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Contents

CORE FACTS AND SKILLS TASKS
Part A1: Algebra and Functions

Question 1

Solve the equation $2\log_e(x) - \log_e(x+3) = \log_e\left(\frac{1}{2}\right)$ for x.

[4 marks]
[VCAA 2009 MM (CAS)]

Question 2

Solve the equation $\tan(2x) = \sqrt{3}$ for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

[3 marks]
[VCAA 2009 MM (CAS)]

Question 3

Let $f: R\setminus\{0\} \to R$ where $f(x) = \frac{3}{x} - 4$. Find f^{-1}, the inverse function of f.

[VCAA 2009 MM (CAS)]

Question 4

The simultaneous linear equations

$kx - 3y = 0$
$5x - (k+2)y = 0$

where k is a real constant, have a unique solution provided

A. $k \in \{-5, 3\}$
B. $k \in R\setminus\{-5, 3\}$
C. $k \in \{-3, 5\}$
D. $k \in R\setminus\{-3, 5\}$
E. $k \in R\setminus\{0\}$

[VCAA 2009 MM (CAS)]

Question 5

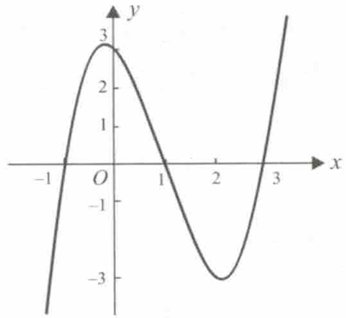
The maximal domain D of the function $f: D \to R$ with rule $f(x) = \log_e(2x+1)$ is

A. $R\setminus\left\{-\frac{1}{2}\right\}$
B. $\left(-\frac{1}{2}, \infty\right)$
C. R
D. $(0, \infty)$
E. $\left(-\infty, -\frac{1}{2}\right)$

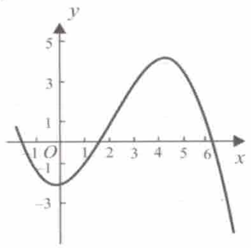
[VCAA 2009 MM (CAS)]

Question 6

The graph of a function *f*, with domain *R*, is as shown.
The graph which best represents $1 - f(2x)$ is



A.



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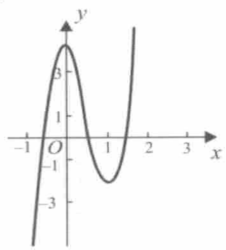

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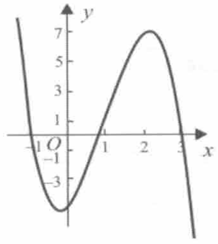

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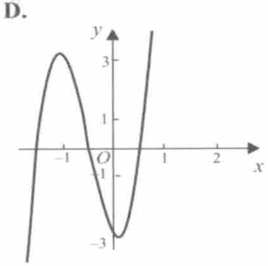
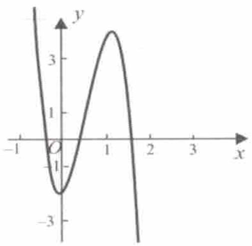


E.



[VCAA 2009 MM (CAS)]

Question 7

At the point (1, 1) on the graph of the function with rule $y = (x-1)^3 + 1$

A. there is a local maximum.
B. there is a local minimum.
C. there is a stationary point of inflection.
D. the gradient is not defined.
E. there is a point of discontinuity.

[VCAA 2009 MM (CAS)]

Part A1: Algebra and Functions

Question 8

The general solution to the equation $\sin(2x) = -1$ is

A. $x = n\pi - \frac{\pi}{4},\ n \in Z$

B. $x = 2n\pi + \frac{\pi}{4}$ or $2n\pi - \frac{\pi}{4},\ n \in Z$

C. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{2},\ n \in Z$

D. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{4},\ n \in Z$

E. $x = n\pi + \frac{\pi}{4}$ or $x = 2n\pi + \frac{\pi}{4},\ n \in Z$

[VCAA 2009 MM (CAS)]

Question 9

Let $f: R \to R, f(x) = x^2$.
Which one of the following is **not** true?

A. $f(xy) = f(x)f(y)$
B. $f(x) - f(-x) = 0$
C. $f(2x) = 4f(x)$
D. $f(x-y) = f(x) - f(y)$
E. $f(x+y) + f(x-y) = 2(f(x) + f(y))$

[VCAA 2009 MM (CAS)]

Question 10

A transformation $T: R^2 \to R^2$ that maps the curve with equation $y = \sin(x)$ onto the curve with equation $y = 1 - 3\sin(2x + \pi)$ is given by

A. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \pi \\ 1 \end{bmatrix}$

B. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{2} \\ 1 \end{bmatrix}$

C. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \pi \\ 1 \end{bmatrix}$

D. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ 1 \end{bmatrix}$

E. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ -1 \end{bmatrix}$

[VCAA 2009 MM (CAS)]

Question 11

The inverse of the function $f: R^+ \to R,\ f(x) = e^{2x+3}$ is

A. $f^{-1}: R^+ \to R, f^{-1}(x) = e^{-2x-3}$

B. $f^{-1}: R^+ \to R, f^{-1}(x) = e^{\frac{x-3}{2}}$

C. $f^{-1}: (e^3, \infty) \to R, f^{-1}(x) = \log_e(\sqrt{x}) - \frac{3}{2}$

D. $f^{-1}: (e^3, \infty) \to R, f^{-1}(x) = e^{\frac{x-3}{2}}$

E. $f^{-1}: (e^3, \infty) \to R, f^{-1}(x) = -\log_e(2x-3)$

[VCAA 2009 MM (CAS)]

Question 12

Write down the amplitude and period of the function

$$f: R \to R, f(x) = 4\sin\left(\frac{x+\pi}{3}\right)$$

[2 marks]
[VCAA 2010 MM (CAS)]

Question 13

Solve the equation $\sqrt{3}\sin(x) = \cos(x)$ for $x \in [-\pi, \pi]$.

[2 marks]
[VCAA 2010 MM (CAS)]

Question 14

Let $f: R^+ \to R$ where $f(x) = \frac{1}{x^2}$.

a. Find $g(x) = f(f(x))$.

b. Evaluate $g^{-1}(16)$, where g^{-1} is the inverse function of g.

[1 + 1 = 2 marks]
[VCAA 2010 MM (CAS)]

Question 15

The simultaneous linear equations $(m-1)x + 5y = 7$ and $3x + (m-3)y = 0.7m$ have infinitely many solutions for

A. $m \in R \setminus \{0, -2\}$ **B.** $m \in R \setminus \{0\}$ **C.** $m \in R \setminus \{6\}$ **D.** $m = 6$ **E.** $m = -2$

[VCAA 2010 MM (CAS)]

Part A1: Algebra and Functions

Question 16

The function with rule $f(x) = 4\tan\left(\frac{x}{3}\right)$ has period

A. $\frac{\pi}{3}$ **B.** 6π **C.** 3 **D.** 3π **E.** $\frac{2\pi}{3}$

[VCAA 2010 MM (CAS)]

Question 17

If $f(x) = \frac{1}{2}e^{3x}$ and $g(x) = \log_e(2x) + 3$ then $g(f(x))$ is equal to

A. $2x^3 + 3$ **B.** $e^{3x} + 3$ **C.** e^{8x+9} **D.** $3(x+1)$ **E.** $\log_e(3x) + 3$

[VCAA 2010 MM (CAS)]

Question 18

The function f has rule $f(x) = 3\log_e(2x)$. If $f(5x) = \log_e(y)$ then y is equal to

A. $30x$ **B.** $6x$ **C.** $125x^3$ **D.** $50x^3$ **E.** $1000x^3$

[VCAA 2010 MM (CAS)]

Question 19

The function $f: (-\infty, a] \to R$ with rule $f(x) = x^3 - 3x^2 + 3$ will have an inverse function provided

A. $a \le 0$ **B.** $a \ge 2$ **C.** $a \ge 0$ **D.** $a \le 2$ **E.** $a \le 1$

[VCAA 2010 MM (CAS)]

Question 20

Consider the simultaneous linear equations

$$kx - 3y = k + 3$$
$$4x + (k+7)y = 1$$

where k is a real constant.

a. Find the value of k for which there are infinitely many solutions.

b. Find the values of k for which there is a unique solution.

[3 + 1 = 4 marks]
[VCAA 2011 MM (CAS)]

Question 21

Solve the equation $4^x - 15 \times 2^x = 16$ for x.

[3 marks]
[VCAA 2011 MM (CAS)]

Question 22

a. State the range and period of the function $h: R \to R, h(x) = 4 + 3\cos\left(\frac{\pi x}{2}\right)$.

b. Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, \pi]$.

[2 + 2 = 4 marks]
[VCAA 2011 MM (CAS)]

Question 23

If the function f has the rule $f(x) = \sqrt{x^2 - 9}$ and the function g has the rule $g(x) = x + 5$

a. find integers c and d such that $f(g(x)) = \sqrt{(x+c)(x+d)}$.

b. state the maximal domain for which $f(g(x))$ is defined.

[2 + 2 = 4 marks]
[VCAA 2011 MM (CAS)]

Question 24

The midpoint of the line segment joining (0, –5) to (d, 0) is

A. $\left(\frac{d}{2}, -\frac{5}{2}\right)$ B. (0, 0) C. $\left(\frac{d-5}{2}, 0\right)$

D. $\left(0, \frac{5-d}{2}\right)$ E. $\left(\frac{5+d}{2}, 0\right)$

[VCAA 2011 MM (CAS)]

Question 25

The gradient of a line **perpendicular** to the line which passes through (–2, 0) and (0, –4) is

A. $\frac{1}{2}$ B. –2 C. $-\frac{1}{2}$ D. 4 E. 2

[VCAA 2011 MM (CAS)]

Question 26

If $x + a$ is a factor of $4x^3 - 13x^2 - ax$, where $a \in R \setminus \{0\}$, then the value of a is

A. –4 B. –3 C. –1 D. 1 E. 2

[VCAA 2011 MM (CAS)]

Question 27

The expression $\log_c(a) + \log_a(b) + \log_b(c)$ is equal to

A. $\frac{1}{\log_c(a)} + \frac{1}{\log_a(b)} + \frac{1}{\log_b(c)}$

B. $\frac{1}{\log_a(c)} + \frac{1}{\log_b(a)} + \frac{1}{\log_c(b)}$

C. $-\frac{1}{\log_a(b)} - \frac{1}{\log_b(c)} - \frac{1}{\log_c(a)}$

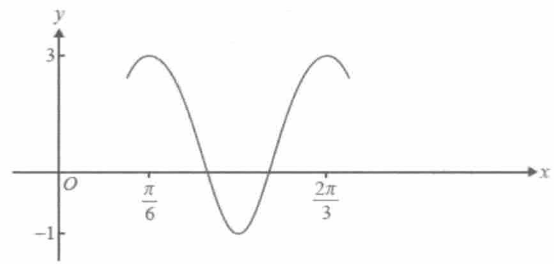
D. $\frac{1}{\log_a(a)} + \frac{1}{\log_b(b)} + \frac{1}{\log_c(c)}$

E. $\frac{1}{\log_c(ab)} + \frac{1}{\log_b(ac)} + \frac{1}{\log_a(cb)}$

[VCAA 2011 MM (CAS)]

Question 28

The graph shown could have equation



A. $y = 2\cos\left(x + \frac{\pi}{6}\right) + 1$

B. $y = 2\cos 4\left(x - \frac{\pi}{6}\right) + 1$

C. $y = 4\sin 2\left(x - \frac{\pi}{12}\right) - 1$

D. $y = 3\cos\left(2x + \frac{\pi}{6}\right) - 1$

E. $y = 2\sin\left(4x + \frac{2\pi}{3}\right) - 1$

[VCAA 2011 MM (CAS)]

Question 29

The inverse function of $g:[2,\infty)\to R, g(x)=\sqrt{2x-4}$ is

A. $g^{-1}:[2,\infty)\to R, g^{-1}(x)=\dfrac{x^2+4}{2}$

B. $g^{-1}:[0,\infty)\to R, g^{-1}(x)=(2x-4)^2$

C. $g^{-1}:[0,\infty)\to R, g^{-1}(x)=\sqrt{\dfrac{x}{2}+4}$

D. $g^{-1}:[0,\infty)\to R, g^{-1}(x)=\dfrac{x^2+4}{2}$

E. $g^{-1}:R\to R, g^{-1}(x)=\dfrac{x^2+4}{2}$

[VCAA 2011 MM (CAS)]

Question 30

Consider the function $f:R\to R, f(x)=x(x-4)$ and the function

$g:\left[\dfrac{3}{2},5\right)\to R, g(x)=x+3$.

If the function $h=f+g$, then the domain of the inverse function of h is

A. $[0,13)$

B. $\left[-\dfrac{3}{4},10\right]$

C. $\left(-\dfrac{3}{4},\dfrac{15}{4}\right]$

D. $\left[\dfrac{3}{4},13\right)$

E. $\left[\dfrac{3}{2},13\right)$

[VCAA 2011 MM (CAS)]

Question 31

Solve the equation $2\log_e(x+2)-\log_e(x)=\log_e(2x+1)$, where $x>0$, for x.

[3 marks]
[VCAA 2012 MM (CAS)]

Question 32

The graphs of $y=\cos(x)$ and $y=a\sin(x)$, where a is a real constant, have a point of intersection at $x=\dfrac{\pi}{3}$.

b Find the value of a.

b. If $x\in[0,2\pi]$, find the x-coordinate of the other point of intersection of the two graphs.

[2 + 1 = 3 marks]
[VCAA 2012 MM (CAS)]

Question 33

The rule for function h is $h(x)=2x^3+1$. Find the rule for the inverse function h^{-1}.

[2 marks]
[VCAA 2012 MM (CAS)]

Question 34

A system of simultaneous linear equations is represented by the matrix equation

$$\begin{bmatrix} m & 3 \\ 1 & m+2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ m \end{bmatrix}.$$

The system of equations will have **no solution** when

A. $m=1$ **B.** $m=-3$ **C.** $m\in\{1,-3\}$ **D.** $m\in R\setminus\{1\}$ **E.** $m\in\{1,3\}$

[VCAA 2012 MM (CAS)]

Question 35

The function with rule $f(x)=-3\sin\left(\dfrac{\pi x}{5}\right)$ has period

A. 3 **B.** 5 **C.** 10 **D.** $\dfrac{\pi}{5}$ **E.** $\dfrac{\pi}{10}$

[VCAA 2012 MM (CAS)]

Question 36

The range of the function $f:[-2,3]\to R, f(x)=x^2-2x-8$ is

A. R **B.** $(-9,-5]$ **C.** $(-5,0)$ **D.** $[-9,0]$ **E.** $[-9,-5)$

[VCAA 2012 MM (CAS)]

Question 37

Let the rule for a function g be $g(x)=\log_e((x-2)^2)$. For the function g, the

A. maximal domain $=R^+$ and range $=R$

B. maximal domain $=R\setminus\{2\}$ and range $=R$

C. maximal domain $=R\setminus\{2\}$ and range $=(-2,\infty)$

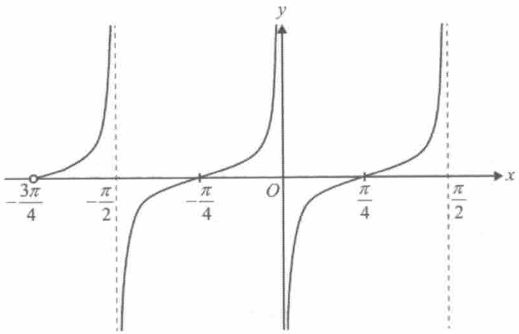
D. maximal domain $=[2,\infty)$ and range $=(0,\infty)$

E. maximal domain $=[2,\infty)$ and range $=[0,\infty)$

[VCAA 2012 MM (CAS)]

Question 38

A section of the graph of f is shown below.



The rule of f could be

A. $f(x)=\tan(x)$

B. $f(x)=\tan\left(x-\frac{\pi}{4}\right)$

C. $f(x)=\tan\left(2\left(x-\frac{\pi}{4}\right)\right)$

D. $f(x)=\tan\left(2\left(x-\frac{\pi}{2}\right)\right)$

E. $f(x)=\tan\left(\frac{1}{2}\left(x-\frac{\pi}{2}\right)\right)$

[VCAA 2012 MM (CAS)]

Question 39

The graph of a cubic function f has a local maximum at $(a, -3)$ and a local minimum at $(b, -8)$.

The values of c, such that the equation $f(x)+c=0$ has exactly one solution, are

A. $3<c<8$ **B.** $c>-3$ or $c<-8$ **C.** $-8<c<-3$

D. $c<3$ or $c>8$ **E.** $c<-8$

[VCAA 2012 MM (CAS)]

Question 40

A function f has the following two properties for all real values of θ.

$$f(\pi-\theta)=-f(\theta) \text{ and } f(\pi-\theta)=-f(-\theta)$$

A possible rule for f is

A. $f(x)=\sin(x)$ **B.** $f(x)=\cos(x)$ **C.** $f(x)=\tan(x)$

D. $f(x)=\sin\left(\frac{x}{2}\right)$ **E.** $f(x)=\tan(2x)$

[VCAA 2012 MM (CAS)]

Question 41

a. Solve the equation $2\log_3(5)-\log_3(2)+\log_3(x)=2$ for x.

b. Solve the equation $3^{-4x}=9^{6-x}$ for x.

[2 + 2 = 4 marks]
[VCAA 2013 MM (CAS)]

Question 42

Solve the equation $\sin\left(\frac{x}{2}\right)=-\frac{1}{2}$ for $x\in[2\pi,4\pi]$.

[2 marks]
[VCAA 2013 MM (CAS)]

Question 43

The midpoint of the line segment which joins $(1,-5)$ to $(d,2)$ is

A. $\left(\frac{d+1}{2},-\frac{3}{2}\right)$

B. $\left(\frac{1-d}{2},-\frac{7}{2}\right)$

C. $\left(\frac{d-4}{2},0\right)$

D. $\left(0,\frac{1-d}{3}\right)$

E. $\left(\frac{5+d}{2},2\right)$

[VCAA 2013 MM (CAS)]

Question 44

If $f:(-\infty,1)\rightarrow R, f(x)=2\log_e(1-x)$ and $g:[-1,\infty)\rightarrow R, g(x)=3\sqrt{x+1}$, then the maximal domain of the function $f+g$ is

A. $[-1,1)$ **B.** $(1,\infty)$ **C.** $(-1,1)$ **D.** $(-\infty,-1]$ **E.** R

\[VCAA 2013 MM (CAS)]

Question 45

If $x+a$ is a factor of $7x^3+9x^2-5ax$, where $a\in R\setminus\{0\}$, then the value of a is

A. -4 **B.** -2 **C.** -1 **D.** 1 **E.** 2

[VCAA 2013 MM (CAS)]

Question 46

A transformation $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ maps the graph of a function f to the graph of $y = x^2$, $x \in R$.

The rule of f is

A. $f(x) = -(x+5)^2$

B. $f(x) = (5-x)^2$

C. $f(x) = -(x-5)^2$

D. $f(x) = -x^2 + 5$

E. $f(x) = x^2 + 5$

[VCAA 2013 MM (CAS)]

Question 47

The function with rule $f(x) = -3\tan(2\pi x)$ has period

A. $\frac{2}{\pi}$ **B.** 2 **C.** $\frac{1}{2}$ **D.** $\frac{1}{4}$ **E.** 2π

[VCAA 2013 MM (CAS)]

Question 48

The function $g: [-a, a] \rightarrow R, g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.

The maximum possible value of a is

A. $\frac{\pi}{12}$ **B.** 1 **C.** $\frac{\pi}{6}$ **D.** $\frac{\pi}{4}$ **E.** $\frac{\pi}{2}$

[VCAA 2013 MM (CAS)]

Question 49

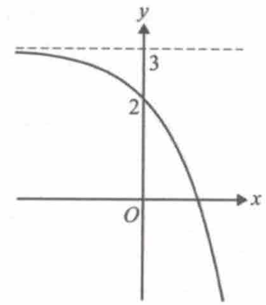
If the equation $f(2x) - 2f(x) = 0$ is true for all real values of x, then the rule for f could be

A. $\frac{x^2}{2}$ **B.** $\sqrt{2x}$ **C.** $2x$ **D.** $\log_e\left(\frac{|x|}{2}\right)$ **E.** $x - 2$

[VCAA 2013 MM (CAS)]

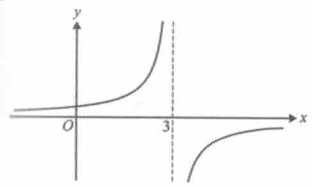
Question 50

Part of the graph of $y = f(x)$, where $f: R \rightarrow R, f(x) = 3 - e^x$, is shown below.



Which one of the following could be the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f?

A.



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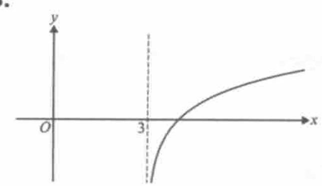

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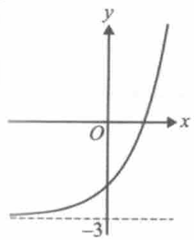

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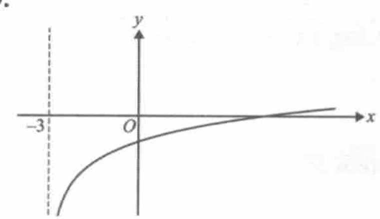

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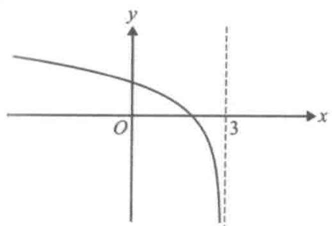


[VCAA 2013 MM (CAS)]

Question 51

Let $g(x)=\log_2(x), x>0$.

Which one of the following equations is true for all positive real values *of x*?

A. $2g(8x)=g(x^2)+8$ **B.** $2g(8x)=g(x^2)+6$ **C.** $2g(8x)=(g(x)+8)^2$

D. $2g(8x)=g(2x)+6$ **E.** $2g(8x)=g(2x)+64$

[VCAA 2013 MM (CAS)]

Question 52

Solve $2\cos(2x)=-\sqrt{3}$ for *x*, where $0\le x\le \pi$.

[2 marks]
[VCAA 2014 MM (CAS)]

Question 53

Solve the equation $2^{3x-3}=8^{2-x}$ for *x*.

[2 marks]
[VCAA 2014 MM (CAS)]

Question 54

Solve $\log_e(x)-3=\log_e(\sqrt{x})$ for *x*, where $x>0$.

[2 marks]
[VCAA 2014 MM (CAS)]

Question 55

The point $P(4,-3)$ lies on the graph of a function *f*. The graph of *f* is translated four units vertically up and then reflected in the *y*-axis.
The coordinates of the final image of *P* are

A. $(-4,1)$ **B.** $(-4,3)$ **C.** $(0,-3)$ **D.** $(4,-6)$ **E.** $(-4,-1)$

[VCAA 2014 MM (CAS)]

Question 56

The linear function $f:D\to R$, $f(x)=4-x$ has range $[-2,6)$.

The domain *D* of the function is

A. $[-2,6)$ **B.** $(-2,2]$ **C.** R **D.** $(-2,6]$ **E.** $[-6,2]$

[VCAA 2014 MM (CAS)]

Question 57

The function $f:D\to R$ with rule $f(x)=2x^3-9x^2-168x$ will have an inverse function for

A. $D=R$ **B.** $D=(7,\infty)$ **C.** $D=(-4,8)$

D. $D=(-\infty,0)$ **E.** $D=\left[-\frac{1}{2},\infty\right)$

[VCAA 2014 MM (CAS)]

Question 58

The inverse of the function $f:R^+\to R$, $f(x)=\frac{1}{\sqrt{x}}+4$ is

A. $f^{-1}:(4,\infty)\to R$ $\quad f^{-1}(x)=\frac{1}{(x-4)^2}$

B. $f^{-1}:R^+\to R$ $\quad f^{-1}(x)=\frac{1}{x^2}+4$

C. $f^{-1}:R^+\to R$ $\quad f^{-1}(x)=(x+4)^2$

D. $f^{-1}:(-4,\infty)\to R$ $\quad f^{-1}(x)=\frac{1}{(x+4)^2}$

E. $f^{-1}:(-\infty,4)\to R$ $\quad f^{-1}(x)=\frac{1}{(x-4)^2}$

[VCAA 2014 MM (CAS)]

Question 59

Which one of the following functions satisfies the functional equation $f(f(x))=x$ for every real number *x*?

A. $f(x)=2x$ **B.** $f(x)=x^2$ **C.** $f(x)=2\sqrt{x}$

D. $f(x)=x-2$ **E.** $f(x)=2-x$

[VCAA 2014 MM (CAS)]

Question 60

The transformation $T: R^2 \to R^2$ with rule

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

maps the line with equation $x - 2y = 3$ onto the line with equation

A. $x + y = 0$ **B.** $x + 4y = 0$ **C.** $-x - y = 4$

D. $x + 4y = -6$ **E.** $x - 2y = 1$

[VCAA 2014 MM (CAS)]

Question 61

The domain of the function h, where $h(x) = \cos(\log_a(x))$ and a is a real number greater than 1, is chosen so that h is a one-to-one function.
Which one of the following could be the domain?

A. $\left(a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$ **B.** $(0, \pi)$ **C.** $\left[1, a^{\frac{\pi}{2}}\right]$ **D.** $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$ **E.** $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right]$

[VCAA 2014 MM (CAS)]

Question 62

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have **no solution** for

A. $a = 3$ **B.** $a = -3$ **C.** both $a = 3$ and $a = -3$

D. $a \in R \setminus \{3\}$ **E.** $a \in R \setminus [-3, 3]$

[VCAA 2014 MM (CAS)]

Question 63

The graph of $y = kx - 4$ intersects the graph of $y = x^2 + 2x$ at two distinct points for

A. $k = 6$

B. $k > 6$ or $k < -2$

C. $-2 \le k \le 6$

D. $6 - 2\sqrt{3} \le k \le 6 + 2\sqrt{3}$

E. $k = -2$

[VCAA 2014 MM (CAS)]

Question 64

On a given day, the depth of the water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right), 0 \le t \le 24$$

where h is the depth of the water, in metres, and t is the time, in hours, after 6 am.

a. Find the minimum depth of water in the river.

b. Find the values of t for which $h(t) = 10$.

[1 + 2 = 3 marks]
[VCAA 2015 MM (CAS)]

Question 65

a. Solve $\log_2(6 - x) - \log_2(4 - x) = 2$ for x, where $x < 4$.

b. Solve $3e^t = 5 + 8e^{-t}$ for t.

[2 + 3 = 5 marks]
[VCAA 2015 MM (CAS)]

Question 66

Let $f: R \to R, f(x) = 2\sin(3x) - 3$.
The period and range of this function are respectively

A. period $= \frac{2\pi}{3}$ and range $= [-5, -1]$

B. period $= \frac{2\pi}{3}$ and range $= [-2, 2]$

C. period $= \frac{\pi}{3}$ and range $= [-1, 5]$

D. period $= 3\pi$ and range $= [-1, 5]$

E. period $= 3\pi$ and range $= [-2, 2]$

[VCAA 2015 MM (CAS)]

Question 67

The inverse function of $f:(-2, \infty) \to R, f(x) = \dfrac{1}{\sqrt{x+2}}$ is

A. $f^{-1}: R^{+} \to R \qquad f^{-1}(x) = \dfrac{1}{x^2} - 2$

B. $f^{-1}: R\backslash\{0\} \to R \qquad f^{-1}(x) = \dfrac{1}{x^2} - 2$

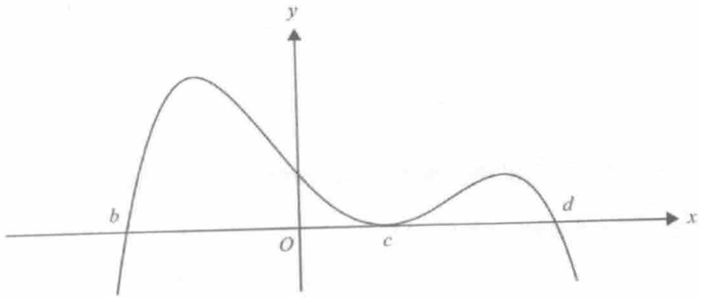
C. $f^{-1}: R^{+} \to R \qquad f^{-1}(x) = \dfrac{1}{x^2} + 2$

D. $f^{-1}: (-2, \infty) \to R \qquad f^{-1}(x) = x^2 + 2$

E. $f^{-1}: (2, \infty) \to R \qquad f^{-1}(x) = \dfrac{1}{x^2 - 2}$

[VCAA 2015 MM (CAS)]

Question 68



The rule for a function with the graph above could be

A. $y = -2(x+b)(x-c)^2(x-d)$

B. $y = 2(x+b)(x-c)^2(x-d)$

C. $y = -2(x-b)(x-c)^2(x-d)$

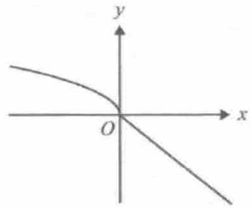
D. $y = 2(x-b)(x-c)(x-d)$

E. $y = -2(x-b)(x+c)^2(x+d)$

[VCAA 2015 MM (CAS)]

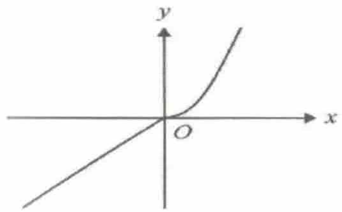
Question 69

Part of the graph of $y = f(x)$ is shown below.

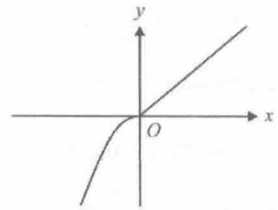


The corresponding part of the graph of the inverse function $y = f^{-1}(x)$ is best represented by

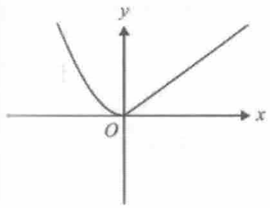
A.



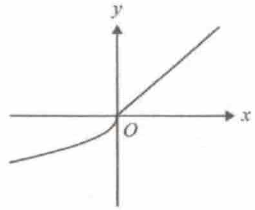
B.



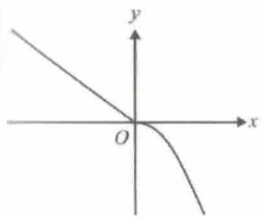
C.



D.



E.



[VCAA 2015 MM (CAS)]

Question 70

For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A. -3 **B.** -1 **C.** 1 **D.** 3 **E.** 10

[VCAA 2015 MM (CAS)]

Question 71

The range of the function $f: (-1, 2] \rightarrow R, f(x) = -x^2 + 2x - 3$ is

A. R **B.** $(-6, -3]$ **C.** $(-6, -2]$ **D.** $[-6, -3]$ **E.** $[-6, -2]$

[VCAA 2015 MM (CAS)]

Question 72

The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A. dilation by a factor of 2 from the y-axis.

B. dilation by a factor of 2 from the x-axis.

C. dilation by a factor of $\frac{1}{2}$ from the x-axis.

D. dilation by a factor of 8 from the y-axis.

E. dilation by a factor of $\frac{1}{2}$ from the y-axis.

[VCAA 2015 MM (CAS)]

Question 73

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts.
The set of all possible values of c is

A. R **B.** R^+ **C.** $\{0, 4\}$ **D.** $(0, 4)$ **E.** $(-\infty, 4)$

[VCAA 2015 MM (CAS)]

Question 74

The graphs of $y = mx + c$ and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

A. $a > 0$ and $c > 0$ **B.** $a > 0$ and $c < 0$ **C.** $a > 0$ and $c > -\frac{m^2}{4a}$

D. $a < 0$ and $c > -\frac{m^2}{4a}$ **E.** $m > 0$ and $c > 0$

[VCAA 2015 MM (CAS)]

Question 75

If $f(x - 1) = x^2 - 2x + 3$, then $f(x)$ is equal to

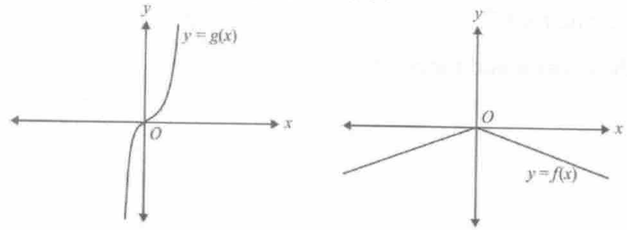
A. $x^2 - 2$ **B.** $x^2 + 2$ **C.** $x^2 - 2x + 2$

D. $x^2 - 2x + 4$ **E.** $x^2 - 4x + 6$

[VCAA 2015 MM (CAS)]

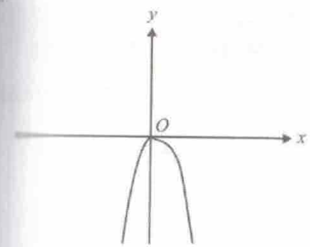
Question 76

The graphs of the functions with rules $f(x)$ and $g(x)$ are shown below.

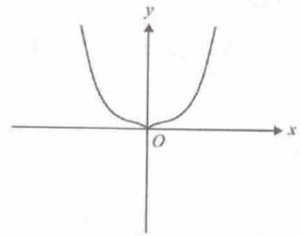


Which one of the following best represents the graph of the function with rule $g(-f(x))$?

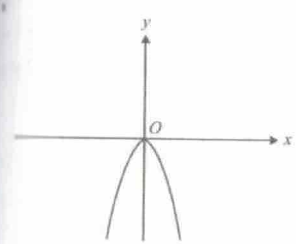
A.



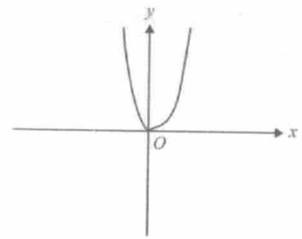
B.



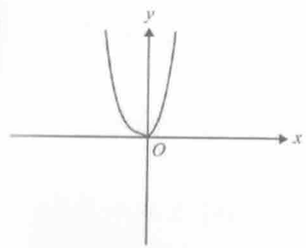
C.



D.



E.



[VCAA 2015 MM (CAS)]

Question 77

Let f: $(0, \infty) \to R$, where $f(x) = \log_e(x)$ and g: $R \to R$, where $g(x) = x^2 + 1$.

a. **i.** Find the rule for h, where $h(x) = f(g(x))$.

ii. State the domain and range of h.

iii. Show that $h(x) + h(-x) = f((g(x))^2)$.

iv. Find the coordinates of the stationary point of h and state its nature.

b. Let k: $(-\infty, 0] \to R$, where $k(x) = \log_e(x^2 + 1)$.

i. Find the rule for k^{-1}.

ii. State the domain and range of k^{-1}.

[1 + 2 + 2 + 2 + 2 + 2 = 11 marks]
[VCAA 2016 MM]

Question 78

The linear function $f: D \to R, f(x) = 5 - x$ has range $[-4, 5)$.

The domain D is

A. $(0, 9]$ **B.** $(0, 1]$ **C.** $[5, -4)$ **D.** $[-9, 0)$ **E.** $[1, 9)$

[VCAA 2016 MM]

Question 79

Let $f: R \to R, f(x) = 1 - 2\cos\left(\frac{\pi x}{2}\right)$.

The period and range of this function are respectively

A. 4 and $[-2, 2]$

B. 4 and $[-1, 3]$

C. 1 and $[-1, 3]$

D. 4π and $[-1, 3]$

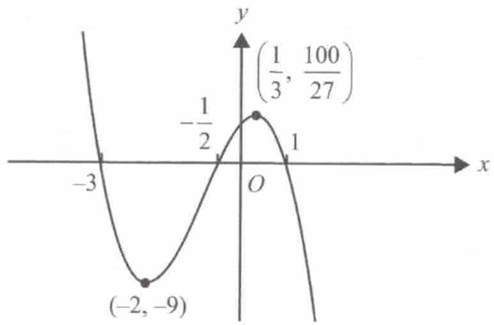
E. 4π and $[-2, 2]$

[VCAA 2016 MM]

Part A1: Algebra and Functions

Question 80

Part of the graph $y = f(x)$ of the polynomial function f is shown below.



$f'(x) < 0$ for

A. $x \in (-2, 0) \cup \left(\frac{1}{3}, \infty\right)$

B. $x \in \left(-9, \frac{100}{27}\right)$

C. $x \in (-\infty, -2) \cup \left(\frac{1}{3}, \infty\right)$

D. $x \in \left(-2, \frac{1}{3}\right)$

E. $x \in (-\infty, -2] \cup (1, \infty)$

[VCAA 2016 MM]

Question 81

Which one of the following is the inverse function of $g: [3, \infty) \to R, g(x) = \sqrt{2x - 6}$?

A. $g^{-1}: [3, \infty) \to R, g^{-1}(x) = \frac{x^2 + 6}{2}$

B. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = (2x - 6)^2$

C. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \sqrt{\frac{x}{2} + 6}$

D. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \frac{x^2 + 6}{2}$

E. $g^{-1}: R \to R, g^{-1}(x) = \frac{x^2 + 6}{2}$

[VCAA 2016 MM]

Question 82

Consider the graph of the function defined by $f: [0, 2\pi] \to R, f(x) = \sin(2x)$.

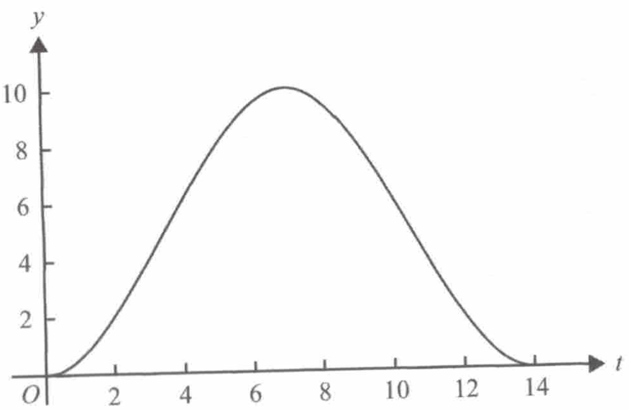
The square of the length of the line segment joining the points on the graph for which $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is

A. $\frac{\pi^2 + 16}{4}$ **B.** $\pi + 4$ **C.** 4 **D.** $\frac{3\pi^2 + 16\pi}{4}$ **E.** $\frac{10\pi^2}{16}$

[VCAA 2016 MM]

Question 83

The UV index, y, for a summer day in Melbourne is illustrated in the graph below, where t is the number of hours after 6 am.



The graph is most likely to be the graph of

A. $y = 5 + 5\cos\left(\frac{\pi t}{7}\right)$

B. $y = 5 - 5\cos\left(\frac{\pi t}{7}\right)$

C. $y = 5 + 5\cos\left(\frac{\pi t}{14}\right)$

D. $y = 5 - 5\cos\left(\frac{\pi t}{14}\right)$

E. $y = 5 + 5\sin\left(\frac{\pi t}{14}\right)$

[VCAA 2016 MM]

Question 84

The function f has the property $f(x) - f(y) = (y - x)f(xy)$ for all non-zero real numbers x and y.

Which one of the following is a possible rule for the function?

A. $f(x) = x^2$

B. $f(x) = x^2 + x^4$

C. $f(x) = x\log_e(x)$

D. $f(x) = \frac{1}{x}$

E. $f(x) = \frac{1}{x^2}$

[VCAA 2016 MM]

Question 85

The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x - 5}$ by a reflection in the x-axis followed by a dilation from the y-axis by a factor of $\frac{1}{2}$.

Which one of the following is the rule for the function f?

A. $f(x) = \sqrt{5 - 4x}$

B. $f(x) = -\sqrt{x - 5}$

C. $f(x) = \sqrt{x + 5}$

D. $f(x) = -\sqrt{4x - 5}$

E. $f(x) = -\sqrt{4x - 10}$

[VCAA 2016 MM]

Question 86

Let $(\tan(\theta) - 1)(\sin(\theta) - \sqrt{3}\cos(\theta))(\sin(\theta) + \sqrt{3}\cos(\theta)) = 0$.

a. State all possible values of $\tan(\theta)$.

b. Hence, find all possible solutions for $(\tan(\theta) - 1)(\sin^2(\theta) - 3\cos^2(\theta)) = 0$, where $0 \le \theta \le \pi$.

[1 + 2 = 3 marks]
[VCAA 2017 MM]

Question 87

Let $f: [0, \infty) \to R,\ f(x) = \sqrt{x+1}$.

a. State the range of f.

b. Let $g: (-\infty, c] \to R,\ g(x) = x^2 + 4x + 3$, where $c < 0$.

  i. Find the largest possible value of c such that the range of g is a subset of the domain of f.

  ii. For the value of c found in part **b.i.**, state the range of $f(g(x))$.

c. Let $h: R \to R,\ h(x) = x^2 + 3$.
State the range of $f(h(x))$.

[1 + 2 + 1 + 1 = 5 marks]
[VCAA 2017 MM]

Question 88

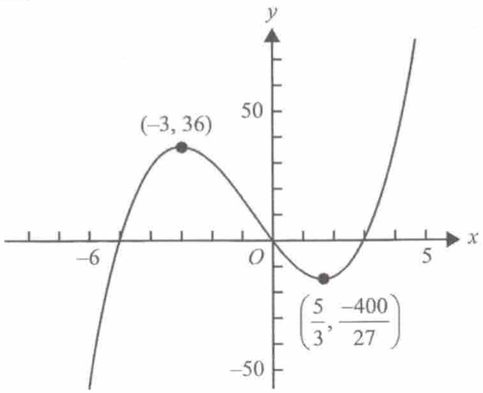
Let $f: R \to R,\ f(x) = 5\sin(2x) - 1$.
The period and range of this function are respectively

A. π and $[-1, 4]$ **B.** 2π and $[-1, 5]$ **C.** π and $[-6, 4]$
D. 2π and $[-6, 4]$ **E.** 4π and $[-6, 4]$

[VCAA 2017 MM]

Question 89

Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.



$f'(x) < 0$ for the interval

A. $(0, 3)$ **B.** $(-\infty, -5) \cup (0, 3)$ **C.** $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D. $\left(-3, \frac{5}{3}\right)$ **E.** $\left(\frac{-400}{27}, 36\right)$

[VCAA 2017 MM]

Part A1: Algebra and Functions

Question 90

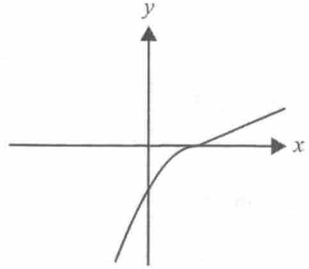
Let f and g be functions such that $f(2) = 5, f(3) = 4, g(2) = 5, g(3) = 2$ and $g(4) = 1$.
The value of $f(g(3))$ is

A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 5

[VCAA 2017 MM]

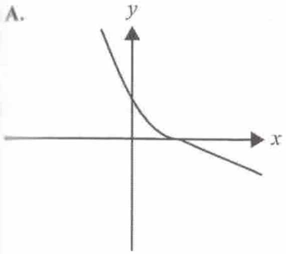
Question 91

Part of the graph of the function f is shown below. The same scale has been used on both axes.

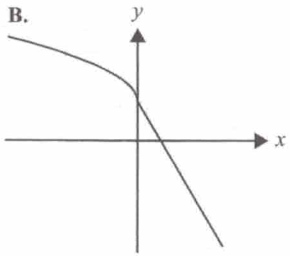


The corresponding part of the graph of the inverse function f^{-1} is best represented by

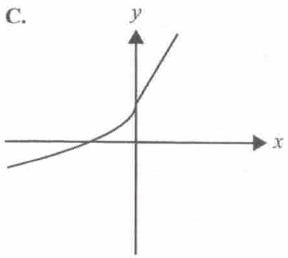
A.



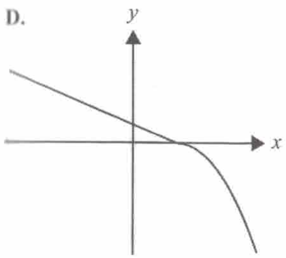
B.



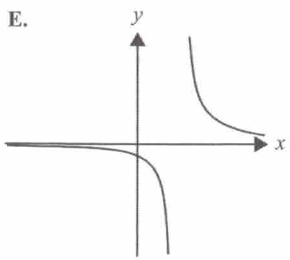
C.



D.



E.



[VCAA 2017 MM]

Question 92

The equation $(p-1)x^2 + 4x = 5 - p$ has no real roots when

A. $p^2 - 6p + 6 < 0$ **B.** $p^2 - 6p + 1 > 0$ **C.** $p^2 - 6p - 6 < 0$
D. $p^2 - 6p + 1 < 0$ **E.** $p^2 - 6p + 6 > 0$

[VCAA 2017 MM]

Part A1: Algebra and Functions

Question 93

If $y = a^{b-4x} + 2$, where $a > 0$, then x is equal to

A. $\frac{1}{4}(b - \log_a(y-2))$ **B.** $\frac{1}{4}(b - \log_a(y+2))$ **C.** $b - \log_a\left(\frac{1}{4}(y+2)\right)$

D. $\frac{b}{4} - \log_a(y-2)$ **E.** $\frac{1}{4}(b + 2 - \log_a(y))$

[VCAA 2017 MM]

Question 94

A transformation $T: R^2 \to R^2$ with rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps the graph of

$y = 3\sin\left(2\left(x + \frac{\pi}{4}\right)\right)$ onto the graph of

A. $y = \sin(x + \pi)$ **B.** $y = \sin\left(x - \frac{\pi}{2}\right)$ **C.** $y = \cos(x + \pi)$

D. $y = \cos(x)$ **E.** $y = \cos\left(x - \frac{\pi}{2}\right)$

[VCAA 2017 MM]

Question 95

The sum of the solutions of $\sin(2x) = \frac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$.

The value of d could be

A. 0 **B.** $\frac{\pi}{6}$ **C.** $\frac{3\pi}{4}$ **D.** $\frac{7\pi}{6}$ **E.** $\frac{3\pi}{2}$

[VCAA 2017 MM]

Question 96

Let $h: (-1, 1) \to R, h(x) = \frac{1}{x-1}$.

Which one of the following statements about h is **not** true?

A. $h(x)h(-x) = -h(x^2)$ **B.** $h(x) + h(-x) = 2h(x^2)$ **C.** $h(x) - h(0) = xh(x)$

D. $h(x) - h(-x) = 2xh(x^2)$ **E.** $(h(x))^2 = h(x^2)$

[VCAA 2017 MM]

Part A2: Differentiation

Question 97

a. Differentiate $x\log_e(x)$ with respect to x.

b. For $f(x) = \frac{\cos(x)}{2x+2}$, find $f'(\pi)$.

[2 + 3 = 5 marks]
[VCAA 2009 MM (CAS)]

Question 98

Let $f: R \to R, f(x) = e^x + k$, where k is a real number. The tangent to the graph of f at the point where $x = a$ passes through the point (0, 0). Find the value of k in terms of a.

[3 marks]
[VCAA 2009 MM (CAS)]

Question 99

For $y = e^{2x}\cos(3x)$ the rate of change of y with respect to x when $x = 0$ is

A. 0 **B.** 2 **C.** 3 **D.** –6 **E.** –1

[VCAA 2009 MM (CAS)]

Question 100

For the function $f: R \to R, f(x) = (x+5)^2(x-1)$, the subset of R for which the gradient of f is negative is

A. $(-\infty, 1)$ **B.** $(-5, 1)$ **C.** $(-5, -1)$ **D.** $(-\infty, -5)$ **E.** $(-5, 0)$

[VCAA 2009 MM (CAS)]

Question 101

A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at (2, 0) and (–3, 0). The maximum value of the derivative function is 10. The value of x for which the graph of $y = f(x)$ has a local maximum is

A. –2 **B.** 2 **C.** –3 **D.** 3 **E.** $-\frac{1}{2}$

[VCAA 2009 MM (CAS)]

Question 102

The tangent at the point (1, 5) on the graph of the curve $y = f(x)$ has equation $y = 3 + 2x$.
The tangent at the point (3, 8) on the curve $y = f(x-2) + 3$ has equation

A. $y = 2x - 4$ **B.** $y = x + 5$ **C.** $y = -2x + 14$

D. $y = 2x + 4$ **E.** $y = 2x + 2$

[VCAA 2009 MM (CAS)]

Question 103

For $y = \sqrt{1 - f(x)}$, $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{2f'(x)}{\sqrt{1-f(x)}}$ **B.** $\dfrac{-1}{2\sqrt{1-f'(x)}}$ **C.** $\dfrac{1}{2}\sqrt{1-f'(x)}$

D. $\dfrac{3}{2(1-f'(x))}$ **E.** $\dfrac{-f'(x)}{2\sqrt{1-f(x)}}$

[VCAA 2009 MM (CAS)]

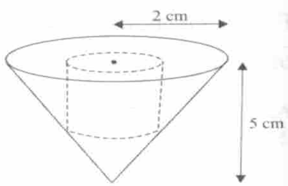
Question 104

The line $y = ax - 1$ is a tangent to the curve $y = x^{\frac{1}{2}} + d$ at the point $(9, c)$ where a, c and d are real constants. Find the values of a, c and d.

[4 marks]
[VCAA 2010 MM (CAS)]

Question 105

A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane as shown in the diagram. The height of the cone is 5 cm and the radius of the cone is 2 cm. The radius of the cylinder is r cm and the height of the cylinder is h cm. For the cylinder inscribed in the cone as shown here,



a. find h in terms of r.

The total surface area, S cm^2, of a cylinder of height h cm and radius r cm is given by the formula $S = 2\pi rh + 2\pi r^2$.

b. Find S in terms of r.

c. Find the value of r for which S is a maximum.

[2 + 1 + 2 = 5 marks]
[VCAA 2010 MM (CAS)]

Part A2: Differentiation

Question 106

Differentiate $x^3 e^{2x}$ with respect to x.

[2 marks]
[VCAA 2010 MM (CAS)]

Question 107

For $f(x) = \log_e(x^2 + 1)$, find $f'(2)$.

[2 marks]
[VCAA 2010 MM (CAS)]

Question 108

For $f(x) = x^3 + 2x$, the average rate of change with respect to x for the interval [1, 5] is

A. 18 **B.** 20.5 **C.** 24 **D.** 32.5 **E.** 33

[VCAA 2010 MM (CAS)]

Question 109

The function f is differentiable for all $x \in R$ and satisfies the following conditions:

- $f'(x) < 0$ where $x < 2$
- $f'(x) = 0$ where $x = 2$
- $f'(x) = 0$ where $x = 4$
- $f'(x) > 0$ where $2 < x < 4$
- $f'(x) > 0$ where $x > 4$

Which one of the following is true?

A. The graph of f has a local maximum point where $x = 4$.
B. The graph of f has a stationary point of inflection where $x = 4$.
C. The graph of f has a local maximum point where $x = 2$.
D. The graph of f has a local minimum point where $x = 4$.
E. The graph of f has a stationary point of inflection where $x = 2$.

[VCAA 2010 MM (CAS)]

Question 110

The gradient of the function $f: R \to R,\ f(x) = \dfrac{5x}{x^2 + 3}$, is negative for

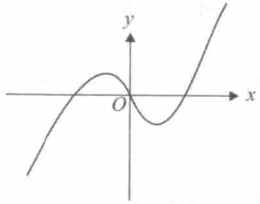
A. $-\sqrt{3} < x < \sqrt{3}$ **B.** $x > 3$ **C.** $x \in R$

D. $x < -\sqrt{3}$ and $x > \sqrt{3}$ **E.** $x < 0$

[VCAA 2010 MM (CAS)]

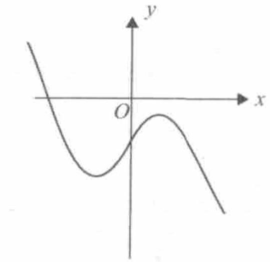
Question 111

The graph of the gradient function $y = f'(x)$ is shown here.

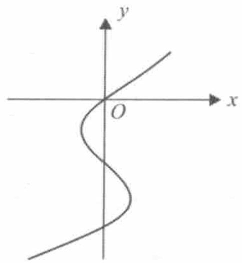


Which of the following could represent the graph of the function $y = f(x)$?

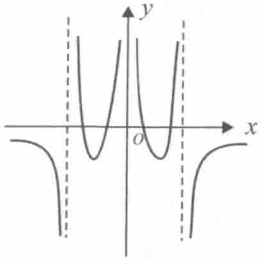
A.



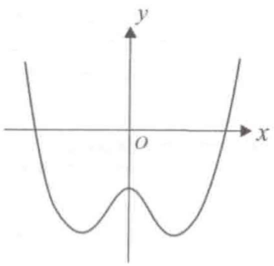
B.



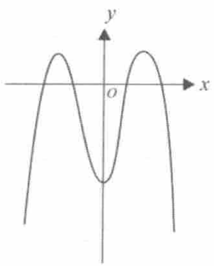
C.



D.



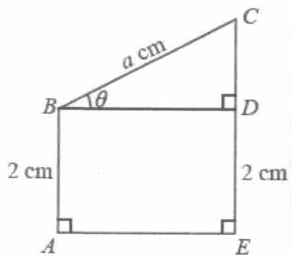
E.



[VCAA 2010 MM (CAS)]

Question 112

The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. The point D is on line segment EC with $AB = ED = 2$ cm and $BC = a$ cm, where a is a positive constant.



$\angle BAE = \angle CEA = \frac{\pi}{2}$

Let $\angle CBD = \theta$ where $0 < \theta < \frac{\pi}{2}$.

a. Find BD and CD in terms of a and θ.

b. Find the length, L cm, of the wire in the frame, including length BD in terms of a and θ.

c. Find $\frac{dL}{d\theta}$, and **hence** show that $\frac{dL}{d\theta} = 0$ when $BD = 2CD$.

d. Find the maximum value of L if $a = 3\sqrt{5}$.

[2 + 1 + 2 + 1 = 6 marks]
[VCAA 2011 MM (CAS)]

Part A2: Differentiation

Question 113

a. Differentiate $\sqrt{4-x}$ with respect to x.

b. If $g(x) = x^2 \sin(2x)$, find $g'\left(\frac{\pi}{6}\right)$.

[1 + 2 = 3 marks]
[VCAA 2011 MM (CAS)]

Question 114

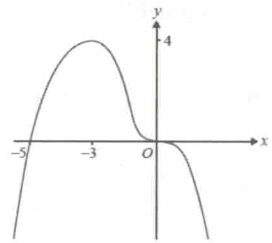
The derivative of $\log_e(2f(x))$ with respect to x is

A. $\frac{f'(x)}{f(x)}$ **B.** $2\frac{f'(x)}{f(x)}$ **C.** $\frac{f'(x)}{2f(x)}$ **D.** $\log_e(2f'(x))$ **E.** $2\log_e(2f'(x))$

[VCAA 2011 MM (CAS)]

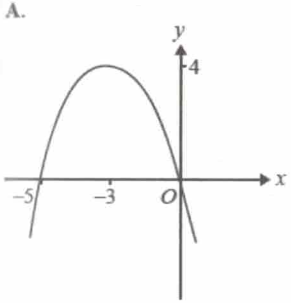
Question 115

The graph of the function $y = f(x)$ is shown here.

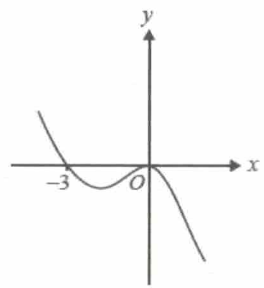


Which of the following could be the graph of the derivative function $y = f'(x)$?

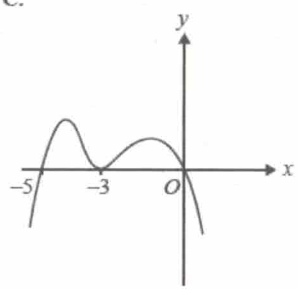
A.



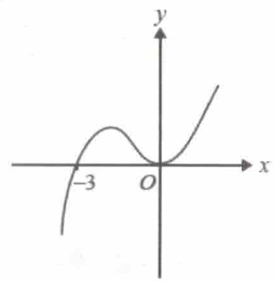
B.



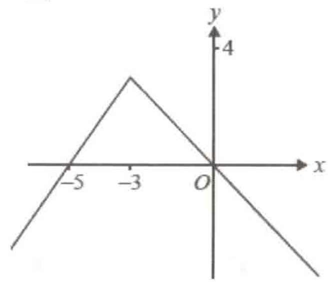
C.



D.



E.



[VCAA 2011 MM (CAS)]

Question 116

The normal to the curve with equation $y = x^{\frac{3}{2}} + x$ at the point (4, 12) is parallel to the straight line with equation

A. $4x = y$ **B.** $4y + x = 7$ **C.** $y = \frac{x}{4} + 1$

D. $x - 4y = -5$ **E.** $4y + 4x = 20$

[VCAA 2011 MM (CAS)]

Question 117

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A. $-7 < w < 25$ **B.** $w \le -7$ **C.** $w \ge 25$

D. $w < -7$ or $w > 25$ **E.** $w > 1$

[VCAA 2011 MM (CAS)]

Question 118

a. If $y = (x^2 - 5x)^4$, find $\frac{dy}{dx}$.

b. If $f(x) = \frac{x}{\sin(x)}$, find $f'\left(\frac{\pi}{2}\right)$.

[1 + 2 = 3 marks]
[VCAA 2012 MM (CAS)]

Question 119

Let $f: R \to R, f(x) = e^{-mx} + 3x$, where m is a positive rational number.

a. **i.** Find, in terms of m, the x-coordinate of the stationary point of the graph of $y = f(x)$.

ii. State the values of m such that the x-coordinate of this stationary point is a positive number.

b. For a particular value of m, the tangent to the graph of $y = f(x)$ at $x = -6$ passes through the origin. Find this value of m.

[2 + 1 + 3 = 6 marks]
[VCAA 2012 MM (CAS)]

Question 120

For the function with rule $f(x) = x^3 - 4x$, the average rate of change of $f(x)$ with respect to x on the interval [1, 3] is

A. 1 **B.** 3 **C.** 5 **D.** 6 **E.** 9

[VCAA 2012 MM (CAS)]

Question 121

Given that g is a differentiable function and k is a real number, the derivative of the composite function $g(e^{kx})$ is

A. $kg'(e^{kx})e^{kx}$ **B.** $kg(e^{kx})$ **C.** $ke^{kx}g(e^{kx})$

D. $ke^{kx}g'(e^{x})$ **E.** $\frac{1}{k}e^{kx}g'(e^{kx})$

[VCAA 2012 MM (CAS)]

Question 122

The function $f: R \to R, f(x) = ax^3 + bx^2 + cx$, where a is a negative real number and b and c are real numbers.

For the real numbers $p < m < 0 < n < q$, we have $f(p) = f(q) = 0$ and $f'(m) = f'(n) = 0$.

The gradient of the graph of $y = f(x)$ is negative for

A. $(-\infty, m) \cup (n, \infty)$ **B.** (m, n) **C.** $(p, 0) \cup (q, \infty)$

D. $(p, m) \cup (0, q)$ **E.** (p, q)

[VCAA 2012 MM (CAS)]

Question 123

The normal to the graph of $y = \sqrt{b - x^2}$ has a gradient of 3 when $x = 1$.

The value of b is

A. $-\frac{10}{9}$ **B.** $\frac{10}{9}$ **C.** 4 **D.** 10 **E.** 11

[VCAA 2012 MM (CAS)]

Question 124

The tangent to the graph of $y = \log_e(x)$ at the point $(a, \log_e(a))$ crosses the x-axis at the point $(b, 0)$, where $b < 0$.

Which of the following is **false**?

A. $1 < a < e$ **B.** The gradient of the tangent is positive

C. $a > e$ **D.** The gradient of the tangent is $\frac{1}{a}$

E. $a > 0$

[VCAA 2012 MM (CAS)]

Question 125

a. If $y = x^2 \log_e(x)$, find $\dfrac{dy}{dx}$.

b. Let $f(x) = e^{x^2}$. Find $f'(3)$.

[2 + 3 = 5 marks]
[VCAA 2013 MM (CAS)]

Question 126

If the tangent to the graph of $y = e^{ax}$, $a \neq 0$, at $x = c$ passes through the origin, then c is equal to

A. 0 **B.** $\dfrac{1}{a}$ **C.** 1 **D.** a **E.** $-\dfrac{1}{a}$

[VCAA 2013 MM (CAS)]

Question 127

For the function $f(x) = \sin(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\dfrac{1}{4}, 5\right]$ is

A. 0 **B.** $\dfrac{34}{19}$ **C.** $\dfrac{7}{2}$ **D.** $\dfrac{2\pi + 10}{4}$ **E.** $\dfrac{23}{4}$

[VCAA 2013 MM (CAS)]

Question 128

Let $y = 4\cos(x)$ and x be a function of t such that $\dfrac{dx}{dt} = 3e^{2t}$ and $x = \dfrac{3}{2}$ when $t = 0$.

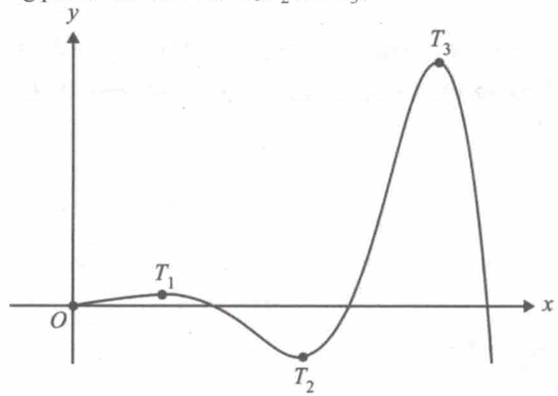
The value of $\dfrac{dy}{dt}$ when $x = \dfrac{\pi}{2}$ is

A. 0 **B.** $3\pi \log_e\left(\dfrac{\pi}{2}\right)$ **C.** -4π **D.** -2π **E.** $-12e$

[VCAA 2013 MM (CAS)]

Question 129

Part of the graph of a function $f:[0, \infty) \to R, f(x) = e^{x\sqrt{3}} \sin(x)$ is shown below.
The first three turning points are labelled T_1, T_2 and T_3.



The x-coordinate of T_3 is

A. $\dfrac{8\pi}{3}$ **B.** $\dfrac{16\pi}{3}$ **C.** $\dfrac{13\pi}{6}$ **D.** $\dfrac{17\pi}{6}$ **E.** $\dfrac{29\pi}{6}$

[VCAA 2013 MM (CAS)]

Question 130

The cubic function $f: R \to R, f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A. $c > \dfrac{b^2}{4a}$ **B.** $c < \dfrac{b^2}{4a}$ **C.** $c < 4b^2 a$ **D.** $c > \dfrac{b^2}{3a}$ **E.** $c < \dfrac{b^2}{3a}$

[VCAA 2013 MM (CAS)]

Question 131

a. If $y = x^2 \sin(x)$, find $\dfrac{dy}{dx}$.

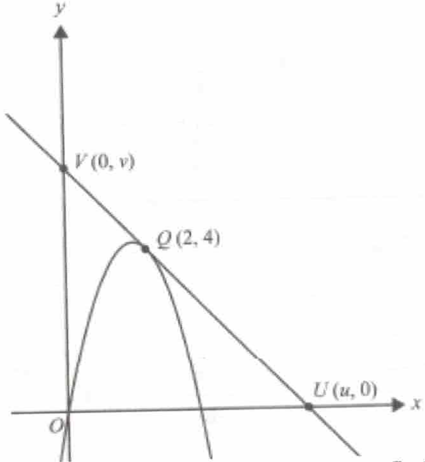
b. If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$.

[2 + 3 = 5 marks]
[VCAA 2014 MM (CAS)]

Question 132

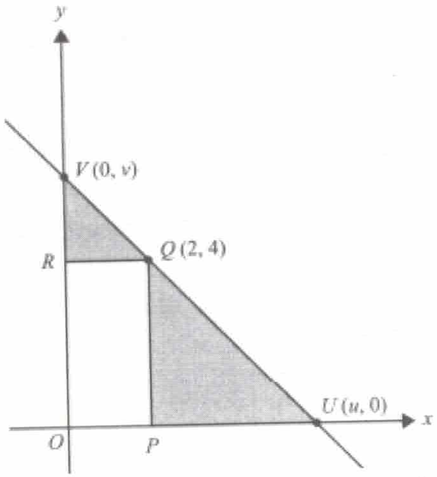
A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \le u \le 6$.

a. When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates (2, 4), as shown.



If a and b are non-zero real numbers, find the values of a and b.

b. The rectangle $OPQR$ has a vertex at Q on the line. The coordinates of Q are (2, 4), as shown.



i. Find an expression for v in terms of u.

ii. Find the **minimum** total shaded area and the value of u for which the area is a minimum.

iii. Find the **maximum** total shaded area and the value of u for which the area is a maximum.

[3 + 1 + 2 + 1 = 7 marks]
[VCAA 2014 MM (CAS)]

Part A2: Differentiation

Question 133

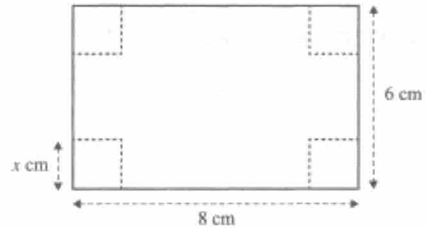
Let f be a function with domain R such that $f'(5) = 0$ and $f'(x) < 0$ when $x \ne 5$.
At $x = 5$, the graph of f has a

A. local minimum.
B. local maximum.
C. gradient of 5.
D. gradient of –5.
E. stationary point of inflection.

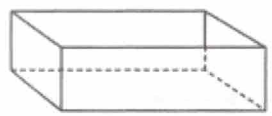
[VCAA 2014 MM (CAS)]

Question 134

Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.



Zoe turns up the sides to form an open box.



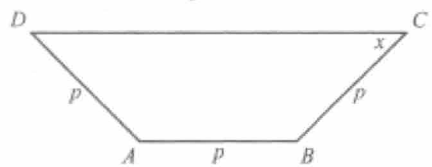
The value of x for which the volume of the box is a maximum is closest to

A. 0.8 **B.** 1.1 **C.** 1.6 **D.** 2.0 **E.** 3.6

[VCAA 2014 MM (CAS)]

Question 135

The trapezium $ABCD$ is shown below. The sides AB, BC and DA are of equal length, p.
The size of the acute angle BCD is x radians.



The area of the trapezium is a maximum when the value of x is

A. $\frac{\pi}{12}$ **B.** $\frac{\pi}{6}$ **C.** $\frac{\pi}{4}$ **D.** $\frac{\pi}{3}$ **E.** $\frac{5\pi}{12}$

[VCAA 2014 MM (CAS)]

Question 136

a. Let $y = (5x + 1)^7$. Find $\frac{dy}{dx}$.

b. Let $f(x) = \frac{\log_e(x)}{x^2}$.

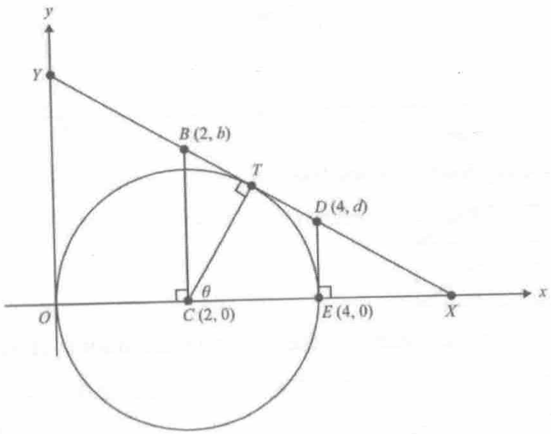
i. Find $f'(x)$.

ii. Evaluate $f'(1)$.

[1 + 2 + 1 = 4 marks]
[VCAA 2015 MM (CAS)]

Question 137

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates (2, 0). The angle ECT is θ, where $0 < \theta \le \frac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.

a. Find the coordinates of T in terms of θ.

b. Find the gradient of the tangent to the circle at T in terms of θ.

c. The equation of the tangent to the circle at T can be expressed as

$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XY.
Find b in terms of θ.

ii. Point D, with coordinates $(4, d)$, is on the line segment XY.
Find d in terms of θ.

d. Consider the trapezium $CEDB$ with parallel sides of length b and d.
Find the value of θ for which the area of the trapezium $CEDB$ is a minimum. Also find the minimum value of the area.

[1 + 1 + 1 + 1 + 3 = 7 marks]
[VCAA 2015 MM (CAS)]

Question 138

Consider the tangent to the graph of $y = x^2$ at the point (2, 4).
Which of the following points lies on this tangent?

A. (1, −4) **B.** (3, 8) **C.** (−2, 6) **D.** (1, 8) **E.** (4, −4)

[VCAA 2015 MM (CAS)]

Question 139

a. Let $y = \frac{\cos(x)}{x^2 + 2}$.

Find $\frac{dy}{dx}$.

b. Let $f(x) = x^2 e^{5x}$.

Evaluate $f'(1)$.

[2 + 2 = 4 marks]
[VCAA 2016 MM]

Question 140

Let $f: \left(-\infty, \frac{1}{2}\right] \to R$, where $f(x) = \sqrt{1 - 2x}$.

a. Find $f'(x)$.

b. Find the angle θ from the positive direction of the x-axis to the tangent to the graph of f at $x = -1$, measured in the anticlockwise direction.

[1 + 2 = 3 marks]
[VCAA 2016 MM]

Question 141

The average rate of change of the function f with rule $f(x) = 3x^2 - 2\sqrt{x + 1}$ between $x = 0$ and $x = 3$, is

A. 8 **B.** 25 **C.** $\frac{53}{9}$ **D.** $\frac{25}{3}$ **E.** $\frac{13}{9}$

[VCAA 2016 MM]

Question 142

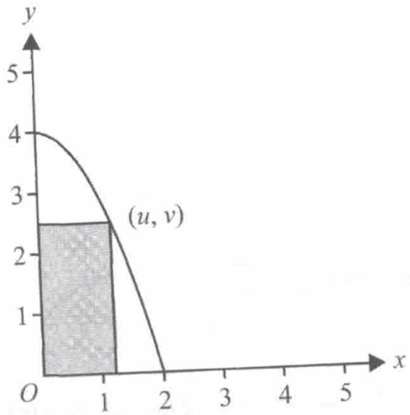
For the curve $y = x^2 - 5$, the tangent to the curve will be parallel to the line connecting the positive x-intercept and the y-intercept when x is equal to

A. $\sqrt{5}$ **B.** 5 **C.** -5 **D.** $\frac{\sqrt{5}}{2}$ **E.** $\frac{1}{\sqrt{5}}$

[VCAA 2016 MM]

Question 143

A rectangle is formed by using part of the coordinate axes and a point (u, v), where $u > 0$ on the parabola $y = 4 - x^2$.



Which one of the following is the maximum area of the rectangle?

A. 4 **B.** $\frac{2\sqrt{3}}{3}$ **C.** $\frac{8\sqrt{3}-4}{3}$ **D.** $\frac{8}{3}$ **E.** $\frac{16\sqrt{3}}{9}$

[VCAA 2016 MM]

Question 144

a. Let $f: (-2, \infty) \rightarrow R,\ f(x) = \frac{x}{x+2}$.
Differentiate f with respect to x.

b. Let $g(x) = (2 - x^3)^3$.
Evaluate $g'(1)$.

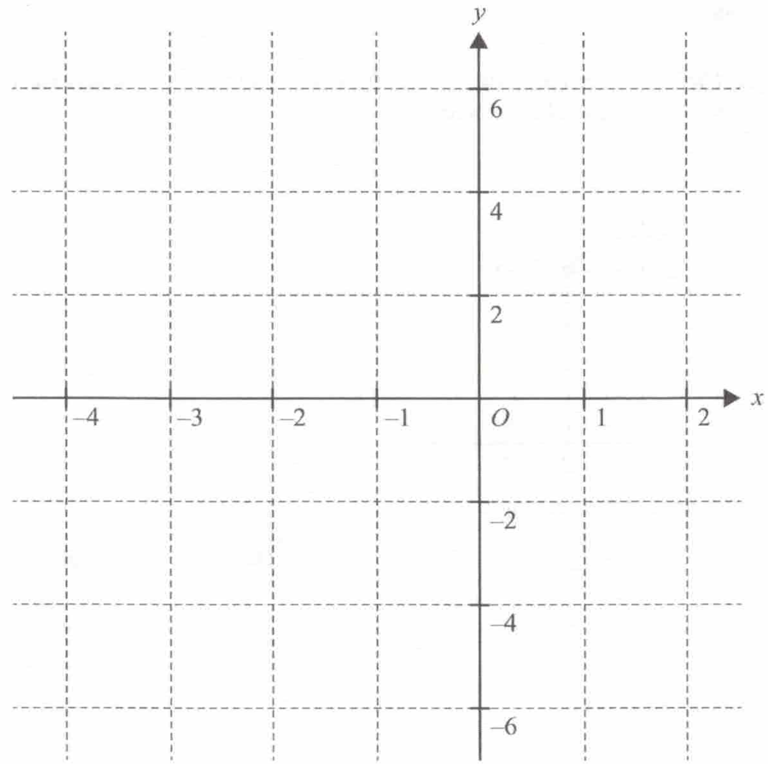
[2 + 2 = 4 marks]
[VCAA 2017 MM]

Question 145

Let $f: [-3, 0] \rightarrow R, f(x) = (x + 2)^2(x - 1)$.

a. Show that $(x + 2)^2(x - 1) = x^3 + 3x^2 - 4$.

a. Sketch the graph of f on the axes below. Label the axis intercepts and any stationary points with their coordinates.



[1 + 3 = 4 marks]
[VCAA 2017 MM]

Question 146

The average rate of change of the function with the rule $f(x) = x^2 - 2x$ over the interval $[1, a]$, where $a > 1$, is 8. The value of a is

A. 9 **B.** 8 **C.** 7 **D.** 4 **E.** $1+\sqrt{2}$

[VCAA 2017 MM]

Question 147

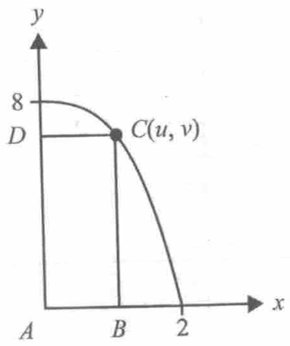
The function $f: R \to R, f(x) = x^3 + ax^2 + bx$ has a local maximum at $x = -1$ and a local minimum at $x = 3$. The values of a and b are respectively

A. -2 and -3 **B.** 2 and 1 **C.** 3 and -9 **D.** -3 and -9 **E.** -6 and -15

[VCAA 2017 MM]

Question 148

A rectangle $ABCD$ has vertices $A(0, 0)$, $B(u, 0)$, $C(u, v)$ and $D(0, v)$, where (u, v) lies on the graph of $y = -x^3 + 8$, as shown below.



The maximum area of the rectangle is

A. $\sqrt[3]{2}$ **B.** $6\sqrt[3]{2}$ **C.** 16 **D.** 8 **E.** $3\sqrt[3]{2}$

[VCAA 2017 MM]

Part A3: Integration

Question 149

a. Find an antiderivative of $\dfrac{1}{1-2x}$ with respect to x.

b. Evaluate $\displaystyle\int_1^4 (\sqrt{x}+1)\,dx$.

[2 + 3 = 5 marks]
[VCAA 2009 MM (CAS)]

Question 150

The average value of the function $f: R\setminus\left\{-\frac{1}{2}\right\} \to R, f(x) = \dfrac{1}{2x+1}$ over the interval $[0, k]$ is $\frac{1}{6}\log_e(7)$. The value of k is

A. $\dfrac{-6}{\log_e(7)} - \dfrac{1}{2}$ **B.** 3 **C.** e^3

D. $\dfrac{-\log_e(7)}{2(\log(7)+6)}$ **E.** 171

[VCAA 2009 MM (CAS)]

Question 151

Consider the region bounded by the x-axis, the y-axis, the line with equation $y = 3$ and the curve with equation $y = \log_e(x-1)$.
The exact value of the area of this region is

A. $e^{-3}-1$ **B.** $16+3\log_e(2)$ **C.** $3e^3-e^{-3}+2$

D. e^3+2 **E.** $3e^2$

[VCAA 2009 MM (CAS)]

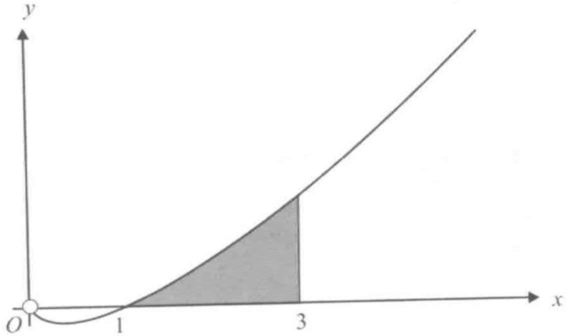
Question 152

Find an antiderivative of $\cos(2x+1)$ with respect to x.

[1 mark]
[VCAA 2010 MM (CAS)]

Question 153

The graph of $f: R^+ \to R, f(x) = x\log_e(x)$ is shown below.



a. Find the derivative of $x^2\log_e(x)$.

b. Use your answer to part **a.** to find the area of the shaded region in the form $a\log_e(b) + c$, where a, b and c are non-zero real constants.

[1 + 3 = 4 marks]
[VCAA 2010 MM (CAS)]

Question 154

The average value of the function $f(x) = e^{2x}\cos(3x)$ for $0 \le x \le \pi$ is closest to

A. –82.5 **B.** 26.3 **C.** –26.3 **D.** –274.7 **E.** π

[VCAA 2010 MM (CAS)]

Question 155

Let f be a differentiable function defined for all real x, where $f(x) \ge 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)dx = a$, then $2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)dx$ is equal to

A. $2a + 6$ **B.** $10a + 6$ **C.** $20a$

D. $40a$ **E.** $50a$

[VCAA 2010 MM (CAS)]

Question 156

Let f be a differentiable function defined for $x > 2$ such that

$$\int_3^{ab+2} f(x)dx = \int_3^{a+2} f(x)dx + \int_3^{b+2} f(x)dx \text{ where } a > 1 \text{ and } b > 1.$$

The rule for $f(x)$ is

A. $\sqrt{x-2}$ **B.** $\log_e(x-2)$ **C.** $\sqrt{2x-4}$

D. $\log_e(2x-4)$ **E.** $\frac{1}{x-2}$

[VCAA 2010 MM (CAS)]

Question 157

A function g with domain R has the following properties:

- $g'(x) = x^2 - 2x$
- The graph of $g(x)$ passes through the point (1, 0).

$g(x)$ is equal to

A. $2x-2$ **B.** $\frac{x^3}{3} - x^2$ **C.** $\frac{x^3}{3} - x^2 + \frac{2}{3}$

D. $x^2 - 2x + 2$ **E.** $3x^3 - x^2 - 1$

[VCAA 2010 MM (CAS)]

Question 158

Find an antiderivative of $\frac{1}{3x-4}$ with respect to x.

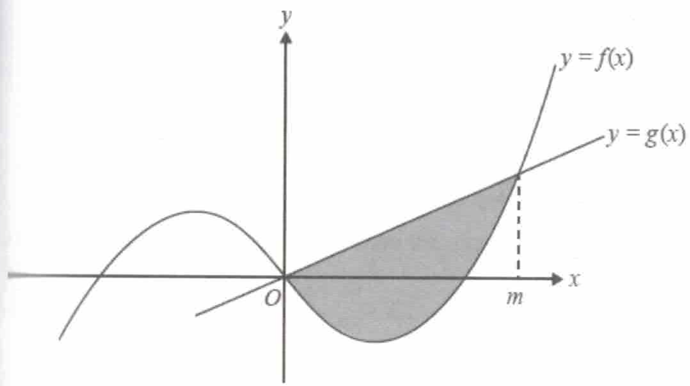
[1 mark]
[VCAA 2011 MM (CAS)]

Question 159

Parts of the graphs of the functions

$$f: R \to R, f(x) = x^3 - ax \quad a > 0$$
$$g: R \to R, g(x) = ax \quad a > 0$$

are shown in the diagram below. The graphs intersect when $x = 0$ and when $x = m$.



The area of the shaded region is 64.
Find the value of a and the value of m.

[4 marks]
[VCAA 2011 MM (CAS)]

Question 160

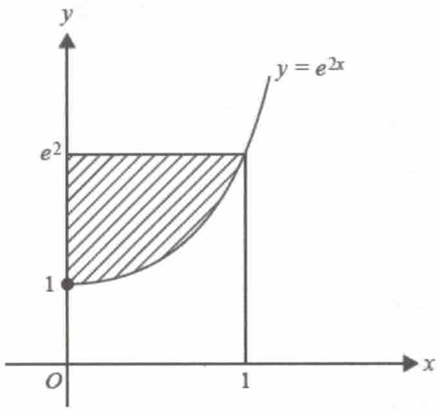
The average value of the function with rule $f(x)=\log_e(x+2)$ over the interval [0, 3] is

A. $\log_e(2)$ **B.** $\frac{1}{3}\log_e(6)$ **C.** $\log_e\left(\frac{3125}{4}\right)-3$

D. $\frac{1}{3}\log_e\left(\frac{3125}{4}\right)-3$ **E.** $\frac{5\log_e(5)-2\log_e(2)-3}{3}$

[VCAA 2011 MM (CAS)]

Question 161



To find the area of the shaded region in the diagram shown, four different students proposed the following calculations.

i. $\int_0^1 e^{2x}\,dx$

ii. $e^2-\int_0^1 e^{2x}\,dx$

iii. $\int_1^{e^2} e^{2y}\,dy$

iv. $\int_1^{e^2} \frac{\log_e(x)}{2}\,dx$

Which of the following is correct?

A. **ii.** only **B.** **ii.** and **iii.** only **C.** **i.**, **ii.**, **iii.** and **iv**

D. **ii.** and **iv.** only **E.** **i.** and **iv.** only

[VCAA 2011 MM (CAS)]

Question 162

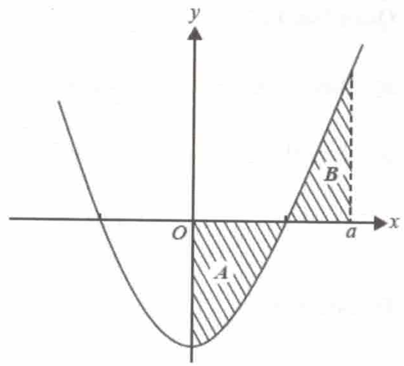
A part of the graph of $g: R\to R, g(x)=x^2-4$ is shown here.

The area of the region marked A is the same as the area of the region marked B.

The exact value of a is

A. 0 **B.** 6 **C.** $\sqrt{6}$

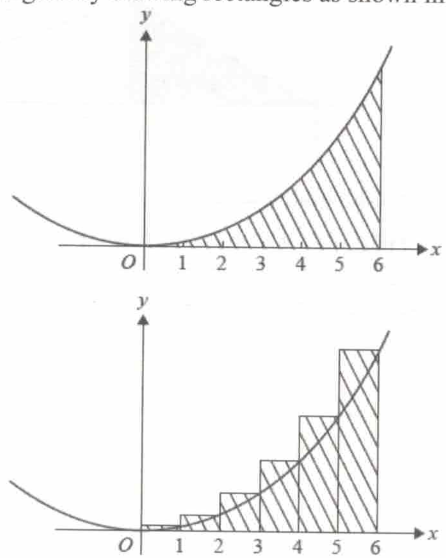
D. 12 **E.** $2\sqrt{3}$



[VCAA 2011 MM (CAS)]

Question 163

A part of the graph of $f: R\to R, f(x)=x^2$ is shown below. Zoe finds the approximate area of the shaded region by drawing rectangles as shown in the second diagram.



Zoe's approximation is p% more than the exact value of the area. The value of p is closest to

A. 10 **B.** 15 **C.** 20 **D.** 25 **E.** 30

[VCAA 2011 MM (CAS)]

Question 164

Find an antiderivative of $\frac{1}{(2x-1)^3}$ with respect to x.

[2 marks]

[VCAA 2012 MM (CAS)]

Question 165

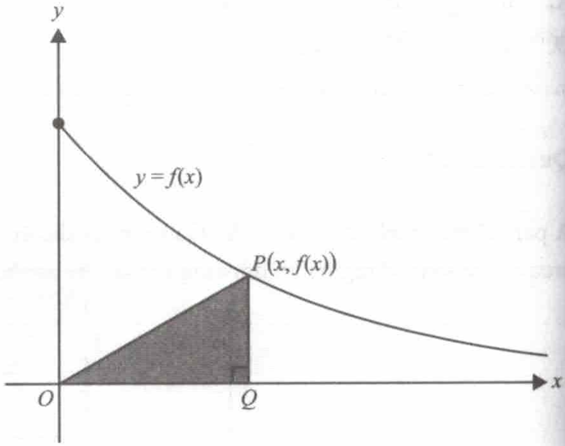
a. Let $f: R \to R, f(x) = x\sin(x)$. Find $f'(x)$.

b. Use the result of **part a.** to find the value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos(x)\,dx$ in the form $a\pi + b$.

[1 + 3 = 4 marks]
[VCAA 2012 MM (CAS)]

Question 166

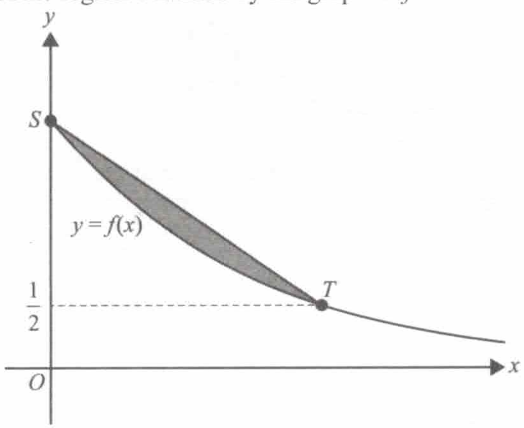
Let $f:[0,\infty) \to R, f(x) = 2e^{-\frac{x}{5}}$.
A right-angled triangle *OQP* has vertex *O* at the origin, vertex *Q* on the *x*-axis and vertex *P* on the graph of *f*, as shown. The coordinates of *P* are $(x, f(x))$.



a. Find the area, *A*, of the triangle *OQP* in terms of *x*.

b. Find the maximum area of triangle *OQP* and the value of *x* for which the maximum occurs.

c. Let *S* be the point on the graph of *f* on the *y*-axis and let *T* be the point on the graph of *f* with the *y*-coordinate $\frac{1}{2}$.
Find the area of the region bounded by the graph of *f* and the line segment *ST*.



[1 + 3 + 3 = 7 marks]
[VCAA 2013 MM (CAS)]

Question 167

The temperature, $T°$ C, inside a building t hours after midnight is given by the function

$$f:[0, 24] \to R, T(t) = 22 - 10\cos\left(\frac{\pi}{12}(t-2)\right)$$

The average temperature inside the building between 2 am and 2 pm is

A. 10° C **B.** 12° C **C.** 20° C **D.** 22° C **E.** 32° C

[VCAA 2012 MM (CAS)]

Question 168

The average value of the function $f: [0, 2\pi] \to R, f(x) = \sin^2(x)$ over the interval $[0, a]$ is 0.4.

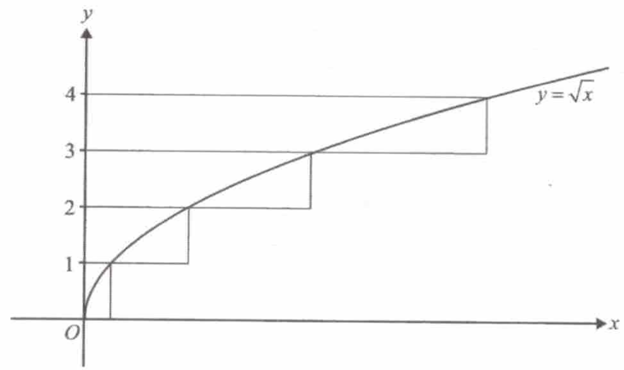
The value of a, to three decimal places, is

A. 0.850 **B.** 1.164 **C.** 1.298 **D.** 1.339 **E.** 4.046

[VCAA 2012 MM (CAS)]

Question 169

The graph of $f: R^+ \cup \{0\} \to R, f(x) = \sqrt{x}$ is shown below.
In order to find an approximation to the area of the region bounded by the graph of *f*, the *y*-axis and the line $y = 4$, Zoe draws four rectangles, as shown, and calculates their total area.



Zoe's approximation to the area of the region is

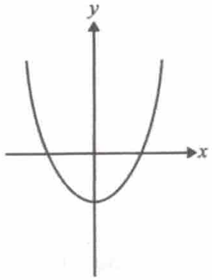
A. 14 **B.** 21 **C.** 29 **D.** 30 **E.** $\frac{64}{3}$

[VCAA 2012 MM (CAS)]

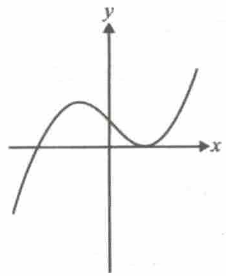
Question 170

If $f'(x) = 3x^2 - 4$, which one of the following graphs could represent the graph of $y = f(x)$?

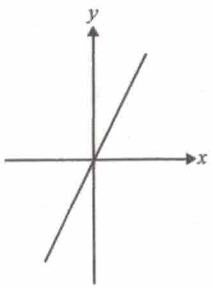
A.



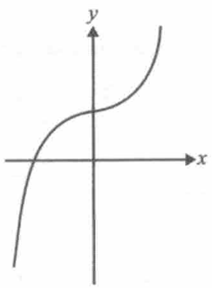
B.



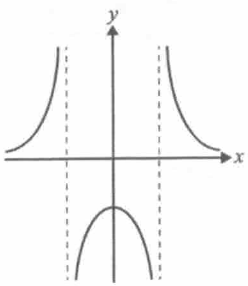
C.



D.



E.



[VCAA 2012 MM (CAS)]

Question 171

Find an anti-derivative of $(4-2x)^{-5}$ with respect to x.

[2 marks]
[VCAA 2013 MM (CAS)]

Question 172

The function with rule $g(x)$ has derivative $g'(x) = \sin(2\pi x)$.

Given that $g(1) = \frac{1}{\pi}$, find $g(x)$.

[2 marks]
[VCAA 2013 MM (CAS)]

Question 173

Let $g: R \to R,\ g(x) = (a-x)^2$, where a is a real constant.

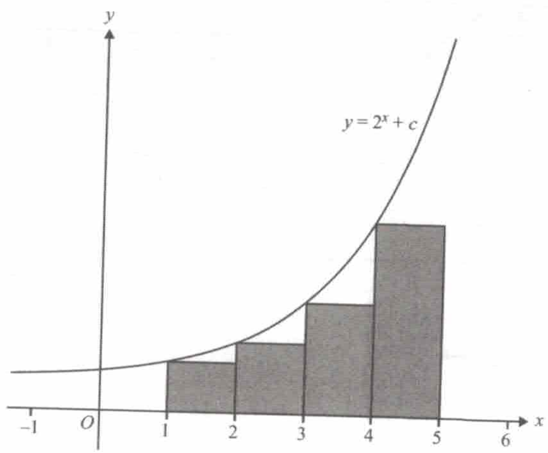
The average value of g on the interval $[-1, 1]$ is $\frac{31}{12}$.

Find all possible values of a.

[3 marks]
[VCAA 2013 MM (CAS)]

Question 174

Consider the graph of $y = 2^x + c$, where c is a real number. The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x-axis and the lines $x = 1$ and $x = 5$.



If the total area of the shaded rectangles is 44, then the value of c is

A. 14

B. –4

C. $\frac{14}{5}$

D. $\frac{7}{2}$

E. $-\frac{16}{5}$

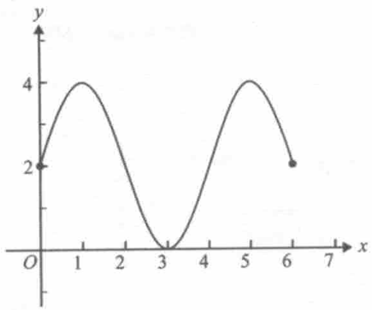
[VCAA 2013 MM (CAS)]

Question 175

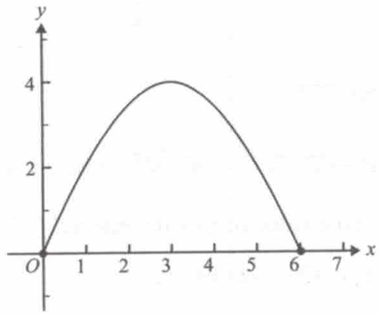
Let h be a function with an average value of 2 over the interval $[0, 6]$.

The graph of h over this interval could be

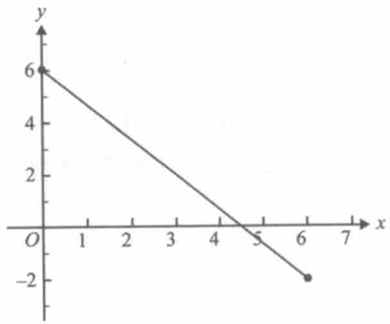
A.



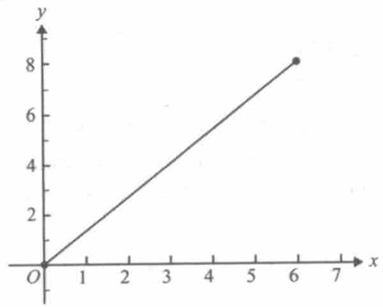
B.



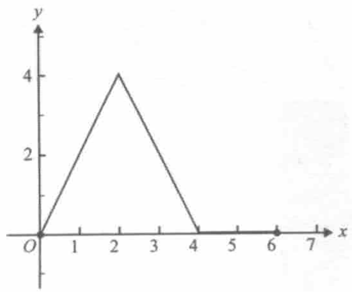
C.



D.



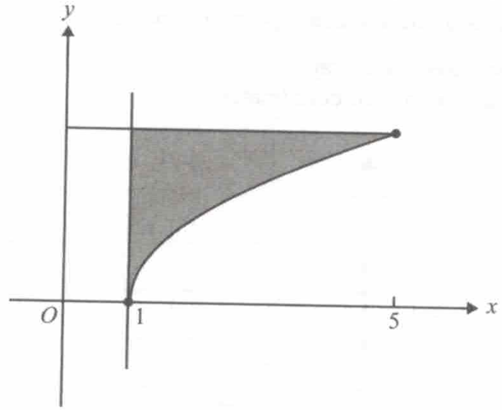
E.



[VCAA 2013 MM (CAS)]

Question 176

The graph of $f:[1, 5] \rightarrow R, f(x) = \sqrt{x-1}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_1^5 \left(\sqrt{x-1}\right)dx$

B. $\int_0^2 \left(\sqrt{x-1}\right)dx$

C. $\int_1^5 \left(2-\sqrt{x-1}\right)dx$

D. $\int_0^2 \left(x^2+1\right)dx$

E. $\int_0^2 \left(x^2\right)dx$

[VCAA 2013 MM (CAS)]

Question 177

Let $\int_4^5 \frac{2}{2x-1}dx = \log_e(b)$.

Find the value of b.

[2 marks]
[VCAA 2014 MM (CAS)]

Question 178

If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

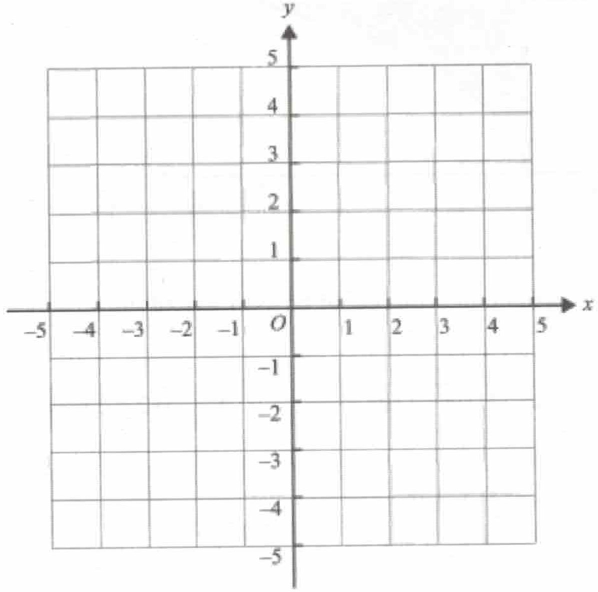
[3 marks]
[VCAA 2014 MM (CAS)]

Question 179

Consider the function $f:[-1,3]\to R,\ f(x)=3x^2-x^3$.

a. Find the coordinates of the stationary points of the function.

b. On the axes below, sketch the graph of f.
Label any end points with their coordinates.



c. Find the area enclosed by the graph of the function and the horizontal line given by $y=4$.

[2 + 2 + 3 = 7 marks]
[VCAA 2014 MM (CAS)]

Question 180

The area of the region enclosed by the graph of $y=x(x+2)(x-4)$ and the x-axis is

A. $\frac{128}{3}$ **B.** $\frac{20}{3}$ **C.** $\frac{236}{3}$ **D.** $\frac{148}{3}$ **E.** 36

[VCAA 2014 MM (CAS)]

Question 181

If $\int_1^4 f(x)\,dx=6$, then $\int_1^4 \left(5-2f(x)\right)dx$ is equal to

A. 3 **B.** 4 **C.** 5 **D.** 6 **E.** 16

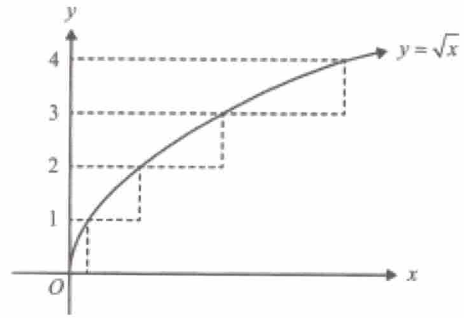
[VCAA 2014 MM (CAS)]

Part A3: Integration

Question 182

Jake and Anita are calculating the area between the graph of $y=\sqrt{x}$ and the y-axis between $y=0$ and $y=4$.

Jake uses a partitioning, shown in the diagram below, while Anita uses a definite integral to find the exact area.



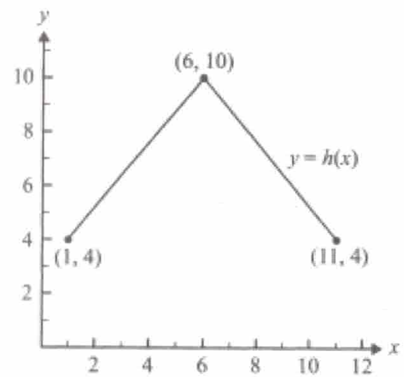
The difference between the results obtained by Jake and Anita is

A. 0 **B.** $\frac{22}{3}$ **C.** $\frac{26}{3}$ **D.** 14 **E.** 35

[VCAA 2014 MM (CAS)]

Question 183

The graph of a function, h, is shown below.



The average value of h is

A. 4 **B.** 5 **C.** 6 **D.** 7 **E.** 10

[VCAA 2014 MM (CAS)]

Question 184

Let $f'(x) = 1 - \frac{3}{x}$, where $x \neq 0$. Given that $f(e) = -2$, find $f(x)$.

[3 marks]
[VCAA 2015 MM (CAS)]

Question 185

Evaluate $\int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx$.

[2 marks]
[VCAA 2015 MM (CAS)]

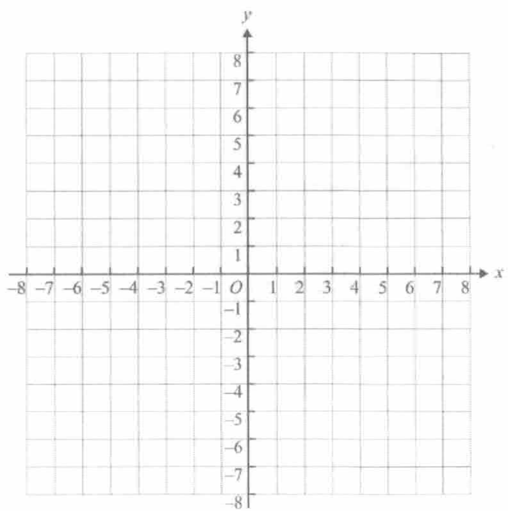
Question 186

Consider the function $f : [-3, 2] \to R$, $f(x) = \frac{1}{2}(x^3 + 3x^2 - 4)$.

a. Find the coordinates of the stationary points of the function.

The rule for f can also be expressed as $f(x) = \frac{1}{2}(x-1)(x+2)^2$.

b. On the axes below, sketch the graph of f, clearly indicating axis intercepts and turning points.
Label the end points with their coordinates.



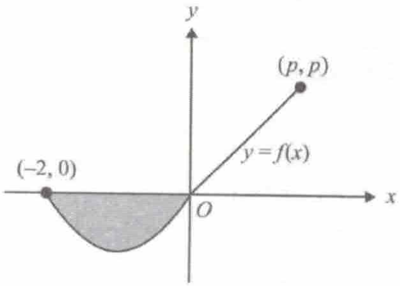
c. Find the average value of f over the interval $0 \leq x \leq 2$.

[2 + 2 + 2 = 6 marks]
[VCAA 2015 MM (CAS)]

Part A3: Integration

Question 187

The graph of a function f: $[-2, p] \to R$ is shown below.



The average value of f over the interval $[-2, p]$ is zero.

The area of the shaded region is $\frac{25}{8}$.

If the graph is a straight line, for $0 \leq x \leq p$, then the value of p is

A. 2 **B.** 5 **C.** $\frac{5}{4}$ **D.** $\frac{5}{2}$ **E.** $\frac{25}{4}$

[VCAA 2015 MM (CAS)]

Question 188

If $\int_0^5 g(x)\,dx = 20$ and $\int_0^5 (2g(x) + ax)\,dx = 90$, then the value of a is

A. 0 **B.** 4 **C.** 2 **D.** −3 **E.** 1

[VCAA 2015 MM (CAS)]

Question 189

Let $f(x) = ax^m$ and $g(x) = bx^n$, where a, b, m and n are positive integers.
The domain of f = domain of $g = R$.
If $f'(x)$ is an antiderivative of $g(x)$, then which one of the following must be true?

A. $\frac{m}{n}$ is an integer **B.** $\frac{n}{m}$ is an integer **C.** $\frac{a}{b}$ is an integer

D. $\frac{b}{a}$ is an integer **E.** $n - m = 2$

[VCAA 2015 MM (CAS)]

Question 190

If $f(x) = \int_0^x \left(\sqrt{t^2 + 4}\right) dt$, then $f'(-2)$ is equal to

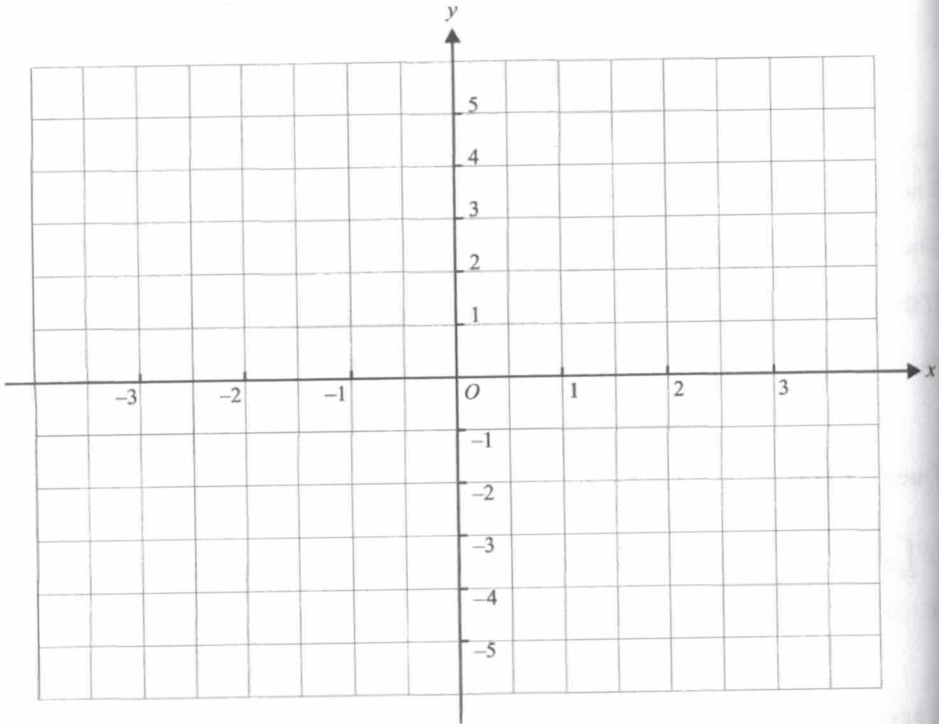
A. $\sqrt{2}$ **B.** $-\sqrt{2}$ **C.** $2\sqrt{2}$ **D.** $-2\sqrt{2}$ **E.** $4\sqrt{2}$

[VCAA 2015 MM (CAS)]

Question 191

Let $f: R\setminus\{1\} \to R$, where $f(x) = 2 + \dfrac{3}{x-1}$.

a. Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation.



b. Find the area enclosed by the graph of f, the lines $x = 2$ and $x = 4$, and the x-axis.

[3 + 2 = 5 marks]
[VCAA 2016 MM]

Question 192

Let $f: [-\pi, \pi] \to R$, where $f(x) = 2\sin(2x) - 1$.

a. Calculate the average rate of change of f between $x = -\dfrac{\pi}{3}$ and $x = \dfrac{\pi}{6}$.

b. Calculate the average value of f over the interval $-\dfrac{\pi}{3} \le x \le \dfrac{\pi}{6}$.

[2 + 3 = 5 marks]
[VCAA 2016 MM]

Part A3: Integration

Question 193

Given that $\dfrac{d(xe^{kx})}{dx} = (kx+1)e^{kx}$, then $\int xe^{kx}\,dx$ is equal to

A. $\dfrac{xe^{kx}}{kx+1} + c$

B. $\left(\dfrac{kx+1}{k}\right)e^{kx} + c$

C. $\dfrac{1}{k}\int e^{kx}\,dx$

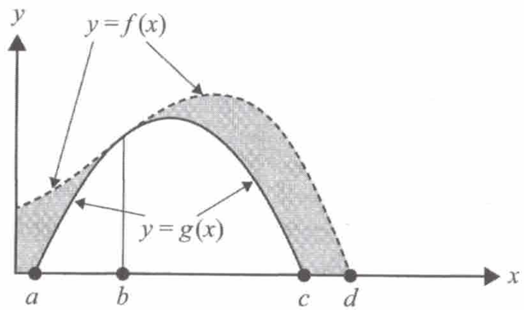
D. $\dfrac{1}{k}\left(xe^{kx} - \int e^{kx}\,dx\right) + c$

E. $\dfrac{1}{k^2}\left(xe^{kx} - e^{kx}\right) + c$

[VCAA 2016 MM]

Question 194

Consider the graphs of the functions f and g shown below.



The area of the shaded region could be represented by

A. $\int_a^d (f(x) - g(x))\,dx$

B. $\int_0^d (f(x) - g(x))\,dx$

C. $\int_0^b (f(x) - g(x))\,dx + \int_b^c (f(x) - g(x))\,dx$

D. $\int_0^a f(x)\,dx + \int_a^c (f(x) - g(x))\,dx + \int_b^d f(x)\,dx$

E. $\int_0^d f(x)\,dx - \int_a^c g(x)\,dx$

[VCAA 2016 MM]

Question 195

Consider the transformation T, defined as

$$T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

The transformation T maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

If $\int_0^3 f(x)\,dx = 5$ then $\int_{-3}^0 g(x)\,dx$ is equal to

A. 0 **B.** 15 **C.** 20 **D.** 25 **E.** 30

[VCAA 2016 MM]

Question 196

Let $y = x\log_e(3x)$.

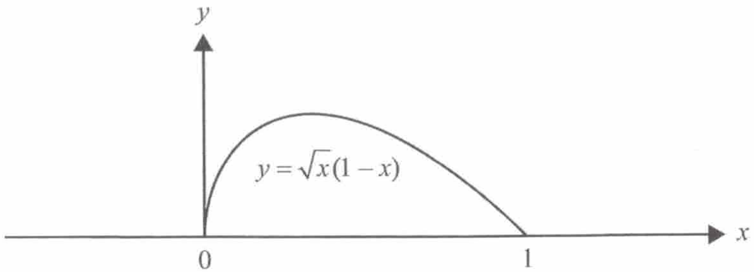
a. Find $\frac{dy}{dx}$.

b. Hence, calculate $\int_1^2 (\log_e(3x)+1)\,dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer.

[2 + 2 = 4 marks]
[VCAA 2017 MM]

Question 197

The graph of f: $[0, 1] \to R$, $f(x) = \sqrt{x}(1-x)$ is shown below.

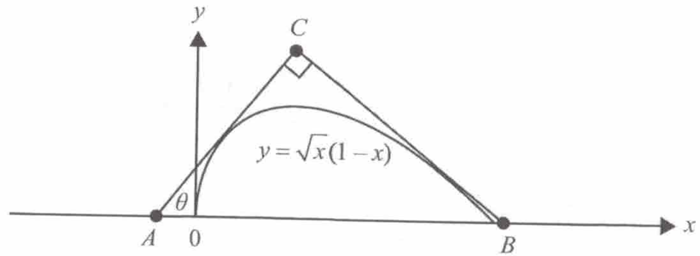


a. Calculate the area between the graph of f and the x-axis.

b. For x in the interval $(0, 1)$, show that the gradient of the tangent to the graph of f is $\frac{1-3x}{2\sqrt{x}}$.

The edges of the **right-angled** triangle ABC are the line segments AC and BC, which are tangent to the graph of f, and the line segment AB, which is part of the horizontal axis, as shown below.

Let θ be the angle that AC makes with the positive direction of the horizontal axis, where $45° \le \theta \le 90°$.



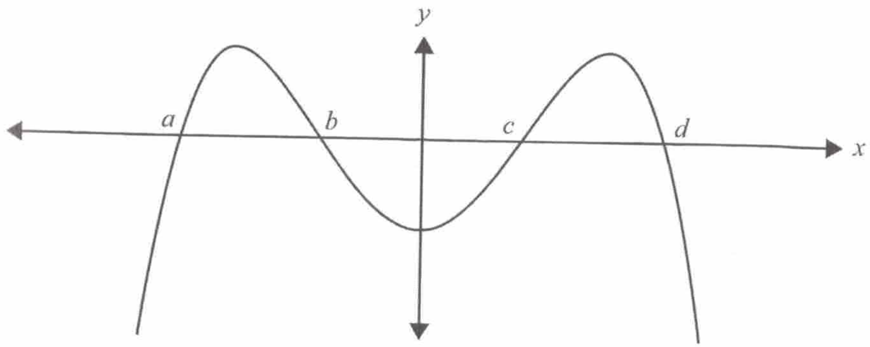
c. Find the equation of the line through B and C in the form $y = mx + c$, for $\theta = 45°$.

d. Find the coordinates of C when $\theta = 45°$.

[2 + 1 + 2 + 4 = 9 marks]
[VCAA 2017 MM]

Question 198

The graph of a function f, where $f(-x) = f(x)$, is shown below.



The graph has x-intercepts at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$ only.

The area bound by the curve and the x-axis on the interval $[a, d]$ is

A. $\int_a^d f(x)dx$

B. $\int_a^b f(x)dx - \int_c^b f(x)dx + \int_c^d f(x)dx$

C. $2\int_a^b f(x)dx + \int_b^c f(x)dx$

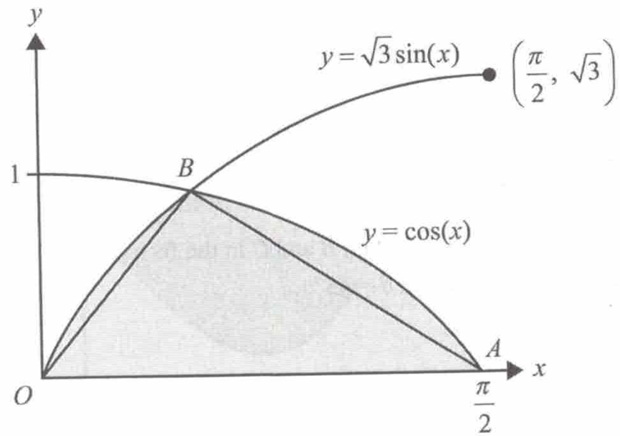
D. $2\int_a^b f(x)dx - 2\int_b^{\frac{b+c}{2}} f(x)dx$

E. $\int_a^b f(x)dx + \int_c^b f(x)dx + \int_d^c f(x)dx$

[VCAA 2017 MM]

Question 199

The graphs of $f:\left[0,\frac{\pi}{2}\right]\rightarrow R$, $f(x)=\cos(x)$ and $g:\left[0,\frac{\pi}{2}\right]\rightarrow R$, $g(x)=\sqrt{3}\sin(x)$ are shown below. The graphs intersect at B.



The ratio of the area of the shaded region to the area of triangle OAB is

A. $9:8$

B. $\sqrt{3}-1:\frac{\sqrt{3}\pi}{8}$

C. $8\sqrt{3}-3:3\pi$

D. $\sqrt{3}-1:\frac{\sqrt{3}\pi}{4}$

E. $1:\frac{\sqrt{3}\pi}{8}$

[VCAA 2017 MM]

Part A4: Discrete probability

Question 200

Four identical balls are numbered 1, 2, 3 and 4 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box.

a. What is the probability that the first ball drawn is numbered 4 and the second ball drawn is numbered 1?

b. What is the probability that the sum of the numbers on the two balls is 5?

c. Given that the sum of the numbers on the two balls is 5, what is the probability that the second ball drawn is numbered 1?

[1 + 1 + 2 = 4 marks]
[VCAA 2009 MM (CAS)]

Question 201

The random variable X has this probability distribution.

x	0	1	2	3	4
$\Pr(X=x)$	0.1	0.2	0.4	0.2	0.1

Find

a. $\Pr(X>1 \mid X\le 3)$

b. $\text{var}(X)$, the variance of X.

[2 + 3 = 5 marks]
[VCAA 2009 MM (CAS)]

Question 202

The discrete random variable X has a probability distribution as shown.

x	0	1	2	3
$\Pr(X=x)$	0.4	0.2	0.3	0.1

The median of X is

A. 0 **B.** 1 **C.** 1.1 **D.** 1.2 **E.** 2

[VCAA 2009 MM (CAS)]

Question 203

A fair coin is tossed twelve times.
The probability (correct to four decimal places) that at most 4 heads are obtained is

A. 0.0730 **B.** 0.1209 **C.** 0.1938 **D.** 0.8062 **E.** 0.9270

[VCAA 2009 MM (CAS)]

Question 204

The sample space when a fair twelve-sided die is rolled is
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
Each outcome is equally likely.
For which one of the following pairs of events are the events independent?

A. {1, 3, 5, 7, 9, 11} and {1, 4, 7, 10}
B. {1, 3, 5, 7, 9, 11} and {2, 4, 6, 8, 10, 12}
C. {4, 8, 12} and {6, 12}
D. {6, 12} and {1, 12}
E. {2, 4, 6, 8, 10, 12} and {1, 2, 3}

[VCAA 2009 MM (CAS)]

Question 205

The discrete random variable X has the probability function

x	-1	0	1	2
$\Pr(X = x)$	p^2	p^2	$\frac{p}{4}$	$\frac{4p+1}{8}$

Find the value of p.

[3 marks]
[VCAA 2010 MM (CAS)]

Question 206

A soccer player is practising her goal kicking. She has a probability of $\frac{3}{5}$ of scoring a goal with each attempt. She has 15 attempts.
The probability that the number of goals she scores is less than 7 is closest to

A. 0.0612 **B.** 0.0950 **C.** 0.1181 **D.** 0.2131 **E.** 0.7869

[VCAA 2010 MM (CAS)]

Question 207

A bag contains four white balls and six black balls. Three balls are drawn from the bag without replacement. The probability that they are all black is

A. $\frac{1}{6}$ **B.** $\frac{27}{125}$ **C.** $\frac{24}{29}$ **D.** $\frac{3}{500}$ **E.** $\frac{8}{125}$

[VCAA 2010 MM (CAS)]

Question 208

The discrete random variable X has the following probability distribution.

x	0	1	2
$\Pr(X = x)$	a	b	0.4

If the mean of X is 1, then

A. $a = 0.3$ and $b = 0.1$ **B.** $a = 0.2$ and $b = 0.2$ **C.** $a = 0.4$ and $b = 0.2$
D. $a = 0.1$ and $b = 0.5$ **E.** $a = 0.1$ and $b = 0.3$

[VCAA 2010 MM (CAS)]

Question 209

Events A and B are mutually exclusive events of a sample space with
$\Pr(A) = p$ and $\Pr(B) = q$ where $0 < p < 1$ and $0 < q < 1$.
$\Pr(A' \cap B')$ is equal to

A. $(1-p)(1-q)$ **B.** $1-pq$ **C.** $1-(p+q)$
D. $2-p-q$ **E.** $1-(p+q-pq)$

[VCAA 2010 MM (CAS)]

Question 210

A biased coin is tossed three times. The probability of a head from a toss of this coin is p.

a. Find, in terms of p, the probability of obtaining
  i. three heads from the three tosses
  ii. two heads and a tail from the three tosses.

b. If the probability of obtaining three heads equals the probability of obtaining two heads and a tail, find p.

[2 + 2 = 4 marks]
[VCAA 2011 MM (CAS)]

Question 211

Two events, A and B, are such that $\Pr(A)=\frac{3}{5}$ and $\Pr(B)=\frac{1}{4}$.

If A' denotes the complement of A, calculate $\Pr(A'\cap B)$ when

a. $\Pr(A\cup B)=\frac{3}{4}$

b. A and B are mutually exclusive.

[2 + 1 = 3 marks]
[VCAA 2011 MM (CAS)]

Question 212

For two events, P and Q, $\Pr(P\cap Q)=\Pr(P'\cap Q)$. P and Q will be independent events exactly when

A. $\Pr(P')=\Pr(Q)$ **B.** $\Pr(P\cap Q')=\Pr(P'\cap Q)$

C. $\Pr(P\cap Q)=\Pr(P)+\Pr(Q)$ **D.** $\Pr(P\cap Q')=\Pr(P\cap Q)$

E. $\Pr(P)=\frac{1}{2}$

[VCAA 2011 MM (CAS)]

Question 213

On any given day, the number X of telephone calls that Daniel receives is a random variable with probability distribution given by

x	0	1	2	3
$\Pr(X=x)$	0.2	0.2	0.5	0.1

a. Find the mean of X.

b. What is the probability that Daniel receives only one telephone call on each of three consecutive days?

c. Daniel receives telephone calls on both Monday and Tuesday.
What is the probability that Daniel receives a total of four calls over these two days?

[2 + 1 + 3 = 6 marks]
[VCAA 2012 MM (CAS)]

Question 214

Demelza is a badminton player. If she wins a game, the probability that she will win the next game is 0.7. If she loses a game, the probability that she will lose the next game is 0.6. Demelza has just won a game.
The probability that she will win exactly one of her next two games is

A. 0.33 **B.** 0.35 **C.** 0.42 **D.** 0.49 **E.** 0.82

[VCAA 2012 MM (CAS)]

Part A4: Discrete probability

Question 215

A and B are events of a sample space S.

$$\Pr(A\cap B)=\frac{2}{5} \text{ and } \Pr(A\cap B')=\frac{3}{7}.$$

$\Pr(B'\mid A)$ is equal to

A. $\frac{6}{35}$ **B.** $\frac{15}{29}$ **C.** $\frac{14}{35}$ **D.** $\frac{29}{35}$ **E.** $\frac{2}{3}$

[VCAA 2012 MM (CAS)]

Question 216

A discrete random variable X has the probability function $\Pr(X=k)=(1-p)^k p$, where k is a non-negative integer. $\Pr(X>1)$ is equal to

A. $1-p+p^2$ **B.** $1-p^2$ **C.** $p-p^2$ **D.** $2p-p^2$ **E.** $(1-p)^2$

[VCAA 2012 MM (CAS)]

Question 217

The probability distribution of a discrete random variable, X, is given by the table below.

x	0	1	2	3	4
$\Pr(X=x)$	0.2	$0.6p^2$	0.1	$1-p$	0.1

a. Show that $p=\frac{2}{3}$ or $p=1$.

b. Let $p=\frac{2}{3}$.

i. Calculate $E(X)$.

ii. Find $\Pr(X\geq E(X))$.

[3 + 2 + 1 = 6 marks]
[VCAA 2013 MM (CAS)]

Question 218

Harry is a soccer player who practises penalty kicks many times each day.
Each time Harry takes a penalty kick, the probability that he scores a goal is 0.7, independent of any other penalty kick.
One day Harry took 20 penalty kicks.
Given that he scored at least 12 goals, the probability that Harry scored exactly 15 goals is closest to

A. 0.1789 **B.** 0.8867 **C.** 0.8 **D.** 0.6396 **E.** 0.2017

[VCAA 2013 MM (CAS)]

Question 219

For events A and B, $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is

A. 0 **B.** $\frac{1}{4}$ **C.** $\frac{3}{8}$ **D.** $\frac{1}{2}$ **E.** $\frac{3}{5}$

[VCAA 2013 MM (CAS)]

Question 220

A and B are events of a sample space.

Given that $\Pr(A \mid B) = p$, $\Pr(B) = p^2$ and $\Pr(A) = p^{\frac{1}{3}}$, $\Pr(B \mid A)$ is equal to

A. p **B.** $p^{\frac{4}{3}}$ **C.** $p^{\frac{7}{3}}$ **D.** $p^{\frac{8}{3}}$ **E.** p^3

[VCAA 2013 MM (CAS)]

Question 221

Sally aims to walk her dog, Mack, most mornings. If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$, and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning.

a. In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning.

Find the probability that Sally walked Mack on at least one of these two mornings.

b. In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$.

i. Find the probability that on a particular morning in April, Sally walked Mack.

ii. Using your answer from **part b.i.**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning.

[2 + 2 + 2 = 6 marks]
[VCAA 2014 MM (CAS)]

Part A4: Discrete probability

Question 222

A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.
The probability that the marbles are different colours is

A. $\frac{20}{81}$ **B.** $\frac{5}{18}$ **C.** $\frac{4}{9}$ **D.** $\frac{40}{81}$ **E.** $\frac{5}{9}$

[VCAA 2014 MM (CAS)]

Question 223

John and Rebecca are playing darts. The result of each of their throws is independent of the result of any other throw.

The probability that John hits the bullseye with a single throw is $\frac{1}{4}$.

The probability that Rebecca hits the bullseye with a single throw is $\frac{1}{2}$. John has four throws and Rebecca has two throws.

The ratio of the probability of Rebecca hitting the bullseye at least once to the probability of John hitting the bullseye at least once is

A. 1:1 **B.** 32:27 **C.** 64:85 **D.** 2:1 **E.** 192:175

[VCAA 2014 MM (CAS)]

Question 224

An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse.

Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

a. An egg is selected at random from the set of all eggs at the warehouse.
Find, in terms of p, the probability that the egg has a white eggshell.

b. Another egg is selected at random from the set of all eggs at the warehouse.

i. Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B.

ii. If the probability that this egg came from farm B is 0.3, find the value of p.

[1 + 2 + 1 = 4 marks]
[VCAA 2015 MM (CAS)]

Question 225

For events A and B from a sample space, $\Pr(A \mid B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

a. Calculate $\Pr(A \cap B)$.

b. Calculate $\Pr(A' \cap B)$, where A' denotes the complement of A.

c. If events A and B are independent, calculate $\Pr(A \cup B)$.

[1 + 1 + 1 = 3 marks]
[VCAA 2015 MM (CAS)]

Question 226

A box contains five red balls and three blue balls. John selects three balls from the box, without replacing them.

The probability that at least one of the balls that John selected is red is

A. $\frac{5}{7}$ **B.** $\frac{5}{14}$ **C.** $\frac{7}{28}$ **D.** $\frac{15}{56}$ **E.** $\frac{55}{56}$

[VCAA 2015 MM (CAS)]

Question 227

The binomial random variable, X, has $\mathrm{E}(X) = 2$ and $\mathrm{var}(X) = \frac{4}{3}$.

$\Pr(X = 1)$ is equal to

A. $\left(\frac{1}{3}\right)^6$ **B.** $\left(\frac{2}{3}\right)^6$ **C.** $\frac{1}{3} \times \left(\frac{2}{3}\right)^2$ **D.** $6 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^5$ **E.** $6 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^5$

[VCAA 2015 MM (CAS)]

Question 228

Consider the following discrete probability distribution for the random variable X.

x	1	2	3	4	5
$\Pr(X = x)$	p	$2p$	$3p$	$4p$	$5p$

The mean of this distribution is

A. 2 **B.** 3 **C.** $\frac{7}{2}$ **D.** $\frac{11}{3}$ **E.** 4

[VCAA 2015 MM (CAS)]

Part A4: Discrete probability

Question 229

A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then returned to the paddock.

a. What is the probability that the number of tagged sheep selected on a given day is zero?

b. What is the probability that at least one tagged sheep is selected on a given day?

c. What is the probability that no tagged sheep are selected on each of six consecutive days?

Express your answer in the form $\left(\frac{a}{b}\right)^c$, where a, b and c are positive integers.

[1 + 1 + 1 = 3 marks]
[VCAA 2016 MM]

Question 230

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B.
At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a. What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer.

b. The selected motor is found to be faulty.
What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer.

[2 + 1 = 3 marks]
[VCAA 2016 MM]

Question 231

The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$\Pr(X = x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

A. 0.3 **B.** 0.305 **C.** 0.355 **D.** 0.405 **E.** 0.8

[VCAA 2016 MM]

Question 232

A box contains six red marbles and four blue marbles. Two marbles are drawn from the box, without replacement.
The probability that they are the same colour is

A. $\frac{1}{2}$ **B.** $\frac{28}{45}$ **C.** $\frac{7}{15}$ **D.** $\frac{3}{5}$ **E.** $\frac{1}{3}$

[VCAA 2016 MM]

Question 233

Consider the discrete probability distribution with random variable X shown in the table below.

x	-1	0	b	$2b$	4
$\Pr(X = x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of $E(X)$ are respectively

A. -0.8 and 1

B. -0.8 and 1.6

C. 0 and 2.4

D. 0.2125 and 1

E. 0 and 1

[VCAA 2016 MM]

Part A4: Discrete probability

Question 234

For Jac to log on to a computer successfully, Jac must type the correct password. Unfortunately, Jac has forgotten the password. If Jac types the wrong password, Jac can make another attempt. The probability of success on any attempt is $\frac{2}{5}$. Assume that the result of each attempt is independent of the result of any other attempt. A maximum of three attempts can be made.

a. What is the probability that Jac does not log on to the computer successfully?

b. Calculate the probability that Jac logs on to the computer successfully. Express your answer in the form $\frac{a}{b}$, where a and b are positive integers.

c. Calculate the probability that Jac logs on to the computer successfully on the second or on the third attempt. Express your answer in the form $\frac{c}{d}$, where c and d are positive integers.

[1 + 1 + 2 = 4 marks]
[VCAA 2017 MM]

Question 235

For events A and B from a sample space, $\Pr(A|B) = \frac{1}{5}$ and $\Pr(B|A) = \frac{1}{4}$.

Let $\Pr(A \cap B) = p$.

a. Find $\Pr(A)$ in terms of p.

b. Find $\Pr(A' \cap B')$ in terms of p.

c. Given that $\Pr(A \cup B) \leq \frac{1}{5}$, state the largest possible interval for p.

[1 + 1 + 2 = 4 marks]
[VCAA 2017 MM]

Question 236

A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement.
The probability that the marbles are of **different** colours is

A. $\frac{5}{8}$ **B.** $\frac{3}{5}$ **C.** $\frac{15}{28}$ **D.** $\frac{15}{56}$ **E.** $\frac{30}{28}$

[VCAA 2017 MM]

Question 237

The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

x	-1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

A. $2p(1 - 3p)$ **B.** $1 - 4p$ **C.** $(1 - 3p)^2$ **D.** $6p - 16p^2$ **E.** $p(5 - 9p)$

[VCAA 2017 MM]

Question 238

Let X be a discrete random variable with binomial distribution $X \sim \text{Bi}(n, p)$. The mean and the standard deviation of this distribution are equal.
Given that $0 < p < 1$, the smallest number of trials, n, such that $p \leq 0.01$ is

A. 37 **B.** 49 **C.** 98 **D.** 99 **E.** 101

[VCAA 2017 MM]

Part A5: Continuous probability and Statistics

Question 239

The continuous random variable X has a normal distribution with mean 14 and standard deviation 2.
If the random variable Z has the standard normal distribution, then the probability that X is greater than 17 is equal to

A. $\Pr(Z > 3)$ **B.** $\Pr(Z < 2)$ **C.** $\Pr(Z < 1.5)$
D. $\Pr(Z < -1.5)$ **E.** $\Pr(Z > 2)$

[VCAA 2009 MM (CAS)]

Question 240

The continuous random variable X has a probability density function given by

$$y = \begin{cases} \pi \sin(2\pi x) & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X > a) = 0.2$ is closest to

A. 0.26 **B.** 0.30 **C.** 0.32 **D.** 0.35 **E.** 0.40

[VCAA 2009 MM (CAS)]

Question 241

The continuous random variable X has a distribution with probability density function given by

$$f(x) = \begin{cases} ax(5 - x) & \text{if } 0 \leq x \leq 5 \\ 0 & \text{if } x < 0 \text{ or if } x > 5 \end{cases}$$

where a is a positive constant.

a. Find the value of a.

b. Express $\Pr(X < 3)$ as a definite integral. (Do **not** evaluate the definite integral.)

[3 + 1 = 4 marks]
[VCAA 2010 MM (CAS)]

Question 242

Let X be a normally distributed random variable with mean 5 and variance 9 and let Z be the random variable with the standard normal distribution.

a. Find $\Pr(X > 5)$.

b. Find b such that $\Pr(X > 7) = \Pr(Z < b)$.

[1 + 2 = 3 marks]
[VCAA 2010 MM (CAS)]

Question 243

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases}\cos(2x) & \text{if } \frac{3\pi}{4}<x<\frac{5\pi}{4}\\ 0 & \text{elsewhere}\end{cases}$$

The value of a such that $\Pr(X < a) = 0.25$ is closest to

A. 2.25 **B.** 2.75 **C.** 2.88 **D.** 3.06 **E.** 3.41

[VCAA 2010 MM (CAS)]

Question 244

The continuous random variable X has a normal distribution with mean 20 and standard deviation 6. The continuous random variable Z has the standard normal distribution. The probability that Z is between –2 and 1 is equal to

A. $\Pr(18 < X < 21)$ **B.** $\Pr(14 < X < 32)$ **C.** $\Pr(14 < X < 26)$

D. $\Pr(8 < X < 32)$ **E.** $\Pr(X > 14) + \Pr(X < 26)$

[VCAA 2010 MM (CAS)]

Question 245

For the continuous random variable X with probability density function

$$f(x)=\begin{cases}\log_e(x) & 1\le x\le e\\ 0 & \text{elsewhere}\end{cases}$$

the expected value of X, $\mathrm{E}(X)$, is closest to

A. 0.358 **B.** 0.5 **C.** 1 **D.** 1.859 **E.** 2.097

[VCAA 2011 MM (CAS)]

Part A5: Continuous probability and Statistics

Question 246

The continuous random variable X has a normal distribution with mean 30 and standard deviation 5. For a given number a, $\Pr(X > a) = 0.20$.
Correct to two decimal places, a is equal to

A. 23.59 **B.** 24.00 **C.** 25.79 **D.** 34.21 **E.** 36.41

[VCAA 2011 MM (CAS)]

Question 247

In an orchard of 2000 apple trees it is found that 1735 trees have a height greater than 2.8 metres. The heights are distributed normally with a mean μ and standard deviation 0.2 metres. The value of μ is closest to

A. 3.023 **B.** 2.577 **C.** 2.230 **D.** 1.115 **E.** 0.223

[VCAA 2011 MM (CAS)]

Question 248

a. The random variable X is normally distributed with mean 100 and standard deviation 4.
If $\Pr(X < 106) = q$, find $\Pr(94 < X < 100)$ in terms of q.

b. The probability density function f of a random variable X is given by

$$f(x)=\begin{cases}\frac{x+1}{12} & 0\le x\le 4\\ 0 & \text{otherwise}\end{cases}$$

Find the value of b such that $\Pr(X \le b) = \frac{5}{8}$.

[2 + 3 = 5 marks]
[VCAA 2012 MM (CAS)]

Question 249

The weights of bags of flour are normally distributed with mean 252 g and standard deviation 12 g. The manufacturer says that 40% of bags weigh more than x g.
The maximum possible value of x is closest to

A. 249.0 **B.** 251.5 **C.** 253.5 **D.** 254.5 **E.** 255.0

[VCAA 2012 MM (CAS)]

Question 250

A continuous random variable, X, has a probability density function

$$f(x)=\begin{cases}\frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) & \text{if } x\in[0,\ 2]\\ 0 & \text{otherwise}\end{cases}$$

Given that $\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)=\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)+\sin\left(\frac{\pi x}{4}\right)$, find E($X$).

[3 marks]
[VCAA 2013 MM (CAS)]

Question 251

Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

A. 7 days **B.** 13 days **C.** 17 days **D.** 21 days **E.** 37 days

[VCAA 2013 MM (CAS)]

Question 252

A continuous random variable, X, has a probability density function given by

$$f(x)=\begin{cases}\frac{1}{5}e^{-\frac{x}{5}} & x\geq 0\\ 0 & x<0\end{cases}$$

The median of X is m.

a. Determine the value of m.

b. The value of m is a number greater than 1. Find $\Pr(X<1\mid X\leq m)$.

[2 + 2 = 4 marks]
[VCAA 2014 MM (CAS)]

Question 253

The continuous random variable X, with probability density function $p(x)$, has mean 2 and variance 5. The value of $\int_{-\infty}^{\infty}x^2p(x)\,dx$ is

A. 1 **B.** 7 **C.** 9 **D.** 21 **E.** 29

[VCAA 2014 MM (CAS)]

Question 254

The random variable X has a normal distribution with mean 12 and standard deviation 0.5.

If Z has the standard normal distribution, then the probability that X is less than 11.5 is equal to

A. $\Pr(Z>-1)$ **B.** $\Pr(Z<-0.5)$ **C.** $\Pr(Z>1)$

D. $\Pr(Z\geq 0.5)$ **E.** $\Pr(Z<1)$

[VCAA 2014 MM (CAS)]

Question 255

If X is a random variable such that $\Pr(X>5)=a$ and $\Pr(X>8)=b$, then $\Pr(X<5\mid X<8)$ is

A. $\frac{a}{b}$ **B.** $\frac{a-b}{1-b}$ **C.** $\frac{1-b}{1-a}$ **D.** $\frac{ab}{1-b}$ **E.** $\frac{a-1}{b-1}$

[VCAA 2014 MM (CAS)]

Question 256

A random sample of 100 people is selected from the population of a country. Of this sample 30 people believed that their president was doing an excellent job. The approximate 95 per cent confidence interval estimate for the proportion of the population, p, who believed that their president was doing an excellent job is

A. $0.21\leq p\leq 0.39$ **B.** $0.25\leq p\leq 0.35$ **C.** $0.29\leq p\leq 0.31$

D. $0.61\leq p\leq 0.79$ **E.** $0.65\leq p\leq 0.75$

[adapted from VCAA 1995 MM]

Question 257

A randomly selected group of 100 Victorian voters was surveyed about a proposal to increase the minimum driving age to 21. Seventy percent of the sample agreed with the proposal. The approximate 95 per cent confidence interval for the proportion, p, of the population of all voters who agreed with the proposal is

A. $0.21\leq p\leq 0.39$ **B.** $0.56\leq p\leq 0.84$ **C.** $0.61\leq p\leq 0.79$

D. $0.65\leq p\leq 0.75$ **E.** $0.63\leq p\leq 0.77$

[VCAA 1996 MM]

Question 258

Rodney rides a bicycle to work. Over a three-year period, he records the time it took him to ride to work on 1000 occasions. His results are given in the table below.

time (t minutes)	number of occasions
$t \le 20$	0
$20 < t \le 21$	3
$21 < t \le 22$	12
$22 < t \le 23$	122
$23 < t \le 24$	347
$24 < t \le 25$	355
$25 < t \le 26$	141
$26 < t \le 27$	18
$27 < t \le 28$	2
$t > 28$	0

If Rodney's trip takes longer than 25 minutes he is in danger of being late for work. The approximate 95% confidence interval for the proportion, p, of occasions when Rodney takes longer than 25 minutes to ride to work is

A. $0.14 \le p \le 0.18$ **B.** $0.15 \le p \le 0.17$ **C.** $0.45 \le p \le 0.51$
D. $0.48 \le p \le 0.55$ **E.** $0.82 \le p \le 0.86$

[adapted from VCAA 1997 MM]

Question 259

Two hundred people were given a taste test to find their preference between two brands of cola. 120 preferred Brand X and 80 preferred Brand Y.

If we think of this a sample of the preference of the population, then the approximate 95% confidence interval for the proportion of the population who prefer Brand X is

A. $[0.30, 0.50]$ **B.** $[0.33, 0.67]$ **C.** $[0.50, 0.70]$
D. $[0.53, 0.67]$ **E.** $[0.33, 0.47]$

[VCAA 1998 MM]

Question 260

A random sample of 300 is selected from the population of Australia. From this sample, 225 expressed concern about global warming.
The approximate 95% confidence interval estimate for the proportion of the population p that expressed concern about global warming is

A. $[0.20, 0.30]$ **B.** $[0.23, 0.28]$ **C.** $[0.69, 0.81]$
D. $[0.70, 0.80]$ **E.** $[0.71, 0.79]$

[adapted from VCAA 1999 MM]

Question 261

In a particular electorate of several thousand voters, 60% of the voters favour the Politically Correct Party. From one of many random samples of 100 voters, the proportion of voters in the sample who favour the Politically Correct Party is recorded. The variance for such a sample proportion is

A. 0.24 **B.** 0.024 **C.** 0.0024 **D.** $\sqrt{0.024}$ **E.** $\sqrt{0.0024}$

[VCAA 1999 MM]

Question 262

Let the random variable X be normally distributed with mean 2.5 and standard deviation 0.3. Let Z be the standard normal random variable, such that $Z \sim N(0, 1)$.

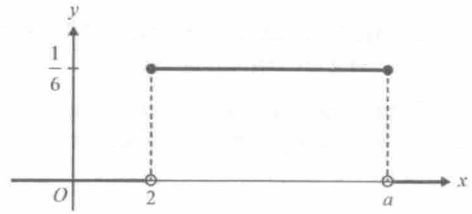
a. Find b such that $\Pr(X > 3.1) = \Pr(Z < b)$.

b. Using the fact that, correct to two decimal places, $\Pr(Z < -1) = 0.16$, find $\Pr(X < 2.8 \mid X > 2.5)$. Write the answer correct to two decimal places.

[1 + 2 = 3 marks]
[VCAA 2015 MM (CAS)]

Question 263

The graph of the probability density function of a continuous random variable, X, is shown below.



If $a > 2$, then $E(X)$ is equal to

A. 8 **B.** 5 **C.** 4 **D.** 3 **E.** 2

[VCAA 2015 MM (CAS)]

Question 264

The function f is a probability density function with rule $f(x) = \begin{cases} ae^x & 0 \le x \le 1 \\ ae & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$.

The value of a is

A. 1 **B.** e **C.** $\frac{1}{e}$ **D.** $\frac{1}{2e}$ **E.** $\frac{1}{2e-1}$

[VCAA 2015 MM (CAS)]

Question 265

A student performs an experiment in which a computer is used to simulate drawing a random sample of size n from a large population. The proportion of the population with the characteristic of interest to the student is p.

a. Let the random variable $\hat{P}$ represent the sample proportion observed in the experiment.

If $p = \frac{1}{5}$, find the smallest integer value of the sample size such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

Each of 23 students in a class independently performs the experiment described above and each student calculates an approximate 95% confidence interval for p using the sample proportions for their sample. It is subsequently found that exactly one of the 23 confidence intervals calculated by the class does not contain the value of p.

b. Two of the confidence intervals calculated by the class are selected at random without replacement.
Find the probability that exactly one of the selected confidence intervals does not contain the value of p.

[2 + 2 = 4 marks]
[VCAA 2016 SAMPLE MM]

Question 266

An opinion pollster reported that for a random sample of 574 voters in a town, 76% indicated a preference for retaining the current council.

An approximate 90% confidence interval for the proportion of the total voting population with a preference for retaining the current council can be found by evaluating

A. $\left(0.76 - \sqrt{\frac{0.76 \times 0.24}{574}}, 0.76 + \sqrt{\frac{0.76 \times 0.24}{574}}\right)$

B. $\left(0.76 - 1.65\sqrt{\frac{0.76 \times 0.24}{574}}, 0.76 + 1.65\sqrt{\frac{0.76 \times 0.24}{574}}\right)$

C. $\left(0.76 - 2.58\sqrt{\frac{0.76 \times 0.24}{574}}, 0.76 + 2.58\sqrt{\frac{0.76 \times 0.24}{574}}\right)$

D. $\left(0.76 - 1.96\sqrt{0.76 \times 0.24 \times 574}, 0.76 + 1.96\sqrt{0.76 \times 0.24 \times 574}\right)$

E. $\left(0.76 - 2\sqrt{0.76 \times 0.24 \times 574}, 0.76 + 2\sqrt{0.76 \times 0.24 \times 574}\right)$

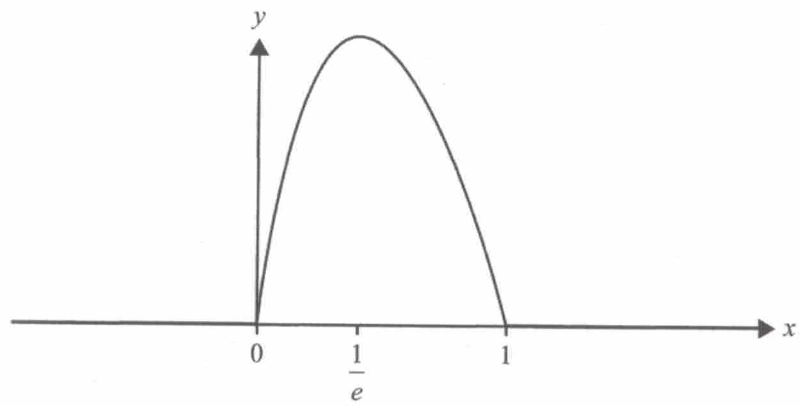
[VCAA 2016 SAMPLE MM]

Question 267

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} -4x\log_e(x) & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Part of the graph of f is shown below. The graph has a turning point at $x = \frac{1}{e}$.



a. Show by differentiation that

$$\frac{x^k}{k^2}\left(k\log_e(x) - 1\right)$$

is an antiderivative of $x^{k-1}\log_e(x)$, where k is a positive real number.

b. **i.** Calculate $\Pr\left(X > \frac{1}{e}\right)$.

ii. Hence, explain whether the median of X is greater than or less than $\frac{1}{e}$, given that $e > \frac{5}{2}$.

[2 + 2 + 2 = 6 marks]
[VCAA 2016 MM]

Question 268

The random variable, X, has a normal distribution with mean 12 and standard deviation 0.25.
If the random variable, Z, has the standard normal distribution, then the probability that X is greater than 12.5 is equal to

A. $\Pr(Z < -4)$

B. $\Pr(Z < -1.5)$

C. $\Pr(Z < 1)$

D. $\Pr(Z \geq 1.5)$

E. $\Pr(Z > 2)$

[VCAA 2016 MM]

Question 269

Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. (Do not use a normal approximation.)

$\Pr\left(\hat{P} \geq \frac{3}{16}\right)$ is closest to

A. 0.6482 **B.** 0.8593 **C.** 0.7543 **D.** 0.6542 **E.** 0.3211

[VCAA 2016 MM]

Question 270

The continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi \leq x \leq 5\pi \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X < a) = \frac{\sqrt{3}+2}{4}$ is

A. $\frac{19\pi}{6}$ **B.** $\frac{14\pi}{3}$ **C.** $\frac{10\pi}{3}$ **D.** $\frac{29\pi}{6}$ **E.** $\frac{17\pi}{3}$

[VCAA 2016 MM]

Part A5: Continuous probability and Statistics

Question 271

At a large sporting arena there are a number of food outlets, including a cafe.

a. The cafe employs five men and four women. Four of these people are rostered at random to work each day. Let $\hat{P}$ represent the sample proportion of men rostered to work on a particular day.

   i. List the possible values that $\hat{P}$ can take.

   ii. Find $\Pr(\hat{P} = 0)$.

b. There are over 80 000 spectators at a sporting match at the arena. Five in nine of these spectators support the Goannas team. A simple random sample of 2000 spectators is selected.
What is the standard deviation of the distribution of $\hat{P}$, the sample proportion of spectators who support the Goannas team?

[1 + 1 + 1 = 3 marks]
[VCAA 2017 NH MM]

Question 272

A bag contains five blue marbles and four red marbles. A sample of four marbles is taken from the bag, without replacement.

The probability that the proportion of blue marbles in the sample is greater than $\frac{1}{2}$ is

A. $\frac{1}{2}$ **B.** $\frac{2}{9}$ **C.** $\frac{5}{14}$ **D.** $\frac{5}{9}$ **E.** $\frac{25}{63}$

[VCAA 2017 NH MM]

Question 273

In a large population of fish, the proportion of angel fish is $\frac{1}{4}$.

Let $\hat{P}$ be the random variable that represents the sample proportion of angel fish for samples of size n drawn from the population.
Find the smallest integer value of n such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

[2 marks]
[VCAA 2017 MM]

Question 274

The 95% confidence interval for the proportion of ferry tickets that are cancelled on the intended departure day is calculated from a large sample to be (0.039, 0.121).

The sample proportion from which this interval was constructed is

A. 0.080 **B.** 0.041 **C.** 0.100 **D.** 0.062 **E.** 0.059

[VCAA 2017 MM]

Question 275

For random samples of five Australians, $\hat{P}$ is the random variable that represents the proportion who live in a capital city.

Given that $\Pr(\hat{P}=0)=\dfrac{1}{243}$, then $\Pr(\hat{P}>0.6)$, correct to four decimal places, is

A. 0.0453 **B.** 0.3209 **C.** 0.4609 **D.** 0.5390 **E.** 0.7901

[VCAA 2017 MM]

Question 276

Let $\hat{P}$ be the random variable that represents the sample proportions of customers who bring their own shopping bags to a large shopping centre.

From a sample consisting of all customers on a particular day, an approximate 95% confidence interval for the proportion of who bring their own shopping bags to this large shopping centre was determined to be $\left(\dfrac{4853}{50\,000}, \dfrac{5147}{50\,000}\right)$.

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval.

b. Use the fact that $1.96=\dfrac{49}{25}$ to find the size of the sample from which this approximate 95% confidence interval was obtained.

[1 + 2 = 3 marks]
[VCAA 2018 NH MM]

Question 277

A box contains 20 000 marbles that are either blue or red. There are more blue marbles than red marbles. Random samples of 100 marbles are taken from the box. Each random sample is obtained by sampling with replacement.

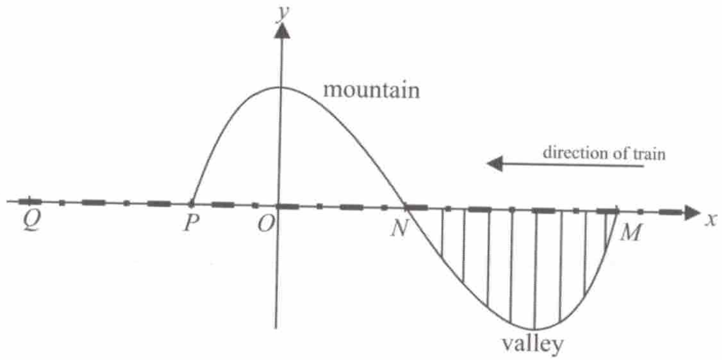
If the standard deviation of the sampling distribution for the proportion of blue marbles is 0.03, then the number of blue marbles in the box is

A. 11 000 **B.** 16 000 **C.** 17 000 **D.** 18 000 **E.** 19 000

[VCAA 2018 NH MM]

Extended-response questions

Question 278



A train is travelling at a constant speed of w km/h along a straight level track from M towards Q. The train will travel along a section of track $MNPQ$.
Section MN passes along a bridge over a valley.
Section NP passes through a tunnel in a mountain.
Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of

$$y=\frac{1}{200}(ax^3+bx^2+c)$$

where a, b and c are real numbers.
All measurements are in kilometres.

a. The curve defined from M to P passes through $N(2, 0)$. The gradient of the curve at N is –0.06 and the curve has a turning point at $x = 4$.

i. From this information write down three simultaneous equations in a, b and c.

ii. Hence show that $a = 1$, $b = -6$ and $c = 16$.

[3 + 2 = 5 marks]

b. Find, giving exact values

i. the coordinates of M and P

ii. the length of the tunnel

iii. the maximum depth of the valley below the train track.

[2 + 1 + 1 = 4 marks]

… continued

The driver sees a large rock on the track at a point Q, 6.2 km from P. The driver puts on the brakes at the instant that the front of the train comes out of the tunnel at P.
From its initial speed of w km/h, the train slows down from point P so that its speed v km/h is given by

$$v = k\log_e\left(\frac{(d+1)}{7}\right)$$

where d km is the distance of the front of the train from P and k is a real constant.

c. Find the value of k in terms of w.

[1 mark]

d. If $v = \dfrac{120\log_e(2)}{\log_e(7)}$ when $d = 2.5$, find the value of w.

[2 marks]

e. Find the exact distance from the front of the train to the large rock when the train finally stops.

[2 marks]
Total 14 marks
[VCAA 2009 MM (CAS)]

Question 279

The Bouncy Ball Company (BBC) makes tennis balls whose diameters are normally distributed with mean 67 mm and standard deviation 1 mm. The tennis balls are packed and sold in cylindrical tins that each hold four balls. A tennis ball fits into such a tin if the diameter of the ball is less than 68.5 mm.

a. What is the probability, correct to four decimal places, that a randomly selected tennis ball produced by BBC fits into a tin?

[2 marks]

BBC management would like each ball produced to have diameter between 65.6 and 68.4 mm.

b. What is the probability, correct to four decimal places, that the diameter of a randomly selected tennis ball made by BBC is in this range?

[2 marks]

c. **i.** What is the probability, correct to four decimal places, that the diameter of a tennis ball which fits into a tin is between 65.6 and 68.4 mm?

ii. A tin of four balls is selected at random. What is the probability, correct to four decimal places, that at least one of these balls has diameter outside the desired range of 65.6 to 68.4 mm?

[1 + 2 = 3 marks]

BBC management wants engineers to change the manufacturing process so that 99% of all balls produced have diameter between 65.6 and 68.4 mm. The mean is to stay at 67 mm but the standard deviation is to be changed.

d. What should the new standard deviation be (correct to two decimal places)?

[3 marks]

BBC sells tennis balls directly to tennis clubs once a year. If a tennis club buys its balls from BBC one year, there is an 80% chance it will buy its balls from BBC the next year. If a tennis club does not buy its balls from BBC one year, there is a 15% chance it will buy its balls from BBC the next year.

Suppose the Melbourne Tennis Club buys its tennis balls from BBC this year.

e. What is the exact probability that it will buy its tennis balls from BBC for the next three years?

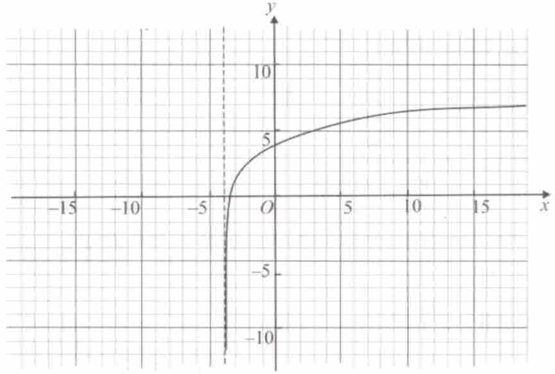
[2 marks]

f. What is the exact probability that it will buy its tennis balls from BBC for exactly two of the next three years?

[3 marks]
Total 15 marks
[adapted from VCAA 2009 MM (CAS)]

Question 280

a. Part of the graph of $g: (-4, \infty) \to R, g(x) = 2\log_e(x+4)+1$ is shown on the axes below.



i. Find the rule and domain of g^{-1}, the inverse function of g.

ii. On the set of axes above, sketch the graph of g^{-1}. Label the axes intercepts with their exact values.

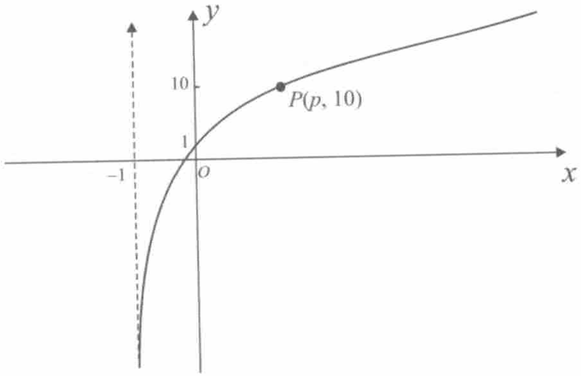
iii. Find the values of x, correct to three decimal places, for which $g^{-1}(x) = g(x)$.

iv. Calculate the area enclosed by the graphs of g and g^{-1}. Give your answer correct to two decimal places.

[3 + 3 + 2 + 2 = 10 marks]
… continued

b. The diagram below shows part of the graph of the function with rule $f(x) = k\log_e(x + a) + c$, where k, a and c are real constants.

- The graph has a vertical asymptote with equation $x = -1$.
- The graph has a y-axis intercept at 1.
- The point P on the graph has coordinates $(p, 10)$, where p is another real constant.



i. State the value of a.

ii. Find the value of c.

iii. Show that $k = \dfrac{9}{\log_e(p+1)}$.

iv. Show that the gradient of the tangent to the graph of f at the point P is $\dfrac{9}{(p+1)\log_e(p+1)}$.

v. If the point $(-1, 0)$ lies on the tangent referred to in part **b. iv.**, find the exact value of p.

[1 + 1 + 2 + 1 + 2 = 7 marks]
Total 17 marks
[VCAA 2010 MM (CAS)]

Question 281

An ancient civilization buried its kings and queens in tombs in the shape of a square-based pyramid, $WABCD$.

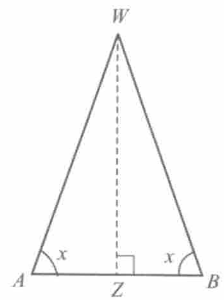
The kings and queens were each buried in a pyramid with

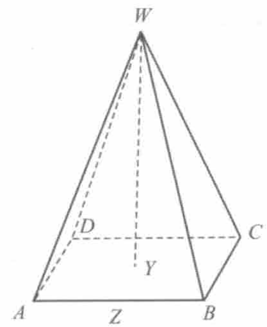
$$WA = WB = WC = WD = 10 \text{ m.}$$

Each of the isosceles triangle faces is congruent to each of the other triangular faces.

The base angle of each of these triangles is x, where $\dfrac{\pi}{4} < x < \dfrac{\pi}{2}$.

Pyramid $WABCD$ and a face of the pyramid, WAB, are shown here.





Z is the midpoint of AB.

a. **i.** Find AB in terms of x.

ii. Find WZ in terms of x.

[1 + 1 = 2 marks]

b. Show that the total surface area (including the base), S m^2, of the pyramid, $WABCD$, is given by $S = 400(\cos^2 x + \cos x \sin x)$.

[2 marks]

c. Find WY, the height of the pyramid $WABCD$, in terms of x.

[2 marks]

d. The volume of any pyramid is given by the formula

$$\text{Volume} = \frac{1}{3} \times \text{area of base} \times \text{vertical height.}$$

Show that the volume, T m^3, of the pyramid $WABCD$ is $\dfrac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}$.

[1 mark]

… continued

Queen Hepzabah's pyramid was designed so that it had the **maximum possible volume**.

e. Find $\frac{dT}{dx}$ and hence find the exact volume of Queen Hepzabah's pyramid and the corresponding value of x.

[4 marks]

Queen Hepzabah's daughter, Queen Jepzibah, was also buried in a pyramid. It also had

$WA = WB = WC = WD = 10$ m.

The volume of Jepzibah's pyramid is exactly one half of the volume of Queen Hepzabah's pyramid. The volume of Queen Jepzibah's pyramid is also given by the formula for T obtained in **part d.**

f. Find the possible values of x, for Jepzibah's pyramid, correct to two decimal places.

[2 marks]
Total 13 marks
[VCAA 2010 MM (CAS)]

Question 282

Consider the function $f: R \to R, f(x) = \frac{1}{27}(2x-1)^3(6-3x)+1$.

a. Find the x-coordinate of each of the stationary points of f and state the nature of each of these stationary points.

[4 marks]

In the following, f is the function $f: R \to R, f(x) = \frac{1}{27}(ax-1)^3(b-3x)+1$ where a and b are real constants.

b. Write down, in terms of a and b, the possible values of x for which $(x, f(x))$ is a stationary point of f.

[3 marks]

c. For what value of a does f have no stationary points?

[1 mark]

d. Find a in terms of b if f has one stationary point.

[2 marks]

e. What is the maximum number of stationary points that f can have?

[1 mark]

f. Assume that there is a stationary point at (1, 1) and another stationary point (p, p) where $p \neq 1$. Find the value of p.

[3 marks]
Total 14 marks
[VCAA 2010 MM (CAS)]

Question 283

In a chocolate factory, the material for making each chocolate is sent to one of two machines, machine A or machine B.

The time, X seconds, taken to produce a chocolate by machine A, is normally distributed with mean 3 and standard deviation of 0.8.

The time, Y seconds, taken to produce a chocolate by machine B, has the following probability density function.

$$f(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{16} & 0 \le y \le 4 \\ 0.25e^{-0.5(y-4)} & y > 4 \end{cases}$$

a. Find, correct to four decimal places

i. $\Pr(3 \le X \le 5)$

ii. $\Pr(3 \le Y \le 5)$

[1 + 3 = 4 marks]

b. Find the mean of Y, correct to three decimal places.

[3 marks]

c. **i.** Find the median of Y.

ii. Find the value of a, correct to two decimal places, such that $\Pr(Y \le a) = 0.7$.

[1 + 2 = 3 marks]

d. It can be shown that $\Pr(Y \le 3) = \frac{9}{32}$. A random sample of 10 chocolates **produced by machine *B*** is chosen. Find the probability, correct to four decimal places, that exactly 4 of these 10 chocolates took 3 or less seconds to produce.

[2 marks]

All of the chocolates produced by machine A and machine B are stored in a large bin. There is an equal number of chocolates from each machine in the bin.
It is found that if a chocolate, produced by either machine, takes longer than 3 seconds to produce then it can easily be identified by its darker colour.

e. A chocolate is selected at random from the bin. It is found to have taken longer than 3 seconds to produce. Find, correct to four decimal places, the probability that it was produced by machine A.

[3 marks]
Total 15 marks
[VCE 2011 MM (CAS)]

Question 284

a. Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9$.

i. Find $f'(x)$.

ii. Explain why $f'(x) \geq 5$ for all x.

[1 + 1 = 2 marks]

b. The cubic function p is defined by $p: R \to R, p(x) = ax^3 + bx^2 + cx + k$, where a, b, c and k are real numbers.

i. If p has m stationary points, what possible values can m have?

ii. If p has an inverse function, what possible values can m have?

[1 + 1 = 2 marks]

c. The cubic function q is defined by $q: R \to R, q(x) = 3 - 2x^3$.

i. Write down an expression for $q^{-1}(x)$.

ii. Determine the coordinates of the point(s) of intersection of the graphs of $y = q(x)$ and $y = q^{-1}(x)$.

[2 + 2 = 4 marks]

d. The cubic function g is defined by $g: R \to R, g(x) = x^3 + 2x^2 + cx + k$, where c and k are real numbers.

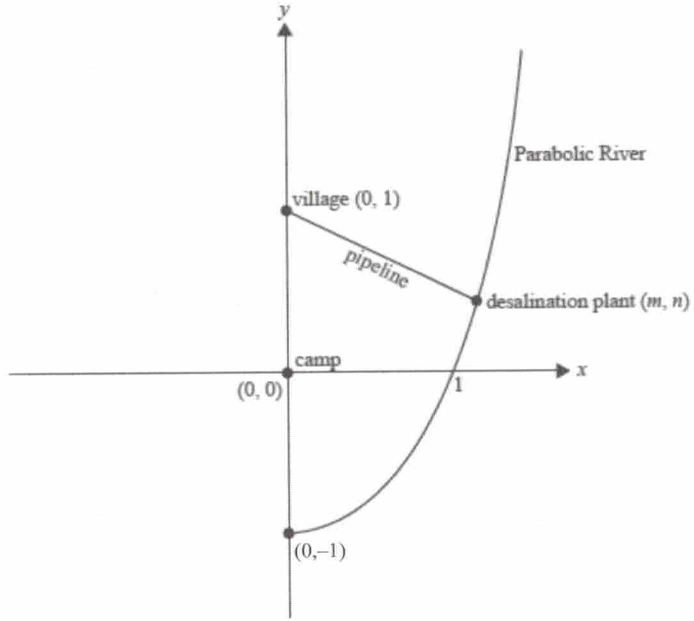
i. If g has exactly one stationary point, find the value of c.

ii. If this stationary point occurs at a point of intersection of $y = g(x)$ and $y = g^{-1}(x)$, find the value of k.

[3 + 3 = 6 marks]
Total 14 marks
[VCE 2011 MM (CAS)]

Question 285

Deep in the South American jungle, Tasmania Jones has been working to help the Quetzacotl tribe to get drinking water from the very salty water of the Parabolic River. The river follows the curve with equation $y = x^2 - 1, x \geq 0$ as shown below. All lengths are measured in kilometres. Tasmania has his camp site at (0, 0) and the Quetzalcotl tribe's village is at (0, 1). Tasmania builds a desalination plant, which is connected to the village by a straight pipeline.



a. If the desalination plant is at the point (m, n), show that the length, L kilometres, of the straight pipeline that carries the water from the desalination plant to the village is given by

$$L = \sqrt{m^4 - 3m^2 + 4}.$$

[3 marks]

b. If the desalination plant is built at the point on the river that is closest to the village

i. find $\frac{dL}{dm}$ and hence find the coordinates of the desalination plant.

ii. find the length, in kilometres, of the pipeline from the desalination plant to the village.

[3 + 2 = 5 marks]
... continued

The desalination plant is actually built at $\left(\frac{\sqrt{7}}{2}, \frac{3}{4}\right)$.

If the desalination plant stops working, Tasmania needs to get to the plant in the minimum time.
Tasmania runs in a straight line from his camp to a point (x, y) on the river bank where $x \le \frac{\sqrt{7}}{2}$. He then swims up the river to the desalination plant.
Tasmania runs from his camp to the river at 2 km per hour. The time that he takes to swim to the desalination plant is proportional to the difference between the y-coordinates of the desalination plant and the point where he enters the river.

c. Show that the total time taken to get to the desalination plant is given by $T = \frac{1}{2}\sqrt{x^4 - x^2 + 1} + \frac{1}{4}k\left(7 - 4x^2\right)$ hours, where k is a positive constant of proportionality.

[3 marks]

The value of k varies from day to day, depending on the weather conditions.

d. If $k = \frac{1}{2\sqrt{13}}$

i. find $\frac{dT}{dx}$

ii. hence find the coordinates of the point where Tasmania should reach the river if he is to get to the desalination plant in the minimum time.

[1 + 2 = 3 marks]

e. On one particular day, the value of k is such that Tasmania should run directly from his camp to the point (1, 0) on the river to get to the desalination plant in the minimum time. Find the value of k on that particular day.

[2 marks]

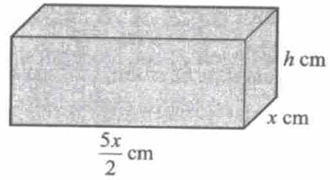
f. Find the values of k for which Tasmania should run directly from his camp towards the desalination plant to reach it in the minimum time.

[2 marks]
Total 18 marks
[VCE 2011 MM (CAS)]

Extended-response questions

Question 286

A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 sq cm.

a. Show that if the height of the block is h cm, $h = \frac{6480 - 5x^2}{7x}$.

[2 marks]

b. The volume, V cm^3, of the block is given by $V(x) = \frac{5x(6480 - 5x^2)}{14}$.

Given that $V(x) > 0$ and $x > 0$, find the possible values of x.

[2 marks]

c. Find $\frac{dV}{dx}$, expressing your answer in the form $\frac{dV}{dx} = ax^2 + b$, where a and b are real numbers.

[3 marks]

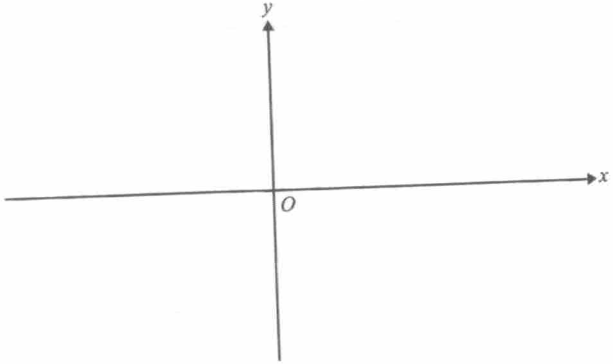
d. Find the exact values of x and h if the block is to have maximum volume.

[2 marks]
Total 9 marks
[VCAA 2012 MM (CAS)]

Question 287

Let $f: R\setminus\{2\} \to R, f(x) = \dfrac{1}{2x-4} + 3.$

a. Sketch the graph of $y = f(x)$ on the set of axes below. Label the axes intercepts with their coordinates and label each of the asymptotes with its equation.



[3 marks]

b. **i.** Find $f'(x)$.

ii. State the range of f'.

iii. Using the result of **part ii.**, explain why f has no stationary points.

[1 + 1 + 1 = 3 marks]

c. If (p, q) is any point on the graph of $y = f(x)$, show that the equation of the tangent to $y = f(x)$ at this point can be written as

$$(2p-4)^2(y-3) = -2x + 4p - 4.$$

[2 marks]

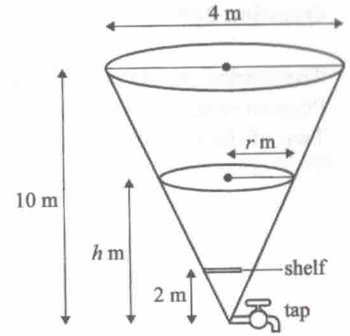
d. Find the coordinates of the points on the graph of $y = f(x)$ such that the tangents to the graph at these points intersect at $\left(-1, \dfrac{7}{2}\right)$.

[4 marks]

e. A transformation $T: R^2 \to R^2$ that maps the graph of f to the graph of the function $g: R\setminus\{0\} \to R, g(x) = \dfrac{1}{x}$ has rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$, where a, c and d are non-zero real numbers.
Find the values of a, c and d.

[2 marks]
Total 14 marks
[VCAA 2012 MM (CAS)]

Question 288



Tasmania Jones is in the jungle, searching for the Quetzalotl tribe's valuable emerald that has been stolen and hidden by a neighbouring tribe. Tasmania has heard that the emerald has been hidden in a tank shaped like an inverted cone, with a height of 10 metres and a diameter of 4 metres (as shown).
The emerald is on a shelf. The tank has a poisonous liquid in it.

a. If the depth of the liquid in the tank is h metres

i. find the radius, r metres, of the surface of the liquid in terms of h

ii. show that the volume of the liquid in the tank is $\dfrac{\pi h^3}{75}$ m^3.

[1 + 1 = 2 marks]

The tank has a tap at its base that allows the liquid to run out of it. The tank is initially full. When the tap is turned on, the liquid flows out of the tank at such a rate that the depth, h metres, of the liquid in the tank is given by

$$h = 10 + \frac{1}{1600}(t^3 - 1200t),$$

where t minutes is the length of time after the tap is turned on until the tank is empty.

b. Show that the tank is empty when $t = 20$.

[1 mark]

c. When $t = 5$ minutes, find the depth of the liquid in the tank.

[1 mark]

d. The shelf on which the emerald is placed is 2 metres above the vertex of the cone. From the moment the liquid starts to flow from the tank, find how long, in minutes, it takes until $h = 2$. (Give your answer correct to one decimal place.)

[2 marks]

e. As soon as the tank is empty, the tap turns itself off and poisonous liquid starts to flow into the tank at a rate of 0.2 m^3/minute.
How long, in minutes, after the tank is first empty will the liquid once again reach a depth of 2 metres?

[2 marks]

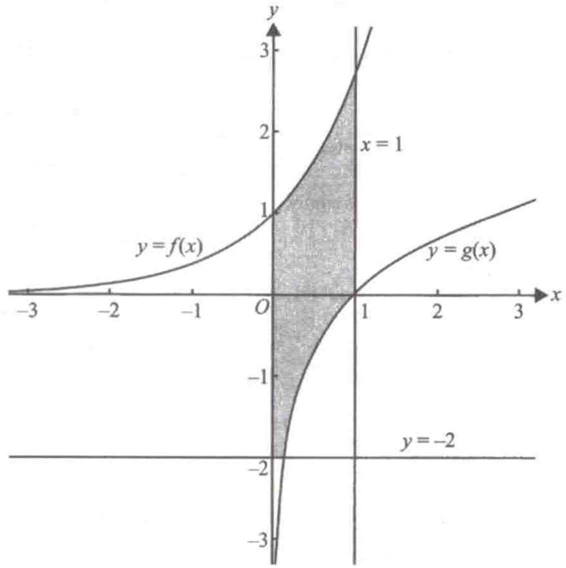
f. In order to obtain the emerald, Tasmania Jones enters the tank using a vine to climb down the wall of the tank as soon as the depth of the liquid is first 2 metres. He must leave the tank before the depth is again greater than 2 metres.
Find the length of time, in minutes, correct to one decimal place, that Tasmania Jones has from the time he enters the tank to the time he leaves the tank.

[1 mark]
Total 9 marks
[adapted from VCAA 2012 MM (CAS)]

Question 289

The shaded region in the following diagram is the plan of a mine site for the Black Possum mining company. All distances are in kilometres.
Two of the boundaries of the mine site are in the shape of the graphs of the functions

$$f: R \to R, f(x) = e^x \text{ and } g: R^+ \to R, g(x) = \log_e(x).$$



a. **i.** Evaluate $\int_{-2}^{0} f(x)dx$.

ii. Hence, or otherwise, find the area of the region bounded by the graph of g, the x and y axes, and the line $y = -2$.

iii. Find the **total** area of the shaded region.

[1 + 1 + 1 = 3 marks]

b. The mining engineer, Victoria, decides that a better site for the mine is the region bounded by the graph of g and that of a new function
$k: (-\infty, a) \to R, k(x) = -\log_e(a - x)$, where a is a positive real number.

i. Find, in terms of a, the x-coordinates of the points of intersection of the graphs of g and k.

ii. Hence, find the set of values of a, for which the graphs of g and k have two distinct points of intersection.

[2 + 1 = 3 marks]

c. For the new mine site, the graphs of g and k intersect at two distinct points, A and B. It is proposed to start mining operations along the line segment AB, which joins the two points of intersection. Victoria decides that the graph of k will be such that the x-coordinate of the midpoint of AB is $\sqrt{2}$.
Find the value of a in this case.

[2 marks]
Total 8 marks
[VCAA 2012 MM (CAS)]

Question 290

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature (T °C) is given by

$T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right), 0 \le t \le 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a. State the maximum temperature in the greenhouse and the values of t when this occurs.

[2 marks]

b. State the period of the function T.

[1 mark]

c. Find the smallest value of t for which $T = 26$.

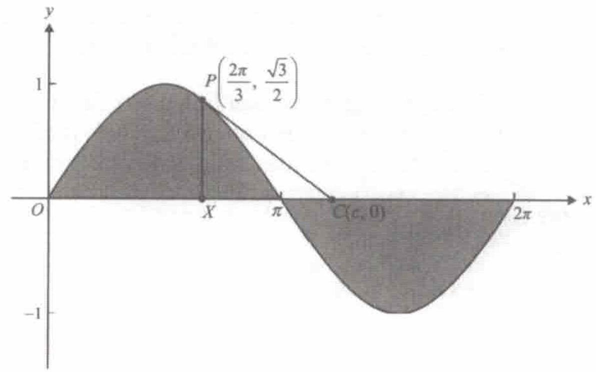
[2 marks]

d. For how many hours during the 24-hour time interval is $T \ge 26$?

[2 marks]

Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \le x \le 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram below. He includes a garden path, which is shown as line segment PC.

The line through $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ points $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P.



e. **i.** Find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$.

ii. Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$.

[1 + 1 = 2 marks]
... continued

In further planning for the garden, Trigg uses a transformation of the plane defined as a dilation of factor k from the x-axis and a dilation of factor m from the y-axis, where k and m are positive real numbers.

f. Let X', P' and C' be the image, under this transformation, of the points X, P and C respectively.

i. Find the values of k and m if $X'P' = 10$ and $X'C' = 30$.

ii. Find the coordinates of the point P'.

[2 + 1 = 3 marks]
Total 12 marks
[VCAA 2013 MM (CAS)]

Question 291

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes **S** in less than three minutes, they are considered fit.

a. At FullyFit's Melbourne gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. In a particular week, 20 members of this gym attempt **S**.

i. Find the probability, correct to four decimal places, that at least 10 of these 20 members will complete **S** in less than three minutes.

ii. Given that at least 10 of these 20 members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that more than 15 of them complete **S** in less than three minutes?

[2 + 3 = 5 marks]

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

[2 marks]

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete **S** is a continuous random variable X, with a probability density function g, as defined below.

$$g(x) = \begin{cases} \dfrac{(x-3)^3+64}{256} & 1 \le x \le 3 \\ \dfrac{x+29}{128} & 3 < x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

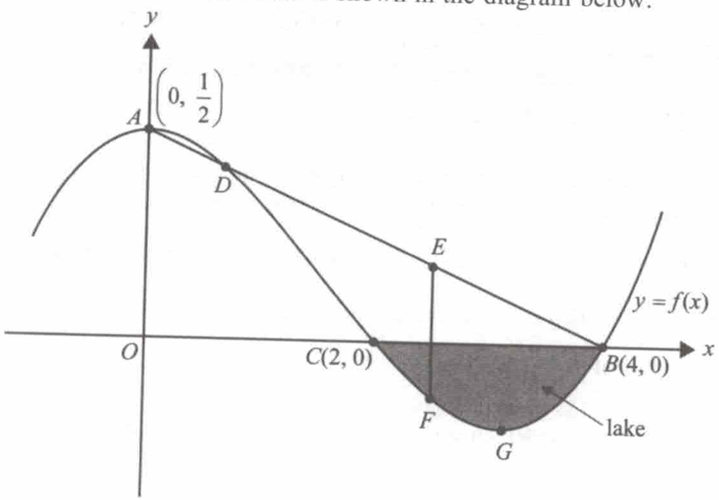
i. Find E(X), correct to four decimal places.

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **S**? Give your answer to the nearest integer.

[2 + 2 = 4 marks]
Total 11 marks
[VCAA 2013 MM (CAS)]

Question 292

Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains. He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram below.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}.$$

... continued

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2,0)$ and $B(4,0)$ are the points at the edge of the lake, situated in the valley. All distances are measured in kilometres.

a. Find the coordinates of G, the deepest point in the lake.

[3 marks]

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A, to the edge of the lake, B. The section of the railway line from A to D passes through a tunnel in the mountain.

b. Write down the equation of the line that passes through A and B.

[2 marks]

c. **i.** Show that the x-coordinate of D, the end point of the tunnel, is $\frac{2}{3}$.

ii. Find the length of the tunnel AD.

[1 + 2 = 3 marks]

In order to ensure that the section of the railway line from D to B remains stable, Tasmania constructs vertical columns from the lake bed to the railway line. The column EF is the longest of all possible columns. (Refer to the diagram.)

d. **i.** Find the x-coordinate of E.

ii. Find the length of the column EF in metres, correct to the nearest metre.

[2 + 2 = 4 marks]

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V:[0,4] \to R, V(x) = k\sqrt{x} - mx^2,$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line. The train then slows to a stop at $B(4, 0)$, that is $V(4) = 0$.

e. Find k in terms of m.

[1 mark]

f. Find the value of x for which the speed, V, is a maximum.

[2 marks]

Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m remains the same.

g. If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place.

[2 marks]

h. If, on another day, the maximum value of V is 120, find the value of m.

[2 marks]

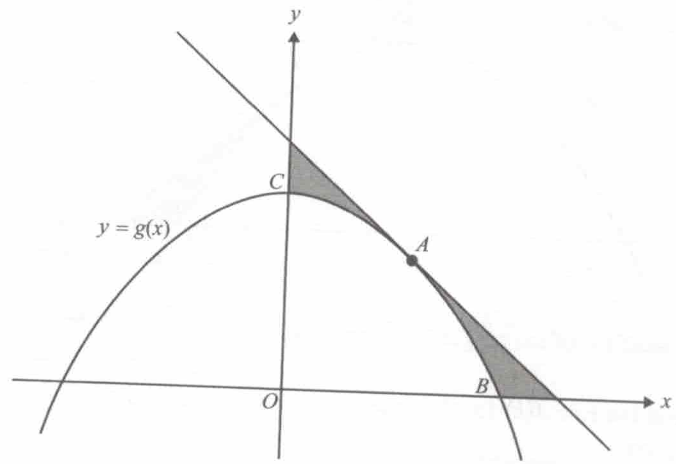
Total 19 marks

[VCAA 2013 MM (CAS)]

Extended-response questions

Question 293

Part of the graph of a function $g: R \to R$, $g(x) = \frac{16 - x^2}{4}$ is shown below.



a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram above. The tangent to the graph of g at the point A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A.

ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the point A, and the x-axis and y-axis.

Evaluate the area of this shaded region.

[2 + 3 = 5 marks]

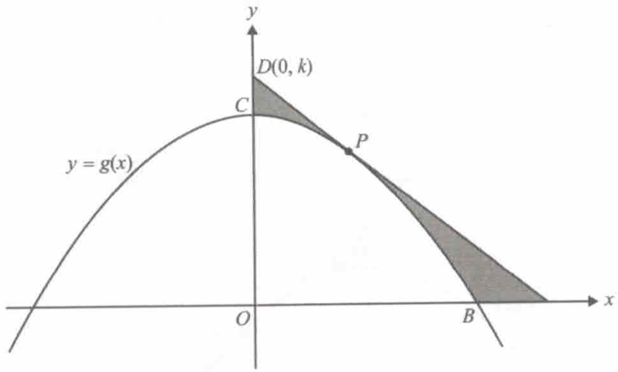
b. Let Q be a point on the graph of $y = g(x)$.

Find the positive value of the x-coordinate of Q, for which the distance OQ is a minimum and find the minimum distance.

[3 marks]

... continued

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y-axis at point $D(O,k)$, where $5 \le k \le 8$.



c. Find the gradient of the tangent in terms of k.

[2 marks]

d. **i.** Find the rule $A(k)$ for the function of k that gives the area of the shaded region.

ii. Find the **maximum** area of the shaded region and the value of k for which this occurs.

iii. Find the **minimum** area of the shaded region and the value of k for which this occurs.

[2 + 2 + 2 = 6 marks]
Total 16 marks
[VCAA 2013 MM (CAS)]

Question 294

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n.

[2 marks]

b. Find the maximum and minimum populations of wombats in this location.

[2 marks]

c. Find $n(10)$.

[1 mark]

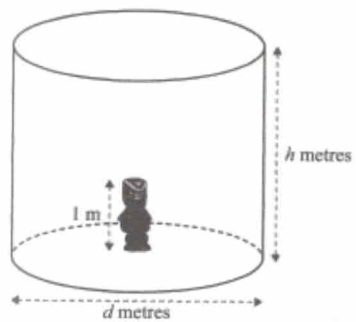
d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$.

[2 marks]
Total 7 marks
[VCAA 2014 MM (CAS)]

Extended-response questions

Question 295

On 1 January 2010, Tasmania Jones was walking through an ice-covered region of Greenland when he found a large ice cylinder that was made a thousand years ago by the Vikings.
A statue was inside the ice cylinder. The statue was 1 m tall and its base was at the centre of the base of the cylinder.



The cylinder had a height of h metres and a diameter of d metres. Tasmania Jones found that the volume of the cylinder was 216 m^3. At that time, 1 January 2010, the cylinder had not changed in a thousand years. It was exactly as it was when the Vikings made it.

a. Write an expression for h in terms of d.

[2 marks]

b. Show that the surface area of the cylinder excluding the base, S square metres, is given by the rule $S = \frac{\pi d^2}{4} + \frac{864}{d}$.

[1 mark]

Tasmania found that the Vikings made the cylinder so that S is a minimum.

c. Find the value of d for which S is a minimum and find this minimum value of S.

[2 marks]

d. Find the value of h when S is a minimum.

[1 mark]

On 1 January 2010, Tasmania believed that due to recent temperature changes in Greenland, the ice of the cylinder had just started melting. Therefore, he decided to return on 1 January each year to measure the ice cylinder. He observes that the volume of the ice cylinder decreases by a constant rate of 10 m^3 per year. Assume that the cylindrical shape is retained and $d = 2h$ at the beginning and as the cylinder melts.

e. Write down an expression for V in terms of h.

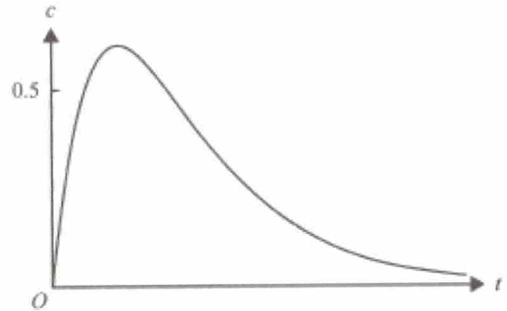
[1 mark]

f. Find the year in which the top of the statue will just be exposed. (Assume that the melting started on 1 January 2010.)

[2 marks]
Total 9 marks
[adapted from VCAA 2014 MM (CAS)]

Question 296

In a controlled experiment, Juan took some medicine at 8 pm. The concentration of medicine in his blood was then measured at regular intervals. The concentration of medicine in Juan's blood is modelled by the function $c(t)=\frac{5}{2}te^{-\frac{3t}{2}}$, $t\geq 0$, where c is the concentration of medicine in his blood, in milligrams per litre, t hours after 8 pm. Part of the graph of the function c is shown below.



a. What was the maximum value of the concentration of medicine in Juan's blood, in milligrams per litre, correct to two decimal places?

[1 mark]

b. **i.** Find the value of t, in hours, correct to two decimal places, when the concentration of medicine in Juan's blood first reached 0.5 milligrams per litre.

ii. Find the length of time that the concentration of medicine in Juan's blood was above 0.5 milligrams per litre. Express the answer in hours, correct to two decimal places.

[1 + 2 = 3 marks]

c. **i.** What was the value of the average rate of change of the concentration of medicine in Juan's blood over the interval $\left[\frac{2}{3}, 3\right]$? Express the answer in milligrams per litre per hour, correct to two decimal places.

ii. At times t_1 and t_2, the instantaneous rate of change of the concentration of medicine in Juan's blood was equal to the average rate of change over the interval $\left[\frac{2}{3}, 3\right]$.

Find the values of t_1 and t_2, in hours, correct to two decimal places.

[2 + 2 = 4 marks]

Alicia took part in a similar controlled experiment. However, she used a different medicine. The concentration of this different medicine was modelled by the function $n(t)=Ate^{-kt}$, $t\geq 0$, where A and $k\in R^+$.

d. If the **maximum** concentration of medicine in Alicia's blood was 0.74 milligrams per litre at $t=0.5$ hours, find the value of A, correct to the nearest integer.

[3 marks]
Total 11 marks
[VCAA 2014 MM (CAS)]

Question 297

Patricia is a gardener and she owns a garden nursery. She grows and sells basil plants and coriander plants.
The heights, in centimetres, of the basil plants that Patricia is selling are distributed normally with a mean of 14 cm and a standard deviation of 4 cm. There are 2000 basil plants in the nursery.

a. Patricia classifies the tallest 10 per cent of her basil plants as **super**.
What is the minimum height of a super basil plant, correct to the nearest millimetre?

[1 mark]

Patricia decides that some of her basil plants are not growing quickly enough, so she plans to move them to a special greenhouse. She will move the basil plants that are less than 9 cm in height.

b. How many basil plants will Patricia move to the greenhouse, correct to the nearest whole number?

[2 marks]

The heights of the coriander plants, x centimetres, follow the probability density function $h(x)$, where

$$h(x)=\begin{cases}\frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right) & 0<x<50\\ 0 & \text{otherwise}\end{cases}$$

c. State the mean height of the coriander plants.

[1 mark]

Patricia thinks that the smallest 15 per cent of her coriander plants should be given a new type of plant food.

d. Find the maximum height, correct to the nearest millimetre, of a coriander plant if it is to be given the new type of plant food.

[2 marks]

Patricia also grows and sells tomato plants that she classifies as either **tall** or **regular**.
She finds that 20 per cent of her tomato plants are tall.
A customer, Jack, selects n tomato plants at random.

e. Let q be the probability that at least one of Jack's n tomato plants is tall.
Find the minimum value of n so that q is greater than 0.95.

[2 marks]
Total 8 marks
[adapted from VCAA 2014 MM (CAS)]

Question 298

Let $f: R \to R$, $f(x) = (x-3)(x-1)(x^2+3)$ and $g: R \to R$, $g(x) = x^4 - 8x$.

a. Express $x^4 - 8x$ in the form $x(x-a)\left((x+b)^2 + c\right)$.

[2 marks]

b. Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

[1 mark]

c. Find the values of d such that the graph of $y = f(x+d)$ has

i. one positive x-axis intercept

ii. two positive x-axis intercepts.

[1 + 1 = 2 marks]

d. Find the value of n for which the equation $g(x) = n$ has one solution.

[1 mark]

e. At the point $(u, g(u))$, the gradient of $y = g(x)$ is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i. Find the value of $u^3 + v^3$.

ii. Find u and v if $u + v = 1$.

[2 + 1 = 3 marks]

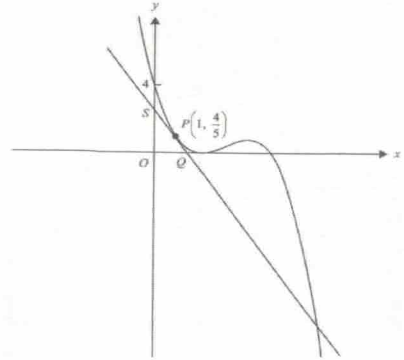
f. i. Find the equation of the tangent to the graph of $y = g(x)$ at the point $(p, g(p))$.

ii. Find the equations of the tangents to the graph of $y = g(x)$ that pass through the point with coordinates $\left(\frac{3}{2}, -12\right)$.

[1 + 3 = 4 marks]
Total 13 marks
[VCAA 2014 MM (CAS)]

Question 299

Let $f: R \to R, f(x) = \frac{1}{5}(x-2)^2(5-x)$. The point $P\left(1, \frac{4}{5}\right)$ is on the graph of f, as shown below. The tangent at P cuts the y-axis at S and the x-axis at Q.



a. Write down the derivative $f'(x)$ of $f(x)$.

[1 mark]

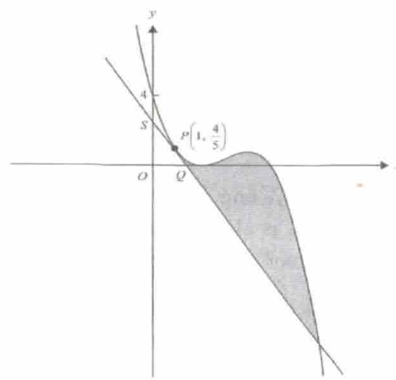
b. i. Find the equation of the tangent to the graph of f at the point $P\left(1, \frac{4}{5}\right)$.

ii. Find the coordinates of points Q and S.

[1 + 2 = 3 marks]

c. Find the distance PS and express it in the form $\frac{\sqrt{b}}{c}$, where b and c are positive integers.

[2 marks]

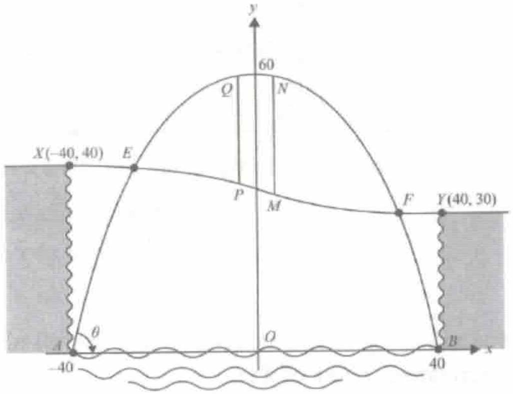


d. Find the area of the shaded region in the graph above.

[3 marks]
Total 9 marks
[VCAA 2015 MM (CAS)]

Question 300

A city is located on a river that runs through a gorge.
The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.
A bridge is to be built that crosses the river and the gorge.
A diagram for the design of the bridge is shown below.



The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by

$y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at B (40, 0) and $A(-40, 0)$.

The road across the gorge is modelled by a cubic polynomial function.

a. Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree.

[2 marks]

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$.

b. Find the maximum downwards slope of the road. Give your answer in the form $-\frac{m}{n}$ where m and n are positive integers.

[2 marks]

Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame. The supporting column, MN, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c. Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places.

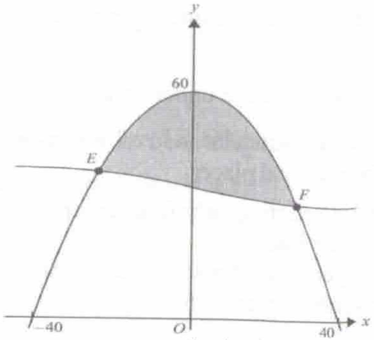
[3 marks]

The second supporting column, PQ, has its lowest point at $P(-u, w)$.

d. Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ.

[3 marks]

For the opening of the bridge, a banner is erected on the bridge, as shown by the shaded region in the diagram below.



e. Find the x-coordinates, correct to two decimal places, of E and F, the points at which the road meets the parabolic frame of the bridge.

[3 marks]

f. Find the area of the banner (shaded region), giving your answer to the nearest square metre.

[1 mark]

Total 14 marks

[VCAA 2015 MM (CAS)]

Question 301

Mani is a fruit grower. After his oranges have been picked, they are sorted by a machine, according to size. Oranges classified as **medium** are sold to fruit shops and the remainder are made into orange juice. The distribution of the diameter, in centimetres, of medium oranges is modelled by a continuous random variable, X, with probability density function

$$f(x) = \begin{cases} \frac{3}{4}(x-6)^2(8-x) & 6 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

a. **i.** Find the probability that a randomly selected medium orange has a diameter greater than 7 cm.

ii. Mani randomly selects three medium oranges. Find the probability that exactly one of the oranges has a diameter greater than 7 cm. Express the answer in the form $\frac{a}{b}$, where a and b are positive integers

[2 + 2 = 4 marks]

b. Find the mean diameter of medium oranges, in centimetres.

[1 mark]

For oranges classified as **large**, the quantity of juice obtained from each orange is a normally distributed random variable with a mean of 74 mL and a standard deviation of 9 mL.

c. What is the probability, correct to three decimal places, that a randomly selected large orange produces less than 85 mL of juice, given that it produces more than 74 mL of juice?

[2 marks]

continued

Mani also grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.

It is known that 3% of Mani's lemons are underweight.

d. i. Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places.

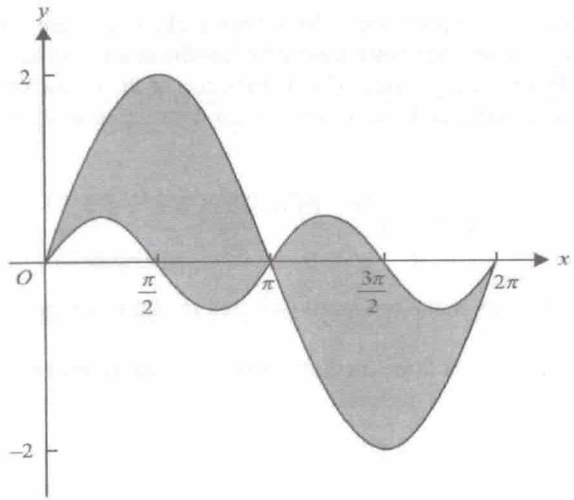
ii. Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load.
Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5

[2 + 2 = 4 marks]
Total 11 marks
[VCAA 2015 MM (CAS)]

Question 302

An electronics company is designing a new logo, based initially on the graphs of the functions $f(x) = 2\sin(x)$ and $g(x) = \frac{1}{2}\sin(2x)$, for $0 \le x \le 2\pi$.

These graphs are shown in the diagram below, in which the measurements in the x and y directions are in metres.



The logo is to be painted onto a large sign, with the area enclosed by the graphs of the two functions (shaded in the diagram) to be painted red.

a. The total area of the shaded regions, in square metres, can be calculated as $a\int_0^{\pi} \sin(x)\,dx$. What is the value of a?

[1 mark]

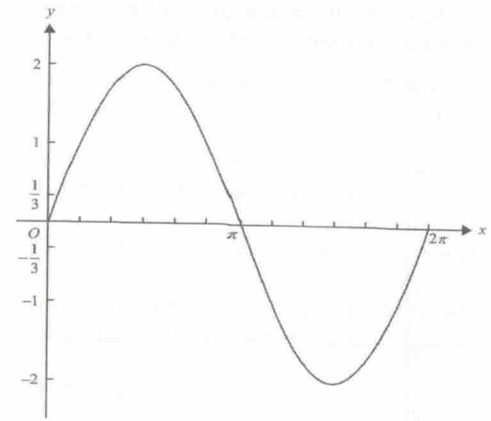
Extended-response questions

The electronics company considers changing the circular functions used in the design of the logo.

Its next attempt uses the graphs of the functions $f(x) = 2\sin(x)$ and $h(x) = \frac{1}{3}\sin(3x)$, for $0 \le x \le 2\pi$.

b. On the axes below, the graph of $y = f(x)$ has been drawn.

On the same axes, draw the graph of $y = h(x)$.



[2 marks]

c. State a sequence of two transformations that maps the graph of $y = f(x)$ to the graph of $y = h(x)$.

[2 marks]

The electronics company now considers using the graphs of the functions $k(x) = m\sin(x)$ and $q(x) = \frac{1}{n}\sin(nx)$, where m and n are positive integers with $m \ge 2$ and $0 \le x \le 2\pi$.

d. i. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if n is even.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers.

ii. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if n is odd.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers.

[2 + 2 = 4 marks]
Total 9 marks
[VCAA 2015 MM (CAS)]

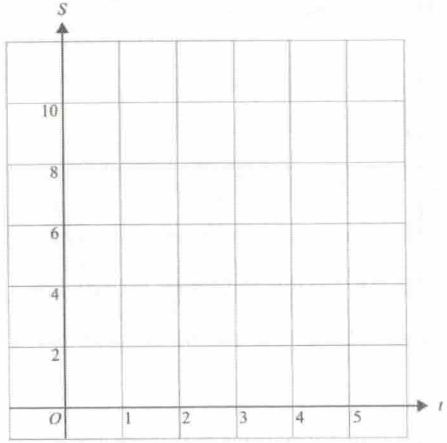
Question 303

a. Let $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, where $0 \le t \le 5$.

i. Find $S(0)$ and $S(5)$.

ii. The minimum value of S occurs when $t = \log_e(c)$. State the value of c and the minimum value of S.

iii. On the axes below, sketch the graph of S against t for $0 \le t \le 5$. Label the end points and the minimum point with their coordinates.



[1 + 2 + 2 + 2 = 7 marks]

iv. Find the value of the average rate of change of the function S over the interval $[0, \log_e(c)]$.

Let $V: [0, 5] \to R, V(t) = de^{\frac{t}{3}} + (10 - d)e^{-\frac{2t}{3}}$, where d is a real number and $d \in (0, 10)$.

b. If the minimum value of the function occurs when $t = \log_e(9)$, find the value of d.

[2 marks]

c. i. Find the set of possible values of d such that the minimum value of the function occurs when $t = 0$.

ii. Find the set of possible values of d such that the minimum value of the function occurs when $t = 5$.

[2 + 2 = 4 marks]

d. If the function V has a local minimum (a, m), where $0 \le a \le 5$, it can be shown that $m = \frac{k}{2}d^{\frac{2}{3}}(10 - d)^{\frac{1}{3}}$. Find the value of k.

[2 marks]

Total 15 marks

[VCAA 2015 MM (CAS)]

Question 304

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. It has more than 100 000 members worldwide. At every one of FullyFit's gyms, each member agrees to have their fitness assessed every month by undertaking a set of exercises called **S**. If someone completes **S** in less than three minutes, they are considered fit.

a. It has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. A random sample of 20 FullyFit members is taken. For a sample of 20 members, let X be the random variable that represents the number of members who complete **S** in less than three minutes.

i. Find $\Pr(X \ge 10)$ correct to four decimal places.

ii. Find $\Pr(X \ge 15 \mid X \ge 10)$ correct to three decimal places.

For samples of 20 members, $\hat{P}$ is the random variable of the distribution of sample proportions of people who complete **S** in less than three minutes.

iii. Find the expected value and variance of $\hat{P}$.

iv. Find the probability that a sample proportion lies within two standard deviations of $\frac{5}{8}$. Give your answer correct to three decimal places. Do not use a normal approximation.

v. Find $\Pr\left(\hat{P} \ge \frac{3}{4} \mid \hat{P} \ge \frac{5}{8}\right)$. Give your answer correct to three decimal places. Do not use a normal approximation.

[2 + 3 + 3 + 3 + 2 = 13 marks]

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

[2 marks]

… continued

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete another exercise routine, **T**, is a continuous random variable W with a probability density function g, as defined below.

$$g(w)=\begin{cases}\dfrac{(w-3)^3+64}{256} & 1\leq w\leq 3\\ \dfrac{w+29}{128} & 3<w\leq 5\\ 0 & \text{elsewhere}\end{cases}$$

i. Find $\mathrm{E}(W)$ correct to four decimal places.

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **T**? Give your answer to the nearest integer.

[2 + 2 = 4 marks]

d. From a random sample of 100 members, it was found that the sample proportion of people who spent more than two hours per week in the gym was 0.6.
Find an approximate 95% confidence interval for the population proportion corresponding to this sample proportion. Give values correct to three decimal places.

[1 mark]
Total 20 marks
[VCAA 2016 SAMPLE MM]

Question 305

Let $f: [0, 8\pi] \to R,\ f(x) = 2\cos\left(\dfrac{x}{2}\right) + \pi$.

a. Find the period and range of f.

[2 marks]

b. State the rule for the derivative function f'.

[1 mark]

c. Find the equation of the tangent to the graph of f at $x = \pi$.

[1 mark]

d. Find the equations of the tangents to the graph of $f: [0, 8\pi] \to R,\ f(x) = 2\cos\left(\dfrac{x}{2}\right) + \pi$ that have a gradient of 1.

[2 marks]

Extended-response questions

e. The rule of f' can be obtained from the rule of f under a transformation T, such that

$$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x\\ y\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ 0 & a\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}+\begin{bmatrix}-\pi\\ b\end{bmatrix}$$

Find the value of a and the value of b.

[3 marks]

f. Find the values of x, $0 \leq x \leq 8\pi$, such that $f(x) = 2f'(x) + \pi$.

[2 marks]
Total 11 marks
[VCAA 2016 MM]

Question 306

Consider the function $f(x) = -\dfrac{1}{3}(x + 2)(x - 1)^2$.

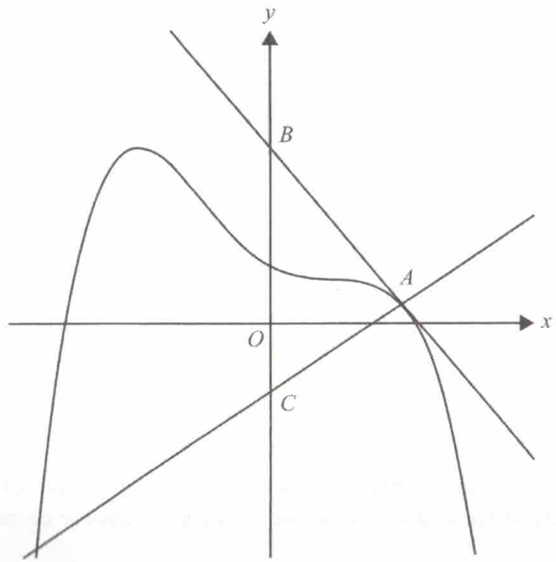
a. i. Given that $g'(x) = f(x)$ and $g(0) = 1$, show that $g(x) = -\dfrac{x^4}{12} + \dfrac{x^2}{2} - \dfrac{2x}{3} + 1$.

ii. Find the values of x for which the graph of $y = g(x)$ has a stationary point.

[1 + 1 = 2 marks]

The diagram below shows part of the graph of $y = g(x)$, the tangent to the graph at $x = 2$ and a straight line drawn perpendicular to the tangent to the graph at $x = 2$. The equation of the tangent at the point A with coordinates $(2, g(2))$ is $y = 3 - \dfrac{4x}{3}$.

The tangent cuts the y-axis at B. The line perpendicular to the tangent cuts the y-axis at C.



… continued

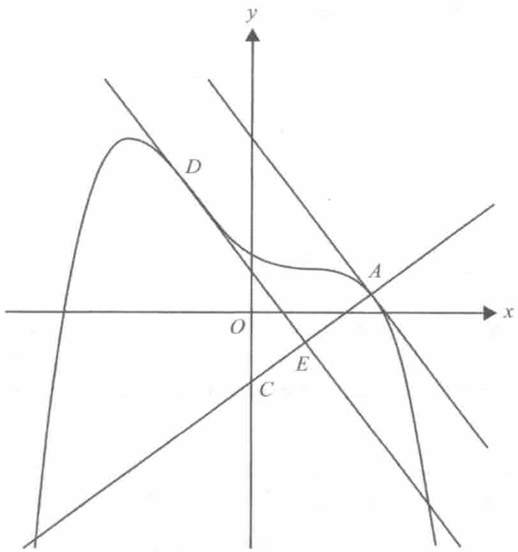
b. **i.** Find the coordinates of B.

ii. Find the equation of the line that passes through A and C and, hence, find the coordinates of C.

iii. Find the area of triangle ABC.

[1 + 2 + 2 = 5 marks]

c. The tangent at D is parallel to the tangent at A. It intersects the line passing through A and C at E.



i. Find the coordinates of D.

ii. Find the length of AE.

[2 + 3 = 5 marks]
Total 12 marks
[VCAA 2016 MM]

Question 307

A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a. Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places.

[2 marks]

Extended-response questions

b. A teacher observes that at least one of the retuned laptops is not correctly plugged into the trolley.
Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places.

[2 marks]

The time for which a laptop will work without recharging (the battery life) is normally distributed, with a mean of three hours and 10 minutes and standard deviation of six minutes. Suppose that the laptops remain out of the recharging trolley for three hours.

c. For any one laptop, find the probability that it will stop working by the end of these three hours. Give your answer correct to four decimal places.

[2 marks]

A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population of laptops with a mean battery life of three hours and 10 minutes and standard deviation of six minutes, $\hat{P}$ is the random variable of the distribution of sample proportions of laptops with a battery life of less than three hours.

d. Find the probability that $\Pr\left(\hat{P} \geq 0.06 \mid \hat{P} \geq 0.05\right)$. Give your answer correct to three decimal places. Do not use a normal approximation.

[3 marks]

It is known that when laptops have been used regularly in a school for six months, their battery life is still normally distributed but the mean battery life drops to three hours. It is also known that only 12% of such laptops work for more than three hours and 10 minutes.

e. Find the standard deviation for the normal distribution that applies to the battery life of laptops that have been used regularly in a school for six months, correct to four decimal places.

[2 marks]

The laptop supplier collects a sample of 100 laptops that have been used for six months from a number of different schools and tests their battery life. The laptop supplier wishes to estimate the proportion of such laptops with a battery life of less than three hours.

f. Suppose the supplier tests the battery life of the laptops one at a time.
Find the probability that the first laptop found to have a battery life of less than three hours is the third one.

[1 mark]

The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.

g. Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give values correct to two decimal places.

[1 mark]
… continued

h. The supplier also provides laptops to businesses. The probability density function for battery life, x (in minutes), of a laptop after six months of use in a business is

$$f(x) = \begin{cases} \dfrac{(210 - x)e^{\frac{x-210}{20}}}{400} & 0 \le x \le 210 \\ 0 & \text{elsewhere} \end{cases}$$

i. Find the **mean** battery life, in minutes, of a laptop with six months of business use, correct to two decimal places.

ii. Find the **median** battery life, in minutes, of a laptop with six months of business use, correct to two decimal places.

[1 + 2 = 3 marks]
Total 16 marks
[VCAA 2016 MM]

Question 308

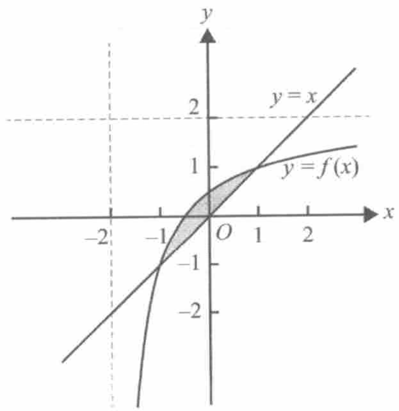
a. Express $\dfrac{2x+1}{x+2}$ in the form $a + \dfrac{b}{x+2}$, where a and b are non-zero integers.

[2 marks]

b. Let $f: R \setminus \{-2\} \to R$, $f(x) = \dfrac{2x+1}{x+2}$.

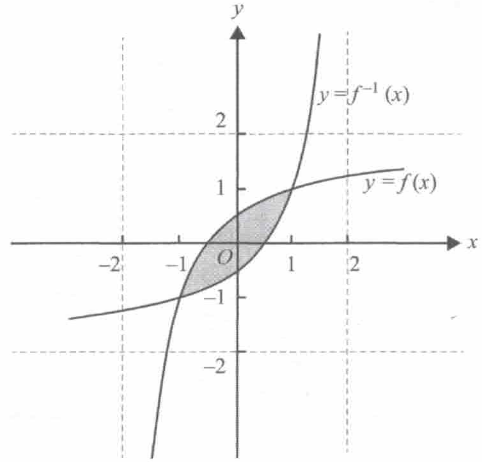
i. Find the rule and domain of f^{1}, the inverse function of f.

ii. Part of the graphs of f and $y = x$ are shown in the diagram below.



Find the area of the shaded region.

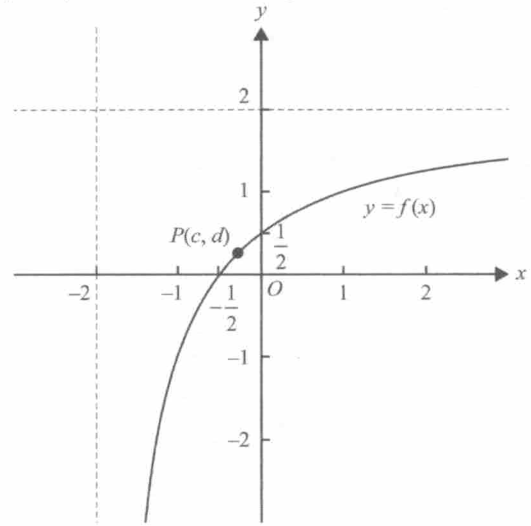
iii. Part of the graphs of f and f^{1} are shown in the diagram below.



Find the area of the shaded region.

[2 + 1 + 1 = 4 marks]

c. Part of the graph of f is shown in the diagram below.



The point $P(c, d)$ is on the graph of f.

Find the exact values of c and d such that the distance of this point to the origin is a minimum, and find this minimum distance.

[3 marks]

Let $g: (-k, \infty) \to R$, $g(x) = \dfrac{kx+1}{x+k}$, where $k > 1$.

d. Show that $x_1 < x_2$ implies that $g(x_1) < g(x_2)$, where $x_1 \in (-k, \infty)$ and $x_2 \in (-k, \infty)$.

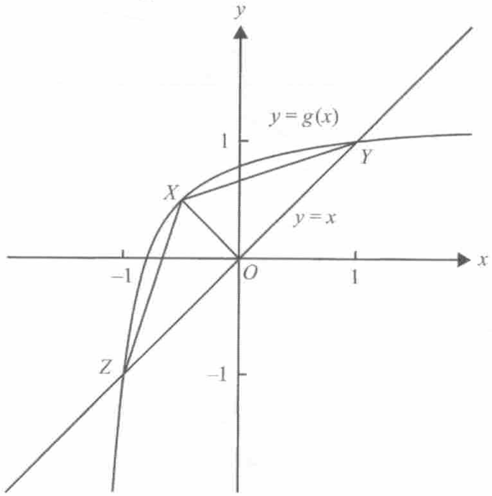
[2 marks]

… continued

e. **i.** Let X be the point of intersection of the graphs of $y = g(x)$ and $y = -x$. Find the coordinates of X in terms of k.

ii. Find the value of k for which the coordinates of X are $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

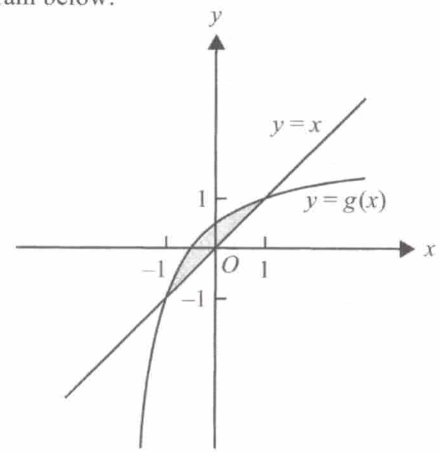
iii. Let $Z(-1, -1)$, $Y(1, 1)$ and X be the vertices of the triangle XYZ. Let $s(k)$ be the square of the area of triangle XYZ.



Find the values of k such that $s(k) \geq 1$.

[2 +2 + 2 = 6 marks]

f. The graph of g and the line $y = x$ enclose a region of the plane. The region is shown shaded in the diagram below.



Let $A(k)$ be the rule of the function A that gives the area of this enclosed region. The domain of A is $(1, \infty)$.

i. Give the rule for $A(k)$.

ii. Show that $0 < A(k) < 2$ for all $k > 1$.

[2 + 2 = 4 marks]
Total 21 marks
[VCAA 2016 MM]

Question 309

A company supplies schools with whiteboard pens.
The total length of time for which a whiteboard pen can be used for writing before it stops working is called its use-time.
There are two types of whiteboard pens: Grade A and Grade B.
The use-time of Grade A pens is normally distributed with a mean of 11 hours and a standard deviation of 15 minutes.

a. Find the probability that a Grade A whiteboard pen will have a use-time that is greater than 10.5 hours, correct to three decimal places.

[1 mark]

The use-time of Grade B whiteboard pens is described by the probability density function

$$f(x) = \begin{cases} \dfrac{x}{576}(12 - x)(e^{\frac{x}{6}} - 1) & 0 \leq x \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

where x is the use-time in hours.

b. Determine the expected use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places.

[2 marks]

c. Determine the standard deviation of the use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places.

[2 marks]

d. Find the probability that a randomly chosen Grade B whiteboard pen will have a use-time that is greater than 10.5 hours, correct to four decimal places.

[2 marks]

A worker at the company finds two boxes of whiteboard pens that are not labelled, but knows that one box contains only Grade A whiteboard pens and the other box contains only Grade B whiteboard pens.
The worker decides to randomly select a whiteboard pen from one of the boxes. If the selected whiteboard pen has a use-time that is greater than 10.5 hours, then the box that it came from will be labelled Grade A and the other box will be labelled Grade B. Otherwise, the box that it came from will be labelled Grade B and the other box will be labelled Grade A.

e. Find the probability, correct to three decimal places, that the worker labels the boxes incorrectly.

[2 marks]

f. Find the probability, correct to three decimal places, that the whiteboard pen selected was Grade B, given that the boxes had been labelled incorrectly.

[2 marks]
… continued

As a whiteboard pen ages, its tip may dry to the point that the whiteboard pen becomes defective (unusable). The company has stock that is two years old and, at that age, it is known that 5% of Grade A whiteboard pens will be defective.

g. A school purchases a box of Grade A whiteboard pens that is two years old and a class of 26 students is the first to use them.

If every student receives a whiteboard pen from this box, find the probability, correct to four decimal places, that at least one student will receive a defective whiteboard pen.

[2 marks]

h. Let $\hat{P}_A$ be the random variable of the distribution of sample proportions of defective Grade A whiteboard pens in boxes of 100. The boxes come from stock that is two years old.

Find $\Pr\left(\hat{P}_A > 0.04 \mid \hat{P}_A < 0.08\right)$. Give your answer correct to four decimal places.

Do not use a normal approximation.

[3 marks]

i. A box of 100 Grade A whiteboard pens that is two years old is selected and it is found that six of the whiteboard pens are defective.

Determine a 90% confidence interval for the population proportion from this sample, correct to two decimal places.

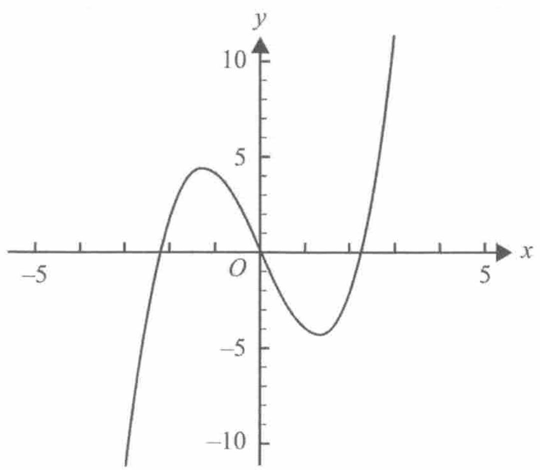
[2 marks]

Total 18 marks

[VCAA 2017 NH MM]

Question 310

Let $f : R \to R, f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a. Find the coordinates of the turning points.

[2 marks]

b. $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

i. Find the equation of the straight line through A and B.

ii. Find the distance AB.

[2 + 1 = 3 marks]

Let $g : R \to R, g(x) = x^3 - kx, k \in R^+$.

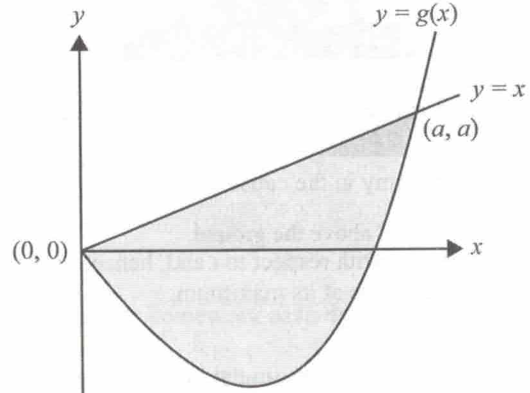
c. Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

i. Find the distance CD in terms of k.

ii. Find the values of k such that the distance CD is equal to $k + 1$.

[2 + 1 = 3 marks]

d. The diagram below shows part of the graphs of g and $y = x$. These graphs intersect at the points with the coordinates (0, 0) and (a, a).



i. Find the value of a in terms of k.

ii. Find the area of the shaded region in terms of k.

[1 + 2 = 3 marks]

Total 11 marks

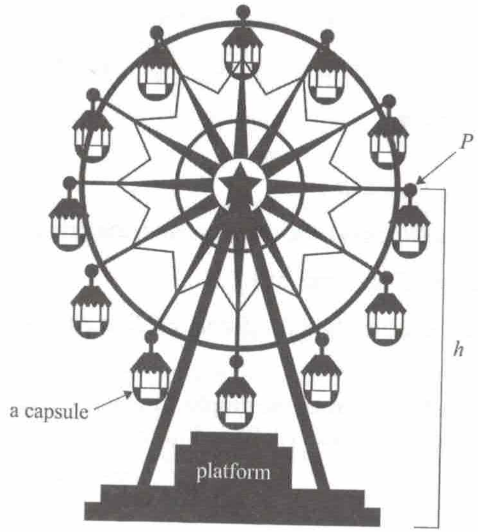
[VCAA 2017 MM]

Question 311

Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes after Sammy enters the capsule and h is measured in metres.

Sammy exits the capsule after one complete rotation of the Ferris wheel.

… continued



a. State the minimum and maximum heights of P above the ground.

[1 mark]

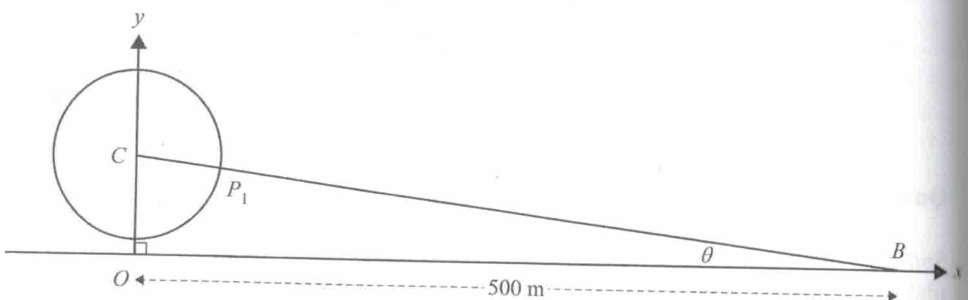
b. For how much time is Sammy in the capsule?

[1 mark]

c. Find the rate of change of h with respect to t and, hence, state the value of t at which the rate of change of h is at its maximum.

[2 marks]

As the Ferris wheel rotates, a stationary boat at B, on a nearby river, first becomes visible at point P_1. B is 500 m horizontally from the vertical axis through the centre C of the Ferris wheel and angle $CBO = \theta$, as shown below.



d. Find θ in degrees, correct to two decimal places.

[1 mark]

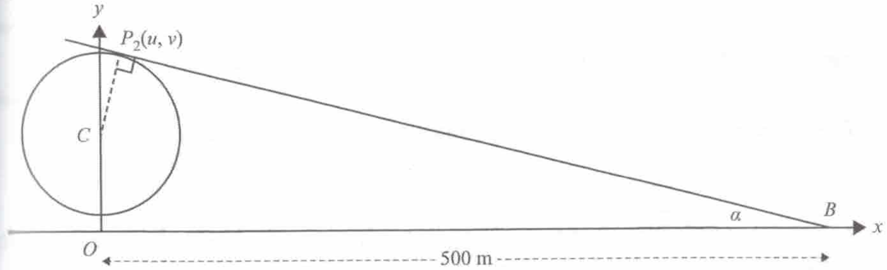
Part of the path of P is given by $y = \sqrt{3025 - x^2} + 65$, $x \in [-55, 55]$, where x and y are in metres.

e. Find $\frac{dy}{dx}$.

[1 mark]

Extended-response questions

As the Ferris wheel continues to rotate, the boat at B is no longer visible from the point $P_2(u, v)$ onwards. The line through B and P_2 is tangent to the path of P, where angle $OBP_2 = \alpha$.



f. Find the gradient of the line segment P_2B in terms of u and, hence, find the coordinates of P_2, correct to two decimal places.

[3 marks]

g. Find α in degrees, correct to two decimal places.

[1 mark]

h. Hence or otherwise, find the length of time, to the nearest minute, during which the boat at B is visible.

[2 marks]

Total 12 marks

[VCAA 2017 MM]

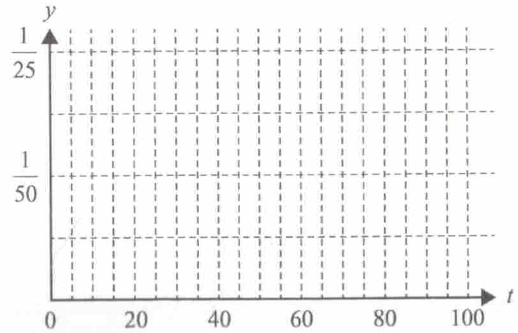
Question 312

The time Jennifer spends on her homework each day varies, but she does some homework every day.

The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f, where

$$f(t) = \begin{cases} \frac{1}{625}(t-20) & 20 \le t < 45 \\ \frac{1}{625}(70-t) & 45 \le t \le 70 \\ 0 & \text{elsewhere} \end{cases}$$

a. Sketch the graph of f on the axes provided below.



[3 marks]

... continued

b. Find $\Pr(25 \le T \le 55)$.

[2 marks]

c. Find $\Pr(T \le 25 \mid T \le 55)$.

[2 marks]

d. Find a such that $\Pr(T \ge a) = 0.7$, correct to four decimal places.

[2 marks]

e. The probability that Jennifer spends more than 50 minutes on her homework on any given day is $\frac{8}{25}$.

Assume that the amount of time spent on her homework on any day is independent of the time spent on her homework on any other day.

i. Find the probability that Jennifer spends more than 50 minutes on her homework on more than three of seven randomly chosen days, correct to four decimal places.

ii. Find the probability that Jennifer spends more than 50 minutes on her homework on at least two of seven randomly chosen days, given that she spends more than 50 minutes on her homework on at least one of those days, correct to four decimal places.

[2 + 2 = 4 marks]

Let p be the probability that on any given day Jennifer spends more than d minutes on her homework.

Let q be the probability that on two or three days out of seven randomly chosen days she spends more than d minutes on her homework.

f. Express q as a polynomial in terms of p.

[2 marks]

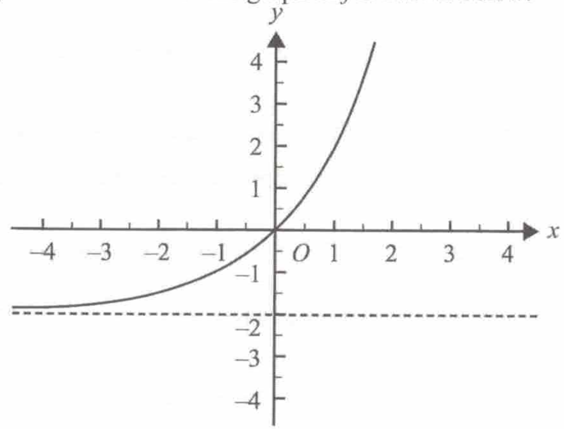
g. i. Find the maximum value of q, correct to four decimal places, and the value of p for which this maximum occurs, correct to four decimal places.

ii. Find the value of d for which the maximum found in **part g.i.** occurs, correct to the nearest minute.

[2 + 2 = 4 marks]
Total 19 marks
[VCAA 2017 MM]

Question 313

Let $f: R \to R, f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below.



a. The transformation $T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps the graph of $y = 2^x$ onto the graph of f. State the values of c and d.

[2 marks]

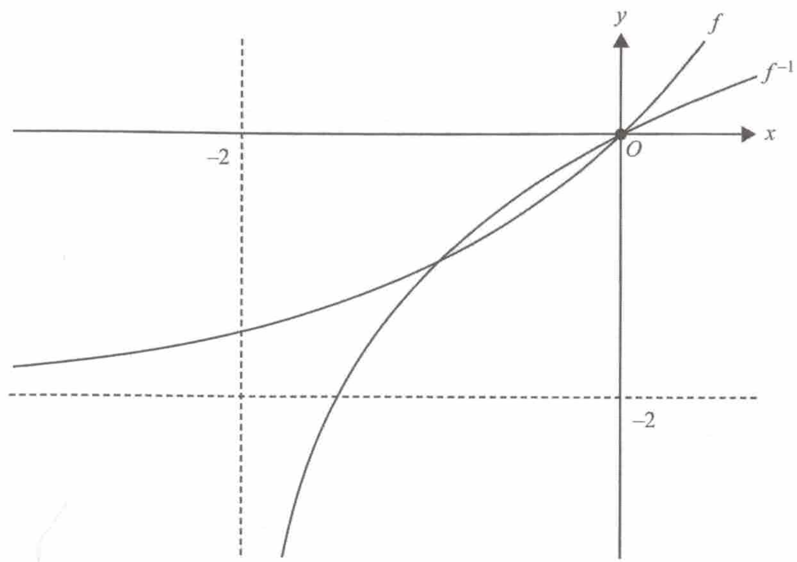
b. Find the rule and domain for f^{-1}, the inverse function of f.

[2 marks]

c. Find the area bounded by the graphs of f and f^{-1}.

[3 marks]

d. Part of the graphs of f and f^{-1} are shown below.



Find the gradient of f and the gradient of f^{-1} at $x = 0$.

[2 marks]
… continued

Extended-response questions

The functions of g_k, where $k \in R^+$, are defined with domain R such that $g_k(x) = 2e^{kx} - 2$.

e. Find the value of k such that $g_k(x) = f(x)$.

[1 mark]

f. Find the rule for the inverse functions g_k^{-1} of g_k, where $k \in R^+$.

[1 mark]

g. **i.** Describe the transformation that maps the graph of g_1 onto the graph of g_k.

ii. Describe the transformation that maps the graph of g_1^{-1} onto the graph of g_k^{-1}.

[1 + 1 = 2 marks]

h. The lines L_1 and L_2 are the tangents at the origin to the graphs of g_k and g_k^{-1} respectively.
Find the value(s) of k for which the angle between L_1 and L_2 is 30°.

[2 marks]

i. Let p be the value of k for which $g_k(x) = g_k^{-1}(x)$ has only one solution.

i. Find p.

ii. Let $A(k)$ be the area bounded by the graphs of g_k and g_k^{-1} for all $k > p$.

State the smallest value of b such that $A(k) < b$.

[2 + 1 = 3 marks]
Total 18 marks
[VCAA 2017 MM]

Solutions: A1

Question 1

$$2\log_e(x) - \log_e(x+3) = \log_e\left(\frac{1}{2}\right)$$

$$\log_e(x^2) - \log_e(x+3) = \log_e\left(\frac{1}{2}\right)$$

$$\log_e\left(\frac{x^2}{x+3}\right) = \log_e\left(\frac{1}{2}\right)$$

So $\frac{x^2}{x+3} = \left(\frac{1}{2}\right)$

$$2x^2 = x+3$$

$$2x^2 - x - 3 = 0$$

$$(x+1)(2x-3) = 0$$

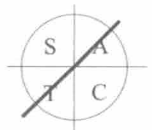
$$x = -1 \text{ or } x = \frac{3}{2}$$

We must reject $x = -1$ since the original equation has $\log_e(x)$ requiring $x > 0$.

Answer: $x = \frac{3}{2}$.

Question 2

$\tan(2x) = \sqrt{3}$ means that the angle $2x$ is in the first or third quadrants:



We are told that

$x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$,

So that $2x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

In the first quadrant $2x$ can be $\frac{\pi}{3}$.

It also means that $2x$ can be $\frac{4\pi}{3}$ in the third quadrant.

Thus, $x = \frac{\pi}{6}$ or $x = \frac{2\pi}{3}$.

Question 3

Let $y = \frac{3}{x} - 4$.

Then the inverse rule is given by

$$x = \frac{3}{y} - 4.$$

$$x + 4 = \frac{3}{y}$$

$$y = \frac{3}{x+4}$$

$$f^{-1}(x) = \frac{3}{x+4}$$

$\text{dom} f^{-1} = \text{ran} f = R \setminus \{-4\}$.

Question 4 B

The simultaneous linear equations

$kx - 3y = 0$

$5x - (k + 2)y = 0$

can be solved using matrices. If the 2 × 2 matrix determinant is zero, then the equations have no unique solution.

$$\begin{bmatrix} k & -3 \\ 5 & -(k+2) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The determinant is $-k(k+2) - (-3)(5)$.

Now find when this is zero.

$$-k^2 - 2k + 15 = 0$$

$$k^2 + 2k - 15 = 0$$

$$(k+5)(k-3) = 0$$

$$k = -5 \text{ or } 3$$

These values give no unique solution, so for a unique solution, $k \in R \setminus \{-5, 3\}$.

Question 5 B

Consider the function $f(x) = \log_e(x)$. The maximal domain is $\{x : x > 0\}$. Here, we require

$$2x + 1 > 0$$

$$x > -\frac{1}{2}$$

or $x \in \left(-\frac{1}{2}, \infty\right)$.