Investigation 1 Unit 1: **Linear, Quadratic and Cubic functions**

Techniques to be used: polynomial functions,

Technology: CAS calculator, spreadsheet, Desmos

Formulation

Look at features of the graphs of a linear, quadratic or cubic function. There are many possibilities here. In the following some very specific functions are considered. Some suggestions are made here. The question is to ask general questions about linear, quadratic and cubic graphs. Some specific questions about cubic functions are made here to indicate the type of questions which can be asked.

Exploration

The general polynomial form of a cubic equation is $f(x) = ax^3 + bx^2 + cx + d$

The graphs of cubic functions vary. Some have two turning points, others have a point of inflection; some have one x-intercept, some have two x-intercepts and some have three.

1a Consider the graphs of the cubic functions of the form

$f(x) = ax^3 + bx^2 + 4x.$

i Find the values of b for which the graph of $y = f(x)$ will have only two two x-intercepts.

ii Find the values of b for which the graph of $y = f(x)$ will have three two x-intercepts.

b Consider the graphs of the cubic functions of the form

$f(x) = ax^3 + bx^2 + bx, \qquad b \neq 0.$

I Find the values of b for which the graph of $y = f(x)$ will have only two two x-intercepts.

ii Find the values of b for which the graph of $y = f(x)$ will have three two x-intercepts.

c Consider the graphs of the cubic functions of the form

$f(x) = ax^3 + bx^2 + cx, \quad b \neq 0$ and $c \neq 0$.

i Find a relationship between b and c for which the graph of $y = f(x)$ will have only two x-intercepts.

ii Find a relationship between b and c for which the graph of $y = f(x)$ will have three x-intercepts.

2 In this question we consider which cubic functions have graphs which are transformations of the graph of $y = x^3$

a Consider the graphs of the cubic functions of the form

$f(x) = x^3 + x^2 + cx + d, \quad d \neq 0$ and $c \neq 0$.

i If $x^3 + x^2 + cx + d = (x + h)^3 + k$ for all x, then find the values of c and h and find k in terms of d.

ii If $d = 1$, what is the value of k?

iii If $x^3 + x^2 + cx + d = (x + h)^3 + k$ for all x, and $f(1) = 1$, find the values of d and k.

b Consider the graphs of the cubic functions of the form

$f(x) = x^3 + bx^2 + x + 1, \quad b \neq 0$

Ask similar questions to those asked in **a.**

3 (Repeated factor) If $x^3 + 3x^2 + ax + b = (x + m)^2(x + n)$, $m \neq 0$ and $n \neq 0$ for all x, find the values of m, n and a, b.

Conclusions

Discuss and list the key results of your investigation. Comment on other directions to investigate. For example looking at transformations. Consider what happens with quartic functions.

Solution suggestions.

(a) Consider the graphs of the cubic functions of the form $f(x) = x^3 + bx^2 + 4x$

$x - intercepts \Rightarrow y = 0$

$0 = x^3 + bx^2 + 4x$
$0 = x(x^2 + bx + 4)$

$f(x)$ therefore has one intercept at $x = 0$

To find the number of other x – intercepts, consider the discriminant of the quadratic factor.

$\Delta = b^2 - 4(1)(4)$
$= b^2 - 16$

(i) One further x – intercept (two in total) if $\Delta = 0$

$\therefore b = \pm 4$

(ii) Two further x – intercepts (three in total) if $\Delta > 0$

$b < -4 \quad or \quad b > +4$

(b) Consider the graphs of the cubic functions of the form $f(x) = x^3 + bx^2 + bx, \quad b \neq 0$.

$x - intercepts \Rightarrow y = 0$

$0 = x^3 + bx^2 + bx$
$0 = x(x^2 + bx + b)$

$f(x)$ therefore has one intercept at $x = 0$

To find the number of other x – intercepts, consider the discriminant of the quadratic factor.

$\Delta = b^2 - 4(1)(b)$
$= b^2 - 4b$

(j) One further x – intercept (two in total) if $\Delta = 0$

$\therefore b = 4$ [note that $b \neq 0$]

(ii) Two further x – intercepts (three in total) if $\Delta > 0$

$\therefore b < 0 \quad or \quad b > 4$

2 ai $h = \frac{1}{3}, h = \frac{1}{3}, k = d - \frac{1}{27}$

ii $k = \frac{26}{27}$

iii $d = -\frac{4}{3}, k = -\frac{37}{27}$

(c) Consider the graphs of the cubic functions of the form.

$x - intercepts \Rightarrow y \;\; f(x) = x^3 + bx^2 + cx, \quad b, c \neq 0 = 0$

$0 = x^3 + bx^2 + cx$

$0 = x(x^2 + bx + c)$

$f(x)$ therefore has one intercept at $x = 0$

To find the number of other x – intercepts, consider the discriminant of the quadratic factor.

$\Delta = b^2 - 4(1)(c)$

$= b^2 - 4c$

(i) One further x – intercept (two in total) if $\Delta = 0$

$b^2 - 4c = 0$

$b^2 = 4c$

$b = \pm 2\sqrt{c}$

(ii) Two further x – intercepts (three in total) if $\Delta > 0$

$b^2 - 4c > 0$

If $c > 0$ two further solutions if $b < -2\sqrt{c} \quad or \quad b > 2\sqrt{c}$

If $c < 0$ two further solutions for $b \in R \setminus \{0\}$

Investigation 1 Unit 2: **Growth of fish**

Techniques to be used: exponential functions, polynomial functions, fitting data

Technology: CAS calculator, spreadsheet, Desmos

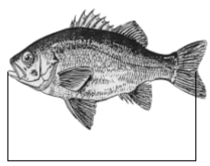
Formulation

When considering the growth and life of particular fish you can often find suitable functions which model the weight of the fish versus the length of the fish, the length of the fish versus the age of the fish and the number of fish which are born or introduced at the one particular time that survive for a given time. These models often involve exponential functions.
A fish farm stocks a pond with small fish. They of course grow with time. What is the best time to harvest the fish to maximise the weight of fish obtained.
In the starting problem different techniques are used. You make like to use models similar to these or develop your own. Different sets of data for length and weight of fish are given in the Excel sheet.

Exploration

1 Weight versus length

As fish grow in length, they increase in weight. The relationship between weight and length is not linear. The relationship between length (L) and weight (W) is often modelled by $W = aL^b$ where W is the weight of a fish in kilograms, L is the length of a fish in centimetre and a and b are constants. The constants a and b can be found for different species of fish. The length measured is the fork length as shown in the diagram. The following measurements are of small species of fish (Species 1).



Fork length

Species 1

Length (cm)	Weight (kg)
25	0.4
26	0.4
27	0.5
28	0.5
29	0.6
30	0.7
31	0.7
32	0.8
33	0.9
34	1
35	1

Length (cm)	Weight (kg)
37	1.2
38	1.3
39	1.5
40	1.6
41	1.7
42	1.8
43	1.9
44	2.1
45	2.2
46	2.4
47	2.5

a Find several different functions to fit this data which connects weight and length. There are other sets of data in the spreadsheet available in the same location as this document including more extensive data on Species 1.

b Discuss these models and see how they fit with the larger set of data for species.

c Try a piece-wise defined function for the larger set of data.

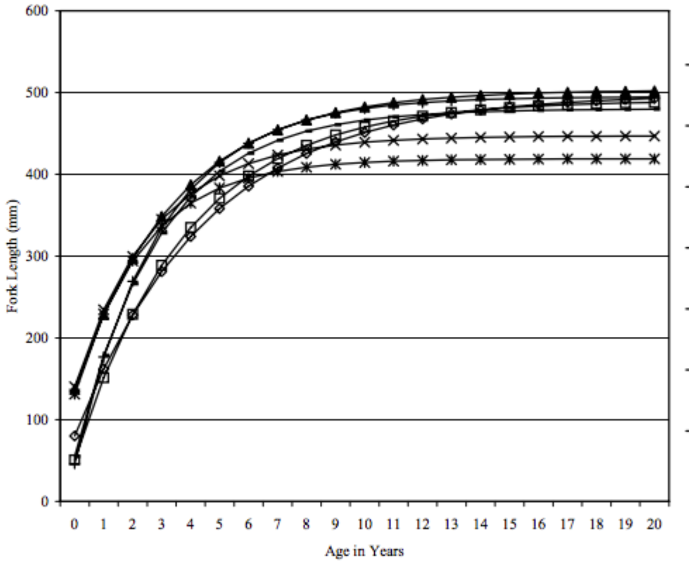
2 Length versus age

The following model for fish growth is called the **von Bertalanffy growth function** (**VBGF**).

$$L = L_\infty (1 - Ka^t)$$

where L is the length at age t(years), L_∞ is the theoretical maximum length, K is a positive constant and $0 < a < 1$. We can also put some restrictions on the model in terms of age as the model does not probably work for very young fish. We suggest $1 < t <$ **Max age.**
If we assume that the length increases according to this formula we can determine the values of K, L_∞ and a given sufficient information. Suppose that we know that the maximum length is 50 cm .
$L = 50\,(1 - Ka^t)$. If we know two pieces of data we can find K and a.

Shown below are the length against age curves of several species (in a couple of cases same species but different gender)



Find suitable growth functions for one of these graphs.

3 Weight of fish in an acqua-culture pond

The following formula gives a relationship between fish length (L) in centimetres and time elapsed (t) in years since small fish were placed in an acqua-culture centre pond.

$$L = 50(1 - 0.6 \times 0.8^t)$$

For every 1000 small fish of this species of fish placed in the pond, the number (n) still alive after t years is given by the formula $n = 1000 \times 0.6^{t}$.

Based on this information and using the result that $W = 0.05 \times 1.09^L$, find the best time to harvest all of the fish remaining in the pond in order to get the maximum weight (S) of live fish.

Conclusions

Discuss the key results of your investigation. Comment on other directions to investigate. Discuss the strength and weaknesses of your findings and raise questions that still remain if you do not believe that you have a complete solution or if you would like to modify your model.

Solution suggestions

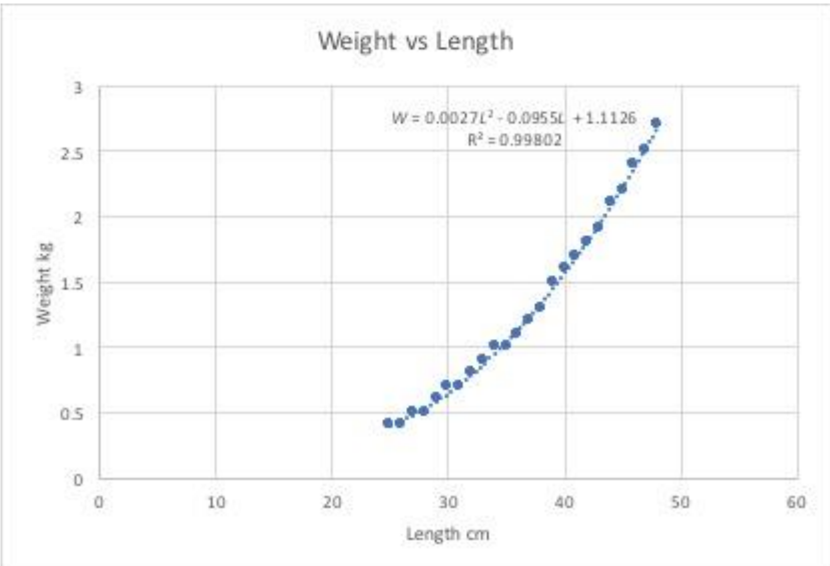
1 Weight versus length

Modelling with an exponential function

$W = 0.0521e^{0.084L} = 0.0521(e^{(0.084)})^L = 0.0521 \times 1.0876^L$

We find that this does not fit the data at all well for $L \geq 48$. For example a fish of length 66 cm is predicted to have a weight of 13.32 kg but our data is that it is actually 7.1 kg.
A suitable piece-wise defined function is
$\boldsymbol{W} = 0.0521 \times 1.0876^L$ for $25 \leq L < 48$ and $\boldsymbol{W} = 0.2389 \times 1.0526^L$ for $L \geq 48$
This predicts a value of 7.04 kg for the fish of length 66 kg.
If the exponential equation for the whole set of data is taken the equation is
$\boldsymbol{W} = 0.0975\times 1.0689^L$ and the predicted value is 7.97 kg. The piecewise defined function gives the best result. This can be confirmed with other values of L.

Modelling with a quadratic



$W = 0.0027L^2 - 0.0955L + 1.1126$

The quadratic model achieved from this small set of data gives quite good results in relation to our larger data set. For example from our equation a length of 66 cm predicts a weight of 6.57 kg.
The quadratic trendline $W = 0.0035L^2 - 0.1586L + 1.1126$ from the whole set of data gives very good results with $R^2 = 0.99945$

2 Length versus age
The graph which uses▲.
$L_\infty = 50$. When $t = 2$, $L = 30$ and when $t = 5$, $L = 40$

$$L = L_\infty(1 - Ka^t)$$

$30 = 50(1 - Ka^2)$ ….(1)
$40 = 50(1 - Ka^5)$ ….(2)
On simplification the equations become

$Ka^2 = \frac{2}{5}$ ….(1′)

$Ka^5 = \frac{1}{5}$ ….(2′)

Divide (2′) by (1′)

$a^3 = \frac{1}{2}$

$a = 0.7937\ldots$ and $K = 0.6349\ldots$
$L = 50(1 - 0.635 \times 0.794^t)$

3 Let S kilograms denote the total weight of fish in the pond.

$S = nw$, where n is the number of fish alive at time t years and w kilograms is the weight of fish after t months.

Given $n = 1000 \times 0.6^t$

Therefore $S = (1000 \times 0.96^t)W$

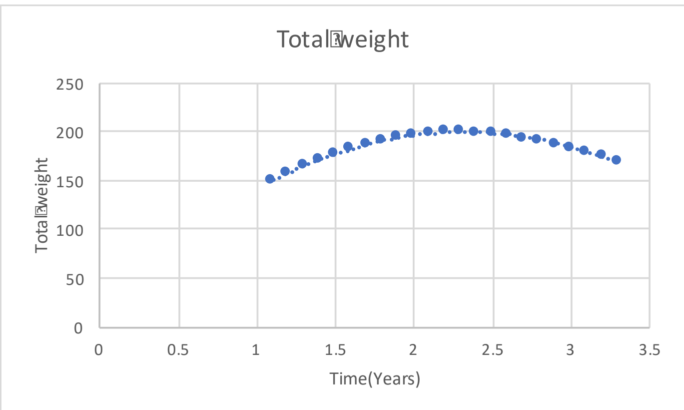
But the length, L cm, of a fish after t months is given by $L = 50(1 - 0.6^t)$ and

$W = 10000.05 \times 1.09^L$

With these formula we can form the spreadsheet shown.

time	length	weight	n	S=nw
1.1	21.4939935	0.3187241	466.516496	148.690052
1.2	22.9135865	0.36020132	435.275282	156.786731
1.3	24.262484	0.40460366	406.126198	164.320148
1.4	25.5442067	0.45185616	378.929142	171.221466
1.5	26.7620999	0.50185896	353.553391	177.433937
1.6	27.9193423	0.55448935	329.876978	182.913272
1.7	29.0189543	0.60960404	307.786103	187.627652
1.8	30.0638058	0.66704168	287.174589	191.55742
1.9	31.056624	0.72662557	267.943366	194.694501
2	32	0.78816644	250	197.04161
2.1	32.8963961	0.85146521	233.258248	198.611282
2.2	33.7481519	0.91631573	217.637641	199.424793
2.3	34.5574904	0.98250744	203.063099	199.511005
2.4	35.326524	1.04982783	189.464571	198.90518
2.5	36.05726	1.11806479	176.776695	197.647798
2.6	36.7516054	1.18700866	164.938489	195.783415
2.7	37.4113726	1.2564542	153.893052	193.359571
2.8	38.0382835	1.3262022	143.587294	190.425785
2.9	38.6339744	1.39606091	133.971683	187.03263
3	39.2	1.46584727	125	183.230909
3.1	39.7378377	1.53538786	116.629124	179.070942
3.2	40.2488911	1.60451967	108.81882	174.601938
3.3	40.7344942	1.67309064	101.53155	169.871485

The scatter diagram for *S* against *t* is shown.



The best time to harvest the fish appears to be after 2.3 years. A parabola can be fitted to this data with equation $S = -33.402t^2 + 154.99t + 19.732$. Maximum occurs when $t \approx 2.32$. The weakest part of this modelling is the length versus age. There are many factors that influence length not only age. It is hoped that in the artificial environment of the pond age is the dominant factor.

Investigation 2 Unit 1: **Navigable river**

Techniques to be used: Quadratic and linear equations other formula

Technology: Spreadsheets, CAS calculator or Desmos.

Formulation

A tourist company runs a tour boat on a navigable river. The tour company would like the return trip to last 4 hours. The modelling exercise here is to find how far up the river the boat can go and return in the required time. The trip goes up the river first, that is in the opposite direction to the flow of the river.

Choose variables and units for distance, speed of boat, speed of current in the river. Use the relationships between variables assuming constant speed of both current and the boat. Initially assume a speed of 12 km/h of the boat and 3 km/h for the current but these should be varied during your modelling process.

Exploration

With the initial figures calculate how far up the river the boat can go. Proceed to factor in other factors such as stops for sightseeing and recalculate the distance. Also

- Assume the current varies from 0 km/h to 5km/h. Form a quadratic expression for the distance travelled with the current speed being a variable. Investigate for different current speed.
- Sometimes the boat is forced to take different paths in the river on the outward and inward parts of the journey so that the current speed is a km/h more on the outward journey than for the inward journey. Again consider this for current speeds of 0 km/h to 5 km/h. This will involve a quadratic expression for the distance travelled.
- Now vary the speed that the boat can travel at in still water and investigate.

Conclusions

Exploring the strengths and limitations of model you have established. Are there ways in which you would change your solution or the questions asked. Modify your model further. There are many possibilities.

Discuss the key results. Discuss the limitations of the model you have established and your solution.

Solution suggestions.

These are some of the ideas that may be used.

A Let v km/h be the speed of the current and w km/h be the speed of the boat.

Resultant speed travelling up river is $(w - v)$ and travelling down the river $(w + v)$

Let d km be the distance travelled up the river.

Then $\frac{d}{w-v} + \frac{d}{w+v} = 4$

B Assume boat speed is 12 km/h .

- Then $\frac{d}{12-v} + \frac{d}{12+v} = 4$
- With common denominator $d = \frac{144 - v^2}{6}$.

C Assume that the current coming back is 2 km/h faster

- Then $\frac{d}{12-v} + \frac{d}{14+v} = 4$
- With common denominator $d = \frac{2(12-v)(14+v)}{13}$

D Assume now that the boat speed is variable and the current is 2 km/h both ways.

$\frac{d}{w-2} + \frac{d}{w+2} = 4.$

Then $2dw = 4(w^2 - 4)$ which can be rearranged to $4w^2 - 2dw - 16 = 0$. Solve for w in terms of d. Of course, you can also solve for d in terms of w.

Investigation 2 Unit 2: **Roller-coaster**

Techniques to be used: polynomial functions, differentiation, gradient – tan *a* is the value of the value of the gradient at a particular point where *a* is the angle with the horizontal.

Technology: CAS calculator, spreadsheet, Desmos

Formulation

You are to design a roller-coaster using the techniques of the starting problem

You are to use at least 4 curves that are continuous at all transition points and your roller

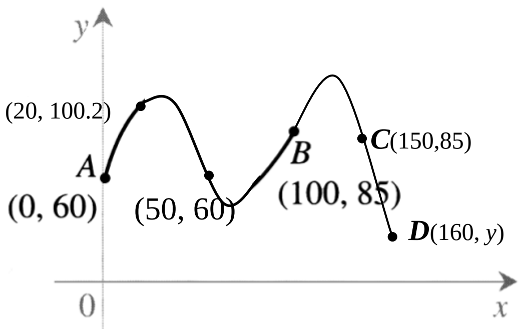
- Your roller coaster must begin and end at the same height.
- The roller-coaster never gets as high as the previous high point.
- No ascent or descent can be steeper than 60 degrees from the horizontal.

Find the functions that satisfy this, Measurements are to be in metres. It would be appropriate to describe your final curve as a piecewise defined.

Exploration

A starting problem 1-5

1 The graph from A to B is that of a cubic function with rule $y = ax^3 + bx^2 + cx + d$. The points (0, 60) (20, 100.2), (50, 60) and (100, 85) lie on this section.



a Find the values of a, b, c and d.

b Find the coordinates of the local maximum and local minimum.

c Find the gradient of the cubic when $x = 100$.

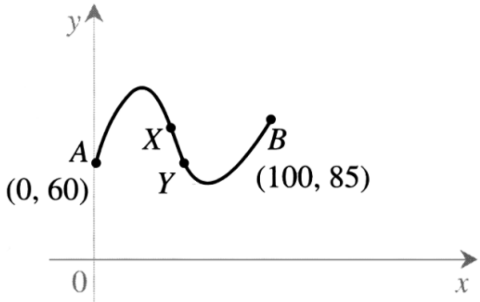
2 At point B, the curve of a quadratic function is joined smoothly so that the gradient of the quadratic and the cubic are the same at B. The quadratic has a rule $y = k(x - a)^2 + b$.

a Find the values of k, a and b.

b Find the gradient at C, which is also on the graph of the quadratic.

3 At the point C a straight line joins the curve of the quadratic. The join is smooth so that the gradient of the straight line and the quadratic are equal at C. Find the equation of the straight line and the value of the y-coordinate, y_1, at the endpoint of the straight line.

4 For the section of the graph of the cubic shown, approximate with graphs of quadratic functions for AX and YB and a linear function for XY. Smoothness at joins is important. Vary the positions of X and Y to improve the fit.



5 Try approximating other cubic functions with combinations of quadratic and linear graphs.

6 Design your roller coaster.

Conclusions

Discuss the key results of your investigation. Comment on other directions to investigate. Discuss the strength and weaknesses of your findings and raise questions that still remain if you do not believe that you have a complete solution or if you would like to modify your model.

Solution suggestions

1 **a** $a = 0.0009$, $b = –0.13$, $c = 4.25$, $d = 60$

b $x = \dfrac{50(26 - \sqrt{217})}{27}$ or $x = \dfrac{50(26 + \sqrt{217})}{27}$

Corresponding y values: $y = 100.26$, and 27.17 (2 decimal places)

c $\dfrac{21}{4}$

2 **a** Gradient of quadratic is $\dfrac{21}{4}$ at $x = 100$.The derivative of the quadratic is $2k(x – a)$.

Hence equations: $85 = k(100 – a)^2 + b$ and $2k(100 – a) = \dfrac{21}{4}$

The quadratic also goes through the point (150, 85).

This tells us the turning point is at $x = 125$.

Thus: $k = -\dfrac{21}{200}$, $a = 125$ and $b = 85 + \dfrac{21}{200}(-25)^2 = 150.625$

b Gradient is $-\dfrac{21}{4}$.

3 The gradient of the straight line is $-\dfrac{21}{4}$.

The equation is given by $y – 85 = -\dfrac{21}{4}(x – 150)$.

At the endpoint: $x = 160$, and $y = \dfrac{65}{2}$

4 Answers will vary.

Investigation 3 Unit 1: **Bisection method and fixed-point iteration**

Techniques to be used: Bisection method with polynomials and power functions

Technology: CAS calculator, Python or spreadsheet, Desmos

Formulation

We use a program through Python, TI basic or Casio basic (See appendices) to implement the bisection method and consider how many iterations it will take to reach the stated accuracy. We also use fixed pint iteration and compare the two methods.

Exploration

Bisection algorithm

We use the Bisection algorithm to solve $x^3 - 2 = 0$. See Section 6K of CSM Mathematical Methods units 1 & 2 for a discussion and a pseudocode expression of the algorithm. Programs in Python, TI basic and Casio basic are presented in the Appendices.

Excel provides the structure for the *if* statement which you can see in columns A and B and *fill down* enables the iterations to continue.

	A	B	C	G	H	I
1	1	2	=(A1+B1)/2	=A1^3-2	=B1^3-2	=C1^3-2
2	=IF(G1*I1<0, A1,C1)	=IF(G1*I1<0, C1,B1)	=(A2+B2)/2	=A2^3-2	=B2^3-2	=C2^3-2
3	=IF(G2*I2<0, A2,C2)	=IF(G2*I2<0, C2,B2)	=(A3+B3)/2	=A3^3-2	=B3^3-2	=C3^3-2

When implemented you obtain the opposite.

The columns are allocated as shown here

Column A: *a*
Column B : *b*
Column C: *m*
Column G: *f*(a)
Column H: *f*(*b*)
Clolumn I: *f*(*m*)

	A	B	C	G	H	I
1	1	2	1.5	-1	6	1.375
2	1	1.5	1.25	-1	1.375	-0.046875
3	1.25	1.5	1.375	-0.046875	1.375	0.59960938
4	1.25	1.375	1.3125	-0.046875	0.59960938	0.26098633
5	1.25	1.3125	1.28125	-0.046875	0.26098633	0.103302
6	1.25	1.28125	1.265625	-0.046875	0.103302	0.02728653
7	1.25	1.265625	1.2578125	-0.046875	0.02728653	-0.0100245
8	1.2578125	1.265625	1.26171875	-0.0100245	0.02728653	0.00857323
9	1.2578125	1.26171875	1.25976563	-0.0100245	0.00857323	-0.0007401
10	1.259765625	1.26171875	1.26074219	-0.0007401	0.00857323	0.00391297
11	1.259765625	1.260742188	1.26025391	-0.0007401	0.00391297	0.00158555
12	1.259765625	1.260253906	1.26000977	-0.0007401	0.00158555	0.00042251
13	1.259765625	1.260009766	1.2598877	-0.0007401	0.00042251	-0.0001588
14	1.259887695	1.260009766	1.25994873	-0.0001588	0.00042251	0.00013182
15	1.259887695	1.25994873	1.25991821	-0.0001588	0.00013182	-1.351E-05
16	1.259918213	1.25994873	1.25993347	-1.351E-05	0.00013182	5.9156E-05
17	1.259918213	1.259933472	1.25992584	-1.351E-05	5.9156E-05	2.2822E-05
18	1.259918213	1.259925842	1.25992203	-1.351E-05	2.2822E-05	4.656E-06

Use any technology in the following:

Implement the bisection algorithm for different functions. Examples should include examples of both strictly increasing and strictly decreasing in the interval $[a, b]$.

How many steps does it take for each of these to reach an error of less than 0.0001?

Number of iterations for a given accuracy

You can get a good idea of this by considering the width of the intervals $[a, b]$ at each stage. We use the example of $x^3 - 2 = 0$

Iteration (i)	a	b	Width of $[a, b]$
1	1	2	1
2	1	1.5	0.5
3	1.25	1.5	0.25
4	1.25	1.375	0.125

You can see that the **maximum error** halves at each iteration. We have started with an interval of width 1 which makes it easy to see the pattern.

At iteration 1 the maximum error is $\left(\frac{1}{2}\right)^0 = 1$

At iteration 2 the maximum error is $\left(\frac{1}{2}\right)^1 = \frac{1}{2}$

At iteration 3 the maximum error is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

At iteration 4 the maximum error is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

At iteration n the maximum error is $\left(\frac{1}{2}\right)^{n-1}$

If we want a solution correct to 3 decimal places you want $\left(\frac{1}{2}\right)^{n-1} < 0.5 \times 10^{-3}$

We obtain $n \geq 12$. We see from the spreadsheet above that we have the solution as 1.260

Carry this without with different functions and intervals of different widths.

Fixed-point iteration

We talk about this in this investigation as it provides an alternative method to contrast the bisection algorithm. Information for this starts in section 8D.

We saw that with some functions continually applying the function from a chosen starting point gives an approximation. The notation for 5 applications of function f is $f^{(5)}(x) = f(f(f(f(f(x)))))$.

This is equivalent to providing iterations for solving $f(x) = x$. The function has to be carefully chosen.

For example, in section 8D we chose to consider $f(x) = 1 + \frac{1}{x}$ and then $f^{(n)}(x)$ with starting point $x = 1$ or equivalently iterating through $f(x) = x$ which gave an approximation to one of the solutions of $x^2 - x - 1 = 0$.

To solve the equation $x^3 = 2$ using this equation we choose $f(x) = \frac{2x^3+2}{3x^2}$.

First prove that $f(x) = x$ simplifies to $x^3 = 2$. Then use a spreadsheet

	A	B
1	1	=(2*A1^3+2)/(3*A1^2)
2	=B1	=(2*A2^3+2)/(3*A2^2)
3	=B2	=(2*A3^3+2)/(3*A3^2)
4	=B3	=(2*A4^3+2)/(3*A4^2)
5	=B4	=(2*A5^3+2)/(3*A5^2)
6	=B5	=(2*A6^3+2)/(3*A6^2)

	A	B
1	1	1.33333333
2	1.3333333	1.26388889
3	1.2638889	1.25993349
4	1.2599335	1.25992105
5	1.2599211	1.25992105
6	1.259921	1.25992105

You get high accuracy after 4 iterations. You could also have considered $f^{(5)}(1) = 1.25992$.

This is much faster than the Bisection method but you must choose the right function first.

Try $g(x) = \frac{x^3+2}{2x^2}$. The equation $g(x) = x$ again simplifies to $x^3 = 2$. This function also works but takes longer

For the square root of 2, try $f(x) = \frac{x^2+2}{2x}$

Choose your own functions and compare. Start with fourth roots, fifth roots etc.

Consider why some functions work and others don't.

When does fixed point iteration work

You might like to add this to your investigation. There is a way to see if a particular equation works. You will need to use differentiation but this can easily be accomplished with your calculator in Unit 1.

Suppose that your iterative formula is $x_{n+1} = f(x_n)$. This takes you from row n to row $n + 1$ in your spreadsheet.

For example, solving the equation $x^2 = 2$ we can use $f(x) = \frac{x^2+2}{2x}$.

We don't know the solution but we can let a denote the solution then we know that $a = f(a)$.

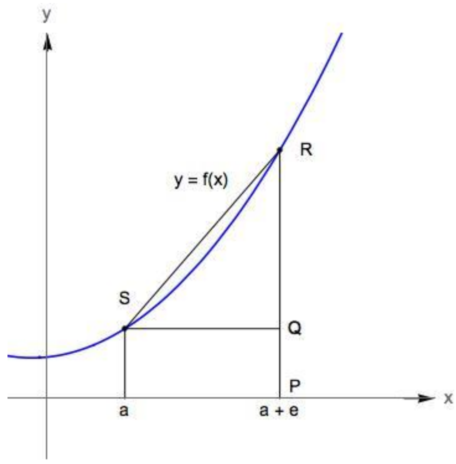
Let x_r be the value at the r th stage. Then

$$x_r = a + e$$

where e can be considered the error at the r th stage.

We have

$$a + e_{r+1} = f(a + e_r). \quad (1)$$



From the diagram

$f(a + e) = PQ + QR = f(a) + QR$

If e is small,

$\frac{QR}{SQ} = \tan \angle QSR$

$\approx$ slope of $y = f(x)$ at $x = a$

$\approx f'(a)$ the derivative of f at a.

Therefore,

$QR \approx e_r f'(a)$ and from (1)

$a + e_{r+1} = f(a + e_r)$

$= f(a) + e_r f'(a)$

But $f(a) = a$, hence $e_{r+1} = e_r f'(a)$.

Assuming all quantities are positive, if $f'(a) < 1$, then $e_{r+1} < e_r$.

That is if $f'(a) < 1$ the error will decrease as more iterations are taken and the approximation improved. It seems from these calculations that the smaller $f'(a)$ is the more rapid the convergence to the solution.

You should try this for different functions. There are other factors involved such as the starting point.

Conclusions

Discuss and list the key results of your investigation. Comment on other directions to investigate. Compare the two methods and the comment on their strengths and weaknesses.

Investigation 3 Unit 2: **Simulating queue waiting times.**

Techniques to be used: probability, probability distributions, simulation.

Technology: CAS calculator, computer spreadsheet

Formulation

Organisations which regularly deal with varying numbers of customers, such as banks and supermarkets, are often faced with the challenge of customer waiting times in queues. The organisation needs to ensure that they have enough staff on hand so that customers are not faced with long queues, but they also don't want to have too many staff allocated to dealing with customers which would not be an efficient use of their resources.

We will start by considering a small bank. When the bank opens there is only one teller in place. The manager determines that when average customer waiting time since the start of the day is 3 or more minutes, she will open a second teller. We wish to estimate how long after the bank opens the second teller will be required.

Exploration

Part 1: We will begin by considering the situation where the time it takes to be served in the bank is constant.

Suppose that the time between arrival of customers at the bank, T minutes, is a random variable with the following probability distribution[1]:

t	2	3	4	5	6
$\Pr(T=t)$	0.2	0.2	0.2	0.2	0.2

There is only one teller open, and the time it takes to be served by the teller is 5 minutes.

1. Use a table of random numbers, a calculator, or a computer to simulate the arrival of customers at the bank when it opens (there is no one in the queue at the start). Record in a table the customer's time of arrival, the time at which they are served, and hence the time that they spent in the queue. Keep a running average of the queuing time and determine how long after the bank opens the average queuing time becomes 3 or more minutes.
2. The simulation carried out in part one is ONE repetition of the simulation model. We require several repetitions to find an estimate which we can have some level of confidence in.

   Either:

   - Combine your estimate of the length of time after the bank opens the second teller is required with the estimates from the other members of you class and use the average of these as an estimate.

[1] For a definition of a probability distribution see page 388 in the textbook.

or

- Use technology to carry our several simulations, the results of which can be combined. A spreadsheet can be used for this purpose.

Part 2: Consider now the situation where the time it takes to be served in the bank is also a random variable, *S* minutes, which has the following probability distribution.

s	3	4	5	6
Pr(*S*=s)	0.25	0.25	0.25	0.25

Assume that there is only one teller open, and the time it takes to be served by the teller is *S* minutes.

3. Use a table of random numbers, a calculator, or a computer to repeat the simulation carried out in Part 1a, but this time two values will need to be randomly generated, the time between arrival of the customers (*t*) and the time it takes for the customer to be served (*s*).
4. The simulation carried out in part one is ONE repetition of the simulation model. We require several repetitions to find an estimate which we can have some level of confidence in.

   Either:

   - Combine your estimate of the length of time after the bank opens that the second teller is required with the estimates from the other members of you class and use the average of these as an estimate.

or

- Use technology to carry our several simulations, the results of which can be combined. A spreadsheet can be used for this purpose.

Part 3: Investigate the effect of varying:

- the probability distribution of the times between customer arrivals
- the probability distribution of service times
- the average waiting time the manager needs to see before a second teller is opened.

Conclusions

Discuss the key results of your investigation. Comment on other directions to investigate. Discuss the strength and weaknesses of your findings and raise questions that still remain if you do not believe that you have a complete solution or if you would like to modify your model.

Solution suggestions

Part 1
Since the values of *t* are all equally likely then the command randInt(2,6) can be used to on a CAS calculator to generate random values of the time between customer arrivals *t*.

1. One simulation is shown in the following table (it has been assumned in this table that the bank opens at 9.30 am).

Customer	*t* (randomly generated time between customers)	arrival time	time served by the teller	service time	finish time with the teller	queuing time	running average of queuing times
1		9.30	9.30	5	9.35	0	0.00
2	5	9.35	9.35	5	9.40	0	0.00
3	5	9.40	9.40	5	9.45	0	0.00
4	3	9.43	9.45	5	9.50	2	0.50
5	6	9.49	9.50	5	9.55	1	0.60
6	5	9.54	9.55	5	10.00	1	0.67
7	5	9.59	10.00	5	10.05	1	0.71
8	2	10.01	10.05	5	10.10	4	1.13
9	4	10.05	10.10	5	10.15	5	1.56
10	4	10.09	10.15	5	10.20	6	2.00
11	5	10.14	10.20	5	10.25	6	2.36
12	3	10.17	10.25	5	10.30	8	2.83
13	5	10.23	10.30	5	10.35	7	3.15

We can see from the table that the average waiting time after the 13^{th} customer, who finishes at 10.35am, is 3.15 minutes. Thus, the manager should open the second teller at 10.35 am, which is 65 minutes after the bank opens.

2. The table above can be set up on a spreadsheet as follows:
   - Set the *arrival time* as 0 (it can represent any time)
   - To determine the customer *arrival time*, add the simulated value of t to the previous customer arrival time.
   - The *time served* is the larger of the *finish time* for the previous customer, and the *arrival time*.
   - The *service time* is 5 minutes.
   - The *finish time* is the *time served* plus 5.
   - The *queuing time* is the larger of the *finish time* for the previous customer minus the *arrival time* for that customer, and 0 (meaning that the previous customer finished before the current customer arrived).

A second simulation using Excel gave the following results:

Customer	*t* (randomly generated time between customers)	arrival time	time served	service time	finish time	customer queuing time	average queuing time
1		0	0	5	5	0	0.00
2	4	4	5	5	10	1	0.50
3	5	9	10	5	15	1	0.67
4	3	12	15	5	20	3	1.25
5	6	18	20	5	25	2	1.40
6	5	23	25	5	30	2	1.50
7	5	28	30	5	35	2	1.57
8	2	30	35	5	40	5	2.00
9	4	34	40	5	45	6	2.44
10	4	38	45	5	50	7	2.90
11	5	43	50	5	55	7	3.27

In this simulation the average waiting time first equals or exceeds 3 minutes after the 11th customer, which 55 minutes after the bank. Thus the bank manager should open the second teller 55 minutes after opening time.

Once the spreadsheet is set up, a new repetition of the simulation is achieved by generated a new set of values for *t*, and observing how long after opening the average queuing time is first equal to or more than 3 minutes.

Thirty such repetitions of the simulation gave the following results:

40	35	55
25	25	35
35	60	25
30	20	25
20	30	35
65	35	25
55	25	40
35	70	95
65	45	25
55	70	60

The average of these 30 values is 42 minutes. So based on our simulation model, we would advise the manager will need to open the second teller 42 minutes after opening time.

Part 2

Since the values of *s* are all equally likely then the command randInt(3,6) can be used on a CAS calculator to generate random values of the time between customer arrivals *t*.

One simulation is shown in the following table:

Customer	*t* (randomly generated time between customers)	arrival time	time served	*s* (randomly generated service time)	finish time	customer queuing time	average queuing time
1		0	0	3	3	0	0.00
2	3	3	3	4	7	0	0.00
3	2	5	7	6	13	2	0.67
4	2	7	13	5	18	6	2.00
5	5	12	18	6	24	6	2.80
6	5	17	24	6	30	7	3.50

In this simulation the average waiting time first equals or exceeds 3 minutes after the 6th customer, which 30 minutes after the bank opens. Thus in this simulation the bank manager should open the second teller 30 minutes after opening time.

Once the spreadsheet is set up, a new repetition of the simulation is achieved by generating new sets of values for *t* and *s*, and observing how long after opening the average queuing time is first equal to or more than 3 minutes.

Thirty such repetitions of the simulation gave the following results:

30	80	20
35	122	77
79	26	61
56	37	18
63	72	65
66	26	25
30	67	24
43	56	34
38	23	93
55	24	51

The average of these 30 values is 49.9 minutes. So based on our simulation model, we would advise the manager will need to open the second teller 50 minutes after opening time.