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Contents

Multiple-choice and short-answer tasks

A1. Algebra and Functions

Question 1

Solve the equation $2\log_e(x+2)-\log_e(x)=\log_e(2x+1)$, where $x>0$, for x.

[3 marks (2.0)]
[VCAA 2012 MM (CAS)]

Question 2

The graphs of $y=\cos(x)$ and $y=a\sin(x)$, where a is a real constant, have a point of intersection at $x=\frac{\pi}{3}$.

a. Find the value of a.

b. If $x\in[0,2\pi]$, find the x-coordinate of the other point of intersection of the two graphs.

[2 + 1 = 3 marks (1.5, 0.5)]
[VCAA 2012 MM (CAS)]

Question 3

The rule for function h is $h(x)=2x^3+1$. Find the rule for the inverse function h^{-1}.

[2 marks (1.5)]
[VCAA 2012 MM (CAS)]

Question 4

The function with rule $f(x)=-3\sin\left(\frac{\pi x}{5}\right)$ has period

A. 3 **B.** 5 **C.** 10 **D.** $\frac{\pi}{5}$ **E.** $\frac{\pi}{10}$

[VCAA 2012 MM (CAS) (91%)]

Question 5

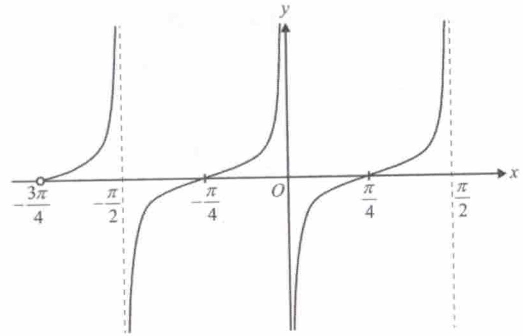
The range of the function $f:[-2,3]\rightarrow R, f(x)=x^2-2x-8$ is

A. R **B.** $(-9,-5]$ **C.** $(-5,0)$ **D.** $[-9,0]$ **E.** $[-9,-5)$

[VCAA 2012 MM (CAS) (69%)]

Question 6

A section of the graph of f is shown below.



The rule of f could be

A. $f(x)=\tan(x)$

B. $f(x)=\tan\left(x-\frac{\pi}{4}\right)$

C. $f(x)=\tan\left(2\left(x-\frac{\pi}{4}\right)\right)$

D. $f(x)=\tan\left(2\left(x-\frac{\pi}{2}\right)\right)$

E. $f(x)=\tan\left(\frac{1}{2}\left(x-\frac{\pi}{2}\right)\right)$

[VCAA 2012 MM (CAS) (72%)]

Question 7

Let the rule for a function g be $g(x)=\log_e\left((x-2)^2\right)$. For the function g, the

A. maximal domain $=R^+$ and range $=R$

B. maximal domain $=R\backslash\{2\}$ and range $=R$

C. maximal domain $=R\backslash\{2\}$ and range $=(-2,\infty)$

D. maximal domain $=[2,\infty)$ and range $=(0,\infty)$

E. maximal domain $=[2,\infty)$ and range $=[0,\infty)$

[VCAA 2012 MM (CAS) (66%)]

Question 8

The graph of a cubic function f has a local maximum at $(a,-3)$ and a local minimum at $(b,-8)$.

The values of c, such that the equation $f(x)+c=0$ has exactly one solution, are

A. $3<c<8$ **B.** $c>-3$ or $c<-8$ **C.** $-8<c<-3$

D. $c<3$ or $c>8$ **E.** $c<-8$

[VCAA 2012 MM (CAS) (34%)]

Question 9

A function f has the following two properties for all real values of θ.

$$f(\pi-\theta)=-f(\theta) \text{ and } f(\pi-\theta)=-f(-\theta)$$

A possible rule for f is

A. $f(x)=\sin(x)$ **B.** $f(x)=\cos(x)$ **C.** $f(x)=\tan(x)$

D. $f(x)=\sin\left(\dfrac{x}{2}\right)$ **E.** $f(x)=\tan(2x)$

[VCAA 2012 MM (CAS) (45%)]

Question 10

Solve the equation $\sin\left(\dfrac{x}{2}\right)=-\dfrac{1}{2}$ for $x\in[2\pi, 4\pi]$.

[2 marks (1.3)]
[VCAA 2013 MM (CAS)]

Question 11

a. Solve the equation $2\log_3(5)-\log_3(2)+\log_3(x)=2$ for x.

b. Solve the equation $3^{-4x}=9^{6-x}$ for x.

[2 + 2 = 4 marks (1.4, 1.5)]
[VCAA 2013 MM (CAS)]

Question 12

If $f:(-\infty,1)\to R, f(x)=2\log_e(1-x)$ and $g:[-1,\infty)\to R, g(x)=3\sqrt{x+1}$, then the maximal domain of the function $f+g$ is

A. $[-1, 1)$ **B.** $(1, \infty)$ **C.** $(-1, 1)$ **D.** $(-\infty, -1]$ **E.** R

[VCAA 2013 MM (CAS) (75%)]

Question 13

If $x+a$ is a factor of $7x^3+9x^2-5ax$, where $a\in R\setminus\{0\}$, then the value of a is

A. -4 **B.** -2 **C.** -1 **D.** 1 **E.** 2

[VCAA 2013 MM (CAS) (59%)]

Question 14

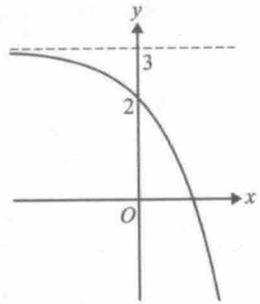
The function with rule $f(x)=-3\tan(2\pi x)$ has period

A. $\dfrac{2}{\pi}$ **B.** 2 **C.** $\dfrac{1}{2}$ **D.** $\dfrac{1}{4}$ **E.** 2π

[VCAA 2013 MM (CAS) (85%)]

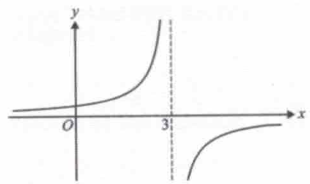
Question 15

Part of the graph of $y=f(x)$, where $f:R\to R, f(x)=3-e^x$, is shown below.

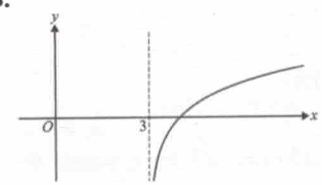


Which one of the following could be the graph of $y=f^{-1}(x)$, where f^{-1} is the inverse of f?

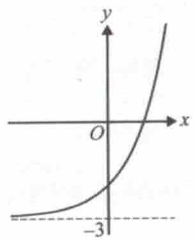
A.



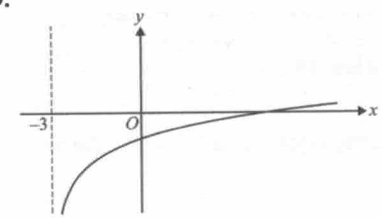
B.



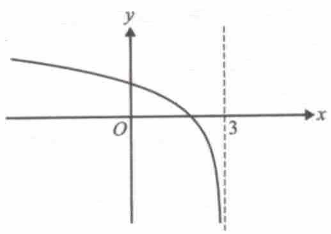
C.



D.



E.



[VCAA 2013 MM (CAS) (84%)]

Question 16

The function $g:[-a, a]\to R, g(x)=\sin\left(2\left(x-\frac{\pi}{6}\right)\right)$ has an inverse function.

The maximum possible value of a is

A. $\frac{\pi}{12}$ **B.** 1 **C.** $\frac{\pi}{6}$ **D.** $\frac{\pi}{4}$ **E.** $\frac{\pi}{2}$

[VCAA 2013 MM (CAS) (37%)]

Question 17

Let $g(x)=\log_2(x), x>0$.

Which one of the following equations is true for all positive real values of x?

A. $2g(8x)=g(x^2)+8$ **B.** $2g(8x)=g(x^2)+6$ **C.** $2g(8x)=\left(g(x)+8\right)^2$

D. $2g(8x)=g(2x)+6$ **E.** $2g(8x)=g(2x)+64$

[VCAA 2013 MM (CAS) (35%)]

Question 18

Solve $2\cos(2x)=-\sqrt{3}$ for x, where $0\le x\le\pi$.

[2 marks (1.4)]
[VCAA 2014 MM (CAS)]

Question 19

Solve the equation $2^{3x-3}=8^{2-x}$ for x.

[2 marks (1.7)]
[VCAA 2014 MM (CAS)]

Question 20

Solve $\log_e(x)-3=\log_e\left(\sqrt{x}\right)$ for x, where $x>0$.

[2 marks (1.2)]
[VCAA 2014 MM (CAS)]

Question 21

The point $P(4, -3)$ lies on the graph of a function f. The graph of f is translated four units vertically up and then reflected in the y-axis.

The coordinates of the final image of P are

A. $(-4, 1)$ **B.** $(-4, 3)$ **C.** $(0, -3)$ **D.** $(4, -6)$ **E.** $(-4, -1)$

[VCAA 2014 MM (CAS) (89%)]

A1. Algebra and Functions

Question 22

The linear function $f: D\to R$, $f(x)=4-x$ has range $[-2, 6)$.

The domain D of the function is

A. $[-2, 6)$ **B.** $(-2, 2]$ **C.** R **D.** $(-2, 6]$ **E.** $[-6, 2]$

[VCAA 2014 MM (CAS) (80%)]

Question 23

The function $f: D\to R$ with rule $f(x)=2x^3-9x^2-168x$ will have an inverse function for

A. $D=R$ **B.** $D=(7,\infty)$ **C.** $D=(-4, 8)$

D. $D=(-\infty, 0)$ **E.** $D=\left[-\frac{1}{2},\infty\right)$

[VCAA 2014 MM (CAS) (55%)]

Question 24

The inverse of the function $f: R^+\to R$, $f(x)=\frac{1}{\sqrt{x}}+4$ is

A. $f^{-1}:(4,\infty)\to R$ $\quad f^{-1}(x)=\frac{1}{(x-4)^2}$

B. $f^{-1}:R^+\to R$ $\quad f^{-1}(x)=\frac{1}{x^2}+4$

C. $f^{-1}:R^+\to R$ $\quad f^{-1}(x)=(x+4)^2$

D. $f^{-1}:(-4,\infty)\to R$ $\quad f^{-1}(x)=\frac{1}{(x+4)^2}$

E. $f^{-1}:(-\infty, 4)\to R$ $\quad f^{-1}(x)=\frac{1}{(x-4)^2}$

[VCAA 2014 MM (CAS) (76%)]

Question 25

The simultaneous linear equations $ax-3y=5$ and $3x-ay=8-a$ have **no solution** for

A. $a=3$ **B.** $a=-3$ **C.** both $a=3$ and $a=-3$

D. $a\in R\backslash\{3\}$ **E.** $a\in R\backslash[-3, 3]$

[VCAA 2014 MM (CAS) (50%)]

Question 26

The domain of the function h, where $h(x)=\cos(\log_a(x))$ and a is a real number greater than 1, is chosen so that h is a one-to-one function. Which one of the following could be the domain?

A. $\left(a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$

B. $(0, \pi)$

C. $\left[1, a^{\frac{\pi}{2}}\right]$

D. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$

E. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right]$

[VCAA 2014 MM (CAS) (42%)]

Question 27

The graph of $y=kx-4$ intersects the graph of $y=x^2+2x$ at two distinct points for

A. $k=6$

B. $k>6$ or $k<-2$

C. $-2\le k\le 6$

D. $6-2\sqrt{3}\le k\le 6+2\sqrt{3}$

E. $k=-2$

[VCAA 2014 MM (CAS) (53%)]

Question 28

On a given day, the depth of the water in a river is modelled by the function

$$h(t)=14+8\sin\left(\frac{\pi t}{12}\right), 0\le t\le 24$$

where h is the depth of the water, in metres, and t is the time, in hours, after 6 am.

a. Find the minimum depth of water in the river.

b. Find the values of t for which $h(t)=10$.

[1 + 2 = 3 marks (0.7, 1.3)]
[VCAA 2015 MM (CAS)]

Question 29

a. Solve $\log_2(6-x)-\log_2(4-x)=2$ for x, where $x<4$.

b. Solve $3e^t=5+8e^{-t}$ for t.

[2 + 3 = 5 marks (1.5, 1.3)]
[VCAA 2015 MM (CAS)]

Question 30

Let $f: R\to R, f(x)=2\sin(3x)-3$. The period and range of this function are respectively

A. period $=\frac{2\pi}{3}$ and range $=[-5,-1]$

B. period $=\frac{2\pi}{3}$ and range $=[-2, 2]$

C. period $=\frac{\pi}{3}$ and range $=[-1, 5]$

D. period $=3\pi$ and range $=[-1, 5]$

E. period $=3\pi$ and range $=[-2, 2]$

[VCAA 2015 MM (CAS) (95%)]

Question 31

The inverse function of $f:(-2,\infty)\to R, f(x)=\frac{1}{\sqrt{x+2}}$ is

A. $f^{-1}: R^+\to R \quad f^{-1}(x)=\frac{1}{x^2}-2$

B. $f^{-1}: R\backslash\{0\}\to R \quad f^{-1}(x)=\frac{1}{x^2}-2$

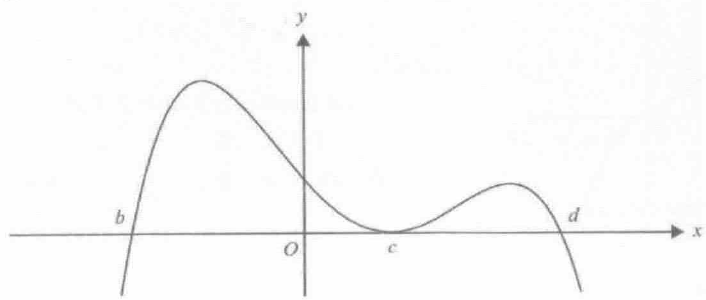
C. $f^{-1}: R^+\to R \quad f^{-1}(x)=\frac{1}{x^2}+2$

D. $f^{-1}: (-2,\infty)\to R \quad f^{-1}(x)=x^2+2$

E. $f^{-1}: (2,\infty)\to R \quad f^{-1}(x)=\frac{1}{x^2-2}$

[VCAA 2015 MM (CAS) (50%)]

Question 32



The rule for a function with the graph above could be

A. $y=-2(x+b)(x-c)^2(x-d)$

B. $y=2(x+b)(x-c)^2(x-d)$

C. $y=-2(x-b)(x-c)^2(x-d)$

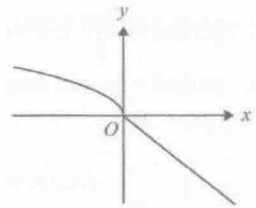
D. $y=2(x-b)(x-c)(x-d)$

E. $y=-2(x-b)(x+c)^2(x+d)$

[VCAA 2015 MM (CAS) (20%)]

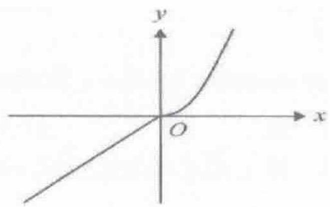
Question 33

Part of the graph of $y = f(x)$ is shown below.

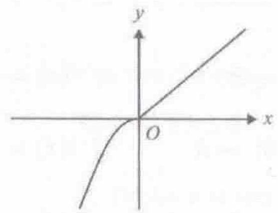


The corresponding part of the graph of the inverse function $y = f^{-1}(x)$ is best represented by

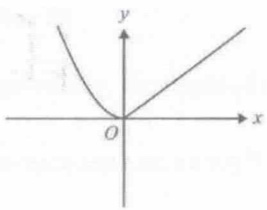
A.



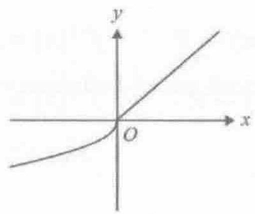
B.



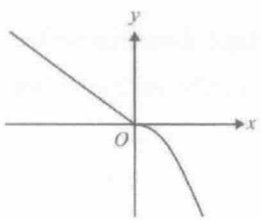
C.



D.



E.



[VCAA 2015 MM (CAS) (71%)]

Question 34

For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A. -3 **B.** -1 **C.** 1 **D.** 3 **E.** 10

[VCAA 2015 MM (CAS) (91%)]

A1. Algebra and Functions

Question 35

The range of the function $f: (-1, 2] \to R, f(x) = -x^2 + 2x - 3$ is

A. R **B.** $(-6, -3]$ **C.** $(-6, -2]$ **D.** $[-6, -3]$ **E.** $[-6, -2]$

[VCAA 2015 MM (CAS) (56%)]

Question 36

The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A. dilation by a factor of 2 from the y-axis.

B. dilation by a factor of 2 from the x-axis.

C. dilation by a factor of $\frac{1}{2}$ from the x-axis.

D. dilation by a factor of 8 from the y-axis.

E. dilation by a factor of $\frac{1}{2}$ from the y-axis.

[VCAA 2015 MM (CAS) (24%)]

Question 37

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts. The set of all possible values of c is

A. R **B.** R^+ **C.** $\{0, 4\}$ **D.** $(0, 4)$ **E.** $(-\infty, 4)$

[VCAA 2015 MM (CAS) (60%)]

Question 38

If $f(x - 1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A. $x^2 - 2$ **B.** $x^2 + 2$ **C.** $x^2 - 2x + 2$

D. $x^2 - 2x + 4$ **E.** $x^2 - 4x + 6$

[VCAA 2015 MM (CAS) (61%)]

Question 39

The graphs of $y = mx + c$ and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

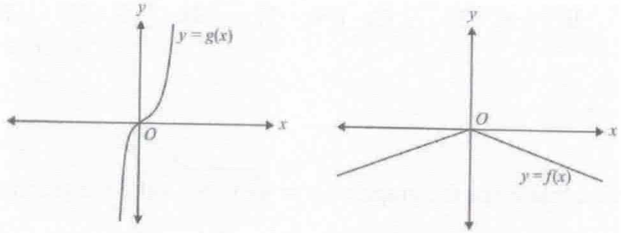
A. $a > 0$ and $c > 0$ **B.** $a > 0$ and $c < 0$ **C.** $a > 0$ and $c > -\frac{m^2}{4a}$

D. $a < 0$ and $c > -\frac{m^2}{4a}$ **E.** $m > 0$ and $c > 0$

[VCAA 2015 MM (CAS) (37%)]

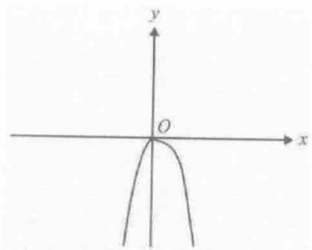
Question 40

The graphs of the functions with rules $f(x)$ and $g(x)$ are shown below.

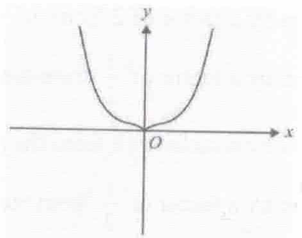


Which one of the following best represents the graph of the function with rule $g(-f(x))$?

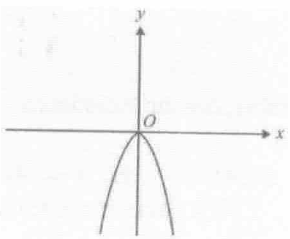
A.



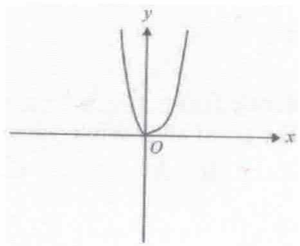
B.



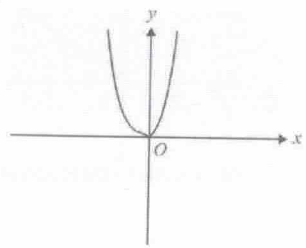
C.



D.



E.



[VCAA 2015 MM (CAS) (35%)]

Question 41

The linear function $f: D \to R, f(x) = 5 - x$ has range $[-4, 5)$. The domain D is

A. $(0, 9]$ **B.** $(0, 1]$ **C.** $[5, -4)$ **D.** $[-9, 0)$ **E.** $[1, 9)$

[VCAA 2016 MM (92%)]

Question 42

Let $f: R \to R, f(x) = 1 - 2\cos\left(\frac{\pi x}{2}\right)$.

The period and range of this function are respectively

A. 4 and $[-2, 2]$ **B.** 4 and $[-1, 3]$ **C.** 1 and $[-1, 3]$

D. 4π and $[-1, 3]$ **E.** 4π and $[-2, 2]$

[VCAA 2016 MM (90%)]

Question 43

Which one of the following is the inverse function of $g: [3, \infty) \to R, g(x) = \sqrt{2x - 6}$?

A. $g^{-1}: [3, \infty) \to R, g^{-1}(x) = \frac{x^2 + 6}{2}$

B. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = (2x - 6)^2$

C. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \sqrt{\frac{x}{2} + 6}$

D. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \frac{x^2 + 6}{2}$

E. $g^{-1}: R \to R, g^{-1}(x) = \frac{x^2 + 6}{2}$

[VCAA 2016 MM (75%)]

Question 44

Consider the graph of the function defined by $f: [0, 2\pi] \to R, f(x) = \sin(2x)$.

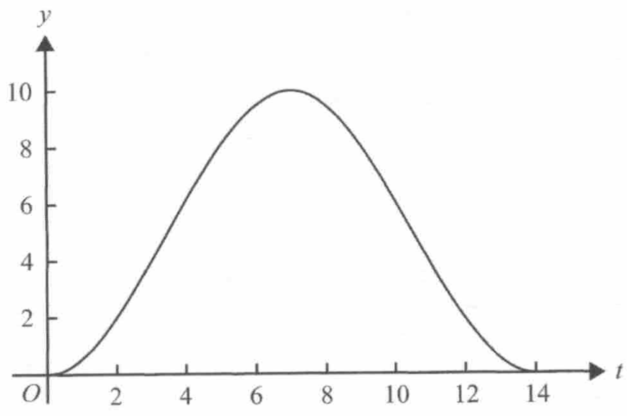
The square of the length of the line segment joining the points on the graph for which $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is

A. $\frac{\pi^2 + 16}{4}$ **B.** $\pi + 4$ **C.** 4 **D.** $\frac{3\pi^2 + 16\pi}{4}$ **E.** $\frac{10\pi^2}{16}$

[VCAA 2016 MM (67%)]

Question 45

The UV index, y, for a summer day in Melbourne is illustrated in the graph below, where t is the number of hours after 6 am.



The graph is most likely to be the graph of

A. $y = 5 + 5\cos\left(\frac{\pi t}{7}\right)$

B. $y = 5 - 5\cos\left(\frac{\pi t}{7}\right)$

C. $y = 5 + 5\cos\left(\frac{\pi t}{14}\right)$

D. $y = 5 - 5\cos\left(\frac{\pi t}{14}\right)$

E. $y = 5 + 5\sin\left(\frac{\pi t}{14}\right)$

[VCAA 2016 MM (77%)]

Question 46

The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x-5}$ by a reflection in the x-axis followed by a dilation from the y-axis by a factor of $\frac{1}{2}$. Which one of the following is the rule for the function f?

A. $f(x) = \sqrt{5-4x}$ **B.** $f(x) = -\sqrt{x-5}$ **C.** $f(x) = \sqrt{x+5}$

D. $f(x) = -\sqrt{4x-5}$ **E.** $f(x) = -\sqrt{4x-10}$

[VCAA 2016 MM (52%)]

Question 47

Let $\left(\tan(\theta)-1\right)\left(\sin(\theta)-\sqrt{3}\cos(\theta)\right)\left(\sin(\theta)+\sqrt{3}\cos(\theta)\right)=0$.

a. State all possible values of $\tan(\theta)$.

b. Hence, find all possible solutions for $\left(\tan(\theta)-1\right)\left(\sin^2(\theta)-3\cos^2(\theta)\right)=0$, where $0 \le \theta \le \pi$.

[1 + 2 = 3 marks (0.2, 0.5)]
[VCAA 2017 MM]

Question 48

Let $f\colon [0, \infty) \to R$, $f(x) = \sqrt{x+1}$.

a. State the range of f.

b. Let $g\colon (-\infty, c] \to R$, $g(x) = x^2 + 4x + 3$, where $c < 0$.

  i. Find the largest possible value of c such that the range of g is a subset of the domain of f.

  ii. For the value of c found in **part b. i.**, state the range of $f(g(x))$.

c. Let $h\colon R \to R$, $h(x) = x^2 + 3$.
State the range of $f(h(x))$.

[1 + 2 + 1 + 1 = 5 marks (0.6, 0.4, 0.1, 0.3)]
[VCAA 2017 MM]

Question 49

Let $f\colon R \to R$, $f(x) = 5\sin(2x) - 1$.
The period and range of this function are respectively

A. π and $[-1, 4]$ **B.** 2π and $[-1, 5]$ **C.** π and $[-6, 4]$

D. 2π and $[-6, 4]$ **E.** 4π and $[-6, 4]$

[VCAA 2017 MM (92%)]

Question 50

Let f and g be functions such that $f(2) = 5$, $f(3) = 4$, $g(2) = 5$, $g(3) = 2$ and $g(4) = 1$.
The value of $f(g(3))$ is

A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 5

[VCAA 2017 MM (75%)]

Question 51

The equation $(p-1)x^2+4x=5-p$ has no real roots when

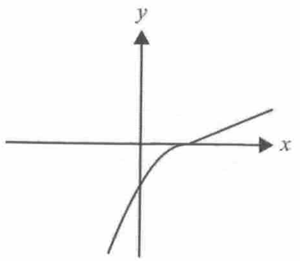
A. $p^2-6p+6<0$ **B.** $p^2-6p+1>0$ **C.** $p^2-6p-6<0$

D. $p^2-6p+1<0$ **E.** $p^2-6p+6>0$

[VCAA 2017 MM (32%)]

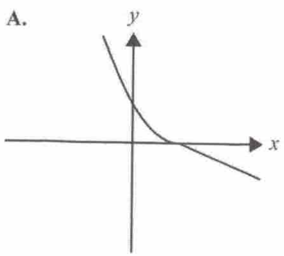
Question 52

Part of the graph of the function *f* is shown below. The same scale has been used on both axes.

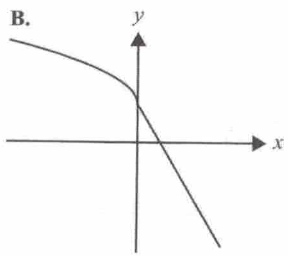


The corresponding part of the graph of the inverse function f^{-1} is best represented by

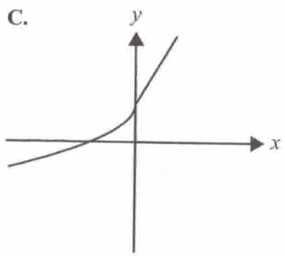
A.



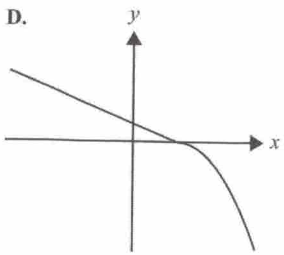
B.



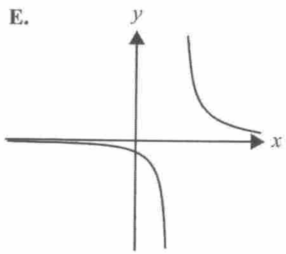
C.



D.



E.



[VCAA 2017 MM (88%)]

Question 53

The sum of the solutions of $\sin(2x)=\frac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$.

The value of d could be

A. 0 **B.** $\frac{\pi}{6}$ **C.** $\frac{3\pi}{4}$ **D.** $\frac{7\pi}{6}$ **E.** $\frac{3\pi}{2}$

[VCAA 2017 MM (45%)]

Question 54

If $y=a^{b-4x}+2$, where $a>0$, then x is equal to

A. $\frac{1}{4}(b-\log_a(y-2))$ **B.** $\frac{1}{4}(b-\log_a(y+2))$ **C.** $b-\log_a\left(\frac{1}{4}(y+2)\right)$

D. $\frac{b}{4}-\log_a(y-2)$ **E.** $\frac{1}{4}(b+2-\log_a(y))$

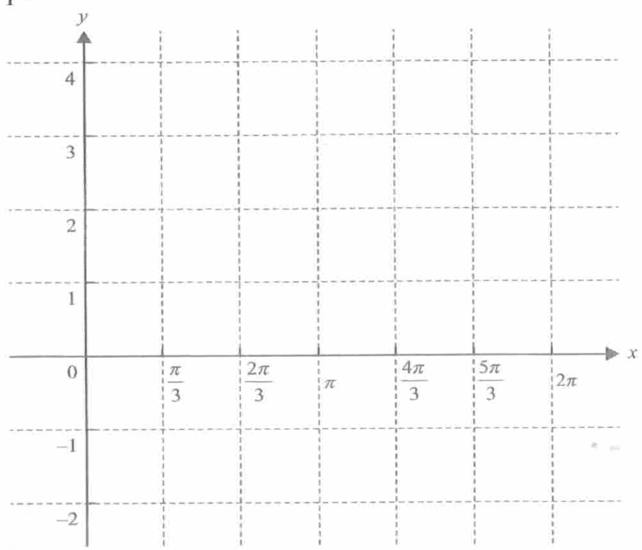
[VCAA 2017 MM (64%)]

Question 55

Let $f:[0, 2\pi]\rightarrow R, f(x)=2\cos(x)+1$.

a. Solve the equation $2\cos(x)+1=0$ for $0\le x\le 2\pi$.

b. Sketch the graph of the function *f* on the axes below. Label the endpoints and local minimum point with their coordinates.



[2 + 3 = 5 marks (1.6, 2.4)]
[VCAA 2018 MM]

Question 56

Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

a. Find the coordinates of P.

b. Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers.

[3 + 2 = 5 marks (1.2, 0.8)]
[VCAA 2018 MM]

Question 57

Let $f:(2, \infty) \to R$, where $f(x) = \frac{1}{(x-2)^2}$. State the rule and domain of f^{-1}.

[3 marks (2.2)]
[VCAA 2018 MM]

Question 58

Let $f: R \to R, f(x) = 4\cos\left(\frac{2\pi x}{3}\right) + 1$. The period of this function is

A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 5

[VCAA 2018 MM (95%)]

Question 59

The maximal domain of the function f is $R\setminus\{1\}$. A possible rule for f is

A. $f(x) = \frac{x^2-5}{x-1}$

B. $f(x) = \frac{x+4}{x-5}$

C. $f(x) = \frac{x^2+x+4}{x^2+1}$

D. $f(x) = \frac{5-x^2}{1+x}$

E. $f(x) = \sqrt{x-1}$

[VCAA 2018 MM (88%)]

Question 60

Consider the function $f:[a, b) \to R, f(x) = \frac{1}{x}$, where a and b are positive real numbers.

The range of f is

A. $\left[\frac{1}{a}, \frac{1}{b}\right)$

B. $\left(\frac{1}{a}, \frac{1}{b}\right]$

C. $\left[\frac{1}{b}, \frac{1}{a}\right)$

D. $\left(\frac{1}{b}, \frac{1}{a}\right]$

E. $[a, b)$

[VCAA 2018 MM (48%)]

Question 61

The point $A(3, 2)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x) = \frac{1}{2}f(x-1)$. The same transformation maps the point A to the point P. The coordinates of the point P are

A. (2, 1) **B.** (2, 4) **C.** (4, 1) **D.** (4, 2) **E.** (4, 4)

[VCAA 2018 MM (48%)]

Question 62

Let $f: R^+ \to R, f(x) = k\log_2(x), k \in R$. Given that $f^{-1}(1) = 8$, the value of k is

A. 0 **B.** $\frac{1}{3}$ **C.** 3 **D.** 8 **E.** 12

[VCAA 2018 MM (83%)]

Question 63

Let f and g be two functions such that $f(x) = 2x$ and $g(x+2) = 3x+1$. The function $f(g(x))$ is

A. $6x-5$ **B.** $6x+1$ **C.** $6x^2+1$ **D.** $6x-10$ **E.** $6x+2$

[VCAA 2018 MM (58%)]

Question 64

The graph of $y = \tan(ax)$, where $a \in R^+$, has a vertical asymptote $x = 3\pi$ and has exactly one x-intercept in the region $(0, 3\pi)$. The value of a is

A. $\frac{1}{6}$ **B.** $\frac{1}{3}$ **C.** $\frac{1}{2}$ **D.** 1 **E.** 2

[VCAA 2018 MM (26%)]

Question 65

Let $f: R\setminus\left\{\frac{1}{3}\right\} \to R, f(x) = \frac{1}{3x-1}$.

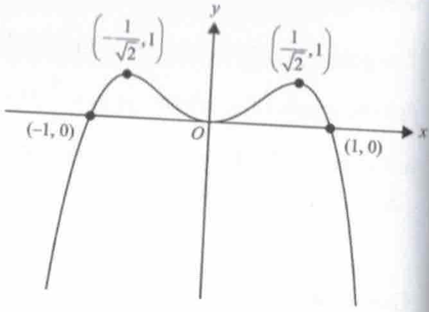
a. Find the rule of f^{-1}.

b. State the domain of f^{-1}.

c. The graph of f is translated c units horizontally and d units vertically, where $c, d \in R$. Let g be the function corresponding to the translated graph. Find the values of c and d given that $g = f^{-1}$.

[2 + 1 + 1 = 4 marks (1.5, 0.7, 0.3)]
[adapted from VCAA 2019 MM]

The function $f: R \to R$, $f(x)$ is a polynomial function of degree 4. Part of the graph of f is shown here. The graph of f touches the x-axis at the origin.



a. Find the rule of f.

Let g be a function with the same rule as f.

Let $h: D \to R$, $h(x) = \log_e(g(x)) - \log_e(x^3 + x^2)$, where D is the maximal domain of h.

b. State D.

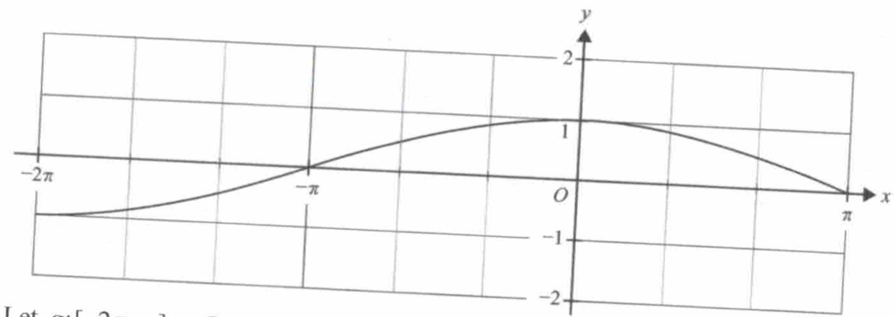
c. State the range of h.

[1 + 1 + 2 = 4 marks (0.2, 0.1, 0.2)]
[VCAA 2019 MM]

Question 67

a. Solve $1 - \cos\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)$ for $x \in [-2\pi, \pi]$.

b. The function $f:[-2\pi, \pi] \to R, f(x) = \cos\left(\frac{x}{2}\right)$ is shown on the axes below.



Let $g:[-2\pi, \pi] \to R, g(x) = 1 - f(x)$.

Sketch the graph of g on the axes above. Label all points of intersection of the graphs of f and g, and the endpoints of g, with their coordinates.

[2 + 2 = 4 marks (1.3, 1.0)]
[VCAA 2019 MM]

Question 68

The set of values of k for which $x^2 + 2x - k = 0$ has two real solutions is

A. $\{-1, 1\}$ **B.** $(-1, \infty)$ **C.** $(-\infty, -1)$ **D.** $\{-1\}$ **E.** $[-1, \infty)$

[VCAA 2019 MM (59%)]

Question 69

Let $f: R \to R$, $f(x) = 3\sin\left(\frac{2x}{5}\right) - 2$.

The period and range of f are respectively

A. 5π and $[-3, 3]$ **B.** 5π and $[-5, 1]$ **C.** 5π and $[-1, 5]$

D. $\frac{5\pi}{2}$ and $[-5, 1]$ **E.** $\frac{5\pi}{2}$ and $[-3, 3]$

[VCAA 2019 MM (89%)]

Question 70

The graph of the function f passes through the point $(-2, 7)$.

If $h(x) = f\left(\frac{x}{2}\right) + 5$, then the graph of the function h must pass through the point

A. $(-1, -12)$ **B.** $(-1, 19)$ **C.** $(-4, 12)$ **D.** $(-4, -14)$ **E.** $(3, 3.5)$

[VCAA 2019 MM (65%)]

Question 71

Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \frac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \frac{5\pi}{4}$, in terms of α, is

A. 0 **B.** 2α **C.** $\pi + 2\alpha$ **D.** $\frac{\pi}{2} + \alpha$ **E.** $\frac{3(\pi + \alpha)}{2}$

[VCAA 2019 MM (25%)]

Question 72

The expression $\log_x(y) + \log_y(z)$, where x, y and z are all real numbers greater than 1, is equal to

A. $-\frac{1}{\log_y(x)} - \frac{1}{\log_z(y)}$ **B.** $\frac{1}{\log_x(y)} + \frac{1}{\log_y(z)}$

C. $-\frac{1}{\log_x(y)} - \frac{1}{\log_y(z)}$ **D.** $\frac{1}{\log_y(x)} + \frac{1}{\log_z(y)}$

E. $\log_y(x) + \log_z(y)$

[VCAA 2019 MM (47%)]

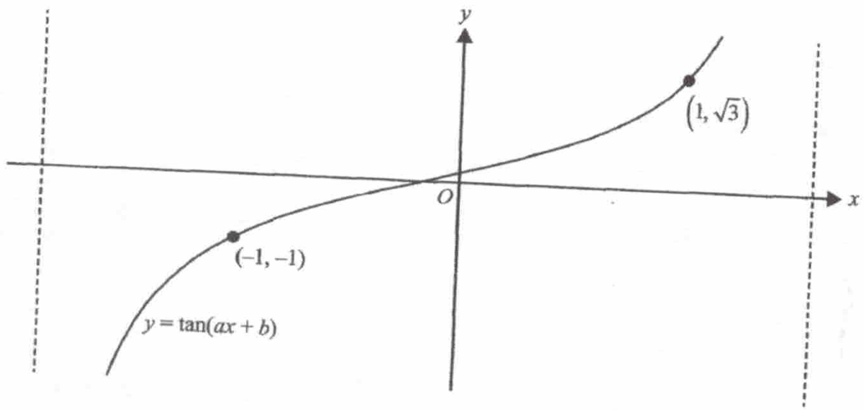
Question 73

Solve the equation $2\log_2(x+5)-\log_2(x+9)=1$.

[3 marks (1.8)]
[VCAA 2020 MM]

Question 74

Shown below is part of the graph of a period of the function of the form $y=\tan(ax+b)$.



The graph is continuous for $x\in[-1,1]$.
Find the value of a and the value of b, where $a>0$ and $0<b<1$.

[3 marks (1.5)]
[VCAA 2020 MM]

Question 75

Let f and g be functions such that $f(-1)=4$, $f(2)=5$, $g(-1)=2$, $g(2)=7$ and $g(4)=6$. The value of $g\left(f(-1)\right)$ is

A. 2 **B.** 4 **C.** 5 **D.** 6 **E.** 7

[VCAA 2020 MM (84%)]

Question 76

Let $p(x)=x^3-2ax^2+x-1$, where $a\in R$. When p is divided by $x+2$, the remainder is 5. The value of a is

A. 2 **B.** $-\frac{7}{4}$ **C.** $\frac{1}{2}$ **D.** $-\frac{3}{2}$ **E.** -2

[VCAA 2020 MM (56%)]

Question 77

Given that $\log_2(n+1)=x$, the values of n for which x is a positive integer are

A. $n=2^k,\ k\in Z^+$ **B.** $n=2^k-1,\ k\in Z^+$ **C.** $n=2^{k-1},\ k\in Z^+$
D. $n=2k-1,\ k\in Z^+$ **E.** $n=2k,\ k\in Z^+$

[VCAA 2020 MM (62%)]

Question 78

The solutions of the equation $2\cos\left(2x-\frac{\pi}{3}\right)+1=0$ are

A. $x=\frac{\pi(6k-2)}{6}$ or $x=\frac{\pi(6k-3)}{6}$, for $k\in Z$

B. $x=\frac{\pi(6k-2)}{6}$ or $x=\frac{\pi(6k+5)}{6}$, for $k\in Z$

C. $x=\frac{\pi(6k-1)}{6}$ or $x=\frac{\pi(6k+2)}{6}$, for $k\in Z$

D. $x=\frac{\pi(6k-1)}{6}$ or $x=\frac{\pi(6k+3)}{6}$, for $k\in Z$

E. $x=\pi$ or $x\frac{\pi(6k+2)}{6}$, for $k\in Z$

[VCAA 2020 MM (67%)]

Question 79

The graph of the function $f:D\to R, f(x)=\frac{3x+2}{5-x}$, where D is the maximal domain, has asymptotes

A. $x=-5,\ y=\frac{3}{2}$ **B.** $x=-3, y=5$ **C.** $x=\frac{2}{3}, y=-3$
D. $x=5, y=3$ **E.** $x=5, y=-3$

[VCAA 2020 MM (83%)]

Question 80

Let $a\in(0,\infty)$ and $b\in R$. Consider the function $h:[-a,0)\cup(0,a]\to R, h(x)=\frac{a}{x}+b$.

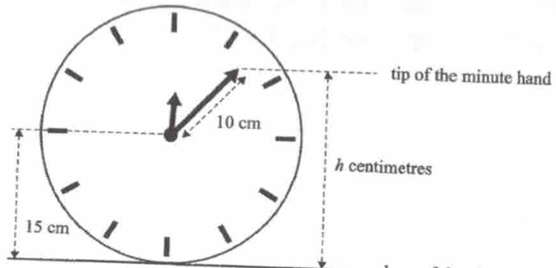
The range of h is

A. $[b-1, b+1]$ **B.** $(b-1, b+1)$ **C.** $(-\infty,b-1)\cup(b+1,\infty)$
D. $(-\infty,b-1]\cup[b+1,\infty)$ **E.** $[b-1,\infty)$

[VCAA 2020 MM (43%)]

Question 81

A clock has a minute hand that is 10 cm long and a clock face with a radius of 15 cm, as shown below.



At 12.00 noon, both hands of the clock point vertically upwards and the tip of the minute hand is at its maximum distance above the base of the clock face.
The height, h centimetres, of the tip of the minute hand above the base of the clock face t minutes after 12.00 noon is given by

A. $h(t) = 15 + 10\sin\left(\frac{\pi t}{30}\right)$ **B.** $h(t) = 15 - 10\sin\left(\frac{\pi t}{30}\right)$ **C.** $h(t) = 15 + 10\sin\left(\frac{\pi t}{60}\right)$

D. $h(t) = 15 + 10\cos\left(\frac{\pi t}{60}\right)$ **E.** $h(t) = 15 + 10\cos\left(\frac{\pi t}{30}\right)$

[VCAA 2020 MM (45%)]

Question 82

Let $f: R \to R$, $f(x) = \cos(ax)$, where $a \in R \setminus \{0\}$, be a function with the property

$$f(x) = f(x+h), \text{ for all } h \in Z$$

Let $g: D \to R$, $g(x) = \log_2(f(x))$ be a function where the range of g is $[-1, 0]$.
A possible interval for D is

A. $\left[\frac{1}{4}, \frac{5}{12}\right]$ **B.** $\left[1, \frac{7}{6}\right]$ **C.** $\left[\frac{5}{3}, 2\right]$ **D.** $\left[-\frac{1}{3}, 0\right]$ **E.** $\left[-\frac{1}{3}, 0\right]$

[VCAA 2020 MM (18%)]

Question 83

Consider the function $g: R \to R$, $g(x) = 2\sin(2x)$.

a. State the range of g.

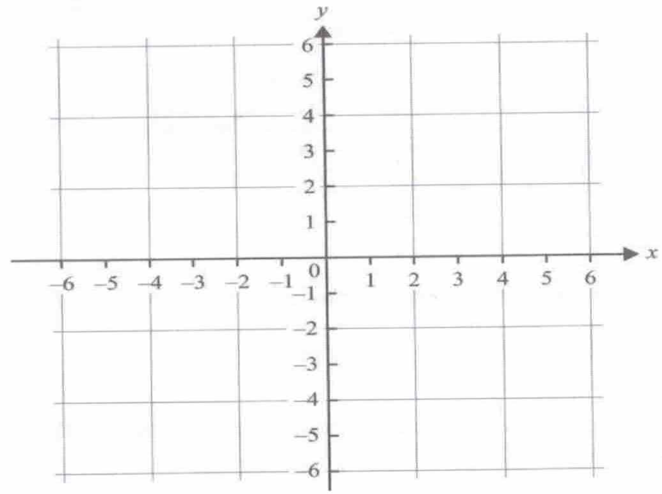
b. State the period of g.

c. Solve $2\sin(2x) = \sqrt{3}$ for $x \in R$.

[1 + 1 + 3 = 5 marks (0.8, 0.9, 1.7)]
[VCAA 2021 MM]

Question 84

a. Sketch the graph of $y = 1 - \frac{2}{x-2}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates.



b. Find the values of x for which $1 - \frac{2}{x-2} \geq 3$.

[3 + 1 = 4 marks (2.3, 0.3)]
[VCAA 2021 MM]

Question 85

Let $f: R \to R$, $f(x) = x^2 - 4$ and $g: R \to R$, $g(x) = 4(x-1)^2 - 4$.

a. The graphs of f and g have a common horizontal axis intercept at $(2, 0)$.
Find the coordinates of the other horizontal axis intercept of the graph of g.

b. Let the graph of h be a transformation of the graph of f where the transformations have been applied in the following order:

- dilation by a factor of $\frac{1}{2}$ from the vertical axis (parallel to the horizontal axis)
- translation by two units to the right (in the direction of the positive horizontal axis)

State the rule of h and the coordinates of the horizontal axis intercepts of the graph of h.

[2 + 2 = 4 marks (1.4, 0.5)]
[VCAA 2021 MM]

Let $\cos(x) = \frac{3}{5}$ and $\sin^2(y) = \frac{25}{169}$, where $x \in \left[\frac{3\pi}{2}, 2\pi\right]$ and $y \in \left[\frac{3\pi}{2}, 2\pi\right]$.

The value of $\sin(x) + \cos(y)$ is

A. $\frac{8}{65}$ **B.** $-\frac{112}{65}$ **C.** $\frac{112}{65}$ **D.** $-\frac{8}{65}$ **E.** $\frac{64}{65}$

[VCAA 2021 MM (70%)]

Question 93

Let $f: R \to R, f(x) = (2x-1)(2x+1)(3x-1)$ and $g: (-\infty, 0) \to R, g(x) = x\log_e(-x)$.
The maximum number of solutions for the equation $f(x-k) = g(x)$, where $k \in R$, is

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

[VCAA 2021 MM (39%)]

A2. Differentiation

Question 94

a. If $y = (x^2 - 5x)^4$, find $\frac{dy}{dx}$.

b. If $f(x) = \frac{x}{\sin(x)}$, find $f'\left(\frac{\pi}{2}\right)$.

[1 + 2 = 3 marks (0.6, 1.5)]
[VCAA 2012 MM (CAS)]

Question 95

Let $f: R \to R, f(x) = e^{-mx} + 3x$, where m is a positive rational number.

a. **i.** Find, in terms of m, the x-coordinate of the stationary point of the graph of $y = f(x)$.

ii. State the values of m such that the x-coordinate of this stationary point is a positive number.

b. For a particular value of m, the tangent to the graph of $y = f(x)$ at $x = -6$ passes through the origin. Find this value of m.

[2 + 1 + 3 = 6 marks (1.4, 0.2, 1.0)]
[VCAA 2012 MM (CAS)]

Question 96

For the function with rule $f(x) = x^3 - 4x$, the average rate of change of $f(x)$ with respect to x on the interval [1, 3] is

A. 1 **B.** 3 **C.** 5 **D.** 6 **E.** 9

[VCAA 2012 MM (CAS) (70%)]

Question 97

Given that g is a differentiable function and k is a real number, the derivative of the composite function $g(e^{kx})$ is

A. $kg'(e^{kx})e^{kx}$ **B.** $kg(e^{kx})$ **C.** $ke^{kx}g(e^{kx})$

D. $ke^{kx}g'(e^{x})$ **E.** $\frac{1}{k}e^{kx}g'(e^{kx})$

[VCAA 2012 MM (CAS) (45%)]

The normal to the graph of $y=\sqrt{b-x^2}$ has a gradient of 3 when $x = 1$.
The value of b is

A. $-\frac{10}{9}$ **B.** $\frac{10}{9}$ **C.** 4 **D.** 10 **E.** 11

[VCAA 2012 MM (CAS) (57%)]

Question 99

The function $f: R \to R, f(x)=ax^3+bx^2+cx$, where a is a negative real number and b and c are real numbers. For the real numbers $p<m<0<n<q$, we have $f(p)=f(q)=0$ and $f'(m)=f'(n)=0$.
The gradient of the graph of $y=f(x)$ is negative for

A. $(-\infty, m)\cup(n, \infty)$ **B.** (m, n) **C.** $(p, 0)\cup(q, \infty)$
D. $(p, m)\cup(0, q)$ **E.** (p, q)

[VCAA 2012 MM (CAS) (49%)]

Question 100

The tangent to the graph of $y=\log_e(x)$ at the point $(a, \log_e(a))$ crosses the x-axis at the point $(b, 0)$, where $b<0$. Which of the following is **false**?

A. $1<a<e$
B. The gradient of the tangent is positive
C. $a>e$
D. The gradient of the tangent is $\frac{1}{a}$
E. $a>0$

[VCAA 2012 MM (CAS) (30%)]

Question 101

a. If $y=x^2\log_e(x)$, find $\frac{dy}{dx}$.

b. Let $f(x)=e^{x^2}$. Find $f'(3)$.

[2 + 3 = 5 marks (1.7, 2.3)]
[VCAA 2013 MM (CAS)]

Question 102

If the tangent to the graph of $y=e^{ax}$, $a\neq 0$, at $x=c$ passes through the origin, then c is equal to

A. 0 **B.** $\frac{1}{a}$ **C.** 1 **D.** a **E.** $-\frac{1}{a}$

[VCAA 2013 MM (CAS) (47%)]

Question 103

For the function $f(x)=\sin(2\pi x)+2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\frac{1}{4}, 5\right]$ is

A. 0 **B.** $\frac{34}{19}$ **C.** $\frac{7}{2}$ **D.** $\frac{2\pi+10}{4}$ **E.** $\frac{23}{4}$

[VCAA 2013 MM (CAS) (73%)]

Question 104

Let $y=4\cos(x)$ and x be a function of t such that $\frac{dx}{dt}=3e^{2t}$ and $x=\frac{3}{2}$ when $t=0$.

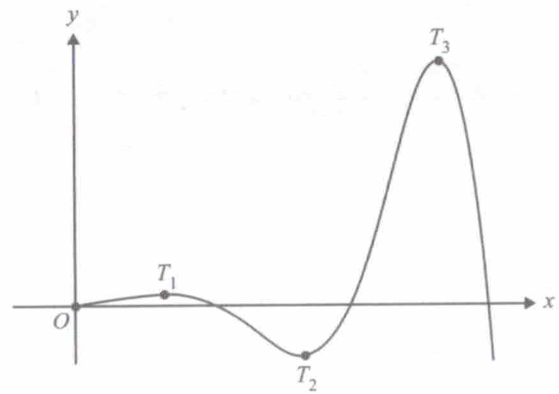
The value of $\frac{dy}{dt}$ when $x=\frac{\pi}{2}$ is

A. 0 **B.** $3\pi\log_e\left(\frac{\pi}{2}\right)$ **C.** -4π **D.** -2π **E.** $-12e$

[VCAA 2013 MM (CAS) (35%)]

Question 105

Part of the graph of a function $f:[0,\infty)\to R, f(x)=e^{x\sqrt{3}}\sin(x)$ is shown below.
The first three turning points are labelled T_1, T_2 and T_3.



The x-coordinate of T_3 is

A. $\frac{8\pi}{3}$ **B.** $\frac{16\pi}{3}$ **C.** $\frac{13\pi}{6}$ **D.** $\frac{17\pi}{6}$ **E.** $\frac{29\pi}{6}$

[VCAA 2013 MM (CAS) (50%)]

Question 106

The cubic function $f: R \to R, f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A. $c > \frac{b^2}{4a}$ **B.** $c < \frac{b^2}{4a}$ **C.** $c < 4b^2a$ **D.** $c > \frac{b^2}{3a}$ **E.** $c < \frac{b^2}{3a}$

[VCAA 2013 MM (CAS) (29%)]

Question 107

a. If $y = x^2 \sin(x)$, find $\frac{dy}{dx}$.

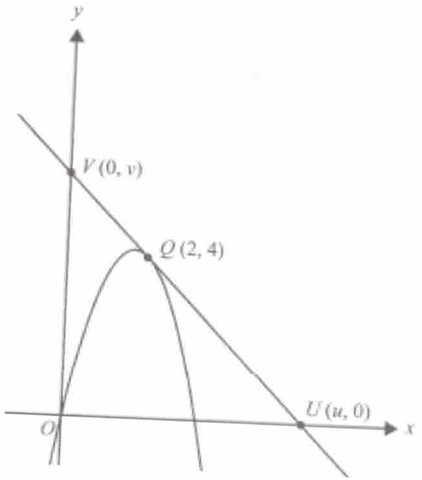
b. If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$.

[2 + 3 = 5 marks (1.8, 2.2)]
[VCAA 2014 MM (CAS)]

Question 108

A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \le u \le 6$.

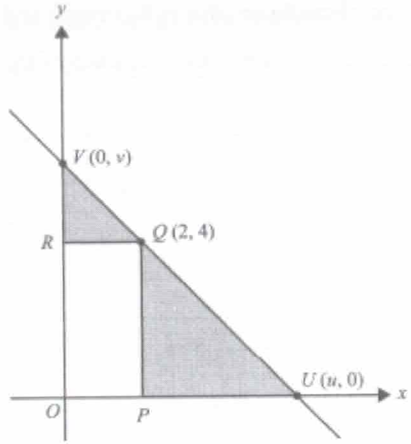
a. When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates (2, 4), as shown.



If a and b are non-zero real numbers, find the values of a and b.

… continued

b. The rectangle $OPQR$ has a vertex at Q on the line. The coordinates of Q are (2, 4), as shown.



i. Find an expression for v in terms of u.

ii. Find the **minimum** total shaded area and the value of u for which the area is a minimum.

iii. Find the **maximum** total shaded area and the value of u for which the area is a maximum.

[3 + 1 + 2 + 1 = 7 marks (1.5, 0.3, 0.4, 0.1)]
[VCAA 2014 MM (CAS)]

Question 109

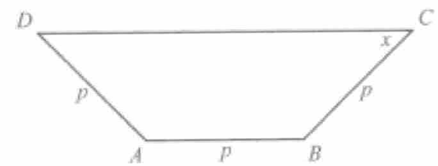
Let f be a function with domain R such that $f'(5) = 0$ and $f'(x) < 0$ when $x \neq 5$.
At $x = 5$, the graph of f has a

A. local minimum. **B.** local maximum.
C. gradient of 5. **D.** gradient of –5.
E. stationary point of inflection.

[VCAA 2014 MM (CAS) (65%)]

Question 110

The trapezium $ABCD$ is shown here.
The sides AB, BC and DA are of equal length, p. The size of the acute angle BCD is x radians.



The area of the trapezium is a maximum when the value of x is

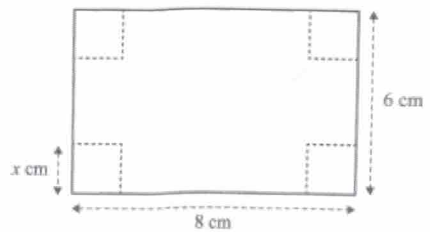
A. $\frac{\pi}{12}$ **B.** $\frac{\pi}{6}$ **C.** $\frac{\pi}{4}$ **D.** $\frac{\pi}{3}$ **E.** $\frac{5\pi}{12}$

[VCAA 2014 MM (CAS) (28%)]

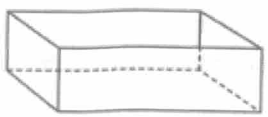
Question 111

Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide.

Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.



Zoe turns up the sides to form an open box.



The value of x for which the volume of the box is a maximum is closest to

A. 0.8 **B.** 1.1 **C.** 1.6 **D.** 2.0 **E.** 3.6

[VCAA 2014 MM (CAS) (44%)]

Question 112

a. Let $y = (5x + 1)^7$.

Find $\dfrac{dy}{dx}$.

b. Let $f(x) = \dfrac{\log_e(x)}{x^2}$.

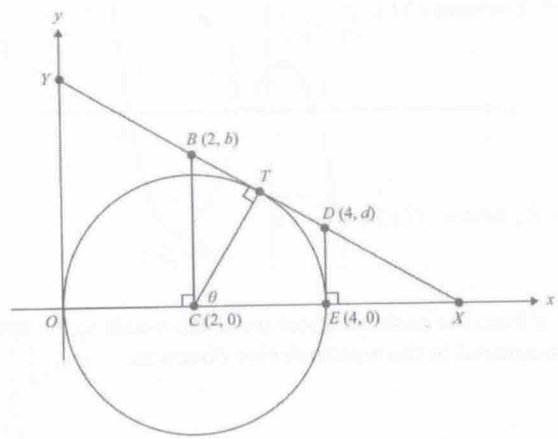
i. Find $f'(x)$.

ii. Evaluate $f'(1)$.

[1 + 2 + 1 = 4 marks (0.9, 1.6, 0.7)]
[VCAA 2015 MM (CAS)]

Question 113

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates (2, 0). The angle ECT is θ, where $0 < \theta \leq \dfrac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.

a. Find the coordinates of T in terms of θ.

b. Find the gradient of the tangent to the circle at T in terms of θ.

c. The equation of the tangent to the circle at T can be expressed as
$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XY.
Find b in terms of θ.

ii. Point D, with coordinates $(4, d)$, is on the line segment XY.
Find d in terms of θ.

d. Consider the trapezium $CEDB$ with parallel sides of length b and d.
Find the value of θ for which the area of the trapezium $CEDB$ is a minimum. Also find the minimum value of the area.

[1 + 1 + 1 + 1 + 3 = 7 marks (0.2, 0.2, 0.6, 0.5, 0.6)]
[VCAA 2015 MM (CAS)]

Question 114

Consider the tangent to the graph of $y = x^2$ at the point (2, 4).
Which of the following points lies on this tangent?

A. (1, −4) **B.** (3, 8) **C.** (−2, 6) **D.** (1, 8) **E.** (4, −4)

[VCAA 2015 MM (CAS) (77%)]

a. Let $y = \dfrac{\cos(x)}{x^2+2}$. Find $\dfrac{dy}{dx}$.

b. Let $f(x) = x^2 e^{5x}$. Evaluate $f'(1)$.

[2 + 2 = 4 marks (1.5, 1.6)]
[VCAA 2016 MM]

Question 116

Let $f : \left(-\infty, \dfrac{1}{2}\right] \to R$, where $f(x) = \sqrt{1-2x}$.

a. Find $f'(x)$.

b. Find the angle θ from the positive direction of the x-axis to the tangent to the graph of f at $x = -1$, measured in the anticlockwise direction.

[1 + 2 = 3 marks (0.7, 0.8)]
[VCAA 2016 MM]

Question 117

Let f: $(0, \infty) \to R$, where $f(x) = \log_e(x)$ and g: $R \to R$, where $g(x) = x^2 + 1$.

a. **i.** Find the rule for h, where $h(x) = f(g(x))$.

ii. State the domain and range of h.

iii. Show that $h(x) + h(-x) = f((g(x))^2)$.

iv. Find the coordinates of the stationary point of h and state its nature.

b. Let k: $(-\infty, 0] \to R$, where $k(x) = \log_e(x^2 + 1)$.

i. Find the rule for k^{-1}.

ii. State the domain and range of k^{-1}.

[1 + 2 + 2 + 2 + 2 + 2 = 11 marks (1.0, 0.6, 1.1, 0.8, 1.0, 0.9)]
[VCAA 2016 MM]

Question 118

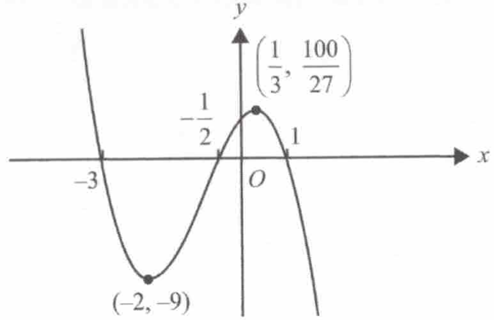
The average rate of change of the function f with rule $f(x) = 3x^2 - 2\sqrt{x+1}$ between $x = 0$ and $x = 3$, is

A. 8 **B.** 25 **C.** $\dfrac{53}{9}$ **D.** $\dfrac{25}{3}$ **E.** $\dfrac{13}{9}$

[VCAA 2016 MM (85%)]

Question 119

Part of the graph $y = f(x)$ of the polynomial function f is shown below.



$f'(x) < 0$ for

A. $x \in (-2, 0) \cup \left(\dfrac{1}{3}, \infty\right)$

B. $x \in \left(-9, \dfrac{100}{27}\right)$

C. $x \in (-\infty, -2) \cup \left(\dfrac{1}{3}, \infty\right)$

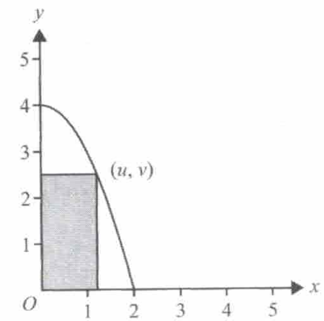
D. $x \in \left(-2, \dfrac{1}{3}\right)$

E. $x \in (-\infty, -2] \cup (1, \infty)$

[VCAA 2016 MM (77%)]

Question 120

A rectangle is formed by using part of the coordinate axes and a point (u, v), where $u > 0$ on the parabola $y = 4 - x^2$.



Which one of the following is the maximum area of the rectangle?

A. 4 **B.** $\dfrac{2\sqrt{3}}{3}$ **C.** $\dfrac{8\sqrt{3}-4}{3}$ **D.** $\dfrac{8}{3}$ **E.** $\dfrac{16\sqrt{3}}{9}$

[VCAA 2016 MM (37%)]

Question 121

For the curve $y = x^2 - 5$, the tangent to the curve will be parallel to the line connecting the positive x-intercept and the y-intercept when x is equal to

A. $\sqrt{5}$ **B.** 5 **C.** -5 **D.** $\frac{\sqrt{5}}{2}$ **E.** $\frac{1}{\sqrt{5}}$

[VCAA 2016 MM (52%)]

Question 122

a. Let $f: (-2, \infty) \to R,\ f(x) = \frac{x}{x+2}$. Differentiate f with respect to x.

b. Let $g(x) = (2 - x^3)^3$. Evaluate $g'(1)$.

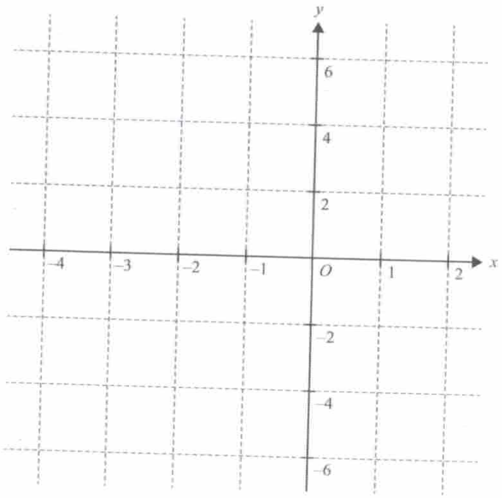
[2 + 2 = 4 marks (1.3, 1.1)]
[VCAA 2017 MM]

Question 123

Let $f: [-3, 0] \to R, f(x) = (x + 2)^2(x - 1)$.

a. Show that $(x + 2)^2(x - 1) = x^3 + 3x^2 - 4$.

b. Sketch the graph of f on the axes below. Label the axis intercepts and any stationary points with their coordinates.



[1 + 3 = 4 marks (0.8, 1.6)]
[VCAA 2017 MM]

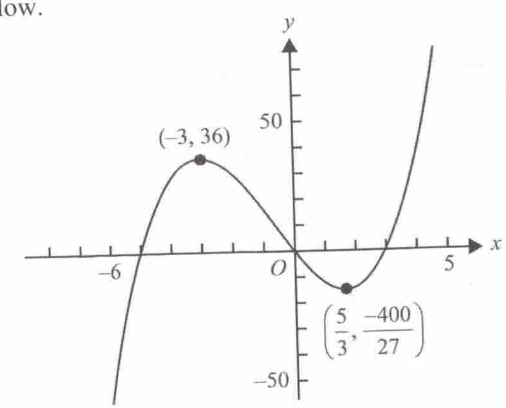
Question 124

The average rate of change of the function with the rule $f(x) = x^2 - 2x$ over the interval $[1, a]$, where $a > 1$, is 8. The value of a is

A. 9 **B.** 8 **C.** 7 **D.** 4 **E.** $1+\sqrt{2}$

Question 125

Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.



$f'(x) < 0$ for the interval

A. $(0, 3)$ **B.** $(-\infty, -5) \cup (0, 3)$ **C.** $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D. $\left(-3, \frac{5}{3}\right)$ **E.** $\left(\frac{-400}{27}, 36\right)$

[VCAA 2017 MM (80%)]

Question 126

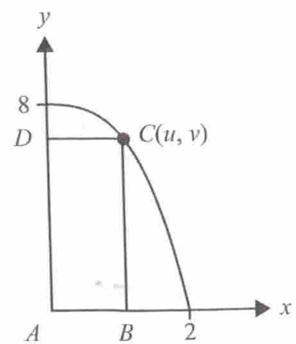
The function $f: R \to R, f(x) = x^3 + ax^2 + bx$ has a local maximum at $x = -1$ and a local minimum at $x = 3$. The values of a and b are respectively

A. -2 and -3 **B.** 2 and 1 **C.** 3 and -9 **D.** -3 and -9 **E.** -6 and -15

[VCAA 2017 MM (72%)]

Question 127

A rectangle $ABCD$ has vertices $A(0, 0)$, $B(u, 0)$, $C(u, v)$ and $D(0, v)$, where (u, v) lies on the graph of $y = -x^3 + 8$, as shown.



The maximum area of the rectangle is

A. $\sqrt[3]{2}$ **B.** $6\sqrt[3]{2}$ **C.** 16 **D.** 8 **E.** $3\sqrt[3]{2}$

[VCAA 2017 MM (58%)]

Question 128

a. If $y=(-3x^3+x^2-64)^3$, find $\dfrac{dy}{dx}$.

b. Let $f(x)=\dfrac{e^x}{\cos(x)}$. Evaluate $f'(\pi)$.

[1 + 2 = 3 marks (0.6, 1.4)]
[VCAA 2018 MM]

Question 129

Consider $f(x)=x^2+\dfrac{p}{x}, x\neq 0, p\in R$. There is a stationary point on the graph of f when $x=-2$. The value of p is

A. -16 B. -8 C. 2 D. 8 E. 16

[VCAA 2018 MM (67%)]

Question 130

A tangent to the graph of $y=\log_e(2x)$ has a gradient of 2. This tangent will cross the y-axis at

A. 0 B. -0.5 C. -1 D. $-1-\log_e(2)$ E. $-2\log_e(2)$

[VCAA 2018 MM (57%)]

Question 131

The turning point of the parabola $y=x^2-2bx+1$ is closest to the origin when

A. $b=0$

B. $b=-1$ or $b=1$

C. $b=-\dfrac{1}{\sqrt{2}}$ or $b=\dfrac{1}{\sqrt{2}}$

D. $b=\dfrac{1}{2}$ or $b=-\dfrac{1}{2}$

E. $b=\dfrac{1}{4}$ or $b=-\dfrac{1}{4}$

[VCAA 2018 MM (45%)]

Question 132

Consider the functions $f:R^+\to R, f(x)=x^{\frac{p}{q}}$ and $g:R^+\to R, g(x)=x^{\frac{m}{n}}$, where p, q, m and n are positive integers, and $\dfrac{p}{q}$ and $\dfrac{m}{n}$ are fractions in simplest form.

If $\{x:f(x)>g(x)\}=(0,1)$ and $\{x:g(x)>f(x)\}=(1,\infty)$, which of the following must be **false**?

A. $q>n$ and $p=m$

B. $m>p$ and $q=n$

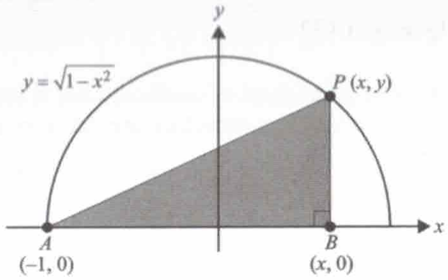
C. $pn<qm$

D. $f'(c)=g'(c)$ for some $c\in(0,1)$

E. $f'(d)=g'(d)$ for some $d\in(1,\infty)$

[VCAA 2018 MM (14%)]

Question 133



The graph of the relation $y=\sqrt{1-x^2}$ is shown on the axes here.

P is a point on the graph of this relation, A is the point $(-1, 0)$ and B is the point $(x, 0)$.

a. Find an expression for the length PB in terms of x only.

b. Find the maximum area of the triangle ABP.

[1 + 3 = 4 marks (0.6, 1.0)]
[VCAA 2019 MM]

Question 134

Consider the functions $f: R\to R, f(x) = 3 + 2x - x^2$ and $g: R\to R, g(x) = e^x$.

a. State the rule of $g(f(x))$.

b. Find the values of x for which the derivative of $g(f(x))$ is negative.

c. State the rule of $f(g(x))$.

d. Solve $f(g(x)) = 0$.

e. Find the coordinates of the stationary point of the graph of $f(g(x))$.

f. State the number of solutions to $g(f(x)) + f(g(x)) = 0$.

[1 + 2 + 1 + 2 + 2 + 1 = 9 marks (1.0, 0.7, 0.9, 1.2, 0.9, 0.2)]
[VCAA 2019 MM]

Question 135

Let $f: R\backslash\{4\} \to R, f(x) = \dfrac{a}{x-4}$, where $a > 0$.

The average rate of change of f from $x = 6$ to $x = 8$ is

A. $a\log_e(2)$ B. $\dfrac{a}{2}\log_e(2)$ C. $2a$ D. $-\dfrac{a}{4}$ E. $-\dfrac{a}{8}$

[VCAA 2019 MM (80%)]

Question 136

Let $f: [2, \infty) \to R, f(x) = x^2 - 4x + 2$ and $f(5) = 7$. The function g is the inverse function of f.

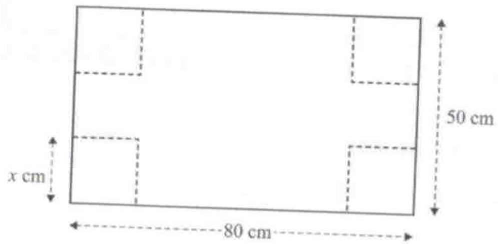
$g'(7)$ is equal to

A. $\dfrac{1}{6}$ B. 5 C. $\dfrac{\sqrt{7}}{14}$ D. 6 E. $\dfrac{1}{7}$

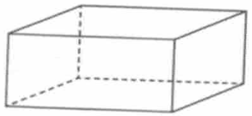
[VCAA 2019 MM (55%)]

Question 137

A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the corners, as shown in the diagram below.



A rectangular box with an open top is then constructed, as shown in the diagram below.



The volume of the box is a maximum when x is equal to

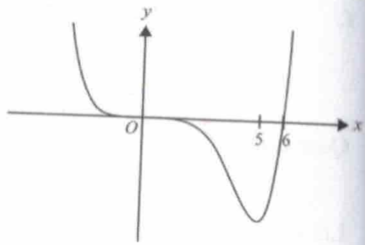
A. 10 **B.** 20 **C.** 25 **D.** $\frac{100}{3}$ **E.** $\frac{200}{3}$

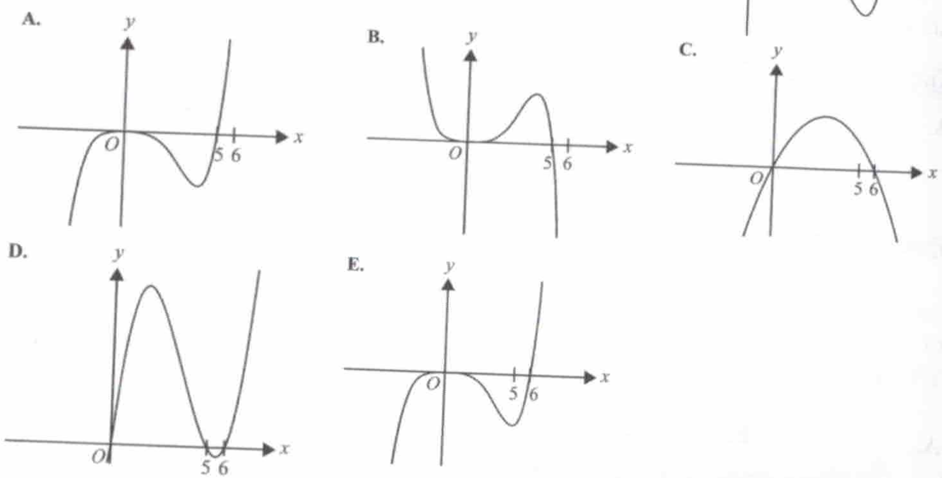
[VCAA 2019 MM (63%)]

Question 138

Part of the graph of $y = f(x)$ is shown here.

The corresponding part of the graph of $y = f'(x)$ is best represented by





[VCAA 2019 MM (63%)]

A2. Differentiation

Question 139

Which one of the following statements is true for $f: R \to R, f(x) = x + \sin(x)$?

A. The graph of f has a horizontal asymptote

B. There are infinitely many solutions to $f(x) = 4$

C. f has a period of 2π

D. $f'(x) \geq 0$ for $x \in R$

E. $f'(x) = \cos(x)$

[VCAA 2019 MM (55%)]

Question 140

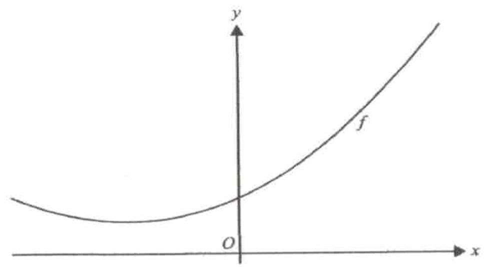
a. Let $y = x^2\sin(x)$. Find $\frac{dy}{dx}$.

b. Evaluate $f'(1)$, where $f: R \to R$, $f(x) = e^{x^2-x+3}$.

[1 + 2 = 3 marks (0.9, 1.4)]
[VCAA 2020 MM]

Question 141

Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.



a. Show that point P is not on the graph of $y = f(x)$.

b. Consider a point $Q(a, f(a))$ to be a point on the graph of f.

- **i.** Find the slope of the line connecting points P and Q in terms of a.
- **ii.** Find the slope of the tangent to the graph of f at point Q in terms of a.
- **iii.** Let the tangent to the graph of f at $x = a$ pass through point P. Find the values of a.
- **iv.** Give the equation of one of the lines passing through point P that is tangent to the graph of f.

c. Find the value, k, that gives the shortest possible distance between the graph of the function of $y = f(x - k)$ and point P.

[1 + 1 + 1 + 2 + 1 + 2 = 8 marks (0.9, 0.5, 0.7, 0.8, 0.3, 0.2)]
[VCAA 2020 MM]

Question 142

If $f(x) = e^{g(x^2)}$, where g is a differentiable function, then $f'(x)$ is equal to

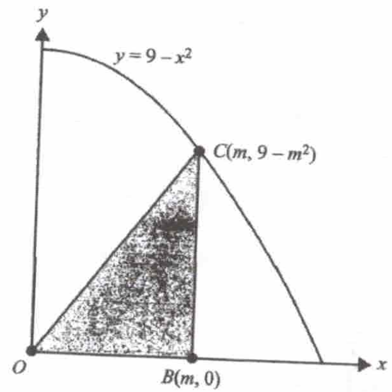
A. $2xe^{g(x^2)}$ **B.** $2xg(x^2)e^{g(x^2)}$ **C.** $2xg'(x^2)e^{g(x^2)}$

D. $2xg'(2x)e^{g(x^2)}$ **E.** $2xg'(x^2)e^{g(2x)}$

[VCAA 2020 MM (70%)]

Question 143

A right-angled triangle, *OBC*, is formed using the horizontal axis and the point $C(m, 9-m^2)$, where $m \in (0, 3)$, on the parabola $y = 9 - x^2$, as shown here.



The maximum area of the triangle *OBC* is

A. $\dfrac{\sqrt{3}}{3}$ **B.** $\dfrac{2\sqrt{3}}{3}$ **C.** $\sqrt{3}$ **D.** $3\sqrt{3}$ **E.** $9\sqrt{3}$

[VCAA 2020 MM (53%)]

Question 144

Let $f(x) = -\log_e(x+2)$. A tangent to the graph of f has a vertical axis intercept at $(0, c)$. The maximum value of c is

A. -1 **B.** $-1+\log_e(2)$ **C.** $-\log_e(2)$ **D.** $-1-\log_e(2)$ **E.** $\log_e(2)$

[VCAA 2020 MM (42%)]

Question 145

a. Differentiate $y = 2e^{-3x}$ with respect to x.

b. Evaluate $f'(4)$, where $f(x) = x\sqrt{2x+1}$.

[1 + 2 = 3 marks (0.9, 1.2)]
[VCAA 2021 MM]

Question 146

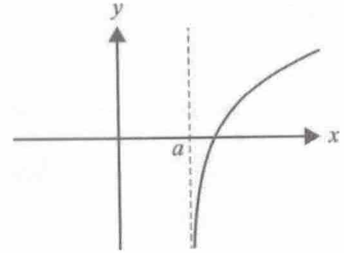
The tangent to the graph of $y = x^3 - ax^2 + 1$ at $x = 1$ passes through the origin. The value of a is

A. $\dfrac{1}{2}$ **B.** 1 **C.** $\dfrac{3}{2}$ **D.** 2 **E.** $\dfrac{5}{2}$

[VCAA 2021 MM (56%)]

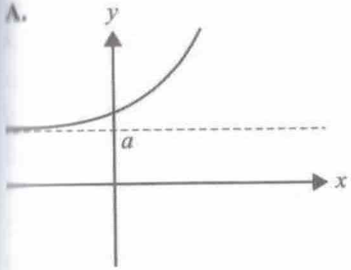
Question 147

The graph of the function f is shown below.

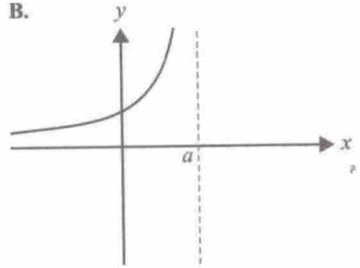


The graph corresponding to f' is

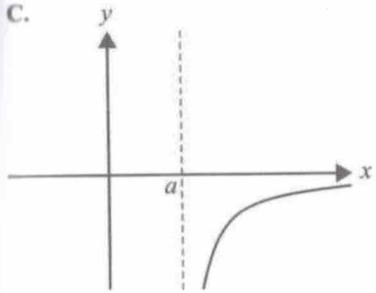
A.



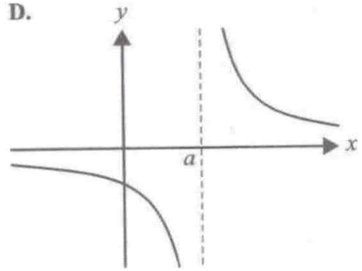
B.



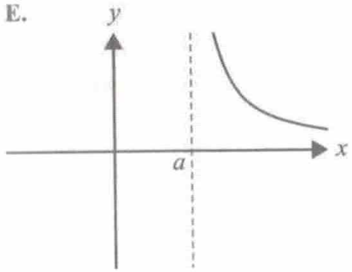
C.



D.



E.



[VCAA 2021 MM (40%)]

Question 148

The value of an investment, in dollars, after n months can be modelled by the function

$$f(n) = 2500 \times (1.004)^n$$

where $n \in \{0, 1, 2, \ldots\}$.

The average rate of change of the value of the investment over the first 12 months is closest to

A. \$10.00 per month. **B.** \$10.20 per month. **C.** \$10.50 per month.

D. \$125.00 per month. **E.** \$127.00 per month.

[VCAA 2021 MM (80%)]

Question 149

Which one of the following functions is differentiable for all real values of x?

A. $f(x) = \begin{cases} x & x < 0 \\ -x & x \geq 0 \end{cases}$

B. $f(x) = \begin{cases} x & x < 0 \\ -x & x > 0 \end{cases}$

C. $f(x) = \begin{cases} 8x+4 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

D. $f(x) = \begin{cases} 2x+1 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

E. $f(x) = \begin{cases} 4x+1 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

[VCAA 2021 MM (35%)]

A3. Integration

Question 150

Find an antiderivative of $\frac{1}{(2x-1)^3}$ with respect to x.

[2 marks (0.8)]
[VCAA 2012 MM (CAS)]

Question 151

a. Let $f: R \to R, f(x) = x\sin(x)$. Find $f'(x)$.

b. Use the result of **part a.** to find the value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos(x)\,dx$ in the form $a\pi + b$.

[1 + 3 = 4 marks (0.8, 1.3)]
[VCAA 2012 MM (CAS)]

Question 152

The average value of the function $f: [0, 2\pi] \to R, f(x) = \sin^2(x)$ over the interval $[0, a]$ is 0.4. The value of a, to three decimal places, is

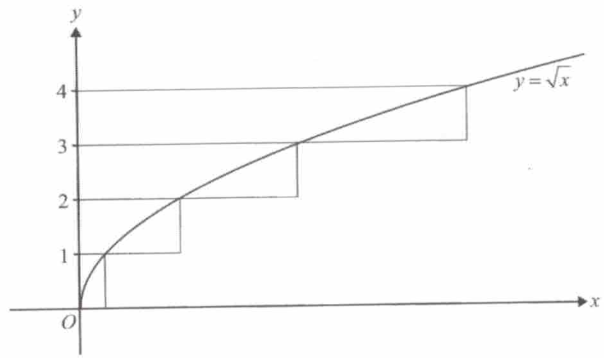
A. 0.850 **B.** 1.164 **C.** 1.298 **D.** 1.339 **E.** 4.046

[VCAA 2012 MM (CAS) (60%)]

Question 153

The graph of $f: R^+ \cup \{0\} \to R, f(x) = \sqrt{x}$ is shown here.

In order to find an approximation to the area of the region bounded by the graph of f, the y-axis and the line $y = 4$, Zoe draws four rectangles, as shown, and calculates their total area.



Zoe's approximation to the area of the region is

A. 14 **B.** 21 **C.** 29 **D.** 30 **E.** $\frac{64}{3}$

[VCAA 2012 MM (CAS) (64%)]

Question 154

The temperature, $T°$ C, inside a building t hours after midnight is given by the function

$$f:[0, 24] \to R, T(t) = 22 - 10\cos\left(\frac{\pi}{12}(t-2)\right)$$

The average temperature inside the building between 2 am and 2 pm is

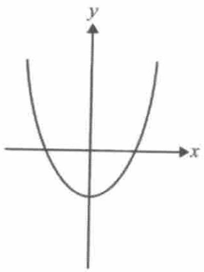
A. 10° C **B.** 12° C **C.** 20° C **D.** 22° C **E.** 32° C

[VCAA 2012 MM (CAS) (67%)]

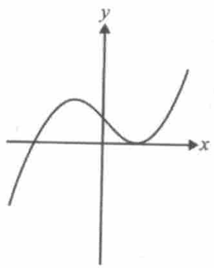
Question 155

If $f'(x) = 3x^2 - 4$, which one of the following graphs could represent the graph of $y = f(x)$?

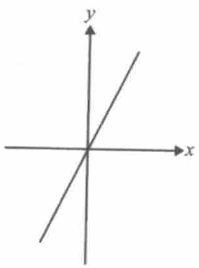
A.



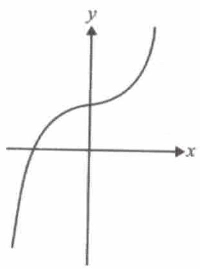
B.



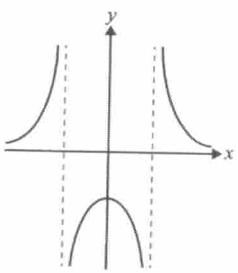
C.



D.



E.



[VCAA 2012 MM (CAS) (81%)]

Question 156

Find an anti-derivative of $(4-2x)^{-5}$ with respect to x.

[2 marks (1.3)]
[VCAA 2013 MM (CAS)]

Question 157

The function with rule $g(x)$ has derivative $g'(x) = \sin(2\pi x)$.

Given that $g(1) = \frac{1}{\pi}$, find $g(x)$.

[2 marks (1.2)]
[VCAA 2013 MM (CAS)]

Question 158

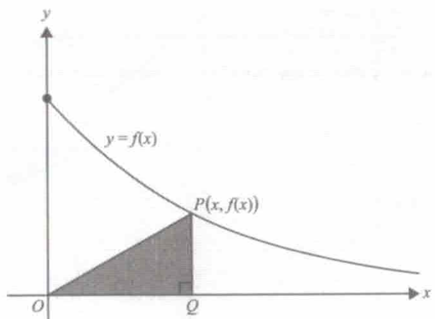
Let $g: R \to R$, $g(x) = (a-x)^2$, where a is a real constant. The average value of g on the interval $[-1, 1]$ is $\frac{31}{12}$. Find all possible values of a.

[3 marks (1.2)]
[VCAA 2013 MM (CAS)]

Question 159

Let $f:[0,\infty) \to R, f(x) = 2e^{-\frac{x}{5}}$.

A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.

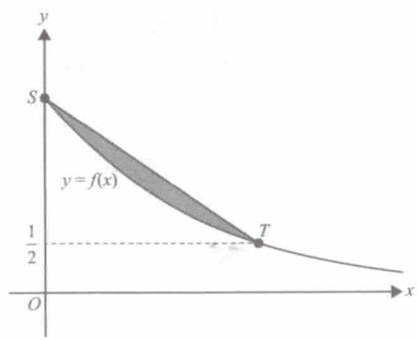


a. Find the area, A, of the triangle OQP in terms of x.

b. Find the maximum area of triangle OQP and the value of x for which the maximum occurs.

c. Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$.

Find the area of the region bounded by the graph of f and the line segment ST.



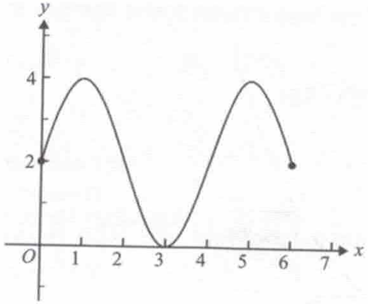
[1 + 3 + 3 = 7 marks (1.0, 1.2, 1.0)]
[VCAA 2013 MM (CAS)]

Question 160

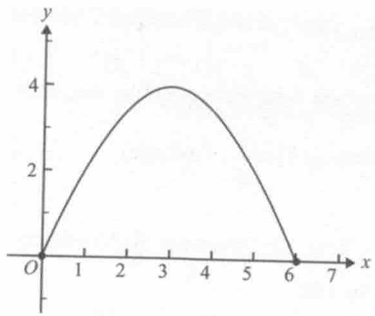
Let h be a function with an average value of 2 over the interval $[0, 6]$.

The graph of h over this interval could be

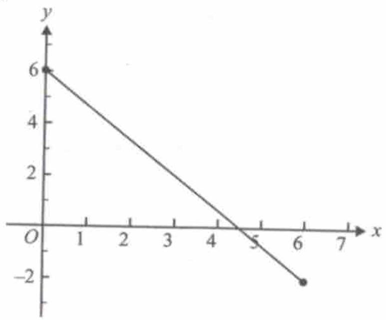
A.



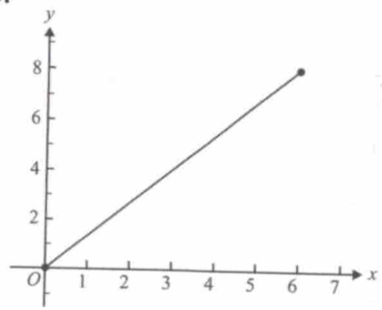
B.



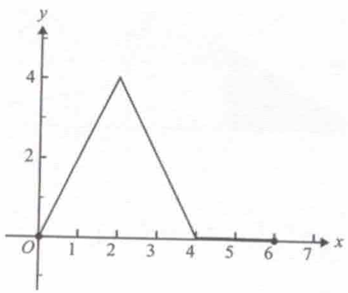
C.



D.



E.



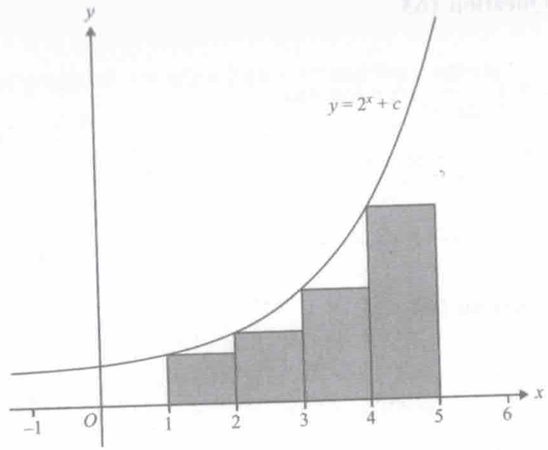
[VCAA 2013 MM (CAS) (25%)]

Question 161

Consider the graph of $y = 2^x + c$, where c is a real number.

The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x-axis and the lines $x = 1$ and $x = 5$.

If the total area of the shaded rectangles is 44, then the value of c is

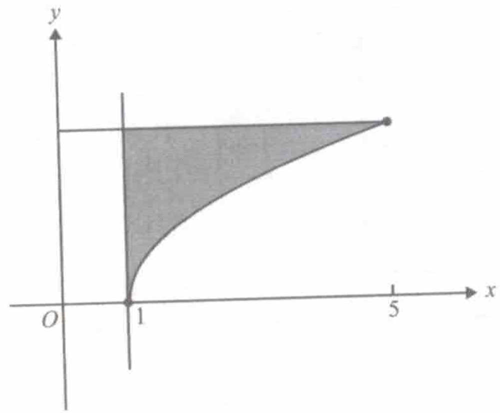


A. 14 **B.** −4 **C.** $\frac{14}{5}$ **D.** $\frac{7}{2}$ **E.** $-\frac{16}{5}$

[VCAA 2013 MM (CAS) (64%)]

Question 162

The graph of $f:[1, 5] \rightarrow R, f(x) = \sqrt{x-1}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_1^5 \left(\sqrt{x-1}\right)dx$ **B.** $\int_0^2 \left(\sqrt{x-1}\right)dx$ **C.** $\int_0^5 \left(2-\sqrt{x-1}\right)dx$

D. $\int_0^2 (x^2+1)dx$ **E.** $\int_0^2 (x^2)dx$

[VCAA 2013 MM (CAS) (21%)]

Question 163

Let $\int_4^5 \frac{2}{2x-1}\,dx = \log_e(b)$.

Find the value of b.

[2 marks (1.4)]
[VCAA 2014 MM (CAS)]

Question 164

If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

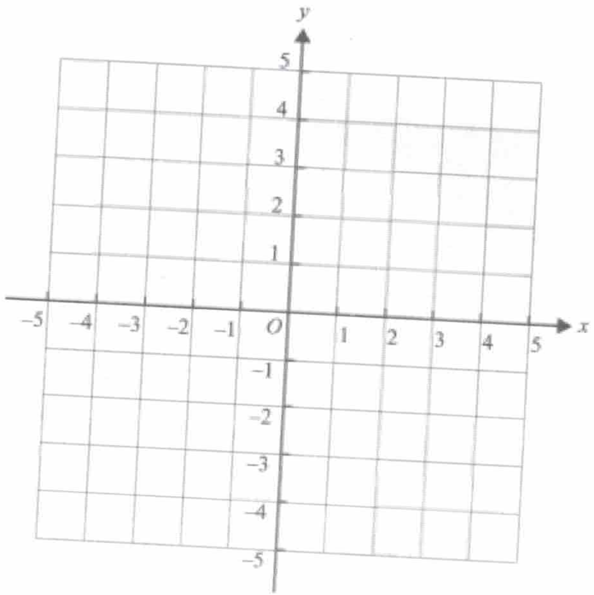
[3 marks (1.9)]
[VCAA 2014 MM (CAS)]

Question 165

Consider the function $f:[-1, 3] \to R$, $f(x) = 3x^2 - x^3$.

a. Find the coordinates of the stationary points of the function.

b. On the axes below, sketch the graph of f.
Label any end points with their coordinates.



c. Find the area enclosed by the graph of the function and the horizontal line given by $y = 4$.

[2 + 2 + 3 = 7 marks (1.5, 1.4, 1.5)]
[VCAA 2014 MM (CAS)]

Question 166

The area of the region enclosed by the graph of $y = x(x+2)(x-4)$ and the x-axis is

A. $\frac{128}{3}$ **B.** $\frac{20}{3}$ **C.** $\frac{236}{3}$ **D.** $\frac{148}{3}$ **E.** 36

[VCAA 2014 MM (CAS) (60%)]

Question 167

If $\int_1^4 f(x)\,dx = 6$, then $\int_1^4 (5 - 2f(x))\,dx$ is equal to

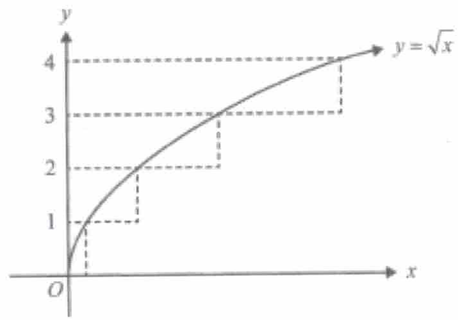
A. 3 **B.** 4 **C.** 5 **D.** 6 **E.** 16

[VCAA 2014 MM (CAS) (59%)]

Question 168

Jake and Anita are calculating the area between the graph of $y = \sqrt{x}$ and the y-axis between $y = 0$ and $y = 4$.

Jake uses a partitioning, shown in the diagram, while Anita uses a definite integral to find the exact area.



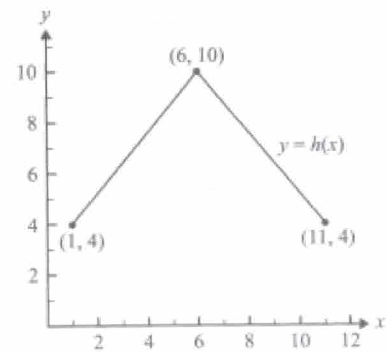
The difference between the results obtained by Jake and Anita is

A. 0 **B.** $\frac{22}{3}$ **C.** $\frac{26}{3}$ **D.** 14 **E.** 35

[VCAA 2014 MM (CAS) (61%)]

Question 169

The graph of a function, h, is shown here.



The average value of h is

A. 4 **B.** 5 **C.** 6 **D.** 7 **E.** 10

[VCAA 2014 MM (CAS) (44%)]

Question 170

Let $f'(x) = 1 - \dfrac{3}{x}$, where $x \neq 0$. Given that $f(e) = -2$, find $f(x)$.

[3 marks (2.1)]
[VCAA 2015 MM (CAS)]

Question 171

Evaluate $\displaystyle\int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx$.

[2 marks (1.2)]
[VCAA 2015 MM (CAS)]

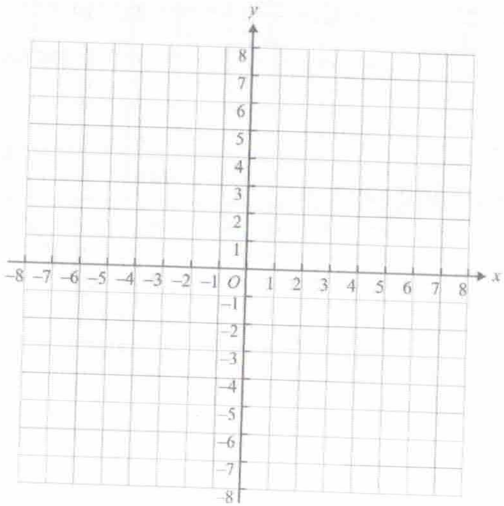
Question 172

Consider the function $f : [-3, 2] \to R,\ f(x) = \frac{1}{2}(x^3 + 3x^2 - 4)$.

a. Find the coordinates of the stationary points of the function.

The rule for f can also be expressed as $f(x) = \frac{1}{2}(x-1)(x+2)^2$.

b. On the axes below, sketch the graph of f, clearly indicating axis intercepts and turning points.
Label the end points with their coordinates.

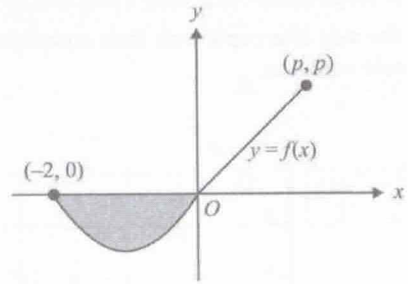


c. Find the average value of f over the interval $0 \le x \le 2$.

[2 + 2 + 2 = 6 marks (1.5, 1.4, 1.0)]
[VCAA 2015 MM (CAS)]

Question 173

The graph of a function $f: [-2, p] \to R$ is shown below.



The average value of f over the interval $[-2, p]$ is zero.

The area of the shaded region is $\frac{25}{8}$.

If the graph is a straight line, for $0 \le x \le p$, then the value of p is

A. 2 **B.** 5 **C.** $\frac{5}{4}$ **D.** $\frac{5}{2}$ **E.** $\frac{25}{4}$

[VCAA 2015 MM (CAS) (53%)]

Question 174

If $\int_0^5 g(x)\,dx = 20$ and $\int_0^5 \left(2g(x) + ax\right) dx = 90$, then the value of a is

A. 0 **B.** 4 **C.** 2 **D.** −3 **E.** 1

[VCAA 2015 MM (CAS) (69%)]

Question 175

Let $f(x) = ax^m$ and $g(x) = bx^n$, where a, b, m and n are positive integers.
The domain of f = domain of $g = R$.
If $f'(x)$ is an antiderivative of $g(x)$, then which one of the following must be true?

A. $\frac{m}{n}$ is an integer **B.** $\frac{n}{m}$ is an integer **C.** $\frac{a}{b}$ is an integer

D. $\frac{b}{a}$ is an integer **E.** $n - m = 2$

[VCAA 2015 MM (CAS) (22%)]

Question 176

If $f(x) = \int_0^x \left(\sqrt{t^2 + 4}\right) dt$, then $f'(-2)$ is equal to

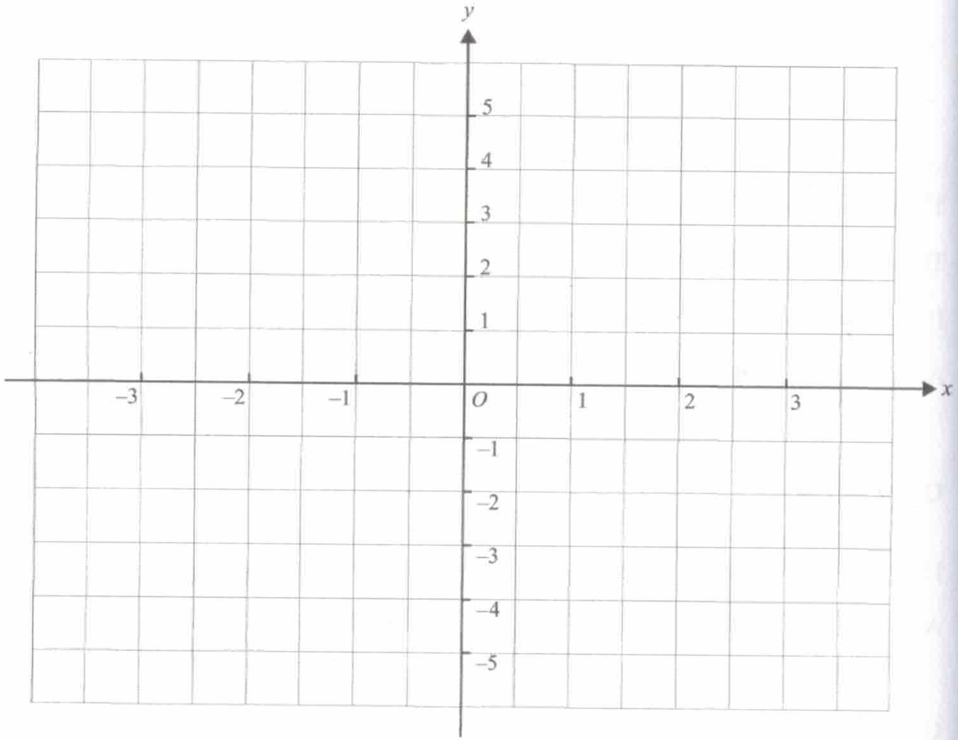
A. $\sqrt{2}$ **B.** $-\sqrt{2}$ **C.** $2\sqrt{2}$ **D.** $-2\sqrt{2}$ **E.** $4\sqrt{2}$

[VCAA 2015 MM (CAS) (68%)]

Question 177

Let $f: R\setminus\{1\} \to R$, where $f(x) = 2 + \dfrac{3}{x-1}$.

a. Sketch the graph of *f*. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation.



b. Find the area enclosed by the graph of *f*, the lines $x = 2$ and $x = 4$, and the *x*-axis.

[3 + 2 = 5 marks (2.3, 1.2)]
[VCAA 2016 MM]

Question 178

Let $f: [-\pi, \pi] \to R$, where $f(x) = 2\sin(2x) - 1$.

a. Calculate the average rate of change of *f* between $x = -\dfrac{\pi}{3}$ and $x = \dfrac{\pi}{6}$.

b. Calculate the average value of *f* over the interval $-\dfrac{\pi}{3} \le x \le \dfrac{\pi}{6}$.

[2 + 3 = 5 marks (1.0, 1.3)]
[VCAA 2016 MM]

A3. Integration

Question 179

Given that $\dfrac{d(xe^{kx})}{dx} = (kx+1)e^{kx}$, then $\int xe^{kx}\,dx$ is equal to

A. $\dfrac{xe^{kx}}{kx+1} + c$

B. $\left(\dfrac{kx+1}{k}\right)e^{kx} + c$

C. $\dfrac{1}{k}\int e^{kx}\,dx$

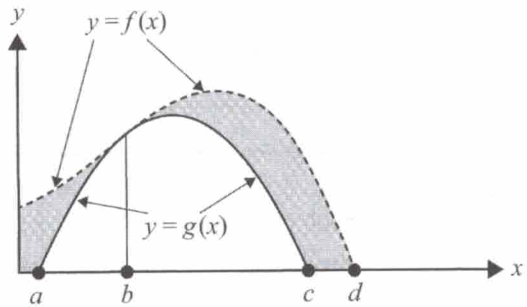
D. $\dfrac{1}{k}\left(xe^{kx} - \int e^{kx}\,dx\right) + c$

E. $\dfrac{1}{k^2}\left(xe^{kx} - e^{kx}\right) + c$

[VCAA 2016 MM (41%)]

Question 180

Consider the graphs of the functions *f* and *g* shown below.



The area of the shaded region could be represented by

A. $\int_a^d (f(x) - g(x))\,dx$

B. $\int_0^d (f(x) - g(x))\,dx$

C. $\int_0^b (f(x) - g(x))\,dx + \int_b^c (f(x) - g(x))\,dx$

D. $\int_0^a f(x)\,dx + \int_a^c (f(x) - g(x))\,dx + \int_b^d f(x)\,dx$

E. $\int_0^d f(x)\,dx - \int_a^c g(x)\,dx$

[VCAA 2016 MM (69%)]

Question 181

Let $y = x\log_e(3x)$.

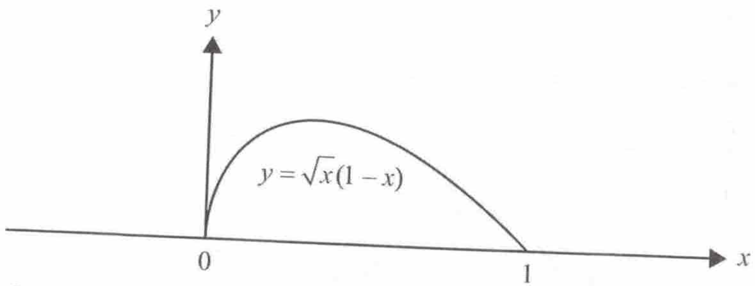
a. Find $\frac{dy}{dx}$.

b. Hence, calculate $\int_1^2 (\log_e(3x)+1)\,dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer.

[2 + 2 = 4 marks (1.1, 0.7)]
[VCAA 2017 MM]

Question 182

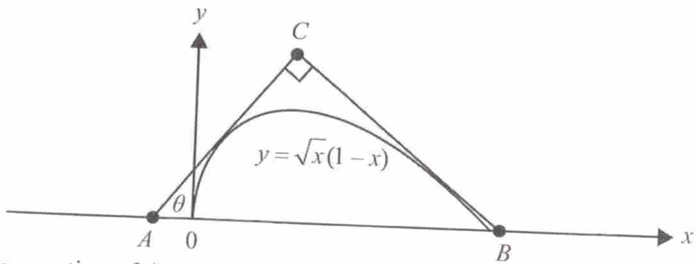
The graph of f: $[0, 1] \to R$, $f(x) = \sqrt{x}(1-x)$ is shown below.



a. Calculate the area between the graph of f and the x-axis.

b. For x in the interval $(0, 1)$, show that the gradient of the tangent to the graph of f is $\frac{1-3x}{2\sqrt{x}}$.

The edges of the **right-angled** triangle ABC are the line segments AC and BC, which are tangent to the graph of f, and the line segment AB, which is part of the horizontal axis, as shown below.
Let θ be the angle that AC makes with the positive direction of the horizontal axis, where $45° \le \theta \le 90°$.



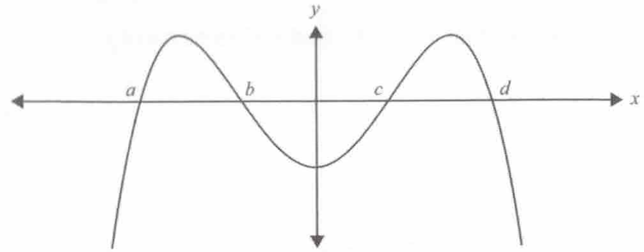
c. Find the equation of the line through B and C in the form $y = mx + c$, for $\theta = 45°$.

d. Find the coordinates of C when $\theta = 45°$.

[2 + 1 + 2 + 4 = 9 marks (0.6, 0.3, 0.2, 0.2)]
[VCAA 2017 MM]

Question 183

The graph of a function f, where $f(-x) = f(x)$, is shown below.



The graph has x-intercepts at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$ only.
The area bound by the curve and the x-axis on the interval $[a, d]$ is

A. $\int_a^d f(x)dx$

B. $\int_a^b f(x)dx - \int_c^b f(x)dx + \int_c^d f(x)dx$

C. $2\int_a^b f(x)dx + \int_b^c f(x)dx$

D. $2\int_a^b f(x)dx - 2\int_b^{b+c} f(x)dx$

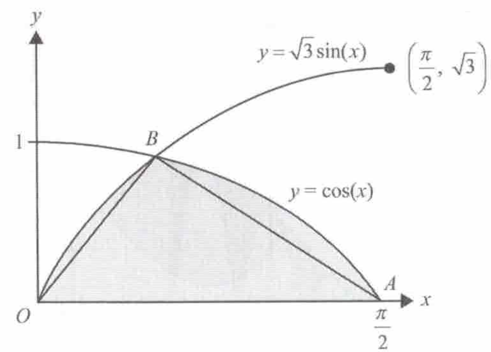
E. $\int_a^b f(x)dx + \int_c^b f(x)dx + \int_d^c f(x)dx$

[VCAA 2017 MM (21%)]

Question 184

The graphs of f: $\left[0, \frac{\pi}{2}\right] \to R$, $f(x) = \cos(x)$ and g: $\left[0, \frac{\pi}{2}\right] \to R$, $g(x) = \sqrt{3}\sin(x)$ are shown here.

The graphs intersect at B.



The ratio of the area of the shaded region to the area of triangle OAB is

A. $9:8$

B. $\sqrt{3}-1 : \frac{\sqrt{3}\pi}{8}$

C. $8\sqrt{3}-3 : 3\pi$

D. $\sqrt{3}-1 : \frac{\sqrt{3}\pi}{4}$

E. $1 : \frac{\sqrt{3}\pi}{8}$

[VCAA 2017 MM (47%)]

Question 185

The derivative with respect to x of the function $f:(1,\infty)\to R$ has the rule

$f'(x)=\frac{1}{2}-\frac{1}{(2x-2)}$. Given that $f(2)=0$, find $f(x)$ in terms of x.

[3 marks (1.8)]
[VCAA 2018 MM]

Question 186

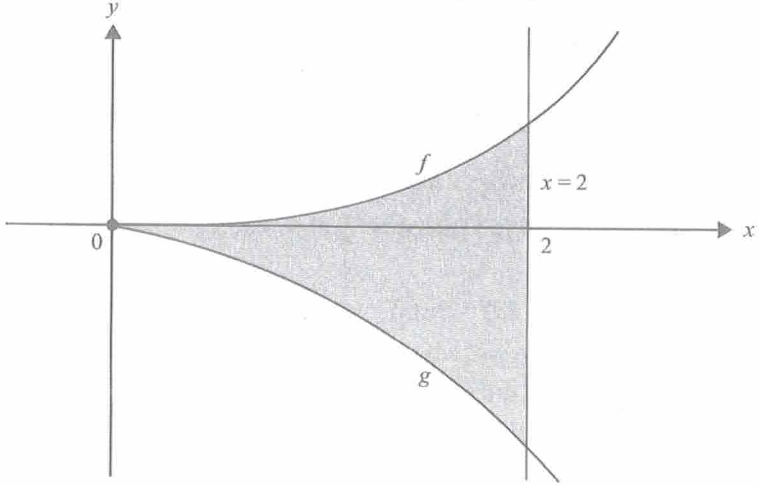
Let $f:R\to R,\ f(x)=x^2e^{kx}$, where k is a positive real constant.

a. Show that $f'(x)=xe^{kx}(kx+2)$.

b. Find the value of k for which the graphs of $y=f(x)$ and $y=f'(x)$ have exactly one point of intersection.

Let $g(x)=-\frac{2xe^{kx}}{k}$.

The diagram below shows sections of the graphs of f and g for $x\ge 0$.



Let A be the area of the region bounded by the curves $y=f(x)$, $y=g(x)$ and the line $x=2$.

c. Write down a definite integral that gives the value of A.

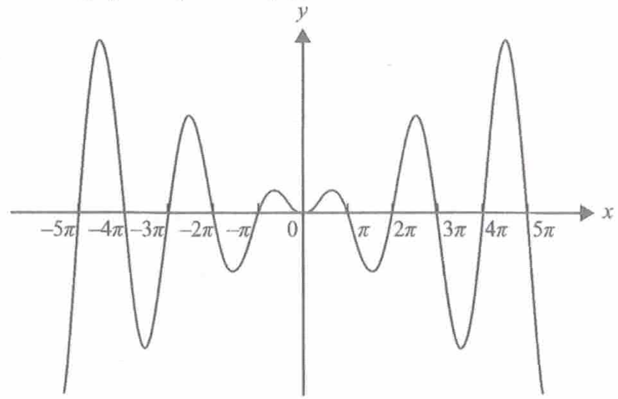
d. Using your result from **part a.**, or otherwise, find the value of k such that $A=\frac{16}{k}$.

[1 + 2 + 1 + 3 = 7 marks (0.9, 0.2, 0.7, 0.9)]
[VCAA 2018 MM]

A3. Integration

Question 187

Consider a part of the graph of $y=x\sin(x)$, as shown below.



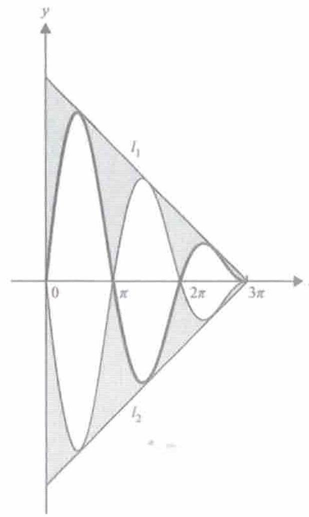
a. **i.** Given that $\int (x\sin(x))\,dx=\sin(x)-x\cos(x)+c$, evaluate $\int_{n\pi}^{(n+1)\pi}(x\sin(x))\,dx$ when n is a positive **even** integer or 0. Give your answer in simplest form.

ii. Given that $\int (x\sin(x))\,dx=\sin(x)-x\cos(x)+c$, evaluate $\int_{n\pi}^{(n+1)\pi}(x\sin(x))\,dx$ when n is a positive **odd** integer. Give your answer in simplest form.

b. Find the equation of the tangent to $y=x\sin(x)$ at the point $\left(-\frac{5\pi}{2},\frac{5\pi}{2}\right)$.

c. The graph of $y=x\sin(x)$ is mapped onto the graph of $y=(3\pi-x)\sin(x)$ by a horizontal translation of a units where a is a real constant.
State the value of a.

d. Let $f:[0,3\pi]\to R,\ f(x)=(3\pi-x)\sin(x)$ and $g:[0,3\pi]\to R,\ g(x)=(x-3\pi)\sin(x)$.
The line l_1 is the tangent to the graph of f at the point $\left(\frac{\pi}{2},\frac{5\pi}{2}\right)$ and the line l_2 is the tangent to the graph of g at $\left(\frac{\pi}{2},-\frac{5\pi}{2}\right)$, as shown in the diagram here.



Find the total area of the shaded regions shown in the diagram above.

[2 + 1 + 2 + 1 + 2 = 8 marks (0.6, 0.2, 1.0, 0.4, 0.2)]
[adapted from VCAA 2018 MM]

Question 188

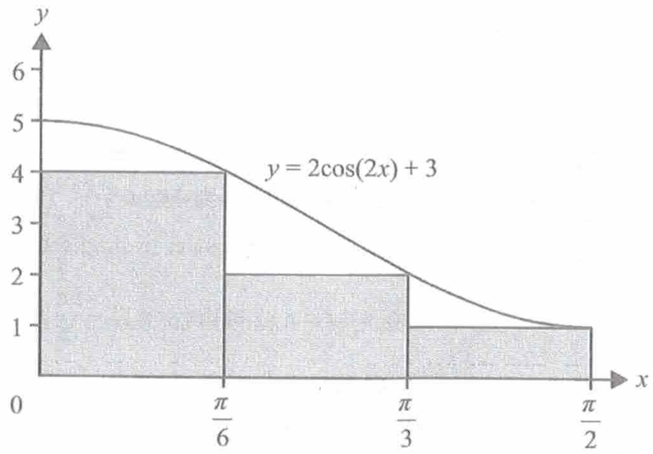
If $\int_1^{12} g(x)\,dx = 5$ and $\int_{12}^{5} g(x)\,dx = -6$, then $\int_1^{5} g(x)\,dx$ is equal to

A. -11 **B.** -1 **C.** 1 **D.** 3 **E.** 11

[VCAA 2018 MM (41%)]

Question 189

Jamie approximates the area between the x-axis and the graph of $y = 2\cos(2x) + 3$, over the interval $\left[0, \frac{\pi}{2}\right]$, using the three rectangles shown below.



Jamie's approximation as a fraction of the exact area is

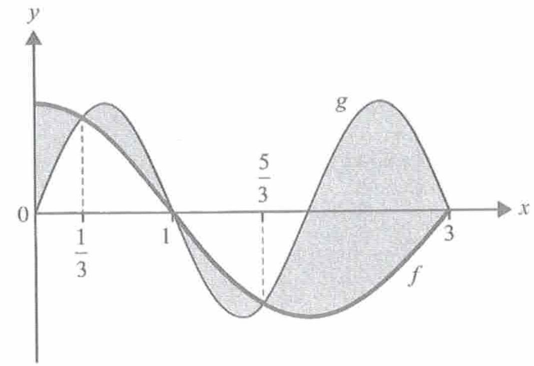
A. $\frac{5}{9}$ **B.** $\frac{7}{9}$ **C.** $\frac{9}{11}$ **D.** $\frac{11}{18}$ **E.** $\frac{7}{3}$

[VCAA 2018 MM (49%)]

A3. Integration

Question 190

The graphs $f: R \to R, f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g: R \to R, g(x) = \sin(\pi x)$ are shown in the diagram below.



An integral expression that gives the total area of the shaded regions is

A. $\int_0^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

B. $2\int_{\frac{5}{3}}^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

C. $\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{1}{3}}^{1} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - \int_{\frac{5}{3}}^{3} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

D. $2\int_1^{\frac{5}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{5}{3}}^{3} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

E. $\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx + 2\int_{\frac{1}{3}}^{1} \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx + \int_{\frac{5}{3}}^{3} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

[VCAA 2018 MM (41%)]

Question 191

Let $f: \left(\frac{1}{3}, \infty\right) \to R, f(x) = \frac{1}{3x-1}$.

a. **i.** Find $f'(x)$.

ii. Find an antiderivative of $f(x)$.

b. Let $g: R\setminus\{-1\} \to R, g(x) = \frac{\sin(\pi x)}{x+1}$. Evaluate $g'(1)$.

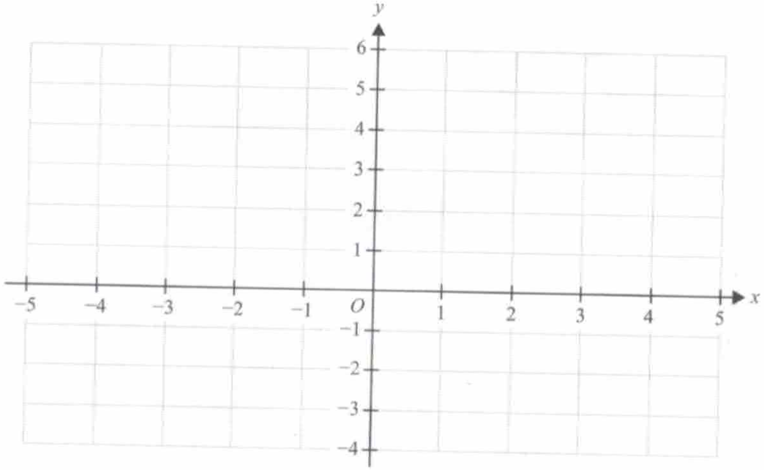
[1 + 1 + 2 = 4 marks (0.7, 0.5, 1.4)]
[VCAA 2019 MM]

Question 192

Let $f: R\setminus\{1\} \to R, f(x) = \dfrac{2}{(x-1)^2} + 1$.

a. **i.** Evaluate $f(-1)$.

ii. Sketch the graph of f on the axes below, labelling all asymptotes with their equations.



b. Find the area bounded by the graph of f, the x-axis, the line $x = -1$ and the line $x = 0$.

[1 + 2 + 2 = 5 marks (1.0, 1.6, 1.1)]
[VCAA 2019 MM]

Question 193

$\int_0^{\frac{\pi}{6}} (a\sin(x) + b\cos(x))\,dx$ is equal to

A. $\dfrac{(2-\sqrt{3})a-b}{2}$

B. $\dfrac{b-(2-\sqrt{3})a}{2}$

C. $\dfrac{(2-\sqrt{3})a+b}{2}$

D. $\dfrac{(2-\sqrt{3})b-a}{2}$

E. $\dfrac{(2-\sqrt{3})b+a}{2}$

[VCAA 2019 MM (75%)]

Question 194

Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$. The rule of f is

A. $f(x) = x^3 - x^2$

B. $f(x) = x^3 - x^2 + 48$

C. $f(x) = x^3 - x^2 - 48$

D. $f(x) = 6x - 2$

E. $f(x) = 6x - 24$

[VCAA 2019 MM (90%)]

Question 195

If $\int_1^4 f(x)\,dx = 4$ and $\int_2^4 f(x)\,dx = -2$, then $\int_1^2 (f(x)+x)\,dx$ is equal to

A. 2

B. 6

C. 8

D. $\dfrac{7}{2}$

E. $\dfrac{15}{2}$

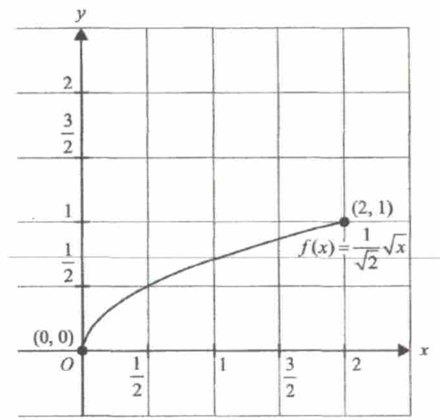
[VCAA 2019 MM (38%)]

Question 196

Let $f: [0, 2] \to R$, where $f(x) = \dfrac{1}{\sqrt{2}}\sqrt{x}$.

a. Find the domain and the rule for f^{-1}, the inverse function of f.

The graph of $y = f(x)$, where $x \in [0, 2]$, is shown on the axes below.



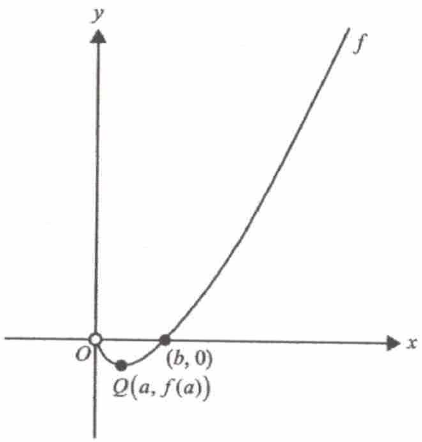
b. On the axes above, sketch the graph of f^{-1} over its domain. Label the endpoints and point(s) of intersection with the function f, giving their coordinates.

c. Find the total area of the two regions: one region bounded by the functions f and f^{-1}, and the other region bounded by f, f^{-1} and the line $x = 1$. Give your answer in the form $\dfrac{a-b\sqrt{b}}{6}$, where $a, b \in Z^+$.

[2 + 2 + 4 = 8 marks (1.4, 1.5, 1.6)]
[VCAA 2020 MM]

Question 197

Part of the graph of $y = f(x)$, where $f: (0, \infty) \to R$, $f(x) = x\log_e(x)$, is shown below.



The graph of f has a minimum at the point $Q(a, f(a))$, as shown above.

a. Find the coordinates of the point Q.

b. Using $\dfrac{d\left(x^2\log_e(x)\right)}{dx} = 2x\log_e(x) + x$, show that $x\log_e(x)$ has an antiderivative $\dfrac{x^2\log_e(x)}{2} - \dfrac{x^2}{4}$.

c. Find the area of the region that is bounded by f, the line $x = a$ and the horizontal axis for $x \in [a, b]$, where b is the x-intercept of f.

d. Let $g: (a, \infty) \to R$, $g(x) = f(x) + k$ for $k \in R$.

i. Find the value of k for which $y = 2x$ is a tangent to the graph of g.

ii. Find all values of k for which the graphs of g and g^{-1} do not intersect.

[2 + 1 + 2 + 1 + 2 = 8 marks (1.3, 0.4, 0.5, 0.2, 0.1)]
[VCAA 2020 MM]

Question 198

Let $f'(x) = \dfrac{2}{\sqrt{2x-3}}$.

If $f(6) = 4$, then

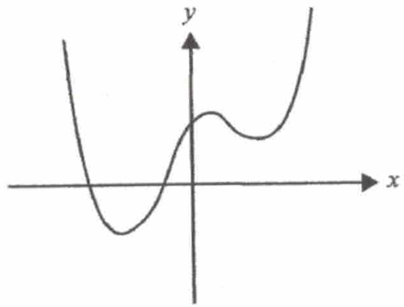
A. $f(x) = 2\sqrt{2x-3}$ **B.** $f(x) = \sqrt{2x-3} - 2$ **C.** $f(x) = 2\sqrt{2x-3} - 2$

D. $f(x) = \sqrt{2x-3} + 2$ **E.** $f(x) = \sqrt{2x-3}$

[VCAA 2020 MM (86%)]

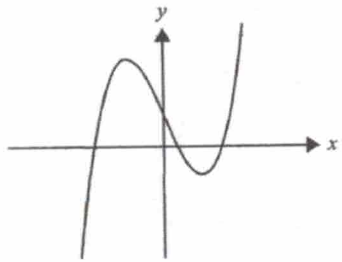
Question 199

Part of the graph of $y = f'(x)$ is shown below.

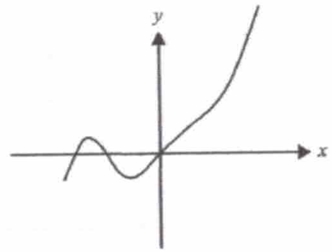


The corresponding part of the graph of $y = f(x)$ is best represented by

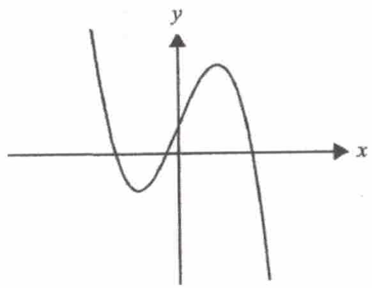
A.



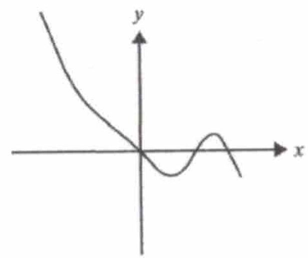
B.



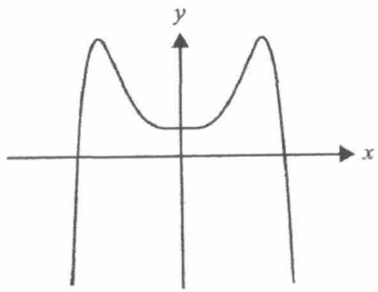
C.



D.



E.



[VCAA 2020 MM (61%)]

Question 200

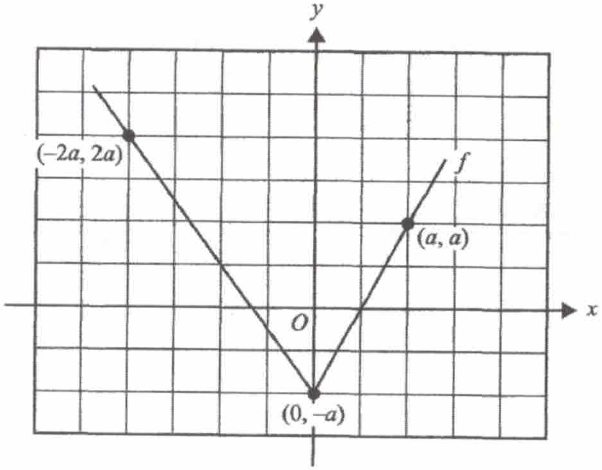
If $\int_4^8 f(x)\,dx = 5$, then $\int_0^2 f\big(2(x+2)\big)\,dx$ is equal to

A. 12 **B.** 10 **C.** 8 **D.** $\frac{1}{2}$ **E.** $\frac{5}{2}$

[VCAA 2020 MM (35%)]

Question 201

Part of the graph of a function *f*, where $a > 0$, is shown below.



The average value of the function *f* over the interval $[-2a, a]$ is

A. 0 **B.** $\frac{a}{3}$ **C.** $\frac{a}{2}$ **D.** $\frac{3a}{4}$ **E.** a

[VCAA 2020 MM (32%)]

Question 202

Let $f'(x) = x^3 + x$.

Find $f(x)$ given that $f(1) = 2$.

[2 marks (1.7)]
[VCAA 2021 MM]

Question 203

The gradient of a function is given by $\frac{dy}{dx} = \sqrt{x+6} - \frac{x}{2} - \frac{3}{2}$.

The graph of the function has a single stationary point at $\left(3, \frac{29}{4}\right)$.

a. Find the rule of the function.

b. Determine the nature of the stationary point.

[3 + 2 = 5 marks (1.6, 0.8)]
[VCAA 2021 MM]

Question 204

If $\int_0^a f(x)\,dx = k$, then $\int_0^a \big(3f(x)+2\big)\,dx$ is

A. $3k + 2a$ **B.** $3k$ **C.** $k + 2a$ **D.** $k + 2$ **E.** $3k + 2$

[VCAA 2021 MM (67%)]

Question 205

A value of k for which the average value of $y = \cos\left(kx - \frac{\pi}{2}\right)$ over the interval $[0, \pi]$ is equal to the average value of $y = \sin(x)$ over the same interval is

A. $\frac{1}{6}$ **B.** $\frac{1}{5}$ **C.** $\frac{1}{4}$ **D.** $\frac{1}{3}$ **E.** $\frac{1}{2}$

[VCAA 2021 MM (63%)]

Question 206

An approximation to $\int_0^1 (6-4x^2)\,dx$ using the trapezium rule with two equal intervals is

A. $\frac{7}{2}$ **B.** 4 **C.** $\frac{9}{2}$ **D.** $\frac{14}{3}$ **E.** $\frac{11}{2}$

[Extra Question]

A4. Discrete probability

Question 207

On any given day, the number X of telephone calls that Daniel receives is a random variable with probability distribution given by

x	0	1	2	3
$\Pr(X=x)$	0.2	0.2	0.5	0.1

a. Find the mean of X.

b. What is the probability that Daniel receives only one telephone call on each of three consecutive days?

c. Daniel receives telephone calls on both Monday and Tuesday.
What is the probability that Daniel receives a total of four calls over these two days?

[2 + 1 + 3 = 6 marks (1.7, 0.6, 1.1)]
[VCAA 2012 MM (CAS)]

Question 208

Demelza is a badminton player. If she wins a game, the probability that she will win the next game is 0.7. If she loses a game, the probability that she will lose the next game is 0.6. Demelza has just won a game.

The probability that she will win exactly one of her next two games is

A. 0.33 **B.** 0.35 **C.** 0.42 **D.** 0.49 **E.** 0.82

[VCAA 2012 MM (CAS) (72%)]

Question 209

A and B are events of a sample space S.

$$\Pr(A \cap B) = \frac{2}{5} \text{ and } \Pr(A \cap B') = \frac{3}{7}.$$

$\Pr(B' \mid A)$ is equal to

A. $\frac{6}{35}$ **B.** $\frac{15}{29}$ **C.** $\frac{14}{35}$ **D.** $\frac{29}{35}$ **E.** $\frac{2}{3}$

[VCAA 2012 MM (CAS) (52%)]

Question 210

A discrete random variable X has the probability function $\Pr(X = k) = (1-p)^k p$, where k is a non-negative integer.

$\Pr(X > 1)$ is equal to

A. $1-p+p^2$ **B.** $1-p^2$ **C.** $p-p^2$ **D.** $2p-p^2$ **E.** $(1-p)^2$

[VCAA 2012 MM (CAS) (19%)]

Question 211

The probability distribution of a discrete random variable, X, is given by the table below.

x	0	1	2	3	4
$\Pr(X=x)$	0.2	$0.6p^2$	0.1	$1-p$	0.1

a. Show that $p = \frac{2}{3}$ or $p = 1$.

b. Let $p = \frac{2}{3}$.

i. Calculate $E(X)$.

ii. Find $\Pr(X \geq E(X))$.

[3 + 2 + 1 = 6 marks (2.2, 1.2, 0.3)]
[VCAA 2013 MM (CAS)]

Question 212

Harry is a soccer player who practises penalty kicks many times each day.
Each time Harry takes a penalty kick, the probability that he scores a goal is 0.7, independent of any other penalty kick.
One day Harry took 20 penalty kicks.
Given that he scored at least 12 goals, the probability that Harry scored exactly 15 goals is closest to

A. 0.1789 **B.** 0.8867 **C.** 0.8 **D.** 0.6396 **E.** 0.2017

[VCAA 2013 MM (CAS) (58%)]

Question 213

For events A and B, $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is

A. 0 **B.** $\frac{1}{4}$ **C.** $\frac{3}{8}$ **D.** $\frac{1}{2}$ **E.** $\frac{3}{5}$

[VCAA 2013 MM (CAS) (51%)]

Question 214

A and B are events of a sample space.

Given that $\Pr(A \mid B) = p$, $\Pr(B) = p^2$ and $\Pr(A) = p^{\frac{1}{3}}$, $\Pr(B \mid A)$ is equal to

A. p **B.** $p^{\frac{4}{3}}$ **C.** $p^{\frac{7}{3}}$ **D.** $p^{\frac{8}{3}}$ **E.** p^3

[VCAA 2013 MM (CAS) (49%)]

Question 215

Sally aims to walk her dog, Mack, most mornings. If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$, and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning.

a. In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning.

Find the probability that Sally walked Mack on at least one of these two mornings.

b. In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$.

i. Find the probability that on a particular morning in April, Sally walked Mack.

ii. Using your answer from **part b.i.**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning.

[2 + 2 + 2 = 6 marks (1.1, 1.1, 0.8)]
[VCAA 2014 MM (CAS)]

Question 216

A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are different colours is

A. $\frac{20}{81}$ **B.** $\frac{5}{18}$ **C.** $\frac{4}{9}$ **D.** $\frac{40}{81}$ **E.** $\frac{5}{9}$

[VCAA 2014 MM (CAS) (62%)]

Question 217

John and Rebecca are playing darts. The result of each of their throws is independent of the result of any other throw.

The probability that John hits the bullseye with a single throw is $\frac{1}{4}$.

The probability that Rebecca hits the bullseye with a single throw is $\frac{1}{2}$. John has four throws and Rebecca has two throws.

The ratio of the probability of Rebecca hitting the bullseye at least once to the probability of John hitting the bullseye at least once is

A. 1:1 **B.** 32:27 **C.** 64:85 **D.** 2:1 **E.** 192:175

[VCAA 2014 MM (CAS) (37%)]

Question 218

An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse.

Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

a. An egg is selected at random from the set of all eggs at the warehouse. Find, in terms of p, the probability that the egg has a white eggshell.

b. Another egg is selected at random from the set of all eggs at the warehouse.

i. Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B.

ii. If the probability that this egg came from farm B is 0.3, find the value of p.

[1 + 2 + 1 = 4 marks (0.5, 0.8, 0.2)]
[VCAA 2015 MM (CAS)]

Question 219

For events A and B from a sample space, $\Pr(A \mid B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

a. Calculate $\Pr(A \cap B)$.

b. Calculate $\Pr(A' \cap B)$, where A' denotes the complement of A.

c. If events A and B are independent, calculate $\Pr(A \cup B)$.

[1 + 1 + 1 = 3 marks (0.9, 0.6, 0.3)]
[VCAA 2015 MM (CAS)]

Question 220

A box contains five red balls and three blue balls. John selects three balls from the box, without replacing them. The probability that at least one of the balls that John selected is red is

A. $\frac{5}{7}$ **B.** $\frac{5}{14}$ **C.** $\frac{7}{28}$ **D.** $\frac{15}{56}$ **E.** $\frac{55}{56}$

[VCAA 2015 MM (CAS) (60%)]

Question 221

The binomial random variable, X, has $\mathrm{E}(X) = 2$ and $\mathrm{var}(X) = \frac{4}{3}$. $\Pr(X = 1)$ is equal to

A. $\left(\frac{1}{3}\right)^6$ **B.** $\left(\frac{2}{3}\right)^6$ **C.** $\frac{1}{3} \times \left(\frac{2}{3}\right)^2$

D. $6 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^5$ **E.** $6 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^5$

[VCAA 2015 MM (CAS) (59%)]

Question 222

Consider the following discrete probability distribution for the random variable X.

x	1	2	3	4	5
$\Pr(X = x)$	p	$2p$	$3p$	$4p$	$5p$

The mean of this distribution is

A. 2 **B.** 3 **C.** $\frac{7}{2}$ **D.** $\frac{11}{3}$ **E.** 4

[VCAA 2015 MM (CAS) (75%)]

Question 223

A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then returned to the paddock.

a. What is the probability that the number of tagged sheep selected on a given day is zero?

b. What is the probability that at least one tagged sheep is selected on a given day?

c. What is the probability that no tagged sheep are selected on each of six consecutive days?

Express your answer in the form $\left(\frac{a}{b}\right)^c$, where a, b and c are positive integers.

[1 + 1 + 1 = 3 marks (0.6, 0.6, 0.6)]
[VCAA 2016 MM]

Question 224

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B.
At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a. What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer.

b. The selected motor is found to be faulty.
What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer.

[2 + 1 = 3 marks (1.1, 0.3)]
[VCAA 2016 MM]

Question 225

The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$\Pr(X = x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

A. 0.3 **B.** 0.305 **C.** 0.355 **D.** 0.405 **E.** 0.8

[VCAA 2016 MM (74%)]

Question 226

A box contains six red marbles and four blue marbles. Two marbles are drawn from the box, without replacement.
The probability that they are the same colour is

A. $\frac{1}{2}$ **B.** $\frac{28}{45}$ **C.** $\frac{7}{15}$ **D.** $\frac{3}{5}$ **E.** $\frac{1}{3}$

[VCAA 2016 MM (84%)]

Question 227

Consider the discrete probability distribution with random variable X shown in the table below.

x	-1	0	b	$2b$	4
$\Pr(X = x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of $E(X)$ are respectively

A. -0.8 and 1 **B.** -0.8 and 1.6 **C.** 0 and 2.4
D. 0.2125 and 1 **E.** 0 and 1

[VCAA 2016 MM (15%)]

Question 228

For Jac to log on to a computer successfully, Jac must type the correct password. Unfortunately, Jac has forgotten the password. If Jac types the wrong password, Jac can make another attempt. The probability of success on any attempt is $\frac{2}{5}$. Assume that the result of each attempt is independent of the result of any other attempt. A maximum of three attempts can be made.

a. What is the probability that Jac does not log on to the computer successfully?

b. Calculate the probability that Jac logs on to the computer successfully. Express your answer in the form $\frac{a}{b}$, where a and b are positive integers.

c. Calculate the probability that Jac logs on to the computer successfully on the second or on the third attempt. Express your answer in the form $\frac{c}{d}$, where c and d are positive integers.

[1 + 1 + 2 = 4 marks (0.6, 0.6, 1.1)]
[VCAA 2017 MM]

Question 229

For events A and B from a sample space, $\Pr(A|B) = \frac{1}{5}$ and $\Pr(B|A) = \frac{1}{4}$.

Let $\Pr(A \cap B) = p$.

a. Find $\Pr(A)$ in terms of p.

b. Find $\Pr(A' \cap B')$ in terms of p.

c. Given that $\Pr(A \cup B) \le \frac{1}{5}$, state the largest possible interval for p.

[1 + 1 + 2 = 4 marks (0.6, 0.3, 0.4)]
[VCAA 2017 MM]

Question 230

A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement.

The probability that the marbles are of **different** colours is

A. $\frac{5}{8}$ **B.** $\frac{3}{5}$ **C.** $\frac{15}{28}$ **D.** $\frac{15}{56}$ **E.** $\frac{30}{28}$

[VCAA 2017 MM (83%)]

Question 231

The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

x	-1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

A. $2p(1 - 3p)$ **B.** $1 - 4p$ **C.** $(1 - 3p)^2$ **D.** $6p - 16p^2$ **E.** $p(5 - 9p)$

[VCAA 2017 MM (62%)]

Question 232

Let X be a discrete random variable with binomial distribution $X \sim \text{Bi}(n, p)$. The mean and the standard deviation of this distribution are equal.

Given that $0 < p < 1$, the smallest number of trials, n, such that $p \le 0.01$ is

A. 37 **B.** 49 **C.** 98 **D.** 99 **E.** 101

[VCAA 2017 MM (38%)]

Question 233

Two boxes each contain four stones that differ only in colour.
Box 1 contains four black stones.
Box 2 contains two black stones and two white stones.
A box is chosen randomly and one stone is drawn randomly from it.
Each box is equally likely to be chosen, as is each stone.

a. What is the probability that the randomly drawn stone is black?

b. It is not known from which box the stone has been drawn. Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1?

[2 + 2 = 4 marks (1.7, 1.4)]
[VCAA 2018 MM]

Question 234

The discrete random variable X has the following probability distribution.

x	0	1	2	3	6
$\Pr(X=x)$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

Let μ be the mean of X. $\Pr(X<\mu)$ is

A. $\frac{1}{2}$ **B.** $\frac{1}{4}$ **C.** $\frac{17}{20}$ **D.** $\frac{4}{5}$ **E.** $\frac{7}{10}$

[VCAA 2018 MM (58%)]

Question 235

In a particular scoring game, there are two boxes of marbles and a player must randomly select one marble from each box. The first box contains four white marbles and two red marbles. The second box contains two white marbles and three red marbles. Each white marble scores –2 points and each red marble scores +3 points. The points obtained from the two marbles randomly selected by a player are added together to obtain a final score.

What is the probability that the final score will equal +1?

A. $\frac{2}{3}$ **B.** $\frac{1}{5}$ **C.** $\frac{2}{5}$ **D.** $\frac{2}{15}$ **E.** $\frac{8}{15}$

[VCAA 2018 MM (59%)]

Question 236

Two events, A and B, are independent, where $\Pr(B) = 2\Pr(A)$ and $\Pr(A \cup B) = 0.52$. $\Pr(A)$ is equal to

A. 0.1 **B.** 0.2 **C.** 0.3 **D.** 0.4 **E.** 0.5

[VCAA 2018 MM (60%)]

Question 237

The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail. Jo has three coins in her pocket; two are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$. Jo randomly selects a coin from her pocket and tosses it.

a. Find the probability that she tosses a head.

b. Find the probability that she selected an unbiased coin, given that she tossed a head.

[2 + 1 = 3 marks (1.5, 0.5)]
[VCAA 2019 MM]

A4. Discrete probability

Question 238

The discrete random variable X has the following probability distribution.

x	0	1	2	3
$\Pr(X=x)$	a	$3a$	$5a$	$7a$

The mean of X is

A. $\frac{1}{16}$ **B.** 1 **C.** $\frac{35}{16}$ **D.** $\frac{17}{8}$ **E.** 2

[VCAA 2019 MM (82%)]

Question 239

An archer can successfully hit a target with a probability of 0.9. The archer attempts to hit the target 80 times. The outcome of each attempt is independent of any other attempt.

Given that the archer successfully hits the target at least 70 times, the probability that the archer successfully hits the target exactly 74 times, correct to four decimal places, is

A. 0.3635 **B.** 0.8266 **C.** 0.1494 **D.** 0.3005 **E.** 0.1701

[VCAA 2019 MM (71%)]

Question 240

A and B are events from a sample space such that $\Pr(A) = p$, where $p > 0$, $\Pr(B|A) = m$ and $\Pr(B|A') = n$.

A and B are independent events when

A. $m = n$ **B.** $m = 1 - p$ **C.** $m + n = 1$
D. $m = p$ **E.** $m + n = 1 - p$

[VCAA 2019 MM (30%)]

Question 241

A box contains n marbles that are identical in every way except colour, of which k marbles are coloured red and the remainder of the marbles are coloured green. Two marbles are drawn randomly from the box.

If the first marble is **not** replaced into the box before the second marble is drawn, then the probability that the two marbles drawn are the same colour is

A. $\frac{k^2+(n-k)^2}{n^2}$ **B.** $\frac{k^2+(n-k-1)^2}{n^2}$ **C.** $\frac{2k(n-k-1)}{n(n-1)}$

D. $\frac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}$ **E.** ${}^nC_2\left(\frac{k}{n}\right)^2\left(1-\frac{k}{n}\right)^{n-2}$

[VCAA 2019 MM (43%)]

Question 242

A car manufacturer is reviewing the performance of its car model X. It is known that at any given six-month service, the probability of model X requiring an oil change is $\frac{17}{20}$, the probability of model X requiring an air filter change is $\frac{3}{20}$ and the probability of model X requiring both is $\frac{1}{20}$.

a. State the probability that at any given six-month service model X will require an air filter change without an oil change.

b. The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where $m, n \in Z^+$.

Determine m in terms of n if the probability of model Y requiring an air filter change without an oil change at any given six-month service is 0.05.

[1 + 2 = 3 marks (0.5, 1.0)]
[VCAA 2020 MM]

Question 243

For a certain population the probability of a person being born with the specific gene SPGEI is $\frac{3}{5}$. The probability of a person having this gene is independent of any other person in the population having this gene.

a. In a randomly selected group of four people, what is the probability that three or more people have the SPGEI gene?

b. In a randomly selected group of four people, what is the probability that exactly two people have the SPGEI gene, given that at least one of those people has the SPGEI gene? Express your answer in the form $\frac{a^3}{b^4-c^4}$, where $a, b, c \in Z^+$.

[2 + 2 = 4 marks (1.0, 0.5)]
[VCAA 2020 MM]

Question 244

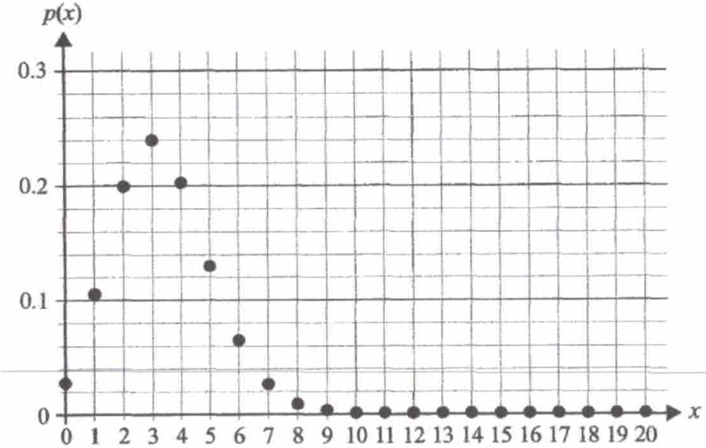
Items are packed in boxes of 25 and the mean number of defective items per box is 1.4. Assuming that the probability of an item being defective is binomially distributed, the probability that a box contains more than three defective items, correct to three decimal places, is

A. 0.037 **B.** 0.048 **C.** 0.056 **D.** 0.114 **E.** 0.162

[VCAA 2020 MM (50%)]

Question 245

Shown below is the graph of p, which is the probability function for the number of times, x, that a '6' is rolled on a fair six-sided die in 20 trials.



Let q be the probability function for the number of times, w, that a '6' is **not** rolled on a fair six-sided die in 20 trials. $q(w)$ is given by

A. $p(20-w)$ **B.** $p\left(1-\frac{w}{20}\right)$ **C.** $p\left(\frac{w}{20}\right)$ **D.** $p(w-20)$ **E.** $1-p(w)$

[VCAA 2020 MM (15%)]

Question 246

The probability of winning a game is 0.25. The probability of winning a game is independent of winning any other game.

If Ben plays 10 games, the probability that he will win exactly four times is closest to

A. 0.1460 **B.** 0.2241 **C.** 0.9219 **D.** 0.0781 **E.** 0.7759

[VCAA 2021 MM (88%)]

Question 247

Four fair coins are tossed at the same time. The outcome for each coin is independent of the outcome for any other coin.

The probability that there is an equal number of heads and tails, given that there is at least one head, is

A. $\frac{1}{2}$ **B.** $\frac{1}{3}$ **C.** $\frac{3}{4}$ **D.** $\frac{2}{5}$ **E.** $\frac{4}{7}$

[VCAA 2021 MM (48%)]

Question 248

A discrete random variable X has a binomial distribution with a probability of success of $p = 0.1$ for n trials, where $n > 2$.

If the probability of obtaining at least two successes after n trials is at least 0.5, then the smallest possible value of n is

A. 15 **B.** 16 **C.** 17 **D.** 18 **E.** 19

[VCAA 2021 MM (57%)]

Question 249

Let A and B be two independent events from a sample space.

If $\Pr(A) = p$, $\Pr(B) = p^2$ and $\Pr(A) + \Pr(B) = 1$, then $\Pr(A' \cup B)$ is equal to

A. $1 - p - p^2$ **B.** $p^2 - p^3$ **C.** $p - p^3$

D. $1 - p + p^3$ **E.** $1 - p - p^2 + p^3$

[VCAA 2021 MM (39%)]

A5. Continuous probability and Statistics

Question 250

a. The random variable X is normally distributed with mean 100 and standard deviation 4.
If $\Pr(X < 106) = q$, find $\Pr(94 < X < 100)$ in terms of q.

b. The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{x+1}{12} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of b such that $\Pr(X \le b) = \dfrac{5}{8}$.

[2 + 3 = 5 marks (0.8, 1.7)]
[VCAA 2012 MM (CAS)]

Question 251

The weights of bags of flour are normally distributed with mean 252 g and standard deviation 12 g. The manufacturer says that 40% of bags weigh more than x g.
The maximum possible value of x is closest to

A. 249.0 **B.** 251.5 **C.** 253.5 **D.** 254.5 **E.** 255.0

[VCAA 2012 MM (CAS) (62%)]

Question 252

A continuous random variable, X, has a probability density function

$$f(x) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi x}{4}\right) & \text{if } x \in [0,\ 2] \\ 0 & \text{otherwise} \end{cases}$$

Given that $\dfrac{d}{dx}\left(x\sin\left(\dfrac{\pi x}{4}\right)\right) = \dfrac{\pi x}{4}\cos\left(\dfrac{\pi x}{4}\right) + \sin\left(\dfrac{\pi x}{4}\right)$, find $E(X)$.

[3 marks (1.1)]
[VCAA 2013 MM (CAS)]

Question 253

Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

A. 7 days **B.** 13 days **C.** 17 days **D.** 21 days **E.** 37 days

[VCAA 2013 MM (CAS) (47%)]

Question 254

The continuous random variable X, with probability density function $p(x)$, has mean 2 and variance 5. The value of $\int_{-\infty}^{\infty} x^2 p(x)\,dx$ is

A. 1 **B.** 7 **C.** 9 **D.** 21 **E.** 29

[VCAA 2014 MM (CAS) (46%)]

Question 255

The random variable X has a normal distribution with mean 12 and standard deviation 0.5. If Z has the standard normal distribution, then the probability that X is less than 11.5 is equal to

A. $\Pr(Z > -1)$ **B.** $\Pr(Z < -0.5)$ **C.** $\Pr(Z > 1)$

D. $\Pr(Z \geq 0.5)$ **E.** $\Pr(Z < 1)$

[VCAA 2014 MM (CAS) (58%)]

Question 256

If X is a random variable such that $\Pr(X > 5) = a$ and $\Pr(X > 8) = b$, then $\Pr(X < 5 \mid X < 8)$ is

A. $\dfrac{a}{b}$ **B.** $\dfrac{a-b}{1-b}$ **C.** $\dfrac{1-b}{1-a}$ **D.** $\dfrac{ab}{1-b}$ **E.** $\dfrac{a-1}{b-1}$

[VCAA 2014 MM (CAS) (45%)]

Question 257

Let the random variable X be normally distributed with mean 2.5 and standard deviation 0.3. Let Z be the standard normal random variable, such that $Z \sim N(0,\ 1)$.

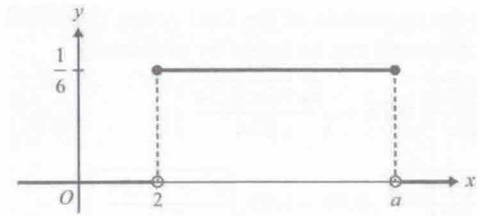
a. Find b such that $\Pr(X > 3.1) = \Pr(Z < b)$.

b. Using the fact that, correct to two decimal places, $\Pr(Z < -1) = 0.16$, find $\Pr(X < 2.8 \mid X > 2.5)$. Write the answer correct to two decimal places.

[1 + 2 = 3 marks (0.5, 1.0)]
[VCAA 2015 MM (CAS)]

Question 258

The graph of the probability density function of a continuous random variable, X, is shown below.



If $a > 2$, then $E(X)$ is equal to

A. 8 **B.** 5 **C.** 4 **D.** 3 **E.** 2

[VCAA 2015 MM (CAS) (37%)]

Question 259

The function f is a probability density function with rule $f(x) = \begin{cases} ae^x & 0 \leq x \leq 1 \\ ae & 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$.

The value of a is

A. 1 **B.** e **C.** $\dfrac{1}{e}$ **D.** $\dfrac{1}{2e}$ **E.** $\dfrac{1}{2e-1}$

[VCAA 2015 MM (CAS) (63%)]

Question 260

A student performs an experiment in which a computer is used to simulate drawing a random sample of size n from a large population. The proportion of the population with the characteristic of interest to the student is p.

a. Let the random variable $\hat{P}$ represent the sample proportion observed in the experiment.

If $p = \dfrac{1}{5}$, find the smallest integer value of the sample size such that the standard deviation of $\hat{P}$ is less than or equal to $\dfrac{1}{100}$.

Each of 23 students in a class independently performs the experiment described above and each student calculates an approximate 95% confidence interval for p using the sample proportions for their sample. It is subsequently found that exactly one of the 23 confidence intervals calculated by the class does not contain the value of p.

b. Two of the confidence intervals calculated by the class are selected at random without replacement.

Find the probability that exactly one of the selected confidence intervals does not contain the value of p.

[2 + 2 = 4 marks]
[VCAA sample examination 2016 MM]

Question 261

An opinion pollster reported that for a random sample of 574 voters in a town, 76% indicated a preference for retaining the current council. An approximate 90% confidence interval for the proportion of the total voting population with a preference for retaining the current council can be found by evaluating

A. $\left(0.76-\sqrt{\dfrac{0.76\times 0.24}{574}},\ 0.76+\sqrt{\dfrac{0.76\times 0.24}{574}}\right)$

B. $\left(0.76-1.65\sqrt{\dfrac{0.76\times 0.24}{574}},\ 0.76+1.65\sqrt{\dfrac{0.76\times 0.24}{574}}\right)$

C. $\left(0.76-2.58\sqrt{\dfrac{0.76\times 0.24}{574}},\ 0.76+2.58\sqrt{\dfrac{0.76\times 0.24}{574}}\right)$

D. $\left(0.76-1.96\sqrt{0.76\times 0.24\times 574},\ 0.76+1.96\sqrt{0.76\times 0.24\times 574}\right)$

E. $\left(0.76-2\sqrt{0.76\times 0.24\times 574},\ 0.76+2\sqrt{0.76\times 0.24\times 574}\right)$

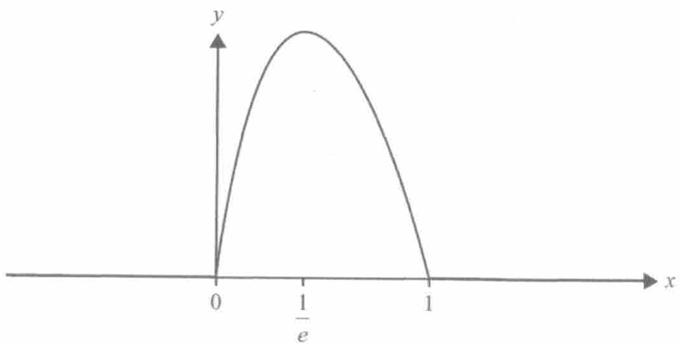
[VCAA Sample examination 2016 MM]

Question 262

Let X be a continuous random variable with probability density function

$$f(x)=\begin{cases}-4x\log_e(x) & 0<x\le 1\\ 0 & \text{elsewhere}\end{cases}$$

Part of the graph of f is shown below. The graph has a turning point at $x=\frac{1}{e}$.



a. Show by differentiation that $\dfrac{x^k}{k^2}\left(k\log_e(x)-1\right)$ is an antiderivative of $x^{k-1}\log_e(x)$, where k is a positive real number.

b. Calculate $\Pr\left(X>\frac{1}{e}\right)$.

[2 + 2 = 4 marks (0.6, 0.3)]
[adapted from VCAA 2016 MM]

Question 263

The random variable, X, has a normal distribution with mean 12 and standard deviation 0.25. If the random variable, Z, has the standard normal distribution, then the probability that X is greater than 12.5 is equal to

A. $\Pr(Z<-4)$ **B.** $\Pr(Z<-1.5)$ **C.** $\Pr(Z<1)$
D. $\Pr(Z\ge 1.5)$ **E.** $\Pr(Z>2)$

[VCAA 2016 MM (78%)]

Question 264

Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. (Do not use a normal approximation.) $\Pr\left(\hat{P}\ge\frac{3}{16}\right)$ is closest to

A. 0.6482 **B.** 0.8593 **C.** 0.7543 **D.** 0.6542 **E.** 0.3211

[VCAA 2016 MM (56%)]

Question 265

The continuous random variable, X, has a probability density function given by

$$f(x)=\begin{cases}\frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi\le x\le 5\pi\\ 0 & \text{elsewhere}\end{cases}$$

The value of a such that $\Pr(X<a)=\dfrac{\sqrt{3}+2}{4}$ is

A. $\frac{19\pi}{6}$ **B.** $\frac{14\pi}{3}$ **C.** $\frac{10\pi}{3}$ **D.** $\frac{29\pi}{6}$ **E.** $\frac{17\pi}{3}$

[VCAA 2016 MM (62%)]

Question 266

At a large sporting arena there are a number of food outlets, including a cafe.

a. The cafe employs five men and four women. Four of these people are rostered at random to work each day. Let $\hat{P}$ represent the sample proportion of men rostered to work on a particular day.

i. List the possible values that $\hat{P}$ can take.

ii. Find $\Pr(\hat{P}=0)$.

b. There are over 80 000 spectators at a sporting match at the arena. Five in nine of these spectators support the Goannas team. A simple random sample of 2000 spectators is selected. What is the standard deviation of the distribution of $\hat{P}$, the sample proportion of spectators who support the Goannas team?

[1 + 1 + 1 = 3 marks]
[VCAA 2017 NH MM]

Question 267

A bag contains five blue marbles and four red marbles. A sample of four marbles is taken from the bag, without replacement.

The probability that the proportion of blue marbles in the sample is greater than $\frac{1}{2}$ is

A. $\frac{1}{2}$ **B.** $\frac{2}{9}$ **C.** $\frac{5}{14}$ **D.** $\frac{5}{9}$ **E.** $\frac{25}{63}$

[VCAA 2017 NH MM]

Question 268

In a large population of fish, the proportion of angel fish is $\frac{1}{4}$.

Let $\hat{P}$ be the random variable that represents the sample proportion of angel fish for samples of size n drawn from the population.

Find the smallest integer value of n such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

[2 marks (0.7)]
[VCAA 2017 MM]

Question 269

The 95% confidence interval for the proportion of ferry tickets that are cancelled on the intended departure day is calculated from a large sample to be (0.039, 0.121).

The sample proportion from which this interval was constructed is

A. 0.080 **B.** 0.041 **C.** 0.100 **D.** 0.062 **E.** 0.059

[VCAA 2017 MM (47%)]

Question 270

For random samples of five Australians, $\hat{P}$ is the random variable that represents the proportion who live in a capital city.

Given that $\Pr(\hat{P}=0)=\frac{1}{243}$, then $\Pr(\hat{P}>0.6)$, correct to four decimal places, is

A. 0.0453 **B.** 0.3209 **C.** 0.4609 **D.** 0.5390 **E.** 0.7901

[VCAA 2017 MM (41%)]

A5. Continuous probability and Statistics

Question 271

A probability density function f is given by

$$f(x)=\begin{cases}\cos(x)+1 & k<x<(k+1)\\ 0 & \text{elsewhere}\end{cases}$$

where $0<k<2$. The value of k is

A. 1 **B.** $\frac{3\pi-1}{2}$ **C.** $\pi-1$ **D.** $\frac{\pi-1}{2}$ **E.** $\frac{\pi}{2}$

[VCAA 2017 MM (59%)]

Question 272

Let $\hat{P}$ be the random variable that represents the sample proportions of customers who bring their own shopping bags to a large shopping centre.
From a sample consisting of all customers on a particular day, an approximate 95% confidence interval for the proportion of who bring their own shopping bags to this large shopping centre was determined to be $\left(\frac{4853}{50\,000}, \frac{5147}{50\,000}\right)$.

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval.

b. Use the fact that $1.96=\frac{49}{25}$ to find the size of the sample from which this approximate 95% confidence interval was obtained.

[1 + 2 = 3 marks]
[VCAA 2018 NH MM]

Question 273

A box contains 20 000 marbles that are either blue or red. There are more blue marbles than red marbles. Random samples of 100 marbles are taken from the box. Each random sample is obtained by sampling with replacement.
If the standard deviation of the sampling distribution for the proportion of blue marbles is 0.03, then the number of blue marbles in the box is

A. 11 000 **B.** 16 000 **C.** 17 000 **D.** 18 000 **E.** 19 000

[VCAA 2018 NH MM]

Question 274

Let X be a normally distributed random variable with a mean of 6 and a variance of 4.

Let Z be a random variable with the standard normal distribution.

a. Find $\Pr(X>6)$.

b. Find b such that $\Pr(X>7)=\Pr(Z<b)$.

[1 + 1 = 2 marks (0.8, 0.4)]
[VCAA 2018 MM]

Question 275

Jacinta tosses a coin five times.

a. Assuming that the coin is fair and given that Jacinta observes a head on the first two tosses, find the probability that she observes a total of either four or five heads.

b. Albin suspects that the coin Jacinta tossed is not actually a fair coin and he tosses it 18 times. Albin observes a total of 12 heads from the 18 tosses.
Based on this sample, find the approximate 90% confidence interval for the probability of observing a head when this coin is tossed. Use the z value $\frac{33}{20}$.

[2 + 2 = 4 marks]
[VCAA 2019 NH MM]

Question 276

Let f be the probability density function

$$f:\left[0, \frac{2}{3}\right] \to R,\ f(x) = kx(2x+1)(3x-2)(3x+2).$$

The value of k is

A. $\frac{308}{405}$ **B.** $-\frac{308}{405}$ **C.** $-\frac{405}{308}$ **D.** $\frac{405}{308}$ **E.** $\frac{960}{133}$

[VCAA 2019 NH MM]

Question 277

A random sample of computer users was surveyed about whether the users had played a particular computer game. An approximate 95% confidence interval for the proportion of computer users who had played this game was calculated from this random sample to be (0.6668, 0.8147). The number of computer users in the sample is closest to

A. 5 **B.** 33 **C.** 135 **D.** 150 **E.** 180

[VCAA 2019 NH MM]

Question 278

Fred owns a company that produces thousands of pegs each day. He randomly selects 41 pegs that are produced on one day and finds eight faulty pegs.

a. What is the proportion of faulty pegs in this sample?

b. Pegs are packed each day in boxes. Each box holds 12 pegs. Let $\hat{P}$ be the random variable that represents the proportion of faulty pegs in a box. The actual proportion of faulty pegs produced by the company each day is $\frac{1}{6}$. Find $\Pr\left(\hat{P} < \frac{1}{6}\right)$.

Express your answer in the form $a(b)^n$, where a and b are positive rational numbers and n is a positive integer.

[1 + 2 = 3 marks (1.0, 0.6)]
[VCAA 2019 MM]

Question 279

The weights of packets of lollies are normally distributed with a mean of 200 g.
If 97% of these packets of lollies have a weight of more than 190 g, then the standard deviation of the distribution, correct to one decimal place, is

A. 3.3 g **B.** 5.3 g **C.** 6.1 g **D.** 9.4 g **E.** 12.1 g

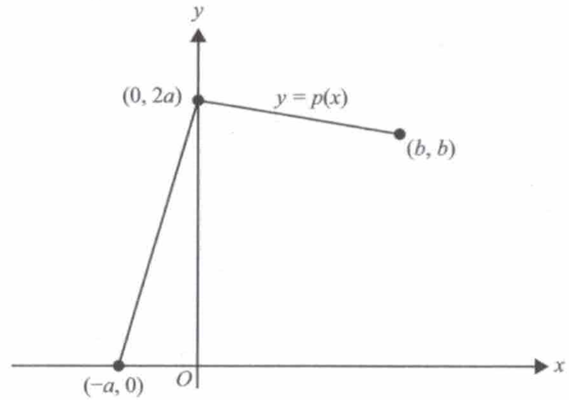
[VCAA 2019 MM (67%)]

Question 280

The distribution of a continuous random variable, X, is defined by the probability density function f, where

$$f(x) = \begin{cases} p(x) & -a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and $a, b \in R^+$.
The graph of the function p is shown below.



It is known that the average value of p over the interval $[-a, b]$ is $\frac{3}{4}$. $\Pr(X > 0)$ is

A. $\frac{2}{3}$ **B.** $\frac{3}{4}$ **C.** $\frac{4}{7}$ **D.** $\frac{7}{9}$ **E.** $\frac{5}{6}$

[VCAA 2019 MM (27%)]

Question 281

The random variable X is normally distributed. The mean of X is twice the standard deviation of X. If $\Pr(X > 5.2) = 0.9$, then the standard deviation of X is closest to

A. 7.238 **B.** 14.476 **C.** 3.327 **D.** 1.585 **E.** 3.169

[VCAA 2020 MM (44%)]

Question 282

The lengths of plastic pipes that are cut by a particular machine are a normally distributed random variable, X, with a mean of 250 mm.
Z is the standard normal random variable.

If $\Pr(X < 259) = 1 - \Pr(Z > 1.5)$, then the standard deviation of the lengths of plastic pipes, in millimetres, is

A. 1.5 **B.** 3 **C.** 6 **D.** 9 **E.** 12

[VCAA 2020 MM (60%)]

Question 283

An online shopping site sells boxes of doughnuts. A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are:

- glazed, with custard
- glazed, with no custard
- not glazed, with custard
- not glazed, with no custard.

It is known that, in the box:

- $\frac{1}{2}$ of the doughnuts are with custard
- $\frac{7}{10}$ of the doughnuts are not glazed
- $\frac{1}{10}$ of the doughnuts are glazed, with custard.

a. A doughnut is chosen at random from the box. Find the probability that it is not glazed, with custard.

b. The 20 doughnuts in the box are randomly allocated to two new boxes, Box A and Box B. Each new box contains 10 doughnuts. One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random.

Let g be the number of glazed doughnuts in Box A.

Find the probability, in terms of g, that the doughnut comes from Box B given that it is glazed.

c. The online shopping site has over one million visitors per day. It is known that half of these visitors are less than 25 years old.

Let $\hat{P}$ be the random variable representing the proportion of visitors who are less than 25 years old in a random sample of five visitors.

Find $\Pr(\hat{P} \geq 0.8)$. Do not use a normal approximation.

[1 + 2 + 3 = 6 marks (0.7, 0.4, 1.2)]
[VCAA 2021 MM]

A5. Continuous probability and Statistics

Question 284

A random variable X has the probability density function f given by

$$f(x) = \begin{cases} \dfrac{k}{x^2} & 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a positive real number.

a. Show that $k = 2$.

b. Find $E(X)$.

[1 + 2 = 3 marks (0.5, 0.9)]
[VCAA 2021 MM]

Question 285

A box contains many coloured glass beads.
A random sample of 48 beads is selected and it is found that the proportion of blue-coloured beads in this sample is 0.125.

Based on this sample, a 95% confidence interval for the proportion of blue-coloured glass beads is

A. (0.0314, 0.2186) **B.** (0.0465, 0.2035) **C.** (0.0018, 0.2482)
D. (0.0896, 0.1604) **E.** (0.0264, 0.2136)

[VCAA 2021 MM (72%)]

Question 286

For a certain species of bird, the proportion of birds with a crest is known to be $\frac{3}{5}$.

Let $\hat{P}$ be the random variable representing the proportion of birds with a crest in samples of size n for this specific bird.

The smallest sample size for which the standard deviation of $\hat{P}$ is less than 0.08 is

A. 7 **B.** 27 **C.** 37 **D.** 38 **E.** 43

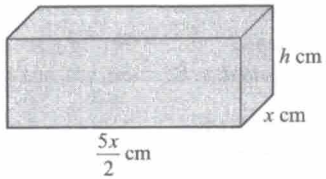
[VCAA 2021 MM (54%)]

Extended-response tasks

B. Extended-response questions

Question 287

A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 sq cm.

a. Show that if the height of the block is h cm, $h = \dfrac{6480 - 5x^2}{7x}$.

[2 marks (1.3)]

b. The volume, V cm^3, of the block is given by $V(x) = \dfrac{5x(6480 - 5x^2)}{14}$.

Given that $V(x) > 0$ and $x > 0$, find the possible values of x.

[2 marks (1.0)]

c. Find $\dfrac{dV}{dx}$, expressing your answer in the form $\dfrac{dV}{dx} = ax^2 + b$, where a and b are real numbers.

[3 marks (2.2)]

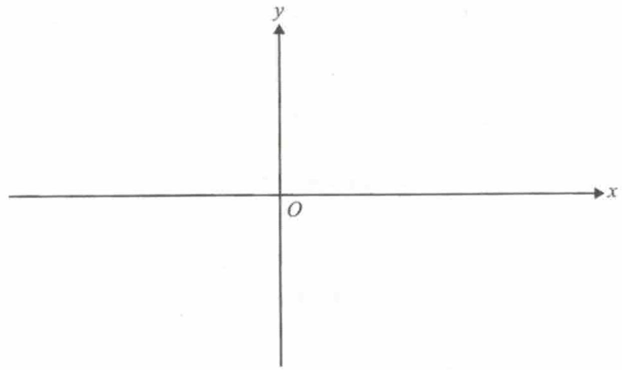
d. Find the exact values of x and h if the block is to have maximum volume.

[2 marks (1.1)]
Total 9 marks
[VCAA 2012 MM (CAS)]

Question 288

Let $f: R \setminus \{2\} \to R, f(x) = \dfrac{1}{2x-4} + 3$.

a. Sketch the graph of $y = f(x)$ on the set of axes below. Label the axes intercepts with their coordinates and label each of the asymptotes with its equation.



[3 marks (2.2)]

b. **i.** Find $f'(x)$.

ii. State the range of f'.

iii. Using the result of **part ii.**, explain why f has no stationary points.

[1 + 1 + 1 = 3 marks (0.9, 0.6, 0.5)]

c. If (p, q) is any point on the graph of $y = f(x)$, show that the equation of the tangent to $y = f(x)$ at this point can be written as $(2p-4)^2(y-3) = -2x + 4p - 4$.

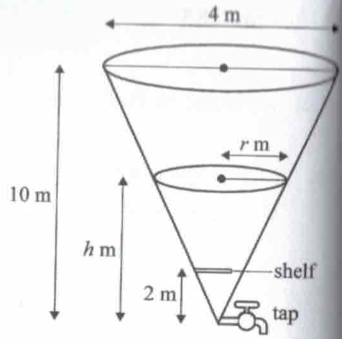
[2 marks (0.6)]

d. Find the coordinates of the points on the graph of $y = f(x)$ such that the tangents to the graph at these points intersect at $\left(-1, \dfrac{7}{2}\right)$.

[4 marks (0.8)]
Total 12 marks
[adapted from VCAA 2012 MM (CAS)]

Question 289

Tasmania Jones is in the jungle, searching for the Quetzalotl tribe's valuable emerald that has been stolen and hidden by a neighbouring tribe. Tasmania has heard that the emerald has been hidden in a tank shaped like an inverted cone, with a height of 10 metres and a diameter of 4 metres (as shown).
The emerald is on a shelf. The tank has a poisonous liquid in it.



a. If the depth of the liquid in the tank is h metres

i. find the radius, r metres, of the surface of the liquid in terms of h

ii. show that the volume of the liquid in the tank is $\frac{\pi h^3}{75}$ m^3.

[1 + 1 = 2 marks (0.5, 0.5)]

The tank has a tap at its base that allows the liquid to run out of it. The tank is initially full. When the tap is turned on, the liquid flows out of the tank at such a rate that the depth, h metres, of the liquid in the tank is given by

$$h = 10 + \frac{1}{1600}(t^3 - 1200t),$$

where t minutes is the length of time after the tap is turned on until the tank is empty.

b. Show that the tank is empty when $t = 20$.

[1 mark (0.8)]

c. When $t = 5$ minutes, find the depth of the liquid in the tank.

[1 mark (0.6)]

d. The shelf on which the emerald is placed is 2 metres above the vertex of the cone. From the moment the liquid starts to flow from the tank, find how long, in minutes, it takes until $h = 2$. (Give your answer correct to one decimal place.)

[2 marks (1.2)]

e. As soon as the tank is empty, the tap turns itself off and poisonous liquid starts to flow into the tank at a rate of 0.2 m^3/minute.
How long, in minutes, after the tank is first empty will the liquid once again reach a depth of 2 metres?

[2 marks (0.3)]

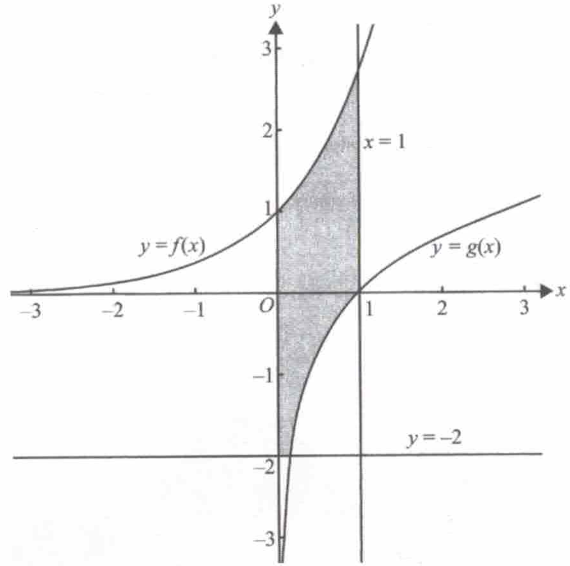
f. In order to obtain the emerald, Tasmania Jones enters the tank using a vine to climb down the wall of the tank as soon as the depth of the liquid is first 2 metres. He must leave the tank before the depth is again greater than 2 metres.
Find the length of time, in minutes, correct to one decimal place, that Tasmania Jones has from the time he enters the tank to the time he leaves the tank.

[1 mark (0.1)]
Total 9 marks
[adapted from VCAA 2012 MM (CAS)]

Question 290

The shaded region in the following diagram is the plan of a mine site for the Black Possum mining company. All distances are in kilometres.
Two of the boundaries of the mine site are in the shape of the graphs of the functions

$$f: R \to R, f(x) = e^x \text{ and } g: R^+ \to R, g(x) = \log_e(x).$$



a. **i.** Evaluate $\int_{-2}^{0} f(x)dx$.

ii. **Hence**, or otherwise, find the area of the region bounded by the graph of g, the x and y axes, and the line $y = -2$.

iii. Find the **total** area of the shaded region.

[1 + 1 + 1 = 3 marks (0.8, 0.4, 0.4)]

b. The mining engineer, Victoria, decides that a better site for the mine is the region bounded by the graph of g and that of a new function
$k: (-\infty, a) \to R, k(x) = -\log_e(a - x)$, where a is a positive real number.

i. Find, in terms of a, the x-coordinates of the points of intersection of the graphs of g and k.

ii. **Hence**, find the set of values of a, for which the graphs of g and k have two distinct points of intersection.

[2 + 1 = 3 marks (0.9, 0.1)]

c. For the new mine site, the graphs of g and k intersect at two distinct points, A and B. It is proposed to start mining operations along the line segment AB, which joins the two points of intersection. Victoria decides that the graph of k will be such that the x-coordinate of the midpoint of AB is $\sqrt{2}$.
Find the value of a in this case.

[2 marks (0.3)]
Total 8 marks
[VCAA 2012 MM (CAS)]

Question 291

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature (T °C) is given by

$T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right), 0 \le t \le 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a. State the maximum temperature in the greenhouse and the values of t when this occurs.

[2 marks (1.5)]

b. State the period of the function T.

[1 mark (0.9)]

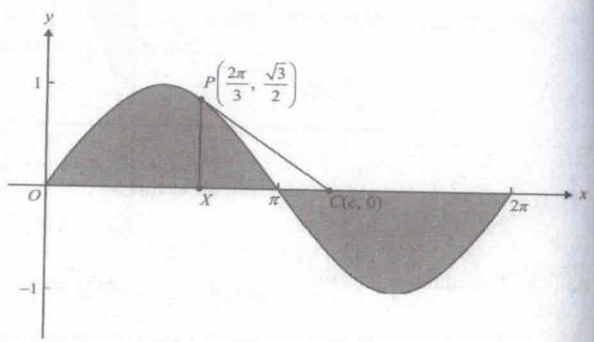
c. Find the smallest value of t for which $T = 26$.

[2 marks (1.5)]

d. For how many hours during the 24-hour time interval is $T \ge 26$?

[2 marks (1.1)]

Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \le x \le 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram here. He includes a garden path, which is shown as line segment PC.



The line through $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P.

e. **i.** Find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$.

ii. Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$.

[1 + 1 = 2 marks (0.8, 0.5)]

In further planning for the garden, Trigg uses a transformation of the plane defined as a dilation of factor k from the x-axis and a dilation of factor m from the y-axis, where k and m are positive real numbers.

f. Let X', P' and C' be the image, under this transformation, of the points X, P and C respectively.

i. Find the values of k and m if $X'P' = 10$ and $X'C' = 30$.

ii. Find the coordinates of the point P'.

[2 + 1 = 3 marks (0.3, 0.1)]
Total 12 marks
[VCAA 2013 MM (CAS)]

Question 292

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes **S** in less than three minutes, they are considered fit.

a. At FullyFit's Melbourne gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. In a particular week, 20 members of this gym attempt **S**.

i. Find the probability, correct to four decimal places, that at least 10 of these 20 members will complete **S** in less than three minutes.

ii. Given that at least 10 of these 20 members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that more than 15 of them complete **S** in less than three minutes?

[2 + 3 = 5 marks (1.6, 1.8)]

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

[2 marks (1.3)]

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete **S** is a continuous random variable X, with a probability density function g, as defined below.

$$g(x) = \begin{cases} \frac{(x-3)^3 + 64}{256} & 1 \le x \le 3 \\ \frac{x+29}{128} & 3 < x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

i Find E(X), correct to four decimal places.

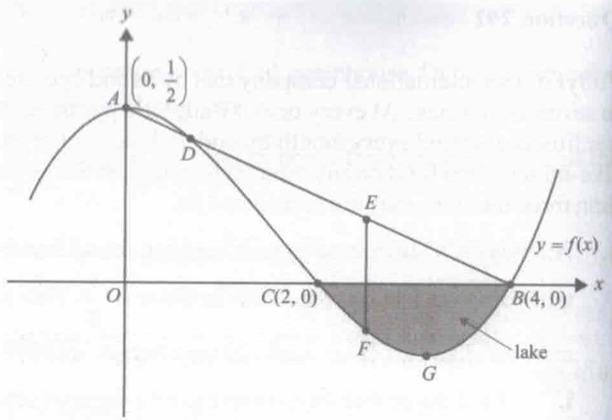
ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **S**? Give your answer to the nearest integer.

[2 + 2 = 4 marks (1.1, 1.0)]
Total 11 marks
[VCAA 2013 MM (CAS)]

Question 293

Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains.

He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram here.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}.$$

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2, 0)$ and $B(4, 0)$ are the points at the edge of the lake, situated in the valley.
All distances are measured in kilometres.

a. Find the coordinates of G, the deepest point in the lake.

[3 marks (2.4)]

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A, to the edge of the lake, B. The section of the railway line from A to D passes through a tunnel in the mountain.

b. Write down the equation of the line that passes through A and B.

[2 marks (1.7)]

c. **i.** Show that the x-coordinate of D, the end point of the tunnel, is $\frac{2}{3}$.

ii. Find the length of the tunnel AD.

[1 + 2 = 3 marks (0.6, 1.2)]

In order to ensure that the section of the railway line from D to B remains stable, Tasmania constructs vertical columns from the lake bed to the railway line. The column EF is the longest of all possible columns. (Refer to the diagram.)

d. **i.** Find the x-coordinate of E.

ii. Find the length of the column EF in metres, correct to the nearest metre.

[2 + 2 = 4 marks (0.6, 0.5)]

… continued

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V : [0, 4] \to R, V(x) = k\sqrt{x} - mx^2,$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line. The train then slows to a stop at $B(4, 0)$, that is $V(4) = 0$.

e. Find k in terms of m.

[1 mark (0.7)]

f. Find the value of x for which the speed, V, is a maximum.

[2 marks (1.1)]

Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m remains the same.

g. If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place.

[2 marks (0.9)]

h. If, on another day, the maximum value of V is 120, find the value of m.

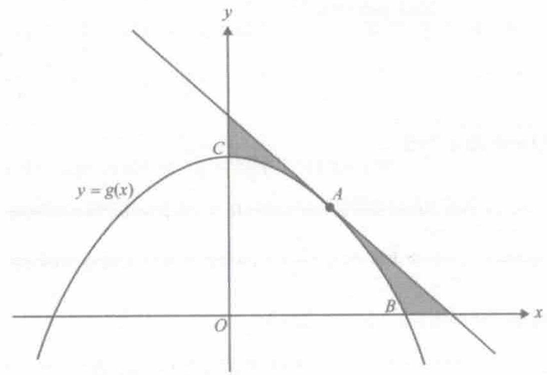
[2 marks (0.8)]

Total 19 marks

[VCAA 2013 MM (CAS)]

Question 294

Part of the graph of a function $g : R \to R$, $g(x) = \frac{16 - x^2}{4}$ is shown here.



a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram. The tangent to the graph of g at the point A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A.

ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the point A, and the x-axis and y-axis.

Evaluate the area of this shaded region.

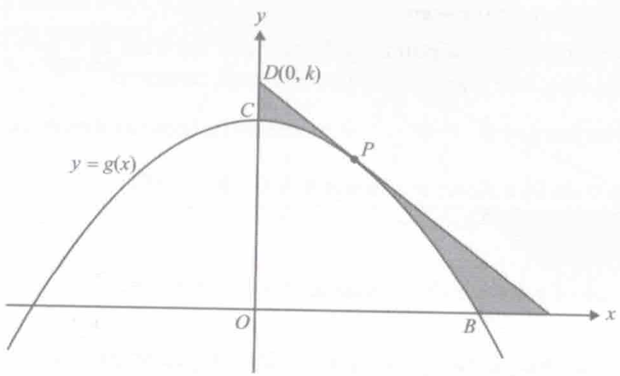
[2 + 3 = 5 marks (1.1, 1.1)]

b. Let Q be a point on the graph of $y = g(x)$.

Find the positive value of the x-coordinate of Q, for which the distance OQ is a minimum and find the minimum distance.

[3 marks (0.6)]

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y-axis at point $D(0, k)$, where $5 \le k \le 8$.



c. Find the gradient of the tangent in terms of k.

[2 marks (0.2)]

d. i. Find the rule $A(k)$ for the function of k that gives the area of the shaded region.

ii. Find the **maximum** area of the shaded region and the value of k for which this occurs.

iii. Find the **minimum** area of the shaded region and the value of k for which this occurs.

[2 + 2 + 2 = 6 marks (0.2, 0.1, 0.1)]
Total 16 marks
[VCAA 2013 MM (CAS)]

Question 295

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n.

[2 marks (1.8)]

b. Find the maximum and minimum populations of wombats in this location.

[2 marks (1.8)]

c. Find $n(10)$.

[1 mark (0.9)]

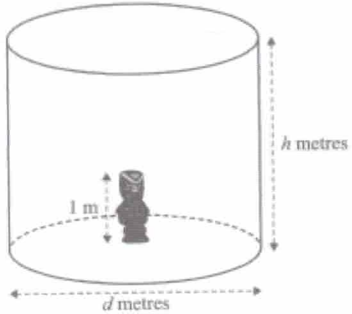
d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$.

[2 marks (1.0)]
Total 7 marks
[VCAA 2014 MM (CAS)]

Question 296

On 1 January 2010, Tasmania Jones was walking through an ice-covered region of Greenland when he found a large ice cylinder that was made a thousand years ago by the Vikings.

A statue was inside the ice cylinder. The statue was 1 m tall and its base was at the centre of the base of the cylinder.



The cylinder had a height of h metres and a diameter of d metres. Tasmania Jones found that the volume of the cylinder was 216 m^3. At that time, 1 January 2010, the cylinder had not changed in a thousand years. It was exactly as it was when the Vikings made it.

a. Write an expression for h in terms of d.

[2 marks (1.6)]

b. Show that the surface area of the cylinder excluding the base, S square metres, is given by the rule $S = \frac{\pi d^2}{4} + \frac{864}{d}$.

[1 mark (0.6)]

Tasmania found that the Vikings made the cylinder so that S is a minimum.

c. Find the value of d for which S is a minimum and find this minimum value of S.

[2 marks (1.3)]

d. Find the value of h when S is a minimum.

[1 mark (0.4)]

On 1 January 2010, Tasmania believed that due to recent temperature changes in Greenland, the ice of the cylinder had just started melting. Therefore, he decided to return on 1 January each year to measure the ice cylinder. He observes that the volume of the ice cylinder decreases by a constant rate of 10 m^3 per year. Assume that the cylindrical shape is retained and $d = 2h$ at the beginning and as the cylinder melts.

e. Write down an expression for V in terms of h.

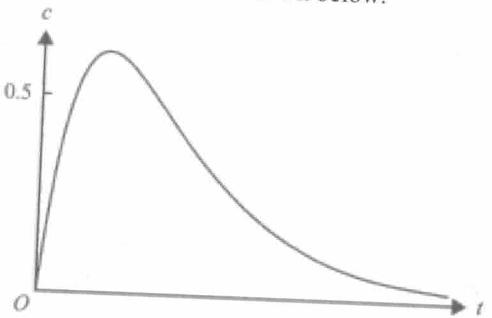
[1 mark (0.6)]

f. Find the year in which the top of the statue will just be exposed. (Assume that the melting started on 1 January 2010.)

[2 marks (0.3)]
Total 9 marks
[adapted from VCAA 2014 MM (CAS)]

Question 297

In a controlled experiment, Juan took some medicine at 8 pm. The concentration of medicine in his blood was then measured at regular intervals. The concentration of medicine in Juan's blood is modelled by the function $c(t)=\frac{5}{2}te^{-\frac{3t}{2}}$, $t \geq 0$, where c is the concentration of medicine in his blood, in milligrams per litre, t hours after 8 pm. Part of the graph of the function c is shown below.



a. What was the maximum value of the concentration of medicine in Juan's blood, in milligrams per litre, correct to two decimal places?

[1 mark (0.8)]

b. **i.** Find the value of t, in hours, correct to two decimal places, when the concentration of medicine in Juan's blood first reached 0.5 milligrams per litre.

ii. Find the length of time that the concentration of medicine in Juan's blood was above 0.5 milligrams per litre. Express the answer in hours, correct to two decimal places.

[1 + 2 = 3 marks (0.8, 1.6)]

c. **i.** What was the value of the average rate of change of the concentration of medicine in Juan's blood over the interval $\left[\frac{2}{3}, 3\right]$? Express the answer in milligrams per litre per hour, correct to two decimal places.

ii. At times t_1 and t_2, the instantaneous rate of change of the concentration of medicine in Juan's blood was equal to the average rate of change over the interval $\left[\frac{2}{3}, 3\right]$.

Find the values of t_1 and t_2, in hours, correct to two decimal places.

[2 + 2 = 4 marks (1.3, 0.8)]

Alicia took part in a similar controlled experiment. However, she used a different medicine. The concentration of this different medicine was modelled by the function $n(t)=Ate^{-kt}$, $t \geq 0$, where A and $k \in R^+$.

d. If the **maximum** concentration of medicine in Alicia's blood was 0.74 milligrams per litre at $t = 0.5$ hours, find the value of A, correct to the nearest integer.

[3 marks (1.4)]

Total 11 marks

[VCAA 2014 MM (CAS)]

Question 298

Patricia is a gardener and she owns a garden nursery. She grows and sells basil plants and coriander plants.

The heights, in centimetres, of the basil plants that Patricia is selling are distributed normally with a mean of 14 cm and a standard deviation of 4 cm. There are 2000 basil plants in the nursery.

a. Patricia classifies the tallest 10 per cent of her basil plants as **super**. What is the minimum height of a super basil plant, correct to the nearest millimetre?

[1 mark (0.5)]

Patricia decides that some of her basil plants are not growing quickly enough, so she plans to move them to a special greenhouse. She will move the basil plants that are less than 9 cm in height.

b. How many basil plants will Patricia move to the greenhouse, correct to the nearest whole number?

[2 marks (1.1)]

The heights of the coriander plants, x centimetres, follow the probability density function $h(x)$, where

$$h(x)=\begin{cases}\frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right) & 0 < x < 50\\ 0 & \text{otherwise}\end{cases}$$

c. State the mean height of the coriander plants.

[1 mark (0.8)]

Patricia thinks that the smallest 15 per cent of her coriander plants should be given a new type of plant food.

d. Find the maximum height, correct to the nearest millimetre, of a coriander plant if it is to be given the new type of plant food.

[2 marks (0.7)]

Patricia also grows and sells tomato plants that she classifies as either **tall** or **regular**. She finds that 20 per cent of her tomato plants are tall.
A customer, Jack, selects n tomato plants at random.

e. Let q be the probability that at least one of Jack's n tomato plants is tall. Find the minimum value of n so that q is greater than 0.95.

[2 marks (0.6)]

Total 8 marks

[adapted from VCAA 2014 MM (CAS)]

Question 299

Let $f : R \to R$, $f(x) = (x-3)(x-1)(x^2+3)$ and $g : R \to R$, $g(x) = x^4 - 8x$.

a. Express $x^4 - 8x$ in the form $x(x-a)\left((x+b)^2 + c\right)$.

[2 marks (1.2)]

b. Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

[1 mark (0.4)]

c. Find the values of d such that the graph of $y = f(x+d)$ has

i. one positive x-axis intercept

ii. two positive x-axis intercepts.

[1 + 1 = 2 marks (0.1, 0.1)]

d. Find the value of n for which the equation $g(x) = n$ has one solution.

[1 mark (0.2)]

e. At the point $(u, g(u))$, the gradient of $y = g(x)$ is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i. Find the value of $u^3 + v^3$.

ii. Find u and v if $u + v = 1$.

[2 + 1 = 3 marks (0.6, 0.1)]

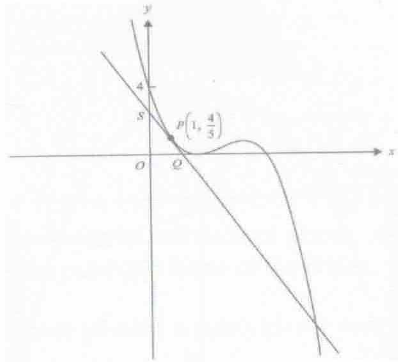
f. i. Find the equation of the tangent to the graph of $y = g(x)$ at the point $(p, g(p))$.

ii. Find the equations of the tangents to the graph of $y = g(x)$ that pass through the point with coordinates $\left(\frac{3}{2}, -12\right)$.

[1 + 3 = 4 marks (0.2, 0.5)]
Total 13 marks
[VCAA 2014 MM (CAS)]

Question 300

Let $f: R \to R, f(x) = \frac{1}{5}(x-2)^2(5-x)$. The point $P\left(1, \frac{4}{5}\right)$ is on the graph of f, as shown below. The tangent at P cuts the y-axis at S and the x-axis at Q.



a. Write down the derivative $f'(x)$ of $f(x)$.

[1 mark (1.0)]

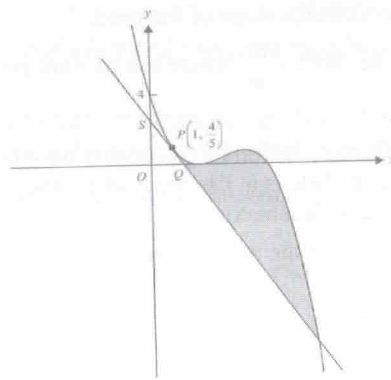
b. i. Find the equation of the tangent to the graph of f at the point $P\left(1, \frac{4}{5}\right)$.

ii. Find the coordinates of points Q and S.

[1 + 2 = 3 marks (0.8, 1.5)]

c. Find the distance PS and express it in the form $\frac{\sqrt{b}}{c}$, where b and c are positive integers.

[2 marks (1.2)]

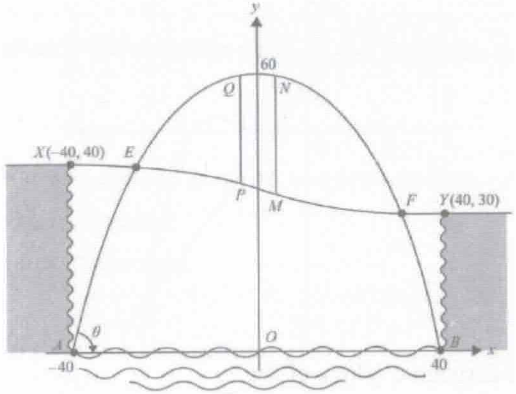


d. Find the area of the shaded region in the graph above.

[3 marks (1.9)]
Total 9 marks
[VCAA 2015 MM (CAS)]

Question 301

A city is located on a river that runs through a gorge.
The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.
A bridge is to be built that crosses the river and the gorge.
A diagram for the design of the bridge is shown below.



The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by $y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at B (40, 0) and $A(-40, 0)$. The road across the gorge is modelled by a cubic polynomial function.

a. Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree.

[2 marks (0.8)]

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$.

b. Find the maximum downwards slope of the road.

Give your answer in the form $-\frac{m}{n}$ where m and n are positive integers.

[2 marks (1.1)]

Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame. The supporting column, MN, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c. Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places.

[3 marks (1.2)]

The second supporting column, PQ, has its lowest point at $P\ (-u, w)$.

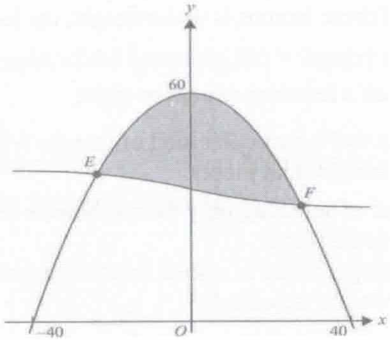
d. Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ.

[3 marks (0.9)]

… continued

B. Extended-response questions

For the opening of the bridge, a banner is erected on the bridge, as shown by the shaded region in the diagram below.



e. Find the x-coordinates, correct to two decimal places, of E and F, the points at which the road meets the parabolic frame of the bridge.

[3 marks (1.9)]

f. Find the area of the banner (shaded region), giving your answer to the nearest square metre.

[1 mark (0.6)]

Total 14 marks

[VCAA 2015 MM (CAS)]

Question 302

Mani is a fruit grower. After his oranges have been picked, they are sorted by a machine, according to size. Oranges classified as **medium** are sold to fruit shops and the remainder are made into orange juice. The distribution of the diameter, in centimetres, of medium oranges is modelled by a continuous random variable, X, with probability density function

$$f(x) = \begin{cases} \frac{3}{4}(x-6)^2(8-x) & 6 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

a. i. Find the probability that a randomly selected medium orange has a diameter greater than 7 cm.

ii. Mani randomly selects three medium oranges. Find the probability that exactly one of the oranges has a diameter greater than 7 cm. Express the answer in the form $\frac{a}{b}$, where a and b are positive integers.

[2 + 2 = 4 marks (1.7, 1.1)]

b. Find the mean diameter of medium oranges, in centimetres.

[1 mark (0.8)]

For oranges classified as **large**, the quantity of juice obtained from each orange is a normally distributed random variable with a mean of 74 mL and a standard deviation of 9 mL.

c. What is the probability, correct to three decimal places, that a randomly selected large orange produces less than 85 mL of juice, given that it produces more than 74 mL of juice?

[2 marks (1.1)]

Mani also grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.

It is known that 3% of Mani's lemons are underweight.

d. **i.** Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places.

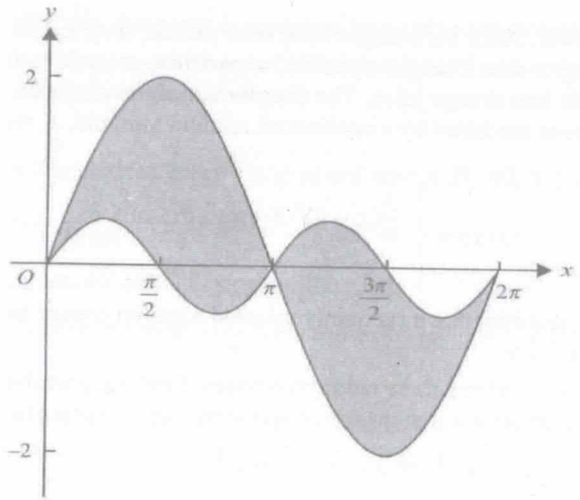
ii. Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load.
Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5

[2 + 2 = 4 marks (1.1, 0.9)]
Total 11 marks
[VCAA 2015 MM (CAS)]

Question 303

An electronics company is designing a new logo, based initially on the graphs of the functions $f(x) = 2\sin(x)$ and $g(x) = \frac{1}{2}\sin(2x)$, for $0 \le x \le 2\pi$.

These graphs are shown in the diagram below, in which the measurements in the x and y directions are in metres.



The logo is to be painted onto a large sign, with the area enclosed by the graphs of the two functions (shaded in the diagram) to be painted red.

a. The total area of the shaded regions, in square metres, can be calculated as $a\int_0^{\pi} \sin(x)\,dx$. What is the value of a?

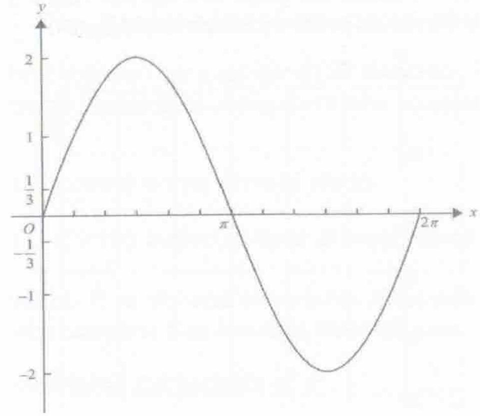
[1 mark (0.4)]
… continued

B. Extended-response questions

The electronics company considers changing the circular functions used in the design of the logo.

Its next attempt uses the graphs of the functions $f(x) = 2\sin(x)$ and $h(x) = \frac{1}{3}\sin(3x)$, for $0 \le x \le 2\pi$.

b. On the axes below, the graph of $y = f(x)$ has been drawn.
On the same axes, draw the graph of $y = h(x)$.



[2 marks (1.5)]

c. State a sequence of two transformations that maps the graph of $y = f(x)$ to the graph of $y = h(x)$.

[2 marks (1.1)]

The electronics company now considers using the graphs of the functions $k(x) = m\sin(x)$ and $q(x) = \frac{1}{n}\sin(nx)$, where m and n are positive integers with $m \ge 2$ and $0 \le x \le 2\pi$.

d. **i.** Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if n is even.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers.

ii. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if n is odd.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers.

[2 + 2 = 4 marks (0.4, 0.3)]
Total 9 marks
[VCAA 2015 MM (CAS)]

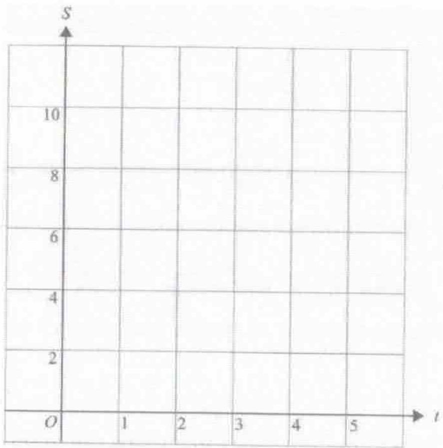
Question 304

a. Let $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, where $0 \le t \le 5$.

i. Find $S(0)$ and $S(5)$.

ii. The minimum value of S occurs when $t = \log_e(c)$. State the value of c and the minimum value of S.

iii. On the axes below, sketch the graph of S against t for $0 \le t \le 5$. Label the end points and the minimum point with their coordinates.



iv. Find the value of the average rate of change of the function S over the interval $[0, \log_e(c)]$.

[1 + 2 + 2 + 2 = 7 marks (0.7, 1.3, 1.5, 1.0)]

Let $V: [0, 5] \to R, V(t) = de^{\frac{t}{3}} + (10 - d)e^{-\frac{2t}{3}}$, where d is a real number and $d \in (0, 10)$.

b. If the minimum value of the function occurs when $t = \log_e(9)$, find the value of d.

[2 marks (1.3)]

c. i. Find the set of possible values of d such that the minimum value of the function occurs when $t = 0$.

ii. Find the set of possible values of d such that the minimum value of the function occurs when $t = 5$.

[2 + 2 = 4 marks (0.6, 0.6)]

d. If the function V has a local minimum (a, m), where $0 \le a \le 5$, it can be shown that $m = \frac{k}{2}d^{\frac{2}{3}}(10 - d)^{\frac{1}{3}}$.

Find the value of k.

[2 marks (0.3)]
Total 15 marks
[VCAA 2015 MM (CAS)]

Question 305

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. It has more than 100 000 members worldwide. At every one of FullyFit's gyms, each member agrees to have their fitness assessed every month by undertaking a set of exercises called **S**. If someone completes **S** in less than three minutes, they are considered fit.

a. It has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. A random sample of 20 FullyFit members is taken. For a sample of 20 members, let X be the random variable that represents the number of members who complete **S** in less than three minutes.

i. Find $\Pr(X \ge 10)$ correct to four decimal places.

ii. Find $\Pr(X \ge 15 \mid X \ge 10)$ correct to three decimal places.

For samples of 20 members, $\hat{P}$ is the random variable of the distribution of sample proportions of people who complete **S** in less than three minutes.

iii. Find the expected value and variance of $\hat{P}$.

iv. Find the probability that a sample proportion lies within two standard deviations of $\frac{5}{8}$. Give your answer correct to three decimal places. Do not use a normal approximation.

v. Find $\Pr\left(\hat{P} \ge \frac{3}{4} \mid \hat{P} \ge \frac{5}{8}\right)$. Give your answer correct to three decimal places. Do not use a normal approximation.

[2 + 3 + 3 + 3 + 2 = 13 marks]

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

[2 marks]

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete another exercise routine, **T**, is a continuous random variable W with a probability density function g, as defined below.

$$g(w) = \begin{cases} \dfrac{(w-3)^3+64}{256} & 1 \le w \le 3 \\ \dfrac{w+29}{128} & 3 < w \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

i. Find $\text{E}(W)$ correct to four decimal places.

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **T**? Give your answer to the nearest integer.

[2 + 2 = 4 marks]

d. From a random sample of 100 members, it was found that the sample proportion of people who spent more than two hours per week in the gym was 0.6. Find an approximate 95% confidence interval for the population proportion corresponding to this sample proportion. Give values correct to three decimal places.

[1 mark]

Total 20 marks

[VCAA Sample examination 2016 MM]

Question 306

Let $f: [0, 8\pi] \to R,\ f(x) = 2\cos\left(\dfrac{x}{2}\right) + \pi$.

a. Find the period and range of f.

[2 marks (1.5)]

b. State the rule for the derivative function f'.

[1 mark (0.9)]

c. Find the equation of the tangent to the graph of f at $x = \pi$.

[1 mark (0.7)]

d. Find the equations of the tangents to the graph of

$f: [0, 8\pi] \to R,\ f(x) = 2\cos\left(\dfrac{x}{2}\right) + \pi$ that have a gradient of 1.

[2 marks (1.0)]

e. Find the values of x, $0 \le x \le 8\pi$, such that $f(x) = 2f'(x) + \pi$.

[2 marks (1.0)]

Total 8 marks

[adapted from VCAA 2016 MM]

Question 307

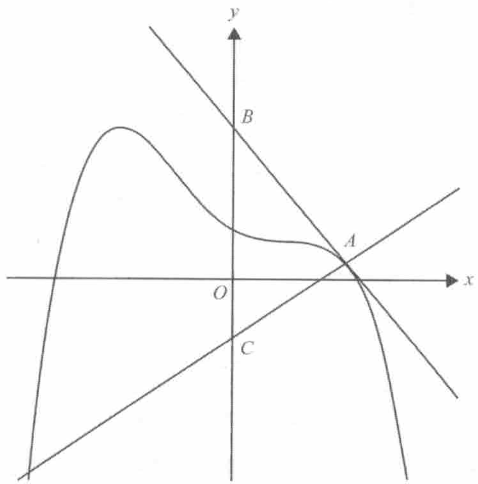
Consider the function $f(x) = -\frac{1}{3}(x+2)(x-1)^2$.

a. **i.** Given that $g'(x) = f(x)$ and $g(0) = 1$, show that $g(x) = -\dfrac{x^4}{12} + \dfrac{x^2}{2} - \dfrac{2x}{3} + 1$.

ii. Find the values of x for which the graph of $y = g(x)$ has a stationary point.

[1 + 1 = 2 marks (0.7, 0.8)]

The diagram here shows part of the graph of $y = g(x)$, the tangent to the graph at $x = 2$ and a straight line drawn perpendicular to the tangent to the graph at $x = 2$.



The equation of the tangent at the point A with coordinates $(2, g(2))$ is

$y = 3 - \dfrac{4x}{3}$.

The tangent cuts the y-axis at B.
The line perpendicular to the tangent cuts the y-axis at C.

b. **i.** Find the coordinates of B.

ii. Find the equation of the line that passes through A and C and, hence, find the coordinates of C.

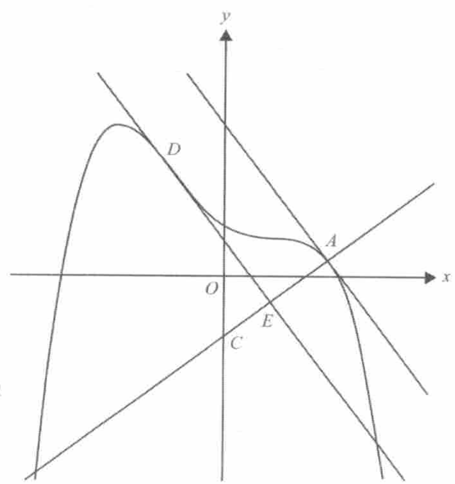
iii. Find the area of triangle ABC.

[1 + 2 + 2 = 5 marks (0.9, 1.3, 1.2)]

c. The tangent at D is parallel to the tangent at A. It intersects the line passing through A and C at E.

i. Find the coordinates of D.

ii. Find the length of AE.



[2 + 3 = 5 marks (0.9, 1.0)]

Total 12 marks

[VCAA 2016 MM]

Question 308

A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a. Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places.

[2 marks (1.6)]

b. A teacher observes that at least one of the retuned laptops is not correctly plugged into the trolley.
Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places.

[2 marks (1.0)]

The time for which a laptop will work without recharging (the battery life) is normally distributed, with a mean of three hours and 10 minutes and standard deviation of six minutes. Suppose that the laptops remain out of the recharging trolley for three hours.

c. For any one laptop, find the probability that it will stop working by the end of these three hours. Give your answer correct to four decimal places.

[2 marks (1.2)]

A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population of laptops with a mean battery life of three hours and 10 minutes and standard deviation of six minutes, $\hat{P}$ is the random variable of the distribution of sample proportions of laptops with a battery life of less than three hours.

d. Find the probability that $\Pr\left(\hat{P} \geq 0.06 \mid \hat{P} \geq 0.05\right)$. Give your answer correct to three decimal places. Do not use a normal approximation.

[3 marks (1.0)]

It is known that when laptops have been used regularly in a school for six months, their battery life is still normally distributed but the mean battery life drops to three hours. It is also known that only 12% of such laptops work for more than three hours and 10 minutes.

e. Find the standard deviation for the normal distribution that applies to the battery life of laptops that have been used regularly in a school for six months, correct to four decimal places.

[2 marks (0.8)]

… continued

The laptop supplier collects a sample of 100 laptops that have been used for six months from a number of different schools and tests their battery life. The laptop supplier wishes to estimate the proportion of such laptops with a battery life of less than three hours.

f. Suppose the supplier tests the battery life of the laptops one at a time.
Find the probability that the first laptop found to have a battery life of less than three hours is the third one.

[1 mark (0.2)]

The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.

g. Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give values correct to two decimal places.

[1 mark (0.4)]
Total 13 marks
[adapted from VCAA 2016 MM]

Question 309

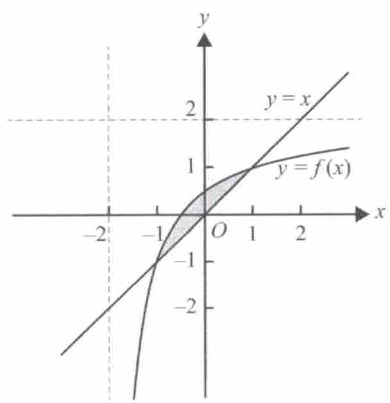
a. Express $\dfrac{2x+1}{x+2}$ in the form $a + \dfrac{b}{x+2}$, where a and b are non-zero integers.

[2 marks (1.1)]

b. Let $f\colon R \setminus \{-2\} \to R,\ f(x) = \dfrac{2x+1}{x+2}$.

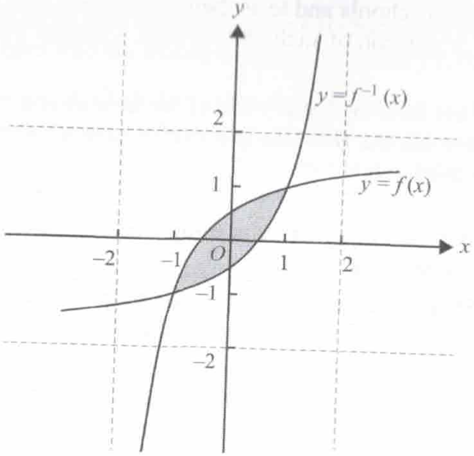
i. Find the rule and domain of f^{-1}, the inverse function of f.

ii. Part of the graphs of f and $y = x$ are shown in the diagram below.



Find the area of the shaded region.

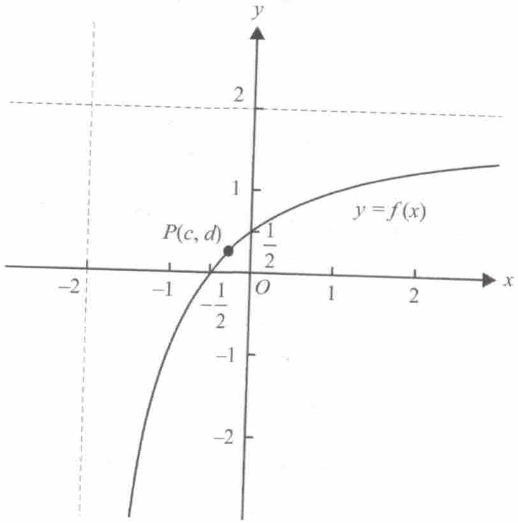
iii. Part of the graphs of f and f^{-1} are shown in the diagram below.



Find the area of the shaded region.

[2 + 1 + 1 = 4 marks (1.4, 0.6, 0.6)]

c. Part of the graph of f is shown in the diagram below.



The point $P(c, d)$ is on the graph of f.

Find the exact values of c and d such that the distance of this point to the origin is a minimum, and find this minimum distance.

[3 marks (0.7)]

Let $g: (-k, \infty) \to R,\ g(x) = \dfrac{kx + 1}{x + k}$, where $k > 1$.

d. Show that $x_1 < x_2$ implies that $g(x_1) < g(x_2)$, where $x_1 \in (-k, \infty)$ and $x_2 \in (-k, \infty)$.

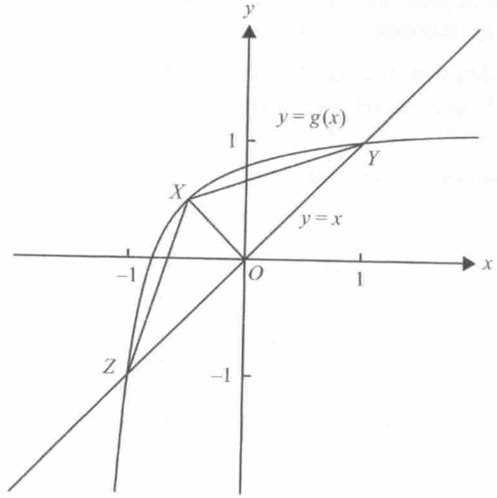
[2 marks (0.2)]

… continued

e. **i.** Let X be the point of intersection of the graphs of $y = g(x)$ and $y = -x$. Find the coordinates of X in terms of k.

ii. Find the value of k for which the coordinates of X are $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

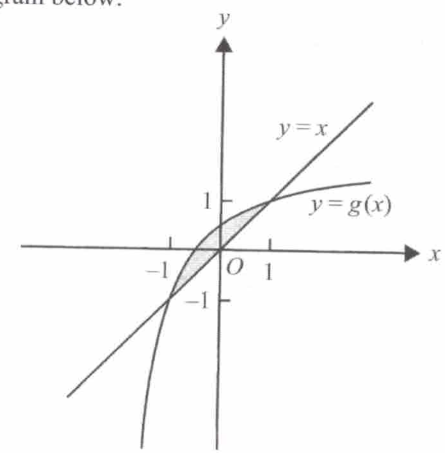
iii. Let $Z(-1, -1)$, $Y(1, 1)$ and X be the vertices of the triangle XYZ. Let $s(k)$ be the square of the area of triangle XYZ.



Find the values of k such that $s(k) \geq 1$.

[2 + 2 + 2 = 6 marks (0.6, 0.9, 0.1)]

f. The graph of g and the line $y = x$ enclose a region of the plane. The region is shown shaded in the diagram below.



Let $A(k)$ be the rule of the function A that gives the area of this enclosed region. The domain of A is $(1, \infty)$.

i. Give the rule for $A(k)$.

ii. Show that $0 < A(k) < 2$ for all $k > 1$.

[2 + 2 = 4 marks (0.6, 0.1)]

Total 21 marks

[VCAA 2016 MM]

Question 310

A company supplies schools with whiteboard pens.
The total length of time for which a whiteboard pen can be used for writing before it stops working is called its use-time.
There are two types of whiteboard pens: Grade A and Grade B.
The use-time of Grade A pens is normally distributed with a mean of 11 hours and a standard deviation of 15 minutes.

a. Find the probability that a Grade A whiteboard pen will have a use-time that is greater than 10.5 hours, correct to three decimal places.

[1 mark]

The use-time of Grade B whiteboard pens is described by the probability density function

$$f(x) = \begin{cases} \frac{x}{576}(12 - x)(e^{\frac{x}{6}} - 1) & 0 \le x \le 12 \\ 0 & \text{otherwise} \end{cases}$$

where x is the use-time in hours.

b. Determine the expected use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places.

[2 marks]

c. Determine the standard deviation of the use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places.

[2 marks]

d. Find the probability that a randomly chosen Grade B whiteboard pen will have a use-time that is greater than 10.5 hours, correct to four decimal places.

[2 marks]

A worker at the company finds two boxes of whiteboard pens that are not labelled, but knows that one box contains only Grade A whiteboard pens and the other box contains only Grade B whiteboard pens.
The worker decides to randomly select a whiteboard pen from one of the boxes. If the selected whiteboard pen has a use-time that is greater than 10.5 hours, then the box that it came from will be labelled Grade A and the other box will be labelled Grade B.
Otherwise, the box that it came from will be labelled Grade B and the other box will be labelled Grade A.

e. Find the probability, correct to three decimal places, that the worker labels the boxes incorrectly.

[2 marks]

f. Find the probability, correct to three decimal places, that the whiteboard pen selected was Grade B, given that the boxes had been labelled incorrectly.

[2 marks]

... continued

As a whiteboard pen ages, its tip may dry to the point that the whiteboard pen becomes defective (unusable). The company has stock that is two years old and, at that age, it is known that 5% of Grade A whiteboard pens will be defective.

g. A school purchases a box of Grade A whiteboard pens that is two years old and a class of 26 students is the first to use them.

If every student receives a whiteboard pen from this box, find the probability, correct to four decimal places, that at least one student will receive a defective whiteboard pen.

[2 marks]

h. Let $\hat{P}_A$ be the random variable of the distribution of sample proportions of defective Grade A whiteboard pens in boxes of 100. The boxes come from stock that is two years old.

Find $\Pr\left(\hat{P}_A > 0.04 \mid \hat{P}_A < 0.08\right)$. Give your answer correct to four decimal places.

Do not use a normal approximation.

[3 marks]

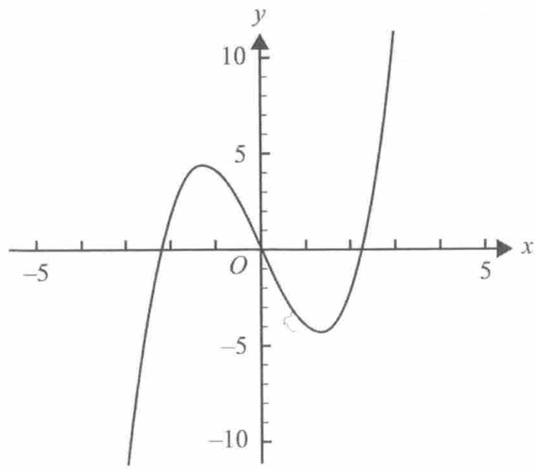
i. A box of 100 Grade A whiteboard pens that is two years old is selected and it is found that six of the whiteboard pens are defective.

Determine a 90% confidence interval for the population proportion from this sample, correct to two decimal places.

[2 marks]
Total 18 marks
[VCAA 2017 NH MM]

Question 311

Let $f: R \to R, f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a. Find the coordinates of the turning points.

[2 marks (1.7)]

b. $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

i. Find the equation of the straight line through A and B.

ii. Find the distance AB.

[2 + 1 = 3 marks (1.6, 0.8)]

Let $g: R \to R, g(x) = x^3 - kx, k \in R^+$.

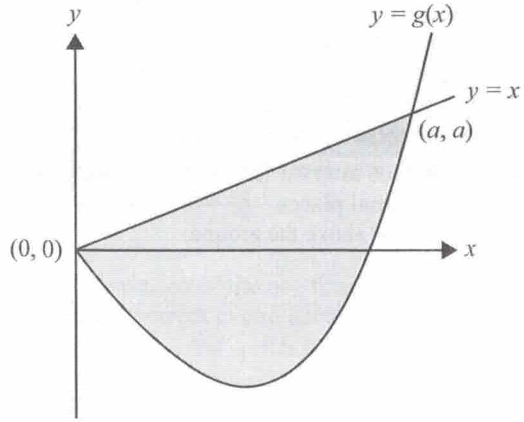
c. Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

i. Find the distance CD in terms of k.

ii. Find the values of k such that the distance CD is equal to $k + 1$.

[2 + 1 = 3 marks (1.5, 0.7)]

d. The diagram below shows part of the graphs of g and $y = x$. These graphs intersect at the points with the coordinates $(0, 0)$ and (a, a).



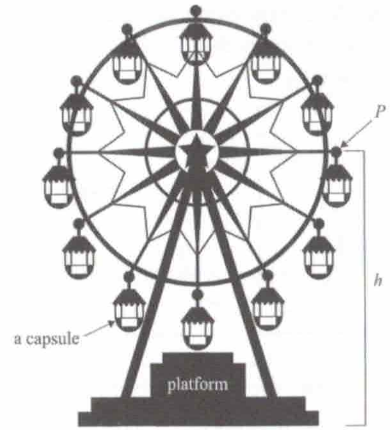
i. Find the value of a in terms of k.

ii. Find the area of the shaded region in terms of k.

[1 + 2 = 3 marks (0.6, 1.0)]
Total 11 marks
[VCAA 2017 MM]

Question 312

Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes after Sammy enters the capsule and h is measured in metres. Sammy exits the capsule after one complete rotation of the Ferris wheel.



a. State the minimum and maximum heights of P above the ground.

[1 mark (0.9)]

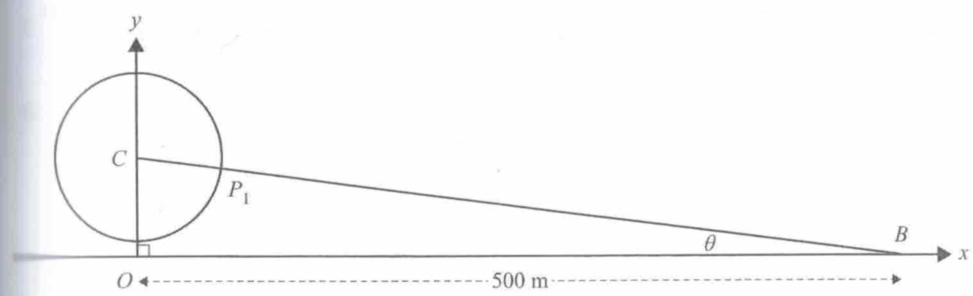
b. For how much time is Sammy in the capsule?

[1 mark (0.9)]

c. Find the rate of change of h with respect to t and, hence, state the value of t at which the rate of change of h is at its maximum.

[2 marks (1.0)]

As the Ferris wheel rotates, a stationary boat at B, on a nearby river, first becomes visible at point P_1. B is 500 m horizontally from the vertical axis through the centre C of the Ferris wheel and angle $CBO = \theta$, as shown below.



d. Find θ in degrees, correct to two decimal places.

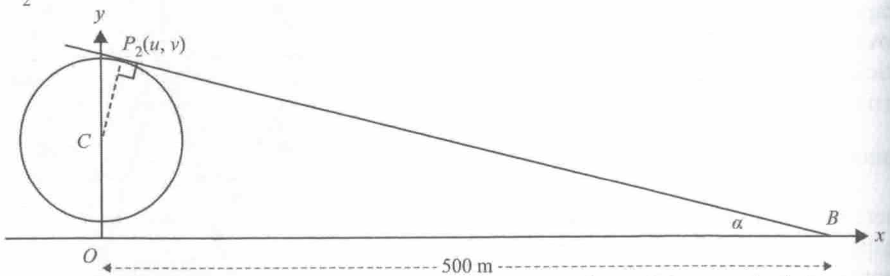
[1 mark (0.4)]

Part of the path of P is given by $y = \sqrt{3025 - x^2} + 65$, $x \in [-55, 55]$, where x and y are in metres.

e. Find $\frac{dy}{dx}$.

[1 mark (0.9)]

As the Ferris wheel continues to rotate, the boat at B is no longer visible from the point $P_2(u, v)$ onwards. The line through B and P_2 is tangent to the path of P, where angle $OBP_2 = \alpha$.



f. Find the gradient of the line segment P_2B in terms of u and, hence, find the coordinates of P_2, correct to two decimal places.

[3 marks (0.5)]

g. Find α in degrees, correct to two decimal places.

[1 mark (0.1)]

h. Hence or otherwise, find the length of time, to the nearest minute, during which the boat at B is visible.

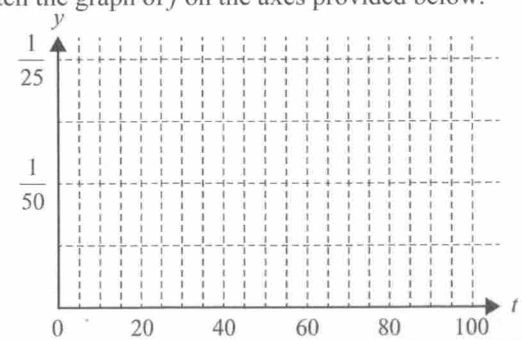
[2 marks (0.1)]
Total 12 marks
[VCAA 2017 MM]

Question 313

The time Jennifer spends on her homework each day varies, but she does some homework every day.
The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f, where

$$f(t) = \begin{cases} \dfrac{1}{625}(t-20) & 20 \le t < 45 \\ \dfrac{1}{625}(70-t) & 45 \le t \le 70 \\ 0 & \text{elsewhere} \end{cases}$$

a. Sketch the graph of f on the axes provided below.



[3 marks (1.9)]

… continued

b. Find $\Pr(25 \le T \le 55)$.

[2 marks (1.5)]

c. Find $\Pr(T \le 25 \mid T \le 55)$.

[2 marks (1.3)]

d. Find a such that $\Pr(T \ge a) = 0.7$, correct to four decimal places.

[2 marks (0.7)]

e. The probability that Jennifer spends more than 50 minutes on her homework on any given day is $\dfrac{8}{25}$.

Assume that the amount of time spent on her homework on any day is independent of the time spent on her homework on any other day.

i. Find the probability that Jennifer spends more than 50 minutes on her homework on more than three of seven randomly chosen days, correct to four decimal places.

ii. Find the probability that Jennifer spends more than 50 minutes on her homework on at least two of seven randomly chosen days, given that she spends more than 50 minutes on her homework on at least one of those days, correct to four decimal places.

[2 + 2 = 4 marks (1.3, 1.3)]

Let p be the probability that on any given day Jennifer spends more than d minutes on her homework.
Let q be the probability that on two or three days out of seven randomly chosen days she spends more than d minutes on her homework.

f. Express q as a polynomial in terms of p.

[2 marks (0.7)]

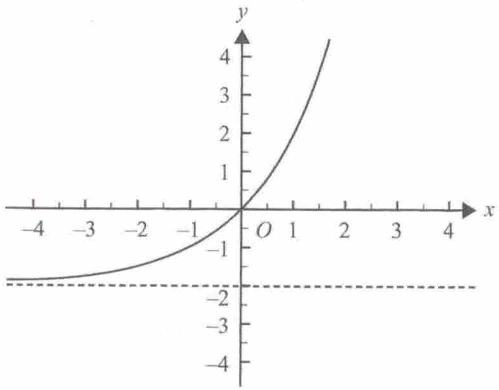
g. i. Find the maximum value of q, correct to four decimal places, and the value of p for which this maximum occurs, correct to four decimal places.

ii. Find the value of d for which the maximum found in **part g.i.** occurs, correct to the nearest minute.

[2 + 2 = 4 marks (0.6, 0.2)]
Total 19 marks
[VCAA 2017 MM]

Question 314

Let $f: R \to R, f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below.



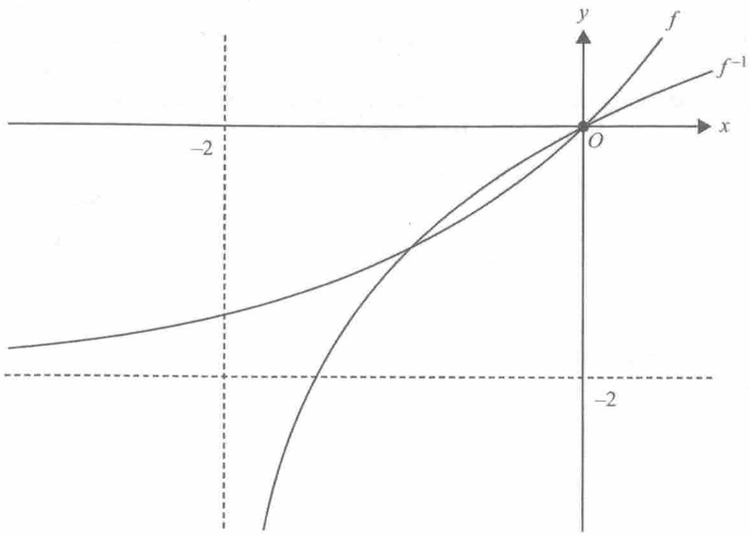
a. Find the rule and domain for f^{-1}, the inverse function of f.

[2 marks (1.5)]

b. Find the area bounded by the graphs of f and f^{-1}.

[3 marks (2.0)]

c. Part of the graphs of f and f^{-1} are shown below.



Find the gradient of f and the gradient of f^{-1} at $x = 0$.

[2 marks (1.4)]

The functions of g_k, where $k \in R^+$, are defined with domain R such that $g_k(x) = 2e^{kx} - 2$.

d. Find the value of k such that $g_k(x) = f(x)$.

[1 mark (0.7)]

e. Find the rule for the inverse functions g_k^{-1} of g_k, where $k \in R^+$.

[1 mark (0.6)]

B. Extended-response questions

f. **i.** Describe the transformation that maps the graph of g_1 onto the graph of g_k.

ii. Describe the transformation that maps the graph of g_1^{-1} onto the graph of g_k^{-1}.

[1 + 1 = 2 marks (0.3, 0.3)]

g. The lines L_1 and L_2 are the tangents at the origin to the graphs of g_k and g_k^{-1} respectively. Find the value(s) of k for which the angle between L_1 and L_2 is 30°.

[2 marks (0.3)]

h. Let p be the value of k for which $g_k(x) = g_k^{-1}(x)$ has only one solution.

i. Find p.

ii. Let $A(k)$ be the area bounded by the graphs of g_k and g_k^{-1} for all $k > p$.

State the smallest value of b such that $A(k) < b$.

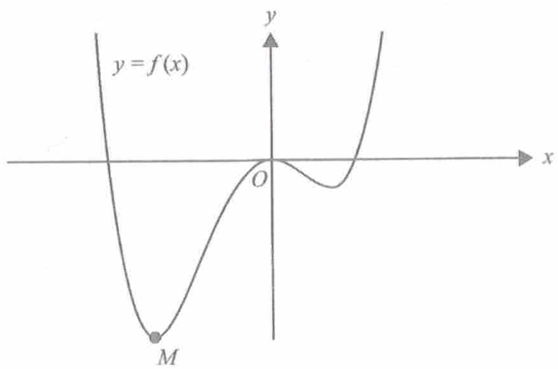
[2 + 1 = 3 marks (0.1, 0.0)]

Total 16 marks

[adapted from VCAA 2017 MM]

Question 315

Consider the quartic $f: R \to R, f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below.



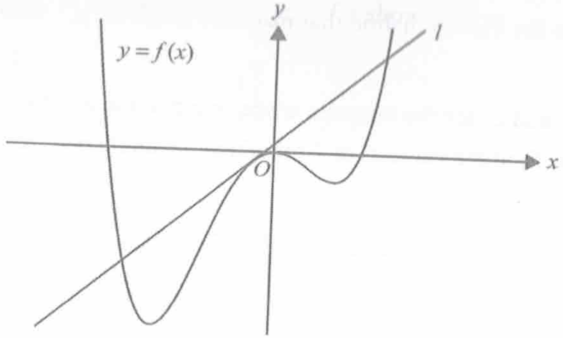
a. Find the coordinates of the point M, at which the minimum value of the function f occurs.

[1 mark (1.0)]

b. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts.

[1 mark (0.7)]

Part of the tangent, l, to $y=f(x)$ at $x=-\frac{1}{3}$ is shown below.



c. Find the equation of the tangent l.

[1 mark (0.8)]

d. The tangent l intersects $y=f(x)$ at $x=-\frac{1}{3}$ and at two other points.
State the x-values of the two other points of intersection. Express your answers in the form $\frac{a\pm\sqrt{b}}{c}$, where a, b and c are integers.

[2 marks (1.5)]

e. Find the total area of the regions bounded by the tangent l and $y=f(x)$. Express your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are integers.

[2 marks (1.1)]

Let $p: R\to R$, $p(x)=3x^4+4x^3+6(a-2)x^2-12ax+a^2$, $a\in R$.

f. State the value of a for which $f(x)=p(x)$ for all x.

[1 mark (0.5)]

g. Find all solutions to $p'(x)=0$, in terms of a where appropriate.

[1 mark (0.6)]

h. i. Find the values of a for which p has only one stationary point.

ii. Find the minimum value of p when $a=2$.

iii. If p has only one stationary point, find the values of a for which $p(x)=0$ has no solutions.

[1 + 1 + 2 = 4 marks (0.2, 0.6, 0.2)]
Total 13 marks
[VCAA 2018 MM]

Question 316

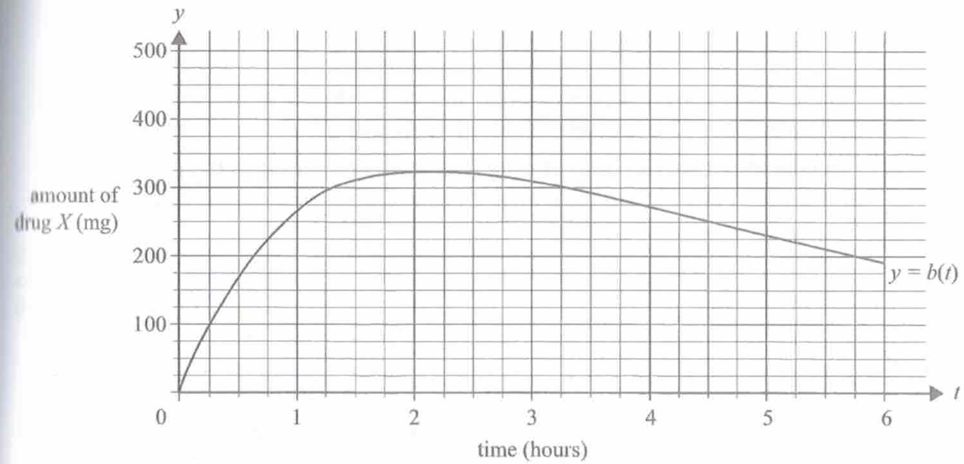
A drug, X, comes in 500 milligram (mg) tablets.
The amount, b, of drug X in the bloodstream, in milligrams, t hours after one tablet is consumed is given by the function

$$b(t)=\frac{4500}{7}\left(e^{\left(-\frac{t}{5}\right)}-e^{\left(-\frac{9t}{10}\right)}\right)$$

a. Find the time, in hours, it takes for drug X to reach a maximum amount in the bloodstream after one tablet is consumed. Express your answer in the form $a\log_e(c)$, where $a, c\in R$.

[2 marks (1.6)]

The graph of $y=b(t)$ is shown below for $0\le t\le 6$.



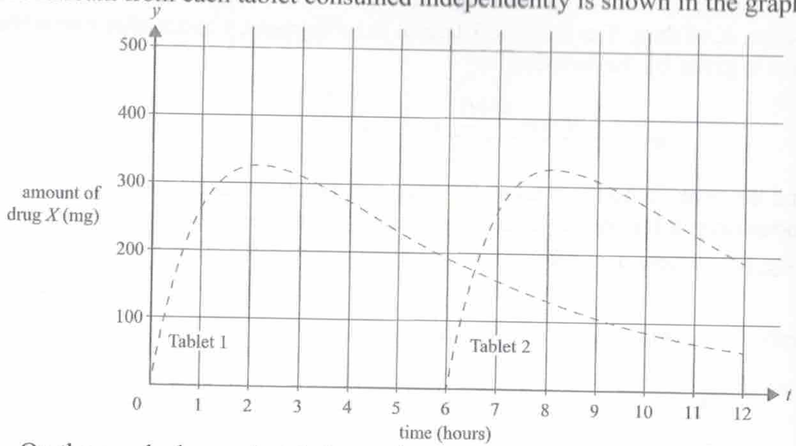
b. Find the average rate of change of the amount of drug X in the bloodstream, in milligrams per hour, over the interval [2, 6]. Give your answer correct to one decimal place.

[2 marks (1.6)]

c. Find the average amount of drug X in the bloodstream, in milligrams, during the first six hours after one tablet is consumed. Give your answer correct to the nearest milligram.

[2 marks (1.2)]

d. Six hours after one 500 milligram tablet of drug X is consumed (Tablet 1), a second identical tablet is consumed (Tablet 2). The amount of drug X in the bloodstream from each tablet consumed independently is shown in the graph.



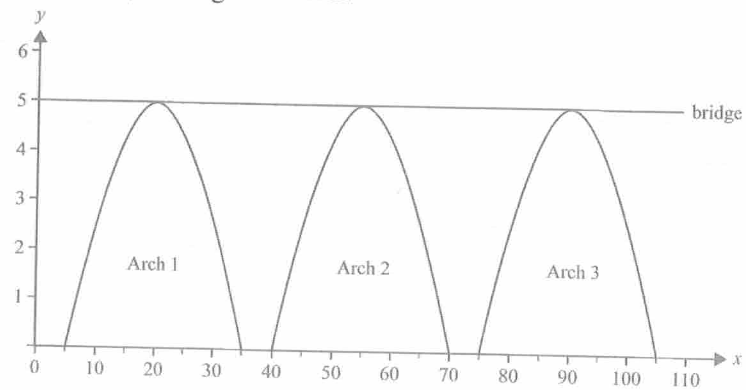
i. On the graph above, sketch the total amount of drug X in the bloodstream during the first 12 hours after Tablet 1 is consumed.

ii. Find the maximum amount of drug X in the bloodstream in the first 12 hours and the time at which this maximum occurs. Give your answers correct to two decimal places.

[2 + 2 = 4 marks (1.0, 0.5)]
Total 10 marks
[VCAA 2018 MM]

Question 317

A horizontal bridge positioned 5 m above level ground is 110 m in length. The bridge also touches the top of three arches. Each arch begins and ends at ground level. The arches are 5 m apart at the base, as shown in the diagram below.
Let x be the horizontal distance, in metres, from the left side of the bridge and let y be the height, in metres, above ground level.



Arch 1 can be modelled by the function $h_1:[5,35]\to R,\ h_1(x)=5\sin\left(\dfrac{(x-5)\pi}{30}\right)$.

... continued

Arch 2 can be modelled by the function $h_2:[40,70]\to R,\ h_2(x)=5\sin\left(\dfrac{(x-40)\pi}{30}\right)$.

Arch 3 can be modelled by the function $h_3:[a,105]\to R,\ h_3(x)=5\sin\left(\dfrac{(x-a)\pi}{30}\right)$.

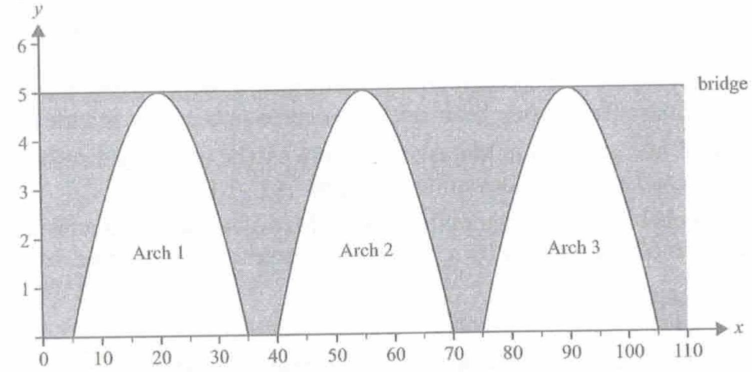
a. State the value of a, where $a \in R$.

[1 mark (1.0)]

b. Describe the transformation that maps the graph of $y=h_2(x)$ to $y=h_3(x)$.

[1 mark (0.8)]

The area above ground level between the arches and the bridge is filled with stone. The stone is represented by the shaded regions shown in the diagram below.



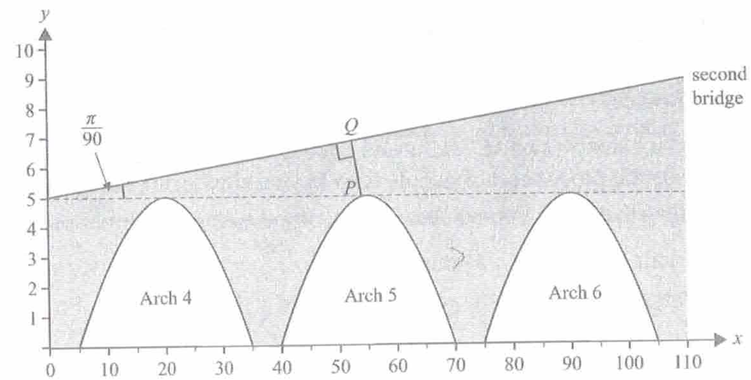
c. Find the total area of the shaded regions, correct to the nearest square metre.

[3 marks (2.4)]

A second bridge has a height of 5 m above the ground at its left-most point and is inclined at a constant angle of elevation of $\dfrac{\pi}{90}$ radians, as shown in the diagram below.

The second bridge also has three arches below it, which are identical to the arches below the first bridge, and spans a horizontal distance of 110 m.

Let x be the horizontal distance, in metres, from the left side of the second bridge and let y be the height, in metres, above ground level.



d. State the gradient of the second bridge, correct to three decimal places.

[1 mark (0.6)]

P is a point on Arch 5. The tangent to Arch 5 at point P has the same gradient as the second bridge.

e. Find the coordinates of P, correct to two decimal places.

[2 marks (1.0)]

f. A supporting rod connects a point Q on the second bridge to point P on Arch 5. The rod follows a straight line and runs perpendicular to the second bridge, as shown in the diagram on the previous page.
Find the distance PQ, in metres, correct to two decimal places.

[3 marks (0.9)]
Total 11 marks
[VCAA 2018 MM]

Question 318

Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm).

The resting heart rate of adults in Mathsland is known to be normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm.

a. Find the probability that a randomly selected Mathsland adult has a resting heart rate between 60 bpm and 90 bpm. Give your answer correct to three decimal places.

[1 mark (0.9)]

The doctors consider a person to have a slow heart rate if the person's resting heart rate is less than 60 bpm. The probability that a randomly chosen Mathsland adult has a slow heart rate is 0.1587.

It is known that 29% of Mathsland adults play sport regularly. It is also known that 9% of Mathsland adults play sport regularly and have a slow heart rate.

Let S be the event that a randomly selected Mathsland adult plays sport regularly and let H be the event that a randomly selected Mathsland adult has a slow heart rate.

b. **i.** Find $\Pr(H|S)$, correct to three decimal places.

ii. Are the events H and S independent? Justify your answer.

[1 + 1 = 2 marks (0.6, 0.5)]

c. **i.** Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places.

ii. For random samples of 16 Mathsland adults, $\hat{P}$ is the random variable that represents the proportion of people who have a slow heart rate. Find the probability that $\hat{P}$ is greater than 10%, correct to three decimal places.

iii. For random samples of n Mathsland adults, $\hat{P}_n$ is the random variable that represents the proportion of people who have a slow heart rate. Find the least value of n for which $\Pr\left(\hat{P}_n > \frac{1}{n}\right) > 0.99$.

[2 + 2 + 2 = 6 marks (1.4, 0.8, 0.2)]
… continued

The doctors took a large random sample of adults from the population of Statsville and calculated an approximate 95% confidence interval for the proportion of Statsville adults who have a slow heart rate. The confidence interval they obtained was (0.102, 0.145).

d. **i.** Determine the sample proportion used in the calculation of this confidence interval.

ii. Explain why this confidence interval suggests that the proportion of adults with a slow heart rate in Statsville could be different from the proportion in Mathsland.

[1 + 1 = 2 marks (0.5, 0.1)]

Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school. The time taken by a randomly selected student to reach the top of the hill has the probability density function M with the rule

$$M(t) = \begin{cases} \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where t is given in minutes.

e. Find the expected time, in minutes, for a randomly selected student from Mathsland Secondary College to reach the top of the hill. Give your answer correct to one decimal place.

[2 marks (1.2)]

Students who take less than 15 minutes to get to the top of the hill are categorised as 'elite'.

f. Find the probability that a randomly selected student from Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

[1 mark (0.6)]

g. The Year 12 students at Mathsland Secondary College make up $\frac{1}{7}$ of the total number of students at the school. Of the Year 12 students at Mathsland Secondary College, 5% are categorised as elite.

Find the probability that a randomly selected non-Year 12 student at Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

[2 marks (0.2)]
Total 16 marks
[VCAA 2018 MM]

Question 319

Consider functions of the form

$$f: R \to R,\ f(x) = \frac{81x^2(a-x)}{4a^4}$$

and

$$h: R \to R,\ h(x) = \frac{9x}{2a^2}$$

where a is a positive real number.

a. Find the coordinates of the local maximum of f in terms of a.

[2 marks (1.2)]

b. Find the x-values of all of the points of intersection between the graphs of f and h, in terms of a where appropriate.

[1 mark (0.6)]

c. Determine the total area of the regions bounded by the graphs of $y = f(x)$ and $y = h(x)$.

[2 marks (0.8)]

Consider the function $g: \left[0, \frac{2a}{3}\right] \to R,\ g(x) = \frac{81x^2(a-x)}{4a^4}$, where a is a positive real number.

d. Evaluate $\frac{2a}{3} \times g\left(\frac{2a}{3}\right)$.

[1 mark (0.7)]

e. Find the area bounded by the graph of g^{-1}, the x-axis and the line $x = g\left(\frac{2a}{3}\right)$.

[2 marks (0.2)]

f. Find the value of a for which the graphs of g and g^{-1} have the same endpoints.

[1 mark (0.1)]

g. Find the area enclosed by the graphs of g and g^{-1} when they have the same endpoints.

[1 mark (0.1)]
Total 10 marks
[VCAA 2018 MM]

Question 320

Let $f: R \to R,\ f(x) = x^2 e^{-x^2}$.

a. Find $f'(x)$.

[1 mark (1.0)]

b. **i.** State the nature of the stationary point on the graph of f at the origin.
ii. Find the maximum value of the function f and the values of x for which the maximum occurs.
iii. Find the values of $d \in R$ for which $f(x) + d$ is always negative.

[1 + 2 + 1 = 4 marks (0.7, 1.6, 0.4)]

c. **i.** Find the equation of the tangent to the graph of f at $x = -1$.
ii. Find the area enclosed by the graph of f and the tangent to the graph of f at $x = -1$, correct to four decimal places.

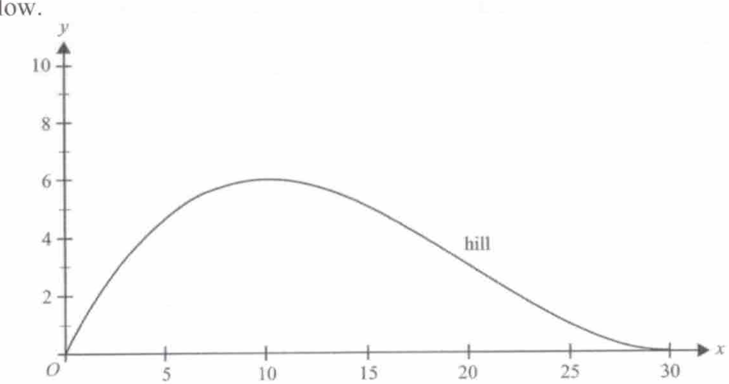
[1 + 2 = 3 marks (0.8, 1.3)]

d. Let $M(m, n)$ be a point on the graph of f, where $m \in [0, 1]$.
Find the minimum distance between M and the point $(0, e)$, and the value of m for which this occurs, correct to three decimal places.

[3 marks (1.1)]
Total 11 marks
[VCAA 2019 MM]

Question 321

An amusement park is planning to build a zip-line above a hill on its property. The hill is modelled by $y = \frac{3x(x-30)^2}{2000}$, $x \in [0, 30]$, where x is the horizontal distance, in metres, from an origin and y is the height, in metres, above this origin, as shown in the graph below.

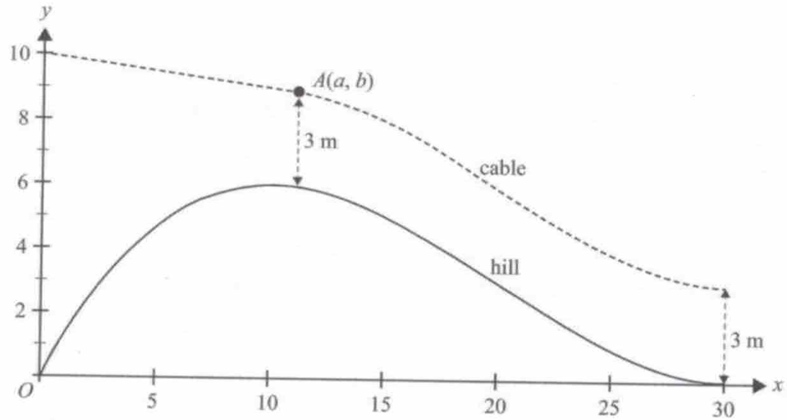


a. Find $\frac{dy}{dx}$.

[1 mark (1.0)]

b. State the set of values for which the gradient of the hill is strictly decreasing.
[1 mark (0.1)]

The cable for the zip-line is connected to a pole at the origin at a height of 10 m and is straight for $0 \le x \le a$, where $10 \le a \le 20$. The straight section joins the curved section at $A(a, b)$. The cable is then exactly 3 m vertically above the hill from $a \le x \le 30$, as shown in the graph.



c. State the rule, in terms of x, for the height of the cable above the horizontal axis for $x \in [a, 30]$.

[1 mark (0.6)]

d. Find the values of x for which the gradient of the cable is equal to the average gradient of the hill for $x \in [10, 30]$.

[3 marks (1.4)]

The gradients of the straight and curved sections of the cable approach the same value at $x = a$, so there is a continuous and smooth join at A.

e. **i.** State the gradient of the cable at A, in terms of a.

ii. Find the coordinates of A, with each value correct to two decimal places.

iii. Find the value of the gradient at A, correct to one decimal place.

[1 + 3 + 1 = 5 marks (0.5, 0.7, 0.2)]
Total 11 marks
[VCAA 2019 MM]

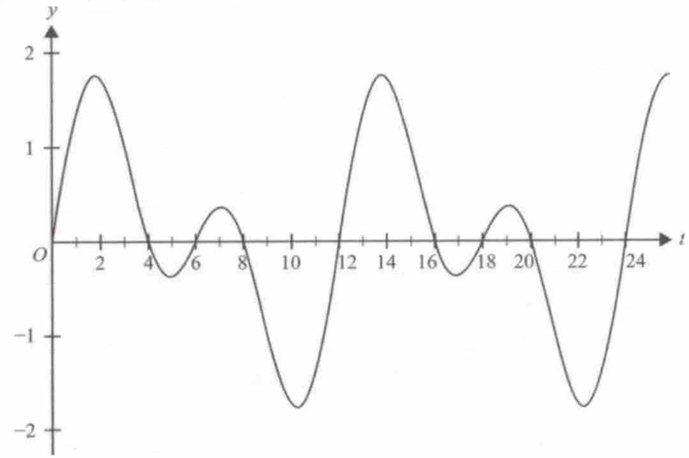
B. Extended-response questions

Question 322

During a telephone call, a phone uses a dual-tone frequency electrical signal to communicate with the telephone exchange.
The strength, f, of a simple dual-tone frequency signal is given by the function

$f(t) = \sin\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{6}\right)$, where t is a measure of time and $t \ge 0$.

Part of the graph of $y = f(t)$ is shown below.



a. State the period of the function.

[1 mark (0.8)]

b. Find the values of t where $f(t) = 0$ for the interval $t \in [0, 6]$.

[1 mark (0.8)]

c. Find the maximum strength of the dual-tone frequency signal, correct to two decimal places.

[1 mark (0.8)]

d. Find the area between the graph of f and the horizontal axis for $t \in [0, 6]$.

[2 marks (1.3)]

e. The rectangle bounded by the line $y = k$, $k \in R^+$, the horizontal axis, and the lines $x = 0$ and $x = 12$ has the same area as the area between the graph of f and the horizontal axis for one period of the dual-tone frequency signal.
Find the value of k.

[2 marks (0.8)]
Total 7 marks
[adapted from VCAA 2019 MM]

Question 323

The Lorenz birdwing is the largest butterfly in Town A.
The probability density function that describes its life span, X, in weeks, is given by

$$f(x)=\begin{cases}\frac{4}{625}(5x^3-x^4) & 0\le x\le 5\\ 0 & \text{elsewhere}\end{cases}$$

a. Find the mean life span of the Lorenz birdwing butterfly.

[2 marks (1.6)]

b. In a sample of 80 Lorenz birdwing butterflies, how many butterflies are expected to live longer than two weeks, correct to the nearest integer?

[2 marks (1.2)]

c. What is the probability that a Lorenz birdwing butterfly lives for at least four weeks, given that it lives for at least two weeks, correct to four decimal places?

[2 marks (1.3)]

The wingspans of Lorenz birdwing butterflies in Town A are normally distributed with a mean of 14.1 cm and a standard deviation of 2.1 cm.

d. Find the probability that a randomly selected Lorenz birdwing butterfly in Town A has a wingspan between 16 cm and 18 cm, correct to four decimal places.

[1 mark (0.8)]

e. A Lorenz birdwing butterfly is considered to be **very small** if its wingspan is in the smallest 5% of all the Lorenz birdwing butterflies in Town A.
Find the greatest possible wingspan, in centimetres, for a **very small** Lorenz birdwing butterfly in Town A, correct to one decimal place.

[1 mark (0.6)]

Each year, a detailed study is conducted on a random sample of 36 Lorenz birdwing butterflies in Town A. A Lorenz birdwing butterfly is considered to be **very large** if its wingspan is greater than 17.5 cm. The probability that the wingspan of any Lorenz birdwing butterfly in Town A is greater than 17.5 cm is 0.0527, correct to four decimal places.

f. i. Find the probability that three or more of the butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large**, correct to four decimal places.

ii. The probability that n or more butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large** is less than 1%.
Find the smallest value of n, where n is an integer.

iii. For random samples of 36 Lorenz birdwing butterflies in Town A, $\hat{P}$ is the random variable that represents the proportion of butterflies that are **very large**. Find the expected value and the standard deviation of $\hat{P}$, correct to four decimal places.

iv. What is the probability that a sample proportion of butterflies that are **very large** lies within one standard deviation of 0.0527, correct to four decimal places? Do not use a normal approximation.

[1 + 2 + 2 + 2 = 7 marks (0.8, 0.7, 1.0, 0.5)]

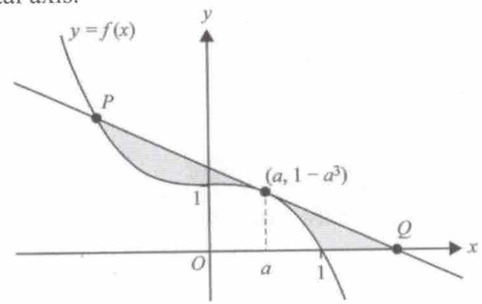
... continued

g. The Lorenz birdwing butterfly also lives in Town B.
In a particular sample of Lorenz birdwing butterflies from Town B, an approximate 95% confidence interval for the proportion of butterflies that are **very large** was calculated to be (0.0234, 0.0866), correct to four decimal places.
Determine the sample size used in the calculation of this confidence interval.

[2 marks (0.6)]
Total 17 marks
[VCAA 2019 MM]

Question 324

Let $f: R \to R,\ f(x)=1-x^3$. The tangent to the graph of f at $x = a$, where $0 < a < 1$, intersects the graph of f again at P and intersects the horizontal axis at Q. The shaded regions shown in the diagram below are bounded by the graph of f, its tangent at $x = a$ and the horizontal axis.



a. Find the equation of the tangent to the graph of f at $x = a$, in terms of a.

[1 mark (0.7)]

b. Find the x-coordinate of Q, in terms of a.

[1 mark (0.7)]

c. Find the x-coordinate of P, in terms of a.

[2 marks (1.3)]

Let A be the function that determines the total area of the shaded regions.

d. Find the rule of A, in terms of a.

[3 marks (1.2)]

e. Find the value of a for which A is a minimum.

[2 marks (0.7)]

Consider the regions bounded by the graph of f^{-1}, the tangent to the graph of f^{-1} at $x = b$, where $0 < b < 1$, and the horizontal axis.

f. Find the value of b for which the total area of these regions is a minimum.

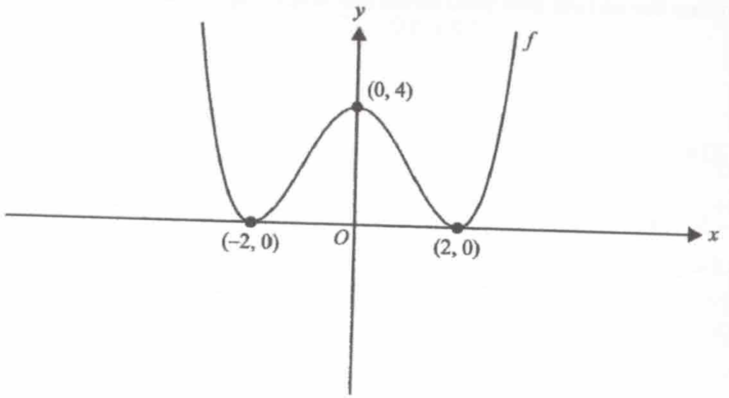
[2 marks (0.1)]

g. Find the value of the acute angle between the tangent to the graph of f and the tangent to the graph of f^{-1} at $x = 1$.

[1 mark (0.1)]
Total 12 marks
[VCAA 2019 MM]

Question 325

Let $f: R \to R, f(x) = a(x+2)^2(x-2)^2$, where $a \in R$. Part of the graph of f is shown below.



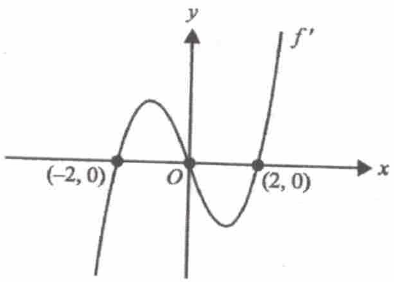
a. Show that $a = \frac{1}{4}$.

[1 mark (0.8)]

b. Express $f(x) = \frac{1}{4}(x+2)^2(x-2)^2$ in the form $f(x) = \frac{1}{4}x^4 + bx^2 + c$, where b and c are integers.

[1 mark (0.8)]

Part of the graph of the derivative function f' is shown below.



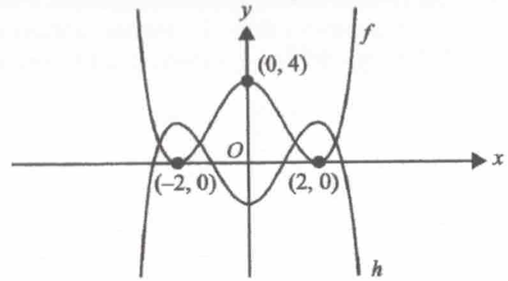
c. **i.** Write the rule for f' in terms of x.

ii. Find the minimum value of the graph of f' on the interval $x \in (0, 2)$.

[1 + 2 = 3 marks (0.8, 1.4)]

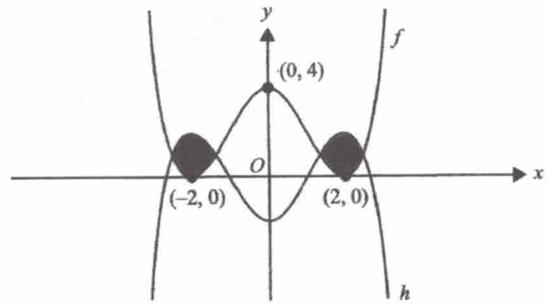
… continued

Let $h: R \to R, h(x) = -\frac{1}{4}(x+2)^2(x-2)^2 + 2$. Parts of the graphs of f and h are shown below.



d. Write a sequence of two transformations that map the graph of f onto the graph of h.

[1 mark (0.8)]



e. **i.** State the values of x for which the graphs of f and h intersect.

ii. Write down a definite integral that will give the total area of the shaded regions in the graph above.

iii. Find the total area of the shaded regions in the graph above. Give your answer correct to two decimal places.

[1 + 1 + 1 = 3 marks (0.8, 0.8, 0.7)]

f. Let D be the vertical distance between the graphs of f and h.

Find all values of x for which D is at most 2 units. Give your answers correct to two decimal places.

[2 marks (0.5)]

Total 11 marks

[VCAA 2020 MM]

Question 326

An area of parkland has a river running through it, as shown below. The river is shown shaded. The north bank of the river is modelled by the function

$f_1:[0,200]\to R, f_1(x)=20\cos\left(\dfrac{\pi x}{100}\right)+40$.

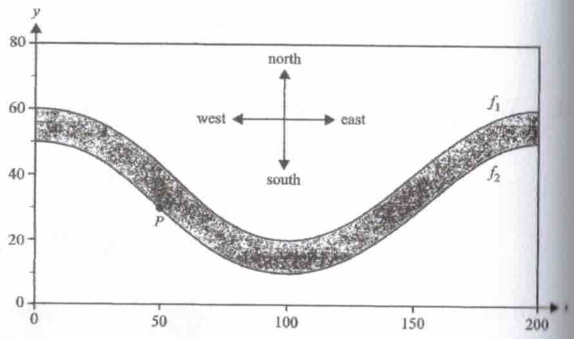
The south bank of the river is modelled by the function

$f_2:[0,200]\to R, f_2(x)=20\cos\left(\dfrac{\pi x}{100}\right)+30$.

The horizontal axis points east and the vertical axis points north. All distances are measured in metres.

A swimmer always starts at point P, which has coordinates (50, 30).

Assume that no movement of water in the river affects the motion or path of the swimmer, which is always a straight line.



a. The swimmer swims north from point P. Find the distance, in metres, that the swimmer needs to swim to get to the north bank of the river.

[1 mark (0.9)]

b. The swimmer swims east from point P. Find the distance, in metres, that the swimmer needs to swim to get to the north bank of the river.

[2 marks (1.3)]

c. On another occasion, the swimmer swims the minimum distance from point P to the north bank of the river. Find this minimum distance.
Give your answer in metres, correct to one decimal place.

[2 marks (0.8)]

d. Calculate the surface area of the section of the river shown on the graph above, in square metres.

[1 mark (0.8)]

e. A horizontal line is drawn through point P. The section of the river that is south of the line is declared a 'no swimming' zone. Find the area of the 'no swimming' zone, correct to the nearest square metre.

[3 marks (1.0)]

f. Scientists observe that the north bank of the river is changing over time. It is moving further north from its current position. They model its predicted new location using the function with rule $y = kf_1(x)$, where $k \geq 1$.

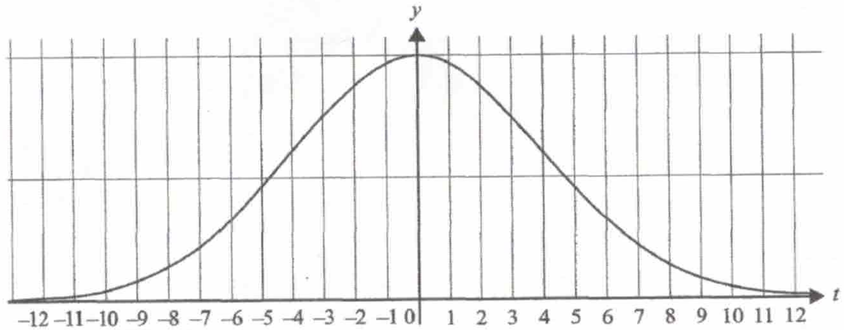
Find the values of k for which the distance **north** across the river, for all parts of the river, is strictly less than 20 m.

[2 marks (0.3)]
Total 11 marks
[VCAA 2020 MM]

B. Extended-response questions

Question 327

A transport company has detailed records of all its deliveries. The number of minutes a delivery is made before or after its scheduled delivery time can be modelled as a normally distributed random variable, T, with a mean of zero and a standard deviation of four minutes. A graph of the probability distribution of T is shown below.



a. If $\Pr(T \leq a) = 0.6$, find a to the nearest minute.

[1 mark (0.7)]

b. Find the probability, correct to three decimal places, of a delivery being no later than three minutes after its scheduled delivery time, given that it arrives after its scheduled delivery time.

[2 marks (1.0)]

c. Using the model described above, the transport company can make 46.48% of its deliveries over the interval $-3 \leq t \leq 2$.
It has an improved delivery model with a mean of k and a standard deviation of four minutes.

Find the values of k, correct to one decimal place, so that 46.48% of the transport company's deliveries can be made over the interval $-4.5 \leq t \leq 0.5$.

[3 marks (0.7)]

A rival transport company claims that there is a 0.85 probability that each delivery it makes will arrive on time or earlier. Assume that whether each delivery is on time or earlier is independent of other deliveries.

d. Assuming that the rival company's claim is true, find the probability that on a day in which the rival company makes eight deliveries, fewer than half of them arrive on time or earlier. Give your answer correct to three decimal places.

[2 marks (1.0)]

e. Assuming that the rival company's claim is true, consider a day in which it makes n deliveries.

i. Express, in terms of n, the probability that one or more deliveries will **not** arrive on time or earlier.

ii. Hence, or otherwise, find the minimum value of n such that there is at least a 0.95 probability that one or more deliveries will **not** arrive on time or earlier.

[1 + 1 = 2 marks (0.2, 0.2)]

f. An analyst from a government department believes the rival transport company's claim is only true for deliveries made before 4 pm. For deliveries made after 4 pm, the analyst believes the probability of a delivery arriving on time or earlier is x, where $0.3 \le x \le 0.7$.

After observing a large number of the rival transport company's deliveries, the analyst believes that the overall probability that a delivery arrives on time or earlier is actually 0.75.

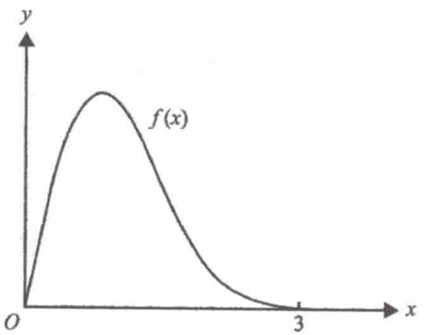
Let the probability that a delivery is made after 4 pm be y.

Assuming that the analyst's beliefs are true, find the minimum and maximum values of y.

[2 marks (0.1)]
Total 12 marks
[VCAA 2020 MM]

Question 328

The graph of the function $f(x) = 2xe^{(1-x^2)}$, where $0 \le x \le 3$, is shown below.



a. Find the slope of the tangent to f at $x = 1$.

[1 mark (0.8)]

b. Find the obtuse angle that the tangent to f at $x = 1$ makes with the positive direction of the horizontal axis. Give your answer correct to the nearest degree.

[1 mark (0.4)]

c. Find the slope of the tangent to f at a point $x = p$. Give your answer in terms of p.

[1 mark (0.7)]

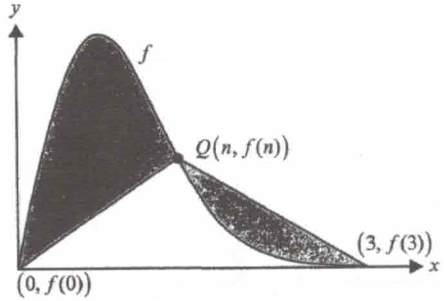
d. **i.** Find the value of p for which the tangent to f at $x = 1$ and the tangent to f at $x = p$ are perpendicular to each other. Give your answer correct to three decimal places.

ii. Hence, find the coordinates of the point where the tangents to the graph of f at $x = 1$ and $x = p$ intersect when they are perpendicular. Give your answer correct to two decimal places.

[2 + 3 = 5 marks (1.2, 1.2)]

… continued

Two line segments connect the points $(0, f(0))$ and $(3, f(3))$ to a single point $Q(n, f(n))$, where $1 < n < 3$, as shown in the graph below.



e. **i.** The first line segment connects the point $(0, f(0))$ and the point $Q(n, f(n))$, where $1 < n < 3$. Find the equation of this line segment in terms of n.

ii. The second line segment connects the point $Q(n, f(n))$ and the point $(3, f(3))$, where $1 < n < 3$. Find the equation of this line segment in terms of n.

iii. Find the value of n, where $1 < n < 3$, if there are equal areas between the function f and each line segment. Give your answer correct to three decimal places.

[1 + 1 + 3 = 5 marks (0.4, 0.3, 0.8)]
Total 13 marks
[VCAA 2020 MM]

Question 329

Let $f: R \to R, f(x) = x^3 - x$.

Let $g_a: R \to R$ be the function representing the tangent to the graph of f at $x = a$, where $a \in R$.

Let $(b, 0)$ be the x-intercept of the graph of g_a.

a. Show that $b = \dfrac{2a^3}{3a^2 - 1}$.

[3 marks (1.7)]

b. State the values of a for which b does not exist.

[1 mark (0.5)]

c. State the nature of the graph of g_a when b does not exist.

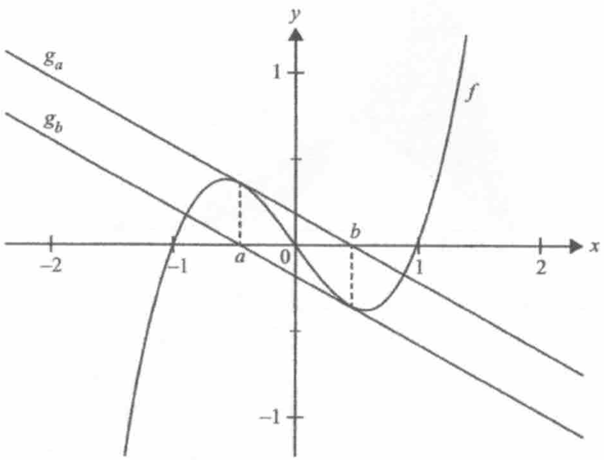
[1 mark (0.2)]

d. **i.** State all values of a for which $b = 1.1$. Give your answers correct to four decimal places.

ii. The graph of f has an x-intercept at $(1, 0)$.
State the values of a for which $1 \le b < 1.1$. Give your answers correct to three decimal places.

[1 + 1 = 2 marks (0.6, 0.1)]

The coordinate $(b, 0)$ is the horizontal axis intercept of g_a. Let g_b be the function representing the tangent to the graph of f at $x = b$, as shown in the graph below.



e. Find the values of a for which the graphs of g_a and g_b, where b exists, are parallel and where $b \neq a$.

[3 marks (0.4)]

Let $p: R \to R, p(x) = x^3 + wx$, where $w \in R$.

f. Show that $p(-x) = -p(x)$ for all $w \in R$.

[1 mark (0.6)]

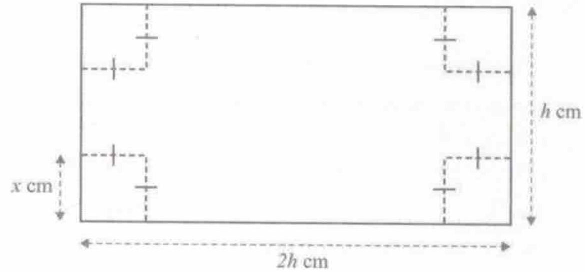
A property of the graphs of p is that two distinct parallel tangents will always occur at $(t, p(t))$ and $(-t, p(-t))$ for all $t \neq 0$.

g. Find all values of w such that a tangent to the graph of p at $(t, p(t))$, for some $t > 0$, will have an x-intercept at $(-t, 0)$.

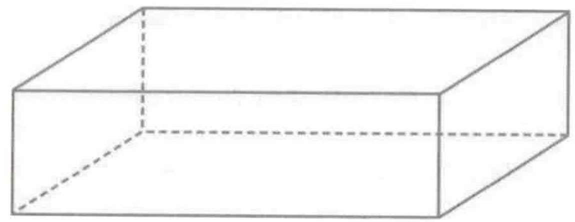
[1 mark (0.03)]
Total 12 marks
[adapted from VCAA 2020 MM]

Question 330

A rectangular sheet of cardboard has a width of h centimetres. Its length is twice its width. Squares of side length x centimetres, where $x > 0$, are cut from each of the corners, as shown in the diagram.



The sides of this sheet of cardboard are then folded up to make a rectangular box with an open top, as shown here. Assume that the thickness of the cardboard is negligible and that $V_{box} > 0$.



A box is to be made from a sheet of cardboard with $h = 25$.

a. Show that the volume, V_{box}, in cubic centimetres, is given by $V_{box} = 2x(25-2x)(25-x)$.

[1 mark (0.7)]

b. State the domain of V_{box}.

[1 mark (0.4)]

c. Find the derivative of V_{box} with respect to x.

[1 mark (0.9)]

d. Calculate the maximum possible volume of the box and for which value of x this occurs.

[3 marks (2.0)]

e. Waste minimisation is a goal when making cardboard boxes. Percentage wasted is based on the area of the sheet of cardboard that is cut out before the box is made. Find the percentage of the sheet of cardboard that is wasted when $x = 5$.

[2 marks (1.1)]

Now consider a box made from a rectangular sheet of cardboard where $h > 0$ and the box's length is still twice its width.

f. **i.** Let V_{box} be the function that gives the volume of the box. State the domain of V_{box} in terms of h.

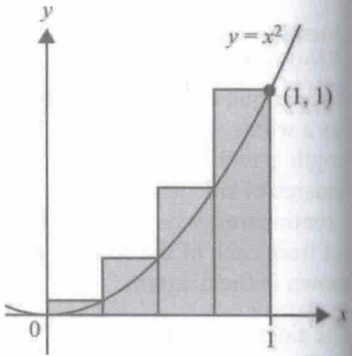
ii. Find the maximum volume for any such rectangular box, V_{box}, in terms of h.

[1 + 3 = 4 marks (0.4, 1.4)]

g. Now consider making a box from a square sheet of cardboard with side lengths of h centimetres. Show that the maximum volume of the box occurs when $x = \frac{h}{6}$.

[2 marks (0.8)]
Total 14 marks
[VCAA 2021 MM]

Question 331



Four rectangles of equal width are drawn and used to approximate the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$.
The heights of the rectangles are the values of the graph of $y = x^2$ at the right endpoint of each rectangle, as shown in the graph.

a. State the width of each of the rectangles shown.

[1 mark (1.0)]

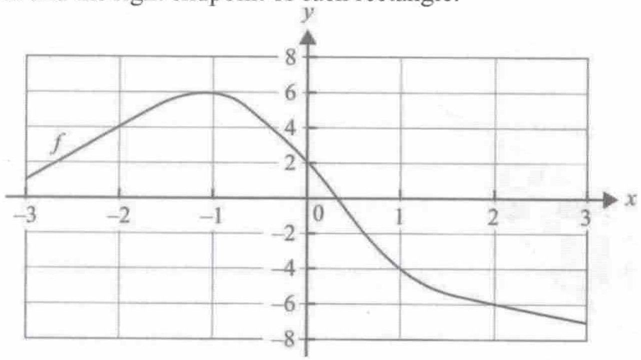
b. Find the total area of the four rectangles shown above.

[1 mark (0.6)]

c. Find the area between the graph of $y = x^2$, the x-axis and the line $x = 1$.

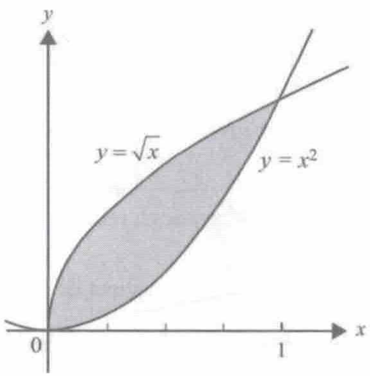
[2 marks (1.7)]

d. The graph of f is shown below. Approximate $\int_{-2}^{2} f(x)\,dx$ using four rectangles of equal width and the right endpoint of each rectangle.



[1 mark (0.2)]

Parts of the graphs of $y = x^2$ and $y = \sqrt{x}$ are shown below.



e. Find the area of the shaded region.

[1 mark (0.9)]

... continued

f. The graph of $y = x^2$ is transformed to the graph of $y = ax^2$, where $a \in (0, 2]$. Find the values of a such that the area defined by the region(s) bounded by the graphs of $y = ax^2$ and $y = \sqrt{x}$ and the lines $x = 0$ and $x = a$ is equal to $\frac{1}{3}$. Give your answer correct to two decimal places.

[4 marks (0.7)]
Total 10 marks
[VCAA 2021 MM]

Question 332

Let $q(x) = \log_e(x^2 - 1) - \log_e(1 - x)$.

a. State the maximal domain and the range of q.

[2 marks (1.1)]

b. **i.** Find the equation of the tangent to the graph of q when $x = -2$.

ii. Find the equation of the line that is perpendicular to the graph of q when $x = -2$ and passes through the point $(-2, 0)$.

[1 + 1 = 2 marks (0.8, 0.7)]

Let $p(x) = e^{-2x} - 2e^{-x} + 1$.

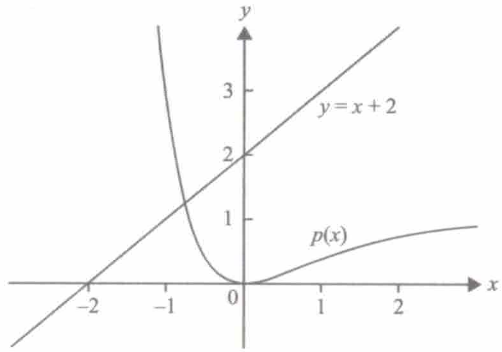
c. Explain why p is not a one-to-one function.

[1 mark (0.7)]

d. Find the gradient of the tangent to the graph of p at $x = a$.

[1 mark (0.7)]

The diagram below shows parts of the graph of p and the line $y = x + 2$.



The line $y = x + 2$ and the tangent to the graph of p at $x = a$ intersect with an acute angle of θ between them.

e. Find the value(s) of a for which $\theta = 60°$. Give your answer(s) correct to two decimal places.

[3 marks (0.4)]

f. Find the x-coordinate of the point of intersection between the line $y = x + 2$ and the graph of p, and hence find the area bounded by $y = x + 2$, the graph of p and the x-axis, both correct to three decimal places.

[3 marks (1.3)]
Total 12 marks
[VCAA 2021 MM]

Question 333

A teacher coaches their school's table tennis team. The teacher has an adjustable ball machine that they use to help the players practise. The speed, measured in metres per second, of the balls shot by the ball machine is a normally distributed random variable W. The teacher sets the ball machine with a mean speed of 10 metres per second and a standard deviation of 0.8 metres per second.

a. Determine $\Pr(W \geq 11)$, correct to three decimal places.

[1 mark (0.8)]

b. Find the value of k, in metres per second, which 80% of ball speeds are below. Give your answer in metres per second, correct to one decimal place.

[1 mark (0.7)]

The teacher adjusts the height setting for the ball machine. The machine now shoots balls high above the table tennis table. Unfortunately, with the new height setting, 8% of balls do not land on the table. Let $\hat{P}$ be the random variable representing the sample proportion of balls that do not land on the table in random samples of 25 balls.

c. Find the mean and the standard deviation of $\hat{P}$.

[2 marks (0.9)]

d. Use the binomial distribution to find $\Pr(\hat{P} > 0.1)$, correct to three decimal places.

[2 marks (1.0)]

The teacher can also adjust the spin setting on the ball machine. The spin, measured in revolutions per second, is a continuous random variable X with the probability density function

$$f(x)=\begin{cases} \dfrac{x}{500} & 0 \leq x < 20 \\ \dfrac{50-x}{750} & 20 \leq x \leq 50 \\ 0 & \text{elsewhere} \end{cases}$$

e. Find the maximum possible spin applied by the ball machine, in revolutions per second.

[1 mark (0.2)]

f. Find the standard deviation of the spin, in revolutions per second, correct to one decimal place.

[3 marks (1.2)]

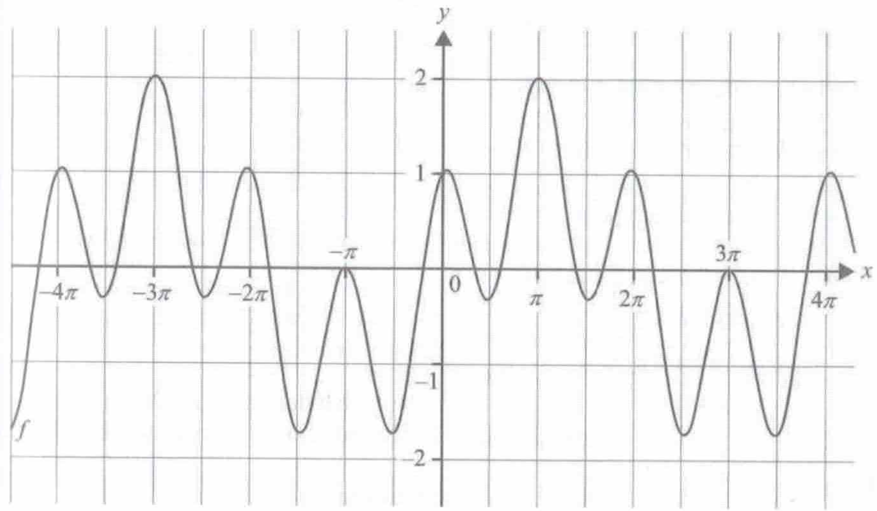
Total 10 marks

[adapted from VCAA 2021 MM]

B. Extended-response questions

Question 334

Part of the graph of $f: R \to R, f(x) = \sin\left(\frac{x}{2}\right) + \cos(2x)$ is shown below.



a. State the period of f.

[1 mark (0.7)]

b. State the minimum value of f correct to three decimal places.

[1 mark (0.6)]

c. Find the smallest positive value of h for which $f(h - x) = f(x)$.

[1 mark (0.2)]

Consider the set of functions of the form $g_a: R \to R, g_a(x) = \sin\left(\frac{x}{a}\right) + \cos(ax)$, where a is a positive integer.

d. State the value of a such that $g_a(x) = f(x)$ for all x.

[1 mark (0.7)]

e. **i.** Find an antiderivative of g_a in terms of a.

ii. Use a definite integral to show that the area bounded by g_a and the x-axis over the interval $[0, 2a\pi]$ is equal above and below the x-axis for all values of a.

[1 + 3 = 4 marks (0.5, 0.9)]

f. Explain why the maximum value of g_a cannot be greater than 2 for all values of a and why the minimum value of g_a cannot be less than −2 for all values of a.

[1 mark (0.2)]

g. Find the greatest possible minimum value of g_a.

[1 mark (0.0)]

Total 10 marks

[VCAA 2021 MM]