 **a** $\Pr(S) = \dfrac{\text{total speed}}{\text{total}}$
$= \dfrac{130}{448} = \dfrac{65}{224}$

b $\Pr(F) = \dfrac{\text{total fatal}}{\text{total}} = \dfrac{115}{448}$

c Look only at the Speed column:
$\Pr(F|S) = \dfrac{42}{130} = \dfrac{21}{65}$

d Look only at the Alcohol column:
$\Pr(F|A) = \dfrac{61}{246}$

8 **a** $\Pr(J \cap S) = 0.8 \times 0.3 = 0.24$

b $Pr(J \cup S) = 0.8 + 0.3 - 0.24 = 0.86$

9 $\Pr(A) = 0.6, \Pr(B) = 0.5, \Pr(C) = 0.4$

a $\Pr(A) \times \Pr(B) = 0.6 \times 0.5 = 0.3$
$A \cap B = \{1, 3, 5\}$ $\quad \Pr(A \cap B) = 0.3$
$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
$\therefore$ A and B are independent

b $Pr(A) \times Pr(C) = 0.6 \times 0.4 = 0.24$
$A \cap C = \{2, 6\}$ $\quad \Pr(A \cap C) = 0.2$
$\Pr(A \cap C) \neq \Pr(A) \times \Pr(C)$
$\therefore$ A and C are not independent

c $Pr(B) \times Pr(C) = 0.5 \times 0.4 = 0.2$
$B \cap C = \{9\}$ $\quad \Pr(B \cap C) = 0.1$

$\Pr(B \cap C) \neq \Pr(B) \times \Pr(C)$
$\therefore$ B and C are not independent

10 $\Pr(A) = 0.5, \Pr(B) = 0.4$

a $$Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$
$$= \frac{\Pr(A) \times \Pr(B)}{\Pr(B)}$$
$$= \Pr(A)$$
$$= 0.5$$

b $\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.2$

c
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 0.5 + 0.4 - 0.2$$
$$= 0.7$$

11 $0.3 \times 0.7 + 0.6 \times 0.3 = 1.3 \times 0.3 = 0.39$

12 $$\Pr(A \cap B) = \Pr(A)\Pr(B)$$
$$0.1452 = 3[\Pr(A)]^2$$
$$0.0484 = [\Pr(A)]^2$$
$$\Pr(A) = 0.22$$

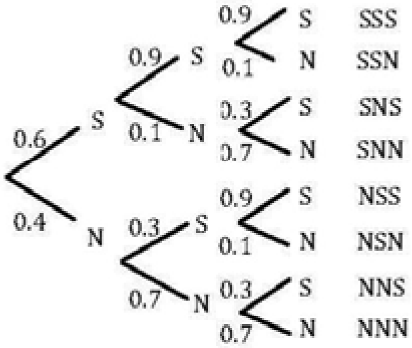
13 $\Pr(A) = \frac{1}{2}\Pr(B)$
$\therefore \Pr(B) = 2\Pr(A)$
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$0.28 = 3\Pr(A) - 2[\Pr(A)]^2$$
$$2[\Pr(A)]^2 - 3\Pr(A) + 0.28 = 0$$
$$(2P(A) - 2.8)(\Pr(A) - 0.1) = 0$$
$$\therefore \Pr(A) = 0.1$$

14 $$\frac{\Pr(HHH)}{1 - \Pr(TTT)} = \frac{\frac{1}{8}}{\frac{7}{8}} = \frac{1}{7}$$

15 $(1, 1) \ldots (1, 6), (3, 1) \ldots (3, 6),$
$(5, 1) \ldots (1, 6).$
There are 18 outcomes to consider.
$(5, 3)$ and $(3, 5)$ satisfy the property.
Therefore required probability is
$\frac{2}{18} = \frac{1}{9}$

16 $0.03 \times 0.95 + 0.97 \times 0.02 = 0.0479$

17 $S = stop \quad N = no\ stop$



a $\Pr(SSS) = 0.6 \times 0.9 \times 0.9 = 0.486$

b $\Pr(NSN) = 0.4 \times 0.3 \times 0.1 = 0.012$

c $\Pr(SNN) = 0.6 \times 0.1 \times 0.7 = 0.042$

$\Pr(NSN) = 0.012$

$\Pr(NNS) = 0.4 \times 0.7 \times 0.3 = 0.084$
$\Pr(SNN) + \Pr(NSN) + \Pr(NNS) =$
0.138

18 **a** $\frac{4}{6} \times \frac{3}{5} = \frac{2}{5}$

b $\frac{2}{6} \times \frac{1}{5} = \frac{1}{15}$

c $\frac{4}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{4}{5} = \frac{16}{30} = \frac{8}{15}$

19 **a** $\frac{160}{400} = \frac{2}{5}$

b $\frac{70}{400} = \frac{7}{40}$

c $\frac{\frac{7}{40}}{\frac{2}{5}} = \frac{7}{16}$

d $\frac{\frac{7}{40}}{\frac{150}{400}} = \frac{7}{15}$

20 **a** $\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{2}{7} = \frac{5}{14}$

b $\frac{\frac{3}{14}}{\frac{5}{14}} = \frac{3}{5}$

21 $\frac{3}{12} \times \frac{2}{11} \times \frac{1}{10} + \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$

$+ \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10}$

$= \frac{1}{220} + \frac{1}{55} + \frac{1}{22}$

$= \frac{1}{220} + \frac{4}{220} + \frac{10}{220}$

$= \frac{15}{220}$

$= \frac{3}{44}$

22 **a** $0.3 \times 0.75 + 0.6 \times 0.8 + 0.1 \times 0.3$

$= 0.735$

b $\frac{0.6 \times 0.2}{1 - 0.735} = \frac{0.12}{0.265} = \frac{24}{53}$

Solutions to Exercise 13C

1 **a** discrete

b continuous

c discrete

d discrete

2 **a** continuous

b discrete

c continuous

d discrete

3 **a** $\{HHH, HHT, HTH, THH,$
$HTT, THT, TTH, TTT\}$

b $X = 0, \{TTT\}$
$X = 1, \{HTT, THT, TTH\}$
$X = 2, \{HHT, HTH, THH\}$
$X = 3, \{HHH\}$

c $\Pr(X \geq 2) = \dfrac{4}{8} = \dfrac{1}{2}$

4 **a** Yes, since $p(x) \geq 0$ for all x, and $\Sigma p(x) = 1$

b $\Pr(X \leq 3) = 0.1 + 0.2 + 0.1 + 0.4 = 0.8$

5 **a** $\Pr(X = 3) = \Pr(RRR) = 4/9 \times 4/9 \times 4/9 = 64/729$
$\Pr(X = 2) = \Pr(RRB) + \Pr(RBR) + \Pr(BRR) = 3 \times 4/9 \times 4/9 \times 5/9 = 240/729$
$\Pr(X = 1) = \Pr(RBB) + \Pr(BBR) + \Pr(BRB) = 3 \times 4/9 \times 5/9 \times 5/9 = 300/729$
$\Pr(X = 0) = \Pr(BBB) = 5/9 \times 5/9 \times 5/9 = 125/729$

b $\Pr(X \geq 1) = 1 - \Pr(X = 0) = \dfrac{604}{729}$

c $\Pr(X > 1) = 1 - \Pr(X = 0) - \Pr(X = 1) = \dfrac{304}{729}$

6 **a** $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b die 2

die 1

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

c **i** $\Pr(Y < 5) = \dfrac{6}{36} = \dfrac{1}{6}$

ii $\Pr(Y = 3|Y < 5) = \dfrac{2}{6} = \dfrac{1}{3}$

iii $\Pr(Y \le 3|Y < 7) = \Pr(Y \le 3)/\Pr(Y < 7) = (3/36)/(15/36) = 3/15 = 1/5$

iv $\Pr(Y \ge 7|Y > 4) = \Pr(Y \ge 7)/\Pr(Y > 4) = (21/36)/(30/36) = 21/30 = 7/10$

v $\Pr(Y = 7|Y > 4) = \Pr(Y = 7)/\Pr(Y > 4) = (6/36)/(30/36) = 6/30 = 1/5$

vi $\Pr(Y = 7|Y < 8) = \Pr(Y = 7)/\Pr(Y < 8) = (6/36)/(21/36) = 6/21 = 2/7$

7 **a**

die 1

die 2

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

b $Y = 1, 2, 3, 4, 5, 6$

c
$$\begin{aligned}\Pr(Y = 1) &= 0.1 + 0.1 - 0.1 \times 0.1\\ &= 0.19\end{aligned}$$

8 **a**
$$\begin{aligned}\Pr(X = 2) &= \Pr(WWB) + \Pr(WBW)\\ &\quad + \Pr(BWW)\end{aligned}$$

where B means 'black ball drawn' and W means 'which ball drawn'.

$$\begin{aligned}\Pr(X = 2) &= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} \\ &\quad + \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} \\ &= 3 \times \frac{12}{125} \\ &= \frac{36}{125} = 0.288\end{aligned}$$

b $\Pr(X = 3) = \Pr(WWW)$

$$= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$$

$$= \frac{8}{125} = 0.064$$

c $\Pr(X \geq 2) = 0.288 + 0.064 = 0.352.$

d $\Pr(X = 3|X \geq 2) = \dfrac{\Pr(X = 3)}{\Pr(X \geq 2)}$

$$= \frac{0.064}{0.352} = \frac{2}{11}$$

$$\approx 0.182$$

9 **a** $\{(1, 1), (1, 2), (1, 3), \ldots, (6, 4), (6, 5), (6, 6)\}$

b $\Pr(A) = \dfrac{1}{6}$

$\Pr(B) = \dfrac{1}{6}$

$\Pr(C) = \dfrac{15}{36}$

(counting possibilities)

$= \dfrac{5}{12}$

$\Pr(D) = \dfrac{6}{36}$

(counting possibilities)

$= \dfrac{1}{6}$

c $\Pr(A \cap B) = \frac{1}{36}$

(counting possibilities)

$$\Pr(A|B) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

(counting possibilities)

$$\Pr(A \cap C) = \frac{3}{36}$$

$$= \frac{1}{12}$$

$$\Pr(A|C) = \frac{\frac{1}{12}}{\frac{5}{12}} = \frac{1}{5}$$

$$\Pr(A \cap D) = \frac{1}{36}$$

(counting possibilities)

$$\Pr(A|D) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

d A & B, A & D since
$\Pr(A|B) = \Pr(A)$&

$\Pr(A|D) = \Pr(A)$

10 **a** Yes, since $p(x) \geq 0$ for all x, and $\Sigma p(x) = 1$

b $\Pr(X \geq 2) = 0.2 + 0.3 = 0.5$

11 **a**, since the sum of the $p(x)$ values > 1; and **c** negative probabilities values is 0.

12 Let x be the number of black balls in the sample.

a $\Pr(X = 0) = \left(\frac{6}{10}\right)^3 = \left(\frac{27}{125}\right)$

$\Pr(X = 3) = \left(\frac{4}{10}\right)^3 = \frac{8}{125}$

$$\begin{aligned}\Pr(X = 1) &= \frac{4}{10} \times \frac{6}{10} \times \frac{6}{10} + \frac{6}{10} \\ &\quad \times \frac{4}{10} \times \frac{6}{10} + \frac{6}{10} \times \frac{6}{10} \\ &= 3 \times \frac{18}{125} \\ &= \frac{54}{125}\end{aligned}$$

$$\begin{aligned}\Pr(X = 2) &= 1 - \frac{27}{125} - \frac{8}{125} - \frac{54}{125} \\ &= \frac{36}{125}\end{aligned}$$

x	0	1	2	3
$\Pr(X = x)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

b $\Pr(X = 0) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$

$\Pr(X = 3) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$

$$\begin{aligned}\Pr(X = 1) &= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} + \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} \\ &\quad + \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} \\ &= 3 \times \frac{1}{6} = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\Pr(X = 2) &= \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} + \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} \\ &\quad + \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} \\ &= 3 \times \frac{1}{10} = \frac{3}{10}\end{aligned}$$

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

13 $\Pr(X = 0) = 0.6^2 = 0.36$
$\Pr(X = 2) = 0.4^2 = 0.16$

$\Pr(X = 1) = 1 - 0.16 - 0.36 = 0.48$

x	0	1	2
$\Pr(X = x)$	0.36	0.48	0.16

14 **a**

x	1	2	3	4	5
$\Pr(X = x)$	0.2	0.2	0.2	0.2	0.2

b $\Pr(X \geq 3) = 0.2 \times 3 = 0.6$

c $\Pr(X \leq 3 | X \geq 3) = \dfrac{0.2}{0.6} = \dfrac{1}{3}$

15 **a** $\{(1,1), (1,2), (1,3), \ldots, (6,4), (6,5), (6,6)\}$

b

die 1

die 2

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$\Pr(X = 2) = \frac{1}{36}$

$\Pr(X = 3) = \frac{2}{36} = \frac{1}{18}$

$\Pr(X = 4) = \frac{3}{36} = \frac{1}{12}$

$\Pr(X = 5) = \frac{4}{36} = \frac{1}{9}$

$\Pr(X = 6) = \frac{5}{36}$

$\Pr(X = 7) = \frac{6}{36} = \frac{1}{6}$

$\Pr(X = 8) = \frac{5}{36}$

$\Pr(X = 9) = \frac{4}{36} = \frac{1}{9}$

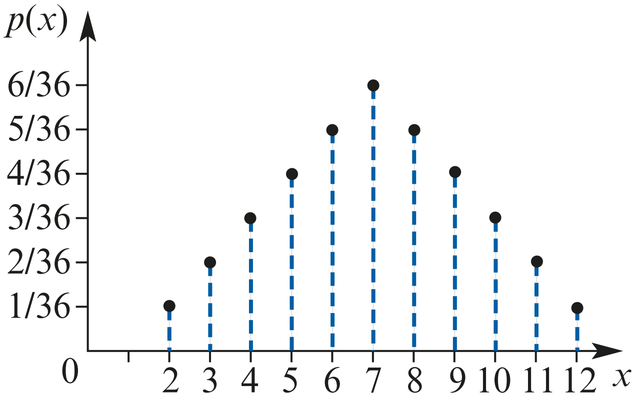
$\Pr(X = 10) = \frac{3}{36} = \frac{1}{12}$

$\Pr(X = 11) = \frac{2}{36} = \frac{1}{18}$

$\Pr(X = 12) = \frac{1}{36}$

x	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

c



d $\Pr(X > 9) = \frac{4+3+2+1}{36} = \frac{10}{36}$

$= \frac{5}{18}$

e $\Pr(X \le 10 | X \ge 9) = \dfrac{\frac{7}{36}}{\frac{10}{36}} = \dfrac{7}{10}$

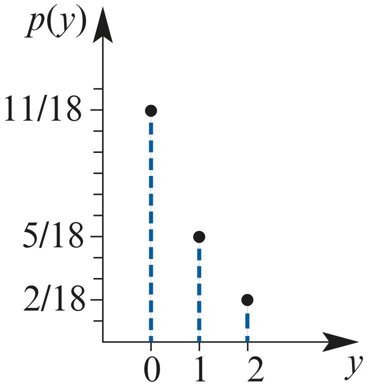
16 **a** $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b

		dice 1					
		1	2	3	4	5	6
die 2	1	0	0	0	0	0	0
	2	0	0	1	0	1	0
	3	0	1	0	1	2	0
	4	0	0	1	0	1	2
	5	0	1	2	1	0	1
	6	0	0	0	2	1	0

y	0	1	2
$\Pr(Y = y)$	$\frac{22}{36}$	$\frac{10}{36}$	$\frac{4}{36}$

c



17 **a** $\Pr(X = 0) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$

$\Pr(X = 2) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$

$\Pr(X = 1) = 1 - \frac{1}{3} - \frac{2}{15} = \frac{8}{15}$

x	0	1	2
$\Pr(X = x)$	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

b $\Pr(X \ne 1) = \frac{7}{15}$

18 centre circle = $\pi(2)^2 = 4\pi$
middle circle = $\pi(10)^2 - \pi(2)^2 = 96\pi$
outer circle = $\pi(20)^2 - \pi(10)^2 = 300\pi$

a $\Pr(X = 100) = \dfrac{4}{400} = \dfrac{1}{100}$

$\Pr(X = 20) = \dfrac{96}{400} = \dfrac{6}{25}$

$\Pr(X = 10) = \dfrac{300}{400} = \dfrac{3}{4}$

x	10	20	100
$\Pr(X = x)$	$\frac{3}{4}$	$\frac{6}{25}$	$\frac{1}{100}$

b $\Pr(Y = 200) = \dfrac{1}{100} \times \dfrac{1}{100} = \dfrac{1}{10\,000}$

$\Pr(Y = 120) = \dfrac{1}{100} \times \dfrac{6}{25} \times 2 = \dfrac{3}{625}$

$\Pr(Y = 110) = \dfrac{3}{100} \times \dfrac{3}{4} \times 2 = \dfrac{3}{200}$

$\Pr(Y = 40) = \dfrac{6}{25} \times \dfrac{6}{25} = \dfrac{36}{625}$

$\Pr(Y = 30) = \dfrac{6}{25} \times \dfrac{3}{4} \times 2 = \dfrac{9}{25}$

$\Pr(Y = 20) = \dfrac{3}{4} \times \dfrac{3}{4} = \dfrac{9}{16}$

y	20	30	40	110	120	200
$\Pr(Y = y)$	$\frac{9}{16}$	$\frac{9}{25}$	$\frac{36}{625}$	$\frac{3}{200}$	$\frac{3}{625}$	$\frac{1}{10\,000}$

19 **a** $\Pr(X = 3) = \Pr(EEE) + \Pr(NNN)$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

b $x = 4$,
{NEEE, ENEE, EENE, ENNN, NENN, NNEN}
$\Pr(X = 4) = 6 \times \dfrac{1}{16} = \dfrac{3}{8}$

c $\Pr(X = 5) = 1 - \Pr(x \neq 5)$

$$= 1 - \frac{2}{8} - \frac{3}{8}$$

$$= \frac{3}{8}$$

20 $\Pr(H_A) = 0.6$ and $\Pr(H_B) = 0.4$

Sample space: $(H_A, H_B), (H_A, T_B), (T_A, H_B), (T_A, T_B)$

Scores: $(H_A, H_B) : -2, (H_A, T_B) : 1, (T_A, H_B) : 1, (T_A, T_B) : 4$

$\Pr(H_A, H_B) = 0.24$ and $\Pr(T_A, T_B) = 0.24$

Probability of a head an a tail is 0.52.

x	-2	1	4
$p(x)$	0.24	0.52	0.24

Solutions to Exercise 13D

1 There is:

30% chance of winning \$0 (\$2 prize less the \$2 cost to play)
10% chance of winning \$18 (\$20 prize less the \$2 cost to play)
60% chance of losing \$2 (the cost to play)
Therefore the expected win/loss per game $= -2 \times 0.6 + 0 \times 0.3 + 18 \times 0.1 = 0.6$ dollars
In 100 games the expected win/loss $= 100 \times 0.6 = 60$ dollars

2 **a** Mean $= 1 \times 0.1 + 3 \times 0.3 + 5 \times 0.3$

$$+ 7 \times 0.3$$

$$= 0.1 + 0.9 + 1.5 + 2.1$$

$$= 4.6$$

b Mean $= 0.25 \times -1 + 0.25 \times 0 + 0.25$

$$\times 1 + 0.25 \times 2$$

$$= 0.5$$

c Mean $= 0 \times 0.09 + 1 \times 0.22 + 2 \times 0.26$

$$+ 3 \times 0.21 + 4 \times 0.13 + 5 \times 0.06$$

$$+ 6 \times 0.02 + 7 \times 0.01$$

$$= 0.22 + 0.52 + 0.63 + 0.52 + 0.30$$

$$+ 0.12 + 0.07$$

$$= 2.38$$

d Mean $= 0.2 \times 0.08 + 0.3 \times 0.13$

$$+ 0.4 \times 0.09 + 0.5 \times 0.19$$

$$+ 0.6 \times 0.7 + 0.7 \times 0.03$$

$$+ 0.8 \times 0.10 + 0.9 \times 0.18$$

$$= 0.569$$

e Mean=7

f Mean=0

3 $\mu = \$10,000 \times 0.13 + \$5,000 \times 0.45$

$$+ \$0 \times 0.25 - \$5,000 \times 0.15$$

$$= \$1,500 + \$2,250 - \$750$$

$$= \$3,000$$

4 *assuming a payout (as opposed to a profit) of* \$5 *for a win,*

$$\mu = \frac{5}{6} \times -\$1 + \frac{1}{6} \times \$4$$

$$= -\$\frac{1}{6} = -\$0.17, \text{ i.e. a loss of 17c.}$$

5 $\mu = 0 \times 0.12 + 1 \times 0.36 + 2 \times 0.38 + 3 \times 0.14$

$$= 0.36 + 0.76 + 0.42$$

$$= 1.54$$

6

x	1	2	3	4	5	7	8	9	10	11	12
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

$$\mu = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6}$$

$$+ 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 7 \times \frac{1}{36}$$

$$+ 8 \times \frac{1}{36} + 9 \times \frac{1}{36} + 10 \times \frac{1}{36}$$

$$+ 11 \times \frac{1}{36} + 12 \times \frac{1}{36}$$

$$= \frac{\begin{array}{c}6 + 12 + 18 + 24 + 30 + 7 + 8 \\ + 9 + 10 + 11 + 12\end{array}}{36}$$

$$= \frac{147}{36}$$

$$= \frac{49}{12}$$

7 **a** $E(X) = 2 \times 0.01 + 3 \times 0.25 + 4 \times 0.40$

$$+ 5 \times 0.30 + 6 \times 0.04$$

$$= 0.02 + 0.75 + 1.60 + 1.50 + 0.24$$

$$= 4.11$$

b $$\begin{aligned}\mathrm{E}(X^3) &= 8 \times 0.01 + 27 \times 0.25 \\ &\quad + 64 \times 0.40 + 125 \times 0.30 \\ &\quad + 216 \times 0.04 \\ &= 0.08 + 6.75 + 25.60 \\ &\quad + 37.50 + 8.64 \\ &= 78.57\end{aligned}$$

c $$\begin{aligned}\mathrm{E}(5X - 4) = 5\mathrm{E}(X) - 4 &= 5 \times 4.11 - 4 \\ &= 16.55\end{aligned}$$

d $$\begin{aligned}\mathrm{E}\left(\frac{1}{X}\right) &= \frac{1}{2} \times 0.01 + \frac{1}{3} \times 0.25 \\ &\quad + \frac{1}{4} \times 0.40 + \frac{1}{5} \times 0.30 + \frac{1}{6} \times 0.04 \\ &= 0.255\end{aligned}$$

8 $\mathrm{E}(X) = \Sigma x \Pr(X = x) = 2.97$
E(commission) $= 2.97 \times \$2000 = \$5940$

9 **a** $$p = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$$
$$p = \frac{1}{16}$$

b $$\begin{aligned}\mu &= 0 \times p + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} \\ &\quad + 8 \times \frac{1}{16} \\ &= 4 \times \frac{1}{2} \\ \mu &= 2\end{aligned}$$

c $$\begin{aligned}\mathrm{E}(X^2) &= 0 \times p + 1 \times \frac{1}{2} + 4 \times \frac{1}{4} \\ &\quad + 16 \times \frac{1}{8} + 64 \times \frac{1}{16} \\ &= 0 + \frac{1}{2} + 1 + 2 + 4 \\ &= \frac{15}{2} \\ \sigma^2 &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= \frac{15}{2} - 4 = \frac{7}{2}\end{aligned}$$

10 **a** $$\begin{aligned} k + 2k + 3k + 4k + 5k + 6k &= 1 \\ 21k &= 1 \\ k &= \frac{1}{21} \end{aligned}$$

b $$\begin{aligned} \mu &= 1 \times k + 2 \times 2k + 3 \times 3k + 4 \times 4k \\ &\quad + 5 \times 5k + 6 \times 6k \\ &= \frac{1 + 4 + 9 + 16 + 25 + 36}{21} \\ &= \frac{91}{21} \end{aligned}$$

c $$\begin{aligned} \sigma^2 &= \mathrm{E}(X^2) - \mu^2 \\ \mathrm{E}(X^2) &= 1 \times k + 4 \times 2k + 8 \times 3k \\ &\quad + 16 \times 4k + 25 \times 5k + 36 \times 6k \\ &= \frac{1 + 8 + 27 + 64 + 125 + 2166}{21} \\ &= \frac{441}{21} \\ &= 21 \\ \sigma^2 &= 21 - \frac{169}{9} \\ &= \frac{20}{9} \approx 2.22 \end{aligned}$$

11

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

a

x	1	2	3	4	6	8	9	12	16
$\Pr(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

b **i** $$\begin{aligned} \Pr(X > 8) &= \frac{1}{16} + \frac{2}{16} + \frac{1}{16} \\ &= \frac{4}{16} = \frac{1}{4} \end{aligned}$$

ii $$\begin{aligned} E(X) = {} & 1 \times \frac{1}{16} + 2 \times \frac{2}{16} \\ & + 3 \times \frac{2}{16} + 4 \times \frac{3}{16} \\ & + 6 \times \frac{2}{16} + 8 \times \frac{2}{16} \\ & + 9 \times \frac{1}{16} + 12 \times \frac{2}{16} \\ & + 16 \times \frac{1}{16} \\ = {} & \frac{\begin{array}{c} 1 + 4 + 6 + 12 + 12 + 16 \\ + 9 + 24 + 16 \end{array}}{16} \\ = {} & \frac{100}{16} \\ = {} & \frac{25}{4} \end{aligned}$$

iii $$\begin{aligned} E(X)^2 = {} & 1 \times \frac{1}{16} + 4 \times \frac{2}{16} \\ & + 9 \times \frac{2}{16} + 16 \times \frac{3}{16} \\ & + 36 \times \frac{2}{16} + 64 \times \frac{2}{16} \\ & + 81 \times \frac{1}{16} + 144 \times \frac{2}{16} \\ & + 256 \times \frac{1}{16} \\ = {} & \frac{\begin{array}{c} 1 + 8 + 18 + 48 + 72 + 128 \\ + 81 + 288 + 256 \end{array}}{16} \\ = {} & \frac{900}{16} \\ = {} & \frac{225}{4} \end{aligned}$$

$$\begin{aligned} \sigma^2 &= E(X^2) - \mu^2 = \frac{225}{4} - \frac{625}{16} \\ &= \frac{275}{16} \end{aligned}$$

12

	H	T
1	1	2
2	2	4
3	3	6
4	4	8
5	5	10
6	6	12

a $\mu = 1 \times \frac{1}{12} + 2 \times \frac{2}{12} + 3 \times \frac{1}{12}$

$+ 4 \times \frac{2}{12} + 5 \times \frac{1}{12} + 6 \times \frac{2}{12}$

$+ 8 \times \frac{1}{12} + 10 \times \frac{1}{12} + 12 \times \frac{1}{12}$

$= \frac{1 + 4 + 3 + 8 + 5 + 12 + 8 + 10 + 12}{12}$

$= \frac{63}{12} = \frac{21}{4}$

b $\Pr(X < \mu) = \frac{7}{12}$ (*counting on the table*)

c $E(X^2) = 1 \times \frac{1}{12} + 4 \times \frac{2}{12}$

$+ 9 \times \frac{1}{12} + 16 \times \frac{2}{12}$

$+ 25 \times \frac{1}{12} + 36 \times \frac{2}{12}$

$+ 64 \times \frac{1}{12} + 100 \times \frac{1}{12}$

$+ 144 \times \frac{1}{12}$

$= \frac{1 + 8 + 9 + 32 + 25 + 72 + 64 + 100 + 144}{12}$

$= \frac{455}{12}$

$\sigma^2 = E(X^2) - \mu^2 = \frac{455}{12} - \frac{441}{16}$

$= \frac{497}{48}$

13 **a** $\mathrm{Var}(2X) = 2^2\mathrm{Var}(X) = 4 \times 16 = 64$

b $\mathrm{Var}(X + 2) = 1^2\mathrm{Var}(X) = 16$

c $\mathrm{Var}(1 - X) = (-1)^2\mathrm{Var}(X) = 16$

d $sd(3X) = \sqrt{\mathrm{Var}(3X)} = \sqrt{3^2\mathrm{Var}(X)}$

$= \sqrt{9 \times 16}$

$= 12$

14 **a** $c = 1 - 0.3 - 0.1 - 0.2 - 0.05$

$= 0.35$

b $\mathrm{E}(X) = 1 \times c + 2 \times 0.3 + 3 \times 0.1$

$+ 4 \times 0.2 + 5 \times 0.05$

$= 0.35 + 0.6 + 0.3 + 0.8 + 0.25$

$= 2.3$

c $\mathrm{E}(X^2) = 1 \times c + 4 \times 0.3 + 9 \times 0.1$

$+ 16 \times 0.2 + 25 \times 0.05$

$= 0.35 + 1.2 + 0.9 + 3.2 + 1.25$

$= 6.9$

$\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= 6.9 - (2.3)^2$

$= 6.9 - 5.29$

$= 1.61$

$\sigma = \sqrt{\sigma^2} \approx 1.27$

15 **a** $k + 2k + 3k + 4k + 5k = 1$

$15k = 1$

$k = \frac{1}{15}$

b $\mu = 1 \times \frac{1}{15} + 2 \times \frac{2}{15} + 3 \times \frac{3}{15}$

$+ 4 \times \frac{4}{15} + 5 \times \frac{5}{15}$

$= \frac{1 + 4 + 9 + 16 + 25}{15}$

$= \frac{55}{15}$

$\mu = \frac{11}{3} \approx 3.667$

$\sigma^2 = \mathrm{E}(X^2) - \mu^2$

c $\mathrm{E}(X^2) = 1\frac{1}{15} + 4 \times \frac{2}{15} + 9 \times \frac{3}{15}$

$+ 16 \times \frac{4}{15} + 25 \times \frac{5}{15}$

$= \frac{1 + 8 + 27 + 64 + 125}{15}$

$= \frac{225}{15}$

$= 15$

$$\sigma^2 = 15 - \frac{121}{9}$$
$$= \frac{14}{9} \approx 1.556$$

16 **a** $$\mathrm{E}(X) = \frac{1}{36} \times 2 + \frac{2}{36} \times 3 + \frac{3}{36} \times 4 + \frac{4}{36} \times 5 + \frac{5}{36} \times 6 + \frac{6}{36} \times 7 + \frac{5}{36} \times 8 + \frac{4}{36} \times 9 + \frac{3}{36} \times 10 + \frac{2}{36} \times 11 + \frac{1}{36} \times 12$$
$$= \frac{2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12}{36}$$
$$= \frac{252}{36}$$
$$= 7$$

alternatively, since we know the probability distribution is symmetrical, we also know that the mean is the central number, *i.e.* $\mathrm{E}(X) = 7$

b $$\begin{aligned}\mathrm{E}(X^2) &= \frac{1}{36}\times 4+\frac{2}{36}\times 9+\frac{3}{36}\times 16\\ &\quad+\frac{4}{36}\times 25+\frac{5}{36}\times 36\\ &\quad+\frac{6}{36}\times 49+\frac{5}{36}\times 64\\ &\quad+\frac{4}{36}\times 81+\frac{3}{36}\times 100\\ &\quad+\frac{2}{36}\times 121+\frac{1}{36}\times 144\\ &= \frac{\begin{array}{l}4+18+48+100+180\\ \;+294+320+324+300\\ \;+242+144\end{array}}{36}\\ &= \frac{1974}{36}\\ &= \frac{329}{6}\approx 54.833\\ \sigma^2 &= \mathrm{E}(X^2)-(\mathrm{E}(X))^2\\ &= \frac{329}{6}-49\\ &= \frac{35}{6}\approx 5.83\end{aligned}$$

17 **a** *by symmetry*, $\mathrm{E}(X) = 3$

b $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$

$$\begin{aligned}\mathrm{E}(X^2) &= 0\times 0.0156+1\times 0.0937\\ &\quad+4\times 0.2344+9\times 0.3126\\ &\quad+16\times 0.2344+5\times 0.0937\\ &\quad+36\times 0.0156\\ &= 0.0937+0.9376+2.8134\\ &\quad+3.7504+2.3425+0.5616\\ &= 10.4992\approx 10.5\\ \mathrm{Var}(X) &= 10.5-9\\ &= 1.5\end{aligned}$$

Solutions to Technology-free questions

1 **a** $\Pr(BW \text{ or } WB) = \frac{5}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{5}{9}$

$= \frac{40}{81}$

$= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8}$

b $\Pr(BW \text{ or } WB) = \frac{5}{9}$

2 m chocolates, q milk chocolates, $m - q$ dark chocolates.

a Probability of dark $= \frac{m-q}{m}$

b $m - 1$ chocolates, q milk chocolates, $m - q - 1$ dark chocolates.

Probability of two dark chocolates

$= \frac{m-q}{m} \times \frac{m-q-1}{m-1}$

$= \frac{(m-q)(m-q-1)}{m(m-1)}$

3 Require Pr(coin A| '$H\&T$' tossed)

$= \frac{\Pr(A \cap \text{'}H\&T'\text{)}}{\Pr(\text{'}H \,\&\, T')}$

$\Pr(\text{selecting } A) = \Pr(\text{selecting } B) = \frac{1}{2}$

$\Pr(H|A) = 0.8,\ \Pr(T|A) = 0.2,$

$\Pr(H|B) = 0.4,\ \Pr(T|B) = 0.6$

$\Pr(3 \cap \text{'}H\&\text{'}T) = \frac{1}{2} \times (0.8 \times 0.2$

$+ 0.2 \times 0.8)$

$= 0.16$

$\Pr(\text{'}H\&T\text{'}) = \Pr(A \cap \text{'H\&T'})$

$+ \Pr(B \cap \text{'}H\&T\text{'})$

$\Pr(A \cap \text{'}K\&T\text{'}) = \frac{1}{2} \times (0.4 \times 0.6$

$+ 0.6 \times 0.4)$

$= 0.24$

So $\Pr(\text{'}H\&T\text{'}) = 0.16 + 0.24$

$= 0.40$

$\Pr(A|\text{'}H\&T\text{'}) = \frac{0.16}{0.40} = 0.4$

4 **a** Machine I 60% of items and Machine II 40% of items.

3% of items produced by machine I are faulty.

2% of items produced by machine II are faulty.

Probability of faulty

$= 0.6 \times 0.03 + 0.4 \times 0.02$

$= 0.6 \times 0.03 + 0.4 \times 0.02$

$= 0.018 + 0.008$

$= 0.026$

b $\Pr(MI|F) = \frac{\Pr(F|MI) \times \Pr(MI)}{\Pr(F)}$

$= \frac{0.03 \times 0.6}{0.026}$

$= \frac{0.018}{0.026}$

$= \frac{9}{13}$

5
$$\begin{aligned} 0.4p^2 + 0.1 + 0.1 + 1 - 0.6p &= 1 \\ 0.4p^2 - 0.6p + 0.2 &= 0 \\ 4p^2 - 6p + 2 &= 0 \\ 2p^2 - 3p + 1 &= 0 \\ (2p - 1)(p - 1) &= 0 \\ \therefore p &= \frac{1}{2} \text{ or } p = 1 \end{aligned}$$

6 **a**
$$\begin{aligned} k + 2k + 3k + 2k + k + k &= 1 \\ 10k &= 1 \\ k &= 0.1 \end{aligned}$$

b
$$\begin{aligned} \text{E}(X) &= \Sigma xp(x) \\ &= -k + 0 + 3k + 4k + 3k + 4k \\ &= 13k \\ &= 1.3 \end{aligned}$$

c
$$\begin{aligned} \text{E}(X^2) &= \Sigma x^2 p(x) \\ &= k + 0 + 3k + 8k + 9k + 16k \\ &= 37k = 3.7 \\ \text{Var}\,(X) &= \text{E}(X^2) - [\text{E}(X)]^2 \\ &= 3.7 - 1.69 = 2.01 \end{aligned}$$

7 **a**
$$\begin{aligned} \text{E}(X) &= \Sigma xp(x) \\ &= 2 \times \frac{1}{4} + 4 \times \frac{1}{4} \\ &\quad + 16 \times \frac{1}{4} + 64 \times \frac{1}{4} \\ &= 21\frac{1}{2}(21.5) \end{aligned}$$

b
$$\begin{aligned} \text{E}(X^2) &= \Sigma x^2 p(x) \\ &= 4 \times \frac{1}{4} + 16 \times \frac{1}{4} + 256 \times \frac{1}{4} \\ &\quad + 64^2 \times \frac{1}{4} \\ &= 1 + 4 + 64 + 1024 \\ &= 1093 \\ \text{Var}\,(X) &= \text{E}(X^2) - [\text{E}(X)]^2 \\ &= 1093 - (21.5) \\ &= 1093 - 462.25 \\ &= 630.75 \\ &= \frac{2523}{4} \end{aligned}$$

c From part *c*, $\text{Var}(X) = \dfrac{2523}{4}$.

$$\begin{aligned} \text{But } 2523 &= 3 \times 841 \\ &= 3 \times 29^2 \\ \text{So Var}(X) &= \frac{3 \times 29^2}{2^2} \\ \Rightarrow \text{sd}(X) &= \frac{29\sqrt{3}}{2} \end{aligned}$$

8 **a** Profit is \$$(x - 2)$ if the cylinder is ok and −\$2 if the cylinder is defective.

p	$x - 2$	2
$\Pr(P = p)$	$\frac{4}{5}$	$\frac{1}{5}$

b
$$\begin{aligned} \text{E}(P) = \Sigma p \Pr(P = p) &= \frac{4}{5}(x - 2) - \frac{2}{5} \\ &= \frac{4}{5}x - 2. \end{aligned}$$

c To make a profit in the long teem, require E(P) > 0, i.e.

$$\frac{4}{5}x - 2 > 0$$

$$\frac{4}{5}x > 2$$

$$x > \frac{5}{2} = 2.5$$

The manufacturer should sell the cylinders for more then \$2.50.

9 **a** $\Pr(\text{‘} < 30\text{’} \cap \text{‘} > 1\ acc\text{’}) = \dfrac{470}{1000}$

$= 0.47$

b $\Pr(\text{‘} < 30\text{’} \cap \text{‘} > 1\ acc\text{’})$

$$= \frac{\Pr(\text{‘} < 30 \cap > 1\ acc\text{’})}{\Pr(\text{‘}{>}1\ acc\text{’})}$$

$$= \frac{0.47}{\left(\dfrac{470 + 230}{100}\right)}$$

$$= \frac{0.47}{0.70}$$

$$= \frac{47}{70}$$

10 **a**

Let $I =$ ‘immunised’, $D =$ ‘get disease’

$$\begin{aligned}\Pr(D) &= \Pr(D \cap I) + \Pr(D \cap I') \\ &= \Pr(I)\Pr(D|I) + \Pr(I')\Pr(D|I') \\ &= 0.7 \times 0.05 + 0.3 \times 0.6 \\ &= 0.035 + 0.18 \\ &= 0.215\end{aligned}$$

So 21.5% are expected to get the disease.

[NOTE: This is a probability way of saying: “5% of the 70% and 60% of the 30% get the disease, i.e. 3.5% + 18% = 21.5% get it]

b $\Pr(I|D) = \dfrac{\Pr(D|I)\Pr(I)}{\Pr(D)}$

$$= \frac{0.05 \times 0.7}{0.215}$$

$$= \frac{0.035}{0.215}$$

$$= \frac{35}{215} = \frac{7}{43}$$

11 $\Pr(A) = \dfrac{1}{2}, \Pr(B) = \dfrac{1}{4}, \Pr\,(A|B) = \dfrac{1}{6}$

a $\Pr(A \cap B) = \Pr(A|B)\Pr(B)$

$$= \frac{1}{6} \times \frac{1}{4}$$

$$= \frac{1}{24}$$

b

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= \frac{1}{2} + \frac{1}{4} - \frac{1}{24} \\ &= \frac{17}{24}\end{aligned}$$

c

$$\Pr(A'|B) = \frac{\Pr(A' \cap B)}{\Pr(B)}$$

But $\Pr(B) = \Pr(A \cap B) + \Pr(A' \cap B)$

So $\Pr(A' \cap\ B) = \Pr(B) - \Pr(A \cap B)$

$$= \frac{1}{4} - \frac{1}{24}$$

$$= \frac{5}{24}$$

So $\Pr(A'|B) = \dfrac{\frac{5}{24}}{\frac{1}{4}}$

$$= \frac{5}{6}$$

d

$$\begin{aligned}
\Pr(A|B') &= \frac{\Pr(A \cap B')}{\Pr(B')} \\
\text{But } \Pr(A') &= \Pr(A \cap B) + \Pr(A \cap B') \\
\text{So } \Pr(A \cap B') &= \Pr(A) - \Pr(A \cap B) \\
&= \frac{1}{2} - \frac{1}{24} \\
&= \frac{11}{24} \\
\Pr(B') &= 1 - \Pr(B) \\
&= 1 - \frac{1}{4} \\
&= \frac{3}{4} \\
\text{So } \Pr(A|B') &= \frac{\frac{11}{24}}{\frac{3}{4}} \\
&= \frac{11}{18}
\end{aligned}$$

Solutions to multiple-choice questions

1 **A** $\Pr(B) = 2\Pr(A)$ and
$\Pr(A \cup B) = 0.405$
$\Pr(A \cup B) = 3\Pr(A) - 2[\Pr(A)]^2$

$$2[\Pr(A)]^2 - 3\Pr(A) + 0.405 = 0$$

$$(2\Pr(A) - 2.7)(\Pr(A) - 0.15) = 0$$

$$\therefore \Pr(A) = 0.15$$

2 **E** Four red and three yellow.
Both red $= \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$
Both yellow $= \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$
Probability the same colour$= \frac{3}{7}$

3 **D** $\Pr(-3 \le X < 0) = \Pr(X = -3)$

$+ \Pr(X = -2) + \Pr(X = -1)$

$= 0.07 + 0.15 + 0.22$

$= 0.44$

4 **C** $2k + 3k + 0.1 + 3k + 2k = 1$

$$10k = 0.9$$

$$k = 0.09$$

$E(X)$

$= 0.09(-1 \times 2 + 2 \times 3 + 3 \times 2) + 0.1$

$= 1$

5

D $E(X) = \Sigma xp(x) = \Pr(X = x))$

$= 1 \times 0.46 + 2 \times 0.26$

$+ 3 \times 0.14 + 4 \times 0.09$

$+ 5 \times 0.07$

$= 0.46 + 0.48 + 0.42 + 0.36 + 0.35$

$= 2.07$

$\Pr(X < \mu) = 0.46 + 0.24 = 0.7$

6 **C** $0.1^2 + 0.25^2 + 0.3^2 + 0.2^2 + 0.1^2 + 0.05^2 = 0.215$

7 **E** $\text{Var}(X) = E(X^2) - [E(X)]^2$

$= 1.69 - (1.20)^2$

$= 1.69 - 1.44$

$= 0.25$

$\text{sd}(X) = \sqrt{0.25}$

$= 0.5$

8 **C** $E(Y) = E(3X + 10)$

$= 3E(X) + 10$

$= 3 \times 100 + 10$

$= 310$

$\text{Var}(Y) = \text{Var}(3X + 10)$

$= 9\,\text{Var}(X)$

$= 9 \times 100$

$= 900$

9 **C** $E(x) = \Sigma xp(x)$

$= -p + 0 + 1 - 3p$

$= 1 - 4p$

10

B/D $a + b + 0.2 = 1 \Rightarrow a + b = 0.8 \ldots$ ①
$E(X) = -2a + 0.4 = 0.2 \ldots$ ②
From ②, $2a = 0.2$ so $a = 0.1$
Substitute in ①: $0.1 + b = 0.8$ so
$b = 0.7$

Solutions to extended-response questions

1 **a**

$$\Sigma \Pr(X = x) = 1$$

$$\therefore \quad c + 2c + 2c + 3c + c^2 + 2c^2 + 7c^2 + c = 1$$

$$\therefore \quad 10c^2 + 9c = 1$$

$$\therefore \quad 10c^2 + 9c - 1 = 0$$

$$\therefore \quad (10c - 1)(c + 1) = 0$$

$$\therefore \quad c = 0.1 \text{ or } c = -1$$

but $c > 0 \quad \therefore \quad c = 0.1$

b $\Pr(X \geq 5) = \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)$

$= 10c^2 + c$

$= 10 \times (0.1)^2 + 0.1$

$= 0.2$

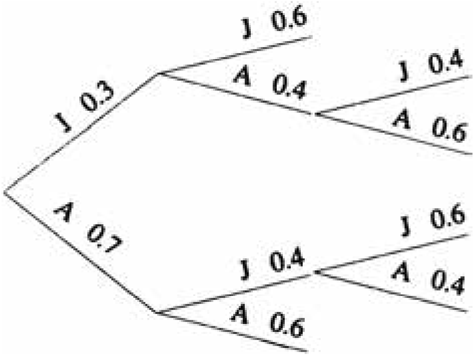
c If $\Pr(X \leq k) > 0.5$

then by considering cumulative probabilities

i.e. $\Pr(X \leq 2) = 0.3; \Pr(X \leq 3) = 0.5$

the minimum value of k is 4.

2 **a**



b **i** Probability of Janet winning

$= 0.3 \times 0.6 + 0.3 \times 0.4 \times 0.4 + 0.7 \times 0.4 \times 0.6$

$= 0.396$

ii Probability of Alan winning

$= 1 - 0.396$

$= 0.604$

c **i** Let X be the number of sets played until match is complete.

$$\begin{aligned}\Pr(X = 2) &= 0.3 \times 0.6 + 0.7 \times 0.6\\ &= 0.6\end{aligned}$$

$$\therefore \ \Pr(X = 3) = 0.4$$

t	2	3
$\Pr(T = t)$	0.6	0.4

ii $E(X) = 2 \times 0.6 + 3 \times 0.4 = 2.4$

d $$\begin{aligned}\Pr(\text{Alan wins} \mid \text{three sets}) &= \frac{\Pr\text{ (Alan wins in three sets)}}{\Pr\text{ (Three sets)}}\\ &= \frac{0.3 \times 0.4 \times 0.6 + 0.7 \times 0.4 \times 0.4}{0.4}\\ &= 0.46\end{aligned}$$

3 Let \$w be the amount a player pays to play.

$\boxed{\$5}\quad\boxed{\$5}\quad\boxed{\$5}\quad\boxed{\$10}\quad\boxed{\$10}$

Let X be the possible value From 2 cards

$\therefore \ X = 10, 15, 20$

A score of 10 is obtained if two \$5 cards are chosen

$\therefore \ \Pr(X = 10) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$ (without replacement)

A score of 15 is obtained with \$10 on the first and \$5 on the second or \$5 on the first and \$10 on the second.

$$\begin{aligned}\therefore \ \Pr(X = 15) &= \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{1}{2}\\ &= \frac{6}{20} + \frac{3}{10}\\ &= \frac{3}{5}\end{aligned}$$

A score of 20 is obtained with a \$10 on each card.

$$\begin{aligned}\Pr(X = 20) &= \frac{2}{5} \times \frac{1}{4}\\ &= \frac{1}{10}\end{aligned}$$

Let Y be the amount a player receives

$Y = 10 - w$ or $15 - w$ or $20 - w$

The probability distribution for Y is as shown

y	$10 - w$	$15 - w$	$20 - w$
$\Pr(Y = y)$	0.3	0.6	0.1

$$\therefore\ \mathrm{E}(Y) = 0.3(10 - w) + 0.6(15 - w) + 0.1(20 - w)$$

$$= 3 - 0.3w + 9 - 0.6w + 2 - 0.1w$$

$$= 14 - w$$

If E(Y) = 0, $w = 14$

i.e. The player should pay $14 to ensure that it is a fair game.

4 Let F denote free from faults

Let N denote not free from faults (defective)

$\Pr(F|A) = 0.95 \quad \Pr(F|B) = 0.98 \quad \Pr(F|C) = 0.99$

$\Pr(A) = 0.5 \quad \Pr(B) = 0.3 \quad \Pr(C) = 0.2$

a $\Pr(A) = 0.5$

b $\Pr(N|A) = 0.05$

c
$$\Pr(N) = \Pr(N|A)\Pr(A) + \Pr(N|B)\Pr(B) + \Pr(N|C)\Pr(C)$$
$$= 0.05 \times 0.5 + 0.02 \times 0.3 + 0.01 \times 0.2$$
$$= 0.033$$

d
$$\Pr(A|D) = \frac{\Pr(\text{produced by } A \text{ and defective})}{\Pr(\text{defective})}$$
$$= \frac{\Pr(N|A)\ \Pr(A)}{\Pr(N)}$$
$$= \frac{0.5 \times 0.5}{0.033}$$
$$= \frac{25}{33}$$

5

p	0	1	2	3	4	5
$\Pr(P = p)$	0.39	0.27	0.16	0.12	0.04	0.02

a **i**
$$\mathrm{E}(P) = 0 \times 3.29 + 1 \times 0.27 + 2 \times 0.16 + 3 \times 0.12 + 4 \times 0.04 + 5 \times 0.02$$
$$= 0.27 + 0.32 + 0.36 + 0.16 + 0.1$$
$$= 1.21$$

The mean number of passengers per car is 1.21

ii $\text{Var}(P) = \text{E}(P^2) - [\text{E}(P)]^2$

$$\text{E}(P^2) = 0^2 \times 0.39 + 1^2 \times 0.27 + 2^2 \times 0.16 + 3^2 \times 0.12 + 4^2 \times 0.04 + 5^2 \times 0.02$$

$$= 0.27 + 0.64 + 1.08 + 0.64 + 0.5 = 3.13$$

$$\text{Var}(P) = 3.13 - 1.4641$$

$$= 1.6659$$

$\text{sd}(P) = \sqrt{1.6659} = 1.2907$ (correct to four decimal places)

iii

$$\sigma = \text{sd}(P) = 1.2907$$

$$\mu - 2\sigma = -1.3714$$

$$\mu + 2\sigma = 3.7914$$

$$\Pr(\mu - 2\sigma \le P \le \mu + 2\sigma) = \Pr(-1.3714 \le P \le 3.7914)$$

$$= \Pr(P = 0) + \Pr(P = 1) + \Pr(P = 2) + \Pr(P = 3)$$

$$= 1 - [\Pr(P = 4) + \Pr(P = 5)]$$

$$= 0.94$$

b **i** Let T be the cost per car in dollars.

$$\Pr(T = 1) = \Pr(P = 0) = 0.39$$

$$\Pr(T = 0.40) = \Pr(P = 1) = 0.27$$

$$\Pr(T = 0) = \Pr(P = 2) + \Pr(P = 3) + \Pr(P = 4) + \Pr(P = 5) = 0.34$$

t	1	0.40	0
$\Pr(T = t)$	0.39	0.27	0.34

ii $\text{E}(T) = 1 \times 0.39 + 0.40 \times 0.27 + 0 \times 0.34$

$$= 0.39 + 0.108$$

$$= 0.498$$

iii $\mathrm{E}(T^2) = 1 \times 0.39 + 0.4^2 \times 0.27$

$= 0.39 + 0.0432$

$= 0.4332$

$\mathrm{Var}(T) = \mathrm{E}(T^2) - [\mathrm{E}(T)]^2$

$= 0.4332 - 0.248004$

$= 0.1852$

$\mathrm{sd}(T) = 0.4303$

$\mu - 2\sigma = 0.498 - 2 \times 0.4304 = -0.3628$

$\mu + 2\sigma = 0.498 + 2 \times 0.4304 = 1.3588$

$\Pr(\mu - 2\sigma \le T \le \mu + 2\sigma) = \Pr(-0.3628 \le T \le 1.3588)$

$= \Pr(T = 0) + \Pr(T = 0.4) + \Pr(T = 1)$

$= 1$

6 **a** $\mathrm{E}(Y) = 0 \times 0.135 + 1 \times 0.271 + 2 \times 0.271 + 3 \times 0.180 + 4 \times 0.090$

$+ 5 \times 0.036 + 6 \times 0.012 + 7 \times 0.003 + 8 \times 0.002$

$= 2.002$

The mean number of sales per week is 2.002.

b $\mathrm{E}(Y^2) = 6.002$

$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - [\mathrm{E}(Y)^2] = 6.022 - 4.008004$

$= 2.013996 \approx 2.014$

$\mathrm{sd}(Y) \approx 1.419$

c **i** Let B be the bonus paid to each salesman.

The possible values for B are 0, 100 and 200

$\Pr(B = 0) = \Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) = 0.677$

$\Pr(B = 100) = \Pr(Y = 3) + \Pr(Y = 4) = 0.27$

$\Pr(B = 200) = \Pr(Y \ge 4) = 0.053$

The probability distribution is

b	0	100	200
$\Pr(B = b)$	0.677	0.27	0.053

ii $\mathrm{E}(B) = 0 \times 0.677 + 100 \times 0.27 + 200 \times 0.053$

$= 27 + 10.6$

$= 37.6$

The mean bonus paid is \$37.60.

7 Let P denote the percentage profit

p	40	30	20	10	0	−10	−20
$\Pr(P = p)$	0.1	0.15	0.25	0.2	0.15	0.1	0.05

a $\mathrm{E}(P) = 40 \times 0.1 + 30 \times 0.15 + 20 \times 0.25 + 10 \times 0.2 + 0 \times 0.15 - 10 \times 0.1 - 20 \times 0.05$

$= 13.5$

The mean return is 13.5%

$$\mathrm{E}(P^2) = 1600 \times 0.1 + 100 \times 0.15 \times 400 \times 0.25 + 100 \times 0.2 + 100 \times 0.1 + 400 \times 0.05$$

$$= 445$$

$$\therefore\ \mathrm{Var}(P) = 445 - 182.25$$

$$= 262.75$$

$$\therefore\ \mathrm{sd}(P) = \sqrt{262.75} \approx 16.2\%$$

b $\Pr(13.5 - 2 \times 16.21 \le P \le 13.5 + 2 \times 16.21) = \Pr(-18.92 \le P \le 45.92)$

$= 1 - \Pr(P = -20)$

$= 1 - 0.05$

$= 0.95$

c Return = Profit–Brokerage

Percentage gain = 0.6 × Return

= 0.6(Profit–Brokerage)

Let G be the percentage gain

Then $G = 0.6(P - 2)$

$$\therefore\ \mathrm{E}(G) = 0.6\mathrm{E}(P) - 1.2$$

$$= 0.6 \times 13.5 - 1.2$$

$$= 6.9\%$$

$$\mathrm{Var}(G) = (0.6)^2\ \mathrm{Var}(P)$$

$$= 0.36 \times 262.75$$

$$= 94.59$$

$$\mathrm{sd}(G) \approx 9.726\%$$

8 Consider the case when the promoter takes out insurance.
If it rains:

($)Profit = 250 000 − 60 000 + 20 000

= 210 000

(assuming the $250 000 is paid by the insurance company and the $20 000 profit is added.) If it does not rain:

($)Profit = 25 0000 − 60 000

= 190 000

The probability distribution for this

p	190 000	210 000
$\Pr(P = p)$	0.67	0.33

$E(P) = 196\ 600$

Then the promoter does not take the insurance, the probability distribution is as shown below:

p	250 000	20 000
$\Pr(P = p)$	0.67	0.33

and $E(P) = 174\ 100$

∴ the promoter should buy the insurance.

9 For the tossing of two dice the sums of the values may be recorded in a table as shown

die A die B	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

The probability of obtaining a sum of 7 $= \frac{6}{36} = \frac{1}{6}$

The probability of obtaining a sum of 11 or 12 $= \frac{3}{36} = \frac{1}{12}$

The probability of any other sum $= \frac{3}{4}$

Let X be the amount obtained from game and let w be the amount obtained from obtaining a sum not equal to 7, 11 or 12

The probability distribution is as shown:

x	−10	11	w
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{4}$

$$\mathrm{E}(X) = -\frac{10}{6} + \frac{11}{12} + \frac{3w}{4}$$

$$\text{If } \mathrm{E}(X) = 0$$

$$-\frac{20}{12} + \frac{11}{12} + \frac{9w}{12} = 0$$

$$\therefore \frac{9w}{12} = \frac{9}{12}$$

$$w = 1$$

There should be a payment of \$1.00 for a sum not equal to 7, 11 or 12.

10 **a** **i** Probability that the first prototype is successful is 0.65.

ii Probability of the first not successful, but the second successful

$= 0.35 \times 0.65$

$= 0.2275$

iii Probability of the first two not successful, but the third successful

$= (0.35)^2 \times 0.65$

$= 0.0796\ 25$

iv Probability that the project is abandoned

$= (0.35)^3$

$= 0.0428\ 75$

b The following cases have to be considered:

	Cost	Probability
A First is successful	\$7 million	0.65
B First is unsuccessful but second is successful	\$10.5 million	0.2275
C First two unsuccessful but third successful	\$12.25 million	0.079625
D The project is abandoned	\$12.25 million	0.042875

Let C be the cost of the project.

C	7	10.5	12.25
$\Pr(C = c)$	0.6	0.2275	0.1225

$\therefore\ \mathrm{E}(C) = 7 \times 0.65 + 10.5 \times 0.2275 + 12.25 \times 0.1225$

$= 8.439\ 375$

$\therefore$ the expected cost is \$8.439 375 million

c Let P denote the profit

P	$20 - 7$	$20 - 10.5$	$20 - 12.25$	-12.25
$\Pr(P = p)$	0.65	0.2275	0.079625	0.042875

$\therefore\ \mathrm{E}(P) = 13 \times 0.65 + 9.5 \times 0.2275 + 7.75 \times 0.079\ 625 - 12.25 \times 0.042\ 875$

$\therefore$ Expected profit is \$10.703 125 million

11 If the score is 5, 6, 7, 8, 9, 10, 11 or 12. Alfred pays \$$x$ to Bertie. Therefore Alfred has $100 - x$ dollars.

If the score is 2, 3 or 4 Alfred has $100 + x + 8 = 108 + x$ dollars.

The tables gives the sum of the scores when the two die are tossed.

die 2 die 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let Y be the score.

$\Pr(Y \geq 5) = \frac{30}{36} = \frac{5}{6}$ and $\Pr(Y \leq 4) = \frac{1}{6}$

a Let A be the amount of Alfred's cash

A	$100 - x$	$108 + x$
$\Pr(A = a)$	$\frac{5}{6}$	$\frac{1}{6}$

$$\begin{aligned} \therefore\ \mathrm{E}(A) &= \frac{5}{6}(100 - x) + \frac{1}{6}(108 + x) \\ &= \frac{1}{6}(608 - 4x) \\ &= \frac{1}{3}(304 - 2x) \end{aligned}$$

b If the game is fair $\quad \mathrm{E}(A) = 100$

$$\begin{aligned} \therefore\ \frac{1}{3}(304 - 2x) &= 100 \\ 304 - 2x &= 300 \\ \therefore\ x &= 2 \end{aligned}$$

c
$$\begin{aligned} \mathrm{E}(A^2) &= 97^2 \times \frac{5}{6} + 111^2 \times \frac{1}{6} \qquad \text{(given } x = 3\text{)} \\ &= 9894\frac{1}{3} \\ \therefore\ \mathrm{Var}\,(A) &= 2894\frac{1}{3} - \left(\frac{1}{3}[298]\right)^2 \\ &= 27\frac{2}{9} \end{aligned}$$

12 Let X be the values of the die

$\Pr(X = 1) = \frac{x}{4} \quad \Pr(X = 2) = \frac{1}{4} \quad \Pr(X = 6) = \frac{1}{4}(1 - x)$

$\Pr(X = 3) = \Pr(X = 4) = \Pr(X = 5) = \frac{1}{6}$

Table for total

2nd						
1st	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a Let Y be the total.

$$\begin{aligned}\Pr(Y = 7) &= 2\Pr(1 \,\&\, 6) + 2\Pr(3 \,\&\, 4) + 2\Pr(5 \,\&\, 2)\\ &= 2 \times \frac{x}{4} \times \frac{1-x}{4} + 2 \times \frac{1}{6} \times \frac{1}{6} + 2 \times \frac{1}{6} \times \frac{1}{4}\\ &= \frac{x(1-x)}{8} + \frac{2}{36} + \frac{2}{24}\\ &= \frac{x(1-x)}{8} + \frac{1}{18} + \frac{1}{12}\\ &= \frac{9x(1-x) + 4 + 6}{72}\\ &= \frac{9x - 9x^2 + 10}{72}\end{aligned}$$

b Let $P = \Pr(Y = 7)$

$$\frac{dP}{dx} = \frac{9 - 18x}{72}$$

and $\dfrac{dP}{dx} = 0 \Rightarrow x = \dfrac{1}{2}$

when $x = \dfrac{1}{2}$, $P = \dfrac{9 \times \frac{1}{2} - 9\left(\frac{1}{2}\right)^2 + 10}{72}$

$$= \frac{\frac{9}{2} - \frac{9}{4} + 10}{72} = \frac{\frac{9}{4} + 10}{72} = \frac{49}{288}$$

Chapter 14 – The binomial distribution

Solutions to Exercise 14A

1 a and b describe a Bernoulli sequence.

2 $n = 7, p = 0.5$

$\Pr(X = 4) = \binom{7}{4}(0.5)^4(0.5)^3 = 0.2734$

3 $n = 4, p = 0.2$

a $\Pr(X = 3) = \binom{4}{3}(0.2)^3(0.8)^1 = 0.0256$

b $\Pr(X = 4) = \binom{4}{4}(0.2)^4(0.8)^0 = 0.0016$

4 $n = 5, p = 0.4$

a $\Pr(X = 0) = \binom{5}{0}(0.4)^0(0.6)^0 = 0.0778$

b $\Pr(X = 3) = \binom{5}{3}(0.4)^3(0.6)^3 = 0.2304$

c $\Pr(X = 5) = \binom{5}{5}(0.4)^5(0.6)^5 =$ 0.01024

5 $n = 3, p = 0.5$

a $\Pr(X = x) = \binom{3}{x}(0.5)^x(0.5)^{3-x}$

$x = 0, 1, 2, 3$

b $\Pr(X = 2) = 0.375$

6 $n = 6, p = 0.48$

a $\Pr(X = x) = \binom{6}{x}(0.48)^x(0.52)^{6-x}$

$x = 0, 1, 2, 3, 4, 5, 6$

b $\Pr(X = 2) = 0.2527$

7 **a** $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 = 20 \times \frac{1}{216} \times \frac{125}{216}$

≈ 0.0536

b $\binom{6}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2 + \binom{6}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right) + \binom{6}{6}\left(\frac{1}{6}\right)^6$

$= \frac{375}{6^6} + \frac{30}{6^6} + \frac{1}{6^6}$

$= \frac{406}{46656} \approx 0.0087$

c $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 + \binom{6}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2$

$+ \binom{6}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right) + \binom{6}{6}\left(\frac{1}{6}\right)^6$

≈ 0.0623

8 $n = 10, p = 0.1$

a $\Pr(X = x) = \binom{10}{x}(0.1)^x(0.9)^{10-x}$

$x = 0, 1, 2, 3, \ldots, 10$

b **i** $\Pr(X = 0) = 0.3487$

ii $\Pr(X \geq 1) = 1 - \Pr(X = 0)$

$= 0.6513$

9 $n = 11, p = 0.2$

a $\Pr(X = x) = \binom{11}{x}(0.2)^x(0.8)^{11-x}$

$x = 0, 1, 2, 3, \ldots, 11$

b **i** $\Pr(X = 2) = 0.2953$

ii $\Pr(X = 0) = 0.0859$

iii $\Pr(X \geq 1) = 1 - \Pr(X = 0)$
$= 0.9141$

10 $n = 7, p = \frac{1}{5}$

a $\Pr(X = x) = \binom{7}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{7-x}$
$x = 0, 1, 2, 3, \ldots, 7$

b **i** $\Pr(X = 7) = 0.000013$

ii $\Pr(X = 0) = 0.2097$

iii $\Pr(X = 2 \text{ or } X = 3) = 0.3899$

11 $1 - \binom{10}{0}(0.2)^0(0.8)^{10} - \binom{10}{1}(0.2)^1(0.8)^9$

$$= 1 - \frac{4^{10}}{5^{10}} - \frac{10 \times 1 \times 4^9}{5^{10}}$$

≈ 0.624

12 $n = 7, p = \frac{x}{100}$

$$\left(\frac{x}{100}\right)^6$$

ii $\frac{6x^5(100 - x)}{100^6}$

iii $\frac{x^6}{100^6} + \frac{6x^5(100 - x)}{100^6} + \frac{15x^4(100 - x)^2}{100^6}$

13 $1 - \binom{4}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^4$

$$= 1 - \left(\frac{3}{4}\right)^4$$

≈ 0.6836

14 *using the CAS calculator*

a 0.1156

b 0.7986

c 0.3170

15 *using the CAS calculator*
0.6791

16 *using the CAS calculator*

a 0.1123

b 0.5561

c 0.00001

d 0.00001

17 $\binom{6}{0}(0.4)^0(0.6)^6 + \binom{6}{1}(0.4)(0.6)^5$

$$+ \binom{6}{2}(0.4)^2(0.6)^4$$

$$= \frac{3^6}{5^6} + \frac{6 \times 2 \times 3^5}{5^6} + \frac{15 \times 2^2 \times 3^4}{5^6}$$

$$= \frac{3^6 + 4 \times 3^6 + 5 \times 4 \times 3^5}{5^6}$$

$$= \frac{3 \times 3^5 + 4 \times 3^5}{5^5} = \frac{7 \times 3^5}{5^5} \approx 0.544$$

18 **a** $\left(\frac{1}{4}\right)^6 \approx 0.00024$

b *using the CAS calculator*
Pr(≥ 3 correct) ≈ 0.1694

19 **a** $\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^3 = \frac{6^3}{5^6} \approx 0.0138$

b $\binom{6}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^3 = 20 \times \frac{6^3}{5^6} \approx 0.2765$

c *using the CAS calculator*
Pr(≥ 3) ≈ 0.8208

d $\Pr(\text{exactly } 3 | \geq 3) = \frac{\Pr(\text{exactly } 3)}{\Pr(\geq 3)} =$
$\frac{(b)}{(c)} = \frac{0.2765}{0.8202} \cong 0.3368$

20 **a** $\left(\frac{4}{5}\right)^8 \approx 0.1678$

b *using the CAS calculator*
Pr(≥ 6 corrent) ≈ 0.00123

c Pr(8 correct | ≥ 6 correct) =
$\frac{\Pr(8 \text{ correct})}{\Pr(\geq 6 \text{ correct})} = \frac{(0.2)^8}{(b)} =$

$\frac{0.00000256}{0.00123} \approx 0.0021$

21 **a** $(0.15)^{10} \approx 0.000\,000\,006$

b $1 - (0.85)^{10} \approx 1 - 0.1969 \approx 0.8031$

c Pr(> | goal | ≥ | goal)

$$= \frac{\Pr(> | \text{goal})}{\Pr(\geq | \text{goal})}$$
$$= \frac{(b) - 10(0.15)(0.85)^9}{(b)}$$
$$= \frac{0.8031 - 0.3474}{0.8031}$$
$$\approx \frac{0.4557}{0.8031} \approx 0.5674$$

22 **a** $\left(\frac{4}{5}\right)^{20} \approx 0.0115$

b *using the CAS calculator*

$p = 0.2,\ n = 20,$

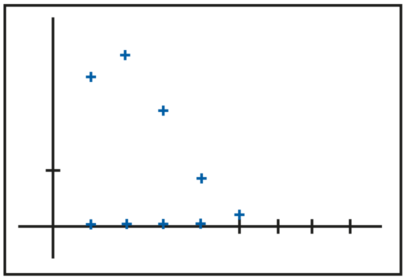
$min = 10,\ max = 20$

Pr(≥ 10 correct) ≈ 0.00259

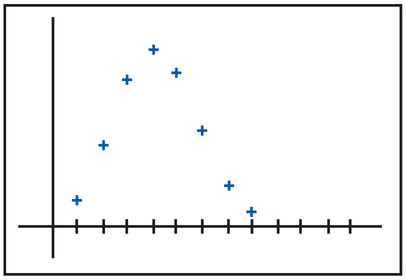
c $\Pr(X \geq 12 | X \geq 10) = \frac{\Pr(X \geq 12)}{\Pr(X \geq 10)} \approx$
0.0393 (using the CAS calculator)

Solutions to Exercise 14B

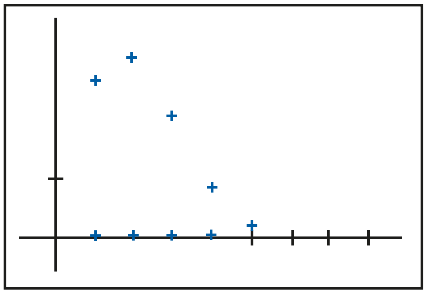
1



2



3 a b



c *the distribution in part* **b** *is a reflection of the distribution in part* **a** *in the line* $X = 5$

4 a $\mu = np = 5$

$$\sigma^2 = np(1-p) = 5 \times (0.8) = 4$$

b $\mu = np = 6$

$$\sigma^2 = np(1-p) = 6 \times (0.4) = \frac{12}{5}$$

c $\mu = np = \dfrac{500}{3}$

$$\sigma^2 = np(1-p) = \frac{1000}{9}$$

d $\mu = np = 8$

$$\sigma^2 = np(1-p) = 8 \times \left(\frac{4}{5}\right) = \frac{32}{5}$$

5 a $\mu = np = 6 \times \dfrac{1}{6} = 1$

b $\Pr(X > 1) = 1 - \Pr(X = 0) - \Pr(X = 1)$

$$= 1 - \left(\frac{5}{6}\right)^6 - 6 \times \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5$$

$$= 1 - \frac{11}{6}\left(\frac{5}{6}\right)^5$$

$$\approx 1 - 0.7368$$

$$\approx 0.2632$$

6 $\mu = np = 50 \times \dfrac{3}{4}$

$= 37.5$ *people will survive on average*

7

$$\mu = np$$

$$\sigma^2 = np(1-p) = \mu(1-p)$$

$$\sigma^2 = 9, \ \mu = 12$$

$$9 = 12(1-p)$$

$$1 - p = \frac{3}{4}$$

$$p = \frac{1}{4}$$

$$12 = n \times \frac{1}{4}$$

$$n = 48$$

$$\Pr(X = 7) = \binom{48}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^{41}$$

$$\approx 0.0339$$

8

$$\begin{aligned} \mu &= 30,\ \sigma^2 = 21 \\ \sigma^2 &= \mu(1-p) \\ 1-p &= \frac{21}{30} \\ p &= \frac{9}{30} = \frac{3}{10} \\ \mu &= np \\ 30 &= n \times \frac{3}{10} \\ n &= 100 \\ \Pr(X=20) &= \binom{100}{20}\left(\frac{3}{10}\right)^{20}\left(\frac{7}{10}\right)^{80} \\ &\approx 0.0076 \end{aligned}$$

9

$$\begin{aligned} n &= 20, p = 0.5 \\ \mu &= np = 10 \\ \sigma^2 &= np(1-p) = 10 \times 0.5 = 5 \\ \sigma &= \sqrt{5} \approx 2.2 \\ \mu \pm 2\sigma &\approx 10 \pm 4.4 \\ &= 5.6, 14.4 \end{aligned}$$

$\therefore$ *the probability of obtaining between* 6 *and* 14 *heads is* ≈ 0.95

10

$$\begin{aligned} n &= 200, p = 0.6 \\ \mu &= np = 200 \times \frac{6}{10} = 120 \\ \sigma^2 &= \mu(1-p) = 120 \times \frac{4}{10} = 48 \\ \sigma &= \sqrt{48} = 4\sqrt{3} \approx 6.9 \\ \mu \pm 2\sigma &\approx 120 \pm 13.8 \\ &= 106.2, 133.8 \end{aligned}$$

$\therefore$ *the probability that between* 107 *and* 133 *students will have attended a government school is* ≈ 0.95

Solutions to Exercise 14C

1 a $n = 5, p = 0.2$

i $\Pr(X = 0) = (0.8)^5 \approx 0.3277$

ii $\Pr(X \geq 1) = 1 - \Pr(X = 0)$

≈ 0.6723

b

$$\begin{aligned} \Pr(X \geq 1) &> 0.95 \\ 1 - \Pr(X = 0) &> 0.95 \\ \Pr(X = 0) &< 0.05 \\ (0.8)^n &< 0.05 \\ n &\approx 13.43 \end{aligned}$$

$\therefore$ *the smallest number of shots is* 14

c

$$\begin{aligned} \Pr(X \geq 1) &> 0.95 \quad \therefore \\ 1 - \Pr(X = 0 - \Pr(X = 1)) &> 0.95 \\ \Pr(X = 0) + \Pr(X = 1) &< 0.05 \\ (0.8)^n + \binom{n}{1} 0.8^{n-1} \times 0.2 &< 0.05 \\ (0.8)^n + n 0.8^{n-1} \times 0.2 &< 0.05 \\ n &\approx 21.77 \end{aligned}$$

the smallest number of shots is 22

2 a **i** $\Pr(X = 2) = \binom{10}{2}(0.1)^2(0.9)^8$

≈ 0.1937

ii $\Pr(X \geq 1) = 1 - \Pr(X = 0)$

$= 1 - (0.9)^{10}$

$\approx 1 - 0.3487$

≈ 0.6513

b

$$\begin{aligned} \Pr(X \geq 1) &> 0.7 \\ 1 - \Pr(X = 0) &> 0.7 \\ \Pr(X = 0) &< 0.3 \\ (0.9)^n &< 0.3 \\ n &\approx 11.43 \end{aligned}$$

$\therefore$ *the smallest number of tickets is* 12

3 $p = 0.6$

$\Pr(X = 5) > 0.25$

$\binom{n}{5}(0.6)^5(0.4)^{n-5} > 0.25$

using CAS calculator, the minimum number of shots is 7

4 $p = 0.2$

$\Pr(X = 3) > 0.1$

$\binom{n}{3}(0.2)^3(0.8)^{n-3} > 0.1$

using the CAS calculator, the minimum number of chocolates is 7

5 $p = 0.35$

$\Pr(X \geq 2) > 0.9$

$1 - \Pr(X = 0) - \Pr(X = 1) > 0.9$

$(0.65)^n + n(0.35)\,(0.65)^{n-1} < 0.1$

using the CAS calculator, the minimum number of games is 10

6 $p = 0.07$

$\Pr(X > 1) > 0.8$

$1 - \Pr(X = 0) - \Pr(X = 1) > 0.8$

$(0.93)^n + n(0.07)(0.93)^{n-1} < 0.2$

using the CAS calculator, the minimum number of balls is 42

7 $p = 0.7$

$\Pr(X \geq 50) > 0.99$

using the CAS calculator the minimum number of shots is 86

Solutions to Technology-free questions

1 $$\begin{aligned}\Pr(X = 2) &= \binom{3}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)\\ &= 3 \times \frac{4}{125}\\ &= \frac{12}{125}\end{aligned}$$

2 $$\begin{aligned}\Pr(X = 1|X \geq 1) &= \frac{\Pr(X = 1)}{1 - \Pr(X = 0)}\\ &= \frac{\binom{5}{1}p(1-p)^4}{1-(1-p)^5}\\ &= \frac{5p(1-p)^4}{1-(1-p)^5}\end{aligned}$$

3 **a** $\Pr(X = 0) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

b

$$\begin{aligned}\Pr(X = 3|X \geq 1) &= \frac{\Pr(X = 3)}{1 - \Pr(X = 0)}\\ &= \frac{\left(\frac{2}{3}\right)^3}{1 - \left(\frac{1}{3}\right)^3}\\ &= \frac{8}{27} \times \frac{27}{26}\\ &= \frac{8}{26}\\ &= \frac{4}{13}\end{aligned}$$

4 **a** $np = 0.1 \times 20 = 2$

b

$$\begin{aligned}\Pr(X < 2) &= \Pr(X = 0) + \Pr(X = 1)\\ &= \left(\frac{19}{20}\right)^{20} + 20 \times \frac{1}{20} \times \left(\frac{19}{20}\right)^{19}\\ &= \left(\frac{19}{20}\right)^{19}\left(\frac{19}{20} + 1\right)\\ &= \left(\frac{19}{20}\right)^{19} \times \frac{39}{20}\\ &= \frac{39 \times 19^{19}}{20^{20}}\end{aligned}$$

5 **a**

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X = 0)\\ \therefore 1 - \Pr(X = 0) &= 0.9984\\ \Pr(X = 0) &= 0.0016(1 - p)^4 \quad = 0.0016\\ 1 - p &= 0.2\\ p &= 0.8\end{aligned}$$

b $$\begin{aligned}\Pr(X = 2) &= \binom{4}{2}p^2(1-p)^2\\ &= 6p^2(1-p)^2\\ &= 6(p(1-p))^2\end{aligned}$$

Maximum when $p = 0.5$

6 **a** $\Pr(x = 5) + \Pr(X = 6) = 0.1$

b $$\begin{aligned}\Pr(X = 1) &= \binom{5}{1} \times \frac{1}{10} \times \left(\frac{9}{10}\right)^4\\ &= \frac{5 \times 9^4}{10^5}\end{aligned}$$

c $200 \times 0.9 = 180$

7 **a** $\Pr(X = 3) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

b $$\begin{aligned}\Pr(Y \geq 1) &= 1 - \Pr(X = 0)\\ &= 1 - \left(\frac{26}{27}\right)^3\\ &= \frac{27^3 - 26^3}{27^3}\end{aligned}$$

Solutions to multiple-choice questions

1 D $n = 5, p = 0.6, X$ is Bi (n, p)

$$\Pr(X = 3) = \binom{5}{3}(0.6)^3(0.4)^2$$
$$= \frac{5 \times 4}{2 \times 1}(0.6)^3(0.4)^2$$
$$= 10 \times (0.6)^3(0.4)^2$$

2 A $n = 5, p = 0.35,\ X$ is Bi (n, p)

Pr(on time at least once)

$= 1 - \Pr(\text{lets all 5 days})$

$= 1 - (0.65)^5$

3 E $\Pr(\text{number} > 4\ ) = \Pr(5 \text{ or } 6) = \frac{1}{3}$

$n = 4, p = \frac{1}{3}, X$ is Bi (n, p)

$$\Pr(X = 2) = \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$$
$$= \frac{4 \times 3}{2 \times 1} \times \frac{1}{9} \times \frac{4}{9}$$
$$= \frac{8}{27}$$

4 B $n = 80, p = 0.4,$

X is Bi (n, p)

$\Pr(X < 30) = \Pr(0 \leq X \leq 29)$

$= 0.2861$

using a CAS Calculator.

5 A $n = 5,$

X is Bi $(5, p)$

$$\Pr(X \leq 1) = \Pr(X = 0) + \Pr(X = 1)$$
$$= (1 - p)^5 + 5p(1 - p)^4$$
$$= (1 - p)^4(1 - p + 5p)$$
$$= (1 - p)^4(1 + 4p)$$

6 A $\mathrm{E}(X) = np = 18 \times \frac{1}{3} = 6$

For $(X) = np(1 - p) = 6 \times \frac{2}{3} = 4$

So $\mu = 6, \sigma^2 = 4$

7 B Since $p = 0.7$, the distribution has a long tail to the left. The greatest probablity will be near the mean, which is $10 \times 0.7 = 7$. Hence the second graph is the best representation.

8 D $\Pr(X = 3) = \binom{10}{3}\left(\frac{1}{2}\right)^{10}$

$m = \binom{10}{3} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$

$= 120$

9 C $\mu = 4 \Rightarrow np = 4 \quad \ldots$ ①

$\sigma = \sqrt{3} \quad \sigma^2 = 3$

$\Rightarrow npq = 3 \quad \ldots$ ②

$(2) \div (1) : q = \frac{3}{4}$

$p = \frac{1}{4}$

10 C Use $(X)npq = np(1 - p) = 1.875n = 10 : p - p^2 = 0.1875$ Using a CAS *solve* command gives $p = 0.25$ or $p = 0.75$. (Automatically, note that $0.1875 = \frac{3}{16}$, so the quadratic can be expened as $p^2 - p + \frac{3}{16} = 0 = \left(p - \frac{1}{4}\right)\left(p - \frac{3}{4}\right) = 0$, so $p = \frac{1}{4}$ or $p = \frac{3}{4}$. Since the coin is biased towards heads, the probability of a head is 0.75.

11 **D** $50p = 1.8 \Rightarrow p = \frac{18}{500} = \frac{9}{250}$

$\Pr(X \le 3) = 0.7316$

12 **A**

$$\Pr(X = 40|X \ge 35) = \frac{\Pr(X = 40)}{\Pr(X \ge 35)}$$
$$= 0.0679$$

13 **E** Pr(Thomas wins at least one set)

1 − Pr(Thomas wins no set,) =

$1 - 0.76n$

$1 - 0.76n > 0.95$

$0.76n < 0.05$

A CAS Calculator show that

$0.76^{10} = 0.065\ldots$ and 0.76

$n = 0.048\ldots$

So that fewest number of days is 11.

(Alternatively, taking $\log_{10}$ of both side) gives $\log_{10} 0.76n < \log_{10} 0.05$

$n \log_{10} 0.76 < \log_{10} 0.05$

$$n > \frac{\log_{10} 0.05}{\log_{10} 076}$$

(Since $\log_{10} 0.76$ is negative)

so $n > 10.91\ldots$ and hence the least number of days is 11)

14 **B** Pr(Thomas wins at least one set)

$$= 1 - \text{Pr(no wins or one win)}$$
$$= 1 - 0.76n - \binom{n}{1}(0.24)^1(0.76)^{4-1}$$
$$= 1 - 0.76n - 0.24n\,(0.76)^{4-1}$$

A CAS calculator shows that this probability i) $0.940\ldots$ when $n = 17$ and $0.952\ldots$

when $n = 18$.

So the fewest number of day is 18.

(Note: An efficient way to use a CAS calculator is to first Define the function $f(n) = 1 - 0.76^n - 0.264(0.76)^{4-1}$ It is then a simple matter to evaluate $f(n)$ for various values of n.)

15 **E**

$$np = 2\sqrt{np(1-p)}$$
$$n^2p^2 = 4np(1-p)$$
$$np = 4(1-p)$$
$$np - 4 + 4p = 0$$
$$p(n+4) = 4$$
$$p = \frac{4}{n+4}$$
$$p \le 0.04 \Rightarrow \frac{4}{n+4} \le \frac{1}{25}$$
$$n + 4 \ge 100$$
$$n \ge 96$$

16 **A** $np = 8.4$ and $np(1-p) = 5.46$

$1 - p = 0.65$

$p = 0.35$

$n = \frac{8.4}{0.35} = 24$

Solutions to extended-response questions

1 **a** For children without disability there is an equal chance of answering A, B or C.
Let X be the number of questions out of 10 which are answered A or B. X is a binomial random variable with $n = 10$ and $p = \frac{2}{3}$

$\Pr(X = 10) = \left(\frac{2}{3}\right)^{10} = 0.0173$

The probability that the answers given by a child without either disability will be all As and Bs is 0.0173

b Pr(Answering C five or more times)

$= $ Pr(Answering A or B 5 or less times)

$= \Pr(X \le 5)$

$= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$

$$= \left(\frac{1}{3}\right)^{10} + \binom{10}{1}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{9} + \binom{10}{2}\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{8} + \binom{10}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{7}$$

$$+ \binom{10}{4}\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{6} + \binom{10}{5}\left(\frac{2}{3}\right)^{5}\left(\frac{1}{5}\right)^{5}$$

$$= \left(\frac{1}{3}\right)^{10} + 10 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^{9} + 45 \times \left(\frac{2}{3}\right)^{2} \times \left(\frac{1}{3}\right)^{8} + 120 \times \left(\frac{2}{3}\right)^{3} \times \left(\frac{1}{3}\right)^{7}$$

$$+ 210\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{6} + 252 \times \left(\frac{2}{3}\right)^{5} \times \left(\frac{1}{3}\right)^{5}$$

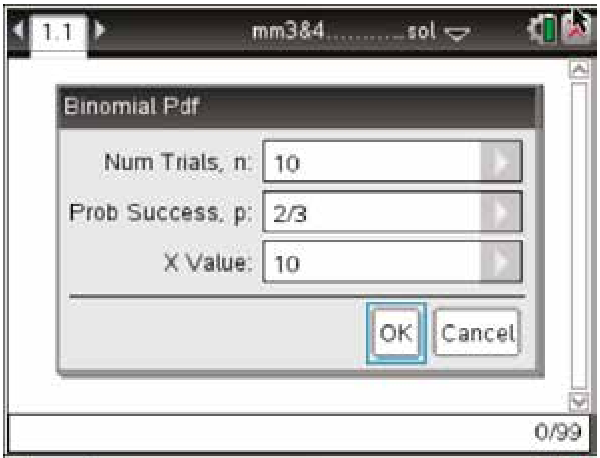
$$= \left(\frac{1}{3}\right)^{10}[1 + 20 + 180 + 960 + 3360 + 8064]$$

$$= \left(\frac{1}{3}\right)^{10}[12585]$$

$= 0.2131$

Graphic calculator techniques for question 1

a In a **Calculator** page select **Binomial Pdf** from the **Probability > Distributions** menu and complete as shown.

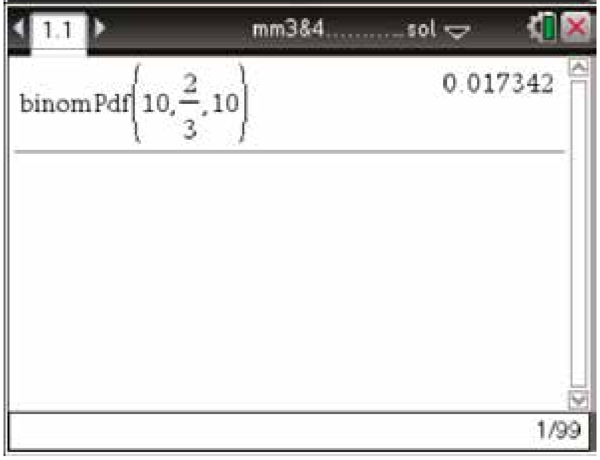


Binomial Pdf

Num Trials, n: 10

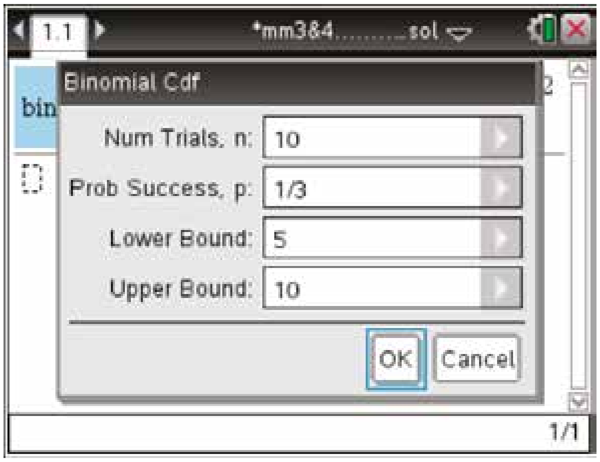
Prob Success, p: 2/3

X Value: 10



$\text{binomPdf}\left(10,\frac{2}{3},10\right)$ 0.017342

b For the cumulative binomial select **Binomial Cdf** from the **Probability > Distributions** menu and complete the dialogue box as shown.



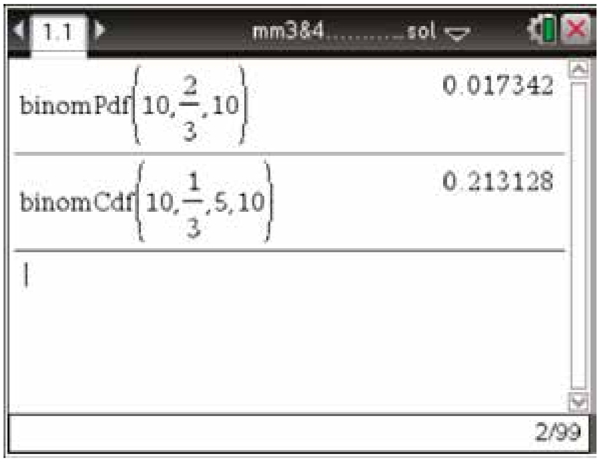
Binomial Cdf

Num Trials, n: 10

Prob Success, p: 1/3

Lower Bound: 5

Upper Bound: 10



binomPdf$\left(10, \frac{2}{3}, 10\right)$ 0.017342

binomCdf$\left(10, \frac{1}{3}, 5, 10\right)$ 0.213128

2 a i $\Pr(X = 1) = \binom{10}{1}(0.1)(0.9)^9 \approx 0.3874$

ii $\Pr(X > 1| \Pr(X \geq 1) = \Pr(X \geq 2|X \geq 1) = \dfrac{\Pr(X \geq 2)}{\Pr(X \geq 1)}$

≈ 0.4052

iii $p = \dfrac{12}{67} \approx 0.18$

iv $\Pr(Y > 1) \leq 0.03$

$1 - \Pr(Y = 0) - \Pr(Y = 1) \leq 0.03$

$1 - [(1 - p)^{50} + 50p(1 - p)^{49}] \leq 0.03$

Use graphical approach with CAS.

$0 \leq p \leq 0.005$

3 a i $p = \dfrac{1}{5}$, $n = 6$ Let X be the number of defectives.

$$\Pr(X = 3) = \binom{6}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3$$

$$= 0.0819$$

ii $\Pr(X < 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$

$$= (0.8)^6 + 6 \times (0.8)^5(0.2) + {}^6C_2(0.8)^4(0.2)^2$$

$$= 0.9011$$

b i Let X be the number of defectives.

$$\Pr(X = 2) = \binom{6}{2}p^2(1 - p)^4$$

$$= 15p^2(1 - p)^4$$

ii Let $P = 15p^2(1-p)^4$

$$\frac{dP}{dp} = 30p(1-p)^4 - 60p^2(1-p)^3$$

$$= 30p[1-p]^3[1-3p]$$

$$\frac{dP}{dp} = 0 \Rightarrow p = 1 \text{ or } p = \frac{1}{3} \text{ or } p = 0$$

Note when $p = 1$ or $p = 0$, $P = 0$

$\therefore \quad p = \frac{1}{3}$ gives a maximum probability.

4 **a** Mean value $= 1 \times \frac{53}{200} + 2 \times \frac{65}{200} + 3 \times \frac{45}{200} + 4 \times \frac{18}{200} + 5 \times \frac{2}{200}$

$= 2$

b $np = 2,\ n = 6 \therefore \quad p = \frac{1}{3}$

c *Using the CAS calculator*

i $\Pr(X = 2) = 0.3292$

ii $\Pr(X > 2|X > 1) = \frac{\Pr(X > 2)}{\Pr(X > 1)} = \frac{0.31962}{0.68834} = 0.4926$

5 **a** Let X be the number of faulty articles in as asample of size 10.
Then X is Bi $(n = 10,\ \text{p} = 0.05)$
Pr(batch accepted after first sample) $= \Pr(X < 2)$

$= \Pr(X = 0) \times \Pr(X = 1)$

Using a CAS calculator given 0.9138616 = 0.9139 correct to 4 decimal places.

b Batch is rejected if 3 or more faulty articles or *if* there are exactly 2 faulty articels and then a second sample of size 10 contains any faulty articles.
$\Pr(X \geq 3) = 0.0115036$

$\Pr(X = 2) = 0.0746348$

In a second sample, Pr($\geq$ 1 faulty articles) = 1 − Pr(o faulty articles)

$= 1 - 0.95^{\circ}$

$= 0.4012631$

Pr(batch rejected) $= 0.0115036 + 0.0746348 \times 0.4012631$

$= 0.0414517$

$= 0.04145$ correct to 4 significant figures.

c Either 10 which are tested or, if 2 of the sample of 10 are faulty, a second 10 (giving a total of 20) are tested.

Let $p' = \Pr(2 \text{ faulty article in first sample}),$ so

$1 - p' = \Pr(0, 1, 3, \ldots 10 \text{ faculty articles in first sample}).$

Thein if y = number of articles tested, this gives:

y	10	20
$\Pr(y - q)$	$1 - p'$	p'

$E(Y) = 10(1 - p') + 20p' = 10p' + 10$

From past b, $p' = 0.0746 \Rightarrow E(Y) = 10(0.0746) + 10 = 10.746$

6 **a** Let X be the number of people with a birthday in January.

$$\Pr(X = 2) = \binom{6}{2}\left(\frac{1}{12}\right)^2\left(\frac{11}{12}\right)^4 = 0.0735$$

b Let Y be the number of people with a birthday in January.

$$\Pr(Y \geq 1) = 1 - \Pr(Y = 0) = 1 - \left(\frac{11}{12}\right)^8$$

$$= 0.5015$$

c Let Z be the number of people with a birthday in January.

$$\Pr(Z \geq 1) = 1 - \Pr(Z = 0) = 1 - \left(\frac{11}{12}\right)^N$$

$$1 - \left(\frac{11}{12}\right)^N > 0.9$$

$$\Leftrightarrow \left(\frac{11}{12}\right)^N < 0.1$$

$$\Leftrightarrow N \log_e\left(\frac{11}{12}\right) < \log_e(0.1)$$

$$\Leftrightarrow N > \frac{\log_e(0.1)}{\log_e\left(\frac{11}{12}\right)}$$

$$\Leftrightarrow N > 26.46304$$

$\therefore$ Least value of $N = 27$.

7 For a two-engine plane

Let X be the number of engines which will fail.

The plane will successfully complete its journey if 0 or 1 engines fail.

$$\Pr(X = 0) + \Pr(X = 1) = (1 - q)^2 + 2q(1 - q)$$

$$= 1 - 2q + q^2 + 2q - 2q^2$$

$$= 1 - q^2$$

For a four-engine plane:

Let Y be the number of engines which will fail.

The plane will successfully complete its journey if 0, 1 or 2 engines fail.

$$\begin{aligned}\Pr(Y=0)+\Pr(Y=1)+\Pr(Y=2) &= (1-q)^4+4q(1-q)^3+6q^2(1-q)^2\\ &= (1-q)^2[(1-q)^2+4q(1-q)+6q^2]\\ &= (1-q)^2[1-2q+q^2+4q-4q^2+6q^2]\\ &= (1-q)^2[1+2q+3q^2]\end{aligned}$$

To find when a two-engine plane is to be preferred to a one-engine consider the inequality

$$1-q^2 > (1-q)^2(1+2q+3q^2)$$

$$(1-q)(1+q) > (1-q)^2(1+2q+3q^2)$$

$$\therefore \quad (1+q) > (1-q)(1+2q+3q^2)$$

$$(1+q) > 1+2q+3q^2-q-2q^2-3q^3q > q+q^2-3q^3$$

$$0 > q^2-3q^3$$

$$0 > q^2(1-3q)$$

$$\therefore \quad \frac{1}{3} \le q \le 1$$

A two-engine plane it to be preferred to a four-engine plane for $\frac{1}{3} \le q \le 1$

Chapter 15 – Continuous random variables and their probability distributions

Solutions to Exercise 15A

1 $$f(x)\begin{cases}\dfrac{24}{x^3} & 3 \le x \le 6\\ 0 & x < 3 \text{ or } x > 6\end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_3^6 \frac{24}{x^3}\,dx$$

$$= \left[\frac{-12}{x^2}\right]_3^6$$

$$= \frac{-12}{6} + \frac{12}{3}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$f(x) \ge 0$$

$\therefore$ *is a probability density function*

2 $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^2 x^2 + kx + 1\,dx = 1$$

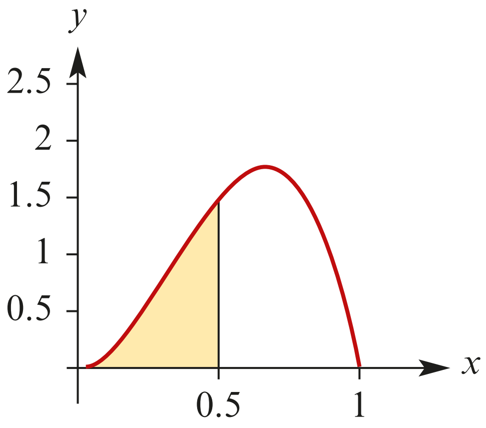
$$\left[\frac{x^3}{3} + \frac{kx^2}{2} + x\right]_0^2 = 1$$

$$\frac{8}{3} + \frac{4k}{2} + 2 = 1$$

$$2k = \frac{-11}{3}$$

$$k = \frac{-11}{6}$$

3 a and c



b

$$\Pr(X < 0.5) = \int_0^{0.5} 12x^2 - 12x^3\,dx$$

$$= \left[4x^3 - 3x^4\right]_0^{0.5}$$

$$= \frac{4}{8} - \frac{3}{16}$$

$$= \frac{8}{16} - \frac{3}{16}$$

$$= \frac{5}{16}$$

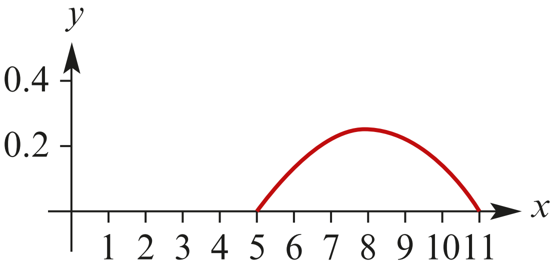
4 a
$$f(y) = \begin{cases} ke^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$
$$\int_{-\infty}^{\infty} f(y)\,dy = 1$$
$$\int_{0}^{\infty} ke^{-y}\,dy = 1$$

consider

$$\begin{aligned} \lim_{a\to\infty} \int_0^a ke^{-y}\,dy &= 1 \\ \lim_{a\to\infty} [-ke^{-y}]_0^a &= 1 \\ \lim_{a\to\infty} (-ke^{-a} + ke^0) &= 1 \\ -k \lim_{a\to\infty} (e^{-a}) + k &= 1 \\ k &= 1 \end{aligned}$$

b
$$f(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$
$$\begin{aligned} \Pr(Y \leq 2) &= \int_0^2 e^{-y}\,dy \\ &= [-e^{-y}]_0^2 \\ &= -e^{-2} + e^0 \\ &= 1 - \frac{1}{e^2} \approx 0.865 \end{aligned}$$

5 a



b
$$\begin{aligned} \Pr(T \leq 7) &= \int_5^7 \frac{1}{36}(t-5)(11-t)\,dt \\ &= \frac{-1}{36}\int_5^7 t^2 - 16t + 55\,dt \\ &= \frac{-1}{36}\left[\frac{t^3}{3} - 8t^2 + 55t\right]_5^7 \\ &= \frac{1}{36}\left(\left(\frac{125}{3} - 200 + 275\right)\right. \\ &\quad \left. - \left(\frac{343}{3} - 392 + 385\right)\right) \\ &= \frac{1}{36}\left(\frac{350}{3} - \frac{322}{3}\right) \\ &= \frac{28}{108} \\ &= \frac{7}{27} \approx 0.259 \end{aligned}$$

c
$$\begin{aligned} \Pr(T < 7 | T > 5.5) &= \frac{\Pr(5.5 < T < 7)}{\Pr(T > 5.5)} \\ &= \frac{\int_{5.5}^{7} \frac{1}{36}(t-5)(11-t)\,dt}{\int_{5.5}^{11} \frac{1}{36}(t-5)(11-t)\,dt} \\ &\approx 0.244 \end{aligned}$$

d
$$\begin{aligned} \Pr(T < 7 | T < 10) &= \frac{\Pr(T < 7)}{\Pr(T < 10)} \\ &= \frac{\int_{5}^{7} \frac{1}{36}(t-5)(11-t)\,dt}{\int_{5}^{10} \frac{1}{36}(t-5)(11-t)\,dt} \\ &\approx 0.28 \end{aligned}$$

6 a

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{7}^{17} k\sin\left(\frac{1}{10}\pi(x-7)\right)dx = 1$$

$$\left[\frac{-10}{\pi}k\cos\left(\frac{\pi}{10}(x-7)\right)\right]_{7}^{17} = 1$$

$$\frac{-10}{\pi}k\cos(\pi) + \frac{10}{\pi}k\cos(0) = 1$$

$$\frac{10}{\pi}k + \frac{10}{\pi}k = 1$$

$$k = \frac{\pi}{20} \quad QED$$

b **i**

$$\Pr(X \geq 16) = \int_{7}^{17} \frac{\pi}{20}\sin\left(\frac{\pi}{10}(x-7)\right)dx$$

$$= \left[\frac{-1}{2}\cos\left(\frac{\pi}{10}(x-7)\right)\right]_{7}^{17}$$

$$= \frac{-1}{2}\cos(\pi) + \frac{1}{2}\cos\left(\frac{9\pi}{10}\right)$$

$$\approx 0.024$$

ii $\Pr(12 \leq X \leq 13)$

$$\int_{12}^{13} \frac{\pi}{20}\sin\left(\frac{\pi}{10}(x-7)\right)dx$$

$$= \left[\frac{-1}{2}\cos\left(\frac{\pi}{10}(x-7)\right)\right]_{12}^{13}$$

$$= \frac{-1}{2}\cos\left(\frac{\pi}{10}\right) + \frac{1}{2}\cos\left(\frac{5\pi}{10}\right)$$

$$= \frac{-1}{2}\cos\left(\frac{3\pi}{5}\right)$$

$$\approx 0.155$$

7 a

$$\int_{-\infty}^{\infty} f(t)\,dt = 1$$

$$\int_{0}^{\infty} ke^{\left(\frac{-t}{200}\right)}\,dt = 1$$

Consider

$$\lim_{a\to\infty}\int_{0}^{a} ke^{\left(\frac{-t}{200}\right)}\,dt = 1$$

$$k \times \lim_{a\to\infty}\left[-200e^{\left(\frac{-t}{200}\right)}\right]_{0}^{a} = 1$$

$$k \times \lim_{a\to\infty}\left(-200e^{\left(\frac{-a}{200}\right)} + 200e^{0}\right) = 1$$

$$200k \times \lim_{a\to\infty}\left(1 - e^{\left(\frac{-a}{200}\right)}\right) = 1$$

$$200k = 1$$

$$k = \frac{1}{200}$$

$$= 0.005$$

b $\Pr(T \geq 1000) = \int_{1000}^{\infty} \frac{1}{200}e^{\left(\frac{-t}{200}\right)}dt$

Consider

$$\lim_{a\to\infty}\int_{1000}^{a} \frac{1}{200}e^{\left(\frac{-t}{200}\right)}dt$$

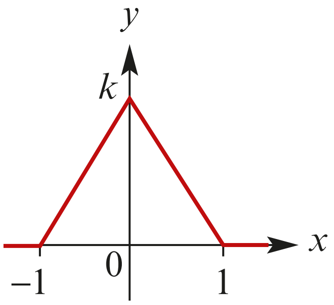
$$= \lim_{a\to\infty}\left[-e^{\left(\frac{-t}{200}\right)}\right]_{1000}^{a}$$

$$= \lim_{a\to\infty}\left(-e^{\left(\frac{-a}{200}\right)} + e^{-5}\right)$$

$$= \frac{1}{e^{5}}$$

$$\therefore \Pr(T \geq 1000) = \frac{1}{e^{5}} \approx 0.007$$

8 **a**



b

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-1}^{0} k(1+x)\,dx + \int_{0}^{1} k(1-x)\,dx = 1$$

$$\left[x+\frac{x^2}{2}\right]_{-1}^{0} + \left[x-\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{k}$$

$$0-\left(-1+\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)-0=\frac{1}{k}$$

$$\frac{1}{2}+\frac{1}{2}=\frac{1}{k}$$

$$k = 1$$

c $\Pr\left(\frac{-1}{2} \le X \le \frac{1}{2}\right)$

$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} f(x)\,dx$$

$$= \int_{\frac{-1}{2}}^{0} 1 + x + \,dx + \int_{0}^{\frac{1}{2}} 1 - x\,dx$$

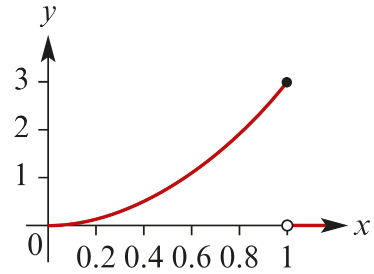
$$= \left[x+\frac{x^2}{2}\right]_{\frac{-1}{2}}^{0} + \left[x-\frac{x^2}{2}\right]_{0}^{\frac{1}{2}}$$

$$= 0-\left(-\frac{1}{2}+\frac{1}{8}\right)+\left(\frac{1}{2}-\frac{1}{8}\right)-0$$

$$= \frac{3}{8}+\frac{3}{8}$$

$$= \frac{3}{4}$$

9 **a**



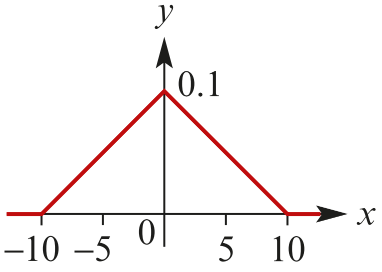
b $\Pr(0.25 < X < 0.75) = \int_{\frac{1}{4}}^{\frac{3}{4}} 3x^2\,dx$

$$= \left[x^3\right]_{\frac{1}{4}}^{\frac{3}{4}}$$

$$= \frac{27}{64} - \frac{1}{64}$$

$$= \frac{13}{32} \approx 0.406$$

10 **a**



b $\Pr(-1 \le X < 1)$

$$= \int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} \frac{1}{100}(10 + x)\,dx$$

$$+ \int_{0}^{1} \frac{1}{100}(10 - x)\,dx$$

$$= \frac{1}{100}\left[10x + \frac{x^2}{2}\right]_{-1}^{0}$$

$$+ \frac{1}{100}\left[10x - \frac{x^2}{2}\right]_{0}^{1}$$

$$= \frac{1}{100}\left(0 - \left(-10 + \frac{1}{2}\right)\right)$$

$$+ \left(10 - \frac{1}{2}\right) - 0$$

$$= \frac{19}{100} = 0.19$$

11 **a** $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{1000}^{\infty} \frac{k}{x^2}\,dx = 1$$

Consider

$$\lim_{a\to\infty} \int_{1000}^{a} \frac{k}{x^2}\,dx = 1$$

$$\lim_{a\to\infty} \left[\frac{-k}{x}\right]_{1000}^{a} = 1$$

$$\lim_{a\to\infty} \left(\frac{-k}{a} + \frac{k}{1000}\right) = 1$$

$$\frac{k}{1000} = 1$$

$$k = 1000$$

b $\Pr(X \ge 2000) = \int_{2000}^{\infty} \dfrac{1000}{x^2}\,dx$

Consider

$$\lim_{a\to\infty} \int_{2000}^{a} \frac{1000}{x^2}\,dx = \lim_{a\to\infty} \left[\frac{-1000}{x}\right]_{2000}^{a}$$

$$= \lim_{a\to\infty} \left(\frac{-1000}{a} + \frac{1000}{2000}\right)$$

$$= \frac{1}{2}$$

$$\therefore \quad \Pr(X \ge 2000) = \frac{1}{2}$$

12 **a** $\Pr(X \ge 1.5) = \displaystyle\int_{1.5}^{\infty} f(x)\,dx$

$$= \int_{1.5}^{2} 2\left(1 - \frac{1}{x^2}\right)dx$$

$$= \left[2\left(x + \frac{1}{x}\right)\right]_{1.5}^{2}$$

$$= 2\left(\left(2 + \frac{1}{2}\right) + \left(\frac{3}{2} + \frac{2}{3}\right)\right)$$

$$= \frac{2}{3}$$

b $\Pr(X \le 1.8 \mid X \ge 1.5)$

$$= \frac{\Pr(1.5 \le X \le 1.8)}{\Pr(X \ge 1.5)}$$

$$= \frac{\int_{1.5}^{1.8} 2\left(1 - \frac{1}{x^2}\right)dx}{\frac{2}{3}}$$

$$= \frac{3}{2}\left[2\left(x + \frac{1}{x}\right)\right]_{1.5}^{1.8}$$

$$= 3\left(\left(\frac{9}{5} + \frac{5}{9}\right) - \left(\frac{3}{2} + \frac{2}{3}\right)\right)$$

$$= \frac{17}{30}$$

13 a $\Pr(X \geq 8) = \int_8^\infty f(x)\,dx$

$$= \int_8^\infty \frac{1}{5}e^{\frac{-x}{5}}\,dx$$

Consider

$$\lim_{a\to\infty}\int_8^a \frac{1}{5}e^{\frac{-x}{5}}\,dx = \lim_{a\to\infty}\left[-e^{\frac{-x}{5}}\right]_8^a$$

$$= \lim_{a\to\infty}\left(-e^{\frac{-a}{5}} + e^{\frac{-8}{5}}\right)$$

$$= e^{\frac{-8}{5}}$$

$\therefore \quad \Pr(X \geq 8) = e^{\frac{-8}{5}} \approx 0.202$

b $\Pr(X \geq 12 \,|\, X \geq 8) = \dfrac{\Pr(X \geq 12)}{\Pr(X \geq 8)}$

$$= \frac{\int_{12}^\infty \frac{1}{5}e^{\frac{-x}{5}}\,dx}{e^{\frac{-8}{5}}}$$

Consider

$$\frac{\lim_{a\to\infty}\int_{12}^a \frac{1}{5}e^{\frac{-x}{5}}\,dx}{e^{\frac{-8}{5}}} = \frac{\lim_{a\to\infty}\left[-e^{\frac{-x}{5}}\right]_{12}^a}{e^{\frac{-8}{5}}}$$

$$= e^{\frac{8}{5}}\lim_{a\to\infty}\left(-e^{\frac{-a}{5}} + e^{\frac{-12}{5}}\right)$$

$$= e^{\frac{8}{5}} \times e^{\frac{-12}{5}}$$

$$= e^{\frac{-4}{5}}$$

$\therefore \quad \Pr(x \geq 12 \,|\, X \geq 8) = e^{\frac{-4}{5}} \approx 0.449$

14 a $\Pr(X \leq 0.5 \,| = 0.45)$

$$\int_{-\infty}^{0.5} f(x)\,dx$$

$$\int_{-1}^{0} 0.2\,dx + \int_{1}^{0.5} 0.2 + 1.2x\,dx$$

$$= \frac{1}{5} + \left[\frac{x}{5} + \frac{3x^2}{5}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{5} + \left(\frac{1}{10} + \frac{3}{20}\right)$$

$$= \frac{9}{20} = 0.45$$

b $\Pr(X > 0.5 \,|\, X > 0.1)$

$$= \frac{\Pr(X > 0.5)}{\Pr(X > 0.1)}$$

$$= \frac{1 - (a)}{\int_{0.1}^\infty f(x)\,dx}$$

$$= \frac{0.55}{\int_{0.1}^1 \frac{1}{5} + \frac{6}{5}x\,dx}$$

$$= \frac{11}{4\int_{0.1}^1 1 + 6x\,dx}$$

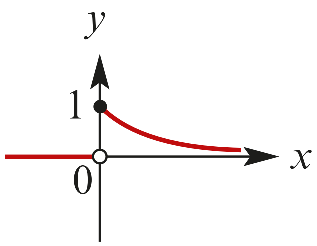
$$= \frac{11}{4\left[x + 3x^2\right]_{0.1}^1}$$

$$= \frac{11}{4\left((1+3) - \left(\frac{1}{10} + \frac{3}{100}\right)\right)}$$

$$= \frac{11}{\frac{387}{25}}$$

$$= \frac{275}{387} \approx 0.711$$

15 **a**



b **i** $\Pr(X < 0.5) = \int_0^{0.5} e^{-x}\,dx$

$$= [-e^{-x}]_0^{0.5}$$

$$= -e^{-0.5} + e^0$$

$$= 1 - \frac{1}{\sqrt{e}}$$

$$= 1 - e^{-\frac{1}{2}}$$

ii $\Pr(X \geq 1)$

$$\int_1^{\infty} e^{-x}\,dx$$

Consider

$$\lim_{a\to\infty} \int_1^{a} e^{-x}\,dx = \lim_{a\to\infty} [-e^{x}]_1^a$$

$$= \lim_{a\to\infty} (-e^{-a} + e^{-1})$$

$$= \frac{1}{e}$$

$$\therefore \quad \Pr(X \geq 1) = \frac{1}{e} = e^{-1}$$

iii

$$\Pr(X \geq 1 \mid X > 0.5) = \frac{\Pr(X \geq 1)}{\Pr(X > 0.5)}$$

$$= \frac{\frac{1}{e}}{\frac{1}{\sqrt{e}}}$$

$$= \frac{1}{\sqrt{e}}$$

$$= e^{-\frac{1}{2}}$$

Solutions to Exercise 15B

1 **a** $f(x) = 2x,\ 0 < x < 1$

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^1 2x^2\, dx \\ &= \left[\frac{2}{3}x^3\right]_0^1 \\ &= \frac{2}{3} \end{aligned}$$

b $f(x) = \dfrac{1}{2\sqrt{x}},\ 0 < x < 1$

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^1 \frac{\sqrt{x}}{2}\, dx \\ &= \left[\frac{1}{3}x^{\frac{3}{2}}\right]_0^1 \\ &= \frac{1}{3} \end{aligned}$$

c $f(x) = 6x - 6x^2,\ 0 < x < 1$

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^1 6x^2 - 6x^3\, dx \\ &= \left[2x^3 - \frac{3}{2}x^4\right]_0^1 \\ &= 2 - \frac{3}{2} = \frac{1}{2} \end{aligned}$$

d $f(x) = \dfrac{1}{x^2},\ x \geq 1$

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^1 \frac{1}{x}\, dx \end{aligned}$$

consider

$$\begin{aligned} \lim_{a\to\infty} \int_1^a \frac{1}{x}\, dx &= \lim_{a\to\infty} [\ln|x|]_1^a \\ &= \lim_{a\to\infty} (\ln a - \ln 1) \\ &= \lim_{a\to\infty} \ln a \end{aligned}$$

$\therefore$ E(X) *does not exist*

2 **a** 1

b 2.097

c 1.132

d 0.4444

3 **a**

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^1 2x^4 - x^2 + x\, dx \\ &= \left[\frac{2}{5}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^1 \\ &= \frac{2}{5} - \frac{1}{3} + \frac{1}{2} \\ &= \frac{12 - 10 + 15}{30} \\ &= \frac{17}{30} \approx 0.567 \end{aligned}$$

b

$$\begin{aligned} \Pr(X \leq \mu) &= \int_{-\infty}^{\mu} f(x)\, dx \\ &= \int_0^{\frac{17}{30}} 2x^3 - x + 1\, dx \\ &= \left[\frac{1}{2}x^4 - \frac{1}{2}x^2 + x\right]_0^{\frac{17}{30}} \\ &= \frac{1}{2}\left(\frac{17}{30}\right)^4 - \frac{1}{2}\left(\frac{17}{30}\right)^2 + \frac{17}{30} \\ &\approx 0.458 \end{aligned}$$

4 $f(x) = \frac{1}{2\pi} + \frac{1}{2\pi}\cos x,\ -\pi \le x \le \pi$

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{x}{2\pi} + \frac{x\cos x}{2\pi}\,dx$$

using the CAS calculator = 0

Alternatively, notice that the function is symmetrical about the *y*-axis, so the mean must be 0.

5 $\int_{-\infty}^{\infty} f(y)\,dy = 1$

$$\int_{0}^{B} Ay\,dy = 1$$

$$\left[\frac{A}{2}y^2\right]_0^B = 1$$

$$\frac{AB^2}{2} = 1$$

① $AB^2 = 2$

$$\mu = \int_{-\infty}^{\infty} yf(y)\,dy$$

$$2 = \int_{0}^{B} Ay^2\,dy$$

$$2 = \left[\frac{A}{3}y^3\right]_0^B$$

② $AB^3 = 6$

$\frac{②}{①} \Rightarrow B = 3$

Sub in ① ⇒ $\mathrm{A}(3)^2 = 2$

$$A = \frac{2}{9}$$

6 a $\mathrm{E}\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\,dx$

$$= \int_{0}^{1} 12x - 12x^2\,dx$$

$$= [6x^2 - 4x^3]_0^1$$

$$= 6 - 4 = 2$$

b $\mathrm{E}(e^x) = \int_{-\infty}^{\infty} e^x f(x)\,dx$

$$= \int_{0}^{1} 12x^2e^x(1 - x)\,dx$$

using the CAS calculator = 1.858

7 a $\Pr(X \le 1)$

$$\int_{0}^{1} e^{-x}\,dx = \left[-e^{-x}\right]_0^1$$

$$= -e^1 + e^{-0}$$

$$= 1 - \frac{1}{e} \approx 0.632$$

b $\Pr(1 \le X \le 2) = \int_{1}^{2} e^{-x}\,dx$

$$= [-e^{-x}]_1^2$$

$$= -e^{-2} + e^{-1}$$

$$= \frac{e - 1}{e^2} \approx 0.233$$

c $\int_{0}^{m} e^{-x}\,dx = \frac{1}{2}$

$$[-e^{-x}]_0^m = \frac{1}{2}$$

$$-e^{-m} + e^0 = \frac{1}{2}$$

$$1 - e^{-m} = \frac{1}{2}$$

$$e^{-m} = \frac{1}{2}$$

$$e^m = 2$$

$$m = \ln 2 \approx 0.693$$

8 **a** $$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= 1\\ \int_0^1 k\,dx &= 1\\ [kx]_0^1 &= 1\\ k &= 1\end{aligned}$$

b $$\begin{aligned}\int_0^m 1\,dx &= \frac{1}{2}\\ [m]_0^1 &= \frac{1}{0}\\ m &= \frac{1}{2}\end{aligned}$$

9 $f(x) = 5(x-1)^4, 0 \le x \le 1$
$$\begin{aligned}\int_0^m 5(x-1)^4\,dx &= 0.25\\ [(x-1)^5]_0^m &= 0.25\\ (m-1)^5 - (-1)^5 &= 0.25\\ (m-1)^5 &= -0.75\\ m-1 &= \left(-0.75\right)^{\frac{1}{5}}\\ m &= (-0.75)^{\frac{1}{5}} + 1 \approx 0.\ 0.0559\end{aligned}$$

10 $$\begin{aligned}\int_0^c \frac{1}{4}e^{\frac{-x}{4}}\,dx &= 0.9\\ \left[-e^{\frac{-x}{4}}\right]_0^c &= 0.9\\ -e^{\frac{-c}{4}} + e^0 &= 0.9\\ e^{\frac{-c}{4}} &= 0.1\\ e^{\frac{c}{4}} &= 10\\ \frac{m}{4} &= \ln 10\\ m &= 4\ln 10 \approx 9.210 \text{ minutes}\end{aligned}$$

11 **a** $$\begin{aligned}\mu &= \int_{-\infty}^{\infty} xf(x)\,dx\\ &= \int_0^1 x^2\,dx + \int_1^2 2x - x^2\,dx\\ &= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2\\ &= \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\\ &= \frac{1}{3} + 4 - \frac{10}{3}\\ &= 1\end{aligned}$$

b $$\int_0^m f(x)\,dx = \frac{1}{2}$$

if $m \le 1$,
$$\begin{aligned}\int_0^m x\,dx &= \frac{1}{2}\\ \left[\frac{x^2}{2}\right]_0^m &= \frac{1}{2}\\ \frac{m^2}{2} &= \frac{1}{2}\\ m^2 &= 1\\ m &= 1\end{aligned}$$

12 $f(x) = 30x^4 - 30x^5,\ 0 < x < 1$

a $$\begin{aligned}\mu &= \int_{-\infty}^{\infty} xf(x)\,dx\\ &= \int_0^1 30x^5 - 30x^6\,dx\\ &= \left[5x^6 - \frac{30}{7}x^7\right]_0^1\\ &= 5 - \frac{30}{7}\\ \mu &= \frac{5}{7} \approx 0.714\end{aligned}$$

b
$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$$
$$\int_{0}^{m} 30x^4 - 30x^5\,dx = \frac{1}{2}$$
$$\left[6x^5 - 5x^6\right]_0^m = \frac{1}{2}$$
$$s^{6m^5} - 5m^6 = \frac{1}{2}$$

using the CAS calculator

$m \approx 0.736$

$\mu = 0.714 < m$ QED

13 $f(x) = \frac{\pi}{20}\sin\left(\frac{\pi}{10}(x-7)\right), 7 \le x \le 17$
$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$$
$$\int_{7}^{m} \frac{\pi}{20}\sin\left(\frac{\pi}{10}(x-7)\right)dx = \frac{1}{2}$$
$$\left[\frac{-1}{2}\cos\left(\frac{\pi}{10}(x-7)\right)\right]_7^m = \frac{1}{2}$$
$$-\cos\left(\frac{\pi}{10}(m-7)\right) + \cos 0 = 1$$
$$\cos\left(\frac{\pi}{10}(m-7)\right) = 0$$
$$\frac{\pi}{10}(m-7) = \frac{\pi}{2}$$
$$m - 7 = 5$$
$$m = 12$$

14 **a** $\mu = \int_{-\infty}^{\infty} xf(x)\,dx$
$$= \int_{-1}^{0} \frac{x}{5}\,dx + \int_{0}^{1} \frac{x}{5} + \frac{6x^2}{5}\,dx$$
$$= \left[\frac{x^2}{10}\right]_{-1}^{0} + \left[\frac{x^2}{10} + \frac{2x^3}{5}\right]_0^1$$
$$= 0 - \frac{1}{10} + \frac{1}{10} + \frac{2}{5} - 0$$
$$= \frac{2}{5}$$

b $\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$

if $m \le 0$,
$$\int_{-1}^{m} \frac{1}{5}\,dx = \frac{1}{2}$$
$$\left[\frac{x}{5}\right]_{-1}^{m} = \frac{1}{2}$$
$$\frac{m}{5} + \frac{1}{5} = \frac{1}{2}$$
$$m > 0$$

$$\therefore \int_{-1}^{0} \frac{1}{5}\,dx + \int_{0}^{m} \frac{1}{5} + \frac{6}{5}x\,dx = \frac{1}{2}$$

$$\frac{1}{5} + \left[\frac{x}{5} + \frac{3x^2}{5}\right]_0^m = \frac{1}{2}$$

$$\frac{m}{5} + \frac{3m^2}{5} - \frac{3}{10} = 0$$

$$3m^2 + m - \frac{3}{2} = 0$$

$$m = \frac{-1 \pm \sqrt{1+18}}{6}$$

since $m > 0$,

$$m = \frac{-1 + \sqrt{19}}{6}$$

15 a $\frac{d}{dx}(kxe^{-kx}) = ke^{-kx} - k^2xe^{-kx}$

$$\int kxe^{-kx}\,dx = \frac{-1}{k}\int -k^2xe^{-kx}$$

$$+ ke^{-kx} - ke^{-kx}\,dx$$

$$= \frac{-1}{k}\int ke^{-kx}$$

$$- k^2xe^{-kx}\,dx$$

$$+ \int e^{-kx}\,dx$$

$$= \frac{-1}{k}(kxe^{-kx}) + \frac{-1}{k}e^{-kx}$$

$$= -xe^{-kx} - \frac{-1}{k}e^{-kx}$$

$$= -\frac{(kx+1)}{k}e^{-kx}$$

b $\mu = \int_{-\infty}^{\infty} xf(x)\,dx$

$$= \int_{0}^{\infty} \frac{x}{\lambda}e^{\frac{-x}{\lambda}}\,dx$$

Consider

$$\lim_{a\to\infty}\int_1^a x\left(\frac{1}{\lambda}\right)e^{-\left(\frac{1}{\lambda}\right)x}\,dx$$

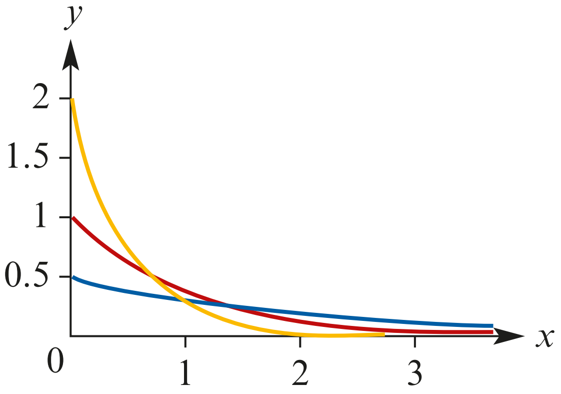
$$a = \lim_{a\to\infty}\left[xe^{-\frac{1}{\lambda}x} - \lambda e^{-\frac{1}{\lambda}x}\right]_0^a$$

$$= \lim_{a\to\infty}\left(-e^{-\frac{a}{\lambda}}(a+\lambda) + e^0(0+\lambda)\right)$$

$$= \lambda + \lim_{a\to\infty}\left(-(a+\lambda) - e^{-\frac{a}{\lambda}}\right)$$

$\mu = \lambda$

c



d $y = e^{-x}$ is dilated by factor $\frac{1}{\lambda}$ from the x-axis and by factor λ from the y-axis

Solutions to Exercise 15C

1

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx \\
&= \int_0^1 2x^2\,dx \\
&= \left[\frac{2}{3}x^3\right]_0^1 \\
&= \frac{2}{3} \\
(\mathrm{E}(X))^2 &= \frac{4}{9} \\
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\
&= \int_0^1 2x^3\,dx \\
&= \left[\frac{1}{2}x^4\right]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\
&= \frac{9}{18} - \frac{8}{18} = \frac{1}{18} \\
\sigma &= \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}
\end{aligned}$$

2 **a**

$$\begin{aligned}
\int_{-\infty}^{a} x f(x)\,dx &= \frac{1}{4} \\
\int_0^a 3x^2\,dx &= \frac{1}{4} \\
[x^3]_0^a &= \frac{1}{4} \\
a^3 &= \frac{1}{4} \\
a &= \left(\frac{1}{4}\right)^{\frac{1}{3}} \approx 0.630
\end{aligned}$$

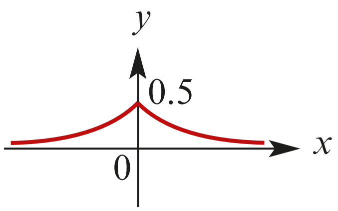
b

$$\begin{aligned}
\int_{-\infty}^{b} f(x)\,dx &= \frac{3}{4} \\
\int_0^b 3x^2\,dx &= \frac{3}{4} \\
[x^3]_0^b &= \frac{3}{4} \\
b^3 &= \frac{3}{4} \\
b &= \left(\frac{3}{4}\right)^{\frac{1}{3}} \approx 0.909
\end{aligned}$$

c

$$\begin{aligned}
\textit{the interquartile range} &= b - a \\
&= \left(\frac{3}{4}\right)^{\frac{1}{3}} - \left(\frac{1}{4}\right)^{\frac{1}{3}} \\
&\approx 0.279
\end{aligned}$$

3 **a**



b

$$\int_{-\infty}^{a} f(x)\,dx = \frac{1}{4}$$

$$\int_{-\infty}^{a} 0.5e^x\,dx = \frac{1}{4}, \textit{ since } a < 0.$$

consider

$$\lim_{k\to-\infty}\int_k^a \frac{1}{2}e^x\,dx = \frac{1}{4}$$

$$\lim_{k\to-\infty}[e^x]_k^a = \frac{1}{2}$$

$$\lim_{k\to-\infty}(e^a - e^k) = \frac{1}{2}$$

$$e^a = \frac{1}{2}$$

$$a = \ln\frac{1}{2}$$

$$a = -\ln 2$$

$$\int_b^\infty \frac{1}{2}e^{-x}\,dx = \frac{1}{4},\ \textit{since } b > 0$$

$$\int_b^\infty e^{-x}\,dx = \frac{1}{2}$$

consider

$$\lim_{h\to\infty}\int_b^h e^{-x}\,dx = \frac{1}{2}$$

$$\lim_{h\to\infty}[-e^{-x}]_b^h = \frac{1}{2}$$

$$\lim_{h\to\infty}(-e^{-h} + e^{-b}) = \frac{1}{2}$$

$$e^{-b} = \frac{1}{2}$$

$$-b = \ln\frac{1}{2}$$

$$b = \ln 2$$

$$\textit{the interquartile range} = b - a$$

$$= \ln 2 - (-\ln 2)$$

$$= 2\ln 2 \approx 1.386$$

4 a $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_1^9 \frac{k}{x}\,dx = 1$$

$$[k\ln x]_1^9 = 1$$

$$k\ln 9 - k\ln 1 = 1$$

$$k\ln 9 = 1$$

$$k = \frac{1}{\ln 9}$$

b $$\mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$\int_1^9 \frac{1}{\ln 9}\,dx = \frac{8}{\ln 9} = \frac{4}{\ln 3} \approx 3.641$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_1^9 \frac{x}{\ln 9}\,dx$$

$$= \left[\frac{x^2}{4\ln 3}\right]_1^9$$

$$= \frac{81}{4\ln 3} - \frac{1}{4\ln 3}$$

$$= \frac{20}{\ln 3}$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2$$

$$= \frac{20}{\ln 3} - \frac{16}{(\ln 3)^2}$$

$$= \frac{20\ln 3 - 16}{(\ln 3)^2} \approx 4.948$$

5 a $$\int_{-\infty}^{a} f(x)\,dx = \frac{1}{4}$$

$$\int_{0}^{a} 2 - 2x\,dx = \frac{1}{4}$$

$$[2x - x^2]_0^a = \frac{1}{4}$$

$$2a - a^2 = \frac{1}{4}$$

$$a^2 - 2a + \frac{1}{4} = 0$$

$$a = \frac{2 \pm \sqrt{4-1}}{2}$$

$$a = 1 - \frac{\sqrt{3}}{2},$$

since $0 \le a \le 1$

$$\int_{-\infty}^{b} f(x)\,dx = \frac{3}{4}$$

$$\int_{0}^{b} 2 - 2x\,dx = \frac{3}{4}$$

$$\left[2x - x^2\right]_0^b = \frac{3}{4}$$

$$2b - b^2 = \frac{3}{4}$$

$$b^2 - 2b + \frac{3}{4} = 0$$

$$b = \frac{2 \pm \sqrt{4-3}}{2}$$

$$b = 1 \pm \frac{1}{2}$$

$$b = \frac{1}{2},\ \textit{since } 0 \le b \le 1$$

$$= b - a$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} - 1$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$\approx 0.366$$

b $$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_0^1 2x - 2x^2\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 2x^2 - 2x^3\,dx$$

$$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6}$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2$$

$$= \frac{1}{6} - \frac{1}{9}$$

$$= \frac{1}{18}$$

6 $$\int_{-\infty}^{a} f(x)\,dx = \frac{1}{4}$$

$$\int_0^a 2xe^{-x^2}\,dx = \frac{1}{4}$$

$$\left[e^{-x2}\right]_0^a = \frac{1}{4}$$

$$-e^{-a^2} + e^0 = \frac{1}{4}$$

$$e^{-a^2} = \frac{3}{4}$$

$$-a^2 = \ln\frac{3}{4}$$

$$a = +\sqrt{\ln\frac{4}{3}} \approx 0.5364,$$

since $a > 0$

$$\int_{-\infty}^{b} f(x)\,dx = \frac{3}{4}$$

$$\int_{0}^{b} 2xe^{-x^2}\,dx = \frac{3}{4}$$

$$\left[e^{-x^2}\right]_0^b = \frac{3}{4}$$

$$e^{-b^2} + e^0 = \frac{3}{4}$$

$$e^{-b^2} = \frac{1}{4}$$

$$-b^2 = \ln\frac{1}{4}$$

$$b = +\sqrt{\ln 4} \approx 1.1774,$$

since $b > 0$

the interquartile range $= b - a$

≈ 0.641

7 a $$\int_{-\infty}^{a} f(x)\,dx = \frac{1}{4}$$

$$\int_{0}^{a} \frac{x}{2}\,dx = \frac{1}{4}$$

$$\int_{0}^{a} 2x\,dx = 1$$

$$\left[x^2\right]_0^a = 1$$

$$a^2 = 1$$

$a = 1,$ *since* $0 \le a \le 2$

$$\int_{-\infty}^{b} f(x)\,dx = \frac{3}{4}$$

$$\int_{0}^{b} \frac{x}{2}\,dx = \frac{3}{4}$$

$$\int_{0}^{b} 2x\,dx = 3$$

$$\left[x^2\right]_0^b = 3$$

$$b^2 = 3$$

$b = \sqrt{3},$ *since* $0 \le b \le 2$

the interquartile range $= b - a$

$= \sqrt{3} - 1$

≈ 0.732

b $$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_{0}^{2} \frac{x^2}{2}\,dx$$

$$= \left[\frac{x^3}{6}\right]_0^2$$

$$= \frac{8}{6} - = \frac{4}{3}$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{0}^{2} \frac{x^3}{3}\,dx$$

$$= \left[\frac{x^4}{8}\right]_0^2$$

$$= \frac{16}{8} = 2$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2$$

$$= 2 - \frac{16}{9} = \frac{2}{9}$$

8 a

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= 1\\
\int_0^{10} kx(100 - x^2)\,dx &= 1\\
k\int_0^{10} 100x - x^3\,dx &= 1\\
k\left[50x^2 - \frac{x^4}{4}\right]_0^{10} &= 1\\
k(5000 - 2500) &= 1\\
k &= \frac{1}{2500}\\
&= 0.0004
\end{aligned}$$

b

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} x\,f(x)\,dx\\
&= \int_0^{10} \frac{100x^2 - x^4}{2500}\,dx\\
&= \frac{1}{2500}\left[\frac{100}{3}x^3 - \frac{x^5}{5}\right]_0^{10}\\
&= \frac{1}{2500}\left(\frac{100000}{3} - \frac{100000}{5}\right)\\
&= \frac{1}{25}\left(\frac{2000}{15}\right)\\
&= \frac{80}{15}\\
&= \frac{16}{3}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx\\
&= \int_0^{10} \frac{1}{2500}(100x^3 - x^5)\,dx\\
&= \frac{1}{2500}\left[25x^4 - \frac{1}{6}x^6\right]_0^{10}\\
&= \frac{1}{2500}\left(250000 - \frac{1000000}{6}\right)\\
&= \frac{1}{25}\left(\frac{15000 - 10000}{6}\right)\\
&= \frac{5000}{150}\\
&= \frac{100}{3}
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \mathrm{E}(X^2) - \mu^2\\
&= \frac{100}{3} - \frac{256}{9}\\
&= \frac{44}{9}\\
\sigma &= \frac{2\sqrt{11}}{3} \approx 2.21
\end{aligned}$$

9 a

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= 1\\
k\int_{-a}^{a} a^2 - x^2\,dx &= 1\\
k\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} &= 1\\
k\left(\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right) &= 1\\
\frac{4}{3}a^3k &= 1\\
k &= \frac{3}{4a^3}
\end{aligned}$$

b $\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{-a}^{a} \frac{3x}{4a} - \frac{3x^3}{4a^3}\, dx$$

$$= \left[\frac{3x^2}{8a} - \frac{3x^4}{16a^3}\right]_{-a}^{a}$$

$$= \left(\frac{3a^2}{8a} - \frac{3a^4}{16a^3}\right) - \left(\frac{3a^2}{8a} - \frac{3a^4}{16a^3}\right)$$

$$= 0$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-a}^{a} \frac{3x^2}{4a} - \frac{3x^4}{4a^3}\, dx$$

$$= \left[\frac{x^3}{4a} - \frac{3x^5}{20a^3}\right]_{-a}^{a}$$

$$= 2\left(\frac{a^3}{4a} - \frac{3a^5}{20a^3}\right)$$

$$= 2\left(\frac{a^2}{4} - \frac{3a^2}{20}\right)$$

$$= \frac{5a^2}{10} - \frac{3a^2}{10}$$

$$= \frac{a^2}{5}$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2$$

$$= \frac{a^2}{5} - 0$$

$$= \frac{a^2}{5}$$

but $\sigma = 2$

$\therefore \quad \sigma^2 = 4$

$$\frac{a^2}{5} = 4$$

$$a^2 = 20$$

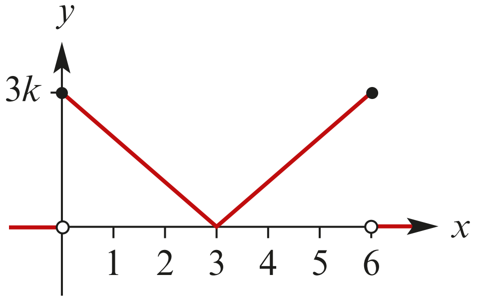
$$a = \pm\sqrt{20}$$

$$a = \pm 2\sqrt{5}$$

note: the way the question is stated implies that a is positive,

$\therefore \quad a = 2\sqrt{5}$

10 a



b

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_0^3 k(3-x)\, dx + \int_3^6 k(3-x)\, dx = 1$$

$$k\left(\left[3x - \frac{x^2}{2}\right]_0^3 + \left[\frac{x^2}{2} - 3x\right]_3^6\right) = 1$$

$$k\left(\left(\left(9 - \frac{9}{2}\right) - 0\right) + \left(\frac{36}{2} - 18\right) - \left(\frac{9}{2} - 9\right)\right) = 1$$

$$k \times 9 = 1$$

$$k = \frac{1}{9}$$

c

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} x f(x)\,dx\\ &= \int_0^3 \frac{x}{9}(3-x)\,dx + \int_3^6 \frac{x}{9}(x-3)\,dx\\ &= \int_0^3 \frac{x}{3} - \frac{x^2}{9}\,dx + \int_3^6 \frac{x^2}{9} - \frac{x}{3}\,dx\\ &= \left[\frac{x^2}{6} - \frac{x^3}{27}\right]_0^3 + \left[\frac{x^3}{27} - \frac{x^2}{6}\right]_3^6\\ &= \left(\left(\frac{9}{6} - \frac{27}{27}\right) - 0\right) + \left(\frac{216}{27} - \frac{36}{6}\right)\\ &\quad - \left(\frac{27}{27} - \frac{9}{6}\right)\\ &= \frac{1}{2} + 2 + \frac{1}{2}\\ &= 3 \quad \mathit{QED}\end{aligned}$$

$$\begin{aligned}\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx\\ &= \int_0^3 \frac{x^2}{3} - \frac{x^3}{9}\,dx\\ &\quad + \int_3^6 \frac{x^3}{9} - \frac{x^2}{3}\,dx\\ &= \left[\frac{x^3}{9} - \frac{x^4}{36}\right]_0^3 + \left[\frac{x^4}{36} - \frac{x^3}{9}\right]_3^6\\ &= \left(\left(\frac{27}{9} - \frac{81}{36}\right) - 0\right)\\ &\quad + \left(\frac{6^4}{36} - \frac{216}{9}\right) - \left(\frac{81}{36} - \frac{27}{9}\right)\\ &= 3 - \frac{9}{4} + 36 - 24 - \frac{9}{4} + 3\\ &= 18 - \frac{9}{2}\\ &= \frac{27}{2}\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \mathrm{E}(X^2) - \mu^2\\ &= \frac{27}{2} - 9\\ &= \frac{9}{2} = 4.5\end{aligned}$$

Solutions to Exercise 15D

1 a

$$\begin{aligned}
\mathrm{E}(X) &= 4\\
C &= 300X + 100\\
\mathrm{E}(C) &= \mathrm{E}(300X + 100)\\
&= 300\mathrm{E}(X) + 100\\
&= 1300
\end{aligned}$$

b

$$\begin{aligned}
\mathrm{Var}(X) &= 0.25\\
C &= 300X + 100\\
\mathrm{Var}(C) &= \mathrm{Var}(300X + 100)\\
&= 90000\mathrm{Var}(X)\\
&= 22500
\end{aligned}$$

2 a

b

$$\begin{aligned}
V &= 2X + 3\\
E(V) &= 2 \times E(X) + 3 = \frac{91}{20}
\end{aligned}$$

3 a

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx\\
&= \int_{5}^{10} \frac{10}{x}\, dx\\
&= \Big[10 \ln x\Big]_{5}^{10}\\
&= 6.93147\\
\mathrm{E}(U) &= 3 \times E(X) + 25\\
&= 45.794
\end{aligned}$$

b

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\, dx\\
&= \int_{5}^{10} 10\, dx\\
&= \Big[10x\Big]_{5}^{10}\\
&= 100 - 50 = 50\\
\mathrm{Var}\,(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2\\
&= 50 - 6.93147^2\\
&\approx 1.95472\\
\mathrm{Var}(U) &= 9 \times Var(X)\\
&= 17.592
\end{aligned}$$

4 a

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx\\
&= \int_{0}^{1} \frac{3x^3}{2} + x^2\, dx\\
&= \left[\frac{3x^4}{8} + \frac{x^3}{3}\right]_{0}^{1}\\
&= \frac{3}{8} + \frac{1}{3} = \frac{17}{24}
\end{aligned}$$

b

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\, dx\\
&= \int_{0}^{1} \frac{3x^4}{2} + x^3\, dx\\
&= \left[\frac{3x^5}{10} + \frac{x^4}{4}\right]_{0}^{1}\\
&= \frac{3}{10} + \frac{1}{4} = \frac{11}{20}\\
\mathrm{Var}\,(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2\\
&= \frac{11}{20} - \frac{17^2}{24^2}\\
&\approx 0.048
\end{aligned}$$

5 **a**

$$\begin{aligned} E(3X) &= \int_{-\infty}^{\infty} 3xf(x)\,dx \\ &= \int_{-1}^{1} \frac{9x^3}{2}\,dx \\ &= \left[\frac{9x^4}{8}\right]_{-1}^{1} \\ &= \frac{9}{8} - \frac{9}{8} = 0 \\ E(9X^2) &= \int_{-\infty}^{\infty} 9x^2 f(x)\,dx \\ &= \int_{-1}^{1} \frac{27x^4}{2}\,dx \\ &= \left[\frac{27x^5}{10}\right]_{-1}^{1} \\ &= \frac{27}{10} + \frac{27}{10} = \frac{27}{5} = 5.4 \\ \text{Var}\,(3X) &= E(9X^2) - E(3X)^2 \\ &= \frac{27}{5} \end{aligned}$$

b

$$\begin{aligned} E(3 - X) &= 3 - E(X) \\ &= 3 - \frac{1}{3}E(3X) \\ &= 3 - 0 = 3 \\ \text{Var}(3 - X) &= \text{Var}\,(X) \\ &= \frac{1}{9}\text{Var}\,(3X) \\ &= \frac{3}{5} = 0.6 \end{aligned}$$

c 1, 5.4

d Let $g(x)$ be the function required we know $g(x) = ax^2$ and since $-1 \le x \le 1,\ -3 \le 3x \le 3$

$$\therefore \quad g(x) = \begin{cases} ax^2 & -3 \le 3x \le 3 \\ 0 & x < -3 \text{ or } x > 3 \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)\,dx &= 1 \\ \int_{-3}^{3} ax^2\,dx &= 1 \\ \left[a\frac{x^3}{3}\right]_{-3}^{3} &= 1 \\ 9a - (-9a) &= 1 \\ a &= \frac{1}{18} \end{aligned}$$

$$\therefore \quad g(x) = \begin{cases} \dfrac{x^2}{18} & -3 \le x \le 3 \\ 0 & x < -3 \text{ or } x > 3 \end{cases}$$

e $h(x) = \begin{cases} \dfrac{(x-1)^2}{18} & -2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

Solutions to Exercise 15E

1 **a** $F(x) = \int_0^x f(t)\,dt$

$= \int_0^x \frac{1}{5}\,dt$

$= \left[\frac{t}{5}\right]_0^x$

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{x}{5} & \text{if } 0 < x \leq 5 \\ 1 & x \geq 5 \end{cases}$$

b $\Pr(X \leq 3) = F(3)$

$= \frac{3}{5}$

2 **a** **First element of hybrid function.**

$\int_0^x \frac{1}{4}\,dt = \frac{x}{4}$

$F(x) = \int_0^1 \frac{1}{4}\,dt = \frac{1}{4}$

where $0 \leq x < 1$

Second element of hybrid function.

$F(x) = \frac{x^4}{20} + c$

When $x = 1, F(x) = \frac{1}{4}$

Therefore, $\frac{1}{4} = \frac{1}{20} + c$

$c = \frac{1}{5}$

$F(x) = \frac{x^4}{20} + \frac{1}{5}$

where $1 \leq x \leq 2$

The hybrid function

$$F(x) = \begin{cases} \frac{x}{4} & \text{if } 0 \leq x < 1 \\ \frac{x^4}{20} + \frac{1}{5} & \text{if } 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

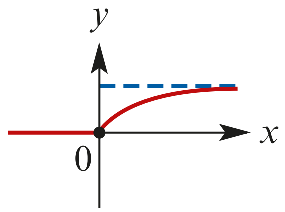
b $F(x) = 0.5$

$\frac{x^4}{20} + \frac{1}{5} = 0.5$

$\frac{x^4}{20} = \frac{3}{10}\ x^4 = 6$

$x = 6^{\frac{1}{4}}$

3 **a**



b $\Pr(X \geq 2) = 1 - \Pr(X < 2)$

$= 1 - F(2)$

$= 1 - (1 - e^{-4})$

$= e^{-4}$

c

$\Pr(X \geq 2 \mid X < 3) = \frac{\Pr(X \geq 2)}{\Pr(X < 3)}$

$= \frac{e^{-4}}{F(3)}$

$= \frac{e^{-4}}{1 - e^{-9}} \approx 0.0183$

4 **a** $F(6) = 1$

$k(6)^2 = 1$

$k = \frac{1}{36}$

b

$\Pr\left(\frac{1}{2} \leq X \leq 1\right) = \Pr(X \leq 1) - \Pr\left(X < \frac{1}{2}\right)$

$= F(1) - F\left(\frac{1}{2}\right)$

$= \frac{1}{36}(1) - \frac{1}{36}\left(\frac{1}{4}\right)$

$= \frac{1}{48}$

5 **a** $F(30) = 1 - \frac{10}{30} = \frac{2}{3}$

b $F(m) = 0.5$

$$1 - \frac{10}{m} = 0.5$$

$$\frac{10}{m} = 0.5$$

$$\frac{m}{10} = 2\ x \quad = 20$$

c $F(a) = 0.025$

$$1 - \frac{10}{a} = 0.025$$

$$\frac{10}{a} = 0.975$$

$$a = \frac{10}{0.975} \quad = \frac{400}{39}$$

$$F(m) = 0.975$$

$$1 - \frac{10}{b} = 0.975$$

$$\frac{10}{b} = 0.025$$

$$b = \frac{10}{0.025} \quad = 400$$

6 $$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 4x^3 - 3x^4 & \text{if } 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 12x^2 - 12x^3 & \text{if } 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

7 $$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1 - x)^5 & \text{if } 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 5(1 - x)^4 & \text{if } 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

8 $$F(x) = \begin{cases} 0.5e^{x} & \text{if } x \le 0 \\ 1 - 0.5e^{-x} & \text{if } x > 0 \end{cases}$$

$$f(x) = \begin{cases} 0.5e^{x} & \text{if } x \le 0 \\ 0.5e^{-x} & \text{if } x > 0 \end{cases}$$

Solutions to Technology-free questions

1 a

$$k\int_1^{\sqrt{2}} x\,dx = 1$$

$$k\left[\frac{1}{2}x^2\right]_1^{\sqrt{2}} = 1$$

$$k\left(\frac{1}{2}\times 2 - \frac{1}{2}\times 1\right) = 1$$

$$\frac{1}{2}k = 1$$

$$k = 2$$

b $\Pr(1 < X < 1.1) = \int_1^{1.1} 2x\,dx$

$$= \left[x^2\right]_1^{1.1}$$

$$= 1.21 - 1$$

$$= 0.21$$

c $\Pr(1 < X < 1.2) = \int_1^{1.2} 2x\,dx$

$$= \left[x^2\right]_1^{1.2}$$

$$= 1.44 - 1$$

$$= 0.44$$

2 $\int_0^1 (a + bx^2)\,dx = 1$

$$\left[ax + \frac{1}{3}bx^3\right]_0^1 = 1$$

$$a + \frac{1}{3}b = 1 \qquad ①$$

$\mathrm{E}(X) = \frac{2}{3}$, *so:*

$$\int_0^1 x(a + bx^2)\,dx = \frac{2}{3}$$

$$\int_0^1 x(ax + bx^3)\,dx = \frac{2}{3}$$

$$\left[\frac{1}{2}ax^2 + \frac{1}{4}bx^4\right]_0^1 = \frac{2}{3}$$

$$\frac{1}{2}a + \frac{1}{4}b = \frac{2}{3} \qquad ②$$

$$2 \times (2) - (1): \left(\frac{1}{2} - \frac{1}{3}\right)b = \frac{4}{3} - 1$$

$$\frac{1}{6}b = \frac{1}{3}$$

$$b = 2$$

Substituting in (1) gives $a = \frac{1}{3}$.

3 The graph of $\frac{1}{2}\sin x,\ 0 \le x \le \pi$, has x-intercepts at (0,0) and $(\pi, 0)$ and a maximum at $\left(\frac{\pi}{2}, \frac{1}{2}\right)$.

Also, the graph is symmetrical about the live $x = \frac{\pi}{2}$, so the area under the curve from 0 to $\frac{\pi}{2}$ is $\frac{1}{2}$.

(Alternatively, solve $\int_0^m \frac{1}{2}\sin x\,dx = \frac{1}{2}$.)

(Note that the symmetry also implies that the mean is $\frac{\pi}{2}$.)

4 a $\Pr(1 < x < 3) = \frac{1}{4}(3 - 1) - \frac{1}{4}(1 - 1)$

$$= \frac{1}{2}$$

b $\Pr(X > 2 \mid 1 < X < 3)$

$$= \frac{\Pr(X > 2 \cap | < X < 3)}{\Pr(1 < X < 3)}$$

$$= \frac{\Pr(2 < X < 3)}{\Pr(1 < X < 3)}$$

$$= \frac{\Pr(X < 3) - \Pr(X < 2)}{\left(\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{4}(3-1) - \frac{1}{4}(2-1)}{\frac{1}{2}}$$

$$= \frac{1}{2}(2-1)$$

$$= \frac{1}{2}$$

c

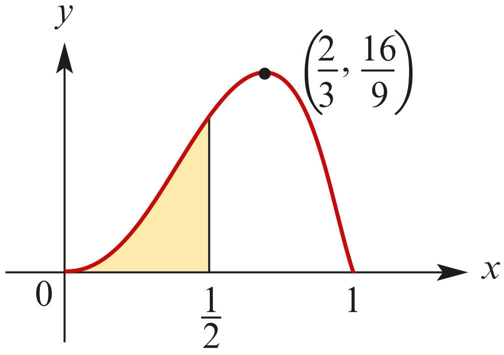
$$\Pr(X > 4 \mid X > 2) = \frac{\Pr(X > 4 \cap X > 2)}{P(X > 2)}$$

$$= \frac{\Pr(X > 4)}{\Pr(X > 2)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{1}{3}$$

5 a



b $P(X < 0.5) = \int_0^{0.5} (12x^2 - 12x^3)\,dx$

$$= \left[4x^3 - 3x^4\right]_0^{0.5}$$

$$= 4 \times \frac{1}{8} - 3 \times \frac{1}{16}$$

$$= \frac{5}{16}$$

6 a $k\int_0^1 (x^2 - x^3)\,dx = 1$

$$k\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 1$$

$$k\left(\frac{1}{3} - \frac{1}{4}\right) = 1$$

$$\frac{1}{12}k = 1$$

$$k = 12$$

(Note that this agrees with the function given in Qn. 5 above.)

b $\Pr\left(X < \frac{2}{3}\right) = \int_0^{\frac{2}{3}} (12x^2 - 12x^3)\,dx$

$$= \left[4x^3 - 3x^4\right]_0^{\frac{2}{3}}$$

$$= 4 \times \frac{8}{27} - 3 \times \frac{16}{81}$$

$$= \frac{16}{27}$$

c $\Pr\left(X < \frac{1}{3} \middle| X < \frac{2}{3}\right) = \dfrac{\Pr\left(X < \frac{1}{3}\right)}{\Pr\left(X < \frac{2}{3}\right)}$

$$\Pr\left(X < \frac{1}{3}\right) = \left[4x^3 - 3x^4\right]_0^{\frac{1}{3}}$$

$$= \frac{1}{9}, \text{ so}$$

$$\Pr\left(X < \frac{1}{3} \middle| X < \frac{2}{3}\right) = \frac{\frac{1}{9}}{\frac{16}{27}} = \frac{3}{16}$$

7 **a** $\Pr(X < 0.2) = \int_0^{0.2} 3x^2\,dx$

$$= \left[x^3\right]_0^{0.2}$$

$$= 0.008$$

b $\Pr(X < 0.2|X < 0.3) = \dfrac{\int_0^{0.2} 3x^2\,dx}{\int_0^{0.3} 3x^2\,dx}$

$$= \frac{0.008}{0.027}$$

$$= \frac{8}{27}$$

8 $\Pr(X < m) = \int_0^m \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right)dx$

$$= \left[\sin\left(\frac{\pi x}{4}\right)\right]_0^m$$

$$= \sin\left(\frac{\pi m}{4}\right)$$

$$\sin\left(\frac{\pi m}{4}\right) = 0.5$$

$$\frac{\pi m}{4} = \frac{\pi}{6}$$

$$m = \frac{2}{3}$$

9 **a** $\mathrm{E}(X) = \int_0^4 \frac{x(x+2)}{16}\,dx$

$$= \frac{1}{16}\int_0^4 x^2 + 2x\,dx$$

$$= \frac{1}{16}\left[\frac{x^3}{3} + x^2\right]_0^4$$

$$= \frac{7}{3}$$

b $\int_0^a \frac{(x+2)}{16}\,dx = \frac{5}{32}$

$$\int_0^a (x+2)\,dx = \frac{5}{2}$$

$$\left[\frac{x^2}{2} + 2x\right]_0^a = \frac{5}{2}$$

$$\frac{a^2}{2} + 2a = \frac{5}{2}$$

$$a^2 + 4a - 5 = 0$$

$$(a+5)(a-1) = 0$$

$$a = -5 \text{ or } a = 1$$

$$\therefore a = 1$$

10 **a** $\int_{-1}^1 c(1-x^2)\,dx = c\left[x - \frac{x^3}{3}\right]_{-1}^1$

$$= \frac{4c}{3}$$

For PDF $\quad \frac{4c}{3} = 1$

$$\therefore c = \frac{3}{4}$$

b 0

11 $\int_0^1 n(1-x)^{n-1}\,dx = \left[\frac{-n(1-x)^n}{n}\right]_0^1$

$$= 1$$

12 **a**
$$\int_0^m \frac{1}{x}\,dx = \Big[\log_e(x)\Big]_1^m$$
$$= \log_e m$$
$$\log_e m = \frac{1}{2}$$
$$m = e^{\frac{1}{2}}$$

b
$$\Pr(X > b) = \frac{1}{4}$$
$$\int_b^e \frac{1}{x}\,dx = \frac{1}{4}$$
$$1 - \log_e b = \frac{1}{4}$$
$$b = e^{\frac{3}{4}}$$

13 **a** Let $g(x) = -\frac{1}{2}\cos x^2$.

Then $g'(x) = -\frac{1}{2} \times 2x\sin x^2 = x\sin x^2$

b
$$\Pr\left(\sqrt{\frac{\pi}{3}} < X < \sqrt{\frac{\pi}{2}}\right) = \int_{\sqrt{\frac{\pi}{3}}}^{\sqrt{\frac{\pi}{2}}} x\sin x^2\,dx$$
$$= \left[-\frac{1}{2}\cos x^2\right]_{\sqrt{\frac{\pi}{3}}}^{\sqrt{\frac{\pi}{2}}}$$
$$= -\frac{1}{2}\left(0 - \frac{1}{2}\right)$$
$$= \frac{1}{4}$$

c
$$\Pr(X < m) = \frac{1}{2}$$
$$\int_0^m x\sin x^2\,dx = \frac{1}{2}$$
$$\left[-\frac{1}{2}\cos x^2\right]_0^m = \frac{1}{2}$$
$$\frac{1}{2} - \cos\left(\frac{m^2}{2}\right) = \frac{1}{2}$$
$$\cos\left(\frac{m^2}{2}\right) = 0$$
$$m^2 = \frac{\pi}{2}$$
$$m = \sqrt{\frac{\pi}{2}}$$

Solutions to multiple-choice questions

1 **B** The second graph partly lies below the x-axis. Since $f(x) \geq 0$ for all x, this could not requirement a probability density function.

2 **D** An antiderivative of $4x$ is $2x^2$. If the domain is of the from $0 < x < a$, then

$$\int_0^a 4x\,dx = \left[2x^2\right]_0^a$$
$$= 2a^2$$
$$= 1$$
$$\Rightarrow a = \frac{1}{\sqrt{2}} \text{ (since } a > 0),$$

so option **D** fits.

(Note that the above shows options $A \rightarrow C$ are not posible.

For option E: $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\frac{2}{\sqrt{2}}} 4x\,dx$

$$= \left[2x^2\right]_{\frac{1}{\sqrt{2}}}^{\frac{2}{\sqrt{2}}}$$
$$= 2\left(\frac{4}{2}\right) - 2\left(\frac{1}{2}\right)$$
$$= 3$$

3 **D** $\displaystyle\int_0^L \frac{1}{2}\sin x\,dx = \left[-\frac{1}{2}\cos x\right]_0^k$

$$= -\frac{1}{2}\cos k + \frac{1}{2}$$
$$= 1$$

if $\cos k = -1$

$k = \pi$

4 **A** $\Pr(X \leq 1.3) = \displaystyle\int_1^{1.3} \frac{3}{4}(x^2 - 1)\,dx$

$$= \left[\frac{1}{4}x^3 - \frac{3}{4}x\right]_1^{1.3}$$
$$\approx 0.0743$$

5 **E** $\mathrm{E}(X) = \displaystyle\int_1^2 x \times \frac{3}{4}(x^2 - 1)\,dx$

$$= \int_1^2 \left(\frac{3}{4}x^3 - \frac{3}{4}x\right)dx$$
$$= \left[\frac{3}{16}x^4 - \frac{3}{8}x^2\right]_1^2$$
$$= \frac{27}{16}$$

6 **B** $\mathrm{E}(x^2) = \displaystyle\int_1^2 x^2 \times \frac{3}{4}(x^2 - 1)\,dx$

$$= \int_1^2 \left(\frac{3}{4}x^4 - \frac{3}{4}x^2\right)dx$$
$$= \left[\frac{3}{20}x^5 - \frac{1}{4}x^3\right]_1^2$$
$$= \frac{29}{10}$$

$$\mathrm{var}(X) = \mathrm{E}(X)^2 - [\mathrm{E}(X)]^2$$
$$= \frac{29}{10} - \left(\frac{27}{16}\right)^2$$
$$= \frac{67}{1280}$$

7 C $$\int_0^m \frac{x^3}{4}\,dx = \frac{1}{2}$$

$$\left[\frac{x^4}{16}\right]_0^m = \frac{1}{2}$$

$$\frac{m^4}{16} = \frac{1}{2}$$

$$m^4 = 8$$

$$m = \sqrt[4]{8}$$

$$\approx 1.6818$$

8 D

$$\int_1^3 xf(x)\,dx = 2.6$$

9 C $$\text{E}(X) = \int_0^{20} x \times \frac{x}{40\,000}(400 - x^2)\,dx$$

$$= \frac{32}{3}$$

(using a CAS calculator)

Then the expected consulations time for three patients is $3 \times \frac{32}{3} = 32$ min.

10 A Let s be the minimum score for an 'A'.

Then $\Pr(X \geq S) = 0.10$ or equivalently $\Pr(X < S) = 0.909$.

Hence $$\int_0^s \frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right)dx = 0.90$$

$$\left[-\frac{1}{2}\cos\left(\frac{\pi x}{50}\right)\right]_0^s = 0.90$$

$$-\frac{1}{2}\cos\left(\frac{\pi s}{50}\right) + \frac{1}{2} = 0.90$$

$$\cos\left(\frac{\pi s}{50}\right) = -0.80$$

$$\frac{\pi s}{50} = \cos^{-1}(-0.80)$$

$$s = \frac{50}{\pi}\cos^{-1}(-0.80)$$

$$\approx 39.8$$

so the minimum score required is closest to 40.

11 B

12 C $$F(2.5) - F(1) = \frac{3}{4} - \frac{1}{8}$$

$$= \frac{5}{8}$$

Solutions to extended-response questions

1 $f(x) = \begin{cases} \dfrac{k}{12(x-1)^3} & \text{if } 0 \le x \le 4 \\ 0 & \text{if } x < 0 \text{ or } x > 4 \end{cases}$

a $$\int_0^4 f(x)\,dx = 1 \Rightarrow \frac{k}{12}\int_0^4 \frac{1}{(x+1)^3}\,dx = 1$$
$$\frac{k}{12}\left[-\frac{1}{2(x+1)^2}\right]_0^k = 1$$
$$\frac{k}{12}\left(-\frac{1}{50}+\frac{1}{2}\right) = 1$$
$$\frac{k}{25} = 1$$
$$k = 25$$

b $$\text{E}(X+1) = \frac{25}{12}\int_0^4 \frac{1}{(x+1)^2}\,dy$$
$$= \frac{25}{12}\left[-\frac{1}{(x+1)}\right]_0^4$$
$$= \frac{25}{12}\left(-\frac{1}{5}+1\right)$$
$$= \frac{5}{3}$$
$$\text{E}(X+1) = \text{E}(X) + 1$$
$$\text{E}(X) = \text{E}(X+1) - 1$$
$$= \frac{5}{3} - 1 = \frac{2}{3}$$

d $$P(X \le c) = c$$
$$\frac{25}{12}\int_0^c \frac{1}{(x+1)^3}\,dx = c$$
$$-\frac{25}{24}\left[\frac{1}{(x+1)^2}\right]_0^c = c$$
$$-\frac{25}{24}\left(\frac{1}{(c+1)^2} - 1\right) = c$$

Using the ‘solve’ command of a CAS calculator gives $c = \dfrac{-13}{6}$ or $c = 0$ or $c = \dfrac{2}{3}$.

But $c > 0$, so $c = \frac{2}{3}$.

2 $f(x) = \begin{cases} \frac{a}{100}\left(1 - \frac{x}{100}\right) & \text{if } 100 \leq x \leq 1000 \\ 0 & \text{otherwise} \end{cases}$

a $\int_{100}^{1000} f(x)\,dx = \left[\frac{a}{100}\left(x - \frac{x^2}{200}\right)\right]_{100}^{1000} = -\frac{81a}{2}$

For f to be a probability density function $-\frac{81a}{2} = 1$ and hence $a = -\frac{2}{81}$

b $\int_{950}^{1000} f(x)\,dx = \left[\frac{a}{100}\left(x - \frac{x^2}{200}\right)\right]_{950}^{1000} \approx 0.108025\ldots$

c $\mathrm{E}(X) = \int_{100}^{1000} xf(x)\,dx = \left[-\frac{2}{8100}\left(\frac{x^2}{2} - \frac{x^3}{300}\right)\right]_{100}^{1000} = 700$ hours

d $\int_{c}^{1000} xf(x)\,dx = 0.5$

Solve for c.

$c \approx 736.396$ hours

e Let X be the number of components in a box that lasts less tha 950 hours. Binomial with $p \approx 0.108025, n = 50$

$\Pr(X \leq 1) = 0.232\ldots$

3 **a** $f(x) = \begin{cases} \frac{\pi}{20}\cos\left(\frac{\pi}{10}(x - 6)\right) & \text{if } 1 \leq x \leq 11 \\ 0 & \text{if } x < 1 \text{ or } x > 11 \end{cases}$

$$\begin{aligned}\Pr(E_A > 10) &= \int_{10}^{11} f(x)\,dx \\ &= 0.024472\ldots\end{aligned}$$

b $\Pr(E_A > c) = \int_{c}^{11} f(x)\,dx$

$$\int_{c}^{11} f(x)\,dx = 0.01$$

$$c = 10.3623\ldots$$

c $\mathrm{E}(X) = \int_{1}^{10} x b(x)\, dx$

$= \frac{32}{5}$

d $\int_{k}^{10} b(x)\, dx = 0.05$

$k \approx 9.1215\ldots$

e $\Pr(\text{Pipe unacceptable}) = \frac{3}{5} \times 0.054496 + = \frac{2}{5} \times 0.017175 \approx 0.040$

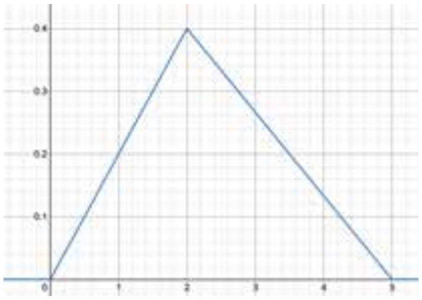
f Probability of machine A given unacceptable

$= \frac{\text{probability of unacceptable given A} \times \text{probability of machine A}}{\text{probability of unacceptable}}$

$= \frac{\frac{3}{5} \times 0.56697}{0.039568}$

≈ 0.826372

4 **a**



b $\Pr(1 < X < 3) = \int_{1}^{2} \frac{x}{5}\, dx + \int_{2}^{3} \frac{1}{15}(10 - 2x)\, dx \quad = \frac{19}{30}$

c $\Pr(X \geq 1.5 | x < 3) = \frac{\Pr(1.5 \leq X < 3)}{\Pr(X < 3)}$

$= \frac{\int_{1.5}^{2} \frac{x}{5}\, dx + \int_{2}^{3} \frac{1}{15}(10 - 2x)\, dx}{\int_{0}^{2} \frac{x}{5}\, dx + \int_{2}^{3} \frac{1}{15}(10 - 2x)\, dx}$

$= \frac{61}{88}$

d $\mathrm{E}(X) = \int_{0}^{2} \frac{x^2}{5}\, dx + \int_{2}^{5} \frac{x}{15}(10 - 2x)\, dx$

$= \frac{7}{3}\, kg$

e **i** $\int_0^c \frac{x}{5}\,dx = 0.1$

$\Rightarrow 0.1c^2 = 0.1$

$\Rightarrow c = 1$

ii $\Pr(X > 1)$, X is binomial with $n = 20$ and $p = 0.1$

$\Pr(X > 1) = 0.6083$

5
$$\begin{aligned} \mathrm{E}(X - c)^2 &= \int_2^4 (x^2 - 2cx + c^2) f(x)\,dx \\ &= \frac{1}{2}\int_2^4 x^2 - 2cx + c^2)(x - 2)\,dx \\ &= \frac{1}{2}\int_2^4 (x^3 - (2c + 2)^2 + (c^2 4c)x - 2c^2)\,dx \\ &= \frac{1}{3}(3c^2 - 20c + 34) \end{aligned}$$

$$\text{If } \mathrm{E}(X - c)^2 = \frac{2}{3}$$

implies $3c^2 - 20c + 34 = 2$

$3c^2 - 20c + 32 = 0$

$(3c - 8)(c - 4) = 0$

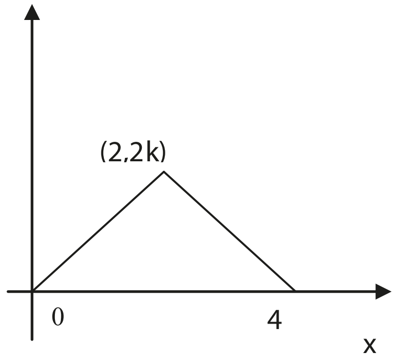
Therefore $c = \frac{8}{3}$ or $c = 4$

6
$$f(x) = \begin{cases} kx & \text{if if } 0 \le x < 2 \\ k(4 - x) & \text{if } 2 \le x < 4 \\ 0 & \text{if } x < 0 \text{ or } x > 4 \end{cases}$$

a The area of the triangle $= \frac{1}{2} \times 4 \times 2k$

$= 4k$

Therefore $k = \frac{1}{4}$



b Since the graph of $y = f(x)$ is symmetrical about $x = 2$, $\mathrm{E}(X) = 2$.

$$\begin{aligned}\operatorname{var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= \frac{1}{4}\int_0^2 x^3\,dx + \frac{1}{4}\int_2^4 x^2(4-x)\,dx - 4 \\ &= \frac{14}{3} - 4 \\ &= \frac{2}{3}\end{aligned}$$

c
$$\begin{aligned}\Pr(|X-\mu|<1) &= \Pr(|X-2|<1) \\ &= \Pr(1 < X < 3) \\ &= \frac{1}{4}\int_1^2 x\,dx + \frac{1}{4}\int_2^3 (4-x)\,dx \\ &= \frac{3}{4}\end{aligned}$$

d $\Pr(X > a) = 0.6$ or equivalently $\Pr(X \le a) = 0.4$.

Since the graph of $y = f(x)$ is symmetrical about $x = 2$, $\Pr(X \le 2) = 0.5$.

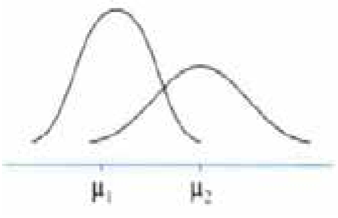
Hence $0 < a < 2$.

$$\begin{aligned}\int_0^a \frac{1}{4}x\,dx &= 0.4 = \frac{4}{10} \\ \left[\frac{1}{8}x^2\right]_0^a &= \frac{2}{5} \\ \frac{1}{8}a^2 &= \frac{2}{5} \\ a^2 &= \frac{16}{5} \\ a &= \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}\ (a > 0).\end{aligned}$$

Chapter 16 – The normal distribution

Solutions to Exercise 16A

1



2 *(c) appears to be the only normally distributed curve*

3 **a** *using CAS calculator, integral* = 1

b **i** $$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx$$

ii using CAS calculator, integral=2

c **i** $$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{3\sqrt{2\pi}}$$

$$\times\ e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx$$

ii *using CAS calculator, integral* = 13

iii $$\sigma = \sqrt{\sigma^2}$$

$$= \sqrt{E(X^2) - [E(X)]^2}$$

$$= \sqrt{13 - 4}$$

$$= 3$$

4 **a** *using CAS calculator, integral* = 1

b **i** $$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} dx$$

ii *using CAS calculator, integral* = −4

c **i** $$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{5\sqrt{2\pi}}$$

$$\times\ e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} dx$$

ii $\mathrm{E}(X^2) = 41$

iii $$\sigma = \sqrt{\sigma^2}$$

$$= \sqrt{E(X^2) - [E(X)]^2}$$

$$= \sqrt{41 - 16}$$

$$= 5$$

5 $$f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{10}\right)^2}$$

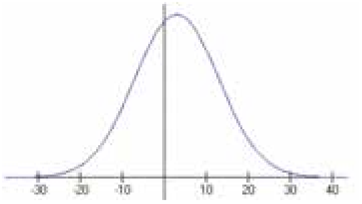
a $\mu = 3$

$\sigma = 10$

$\left(\textit{read off} \left(\frac{x-3}{10}\right)\right.$

$\left.\textit{section of the equation}\right)$

b



6 $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+3}{1}\right)^2}$

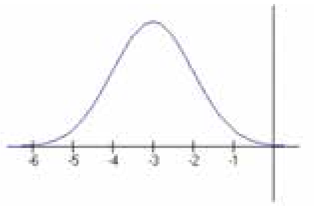
a $\mu = -3,$

$\sigma = 1$

$\left(\textit{read off}\left(\dfrac{x+3}{1}\right)\right.$

$\left.\textit{section of the equation}\right)$

b



7 $f(x) = \dfrac{1}{9\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+0}{3}\right)^2}$

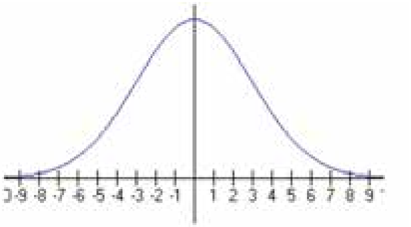
a $\mu = 0$

$\sigma = 3$

$\left(\textit{read off}\left(\dfrac{x+0}{3}\right)\right.$

$\left.\textit{section of the equation}\right)$

b



8 **a** *translation* + 3 *along the x-axis*
($\mu = 3$)
dilation factor 2 from the *y*-axis
($\sigma = 2$)
dilation factor $\frac{1}{2}$ *from the x-axis*

b *translation* + 3 *along the x-axis*
($\mu = 3$)
dilation factor $\frac{1}{2}$ *from the y-axis*
$\left(\sigma = \frac{1}{2}\right)$
dilation factor 2 *from the x-axis*

c *translation* −3 *along the x-axis*
($\mu = -3$)
dilation factor 2 *from the y-axis*
($\sigma = 2$)
dilation factor $\frac{1}{2}$ *from the x-axis*

9 **a** *translation* −3 *along the x-axis*
($\mu = 3$)
dilation factor 2 *from the x-axis*
dilation factor $\frac{1}{2}$ *from the y-axis*
($\sigma = 2$)

b *translation* −3 *along the x-axis*
($\mu = 3$)
dilation factor $\frac{1}{2}$ *from the x-axis*
dilation factor 2 *from the y-axis*
$\left(\sigma = \frac{1}{2}\right)$

c *translation* +3 *along the x-axis*
($\mu = -3$)
dilation factor 2 *from the x-axis*
dilation factor $\frac{1}{2}$ *from the y-axis*
($\sigma = 2$)

Solutions to Exercise 16B

1 **a** 16%

b 16 %

c 2.5 %

d 2.5 %

2 **a** $\mu \approx 135$

$3\sigma \approx 15$

$\sigma \approx 5$

b $\mu \approx 10$

$3\sigma \approx 4$

$\sigma \approx \frac{4}{3}$

3 **a** ≈ 68%

b $\approx \dfrac{100\% - 68\%}{2} = 16\%$

c $\approx \dfrac{100\% - 99.7\%}{2} = 0.15\%$

4 $\mu - 2\sigma$ *and* $\mu + 2\sigma$
27.3 − 6.2 *and* 27.3 + 6.2
answer:
21.1 *and* 33.5

5 *one*; 95; 99.7; *three*

6 $\approx \dfrac{1 - 0.95}{2} = 0.025$, i.e. 2.5%

7 **a** $\approx \dfrac{1 - 0.68}{2} = 0.16$, i.e. 16%

b $\approx \dfrac{1 - 0.68}{2} = 0.16$ i.e. 16%

8 **a** ≈ 68%

b $\approx \dfrac{100\% - 68\%}{2} = 16\%$

c $\approx \dfrac{100\% - 95\%}{2} = 2.5\%$

9 **a** ≈ 95%

b $\approx \dfrac{100\% - 68\%}{2} = 16\%$

c = 50%, *since the mean = the median for normal distributions*

d ≈ 99.7%

10 **a** $\dfrac{160 - 160}{8} = 0$

b $\dfrac{150 - 160}{8} = -1.25$

c $\dfrac{172 - 160}{8} = 1.5$

11 **a** $\dfrac{256 - 270}{10} = -1.4$

b $\dfrac{281 - 270}{10} = 1.1$

c $\dfrac{305 - 270}{10} = 3.5$

12 *Michael has a score of* $\dfrac{85 - 78}{5} = 1.4$ *standard deviations*
Cheryl has a score of $\dfrac{27 - 18}{6} = 1.5$ *standard deviations*
∴ *Cheryl performed better*

13 *Biology score is* $\frac{77 - 68.5}{4.9} \approx 1.73$ *standard deviations*

History score is $\frac{79 - 75.3}{4.1} \approx 0.90$ *standard deviations*

$\therefore$ *the student did better in Biology*

14 **a**

Mary:

French: $\frac{19 - 15}{4} = 1$

English: $\frac{42 - 35}{8} = 0.875$

Mathematics: $\frac{20 - 20}{5} = 0$

Steve:

French: $\frac{21 - 23}{4} = -0.5$

English: $\frac{39 - 42}{3} = -1$

Mathematics: $\frac{23 - 18}{4} = 1.25$

Sue:

French: $\frac{15 - 15}{5} = 0$

English: $\frac{42 - 35}{10} = 0.7$

Mathematics: $\frac{19 - 20}{5} = -0.2$

b **i** *Mary*

ii *Mary*

iii *Steve*

c *if all the subjects are weighted equally, Mary is the best student, since her total standardised mark is higher*

Solutions to Exercise 16C

1
- **a** 0.9772
- **b** 0.9938
- **c** 0.9938
- **d** 0.9943
- **e** 0.0228
- **f** 0.0668
- **g** 0.3669
- **h** 0.1562

2
- **a** 0.9772
- **b** 0.6915
- **c** 0.9938
- **d** 0.9003
- **e** 0.0228
- **f** 0.0099
- **g** 0.0359
- **h** 0.1711

3
- **a** 0.6826
- **b** 0.9544
- **c** 0.9974

These results are vary close to the '68%–95%–99.7%' rule.

4
- **a** 0.0214
- **b** 0.9270
- **c** 0.0441
- **d** 0.1311

5 $c = 1.2816$

6 $c = 0.6745$

7 $c = 1.96$

8 -1.6449

9 -0.8416

10 -1.2816

11 -1.9600

12
- **a** 0.9522
- **b** 0.7977
- **c** 0.0478
- **d** 0.1547

13 **a** 0.9452

b 0.2119

c 0.9452

d 0.1571

14 **a** 9.2897

b 8.5631

15 **a** $c = 10$

b $k = 15.88$

16 **a** $a = 0.994$

b $b = 1.96$

c $c = 2.968$

17 **a** 0.7161

b 0.0966

c $\Pr(x < 26 \mid 25 < x < 27)$

$$= \frac{\Pr((x < 26) \cap (25 < x < 27))}{\Pr(25 < x < 27)}$$

$$= \frac{\Pr(25 < x < 26)}{0.096\cdots}$$

$$= 0.5204$$

d $c = 33.5143$

e $k = 13.02913$

f $c_1 = 8.28; c_2 = 35.72$
(assumed symmetrical about the mean)

18 **a** 0.9772

b $\Pr(x < 11 \mid x < 13)$

$$= \frac{\Pr('x < 11 \mid ' \cap 'x < 13')}{\Pr(x < |3)}$$

$$= \frac{\Pr(x < 11)}{\Pr(x < 13)}$$

$$= \frac{0.9772}{0.9999}$$

$$= 0.9772$$

c 10.822

d 9.5792

e $c_2 = 10.98; c_1 = 9.02$
(assumed symmetrical about the mean)

Solutions to Exercise 16D

1 **a** **i** 0.2525

ii 0.0478

iii Pr(IQ > 130 | IQ > 110)

$$= \frac{\Pr(\text{IQ} > 130)}{\Pr(\text{IQ} > 110)}$$

$$= \frac{0.0227\cdots}{0.2524\cdots}$$

$$= 0.0901$$

b 124.7

2 **a** **i** 0.7340

ii 0.8944

iii Pr(> 170 | between | 68 &174)

$$= \frac{\Pr(\text{between } 170 \ \&174)}{\Pr(\text{between } 168 \ \&174)}$$

$$= \frac{0.0655\cdots}{0.1185\cdots}$$

$$= 0.5531$$

b 170.25 cm

c 153.267 cm

3 **a** **i** 0.0766

ii 0.9998

iii 0.1531

b 57.3

4 **a** 10.56%

b 78.51%

5 *mean* = 1.55 kg; *sd* = 0.194 kg

6 **a** 36.9%

b 69

7 **a** 0.0228

b 0.0005

c If Y is the number with heights exceeding 190 cm, then Y is Binomial with $n = 10, P = 0.02275\ldots$

$\Pr(Y \geq 2) = \Pr(2 \leq Y \leq 10)$

$= 0.0206$

using a CAS calculator's 'bimom CAS' function.

8 **a** $\Pr(X < 295) = 0.05$

$$Pr(Z < \frac{295 - 300}{\sigma} = 0.05$$

$$\frac{5}{\sigma} = -1.6449$$

$\sigma = 3.04$ grams

b $\Pr(X < 340) = 0.02$

$$Pr(Z < \frac{340 - \mu}{5} = 0.02$$

$$\frac{340 - \mu}{5} = -2.0537$$

$\mu = 350.27$ grams

9 1004 ml

10 **a** *small* 0.1587
medium 0.7745
large 0.0668

b Expected cost

$= 100 \times \$(2.80 \times 0.1587 + 3.50$

$\times\ 0.7745 + 5.00 \times 0.0688)$

$= \$348.92$

11 **a** **i** 0.1169

ii 17.7

b 0.0284

12 **a** 0.0228, 0.1587

b Let x be the amount of chemical in a type A call, so x is normal with mean 10 and sd 1 Let Y be the amount of chemical in a type 1 cell, so y is normal with mean 14 and sd 2

$$\Pr(x < c) = \Pr(y > c)$$

$$\Pr\left(\frac{x - 10}{1} < \frac{c - 10}{1}\right) = \Pr\left(\frac{y - 14}{2} > \frac{c - 14}{2}\right)$$

$$\text{i.e. } \Pr(z < c - 10) = \Pr\left(z > \frac{c - 14}{2}\right)$$

where z has a standard normal distribution. Since the graph of $y = f(z)$ is symmetrical about the y-axis, the number $c - 10$ cm $\frac{c - 14}{2}$ are equidistant from the original.

Hence $\frac{c - 14}{2} = -(c - 10)$

$$c - 14 = -2c + 20$$

$$3c = 34$$

$$c = \frac{34}{3}.$$

Solutions to Exercise 16E

1 $n = 100, p = \frac{1}{6}$

$\mu = np = \frac{100}{6} \approx 16.667$

$\sigma = \sqrt{np(1-p)} \approx 3.727$

$\Pr(X > 10) = 0.9632$ calculator

2 $n = 300, p = 0.5$

$\mu = np = 150$

$\sigma = \sqrt{np(1-p)} \approx 8.660$

$\Pr(X > 156) = 0.2442$ calculator

3 $n = 100, p = 0.1$

$\mu = np = 10$

$\sigma = \sqrt{np(1-p)} = 3$

a $\Pr(X \geq 15) = 0.0478$ calculator

b $\Pr(X \leq 15) = 0.2525$ calculator

4 $n = 400, p = 0.4$

$\mu = np = 16$

$\sigma = \sqrt{np(1-p)} \approx 6.898$

a $\Pr(10 \leq X < 20) = 0.7834$ calculator

b $\Pr(X \geq 15) = 0.0108$ calculator

5 $n = 200, p = 0.4$

$\mu = np = 80$

$\sigma = \sqrt{np(1-p)} \approx 6.928$

$\Pr(X < 76) = 0.2819$ calculator

6 $n = 25, p = 0.25$

$\mu = np = 6.25$

$\sigma = \sqrt{np(1-p)} \approx 2.165$

a $\Pr(X \geq 10) = 0.0416$ calculator

b $\Pr(12 \leq X \leq 14) = 0.0038$ calculator

Solutions to Technology-free questions

1 **a** $\Pr(Z > a) = 1 - \Pr(Z \le a)$

$= 1 - p$

b $\Pr(Z < -a) = \Pr(Z > a)$

$= 1 - p$

c $\Pr(-a \le Z \le a) = \Pr(Z \le a)$

$- \Pr(Z < -a)$

$= P - (1 - p)$

$= 2p - 1$

2 **a** $\Pr(X < 3) = \Pr\left(\dfrac{X-4}{1} < \dfrac{3-4}{1}\right)$

$= \Pr(Z < -1)$

So $\quad a = -1$

b $\Pr(X > 5) = \Pr\left(\dfrac{X-4}{1} > \dfrac{5-4}{1}\right)$

$= \Pr(Z > 1)$

So $\quad b = 1$

c $\Pr(x > 4) = \Pr(Z > 0)$

$= 0.5$

3 $(x, y) \rightarrow \left(\dfrac{x-\mu}{\sigma}, \sigma y\right)$

Since $\mu = 8$ and $\sigma = 3$, then

$(x, y) \rightarrow \left(\dfrac{x-8}{3}, 3y\right)$

4 **a** $\Pr(x < a | x < b)$

$= \dfrac{\Pr('x < a' \cap 'x < b')}{\Pr(x < b)}$

$= \dfrac{\Pr(x < a)}{\Pr(x < b)}$

$= \dfrac{q}{p}$

b $\Pr(X < 2\mu - a) = \Pr\left(\dfrac{x-\mu}{\sigma} < \dfrac{\mu - a}{\sigma}\right)$

$= \Pr\left(Z < \dfrac{\mu - a}{\sigma}\right)$

$= \Pr\left(Z > \dfrac{a - \mu}{\sigma}\right)$

$= 1 - \Pr\left(Z - \dfrac{a - \mu}{\sigma}\right)$

Also, $\quad \Pr(X < a) = q$

$\Pr\left(\dfrac{x-\mu}{\sigma} < \dfrac{a-\mu}{\sigma}\right) = q$

$\Pr\left(Z < \dfrac{a-\mu}{\sigma}\right) = q$

Hence $\Pr(X < 2\mu - a)$

$= 1 - \Pr\left(Z < \dfrac{a-\mu}{\sigma}\right)$

$= 1 - q$

c $\Pr(X > b \mid X > a)$

$= \dfrac{\Pr('x > b' \cap 'x > a')}{\Pr(x > a)}$

$= \dfrac{\Pr(x > b)}{\Pr(x > a)}$

$= \dfrac{1 - \Pr(x < b)}{1 - \Pr(x < a)}$

$= \dfrac{1 - p}{1 - q}$

5 **a** $\Pr(X < 5) = \Pr\left(\frac{x-4}{2} < \frac{5-4}{2}\right)$

$= \Pr\left(Z < \frac{1}{2}\right)$

b $\Pr(X < 3) = \Pr\left(\frac{x-4}{2} < \frac{3-4}{2}\right)$

$= \Pr\left(Z < -\frac{1}{2}\right)$

c $\Pr(X > 5) = \Pr\left(Z > \frac{1}{2}\right)$

d $\Pr(3 < X < 5) = \Pr\left(-\frac{1}{2} < Z < \frac{1}{2}\right)$

e $\Pr(3 < X < 6) = \Pr(-\frac{1}{2} < Z < 1)$

6 **a** $\Pr(X < 2.55) = \Pr\left(Z < \frac{2.55 - 2.5}{0.05}\right)$

$= \Pr(Z < 1)$

$= 0.84$

b $\Pr(X < 2.5) = 0.5 \quad \text{since } \mu = 2.5$

c $\Pr(X < 2.45) = \Pr\left(Z < \frac{2.45 - 2.5}{0.05}\right)$

$= \Pr(Z < -1)$

$= \Pr(Z > 1)$

$= 1 - \Pr(Z < 1)$

$= 0.16$

d $\Pr(2.45 < X < 2.55)$

$= \Pr(-1 < Z < 1)$

$= \Pr(Z < 1) - \Pr(Z < -1)$

$= 0.84 - 0.16$

$= 0.68$

7 **a** $\Pr(\mathrm{W} > 505) = \Pr\left(Z > \frac{505 - 500}{5}\right)$

$= \Pr(Z > 1)$

$= 1 - \Pr(Z < 1)$

$= 1 - 0.84$

$= 0.16$

b $\Pr(500 < \mathrm{W} < 505) = \Pr(0 < Z < 1)$

$= \Pr(Z < 1)$

$= -\Pr(Z < 0)$

$= 0.84 - 0.5$

$= 0.34$

c $\Pr(\mathrm{W} > 505 | \mathrm{W} > 500)$

$= \frac{\Pr(W > 505)}{\Pr(W > 500)}$

$= \frac{0.16}{0.5}$

$= 0.32$

d $\Pr(W > 510) = \Pr\left(Z > \frac{510 - 500}{5}\right)$

$= \Pr(Z > 2)$

$= 1 - \Pr(Z < 2)$

$= 1 - 0.98$

$= 0.02$

8 **a** $\Pr(X < 6.5) = \Pr(Z < 0.5)$

$= 0.69$

b $\Pr(6 < X < 6.5) = \Pr(0 < Z < 0.5)$

$= 0.69 - 0.5$

$= 0.19$

c $\Pr(6.5 < X < 7)$

$= \Pr(0.5 < Z < 1)$

$= \Pr(Z < 1) - \Pr(Z < 0.5)$

$= 0.84 - 0.69$

$= 0.15$

d $\Pr(5 < X < 7) = \Pr(-1 < Z < 1)$

$= \Pr(Z < 1)$

$- \Pr(Z < -1)$

$= 0.84 - (1 - 0.84)$

$= 0.84 - 0.16$

$= 0.68$

9 The standardised scores are as follows.

Test A: $\dfrac{62 - 50}{11} = \dfrac{12}{11} = 1.0909\cdots$

Test B: $\dfrac{64 - 48}{17} = 1$

Test C: $\dfrac{73 - 63}{8} = \dfrac{10}{8} = 1.25$

So the best test was test C and the worst test was test *B*.

10 **a** $\Pr(X > 10) = \Pr(Z > 0) = 0.5$

b $\Pr(X > 13) = \Pr(Z > b)$

$\therefore \Pr(Z > \frac{13 - 10}{2}) = \Pr(Z < b)$

$\therefore \Pr(Z > 1.5) = \Pr(Z < b)$

$\therefore \Pr(Z < -1.5) = \Pr(Z < b)$

$\therefore b = -1.5$

Solutions to multiple-choice questions

1 **A** The graph is symmetrical about the line $x = 4$, so $\mu = 4$.
Almost all of the distribution lies between -5 and 13, i.e. 18 unit, so
$6\sigma = 18$

$\sigma = 3$

2 **B** Use the *invNom* command of a CAS calculator with *Area* set to 0.25. This gives -0.6745 correct to 4 decimal places.

3 **C** X has mean 12 and variance 9, so the standard deviation is 3.

$$\Pr(X > 15) = \Pr\left(\frac{x-12}{3} > \frac{15-12}{3}\right)$$
$$= \Pr(Z > 1)$$

4 **C** $\Pr(X > 110) = 0.0038$

$$\therefore \Pr(Z > \frac{110-102}{\sigma}) = 0.0038$$

$\therefore \dfrac{8}{\sigma} = 2.6693$ (Use the *Inverse normal* command)

$\therefore \sigma = 3$

5 **C** $(x, y) \rightarrow \left(\dfrac{x-\mu}{\sigma}, \sigma y\right)$

Here $\mu = 6$ and $\sigma = 3$, so

$(x, y) \rightarrow \left(\dfrac{x-6}{3}, 3y\right)$

6 **D** The given information means that
$\Pr(X > k) = 0.20$
where x is normally distributed with $\mu = 100$ and $\sigma = 14$.
This can be re-written in the form $\Pr(X \le k) = 0.80$.
Use the *invNorm* command of a CAS calculator with *Area* set to 0.80 and the values 100 and 14 for the mean and standard deviation respectively.
This gives $k = 111.8$, correct to one decimal place.

7 **A** Angie's standardised scores are follows.

Mathematics: $\dfrac{72-72}{5} = 0$

Indonesian: $\dfrac{57-59}{4} = -1$

Politics: $\dfrac{68-64}{4} = 1$

So her best subject was Politics, followed by Mathematics and then Indonesian.

8 **A**

$$\Pr(-1 < X < 2) = \Pr(\mu - \sigma < Z < \mu + 2\sigma)$$
$$= \Pr(8.4 < X < 17.1)$$

9 **D** $\mu = ?, \sigma = 0.005$

$\Pr(X > 1) \approx 0.999$

$$\therefore \Pr\left(Z > \frac{1-\mu}{0.005}\right) \approx 0.999$$

$$\therefore \frac{1-\mu}{0.005} = 3.0902$$

$\therefore \mu = 1.015$

10 **C** $\mu = 272, \sigma = ?$

$\Pr(X < 260) \approx 0.091$

$$\therefore \Pr\left(Z < \frac{260-272}{\sigma}\right) \approx 0.091$$

$$\therefore \frac{12}{\sigma} = -1.3346$$

$\therefore \sigma \approx 8.99$

Solutions to extended-response questions

1 $\mu = 50, \sigma = 10$

Let X be the score.

For the top 10% consider

$$\Pr(X > k_1) = 0.1$$

$$\therefore \Pr(X \le k_1) = 0.9$$

$$\Pr\left(Z \le \frac{k_1 - 50}{10}\right) = 0.9$$

$$\therefore \frac{k_1 - 50}{10} = 1.2816$$

$$\therefore k_1 = 10 \times 1.2816 + 50$$

$$= 12.816 + 50 = 62.816$$

∴ 63 and above indicate high aptitude.

For the next 20% consider

$$\Pr(X > k_2) = 0.3$$

$$\therefore \Pr(X \le k_2) = 0.7$$

$$\therefore \Pr\left(Z \le \frac{k_2 - 50}{10}\right) = 0.7$$

$$\frac{k_2 - 50}{10} = 0.5244$$

$$\therefore k_2 = 50 + 5.244 = 55.244$$

∴ Scores from 56 to 62 indicate moderate aptitude.

For the middle 40% consider

$$\Pr(X > k_3) = 0.7$$

$$\therefore \Pr(X \le k_3) = 0.3$$

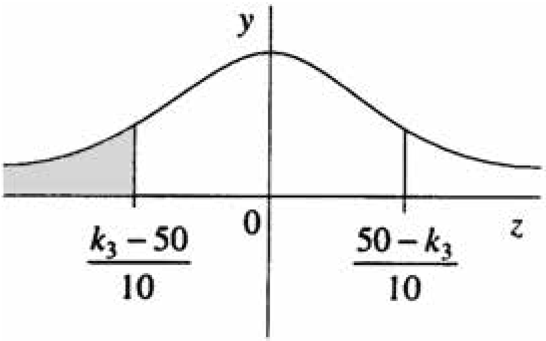
$$\Pr\left(Z \le \frac{k_3 - 50}{10}\right) = 0.3$$

∴ From the diagram

$$\Pr\left(Z \le \frac{50 - k_3}{10}\right) = 0.7$$

$$\therefore \frac{50 - k_3}{10} = 0.5244$$

$$\therefore k_3 = 50 - 5.244 = 44.756$$



∴ Scores from 45 to 55 indicate average aptitude.

For the category of little aptitude consider

$$\Pr(X > k_4) = 0.1$$

$$\therefore \Pr\left(Z > \frac{k_4 - 50}{10}\right) = 0.1$$

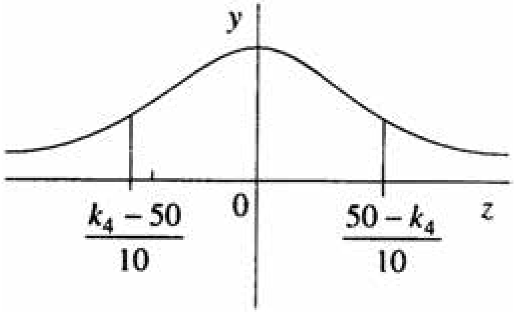
$$\therefore \Pr\left(Z \le \frac{50 - k_4}{10}\right) = 0.9$$

$$\frac{50 - k_4}{10} = 1.2816$$

$$50 - k_4 = 12.816$$

$$\text{and } k_4 = 10 - 12.816$$

$$= 37.184$$



Scores from 37 to 44 indicate little aptitude.
Scores less than 37 indicate no aptitude.
i.e. Scores 63 and above indicate high aptitude.
Scores from 56 to 62 indicate moderate aptitude.
Scores from 45 to 55 indicate average aptitude.
Scores from 37 to 44 indicate little aptitude.
Scores < 37 indicate no aptitude.

2 Let L be the amount (mg) for a lethal dose
$\mu = 110, \sigma = 20$
Let D be the amount (mg) for a surgical anaesthesia
$\mu = 50, \sigma = 10$
Let c mg be the dose such that 90% of patients need less than this amount for surgical anaesthesia
i.e. $\Pr(D \le c) = 0.9$
Transforming to the standard normal

$$\Pr\left(Z \le \frac{c - 50}{10}\right) = 0.9$$

$$\frac{c - 50}{10} = 1.2816$$

$$\therefore c = 10 \times 1.2816 + 50$$

$$= 12.816 + 50$$

$$= 62.816$$

To find what percentage of patients would be killed by these amounts consider

$$\begin{aligned}\Pr(L \leq 62.816) &= \Pr\left(Z \leq \frac{62.816 - 110}{20}\right)\\ &= \Pr(Z \leq -2.3592)\\ &= 1 - \Pr(Z \leq 2.3592)\\ &= 1 - 0.9908\\ &= 0.0092\end{aligned}$$

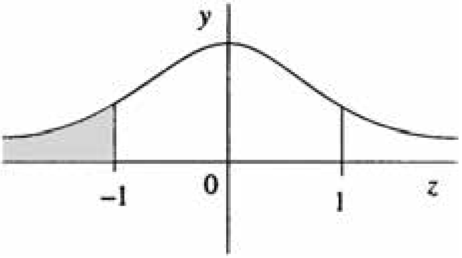
i.e. 0.92% of patients would be killed by a dose of 62.816 mg or less.

3 $\mu = 60\,000, \sigma = 5000$

a **i** Let X be the mileage for a tyre

$$\Pr(X \leq 55\,000) = \Pr\left(Z \leq \frac{55\,000 - 60\,000}{5000}\right)$$

where Z is the standard normal variable



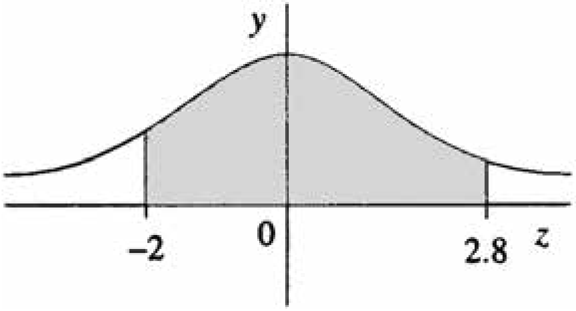
$$\begin{aligned}&= \Pr\left(Z \leq -\frac{5000}{5000}\right)\\ &= \Pr(Z \leq -1)\\ &= 1 - 0.8413\\ &= 0.1587\end{aligned}$$

The proportion of the tyres which last less than 55 000 kilometres is 0.1587 or 15.87%

ii $\Pr(50\,000 \leq X \leq 74\,000) = \Pr\left(\frac{50\,000 - 60\,000}{5000} \leq Z \leq \frac{74\,000 - 60\,000}{5000}\right)$

$$= \Pr(-2 \leq Z \leq 2.8)$$

The required region is shown:

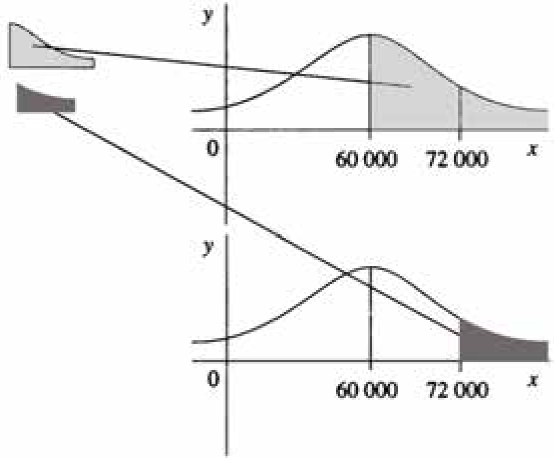


$$\begin{aligned}&\Pr(-2 \leq Z \leq 2.8)\\ &= \Pr(Z \leq 2.8) - \Pr(Z \leq -2)\\ &= \Pr(Z \leq 2.8) - [1 - \Pr(Z \leq 2)]\\ &= \Pr(Z \leq 2.8) + \Pr(Z \leq 2) - 1\\ &= 0.99744 + 0.97725 - 1\\ &= 0.9747\end{aligned}$$

The proportion of tyres which last less than 74 000 kilometres but more than 50 000 is 0.9746 or 97.46%

iii $\Pr(X \geq 72\,000 \mid X \geq 60\,000) = \dfrac{\Pr(X \geq 72\,000)}{\Pr(X \geq 60\,000)}$ (conditional probability)

The diagrams show that the required probability is given by Area divided by Area and transforming to the standard normal



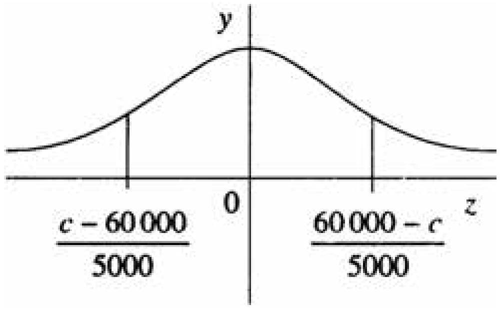
$$\begin{aligned}\frac{\Pr(X \geq 72\,000)}{\Pr(X \geq 60\,000)} &= \frac{\Pr(Z \geq 2.4)}{\Pr(Z \geq 0)}\\ &= \frac{1 - \Pr(Z < 2.4)}{0.5}\\ &= \frac{1 - 0.9918}{0.5}\\ &= 0.0164\end{aligned}$$

b $\Pr(X \geq c) = 0.9$

Transforming to the standard normal

$$\Pr\left(Z \geq \frac{c - 60\,000}{5000}\right) = 0.9$$

A graph of the standard normal curve helps:



$$\therefore\ \Pr\left(Z \geq \frac{c - 60\,000}{5000}\right) = \Pr\left(Z \leq \frac{60\,000 - c}{5000}\right)$$

$$\therefore\ \Pr\left(Z \leq \frac{60\,000 - c}{5000}\right) = 0.9$$

$$\therefore\ \frac{60\,000 - c}{5000} = 1.2816$$

$$\therefore\ 60\,000 - 5000 \times 1.2816 = c$$

$$\therefore\ c = 53\,592$$

The company's advertising manager can claim that 90% of their tyres last more than 53 592 kilometres.

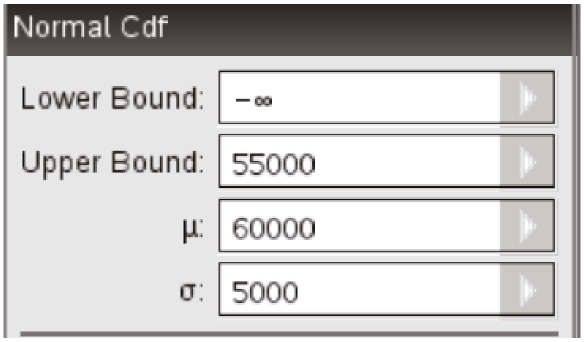
c $$\begin{aligned}\Pr(X \geq 72\,000) &= \Pr\left(Z \geq \frac{72\,000 - 60\,000}{5000}\right)\\ &= \Pr\left(Z \geq \frac{12\,000}{5000}\right)\\ &= \Pr(Z \geq 2.4)\\ &= 1 - \Pr(Z < 2.4)\\ &= 1 - 0.9918\\ &= 0.0082\end{aligned}$$

The probability of one tyre lasting more than 72 000 kilometres is 0.0082

The probability of 5 tyres lasting longer than 72 000 kilometres is $(0.0082)^5 \approx 3.7 \times 10^{-11}$

Graphic calculator techniques for question 3

a **i** Choose **Normal Cdf** from the **Probability**>**Distributions** menu. Complete as shown.



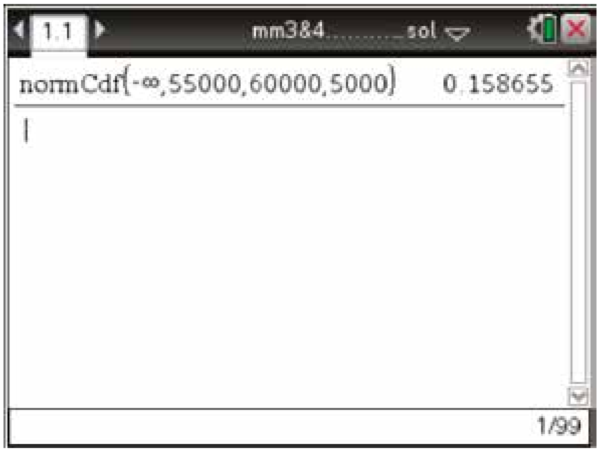
Normal Cdf

Lower Bound: −∞

Upper Bound: 55000

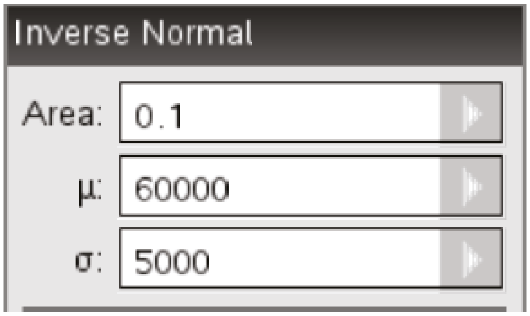
μ: 60000

σ: 5000



```
normCdf(-∞,55000,60000,5000)     0.158655
```

ii $\Pr(X \geq c) = 0.9 \Leftrightarrow \Pr(X \leq c) = 0.1$
Choose **Inverse Normal** from the **Probability**>**Distributions** menu. Complete as shown.

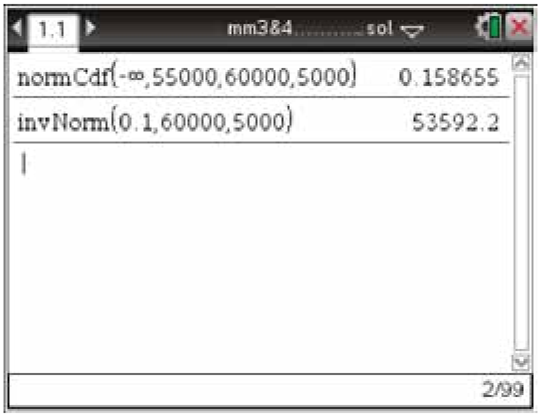


Inverse Normal

Area: 0.1

μ: 60000

σ: 5000



```
normCdf(-∞,55000,60000,5000)     0.158655
invNorm(0.1,60000,5000)          53592.2
```

4 **a** Let L be the useful life of a fluorescent tube

$$\mu = 600, \sigma = 4$$

$$\begin{aligned}
\Pr(L \geq 605) &= \Pr\left(Z \geq \frac{605 - 600}{4}\right) \\
&= \Pr\left(Z \geq \frac{5}{4}\right) \\
&= \Pr(Z \geq 1.25) \\
&= 1 - \Pr(Z \leq 1.25) \\
&= 1 - 0.8944 \\
&= 0.1056
\end{aligned}$$

b $\Pr(L > 607 | L > 605)$

$$= \frac{\Pr(L > 607)}{\Pr(L > 605)} = 0.379169\ldots$$
$$\approx 0.3792$$

c $\Pr(B > 605) = \int_{605}^{612} f(x)\,dx \approx 0.077537\ldots$

Binomial , $n = 10$, $p = 0.077537\ldots$

$\Pr(Y \geq 3) = 0.036978\ldots$

d Assume equally likely to select globe from either company.

$\Pr(W > 605) = 0.5 \times 0.1056 + 0.5 \times 0.0775 \approx 0.09155$

≈ 0.092

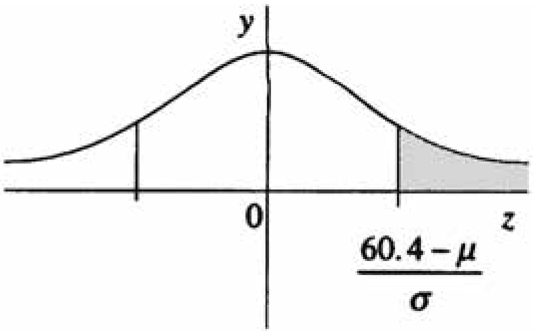
e Pr(Company B|lasts longer than 605 hours)

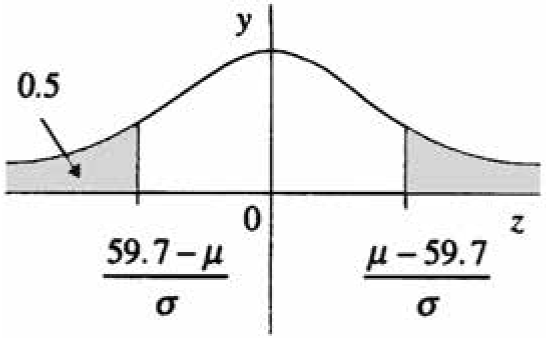
$$= \frac{\text{Pr(lasts longer than 605 hours|Company B) Pr(Company B)}}{\Pr(W > 605)}$$
$$= \frac{0.077537 \times 0.5}{0.09155}$$
$$= 0.423$$

5 **a** Let X be the length of the dimension

$\Pr(X > 60.4) = 0.03$

$\Pr(X < 59.7) = 0.05$





$$\Pr\left(Z < \frac{60.4 - \mu}{\sigma}\right) = 0.97 \qquad \Pr\left(Z < \frac{\mu - 59.7}{\sigma}\right) = 0.95$$

$$\therefore \frac{60.4 - \mu}{\sigma} = 1.88079 \quad \text{and} \quad \frac{\mu - 597}{\sigma} = 1.64485$$

$$\therefore 60.4 - \mu = 1.88079\sigma \quad ① \quad \text{and} \quad \mu - 59.7 = 1.64485\sigma \quad ②$$

Add equations ① and ②

$0.7 = 3.52564\sigma$

$\therefore \sigma = 0.19854$, i.e. $\sigma = 0.2$, correct to one decimal plea.

Substitute in ①

$60.4 - \mu = 1.88079\sigma$

$\therefore \mu = 60.4 - 1.88079\sigma$

$= 60.02658$, i.e. $\mu = 60.0$, correct to one decimal place.

b
$$\begin{aligned}\Pr(X > 60.3) + \Pr(X < 59.6) &= \Pr\left(Z > \frac{60.3 - 60.02658}{0.19854}\right) \\ &\quad + \Pr\left(Z < \frac{59.6 - 60.02658}{0.19854}\right) \\ &= \Pr(Z > 1.37715) + \Pr(Z < -2.14858) \\ &= 1 - \Pr(Z < 1.37715) + 1 - \Pr(Z < 2.14858) \\ &= 2 - \Pr(Z < 1.37715) - \Pr(Z < 2.14858) \\ &= 2 - 0.915767 - 0.98416 \\ &= 0.1\end{aligned}$$

These the percentage of rejects is 10%.

6 Let H denote the hardness of the metal

$\mu = 70$ and $\sigma = 3$

a
$$\begin{aligned}\Pr(65 \leq H \leq 75) &= \Pr\left(\frac{65 - 70}{3} \leq Z \leq \frac{75 - 70}{3}\right) \\ &= \Pr\left(-\frac{5}{3} \leq Z \leq \frac{5}{3}\right) \\ &= \Pr\left(Z \leq \frac{5}{3}\right) - \Pr\left(Z \leq -\frac{5}{3}\right) \\ &= \Pr\left(Z \leq \frac{5}{3}\right) - \left[1 - \Pr\left(Z \leq \frac{5}{3}\right)\right] \\ &= 2\Pr\left(Z \leq \frac{5}{3}\right) - 1 \\ &= 0.9044\end{aligned}$$

The probability that a randomly chosen specimen has acceptable hardness is 0.9044.

b $$\Pr(70 - c \le H \le 70 + c) = 0.95$$

implies $$\Pr\left(\frac{70 - c - 70}{3} \le Z \le \frac{70 + c - 70}{3}\right) = 0.95$$

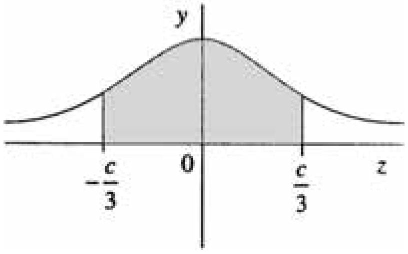
$$\Pr\left(-\frac{c}{3} \le Z \le \frac{c}{3}\right) = 0.95$$

$$\therefore\ 2\Pr\left(Z \le \frac{c}{3}\right) - 1 = 0.95$$

$$\Pr\left(Z \le \frac{c}{3}\right) = 0.975$$

$$\therefore\ \frac{c}{3} = 1.96$$

$$c = 5.88$$



c Let X be the number of acceptable specimens out of 10 randomly selected specimens.
X is a binomial random variable with $n = 10$ and $p = 0.9044$
$E(X) = np = 9.044$
The expected number of acceptable specimens is 9.044.

d $$\Pr(H < 73.84) = \Pr\left(Z < \frac{73.84 - 70}{3}\right)$$
$$= \Pr\left(Z < \frac{3.84}{3}\right)$$
$$= \Pr(Z < 1.28)$$
$$= 0.8997$$

Let X be the number of specimens out of the ten selected which have a hardness less than 73.84.

$$\Pr(X \le 8) = 1 - [\Pr(X = 9) + \Pr(X = 10)]$$
$$= 1 - \binom{10}{9}(0.8997)^9(0.1003) - (0.8997)^{10}$$
$$= 0.2651 \text{ (to four decimal places)}$$

e Let P be profit. The probability distribution for P

P	20	−5
$\Pr(P = p)$	0.9044	0.0956

$$\therefore\ E(P) = 20 \times 0.9044 - 5 \times 0.0956$$
$$= 17.61$$

The expected profit is \$17.61.

$$\begin{aligned} E(P^2) &= 400 \times 0.9044 + 25 \times 0.0956 \\ &= 364.15 \\ \therefore \ \mathrm{Var}(P) &= E(P^2) - [E(P)]^2 \\ &= 364.15 - 310.1121 \\ &= 54.04 \end{aligned}$$

7 Let μ be the mean lifetime for a watch and σ the standard deviation.

a The mean error is 0
Let X be the error

$$\Pr(-5 \le X \le 5) = 0.94$$

$$\therefore \ \Pr\left(\frac{-5}{\sigma} \le Z \le \frac{5}{\sigma}\right) = 0.94$$

$$\therefore \ 2\Pr\left(Z \le \frac{5}{\sigma}\right) - 1 = 0.94$$

$$\Pr\left(Z \le \frac{5}{\sigma}\right) = \frac{1.94}{2}$$

$$\frac{5}{\sigma} = 1.8808$$

$$\therefore \ \sigma = \frac{5}{1.8808}$$

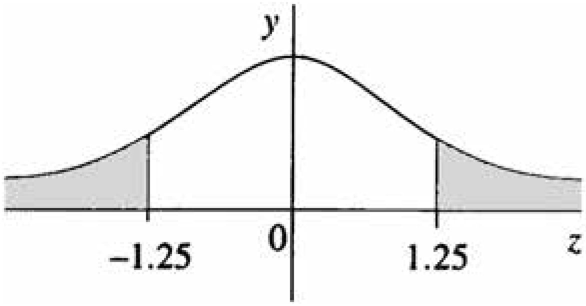
$$= 2.658$$

b Let Y be the number of watches rejected out of a batch of 10 watches.
This is a Binomial distribution with $p = 0.06$ and $n = 10$

$$\begin{aligned} \Pr(Y < 2) &= \Pr(Y = 0) + \Pr(Y = 1) \\ &= (0.94)^{10} + \binom{10}{1}(0.06)(0.94)^9 \\ &= 0.5386 + 0.3438 \\ &= 0.882 \end{aligned}$$

8 a Let X be the number of litres in a standard bottle.

$$\begin{aligned} \Pr(X < 0.75) &= \Pr\left(Z < \frac{0.75 - 0.76}{0.008}\right) \\ &= \Pr(Z < -1.25) \\ &= 1 - \Pr(Z < 1.25) \\ &= 0.1056 \end{aligned}$$



b Let N be the number of bottles out of ten which contain less than 0.75 1itres. This is a binomial random variable with $n = 10$ and $p = 0.10565$.

$$\begin{aligned}\therefore\ \Pr(N \geq 3) &= 1 - [\Pr(N = 0) + \Pr(N = 1) + \Pr(N = 2)]\\ &= 1 - 0.9197\\ &= 0.0803\end{aligned}$$

c Let Y be the number of litres in a large bottle.

Define $W = 4X - 3Y$

We require $\Pr(W) \geq 0$

i.e. $\Pr(4X - 3Y \geq 0)$

$$\begin{aligned}\text{Note: } \mathrm{E}(W) &= 4\mathrm{E}(X) - 3\mathrm{E}(Y)\\ &= 0.01\\ \mathrm{Var}(W) &= 16\mathrm{Var}(X) + 9\mathrm{Var}(Y)\\ &= 16 \times (0.008)^2 + 9 \times (0.009)^2\\ &= 0.001753\\ \therefore\ \mathrm{sd}(W) &= 0.04187\end{aligned}$$

$$\begin{aligned}\therefore\ \Pr(W > 0) &= \Pr\left(Z > \frac{0.01}{0.04187}\right)\\ &= \Pr(Z > -0.23883)\\ &= \Pr(Z < 0.23883)\\ &= 0.5944\end{aligned}$$

Chapter 17 – Sampling and estimation

Solutions to Exercise 17A

1 No; sample will be biased towards the type of movie being shown.

2 **a** No; biased towards shoppers.

b Randomly select a sample from telephone lists or an electoral roll.

3 No; only interested people will call, and they may call more than once.

4 **a** No; biased towards older, friendly or sick guinea pigs which may be easier to catch.

b Number guinea pigs and then generate random numbers to select a sample.

5 No; a student from a large school has less chance of being selected than a student from a small school.

7 **a** Unemployed will be under represented.

b Unemployed or employed may be under represented, depending on time of day.

c Unemployed will be over represented.

Use random sampling based on the whole population (e.g. electoral roll).

8 **a** Divide platform into a grid of 1 m^2 squares. Select squares using a random number generator to give two digits, one a vertical reference and one a horizontal reference.

b Yes, if crabs are fairly evenly distributed; otherwise, five squares may not be enough.

9 No; a parent's chance of selection depends on how many children they have at the school.

10 Not a random sample; only interested people will call, and they may call more than once.

11 People who go out in the evenings will not be included in the sample.

12 **a** All students at this school

b $p = 0.35$

c $\hat{p} = 0.42$

13 **a** 0.22

b $\hat{p}$

Solutions to Exercise 17B

1 **a** $p = \frac{5}{10} = \frac{1}{2}$

b $0, \frac{1}{3}, \frac{2}{3}, 1$

c $\Pr(\hat{P} = 0) = \dfrac{\binom{5}{0}\binom{5}{3}}{\binom{10}{3}} = \dfrac{1}{12}$

$\Pr(\hat{P} = \frac{1}{3}) = \dfrac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \dfrac{5}{12}$

$\Pr(\hat{P} = \frac{2}{3}) = \dfrac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \dfrac{5}{12}$

$\Pr(\hat{P} = 1) = \dfrac{\binom{5}{3}\binom{5}{0}}{\binom{10}{3}} = \dfrac{1}{12}$

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

d $\Pr(\hat{P} > 0.5) = \frac{5}{12} + \frac{1}{12} = \frac{1}{2}$

2 **a** $p = \frac{12}{20} = \frac{3}{5}$

b Values of $\hat{P} : 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

c

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$\Pr(\hat{P} = \hat{p})$	0.0036	0.0542	0.2384
$\hat{p}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	0.3973	0.2554	0.0511

d $\Pr(\hat{P} > 0.7) = 0.2554 + 0.0511$

$= 0.3065$

e $\Pr(\hat{P} < 0.7|\hat{P} > 0) = \dfrac{\Pr(0 < \hat{P} < 0.7)}{\Pr(\hat{P} > 0)} = 0.6924$

3 **a** $p = 0.5$

b Values of $\hat{P}$: $0, \frac{1}{2}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.1	0.4	0.4	0.1

d $\Pr(\hat{P} > 0.25) = 0.9$

4 **a** $p = 0.4$

b Values of $\hat{P}$: $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

d $\Pr(\hat{P} > 0.5) = \dfrac{1}{3}$

e $\Pr(\hat{P} < 0.5|\hat{P} > 0)$

$= \dfrac{\Pr(0 < \hat{P} < 0.5)}{\Pr(\hat{P} > 0)} = \dfrac{3}{5}$

5 **a** $p = 0.5$

b Values of $\hat{P}$: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

c

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

d $\Pr(\hat{P} > 0.7) = \dfrac{5}{16}$

6 **a** Values of $\hat{P}$: $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

b

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

c $\Pr(\hat{P} < 0.4) = \frac{3}{16}$

d $\Pr(\hat{P} > 0 | \hat{P} < 0.8)$

$$= \frac{\Pr(0 < \hat{P} < 0.8)}{\Pr(\hat{P} < 0.8)} = \frac{25}{26}$$

7 **a** Values of $\hat{P}$: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

b

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

c $\Pr(\hat{P} \geq 0.5) = \frac{113}{625}$

8

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$\mathrm{E}(X) = 0 \times \frac{1}{16} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{8} + \frac{3}{4} \times \frac{1}{4} + 1 \times \frac{3}{16} = 0.5$$

$$\mathrm{E}(X^2) = 0^2 \times \frac{1}{16} + \left(\frac{1}{4}\right)^2 \times \frac{1}{4} + \left(\frac{1}{2}\right)^2 \times \frac{3}{8} + \left(\frac{3}{4}\right)^2 \times \frac{1}{4} + \left(1\right)^2 \times \frac{1}{16} = \frac{5}{16}$$

$$\therefore \mathrm{Var}(X) = \left(\frac{5}{16}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{16}$$

$$\therefore \mathrm{sd}(x) = \frac{1}{4}$$

9

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

$$\mathrm{E}(X) = 0 \times \frac{1}{32} + \frac{1}{5} \times \frac{5}{32} + \frac{2}{5} \times \frac{5}{16} + \frac{3}{5} \times \frac{5}{16} + \frac{4}{5} \times \frac{5}{32} + 1 \times \frac{1}{32} = 0.5$$

$$\mathrm{E}(X^2) = 0^2 \times \frac{1}{32} + \left(\frac{1}{5}\right)^2 \times \frac{5}{32} + \left(\frac{2}{5}\right)^2 \times \frac{5}{16} + \left(\frac{3}{5}\right)^2 \times \frac{5}{16} + \left(\frac{4}{5}\right)^2 \times \frac{5}{32} + \left(1\right)^2 \times \frac{1}{32} = 0.340176$$

$\therefore \mathrm{Var}(X) = 0.340176 - \left(\frac{1}{2}\right)^2 = 0.050176$

$\therefore \mathrm{sd}(X) = 0.224$

$\mu = 0.5, \ \sigma = 0.224$

10

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

$$\mathrm{E}(X) = 0 \times \frac{256}{625} + \frac{1}{4} \times \frac{256}{625} + \frac{3}{4} \times \frac{16}{625} + 1 \times \frac{1}{625} = 0.2$$

$$\mathrm{E}(X^2) = 0^2 \times \frac{256}{625} + \left(\frac{1}{4}\right)^2 \times \frac{256}{625} + \left(\frac{3}{4}\right)^2 \times \frac{16}{625} + \left(1\right)^2 \times \frac{1}{32} = 0.08$$

$\therefore \mathrm{Var}(X) = 0.08 - \left(0.02\right)^2 = 0.04$

$\therefore \mathrm{sd}(X) = 0.2$

$\mu = 0.2, \ \sigma = 0.2$

11 $n = 30, p = 0.4$. Let $\hat{P} = \frac{X}{30}$

a $\Pr(\hat{P} > 0.4 = \Pr(X > 12) = 0.0845$

b $\mu = p = 0.3, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.3 \times 0.7}{30}} = 0.084$

12 $n = 100, p = 0.4,$

a $\Pr(\hat{P} > 0.45) \Leftrightarrow \Pr\left(\frac{X}{100}\right) > 0.45) \Leftrightarrow \Pr(X > 45) = 0.1311$

b $\Pr(\hat{P} > 0.45) \Leftrightarrow \Pr\left(\frac{X}{200}\right) > 0.45) \Leftrightarrow \Pr(X > 90) = 0.0655$

13 **a** $n = 16, p = \frac{1}{4}$

$\Pr\left(\hat{P} \geq \frac{5}{16}\right) \Leftrightarrow \Pr\left(\frac{X}{16} \geq \frac{5}{16}\right) \Leftrightarrow \Pr(X \geq 5) = 0.3698$

b $\Pr\left(\hat{P} \geq \frac{5}{16} | \hat{P} \geq \frac{3}{16}\right) = \dfrac{\Pr\left(\hat{P} \geq \frac{5}{16}\right)}{\Pr\left(\hat{P} \geq \frac{3}{16}\right)}$

$$= \frac{\Pr(X \geq 5)}{\Pr(X \geq 3)}$$

$$= 0.4606$$

c $\mu = p = \dfrac{1}{4}$ and $\sigma = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.25 \times 0.75}{16}} = \dfrac{\sqrt{3}}{16} \approx 0.1083$

14 **a** $p = 0.65, n = 20$

$\Pr(\hat{P} = 0.65) = \Pr(X = 13) = 0.1844$

b $\mu = 0.65, \sigma = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.65 \times 0.35}{20}} = 0.1066$

$\mu - \sigma = 0.543$

$\mu + \sigma = 0.757$

$$\Pr(0.543 < \hat{P} < 0.757) = \Pr(10.86 < X < 15.14)$$

$$= \Pr(11 \leq X \leq 15)$$

$$= 0.7600$$

c $\mu - 2\sigma = 0.4368$

$\mu + 2\sigma = 0.8632$

$$\Pr(0.4368 < \hat{P} < 0.8632) = \Pr(8.74 < X < 17.26)$$

$$= \Pr(9 \leq X \leq 17)$$

$$= 0.9683$$

Solutions to Exercise 17C

1 $p = 0.5, n = 50$

$\mu = 0.5, \sigma = \sqrt{\dfrac{p(1-p)}{n}} =$

$\sqrt{\dfrac{0.5 \times 0.5}{50}} = 0.0707$

$\Pr(\hat{P} < 0.46) \approx \Pr(Z \le \dfrac{0.46 - 0.5}{0.7070}) =$

$Pr(Z \le -0.5658)$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} < 0.46) \approx 0.2858$

2 $p = 0.12, n = 300$

$\mu = 0.12, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.12 \times 0.88}{300}} = 0.018762$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} > 0.1) \approx 0.8568$

3 $p = 0.5, n = 25$

$\mu = 0.5, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.5 \times 0.5}{25}} = 0.1$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} > 0.6) \approx 0.1587$

4 $p = 0.1, n = 200$

$\mu = 0.1, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.1 \times 0.9}{200}} = 0.0212$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} > 0.15) \approx 0.0092$

5 $p = 0.3, n = 50$

$\mu = 0.3, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.3 \times 0.7}{50}} = 0.0648$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} < 0.2) \approx 0.0614$

6 $p = 0.6, n = 100$

$\mu = 0.6, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.6 \times 0.4}{100}} = 0.0490$

a $\Pr(\hat{P} < 0.8) \approx 1$

b $\Pr(0.6 < \hat{P} < 0.8) \approx 0.5$

c $\Pr(0.7 < \hat{P} < 0.8 | \hat{P} > 0.6)$

$= \dfrac{\Pr(0.7 < \hat{P} < 0.8)}{\Pr(\hat{P} > 0.6)} \approx 0.0412$

7 $p = 0.5, n = 100$

a $\mu = 0.5, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.5 \times 0.5}{100}} = 0.05$

The calculation can also be done directly with calculator:

$\Pr(0.4 < \hat{P} < 0.6) \approx 0.9545$

b $\Pr(\hat{P} > 0.55) = 0.1$

Let X be the corresponding normal approximation with

$\mu = 0.5$ and $\sigma = \sqrt{\dfrac{0.25}{n}}$

$$\begin{aligned}
\Pr(X > 0.55) &= 0.1\\
\Leftrightarrow \Pr\left(Z > \frac{0.55-0.5}{\sigma}\right) &= 0.1\\
\Leftrightarrow \Pr\left(Z \leq \frac{0.05}{\sigma}\right) &= 0.9\\
\Rightarrow \frac{0.05}{\sigma} &= 1.28155\ldots\\
\Rightarrow \sqrt{\frac{0.25}{n}} &= \frac{0.05}{1.28155\ldots}\\
\Rightarrow \sqrt{\frac{0.25}{n}} &= 0.0390\\
\Rightarrow \frac{0.25}{n} &= 0.001522\ldots\\
\Rightarrow n &= 164.237 \approx 164
\end{aligned}$$

8 $p = 0.1, n = 1000$

$\mu = 0.1, \sigma = \sqrt{\frac{p(1-p)}{n}}$

$= \sqrt{\frac{0.1 \times 0.9}{1000}} = 0.0095$

a $\Pr(0.08 < \hat{P} < 0.12) \approx 0.9650$

b $\Pr(0.08 < \hat{P} < 0.12 | \hat{P} > 0.10) =$
$\frac{\Pr(0.08 < \hat{P} < 0.12)}{\Pr(\hat{P} < 0.12)} \approx 0.9650$

9 $p = 0.52, n = 400$

$\mu = 0.1, \sigma = \sqrt{\frac{p(1-p)}{n}}$

$= \sqrt{\frac{0.1 \times 0.9}{1000}} = 0.0095$

a $\hat{p} = \frac{230}{400} = 0.575$

b $\mu = 0.52, \sigma = \sqrt{\frac{p(1-p)}{n}} = 0.0350$
$\Pr(\hat{P} \geq 0.575) \approx 0.0139$

10 $p = 0.9, n = 250$

a $\hat{p} = \frac{212}{250} = 0.848$

b $\mu = 0.9, \sigma = \sqrt{\frac{p(1-p)}{n}} = 0.0190$
$\Pr(\hat{P} \leq 0.848) \approx 0.0031$

c Yes, because the chance of the battery lasting only this short period of time is very small if the manufacturers claim is correct.

11 $\Pr(\hat{P} < 0.32) = 0.2445$
Let X be the corresponding normal approximation with
$\mu = 0.35$ and $\sigma = \sqrt{\frac{0.2275}{n}}$

$$\begin{aligned}
\Pr(X < 0.32) &= 0.2245\\
\Leftrightarrow \Pr\left(Z < \frac{0.32-0.35}{\sigma}\right) &= 0.2445\\
\Leftrightarrow \Pr\left(Z \leq \frac{-0.03}{\sigma}\right) &= 0.2445\\
\Rightarrow \frac{-0.03}{\sigma} &= -0.6919\ldots\\
\Rightarrow \sqrt{\frac{0.2275}{n}} &= \frac{0.03}{0.6919\ldots}\\
\Rightarrow \sqrt{\frac{0.2275}{n}} &= 0.0390\\
\Rightarrow \frac{0.2275}{n} &= 0.00188\ldots\\
\Rightarrow n &= 121.011 \approx 121
\end{aligned}$$

Solutions to Exercise 17D

1 **a** 0.08

b 90%: $(0.0354, 0.1246)$,
95%: $(0.0268, 0.1332)$,
99%: $(0.0101, 0.1499)$ Interval width increases as confidence level increases

2 **a** 0.192

b 90%: $(0.1510, 0.2330)$,
95%: $(0.1432, 0.2408)$,
99%: $(0.1278, 0.2562)$ Interval width increases as confidence level increases

3 **a** 0.2

b $(0.1069, 0.2931)$

4 $(0.2888, 0.3712)$

5 **a** $(0.4761, 0.5739)$

b $(0.5095, 0.5405)$

c The second interval is narrower because the sample size is larger

6 **a** $(0.7895, 0.9065)$

b $(0.8295, 0.8665)$

c The point estimates for both samples are the same, but the second confidence interval is narrower because the sample size is larger. This interval does not contain 0.9, and would cause us to doubt the manufacturers claim.

7 90%: $(0.5194, 0.6801)$,
95%: $(0.5040, 0.6960)$,
99%: $(0.4738, 0.7262)$; Interval width increases as confidence level increases

8 90%: $(0.5111, 0.5629)$,
95%: $(0.5061, 0.5679)$,
99%: $(0.4964, 0.5776)$; Interval width increases as confidence level increases

9 $M = 0.02, \hat{p} = 0.8$

$$n = \left(\frac{1.96}{0.02}\right)^2 \times 0.8 \times 0.3 = 1536.64$$

Since n must be an integer larger than the calculated value to ensure the margin of error is no more than 0.02, $n = 1537$

10 $M = 0.05, \hat{p} = 0.2$ $n =$

$$\left(\frac{1.6449}{0.05}\right)^2 \times 0.2 \times 0.8 = 173.165$$

Since n must be an integer larger than the calculated value to ensure the margin of error is no more than 0.05, $n = 174$

11 $p^* = 0.30$

a $M = 0.03$

$$n = \left(\frac{2.5758}{0.03}\right)^2 \times 0.3 \times 0.7 = 1548.11$$

Since n must be an integer larger than the calculated value to ensure the margin of error is no more than 0.03, $n = 1549$

b $M = 0.02, \left(\frac{2.5758}{0.02}\right)^2 \times 0.3 \times 0.7 \approx$ 3484

c Reducing margin of error by 1% requires the sample size to be more than doubled

12 a $p^* = 0.3, M = 0.02\ n =$ $\left(\frac{1.96}{0.02}\right)^2 \times 0.3 \times 0.7 = 2016.94 \approx 2017$

b $p^* = 0.5, M = 0.02$
$n = \left(\frac{1.96}{0.02}\right)^2 \times 0.5 \times 0.5 \approx 2401$

c **i** $p^* = 0.3, n = 2401$
$M = 1.96\sqrt{\frac{0.3 \times 0.7}{2401}} \approx 1.8$
The margin of error is less than 2%

ii $p^* = 0.5, n = 2017$
$M = 1.96\sqrt{\frac{0.5 \times 0.5}{2017}} \approx 2.2$
The margin of error is greater than 2%

d 2401, as this ensures that M is 2% or less, whoever is correct

Solutions to Technology-free questions

1 **a** All employees of the company

b $p = 0.35$

c $\hat{p} = 0.40$

2 **a** No; only people already interested in yoga

b Use electoral roll

3 **a** $\dfrac{3}{5}$

b $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$

c $\Pr\left(\hat{P} = \dfrac{1}{3}\right) = \Pr(X = 1)$

Possible ways:

BRR, RBR, RRB

$\Pr(X = 1) =$

$$\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} + \frac{2}{5} \times \frac{3}{4} \times \frac{1}{3} + \frac{2}{5} \times \frac{1}{4} \times 1$$

$$= \frac{3}{10}$$

4
$$\sqrt{\frac{\frac{1}{8} \times \frac{7}{8}}{n}} = \frac{1}{80}$$

$$\sqrt{\frac{\frac{7}{64}}{n}} = \frac{1}{80}$$

$$\frac{1}{8}\sqrt{\frac{7}{n}} = \frac{1}{80}$$

$$\sqrt{\frac{7}{n}} = \frac{1}{10}$$

$$\frac{7}{n} = \frac{1}{100}$$

$$n = 700$$

5 **a** $\dfrac{k}{100}$

b $\hat{p} \pm 1.96\sqrt{\dfrac{\hat{p}(1 - \hat{p}}{n}}$

$$= \frac{k}{100} \pm 1.96\sqrt{\frac{\frac{k}{100}(1 - \frac{k}{100}}{100}}$$

$$= \frac{k}{100} \pm \frac{1.96\sqrt{k(100 - k)}}{1000}$$

6 **a** $\hat{p} = 0.9$

b $M = 1.96\sqrt{\dfrac{\hat{p}(1 - \hat{p}}{100}}$

$$= 1.96 \times \sqrt{\frac{0.9 \times 0.1}{n}}$$

$$= 1.96 \times \frac{0.3}{\sqrt{n}}$$

$$\therefore M = \frac{0.588}{\sqrt{n}}$$

c Margin of error would decrease by a factor of $\sqrt{2}$

7 **a** Confidence interval

$$\left(\frac{576}{1250}, \frac{674}{1250}\right)$$

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1 - \hat{p}}{n}}, \hat{p} + 1.96\sqrt{\frac{\hat{p}(1 - \hat{p}}{n}}\right)$$

Adding the boundary values:

$$2\hat{p} = 1 \Rightarrow \hat{p} = \frac{1}{2}$$

b $\hat{p} + 1.96\sqrt{\dfrac{\hat{p}(1 - \hat{p}}{n}} = \dfrac{674}{1250}$

$$
\begin{aligned}
\frac{1}{2} + 1.96\sqrt{\frac{1}{4n}} &= \frac{674}{1250} \\
\frac{49}{25}\sqrt{\frac{1}{4n}} &= \frac{49}{1250} \\
\sqrt{\frac{1}{4n}} &= \frac{1}{50} \\
\frac{1}{4n} &= \frac{1}{2500} \\
4n &= 2500 \\
n &= 625
\end{aligned}
$$

8 **a** $40 \times 0.95 = 38$

b $\Pr(Y = 40) = \binom{40}{40}(0.95)^{40}(0.05)^{0} = (0.95)^{40}$

9 **a** $50 \times 0.95 = 45$

b

$$
\begin{aligned}
\Pr(Y \geq 49) &= \Pr(Y = 49) + \Pr(Y = 50) \\
&= \binom{50}{49}(0.1)^{1}(0.9)^{49} + \binom{50}{50}(0.1)^{0}(0.9)^{50} \\
&= 5(0.9)^{49} + (0.9)^{50} \\
&= 5.9(0.9)^{49}
\end{aligned}
$$

10 **a** $\hat{p} = 0.60$

b $M = 0.10$

c Increase sample size

Solutions to multiple-choice questions

1 **B** This class is a sample of the whole school population, so any statistics determined from this sample is called a sample statistic.

2 **C** When the statistics is calculated from the whole population it is known as a population parameter.

3 **D** All we can say about a 95% confidence interval is that 95% of such intervals will capture the true mean. Statement B is a common incorrect interpretation of a confidence interval.

4 **E** $M = 1.96\sqrt{\frac{\hat{p}(1-\hat{p}}{100}}$

$= 1.96 \times \sqrt{\frac{0.3 \times 0.7}{50}}$

$= 0.1270$

5 **C** $\Pr\left(\hat{P} \geq \frac{3}{10}\right) = \Pr(X \geq 6)$

Binomial: $n = 20, p = 0.15$

$\Pr(X \geq 6) = 0.067308\ldots$

6 **E** $2\hat{p} = 0.084 + 0.236$

$\hat{p} = 0.16$

7 **B** To be more confidence of capturing the true mean the interval will be wider.

8 **E** I the centre of a confidence interval is a sample parameter not a population parameter

II the bigger the margin of error the bigger the confidence interval

III a point estimate is a single value estimate like $\hat{p}$

IV the sample proportion a point estimate

9 **C** Since the width of the confidence interval is inversely proportional to the square root of the sample size, increasing the sample size by a factor of 4 decreases the width by a factor of 2.

10 **E** $M = 0.03$ $n = \left(\frac{1.96}{0.03}\right)^2 \times 0.3 \times 0.7 = 896.37 \approx 897$

11 **A** See definitions

12 **B** A sampling distribution is the distribution of a sample statistic, and as such shows how this statistic varies from sample to sample.

13 **C** Let b be the number of red bricks. Take $\hat{p} = \frac{b}{10\,000}$

$\sqrt{\frac{\hat{p}(1-\hat{p}}{100}} = 0.04 \quad \hat{p} = 0.2$ or

$\frac{\hat{p}(1-\hat{p})}{100} = 0.0016$

$\hat{p}(1-\hat{p}) = 0.16$

$\hat{p}^2 - \hat{p} + 0.16 = 0$

$\hat{p} = 0.8$

More red bricks than grey. Hence $\hat{p} = 0.8$ and $b = 8000$

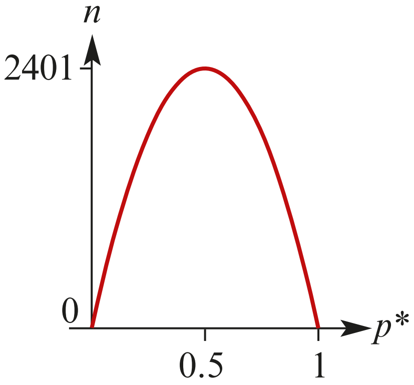
14 **D** The width of a confidence interval will decrease if the sample size is increased, or if the level of confidence is decreased.

Solutions to extended-response questions

1 **a** $n = \left(\frac{1.96}{M}\right)^2 p^*(1 - p^*) \qquad 0 \le p^* \le 1$

$= \left(\frac{1.96}{0.02}\right)^2 p^*(1 - p^*)$

$= 9604^2 p^*(1 - p^*)$



b From the graph, the maximum occurs when $p^* = 0.5$

c If they use the maximum samples size (2401) then they will ensure the margin of error stays within the desired range of ±2%

2 $p = 0.7, n = 100$

a $\Pr(\hat{P} > 0.75) = \Pr(X > 75) = 0.1136$ (using CAS calculator)

b $\Pr(0.68 \le \hat{P} < 0.75) = \dfrac{\Pr(69 \le X \le 74)}{\Pr(X \ge 69)} = 0.7423$

3 **a** $\hat{p} = 0.57, n = 100$

95% CI = (0.4730, 0.6670)

b **i** $\Pr(Y = 5) = \binom{5}{5}(0.95)^5(0.05)^0 = 0.7738$

ii $\Pr(Y = 0) = \binom{5}{0}(0.95)^0(0.05)^5 = 0.0000003$

iii $\Pr(Y \le 4) = 0.2262$

iv $0.95 \times 5 = 4.75$

c $n = 500$

$X = 57 + 67 + 72 + 55 + 60 = 311$

$\hat{p} = \dfrac{311}{500} = 0.622$

CI = (0.5795, 0.6645)

4 **a** $p = \dfrac{500}{N}$

b $\hat{p} = \dfrac{60}{400} = 0.15$

c $\dfrac{500}{N} = 0.15\ N \approx \dfrac{500}{0.15} = 3333.33 \approx 3333$

d 95% CI for $\hat{p}$

$$0.15 - 1.96\sqrt{\frac{0.15 \times 0.85}{400}} < p < 0.15 + 1.96\sqrt{\frac{0.15 \times 0.85}{400}}$$

$$0.15 - 1.96\sqrt{\frac{0.1275}{400}} < p < 0.15 + 1.96\sqrt{\frac{0.1275}{400}}$$

e $0.1150 < \dfrac{500}{N} < 0.1850$

$5.4056 < \dfrac{N}{500} < 8.6951$

$2703 < N < 4348$

Chapter 18 – Revision of chapters 13–17

Solutions to technology-free questions

1 a Pr(faulty)

$$= \frac{8}{15} \times \frac{20}{800} + \frac{7}{15} \times \frac{14}{700}$$

$$= \frac{8}{15} \times \frac{1}{40} + \frac{7}{15} \times \frac{1}{50}$$

$$= \frac{17}{750}$$

b

Pr(Machine A| faulty

$$= \frac{\text{Pr(faulty|Machine A)} \times \text{Pr(Machine A)}}{\text{Pr(faulty)}}$$

$$= \frac{\frac{7}{15} \times \frac{1}{50}}{\frac{17}{50}}$$

$$= \frac{7}{17}$$

2 a Let X be the number of plants which do not survive. X is binomial $n = 4$, $p = \frac{1}{5}$

b $\Pr(X = 4) = \left(\frac{1}{5}\right)^4 = \frac{1}{625}$

c $\Pr(X \geq 1) = 1 - \frac{1}{625}$

$$= \frac{624}{625}$$

d Let Y represent the number of boxes in the sample of six in which all plants survive. Y is binomial $n = 6$, $p = \left(\frac{4}{5}\right)^4$

$$\Pr(Y = 6) = \left(\left(\frac{4}{5}\right)^4\right)^6$$

$$= \left(\frac{4}{5}\right)^{24}$$

3 a

$$\int_{\frac{3}{2}}^{\frac{5}{2}} k\cos(\pi x)\,dx = \left[\frac{k}{\pi}\sin \pi x\right]_{\frac{3}{2}}^{\frac{5}{2}}$$

$$= \frac{k}{\pi}\left(\sin\frac{5\pi}{2} - \sin\frac{3\pi}{2}\right)$$

$$= \frac{2k}{\pi}$$

Since area= 1, $\frac{2k}{\pi} = 1 \Rightarrow k = \frac{\pi}{2}$

b

$$\int_{\frac{3}{2}}^{m} \cos(\pi x)\,dx = \left[\frac{\pi}{2} \times \frac{1}{\pi}\sin(\pi x)\right]_{\frac{3}{2}}^{m}$$

$$= \frac{1}{2}\left(\sin(m\pi) - \sin\frac{3\pi}{2}\right)$$

$$= \frac{1}{2}\left(\sin(m\pi) + 1\right)$$

For the median;

$$\frac{1}{2}\left(\sin(m\pi) + 1\right) = 0$$

$$m\pi = 0 \text{ or } m\pi = 2\pi$$

$$\therefore m = 2$$

$$\left(\text{since } \frac{3}{2} < m < \frac{5}{2}\right)$$

c $\Pr\left(X < \frac{7}{4} \middle| X < 2\right) = \frac{\Pr\left(X < \frac{7}{4}\right)}{\Pr(X < 2)}$

$$= \frac{2 - \sqrt{2}}{2}$$

d

$$\Pr\left(X > \frac{9}{4}\middle| X > \frac{7}{4}\right) = \frac{\Pr\left(X > \frac{9}{4}\right)}{\Pr(X > \frac{7}{4})}$$

$$= \left(\frac{2 - \sqrt{2}}{2}\right) \div \left(1 - \frac{2 - \sqrt{2}}{2}\right)$$

$$= 2\sqrt{2} - 2$$

4 **a** $\Pr(X > 3|X > 1) = \dfrac{\Pr(X > 3)}{\Pr(X > 1)}$

$$= \frac{0.1}{0.5}$$

$$= \frac{1}{5}$$

b $\Pr(X > 1|X \leq 3) = \dfrac{\Pr(1 < X \leq 3)}{\Pr(X \leq 3)}$

$$= \frac{0.4}{0.9}$$

$$= \frac{4}{9}$$

c $\mathrm{E}(X) = \Sigma x \Pr(X = x) = 1.7$

d $\mathrm{E}(X^2) = \Sigma x^2 \Pr(X = x) = 4.9$

$\mathrm{Var}(X) = 4.9 - 1.7^2 = 2.01$

5 **a**

$$\int_0^6 kx(6 - x)\,dx = k\left[3x^2 - \frac{x^3}{3}\right]_0^6$$

$$= k(108 - 72) \quad = 36k$$

$36k = 1 \Rightarrow k = \dfrac{1}{36}$

b $\dfrac{1}{36}\displaystyle\int_0^4 x(6 - x)\,dx = \dfrac{20}{27}$

c $\displaystyle\int_0^m \frac{x}{36}(6 - x)\,dx = 0.5$

$$\frac{x}{36}\left[3x^2 - \frac{x^3}{3}\right]_0^m = 0.5$$

$$3m^2 - \frac{1}{3}m^3 - 18 = 0$$

$$9m^2 - m^3 - 54 = 0$$

$$-(m - 3)(m^2 - 6m - 18) = 0$$

$\therefore m = 3$

Other solutions are outside $[0, 6]$

d $\dfrac{1}{36}\displaystyle\int_0^6 x^2(6 - x)\,dx = 3$ Symmetry can be used for this.

e

$$\Pr(X < 2|X < 3) = \frac{\Pr(X < 2)}{\Pr(X < 3)}$$

$$= \frac{\frac{1}{36}\int_0^2 x(6 - x)\,dx}{\frac{1}{36}\int_0^3 x(6 - x)\,dx}$$

$$= \frac{\int_0^2 x(6 - x)\,dx}{\int_0^3 x(6 - x)\,dx}$$

$$= \frac{\frac{28}{3}}{18}$$

$$= \frac{14}{27}$$

f

$$\Pr(X > 2|X < 4) = \frac{\Pr(2 < X < 4)}{\Pr(X < 4)}$$
$$= \frac{\frac{1}{36}\int_2^4 x(6-x)\,dx}{\frac{1}{36}\int_0^4 x(6-x)\,dx}$$
$$= \frac{\int_2^4 x(6-x)\,dx}{\int_0^4 x(6-x)\,dx}$$
$$= \frac{\frac{52}{3}}{\frac{80}{3}}$$
$$= \frac{13}{20}$$

6 **a** $\Pr(RG) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

b $\Pr(RG) + \Pr(GR) = \frac{3}{28} + \frac{3}{28} = \frac{3}{14}$

c $\Pr(G_2|R_1) + \Pr(B_2|R_1) + \Pr(Y_2|R_1) =$
$\frac{2}{7} + \frac{2}{7} + \frac{1}{7} = \frac{5}{7}$

d $\Pr(R_1' \cap R_2') = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$

e $\Pr(R_1 \cap R_2) + \Pr(B_1 \cap B_2) + \Pr(G_1 \cap G_2)$
$= \frac{3}{8} \times \frac{2}{7} + \frac{2}{8} \times \frac{1}{7} + \frac{2}{8} \times \frac{1}{7}$
$= \frac{5}{28}$

7 $\Pr(A) = \frac{4}{7}, \Pr(B) = \frac{1}{3}, \Pr(A' \cap B) = ?$

a $\Pr(A' \cap B) + \Pr(A \cap B) = \Pr(B)$
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$\therefore$
$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$
$\therefore \Pr(A \cap B) = \frac{4}{7} + \frac{1}{3} - \frac{5}{7} = \frac{4}{21}$

Also
$\Pr(A' \cap B) + \Pr(A \cap B) = \Pr(B)$
$\therefore \Pr(A' \cap B) = \frac{1}{3} - \frac{4}{21} = \frac{1}{7}$

b $\Pr(A' \cap B) = \Pr(B) = \frac{1}{3}$

8 **a** $\Pr(A \cap B) = \Pr(B|A)\Pr(A)$
$= \frac{1}{5} \times \frac{3}{4}$
$= \frac{3}{20}$
$\Pr(A' \cap B') = \Pr(B'|A')\Pr(A')$
$= p \times \frac{1}{4}$
$= \frac{p}{4}$
$\Pr(B) = \Pr(A \cap B) + \Pr(A' \cap B)$
$= \frac{3}{20} + \frac{1-p}{4}$
$= \frac{8-5p}{20}$

b $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$
$= \frac{3}{8-5p}$

9 **a**

$$\Sigma \Pr(X = x) = 1$$

$$\therefore a + 0.3 + 0.1 + 0.2 + b = 1$$

$$\therefore a + b = 0.4 \ldots (1)$$

$$\mathrm{E}(X) = \Sigma x \Pr(X = x) = 2.34$$

$$\therefore a + 0.6 + 0.3 + 0.8 + 5b = 2.34$$

$$\therefore a + 5b = 0.64 \ldots (2)$$

$$(2) - (1)$$

$$4b = 0.24$$

$$b = 0.06$$

$$\text{From}(1)\ a = 0.34$$

b $\mathrm{E}(X^2) = 6.54$

$\mathrm{Var}(X) = 1.0644$

10 a Pr(win)

$$= 0.7 \times 0.9 + 0.3 \times 0.4$$

$$= 0.63 + 0.12$$

$$= 0.75$$

b Pr(Fully fit| Did not win)

$$= \frac{\Pr(\text{Fully fit} \cap \text{Did not win})}{\Pr(\text{Did not win})}$$

$$= \frac{0.1 \times 0.7}{0.25}$$

$$= \frac{7}{25}$$

11 a $\displaystyle\int_a^{2a} (x - a)(2a - x)\,dx$

$$= \left[-\frac{x^3}{3} + \frac{3ax^2}{2} - 2xa^2\right]_a^{2a}$$

$$= \frac{a^3}{6}$$

Since the area = 1

$$\frac{a^3}{6} = 1$$

$$\therefore a^3 = 6$$

b $\displaystyle\mathrm{E}(X) = \int_a^{2a} x(x - a)(2a - x)\,dx$

$$= \left[-\frac{x^4}{4} + \frac{3ax^3}{3} - (xa)^2\right]_a^{2a}$$

$$= \frac{a^4}{4}$$

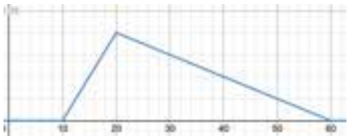
$$= \frac{6^{\frac{4}{3}}}{4}$$

12 a $\mathrm{E}(X) = 2$

b $\displaystyle\Pr(X < 2) = \left(\frac{7}{8}\right)^1 6 + 16 \times \left(\frac{7}{8}\right)^1 6 \times \frac{1}{8}$

$$= \frac{23 \times 7^{15}}{8^{16}}$$

13



$$0.004x - 0.04 = -0.001x + 0.06$$

$$0.005x = 0.1$$

$$x = 20$$

$$f(20) = \frac{1}{25}$$

$$f(15) = \frac{1}{50}$$

$$f(30 = \frac{2}{25}$$

Area of the whole triangle is 1.

Therefore area of required region

$$= 1 - \left(\frac{1}{2} \times 5 \times \frac{1}{25} + \frac{1}{2} \times 10 \times \frac{2}{25}\right)$$

$$= 1 - \frac{1}{2}\left(\frac{1}{5} + \frac{4}{5}\right)$$

$$= \frac{1}{2}$$

14 **a**

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$\therefore 1 - \Pr(X = 0) = 0.99968$$

$$\Pr(X = 0) = 0.00032$$

$$(1 - p)^5 = 0.00032$$

$$1 - p = 0.2$$

$$p = 0.8$$

b $\Pr(X = 3) = \binom{5}{3} p^3(1 - p)^2$

$$= 10p^3(1 - p)^2$$

$$= 10(p^3 - 2p^4 + p^5)$$

$$\text{Let} P = p^3(1 - p)^2$$

$$\frac{dP}{dp} = 3p^2 - 8p^3 + 5p^4$$

$$= p^2(3 - 8p + 5p^2$$

$$= p^2(5p - 3)(p - 1)$$

$$\frac{dP}{dp} = 0 \Rightarrow p = \frac{3}{5}$$

15 $\mu = 40, \sigma = 2$

$Pr(36 < X < 44) = q$

$Pr(X > 44) = \dfrac{1 - q}{2}$

16 $\int_a^0 2(1 - x)\, dx = \frac{3}{4}$ since

$$\left[2(x - \frac{x^2}{2})\right]_0^a = \frac{3}{4}$$

$$2a - a^2 = \frac{3}{4}$$

$$8a - 4a^2 = 3$$

$$4a^2 - 8a + 3 = 0$$

$$(2a - 1)(2a - 3) = 0$$

$$a = \frac{1}{2} \text{ or } a = \frac{3}{2}$$

$$0 \leq x \leq 1, a = \frac{1}{2}$$

17 $n = 3, =?$

a $\Pr(X = 0) = (1 - p)^3$

b $\Pr(X = 0) = p^3$

$$\begin{aligned} (1 - p)^3 &= 8p^3 \\ 1 - p &= 2p \\ 1 &= 3p \\ p &= \frac{1}{3} \end{aligned}$$

18 **a** Let $y = \sin x^2$.

Using the chain rule $\dfrac{dy}{dx} = 2x\cos x^2$

b

$$\begin{aligned} \Pr\left(\sqrt{\frac{\pi}{3}} < X < \sqrt{\frac{\pi}{2}}\right) &= \int_{\sqrt{\frac{\pi}{3}}}^{\sqrt{\frac{\pi}{2}}} 2x\cos x^2\, dx \\ &= \left[\sin x^2\right]_{\sqrt{\frac{\pi}{3}}}^{\sqrt{\frac{\pi}{2}}} \\ &= \sin\frac{\pi}{2} - \sin\frac{\pi}{3} \\ &= 1 - \frac{\sqrt{3}}{2} \end{aligned}$$

c

$$\begin{aligned} \Pr(X \le m) &= \frac{1}{2} \\ \int_0^m 2x\cos x^2\, dx &= \frac{1}{2} \\ \left[\sin x^2\right]_0^m &= \frac{1}{2} \\ \sin m^2 &= \frac{1}{2} \\ m^2 &= \frac{\pi}{6} \\ m &= \sqrt{\frac{\pi}{6}} \end{aligned}$$

19 **a** $p = \frac{3}{10}$

b $0, \frac{1}{3}, \frac{2}{3}, 1$

c $\Pr(\hat{P} = 0)$ =probability of all red balls.

$$= \frac{7}{10} \times \frac{6}{9} \times 58$$
$$= \frac{7}{2} \times \frac{1}{3} \times 14$$
$$= \frac{7}{24}$$

20 Let $p = \frac{1}{5}$

$$\sqrt{\frac{p(1-p)}{n}} \leq \frac{1}{20}$$
$$\frac{p(1-p)}{n} \leq \frac{1}{400}$$
$$400p(1-p) \leq n$$
$$n \geq 400 \times \frac{1}{5} \times \frac{4}{5}$$
$$n \geq 64$$

21 **a** Consider black out of the red box and then white out of the red box.

i $\frac{n-3}{n} \times \frac{4}{n+1} + \frac{3}{n} \times \frac{3}{n+1}$

$$= \frac{4n - 12 + 9}{n(n+1)}$$
$$= \frac{4n-3}{n(n+1)}$$

ii $\frac{n-3}{n} \times \frac{n-3}{n+1} + \frac{3}{n} \times \frac{n-2}{n+1}$

$$= \frac{n^2 - 3n + 3}{n(n+1)}$$

b Pr(first is black | white is second)

$$= \frac{n-3}{n} \times \frac{n-3}{n+1} \div \frac{n^2-3n+3}{n(n+1)}$$
$$= \frac{(n-3)^2}{n^2-3n+3}$$

22

$$\mathrm{E}(Z) = \mathrm{E}\left(\frac{X-\mu}{\sigma}\right)$$
$$= \frac{1}{\sigma}\mathrm{E}(X-\mu)(\text{Using } \mathrm{E}(aX+b) = a\mathrm{E}(X)+b)$$
$$= \frac{1}{\sigma}(\mu-\mu)$$
$$= 0$$

$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{X-\mu}{\sigma}\right)$$
$$= \frac{1}{\sigma^2}\mathrm{Var}(X) \quad (\text{Using } \mathrm{Var}(aX+b) = a^2\mathrm{Var}(X))$$
$$= \frac{\sigma^2}{\sigma^2}$$
$$= 1$$

Solutions to multiple-choice questions

1 E

$$\begin{aligned}\text{Pr (same color)} &= \frac{4}{16}\times\frac{3}{15}+\frac{12}{16}\times\frac{11}{15}\\ &= \frac{1}{4}\times\frac{1}{5}+\frac{3}{4}\times\frac{11}{15}\\ &= \frac{1}{20}+\frac{11}{20}\\ &= \frac{12}{20}\\ &= \frac{3}{5}\end{aligned}$$

2 B

$$\begin{aligned}\Pr(A\cup B) &= \Pr(A)+\Pr(B)-\Pr(A\cap B)\\ 0.5325 &= \Pr(A)+3\Pr(A)-\Pr(A)\times 3\Pr(A)\\ 0.5325 &= 4\Pr(A)-3[\Pr(A)]^2\\ \Pr(A) &= 0.15\end{aligned}$$

3 D

$$\begin{aligned}\text{Pr (six correct)} &= \left(\frac{1}{2}\right)^6\\ &= \frac{1}{64}\\ &= 0.0156\end{aligned}$$

4 B

$$\begin{aligned}4c^2+5c^2+4c^2+3c^2 &= 1\\ 16c^2 &= 1\\ c^2 &= \frac{1}{16}\end{aligned}$$

$$\begin{aligned}E(X) &= \Sigma x\Pr(X=x)\\ &= 1\times\frac{4}{16}+2\times\frac{5}{16}+3\times\frac{4}{16}+4\times\frac{3}{16}\\ &= \frac{38}{16}\\ &= 2.375\end{aligned}$$

$$\Pr(X<\mu)=\frac{4}{16}+\frac{5}{16}=0.5625$$

5 C If X is the number of spins if takes to get a ‘3’, then it could take 1 spin or 2 spins or . . . .; there is no theoretical upper limit. So the sample space is $\{1, 2, 3, 4, \ldots\}$.

6 A

$$\begin{aligned}E(X) &= \Sigma x\Pr(X=x)\\ &= 4\times 0.3+6\times 0.2\\ &\quad +7\times 0.1+9\times 0.4\\ &= 1.2+1.2+0.7+3.6\\ &= 6.7\end{aligned}$$

7 B

$$\begin{aligned}E(X^2) &= \Sigma x^2\Pr(X=x)\\ &= 16\times 0.3+36\times 0.2\\ &\quad +49\times 0.1+81\times 0.4\\ &= 4.8+7.2+4.9+32.4\\ &= 49.3\end{aligned}$$

$$\begin{aligned}\text{var}(X) &= E(X^2)-[E(X)]^2\\ &= 49.3-6.7^2\\ &= 4.41\end{aligned}$$

8 E The values of X are 4, 6, 7, 9. Since $Y = 2X - 1$, the corresponding values of Y are 7, 11, 13, 17. The probabilities are uncharged, so the fifth option fits.

9 D If $Z = aX + b$, then

$$\text{var}(Z) = a^2\ \text{var}(x)$$

Here, $a = -1$ and $b = 4$, so

$$\begin{aligned}\text{var}(Z) &= (1-)^2\ \text{var}(X)\\ &= \text{var}(X)\\ &= 4.41\end{aligned}$$

10 E Let H be the even that the temperature exceeds 30. Using a tree diagram:

$$\Pr(H_{Wed}|H_{Mon}) = 0.6 \times 0.6 + 0.4 \times 0.25$$
$$= 0.46$$

11 C
$$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$$
$$= 202 - 11^2$$
$$= 202 - 121$$
$$= 81$$
$$\text{sd}(X) = 9$$

12 B 95% of scores, assuming an approximate normal distribution, will lie in the internal $(\mu - 2\sigma,\ \mu + 2\sigma)$.

$\mu - 2\sigma = 50 - 20 = 30$

$\mu - 2\sigma = 50 + 20 = 70$

So the required interval is (30, 70).

13 D Pr(at least 2 heads)

$$= {}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 + \left(\frac{1}{3}\right)^3$$
$$= 3 \times \frac{1}{8} + \frac{1}{8}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2}$$

14 C
$$\text{E}(X) = np$$
$$= 400 \times 0.1$$
$$= 40$$

15 B If a die is rolled until a six is obtained the sample space is $\{1, 2, 3, \ldots\}$. This can not be a binominal variable since there is no theoretical limit to the number of rolls.

16 E
$$\text{var}(X) = np(1 - p)$$
$$= 900 \times 0.2 \times 0.8$$
$$= 9 \times 16$$
$$\text{sd}(X) = 3 \times 4$$
$$= 12$$

17 E
$$\text{var}(X) = np(1 - p)$$
$$= 42p(1 - p)$$
$$= 9.4248$$
$$p(1 - p) = 0.2244$$

Use the *solve* command of a CAS calculator, giving $p = 0.34$ or $p = 0.66$. (Alternatively, solve the equation formula)

18 A If p is the probability of success, then Pr(5 sucesses) = $\binom{7}{5}p^5(1 - p)^2$. But 5 successes is the same as 2 fails. So this represents the probability of exactly two failures.

19 A
$$\Pr(4 \text{ females}) = {}^{10}C_4(0.2)^4(0.8)^6$$
$$\approx 0.0881$$

20 D
$$\Pr(\geq 1) = 1 - (0 \text{ at home})$$
$$= 1 - (p)^5$$

21 D
$$\int_0^2 \left(kx^3 + \frac{3}{4}x\right)dx = 1$$
$$\left[\frac{1}{4}kx^4 + \frac{3}{8}x^2\right]_0^2 = 1$$
$$4k + \frac{3}{2} = 1$$
$$4k = -\frac{1}{2}$$
$$k = -\frac{1}{8}$$

22 C $\Pr(X \le 2) = \frac{1}{9}\int_0^2 (4x - x^2)dx$

$$= \frac{1}{9}\left[2x^2 - \frac{1}{3}x^3\right]_0^2$$

$$= \frac{1}{9}\left(8 - \frac{8}{3}\right)$$

$$= \frac{16}{29}$$

$$\approx 0.5926$$

23 E $\frac{8}{3}\int_0^m (1 - x)dx = \frac{1}{2}$ where m is the median.

$$\frac{8}{3}\left[x - \frac{x^2}{2}\right]_0^m = \frac{1}{2}$$

$$m - \frac{1}{2}m^2 = \frac{3}{16}$$

Solving this quadratic with a CAS 'Solve' command (or by use of the quadratic formula) gives $m \approx 0.209$ or $m \approx 1.791$

But $0 < m < \frac{1}{2}$, so $m \approx 0.209$

24 B $E(X) = \int_1^2 2x\left(1 - \frac{1}{x^2}\right)dx$

$$= \int_1^2 \left(2x - \frac{2}{x}\right)dx$$

$$= \left[x^2 - 2\log_e x\right]_1^2$$

$$= (4 - 2\log_e 2) - (1 - 0)$$

$$= 3 - 2\log_e 2$$

$$\approx 1.614$$

25 C $\Pr(-1.0 < Z < 0)$

$$= \frac{1}{2}\Pr(-1.0 < Z < 1.0)$$

$$\approx \frac{1}{2} \times 0.68$$

$$= 0.34$$

26 D From the definition of standard deviation, it is always positive for any distribution, including. (Checking the other options:

A: a mean can be negative

B: values for any normal distribution be any number in the interval $(-\infty, \infty)$

C: the area is *exactly* 1

E: the standard deviation could be greater than the mean (it is for a standard normal distribution))

27 C $\Pr(X > 2.6) \approx 0.1151$, using the 'normCdf' command of a CAS calculator. (You do not need to standardise.)

28 E $\Pr(X < -2), \approx 0.0228$, using the 'normCdf' command of a CAS calculator. (In this case, you might note that:

$$\Pr(X < -2) = \Pr\left(Z < \frac{-2 - 2}{2}\right)$$

$$= \Pr(Z < -2)$$

$$\approx \frac{1}{2} \times 0.05 = 0.025$$

using the 2σ limits. the only close option is the last option.)

29 A Since $\sigma^2 = 0.4$, $\sigma = \sqrt{0.4}$

$\Pr(X > -2.73) = 1$, using the 'normCdf' command of a CAS calculator.

30 B Since $\sigma^2 = 4$, $\sigma = 2$.

$\Pr(1 < X < 2.5) \approx 0.2902$, using the 'normCdf' command of a CAS calculator. (If you mistakenly used 4 for σ, you would get 0.1484, which is not one of the option!)

31 E If X is the amount of cordial in a cup, then X is normal with $\mu = 50$ and $\sigma = 2$.
$\Pr(X > c) = 0.90$, so
$\Pr(X < c) = 0.10$, giving
$c = 47.44$ mL using the 'invnorm' command of CAS calculator.

32 C If X cm is the length of a lock of cheese, then X is normal with $\mu = 10$ and $\sigma = \sqrt{0.5}$.
$\Pr(X < c) = 0.95$, giving $c \approx 11.16$ cm using the 'invNorm' command of a CAS calculator.

33 A $\Pr(\mu - k < x < \mu \ + k) = 0.7$

$$\Pr\left(-\frac{k}{\sigma} < \frac{X-\mu}{\sigma} < \frac{k}{\sigma}\right) = 0.7$$

$$\Pr\left(-\frac{k}{\sigma} < Z < \frac{k}{\sigma}\right) = 0.7$$

Thus an area of 0.3 remains in the two tail, or 0.15 in each tail.

$$\text{So } \Pr\left(Z < \frac{k}{\sigma}\right) = 0.7 + 0.15$$

$$= 0.85$$

Using the 'invNorm' command of a CAS calculator shown that
$\Pr(Z < 1.03643) = 0.85$

$$\Rightarrow \frac{k}{\sigma} = 1.03643$$

$$\text{Now } \sigma^2 = 2.25, \text{ so } \sigma = 1.5.$$

Hence $k = 1.555$. to 3 decimal places.
(Note that the value of the mean μ is not actually needed.)

34 B If X kg is the weight of a pocket, then X is normal $\mu = 1$.
More than 0.05 kg underweight means $X < 0.95$ and 3% are underweight.

$\Pr(X < 0.95) = 0.03$

$$\Pr\left(Z < \frac{-0.05}{\sigma}\right) = 0.03$$

Using the 'invNorm' command of a CAS calculator shows that
$\Pr(Z < -1.88079) = 0.03$

$$\Rightarrow -\frac{0.05}{\sigma} = -1.88079$$

$$\sigma = \frac{0.05}{1.88079}$$

$$\approx 0.027$$

35 B The graphs have the some centre so $\mu_1 = \mu_2$.
The lower graph is more spread out then the upper graph so $\sigma_1 > \sigma_2$.

36 B The standard deviation is $\sqrt{25} = 5$.
About 68% represent, ± 1 standard deviation from the mean of 173.
$173 - 5 = 168$
$173 + 5 = 178$
So the interval is (168, 178)

37 B $n = 200, p = 0.38$
95% CI = (0.313, 0.447) (Calculator)

38 D Interval is

$$\left(p - 2.55\sqrt{\frac{p(1-p)}{n}}, p + 2.55\sqrt{\frac{p(1-p)}{n}}\right)$$

Adding the left and right boundary values gives $2p$.
Hence, $2p = 0.620$
$p = 0.310$

39 B Increasing the level of confidence means that the interval will be wider

40 C Only statement II is correct

41 B Since the width of the confidence interval is inversely proportional to the square root of the sample size, decreasing the sample size by a factor of 2 will increase the width of the interval by a factor of $\sqrt{2}$

42 **C** $\Pr\left(\hat{P} \geq \frac{3}{10}\right) = \Pr\left(\frac{X}{30} \geq \frac{3}{10}\right)$

$= \Pr(X \geq 9)$

$= 0.3264$

Solutions to extended-response questions

1 CAS calculator is used throughout this question

a Let X be the weight of a trout from lake A. $\mu = 3.6$ kg, $\sigma = 0.5$ kg
$\Pr(X > 4.25) = 0.0968$

b $\Pr(X > k) = 0.9$
$k = 2.96$

c Let Y be the weight of a trout from lake B. $b(x) = \begin{cases} \frac{\pi}{6}\cos\left(\frac{\pi(2x-7)}{6}\right) & 2 \le x \le 5 \\ 0 & elsewhere \end{cases}$

$$\mathrm{E}(Y) = \mu = \int_2^5 xb(x)\,dx = \frac{7}{2}\text{ kg}$$

d $\Pr(Y > 4.25) = \int_{4.25}^5 b(x)\,dx = 0.1464$

e $\Pr(Y > c) = 0.9$
$\int_c^5 b(x)\,dx = 0.9$
$c = 2.6145$

f $\Pr(W > 4) = 0.6 \times \Pr(X > 4) + 0.4 \times \Pr(Y > 4)$
$= 0.227$

g Pr(caught in lake A|weight more than 4 kg)

$$= \frac{\text{Pr(weight more than 4 kg|(caught in lake A)} \times \text{Pr(caught in lake A)}}{\text{Pr(weight more than 4 kg)}}$$

$= 0.560$

h Binomial $n = 6$ and $p = 0.227$
Let X_1 be the random variable.
$\Pr(X_1 > 0) = 1 - \Pr(X_1 = 0)$
$\approx 1 - (1 - 0.227)^6$
≈ 0.79

i Let X_n be the number of fish in a box of size n that weighs more than 4 kg.

$$\Pr(\hat{P}_n < \frac{1}{n}) \Leftrightarrow \Pr(\frac{X_n}{n} < \frac{1}{n}) \Leftrightarrow \Pr(X_n < 1) \Leftrightarrow \Pr(X_n = 0)$$

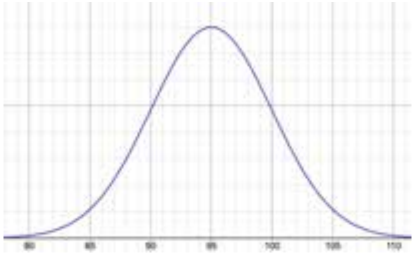
$$\Pr(\hat{P}_n < \frac{1}{n}) < 0.1$$

$$\Pr(X_n = 0) < 0.1$$

$$(1 - 0.227)^n < 0.1$$

$$n \geq 9$$

2 **a**



$\mu = 95, \sigma = 5$

$\Pr(T < a) = 0.8 \Rightarrow a = 99.2°$ C

b $$\Pr(T < 112 | T \geq 108) = \frac{108 \leq T < 112}{\Pr(T \geq 108)}$$

$$= 0.928$$

c $\Pr(90 < T < 105) = 0.8186$

$\Pr(95 < T_1 < 110) = 0.8186$ The standard deviation is the same and the interval has moved 5 to the right. So an immediate answer is $k = 100$, but by the symmetry of the graph, we see that $k = 105$ is a second value.

d $\Pr(T \geq 80) = 0.98$ and $\Pr(T \geq 110) = 0.04$

$$\Pr\left(Z \geq \frac{80 - \mu}{\sigma}\right) = 0.98 \text{ and } \Pr\left(Z \geq \frac{110 - \mu}{\sigma}\right) = 0.04$$

$$\Pr\left(Z < \frac{80 - \mu}{\sigma}\right) = 0.02 \text{ and } \Pr\left(Z < \frac{110 - \mu}{\sigma}\right) = 0.96$$

$$\frac{80 - \mu}{\sigma} = -2.05375 \text{ and } \frac{110 - \mu}{\sigma} = 1.75069$$

$\mu = 96.2$ and $\sigma = 7.9$ correct to one decimal place.

e probability of working at 110° C = 0.1

Binomial, $n = 20$, $p = 0.1$. Let X be the number working in box.

$\Pr(X \geq 2) = 0.6080$.

f $\Pr(X_n \geq 1) = 1 - \Pr(X_n = 0)$

$\Pr(X_n \geq 1) \geq 0.9$

$\Leftrightarrow 1 - \Pr(X_n = 0) \geq 0.9$

$\Leftrightarrow \Pr(X_n = 0) \leq 0.1$

$\Leftrightarrow 0.9^n \leq 0.1$

$n > 21.8543$
$n = 22$

g CI (0.0186, 0414)
$$\left(p - 1.96\sqrt{\frac{p(1-p)}{n}}, p + 1.96\sqrt{\frac{p(1-p)}{n}}\right)$$
Adding the left and righ boundaries of the CI
$2p = 0.16$
$p = 0.08$

h $p + 1.96\sqrt{\frac{p(1-p)}{n}} = 01414$
$p = 0.08$
Solve for n
$n = 74.9986\ldots$
Therefore take $n = 75$

3 **a** $\mu = 3$ min and $\sigma = 3$ minutes.

i $\Pr(-5 < T < 5) = 0.743677\cdots \approx 0.7437$

ii $\Pr(T > 5) = 0.252592\cdots \approx 0.2525$

b Binomial $n = 7, p = 0.2525$
Late less than twice
$\Pr(T < 2) = \Pr(T = 0) + \Pr(T = 1)$
$= 0.438738\cdots \approx 0.4387$
Late twice
$\Pr(T = 2) = 0.312463\cdots \approx 0.3125$
Late more than twice
$\Pr(T > 2) = 0.248799\cdots \approx 0.2488$

f	0	3000	10 000
$\Pr(F = f)$	0.4387	0.3125	0.2488

c $E(F) = 3000 \times 0.3125 + 10\,000 \times 0.2488 = \$3425.38$
If previous approximations are used \$3425.50
$E(F^2) = 3000^2 \times 0.3125 + 10\,000^2 \times 0.2488 = 2.7692\cdots \times 10^7$

$E(F^2) - [E(F)]^2 = 1.9589\cdots \times 10^7$

Therefore sd $\approx$ \$3994.85

If previous approximations are used \$3425.8

d $p = \dfrac{33}{268}$

CI $(0.084, 0.162)$

e $\displaystyle\int_{\pi}^{3\pi} df(d)\,dd = \frac{2(\pi^2 - 2)}{\pi} \approx 5.0$

f $\displaystyle\int_{8}^{3\pi} df(d)\,dd \approx 0.0187$

4 a $\displaystyle\int_{8}^{12} k(x-8)(12-x)^2\,dx = 1$

$$64k = 3$$

$$k = \frac{3}{64}$$

b $\displaystyle\frac{3}{64}\int_{8}^{12} x(x-8)(12-x)^2\,dx = 9.6$

c $\displaystyle\Pr(X > 10) = \frac{3}{64}\int_{10}^{12} x(x-8)(12-x)^2\,dx = 0.3125$

$E(Y) = 80 \times 0.3125 = 25$

d $\displaystyle\Pr(X > 11|x \geq 10) = \frac{\Pr(10 \leq X < 11)}{\Pr(X \geq 10)}$

$$= \frac{\dfrac{3}{64}\displaystyle\int_{10}^{11} x(x-8)(12-x)^2\,dx}{\dfrac{3}{64}\displaystyle\int_{10}^{12} x(x-8)(12-x)^2\,dx}$$

$= 0.1625$

e $\Pr(8.5 < Y < 10.5) = 0.555889\cdots \approx 0.5559$

f $\Pr(Y \leq a) = 0.95$

$a = 11.9738 \approx 12.0$

g **i** $\Pr(Y > 12.2) = 0.033376\ldots$

Binomial $n = 64$ and $p = 0.033376\ldots$

$\Pr(P \geq 1) = 1 - \Pr(X = 0)$

$= 1 - (0.96662)^{64}$

$= 0.88611\cdots \approx 0.8861$

ii $\Pr(Y \geq m) \leq 0.00055 \Leftrightarrow 1 - \Pr(Y \leq m) \leq 0.00055$
$\Pr(Y \leq m) \geq 0.9995$
Using calculator $m = 8$

iii $\mathrm{E}(\hat{P}) = p = 0.033376$
and $\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}} = 0.02245\ldots$

h Let m be the number from machine B and T the total number. Therefore $T - m$ produced by machine A.

$$0.6125 = \frac{m}{T} \times 0.5 + \frac{T-m}{T} \times \frac{11}{16}$$

$$0.6125 = \frac{m}{T} \times 0.5 + (1 - \frac{m}{T}) \times \frac{11}{16}$$

$$\text{Let} x = \frac{m}{T}$$

$$0.6125 = 0.5x + (1-x) \times \frac{11}{16}$$

$$\therefore x = 0.4$$

$$1 - x = 0.6$$

5 In the following E denotes the event occurring, N the event not occurring.
Three trials are considered first.

a **i** The outcomes to consider are

(E, E, E) $\quad \Pr(E, E, E) = 0$

(E, E, N) $\quad \Pr(E, E, N) = 0$

(E, N, E) $\quad \Pr(E, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2}$

(N, E, E) $\quad \Pr(N, E, E) = 0$

(N, N, E)

(N, E, N) Note: Remember the event cannot occur in consecutive trials.

(E, N, N)

(N, N, N)

$\therefore$ Probability of it occurring just twice $= \Pr\{(E, N, E)\} = \dfrac{1}{4}$

ii Consider the following outcomes

(E, E, N, N) $\quad \Pr(E, E, N, N) = 0$

(E, N, E, N) $\quad \Pr(E, N, E, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 = \frac{1}{4}$

(E, N, N, E) $\quad \Pr(E, N, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

(N, E, E, N) $\quad \Pr(N, E, E, N) = 0$

(N, E, N, E) $\quad \Pr(N, E, N, E) = \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{8}$

(N, N, E, E) $\quad \Pr(N, N, E, E) = 0$

$\therefore$ Pr(the event occurs exactly twice) $= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

b For 5 trials there are 10 possible outcomes to consider

(E, E, N, N, N) $\quad \Pr(E, E, N, N, N) = 0$

(E, N, E, N, N) $\quad \Pr(E, N, E, N, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{8}$

(E, N, N, E, N) $\quad \Pr(E, N, N, E, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{8}$

(E, N, N, N, E) $\quad \Pr(E, N, N, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

(N, E, E, N, N) $\quad \Pr(N, E, E, N, N) = 0$

(N, E, N, E, N) $\quad \Pr(N, E, N, E, N) = \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 = \frac{1}{8}$

(N, E, N, N, E) $\quad \Pr(N, E, N, N, E) = \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

(N, N, E, E, N) $\quad \Pr(N, N, E, E, N) = 0$

(N, N, E, N, E) $\quad \Pr(N, N, E, N, E) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{16}$

(N, N, N, E, E) $\quad \Pr(N, N, N, E, E) = 0$

$\therefore$ Pr(the event occurs exactly twice) $= \frac{9}{16}$

6 Let X be the number of sixes obtained in 5 tosses of a die

$$\begin{aligned}
\Pr(\text{an even number of sixes}) &= \Pr(X = 0) + \Pr(X = 2) + \Pr(X = 4)\\
&= \left(\frac{5}{6}\right)^5 + 10\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 + 10\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)\\
&= \left(\frac{1}{6}\right)^5[5^5 + 10.5^3 + 10.5]\\
&= \left(\frac{1}{6}\right)^5[4425]\\
&= 0.5692
\end{aligned}$$

$$\begin{aligned}
\Pr(\text{an odd number of sixes}) &= 1 - \Pr(\text{an even number of sixes})\\
&= 0.4309
\end{aligned}$$

Let Y be the amount won by Katia. The probability distribution is as shown

y	1	$-x$
$\Pr(Y = y)$	0.4309	0.5691

The game is fair if $E(Y) = 0$

i.e. if $1 \times 0.4308 - 0.5691x = 0$

This implies $x = 0.7570$

Therefore Mikki should receive 76 cents from Katia if there is an even number of sixes.

7 **a** Let x be the daily demand

Let s be the number of newspapers stocked

If the demand is less than the number stocked

$P = 0.75x - 0.5s$

If the demand is greater than the number stocked

$$\begin{aligned}
P &= 0.25s - 0.25(x - s)\\
&= 0.5s - 0.25x
\end{aligned}$$

(Note: the newspaper seller is considered to lose money by not ordering enough.)

$$\therefore \quad P = \begin{cases} 0.75x - 0.5s & x \le s \\ 0.5s - 0.25x & x > s \end{cases}$$

b Using the result of **a** a probability distribution for P is obtained with $s = 26$

p	5	5.75	6.50	6.25	6	5.75	5.50
$\Pr(P = p)$	0.05	0.10	0.10	0.25	0.25	0.15	0.10

The computations are as follows

$x = 24 \qquad p = 0.75 \times 24 - 0.5 \times 26 = 5$

$x = 25 \qquad p = 0.75 \times 25 - 0.5 \times 26 = 5.75$

$x = 26 \qquad p = 0.75 \times 26 - 0.5 \times 26 = 6.5$

$x = 27 \qquad p = 0.5 \times 26 - 0.25 \times 27 = 6.25$

etc.

Reorganising the table

p	5	5.50	5.75	6	6.25	6.50
$\Pr(P = p)$	0.05	0.10	0.25	0.25	0.25	0.1

$$\therefore \quad E(P) = 5 \times 0.05 + 5.50 \times 0.10 + 5.75 \times 0.25 + 6 \times 0.25 + 6.25 \times 0.25 + 6.25 \times 0.1$$

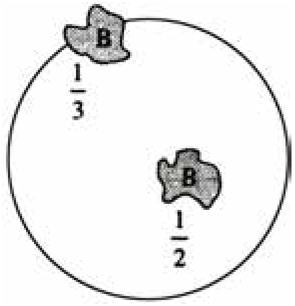
$$= 5.95$$

The expected profit is \$5.95.

c $E(P) = \sum_{x=24}^{s} (0.75x - 0.5s)p(x) + \sum_{x=s+1}^{30} (0.5s - 0.25x)p(x)$

d The newspaper seller should stock 27 (computation not shown).

8 a **i** Probability bean bag lands outside $= 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$



ii Probability of two consecutive throws landing outside the circle $= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

iii Probability of first on the rim and second inside the circle $= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

b Let X be the score.

X	0	5	10
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{2}{5}$

i With two shots to score a 20 requires two 10's. $\therefore$ Probability of score 20 $= \frac{4}{25}$

ii In other to score 10 the score could have resulted through 0 and 10 or 10 and 0 or 5 and 5.

$$\therefore \quad \text{Probability of score } 10 = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{2}{5} + \frac{1}{10} \times \frac{1}{10}$$

$$= \frac{1}{5} + \frac{1}{5} + \frac{1}{100}$$

$$= \frac{2}{5} + \frac{1}{100}$$

$$= \frac{41}{100}$$

c For Jane to score a 10:

- It can be a ten from 2 shots (bean bag of Anne; outside).

Probability of this $= \frac{41}{100} \times \frac{1}{6} = \frac{41}{600}$

- It can be a ten from one throw (bean bag of Anne: rim)

$$\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$$

$\therefore$ Probability of a ten $= \frac{41}{600} + \frac{2}{15} = \frac{121}{600}$

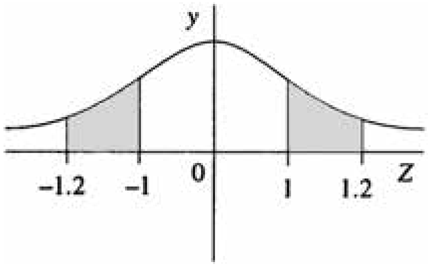
9 $\mu = 80\,000,\ \sigma = 20\,000$

a Let X be the distance travelled annually

$\Pr(56\,000 \le X \le 60\,000)$

$= \Pr\left(\dfrac{56\,000 - 80\,000}{20\,000} \le Z \le \dfrac{60\,000 - 80\,000}{20\,000}\right)$

$= \Pr(-1.2 \le Z \le -1)$



From the graph it can be seen

$\Pr(-1.2 \le Z \le -1)$

$= \Pr(1 \le Z \le 1.2)$

$= \Pr(Z \le 1.2) - \Pr(Z \le 1)$

$= 0.8849 - 0.8413$

$= 0.0436$

The probability that a randomly selected taxi will travel between 50 000 km and 60 000 km is 0.0436.

b Pr(Below 48 000 or above 96 000)

$= \Pr(X \le 48\,000) + \Pr(X \ge 96\,000)$

$= \Pr\left(Z \le \dfrac{48\,000 - 80\,000}{20\,000}\right) + \Pr\left(Z \ge \dfrac{96\,000 - 80\,000}{20\,000}\right)$

$= \Pr(Z \le -1.6) + \Pr(Z \ge 0.8)$

$= 1 - \Pr(Z \le 1.6) + 1 - \Pr(Z \ge 0.8)$

$= 2 - \Pr(Z \le 1.6) - \Pr(Z \le 0.8)$

$= 2 - 0.9452 - 0.7881$

$= 0.2667$

The percentage of taxis which travel below 48 000 km or have 96 000 km is 26.67%

c $\Pr(48\,000 \le X \le 96\,000)$

$= 1 - [\Pr(X \le 48\,000) + \Pr(X \ge 96\,000)]$

$= 1 - 0.2667$

$= 0.7333$

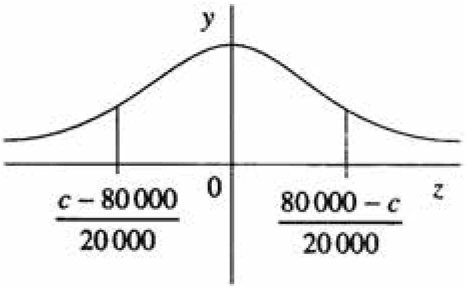
Let Y be the number of taxis out of the 250 which will travel between 48 000 and 96 000 km.

Y is a Binomial random variable with $n = 250$ and $p = 0.7333$

$\therefore \quad E(Y) = np = 183.325$

i.e. 183 taxis out of the 250 are expected to travel between 48 000 and 96 000 km.

d Let c be such that



$$\Pr(X \geq c) = 0.85$$

Then $\Pr\left(Z \geq \frac{c - 80\,000}{20\,000}\right) = 0.85$

From the graph

$$\Pr\left(Z \leq \frac{80\,000 - c}{20\,000}\right) = 0.85$$

$$\therefore \quad \frac{80\,000 - c}{20\,000} = 1.03643$$

$$\therefore \quad c = 80\,000 - 20\,000 \times 1.03643$$

$$= 59\,271$$

85% of taxis travel at least 59 271 kilometres.

10 a **i** Let X be the weight of cereal in a box

$9\mu = 505, \ \sigma = 5$

$\Pr(X \leq 500)$

$$= \Pr\left(Z \leq \frac{500 - 505}{5}\right)$$

$$= \Pr(Z \leq -1)$$

$$= 1 - \Pr(Z \leq 1)$$

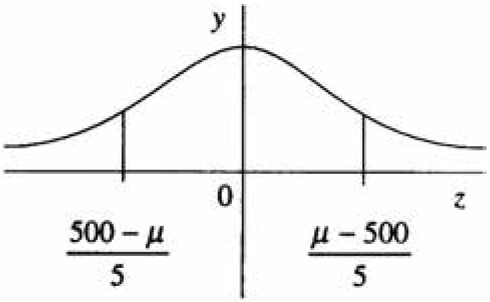
$$= 1 - 0.8413$$

$$= 0.1587$$

ii $\mu =?, \ \sigma = 5$

$$\Pr(X \leq 500) = 0.1$$

implies $\Pr\left(Z \leq \frac{500 - \mu}{5}\right) = 0.1$



From the graph

$$\Pr\left(Z \leq \frac{500 - \mu}{5}\right)$$

$$= \Pr\left(Z \geq \frac{\mu - 500}{5}\right)$$

$$= 1 - \Pr\left(Z \leq \frac{\mu - 500}{5}\right)$$

$$\therefore \quad 1 - \Pr\left(Z \le \frac{\mu - 500}{5}\right) = 0.01$$

$$0.99 = \Pr\left(Z \le \frac{\mu - 500}{5}\right)$$

$$2.3263 = \frac{\mu - 500}{5}$$

$$\therefore \quad \mu = 5 \times 2.3263 + 500$$

$$= 511.63$$

b Let Y be the number of boxes under weight. Y is a Binomial random variable with $n = 5,\ p = 0.158655$

$\Pr(Y > 1) = 1 - (\Pr(Y = 0) + \Pr(Y = 1))$

$$= 1 - (0.841345)^5 - \binom{5}{1}(0.158655)(0.841345)^4$$

$$= 0.1809$$

11 a i
$$f(x) = \begin{cases} kx(100 - x^2) & \text{if } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^{10} f(x)dx = \left[k\left(50x^2 - \frac{x^4}{4}\right)\right]_0^{10} = 2500k$$

For f to be the function of a probability density $k = \dfrac{1}{2500}$

ii $\mathrm{E}(X) = \int_0^{10} xf(x)dx = \dfrac{1}{2500}\int_0^{10} x^2(100 - x^2)dx = \dfrac{16}{3}$

iii $\Pr(X > 3) = \int_3^{10} f(x)dx == \left[k\left(50x^2 - \dfrac{x^4}{4}\right)\right]_3^{10} = 0.8281$

iv $\Pr(X > 3 \mid X < 7) = \dfrac{\Pr(3 < X < 7)}{\Pr(X < 7)} = \dfrac{\int_3^7 f(x)dx}{\int_0^7 f(x)dx} = 0.7677$

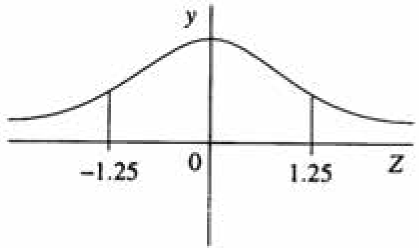
b This is a binomial distribution. Let W be the number of moviegoers who have to queue for more than 3 minutes

$$\Pr(W \ge 5) = ?$$

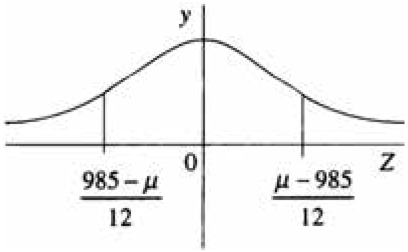
In this situation $n = 10$ and $p = 0.8281$

Using a calculator gives $\Pr(W \ge 5) = 0.9971$

12 a $$\begin{aligned}\Pr(X \le 985) &= \Pr\left(Z \le \frac{985-1000}{12}\right)\\ &= \Pr\left(Z \le \frac{-15}{12}\right)\\ &= \Pr(Z \le -1.25)\\ &= 1 - \Pr(Z \le 1.25)\\ &= 0.1056\end{aligned}$$

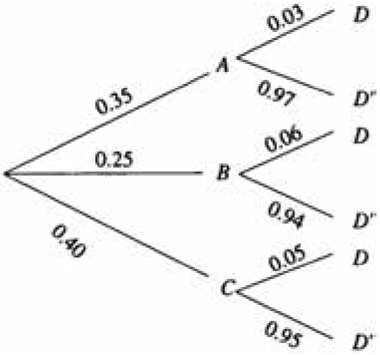


b $$\begin{aligned} \Pr(X \le 985) &= 0.01\\ \therefore \quad \Pr\left(Z \le \frac{985-\mu}{12}\right) &= 0.01\\ \therefore \quad 1-\Pr\left(Z \le \frac{\mu-985}{12}\right) &= 0.01\\ \Pr\left(Z \le \frac{\mu-985}{12}\right) &= 0.99\\ \therefore \quad \frac{\mu-985}{12} &= 2.3263\\ \therefore \quad \mu &= 12 \times 2.3263 + 985\\ \therefore \quad &= 1012.92\end{aligned}$$



The machine should be set at 1012.92

13 In the tree diagram A, B and C are the machines. D is defective. D′ is not defective.



a **i** $\Pr(A \cap D) = 0.35 \times 0.03 = 0.0105$

ii $$\begin{aligned}\Pr(D) &= \Pr(A \cap D) + \Pr(B \cap D) + \Pr(C \cap D)\\ &= 0.0105 + 0.25 \times 0.06 + 0.40 \times 0.05\\ &= 0.0455\end{aligned}$$

b $\Pr(C \mid D) = \dfrac{\Pr(C \cap D)}{\Pr(D)} = \dfrac{0.4. \times 0.05}{0.0455} = 0.4396$

c $\Pr(A \cup B \mid D') = \dfrac{\Pr((A \cup B) \cap D')}{\Pr(D')} = \dfrac{\Pr(A \cap D') + \Pr(B \cap D')}{\Pr(D')} = \dfrac{0.5745}{0.9545} = \dfrac{1149}{1909}$

14 a i $\mu = \mathrm{E}(X) = \Sigma x\Pr(X = x)$

$= 4.25$

ii $\sigma = \sqrt{\mathrm{Var}(X)}$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

$= 1 \times 0.02 + 4 \times 0.03 + 9 \times 0.04 + 16 \times 0.45 + 25 \times 0.45 - (4.25)^2$

$= 0.02 + 0.12 + 0.36 + 7.2 + 11.25 - (4.25)^2$

$= 18.95 - 18.0625$

$= 0.8875$

$\therefore \quad \sigma = \sqrt{0.8875} = 0.9421$

iii $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

$= \Pr(2.366 \le X \le 6.134)$

$= 0.94$

iv $\Pr(X \ge 4) = 0.45 + 0.45$

$= 0.9$

b i Binomial

ii Expected number of working games in box

$= \mathrm{E}(Y)$

$= 20 \times 0.9$

$= 18$

iii $\mathrm{Var}(Y) = 20 \times 0.9 \times 0.1$

$= 1.8$

$\sigma = \sqrt{1.8}$

$= 1.342$

iv $\Pr(Y \geq 19) = \Pr(Y = 19) + \Pr(Y = 20)$

$= 0.27017\ldots + 0.12158\ldots$

$= 0.3917$ (correct to 4 decimal places)

15 **a** $n = 1000, X = 100$ CI= $(0.0814, 0.1186)$

b $m = 800, Y = 80$ CI= $(0.0792, 0.1208)$

c width female = 0.0372
width male = 0.0416
The confidence interval is narrower because the sample size for the females is larger.

d 900 of each sex

e

$$\frac{\hat{p}_1(1 - \hat{p}_1)}{n} = \frac{\hat{p}_2(1 - \hat{p}_2)}{m}$$

$$\frac{0.1 \times 0.9}{1000} = \frac{\hat{p}_2(1 - \hat{p}_2)}{800}$$

$$0.072 = \hat{p}_2(1 - \hat{p_2})$$

$\hat{p}_2 = 0.078$ or 0.922

Solutions for algorithms and pseudocode

1 a i

```
input $N$
$count \leftarrow 0$
for $i$ from 1 to $N$
    $outcome \leftarrow randint(1, 6)$
    if $outcome = 2$ or $outcome = 4$ or $outcome = 6$ then
        $count \leftarrow count + 1$
    end if
end for
$estimate \leftarrow \frac{count}{N}$
print $estimate$
```

ii

```
input $N$
$count \leftarrow 0$
for $i$ from 1 to $N$
    $outcome \leftarrow randint(1, 6)$
    if $outcome \leq 4$ then
        $count \leftarrow count + 1$
    end if
end for
$estimate \leftarrow \frac{count}{N}$
print $estimate$
```

b

```
$total \leftarrow 0$
$count \leftarrow 0$
for $i$ from 1 to 6
    for $j$ from 1 to 6
        $total \leftarrow total + 1$
        if $6 \leq i + j \leq 10$ then
            $count \leftarrow count + 1$
        end if
    end for
end for
print $\frac{count}{total}$
```

2 a

```
$count \leftarrow 0$
for $i$ from 1 to 1000
    $outcome \leftarrow randint(1, 9)$
```

```
        if $outcome = 9$ then
            $count \leftarrow count + 1$
        end if
    end for
    $estimate \leftarrow \dfrac{count}{9}$
    print $estimate$
```

b

```
    $count \leftarrow 0$
    for $i$ from 1 to 1000
        $outcome \leftarrow randint(1, 9)$
        if $outcome = 2$ or $outcome = 4$ or $outcome = 6$ or $outcome = 8$ then
            $count \leftarrow count + 1$
        end if
    end for
    $estimate \leftarrow \dfrac{count}{9}$
    print $estimate$
```

c

```
    $count \leftarrow 0$
    for $i$ from 1 to 1000
        $outcome \leftarrow randint(1, 9)$
        if $outcome \geq 3$ and $outcome \leq 7$ then
            $count \leftarrow count + 1$
        end if
    end for
    $estimate \leftarrow \dfrac{count}{9}$
    print $estimate$
```

d

```
    $count \leftarrow 0$
    for $i$ from 1 to 000
        $outcome \leftarrow randint(1, 9)$
        if $outcome \geq 3$ and $outcome \leq 7$ then
            $count \leftarrow count + 1$
        end if
    end for
    $estimate \leftarrow \dfrac{count}{9}$
    print $estimate$
```

e

```
$count \leftarrow 0$
for $j$ from 1 to 1000
    $number \leftarrow randint(1,9)$
    $outcome \leftarrow randint(1,9)$
    if $outcome = number$ then
        $count \leftarrow count + 1$
    end if
end for
$estimate \leftarrow \dfrac{count}{1000}$
print $estimate$
```

3 a i

```
$counta \leftarrow 0$
$countb \leftarrow 0$
$countc \leftarrow 0$
$validthrow \leftarrow 0$
for $i$ from 1 to 1000
    $x \leftarrow random(-20, 20)$
    $y \leftarrow random(-20, 20)$
    if $x^2 + y^2 \leq 400$ then
        $validthrow = validthrow + 1$
        if $x^2 + y^2 \geq 100$ then
            $counta = counta + 1$
        else if $x^2 + y^2 \geq 4$ then
            $countb = countb + 1$
        else
            $countc = countc + 1$
        end if
    end if
end for
print $\dfrac{validthrow}{1000}$
```

ii Change the print statement to: `print` $\dfrac{countc}{1000}$

iii Change the print statement to: `print` $\dfrac{countb}{1000}$

iv Change the print statement to: `print` $\dfrac{counta}{1000}$

b Here is a program to give the probability a score of 200 and a score of 40 points. You can obtain the other probabilities of scores by using other if statements. You are working with two throws at a time in the loop.

```
$counta \leftarrow 0$
$countb \leftarrow 0$
for $i$ from 1 to 10000
    $x1 \leftarrow random(-20, 20)$
    $y1 \leftarrow random(-20, 20)$
    $x2 \leftarrow random(-20, 20)$
    $y2 \leftarrow random(-20, 20)$
    if $x1^2 + y1^2 \leq 400$ and $x2^2 + y2^2 \leq 400$ then
        $twovalidthrow = twovalidthrow + 1$
        end if
        if $x1^2 + y1^2 \leq 4$ and $x2^2 + y2^2 \leq 4$ then
            $counta = counta + 1$
        end if
        if $4 < x1^2 + y1^2 \leq 100$ and $4 < x2^2 + y2^2 \leq 100$ then
            $countb = countb + 1$
        end if
    end if
end for
print $\frac{twovalidthrow}{1000}, \frac{counta}{twovalidthrow}, \frac{countb}{twovalidthrow}$
```

4

```
$counta \leftarrow 0$
$countb \leftarrow 0$
$countc \leftarrow 0$
$countd \leftarrow 0$
$counte \leftarrow 0$
$countf \leftarrow 0$
$countg \leftarrow 0$
$counth \leftarrow 0$
Input $R$
for $i$ from 1 to 10000
    $x1 \leftarrow random(-25, 25)$
    $y1 \leftarrow random(-25, 25)$
    $x2 \leftarrow random(-25, 25)$
    $y2 \leftarrow random(-25, 25)$
    $x3 \leftarrow random(-25, 25)$
    $y3 \leftarrow random(-25, 25)$
    if $x1^2 + y1^2 \leq R^2$ and $x2^2 + y2^2 \leq R^2$ and $x3^2 + y3^2 \leq R^2$ then
```

```
        $counta = counta + 1$
        end if
    if $x1^2 + y1^2 \geq R^2$ and $x2^2 + y2^2 \geq R^2$ and $x2^2 + y2^2 \geq R^2$ then
        $countb = countb + 1$
        end if
    if $x1^2 + y1^2 \geq R^2$ and $x2^2 + y2^2 \geq R^2$ and $x2^2 + y2^2 \leq R^2$ then
        $countc = countc + 1$
        end if
    if $x1^2 + y1^2 \geq R^2$ and $x2^2 + y2^2 \leq R^2$ and $x2^2 + y2^2 \geq R^2$ then
        $countd = countd + 1$
        end if
    if $x1^2 + y1^2 \leq R^2$ and $x2^2 + y2^2 \geq R^2$ and $x2^2 + y2^2 \geq R^2$ then
        $counte = counte + 1$
        end if
    if $x1^2 + y1^2 \leq R^2$ and $x2^2 + y2^2 \leq R^2$ and $x2^2 + y2^2 \geq R^2$ then
        $countf = countf + 1$
        end if
    if $x1^2 + y1^2 \leq R^2$ and $x2^2 + y2^2 \geq R^2$ and $x2^2 + y2^2 \leq R^2$ then
        $countg = countg + 1$
        end if
    if $x1^2 + y1^2 \geq R^2$ and $x2^2 + y2^2 \leq R^2$ and $x2^2 + y2^2 \leq R^2$ then
        $counth = counth + 1$
        end if
end for
$score150 = counta$
$score30 = countb$
$score70 = countc + countd + counte$
$score110 = countf + countg + counth$
print $\frac{score150}{10000}, \frac{score30}{10000}, \frac{score70}{10000}, \frac{score110}{10000}$
```

5 **a**

```
$sum \leftarrow 0$
for $j$ from 1 to 1000
    $count \leftarrow 0$
    $x \leftarrow 0$
    while $x \leq 5$ and $x \geq -5$
        $A \leftarrow random(\,)$
            if $A < 0.6$ then
                $x \leftarrow x + 1$
            else
                $x \leftarrow x - 1$
```

```
            end if
            $count \leftarrow count + 1$
        end while
        $sum \leftarrow sum + count$
        $average \leftarrow sum/j$
end for

print $average$
```

b Here we take 50 steps.

```
$x \leftarrow 0$
for $j$ from $1$ to $50$
    $A \leftarrow random()$
    if $A < 0.5$ then
        $x \leftarrow x + 1$
    else
        $x \leftarrow x - 1$
    end if
end for

print $x$
```

6 For this program you need to randomly choose a direction N, S, E or W. For this we will name a command: $randomchoice[N, S, E, W]$ which will return one of these with equal probability.

Here we take 50 steps.

```
$x \leftarrow 0$
$y \leftarrow 0$
for $j$ from $1$ to $50$
    $direction \leftarrow randomchoice[N, S, E, W]$
    if $direction = N$ then
        $y \leftarrow y + 1$
    else if $direction = S$ then
        $y \leftarrow y - 1$
    else if $direction = E$ then
        $x \leftarrow x + 1$
    else
        $x \leftarrow x - 1$
    end if
end for

print $(x, y)$
```

7

```
$x \leftarrow 0; y \leftarrow 0; u \leftarrow 0; z \leftarrow 0$
$average \leftarrow 0$
for $j$ from 1 to $N$
    $x \leftarrow 2\times$ random( )$-1$
    $y \leftarrow 2\times$ random( )$-1$
    $u \leftarrow 2\times$ random( )$-1$
    $z \leftarrow 2\times$ random( )$-1$
    if $x^2 + y^2 \leq 1$ and $u^2 + z^2 \leq 1$ then
        $count \leftarrow count + 1$
        $d \leftarrow \sqrt{(x-u)^2 + (y-z)^2}$
        $average \leftarrow (average \times (count - 1) + d)/count$
    end if
end for
print $average$
```

8 a i

```
$total \leftarrow 0$
$count \leftarrow 0$
for $i$ from 1 to 9
    for $j$ from 1 to 9
        for $k$ from 1 to 9
            $total \leftarrow total + 1$
            if $5 < i + j + k \leq 20$ then
                $count \leftarrow count + 1$
            end if
        end for
    end for
end for
print $\dfrac{count}{total}$
```

ii

```
$total \leftarrow 0$
$count \leftarrow 0$
for $i$ from 1 to 9
    for $j$ from 1 to 9
        for $k$ from 1 to 9
            if $i + j + k > 15$ then
                $total \leftarrow total + 1$
                if $i + j + k > 20$ then
                    $count \leftarrow count + 1$
                end if
            end if
        end for
    end for
end for
print $\dfrac{count}{total}$
```

iii

```
$total \leftarrow 0$
$count \leftarrow 0$
for $i$ from 1 to 9
    for $j$ from 1 to 9
        for $k$ from 1 to 9
            $total \leftarrow total + 1$
            if $i + 2j + 3k > 40$ then
                $count \leftarrow count + 1$
            end if
        end for
    end for
end for
print $\dfrac{count}{total}$
```

b **i**

```
$total \leftarrow 0$
$count \leftarrow 0$
for $i$ from 0 to 9
    for $j$ from 12 to 16
        $total \leftarrow total + 1$
        if $16 < i + j < 20$ then
            $count \leftarrow count + 1$
        end if
    end for
end for
```

```
print $\dfrac{count}{total}$
```

ii

```
$total \leftarrow 0$
$count \leftarrow 0$
for $i$ from $0$ to $9$
    for $j$ from $12$ to $16$
        if $i + j > 16$ then
            $total \leftarrow total + 1$
            if $i + j > 20$ then
                $count \leftarrow count + 1$
            end if
        end if
    end for
end for
print $\dfrac{count}{total}$
```

Chapter 19 – Revision of Chapters 1–18

Solutions to Technology-free questions

1 $$\begin{aligned} f(g(x)) &= f(3x+1) \\ &= (3x+1)^2+6 \\ &= 9x^2+6x+7 \end{aligned}$$

2 Infinitely many solutions if the determined of the coefficients matrix is zero, i.e.

$$\begin{vmatrix} l_c & 3 \\ 4 & (l_c \;\; + 2) \end{vmatrix} = 0$$

$$l_c(l_c+2) - 12 = 0$$

$$l_c^2 + 2l_c - 12 = 0$$

$$(l_c+1)^2 - 13 = 0$$

$$l_c = -1 \pm \sqrt{13}$$

3 $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 2x \\ -3y \end{bmatrix}$$

$$x' = 2x \text{ and } y' = -3y$$

$$x = \frac{1}{2}x' \text{ and } y = -\frac{1}{3}y'$$

$$y = \frac{1}{x} \text{ becomes } -\frac{1}{3}y' = \frac{1}{\frac{1}{2}x'}$$

i.e. $y' = -\dfrac{6}{x'}$ or in terms of x, y; $y = -\dfrac{6}{x}$

Reflection in x-axis, dilation by factor 2 from y-axis, dilation by factor 3 from x-axis, OR (using the final rule) reflection in x-axis, then dilation by factor 6 from the x-(or y-) axis.

4 **a** $$\begin{aligned} f'(x) &= 7(5x^3-3x)^6 \times (15x^2-3) \\ &= 21(5x^2-1)(5x^2-3x)^6 \\ &= 21x^6(5x^2-1)(5x^2-3)^6 \end{aligned}$$

b $$\begin{aligned} f'(x) &= 2e^{4x} + 2x \times 4e^{4x} \\ &= 2e^{4x}(1+4x) \\ f'(0) &= 2 \times 1 \times 1 \\ &= 2 \end{aligned}$$

5 **a**

$$\begin{aligned} \frac{d}{dx}\left(x^2 \log_e(2x)\right) &= 2x\log_e(2x) + x^2 \times \frac{1}{x} \\ &= x\left(1 + 2\log_e\left(2x\right)\right) \end{aligned}$$

b

$$f'(x) = \frac{(2x+1)\;\cos(x) - 2\sin(x)}{(2x+1)^2}$$

$$f'\left(\frac{\pi}{2}\right) = \frac{\left(2 \times \frac{\pi}{2} + 1\right)\;\cos\left(\frac{\pi}{2}\right) - 1\sin\left(\frac{\pi}{2}\right)}{\left(2 \times \frac{\pi}{2} + 1\right)^2}$$

$$= -\frac{2}{(\pi+1)^2}$$

6 **a** $$\begin{aligned} f'(x) &= e^{\sin(2x)} \times 2\cos(2x) \\ &= 2\cos(2x)e^{\sin(2x)} \end{aligned}$$

b $f'(x) = 3\tan(2x) + 3x \times 2\sec^2(2x)$

$$= 3\tan(2x) + 6x\sec^2(2x)$$

$$f'\left(\frac{\pi}{3}\right) = 3\tan\left(\frac{2\pi}{3}\right) + 2\pi\sec^2\left(\frac{2\pi}{3}\right)$$

$$= -3\sqrt{3} + 2\pi \times \frac{1}{\cos^2\left(\frac{2\pi}{3}\right)}$$

$$= -3\sqrt{3} + 2\pi \times 4$$

$$= 8\pi - 3\sqrt{3}$$

7 $\sin(2x) - \cos(2x) = 0$

$$\sin(2x) = \cos(2x)$$

$$\tan(2x) = 1$$

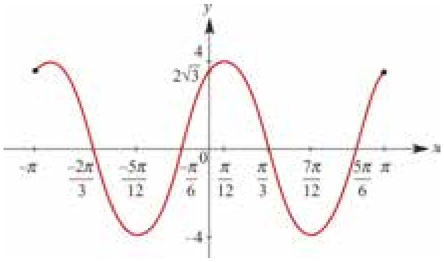
$$2x = \frac{\pi}{4} + n\pi$$

$$= \frac{\pi(4n+1)}{4}$$

$$x = \frac{\pi(4n+1)}{8}, n \in \mathbb{Z}$$

8 **a** Amplitude = 4, period = $\frac{2\pi}{2} = \pi$

b



9 $y = f(x) = 1 - \frac{4}{x-2}$

$x \to \pm\infty,\ y \to 1;\ x \to 2,\ y \to \pm\infty$

S, the asymptotes have equations

$x = 2$ and $y = 1$

$$x = 0:\ y = 1 - \frac{4}{0-2}$$

$$= 1 - (-2)$$

$$= 3$$

$$y = 0:\ \frac{4}{x-2} = 1$$

$$= x - 2 = 4$$

$x = 6$

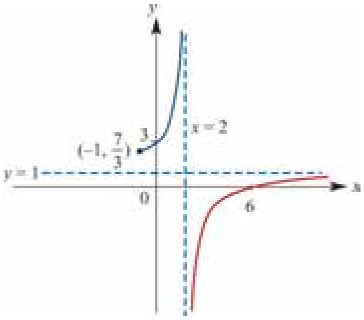
The intercepts are (6, 0) and (0, 3)

check the endpoint of the domain:

$$x = -1: y = 1 - \frac{4}{-1-2}$$

$$= 1 + \frac{4}{3}$$

$$= \frac{7}{3}$$



10 **a** $y = 5e^{x-1} - 3$

interchange x and y and solve for y:

$$x = 5e^{y-1} - 3$$

$$5e^{y-1} = x + 3$$

$$e^{y-1} = \log_e \frac{x+3}{5}$$

$$y - 1 = \log_e\left(\frac{x+3}{5}\right)$$

$$y = f^{-1}(x) = \log_e\left(\frac{x+3}{5}\right) + 1$$

b range of $f = (-3, \infty)$ = domain of f^{-1}

11 $\cos\left(\frac{5x}{2}\right) = \frac{1}{2}, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

General solution is given by

$$\frac{5x}{2} = \pm\frac{\pi}{3} + 2n\pi$$

$$x = \pm\frac{2\pi}{15} + \frac{4n\pi}{15}$$

$$n = 0: x = \pm\frac{2\pi}{15}$$

$$n = 1: x = \pm\frac{2\pi}{15} + \frac{4\pi}{5}$$

$$= \frac{2\pi}{3}, \frac{14\pi}{15}\text{(outside internal)}$$

$$n = -1: x = \pm\frac{2\pi}{3} - \frac{4\pi}{5}$$

$$= -\frac{14\pi}{15}, -\frac{2\pi}{3}\text{(outside internal)}$$

Solutions are $x = -\frac{2\pi}{15}, \frac{2\pi}{15}$

12

$$\begin{aligned} g(u+v) &= 5(u+v)^2 \\ &= 5(u^2 + 2uv + v^2) \\ g(u+v) &= 5(u-v)^2 \\ &= 5(u^2 - 2uv + v^2) \\ g(u+v) + g(u-v) &= 10(u^2 + v^2) \\ &= 2(5u^2 + 5v^2) \\ &= 2(9(u) + g(v)) \end{aligned}$$

13 Average value $= \frac{1}{4-0}\int_0^4 e^x\,dx$

$$= \frac{1}{4}\left[e^x\right]_0^4$$

$$= \frac{1}{4}(e^4 - 1)$$

14 a $x = 0,\ y = 6: \quad 6 = 0 + 0 + c$

$$c = 6$$

$$x = -2,\ y = 0: 0 = -8a - 2b + 6$$

$$4a + b = 3 \qquad \ldots①$$

$\frac{dy}{dx} = 0$ when $x = -1$

b $\frac{dy}{dx} = 3ax^2 + b$

$= 0$ when $x = -1$, so

$$3a + b = 0 \qquad \ldots②$$

c ①-②: $a = 3$

Substitute into ②: $b = -3 \times 3$

$= -9$

15 a $y = g(x) = 3 - e^{2x}$

Interchange x and y and solve for y.

$$x = 3 - e^{2y}$$

$$e^{2y} = 3 - x$$

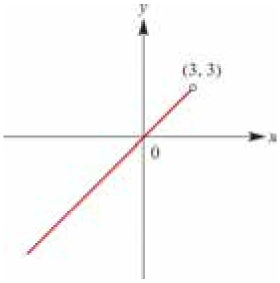
$$2\,y = \log_e(3 - x)$$

$$y = g^{-1}(x) = \frac{1}{2}\log_e(3 - x)$$

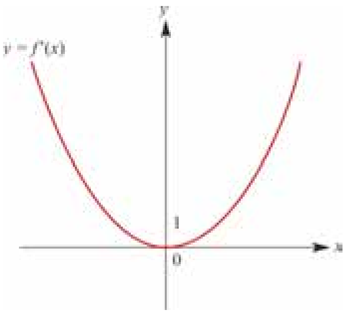
domain of g^{-1} = range of $g = (-\infty,\ 3)$

b $y = g(g^{-1}(x))$

$= x$, with domain$(-\infty,\ 3)$



16 a The graph of $y = f(x)$ is continuous and appears to be 'smooth' at (0, 1), so the derivative exists of $x = 0$ where the gradient appears to be zero. The gradient is positive for all other value of x. The graph of $y = f'(x)$ is shown below.



b $f'(x) = \begin{cases} -8x^3 & x \le 0 \\ 8x^3 & \text{otherwise} \end{cases}$

(Note that $f'(0) = 0$ as expected.)

17 $f(x) = \frac{1}{-3}\log_e(1-3x) + c$

$$= -\frac{1}{3}\log_e(1-3x) + c$$

18 $y = f(x) = \frac{3}{2x-1} + 3$

Intercharge x and y and solve for y

$$x = \frac{3}{2y-1} + 3$$

$$\frac{3}{2y-1} = x - 3$$

$$\frac{2y-1}{3} = \frac{1}{x-3}$$

$$2y - 1 = \frac{3}{x-3}$$

$$2y = \frac{3}{x-3} + 1$$

$$= \frac{3 + x - 3}{x-3}$$

$$= \frac{x}{x-3}$$

$$y = f^{-1}(x) = \frac{x}{2(x-3)}$$

19 $\tan(2x) = -\sqrt{3}$

$$2x = \cdots - \frac{\pi}{3} - \pi, \frac{\pi}{3}, -\frac{\pi}{3} + \pi, -\frac{\pi}{3} + 2\pi, \cdots$$

$$2x = \cdots - \frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \cdots$$

$$x = \cdots - \frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \cdots$$

since $x \in \left(\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, the solution are $x = -\frac{\pi}{6}, \frac{\pi}{3}$.

20 X is normal with mean 84 and standard derviation 6.

a $\Pr(X > 84) = \Pr(Z > 0)$

$$= 0.5$$

b $\Pr(78 < X < 90)$

$$= \Pr\left(\frac{78-84}{6} < Z < \frac{90-84}{6}\right)$$

$$= \Pr(-1 < Z < 1)$$

$$= \Pr(Z < 1) - \Pr(Z > 1)$$

$$= \Pr(Z < 1) - \Pr(Z > 1)$$

$$= \Pr(Z < 1) - (1 - \Pr(Z < 1)$$

$$= 2\ \Pr(Z < 1) - 1$$

$$= 2 \times 0.84 - 1$$

$$= 0.68$$

c $\Pr(X < 78 \mid X < 84)$

$$= \frac{\Pr('X < 78' \cap' X < 84')}{\Pr(X < 84')}$$

$$= \frac{\Pr(X < 78)}{\Pr(X < 84)}$$

$$\Pr(X < 78) = \Pr(Z < -1)$$

$$= 1 - \Pr(Z < 1)$$

$$= 1 - 0.84$$

$$= 0.16$$

$$\Pr(X < 84) = 0.5$$

$$\Pr(X < 78 \,|\, X < 84) = \frac{0.16}{0.5}$$

$$= 0.32$$

21 **a** $\Pr(X < 3) = \int_1^3 \frac{x}{24}\, dx$

$$= \left[\frac{x^2}{48}\right]_1^3$$

$$= \frac{9 - 1}{48}$$

$$= \frac{1}{6}$$

b $\Pr(X \geq b) = \int_b^7 \frac{x}{24}\, dx$

$$= \left[\frac{x^2}{48}\right]_b^7$$

$$= \frac{49 - b^2}{48}$$

$$= \frac{3}{8}$$

if $49 - b^2 = 18$

$$b^2 = 31$$

$b = \sqrt{31}$, since $b \in [1, 7]$

22 The gradient of the tangent is $\frac{1}{3}$

Also, $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

$$= \frac{1}{3}$$

if $x^{-\frac{2}{3}} = 1$

$$x^{-2} = 1$$

$$x^2 = 1$$

$$x = \pm 1$$

$x = -1,\ y = -1{:}y + 1 = \frac{1}{3}(x + 1)$

$$\Rightarrow y = \frac{1}{3}x - \frac{2}{3}$$

$x = -1,\ y = -1{:}y + 1 = \frac{1}{3}(x + 1)$

$$\Rightarrow y = \frac{1}{3}x - \frac{2}{3}$$

Hence $a = \pm\frac{2}{3}$

23 **a** $b = 16 - 4a^2$

A = area $XYZW$

$$= 2\, ab$$

$$= 2\, a(16 - 4\, a^2)$$

$$= 32a - 8a^3$$

b $\frac{dA}{da} = 32 - 24a^2$

$$= 0$$

If $a^2 = \frac{32}{24}$

$$= \frac{4}{3}$$

$$a = \pm\frac{2}{\sqrt{3}}$$

$$= \pm\frac{2\sqrt{3}}{3}$$

But $a > 0$, so $a = \frac{2\sqrt{3}}{3}$, and

$A = \dfrac{128\sqrt{3}}{9}$.

(This clearly correspond, to a maximum since $a \in [0, 2]$ and $A = 0$ for $a = 0$ or $a = 2$. Alternately check the sign of the derivative.)

24 $\displaystyle\int_{-1}^{3} (-3x^2 + 2bx + 9)\,dx = 32$

$$\Big[-x^3 + bx^2 + 9x\Big]_{-1}^{3} = 32$$

$$(-27 + 9b + 27) - (1 + b - 9) = 32$$

$$8b + 8 = 32$$

$$8b = 24$$

$$b = 3$$

25 **a** Using a tree diagram

$0.4 \times 0.3 + 0.6 \times 0.4 = 0.36$

b

$$\Pr(S\,an_{Tue}|S\,an_{Wed}) = \frac{S\,an_{Tue} \cap S\,an_{Wed}}{S\,an_{Wed}}$$

$$= \frac{0.36}{0.64}$$

$$= 0.5625$$

26 **a** Mean of $X = \mathrm{E}(X)$

$$= 0 \times 0.6 + | \times 0.2 + 2 \times 0.15$$

$$+ 3 \times 0.0$$

$$= 0.65$$

the mean is \$0.65.

b Pr(same amount) = Pr(0& 0 or 1&1

or 2 & 2 or 3 & 3)

$$= 0.6^2 + 0.2^2$$

$$+ 0.15^2 + 0.05^2$$

$$= 0.36 + 0.04$$

$$+ 0.0225 + 0.0025$$

$$= 0.425$$

27 The possible sequences are:

G → G → R → R or

G → R → G → R or

G → R → R → G

where G = goes to gym, and

R = goes for run

Required probability $= 0.5 \times 0.5 \times 0.6$

$$+ 0.5 \times 0.4 \times 0.5$$

$$+ 0.5 \times 0.6 \times 0.4$$

$$= 0.15 + 0.10$$

$$+ 0.12$$

$$= 0.37$$

28 **a** Volume = area cross-section × height

$$= \frac{1}{2}x^2h$$

$$= 2000$$

$$x^2h = 4000$$

$$h = \frac{4000}{x^2}$$

b The hypotenuse of the right-angled triangle cross-section has length $\sqrt{2}x$. The surface area is made up of three vertical rectangles and two equal triangular ends.

$$\begin{aligned} A &= \sqrt{2}xh + xh + xh + 2 \times \frac{1}{2}x^2 \\ &= xh(2 + \sqrt{2}) + x^2 \\ &= x \times \frac{4000}{x^2} \times (2 + \sqrt{2}) + x^2 \\ &= \frac{4000\sqrt{2} + 8000}{x} + x^2 \end{aligned}$$

c

$$\begin{aligned} \frac{dA}{dx} &= -\frac{4000\sqrt{2} + 8000}{x^2} + 2x \\ &= 0 \\ \text{if } 2x^3 &= 4000\sqrt{2} + 8000 \\ \text{i.e. } x^3 &= 2000\sqrt{2} + 4000 \\ &= 2000(2 + \sqrt{2}) \end{aligned}$$

29 a E(X) = 1

b i

$$\{0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, 1\}$$

ii

$$\begin{aligned} \Pr(\hat{P} < 0.2) &= \Pr(X < 2) \\ &= \Pr(X = 0) + \Pr(X = 1) \\ &= \left(\frac{9}{10}\right)^{10} + 10 \times \left(\frac{9}{10}\right)^9 \times \left(\frac{9}{10}\right) \\ &= \left(\frac{9}{10}\right)^9 \left(\frac{9}{10} + 1\right) \\ &= \left(\frac{9}{10}\right)^9 \times \frac{19}{10} \\ &= \frac{19 \times 9^9}{10^{10}} \end{aligned}$$

30

$$\begin{aligned} \Pr(X = 5 | X \geq 5) &= \frac{14}{15} \\ \frac{\Pr(X = 5)}{\Pr(X = 5) + \Pr(X = 6)} &= \frac{14}{15} \\ \frac{6p^5(1 - p)}{6p^5(1 - p) + p^6} &= \frac{14}{15} \\ \frac{6(1 - p)}{6(1 - p) + p} &= \frac{14}{15} \\ 45 - 45p &= 42 - 35p \\ 3 &= 10p \\ p &= \frac{3}{10} \end{aligned}$$

31

$$\begin{aligned} \Pr(46 < X < 54) &= \Pr\left(\frac{46 - 50}{5} < Z < \frac{54 - 50}{5}\right) \\ &= \Pr\left(-\frac{4}{5} < Z < \frac{4}{5}\right) \\ &= 2\Pr\left(Z < \frac{4}{5}\right) - 1 \\ \therefore 2\Pr\left(Z < \frac{4}{5}\right) - 1 &= q \\ \Pr\left(Z < \frac{4}{5}\right) &= \frac{1 + q}{2} \end{aligned}$$

Solutions to multiple-choice questions

1 B Write the equations in matrix form:

$$y = \frac{mx}{2}$$

$$y = \frac{6x}{m+4}$$

$$\therefore \frac{mx}{2} = \frac{6x}{m+4}$$

$$m(m+4)x = 12x$$

$(m^2 + 4m - 12)x = 0$

Solution is unique if

$(m+6)(m-2) \neq 0$

That is if, $m \neq -6$ or $m \neq 2$

2 A Since $\sin\left(\frac{\pi}{2}\right) = 1$, then

$$2x = \frac{\pi}{2} + 2n\pi$$

$$x = n\pi + \frac{\pi}{4}$$

3 B The graph of f has a sharp point at $x = -\frac{4}{5}$, so $f'\left(-\frac{4}{5}\right)$ is not defined.

Hence the graph of f' is discontinuous at $x = -\frac{4}{5}$.

Checking the other points shows that each one is true.

4 C $k = \int_{-6}^{-2} \frac{2}{x}\,dx$

$$= \Big[2\log_e|x|\Big]_{-6}^{-2}$$

$$= 2\log_e 2 - 2\log_e 6$$

$$= 2\log_e \frac{x}{6}$$

$$= \log_e\left(\frac{1^2}{3}\right)$$

$$= \log_e \frac{1}{9}$$

$$e^k = \frac{1}{9}$$

5 D Average value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$$= \frac{1}{3-(-1)}$$

$$\int_{-1}^{3} \log_e(x+2)\,dx$$

$$= \frac{5\log_e 5 - 4}{4}$$

using the integral command of a CAS calculator.

6 A Average value $= \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \sin(2x)\,dx$

$$= \frac{2}{\pi}\left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(-\frac{1}{2}\cos(\pi)\right) + \frac{1}{2}\cos(0)$$

$$= \frac{2}{\pi}$$

7 D $(x', y') = (3x + 5, y + 1)]$

Hence, $x' = 3x + 5$ and $y' = y + 1$

$x = \dfrac{x' - 5}{3}$ and $y = y' - 1$

The image of $y = x^2$ has equation

$$y' - 1 = \frac{1}{9}(x' - 5)^2$$

$$9y' = (x' - 5)^2 + 9$$

That is, $9y = (x - 5)^2 + 9$

8 C

$$\begin{aligned}[f(x)]^3 &= f(y)\\ (e^{3x})^3 &= e^{3y}\\ e^{3y} &= e^{9x}\\ 3y &= 9x\\ y &= 3x\end{aligned}$$

9 D

$$\begin{aligned}\Pr(X > a) &= 0.25\\ \int_a^{\frac{\pi}{2}} \sin(2x)\,dx &= 0.25\\ \left[-\frac{1}{2}\cos(2x)\right]_a^{\frac{\pi}{2}} &= 0.25\\ -\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(2a) &= 0.25 - (-1) + \cos(2a)\\ &= 0.25 \cos(2a) = -0.5\\ 2a &= \frac{2\pi}{3}\\ a &= \frac{\pi}{3}\\ &\approx 1.05\end{aligned}$$

10 A

$$\begin{aligned}\int_0^{2k} (1 + 2e^{\frac{x}{k}})\,dx &= 1\\ \left[x + 2ke^{\frac{x}{k}}\right]_0^{2k} &= 1\\ (2k + 2ke^2) - (2k) &= 1\\ 2ke^2 &= 1\\ k &= \frac{1}{2e^2}\\ &= \frac{1}{2}e^{-2}\end{aligned}$$

11 A

$$\begin{aligned}\Pr(X < 7.5) &= \Pr(Z < \frac{7.5 - 8}{0.25})\\ &= \Pr(Z < -2)\\ &= \Pr(Z > 2)\end{aligned}$$

12 B $x^2 + 12x = 2kx - 2$

$x^2 + (12 - 2k)x + 2 = 0$

Quadratic has two solutions if

$(12 - 2k)^2 - 4(1)(2) > 0$

$4k^2 - 48k + 144 - 8 > 0$

$k^2 - 12k + 34 > 0$

$(k - 6)^2 - 2 > 0$

$(k - 6 - \sqrt{2})(k - 6 + \sqrt{2}) > 0$

$k < 6 - \sqrt{2}$ or $k > 6 + \sqrt{2}$

13

D

$$\begin{aligned}e^{4x} - 7e^{2x} + 12 &= 0\\ (e^{2x} - 3)(e^{2x} - 4) &= 0\\ e^{2x} &= 3, 4\\ 2x &= \log_e 3,\ \log_e 4\\ x &= \frac{1}{2}\log_e 3,\ \frac{1}{2}\log_e 4\\ &= \log_e \sqrt{3},\ \log_e 2\\ \textit{Solution set} &= \left\{\log_e \sqrt{3},\ \log_e 2\right\}\end{aligned}$$

14 B Reflection in x – axis: $-7x^{\frac{3}{2}}$

Translated 3 units right: $-7(x - 3)^{\frac{3}{2}}$

Translated 4 units down:

$-7(x-3)^{\frac{3}{2}}-4$

The equation of the new graph is

$y=-7(x-3)^{\frac{3}{2}}-4$

15 C $E(X)=\frac{1}{8}\int_0^4 x^2\,dx$

$$=\frac{1}{8}\left[\frac{1}{3}x^3\right]_0^4$$

$$=\frac{1}{8}\left(\frac{64}{3}\right)$$

$$=\frac{8}{3}$$

16 A Since $f(2)$ does not exist, since $\log_e 0$ is undefined, the graph of $y=f(x)=4\log_e(x-2)^4$ is symmetrical about the asymptote $x=2$. For a one-one function, the domain must be restricted and for a domain of $[a, \infty)$, we must have $a>2$ of the available options, only the first fits.

17 A $e^{2x+4}-3=e^{2(x+2)}-3$

$$=f(2(x+2))-3$$

So transform the graph of $y=f(x)$ using this sequences:

- dilation of factor $\frac{1}{2}=0.5$ from the y-axis
- translations of 2 left and 3 down

18 E

$$f'(x)=g'(x), \text{so}$$

$$f(x)=g(x)+c$$

Now $f(1)=2$ and $g(x)=-xf(x)$, so

So $f(x)=g(x)+4$

19 E Let $\Pr(B)=p$

Hence $\Pr(A)=5p-0.1$

$$\Pr(A\cup B)=(5p-1)+p-p(5p-1)$$

$$0.7075=-5p^2+6.1p-0.1$$

$$0=5p^2-6.1p+0.8025$$

$$p=0.15$$

$$\Pr(A)=5\times0.15-0.1=0.65$$

20 E $E(X)=0\times a+1\times b+2\times 0.6$

$$=b+1.2$$

$$=1.6 \text{ if } b=0.4$$

Then $a+0.4+0.6=1\rightarrow a=0$

21 C

Adding the boundaries of the interval:

$0.1723+0.3277=0.5$

In the usual way,

$2p=0.5\Rightarrow p=0.25$

Hence

$$p+1.645\sqrt{\frac{p(1-p)}{n}}=0.3277$$

$$0.25+1.645\sqrt{\frac{0.25\times0.75}{n}}=0.3277$$

$$\sqrt{\frac{0.25\times0.75}{n}}=0.047234$$

$$\frac{0.25\times0.75}{n}=0.002231$$

$$\frac{1}{n}=0.011899$$

$$n=84.041$$

22 E

$$(m-4)x+6y=6\ldots(1)$$

$$2x+(m-3)y=2m-10\ldots(2)$$

$$(1)\times(m-3)\text{ and }(2)\times 6$$

$$(m-4)(m-3)x+6(m-3)y=6(m-3)\ldots(1')$$

$$12x+6(m-3)y=6(2m-10)\ldots(2')$$

$$(1')-(2')$$

$$(m-4)(m-3)x-12x=6(m-3)-6(2m-10)$$

$$m^2-7m+12x-12x=6m-18-12m+60$$

$$(m^2-7m)x=-6m+42$$

$$m(m-7)x=-6(m-7)$$

Infinitely many solutions if $m = 7$
and no solutions for $m = 0$

23 D $n = 1000, \hat{p} = 0.52$
95% CI = (0.489,0.551)

24 C The candidate needs more than 50 % of the vote to win. Based on the confidence interval they will get between 48.9% and 55.1% of the vote- they might win but its too close to tell.

Solutions to extended-response questions

1 a i $y = \dfrac{16x^3 + 4x^2 + 1}{2x^2}$

$$\frac{dy}{dx} = \frac{2x^2(48x^2 + 8x) - 4x(16x^3 + 4x^2 + 1)}{(2x^2)^2} \text{(quotient rule)}$$

$$= \frac{96x^4 + 16x^3 - 64x^4 - 16x^3 - 4x}{4x^4}$$

$$= \frac{32x^4 - 4x}{4x^4} = \frac{8x^3 - 1}{x^3}$$

$$\frac{dy}{dx} = 0 \text{ implies } 8x^3 - 1 = 0$$

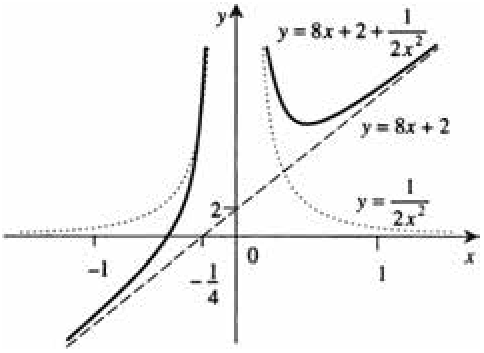
$$x^3 = \frac{1}{8}$$

$$\therefore \quad x = \frac{1}{2}$$

$\therefore$ Stationary point at $\left(\dfrac{1}{2}, 8\right)$

ii $y = 8x + 2 + \dfrac{1}{2}x^{-2}$ (achieved by dividing by $2x^2$)

Addition of coordinates gives the shape of the graph.



To establish minimum:

when $x = 0.25$, $y = 12$

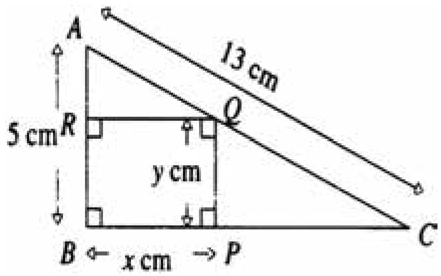
when $x = 0.75$, $y = \dfrac{80}{9} = 8\dfrac{8}{9}$

or gradient

when $x = 0.25$, $\dfrac{dy}{dx} = -56$

when $x = 0.75$, $\dfrac{dy}{dx} = \dfrac{152}{27}$

b



i $\Delta \dot{Q}PC \sim \Delta ABC$

and both are right angled triangles. By Pythagoras' Theorem

$$BC = \sqrt{13^2 - 5^2} = 12$$

$$\text{and } \frac{PC}{BC} = \frac{QP}{AB}$$

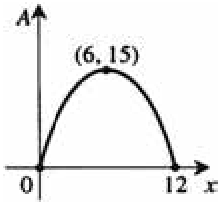
$$\therefore \quad \frac{12 - x}{12} = \frac{y}{5}$$

$$\text{and } y = \frac{60 - 5x}{12}$$

ii Area of the rectangle $A = xy = \dfrac{x(60 - 5x)}{12}$

iii The practical domain for A is $0 \le x \le 12$

By the properties of parabolas for which the coefficient of x^2 is negative, maximum point has coordinates (6, 15).



Alternately: $A = 5x - \dfrac{5x^2}{12}$

$$\frac{dA}{dx} = 5 - \frac{5x}{6}$$

$$\frac{dA}{dx} = 0, \text{implies } x = 6$$

$$\text{when } x = 6,\ A = \frac{6(60 - 5 \times 6)}{12} = 15$$

$\therefore$ maximum area is 15 cm^2

2 a

x	0	1	3
y	6	0	0

$y = k(x - p)(x - q)$

Since $y = 6$ when $x = 0$, $6 = kpq$ ①

Also $0 = k(x - p)(x - q)$

implies $x = p$ or $x = q$

hence $p = 1$ and $q = 3$ as $p<q$

From equation ① $k = 2$

b **i** For $y = m(x - p)^2(x - q)$

As before $p = 1$ and $q = 3$

Now as then $x = 2,\ y = 2$

$2 = m(2 - 1)^2(2 - 3)$

$\therefore \quad m = -2$ (Note: when $x = 0, y = 6$)

ii $y = -2(x-1)^2(x-3)$

$$= -2\left[(x^2 - 2x + 1)(x - 3)\right]$$

$$= -2[x^3 - 2x^2 + x - 3x^2 + 6x - 3]$$

$$= -2[x^3 - 5x^2 + 7x - 3]$$

$$= -2x^3 + 10x^2 - 14x + 6$$

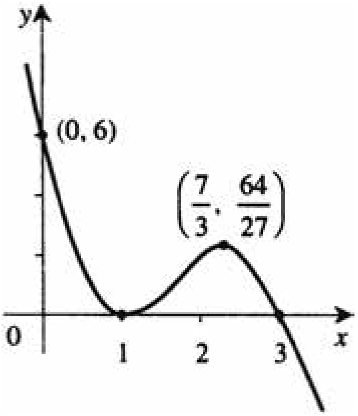
iii $\frac{dy}{dx} = -6x^2 + 20x - 14$

$\frac{dy}{dx} = 0$ implies $-6x^2 + 20x - 14 = 0$

$\rightarrow -2(3x^2 - 10x + 7) = 0$

$\rightarrow (3x - 7)(x - 1) = 0$

$\rightarrow x = \frac{7}{3}$ or $x = 1$



When $x = 1, y = 0$, When $x = \frac{7}{3}$, $y = \frac{64}{27}$.

There is a local min at (1,0) and a local max at $\left(\frac{7}{3}, \frac{64}{27}\right)$.

Note: $\frac{dy}{dx} = -\frac{11}{2}$ when $x = \frac{1}{2}$

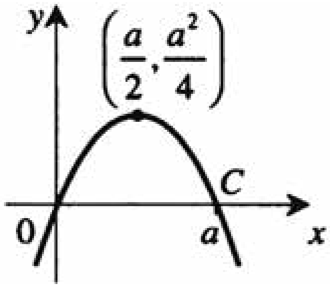
$\frac{dy}{dx} = 2$ when $x = 2$

$\frac{dy}{dx} = -1.5$ when $x = 2.5$

A gradient chart illustrates the nature of the stationary points

	$x < 1$	1	$1 < x < 2\frac{1}{3}$	$2\frac{1}{3}$	$x > 2\frac{1}{3}$
sign of $\frac{dy}{dx}$	–ve	0	+ve	0	–ve
shape	\	–	/	–	\

3 **a** $y = ax - x^2$



When $y = 0$

$x(a - x) = 0$

$\therefore \quad x = 0$, or $x = a$

By symmetry turning point occurs when

$$x = \frac{a}{2}$$

$x = \frac{a}{2}$ When $y = \frac{a}{2}(a - \frac{a}{2})$

$$= \frac{a^2}{4}$$

b $\int_0^a ax - x^2 \, dx = \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a$

$= \frac{a^3}{2} - \frac{a^3}{3}$

$= \frac{a^3}{6}$

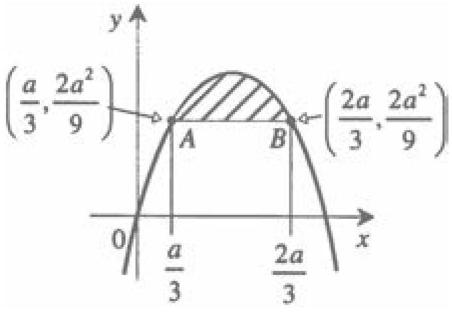
$\therefore$ The area is $\frac{a^3}{6}$ square units.

c **i** When $x = \frac{a}{3}$, $y = a \times \frac{a}{3} - \left(\frac{a}{3}\right)^2$

$= \frac{a^2}{3} - \frac{a^2}{9}$

$= \frac{2a^2}{9}$

when $x = \frac{2a}{3}$, $y = \frac{2a^2}{9}$ by symmetry



ii $\int_{\frac{1}{3}a}^{\frac{2}{3}a} ax - x^2\, dx$

$$= \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_{\frac{1}{3}a}^{\frac{2}{3}a}$$

$$= \frac{a}{2} \times \frac{4}{9}a^2 - \frac{8a^3}{81} - \left(\frac{a}{2} \times \frac{1}{9}a^2 - \frac{1}{81}a^3\right)$$

$$= \frac{2a^3}{9} - \frac{a^3}{18} - \frac{8a^3}{81} + \frac{1}{81}a^1$$

$$= \frac{a^3}{81}\left[18 - \frac{9}{2} - 8 + 1\right]$$

$$= \frac{13a^3}{162}$$

Area of the rectangle $= \frac{a}{3} \times \frac{2a^2}{9}$

$$= \frac{12a^3}{162}$$

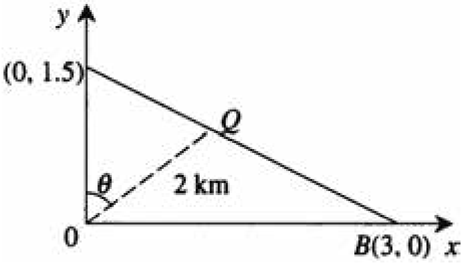
$\therefore$ required area $= \frac{a^3}{162}$ square units

Note: This area may also be found by evaluating $\int_{\frac{1}{3}a}^{\frac{2}{3}a} ax - x^2 - \frac{2a^2}{9}\, dx$

4 a Equation of line

$$y = \left(\frac{1.5 - 0}{0 - 3}\right)x + 1.5$$

$$= -\frac{1}{2}x + \frac{3}{2}$$



b i $y = \sin\theta + 2\cos\theta$

$\therefore \quad \frac{dy}{d\theta} = \cos\theta - 2\sin\theta$

ii $\frac{dy}{d\theta} = 0$ implies $\cos\theta = 2\sin\theta$

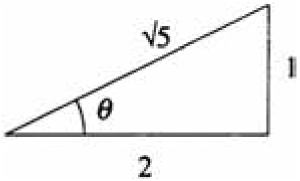
which implies $\tan\theta = \frac{1}{2}(\cos\theta \neq 0)$

$\therefore \theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.57^0$

iii $y \approx 2.2361$ when $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\therefore$ (26.57, 2.2361) are the coordinates of the stationary point. The following shows the exact coordinates to be $\left(\tan^{-1}(\frac{1}{2}),\ \sqrt{5}\right)$.

iv A maximum occurs when $\theta = \tan^{-1}\left(\frac{1}{2}\right)$



Note: $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) + 2\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$

$= \frac{1}{\sqrt{5}} + 2 \times \frac{2}{\sqrt{5}}$

$= \frac{5}{\sqrt{5}} = \sqrt{5}$

$\therefore \quad r = \sqrt{5}$

$\therefore \quad y = \sqrt{5}\sin(\theta + \alpha)$

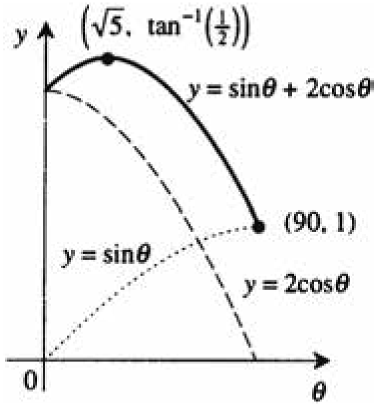
when $\theta = 0, y = 2 \quad \therefore \ \sin\ \alpha = \frac{2}{\sqrt{5}}$

$\therefore \quad \alpha = 63.435°$ [The smallest positive solution is chosen.

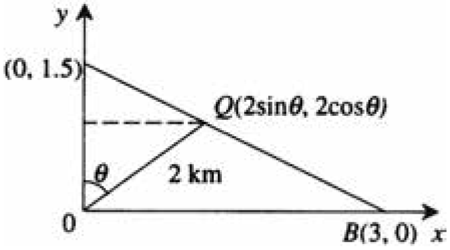
Any solution will work.]

$\therefore \quad y = \sqrt{5}\sin(\theta + 63.435)$

v



c **i** coordinates of $Q = (2\sin\theta,\ 2\cos\theta)$



ii Q is on the line with equation

$y = -\frac{1}{2}x + \frac{3}{2}$

$\therefore \quad 2\ \cos\theta = -\sin\theta + \frac{3}{2}$

$\therefore \quad 2\ \cos\theta + \sin\theta + \frac{3}{2}$

i.e. $4\ \cos\theta + 2\sin\theta = 3$

iii $2\ \cos\theta + \sin\theta = \frac{3}{2}$

From (b)(iv)

$$\sqrt{5}\ \sin\left(\theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \frac{3}{2}$$

$$\sin\left(\theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \frac{3}{2\sqrt{5}}$$

$$\therefore \quad \theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right) \text{or } 180 - \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right)$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{2}{2\sqrt{5}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \text{or } 180 - \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

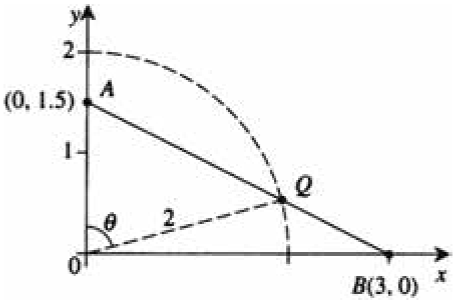
$= -21.3045$ or 74.4346

for $0° < \theta \leq 90°$, required answer $\theta = 74.4346°$

Alternative Method to find the point Q

Q can be considered to be on a circle radius 2 km centre 0.

The equation of this circle is $x^2 + y^2 = 4$



$\therefore$ Solve simultaneously the equations

$x^2 + y^2 = 4$ ①

and $y = -\frac{1}{2}x + \frac{3}{2}$ ②

Substitute from ② into ①

$$\therefore \quad \left(-\frac{1}{2}x + \frac{3}{2}\right)^2 + x^2 = 4$$

$$\therefore \quad 9 - 6x + x^2 + 4x^2 = 16$$

$$5x^2 - 6x - 7 = 0$$

$$\therefore x = \frac{6 \pm \sqrt{36 - 4 \times -7 \times 5}}{10}$$

$$= \frac{6 \pm \sqrt{176}}{10}$$

$$x = \frac{6 \pm 4\sqrt{11}}{10} = \frac{3 \pm 2\sqrt{11}}{5}$$

x must be positive $\therefore\ x = \dfrac{3 + 2\sqrt{11}}{5}$

i.e. $2\sin\theta = \dfrac{3 + 2\sqrt{11}}{5}$

i.e. $\sin\theta = \dfrac{3 + 2\sqrt{11}}{10}$

$\theta = 74.4346°$

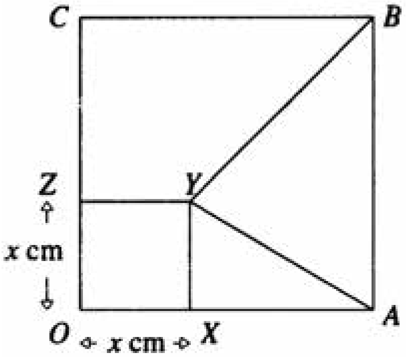
5 a i Area of $OXYZ = x^2$ cm^2

Area of $ABY = \dfrac{1}{2} \times 10 \times (10 - x)$

$= 5(10 - x)$ cm^2

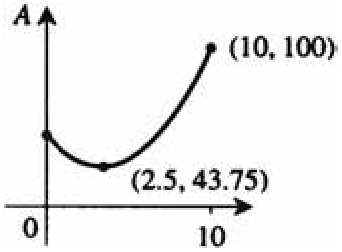
$\therefore$ total area $A = x^2 + 50 - 5x$

$= x^2 - 5x + 50$



ii domain = (0, 10)

iii

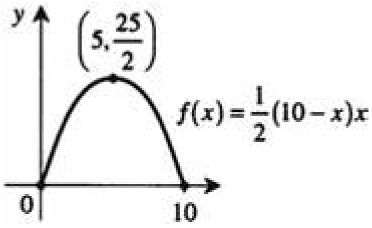


iv minimum area = 43.75 cm^2

b i $f(x) = \dfrac{1}{2} \times \text{base} \times \text{height}$

$= \dfrac{1}{2} \times AX \times XY$

$= \dfrac{1}{2}(10 - x)x$ domain = (0, 10)



ii Maximum area of $AYX = 12.5\text{cm}^2$
This occurs when $x = 5$
When $x = 5$
Area of square $OXYZ = 25\text{ cm}^2$
Area of triangle $ABY = 25\text{ cm}^2$
Area of trapezium $CBYZ = 37.5\text{ cm}^2$
$\therefore$ ratio of areas AYX: $OXYZ$:ABY : $CBYZ$
$= 12.5 : 25 : 25 : 37.5$

$= 25 : 50 : 50 : 75$

$= 1 : 2 : 2 : 3$

6 $f(t) = 1000(t^2 - 10t + 44)e^{-\frac{t}{10}} 0 \leq t \leq 35$
Using a CAS calculator it is interesting to graph the function for $t \in [0, 35]$.

a **i** $f'(t) = 1000(2t - 10)e^{-\frac{t}{10}} - \frac{1}{10}\left(1000\left(t^2 - 10t + 44\right)\right)e^{-\frac{t}{10}}$

$= 100e^{-\frac{t}{10}}[20t - 100 - t^2 + 10t - 44]$

$= 100e^{-\frac{t}{10}}[30t - 144 - t^2]$

$= -100e^{-\frac{t}{10}}[t^2 - 30t + 144]$

ii $f''(t) = -100\left[-\frac{1}{10}e^{-\frac{t}{10}}(t^2 - 30t + 144) + (2t - 30)e^{-\frac{t}{10}}\right]$

$= 10e^{-\frac{t}{10}}\left[t^2 - 30t + 144 - 20t + 300\right]$

$= 10e^{-\frac{t}{10}}\left[t^2 - 50t + 444\right]$

b **i** Increasing if $f'(t) > 0$
i.e. $-100e^{-\frac{t}{10}}[t^2 - 30t + 144] > 0$
is equivalent to $t^2 - 30t + 144 < 0$ as $-100e^{-\frac{t}{10}} < 0$ for all t
$\therefore$ $(t - 24)(t - 6) < 0$
$\therefore$ $t \in (6, 24)$
The number of unemployed was increasing for $6 < t < 24$.

ii $f''(t) < 0$
$10\, e^{-\frac{t}{10}}[t^2 - 50t + 444] < 0$
is equivalent to $t^2 - 50t + 444 < 0$
First consider the equation

$t^2 - 50t + 444 = 0$

$$t = \frac{50 \pm \sqrt{50^2 - 4 \times 444}}{2}$$

$$= \frac{50 \pm \sqrt{724}}{2}$$

$$= 25 \pm \sqrt{181}$$

$\therefore\ t^2 - 50t + 444 < 0$ for $t \in (25 - \sqrt{181}, 25 + \sqrt{181})$

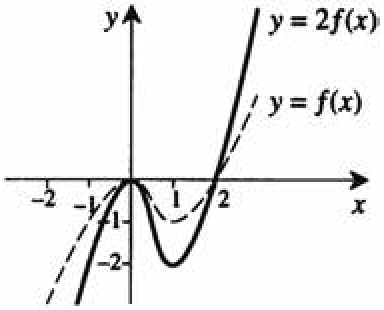
However, the domain of the function is $[0, 35]$.

So $f''(t) < 0$ for $t \in (11.546, 35)$

iii $(6.\ 24) \cap (11.546,\ 38.454)$

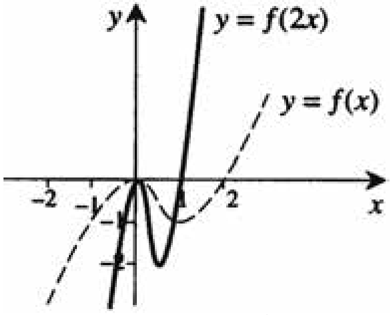
$= (11.546, 24)$

7 a i



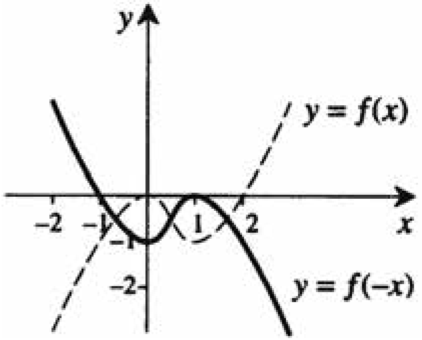
A dilation of factor 2 from the x-axis.

ii



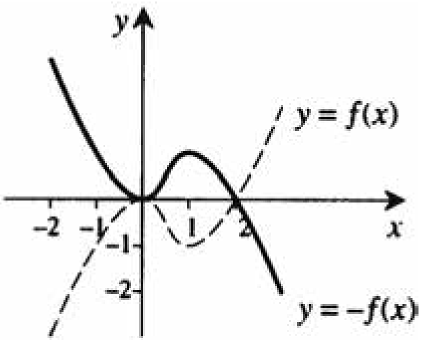
A dilation of factor $\frac{1}{2}$ from the y-axis.

iii



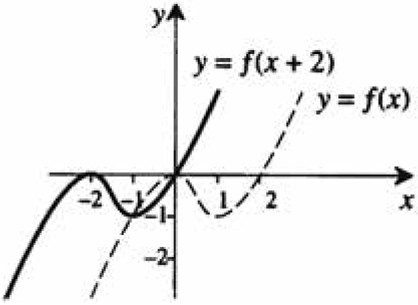
A reflection in the y-axis.

iv



A reflection in the x-axis.

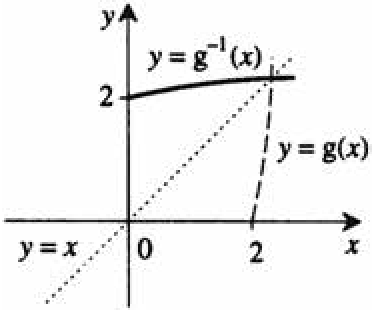
v



A translation of 2 to the left.

b f does not have an inverse function as it is not one-to-one.

c



d **i** $g(x) = x^2(x-2)$ and $g : (2, \infty) \to R$

$g'(x) = 2x(x-2) + x^2 = x(2x-4+x) = x(3x-4)$

When $x = 3g'(x) = 15$

ii $(g \circ g^{-1})'(x) = 1$

$$\therefore g'\left(g^{-1}(x)\right)\left((g^{-1})'(x)\right) = 1 \text{ (by the chain rule)}$$

$$\therefore \left(g^{-1}\right)'(x) = \frac{1}{g'\left(g^{-1}(x)\right)}$$

$$\therefore \left(g^{-1}\right)'(9) = \frac{1}{g'\left(g^{-1}(9)\right)} = \frac{1}{g'(3)} = \frac{1}{15}$$

This can also be shown by the result $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}, \dfrac{dy}{dx} \neq 0$

8 a i $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

Actual value, correct to three decimal places = 0.995

ii $\cos x = 0.98$

Consider the equation

$1 - \frac{1}{2}x^2 = 0.98$

$\therefore\ 2 - x^2 = 1.96$

$0.04 = x^2$

$x = \pm 0.2$

Actual value correct to three decimal places $x = \pm 0.2.00 \leftarrow (\pm 0.200)$

b i Let $f(x) = 1 - \frac{1}{2}x^2$

A reflection in the x-axis is given by

$g(x) = -f(x) = \frac{1}{2}x^2 - 1$

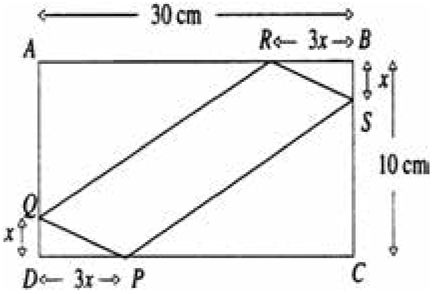
A translation of π units in the positive direction of the x-axis if given by

$h(x) = g(x - \pi) = \frac{1}{2}(x - \pi)^2 - 1$

ii $h(3) = \frac{1}{2}(3 - \pi)^2 - 1 \approx -0.98998$

(Actual $\cos(3) = -0.98999$ correct to five decimal places.)

9



a Area of a triangle RBS = area of triangle $PDQ = \dfrac{3x^2}{2}$ cm^2

Area of a triangle CPS = area of triangle $ARQ = \dfrac{1}{2} \times (30 - 3x)(10 - x)$

$$\therefore \text{ Area of parallelogram} = 300 - 3x^2 - 3(10 - x)^2$$
$$= [300 - 3x^2 - 3(100 - 20x + x^2)]$$
$$= (60x - 6x^2) \text{ cm}^2$$

b $0 < 3x < 30$ and $0 < x < 10$

$\therefore 0 < x < 10$

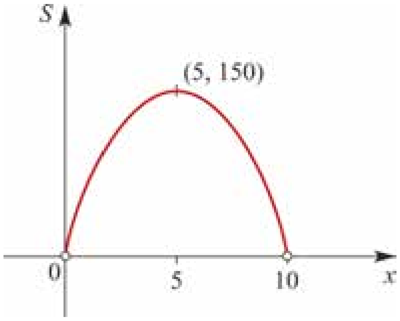
c $A = 60x - 6x^2$

and $\dfrac{dA}{dx} = 60 - 12x$

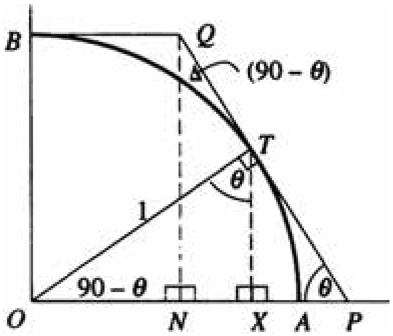
$\dfrac{dA}{dx} = 0$ implies $x = 5$

Since the expression is quadratic with negative coefficient of x^2 a local maximum at $(5, 150)$.

d



10 a i



From triangle OTP

$$\frac{1}{OP} = \sin\theta$$

$$\therefore\ OP = \frac{1}{\sin\theta}$$

ii $BQ = OP - NP$

$NP = TP$ as ΔQNP is congment to ΔOTP

and $TP = \dfrac{1}{\tan\theta}$

$$\therefore BQ = \frac{1}{\sin\theta} - \frac{1}{\tan\theta}$$

$$= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$$

$$= \frac{1 - \cos\theta}{\sin\theta}$$

b Area of the trapezium $= \frac{1}{2}\left(\frac{1 - \cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right)$

$$= \frac{2 - \cos\theta}{2\sin\theta}$$

c $S = \frac{2 - \cos\theta}{2\ \sin\theta}$

$$\frac{dS}{d\theta} = \frac{\sin\theta \times 2\ \sin\theta - 2\cos\theta(2 - \cos\theta)}{(2\ \sin\theta)^2}$$

$$= \frac{2\ \sin^2\theta - 4\ \cos\theta + 2\ \cos^2\theta}{(2\sin\theta)^2}$$

$$= \frac{2 - 4\ \cos\theta}{4\ \sin\theta}$$

d $\frac{dS}{d\theta} = 0$ implies, $\frac{2 - 4\cos\theta}{2 - 4\cos\theta = 0}$ $\quad 0 < \theta < \frac{\pi}{2}$

$\therefore \quad \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = \frac{\pi}{3}$

and $\frac{dS}{d\theta} < 0$ when $\quad \theta = \frac{\pi}{4}$

and $\frac{dS}{d\theta} > 0$ when $\quad \theta = \frac{\pi}{2}$

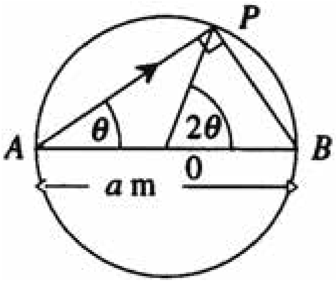
$\therefore$ a minimum when $\theta = \frac{\pi}{3}$

When $\theta = \frac{\pi}{3}$ $\quad S = \dfrac{2 - \frac{1}{2}}{2 \times \frac{\sqrt{3}}{2}} = \dfrac{3}{2} \times \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

and $\quad AP = OP - 1$

$$\begin{aligned} &= \frac{1}{\sin(\frac{\pi}{3})} - 1 \\ &= \frac{1}{\frac{\sqrt{3}}{2}} - 1 \\ &= \frac{2}{\sqrt{3}} - 1 \\ &= \frac{2 - \sqrt{3}}{\sqrt{3}} \\ &= \frac{2\sqrt{3} - 3}{3} \end{aligned}$$

11



a **i** distance $AP = a\cos\theta$

distance $PB = \dfrac{a}{2} \times 2\theta = a\theta$(for arc PB)

time for $AP = \dfrac{a\cos\theta}{\frac{1}{2}} = 2a\cos\theta$

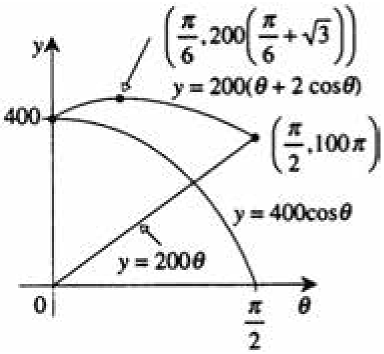
time for $PB = \dfrac{a\theta}{1} = a\theta$

$\therefore$ total time, $T = a(\theta + 2\cos\theta)$

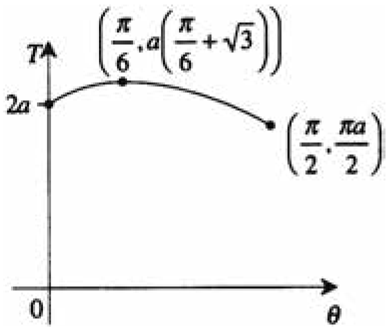
b $\dfrac{dy}{d\theta} = 200(1 - 2\sin\theta)$

$\dfrac{dy}{d\theta} = 0$ implies $\sin\theta = \dfrac{1}{2}$

$\therefore$ maximum when $\theta = \dfrac{\pi}{6}$



c The minimum value for T is $\dfrac{\pi a}{2}$. This is obtained by the dog running around outside of the lake.



12 a i $f(x) = (x - 1)\,g(x)$ and $f'(x) = (x - 1)h(x)$

$f'(x) = g(x) + (x - 1)g'(x)$ (product rule)

$\therefore\quad g(x) + (x - 1)g'(x) = (x - 1)h(x)$

$\therefore\quad g(x) = (x - 1)h(x) - (x - 1)g'(x)$

$= (x - 1)[h(x) - g'(x)]$

$\therefore\quad (x - 1)$ is a factor of $g(x)$

ii $F(1) = 1 - k - 3 + 2k - k + 2 = 0$ where $F(x) = x^3 - kx^2 - (3 - 2k)x - (k - 2)$

$F'(x) = 3x^2 - 2kx - (3 - 2k)$

$\therefore\quad F'(1) = 3 - 2k - 3 + 2k = 0$

iii By the factor theorem $x - 1$ is a factor of $F(x)$ and $F'(x)$.

$\therefore F(x) = (x - 1)g(x)$ and $F'(x) = (x - 1)h(x)$ where $g(x)$ and $h(x)$ are polynomials.

$\therefore x - 1$ is a factor of $g(x)$

$\therefore F(x) = (x - 1)^2 w(x)$ where $w(x)$ is a linear polynomial)

$(x^2 - 2x + 1)(x - p) = x^3 - kx^2 - (3 - 2k)x - (k - 2)$

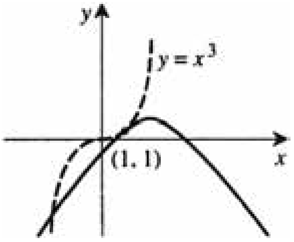
$\therefore -p = -(k - 2)$

i.e. $p = k - 2$

$\therefore F(x) = (x - 1)^2(x - (k - 2))$

and $F(x) = 0$ implies $x = 1$ or $x = k - 2$

b **i** For $y = ax^2 + bx + c$



$$\frac{dy}{dx} = 2ax + b$$

Given that (1, 1) is on the parabola and the gradient is the same as $y = x^3$ at $x = 1$ we have

$a + b + c(1) \;\; = 1$

$2a + b = 3(2) \left(y = x^3,\ \frac{dy}{dx} = 3x^2 \text{ and when } x = 1,\ \frac{dy}{dx} = 3\right)$

From ② $\quad b = 3 - 2a$

From ① $\quad c = 1 - a - b$

$\quad = 1 - a - (3 - 2a)$

$\quad = 1 - a - 3 + 2a$

$\quad = -2 + a$

$\quad = a - 2$

ii $y = ax^2 + (3 - 2a)x + a - 2$

$y = x^3$

to find Q consider

$ax^2 + (3 - 2a)x + a - 2 = x^3$

i.e. $x^3 - ax^2 + (2a - 3)x + (2 - a) = 0$

Let $F(x) = x^3 - ax^2 + (2a - 3)x + (2 - a)$ (the polynomial of a)

$\therefore F(x) = (x - 1)^2(x - (a - 2))$

$\therefore$ The parabola meets the curve $y = x^3$ at the point $((a - 2),\ (a - 2)^3)$ and $h = a - 2$.

iii If $a - 2 = -2,\ a = 0,\ b = 3$ and $c = -2$

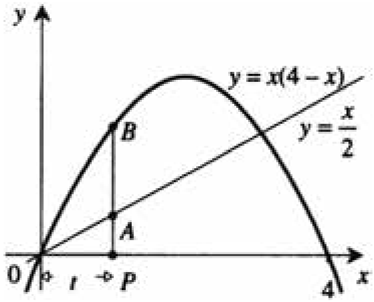
Q is the point of intersection of $y = x^3$ with the straight line $y = 3x - 2$.

Note: $y = 3x - 2$ is the equation of the tangent to the curve $y = x^3$ at the point with coordinates (1, 1)

iv If $a - 2 = -3,\ a = -1,\ b = 5$ and $c = -3$

Q is the point of intersection of $y = -x^2 + 5x - 3$ and $y = x^3$

13



a coordinates of $A\left(t, \dfrac{t}{2}\right)$

coordinates of $B(t, t(4 - t))$

Length of $AB = Z = t(4 - t) - \dfrac{t}{2} = \dfrac{1}{2}(8t - 2t^2 - t) = \dfrac{1}{2}(7t - 2t^2)$

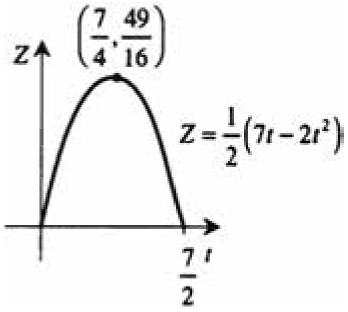
b For the intercepts consider:

$\dfrac{1}{2}\left(7t - 2t^2\right) = 0$

$\therefore\ t(7 - 2t) = 0$

$\therefore\quad t = 0 \text{ or } t = \dfrac{7}{2}$

Note: $\left(\dfrac{7}{2}, \dfrac{7}{4}\right)$ is the point of intersection of $y = \dfrac{x}{2}$ and $y = x(4 - x)$



c The maximum value of $Z = \dfrac{49}{16}$ and this occurs when $t = \dfrac{7}{4}$.

14 **a** Let X be the number of boys.

X is the random variable of a Binomial distribution.

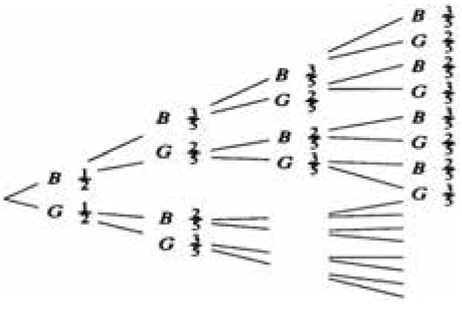
i $\Pr(X = 2) = \dbinom{4}{2}(0.5)^2(0.5)^2 \quad = \dfrac{3}{8}$

ii $\Pr(X = 1 \mid X \geq 1) = ?$

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X = 0)\\ &= 1 - (0.5)^4\\ &= \frac{15}{16}\end{aligned}$$

$$\Pr(X = 1 \mid X \geq 1) = \frac{\Pr(X = 1)}{\Pr(X \geq 1)} = \frac{\binom{4}{1}(0.5)^4}{\frac{15}{16}} = 4 \times \frac{1}{16} \times \frac{16}{15} = \frac{4}{15}$$

b *Child* 1 *Child* 2 *Child* 3 *Child* 4



i $\Pr(\text{all boys}) = \dfrac{1}{2} \times \dfrac{3}{5} \times \dfrac{3}{5} \times \dfrac{3}{5} = \dfrac{27}{250}$

$$\text{Pr (all girls)} = \frac{1}{2} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{25} = \frac{27}{250}$$

$$\therefore \quad \text{Pr (same sex)} = \frac{27}{250} + \frac{27}{250} = \frac{27}{125}$$

ii $\text{Pr}(BGBG) = \frac{1}{2} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{250} = \frac{4}{125}$

$\text{Pr}(GBGB) = \frac{1}{2} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{250} = \frac{4}{125}$

$\therefore$ Pr(no two consecutive children will be of the same sex) $= \frac{8}{125}$

iii Two males and two females. The possible combinations are *BBGG GGBB*

BGBG Note: No. of ways of arranging $= \frac{4!}{2!\,2!} = \frac{24}{4} = 6$

GBGB

BGGB

GBBG

$\text{Pr}(BBGG) = \frac{1}{2}\times\frac{3}{5}\times\frac{2}{5}\times\frac{3}{5} = \frac{18}{250}$

$\text{Pr}(GGBB) = \frac{18}{250}$

$\text{Pr}(BGBG) = \frac{4}{125}$ (see part **ii**)

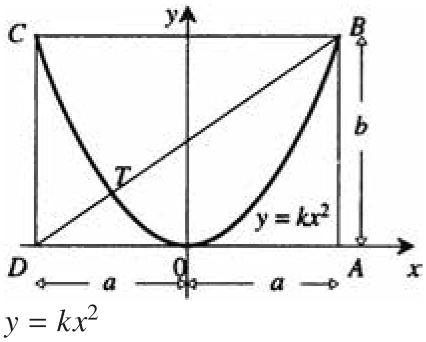
$\text{Pr}(GBGB) = \frac{4}{125}$ (see part **ii**)

$\text{Pr}(BGGB) = \frac{1}{2} \times \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{12}{250}$

$\text{Pr}(GBBG) = \frac{12}{250}$

$$\therefore \text{Pr(two males and two females)} = \frac{18 + 18 + 8 + 8 + 12 + 12}{250} = \frac{76}{250} = \frac{38}{125}$$

15 a



$y = kx^2$

$\therefore \quad b = ka^2$

$\therefore \quad k = \dfrac{b}{a^2}$

b **i** Gradient of $DB = \dfrac{b}{2a}$ and it passes through $(-a,\ 0)$

$\therefore y - 0 = \dfrac{b}{2a}(x + a)$

i.e. $y = \dfrac{b}{2a}x + \dfrac{b}{2}$

ii crosses $y = \dfrac{b}{a^2}x^2$

where $\dfrac{b}{2a}x + \dfrac{b}{2} = \dfrac{b}{a^2}x^2$

Multiply both sides by $2a^2$

$bax + ba^2 = 2bx^2$

i.e. $2bx^2 - bax - ba^2 = 0$

which implies

$2x^2 - ax - a^2 = 0$

$\therefore (2x + a)(x - a) = 0$

$\therefore x = -\dfrac{a}{2}$ or $x = a$

$\therefore$ at $T \qquad x = -\dfrac{a}{2}$

and $y = \dfrac{b}{a^2}\left(\dfrac{-a}{2}\right)^2 = \dfrac{b}{4}$

$\therefore$ coordinates of T are $\left(-\dfrac{a}{2},\ \dfrac{b}{4}\right)$

c Area $= \int_{-a}^{a} b - \dfrac{b}{a^2}x^2\,dx$

$= 2b\int_0^a 1 - \dfrac{x^2}{a^2}\,dx$ (by symmetry)

$= 2b\left[x - \dfrac{x^3}{3a^2}\right]_0^a$

$= 2b\left[a - \dfrac{a^3}{3a^2}\right]$

$= 2b\left[a - \dfrac{a}{3}\right]$

$= \dfrac{4}{3}\,ab$

d $S_1 = \int_{\frac{-a}{2}}^{a} \frac{b}{2a}x + \frac{b}{2} - \left(\frac{b}{a^2}x^2\right) dx$

$$= b\int_{\frac{-a}{2}}^{a} \frac{x}{2a} + \frac{1}{2} - \frac{x^2}{a^2}\,dx$$

$$= b\left[\frac{x^2}{4a} + \frac{x}{2} - \frac{x^3}{3a^2}\right]_{-\frac{a}{2}}^{a}$$

$$= b\left[\left(\frac{a^2}{4a} + \frac{a}{2} - \frac{a^3}{3a^2}\right) - \left(\frac{a^2}{4} \times \frac{1}{4a} - \frac{a}{4} + \frac{a^3}{8} \times \frac{1}{3a^2}\right)\right]$$

$$= b\left[\frac{a}{4} + \frac{a}{2} - \frac{a}{3} - \left(\frac{a}{16} - \frac{a}{4} + \frac{a}{24}\right)\right]$$

$$= \frac{ba}{48}[12 + 24 - 16 - (3 - 12 + 2)]$$

$$= \frac{ba}{8}[20 + 7]$$

$$= \frac{27ba}{48} = \frac{9ba}{16}$$

Now $S_2 = \frac{4}{3}ab - \frac{9ba}{16}$ from **c**

$$= \frac{(64 - 27)ba}{48} = \frac{37ba}{48}$$

$\therefore$ ratio $S_1 : S_2 = 27{:}37$

16 **a** Let X be the thickness of the washer

Let Y be the diameter of the hole

For X: $\mu = 0.25, \sigma = 0.002$

For Y: $\mu = 0.5, \sigma = 0.05$

i $\Pr(X < 0.253)$

$$= \Pr\left(z < \frac{0.253 - 0.25}{0.002}\right)$$

$$= \Pr\left(Z < \frac{3}{2}\right)$$

$$= 0.9332$$

ii $\Pr(X < 0.247)$

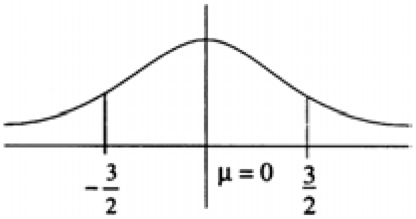
$$= \Pr\left(Z < \frac{0.247 - 0.25}{0.002}\right)$$

$$= \Pr\left(Z < -\frac{3}{2}\right)$$

$$= 1 - \Pr\left(Z < -\frac{3}{2}\right)$$

$$= 1 - 0.9332$$

$$= 0.0668$$



iii $\Pr(Y > 0.56)$

$$= \Pr\left(Z > \frac{0.56 - 0.5}{0.05}\right)$$

$$= \Pr\left(Z > -\frac{6}{5}\right)$$

$$= \Pr(Z > 1.2)$$

$$= 1 - \Pr(Z < 1.2)$$

$$= 0.1151$$

iv $\Pr(Y < 0.44)$

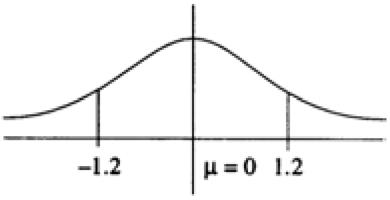
$$= \Pr\left(Z < \frac{0.44 - 0.5}{0.05}\right)$$

$$= \Pr\left(Z < -\frac{0.06}{0.05}\right)$$

$$= \Pr(Z < -1.2)$$

$$= \Pr(Z > 1.2)$$

$$= 0.1151$$



b **i** Let A be the event $0.247 < X \leq 0.253$

Let B the event $0.44 \leq Y \leq 0.56$

$\Pr(A) = \Pr(X \leq 0.253) - \Pr(X \leq 0.247)$

$= 0.9332 - 0.0668$

$= 0.8664$

$\Pr(B) = \Pr(Y \leq 0.56) - \Pr(Y \leq 0.44)$

$= 0.8849 - 0.1151$

$= 0.7698$

$\Pr(A \cap B) = \Pr(A)\ \Pr(B)$ X and Y are independent and therefore

$= 0.8664 \times 0.7698$ A and B are independent events.

$= 0.6670$

∴ Probability of rejecting a washer is 0.333.

∴ 33.3% of washers are rejected.

ii $\Pr(A) = 0.8664$

∴ expected number of washers of acceptable thickness in a batch of 1000 is 866.4.

iii $\Pr(A \cap B') = \Pr(A)\ \Pr(B')$

$= 0.8664 \times (1 - 0.7698)$

$= 0.8664 \times 0.2302$

$= 0.1994$

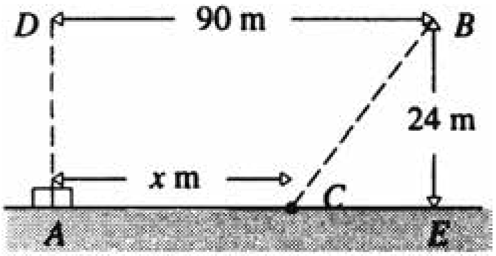
∴ Expected number with acceptable thickness but not acceptable diameter is 199.4.

17 Let $AC = x$

Then $CE = 90 - x$

and $CB = \sqrt{(90 - x)^2 + 24^2}$

∴ total cost $C = 100(90 - x) + 200(90 - x)^2 + 576)^{\frac{1}{2}}$



$$\frac{dC}{dx} = -100 + 200\left[\frac{1}{2} \times ((90 - x)^2 + 576)^{-\frac{1}{2}} \times -2(90 - x)\right]$$

$$\frac{dC}{dx} = 0 \text{ implies } \quad 100 = \frac{200(x - 90)}{((90 - x)^2 + 576)^{\frac{1}{2}}}$$

which implies $[(90 - x)^2 + 576]^{\frac{1}{2}} = 2(x - 90)$

$\therefore [(90 - x)^2 + 576] = 4(x - 90)^2$

$\therefore 3(x - 90)^2 = 576$

$\therefore x - 90 = \pm\frac{24}{\sqrt{3}}$

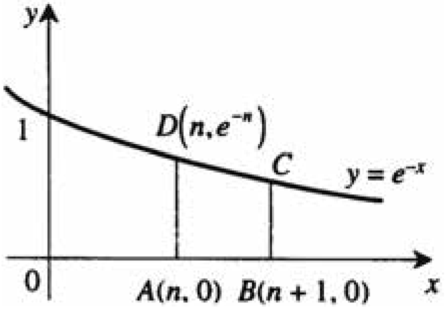
$\therefore x = 90 \pm 8\sqrt{3}$

as $0 \leq x \leq 90$

$x = 90 - 8\sqrt{3}$, $[x = 90 - 8\sqrt{3},\ as\ 0 \leq x \leq 9.]$

$\frac{dC}{dx} > 0$ when $x > 90 - 8\sqrt{3}$ and $\frac{dC}{dx} < 0$ when $x < 90 - 8\sqrt{3}$

$\therefore$ a minimum when $x = 90 - 8\sqrt{3} \approx 76.1436$ m



18 a i $\frac{dy}{dx} = -e^{-x}$

When $x = n$, $\frac{dy}{dx} = -e^{-n}$

$\therefore$ equation of tangent is $y - e^{-n} = -e^{-n}(x - n)$

$\therefore y = -e^{-n}x + e^{-n}n + e^{-n}$

ii When $y = 0$

$\therefore e^{-n}x = e^{-n}n + e^{-n}$

$x = n + 1(e^{-n} \neq 0)$

The line *DB* is a segment of the tangent at *D*.

b i

$$\int_n^{n+1} e^{-x}\,dx = [-e^{-x}]_n^{n+1}$$

$$= -(e^{-(n+1)} - e^{-n})$$

$$= -e^{-n}(e^{-1} - 1)$$

$\therefore$ area of region $ABCD = \frac{1}{e^n}\left(1 - \frac{1}{e}\right)$

$\therefore$ The area under the curve $y = e^{-x}$ between $x = n$ and $x = n + 1$ is $\frac{1}{e^n}\left(1 - \frac{1}{e}\right)$.

$$\therefore \text{ The area of the second part } = \frac{1}{e^n} - \frac{1}{e^{n+1}} - \frac{1}{2e^n}$$

$$= \frac{1}{2e^n} - \frac{1}{e^{n+1}}$$

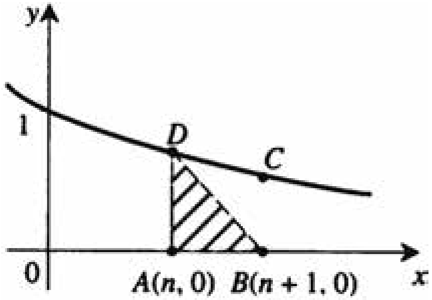
$$= \frac{1}{e^n}\left(\frac{1}{2} - \frac{1}{e}\right)$$

$$\therefore \text{ The ratio of the two parts } = \frac{1}{2e^n} : \frac{1}{e^n}\left(\frac{1}{2} - \frac{1}{e}\right)$$

$$= \frac{1}{2} : \frac{1}{2} - \frac{1}{e}$$

$$= e{:}e - 2$$

ii



$$\text{The shaded area } = \int_n^{n+1} -e^{-n}x + e^{-n}n + e^{-n}\,dx$$

$$= \left[-\frac{e^{-n}x^2}{2} + (e^{-n}n + e^{-n})x\right]_n^{n+1}$$

$$= \left(-\frac{e^{-n}(n+1)^2}{2} + e^{-n}(n+1)(n+1)\right) - \left(-\frac{e^{-n}n^2}{2} + e^{-n}(n+1)n\right)$$

$$= \frac{e^{-n}(n+1)^2}{2} - e^{-n}\left[-\frac{n^2}{2} + n^2 + n\right]$$

$$= \frac{e^{-n}}{2}[(n+1)^2 - (n^2 + 2n)]$$

$$= \frac{e^{-n}}{2}[n^2 + 2n + 1 - n^2 - 2n]$$

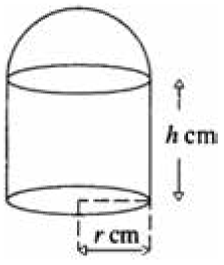
$$= \frac{e^{-n}}{2}$$

19 a i Volume of cylinder = $\pi r^2 h$

Volume of hemisphere = $\frac{2}{3}\pi r^3$

$\therefore$ total volume, $V = \frac{2}{3}\pi r^3 + \pi r^2 h$

$= \frac{\pi r^2}{3}(2r + 3h)$



ii Surface areas of capsule = surface area of hemisphere

$$\begin{aligned} &+ \text{curved surface of cylinder} + \text{base} \\ &= 2\pi r^2 + 2\pi rh + \pi r^2 \\ &= \pi r(3r + 2h) \end{aligned}$$

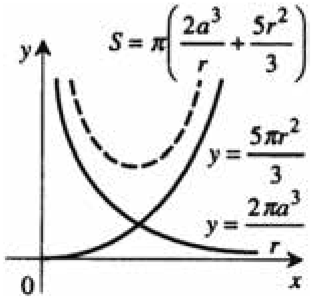
b **i** If

$$\begin{aligned} V &= \pi a^3 \\ \pi a^3 &= \frac{\pi r^2}{3}(2r + 3h) \\ 3\pi a^3 - 2\pi r^3 &= 3\pi r^2 h \\ \therefore h &= \frac{a^3}{r^2} - \frac{2}{3}r \\ &= \frac{3a^3 - 2r^3}{3r^2} \end{aligned}$$

ii

$$\begin{aligned} S &= \pi r(2h + 3r) \\ &= \pi r\left(\frac{2 \times (3a^3 - 2r^3)}{3r^2} + 3r\right) \\ &= \pi\left(\frac{2 \times (3a^3 - 2r^3)}{3r} + 3r^2\right) \\ &= \pi\left(\frac{2a^3}{r} - \frac{4r^2}{3} + 3r^2\right) \\ &= \pi\left(\frac{2a^3}{r} + \frac{5r^2}{3}\right) \end{aligned}$$

c **i**



ii

$$S = \pi\left(\frac{2a^3}{r} + \frac{5r^2}{3}\right)$$

$$\frac{dS}{dr} = \pi\left(-\frac{2a^3}{r^2} + \frac{10r}{3}\right)$$

$$\frac{dS}{dr} = 0 \text{ implies } \frac{2a^3}{r^2} = \frac{10r}{3}$$

$$\therefore r^3 = 0.6a^3$$
$$\therefore r = (\sqrt[3]{0.6})a$$
$$s_{\min} = \pi\left(\frac{2a^3}{\sqrt[3]{0.6a}} + \frac{5}{3}\left[\sqrt[3]{0.6a}\right]^2\right)$$
$$= \pi a^2\left(\frac{2}{\sqrt[3]{0.6}} + \frac{5}{3}(\sqrt[3]{0.6})^2\right)$$

20 **a** Let X be the cylinder diameter.

$$\Pr(3 - d < X < 3 + d) = 0.75$$
$$\therefore \quad \Pr\left(\frac{-d}{0.002} < Z < \frac{d}{0.002}\right) = 0.75$$
$$\therefore \quad 2\Pr\left(Z < \frac{d}{0.002}\right) - 1 = 0.75$$
$$\Pr\left(Z < \frac{d}{0.002}\right) = 0.875$$
$$\therefore \quad \frac{d}{0.002} = 1.15$$
$$d = 0.0023$$

b

q	$s - 1$	-1
$\Pr(Q = q)$	$\frac{3}{4}$	$\frac{1}{4}$

c

$$\mathrm{E}(Q) = \frac{3}{4}(s-1) - 1 \times \frac{1}{4} = \frac{3}{4}s - 1$$
$$\mathrm{E}(Q^2) = (s-1)^2 \times \frac{3}{4} + 1 \times \frac{1}{4}$$
$$\mathrm{Var}\,(Q) = (s-1)^2 \times \frac{3}{4} + \frac{1}{4} - \left(\frac{3s-4}{4}\right)^2$$
$$= \frac{3}{4}(s^2 - 2s + 1) + \frac{1}{4} - \frac{1}{16}(9s^2 - 24s + 16)$$
$$= \left(\frac{3}{4} - \frac{9}{16}\right)s^2 + \left(\frac{24}{16} - \frac{6}{4}\right)s + \frac{3}{4} + \frac{1}{4} - 1$$
$$= \left(\frac{3}{16}\right)s^2$$
$$\therefore \mathrm{sd}\,(Q) = \frac{\sqrt{3}}{4}s$$

21 Let X be the length of a worm.

$\mu = 20$ and $\sigma = 1.5$.

a $\Pr(X \geq 22) = \Pr\left(Z \geq \frac{22 - 20}{1.5}\right)$

$= \Pr\left(Z \geq \frac{2}{1.5}\right)$

$= \Pr(Z \geq 1.3333)$

$= 1 - \Pr(Z \leq 1.3333)$

$= 0.09121$

b $\Pr(19.5 \leq X \leq 20.5) = \Pr\left(\frac{19.5 - 20}{1.5} \leq Z \leq \frac{20.5 - 20}{1.5}\right)$

$= \Pr\left(-\frac{1}{3} \leq Z \leq \frac{1}{3}\right)$

$= \Pr(0.3333 \leq Z \leq 0.3333)$

$= 2\Pr(Z \leq 0.3333) - 1$

$= 2 \times 0.63056 - 1$

$= 0.2611$

c Let Y be the number of worms out of five of 20 cm in length, So Y has a binomial distribution with $n = 5$ and $p = 0.2611$.

$\Pr(Y = 2) = \binom{5}{2}(0.2612)^2\ (0.7392)^3$

$= 10 \times (0.2611)^2\ (0.7389)^3$

$= 10 \times 0.0682 \times 0.4039$

$= 0.275$

22 **a** $P = \frac{x^2}{90}(56 - x)\ x \in [1, 40]$

$= \frac{1}{90}(56x^2 - x^3)$

$\frac{dP}{dx} = \frac{1}{90}(112x - 3x^2)$

b **i**

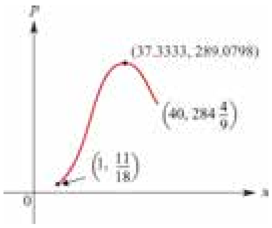
ii $P(1) = \frac{1}{90} \times (56 - 1)\ P(40) = \frac{40^2}{90} \times [56 - 40]$

$= \frac{11}{18} \qquad = 284\frac{4}{9}$

$\frac{dP}{dx} = 0 \Rightarrow \frac{1}{90}(112x - 3x^2) = 0$

$\therefore x(112 - 3x) = 0$

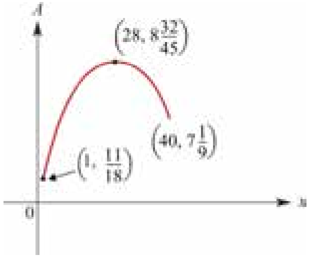
$\therefore x = 0$ or $x = 37\frac{1}{3}$



when $x = 37\frac{1}{3}$ $P(x) = \frac{351232}{1215} \approx 289.0798$

The maximum value of P is 289.0798 tonnes

c **i** $A = \frac{1}{x} \times \frac{x^2}{90}(56 - x) = \frac{x}{90}(56 - x)$

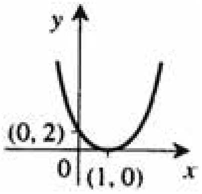


ii The maximum value of A is $8\,\frac{32}{45}$ tonnes/man, when $x = 28$.

23 $f(x) = (k + 2)x^2 + (6k - 4)x + 2$

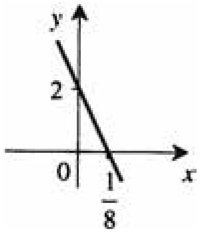
a **i** When $k = 0$

$$\begin{aligned} f(x) &= 2x^2 - 4x + 2 \\ &= 2(x^2 - 2x + 1) \\ &= 2(x - 1)^2 \end{aligned}$$



ii When $k = -2$

$f(x) = -16x + 2$



iii When $k = -4$

$$f(x) = -2x^2 - 28x + 2$$
$$= -2x[x^2 + 14x - 1]$$
$$= -2[x^2 + 14x + 49) - 1 - 49]$$
$$= -2[(x + 7)^2 - 50]$$

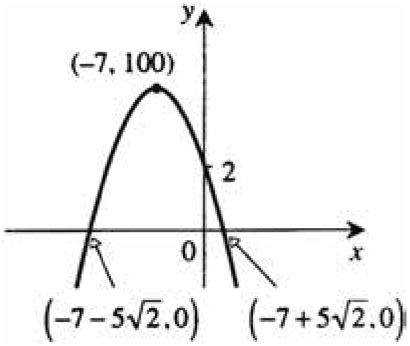
$f(0) = 2$; when $f(x) = 0$, $(x + 7)^2 = 50$

$$\therefore \quad x = -7 \pm \sqrt{50}$$
$$= -7 \pm 5\sqrt{2}$$

$$f(x) = -2((x + 7)^2) + 100$$

$\therefore$ axes intercepts are $(0, 2)$ $(-7 - 5\sqrt{2}, 0)$ and $(-7 + 5\sqrt{2}, 0)$

Vertex is at $(-7, 100)$



b $f'(x) = 2(k + 2)x + (6k - 4)$

$f'(x) = 0$ implies

$$x = \frac{4 - 6k}{2\,(k + 2)} = \frac{2 - 3k}{k + 2}$$

$$f\left(\frac{2 - 3k}{k + 2}\right) = (k + 2) \times \left(\frac{2 - 3k}{k + 2}\right)^2 + (6k - 4)\frac{(2 - 3k)}{k + 2} + 2$$

$$= \frac{(2 - 3k)^2}{k + 2} + 2\frac{(3k - 2)(2 - 3k)}{k + 2} + 2$$

$$= \frac{(2 - 3k)^2 - 2(2 - 3k)^2 + 2(k + 2)}{k + 2}$$

$$= \frac{-(2 - 3k)^2 + 2(k + 2)}{k + 2}$$

A check from previous results

When $k = 0$, $x = 1$, $f(1) = \dfrac{-4 + 4}{2} = 0$

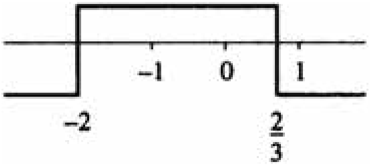
When $k = -2$, f is undefined

When $k = -4$, $x = \dfrac{14}{-2} = -7$

$$f(-7) = -\frac{-(2 + 12)^2 + 2(-2)}{-2}$$
$$= -\frac{-196 - 4}{-2}$$
$$= 100$$

i If $a > 0$, $\dfrac{2 - 3k}{k + 2} > 0$

Multiply both sides of the inequality by $(k + 2)^2$ $(2 - 3k)(k + 2) > 0$



A sign diagram reveals $\{k : a > 0\} = \left\{k : -2 < k < \frac{2}{3}\right\}$

ii $a = 0$ implies $k = \frac{2}{3}$

$\{k : a = 0\} = \left\{\frac{2}{3}\right\}$

iii If $b > 0$, $\frac{-(2-3k)^2 + 2(k+2)}{k+2} > 0$

Multiply both sides of inequality by $(k+2)^2$

$(-(2-3k)^2 + 2(k+2))(k+2) > 0$

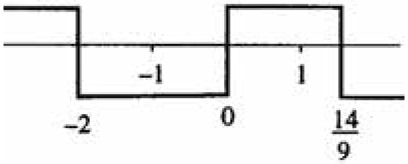
$(-(4-12k+9k^2) + 2k + 4)(k+2) > 0$

$\therefore (-4 + 12k - 9k^2 + 2k + 4)(k+2) > 0$

$(14k - 9k^2)(k+2) > 0$

$k(14-9k)(k+2) > 0$

Consider the sign diagram



$\therefore \{k : b > 0\} = \left\{k : 0 < k < \frac{14}{9}\right\} \cup \{k : k < -2\}$

iv $\{k : b < 0\} = \{k : -2 < k < 0\} \cup \left\{k : k > \frac{14}{9}\right\}$

c f has a local maximum when $k + 2 < 0$

i.e. when $k < -2$

d For $f(x) = (k+2)x^2 + (6k-4)x + 2$

$\Delta = (6k-4)^2 - 4(k+2)2$

$= 36k^2 - 48k + 16 - 8k - 16$

$= 36k^2 - 56k$

i $f(x)$ is a perfect square if $\Delta = 0$

i.e. $36k^2 - 56k = 0$

$4k(9k - 14) = 0$

$k = 0$ or $k = \frac{14}{9}$

ii If there are no solutions $\Delta < 0$

i.e. $4k(9k - 14) < 0$

$0 < k < \frac{14}{9}$

24 **a** $e^{2-2x} = 2e^{-x}$

$\therefore e^2 = 2e^x$

$\therefore e^x \frac{e^2}{2}$

$\therefore x = \log_e\left(\frac{e^2}{2}\right)$

$\therefore = \log_e (e^2) - \log_e$ ②

$= 2 - \log_e 2$

b **i** $y = e^{2-2x} - 2e^{-x}$

$\frac{dy}{dx} = -2e^{2-2x} + 2e^{-x}$

ii $\frac{dy}{dx} = 0$ implies $e^{2-2x} = e^{-x}$

$\therefore e^x = e^2$

$x = 2$

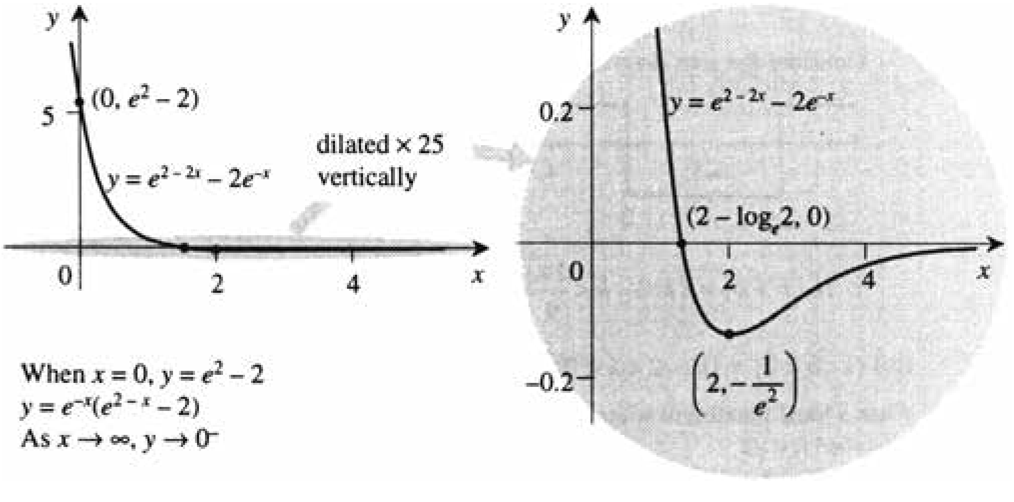
iii When $x = 2$, $y = e^{2-4} - 2e^{-2}$

$= e^{-2} - 2e^{-2}$

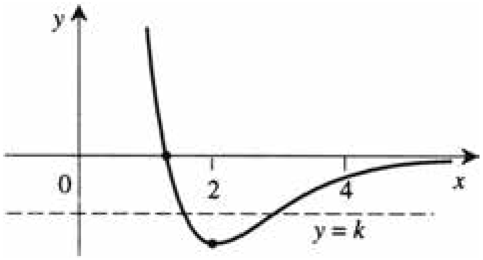
$= -\frac{1}{e^2}$

$\therefore$ coordinates of turning point $(2, -\frac{1}{e^2})$

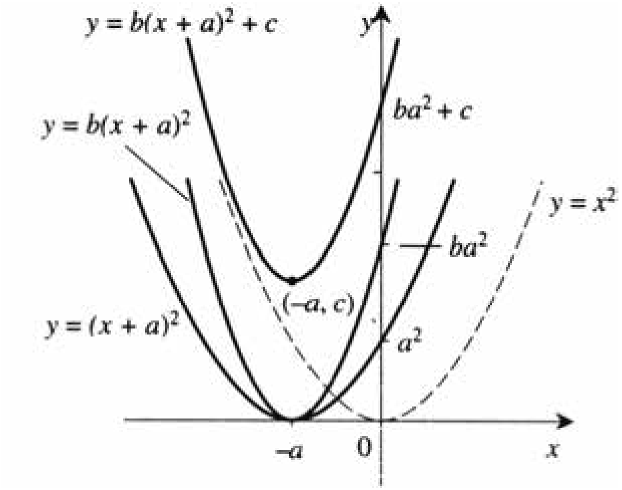
iv



c The equation $e^{2-2x} - 2e^{-x} = k$ has two distinct positive solutions for $k \in \left(-\frac{1}{e^2}, 0\right)$



25 a

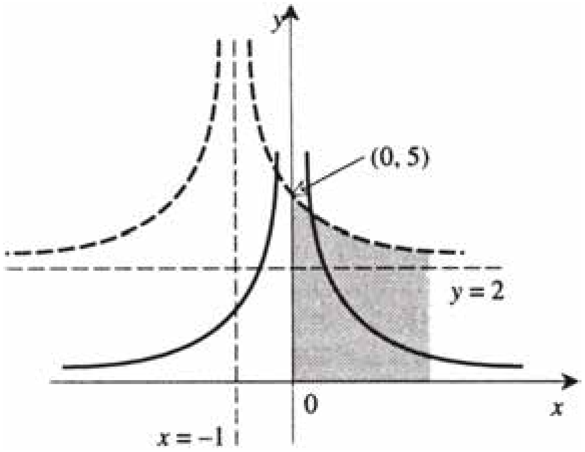


b $$\frac{3}{(x+1)^2} + 2 = \frac{3}{x^2 + 2x + 1} + 2$$
$$= \frac{3 + 2x^2 + 4x + 2}{x^2 + 2x + 1}$$
$$= \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}, x \neq -1$$

c
- A dilation of factor 3 from the x-axis
- A translation of 1 unit in the negative direction of the x-axis.
- A translation of 2 units in the positive direction of the y-axis

d $$\int_0^1 \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}\, dx = \int_0^1 \frac{3}{(x+1)^2} + 2\, dx$$
$$= \int_0^1 3(x+1)^{-2} + 2\, dx$$
$$= [-3(x+1)^{-1} + 2x]_0^1$$
$$= -\frac{3}{2} + 2 - (-3)$$
$$= \frac{7}{2}$$

e



26 **a** **i** $y = 50$

ii $y - 25 = \frac{25 - 0}{50 - 25}(x - 50)$

$\therefore y - 25 = x - 50$

$\therefore y = x - 25$

b $y = ax^2 + 4x + c$

$\therefore 50 = 25^2a + 100 + c$ ①

$25 = 50^2a + 200 + c$ ②

Subtract ② from ①

$25 = (25^2 - 50^2)a - 100$

$\frac{125}{25^2 - 50^2} = a$

$a = \frac{125}{75 \times -25} = \frac{-1}{15}$

Substitute in ①

$50 = 625 \times -\frac{1}{15} + 100 + c$

$\therefore c = -50 + 625 \times \frac{1}{15}$

$= -\frac{25}{3}$

$\therefore$ equation of parabola

$y = -\frac{1}{15}x^2 + 4x - \frac{25}{3}$

$= -\frac{1}{15}(x^2 - 60x + 125)$

c **i** area of rectangle $OABE$

$= 25 \times 50$

$= 1250$ square units

ii area of region EBC $$= \int_{25}^{50} -\frac{1}{15}(x^2 - 60x + 125) - (x - 25)\,dx$$

$$= -\frac{1}{15}\int_{25}^{50} x^2 - 45x - 250\,dx$$

$$= -\frac{1}{15}\left[\frac{x^3}{3} - \frac{45x^2}{2} - 250x\right]_{25}^{50}$$

$$= \frac{14375}{18}$$

iii total area $= \dfrac{36875}{18}$ square units

27 a Area of rectangle $PQST = (4\cos\theta + 4\cos\theta) \times 2$

$$= 16\cos\theta$$

Area of triangle $QRS = \dfrac{1}{2} \times 8\cos\theta \times 4\sin\theta$

$$= 16\cos\theta\ \sin\theta$$

$\therefore$ Area of metal plate $= 16(\cos\theta + \cos\theta\ \sin\theta), 0 < \theta < \dfrac{\pi}{2}$

b $\dfrac{dA}{d\theta} = 16[-\sin\theta + \sin\theta(-\sin\theta) + \cos\theta\cos\theta]$

$$= 16[-\sin\theta + \cos^2\theta - \sin^2\theta]$$

$$= 16[-\sin\theta + (1 - \sin^2\theta) - \sin^2\theta]$$

$$= 16[1 - \sin\theta - 2\sin^2\theta]$$

c $\dfrac{dA}{d\theta} = 0$ implies $= 16[1 - a - 2a^2] = 0$ (where $a = \sin\theta$)

$$\Leftrightarrow 2a^2 + a - 1 = 0$$

$$\Leftrightarrow (2a - 1)(a + 1) = 0$$

$$\Leftrightarrow a = \frac{1}{2} \text{ or } a = -1$$

$\therefore \sin\theta = \dfrac{1}{2}$ or $\sin\theta = -1$

$\theta = \dfrac{\pi}{6}$ since $O < Q < \dfrac{\pi}{2}$

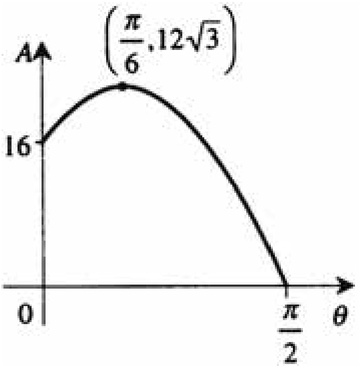
d $A(0) = 16$

$$A\left(\frac{\pi}{2}\right) = 0$$

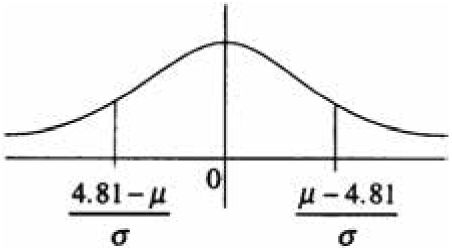
$$A\left(\frac{\pi}{6}\right) = 16\left(\cos\frac{\pi}{6} + \cos\frac{\pi}{6}\sin\frac{\pi}{6}\right)$$

$$= 16 \times \frac{\sqrt{3}}{2} \times \frac{3}{2}$$

$$= 12\sqrt{3}$$



28 **a** Let X be the length of the engine part.
The engine part must be between 4.81 cm and 5.20 cm.
$\Pr(X < 4.81) = 0.008$
$\Pr(X > 5.20) = 0.03$



$$\Pr\left(Z < \frac{4.81-\mu}{\sigma}\right) = 0.008$$

$$\therefore \Pr\left(Z < \frac{\mu-4.81}{\sigma}\right) = 0.992$$

$$\Pr\left(Z > \frac{5.20-\mu}{\sigma}\right) = 0.03$$

$$\therefore \Pr\left(Z < \frac{5.20-\mu}{\sigma}\right) = 0.97$$

$\therefore$ we have the equations

$$\frac{\mu-481}{\sigma} = 2.41 \text{ and } \frac{5.20-\mu}{\sigma} = 1.881$$

$\mu - 4.81 = 2.41\sigma$ ① and $5.20 - \mu = 1.881\sigma$ ②

Add ① and ②

$$5.20 - 4.81 = (2.41 + 1.881)\sigma$$

$$0.39 = 4.291\sigma$$

$0.0909 = \sigma$ (correct to four decimal places)

Substitute in ①

$$\mu - 4.81 = 2.41 \times 0.0909$$

$$\mu = 5.0290$$

b Let \$C be the cost to produce a part that meets the specifications. Then with probability 0.962, the cost is \$4; with probability 0.03, the part is priced at a cost of \$(4+2)= \$6 ;with probability 0.008, the part is rejected and the process begins again: so with probability 0.008 × 0.962 = 0.007697, a good part is made at a cost of

\$(4+4) = \$8, and with probability $0.008 \times 0.03 = 0.000\,24$, a good part is made at a cost of \$$(4+4+2)$ = \$10.
But with probability $(0.008)^2 = 0.000064$, it is rejected and the process repeats ad infinitum.

METHOD 1

C	4	6	8	10	12	> 12
$\Pr(C = c)$	0.962	0.03	0.001696	0.00024	0.0000616	insignificant

$\mathrm{E}(C) = 4 \times 0.962 + 6 \times 0.03 + 8 \times 0.007696 + 10 \times 0.00024 + 12 \times 0.0000616 =$ 4.092707

The expected cost of producing 100 parts is \$409.27.

29 **a** $\theta = 21$

$T = 21 + Ae^{-kt}$

When $t = 0,\ T = 100$

$\therefore 100 = 21 + A$

$\therefore A = 79$

$\therefore T = 21 + 79e^{-kt}$

When $t = 10, T = 84$

$\therefore 84 = 21 + 79e^{-10t}$

$$\frac{63}{79} = e^{-10k}$$

$$\therefore -10k = \log_e \frac{63}{79}$$

$$\therefore k = \frac{1}{10} \log_e \frac{79}{63} \approx 0.02$$

b $70 = 21 + 79e^{-kt}$

$$\therefore \frac{49}{79} = e^{-kt}$$

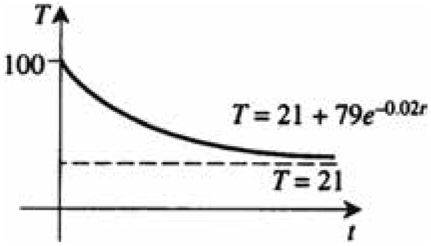
$$\therefore -kt = \log_e \frac{49}{79}$$

$$\therefore t = -\frac{1}{k} \log_e \frac{49}{79}$$

As $\dfrac{1}{k} = \dfrac{10}{\log_e \dfrac{79}{63}}, t = \dfrac{10}{\log_e \left(\dfrac{79}{63}\right)} \times \log_e\left(\dfrac{79}{49}\right) \approx 21.1$

The temperature of the kettle will be 70°C after 21.1 minutes i.e. at approximately 2.44 pm.

c



d When $t = 0,\ T = 100$

When $t = 10,\ T = 84$

$\therefore$ the average rate of change $= \dfrac{84 - 100}{10}$ °C min

$= \dfrac{-16}{10}$ °C/min

$= -1.6$ °C/min

e $\dfrac{dT}{dt} = -kAe^{-kt}$

i When $t = 6$

$$\frac{dT}{dt} = -k \times 79 \times e^{-6k}$$

$$\approx -2.0479\ ^\circ C/min$$

ii

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T - \theta)$$

$$= -k(60 - 21)$$

$$= -k(39)$$

$$= -39k$$

$$= -0.8826\ ^\circ\text{C/min}$$

30 Let X be the number of good components

a Probability of a batch being accepted

$$= \Pr(X = 4) + \Pr(X = 5) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5$$

$$= 5 \times \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5$$

$$= 6 \times \left(\frac{1}{2}\right)^5$$

$$= \frac{6}{32}$$

$$= \frac{3}{16} = 0.1875$$

b

$$A(p) = \Pr(X = 4) + \Pr(X = 5)$$

$$= \binom{5}{4}(1 - p)^4 p + (1 - p)^5$$

$$= (1 - p)^4[5p + (1 - p)]$$

$$= (1 - p)^4[4p + 1]$$

$\therefore \quad b = 4$

c

$$A'(p) = -4(1 - p)^3(1 + 4p) + 4(1 - p)^4$$

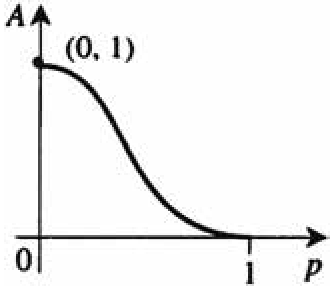
$$= (1 - p)^3[-4(1 + 4p) + 4(1 - p)]$$

$$= (1 - p)^3[-4 - 16p + 4 - 4p]$$

$$= (1 - p)^3[-20p]$$

$$= -20p(1 - p)^3$$

Note: no stationary point for $p \in (0, 1)$

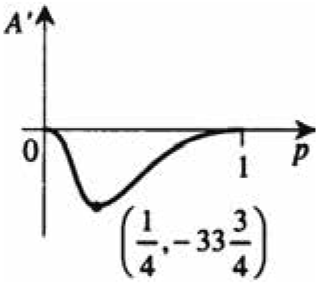


d **i** $A(p) = 0.95$: using the 'solve' command of a CAS calculate with $0 < P < 1$ gives $P \approx 0.076$

ii $A(p) = 0.05$: again using 'solve' gives $P \approx 0.657$

e **i** $A'(p) = -20p(1 - p)^3$

ii



iii

$$\begin{aligned} A''(p) &= -3(1-p)^2(-20p) - 20(1-p)^3 \\ &= -20[1-p]^2[-3p + (1-p)] \\ &= -20(1-p)^2(1-4p) \end{aligned}$$

$A''(p) = 0$ implies $p = 1$ or $p = \frac{1}{4}$ so $A'(p)$ is a minimum in $p = \frac{1}{4}$.

iv Most rapid rate of change of probabilities occurs when $p = \frac{1}{4}$.

31 $h(t) = (4.5 - 0.3t)^3$

a When $t = 0$, $h(0) = 4.5^3 = 91.125$ cm

b $h(t) \geq$ and $t \geq 0$

$\therefore (4.5 - 0.3t)^3 \geq 0$ and $t \geq 0$

equivalently $4.5 - 0.3t \geq$ and $t \geq 0$

$\therefore \frac{4.5}{0.3} \geq t$ and $t \geq 0$

$\therefore t \leq 15$ and $t \geq 0$

i.e. $t \in [0, 15]$

c $V = (0.8)^2(4.5 - 0.3t)^3$

$= 0.64(4.5 - 0.3t)^3$

d h is a 1 to 1 function

domain of h is $[0, 15]$

range of $h = [0, 91.125]$

Consider $x = (4.5 - 0.3y)^3$

$x^{\frac{1}{3}} = 4.5 - 0.3y$

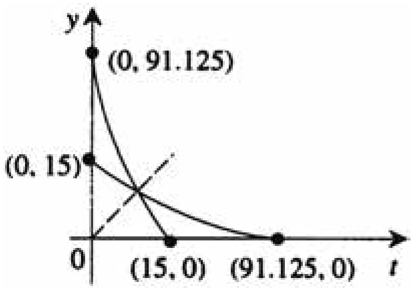
$\therefore 0.3y = 4.5 - x^{\frac{1}{3}}$

$\therefore y = 15 - \frac{10x^{\frac{1}{3}}}{3}$

$\therefore$ inverse function is $h^{-1}(t) = 15 - \dfrac{10t^{\frac{1}{3}}}{3}$

domain = $[0, 91.125]$

e



32 $\mu = 3$ mm

Let X be the diameter

$\Pr(X < 2.9) = 0.063$

$\Pr(X > 3.1) = 0.063$

a $\Pr\left(Z > \dfrac{3.1 - 3}{\sigma}\right) = 0.063$

$$\therefore \quad \Pr\left(Z > \frac{0.1}{\sigma}\right) = 0.063$$

$$\therefore \quad \Pr\left(Z \leq \frac{0.1}{\sigma}\right) = 0.937$$

$$\therefore \quad \frac{0.1}{\sigma} = 1.53$$

$$\sigma = \frac{0.1}{1.53}$$

$$= 0.06536$$

b Let Y be the number of ball bearings accepted out of 8.

The probability of rejection = $\Pr(X < 2.9) + \Pr(X > 3.1)$

$= 0.063 \times 2$

$= 0.126$

For the binomial distribution, $p = 0.126$ and $n = 8$

i

$$\begin{aligned} \Pr(Y \geq 1) &= 1 - \Pr(Y = 0) \\ &= 1 - (0.874)^8 \\ &= 1 - 0.34047 \\ &= 0.6595 \end{aligned}$$

ii $\Pr(Y = 2) = \begin{pmatrix} 8 \\ 2 \end{pmatrix}(0.126)^2(0.874)^6$

$= 0.198\ 14$

c **i** $\mu = 3.05, \sigma = 0.065\ 36$

$\Pr(X \leq 2.9) + \Pr(X \geq 3.1) = \Pr\left(Z \leq \dfrac{2.9 - 3.05}{0.065.36}\right) + 1 - \Pr\left(Z \leq \dfrac{3.1 - 305}{0.065.36}\right)$

$= \Pr(Z \leq -2.295) + 1 - \Pr(Z \leq 0.765)$

$= 2 - \Pr(Z \leq 2.295) - \Pr(Z \leq 0.765)$

$= 2 - 0.9891 - 0.7779$

$= 0.233$

So 23.3% will now filloutside the given range.

ii $\Pr(3.05 - c \leq X \leq 3.05 + c) = 0.9$

$\therefore \Pr\left(\dfrac{-c}{0.06\ 536} \leq Z \leq \dfrac{c}{0.06\ 536}\right) = 0.9$

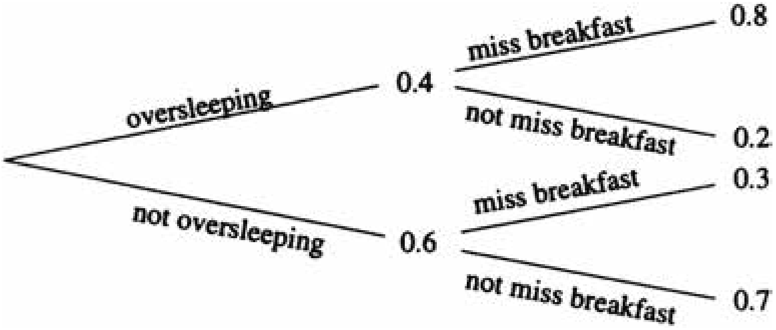
$\therefore 2\ \Pr\left(Z \leq \dfrac{c}{0.06\ 536}\right) - 1 = 0.9$

$\therefore \Pr\left(Z \leq \dfrac{c}{0.065\ 36}\right) = 0.95$

$\therefore \dfrac{c}{0.065\ 36} = 1.6449$

$\therefore c = 0.1075$

33



a From tree diagram

i Pr(oversleeping ∩ missing breakfast) = 0.4 × 0.8 = 0.32

ii Pr(not oversleeping ∩ missing breakfast) = 0.6 × 0.3 = 0.18

iii Pr(oversleeping ∩ missing breakfast) + Pr (not oversleeping ∩ missing breakfast) = 0.32 + 0.18 = 0.5

$$\Pr(\text{overslept} \mid \text{missing breakfast}) = \frac{\Pr(\text{overslept} \cap \text{missing breakfast})}{\Pr(\text{missing breakfast})}$$

b
$$= \frac{0.32}{0.5}$$
$$= 0.64$$

c This is a binomial distribution problem if it is assumed that a student's behaviour is independent of any other students behaviour.
Let X be the number of students who miss breakfast

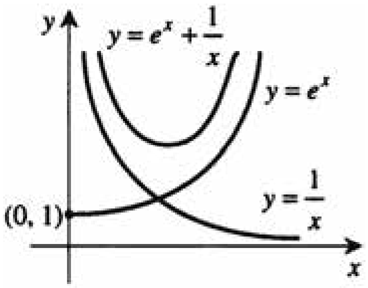
i $\Pr(X = 2) =^{10} C_2(0.5)^2(0.5)^8 = 0.043955$

ii $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - (0.5)^{10} = 0.999$

iii Probability of at least 8 not missing breakfast
$= \Pr(X \leq 2) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$
$= (0.5)^{10} + 10 \times (0.5)^{10} +^{10} C_2(0.5)^2(0.8)^8$
$= \dfrac{7}{128}$

34 a

b



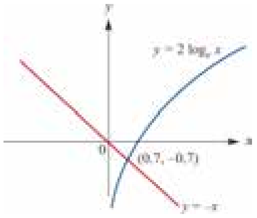
c $y = \dfrac{1}{x} + e^x$
$\dfrac{dy}{dx} = -\dfrac{1}{x^2} + e^x$

d **i** $\dfrac{dy}{dx} = 0 \Leftrightarrow -\dfrac{1}{x^2} + e^x = 0$
which implies $\dfrac{1}{x^2} = e^x$
$\therefore x^2 = e^{-x}$
$\therefore \log_e(x^2) = -x$
i.e. $2 \log_e x = -x$

ii As $x > 0,\ 2\log_e x = -x < 0$
$\therefore 2\ \log_e x < 0$

$\therefore x < 1$

$\therefore$ local minimum lies in the interval (0, 1)



iii

iv local minimum occurs when $x = 0.7$

$$\therefore y = \frac{1}{0.7} + e^{0.7}$$

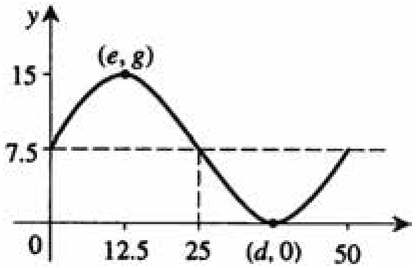
$= 3.4$ correct to one decimal place

i.e. coordinates local minimum are (0.7, 3.4)

35 a i From the diagram

amplitude = 7.5 $\therefore b = 7.5$

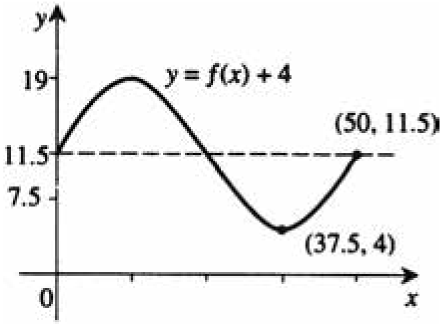
and centre is at $y = 7.5 \therefore a = 7.5$



$$\text{period} = 2\pi \div \frac{2\pi}{50} = 50$$

$\therefore m = 12.5,\ n = 15$ and $d = 37.5$

ii



b $10 = 7.5 + 7.5 \sin \frac{\left(2\pi x\right)}{50}$

$$\therefore \quad \frac{2.5}{7.5} = \sin\left(\frac{2\pi x}{50}\right)$$

$$\frac{1}{3} = \sin\left(\frac{2\pi x}{50}\right)$$

Let $\theta = \dfrac{2\pi x}{50}$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{3}\right) \text{ or } \pi - \sin^{-1}\left(\frac{1}{3}\right)$$

$$\therefore \quad x = \frac{50}{2\pi}\sin^{-1}\left(\frac{1}{3}\right) \text{ or } \frac{50}{2\pi}\left(\pi - \sin^{-1}\left(\frac{1}{3}\right)\right)$$

$= 2.704$ or 22.296

c $g(x) = 2f\left(\dfrac{x}{5}\right) = 2\left(7.5 + 7.5\sin\left(\dfrac{2\pi}{50}\left(\dfrac{x}{5}\right)\right)\right)$

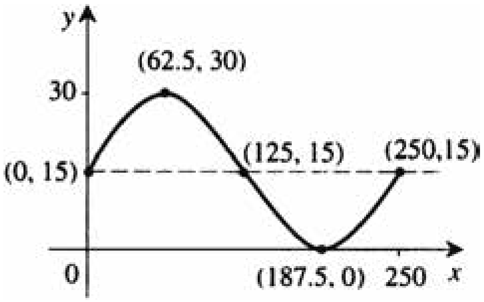
$$= 15 + 15\sin\left(\frac{2\pi x}{250}\right)$$

$$= 15 + 15\sin\left(\frac{\pi x}{125}\right)$$

$\therefore$ amplitude = 15

centre $y = 15$

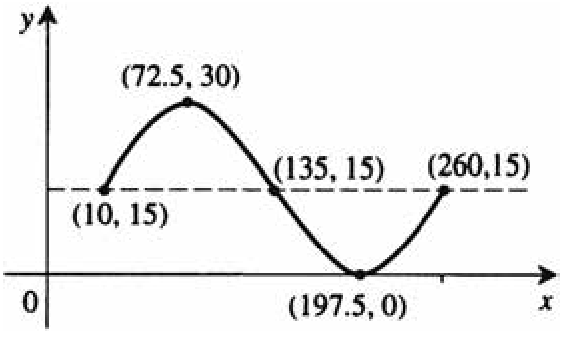
period $= 2\pi \div \dfrac{\pi}{125} = 250$



d **i** the new function has rule $h(x) = g(x - 10)$

$$= 15 + 15\sin\left(\frac{\pi}{125}(x - 10)\right)$$

ii



36 **a** $f(x) = \begin{cases} 0 & if\ x < 20 \\ k(5 - 2x) & 2 < x \le \dfrac{5}{2} \\ 0 & x > \dfrac{5}{2} \end{cases}$

$\int_2^{\frac{5}{2}} f(x)\,dx = [k(5x - x^2)]_2^{\frac{5}{2}} = \frac{k}{4}$

For f to be a probability density function $k = 4$.

b **i** $\mathrm{E}(X) = \int_2^{\frac{5}{2}} xf(x)\,dx = 4\int_2^{\frac{5}{2}} 5x - 2x^2\,dx = \frac{13}{6}$

ii Solve $\int_0^a f(x)\,dx = 0.5$ for a

$4(5a - a^2 - (10 - 4)) = 0.5$

$8(-a^2 + 5a - 6) = 1$

$-8a^2 + 40a - 49 = 0$

Therefore $a = \frac{10 - \sqrt{2}}{4}$ as $2 < \mathrm{a}\frac{5}{2}$

The median is $\frac{10 - \sqrt{2}}{4}$

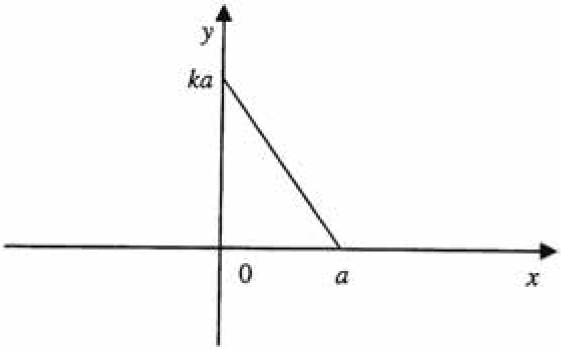
iii $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{1}{72}$

Therefore $\sigma = \frac{\sqrt{2}}{12}$

iv $\Pr(X < \mu - \sigma) = \Pr\left(x < \frac{13}{6} - \frac{\sqrt{2}}{12}\right)$

$= 0.1857$

37



a $\int_0^a f(x)\,dx = 1$

Therefore $\frac{1}{2}ka^2 = 1$

and $k = \frac{2}{a^2}$

$$\begin{aligned} \mathrm{E}(X) &= \int_0^a xf(x)\,dx \\ &= \frac{a^3k}{6} \\ &= \frac{a}{3} \end{aligned}$$

b $Var(\mathrm{X}) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

$$\begin{aligned} &= \int_0^a x^2 f(x)\,dx - \frac{a^2}{9} \\ &= \frac{a^4k}{12} - \frac{a^2}{9} \\ &= \frac{a^2}{18} \end{aligned}$$

c $\Pr(X > \mu + 2\sigma) = \Pr\left(x > \dfrac{a}{3} + \dfrac{2a}{3\sqrt{2}}\right) = \dfrac{6 - 4\sqrt{2}}{9}$

d Solve $\int_0^{1000} f(x)\,dx = 0.5$ for a

$a = 1000(\sqrt{2} + 2)$

38 $y = \dfrac{x}{10} - \log_e(x + 3)$, $x > -3$

a $\dfrac{dy}{dx} = \dfrac{1}{10} - \dfrac{1}{x+3}$

and $\dfrac{dy}{dx} = 0$ implies $x + 3 = 10$. Hence $x = 7$

b $\dfrac{dy}{dx} = \dfrac{1}{10} - \dfrac{1}{x+3} > \dfrac{1}{10}$ for $x > -3$

c The coordinates of M are $\left(7, \dfrac{7}{10} - \log_e(10)\right)$

Equation of line is $y - \left(\frac{7}{10} - \log_e(10)\right) = \frac{1}{10}(x - 7)$, i.e. $y = \frac{1}{10}x - \log_e 10$

d **i** "The line in *c* has gradient $\frac{1}{10}$ and hence if corners the x-axis at a point to the left of P (since the gradient of the curve $\pi < \frac{1}{10}$).

For the line, when $y = 0$,

$x = 10 \log_e 10$.

Hence the x-axis intercept at P is granter than $10 \log_e 10$."

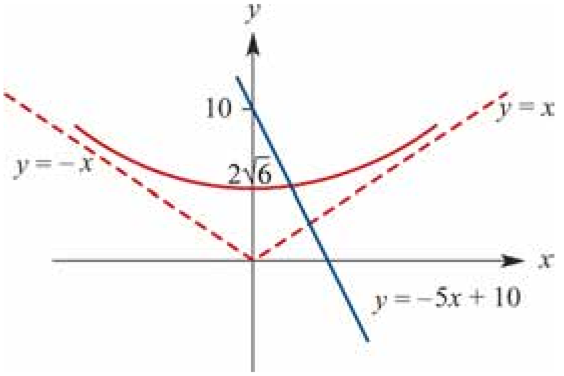
ii Using the 'solve' command of a CAS calculator shows that the intercept at P has x coordinate 36.852

39 **a** $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 + 24}}$

b $\dfrac{dy}{dx} = 0 \rightarrow x = 0$; then $y = \sqrt{24} = 2\sqrt{6}$.
So the coordinates of the local minimum are $(0, 2\sqrt{6})$.

c $f(-x) = \sqrt{(-x)^2 + 24} = \sqrt{x^2 + 24} = f(x)$, so the function is even.

d



e When $x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{5}$.
So the gradient of the normal at (1, 5) is −5.
Its equation is $y - 5 = -5(x - 1)$
$y = -5x + 10$:

f $\dfrac{dy}{dt} = 10$ at the point (5, 7):

$\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$ (chain rule)

$10 = \dfrac{x}{\sqrt{x^2 + 24}}\dfrac{dx}{dt}$

$10 = \dfrac{5}{7}\dfrac{dx}{dt}$ at (5, 7)

$\dfrac{dx}{dt} = 14$ units/second

g $\dfrac{d}{dx}\left(12\log_e\left|\sqrt{x^2 + 24} + x\right| + \dfrac{\sqrt[x]{x^2 + 24}}{2}\right)$

$$= 12 \times \frac{\dfrac{x}{\sqrt{x^2 + 24}} + 1}{\sqrt{x^2 + 24} + x} + \frac{\sqrt{x^2 + 24}}{2} + \frac{x^2}{\sqrt[2]{x^2 + 24}}$$

$$= \frac{12}{\sqrt{x^2 + 24}} + \frac{x^2 + 24}{2\sqrt{x^2 + 24}} + \frac{x^2}{2\sqrt{x^2 + 24}}$$

$$= \frac{x^2 + 24}{\sqrt{x^2 + 24}} = \sqrt{x^2 + 24} \text{ as required.}$$

h Area $= \int_2^5 \sqrt{x^2 + 24}\ dx$

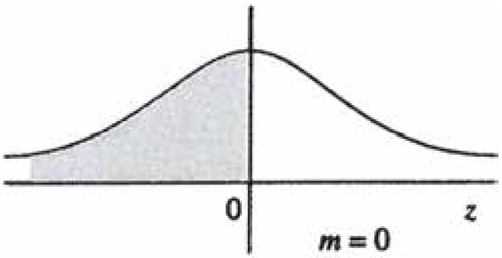
$$= \left[12 \log_e \left| \sqrt{x^2 + 24} + x \right| + \frac{x\sqrt{x^2 + 24}}{2}\right]_2^5$$

$$= \left(12 \log_e^{12} + \frac{35}{2}\right) - (12 \log_e(2\sqrt{7} + 2) + 2\sqrt{7})$$

$$= 12 \log_e\left(\frac{6}{\sqrt{7} + 1}\right) - 2\sqrt{7} + \frac{35}{2}$$

$$= 12 \log_e(\sqrt{7} - 1) - 2\sqrt{7} + \frac{35}{2}$$

40 a i



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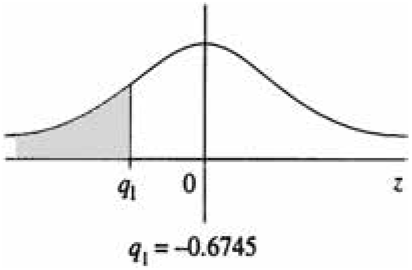
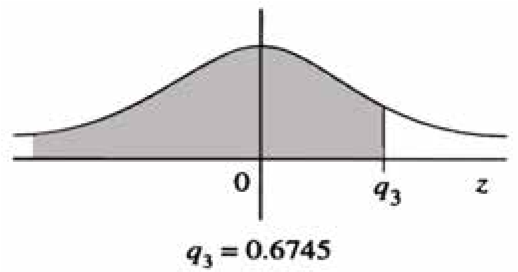


iii



iv interquartile range = 1.3490

v $\Pr(q_1 - 1.5 \times IQR < Z \leq q_3 + 1.5 \times IQR)$

$= \Pr(-0.6745 - 1.5 \times 1.3490 < Z < 0.6745 + 1.5 \times 1.3490)$

$= \Pr(-2.698 < Z < 2.698)$

$= 0.993$ or 99.3%

vi 0.7%

b **i** μ

ii $\mu - 0.6745\sigma$

iii $\mu + 0.6745\sigma$

iv 1.3490σ

v 0.993 or 99.3%

vi 0.7%

41 **a** $\int_0^1 f(x)\,dx = 1$

$\dfrac{k}{n+1} = 1$

Therefore $k = n + 1$

b $\mathrm{E}(X) = \int_0^1 xf(x)\,dx = \dfrac{n+1}{n+2}$

c $\mathrm{E}(X^2) = \int_0^1 x^2 f(x)\,dx = \dfrac{n+1}{n+3}$

$\mathrm{Var(X)} = \dfrac{n+1}{(n+2)^2(n+3)}$

d If m is the median, then

$k\int_0^m x^4\,dx = \dfrac{1}{2}$

$k\left[\dfrac{1}{n+1}x^{n+1}\right]_0^m = \dfrac{1}{2}$

$k\left(\dfrac{m^{n+1}}{n+1}\right) = \dfrac{1}{2}$

Since $k = n + 1$

$m^{n+1} = \dfrac{n+1}{2k} = \dfrac{1}{2}$

$m = \left(\dfrac{1}{2}\right)^{\frac{1}{n+1}}$

42 **a** **i** Gradient $AB = \dfrac{\dfrac{1}{b-1} - 1}{b-2}$

$$= \frac{2-b}{(b-1)(b-2)}$$

$$= -\frac{1}{b-1}$$

$$= \frac{1}{1-b}$$

ii $$g'(x) = -\frac{1}{(x-1)^2}$$

$$= \frac{1}{1-b}$$

if $(x-1)^2 = b-1$

$x - 1 = \sqrt{b-1}$ (positive square root since $x > 1$)

$$x = 1 + \sqrt{b-1}$$

b **i** $$\int_2^{e+1} \frac{1}{x-1}\,dx = [\log_e(x-1)]_2^{e+1}$$

$$= \log_e e - \log_e 1$$

$$= 1 - 0$$

$$= 1$$

ii $$\int_c^{1+e} \frac{1}{x-1}\,dx = 8$$

$$[\log_e (x-1)]_c^{1+e} = 8$$

$$\log_e e - log_e (c-1) = 8$$

$$1 - \log_e (c-1) = 8$$

$$\log_e (c-1) = -7$$

$$c - 1 = e^{-7}$$

$$c = 1 + e^{-7}$$

c **i** Area of trapezium $= \dfrac{1}{2}(b-2)\left(1 + \dfrac{1}{b-1}\right)$

$$= \frac{1}{2}(b-2)\left(\frac{b}{b-1}\right)$$

$$= \frac{b(b-2)}{2(b-1)}$$

ii $$\frac{b(b-2)}{2(b-1)} = 8$$

$$b^2 - 2b = 16b - 16$$

$$b^2 - 18b + 16 = 0$$

Solving by the formula or completing the square gives $b = 9 \pm \sqrt{65}$ but $b > 2$. so $b - 9 + \sqrt{65}$.

d $$\int_2^{mn+1} \frac{1}{x-1}\,dx + \int_2^{\frac{m}{n}+1} \frac{1}{x-1}\,dx = 2$$

Now the upper terminals must be greater than 1 since we can not integrate over the discontinuity at $x = 1$. Hence:

$$[\log_e(x-1)]_2^{mn+1} + [\log_e(x-1)]_2^{\frac{m}{n}+1} = 2$$

$\left(\log_e(mn) + \log_e(\frac{m}{n})\right) = 2$ (n positive so m positive)

$$\log_e\left[(mn) \times (\frac{m}{n})\right] = 2$$

$$\log_e m^2 = 2$$

$$m^2 = e^2$$

$$m = e \quad (m > 0)$$

43 a i Gradient $AB = \dfrac{\frac{1}{b^2} - 1}{b - 1}$

$$= \frac{1 - b^2}{b^2(b-1)}$$

$$= \frac{(1-b)(1+b)}{b^2(b-1)}$$

$$= -\frac{b+1}{b^2}$$

$$f'(x) = -\frac{2}{x^3}$$

$$= -\frac{b+1}{b^2}$$

ii if $x^3 = -\dfrac{2b^2}{(b+1)}$

$$x = \left(\frac{2b^2}{b+1}\right)^{\frac{1}{3}}$$

b i Area of trapeziums $= \dfrac{1}{2}(b-1)\left(\dfrac{1}{b^2} + 1\right)$

$$S(b) = \frac{(b^2 + 1)(b - 1)}{2b^2}$$

ii $\dfrac{(b^2 + 1)(b - 1)}{2b^2} = \dfrac{10}{9}$

$9(b^3 - b^2 + b - 1) = 20b^2$

$9b^3 - 29b^2 + 9b - 9 = 0$

Using the factor theorem or a CAS calculator shows that $b - 3$ is a factor of the cubic, giving $(b - 3)(9b^2 - 2b + 3) = 0$

The quadratic has no zeroes ($B^2 - 4AC < 0$) , so $b = 3$ is the only solution.

iii
$$\begin{aligned}\int_1^b f(x)\,dx &= \int_1^b \frac{1}{x^2}\,dx \\ &= \left[-\frac{1}{x}\right]_1^b \\ &= \frac{-1}{b} + 1 \\ &= 1 - \frac{1}{b}\end{aligned}$$

< 1 since $b > 1$ and so $0 < \dfrac{1}{b} < 1$

c
$$\begin{aligned}D(b) &= S(b) - \int_1^b f(x)\,dx \\ &= \frac{(b^2 + 1)(b - 1)}{2b^2} - \left(1 - \frac{1}{b}\right) \text{ from } b\ i \text{ and } b\ iii \\ &= \frac{(b^2 + 1)(b - 1)}{2b^2} - \left(\frac{b - 1}{b}\right) \qquad [t] \\ &= \frac{b - 1}{2b^2}(b^2 + 1 - 2b) \\ &= \frac{b - 1}{2b^2}(b - 1)^2 \\ &= \frac{(b - 1)^3}{2b^2}\end{aligned}$$

To show that the function is strictly increasing for $b > 1$, it is sufficient to show that $D'(b) > 0$ for $b > 1$.

$$D'(b) = \frac{\left(2b^2)(3(b-1)^2\right) - \left((b-1)^3\right)\left(4b\right)}{4b^4}$$
$$= \frac{3b(b-1)^2 - 2(b-1)^3}{2b^3}$$
$$= \frac{(b-1)^2(3b - 2(b-1))}{2b^3}$$
$$= \frac{(b-1)^2(b+2)}{2b^3}$$
> 0 for all $b > 1$

44 a $f'(x) = x^m(-ne^{-nx+n}) + mx^{m-1}e^{-nx+n}$

$$= x^{m-1}e^{-nx+n}(-nx + m)$$
$$= 0$$

if $x = 0$ or $x = \frac{m}{n}$.

So for the stationary point not at the origin, $x = \frac{m}{n}$ and then

$$f\left(\frac{m}{n}\right) = \left(\frac{m}{n}\right)^m e^{-m+n}$$

The Point with coordinates $\left(\frac{m}{n}, \left(\frac{m}{n}\right)^m e^{-m+n}\right)$ is a local maximum tuning point (by reference to the given graph or by checking the sign of the first derivative which goes from positive to negative through $x = \frac{m}{n}$).

b Find the equation of the tangent at a general point $x = a$ on the curve.

$x = a : f(a) = a^m e^{-an+n}$

$$f'(a) = a^{m-1}e^{-an+n}(-an + m)$$

using $y - y_1 = m(x - x_1)$, the equation of the tangent is

$y - a^m e^{-an+n} = a^{m-1}e^{-an+n}(-an + m)(x - a)$

The tangent passes through the origin, so $(0, 0)$ satisfies the equation.

$-a^m e^{-an+n} = a^{m-1}e^{-an+n}(-an + m)(-a)$

$$1 = (-an + m)\ (a\ m + 0,\ e^{-an+n} + 0)$$
$$an = m - 1$$
$$a = \frac{m-1}{n}$$

substitute to find the y-coordinate:

$$f(a) = f\left(\frac{m-1}{n}\right)$$
$$= \left(\frac{m-1}{n}\right)^m e^{n-m+1}$$

So the tangent at $\left(\frac{m-1}{n}, (\frac{m-1}{n})^m e^{n-m+1}\right)$ passes through the origin.
(Note: the tangent at $(0, 0)$ also passes though the origin!)

c **i** Using CAS calculator, we find that

$\int_0^\infty x^2 e^{-2x+2}\,dx = \frac{e^2}{4}$

So: $\frac{4}{e^2}\int_0^\infty x^2 e^{-2x+2}\,dx = 1$ and $k = \frac{4}{e^2}$

ii

$$\begin{aligned}\Pr(X < 1) &= \int_0^1 \frac{4}{e^2} x^2 e^{-2x+2}\,dx \\ &= (e^2 - 5)e^{-2} \quad \text{(using a CAS calculator)} \\ &= 1 - 5e^{-2} \\ &= 1 - \frac{5}{e^2}\end{aligned}$$

iii The made is the value for which f is a maximum. Use calculus to solve $f'(x) = 0$.

$$\begin{aligned}f'(x) &= x^2(-2e^{-2x+2}) + 2xe^{-2x+2}\,dx \\ &= 2xe^{-2x+2}(-x + 1) \\ &= 0\end{aligned}$$

if $x = 1$

So the mode is 1.

Alternatively, note that x^2e^{-2x+2} is the function from part **a** with m = 2 and n = 2
From That question, the x-coordination of the stationary point is $x = \frac{m}{n} = \frac{2}{2} = 1$ in this case.

45 **a** **i**

$$\begin{aligned}\int_0^\infty e^{-qx}\,dx &= \lim_{a\to\infty}\int_0^\infty e^{-qx}\,dx \\ &= \lim_{a\to\infty}\left[-\frac{1}{q}e^{-qa}\right]_0^a \\ &= \lim_{a\to\infty}\left(-\frac{1}{q}e^{-qa} + \frac{1}{q}\right) \\ &= 0 + \frac{1}{q} \\ &= \frac{1}{q}(\text{ since } e^{-qa} \to 0 \text{ as } a \to \infty)\end{aligned}$$

$$\begin{aligned}\text{Hence } \int_0^\infty ke^{-qx}\,dx &= \frac{k}{q} \\ &= 1 \text{ if } k = q\end{aligned}$$

ii $$\mathrm{E}(X) = \int_0^\infty x \times qe^{-qx}\,dx$$

$$= \frac{1}{q} \text{ (using a CAS calculator)}$$

iii $$\mathrm{E}\,(X^2) = \int_0^\infty x^2 \times qe^{-qx}\,dx$$

$$= \frac{2}{q^2}\text{(using a CAS calculator)}$$

$$\mathrm{var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

$$= \frac{2}{q^2} - \left(\frac{1}{q}\right)^2$$

$$= \frac{1}{q^2}$$

iv If $m = \frac{1}{2}\log_e(2)$, then

$$\int_0^m qe^{-qx}\,dx = [-e^{-qx}]_0^m$$

$$= -e^{-qm} + 1$$

$$= -e^{-\log_e^2} + 1$$

$$= -e^{-\log_e \frac{1}{2}} + 1$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}$$

So m is the medium.

Alternatively, solve $\int_0^m f(x)\,dx = \frac{1}{2}$ for m.

b $$\Pr\left(X > \frac{1}{q}\log_e(3) \mid X > \frac{1}{q}\log_e(2)\right) = \frac{\Pr\left(X > \frac{1}{q}\log_e(3)\right)}{\Pr\left(X > \frac{1}{q}\log_e(2)\right)}$$

Since the median is $\frac{1}{q}\log_e(2)$ from part **a iv**, the denominator is $\frac{1}{2}$.

$$\Pr\left(X > \frac{1}{q}\log_e 3\right) = 1 - \Pr(X \le \frac{1}{q}\log_e(3))$$

$$= 1 - \int_0^{\frac{1}{q}\log_e(3)} qe^{-qx}\,dx$$

$$= 1 - [-e^{-qx}]_0^{\frac{1}{q}\log_e(3)}$$

$$= 1 + e^{\log_e^3} - 1$$

$$= e^{\log_e \frac{1}{3}}$$

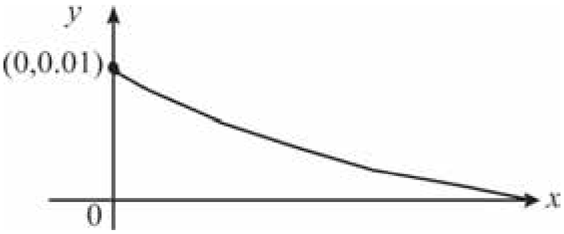
For the numerator: $= \frac{1}{3}$

So

$$\Pr(X > \frac{1}{q}\log_e(3) \mid X > \frac{1}{q}\log_e(2)) = \frac{\frac{1}{3}}{\frac{1}{2}}$$

$$= \frac{2}{3}$$

c **i** The graph of $y = f(x) = 0.01e^{-0.01x},\ x \ge 0$, is that of an exponential function with y-axis intercept $(0, 0.01)$ and horizontal asymptote $y = 0$ (the x-axis).



ii $\Pr(X > 100) = 1 - \Pr(X \le 100)$

$$= 1 - \int_0^{100} 0.01e^{-0.01x}\,dx$$

$$= 1 - \left[-e^{-0.01x}\right]_0^{100}$$

$$= 1 + e^{-1} - 1$$

$$= e^{-1} \approx 0.37$$

iii From part **a iv**, $m = \frac{1}{0.01}\log_e(2)$

$$= 100\log_e(2) \approx 69.31$$

46 **a** 0.527

b (0.4961, 0.5580)

c For a 95% CI, $M = 1.96 \times \sqrt{\frac{0.527 \times 0.473}{1000}} \approx 0.0309$ Half this width is 0.0155
Thus, we need to find a such that
$a \times \sqrt{\frac{0.527 \times 0.473}{1000}} = 0.0155$
$a = 0.981$
To find the level of confidence associated with $a = 0.981$ we use the normal cdf function.
Level of confidence = $\Pr(-0.981 < Z < 0.981) = 0.6734$
Ie, 67.34% confidence interval

d Twice this width is 0.0618 Thus, we need to find a such that Thus, we need to find a such that
$a \times \sqrt{\frac{0.527 \times 0.473}{1000}} = 0.0618$
$a = 3.914$
To find the level of confidence associated with $a = 3.914$ we use the normal cdf function. Level of confidence = $\Pr(-3.914 < Z < 3.914) = 0.9999$
ie, 99.99% confidence interval