MATHSQUEST 11

MATHEMATICAL METHODS

SOLUTIONS MANUAL

VCE UNITS 1 AND 2

MATHSQUEST 11
MATHEMATICAL METHODS

SOLUTIONS MANUAL

VCE UNITS 1 AND 2

AUTHOR
SUE MICHELL

CONTRIBUTING AUTHORS
KATE BALKA | ISABELLE LAM

jacaranda
A Wiley Brand

First published 2016 by
John Wiley & Sons Australia, Ltd
42 McDougall Street, Milton, Qld 4064

Typeset in 9/11 pt Times LT Std

© John Wiley & Sons Australia, Ltd 2016

The moral rights of the authors have been asserted.

ISBN 978 0 7303 2297 9

Reproduction and communication for educational purposes
The Australian *Copyright Act 1968* (the Act) allows a maximum of one chapter or 10% of the pages of this work, whichever is the greater, to be reproduced and/or communicated by any educational institution for its educational purposes provided that the educational institution (or the body that administers it) has given a remuneration notice to Copyright Agency Limited (CAL).

Reproduction and communication for other purposes
Except as permitted under the Act (for example, a fair dealing for the purposes of study, research, criticism or review), no part of this book may be reproduced, stored in a retrieval system, communicated or transmitted in any form or by any means without prior written permission. All inquiries should be made to the publisher.

Trademarks
Jacaranda, the JacPLUS logo, the learnON, assessON and studyON logos, Wiley and the Wiley logo, and any related trade dress are trademarks or registered trademarks of John Wiley & Sons Inc. and/or its affiliates in the United States, Australia and in other countries, and may not be used without written permission. All other trademarks are the property of their respective owners.

Cover and internal design images: © John Wiley and Sons, Australia Ltd

Illustrated by diacriTech and Wiley Composition Services

Typeset in India by diacriTech

Printed in Singapore by
Printplus Limited

10 9 8 7 6 5 4 3 2 1

Table of contents

About eBookPLUS

jacaranda *plus*

This book features eBookPLUS: an electronic version of the entire textbook and supporting digital resources. It is available for you online at the JacarandaPLUS website (**www.jacplus.com.au**).

Using JacarandaPLUS

To access your eBookPLUS resources, simply log on to **www.jacplus.com.au** using your existing JacarandaPLUS login and enter the registration code. If you are new to JacarandaPLUS, follow the three easy steps below.

Step 1. Create a user account
The first time you use the JacarandaPLUS system, you will need to create a user account. Go to the JacarandaPLUS home page (**www.jacplus.com.au**), click on the button to create a new account and follow the instructions on screen. You can then use your nominated email address and password to log in to the JacarandaPLUS system.

Step 2. Enter your registration code
Once you have logged in, enter your unique registration code for this book, which is printed on the inside front cover of your textbook. The title of your textbook will appear in your bookshelf. Click on the link to open your eBookPLUS.

Step 3. Access your eBookPLUS resources
Your eBookPLUS and supporting resources are provided in a chapter-by-chapter format. Simply select the desired chapter from the table of contents. Digital resources are accessed within each chapter via the resources tab.

Once you have created your account, you can use the same email address and password in the future to register any JacarandaPLUS titles you own.

Using eBookPLUS references

eBookPLUS logos are used throughout the printed books to inform you that a digital resource is available to complement the content you are studying.

Searchlight IDs (e.g. **INT-0001**) give you instant access to digital resources. Once you are logged in, simply enter the Searchlight ID for that resource and it will open immediately.

Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. The complete list of JacPLUS minimum system requirements can be found at **http://jacplus.desk.com**.

Troubleshooting

- Go to **www.jacplus.com.au** and click on the Help link.
- Visit the JacarandaPLUS Support Centre at **http://jacplus.desk.com** to access a range of step-by-step user guides, ask questions or search for information.
- Contact John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
  Phone: 1800 JAC PLUS (1800 522 7587)

Topic 1 — Lines and linear relationships

Exercise 1.2 — Linearly related variables, linear equations and inequations

1 Let the volume of the cone be V cm^3 and its height h cm.

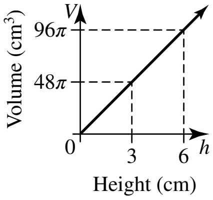
a Let k be the constant of proportionality

$$V = kh$$
$$\therefore 96\pi = k(6)$$
$$\therefore k = 16\pi$$

b $V = 16\pi h,\ V = 48\pi$

$$\therefore 48\pi = 16\pi h$$
$$\therefore h = 3$$

the height is also halved and equals 3 cm



2 **a** Let $C = 210 + kn$

$n = 20, C = 330 \Rightarrow 330 = 210 + k(20)$

$$\therefore 330 - 210 = 20k$$
$$\therefore k = \frac{120}{20}$$
$$\therefore k = 6$$

The linear model is $C = 210 + 6n$

b When $n = 25, C = 210 + 6(25) = 360$

Therefore the cost would be $360

3 **a** $3(5x - 1) = 4x - 14$

$$\therefore 15x - 3 = 4x - 14$$
$$\therefore 11x = -14 + 3$$
$$\therefore 11x = -11$$
$$\therefore x = -1$$

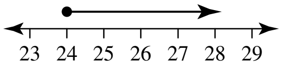
b $$\frac{4-x}{3} + \frac{3x-2}{4} = 5$$
$$\therefore \frac{4(4-x)+3(3x-2)}{12} = 5$$
$$\therefore \frac{16-4x+9x-6}{12} = 5$$
$$\therefore \frac{10+5x}{12} = 5$$
$$\therefore 10 + 5x = 60$$
$$\therefore 5x = 50$$
$$\therefore x = 10$$

4 $$\frac{7(x-3)}{8} + \frac{3(2x+5)}{4} = \frac{3x}{2} + 1$$
$$\therefore \frac{7(x-3)}{8} + \frac{3(2x+5)}{4} - \frac{3x}{2} = 1$$
$$\therefore \frac{7(x-3)+6(2x+5)-12x}{8} = 1$$
$$\therefore \frac{7x-21+12x+30-12x}{8} = 1$$
$$\therefore \frac{7x+9}{8} = 1$$
$$\therefore 7x + 9 = 8$$
$$\therefore 7x = -1$$
$$\therefore x = -\frac{1}{7}$$

5 $$\frac{d-x}{a} = \frac{a-x}{d}$$
$$\therefore d(d-x) = a(a-x)$$
$$\therefore d^2 - dx = a^2 - ax$$
$$\therefore ax - dx = a^2 - d^2$$
$$\therefore x(a-d) = a^2 - d^2$$
$$\therefore x = \frac{a^2-d^2}{a-d}$$
$$\therefore x = \frac{(a-d)(a+d)}{a-d}$$
$$\therefore x = a + d$$

6 $$b(x+c) = a(x-c) + 2bc$$
$$\therefore bx + bc = ax - ac + 2bc$$
$$\therefore bx - ax = -ac + bc$$
$$\therefore x(b-a) = c(b-a)$$
$$\therefore x = \frac{c(b-a)}{b-a}$$
$$\therefore x = c$$

7 $$7 - \frac{3x}{8} \le -2$$
$$\therefore -\frac{3x}{8} \le -2 - 7$$
$$\therefore -\frac{3x}{8} \le -9$$
$$\therefore -3x \le -72$$
$$\therefore x \ge \frac{-72}{-3}$$
$$\therefore x \ge 24$$



8 $$4(2+3x) > 8 - 3(2x+1)$$
$$\therefore 8 + 12x > 8 - 6x - 3$$
$$\therefore 12x + 6x > 8 - 3 - 8$$
$$\therefore 18x > -3$$
$$\therefore x > \frac{-3}{18}$$
$$\therefore x > -\frac{1}{6}$$

9 **a** $x = 2y + 5$

$4x - 3y = 25$

Substitute the first equation into the second equation

$$4(2y+5) - 3y = 25$$
$$\therefore 8y + 20 - 3y = 25$$
$$\therefore 5y + 20 = 25$$
$$\therefore 5y = 5$$
$$\therefore y = 1$$

In the first equation replace y by 1 to obtain $x = 7$.

Therefore $x = 7,\ y = 1$

b $5x+9y=-38....(1)$

$-3x+2y=8....(2)$

Choosing to eliminate x.

Multiply equation (1) by 3 and equation (2) by 5

$15x+27y=-114....(3)$

$-15x+10y=40....(4)$

Add equations (3) and (4)

$37y=-74$

$\therefore y=-2$

Substitute $y=-2$ in equation (2)

$\therefore -3x-4=8$

$\therefore -3x=12$

$\therefore x=-4$

Therefore $x=-4, y=-2$

10 **a** $2x-y=7....(1)$

$7x-5y=42...(2)$

Either of the substitution or elimination methods could be used.

Using substitution:

From equation (1), $2x-7=y$. Substitute in equation (2)

$\therefore 7x-5(2x-7)=42$

$\therefore 7x-10x+35=42$

$\therefore -3x=7$

$\therefore x=-\frac{7}{3}$

Substitute $x=-\frac{7}{3}$ in equation (1)

$\therefore y=-\frac{14}{3}-7$

$\therefore y=-\frac{14}{3}-\frac{21}{3}$

$\therefore y=-\frac{35}{3}$

Answer $x=-\frac{7}{3}, y=-\frac{35}{3}$

b $ax-by=a....(1)$

$bx+ay=b....(2)$

Multiply equation (1) by a and equation (2) by b

$a^2x-aby=a^2....(3)$

$b^2x+aby=b^2....(4)$

Adding equations (3) and (4),

$(a^2+b^2)x=a^2+b^2$

$\therefore x=1$

Substitute $x=1$ in equation (2)

$b+ay=b$

$\therefore ay=0$

$\therefore y=0$

Therefore $x=1,\ y=0$

11 Let n be the number of books sold and $\$P$ be the profit made.

a Revenue = $2.5n$ dollars and costs $=100+0.2n$ dollars.

$\therefore P=2.5n-(100+0.2n)$

$\therefore P=2.3n-100$

b To make a profit, $P>0$

$\therefore 2.3n-100>0$

$\therefore 2.3n>100$

$\therefore n>\frac{100}{2.3}$

$\therefore n>43.478$

At least 44 books must be sold to ensure a profit is made,

c Let h be the number of hardcover books sold and p be the number of paperbacks sold.

$40h+65p=330.........(1)$

$30h+110p=370........(2)$

Multiply the first equation by 3 and the second equation by 4

$120h+195p=990$

$120h+440p=1480$

Eliminate h by subtracting

$(440-195)p=1480-990$

$\therefore 245p=490$

$\therefore p=\frac{490}{245}$

$\therefore p=2$

Substitute $p=2$ in the first equation

$40h+65(2)=330$

$\therefore 40h+130=330$

$\therefore 40h=200$

$\therefore h=5$

Hardback books cost \$5 and paperbacks cost \$2.

12 Let the cyclist travel a distance of x km in a time of t hours when the motorcyclist catches up.

The motorcyclist travels the same distance but in a time of $\left(t-\frac{15}{60}\right)=\left(t-\frac{1}{4}\right)$ hours.

Since distance = speed × time,

For the cyclist, $x=16t$..........equation (1)

And for the motorcyclist, $x=48\left(t-\frac{1}{4}\right)$..........equation (2)

Substituting equation (1) into equation (2) gives

$16t=48\left(t-\frac{1}{4}\right)$

$\therefore 16t=48t-12$

$\therefore 12=32t$

$\therefore t=\frac{12}{32}$

$\therefore t=\frac{3}{8}$

The cyclist has travelled for $\frac{3}{8}\times 60=22.5$ minutes before being overtaken.

13 **a** $T=kx$ where k is the constant of proportionality

When $x=0.2,\ T=0.7$,

$0.7=k(0.2)$

$\therefore k=\frac{0.7}{0.2}$

$\therefore k=3.5$

The linear model is $T=3.5x$

If $x=0.2+0.1=0.3$, then

$T=3.5(0.3)$

$=1.05$

Therefore, the tension is 1.05 newton.

b $C=kl$ where k is the constant of proportionality

When $l=36.2,\ C=52.49$,

$52.49=k(36.2)$

$\therefore k=\frac{52.49}{36.2}$

$\therefore k=1.45$

The linear model is $C=1.45l$

If $C = 43.50$, then

$43.50 = 1.45l$

$\therefore l = \dfrac{43.50}{1.45}$

$\therefore l = 30$

Therefore, 30 litres of petrol can be purchased.

c $v = 12 - kt$ where k is the constant of proportionality

When $t = 0.5,\ v = 7.1$,

$7.1 = 12 - 0.5k$

$\therefore 0.5k = 12 - 7.1$

$\therefore k = \dfrac{4.9}{0.5}$

$\therefore k = 9.8$

The linear model is $v = 12 - 9.8t$

The ball will start to fall back to the ground once its speed upwards becomes zero.

When $v = 0$, then

$0 = 12 - 9.8t$

$\therefore t = \dfrac{12}{9.8}$

$\therefore t = \dfrac{60}{49} \simeq 1.22$

Therefore, the ball starts to fall back after approximately 1.22 seconds.

14 a The simple interest, I, earned on the investment of \$100 is directly proportional to the number of years, T, that the money is invested according to the rule $I = 46T$.

b The variables are not linearly related since the graph is not a straight line.

c The weekly wage, W, is \$400 plus an amount directly proportional to n, the number of hours of overtime worked according to the rule $W = 400 + 50n$.

15 a $7(2x - 3) = 5(3 + 2x)$

$\therefore 14x - 21 = 15 + 10x$

$\therefore 14x - 10x = 15 + 21$

$\therefore 4x = 36$

$\therefore x = 9$

b $\dfrac{4x}{5} - 9 = 7$

$\therefore \dfrac{4x}{5} = 16$

$\therefore 4x = 80$

$\therefore\ x = 20$

c $4 - 2(x - 6) = \dfrac{2x}{3}$

$\therefore 4 - 2x + 12 = \dfrac{2x}{3}$

$\therefore 16 - 2x = \dfrac{2x}{3}$

$\therefore 48 - 6x = 2x$

$\therefore 48 = 8x$

$\therefore x = 6$

d $\dfrac{3x + 5}{9} = \dfrac{4 - 2x}{5}$

$\therefore 5(3x + 5) = 9(4 - 2x)$

$\therefore 15x + 25 = 36 - 18x$

$\therefore 33x = 11$

$\therefore x = \dfrac{11}{33}$

$\therefore x = \dfrac{1}{3}$

e $\dfrac{x + 2}{3} + \dfrac{x}{2} - \dfrac{x + 1}{4} = 1$

$\therefore \dfrac{4(x + 2) + 6x - 3(x + 1)}{12} = 1$

$\therefore 4x + 8 + 6x - 3x - 3 = 12$

$\therefore 7x = 7$

$\therefore x = 1$

f $\dfrac{7x}{5} - \dfrac{3x}{10} = 2\left(x + \dfrac{9}{2}\right)$

$\therefore \dfrac{14x - 3x}{10} = 2x + 9$

$\therefore 11x = 10(2x + 9)$

$\therefore 11x = 20x + 90$

$\therefore -9x = 90$

$\therefore x = -10$

16 a $ax + b = c$

$\therefore ax = c - b$

$\therefore x = \dfrac{c - b}{a}$

b $a(x - b) = bx$

$\therefore ax - ab = bx$

$\therefore ax - bx = ab$

$\therefore x(a - b) = ab$

$\therefore x = \dfrac{ab}{a - b}$

c $a^2x + a^2 = ab + abx$

$\therefore a^2x - abx = ab - a^2$

$\therefore x(a^2 - ab) = -(a^2 - ab)$

$\therefore x = -1$

d $\dfrac{x}{a} + \dfrac{x}{b} = a + b$

$\therefore \dfrac{bx + ax}{ab} = a + b$

$\therefore bx + ax = ab(a + b)$

$\therefore x(b + a) = ab(a + b)$

$\therefore x = ab$

e $\dfrac{bx - a}{c} = \dfrac{cx + a}{b}$

$\therefore b(bx - a) = c(cx + a)$

$\therefore b^2x - ba = c^2x + ca$

$\therefore b^2x - c^2x = ca + ba$

$\therefore x(b^2 - c^2) = a(c + b)$

$\therefore x = \dfrac{a(c + b)}{b^2 - c^2}$

$\therefore x = \dfrac{a(c + b)}{(b + c)(b - c)}$

$\therefore x = \dfrac{a}{b - c}$

f $\dfrac{x+a}{b}-2=\dfrac{x-b}{a}$

$\therefore \dfrac{x+a}{b}-\dfrac{x-b}{a}=2$

$\therefore \dfrac{a(x+a)-b(x-b)}{ba}=2$

$\therefore ax+a^2-bx+b^2=2ba$

$\therefore ax-bx=2ba-a^2-b^2$

$\therefore x(a-b)=-(a^2-2ab+b^2)$

$\therefore x=\dfrac{-(a-b)^2}{a-b}$

$\therefore x=-(a-b)$

$\therefore x=b-a$

17 **a** $3x-5\le -8$

$\therefore 3x\le -8+5$

$\therefore 3x\le -3$

$\therefore x\le -1$

b $4(x-6)+3(2-2x)<0$

$\therefore 4x-24+6-6x<0$

$\therefore -2x-18<0$

$\therefore -2x<18$

$\therefore x>-9$

c $1-\dfrac{2x}{3}\ge -11$

$\therefore -\dfrac{2x}{3}\ge -12$

$\therefore -2x\ge -36$

$\therefore x\le 18$

d $\dfrac{5x}{6}-\dfrac{4-x}{2}>2$

$\therefore \dfrac{5x-3(4-x)}{6}>2$

$\therefore 5x-12+3x>12$

$\therefore 8x>24$

$\therefore x>3$

e $8x+7(1-4x)\le 7x-3(x+3)$

$\therefore 8x+7-28x\le 7x-3x-9$

$\therefore -20x+7\le 4x-9$

$\therefore -24x\le -16$

$\therefore x\ge \dfrac{16}{24}$

$\therefore x\ge \dfrac{2}{3}$

f $\dfrac{2}{3}(x-6)-\dfrac{3}{2}(x+4)>1+x$

$\therefore \dfrac{2(x-6)}{3}-\dfrac{3(x+4)}{2}>1+x$

$\therefore \dfrac{4(x-6)-9(x+4)}{6}>1+x$

$\therefore 4x-24-9x-36>6(1+x)$

$\therefore -5x-60>6+6x$

$\therefore -11x>66$

$\therefore x<-6$

18 **a** $y=5x-1......(1)$

$x+2y=9....(2)$

Substitute equation (1) in equation (2)

$\therefore x+2(5x-1)=9$

$\therefore x+10x-2=9$

$\therefore 11x=11$

$\therefore x=1$

Substitute $x=1$ in equation (1)

$\therefore y=4$

Answer $x=1, y=4$

b $3x+5y=4......(1)$

$8x+2y=-12.....(2)$

Multiply equation (2) by $\dfrac{5}{2}$

$\therefore 20x+5y=-30....(3)$

Subtract equation (1) from equation (3)

$\therefore 17x=-34$

$\therefore x=-2$

Substitute $x=-2$ in equation (1)

$\therefore -6+5y=4$

$\therefore 5y=10$

$\therefore y=2$

Answer $x=-2, y=2$

c $x=5+\dfrac{y}{2}......(1)$

$-4x-3y=35....(2)$

Substitute equation (1) in equation (2)

$\therefore -4\left(5+\dfrac{y}{2}\right)-3y=35$

$\therefore -20-2y-3y=35$

$\therefore -5y=55$

$\therefore y=-11$

Substitute $y=-11$ in equation (1)

$\therefore x=5-\dfrac{11}{2}$

$\therefore x=\dfrac{10}{2}-\dfrac{11}{2}$

$\therefore x=-\dfrac{1}{2}$

Answer $x=-\dfrac{1}{2}, y=-11$

d $8x+3y=8......(1)$

$-2x+11y=\dfrac{35}{6}.....(2)$

Multiply equation (2) by 4

$\therefore -8x+44y=\dfrac{70}{3}....(3)$

Add equation (3) to equation (1)

$\therefore 47y=\dfrac{24}{3}+\dfrac{70}{3}$

$\therefore 47y=\dfrac{94}{3}$

$\therefore y=\dfrac{2}{3}$

Substitute $y=\dfrac{2}{3}$ in equation (1)

$\therefore 8x+2=8$

$\therefore 8x=6$

$\therefore x=\dfrac{3}{4}$

Answer $x=\dfrac{3}{4}, y=\dfrac{2}{3}$

e $\frac{x}{2}+\frac{y}{3}=8......(1)$

$\frac{x}{3}+\frac{y}{2}=7.....(2)$

Rearranging equation (1)

$\frac{3x+2y}{6}=8$

$\therefore 3x+2y=48....(3)$

Rearranging equation (2)

$\frac{2x+3y}{6}=7$

$\therefore 2x+3y=42....(4)$

Multiply equation (3) by 3 and equation (4) by 2

$9x+6y=144....(5)$

$4x+6y=84......(6)$

Equation (5) subtract equation (6)

$\therefore 5x=60$

$\therefore x=12$

Substitute $x=12$ in equation (1)

$\therefore 6+\frac{y}{3}=8$

$\therefore \frac{y}{3}=2$

$\therefore y=6$

Answer $x=12, y=6$

f $ax+by=4ab......(1)$

$ax-by=2ab.....(2)$

Add the equations together	Subtract the equations
$\therefore 2ax=6ab$	$\therefore 2by=2ab$
$\therefore x=3b$	$\therefore y=a$

Answer $x=3b, y=a$

19 a Let the numbers be x and $x+2$

Then $x+(x+2)=9((x+2)-x)$.

Solving,

$\therefore 2x+2=9(2)$

$\therefore 2x=16$

$\therefore x=8$

The numbers are 8 and 10.

b Let the number be x

Then $4(x-3)=72$.

Solving,

$x-3=18$

$\therefore x=21$

The number is 21.

c Let the three consecutive numbers be x, $x+1$ and $x+2$.

Then $x+(x+1)+(x+2)=36+\frac{1}{4}x$.

Solving,

$3x+3=36+\frac{x}{4}$

$\therefore 3x-\frac{x}{4}=33$

$\therefore \frac{12x-x}{4}=33$

$\therefore 11x=132$

$\therefore x=12$

The numbers are 12, 13 and 14.

d Let the width of the rectangle be x cm. Therefore, its length is $(2x+12)$ cm.

Perimeter is 48 cm.

$\therefore 2x+2(2x+12)=48$

$\therefore 2x+4x+24=48$

$\therefore 6x=24$

$\therefore x=4$

The width is 4 cm and the length is 20 cm.

e Let the width be x cm.

As length : width : height $=2:1:3$, the length is $2x$ cm and the height is $3x$ cm.

The sum of the lengths of the 12 edges of the rectangular prism is 360 cm.

Therefore, $4(2x)+4(x)+4(3x)=360$.

$\therefore 24x=360$

$\therefore x=15$

Since the height is $3x$ cm, the height is 45 cm.

20 a Let the distance from A to B be x km.

Speed (km/h)	Time (h)	Distance (km)
60	t	x
50	$t+\frac{1}{2}$	x

$\text{speed}=\frac{\text{distance}}{\text{time}}\Rightarrow \text{distance}=\text{speed}\times\text{time}.$

Therefore,

$x=60t..............(1)$

$x=50\left(t+\frac{1}{2}\right)....(2)$

Equating, $60t=50\left(t+\frac{1}{2}\right)$

$\therefore 60t=50t+25$

$\therefore 10t=25$

$\therefore t=2.5$

Hence,

$x=60\times 2.5$

$=150$

The value of t is 2.5 and the distance between A and B is 150 km.

b Time to travel A to B is t hours.

Time to return to A is rest time plus time to travel B to A.

Total time is $2t$ hours, so $2t=t+0.1t+\text{travel time}_{B\to A}$.

Therefore, the time to travel B to A is $0.9t$ hours.

Let the distance from A to B be x km and let the average speed of the cyclist from B to A be v km/h.

Average speed (km/h)	Time (h)	Distance (km)
u	t	x
v	$0.9t$	x

$x=ut....(1)$

$x=v(0.9t)....(2)$

Therefore $ut=v(0.9t)$

$\therefore v=\frac{ut}{0.9t}$

$\therefore v=\frac{u}{0.9}$

$\therefore v=\frac{10u}{9}$

The average speed for the return journey is $\frac{10}{9}u$ km/h.

21 **a** Let \$$a$ be the cost of an adult ticket and \$$c$ be the cost of a child's ticket.

$4a+5c=160....(1)$

$3a+7c=159....(2)$

$3\times$ equation (1) $\quad 12a+15c=480....(3)$

$4\times$ equation (2) $\quad 12a+28c=636....(4)$

Subtract equation (3) from equation (4)

$13c=156$

$\therefore c=12$

Substitute $c=12$ in equation (1)

$\therefore 4a+5(12)=160$

$\therefore 4a+60=160$

$\therefore 4a=100$

$\therefore a=25$

An adult ticket costs \$25 and a child's ticket costs \$12.

b Let the cost \$$C$ of the bill for the use of n units of electricity be $C=a+kn$ where \$$a$ is the fixed amount and \$$k$ the charge per unit.

$n=1428, C=235.90 \Rightarrow 235.90=a+1428k.......(1)$

$n=2240, C=353.64 \Rightarrow 353.64=a+2240k.......(2)$

equations $(2)-(1) \Rightarrow 117.74=812k$

$\therefore k=\dfrac{117.74}{812}$

$\therefore k=0.145$

Substitute in equation (1)

$\therefore 235.90=a+1428(0.145)$

$\therefore a=235.90-1428\times 0.145$

$\therefore a=28.84$

Therefore, $C=28.84+0.145n$.

When $n=3050$,

$C=28.84+0.145\times 3050$

$=471.09$

The bill will be \$471.09.

c Let the linear equation be $C=a+bF$ where a,b are constants.

$C=100, F=212 \Rightarrow 100=a+212b...(1)$

$C=0, F=32 \Rightarrow 0=a+32b...(2)$

equations $(1)-(2)$

$100=180b$

$\therefore b=\dfrac{100}{180}$

$=\dfrac{5}{9}$

Substitute in equation (2)

$\therefore 0=a+32\times\dfrac{5}{9}$

$\therefore a=-32\times\dfrac{5}{9}$

Therefore,

$C=-32\times\dfrac{5}{9}+\dfrac{5}{9}F$

$\therefore C=\dfrac{5}{9}(F-32)$

22 **a** Four tables with three or four people at each means the number of people lies between 12 and 16.

If n is the number of people, then $12\le n\le 16,\ n\in N$.

b Let \$$c$ be the fixed charge per person, where c can have a maximum of two decimal places and nc must exactly equal 418.50.

Since n is equal to either 12,13,14,15 or 16, use trial and error to obtain c from $c=\dfrac{418.50}{n}$ for each possibility.

n	12	13	14	15	16
$c=\dfrac{418.50}{n}$	34.875	32.1923..	29.8928..	27.9	26.15625

Only $n=15$ gives an appropriate value for c.

Therefore, 15 people attended the lunch at a fixed charge of \$27.90 per person.

c Let t be the number who drank tea and f be the number who drank coffee.

$t=\dfrac{1}{2}f....(1)$ and $t+f=15...(2)$ since there were 15 people

Substitute (1) in (2)

$\dfrac{1}{2}f+f=15$

$\therefore \dfrac{3f}{2}=15$

$\therefore 3f=30$

$\therefore f=10$

If $f=10$ then $t=5$.

10 people drank coffee and 5 drank tea.

23 **a** Use the Casio ClassPad equation solver: Interactive → equation/inequality → solve to obtain the solutions for parts a and b.

a $\{x=0\}$ (this gives $x=0$ as the solution).

b The inequality sign is in keyboard → mth → OPTN

The answer is given as $\{x>4\}$.

Therefore, $x>4$

c Use the Casio ClassPad simultaneous equation template: keyboard → $2D$ and select the template to obtain the solutions for parts c and d.

c $\{x=5, y=-1\}$

d $x=-82, y=-120$

24 **a** The independent variable is T so the model is of the form $C=aT+b$ where a,b are constants.

One method to use is the ClassPad statistics menu.

Enter the values of T as $List1$ and the corresponding values of C in $List2$. Select Calc → Linear Reg

This gives

$a=10$

$b=-97$

$r=1$

$r^2=1$

So the linear model is $C=10T-97$.

Alternatively, use the simultaneous equations template to obtain the values of a and b for the system:

$113=21a+b$

$173=27a+b$

b keyboard → $2D$ → select the simultaneous equations template. Use the abc Tab for the constants, entering any product of constants with a multiplication sign between them. For example, as $ax+b\times cy=c$.

$\left\{x=\dfrac{c}{a+b}, y=\dfrac{1}{a+b}\right\}$

Exercise 1.3 — Systems of 3 × 3 simultaneous linear equations

1 $5x-2y+z=3$

$3x+y+3z=5$

$6x+y-4z=62$

Using eqn 1 and eqn 2 multiplied by 2	Using eqn 2 and eqn 3
$5x-2y+z=3$	$3x+y+3z=5$
$6x+2y+6z=10$	$6x+y-4z=62$
Add	Subtract
$11x+7z=13$ …eqn4	$3x-7z=57$ …eqn5

Add eqn 4 and eqn 5

$14x=70$

$\therefore x=5$

Substitute $x=5$ in eqn 4

$55+7z=13$

$\therefore 7z=-42$

$\therefore z=-6$

Substitute $x=5$, $z=-6$ in eqn 2

$15+y-18=5$

$\therefore y=8$

Answer: $x=5$, $y=8$, $z=-6$

2 Rearranging the equations (not essential, however)

$x-4y+z=12$

$5x+3y-z=-4$

$12x+5y+z=5$

Add eqn 1 and eqn 2	Add eqn 2 and eqn 3
$6x-y=8$ …eqn4	$17x+8y=1$ …eqn 5

From eqn 4, $y=6x-8$. Substitute this into eqn 5

$17x+8(6x-8)=1$

$\therefore 17x+48x-64=1$

$\therefore 65x=65$

$\therefore x=1$

Hence $y=-2$.

Substitute $x=1$, $y=-2$ in eqn 1

$1+8+z=12$

$\therefore z=3$

Answer: $x=1$, $y=-2$, $z=3$

3 $2x+3y-z=3$....(1)

$5x+y+z=15$....(2)

$4x-6y+z=6$....(3)

Eliminate z

equations $(1)+(2)$ $7x+4y=18$....(4)

equations $(1)+(3)$ $6x-3y=9$....(5)

Multiply equation (5) by $\frac{4}{3}$

$\therefore 8x-4y=12$.....(6)

equations $(4)+(6)$

$15x=30$

$\therefore x=2$

Substitute $x=2$ in equation (4)

$\therefore 14+4y=18$

$\therefore 4y=4$

$\therefore y=1$

Substitute $x=2$ and $y=1$ in equation (2)

$\therefore 10+1+z=15$

$\therefore z=4$

Answer $x=2, y=1, z=4$

4 $x-2y+z=-1$....(1)

$x+4y+3z=9$....(2)

$x-7y-z=-9$....(3)

Eliminate x

equations $(2)-(1)$ $6y+2z=10$....(4)

equations $(1)-(3)$ $5y+2z=8$....(5)

equations $(5)-(4)$ $y=2$

Substitute $y=2$ in equation (5)

$\therefore 10+2z=8$

$\therefore 2z=-2$

$\therefore z=-1$

Substitute $y=2$ and $z=-1$ in equation (1)

$\therefore x-4-1=-1$

$\therefore x=4$

Answer $x=4, y=2, z=-1$

5 $2x-y+z=-19$....(1)

$3x+y+9z=-1$....(2)

$4x+3y-5z=-5$....(3)

Eliminate y

equations $(1)+(2)$ $5x+10z=-20$....(4)

$3\times$ equation $(1)+$ equation (3) $10x-2z=-62$....(5)

Divide equation (4) by 5 $\therefore x+2z=-4$.....(6)

equations $(5)+(6)$

$11x=-66$

$\therefore x=-6$

Substitute $x=-6$ in equation (6)

$\therefore -6+2z=-4$

$\therefore 2z=2$

$\therefore z=1$

Substitute $x=-6$ and $z=1$ in equation (2)

$\therefore -18+y+9=-1$

$\therefore y=8$

Answer $x=-6, y=8, z=1$

6 $2x+3y-4z=-29$....(1)

$-5x-2y+4z=40$....(2)

$7x+5y+z=21$....(3)

Eliminate z

equations $(1)+(2)$ $-3x+y=11$....(4)

equation $(1)+4\times$ equation (3) $30x+23y=55$....(5)

Multiply equation (4) by 10

$\therefore -30x+10y=110$.....(6)

equations $(5)+(6)$

$33y=165$

$\therefore y=5$

Substitute $y=5$ in equation (4)

$\therefore -3x+5=11$

$\therefore -3x=6$

$\therefore x=-2$

Substitute $x=-2$ and $y=5$ in equation (3)

$\therefore -14+25+z=21$

$\therefore z=10$

Answer $x=-2, y=5, z=10$

7 $3x-2y+z=8$....(1)

$3x+6y+z=32$....(2)

$3x+4y-5z=14$....(3)

Eliminate x

equations $(2)-(1)$ $8y=24\Rightarrow y=3$

equations $(3)-(1)$ $6y-6z=6\Rightarrow y-z=1$....(4)

Substitute $y=3$ in equation (4)
$\therefore 3-z=1$
$\therefore z=2$
Substitute $y=3$ and $z=2$ in equation (1)
$\therefore 3x-6+2=8$
$\therefore 3x=12$
$\therefore x=4$
Answer $x=4, y=3, z=2$

8 $y=3x-5....(1)$
$\frac{x-z}{2}-y+10=0....(2)$
$9x+2y+z=0....(3)$
Substitute equation (1) into equations (2) and (3)
$(2) \Rightarrow \frac{x-z}{2}-(3x-5)+10=0$
$\therefore \frac{x-z}{2}=3x-15$
$\therefore x-z=6x-30$
$\therefore z=30-5x.....(4)$
$(3) \Rightarrow 9x+2(3x-5)+z=0$
$\therefore 9x+6x-10+z=0$
$\therefore z=10-15x.....(5)$
Substitute equation (4) into equation (5)
$\therefore 30-5x=10-15x$
$\therefore 10x=-20$
$\therefore x=-2$
Substitute $x=-2$ into equation (5)
$\therefore z=10+30$
$\therefore z=40$
Substitute $x=-2$ into equation (1)
$\therefore y=-6-5$
$\therefore y=-11$
Answer $x=-2, y=-11, z=40$

9 Let each adult, concession and child's ticket cost \$$a$, \$$b$ and \$$c$ respectively.
$3a+2b+3c=96....(1)$
$2a+b+6c=100....(2)$
$a+4b+c=72....(3)$
Eliminate c
2 × equation (1) – equation (2) $4a+3b=92....(4)$
3 × equation (3) – equation (1) $10b=3\times 72-96$
$\therefore 10b=120$
$\therefore b=12$
Substitute $b=12$ in equation (4)
$\therefore 4a+36=92$
$\therefore 4a=56$
$\therefore a=14$
Substitute $b=12$ and $a=14$ in equation (3)
$\therefore 14+48+c=72$
$\therefore c=10$
An adult ticket costs \$14, a concession ticket costs \$12 and a children's ticket costs \$10.

10 Let the hourly rate of pay for Agnes, Bjork and Chi be \$$a$, \$$b$ and \$$c$ respectively.
$2a+3b+4c=194....(1)$
$4a+2b+3c=191....(2)$
$2a+5b+2c=180....(3)$
Eliminate a
equation (1) – equation (3) $-2b+2c=14$
$\therefore -b+c=7....(4)$
2 × equation (3) – equation (2) $8b+c=360-191$
$\therefore 8b+c=169....(5)$
equation (5) – equation (4)
$\therefore 9b=162$
$\therefore b=18$
Substitute $b=18$ in equation (4)
$\therefore -18+c=7$
$\therefore c=25$
Substitute $b=18$ and $c=25$ in equation (3)
$\therefore 2a+90+50=180$
$\therefore 2a=40$
$\therefore a=20$
Agnes earns \$20 / hour, Bjork earns \$18 / *hour* and Chi earns \$25 / *hour*.

11 Let the amount of each of the Supplements X, Y and Z used be x kg, y kg and z kg respectively.
Unsaturated fat: $0.06x+0.10y+0.08z=6.8....(1)$
Saturated fat: $0.03x+0.04y+0.04z=3.1....(2)$
Trans fat: $0.01x+0.02y+0.03z=1.4....(3)$
Eliminate y
equation (2) – 2 × equation (3) $0.01x-0.02z=0.3....(4)$
equation (1) – 5 × equation (3) $0.01x-0.07z=-0.2....(5)$
equation (4) – equation (5)
$\therefore 0.05z=0.5$
$\therefore z=10$
Substitute $z=10$ in equation (4)
$\therefore 0.01x-0.2=0.3$
$\therefore 0.01x=0.5$
$\therefore x=50$
Substitute $z=10$ and $x=50$ in equation (1)
$\therefore 3+0.1y+0.8=6.8$
$\therefore 0.1y=3$
$\therefore y=30$
The food compound needs to use 50 kg of Supplement X, 30 kg of Supplement Y and 10 kg of Supplement Z.

12 Let the student use x fifty cent coins, y twenty cent coins and z ten cent coins.
Student spends \$4.20: $0.50x+0.20y+0.10z=4.20$
$\therefore 5x+2y+z=42....(1)$
The number of twenty cent coins: $y=\frac{1}{2}z+4x$
$\therefore 2y=z+8x$
$\therefore 8x-2y+z=0....(2)$
Total number of coins: $x+y+z=22....(3)$
Eliminate z
equation (1) – equation (2) $-3x+4y=42....(4)$
equation (1) – equation (3) $4x+y=20....(5)$
equation (4) – 4 × equation (5)
$-19x=-38$
$\therefore x=2$
Substitute $x=2$ in equation (5)
$\therefore 8+y=20$
$\therefore y=12$
Substitute $x=2$ and $y=12$ in equation (3)
$\therefore 2+12+z=22$
$\therefore z=8$
The student uses 2 fifty cent coins, 12 twenty cent coins and 8 ten cent coins.

13 Use the Casio ClassPad simultaneous equation template: keyboard → $2D$ and select the template, tapping twice in order for the template to allow for a 3×3 system of equations.
$\{x=3, y=1.5, z=-2.6\}$

14 **a** $z=a+bx+cy$
Using the given data
$23=a+30b+320c$
$28=a+50b+360c$
$30=a+40b+400c$
Solve on CAS to obtain $a=-4, b=\dfrac{1}{10}, c=\dfrac{3}{40}$
$\therefore z=-4+0.1x+0.075y$

b When $x=40$ and $y=200$,
$z=-4+0.1(40)+0.075(200)$
$\therefore z=15$
The yield is 15 kg/hectare of zucchini.

Exercise 1.4 — Linear graphs and their equations

1 Points are $(-3,0)$ and $(0,-4)$

$$m=\frac{-4-0}{0-(-3)}$$

$$\therefore m=-\frac{4}{3}$$

2 Gradient of line through points (a,b) and $(-b,-a)$

$$m=\frac{-a-b}{-b-a}$$

$\therefore m=1$

Gradient of line through points $(-c,d)$ and $(-d,c)$

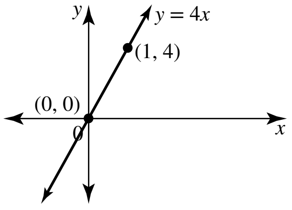
$$m=\frac{c-d}{-d-(-c)}$$

$$\therefore m=\frac{c-d}{-d+c}$$

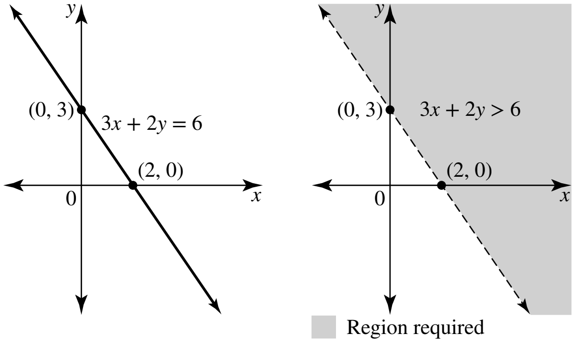
$\therefore m=1$

As the gradients are the same, the lines are parallel

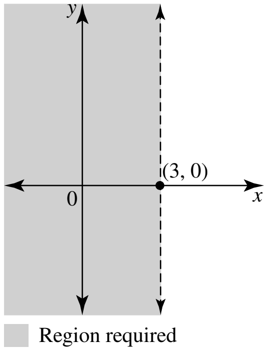
3 (a)



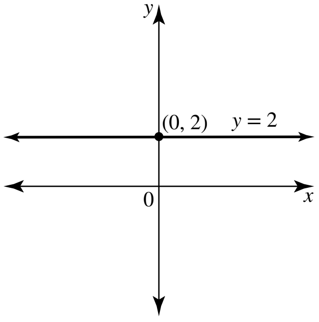
(b)



(c)



(d)



4 $y=4.4-10x$

Substitute $x=-2.2, y=17.6$

LHS 17.6 RHS $4.4-10(-2.2)=27$

Since $17.6<27$, the point lies in the region $y<4.4-10x$

Therefore the point lies below the line.

5 **a** gradient −2, point $(-8,3)$

$y-3=-2(x+8)$

$\therefore y=-2x-13$

b points $(4,-1)$ and $(-3,1)$

$m=\dfrac{1+1}{-3-4}$

$\therefore m=-\dfrac{2}{7}$

Equation of the line

$y+1=-\dfrac{2}{7}(x-4)$

$\therefore 7y+7=-2x+8$

$\therefore 7y+2x=1$

c The gradient of the line is $-\frac{6}{4}=-\frac{3}{2}$ and its y intercept gives $c=6$. Therefore the equation is $y=-\frac{3}{2}x+6$

d $6y-5x-18=0$

$\therefore 6y=5x+18$

$\therefore y=\dfrac{5x}{6}+3$

$m=\dfrac{5}{6},\ c=3$

Gradient is $\frac{5}{6}$, co-ordinates of y intercept are $(0,3)$

6 Horizontal lines are parallel to the x axis.

The horizontal line through the point $(2,10)$ has the equation $y=10$

7 **a** $(-3,8),(-7,18)$

$m=\dfrac{y_2-y_1}{x_2-x_1}$

$\therefore m=\dfrac{18-8}{-7-(-3)}$

$=\dfrac{10}{-4}$

$=-2.5$

b $(0,-4),(12,56)$

$\therefore m=\dfrac{56-(-4)}{12-0}$

$=\dfrac{60}{12}$

$=5$

c $(-2,-5),(10,-5)$ Note that the line joining these points is horizontal.

$\therefore m=\dfrac{-5-(-5)}{10-(-2)}$

$=\dfrac{0}{12}$

$=0$

d $(3,-3),(3,15)$ Note that the line joining these points is vertical.

$\therefore m=\dfrac{15-(-3)}{3-3}$

$=\dfrac{18}{0}$

Therefore the gradient is undefined.

8 **a** **i** $y=kx$ where k is the constant of proportionality

When $x=15,\ y=52.5$,

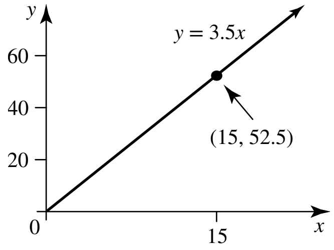
$\therefore 52.5=15k$

$\therefore k=\dfrac{52.5}{15}$

$\therefore k=3.5$

The rule is $y=3.5x$

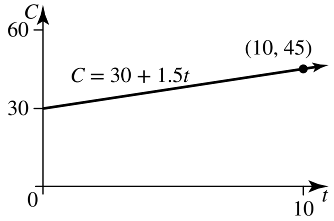
ii



gradient is 3.5

b **i** $C=30+1.5t$

ii



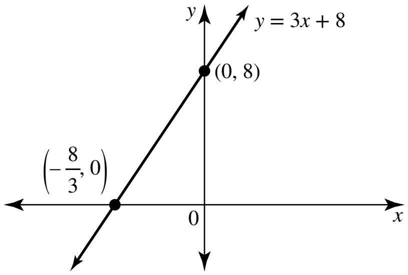
gradient is 1.5

9 **a** $y=3x+8$

y intercept $(0,8)$ x intercept: When $y=0,\ 3x+8=0$

$\therefore x=-\dfrac{8}{3}$

x intercept $\left(-\dfrac{8}{3},0\right)$



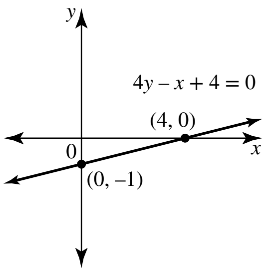
b $4y-x+4=0$

y intercept When $x=0,\ 4y+4=0\Rightarrow y=-1$

y intercept $(0,-1)$

x intercept: When $y=0,\ -x+4=0\Rightarrow x=4$

x intercept $(4,0)$



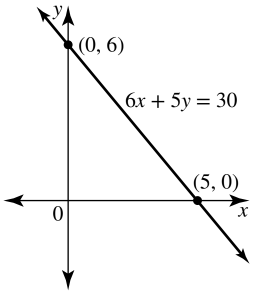
c $6x+5y=30$

y intercept When $x=0,\ 5y=30\Rightarrow y=6$

y intercept $(0,6)$

x intercept: When $y=0,\ 6x=30\Rightarrow x=5$

x intercept $(5,0)$

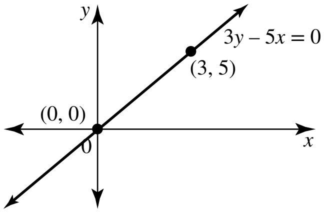


d $3y - 5x = 0$

When $x = 0$, $y = 0$ so line contains the origin $(0,0)$

Second point: Let $x = 3$ so $3y - 15 = 0 \Rightarrow y = 5$

A second point is $(3,5)$



e $\frac{x}{2} - \frac{3y}{4} = 6$

y intercept When $x = 0$, $-\frac{3y}{4} = 6$

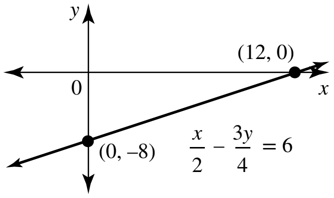
$\therefore -3y = 24$

$\therefore y = -8$

y intercept $(0,-8)$

x intercept: When $y = 0$, $\frac{x}{2} = 6 \Rightarrow x = 12$

x intercept $(12,0)$

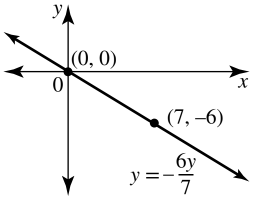


f $y = -\frac{6x}{7}$

When $x = 0$, $y = 0$ so line contains the origin $(0,0)$

Second point: Let $x = 7 \Rightarrow y = -6$

A second point is $(7,-6)$



10 a $y \le 3 - 3x$ is the region below the closed line $y = 3 - 3x$

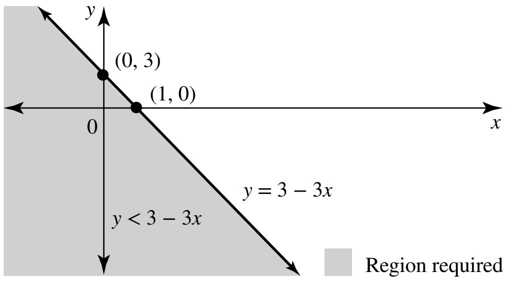
For $y = 3 - 3x$,

y intercept $(0,3)$

x intercept: When $y = 0$, $0 = 3 - 3x \Rightarrow x = 1$

x intercept $(1,0)$

The region is shaded in the diagram



b $4x + y > 12$ is the region above the open line $4x + y = 12$

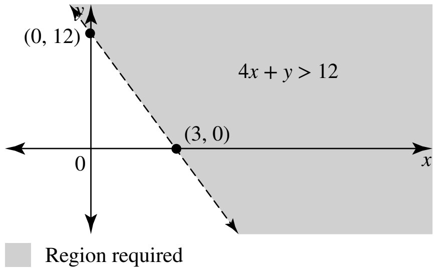
For the line $4x + y = 12$,

y intercept $(0,12)$

x intercept: When $y = 0$, $4x = 12 \Rightarrow x = 3$

x intercept $(3,0)$

The region is shaded in the diagram



c $5x - 2y \le 8$

Rearranging,

$-2y \le 8 - 5x$

$\therefore y \ge \frac{8 - 2x}{-2}$

Therefore, the region above the closed line $5x - 2y = 8$ is required

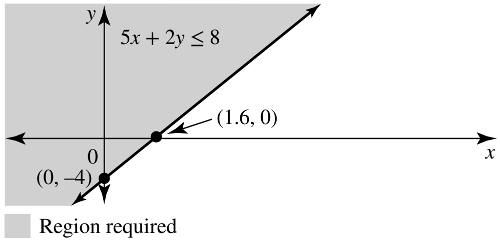
For the line $5x - 2y = 8$.

y intercept When $x = 0$, $-2y = 8 \Rightarrow y = -4$

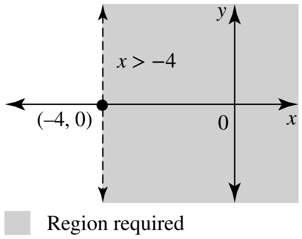
y intercept $(0,-4)$

x intercept: When $y = 0$, $5x = 8 \Rightarrow x = \frac{8}{5}$

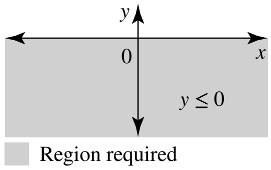
x intercept $\left(\frac{8}{5},0\right)$



d $x > -4$ is the open region to the right of the vertical line $x = -4$.



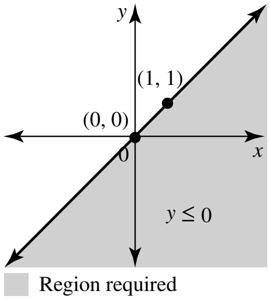
e $y \le 0$ is the region below the closed horizontal line $y = 0$ (the x axis)



f $y \le x$ is the region below the closed line $y = x$.

The line $y = x$ passes through the origin $(0,0)$.

For a second point, let $x = 1$ which means $y = 1$ so line passes through $(1,1)$.



11 a $y - y_1 = m(x - x_1), m = -5$ and point $(7, 2)$

$\therefore y - 2 = -5(x - 7)$

$\therefore y = -5x + 37$

b $y - y_1 = m(x - x_1), m = \frac{2}{3}$ and point $(-4, -6)$

$\therefore y + 6 = \frac{2}{3}(x + 4)$

$\therefore 3(y + 6) = 2(x + 4)$

$\therefore 3y + 18 = 2x + 8$

$\therefore 3y - 2x + 10 = 0$

c $y = mx + c, m = -\frac{7}{4}, c = -9$

$\therefore y = -\frac{7}{4}x - 9$

$\therefore 4y = -7x - 36$

$\therefore 4y + 7x + 36 = 0$

d $y - y_1 = m(x - x_1), m = -0.8$ and point $(0.5, -0.2)$

$\therefore y + 0.2 = -0.8(x - 0.5)$

$\therefore y = -0.8x + 0.4 - 0.2$

$\therefore y = -0.8x + 0.2$

e Points $(-1, 8), (-4, -2)$

gradient: $m = \frac{-2 - 8}{-4 + 1} = \frac{10}{3}$

equation $y - y_1 = m(x - x_1), m = \frac{10}{3}$ and point $(-1, 8)$

$\therefore y - 8 = \frac{10}{3}(x + 1)$

$\therefore 3(y - 8) = 10(x + 1)$

$\therefore 3y - 24 = 10x + 10$

$\therefore 3y - 10x = 34$

f Points $(0, 10), (10, -10)$

gradient: $m = \frac{-10 - 10}{10 - 0} = -2$

equation $y = mx + c, m = -2, c = 10$

$\therefore y = -2x + 10$

12 a i $m = \frac{\text{rise}}{\text{run}}$

$\therefore m = \frac{5}{4}$

Line contains the origin so $c = 0$

Equation is $y = \frac{5x}{4}$.

ii $m = \frac{\text{rise}}{\text{run}}$

$\therefore m = \frac{-9}{3}$

$\therefore m = -3$

Line cuts y axis at $(0, 9)$ so $c = 9$

Equation is $y = -3x + 9$.

iii $m = \frac{\text{rise}}{\text{run}}$

$\therefore m = \frac{2}{3}$

Line cuts y axis at $(0, -2)$ so $c = -2$

Equation is $y = \frac{2x}{3} - 2$.

iv $m = \frac{\text{rise}}{\text{run}}$

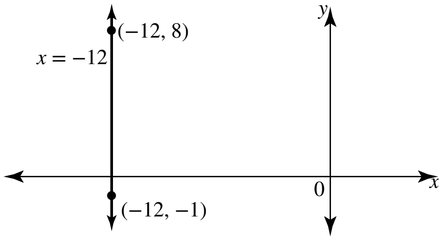
$\therefore m = \frac{-1}{2}$

Line cuts y axis at $(0, -1)$ so $c = -1$

Equation is $y = -\frac{x}{2} - 1$.

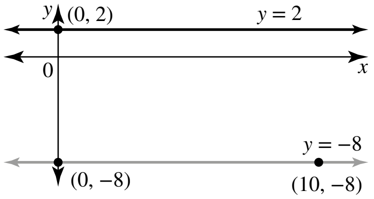
b Points $(-12, 8), (-12, -1)$

The vertical line with equation $x = -12$ contains these two points.



c The line $y = 2$ is horizontal so any parallel line to it must also be horizontal.

The equation of the horizontal line containing $(10, -8)$ is $y = -8$.



13 a $4x + 5y = 20$

Rearrange to the $y = mx + c$ form.

$\therefore 5y = -4x + 20$

$\therefore y = \frac{-4x}{5} + \frac{20}{5}$

$\therefore y = -\frac{4}{5}x + 4$

$m = -\frac{4}{5}, c = 4$ so the gradient is $-\frac{4}{5}$ and the y intercept is $(0, 4)$.

b $\frac{2x}{3} - \frac{y}{4} = -5$

Rearrange to the $y = mx + c$ form.

$\therefore \frac{2x}{3} + 5 = \frac{y}{4}$

$\therefore y = 4\left(\frac{2x}{3} + 5\right)$

$\therefore y = \frac{8}{3}x + 20$

$m = \frac{8}{3}, c = 20$ so the gradient is $\frac{8}{3}$ and the y intercept is $(0, 20)$.

c $x - 6y + 9 = 0$

Rearrange to the $y = mx + c$ form.

$\therefore x + 9 = 6y$

$\therefore y = \frac{x}{6} + \frac{9}{6}$

$\therefore y = \frac{1}{6}x + \frac{3}{2}$

$m=\frac{1}{6}, c=\frac{3}{2}$ so the gradient is $\frac{1}{6}$ and the y intercept is $\left(0,\frac{3}{2}\right)$.

d $2y-3=0$

$\therefore y=\frac{3}{2}$ which is a horizontal line.

gradient is 0 and y intercept is $\left(0,\frac{3}{2}\right)$.

14 a $5y=-3x+4$

The point $(2a,2-a)$ lies on the line.

Substitute $x=2a, y=2-a$ in $5y=-3x+4$.

$\therefore 5(2-a)=-3(2a)+4$

$\therefore 10-5a=-6a+4$

$\therefore a=-6$

b $7y-3x=25$

Substitute the point $(-22,13)$

LHS $=7(13)-3(-22)$ RHS $=25$

$=91+66$

$=157$

Since LHS > RHS the point lies above the line.

c Points $(p,q),(-p,-q)$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$\therefore m=\frac{-q-q}{-p-p}$$

$$=\frac{-2q}{-2p}$$

$$=\frac{q}{p}$$

equation $y-y_1=m(x-x_1)$, $m=\frac{q}{p}$ and point (p,q)

$$\therefore y-q=\frac{q}{p}(x-p)$$

$$\therefore p(y-q)=q(x-p)$$

$$\therefore py-pq=qx-qp$$

$$\therefore py=qx$$

$$\therefore y=\frac{qx}{p}$$

15 a $p=180-kt$ where k is the constant of proportionality.

b The girl's pulse rate has to drop from 180 beats per minute to 60 beats per minute, a drop of 120 beats. At the rate of decrease of 10 beats every minute, this will take 12 minutes for her pulse rate to return to its normal rest rate.

c $p=180-kt$

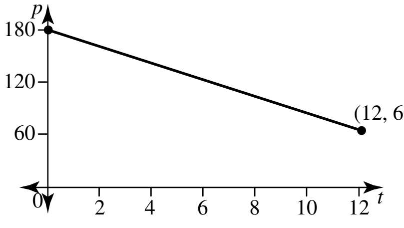
Substitute $t=12, p=60$

$\therefore 60=180-12k$

$\therefore 12k=120$

$\therefore k=10$

$\therefore p=180-10t$



The graph is drawn over the interval $0\le t\le 12$.

d The gradient measures the rate at which the pulse rate decreases, so the gradient is -10.

16 $3x-2y=k, k\in R$

a Rearrange to the $y=mx+c$ form.

$\therefore 3x-k=2y$

$\therefore y=\frac{3x}{2}-\frac{k}{2}$

The gradient of each member in this family of lines is $\frac{3}{2}$.

b Let $x=k$

$\therefore 3k-2y=k$

$\therefore 3k-k=2y$

$\therefore 2y=2k$

$\therefore y=k$

Therefore every line contains a point (k,k).

c The point $(-3,-8)$ lies on the line. Substitute $x=-3, y=-8$ in the equation $3x-2y=k$ to determine the value of k.

$\therefore -9+16=k$

$\therefore k=7$

The line $3x-2y=7$ contains the point $(-3,-8)$.

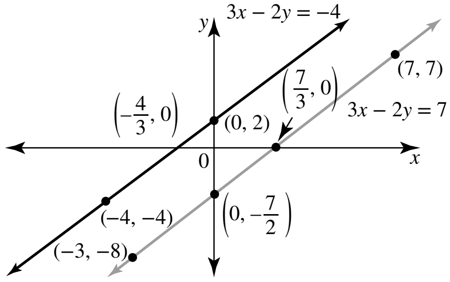
d The point $(0,2)$ lies on the line. Substitute $x=0, y=2$ in the equation $3x-2y=k$.

$0-4=k$

$\therefore k=-4$

The line $3x-2y=-4$ has a y intercept of 2.

e $3x-2y=7$	$3x-2y=-4$
y intercept: When $x=0$, $y=-\frac{7}{2}$	y intercept: $(0,2)$
x intercept: When $y=0$, $x=\frac{7}{3}$	x intercept: When $y=0$, $x=-\frac{4}{3}$
$k=7$, the point $(7,7)$ is on the graph	$k=-4$, the point $(-4,-4)$ is on the graph point $(-3,-8)$ is also on the graph



f The region between the lines lies below $3x-2y=-4$ and above $3x-2y=7$.

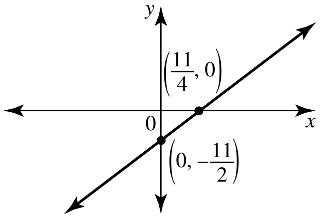
Due to the negative coefficient of y in the equations, this means $3x-2y\ge -4$ and $3x-2y\le 7$.

Therefore, $-4\le 3x-2y\le 7$.

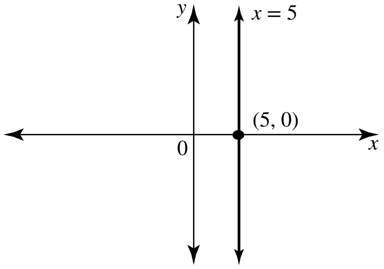
17 a To sketch the line $2y-4x=-11$ one method is to type the equation in the Main Home screen, tap the graphing window so this screen is split in two areas and then highlight and drag the equation to the graphing area. Alternatively, open the Graph and Tab menu.

The equation $2y-4x=-11$ needs to be expressed with y as the subject. This can be done using solve $2y-4x=-11$ on main screen for y (or do it yourself).

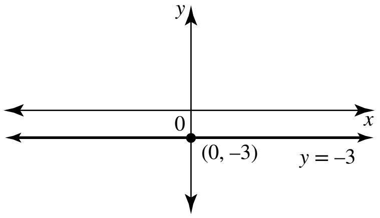
Enter the equation as $y_1=2x-\frac{11}{2}$ and tap the box to the left of y_1 to signal this is the graph to be sketched. Tap the graph symbol to activate the sketch, adjusting the window view if necessary.



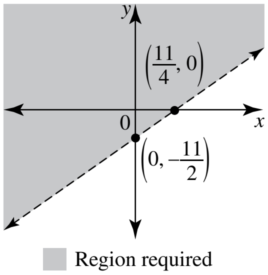
b To sketch a vertical line, tap Type and select $x =$ Type. The equation $x = 5$ can then be entered and the graph obtained.



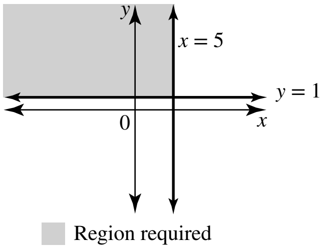
c Use $y =$ Type and enter the equation $y = 3$ and proceed to obtain the graph.



18 **a** To sketch the open region defined by the inequation $2y - 4x > -11$, change to Type to $y >$.
The inequation will need to be rearranged to have y as the subject before it can be entered.
Enter as $y > 2x - \frac{11}{2}$.
The calculator shades the region required.



b $x \le 5$. Alter TYPE to $x \le$ and enter the inequation at x_1.
$y \ge 1$ At x_2 position, alter TYPE to $y \ge$ and enter the inequation at what will now be y_2.
Tap both left end boxes and graph.
The calculator will shade the region where both inequations hold.



Exercise 1.5 — Intersections of lines and their applications

1 revenue $d_R = 25n$ and cost $d_C = 260 + 12n$

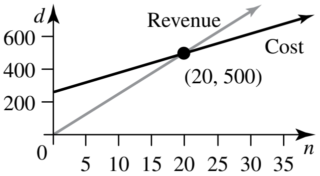
a At point of intersection (break-even point), $d_R = d_C$

$\therefore 25n = 260 + 12n$

$\therefore 13n = 260$

$\therefore n = 20$

If $n = 20, d = 500$, so the point of intersection is (20,500)
Revenue: (0,0),(20,500) Cost: (0,260),(20,500)



b As the gradient of the revenue graph is larger than the gradient of the cost graph, if $n > 20, d_R > d_C$.
Therefore at least 21 items must be sold for a profit to be made.
Alternative method is to solve the inequation $d_R > d_C$

$\therefore 25n > 260 + 12n$

$\therefore 13n > 260$

$\therefore n > 20$

2 $3x - 2y = 15 \ldots \text{eqn}1$
$x + 4y = 54 \ldots \text{eqn}2$
$2 \times (1) + (2)$
$6x - 4y = 30$
$x + 4y = 54$
$\therefore 7x = 84$
$\therefore x = 12$
Substitute in eqn2
$\therefore 12 + 4y = 54$
$\therefore 4y = 42$
$\therefore y = 10.5$
Point of intersection is (12,10.5)

3 $2mx + 3y = 2m$
$4x + y = 5$
For no solutions, the two equations must represent parallel lines and therefore have the same gradient.
The system becomes

$2mx + 3y = 2m \Rightarrow y = -\frac{2mx}{3} + \frac{2m}{3}$

$4x + y = 5 \Rightarrow y = -4x + 5$

Hence, equating gradients $-\frac{2m}{3} = -4$

$\therefore m = 6$

Checking y intercepts are not equal, $\frac{2m}{3} = 4 \ne 5$

So lines are parallel, not the same line.
Therefore $m = 6$ for the system not to have any solutions

4 $ax + y = b$
$3x - 2y = 4$
For infinitely many solutions, the two equations must be identical.
The system becomes
$y = -ax + b$

$y = \frac{3x}{2} - 2$

Equating gradients and y intercepts,

$-a=\frac{3}{2}$

$\therefore a=-1.5$ and $b=-2$ for infinite solutions

5 $2x+3y=0$
$x-8y=19$

Multiply second equation by 2, then subtract from first equation

$$2x+3y=0$$
$$2x-16y=38$$
$$\therefore 19y=-38$$
$$\therefore y=-2$$
$$\therefore x=3$$

Two of the lines intersect at $(3,-2)$. Test if this point lies on the third line $9x+5y=17$

$$\text{LHS}=9(3)+5(-2)$$
$$=27-10$$
$$=17$$
$$=\text{RHS}$$

Therefore the three lines are concurrent since they meet at the point $(3,-2)$

6 Point of intersection of $x+4y=13$ and $5x-4y=17$

$$x+4y=13$$
$$5x-4y=17$$

Add

$$6x=30$$
$$\therefore x=5$$
$$\therefore 5+4y=13$$
$$\therefore 4y=8$$
$$\therefore y=2$$

This point $(5,-2)$ must lie on the third line $-3x+ay=5$ since lines are concurrent

$$-3(5)+a(2)=5$$
$$\therefore 2a=20$$
$$\therefore a=10$$

7 a Use simultaneous equations to obtain the point of intersection.

$4x-3y=13....(1)$

$2y-6x=-7....(2)$

Eliminate y

$2\times$ equation (1) $8x-6y=26....(3)$

$3\times$ equation (2) $-18x+6y=-21....(4)$

Add equations (3) and (4)

$$\therefore -10x=5$$
$$\therefore x=-\frac{5}{10}$$
$$\therefore x=-\frac{1}{2}$$

Substitute $x=-\frac{1}{2}$ in equation (2)

$$\therefore 2y+3=-7$$
$$\therefore 2y=-10$$
$$\therefore y=-5$$

The point of intersection is $\left(-\frac{1}{2},-5\right)$.

b $y=\frac{3x}{4}-9....(1)$

$x+5y+7=0....(2)$

Substitute equation (1) in equation (2)

$$\therefore x+5\left(\frac{3x}{4}-9\right)+7=0$$
$$\therefore x+\frac{15x}{4}-45+7=0$$
$$\therefore \frac{19x}{4}=38$$
$$\therefore x=38\times\frac{4}{19}$$
$$\therefore x=8$$

Substitute $x=8$ in equation (1)

$$\therefore y=\frac{3}{4}\times 8-9$$
$$\therefore y=-3$$

The point of intersection is $(8,-3)$.

c The horizontal line $y=-5$ and the vertical line $x=7$ would intersect at the point $(7,-5)$.

8 The equations of each line need to be formed before using simultaneous equations to obtain their point of intersection.

Line 1: $y-y_1=m(x-x_1)$, $m=-2$ and point is $(4,-8)$

$\therefore y+8=-2(x-4)$

$\therefore y=-2x....(1)$

Line 2: $y=mx+c$, $m=3, c=5$

$\therefore y=3x+5....(2)$

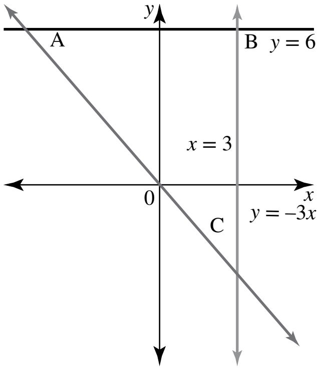
At intersection at point Q,

$$-2x=3x+5$$
$$\therefore -5x=5$$
$$\therefore x=-1$$

When $x=-1$, equation (1) gives $y=2$

The point Q has co-ordinates $(-1,2)$.

9 a



Let the triangle enclosed by the horizontal, vertical and oblique line through the origin have vertices A, B and C.

Point B has co-ordinates $(3,6)$.

Point A is the intersection of $y=-3x$ and $y=6$

$\therefore 6=-3x$

$\therefore x=-2$

A has co-ordinates $(-2,6)$.

Point C is the intersection of $y=-3x$ and $x=3$

$\therefore y=-3\times 3$

$\therefore y=-9$

C has co-ordinates $(3,-9)$.

The vertices of the triangle are $(-2,6)$, $(3,6)$ and $(3,-9)$.

b The triangle is right angled at B with AB of length 5 units and BC of length 15 units.

Area of triangle ABC: $A=\frac{1}{2}bh, b=5, h=15$

$\therefore A = \frac{1}{2} \times 5 \times 15$

$= 37.5$

The area of the triangle is 37.5 square units.

10 The lines will not intersect if they are parallel, in which case they will have the same gradients but different y intercepts.

Rearranging $2x + 3y = 23$,

$3y = -2x + 23$

$\therefore y = \frac{-2}{3}x + \frac{23}{3}$

$\therefore m_1 = -\frac{2}{3},\ c_1 = \frac{23}{3}$

Rearranging $7x + py = 8$,

$py = -7x + 8$

$\therefore y = \frac{-7}{p}x + \frac{8}{p}$

$\therefore m_2 = -\frac{7}{p},\ c_2 = \frac{8}{p}$

Same gradients

$\therefore -\frac{2}{3} = -\frac{7}{p}$

$\therefore 2p = 21$

$\therefore p = \frac{21}{2}$

With $p = \frac{21}{2}, c_2 = \frac{8}{21/2}$ and $c_1 = \frac{23}{3}$

$\therefore c_2 = \frac{16}{21}$

$\therefore c_2 \neq c_1$

If $p = \frac{21}{2}$, the lines will not intersect.

11 **a** $px + 5y = q$

$\therefore 5y = -px + q$

$\therefore y = -\frac{px}{5} + \frac{q}{5}$

$3x - qy = 5q$

$\therefore 3x - 5q = qy$

$\therefore y = \frac{3x}{q} - 5$

b For infinitely many solutions the two equations must be identical. This means they will have the same gradients and the same y intercepts.

Equating gradients $-\frac{p}{5} = \frac{3}{q}$.....(1)

Equating y intercepts $\frac{q}{5} = -5$.....(2)

Equation (2) gives $q = -25$. Substitute this in equation (1)

$\therefore -\frac{p}{5} = \frac{3}{25}$

$\therefore p = \frac{3}{25} \times -5$

$\therefore p = -\frac{3}{5}$

For infinitely many solutions, $p = -\frac{3}{5}$ and $q = -25$.

c Two lines will intersect if they do not have the same gradients.

Therefore, $-\frac{p}{5} \neq \frac{3}{q}$ or $pq \neq -15$.

12 **a** If the three lines are concurrent they must intersect at the same point.

$3x - y + 3 = 0$....(1)

$5x + 2y + 16 = 0$....(2)

$9x - 5y + 3 = 0$....(3)

Point of intersection of lines given by equations (1) and (2).

$2 \times$ equation (1) + equation (2)

$11x + 22 = 0$

$\therefore x = -2$

Substitute $x = -2$ in equation (1)

$\therefore -6 - y + 3 = 0$

$\therefore y = -3$

Therefore the lines (1) and (2) intersect at $(-2, -3)$.

Test if this point satisfies equation (3)

$\text{LHS} = 9x - 5y + 3$

$= 9 \times -2 - 5 \times -3 + 3$

$= -18 + 15 + 3$

$= 0$

$= \text{RHS}$

Since all three lines contain the point $(-2, -3)$, the lines are concurrent and their point of concurrency is $(-2, -3)$.

b Obtain the point of intersection of two of the lines.

$x + 4y = 9$....(1)

$3x - 2y = -1$....(2)

equation (1) + $2 \times$ equation (2)

$7x = 7$

$\therefore x = 1$

Substitute $x = 1$ in equation (1)

$\therefore 1 + 4y = 9$

$\therefore 4y = 8$

$\therefore y = 2$

The two lines intersect at $(1, 2)$.

For the three lines not to be concurrent, the point $(1, 2)$ cannot be on the line $4x + 3y = d$.

$\therefore d \neq 4 \times 1 + 3 \times 2$

$\therefore d \neq 10$

The lines will not be concurrent if d is any real number except 10.

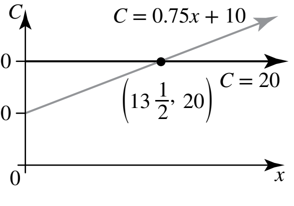
13 **a** Let cost be $\$C$ for a distance of x km.

"Pedal on": $C = 10 + 0.75x$

"Bikes r gr8": $C = 20$

b Points on line $C = 10 + 0.75x$ could be $(0, 10)$ and $(4, 13)$

$C = 20$ is a horizontal line.



c When costs are equal, $10 + 0.75x = 20$

$\therefore 0.75x = 10$

$\therefore \frac{3}{4}x = 10$

$\therefore x = 10 \times \frac{4}{3}$

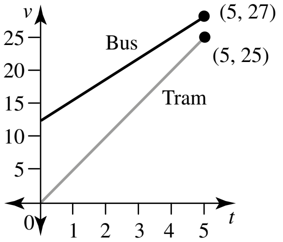
$\therefore x = \frac{40}{3}$

After $13\frac{1}{3}$ km, the costs are equal.

d Shay needs to estimate whether the number of kilometres to be ridden will be greater than, or less than, $13\frac{1}{3}$ km.
If the distance is less than $13\frac{1}{3}$ km, “Pedal on” is cheaper but if the distance is greater than $13\frac{1}{3}$ km then “Bikes r gr8” is cheaper.

14 a Let speed be v km/h and time of travel be t minutes.
Tram: $v = k_1 t$ where k_1 is the constant of proportionality
When $t = 2, v = 10 \Rightarrow 10 = 2k_1$.
$\therefore k_1 = 5$
For the tram, $v = 5t$.
Bus: $v = a + k_2 t$ where k_2 is the constant of proportionality and a is a constant.
When $t = 2, v = 18 \Rightarrow 18 = a + 2k_2$....(1)
When $t = 5, v = 27 \Rightarrow 27 = a + 5k_2$....(2)
equation (2) − equation (1)
$9 = 3k_2$
$\therefore k_2 = 3$
Substitute $k_2 = 3$ in equation (1)
$\therefore 18 = a + 6$
$\therefore a = 12$
For the bus, $v = 12 + 3t$.

b When $t = 0, v_{\text{tram}} = 0, v_{\text{bus}} = 12$
When $t = 5, v_{\text{tram}} = 25, v_{\text{bus}} = 27$



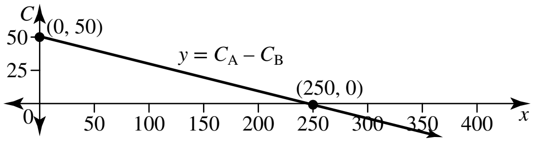
c The tram’s speed will be faster than the bus’s speed when
$5t > 12 + 3t$
$\therefore 2t > 12$
$\therefore t > 6$
After 6 minutes, the tram’s speed is faster than that of the bus.

15 a Model A has the higher y intercept.
Therefore, the cost model for A is $C = 300 + 0.05x$ and the cost model B is $C = 250 + 0.25x$.

b The gradients give the cost per kilometre.

c
$$\begin{aligned} y &= C_A - C_B \\ &= (300 + 0.05x) - (250 + 0.25x) \\ &= 50 - 0.2x \end{aligned}$$
$\therefore y = 50 - 0.2x$

d y intercept: $(0,50)$
x intercept: Let $y = 0$
$\therefore 0 = 50 - 0.2x$
$\therefore 0.2x = 50$
$\therefore x = 250 \Rightarrow (250,0)$
The graph of $y = C_A - C_B$ is shown in the diagram.



e i If $C_A = C_B$ then $y = 0$. Since $y = 0$ when $x = 250$, the costs are equal when the number of kilometres travelled is 250 km.

ii If $C_A < C_B$ then $C_A - C_B < 0$ and so $y < 0$. This occurs when $x > 250$. Therefore, company A is cheaper if the distance travelled is more than 250 km.

16 a At 6 am the boat’s position is at $(0,2)$ and at 7 am its position is at $(6,3)$. It travels on a straight line through these points. For this line $m = \frac{3-2}{6-0} = \frac{1}{6}$, $c = 2$.
The equation of the boat’s path is $y = \frac{x}{6} + 2,\ x \geq 0$.

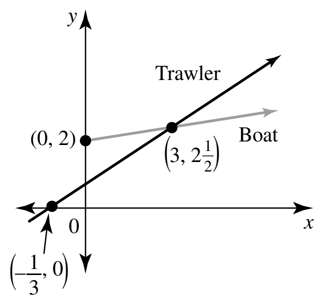
b When $x = 6t$,
$$\begin{aligned} y &= \frac{6t}{6} + 2 \\ &= t + 2 \end{aligned}$$
The boat’s position north of the lookout is $(t + 2)$ km.

c As t measures the time after 6 am, at 9 : 30am, $t = 3.5$.
When $t = 3.5$,
$$\begin{aligned} x &= 6(3.5) \\ &= 21 \end{aligned} \quad \text{and} \quad \begin{aligned} y &= 3.5 + 2 \\ &= 5.5 \end{aligned}$$
Therefore, the boat is at the position $(21,5.5)$.

d At t hours after 6 am the trawler’s position is given by $x = \frac{4t-1}{3}, y = t$.
At 6am, $t = 0$ so its position is $\left(-\frac{1}{3},0\right)$.
At 7am, $t = 1$ so its position is $(1,1)$.
The line joining these points has gradient
$$\begin{aligned} m &= \frac{1-0}{1+\frac{1}{3}} \\ &= \frac{1}{\frac{4}{3}} \\ &= \frac{3}{4} \end{aligned}$$
Its equation is $y - y_1 = m(x - x_1)$ with $m = \frac{3}{4}$ and a point $\left(-\frac{1}{3},0\right)$
$\therefore y = \frac{3}{4}\left(x + \frac{1}{3}\right)$
$\therefore y = \frac{3x}{4} + \frac{1}{4}$
The equation of the trawler’s path is $y = \frac{3x}{4} + \frac{1}{4},\ x \geq -\frac{1}{3}$

d The common point of the two paths is their point of intersection.
$y = \frac{x}{6} + 2$....(1)
$y = \frac{3x}{4} + \frac{1}{4}$....(2)
At the intersection,
$\frac{x}{6} + 2 = \frac{3x}{4} + \frac{1}{4}$
Multiply both sides by 12
$\therefore 2x + 24 = 9x + 3$
$\therefore 21 = 7x$
$\therefore x = 3$
Substitute $x = 3$ in equation (1)
$\therefore y = \frac{3}{6} + 2$
$\therefore y = 2\frac{1}{2}$
The common point of the paths is $\left(3, 2\frac{1}{2}\right)$.

f



Although there is a common point to the paths, the boat and the trawler will only collide if they are at this common point at the same time.

Boat: $x = 6t,\ y = t + 2$

At the common point where $x = 3, y = 2.5, t = 0.5$ so the boat is at this point at 6:30am.

Trawler: $x = \dfrac{4t-1}{3}, y = t$

At the common point where $x = 3, y = 2.5, t = 2.5$ so the trawler is not at this point until 8:30am.

Therefore the two boats do not collide.

17 Graph the two lines $y = \dfrac{17+9x}{5}$ and $y = 8 - \dfrac{3x}{2}$.

Then tap Analysis → G-Solve → intersect to obtain the co-ordinates of the point of intersection to two decimal places as (1.39,5.91).

18 **a** In the graphing screen select Param Type and then enter $xt1 = t, yt1 = 3 + 2t$ and $xt2 = t + 1, yt2 = 4t - 1$. Graph both lines. To graph the lines simultaneously, alter the t step to 1, for example.

Tap Analysis → G-Solve → trace gives an estimate of the common point as occurring when $t = 4$.

Tap the table of values symbol to compare the positions of each graph around $t = 4$. The table of values shows that when $t = 4$, particle P_1 is at (4,11) but the second particle P_2 is at (4,11) when $t = 3$. This identifies the common point on the paths to be (4,11) but that the two particles are in this position at different times and therefore do not collide.

b From the table of values, the common point is (4,11)

Exercise 1.6 — Coordinate geometry of the straight line

1 A (−3,−12), B (0,3)

$$m = \frac{3-(-12)}{0-(-3)}$$

$\therefore m_{AB} = 5$

B (0,3), C (4,23)

$$m = \frac{23-3}{4-0}$$

$\therefore m_{BC} = 5$

Since $m_{AB} = m_{BC}$ and point B is common, the points A, B and C are collinear

2 P (−6,−8), Q (6,4)

$$m_{PQ} = \frac{-8-4}{-6-6}$$

$\therefore m_{PQ} = 1$

Equation of PQ:

$y - 4 = 1(x - 6)$

$\therefore y = x - 2$

Test point R (−32,34)

If $x = -32$,

$y = -32 - 2$

$= -34$

$\neq 34$

Therefore P,Q and R are not collinear

3 **a** points (1,−8) and (5,−2)

$$m = \frac{-2+8}{5-1}$$

$$\therefore m = \frac{3}{2}$$

$$\therefore \tan\theta = \frac{3}{2}$$

$$\therefore \theta = \tan^{-1}(1.5)$$

$$\therefore \theta = 56.31^{\circ}$$

b gradient of −2

$\tan\theta = -2$

$$\therefore \theta = 180^{\circ} - \tan^{-1}(2)$$

$$\therefore \theta = 116.57^{\circ}$$

c $m = \tan(135)^{\circ}$ Point (2,7)

$\therefore m = -1$

Equation of line is

$y - 7 = -(x - 2)$

$\therefore y = -x + 9$

4 Angle of inclination to horizontal of line with gradient of 5

$\tan\theta = 5$

$\therefore \theta = \tan^{-1}(5)$

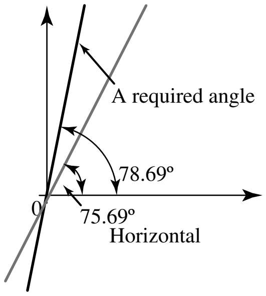
$\therefore \theta = 78.69^{\circ}$

Angle of inclination to horizontal of line with gradient of 4

$\tan\theta = 4$

$\therefore \theta = \tan^{-1}(4)$

$\therefore \theta = 75.96^{\circ}$



Therefore the angle between the two lines is the difference between the larger and the smaller angle, $78.69^{\circ} - 75.96^{\circ} = 2.73^{\circ}$.

5 **a** $3y - 6x = 1$

$$\therefore 3y = 6x + 1$$

$$\therefore y = 2x + \frac{1}{3}$$

$$m = 2$$

i Parallel line has gradient 2

ii perpendicular line has gradient $-\dfrac{1}{2}$

b Gradient of $y = x$ is $m_1 = 1$, gradient of $y = -x$ is $m_2 = -1$

Since $m_1 m_2 = -1$, the lines are perpendicular

c Gradient of $y = 5x + 10$ is 5, so gradient of perpendicular line is $-\dfrac{1}{5}$. Point is (1,1)

Equation of line is

$$y-1=-\frac{1}{5}(x-1)$$

$$\therefore 5y-5=-x+1$$

$$\therefore 5y+x=6$$

6 $2y-4x=7$

$$2y=4x+7$$

$$\therefore y=2x+\frac{7}{2}$$

$$m=2$$

Parallel line has gradient 2, and point $(8,-2)$

Equation is

$y+2=2(x-8)$

$\therefore y=2x-18$

For x intercept, let $y=0$

$\therefore 0=2x-18$

$\therefore x=9$

Co-ordinates of the x intercept are $(9,0)$

7 Points $(12,5)$ and $(-9,-1)$

Midpoint:

$$x=\frac{12+(-9)}{2}=\frac{3}{2} \qquad y=\frac{5+(-1)}{2}=\frac{4}{2}=2$$

midpoint is $(1.5,2)$

8 Midpoint of PQ is $(3,0)$, Q is $(-10,10)$

$$x=\frac{x_1+x_2}{2} \qquad y=\frac{y_1+y_2}{2}$$

$$\therefore 3=\frac{x_1+(-10)}{2} \quad \text{and} \quad \therefore 0=\frac{y_1+10}{2}$$

$$\therefore x_1-10=6 \qquad \therefore y_1+10=0$$

$$\therefore x_1=16 \qquad \therefore y_1=-10$$

Therefore point P has co-ordinates $(16,-10)$

9 A $(-4,4)$, B $(-3,10)$

Point: Midpoint of AB is $\left(\frac{-4+(-3)}{2},\frac{4+10}{2}\right)=\left(-\frac{7}{2},7\right)$

Gradient: Gradient of AB is $\frac{10-4}{-3-(-4)}=6$. Therefore gradient of perpendicular is $-\frac{1}{6}$

Equation: point $\left(-\frac{7}{2},7\right)$ gradient $-\frac{1}{6}$

$$y-7=-\frac{1}{6}\left(x+\frac{7}{2}\right)$$

$$\therefore 6y-42=-\left(x+\frac{7}{2}\right)$$

$$\therefore 12y-84=-2x-7$$

$$\therefore 12y+2x=77$$

10 $ax+by=c$ is the perpendicular bisector of CD where C $(-2,-5)$ and D $(2,5)$

Perpendicular bisector of CD

Point: Midpoint of CD is $\left(\frac{-2+2}{2},\frac{-5+5}{2}\right)=(0,0)$

Gradient: Gradient of CD is $\frac{5-(-5)}{2-(-2)}=\frac{5}{2}$. Therefore gradient of perpendicular is $-\frac{2}{5}$

Equation: point $(0,0)$ gradient $-\frac{2}{5}$

Therefore the perpendicular bisector has the equation $y=-\frac{2}{5}x$

Rearranging,

$$5y=-2x$$

$$\therefore 2x+5y=0$$

Comparing with $ax+by=c$, $a=2, b=5, c=0$

11 Points $(6,-8)$ and $(-4,-5)$

$$d=\sqrt{(6-(-4))^2+(-8-(-5))^2}$$

$$=\sqrt{(10)^2+(-3)^2}$$

$$=\sqrt{109}$$

$$\simeq 10.44$$

12 A $(-1,1)$ and B $(6,-1)$

Midpoint of AB is $\left(\frac{-1+6}{2},\frac{1+(-1)}{2}\right)=\left(\frac{5}{2},0\right)$

Distance between point $(3,10)$ and $\left(\frac{5}{2},0\right)$

$$=\sqrt{\left(3-\frac{5}{2}\right)^2+(10-0)^2}$$

$$=\sqrt{(0.5)^2+(10)^2}$$

$$=\sqrt{1.25+100}$$

$$=\sqrt{101.25}$$

$$=10.06 \ \text{(2dp)}$$

13 **a** The gradient of the line is obtained using the two given points $(-2,0)$ and $(3,9)$.

$$m=\frac{9-0}{3+2}=\frac{9}{5}=1.8$$

$$\therefore \tan\theta=1.8$$

$$\therefore \theta=\tan^{-1}(1.8)$$

$$\therefore \theta\simeq 60.95^{\circ}$$

The angle is 60.95° to two decimal places.

Points $(4,0)$ and $(0,3)$

$$m=\frac{3-0}{0-4}=-\frac{3}{4}$$

$$\therefore \tan\theta=-\frac{3}{4}$$

$$\therefore \theta=180^{\circ}-\tan^{-1}(0.75)$$

$$\therefore \theta\simeq 143.13^{\circ}$$

The angle is 143.13° to two decimal places.

b A line parallel to the y axis must be a vertical line. Therefore the angle of inclination to the x axis is 90°.

c Given $m=-7$, $\tan\theta=-7$

$$\therefore \theta=180^{\circ}-\tan^{-1}(7)$$

$$\therefore \theta\simeq 98.13^{\circ}$$

14 **a** The line $7y-5x=0$ rearranges to $y=\frac{5x}{7}$. Its gradient is $\frac{5}{7}$.

The parallel line has the same gradient of $\frac{5}{7}$ and a y intercept at $(0,6)$. Its equation is $y=mx+c$, $m=\frac{5}{7}, c=6$

$$\therefore y=\frac{5x}{7}+6$$

Rearranging to the required form,

$\therefore 7y = 5x + 42$

$\therefore 5x - 7y = -42$

b $3y + 4x = 2$

$3y + 4x = 2$

$\therefore 3y = -4x + 2$

$\therefore y = -\frac{4x}{3} + \frac{2}{3}$

The parallel line has $m = -\frac{4}{3}$.

Equation of the parallel line: $y - y_1 = m(x - x_1)$, $m = -\frac{4}{3}$, point $\left(-2, \frac{4}{5}\right)$

$\therefore y - \frac{4}{5} = -\frac{4}{3}(x + 2)$

Multiply both sides by 15

$\therefore 15y - 12 = -20(x + 2)$

$\therefore 15y - 12 = -20x - 40$

$\therefore 20x + 15y = -28$

c $2x - 3y + 7 = 0$

$\therefore 2x + 7 = 3y$

$\therefore y = \frac{2x}{3} + \frac{7}{3}$

$m_1 = \frac{2}{3}$ so the gradient of the perpendicular line $m_2 = -\frac{3}{2}$ since $m_1 m_2 = -1$.

Equation of the perpendicular line: $y - y_1 = m(x - x_1)$, $m = -\frac{3}{2}$, point $\left(-\frac{3}{4}, 1\right)$

$\therefore y - 1 = -\frac{3}{2}\left(x + \frac{3}{4}\right)$

$\therefore 2y - 2 = -3\left(x + \frac{3}{4}\right)$

$\therefore 2y - 2 = -3x - \frac{9}{4}$

$\therefore 8y - 8 = -12x - 9$

$\therefore 12x + 8y = -1$

d $3x - y = 2$

$\therefore y = 3x - 2$

$m_1 = 3 \Rightarrow m_2 = -\frac{1}{3}$

equation of perpendicular line through the origin:

$y = mx + c$, $m = -\frac{1}{3}, c = 0$

$\therefore y = -\frac{1}{3}x$

$\therefore 3y = -x$

$\therefore x + 3y = 0$

e point $(-6, 12)$

$m = \tan\theta$ where $\theta = \tan^{-1}(1.5)$

$\therefore m = 1.5$

equation: $y - y_1 = m(x - x_1)$

$\therefore y - 12 = 1.5(x + 6)$

$\therefore y = 1.5x + 9 + 12$

$\therefore y = 1.5x + 21$

$\therefore 2y = 3x + 42$

$\therefore 3x - 2y = -42$

f Point of intersection obtained by solving simultaneous equations.

$2x - 3y = 18....(1)$

$5x + y = 11....(2)$

equation $(1) + 3 \times$ equation (2)

$17x = 51$

$\therefore x = 3$

Substitute $x = 3$ in equation (2)

$\therefore 15 + y = 11$

$\therefore y = -4$

Point is $(3, -4)$.

If the line is perpendicular to the horizontal line $y = 8$ then the vertical line through the point $(3, -4)$ is required.

Therefore, the equation is $x = 3$.

15 A $(-7, 2)$ and B $(-13, 10)$

a $d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(-13 + 7)^2 + (10 - 2)^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100}$

$= 10$

The distance is 10 units.

b The co-ordinates of the midpoint are given by

$x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}$.

$\therefore x = \frac{-7 - 13}{2}, y = \frac{2 + 10}{2}$

$\therefore x = -10, \quad y = 6$

Midpoint is $(-10, 6)$.

c The perpendicular bisector of AB contains the midpoint $(-10, 6)$.

$m_{AB} = \frac{10 - 2}{-13 + 7}$

$= \frac{8}{-6}$

$= -\frac{4}{3}$

Therefore the gradient of the perpendicular bisector is $\frac{3}{4}$.

Equation of perpendicular bisector:

$y - 6 = \frac{3}{4}(x + 10)$

$\therefore 4y - 24 = 3x + 30$

$\therefore 4y - 3x = 54$

d $4y - 3x = 54....(1)$

$4y + 3x = 24....(2)$

Add equations (1) and (2)

$\therefore 8y = 78$

$\therefore y = \frac{39}{4}$

Subtract equation (1) from equation (2)

$\therefore 6x = -30$

$\therefore x = -5$

Point of intersection is $(-5, 9.75)$.

16 **a** point $(4, 0)$.

gradient: $m = \tan(123.69^\circ)$

$\therefore m \simeq -1.5$

equation: $y - 0 = -1.5(x - 4)$

$\therefore y = -1.5x + 6$ is the equation of line L.

b K and L are perpendicular lines.

$$m_L = -1.5$$
$$= -\frac{3}{2}$$
$$\therefore m_K = \frac{2}{3}$$

equation of K: $m = \frac{2}{3}$, point $(4,0)$

$$\therefore y = \frac{2}{3}(x-4)$$
$$\therefore y = \frac{2x}{3} - \frac{8}{3}$$

or $2x - 3y = 8$.

c y intercepts of lines K and L are $\left(0, -\frac{8}{3}\right)$ and $(0,6)$ respectively so the distance between these is $6 + \frac{8}{3} = \frac{26}{3}$ units.

17 **a** A $(-4,13)$, B $(7,-9)$ and C $(12,-19)$.

$$m_{AB} = \frac{-9-13}{7+4} = -\frac{22}{11} = -2 \quad \text{and} \quad m_{BC} = \frac{-19+9}{12-7} = -\frac{10}{5} = -2$$

Since $m_{AB} = m_{BC}$ and point B is common, the three points are collinear.

b A $(-15,-95)$, B $(12,40)$ and C $(20,75)$.

If the points can be joined to form a triangle then they cannot be collinear.

$$m_{AB} = \frac{40+95}{12+15} = \frac{135}{27} = 5 \quad \text{and} \quad m_{BC} = \frac{75-40}{20-12} = \frac{35}{8} \neq 5$$

Since $m_{AB} \neq m_{BC}$, the points A, B and C are not collinear and so they may be joined to form a triangle ABC.

c A $(3,0)$, B $(9,4)$, C $(5,6)$ and D $(-1,2)$.

If the two line segments AC and BD bisect each other they must have the same midpoint.

Midpoint of AC

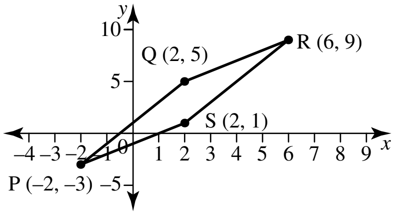
$x = \frac{3+5}{2} = 4$, $y = \frac{0+6}{2} = 3$ $\therefore$ midpoint of AC is $(4,3)$.

Midpoint of BD

$x = \frac{9-1}{2} = 4$, $y = \frac{4+2}{2} = 3$ $\therefore$ midpoint of BD is $(4,3)$.

Hence AC and BD bisect each other.

d P $(-2,-3)$, Q $(2,5)$, R $(6,9)$ and S $(2,1)$.



PQRS will be a parallelogram if its opposite sides are parallel, that is, if PQ and SR are parallel and PS and QR are parallel.

$$m_{PQ} = \frac{5+3}{2+2} = 2 \quad \text{and} \quad m_{SR} = \frac{9-1}{6-2} = 2$$

Since $m_{PQ} = m_{SR}$, PQ is parallel to SR which is sufficient for PQRS to be a parallelogram. However, to show the other pair of opposite sides are parallel:

$$m_{PS} = \frac{1+3}{2+2} = 1 \quad \text{and} \quad m_{QR} = \frac{9-5}{6-2} = 1$$

Since $m_{PS} = m_{QR}$, PS is parallel to QR and therefore PQRS is a parallelogram.

To be a rectangle, the adjacent sides need to be perpendicular. Testing PQ and QR gives

$$m_{PQ} \times m_{QR}$$
$$= 2 \times 1$$
$$= 2$$
$$\neq -1$$

The sides PQ and QR are not perpendicular so PQRS is not a rectangle.

18 C $(-8,5)$, D $(2,4)$ and E $(0.4,0.8)$.

a Perimeter is the sum of the side lengths.

$$CD = \sqrt{(2+8)^2 + (4-5)^2} = \sqrt{100+1} = \sqrt{101}$$

$$DE = \sqrt{(2-0.4)^2 + (4-0.8)^2} = \sqrt{1.6^2 + 3.2^2} = \sqrt{12.8}$$

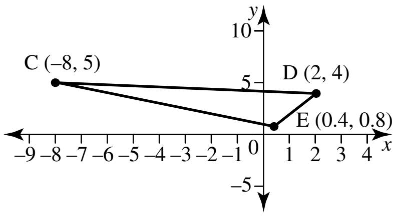
$$EC = \sqrt{(-8-0.4)^2 + (5-0.8)^2} = \sqrt{8.4^2 + 4.2^2} = \sqrt{88.2}$$

Perimeter

$$= \sqrt{101} + \sqrt{12.8} + \sqrt{88.2}$$
$$\simeq 23$$

The perimeter is 23 units.

b If angle CED is a right angle, CE is perpendicular to ED and CD is the hypotenuse.



Test if Pythagoras's theorem $CD^2 = CE^2 + ED^2$ holds.

$$\text{LHS} = CD^2 = 101$$

$$\text{RHS} = CE^2 + ED^2 = 88.2 + 12.8 = 101 = \text{LHS}$$

Therefore angle CED is a right angle.

An alternative method is to show CE and ED are perpendicular.

$$m_{CE}=\frac{5-0.8}{-8-0.4}\qquad m_{ED}=\frac{4-0.8}{2-0.4}$$

$$=-\frac{4.2}{8.4}\quad\text{and}\quad=\frac{3.2}{1.6}$$

$$=-\frac{1}{2}\qquad\qquad=2$$

Since $m_{CE}\times m_{ED}=-1$, CE is perpendicular to ED and therefore angle CED is 90°.

c Midpoint, M, of the hypotenuse CD is

$$\left(\frac{-8+2}{2},\frac{5+4}{2}\right)=(-3,4.5).$$

d M is equidistant from C and D so $MC=MD=\frac{1}{2}CD$.

$$\therefore MC=MD=\frac{1}{2}\sqrt{101}$$

$$ME=\sqrt{(-3-0.4)^2+(4.5-0.8)^2}$$

$$=\sqrt{25.25}$$

$$=\sqrt{\frac{101}{4}}$$

$$=\frac{1}{2}\sqrt{101}$$

$\therefore ME=MC=MD$ which shows M is equidistant from each of the vertices of the triangle.

19 a The centre of the circle is the midpoint of the diameter.

$\therefore 4=\frac{-2+x_2}{2}$, and $8=\frac{-2+y_2}{2}$ where (x_2,y_2) are the co-ordinates of the other endpoint of the diameter.

$$\therefore 8=-2+x_2\quad\text{and}\quad\therefore 16=-2+y_2$$

$$\therefore x_2=10\qquad\qquad\therefore y_2=18$$

The other end of the diameter is the point (10,18).

b Area formula: $A=\pi r^2$

The radius is the distance between the centre (4,8) and (−2,−2), an endpoint of the diameter.

$$r=\sqrt{(4+2)^2+(8+2)^2}$$

$$=\sqrt{36+100}$$

$$=\sqrt{136}$$

$$\therefore A=\pi\left(\sqrt{136}\right)^2$$

$$=\pi\times 136$$

$$=136\pi$$

Area is 136π square units.

20 a i $ax-7y=8$ is parallel to $3y+6x=7$.

Rearranging each equation,

$$ax-7y=8$$

$$\therefore ax-8=7y$$

$$\therefore y=\frac{ax}{7}-\frac{8}{7}$$

$$\therefore m_1=\frac{a}{7}$$

$$3y+6x=7$$

$$\therefore 3y=-6x+7$$

$$\therefore y=-2x+\frac{7}{3}$$

$$\therefore m_2=-2$$

For parallel lines, $m_1=m_2$

$$\therefore \frac{a}{7}=-2$$

$$\therefore a=-14$$

ii For perpendicular lines, $m_1m_2=-1$

$$\therefore \frac{a}{7}\times -2=-1$$

$$\therefore -2a=-7$$

$$\therefore a=\frac{7}{2}$$

b Let the points be A $(3,b)$, B $(4,2b)$ and C $(8,5-b)$.

The points are collinear if $m_{AB}=m_{BC}$

$$\therefore \frac{2b-b}{4-3}=\frac{5-b-2b}{8-4}$$

$$\therefore b=\frac{5-3b}{4}$$

$$\therefore 4b=5-3b$$

$$\therefore 7b=5$$

$$\therefore b=\frac{5}{7}$$

c The gradient of the line through $(2c,-c)$ and $(c,-c-2)$ is:

$$m=\frac{-c-2+c}{c-2c}$$

$$=\frac{-2}{-c}$$

$$=\frac{2}{c}$$

Since $m=\tan(45^{\circ})$

$$\frac{2}{c}=1$$

$$\therefore c=2$$

d Points $(d+1,d-1)$ and $(4,8)$

$$m=\frac{d-1-8}{d+1-4}$$

$$=\frac{d-9}{d-3}$$

i parallel to line through (7,0) and (0,−2) with gradient $=\frac{\text{rise}}{\text{run}}=\frac{2}{7}$.

$$\therefore \frac{d-9}{d-3}=\frac{2}{7}$$

$$\therefore 7d-63=2d-6$$

$$\therefore 5d=57$$

$$\therefore d=\frac{57}{5}$$

$$\therefore d=11.4$$

ii Parallel to the x axis $\Rightarrow m=0$

$$\therefore \frac{d-9}{d-3}=0$$

$$\therefore d-9=0$$

$$\therefore d=9$$

Alternatively, a horizontal line through (4,8) has equation $y=8$. As this line passes through $(d+1,d-1)$, then

$$d-1=8$$

$$\therefore d=9$$

iii Perpendicular to the x axis $\Rightarrow m$ is undefined

$\frac{d-9}{d-3}$ is undefined when its denominator is zero

$$\therefore d=3$$

Alternatively, a vertical line through $(4,8)$ has equation $x = 4$. As this line passes through $(d+1, d-1)$, then

$d + 1 = 4$

$\therefore d = 3$

e Distance between the points $(p,8)$ and $(0,-4)$ is 13 units.

$$\therefore 13 = \sqrt{(p-0)^2 + (8+4)^2}$$

$$\therefore 13 = \sqrt{p^2 + 144}$$

$$\therefore 169 = p^2 + 144$$

$$\therefore p^2 = 25$$

$$\therefore p = \pm 5$$

f Let the lines with gradients $m_1 = -1.25$ and $m_2 = 0.8$ be inclined at θ_1 and θ_2 with the horizontal.

$\tan\theta_1 = -1.25$

$$\therefore \theta_1 = 180^{\circ} - \tan^{-1}(1.25)$$

$$\therefore \theta_1 \simeq 128.66^{\circ}$$

$\tan\theta_2 = 0.8$

$$\therefore \theta_2 = \tan^{-1}(0.8)$$

$$\therefore \theta_2 \simeq 38.66^{\circ}$$

The angle between the two lines is

$$\alpha = \theta_1 - \theta_2$$

$$= 128.66^{\circ} - 38.66^{\circ}$$

$$= 90^{\circ}$$

The value of α, the angle between the two lines, is 90°.

21 Anna: The distance of $(-2.3, 1.5)$ from $(0,0)$ is

$$d_A = \sqrt{2.3^2 + 1.5^2}$$

$$= \sqrt{7.54}$$

Liam: The distance of $(1.7, 2.1)$ from $(0,0)$ is

$$d_L = \sqrt{1.7^2 + 2.1^2}$$

$$= \sqrt{7.3}$$

Since $d_A > d_L$, Anna carries the rucksack.

22 P $(3,7)$ and R $(5,3)$

a Distance PR

$$d(P,R) = \sqrt{(5-3)^2 + (3-7)^2}$$

$$= \sqrt{4+16}$$

$$= \sqrt{20}$$

$$\simeq 4.5$$

The petrol station and the restaurant are approximately 4.5 km apart.

b $m_{PR} = \dfrac{3-7}{5-3}$, point $(5,3)$

$$= -2$$

Equation of PR:

$$y - 3 = -2(x-5)$$

$$\therefore y = -2x + 13$$

c H $(2, 3.5)$ Since PR has gradient -2, the perpendicular line through H has gradient $\dfrac{1}{2}$.

Its equation is

$$y - 3.5 = 0.5(x-2)$$

$$\therefore y = 0.5x + 2.5$$

d B is the intersection of lines PR and HB

$y = -2x + 13 \ldots.(1)$

$y = 0.5x + 2.5 \ldots.(2)$

At intersection,

$$-2x + 13 = 0.5x + 2.5$$

$$\therefore 10.5 = 2.5x$$

$$\therefore x = \frac{10.5}{2.5}$$

$$\therefore x = 4.2$$

Substitute $x = 4.2$ in equation (1)

$$\therefore y = -8.4 + 13$$

$$\therefore y = 4.6$$

The co-ordinates of B are $(4.2, 4.6)$.

e Total distance Ada cycles is HB + BR

$$d(H,B) = \sqrt{(4.2-2)^2 + (4.6-3.5)^2}$$

$$\simeq 2.460$$

$$d(B,R) = \sqrt{(4.2-5)^2 + (4.6-3)^2}$$

$$\simeq 1.789$$

Ada cycles 4.249 km at an average speed of 10 km/h.

Time taken is $\dfrac{4.249}{10} = 0.4249$ hours

In minutes, the time is $0.4249 \times 60 = 25.494$ minutes. This figure when rounded to the nearest minute, is 25 minutes. However, Ada would not be at R at this time. To reach the restaurant, Ada takes 26 minutes, to the nearest minute.

23 **a** Select Geometry from the menu.

Tap Draw → Point three times to obtain points A, B and C.

Tap Draw → Line Segment and join the points to form triangle ABC.

Tap a line segment then Draw → Construct → Perp. Bisector. Do this for each side to create the three perpendicular bisectors.

Move a vertex around to create different triangles.

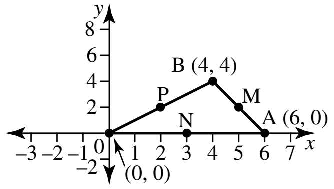
For every case, the perpendicular bisectors are concurrent.

b In the geometry menu select the grid template.

Plot the three points A $(0,0)$, B $(4,4)$ and $C(6,0)$ checking they are accurately placed using the co-ordinate tab. Then construct the triangle ABC and the perpendicular bisectors of each side. Select Intersection to mark the point D where these perpendicular bisectors meet.

Tap the co-ordinate button for point D. This gives D as the point $(3,1)$.

Checking algebraically:



BA: Midpoint M is $\left(\dfrac{4+6}{2}, \dfrac{4+0}{2}\right) = (5,2)$.

$$m_{BA} = \frac{4-0}{4-6}$$

$$= -2$$

Therefore the gradient of the perpendicular bisector of BA is $\dfrac{1}{2}$.

Equation of perpendicular bisector:

$y-2=\frac{1}{2}(x-5)$

$\therefore y=\frac{x}{2}-\frac{1}{2}$....(1)

OA: Midpoint is N $(3,0)$ so equation of perpendicular bisector of OA is $x=3$.....(2)

OB: Midpoint is P $(2,2)$ and gradient is 1. The perpendicular bisector of OB has gradient -1.

Equation of perpendicular bisector:

$y-2=-(x-2)$

$\therefore y=-x+4$....(3)

Substitute $x=3$ from equation (2) into each of equation (1) and (3).

equation (1): $y=\frac{3}{2}-\frac{1}{2}=1$

equation (3): $y=-3+4=1$

All three lines contain the point $(3,1)$ so this is the point of intersection of the perpendicular bisectors.

24 a Create three points and join each pair by a line segment to create the triangle ABC.

Mark the midpoints of each side using Draw → Construct → Midpoint.

Join each vertex to the midpoint of the opposite side using Draw → Line Segment.

Move a vertex around to create different triangles.

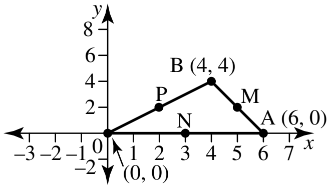
For every case, the medians are concurrent.

b In the geometry menu select the grid template.

Plot the three points A $(0,0)$, B $(4,4)$ and $C(6,0)$ checking they are accurately placed using the co-ordinate tab. Then construct the triangle ABC and the midpoints of each side. Draw the medians from each vertex to the midpoint of the opposite side and select Intersection to mark the point P where these medians meet.

Tap the co-ordinate button for point P. This gives the co-ordinates as repeated decimals $(3.33333, 1.33333)$ or $(3.\overline{3}, 1.\overline{3})$ giving the point of intersection of the medians as $\left(\frac{10}{3},\frac{4}{3}\right)$.

Check algebraically:



M $(5,2)$, N $(3,0)$ and P $(2,2)$

Median OM: $m=\frac{2}{5}, c=0$ so equation of OM is $y=\frac{2x}{5}$....(1)

Median NB: $m=4$, point $(3,0)$

equation of NB

$y=4(x-3)$

$\therefore y=4x-12$....(2)

Median AP: point $(6,0)$

$m=\frac{2}{-4}$

$\therefore m=-\frac{1}{2}$

equation of AP:

$y=-\frac{1}{2}(x-6)$

$\therefore y=-\frac{x}{2}+3$....(3)

Substitute equation (1) in equation (2)

$\therefore \frac{2x}{5}=4x-12$

$\therefore 2x=20x-60$

$\therefore 60=18x$

$\therefore x=\frac{60}{18}$

$\therefore x=\frac{10}{3}$

Substitute $x=\frac{10}{3}$ in equation (1)

$\therefore y=\frac{2}{5}\times\frac{10}{3}$

$\therefore y=\frac{4}{3}$

Test if $x=\frac{10}{3}$, $y=\frac{4}{3}$ satisfies equation (3)

$y=-\frac{x}{2}+3$

$\text{LHS}=\frac{4}{3} \qquad \text{RHS}=-\frac{1}{2}\times\frac{10}{3}+3$

$=-\frac{5}{3}+\frac{9}{3}$

$=\frac{4}{3}$

$=\text{LHS}$

Therefore the three medians meet at $\left(\frac{10}{3},\frac{4}{3}\right)$.

Topic 2 — Algebraic foundations

Exercise 2.2 — Algebraic skills

1 $3(2x+1)^2+(7x+11)(7x-11)-(3x+4)(2x-1)$
$=3(4x^2+4x+1)+(49x^2-121)-(6x^2-3x+8x-4)$
$=12x^2+12x+3+49x^2-121-6x^2-5x+4$
$=55x^2+7x-114$
Coefficient of x is 7.

2 $(2+3x)(x+6)(3x-2)(6-x)$
$=(3x+2)(3x-2)(6+x)(6-x)$
$=(9x^2-4)(36-x^2)$
$=324x^2-9x^4-144+4x^2$
$=-9x^4+328x^2-144$

3 **a** $4x^3-8x^2y-12y^2x$
$=4x(x^2-2xy-3y^2)$
$=4x(x-3y)(x+y)$

b $9y^2-x^2-8x-16$
$=9y^2-(x^2+8x+16)$
$=9y^2-(x+4)^2$
$=[3y-(x+4)][3y+(x+4)]$
$=(3y-x-4)(3y+x+4)$

c $4(x-3)^2-3(x-3)-22$
$=4a^2-3a-22$ where $a=(x-3)$
$=(4a-11)(a+2)$
$=[4(x-3)-11][(x-3)+2]$
$=(4x-23)(x-1)$

4 $x^2-6x+9-xy+3y$
$=(x^2-6x+9)+(-xy+3y)$
$=(x-3)^2-y(x-3)$
$=(x-3)[(x-3)-y]$
$=(x-3)(x-3-y)$

5 **a** $x^3-125=x^3-5^3$
$=(x-5)(x^2+5x+25)$

b $3+3x^3=3(1^3+x^3)$
$=3(1+x)(1-x+x^2)$

6 First take out the common factors
$2y^4+2y(x-y)^3$
$=2y[y^3+(x-y)^3]$

The term in the square brackets is a sum of two cubes $[y^3+b^3]$ where $b=(x-y)$ so the next step is to factorise this sum of two cubes

$=2y(y+b)(y^2-yb+b^2)$
$=2y[y+(x-y)][y^2-y(x-y)+(x-y)^2]$
$=2y(x)(y^2-yx+y^2+x^2-2xy+y^2)$
$=2xy(x^2-3xy+3y^2)$

Finally the inner brackets in each factor need to be expanded and like terms collected

$=2y(x)(y^2-yx+y^2+x^2-2xy+y^2)$
$=2xy(x^2-3xy+3y^2)$

7 **a** $\frac{x^2+4x}{x^2+2x-8}$
$=\frac{x(x+4)}{(x+4)(x-2)}$
$=\frac{x}{x-2}$

b $\frac{x^4-64}{5-x}\div\frac{x^2+8}{x-5}$
$=\frac{x^4-64}{5-x}\times\frac{x-5}{x^2+8}$
$=\frac{(x^2+8)(x^2-8)}{-(x-5)}\times\frac{x-5}{x^2+8}$
$=-(x^2-8)$
$=8-x^2$

8 $\frac{x^3-125}{x^2-25}\times\frac{5}{x^3+5x^2+25x}$
$=\frac{(x-5)(x^2+5x+25)}{(x-5)(x+5)}\times\frac{5}{x(x^2+5x+25)}$
$=\frac{5}{x(x+5)}$

9 $\frac{6}{5x-25}+\frac{1}{x-1}-\frac{2x}{x^2-6x+5}$
$=\frac{6}{5(x-5)}+\frac{1}{(x-1)}-\frac{2x}{(x-5)(x-1)}$
$=\frac{6(x-1)+5(x-5)-10x}{5(x-5)(x-1)}$
$=\frac{x-31}{5(x-5)(x-1)}$

10 $\left(\frac{4}{x+1}-\frac{3}{(x+1)^2}\right)\div\frac{16x^2-1}{x^2+2x+1}$
$=\left(\frac{4(x+1)-3}{(x+1)^2}\right)\times\frac{x^2+2x+1}{16x^2-1}$
$=\frac{4x+1}{(x+1)^2}\times\frac{(x+1)^2}{(4x+1)(4x-1)}$
$=\frac{1}{4x-1}$

11 **a** $(2x+3)^2=(2x)^2+2(2x)(3)+(3)^2$
$\therefore (2x+3)^2=4x^2+12x+9$

b $4a(b-3a)(b+3a)$
$=4a\left(b^2-9a^2\right)$
$=4ab^2-36a^3$

c $10-(c+2)(4c-5)$
$=10-\left(4c^2-5c+8c-10\right)$
$=10-\left(4c^2+3c-10\right)$
$=10-4c^2-3c+10$
$=20-3c-4c^2$

d $(5-7y)^2=25-70y+49y^2$

e $\left(3m^3+4n\right)\left(3m^3-4n\right)$

$=\left(3m^3\right)^2-(4n)^2$

$=9m^6-16n^2$

f $(x+1)^3$

$=(x+1)(x+1)^2$

$=(x+1)(x^2+2x+1)$

$=x^3+2x^2+x+x^2+2x+1$

$=x^3+3x^2+3x+1$

12 a $(g+12+h)^2$

$=\left((g+12)+h\right)^2$

$=(g+12)^2+2h(g+12)+h^2$

$=g^2+24g+144+2gh+24h+h^2$

$=g^2+h^2+2gh+24g+24h+144$

b $(2p+7q)^2(7q-2p)$

$=(7q+2p)^2(7q-2p)$

$=(7q+2p)\left((7q+2p)(7q-2p)\right)$

$=(7q+2p)(49q^2-4p^2)$

$=343q^3-28qp^2+98pq^2-8p^3$

c $(x+10)(5+2x)(10-x)(2x-5)$

$=(10+x)(10-x)(2x+5)(2x-5)$

$=(100-x^2)(4x^2-25)$

$=400x^2-2500-4x^4+25x^2$

$=-4x^4+425x^2-2500$

13 a $2(2x-3)(x-2)+(x+5)(2x-1)$

$=2(2x^2-7x+6)+(2x^2+9x-5)$

$=4x^2-14x+12+2x^2+9x-5$

$=6x^2-5x+7$

Coefficient of x is -5

b $(2+3x)(4-6x-5x^2)-(x-6)(x+6)$

$=8-12x-10x^2+12x-18x^2-15x^3-(x^2-36)$

$=8-28x^2-15x^3-x^2+36$

$=44-29x^2-15x^3$

Coefficient of x is 0

c $(4x+7)(4x-7)(1-x)$

$=(16x^2-49)(1-x)$

$=16x^2-16x^3-49+49x$

$=-16x^3+16x^2+49x-49$

Coefficient of x is 49

d $(x+1-2y)(x+1+2y)+(x-1)^2$

$=\left((x+1)-2y\right)\left((x+1)+2y\right)+(x-1)^2$

$=(x+1)^2-4y^2+(x-1)^2$

$=x^2+2x+1-4y^2+x^2-2x+1$

$=2x^2+2-4y^2$

Coefficient of x is 0

e $(3-2x)(2x+9)-3(5x-1)(4-x)$

$=6x+27-4x^2-18x-3(20x-5x^2-4+x)$

$=-12x+27-4x^2-60x+15x^2+12-3x$

$=11x^2-75x+39$

Coefficient of x is -75

f $x^2+x-4(x^2+x-4)$

$=x^2+x-4x^2-4x+16$

$=16-3x-3x^2$

Coefficient of x is -3

14 a $x^2+7x-60=(x+12)(x-5)$

b $4a^2-64$

$=4(a^2-16)$

$=4(a-4)(a+4)$

c $2bc+2b+1+c$

$=2b(c+1)+(1+c)$

$=(c+1)(2b+1)$

d $15x+27-2x^2$

$=27+15x-2x^2$

$=(9-x)(3+2x)$

e $1-9(1-m)^2$

$=1-\left(3(1-m)\right)^2$

$=\left(1-3(1-m)\right)\left(1+3(1-m)\right)$

$=(1-3+3m)(1+3-3m)$

$=(3m-2)(4-3m)$

f $8x^2-48xy+72y^2$

$=8\left(x^2-6xy+9y^2\right)$

$=8(x-3y)^2$

15 a $x^3+2x^2-25x-50$

$=x^2(x+2)-25(x+2)$

$=(x+2)(x^2-25)$

$=(x+2)(x-5)(x+5)$

b $100p^3-81pq^2$

$=p(100p^2-81q^2)$

$=p(10p-9q)(10p+9q)$

c $4n^2+4n+1-4p^2$

$=\left(4n^2+4n+1\right)-4p^2$

$=(2n+1)^2-(2p)^2$

$=(2n+1-2p)(2n+1+2p)$

d $49(m+2n)^2-81(2m-n)^2$

$=[7(m+2n)-9(2m-n)][7(m+2n)+9(2m-n)]$

$=(7m+14n-18m+9n)(7m+14n+18m-9n)$

$=(23n-11m)(5n+25m)$

$=(23n-11m)\times5(n+5m)$

$=5(23n-11m)(n+5m)$

e $13(a-1)+52(1-a)^3$

$=13(a-1)-52(a-1)^3$

$=13(a-1)\left(1-4(a-1)^2\right)$

$=13(a-1)[1-2(a-1)][1+2(a-1)]$

$=13(a-1)(1-2a+2)(1+2a-2)$

$=13(a-1)(3-2a)(2a-1)$

f $a^2-b^2-a+b+(a+b-1)^2$

$=(a-b)(a+b)-(a-b)+(a+b-1)^2$

$=(a-b)\left((a+b)-1\right)+(a+b-1)^2$

$=(a+b-1)[(a-b)+(a+b-1)]$

$=(a+b-1)(a-b+a+b-1)$

$=(a+b-1)(2a-1)$

16 a $(x+5)^2+(x+5)-56$

$=a^2+a-56$ where $a=(x+5)$

$=(a+8)(a-7)$

$=((x+5)+8)((x+5)-7)$

$=(x+13)(x-2)$

b $2(x+3)^2-7(x+3)-9$

$=2a^2-7a-9$ where $a=(x+3)$

$=(2a-9)(a+1)$

$=(2(x+3)-9)((x+3)+1)$

$=(2x-3)(x+4)$

c $70(x+y)^2-y(x+y)-6y^2$

$=70a^2-ya-6y^2$ where $a=(x+y)$

$=(7a+2y)(10a-3y)$

$=(7(x+y)+2y)(10(x+y)-3y)$

$=(7x+9y)(10x+7y)$

d x^4-8x^2-9

$=\left(x^2\right)^2-8\left(x^2\right)-9$

$=a^2-8a-9$ where $a=x^2$

$=(a-9)(a+1)$

$=\left(x^2-9\right)\left(x^2+1\right)$

$=(x-3)(x+3)(x^2+1)$

e $9(p-q)^2+12(p^2-q^2)+4(p+q)^2$

$=9(p-q)^2+12(p-q)(p+q)+4(p+q)^2$

$=9a^2+12ab+4b^2$ where $a=(p-q)$, $b=(p+q)$

$=(3a+2b)^2$

$=(3(p-q)+2(p+q))^2$

$=(3p-3q+2p+2q)^2$

$=(5p-q)^2$

f $a^2\left(a+\dfrac{1}{a}\right)^2-4a^2\left(a+\dfrac{1}{a}\right)+4a^2$

$=a^2x^2-4a^2x+4a^2$ where $x=\left(a+\dfrac{1}{a}\right)$

$=a^2(x^2-4x+4)$

$=a^2(x-2)^2$

$=(a(x-2))^2$

$=\left(a\left(a+\dfrac{1}{a}-2\right)\right)^2$

$=(a^2+1-2a)^2$

$=\left((a-1)^2\right)^2$

$=(a-1)^4$

17 a x^3-8

$=x^3-2^3$

$=(x-2)(x^2+2x+4)$

b x^3+1000

$=x^3+10^3$

$=(x+10)(x^2-10x+100)$

c $1-x^3$

$=1^3-x^3$

$=(1-x)(1+x+x^2)$

d $27x^3+64y^3$

$=(3x)^3+(4y)^3$

$=(3x+4y)(9x^2-12xy+16y^2)$

e x^4-125x

$=x(x^3-125)$

$=x\left(x^3-5^3\right)$

$=x(x-5)(x^2+5x+25)$

f $(x-1)^3+216$

$=(x-1)^3+6^3$

$=((x-1)+6)\left((x-1)^2-6(x-1)+36\right)$

$=(x+5)(x^2-2x+1-6x+6+36)$

$=(x+5)(x^2-8x+43)$

18 a $24x^3-81y^3$

$=3\left(8x^3-27y^3\right)$

$=3\left((2x)^3-(3y)^3\right)$

$=3(2x-3y)(4x^2+6xy+9y^2)$

b $8x^4y^4+xy$

$=xy\left(8x^3y^3+1\right)$

$=xy\left((2xy)^3+1^3\right)$

$=xy(2xy+1)(4x^2y^2-2xy+1)$

c $125(x+2)^3+64(x-5)^3$

$=[5(x+2)+4(x-5)]\big[25(x+2)^2-20(x+2)(x-5)$

$+16(x-5)^2\big]$

$=(5x+10+4x-20)\big(25(x^2+4x+4)-20(x^2-3x-10)$

$+16(x^2-10x+25)\big)$

$=(9x-10)(25x^2+100x+100-20x^2+60x+200+16x^2$

$-160x+400)$

$=7(9x-10)\ (3x^2+100)$

d $2(x-y)^3-54(2x+y)^3$

$=2\left((x-y)^3-27(2x+y)^3\right)$

$=2\left((x-y)-3(2x+y)\right)\big((x-y)^2+3(x-y)(2x+y)$

$+9(2x+y)^2\big)$

$=2(x-y-6x-3y)\big(x^2-2xy+y^2+3(2x^2-xy-y^2)$

$+9(4x^2+4xy+y^2)\big)$

$=2(-5x-4y)(43x^2+31xy+7y^2)$

$=-2(5x+4y)(43x^2+31xy+7y^2)$

e $a^5-a^3b^2+a^2b^3-b^5$

$=a^3\left(a^2-b^2\right)+b^3\left(a^2-b^2\right)$

$=(a^2-b^2)(a^3+b^3)$

$=(a-b)(a+b)(a+b)(a^2-ab+b^2)$

$=(a-b)(a+b)^2(a^2-ab+b^2)$

f x^6-y^6

$=\left(x^3\right)^2-\left(y^3\right)^2$

$=(x^3-y^3)(x^3+y^3)$

$=(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)$

$=(x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$

19 a $$\frac{3x^2-7x-20}{25-9x^2}$$
$$=\frac{\cancel{(3x+5)}(x-4)}{(5-3x)\cancel{(5+3x)}}$$
$$=\frac{x-4}{5-3x}$$

b $$\frac{x^3+4x^2-9x-36}{x^2+x-12}$$
$$=\frac{x^2(x+4)-9(x+4)}{(x+4)(x-3)}$$
$$=\frac{(x+4)(x^2-9)}{(x+4)(x-3)}$$
$$=\frac{(x+4)(x-3)(x+3)}{(x+4)(x-3)}$$
$$=x+3$$

c $$\frac{(x+h)^3-x^3}{h}$$
$$=\frac{((x+h)-x)\left((x+h)^2+x(x+h)+x^2\right)}{h}$$
$$=\frac{\cancel{(h)}(x^2+2xh+h^2+x^2+xh+x^2)}{\cancel{h}}$$
$$=3x^2+3xh+h^2$$

d $$\frac{2x^2}{9x^3+3x^2}\times\frac{1-9x^2}{18x^2-12x+2}$$
$$=\frac{2x^2}{3x^2(3x+1)}\times\frac{(1-3x)(1+3x)}{2(9x^2-6x+1)}$$
$$=\frac{\cancel{2}}{3\cancel{(3x+1)}}\times\frac{(1-3x)\cancel{(1+3x)}}{\cancel{2}(3x-1)^2}$$
$$=\frac{1}{3}\times\frac{(1-3x)}{(1-3x)^2}$$
$$=\frac{1}{3}\times\frac{1}{1-3x}$$
$$=\frac{1}{3(1-3x)}$$

e $$\frac{m^3-2m^2n}{m^3+n^3}\div\frac{m^2-4n^2}{m^2+3mn+2n^2}$$
$$=\frac{m^3-2m^2n}{m^3+n^3}\times\frac{m^2+3mn+2n^2}{m^2-4n^2}$$
$$=\frac{m^2(m-2n)}{(m+n)(m^2-mn+n^2)}\times\frac{(m+2n)(m+n)}{(m-2n)(m+2n)}$$
$$=\frac{m^2}{m^2-mn+n^2}$$

f $$\frac{1-x^3}{1+x^3}\times\frac{1-x^2}{1+x^2}\div\frac{1+x+x^2}{1-x+x^2}$$
$$=\frac{1-x^3}{1+x^3}\times\frac{1-x^2}{1+x^2}\times\frac{1-x+x^2}{1+x+x^2}$$
$$=\frac{(1-x)(1+x+x^2)}{(1+x)(1-x+x^2)}\times\frac{(1-x)(1+x)}{1+x^2}\times\frac{1-x+x^2}{1+x+x^2}$$
$$=\frac{(1-x)}{1}\times\frac{(1-x)}{1+x^2}\times\frac{1}{1}$$
$$=\frac{(1-x)^2}{1+x^2}$$

20 a $$\frac{4}{x^2+1}+\frac{4}{x-x^2}$$
$$=\frac{4}{x^2+1}+\frac{4}{x(1-x)}$$
$$=\frac{4x(1-x)+4(x^2+1)}{(x^2+1)x(1-x)}$$
$$=\frac{4x-4x^2+4x^2+4}{x(1-x)(x^2+1)}$$
$$=\frac{4(x+1)}{x(1-x)(x^2+1)}$$

b $$\frac{4}{x^2-4}-\frac{3}{x+2}+\frac{5}{x-2}$$
$$=\frac{4}{(x-2)(x+2)}-\frac{3}{x+2}+\frac{5}{x-2}$$
$$=\frac{4-3(x-2)+5(x+2)}{(x-2)(x+2)}$$
$$=\frac{4-3x+6+5x+10}{(x-2)(x+2)}$$
$$=\frac{2x+20}{(x-2)(x+2)}$$
$$=\frac{2(x+10)}{(x-2)(x+2)}$$

c $$\frac{5}{x+6}+\frac{4}{5-x}+\frac{3}{x^2+x-30}$$
$$=\frac{5}{x+6}-\frac{4}{x-5}+\frac{3}{(x+6)(x-5)}$$
$$=\frac{5(x-5)-4(x+6)+3}{(x+6)(x-5)}$$
$$=\frac{5x-25-4x-24+3}{(x+6)(x-5)}$$
$$=\frac{x-46}{(x+6)(x-5)}$$

d $$\frac{1}{4y^2-36y+81}+\frac{2}{4y^2-81}-\frac{1}{2y^2-9y}$$
$$=\frac{1}{(2y-9)^2}+\frac{2}{(2y-9)(2y+9)}-\frac{1}{y(2y-9)}$$
$$=\frac{y(2y+9)+2y(2y-9)-(2y-9)(2y+9)}{(2y-9)^2(2y+9)y}$$
$$=\frac{2y^2+9y+4y^2-18y-(4y^2-81)}{y(2y+9)(2y-9)^2}$$
$$=\frac{2y^2-9y+81}{y(2y-9)^2(2y+9)}$$

e $$\frac{1}{p-q}-\frac{p}{p^2-q^2}-\frac{q^3}{p^4-q^4}$$
$$=\frac{1}{p-q}-\frac{p}{(p-q)(p+q)}-\frac{q^3}{(p^2-q^2)(p^2+q^2)}$$
$$=\frac{1}{p-q}-\frac{p}{(p-q)(p+q)}-\frac{q^3}{(p-q)(p+q)(p^2+q^2)}$$
$$=\frac{(p+q)(p^2+q^2)-p(p^2+q^2)-q^3}{(p-q)(p+q)(p^2+q^2)}$$
$$=\frac{p^3+pq^2+qp^2+q^3-p^3-pq^2-q^3}{(p-q)(p+q)(p^2+q^2)}$$
$$=\frac{qp^2}{(p-q)(p+q)(p^2+q^2)}$$
$$=\frac{p^2q}{p^4-q^4}$$

f $(a+6b)\div\left(\frac{7}{a^2-3ab+2b^2}-\frac{5}{a^2-ab-2b^2}\right)$

$=(a+6b)\div\left(\frac{7}{(a-b)(a-2b)}-\frac{5}{(a-2b)(a+b)}\right)$

$=(a+6b)\div\frac{7(a+b)-5(a-b)}{(a-b)(a-2b)(a+b)}$

$=(a+6b)\div\frac{2a+12b}{(a-b)(a-2b)(a+b)}$

$=(a+6b)\times\frac{(a-b)(a-2b)(a+b)}{2(a+6b)}$

$=\frac{(a-b)(a-2b)(a+b)}{2}$

$=\frac{1}{2}(a-b)(a-2b)(a+b)$

21 a On the Main window of a CAS calculator, tap Interactive → Transformation → expand and enter the expression $(x+5)(2-x)(3x+7)$. Then tap OK.

$(x+5)(2-x)(3x+7)=-3x^3-16x^2+9x+70$

b Tap Interactive → Transformation → factor and enter the expression $27(x-2)^3+64(x+2)^3$.

$27(x-2)^3+64(x+2)^3=(13x^2+28x+148)(7x+2)$

22 a Tap Interactive → Transformation → combine and enter the expression as $3/(x-1)+8/(x+8)$.

$\frac{3}{x-1}+\frac{8}{x+8}=\frac{11x+16}{(x+8)(x-1)}$

b Left to the student.

Exercise 2.3 — Pascal's triangle and binomial expansions

1 Use $a=3x$ and $b=2$ in the expansion of $(a-b)^3$

$(3x-2)^3$

$=(3x)^3-3(3x)^2(2)+3(3x)(2)^2-(2)^3$

$=27x^3-54x^2+36x-8$

2 In the rule for expanding $(a+b)^3$, replace a by $\frac{a}{3}$ and b by b^2.

$\left(\frac{a}{3}+b^2\right)^3$

$=\left(\frac{a}{3}\right)^3+3\left(\frac{a}{3}\right)^2(b^2)+3\left(\frac{a}{3}\right)(b^2)^2+(b^2)^3$

$=\frac{a^3}{27}+3\times\frac{a^2}{9}\times b^2+3\times\frac{a}{3}\times b^4+b^6$

$=\frac{a^3}{27}+\frac{a^2b^2}{3}+ab^4+b^6$

The coefficient of a^2b^2 is $\frac{1}{3}$

3 $(a-b)^6$

The Binomial coefficients for Row 6 are:
1, 6, 15, 20, 15, 6, 1

Thus, $(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$

$(a-b)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$

$(2x-1)^6$

$=(2x)^6-6(2x)^5(1)+15(2x)^4(1)^2-20(2x)^3(1)^3+15(2x)^2(1)^4-6(2x)(1)^5+(1)^6$

$=64x^6-192x^5+240x^4-160x^3+60x^2-12x+1$

4 $(3x+2y)^4$

$=(3x)^4+4(3x)^3(2y)+6(3x)^2(2y)^2+4(3x)(2y)^3+(2y)^4$

$=81x^4+216x^3y+216x^2y^2+96xy^3+16y^4$

5 a $(3x+1)^3$

$=(3x)^3+3(3x)^2(1)+3(3x)(1)^2+(1)^3$

$=27x^3+27x^2+9x+1$

b $(1-2x)^3$

$=(1)^3-3(1)^2(2x)+3(1)(2x)^2-(2x)^3$

$=1-6x+12x^2-8x^3$

c $(5x+2y)^3$

$=(5x)^3+3(5x)^2(2y)+3(5x)(2y)^2+(2y)^3$

$=125x^3+150x^2y+60xy^2+8y^3$

d $\left(\frac{x}{2}-\frac{y}{3}\right)^3$

$=\left(\frac{x}{2}\right)^3-3\left(\frac{x}{2}\right)^2\left(\frac{y}{3}\right)+3\left(\frac{x}{2}\right)\left(\frac{y}{3}\right)^2-\left(\frac{y}{3}\right)^3$

$=\frac{x^3}{8}-\frac{x^2y}{4}+\frac{xy^2}{6}-\frac{y^3}{27}$

6 $(x+2)^3$

$=x^3+3x^2(2)+3x(2)^2+(2)^3$

$=x^3+6x^2+12x+8$

Statement A is the only correct statement.

7 a $(x+1)^3-3x(x+2)^2$

$=x^3+3x^2+3x+1-3x(x^2+4x+4)$

$=-2x^3-9x^2-9x+1$

Coefficient of x^2 is -9

b $3x^2(x+5)(x-5)+4(5x-3)^3$

$=3x^2(x^2-25)+4(125x^3-3\times25x^2\times3+3\times5x\times9-27)$

$=3x^4-75x^2+500x^3-900x^2+540x-108$

$=3x^4+500x^3-975x^2+540x-108$

Coefficient of x^2 is -975

c $(x-1)(x+2)(x-3)-(x-1)^3$

$=(x-1)(x^2-x-6)-(x^3-3x^2+3x-1)$

$=x^3-x^2-6x-x^2+x+6-x^3+3x^2-3x+1$

$=x^2-8x+7$

Coefficient of x^2 is 1

d $(2x^2-3)^3+2(4-x^2)^3$

$=(2x^2)^3-3(2x^2)^2(3)+3(2x^2)(3)^2-(3)^3+2\left(4^3-3(4)^2(x^2)+3(4)(x^2)^2-(x^2)^3\right)$

$=8x^6-36x^4+54x^2-27+2(64-48x^2+12x^4-x^6)$

$=8x^6-36x^4+54x^2-27+128-96x^2+24x^4-2x^6$

$=6x^6-12x^4-42x^2+101$

Coefficient of x^2 is -42

8 a Starting from Row 5, generate Row 7.

Row 5			1		5		10		10		5		1		
Row 6		1		6		15		20		15		6		1	
Row 7	1		7		21		35		35		21		7		1

b The second term matches the row number, so 1,9,.... are the terms of Row 9.

c The number of terms is one greater than the row number. Row n has $n+1$ terms.

9

Binomial power	Expansion	Number of terms in the expansion	Sum of indices in each term
$(x+a)^2$	$x^2+2xa+a^2$	3	2
$(x+a)^3$	$x^3+3x^2a+3xa^2+a^3$	4	3
$(x+a)^4$	$x^4+4x^3a+6x^2a^2$ $+4xa^3+a^4$	5	4
$(x+a)^5$	$x^5+5x^4a+10x^3a^2$ $+10x^2a^3+5xa^4+a^5$	6	5

10 a $(x+4)^5$

$$\begin{aligned}&=x^5+5x^4(4)+10x^3(4)^2+10x^2(4)^3+5x(4)^4+(4)^5\\&=x^5+20x^4+160x^3+640x^2+1280x+1024\end{aligned}$$

b Alternating the signs in part a gives

$(x-4)^5=x^5-20x^4+160x^3-640x^2+1280x-1024$

c $(xy+2)^5$

$$\begin{aligned}(xy+2)^5&=(xy)^5+5(xy)^4(2)+10(xy)^3(2)^2+10(xy)^2(2)^3\\&\quad+5(xy)(2)^4+(2)^5\\&=x^5y^5+10x^4y^4+40x^3y^3+80x^2y^2+80xy+32\end{aligned}$$

d $(3x-5y)^4$

$$\begin{aligned}&=(3x)^4-4(3x)^3(5y)+6(3x)^2(5y)^2-4(3x)(5y)^3+(5y)^4\\&=81x^4-540x^3y+1350x^2y^2-1500xy^3+625y^4\end{aligned}$$

e $(3-x^2)^4$

$$\begin{aligned}&=(3)^4-4(3)^3(x^2)+6(3)^2(x^2)^2-4(3)(x^2)^3+(x^2)^4\\&=81-108x^2+54x^4-12x^6+x^8\end{aligned}$$

f $(1+x)^6-(1-x)^6$

$$\begin{aligned}&=1+6x+15x^2+20x^3+15x^4+6x^5+x^6-\big(1-6x+15x^2\\&\quad-20x^3+15x^4-6x^5+x^6\big)\\&=12x+40x^3+12x^5\end{aligned}$$

11 a $[(x-1)+y]^4$

$$\begin{aligned}&=(x-1)^4+4(x-1)^3y+6(x-1)^2y^2+4(x-1)y^3+y^4\\&=\left(x^4-4x^3+6x^2-4x+1\right)+4y\left(x^3-3x^2+3x-1\right)\\&\quad+6y^2(x^2-2x+1)+4y^3(x-1)+y^4\\&=\left(x^4-4x^3+6x^2-4x+1\right)\\&\quad+\left(4yx^3-12yx^2+12yx-4y\right)\\&\quad+\left(6y^2x^2-12y^2x+6y^2\right)\\&\quad+\left(4y^3x-4y^3\right)+y^4\\&=x^4+y^4-4x^3-4y^3+6x^2+6y^2-4x-4y+4x^3y+4y^3x\\&\quad+6x^2y^2-12x^2y-12xy^2+12xy+1\end{aligned}$$

b $\left(\dfrac{x}{2}+\dfrac{2}{x}\right)^6$

$$\begin{aligned}&=\left(\frac{x}{2}\right)^6+6\left(\frac{x}{2}\right)^5\left(\frac{2}{x}\right)+15\left(\frac{x}{2}\right)^4\left(\frac{2}{x}\right)^2+20\left(\frac{x}{2}\right)^3\left(\frac{2}{x}\right)^3\\&\quad+15\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^4+6\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^5+\left(\frac{2}{x}\right)^6\end{aligned}$$

For the term independent of x, the x power in the numerator and denominator must be the same. This will occur in the fourth term, $20\left(\dfrac{x}{2}\right)^3\left(\dfrac{2}{x}\right)^3$.

$$\begin{aligned}&20\left(\frac{x}{2}\right)^3\left(\frac{2}{x}\right)^3\\&=20\times\frac{x^3}{8}\times\frac{8}{x^3}\\&=20\end{aligned}$$

c $(x+ay)^4=x^4+4x^3ay+6x^2a^2y^2+4xa^3y^3+a^4y^4$

The coefficient of x^2y^2 is $6a^2$.

$$\begin{aligned}(ax^2-y)^4&=\left(ax^2\right)^4-4\left(ax^2\right)^3y+6\left(ax^2\right)^2\\&\quad y^2-4\left(ax^2\right)y^3+y^4\\&=a^4x^8-4a^3x^6y+6a^2x^4y^2-4ax^2y^3+y^4\end{aligned}$$

The coefficient of x^2y^3 is $-4a$.

$$\begin{aligned}&\therefore 6a^2=3\times(-4a)\\&\therefore 6a^2=-12a\\&\therefore 6a=-12\ \text{ since } a\neq 0\\&\therefore a=-2\end{aligned}$$

d $(1+2x)(1-x)^5$

$$\begin{aligned}&=(1+2x)(1-5x+10x^2-10x^3+5x^4-x^5)\\&=1-5x+10x^2-10x^3+5x^4-x^5+2x(1-5x+10x^2\\&\quad-10x^3+5x^4-x^5)\\&=1-5x+10x^2-10x^3+5x^4-x^5+2x-10x^2+20x^3\\&\quad-20x^4+10x^5-2x^6\\&=1-3x+10x^3-15x^4+9x^5-2x^6\end{aligned}$$

The coefficient of x is -3.

12 a Row 3: Sum is $1+3+3+1=8$

Row 4: Sum is $1+4+6+4+1=16$

Row 5: Sum is $1+5+10+10+5+1=32$

b Sum of Row 3 equals $8=2^3$

Sum of Row 4 equals $16=2^4$

Sum of Row 5 equals $32=2^5$

Therefore, the sum of Row n equals 2^n.

c $(1+x)^4=1+4x+6x^2+4x^3+x^4$

d Put $x=1$ in part c:

$$\begin{aligned}\therefore (1+1)^4&=1+4+6+4+1\\\therefore 2^4&=1+4+6+4+1\end{aligned}$$

The result illustrates again that the sum of the coefficients in Row 4 equals 2^4 since $1+4+6+4+1=2^4$.

e $1.1^4=(1+0.1)^4$

$\quad\;=(1+x)^4$ for $x=0.1$

$$\begin{aligned}(1+0.1)^4&=1+4(0.1)+6(0.1)^2+4(0.1)^3+(0.1)^4\\\therefore 1.1^4&=1+0.4+0.06+0.004+0.0001\\\therefore 1.1^4&=1.4641\end{aligned}$$

13 a $(x+1)^5-(x+1)^4$

$$\begin{aligned}&=x^5+5x^4+10x^3+10x^2+5x+1-\left(x^4+4x^3+6x^2+4x+1\right)\\&=x^5+4x^4+6x^3+4x^2+x\end{aligned}$$

Hence,

$$\begin{aligned}&(x+1)^5-(x+1)^4\\&=x^5+4x^4+6x^3+4x^2+x\\&=x\left(x^4+4x^3+6x^2+4x+1\right)\\&=x(x+1)^4\end{aligned}$$

b $$\begin{aligned}(x+1)^{n+1}-(x+1)^n &= (x+1)(x+1)^n-(x+1)^n\\ &= (x+1)^n\left[(x+1)-1\right]\\ &= (x+1)^n[x]\end{aligned}$$

$$\therefore (x+1)^{n+1}-(x+1)^n = x(x+1)^n$$

14 Since each term is formed by adding the two terms to its left and right from the preceding row,

$$\begin{aligned}b+165&=220\\ \therefore b&=55\\ 165+330&=c\\ \therefore c&=495\\ 45+a&=165\\ \therefore a&=120\end{aligned}$$

15 a The second column gives the natural numbers 1,2,3,4, ... where each number is one more than the preceding number.

b Each entry in the third column can be obtained by adding the entries from the second and third columns in the row above. The 5th entry in the third column is 15 and therefore the 6th entry in the third column would be $6+15=21$.

c Enter the table of values into the Statistics menu.

List 1(n)	1	2	3	4	5	6
List 2	1	3	6	10	15	21

Fit the data to a rule using the Calc menu.
A perfect fit is obtained for Quadratic Reg as $y=0.5x^2+0.5x$.
Replacing x by n and y by t_n, the value of the nth entry gives the formula $t_n=0.5n^2+0.5n$.

This formula could also be expressed as $t_n=\dfrac{n(n+1)}{2}$ and could be deduced from the table.

d Extend the fourth column and enter the table of values by into the Statistics menu.

List 1(n)	1	2	3	4	5
List 2	1	4	10	20	35

Fit the data to a rule using the Calc menu.
A perfect fit is obtained for Cubic Reg as $y=0.1\dot{6}x^3+0.5x^2+0.\dot{3}x+0$. This is $y=\dfrac{1}{6}x^3+\dfrac{1}{2}x^2+\dfrac{1}{3}x$.
Replacing x by n and y by t_n, the value of the nth entry gives the formula $$\begin{aligned}t_n&=\frac{1}{6}n^3+\frac{1}{2}n^2+\frac{1}{3}n\\ &=\frac{n}{6}\left(n^2+3n+2\right)\\ &=\frac{n(n+1)(n+2)}{6}\end{aligned}$$

16 Expanding $(1+x+x^2)^4$ in the Main menu gives
$x^8+4x^7+10x^6+16x^5+19x^4+16x^3+10x^2+4x+1$

Require x so that $1+x+x^2=0.91$. Using Interactive → Equation/Inequality → solve gives $x=-0.9, x=-0.1$.

Evaluate $x^8+4x^7+10x^6+16x^5+19x^4+16x^3+10x^2+4x+1$ with $x=-0.1$ as

$x^8+4x^7+10x^6+16x^5+19x^4+16x^3+10x^2+4x+1 \mid x=-0.1$ to obtain 0.68574961.

(The conditional or ‘given’ symbol, |, is obtained from the keyboard → mth → OPTN).

$\therefore 0.91^4=0.68574961$

Exercise 2.4 — The binomial theorem

1 $$\begin{aligned}&6!+4!-\frac{10!}{9!}\\ &=6\times5!+4\times3!-\frac{10\times9!}{9!}\\ &=6\times120+4\times6-10\\ &=734\end{aligned}$$

2 $$\begin{aligned}&\frac{n!}{(n-2)!}\\ &=\frac{n(n-1)(n-2)!}{(n-2)!}\\ &=n(n-1)\end{aligned}$$

3 $$\begin{aligned}&\binom{7}{4}\\ &=\frac{7!}{4!(7-4)!}\\ &=\frac{7!}{4!3!}\\ &=\frac{7\times6\times5\times4!}{4!\times3\times2\times1}\\ &=\frac{7\times6\times5}{3\times2}\\ &=35\end{aligned}$$

4 $$\begin{aligned}\binom{n}{2}&=\frac{n!}{2!(n-2)!}\\ &=\frac{n(n-1)(n-2)!}{2\times1\times(n-2)!}\\ &=\frac{n(n-1)}{2}\end{aligned}$$

$$\binom{21}{2}=\frac{21\times20}{2}=210$$

5 $$\begin{aligned}&(2x+3)^5\\ &=(2x)^5+\binom{5}{1}(2x)^4(3)+\binom{5}{2}(2x)^3(3)^2+\binom{5}{3}(2x)^2(3)^3\\ &\quad+\binom{5}{4}(2x)(3)^4+(3)^5\\ &=32x^5+5\times16x^4\times3+10\times8x^3\times9+10\times4x^2\times27\\ &\quad+5\times2x\times81+243\\ &=32x^5+240x^4+720x^3+1080x^2+810x+243\end{aligned}$$

6 $$\begin{aligned}&(x-2)^7\\ &=x^7-\binom{7}{1}x^6(2)+\binom{7}{2}x^5(2)^2-\binom{7}{3}x^4(2)^3+\binom{7}{4}x^3(2)^4\\ &\quad-\binom{7}{5}x^2(2)^5+\binom{7}{6}x(2)^6-(2)^7\\ &=x^7-7\times x^6\times2+21\times x^5\times4-35\times x^4\times8+35\times x^3\\ &\quad\times16-21\times x^2\times32+7\times x\times64-128\\ &=x^7-14x^6+84x^5-280x^4+560x^3-672x^2+448x-128\end{aligned}$$

7 In the general term let $r=3$ and $n=7$.

$$\begin{aligned}t_4&=\binom{7}{3}\left(\frac{x}{3}\right)^4\left(-\frac{y}{2}\right)^3\\ &=-35\times\frac{x^4}{81}\times\frac{y^3}{8}\\ &=-\frac{35}{648}x^4y^3\end{aligned}$$

Checking, the fourth term is an even term so its coefficients is _.

8 There are 11 terms so the middle term is the sixth term.
For t_6 let $r = 5$ and $n = 10$.

$$\begin{aligned} t_6 &= \binom{10}{5}(x^2)^5\left(\frac{y}{2}\right)^5 \\ &= 252x^{10}\times\frac{y^5}{32} \\ &= \frac{63x^{10}y^5}{8} \end{aligned}$$

9 $(4+3x^3)^8$

$$\begin{aligned} t_{r+1} &= \binom{8}{r}(4)^{8-r}(3x^3)^r \\ &= \binom{8}{r}(4)^{8-r}(3)^r x^{3r} \end{aligned}$$

For x^{15} we require $3r = 15$, so $r = 5$.
Hence $t_{r+1} = t_6$ so the sixth term contains x^{15}
$t_6 = \binom{8}{5}(4)^3(3)^5x^{15}$
The coefficient of x^{15} is $\binom{8}{5}(4)^3(3)^5$

$$\begin{aligned} &\binom{8}{5}(4)^3(3)^5 \\ &= \frac{8\times7\times6}{3\times2\times1}\times(2^2)^3\times3^5 \\ &= 8\times7\times2^6\times3^5 \\ &= 2^3\times7\times2^6\times3^5 \\ &= 2^9\times3^5\times7 \end{aligned}$$

Therefore, the coefficient of x^{15} is $2^9\times3^5\times7$

10 Term independent of x in $\left(x+\frac{2}{x}\right)^6$.

The question is equivalent to finding the coefficient of x^0.

$$\begin{aligned} t_{r+1} &= \binom{6}{r}x^{6-r}\left(\frac{2}{x}\right)^r \\ &= \binom{6}{r}2^r\frac{x^{6-r}}{x^r} \\ &= \binom{6}{r}2^r x^{6-2r} \end{aligned}$$

For x^0, $6-2r = 0$ which means $r = 3$.
So,

$$\begin{aligned} t_4 &= \binom{6}{3}2^3 \\ &= 160 \end{aligned}$$

11 **a** $6! = 6\times5\times4\times3\times2\times1 = 720$

b $4!+2! = 24+2 = 26$

c $7\times6\times5! = 7! = 5040$

d
$$\begin{aligned} &\frac{6!}{3!} \\ &= \frac{6\times5\times4\times3!}{3!} \\ &= 6\times5\times4 \\ &= 120 \end{aligned}$$

e
$$\begin{aligned} &10!-9! \\ &= 9!(10-1) \\ &= 9\times9! \\ &= 3,265,920 \end{aligned}$$

f
$$\begin{aligned} &(4!+3!)^2 \\ &= (24+6)^2 \\ &= 30^2 \\ &= 900 \end{aligned}$$

12 **a**
$$\begin{aligned} &\frac{26!}{24!} \\ &= \frac{26\times25\times24!}{24!} \\ &= 26\times25 \\ &= 650 \end{aligned}$$

b
$$\begin{aligned} &\frac{42!}{43!} \\ &= \frac{42!}{43\times42!} \\ &= \frac{1}{43} \end{aligned}$$

c
$$\begin{aligned} &\frac{49!}{50!}\div\frac{69!}{70!} \\ &\frac{49!}{50!}\times\frac{70!}{69!} \\ &= \frac{49!}{50\times49!}\times\frac{70\times69!}{69!} \\ &= \frac{1}{50}\times\frac{70}{1} \\ &= \frac{7}{5} \end{aligned}$$

d
$$\begin{aligned} &\frac{11!+10!}{11!-10!} \\ &= \frac{10!(11+1))}{10!(11-1)} \\ &= \frac{12}{10} \\ &= \frac{6}{5} \end{aligned}$$

13 **a** $(n+1)\times n! = (n+1)!$

b $(n-1)\times(n-2)\times(n-3)! = (n-1)!$

c
$$\begin{aligned} &\frac{n!}{(n-3)!} \\ &= \frac{n\times(n-1)\times(n-2)\times(n-3)!}{(n-3)!} \\ &= n(n-1)(n-2) \end{aligned}$$

d
$$\begin{aligned} &\frac{(n-1)!}{(n+1)!} \\ &= \frac{(n-1)!}{(n+1)\times n\times(n-1)!} \\ &= \frac{1}{n(n+1)} \end{aligned}$$

e
$$\begin{aligned} &\frac{(n-1)!}{n!}-\frac{(n+1)!}{(n+2)!} \\ &= \frac{(n-1)!}{n\times(n-1)!}-\frac{(n+1)!}{(n+2)\times(n+1)!} \\ &= \frac{1}{n}-\frac{1}{n+2} \\ &= \frac{(n+2)-n}{n(n+2)} \\ &= \frac{2}{n(n+2)} \end{aligned}$$

f
$$\begin{aligned} &\frac{n^3-n^2-2n}{(n+1)!}\times\frac{(n-2)!}{n-2} \\ &= \frac{n(n^2-n-2)}{(n+1)\times n\times(n-1)\times(n-2)!}\times\frac{(n-2)!}{n-2} \\ &= \frac{n(n-2)(n+1)}{(n+1)n(n-1)}\times\frac{1}{n-2} \\ &= \frac{1}{n-1} \end{aligned}$$

14 a $\binom{5}{2}$

$$\begin{aligned}&=\frac{5!}{2!\times(5-2)!}\\&=\frac{5!}{2!\times3!}\\&=\frac{5\times4\times3!}{2\times3!}\\&=\frac{20}{2}\\&=10\end{aligned}$$

b $\binom{5}{3}$

$$\begin{aligned}&=\frac{5!}{3!\times(5-3)!}\\&=\frac{5!}{3!\times2!}\\&=10\end{aligned}$$

c $\binom{12}{12}$

$$\begin{aligned}&=\frac{12!}{12!\times0!}\\&=\frac{12!}{12!}\\&=1\end{aligned}$$

d ${}^{20}C_3$

$$\begin{aligned}&=\binom{20}{3}\\&=\frac{20!}{3!\times17!}\\&=\frac{20\times19\times18\times17!}{3!\times17!}\\&=\frac{20\times19\times18}{3\times2\times1}\\&=1140\end{aligned}$$

e $\binom{7}{0}$

$$\begin{aligned}&=\frac{7!}{0!\times7!}\\&=1\end{aligned}$$

f $\binom{13}{10}$

$$\begin{aligned}&=\frac{13!}{10!\times3!}\\&=\frac{13\times12\times11\times10!}{10!\times3!}\\&=\frac{13\times12\times11}{3\times2\times1}\\&=286\end{aligned}$$

15 a $\binom{n}{3}$

$$\begin{aligned}&=\frac{n!}{3!\times(n-3)!}\\&=\frac{n\times(n-1)\times(n-2)\times\cancel{(n-3)!}}{3!\times\cancel{(n-3)!}}\\&=\frac{n(n-1)(n-2)}{6}\end{aligned}$$

b $\binom{n}{n-3}$

$$\begin{aligned}&=\frac{n!}{(n-3)!\times3!}\\&=\frac{n(n-1)(n-2)}{6}\end{aligned}$$

c $\binom{n+3}{3}$

$$\begin{aligned}&=\frac{(n+3)!}{3!\times((n+3)-3)!}\\&=\frac{(n+3)!}{3!\times n!}\\&=\frac{(n+3)\times(n+2)\times(n+1)\times n!}{3!\times n!}\\&=\frac{(n+3)(n+2)(n+1)}{6}\end{aligned}$$

d $\binom{2n+1}{2n-1}$

$$\begin{aligned}&=\frac{(2n+1)!}{(2n-1)!\times((2n+1)-(2n-1))!}\\&=\frac{(2n+1)!}{(2n-1)!\times2!}\\&=\frac{(2n+1)\times(2n)\times(2n-1)!}{(2n-1)!\times2!}\\&=\frac{2n(2n+1)}{2}\\&=n(2n+1)\end{aligned}$$

e $\binom{n}{2}+\binom{n}{3}$

$$\begin{aligned}&=\frac{n!}{2!\times(n-2)!}+\frac{n!}{3!\times(n-3)!}\\&=\frac{n(n-1)(n-2)!}{2!\times(n-2)!}+\frac{n(n-1)(n-2)(n-3)!}{3!\times(n-3)!}\\&=\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{6}\\&=\frac{3n(n-1)+n(n-1)(n-2)}{6}\\&=\frac{n(n-1)\left[3+(n-2)\right]}{6}\\&=\frac{n(n-1)(n+1)}{6}\end{aligned}$$

f $\binom{n+1}{3}$

$$\begin{aligned}&=\frac{(n+1)!}{3!\times((n+1)-3)!}\\&=\frac{(n+1)!}{3!\times(n-2)!}\\&=\frac{(n+1)n(n-1)(n-2)!}{6(n-2)!}\\&=\frac{n(n-1)(n+1)}{6}\end{aligned}$$

16 a $(x+1)^5$

$$\begin{aligned}&=x^5+\binom{5}{1}x^4+\binom{5}{2}x^3+\binom{5}{3}x^2+\binom{5}{4}x+1\\&=x^5+5x^4+10x^3+10x^2+5x+1\end{aligned}$$

b $(2-x)^5$

$$\begin{aligned}&=(2)^5-\binom{5}{1}(2)^4x+\binom{5}{2}(2)^3x^2-\binom{5}{3}(2)^2x^3+\binom{5}{4}(2)x^4-x^5\\&=32-5\times16x+10\times8x^2-10\times4x^3+5\times2x-x^5\\&=32-80x+80x^2-40x^3+10x^4-x^5\end{aligned}$$

c $(2x+3y)^6$

$$\begin{aligned}&=(2x)^6+\binom{6}{1}(2x)^5(3y)+\binom{6}{2}(2x)^4(3y)^2+\binom{6}{3}(2x)^3(3y)^3\\&\quad+\binom{6}{4}(2x)^2(3y)^4+\binom{6}{5}(2x)(3y)^5+(3y)^6\\&=64x^6+6\times32x^5\times3y+15\times16x^4\times9y^2+20\times8x^3\times27y^3\\&\quad+15\times4x^2\times81y^4+6\times2x\times243y^5+729y^6\\&=64x^6+576x^5y+2160x^4y^2+4320x^3y^3+4860x^2y^4\\&\quad+2916xy^5+729y^6\end{aligned}$$

d $\left(\frac{x}{2}+2\right)^7$

$$=\left(\frac{x}{2}\right)^7+\binom{7}{1}\left(\frac{x}{2}\right)^6(2)+\binom{7}{2}\left(\frac{x}{2}\right)^5(2)^2+\binom{7}{3}\left(\frac{x}{2}\right)^4(2)^3$$

$$+\binom{7}{4}\left(\frac{x}{2}\right)^3(2)^4+\binom{7}{5}\left(\frac{x}{2}\right)^2(2)^5+\binom{7}{6}\left(\frac{x}{2}\right)(2)^6+(2)^7$$

$$=\frac{x^7}{128}+7\times\frac{x^6}{64}\times 2+21\times\frac{x^5}{32}\times 4+35\times\frac{x^4}{16}\times 8+35\times\frac{x^3}{8}$$

$$\times 16+21\times\frac{x^2}{4}\times 32+7\times\frac{x}{2}\times 64+128$$

$$=\frac{x^7}{128}+\frac{7x^6}{32}+\frac{21x^5}{8}+\frac{35x^4}{2}+70x^3+168x^2+224x+128$$

e $\left(x-\frac{1}{x}\right)^8$

$$=x^8-\binom{8}{1}x^7\left(\frac{1}{x}\right)+\binom{8}{2}x^6\left(\frac{1}{x}\right)^2-\binom{8}{3}x^5\left(\frac{1}{x}\right)^3+\binom{8}{4}x^4\left(\frac{1}{x}\right)^4$$

$$-\binom{8}{5}x^3\left(\frac{1}{x}\right)^5+\binom{8}{6}x^2\left(\frac{1}{x}\right)^6-\binom{8}{7}x\left(\frac{1}{x}\right)^7+\left(\frac{1}{x}\right)^8$$

$$=x^8-\frac{8x^7}{x}+\frac{28x^6}{x^2}-\frac{56x^5}{x^3}+\frac{70x^4}{x^4}-\frac{56x^3}{x^5}+\frac{28x^2}{x^6}-\frac{8x}{x^7}+\frac{1}{x^8}$$

$$=x^8-8x^6+28x^4-56x^2+70-\frac{56}{x^2}+\frac{28}{x^4}-\frac{8}{x^6}+\frac{1}{x^8}$$

f $\left(x^2+1\right)^{10}$

$$=\left(x^2\right)^{10}+\binom{10}{1}\left(x^2\right)^9+\binom{10}{2}\left(x^2\right)^8+\binom{10}{3}\left(x^2\right)^7+\binom{10}{4}\left(x^2\right)^6$$

$$+\binom{10}{5}\left(x^2\right)^5+\binom{10}{6}\left(x^2\right)^4+\binom{10}{7}\left(x^2\right)^3$$

$$+\binom{10}{8}\left(x^2\right)^2+\binom{10}{9}\left(x^2\right)+1$$

$$=x^{20}+10x^{18}+45x^{16}+120x^{14}+210x^{12}+252x^{10}+210x^8$$

$$+120x^6+45x^4+10x^2+1$$

17 a $(5x+2)^6$ has general term $t_{r+1}=\binom{n}{r}(5x)^{n-r}(2)^r$ with $n=6$

For t_4, $r=3$

$$\therefore t_4=\binom{6}{3}(5x)^{6-3}(2)^3$$

$$=20\times(5x)^3\times 8$$

$$=160\times 125x^3$$

$$\therefore t_4=20\ 000x^3$$

b $(1+2x)^{12}$

$$t_{r+1}=\binom{12}{r}(1)^{12-r}(2x)^r$$

Put $r=9$

$$\therefore t_{10}=\binom{12}{9}(1)^3(2x)^9$$

$$=220\times 512x^9$$

$$\therefore t_{10}=112\ 640x^9$$

c $(2x+3)^{10}$

$$t_{r+1}=\binom{10}{r}(2x)^{10-r}(3)^r$$

Put $r=5$

$$\therefore t_6=\binom{10}{5}(2x)^5(3)^5$$

$$=252\times 32x^5\times 243$$

$$\therefore t_6=195\ 9552x^5$$

d $\left(3x^2-1\right)^6$

$$t_{r+1}=\binom{6}{r}\left(3x^2\right)^{6-r}(-1)^r$$

Put $r=2$

$$\therefore t_3=\binom{6}{2}\left(3x^2\right)^4(-1)^2$$

$$=15\times 81x^8\times 1$$

$$\therefore t_3=1215x^8$$

e $(x-5)^6$ has 7 terms in its expansion so the middle term is t_4.

$$t_{r+1}=\binom{6}{r}(x)^{6-r}(-5)^r$$

Put $r=3$

$$\therefore t_4=\binom{6}{3}(x)^3(-5)^3$$

$$=20\times x^3\times(-125)$$

$$\therefore t_4=-2500x^3$$

The middle term is $-2500x^3$

f $(x+2y)^7$ has 8 terms in its expansion so there are two middle terms: t_4 and t_5.

$$t_{r+1}=\binom{7}{r}(x)^{7-r}(2y)^r$$

For t_4, put $r=3$

$$\therefore t_4=\binom{7}{3}(x)^4(2y)^3$$

$$=35\times x^4\times 8y^3$$

$$\therefore t_4=280x^4y^3$$

For t_5, put $r=4$

$$\therefore t_5=\binom{7}{4}(x)^3(2y)^4$$

$$=35\times x^3\times 16y^4$$

$$\therefore t_5=560x^3y^4$$

The middle terms are $280x^4y^3$ and $560x^3y^4$.

18 a $(x+3)^{12}$

$$t_{r+1}=\binom{12}{r}(x)^{12-r}(3)^r$$

For the term in x^4, $12-r=4\Rightarrow r=8$

$$\therefore t_9=\binom{12}{8}(x)^4(3)^8$$

$$=495\times x^4\times 6561$$

$$\therefore t_9=324\ 7695x^4$$

b $\left(1-2x^2\right)^9$

$$t_{r+1}=\binom{9}{r}(1)^{9-r}\left(-2x^2\right)^r$$

$$=\binom{9}{r}(-2)^r x^{2r}$$

For the term in x^6, $2r=6\Rightarrow r=3$

$$\therefore t_4=\binom{9}{3}(-2)^3x^6$$

$$=84\times(-8)x^6$$

$$\therefore t_4=-672x^6$$

The coefficient of x^6 is -672.

c $(3+4x)^{11}$

$$t_{r+1}=\binom{11}{r}(3)^{11-r}(4x)^r$$

$$=\binom{11}{r}(3)^{11-r}(4)^rx^r$$

For the term in x^5, $r=5$

$$\therefore t_6=\binom{11}{5}(3)^6(4)^5x^5$$

The coefficient of x^5 is $\binom{11}{5}(3)^6(4)^5$ which needs to be expressed as the product of its prime factors.

$$\binom{11}{5}(3)^6(4)^5$$

$$=\frac{11\times\cancel{10}^{2}\times\cancel{9}^{3}\times\cancel{8}\times 7}{\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times 1}\times 3^6\times\left(2^2\right)^5$$

$$=11\times 2\times 3\times 7\times 3^6\times 2^{10}$$

$$=11\times 7\times 2^{11}\times 3^7$$

d $\left(\frac{x}{2}-\frac{2}{x}\right)^8$

$$t_{r+1}=\binom{8}{r}\left(\frac{x}{2}\right)^{8-r}\left(\frac{-2}{x}\right)^r$$
$$=\binom{8}{r}\times\frac{x^{8-r}}{2^{8-r}}\times\frac{(-2)^r}{x^r}$$
$$=\binom{8}{r}\times\frac{x^{8-2r}\times(-2)^r}{2^{8-r}}$$

For the term in x^2, $8-2r=2\Rightarrow r=3$

$$\therefore t_4=\binom{8}{3}\times\frac{x^2\times(-2)^3}{2^5}$$
$$=56\times\frac{-8x^2}{32}$$
$$\therefore t_4=-14x^2$$

The coefficient of x^2 is –14.

e $\left(x^2+\frac{1}{x^3}\right)^{10}$

$$t_{r+1}=\binom{10}{r}\left(x^2\right)^{10-r}\left(\frac{1}{x^3}\right)^r$$
$$=\binom{10}{r}x^{20-2r}\times\frac{1}{x^{3r}}$$
$$=\binom{10}{r}x^{20-5r}$$

For the term independent of x, $20-5r=0\Rightarrow r=4$

$$\therefore t_5=\binom{10}{4}x^0$$
$$=210$$

The term independent of x is $t_5=210$

f $\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^6$

$$\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^6$$
$$=\left(x^2-\frac{1}{x^2}\right)^6$$

$$t_{r+1}=\binom{6}{r}\left(x^2\right)^{6-r}\left(\frac{-1}{x^2}\right)^r$$
$$=\binom{6}{r}x^{12-2r}\times\frac{(-1)^r}{x^{2r}}$$
$$=\binom{6}{r}(-1)^r x^{12-4r}$$

For the term independent of x, $12-4r=0\Rightarrow r=3$

$$\therefore t_4=\binom{6}{3}(-1)^3x^0$$
$$\therefore t_4=-20$$

The term independent of x is $t_4=-20$

19 a $(1+ax)^{10}$

$$t_{r+1}=\binom{10}{r}(1)^{10-r}(ax)^r$$
$$=\binom{10}{r}a^rx^r$$

For t_4, put $r=3$

$\therefore t_4=\binom{10}{3}a^3x^3$ so the coefficient of the fourth term is $120a^3$.

For t_5, put $r=4$

$\therefore t_5=\binom{10}{4}a^4x^4$ so the coefficient of the fifth term is $210a^4$.

Equating the coefficients of the fourth and fifth terms,

$$210a^4=120a^3$$
$$\therefore\frac{a^4}{a^3}=\frac{120}{210}$$
$$\therefore a=\frac{4}{7}$$

b Consider the expansion of $(1+x+x^2)^4$ by expanding as $\left((1+x)+x^2\right)^4$.

$$\left((1+x)+x^2\right)^4=(1+x)^4+\binom{4}{1}(1+x)^3\left(x^2\right)$$
$$+\binom{4}{2}(1+x)^2\left(x^2\right)^2+\binom{4}{3}(1+x)\left(x^2\right)^3+\left(x^2\right)^4$$
$$\therefore\left((1+x)+x^2\right)^4=(1+x)^4+4x^2(1+x)^3+6x^4(1+x)^2$$
$$+4x^6(1+x)+x^8$$

Only the x^2 term is of interest so not all of the expansion needs to be obtained.

$$\therefore\left((1+x)+x^2\right)^4$$
$$=1+\binom{4}{1}x+\binom{4}{2}x^2+\binom{4}{3}x^3+x^4+4x^2(1+3x+...)+....+x^8$$
$$=1+4x+6x^2+...+4x^2+12x^3+...+x^8$$

The coefficient of x^2 is 10.

Consider the expansion of $(1+x)^n$ to determine the coefficient of x.

$$(1+x)^n=1+\binom{n}{1}x+\binom{n}{2}x^2+....+x^n$$

The coefficient of x is $\binom{n}{1}$.

Hence, $\binom{n}{1}=10$

$$\therefore\frac{n!}{1!(n-1)!}=10$$
$$\therefore\frac{n(n-1)!}{(n-1)!}=10$$
$$\therefore n=10$$

20 a $(1+x)^{10}=1+\binom{10}{1}x+\binom{10}{2}x^2+....+x^{10}$

$$=\binom{10}{0}+\binom{10}{1}x+\binom{10}{2}x^2+....+\binom{10}{10}x^{10}$$

Put $x=1$

$$\therefore(1+1)^{10}=\binom{10}{0}+\binom{10}{1}(1)+\binom{10}{2}(1)^2+....+\binom{10}{10}(1)^{10}$$
$$\therefore 2^{10}=\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+....+\binom{10}{10}$$

The right hand expression is the sum of the coefficients in row 10 of Pascal's triangle. Hence the sum of the coefficients in row 10 of Pascal's triangle is equal to 2^{10}.

b $\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$

LHS

$$\binom{n+1}{r}$$
$$=\frac{(n+1)!}{r!(n+1-r)!}$$

RHS

$$\binom{n}{r-1}+\binom{n}{r}$$
$$=\frac{n!}{(r-1)!(n-(r-1))!}+\frac{n!}{r!(n-r)!}$$
$$=\frac{n!}{(r-1)!(n-r+1))!}+\frac{n!}{r!(n-r)!}$$
$$=\frac{n!}{(r-1)!(n-r+1)(n-r)!}+\frac{n!}{r(r-1)!(n-r)!}$$
$$=\frac{n!\times r+n!\times(n-r+1)}{(r-1)!(n-r)!r(n-r+1)}$$
$$=\frac{n!(r+n-r+1)}{r!(n-r+1)!}$$
$$=\frac{n!(n+1)}{r!(n-r+1)!}$$
$$=\frac{(n+1)!}{r!(n-r+1)!}$$

$\therefore$ LHS = RHS

The terms in row n of Pascal's triangle are $\binom{n}{0},\binom{n}{1},\ldots\binom{n}{r-1},\binom{n}{r},\ldots\binom{n}{n}$ and the terms in row $n+1$ are $\binom{n+1}{0},\binom{n+1}{1},\ldots,\binom{n+1}{r},\ldots\binom{n+1}{n+1}$.

The relationship $\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$ is saying that the terms in row $n+1$ are formed by adding the two terms to the left and right in the preceding row n.

Row n		$\binom{n}{r-1}$		$\binom{n}{r}$	
Row $n+1$			$\binom{n+1}{r}$		

21 a The factorial symbol is found in the Main menu → Keyboard → mth → CALC
$15! = 1\,307\,674\,368\,000$

b $\binom{15}{10}$ is evaluated using the nCr key obtained from the main menu by tapping
Keyboard → mth → CALC → nCr and entering 15, 10).
$\binom{15}{10}=3003$

22 a $\binom{n}{2}=1770$
First use Simplify to express nCr$(n,2)$ as $\frac{n(n-1)}{2}$.
Drag this expression down to the next prompt position and complete the equation $\frac{n(n-1)}{2}=1770$.
Highlight the equation and drop it into the equation solver, solving for n.
This returns $n=-59, n=60$.
Since n must be a positive integer, $n=60$

b $\binom{12}{r}=220$
As the method used in part a does not enable a solution to be obtained, an alternative approach is through the Table of values in the graphing menu.
In the Main window, define $f(x)=\text{nCr}(12,x)$ by tapping Interactive → Define and completing the boxes as follows:
Func name: f obtained from Keyboard → abc
Variable/s: x
Expression: nCr$(12,x)$
In the graphing window, enter $f(x)$ at y_1:
Tap the table symbol to open Table Input and complete the boxes as follows:
Start: 1
End: 12
Step: 1
Tap the symbol to open the table of values. Scroll down the y_1 column to locate 220. This will occur at both the values $x=3$ and $x=9$.
Hence the solutions to $\binom{12}{r}=220$ are $r=3, r=9$.
There are two solutions since $\binom{12}{r}=\binom{12}{12-r}$.

Exercise 2.5 — Sets of real numbers

1 a i $\frac{6}{11}\in Q$

ii $\sqrt{27}\in Q'$ as 27 is not a perfect square

iii $(6-2)\times 3\in N$ since $(6-2)\times 3=12$ Other possible answers are $12\in Z$ or $12\in Q$

iv $\sqrt{0.25}\in Q$ since $\sqrt{0.25}=0.5=\frac{1}{2}$

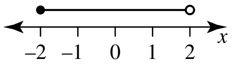
b i $17\in N$ is correct since 17 is a natural number.

ii $Q\subset N$ is incorrect since $N\subset Q$

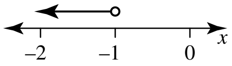
iii $Q\cup Q'=R$ is correct

2 $\frac{x-5}{(x+1)(x-3)}$ is undefined if its denominator is zero.
Since $(x+1)(x-3)=0$ when $x=-1$ or $x=3$, the expression is undefined for these values.
Note that if $x=5$, which makes the numerator zero, then $\frac{0}{(1)(-3)}=\frac{0}{-3}=0$ so the expression also equals zero (and is defined).

3 a i $[-2,2]\ =\{x:-2\le x<2\}$

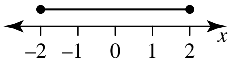


ii $\{x:x<-1\}=(-\infty,-1)$



iii $\{-2,-1,0,1,2\}=Z\cap[-2,2]$ or $\{x:-2\le x\le 2, x\in Z\}$

iv



b i $[3,5]$

ii R \ (3,5) OR $(-\infty,3)\cup(5,\infty)$

4 $R\setminus\{x:1<x\le 4\}$
This is the set of real numbers excluding those numbers in the set $(1,4]$.
This is equivalent to $(-\infty,1]\cup(4,\infty)$

5 Option e does not represent a real number since $\frac{(8-4)\times 2}{8-4\times 2}=\frac{8}{0}$ and division by zero is not defined.

6 a The statement $\sqrt{16+25}\in Q$ is false because $\sqrt{16+25}=\sqrt{41}$ and this surd is not a rational number since 41 is not a perfect square.
A correct statement is $\sqrt{16+25}\in Q'$.

b The statement $\left(\frac{4}{9}-1\right)\in Z$ is false because $\frac{4}{9}-1=-\frac{5}{9}$ and this fraction is not an integer.
A correct statement is $\left(\frac{4}{9}-1\right)\in Q$.

c The statement $R^+=\{x:x\ge 0\}$ is false because 0 is included in the set.
A correct statement is $R^+=\{x:x>0\}$.

d The statement $\sqrt{2.25}\in Q'$ is false because $\sqrt{2.25}=1.5$ which is rational.
A correct statement is $\sqrt{2.25}\in Q$.

7 $\sqrt{11}$ is not a rational number; $\frac{2}{11}$ is a rational number; 11^{11} is a large positive integer;
11π and 2^π are irrational as π is irrational; and $\sqrt{121}=11$ which is a positive integer.
The irrational numbers are $\sqrt{11}$, 11π and 2^π.

8 a $R^-\subset R$ is a true statement since the negative reals are a subset of the real numbers.

b $N\subset R^+$ is a true statement since the set of natural numbers $\{1,2,3,....\}$ is a subset of the positive real numbers.

c $Z\cup N=R$ is a false statement since the union of the integers and the natural numbers is just the set of integers: $Z\cup N=Z$ not R.

d $Q\cap Z=Z$ is a true statement since the intersection of the rational numbers with the integers is the set of integers.

e $Q'\cup Z=R\setminus Q$ is a false statement. Excluding the rationals from the real numbers leaves the set of irrationals so $R\setminus Q=Q'$. However, the union of the irrationals with the integers does not form only the set of irrationals so $Q'\cup Z\ne Q'$.

f $Z \setminus N = Z^-$ is a false statement since excluding the natural numbers from the integers gives $Z^- \cup \{0\}$, not Z^-.

9 a $\frac{1}{x+5}$ is undefined when its denominator is zero. Therefore, it is undefined if $x = -5$.

b $\frac{x+2}{x-2}$ is undefined when its denominator is zero. Therefore, it is undefined if $x = 2$.

c $\frac{x+8}{(2x+3)(5-x)}$ is undefined if either of the factor terms in its denominator are zero.

If $2x+3=0$ then $x = -\frac{3}{2}$.

If $5-x=0$ then $x=5$.

Therefore $\frac{x+8}{(2x+3)(5-x)}$ is undefined if $x = -\frac{3}{2}$ or 5.

d As $\frac{4}{x^2-4x} = \frac{4}{x(x-4)}$, it is undefined if either of the factor terms in its denominator are zero. Therefore it is undefined if $x = 0$ or 4.

10 a $[-2,3)$

b $(1,9)$

c $(-\infty,5)$

d $(0,4]$

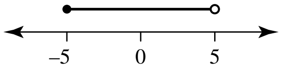
11 a $\{x: 4 < x \le 8\} = (4,8]$

b $\{x: x > -3\} = (-3,\infty)$

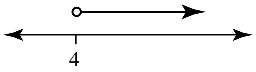
c $\{x: x \le 0\} = (-\infty,0]$

d $\{x: -2 \le x \le 0\} = [-2,0]$

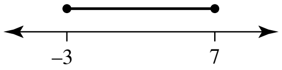
12 a $[-5,5)$



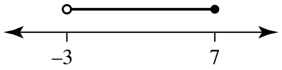
b $(4,\infty)$



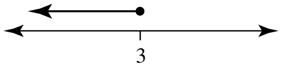
c $[-3,7]$



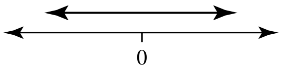
d $(-3,7]$



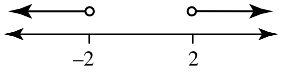
e $(-\infty,3]$



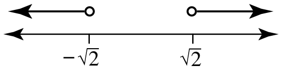
f $(-\infty,\infty) = R$



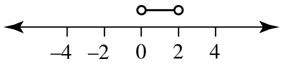
13 a $R \setminus [-2,2]$



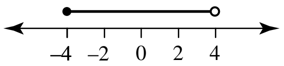
b $(-\infty,-\sqrt{2}) \cup (\sqrt{2},\infty)$



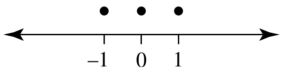
c $[-4,2) \cap (0,4) = (0,2)$



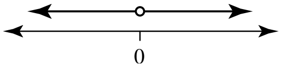
d $[-4,2) \cup (0,4) = [-4,4)$



e $\{-1,0,1\}$



f $R \setminus \{0\}$



14 a $\{x: 2 < x < 6, x \in Z\}$ is the set of integers between 2 and 6 not including 2 and 6.

Therefore, $\{x: 2 < x < 6, x \in Z\} = \{3,4,5\}$.

b $R \setminus (-1,5]$ is the set of real numbers excluding those which lie between -1 and 5 but not excluding -1 since $-1 \notin (-1,5]$.

Therefore, $R \setminus (-1,5] = (-\infty,-1] \cup (5,\infty)$.

c R^- is the set of negative real numbers so $R^- = (-\infty,0)$.

d $(-\infty,-4) \cup [2,\infty)$ is the set of all real numbers excluding those which lie in $[-4,2)$. Therefore, $(-\infty,-4) \cup [2,\infty) = R \setminus [-4,2)$.

15 a With the main menu on Standard mode, enter $\sqrt{7225}$ (the square root symbol is found in the Keyboard mth.

$\sqrt{7225} = 85$ so it is rational.

b CAS on Standard mode gives $\sqrt{75600} = 60\sqrt{21}$ so $\sqrt{75600}$ is irrational.

c 0.234234234..... is a recurring decimal. Enter it as 0.234234234234234234234234 so it occupies the entire row.

On Standard mode this gives the value as $\frac{26}{111}$. so it is rational.

16 The formula for the area of a circle is $A = \pi r^2$.

With $r = \frac{1}{2}d$, $A = \frac{d^2\pi}{4}$.

The actual area of the circle is $A = \frac{\pi}{4}d^2$.

The Eygptian formula gives $A = \frac{64}{81}d^2$.

$\therefore \frac{\pi}{4} = \frac{64}{81}$

$\therefore \pi = \frac{64}{81} \times 4$

$\therefore \pi = \frac{256}{81}$

Evaluating on Decimal mode, gives the estimate $\pi = 3.160493827$.

CAS on Decimal mode gives $\frac{22}{7} = 3.142857143$ to 9 decimal places and $\pi = 3.141592654$ to 9 decimal places.

This shows $\frac{22}{7}$ is a better approximation.

Exercise 2.6 — Surds

1 a $3\sqrt{3} = \sqrt{9\times 3} = \sqrt{27}$, $4\sqrt{5} = \sqrt{16\times 5} = \sqrt{80}$,
$5\sqrt{2} = \sqrt{25\times 2} = \sqrt{50}$ and $5 = \sqrt{25}$.
In increasing order the set is written as $\left\{5, 3\sqrt{3}, 5\sqrt{2}, 4\sqrt{5}\right\}$

b **i** $\sqrt{84}=\sqrt{4\times21}=2\sqrt{21}$

ii $2\sqrt{108ab^2}$

$=2\sqrt{36\times3ab^2}$

$=2\times6\times b\times\sqrt{3a}$

$=12b\sqrt{3a}$

2 Use perfect cube factors

$\sqrt[3]{384}$

$=\sqrt[3]{4\times4\times4\times6}$

$=\sqrt[3]{4^3\times6}$

$=4\sqrt[3]{6}$

3 **a** $\sqrt{5}-4\sqrt{7}-7\sqrt{5}+3\sqrt{7}$

$\sqrt{5}-7\sqrt{5}+3\sqrt{7}-4\sqrt{7}$

$=-6\sqrt{5}-\sqrt{7}$

b $3\sqrt{48}-4\sqrt{27}+3\sqrt{32}$

$=3\sqrt{16\times3}-4\sqrt{9\times3}+3\sqrt{16\times2}$

$=3\times4\sqrt{3}-4\times3\sqrt{3}+3\times4\sqrt{2}$

$=12\sqrt{3}-12\sqrt{3}+12\sqrt{2}$

$=12\sqrt{2}$

c $3\sqrt{5}\times7\sqrt{15}$

$=21\sqrt{75}$

$=21\sqrt{25\times3}$

$=21\times5\sqrt{3}$

$=105\sqrt{3}$

4 $\frac{1}{2}\sqrt{12}-\frac{1}{5}\sqrt{80}+\frac{\sqrt{10}}{\sqrt{2}}+\sqrt{243}+5$

$=\frac{1}{2}\sqrt{4\times3}-\frac{1}{5}\sqrt{16\times5}+\sqrt{\frac{10}{2}}+\sqrt{81\times3}+5$

$=\frac{1}{2}\times2\sqrt{3}-\frac{1}{5}\times4\sqrt{5}+\sqrt{5}+9\sqrt{3}+5$

$=\sqrt{3}-\frac{4}{5}\sqrt{5}+\sqrt{5}+9\sqrt{3}+5$

$=10\sqrt{3}+\frac{1}{5}\sqrt{5}+5$

5 **a** $2\sqrt{3}\left(4\sqrt{15}+5\sqrt{3}\right)$

$=8\sqrt{45}+10\sqrt{9}$

$=8\times3\sqrt{5}+10\times3$

$=24\sqrt{5}+30$

b $\left(\sqrt{3}-8\sqrt{2}\right)\left(5\sqrt{5}-2\sqrt{21}\right)$

$=5\sqrt{15}-2\sqrt{63}-40\sqrt{10}+16\sqrt{42}$

$=5\sqrt{15}-2\times3\sqrt{7}-40\sqrt{10}+16\sqrt{42}$

$=5\sqrt{15}-6\sqrt{7}-40\sqrt{10}+16\sqrt{42}$

c $\left(4\sqrt{3}-5\sqrt{2}\right)^2$

$=16\times3-2\times20\sqrt{6}+25\times2$

$=48-40\sqrt{6}+50$

$=98-40\sqrt{6}$

d $\left(3\sqrt{5}-2\sqrt{11}\right)\left(3\sqrt{5}+2\sqrt{11}\right)$

$=9\times5-4\times11$

$=45-44$

$=1$

6 **a** $\left(\sqrt{2}+1\right)^3$

$=\left(\sqrt{2}\right)^3+3\left(\sqrt{2}\right)^2(1)+3\left(\sqrt{2}\right)(1)^2+(1)^3$

$=2\sqrt{2}+6+3\sqrt{2}+1$

$=7+5\sqrt{2}$

b $\left(\sqrt{2}+\sqrt{6}\right)^2-2\sqrt{3}\left(\sqrt{2}+\sqrt{6}\right)\left(\sqrt{2}-\sqrt{6}\right)=a+b\sqrt{3}$

Consider

$\left(\sqrt{2}+\sqrt{6}\right)^2-2\sqrt{3}\left(\sqrt{2}+\sqrt{6}\right)\left(\sqrt{2}-\sqrt{6}\right)$

$=2+2\sqrt{12}+6-2\sqrt{3}(2-6)$

$=8+4\sqrt{3}-2\sqrt{3}\times-4$

$=8+4\sqrt{3}+8\sqrt{3}$

$=8+12\sqrt{3}$

$\therefore 8+12\sqrt{3}=a+b\sqrt{3}$

$\therefore a=8,\ b=12$

7 **a** **i** $\frac{6}{7\sqrt{2}}$

$=\frac{6}{7\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$

$=\frac{6\sqrt{2}}{7\times2}$

$=\frac{3\sqrt{2}}{7}$

ii $\frac{3\sqrt{5}+7\sqrt{15}}{2\sqrt{3}}$

$=\frac{3\sqrt{5}+7\sqrt{15}}{2\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$

$=\frac{3\sqrt{15}+7\sqrt{45}}{6}$

$=\frac{3\sqrt{15}+21\sqrt{5}}{6}$

$=\frac{3\left(\sqrt{15}+7\sqrt{5}\right)}{6}$

$=\frac{\sqrt{15}+7\sqrt{5}}{2}$

b $14\sqrt{6}+\frac{12}{\sqrt{6}}-5\sqrt{24}$

$=14\sqrt{6}+\frac{12}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}-5\times2\sqrt{6}$

$=14\sqrt{6}+\frac{12\sqrt{6}}{6}-10\sqrt{6}$

$=14\sqrt{6}+2\sqrt{6}-10\sqrt{6}$

$=6\sqrt{6}$

c $\frac{10}{4\sqrt{3}+3\sqrt{2}}$

$=\frac{10}{4\sqrt{3}+3\sqrt{2}}\times\frac{4\sqrt{3}-3\sqrt{2}}{4\sqrt{3}-3\sqrt{2}}$

$=\frac{10\left(4\sqrt{3}-3\sqrt{2}\right)}{48-18}$

$=\frac{10\left(4\sqrt{3}-3\sqrt{2}\right)}{30}$

$=\frac{4\sqrt{3}-3\sqrt{2}}{3}$

d $\frac{1}{p^2-2}, \; p = 4\sqrt{3}+1$

$$
\begin{aligned}
&= \frac{1}{\left(4\sqrt{3}+1\right)^2-1} \\
&= \frac{1}{\left(48+8\sqrt{3}+1\right)-1} \\
&= \frac{1}{48+8\sqrt{3}} \\
&= \frac{1}{8\left(6+\sqrt{3}\right)} \\
&= \frac{1}{8\left(6+\sqrt{3}\right)} \times \frac{6-\sqrt{3}}{6-\sqrt{3}} \\
&= \frac{6-\sqrt{3}}{8(36-3)} \\
&= \frac{6-\sqrt{3}}{264}
\end{aligned}
$$

8 **a** $\frac{2\sqrt{3}-1}{\sqrt{3}+1} - \frac{\sqrt{3}}{\sqrt{3}+2}$

$$
\begin{aligned}
&= \frac{2\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} - \frac{\sqrt{3}}{\sqrt{3}+2} \times \frac{\sqrt{3}-2}{\sqrt{3}-2} \\
&= \frac{\left(2\sqrt{3}-1\right)\left(\sqrt{3}-1\right)}{3-1} - \frac{\sqrt{3}\left(\sqrt{3}-2\right)}{3-4} \\
&= \frac{6-3\sqrt{3}+1}{2} - \frac{3-2\sqrt{3}}{-1} \\
&= \frac{7-3\sqrt{3}}{2} + \frac{3-2\sqrt{3}}{1} \\
&= \frac{7-3\sqrt{3}+6-4\sqrt{3}}{2} \\
&= \frac{13-7\sqrt{3}}{2}
\end{aligned}
$$

b $\frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$$
\begin{aligned}
&= \frac{\left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right)+\left(\sqrt{3}+1\right)\left(\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} \\
&= \frac{\left(3-2\sqrt{3}+1\right)+\left(3+2\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} \\
&= \frac{8}{3-1} \\
&= 4
\end{aligned}
$$

This is a rational number.

9 $\sqrt{900} = 30$, $\sqrt{\frac{4}{9}} = \frac{2}{3}$, $\sqrt{1.44} = 1.2$ and $\sqrt[3]{27} = 3$ so these are all rational.

While π is irrational it does not contain a radical sign so it is not a surd.

The surds are $\sqrt{8}$, $\sqrt{10^3}$ and $\sqrt[3]{36}$.

10 **a** $4\sqrt{5}$

$$
\begin{aligned}
&= \sqrt{16\times 5} \\
&= \sqrt{80}
\end{aligned}
$$

b $2\sqrt[3]{6}$

$$
\begin{aligned}
&= \sqrt[3]{8} \times \sqrt[3]{6} \\
&= \sqrt[3]{48}
\end{aligned}
$$

c $\frac{9\sqrt{7}}{4}$

$$
\begin{aligned}
&= \frac{\sqrt{81}\times\sqrt{7}}{\sqrt{16}} \\
&= \frac{\sqrt{567}}{\sqrt{16}} \\
&= \sqrt{\frac{567}{16}}
\end{aligned}
$$

d $\frac{3}{\sqrt{3}}$

$$
\begin{aligned}
&= \frac{\sqrt{9}}{\sqrt{3}} \\
&= \sqrt{\frac{9}{3}} \\
&= \sqrt{3}
\end{aligned}
$$

e $ab\sqrt{c} = \sqrt{a^2b^2c}$

f $m\sqrt[3]{n} = \sqrt[3]{m^3n}$

11 **a** $\sqrt{75}$

$$
\begin{aligned}
&= \sqrt{25\times 3} \\
&= 5\sqrt{3}
\end{aligned}
$$

b $5\sqrt{48}$

$$
\begin{aligned}
&= 5\sqrt{16\times 3} \\
&= 5\times 4\sqrt{3} \\
&= 20\sqrt{3}
\end{aligned}
$$

c $\sqrt{2000}$

$$
\begin{aligned}
&= \sqrt{20\times 100} \\
&= 10\sqrt{20} \\
&= 10\sqrt{4\times 5} \\
&= 10\times 2\sqrt{5} \\
&= 20\sqrt{5}
\end{aligned}
$$

d $3\sqrt{288}$

$$
\begin{aligned}
&= 3\sqrt{2\times 144} \\
&= 3\times 12\sqrt{2} \\
&= 36\sqrt{2}
\end{aligned}
$$

e $2\sqrt{72}$

$$
\begin{aligned}
&= 2\sqrt{36\times 2} \\
&= 2\times 6\sqrt{2} \\
&= 12\sqrt{2}
\end{aligned}
$$

f $\sqrt[3]{54}$

$$
\begin{aligned}
&= \sqrt[3]{27\times 2} \\
&= 3\sqrt[3]{2}
\end{aligned}
$$

12 **a** $3\sqrt{7}+8\sqrt{3}+12\sqrt{7}-9\sqrt{3}$

$$
\begin{aligned}
&= 3\sqrt{7}+12\sqrt{7}+8\sqrt{3}-9\sqrt{3} \\
&= 15\sqrt{7}-\sqrt{3}
\end{aligned}
$$

b $10\sqrt{2}-12\sqrt{6}+4\sqrt{6}-8\sqrt{2}$

$$
\begin{aligned}
&= 10\sqrt{2}-8\sqrt{2}-12\sqrt{6}+4\sqrt{6} \\
&= 2\sqrt{2}-8\sqrt{6}
\end{aligned}
$$

c $3\sqrt{50}-\sqrt{18}$

$$
\begin{aligned}
&= 3\times 5\sqrt{2}-3\sqrt{2} \\
&= 15\sqrt{2}-3\sqrt{2} \\
&= 12\sqrt{2}
\end{aligned}
$$

d $8\sqrt{45}+2\sqrt{125}$
$=8\times3\sqrt{5}+2\times5\sqrt{5}$
$=24\sqrt{5}+10\sqrt{5}$
$=34\sqrt{5}$

e $\sqrt{6}+7\sqrt{5}+4\sqrt{24}-8\sqrt{20}$
$=\sqrt{6}+7\sqrt{5}+4\times2\sqrt{6}-8\times2\sqrt{5}$
$=\sqrt{6}+7\sqrt{5}+8\sqrt{6}-16\sqrt{5}$
$=9\sqrt{6}-9\sqrt{5}$

f $2\sqrt{12}-7\sqrt{243}+\frac{1}{2}\sqrt{8}-\frac{2}{3}\sqrt{162}$
$=2\times2\sqrt{3}-7\times9\sqrt{3}+\frac{1}{2}\times2\sqrt{2}-\frac{2}{3}\times9\sqrt{2}$
$=4\sqrt{3}-63\sqrt{3}+\sqrt{2}-6\sqrt{2}$
$=-59\sqrt{3}-5\sqrt{2}$

13 a $4\sqrt{5}\times2\sqrt{7}=8\sqrt{35}$

b $-10\sqrt{6}\times\left(-8\sqrt{10}\right)$
$=80\sqrt{60}$
$=80\times2\sqrt{15}$
$=160\sqrt{15}$

c $3\sqrt{8}\times2\sqrt{5}$
$=6\sqrt{40}$
$=6\times2\sqrt{10}$
$=12\sqrt{10}$

d $\sqrt{18}\times\sqrt{72}$
As the numbers are large, simplify each surd before multiplying them together.
$=3\sqrt{2}\times6\sqrt{2}$
$=18\sqrt{4}$
$=18\times2$
$=36$

e $\dfrac{4\sqrt{27}\times\sqrt{147}}{2\sqrt{3}}$
$=\dfrac{4\times3\sqrt{3}\times\sqrt{49\times3}}{2\sqrt{3}}$
$=\dfrac{\overset{6}{\cancel{12}}\sqrt{3}\times7\cancel{\sqrt{3}}}{\cancel{2}\cancel{\sqrt{3}}}$
$=6\sqrt{3}\times7$
$=42\sqrt{3}$

f $5\sqrt{2}\times\sqrt{3}\times4\sqrt{5}\times\dfrac{\sqrt{6}}{6}+3\sqrt{2}\times7\sqrt{10}$
$=\dfrac{20\sqrt{2\times3\times5\times6}}{6}+21\sqrt{20}$
$=\dfrac{10\sqrt{180}}{3}+21\times2\sqrt{5}$
$=\dfrac{10\times6\sqrt{5}}{3}+42\sqrt{5}$
$=20\sqrt{5}+42\sqrt{5}$
$=62\sqrt{5}$

14 a $\sqrt{2}\left(3\sqrt{5}-7\sqrt{6}\right)$
$=3\sqrt{10}-7\sqrt{12}$
$=3\sqrt{10}-14\sqrt{3}$

b $5\sqrt{3}\left(7-3\sqrt{3}+2\sqrt{6}\right)$
$=35\sqrt{3}-15\sqrt{9}+10\sqrt{18}$
$=35\sqrt{3}-15\times3+30\sqrt{2}$
$=35\sqrt{3}-45+30\sqrt{2}$

c $2\sqrt{10}-3\sqrt{6}\left(3\sqrt{15}+2\sqrt{6}\right)$
$=2\sqrt{10}-9\sqrt{90}-6\times6$
$=2\sqrt{10}-27\sqrt{10}-36$
$=-25\sqrt{10}-36$

d $\left(2\sqrt{3}+\sqrt{5}\right)\left(3\sqrt{2}+4\sqrt{7}\right)$
$=6\sqrt{6}+8\sqrt{21}+3\sqrt{10}+4\sqrt{35}$

e $\left(5\sqrt{2}-3\sqrt{6}\right)\left(2\sqrt{3}+3\sqrt{10}\right)$
$=10\sqrt{6}+15\sqrt{20}-6\sqrt{18}-9\sqrt{60}$
$=10\sqrt{6}+30\sqrt{5}-18\sqrt{2}-18\sqrt{15}$

f $\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\left(\sqrt[3]{x}\right)^2+\sqrt[3]{xy}+\left(\sqrt[3]{y}\right)^2\right)$
Letting $a=\sqrt[3]{x}$, $b=\sqrt[3]{y}$, the expression becomes
$(a-b)(a^2+ab+b^2)$
$=a^3-b^3$
$=\left(\sqrt[3]{x}\right)^3-\left(\sqrt[3]{y}\right)^3$
$=x-y$

15 a $\left(2\sqrt{2}+3\right)^2$
$=\left(2\sqrt{2}\right)^2+2\times\left(2\sqrt{2}\right)\times3+3^2$
$=4\times2+12\sqrt{2}+9$
$=17+12\sqrt{2}$

b $\left(3\sqrt{6}-2\sqrt{3}\right)^2$
$=9\times6-2\times6\sqrt{18}+4\times3$
$=54-2\times18\sqrt{2}+12$
$=66-36\sqrt{2}$

c $\left(\sqrt{7}-\sqrt{5}\right)^3$
$=\left(\sqrt{7}\right)^3-3\left(\sqrt{7}\right)^2\left(\sqrt{5}\right)+3\left(\sqrt{7}\right)\left(\sqrt{5}\right)^2-\left(\sqrt{5}\right)^3$
$=7\sqrt{7}-3\times7\times\sqrt{5}+3\sqrt{7}\times5-5\sqrt{5}$
$=7\sqrt{7}-21\sqrt{5}+15\sqrt{7}-5\sqrt{5}$
$=22\sqrt{7}-26\sqrt{5}$

d $\left(2\sqrt{5}+\sqrt{3}\right)\left(2\sqrt{5}-\sqrt{3}\right)$
$=\left(2\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2$
$=20-3$
$=17$

e $\left(10\sqrt{2}-3\sqrt{5}\right)\left(10\sqrt{2}+3\sqrt{5}\right)$
$=\left(10\sqrt{2}\right)^2-\left(3\sqrt{5}\right)^2$
$=200-45$
$=155$

f $\left(\sqrt{3}+\sqrt{2}+1\right)\left(\sqrt{3}+\sqrt{2}-1\right)$
$=\left(\left(\sqrt{3}+\sqrt{2}\right)+1\right)\left(\left(\sqrt{3}+\sqrt{2}\right)-1\right)$
$=\left(\sqrt{3}+\sqrt{2}\right)^2-(1)^2$
$=\left(3+2\sqrt{6}+2\right)-1$
$=4+2\sqrt{6}$

16 a $\dfrac{3\sqrt{2}}{4\sqrt{3}}$

$$=\frac{3\sqrt{2}}{4\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$=\frac{3\sqrt{6}}{4\times 3}$$
$$=\frac{\sqrt{6}}{4}$$

b $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{2}}$

$$=\frac{\sqrt{5}+\sqrt{2}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{\sqrt{10}+2}{2}$$

c $\dfrac{\sqrt{12}-3\sqrt{2}}{2\sqrt{18}}$

$$=\frac{2\sqrt{3}-3\sqrt{2}}{6\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{2\sqrt{6}-3\times 2}{6\times 2}$$
$$=\frac{2\left(\sqrt{6}-3\right)}{12}$$
$$=\frac{\sqrt{6}-3}{6}$$

d $\dfrac{1}{\sqrt{6}+\sqrt{2}}$

$$=\frac{1}{\sqrt{6}+\sqrt{2}}\times\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}$$
$$=\frac{\sqrt{6}-\sqrt{2}}{6-2}$$
$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

e $\dfrac{2\sqrt{10}+1}{5-\sqrt{10}}$

$$=\frac{2\sqrt{10}+1}{5-\sqrt{10}}\times\frac{5+\sqrt{10}}{5+\sqrt{10}}$$
$$=\frac{\left(2\sqrt{10}+1\right)\left(5+\sqrt{10}\right)}{25-10}$$
$$=\frac{10\sqrt{10}+2\times 10+5+\sqrt{10}}{15}$$
$$=\frac{11\sqrt{10}+25}{15}$$

f $\dfrac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

$$=\frac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$
$$=\frac{\left(3\sqrt{3}+2\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}{3-2}$$
$$=\left(3\sqrt{3}+2\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$$
$$=3\times 3-3\sqrt{6}+2\sqrt{6}-2\times 2$$
$$=5-\sqrt{6}$$

17 a $4\sqrt{5}-2\sqrt{6}+\dfrac{3}{\sqrt{6}}-\dfrac{10}{3\sqrt{5}}$

$$=4\sqrt{5}-2\sqrt{6}+\frac{3}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}-\frac{10}{3\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}$$
$$=4\sqrt{5}-2\sqrt{6}+\frac{3\sqrt{6}}{6}-\frac{10\sqrt{5}}{15}$$
$$=4\sqrt{5}-2\sqrt{6}+\frac{\sqrt{6}}{2}-\frac{2\sqrt{5}}{3}$$
$$=\frac{24\sqrt{5}-12\sqrt{6}+3\sqrt{6}-4\sqrt{5}}{6}$$
$$=\frac{20\sqrt{5}-9\sqrt{6}}{6}$$

b $\sqrt{2}\left(2\sqrt{10}+9\sqrt{8}\right)-\dfrac{\sqrt{5}}{\sqrt{5}-2}$

$$=2\sqrt{20}+9\sqrt{16}-\frac{\sqrt{5}}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}$$
$$=4\sqrt{5}+36-\frac{\sqrt{5}\left(\sqrt{5}+2\right)}{5-4}$$
$$=4\sqrt{5}+36-\sqrt{5}\left(\sqrt{5}+2\right)$$
$$=4\sqrt{5}+36-5-2\sqrt{5}$$
$$=2\sqrt{5}+31 \text{ or } \frac{2\sqrt{5}+31}{1}$$

c $\dfrac{3}{2\sqrt{3}\left(\sqrt{2}+\sqrt{3}\right)^2}+\dfrac{2}{\sqrt{3}}$

$$=\frac{3}{2\sqrt{3}\left(2+2\sqrt{6}+3\right)}+\frac{2}{\sqrt{3}}$$
$$=\frac{3}{2\sqrt{3}\left(5+2\sqrt{6}\right)}+\frac{2}{\sqrt{3}}$$
$$=\frac{3}{10\sqrt{3}+4\sqrt{18}}+\frac{2}{\sqrt{3}}$$
$$=\frac{3}{10\sqrt{3}+12\sqrt{2}}+\frac{2}{\sqrt{3}}$$
$$=\frac{3}{10\sqrt{3}+12\sqrt{2}}\times\frac{10\sqrt{3}-12\sqrt{2}}{10\sqrt{3}-12\sqrt{2}}+\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$=\frac{3\left(10\sqrt{3}-12\sqrt{2}\right)}{300-288}+\frac{2\sqrt{3}}{3}$$
$$=\frac{3\left(10\sqrt{3}-12\sqrt{2}\right)}{12}+\frac{2\sqrt{3}}{3}$$
$$=\frac{3\left(10\sqrt{3}-12\sqrt{2}\right)+8\sqrt{3}}{12}$$
$$=\frac{30\sqrt{3}-36\sqrt{2}+8\sqrt{3}}{12}$$
$$=\frac{38\sqrt{3}-36\sqrt{2}}{12}$$
$$=\frac{2\left(19\sqrt{3}-18\sqrt{2}\right)}{12}$$
$$=\frac{19\sqrt{3}-18\sqrt{2}}{6}$$

d $\frac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}}+\frac{2\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$

As denominators are a pair of conjugates, express each fraction on the common denominator rather than rationalise each denominator.

$$=\frac{\left(2\sqrt{3}-\sqrt{2}\right)^2+\left(2\sqrt{3}+\sqrt{2}\right)^2}{\left(2\sqrt{3}+\sqrt{2}\right)\left(2\sqrt{3}-\sqrt{2}\right)}$$

$$=\frac{\left(12-4\sqrt{6}+2\right)+\left(12+4\sqrt{6}+2\right)}{4\times 3-2}$$

$$=\frac{28}{10}$$

$$=\frac{14}{5}$$

e $\frac{\sqrt{2}}{16-4\sqrt{7}}-\frac{\sqrt{14}+2\sqrt{2}}{9\sqrt{7}}$

$$=\frac{\sqrt{2}}{4\left(4-\sqrt{7}\right)}\times\frac{4+\sqrt{7}}{4+\sqrt{7}}-\frac{\sqrt{14}+2\sqrt{2}}{9\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}$$

$$=\frac{\sqrt{2}\left(4+\sqrt{7}\right)}{4(16-7)}-\frac{\sqrt{7}\left(\sqrt{14}+2\sqrt{2}\right)}{63}$$

$$=\frac{4\sqrt{2}+\sqrt{14}}{36}-\frac{\sqrt{7}\times\sqrt{2}\left(\sqrt{7}+2\right)}{63}$$

$$=\frac{7\left(4\sqrt{2}+\sqrt{14}\right)-4\sqrt{7}\times\sqrt{2}\left(\sqrt{7}+2\right)}{4\times 9\times 7}$$

$$=\frac{28\sqrt{2}+7\sqrt{14}-4\times 7\times\sqrt{2}-8\sqrt{14}}{252}$$

$$=-\frac{\sqrt{14}}{252}$$

f $\frac{\left(2-\sqrt{3}\right)^2}{2+\sqrt{3}}+\frac{2\sqrt{3}}{4-3\sqrt{2}}$

$$=\frac{4-4\sqrt{3}+3}{2+\sqrt{3}}+\frac{2\sqrt{3}}{4-3\sqrt{2}}$$

$$=\frac{7-4\sqrt{3}}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}+\frac{2\sqrt{3}}{4-3\sqrt{2}}\times\frac{4+3\sqrt{2}}{4+3\sqrt{2}}$$

$$=\frac{\left(7-4\sqrt{3}\right)\left(2-\sqrt{3}\right)}{4-3}+\frac{2\sqrt{3}\left(4+3\sqrt{2}\right)}{16-9\times 2}$$

$$=\left(7-4\sqrt{3}\right)\left(2-\sqrt{3}\right)+\frac{2\sqrt{3}\left(4+3\sqrt{2}\right)}{-2}$$

$$=\left(7-4\sqrt{3}\right)\left(2-\sqrt{3}\right)-\sqrt{3}\left(4+3\sqrt{2}\right)$$

$$=14-15\sqrt{3}+12-4\sqrt{3}-3\sqrt{6}$$

$$=26-19\sqrt{3}-3\sqrt{6}\text{ or }\frac{26-19\sqrt{3}-3\sqrt{6}}{1}$$

18 a $x=2\sqrt{3}-\sqrt{10}$

i $x+\frac{1}{x}$

$$=2\sqrt{3}-\sqrt{10}+\frac{1}{2\sqrt{3}-\sqrt{10}}$$

$$=2\sqrt{3}-\sqrt{10}+\frac{1}{2\sqrt{3}-\sqrt{10}}\times\frac{2\sqrt{3}+\sqrt{10}}{2\sqrt{3}-\sqrt{10}}$$

$$=2\sqrt{3}-\sqrt{10}+\frac{2\sqrt{3}+\sqrt{10}}{12-10}$$

$$=2\sqrt{3}-\sqrt{10}+\frac{2\sqrt{3}+\sqrt{10}}{2}$$

$$=\frac{4\sqrt{3}-2\sqrt{10}+2\sqrt{3}+\sqrt{10}}{2}$$

$$=\frac{6\sqrt{3}-\sqrt{10}}{2}$$

ii $x^2-4\sqrt{3}x$

$$=\left(2\sqrt{3}-\sqrt{10}\right)^2-4\sqrt{3}\left(2\sqrt{3}-\sqrt{10}\right)$$

$$=12-4\sqrt{30}+10-8\times 3+4\sqrt{30}$$

$$=-2$$

b $y=\frac{\sqrt{7}+2}{\sqrt{7}-2}$

i $y-\frac{1}{y}$

$$=\frac{\sqrt{7}+2}{\sqrt{7}-2}-\frac{\sqrt{7}-2}{\sqrt{7}+2}$$

$$=\frac{\left(\sqrt{7}+2\right)^2-\left(\sqrt{7}-2\right)^2}{\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)}$$

$$=\frac{7+4\sqrt{7}+4-\left(7-4\sqrt{7}+4\right)}{7-4}$$

$$=\frac{8\sqrt{7}}{3}$$

ii $\frac{1}{y^2-1}$

$$=\frac{1}{\frac{\left(\sqrt{7}+2\right)^2}{\left(\sqrt{7}-2\right)^2}-1}$$

$$=1\div\left(\frac{\left(\sqrt{7}+2\right)^2-\left(\sqrt{7}-2\right)^2}{\left(\sqrt{7}-2\right)^2}\right)$$

$$=\frac{\left(\sqrt{7}-2\right)^2}{\left(\sqrt{7}+2\right)^2-\left(\sqrt{7}-2\right)^2}$$

$$=\frac{7-4\sqrt{7}+4}{11+4\sqrt{7}-\left(11-4\sqrt{7}\right)}$$

$$=\frac{11-4\sqrt{7}}{8\sqrt{7}}$$

$$=\frac{11-4\sqrt{7}}{8\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}$$

$$=\frac{11\sqrt{7}-28}{56}$$

c i $\frac{1}{\sqrt{7}+\sqrt{3}}-\frac{1}{\sqrt{7}-\sqrt{3}}=m\sqrt{7}+n\sqrt{3}$

$$\therefore\frac{\left(\sqrt{7}-\sqrt{3}\right)-\left(\sqrt{7}+\sqrt{3}\right)}{\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)}=m\sqrt{7}+n\sqrt{3}$$

$$\therefore\frac{-2\sqrt{3}}{7-3}=m\sqrt{7}+n\sqrt{3}$$

$$\therefore-\frac{1}{2}\sqrt{3}=m\sqrt{7}+n\sqrt{3}$$

$$\therefore m=0, n=-\frac{1}{2}$$

ii $\left(2+\sqrt{3}\right)^4-\frac{7\sqrt{3}}{\left(2+\sqrt{3}\right)^2}=m+\sqrt{n}$

$$\left(2+\sqrt{3}\right)^2=4+4\sqrt{3}+3$$

$$=7+4\sqrt{3}$$

$$\therefore\left(2+\sqrt{3}\right)^4=\left(7+4\sqrt{3}\right)^2$$

$$=49+56\sqrt{3}+48$$

$$=97+56\sqrt{3}$$

Hence,

$$97+56\sqrt{3}-\frac{7\sqrt{3}}{7+4\sqrt{3}}=m+\sqrt{n}$$

$$\therefore 97+56\sqrt{3}-\frac{7\sqrt{3}}{7+4\sqrt{3}}\times\frac{7-4\sqrt{3}}{7-4\sqrt{3}}=m+\sqrt{n}$$

$$\therefore 97+56\sqrt{3}-\frac{7\sqrt{3}\left(7-4\sqrt{3}\right)}{49-48}=m+\sqrt{n}$$

$$\therefore 97+56\sqrt{3}-49\sqrt{3}+28\times 3=m+\sqrt{n}$$

$$\therefore 181+7\sqrt{3}=m+\sqrt{n}$$

$$\therefore 181+\sqrt{49\times 3}=m+\sqrt{n}$$

$$\therefore 181+\sqrt{147}=m+\sqrt{n}$$

$$\therefore m=181,\ n=147$$

d $x_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$

i $x_1=-\dfrac{b}{2a}+\dfrac{\sqrt{b^2-4ac}}{2a}$ so the conjugate is

$$\begin{aligned}x_2&=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\\&=\frac{-b-\sqrt{b^2-4ac}}{2a}\end{aligned}$$

ii x_1+x_2

$$\begin{aligned}&=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}+-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\\&=-\frac{2b}{2a}\\&=-\frac{b}{a}\end{aligned}$$

iii x_1x_2

$$\begin{aligned}&=\left(-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)\\&=\left(-\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\\&=\frac{b^2}{4a^2}-\frac{b^2-4ac}{4a^2}\\&=\frac{b^2-b^2+4ac}{4a^2}\\&=\frac{4ac}{4a^2}\\&=\frac{c}{a}\end{aligned}$$

19 $A\left(\sqrt{2},-1\right),B\left(\sqrt{5},\sqrt{10}\right),C\left(\sqrt{10},\sqrt{5}\right)$

a $d(A,B)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$$\therefore AB=\sqrt{\left(\sqrt{5}-\sqrt{2}\right)^2+\left(\sqrt{10}+1\right)^2}$$

On the CAS calculator, use Simplify to obtain $AB=3\sqrt{2}$.

$$\begin{aligned}BC&=\sqrt{\left(\sqrt{10}-\sqrt{5}\right)^2+\left(\sqrt{5}-\sqrt{10}\right)^2}\\&=\sqrt{2\left(\sqrt{10}-\sqrt{5}\right)^2}\\&=\sqrt{2}\left(\sqrt{10}-\sqrt{5}\right)\\&=\sqrt{20}-\sqrt{10}\\&=2\sqrt{5}-\sqrt{10}\end{aligned}$$

(or use CAS to obtain $-\sqrt{10}+2\sqrt{5}$)

$$AC=\sqrt{\left(\sqrt{10}-\sqrt{2}\right)^2+\left(\sqrt{5}+1\right)^2}$$

$$\therefore AC=\sqrt{-2\sqrt{5}+18}$$

b $\sqrt{a^2\pm b}\simeq a\pm\dfrac{b}{2a}$

The surds present in the lengths of the three sides are $\sqrt{2},\sqrt{5},\sqrt{10}.$

$$\begin{aligned}\sqrt{2}&=\sqrt{1^2+1},\ a=1,b=1\\&\simeq 1+\frac{1}{2\times 1}\\&=1.5\end{aligned}$$

$$\begin{aligned}\sqrt{5}&=\sqrt{2^2+1},\ a=2,b=1\\&\simeq 2+\frac{1}{2\times 2}\\&=2.25\end{aligned}$$

$$\begin{aligned}\sqrt{10}&=\sqrt{3^2+1},\ a=3,b=1\\&\simeq 3+\frac{1}{2\times 3}\\&=3+\frac{1}{6}\\&\simeq 3.167\end{aligned}$$

$$\begin{aligned}AB&=3\sqrt{2}\\\therefore AB&\simeq 3\times 1.5\\\therefore AB&=4.5\end{aligned}$$

$$\begin{aligned}BC&=-\sqrt{10}+2\sqrt{5}\\\therefore BC&\simeq -3.167+2\times 2.25\\\therefore BC&=4.5-3.167\\\therefore BC&\simeq 1.333\end{aligned}$$

$$\begin{aligned}AC&=\sqrt{18-2\sqrt{5}}\\&\simeq\sqrt{18-4.5}\\&=\sqrt{13.5}\end{aligned}$$

Using the approximation formula,

$$\begin{aligned}\sqrt{13.5}&=\sqrt{4^2-2.5},\ a=4,b=2.5\\&\simeq 4-\frac{2.5}{2\times 4}\\&=4-\frac{2.5}{8}\\&=4-0.3125\\&\simeq 3.6875\end{aligned}$$

$$\therefore AC\simeq 3.6875$$

c From the approximation values, the side AB is the longest. Switch calculator to decimal mode and calculate $3\sqrt{2}$. The longest side is $AB\simeq 4.2$ units.

20 **a** Formula for the area of a rectangle is $A=lw$.

$$A=\left(\sqrt{6}+\sqrt{3}+1\right)\left(\sqrt{3}+2\right)$$

Expand using CAS

$$\therefore A=2\sqrt{6}+3\sqrt{3}+3\sqrt{2}+5$$

The area is $\left(2\sqrt{6}+3\sqrt{3}+3\sqrt{2}+5\right)$ square metres.

b Cost for mowing the area is $\$50-\$23.35=\$26.65$.

Cost per square metre is $\dfrac{26.65}{2\sqrt{6}+3\sqrt{3}+3\sqrt{2}+5}$ dollars. This evaluates to \$1.38 per sq m.

Topic 3 — Quadratic relationships

Exercise 3.2 — Quadratic equations with rational roots

1 **a** $10x^2+23x=21$

$10x^2+23x-21=0$

$\therefore (10x-7)(x+3)=0$

$\therefore x=\frac{7}{10}, x=-3$

b Zero of $x=-5 \Rightarrow (x-(-5))=(x+5)$ is a factor and zero of $x=0 \Rightarrow (x-0)=x$ is a factor

The quadratic takes the form $(x+5)x=x^2+5x$

2 Roots are the solutions

$32x^2-96x+72=0$

$\therefore 8(4x^2-12x+9)=0$

$\therefore 4x^2-12x+9=0$

$\therefore (2x-3)^2=0$

$\therefore 2x-3=0$

$\therefore x=\frac{3}{2}$

3 $(5x-1)^2-16=0$

$\therefore (5x-1)^2=16$

$\therefore 5x-1=\pm\sqrt{16}$

$\therefore 5x-1=4$ or $5x-1=-4$

$\therefore 5x=5 \qquad 5x=-3$

$\therefore x=1,-\frac{3}{5}$

4 $(px+q)^2=r^2$

$\therefore (px+q)=\pm\sqrt{r^2}$

$\therefore px+q=r$ or $px+q=-r$

$\therefore px=r-q \qquad px=-r-q$

$\therefore x=\frac{r-q}{p}, x=-\frac{r+q}{p}$

5 $9x^4+17x^2-2=0$

Let $a=x^2$

$9a^2+17a-2=0$

$\therefore (9a-1)(a+2)=0$

$\therefore a=\frac{1}{9}, a=-2$

Substitute back for x^2

$x^2=\frac{1}{9}$

$\therefore x=\pm\sqrt{\frac{1}{9}}$ or $x^2=-2$ no real solutions, therefore reject

$\therefore x=\pm\frac{1}{3}$

6 $\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)+4=0$

Let $a=x+\frac{1}{x}$

$\therefore a^2-4a+4=0$

$\therefore (a-2)^2=0$

$\therefore a=2$

Substitute back

$\therefore x+\frac{1}{x}=2$

$\therefore x^2+1=2x$

$\therefore x^2-2x+1=0$

$\therefore (x-1)^2=0$

$\therefore x=1$

7 **a** $3x(5-x)=0$

$\therefore 3x=0$ or $5-x=0$

$\therefore x=0, x=5$

b $(3-x)(7x-1)=0$

$\therefore 3-x=0$ or $7x-1=0$

$\therefore x=3, x=\frac{1}{7}$

c $(x+8)^2=0$

$\therefore x=-8$

d $2(x+4)(6+x)=0$

$\therefore x=-4, x=-6$

8 **a** $6x^2+5x+1=0$

$\therefore (3x+1)(2x+1)=0$

$\therefore 3x+1=0$ or $2x+1=0$

$\therefore x=-\frac{1}{3}, x=-\frac{1}{2}$

b $12x^2-7x=10$

$\therefore 12x^2-7x-10=0$

$\therefore (4x-5)(3x+2)=0$

$\therefore 4x-5=0$ or $3x+2=0$

$\therefore x=\frac{5}{4}, x=-\frac{2}{3}$

c $49=14x-x^2$

$\therefore x^2-14x+49=0$

$\therefore (x-7)^2=0$

$\therefore x=7$

d $5x+25-30x^2=0$

$\therefore -5(6x^2-x-5)=0$

$\therefore 6x^2-x-5=0$

$\therefore (6x+5)(x-1)=0$

$\therefore 6x+5=0$ or $x-1=0$

$\therefore x=-\frac{5}{6}, x=1$

e $44+44x^2=250x$

$\therefore 44x^2-250x+44=0$

$\therefore 2(22x^2-125x+22)=0$

$\therefore 22x^2-125x+22=0$

$\therefore (11x-2)(2x-11)=0$

$\therefore x=\frac{2}{11}, x=\frac{11}{2}$

f "Obtain the side of a square if the 'area' less the 'side' is 870." Let 'side' be x so 'area' is x^2. The statement can now be expressed as $x^2-x=870$.

Solving this equation:

$x^2-x=870$

$\therefore x^2-x-870=0$

$\therefore (x-30)(x+29)=0$

$\therefore x=30, x=-29$

As the side of a square cannot be negative, reject $x=-29$. The side of the square is 30 units.

9 **a** $x(x-7)=8$

$\therefore x^2-7x=8$

$\therefore x^2-7x-8=0$

$\therefore (x-8)(x+1)=0$

$\therefore x=8, x=-1$

b $4x(3x-16)=3(4x-33)$

$\therefore 12x^2-64x=12x-99$

$\therefore 12x^2-76x+99=0$

$\therefore (6x-11)(2x-9)=0$

$\therefore x=\frac{11}{6}, x=\frac{9}{2}$

c $(x+4)^2+2x=0$

$\therefore x^2+8x+16+2x=0$

$\therefore x^2+10x+16=0$

$\therefore (x+8)(x+2)=0$

$\therefore x=-8, x=-2$

d $(2x+5)(2x-5)+25=2x$

$\therefore 4x^2-25+25=2x$

$\therefore 4x^2-2x=0$

$\therefore 2x(2x-1)=0$

$\therefore x=0, x=\frac{1}{2}$

10 **a** $2-3x=\frac{1}{3x}$

$\therefore 3x(2-3x)=1$

$\therefore 6x-9x^2=1$

$\therefore 9x^2-6x+1=0$

$\therefore (3x-1)^2=0$

$\therefore x=\frac{1}{3}$

b $\frac{4x+5}{x+125}=\frac{5}{x}$

$\therefore x(4x+5)=5(x+125)$

$\therefore 4x^2+5x=5x+625$

$\therefore 4x^2-625=0$

$\therefore (2x-25)(2x+25)=0$

$\therefore x=\frac{25}{2}, x=-\frac{25}{2}$

c $7x-\frac{2}{x}+\frac{11}{5}=0$

$\therefore \frac{35x^2-10+11x}{5x}=0$

$\therefore 35x^2+11x-10=0$

$\therefore (7x+5)(5x-2)=0$

$\therefore x=-\frac{5}{7}, x=\frac{2}{5}$

d $\frac{12}{x+1}-\frac{14}{x-2}=19$

$\therefore \frac{12(x-2)-14(x+1)}{(x+1)(x-2)}=19$

$\therefore 12(x-2)-14(x+1)=19(x+1)(x-2)$

$\therefore 12x-24-14x-14=19(x^2-x-2)$

$\therefore -2x-38=19x^2-19x-38$

$\therefore 0=19x^2-17x$

$\therefore x(19x-17)=0$

$\therefore x=0, x=\frac{17}{19}$

11 **a** $x^2=121$

$\therefore x=\pm 11$

b $9x^2=16$

$\therefore x^2=\frac{16}{9}$

$\therefore x=\pm\frac{4}{3}$

c $(x-5)^2=1$

$\therefore x-5=\pm 1$

$\therefore x=1+5$ or $x=-1+5$

$\therefore x=6, x=4$

d $(5-2x)^2-49=0$

$\therefore [(5-2x)-7][(5-2x)+7]=0$

$\therefore (-2-2x)(12-2x)=0$

$\therefore -2-2x=0$ or $12-2x=0$

$\therefore -2=2x$ or $12=2x$

$\therefore x=-1, x=6$

e $2(3x-1)^2-8=0$

$\therefore 2(3x-1)^2=8$

$\therefore (3x-1)^2=4$

$\therefore 3x-1=\pm 2$

$\therefore 3x=3$ or $3x=-1$

$\therefore x=1, x=-\frac{1}{3}$

f $\left(x^2+1\right)^2=100$

$\therefore \left(x^2+1\right)=\pm 10$

$\therefore x^2+1=10$ or $x^2+1=-10$

$\therefore x^2=9$ or $x^2=-11$

Reject $x^2=-11$ since x^2 cannot be negative

$\therefore x^2=9$

$\therefore x=\pm 3$

12 **a** $(3x+4)^2+9(3x+4)-10=0$

Let $a=3x+4$

$\therefore a^2+9a-10=0$

$\therefore (a+10)(a-1)=0$

$\therefore a=-10$ or $a=1$

$\therefore 3x+4=-10$ or $3x+4=1$

$\therefore 3x=-14$ or $3x=-3$

$\therefore x=-\frac{14}{3}, x=-1$

b $2(1+2x)^2+9(1+2x)=18$

Let $a=1+2x$

$\therefore 2a^2+9a=18$

$\therefore 2a^2+9a-18=0$

$\therefore (2a-3)(a+6)=0$

$\therefore a=\frac{3}{2}$ or $a=-6$

$\therefore 1+2x=\frac{3}{2}$ or $1+2x=-6$

$\therefore 2x=\frac{1}{2}$ or $2x=-7$

$\therefore x=\frac{1}{4}, x=-\frac{7}{2}$

c $x^4-29x^2+100=0$

Let $a=x^2$

$\therefore a^2-29a+100=0$

$\therefore (a-25)(a-4)=0$

$\therefore a=25$ or $a=4$

$\therefore x^2=25$ or $x^2=4$

$\therefore x=\pm 5, x=\pm 2$

d $2x^4 = 31x^2 + 16$
Let $a = x^2$
$\therefore 2a^2 = 31a + 16$
$\therefore 2a^2 - 31a - 16 = 0$
$\therefore (2a+1)(a-16) = 0$
$\therefore a = -\frac{1}{2}$ or $a = 16$
$\therefore x^2 = -\frac{1}{2}$ or $x^2 = 16$
Reject $x^2 = -\frac{1}{2}$
$\therefore x^2 = 16$
$\therefore x = \pm 4$

e $36x^2 = \frac{9}{x^2} - 77$
Let $a = x^2$
$\therefore 36a = \frac{9}{a} - 77$
$\therefore 36a^2 = 9 - 77a$
$\therefore 36a^2 + 77a - 9 = 0$
$\therefore (4a+9)(9a-1) = 0$
$\therefore a = -\frac{9}{4}$ or $a = \frac{1}{9}$
$\therefore x^2 = -\frac{9}{4}$ or $x^2 = \frac{1}{9}$
Reject $x^2 = -\frac{9}{4}$
$\therefore x^2 = \frac{1}{9}$
$\therefore x = \pm\frac{1}{3}$

f $\left(x^2+4x\right)^2 + 7\left(x^2+4x\right) + 12 = 0$
Let $a = x^2 + 4x$
$\therefore a^2 + 7a + 12 = 0$
$\therefore (a+3)(a+4) = 0$
$\therefore a = -3$ or $a = -4$
$\therefore x^2 + 4x = -3$ or $x^2 + 4x = -4$
$\therefore x^2 + 4x + 3 = 0$ or $x^2 + 4x + 4 = 0$
$\therefore (x+3)(x+1) = 0$ or $(x+2)^2 = 0$
$\therefore x = -3, x = -1, x = -2$

13 a $x^4 = 81$
$\therefore x^2 = \pm 9$
Reject $x^2 = -9$
$\therefore x^2 = 9$
$\therefore x = \pm 3$

b $(9x^2 - 16)^2 = 20(9x^2 - 16)$
Let $a = 9x^2 - 16$
$\therefore a^2 = 20a$
$\therefore a^2 - 20a = 0$
$\therefore a(a-20) = 0$
$\therefore a = 0$ or $a = 20$
$\therefore 9x^2 - 16 = 0$ or $9x^2 - 16 = 20$
$\therefore (3x-4)(3x+4) = 0$ or $9x^2 = 36$
$\therefore x = \frac{4}{3}, x = -\frac{4}{3}$ or $x^2 = 4$
$\therefore x = \pm\frac{4}{3}, x = \pm 2$

c $\left(x - \frac{2}{x}\right)^2 - 2\left(x - \frac{2}{x}\right) + 1 = 0$
Let $a = x - \frac{2}{x}$
$\therefore a^2 - 2a + 1 = 0$
$\therefore (a-1)^2 = 0$
$\therefore a = 1$
$\therefore x - \frac{2}{x} = 1$
$\therefore x^2 - 2 = x$
$\therefore x^2 - x - 2 = 0$
$\therefore (x-2)(x+1) = 0$
$\therefore x = 2, x = -1$

d $2\left(1 + \frac{3}{x}\right)^2 + 5\left(1 + \frac{3}{x}\right) + 3 = 0$
Let $a = 1 + \frac{3}{x}$
$\therefore 2a^2 + 5a + 3 = 0$
$\therefore (2a+3)(a+1) = 0$
$\therefore a = -\frac{3}{2}$ or $a = -1$
$\therefore 1 + \frac{3}{x} = -\frac{3}{2}$ or $1 + \frac{3}{x} = -1$
$\therefore \frac{3}{x} = -\frac{5}{2}$ or $\frac{3}{x} = -2$
$\therefore 6 = -5x$ or $3 = -2x$
$\therefore x = -\frac{6}{5}, x = -\frac{3}{2}$

14 a $(x-2b)(x+3a) = 0$
$\therefore x = 2b, x = -3a$

b $2x^2 - 13ax + 15a^2 = 0$
$\therefore (2x-3a)(x-5a) = 0$
$\therefore x = \frac{3a}{2}, x = 5a$

c $(x-b)^4 - 5(x-b)^2 + 4 = 0$
Let $u = (x-b)^2$
$\therefore u^2 - 5u + 4 = 0$
$\therefore (u-4)(u-1) = 0$
$\therefore u = 4$ or $u = 1$
$\therefore (x-b)^2 = 4$ or $(x-b)^2 = 1$
$\therefore x - b = \pm 2$ or $x - b = \pm 1$
$\therefore x = 2 + b, x = -2 + b, x = 1 + b, x = -1 + b$
$\therefore x = b + 2, x = b - 2, x = b + 1, x = b - 1$

d $(x - a - b)^2 = 4b^2$
$\therefore x - a - b = \pm 2b$
$\therefore x = 2b + a + b$ or $x = -2b + a + b$
$\therefore x = a + 3b, x = a - b$

e $(x+a)^2 - 3b(x+a) + 2b^2 = 0$
Let $u = x + a$
$\therefore u^2 - 3bu + 2b^2 = 0$
$\therefore (u-2b)(u-b) = 0$
$\therefore u = 2b$ or $u = b$
$\therefore x + a = 2b$ or $x + a = b$
$\therefore x = 2b - a, x = b - a$

f
$$ab\left(x+\frac{a}{b}\right)\left(x+\frac{b}{a}\right)=(a+b)^2x$$
$$\therefore ab\left(\frac{bx+a}{b}\right)\left(\frac{ax+b}{a}\right)=(a+b)^2x$$
$$\therefore \frac{ab(bx+a)(ax+b)}{ab}=x(a^2+2ab+b^2)$$
$$\therefore (bx+a)(ax+b)=x(a^2+2ab+b^2)$$
$$\therefore abx^2+b^2x+a^2x+ab=a^2x+2abx+b^2x$$
$$\therefore abx^2+ab=2abx$$
$$\therefore abx^2-2abx+ab=0$$
$$\therefore ab(x^2-2x+1)=0$$
$$\therefore ab(x-1)^2=0$$
$$\therefore x=1$$

15 $(x-\alpha)(x-\beta)=0$

a If the roots are $x=1, x=7$, the equation must be $(x-1)(x-7)=0$.

b If the roots are $x=-5, x=4$, the equation must be $(x+5)(x-4)=0$.

c If the roots are $x=0, x=10$, the equation must be $(x-0)(x-10)=0$
$\therefore x(x-10)=0$.

d If the quadratic equation has one root only of $x=2$, the equation must be $(x-2)(x-2)=0$
$\therefore (x-2)^2=0$.

16 **a** As the zeros of $4x^2+bx+c$ are $x=-4$ and $x=\frac{3}{4}$ then $(x+4)$ and $\left(x-\frac{3}{4}\right)$ are factors of $4x^2+bx+c$.

However, the coefficient of x^2 must be 4 so
$$4x^2+bx+c=4(x+4)\left(x-\frac{3}{4}\right).$$
$$\therefore 4x^2+bx+c=4\left(x-\frac{3}{4}\right)(x+4)$$
$$=(4x-3)(x+4)$$
$$\therefore 4x^2+bx+c=4x^2+13x-12$$
Hence $b=13$ and $c=-12$.

b
$$px^2+(p+q)x+q=0$$
$$\therefore px^2+px+qx+q=0$$
$$\therefore px(x+1)+q(x+1)=0$$
$$\therefore (x+1)(px+q)=0$$
$$\therefore x=-1.x=-\frac{q}{p}$$
Consider $p(x-1)^2+(p+q)(x-1)+q=0$

Let $a=x-1$

$\therefore pa^2+(p+q)a+q=0$

Using the above roots for an equation in this form gives
$$a=-1, a=-\frac{q}{p}$$
$$\therefore x-1=-1 \text{ or } x-1=-\frac{q}{p}$$
$$\therefore x=0 \text{ or } x=-\frac{q}{p}+1$$
$$\therefore x=0, x=\frac{p-q}{p}$$

17 $60x^2+113x-63=0$

Solve using Interactive → Equation/inequality on Standard mode
$$\therefore x=-\frac{7}{3}, x=\frac{9}{20}$$

18 $4x(x-7)+8(x-3)^2=x-26$
$$\therefore x=\frac{7}{4}, x=\frac{14}{3}$$

Exercise 3.3 — Quadratics over *R*

1 **a** $x^2-10x-7$
$$=(x^2-10x+25)-25-7$$
$$=(x-5)^2-32$$
$$=\left(x-5-\sqrt{32}\right)\left(x-5+\sqrt{32}\right)$$
$$=\left(x-5-4\sqrt{2}\right)\left(x-5+4\sqrt{2}\right)$$

b $3x^2+7x+3$
$$=3\left(x^2+\frac{7}{3}x+1\right)$$
$$=3\left(x^2+\frac{7}{3}+\left(\frac{7}{6}\right)^2-\left(\frac{7}{6}\right)^2+1\right)$$
$$=3\left[\left(x+\frac{7}{6}\right)^2-\frac{49}{36}+1\right]$$
$$=3\left[\left(x+\frac{7}{6}\right)^2-\frac{49}{36}+\frac{36}{36}\right]$$
$$=3\left[\left(x+\frac{7}{6}\right)^2-\frac{13}{36}\right]$$
$$=3\left(x+\frac{7}{6}-\sqrt{\frac{13}{36}}\right)\left(x+\frac{7}{6}+\sqrt{\frac{13}{36}}\right)$$
$$=3\left(x+\frac{7}{6}-\frac{\sqrt{13}}{6}\right)\left(x+\frac{7}{6}+\frac{\sqrt{13}}{6}\right)$$
$$=3\left(x+\frac{7-\sqrt{13}}{6}\right)\left(x+\frac{7+\sqrt{13}}{6}\right)$$

c $5x^2-9$
$$=\left(\sqrt{5}x\right)^2-3^2$$
$$=\left(\sqrt{5}x-3\right)\left(\sqrt{5}x+3\right)$$

2 $-3x^2+8x-5$
$$=-3\left(x^2-\frac{8x}{3}+\frac{5}{3}\right)$$
$$=-3\left(\left(x^2-\frac{8x}{3}+\left(\frac{4}{3}\right)^2\right)-\left(\frac{4}{3}\right)^2+\frac{5}{3}\right)$$
$$=-3\left(\left(x-\frac{4}{3}\right)^2-\frac{16}{9}+\frac{5}{3}\right)$$
$$=-3\left(\left(x-\frac{4}{3}\right)^2-\frac{16}{9}+\frac{15}{9}\right)$$
$$=-3\left(\left(x-\frac{4}{3}\right)^2-\frac{1}{9}\right)$$
$$=-3\left[\left(x-\frac{4}{3}-\frac{1}{3}\right)\right]\left[\left(x-\frac{4}{3}+\frac{1}{3}\right)\right]$$
$$=-3\left(x-\frac{5}{3}\right)\left(x-\frac{3}{3}\right)$$
$$=(-3x+5)(x-1)$$
Factorisation by inspection gives

$-3x^2+8x-5$

$=-(3x^2-8x+5)$ which is equivalent to $(-3x+5)(x+1)$

$=-(3x-5)(x-1)$

3 a $4x^2+5x+10$

$\Delta=b^2-4ac,\ a=4, b=5, c=10$

$\therefore \Delta=(5)^2-4\times4\times10$

$=25-160$

$=-135$

Since $\Delta<0$, there are no real factors

b $169x^2-78x+9$

$\Delta=b^2-4ac,\ a=169, b=-78, c=9$

$\therefore \Delta=(-78)^2-4\times169\times9$

$=6084-6084$

$=0$

Since $\Delta=0$, the quadratic is a perfect square and factorises over Q. There are two identical rational factors. Completing the square is not essential to obtain the factors.

Check: $169x^2-78x+9=(13x-3)^2$

c $-3x^2+11x-10$

$\Delta=b^2-4ac,\ a=-3, b=11, c=-10$

$\therefore \Delta=(11)^2-4\times(-3)(-10)$

$=121-120$

$=1$

Since $\Delta>0$ and a perfect square, the quadratic factorises over Q and has two rational factors. Completing the square is not essential to obtain the factors.

Check: $-3x^2+11x-10=(-3x+5)(x-2)$

d $\frac{1}{3}x^2-\frac{8}{3}x+2$

$\Delta=b^2-4ac,\ a=\frac{1}{3}, b=-\frac{8}{3}, c=2$

$$\therefore \Delta=\left(-\frac{8}{3}\right)^2-4\times\frac{1}{3}\times2$$

$$=\frac{64}{9}-\frac{8}{3}$$

$$=\frac{64-24}{9}$$

$$=\frac{40}{9}$$

Since $\Delta>0$ but is not a perfect square, the quadratic factorises over R and has two real linear factors. Completing the square is needed to obtain the factors.

4 a $3(x-8)^2-6$

$=3\left[(x-8)^2-2\right]$

$=3\left(x-8-\sqrt{2}\right)\left(x-8+\sqrt{2}\right)$

b $(xy-7)^2+9$ is the sum of two squares so does not factorise over R

5 $(2x+1)(x+5)-1=0$

$\therefore 2x^2+11x+5-1=0$

$\therefore 2x^2+11x+4=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},\ a=2, b=11, c=4$$

$$\therefore x=\frac{-11\pm\sqrt{(11)^2-4\times(2)\times(4)}}{2\times(2)}$$

$$=\frac{-11\pm\sqrt{121-32}}{4}$$

$$=\frac{-11\pm\sqrt{89}}{4}$$

6 $3(2x+1)^4-16(2x+1)^2-35=0$

Let $a=(2x+1)^2$

$3a^2-16a-35=0$

$\therefore (3a+5)(a-7)=0$

$$\therefore a=-\frac{5}{3},\ a=7$$

$\therefore (2x+1)^2=-\frac{5}{3}$ which is not possible since a perfect square cannot be negative

or $(2x+1)^2=7$

$\therefore 2x+1=\pm\sqrt{7}$

$\therefore 2x=-1\pm\sqrt{7}$

$$\therefore x=\frac{-1\pm\sqrt{7}}{2}$$

7 a $0.2x^2-2.5x+10=0$

$\Delta=b^2-4ac,\ a=0.2, b=-2.5, c=10$

$\therefore \Delta=(-2.5)^2-4\times(0.2)\times(10)$

$=6.25-8$

$=-1.75$

Since $\Delta<0$, there are no real roots to the equation.

b $kx^2-(k+3)x+k=0$

$\Delta=b^2-4ac,\ a=k, b=-(k+3), c=k$

$\therefore \Delta=\left(-(k+3)\right)^2-4\times(k)\times(k)$

$=(k+3)^2-4k^2$

$=(k+3-2k)(k+3+2k)$

$=(3-k)(3k+3)$

For two equal solutions, $\Delta=0$

$\therefore (3-k)(3k+3)=0$

$\therefore k=3,\ k=-1$

8 $mx^2+(m-4)x=4$

$\therefore mx^2+(m-4)x-4=0$

$\Delta=b^2-4ac,\ a=m, b=(m-4), c=-4$

$\Delta=(m-4)^2-4\times(m)\times(-4)$

$=(m-4)^2+16m$

$=m^2-8m+16+16m$

$=m^2+8m+16$

$=(m+4)^2$

Since $\Delta\geq0$ for all m, the equation will always have real roots

9 a $x^2-20\sqrt{5}x+100=0$

$$\therefore\left(x^2-20\sqrt{5}x+\left(10\sqrt{5}\right)^2\right)-\left(10\sqrt{5}\right)^2+100=0$$

$$\therefore\left(x-10\sqrt{5}\right)^2-500+100=0$$

$$\therefore\left(x-10\sqrt{5}\right)^2=400$$

$$\therefore x-10\sqrt{5}=\pm20$$

$$\therefore x=10\sqrt{5}\pm20$$

b Since one root is $x=1-\sqrt{2}$ the other root is $x=1+\sqrt{2}$ since the roots occur in conjugate surd pairs.

Since $x=1-\sqrt{2}$ is a root, then $\left(x-(1-\sqrt{2})\right)$ is a factor of the equation, and,

since $x=1+\sqrt{2}$ is a root, then $\left(x-(1+\sqrt{2})\right)$ is a factor of the equation.

Therefore the equation is $\left(x-1+\sqrt{2}\right)\left(x-1-\sqrt{2}\right)=0$

Expanding,

$$\begin{aligned}(x-1+\sqrt{2})(x-1-\sqrt{2})&=0\\ \therefore\big((x-1)+\sqrt{2}\big)\big((x-1)-\sqrt{2}\big)&=0\\ \therefore (x-1)^2-\left(\sqrt{2}\right)^2&=0\\ \therefore x^2-2x+1-2&=0\\ \therefore x^2-2x-1&=0\end{aligned}$$

$\therefore b=-2, c=-1$

10 $\sqrt{2}x^2+4\sqrt{3}x-8\sqrt{2}=0$

$$\begin{aligned}x&=\frac{-4\sqrt{3}\pm\sqrt{\left(4\sqrt{3}\right)^2-4\times\sqrt{2}\times-8\sqrt{2}}}{2\times\sqrt{2}}\\ &=\frac{-4\sqrt{3}\pm\sqrt{48+64}}{2\sqrt{2}}\\ &=\frac{-4\sqrt{3}\pm\sqrt{112}}{2\sqrt{2}}\\ &=\frac{-4\sqrt{3}\pm4\sqrt{7}}{2\sqrt{2}}\\ &=\frac{-2\sqrt{3}\pm2\sqrt{7}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\ &=\frac{-2\sqrt{6}\pm2\sqrt{14}}{2}\\ &=-\sqrt{6}\pm\sqrt{14}\end{aligned}$$

11 $4x-3\sqrt{x}=1$

$$\begin{aligned}\therefore 4x-1&=3\sqrt{x}\\ \therefore (4x-1)^2&=9x\\ \therefore 16x^2-8x+1&=9x\\ \therefore 16x^2-17x+1&=0\\ \therefore (16x-1)(x-1)&=0\\ \therefore x&=\frac{1}{16}, x=1\end{aligned}$$

Check: Substitute $x=\dfrac{1}{16}$ in $4x-3\sqrt{x}=1$

$$\begin{aligned}\text{LHS}&=\frac{1}{4}-3\times\frac{1}{4}\\ &=-0.5\\ &\neq\text{RHS}\end{aligned}$$

Substitute $x=1$ in $4x-3\sqrt{x}=1$

$$\begin{aligned}\text{LHS}&=4-3\times1\\ &=1\\ &=\text{RHS}\end{aligned}$$

Reject $x=\dfrac{1}{16}$

Answer is $x=1$

12 Let $u=\sqrt{x}$ in $4x-3\sqrt{x}=1$

$$\begin{aligned}\therefore 4u^2-3u&=1\\ \therefore 4u^2-3u-1&=0\\ \therefore (4u+1)(u-1)&=0\\ \therefore u&=-\frac{1}{4}, u=1\\ \therefore \sqrt{x}&=-1 \text{ or } \sqrt{x}=1\end{aligned}$$

Since $\sqrt{x}$ cannot be negative, reject $\sqrt{x}=-1$

$\therefore \sqrt{x}=1$

$\therefore x=1$

13 **a** $x^2+10x+25=(x+5)^2$

b $x^2-7x+\left(\dfrac{7}{2}\right)^2=\left(x-\dfrac{7}{2}\right)^2$

$\therefore x^2-7x+\dfrac{49}{4}=\left(x-\dfrac{7}{2}\right)^2$

c $x^2+x+\left(\dfrac{1}{2}\right)^2=\left(x+\dfrac{1}{2}\right)^2$

$\therefore x^2+x+\dfrac{1}{4}=\left(x+\dfrac{1}{2}\right)^2$

d $x^2-\dfrac{4}{5}x+\left(\dfrac{2}{5}\right)^2=\left(x-\dfrac{2}{5}\right)^2$

$\therefore x^2-\dfrac{4}{5}x+\dfrac{4}{25}=\left(x-\dfrac{2}{5}\right)^2$

14 **a** x^2-12

$$\begin{aligned}&=\left(x-\sqrt{12}\right)\left(x+\sqrt{12}\right)\\ &=\left(x-2\sqrt{3}\right)\left(x+2\sqrt{3}\right)\end{aligned}$$

b $x^2-12x+4$

$$\begin{aligned}&=\left(x^2-12x+36\right)-36+4\\ &=(x-6)^2-32\\ &=\left(x-6-\sqrt{32}\right)\left(x-6+\sqrt{32}\right)\\ &=\left(x-6-4\sqrt{2}\right)\left(x-6+4\sqrt{2}\right)\end{aligned}$$

c x^2+9x-3

$$\begin{aligned}&=\left(x^2+9x+\left(\frac{9}{2}\right)^2\right)-\left(\frac{9}{2}\right)^2-3\\ &=\left(x+\frac{9}{2}\right)^2-\frac{81}{4}-\frac{12}{4}\\ &=\left(x+\frac{9}{2}\right)^2-\frac{93}{4}\\ &=\left(x+\frac{9}{2}-\sqrt{\frac{93}{4}}\right)\left(x+\frac{9}{2}+\sqrt{\frac{93}{4}}\right)\\ &=\left(x+\frac{9-\sqrt{93}}{2}\right)\left(x+\frac{9+\sqrt{93}}{2}\right)\end{aligned}$$

d $2x^2+5x+1$

$$\begin{aligned}&=2\left(x^2+\frac{5x}{2}+\frac{1}{2}\right)\\ &=2\left(\left(x^2+\frac{5x}{2}+\left(\frac{5}{4}\right)^2\right)-\left(\frac{5}{4}\right)^2+\frac{1}{2}\right)\\ &=2\left(\left(x+\frac{5}{4}\right)^2-\frac{25}{16}+\frac{8}{16}\right)\\ &=2\left(\left(x+\frac{5}{4}\right)^2-\frac{17}{16}\right)\\ &=2\left(x+\frac{5-\sqrt{17}}{4}\right)\left(x+\frac{5+\sqrt{17}}{4}\right)\end{aligned}$$

e $3x^2+4x+3$

$$\begin{aligned}&=3\left(x^2+\frac{4x}{3}+1\right)\\ &=3\left(\left(x^2+\frac{4x}{3}+\left(\frac{2}{3}\right)^2\right)-\left(\frac{2}{3}\right)^2+1\right)\\ &=3\left(\left(x+\frac{2}{3}\right)^2-\frac{4}{9}+\frac{9}{9}\right)\\ &=3\left(\left(x+\frac{2}{3}\right)^2+\frac{5}{9}\right)\end{aligned}$$

Since the sum of two squares does not factorise over *R*, there are no linear factors.

f $1+40x-5x^2$

$$\begin{aligned}&=-5\left(x^2-8x-\frac{1}{5}\right)\\&=-5\left[\left(x^2-8x+16\right)-16-\frac{1}{5}\right]\\&=-5\left[(x-4)^2-\frac{80}{5}-\frac{1}{5}\right]\\&=-5\left[(x-4)^2-\frac{81}{5}\right]\\&=-5\left(x-4-\frac{9}{\sqrt{5}}\right)\left(x-4+\frac{9}{\sqrt{5}}\right)\\&=-5\left(x-4-\frac{9\sqrt{5}}{5}\right)\left(x-4+\frac{9\sqrt{5}}{5}\right)\end{aligned}$$

15 a $5x^2+9x-2$

$$\begin{aligned}&\Delta=b^2-4ac\\&a=5,\ b=9,\ c=-2\\&\therefore\Delta=(9)^2-4\times5\times(-2)\\&\quad=81+40\\&\therefore\Delta=121\end{aligned}$$

Since Δ is a perfect square, there are 2 rational factors

b $12x^2-3x+1$

$$\begin{aligned}&\Delta=b^2-4ac\\&a=12,b=-3,c=1\\&\therefore\Delta=(-3)^2-4\times12\times1\\&\quad=9-48\\&\therefore\Delta=-39\end{aligned}$$

Since $\Delta<0$, there are no real linear factors

c $121x^2+110x+25$

$$\begin{aligned}&\Delta=b^2-4ac\\&a=121,\ b=110,\ c=25\\&\therefore\Delta=(110)^2-4\times121\times25\\&\quad=12100-12100\\&\therefore\Delta=0\end{aligned}$$

Since $\Delta=0$ there is one repeated rational factor

d $x^2+10x+23$

$$\begin{aligned}&\Delta=b^2-4ac\\&a=1,\ b=10,\ c=23\\&\therefore\Delta=(10)^2-4\times1\times23\\&\quad=100-92\\&\therefore\Delta=8\end{aligned}$$

Since $\Delta>0$ but not a perfect square, there are two irrational factors

16 a $x^3-8=(x-2)(x^2+2x+4)$

Consider the discriminant of the quadratic factor x^2+2x+4

$$\begin{aligned}&\Delta=b^2-4ac\\&a=1,b=2,c=4\\&\therefore\Delta=(2)^2-4\times1\times4\\&\quad=4-16\\&\therefore\Delta=-12\end{aligned}$$

Since $\Delta<0$, the quadratic factor cannot be expressed as a product of linear factors.

Therefore, $(x-2)$ is the only linear factor of x^3-8 over R.

b i If $x=\sqrt{2}$ is a zero then $\left(x-\sqrt{2}\right)$ is a factor and if $x=-\sqrt{2}$ is a zero then $\left(x+\sqrt{2}\right)$ is a factor.

The product of these factors give:

$$\begin{aligned}&\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)\\&=x^2-\left(\sqrt{2}\right)^2\\&=x^2-2\end{aligned}$$

ii If $x=-4+\sqrt{2}$ is a zero then $\left(x-\left(-4+\sqrt{2}\right)\right)=\left(x+4-\sqrt{2}\right)$ is a factor and if $x=-4-\sqrt{2}$ is a zero then $\left(x+4+\sqrt{2}\right)$ is a factor.

The product of these factors give:

$$\begin{aligned}&\left(x+4-\sqrt{2}\right)\left(x+4+\sqrt{2}\right)\\&=\left((x+4)-\sqrt{2}\right)\left((x+4)+\sqrt{2}\right)\\&=(x+4)^2-\left(\sqrt{2}\right)^2\\&=x^2+8x+16-2\\&=x^2+8x+14\end{aligned}$$

17 a $9x^2-3x-4=0$

$$\begin{aligned}&x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\&a=9,b=-3,c=-4\\&\therefore x=\frac{-(-3)\pm\sqrt{(-3)^2-4\times9\times(-4)}}{2\times9}\\&=\frac{3\pm\sqrt{9+144}}{18}\\&=\frac{3\pm\sqrt{153}}{18}\\&=\frac{3\pm3\sqrt{17}}{18}\\&=\frac{3\left(1\pm\sqrt{17}\right)}{18}\\&\therefore x=\frac{1\pm\sqrt{17}}{6}\end{aligned}$$

b $5x(4-x)=12$

$$\begin{aligned}&\therefore 20x-5x^2=12\\&\therefore 5x^2-20x+12=0\\&x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},\ a=5,\ b=-20,\ c=12\\&\therefore x=\frac{-(-20)\pm\sqrt{(-20)^2-4\times5\times12}}{2\times5}\\&=\frac{20\pm\sqrt{400-240}}{10}\\&=\frac{20\pm\sqrt{160}}{10}\\&=\frac{20\pm4\sqrt{10}}{10}\\&=\frac{4\left(5\pm\sqrt{10}\right)}{10}\\&\therefore x=\frac{10\pm2\sqrt{10}}{5}\end{aligned}$$

c $(x-10)^2=20$

$$\begin{aligned}&\therefore x-10=\pm\sqrt{20}\\&\therefore x=10\pm2\sqrt{5}\end{aligned}$$

d $x^2+6x-3=0$

Completing the square

$\therefore \left(x^2+6x+9\right)-9-3=0$

$\therefore (x+3)^2=12$

$\therefore x+3=\pm\sqrt{12}$

$\therefore x=-3\pm 2\sqrt{3}$

e $56x^2+51x-27=0$

$\therefore (7x+9)(8x-3)=0$

$\therefore x=-\dfrac{9}{7}, x=\dfrac{3}{8}$

f $5x-x(7+2x)=(x+5)(2x-1)$

$\therefore 5x-7x-2x^2=2x^2+9x-5$

$\therefore 4x^2+11x-5=0$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, $a=4$, $b=11$, $c=-5$

$\therefore x=\dfrac{-11\pm\sqrt{(11)^2-4\times 4\times(-5)}}{2\times 4}$

$=\dfrac{-11\pm\sqrt{121+80}}{8}$

$\therefore x=\dfrac{-11\pm\sqrt{201}}{8}$

18 a $-5x^2-8x+9=0$

$\Delta=b^2-4ac,\quad a=-5, b=-8, c=9$

$\therefore \Delta=(-8)^2-4\times(-5)\times 9$

$=64+180$

$\therefore \Delta=244$

Since $\Delta>0$ but not a perfect square, there are two irrational roots

b $4x^2+3x-7=0$

$\Delta=b^2-4ac,\quad a=4, b=3, c=-7$

$\therefore \Delta=(3)^2-4\times 4\times(-7)$

$=9+112$

$\therefore \Delta=121$

Since Δ is a perfect square, there are two rational roots

c $4x^2+x+2=0$

$\Delta=b^2-4ac,\quad a=4, b=1, c=2$

$\therefore \Delta=(1)^2-4\times 4\times 2$

$=1-32$

$\therefore \Delta=-31$

Since $\Delta<0$ there are no real roots

d $28x-4-49x^2=0$

$\Delta=b^2-4ac,\quad a=-49, b=28, c=-4$

$\therefore \Delta=(28)^2-4\times(-49)\times(-4)$

$=784-784$

$\therefore \Delta=0$

Since $\Delta=0$ there is one rational root (or two equal roots)

e $4x^2+25=0$

As $4x^2\geq 0$ then the sum $4x^2+25$ cannot equal zero. Therefore there are no real roots.

f $3\sqrt{2}x^2+5x+\sqrt{2}=0$

$\Delta=b^2-4ac,\quad a=3\sqrt{2}, b=5, c=\sqrt{2}$

$\therefore \Delta=(5)^2-4\times 3\sqrt{2}\times\sqrt{2}$

$=25-24$

$\therefore \Delta=1$

Since $\Delta>0$ there are two roots. However, despite Δ being a perfect square, the coefficient of x^2 in the quadratic equation is irrational so the two roots are irrational.

19 a $(x^2-3)^2-4(x^2-3)+4=0$

Let $a=x^2-3$

$\therefore a^2-4a+4=0$

$\therefore (a-2)^2=0$

$\therefore a=2$

$\therefore x^2-3=2$

$\therefore x^2=5$

$\therefore x=\pm\sqrt{5}$

b $5x^4-39x^2-8=0$

Let $a=x^2$

$\therefore 5a^2-39a-8=0$

$\therefore (5a+1)(a-8)=0$

$\therefore a=-\dfrac{1}{5}$ or $a=8$

$\therefore x^2=-\dfrac{1}{5}$ (reject) or $x^2=8$

$\therefore x^2=8$

$\therefore x=\pm 2\sqrt{2}$

c $x^2(x^2-12)+11=0$

Let $a=x^2$

$\therefore a(a-12)+11=0$

$\therefore a^2-12a+11=0$

$\therefore (a-1)(a-11)=0$

$\therefore a=1$ or $a=11$

$\therefore x^2=1$ or $x^2=11$

$\therefore x=\pm 1, x=\pm\sqrt{11}$

d $\left(x+\dfrac{1}{x}\right)^2+2\left(x+\dfrac{1}{x}\right)-3=0$

Let $a=x+\dfrac{1}{x}$

$\therefore a^2+2a-3=0$

$\therefore (a+3)(a-1)=0$

$\therefore a=-3$ or $a=1$

$\therefore x+\dfrac{1}{x}=-3$ or $x+\dfrac{1}{x}=1$

$\therefore x^2+1=-3x$ or $x^2+1=x$

$\therefore x^2+3x+1=0$ or $x^2-x+1=0$

$\therefore x=\dfrac{-3\pm\sqrt{9-4(1)(1)}}{2}$ or $x=\dfrac{1\pm\sqrt{1-4(1)(1)}}{2}$

$\therefore x=\dfrac{-3\pm\sqrt{5}}{2}$ or $x=\dfrac{1\pm\sqrt{-3}}{2}$

$\therefore x=\dfrac{-3\pm\sqrt{5}}{2}$

(since $x=\dfrac{1\pm\sqrt{-3}}{2}$ is not real).

e $(x^2-7x-8)^2=3(x^2-7x-8)$

Let $a=x^2-7x-8$

$\therefore a^2=3a$

$\therefore a^2-3a=0$

$\therefore a(a-3)=0$

$\therefore a=0$ or $a=3$

$\therefore x^2-7x-8=0$ or $x^2-7x-8=3$

$\therefore (x+1)(x-8)=0$ or $x^2-7x-11=0$

$\therefore x=-1, x=8$ or $x=\dfrac{7\pm\sqrt{49-4(1)(-11)}}{2}$

$\therefore x=-1, x=8$ or $x=\dfrac{7\pm\sqrt{93}}{2}$

f $3\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-2=0$ given that

$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$

Let $a=x+\frac{1}{x}$ so $x^2+\frac{1}{x^2}=a^2-2$

$\therefore 3(a^2-2)+2a-2=0$

$\therefore 3a^2+2a-8=0$

$\therefore (3a-4)(a+2)=0$

$\therefore a=\frac{4}{3}$ or $a=-2$

$\therefore x+\frac{1}{x}=\frac{4}{3}$ or $x+\frac{1}{x}=-2$

$\therefore 3x^2+3=4x$ or $x^2+1=-2x$

$\therefore 3x^2-4x+3=0$ or $x^2+2x+1=0$

Consider $3x^2-4x+3=0$

$\Delta=16-4(3)(3)$

$=-20$

Since $\Delta<0$ there are no real roots

$\therefore x^2+2x+1=0$

$\therefore (x+1)^2=0$

$\therefore x=-1$

20 a $x^2+(m+2)x-m+5=0$

For one root, $\Delta=0$.

$\Delta=b^2-4ac,\quad a=1,\ b=m+2,\ c=-m+5$

$\therefore \Delta=(m+2)^2-4\times1\times(-m+5)$

$=m^2+4m+4+4m-20$

$\therefore \Delta=m^2+8m-16$

Therefore, for one root, $m^2+8m-16=0$.

$\therefore \left(m^2+8m+16\right)-16-16=0$

$\therefore (m+4)^2=32$

$\therefore m+4=\pm\sqrt{32}$

$\therefore m=-4\pm4\sqrt{2}$

b $(m+2)x^2-2mx+4=0$

For one root, $\Delta=0$.

$\therefore (-2m)^2-4(m+2)(4)=0$

$\therefore 4m^2-16m-32=0$

$\therefore m^2-4m-8=0$

$\therefore \left(m^2-4m+4\right)-4-8=0$

$\therefore (m-2)^2-12=0$

$\therefore m-2=\pm\sqrt{12}$

$\therefore m=2\pm2\sqrt{3}$

c $3x^2+4x-2(p-1)=0$

For no roots, $\Delta<0$

$\therefore 16-4(3)\left(-2(p-1)\right)<0$

$\therefore 16+24(p-1)<0$

$\therefore 24p-8<0$

$\therefore 24p<8$

$\therefore p<\frac{1}{3}$

d $kx^2-4x-k=0$

The discriminant determines the number of solutions

$\Delta=(-4)^2-4(k)(-k)$

$=16+4k^2$

Since $k\in R\setminus\{0\}$, $k^2>0$

$\therefore 16+4k^2>16$

Thus Δ is always positive. Therefore the equation always has two solutions.

e $px^2+(p+q)x+q=0$

$\Delta=(p+q)^2-4pq$

$=p^2+2pq+q^2-4pq$

$=p^2-2pq+q^2$

$=(p-q)^2$

As Δ is a perfect square and $p,q\in Q$, the roots are always rational.

21 a i $x^2+6\sqrt{2}x+18=0$

$\therefore x^2+6\sqrt{2}x+\left(3\sqrt{2}\right)^2-\left(3\sqrt{2}\right)^2+18=0$

$\therefore \left(x^2+6\sqrt{2}x+\left(3\sqrt{2}\right)^2\right)-18+18=0$

$\therefore \left(x+3\sqrt{2}\right)^2=0$

$\therefore x=-3\sqrt{2}$

ii $2\sqrt{5}x^2-3\sqrt{10}x+\sqrt{5}=0$

Divide both sides by $\sqrt{5}$

$\therefore 2x^2-3\sqrt{2}+1=0$

$\therefore x=\frac{3\sqrt{2}\pm\sqrt{\left(3\sqrt{2}\right)^2-4\times2\times1}}{4}$

$\therefore x=\frac{3\sqrt{2}\pm\sqrt{18-8}}{4}$

$\therefore x=\frac{3\sqrt{2}\pm\sqrt{10}}{4}$

iii $\sqrt{3}x^2-\left(2\sqrt{2}-\sqrt{3}\right)x-\sqrt{2}=0$

$\therefore x=\frac{\left(2\sqrt{2}-\sqrt{3}\right)\pm\sqrt{\left(2\sqrt{2}-\sqrt{3}\right)^2-4\times\sqrt{3}\times-\sqrt{2}}}{2\sqrt{3}}$

$\therefore x=\frac{2\sqrt{2}-\sqrt{3}\pm\sqrt{8-4\sqrt{6}+3+4\sqrt{6}}}{2\sqrt{3}}$

$\therefore x=\frac{2\sqrt{2}-\sqrt{3}\pm\sqrt{11}}{2\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$

$\therefore x=\frac{\sqrt{3}\left(2\sqrt{2}-\sqrt{3}\pm\sqrt{11}\right)}{6}$

$\therefore x=\frac{2\sqrt{6}-3\pm\sqrt{33}}{6}$

$\therefore x=\frac{-3+2\sqrt{6}\pm\sqrt{33}}{6}$

b i If $x=\frac{-1+\sqrt{5}}{2}$ is a root of a quadratic equation $x^2+bx+c=0$ with rational coefficients then its conjugate $x=\frac{-1-\sqrt{5}}{2}$ is also a root.

ii In factorised form, the equation is

$\left(x-\frac{-1+\sqrt{5}}{2}\right)\left(x-\frac{-1-\sqrt{5}}{2}\right)=0$

$\therefore \left(x+\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(x+\frac{1}{2}+\frac{\sqrt{5}}{2}\right)=0$

$\therefore \left(\left(x+\frac{1}{2}\right)-\frac{\sqrt{5}}{2}\right)\left(\left(x+\frac{1}{2}\right)+\frac{\sqrt{5}}{2}\right)=0$

$\therefore \left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2=0$

$\therefore x^2+x+\frac{1}{4}-\frac{5}{4}=0$

$\therefore x^2+x-1=0$

Therefore, $b=1,\ c=-1$.

c Since the roots of the quadratic equation $x^2+bx+c=0$ with real coefficients are $x=4\sqrt{3}\pm5\sqrt{6}$, the equation in factorised form is $\left(x-\left(4\sqrt{3}+5\sqrt{6}\right)\right)\left(x-\left(4\sqrt{3}-5\sqrt{6}\right)\right)=0.$

$$\begin{aligned}\left(x-\left(4\sqrt{3}+5\sqrt{6}\right)\right)\left(x-\left(4\sqrt{3}-5\sqrt{6}\right)\right)&=0\\ \therefore\left(x-4\sqrt{3}-5\sqrt{6}\right)\left(x-4\sqrt{3}+5\sqrt{6}\right)&=0\\ \therefore\left(\left(x-4\sqrt{3}\right)-5\sqrt{6}\right)\left(\left(x-4\sqrt{3}\right)+5\sqrt{6}\right)&=0\\ \therefore\left(x-4\sqrt{3}\right)^2-\left(5\sqrt{6}\right)^2&=0\\ \therefore x^2-8\sqrt{3}x+48-25\times6&=0\\ \therefore x^2-8\sqrt{3}x-102&=0\end{aligned}$$

Therefore, $b=-8\sqrt{3},\ c=-102$

22 a $2\sqrt{x}=8-x$

i Squaring both sides

$$\begin{aligned}\left(2\sqrt{x}\right)^2&=(8-x)^2\\ \therefore 4x&=64-16x+x^2\\ \therefore x^2-20x+64&=0\\ \therefore (x-4)(x-16)&=0\\ \therefore x&=4,x=16\end{aligned}$$

Checking: Substitute $x=4$ in $2\sqrt{x}=8-x$

$$\begin{aligned}\text{LHS}&=2\sqrt{4} & \text{RHS}&=8-4\\ &=4 & &=4\end{aligned}$$

$\therefore$ LHS = RHS

Hence $x=4$ is a solution

Substitute $x=16$ in $2\sqrt{x}=8-x$

$$\begin{aligned}\text{LHS}&=2\sqrt{16} & \text{RHS}&=8-16\\ &=8 & &=-8\end{aligned}$$

$\therefore$ LHS $\neq$ RHS

Hence, reject $x=16$

Answer is $x=4$.

ii Let $a=\sqrt{x}$

$$\begin{aligned}\therefore 2a&=8-a^2\\ \therefore a^2+2a-8&=0\\ \therefore (a-2)(a+4)&=0\\ \therefore a&=2\text{ or }a=-4\\ \therefore \sqrt{x}&=2\text{ or }\sqrt{x}=-4\end{aligned}$$

Reject $\sqrt{x}=-4$ since $\sqrt{x}\geq0$

$$\begin{aligned}\therefore \sqrt{x}&=2\\ \therefore x&=2^2\\ \therefore x&=4\end{aligned}$$

b $1+\sqrt{x+1}=2x$

$$\begin{aligned}\therefore \sqrt{x+1}&=2x-1\\ \therefore x+1&=(2x-1)^2\\ \therefore x+1&=4x^2-4x+1\\ \therefore 4x^2-5x&=0\\ \therefore x(4x-5)&=0\\ \therefore x&=0,x=\frac{5}{4}\end{aligned}$$

Check: Substitute $x=0$ in $1+\sqrt{x+1}=2x$

$$\begin{aligned}\text{LHS}&=1+\sqrt{1} & \text{RHS}&=2\times0\\ &=2 & &=0\end{aligned}$$

$\therefore$ LHS $\neq$ RHS

Reject $x=0$

Substitute $x=\frac{5}{4}$ in $1+\sqrt{x+1}=2x$

$$\begin{aligned}\text{LHS}&=1+\sqrt{\frac{5}{4}+1} & \text{RHS}&=2\times\frac{5}{4}\\ &=1+\sqrt{\frac{9}{4}} & &=\frac{5}{2}\\ &=\frac{5}{2}\end{aligned}$$

$\therefore$ LHS = RHS

The solution is $x=\frac{5}{4}$

23 $12x^2+4x-9$

Use Interactive → Transformation → Factor to factorise over R.

This gives $12\left(x+\frac{\sqrt{7}}{3}+\frac{1}{6}\right)\left(x-\frac{\sqrt{7}}{3}+\frac{1}{6}\right)$ as the factors.

24 a The simultaneous equations are

$x+y=p....(1)$

$xy=q....(2)$

From equation (1), $y=p-x$.

Substitute in equation (2)

$$\begin{aligned}&\therefore x(p-x)=q\\ &\therefore px-x^2=q\\ &\therefore x^2+q=px\end{aligned}$$

Therefore, x is a solution of $x^2+q=px$

From equation (1), $x=p-y$.

Substitute in equation (2)

$$\begin{aligned}&\therefore y(p-y)=q\\ &\therefore py-y^2=q\\ &\therefore y^2+q=py\end{aligned}$$

Therefore, y is a solution of $x^2+q=px$.

Hence if $x=a,y=b$ are the solutions to the simultaneous equations, then $x=a,y=b$ are also solutions to the quadratic equation $x^2+q=px$.

b Either use the template to solve the simultaneous equations or Equation/inequality to solve the quadratic equation. Solving the quadratic equation gives the two solutions

$x=\dfrac{p-\sqrt{p^2-4q}}{2},x=\dfrac{p+\sqrt{p^2-4q}}{2}.$

If $x=\dfrac{p+\sqrt{p^2-4q}}{2}$ then $y=\dfrac{p-\sqrt{p^2-4q}}{2}$ or vice versa.

Hence, the solutions are $x=\dfrac{p\pm\sqrt{p^2-4q}}{2},y=\dfrac{p\pm\sqrt{p^2-4q}}{2}.$

Exercise 3.4 — Applications of quadratic equations

1 The salmon costs $\frac{400}{x}$ dollars per kilogram at the market.

$(x-2)$ kilograms of salmon is sold at $\left(\frac{400}{x}+10\right)$ dollars per kilogram for \$540

$$\begin{aligned}\therefore (x-2)\times\left(\frac{400}{x}+10\right)&=540\\ \therefore 400+10x-\frac{800}{x}-20&=540\\ \therefore 10x-\frac{800}{x}&=160\\ \therefore 10x^2-800&=160x\\ \therefore x^2-16x-80&=0\\ \therefore (x-20)(x+4)&=0\\ \therefore x&=20,\ x=-4\end{aligned}$$

Reject $x = -4$ since $x \in N$.

Therefore 20 kilograms of salmon were bought at the market.

2 Let the numbers be n and $n+2$

The product is 440.

$\therefore n(n+2) = 440$

$\therefore n^2 + 2n - 440 = 0$

$\therefore (n-20)(n+22) = 0$

$\therefore n = 20, -22$

Since $n \in N,\ n = 20$

Therefore the numbers are 20 and 22.

3 Let area of a sphere of radius r cm be A cm^2.

$A = kr^2$

Substitute $r = 5, A = 100\pi$

$\therefore 100\pi = k(25)$

$\therefore k = \frac{100\pi}{25}$

$\therefore k = 4\pi$

$\therefore A = 4\pi r^2$

When $A = 360\pi$,

$360\pi = 4\pi r^2$

$\therefore r^2 = 90$

$\therefore r = \pm 3\sqrt{10}$

$r > 0, \therefore r = 3\sqrt{10}$

Radius is 3 cm

4 Let C dollars be the cost of hire for t hours

$C = 10 + kt^2$

$t = 3, C = 32.50$

$\Rightarrow 32.5 = 10 + k(9)$

Solving,

$\therefore 9k = 22.5$

$\therefore k = 2.5$

$\therefore C = 10 + 2.5t^2$

When $C = 60$,

$60 = 10 + 2.5t^2$

$\therefore 2.5t^2 = 50$

$\therefore t^2 = \frac{50}{2.5}$

$\therefore t^2 = 20$

Since $t > 0$, $t = \sqrt{20}$

$\therefore t \simeq 4.472$

The chainsaw was hired for approximately $4\frac{1}{2}$ hours

5 **a** $A = kx^2$ where A is the area of an equilateral triangle of side length x and k is the constant of proportionality.

When $x = 2\sqrt{3}, A = 3\sqrt{3}$

$\therefore 3\sqrt{3} = k\left(2\sqrt{3}\right)^2$

$\therefore 3\sqrt{3} = 12k$

$\therefore k = \frac{\sqrt{3}}{4}$

Hence $A = \frac{\sqrt{3}}{4}x^2$

If $A = 12\sqrt{3}$,

$12\sqrt{3} = \frac{\sqrt{3}}{4}x^2$

$\therefore x^2 = 48$

$\therefore x = 4\sqrt{3}$

(the negative square root is not appropriate for the length).

The side length is $4\sqrt{3}$ cm.

b $d = kt^2$ where d is the distance fallen after time t and k is the constant of proportionality.

Replace t by $2t$

$\therefore d \rightarrow k(2t)^2$

$\therefore d = 4\left(kt^2\right)$

The distance is quadrupled.

c $H = kV^2$ where H is the number of calories of heat in a wire with voltage V and k is the constant of proportionality.

If the voltage is reduced by 20% then 80% of it remains.

Replace V by $0.80V$

$\therefore H \rightarrow k(0.80V)^2$

$\therefore H = 0.64\left(kV^2\right)$

This means H is now 64% of what it was, so the effect of reducing the voltage by 20% is to reduce the number of calories of heat by 36%.

6 **a** $S = k_1 n + k_2 n^2$

b $1+2+3+4 = 10 \Rightarrow S = 10$ when $n = 4$

$1+2+3+4+5 = 15 \Rightarrow S = 15$ when $n = 5$

This allows simultaneous equations to be formed.

$10 = 4k_1 + 16k_2 \Rightarrow 5 = 2k_1 + 8k_2$...(1)

$15 = 5k_1 + 25k_2 \Rightarrow 3 = k_1 + 5k_2$....(2)

equation (1) $-2\times$ equation (2)

$-1 = -2k_2$

$\therefore k_2 = \frac{1}{2}$

Substitute $k_2 = \frac{1}{2}$ in equation (2)

$\therefore 3 = k_1 + \frac{5}{2}$

$\therefore k_1 = \frac{1}{2}$

Answer $k_1 = \frac{1}{2} = k_2$

c The formula for the sum becomes $S = \frac{1}{2}n + \frac{1}{2}n^2$.

When $S = 1275$

$1275 = \frac{1}{2}n + \frac{1}{2}n^2$

$\therefore n^2 + n = 2550$

$\therefore n^2 + n - 2550 = 0$

$\therefore (n+51)(n-50) = 0$

$\therefore n = -51, n = 50$

Reject $n = -51$ since $n \in N$

$\therefore n = 50$

7 **a** $y = c + k_1 x + k_2 x^2$ where c is a constant and k_1, k_2 are constants of proportionality.

$x = 0, y = 2 \Rightarrow 2 = c$

$x = 1, y = 9 \Rightarrow 9 = c + k_1 + k_2$

$\therefore k_1 + k_2 = 7$....(1)

$x = 2, y = 24 \Rightarrow 24 = c + 2k_1 + 4k_2$

$\therefore 2k_1 + 4k_2 = 22$

$\therefore k_1 + 2k_2 = 11$....(2)

equation (2) – equation (1)

$\therefore k_2 = 4$

Substitute $k_2 = 4$ in equation (1)

$\therefore k_1 + 4 = 7$

$\therefore k_1 = 3$

The rule is $y = 2 + 3x + 4x^2$

b When $y = 117$,

$$117 = 2 + 3x + 4x^2$$
$$\therefore 4x^2 + 3x - 115 = 0$$
$$\therefore (4x + 23)(x - 5) = 0$$
$$\therefore x = -\frac{23}{4}, x = 5$$

The positive value of x is 5.

8 Cost in dollars, $C = 20 + 5x$
Revenue in dollars, $R = 1.5x^2$
Profit in dollars, $P = R - C$
$\therefore P = 1.5x^2 - 5x - 20$
If $P = 800$,

$$800 = 1.5x^2 - 5x - 20$$
$$\therefore 1.5x^2 - 5x - 820 = 0$$
$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1.5 \times (-820)}}{2 \times 1.5}$$
$$\therefore x = \frac{5 \pm \sqrt{25 + 4920}}{3}$$
$$\therefore x = \frac{5 \pm \sqrt{4945}}{3}$$
$$\therefore x \simeq 25.107, x \simeq -21.773$$

Reject the negative value so $x \simeq 25.107$.
The number of litres is $100x = 2510.7$. To the nearest litre, 2511 litres must be sold.

9 Let the natural numbers be n and $n + 1$.

$$n^2 + (n+1)^2 + (n + (n+1))^2 = 662$$
$$\therefore n^2 + (n+1)^2 + (2n+1)^2 = 662$$
$$\therefore n^2 + n^2 + 2n + 1 + 4n^2 + 4n + 1 = 662$$
$$\therefore 6n^2 + 6n - 660 = 0$$
$$\therefore n^2 + n - 110 = 0$$
$$\therefore (n + 11)(n - 10) = 0$$
$$\therefore n = -11 (\text{reject}), n = 10$$
$$\therefore n = 10$$

The two consecutive natural numbers are 10 and 11.

10 $A = \frac{1}{2}bh$ where A is the area of a triangle with base b and height h.
Given $h : b = \sqrt{2} : 1$, then

$$\frac{h}{b} = \frac{\sqrt{2}}{1}$$
$$\therefore h = \sqrt{2}b$$
$$\therefore A = \frac{1}{2}\sqrt{2}b^2$$

When $A = \sqrt{32}$

$$\sqrt{32} = \frac{\sqrt{2}}{2}b^2$$
$$\therefore b^2 = 4\sqrt{2} \times \frac{2}{\sqrt{2}}$$
$$\therefore b^2 = 8$$
$$\therefore b = 2\sqrt{2}$$

(reject negative square root)
With $b = 2\sqrt{2}$,

$$h = \sqrt{2} \times 2\sqrt{2}$$
$$\therefore h = 4$$

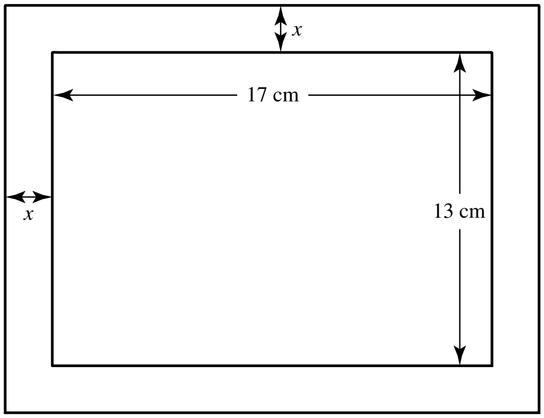
The base is $2\sqrt{2}$ cm and the height is 4 cm.

11 Using Pythagoras's theorem,

$$(3x + 3)^2 = (3x)^2 + (x - 3)^2$$
$$\therefore 9x^2 + 18x + 9 = 9x^2 + x^2 - 6x + 9$$
$$\therefore x^2 - 24x = 0$$
$$\therefore x(x - 24) = 0$$
$$\therefore x = 0, x = 24$$

Reject $x = 0$ because $3x$ would be zero and $x - 3$ would be negative.
$\therefore x = 24$
The three side lengths are 72 cm, 21 cm and 75 cm so the perimeter is 168 cm.

12



Let the width of the border be x cm.
The frame has length $(17 + 2x)$ cm and width $(13 + 2x)$.
Area of border is the difference between the area of the frame and the area of the photo

$$\therefore 260 = (17 + 2x)(13 + 2x) - 17 \times 13$$
$$\therefore 260 = 221 + 60x + 4x^2 - 221$$
$$\therefore 4x^2 + 60x - 260 = 0$$
$$\therefore x^2 + 15x - 65 = 0$$
$$\therefore x = \frac{-15 \pm \sqrt{15^2 - 4 \times 1 \times (-65)}}{2}$$
$$\therefore x = \frac{-15 \pm \sqrt{225 + 260}}{2}$$
$$\therefore x = \frac{-15 \pm \sqrt{485}}{2}$$
$$\therefore x \simeq 3.511, x \simeq -18.51$$

Reject negative value
$\therefore x \simeq 3.511$
Length of frame is $17 + 2 \times 3.511 = 24.0$ cm or 240 mm
Width of frame is $13 + 2 \times 3.511 = 20.0$ cm or 200 mm.

13 **a** Let the length be l metres.

$$x + l + x = 16$$
$$\therefore 2x + l = 16$$
$$\therefore l = 16 - 2x$$

b Area is the product of the length and width

$$k = x(16 - 2x)$$
$$\therefore k = 16x - 2x^2$$
$$\therefore 2x^2 - 16x + k = 0$$

c Discriminant,

$$\Delta = (-16)^2 - 4 \times 2 \times k$$
$$= 256 - 8k$$

i No solutions if $\Delta < 0$

$$\therefore 256 - 8k < 0$$
$$\therefore 256 < 8k$$
$$\therefore k > 32$$

ii One solution if $\Delta = 0$

$\therefore 256 - 8k = 0$

$\therefore k = 32$

iii Two solutions if $\Delta > 0$

$\therefore 256 - 8k > 0$

$\therefore k < 32$

However, k is the area measure so there are two solutions if $0 < k < 32$.

d There can only be solutions to $2x^2 - 16x + k = 0$ if $\Delta \geq 0$. This means $0 < k \leq 32$. The greatest value of k is therefore 32.

The largest area is 32 square metres.

To find the dimensions, the value of x needs to be found when $k = 32$.

$$2x^2 - 16x + 32 = 0$$
$$\therefore x^2 - 8x + 16 = 0$$
$$\therefore (x-4)^2 = 0$$
$$\therefore x = 4$$

Length is $16 - 2 \times 4 = 8$ metres and width is 4 metres

e Put $k = 15$ in the equation $2x^2 - 16x + k = 0$

$$\therefore 2x^2 - 16x + 15 = 0$$
$$\therefore x^2 - 8x + 7.5 = 0$$
$$\therefore (x^2 - 8x + 16) - 16 + 7.5 = 0$$
$$\therefore (x-4)^2 - 8.5 = 0$$
$$\therefore x - 4 = \pm\sqrt{8.5}$$
$$\therefore x = 4 \pm \sqrt{8.5}$$
$$\therefore x \simeq 6.91 \text{ or } x \simeq 1.08$$

If $x = 6.91$, width is 6.91 m and length is $16 - 2 \times 6.91 = 2.18$ m

If $x = 1.08$, width is 1.08 m and length is $16 - 2 \times 1.08 = 13.84$ m.

To use as much of the back fence as possible $x = 1.08$.

The dimensions of the rectangle are width 1.1 metres and length 13.8 metres,

14 Let the parcel of cards purchased for \$10 at the fete contain x cards.

Therefore, the cost per card is $\frac{10}{x}$ dollars.

The collector sells $(x - 2)$ cards to the friend at a cost per card of $\left(\frac{10}{x} + 1\right)$ dollars for a total of \$16 since the collector makes a profit of \$6.

$$\therefore (x-2) \times \left(\frac{10}{x} + 1\right) = 16$$
$$\therefore 10 + x - \frac{20}{x} - 2 = 16$$
$$\therefore x - 8 = \frac{20}{x}$$
$$\therefore x^2 - 8x = 20$$
$$\therefore x^2 - 8x - 20 = 0$$
$$\therefore (x-10)(x+2) = 0$$
$$\therefore x = 10$$

(reject $x = -2$)

The friend receives 8 cards.

15 One method is to Define $A = \pi r^2 + \pi r l$ using Func name: A, Variable/s : r, l and Expression $\pi r^2 + \pi r l$.

Then use Equation/inequality to solve $20 = A(r, 5)$ for r.

This gives, to three decimal places, $\{r = -6.0525, r = 1.052\}$.

Rejecting the negative value, the radius of the base of the cone is 1.052 m.

16 Use Equation/inequality to solve $20 = A(r, l)$ for r using Standard mode gives

$$\left\{ r = \frac{-\left(l\pi - \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi}, r = \frac{-\left(l\pi + \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi} \right\}$$

Rejecting the negative value, $r = \dfrac{-l\pi + \sqrt{l^2\pi^2 + 80\pi}}{2\pi}$

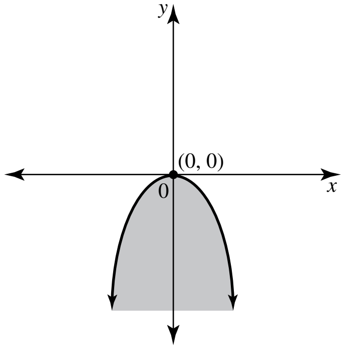
Highlight this expression and drag down to the next prompt. Add in the condition $l = 5$ from the keyboard Math Optn and evaluate on Decimal mode to obtain the same value for r as in Question 11.

$$\frac{-\left(l\pi - \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi} \mid l = 5$$
$$= 1.052$$

Exercise 3.5 — Graphs of quadratic polynomials

1 **A i** $y = x^2 - 2$ **B ii** $y = -2x^2$ **C iii** $y = -(x+2)^2$

2



3 $y = \frac{1}{3}x^2 + x - 6$

Axis of symmetry equation:

$$x = -\frac{1}{2/3}$$
$$\therefore x = -\frac{3}{2}$$

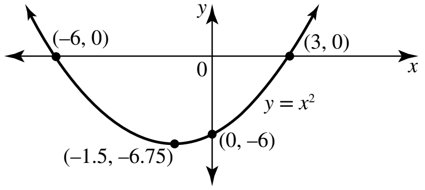
Turning point: Substitute $x = -\frac{3}{2}$

$$\therefore y = \frac{1}{3} \times \left(-\frac{3}{2}\right)^2 + -\frac{3}{2} - 6$$
$$\therefore y = \frac{3}{4} - \frac{3}{2} - 6$$
$$\therefore y = -\frac{27}{4}$$
$$\Rightarrow (-1.5, -6.75)$$

y intercept: $(0, -6)$

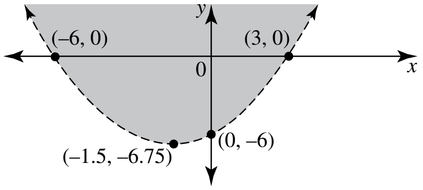
x intercepts: Put $y = 0$

$$\therefore \frac{1}{3}x^2 + x - 6 = 0$$
$$\therefore x^2 + 3x - 18 = 0$$
$$\therefore (x+6)(x-3) = 0$$
$$\therefore x = -6, 3$$
$$\Rightarrow (-6, 0), (3, 0)$$



4 $y > \frac{1}{3}x^2 + x - 6$

Boundary graph is not included as region is open.



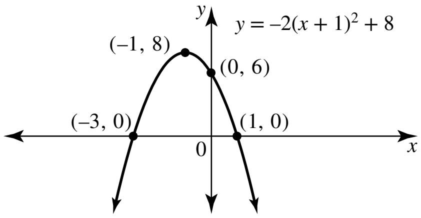
5 **a** $y = -2(x+1)^2 + 8$

Turning point: $(-1, 8)$ Type: maximum

y intercept: Put $x = 0 \therefore y = 6 \Rightarrow (0, 6)$

x intercepts: Put $y = 0$

$$\therefore -2(x+1)^2 + 8 = 0$$
$$\therefore (x+1)^2 = 4$$
$$\therefore x + 1 = \pm 2$$
$$\therefore x = -3, x = 1$$
$$\Rightarrow (-3, 0), (1, 0)$$



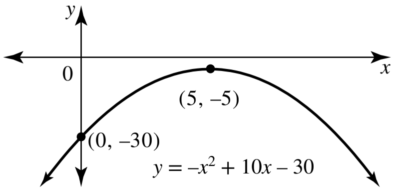
b **i** $y = -x^2 + 10x - 30$

$$= -(x^2 - 10x + 30)$$
$$= -[(x^2 - 10x + 25) - 25 + 30]$$
$$= -\left[(x-5)^2 + 5\right]$$
$$= -(x-5)^2 - 5$$

Vertex is $(5, -5)$, a maximum turning point.

ii y intercept: $(0, -30)$

no x intercepts since concave down with maximum y value of -5.



6 **a** $y = 4 - 3x^2$

$\therefore y = -3x^2 + 4$

maximum turning point at $(0, 4)$

b $y = (4 - 3x)^2$

$(4 - 3x) = 0 \Rightarrow x = \frac{4}{3}$ Therefore minimum turning point at $\left(\frac{4}{3}, 0\right)$

or

$$y = (4 - 3x)^2$$
$$= (3x - 4)^2$$
$$= \left(3\left(x - \frac{4}{3}\right)\right)^2$$
$$= 9\left(x - \frac{4}{3}\right)^2$$

minimum turning point at $\left(\frac{4}{3}, 0\right)$

7 $y = 2x(4 - x)$

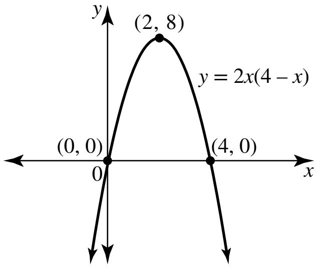
x intercepts: $2x(4 - x) = 0$

$\therefore x = 0, x = 4$

$\Rightarrow (0, 0), (4, 0)$

Turning point:

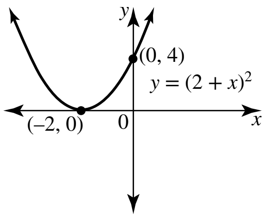
$$x = \frac{0 + 4}{2}$$
$$\therefore x = 2$$
$$\therefore y = 4(2)$$
$$\therefore y = 8$$
$$\Rightarrow (2, 8)$$



8 $y = (2 + x)^2$

x intercept and turning point: $(-2, 0)$

y intercept: $(0, 4)$



9 **a** $y = 42x - 18x^2$

$\Delta = b^2 - 4ac,\ a = -18, b = 42, c = 0$

$$\therefore \Delta = 42^2 - 4 \times (-18) \times 0$$
$$= 42^2$$

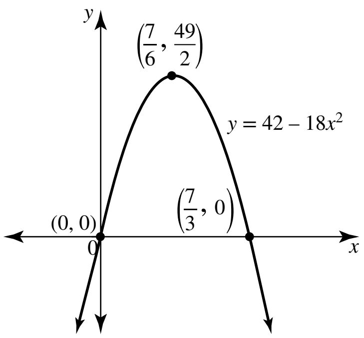
Since the discriminant is a perfect square there are 2 rational x intercepts. (Obvious from the factors of the equation).

b Factored form is $y = 6x(7 - 3x)$

x intercepts: $(0, 0), \left(\frac{7}{3}, 0\right)$

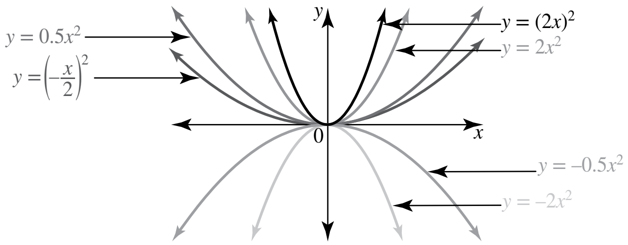
Turning point:

$$x = \frac{0 + 7/3}{2}$$
$$\therefore x = \frac{7}{6}$$
$$\therefore y = 6 \times \frac{7}{6}\left(7 - 3 \times \frac{7}{6}\right)$$
$$\therefore y = 7\left(\frac{7}{2}\right)$$
$$\therefore y = \frac{49}{2}$$
$$\Rightarrow \left(\frac{7}{6}, \frac{49}{2}\right)$$

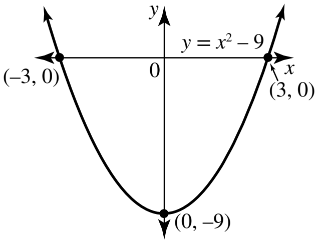


10 $7x^2 - 4x + 9$
$\Delta = b^2 - 4ac,\ a = 7, b = -4, c = 9$
$\therefore \Delta = (-4)^2 - 4 \times 7 \times 9$
$= -236$
Since $\Delta < 0$ and $a > 0$, $7x^2 - 4x + 9$ is positive definite

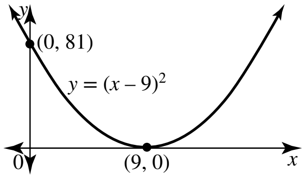
11



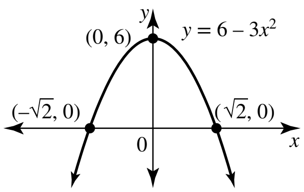
12 **a** $y = x^2 - 9$
Min TP $(0, -9)$ and this is also the y intercept.
x intercepts: $0 = x^2 - 9$
$\therefore 0 = (x - 3)(x + 3)$
$\therefore x = 3, x = -3$
$(3, 0), (-3, 0)$



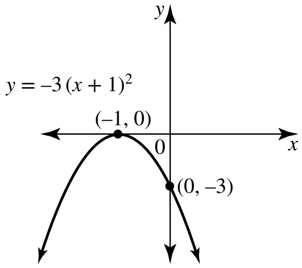
b $y = (x - 9)^2$
Min TP $(9, 0)$ and this is also the x intercept.
y intercept $y = (-9)^2 \Rightarrow (0, 81)$



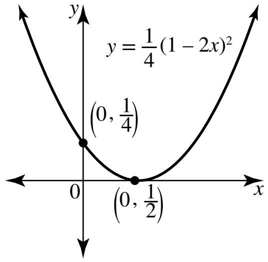
c $y = 6 - 3x^2$
$\therefore y = -3x^2 + 6$
Max TP $(0, 6)$ which is also the y intercept.
x intercept: $0 = 6 - 3x^2$
$\therefore 3x^2 = 6$
$\therefore x^2 = 2$
$\therefore x = \pm\sqrt{2}$
$\left(\sqrt{2}, 0\right), \left(-\sqrt{2}, 0\right)$



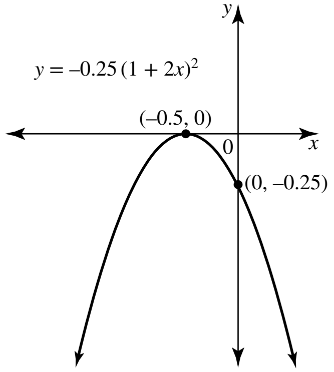
d $y = -3(x + 1)^2$
Max TP $(-1, 0)$ and this is also x intercept.
y intercept: $y = -3(1)^2 \Rightarrow (0, -3)$



e $y = \frac{1}{4}(1 - 2x)^2$
TP occurs when
$1 - 2x = 0$
$\therefore x = \frac{1}{2}$
Min TP $\left(\frac{1}{2}, 0\right)$ is also x intercept.
y intercept $y = \frac{1}{4}(1)^2 \Leftarrow \left(0, \frac{1}{4}\right)$



f $y = -0.25(1 + 2x)^2$
TP occurs when
$1 + 2x = 0$
$\therefore x = -0.5$
Max TP $(-0.5, 0)$ is also the x intercept.
y intercept: $y = -0.25(1)^2 \Leftarrow (0, -0.25)$



13 **a** $y = x^2 + 6x - 7$
TP $x = \frac{-b}{2a}$
$\therefore x = \frac{-6}{2} = -3$
$\therefore y = 9 - 18 - 7 = -16$

Min TP $(-3,-16)$

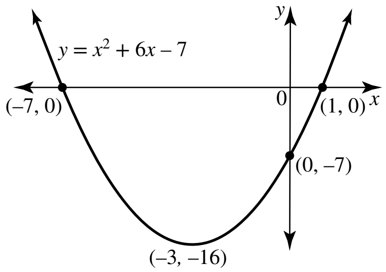
y intercept $(0,-7)$

x intercepts: $0 = x^2 + 6x - 7$

$\therefore (x+7)(x-1) = 0$

$\therefore x = -7, x = 1$

$(-7,0),(1,0)$



b $y = 3x^2 - 6x - 7$

TP $x = \dfrac{-b}{2a}$

$\therefore x = \dfrac{6}{6}$

$\therefore x = 1$

$y = 3 - 6 - 7$

$\therefore y = -10$

Min TP $(1,-10)$

y intercept $(0,-7)$

x intercept $0 = 3x^2 - 6x - 7$

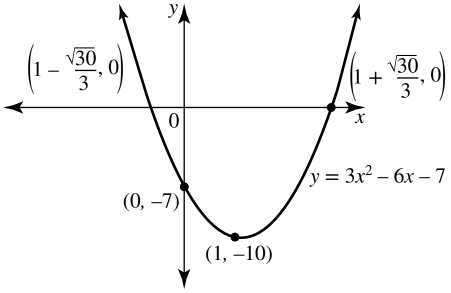
$$\therefore x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\times 3\times(-7)}}{6}$$

$$= \frac{6 \pm \sqrt{120}}{6}$$

$$= \frac{6 \pm 2\sqrt{30}}{6}$$

$$\therefore x = \frac{3 \pm \sqrt{30}}{3}$$

$\left(\dfrac{3+\sqrt{30}}{3},0\right), \left(\dfrac{3-\sqrt{30}}{3},0\right)$



c $y = 5 + 4x - 3x^2$

$\therefore y = -3x^2 + 4x + 5$

TP $x = \dfrac{-b}{2a}$

$\therefore x = \dfrac{-4}{-6}$

$\therefore x = \dfrac{2}{3}$

$y = -3\times\dfrac{4}{9} + 4\times\dfrac{2}{3} + 5$

$\therefore y = \dfrac{19}{3}$

Max TP $\left(\dfrac{2}{3},\dfrac{19}{3}\right)$

y intercept $(0,5)$

x intercept: $0 = -3x^2 + 4x + 5$

$\therefore 3x^2 - 4x - 5 = 0$

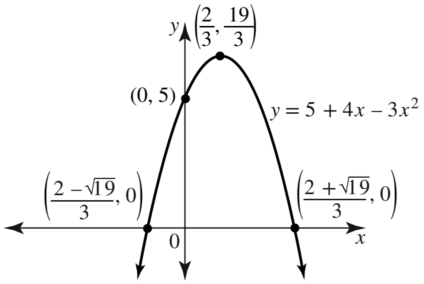
$$\therefore x = \frac{4 \pm \sqrt{4^2 - 4\times 3\times(-5)}}{6}$$

$$= \frac{4 \pm \sqrt{76}}{6}$$

$$= \frac{4 \pm 2\sqrt{19}}{6}$$

$$\therefore x = \frac{2 \pm \sqrt{19}}{3}$$

$\left(\dfrac{2\pm\sqrt{19}}{3},0\right)$



d $y = 2x^2 - x - 4$

Min TP $x = \dfrac{-b}{2a}$

$\therefore x = \dfrac{1}{4}$

$y = 2\times\dfrac{1}{16} - \dfrac{1}{4} - 4$

$= -\dfrac{33}{8}$

$\left(\dfrac{1}{4},-\dfrac{33}{8}\right)$

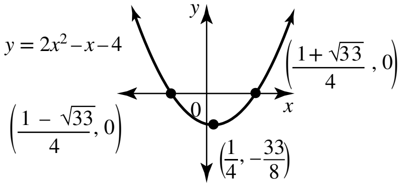
y intercept $(0,-4)$

x intercepts $0 = 2x^2 - x - 4$

$$\therefore x = \frac{1 \pm \sqrt{1+32}}{4}$$

$$\therefore x = \frac{1 \pm \sqrt{33}}{4}$$

$\left(\dfrac{1\pm\sqrt{33}}{4},0\right)$



e $y = -2x^2 + 3x - 4$

MaxTP $x = \dfrac{-b}{2a}$

$\therefore x = \dfrac{3}{4}$

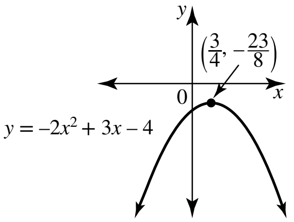
$y = -2\times\dfrac{9}{16} + 3\times\dfrac{3}{4} - 4$

$= -\dfrac{23}{8}$

$\left(\dfrac{3}{4},-\dfrac{23}{8}\right)$

y intercept $(0,-4)$

As TP $\left(\frac{3}{4},-\frac{23}{8}\right)$ is a maximum, there are no x intercepts



f $y=10-2x^2+8x$

$\therefore y=-2x^2+8x+10$

Max TP $x=\frac{-b}{2a}$

$\therefore x=\frac{-8}{-4}$

$\therefore x=2$

$y=-2(2)^2+8(2)+10$

$\therefore y=18$

$(2,18)$

y intercept $(0,10)$

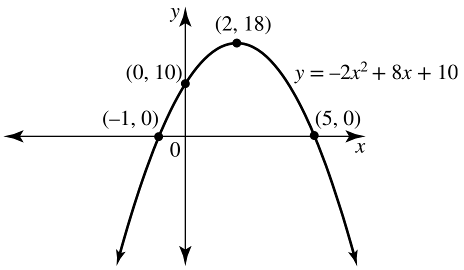
x intercept: $0=-2x^2+8x+10$

$\therefore x^2-4x-5=0$

$\therefore (x-5)(x+1)=0$

$\therefore x=5, x=-1$

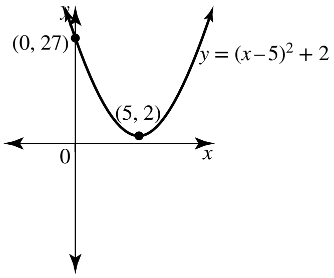
$(5,0),(-1,0)$



14 a $y=(x-5)^2+2$

Min TP $(5,2)$ so there are no x intercepts

y intercept $(0,27)$



b $y=2(x+1)^2-2$

Min TP $(-1,-2)$

y intercept $(0,0)$

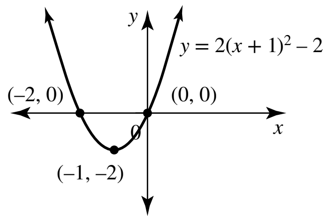
x intercepts: $0=2(x+1)^2-2$

$\therefore (x+1)^2=1$

$\therefore x+1=\pm 1$

$\therefore x=-2, x=0$

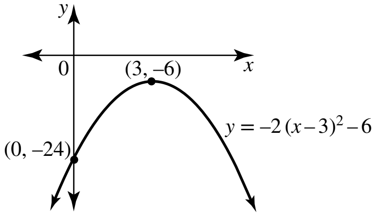
$(-2,0),(0,0)$



c $y=-2(x-3)^2-6$

Max TP $(3,-6)$ so no x intercepts

y intercept $(0,-24)$



d $y=-(x-4)^2+1$

Max TP $(4,1)$

y intercept $(0,-15)$

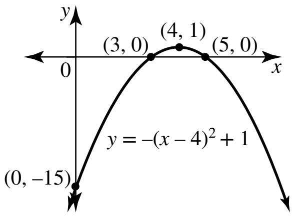
x intercepts: $0=-(x-4)^2+1$

$\therefore (x-4)^2=1$

$\therefore x-4=\pm 1$

$\therefore x=3, x=5$

$(3,0),(5,0)$



e $y+2=\frac{(x+4)^2}{2}$

$\therefore y=\frac{1}{2}(x+4)^2-2$

Min TP $(-4,-2)$

y intercept $(0,6)$

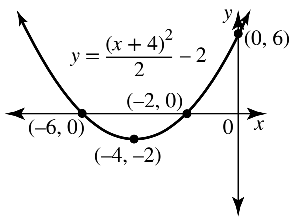
x intercepts: $2=\frac{(x+4)^2}{2}$

$\therefore (x+4)^2=4$

$\therefore x+4=\pm 2$

$\therefore x=-6, x=-2$

$(-6,0),(-2,0)$



f $9y=1-\frac{1}{3}(2x-1)^2$

$\therefore y=-\frac{1}{27}(2x-1)^2+\frac{1}{9}$

Max TP $\left(\frac{1}{2},\frac{1}{9}\right)$

y intercept: $y=-\frac{1}{27}+\frac{1}{9}\Rightarrow\left(0,\frac{2}{27}\right)$

x intercepts: $0=1-\frac{1}{3}(2x-1)^2$

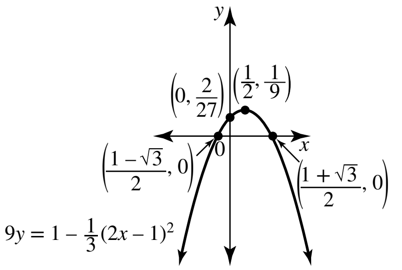
$\therefore (2x-1)^2=3$

$\therefore 2x-1=\pm\sqrt{3}$

$\therefore 2x=1\pm\sqrt{3}$

$\therefore x=\frac{1\pm\sqrt{3}}{2}$

$\left(\frac{1\pm\sqrt{3}}{2},0\right)$



15 a $y=-5x(x+6)$

x intercepts: $5x=0, x+6=0\Rightarrow(0,0), (-6,0)$

TP The x intercepts are symmetric with the turning point.

$\therefore x=\frac{-6+0}{2}$

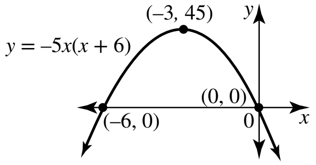
$\therefore x=-3$

$y=-5\times(-3)\times(-3+6)$

$\therefore y=45$

Max TP $(-3,45)$

y intercept $(0,0)$



b $y=(4x-1)(x+2)$

x intercepts: $\left(\frac{1}{4},0\right),(-2,0)$

Min TP $x=\frac{\frac{1}{4}+(-2)}{2}$

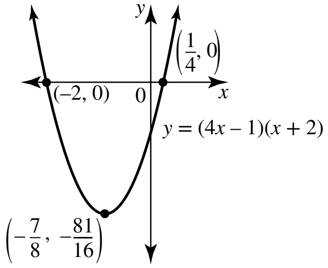
$\therefore x=\frac{-7}{8}$

$y=\left(-\frac{7}{2}-1\right)\left(-\frac{7}{8}+2\right)$

$\therefore y=-\frac{81}{16}$

$\left(-\frac{7}{8},-\frac{81}{16}\right)$

y intercept $(0,-2)$



c $y=-2(1+x)(2-x)$

x intercepts: $(-1,0),(2,0)$

Min TP $x=\frac{-1+2}{2}$

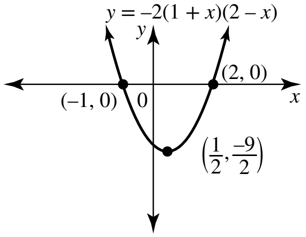
$\therefore x=\frac{1}{2}$

$y=-2\times\frac{3}{2}\times\frac{3}{2}$

$\therefore y=-\frac{9}{2}$

$\left(\frac{1}{2},-\frac{9}{2}\right)$

y intercept $(0,-4)$



d $y=(2x+1)(2-3x)$

x intercepts $2x+1=0, 2-3x=0$

$\therefore x=-\frac{1}{2},\ x=\frac{2}{3}$

$\left(-\frac{1}{2},0\right),\left(\frac{2}{3},0\right)$

Max TP $x=\frac{-\frac{1}{2}+\frac{2}{3}}{2}$

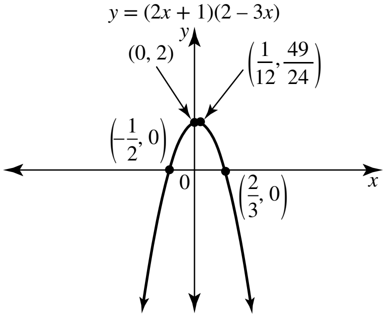
$\therefore x=\frac{1}{12}$

$y=\left(\frac{1}{6}+1\right)\times\left(2-\frac{1}{4}\right)$

$\therefore y=\frac{49}{24}$

$\left(\frac{1}{12},\frac{49}{24}\right)$

y intercept $(0,2)$



e $y=0.8x(10x-27)$

x intercepts: $(0,0),(2.7,0)$

Min TP $x=\frac{0+2.7}{2}$

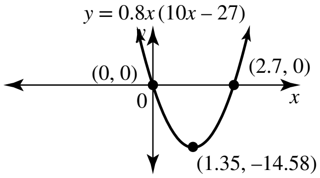
$\therefore x=1.35$

$y=0.8\times1.35\times(13.5-27)$

$\therefore y=-14.58$

$(1.35,-14.58)$

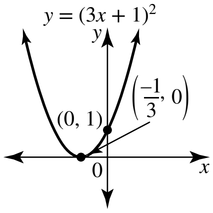
y intercept $(0,0)$



f $y = (3x+1)^2$

x intercept and minimum turning point at $\left(-\frac{1}{3}, 0\right)$

y intercept $(0,1)$



16 a $y = 9x^2 + 17x - 12$

$\Delta = b^2 - 4ac \quad a = 9,\ b = 17,\ c = -12$

$\therefore \Delta = 289 - 4 \times 9 \times (-12)$

$\therefore \Delta = 721$

Since $\Delta > 0$ but not a perfect square, there are two irrational intercepts with the x axis.

b $y = -5x^2 + 20x - 21$

$\Delta = b^2 - 4ac \quad a = -5,\ b = 20,\ c = -21$

$\therefore \Delta = 400 - 4 \times (-5) \times (-21)$

$\therefore \Delta = -20$

Since $\Delta < 0$, there are no intercepts with the x axis.

c $y = -3x^2 - 30x - 75$

$\Delta = b^2 - 4ac \quad a = -3,\ b = -30,\ c = -75$

$\therefore \Delta = 900 - 4 \times (-3) \times (-75)$

$\therefore \Delta = 0$

Since $\Delta = 0$, there is one rational intercept with the x axis.

d $y = 0.02x^2 + 0.5x + 2$

$\Delta = b^2 - 4ac \quad a = 0.02,\ b = 0.5,\ c = 2$

$\therefore \Delta = 0.25 - 4 \times 0.02 \times 2$

$\therefore \Delta = 0.09$

$\therefore \Delta = (0.3)^2$

Since $\Delta > 0$ and it is a perfect square, there are two rational intercepts with the x axis.

17 a i $2x^2 - 12x + 9$

$$\begin{aligned}
&= 2\left[x^2 - 6x + \frac{9}{2}\right] \\
&= 2\left[\left(x^2 - 6x + 9\right) - 9 + \frac{9}{2}\right] \\
&= 2\left[(x-3)^2 - \frac{18}{2} + \frac{9}{2}\right] \\
&= 2\left[(x-3)^2 - \frac{9}{2}\right] \\
&= 2(x-3)^2 - 9
\end{aligned}$$

ii TP $(3, -9)$

iii minimum value is given by the y co-ordinate of the minimum turning point.
The minimum value is –9.

b i $-x^2 - 18x + 5$

$$\begin{aligned}
&= -\left[x^2 + 18x - 5\right] \\
&= -\left[\left(x^2 + 18x + 81\right) - 81 - 5\right] \\
&= -\left[(x+9)^2 - 86\right] \\
&= -(x+9)^2 + 86
\end{aligned}$$

ii maximum TP $(-9, 86)$.

iii Maximum value is 86.

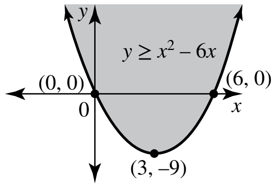
18 a $y \geq x^2 - 6x$ is the closed region above $y = x^2 - 6x$.

Consider $y = x^2 - 6x$

$y = x(x-6)$

intercepts with the axes: $(0,0), (6,0)$

Min TP $x = 3, y = -9 \Rightarrow (3, -9)$



b $y \leq 8 - 2x^2$ is the closed region below $y = 8 - 2x^2$.

Consider $y = 8 - 2x^2$

$y = -2x^2 + 8$ has maximum turning point and y intercept $(0, 8)$.

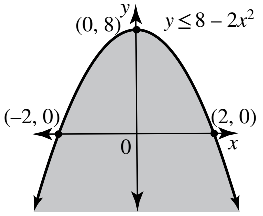
x intercepts: $0 = -2x^2 + 8$

$\therefore 2x^2 = 8$

$\therefore x^2 = 4$

$\therefore x = \pm 2$

$(\pm 2, 0)$



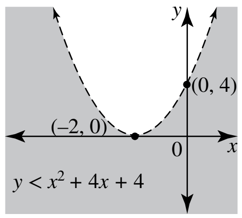
c $y < x^2 + 4x + 4$ is the open region below $y = x^2 + 4x + 4$.

Consider $y = x^2 + 4x + 4$

$y = (x+2)^2$

Minimum turning point and x intercept $(-2, 0)$.

y intercept $(0, 4)$

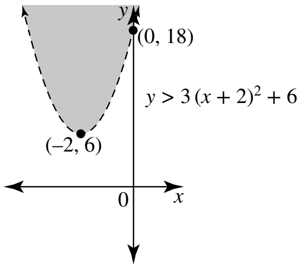


d $y > 3(x+2)^2 + 6$ is the open region above $y = 3(x+2)^2 + 6$.

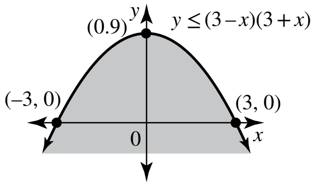
Consider $y = 3(x+2)^2 + 6$

Min TP $(-2, 6)$ so no x intercepts

y intercept $(0, 18)$



e $y \le (3-x)(3+x)$ is the closed region below $y=(3-x)(3+x)$.
Consider $y=(3-x)(3+x)$, which is $y=9-x^2$.
Max TP and y intercept $(0,9)$.
x intercepts $(\pm 3, 0)$



f $y < 7x+14x^2$ is the open region below $y=7x+14x^2$.
Consider $y=7x+14x^2$ which is the same as
$y=7x(1+2x)$

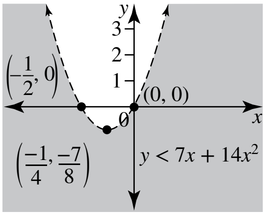
intercepts with axes $(0,0), \left(-\frac{1}{2},0\right)$

Min TP $x=-\frac{1}{4}$

$$y=-\frac{7}{4}\left(1-\frac{1}{2}\right)$$

$$\therefore y=-\frac{7}{8}$$

$$\left(-\frac{1}{4},-\frac{7}{8}\right)$$



19 $y=5x^2+10x-k$

i For one x intercept, $\Delta=0$.
$\Delta=(10)^2-4\times(5)\times(-k)$
$=100+20k$
Therefore, $100+20k=0 \Rightarrow k=-5$

ii For two x intercepts, $\Delta>0$.
Therefore, $100+20k>0 \Rightarrow k>-5$

iii For no x intercepts, $\Delta<0$.
Therefore, $100+20k<0 \Rightarrow k<-5$.

20 **a** mx^2-2x+4 is positive definite if $\Delta<0$ and $m>0$.
$\Delta=(-2)^2-4\times m\times 4$
$=4-16m$
$\Delta<0 \Rightarrow 4-16m<0$

$$\therefore m>\frac{1}{4}$$

Positive definite for $m>\frac{1}{4}$.

b **i** px^2+3x-9 is positive definite if $\Delta<0$ and $p>0$.
$\Delta=(3)^2-4\times p\times(-9)$
$=9+36p$

$\Delta<0 \Rightarrow 9+36p<0$

$$\therefore p<-\frac{1}{4}$$

There is no positive value of p for which $\Delta<0$. Hence, there is no real value of p for which px^2+3x-9 is positive definite.

ii If $p=3$, $y=3x^2+3x-9$.

The equation of the axis of symmetry is $x=-\frac{b}{2a}$.

$$\therefore x=-\frac{3}{6}$$

$$\therefore x=-\frac{1}{2}$$

The axis of symmetry has equation $x=-\frac{1}{2}$.

c **i** $y=2x^2-3tx+12$
If the turning point lies on the x axis, there is only one x intercept so $\Delta=0$.

$$\therefore (-3t)^2-4\times 2\times 12=0$$

$$\therefore 9t^2=96$$

$$\therefore t^2=\frac{32}{3}$$

$$\therefore t=\pm\sqrt{\frac{32}{3}}$$

$$\therefore t=\pm\frac{4\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$\therefore t=\pm\frac{4\sqrt{6}}{3}$$

ii Axis of symmetry equation is $x=-\frac{b}{2a}$

$$\therefore x=-\frac{-3t}{4}$$

$$\therefore x=\frac{3t}{4}$$

For axis of symmetry equation to be $x=3t^2$,

$$3t^2=\frac{3t}{4}$$

$$\therefore 12t^2-3t=0$$

$$\therefore 3t(4t-1)=0$$

$$\therefore t=0 \text{ or } t=\frac{1}{4}$$

21 **a** **i** As the x co-ordinate of the turning point is $x=-\frac{b}{2a}$, one method is to enter $y=12x^2-18x+5|x=-(-18)/24$ in the main screen on Standard mode and press EXE to calculate the y co-ordinate.

The turning point has $y=\frac{7}{4}$ and simplifying the x co-ordinate gives $x=\frac{3}{4}$.

Turning point is $\left(\frac{3}{4},-\frac{7}{4}\right)$

ii In the main screen use Equation/Inequality to solve $12x^2-18x+5=0$. This gives the x intercepts as

$$x=\frac{-\sqrt{21}+9}{12}, x=\frac{\sqrt{21}+9}{12}.$$

b **i** Entering $y=-8x^2+9x+12|x=-9/-16$ gives $y=\frac{465}{32}$.

Turning point is $\left(\frac{9}{16},\frac{465}{32}\right)$.

ii Solving $-8x^2+9x+12=0$ gives the x intercepts as

$x=\frac{-\left(\sqrt{465}-9\right)}{16}$, $x=\frac{\sqrt{465}+9}{16}$.

22 As decimals are required, the Graphing screen could be used. Enter $y1=-2.1x^2+52x+10$, tick the box and graph. It is likely the screen scale will need adjusting to show the key aspects of the graph. Zoom → Auto or Zoom Out may enable the details of the graph to be visible.

Tap Analysis → G-Solve → Root to obtain x intercepts. This gives the smaller root as $x=-0.19$ (correct to two decimal places). Tap the central cursor on the right to obtain the second root as $x=24.95$.

Tap Analysis → G-Solve → Max to obtain the turning point as $x=12.38, y=331.90$.

Tap Analysis → G-Solve → y-intercept to obtain, of course, $x=0, y=10$.

The key points are: TP $(12.38, 331.90)$, x intercepts $(-0.19,0),(24.95,0)$, y intercept $(0,10)$.

Exercise 3.6 — Determining the rule from a graph of a quadratic polynomial

1 **a** Given information: minimum turning point $(-2,1)$ and point $(0,5)$.

Let $y=a(x-h)^2+k$

$\therefore y=a(x+2)^2+1$

Substitute $(0,5)$

$\therefore 5=a(2)^2+1$

$\therefore a=1$

Therefore the equation of the parabola is $y=(x+2)^2+1$

b Given information: x intercepts at $x=0, x=2$ and point $(-1,6)$

Let $y=a(x-x_1)(x-x_2)$

Since x intercepts at $x=0, x=2$,

$\therefore y=ax(x-2)$

Substitute $(-1,6)$

$\therefore 6=a(-1)(-1-2)$

$\therefore 6=3a$

$\therefore a=2$

Therefore the equation of the parabola is $y=2x(x-2)$

2 Given information: minimum turning point and x intercept of $(-2,0)$, point $(2,2)$

Let $y=a(x-h)^2+k$ or $y=a(x-x_1)(x-x_2)$

$\therefore y=a(x+2)^2$

Substitute $(2,2)$

$\therefore 2=a(2+2)^2$

$\therefore 2=a(16)$

$\therefore a=\frac{1}{8}$

Therefore the equation is $y=\frac{1}{8}(x+2)^2$

To obtain polynomial form:

$y=\frac{1}{8}(x+2)^2$

$\therefore y=\frac{1}{8}(x^2+4x+4)$

$\therefore y=\frac{1}{8}x^2+\frac{1}{2}x+\frac{1}{2}$

3 Let $y=ax^2+bx+c$

$(-1,-7)\Rightarrow -7=a(-1)^2+b(-1)+c$

$\therefore a-b+c=-7$.........(1)

$(2,-10)\Rightarrow -10=a(2)^2+b(2)+c$

$\therefore 4a+2b+c=-10$.........(2)

$(4,-32)\Rightarrow -32=a(4)^2+b(4)+c$

$\therefore 16a+4b+c=-32$.........(3)

Eliminating c

$(2)-(1)$

$\therefore 3a+3b=-3$

$\therefore a+b=-1$.........(4)

$(3)-(1)$

$\therefore 15a+5b=-25$

$\therefore 3a+b=-5$.........(5)

Solving equations (4) and (5)

$(5)-(4)$

$\therefore 2a=-4$

$\therefore a=-2$

$\therefore b=1$

In equation (1)

$-2-1+c=-7$

$\therefore c=-4$

Therefore the equation of the parabola is $y=-2x^2+x-4$

4 Let $y=ax^2+bx+c$

$(0,-2)\Rightarrow -2=a(0)^2+b(0)+c$

$\therefore c=-2$

$\therefore y=ax^2+bx-2$

$(-1,0)\Rightarrow 0=a(-1)^2+b(-1)-2$

$\therefore a-b-2=0$

$\therefore a-b=2$......(1)

$(4,0)\Rightarrow 0=a(4)^2+b(4)-2$

$\therefore 16a+4b-2=0$

$\therefore 8a+2b=1$......(2)

Solving equations (1) and (2),

$(2)+2\times(1)$

$10a=5$

$\therefore a=\frac{1}{2}$

$\therefore \frac{1}{2}-b=2$

$\therefore b=-\frac{3}{2}$

Therefore the equation is $y=\frac{1}{2}x^2-\frac{3}{2}x-2$

In worked example 17b, the equation $y=\frac{1}{2}(x+1)(x-4)$ was obtained.

$y=\frac{1}{2}x^2-\frac{3}{2}x-2$

$\therefore y=\frac{1}{2}(x^2-3x-4)$

$\therefore y=\frac{1}{2}(x+1)(x-4)$

So the equations represent the same parabola.

5 **a** From the diagram, the maximum turning point is $(0,6)$.

$\therefore y=ax^2+6$

The point $(1,4)$ lies on the graph,

$\therefore 4=a(1)^2+6$

$\therefore a=-2$

Equation is $y=-2x^2+6$.

b From the diagram the x intercepts are $x = -6$ and $x = -1$.

$\therefore y = a(x+6)(x+1)$

Point $(-9, 4.8)$ lies on the graph

$\therefore 4.8 = a(-9+6)(-9+1)$

$\therefore 4.8 = 24a$

$\therefore a = 0.2$

Equation is $y = 0.2(x+6)(x+1)$

6 a TP $(2,-4) \Rightarrow y = a(x-2)^2 - 4$

Point $(0,-2) \Rightarrow -2 = a(4) - 4$

$\therefore 4a = 2$

$\therefore a = \dfrac{1}{2}$

Equation is $y = \dfrac{1}{2}(x-2)^2 - 4$

b x intercepts: $0 = \dfrac{1}{2}(x-2)^2 - 4$

$\therefore (x-2)^2 = 8$

$\therefore x - 2 = \pm\sqrt{8}$

$\therefore x = 2 \pm 2\sqrt{2}$

The distance between $x = 2 - 2\sqrt{2}$ and $x = 2 + 2\sqrt{2}$ is $2 \times 2\sqrt{2} = 4\sqrt{2}$.

The length of the intercept cut off on the x axis is $4\sqrt{2}$ units.

7 $A(-1,10)$, $B(1,0)$, $C(2,4)$

a Let the equation be $y = ax^2 + bx + c$

$(-1,10) \Rightarrow 10 = a - b + c....(1)$

$(1,0) \Rightarrow \quad 0 = a + b + c....(2)$

$(2,4) \Rightarrow \quad 4 = 4a + 2b + c....(3)$

equation (2) – equation (1)

$-10 = 2b$

$\therefore b = -5$

Substitute $b = -5$ in equations (2) and (3)

equation $(2) \Rightarrow 5 = a + c....(4)$

equation $(3) \Rightarrow 4 = 4a - 10 + c$

$\therefore 14 = 4a + c....(5)$

equation (5) – equation (4)

$9 = 3a$

$\therefore a = 3$

Substitute $a = 3$ in equation (4)

$\therefore c = 2$

The equation is $y = 3x^2 - 5x + 2$

b y intercept $(0,2)$

x intercepts: $3x^2 - 5x + 2 = 0$

$\therefore (3x-2)(x-1) = 0$

$\therefore x = \dfrac{2}{3}, x = 1$

x intercepts $\left(\dfrac{2}{3}, 0\right), (1,0)$

c TP $x = \dfrac{2/3 + 1}{2}$

$\therefore x = \dfrac{5}{6}$

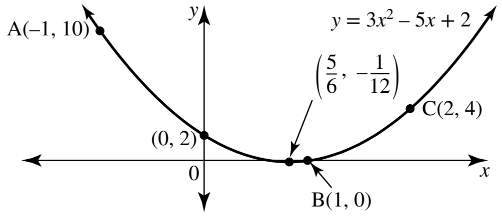
$y = \left(3 \times \dfrac{5}{6} - 2\right)\left(\dfrac{5}{6} - 1\right)$

$\therefore y = \dfrac{1}{2} \times \dfrac{-1}{6}$

$\therefore y = -\dfrac{1}{12}$

Vertex is $\left(\dfrac{5}{6}, -\dfrac{1}{12}\right)$

d



8 a Answers may vary but all are of the form $y = a(x+3)(x-5)$.

Three possible answers are $y = (x+3)(x-5)$, $y = 2(x+3)(x-5)$ and $y = -3(x+3)(x-5)$.

b The point $(0,45)$ also lies on the parabola.

$\therefore 45 = a(3)(-5)$

$\therefore -15a = 45$

$\therefore a = -3$

The equation is $y = -3(x+3)(x-5)$

Vertex $x = \dfrac{-3+5}{2}$

$\therefore x = 1$

$y = -3(4)(-4)$

$\therefore y = 48$

Vertex is $(1, 48)$

9 a The boundary parabola has vertex $(0,0)$

$\therefore y = ax^2$

Point $(5,-20) \Rightarrow -20 = a(5)^2$

$\therefore a = -\dfrac{4}{5}$

The equation of the boundary is $y = -\dfrac{4}{5}x^2$

The equation of the closed region below $y = -\dfrac{4}{5}x^2$ is $y \le -\dfrac{4}{5}x^2$.

b The boundary graph has x intercepts at $(0,0), (-4,0)$

$\therefore y = ax(x+4)$

Point $(-3,6) \Rightarrow 6 = a(-3)(1)$

$\therefore a = -2$

The equation of the boundary is $y = -2x(x+4)$.

The equation of the open region above $y = -2x(x+4)$ is $y > -2x(x+4)$.

10 a $y = 3x^2 \rightarrow y = 3(x+1)^2 - 5$ or $y = 3x^2 + 6x - 2$

b $y = (x-3)^2$ has vertex $(3,0)$. Translating 8 units to the left, $(3,0) \rightarrow (-5,0)$. The equation of the image is $y = (x+5)^2$.

11 a Axis of symmetry at $x = 4$ means the equation is of the form $y = a(x-4)^2 + k$.

Point $(0,6) \Rightarrow 6 = 16a + k....(1)$

Point $(6,0) \Rightarrow 0 = 4a + k....(2)$

equation (1) – equation (2)

$6 = 12a$

$\therefore a = \dfrac{1}{2}$

Substitute $a = \dfrac{1}{2}$ in equation (2)

$\therefore k = -2$

The equation is $y = \dfrac{1}{2}(x-4)^2 - 2$

$\therefore y = \dfrac{1}{2}(x^2 - 8x + 16) - 2$

$\therefore y = \dfrac{1}{2}x^2 - 4x + 6$

12 a $y=(ax+b)(x+c)$

Point $(5,0)\Rightarrow 0=(5a+b)(5+c)$

$\therefore 5a+b=0$ or $5+c=0$

$\therefore b=-5a$ or $c=-5$

Point $(0,-10)\Rightarrow -10=(b)(c)$ ....(1)

Consider the case where $c=-5$. Equation (1) becomes $-10=-5b$

$\therefore b=2$

Substitute in $y=(ax+b)(x+c)$

$\therefore y=(ax+2)(x-5)$

$\therefore y=ax^2-5ax+2x-10$

$\therefore y=ax^2+x(2-5a)-10$

Consider the case $b=-5a$. Equation (1) becomes $-10=-5ac$

$\therefore c=\frac{2}{a}$

Substitute in $y=(ax+b)(x+c)$

$\therefore y=(ax-5a)\left(x+\frac{2}{a}\right)$

$\therefore y=ax^2+2x-5ax-10$

$\therefore y=ax^2+(2-5a)x-10$

Either way, $y=ax^2+(2-5a)x-10$.

b $\Delta=(2-5a)^2-4\times a\times(-10)$

$\therefore \Delta=(2-5a)^2+40a$

$\therefore \Delta=4-20a+25a^2+40a$

$\therefore \Delta=4+20a+25a^2$

$\therefore \Delta=(2+5a)^2$

c Given $\Delta=4$

$\therefore (2+5a)^2=4$

$\therefore 2+5a=\pm 2$

$\therefore 5a=-4$ or $5a=0$

$\therefore a=-\frac{4}{5}, a=0$

As a is the coefficient of x^2, for a parabola $a\neq 0$. Therefore, reject $a=0$.

$\therefore a=-\frac{4}{5}$

With $a=-\frac{4}{5}$, the equation of the parabola becomes

$$y=-\frac{4}{5}x^2+\left(2-5\times\frac{-4}{5}\right)x-10$$

$$\therefore y=-\frac{4}{5}x^2+6x-10$$

Factorising,

$$y=-\frac{2}{5}(2x^2-15x+25)$$

$$\therefore y=-\frac{2}{5}(2x-5)(x-5)$$

The x intercepts are at $x=\frac{5}{2}, x=5$. Since $x=5$ was given, the other x intercept is $\left(\frac{5}{2},0\right)$.

13 a The point $(-4,0)$ is the turning point.

$\therefore y=a(x+4)^2$

Point $(2,9)\Rightarrow 9=a(6)^2$

$\therefore a=\frac{9}{36}$

$\therefore a=\frac{1}{4}$

The equation is $y=\frac{1}{4}(x+4)^2$.

b The point $(p,0)$ is the turning point.

$\therefore y=a(x-p)^2$

Point $(2,9)\Rightarrow 9=a(2-p)^2$....(1)

Point $(0,36)\Rightarrow 36=a(-p)^2$

$\therefore 36=ap^2$....(2)

equation (2) ÷ equation (1)

$$\frac{36}{9}=\frac{ap^2}{a(2-p)^2}$$

$$\therefore 4=\frac{p^2}{(2-p)^2}$$

$\therefore 4\left(4-4p+p^2\right)=p^2$

$\therefore 3p^2-16p+16=0$

$\therefore (3p-4)(p-4)=0$

$\therefore p=\frac{4}{3}, p=4$

Therefore there are two possible values for p.

If $p=\frac{4}{3}$, equation (2) $\Rightarrow 36=a\times\frac{16}{9}$

$36=a\times\frac{16}{9}$

$\therefore a=\frac{36\times 9}{16}$

$\therefore a=\frac{81}{4}$

The equation of the parabola $y=a(x-p)^2$ becomes

$$y=\frac{81}{4}\left(x-\frac{4}{3}\right)^2$$

$$\therefore y=\frac{81}{4}\left(\frac{3x-4}{3}\right)^2$$

$$\therefore y=\frac{81}{4}\times\frac{(3x-4)^2}{9}$$

$$\therefore y=\frac{9}{4}(3x-4)^2$$

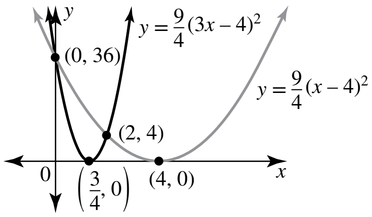
If $p=4$, equation (2) $\Rightarrow 36=a\times 16$

$\therefore a=\frac{9}{4}$

The equation of the parabola would be $y=\frac{9}{4}(x-4)^2$

The graphs of both these parabolas contain the points $(2,9)$ and $(0,10)$.

The turning point of $y=\frac{9}{4}(3x-4)^2$ is $\left(\frac{4}{3},0\right)$ and the turning point of $y=\frac{9}{4}(x-4)^2$ is $(4,0)$.



14 a Since the points A $(-1,15)$ and B $(5,15)$ have the same y co-ordinate, they are placed symmetrically to the axis of symmetry of the parabola.

The axis of symmetry has the equation $x=\frac{-1+5}{2}$ giving the x co-ordinate of the vertex V as $x=2$.

Let V be the point $(2,k)$.

$$d(V,B)=\sqrt{(2-5)^2+(k-15)^2}$$
$$=\sqrt{9+(k-15)^2}$$

As the distance VB is $\sqrt{90}$ units,

$\sqrt{9+(k-15)^2}=\sqrt{90}$

$\therefore 9+(k-15)^2=90$

$\therefore (k-15)^2=81$

$\therefore k-15=\pm 9$

$\therefore k=6, k=24$

If $k=24$ then the vertex V $(2,24)$ would lie above the points A and B and the parabola would be concave down. As the parabola is concave up, reject $k=24$.

$\therefore k=6$

The vertex V has co-ordinates $(2,6)$.

b The equation of the parabola is $y=a(x-2)^2+6$

Substitute the point B $(5,15)$

$\therefore 15=a(3)^2+6$

$\therefore a=1$

The equation is $y=(x-2)^2+6$

c V $(2,6)$, B $(5,15)$, O $(0,0)$

Gradient of VB, $m=\dfrac{15-6}{5-2}$

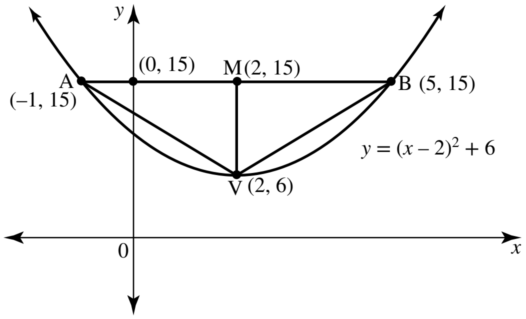
$\therefore m_{VB}=3$

Gradient of OV, $m=\dfrac{6}{2}$

$\therefore m_{OV}=3$

Since $m_{VB}=m_{OV}$ and point V is common, the three points O,V and B are collinear.

d



AB is the base of triangle VAB and VM is its height.

AB = 6 units and VM = 9 units.

The area of triangle VAB:

$\dfrac{1}{2}bh$

$=\dfrac{1}{2}\times 6\times 9$

$=27$

The area of VAB is 27 square units.

15 One method is to use the Statistics facility. Enter the x co-ordinates in List 1 and the y co-ordinates in List 2 in the Statistics screen. Tap Calc → Quadratic Reg and tap OK to accept the entry on the Stat Calculation window that appears. This then gives

$a=4.2$

$b=6$

$c=-10$

The equation of the parabola is $y=4.2x^2+6x-10$.

16 **a** $y=ax^2+b$

Substitute the two points to form a pair of simultaneous equations

$(20.5, 47.595)\Rightarrow 47.595=a(20.5)^2+b$

$(42, 20.72)\Rightarrow 20.72=a(42)^2+b$

Solving using the simultaneous equation template gives $a=-0.02, b=56$.

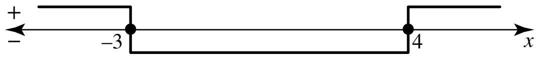
b The equation of the parabola is $y=-0.02x^2+56$.

Solving the equation $0=-0.02x^2+56$ in Equation/Inequality gives $x=-52.915, x=52.915$ correct to three decimal places.

The x intercepts are $(\pm 52.915, 0)$.

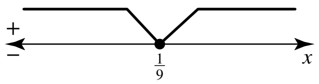
Exercise 3.7 — Quadratic inequations

1 zeros $x=-3, x=4$, concave up



2 $81x^2-18x+1=(9x-1)^2$

Zero $x=\dfrac{1}{9}$ multiplicity 2



3 **a** $(4-x)(2x-3)\le 0$

Solution is $x\le\dfrac{3}{2}$ or $x\ge 4$

Check: Let $x=-2$

$(4-(-2))(2(-2)-3)$

$=6\times -7$

$=-42$

<0

Let $x=5$

$(4-5)(10-3)$

$=-1\times 7$

$=-7$

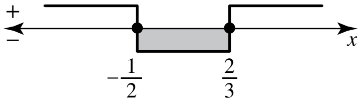
<0

b $6x^2<x+2$

$\therefore 6x^2-x-2<0$

$\therefore (3x-2)(2x+1)<0$

Zeros are $x=\dfrac{2}{3}, x=-\dfrac{1}{2}$



The solution set is $\left\{x: -\dfrac{1}{2}<x<\dfrac{2}{3}\right\}$

4 $6x<x^2+9$

$\therefore 0<x^2-6x+9$

$\therefore (x-3)^2>0$

This will always be true except if $x=3$

Therefore, $x\in R\setminus\{3\}$

5 **a** $y=x^2+3x-10 .....(1)$

$y+x=2 .....(2)$

From equation (2), $y=2-x$

Substitute in (1)

$2-x=x^2+3x-10$

$\therefore x^2+4x-12=0$

$\therefore (x+6)(x-2)=0$

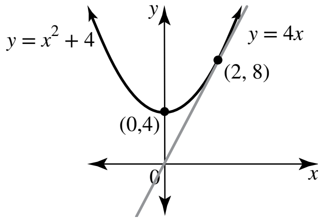
$\therefore x=-6, x=2$

If $x=-6, y=8$, if $x=2, y=0$

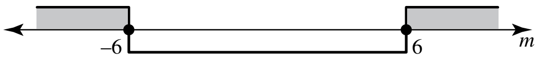
Points of intersection are $(-6,8)$ and $(2,0)$

b $y = 6x + 1$ and $y = -x^2 + 9x - 5$
For intersection,
$$6x + 1 = -x^2 + 9x - 5$$
$$\therefore x^2 - 3x + 6 = 0$$
$$\begin{aligned}\Delta &= (-3)^2 - 4(1)(6)\\ &= 9 - 24\\ &= -15\end{aligned}$$
Since $\Delta < 0$ there are no intersections

6 $y = 4x$ and $y = x^2 + 4$
At intersection,
$$4x = x^2 + 4$$
$$\therefore 0 = x^2 - 4x + 4$$
$$\therefore (x-2)^2 = 0$$
$$\therefore x = 2$$
Since there is only one value the line is a tangent to the parabola
When $x = 2$, $y = 8$, so the point of contact is $(2,8)$

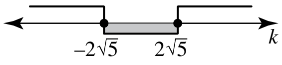


7 $y = mx - 7$ and $y = 3x^2 + 6x + 5$
For intersection,
$$3x^2 + 6x + 5 = mx - 7$$
$$\therefore 3x^2 + x(6-m) + 12 = 0$$
$$\begin{aligned}\Delta &= (6-m)^2 - 4(3)(12)\\ &= (6-m)^2 - 144\\ &= (6-m-12)(6-m+12)\\ &= (-6-m)(6-m)\\ &= -(6+m)(6-m)\end{aligned}$$
For at least one intersection, $\Delta \geq 0$
Sign diagram of the discriminant



Therefore there will be at least one intersection if $m \leq -6$ or $m \geq 6$

8 $y = kx + 9$ and $y = x^2 + 14$
For intersection,
$$x^2 + 14 = kx + 9$$
$$\therefore x^2 - kx + 5 = 0$$
$$\begin{aligned}\Delta &= (-k)^2 - 4(1)(5)\\ &= k^2 - 20\end{aligned}$$
For no intersection, $\Delta < 0$
$$\therefore k^2 - 20 < 0$$
$$\therefore \left(k + \sqrt{20}\right)\left(k - \sqrt{20}\right) < 0$$
Zeros are $k = \pm 2\sqrt{5}$
Sign diagram of the discriminant

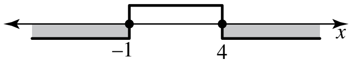


Therefore no intersections if $-2\sqrt{5} < k < 2\sqrt{5}$

9 **a** $x^2 + 8x - 48 \leq 0$
$\therefore (x+12)(x-4) \leq 0$

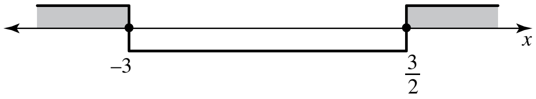
$\therefore -12 \leq x \leq 4$

b $-x^2 + 3x + 4 \leq 0$
$\therefore (-x+4)(x+1) \leq 0$



$\therefore x \leq -1$ or $x \geq 4$

c $3(3-x) < 2x^2$
$$\therefore 9 - 3x < 2x^2$$
$$\therefore 2x^2 + 3x - 9 > 0$$
$$\therefore (2x-3)(x+3) > 0$$

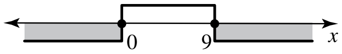


$\therefore x < -3$ or $x > \dfrac{3}{2}$

d $(x+5)^2 > 9$
$$\therefore (x+5)^2 - 9 < 0$$
$$\therefore (x+5-3)(x+5+3) < 0$$
$$\therefore (x+2)(x+8) < 0$$

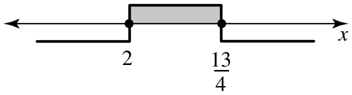
$\therefore -8 < x < -2$

e $9x < x^2$
$$\therefore 9x - x^2 < 0$$
$$\therefore x(9-x) < 0$$



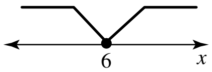
$\therefore x < 0$ or $x > 9$

f $5(x-2) \geq 4(x-2)^2$
$$\therefore 5(x-2) - 4(x-2)^2 \geq 0$$
$$\therefore (x-2)[5 - 4(x-2)] \geq 0$$
$$\therefore (x-2)(13-4x) \geq 0$$



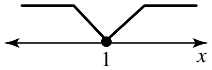
$\therefore 2 \leq x \leq \dfrac{13}{4}$

10 **a** $36 - 12x + x^2 > 0$
$$36 - 12x + x^2 > 0$$
$$\therefore (6-x)^2 > 0$$



$\therefore x \in R \setminus \{6\}$
The solution set is $R \setminus \{6\}$

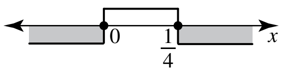
b $6x^2 - 12x + 6 \leq 0$
$$\therefore 6(x^2 - 2x + 1) \leq 0$$
$$\therefore 6(x-1)^2 \leq 0$$



Solution set is $\{1\}$

c $-8x^2+2x<0$

$\therefore -2x(4x-1)<0$



$\therefore x<0$ or $x>\frac{1}{4}$

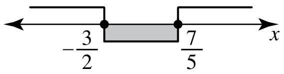
Solution set is $\{x : x<0\}\cup\left\{x : x>\frac{1}{4}\right\}$

d $x(1+10x)\le 21$

$\therefore x+10x^2\le 21$

$\therefore 10x^2+x-21\le 0$

$\therefore (5x-7)(2x+3)\le 0$



$\therefore -\frac{3}{2}\le x\le \frac{7}{5}$

Solution set is $\left\{x : -\frac{3}{2}\le x\le \frac{7}{5}\right\}$

11 a $y=5x+2....(1)$

$y=x^2-4....(2)$

Substitute (1) in (2)

$\therefore 5x+2=x^2-4$

$\therefore x^2-5x-6=0$

$\therefore (x-6)(x+1)=0$

$\therefore x=6, x=-1$

In (1), when $x=6$, $y=32$ and when $x=-1$, $y=-3$.

Answer $x=6$, $y=32$ or $x=-1$, $y=-3$.

b $4x+y=3....(1)$

$y=x^2+3x-5....(2)$

From (1), $y=3-4x$. Substitute in (2)

$\therefore 3-4x=x^2+3x-5$

$\therefore x^2+7x-8=0$

$\therefore (x-1)(x+8)=0$

$\therefore x=1, x=-8$

In (1), when $x=1$, $y=-1$ and when $x=-8$, $y=35$.

Answer $x=1$, $y=-1$ or $x=-8$, $y=35$.

c $2y+x-4=0....(1)$

$y=(x-3)^2+4....(2)$

From (1), $x=4-2y$. Substitute in (2)

$\therefore y=(4-2y-3)^2+4$

$\therefore y=(1-2y)^2+4$

$\therefore y=1-4y+4y^2+4$

$\therefore 4y^2-5y+5=0$

Test discriminant

$\Delta=(-5)^2-4\times 4\times 5$

$=-55$

Since $\Delta<0$, there are no solutions.

d $\frac{x}{3}+\frac{y}{5}=1....(1)$

$x^2-y+5=0....(2)$

From (2), $y=x^2+5$. Substitute in (1)

$\therefore \frac{x}{3}+\frac{x^2+5}{5}=1$

$\therefore \frac{5x+3(x^2+5)}{15}=1$

$\therefore 3x^2+5x+15=15$

$\therefore 3x^2+5x=0$

$\therefore x(3x+5)=0$

$\therefore x=0, x=-\frac{5}{3}$

In (2) when $x=0$, $y=5$ and when $x=-\frac{5}{3}$, $y=\frac{25}{9}+5$

Answer $x=0$, $y=5$ or $x=-\frac{5}{3}$, $y=\frac{70}{9}$

12 a $y=2x+5....(1)$

$y=-5x^2+10x+2....(2)$

At intersection, $2x+5=-5x^2+10x+2$

$\therefore 5x^2-8x+3=0$

$\therefore (5x-3)(x-1)=0$

$\therefore x=\frac{3}{5}, x=1$

In (1) when $x=\frac{3}{5}$, $y=\frac{6}{5}+5$

$\therefore x=\frac{3}{5}, y=\frac{31}{5}$

In (1) when $x=1$, $y=7$

The points of intersection are $\left(\frac{3}{5},\frac{31}{5}\right),(1,7)$.

b $y=-5x-13....(1)$

$y=2x^2+3x-5....(2)$

At intersection $-5x-13=2x^2+3x-5$

$\therefore 2x^2+8x+8=0$

$\therefore x^2+4x+4=0$

$\therefore (x+2)^2=0$

$\therefore x=-2$

In (1) when $x=-2$, $y=-3$

Point of intersection is $(-2,-3)$.

c $y=10....(1)$

$y=(5-x)(6+x)....(2)$

At intersection, $10=(5-x)(6+x)$

$\therefore 10=30-x-x^2$

$\therefore x^2+x-20=0$

$\therefore (x-4)(x+5)=0$

$\therefore x=4, x=-5$

Points of intersection are $(4,10),(-5,10)$

d $19x-y=46....(1)$

$y=3x^2-5x+2....(2)$

From (1), $y=19x-46$. Substitute in (2)

$\therefore 19x-46=3x^2-5x+2$

$\therefore 3x^2-24x+48=0$

$\therefore x^2-8x+16=0$

$\therefore (x-4)^2=0$

$\therefore x=4$

Substitute $x=4$ in (1)

$\therefore y=19\times 4-46$

$\therefore y=30$

Point of intersection is $(4,30)$.

13 a $y = 4 - 2x....(1)$

$y = 3x^2 + 8....(2)$

At intersection, $4 - 2x = 3x^2 + 8$

$\therefore 3x^2 + 2x + 4 = 0$

$\Delta = 2^2 - 4 \times 3 \times 4$

$\therefore \Delta = -44$

Since $\Delta < 0$, there are no intersections.

b $y = 2x + 1....(1)$

$y = -x^2 - x + 2....(2)$

At intersection, $2x + 1 = -x^2 - x + 2$

$\therefore x^2 + 3x - 1 = 0$

$\Delta = 3^2 - 4 \times 1 \times (-1)$

$\therefore \Delta = 13$

Since $\Delta > 0$, there are two intersections.

c $y = 0....(1)$

$y = -2x^2 + 3x - 2....(2)$

At intersection $0 = -2x^2 + 3x - 2$

$\Delta = 3^2 - 4 \times (-2) \times (-2)$

$\therefore \Delta = -7$

Since $\Delta < 0$, there are no intersections.

14 $y = (k - 2)x + k....(1)$

$y = x^2 - 5x....(2)$

At intersection, $(k - 2)x + k = x^2 - 5x$

$\therefore x^2 - 5x - (k - 2)x - k = 0$

$\therefore x^2 - x(5 + k - 2) - k = 0$

$\therefore x^2 - (3 + k)x - k = 0$

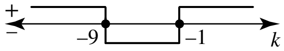
$\Delta = [-(3 + k)]^2 - 4 \times (1) \times (-k)$

$\therefore \Delta = 9 + 6k + k^2 + 4k$

$\therefore \Delta = k^2 + 10k + 9$

$\therefore \Delta = (k + 1)(k + 9)$

Sign diagram of Δ



a For no intersections, $\Delta < 0$

$\therefore -9 < k < -1$

b For one point of intersection, $\Delta = 0$

$\therefore k = -1, k = -9$

c For two points of intersection, $\Delta > 0$

$\therefore k < -9$ or $k > -1$

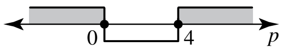
15 a The equation $px^2 - 2px + 4 = 0$ will have real roots if $\Delta \geq 0$ and $p \neq 0$.

$\Delta = (-2p)^2 - 4 \times p \times 4$

$= 4p^2 - 16p$

$= 4p(p - 4)$

Sign diagram of Δ



$\Delta \geq 0 \Rightarrow 0 \leq p \leq 4.$

The equation will have real roots if $0 < p \leq 4$ since $p \neq 0$.

b $y = tx + 1....(1)$

$y = 2x^2 + 5x + 11....(2)$

At intersection $tx + 1 = 2x^2 + 5x + 11$

$\therefore 2x^2 + 5x - tx + 10 = 0$

$\therefore 2x^2 + (5 - t)x + 10 = 0$

For no intersection, there are no solutions to $2x^2 + (5 - t)x + 10 = 0$ and its discriminant must be negative.

$\Delta = (5 - t)^2 - 4 \times 2 \times 10$

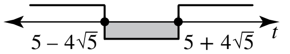
$= (5 - t)^2 - 80$

$= \left(5 - t - \sqrt{80}\right)\left(5 - t + \sqrt{80}\right)$

$\therefore \Delta = \left(5 - 4\sqrt{5} - t\right)\left(5 + 4\sqrt{5} - t\right)$

Zeros are $t = 5 - 4\sqrt{5}$ and $t = 5 + 4\sqrt{5}$

Sign diagram of Δ



$\Delta < 0$ when $5 - 4\sqrt{5} < t < 5 + 4\sqrt{5}$

The line does not intersect the parabola for $5 - 4\sqrt{5} < t < 5 + 4\sqrt{5}$.

c $y = x....(1)$

$y = 9x^2 + nx + 1....(2)$

At intersection $x = 9x^2 + nx + 1$

$\therefore 9x^2 + nx - x + 1 = 0$

$\therefore 9x^2 + (n - 1)x + 1 = 0$

If the line is a tangent to the parabola, there is one solution so $\Delta = 0$.

$\Delta = (n - 1)^2 - 4 \times 9 \times 1$

$= (n - 1)^2 - 36$

$= (n - 1 - 6)(n - 1 + 6)$

$\therefore \Delta = (n - 7)(n + 5)$

$\Delta = 0$ when $n = 7, n = -5$.

The line is a tangent to the parabola if $n = 7, n = -5$.

d $x^4 - x^2 < 12$

Let $a = x^2$

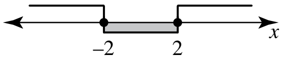
$\therefore a^2 - a - 12 < 0$

$\therefore (a - 4)(a + 3) < 0$

$\therefore (x^2 - 4)(x^2 + 3) < 0$

$\therefore (x - 2)(x + 2)(x^2 + 3) < 0$

Since $x^2 + 3 > 0$ for all values of x, the inequation becomes $(x - 2)(x + 2) < 0$



$\therefore -2 < x < 2$

16 a $2y - 3x = 6....(1)$

$y = x^2....(2)$

Substitute (2) in (1)

$\therefore 2x^2 - 3x = 6$

$\therefore 2x^2 - 3x - 6 = 0$

$$\therefore x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 2 \times (-6)}}{4}$$

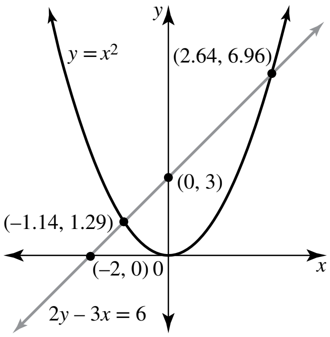
$$\therefore x = \frac{3 \pm \sqrt{57}}{4}$$

Therefore, $x = -1.137...$ or $x = 2.637...$

Substitute each into equation (2) to obtain $y = 1.29$ or $y = 6.96$ respectively.

To two decimal places, the points of intersection are $(-1.14, 1.29), (2.64, 6.96)$.

b The line $2y - 3x = 6$ intersects the axes at $(-2, 0), (0, 3)$. The parabola $y = x^2$ has minimum turning point at $(0, 0)$. Both graphs contain the points $(-1.14, 1.29), (2.64, 6.96)$.



c The region enclosed between the two graphs lies below the line and above the parabola. The closed region can be described by the inequalities $2y-3x \le 6$ and $y \ge x^2$. That is, the region is $\{(x,y): 2y-3x \le 6\} \cap \left\{(x,y): y \ge x^2\right\}$.

d $2y-3x=6 \Rightarrow y=\frac{3}{2}x+3$/ Its gradient is $\frac{3}{2}$.

Let the parallel line which is to be a tangent to the parabola, have the equation $y=\frac{3}{2}x+c$.

At intersection with the parabola, $x^2=\frac{3}{2}x+c$.

$\therefore 2x^2-3x-2c=0$

For line to be a tangent, $\Delta=0$

$\therefore 9-4\times 2\times(-2c)=0$

$\therefore 9+16c=0$

$\therefore c=-\frac{9}{16}$

The tangent line has the equation $y=\frac{3}{2}x-\frac{9}{16}$. Its y intercept is $\left(0,-\frac{9}{16}\right)$.

17 a The equation $x^2-5x+4=0$ can be expressed as $x^2=5x-4$. The intersection of $y=x^2$ with the line $y=5x-4$ would form this equation.

b Rearranging the equation $3x^2+9x-2=0$ gives

$9x=2-3x^2$

$\therefore 9x+3=2-3x^2+3$

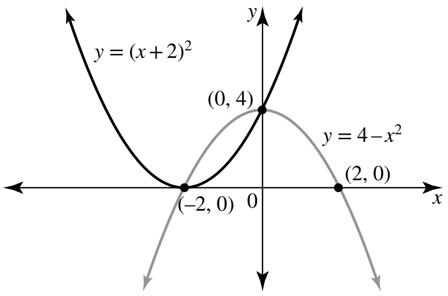
$\therefore 3(3x+1)=-3x^2+5$

$\therefore 3x+1=-x^2+\frac{5}{3}$

The equation gives the x co-ordinates of the points of intersection of the line $y=3x+1$ with the parabola $y=-x^2+\frac{5}{3}$.

18 a $y=(x+2)^2$ has minimum turning point and x intercept at $(-2,0)$; y intercept at $(0,4)$.

$y=4-x^2$ has maximum turning point and y intercept at $(0,4)$; its x intercepts are at $(-2,0)$ and $(2,0)$.



Points of intersection are $(-2,0)$ and $(0,4)$.

b i $y=(x+2)^2$ and $y=k-x^2$ intersect when $(x+2)^2=k-x^2$.

$\therefore x^2+4x+4=k-x^2$

$\therefore 2x^2+4x+4-k=0$

$\Delta=4^2-4\times 2\times(4-k)$

$\therefore \Delta=-16+8k$

For one point of intersection, $\Delta=0$

$\therefore -16+8k=0$

$\therefore k=2$

ii $y=2-x^2$ has maximum turning point and y intercept at $(0,2)$; its x intercepts are at $\left(-\sqrt{2},0\right)$ and $\left(\sqrt{2},0\right)$.

Point of intersection: Substitute $k=2$ in

$2x^2+4x+4-k=0$

$\therefore 2x^2+4x+2=0$

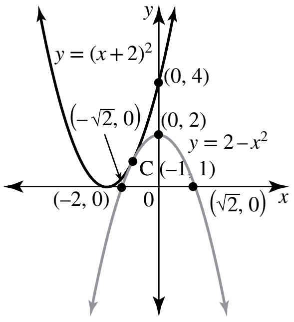
$\therefore x^2+2x+1=0$

$\therefore (x+1)^2=0$

$\therefore x=-1$

When $x=-1, y=1$. Common point C has co-ordinates $(-1,1)$.

iii



C $(-1,1)$ lies on $y=ax+b$.

$\therefore 1=-a+b$

$\therefore b=a+1$

The equation of the tangent line becomes $y=ax+a+1$.

There is one solution to the simultaneous equations $y=ax+a+1$ and $y=2-x^2$.

$\therefore ax+a+1=2-x^2$

$\therefore x^2+ax+a-1=0$

$$\begin{aligned}\Delta&=a^2-4\times 1\times(a-1)\\&=a^2-4a+4\\&=(a-2)^2\end{aligned}$$

$\Delta=0 \Rightarrow a=2$ and $a=2 \Rightarrow b=3$

The tangent to $y=2-x^2$ is the line $y=2x+3$.

To show this line is also a tangent to $y=(x+2)^2$:

$2x+3=(x+2)^2$

$\therefore 2x+3=x^2+4x+4$

$\therefore x^2+2x+1=0$

$\Delta=4-4\times 1\times 1$

$\therefore \Delta=0$

The line $y=2x+3$ is a tangent to $y=(x+2)^2$.

The equation of the common tangent is $y=2x+3$.

19 a On the main screen in Standard mode, tap Interactive → Equation/Inequality → solve and enter $19-3x-5x^2<0$.

The < symbol is found in Keyboard → Mth → OPTN.

The answer given is $x<\frac{-\left(\sqrt{389}+3\right)}{10}, x>\frac{\sqrt{389}-3}{10}$.

b Use the procedure in part **a**. The solution to $6x^2+15x \le 10$ is $\frac{-\left(\sqrt{465}+15\right)}{12} \le x \le \frac{\sqrt{465}-15}{12}$.

20 a In the Graph & Tab screen enter $y1: 2x^2-10x$. The family of lines could be graphed using the Dynamic Graph facility.

enter $y2: bx + a$ and tap the diamond shape on the ribbon → Dynamic Graph.
Enter for a
Start: –6
End: 0
Step: 2
Although $b = -4$ enter for b
Start: –4
End: –3
Step: 1
Then graph by ticking the boxes and tapping the graphing symbol.
To alter the values of a the cursor is moved sideways. (The values of b are contolled by the vertical cursor but this is not relevant for the line $y = -4x + a$).
For each line, tap Analysis → G-Solve → Intersect to obtain the points of intersection.
$a = -6$: Intersections are Not Found
$a = -4$: $(1, -8), (2, -12)$
$a = -2$: $(0.38, -3.53), (2.62, -12.47)$ to two decimal places
$a = 0$: $(0, 0), (3, -12)$

b Solve $2x^2 - 10x = -4x + a$ for x using Equation/Inequality having cleared All Variables first.
Solution is $x = \frac{-\left(\sqrt{2a+9} - 3\right)}{2}, x = \frac{\sqrt{2a+9} + 3}{2}$

c For the line $y = -4x + a$ to be a tangent, there should be only one solution.

$$\therefore \sqrt{2a+9} = 0$$
$$\therefore 2a + 9 = 0$$
$$\therefore a = -\frac{9}{2}$$

For the point of contact, $x = \frac{0+3}{2}$,
When $x = \frac{3}{2}$ and $a = -\frac{9}{2}$,

$$y = -4 \times \frac{3}{2} - \frac{9}{2}$$
$$\therefore y = -\frac{21}{2}$$

For the line to be a tangent, $a = -4.5$ and the point of contact is $(1.5, -10.5)$.

Exercise 3.8 — Quadratic models and applications

1 $h = 100 + 38t - \frac{19}{12}t^2$

a At turning point $t = -\frac{b}{2a}$

$$\therefore t = -\frac{38}{2\left(-\frac{19}{12}\right)}$$
$$\therefore t = 12$$
$$\therefore h = 100 + 38(12) - \frac{19}{12} \times 12^2$$
$$\therefore h = 100 + 19 \times 12$$
$$\therefore h = 328$$

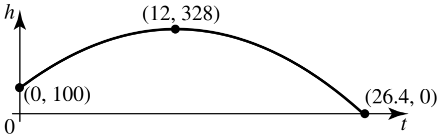
Therefore the greatest height the missile reaches is 328 metres

b It reaches its greatest height after 12 seconds

c time to return to the ground: $0 = 100 + 38t - \frac{19}{12}t^2$

$$\therefore 19t^2 - 456t - 1200 = 0$$
$$\therefore t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore t = \frac{456 \pm \sqrt{456^2 - 4 \times 19 \times (-1200)}}{38}$$
$$\therefore t = -2.39, 26.39$$
$$\therefore t = 26.4$$

(reject negative value)



2 30 metres of edging using the back fence as one edge.

a width x metres, length $30 - 2x$ metres
$\therefore A = x(30 - 2x)$
$\therefore A = 30x - 2x^2$

b $A = 0 \Rightarrow x(30 - 2x) = 0$
$\therefore x = 0, x = 15$
Therefore, turning point is
$x = 7.5, A = 7.5(30 - 15)$
$\Rightarrow (7.5, 112.5)$
Dimensions of garden for maximum area are width 7.5 m, length 15 m

c Greatest area is 112.5 square metres

3 $10h = 16t + 4 - 9t^2$

a The ball reaches the ground when $h = 0$.

$$\therefore 0 = 16t + 4 - 9t^2$$
$$\therefore 9t^2 - 16t - 4 = 0$$
$$\therefore (9t + 2)(t - 2) = 0$$
$$\therefore t = -\frac{2}{9}, t = 2$$

Reject $t = -\frac{2}{9}$ since $t > 0$
$\therefore t = 2$
It takes the ball 2 seconds to reach the ground.

b Let $h = 1.6$

$$\therefore 10 \times 1.6 = 16t + 4 - 9t^2$$
$$\therefore 9t^2 - 16t + 12 = 0$$
$$\Delta = (-16)^2 - 4 \times 9 \times 12$$
$$\therefore \Delta = -176$$

Since $\Delta < 0$ there is no value of t for which $h = 1.6$.
The ball does not strike the overhanging foliage.

c $10h = 16t + 4 - 9t^2$

$$10h = 16t + 4 - 9t^2$$
$$\therefore h = 1.6t + 0.4 - 0.9t^2$$
$$\therefore h = -0.9t^2 + 1.6t + 0.4$$

At turning point, $t = -\frac{b}{2a}$

$$\therefore t = -\frac{1.6}{2 \times (-0.9)}$$
$$\therefore t = \frac{1.6}{1.8}$$
$$\therefore t = \frac{8}{9}$$

When $t = \frac{8}{9}$,

$$10h = -9\times\frac{64}{81}+16\times\frac{8}{9}+4$$
$$\therefore 10h = \frac{100}{9}$$
$$\therefore h = \frac{10}{9}$$

The greatest height the ball reaches is $\frac{10}{9}$ metres.

4 a $y = 1.2 + 2.2x - 0.2x^2$

$$\therefore y = -0.2(x^2 - 11x - 6)$$
$$\therefore y = -0.2\left[\left(x^2 - 11x + \left(\frac{11}{2}\right)^2\right) - \left(\frac{11}{2}\right)^2 - 6\right]$$
$$\therefore y = -0.2\left[\left(x - \frac{11}{2}\right)^2 - \frac{121}{4} - \frac{24}{4}\right]$$
$$\therefore y = -0.2\left[\left(x - \frac{11}{2}\right)^2 - \frac{145}{4}\right]$$
$$\therefore y = -0.2(x-5.5)^2 + 0.2\times\frac{145}{4}$$
$$\therefore y = -0.2(x-5.5)^2 + 7.25$$

b Since the maximum turning point is $(5.5, 7.25)$, the greatest height the volleyball reaches is 7.25 metres.

c The court is 18 metres in length so the net is 9 metres horizontally from the back of the court.
When $x = 9$, the height of the volleyball is

$$y = 1.2 + 2.2\times 9 - 0.2\times 81$$
$$\therefore y = 4.8$$

The volleyball is 4.8 metres high and the net is 2.43 metres high. Therefore, the ball clears the net by 2.37 metres.

5 a The total length of hosing for the edges is 120 metres.

$$\therefore 2l + 4w = 120$$
$$\therefore l + 2w = 60$$
$$\therefore l = 60 - 2w$$

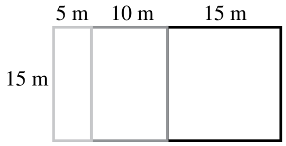
The total area of the garden is $A = l \times w$

$$\therefore A = (60 - 2w)\times w$$
$$\therefore A = 60w - 2w^2$$

b Completing the square,

$$A = -2(w^2 - 30w)$$
$$\therefore A = -2\left[\left(w^2 - 30w + 15^2\right) - 15^2\right]$$
$$\therefore A = -2\left[(w-15)^2 - 225\right]$$
$$\therefore A = -2(w-15)^2 + 450$$

Maximum area of 450 sq m when $w = 15$, and if $w = 15$ then $l = 30$.
The total area is divided into three sections in the ratio $1:2:3$.
Dividing the length $l = 30$ into 6 parts gives each part 5.
The lengths of each section are 5,10,15 metres respectively.



The smallest section has width 15 metres and length 5 metres with area 75 sq m. The amount of hosing required for its four sides is its perimeter of 40 m.
The middle section has width 15 metres and length 10 metres with area 150 sq m. The amount of hosing required is for three sides since it shares one side with the smallest section. Therefore the amount of hosing is $2\times 10 + 15 = 35$ metres.
The largest section has width 15 metres and length 15 metres with area 225 sq m. The amount of hosing required is for three sides since it shares one side with the middle section. Therefore the amount of hosing is $2\times 15 + 15 = 45$ metres.

6 a $N = 100 + 46t + 2t^2$
Initially, $t = 0 \Rightarrow N = 100$
When $N = 200$,

$$200 = 100 + 46t + 2t^2$$
$$\therefore 2t^2 + 46t - 100 = 0$$
$$\therefore t^2 + 23t - 50 = 0$$
$$\therefore (t+25)(t-2) = 0$$
$$\therefore t = -25(\text{reject}) \text{ or } t = 2$$
$$\therefore t = 2$$

It takes 2 seconds for the initial number of bacteria to double.

b At $1pm$, $t = 5$

$$\therefore N = 100 + 46\times 5 + 2\times 25$$
$$\therefore N = 380$$

At $1pm$ there are 380 bacteria present.

c $N = 380 - 180t + 30t^2$ where t is the time since $1pm$.
The minimum number of bacteria occurs at the minimum turning point.
At the turning point,

$$t = -\frac{-180}{2\times 30}$$
$$\therefore t = 3$$
$$N = 380 - 180\times 3 + 30\times 9$$
$$\therefore N = 110$$

The minimum number of bacteria is 110 reached at $4pm$.

7 $z = 5x^2 + 4xy + 6y^2 ....(1)$
$x + y = 2....(2)$
From equation (2), $y = 2 - x$. Substitute in equation (1).

$$\therefore z = 5x^2 + 4x(2-x) + 6(2-x)^2$$
$$\therefore z = 5x^2 + 8x - 4x^2 + 6(4 - 4x + x^2)$$
$$\therefore z = 7x^2 - 16x + 24$$

At the turning point,

$$x = -\frac{-16}{2\times 7}$$
$$\therefore x = \frac{8}{7}$$
$$z = 7\times\frac{64}{49} - 16\times\frac{8}{7} + 24$$
$$= \frac{64}{7} - \frac{128}{7} + \frac{168}{7}$$
$$\therefore z = \frac{104}{7}$$

Minimum turning point is $\left(\frac{8}{7}, \frac{104}{7}\right)$.

The minimum value of z is $\frac{104}{7}$. This occurs when $x = \frac{8}{7}$ and

$$y = 2 - \frac{8}{7}$$
$$= \frac{6}{7}$$

8 a Each piece of wire forms the perimeter of each square.
The perimeter of a square of side length 4 cm is 16 cm.
As the total length of the original piece of wire is 20 cm,

the perimeter of the second square is $20-16=4$ cm. This makes the side length of the second square 1 cm.
The sum of the areas of the two squares gives the value of S.
$\therefore S=4\times 4+1\times 1$
$\therefore S=17$

b The square of side length x cm has area x^2 sq cm and perimeter $4x$ cm.

The second square has perimeter $(20-4x)$ cm. Each of its sides is $\frac{20-4x}{4}=5-x$ cm. hence the area of the second square is $(5-x)^2$ sq cm.

$\therefore S=x^2+(5-x)^2$
$\quad =x^2+25-10x+x^2$
$\therefore S=2x^2-10x+25$

The minimum turning point occurs when
$x=-\frac{-10}{2\times 2}\Rightarrow x=\frac{5}{2}.$

If $x=\frac{5}{2}$, the perimeter of the first square is $4\times\frac{5}{2}=10$ cm.
For the sum of the areas of the two squares to be a minimum, the piece of wire needs to be cut into two equal pieces of 10 cm.

9 a $C=c+k_1n+k_2n^2$ where k_1 and k_2 are the constants of proportionality.
$n=5, C=195\Rightarrow 195=c+5k_1+25k_2....(1)$
$n=8, C=420\Rightarrow 420=c+8k_1+64k_2....(2)$
$n=10, C=620\Rightarrow 620=c+10k_1+100k_2....(3)$

Eliminate c
equation (2) – equation (1)
$225=3k_1+39k_2$
$\therefore 75=k_1+13k_2....(4)$

equation (3) – equation (2)
$200=2k_1+36k_2$
$\therefore 100=k_1+18k_2....(5)$

equation (5) – equation (4)
$25=5k_2$
$\therefore k_2=5$

Substitute $k_2=5$ in equation (4)
$\therefore 75=k_1+65$
$\therefore k_1=10$

Substitute $k_1=10$ and $k_2=5$ in equation (1)
$\therefore 195=c+50+125$
$\therefore c=20$
Answer The relationship is $C=20+10n+5n^2$

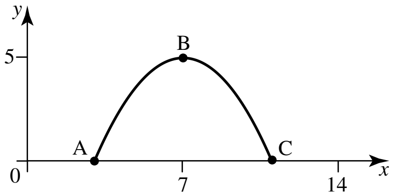
b $C\le 1000$
$\therefore 20+10n+5n^2\le 1000$
$\therefore n^2+2n+4\le 200$
$\therefore n^2+2n-196\le 0$
Consider $n^2+2n-196=0$
$\therefore \left(n^2+2n+1\right)-1-196=0$
$\therefore (n+1)^2=197$
$\therefore n+1=\pm\sqrt{197}$
$\therefore n=-1\pm\sqrt{197}$

Sign diagram of $n^2+2n-196$

Since $n\in N$, and $-1+\sqrt{197}\simeq 13.04$, $n^2+2n-196\le 0$ when $0<n<-1+\sqrt{197}$
$\therefore 1\le n\le 13$

Therefore 13 is the maximum number of tables that can be manufactured if the costs are not to exceed \$1000.

10 a



The width of the bridge is 14 metres so the turning point B has co-ordinates $(7,5)$.
As AC = 8 metres, by symmetry, OA = 3 metres and OC = 11 metres.
The co-ordinates of A are $(3,0)$ and the co-ordinates of C are $(11,0)$.

b The equation has the form $y=a(x-7)^2+5$.
C $(11,0)\Rightarrow 0=a(11-7)^2+5$
$\therefore 16a+5=0$
$\therefore a=-\frac{5}{16}$

The equation of the parabola is $y=-\frac{5}{16}(x-7)^2+5$.

c When $y=1.5$,
$\frac{3}{2}=-\frac{5}{16}(x-7)^2+5$
$\therefore 24=-5(x-7)^2+80$
$\therefore 5(x-7)^2=56$
$\therefore (x-7)^2=\frac{56}{5}$
$\therefore x-7=\pm\sqrt{\frac{56}{5}}$
$\therefore x\simeq 7\pm 3.35$
$\therefore x=3.65, x=10.35$

The width of the water level is $10.35-3.65=6.7$ metres, to one decimal place.

11 Cost: $C=15+10x$
Revenue: $R=vx$
$\therefore R=(50-x)x$
Profit = Revenue – Cost
$\therefore P=(50-x)x-(15+10x)$ where P dollars is the profit from a sale of x kg of fertiliser.
$\therefore P=50x-x^2-15-10x$
$\therefore P=-x^2+40x-15$
Completing the square,
$P=-\left[\left(x^2-40x+400\right)-400+15\right]$
$\therefore P=-\left[(x-20)^2-385\right]$
$\therefore P=-(x-20)^2+385$

Maximum turning point is $(20,385)$. The profit is greatest when $x=20$.
If $x=20$, the cost per kilogram, v, equals 30.
For maximum profit, the cost per kilogram is \$30 per kilogram.

12 **a** Let the two numbers be x and y.
Given $x+y=16$, then $y=16-x$.

i Let P be the product of the two numbers.

$P=xy$

$\therefore P=x(16-x)$

The x intercepts of the graph of P against x are $x=0, x=16$ so the axis of symmetry is $x=8$.
When $x=8$, $P=64$ so $(8,64)$ is the maximum turning point. And, when $x=8$, $y=8$.
The product is greatest when the numbers are both 8.

ii Let S be the sum of the squares of the numbers

$\therefore S=x^2+(16-x)^2$

$\therefore S=x^2+256-32x+x^2$

$\therefore S=2x^2-32x+256$

The graph of this function would have a minimum turning point.
C-ordinates of the turning point: $x=-\dfrac{-32}{4}\Rightarrow x=8$

$S=8^2+8^2$

$\therefore S=128$

TP (8,128) so S is least when $x=8$ and therefore $y=8$.
The sum of the squares of the two numbers is least when both are 8.

b $x+y=k$ so $y=k-x$

i Product, $P=x(k-x)$

Greatest when $x=\dfrac{0+k}{2}$.

When $x=\dfrac{k}{2}$,

$P=\dfrac{k}{2}\times\dfrac{k}{2}$

$\therefore P=\dfrac{k^2}{4}$

The greatest product of the two numbers is $\dfrac{k^2}{4}$.

ii Sum of squares, $S=x^2+(k-x)^2$.
If $S=P$, then $x^2+(k-x)^2=x(k-x)....(1)$

$\therefore x^2+k^2-2kx+x^2=kx-x^2$

$\therefore 3x^2-3kx+k^2=0$

Use the discriminant to test if there are solutions.

$\Delta=(-3k)^2-4\times3\times k^2$

$=9k^2-12k^2$

$=-3k^2$

$\Delta<0$ unless $k=0$.
Substitute $k=0$ in equation (1)

$2x^2=-x^2$

$\therefore 3x^2=0$

$\therefore x=0$

But the numbers were non zero so $k\neq0$
There are no non zero numbers for which the sum of their squares and their product are equal.

13 **a** The times of day need to be converted to the number of hours after midnight.
$4:21pm$ is 16 hours plus $\dfrac{21}{60}$ hours after midnight, giving the t value of 16.35.
Enter the values into the Statistics menu as

List 1	List 2
10.25	1.05
16.35	3.26
22.5	0.94

Tap Calc → Quadratic Reg to obtain, to two decimal places,

$a=-0.06$

$b=1.97$

$c=-12.78$

The equation of the quadratic model is
$h=-0.06t^2+1.97t-12.78$.

b Enter the equation as $y1=0.06x^2+1.97x-12.78$ in the Graph&Tab screen and sketch the graph.
Tap Analysis → G-Solve → Max to obtain the maximum turning point at $x=16.42, y=3.39$ to two decimal places.
The greatest height the tide reaches is 3.39 metres above sea level.
The time of day is 16.42 hours after midnight which is 4 pm plus $0.42\times60=25.2$ minutes.
The greatest tide is predicted to occur at $4:25pm$.

14 **a** The two sections of the wire form the perimeters of the two shapes. For the square of side length x cm, its perimeter is $4x$ cm. Therefore, the perimeter of the circle is $20-4x$ cm.
The area of the square is x^2 sq cm.
The area of the circle is given by $A=\pi r^2$,
The circle's circumference, or perimeter, is $20-4x$ cm.

$\therefore 2\pi r=20-4x$

$\therefore r=\dfrac{20-4x}{2\pi}$

$\therefore r=\dfrac{10-2x}{\pi}$

The area of the circle is $\pi\left(\dfrac{10-2x}{\pi}\right)^2=\dfrac{(10-2x)^2}{\pi}$ sq cm.

The sum of the areas: $S=x^2+\dfrac{(10-2x)^2}{\pi}$.

b Use the graphing screen to obtain the graph and the Analysis menu to obtain the co-ordinates of the minimum turning point as (2.8,14.0) to one decimal place.
For minimum S, $x=2.8$ and therefore the perimeter of the square is $4\times2.8=11.2$ cm.
Hence, the wire should be cut into two sections with one of length 11.2 cm and the other 8.8 cm in order to minimise the sum of the areas of the square and the circle these sections respectively enclose.

c Since the perimeters of the square and the circle are $4x$ and $20-4x$ respectively,

$4x\geq0$ and $20-4x\geq0$

$\therefore x\geq0$ and $x\leq5$

$\therefore 0\leq x\leq5$

Sketch the graph of $S=x^2+\dfrac{(10-2x)^2}{\pi}$ for $0\leq x\leq5$.

The screen suggests the greatest value for S occurs when $x=0$. Check this by tapping Analysis → G-Solve → y-Intercept to obtain $S=31.83$.
Then tap Analysis → G-Solve → y-Cal, and enter the value 5 for x. This gives $S=25$.
The value of x for which S is a minimum is $x=0$. (This means all the wire is used to form a circle).

Topic 4 — Cubic polynomials

Exercise 4.2 — Polynomials

1 A and C are polynomials, B is not a polynomial due to the term $-\frac{2}{x}$.

For A: degree is 5 (a quintic), leading term coefficient is 4 and the constant term is 12. The coefficients are integers so A is a polynomial over Z, the set of integers.

For C: degree is 2, leading term coefficient is –0.2 and the constant term is 5.6. The coefficients are rational numbers so C is a polynomial over Q, the set of rationals.

2 Many answers are possible, however they must contain $y^7-\sqrt{2}y^2+4$ and the fourth term chosen must have the power of y as one of the whole numbers 1, 3, 4, 5 or 6. One answer could be $y^7+2y^5-\sqrt{2}y^2+4$, for example.

3 **a** $P(x)=7x^3-8x^2-4x-1$

$$\begin{aligned}P(2)&=7(2)^3-8(2)^2-4(2)-1\\&=56-32-8-1\\&=15\end{aligned}$$

b $P(x)=2x^2+kx+12$

$$\begin{aligned}P(-3)=0\Rightarrow 0&=2(-3)^2+k(-3)+12\\\therefore 0&=18-3k+12\\0&=30-3k\\3k&=30\\k&=10\end{aligned}$$

4 $P(x)=-2x^3+9x+m$

$$\begin{aligned}P(1)&=-2(1)^3+9(1)+m\\P(1)&=7+m\\P(-1)&=-2(-1)^3+9(-1)+m\\P(-1)&=-7+m\end{aligned}$$

$$\begin{aligned}\therefore P(1)=2P(-1)\Rightarrow 7+m&=2(-7+m)\\7+m&=-14+2m\\21&=m\\m&=21\end{aligned}$$

5 $(2x+1)(x-5)\equiv a(x+1)^2+b(x+1)+c$

Expanding,

$$2x^2-9x-5=ax^2+2ax+a+bx+b+c$$
$$2x^2-9x-5=ax^2+(2a+b)x+(a+b+c)$$

$$\begin{array}{lll}\therefore 2=a, & -9=2a+b, & -5=a+b+c\\ a=2, & -9=4+b, & -5=2+b+c\\ & \therefore b=-13 & -5=2-13+c\\ & & \therefore c=6\end{array}$$

6 Let $(x+2)^3=px^2(x+1)+qx(x+2)+r(x+3)+t$

Expanding,

$$x^3+3x^2(2)+3x(2)^2+(2)^3=px^3+px^2+qx^2+2qx+rx+3r+t$$
$$x^3+6x^2+12x+8=px^3+(p+q)x^2+(2q+r)x+(3r+t)$$

$$\begin{array}{llll}\therefore 1=p, & 6=p+q, & 12=2q+r, & 8=3r+t\\ p=1, & 6=1+q, & 12=2q+r, & 8=6+t\\ p=1 & q=5 & 12=10+r & t=2\\ & & r=2 & \end{array}$$

Therefore $(x+2)^3=x^2(x+1)+5x(x+2)+2(x+3)+2$

7 $P(x)+2Q(x)$

$$\begin{aligned}&=4x^3-px^2+8+2(3x^2+qx-7)\\&=4x^3+(6-p)x^2+2qx-6\end{aligned}$$

Therefore $4x^3+(6-p)x^2+2qx-6=4x^3+x^2-8x-6$

Equating coefficients:

(x^2) $6-p=1$ and (x) $2q=-8$

Hence $p=5$ and $q=-4$.

8 $P(x)Q(x)$

$$\begin{aligned}&=(x^4+3x^2-7x+2)(x^3+x+1)\\&=x^4(x^3+x+1)+3x^2(x^3+x+1)-7x(x^3+x+1)+2(x^3+x+1)\\&=x^7+x^5+x^4+3x^5+3x^3+3x^2-7x^4-7x^2-7x+2x^3+2x+2\\&=x^7+4x^5-6x^4+5x^3-4x^2-5x+2\end{aligned}$$

Degree is 7

9 **a** $\frac{x-12}{x+3}$

$$\begin{aligned}&=\frac{(x+3)-3-12}{x+3}\\&=\frac{(x+3)-15}{x+3}\\&=\frac{x+3}{x+3}-\frac{15}{x+3}\\&=1-\frac{15}{x+3}\end{aligned}$$

Quotient is 1, remainder is –15

b $\frac{4x+7}{2x+1}$

$$\begin{aligned}&=\frac{2(2x+1)+5}{2x+1}\\&=2+\frac{5}{2x+1}\end{aligned}$$

10 $3x^2-6x+5\equiv ax(x+2)+b(x+2)+c$

$$\therefore 3x^2-6x+5=ax^2+2ax+bx+2b+c$$
$$\therefore 3x^2-6x+5=ax^2+x(2a+b)+2b+c$$

Equating coefficients of like terms

$x^2: 3=a....(1)$

$x: -6=2a+b....(2)$

constant: $5=2b+c....(3)$

Substitute equation (1) in equation (2)

$\therefore -6=2\times 3+b$

$\therefore b=-12$

Substitute $b=-12$ in equation (3)

$\therefore 5=2\times -12+c$

$\therefore c=29$

$\therefore a=3, b=-12, c=29$

$\therefore 3x^2-6x+5\equiv 3x(x+2)-12(x+2)+29$

$$\begin{aligned}&\frac{3x^2-6x+5}{x+2}\\&=\frac{3x(x+2)-12(x+2)+29}{x+2}\\&=\frac{3x(x+2)}{x+2}-\frac{12(x+2)}{x+2}+\frac{29}{x+2}\\&=3x-12+\frac{29}{x+2}\end{aligned}$$

$P(x)=3x-12,\ a=29$

11 a Long division method gives

$$\begin{array}{r}
2x^2-\ \ x+6\\
x-2\overline{)2x^3-5x^2+8x+6}\\
2x^3-4x^2\qquad\qquad\\
\hdashline
-x^2+8x+6\\
-x^2+2x\qquad\\
\hdashline
6x+6\\
6x-12\\
\hdashline
18\\
\hdashline
\end{array}$$

The remainder is 18.

Alternatively,

$$\frac{2x^3-5x^2+8x+6}{x-2}$$
$$=\frac{2x^2(x-2)-x(x-2)+6(x-2)+18}{x-2}$$
$$=2x^2-x+6+\frac{18}{x-2}$$

The quotient is $2x^2-x+6$ and the remainder is 18.

b

$$\begin{array}{r}
-\tfrac{1}{2}x^2-\tfrac{1}{4}x-\tfrac{1}{8}\\
-2x+1\overline{)x^3+0x^2+0x+10}\\
x^3-\tfrac{1}{2}x^2\qquad\qquad\qquad\\
\hdashline
\tfrac{1}{2}x^2+0x+10\\
\tfrac{1}{2}x^2-\tfrac{1}{4}x\qquad\\
\hdashline
\tfrac{1}{4}x+10\\
\tfrac{1}{4}x-\tfrac{1}{8}\\
\hdashline
\tfrac{81}{8}\\
\hdashline
\end{array}$$

$$\therefore \frac{x^3+10}{1-2x}=-\frac{1}{2}x^2-\frac{1}{4}x-\frac{1}{8}+\frac{81}{8(1-2x)}$$

The remainder is $\frac{81}{8}$.

12 Divisor $(x-1)^2=x^2-2x+1$

Long division method would give:

$$\begin{array}{r}
x^2-\ \ x+3\\
x^2-2x+1\overline{)x^4-3x^3+6x^2-7x+3}\\
x^4-2x^3+\ \ x^2\qquad\qquad\qquad\\
\hdashline
-x^3+5x^2-7x+3\\
-x^3+2x^2-\ \ x\qquad\\
\hdashline
3x^2-6x+3\\
3x^2-6x+3\\
\hdashline
0\\
\hdashline
\end{array}$$

Quotient $=x^2-x+3$, remainder $=0$.

Alternatively,

$$\frac{x^4-3x^3+6x^2-7x+3}{(x-1)^2}$$
$$=\frac{x^4-3x^3+6x^2-7x+3}{x^2-2x+1}$$
$$=\frac{x^2(x^2-2x+1)-x(x^2-2x+1)+3(x^2-2x+1)}{x^2-2x+1}$$
$$=x^2-x+3$$

13 a The polynomials are A, B, D and F.

A: $3x^5+7x^4-\frac{x^3}{6}+x^2-8x+12$

B: $9-5x^4+7x^2-\sqrt{5}x+x^3=9-\sqrt{5}x+7x^2+x^3-5x^4$

D: $2x^2(4x-9x^2)=8x^3-18x^4$

F: $\left(4x^2+3+7x^3\right)^2$

$$=\left((4x^2+3)+7x^3\right)^2$$
$$=(4x^2+3)^2+2(4x^2+3)(7x^3)+(7x^3)^2$$
$$=16x^4+24x^2+9+14x^3(4x^2+3)+49x^6$$
$$=16x^4+24x^2+9+56x^5+42x^3+49x^6$$
$$=49x^6+56x^5+16x^4+42x^3+24x^2+9$$

	Degree	**Type of coefficient**	**Leading term**	**Constant term**
A	5	Q	$3x^5$	12
B	4	R	$-5x^4$	9
D	4	Z	$-18x^4$	0
F	6	N	$49x^6$	9

b **C:** $\sqrt{4x^5}-\sqrt{5}x^3+\sqrt{3}x-1$ is not a polynomial due to $\sqrt{4x^5}=2x^{\frac{1}{2}}$ term: $\frac{1}{2}\notin N$.

E: $\frac{x^6}{10}-\frac{2x^5}{7}+\frac{5}{3x^2}-\frac{7x}{5}+\frac{4}{9}$ is not a polynomial due to $\frac{5}{3x^2}=\frac{5}{3}x^{-2}$ term: $-2\notin N$

14 $P(x)=2x^3+3x^2+x-6$

a $P(3)=2(3)^3+3(3)^2+(3)-6$
$=54+27+3-6$
$=78$

b $P(-2)=2(-2)^3+3(-2)^2+(-2)-6$
$=-16+12-2-6$
$=-12$

c $P(1)=2(1)^3+3(1)^2+(1)-6$
$=2+3+1-6$
$=0$

d $P(0)=-6$

e $P\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)-6$
$=-\frac{1}{4}+\frac{3}{4}-\frac{1}{2}-6$
$=-6$

f $P(0.1)=2(0.1)^3+3(0.1)^2+(0.1)-6$
$=0.002+0.03+0.1-6$
$=-5.868$

15 $P(x)=x^2-7x+2$

a $P(a)-P(-a)=\left((a)^2-7(a)+2\right)-\left((-a)^2-7(-a)+2\right)$
$=\left(a^2-7a+2\right)-\left(a^2+7a+2\right)$
$=-14a$

b $P(1+h)=(1+h)^2-7(1+h)+2$

$=1+2h+h^2-7-7h+2$

$=h^2-5h-4$

c $P(x+h)-P(x)$

$=\left[(x+h)^2-7(x+h)+2\right]-\left[x^2-7x+2\right]$

$=\left[x^2+2xh+h^2-7x-7h+2\right]-x^2+7x-2$

$=x^2+2xh+h^2-7x-7h+2-x^2+7x-2$

$=2xh+h^2-7h$

16 a $P(x)=4x^3+kx^2-10x-4$

$P(1)=4+k-10-4$

$=k-10$

$P(1)=15$

$\Rightarrow k-10=15$

$\therefore k=25$

b $Q(x)=ax^2-12x+7$

$Q(-2)=-5$

$\Rightarrow a(-2)^2-12(-2)+7=-5$

$\therefore 4a+24+7=-5$

$\therefore 4a=-36$

$\therefore a=-9$

c $P(x)=x^3-6x^2+nx+2$

$P(2)=3P(-1)$

$\Rightarrow (2)^3-6(2)^2+n(2)+2=3\left[(-1)^3-6(-1)^2+n(-1)+2\right]$

$\therefore 8-24+2n+2=3(-1-6-n+2)$

$\therefore 2n-14=3(-5-n)$

$\therefore 2n-14=-15-3n$

$\therefore 2n+3n=-15+14$

$\therefore 5n=-1$

$\therefore n=-\dfrac{1}{5}$

d $Q(x)=-x^2+bx+c$

$Q(0)=5\Rightarrow c=5$

$Q(5)=0$

$\Rightarrow -(5)^2+b(5)+c=0$

Substitute $c=5$

$\therefore -25+5b+5=0$

$\therefore 5b=20$

$\therefore b=4$

Answer $b=4, c=5$

17 a $3x^2+4x-7\equiv a(x+1)^2+b(x+1)+c$

Expand the right hand side

$\therefore 3x^2+4x-7=a(x^2+2x+1)+bx+b+c$

$=ax^2+2ax+bx+a+b+c$

$\therefore 3x^2+4x-7=ax^2+(2a+b)x+(a+b+c)$

Equate coefficients of like terms

$x^2: 3=a....(1)$

$x: 4=2a+b....(2)$

constant: $-7=a+b+c....(3)$

Substitute $a=3$ in equation (2)

$\therefore 4=6+b$

$\therefore b=-2$

Substitute $a=3$, $b=-2$ in equation (3)

$\therefore -7=3-2+c$

$\therefore c=-8$

Answer $a=3, b=-2, c=-8$

b $x^3+mx^2+nx+p\equiv(x-2)(x+3)(x-4)$

Expand

$\therefore x^3+mx^2+nx+p=(x-2)(x^2-x-12)$

$=x^3-x^2-12x-2x^2+2x+24$

$\therefore x^3+mx^2+nx+p=x^3-3x^2-10x+24$

Equate coefficients of like terms

$x^2: m=-3$

$x: n=-10$

constant: $p=24$

Answer: $m=-3, n=-10, p=24$

c $x^2-14x+8\equiv a(x-b)^2+c$

$\therefore x^2-14x+8=a(x^2-2xb+b^2)+c$

$=ax^2-2abx+ab^2+c$

Equating coefficients of like terms

$x^2: 1=a....(1)$

$x: -14=-2ab....(2)$

constant: $8=ab^2+c....(3)$

Substitute $a=1$ in equation (2)

$\therefore -14=-2b$

$\therefore b=7$

Substitute $a=1$, $b=7$ in equation (3)

$\therefore 8=49+c$

$\therefore c=-41$

$\therefore a=1, b=7, c=-41$

$\therefore x^2-14x+8=(x-7)^2-41$

d Let $4x^3+2x^2-7x+1=ax^2(x+1)+bx(x+1)+c(x+1)+d$

$\therefore 4x^3+2x^2-7x+1=ax^3+ax^2+bx^2+bx+cx+c+d$

$=ax^3+(a+b)x^2+(b+c)x+(c+d)$

Equating coefficients of like terms

$x^3: 4=a....(1)$

$x^2: 2=a+b....(2)$

$x: -7=b+c....(3)$

constant: $1=c+d....(4)$

Substitute $a=4$ in equation (2)

$\therefore 2=4+b$

$\therefore b=-2$

Substitute $b=-2$ in equation (3)

$\therefore -7=-2+c$

$\therefore c=-5$

Substitute $c=-5$ in equation (4)

$\therefore 1=-5+d$

$\therefore d=6$

Hence, $4x^3+2x^2-7x+1=4x^2(x+1)-2x(x+1)-5(x+1)+6$

18 a $P(x)=2x^2-7x-11$ and $Q(x)=3x^4+2x^2+1$

i $Q(x)-P(x)$

$=3x^4+2x^2+1-(2x^2-7x-11)$

$=3x^4+7x+12$

ii $3P(x)+2Q(x)$

$=3(2x^2-7x-11)+2(3x^4+2x^2+1)$

$=6x^2-21x-33+6x^4+4x^2+2$

$=6x^4+10x^2-21x-31$

iii $P(x)Q(x)$

$=(2x^2-7x-11)(3x^4+2x^2+1)$

$=6x^6+4x^4+2x^2-21x^5-14x^3-7x-33x^4-22x^2-11$

$=6x^6-21x^5-29x^4-14x^3-20x^2-7x-11$

b $P(x)$ has degree m, $Q(x)$ has degree n and $m > n$

i The leading term of $P(x)+Q(x)$ must be the x^m term so the degree is m.

ii Similarly, the degree of $P(x)-Q(x)$ is m.

iii The leading term of $P(x)Q(x)$ must be the $x^m \times x^n = x^{m+n}$ term so the degree is $m+n$.

19 a $P(x)=x^3-3x^2+px-2$, $Q(x)=ax^3+bx^2+3x-2a$

Given $2P(x)-Q(x)=5(x^3-x^2+x+q)$

$\therefore 2(x^3-3x^2+px-2)-(ax^3+bx^2+3x-2a)=5(x^3-x^2+x+q)$

$\therefore 2x^3-6x^2+2px-4-ax^3-bx^2-3x+2a=5x^3-5x^2+5x+5q$

$\therefore (2-a)x^3+(-6-b)x^2+(2p-3)x+(-4+2a)=5x^3-5x^2+5x+5q$

Equating coefficients of like terms

$x^3: 2-a=5....(1)$

$x^2: -6-b=-5....(2)$

$x: 2p-3=5....(3)$

constant: $-4+2a=5q....(4)$

equation (1) $\Rightarrow a=-3$, equation (2) $\Rightarrow b=-1$, equation (3) $\Rightarrow p=4$.

Substitute $a=-3$ in equation (4)

$\therefore -4-6=5q$

$\therefore q=-2$

Answer $a=-3, b=-1, p=4, q=-2$

b i Let $4x^4+12x^3+13x^2+6x+1=\left(ax^2+bx+c\right)^2$

$\therefore 4x^4+12x^3+13x^2+6x+1$

$=\left(ax^2+(bx+c)\right)^2$

$=\left(ax^2\right)^2+2ax^2(bx+c)+(bx+c)^2$

$=a^2x^4+2abx^3+2acx^2+b^2x^2+2bcx+c^2$

$=a^2x^4+2abx^3+(2ac+b^2)x^2+2bcx+c^2$

Equating coefficients of like terms

$x^4: 4=a^2....(1)$

$x^3: 12=2ab....(2)$

$x^2: 13=2ac+b^2....(3)$

$x: 6=2bc....(4)$

constant: $1=c^2....(5)$

Since $a>0$, equation (1) $\Rightarrow a=2$

Substitute $a=2$ in equation (2)

$\therefore 12=2\times 2\times b$

$\therefore b=3$

Substitute $b=3$ in equation (4)

$\therefore 6=2\times 3\times c$

$\therefore c=1$

Check $a=2, b=3, c=1$ satisfies the remaining two equations

equation (3): $2ac+b^2=2(2)(1)+(3)^2=13$ as required

equation (5): $c^2=(1)^2=1$ as required

$\therefore 4x^4+12x^3+13x^2+6x+1=\left(2x^2+3x+1\right)^2$

ii Hence, $2x^2+3x+1$ is a square root of $4x^4+12x^3+13x^2+6x+1$ as

$$\sqrt{4x^4+12x^3+13x^2+6x+1}=\sqrt{\left(2x^2+3x+1\right)^2}$$

$$=2x^2+3x+1$$

20 a $P(x)=x^4+kx^2+n^2$, $Q(x)=x^2+mx+n$

$P(x)Q(x)$

$=\left(x^4+kx^2+n^2\right)\left(x^2+mx+n\right)$

$=x^6+mx^5+nx^4+kx^4+kmx^3+knx^2+n^2x^2+n^2mx+n^3$

$=x^6+mx^5+(n+k)x^4+kmx^3+(kn+n^2)x^2+n^2mx+n^3$

Since $P(x)Q(x)=x^6-5x^5-7x^4+65x^3-42x^2-180x+216$ (given), equating coefficients of like terms gives

$x^5: m=-5....(1)$

$x^4: n+k=-7....(2)$

$x^3: km=65....(3)$

$x^2: kn+n^2=-42....(4)$

$x: n^2m=-180....(5)$

$\text{constant}: n^3=216....(6)$

Substitute $m=-5$ in equation (3)

$\therefore -5k=65$

$\therefore k=-13$

Substitute $k=-13$ in equation (2)

$\therefore n-13=-7$

$\therefore n=6$

Check values $m=-5, k=-13, n=6$ in remaining equations

equation (4): $kn+n^2=-13\times 6+6^2$

$=-78+36$

$=-42$

equation (5): $n^2m=36\times -5=-180$ as required

equation (6): $n^3=6^3=216$ as required

Answer $m=-5, k=-13, n=6$

b $P(x)Q(x)=\left(x^4-13x^2+36\right)\left(x^2-5x+6\right)$

$P(x)=x^4-13x^2+36$

$=(x^2-4)(x^2-9)$

$=(x-2)(x+2)(x-3)(x+3)$

$Q(x)=x^2-5x+6$

$=(x-2)(x-3)$

$\therefore P(x)Q(x)=(x-2)^2(x-3)^2(x+2)(x+3)$

21 a $\dfrac{x+7}{x-2}$

$=\dfrac{(x-2)+9}{x-2}$

$=\dfrac{x-2}{x-2}+\dfrac{9}{x-2}$

$=1+\dfrac{9}{x-2}$

Quotient 1, remainder 9

b $\dfrac{8x+5}{2x+1}$

$=\dfrac{4(2x+1)+1}{2x+1}$

$=\dfrac{4(2x+1)}{2x+1}+\dfrac{1}{2x+1}$

$=4+\dfrac{1}{2x+1}$

Quotient 4, remainder 1

c $\dfrac{x^2+6x-17}{x-1}$

$=\dfrac{x(x-1)+7x-17}{x-1}$

$=\dfrac{x(x-1)+7(x-1)-10}{x-1}$

$=\dfrac{x(x-1)}{x-1}+\dfrac{7(x-1)}{x-1}+\dfrac{-10}{x-1}$

$=x+7-\dfrac{10}{x-1}$

Quotient $x+7$, remainder -10

d $\dfrac{2x^2-8x+3}{x+2}$

$=\dfrac{2x(x+2)-12x+3}{x+2}$

$=\dfrac{2x(x+2)-12(x+2)+27}{x+2}$

$=\dfrac{2x(x+2)}{x+2}+\dfrac{-12(x+2)}{x+2}+\dfrac{27}{x+2}$

$=2x-12+\dfrac{27}{x+2}$

Quotient $2x-12$, remainder 27

e $\dfrac{x^3+2x^2-3x+5}{x-3}$

$=\dfrac{x^2(x-3)+5x^2-3x+5}{x-3}$

$=\dfrac{x^2(x-3)+5x(x-3)+12x+5}{x-3}$

$=\dfrac{x^2(x-3)+5x(x-3)+12(x-3)+41}{x-3}$

$=\dfrac{x^2(x-3)}{x-3}+\dfrac{5x(x-3)}{x-3}+\dfrac{12(x-3)}{x-3}+\dfrac{41}{x-3}$

$=x^2+5x+12+\dfrac{41}{x-3}$

Quotient $x^2+5x+12$, remainder 41

f $\dfrac{x^3-8x^2+9x-2}{x-1}$

$=\dfrac{x^2(x-1)-7x^2+9x-2}{x-1}$

$=\dfrac{x^2(x-1)-7x(x-1)+2x-2}{x-1}$

$=\dfrac{x^2(x-1)-7x(x-1)+2(x-1)}{x-1}$

$=\dfrac{x^2(x-1)}{x-1}+\dfrac{-7x(x-1)}{x-1}+\dfrac{2(x-1)}{x-1}$

$=x^2-7x+2$

Quotient x^2-7x+2, remainder 0

22 a $(8x^3+6x^2-5x+15)$ divided by $(1+2x)$

$\Rightarrow (8x^3+6x^2-5x+15)$ divided by $(2x+1)$

$$\begin{array}{r}
4x^2+\ \ x-3 \\
2x+1\overline{)8x^3+6x^2-5x+15} \\
4x^3+4x^2 \\
\hdashline
2x^2-5x+15 \\
2x^2+\ \ x \\
\hdashline
-6x+15 \\
-6x-3 \\
\hdashline
18 \\
\hdashline
\end{array}$$

$\therefore \dfrac{8x^3+6x^2-5x+15}{2x+1}=4x^2+x-3+\dfrac{18}{2x+1}$

b $(4x^3+x+5)$ divided by $(2x-3)$

$$\begin{array}{r}
2x^2+3x+5 \\
2x-3\overline{\big)4x^3+0x^2+\ \ x+5} \\
4x^3-6x^2 \qquad\qquad \\
\hline
6x^2+\ \ x+5 \\
6x^2-9x \quad\ \\
\hline
10x+5 \\
10x-15 \\
\hline
20 \\
\hline
\end{array}$$

$$\therefore \frac{4x^3+x+5}{2x-3}=2x^2+3x+5+\frac{20}{2x-3}$$

c $(x^3+6x^2+6x-12)\div(x+6)$

$$\begin{array}{r}
x^2 \qquad +6 \quad \\
x+6\overline{\big)x^3+6x^2+6x-12} \\
x^3+6x^2 \qquad\qquad \\
\hline
6x-12 \\
6x+36 \\
\hline
-48 \\
\hline
\end{array}$$

$$\therefore \frac{x^3+6x^2+x-12}{x+6}=x^2+6-\frac{48}{x+6}$$

d $(2+x^3)\div(x+1)\Rightarrow(x^3+2)\div(x+1)$

$$\begin{array}{r}
x^2-\ \ x+1 \\
x+1\overline{\big)x^3+0x^2+0x+2} \\
x^3+\ \ x^2 \qquad\qquad \\
\hline
-x^2+0x+2 \\
-x^2-\ \ x \quad\ \ \\
\hline
x+2 \\
x+1 \\
\hline
1 \\
\hline
\end{array}$$

$$\therefore \frac{x^3+2}{x+1}=x^2-x+1+\frac{1}{x+1}$$

e $\dfrac{x^4+x^3-x^2+2x+5}{x^2-1}$

$$\begin{array}{r}
x^2+\ \ x \qquad\qquad \\
x^2-1\overline{\big)x^4+x^3-\ \ x^2+2x+5} \\
x^4 \qquad -\ \ x^2 \qquad\qquad \\
\hline
x^3+0x^2+2x+5 \\
x^3 \qquad -\ \ x \quad\ \ \\
\hline
3x+5 \\
\hline
\end{array}$$

$$\therefore \frac{x^4+x^3-x^2+2x+5}{x^2-1}=x^2+x+\frac{3x+5}{x^2-1}$$

f $\dfrac{x(7-2x^2)}{(x+2)(x-3)}=\dfrac{-2x^3+7x}{x^2-x-6}$

$$\begin{array}{r}
-\ \ 2x-2 \quad \\
x^2-x-6\overline{\big)-2x^3+0x^2+\ \ 7x+0} \\
-2x^3+2x^2+12x \quad\ \\
\hline
-2x^2-\ \ 5x+0 \\
-2x^2+\ \ 2x+12 \\
\hline
-\ \ 7x-12 \\
\hline
\end{array}$$

$$\therefore \frac{-2x^3+7x}{x^2-x-6}=-2x-2-\frac{7x+12}{x^2-x-6}$$

23 a $\dfrac{4x^3-7x^2+5x+2}{2x+3}$

In the Main menu → Interactive → Transformation → propFrac and enter the expression to obtain

propFrac

$$\left(\frac{4x^3-7x^2+5x+2}{2x+3}\right)=2x^2-\frac{13}{2}x-\frac{139}{4(2x+3)}+\frac{49}{4}.$$

b Remainder is $-\dfrac{139}{4}$, quotient is $2x^2-\dfrac{13}{2}x+\dfrac{49}{4}$

c The dividend is $4x^3-7x^2+5x+2$.

If $x=-\dfrac{3}{2}$, $4\left(-\dfrac{3}{2}\right)^3-7\left(-\dfrac{3}{2}\right)^2+5\left(-\dfrac{3}{2}\right)+2=-\dfrac{139}{4}$

d The divisor is $2x+3$.

If $x=-\dfrac{3}{2}$, $2\left(-\dfrac{3}{2}\right)+3=0$

24 a In the Main menu → Interactive → Define

Func name: P

Variable/s: x

Expression: $3x^3+6x^2-8x-10$

Tap OK to define $P(x)=3x^3+6x^2-8x-10$.

Repeat the steps to define $Q(x)=-x^3+ax-6$

b Enter $P(-4)+P(3)-P\left(\dfrac{2}{3}\right)$ to obtain $\dfrac{349}{9}$

c Enter $P(2n)+24Q(n)$ to obtain $24n^3+24n^2-24(n^3-an+6)-16n-10$. Highlight and drop into Simplify to obtain $24n^2+24an-16n-154$.

d Enter $Q(-2)=-16$ in Equation/Inequality and solve for a to obtain $a=9$.

Exercise 4.3 — The remainder and factor theorems

1 $P(x)=x^3+4x^2-3x+5$

a Remainder $=P(-2)$.

$$\begin{aligned}P(-2)&=(-2)^3+4(-2)^2-3(-2)+5\\&=-8+16+6+5\\&=19\end{aligned}$$

Remainder is 19.

b Remainder $= P\left(\frac{1}{2}\right)$.

$$\begin{aligned}P\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 5\\ &= \frac{1}{8} + 1 - \frac{3}{2} + 5\\ &= 6 - \frac{11}{8}\\ &= \frac{37}{8}\end{aligned}$$

Remainder is $\frac{37}{8}$.

2 Let $P(x) = x^3 - kx^2 + 4x + 8$

Remainder = 29 when $P(x)$ is divided by $(x-3) \Rightarrow P(3) = 29$

$$\begin{aligned}\therefore (3)^3 - k(3)^2 + 4(3) + 8 &= 29\\ \therefore 47 - 9k &= 29\\ \therefore 18 &= 9k\\ \therefore k &= 2\end{aligned}$$

3 **a** $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$

$$\begin{aligned}Q(2) &= 4(2)^4 + 4(2)^3 - 25(2)^2 - (2) + 6\\ &= 64 + 32 - 100 - 2 + 6\\ &= 0\end{aligned}$$

Therefore $(x-2)$ is a factor

b Let $P(x) = 3x^3 + ax^2 + bx - 2$

Remainder of -22 when divided by $(x+1) \Rightarrow P(-1) = -22$

$$\begin{aligned}\therefore 3(-1)^3 + a(-1)^2 + b(-1) - 2 &= -22\\ \therefore a - b - 5 &= -22\\ \therefore a - b &= -17\end{aligned}$$

Exactly divisible by $(x-1) \Rightarrow P(1) = 0$

$$\begin{aligned}\therefore 3 + a + b - 2 &= 0\\ \therefore a + b &= -1\end{aligned}$$

Solve the simultaneous equations $\begin{aligned}a - b &= -17\\ a + b &= -1\end{aligned}$

by adding	and	by subtracting
$2a = -18$		$-2b = -16$
$\therefore a = -9$		$\therefore b = 8$

Therefore $P(x) = 3x^3 - 9x^2 + 8x - 2$

4 Let $P(x) = 12x^2 - 4x + a$

Since $(2x + a)$ is a factor, $P\left(-\frac{a}{2}\right) = 0$

$$\begin{aligned}\therefore 12\left(-\frac{a}{2}\right)^2 - 4\left(-\frac{a}{2}\right) + a &= 0\\ \therefore 3a^2 + 2a + a &= 0\\ \therefore 3a^2 + 3a &= 0\\ \therefore 3a(a+1) &= 0\\ \therefore a &= 0, a = -1\end{aligned}$$

5 **a** $P(x) = x^3 + 3x^2 - 13x - 15$

$P(-1) = -1 + 3 + 13 - 15 = 0 \Rightarrow (x+1)$ is a factor

$$\begin{aligned}x^3 + 3x^2 - 13x - 15 &= (x+1)(x^2 + bx - 15)\\ &= (x+1)(x^2 + 2x - 15)\\ &= (x+1)(x+5)(x-3)\end{aligned}$$

b $P(x) = 12x^3 + 41x^2 + 43x + 14$

Since $(x+1)$ and $(3x+2)$ are factors, then $(x+1)\,(3x+2)$ is a quadratic factor

$$\begin{aligned}P(x) &= 12x^3 + 41x^2 + 43x + 14\\ &= (x+1)(3x+2)(ax+b)\\ &= (3x^2 + 5x + 2)(ax+b)\\ &= (3x^2 + 5x + 2)(4x+7)\end{aligned}$$

$\therefore P(x) = (x+1)(3x+2)(4x+7)$

6 $P(x) = 12x^3 + 8x^2 - 3x - 2$

The zeros are of the form $\frac{p}{q}$ where p is a factor of 2 and q is a factor of 12.

Try $p = 1$ and $q = 2$

$$\begin{aligned}P\left(\frac{1}{2}\right) &= 12\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) - 2\\ &= \frac{12}{8} + 2 - \frac{3}{2} - 2\\ &= 0\end{aligned}$$

$\therefore (2x-1)$ is a factor

Hence,

$$\begin{aligned}12x^3 + 8x^2 - 3x - 2 &= (2x-1)(6x^2 + bx + 2)\\ &= (2x-1)(6x^2 + 7x + 2)\\ &= (2x-1)(2x+1)(3x+2)\end{aligned}$$

The three linear factors are $(2x-1), (2x+1)$ and $(3x+2)$.

7 $6x^3 + 13x^2 = 2 - x$

$\therefore 6x^3 + 13x^2 + x - 2 = 0$

Let $P(x) = 6x^3 + 13x^2 + x - 2$

$P(-1) = -6 + 13 - 1 - 2 \neq 0$

$P(-2) = -48 + 52 - 2 - 2 = 0$

$\therefore (x+2)$ is a factor

$$\begin{aligned}&6x^3 + 13x^2 + x - 2\\ &= (x+2)(6x^2 + bx - 1)\\ &= (x+2)(6x^2 + x - 1)\\ &= (x+2)(3x-1)(2x+1)\end{aligned}$$

For $6x^3 + 13x^2 + x - 2 = 0$,

$(x+2)(3x-1)(2x+1) = 0$

$\therefore x = -2, x = \frac{1}{3}, x = -\frac{1}{2}$

8
$$\begin{aligned}2x^4 + 3x^3 - 8x^2 - 12x &= 0\\ \therefore x(2x^3 + 3x^2 - 8x - 12) &= 0\\ \therefore x\left[x^2(2x+3) - 4(x+3)\right] &= 0\\ \therefore x(2x+3)(x^2 - 4) &= 0\\ \therefore x(2x+3)(x-2)(x+2) &= 0\end{aligned}$$

$\therefore x = 0, x = -\frac{3}{2}, x = 2, x = -2$

9 **a** Let $P(x) = x^3 - 4x^2 - 5x + 3$

Remainder $= P(1)$ when $P(x)$ is divided by $(x-1)$

$P(1) = 1 - 4 - 5 + 3 = -5$

Remainder is -5

b Let $P(x) = 6x^3 + 7x^2 + x + 2$

Remainder $= P(-1)$ when $P(x)$ is divided by $(x+1)$

$P(-1) = -6 + 7 - 1 + 2 = 2$

Remainder is 2

c Let $P(x) = -2x^3 + 2x^2 - x - 1$

Remainder $= P(4)$ when $P(x)$ is divided by $(x-4)$

$$\begin{aligned}P(4) &= -2(4)^3 + 2(4)^2 - (4) - 1\\ &= -128 + 32 - 4 - 1\\ &= -101\end{aligned}$$

Remainder is -101

d Let $P(x) = x^3 + x^2 + x - 10$

Remainder $= P\left(-\frac{1}{2}\right)$ when $P(x)$ is divided by $(2x+1)$

$$P\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)-10$$
$$=-\frac{1}{8}+\frac{1}{4}-\frac{1}{2}-10$$
$$=-10\tfrac{3}{8}$$

Remainder is $-10\tfrac{3}{8}$

e Let $P(x)=27x^3-9x^2-9x+2$

Remainder $=P\left(\frac{2}{3}\right)$ when $P(x)$ is divided by $(3x-2)$

$$P\left(\frac{2}{3}\right)=27\left(\frac{2}{3}\right)^3-9\left(\frac{2}{3}\right)^2-9\left(\frac{2}{3}\right)+2$$
$$=27\times\frac{8}{27}-9\times\frac{4}{9}-9\times\frac{2}{3}+2$$
$$=8-4-6+2$$
$$=0$$

Remainder is 0

f Let $P(x)=4x^4-5x^3+2x^2-7x+8$

Remainder $=P(2)$ when $P(x)$ is divided by $(x-2)$

$$P(2)=4(2)^4-5(2)^3+2(2)^2-7(2)+8$$
$$=64-40+8-14+8$$
$$=26$$

Remainder is 26

10 a $P(x)=x^3-2x^2+ax+7$

Remainder $=P(-2)$ when $P(x)$ is divided by $(x+2)$

$$\therefore P(-2)=11$$
$$\therefore (-2)^3-2(-2)^2+a(-2)+7=11$$
$$\therefore -8-8-2a+7=11$$
$$\therefore -2a-9=11$$
$$\therefore -2a=20$$
$$\therefore a=-10$$

b $P(x)=4-x^2+5x^3-bx^4$

Remainder $=P(1)$ when $P(x)$ is divided by $(x-1)$

Remainder $=0$ since $P(x)$ is exactly divisible by $(x-1)$

$$\therefore P(1)=0$$
$$\therefore 4-1+5-b=0$$
$$\therefore 8-b=0$$
$$\therefore b=8$$

c Let $P(x)=2x^3+cx^2+5x+8$

Remainder $=P\left(\frac{1}{2}\right)$ when $P(x)$ is divided by $(2x-1)$

$$\therefore P\left(\frac{1}{2}\right)=6$$
$$\therefore 2\left(\frac{1}{2}\right)^3+c\left(\frac{1}{2}\right)^2+5\left(\frac{1}{2}\right)+8=6$$
$$\therefore \frac{1}{4}+\frac{c}{4}+\frac{5}{2}+8=6$$
$$\therefore 1+c+10+32=24$$
$$\therefore c=-19$$

d Let $P(x)=x^3+3x^2-4x+d$ and $Q(x)=x^4-9x^2-7$

Same remainder when divided by $(x+3)$

$$\therefore P(-3)=Q(-3)$$
$$\therefore (-3)^3+3(-3)^2-4(-3)+d=(-3)^4-9(-3)^2-7$$
$$\therefore -27+27+12+d=81-81-7$$
$$\therefore 12+d=-7$$
$$\therefore d=-19$$

11 a $Q(x)=ax^3+4x^2+bx+1$

Remainder, $Q(2)=39$

$$\therefore a(2)^3+4(2)^2+b(2)+1=39$$
$$\therefore 8a+2b+17=39$$
$$\therefore 8a+2b=22$$
$$\therefore 4a+b=11....(1)$$

$Q(-1)=0$ since $(x+1)$ is a factor

$$\therefore a(-1)^3+4(-1)^2+b(-1)+1=0$$
$$\therefore -a-b+5=0$$
$$\therefore a+b=5....(2)$$

Solving

$4a+b=11....(1)$

$a+b=5....(2)$

equation (1) subtract equation (2)

$\therefore 3a=6$

$\therefore a=2$

equation (2) $\Rightarrow b=3$

Answer $a=2, b=3$

b $P(x)=\frac{1}{3}x^3+mx^2+nx+2$

Remainders, $P(3)=P(-3)$

$$\therefore \frac{1}{3}(3)^3+m(3)^2+n(3)+2=\frac{1}{3}(-3)^3+m(-3)^2+n(-3)+2$$
$$\therefore 9+9m+3n+2=-9+9m-3n+2$$
$$\therefore 9+3n=-9-3n$$
$$\therefore 6n=-18$$
$$\therefore n=-3$$

$$\therefore P(x)=\frac{1}{3}x^3+mx^2-3x+2$$

$P(3)=3P(1)$

$$\therefore 9+9m-9+2=3\left(\frac{1}{3}+m-3+2\right)$$
$$\therefore 9m+2=1+3m-9+6$$
$$\therefore 6m=-4$$
$$\therefore m=-\frac{2}{3}$$

Answer $m=-\frac{2}{3}, n=-3$

12 a Zero $x=5\Rightarrow(x-5)$ is a factor of the polynomial

Zero $x=9\Rightarrow(x-9)$ is a factor of the polynomial

Zero $x=-2\Rightarrow(x+2)$ is a factor of the polynomial

- **i** Therefore, the degree 3 monic polynomial is $(x-5)(x-9)(x+2)$ in factorised form.
- **ii** Expanding, $(x-5)(x-9)(x+2)$

$$=(x-5)(x^2-7x-18)$$
$$=x^3-7x^2-18x-5x^2+35x+90$$
$$=x^3-12x^2+17x+90$$

b Zero $x=-4\Rightarrow(x+4)$ is a factor of the polynomial

Zero $x=-1\Rightarrow(x+1)$ is a factor of the polynomial

Zero $x=\frac{1}{2}\Rightarrow\left(x-\frac{1}{2}\right)$ is a factor of the polynomial

- **i** Therefore, in factorised form, the degree 3 polynomial with leading coefficient -2 is

$$-2(x+4)(x+1)\left(x-\frac{1}{2}\right)=(x+4)(x+1)(1-2x)$$

- **ii** Expanding,

$$(x+4)(x+1)(1-2x)$$
$$=(x^2+5x+4)(1-2x)$$
$$=x^2-2x^3+5x-10x^2+4-8x$$
$$=-2x^3-9x^2-3x+4$$

13 a $P(x)=x^3-x^2-10x-8$

$\therefore P(x)=(x-4)(x^2+3x+2)$

$\therefore P(x)=(x-4)(x+2)(x+1)$

b $P(x)=3x^3+40x^2+49x+12$

$=(x+12)(3x^2+4x+1)$

$\therefore P(x)=(x+12)(3x+1)(x+1)$

c $P(x)=20x^3+44x^2+23x+3$

$=(5x+1)(4x^2+8x+3)$

$\therefore P(x)=(5x+1)(2x+3)(2x+1)$

d $P(x)=-16x^3+12x^2+100x-75$

$=(4x-3)(-4x^2+0x+25)$

$=(4x-3)(25-4x^2)$

$\therefore P(x)=(4x-3)(5+2x)(5-2x)$

e $P(x)=-8x^3+59x^2-138x+99$

$\therefore P(x)=(8x-11)(x-3)(ax+b)$

$=(8x^2-35x+33)(ax+b)$

$=(8x^2-35x+33)(-x+3)$

$=(8x-11)(x-3)\big(-(x-3)\big)$

$\therefore P(x)=-(x-3)^2(8x-11)$

f $P(x)=9x^3-75x^2+175x-125$

$\therefore P(x)=(3x-5)(3x^2-20x+25)$

$=(3x-5)(3x-5)(x-5)$

$\therefore P(x)=(3x-5)^2(x-5)$

14 a Let $P(x)=x^3+5x^2+2x-8$

$P(1)=1+5+2-8=0$

$\therefore (x-1)$ is a factor

$\therefore x^3+5x^2+2x-8=(x-1)(x^2+6x+8)$

$=(x-1)(x+2)(x+4)$

b Let $P(x)=x^3+10x^2+31x+30$

$P(-2)=(-2)^3+10(-2)^2+31(-2)+30$

$=-8+40-62+30$

$=0$

$\therefore (x+2)$ is a factor

$\therefore x^3+10x^2+31x+30=(x+2)(x^2+8x+15)$

$=(x+2)(x+3)(x+5)$

c Let $-7=b$

$P(2)=2(2)^3-13(2)^2+13(2)+10$

$=16-52+26+10$

$=0$

$\therefore (x-2)$ is a factor

$\therefore 2x^3-13x^2+13x+10=(x-2)(2x^2-9x-5)$

$=(x-2)(2x+1)(x-5)$

d Let $P(x)=-18x^3+9x^2+23x-4$

$P(-1)=-18(-1)^3+9(-1)^2+23(-1)-4$

$=18+9-23-4$

$=0$

$\therefore (x+1)$ is a factor

$\therefore -18x^3+9x^2+23x-4=(x+1)(-18x^2+27x-4)$

$=(x+1)(-6x+1)(3x-4)$

$=(x+1)(1-6x)(3x-4)$

e Let $P(x)=x^3-7x+6$

$P(1)=1-7+6=0$

$\therefore (x-1)$ is a factor

$\therefore x^3+0x^2-7x+6=(x-1)(x^2+x-6)$

$\therefore x^3-7x+6=(x-1)(x-2)(x+3)$

f $x^3+x^2-49x-49$

$=x^2(x+1)-49(x+1)$

$=(x+1)(x^2-49)$

$=(x+1)(x-7)(x+7)$

15 a Let $P(x)=24x^3+34x^2+x-5$

As the zeros are not integers they must be of the form $\frac{p}{q}$ where p is a factor of 5 and q is a factor 24.

$P\left(\frac{1}{2}\right)=24\left(\frac{1}{2}\right)^3+34\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)-5$

$=3+\frac{17}{2}+\frac{1}{2}-5$

$\neq 0$

$P\left(-\frac{1}{2}\right)=-3+\frac{17}{2}-\frac{1}{2}-5$

$=0$

$\therefore (2x+1)$ is a factor

Hence,

$24x^3+34x^2+x-5=(2x+1)(12x^2+bx-5)$

$=(2x+1)(12x^2+11x-5)$

$=(2x+1)(3x-1)(4x+5)$

b $P(x)=8x^3+mx+13x+5$

i A zero of $\frac{5}{2}$ means that $(2x-5)$ is a factor.

ii $\therefore 8x^3+mx^2+13x+5=(2x-5)(4x^2+bx-1)$

Equate coefficients of x:

$13=-2-5b$

$\therefore 5b=-15$

$\therefore b=-3$

Hence,

$8x^3+mx^2+13x+5=(2x-5)(4x^2-3x-1)$

$=(2x-5)(4x+1)(x-1)$

iii Consider $8x^3+mx^2+13x+5=(2x-5)(4x^2-3x-1)$ and equate coefficients of x^2.

$\therefore m=-6-20$

$\therefore m=-26$

c i $P(x)=x^3-12x^2+48x-64$

$P(4)=64-12(16)+48(4)-64$

$=64-192+193-64$

$=0$

$\therefore (x-4)$ is a factor

Hence,

$x^3-12x^2+48x-64=(x-4)(x^2+bx+16)$

$=(x-4)(x^2-8x+16)$

$=(x-4)(x-4)^2$

$=(x-4)^3$

$\therefore P(x)=(x-4)^3$

$Q(x)=x^3-64$

$Q(x)=x^3-4^3$

$=(x-4)(x^2+4x+16)$

$\therefore Q(x)=(x-4)(x^2+4x+16)$

ii $\frac{P(x)}{Q(x)}$

$= \frac{(x-4)^3}{(x-4)(x^2+4x+16)}$

$= \frac{(x-4)^2}{x^2+4x+16}$

$= \frac{x^2-8x+16}{x^2+4x+16}$

$= \frac{\left(x^2+4x+16\right)-12x}{x^2+4x+16}$

$= \frac{x^2+4x+16}{x^2+4x+16} - \frac{12x}{x^2+4x+16}$

$= 1 - \frac{12x}{x^2+4x+16}$

d $P(x) = ax^3+bx^2+cx+d$

$P(0) = 9 \Rightarrow d = 9$

Given factors $\Rightarrow \left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right) = x^2-3$ is a quadratic factor.

$\therefore x^3+bx^2+cx+9 = (x^2-3)(x-3)$

The third factor is $(x-3)$.

Expanding,

$x^3+bx^2+cx+9 = x^3-3x^2-3x+9$

$\therefore b = -3 = c$

16 a $(x+4)(x-3)(x+5) = 0$

$\therefore x = -4, x = 3, x = -5$

b $2(x-7)(3x+5)(x-9) = 0$

$\therefore (x-7)(3x+5)(x-9) = 0$

$\therefore x = 7, x = -\frac{5}{3}, x = 9$

c $x^3-13x^2+34x+48 = 0$

Let $P(x) = x^3-13x^2+34x+48$

$P(-1) = -1-13-34+48 = 0$

$\therefore (x+1)$ is a factor

$\therefore x^3-13x^2+34x+48 = 0$

$\Rightarrow (x+1)(x^2-14x+48) = 0$

$\therefore (x+1)(x-6)(x-8) = 0$

$\therefore x = -1, x = 6, x = 8$

d $2x^3+7x^2 = 9$

$\therefore 2x^3+7x^2-9 = 0$

Let $P(x) = 2x^3+7x^2-9$

$P(1) = 2+7-9 = 0$

$\therefore (x-1)$ is a factor

$\therefore 2x^3+7x^2-9 = 0$

$\Rightarrow (x-1)(2x^2+9x+9) = 0$

$\therefore (x-1)(2x+3)(x+3) = 0$

$\therefore x = 1, x = -\frac{3}{2}, x = -3$

e $3x^2(3x+1) = 4(2x+1)$

$\therefore 9x^3+3x^2 = 8x+4$

$\therefore 9x^3+3x^2-8x-4 = 0$

Let $P(x) = 9x^3+3x^2-8x-4$

$P(1) = 9+3-8-4 = 0$

$\therefore (x-1)$ is a factor

$\therefore 9x^3+3x^2-8x-4 = 0$

$\Rightarrow (x-1)(9x^2+12x+4) = 0$

$\therefore (x-1)(3x+2)^2 = 0$

$\therefore x = 1, x = -\frac{2}{3}$

f $8x^4+158x^3-46x^2-120x = 0$

$\therefore 2x(4x^3+79x^2-23x-60) = 0$

Let $P(x) = 4x^3+79x^2-23x-60$

$P(1) = 4+79-23-60 = 0$

$\therefore (x-1)$ is a factor

$\therefore P(x) = (x-1)(4x^2+83x+60)$

$\therefore 2x(4x^3+79x^2-23x-60) = 0$

$\Rightarrow 2x(x-1)(4x^2+83x+60) = 0$

$\therefore 2x(x-1)(4x+3)(x+20) = 0$

$\therefore x = 0, x = 1, x = -\frac{3}{4}, x = -20$

17 a $P(x) = x^3+6x^2-7x-18$

$P(2) = (2)^3+6(2)^2-7(2)-18$

$= 8+24-14-18$

$= 0$

Since $P(2) = 0$, $(x-2)$ is a factor of $P(x)$.

$\therefore x^3+6x^2-7x-18$

$= (x-2)(x^2+8x+9)$

$= (x-2)\left[(x^2+8x+16)-16+9\right]$

$= (x-2)\left[(x+4)^2-7\right]$

$= (x-2)\left(x+4+\sqrt{7}\right)\left(x+4-\sqrt{7}\right)$

b Let $P(x) = 3x^3+5x^2+10x-4$

If $(3x-1)$ is a factor then $P\left(\frac{1}{3}\right) = 0$

$P\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^2 + 10\left(\frac{1}{3}\right) - 4$

$= \frac{1}{9}+\frac{5}{9}+\frac{10}{3}-4$

$= \frac{2}{3}+\frac{10}{3}-4$

$= 0$

$\therefore (3x-1)$ is a factor

$\therefore 3x^3+5x^2+10x-4 = (3x-1)(x^2+2x+4)$

Consider the quadratic factor x^2+2x+4

$\Delta = b^2-4ac,\ a = 1, b = 2, c = 4$

$= 2^2-4\times1\times4$

$= -12$

Since $\Delta < 0$, the quadratic has no real linear factors.

Hence, $(3x-1)$ is the only real linear factor of $P(x) = 3x^3+5x^2+10x-4$.

c Let $P(x) = 2x^3-21x^2+60x-25$

Since $2x^2-11x+5 = (2x-1)(x-5)$, $2x^2-11x+5$ will be a factor of $P(x)$ if both $(2x-1)$ and $(x-5)$ are factors.

$P\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 21\left(\frac{1}{2}\right)^2 + 60\left(\frac{1}{2}\right) - 25$

$= \frac{1}{4}-\frac{21}{4}+30-25$

$= -5+30-25$

$= 0$

$\therefore (2x-1)$ is a factor

$P(5) = 2(5)^3-21(5)^2+60(5)-25$

$= 250-525+300-25$

$= 0$

$\therefore (x-5)$ is a factor.

Hence, $2x^2-11x+5$ is a factor.

$\therefore 2x^3 - 21x^2 + 60x - 25 = 0$

$\Rightarrow (2x^2 - 11x + 5)(x - 5) = 0$

$\therefore (2x - 1)(x - 5) = 0$

$\therefore x = \frac{1}{2}, x = 5$

18 a $P(x) = 5x^3 + kx^2 - 20x - 36$

As $(x^2 - 4)$ is a factor,

$$\begin{aligned} 5x^3 + kx^2 - 20x - 36 &= (x^2 - 4)(ax + b) \\ &= (x^2 - 4)(5x + 9) \\ &= (x - 2)(x + 2)(5x + 9) \end{aligned}$$

Expanding the factorised form,

$$\begin{aligned} 5x^3 + kx^2 - 20x - 36 &= (x^2 - 4)(5x + 9) \\ &= 5x^3 + 9x^2 - 20x - 36 \end{aligned}$$

Equating coefficients of x^2: $k = 9$.

b $ax^2 - 5ax + 4(2a - 1) = 0$

Since $x = a$ is a solution, substitute $x = a$ in the equation.

$\therefore a(a)^2 - 5a(a) + 4(2a - 1) = 0$

$\therefore a^3 - 5a^2 + 8a - 4 = 0$

Let $P(a) = a^3 - 5a^2 + 8a - 4$

$P(1) = 1 - 5 + 8 - 4 = 0$

$\therefore (a - 1)$ is a factor of $P(a)$.

$\therefore a^3 - 5a^2 + 8a - 4 = 0$

$\Rightarrow (a - 1)(a^2 - 4a + 4) = 0$

$\therefore (a - 1)(a - 2)^2 = 0$

$\therefore a = 1, a = 2$

c $P(x) = x^3 + ax^2 + bx - 3$ and $Q(x) = x^3 + bx^2 + 3ax - 9$

Since $(x + a)$ is a common factor, $P(-a) = 0$ and $Q(-a) = 0$.

$P(-a) = 0$

$\therefore -a^3 + a^3 - ba - 3 = 0$

$\therefore ab = -3....(1)$

$Q(-a) = 0$

$\therefore -a^3 + ba^2 - 3a^2 - 9 = 0$

$\therefore a^3 - a^2b + 3a^2 + 9 = 0....(2)$

Solve the simultaneous equations:

$ab = -3....(1)$

$a^3 - a(ab) + 3a^2 + 9 = 0....(2)$

Substitute equation (1) in equation (2)

$\therefore a^3 - a(-3) + 3a^2 + 9 = 0$

$\therefore a^3 + 3a^2 + 3a + 9 = 0$

$\therefore a^2(a + 3) + 3(a + 3) = 0$

$\therefore (a + 3)(a^2 + 3) = 0$

$\therefore a = -3$ or $a^2 = -3$ (reject)

$\therefore a = -3$

Substitute $a = -3$ in equation (1)

$\therefore -3b = -3$

$\therefore b = 1$

With $a = -3, b = 1$,

$$\begin{aligned} P(x) &= x^3 - 3x^2 + x - 3 \\ &= x^2(x - 3) + (x - 3) \\ &= (x - 3)(x^2 + 1) \end{aligned}$$

$$\begin{aligned} Q(x) &= x^3 + x^2 - 9x - 9 \\ &= x^2(x + 1) - 9(x + 1) \\ &= (x + 1)(x^2 - 9) \\ &= (x + 1)(x - 3)(x + 3) \end{aligned}$$

$(x + a) = (x - 3)$ is a factor of each polynomial.

d $P(x) = x^3 + px^2 + 15x + a^2$

Since $(x + a)^2 = x^2 + 2ax + a^2$ is a factor,

$x^3 + px^2 + 15x + a^2 = (x^2 + 2ax + a^2)(x + 1)$

Expanding the factorised form

$$\begin{aligned} &x^3 + px^2 + 15x + a^2 \\ &= (x^2 + 2ax + a^2)(x + 1) \\ &= x^3 + x^2 + 2ax^2 + 2ax + a^2x + a^2 \\ &= x^3 + (1 + 2a)x^2 + (2a + a^2)x + a^2 \end{aligned}$$

Equating coefficients of like terms

$x^2 : p = 1 + 2a....(1)$

$x : 15 = 2a + a^2....(2)$

Solving equation (2),

$a^2 + 2a - 15 = 0$

$\therefore (a + 5)(a - 3) = 0$

$\therefore a = -5$ or $a = 3$

Substitute $a = -5$ in equation (1)

$\therefore p = 1 + 2 \times -5$

$\therefore p = -9$

Substitute $a = 3$ in equation (1)

$\therefore p = 1 + 2 \times 3$

$\therefore p = 7$

There are two possible polynomials, one for which $a = -5, p = -9$ and one for which $a = 3, p = 7$.

$x^3 + px^2 + 15x + a^2 = 0$

If $a = -5, p = -9$, then $x^3 - 9x^2 + 15x + 25 = 0$

As $(x + a)^2 = (x - 5)^2$ is a factor of the polynomial,

$(x^2 - 10x + 25)(x + 1) = 0$

$\therefore (x - 5)^2(x + 1) = 0$

$\therefore x = 5, x = -1$

If $a = 3, p = 7$, then $x^3 + 7x^2 + 15x + 9 = 0$

As $(x + a)^2 = (x + 3)^2$ is a factor of the polynomial,

$(x^2 + 6x + 9)(x + 1) = 0$

$\therefore (x + 3)^2(x + 1) = 0$

$\therefore x = -3, x = -1$

19 Let $P(x) = 9 + 19x - 2x^2 - 7x^3$

Divisor $\left(x - \sqrt{2} + 1\right)$ is zero when $x = \sqrt{2} - 1$.

Hence, remainder is $P\left(\sqrt{2} - 1\right)$

$P\left(\sqrt{2} - 1\right) = 9 + 19\left(\sqrt{2} - 1\right) - 2\left(\sqrt{2} - 1\right)^2 - 7\left(\sqrt{2} - 1\right)^3$

Evaluate in CAS main screen → interactive → transformation → simplify in Standard mode gives the remainder as $-12\sqrt{2} + 33$.

Alternatively, $propFrac\left(\dfrac{9 + 19x - 2x^2 - 7x^3}{x - \sqrt{2} + 1}\right)$ gives the quotient and the remainder.

20 $10x^3 - 5x^2 + 21x + 12 = 0$

Solve for x on CAS in Decimal mode using Interactive → Equation/Inequality → solve to obtain one solution $x = -0.4696$, correct to 4 decimal places.

Exercise 4.4 — Graphs of cubic polynomials

1 a $y = (x - 1)^3 - 8$

Point of inflection $(1, -8)$

y intercept $(0, -9)$

x intercept Let $y = 0$

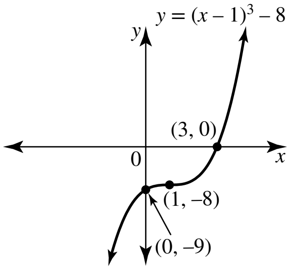
$\therefore (x-1)^3 - 8 = 0$

$\therefore (x-1)^3 = 8$

$\therefore x - 1 = 2$

$\therefore x = 3$

$\Rightarrow (3,0)$



b $y = 1 - \frac{1}{36}(x+6)^3$

Point of inflection $(-6,1)$

y intercept $(0,-5)$

x intercept Let $y = 0$

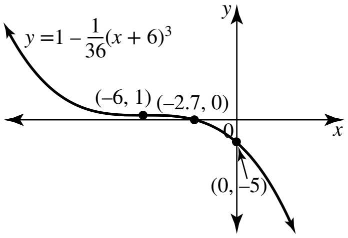
$\therefore 0 = 1 - \frac{1}{36}(x+6)^3$

$\therefore (x+6)^3 = 36$

$\therefore x + 6 = \sqrt[3]{36}$

$\therefore x = \sqrt[3]{36} - 6$

$\Rightarrow (\sqrt[3]{36} - 6, 0) \simeq (-2.7, 0)$



2 a $y = \left(\frac{x}{2} - 3\right)^3$

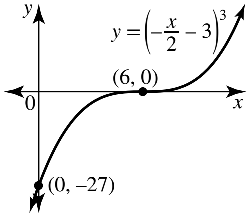
Point of inflection and x intercept

$\frac{x}{2} - 3 = 0$

$\therefore x = 6$

$\Rightarrow (6,0)$

y intercept $(0,-27)$



b $y = 2x^3 - 2$

Point of inflection and y intercept $(0,-2)$

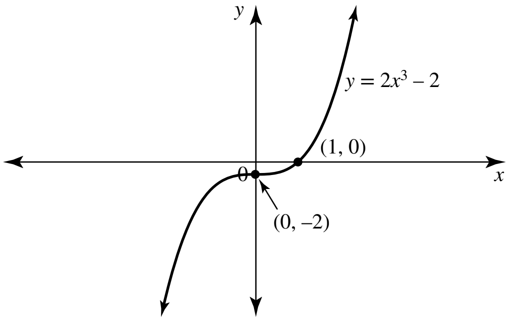
x intercept Let $y = 0$

$\therefore 2x^3 - 2 = 0$

$\therefore x^3 = 1$

$\therefore x = 1$

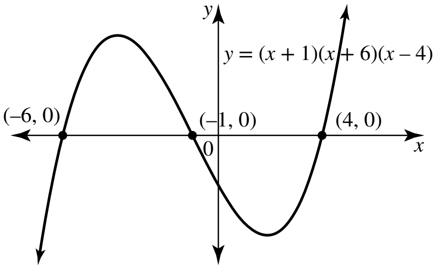
$\Rightarrow (1,0)$



3 a $y = (x+1)(x+6)(x-4)$

x intercepts at $x = -1, x = -6, x = 4$

y intercept at $y = (1)(6)(-4) = -24$



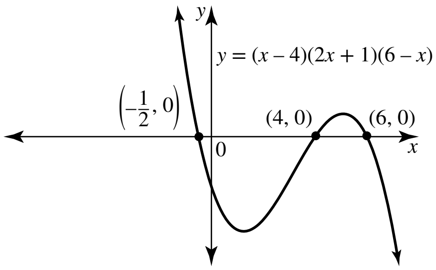
b $y = (x-4)(2x+1)(2-x)$

x intercepts when

$x - 4 = 0,\ 2x + 1 = 0,\ 6 - x = 0$

$\therefore x = 4, -0.5, 6$

y intercept at $y = (-4)(1)(6) = -24$



4 Factorise $y = 3x(x^2 - 4)$

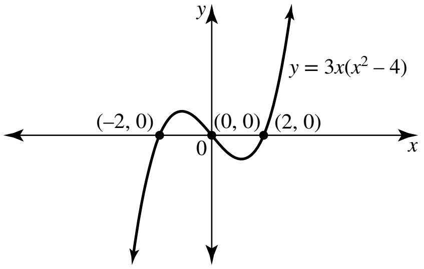
$y = 3x(x^2 - 4)$

$\therefore y = x(x-2)(x+2)$

x intercepts at $x = 0, x = 2, x = -2$

y intercept $(0,0)$

Shape: Positive x^3 shape



5 a $y = \frac{1}{9}(x-3)^2(x+6)$

x intercepts: $x = 3$ (touch), $x = -6$ (cut)

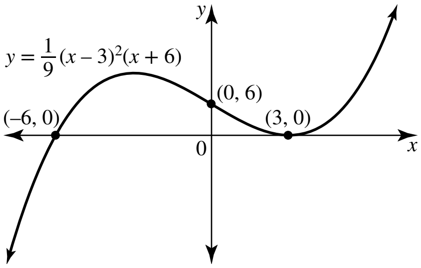
turning point at $(3,0)$

y intercept: When $x = 0$,

$y = \frac{1}{9}(-3)^2(6)$

$= 6$

$\Rightarrow (0,6)$



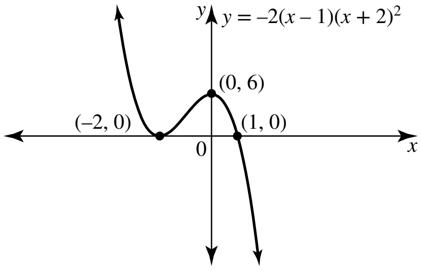
b $y = -2(x-1)(x+2)^2$

x intercepts: $x = 1$ (cut), $x = -2$ (touch)

turning point at $(-2, 0)$

y intercept: When $x = 0$,

$$y = -2(-1)(2)^2$$
$$= 8$$
$$\Rightarrow (0, 8)$$



6 $y = 0.1x(10-x)^2$

x intercepts: $x = 0$ (cut), $x = 10$ (touch)

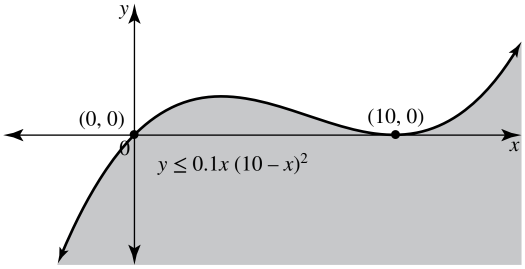
turning point at $(10, 0)$

y intercept: $(0, 0)$

Shape determined by x^3 term

$y = 0.1x(100 - 20x + x^2) \rightarrow 0.1x^3$ is x^3 term, so a positive cubic shape

Shade the region below the graph for $y \le 0.1x(10-x)^2$



7 $y = x^3 - 3x^2 - 10x + 24$

y intercept: $(0, 24)$

x intercepts: Let $P(x) = x^3 - 3x^2 - 10x + 24$

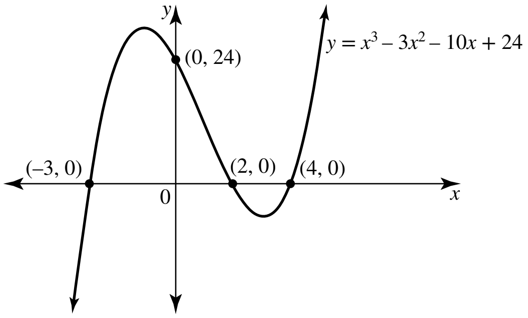
$P(1) \ne 0$

$P(2) = 8 - 12 - 20 + 24 = 0$

$\therefore (x-2)$ is a factor

$$\therefore x^3 - 3x^2 - 10x + 24$$
$$= (x-2)(x^2 + bx - 12)$$
$$= (x-2)(x^2 - x - 12)$$
$$= (x-2)(x-4)(x+3)$$

Therefore x intercepts at $x = 2, x = 4, x = -3$ (all cuts)



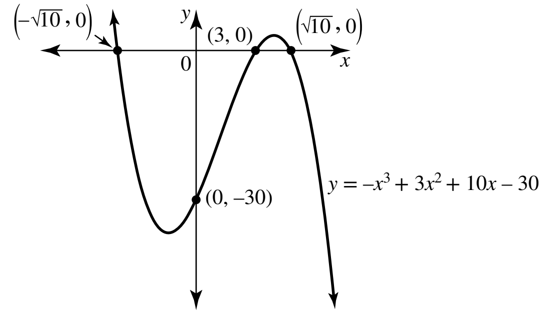
8 a $y = -x^3 + 3x^2 + 10x - 30$

y intercept: $(0, -30)$

x intercepts: Factorise by grouping '2&2'

$$y = -x^3 + 3x^2 + 10x - 30$$
$$= -x^2(x-3) + 10(x-3)$$
$$= (x-3)(-x^2 + 10)$$
$$= -(x-3)\left(x - \sqrt{10}\right)\left(x + \sqrt{10}\right)$$

Therefore x intercepts at $x = 3, x = \pm\sqrt{10}$ (all cuts)



b $y = x^3 + 3x^2 + 3x + 2$

$$\therefore y = x^3 + 3x^2 + 3x + 1 + 1$$
$$\therefore y = (x^3 + 3x^2 + 3x + 1) + 1$$
$$\therefore y = (x+1)^3 + 1$$

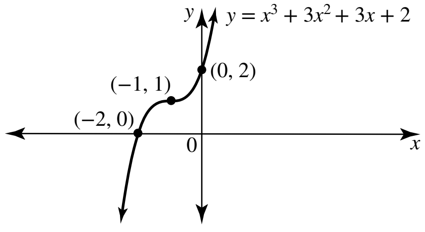
The stationary point of inflection has co-ordinates $(-1, 1)$.

y intercept $(0, 2)$

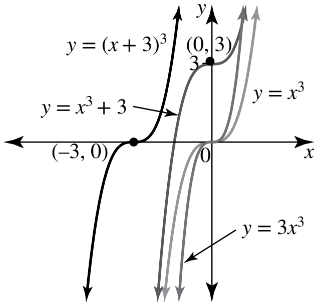
x intercept Let $y = 0$

$$\therefore (x+1)^3 + 1 = 0$$
$$\therefore (x+1)^3 = -1$$
$$\therefore x + 1 = -1$$
$$\therefore x = -2$$

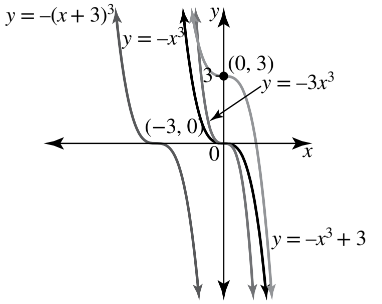
$(-2, 0)$



9 a graphs of $y = x^3$, $y = 3x^3$, $y = x^3 + 3$, $y = (x+3)^3$



b graphs of $y=-x^3$, $y=-3x^3$, $y=-x^3+3$, $y=-(x+3)^3$



10 **a** $y=(x+4)^3-27$

POI: $(-4,-27)$

y intercept: Let $x=0$

$\therefore y=(4)^3-27=37$

$(0,37)$

x intercept: Let $y=0$

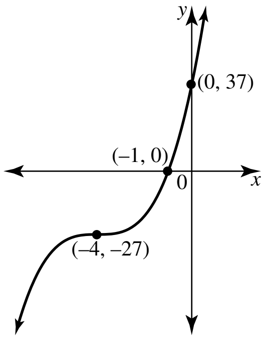
$\therefore (x+4)^3-27=0$

$\therefore (x+4)^3=27$

$\therefore x+4=3$

$\therefore x=-1$

$(-1,0)$



b $y=2(x-1)^3+10$

POI: $(1,10)$

y intercept: Let $x=0$, $y=2(-1)^3+10=8\Rightarrow(0,8)$

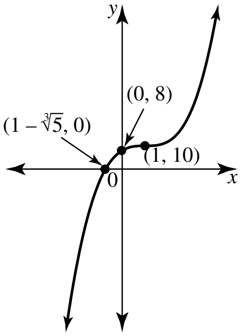
x intercept: Let $y=0$

$2(x-1)^3+10=0$

$\therefore (x-1)^3=-5$

$\therefore x=1-\sqrt[3]{5}\simeq-0.7$

$\left(1-\sqrt[3]{5},0\right)$



c $y=27+2(x-3)^3$

POI: $(3,27)$

y intercept: Let $x=0$, $y=27+2(-3)^3=-27\Rightarrow(0,-27)$

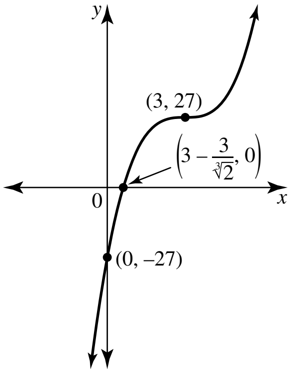
x intercept: Let $y=0$

$2(x-3)^3+27=0$

$\therefore (x-3)^3=-\frac{27}{2}$

$\therefore x=3-\frac{3}{\sqrt[3]{2}}\simeq0.6$

$\left(3-\frac{3}{\sqrt[3]{2}},0\right)$

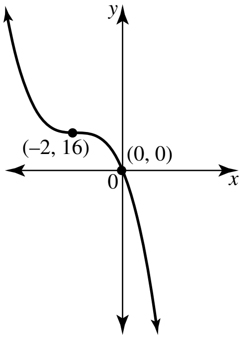


d $y=16-2(x+2)^3$

POI: $(-2,16)$

y intercept: Let $x=0$, $y=16-2(2)^3=0$

graph passes through the origin



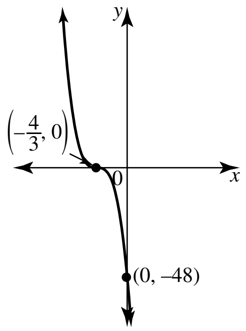
e $y=-\frac{3}{4}(3x+4)^3$

POI: $3x+4=0\Rightarrow x=-\frac{4}{3}$, so POI and x intercept is $\left(-\frac{4}{3},0\right)$

y intercept: Let $x=0$,

$y=-\frac{3}{4}(4)^3$

$=-48\Rightarrow(0,-48)$



f $y=9+\frac{x^3}{3}$

POI: $(0,9)$

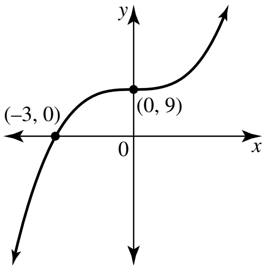
x intercept: Let $y=0$

$\frac{x^3}{3}+9=0$

$\therefore x^3=-27$

$\therefore x=-3$

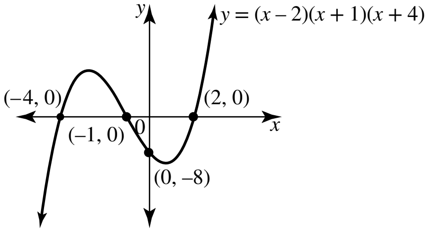
$(-3,0)$



11 a $y=(x-2)(x+1)(x+4)$

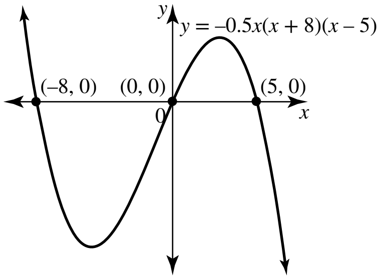
x intercepts occur at $x=2, x=-1, x=-4$

y intercept: Let $x=0$, $y=(-2)(1)(4)=-8\Rightarrow(0,-8)$



b $y=-0.5x(x+8)(x-5)$

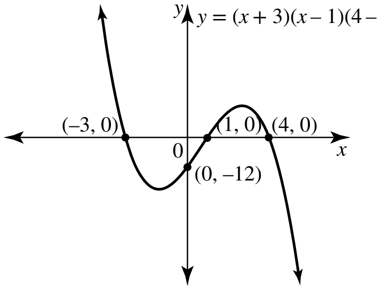
x intercepts occur at $x=0, x=-8, x=5$



c $y=(x+3)(x-1)(4-x)$

x intercepts occur when $x=-3, x=1, x=4$

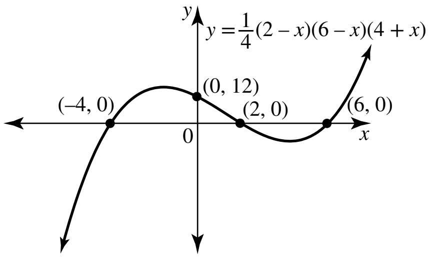
y intercept: Let $x=0$, $y=(3)(-1)(4)=-12\Rightarrow(0,-12)$



d $y=\frac{1}{4}(2-x)(6-x)(4+x)$

x intercepts occur when $x=2, x=6, x=-4$

y intercept: Let $x=0$, $y=\frac{1}{4}(2)(6)(4)=12\Rightarrow(0,12)$



e $y=0.1(2x-7)(x-10)(4x+1)$

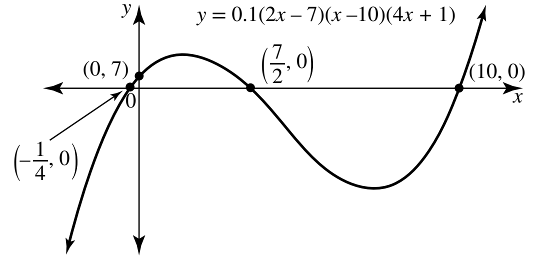
x intercepts: Let $y=0$

$2x-7=0,\ x-10=0,\ 4x+1=0$

$\therefore x=\frac{7}{2}, x=10, x=-\frac{1}{4}$

$\left(\frac{7}{2},0\right),(10,0),\left(-\frac{1}{4},0\right)$

y intercept: Let $x=0$, $y=0.1(-7)(-10)(1)=7\Rightarrow(0,7)$

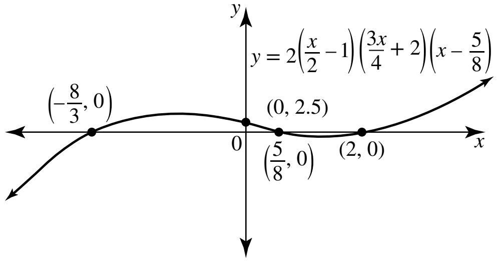


f $y=2\left(\frac{x}{2}-1\right)\left(\frac{3x}{4}+2\right)\left(x-\frac{5}{8}\right)$

x intercepts occur when $\frac{x}{2}-1=0,\ \frac{3x}{4}+2=0,\ x-\frac{5}{8}=0$

$\therefore x=2, x=-\frac{8}{3}, x=\frac{5}{8}$

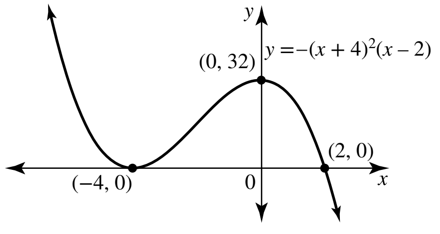
y intercept: Let $x=0$, $y=2(-1)(2)\left(-\frac{5}{8}\right)=\frac{5}{2}\Rightarrow\left(\frac{5}{2},0\right)$



12 a $y=-(x+4)^2(x-2)$

x intercepts occur when $x=-4$ (touch), $x=2$ (cut)

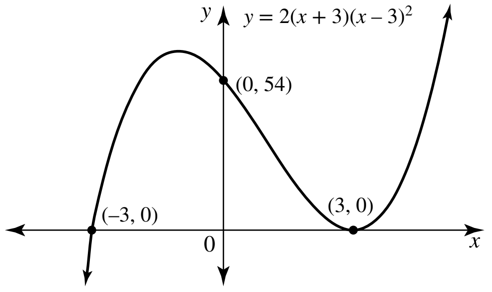
y intercept: Let $x=0$, $y=-(4)^2(-2)=32\Rightarrow(0,32)$



b $y=2(x+3)(x-3)^2$

x intercepts occur when $x=-3$ (cut), $x=3$ (touch)

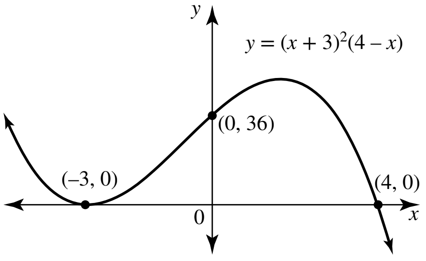
y intercept: Let $x=0$, $y=2(3)(-3)^2=54\Rightarrow(0,54)$



c $y=(x+3)^2(4-x)$

x intercepts occur when $x=-3$ (touch), $x=4$ (cut)

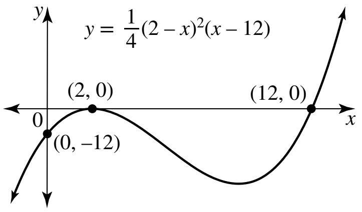
y intercept: Let $x=0$, $y=(3)^2(4)=36\Rightarrow(0,36)$



d $y = \frac{1}{4}(2-x)^2(x-12)$

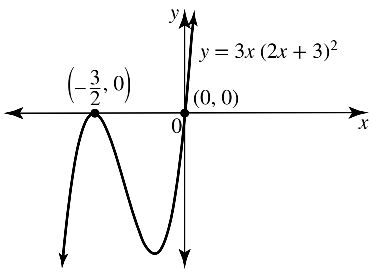
x intercepts occur when $x = 2$ (touch), $x = 12$ (cut)

y intercept: Let $x = 0$, $y = \frac{1}{4}(2)^2(-12) = -12 \Rightarrow (0, -12)$



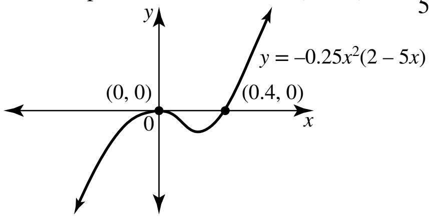
e $y = 3x(2x+3)^2$

x intercepts occur when $x = 0$ (cut), $x = -\frac{3}{2}$ (touch)



f $y = -0.25x^2(2-5x)$

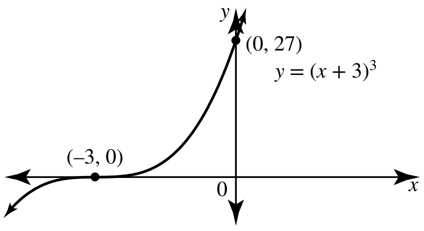
x intercepts occur when $x = 0$ (touch), $x = \frac{2}{5}$ (cut)



13 a $y = (x+3)^3$

POI and x intercept $(-3, 0)$

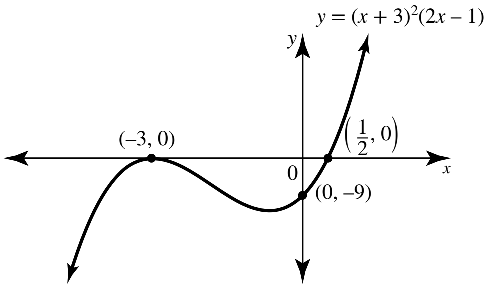
y intercept $(0, 27)$



b $y = (x+3)^2(2x-1)$

x intercepts at $x = -3$ (touch), $x = \frac{1}{2}$ (cut)

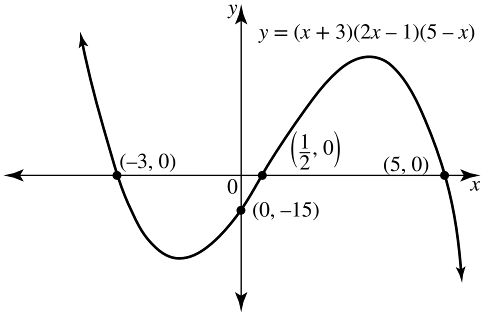
y intercept at $y = (3)^2(-1) = -9$



c $y = (x+3)(2x-1)(5-x)$

x intercepts at $x = -3, x = \frac{1}{2}, x = 5$

y intercept at $y = (3)(-1)(5) = -15$



d $2(y-1) = (1-2x)^3$

$$\therefore y - 1 = \frac{1}{2}(1-2x)^3$$

$$\therefore y = \frac{1}{2}(1-2x)^3 + 1$$

POI: When $1 - 2x = 0, x = \frac{1}{2} \Rightarrow \left(\frac{1}{2}, 1\right)$

x intercept: Let $y = 0$

$$\therefore 2(-1) = (1-2x)^3$$

$$\therefore (1-2x)^3 = -2$$

$$\therefore 1 - 2x = -\sqrt[3]{2}$$

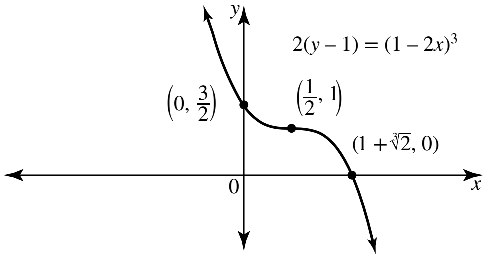
$$\therefore 1 + \sqrt[3]{2} = 2x$$

$$\therefore x = \frac{1+\sqrt[3]{2}}{2} \simeq 1.1$$

$$\left(\frac{1+\sqrt[3]{2}}{2}, 0\right)$$

y intercept: Let $x = 0$

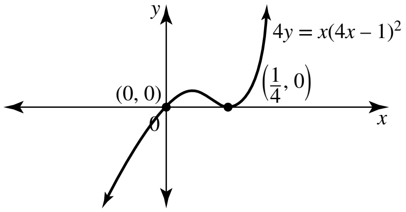
$$y = \frac{1}{2}(1)^3 + 1 = \frac{3}{2} \Rightarrow \left(0, \frac{3}{2}\right)$$



e $4y = x(4x-1)^2$

$\therefore y = \frac{1}{4}x(4x-1)^2$

x intercepts at $x = 0$ (cut), $x = \frac{1}{4}$ (touch)



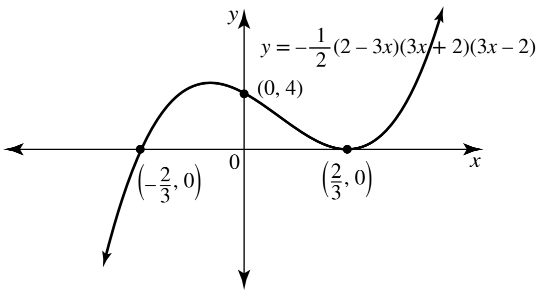
f $y = -\frac{1}{2}(2-3x)(3x+2)(3x-2)$

$\therefore y = -\frac{1}{2} \times -1(3x-2)(3x+2)(3x-2)$

$\therefore y = \frac{1}{2}(3x-2)^2(3x+2)$

x intercepts when $x = \frac{2}{3}$ (touch), $x = -\frac{2}{3}$ (cut)

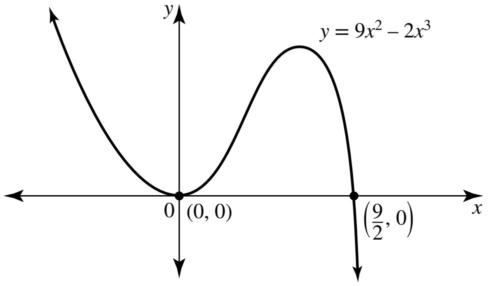
y intercept at $y = \frac{1}{2}(-2)^2(2) = 4$



14 a $y = 9x^2 - 2x^3$

$\therefore y = x^2(9-2x)$

x intercepts when $x = 0$ (touch), $x = \frac{9}{2}$ (cut)

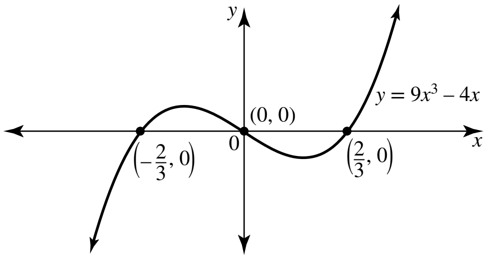


b $y = 9x^3 - 4x$

$\therefore y = x(9x^2 - 4)$

$\therefore y = x(3x-2)(3x+2)$

x intercepts when $x = 0, x = \pm\frac{2}{3}$



c $y = 9x^2 - 3x^3 + x - 3$

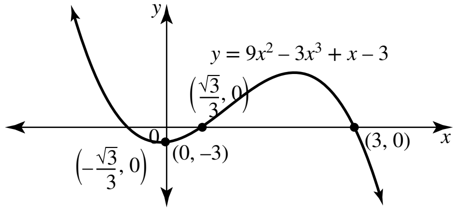
$\therefore y = 3x^2(3-x) - (3-x)$

$= (3-x)(3x^2-1)$

$\therefore y = (3-x)\left(\sqrt{3}x-1\right)\left(\sqrt{3}x+1\right)$

x intercepts when $x = 3, x = \frac{1}{\sqrt{3}}, x = -\frac{1}{\sqrt{3}}$

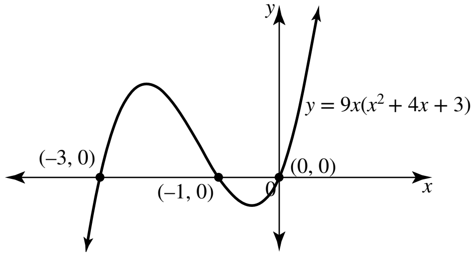
y intercept: $(0, -3)$



d $y = 9x(x^2 + 4x + 3)$

$\therefore y = 9x(x+3)(x+1)$

x intercepts when $x = 0, x = -3, x = -1$



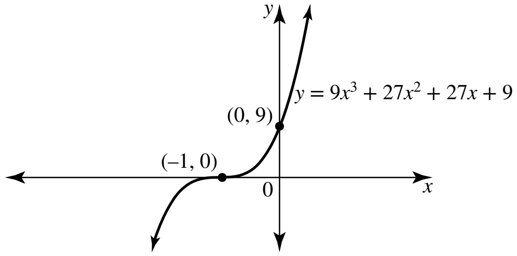
e $y = 9x^3 + 27x^2 + 27x + 9$

$\therefore y = 9(x^3 + 3x^2 + 3x + 1)$

$\therefore y = 9(x+1)^3$

POI $(-1, 0)$

y intercept $(0, 9)$



f $y = -9x^3 - 9x^2 + 9x + 9$

$\therefore y = -9(x^3 + x^2 - x - 1)$

$= -9\left[x^2(x+1) - (x+1)\right]$

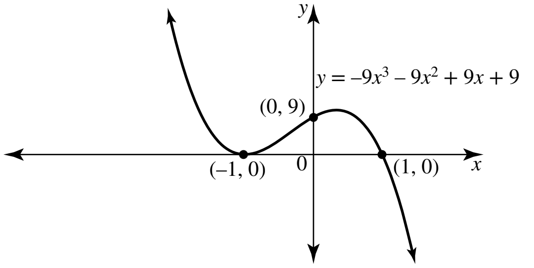
$= -9(x+1)(x^2-1)$

$= -9(x+1)(x+1)(x-1)$

$\therefore y = -9(x+1)^2(x-1)$

x intercepts when $x = -1$ (touch), $x = 1$ (cut)

y intercept $(0, 9)$



15 **a** $y = 2x^3 - 3x^2 - 17x - 12$

x intercepts when $2x^3 - 3x^2 - 17x - 12 = 0$

Let $P(x) = 2x^3 - 3x^2 - 17x - 12$

$P(-1) = -2 - 3 + 17 - 12 = 0$

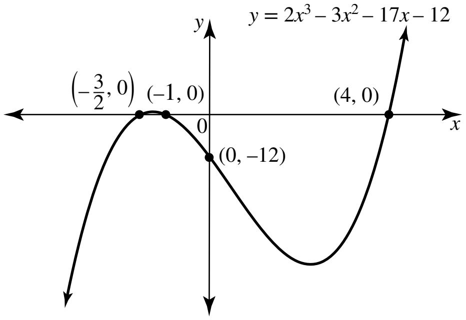
$\therefore (x+1)$ is a factor

$$\begin{aligned}\therefore 2x^3 - 3x^2 - 17x - 12 &= (x+1)(2x^2 - 5x - 12)\\ &= (x+1)(2x+3)(x-4)\end{aligned}$$

$\therefore (x+1)(2x+3)(x-4) = 0$

$$\therefore x = -1, -\frac{3}{2}, 4$$

y intercept $(0, -12)$



b $y = 6 - 55x + 57x^2 - 8x^3$

x intercepts when $P(x) = 6 - 55x + 57x^2 - 8x^3 = 0$

$P(1) = 6 - 55 + 57 - 8 = 0$

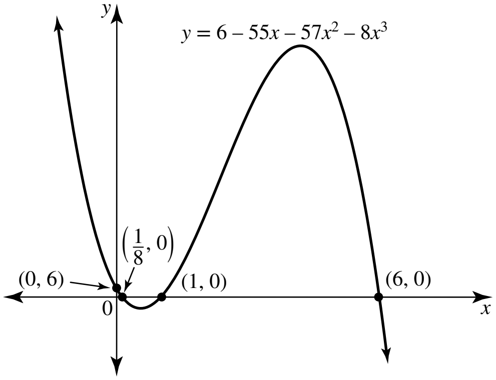
$\therefore (x-1)$ is a factor

$\therefore (x-1)(-8x^2 + 49x - 6) = 0$

$\therefore (x-1)(-8x+1)(x-6) = 0$

$$\therefore x = 1, x = \frac{1}{8}, x = 6$$

y intercept $(0, 6)$



c $y = x^3 - 17x + 4$

x intercepts when $P(x) = x^3 - 17x + 4 = 0$

$P(4) = 64 - 68 + 4 = 0$

$\therefore (x-4)$ is a factor

$\therefore (x-4)(x^2 + 4x - 1) = 0$

$\therefore (x-4)\left[(x^2 + 4x + 4) - 4 - 1\right] = 0$

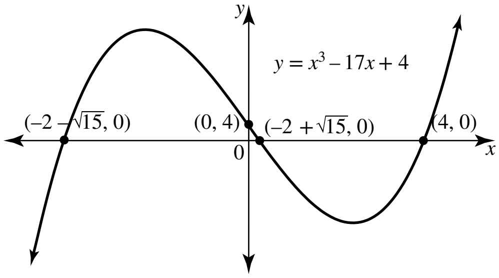
$\therefore (x-4)\left[(x+2)^2 - 5\right] = 0$

$\therefore (x-4)\left(x + 2 - \sqrt{5}\right)\left(x + 2 + \sqrt{5}\right) = 0$

$\therefore x = 4, x = -2 + \sqrt{5}, x = -2 - \sqrt{5}$

$\therefore x = 4, x \simeq 0.2, x \simeq -4.2$

y intercept $(0, 4)$



d $y = 6x^3 - 13x^2 - 59x - 18$

x intercepts when $P(x) = 6x^3 - 13x^2 - 59x - 18 = 0$

$$\begin{aligned}P(-2) &= 6 \times -8 - 13 \times 4 + 59 \times 2 - 18\\ &= -48 - 52 + 118 - 18\\ &= 0\end{aligned}$$

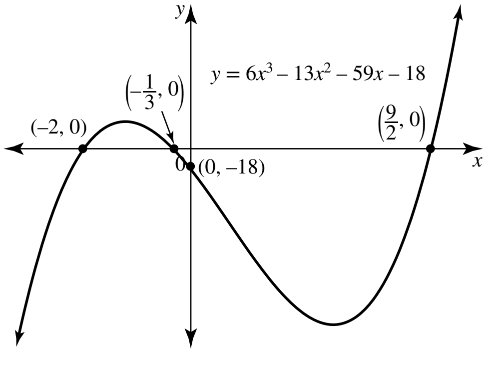
$\therefore (x+2)$ is a factor

$\therefore (x+2)(6x^2 - 25x - 9) = 0$

$\therefore (x+2)(3x+1)(2x-9) = 0$

$$\therefore x = -2, x = -\frac{1}{3}, x = \frac{9}{2}$$

y intercept $(0, -18)$



e $y = -5x^3 - 7x^2 + 10x + 14$

x intercepts when $-5x^3 - 7x^2 + 10x + 14 = 0$

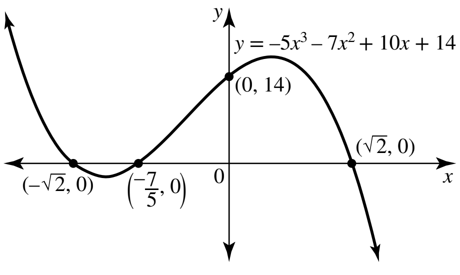
$\therefore -x^2(5x+7) + 2(5x+7) = 0$

$\therefore (5x+7)(2 - x^2) = 0$

$\therefore (5x+7)\left(\sqrt{2} - x\right)\left(\sqrt{2} + x\right) = 0$

$$\therefore x = -\frac{7}{5}, x = \sqrt{2}, x = -\sqrt{2}$$

y intercept $(0, 14)$



f $y=-\frac{1}{2}x^3+14x-24$

x intercepts when $-\frac{1}{2}x^3+14x-24=0$

Multiply by -2

$\therefore P(x)=x^3-28x+48=0$

$P(2)=8-56+48=0$

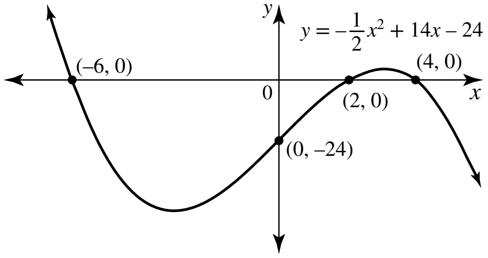
$\therefore (x-2)$ is a factor

$\therefore (x-2)(x^2+2x-24)=0$

$\therefore (x-2)(x+6)(x-4)=0$

$\therefore x=2, x=-6, x=4$

y intercept $(0,-24)$



16 $P(x)=30x^3+kx^2+1$

a Since $(3x-1)$ is a factor, $P\left(\frac{1}{3}\right)=0$

$$\therefore 30\left(\frac{1}{3}\right)^3+k\left(\frac{1}{3}\right)^2+1=0$$

$$\therefore \frac{30}{27}+\frac{k}{9}=-1$$

$$\therefore 30+3k=-27$$

$$\therefore 3k=-57$$

$$\therefore k=-19$$

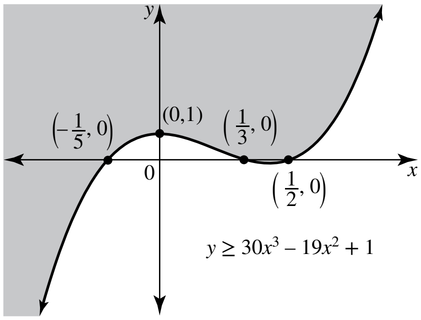
b $P(x)=30x^3-19x^2+1$

$=(3x-1)(10x^2-3x-1)$

$=(3x-1)(5x+1)(2x-1)$

c $P(x)=0 \Rightarrow x=\frac{1}{3}, x=-\frac{1}{5}, x=\frac{1}{2}$

d y intercept $(0,1)$



e If $x=-1$,

$y=30(-1)^3-19(-1)^2+1$

$=-48$

$\neq -40$

The point $(-1, 40)$ does not lie on the graph.

f shaded region above the graph, including the curve.

17 **a** $-\frac{1}{2}x^3+6x^2-24x+38 \equiv a(x-b)^3+c$

$$\therefore -\frac{1}{2}x^3+6x^2-24x+38=a(x^3-3x^2b+3xb^2-b^3)+c$$
$$=ax^3-3abx^2+3ab^2x-ab^3+c$$

Equating coefficients of like terms

$x^3: -\frac{1}{2}=a$

$x^2: 6=-3ab$

$\therefore 6=-3\times\left(-\frac{1}{2}\right)b$

$\therefore b=4$

constant: $38=-ab^3+c$

$\therefore 38=-\left(-\frac{1}{2}\right)(4)^3+c$

$\therefore 38=32+c$

$\therefore c=6$

Check coefficient of $x: -24=3ab^2$

$3ab^2=3\left(-\frac{1}{2}\right)(4)^2$

$=3\times -8$

$=-24$

as required

$\therefore -\frac{1}{2}x^3+6x^2-24x+38 \equiv -\frac{1}{2}(x-4)^3+6$

b $y=-\frac{1}{2}x^3+6x^2-24x+38 \Rightarrow y=-\frac{1}{2}(x-4)^3+6$

POI: $(4,6)$

y intercept $(0,38)$

x intercept: Let $y=0$

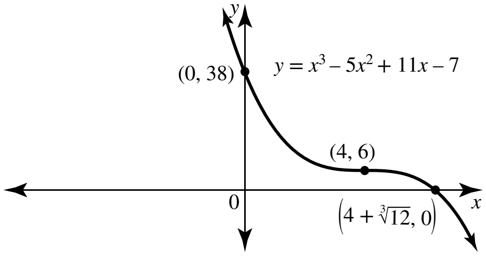
$\therefore 0=-\frac{1}{2}(x-4)^3+6$

$\therefore \frac{1}{2}(x-4)^3=6$

$\therefore (x-4)^3=12$

$\therefore x=4+\sqrt[3]{12}\simeq 6.3$

$\left(4+\sqrt[3]{12},0\right)$



18 $y=x^3-5x^2+11x-7$

a Let $P(x)=x^3-5x^2+11x-7$

$P(1)=1-5+11-7=0$

$\therefore (x-1)$ is a factor

$\therefore x^3-5x^2+11x-7=(x-1)(x^2-4x+7)$

Consider x^2-4x+7

$\Delta=b^2-4ac,\ \ a=1, b=-4, c=7$

$=16-28$

$=-12$

$\therefore \Delta<0$

No real linear factors for the quadratic term

$\therefore y=x^3-5x^2+11x-7$ has only one linear factor and therefore the graph has only one x intercept at $x=1$.

b Let $x^3-5x^2+11x-7 \equiv a(x-b)^3+c$

$\therefore x^3-5x^2+11x-7=ax^3-3ax^2b+3ab^2x-ab^3+c$

Equating coefficients of like terms

$x^3: 1=a$

$x^2: -5=-3ab$

$\therefore -5=-3(1)b$

$\therefore b=\frac{5}{3}$

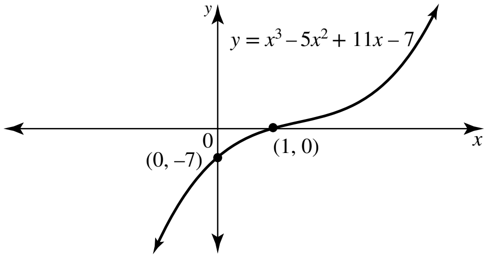
$$x:11=3ab^2$$

$$\begin{aligned}3ab^2&=3(1)\left(\frac{5}{3}\right)^2\\&=\frac{25}{3}\\&\neq 11\end{aligned}$$

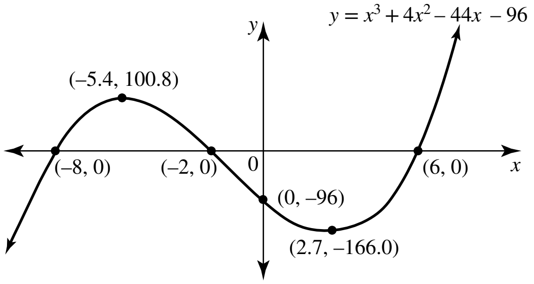
Hence, $x^3-5x^2+11x-7$ cannot be expressed in the form $a(x-b)^3+c$.

c The coefficient of x^3 is positive; therefore, as $x\to\infty$ the y values of the graph also $\to\infty$.

d No POI, no turning points.
y intercept $(0,-7)$
one x intercept $(1,0)$
As $x\to\pm\infty, y\to\pm\infty$

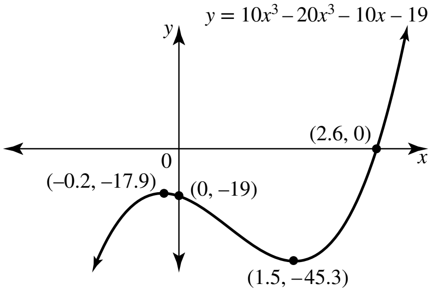


19 a $y=x^3+4x^2-44x-96$

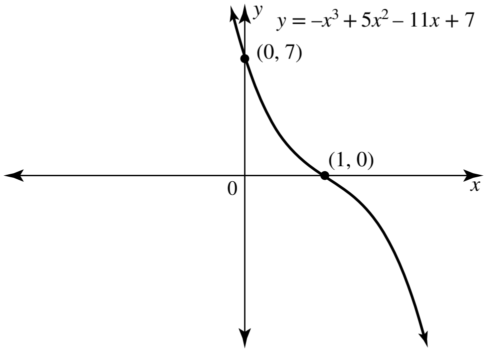


b The maximum turning point $(-5.4,100.8)$ lies between the x intercepts of $(-8,0)$ and $(-2,0)$. The midpoint of the interval where $x\in[-8,-2]$ is $x=-5$ but for the turning point, $x_{tp}=-5.4$. The turning point is not symmetrically placed between the two intercepts.
Similarly, for the minimum turning point $(2.7,-166.0)$, $2.7\neq\frac{-2+6}{2}$, that is, $2.7\neq 2$.
Neither turning point is placed halfway between its adjoining x intercepts.

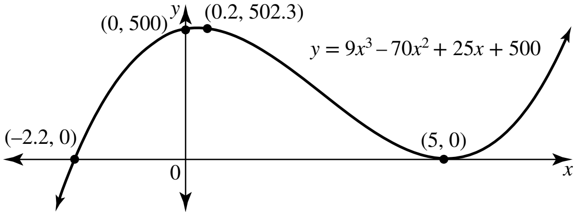
20 a $y=10x^3-20x^2-10x-19$



b $y=-x^3+5x^2-11x+7$



c $y=9x^3-70x^2+25x+500$



Exercise 4.5 — Equations of cubic polynomials

1 a Inflection at $(3,-7)$
Therefore equation is $y=a(x-3)^3-7$
Substitute $(10,0)$

$$\therefore 0=a(7)^3-7$$
$$\therefore a=\frac{7}{7^3}$$
$$\therefore a=\frac{1}{49}$$

Equation is $y=\frac{1}{49}(x-3)^3-7$

b Information: cuts at x intercepts at $x=-5,0,4$
Therefore equation is $y=a(x+5)(x)(x-4)$
$\therefore y=ax(x+5)(x-4)$
Substitute $(2,-7)$
$-7=a(2)(7)(-2)$
$\therefore a=\frac{1}{4}$
Equation is $y=\frac{1}{4}x(x+5)(x-4)$

c Information: x intercepts at $x=-2,3$ turning point at $x=-2$
Therefore equation is $y=a(x+2)^2(x-3)$
Substitute $(0,12)$
$\therefore 12=a(2)^2(-3)$
$\therefore 12=-12a$
$\therefore a=-1$
Equation is $y=-(x+2)^2(x-3)$

2 Let $y=ax^3+bx^2+cx+d$
$(0,3),(1,4),(-1,8),(-2,7)$
$(0,3)\Rightarrow 3=d$
$\therefore y=ax^3+bx^2+cx+3$
Substitute other points to form simultaneous equations
$(1,4)\Rightarrow 4=a+b+c+3$
$\therefore a+b+c=1$...............(1)
$(-1,8)\Rightarrow 8=-a+b-c+3$
$\therefore -a+b-c=5$.............(2)
$(-2,7)\Rightarrow 7=-8a+4b-2c+3$
$\therefore -8a+4b-2c=4$
$\therefore 4a-2b+c=-2$...........(3)

$(1)+(2)$

$2b=6$

$\therefore b=3$

$(2)+(3)$

$3a-b=3$

$\therefore 3a-3=3$

$\therefore a=2$

Substitute $a=2, b=3$ in (1)

$2+3+c=1$

$\therefore c=-4$

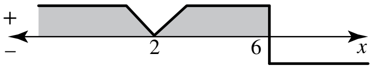
Equation is $y=2x^3+3x^2-4x+3$

3 **a** $(x-2)^2(6-x)>0$

Zeros: $x=2$ (multiplicity 2), $x=6$

Negative coefficient of x^3

Sign diagram



Answer $x<6, x\neq 2$

b $4x\le x^3$

$\therefore 4x-x^3\le 0$

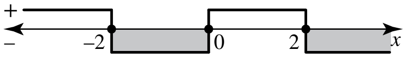
$\therefore x(4-x^2)\le 0$

$\therefore x(2-x)(2+x)\le 0$

Zeros: $x=0, x=2, x=-2$

Negative coefficient of x^3

Sign diagram



Answer $\{x:-2\le x\le 0\}\cup\{x:x\ge 2\}$

c $2(x+4)^3-16<0$

$\therefore (x+4)^3<8$

$\therefore x+4<2$

$\therefore x<-2$

4 $\{x: 3x^3+7>7x^2+3x\}$

$3x^3+7>7x^2+3x$

$\therefore 3x^3-7x^2-3x+7>0$

$\therefore x^2(3x-7)-(3x-7)>0$

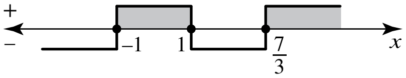
$\therefore (3x-7)(x^2-1)>0$

$\therefore (3x-7)(x-1)(x+1)>0$

Zeros: $x=\frac{7}{3}, x=1, x=-1$

Positive cubic

Sign diagram



Answer $\{x:-1<x<1\}\cup\left\{x:x>\frac{7}{3}\right\}$

5 Points of intersection of $y=(x+2)(x-1)^2$ and $y=-3x$ are found by solving

$(x+2)(x-1)^2=-3x$

Expanding,

$(x+2)(x^2-2x+1)=-3x$

$\therefore x^3-3x+2=-3x$

$\therefore x^3+2=0$

$\therefore x^3=-2$

$\therefore x=-\sqrt[3]{2}$

$\therefore y=3\sqrt[3]{2}$

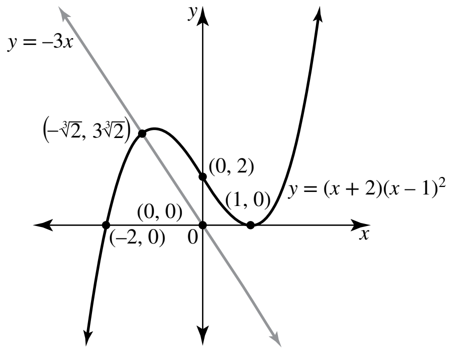
Point of intersection is $\left(-\sqrt[3]{2}, 3\sqrt[3]{2}\right)$

Graphs: Cubic $y=(x+2)(x-1)^2$

x intercepts at $x=-2$ (cut), $x=1$ (touch)

y intercept $(0,2)$

Linear $y=-3x$ passes through $(0,0),(-1,3)$



For $-3x<(x+2)(x-1)^2$, require the line to lie below the cubic.

Answer is $x>-\sqrt[3]{2}$

6 At intersection of $y=4-x^2$ and $y=4x-x^3$

$4-x^2=4x-x^3$

$\therefore x^3-x^2-4x+4=0$

$\therefore x^2(x-1)-4(x-1)=0$

$\therefore (x-1)(x^2-4)=0$

$\therefore (x-1)(x-2)(x+2)=0$

$\therefore x=1, x=2, x=-2$

Substitute in $y=4-x^2$

Points of intersection are $(1,3),(2,0),(-2,0)$

Graphs: Cubic $y=4x-x^3$

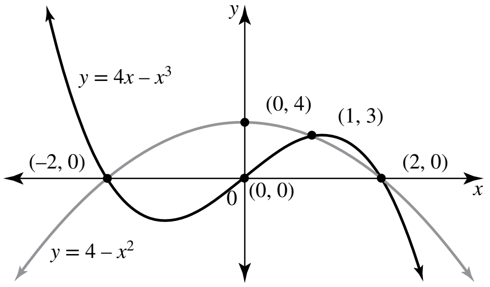
$\therefore y=x(4-x^2)$

$\therefore y=x(x-2)(x+2)$

x intercepts at $x=0, x=\pm 2$

Quadratic $y=4-x^2$

maximum turning point $(0,4)$, x intercepts at $x=\pm 2$



7 **a** x intercepts occur at $x=-8, x=-4, x=-1$

Let equation of graph be $y=a(x+8)(x+4)(x+1)$

Substitute the y intercept $(0,16)$

$\therefore 16=a(8)(4)(1)$

$\therefore 16=32a$

$\therefore a=\frac{1}{2}$

The equation is $y=\frac{1}{2}(x+8)(x+4)(x+1)$

b x intercepts occur at $x=0$ (touch) and $x=5$ (cut)

Let equation of graph be $y=ax^2(x-5)$

Substitute the given point $(2,24)$

$\therefore 24=a(2)^2(-3)$

$\therefore 24=-12a$

$\therefore a=-2$

The equation is $y=-2x^2(x-5)$

c Point of inflection $(1,-3)$

Let equation of graph be $y=a(x-1)^3-3$

Substitute the origin $(0,0)$

$\therefore 0=a(-1)^3-3$

$\therefore 0=-a-3$

$\therefore a=-3$

The equation is $y=-3(x-1)^3-3$

d x intercepts occur at $x=1$ (cut) and $x=5$ (touch)

Let equation of graph be $y=a(x-1)(x-5)^2$

Substitute the y intercept $(0,-20)$

$\therefore -20=a(-1)(-5)^2$

$\therefore -20=-25a$

$\therefore a=\frac{4}{5}$

The equation is $y=\frac{4}{5}(x-1)(x-5)^2$

8 **a** POI $(-6,-7)$

The equation is $y=-2(x+6)^3-7$

b Under the translations, $y=x^3\rightarrow y=(x-2)^3-4$

y intercept: Let $x=0$

$y=(-2)^3-4$

$=-12$

y intercept is $(0,-12)$

c POI $(-5,2)$

Let the equation be $y=a(x+5)^3+2$

Substitute the point $(0,-23)$

$\therefore -23=a(5)^3+2$

$\therefore -23=125a+2$

$\therefore 125a=-25$

$\therefore a=-\frac{1}{5}$

The equation is $y=-\frac{1}{5}(x+5)^3+2$

x intercept: Let $y=0$

$\therefore 0=-\frac{1}{5}(x+5)^3+2$

$\therefore \frac{1}{5}(x+5)^3=2$

$\therefore (x+5)^3=10$

$\therefore x=-5+\sqrt[3]{10}$

x intercept is $\left(-5+\sqrt[3]{10},0\right)$

d $y=ax^3+b$

Point $(1,3)\Rightarrow 3=a+b\ldots.(1)$

Point $(-2,39)\Rightarrow 39=-8a+b\ldots.(2)$

equation (1) subtract equation (2)

$\therefore -36=9a$

$\therefore a=-4$

Substitute $a=-4$ in equation (1)

$\therefore 3=-4+b$

$\therefore b=7$

The equation is $y=-4x^3+7$ so the point of inflection is $(0,7)$

9 **a** x intercepts occur at $x=a$ (touch) and $x=b$ (cut)

Monic polynomial $\Rightarrow$ coefficient of x^3 is 1. After reflection in the x axis, the coefficient of x^3 becomes -1. The shape shows the coefficient is negative.

The equation is $y=-(x-a)^2(x-b)$

b $\{x:P(x)\geq 0\}$ is $\{x:x\leq b\}$

c For both x intercepts to be negative, graph needs to be horizontally translated more than b units to the left.

d For both x intercepts to be positive, graph needs to be horizontally translated more than $|a|$ units to the right; that is, more than $-a$ units to the right.

10 $y=x^3+ax^2+bx+9$

a Turning point and x intercept $(3,0),\Rightarrow(x-3)^2$ is a factor.

b $x^3+ax^2+bx+9=(x-3)^2(cx+d)$

$=(x^2-6x+9)(cx+d)$

$=(x^2-6x+9)(x+1)$

$\therefore x^3+ax^2+bx+9=(x-3)^2(x+1)$

The other x intercept is $(-1,0)$.

c Expanding,

$x^3+ax^2+bx+9=(x-3)^2(x+1)$

$=(x^2-6x+9)(x+1)$

$=x^3+x^2-6x^2-6x+9x+9$

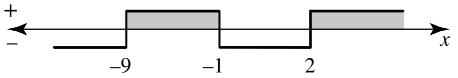
$=x^3-5x^2+3x+9$

Equating coefficients of $x^2:a=-5$

Equating coefficients of $x:b=3$

11 **a** $(x-2)(x+1)(x+9)\geq 0$

Zeros are $x=2,x=-1,x=-9$; coefficient of x^3 is positive.

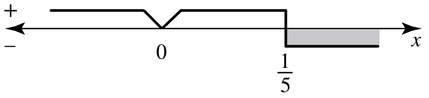


$\therefore -9\leq x\leq -1$ or $x\geq 2$

b $x^2-5x^3<0$

$\therefore x^2(1-5x)<0$

Zeros are $x=0$ (multiplicity 2), $x=\frac{1}{5}$; coefficient of x^3 is negative



Answer: $x>\frac{1}{5}$

c $8(x-2)^3-1>0$

$\therefore (x-2)^3>\frac{1}{8}$

$\therefore x-2>\frac{1}{2}$

$\therefore x>2.5$

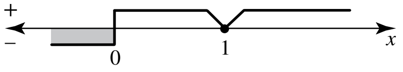
d $x^3+x\leq 2x^2$

$\therefore x^3-2x^2+x\leq 0$

$\therefore x(x^2-2x+1)\leq 0$

$\therefore x(x-1)^2\leq 0$

Zeros: $x=0$ (cut), $x=1$ (touch); coefficient of x^3 is positive



Answer: $x\leq 0$ or $x=1$

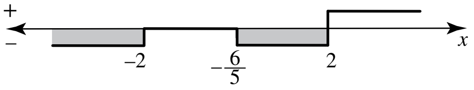
e $5x^3+6x^2-20x-24<0$

$\therefore x^2(5x+6)-4(5x+6)<0$

$\therefore (5x+6)(x^2-4)<0$

$\therefore (5x+6)(x-2)(x+2)<0$

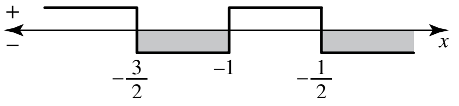
Zeros are $x=-\frac{6}{5},x=2,x=-2$; coefficient of x^3 is positive



Answer: $x < -2$ or $-\frac{6}{5} < x < 2$

f
$$2(x+1)-8(x+1)^3 < 0$$
$$\therefore 2(x+1)\left[1-4(x+1)^2\right] < 0$$
$$\therefore 2(x+1)\left(1-2(x+1)\right)\left(1+2(x+1)\right) < 0$$
$$\therefore 2(x+1)(-2x-1)(2x+3) < 0$$

Zero are $x=-1, x=-\frac{1}{2}, x=-\frac{3}{2}$; coefficient of x^3 is negative



Answer: $-\frac{3}{2} < x < -1$ or $x > -\frac{1}{2}$

12 a $y = 2x^3$ and $y = x^2$

At intersection, $2x^3 = x^2$

$\therefore 2x^3 - x^2 = 0$

$\therefore x^2(2x-1) = 0$

$\therefore x = 0,\ x = \frac{1}{2}$

Substituting $x = 0$ in $y = x^2$ gives $y = 0$

Substituting $x = \frac{1}{2}$ in $y = x^2$ gives $y = \frac{1}{4}$

The points of intersection are $(0,0)$ and $\left(\frac{1}{2},\frac{1}{4}\right)$

b $y = 2x^3$ and $y = x-1$

At intersection, $2x^3 = x-1$

$\therefore 2x^3 - x + 1 = 0$

Let $P(x) = 2x^3 - x + 1$

$P(-1) = -2+1+1 = 0$

$\therefore (x+1)$ is a factor

$\therefore 2x^3 - x + 1 = (x+1)(2x^2-2x+1)$

Consider quadratic factor $2x^2 - 2x + 1$

$$\begin{aligned}\Delta &= (-2)^2 - 4(2)(1)\\ &= 4-8\\ &< 0\end{aligned}$$

There are no real linear factors of this quadratic.

$$\therefore 2x^3 - x + 1 = 0 \Rightarrow (x+1)(2x^2-2x+1) = 0$$
$$\therefore x = -1$$

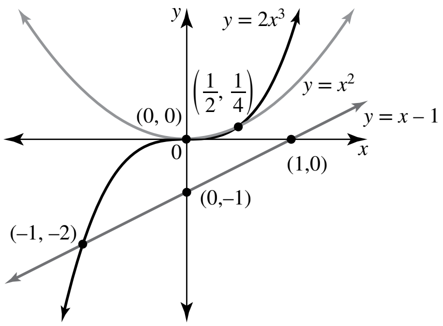
Substituting $x = -1$ in $y = x-1$ gives $y = -2$

Point of intersection is $(-1,-2)$

c Cubic graph of $y = 2x^3$ has POI $(0,0)$ and passes through $\left(\frac{1}{2},\frac{1}{4}\right)$ and $(-1,-2)$

Parabola graph of $y = x^2$ has minimum turning point at $(0,0)$ and passes through $\left(\frac{1}{2},\frac{1}{4}\right)$

Linear graph of $y = x-1$ has x intercept $(1,0)$, y intercept $(0,-1)$ and passes through $(-1,-2)$



d $2x^3 - x^2 \le 0$

$\therefore x^2(2x-1) \le 0$

Since $x^2 \ge 0$, either $x = 0$ or $(2x-1) \le 0$

$\therefore x = 0$ or $x \le \frac{1}{2}$

$\therefore x \le \frac{1}{2}$

(Alternatively, draw a sign diagram to solve the inequation).

Since $2x^3 - x^2 \le 0 \Rightarrow 2x^3 \le x^2$, the inequation can be solved by considering where the graph of $y = 2x^3$ lies on or below the graph of $y = x^2$. From the diagram this occurs when $x \le \frac{1}{2}$, so $x \le \frac{1}{2}$ is the solution to the inequation.

13 a $x^3 + 2x - 5 = 0$

Rearranging, $x^3 = -2x + 5$

The solutions to the equation are the x co-ordinates of the points of intersection of the graphs of $y = x^3$ and $y = -2x + 5$.

Since the line $y = -2x + 5$ has a negative gradient it will intersect the graph of the cubic $y = x^3$ exactly once.

The equation $x^3 + 2x - 5 = 0$ has one solution.

b $x^3 + 3x^2 - 4x = 0$

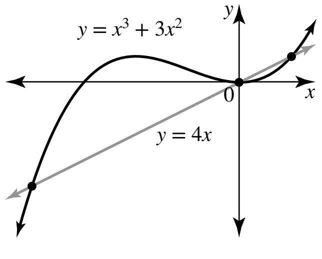
Rearranging, the equation can be written as $x^3 + 3x^2 = 4x$.

The solutions to the equation are the x co-ordinates of the points of intersection of the graphs of $y = x^3 + 3x^2$ and $y = 4x$.

The cubic graph: $y = x^3 + 3x^2 \Rightarrow y = x^2(x+3)$ touches the x axis at $x = 0$ and cuts the x axis at $x = -3$.

The linear graph: $y = 4x$ passes through the origin with a positive gradient. A second point on the graph is $(1,4)$.

For the cubic graph, when $x = 1, y = 4$ so both graphs pass through the origin and the point $(1,4)$.

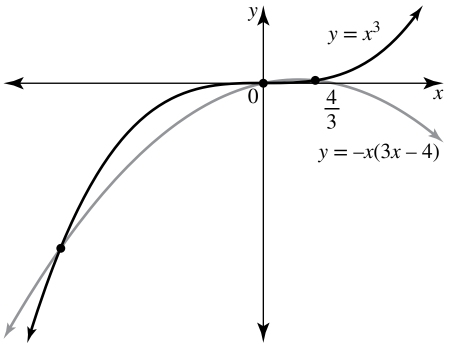


As there are 3 points of intersection, there are 3 solutions to the equation $x^3 + 3x^2 - 4x = 0$.

c $x^3 + 3x^2 - 4x = 0$

One way to interpret the equation as the intersection of a cubic and a quadratic is to rearrange the equation to the form $x^3 = -3x^2 + 4x$.

Graphing $y = x^3$ and $y = -3x^2 + 4x = -x(3x-4)$ shows there are 3 points of intersection.



There are 3 solutions to the equation $x^3 + 3x^2 - 4x = 0$.

d $x^3 + 3x^2 - 4x = 0$

$\therefore x(x^2 + 3x - 4) = 0$

$\therefore x(x + 4)(x - 1) = 0$

$\therefore x = 0, x = -4, x = 1$

14 a Given information about the cubic polynomial are the three points $(1,0), (2,0), (0,12)$. However, the equation $y = ax^3 + bx^2 + cx + d$ contains 4 unknowns, so three points are insufficient to completely determine the equation.

b $y = ax^3 + bx^2 + cx + d$

Substitute the given y intercept $(0,12)$

$\therefore 12 = d$

$\therefore y = ax^3 + bx^2 + cx + 12$

Substitute the given x intercepts

$(1,0) \Rightarrow 0 = a + b + c + 12$

$\therefore a + b + c = -12....(1)$

$(2,0) \Rightarrow 0 = a(8) + b(4) + c(2) + 12$

$\therefore 8a + 4b + 2c = -12$

$\therefore 4a + 2b + c = -6....(2)$

Subtract equation (1) from equation (2)

$\therefore 3a + b = 6$

$\therefore b = 6 - 3a$

Substitute $b = 6 - 3a$ in equation (1)

$\therefore a + 6 - 3a + c = -12$

$\therefore c = 2a - 18$

Hence, $y = ax^3 + (6 - 3a)x^2 + (2a - 18)x + 12$, as required.

c First curve: $a = 1 \Rightarrow y = x^3 + 3x^2 - 16x + 12$

As $(1,0), (2,0)$ are x intercepts, $(x - 1)$ and $(x - 2)$ are factors.

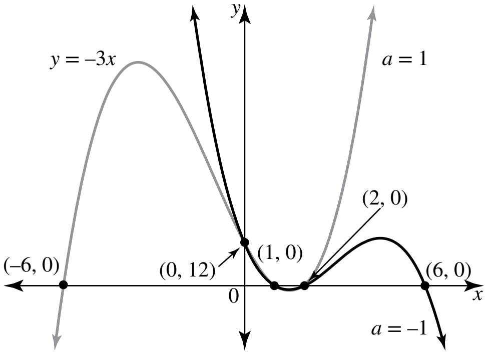
As $(x - 1)(x - 2) = x^2 - 3x + 2$, then $x^3 + 3x^2 - 16x + 12 = (x^2 - 3x + 2)(x + 6)$. The third x intercept is $(-6,0)$.

Second curve: $a = -1 \Rightarrow y = -x^3 + 9x^2 - 20x + 12$

$(1,0), (2,0)$ are also x intercepts for the second curve, so it has $(x - 1)$ and $(x - 2)$ as factors.

$\therefore -x^3 + 9x^2 - 20x + 12 = (x^2 - 3x + 2)(-x + 6)$

The third x intercept is $(6,0)$.



d If the coefficient of x^2 is 15, then $6 - 3a = 15$

$\therefore 3a = -9$

$\therefore a = -3$

$\therefore y = -3x^3 + 15x^2 - 24x + 12$

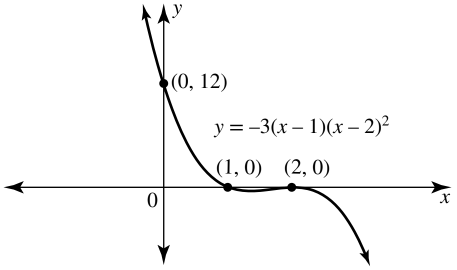
$= (x^2 - 3x + 2)(-3x + 6)$

$= -3(x^2 - 3x + 2)(x - 2)$

$= -3(x - 1)(x - 2)(x - 2)$

$\therefore y = -3(x - 1)(x - 2)^2$

x intercepts are $(1,0)$ (cut) and $(2,0)$ (touch).



15 $y = (x + a)^3 + b$

a Substitute the given points into the equation.

$(0,0) \Rightarrow 0 = a^3 + b....(1)$

$(1,7) \Rightarrow 7 = (1 + a)^3 + b....(2)$

$(2,26) \Rightarrow 26 = (2 + a)^3 + b....(3)$

From equation (1), $b = -a^3$. Substitute this in each of the other two equations.

Equation (2):

$7 = (1 + a)^3 - a^3$

$\therefore 7 = 1 + 3a + 3a^2 + a^3 - a^3$

$\therefore 6 = 3a + 3a^2$

$\therefore a^2 + a - 2 = 0$

$\therefore (a + 2)(a - 1) = 0$

$\therefore a = -2, a = 1$

Equation (3):

$26 = (2 + a)^3 - a^3$

$\therefore 26 = 8 + 12a + 6a^2 + a^3 - a^3$

$\therefore 18 = 12a + 6a^2$

$\therefore a^2 + 2a - 3 = 0$

$\therefore (a + 3)(a - 1) = 0$

$\therefore a = -3, a = 1$

The consistent value for a is $a = 1$

If $a = 1$ then $b = -a^3 = -1$

Answer: $a = 1$ and $b = -1$

b The graph has the equation $y = (x + 1)^3 - 1$.

At the intersection with the line $y = x$,

$(x + 1)^3 - 1 = x$

$\therefore x^3 + 3x^2 + 3x + 1 - 1 - x = 0$

$\therefore x^3 + 3x^2 + 2x = 0$

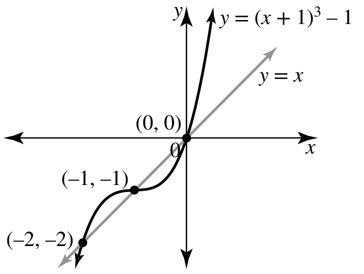
$\therefore x(x^2 + 3x + 2) = 0$

$\therefore x(x + 1)(x + 2) = 0$

$\therefore x = 0, x = -1, x = -2$

Since points lie on $y = x$, the points of intersection are $(0,0), (-1,-1), (-2,-2)$.

c $y=(x+1)^3-1$ has POI $(-1,-1)$ and its graph and the graph of $y=x$ must intersect at $(0,0),(-1,-1),(-2,-2)$.



d $x^3+3x^2+2x>0$

From part **b**, when the graph of $y=(x+1)^3-1$ intersects with the line $y=x$, $(x+1)^3-1=x \Rightarrow x^3+3x^2+2x=0$.

Therefore $x^3+3x^2+2x>0 \Rightarrow (x+1)^3-1>x \Rightarrow$ the graph of $y=(x+1)^3-1$ lies above the line $y=x$.

Using the diagram in part **c**, this occurs for $-2<x<-1$ and for $x>0$.

The solution set answer to the inequation is $\{x:-2<x<-1\}\cup\{x:x>0\}$.

16 a At the intersection of $y=x^3$ with $y=3x+2$,

$$x^3=3x+2$$

$\therefore x^3-3x-2=0$

Let $P(x)=x^3-3x-2$

$P(-1)=-1+3-2=0$

$\therefore (x+1)$ is a factor

$$\begin{aligned}x^3-3x-2&=(x+1)(x^2-x-2)\\&=(x+1)(x+1)(x-2)\\&=(x+1)^2(x-2)\end{aligned}$$

$\therefore x^3-3x-2=0 \Rightarrow (x+1)^2(x-2)=0$

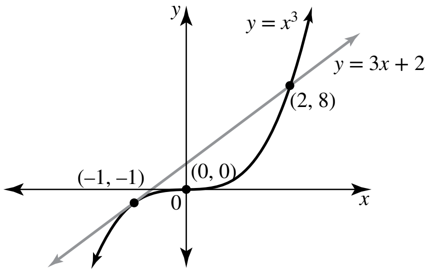
$\therefore x=-1, x=2$

Since $x=-1$ is a root of multiplicity 2, the two graphs touch at $x=-1$.

When $x=-1$, $y=x^3=-1$. The line is a tangent to the cubic curve at the point $(-1,-1)$.

b When $x=2$, $y=x^3=8$, The line cuts the curve at $(2,8)$.

c



d $y=mx+2$ is a line through $(0,2)$ as is $y=3x+2$. Hence, if $m=3$ there will be two points of intersection.

For positive gradient, if the line through $(0,2)$ is steeper than $y=3x+2$, it will no longer be a tangent to the cubic graph, so there will be three points of intersection. However, if the line is less steep than $y=3x+2$, there would only be one point of intersection.

For negative gradient, the line through $(0,2)$ would only have one point of intersection with the cubic graph.

For zero gradient, the line through $(0,2)$ is horizontal and would only intersect the cubic graph once.

Answer: One point of intersection if $m<3$, two points of intersection if $m=3$ and three points of intersection if $m>3$.

17 Define p, variable x, Expression ax^3+bx^2+cx+d

Solve

$$\left.\begin{array}{r}p(-2)=53\\p(-1)=-6\\p(2)=33\\p(4)=-121\end{array}\right|_{a,b,c,d}$$

This gives $a=-6, b=4, c=19, d=-5$

The equation is $y=-6x^3+12x^2+19x-5$

18 a Use the Graph&Tab menu to draw the two graphs; then obtain the points of intersection from Analysis → G-Solve → Intersect.

The graphs of $y=(x+1)^3$ and $y=4x+3$ intersect at $(-3.11,-9.46),(-0.75,0.02),(0.87,6.44)$.

b At intersection, $(x+1)^3=4x+3$ so the x co-ordinates of the points of intersection are the solutions to this equation.

Tap Interactive → Transformation → Expand (or expand by hand) to obtain the equation in the form $x^3+3x^2-x-2=0$.

c The x co-ordinates of the points of intersection of $y=(x+1)^3$ and $y=4x+3$ represent the x intercepts of the graph of $y=x^3+3x^2-x-2$.

Exercise 4.6 — Cubic models and applications

1 **a** Sum of edges is 6 m

$8x+4h=6$

$\therefore 2h=3-4x$

$\therefore h=\frac{3-4x}{2}$

b $V=x^2h$

$\therefore V=x^2\left(\frac{3-4x}{2}\right)$

$\therefore V=\frac{3x^2-4x^3}{2}$

$\therefore V=1.5x^2-2x^3$

c $x\geq 0, h\geq 0$

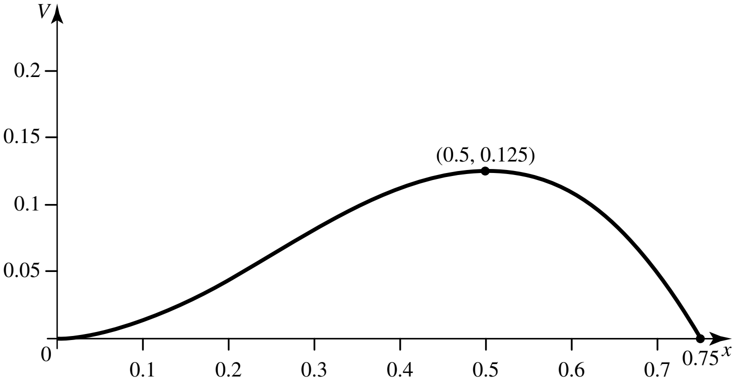
$\therefore \frac{3-4x}{2}\geq 0$

$\therefore 3-4x\geq 0$

$\therefore x\leq \frac{3}{4}$

Therefore restriction on domain is $0\leq x\leq \frac{3}{4}$

d $V=x^2(1.5-2x)$. Graph has x intercepts at $x=0$ (touch), $x=0.75$ (cut) shape of a negative cubic



e Greatest volume occurs when $x=0.5$ and therefore $h=\frac{1}{2}$

Therefore the container with greatest volume is a cube of edge 0.5 m

2 **a** $l=20-2x, w=12-2x$ and $h=x$

Since volume is $V=lwh$, $V=(20-2x)(12-2x)x$

b $x\geq 0$

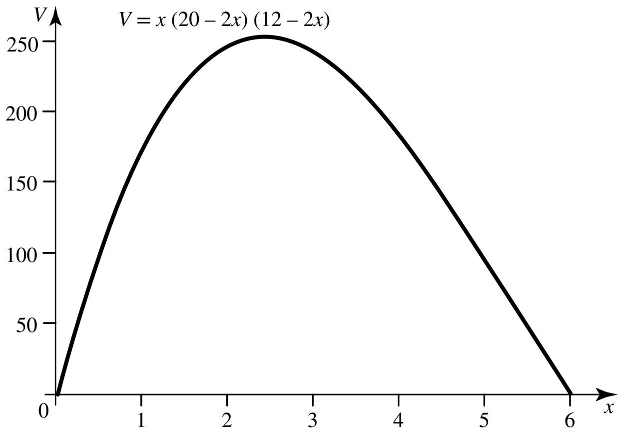
$l\geq 0\Rightarrow 20-2x\geq 0$ $\qquad$ $w\geq 0\Rightarrow 12-2x\geq 0$

$\therefore x\leq 10$ $\qquad$ $\therefore x\leq 6$

Therefore to satisfy all three conditions, $0\leq x\leq 6$

c $V=(20-2x)(12-2x)x$, $0\leq x\leq 6$

x intercepts at $x=10, x=6, x=0$ but since $0\leq x\leq 6$, the graph won't reach $x=10$ shape is of a positive cubic



d Turning points at $x = 2.43$ and $x = 8.24$ The first turning point is a maximum and the graph doesn't reach the second turning point (which would be a minimum) due to the domain restriction.

Therefore the greatest volume occurs when $x = 2.43$

$l = 20 - 2(2.43) \qquad w = 12 - 2(2.43)$

$= 15.14 \qquad\qquad = 7.14$

Box has length 15.14 cm, width 7.14 cm, height 2.43 cm

Greatest volume is $15.14 \times 7.14 \times 2.43 = 263\text{ cm}^3$ to the nearest whole number

3 a Cost model: $C = x^3 + 100x + 2000$

Consider the case when 5 sculptures are produced.

$$C(5) = 5^3 + 100(5) + 2000$$
$$= 2625$$

It costs the artist \$2625 to produce 5 sculptures. The artist earns $500 \times 5 = 2500$ dollars from the sale of the 5 sculptures. This results in a loss of $\$(2625 - 2500) = \$125$.

Consider the case when 6 sculptures are produced.

$$C(6) = 6^3 + 100(6) + 2000$$
$$= 2816$$

It costs the artist \$2816 to produce 6 sculptures. The artist earns $500 \times 6 = 3000$ dollars from the sale of the 6 sculptures. This results in a profit of $\$(3000 - 2816) = \$184$.

b The artist earns $\$500x$ from the sale of x sculptures and the cost of production is given by $C = x^3 + 100x + 2000$.

The profit model is $P = 500x - (x^3 + 100x + 2000)$

$\therefore P = -x^3 + 400x - 2000$ as required.

c As $x \to \infty, P \to -\infty$. Thus, for large numbers of sculptures, the cost of production outweighs the revenue from their sales.

d i If 16 sculptures are produced,

$$P(16) = -(16)^3 + 400(16) - 2000$$
$$= 304$$

A profit of \$304 is made.

ii If 17 sculptures are produced,

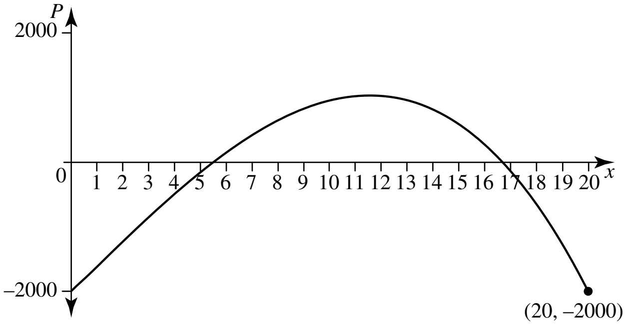
$$P(17) = -(17)^3 + 400(17) - 2000$$
$$= -113$$

A loss of \$113 is made.

e As $P(5) < 0$ and $P(6) > 0$, the graph of the profit model will have an x intercept in the interval $x \in (5, 6)$.

As $P(16) > 0$ and $P(17) < 0$, the graph of the profit model will have another x intercept in the interval $x \in (16, 17)$.

Endpoints: $P(0) = -2000$ and $P(20) = -(20)^3 + 400(20) - 2000 = -2000$.



f For a profit $P > 0$. A profit is made if between 6 and 16 sculptures are produced.

4 $N = 54 + 23t + t^3$

a When $t = 0, N = 54$.

At 9 am there were initially 54 bacteria.

b When the number has doubled, $N = 108$.

$\therefore 54 + 23t + t^3 = 108$

$\therefore t^3 + 23t - 54 = 0$

Let $P(t) = t^3 + 23t - 54$

$P(2) = 8 + 46 - 54 = 0 \Rightarrow (t - 2)$ is a factor

$\therefore t^3 + 23t - 54 = (t - 2)(t^2 + 2t + 27) = 0$

$\therefore t = 2$ or $t^2 + 2t + 27 = 0$

Consider $t^2 + 2t + 27 = 0$

$$\Delta = (2)^2 - 4 \times 1 \times 27$$
$$= -104$$

no real solutions.

$\therefore t = 2$.

The bacteria double the initial number after 2 hours.

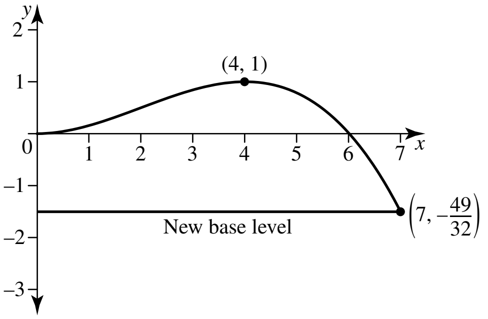
c At 1 pm, $t = 4$ since time is measured from 9 am.
$N = 54 + 23 \times 4 + 4^3$
$= 210$
There are 210 bacteria at 1 pm.

d When $N = 750$,
$54 + 23t + t^3 = 750$
$\therefore t^3 + 23t - 696 = 0$
Let $P(t) = t^3 + 23t - 696$
Trial and error using factors of 696:
$P(4) \neq 0, P(6) \neq 0, P(8) = 0$
$\therefore (t - 8)$ is a factor
$t^3 + 23t - 696 = (t - 8)(t^2 + 8t + 87)$
$\therefore (t - 8)(t^2 + 8t + 87 = 0$
$\therefore t = 8$ or $t^2 + 8t + 87 = 0$
Consider $t^2 + 8t + 87 = 0$
$\Delta = 64 - 4 \times 1 \times 87$
$\therefore \Delta < 0$
No real solutions
$\therefore t = 8$
The number of bacteria reaches 750 after 8 hours after 9 am. The time is 5 pm.

5 $y = ax^2(x - b)$

a Given information $\Rightarrow (6,0)$ is an x intercept and $(4,1)$ is the maximum turning point.
The curve $y = ax^2(x - b)$ has x intercepts at $x = 0$ and $x = b$, which means that $b = 6$.
Substitute the point $(4,1)$ in $y = ax^2(x - 6)$
$\therefore 1 = a(4)^2(4 - 6)$
$\therefore 1 = -32a$
$\therefore a = -\frac{1}{32}$
The equation of the bounding curve is $y = -\frac{1}{32}x^2(x - 6)$

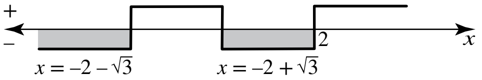
b If $x = 7$ then
$y = -\frac{1}{32}(7)^2(1)$
$= -\frac{49}{32}$
The new base level ends at $\left(7, -\frac{49}{32}\right)$



The greatest height will now be $\frac{49}{32} + 1 = \frac{81}{32}$ km above the base level.

6 Let the number be x with $x \in Z$.
It is required that $(x + 5)^2 - (x + 1)^3 > 22$
$\therefore x^2 + 10x + 25 - (x^3 + 3x^2 + 3x + 1) > 22$
$\therefore -x^3 - 2x^2 + 7x + 24 > 22$
$\therefore -x^3 - 2x^2 + 7x + 2 > 0$
$\therefore x^3 + 2x^2 - 7x - 2 < 0$
Let $P(x) = x^3 + 2x^2 - 7x - 2$
$P(2) = 8 + 8 - 14 - 2 = 0$
$\therefore (x - 2)$ is a factor
$\therefore x^3 + 2x^2 - 7x - 2 = (x - 2)(x^2 + 4x + 1)$
$= (x - 2)\left[(x^2 + 4x + 4) - 4 + 1\right]$
$= (x - 2)\left[(x + 2)^2 - 3\right]$
$= (x - 2)\left(x + 2 + \sqrt{3}\right)\left(x + 2 - \sqrt{3}\right)$

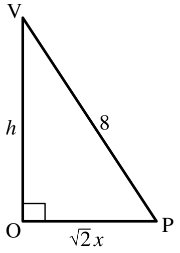
$x^3 + 2x^2 - 7x - 2 < 0$
$\Rightarrow (x - 2)\left(x + 2 + \sqrt{3}\right)\left(x + 2 - \sqrt{3}\right) < 0$
Zeros are $x = 2, x = -2 - \sqrt{3}, x = -2 + \sqrt{3}$, coefficient of x^3 is positive



$\therefore x < -2 - \sqrt{3}$ or $-2 + \sqrt{3} < x < 2$
However, $x \in Z$. The solution intervals are approximately, $x < -3.732$ or $-0.268 < x < 2$. The smallest positive integer which lies in the solution interval is 1 and the largest negative integer is −4.

7 **a** Using Pythagoras' theorem in the right angled triangle MNP with l metres the length of the diagonal MP:
$l^2 = (2x)^2 + (2x)^2$
$= 8x^2$
$\therefore l = 2\sqrt{2}x$

b Consider the right angled triangle VOP:
Since $\text{OP} = \frac{1}{2}l$, $\text{OP} = \sqrt{2}x$



Using Pythagoras' theorem,
$h^2 + \left(\sqrt{2}x\right)^2 = 8^2$
$\therefore 2x^2 = 64 - h^2$

c Area of square base is given by $A = (2x)(2x) = 4x^2$.
Substitute $2x^2 = 64 - h^2$
$\therefore A = 2 \times (64 - h^2)$
$\therefore A = 128 - 2h^2$
Volume: $V = \frac{1}{3}Ah$
$\therefore V = \frac{1}{3}(128 - 2h^2)h$
$\therefore V = \frac{1}{3}(128h - 2h^3)$

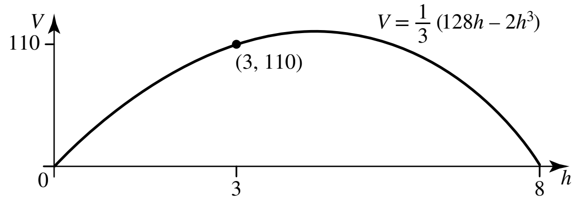
d **i** If $h = 3$, $V = \frac{1}{3}(128 \times 3 - 2 \times 27) = 110$
The volume is 110 m^3.

ii Since the volume cannot be negative, $\frac{1}{3}(128h - 2h^3) \geq 0$
$\therefore 128h - 2h^3 \geq 0$
$\therefore 2h(64 - h^2) \geq 0$
Since the height cannot be negative, $64 - h^2 \geq 0$
$\therefore h^2 \leq 64$
$\therefore 0 \leq h \leq 8$
Note: For practical purposes, the allowable values for the height would need further restriction.

e $V=\frac{1}{3}h(64-h^2),\ 0\le h\le 8$

$\therefore V=\frac{1}{3}h(8-h)(8+h)$

horizontal axis intercepts at $h=0, h=8, h=-8$ (not applicable). Negative cubic shape.



An estimate of the height for which the volume is greatest could be 5 metres but answers will vary.

f Volume is greatest when $h=\frac{1}{2}(2x)\Rightarrow h=x$

Substitute this in the relationship from part **b** that $2x^2=64-h^2$.

$\therefore 2h^2=64-h^2$

$\therefore 3h^2=64$

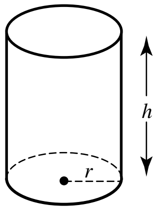
$\therefore h^2=\frac{64}{3}$

$\therefore h=\frac{8}{\sqrt{3}}\simeq 4.6$

Volume is greatest when the height is $\frac{8}{\sqrt{3}}\simeq 4.6$ metres.

To one significant figure, the estimate given in part **e** was accurate.

8



a Surface area of cylinder open at the top is given by $SA=2\pi rh+\pi r^2$.

$\therefore 400\pi=2\pi rh+\pi r^2$

$\therefore 400=2rh+r^2$

$\therefore 400-r^2=2rh$

$\therefore h=\frac{400-r^2}{2r}$

b Volume formula for a cylinder is $V=\pi r^2 h$

Substitute $h=\frac{400-r^2}{2r}$ from part **a**

$\therefore V=\pi r^2\left(\frac{400-r^2}{2r}\right)$

$\therefore V=\frac{\pi r^2(400-r^2)}{2r}$

$\therefore V=\frac{\pi r(400-r^2)}{2}$

$\therefore V=\frac{400\pi r}{2}-\frac{\pi r^3}{2}$

$\therefore V=200\pi r-\frac{1}{2}\pi r^3$

c $r\ge 0,\ h\ge 0,\ V\ge 0$

$\frac{\pi r(400-r^2)}{2}\ge 0$

$\therefore r(400-r^2)\ge 0$

$\therefore 400-r^2\ge 0$

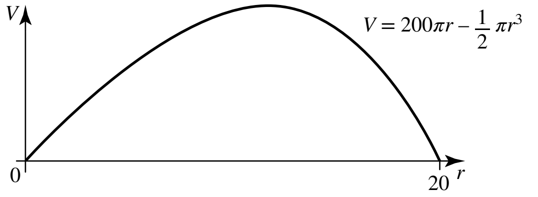
$\therefore r^2\le 400$

$\therefore 0\le r\le 20$

d $V=\frac{\pi r(400-r^2)}{2}, 0\le r\le 20$

$\therefore V=\frac{\pi}{2}r(20-r)(20+r)$

Horizontal axis intercepts occur at $r=0, r=20, r=-20$ (not applicable). Negative cubic shape graph.



e $V=396\pi$

$\therefore 396\pi=200\pi r-\frac{1}{2}\pi r^3$

$\therefore 396=200r-\frac{1}{2}r^3$

$\therefore 792=400r-r^3$

$\therefore r^3-400r+792=0$

Let $P(r)=r^3-400r+792$

Testing factors of 792, $P(2)=8-800+792=0$

$\therefore (r-2)$ is a factor

$\therefore r^3-400r+792=(r-2)(r^2+2r-396)$

Hence,

$(r-2)(r^2+2r-396)=0$

$\therefore r=2$ or $r^2+2r-396=0$

$\therefore \left(r^2+2r+1\right)-1-396=0$

$\therefore (r+1)^2-397=0$

$\therefore (r+1)^2=397$

$\therefore r+1=\pm\sqrt{397}$

$\therefore r=-\sqrt{397}-1, r=\sqrt{397}-1$

As $0\le r\le 20$, $r=2, r=\sqrt{397}-1\simeq 18.925$

Height: $h=\frac{400-r^2}{2r}$

If $r=2$, $h=\frac{400-4}{4}=99$

If $r=18.925$, $h=\frac{400-18.925^2}{2\times 18.925}=1.10$

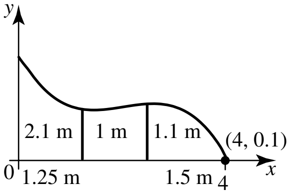
Both the container with height 99 cm and base radius 2 cm and the container with height 1.1 cm and base radius 18.9 cm have a volume of 396π cm^3.

f Substitute $r=\frac{20}{\sqrt{3}}$ into $V=200\pi r-\frac{1}{2}\pi r^3$

$$\begin{aligned}\therefore V &= 200\pi\times\frac{20}{\sqrt{3}}-\frac{1}{2}\pi\times\left(\frac{20}{\sqrt{3}}\right)^3\\ &=\frac{4000\pi}{\sqrt{3}}-\frac{\pi}{2}\times\frac{8000}{3\sqrt{3}}\\ &=\frac{24\,000\pi-8000\pi}{6\sqrt{3}}\\ &=\frac{1600\pi}{6\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\ &=\frac{8000\pi\sqrt{3}}{3(3)}\\ &=\frac{8000\sqrt{3}\pi}{9}\end{aligned}$$

The maximum volume is $\frac{8000\sqrt{3}\pi}{9}$ cm^3 or approximately 4837 cm^3.

9



a Known points on the curve are $(0,2.1),(1.25,1),(2.5,1.1),(4,0.1)$.

b $h=ax^3+bx^2+cx+d$

Substitute $(0,2.1)$

$\therefore 2.1=d$

c $h=ax^3+bx^2+cx+2.1$

Substitute $(1.25,1)$

$$\therefore 1=a(1.25)^3+b(1.25)^2+c(1.25)+2.1$$

$$\therefore\left(\frac{5}{4}\right)^3a+\left(\frac{5}{4}\right)^2b+\left(\frac{5}{4}\right)c=-1.1$$

$$\therefore\frac{125}{64}a+\frac{25}{16}b+\frac{5}{4}c=-1.1$$

$$\therefore 125a+100b+80c=-70.4....(1)$$

Substitute $(2.5,1.1)$

$$\therefore 1.1=a\left(\frac{5}{2}\right)^3+b\left(\frac{5}{2}\right)^2+c\left(\frac{5}{2}\right)+2.1$$

$$\therefore\frac{125}{8}a+\frac{25}{4}b+\frac{5}{2}c=-1$$

$$\therefore 125a+50b+20c=-8....(2)$$

Substitute $(4,0.1)$

$$\therefore 0.1=a(4)^3+b(4)^2+c(4)+2.1$$

$\therefore 64a+16b+4c=-2....(3)$

The system of simultaneous equations is

$125a+100b+80c=-70.4....(1)$

$125a+50b+20c=-8....(2)$

$64a+16b+4c=-2....(3)$

d Given $y=-0.164x^3+x^2-1.872x+2.1$

When $x=3.5$,

$y=-0.164\times3.5^3+3.5^2-1.872\times3.5+2.1$

$=0.7665$

The third strut would have length 0.77 metres.

10 $T(t)=-0.00005(t-6)^3+9.85$

a Time is measured from 1988. Therefore, $t=3$ for 1991

$T(3)=-0.00005(-3)^3+9.85$

$=9.85135$

Model predicts 9.85 seconds, to two decimal places; the actual time was 9.86 seconds.

For 2008, $t=20$

$T(20)=-0.00005(14)^3+9.85$

$=9.7128$

Model predicts 9.71 seconds, to two decimal places; the actual time was 9.72 seconds.

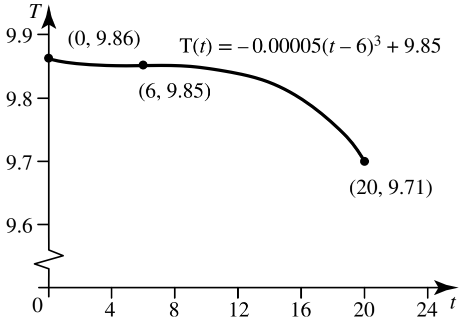
The predictions and the actual times agree to one decimal place.

b $T(t)=-0.00005(t-6)^3+9.85$ for $t\in[0,20]$

POI $(6,9.85)$

Endpoints: If $t=0, T=-0.00005(-6)^3+9.85=9.8608$.

Left endpoint $(0,9.86)$. Right endpoint $(20,9.71)$



c For 2016, $t=28$

$T(28)=-0.00005(22)^3+9.85$

$=9.3176$

The model predicts 9.32 seconds, which seems unlikely although not impossible. The graph shows the time taken starts to decrease quite steeply after 2008, so its predictions are probably not accurate.

11 **a** $y=9-(x-3)^2$

x intercepts: Let $y=0$

$\therefore 0=9-(x-3)^2$

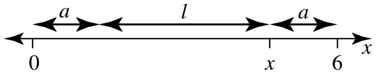
$\therefore (x-3)^2=9$

$\therefore x-3=\pm3$

$\therefore x=0, x=6$

The x intercepts are $(0,0)$ and $(6,0)$.

b length:



$l=x-a$

$=x-(6-x)$

$=2x-6$

width:

$w=y$

$=9-(x-3)^2$

c Area is given by $A=lw$

$$\begin{aligned}\therefore A&=(2x-6)\left(9-(x-3)^2\right)\\ &=(2x-6)\left(9-(x^2-6x+9)\right)\\ &=(2x-6)(9-x^2+6x-9)\\ &=(2x-6)(-x^2+6x)\\ &=-2x^3+12x^2+6x^2-36x\end{aligned}$$

$\therefore A=-2x^3+18x^2-36x$

d $l\ge0\Rightarrow 2x-6\ge0$

$\therefore x\ge3$

$w\ge0\Rightarrow 9-(x-3)^2\ge0$

$\therefore -x^2+6x\ge0$

$\therefore -x(x-6)\ge0$

$\therefore 0\le x\le6$

For both the length and width to be non-negative, the model is valid for $3 \le x \le 6$.

e Let $A = 16$

$$\therefore 16 = -2x^3 + 18x^2 - 36x$$

$\therefore 2x^3 - 18x^2 + 36x + 16 = 0$

$\therefore x^3 - 9x^2 + 18x + 8 = 0$

Let $P(x) = x^3 - 9x^2 + 18x + 8$

$P(4) = 64 - 144 + 72 + 8 = 0$

$\therefore (x - 4)$ is a factor

$\therefore x^3 - 9x^2 + 18x + 8 = (x - 4)(x^2 - 5x - 2)$

$P(x) = 0 \Rightarrow (x - 4)(x^2 - 5x - 2) = 0$

$\therefore x = 4$ or $x^2 - 5x - 2 = 0$

$$\therefore x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times -2}}{2}$$

$$= \frac{5 \pm \sqrt{33}}{2}$$

$$\therefore x = 4, \frac{5 + \sqrt{33}}{2}, \frac{5 - \sqrt{33}}{2}$$

Since $3 \le x \le 6$, reject $x = \dfrac{5 - \sqrt{33}}{2}$

Answer $x = 4, x = \dfrac{5 + \sqrt{33}}{2}$

12 a A cubic graph can have up to 2 turning points and up to 3 x-intercepts.

b Given the x intercepts are $A(-3,0), O(0,0), D(3,0)$, then the equation of the path is $y = ax(x + 3)(x - 3)$.

Substitute the point $C\left(\sqrt{3}, 12\sqrt{3}\right)$

$\therefore 12\sqrt{3} = a\left(\sqrt{3}\right)\left(\sqrt{3} + 3\right)\left(\sqrt{3} - 3\right)$

$\therefore 12\sqrt{3} = a\sqrt{3}\left(\left(\sqrt{3}\right)^2 - 3^2\right)$

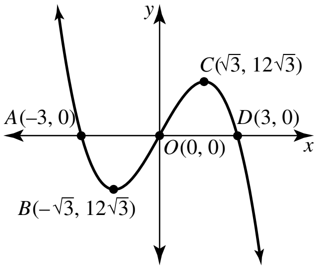
$\therefore 12\sqrt{3} = a\sqrt{3}(3 - 9)$

$\therefore 12\sqrt{3} = -6\sqrt{3}a$

$\therefore a = -2$

The equation of the path is $y = -2x(x + 3)(x - 3)$ or $y = -2x(x^2 - 9)$

c Points A, O and D are x intercepts; points B and C are turning points; shape is of a negative cubic. The value of $12\sqrt{3} \simeq 20.8$.



d Consider the gradient of the straight line through $B\left(-\sqrt{3}, -12\sqrt{3}\right)$ and $C\left(\sqrt{3}, 12\sqrt{3}\right)$.

$$m_{BC} = \frac{12\sqrt{3} - \left(-12\sqrt{3}\right)}{\sqrt{3} - \left(-\sqrt{3}\right)}$$

$$= \frac{24\sqrt{3}}{2\sqrt{3}}$$

$$= 12$$

Consider the gradient of the line through $O(0,0)$ and $C\left(\sqrt{3}, 12\sqrt{3}\right)$:

$$m_{OC} = \frac{12\sqrt{3}}{\sqrt{3}}$$

$$= 12$$

Since $m_{BC} = m_{OC}$ and the point C is common, then the three points B, O and C are collinear. Therefore, a straight line through B and C will pass through O.

Equation of BC:

$y - 12\sqrt{3} = 12(x - \sqrt{3})$

$\therefore y = 12x$

e Let the equation be $y = a(x - h)^3 + k$

POI at $(0,0)$: $\therefore y = ax^3$

Substitute the point $C\left(\sqrt{3}, 12\sqrt{3}\right)$

$\therefore 12\sqrt{3} = a\left(\sqrt{3}\right)^3$

$\therefore 12\sqrt{3} = 3\sqrt{3}a$

$\therefore a = 4$

The equation of the path is $y = 4x^3$

13 a refers to question **5e**

Use the graphing menu to sketch $y = \dfrac{1}{3}(128x - 2x^3)$ and obtain the co-ordinates of the maximum turning point by tapping Analysis → G-Solve → Max.

The maximum turning point is $(4.6188, 131.37926)$, so the height for which the volume is greatest is 4.62 metres.

b refers to question **5f**

We know the relationship between the height h and the length of the base $2x$ is given by $2x^2 = 64 - h^2$.

Substitute $h = 4.6188$

$\therefore 2x^2 = 64 - (4.6188)^2$

Solve for x in the main menu by tapping Interactive → Equation/Inequality → solve

$\therefore x = 4.6188$ (rejecting the negative solution).

The maximum volume occurs when $h = x$.

The length of the base is $2x$. Thus the greatest volume occurs when the height is half the length of the base.

c refers to question **6f**

Use the graphing menu to sketch $y = 200\pi x - \dfrac{1}{2}\pi x^3$ and obtain the co-ordinates of the maximum turning point by tapping Analysis → G-Solve → Max.

The maximum turning point is $(11.547885, 4836.7983)$, so the maximum volume is 4836.8 cm^3.

For the maximum volume, $r = 11.547885$. Substitute this into the relationship between the height h and the radius r that $h = \dfrac{400 - r^2}{2r}$.

$$\therefore h = \frac{400 - (11.547885)^2}{2 \times 11.547885}$$

$\therefore h = 11.5$ to one decimal place.

The dimensions of the cylinder with greatest volume are a base radius of 11.5 cm and a height of 11.5 cm, to one decimal place.

14 a refers to question **7c**

$125a + 100b + 80c = -70.4....(1)$

$125a + 50b + 20c = -8....(2)$

$64a + 16b + 4c = -2....(3)$

The system of simultaneous equations can be solved using the template from keyboard → math → 2D. This gives $a = -0.164, b = 1, c = -1.872$.

The equation of the slide is $y = ax^3 + bx^2 + cx + d$. It is already known that $d = 2.1$.

The equation is $y = -0.164x^3 + x^2 - 1.872x + 2.1$.

Alternatively, the co-ordinates of the four given points can be entered in the Statistics menu and the equation calculated from Calc → Cubic Reg.

b refers to question **9**

Sketch the area graph $A = -2x^3 + 18x^2 - 36x$ and obtain the co-ordinates of the maximum turning point by tapping Analysis → G-Solve → Max.

The maximum turning point is $(4.7320507, 20.784609)$, so the area is greatest when $x = 4.7320507$.

Length, $l = 2x - 6$

$\therefore l = 2 \times 4.7320507 - 6$

$\therefore l = 3.464$

Width, $w = -x^2 + 6x$

$\therefore w = -(4.7320507)^2 + 6 \times 4.7320507$

$\therefore w = 6.000$

To three decimal places, the length and width of the rectangle which has the greatest area are 3.464 units and 6.000 units respectively.

Topic 5 — Higher-degree polynomials

Exercise 5.2 — Quartic polynomials

1 a $y=(x-2)^4-1$

Minimum turning point $(2,-1)$

y intercept: Let $x=0$

$\therefore y=(-2)^4-1$

$=16-1$

$=15$

$(0,15)$

x intercepts: Let $y=0$

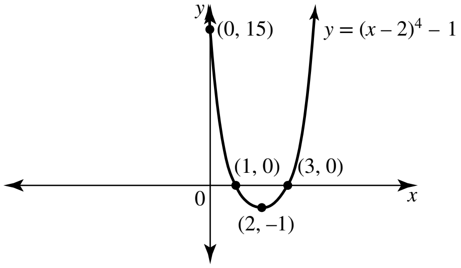
$\therefore (x-2)^4-1=0$

$\therefore (x-2)^4=1$

$\therefore x-2=\pm1$

$\therefore x=1$ or $x=3$

$(1,0),(3,0)$



b $y=-(2x+1)^4$

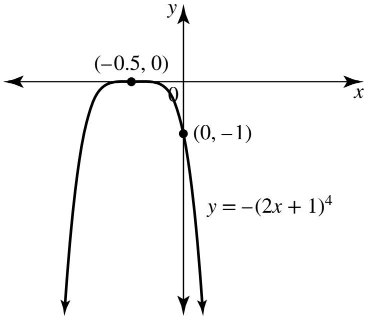
x intercept and maximum turning point $\left(-\frac{1}{2},0\right)$

y intercept: Let $x=0$

$\therefore y=-(1)^4$

$\therefore y=-1$

$(0,-1)$



2 a $y=a(x-b)^4+c$

turning point $(-2,4)\Rightarrow y=a(x+2)^4+4$

point $(0,0)\Rightarrow 0=a(2)^4+4$

$\therefore a=-\frac{4}{16}$

$\therefore a=-\frac{1}{4}$

Hence, the equation is $y=-\frac{1}{4}(x+2)^4+4$

b x intercepts: Let $y=0$

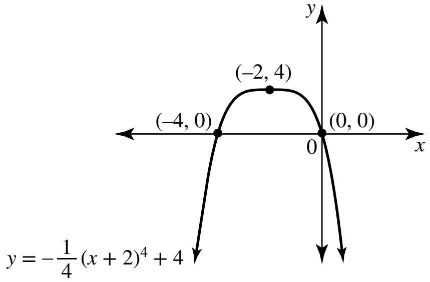
$0=-\frac{1}{4}(x+2)^4+4$

$\therefore (x+2)^4=16$

$\therefore x+2=\pm\sqrt[4]{16}$

$\therefore x+2=\pm2$

$\therefore x=-4, x=0$



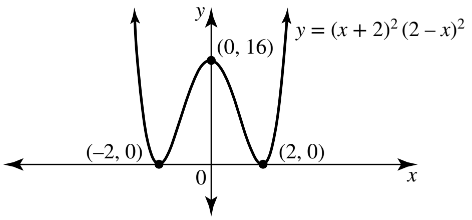
$\{x: -\frac{1}{4}(x+2)^4+4>0\}=\{x: -4<x<0\}$

3 $y=(x+2)^2(2-x)^2$

x intercepts at $x=-2$ (touch) and $x=2$ (touch)

y intercept at $y=16$

Leading term gives positive x^4 shape



4 The x intercepts indicate the linear factors of the polynomial. As the graph cuts the x axis at each of $x=-4, x=0, x=2, x=5$ then the equation of the graph is of the form $y=a(x+4)x(x-2)(x-5)$.

Substitute the given point $(-3,-30)$

$\therefore -30=a(1)(-3)(-5)(-8)$

$\therefore -30=-120a$

$\therefore a=\frac{1}{4}$

The equation of the given graph is $y=\frac{1}{4}x(x+4)(x-2)(x-5)$.

5 $P(x)=x^4+5x^3-6x^2-32x+32$

Using trial and error to find two factors,

$P(1)=1+5-6-32+32=0$

$\Rightarrow (x-1)$ is a factor

$P(2)=16-40-24-64+32$

$=0$

$\Rightarrow (x-2)$ is a factor

Hence $(x-1)(x-2)=x^2-3x+2$ is a quadratic factor.

$\therefore x^4+5x^3-6x^2-32x+32=\left(x^2-3x+2\right)\left(x^2+bx+16\right)$

Equate coefficients of x^3

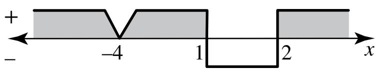
$\therefore 5=b-3$

$\therefore b=8$

$\therefore x^4+5x^3-6x^2-32x+32=\left(x^2-3x+2\right)\left(x^2+8x+16\right)$

$=(x-1)(x-2)(x+4)^2$

To solve the inequation we shall draw the sign diagram. The sign diagram of a positive quartic with cuts at $x=1$ and $x=2$ and a touch at $x=-4$.



The inequation $x^4+5x^3-6x^2-32x+32>0$ holds for $x<-4$, or $-4<x<1$ or $x>2$.

6 $(x+2)^4-13(x+2)^2-48=0$.

Let $a=(x+2)^2$

$\therefore a^2-13a-48=0$

$\therefore (a-16)(a+3)=0$

$\therefore a=16$ or $a=-3$

Substitute back $(x+2)^2$ for a

$\therefore (x+2)^2=16$ or $(x+2)^2=-3$

Reject $(x+2)^2=-3$ since $(x+2)^2$ cannot be negative

$\therefore (x+2)^2=16$

$\therefore x+2=\pm\sqrt{16}$

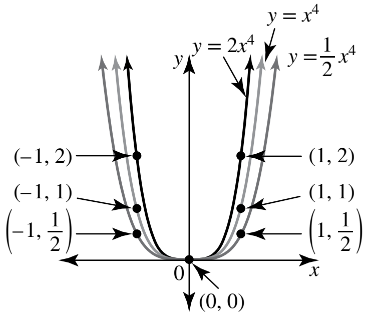
$\therefore x=-2\pm 4$

$\therefore x=-6$ or $x=2$

7 **a** $y=x^4, y=2x^4, y=\dfrac{1}{2}x^4$

All three graphs have a minimum turning point at the origin.

The points $(\pm 1,1)$ lie on $y=x^4$, the points $(\pm 1,2)$ lie on $y=2x^4$ and the points $\left(\pm 1,\dfrac{1}{2}\right)$ lie on $y=\dfrac{1}{2}x^4$.



b $y=x^4, y=-x^4, y=-2x^4, y=(-2x)^4$

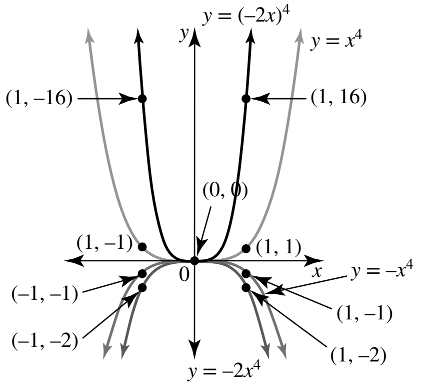
The points $(0,0),(-1,-1),(1,-1)$ lie on $y=-x^4$.

The points $(0,0),(-1,-2),(1,-2)$ lie on $y=-2x^4$.

$(-2x)^4=(-2)^4x^4$

$=16x^4$

Therefore the points $(0,0),(-1,16),(1,16)$ lie on $y=(-2x)^4$.

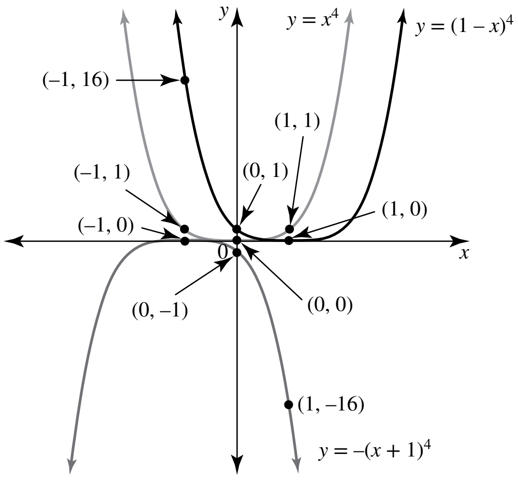


c $y=x^4, y=-(x+1)^4, y=(1-x)^4$

The points $(-1,0),(0,-1),(1,-16)$ lie on $y=-(x+1)^4$.

The points $(-1,16),(0,1),(1,0)$ lie on $y=(1-x)^4$.

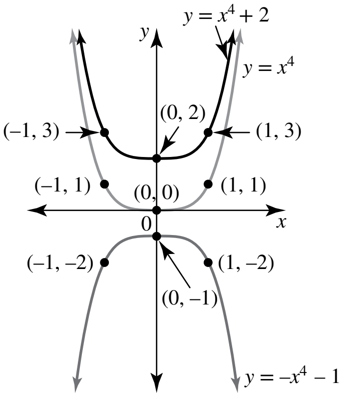
As $(1-x)^4=(x-1)^4$, $y=(1-x)^4$ is the same as $y=(x-1)^4$.



d $y=x^4, y=x^4+2, y=-x^4-1$.

The points $(-1,3),(0,2),(1,3)$ lie on $y=x^4+2$.

The points $(-1,-2),(0,-1),(1,-2)$ lie on $y=-x^4-1$.



8 **a** $y=(x-1)^4-16$

Minimum turning point $(1,-16)$

y intercept: Let $x=0$

$\therefore y=(-1)^4-16$

$\therefore y=-15$

$(0,-15)$

x intercepts: Let $y=0$

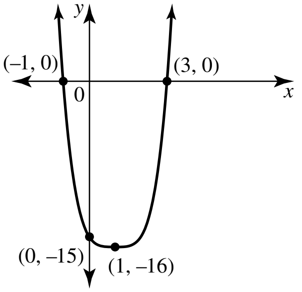
$\therefore (x-1)^4-16=0$

$\therefore (x-1)^4=16$

$\therefore x-1=\pm 2$

$\therefore x=-1, x=3$

$(-1,0),(3,0)$



b $y=\dfrac{1}{9}(x+3)^4+12$

Minimum turning point $(-3,12)$

y intercept: Let $x=0$

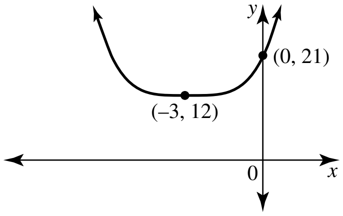
$\therefore y=\frac{1}{9}(3)^4+12$

$\therefore y=9+12$

$\therefore y=21$

$(0,21)$

No x intercepts.



c $y=250-0.4(x+5)^4$

Maximum turning point $(-5,250)$

y intercept: Let $x=0$

$\therefore y=250-0.4(5)^4$

$\therefore y=250-250$

$\therefore y=0$

$(0,0)$

x intercepts: Let $y=0$

$\therefore 250-0.4(x+5)^4=0$

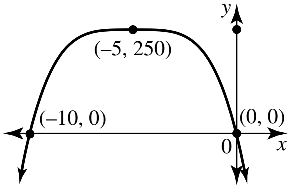
$\therefore (x+5)^4=\frac{250}{0.4}$

$\therefore (x+5)^4=625$

$\therefore x+5=\pm 5$

$\therefore x=-10, x=0$

$(-10,0),(0,0)$



d $y=-\left(6(x-2)^4+11\right)$

$\therefore y=-6(x-2)^4-11$

Maximum turning point $(2,-11)$

y intercept: Let $x=0$

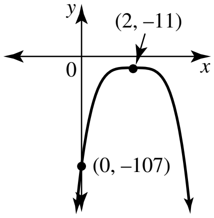
$\therefore y=-6(-2)^4-11$

$\therefore y=-6\times 16-11$

$\therefore y=-107$

$(0,-107)$

no x intercepts.



e $y=\frac{1}{8}(5x-3)^4-2$

Minimum turning point $\left(\frac{3}{5},-2\right)$

y intercept: Let $x=0$

$\therefore y=\frac{1}{8}(-3)^4-2$

$\therefore y=\frac{81}{8}-\frac{16}{8}$

$\therefore y=\frac{65}{81}$

$\left(0,\frac{65}{81}\right)$

x intercepts: Let $y=0$

$\therefore \frac{1}{8}(5x-3)^4-2=0$

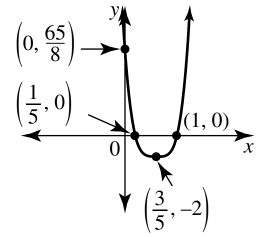
$\therefore (5x-3)^4=16$

$\therefore 5x-3=\pm 2$

$\therefore 5x=1 \text{ or } 5x=5$

$\therefore x=\frac{1}{5}, x=1$

$\left(\frac{1}{5},0\right),(1,0)$



f $y=1-\left(\frac{2-7x}{3}\right)^4$

$\therefore y=1-\frac{(-(7x-2))^4}{(3)^4}$

$\therefore y=1-\frac{(7x-2)^4}{81}$

$\therefore y=-\frac{1}{81}(7x-2)^4+1$

Maximum turning point $\left(\frac{2}{7},1\right)$

y intercept: Let $x=0$

$\therefore y=-\frac{1}{81}(-2)^4+1$

$\therefore y=-\frac{16}{81}+\frac{81}{81}$

$\therefore y=\frac{65}{81}$

$\left(0,\frac{65}{81}\right)$

x intercepts: Let $y=0$

$\therefore 1-\left(\frac{2-7x}{3}\right)^4=0$

$\therefore \left(\frac{2-7x}{3}\right)^4=1$

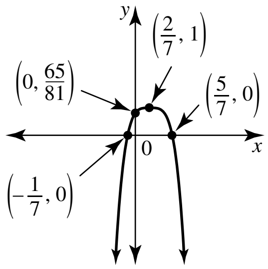
$\therefore \frac{2-7x}{3}=\pm 1$

$\therefore 2-7x=\pm 3$

$\therefore -7x=1, \text{ or } -7x=-5$

$\therefore x=-\frac{1}{7}, x=\frac{5}{7}$

$\left(-\frac{1}{7},0\right), \left(\frac{5}{7},0\right)$



9 **a** A quartic graph with the same shape as $y=\frac{2}{3}x^4$ but whose turning point has the co-ordinates $(-9,-10)$ would have the equation $y=\frac{2}{3}(x+9)^4-10$.

b $y=a(x+b)^4+c$

Turning point $(-3,-8)$

$\therefore y=a(x+3)^4-8$

Substitute the point $(-4,-2)$

$\therefore -2=a(-4+3)^4-8$

$\therefore -2=a(-1)^4-8$

$\therefore -2=a-8$

$\therefore a=6$

The equation is $y=6(x+3)^4-8$

c $y=(ax+b)^4$ where $a>0$ and $b<0$.

Substitute the point $(0,16)$

$\therefore 16=b^4$

$\therefore b=\pm 2$

Hence $b=-2$ since $b<0$

$\therefore y=(ax-2)^4$

Substitute the point $(2,256)$

$\therefore 256=(2a-2)^4$

$\therefore 2a-2=\pm 4$

$\therefore 2a=-2$ or $2a=6$

$\therefore a=-1, a=3$

Hence, $a=3$ since $a>0$.

The equation is $y=(3x-2)^4$.

d $y=a(x-h)^4+k$

From the graph the x intercepts are $(-110,0)$ and $(-90,0)$, so the axis of symmetry has the equation $x=-100$.

The maximum turning point must be at $(-100,10\,000)$.

The equation of the graph becomes $y=a(x+100)^4+10\,000$.

Substitute the point $(-90,0)$

$\therefore 0=a(10)^4+10\,000$

$\therefore a(10\,000)=-10\,000$

$\therefore a=-1$

The equation is $y=-(x+100)^4+10\,000$.

10 **a** $y=ax^4+k$

Substiute the point $(-1,1)$

$\therefore 1=a(-1)^4+k$

$\therefore 1=a+k$ ....(1)

Substitute the point $\left(\frac{1}{2},\frac{3}{8}\right)$

$\therefore \frac{3}{8}=a\left(\frac{1}{2}\right)^4+k$

$\therefore \frac{3}{8}=\frac{1}{16}a+k$ ....(2)

Subtract equation (2) from equation (1)

$\therefore \frac{5}{8}=\frac{15}{16}a$

$\therefore a=\frac{5}{8}\times\frac{16}{15}$

$\therefore a=\frac{2}{3}$

Substitute $a=\frac{2}{3}$ in equation (1)

$\therefore 1=\frac{2}{3}+k$

$\therefore k=\frac{1}{3}$

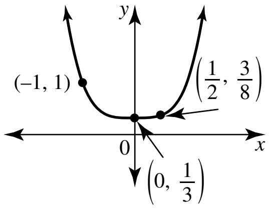
$a=\frac{2}{3}, k=\frac{1}{3}$

b The equation of the curve is $y=\frac{2}{3}x^4+\frac{1}{3}$.

Hence the minimum turning point is $\left(0,\frac{1}{3}\right)$.

c The axis of symmetry is $x=0$.

d There are no x intercepts and the turning point is the y intercept.



11 $y=a(x+b)^4+c$

a As line joining the points $(-2,3)$ and $(4,3)$ is horizontal the axis of symmetry passes through their midpoint.

$$x=\frac{-2+4}{2}$$
$$=1$$

The axis of symmetry has the equation $x=1$.

b Maximum turning point is $(1,10)$.

c The equation is of the form $y=a(x-1)^4+10$.

Substitute the point $(4,3)$

$\therefore 3=a(3)^4+10$

$\therefore 81a=-7$

$\therefore a=-\frac{7}{81}$

The equation is $y=-\frac{7}{81}(x-1)^4+10$.

d y intercept: Let $x=0$

$\therefore y=-\frac{7}{81}(-1)^4+10$

$\therefore y=-\frac{7}{81}+10$

$\therefore y=\frac{-7+810}{81}$

$\therefore y=\frac{803}{81}$

The y intercept is $\left(0,\frac{803}{81}\right)$.

e x intercepts: Let $y=0$

$$\therefore 0=-\frac{7}{81}(x-1)^4+10$$

$$\therefore 7(x-1)^4=810$$

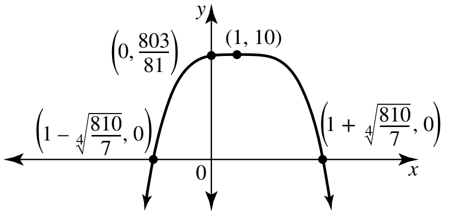
$$\therefore (x-1)^4=\frac{810}{7}$$

$$\therefore x-1=\pm\sqrt[4]{\frac{810}{7}}$$

$$\therefore x=1\pm\sqrt[4]{\frac{810}{7}}$$

The exact x intercepts are $\left(1-\sqrt[4]{\frac{810}{7}},0\right)$ and $\left(1+\sqrt[4]{\frac{810}{7}},0\right)$.

f



12 **a** $y=(x+8)(x+3)(x-4)(x-10)$

x intercepts: Let $y=0$

$\therefore (x+8)(x+3)(x-4)(x-10)=0$

$\therefore x=-8, x=-3, x=4, x=10$

$(-8,0),(-3,0),(4,0),(10,0)$ (all cuts)

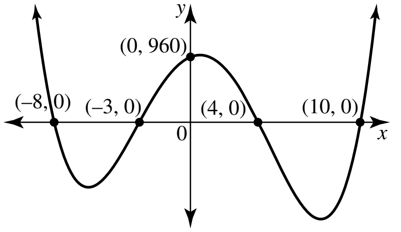
y intercept: Let $x=0$

$\therefore y=(8)(3)(-4)(-10)$

$\therefore y=960$

$(0,960)$

Shape: positive fourth degree polynomial



b $y=-\frac{1}{100}(x+3)(x-2)(2x-15)(3x-10)$

x intercepts: Let $y=0$

$\therefore -\frac{1}{100}(x+3)(x-2)(2x-15)(3x-10)=0$

$\therefore x=-3, x=2, x=\frac{15}{2}, x=\frac{10}{3}$

$(-3,0),(2,0),\left(\frac{15}{2},0\right),\left(\frac{10}{3},0\right)$ (all cuts)

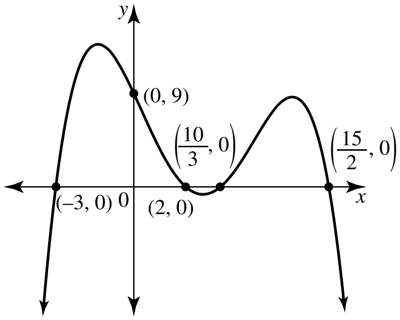
y intercept: Let $x=0$

$\therefore y=-\frac{1}{100}(3)(-2)(-15)(-10)$

$\therefore y=9$

$(0,9)$

Shape: $-\frac{1}{100}(x)(x)(2x)(3x)$ shows a negative fourth degree polynomial



c $y=-2(x+7)(x-1)^2(2x-5)$

x intercepts: Let $y=0$

$\therefore -2(x+7)(x-1)^2(2x-5)=0$

$\therefore x=-7, x=1, x=2.5$

$(-7,0),(2.5,0)$ and turning point $(1,0)$

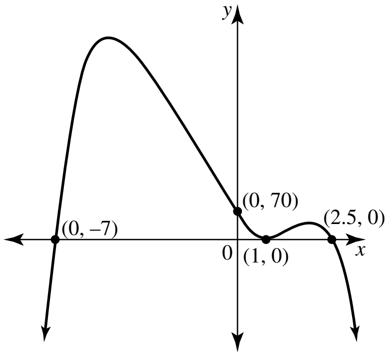
y intercept: Let $x=0$

$\therefore y=-2(7)(-1)^2(-5)$

$\therefore y=70$

$(0,70)$

Shape: $-2(x)(x)^2(2x)$ shows a negative fourth degree polynomial



d $y=\frac{2}{3}x^2(4x-15)^2$

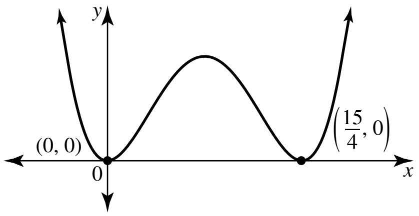
x intercepts: Let $y=0$

$\therefore \frac{2}{3}x^2(4x-15)^2=0$

$\therefore x=0, x=\frac{15}{4}$

$(0,0),\left(\frac{15}{4},0\right)$ (both turning points)

Shape: positive fourth degree polynomial



e $y = 3(1+x)^3(4-x)$

x intercepts: Let $y = 0$

$\therefore y = 3(1+x)^3(4-x) = 0$

$\therefore x = -1, x = 4$

stationary point of inflection $(-1,0)$ and $(4,0)$ (cut)

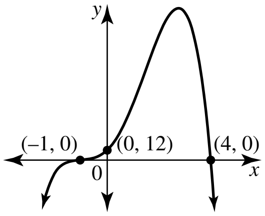
y intercept: Let $x = 0$

$\therefore y = 3(1)^3(4)$

$\therefore y = 12$

$(0,12)$

Shape: $3(x^3)(-x)$ shows a negative fourth degree polynomial



f $y = (3x+10)(3x-10)^3$

x intercepts: Let $y = 0$

$\therefore y = (3x+10)(3x-10)^3 = 0$

$\therefore x = -\dfrac{10}{3}, x = \dfrac{10}{3}$

stationary point of inflection $\left(\dfrac{10}{3}, 0\right)$ and x intercept $\left(-\dfrac{10}{3}, 0\right)$ (cut)

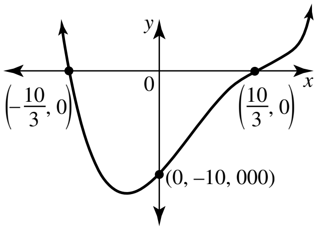
y intercept: Let $x = 0$

$\therefore y = (10)(-10)^3$

$\therefore y = -10\,000$

$(0, -10\,000)$

Shape: $(3x)(3x)^3$ shows a positive fourth degree polynomial



13 a The graph cuts the x axis at $x = -6, x = -5, x = -3$ and $x = 4$. These values identify the linear factors so the equation of the graph must be of the form $y = a(x+6)(x+5)(x+3)(x-4)$.

Substitute the point $(0,5)$

$\therefore 5 = a(6)(5)(3)(-4)$

$\therefore a = -\dfrac{1}{72}$

The equation is $y = -\dfrac{1}{72}(x+6)(x+5)(x+3)(x-4)$

b The x intercepts of the graph occur at:

$x = -2$ (touch) $\Rightarrow (x+2)^2$ is a factor

$x = 0 \Rightarrow x$ is a factor

$x = 4 \Rightarrow (x-4)$ is a factor

The equation is of the form $y = a(x+2)^2 x(x-4)$

Substitute the point $(3,75)$

$\therefore 75 = a(3+2)^2(3)(3-4)$

$\therefore 75 = a(25)(3)(-1)$

$\therefore 75 = -75a$

$\therefore a = -1$

The equation is $y = -x(x-4)(x+2)^2$

c The x intercepts of the graph occur at:

$x = -6 \Rightarrow (x+6)$ is a factor

$x = 0$ (saddle cut) $\Rightarrow$ x^3 is a factor

The equation is of the form $y = a(x+6)x^3$

Substitute the point $(-3,-54)$

$\therefore -54 = a(-3+6)(-3)^3$

$\therefore -54 = a(3)(-27)$

$\therefore -54 = -81a$

$\therefore a = \dfrac{54}{81}$

$\therefore a = \dfrac{2}{3}$

The equation is $y = \dfrac{2}{3}x^3(x+6)$

d The x intercepts of the graph occur at:

$x = -1.5$ (touch) $\Rightarrow (x+1.5)^2$ is a factor

$x = 0.8$ (touch) $\Rightarrow (x-0.8)^2$ is a factor

The equation is of the form $y = a(x+1.5)^2(x-0.8)^2$

Substitute the point $(0,54)$

$\therefore 54 = a(1.5)^2(-0.8)^2$

Using fractions rather than decimals,

$$54 = a\left(\frac{3}{2}\right)^2\left(-\frac{4}{5}\right)^2$$

$$\therefore 54 = a \times \frac{9\times16}{4\times25}$$

$$\therefore 54 = a \times \frac{36}{25}$$

$$\therefore a = \frac{54\times25}{36}$$

$$\therefore a = \frac{3\times25}{2}$$

$$\therefore a = \frac{75}{2}$$

The equation becomes

$$y = \frac{75}{2}\left(x+\frac{3}{2}\right)^2\left(x-\frac{4}{5}\right)^2$$

$$= \frac{75}{2}\times\frac{1}{4}(2x+3)^2\times\frac{1}{25}(5x-4)^2$$

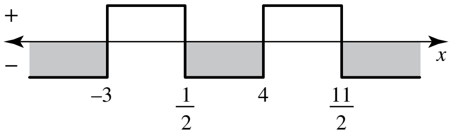
$$= \frac{3}{8}(2x+3)^2(5x-4)^2$$

$$\therefore y = \frac{3}{8}(2x+3)^2(5x-4)^2$$

14 a $(x+3)(2x-1)(4-x)(2x-11) < 0$

Zeros: $x = -3, x = \dfrac{1}{2}, x = 4, x = \dfrac{11}{2}$ (all cuts)

Leading term: $(x)(2x)(-x)(2x) = -4x^4$



Answer: $x < -3$ or $0.5 < x < 4$ or $x > 5.5$

b

$$9x^4 - 49x^2 = 0$$

$$\therefore x^2\left(9x^2-49\right) = 0$$

$$\therefore x^2(3x-7)(3x+7) = 0$$

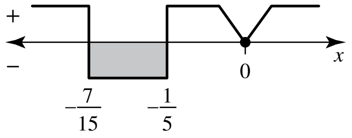
$$\therefore x = 0, x = \frac{7}{3}, x = -\frac{7}{3}$$

c $300x^4+200x^3+28x^2 \le 0$

$\therefore 4x^2\left(75x^2+50x+7\right)\le 0$

$\therefore 4x^2(15x+7)(5x+1) \le 0$

Zeros: $x=0$ (touch), $x=-\frac{7}{15}$ and $x=-\frac{1}{5}$.



Answer: $-\frac{7}{15}\le x\le -\frac{1}{5}$ or $x=0$

d $-3x^4+20x^3+10x^2-20x-7=0$

Let $P(x)=-3x^4+20x^3+10x^2-20x-7$

$P(1)=-3+20+10-20-7=0\Rightarrow(x-1)$ is a factor.

$P(-1)=-3-20+10+20-7=0\Rightarrow(x+1)$ is a factor.

Thus, $(x-1)(x+1)=x^2-1$ is a quadratic factor.

$\therefore -3x^4+20x^3+10x^2-20x-7=\left(x^2-1\right)\left(-3x^2+bx+7\right)$

Equate coefficients of x^3: $20=b$

$\therefore -3x^4+20x^3+10x^2-20x-7=\left(x^2-1\right)\left(-3x^2+20x+7\right)$

The equation becomes

$\left(x^2-1\right)\left(-3x^2+20x+7\right)=0$

$\therefore (x-1)(x+1)(-3x-1)(x-7)=0$

$\therefore x=1, x=-1, x=-\frac{1}{3}, x=7$

e $x^4+x^3-8x=8$

$\therefore x^4+x^3-8x-8=0$

$\therefore x^3(x+1)-8(x+1)=0$

$\therefore (x+1)(x^3-8)=0$

$\therefore x+1=0$ or $x^3-8=0$

$\therefore x=-1$ or $x^3=8$

$\therefore x=-1$ or $x=2$

f $20(2x-1)^4-8(1-2x)^3\ge 0$

$\therefore 20(2x-1)^4+8(2x-1)^3\ge 0$

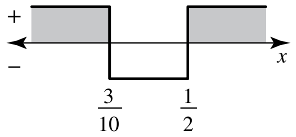
$\therefore 4(2x-1)^3(5(2x-1)+2)\ge 0$

$\therefore 4(2x-1)^3(10x-3)\ge 0$

Zeros are $x=\frac{1}{2}$ (multiplicity 3) and $x=\frac{3}{10}$.

A zero of multiplicity 3 is a just a ‘cut’ on a sign diagram.

Leading term is $4(2x)^3(10x)$ is positive.



Answer: $x\le\frac{3}{10}$ or $x\ge\frac{1}{2}$

15 a $-x^4+18x^2-81.$

Let $a=x^2$

$\therefore -a^2+18a-81$

$=-(a^2-18a+81)$

$=-(a-9)^2$

Substitute back for x

$=-(x^2-9)^2$

$=-\left((x-3)(x+3)\right)^2$

$=-(x-3)^2(x+3)^2$

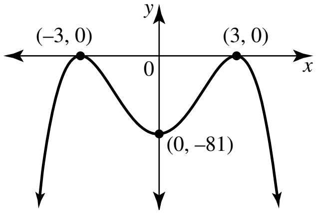
$\therefore -x^4+18x^2-81=-(x-3)^2(x+3)^2$

b $y=-x^4+18x^2-81\Rightarrow y=-(x-3)^2(x+3)^2$

x intercepts occur at $x=\pm3$ (both touch)

y intercept $(0,-81)$

Shape: negative fourth degree polynomial



c As no part of the graph lies above the x axis, $\{x:-x^4+18x^2-81>0\}=\varnothing$

d $\{x:x^4-18x^2+81>0\}$

If $x^4-18x^2+81>0$ then $-1\times\left(x^4-18x^2+81\right)<0$

$\therefore -x^4+18x^2-81<0$

As the graph in part b lies below the x axis at all places except for the x intercepts at $x=\pm3$, then $\{x:x^4-18x^2+81>0\}=R\setminus\{-3,3\}$.

16 a At the intersection of the graphs of $y=x^4$ and $y=2x^3$, $x^4=2x^3$.

$\therefore x^4-2x^3=0$

$\therefore x^3(x-2)=0$

$\therefore x=0, x=2$

When $x=0, y=0\Rightarrow(0,0)$

When $x=2, y=16\Rightarrow(2,16)$

The point P is (2,16).

b Given the point (2,16) lies on $y=ax^2$, then

$16=a(2)^2$

$\therefore 4a=16$

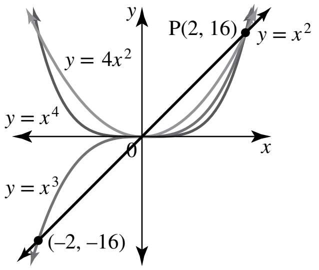
$\therefore a=4$

Given the point (2,16) lies on $y=mx$, then

$16=m(2)$

$\therefore m=8$

c The parabola has the equation $y=4x^2$ and the line has the equation $y=8x$.



d i The graphs of $y=nx^3$ and $y=x^4$ intersect when

$nx^3=x^4$

$\therefore nx^3-x^4=0$

$\therefore x^3(n-x)=0$

$\therefore x=0, x=n$

When $x=0, y=0\Rightarrow(0,0)$

When $x=n, y=n^4\Rightarrow\left(n,n^4\right)$.

The graphs intersect at $(0,0)$ and $\left(n,n^4\right)$.

ii Substitute the point $\left(n,n^4\right)$ in $y=ax^2$

$\therefore n^4=an^2$

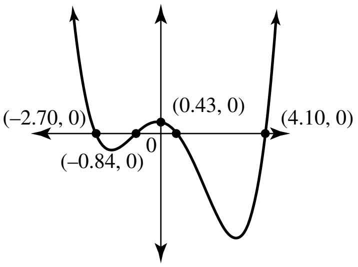
$\therefore a=n^2$

Substitute the point $\left(n,n^4\right)$ in $y=mx$

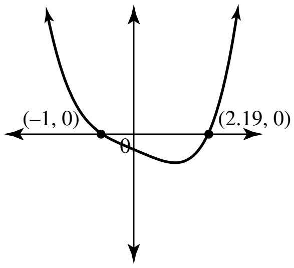
$\therefore n^4=mn$

$\therefore m=n^3$

17 Sketch the graph on the graphing screen and use the Analysis tools to obtain the required points.
The x intercepts are $(-2.70, 0), (-0.84, 0), (0.43, 0), (4.10, 0)$.
The minimum turning points are $(-2, -12), (2.92, -62.19)$ and the maximum turning point is $(-0.17, 4.34)$.



18 **a** $y = x^4 - 7x - 8$



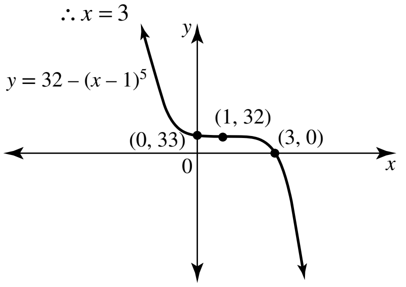
One minimum turning point at $(1.21, -14.33)$ and x intercepts are $(-1, 0), (2.19, 0)$.

b As $(-1, 0)$ is an x intercept, $(x + 1)$ is a factor.
Use CAS to divide $x^4 - 7x - 8$ by $(x + 1)$.
$\therefore x^4 - 7x - 8 = (x + 1)(x^3 - x^2 + x - 8)$.

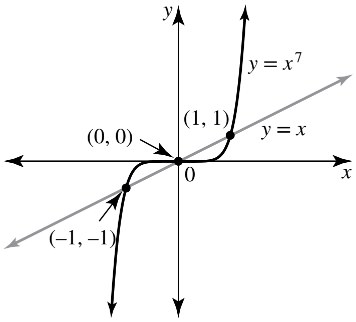
Exercise 5.3 — Families of polynomials

1 $y = 32 - (x - 1)^5$
Stationary point of inflection $(1, -32)$
y intercept $(0, 33)$
x intercept: Let $y = 0$

$$\therefore 0 = 32 - (x - 1)^5$$
$$\therefore (x - 1)^5 = 32$$
$$\therefore x - 1 = \sqrt[5]{32}$$
$$\therefore x - 1 = 2$$
$$\therefore x = 3$$

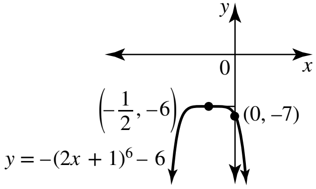


2 $y = x^7$ and $y = x$



Hence $\{x : x^7 \le x\} = \{x : x \le -1\} \cup \{x : 0 \le x \le 1\}$.

3 $y = -(2x + 1)^6 - 6$
Maximum turning point when $2x + 1 = 0 \Rightarrow x = -\dfrac{1}{2}$.
Co-ordinates are $\left(-\dfrac{1}{2}, -6\right)$.
y intercept: Let $x = 0$
$y = -(1)^6 - 6$
$= -7$
$(0, -7)$ is y intercept.
No x intercepts

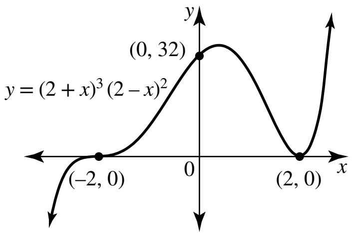


4 **a** $y = a(x - b)^8 + c$
turning point $(-1, -12) \Rightarrow y = a(x + 1)^8 - 12$
point $(0, 0) \Rightarrow 0 = a(1)^8 - 12$
$\therefore a = 12$
Hence, the equation is $y = 12(x + 1)^8 - 12$

b As the turning point is $(-1, -12)$ the axis of symmetry has the equation $x = -1$.

c As one x intercept occurs at $x = 0$ and the axis of symmetry is $x = -1$, the other x intercept occurs at $x = -2$. The other x intercept is $(-2, 0)$.

5 $y = (2 + x)^3(2 - x)^2$
x intercepts at $x = -2$ (cut) and $x = 2$ (touch)
y intercept at $y = 8 \times 4 = 32$
Leading term $(x)^3(-x)^2 = x^5$ gives positive degree 5 shape so the graph starts below the x axis and rises.



6 **a** The given graph shows the graph
cuts the x axis at $x = -4 \Rightarrow (x + 4)$ is a factor
touches the x axis at $x = -1 \Rightarrow (x + 1)^2$ is a factor
cuts the x axis at $x = 2 \Rightarrow (x - 2)$ is a factor
and touches the x axis at $x = 4 \Rightarrow (x - 4)^2$ is a factor.
Adding the multiplicities of the factors gives a degree 6 polynomial.

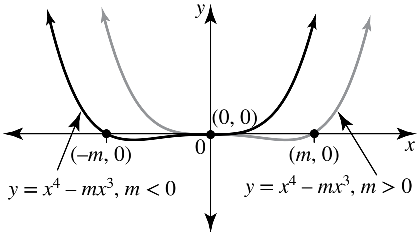
b Let the equation be $y = a(x + 4)(x + 1)^2(x - 2)(x - 4)^2$
Substitute the given point $(5, 24.3)$.
$\therefore 24.3 = a(9)(6)^2(3)(1)^2$
$$\therefore a = \frac{24.3}{9 \times 36 \times 3}$$
$\therefore a = 0.025$
The equation is $y = 0.025(x + 4)(x + 1)^2(x - 2)(x - 4)^2$

7 $y = x^4 - mx^3$

a x intercepts: Let $y = 0$
$\therefore x^4 - mx^3 = 0$
$\therefore x^3(x - m) = 0$
$\therefore x = 0, x = m$
$(0, 0), (m, 0)$

b At the origin there is a stationary point of inflection. If $m < 0$, the point $(m, 0)$ lies to the left of the origin and if $m > 0$ then it will lie to the right of the origin.

The shape of the graph would be one of the forms shown.



If $m < 0$ the stationary point of inflection at the origin increases from negative to positive as x increases while if $m > 0$ it decreases from positive to negative.

c $y = x^4 - mx^3$

Substitute the point $(-1, -16)$

$\therefore -16 = (-1)^4 - m(-1)^3$

$\therefore -16 = 1 + m$

$\therefore m = -17$

The required curve has equation $y = x^4 + 17x^3$.

8 $y = a(x-3)^2 + 5 - 4a,\ a \in R \setminus \{0\}$.

a Let $x = 1$

$\therefore y = a(-2)^2 + 5 - 4a$

$\therefore y = 4a + 5 - 4a$

$\therefore y = 5$

Therefore,the point $(1, 5)$ lies on every member of the family regardless of the value of a.

b The turning point is $(3, 5 - 4a)$. If this lies on the x axis, then $5 - 4a = 0$.

This means $a = \frac{5}{4}$ for the turning point to lie on the x axis.

c The turning point is $(3, 5 - 4a)$ and could be a maximum or a minimum.

If the turning point is a maximum, $a < 0$ and there will be no x intercepts if $5 - 4a < 0$.

But, if $a < 0$ then $-4a > 0$ so $5 - 4a$ could never be negative.

If the turning point is a minimum, $a > 0$ and there will be no x intercepts if $5 - 4a > 0$.

$5 - 4a > 0$

$\therefore 5 > 4a$

$\therefore a < \frac{5}{4}$

Hence, if $0 < a < \frac{5}{4}$ the parabolas will have no x intercepts.

9 a $y = x^6, y = -x^6, y = x, y = -x$

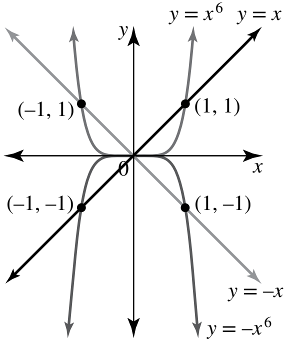
$y = x^6$: Points $(0,0), (1,1), (-1,1)$

$y = -x^6$: Points $(0,0), (1,-1), (-1,-1)$

Turning point at the origin for each graph

$y = x$: Points $(0,0), (1,1), (-1,-1)$

$y = -x$: Points $(0,0), (1,-1), (-1,1)$

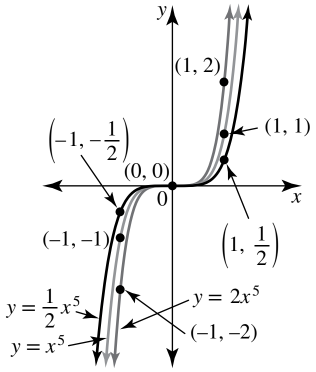


b $y = x^5$: Points $(0,0), (-1,-1), (1,1)$

$y = 2x^5$: Points $(0,0), (-1,-2), (1,2)$

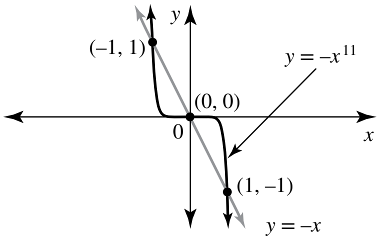
$y = \frac{1}{2}x^5$: Points $(0,0), \left(-1, \frac{1}{2}\right), \left(1, \frac{1}{2}\right)$

Stationary points of inflection at the origin for each graph



10 a i $y = -x^{11}$: points $(0,0), (-1,1), (1,-1)$ with stationary point of inflection at $(0,0)$.

$y = -x$: line through $(0,0), (-1,1), (1,-1)$



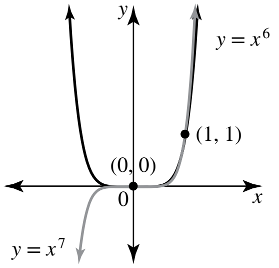
ii For $x^{11} > x, -x^{11} < -x$

From the diagram, the graph of $y = -x^{11}$ lies below that of $y = -x$ for $-1 < x < 0$ and for $x > 1$.

$\{x : x^{11} > x\} = \{x : -1 < x < 0\} \cup \{x : x > 1\}$

b i $y = x^6$ has a minimum turning point at the origin and passes through the points $(\pm 1, 1)$.

$y = x^7$ has a stationary point of inflection at the origin and passes through the points $(-1,-1), (1,1)$.



ii The graphs intersect at $(0,0)$ and at $(1,1)$.

Hence, the solution to the equation $x^6 = x^7$ is $x = 0$ or $x = 1$.

c i $y = (x+1)^4 + 1$ and $y = (x+1)^5 + 1$.

$y = (x+1)^4 + 1$ has a minimum turning point $(-1,1)$ and no x intercepts.

y intercept: Let $x = 0$

$\therefore y = (1)^4 + 1 = 2$

$(0, 2)$

$y = (x+1)^5 + 1$ has a stationary point of inflection at $(-1,1)$.

x intercept: Let $y = 0$

$\therefore (x+1)^5 + 1 = 0$

$\therefore (x+1)^5 = -1$

$\therefore x + 1 = \sqrt[5]{-1}$

$\therefore x + 1 = -1$

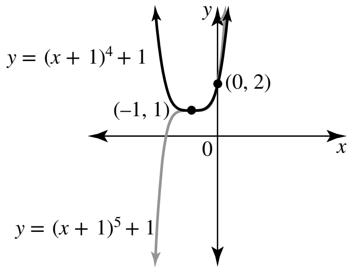
$\therefore x = -2$

$(-2,0)$

y intercept: Let $x=0$

$\therefore y=(1)^5+1=2$

$(0,2)$



ii The graphs intersect at the points $(-1,1)$ and $(0,2)$.

11 a i $y=\frac{1}{16}(x-4)^{10}+3$ is even degree with positive leading term. It has a minimum turning point at $(4,3)$.

ii $y=-\frac{1}{125}\left(\frac{3x}{2}-5\right)^6$ is even degree with negative leading term. It has a maximum turning point with a y co-ordinate of zero and a x co-ordinate for which $\left(\frac{3x}{2}-5\right)=0$.

$\therefore \frac{3x}{2}=5$

$\therefore x=\frac{10}{3}$

The maximum turning point is $\left(\frac{10}{3},0\right)$.

b i $y=\frac{x^5+27}{54}$

$\therefore y=\frac{x^5}{54}+\frac{27}{54}$

$\therefore y=\frac{1}{54}x^5+\frac{1}{2}$

The stationary point of inflection is $\left(0,\frac{1}{2}\right)$.

ii $y=16-(2x+1)^7$

$\therefore y=-(2x+1)^7+16$

When $2x+1=0,\ x=-\frac{1}{2}$

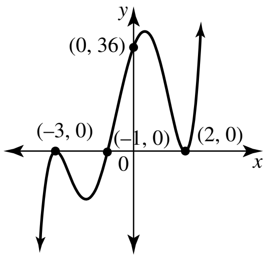
The stationary point of inflection is $\left(-\frac{1}{2},16\right)$.

12 a $y=(x+3)^2(x+1)(x-2)^2$

x intercepts occur at $x=-3$ (touch), $x=-1$ (cut) and $x=2$ (touch)

y intercept: Let $x=0,\ \therefore y=(3)^2(1)(-2)^2=36,\ \ (0,36)$

Shape: positive fifth degree polynomial so graph starts from below x axis and rises.



b $y=\frac{1}{4}(x+2)^3(8-x)$

x intercepts occur at $x=-2$ (saddle cut), $x=8$ (cut)

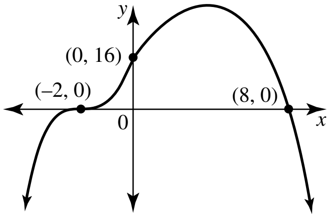
y intercept: Let $x=0$

$\therefore y=\frac{1}{4}(2)^3(8)$

$\therefore y=16$

$(0,16)$

Shape: negative fourth degree polynomial



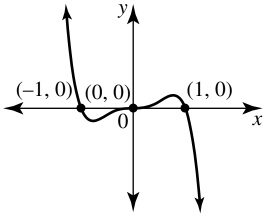
c $y=-x^3(x^2-1)$

$\therefore y=-x^3(x-1)(x+1)$

Stationary point of inflection at the origin

Other x intercepts at $(-1,0)$ and $(1,0)$.

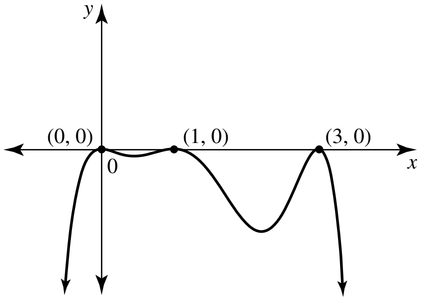
Shape: negative fifth degree polynomial so graph starts from above the x axis and falls.



d $y=-x^2(x-1)^2(x-3)^2$

Turning points at $(0,0),(1,0),(3,0)$

Shape: negative sixth degree polynomial so graph starts from below the x axis.



13 a $y=a(x+b)^5+c$

Stationary point of inflection at $(-1,7)$

$\therefore y=a(x+1)^5+7$

Substitute the point $(-2,-21)$.

$\therefore -21=a(-1)^5+7$

$\therefore -28=-a$

$\therefore a=28$

The equation is $y=28(x+1)^5+7$.

b Turning point at $(-2,0)\Rightarrow(x-2)^2$ is a factor

x intercept $(4,0)\Rightarrow(x-4)$ is a factor

As the polynomial has degree 4, there is one other linear factor.

Let this factor be $(ax+b)$

$\therefore y=(x-2)^2(x-4)(ax+b)$

However, the polynomial is monic so the coefficient of x^4 must be 1.

$\therefore a=1$

$\therefore y=(x-2)^2(x-4)(x+b)$

Substitute $(0,48)$

$\therefore 48=(-2)^2(-4)(b)$

$\therefore -48=-16b$

$\therefore b=3$

The equation is $y=(x+2)^2(x-4)(x-3)$ and the other x intercept is $(3,0)$.

c The x intercepts of the graph occur at:

$x=-5$ (touch) $\Rightarrow (x+5)^2$ is a factor

$x=-1$ (saddle cut) $\Rightarrow (x+1)^3$ is a factor

$x=1 \Rightarrow (x-1)$ is a factor

$x=3 \Rightarrow (x-3)$ is a factor

The equation is of the form $y=a(x+5)^2(x+1)^3(x-1)(x-3)$

Substitute the point $(-3,-76.8)$

$\therefore -76.8=a(-3+5)^2(-3+1)^3(-3-1)(-3-3)$

$\therefore -76.8=a(4)(-8)(-4)(-6)$

$\therefore -76.8=a(-32\times 24)$

$$\therefore a=\frac{-\cancel{76.8}^{9.6}}{-32\times\cancel{24}_{3}}$$

$$\therefore a=\frac{3.2}{32}$$

$\therefore a=0.1$

The equation is $y=0.1(x+5)^2(x+1)^3(x-1)(x-3)$

d $y=(ax+b)^4$

Expanding (or use the general term formula)

$y=(ax)^4+4(ax)^3(b)+6(ax)^2(b)^2+4(ax)(b)^3+(b)^4$

Coefficient of term in x is $4ab^3$; coefficient of term in x^2 is $6a^2b^2$ and these coefficients are equal.

$\therefore 4ab^3=6a^2b^2$

Divide by $2ab^2$ (neither a nor b are zero since the coefficients are non-zero)

$\therefore 2b=3a$....(1)

Coefficient of x^3 is $4a^3b$ and this equals 1536

$\therefore 4a^3b=1536$

$\therefore a^3b=384$....(2)

Divide equation (1) by equation (2)

$$\therefore \frac{2b}{a^3b}=\frac{3a}{384}$$

$$\therefore \frac{2}{a^3}=\frac{a}{128}$$

$\therefore 256=a^4$

$\therefore a=\pm 4$

Substitute $a=\pm 4$ in equation (1)

$\therefore 2b=\pm(3\times 4)$

$\therefore 2b=\pm 12$

$\therefore b=\pm 6$

The equation is $y=(4x+6)^4$ or $y=(-4x-6)^4$.

However, these are the same since

$y=(-4x-6)^4=\left(-(4x+6)\right)^4=(4x+6)^4$.

The equation is $y=(4x+6)^4$ which can be expressed as

$y=\left(2(2x+3)\right)^4$

$=16(2x+3)^4$

14 a i $9x^3-36x^5=0$

$\therefore 9x^3(1-4x^2)=0$

$\therefore 9x^2(1-2x)(1+2x)=0$

$$\therefore x=0, x=\frac{1}{2}, x=-\frac{1}{2}$$

ii $x^6-2x^3+1=0$

$\therefore (x^3)^2-2x^3+1=0$

$\therefore (x^3-1)^2=0$

$\therefore x^3=1$

$\therefore x=1$

iii $x^5+x^4-x^3-x^2-2x-2=0$

Factorise by grouping terms together

$\therefore (x^5+x^4)-(x^3+x^2)-(2x+2)=0$

$\therefore x^4(x+1)-x^2(x+1)-2(x+1)=0$

$\therefore (x+1)(x^4-x^2-2)=0$

$\therefore x=-1$ or $x^4-x^2-2=0$

Consider $x^4-x^2-2=0$

Let $a=x^2$

$\therefore a^2-a-2=0$

$\therefore (a-2)(a+1)=0$

$\therefore a=2$ or $a=-1$

$\therefore x^2=2$ or $x^2=-1$ (reject)

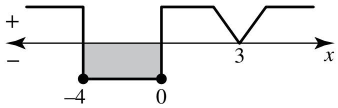
$\therefore x=\pm\sqrt{2}$

Answer: $x=-\sqrt{2},-1,\sqrt{2}$

b i $x(x+4)^3(x-3)^2<0$

Zeros: $x=0$ (cut), $x=-4$ (saddle cut), $x=3$ (touch)

Shape: Positive sixth degree



Answer $x(x+4)^3(x-3)^2<0$ when $-4<x<0$.

ii $\frac{1}{81}(9-10x)^5-3\geq 0$

$$\therefore \frac{1}{81}(9-10x)^5\geq 3$$

$\therefore (9-10x)^5\geq 3\times 81$

$\therefore (9-10x)^5\geq 3\times 3^4$

$\therefore (9-10x)^5\geq 3^5$

Take the fifth root of each side

$\therefore (9-10x)\geq 3$

$\therefore 9-3\geq 10x$

$\therefore 10x\leq 6$

$$\therefore x\leq\frac{3}{5}$$

iii $(x-2)^4\leq 81$

$\therefore (x-2)^4-81\leq 0$

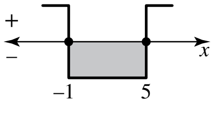
$\therefore \left[(x-2)^2+9\right]\left[(x-2)^2-9\right]\leq 0$

$\therefore \left[(x-2)^2+9\right](x-2+3)(x-2-3)\leq 0$

$\therefore \left[(x-2)^2+9\right](x+1)(x-5)\leq 0$

Zeros: Since $(x-2)^2+9\neq 0$, the zeros are $x=-1, x=5$

Shape: positive fourth degree



Answer: $-1\leq x\leq 5$

15 a i The equation of a line is $y-y_1=m(x-x_1)$

The lines pass through the point $(2,3)$

$\therefore y-3=m(x-2)$

$\therefore y=mx-2m+3$

ii Substitute $(0,0)$

$\therefore y-3=m(0,0)$

$\therefore 0=-2m+3$

$$\therefore m=\frac{3}{2}$$

Substitute $m=\frac{3}{2}$ in $y=mx-2m+3$

$y=\frac{3}{2}x-2\times\frac{3}{2}+3$

$\therefore y=\frac{3}{2}x$

b **i** $y=ax^2+bx$

All of the parabolas pass through the origin.

ii The point $(2,6)$ must lie on the parabola.

$\therefore 6=a(2)^2+b(2)$

$\therefore 6=4a+2b$

$\therefore 2a+b=3$

$\therefore b=3-2a$ ......(1)

The equation of the parabola is $y=ax^2+(3-2a)x$, $a\neq 0$.

c $y=mx^2-x^4,\ m>0$

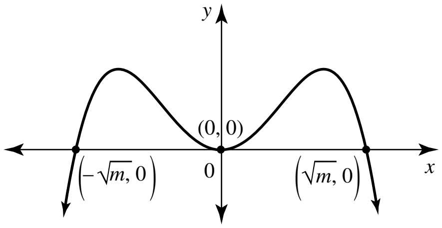
$\therefore y=x^2\left(m-x^2\right)$

Since $m>0, \sqrt{m}$ is real

$\therefore y=x^2(\sqrt{m}-x)(\sqrt{m}+x)$

The x intercepts are $(0,0)$ which is a turning point and $\left(\sqrt{m},0\right)$ and $\left(-\sqrt{m},0\right)$.

Shape: Negative fourth degree polynomial



16 **a** The family for which $y=k$ is a set of horizontal lines. The family for which $y=x^2+bx+10$ is the set of concave up parabolas with y intercept $(0,10)$.

b $y=x^2+bx+10$ for $b=-7$ becomes $y=x^2-7x+10$.

$\therefore y=(x-2)(x-5)$

x intercepts: $(2,0),(5,0)$

axis of symmetry: $x=\frac{2+5}{2}=3.5$

turning point: Let $x=3.5$

$\therefore y=(1.5)(-1.5)$

$\therefore y=-2.25$

Minimum turning point $(3.5,-2.25)$.

For $b=7$

$y=x^2+7x+10$

$\quad=(x+2)(x+5)$

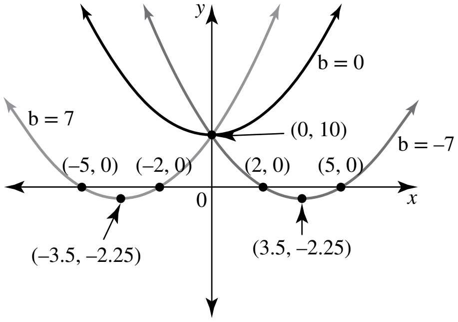
Hence,

x intercepts: $(-2,0),(-5,0)$

Minimum turning point $(-3.5,-2.25)$.

For $b=0,\ y=x^2+10$

No x intercepts and minimum turning point $(0,10)$.



c **i** A horizontal line intersects $y=x^2+7x+10$ once when it is a tangent at the turning point $(-3.5,-2.25)$.

So $k=-2.25$.

ii Any horizontal line for which $k>-2.25$ will intersect the parabola twice.

iii Any horizontal line for which $k<-2.25$ will not intersect the parabola.

d At the intersection of $y=7$ and $y=x^2+bx+10$,

$x^2+bx+10=7$

$\therefore x^2+bx+3=0$

$\Delta=b^2-4\times1\times3$

$\therefore\Delta=b^2-12$

i For one intersection, $\Delta=0$

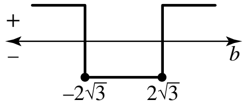
$\therefore b^2-12=0$

$\therefore b=\pm\sqrt{12}$

$\therefore b=\pm2\sqrt{3}$

ii and **iii**

Consider the sign diagram of the discriminant.



For two intersections, $\Delta>0$

$\therefore b<-2\sqrt{3}$ or $b>2\sqrt{3}$

For no intersections, $\Delta<0$

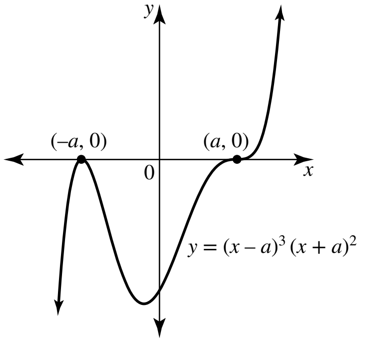
$\therefore -2\sqrt{3}<b<2\sqrt{3}$

17 $y=(x-a)^3(x+a)^2$ where a is a positive real constant.

a Noting the multiplicity of each factor, at $(a,0)$ there is a stationary point of inflection and at $(-a,0)$ there is a turning point.

b Since $a>0$, the point $(-a,0)$ lies to the left of the origin and $(a,0)$ lies to the right.

The polynomial is a positive degree 5 so $(-a,0)$ must be a maximum turning point.



c The line $y=-x$ passes through the origin with a negative gradient so it will intersect the graph once.

d The point $\left(\frac{a}{2},-\frac{a}{2}\right)$ must lie on both the line and the curve.

Substitute $\left(\frac{a}{2},-\frac{a}{2}\right)$ into $y=(x-a)^3(x+a)^2$

$$\therefore -\frac{a}{2} = \left(\frac{a}{2} - a\right)^3 \left(\frac{a}{2} + a\right)^2$$

$$\therefore -\frac{a}{2} = \left(-\frac{a}{2}\right)^3 \left(\frac{3a}{2}\right)^2$$

$$\therefore -\frac{a}{2} = -\frac{a^3}{8} \times \frac{9a^2}{4}$$

$$\therefore -\frac{a}{2} = -\frac{9a^5}{32}$$

$$\therefore 16a = 9a^5$$

$$\therefore 9a^5 - 16a = 0$$

$$\therefore a\left(9a^4 - 16\right) = 0$$

$$\therefore a = 0 \text{ or } a^4 = \frac{16}{9}$$

$$\therefore a = 0 \text{ or } a = \pm\frac{2}{\sqrt[4]{9}}$$

$$\therefore a = 0 \text{ or } a^4 = \frac{16}{9}$$

$$\therefore a = 0 \text{ or } a = \pm\frac{2}{\sqrt[4]{9}}$$

Since a is positive, then $a = \frac{2}{\sqrt[4]{9}}$

$$\therefore a = \frac{2}{\sqrt{3}}$$

$$\therefore a = \frac{2\sqrt{3}}{3}$$

18 $y = ax^3 + (3-2a)x^2 + (3a+1)x - 4 - 2a$ where $a \in R \setminus \{0\}$.

a Let $x = 1$

$$\therefore y = a + (3-2a) + (3a+1) - 4 - 2a$$

$$\therefore y = (a - 2a + 3a - 2a) + (3 + 1 - 4)$$

$$\therefore y = 0$$

Therefore, the point $(1,0)$ is common to all this family.

b Substitute $(0,0)$ in the curve's equation

$$\therefore 0 = 0 + 0 + 0 - 4 - 2a$$

$$\therefore 2a = -4$$

$$\therefore a = -2$$

The equation of the curve with $a = -2$ becomes

$$y = -2x^3 + (3+4)x^3 + (-6+1)x$$

$$\therefore y = -2x^3 + 7x^2 - 5x$$

For the graph, it is known the origin and $(1,0)$ are x intercepts. There is one more to obtain.

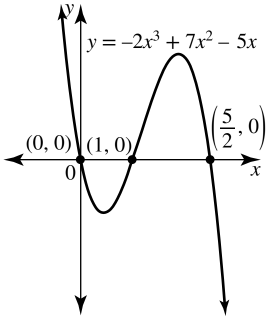
$$y = -2x^3 + 7x^2 - 5x$$

$$= -x\left(2x^2 - 7x + 5\right)$$

$$= -x(x-1)(2x-5)$$

The other x intercept is $\left(\frac{5}{2}, 0\right)$.

The shape of the graph is for a negative third degree polynomial.



c Substitute the point $(-1,-10)$ into the curve's equation to calculate a.

$$-10 = -a + (3-2a)(1) + (3a+1)(-1) - 4 - 2a$$

$$\therefore -10 = (-a - 2a - 3a - 2a) + (3 - 1 - 4)$$

$$\therefore -10 = -8a - 2$$

$$\therefore -8 = -8a$$

$$\therefore a = 1$$

With $a = 1$, the equation becomes

$$y = x^3 + (3-2)x^2 + (3+1)x - 4 - 2$$

$$\therefore y = x^3 + x^2 + 4x - 6$$

As $(1,0)$ is a x intercept, $(x-1)$ is a factor

$$\therefore x^3 + x^2 + 4x - 6 = (x-1)(x^2 + bx + 6)$$

$$= (x-1)(x^2 + 2x + 6)$$

Consider the quadratic factor $x^2 + 2x + 6$

Its discriminant is $\Delta = 4 - 4 \times 1 \times 6 = -20$

Since $\Delta < 0$ there are no real solutions to $x^2 + 2x + 6 = 0$ hence the cubic graph has exactly one x intercept.

d At the intersection of

$y = ax^3 + (3-2a)x^2 + (3a+1)x - 4 - 2a$ and

$y = (a-1)x^3 - 2ax^2 + (3a-2)x - 2a - 5$,

$$ax^3 + (3-2a)x^2 + (3a+1)x - 4 - 2a$$

$$= (a-1)x^3 - 2ax^2 + (3a-2)x - 2a - 5$$

$$\therefore x^3\left(a - (a-1)\right) + x^2\left((3-2a) + 2a\right) +$$

$$x\left((3a+1) - (3a-2)\right) - 4 - 2a + 2a + 5 = 0$$

$$\therefore x^3 + 3x^2 + 3x + 1 = 0$$

The LHS is the expansion of $(x+1)^3$

$$\therefore (x+1)^3 = 0$$

$$\therefore x + 1 = 0$$

$$\therefore x = -1$$

Substitute $x = -1$ in $y = ax^3 + (3-2a)x^2 + (3a+1)x - 4 - 2a$

$$\therefore y = -a + (3-2a) - (3a+1) - 4 - 2a$$

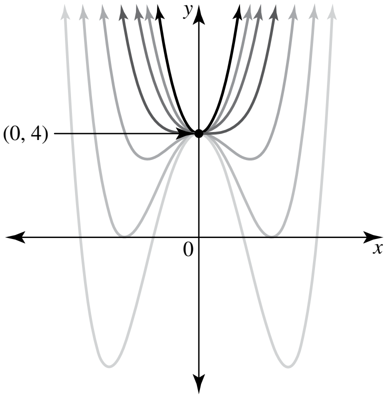
$$\therefore y = -8a - 2$$

The two sets of curves intersect at $(-1, -8a-2)$.

Hence, no matter what the value of a is, the point of intersection lies on the vertical line with equation $x = -1$.

19 $y = x^4 + ax^2 + 4,\ a \in R$ for $a = -6, -4, -2, 0, 2, 4, 6$

Using the graphing screen, the graphs can be seen to resemble those shown in the diagram.



If $a = -4$, $y = x^4 - 4x^2 + 4$. This can be factorised as follows:

$$y = \left(x^2 - 2\right)^2$$

$$= \left(\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)\right)^2$$

$$= \left(x - \sqrt{2}\right)^2 \left(x + \sqrt{2}\right)^2$$

Hence, the turning points of the graph lie on the x axis and the polynomial equation $x^4 - 4x^2 + 4 = 0$ has two roots.

As the family of graphs show, there will be: **a** four roots when $a < -4$, **b** two roots when $a = -4$ and **c** no roots when $a > -4$.

20 $y = \frac{1}{18}(x+2)^6 - 2$

Turning point $(-2, -2)$

y intercept: Let $x = 0$

$$\begin{aligned}\therefore y &= \frac{1}{18}(2)^6 - 2\\ &= \frac{1}{9} \times 32 - 2\\ &= \frac{32}{9} - \frac{18}{9}\end{aligned}$$

$\therefore y = \frac{14}{9}$

$\left(0, \frac{14}{9}\right)$

Let the equation of the parabola be $y = a(x-h)^2 + k$

Axis of symmetry has equation $x = 1$

$\therefore y = a(x-1)^2 + k$

Substitute the point $(-2, -2)$

$\therefore -2 = a(-2-1)^2 + k$

$\therefore -2 = 9a + k....(1)$

Substitute the point $\left(0, \frac{14}{9}\right)$

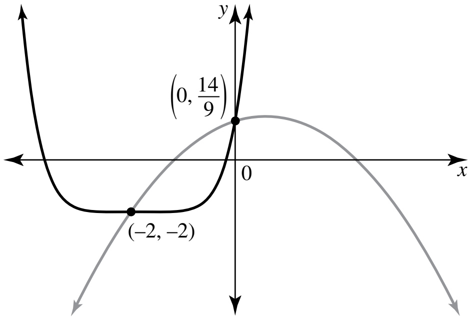
$\therefore \frac{14}{9} = a(-1)^2 + k$

$\therefore \frac{14}{9} = a + k....(2)$

Solve the simultaneous equations using the template on CAS (or do it yourself!)

$a = -\frac{4}{9}, k = 2$

The equation of the parabola is $y = -\frac{4}{9}(x-1)^2 + 2$



Exercise 5.4 — Numerical approximations to roots of polynomial equations

1 a $P(x) = x^3 + 3x^2 - 7x - 4$

$$\begin{aligned}P(1) &= 1 + 3 - 7 - 4\\ &= -7\\ &< 0\\ P(2) &= 8 + 12 - 14 - 4\\ &= 2\\ &> 0\end{aligned}$$

Therefore $P(x) = 0$ for some $x \in [1, 2]$.

Therefore the equation $x^3 + 3x^2 - 7x - 4 = 0$ has a root which lies between $x = 1$ and $x = 2$.

b Since $P(2)$ is closer to zero than $P(1)$, a first estimate of the root is $x = 2$.

c First iteration: Midpoint of interval $[1, 2]$ is $x = 1.5$. This is a second estimate of the root.

Second iteration:

$$\begin{aligned}P(1.5) &= \left(\frac{3}{2}\right)^3 + 3\left(\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) - 4\\ &= \frac{27}{8} + \frac{27}{4} - \frac{21}{2} - 4\\ &= \frac{27 + 54 - 84 - 32}{8}\\ &= -\frac{35}{8}\\ &< 0\end{aligned}$$

The root lies in the interval $[1.5, 2]$.

Midpoint of this interval is $x = 1.75$. This is a third estimate of the root.

d Simplest to use a form of technology to continue the iterations.

Using a calculator gives the following.

Midpoint	Value of $P(x)$	New interval
$x = 1.75$	$-1.703\,125$	$[1.75, 2]$
$x = 1.875$	$0.013\,671\,875 < 0.0$	

An estimate of the solution to the equation is $x = 1.875$.

2 a $y = x^4 - 2x - 12$

x	−3	−2	−1	0	1	2	3
y	75	8	−9	−12	−13	0	63

b An exact solution to $x^4 - 2x - 12 = 0$ is $x = 2$.

c The other root lies in the interval $[-2, -1]$ since the graph changes position from above the x axis to below the axis between the endpoints of this interval.

Midpoint of the interval is $x = \frac{1}{2}((-2) + (-1)) = -1.5$.

When $x = -1.5, y = -3.9375$. The root lies between $x = -2$ and $x = -1.5$.

Midpoint	y value	New interval
$x = -1.5$	$-1.703\,125$	$[-2, -1.5]$
$x = -1.75$	$0.878\,906\,25$	$[-1.75, -1.5]$
$x = -1.625$	−1.777..	$[-1.75, -1.625]$
$x = -1.6875$	−0.516..	$[-1.75, -1.6875]$
$x = -1.71875$	0.164..	$[-1.71875, -1.6875]$
$x = -1.703\,125$		

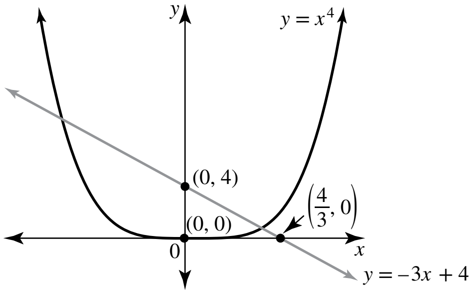
To one decimal place, $x = -1.7$ is a root of the equation.

3 $x^4 + 3x - 4 = 0$

$\therefore x^4 = -3x + 4$

The solutions to the equation can be obtained from the intersection of the graphs of $y = x^4$ and the line $y = -3x + 4$. The quartic graph has a minimum turning point at the origin and passes through the points $(\pm 1, 1)$.

The line has y intercept $(0, 4)$ and x intercept $\left(\frac{4}{3}, 0\right)$.

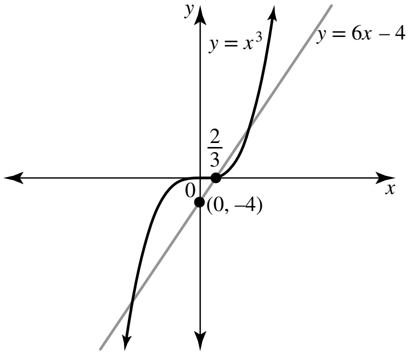


Estimating from the graph, the points of intersection have x co-ordinates of approximately $x=-1.75$ and exactly $x=1$.
Check: Substitute $x=1$ in $x^4+3x-4=0$.

$$\begin{aligned}\text{LHS}&=1^4+3(1)-4\\&=0\\&=\text{RHS}\end{aligned}$$

The equation has an approximate solution of $x=-1.75$ and an exact solution of $x=1$.

4 $x^3-6x+4=0$
$\therefore x^3=6x-4$
The solutions to the equation can be obtained from the intersection of the graphs of $y=x^3$ and the line $y=6x-4$.
The cubic graph has a stationary point of inflection at the origin and passes through the points $(-1,-1)$ and $(1,1)$ and $(2,8)$ and $(-2,-8)$.
The line has y intercept $(0,-4)$ and x intercept $\left(\frac{2}{3},0\right)$. It also passes through $(2,8)$.



Estimating from the graph, the points of intersection have x co-ordinates of $x=2$ and approximately $x=-2.7$ and $x=0.7$.
The equation $x^3-6x+4=0$ has an exact solution of $x=2$ and approximate solutions of $x=-2.7$ and $x=0.7$.

5 a $y=-x(x+2)(x-3)$
The graph cuts the x axis at $x=0, x=-2, x=3$ i.e. at $x=-2, x=0, x=3$. Between successive pairs of these values the graph must have a turning point.
As the graph is of a cubic polynomial with a negative leading coefficient, the first turning point is a minimum and the second is a maximum.
Hence, the maximum turning point must lie in the interval for which $x\in[0,3]$.

b Construct a table of values for the interval $x\in[0,3]$.

x	0	0.5	1	1.5	2	2.5	3
y	0	3.125	6	7.875	8	5.625	0

The maximum turning point is near $(2,8)$. Zoom in on this point.

x	1.6	1.7	1.8	1.9	2	2.1	2.2
y	8.064	8.177	8.208	8.151	8		

An estimate of the maximum turning point is $(1.8, 8.208)$.

6 $y=2x^3-x^2-15x+9$.

a The y intercept is $(0,9)$.

b Let $y=9$

$$\begin{aligned}\therefore 9&=2x^3-x^2-15x+9\\ \therefore 2x^3-x^2-15x&=0\\ \therefore x\left(2x^2-x-15\right)&=0\\ \therefore x(2x+5)(x-3)&=0\\ \therefore x=0, x&=-\frac{5}{2}\\ x&=3\end{aligned}$$

The two other points which have the same y co-ordinate as the y intercept are $\left(-\frac{5}{2},9\right)$, $(3,9)$.

c There must be a turning point between $x=-\frac{5}{2}$ and $x=0$ and a second turning point between $x=0$ and $x=3$.
The graph is of a positive cubic so the first turning point is the maximum turning point. This must lie in the interval for which $-\frac{5}{2}\le x\le 0$.

d $-\frac{5}{3}\simeq -1.67$

x	$-\frac{5}{3}$	−1.5	−1	−0.5	0
y	9	22.5	21	16	9

The turning point is near $(-1.5, 22.5)$. Zoom in around this point.

x	−1.6	−1.5	−1.4	−1.3
y	22.248	22.5	22.552	22.416

An estimate of the maximum turning point is $(-1.4, 22.552)$.

7 a $P(x)=x^2-12x+1$

$$\begin{aligned}P(10)&=100-120+1\\&=-19\\&<0\\P(12)&=144-144+1\\&=1\\&>0\end{aligned}$$

Therefore there is a zero of the polynomial between $x=10$ and $x=12$.

b $P(x)=-2x^3+8x+3$

$$\begin{aligned}P(-2)&=-2(-8)+8(-2)+3\\&=3\\&>0\\P(-1)&=-2(-1)+8(-1)+3\\&=-3\\&<0\end{aligned}$$

There is a zero of the polynomial between $x=-2$ and $x=-1$.

c $P(x)=x^4+9x^3-2x+1$

$$\begin{aligned}P(-2)&=(-2)^4+9(-2)^3-2(-2)+1\\&=16-72+4+1\\&=-51\\&<0\\P(1)&=1+9-2+1\\&=9\\&>0\end{aligned}$$

There is a zero of the polynomial between $x=-2$ and $x=1$.

d $P(x)=x^5-4x^3+2$

$$\begin{aligned}P(0)&=2>0\\P(1)&=1-4+2\\&=-1\\&<0\end{aligned}$$

There is a zero of the polynomial between $x=0$ and $x=1$.

8 a Initial interval is $[10,12]$.
Midpoint of this interval is $x=11$.

$$\begin{aligned}P(11)&=121-132+1\\&=-10\\&<0\end{aligned}$$

Since $P(10)<0, P(12)>0$, the root lies in the interval $[11,12]$.
Midpoint of interval $[11,12]$ is $x=11.5$
$P(11.5)=-4.75<0$

The root lies in the interval [11.5,12].

An estimate is the midpoint of this interval.

$x = 0.5(11.5 + 12)$

$= 11.75$

An estimate of the root is $x = 11.75$

b Initial interval is $[-2, -1]$.

Midpoint of this interval is $x = -1.5$.

$P(-1.5) = -2.25 < 0$

Since $P(-2) > 0, P(-1) < 0$, the root lies in the interval $[-2, -1.5]$.

Midpoint of interval $[-2, -1.5]$ is $x = -1.75$

$P(-1.75) = -0.28125 < 0$

The root lies in the interval $[-2, -1.75]$.

An estimate is the midpoint of this interval.

$x = 0.5(-2 - 1.75)$

$= -1.875$

An estimate of the root is $x = -1.875$

c Initial interval is $[-2, 1]$.

Midpoint of this interval is $x = -0.5$.

$P(-0.5) = 0.9375 > 0$

Since $P(-2) < 0, P(1) > 0$, the root lies in the interval $[-2, -0.5]$.

Midpoint of interval $[-2, -0.5]$ is $x = -1.25$

$P(-1.25) = -34.355.. < 0$

The root lies in the interval $[-1.25, -0.5]$.

An estimate of the root is the midpoint of this interval, $x = -0.875$

d Initial interval is $[0, 1]$.

Midpoint of this interval is $x = 0.5$.

$P(0.5) = 1.53125 > 0$

Since $P(0) > 0, P(1) < 0$, the root lies in the interval $[0.5, 1]$.

Midpoint of interval $[0.5, 1]$ is $x = 0.75$

$P(0.75) = 0.5498.. > 0$

The root lies in the interval $[0.25, 1]$.

An estimate of the root is the midpoint of this interval, $x = 0.875$.

9 $5x^2 - 26x + 24 = 0$

a Given the root is in the interval $1 \le x \le 2$.

Let $P(x) = 5x^2 - 26x + 24$

$P(1) = 3 > 0$ and $P(2) = -8 < 0$.

Midpoint of interval is $x = 1.5$

$P(1.5) = 5(2.25) - 26(1.5) + 24$

$= -3.75$

< 0

Root lies in interval $[1, 1.5]$.

Continuing the method:

Midpoint	**Value of $P(x)$ at midpoint**	**New interval**
		$[1, 1.5]$
$x = 1.25$	$-0.6875 < 0$	$[1, 1.25]$
$x = 1.125$	$1.078125 > 0$	$[1.125, 1.25]$
$x = 1.1875$	$0.17578.. > 0$	$[1.1875, 1.25]$
$x = 1.21875$		

The last 2 estimates have the same value to one decimal place. The Method of bisection estimates the root to be $x = 1.2$ to one decimal place.

b $5x^2 - 26x + 24 = 0$

The equation factorises.

$(5x - 6)(x - 4) = 0$

$\therefore x = \frac{6}{5}, x = 4$

$\therefore x = 1.2, x = 4$

The other root of the equation is $x = 4$.

c As $x = 1.2$ is in fact an exact root of the equation. The Method of bisection was very slow to converge towards the vicinity of this value.

10 a $y = x^4 - 3$

x	-2	-1	0	1	2
y	13	-2	-3	-2	13

b Let $y = 0$

$\therefore x^4 - 3 = 0$

$\therefore x^4 = 3$

$\therefore x = \pm\sqrt[4]{3}$

Hence, an interval in which $\sqrt[4]{3}$ lies is $x \in [1, 2]$.

c Using the Method of bisection with initial interval $x \in [1, 2]$.

Midpoint	***y* value of at midpoint**	**New interval**
		$[1, 2]$
$x = 1.5$	$2.0625 > 0$	$[1, 1.5]$
$x = 1.25$	$-0.558.. < 0$	$[1.25, 1.5]$
$x = 1.375$	$0.574.. > 0$	$[1.25, 1.375]$
$x = 1.3125$	$-0.032.. < 0$	$[1.3125.1.375]$
$x = 1.34375$	$0.260.. > 0$	$[1.3125, 1.34375]$
$x = 1.328125$	$0.1113.. > 0$	$[1.3125, 1.328125]$
$x = 1.3203125$	$0.0388.. > 0$	$[1.3125, 1.3203125]$
$x = 1.31640625$		

The last two estimates are the same value correct to two decimal places.

An estimate of the value of $\sqrt[4]{3}$ is 1.32.

11 $P(x) = x^3 + 5x - 2 = 0$

a Using trial and error,

$P(0) = -2$ and $P(1) = 4$.

As they have opposite signs there is a root of the equation in the interval $[0, 1]$.

b Midpoint of interval is $x = 0.5$.

$P(0.5) = 0.625 > 0$

Root lies in interval $[0, 0.5]$.

Continuing the method until the value of $P(x)$ differs from zero by less than 0.05

Midpoint	**Value of $P(x)$ at midpoint**	**New interval**
		$[0, 0.5]$
$x = 0.25$	$-0.734375 < 0$	$[0.25, 0.5]$
$x = 0.375$	$-0.1 < -0.072.. < 0.1$	

The root is $x = 0.375$ to the required accuracy.

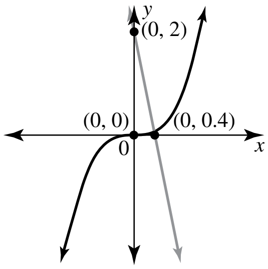
c $x^3 + 5x - 2 = 0$

$\therefore x^3 = -5x + 2$

The intersection of the graphs of $y = x^3$ and $y = -5x + 2$ will allow the solutions to the equation to be found.

d The cubic graph of $y = x^3$ has a stationary point of inflection at the origin and contains the points $(-1, -1)$ and $(1, 1)$.

The line $y = -5x + 2$ has y intercept $(0, 2)$ and x intercept $(0.4, 0)$



There is one point of intersection and therefore the equation $x^3+5x-2=0$ has only one root.
The graphs intersect slightly before $x=0.4$ so the value $x=0.375$ is supported by the diagram.

12 a At the intersection points of $y=3x-2$ and $y=x^3$,

$$x^3=3x-2$$

$\therefore x^3-3x+2=0$

The x co-ordinates of the points A and B are solutions of $x^3-3x+2=0$.

b As one solution occurs when the line touches the point A, the polynomial $P(x)=x^3-3x+2$ has a linear factor of multiplicity 2. The other solution occurs when the line cuts the curve at the point B so the polynomial has a second linear factor of multiplicity 1.
There are two factors, one of multiplicity 2 and one of multiplicity 1.

c $P(x)=x^3-3x+2$
$P(1)=1-3+2=0$
$\therefore (x-1)$ is a factor

$$\begin{aligned}x^3-3x+2&=(x-1)\left(x^2+bx-2\right)\\&=(x-1)\left(x^2+x-2\right)\\&=(x-1)(x+2)(x-1)\\&=(x-1)^2(x+2)\end{aligned}$$

Thus, the solutions of the equation $x^3-3x+2=0$ are $x=1, x=-2$.
The point A has $x=1$ and point B has $x=-2$.
Substitute in $y=3x-2$
For A, $y=1$ and for B, $y=-8$.
A is the point $(1,1)$ and B is the point $(-2,-8)$

d The equation $x^3-3x+1=0$ can be solved by the intersection of $y=x^3$ and $y=3x-1$ since $x^3-3x+1=0$ rearranges to $x^3=3x-1$.
The line $y=3x-1$ is parallel to the line in the diagram but it has a higher y intercept of $(0,-1)$. This means the line will cut the cubic curve in 3 places.
The equation $x^3-3x+1=0$ has three solutions.

13 a $y=(x+4)(x-2)(x-6)$
This positive cubic graph has x intercepts when $x=-4, x=2$ and $x=6$. There is a maximum turning point between $x=-4$ and $x=2$ and a minimum turning point between $x=2$ and $x=6$.
For the maximum turning point, construct a table of values between $x=-4$ and $x=2$.

x	–4	–3	–2	–1	0	1	2
y	0	45	64	63	48	25	0

The maximum turning point is near $(-2,64)$. Zoom in around this point.

x	–2.1	–2	–1.9	–1.8	–1.7	–1.6	–1.5
y	63.099	64	64.701	65.208	65.527	65.664	65.625

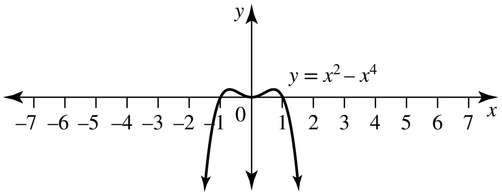
An estimate of the position of the maximum turning point is $(-1.6, 65.664)$.

b $y=x(2x+5)(2x+1)$
x intercepts when $x=0, x=-\frac{5}{2}, x=-\frac{1}{2}$.
Shape is of a positive cubic so maximum turning point between $x=-2.5$ and $x=-0.5$ and minimum turning point between $x=-0.5$ and $x=0$.
For the minimum turning point, construct a table of values between $x=-0.5$ and $x=0$.

x	–0.5	–0.4	–0.3	–0.2	–0.1	0
y	0	–0.336	–0.528	–0.552	–0.384	0

The minimum turning point's estimated position is $(-0.2,-0.552)$

c $y=x^2-x^4$
x intercepts: Let $y=0$
$\therefore y=x^2(1-x^2)$
$\therefore y=x^2(1-x)(1+x)$
x intercepts: Let $y=0$
$x=0, x=1, x=-1$
Due to the multiplicity of the factor, there is a turning point at $(0,0)$.
The shape of the graph is of a negative fourth degree. Therefore there is a maximum turning point between $x=-1$ and $x=0$, $(0,0)$ is a minimum turning point and there is a second maximum turning point between $x=0$ and $x=1$. The two maximum turning points are symmetric about the y axis.



Maximum turning point between $x=0$ and $x=1$.

x	0	0.2	0.4	0.6	0.8	1
y	0	0.0384	0.1344	0.2304	0.2304	0

The maximum turning point lies between $x=0.6$ and $x=0.8$ and therefore, to one decimal place its x co-ordinate must be $x=0.7$. When $x=0.7$, $y=0.2499$
There are maximum turning points at approximately $(0.7, 0.2499)$ and $(-0.7, 0.2499)$, and there is a minimum turning point exactly at $(0,0)$. *Note*: Since minimum turning point is exact we shall choose not to write it as $(0.0, 0)$.

14 a $P(x)=x^3-3x^2-4x+9$
$P(0)=9$

$$\begin{aligned}&\therefore x^3-3x^2-4x+9=9\\&\quad\therefore x^3-3x^2-4x=0\\&\quad\therefore x(x^2-3x-4)=0\\&\quad\therefore x(x-4)(x+1)=0\\&\qquad\therefore x=-1, x=0, x=4\end{aligned}$$

Shape of graph is a positive cubic so there is a maximum turning point in the interval $x\in[-1,0]$ and a minimum turning point in the interval for which $x\in[0,4]$.

b $P(x)=x^3-12x+18$
$P(0)=18$

$\therefore x^3 - 12x + 18 = 18$

$\therefore x^3 - 12x = 0$

$\therefore x(x^2 - 12) = 0$

$\therefore x(x + \sqrt{12})(x - \sqrt{12}) = 0$

$\therefore x(x + 2\sqrt{3})(x - 2\sqrt{3}) = 0$

$\therefore x = -2\sqrt{3}, x = 0, x = 2\sqrt{3}$

Shape of graph is a positive cubic so there is a maximum turning point in the interval $x \in [-2\sqrt{3}, 0]$ and a minimum turning point in the interval for which $x \in [0, 2\sqrt{3}]$.

c $P(x) = -2x^3 + 10x^2 - 8x + 1$

$P(0) = 1$

$\therefore -2x^3 + 10x^2 - 8x + 1 = 1$

$\therefore -2x^3 + 10x^2 - 8x = 0$

$\therefore -2x(x^2 - 5x + 4) = 0$

$\therefore -2x(x - 1)(x - 4) = 0$

$\therefore x = 0, x = 1, x = 4$

Shape of graph is a negative cubic so there is a minimum turning point in the interval $x \in [0, 1]$ and a maximum turning point in the interval $x \in [1, 4]$.

d $P(x) = x^3 + x^2 + 7$

$P(0) = 7$

$\therefore x^3 + x^2 + 7 = 7$

$\therefore x^3 + x^2 = 0$

$\therefore x^2(x + 1) = 0$

$\therefore x = -1, x = 0$

Shape of graph is a positive cubic so there is a maximum turning point in the interval $x \in [-1, 0]$. The multiplicity of the factor indicates the line $y = 7$ touches the graph when $x = 0$ and so there is a turning point at the point $(0, 7)$. This must be a minimum turning point.

15 $y = -x^3 + 7x^2 - 3x - 4,\ x \geq 0$

a The number of containers is $10x$, so for 10 containers $x = 1$ and for 20 containers, $x = 2$.

When $x = 1$, $y = -1 + 7 - 3 - 4 = -1 < 0$ so no profit is made.

When $x = 2$, $y = -8 + 28 - 12 - 4 = 4 > 0$ so a profit is made.

A profit is first made for $x \in (1, 2)$ indicating the number of containers sold was between 10 and 20.

b The midpoint of the interval $[1, 2]$ is $x = 1.5$.

When $x = 1.5$, $y = 3.875 > 0$

A profit is first made for $x \in [1, 1.5]$.

The midpoint of the interval $[1, 1.5]$ is $x = 1.25$.

When $x = 1.25$, $y = 1.234375 > 0$

A profit is first made for $x \in [1, 1.25]$.

c The number of containers is $10x$ and must be a whole number. For $x \in [1, 1.25]$, the number of containers lies between 10 and 12 so either $x = 1.1$ or $x = 1.2$.

x	1	1.1	1.2	1.25
y	< 0	-0.161	0.752	> 0

A profit is first made when 12 containers are sold.

d The graph shows the maximum turning point lies in the interval $x \in [3, 6]$.

e Test values of y for $x \in [3, 6]$

x	4	4.5	5	5.5
y	32	33.125	31	24.875

The maximum turning point is near $(4.5, 33.125)$

Zooming in around this point

x	4.2	4.3	4.4	4.5	4.6
y	32.792	33.023	33.136	33.125	32.984

The maximum turning point is closest to $(4.4, 33.136)$.

Selling 44 containers will give the maximum profit of \$331, to nearest dollar.

f The value $x = 6.5$ is an estimate from the graph of when $y = 0$.

Testing around this value,

x	6.4	6.5
y	1.376	-2.375

When there are 65 or more containers, no profit will be made.

16 a If $P(x_1)$ and $P(x_2)$ have opposite signs then their product must be negative. Therefore, if $P(x_1)P(x_2) < 0$ then there is at least one root of the equation $P(x) = 0$ that lies between x_1 and x_2.

b i $P(x) = 6x^3 - 11x^2 - 4x - 15$

$P(2) = 6(8) - 11(4) - 4(2) - 15$

$= -19$

$P(3) = 6(27) - 11(9) - 4(3) - 15$

$= 36$

$\therefore P(2)P(3) = -19 \times 36$

$= -684$

$\therefore P(2)P(3) < 0$

ii The equation $6x^3 - 11x^2 - 4x - 15 = 0$ has a root in the interval $[2, 3]$.

Midpoint of this interval is $x = 2.5$

$P(2.5) = 6(2.5)^3 - 11(2.5)2 - 4(2.5) - 15$

$= 0$

So, $x = 2.5$ is an exact solution of the equation.

iii $2.5 = \frac{5}{2}$. Therefore, $2x - 5$ is a factor of the polynomial.

$P(x) = 6x^3 - 11x^2 - 4x - 15$

$= (2x - 5)(3x^2 + bx + 3)$

$= (2x - 5)(3x^2 + 2x + 3)$

For the equation

$(2x - 5)(3x^2 + 2x + 3) = 0$

$\therefore x = \frac{5}{2}$ or $3x^2 + 2x + 3 = 0$

Consider the discriminant of $3x^2 + 2x + 3$

$\Delta = b^2 - 4ac \quad a = 3, b = 2, c = 3$

$= 4 - 4 \times 3 \times 3$

$= -32$

Since $\Delta < 0$ there are no real solutions of the quadratic equation.

Hence, $x = 2.5$ is the only solution to the equation $6x^3 - 11x^2 - 4x - 15 = 0$.

iv At the intersection of $y = 6x^3 - 11x^2 - 4x - 15$ and $y = -15$,

$6x^3 - 11x^2 - 4x - 15 = -15$

$\therefore 6x^3 - 11x^2 - 4x = 0$

$\therefore x(6x^2 - 11x - 4) = 0$

$\therefore x = 0$ or $6x^2 - 11x - 4 = 0$

Consider the discriminant of $6x^2 - 11x - 4$

$\Delta = b^2 - 4ac \quad a = 6, b = -11, c = -4$

$= 121 - 4(6)(-4)$

$= 217$

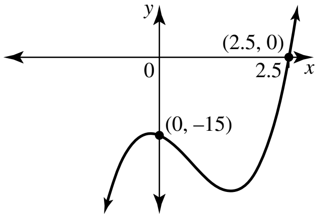
As $\Delta > 0$ there will be two solutions to the quadratic equation $6x^2 - 11x - 4 = 0$.
Therefore there are 3 points of intersection of the graphs of $y = 6x^3 - 11x^2 - 4x - 15$ and $y = -15$. As $y = -15$ is the value of the y intercept of the cubic graph, this means the graph has a maximum and a minimum turning point.
Using the formula to solve the quadratic equation $6x^2 - 11x - 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{11 \pm \sqrt{217}}{12}$$

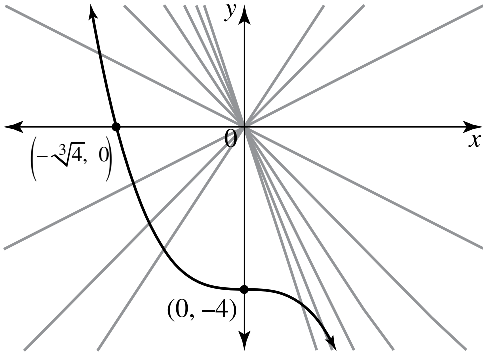
Approximately, $x = -1.87$ or $x = 2.14$.

The graph of $y = 6x^3 - 11x^2 - 4x - 15$ has a maximum turning point between $x = -1.87$ and $x = 0$ and a minimum turning point between $x = 0$ and $x = 2.14$. As there is only one x intercept of the graph at $x = 2.5$ then both of these turning points lie below the x-axis.

v



17 a $y = -x^3 - 4$ and the family of lines $y = ax$. The values of a shown in the diagram are integer values from –6 to 3.



b $x^3 + ax + 4 = 0$
Rearranging this equation to $ax = -x^3 - 4$, shows the solutions are the x co-ordinates of the points of intersection of the graphs in part a.
The line $y = -5x$ is the first to intersect the cubic graph in three places, for integer values of a. The largest integer is $a = -5$.

c The lines with positive gradients i.e. for which $a > 0$ intersect the cubic once. There is 1 solution to the equation $x^3 + ax + 4 = 0$

d $x^3 + x + 4 = 0$
Here $a = 1$ so there will be 1 solution which will have a negative value.
Define the polynomial $p(x) = x^3 + x + 4$.
$p(-2) = -6 < 0$
$p(-1) = 2 > 0$
So an initial interval is the solution lies in the interval $[-2, -1]$. Proceed with the Method of bisection from there (or program the calculator to carry out the procedure).

Midpoint	**Value of $p(x)$ at midpoint**	**New interval**
		$[-2, -1]$
$x = -1.5$	< 0	$[-1.5, -1]$
$x = -1.25$	> 0	$[-1.5, -1, 25]$
$x = -1.375$	> 0	$[-1.5, -1.375]$
$x = -1.4375$	< 0	$[-1.4375, -1.375]$
$x = -1.40625$	< 0	$[-1.40625, -1.375]$
$x = -1.390\,625$	< 0	$[-1.390\,625, -1.375]$
$x = -1.382\,8125$	< 0	$[-1.382\,8125, -1.375]$
$x = -1.378\,906\,25$		

The last two values agree at an accuracy of two decimal places.
To 2 decimal places, the solution is $x = -1.38$.

18 a The dimensions in cm of the box are: length $18 - 2x$, width $14 - 2x$, height x.
Let the volume be V cubic cm.
$V = l \times w \times h$
$\therefore V = x(18 - 2x)(14 - 2x)$

b The graph of the cubic polynomial $y = x(18 - 2x)(14 - 2x)$ would have x intercepts at $x = 0, x = 7, x = 9$. Between $x = 0$ and $x = 7$ there would be a maximum turning point and between $x = 7$ and $x = 9$ there would be a minimum turning point.
However, since neither V nor x can be negative in this practical model, the graph of $V = x(18 - 2x)(14 - 2x)$ is defined for $0 < x < 7$.
The volume is greatest within the interval between $x = 0$ and $x = 7$.

c Tap the Spreadsheet icon and set up the rule $V = x(18 - 2x)(14 - 2x)$ to be evaluated over $[0, 7]$.
$x = 2.6049$ gives the greatest volume.
The side length of the square to be cut out is 2.605 cm to three decimal places.

Topic 6 — Functions and relations

Exercise 6.2 — Functions and relations

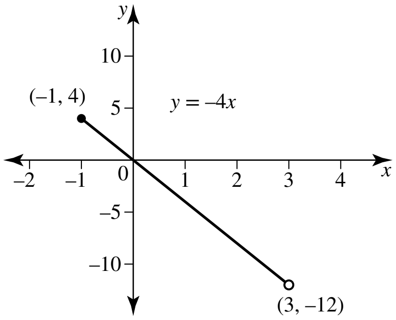
1 a $\{(4,4),(3,0),(2,3),(0,-1)\}$
Domain is $\{0,2,3,4\}$ Range is $\{-1,0,3,4\}$ This is a function since each x co-ordinate is used exactly once.

b Domain is $[-2,\infty)$, Range is R. This is not a function since a vertical line cuts the graph more than once.

c Domain is $[0,3]$, Range is $[0,4]$. This is a function since a vertical line cuts the graph exactly once.

d $\{(x,y): y = 4 - x^2\}$
This is the parabola $y = 4 - x^2$, with maximum turning point at $(0,4)$.
The maximal domain is R and range is $(-\infty,4]$. It is a function since a vertical line would cut its graph exactly once.

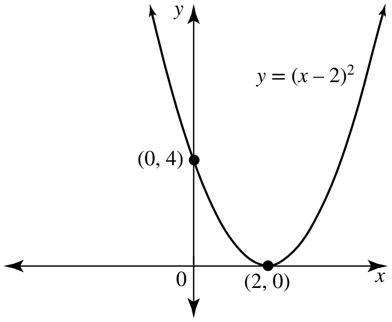
2 $y = -4x, x \in [-1,3)$
End points $x = -1, y = 4 \Rightarrow (-1,4)$ (closed) and
$x = 3, y = -12 \Rightarrow (3,-12)$ (open)
Domain is $[-1,3)$, Range is $(-12,4]$.



3 a $y = 8(x+1)^3 - 1$
This is a cubic with a stationary point of inflection at $(-1,1)$. It has a one-to-one correspondence since a horizontal line would cut the graph exactly once and a vertical line likewise. It is a function since the vertical line cuts the graph once.

b Considering the cuts made by a horizontal and then a vertical line, the graph has a many-to-many correspondence and is therefore not a function.

4 a $y = (x-2)^2$ minimum turning point at $(2,0)$, y intercept at $(0,4)$



Domain R, Range $R^+ \cup \{0\}$, many-to-one correspondence

b An answer is $[2,\infty)$

5 a $f(x) = ax + b$
$f(2) = 7 \Rightarrow 7 = 2a + b........(1)$
$f(3) = 9 \Rightarrow 9 = 3a + b........(2)$
$(2) - (1)$
$2 = a$
$\therefore b = 3$
$\Rightarrow f(x) = 2x + 3$

b $f(x) = 2x + 3$
$\therefore f(0) = 3$

c $f(x) = 0$
$\therefore 2x + 3 = 0$

$$\therefore x = -\frac{3}{2}$$

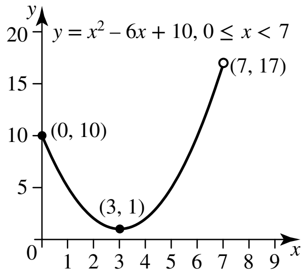
d Image of -3 is $f(-3)$
$f(-3) = 2(-3) + 3$
$= -3$

e $g: (-\infty, 0] \to R, g(x) = 2x + 3$

6 $y = x^2 - 6x + 10,\ 0 \le x < 7$
$f: [0,7) \to R, f(x) = x^2 - 6x + 10$
Domain is $[0,7)$
For the range, a sketch of the graph could be used or the range can be deduced from the position and type of the turning point of a parabola and the end points of the graph.

$$y = x^2 - 6x + 10$$
$$\therefore y = (x^2 - 6x + 9) - 9 + 10$$
$$\therefore y = (x-3)^2 + 1$$

Minimum turning point at $(3,1)$
End points: $x = 0, y = 10 \Rightarrow (0,10)$ (closed) and
$x = 7, y = 17 \Rightarrow (7,17)$ (open)
Therefore range is $[1,17)$. This is confirmed by the graph.



7 Domain refers to x values and range refers to y values.

a Domain $[0,5]$, range $[0,15]$

b Domain $[-4,2) \cup (2,\infty)$, range $(-\infty,10)$

c Domain $[-3,6]$, range $[0,8]$

d Domain $[-2,2]$, range $[-4,4]$

e Domain $\{3\}$, range R

f Domain R, range R

8 a The type of correspondence is determined by the number of intersections of a horizontal line-to-the number of intersections of a vertical line with the graph given.
The relation in part **a** has a one-to-one correspondence; part **b** has a many-to-one correspondence; part **c** has a many-to-one correspondence; part **d** has a one-to-many correspondence; part **e** has a one-to many correspondence; part **f** has a many-to-one correspondence.

b A vertical line would cut the graphs in parts **d** and **e** in more than one place so these are not the graphs of functions.

9 a $\{(-11,2),(-3,8),(-1,0),(5,2)\}$
Domain $\{-11,-3,-1,5\}$, range $\{0,2,8\}$
Both $x = -11$ and $x = 5$ are mapped to $y = 2$, so there is a many-to-one correspondence. Each x value is mapped to a unique y value so the relation is a function.

b $\{(20,6),(20,20),(50,10),(60,10)\}$
Domain $\{20,50,60\}$, range $\{6,10,20\}$
Since $x=20$ is mapped to both $y=6$ and $y=20$ the relation is not a function.

Also, more than one x value is mapped to $y=10$ so the type of correspondence is many-to-many.

c $\{(-14,-7),(0,0),(0,2),(14,7)\}$
Domain $\{-14,0,14\}$, range $\{-7,0,2,7\}$
Since $x=0$ is mapped to more than one y value, the relation is not a function. The type of correspondence is one-to-many.

d $\{(x,y): y=2(x-16)^3+13\}$
This is a cubic polynomial function with a stationary point of inflection at $(16,13)$. Its domain is R and its range is R. It has a one-to-one correspondence.

e $\{(x,y): y=4-(x-6)^2\}$
This is a concave down quadratic function with a maximum turning points at $(6,4)$. Its domain is R and its range is $(-\infty,4]$. It has a many-to-one correspondence.

f $\{(x,y): y=3x^2(x-5)^2\}$
This is a positive fourth degree polynomial function with minimum turning points at $(0,0)$ and $(5,0)$. Its domain is R and its range is $R^+ \cup \{0\}$. Its correspondence is many-to-one.

10 $A=\{(1,2),(2,3),(3,2),(k,3)\}$

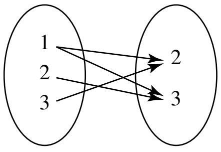
a If $k=1$ then A would contain the two points $(1,2)$ and $(1,3)$. This has one x value mapped to more than one y value and so A would not be a function if $k=1$.

Similarly, if $k=3$ then $x=3$ would be mapped to more than one y value and so A would not be a function.

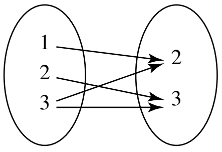
Answer: $k=1, k=3$, with the relation having a many-to-many correspondence for either value of k.

b A will be a function if $k \in R \setminus \{1,3\}$. As A contains the points $(1,2)$ and $(3,2)$ it has a many-to-one correspondence.

c Mapping diagrams for $k=1, k=3$.

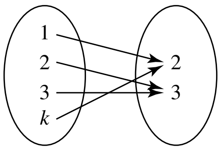


$k=1$

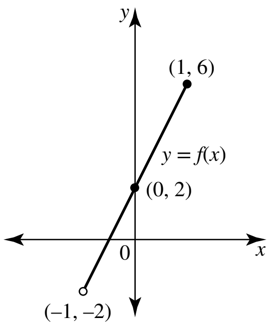


$k=3$

d Mapping diagram for $k \in R | \{1,3\}$.

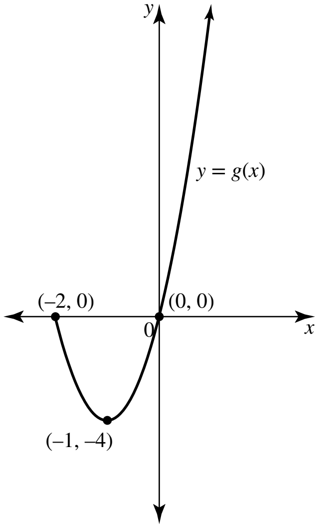


11 a $f(x)=4x+2,\ x \in (-1,1]$
linear function.
Endpoints: $f(-1)=-4+2=-2 \Rightarrow (-1,-2)$ open,
$f(1)=4+2=6 \Rightarrow (1,6)$ closed.
y intercept $(0,2)$ and x intercept $\left(-\frac{1}{2},0\right)$



Domain is $(-1,1]$ and range is $(-2,6]$

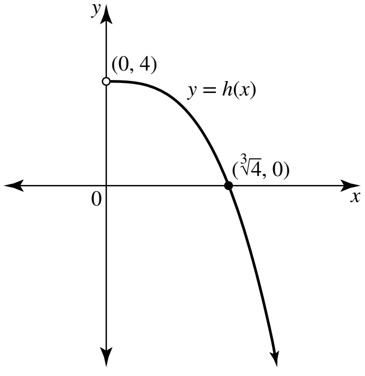
b $g(x)=4x(x+2),\ x \geq -2$
Graph of $y=g(x)$ is a concave up parabola.
x intercepts and closed endpoint $(-2,0)$, x and y intercept $(0,0)$
x co-ordinate of turning point occurs midway between the x intercepts.
When $x=-1$, $y=-4(1)=-4 \Rightarrow (-1,-4)$ is the minimum turning point



Domain $[-2,\infty)$, range $[-4,\infty)$

c $h(x)=4-x^3, x \in R^+$
The graph of $y=h(x)$ is a cubic function with a stationary point of inflection at $(0,4)$. Since $x \in R^+$ this point is open.
x intercept: Let $y=0$

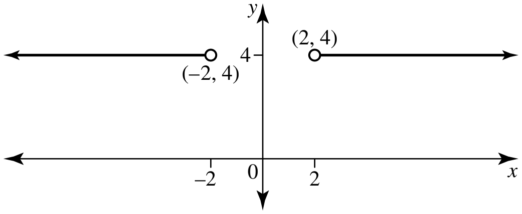
$$\therefore 4-x^3=0$$
$$\therefore x^3=4$$
$$\therefore x=\sqrt[3]{4}$$
$$\left(\sqrt[3]{4},0\right)$$



Domain R^+, range $(-\infty,4)$

d $y = 4, x \in R \setminus [-2, 2]$

Graph is a horizontal line drawn for $x < -2$ and $x > -2$ with open endpoints at $(-2, 4)$ and $(2, 4)$.



Domain $R \setminus [-2, 2]$, range $\{4\}$

12 $f(x) = x^2 + 2x - 3$

a **i** $f(-2) = (-2)^2 + 2(-2) - 3$
$= 4 - 4 - 3$
$f(-2) = -3$

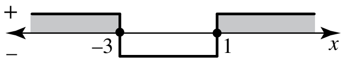
ii $f(9) = (9)^2 + 2(9) - 3$
$= 81 + 18 - 3$
$f(9) = 96$

b **i** $f(2a) = (2a)^2 + 2(2a) - 3$
$= 4a^2 + 4a - 3$

ii $f(1-a) = (1-a)^2 + 2(1-a) - 3$
$= 1 - 2a + a^2 + 2 - 2a - 3$
$f(1-a) = a^2 - 4a$

c $f(x+h) = (x+h)^2 + 2(x+h) - 3$
$= x^2 + 2xh + h^2 + 2x + 2h - 3$
$f(x) = x^2 + 2x - 3$
$\therefore f(x+h) - f(x)$
$= \left(x^2 + 2xh + h^2 + 2x + 2h - 3\right) - (x^2 + 2x - 3)$
$= \cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{3} - \cancel{x^2} - \cancel{2x} + \cancel{3}$
$= 2xh + h^2 + 2h$

d $f(x) > 0$
$\therefore x^2 + 2x - 3 > 0$
$\therefore (x+3)(x-1) > 0$
Zeros $x = -3, x = 1$



Solution set is $\{x : x < -3\} \cup \{x : x > 1\}$

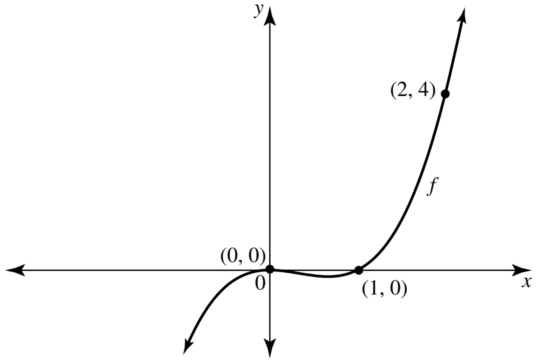
e $f(x) = 12$
$\therefore x^2 + 2x - 3 = 12$
$\therefore x^2 + 2x - 15 = 0$
$\therefore (x+5)(x-3) = 0$
$\therefore x = -5,\ x = 3$

f $f(x) = 1 - x$
$\therefore x^2 + 2x - 3 = 1 - x$
$\therefore x^2 + 3x - 4 = 0$
$\therefore (x+4)(x-1) = 0$
$\therefore x = -4,\ x = 1$

13 $f : R \to R, f(x) = x^3 - x^2$

a The image of 2 is $f(2)$.
$f(2) = 2^3 - 2^2$
$= 8 - 4$
$= 4$
The image of 2 is 4.

b $f(x) = x^3 - x^2$
$= x^2(x - 1)$
x intercepts occur at $x = 0$ (touch) and $x = 1$ (cut)



c Domain R, range R.

d many-to-one correspondence

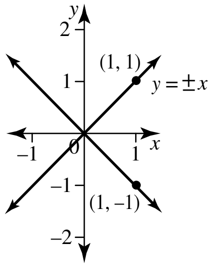
e many answers are possible, One answer is to restrict the domain to $(1, \infty)$ and another is to restrict the domain to R^-.

f $f(x) = 4$
$\therefore x^3 - x^2 = 4$
$\therefore x^3 - x^2 - 4 = 0$
From part **a**, $f(2) = 4$ so $x = 2$ is a solution. Therefore, $(x - 2)$ is a factor.
$\therefore (x-2)(x^2 + x + 2) = 0$
$\therefore x = 2$ or $x^2 + x + 2 = 0$
Consider $x^2 + x + 2 = 0$
$\Delta = (1)^2 - 4 \times 1 \times 2$
$= -7$
$\therefore \Delta < 0$
No real solutions.
The only solution is $x = 2$.
Alternatively, the horizontal line $y = 4$ intersects the graph of $y = f(x)$ exactly once at the point $(2, 4)$, so $x = 2$ is the only solution to $f(x) = 4$.
Answer: $\{x : x = 2\}$

14 **a** $y = x^2, x \in Z^+$ is the rule for the set of points $\{(1,1), (2,4), (3,9), \ldots.\}$
It is the function $f : Z^+ \to R, f(x) = x^2$ with range $\{1, 4, 9, 16 \ldots.\}$.

b $2x + 3y = 6$ is the equation of a straight line.
Rearranging,
$3y = 6 - 2x$
$\therefore y = -\frac{2}{3}x + 2$
It is the function $f : R \to R, f(x) = -\frac{2}{3}x + 2$ with range R.

c $y = \pm x$
Let $x = 1$, $\therefore y = \pm 1$. The points $(1, -1)$ and $(1, 1)$ are part of this relation. It is not a function as the x values are not mapped to a unique y value.



Range is R

d $\{(x,5), x \in R\}$ is the set of points whose y value is always equal to 5. It is the set of points that lie on the horizontal line $y = 5$.
This is a function $f: R \to R, f(x) = 5$ with range $\{5\}$.

e $\{(-1, y), y \in R\}$ is the set of points whose x value is always equal to -1. It is the set of points that lie on the vertical line $x = -1$. This is not a function. The relation has range R.

f $y = -x^5, x \in R^+$ is part of the fifth degree polynomial function with a point of inflection at the origin. Since $x \in R^+$, the values of the function are always negative.
It is the function $f: R^+ \to R, f(x) = -x^5$ with range R^-.

15 $f(x) = a + bx + cx^2$ and $g(x) = f(x-1)$

a The function f is a quadratic polynomial.
Given $f(-2) = 0 \Rightarrow (x+2)$ is a factor
Given $f(5) = 0 \Rightarrow (x-5)$ is a factor.
Let $f(x) = a(x+2)(x-5)$
Given $f(2) = 3 \Rightarrow 3 = a(4)(-3)$

$$\therefore 3 = -12a$$
$$\therefore a = -\frac{3}{12}$$
$$\therefore a = -\frac{1}{4}$$
$$\therefore f(x) = -\frac{1}{4}(x+2)(x-5)$$

Expanding,

$$\begin{aligned} f(x) &= -\frac{1}{4}(x^2 - 3x - 10) \\ &= -\frac{1}{4}x^2 + \frac{3}{4}x + \frac{10}{4} \\ &= \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2 \end{aligned}$$

$\therefore f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$ is the rule for the function f.

b As $g(x) = f(x-1)$, then $g(x) = \frac{5}{2} + \frac{3}{4}(x-1) - \frac{1}{4}(x-1)^2$.

$$\begin{aligned} \therefore g(x) &= \frac{5}{2} + \frac{3}{4}x - \frac{3}{4} - \frac{1}{4}(x^2 - 2x + 1) \\ &= \frac{7}{4} + \frac{3}{4}x - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{4} \\ &= \frac{6}{4} + \frac{5}{4}x - \frac{1}{4}x^2 \end{aligned}$$
$$\therefore g(x) = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$$

c $f(x) = g(x)$

$$\begin{aligned} \therefore \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2 &= \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2 \\ \therefore \frac{5}{2} + \frac{3}{4}x &= \frac{3}{2} + \frac{5}{4}x \\ \therefore \frac{5}{2} - \frac{3}{2} &= \frac{5}{4}x - \frac{3}{4}x \\ \therefore 1 &= \frac{1}{2}x \\ \therefore x &= 2 \end{aligned}$$

d $f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$ or $f(x) = -\frac{1}{4}(x+2)(x-5)$

x intercepts: $(-2, 0), (5, 0)$

y intercept: $\left(0, \frac{5}{2}\right)$

Maximum turning point occurs at $x = \frac{-2+5}{2} = \frac{3}{2}$

$$\begin{aligned} \therefore y &= -\frac{1}{4}\left(\frac{3}{2} + 2\right)\left(\frac{3}{2} - 5\right) \\ &= -\frac{1}{4} \times \frac{7}{2} \times \frac{-7}{2} \\ &= \frac{49}{16} \end{aligned}$$

$\left(\frac{3}{2}, \frac{49}{16}\right)$

$$\begin{aligned} g(x) &= \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2 \\ &= -\frac{1}{4}(x^2 - 5x - 6) \\ &= -\frac{1}{4}(x-6)(x+1) \end{aligned}$$

x intercepts: $(-1, 0)$, $(6, 0)$

y intercept: $\left(0, \frac{3}{2}\right)$

Maximum turning point occurs at $x = \frac{-1+6}{2} = \frac{5}{2}$

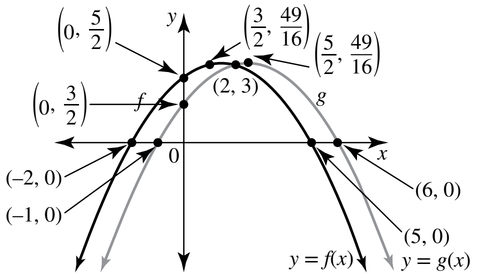
$$\begin{aligned} \therefore y &= -\frac{1}{4}\left(\frac{5}{2} - 6\right)\left(\frac{5}{2} + 1\right) \\ &= -\frac{1}{4} \times \frac{-7}{2} \times \frac{7}{2} \\ &= \frac{49}{16} \end{aligned}$$

$\left(\frac{5}{2}, \frac{49}{16}\right)$

Point of intersection of the two graphs occurs when $x = 2$

$$\begin{aligned} f(2) &= -\frac{1}{4}(2+2)(2-5) \\ &= -\frac{1}{4} \times -12 \\ &= 3 \end{aligned}$$

$(2, 3)$



The graph of function g has the same shape as the graph of function f but g has been horizontally translated 1 unit to the right.

16 a $x(t) = 4 + 5t, t \in [0, 5]$ is a linear function with domain $[0, 5]$.
Endpoints of range: $x(0) = 4$ and $x(5) = 4 + 25 = 29$
Range is $[4, 29]$
The distance travelled is $29 - 4 = 25$ units.

b $h(t) = 10t - 5t^2$
At ground level, $h(t) = 0$
$\therefore 0 = 10t - 5t^2$
$\therefore 0 = 5t(2 - t)$
$\therefore t = 0,\ t = 2$
It takes the hat 2 seconds to return to the ground.
The domain is $[0, 2]$.
For the range, the turning point is required.
Maximum turning point occurs when $t = 1$.
$h(1) = 10 - 5 = 5$
The turning point is $(1, 5)$
The range is $[0, 5]$

c $l(t) = 0.5 + 0.2t^3, 0 \le t \le 2$

i Domain is $[0, 2]$.
$l(0) = 0.5$, $l(2) = 0.5 + 0.2 \times 8 = 2.1$ so the range of the cubic function is $[0.5, 2.1]$.

ii At the end of the two weeks, the leaf is 2.1 units in length.
Find t when $l(t) = 0.5 \times 2.1$.

$$\therefore 0.5 + 0.2t^3 = 0.5 \times 2.1$$

Multiply both sides by 2

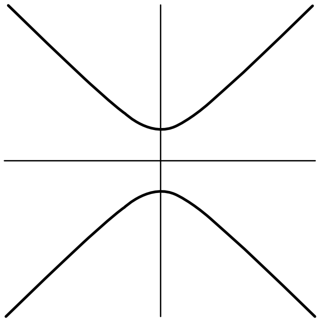
$$\therefore 1 + 0.4t^3 = 2.1$$
$$\therefore 0.4t^3 = 1.1$$
$$\therefore t^3 = \frac{11}{4}$$
$$\therefore t = \sqrt[3]{2.75}$$
$$\therefore t \simeq 1.4$$

It took approximately 1.4 weeks for the leaf to reach half its final length.

17 $y^2 = x^2 + 1$
Enter the equation in the graph and Tab menu as

$y_1 = \sqrt{x^2 + 1}$

$y_2 = -\sqrt{x^2 + 1}$



Domain R.
The y intercepts are $(0, -1)$ and $(0, 1)$ so the range is $R \setminus (-1, 1)$.
This is not the graph of a function as a vertical line can intersect the graph more than once.

18 In the main menu, tap Interactive → Define
Func name: f from Keyboard → abc
variable: x
expression: $x^3 + lx^2 + mx + n$

a Use keyboard → 2D template to solve

$$\left. \begin{array}{l} f(3) = -25 \\ f(5) = 49 \\ f(7) = 243 \end{array} \right|_{l,m,n}$$

This gives $l = 0, m = -12, n = -16$
$\therefore f(x) = x^3 - 12x - 16$

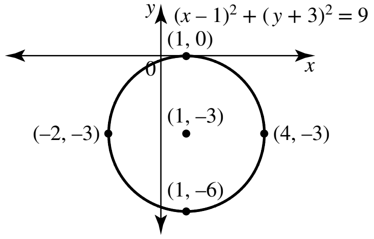
b Redefine $f(x) = x^3 - 12x - 16$
Image of 1.2 is $f(1.2) = -28.672$

c Solve $f(x) = 20$ by tapping Interactive → Equation/Inequality → solve to obtain $x = 4.477$ correct to three decimal places.

d $f: [0, \infty) \to R, f(x) = x^3 - 12x - 16.$
Graph the function by entering $y1 = f(x)$ for $x \ge 0$. Tap Analysis → G-Solve → Min to obtain the minimum turning points $(2, -32)$.
The range is $[-32, \infty)$.

Exercise 6.3 — The circle

1 a $(x - 1)^2 + (y + 3)^2 = 9$
Centre $(1, -3)$, radius $\sqrt{9} = 3$
Domain $[1 - 3, 1 + 3] = [-2, 4]$, range $[-3 - 3, -3 + 3] = [-6, 0]$



b $x^2 + y^2 + 2x + 8y = 0$

$$\therefore (x^2 + 2x + 1) - 1 + (y^2 + 8y + 16) - 16 = 0$$
$$\therefore (x + 1)^2 + (y + 4)^2 = 17$$

Centre $(-1, -4)$, radius $\sqrt{17}$
Domain $[-1 - \sqrt{17}, -1 + \sqrt{17}]$ range $[-4 - \sqrt{17}, -4 + \sqrt{17}]$

2 Centre $(-5, 0)$ passes through the point $(2, 3)$

$\therefore (x + 5)^2 + y^2 = r^2$

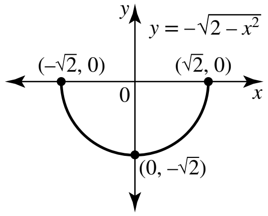
Substitute $(2, 3)$

$$\therefore (2 + 5)^2 + (3)^2 = r^2$$
$$\therefore r^2 = 58$$

So the equation is $(x + 5)^2 + y^2 = 58$
Expanding,

$$x^2 + 10x + 25 + y^2 = 58$$
$$\therefore x^2 + y^2 + 10x - 33 = 0$$

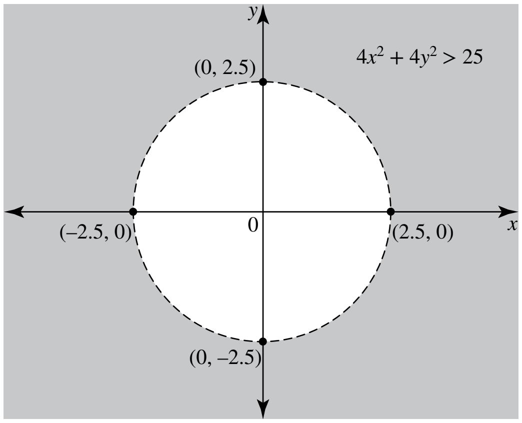
3 a Lower half semi-circle with centre $(0, 0)$ and radius $\sqrt{2}$
Domain $\left[-\sqrt{2}, \sqrt{2}\right]$, range $\left[-\sqrt{2}, 0\right]$



b $4x^2 + 4y^2 > 25$
Open Region outside the circle $4x^2 + 4y^2 = 25$

Dividing by 4 gives the equation $x^2 + y^2 = \dfrac{25}{4}$

Centre $(0, 0)$, radius $\sqrt{\dfrac{25}{4}} = \dfrac{5}{2}$



c Equation of upper semi-circle is

$$y=\sqrt{\left(\frac{5}{2}\right)^2-x^2}$$

$$\therefore y=\sqrt{6.25-x^2}$$

Domain $[-2.5, 2.5]$, range $[0, 2.5]$

4 a $y=2+\sqrt{8-4x-x^2}$

$$\therefore y-2=\sqrt{8-4x-x^2}$$
$$\therefore (y-2)^2=8-4x-x^2$$
$$\therefore x^2+4x+(y-2)^2=8$$
$$\therefore (x^2+4x+4)-4+(y-2)^2=8$$
$$\therefore (x+2)^2+(y-2)^2=12$$

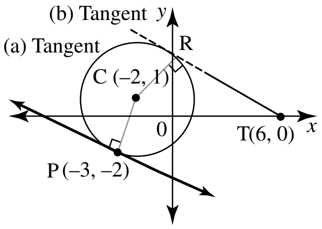
b Centre $(-2, 2)$, radius $\sqrt{12}=2\sqrt{3}$

Due to the positive square root, the semi-circle is the top one.

Domain $\left[-2-2\sqrt{3}, -2+2\sqrt{3}\right]$ Range $\left[2, 2+2\sqrt{3}\right]$

5 $(x+2)^2+(y-1)^2=10$

centre C$(-2, 1)$, radius $\sqrt{10}$, P$(-3, -2)$



a $m_{CP}=3$, therefore gradient of tangent is $-\frac{1}{3}$ equation of tangent

$$y+2=-\frac{1}{3}(x+3)$$
$$\therefore y=-\frac{1}{3}x-3$$

b Let T be the point $(6, 0)$

$$CT=\sqrt{(8)^2+(-1)^2}$$
$$=\sqrt{65}$$

Let R be point of contact tangent makes with the circle and let $RT=t$

$CR=\sqrt{10}$

Triangle CRT is right angled at R since tangent is perpendicular to the radius

Using Pythagoras' theorem

$$\left(\sqrt{65}\right)^2=t^2+\left(\sqrt{10}\right)^2$$
$$\therefore 65=t^2+10$$
$$\therefore t=\sqrt{55}$$

tangent is $\sqrt{55}$ units in length from T to R

c $y+2x-4=0....(1)$

$(x+2)^2+(y-1)^2=10....(2)$

From equation (1), $y=-2x+4$. Substitute this in equation (2)

$$\therefore (x+2)^2+(-2x+5-1)^2=10$$
$$\therefore (x+2)^2+(-2x+4)^2=10$$
$$\therefore x^2+4x+4+4x^2-16x+16=10$$
$$\therefore 5x^2-12x+10=0$$

$$\Delta=b^2-4ac$$
$$=(-12)^2-4(5)(10)$$
$$=144-200$$
$$=-56$$

Since $\Delta<0$, there are no intersections.

6 $x^2+y^2=4$ and $y=mx-3$

Solving as simultaneous equations:

$$x^2+(mx-3)^2=4$$
$$\therefore x^2+m^2x^2-6mx+9=4$$
$$\therefore x^2(1+m^2)-6mx+5=0$$

$$\Delta=b^2-4ac,\ a=(1+m^2),\ b=-6m,\ c=5$$
$$\Delta=(-6m)^2-4(1+m^2)(5)$$
$$=36m^2-20m^2-20$$
$$=16m^2-20$$

For line to be a tangent, $\Delta=0$

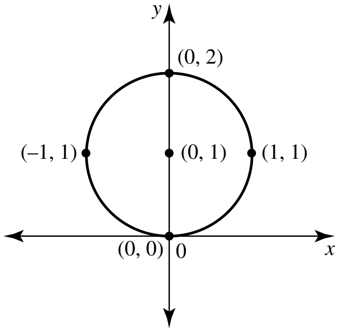
$$\therefore 16m^2-20=0$$
$$\therefore m^2=\frac{5}{4}$$
$$\therefore m=\pm\frac{\sqrt{5}}{2}$$

7 a $x^2+(y-1)^2=1$

Centre $(0, 1)$, radius 1,

Domain: $[0-1, 0+1]=[-1, 1]$

Range: $[1-1, 1+1]=[0, 2]$

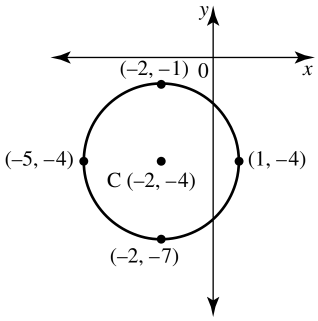


b $(x+2)^2+(y+4)^2=9$

Centre $(-2, -4)$, radius 3

Domain: $[-2-3, -2+3]=[-5, 1]$

Range: $[-4-3, -4+3]=[-7, -1]$

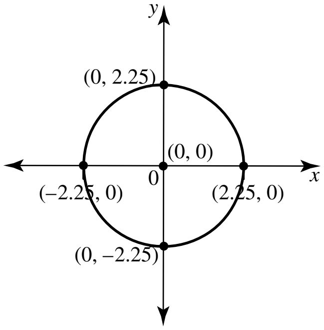


c $16x^2+16y^2=81$

$$\therefore x^2+y^2=\frac{81}{16}$$

Centre $(0, 0)$, radius $\sqrt{\frac{81}{16}}=\frac{9}{4}$

Domain $\left[-\frac{9}{4}, \frac{9}{4}\right]$, range $\left[-\frac{9}{4}, \frac{9}{4}\right]$



d $x^2+y^2-6x+2y+6=0$

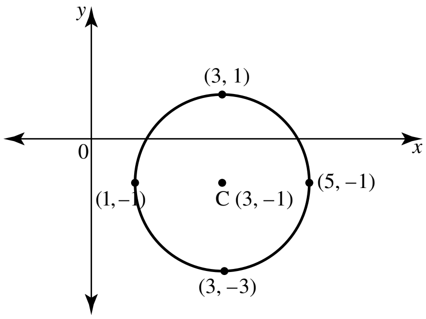
$\therefore \left(x^2-6x+9\right)-9+\left(y^2+2y+1\right)-1+6=0$

$\therefore (x-3)^2+(y+1)^2=4$

Centre $(3,-1)$, radius 2

Domain: $[3-2,3+2]=[1,5]$

Range: $[-1-2,-1+2]=[-3,1]$



e $16x^2+16y^2-16x-16y+7=0$

$$\therefore x^2+y^2-x-y=-\frac{7}{16}$$

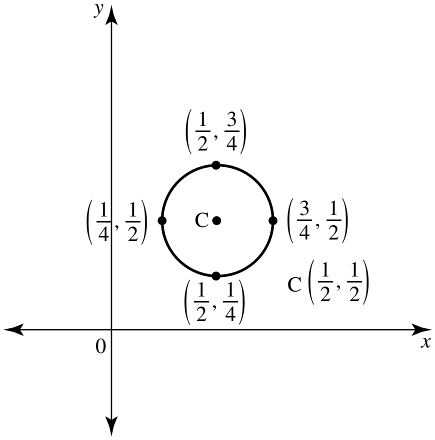
$$\therefore \left(x^2-x+\frac{1}{4}\right)+\left(y^2-y+\frac{1}{4}\right)=-\frac{7}{16}+\frac{1}{4}+\frac{1}{4}$$

$$\therefore \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{-7+4+4}{16}$$

$$\therefore \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{16}$$

Centre $\left(\frac{1}{2},\frac{1}{2}\right)$, radius $\frac{1}{4}$

Domain: $\left[\frac{1}{2}-\frac{1}{4},\frac{1}{2}+\frac{1}{4}\right]=\left[\frac{1}{4},\frac{3}{4}\right]$, range $\left[\frac{1}{4},\frac{3}{4}\right]$



f $(2x+6)^2+(6-2y)^2=4$

$\therefore \left(2(x+3)\right)^2+\left(-2(y-3)\right)^2=4$

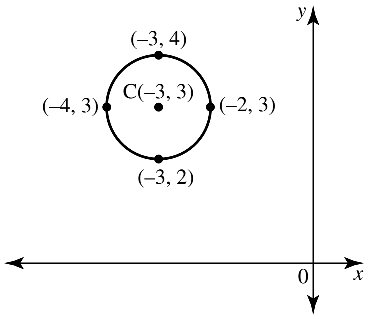
$\therefore 4(x+3)^2+4(y-3)^2=4$

$\therefore (x+3)^2+(y-3)^2=1$

Centre $(-3,3)$, radius 1

Domain: $[-3-1,-3+1]=[-4,-2]$

Range: $[3-1,3+1]=[2,4]$



8 **a** Centre $(-8,9)$, radius 6

equation is $(x+8)^2+(y-9)^2=36$

b Centre $(7,0)$, radius $2\sqrt{2}$

equation is $(x-7)^2+(y-0)^2=\left(2\sqrt{2}\right)^2$

$\therefore (x-7)^2+y^2=8$

c Centre $(1,6)$

Equation has the form $(x-1)^2+(y-6)^2=r^2$

Substitute the given point $(-5,-4)$

$\therefore (-5-1)^2+(-4-6)^2=r^2$

$\therefore r^2=36+100$

$\therefore r^2=136$

Equation is $(x-1)^2+(y-6)^2=136$

d Diameter has endpoints $\left(-\frac{4}{3},2\right)$ and $\left(\frac{4}{3},2\right)$

Centre is the midpoint of the diameter. Centre is $(0,2)$

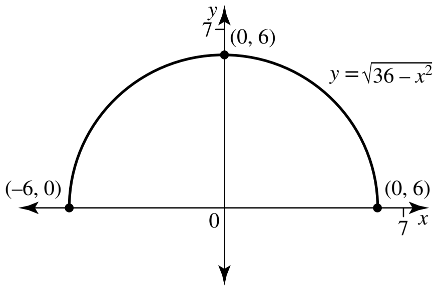
Radius is distance from $(0,2)$ to $\left(\frac{4}{3},2\right)$. Radius is $\frac{4}{3}$.

Equation of circle is $(x-0)^2+(y-2)^2=\left(\frac{4}{3}\right)^2$

$\therefore x^2+(y-2)^2=\frac{16}{9}$ or $9x^2+9(y-2)^2=16$

9 **a** $\left\{(x,y):y=\sqrt{36-x^2}\right\}$ represents the upper half semicircle centre $(0,0)$, radius 6.

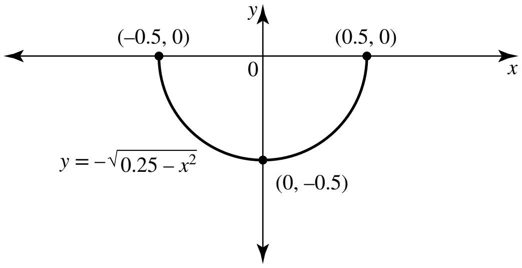
Domain $[-6,6]$, range $[0,6]$



As a mapping this function is $f:[-6,6]\to R, f(x)=\sqrt{36-x^2}$.

b $\left\{(x,y):y=-\sqrt{0.25-x^2}\right\}$ represents the lower half semicircle centre $(0,0)$, radius $\sqrt{0.25}=0.5$.

Domain $[-0.5,0.5]$, range $[-0.5,0]$

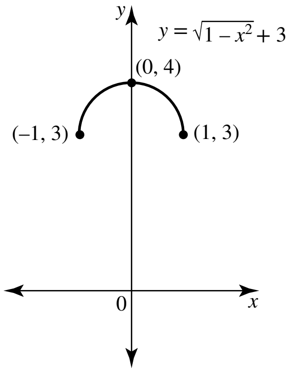


As a mapping this function is
$f:[-0.5,0.5]\to R, f(x)=-\sqrt{0.25-x^2}$

c $y=\sqrt{1-x^2}+3$ Upper half semicircle

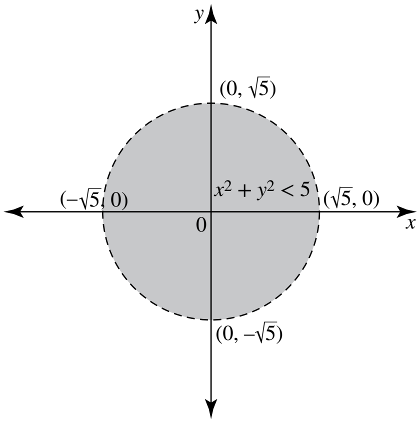
$$\therefore y-3=\sqrt{1-x^2}$$
$$\therefore (y-3)^2=1-x^2$$
$$\therefore x^2+(y-3)^2=1$$

Centre $(0,3)$, radius 1
Domain $[-1,1]$, range $[3,3+1]=[3,4]$



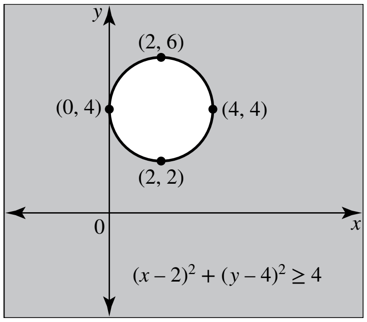
As a mapping this function is
$f:[-1,1]\to R, f(x)=\sqrt{1-x^2}+3$

d $\{(x,y):x^2+y^2<5\}$ is the set of points which lie inside the circle $x^2+y^2=5$ not including the circle itself. The circle has centre $(0,0)$ and radius $\sqrt{5}$.



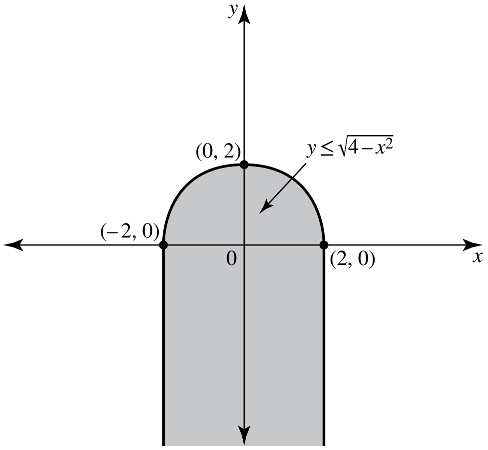
This is not a function. The relation has domain $\left(-\sqrt{5},\sqrt{5}\right)$ and range $\left(-\sqrt{5},\sqrt{5}\right)$.

e $(x-2)^2+(y-4)^2\geq 4$ is the region outside the circle $(x-2)^2+(y-4)^2=4$ including the circle itself. This is not a function. The circle has centre $(2,4)$ and radius 2.



Domain is R and range is R.

f $\{(x,y):y\leq\sqrt{4-x^2}\}$ includes the region on and inside the semicircle $y=\sqrt{4-x^2}$. The semicircle has centre $(0,0)$ and radius 2.
The relation also includes the set of points for which $-2\leq x\leq 2$ and $y\leq 0$.
For example, test the point $(-1,-1)$.
It is a true statement that $-1\leq\sqrt{4-(-1)^2}$ since $-1<\sqrt{3}$



The domain of the relation is $[-2,2]$ and the range is $(-\infty,2]$.

10 a $x^2+y^2-3x+3y+3=0$
Substitute the point $(3,-3)$ into the left hand side of the rule.

$$\begin{aligned}\text{LHS}&=(3)^2+(-3)^2-3(3)+3(-3)+3\\&=9+9-9-9+3\\&=3\end{aligned}$$

$\text{RHS}=0$

Since LHS > RHS, the point $(3,-3)$ lies outside the circle.

b $x^2+y^2+8x-3y+2=0$
Substitute the point $(a,2)$

$$\therefore a^2+4+8a-6+2=0$$
$$\therefore a^2+8a=0$$
$$\therefore a(a+8)=0$$
$$\therefore a=0,\ a=-8$$

The two points are $(0,2)$ and $(-8,2)$.
The circle equation becomes:

$$x^2+8x+y^2-3y=-2$$
$$\therefore (x^2+8x+16)+\left(y^2-3y+\frac{9}{4}\right)=-2+16+\frac{9}{4}$$
$$\therefore (x+4)^2+\left(y-\frac{3}{2}\right)^2=\frac{65}{4}$$

Centre is $\left(-4, \frac{3}{2}\right)$. The y value of the centre is less than the y values of the two points $(0,2)$ and $(-8,2)$ so the two points lie on the upper semicircle.
Rearranging the equation of the circle,

$$\left(y-\frac{3}{2}\right)^2=\frac{65}{4}-(x+4)^2$$
$$\therefore y-\frac{3}{2}=\pm\sqrt{\frac{65}{4}-(x+4)^2}$$

Upper semicircle requires the positive square root
$\therefore y=\sqrt{\frac{65}{4}-(x+4)^2}+\frac{3}{2}$ is the equation of the semicircle on which the two points lie.
This equation could also be expressed as

$$\begin{aligned}y&=\sqrt{\frac{65-4(x+4)^2}{4}}+\frac{3}{2}\\&=\frac{\sqrt{65-4(x^2+8x+16)}}{2}+\frac{3}{2}\\&=\frac{\sqrt{1-32x-4x^2}}{2}+\frac{3}{2}\end{aligned}$$

11 a Circle: $(x-2)^2+(y-2)^2=1$ Line: $y=2x$
Substitute the equation of the line into the equation of the circle.
At intersection,

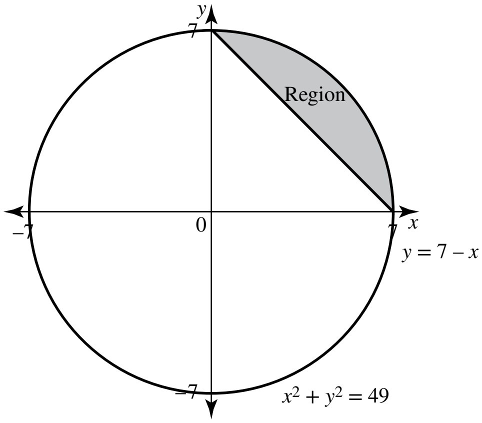
$$\begin{aligned}(x-2)^2+(2x-2)^2&=1\\\therefore x^2-4x+4+4x^2-8x+4&=1\\\therefore 5x^2-12x+7&=0\\\therefore (5x-7)(x-1)&=0\\\therefore x&=\frac{7}{5}\text{ or }x=1\end{aligned}$$

Substitute the x values in $y=2x$
$x=\frac{7}{5}\Rightarrow y=\frac{14}{5}$
$x=1\Rightarrow y=2$
The points of intersection are $\left(\frac{7}{5},\frac{14}{5}\right)$ and $(1,2)$.

b Circle: $x^2+y^2=49$ Line: $y=7-x$
Substitute the equation of the line into the equation of the circle.
At intersection,

$$\begin{aligned}x^2+(7-x)^2&=49\\\therefore x^2+49-14x+x^2&=49\\\therefore 2x^2-14x&=0\\\therefore 2x(x-7)&=0\\\therefore x&=0,\ x=7\end{aligned}$$

Substitute the x values in $y=7-x$
$x=0\Rightarrow y=7$
$x=7\Rightarrow y=0$
The points of intersection are $(0,7)$ and $(7,0)$.
Circle: $x^2+y^2=49$ has centre $(0,0)$ and radius 7.
Both the circle and the line pass through the two points $(0,7)$ and $(7,0)$.
The region $\{(x,y): y\geq 7-x\}\cap\{(x,y): x^2+y^2\leq 49\}$ must lie above the line and inside the circle, with boundaries included.



The required region is the overlap of the two shaded areas.

12 a Circle: $x^2+y^2=5$
Line: $y+2x=5\Rightarrow y=-2x+5$
Substitute $y=-2x+5$ in the equation of the circle
At intersection,

$$\begin{aligned}x^2+(-2x+5)^2&=5\\\therefore x^2+4x^2-20x+25&=5\\\therefore 5x^2-20x+20&=0\\\therefore x^2-4x+4&=0\\\therefore (x-2)^2&=0\\\therefore x&=2\end{aligned}$$

When $x=2$ in the equation of the line, $y=-4+5=1$
There is only one point of intersection, $(2,1)$, so the line is a tangent to the circle at this point of contact.

b Circle: $x^2+y^2+5x-4y+10=0$ Line: $y=kx+2$
Substitute $y=kx+2$ in the equation of the circle.

$$\begin{aligned}\therefore x^2+(kx+2)^2+5x-4(kx+2)+10&=0\\\therefore x^2+k^2x^2+\cancel{4kx}+4+5x-\cancel{4kx}-8+10&=0\\\therefore (1+k^2)x^2+5x+6&=0\end{aligned}$$

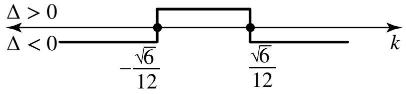
The discriminant of this quadratic equation determines the number of solutions and hence the number of intersections of the line with the circle.

$$\begin{aligned}\Delta&=b^2-4ac,\ a=(1+k^2),\ b=5,\ c=6\\&=25-24(1+k^2)\\&=1-24k^2\end{aligned}$$

i For one intersection, $\Delta=0$

$$\begin{aligned}\therefore 1-24k^2&=0\\\therefore 1&=24k^2\\\therefore k^2&=\frac{1}{24}\\\therefore k&=\pm\frac{1}{2\sqrt{6}}\\\therefore k&=\pm\frac{\sqrt{6}}{12}\end{aligned}$$

ii For two intersections, $\Delta>0$



From the sign diagram of the discriminant, the values of k for two intersections are $-\frac{\sqrt{6}}{12}<k<\frac{\sqrt{6}}{12}$.

iii For no intersections, $\Delta < 0$. The values of k are $k < -\dfrac{\sqrt{6}}{12}$ or $k > \dfrac{\sqrt{6}}{12}$.

13 $x^2 + y^2 - 6x + 4y - 12 = 0$

a Completing the square, the equation becomes

$$x^2 - 6x + y^2 + 4y = 12$$
$$\therefore (x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4$$
$$\therefore (x-3)^2 + (y+2)^2 = 25$$

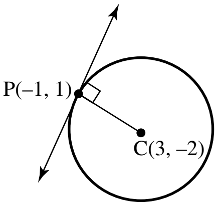
Centre $(3,-2)$, radius 5

b **i** Substitute the point $(-1,1)$ in the left hand side of the expanded equation.

$$\begin{aligned} \text{LHS} &= 1 + 1 + 6 + 4 - 12 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

The point $(-1,1)$ lies on the circle.

ii Let the centre be $C(3,-2)$ and the point of contact $P(-1,1)$



$$\begin{aligned} m_{CP} &= \frac{1-(-2)}{-1-3} \\ &= \frac{3}{-4} \\ &= -\frac{3}{4} \end{aligned}$$

The tangent is perpendicular to CP so $m_{\text{tangent}} = \dfrac{4}{3}$.

The equation of the tangent through $P(-1,1)$ is

$$y - 1 = \frac{4}{3}(x+1)$$
$$\therefore 3(y-1) = 4(x+1)$$
$$\therefore 3y - 3 = 4x + 4$$
$$\therefore 3y - 4x = 7$$

c **i** Substitute the point $(3,3)$ into the centre-radius form of the equation of the circle

$(x-3)^2 + (y+2)^2 = 25$

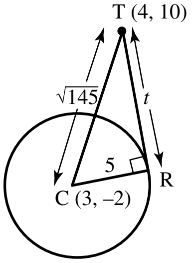
$$\begin{aligned} \text{LHS} &= (3-3)^2 + (3+2)^2 \\ &= 0 + 25 \\ &= 25 \\ &= \text{RHS} \end{aligned}$$

The point $(3,3)$ lies on the circle.

ii The centre of the circle is $(3,-2)$ so the point $(3,3)$ is an endpoint of the vertical diameter of the circle. The tangent to the circle at this point must be horizontal. The horizontal line through $(3,3)$ has equation $y = 3$ so this is the equation of the tangent.

d The range of the circle is $[-2-5, -2+5] = [-7, 3]$ so the point T(4,10) lies outside the circle. The distance of T from the centre C of the circle is

$$\begin{aligned} d_{CT} &= \sqrt{(4-3)^2 + (10-(-2))^2} \\ &= \sqrt{1+144} \\ &= \sqrt{145} \end{aligned}$$

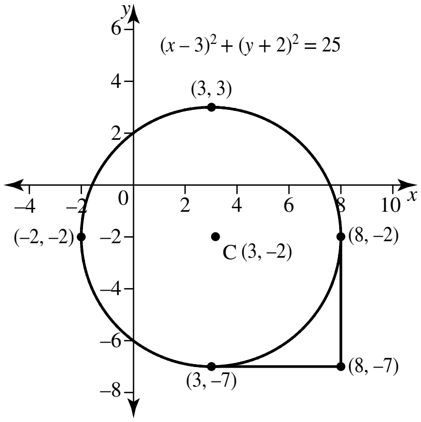


Let the distance between T and R be t.
Using Pythagoras' theorem in triangle CRT

$$t^2 + 5^2 = \left(\sqrt{145}\right)^2$$
$$\therefore t^2 = 145 - 25$$
$$\therefore t^2 = 120$$
$$\therefore t = \sqrt{120}$$
$$\therefore t = 2\sqrt{30}$$

The length TR is $2\sqrt{30}$ units.

e The endpoints of the vertical and horizontal diameters of the circle with centre $(3,-2)$ and radius 5 are $(3,3),(3,-7)$ and $(-2,-2),(8,-2)$ respectively.



Since the point $(8,-7)$ lies vertically below the endpoint $(8,-2)$ and horizontally to the right of the endpoint $(3,-7)$, the tangent from this point to the circle is either vertical with point of contact $(8,-2)$ or horizontal with point of contact $(3,-7)$.

Either way, the length of the tangent from the point to the circle is 5 units.

f The equation of the tangent in part **e** is either $x = 8$ (vertical) or $y = -7$ (horizontal).

At the intersection of $3y - 4x = 7$ and $x = 8$

$$3y - 4 \times 8 = 7$$
$$\therefore 3y = 39$$
$$\therefore y = 13$$

Point of intersection is $(8,13)$

At the intersection of $3y - 4x = 7$ and $y = -7$,

$$3 \times (-7) - 4x = 7$$
$$\therefore -4x = 28$$
$$\therefore x = -7$$

Point of intersection is $(-7,-7)$.

14 $x^2+y^2-2x-4y-20=0$

a Let $y=0$

$$\therefore x^2-2x-20=0$$
$$\therefore (x^2-2x+1)-1-20=0$$
$$\therefore (x-1)^2=21$$
$$\therefore x-1=\pm\sqrt{21}$$
$$\therefore x=1\pm\sqrt{21}$$

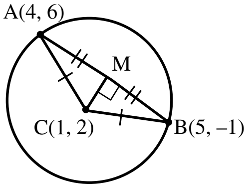
The x intercepts occur at $x=1-\sqrt{21}$ and $x=1+\sqrt{21}$ so the length of the intercept cut off on the x axis is $2\sqrt{21}$ units.

b $x^2+y^2-2x-4y-20=0$

$$\therefore x^2-2x+y^2-4y=20$$
$$(x^2-2x+1)+(y^2-4y+4)=20+1+4$$
$$\therefore (x-1)^2+(y-2)^2=25$$

Centre $(1,2)$ and radius 5.

Let A be the point $(4,6)$, B the point $(5,-1)$, C the centre point $(1,2)$ and M the midpoint of AB.



The length of CM measures the distance of the centre from the chord AB.

Co-ordinates of midpoint M are $\left(\frac{4+5}{2},\frac{6+(-1)}{2}\right)=\left(\frac{9}{2},\frac{5}{2}\right)$.

Distance between M and C:

$$d_{CM}=\sqrt{\left(\frac{9}{2}-1\right)^2+\left(\frac{5}{2}-2\right)^2}$$
$$=\sqrt{\left(\frac{7}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$
$$=\sqrt{\frac{49}{4}+\frac{1}{4}}$$
$$=\frac{\sqrt{50}}{2}$$
$$=\frac{5\sqrt{2}}{2}$$

The required distance is $\frac{5\sqrt{2}}{2}$ units.

15 a $x^2+y^2+ax+by+c=0$

Substitute the given points

$(1,0)\Rightarrow 1+a+c=0....(1)$

$(0,2)\Rightarrow 4+2b+c=0....(2)$

$(0,8)\Rightarrow 64+8b+c=0....(3)$

Subtract equation (2) from equation (3)

$$\therefore 60+6b=0$$
$$\therefore 6b=-60$$
$$\therefore b=-10$$

Substitute $b=-10$ in equation (2)

$$\therefore 4-20+c=0$$
$$\therefore c=16$$

Substitute $c=16$ in equation (1)

$$\therefore 1+a+16=0$$
$$\therefore a=-17$$

Answer: $a=-17,\ b=-10,\ c=16$

b The equation of the circle is $x^2+y^2-17x-10y+16=0$

$$\therefore x^2-17x+y^2-10y=-16$$
$$\therefore \left(x^2-17x+\left(\frac{17}{2}\right)^2\right)+(y^2-10y+25)=-16+\left(\frac{17}{2}\right)^2+25$$
$$\therefore \left(x-\frac{17}{2}\right)^2+(y-5)^2=9+\frac{289}{4}$$
$$\therefore \left(x-\frac{17}{2}\right)^2+(y-5)^2=\frac{36+289}{4}$$
$$\therefore \left(x-\frac{17}{2}\right)^2+(y-5)^2=\frac{325}{4}$$

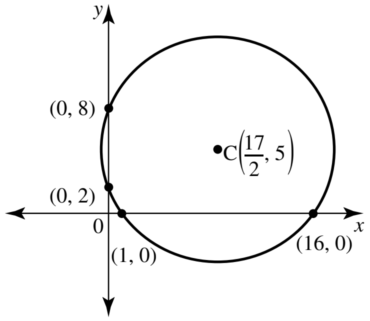
Centre $\left(\frac{17}{2},5\right)$, radius $\sqrt{\frac{325}{4}}=\frac{5\sqrt{13}}{2}$

c x intercepts: Let $y=0$

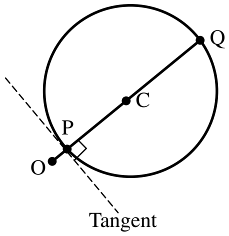
$$\therefore x^2-17x+16=0$$
$$\therefore (x-1)(x-16)=0$$
$$\therefore x=1, x=16$$

$(1,0)$ (given) and $(16,0)$ are the x intercepts.

y intercepts are given as $(0,2),(0,8)$.



d The closest point P on the circle to the origin is where the line OC first intersects the circle.



$OP=OC-PC$ where the radius $PC=\frac{5\sqrt{13}}{2}$.

$$d_{OC}=\sqrt{\left(\frac{17}{2}\right)^2+(5)^2}$$
$$=\sqrt{\frac{289+100}{4}}$$
$$=\frac{\sqrt{389}}{2}$$
$$\therefore OP=\frac{\sqrt{389}}{2}-\frac{5\sqrt{13}}{2}$$
$$\therefore OP\simeq 0.85$$

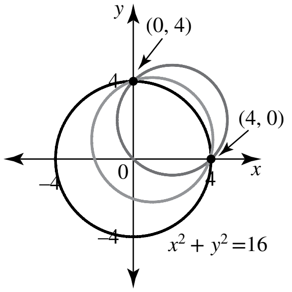
The shortest distance from the origin to the circle is 0.85 units, correct to two decimal places.

e The greatest distance from the origin to the circle is OQ where Q is the second point on the circle intersected by the line OC.

$$OQ=OC+CQ$$
$$\therefore OQ=\frac{\sqrt{389}}{2}+\frac{5\sqrt{13}}{2}$$
$$\therefore OQ\simeq 18.88$$

The greatest distance is 18.88 units correct to two decimal places.

16 a The circle $x^2 + y^2 = 16$ has centre $(0,0)$ and radius 4. The endpoints of its horizontal diameter are $(-4,0),(4,0)$ and the endpoints of its vertical diameter are $(0,-4),(0,4)$.



Two other circles through the points $(0,4)$ and $(4,0)$ are sketched. Three points are required to determine a circle so there can be several circles drawn through two points.

b With the points $(0,4)$ and $(4,0)$ endpoints of a diameter, the centre is the midpoint of the diameter and the radius is half the length of the diameter,

Centre: $\left(\frac{0+4}{2}, \frac{4+0}{2}\right) = (2,2)$

Radius:

$$\begin{aligned} r &= \frac{1}{2} \times \sqrt{(4-0)^2 + (0-4)^2} \\ &= \frac{1}{2} \times \sqrt{32} \\ &= \frac{1}{2} \times 4\sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

radius is $2\sqrt{2}$ units in length.

The equation of the circle centre $(2,2)$, radius $2\sqrt{2}$ is

$$(x-2)^2 + (y-2)^2 = \left(2\sqrt{2}\right)^2$$

$$\therefore (x-2)^2 + (y-2)^2 = 8$$

Test if $(0,0)$ lies on the circle:

$$\begin{aligned} \text{LHS} &= (-2)^2 + (-2)^2 \\ &= 4+4 \\ &= 8 \\ &= \text{RHS} \end{aligned}$$

The circle does pass through the origin.

c i At the intersection of the line $y = x$ with the circle $(x-2)^2 + (y-2)^2 = 8$,

$$\begin{aligned} (x-2)^2 + (x-2)^2 &= 8 \\ \therefore 2(x-2)^2 &= 8 \\ \therefore (x-2)^2 &= 4 \\ \therefore x-2 &= \pm 2 \\ \therefore x &= 0, x = 4 \end{aligned}$$

Substituting the x values in $y = x$ gives the points of intersection as $(0,0)$ and $(4,4)$. Since the friends start their competition at the origin, the kiosk, K has co-ordinates $(4,4)$.

ii Sam: Distance OK is the diameter of the circle. The radius is $2\sqrt{2}$ so the distance Sam swims is $4\sqrt{2}$ km. Sam swims at a speed of 2.5 km/h so the time taken is

$$\begin{aligned} \text{time} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{4\sqrt{2}}{2.5} \\ &\simeq 2.26 \end{aligned}$$

Sam takes 2.26 hours.

Rufus: The distance Rufus walks is half the circumference of the circle.

Circumference, $C = 2\pi r$. Rufus walks a distance of $\frac{1}{2} \times 2\pi \times 2\sqrt{2} = 2\sqrt{2}\pi$ km at a speed of 4 km/h.

$$\begin{aligned} \text{time} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{2\sqrt{2}\pi}{4} \\ &= \frac{\sqrt{2}\pi}{2} \\ &\simeq 2.22 \end{aligned}$$

Rufus takes 2.22 hours so Rufus arrives at the kiosk first.

The time difference is $\left(\frac{4\sqrt{2}}{2.5} - \frac{\sqrt{2}\pi}{2}\right) \times 60 \simeq 2.48$ minutes. (approximately 2 minutes)

17 Select the Conics menu and tap Form and select the Conic form for a circle in expanded form, $Ax^2 + Ay^2 + Bx + Cy + D = 0$. Replace A by 1, B by 4, C by -7 and D by 2 to obtain the equation $x^2 + y^2 + 4x - 7y + 2 = 0$ then press EXE to obtain the shape. From Analysis → G-Solve → x intercept or to y intercept or to Center (sic) or to Radius, the required information about the circle can be obtained.
x intercepts occur at $x = -3.414, x = -0.586$
y intercepts occur at $y = 0.298, y = 6.702$
Centre is $(-2, 3.5)$ and radius is 3.775.

18 In the Conics menu, select the form $(x-H)^2 + (y-K)^2 = R^2$ and use $H = 6, K = -4, R = \sqrt{66}$ to obtain the equation $(x-6)^2 + (y+4)^2 = 66$. (Alternatively, enter the equation in the main menu and highlight and drop into the conics menu). Press EXE to obtain the shape.

Select x-Cal from Analysis and set y equal to -4 to obtain the co-ordinates of the end points of the horizontal diameter as $(-2.124,-4),(14.124,-4)$.

Select y-Cal from Analysis and set x equal to 6 to obtain the co-ordinates of the end points of the vertical diameter as $(6,-12.124),(6,4.124)$.

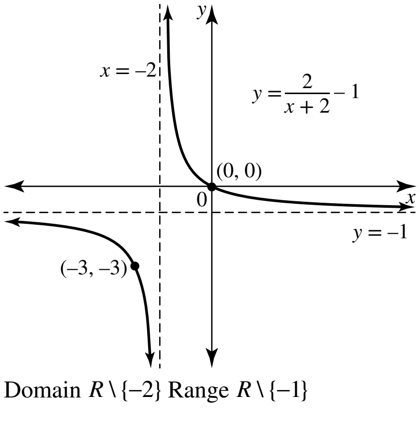
The domain is $[-2.124, 14.124]$ and the range is $[-12.124, 4.124]$ correct to three decimal places.

At the endpoints of the domain, the tangent to the circle would be vertical. Therefore, its gradient is undefined.

Exercise 6.4 — The rectangular hyperbola and the truncus

1 a $y = \frac{2}{x+2} - 1$

vertical asymptote $x = -2$
horizontal asymptote $y = -1$
y intercept $(0,0)$



Domain $R \setminus \{-2\}$ Range $R \setminus \{-1\}$

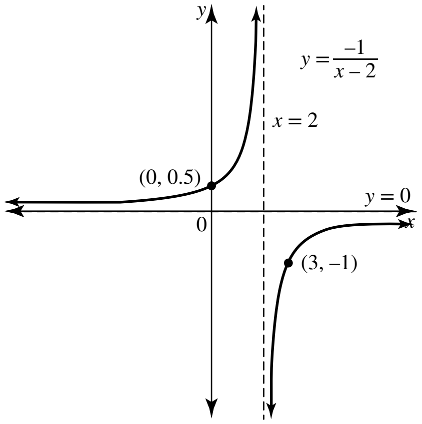
b $y = \dfrac{-1}{x-2}$

vertical asymptote $x = 2$

horizontal asymptote $y = 0$

y intercept $\left(0, \dfrac{1}{2}\right)$

Domain $R \setminus \{2\}$ Range $R \setminus \{0\}$



2 $f : R \setminus \left\{\dfrac{1}{2}\right\} \to R, f(x) = 4 - \dfrac{3}{1-2x}$

Vertical asymptote when $1 - 2x = 0$

Therefore vertical asymptote has equation $x = \dfrac{1}{2}$

Horizontal asymptote is $y = 4$

The equation $y = 4 - \dfrac{3}{1-2x}$ can be expressed as

$$y = -\frac{3}{-2\left(x - \frac{1}{2}\right)} + 4$$

So, $y = \dfrac{3}{2\left(x - \frac{1}{2}\right)} + 4 \Rightarrow y = \dfrac{3/2}{x - 1/2} + 4$

This means the hyperbola lies in quadrants 1 and 3 (quadrants as defined by the asymptotes).

3 **a** $y = \dfrac{6x}{3x+2}$ improper form so divide to obtain proper form

$$y = \frac{2(3x+2) - 4}{3x+2}$$

$$\therefore y = 2 - \frac{4}{3x+2}$$

vertical asymptote $x = -\dfrac{2}{3}$, horizontal asymptote $y = 2$

b Asymptotes shown as $x = 4, y = \dfrac{1}{2}$

Equation becomes $y = \dfrac{a}{x-4} + \dfrac{1}{2}$

Substitute the point $(6, 0)$

$\therefore 0 = \dfrac{a}{2} + \dfrac{1}{2}$

$\therefore a = -1$

Therefore the equation is $y = \dfrac{-1}{x-4} + \dfrac{1}{2}$

4 $xy - 4y + 1 = 0$ needs to be expressed in standard hyperbola form

$xy - 4y + 1 = 0$

$\therefore xy - 4y = -1$

$\therefore y(x-4) = -1$

$\therefore y = \dfrac{-1}{x-4}$

Asymptotes have equations $x = 4, y = 0$, so domain is $R \setminus \{4\}$ and range is $R \setminus \{0\}$.

5 **a** $R \propto \dfrac{1}{I} \Rightarrow R = \dfrac{k}{I}$

$I = 0.6, R = 400$

$\therefore 400 = \dfrac{k}{0.6}$

$\therefore k = 400 \times 0.6$

$\therefore k = 240$

Hence $R = \dfrac{240}{I}$

b Current is increased by 20% $\Rightarrow I = 0.6 \times 1.20 = 0.72$

If $I = 0.72$,

$R = \dfrac{240}{0.72}$

$= \dfrac{1000}{3}$

resistance is $333\frac{1}{3}$ ohm

6 For inverse proportion xy is constant

This is true for the second table **b** where $xy = 7.2$

$\therefore y = \dfrac{7.2}{x}$ is the rule.

If $y = 6.4$, then $x = \dfrac{7.2}{6.4}$

Therefore $x = \dfrac{9}{8} = 1.125$

If $x = 8, y = \dfrac{7.2}{8} = 0.9$

7 **a** $y = \dfrac{1}{(x-3)^2} - 1$

Asymptotes: $x = 3, y = -1$

y intercept: Let $x = 0$

$\therefore y = \dfrac{1}{(-3)^2} - 1$

$\therefore y = -\dfrac{8}{9}$

$\left(0, -\dfrac{8}{9}\right)$

x intercepts: Let $y = 0$

$\therefore \dfrac{1}{(x-3)^2} - 1 = 0$

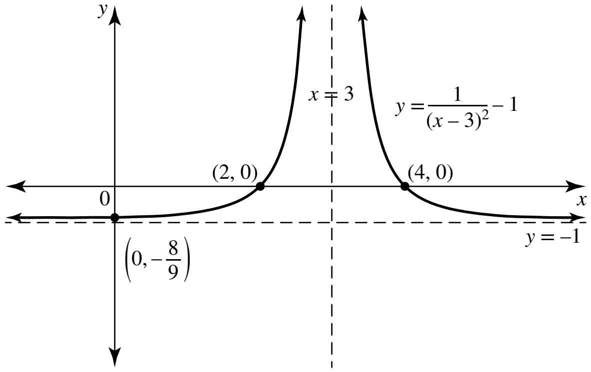
$\therefore \dfrac{1}{(x-3)^2} = 1$

$\therefore (x-3)^2 = 1$

$\therefore x - 3 = \pm 1$

$\therefore x = 2, x = 4$

$(2, 0), (4, 0)$



Domain $R \setminus \{3\}$, range $(-1, \infty)$

b $y=\frac{-8}{(x+2)^2}-4$

Asymptotes: $x=-2, y=-4$

y intercept: Let $x=0$

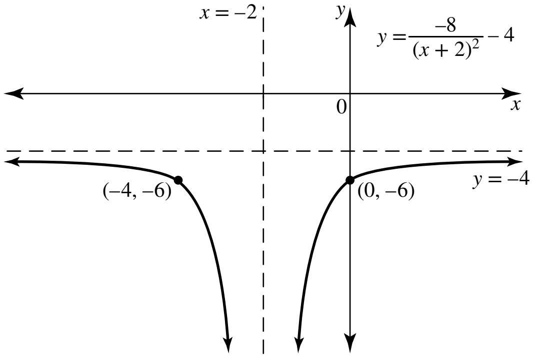
$\therefore y=\frac{-8}{(2)^2}-4$

$\therefore y=-6$

$(0,-6)$

x intercepts: As $a<0$ and horizontal asymptote is $y=-4$, there are no x intercepts.

Point: By symmetry, the point $(-4,-6)$ lies on the graph.



Domain $R\setminus\{-2\}$, range $(-\infty,-4)$

8 $y=\frac{3}{2(1-5x)^2}$

$\therefore y=\frac{\frac{3}{2}}{(1-5x)^2}$

Vertical asymptote occurs when $(1-5x)^2=0$

$\therefore 1-5x=0$

$\therefore x=\frac{1}{5}$

Horizontal asymptote is $y=0$.

Domain is $R\setminus\left\{\frac{1}{5}\right\}$

$a=\frac{3}{2}>0$, so range is $(0,\infty)$

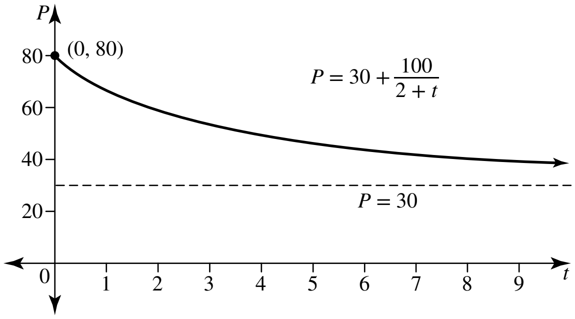
9 **a** $P=30+\frac{100}{2+t}$

$t=0\Rightarrow P=80$

$t=2\Rightarrow P=55$

Therefore the herd has reduced by 25 cattle after the first 2 years.

b Asymptote $t=-2$ (not applicable), $h=50$



Domain $\{t: t\geq 0\}$ Range $(30,80]$

c The number of cattle will never go below 30.

10 $h=25-\frac{100}{(t+2)^2},\ t\geq 0$

a Let $h=12.5$

$\therefore 12.5=25-\frac{100}{(t+2)^2}$

$\therefore \frac{100}{(t+2)^2}=12.5$

$\therefore 100=12.5(t+2)^2$

$\therefore (t+2)^2=\frac{100}{12.5}$

$\therefore (t+2)^2=8$

$\therefore t+2=\pm\sqrt{8}$

$\therefore t=-2\pm\sqrt{8}$

$t>0$, so $t=-2+\sqrt{8}$

The time to reach the height is $0.8284\times 60\simeq 50$ seconds.

b As $t\to\infty$, the graph approaches its horizontal asymptote, $h=25$. The limiting altitude is 25 metres above the ground.

11 **a** $y=\frac{1}{x+5}+2$

Since $x+5=0$ when $x=-5$, the asymptotes have the equations $x=-5$ and $y=2$.

b $y=\frac{8}{x}-3$

The asymptotes have the equations $x=0$ and $y=-3$.

c $y=\frac{-3}{4x}$

Since $4x=0$ when $x=0$, the asymptotes have the equations $x=0$ and $y=0$.

d $y=\frac{-3}{14+x}-\frac{3}{4}$

Since $14+x=0$ when $x=-14$, the asymptotes have the equations $x=-14$ and $y=-\frac{3}{4}$.

12 **a** $y=\frac{1}{x+1}-3$

Asymptotes: $x=-1, y=-3$

y intercept: Let $x=0$, $y=\frac{1}{1}-3=-2$. $(0,-2)$

x intercept: Let $y=0$

$0=\frac{1}{x+1}-3$

$\therefore 3=\frac{1}{x+1}$

$\therefore 3(x+1)=1$

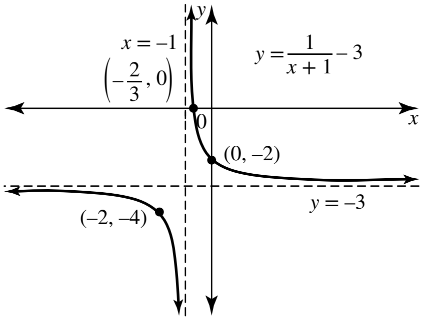
$\therefore x+1=\frac{1}{3}$

$\therefore x=-\frac{2}{3}$

$\left(-\frac{2}{3},0\right)$

Domain $R\setminus\{-1\}$, range $R\setminus\{-3\}$

Point: When $x=-2$, $y=\frac{1}{-1}-3=-4$ $(-2,-4)$



b $y=4-\dfrac{3}{x-3}$ or $y=-\dfrac{3}{x-3}+4$

Asymptotes: $x=3, y=4$

y intercept: Let $x=0$, $y=4-\dfrac{3}{-3}=5\ (0,5)$

x intercept: Let $y=0$

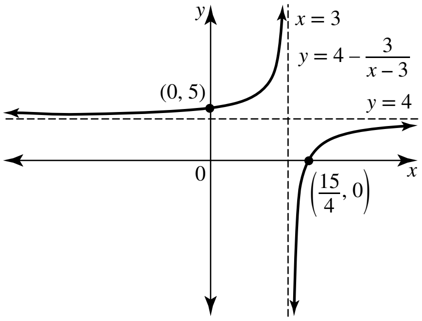
$$0=4-\frac{3}{x-3}$$

$$\therefore \frac{3}{x-3}=4$$

$$\therefore \frac{3}{4}=x-3$$

$$\therefore x=\frac{15}{4} \quad \left(\frac{15}{4},0\right)$$

Domain $R\setminus\{3\}$, range $R\setminus\{4\}$



c $y=-\dfrac{5}{3+x}$

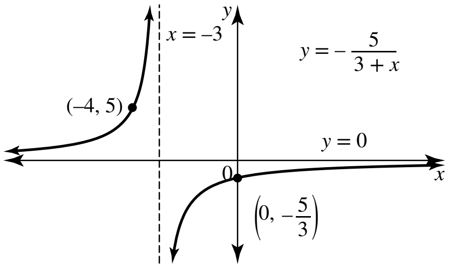
Asymptotes: $x=-3, y=0$

y intercept: Let $x=0$, $y=-\dfrac{5}{3}$ $\left(0,-\dfrac{5}{3}\right)$

No x intercept

Domain $R\setminus\{-3\}$, range $R\setminus\{0\}$

Point: Let $x=-4$, $y=-\dfrac{5}{-1}=5$ $(-4,5)$



d $y=-\left(1+\dfrac{5}{2-x}\right)$

$$\therefore y=-1-\frac{5}{2-x}$$

$$\therefore y=-1+\frac{5}{x-2}$$

Asymptotes: $x=2, y=-1$

y intercept: Let $x=0$, $y=-1+\dfrac{5}{-2}=-\dfrac{7}{2}$ $\left(0,-\dfrac{7}{2}\right)$

x intercept: Let $y=0$

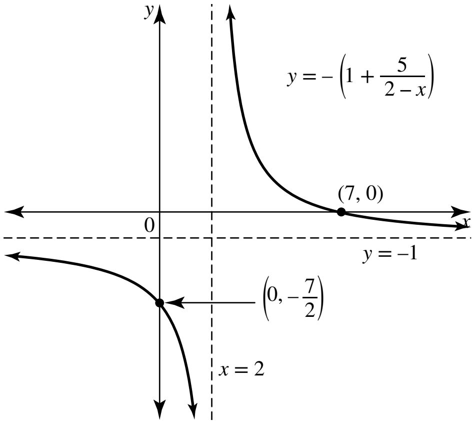
$$0=-1+\frac{5}{x-2}$$

$$\therefore 1=\frac{5}{x-2}$$

$$\therefore x-2=5$$

$$\therefore x=7 \qquad (7,0)$$

Domain $R\setminus\{2\}$, range $R\setminus\{-1\}$



13 a $\dfrac{11-3x}{4-x}=a-\dfrac{b}{4-x}$

$$\therefore \frac{11-3x}{4-x}=\frac{a(4-x)-b}{4-x}$$

$$\therefore \frac{11-3x}{4-x}=\frac{4a-ax-b}{4-x}$$

$$\therefore 11-3x=-ax+4a-b$$

Equating coefficients of like terms:

x: $-3=-a$

$\therefore a=3$

constant: $11=4a-b$

Substitute $a=3$

$\therefore 11=12-b$

$\therefore b=1$

Answer: $a=3,\ b=1$

b $y=\dfrac{11-3x}{4-x}\Rightarrow y=3-\dfrac{1}{4-x}$

Asymptotes: $x=4, y=3$

y intercept: Let $x=0$, $y=3-\dfrac{1}{4}=\dfrac{11}{4}$ $\left(0,\dfrac{11}{4}\right)$

x intercept: Let $y=0$ in $y=\dfrac{11-3x}{4-x}$

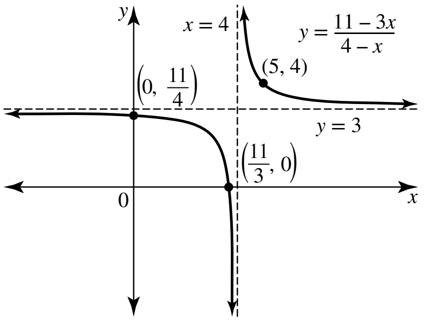
$$\therefore 0=\frac{11-3x}{4-x}$$

$$\therefore 0=11-3x$$

$$\therefore x=\frac{11}{3}$$

$$\left(\frac{11}{3},0\right)$$

Point: Let $x=5$, $y=3-\dfrac{1}{-1}=4$ $(5,4)$



c The y values of the points on the graph are positive when $x<\dfrac{11}{3}$ or $x>4$ so $\dfrac{11-3x}{4-x}>0$ when $x<\dfrac{11}{3}$ or $x>4$.

14 a $y=\frac{x}{4x+1}$

Using the division algorithm,

$$\begin{array}{r} \frac{1}{4} \\ 4x+1\overline{)x+0} \\ x+\frac{1}{4} \\ \hline -\frac{1}{4} \\ \hline \end{array}$$

$\therefore \frac{x}{4x+1}=\frac{1}{4}-\frac{\frac{1}{4}}{4x+1}$

$\therefore y=\frac{-1}{4(4x+1)}+\frac{1}{4}$

$\therefore y=\frac{-1}{16x+4}+\frac{1}{4}$

This is in the form $y=\frac{a}{bx+c}+d$ with

$a=-1, b=16, c=4, d=\frac{1}{4.}$

Since $16x+4=0$ when $x=-\frac{1}{4}$, the equations of the asymptotes are $x=-\frac{1}{4}, y=\frac{1}{4}$.

b $(x-4)(y+2)=4$

$\therefore (y+2)=\frac{4}{(x-4)}$

$\therefore y=\frac{4}{x-4}-2$

The equations of the asymptotes are $x=4, y=-2$.

c $y=\frac{1+2x}{x}$

$\therefore y=\frac{1}{x}+\frac{2x}{x}$

$\therefore y=\frac{1}{x}+2$

The equations of the asymptotes are $x=0, y=2$

d $2xy+3y+2=0$

$\therefore y(2x+3)+2=0$

$\therefore y(2x+3)=-2$

$\therefore y=\frac{-2}{2x+3}$

Since $2x+3=0$ when $x=-\frac{3}{2}$, the equations of the asymptotes are $x=-\frac{3}{2}$, $y=0$.

15 a $y=\frac{12}{(x-2)^2}+5$

Asymptotes: $x=2, y=5$

y intercept: Let $x=0$

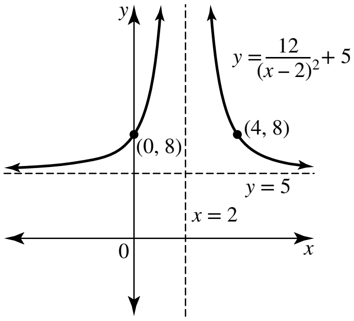
$\therefore y=\frac{12}{(-2)^2}+5$

$\therefore y=8$

$(0,8)$

There are no x intercepts.

Point: By symmetry the point $(4,8)$ lies on the graph.



Domain $R\setminus\{2\}$, range $(5,\infty)$

b $y=\frac{-24}{(x+2)^2}+6$

Asymptotes: $x=-2, y=6$

y intercept: Let $x=0$

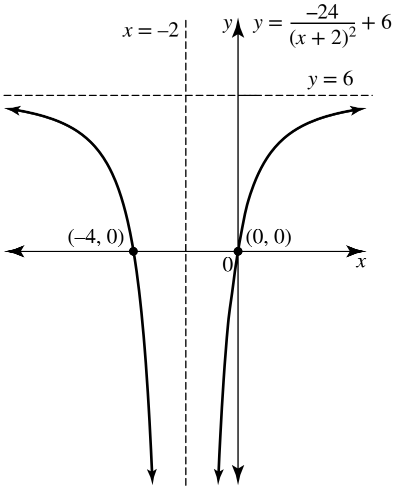
$\therefore y=\frac{-24}{(2)^2}+6$

$\therefore y=0$

Graph passes through the origin.

Other x intercept, by symmetry with the asymptote $x=-2$ must be $(-4,0)$.

Domain $R\setminus\{-2\}$, range $(-\infty,6)$.



c $y=7-\frac{1}{7x^2}$

Asymptotes: $x=0, y=7$

No y intercept

x intercepts: Let $y=0$

$\therefore 7-\frac{1}{7x^2}=0$

$\therefore \frac{1}{7x^2}=7$

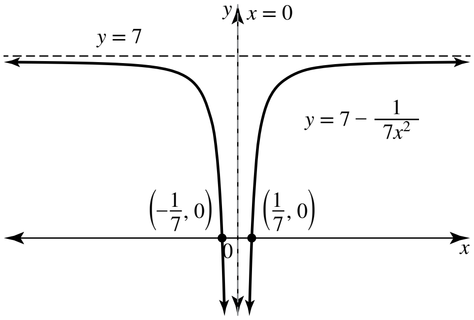
$\therefore 1=49x^2$

$\therefore x^2=\frac{1}{49}$

$\therefore x=\pm\frac{1}{7}$

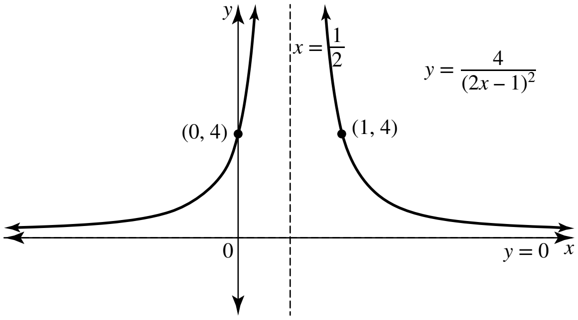
$\left(-\frac{1}{7},0\right),\left(\frac{1}{7},0\right)$

Domain $R\setminus\{0\}$, range $(-\infty,7)$



d $y = \dfrac{4}{(2x-1)^2}$

Asymptotes: $2x - 1 = 0 \Rightarrow x = \dfrac{1}{2}$ and horizontal asymptote is $y = 0$
No x intercepts
y intercept: Let $x = 0$
$\therefore y = \dfrac{4}{(-1)^2}$
$\therefore y = 4$
$(0, 4)$
By symmetry with the asymptote $x = \dfrac{1}{2}$, the point $(1, 4)$ is on the graph.
Domain $R \setminus \left\{\dfrac{1}{2}\right\}$, range R^+

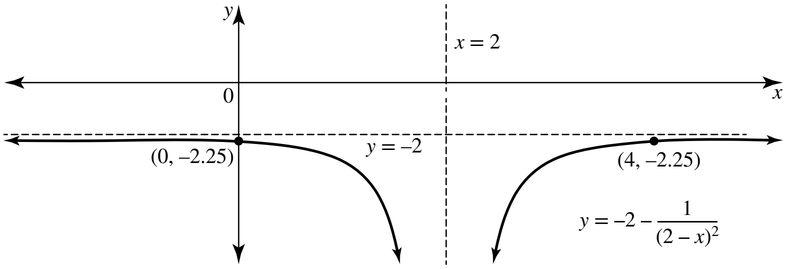


e $y = -2 - \dfrac{1}{(2-x)^2}$

Asymptotes: $(2 - x)^2 = 0 \Rightarrow x = 2$, horizontal asymptote $y = -2$
y intercept: Let $x = 0$
$\therefore y = -2 - \dfrac{1}{(2)^2}$
$\therefore y = -\dfrac{9}{4}$
$\left(0, -\dfrac{9}{4}\right)$

No x intercepts.

Point $\left(4, -\dfrac{9}{4}\right)$ is symmetric to the vertical asymptote with the y intercept.
Domain $R \setminus \{2\}$, range $(-\infty, -2)$

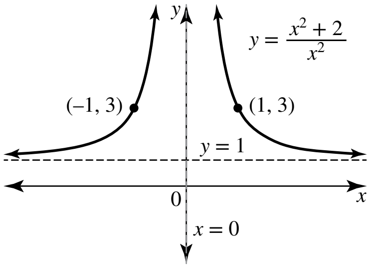


f $y=\dfrac{x^2+2}{x^2}$

$\therefore y=\dfrac{x^2}{x^2}+\dfrac{2}{x^2}$

$\therefore y=1+\dfrac{2}{x^2}$

Asymptotes: $x=0, y=1$
No intercepts with the axes.
Domain $R\setminus\{0\}$, range $(1,\infty)$
Points: Let $x=1$
$\therefore y=3$
$(1,3)$
By symmetry the point $(-1,3)$ is also on the graph.



16 **a** Let the equation be $y=\dfrac{a}{x-h}+k$

Vertical asymptote at $x=3\Rightarrow h=3$
Horizontal asymptote at $y=1\Rightarrow k=1$

$\therefore y=\dfrac{a}{x-3}+1$

Substitute the known point $(1,0)$

$\therefore 0=\dfrac{a}{1-3}+1$

$\therefore 0=\dfrac{a}{-2}+1$

$\therefore \dfrac{a}{2}=1$

$\therefore a=2$

The equation is $y=\dfrac{2}{x-3}+1.$

b Let the equation be $y=\dfrac{a}{x-h}+k$

Vertical asymptote at $x=-3\Rightarrow h=-3$
Horizontal asymptote at $y=1\Rightarrow k=1$

$\therefore y=\dfrac{a}{x+3}+1$

Substitute the known point $(-5,1.75)$

$\therefore 1.75=\dfrac{a}{-5+3}+1$

$\therefore 0.75=\dfrac{a}{-2}$

$\therefore a=-1.50$

The equation is $y=\dfrac{-1.5}{x+3}+1.$

c Let the equation be $y=\dfrac{a}{(x-h)^2}+k$

Vertical asymptote at $x=0\Rightarrow h=0$
Horizontal asymptote at $y=-2\Rightarrow k=-2$

$\therefore y=\dfrac{a}{x^2}-2$

Substitute the point $(1,1)$
$\therefore 1=a-2$
$\therefore a=3$

The equation is $y=\dfrac{3}{x^2}-2.$

d Let the equation be $y=\dfrac{a}{(x-h)^2}+k$

Vertical asymptote at $x=-3\Rightarrow h=-3$
Horizontal asymptote at $y=2\Rightarrow k=2$

$\therefore y=\dfrac{a}{(x+3)^2}+2$

Substitute the point $(0,1)$

$\therefore 1=\dfrac{a}{9}+2$

$\therefore \dfrac{a}{9}=-1$

$\therefore a=-9$

The equation is $y=\dfrac{-9}{(x+3)^2}+2.$

e Graph with the same shape as $y=\dfrac{4}{x^2}$ and vertical asymptote $x=-9$ has the equation $y=\dfrac{4}{(x+2)^2}$. The truncus function is $f:R\setminus\{-2\}\to R, f(x)=\dfrac{4}{(x+2)^2}.$

f The hyperbola has a vertical asymptote $x=\dfrac{1}{4}$ and a horizontal asymptote $y=-\dfrac{1}{2}$. It passes through the point $(1,0)$.

i The equation is of the form $y=\dfrac{a}{x-\frac{1}{4}}-\dfrac{1}{2}$

Substitute the point $(1,0)$

$\therefore 0=\dfrac{a}{1-\frac{1}{4}}-\dfrac{1}{2}$

$\therefore \dfrac{1}{2}=\dfrac{a}{\frac{3}{4}}$

$\therefore a=\dfrac{1}{2}\times\dfrac{3}{4}$

$\therefore a=\dfrac{3}{8}$

The equation is $y=\dfrac{\frac{3}{8}}{x-\frac{1}{4}}-\dfrac{1}{2}$

$$
\begin{aligned}
\therefore y&=\frac{3}{8(x-\frac{1}{4})}-\frac{1}{2}\\
&=\frac{3}{8x-2}-\frac{1}{2}\\
&=\frac{3}{2(4x-1)}-\frac{1}{2}\\
&=\frac{3-(4x-1)}{2(4x-1)}\\
&=\frac{3-4x+1}{2(4x-1)}\\
&=\frac{4-4x}{2(4x-1)}\\
&=\frac{2-2x}{4x-1}
\end{aligned}
$$

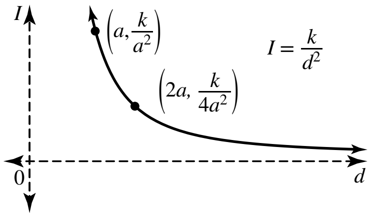
$\therefore y=\dfrac{-2x+2}{4x-1}$

The equation is in the form $y=\dfrac{ax+b}{cx+d}$ with $a=-2, b=2, c=4, d=-1.$

ii As the vertical asymptote is $x=\dfrac{1}{4}$, the domain is $R\setminus\left\{\dfrac{1}{4}\right\}.$

The hyperbola function can be expressed as the mapping $f:R\setminus\left\{\dfrac{1}{4}\right\}\to R, f(x)=\dfrac{-2x+2}{4x-1}.$

17 a The inverse proportion relationship $I = \frac{k}{d^2}$ is the branch of a truncus for $d > 0$.



b To consider the effect on the intensity when the distance from the transmitter is doubled:

Let $d = a$, then $I = \frac{k}{a^2}$.

Let $d = 2a$, then $I = \frac{k}{(2a)^2} \Rightarrow I = \frac{k}{4a^2}$.

The intensity is one quarter of what it was before the distance was doubled. This means doubling the distance has reduced the intensity by 75%.

18 a Since time $= \frac{\text{distance}}{\text{speed}}$, $t = \frac{\text{distance}}{v}$ so the constant of proportionality is the distance travelled. Therefore, $k = 180$.

b The relationship is $t = \frac{180}{v}$.

This represents a hyperbola with independent variable v and dependent variable t.

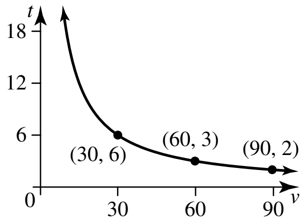
Asymptotes: $v = 0, t = 0$

Points: Let $v = 30$, $t = \frac{180}{30} = 6 \quad (30,6)$

Let $v = 60$, $t = \frac{180}{60} = 3 \quad (60,3)$

Let $v = 90$, $t = \frac{180}{90} = 2 \quad (90,2)$

Only the first quadrant branch is applicable since neither time nor speed can be negative.



c Let $t = 2\frac{1}{4} = \frac{9}{4}$

$\therefore \frac{9}{4} = \frac{180}{v}$

$\therefore \frac{9}{4}v = 180$

$\therefore v = \cancel{180}^{20} \times \frac{4}{\cancel{9}_1}$

$\therefore v = 80$

The speed should be 80 km/h.

19 a $xy = 2$ and $y = \frac{x^2}{4}$

At intersection, $x\left(\frac{x^2}{4}\right) = 2$

$\therefore \frac{x^3}{4} = 2$

$\therefore x^3 = 8$

$\therefore x = 2$

Substitute $x = 2$ in $xy = 2$

$\therefore 2y = 2$

$\therefore y = 1$

The point of intersection is $(2,1)$.

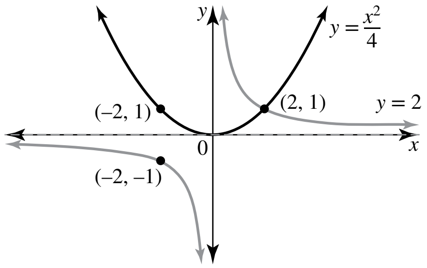
b $xy = 2$ is the hyperbola $y = \frac{2}{x}$.

Asymptotes: $x = 0, y = 0$

Points: $(2,1)$ and when $x = -2, y = -1$ so a second point is $(-2,-1)$.

$y = \frac{x^2}{4}$ is a parabola with minimum turning point at $(0,0)$.

Points: $(2,1)$ and when $x = -2, y = \frac{(-2)^2}{4} = 1$ so another point is $(-2,1)$.



c $y = \frac{4}{x^2}$ and $y = \frac{x^2}{4}$

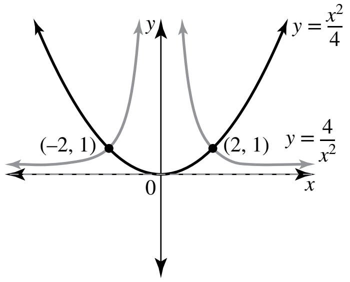
At intersection, $\frac{x^2}{4} = \frac{4}{x^2}$

$\therefore x^4 = 16$

$\therefore x = \pm 2$

When $x = \pm 2$, $y = 1$ so the points of intersection are $(-2,1)$ and $(2,1)$.

$y = \frac{4}{x^2}$ is a truncus with asymptotes $x = 0, y = 0$



d $y = \frac{x^2}{a}$ and $y = \frac{a}{x^2}$

At intersection,

$\frac{x^2}{a} = \frac{a}{x^2}$

$\therefore x^4 = a^2$

$\therefore x = \pm\sqrt[4]{a^2}$

$\therefore x = \pm a^{\frac{1}{2}}$

$\therefore x = \pm\sqrt{a}$

Substitute $x = \pm\sqrt{a}$ in $y = \frac{x^2}{a}$

$\therefore y = \frac{\left(\pm\sqrt{a}\right)^2}{a}$

$\therefore y = \frac{a}{a}$

$\therefore y = 1$

The points of intersection are $\left(-\sqrt{a},1\right)$ and $\left(\sqrt{a},1\right)$.

20 $N: R^+ \cup \{0\} \to R, N(t) = \dfrac{at+b}{t+2}$

a $N(t) = \dfrac{at+b}{t+2}$

$N(0) = 20$

$\Rightarrow 20 = \dfrac{b}{2}$

$\therefore b = 40$

$N(2) = 240$

$\Rightarrow 240 = \dfrac{2a+b}{4}$

Substitute $b = 40$

$\therefore 240 = \dfrac{2a+40}{4}$

$\therefore 960 = 2a + 40$

$\therefore 2a = 920$

$\therefore a = 460$

Answer: $a = 460,\ b = 40$

b The function rule is $N(t) = \dfrac{460t+40}{t+2}$.

When $N = 400$,

$$400 = \frac{460t+40}{t+2}$$

$\therefore 400(t+2) = 460t + 40$

$\therefore 400t + 800 = 460t + 40$

$\therefore 800 - 40 = 460t - 400t$

$\therefore 760 = 60t$

$\therefore t = \dfrac{760}{60}$

$\therefore t = \dfrac{38}{3}$

The time taken is $12\frac{2}{3}$ years which is 12 years and 8 months.

c $N(t) = \dfrac{460t+40}{t+2}$

$\therefore N(t+1) = \dfrac{460(t+1)+40}{(t+1)+2}$

$= \dfrac{460t+500}{t+3}$

$N(t+1) - N(t)$

$= \dfrac{460t+500}{t+3} - \dfrac{460t+40}{t+2}$

$= \dfrac{(460t+500)(t+2) - (460t+40)(t+3)}{(t+3)(t+2)}$

$= \dfrac{[460t(t+2) - 460t(t+3)] + [500(t+2) - 40(t+3)]}{(t+2)(t+3)}$

$= \dfrac{[-460t] + [460t + 1000 - 120]}{(t+2)(t+3)}$

$= \dfrac{880}{(t+2)(t+3)}$

as required.

d The change in population during the 12th year is $N(13) - N(12)$.

From part **c**, $N(t+1) - N(t) = \dfrac{880}{(t+2)(t+3)}$.

Put $t = 12$

$\therefore N(13) - N(12) = \dfrac{880}{(14)(15)}$

$= \dfrac{440}{7 \times 15}$

$= \dfrac{88}{7 \times 3}$

$= \dfrac{88}{21}$

The population increased by $\dfrac{88}{21} \simeq 4$ insects during the 12th year.

The change in population during the 14th year is $N(15) - N(14)$.

Put $t = 14$,

$\therefore N(15) - N(14) = \dfrac{880}{(16)(17)}$

$= \dfrac{110}{2 \times 17}$

$= \dfrac{55}{17} \simeq 3$

During the 14th year the population changed by approximately 3 insects so the growth in population is slowing.

e Let $N = 500$

$\therefore 500 = \dfrac{460t+40}{t+2}$

$\therefore 500(t+2) = 460t + 40$

$\therefore 500t + 1000 = 460t + 40$

$\therefore 40t = 40 - 1000$

$\therefore 40t = -960$

$\therefore t = -24$

However, $t \in R^+ \cup \{0\}$ so there is no value of t for which $N = 500$. The population of insects will never reach 500.

f $N = \dfrac{460t+40}{t+2}$

$= \dfrac{460(t+2) - 920 + 40}{t+2}$

$= \dfrac{460(t+2)}{t+2} - \dfrac{880}{t+2}$

$= 460 - \dfrac{880}{t+2}$

The function N is a hyperbola with horizontal asymptote $N = 460$. This means that as $t \to \infty, N \to 460$ so the population can never exceed 460 insects according to this model.

21 **a** Enter the expression $\dfrac{x+1}{x+2}$ in the Main menu and highlight it. Then tap Interactive → Transformation → Propfrac to obtain $\dfrac{x+1}{x+2} = -\dfrac{1}{x+2} + 1$.

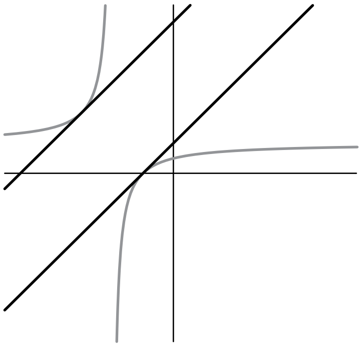
This means that $y = \dfrac{x+1}{x+2}$ has asymptotes $x = -2, y = 1$.

Highlight and drop the equation into the Graph & Tab menu. To show the asymptotes, enter these in the graphing list.

Graph $y = x$ on the same screen and observe there are 2 points of intersection.

b If $k = 0$ the line $y = x + k$ becomes the line $y = x$ so there are two intersections. All lines in the family $y = x + k$ have the same gradient of 1 with the value of k determining the y intercept of each line.

By trial and error and testing the number of points of intersection from the Analysis → G-Solve → Intersect, it can be found that the lines $y = x + 1$ and $y = x + 5$ intersect the right and the left branches respectively of the hyperbola exactly once.

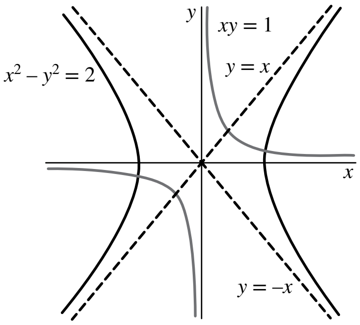


Thus:
One intersection if $k = 1$ or $k = 5$;
two intersections if $k < 1$ or $k > 5$;
no intersection if $1 < k < 5$.

22 a Enter the equation $x^2 - y^2 = 2$ in the Conic editor and graph. From Analysis, select G-Solve → Asymptotes to obtain the asymptotes $y = -x, y = x$.

Repeat for $xy = 1$ which has asymptotes $x = 0, y = 0$.

The shape of each graph is shown in the accompanying diagram.



b For the graph of $xy = 1$, an anticlockwise rotation of the axes by 45° would give a diagram where the hyperbola would have the same appearance as $x^2 - y^2 = 2$.

The asymptotes of $xy = 1$ are $x = 0, y = 0$. Rotating these anticlockwise by 45° gives the asymptotes of $x^2 - y^2 = 2$.

The line $y = 0$ is rotated to the line with gradient $\tan(45°) = 1$ giving an asymptote with equation $y = x$.

The line $x = 0$ is rotated anticlockwise to the line with gradient $\tan(135°) = -1$ giving an asymptote with equation $y = -x$.

The asymptotes of $x^2 - y^2 = 2$ are $y = x$ and $y = -x$.

Exercise 6.5 — The relation $y^2 = x$

1 a $(y+3)^2 = 4(x-1)$

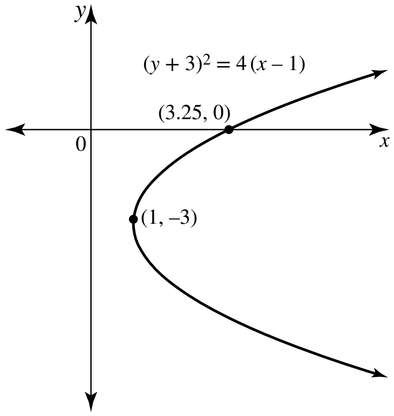
Vertex is the point $(1, -3)$.

x intercept: Substitute $y = 0$

$\therefore 9 = 4x - 4$

$\therefore x = \frac{13}{4}$

There is no y intercept. (Check, if $x = 0, (y+3)^2 = -4$ for which there is no real solution)



Domain $[1, \infty)$, Range R

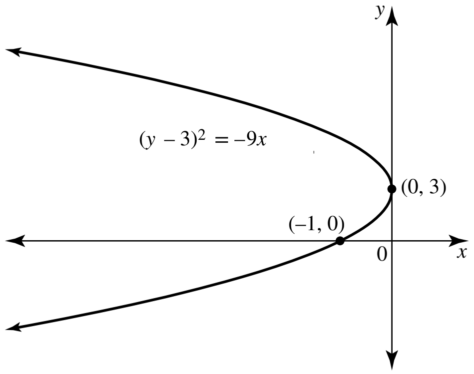
b $(y-3)^2 = -9x$

Vertex $(0, 3)$ which is also the y intercept.

x intercept: When $y = 0$,

$(-3)^2 = -9x$

$\therefore x = -1$



Domain $(-\infty, 0]$, Range R

2 $y^2 + 8y - 3x + 20 = 0$

Completing the square

$(y^2 + 8y + 16) - 16 - 3x + 20 = 0$

$\therefore (y+4)^2 = 3x - 4$

$\therefore (y+4)^2 = 3\left(x - \frac{4}{3}\right)$

Vertex $\left(\frac{4}{3}, -4\right)$ and the axis of symmetry has the equation $y = -4$

3 a $(y-k)^2 = a(x-h)$

Substituting the vertex $(4, -7)$ gives $(y+7)^2 = a(x-4)$

Substitute the given point $(-10, 0)$

$(7)^2 = a(-14)$

$\therefore a = -\frac{49}{14}$

$\therefore a = -3.5$

The equation is $(y+7)^2 = -3.5(x-4)$ or

$(y+7)^2 = -\frac{7}{2}(x-4)$.

b Let the equation be $(y-k)^2 = a(x-h)$

The y intercept points $(0,0), (0,6)$ mean the equation of the axis of symmetry is $y = 3$

$\therefore (y-3)^2 = a(x-h)$

Point $(0,0) \Rightarrow 9 = -ah$........(1)

Point $(9,-3) \Rightarrow (-6)^2 = a(9-h)$

$\therefore 36 = 9a - ah ....(2)$

$(2)-(1)$

$27 = 9a$

$\therefore a = 3$

$(1) \Rightarrow h = -3$

The equation is $(y-3)^2 = 3(x+3)$.

4 As the y axis is vertical, a curve touching the y axis will fail the vertical line test for functions, since this parabola is a sideways one and therefore not a function. The point $(0,3)$ is its vertex so the equation becomes $(y-3)^2 = ax$.

Substitute the point $(2,0)$ into $(y-3)^2 = ax$

$\therefore 9 = 2a$

$\therefore a = 4.5$

The equation is $(y-3)^2 = 4.5x$.

5 **a** $y = \sqrt{x-1} - 3$

endpoint $(1,-3)$, no y intercept since domain is $[1,\infty)$

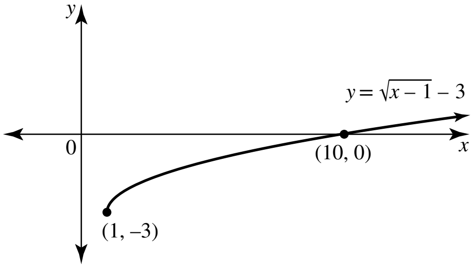
x intercept when $\sqrt{x-1} - 3 = 0$

$\therefore \sqrt{x-1} = 3$

$\therefore x - 1 = 9$

$\therefore x = 10$

$(10,0)$



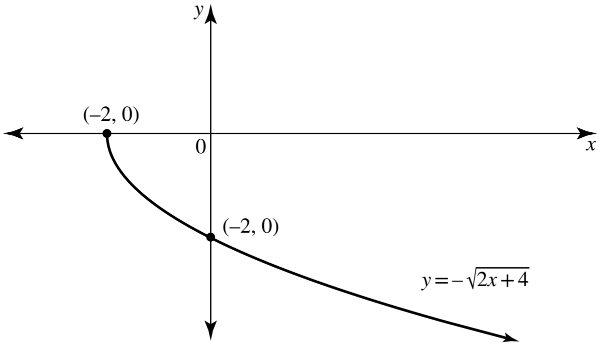
Range $[-3,\infty)$

b **i** $f(x) = -\sqrt{2x+4}$

Maximal domain: $2x + 4 \geq 0$ $\therefore x \geq -2$ Therefore domain is $[-2,\infty)$

ii Endpoint when $2x + 4 = 0$ therefore endpoint is $(-2,0)$

y intercept $(0,-\sqrt{4}) = (0,-2)$



iii $f(x) = -\sqrt{2x+4}$

Let $y = -\sqrt{2x+4}$ and square both sides of this equation

$\therefore y^2 = 2x + 4$

$\therefore y^2 = 2(x+2)$

Alternatively, the graph is the lower half of the sideways parabola which must have both the lower and upper branches

Therefore its equation is

$y = \pm\sqrt{2x+4}$

$\therefore y^2 = 2x + 4$

$\therefore y^2 = 2(x+2)$

6 $S = \{(x, y) : (y+2)^2 = 9(x-1)\}$

a $(y+2)^2 = 9(x-1)$ vertex $(1,-2)$

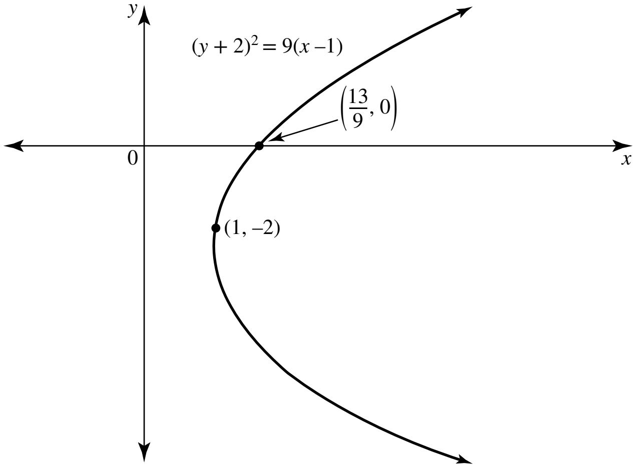
x intercept: When $y = 0$

$(2)^2 = 9(x-1)$

$\therefore \frac{4}{9} = x-1$

$\therefore x = \frac{13}{9}$

x intercept $\left(\frac{13}{9}, 0\right)$



b Equation can be expressed as $y+2 = \pm\sqrt{9(x-1)}$ so for the bottom half square root function the equation is $y+2 = -\sqrt{9(x-1)} \Rightarrow y = -\sqrt{9(x-1)} - 2$

7 Endpoint given as $(-2,1)$ and domain is $[-2,\infty)$ so let equation be $y = a\sqrt{x-h} + k$.

Substitute the end point, so the equation becomes $y = a\sqrt{x+2} + 1$

Substitute the point $(0,3)$

$\therefore 3 = a\sqrt{2} + 1$

$\therefore \sqrt{2}a = 2$

$\therefore a = \frac{2}{\sqrt{2}}$

$\therefore a = \sqrt{2}$

Therefore the equation is $y = \sqrt{2}\sqrt{x+2} + 1 \Rightarrow y = \sqrt{2(x+2)} + 1$

8 $(y-a)^2 = b(x-c)$

Vertex at $(2,5)$, so equation becomes $(y-5)^2 = b(x-2)$

Substitute the point $(-10.5, 0)$

$(-5)^2 = b(-10.5-2)$

$\therefore 25 = b(-12.5)$

$\therefore b = -\frac{25}{12.5}$

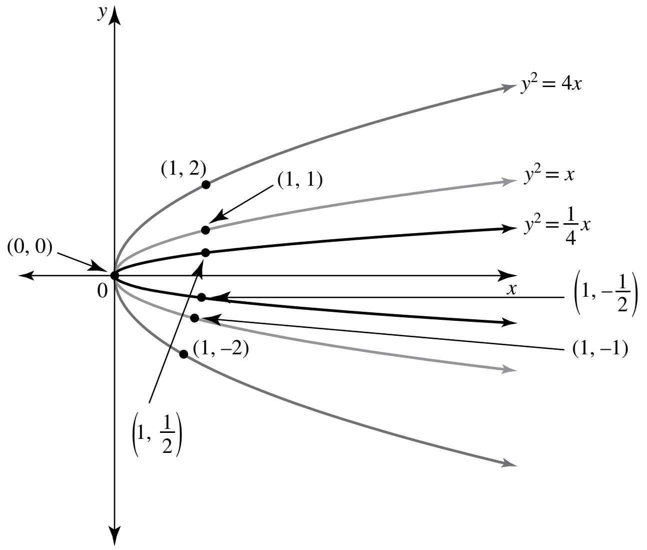
$\therefore b = -2$

The equation is $(y-5)^2 = -2(x-2)$, $a = 5$, $b = -2$, $c = 2$, Domain $(-\infty, 2]$, range R

9 All three graphs have a vertex at the origin.

For each, let $x = 1$

$y^2 = x$	$y^2 = 4x$	$y^2 = \frac{1}{4}x$
$\therefore y^2 = 1$	$\therefore y^2 = 4$	$\therefore y^2 = \frac{1}{4}$
$\therefore y = \pm 1$	$\therefore y = \pm 2$	$\therefore y = \pm\frac{1}{2}$
$(1,1), (1,-1)$	$(1,2), (1,-2)$	$\left(1,\frac{1}{2}\right), \left(1,-\frac{1}{2}\right)$



Increasing the coefficient of the x term makes the graphs wider in the y axis direction and the graphs become more open.

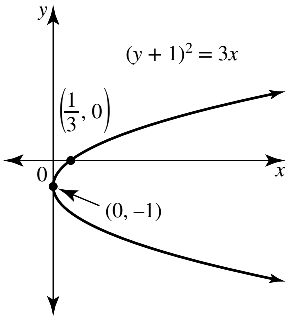
10 **a** $(y+1)^2 = 3x$

Vertex: Since $y+1=0$ when $y=-1$, the vertex is $(0,-1)$.

x intercept: Let $y=0$

$\therefore (1)^2 = 3x$

$\therefore x = \frac{1}{3} \qquad \left(\frac{1}{3}, 0\right)$



b $9y^2 = x+1$

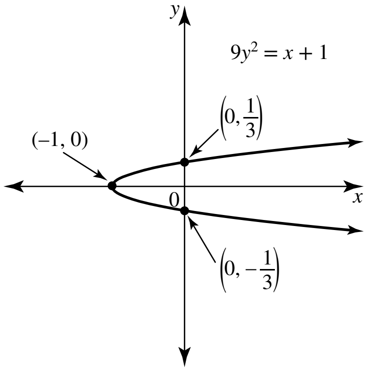
$\therefore y^2 = \frac{1}{9}(x+1)$

Vertex: $(-1,0)$

y intercepts: Let $x=0$

$\therefore y^2 = \frac{1}{9}$

$\therefore y = \pm\frac{1}{3} \qquad \left(0, -\frac{1}{3}\right), \left(0, \frac{1}{3}\right)$



c $(y+2)^2=8(x-3)$

Vertex: $(3,-2)$

x intercept: Let $y=0$

$\therefore (2)^2=8(x-3)$

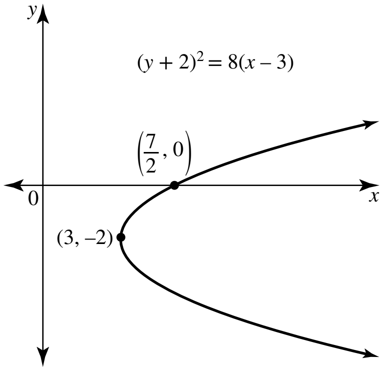
$\therefore 4=8x-24$

$\therefore 8x=28$

$\therefore x=\dfrac{28}{8}$

$\therefore x=\dfrac{7}{2} \qquad \left(\dfrac{7}{2},0\right)$

No y intercepts



d $(y-4)^2=2x+1$

$\therefore (y-4)^2=2\left(x+\dfrac{1}{2}\right)$

Vertex: $\left(-\dfrac{1}{2},4\right)$

y intercepts: Let $x=0$

$\therefore (y-4)^2=1$

$\therefore y-4=\pm 1$

$\therefore y=3, y=5 \qquad (0,3),(0,5)$

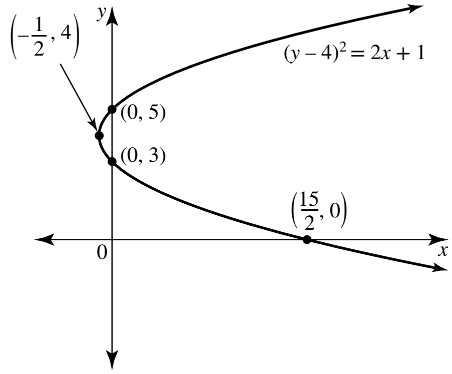
x intercept: Let $y=0$

$\therefore (-4)^2=2x+1$

$\therefore 16=2x+1$

$\therefore 2x=15$

$\therefore x=\dfrac{15}{2} \qquad \left(\dfrac{15}{2},0\right)$



11 a $y^2=-2x$

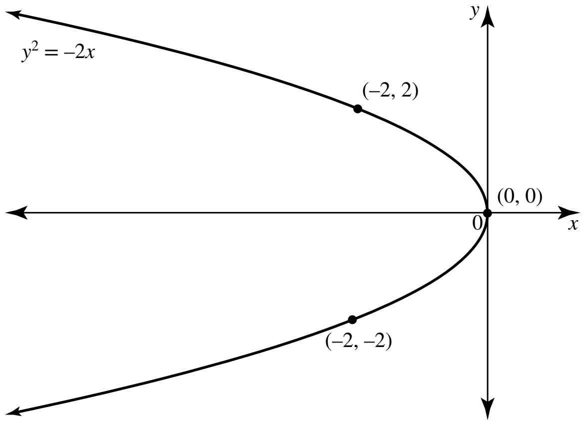
Vertex $(0,0)$

Points: Let $x=-2$

$\therefore y^2=4$

$\therefore y=\pm 2$

$(-2,-2),(-2,2)$



b $(y+1)^2=-2(x-4)$

Vertex: $(4,-1)$

y intercepts: Let $x=0$

$\therefore (y+1)^2=-2(-4)$

$\therefore (y+1)^2=8$

$\therefore y+1=\pm\sqrt{8}$

$\therefore y=-1\pm 2\sqrt{2} \qquad \left(0,-1-2\sqrt{2}\right),\left(0,-1+2\sqrt{2}\right)$

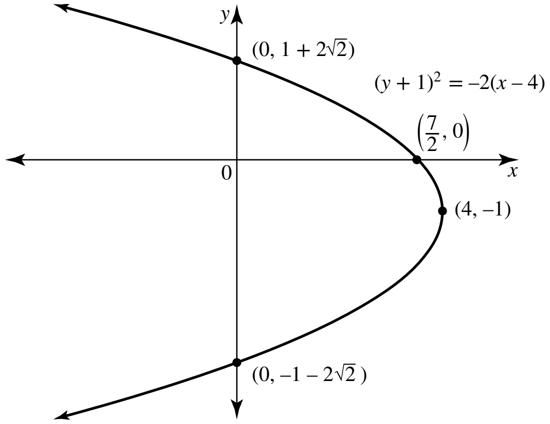
x intercept: Let $y=0$

$\therefore (1)^2=-2(x-4)$

$\therefore 1=-2x+8$

$\therefore 2x=7$

$\therefore x=\dfrac{7}{2} \qquad \left(\dfrac{7}{2},0\right)$



c $(6-y)^2=-8-2x$

$\therefore (6-y)^2=-2(x+4)$

Vertex: $(-4,6)$, No y intercept

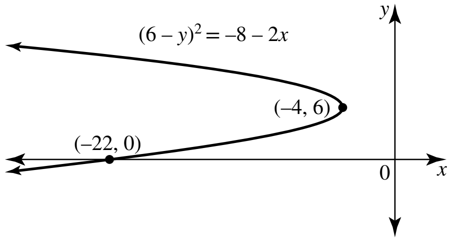
x intercept: Let $y=0$

$\therefore (6)^2=-8-2x$

$\therefore 36=-8-2x$

$\therefore 2x=-44$

$\therefore x=-22 \qquad (-22,0)$



d $x=-(2y-6)^2$

$\therefore x=-\left(2(y-3)\right)^2$

$\therefore x=-4(y-3)^2$

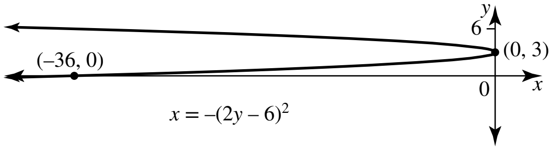
$\therefore (y-3)^2=-\dfrac{1}{4}x$

Vertex: $(0,3)$

x intercept: Let $y=0$

$\therefore x=-(-6)^2$

$\therefore x=-36 \quad (-36,0)$



12 a $y^2+16y-5x+74=0$

Complete the square on the y terms

$\therefore y^2+16y=5x-74$

$\therefore y^2+16y+64=5x-74+64$

$\therefore (y+8)^2=5x-10$

$\therefore (y+8)^2=5(x-2)$

vertex: $(2,-8)$

The coefficient of x is positive so the graph opens to the right. Domain is $[2,\infty)$.

b $y^2-3y+13x-1=0$

$\therefore y^2-3y=-13x+1$

$\therefore y^2-3y+\left(\dfrac{3}{2}\right)^2=-13x+1+\left(\dfrac{3}{2}\right)^2$

$\therefore \left(y-\dfrac{3}{2}\right)^2=-13x+1+\dfrac{9}{4}$

$\therefore \left(y-\dfrac{3}{2}\right)^2=-13x+\dfrac{13}{4}$

$\therefore \left(y-\dfrac{3}{2}\right)^2=-13\left(x-\dfrac{1}{4}\right)$

vertex $\left(\dfrac{1}{4},\dfrac{3}{2}\right)$.

The coefficient of x is negative so the graph opens to the left. Domain is $(-\infty,\frac{1}{4}]$.

c $(5+2y)^2=8-4x$

$\therefore \left[2\left(y+\dfrac{5}{2}\right)\right]^2=-4(x-2)$

$\therefore 4\left(y+\dfrac{5}{2}\right)^2=-4(x-2)$

$\therefore \left(y+\dfrac{5}{2}\right)^2=-(x-2)$

Vertex $\left(2,-\dfrac{5}{2}\right)$

The coefficient of x is negative so the graph opens to the left. Domain is $(-\infty,2]$.

d $(5-y)(1+y)+5(x-1)=0$

$\therefore 5+4y-y^2+5x-5=0$

$\therefore 5x=y^2-4y$

$\therefore y^2-4y+4=5x+4$

$\therefore (y-2)^2=5\left(x+\dfrac{4}{5}\right)$

Vertex $\left(-\dfrac{4}{5},2\right)$.

The coefficient of x is positive so the graph opens to the right. Domain is $[-\frac{4}{5},\infty)$.

13 a Let the equation be $(y-k)^2=a(x-h)$

Vertex $(1,-1)$, so the equation becomes $(y+1)^2=a(x-1)$.

Substitute the known point $(-2,2)$

$\therefore (2+1)^2=a(-2-1)$

$\therefore 9=-3a$

$\therefore a=-3$

The equation is $(y+1)^2=-3(x-1)$.

b Let the equation be $(y-k)^2=a(x-h)$

Vertex $(1,-2)$, so the equation becomes $(y+2)^2=a(x-1)$.

Substitute the known x intercept $(2,0)$

$\therefore (2)^2=a(2-1)$

$\therefore 4=a$

The equation is $(y+2)^2=4(x-1)$.

c **i** The axis of symmetry must pass halfway between the two points $(1,12)$ and $(1,-4)$ since these points have the same x values. The midpoint is $\left(\dfrac{1+1}{2},\dfrac{-4+12}{2}\right)=(1,4)$.

The equation of the axis of symmetry is that of the horizontal line through $(1,4)$. Therefore, the equation of the axis of symmetry is $y=4$.

ii The vertex lies on the y axis and is symmetric with the two points $(1,12)$ and $(1,-4)$. The co-ordinates of the vertex are $(0,4)$.

The equation of the curve has the form

$(y-4)^2=a(x-0)$

Substitute the point $(1,12)$

$\therefore (12-4)^2=a(1)$

$\therefore a=64$

The equation of the curve is $(y-4)^2=64x$.

d The vertex is at $(0,0)$ so the form of the equation is $y^2=ax$.

The diagram indicates that the points $\left(12,\dfrac{11}{2}\right)$ and $\left(12,-\dfrac{11}{2}\right)$ lie on the parabola.

Substitute $\left(12,\dfrac{11}{2}\right)$

$\therefore \left(\dfrac{11}{2}\right)^2=a(12)$

$\therefore 12a=\dfrac{121}{4}$

$\therefore a=\dfrac{121}{48}$

The equation is $y^2=\dfrac{121}{48}x$.

14 $S=\{(x,y):(y-2)^2=1-x\}$

a $(y-2)^2=1-x$

Take the square root of both sides

$\therefore y-2=\pm\sqrt{1-x}$

$\therefore y=2\pm\sqrt{1-x}$

b The relation S is also described by $(y-2)^2=-(x-1)$. Its vertex is $(1,2)$ and as the coefficient of x is negative, the graph opens to the left with domain is $(-\infty,1]$.

The two functions which together form the relation, are the upper half and the lower half square root functions:

$f:(-\infty,1]\to R, f(x)=2+\sqrt{1-x}$ and

$g:(-\infty,1]\to R, g(x)=2-\sqrt{1-x}$.

c Each function has endpoint $(1,2)$.

Graph of f:

y intercept: Let $x=0$

$f(0) = 2 + \sqrt{1}$

$= 3 \qquad (0,3)$

Graph of g:

y intercept: Let $x = 0$

$g(0) = 2 - \sqrt{1}$

$= 1 \qquad (0,1)$

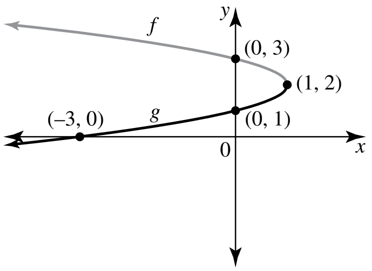
x intercept: Let $y = 0$

$\therefore 0 = 2 - \sqrt{1-x}$

$\therefore \sqrt{1-x} = 2$

$\therefore 1 - x = 2^2$

$\therefore x = -3 \qquad (-3,0)$



d Image of -8 under f is $f(-8)$

$f(-8) = 2 + \sqrt{9}$

$= 5$

Image of -8 under g is $g(-8)$

$g(-8) = 2 - \sqrt{9}$

$= -1$

15 a $y = \sqrt{x+3} - 2$

Endpoint: $(-3,-2)$

Domain: $x + 3 \geq 0$

$\therefore x \geq -3$

Domain is $[-3, \infty)$

The rule contains a positive square root so the graph rises from the endpoint giving a range of $[-2, \infty)$.

y intercept: Let $x = 0$

$\therefore y = \sqrt{3} - 2 \qquad \left(0, \sqrt{3} - 2\right)$

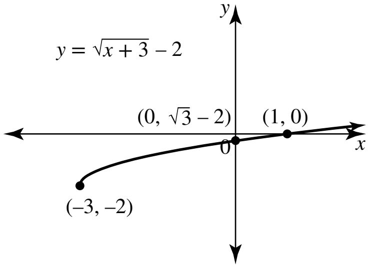
x intercept: Let $y = 0$

$\therefore \sqrt{x+3} - 2 = 0$

$\therefore \sqrt{x+3} = 2$

$\therefore x + 3 = 4$

$\therefore x = 1 \qquad (1,0)$



b $y = 5 - \sqrt{5x}$

Endpoint: $(0,5)$

x intercept: Let $y = 0$

$\therefore 0 = 5 - \sqrt{5x}$

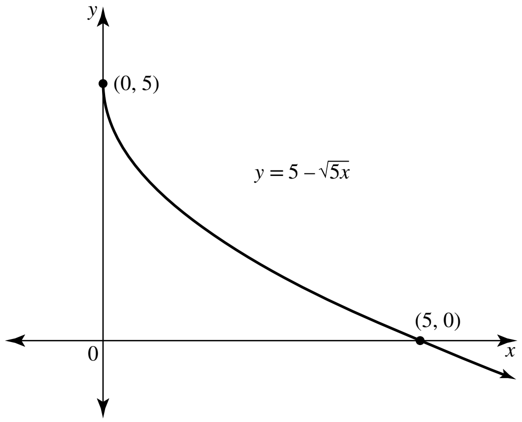
$\therefore \sqrt{5x} = 5$

$\therefore 5x = 25$

$\therefore x = 5 \qquad (5,0)$

Domain: $5x \geq 0 \Rightarrow x \geq 0 \qquad [0, \infty)$

Range: The rule contains a negative square root so the graph falls from the endpoint giving a range of $(-\infty, 5]$.



c $y = 2\sqrt{9-x} + 4$

$\therefore y = 2\sqrt{-(x-9)} + 4$

Endpoint $(9,4)$

Domain: $9 - x \geq 0$

$\therefore 9 \geq x$

$\therefore x \leq 9 \qquad (-\infty, 9]$

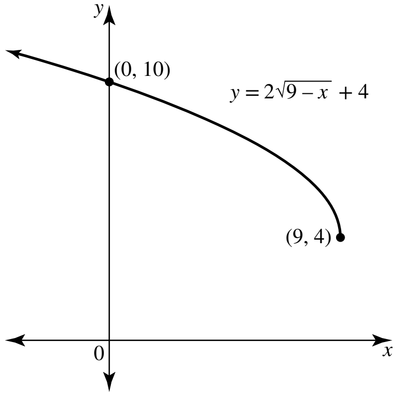
Range: The rule contains a positive square root so the graph lies above its endpoint giving a range of $[4, \infty)$.

y intercept: Let $x = 0$

$\therefore y = 2\sqrt{9} + 4$

$\therefore y = 10 \qquad (0,10)$

No x intercept.



d $y = \sqrt{49 - 7x}$

$\therefore y = \sqrt{-7(x-7)}$

Endpoint: $(7,0)$

y intercept: Let $x = 0$

$\therefore y = \sqrt{49}$

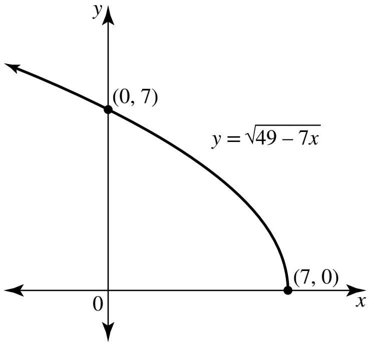
$\therefore y = 7 \qquad (0,7)$

Domain: $49 - 7x \geq 0$

$\therefore 49 \geq 7x$

$\therefore x \leq 7 \qquad (-\infty, 7]$

Range: The rule contains a positive square root so the graph lies above its endpoint giving a range of $[0, \infty)$.



e $y = 2 \pm \sqrt{x+4}$

$\therefore y - 2 = \pm\sqrt{x+4}$

$\therefore (y-2)^2 = x + 4$

Sideways parabola with vertex $(-4, 2)$. Coefficient of x is positive so the graph opens to the right of the vertex. Domain is $[-4, \infty)$ and range is R.

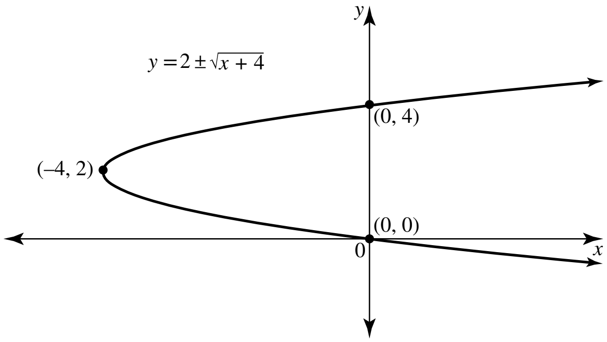
y intercepts: Let $x = 0$

$\therefore y = 2 \pm \sqrt{4}$

$\therefore y = 2 - 2$ or $y = 2 + 2$

$\therefore y = 0, 4$

$(0,0), (0,4)$



f $y + 1 + \sqrt{-2x+3} = 0$

$\therefore y = -\sqrt{-2\left(x - \frac{3}{2}\right)} - 1$

Endpoint: $\left(\frac{3}{2}, -1\right)$

y intercept: Let $x = 0$

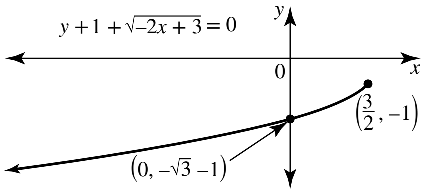
$\therefore y = -\sqrt{3} - 1 \quad \left(0, -\sqrt{3} - 1\right)$

Domain: $-2x + 3 \geq 0$

$\therefore 3 \geq 2x$

$\therefore x \leq \frac{3}{2} \quad (-\infty, \frac{3}{2}]$

Range: The rule contains a negative square root so the graph lies below the endpoint giving a range of $(-\infty, -1]$.



16 a As the graph opens to the right from its endpoint, let the equation be $y = a\sqrt{x-h} + k$.

Endpoint is $(-2, -2)$ so the equation becomes $y = a\sqrt{x+2} - 2$.

Substitute the known point $(1, -1)$

$\therefore -1 = a\sqrt{3} - 2$

$\therefore a\sqrt{3} = 1$

$\therefore a = \frac{1}{\sqrt{3}}$

The equation is $y = \frac{1}{\sqrt{3}}\sqrt{x+2} - 2 \Rightarrow y = \sqrt{\frac{x+2}{3}} - 2$.

b Given that the endpoint is $(4,-1)$ and the point $(0,9)$ lies on the function, then the domain of the function must be $(-\infty,4]$.

Let the equation be $y=a\sqrt{-(x-h)}+k$

Substituting the endpoint, $y=a\sqrt{-(x-4)}-1$.

Substitute the point $(0,9)$

$\therefore 9=a\sqrt{4}-1$

$\therefore 10=2a$

$\therefore a=5$

The function has the equation

$y=5\sqrt{-(x-4)}-1 \Rightarrow y=5\sqrt{4-x}-1$.

x intercept: Let $y=0$

$\therefore 0=5\sqrt{4-x}-1$

$\therefore 1=5\sqrt{4-x}$

$\therefore \sqrt{4-x}=\frac{1}{5}$

$\therefore 4-x=\frac{1}{25}$

$\therefore x=4-\frac{1}{25}$

$\therefore x=\frac{100-1}{25}$

$\therefore x=\frac{99}{25}$

The graph of the function cuts the x axis at the point $\left(\frac{99}{25},0\right)$.

c Required function has the same shape as $y=\sqrt{-x}$ but with endpoint $(4,-4)$.

Its equation is

$y=\sqrt{-(x-4)}-4$

$\therefore y=\sqrt{4-x}-4$

d $t\propto\sqrt{h}$

$\therefore t=k\sqrt{h}$ where k is the constant of proportionality

Given $h=19.6$, $t=2$ then $2=k\sqrt{19.6}$

$\therefore k=\frac{2}{\sqrt{19.6}}$

$\therefore k=\frac{2}{2\sqrt{4.9}}$

$\therefore k=\frac{1}{\sqrt{4.9}}$

Hence the rule is $t=\frac{1}{\sqrt{4.9}}\sqrt{h}$ or $t=\sqrt{\frac{h}{4.9}}$.

17 $f:[0,\infty)\to R, f(x)=\sqrt{mx}+n$

a From the given information,

$f(1)=1\Rightarrow 1=\sqrt{m}+n$ ....(1)

$f(4)=4\Rightarrow 4=\sqrt{4m}+n$....(2)

Subtract equation (1) from equation (2)

$\therefore 3=\sqrt{4m}-\sqrt{m}$

$\therefore 3=2\sqrt{m}-\sqrt{m}$

$\therefore \sqrt{m}=3$

$\therefore m=9$

Substitute $m=9$ in equation (1)

$\therefore 1=\sqrt{9}+n$

$\therefore 3+n=1$

$\therefore n=-2$

Answer: $m=9, n=-2$

b $f(x)=\sqrt{9x}-2$

$\therefore f(x)=3\sqrt{x}-2$

Endpoint: $(0,-2)$

x intercept: Let $y=0$

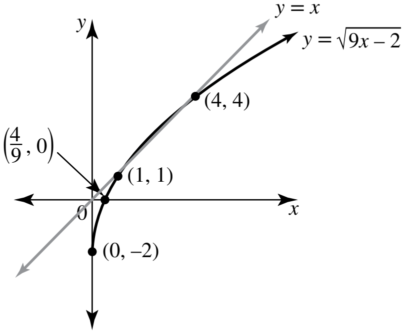
$\therefore 0=3\sqrt{x}-2$

$\therefore 2=3\sqrt{x}$

$\therefore \sqrt{x}=\frac{2}{3}$

$\therefore x=\frac{4}{9}$ $\left(\frac{4}{9},0\right)$

c The line $y=x$ must pass through the origin and the points $(1,1)$ and $(4,4)$.



The square root function lies above the straight line for $1<x<4$.

$\therefore \{x: f(x)>x\}=\{x:1<x<4\}$

d The rule for f is $y=\sqrt{9x}-2$

$\therefore y+2=\sqrt{9x}$

$\therefore (y+2)^2=9x$

The function f is the upper half of the sideways parabola with equation $(y+2)^2=9x$.

18 $y^2=-8x$

a The negative coefficient of x indicates the sideways parabola opens to the left of its vertex $(0,0)$. Its domain is $(-\infty,0]$.

To test if $\text{P}\left(-3,2\sqrt{6}\right)$ lies on the curve, substitute P's co-ordinates into the equation of the curve.

$\text{LHS}=\left(2\sqrt{6}\right)^2$ $\quad$ $\text{RHS}=-8\times-3$

$=24$ $\quad$ $=24$

Since LHS = RHS, P lies on the curve.

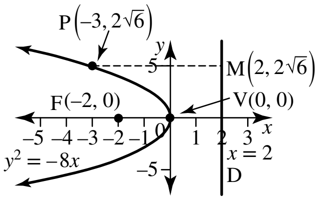
When $x=-3$, $y^2=24$

$\therefore y=\pm2\sqrt{6}$

The points $\left(-3,2\sqrt{6}\right)$ and $\left(-3,-2\sqrt{6}\right)$ both are on the curve. As the y value for P is positive, it lies on the upper branch of the curve.

b The point $\text{V}(0,0)$ lies 2 units from $\text{F}(-2,0)$ and 2 units from the line $x=2$. Therefore, V is equidistant from point F and line D.

Consider $\text{P}\left(-3,2\sqrt{6}\right)$:



The distance of P from the line D is the horizontal distance PM shown in the diagram. The distance PM is 5 units.

Distance PF: $\text{P}\left(-3,2\sqrt{6}\right)$, $\text{F}(-2,0)$

$$\begin{aligned}d_{PF}&=\sqrt{(-2-(-3))^2+\left(0-2\sqrt{6}\right)^2}\\&=\sqrt{(1)^2+\left(2\sqrt{6}\right)^2}\\&=\sqrt{1+24}\\&=\sqrt{25}\\&=5\end{aligned}$$

Therefore P is equidistant from the point F and the line D.

c Point Q lies on the curve: Let $x=a$ in the equation $y^2=-8x$

$\therefore y^2=-8a$

$\therefore y=\pm\sqrt{-8a}$

Since Q lies on the lower branch to P, $y=-\sqrt{-8a}$

The co-ordinates of Q are $\left(a,-\sqrt{-8a}\right)$.

Distance QF:

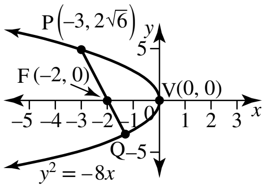
$$\begin{aligned}d_{QF}&=\sqrt{(-2-a)^2+\left(0+\sqrt{-8a}\right)^2}\\&=\sqrt{(-2-a)^2+\left(\sqrt{-8a}\right)^2}\\&=\sqrt{4+4a+a^2-8a}\\&=\sqrt{4-4a+a^2}\\&=\sqrt{(2-a)^2}\\&=2-a\end{aligned}$$

(This is positive since $a<0$).

The distance of Q from the line $x=2$ is the length of the horizontal line from $Q\left(a,-\sqrt{-8a}\right)$ to the point $\left(2,-\sqrt{-8a}\right)$. This distance is $2-a$.

Therefore Q is also equidistant from the point F and the line D.

d



Q is the point where the line PF intersects the sideways parabola.

Equation of the line through P and F:

$$\begin{aligned}m_{PF}&=\frac{0-2\sqrt{6}}{-2+3}\\&=-2\sqrt{6}\end{aligned}$$

$\therefore y-0=-2\sqrt{6}(x+2)$

$\therefore y=-2\sqrt{6}(x+2)$

This line intersects $y^2=-8x$ when $\left(-2\sqrt{6}(x+2)\right)^2=-8x$

$\therefore 24(x+2)^2=-8x$

$\therefore 3(x^2+4x+4)=-x$

$\therefore 3x^2+13x+12=0$

$\therefore (3x+4)(x+3)=0$

$\therefore x=-\frac{4}{3}, x=-3$

P is the point where $x=-3$ so Q is the point where $x=-\frac{4}{3}$

Hence, $a=-\frac{4}{3}$.

19 $f(x)=3\sqrt{x+1}+2$ and $g(x)=\sqrt{4-x^2}+2$

a Function f is a square root function. For its domain, $x+1\ge 0\Rightarrow x\ge -1$ so $d_f=[-1,\infty)$.

Function g is a semicircle. For its domain, $4-x^2\ge 0\Rightarrow -2\le x\le 2$ so $d_g=[-2,2]$.

b One method is to sketch the graphs in the Graph & Tab menu and from the Analysis tools select G-Solve and then Intersection. The point of intersection is (–0.6, 3.9) to one decimal place.

c Relation A: Let $y=3\sqrt{x+1}+2$

$\therefore y-2=3\sqrt{x+1}$

$\therefore (y-2)^2=9(x+1)$

Relation B: Let $y=\sqrt{4-x^2}+2$

$\therefore y-2=\sqrt{4-x^2}$

$\therefore (y-2)^2=4-x^2$

$\therefore x^2+(y-2)^2=4$

d One point of intersection of the relations A and B must be (–0.6, 3.9). As these relations are each made up of two branches, of which f and g are one of these branches respectively, there must be a second point of intersection which will also occur when $x=-0.6$. Substitute $x=-0.6$ in the equation of the lower branch of relation A, $y=-3\sqrt{x+1}+2$ and evaluate in the Main menu to obtain $y=0.1$.

The points of intersection are (–0.6, 3.9) and (–0.6, 0.1).

20 In the Geometry application use the Draw menu and the icons to follow the instructions. The labelling may be automatically done on the ClassPad but if using the Cabri program on a computer, label as instructed.

a The shape of the locus path is a sideways parabola opening to the right.

b The line segments FP and PM are of equal length. This remains the case even when moving F or M. Any point on the parabola is equidistant from the fixed straight line D and the fixed point F.

Exercise 6.6 — Other functions and relations

1 a $y=\frac{1}{16-x^2}$

Denominator is zero when

$16-x^2=0$

$\therefore 16=x^2$

$\therefore x=\pm 4$

These values must be excluded from the domain. Therefore, the maximal domain is $R\setminus\{\pm 4\}$.

b $y=\frac{2-x}{x^2+3}$

If $x^2+3=0$, then $x^2=-3$ for which there are no real values of x.

With no values to exclude, and numerator and denominator polynomials, the maximal domain is R.

c $y=\frac{1}{\sqrt{x^3+1}}$

Denominator cannot be zero, so the domain requires $x^3+1>0$

$\therefore x^3>-1$

$\therefore x>-1$

The maximal domain is $(-1,\infty)$.

2 a $f(x)=\frac{1}{x^2+5x+4}$

Denominator will be zero when

$x^2+5x+4=0$

$\therefore (x+4)(x+1)=0$

$\therefore x=-4, x=-1$

These values must be excluded from the domain. Therefore the maximal domain of the function f is $R\setminus\{-4,-1\}$.

b $g(x)=\sqrt{x+3}$

The domain of this square root function requires $x+3\geq 0$.

Therefore, $x\geq -3$ and the maximal domain of the function g is $[-3,\infty)$.

c $h(x)=f(x)+g(x)$. The domain of h is the intersection of $R\setminus\{-4,-1\}$ and $[-3,\infty)$.

Therefore the maximal domain is $[-3,\infty)\setminus\{-1\}$.

3 $f:(-\infty,2]\rightarrow R, f(x)=6-3x$

a domain is $(-\infty,2]$. Endpoint $(2,0)$ and as the line has a negative gradient for the given domain, the range is $R^{+}\cup\{0\}$.

b Interchanging, domain of inverse is $R^{+}\cup\{0\}$, range is $(-\infty,2]$

c function: $y=6-3x$

inverse: $x=6-3y$

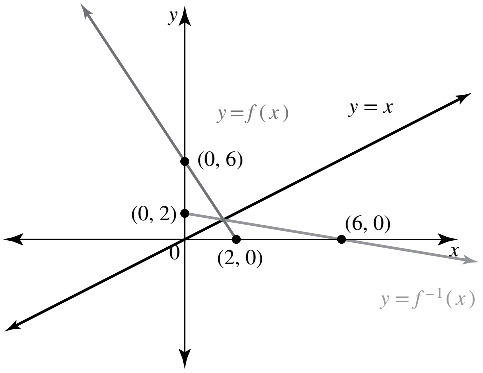
Making y the subject,

$3y=6-x$

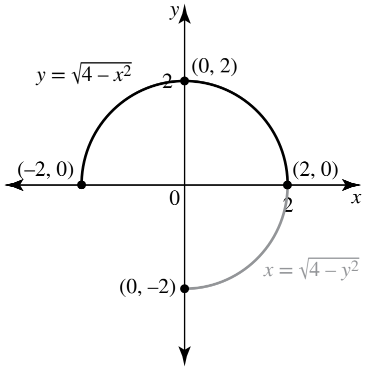
$\therefore y=\dfrac{6-x}{3}$

Hence, $f^{-1}:R^{+}\cup\{0\}\rightarrow R, f^{-1}(x)=\dfrac{6-x}{3}$

d f: points $(2,0)$, $(0,6)$ inverse: points $(0,2),(6,0)$



4 **a**



$y=\sqrt{4-x^{2}}$ has centre $(0,0)$, radius 2, top half semicircle, many-to-one correspondence

Inverse has centre $(0,0)$, radius 2, half semicircle to the right of the y axis, one-to-many correspondence.

b the inverse is not a function since it has a one-to-many correspondence.

5 $y=(x+1)^{2}$

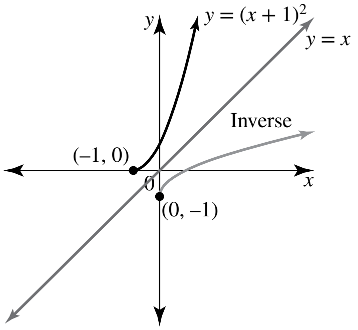
a Inverse: $x=(y+1)^{2}$

$\therefore (y+1)^{2}=x$ or $y=\pm\sqrt{x}-1$ This is the sideways parabola so it fails the vertical line test (it has a one-to-many correspondence), therefore not a function

b Domain restriction $x\in[-1,\infty)$ makes the parabola $y=(x+1)^{2}$ one-to-one, so its inverse will be a function.

Turning point and endpoint $(-1,0)$ (closed), y intercept $(0,1)$

For the inverse, endpoint $(0,-1)$ (closed), x intercept $(1,0)$

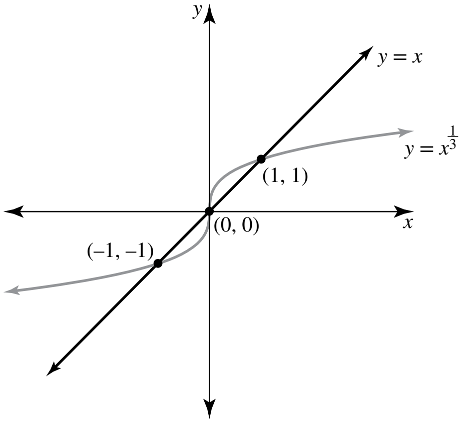


c Function : $y=(x+1)^2,\ x\in[-1,\infty),\ y\geq 0$
Inverse: $x=(y+1)^2,\ y\in[-1,\infty),\ x\geq 0$
$\therefore y+1=\pm\sqrt{x}$
Since range requires $y\in[-1,\infty)$, take the positive square root (as diagram shows)
$\therefore y+1=\sqrt{x}$
$\therefore y=\sqrt{x}-1$

d It can be seen from the graph in part **b** that the functions do not intersect.

6 $f:[a,\infty)\to R, f(x)=x^2-4x+9$
For f^{-1} to exist the parabola must be one-to-one.
$f(x)=x^2-4x+9$
Completing the square to obtain its turning point,
$f(x)=(x^2-4x+4)-4+9$
$\therefore f(x)=(x-2)^2+5$
Turning point at $(2,5)$ so on the domain $[2,\infty)$, the function is one-to-one
Therefore the smallest value of a for f^{-1} to exist is $a=2$.

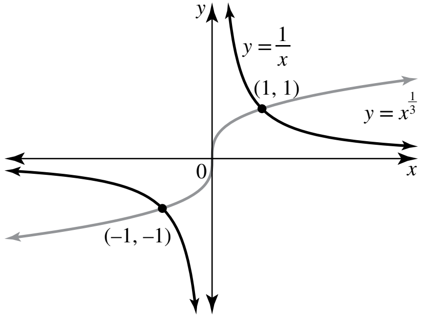
7 $y=x^n$ for $n=1$ and $n=\dfrac{1}{3}$ are the functions $y=x$ and $y=x^{\frac{1}{3}}$.
Both graphs contain the points $(-1,-1),(0,0)$ and $(1,1)$.



Since the cube root of numbers greater than 1 are smaller than the number, the cube root graph lies below the straight line when $x>1$. The cube roots of numbers between zero and 1 are larger than the number so the cube root graph lies above the straight line between $x=0$ and $x=1$.
$\therefore \{x : x^{\frac{1}{3}} > x\}=\{x: 0<x<1\}\cup\{x : x<-1\}$

8 **a** $y=x^n$ for $n=-1$ and $n=\dfrac{1}{3}$ are the functions $y=x^{-1}$ and $y=x^{\frac{1}{3}}$.
The graph of $y=x^{-1}$ is the hyperbola $y=\dfrac{1}{x}$ and the graph of $y=x^{\frac{1}{3}}$ is the cube root function $y=\sqrt[3]{x}$
At the intersection of the two graphs, $\sqrt[3]{x}=\dfrac{1}{x}$.
Cube both sides of the equation
$\therefore x=\left(\dfrac{1}{x}\right)^3$
$\therefore x=\dfrac{1}{x^3}$
$\therefore x^4=1$
$\therefore x=\pm 1$
$\therefore y=\pm 1$
The points of intersection are $(-1,-1)$ and $(1,1)$.

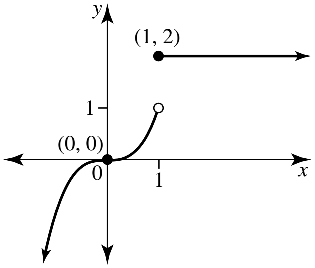
b Sketching the graphs:



9 $f(x)=\begin{cases}x^3, & x<1\\ 2, & x\geq 1\end{cases}$

a **i** $f(-2)=(-2)^3 \therefore f(-2)=-8$
ii $f(1)=2$
iii $f(2)=2$

b



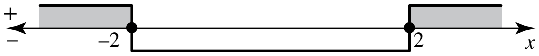
Domain R, range $(-\infty,1)\cup\{2\}$

c Not continuous at $x=1$

10 **a** $f(x)=\begin{cases}4x+a, & x<1\\ \dfrac{2}{x}, & 1\leq x\leq 4\end{cases}$
To be continuous the two branches must join at $x=1$
$\therefore 4(1)+a=\dfrac{2}{(1)}$
$\therefore 4+a=2$
$\therefore a=-2$

b $x=0$ lies in the domain for which the rule is the linear function $y=4x+a$ so the graph will be continuous at this point.

11 **a** $y=\sqrt{x^2-4}$
Require $x^2-4\geq 0$
$\therefore (x+2)(x-2)\geq 0$



$\therefore x\leq -2$ or $x\geq 2$
Domain is $(-\infty,-2]\cup[2,\infty)$ or $R\backslash(-2,2)$.

b $y=\dfrac{2x}{x^2-4}$
If denominator is zero then $x^2-4=0\Rightarrow x=\pm 2$. These values must be excluded from the domain.
Domain is $R\backslash\{-2,2\}$.

c $y=\dfrac{1}{\sqrt{4-x}}$
Require $4-x>0$
$\therefore -x>-4$
$\therefore x<4$
Domain $(-\infty,4)$

d $y=\sqrt{x}+\sqrt{2-x}$
Require $x\geq 0$ and $2-x\geq 0$
$\therefore x\geq 0$ and $x\leq 2$
$\therefore 0\leq x\leq 2$
Domain $[0,2]$

e $y = 4x^6 + 6x^{-1} + 2x^{\frac{1}{2}}$

$\therefore y = 4x^6 + \frac{6}{x} + 2\sqrt{x}$

Require $x \neq 0$ and $x \geq 0$

$\therefore x > 0$

Domain R^+

f $y = \frac{\sqrt{x^2+2}}{x^2+8}$

Require $x^2 + 2 \geq 0$ and $x^2 + 8 \neq 0$

Both statements hold for $x \in R$ so the domain is R.

12 a Given relation: $4x - 8y = 1$

Inverse: $4y - 8x = 1$

b Given relation: $y = -\frac{2}{3}x - 4$

Inverse: $x = -\frac{2}{3}y - 4$

$\therefore \frac{2}{3}y = -4 - x$

$\therefore 2y = -3x - 12$

$\therefore y = -\frac{3x}{2} - 6$

c Given relation: $y^2 = 4x$

Inverse: $x^2 = 4y$

$\therefore y = \frac{x^2}{4}$

d Given relation: $y = 4x^2$

Inverse: $x = 4y^2$

$\therefore y^2 = \frac{x}{4}$ or $y = \pm\frac{\sqrt{x}}{2}$

e Given relation: $x^2 + (y-3)^2 = 1$

Inverse: $y^2 + (x-3)^2 = 1$ or $(x-3)^2 + y^2 = 1$

f Given relation: $y = \sqrt{2x+1}$

Inverse: $x = \sqrt{2y+1}$ which requires that $x \geq 0$

$\therefore x^2 = 2y + 1$

$\therefore 2y = x^2 - 1$

$\therefore y = \frac{x^2-1}{2}, \ x \geq 0$

13 $f(x) = \frac{1}{x-2}$

a Domain: $x - 2 \neq 0 \Rightarrow x \neq 2$

The maximal domain of the function is $R \setminus \{2\}$. Its asymptotes have equations $x = 2$ and $y = 0$.

b Function: $y = \frac{1}{x-2}$

Inverse: $x = \frac{1}{y-2}$

$\therefore y - 2 = \frac{1}{x}$

$\therefore y = \frac{1}{x} + 2$

This is another hyperbola so it is a function.

The rule for the inverse is $f^{-1}(x) = \frac{1}{x} + 2$.

c The asymptotes of the inverse function have equations $x = 0$ and $y = 2$.

d $y = \frac{1}{x-a} + b$

Since this hyperbola has asymptotes with equations $x = a, \ y = b$, its inverse has asymptotes with equations $y = a, \ x = b$.

The equation of the hyperbola which is the inverse of $y = \frac{1}{x-a} + b$ is $y = \frac{1}{x-b} + a$.

14 $f : [-2, 4) \rightarrow R, f(x) = 4 - 2x$

a $d_f = [-2, 4) = r_{f^{-1}}$.

Endpoints for f: Closed at $x = -2$, open at $x = 4$.

$f(-2) = 4 + 4$ and $f(4) = 4 - 8$

$= 8$ $\quad = -4$

$r_f = (-4, 8] = d_{f^{-1}}$

Domain of f^{-1} is $(-4, 8]$, range is $[-2, 4)$.

b function f: $y = 4 - 2x$

inverse function f^{-1} : $x = 4 - 2y$

$\therefore 2y = 4 - x$

$\therefore y = 2 - \frac{x}{2}$

$\therefore f^{-1}(x) = -\frac{x}{2} + 2, \ x \in (-4, 8]$

As a mapping, $f^{-1} : (-4, 8] \rightarrow R, f^{-1}(x) = -\frac{x}{2} + 2$

c f: $y = 4 - 2x$

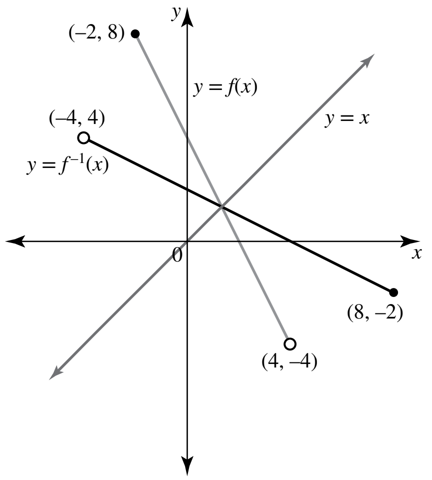
Endpoints $(-2, 8)$ closed, $(4, -4)$ open

y intercept $(0, 4)$, x intercept $(2, 0)$

f^{-1}: $y = 2 - \frac{x}{2}$

Endpoints: $(8, -2)$ closed, $(-4, 4)$ open

y intercept $(0, 2)$, x intercept $(4, 0)$

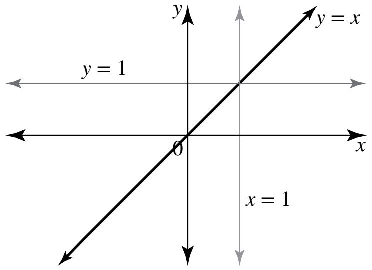


d $f(x) = f^{-1}(x)$ at their point of intersection, which is also their point of intersection with the line $y = x$.

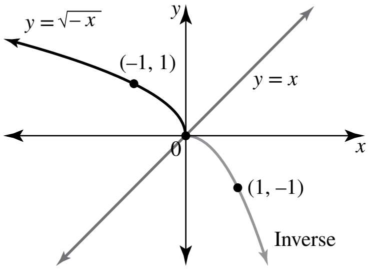
Solving $4 - 2x = x$ gives $4 = 3x$

$\therefore x = \frac{4}{3}$ when $f(x) = f^{-1}(x) = x$

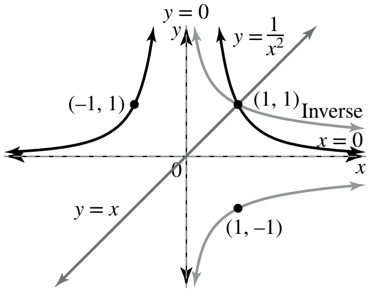
15 a i $y = 1$ represents the horizontal line through $(0, 1)$. Its inverse, $x = 1$ represents the vertical line through $(1, 0)$.



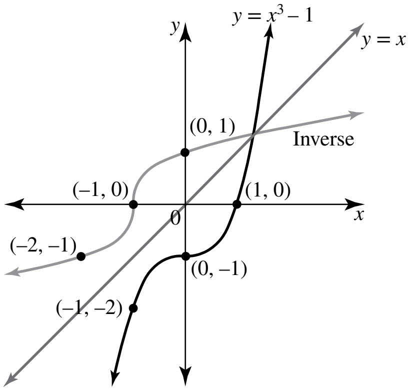
ii $y=\sqrt{-x}$ is the square root function with endpoint $(0,0)$ and domain for which $x\in(-\infty,0]$. The point $(-1,1)$ lies on the graph.
Its inverse has endpoint $(0,0)$ and range for which $y\in(-\infty,0]$. The point $(1,-1)$ lies on its graph.



iii $y=\dfrac{1}{x^2}$ is a truncus with asymptotes $x=0, y=0$. The points $(-1,1)$ and $(1,1)$ lie on its graph.
Its inverse will have asymptotes $y=0, x=0$ and contain the points $(1,-1)$ and $(1,1)$.



iv $y=x^3-1$ has a stationary point of inflection at $(0,-1)$ and cuts the x axis at $(1,0)$.
Its inverse has a point of inflection at $(-1,0)$ and cuts the y axis at $(0,1)$.



v $y=1-\sqrt{1-x^2}$

$\therefore (y-1)=-\sqrt{1-x^2}$

Lower half semicircle

$\therefore (y-1)^2=1-x^2$

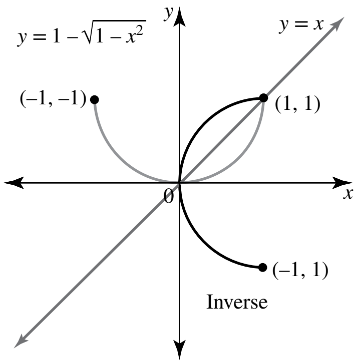
$\therefore x^2+(y-1)^2=1$

centre $(0,1)$ radius 1

Domain $[0-1,0+1]=[-1,1]$. Range of lower semicircle is $[1-1,1]=[0,1]$.

The semicircle contains the points $(-1,1),(0,0),(1,1)$

The inverse must contain the points $(1,-1),(0,0),(1,1)$.



vi $(y-2)^2=1-x$

$\therefore (y-2)^2=-(x-1)$

Sideways parabola with vertex $(1,2)$,

y intercepts: Let $x=0$

$\therefore (y-2)^2=1$

$\therefore y-2=\pm1$

$\therefore y=1, y=3$

$(0,1),(0,3)$

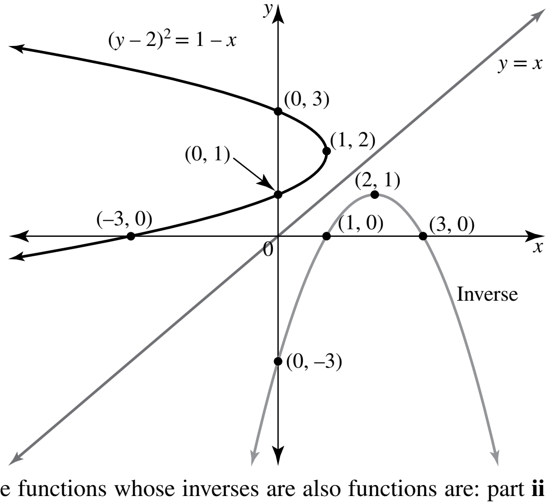
x intercept: Let $y=0$

$\therefore (-2)^2=1-x$

$\therefore 4=1-x$

$\therefore x=-3 \qquad (-3,0)$

The inverse will have vertex $(2,1)$, x intercepts $(1,0),(3,0)$ and y intercept $(0,-3)$.



b The functions whose inverses are also functions are: part **ii** $y=\sqrt{-x}$ and part **iv** $y=x^3-1$.
The rules for their inverses:
function: $y=\sqrt{-x}$
inverse: $x=\sqrt{-y}$

$\therefore x^2=-y$

$\therefore y=-x^2$

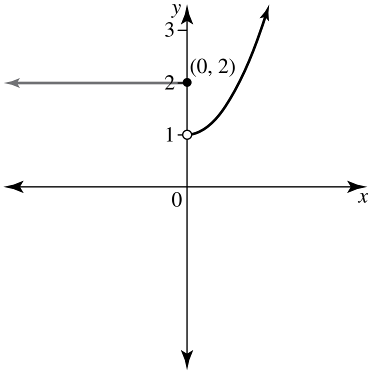
From the diagram there is a domain restriction that $x\ge0$.
The inverse is $y=-x^2,\ x\ge0$
Function: $y=x^3-1$
inverse: $x=y^3-1$

$\therefore y^3=x+1$

$\therefore y=\sqrt[3]{x+1}$ or $y=(x+1)^{\frac{1}{3}}$

16 a $y=\begin{cases}2, & x\le 0\\ 1+x^2, & x>0\end{cases}$

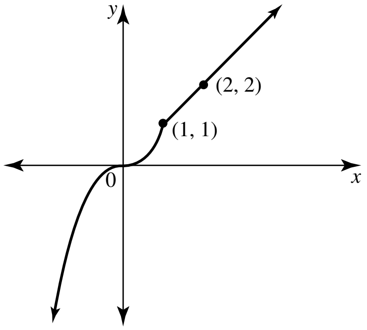
If $x\le 0$, $y=2$. Horizontal line with endpoint $(0,2)$ which is closed.
If $x>0$, $y=1+x^2$. Concave up parabola with minimum turning point $(0,1)$ which is open.



Domain R, range $(1,\infty)$

b $y=\begin{cases}x^3, & x<1\\ x, & x\ge 1\end{cases}$

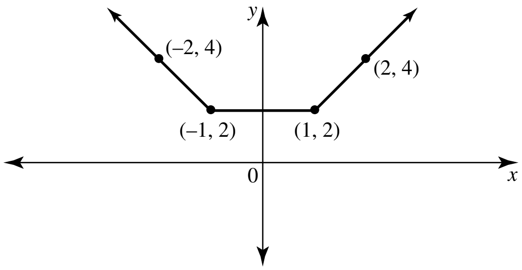
For $x<1$, $y=x^3$ has a stationary point of inflection at $(0,0)$ and an open endpoint at $(1,1)$.
For $x\ge 1$, $y=x$ has a closed endpoint at $(1,1)$ and also passes through the point $(2,2)$.
As the endpoint of both branches has the same co-ordinates, the graph is continuous at $(1,1)$ so no open or closed shading is given at this point.



Domain R, range R.

c $y=\begin{cases}-2x, & x<-1\\ 2, & -1\le x\le 1\\ 2x, & x>1\end{cases}$

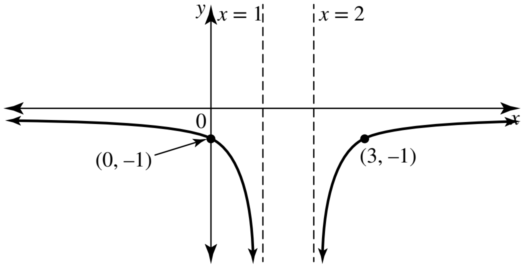
For $x<-1$, $y=-2x$. This line has endpoint $(-1,2)$ and passes through $(-2,4)$.
For $-1\le x\le 1$, $y=2$. This horizontal line has endpoints $(-1,2)$ and $(1,2)$.
For $x>1$, $y=2x$. This line has endpoint $(1,2)$ and passes through $(2,4)$.
The graph is continuous at the points where the branches join.



Domain R, range $[2,\infty)$

d $y=\begin{cases}\dfrac{1}{x-1}, & x<1\\ \dfrac{1}{2-x}, & x>2\end{cases}$

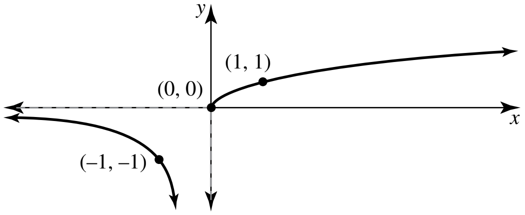
For $x<1$, $y=\dfrac{1}{x-1}$. This hyperbola has asymptotes $x=1, y=0$ and contains the point $(0,-1)$. Only one branch of the hyperbola lies in the domain restriction that $x<1$.
For $x>2$, $y=\dfrac{1}{2-x}$. This hyperbola has asymptotes $x=2, y=0$ and contains the point $(3,-1)$. Only one branch of the hyperbola lies in the domain restriction that $x>2$.



Domain $R\setminus[1,2]$, range R^-.

e $y=\begin{cases}x^{\frac{1}{3}} & x\ge 0\\ -x^{-2} & x<0\end{cases}$

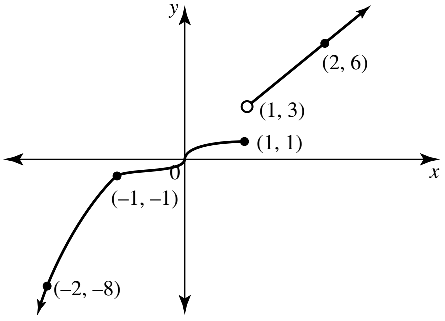
For $x\ge 0$, $y=x^{\frac{1}{3}}$. This cube root function contains the points $(0,0)$ and $(1,1)$.
For $x<0$, $y=-x^{-2}\Rightarrow y=\dfrac{-1}{x^2}$, This is a branch of the truncus which has asymptotes $x=0, y=0$ and contains the point $(-1,-1)$.



Domain R and range R.

f $y=\begin{cases}x^3, & x<-1\\ \sqrt[3]{x}, & -1\le x\le 1\\ 3x, & x>1\end{cases}$

For $x<-1$, $y=x^3$. This is the left hand half of the cubic function. The graph approaches the point $(-1,-1)$.
For $-1\le x\le 1$, $y=\sqrt[3]{x}$. This is the cube root function with a point of inflection at $(0,0)$ and endpoints $(-1,-1)$ and $(1,1)$.
For $x>1$, $y=3x$ is the straight line joining the points $(1,3)$ and $(2,6)$.
The point $(1,3)$ is open, the point $(1,1)$ is closed and the cubic and cube root graphs join at $(-1,-1)$.



Domain R and range $R\setminus(1,3]$.

17 a $f(x)=\begin{cases}-x-1, & x<-1\\ \sqrt{1-x^2}, & -1\le x\le 1\\ x+1, & x>1\end{cases}$

i $f(0)=\sqrt{1-0^2}=1$

ii $f(3)=3+1=4$

iii $f(-2)=-(-2)-1=1$

iv $f(1)=\sqrt{1-1^2}=0$

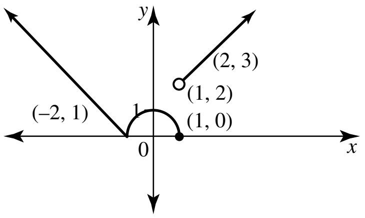
b The two branches around $x=1$ are $y=\sqrt{1-x^2},-1\le x\le 1$ and $y=x+1, x>1$.

For $y=\sqrt{1-x^2}$, there is a closed endpoint $(1,0)$ but for $y=x+1$ there is an open endpoint $(1,2)$. The two branches do not join. Hence the function is not continuous at $x=1$ as there will be a break in its graph.

c For $x<-1$, $y=-x-1$. This line has endpoint $(-1,0)$ and passes through $(-2,1)$.

For $-1\le x\le 1$, $y=\sqrt{1-x^2}$. This semicircle has centre at the origin and a radius of 1 unit. One endpoint is $(-1,0)$ so the graph is continuous at $x=-1$. The other endpoint $(1,0)$ is closed and the graph is discontinuous at $x=1$.

For $x>1$, $y=x+1$. This line has an open endpoint at $(1,2)$ and passes through $(2,1)$.



The function has a many-to-one correspondence.

d If $f(a)=a$ then the point (a,a) must lie on the graph of f. This point must also lie on the graph of $y=x$, a line through the origin with a gradient of 1. This line intersects the semicircle section of the function's graph.

At intersection, $x=\sqrt{1-x^2}$ for $0\le x\le 1$

Squaring both sides of the equation gives

$$x^2=1-x^2$$
$$\therefore 2x^2=1$$
$$\therefore x^2=\frac{1}{2}$$
$$\therefore x=\pm\frac{1}{\sqrt{2}}$$

As $0\le x\le 1$, $x=\frac{1}{\sqrt{2}}$.

$\therefore a=\frac{\sqrt{2}}{2}$ for $f(a)=a$.

18 a Left branch is a line through $(-1,0)$ and $(0,1)$.

$$m=\frac{1-0}{0+1}=1$$
$$\therefore y=x+1$$

Right branch is a line through $(1,0)$ and $(0,1)$.

$$m=-1$$
$$\therefore y=-x+1$$

The rule for the hybrid function is $y=\begin{cases}x+1, & x\le 0\\ -x+1, & x>0\end{cases}$.

b For $x<2$, the left branch is the horizontal line $y=3$

For $x\ge 2$, the right branch is the line through $(2,0)$ and $(4,6)$.

$$m=\frac{6-0}{4-2}=3$$
$$\therefore y-0=3(x-2)$$
$$\therefore y=3x-6$$

The rule for the hybrid function is $y=\begin{cases}3, & x<2\\ 3x-6, & x\ge 2\end{cases}$.

c $f(x)=\begin{cases}a, & x\in(-\infty,-3]\\ x+2, & x\in(-3,3)\\ b, & x\in[3,\infty)\end{cases}$

To be continuous the branches must join.

At the join where $x=-3$, $a=-3+2$

$\therefore a=-1$

At the join where $x=3$, $3+2=b$

$\therefore b=5$

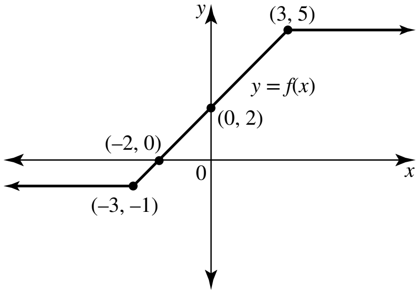
With $a=-1$, $b=5$, the rule for the function becomes

$$f(x)=\begin{cases}-1, & x\in(-\infty,-3]\\ x+2, & x\in(-3,3)\\ 5, & x\in[3,\infty)\end{cases}$$

For $x\in(-\infty,-3]$, $y=-1$ is a horizontal line with endpoint $(-3,-1)$.

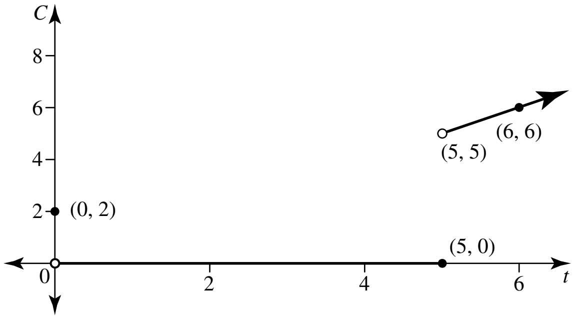
For $x\in(-3,3)$, $y=x+2$ is a line with y intercept $(0,2)$, x intercept $(-2,0)$ and endpoints $(-3,-1)$ and $(3,5)$.

For $x\in[3,\infty)$, $y=5$ is a horizontal line with endpoint $(3,5)$.



d Let the time in the shower be t minutes and the dollar amount of the fine be C.

The rule is $C=\begin{cases}2, & t=0\\ 0, & 0<t\le 5\\ t, & t>5\end{cases}$.



19 $g: D\to R, g(x)=x^2+8x-9$

a g^{-1} exists for a domain where g has a one-to-one correspondence.

$$g(x)=x^2+8x-9$$
$$=(x^2+8x+16)-16-9$$
$$=(x+4)^2-25$$

The turning point of the parabola g is the point $(-4,-25)$.

A possible domain over which g is one-to-one is $D=[-4,\infty)$. Another possibility could be $D=(-\infty,-4]$.

b For the domain $D = R^+$, the x value of the turning point does not lie in the domain.
$g(0) = -9$, so the point $(0, -9)$ is an open endpoint of the graph of g
The range of g is $(-9, \infty)$.

Function g: $y = (x+4)^2 - 25,\ d_g = R^+,\ r_g = (-9, \infty)$

Inverse function g^{-1}: $x = (y+4)^2 - 25,\ d_{g^{-1}} = (-9, \infty), r_{g^{-1}} = R^+$

$$\therefore (y+4)^2 = x + 25$$
$$\therefore y + 4 = \pm\sqrt{x+25}$$

Due to the range, the positive square root is required.

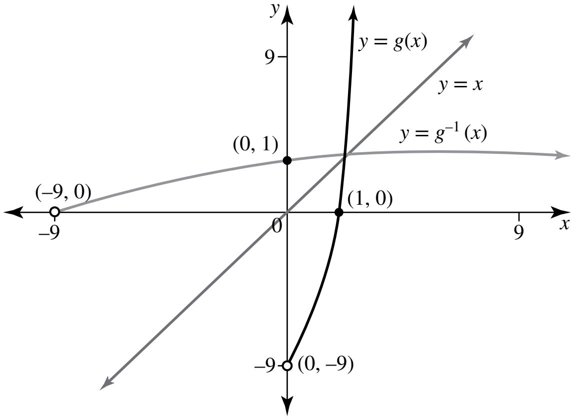
$$\therefore y + 4 = \sqrt{x+25}$$
$$\therefore y = \sqrt{x+25} - 4$$
$$\therefore g^{-1}(x) = \sqrt{x+25} - 4$$

Hence, $g^{-1}: (-9, \infty) \to R, g^{-1}(x) = \sqrt{x+25} - 4$
The range of g^{-1} is R^+.

c **i** Graph of g: $y = x^2 + 8x - 9 \Rightarrow y = (x+4)^2 - 25$
Domain R^+, open endpoint $(0, -9)$
x intercept: Let $y = 0$

$$\therefore 0 = x^2 + 8x - 9$$
$$\therefore (x+9)(x-1) = 0$$
$$\therefore x = -9, x = 1$$

$x = -9$ does not lie in the domain so the x intercept is $(1, 0)$.
Graph of g^{-1}: $y = \sqrt{x+25} - 4$
Domain $(-9, \infty)$, open endpoint $(-9, 0)$, y intercept $(0, 1)$



ii At the point of intersection, $g(x) = g^{-1}(x) = x$
Solving $g(x) = x,\ x > 0$

$$x^2 + 8x - 9 = x$$
$$\therefore x^2 + 7x - 9 = 0$$
$$\therefore x = \frac{-7 \pm \sqrt{49 - 4 \times 1 \times -9}}{2}$$
$$\therefore x = \frac{-7 \pm \sqrt{85}}{2}$$

Since $x > 0$, reject the negative square root

$$x = \frac{-7 + \sqrt{85}}{2}$$

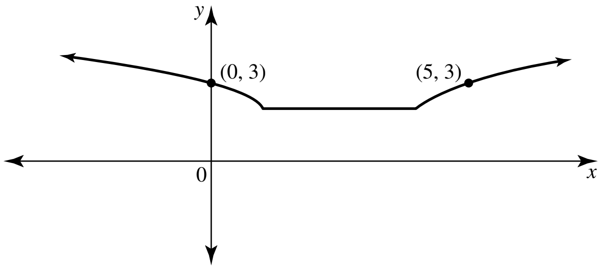
The point of intersection is $\left(\frac{-7+\sqrt{85}}{2}, \frac{-7+\sqrt{85}}{2}\right)$.

d Require the value of x so that $g^{-1}(x) = x$. This is the value from part **c** $\frac{-7+\sqrt{85}}{2}$ is its own image under g^{-1}.

20 $$f(x) = \begin{cases} \sqrt{x-4} + 2, & x > 4 \\ 2, & 1 \le x \le 4 \\ \sqrt{1-x} + 2, & x < 1 \end{cases}$$

a $f(0) = \sqrt{1-0} + 2 = 3$
$f(5) = \sqrt{5-4} + 2 = 3$

b For $x<1$, $y=\sqrt{1-x}+2 \Rightarrow y=\sqrt{-(x-1)}+2$
Square root function with endpoint: $(1,2)$.
y intercept: $(0,3)$, no x intercept
For $1\le x\le 4$, $y=2$ is a horizontal line with endpoints $(1,2),(4,2)$,
For $x>4$, $y=\sqrt{x-4}+2$
Square root function with endpoint $(4,2)$ and passing through $(5,3)$.
As the branches join there are no points of discontinuity.



Domain R, range $[2,\infty)$

c The horizontal line $y=8$ would intersect the graph at a point on each of its left and right branches, but it would not intersect with the middle branch. Therefore, $f(x)=8$ when either $\sqrt{1-x}+2=8,\ x<1$ or $\sqrt{x-4}+2=8,\ x>4$.

Solving $\sqrt{1-x}+2=8,\ x<1$

$$\sqrt{1-x}=6$$
$$\therefore 1-x=36$$
$$\therefore x=-35$$

Solving $\sqrt{x-4}+2=8,\ x>4$

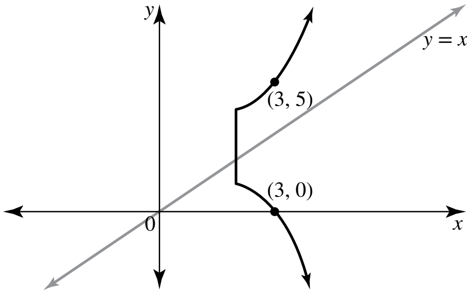
$$\sqrt{x-4}=6$$
$$\therefore x-4=36$$
$$\therefore x=40$$

Therefore, $f(x)=8$ when $x=-35$ or $x=40$.

d The function has a many-to-one correspondence so its inverse is not a function since its correspondence is one-to-many.
The function contains the points $(0,3),(1,2),(4,2),(5,3)$ so its inverse contains the points $(3,0),(2,1),(2,4),(3,5)$. The graph of the inverse is the reflection of the function in the line $y=x$, so the middle branch is $x=2,\ 1\le y\le 4$.

The function has the rule $y=\begin{cases}\sqrt{x-4}+2, & x>4, y>2\\ 2, & 1\le x\le 4\\ \sqrt{1-x}+2, & x<1, y>2\end{cases}$

The inverse has the rule $x=\begin{cases}\sqrt{y-4}+2, & y>4, x>2\\ 2, & 1\le y\le 4\\ \sqrt{1-y}+2, & y<1, x>2\end{cases}$



21 a $y=x^2+5x-2$
To obtain the rule for the inverse, use Equation/Inequality and Solve $(x=y^2+5y-2, y)$. This gives
$\left\{y=0.5\left((4x+33)^{0.5}-5\right), y=-0.5\left((4x+33)^{0.5}+5\right)\right\}$.
The rule for the inverse is $y=\pm 0.5\sqrt{4x+33}-2.5$.

b In Graph&Tab, sketch $y=x^2+5x-2$. The graph of its inverse is obtained by tapping Analysis → Sketch → Inverse.
The graph indicates there are four points of intersection. To obtain their co-ordinates, the two branches of the inverse must be sketched by rule and the points of intersection obtained from Analysis → G-Solve → Intersect.
The upper branch of the inverse $y=0.5\sqrt{4x+33}-2.5$ and $y=x^2+5x-2$ intersect at the two points $(-5.24,-0.76)$ and $(0.45, 0.45)$; the lower branch of the inverse $y=-0.5\sqrt{4x+33}-2.5$ and $y=x^2+5x-2$ intersect at the two points $(-4.45,-4.45)$ and $(-0.76,-5.24)$.
Two of these points are the intersections with $y=x$. Sketching $y=x$ and from Analysis → G-Solve → Intersect confirms their co-ordinates as $(-4.45,-4.45),(0.45,0.45)$.
The other two points are points common to each parabola which are reflections in the line $y=x$ but which do not lie on this line.
Correct to two decimal places the points of intersection are $(-5.24,-0.76)$, $(0.45,0.45)$, $(-4.45,-4.45)$ and $(-0.76,-5.24)$.

22 $y=\begin{cases}\sqrt{2+x^2}, & x\le -2\\ x\sqrt{2+x}, & -2<x<2\\ \dfrac{1}{\sqrt{2+x}}, & x\ge 2\end{cases}$

a At the join $x=-2$,
$\sqrt{2+x^2}=\sqrt{2+4}=\sqrt{6}$ and $x\sqrt{2+x}=-2\sqrt{2-2}=0$.
Point of discontinuity at $x=-2$.
At the join $x=2$,
$x\sqrt{2+x}=2\sqrt{4}=4$ and $\dfrac{1}{\sqrt{2+x}}=\dfrac{1}{\sqrt{4}}=\dfrac{1}{2}$.
Point of discontinuity at $x=2$.
Consider the function rules for each branch in case there are further points of discontinuity.
For $x\le -2$, $y=\sqrt{2+x^2}$
As $2+x^2>0$ for all x values there is no restriction.
For $-2<x<2$, $y=x\sqrt{2+x}$
This requires $2+x\ge 0 \Rightarrow x\ge -2$. As the rule applies for $-2<x<2$, this condition is satisfied.
For $x\ge 2$, $y=\dfrac{1}{\sqrt{2+x}}$
This requires $2+x>0 \Rightarrow x>-2$. This condition is satisfied since the rule applies for $x\ge 2$.
There are points of discontinuity at the joins of the branches for $x=-2,\ x=2$.

b In Graph&Tab enter the function as

$$y1=\sqrt{(2+x^2)}\,\Big|\,x\le -2$$
$$y2=x\sqrt{(2+x)}\,\Big|-2<x<2$$
$$y3=1/\sqrt{(2+x)}\,\Big|\,x\ge 2$$

To obtain the range, the minimum turning point on the branch $y = x\sqrt{2+x}$ needs to be obtained from the Analysis tools. This is the point $(-1.333333, -1.888662)$. The x co-ordinate is $x = -\frac{4}{3}$ and substituting this into $y = x\sqrt{2+x}$ in the main menu gives $y = -\frac{4}{3}\sqrt{2-\frac{4}{3}} = -\frac{4\sqrt{6}}{9}$.

The range is $[-\frac{4\sqrt{6}}{9}, \infty)$.

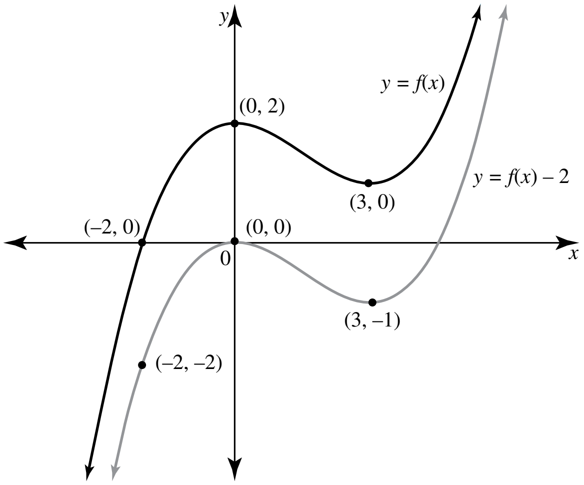
Exercise 6.7 — Transformations of functions

1 $y = f(x) - 2$ is a vertical translation down of 2 units of the graph of $y = f(x)$.

$(-2,0) \rightarrow (-2,-2)$

$(0,2) \rightarrow (0,0)$

$(3,1) \rightarrow (3,-1)$

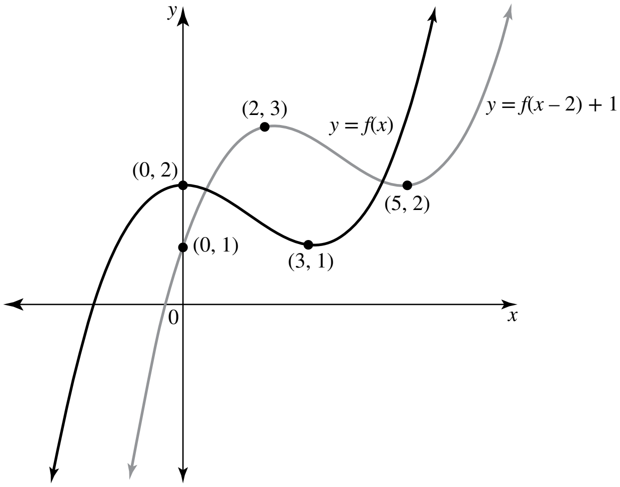


2 $y = f(x-2) + 1$ is a horizontal translation 2 units to right, vertical translation 1 unit up of $y = f(x)$,

$(-2,0) \rightarrow (0,1)$

$(0,2) \rightarrow (2,3)$

$(3,1) \rightarrow (5,2)$

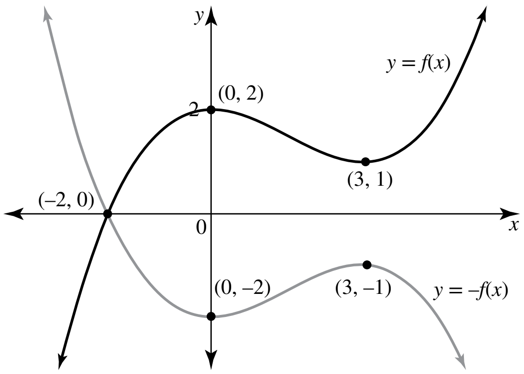


3 $y = -f(x)$ reflection in the x axis of $y = f(x)$.

$(-2,0) \rightarrow (-2,0)$

$(0,2) \rightarrow (0,-2)$

$(3,1) \rightarrow (3,-1)$



4 **a** $y=(x-1)^2$: reflect in the x axis then vertically translate up 3 units.

$y=(x-1)^2 \to y=-(x-1)^2 \to y=-(x-1)^2+3$

Therefore the image has equation $y=-(x-1)^2+3$.

b vertically translate up 3 units then reflect in the x axis

$y=(x-1)^2 \to y=(x-1)^2+3 \to y=-\left((x-1)^2+3\right)$

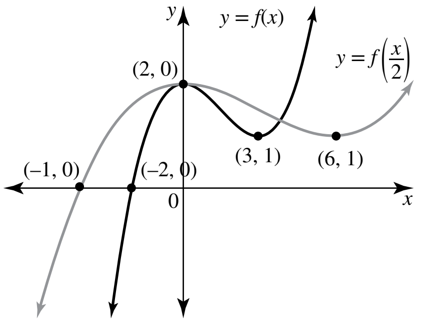
Therefore the image has equation $y=-(x-1)^2-3$ which is not the same as in part **a**.

5 $y=f\left(\dfrac{x}{2}\right)$ dilation of $y=f(x)$ by factor 2 from the y axis

$(-2,0)\to(-4,0)$

$(0,2)\to(0,2)$

$(3,1)\to(6,1)$

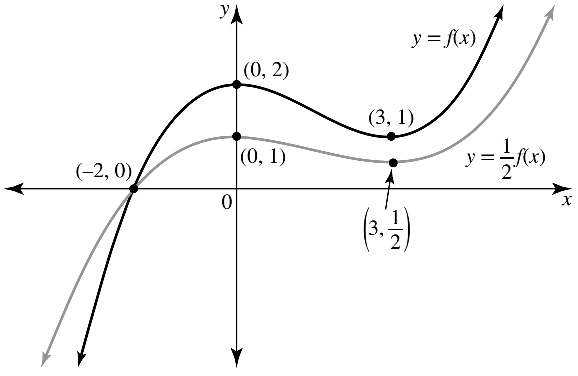


6 $y=\dfrac{1}{2}f(x)$ dilation of $y=f(x)$ by factor $\dfrac{1}{2}$ from x axis

$(-2,0)\to(-2,0)$

$(0,2)\to(0,1)$

$(3,1)\to\left(3,\dfrac{1}{2}\right)$



7 **a** $y=4f\left(\dfrac{x}{2}-1\right)+3$

$\therefore y=4f\left(\dfrac{1}{2}(x-2)\right)+3$

Comparing this with $y=af\left(n(x-h)\right)+k$,

$a=4,\ n=\dfrac{1}{2},\ h=2$ and $k=3$.

The graph of $y=f(x)$ has undergone The following transformations:

Dilation of factor 4 from the x axis, dilation of factor 2 from the y axis, horizontal translation 2 units to the right and vertical translation 3 units upwards.

b $y=\sqrt{3-\dfrac{x}{4}}$

$\therefore y=\sqrt{-\dfrac{1}{4}(x-12)}$

Reflection in y axis, dilation of factor 4 from the y axis followed by horizontal translation 12 units to the right

An alternative answer is obtained by writing the equation as $y=\dfrac{1}{2}\sqrt{-(x-12)}$.

Reflection in y axis, dilation of factor $\dfrac{1}{2}$ from the x axis followed by horizontal translation 12 units to the right

8 **a** Dilation of factor $\dfrac{1}{2}$ from the y axis, horizontal translation 3 units left

$y=\dfrac{1}{x}\to y=\dfrac{1}{(2x)}\to y=\dfrac{1}{2(x+3)}$

Therefore the equation of the image is $y=\dfrac{1}{2(x+3)}$.

b Undoing the transformations requires the image to undergo a horizontal translation 3 units to the right followed by dilation of factor 2 from the y axis.

$y=\dfrac{1}{2(x+3)}\to y=\dfrac{1}{2(x)}\to y=\dfrac{1}{\left(\dfrac{2x}{2}\right)}\to y=\dfrac{1}{x}$

9 **a** $y=x^2\to y=3x^2$ under a dilation of factor 3 from the x axis.

b $y=x^2\to y=-x^2$ under a reflection the x axis.

c $y=x^2\to y=x^2+5$ under a vertical translation of 5 units upwards.

d $y=x^2\to y=(x+5)^2$ under a horizontal translation of 5 units to the left.

10 **a** $y=x^3\to y=\left(\dfrac{x}{3}\right)^3$ under a dilation of factor 3 from the y axis.

b $y=x^3\to y=(2x)^3+1$ under a dilation of factor $\dfrac{1}{2}$ from the y axis followed by a vertical translation of 1 unit upwards.

c $y=x^3\to y=(x-4)^3-4$ under a horizontal translation of 4 units to the right and a vertical translation of 4 units downwards.

d Since $y=(1+2x)^3$ can be expressed as $y=\left[2\left(x+\dfrac{1}{2}\right)\right]^3$, then $y=x^3\to y=(1+2x)^3$ under a dilation of factor $\dfrac{1}{2}$ from the y axis followed by a horizontal translation of $\dfrac{1}{2}$ unit to the left.

11 **i** **a** Under a dilation of factor 2 from the x axis, $y=\sqrt{x}\to y=2\sqrt{x}$

b Under a dilation of factor 2 from the y axis, $y=\sqrt{x}\to y=\sqrt{\dfrac{x}{2}}$

c Under a reflection in the x axis, $y=\sqrt{x}\to y=-\sqrt{x}$; and, after a translation of 2 units vertically upwards $y=-\sqrt{x}\to y=-\sqrt{x}+2$.

The image has equation $y=-\sqrt{x}+2$.

d Under a vertical translation of 2 units upwards, $y=\sqrt{x}\to y=\sqrt{x}+2$. Then, after a reflection in the x axis, $y=\sqrt{x}+2\to y=-\left(\sqrt{x}+2\right)$.
The image has equation $y=-\sqrt{x}-2$.

e Under a reflection in the y axis, $y=\sqrt{x}\to y=\sqrt{(-x)}$; and then after a translation of 2 units to the right, $y=\sqrt{(-x)}\to\sqrt{(-(x-2))}$.
The equation of the image is $y=\sqrt{2-x}$.

f Under a translation of 2 units to the right, $y=\sqrt{x}\to y=\sqrt{(x-2)}$. Then, after a reflection in the y axis, $y=\sqrt{x-2}\to\sqrt{(-x)-2}$.
The equation of the image is $y=\sqrt{-x-2}$.

ii a Under a dilation of factor 2 from the x axis, $y=x^4\to y=2x^4$.

b Under a dilation of factor 2 from the y axis, $y=x^4\to y=\left(\frac{x}{2}\right)^4$.
The equation of the image is $y=\frac{x^4}{16}$.

c Under a reflection in the x axis, $y=x^4\to y=-x^4$; and, after a translation of 2 units vertically upwards $y=-x^4\to y=-x^4+2$.
The image has equation $y=-x^4+2$.

d Under a vertical translation of 2 units upwards, $y=x^4\to y=x^4+2$. Then, after a reflection in the x axis, $y=x^4+2\to y=-\left(x^4+2\right)$.
The image has equation $y=-x^4-2$.

e Under a reflection in the y axis, $y=x^4\to y=(-x)^4$; and then after a translation of 2 units to the right, $y=(-x)^4\to y=\left(-(x-2)\right)^4$.
As $y=\left(-(x-2)\right)^4$ is equivalent to $y=(x-2)^4$, the equation of the image is $y=(x-2)^4$.

f Under a translation of 2 units to the right, $y=x^4\to y=(x-2)^4$. Then, after a reflection in the y axis, $y=(x-2)^4\to y=((-x)-2)^4$

$$\begin{aligned}y&=((-x)-2)^4\\&=(-x-2)^4\\&=\left(-(x+2)\right)^4\\&=(x+2)^4\end{aligned}$$

The equation of the image is $y=(x+2)^4$.

12 a Under a translation 2 units to the left and 4 units down, $(3,-4)\to(1,-8)$.

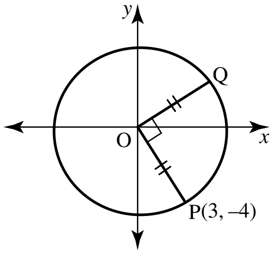
b Under a reflection in the y axis, $(3,-4)\to(-3,-4)$. Then under a reflection in the x axis, $(-3,-4)\to(-3,4)$.
The image is $(-3,4)$.

c Under a dilation of factor $\frac{1}{5}$ from the x axis, acting in the y direction, $(3,-4)\to\left(-3,-4\times\frac{1}{5}\right)$.
The image is $\left(-3,-\frac{4}{5}\right)$.

d Under a dilation of factor $\frac{1}{5}$ from the y axis, acting in the x direction, $(3,-4)\to\left(3\times\frac{1}{5},-4\right)$.
The image is $\left(\frac{3}{5},-4\right)$.

e Under a reflection in the line $y=x$, co-ordinates are interchanged so $(3,-4)\to(-4,3)$.

f Consider a rotation of 90° anticlockwise about the origin of the point P $(3,-4)$. Let its image be the point Q.



The radius of the circle, OR is equal to $\sqrt{(3)^2+(-4)^2}=5$.
Therefore, OQ is also 5 units in length.
The radii OQ and OP are perpendicular.

$$\therefore m_{OQ}\times m_{OP}=-1$$
$$\therefore m_{OQ}\times\frac{-4}{3}=-1$$
$$\therefore m_{OQ}=\frac{3}{4}$$

Since $OQ=5$, and '3, 4, 5' is a Pythagorean triple, the co-ordinates of Q must be $(4,3)$.
Under the transformation, $(3,-4)\to(4,3)$.

13 a i Under a dilation of factor 3 from the y axis followed by a reflection in the y axis,

$$y=\frac{1}{x}\to y=\frac{1}{\left(\frac{x}{3}\right)}=\frac{3}{x}\to y=\frac{3}{(-x)}=-\frac{3}{x}$$

The equation of the image is $y=-\frac{3}{x}$.

ii Under a reflection in the y axis followed by a dilation of factor 3 from the y axis,

$$y=\frac{1}{x}\to y=\frac{1}{(-x)}=\frac{-1}{x}\to y=\frac{-1}{\left(\frac{x}{3}\right)}=-\frac{3}{x}$$

When the order of the transformations is reversed the same image is obtained. The image has the equation $y=-\frac{3}{x}$.

b i Under a dilation of factor 3 from the x axis followed by a vertical translation of 6 units upwards,

$$y=\frac{1}{x^2}\to y=3\times\left(\frac{1}{x^2}\right)=\frac{3}{x^2}\to y=\frac{3}{x^2}+6.$$

The image has equation $y=\frac{3}{x^2}+6$.

ii Under a vertical translation of 6 units upwards followed by a dilation of factor 3 from the x axis,

$$y=\frac{1}{x^2}\to y=\frac{1}{x^2}+6\to y=3\times\left(\frac{1}{x^2}+6\right)=\frac{3}{x^2}+18.$$

The image has the equation $y=\frac{3}{x^2}+18$.

c Equation of image is $y=-\frac{1}{2x+2}+1$. Express this in the form $y=\frac{a}{x-h}+k$.

$$\therefore y=-\frac{1}{2(x+1)}+1$$
$$\therefore y=\frac{-\frac{1}{2}}{x+1}+1$$

The transformations that have been applied to $y=\frac{1}{x}$ to obtain this image are: reflection in the x axis, dilation of factor $\frac{1}{2}$ from the x axis followed by a horizontal translation 1 unit to the left and a vertical translation 1 unit upwards.

d $y = f(x) \to y = -2f(x+1)$ under a reflection in the x axis, dilation of factor 2 from the x axis followed by a horizontal translation 1 unit to the left.

The asymptotes of $y = f(x) = \frac{1}{x^2}$ are $x = 0$ and $y = 0$.

Under the transformations, $x = 0 \to x = 0 \to x = -1$ and $y = 0 \to y = 0 \to y = 0$.

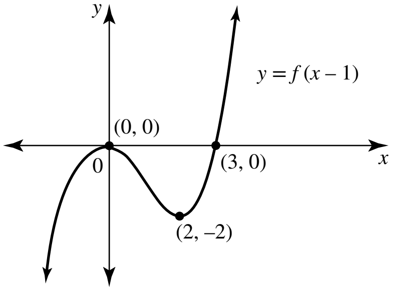
The equations of the asymptotes of the image are $x = -1, y = 0$.

Alternatively, if $f(x) = \frac{1}{x^2}$ then $-2f(x+1) = \frac{-2}{(x+1)^2}$. The asymptotes of $y = \frac{-2}{(x+1)^2}$ are $x = -1, y = 0$.

14 Key points on the given graph are $(-1,0), (0,-1), (1,-2), (2,0)$.

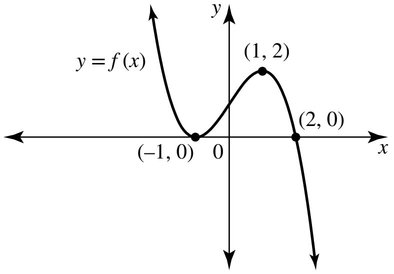
a The graph of $y = f(x-1)$ is obtained from a horizontal translation 1 unit to the right of the given graph.

$(-1,0) \to (0,0)$
$(0,-1) \to (1,-1)$
$(1,-2) \to (2,-2)$
$(2,0) \to (3,0)$



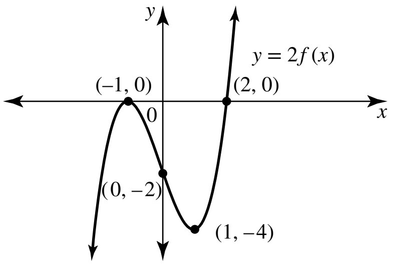
b The graph of $y = -f(x)$ is obtained by a reflection in the x axis of the given graph.

$(-1,0) \to (-1,0)$
$(0,-1) \to (0,1)$
$(1,-2) \to (1,2)$
$(2,0) \to (2,0)$



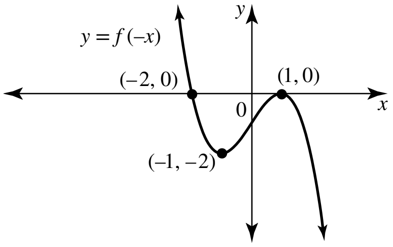
c The graph of $y = 2f(x)$ is obtained by a dilation of factor 2 from the x axis.

$(-1,0) \to (-1,0)$
$(0,-1) \to (0,-2)$
$(1,-2) \to (1,-4)$
$(2,0) \to (2,0)$



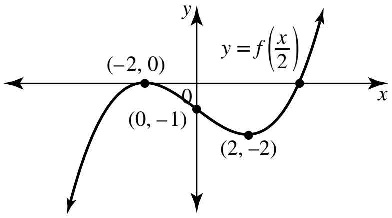
d The graph of $y = f(-x)$ is obtained by a reflection in the y axis.

$(-1,0) \to (1,0)$
$(0,-1) \to (0,-1)$
$(1,-2) \to (-1,-2)$
$(2,0) \to (-2,0)$



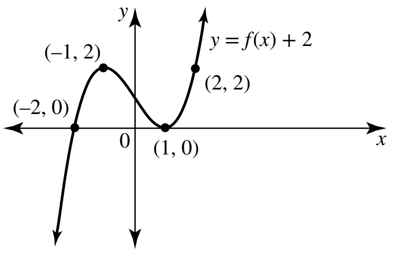
e The graph of $y = f\left(\frac{x}{2}\right)$ is obtained by a dilation of factor 2 from the y axis.

$(-1,0) \to (-2,0)$
$(0,-1) \to (0,-1)$
$(1,-2) \to (2,-2)$
$(2,0) \to (4,0)$



f The graph of $y = f(x) + 2$ is obtained by a vertical translation of 2 units upwards.

$(-1,0) \to (-1,2)$
$(0,-1) \to (0,1)$
$(1,-2) \to (1,0)$
$(2,0) \to (2,2)$



15 a $y = f(x) \to y = 2f(x+3)$ under a dilation of factor 2 from the x axis followed by a horizontal translation of 3 units to the left.

b $y = f(x) \to y = 6f(x-2)+1$ under a dilation of factor 6 from the x axis followed by a horizontal translation of 2 units to the right and a vertical translation of 1 unit upwards.

c $y = f(2x+2)$ can be expressed as $y = f(2(x+1))$.

$\therefore y = f(x) \to y = f(2x+2)$ under a dilation of factor $\frac{1}{2}$ from the y axis followed by a horizontal translation of 1 unit to the left.

d $y = f(-x+3)$ can be expressed as $y = f(-(x-3))$.

$\therefore y = f(x) \to y = f(-x+3)$ under a reflection in the y axis followed by a horizontal translation of 3 units to the right.

e $y=1-f(4x)$ can be rewritten as $y=-f(4x)+1$.
$\therefore y=f(x)\to y=1-f(4x)$ under a reflection in the x axis, a dilation of factor $\frac{1}{4}$ from the y axis and then a vertical translation of 1 unit upwards.

f Write $y=\frac{1}{9}f\left(\frac{x-3}{9}\right)$ as $y=\frac{1}{9}f\left(\frac{1}{9}(x-3)\right)$
$\therefore y=f(x)\to y=\frac{1}{9}f\left(\frac{x-3}{9}\right)$ under a dilation of factor $\frac{1}{9}$ from the x axis, a dilation of factor 9 from the y axis and then a horizontal translation of 3 units to the right.

16 a Under a dilation of factor $\frac{1}{3}$ from the x axis followed by a horizontal translation 3 units to the left, $y=\frac{1}{x^2}\to y=\frac{1}{3x^2}\to y=\frac{1}{3(x+3)^2}$. The equation of the image is $y=\frac{1}{3(x+3)^2}$.

b Under a vertical translation of 3 units down followed by a reflection in the x axis, $y=x^5\to y=x^5-3\to y=-\left(x^5-3\right)$.
The equation of the image is $y=-x^5+3$.

c Under a reflection in the y axis followed by a horizontal translation 1 unit to the right, $y=\frac{1}{x}\to y=\frac{1}{(-x)}\to y=\frac{1}{-(x-1)}$. The equation of the image is $y=\frac{1}{1-x}$.

d Under a horizontal translation 1 unit to the right followed by a dilation of factor 0.5 from the y axis, $y=\sqrt[3]{x}\to y=\sqrt[3]{x-1}\to y=\sqrt[3]{\frac{x}{0.5}-1}$.

$$\begin{aligned}y&=\sqrt[3]{\frac{x}{0.5}-1}\\&=\sqrt[3]{2x-1}\end{aligned}$$

The equation of the image is $y=\sqrt[3]{2x-1}$.

e Under a horizontal translation of 6 units to the right followed by a reflection in the x axis,

$$\begin{aligned}&y=(x+9)(x+3)(x-1)\\\to\ &y=((x-6)+9)((x-6)+3)((x-6)-1)\\&\quad=(x+3)(x-3)(x-7)\\\to\ &y=-(x+3)(x-3)(x-7)\end{aligned}$$

The equation of the image is $y=-(x+3)(x-3)(x-7)$.

f Under a dilation of factor 2 from both the x and y axes,

$$y=x^2(x+2)(x-2)\to y=2\left(\frac{x}{2}\right)^2\left(\frac{x}{2}+2\right)\left(\frac{x}{2}-2\right)$$

$$\begin{aligned}y&=2\left(\frac{x}{2}\right)^2\left(\frac{x}{2}+2\right)\left(\frac{x}{2}-2\right)\\&=2\times\frac{x^2}{4}\times\left(\frac{x+4}{2}\right)\times\left(\frac{x-4}{2}\right)\\&=\frac{1}{8}x^2(x+4)(x-4)\end{aligned}$$

The equation of the image is $y=\frac{1}{8}x^2(x+4)(x-4)$.

17 a $g:R\to R, g(x)=x^2-4$
Under a reflection in the y axis, $y=g(x)\to y=g(-x)$.
Therefore, $y=x^2-4\to y=(-x)^2-4=x^2-4$.
Hence the function g is its own image under this reflection.
It is symmetric about the y axis.

b $f:R\to R, f(x)=x^{\frac{1}{3}}$
Under a reflection in the x axis, $y=f(x)\to y=-f(x)$.
Therefore, $y=x^{\frac{1}{3}}\to y=-x^{\frac{1}{3}}$.
Under a reflection in the y axis, $y=f(x)\to y=f(-x)$.
Therefore, $y=x^{\frac{1}{3}}\to y=(-x)^{\frac{1}{3}}$

$$\begin{aligned}y&=(-x)^{\frac{1}{3}}\\&=(-1)^{\frac{1}{3}}x^{\frac{1}{3}}\\&=-x^{\frac{1}{3}}\end{aligned}$$

The image under reflection in either axis is the same, $y=-x^{\frac{1}{3}}$.

c $h:[-3,3]\to R, h(x)=-\sqrt{9-x^2}$
Under a reflection in the x axis, $y=h(x)\to y=-h(x)$.
Therefore, $y=-\sqrt{9-x^2}\to y=\sqrt{9-x^2}$.
The function h is the lower semicircle, centre $(0,0)$, radius 3. After reflection in the x axis its image is the upper semicircle.
To return the curve back to its original position, reflect in the x axis again.

d $y=(x-2)^2+5$ has a minimum turning point with co-ordinates $(2,5)$.
Under a reflection in the x axis, $(2,5)\to(2,-5)$ and becomes a maximum turning point.
Under reflection in the y axis, $(2,-5)\to(-2,-5)$ and stays as a maximum turning point.
If the reflections were reversed, under reflection in the y axis $(2,5)\to(-2,5)$ and stays as a minimum turning point; then under reflection in the x axis $(-2,5)\to(-2,-5)$ and becomes a maximum turning point.
The image has a maximum turning point with co-ordinates $(-2,-5)$.

e The transformations: vertical translation down 2 units followed by reflection in the y axis, are 'undone' by the inverse transformations: reflection in the y axis followed by vertical translation up 2 units.
Applying the inverse transformations to the image with equation $y^2=(x-3)$ gives
$y^2=(x-3)\to y^2=(-x-3)\to(y-2)^2=(-x-3)$.
The original equation was $(y-2)^2=-(x+3)$
Check: Vertex $(-3,2)\to(-3,0)\to(3,0)$ which agrees with the vertex of the image.

f The transformations applied to $y=f(x)$ are dilation of factor 2 from the x axis, then vertical translation 1 unit upwards and then reflection in the x axis.
$\therefore y=f(x)\to y=2f(x)\to y=2f(x)+1\to y=-2f(x)-1$.
Hence, $-2f(x)-1=6(x-2)^3-1$ since the image has the equation $y=6(x-2)^3-1$.

$$\begin{aligned}&\therefore -2f(x)=6(x-2)^3\\&\quad\therefore f(x)=-3(x-2)^3\end{aligned}$$

Alternatively, apply the inverse transformations to the image. The inverse transformations are: reflection in the x axis, then vertical translation 1 unit down and then dilation of factor $\frac{1}{2}$ from the x axis.

$$\begin{aligned}&y=6(x-2)^3-1\\\to\ &y=-6(x-2)^3+1\\\to\ &y=-6(x-2)^3\\\to\ &y=\frac{1}{2}\times-6(x-2)^3\\\therefore\ &y=-3(x-2)^3\end{aligned}$$

18 **a** The graph of $y = -g(2x)$ is obtained by reflection in the x axis and dilation of factor $\frac{1}{2}$ from the y axis.

Images of key points:

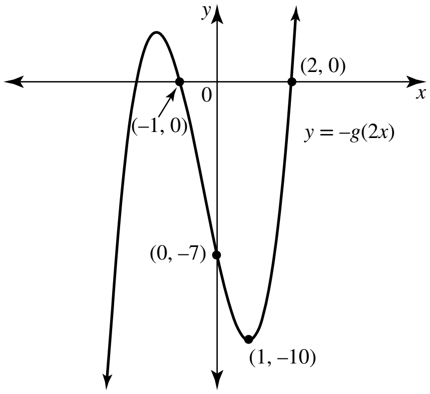
$(-4,0) \rightarrow (-4,0) \rightarrow (-2,0)$

$(-2,0) \rightarrow (-2,0) \rightarrow (-1,0)$

$(0,7) \rightarrow (0,-7) \rightarrow (0,-7)$

$(2,10) \rightarrow (2,-10) \rightarrow (1,-10)$

$(4,0) \rightarrow (4,0) \rightarrow (2,0)$



b $y = g(2-x) \Rightarrow y = g(-(x-2))$

The graph of this function is obtained by reflection in the y axis followed by a horizontal translation of 2 units to the right.

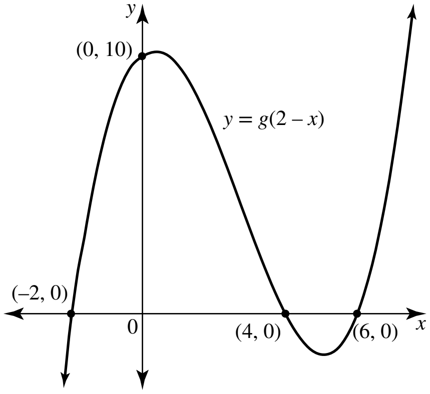
$(-4,0) \rightarrow (-4,0) \rightarrow (-2,0)$

$(-2,0) \rightarrow (2,0) \rightarrow (4,0)$

$(0,7) \rightarrow (0,7) \rightarrow (2,7)$

$(2,10) \rightarrow (-2,10) \rightarrow (0,10)$

$(4,0) \rightarrow (-4,0) \rightarrow (-2,0)$



c If the graph of $y = g(x)$ is shifted more than 4 units horizontally to the left, all three of its x intercepts will be negative.

$y = g(x+h)$ is a horizontal translation of h units to the left.

The required values of h are $h > 4$.

d The given graph has x intercepts at $x = -4, x = -2$ and $x = 4$. Therefore, $(x+4)(x+2)(x-4)$ are factors.

Let the equation be $y = a(x+4)(x+2)(x-4)$

Substitute the point $(0,7)$

$\therefore 7 = a(4)(2)(-4)$

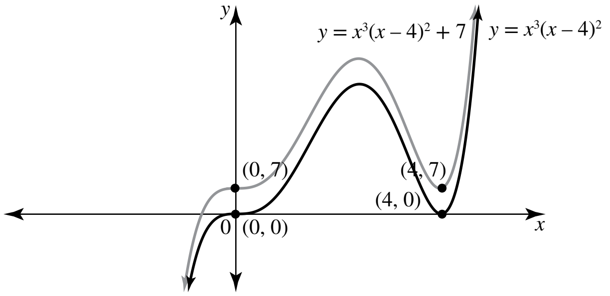
$\therefore 7 = -32a$

$\therefore a = -\frac{7}{32}$

The equation is $g(x) = -\frac{7}{32}(x+4)(x+2)(x-4)$.

$$\therefore g(2x) = -\frac{7}{32}(2x+4)(2x+2)(2x-4)$$
$$= -\frac{7}{32} \times 2(x+2) \times 2(x+1) \times 2(x-2)$$
$$\therefore g(2x) = -\frac{7}{4}(x+2)(x+1)(x-2)$$

e



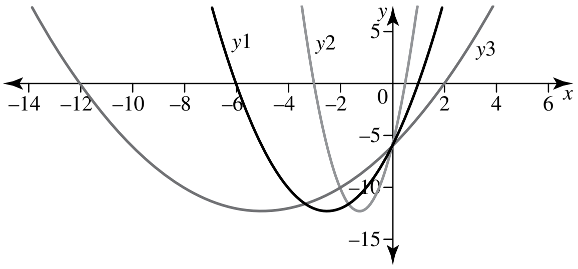
19 Enter the equations in the Graph&Tab editor

$y1 = x^2 + 5x - 6$

$y2 = (2x)^2 + 5(2x) - 6$

$y3 = \left(\frac{x}{2}\right)^2 + 5\left(\frac{x}{2}\right) - 6$

All three are parabolas with y_2 and y_3 dilations of the parabola y_1; y_2 is a dilation of factor $\frac{1}{2}$ from the y axis and y_3 is a dilation of factor 2 from the y axis.



20 $f(x) = \sqrt{16-x^2}$ is the rule for an upper semicircle, centre $(0,0)$, radius 4.

Define $f(x) = \sqrt{16-x^2}$ in the main menu and then enter the equations in the Graph&Tab editor as

$y_1 = f(x)$

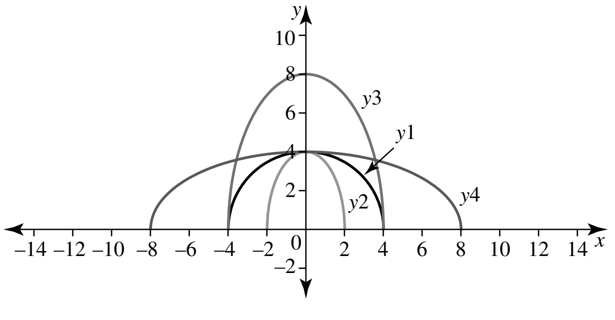
$y_2 = f(2x)$

$y_3 = 2f(x)$

$y_4 = f\left(\frac{x}{2}\right)$

y_2, y_3 and y_4 are dilations of the semicircle y_1.

y_2 is a dilation of factor $\frac{1}{2}$ from the y axis, y_3 is a dilation of factor 2 from the x axis and y_4 is a dilation of factor 2 from the y axis. The dilations produce images which are not semicircles. Only y_1 is a semicircle. The others are parts of ellipses.



Topic 7 — Matrices and applications to transformations

Exercise 7.2 — Addition, subtraction and scalar multiplication of matrices

1 Set up a 1×3 matrix for each player and add them to determine the total number of kicks, marks and handballs.

$$P1 = \begin{bmatrix} 25 & 8 & 10 \end{bmatrix}$$
$$P2 = \begin{bmatrix} 20 & 6 & 8 \end{bmatrix}$$
$$P3 = \begin{bmatrix} 18 & 5 & 7 \end{bmatrix}$$

$$\begin{aligned} P1 + P2 + P3 &= \begin{bmatrix} 25+20+18 & 8+6+5 & 10+8+7 \end{bmatrix} \\ &= \begin{bmatrix} 63 & 19 & 25 \end{bmatrix} \end{aligned}$$

Therefore a total of 63 kicks, 8 marks and 10 handballs.

2 Set up a 2×2 matrix for each player and add them to determine the total number of aces, double faults, forehand winners and backhand winners.

$$P1 = \begin{bmatrix} 2 & 3 \\ 25 & 10 \end{bmatrix}$$
$$P2 = \begin{bmatrix} 4 & 5 \\ 28 & 7 \end{bmatrix}$$

$$\begin{aligned} P1 + P2 &= \begin{bmatrix} 2+4 & 3+5 \\ 25+28 & 10+7 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 8 \\ 53 & 17 \end{bmatrix} \end{aligned}$$

Therefore a total of 6 aces, 8 double faults, 53 forehand winners and 17 backhand winners.

3
$$xA + 2B = C$$
$$x\begin{bmatrix} -3 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$$
$$\begin{bmatrix} -3x \\ 4x \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$$
$$\begin{bmatrix} -3x+8 \\ 4x+10 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$$

Taking the equations out of the matrices gives:

$-3x + 8 = 2$

$4x + 10 = y$

Solve equations simultaneously.

$$\begin{aligned} -3x + 8 &= 2 \\ -3x + 8 - 8 &= 2 - 8 \\ -3x &= -6 \\ \frac{-3x}{-3} &= \frac{-6}{-3} \\ x &= 2 \end{aligned}$$

$$\begin{aligned} 4x + 10 &= y \\ 4 \times 2 + 10 &= y \\ 8 + 10 &= y \\ y &= 18 \end{aligned}$$

4
$$xA - 2B = C$$
$$x\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ y \\ z \end{bmatrix}$$
$$\begin{bmatrix} 4x \\ -2x \\ 3x \end{bmatrix} + \begin{bmatrix} -6 \\ -10 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ y \\ z \end{bmatrix}$$
$$\begin{bmatrix} 4x-6 \\ -2x-10 \\ 3x+2 \end{bmatrix} = \begin{bmatrix} 6 \\ y \\ z \end{bmatrix}$$

Taking the equations out of the matrices gives:

$4x - 6 = 6$

$-2x - 10 = y$

$3x + 2 = z$

Solve equations simultaneously.

$$\begin{aligned} 4x - 6 &= 6 \\ 4x - 6 &= 6 = 6 + 6 \\ 4x &= 12 \\ \frac{4x}{4} &= \frac{12}{4} \\ x &= 3 \end{aligned}$$

$$\begin{aligned} -2x - 10 &= y \\ -2 \times 3 - 10 &= y \\ -6 - 10 &= y \\ y &= -16 \end{aligned}$$

$$\begin{aligned} 3x + 2 &= z \\ 3 \times 3 + 2 &= z \\ 9 + 2 &= z \\ z &= 11 \end{aligned}$$

5 **a** $X = 3A - 2B$

$$\begin{aligned} &= 3\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} - 2\begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 12 \\ 9 & 15 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ 2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} -6-4 & 12-8 \\ 9+2 & 15+6 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 4 \\ 11 & 21 \end{bmatrix} \end{aligned}$$

b $2A + X = O$

$$\begin{aligned} X &= O - 2A \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - 2\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & -8 \\ -6 & -10 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -8 \\ -6 & -10 \end{bmatrix} \end{aligned}$$

c $X = B - 3A + 2I$

$$\begin{aligned}
&= \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} - 3\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} + \begin{bmatrix} 6 & -12 \\ -9 & -15 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2+6+2 & 4-12+0 \\ -1-9+0 & -3-15+2 \end{bmatrix} \\
&= \begin{bmatrix} 10 & -8 \\ -10 & -16 \end{bmatrix}
\end{aligned}$$

6 a $A + 2I - 2B = O$

$$\begin{aligned}
A &= O - 2I + 2B \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ -2 & -6 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 0-2+4 & 0+0+8 \\ 0+0-2 & 0-2-6 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 2 & 8 \\ -2 & -8 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 2 & 8 \\ -2 & -8 \end{bmatrix}
\end{aligned}$$

$a = 2, b = 8, c = -2, d = -8$

b $3I + 4B - 2A = O$

$$\begin{aligned}
3I + 4B &= O + 2A \\
2A &= 3I + 4B \\
2\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \\
2\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 16 \\ -4 & -12 \end{bmatrix} \\
2\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 3+8 & 0+16 \\ 0-4 & 3-12 \end{bmatrix} \\
2\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 11 & 16 \\ -4 & -9 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 11 & 16 \\ -4 & -9 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} \frac{11}{2} & \frac{16}{2} \\ \frac{-4}{2} & \frac{-9}{2} \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} \frac{11}{2} & 8 \\ -2 & \frac{-9}{2} \end{bmatrix}
\end{aligned}$$

$\therefore a = \frac{11}{2}, b = 8, c = -2, d = -\frac{9}{2}$

7 a $C = A + B$

$$\begin{aligned}
&= \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} -1+3 \\ 2+5 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 7 \end{bmatrix}
\end{aligned}$$

b $A + C = B$

$$\begin{aligned}
C &= B - A \\
&= \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 3+1 \\ 5-2 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 3 \end{bmatrix}
\end{aligned}$$

c $3A + 2C = 4B$

$$\begin{aligned}
2C &= 4B - 3A \\
2C &= 4\begin{bmatrix} 3 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
2C &= \begin{bmatrix} 12 \\ 20 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix} \\
2C &= \begin{bmatrix} 12+3 \\ 20-6 \end{bmatrix} \\
2C &= \begin{bmatrix} 15 \\ 14 \end{bmatrix} \\
C &= \frac{1}{2}\begin{bmatrix} 15 \\ 14 \end{bmatrix}
\end{aligned}$$

8 a $C = A + B$

$$\begin{aligned}
&= \begin{bmatrix} 1 & -2 \end{bmatrix} + \begin{bmatrix} 3 & -5 \end{bmatrix} \\
&= \begin{bmatrix} 1+3 & -2-5 \end{bmatrix} \\
&= \begin{bmatrix} 4 & -7 \end{bmatrix}
\end{aligned}$$

b $A + C = B$

$$\begin{aligned}
C &= B - A \\
&= \begin{bmatrix} 3 & -5 \end{bmatrix} - \begin{bmatrix} 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 3-1 & -5+2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -3 \end{bmatrix}
\end{aligned}$$

c $3A + 2C = 4B$

$$\begin{aligned}
2C &= 4B - 3A \\
2C &= 4\begin{bmatrix} 3 & -5 \end{bmatrix} - 3\begin{bmatrix} 1 & -2 \end{bmatrix} \\
2C &= \begin{bmatrix} 12 & -20 \end{bmatrix} + \begin{bmatrix} -3 & 6 \end{bmatrix} \\
2C &= \begin{bmatrix} 12-3 & -20+6 \end{bmatrix} \\
2C &= \begin{bmatrix} 9 & -14 \end{bmatrix} \\
C &= \frac{1}{2}\begin{bmatrix} 9 & -14 \end{bmatrix} \\
C &= \begin{bmatrix} \frac{9}{2} & -7 \end{bmatrix}
\end{aligned}$$

9 a i $B + C = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$

$$\begin{aligned}
&= \begin{bmatrix} 4+1 & 5-2 \\ 2+5 & -3+4 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 3 \\ 7 & 1 \end{bmatrix}
\end{aligned}$$

ii $A + B = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$

$$\begin{aligned}
&= \begin{bmatrix} 2+4 & 3+5 \\ -1+2 & 4-3 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

b $\text{LHS} = A + (B + C)$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \left(\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+5 & 3+3 \\ -1+7 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 6 & 5 \end{bmatrix}$$

$\text{RHS} = (A + B) + C$

$$= \left(\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix} \right) + \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+1 & 8-2 \\ 1+5 & 1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 6 & 5 \end{bmatrix}$$

$\text{LHS} = \text{RHS}$

$\therefore A + (B + C) = (A + B) + C$

Therefore the associate law holds for matrix addition.

10 **a** $3A = C - 2B$

$C = 3A + 2B$

$$C = 3\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} + 2\begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 12 \\ -9 & 6 \end{bmatrix} + \begin{bmatrix} 8 & -4 \\ 6 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 \\ -3 & 16 \end{bmatrix}$$

b $C + 3A - 2B = O$

$C = O - 3A + 2B$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} + 2\begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -12 \\ 9 & -6 \end{bmatrix} + \begin{bmatrix} 8 & -4 \\ 6 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0-3+8 & 0-12-4 \\ 0+9+6 & 0-6+10 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -16 \\ 15 & 4 \end{bmatrix}$$

c $2C + 3A - 2B = O$

$2C = O - 3A + 2B$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} + 2\begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -12 \\ 9 & -6 \end{bmatrix} + \begin{bmatrix} 8 & -4 \\ 6 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0-3+8 & 0-12-4 \\ 0+9+6 & 0-6+10 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -16 \\ 15 & 4 \end{bmatrix}$$

$$C = \frac{1}{2}\begin{bmatrix} 5 & -16 \\ 15 & 4 \end{bmatrix}$$

11 **a** $3A+C-2B+4I=O$

$$\begin{aligned}
C&=O-3A+2B-4I\\
&=\begin{bmatrix}0&0\\0&0\end{bmatrix}-3\begin{bmatrix}1&4\\-3&2\end{bmatrix}+2\begin{bmatrix}4&-2\\3&5\end{bmatrix}-4\begin{bmatrix}1&0\\0&1\end{bmatrix}\\
&=\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}-3&-12\\9&-6\end{bmatrix}+\begin{bmatrix}8&-4\\6&10\end{bmatrix}+\begin{bmatrix}-4&0\\0&-4\end{bmatrix}\\
&=\begin{bmatrix}0-3+8-4&0-12-4-0\\0+9+6-0&0-6+10-4\end{bmatrix}\\
&=\begin{bmatrix}1&-16\\15&0\end{bmatrix}
\end{aligned}$$

b $4A-C+3B-2I=O$

$$\begin{aligned}
-C&=O-4A-3B+2I\\
C&=-O+4A+3B-2I\\
&=-\begin{bmatrix}0&0\\0&0\end{bmatrix}+4\begin{bmatrix}1&4\\-3&2\end{bmatrix}+3\begin{bmatrix}4&-2\\3&5\end{bmatrix}-2\begin{bmatrix}1&0\\0&1\end{bmatrix}\\
&=\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}4&16\\-12&8\end{bmatrix}+\begin{bmatrix}12&-6\\9&15\end{bmatrix}+\begin{bmatrix}-2&0\\0&-2\end{bmatrix}\\
&=\begin{bmatrix}0+4+12-2&0+16-6+0\\0-12+9+0&0+8+15-2\end{bmatrix}\\
&=\begin{bmatrix}14&10\\-3&21\end{bmatrix}
\end{aligned}$$

12 **a**

$$\begin{aligned}
A+B&=\begin{bmatrix}7&4\\9&2\end{bmatrix}\\
\begin{bmatrix}x&-3\\2&x\end{bmatrix}+\begin{bmatrix}2&y\\y&-3\end{bmatrix}&=\begin{bmatrix}7&4\\9&2\end{bmatrix}\\
\begin{bmatrix}x+2&-3+y\\2+y&x-3\end{bmatrix}&=\begin{bmatrix}7&4\\9&2\end{bmatrix}
\end{aligned}$$

Taking the equations out of the matrices gives:

$x+2=7$ (1)
$2+y=9$ (2)
$-3+y=4$ (3)
$x-3=2$ (4)

Only need to solve equations (1) and (2) to find values for x and y

$$\begin{aligned}
x+2&=7\\
x+2-2&=7-2\\
x&=5
\end{aligned}$$

$$\begin{aligned}
2+y&=9\\
2+y-2&=9-2\\
y&=7
\end{aligned}$$

$\therefore x=5,\ y=7$

b

$$\begin{aligned}
A-B&=\begin{bmatrix}2&-9\\-4&7\end{bmatrix}\\
\begin{bmatrix}x&-3\\2&x\end{bmatrix}-\begin{bmatrix}2&y\\y&-3\end{bmatrix}&=\begin{bmatrix}2&-9\\-4&7\end{bmatrix}\\
\begin{bmatrix}x-2&-3-y\\2-y&x+3\end{bmatrix}&=\begin{bmatrix}2&-9\\-4&7\end{bmatrix}
\end{aligned}$$

Taking the equations out of the matrices gives:

$x-2=2$ (1)
$2-y=-4$ (2)
$-3-y=-9$ (3)
$x+3=7$ (4)

Only need to solve equations (1) and (2) to find values for x and y

$$\begin{aligned} x-2&=2\\ x-2+2&=2+2\\ x&=4 \end{aligned}$$

$$\begin{aligned} 2-y&=-4\\ 2-y-2&=-4-2\\ -y&=-6\\ y&=6 \end{aligned}$$

$\therefore x=4,\ y=6$

c $$B-A=\begin{bmatrix} -1 & 1\\ -4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & y\\ y & -3 \end{bmatrix}-\begin{bmatrix} x & -3\\ 2 & x \end{bmatrix}=\begin{bmatrix} -1 & 1\\ -4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2-x & y+3\\ y-2 & -3-x \end{bmatrix}=\begin{bmatrix} -1 & 1\\ -4 & 6 \end{bmatrix}$$

Taking the equations out of the matrices gives:

$2-x=-1$ (1)
$y-2=-4$ (2)
$y+3=1$ (3)
$-3-x=6$ (4)

Only need to solve equations (1) and (2) to find values for x and y

$$\begin{aligned} 2-x&=-1\\ 2-x-2&=-1-2\\ -x&=-3\\ x&=3 \end{aligned}$$

$$\begin{aligned} y-2&=-4\\ y-2+2&=-4+2\\ y&=-2 \end{aligned}$$

$\therefore x=3,\ y=-2$

13 a $C=D+E$

$$\begin{aligned} &=\begin{bmatrix} 1 & 4\\ -3 & 2\\ 2 & 5 \end{bmatrix}+\begin{bmatrix} 2 & -2\\ -1 & 5\\ 3 & -3 \end{bmatrix}\\ &=\begin{bmatrix} 1+2 & 4-2\\ -3-1 & 2+5\\ 2+3 & 5-3 \end{bmatrix}\\ &=\begin{bmatrix} 3 & 2\\ -4 & 7\\ 5 & 2 \end{bmatrix} \end{aligned}$$

b $D+C=E$

$$\begin{aligned} C&=E-D\\ &=\begin{bmatrix} 2 & -2\\ -1 & 5\\ 3 & -3 \end{bmatrix}-\begin{bmatrix} 1 & 4\\ -3 & 2\\ 2 & 5 \end{bmatrix}\\ &=\begin{bmatrix} 2-1 & -2-4\\ -1+3 & 5-2\\ 3-2 & -3-5 \end{bmatrix}\\ &=\begin{bmatrix} 1 & -6\\ 2 & 3\\ 1 & -8 \end{bmatrix} \end{aligned}$$

c $3D+2C=4E$

$$\begin{aligned} 2C&=4E-3D\\ &=4\begin{bmatrix} 2 & -2\\ -1 & 5\\ 3 & -3 \end{bmatrix}-3\begin{bmatrix} 1 & 4\\ -3 & 2\\ 2 & 5 \end{bmatrix}\\ &=\begin{bmatrix} 8 & -8\\ -4 & 20\\ 12 & -12 \end{bmatrix}+\begin{bmatrix} -3 & -12\\ 9 & -6\\ -6 & -15 \end{bmatrix}\\ &=\begin{bmatrix} 5 & -20\\ 5 & 14\\ 6 & -27 \end{bmatrix}\\ C&=\frac{1}{2}\begin{bmatrix} 5 & -20\\ 5 & 14\\ 6 & -27 \end{bmatrix} \end{aligned}$$

14 a $C=D+E$

$$\begin{aligned} &=\begin{bmatrix} 1 & 4 & 5\\ -3 & 2 & -2 \end{bmatrix}+\begin{bmatrix} 2 & -2 & 4\\ 1 & 4 & -3 \end{bmatrix}\\ &=\begin{bmatrix} 1+2 & 4-2 & 5+4\\ -3+1 & 2+4 & -2-3 \end{bmatrix}\\ &=\begin{bmatrix} 3 & 2 & 9\\ -2 & 6 & -5 \end{bmatrix} \end{aligned}$$

b $D+C=E$

$$\begin{aligned} C&=E-D\\ &=\begin{bmatrix} 2 & -2 & 4\\ 1 & 4 & -3 \end{bmatrix}-\begin{bmatrix} 1 & 4 & 5\\ -3 & 2 & -2 \end{bmatrix}\\ &=\begin{bmatrix} 2-1 & -2-4 & 4-5\\ 1+3 & 4-2 & -3+2 \end{bmatrix}\\ &=\begin{bmatrix} 1 & -6 & -1\\ 4 & 2 & -1 \end{bmatrix} \end{aligned}$$

c $3D+2C=4E$

$$\begin{aligned} 2C&=4E-3D\\ &=4\begin{bmatrix} 2 & -2 & 4\\ 1 & 4 & -3 \end{bmatrix}-3\begin{bmatrix} 1 & 4 & 5\\ -3 & 2 & -2 \end{bmatrix}\\ &=\begin{bmatrix} 8 & -8 & 16\\ 4 & 16 & -12 \end{bmatrix}+\begin{bmatrix} -3 & -12 & -15\\ 9 & -6 & 6 \end{bmatrix}\\ &=\begin{bmatrix} 8-3 & -8-12 & 16-15\\ 4+9 & 16-6 & -12+6 \end{bmatrix}\\ &=\begin{bmatrix} 5 & -20 & 1\\ 13 & 10 & -6 \end{bmatrix}\\ C&=\frac{1}{2}\begin{bmatrix} 5 & -20 & 1\\ 13 & 10 & -6 \end{bmatrix} \end{aligned}$$

15 a Given $A=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{bmatrix}$, for the given matrix

$$A=\begin{bmatrix} 2 & 3\\ -1 & 4 \end{bmatrix}$$

$a_{11}=2$
$a_{12}=3$
$a_{21}=-1$
$a_{22}=4$

b If $a_{11}=3,\ a_{12}=-2,\ a_{21}=-3$ and $a_{22}=5$

Since $A=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{bmatrix}$, then $A=\begin{bmatrix} 3 & -2\\ -3 & 5 \end{bmatrix}$

16 a $a_{ij}=2i-j$ for $j\neq i$
$a_{ij}=ij$ for $j=i$
For a_{11}: $i=1,\ j=1$ so $j=i$.

Therefore

$$\begin{aligned} a_{11} &= ij \\ &= 1\times 1 \\ &= 1 \end{aligned}$$

For a_{12}: $i = 1, j = 2$ so $j \neq i$.

Therefore

$$\begin{aligned} a_{12} &= 2i - j \\ &= 2\times 1 - 2 \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

For a_{21}: $i = 2, j = 1$ so $j \neq i$.

Therefore

$$\begin{aligned} a_{21} &= 2i - j \\ &= 2\times 2 - 1 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

For a_{22}: $i = 2,\ j = 2$ so $j = i$.

Therefore

$$\begin{aligned} a_{22} &= ij \\ &= 2\times 2 \\ &= 4 \end{aligned}$$

So, the 2×2 matrix A is

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} \end{aligned}$$

b $a_{ij} = i + j$ for $i < j$

$a_{ij} = i - j + 1$ for $i > j$

$a_{ij} = i + j + 1$ for $i = j$

For a_{11}: $i = 1,\ j = 1$ so $i = j$.

Therefore

$$\begin{aligned} a_{11} &= i + j + 1 \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

For a_{12}: $i = 1, j = 2$ so $i < j$.

Therefore

$$\begin{aligned} a_{12} &= i + j \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

For a_{21}: $i = 2,\ j = 1$ so $i > j$.

Therefore

$$\begin{aligned} a_{21} &= i - j + 1 \\ &= 2 - 1 + 1 \\ &= 2 \end{aligned}$$

For a_{22}: $i = 2,\ j = 2$ so $i = j$.

Therefore

$$\begin{aligned} a_{22} &= i + j + 1 \\ &= 2 + 2 + 1 \\ &= 5 \end{aligned}$$

So, the 2×2 matrix A is

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 2 & 5 \end{bmatrix} \end{aligned}$$

17 a i
$$\begin{aligned} tr(A) &= a_{11} + a_{22} \\ &= 2 + 4 \\ &= 6 \end{aligned}$$

ii
$$\begin{aligned} tr(B) &= b_{11} + b_{22} \\ &= 4 + 5 \\ &= 9 \end{aligned}$$

iii
$$\begin{aligned} tr(C) &= c_{11} + c_{22} \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

b
$$\begin{aligned} \text{LHS} &= tr(A + B + C) \\ &= tr\left(\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}\right) \\ &= tr\left(\begin{bmatrix} 2+4+1 & 3-2-2 \\ -1+3+5 & 4+5+4 \end{bmatrix}\right) \\ &= tr\left(\begin{bmatrix} 7 & -1 \\ 7 & 13 \end{bmatrix}\right) \\ &= 7 + 13 \\ &= 20 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= tr(A) + tr(B) + tr(C) \\ &= 6 + 9 + 5 \\ &= 20 \end{aligned}$$

LHS = RHS

So $tr(A + B + C) = tr(A) + tr(B) + tr(C)$

c
$$\begin{aligned} \text{LHS} &= tr(2A + 3B - 4C) \\ &= tr\left(2\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + 3\begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}\right) \\ &= tr\left(\begin{bmatrix} 4 & 6 \\ -2 & 8 \end{bmatrix} + \begin{bmatrix} 12 & -6 \\ 9 & 15 \end{bmatrix} + \begin{bmatrix} -4 & 8 \\ -20 & -16 \end{bmatrix}\right) \\ &= tr\left(\begin{bmatrix} 4+12-4 & 6-6+8 \\ -2+9-20 & 8+15-16 \end{bmatrix}\right) \\ &= tr\left(\begin{bmatrix} 12 & 8 \\ -1 & 7 \end{bmatrix}\right) \\ &= 12 + 7 \\ &= 19 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 2tr(A) + 3tr(B) - 4tr(C) \\ &= 2\times 6 + 3\times 9 - 4\times 5 \\ &= 12 + 27 - 20 \\ &= 19 \end{aligned}$$

LHS = RHS

So $tr(2A + 3B - 4C) = 2tr(A) + 3tr(B) - 4tr(C)$

18 After multiplying matrix A and matrix B together on the calculator the matrix obtained is:

$$AB = \begin{bmatrix} 216 \\ 164 \\ 274 \end{bmatrix}$$

Therefore since this resultant matrix has 3 rows and 1 column it is a 3×1 matrix.

Exercise 7.3 — Matrix multiplication

1 a
$$\begin{aligned} AX &= \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -2\times -3 + 4\times 2 \\ 3\times -3 + 5\times 2 \end{bmatrix} \\ &= \begin{bmatrix} 14 \\ 1 \end{bmatrix} \end{aligned}$$

b A is a 2×2 matrix and X is a 2×1 matrix.

XA does not exist since the number of columns of X is not equal to the number of rows in A

2 a $$\begin{aligned} AX &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} a\times x+b\times y \\ c\times x+d\times y \end{bmatrix} \\ &= \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \end{aligned}$$

b A is a 2×2 matrix and X is a 2×1 matrix.
XA does not exist since the number of columns of X is not equal to the number of rows in A

3 a $$\begin{aligned} AB &= \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -2\times 2+4\times -1 & -2\times 4+4\times -3 \\ 3\times 2+5\times -1 & 3\times 4+5\times -3 \end{bmatrix} \\ &= \begin{bmatrix} -8 & -20 \\ 1 & -3 \end{bmatrix} \end{aligned}$$

b $$\begin{aligned} BA &= \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2\times -2+4\times 3 & 2\times 4+4\times 5 \\ -1\times -2-3\times 3 & -1\times 4-3\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 28 \\ -7 & -19 \end{bmatrix} \end{aligned}$$

c $$\begin{aligned} AA &= \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -2\times -2+4\times 3 & -2\times 4+4\times 5 \\ 3\times -2+5\times 3 & 3\times 4+5\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix} \end{aligned}$$

4 a $$\begin{aligned} AO &= \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2\times 0+4\times 0 & -2\times 0+4\times 0 \\ 3\times 0+5\times 0 & 3\times 0+5\times 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$
$\therefore AO = O$

b $$\begin{aligned} OA &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0\times -2+0\times 3 & 0\times 4+0\times 5 \\ 0\times -2+0\times 3 & 0\times 4+0\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$
$\therefore OA = O$

c $$\begin{aligned} AI &= \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2\times 1+4\times 0 & -2\times 0+4\times 1 \\ 3\times 1+5\times 0 & 3\times 0+5\times 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \end{aligned}$$
$\therefore AI = A$

d $$\begin{aligned} IA &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1\times -2+0\times 3 & 1\times 4+0\times 5 \\ 0\times -2+1\times 3 & 0\times 4+1\times 5 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \end{aligned}$$
$\therefore IA = A$

5 a $$\begin{aligned} DE &= \begin{bmatrix} 2 & -1 \\ -3 & 5 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 1-1\times 2 & 2\times 2-1\times -4 & 2\times -3-1\times 5 \\ -3\times 1+5\times 2 & -3\times 2+5\times -4 & -3\times -3+5\times 5 \\ -1\times 1-4\times 2 & -1\times 2-4\times -4 & -1\times -3-4\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 8 & -11 \\ 7 & -26 & 34 \\ -9 & 14 & -17 \end{bmatrix} \end{aligned}$$

b $$\begin{aligned} ED &= \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 5 \\ -1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 2+2\times -3-3\times -1 & 1\times -1+2\times 5-3\times -4 \\ 2\times 2-4\times -3+5\times -1 & 2\times -1-4\times 5+5\times -4 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 21 \\ 11 & -42 \end{bmatrix} \end{aligned}$$

6 a $$\begin{aligned} CD &= \begin{bmatrix} 1 \\ -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 3 & 1\times -2 \\ -2\times 3 & -2\times -2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix} \end{aligned}$$

b $$\begin{aligned} DC &= \begin{bmatrix} 3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= [3\times 1-2\times -2] \\ &= [7] \end{aligned}$$

7 a i $$\begin{aligned} AI &= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 1+3\times 0 & 2\times 0+3\times 1 \\ -1\times 1+4\times 0 & -1\times 0+4\times 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \\ &= A \end{aligned}$$

$$\begin{aligned} IA &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 2+0\times 3 & 1\times 3+0\times 4 \\ 0\times 2+1\times -1 & 0\times 3+1\times 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \\ &= A \end{aligned}$$
$\therefore AI = IA = A$

ii $$\begin{aligned} AO &= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 0+3\times 0 & 2\times 0+3\times 0 \\ -1\times 0+4\times 0 & -1\times 0+4\times 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= O \end{aligned}$$

$$\begin{aligned} OA &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0\times 2+0\times -1 & 0\times 3+0\times 4 \\ 0\times 2+0\times -1 & 0\times 3+0\times 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= O \end{aligned}$$
$\therefore AO = OA = O$

b **i** $AI=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$=\begin{bmatrix} a\times1+b\times0 & a\times0+b\times1 \\ c\times1+d\times0 & c\times0+d\times1 \end{bmatrix}$

$=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$=A$

$IA=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$=\begin{bmatrix} 1\times a+0\times c & 1\times b+0\times d \\ 0\times a+1\times c & 0\times b+1\times d \end{bmatrix}$

$=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$=A$

$\therefore AI=IA=A$

ii $AO=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$=\begin{bmatrix} a\times0+b\times0 & a\times0+b\times0 \\ c\times0+d\times0 & c\times0+d\times0 \end{bmatrix}$

$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$=O$

$OA=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$=\begin{bmatrix} 0\times a+0\times c & 0\times b+0\times d \\ 0\times a+0\times c & 0\times b+0\times d \end{bmatrix}$

$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$=O$

$\therefore AO=OA=O$

8 **a** $\text{LHS}=(I-A)(I+A)$

$=\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\right)$

$=\left(\begin{bmatrix} 1-2 & 0-3 \\ 0+1 & 1-4 \end{bmatrix}\right)\left(\begin{bmatrix} 1+2 & 0+3 \\ 0-1 & 1+4 \end{bmatrix}\right)$

$=\begin{bmatrix} -1 & -3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -1 & 5 \end{bmatrix}$

$=\begin{bmatrix} -1\times3-3\times-1 & -1\times3-3\times5 \\ 1\times3-3\times-1 & 1\times3-3\times5 \end{bmatrix}$

$=\begin{bmatrix} 0 & -18 \\ 6 & -12 \end{bmatrix}$

$\text{RHS}=I-A^2$

$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}^2$

$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 2\times2+3\times-1 & 2\times3+3\times4 \\ -1\times2+4\times-1 & -1\times3+4\times4 \end{bmatrix}$

$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 1 & 18 \\ -6 & 13 \end{bmatrix}$

$=\begin{bmatrix} 1-1 & 0-18 \\ 0+6 & 1-13 \end{bmatrix}$

$=\begin{bmatrix} 0 & -18 \\ 6 & -12 \end{bmatrix}$

$\text{LHS}=\text{RHS}$

$\therefore (I-A)(I+A)=I-A^2$

b $$\begin{aligned}AB&=\begin{bmatrix}3&0\\-4&0\end{bmatrix}\begin{bmatrix}0&0\\2&-1\end{bmatrix}\\&=\begin{bmatrix}3\times0+0\times2&3\times0+0\times-1\\-4\times0+0\times2&-4\times0+0\times-1\end{bmatrix}\\&=\begin{bmatrix}0&0\\0&0\end{bmatrix}\\&=O\end{aligned}$$

$$\begin{aligned}BA&=\begin{bmatrix}0&0\\2&-1\end{bmatrix}\begin{bmatrix}3&0\\-4&0\end{bmatrix}\\&=\begin{bmatrix}0\times3+0\times-4&0\times0+0\times0\\2\times3-1\times-4&2\times0-1\times0\end{bmatrix}\\&=\begin{bmatrix}0&0\\10&0\end{bmatrix}\end{aligned}$$

Therefore $AB=O$ but $BA\neq O$

c $$\begin{aligned}AB&=\begin{bmatrix}a&0\\b&0\end{bmatrix}\begin{bmatrix}0&0\\x&y\end{bmatrix}\\&=\begin{bmatrix}a\times0+0\times x&a\times0+0\times y\\b\times0+0\times x&b\times0+0\times y\end{bmatrix}\\&=\begin{bmatrix}0&0\\0&0\end{bmatrix}\\&=O\end{aligned}$$

$$\begin{aligned}BA&=\begin{bmatrix}0&0\\x&y\end{bmatrix}\begin{bmatrix}a&0\\b&0\end{bmatrix}\\&=\begin{bmatrix}0\times a+0\times b&0\times0+0\times0\\x\times a+y\times b&x\times0+y\times0\end{bmatrix}\\&=\begin{bmatrix}0&0\\ax+by&0\end{bmatrix}\end{aligned}$$

Therefore $AB=O$ but $BA\neq O$

9 **a** $$\begin{aligned}\text{LHS}&=A(B+C)\\&=\begin{bmatrix}2&3\\-1&4\end{bmatrix}\left(\begin{bmatrix}4&5\\2&-3\end{bmatrix}+\begin{bmatrix}1&-2\\5&4\end{bmatrix}\right)\\&=\begin{bmatrix}2&3\\-1&4\end{bmatrix}\left(\begin{bmatrix}4+1&5-2\\2+5&-3+4\end{bmatrix}\right)\\&=\begin{bmatrix}2&3\\-1&4\end{bmatrix}\begin{bmatrix}5&3\\7&1\end{bmatrix}\\&=\begin{bmatrix}2\times5+3\times7&2\times3+3\times1\\-1\times5+4\times7&-1\times3+4\times1\end{bmatrix}\\&=\begin{bmatrix}31&9\\23&1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{RHS}&=AB+AC\\&=\begin{bmatrix}2&3\\-1&4\end{bmatrix}\begin{bmatrix}4&5\\2&-3\end{bmatrix}+\begin{bmatrix}2&3\\-1&4\end{bmatrix}\begin{bmatrix}1&-2\\5&4\end{bmatrix}\\&=\begin{bmatrix}2\times4+3\times2&2\times5+3\times-3\\-1\times4+4\times2&-1\times5+4\times-3\end{bmatrix}+\begin{bmatrix}2\times1+3\times5&2\times-2+3\times4\\-1\times1+4\times5&-1\times-2+4\times4\end{bmatrix}\\&=\begin{bmatrix}14&1\\4&-17\end{bmatrix}+\begin{bmatrix}17&8\\19&18\end{bmatrix}\\&=\begin{bmatrix}14+17&1+8\\4+19&-17+18\end{bmatrix}\\&=\begin{bmatrix}31&9\\23&1\end{bmatrix}\end{aligned}$$

LHS = RHS

$\therefore A(B+C)=AB+AC$

b $\text{LHS} = A(BC)$

$$\begin{aligned}
&= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\left(\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\left(\begin{bmatrix} 4\times1+5\times5 & 4\times-2+5\times4 \\ 2\times1-3\times5 & 2\times-2-3\times4 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 29 & 12 \\ -13 & -16 \end{bmatrix} \\
&= \begin{bmatrix} 2\times29+3\times-13 & 2\times12+3\times-16 \\ -1\times29+4\times-13 & -1\times12+4\times-16 \end{bmatrix} \\
&= \begin{bmatrix} 19 & -24 \\ -81 & -76 \end{bmatrix}
\end{aligned}$$

$\text{RHS} = (AB)C$

$$\begin{aligned}
&= \left(\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\right)\begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix} \\
&= \left(\begin{bmatrix} 2\times4+3\times2 & 2\times5+3\times-3 \\ -1\times4+4\times2 & -1\times5+4\times-3 \end{bmatrix}\right)\begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 14 & 1 \\ 4 & -17 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 14\times1+1\times5 & 14\times-2+1\times4 \\ 4\times1-17\times5 & 4\times-2-17\times4 \end{bmatrix} \\
&= \begin{bmatrix} 19 & -24 \\ -81 & -76 \end{bmatrix}
\end{aligned}$$

$\text{LHS} = \text{RHS}$

$\therefore A(BC) = (AB)C$

c $\text{LHS} = (A+B)^2$

$$\begin{aligned}
&= \left(\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\right)^2 \\
&= \left(\begin{bmatrix} 2+4 & 3+5 \\ -1+2 & 4-3 \end{bmatrix}\right)^2 \\
&= \begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix}^2 \\
&= \begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 6\times6+8\times1 & 6\times8+8\times1 \\ 1\times6+1\times1 & 1\times8+1\times1 \end{bmatrix} \\
&= \begin{bmatrix} 44 & 56 \\ 7 & 9 \end{bmatrix}
\end{aligned}$$

$\text{RHS} = A^2 + 2AB + B^2$

$$\begin{aligned}
&= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}^2 + 2\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}+\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}^2 \\
&= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+2\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}+\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix} \\
&= \begin{bmatrix} 2\times2+3\times-1 & 2\times3+3\times4 \\ -1\times2+4\times-1 & -1\times3+4\times4 \end{bmatrix}+2\begin{bmatrix} 2\times4+3\times2 & 2\times5+3\times-3 \\ -1\times4+4\times2 & -1\times5+4\times-3 \end{bmatrix}+\begin{bmatrix} 4\times4+5\times2 & 4\times5+5\times-3 \\ 2\times4-3\times2 & 2\times5-3\times-3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 18 \\ -6 & 13 \end{bmatrix}+2\begin{bmatrix} 14 & 1 \\ 4 & -17 \end{bmatrix}+\begin{bmatrix} 26 & 5 \\ 2 & 19 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 18 \\ -6 & 13 \end{bmatrix}+\begin{bmatrix} 28 & 2 \\ 8 & -34 \end{bmatrix}+\begin{bmatrix} 26 & 5 \\ 2 & 19 \end{bmatrix} \\
&= \begin{bmatrix} 1+28+26 & 18+2+5 \\ -6+8+2 & 13-34+19 \end{bmatrix} \\
&= \begin{bmatrix} 55 & 25 \\ 4 & -2 \end{bmatrix}
\end{aligned}$$

$\text{LHS} \neq \text{RHS}$

Since $AB \neq BA$, $(A+B)^2 \neq A^2 + 2AB + B^2$

When you expand $(A+B)^2$ you get:

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= A^2 + AB + BA + B^2\end{aligned}$$

And since, $AB \neq BA$,

$AB + BA \neq 2AB$

Therefore, $(A+B)^2 \neq A^2 + 2AB + B^2$

d LHS $= (A+B)^2$

$$\begin{aligned}
&= \left(\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\right)^2\\
&= \left(\begin{bmatrix} 2+4 & 3+5 \\ -1+2 & 4-3 \end{bmatrix}\right)^2\\
&= \begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix}^2\\
&= \begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix}\\
&= \begin{bmatrix} 6\times 6+8\times 1 & 6\times 8+8\times 1 \\ 1\times 6+1\times 1 & 1\times 8+1\times 1 \end{bmatrix}\\
&= \begin{bmatrix} 44 & 56 \\ 7 & 9 \end{bmatrix}
\end{aligned}$$

RHS $= A^2 + AB + BA + B^2$

$$\begin{aligned}
&= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}^2 + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}^2\\
&= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\\
&= \begin{bmatrix} 2\times 2+3\times -1 & 2\times 3+3\times 4 \\ -1\times 2+4\times -1 & -1\times 3+4\times 4 \end{bmatrix} + \begin{bmatrix} 2\times 4+3\times 2 & 2\times 5+3\times -3 \\ -1\times 4+4\times 2 & -1\times 5+4\times -3 \end{bmatrix} + \begin{bmatrix} 4\times 2+5\times -1 & 4\times 3+5\times 4 \\ 2\times 2-3\times -1 & 2\times 3-3\times 4 \end{bmatrix}\\
&\quad + \begin{bmatrix} 4\times 4+5\times 2 & 4\times 5+5\times -3 \\ 2\times 4-3\times 2 & 2\times 5-3\times -3 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 18 \\ -6 & 13 \end{bmatrix} + \begin{bmatrix} 14 & 1 \\ 4 & -17 \end{bmatrix} + \begin{bmatrix} 3 & 32 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 26 & 5 \\ 2 & 19 \end{bmatrix}\\
&= \begin{bmatrix} 1+14+3+26 & 18+1+32+5 \\ -6+4+7+2 & 13-17-6+19 \end{bmatrix}\\
&= \begin{bmatrix} 44 & 56 \\ 7 & 9 \end{bmatrix}
\end{aligned}$$

LHS = RHS

$\therefore (A+B)^2 = A^2 + AB + BA + B^2$

10 a

$$\begin{aligned}
AB &= \begin{bmatrix} 3 & 18 \\ 13 & 3 \end{bmatrix}\\
\begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix}\begin{bmatrix} 2 & x \\ x & -3 \end{bmatrix} &= \begin{bmatrix} 3 & 18 \\ 13 & 3 \end{bmatrix}\\
\begin{bmatrix} x\times 2-3\times x & x\times x-3\times -3 \\ 2\times 2+x\times x & 2\times x+x\times -3 \end{bmatrix} &= \begin{bmatrix} 3 & 18 \\ 13 & 3 \end{bmatrix}\\
\begin{bmatrix} -x & x^2+9 \\ x^2+4 & -x \end{bmatrix} &= \begin{bmatrix} 3 & 18 \\ 13 & 3 \end{bmatrix}
\end{aligned}$$

Taking the equations out of the matrices gives:

$-x = 3 \quad (1)$

$x^2 + 9 = 18 \quad (2)$

$x^2 + 4 = 13 \quad (3)$

$-x = 3 \quad (4)$

Only need to solve equation (1) to find a value for x:

$-x = 3$

$x = -3$

b

$$BA = \begin{bmatrix} -16 & 10 \\ 10 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & x \\ x & -3 \end{bmatrix}\begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix} = \begin{bmatrix} -16 & 10 \\ 10 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 2\times x + x\times 2 & 2\times -3 + x\times x \\ x\times x - 3\times 2 & x\times -3 - 3\times x \end{bmatrix} = \begin{bmatrix} -16 & 10 \\ 10 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 4x & x^2-6 \\ x^2-6 & -6x \end{bmatrix} = \begin{bmatrix} -16 & 10 \\ 10 & 24 \end{bmatrix}$$

Taking the equations out of the matrices gives:

$4x = -16 \quad (1)$

$x^2 - 6 = 10 \quad (2)$

$x^2 - 6 = 10 \quad (3)$

$-6x = 24 \quad (4)$

Only need to solve equation (1) to find a value for x:

$4x = -16$

$\frac{4x}{4} = \frac{-16}{4}$

$x = -4$

c

$$A^2 = \begin{bmatrix} -2 & 12 \\ -8 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix}^2 = \begin{bmatrix} -2 & 12 \\ -8 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix}\begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix} = \begin{bmatrix} -2 & 12 \\ -8 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x\times x - 3\times 2 & x\times -3 - 3\times x \\ 2\times x + x\times 2 & 2\times -3 + x\times x \end{bmatrix} = \begin{bmatrix} -2 & 12 \\ -8 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x^2-6 & -6x \\ 4x & x^2-6 \end{bmatrix} = \begin{bmatrix} -2 & 12 \\ -8 & -2 \end{bmatrix}$$

Taking the equations out of the matrices gives:

$x^2 - 6 = -2 \quad (1)$

$4x = -8 \quad (2)$

$-6x = 12 \quad (3)$

$x^2 - 6 = -2 \quad (4)$

Only need to solve equation (2) to find a value for x:

$4x = -8$

$\frac{4x}{4} = \frac{-8}{4}$

$x = -2$

d

$$B^2 = \begin{bmatrix} 8 & -2 \\ -2 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 2 & x \\ x & -3 \end{bmatrix}^2 = \begin{bmatrix} 8 & -2 \\ -2 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 2 & x \\ x & -3 \end{bmatrix}\begin{bmatrix} 2 & x \\ x & -3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -2 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 2\times 2 + x\times x & 2\times x + x\times -3 \\ x\times 2 - 3\times x & x\times x - 3\times -3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -2 & 13 \end{bmatrix}$$

$$\begin{bmatrix} x^2+4 & -x \\ -x & x^2+9 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -2 & 13 \end{bmatrix}$$

Taking the equations out of the matrices gives:

$x^2 + 4 = 8 \quad (1)$

$-x = -2 \quad (2)$

$-x = -2 \quad (3)$

$x^2 + 9 = 13 \quad (4)$

Only need to solve equation (2) to find a value for x:

$-x = -2$

$x = 2$

11 **a** You can only add matrices of the same size. Therefore since A is a 2×1 matrix and B is a 1×2 matrix, $A + B$ does not exist.

b You can only add matrices of the same size. Therefore since A is a 2×1 matrix and C is a 2×2 matrix, $A + C$ does not exist.

c You can only add matrices of the same size. Therefore since B is a 1×2 matrix and C is a 2×2 matrix, $B + C$ does not exist.

d A is a 2×1 matrix and B is a 1×2 matrix. Therefore since the number of columns of A is equal to the number of rows in B, the matrices can be multiplied.

$$\begin{aligned} AB &= \begin{bmatrix} -1 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \end{bmatrix} \\ &= \begin{bmatrix} -1\times 3 & -1\times -5 \\ 2\times 3 & 2\times -5 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 5 \\ 6 & -10 \end{bmatrix} \end{aligned}$$

e B is a 1×2 matrix and A is a 2×1 matrix. Therefore since the number of columns of B is equal to the number of rows in A, the matrices can be multiplied.

$$\begin{aligned} BA &= \begin{bmatrix} 3 & -5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= [3\times -1 - 5\times 2] \\ &= [-13] \end{aligned}$$

f A is a 2×1 matrix and C is a 2×2 matrix. Therefore since the number of columns of A is not equal to the number of rows in C, the matrices cannot be multiplied. Therefore AC does not exist.

g C is a 2×2 matrix and A is a 2×1 matrix. Therefore since the number of columns of C is equal to the number of rows in A, the matrices can be multiplied.

$$\begin{aligned} CA &= \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2\times -1 + 4\times 2 \\ -3\times -1 + 5\times 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 13 \end{bmatrix} \end{aligned}$$

h B is a 1×2 matrix and C is a 2×2 matrix. Therefore since the number of columns of B is equal to the number of rows in C, the matrices can be multiplied.

$$\begin{aligned} BC &= \begin{bmatrix} 3 & -5 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 3\times 2 - 5\times -3 & 3\times 4 - 5\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 21 & -13 \end{bmatrix} \end{aligned}$$

i C is a 2×2 matrix and B is a 1×2 matrix. Therefore since the number of columns of C is not equal to the number of rows in B, the matrices cannot be multiplied. Therefore CB does not exist.

j A is a 2×1 matrix and B is a 1×2 matrix. Therefore since the number of columns of A is equal to the number of rows in B, the matrices can be multiplied. This multiplication will result in a 2×2 matrix. And since C is a 2×2 matrix, ABC can be calculated.

$$\begin{aligned}ABC&=\begin{bmatrix}-1\\2\end{bmatrix}\begin{bmatrix}3&-5\end{bmatrix}\begin{bmatrix}2&4\\-3&5\end{bmatrix}\\&=\begin{bmatrix}-1\times3&-1\times-5\\2\times3&2\times-5\end{bmatrix}\begin{bmatrix}2&4\\-3&5\end{bmatrix}\\&=\begin{bmatrix}-3&5\\6&-10\end{bmatrix}\begin{bmatrix}2&4\\-3&5\end{bmatrix}\\&=\begin{bmatrix}-3\times2+5\times-3&-3\times4+5\times5\\6\times2-10\times-3&6\times4-10\times5\end{bmatrix}\\&=\begin{bmatrix}-21&13\\42&-26\end{bmatrix}\end{aligned}$$

k C is a 2×2 matrix and B is a 1×2 matrix. Therefore since the number of columns of C is not equal to the number of rows in B, the matrices cannot be multiplied. Therefore since CB cannot be multiplied, CBA does not exist.

l C is a 2×2 matrix and A is a 2×1 matrix. Therefore since the number of columns of C is equal to the number of rows in A, the matrices can be multiplied. This multiplication will result in a 2×1 matrix. And since B is a 1×2 matrix, ABC can be calculated.

$$\begin{aligned}CAB&=\begin{bmatrix}2&4\\-3&5\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}\begin{bmatrix}3&-5\end{bmatrix}\\&=\begin{bmatrix}2\times-1+4\times2\\-3\times-1+5\times2\end{bmatrix}\begin{bmatrix}3&-5\end{bmatrix}\\&=\begin{bmatrix}6\\13\end{bmatrix}\begin{bmatrix}3&-5\end{bmatrix}\\&=\begin{bmatrix}6\times3&6\times-5\\13\times3&13\times-5\end{bmatrix}\\&=\begin{bmatrix}18&-30\\39&-65\end{bmatrix}\end{aligned}$$

12 a D is a 3×2 matrix and E is a 2×3 matrix. Therefore since the number of columns of D is equal to the number of rows in E, the matrices can be multiplied.

$$\begin{aligned}DE&=\begin{bmatrix}1&4\\-3&2\\2&5\end{bmatrix}\begin{bmatrix}2&-2&4\\1&4&-3\end{bmatrix}\\&=\begin{bmatrix}1\times2+4\times1&1\times-2+4\times4&1\times4+4\times-3\\-3\times2+2\times1&-3\times-2+2\times4&-3\times4+2\times-3\\2\times2+5\times1&2\times-2+5\times4&2\times4+5\times-3\end{bmatrix}\\&=\begin{bmatrix}6&14&-8\\-4&14&-18\\9&16&-7\end{bmatrix}\end{aligned}$$

b E is a 2×3 matrix and D is a 3×2 matrix. Therefore since the number of columns of E is equal to the number of rows in D, the matrices can be multiplied.

$$\begin{aligned}ED&=\begin{bmatrix}2&-2&4\\1&4&-3\end{bmatrix}\begin{bmatrix}1&4\\-3&2\\2&5\end{bmatrix}\\&=\begin{bmatrix}2\times1-2\times-3+4\times2&2\times4-2\times2+4\times5\\1\times1+4\times-3-3\times2&1\times4+4\times2-3\times5\end{bmatrix}\\&=\begin{bmatrix}16&24\\-17&-3\end{bmatrix}\end{aligned}$$

c You can only add matrices of the same size. Therefore since D is a 3×2 matrix and E is a 2×3 matrix, $E+D$ does not exist.

d D is a 3×2 matrix and D is a 3×2 matrix. Therefore since the number of columns of D is not equal to the number of rows in D (i.e. D is not a square matrix) the matrices cannot be multiplied. Therefore DD or D^2 does not exist.

13 a D is a 2×3 matrix and E is a 3×2 matrix. Therefore since the number of columns of D is equal to the number of rows in E, the matrices can be multiplied.

$$\begin{aligned}DE&=\begin{bmatrix}1&-4&2\\-2&8&-4\end{bmatrix}\begin{bmatrix}6&2\\3&-1\\3&-3\end{bmatrix}\\&=\begin{bmatrix}1\times6-4\times3+2\times3&1\times2-4\times-1+2\times-3\\-2\times6+8\times3-4\times3&-2\times2+8\times-1-4\times-3\end{bmatrix}\\&=\begin{bmatrix}0&0\\0&0\end{bmatrix}\end{aligned}$$

b E is a 3×2 matrix and D is a 2×3 matrix. Therefore since the number of columns of E is equal to the number of rows in D, the matrices can be multiplied.

$$\begin{aligned}ED&=\begin{bmatrix}6&2\\3&-1\\3&-3\end{bmatrix}\begin{bmatrix}1&-4&2\\-2&8&-4\end{bmatrix}\\&=\begin{bmatrix}6\times1+2\times-2&6\times-4+2\times8&6\times2+2\times-4\\3\times1-1\times-2&3\times-4-1\times8&3\times2-1\times-4\\3\times1-3\times-2&3\times-4-3\times8&3\times2-3\times-4\end{bmatrix}\\&=\begin{bmatrix}2&-8&4\\5&-20&10\\9&-36&18\end{bmatrix}\end{aligned}$$

c You can only add matrices of the same size. Therefore since E is a 3×2 matrix and D is a 2×3 matrix, $E+D$ does not exist.

d D is a 2×3 matrix and D is a 2×3 matrix. Therefore since the number of columns of D is not equal to the number of rows in D, the matrices cannot be multiplied. Therefore DD or D^2 does not exist.

14 a

$$\begin{aligned}P^2&=\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}-1&0\\0&4\end{bmatrix}\\&=\begin{bmatrix}-1\times-1+0\times0&-1\times0+0\times4\\0\times-1+4\times0&0\times0+4\times4\end{bmatrix}\\&=\begin{bmatrix}1&0\\0&16\end{bmatrix}\end{aligned}$$

$$\begin{aligned}P^3&=\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}-1&0\\0&4\end{bmatrix}\\&=\begin{bmatrix}1&0\\0&16\end{bmatrix}\begin{bmatrix}-1&0\\0&4\end{bmatrix}\\&=\begin{bmatrix}1\times-1+0\times0&1\times0+0\times4\\0\times-1+16\times0&0\times0+16\times4\end{bmatrix}\\&=\begin{bmatrix}-1&0\\0&64\end{bmatrix}\end{aligned}$$

$$\begin{aligned}P^4&=\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}-1&0\\0&4\end{bmatrix}\\&=\begin{bmatrix}1&0\\0&64\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\\&=\begin{bmatrix}1\times1+0\times0&1\times0+0\times4\\0\times1+64\times0&0\times0+64\times4\end{bmatrix}\\&=\begin{bmatrix}1&0\\0&256\end{bmatrix}\end{aligned}$$

Therefore, you can see that

$P^2=\begin{bmatrix}(-1)^2&0\\0&4^2\end{bmatrix}$ and $P^3=\begin{bmatrix}(-1)^3&0\\0&4^3\end{bmatrix}$ and

$P^4=\begin{bmatrix}(-1)^4&0\\0&4^4\end{bmatrix}$

Therefore in general: $P^n=\begin{bmatrix}(-1)^n&0\\0&4^n\end{bmatrix}$

b $$Q^2=\begin{bmatrix}2&0\\0&-3\end{bmatrix}\begin{bmatrix}2&0\\0&-3\end{bmatrix}$$
$$=\begin{bmatrix}2\times2+0\times0&2\times0+0\times-3\\0\times2-3\times0&0\times0-3\times-3\end{bmatrix}$$
$$=\begin{bmatrix}4&0\\0&9\end{bmatrix}$$

$$Q^3=\begin{bmatrix}2&0\\0&-3\end{bmatrix}\begin{bmatrix}2&0\\0&-3\end{bmatrix}\begin{bmatrix}2&0\\0&-3\end{bmatrix}$$
$$=\begin{bmatrix}4&0\\0&9\end{bmatrix}\begin{bmatrix}2&0\\0&-3\end{bmatrix}$$
$$=\begin{bmatrix}4\times2+0\times0&4\times0+0\times-3\\0\times2+9\times0&0\times0+9\times-3\end{bmatrix}$$
$$=\begin{bmatrix}8&0\\0&-27\end{bmatrix}$$

$$Q^4=\begin{bmatrix}2&0\\0&-3\end{bmatrix}\begin{bmatrix}2&0\\0&-3\end{bmatrix}\begin{bmatrix}2&0\\0&-3\end{bmatrix}\begin{bmatrix}2&0\\0&-3\end{bmatrix}$$
$$=\begin{bmatrix}8&0\\0&-27\end{bmatrix}\begin{bmatrix}2&0\\0&-3\end{bmatrix}$$
$$=\begin{bmatrix}8\times2+0\times0&8\times0+0\times-3\\0\times2-27\times0&0\times0-27\times-3\end{bmatrix}$$
$$=\begin{bmatrix}16&0\\0&81\end{bmatrix}$$

Therefore, you can see that

$$Q^2=\begin{bmatrix}2^2&0\\0&(-3)^2\end{bmatrix}\text{ and }Q^3=\begin{bmatrix}2^3&0\\0&(-3)^3\end{bmatrix}\text{ and }Q^4=\begin{bmatrix}2^4&0\\0&(-3)^4\end{bmatrix}$$

Therefore in general: $Q^n=\begin{bmatrix}2^n&0\\0&(-3)^n\end{bmatrix}$

c $$R^2=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}$$
$$=\begin{bmatrix}1\times1+0\times3&1\times0+0\times1\\3\times1+1\times3&3\times0+1\times1\end{bmatrix}$$
$$=\begin{bmatrix}1&0\\6&1\end{bmatrix}$$

$$R^3=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}$$
$$=\begin{bmatrix}1&0\\6&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}$$
$$=\begin{bmatrix}1\times1+0\times3&1\times0+0\times1\\6\times1+1\times3&6\times0+1\times1\end{bmatrix}$$
$$=\begin{bmatrix}1&0\\9&1\end{bmatrix}$$

$$R^4=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}$$
$$=\begin{bmatrix}1&0\\9&1\end{bmatrix}\begin{bmatrix}1&0\\3&1\end{bmatrix}$$
$$=\begin{bmatrix}1\times1+0\times3&1\times0+0\times1\\9\times1+1\times3&9\times0+1\times1\end{bmatrix}$$
$$=\begin{bmatrix}1&0\\12&1\end{bmatrix}$$

Therefore, you can see that

$R^2 = \begin{bmatrix} 1 & 0 \\ 2\times 3 & 1 \end{bmatrix}$ and $R^3 = \begin{bmatrix} 1 & 0 \\ 3\times 3 & 1 \end{bmatrix}$ and $R^4 = \begin{bmatrix} 1 & 0 \\ 4\times 3 & 1 \end{bmatrix}$

Therefore in general: $R^n = \begin{bmatrix} 1 & 0 \\ 3n & 1 \end{bmatrix}$

d
$$\begin{aligned} S^2 &= \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0\times 0+3\times 2 & 0\times 3+3\times 0 \\ 2\times 0+0\times 2 & 2\times 3+0\times 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} S^3 &= \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 6\times 0+0\times 2 & 6\times 3+0\times 0 \\ 0\times 0+6\times 2 & 0\times 3+6\times 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 18 \\ 12 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} S^4 &= \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 18 \\ 12 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0\times 0+18\times 2 & 0\times 3+18\times 0 \\ 12\times 0+0\times 2 & 12\times 3+0\times 0 \end{bmatrix} \\ &= \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} \end{aligned}$$

15
$$\begin{aligned} A^2-6A+11A &= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}-6\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 2+3\times -1 & 2\times 3+3\times 4 \\ -1\times 2+4\times -1 & -1\times 3+4\times 4 \end{bmatrix}+\begin{bmatrix} -12 & -18 \\ 6 & -24 \end{bmatrix}+\begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 18 \\ -6 & 13 \end{bmatrix}+\begin{bmatrix} -12 & -18 \\ 6 & -24 \end{bmatrix}+\begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} \\ &= \begin{bmatrix} 1-12+11 & 18-18+0 \\ -6+6+0 & 13-24+11 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

16 **a**
$$\begin{aligned} B^2-B-22I &= \begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix}-\begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix}-22\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4\times 4+5\times -2 & 4\times 5+5\times -3 \\ -2\times 4-3\times -2 & -2\times 5-3\times -3 \end{bmatrix}+\begin{bmatrix} -4 & -5 \\ 2 & 3 \end{bmatrix}+\begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 5 \\ -2 & -1 \end{bmatrix}+\begin{bmatrix} -4 & -5 \\ 2 & 3 \end{bmatrix}+\begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix} \\ &= \begin{bmatrix} 6-4-22 & 5-5+0 \\ -2+2+0 & -1+3-22 \end{bmatrix} \\ &= \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix} \end{aligned}$$

b
$$\begin{aligned} C^2-5C+14I &= \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}-5\begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}+14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 1-2\times 5 & 1\times -2-2\times 4 \\ 5\times 1+4\times 5 & 5\times -2+4\times 4 \end{bmatrix}+\begin{bmatrix} -5 & 10 \\ -25 & -20 \end{bmatrix}+\begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} \\ &= \begin{bmatrix} -9 & -10 \\ 25 & 6 \end{bmatrix}+\begin{bmatrix} -5 & 10 \\ -25 & -20 \end{bmatrix}+\begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} \\ &= \begin{bmatrix} -9-5+14 & -10+10+0 \\ 25-25+0 & 6-20+14 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

17
$$\begin{aligned}D^2-9D&=\begin{bmatrix} d & -4\\ -2 & 8\end{bmatrix}\begin{bmatrix} d & -4\\ -2 & 8\end{bmatrix}-9\begin{bmatrix} d & -4\\ -2 & 8\end{bmatrix}\\
&=\begin{bmatrix} d\times d-4\times -2 & d\times -4-4\times 8\\ -2\times d+8\times -2 & -2\times -4+8\times 8\end{bmatrix}+\begin{bmatrix} -9d & 36\\ 18 & -72\end{bmatrix}\\
&=\begin{bmatrix} d^2+8 & -4d-32\\ -2d-16 & 72\end{bmatrix}+\begin{bmatrix} -9d & 36\\ 18 & -72\end{bmatrix}\\
&=\begin{bmatrix} d^2+8-9d & -4d-32+36\\ -2d-16+18 & 72-72\end{bmatrix}\\
&=\begin{bmatrix} d^2-9d+8 & -4d+4\\ -2d+2 & 0\end{bmatrix}\\
&=\begin{bmatrix} d^2-9d+8 & 4-4d\\ 2-2d & 0\end{bmatrix}\end{aligned}$$

18 a i
$$\begin{aligned}AB&=\begin{bmatrix} 2 & 3\\ -1 & 4\end{bmatrix}\begin{bmatrix} 4 & -2\\ 3 & 5\end{bmatrix}\\
&=\begin{bmatrix} 2\times 4+3\times 3 & 2\times -2+3\times 5\\ -1\times 4+4\times 3 & -1\times -2+4\times 5\end{bmatrix}\\
&=\begin{bmatrix} 17 & 11\\ 8 & 22\end{bmatrix}\end{aligned}$$

$$\begin{aligned}tr(AB)&=17+22\\&=39\end{aligned}$$

ii
$$\begin{aligned}BA&=\begin{bmatrix} 4 & -2\\ 3 & 5\end{bmatrix}\begin{bmatrix} 2 & 3\\ -1 & 4\end{bmatrix}\\
&=\begin{bmatrix} 4\times 2-2\times -1 & 4\times 3-2\times 4\\ 3\times 2+5\times -1 & 3\times 3+5\times 4\end{bmatrix}\\
&=\begin{bmatrix} 10 & 4\\ 1 & 29\end{bmatrix}\end{aligned}$$

$$\begin{aligned}tr(BA)&=10+29\\&=39\end{aligned}$$

iii
$$\begin{aligned}tr(A)tr(B)&=(2+4)(4+5)\\&=6\times 9\\&=54\end{aligned}$$

b
$$\begin{aligned}ABC&=\begin{bmatrix} 2 & 3\\ -1 & 4\end{bmatrix}\begin{bmatrix} 4 & -2\\ 3 & 5\end{bmatrix}\begin{bmatrix} 1 & -2\\ 5 & 4\end{bmatrix}\\
&=\begin{bmatrix} 2\times 4+3\times 3 & 2\times -2+3\times 5\\ -1\times 4+4\times 3 & -1\times -2+4\times 5\end{bmatrix}\begin{bmatrix} 1 & -2\\ 5 & 4\end{bmatrix}\\
&=\begin{bmatrix} 17 & 11\\ 8 & 22\end{bmatrix}\begin{bmatrix} 1 & -2\\ 5 & 4\end{bmatrix}\\
&=\begin{bmatrix} 17\times 1+11\times 5 & 17\times -2+11\times 4\\ 8\times 1+22\times 5 & 8\times -2+22\times 4\end{bmatrix}\\
&=\begin{bmatrix} 72 & 10\\ 118 & 72\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{LHS}&=tr(ABC)\\&=72+72\\&=144\end{aligned}$$

$$\begin{aligned}\text{RHS}&=tr(A)tr(B)tr(C)\\&=(2+4)(4+5)(1+4)\\&=6\times 9\times 5\\&=270\end{aligned}$$

$$\text{LHS}\neq\text{RHS}$$
$$\therefore tr(ABC)\neq tr(A)tr(B)tr(C)$$

Exercise 7.4 — Determinants and inverses of 2×2 matrices

1 If $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ then the determinant is given by, $\det(A)=ad-bc$

$$\begin{aligned}\det(G)&=-2\times 5-4\times 3\\&=-10-12\\&=-22\end{aligned}$$

2 Let $A=\begin{bmatrix} x & 5\\ 3 & x+2\end{bmatrix}$

$$\begin{aligned}\det(A)&=9\\
\begin{vmatrix} x & 5\\ 3 & x+2\end{vmatrix}&=9\\
x(x+2)-5\times 3&=9\\
x^2+2x-15-9&=9-9\\
x^2+2x-24&=0\\
(x-4)(x+6)&=0\\
\therefore x&=4,-6\end{aligned}$$

3 If $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ then the inverse matrix A^{-1} is given by,

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & b\end{bmatrix}$$

Let $A=\begin{bmatrix} 4 & -2\\ 5 & 6\end{bmatrix}$

$$\begin{aligned}A^{-1}&=\frac{1}{4\times 6-(-2\times 5)}\begin{bmatrix} 6 & 2\\ -5 & 4\end{bmatrix}\\
&=\frac{1}{24+10}\begin{bmatrix} 6 & 2\\ -5 & 4\end{bmatrix}\\
&=\frac{1}{34}\begin{bmatrix} 6 & 2\\ -5 & 4\end{bmatrix}\end{aligned}$$

4
$$\begin{aligned}A^{-1}&=\frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & b\end{bmatrix}\\
\begin{bmatrix} p & 3\\ 3 & q\end{bmatrix}&=\frac{1}{2\times 4-3\times 3}\begin{bmatrix} 4 & -3\\ -3 & 2\end{bmatrix}\\
\begin{bmatrix} p & 3\\ 3 & q\end{bmatrix}&=\frac{1}{8-9}\begin{bmatrix} 4 & -3\\ -3 & 2\end{bmatrix}\\
\begin{bmatrix} p & 3\\ 3 & q\end{bmatrix}&=\frac{1}{8-9}\begin{bmatrix} 4 & -3\\ -3 & 2\end{bmatrix}\\
\begin{bmatrix} p & 3\\ 3 & q\end{bmatrix}&=-\begin{bmatrix} 4 & -3\\ -3 & 2\end{bmatrix}\\
\begin{bmatrix} p & 3\\ 3 & q\end{bmatrix}&=\begin{bmatrix} -4 & 3\\ 3 & -2\end{bmatrix}\end{aligned}$$

$\therefore p=-4,\ q=-2$

5 Let $A=\begin{bmatrix} 1 & -2\\ -5 & 10\end{bmatrix}$

$$\begin{aligned}\det(A)&=1\times 10-(-5\times -2)\\&=10-10\\&=0\end{aligned}$$

Therefore since the determinant is zero the matrix is singular.

6 Let $A=\begin{bmatrix} x & 4\\ 3 & x+4\end{bmatrix}$

If A is singular, $\det(A)=0$

$$\begin{aligned}\det(A)&=0\\
\begin{vmatrix} x & 4\\ 3 & x+4\end{vmatrix}&=0\\
x(x+4)-4\times 3&=0\\
x^2+4x-12&=0\\
(x+6)(x-2)&=0\\
\therefore x&=-6,2\end{aligned}$$

7

$$\begin{aligned} A-kI &= \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} -k & 0 \\ 0 & -k \end{bmatrix} \\ &= \begin{bmatrix} 2-k & 3 \\ -1 & 5-k \end{bmatrix} \\ \det(A-kI) &= \begin{vmatrix} 2-k & 3 \\ -1 & 5-k \end{vmatrix} \\ &= (2-k)(5-k)-3\times -1 \\ &= 10-2k-5k+k^2+3 \\ &= k^2-7k+13 \end{aligned}$$

$\therefore p=1,\ q=-7,\ r=13$

$$\begin{aligned} A^2 &= \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2\times2+3\times-1 & 2\times3+3\times5 \\ -1\times2+5\times-1 & -1\times3+5\times5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 21 \\ -7 & 22 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} pA^2+qA+rI &= \begin{bmatrix} 1 & 21 \\ -7 & 22 \end{bmatrix} - 7\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} + 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 21 \\ -7 & 22 \end{bmatrix} + \begin{bmatrix} -14 & -21 \\ 7 & -35 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} \\ &= \begin{bmatrix} 1-14+13 & 21-21+0 \\ -7+7+0 & 22-35+13 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

8

$$\begin{aligned} A-kI &= \begin{bmatrix} 4 & -8 \\ -3 & 2 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -8 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} -k & 0 \\ 0 & -k \end{bmatrix} \\ &= \begin{bmatrix} 4-k & -8 \\ -3 & 2-k \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(A-kI) &= 0 \\ \begin{vmatrix} 4-k & -8 \\ -3 & 2-k \end{vmatrix} &= 0 \\ (4-k)(2-k)-(-8\times-3) &= 0 \\ 8-4k-2k+k^2-24 &= 0 \\ k^2-6k-16 &= 0 \\ (k+2)(k-8) &= 0 \end{aligned}$$

$\therefore k=-2,8$

9 **a** **i** If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant is given by,

$$\begin{aligned} \det(A) &= ad-bc \\ \det(P) &= 6\times2-(-2\times4) \\ &= 12+8 \\ &= 20 \end{aligned}$$

ii If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the inverse matrix A^{-1} is given by,

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & b \end{bmatrix} \\ P^{-1} &= \frac{1}{20}\begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} \\ &= \frac{2}{20}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \end{aligned}$$

b

$$\begin{aligned} PP^{-1} &= \begin{bmatrix} 6 & -2 \\ 4 & 2 \end{bmatrix} \times \frac{1}{10}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 6 & -2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 6\times1-2\times-2 & 6\times1-2\times3 \\ 4\times1+2\times-2 & 4\times1+2\times3 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ PP^{-1} &= I \end{aligned}$$

$$\begin{aligned} P^{-1}P &= \frac{1}{10}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ 4 & 2 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 1\times6+1\times4 & 1\times-2+1\times2 \\ -2\times6+3\times4 & -2\times-2+3\times2 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ P^{-1}P &= I \end{aligned}$$

$\therefore PP^{-1}=P^{-1}P=I$

c **i**

$$\begin{aligned} P^{-1} &= \frac{1}{10}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{-2}{10} & \frac{3}{10} \end{bmatrix} \\ \det(P^{-1}) &= \frac{1}{10}\times\frac{3}{10}-\left(\frac{1}{10}\times\frac{-2}{10}\right) \\ &= \frac{3}{100}+\frac{2}{100} \\ &= \frac{5}{100} \\ &= \frac{1}{20} \end{aligned}$$

ii

$$\begin{aligned} \det(P)\det(P^{-1}) &= 20\times\frac{1}{20} \\ &= 1 \end{aligned}$$

10 **a** Let,

$$\begin{aligned} A &= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \\ A^{-1} &= \frac{1}{-1\times4-0\times0}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \\ &= -\frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

b Let,

$$\begin{aligned} B &= \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \\ B^{-1} &= \frac{1}{2\times-3-0\times0}\begin{bmatrix} -3 & -1 \\ 0 & 2 \end{bmatrix} \\ &= -\frac{1}{6}\begin{bmatrix} -3 & -1 \\ 0 & 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

c Let,

$$C=\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\begin{aligned} C^{-1} &= \frac{1}{2\times 1-0\times 3}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \end{aligned}$$

d Let,

$$D=\begin{bmatrix} 0 & -3 \\ 2 & -1 \end{bmatrix}$$

$$\begin{aligned} D^{-1} &= \frac{1}{0\times -1-(-3\times 2)}\begin{bmatrix} -1 & 3 \\ -2 & 0 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} -1 & 3 \\ -2 & 0 \end{bmatrix} \end{aligned}$$

11 a

$$\begin{aligned} \det(A) &= 2\times -4-(-3\times -1) \\ &= -8-3 \\ &= -11 \end{aligned}$$

$$\begin{aligned} \det(B) &= 4\times 3-5\times 2 \\ &= 12-10 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \det(C) &= 1\times 4-(-2\times 3) \\ &= 4+6 \\ &= 10 \end{aligned}$$

b

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 4-3\times 2 & 2\times 5-3\times 3 \\ -1\times 4-4\times 2 & -1\times 5-4\times 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ -12 & -17 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \det(AB) \\ &= 2\times -17-(1\times -12) \\ &= -34+12 \\ &= -22 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \det(A)\det(B) \\ &= (2\times -4-(-3\times -1))\times(4\times 3-5\times 2) \\ &= (-8-3)\times(12-10) \\ &= -11\times 2 \\ &= -22 \end{aligned}$$

LHS = RHS

$\therefore \det(AB)=\det(A)\det(B)$

c

$$\begin{aligned} ABC &= \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 4-3\times 2 & 2\times 5-3\times 3 \\ -1\times 4-4\times 2 & -1\times 5-4\times 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ -12 & -17 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 1+1\times 3 & 2\times -2+1\times 4 \\ -12\times 1-17\times 3 & -12\times -2-17\times 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 \\ -63 & -44 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \det(ABC) \\ &= 5\times -44-0\times -63 \\ &= -220 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \det(A)\det(B)\det(C) \\ &= (2\times -4-(-3\times -1))\times(4\times 3-5\times 2)\times(1\times 4-(-2\times 3)) \\ &= (-8-3)\times(12-10)\times(4+6) \\ &= -11\times 2\times 10 \\ &= -220 \end{aligned}$$

LHS = RHS

$\therefore \det(ABC)=\det(A)\det(B)\det(C)$

12 a

$$\begin{aligned} A^{-1} &= \frac{1}{2\times -4-(-3\times -1)}\begin{bmatrix} -4 & 3 \\ 1 & 2 \end{bmatrix} \\ &= -\frac{1}{11}\begin{bmatrix} -4 & 3 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B^{-1} &= \frac{1}{4\times 3-5\times 2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} C^{-1} &= \frac{1}{1\times 4-(-2\times 3)}\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} \end{aligned}$$

b

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 4-3\times 2 & 2\times 5-3\times 3 \\ -1\times 4-4\times 2 & -1\times 5-4\times 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ -12 & -17 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= (AB)^{-1} \\ &= \frac{1}{2\times -17-1\times -12}\begin{bmatrix} -17 & -1 \\ 12 & 2 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} -17 & -1 \\ 12 & 2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} 17 & 1 \\ -12 & -2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= A^{-1}B^{-1} \\ &= -\frac{1}{11}\begin{bmatrix} -4 & 3 \\ 1 & 2 \end{bmatrix}\times\frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} -4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} -4\times 3+3\times -2 & -4\times -5+3\times 4 \\ 1\times 3+2\times -2 & 1\times -5+2\times 4 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} -18 & 32 \\ -1 & 3 \end{bmatrix} \end{aligned}$$

LHS ≠ RHS

$(AB)^{-1}\neq A^{-1}B^{-1}$

c

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 4-3\times 2 & 2\times 5-3\times 3 \\ -1\times 4-4\times 2 & -1\times 5-4\times 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ -12 & -17 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\text{LHS} &= (AB)^{-1}\\
&= \frac{1}{2\times -17 - 1\times -12}\begin{bmatrix} -17 & -1 \\ 12 & 2\end{bmatrix}\\
&= -\frac{1}{22}\begin{bmatrix} -17 & -1 \\ 12 & 2\end{bmatrix}\\
&= \frac{1}{22}\begin{bmatrix} 17 & 1 \\ -12 & -2\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= A^{-1}B^{-1}\\
&= \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4\end{bmatrix}\times -\frac{1}{11}\begin{bmatrix} -4 & 3 \\ 1 & 2\end{bmatrix}\\
&= -\frac{1}{22}\begin{bmatrix} 3 & -5 \\ -2 & 4\end{bmatrix}\begin{bmatrix} -4 & 3 \\ 1 & 2\end{bmatrix}\\
&= -\frac{1}{22}\begin{bmatrix} 3\times -4 - 5\times 1 & 3\times 3 - 5\times 2 \\ -2\times -4 + 4\times 1 & -2\times 3 + 4\times 2\end{bmatrix}\\
&= -\frac{1}{22}\begin{bmatrix} -17 & -1 \\ 12 & 2\end{bmatrix}\\
&= \frac{1}{22}\begin{bmatrix} 17 & 1 \\ -12 & -2\end{bmatrix}
\end{aligned}$$

LHS = RHS

$(AB)^{-1} = B^{-1}A^{-1}$

d
$$\begin{aligned}
ABC &= \begin{bmatrix} 2 & -3 \\ -1 & -4\end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3\end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4\end{bmatrix}\\
&= \begin{bmatrix} 2\times 4 - 3\times 2 & 2\times 5 - 3\times 3 \\ -1\times 4 - 4\times 2 & -1\times 5 - 4\times 3\end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4\end{bmatrix}\\
&= \begin{bmatrix} 2 & 1 \\ -12 & -17\end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4\end{bmatrix}\\
&= \begin{bmatrix} 2\times 1 + 1\times 3 & 2\times -2 + 1\times 4 \\ -12\times 1 - 17\times 3 & -12\times -2 - 17\times 4\end{bmatrix}\\
&= \begin{bmatrix} 5 & 0 \\ -63 & -44\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= (ABC)^{-1}\\
&= \frac{1}{5\times -44 - 0\times -63}\begin{bmatrix} -44 & 0 \\ 63 & 5\end{bmatrix}\\
&= -\frac{1}{220}\begin{bmatrix} -44 & 0 \\ 63 & 5\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= C^{-1}B^{-1}A^{-1}\\
&= \frac{1}{10}\begin{bmatrix} 4 & 2 \\ -3 & 1\end{bmatrix}\times\frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4\end{bmatrix}\times -\frac{1}{11}\begin{bmatrix} -4 & 3 \\ 1 & 2\end{bmatrix}\\
&= -\frac{1}{220}\begin{bmatrix} 4 & 2 \\ -3 & 1\end{bmatrix}\begin{bmatrix} 3 & -5 \\ -2 & 4\end{bmatrix}\begin{bmatrix} -4 & 3 \\ 1 & 2\end{bmatrix}\\
&= -\frac{1}{220}\begin{bmatrix} 4\times 3 + 2\times -2 & 4\times -5 + 2\times 4 \\ -3\times 3 + 1\times -2 & -3\times -5 + 1\times 4\end{bmatrix}\begin{bmatrix} -4 & 3 \\ 1 & 2\end{bmatrix}\\
&= -\frac{1}{220}\begin{bmatrix} 8 & -12 \\ -11 & 19\end{bmatrix}\begin{bmatrix} -4 & 3 \\ 1 & 2\end{bmatrix}\\
&= -\frac{1}{220}\begin{bmatrix} 8\times -4 - 12\times 1 & 8\times 3 - 12\times 2 \\ -11\times -4 + 19\times 1 & -11\times 3 + 19\times 2\end{bmatrix}\\
&= -\frac{1}{220}\begin{bmatrix} -44 & 0 \\ 63 & -5\end{bmatrix}
\end{aligned}$$

LHS = RHS

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

13 a
$$\begin{aligned}
\begin{vmatrix} x & -3 \\ 4 & 2\end{vmatrix} &= 6\\
x\times 2 - (-3\times 4) &= 6\\
2x + 12 &= 6\\
2x + 12 - 12 &= 6 - 12\\
2x &= -6\\
\frac{2x}{2} &= \frac{-6}{2}\\
x &= -3
\end{aligned}$$

b
$$\begin{aligned}
\begin{vmatrix} x & x \\ 8 & 2\end{vmatrix} &= 12\\
x\times 2 - x\times 8 &= 12\\
2x - 8x &= 12\\
-6x &= 12\\
\frac{-6x}{-6} &= \frac{12}{-6}\\
x &= -2
\end{aligned}$$

c
$$\begin{aligned}
\begin{vmatrix} x & 3 \\ 4 & x\end{vmatrix} &= 4\\
x\times x - 3\times 4 &= 4\\
x^2 - 12 &= 4\\
x^2 &= 16\\
x &= \pm\sqrt{16}\\
x &= \pm 4
\end{aligned}$$

d
$$\begin{aligned}
\begin{vmatrix} \frac{1}{x} & x \\ -2 & 3\end{vmatrix} &= 7\\
\frac{1}{x}\times 3 - x\times -2 &= 7\\
\frac{3}{x} + 2x &= 7\\
x\times\frac{3}{x} + x\times 2x &= x\times 7\\
3 + 2x^2 &= 7x\\
2x^2 - 7x + 3 &= 0\\
(2x-1)(x-3) &= 0\\
x &= \frac{1}{2}, 3
\end{aligned}$$

14 a If a matrix A is singular, then $\det(A) = 0$

$$\begin{aligned}
\begin{vmatrix} x & -3 \\ 4 & 2\end{vmatrix} &= 0\\
x\times 2 - (-3\times 4) &= 0\\
2x + 12 &= 0\\
2x &= -12\\
\frac{2x}{2} &= \frac{-12}{2}\\
x &= -6
\end{aligned}$$

b If a matrix A is singular, then $\det(A)=0$

$$\begin{aligned}
\begin{vmatrix} x & \frac{1}{x} \\ 8 & 2 \end{vmatrix} &= 0 \\
x\times 2-\frac{1}{x}\times 8 &= 0 \\
2x-\frac{8}{x} &= 0 \\
x\times 2x - x\times\frac{8}{x} &= 0 \\
2x^2-8 &= 0 \\
2x^2 &= 8 \\
\frac{2x^2}{2} &= \frac{8}{2} \\
x^2 &= 4 \\
x &= \pm\sqrt{4} \\
x &= \pm 2
\end{aligned}$$

c If a matrix A is singular, then $\det(A)=0$

$$\begin{aligned}
\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} &= 0 \\
x\times x-3\times 4 &= 0 \\
x^2-12 &= 0 \\
x^2 &= 12 \\
x &= \pm\sqrt{12} \\
x &= \pm 2\sqrt{3}
\end{aligned}$$

d If a matrix A is singular, then $\det(A)=0$

$$\begin{aligned}
\begin{vmatrix} x+1 & -3 \\ -2 & x \end{vmatrix} &= 0 \\
(x+1)\times x-(-3\times -2) &= 0 \\
x(x+1)-6 &= 0 \\
x^2+x-6 &= 0 \\
(x+3)(x-2) &= 0 \\
x &= -3, 2
\end{aligned}$$

15 **a**

$$\begin{aligned}
AB &= \begin{bmatrix} -1 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \end{bmatrix} \\
&= \begin{bmatrix} -1\times 3 & -1\times -5 \\ 2\times 3 & 2\times -5 \end{bmatrix} \\
&= \begin{bmatrix} -3 & 5 \\ 6 & -10 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\det(AB) &= -3\times -10-5\times 6 \\
&= 30-30 \\
&= 0
\end{aligned}$$

Therefore since $\det(AB)=0$, matrix AB is singular and therefore $(AB)^{-1}$ does not exist.

b A is not a square matrix so A^{-1} does not exist.

c B is not a square matrix so B^{-1} does not exist.

d

$$\begin{aligned}
\det(C) &= 2\times 5-4\times -3 \\
&= 10+12 \\
&= 22
\end{aligned}$$

$$C^{-1}=\frac{1}{22}\begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix}$$

e $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

Therefore since A^{-1} and B^{-1} do not exist, $(ABC)^{-1}$ does not exist.

Therefore since BA is a square matrix, its inverse $(BA)^{-1}$ can be calculated.

$$\begin{aligned}BA(BA)^{-1}&=I\\ [-13](BA)^{-1}&=[1]\\ [-13]\begin{bmatrix}-\frac{1}{13}\end{bmatrix}&=[1]\\ \therefore (BA)^{-1}&=\begin{bmatrix}-\frac{1}{13}\end{bmatrix}\end{aligned}$$

16 $$\begin{aligned}A-kI&=\begin{bmatrix}2&-3\\-1&-4\end{bmatrix}-k\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ &=\begin{bmatrix}2-k&-3\\-1&-4-k\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\det(A-kI)&=(2-k)(-4-k)-(-3\times-1)\\ &=-(2-k)(k+4)-3\\ &=-(8+2k-k^2-4k)-3\\ &=-(-k^2-2k+8)-3\\ &=k^2+2k-8-3\\ &=k^2+2k-11\\ &\therefore p=1,q=2,r=-11\end{aligned}$$

$$\begin{aligned}pA^2+qA+rI&=A^2+2A-11I\\ &=\begin{bmatrix}2&-3\\-1&-4\end{bmatrix}\begin{bmatrix}2&-3\\-1&-4\end{bmatrix}+2\begin{bmatrix}2&-3\\-1&-4\end{bmatrix}-11\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ &=\begin{bmatrix}2\times2-3\times-1&2\times-3-3\times-4\\-1\times2-4\times-1&-1\times-3-4\times-4\end{bmatrix}+\begin{bmatrix}4&-6\\-2&-8\end{bmatrix}+\begin{bmatrix}-11&0\\0&-11\end{bmatrix}\\ &=\begin{bmatrix}7&6\\2&19\end{bmatrix}+\begin{bmatrix}4&-6\\-2&-8\end{bmatrix}+\begin{bmatrix}-11&0\\0&-11\end{bmatrix}\\ &=\begin{bmatrix}7+4-11&6-6+0\\2-2+0&19-8-11\end{bmatrix}\\ &=\begin{bmatrix}0&0\\0&0\end{bmatrix}\end{aligned}$$

17 $$\begin{aligned}B-kI&=\begin{bmatrix}4&5\\2&3\end{bmatrix}-k\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ &=\begin{bmatrix}4-k&5\\2&3-k\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\det(B-kI)&=(4-k)(3-k)-5\times2\\ &=12-4k-3k+k^2-10\\ &=k^2-7k+2\\ &\therefore p=1,q=-7,r=2\end{aligned}$$

$$\begin{aligned}pB^2+qB+rI&=B^2+-7B+2I\\ &=\begin{bmatrix}4&5\\2&3\end{bmatrix}\begin{bmatrix}4&5\\2&3\end{bmatrix}-7\begin{bmatrix}4&5\\2&3\end{bmatrix}+2\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ &=\begin{bmatrix}4\times4+5\times2&4\times5+5\times3\\2\times4+3\times2&2\times5+3\times3\end{bmatrix}+\begin{bmatrix}-28&-35\\-14&-21\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}\\ &=\begin{bmatrix}26&35\\14&19\end{bmatrix}+\begin{bmatrix}-28&-35\\-14&-21\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}\\ &=\begin{bmatrix}26-28+2&35-35+0\\14-14+0&19-21+2\end{bmatrix}\\ &=\begin{bmatrix}0&0\\0&0\end{bmatrix}\end{aligned}$$

18 a

$$\begin{aligned} A-kI &= \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-k & 3 \\ 2 & 2-k \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(A-kI) &= 0 \\ \begin{vmatrix} 1-k & 3 \\ 2 & 2-k \end{vmatrix} &= 0 \\ (1-k)(2-k)-3\times 2 &= 0 \\ 2-k-2k+k^2-6 &= 0 \\ k^2-3k-4 &= 0 \\ (k+1)(k-4) &= 0 \\ k &= -1, 4 \end{aligned}$$

b

$$\begin{aligned} P &= \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \\ P^1 &= \frac{1}{3\times 1-1\times -2}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} P^{-1}AP &= \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 1\times 1-1\times 2 & 1\times 3-1\times 2 \\ 2\times 1+3\times 2 & 2\times 3+3\times 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 8 & 12 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} -1\times 3+1\times -2 & -1\times 1+1\times 1 \\ 8\times 3+12\times -2 & 8\times 1+12\times 1 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} -5 & 0 \\ 0 & 20 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

Note that $P^{-1}AP$ is a diagonal matrix with k values along the main diagonal.

19 a

$$\begin{aligned} B-kI &= \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -k-3 & 5 \\ -2 & 4-k \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(B-kI) &= 0 \\ \begin{vmatrix} -k-3 & 5 \\ -2 & 4-k \end{vmatrix} &= 0 \\ (-k-3)(4-k)-5\times -2 &= 0 \\ -4k+k^2-12+3k+10 &= 0 \\ k^2-k-2 &= 0 \\ k &= -1, 2 \end{aligned}$$

b Using a CAS calculator:

$$\begin{aligned} Q^{-1}BQ &= \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

20 a i Sub in $\theta = \frac{\pi}{2}$

$$R\left(\frac{\pi}{2}\right) = \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

ii Sub in $\theta = \frac{\pi}{6}$

$$R\left(\frac{\pi}{6}\right) = \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

iii Sub in $\theta = \frac{\pi}{3}$

$$R\left(\frac{\pi}{3}\right) = \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

iv $R^2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

$$= \begin{bmatrix} \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) & -\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) & -\sin(\theta)\sin(\theta) + \cos(\theta)\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & -2\cos(\theta)\sin(\theta) \\ 2\cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

v $R^{-1} = \frac{1}{\cos(\theta)\times\cos(\theta) - (-\sin(\theta)\times\sin(\theta))}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

$$= \frac{1}{\cos^2(\theta) + \sin^2(\theta)}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Remember, $\cos^2(\theta) + \sin^2(\theta) = 1$

b $$\begin{aligned}\text{LHS} &= R\left(\frac{\pi}{6}\right)R\left(\frac{\pi}{3}\right)\\ &= \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right)\\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right)\end{bmatrix}\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right)\\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right)\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix}\times\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\\ &= \frac{1}{4}\begin{bmatrix} \sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix}\begin{bmatrix} 1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\\ &= \frac{1}{4}\begin{bmatrix} \sqrt{3}\times 1-1\times\sqrt{3} & \sqrt{3}\times-\sqrt{3}-1\times 1\\ 1\times 1+\sqrt{3}\times\sqrt{3} & 1\times-\sqrt{3}+\sqrt{3}\times 1\end{bmatrix}\\ &= \frac{1}{4}\begin{bmatrix} 0 & -4\\ 4 & 0\end{bmatrix}\\ &= \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= R\left(\frac{\pi}{2}\right)\\ &= \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right)\\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right)\end{bmatrix}\\ &= \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}\end{aligned}$$

LHS = RHS

$$\therefore R\left(\frac{\pi}{6}\right)R\left(\frac{\pi}{3}\right) = R\left(\frac{\pi}{2}\right)$$

c $$\begin{aligned}\text{LHS} &= R(\alpha)R(\beta)\\ &= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}\begin{bmatrix} \cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{bmatrix}\\ &= \begin{bmatrix} \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta)-\sin(\alpha)\cos(\beta)\\ \sin(\alpha)\cos(\beta)+\cos(\beta)\sin(\beta) & -\sin(\alpha)\sin(\beta)+\cos(\alpha)\cos(\beta)\end{bmatrix}\\ &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}\\ &= \text{RHS}\end{aligned}$$

$$\therefore R(\alpha)R(\beta) = R(\alpha+\beta)$$

Exercise 7.5 — Matrix equations and solving 2×2 linear simultaneous equations

1 $3x-4y=23$

$5x+2y+21$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 3 & -4\\ 5 & 2\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 23\\ 21\end{bmatrix}$$

$$A=\begin{bmatrix} 3 & -4\\ 5 & 2\end{bmatrix}, X=\begin{bmatrix} x\\ y\end{bmatrix}, K=\begin{bmatrix} 23\\ 21\end{bmatrix}$$

$$\begin{aligned}AX &= K\\ A^{-1}AX &= A^{-1}K\\ X &= A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=3\times2-(-4\times5)\\&=6+20\\&=26\end{aligned}$$

$$A^{-1}=\frac{1}{26}\begin{bmatrix}2&4\\-5&3\end{bmatrix}$$

$$\begin{aligned}X&=A^{-1}K\\&=\frac{1}{26}\begin{bmatrix}2&4\\-5&3\end{bmatrix}\begin{bmatrix}23\\21\end{bmatrix}\\&=\frac{1}{26}\begin{bmatrix}2\times23+4\times21\\-5\times23+3\times21\end{bmatrix}\\&=\frac{1}{26}\begin{bmatrix}130\\-52\end{bmatrix}\\&=\begin{bmatrix}5\\-2\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\-2\end{bmatrix}$$

$x=5,\ y=-2$

2 $2x+5y=-7$

$3x-2y=18$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix}2&5\\3&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-7\\18\end{bmatrix}$$

$$A=\begin{bmatrix}2&5\\3&-2\end{bmatrix},\ X=\begin{bmatrix}x\\y\end{bmatrix},\ K=\begin{bmatrix}-7\\18\end{bmatrix}$$

$$\begin{aligned}AX&=K\\A^{-1}AX&=A^{-1}K\\X&=A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=2\times-2-5\times3\\&=-4+-15\\&=-19\end{aligned}$$

$$A^{-1}=-\frac{1}{19}\begin{bmatrix}-2&-5\\-3&2\end{bmatrix}$$

$$\begin{aligned}X&=A^{-1}K\\&=-\frac{1}{19}\begin{bmatrix}-2&-5\\-3&2\end{bmatrix}\begin{bmatrix}-7\\18\end{bmatrix}\\&=-\frac{1}{19}\begin{bmatrix}-2\times-7-5\times18\\-3\times-7+2\times18\end{bmatrix}\\&=-\frac{1}{19}\begin{bmatrix}-76\\57\end{bmatrix}\\&=\begin{bmatrix}4\\-3\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\-3\end{bmatrix}$$

$x=4,\ y=-3$

3 $4x-3y=12$

$-8x+6y=-18$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix}4&-3\\-8&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}12\\-18\end{bmatrix}$$

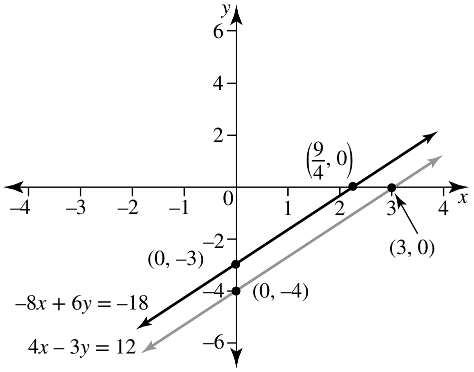
$$A=\begin{bmatrix}4&-3\\-8&6\end{bmatrix},\ X=\begin{bmatrix}x\\y\end{bmatrix},\ K=\begin{bmatrix}12\\-18\end{bmatrix}$$

$$\begin{aligned}AX&=K\\A^{-1}AX&=A^{-1}K\\X&=A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=4\times6-(-3\times-8)\\&=24-24\\&=0\end{aligned}$$

Therefore, matrix A is singular.

It can be seen from the graphs below that the two lines are parallel and therefore have no points of intersection. Therefore, there is no solution.



4 $5x-4y=20$

$kx+2y=-8$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix}5&-4\\k&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}20\\-8\end{bmatrix}$$

$$A=\begin{bmatrix}5&-4\\k&2\end{bmatrix}$$

There is no solution if matrix A is singular, therefore $\det(A)=0$

$$\begin{aligned}\det(A)&=0\\5\times2-(-4\times k)&=0\\10+4k&=0\\4k&=-10\\\frac{4k}{4}&=\frac{-10}{4}\\k&=-\frac{10}{4}\\k&=-\frac{5}{2}\end{aligned}$$

5 $4x-3y=12$

$-8x+6y=-24$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix}4&-3\\-8&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}12\\-24\end{bmatrix}$$

$$A=\begin{bmatrix}4&-3\\-8&6\end{bmatrix},\ X=\begin{bmatrix}x\\y\end{bmatrix},\ K=\begin{bmatrix}12\\-24\end{bmatrix}$$

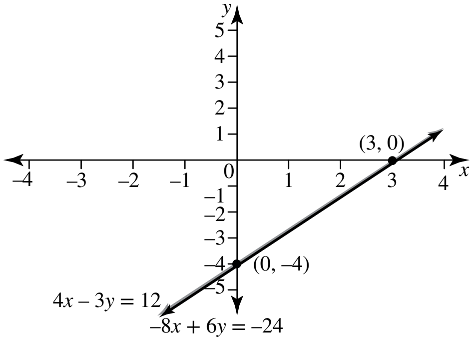
$$\begin{aligned}\det(A)&=4\times6-(-3\times-8)\\&=24-24\\&=0\end{aligned}$$

Therefore, matrix A is singular.

(1) $4x-3y=12$

(2) $-8x+6y=-24$

Sketch the graphs as shown below. Equations (1) and (2) are actually the same line as $(2)=-2\times(1)$



Note that as the lines overlap, there are an infinite number of points of intersection.

In general, let $y=t$ and then substitute for y into equation (1) and solve for x:

$$\begin{aligned} 4x-3t&=12\\ 4x-3t+3t&=12+3t\\ 4x&=12+3t\\ \frac{4x}{4}&=\frac{12+3t}{4}\\ x&=3+\frac{3t}{4} \end{aligned}$$

Therefore, infinite number of solutions are of the form $\left(3+\frac{3t}{4},\ t\right),\ t\in R$

6 $5x-4y=20$

$kx+2y=-10$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix} 5 & -4\\ k & 2 \end{bmatrix}\begin{bmatrix} x\\ y \end{bmatrix}=\begin{bmatrix} 20\\ -10 \end{bmatrix}$$

$$A=\begin{bmatrix} 5 & -4\\ k & 2 \end{bmatrix},\ X=\begin{bmatrix} x\\ y \end{bmatrix},\ K=\begin{bmatrix} 20\\ -10 \end{bmatrix}$$

For an infinite number of solutions, $\det(A)=0$

$$\begin{aligned} \det(A)&=0\\ 5\times 2-(-4\times k)&=0\\ 10+4k&=0\\ 4k&=-10\\ \frac{4k}{4}&=-\frac{10}{4}\\ k&=-\frac{10}{4}\\ k&=-\frac{5}{2} \end{aligned}$$

When $k=-\frac{5}{2}$,

(1) $5x-4y=20$

(2) $-\frac{5}{2}x+2y=-10$

$\Rightarrow (1)=-2\times(2)$

Therefore, equations give the same line and infinitely many solutions.

7 **a** $(k+1)x-2y=2k$

$-6x+2ky=-8$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix} k+1 & -2\\ -6 & 2k \end{bmatrix}\begin{bmatrix} x\\ y \end{bmatrix}=\begin{bmatrix} 2k\\ -8 \end{bmatrix}$$

$$A=\begin{bmatrix} k+1 & -2\\ -6 & 2k \end{bmatrix},X=\begin{bmatrix} x\\ y \end{bmatrix},K=\begin{bmatrix} 2k\\ -8 \end{bmatrix}$$

For a unique solution, $\det(A)\neq 0$

$$\begin{aligned} \det(A)&\neq 0\\ \begin{vmatrix} k+1 & -2\\ -6 & 2k \end{vmatrix}&\neq 0\\ (k+1)\times 2k-(-2\times -6)&\neq 0\\ 2k(k+1)-12&\neq 0\\ 2k^2+2k-12&\neq 0\\ 2(k^2+k-6)&\neq 0\\ (k+3)(k-2)&\neq 0\\ k&\neq -3,2\\ k&\in R\setminus\{-3,2\} \end{aligned}$$

b If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\begin{aligned} \det(A)&=0\\ (k+3)(k-2)&=0\\ k&=-3,2 \end{aligned}$$

$k=-3$

(1) $-2x-2y=-6$

$x+y=3$

(2) $-6x-6y=-8$

$x+y=4$

Therefore there is no solution (lines are parallel) when $k=-3$

c For an infinite number of solutions, $\det(A)=0$ and the lines are the same.

$$\begin{aligned} \det(A)&=0\\ (k+3)(k-2)&=0\\ k&=-3,2 \end{aligned}$$

$k=2$

(1) $3x-2y=4$

(2) $-6x+4y=-8$

$\Rightarrow (2)=-2\times(1)$

Therefore, there is an infinite number of solutions when $k=2$

8 **a** $2x+(k-1)y=4$

$kx+6y=k+4$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix} 2 & k-1\\ k & 6 \end{bmatrix}\begin{bmatrix} x\\ y \end{bmatrix}=\begin{bmatrix} 4\\ k+4 \end{bmatrix}$$

$$A=\begin{bmatrix} 2 & k-1\\ k & 6 \end{bmatrix},\ X=\begin{bmatrix} x\\ y \end{bmatrix},\ K=\begin{bmatrix} 4\\ k+4 \end{bmatrix}$$

For a unique solution, $\det(A)\neq 0$

$$\begin{aligned} \det(A)&\neq 0\\ \begin{vmatrix} 2 & k-1\\ k & 6 \end{vmatrix}&\neq 0\\ 2\times 6-(k-1)\times k&\neq 0\\ 12-k(k-1)&\neq 0\\ 12-k^2+k&\neq 0\\ -(k^2-k-12)&\neq 0\\ (k+3)(k-4)&\neq 0\\ k&\neq -3,4\\ k&\in R\setminus\{-3,4\} \end{aligned}$$

b If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\begin{aligned} \det(A)&=0\\ (k+3)(k-4)&=0\\ k&=-3,4 \end{aligned}$$

$k=-3$

(1) $2x-4y=4$

$x-2y=2$

(2) $-3x+6y=1$

$x-2y=-\frac{1}{3}$

Therefore, there is no solution (lines are parallel) when $k=-3$.

c For an infinite number of solutions, $\det(A)=0$ and the lines are the same.

$$\begin{aligned} \det(A)&=0\\ (k+3)(k-4)&=0\\ k&=-3,4 \end{aligned}$$

$k = 4$

(1) $2x + 3y = 4$

(2) $4x + 6y = 8$

$\Rightarrow (2) = 2 \times (1)$

Therefore there is an infinite number of solutions when $k = 4$.

9 a
$$\begin{aligned}
AX &= B\\
A^{-1}AX &= A^{-1}B\\
X &= A^{-1}B\\
&= \frac{1}{2\times -4-(-3\times 1)}\begin{bmatrix} -4 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix} -1 & 4\\ 3 & 5\end{bmatrix}\\
&= -\frac{1}{5}\begin{bmatrix} -4\times -1+3\times 3 & -4\times 4+3\times 5\\ -1\times -1+2\times 3 & -1\times 4+2\times 5\end{bmatrix}\\
&= -\frac{1}{5}\begin{bmatrix} 13 & -1\\ 7 & 6\end{bmatrix}\\
&= \frac{1}{5}\begin{bmatrix} -13 & 1\\ -7 & -6\end{bmatrix}
\end{aligned}$$

b
$$\begin{aligned}
XA &= B\\
XAA^{-1} &= BA^{-1}\\
X &= BA^{-1}\\
&= \begin{bmatrix} -1 & 4\\ 3 & 5\end{bmatrix}\times -\frac{1}{5}\begin{bmatrix} -4 & 3\\ -1 & 2\end{bmatrix}\\
&= -\frac{1}{5}\begin{bmatrix} -1 & 4\\ 3 & 5\end{bmatrix}\begin{bmatrix} -4 & 3\\ -1 & 2\end{bmatrix}\\
&= -\frac{1}{5}\begin{bmatrix} -1\times -4+4\times -1 & -1\times 3+4\times 2\\ 3\times -4+5\times -1 & 3\times 3+5\times 2\end{bmatrix}\\
&= -\frac{1}{5}\begin{bmatrix} 0 & 5\\ -17 & 19\end{bmatrix}\\
&= \frac{1}{5}\begin{bmatrix} 0 & -5\\ 17 & -19\end{bmatrix}
\end{aligned}$$

10 a
$$\begin{aligned}
XP - Q &= O\\
XP &= O + Q\\
XP &= Q\\
XPP^{-1} &= QP^{-1}\\
X &= QP^{-1}\\
&= \begin{bmatrix} 2 & -1\\ -3 & 6\end{bmatrix}\times\frac{1}{1\times 4-(-2\times 3)}\begin{bmatrix} 4 & 2\\ -3 & 1\end{bmatrix}\\
&= \frac{1}{10}\begin{bmatrix} 2 & -1\\ -3 & 6\end{bmatrix}\begin{bmatrix} 4 & 2\\ -3 & 1\end{bmatrix}\\
&= \frac{1}{10}\begin{bmatrix} 2\times 4-1\times -3 & 2\times 2-1\times 1\\ -3\times 4+6\times -3 & -3\times 2+6\times 1\end{bmatrix}\\
&= \frac{1}{10}\begin{bmatrix} 11 & 3\\ -30 & 0\end{bmatrix}
\end{aligned}$$

b
$$\begin{aligned}
PX - Q &= O\\
PX &= O + Q\\
PX &= Q\\
P^{-1}PX &= P^{-1}Q\\
X &= P^{-1}Q\\
&= \frac{1}{10}\begin{bmatrix} 4 & 2\\ -3 & 1\end{bmatrix}\begin{bmatrix} 2 & -1\\ -3 & 6\end{bmatrix}\\
&= \frac{1}{10}\begin{bmatrix} 4\times 2+2\times -3 & 4\times -1+2\times 6\\ -3\times 2+1\times -3 & -3\times -1+1\times 6\end{bmatrix}\\
&= \frac{1}{10}\begin{bmatrix} 2 & 8\\ -9 & 9\end{bmatrix}
\end{aligned}$$

11 a
$$\begin{aligned}
AX &= C\\
A^{-1}AX &= A^{-1}C\\
X &= A^{-1}C\\
&= \frac{1}{-2\times -5-4\times 3}\begin{bmatrix} -5 & -4\\ -3 & -2\end{bmatrix}\begin{bmatrix} -2\\ 3\end{bmatrix}\\
&= -\frac{1}{2}\begin{bmatrix} -5\times -2-4\times 3\\ -3\times -2-2\times 3\end{bmatrix}\\
&= -\frac{1}{2}\begin{bmatrix} -2\\ 0\end{bmatrix}\\
&= \begin{bmatrix} 1\\ 0\end{bmatrix}
\end{aligned}$$

b
$$\begin{aligned}
XA &= D\\
XAA^{-1} &= DA^{-1}\\
X &= DA^{-1}\\
&= \begin{bmatrix} 2 & -5\end{bmatrix}\times -\frac{1}{2}\begin{bmatrix} -5 & -4\\ -3 & -2\end{bmatrix}\\
&= -\frac{1}{2}\begin{bmatrix} 2\times -5-5\times -3 & 2\times -4-5\times -2\end{bmatrix}\\
&= -\frac{1}{2}\begin{bmatrix} -5 & -2\end{bmatrix}
\end{aligned}$$

12 a
$$\begin{aligned}
BX &= C\\
B^{-1}BX &= B^{-1}C\\
X &= B^{-1}C\\
&= \frac{1}{-5\times 4-(-3\times 3)}\begin{bmatrix} 4 & 3\\ -3 & -5\end{bmatrix}\begin{bmatrix} -1\\ 2\end{bmatrix}\\
&= -\frac{1}{11}\begin{bmatrix} 4\times -1+3\times 2\\ -3\times -1-5\times 2\end{bmatrix}\\
&= -\frac{1}{11}\begin{bmatrix} 2\\ -7\end{bmatrix}\\
&= \frac{1}{11}\begin{bmatrix} -2\\ 7\end{bmatrix}
\end{aligned}$$

b
$$\begin{aligned}
XB &= D\\
XBB^{-1} &= DB^{-1}\\
X &= DB^{-1}\\
&= \begin{bmatrix} 4 & 3\end{bmatrix}\times -\frac{1}{11}\begin{bmatrix} 4 & 3\\ -3 & -5\end{bmatrix}\\
&= -\frac{1}{11}\begin{bmatrix} 4\times 4+3\times -3 & 4\times 3+3\times -5\end{bmatrix}\\
&= -\frac{1}{11}\begin{bmatrix} 7 & -3\end{bmatrix}\\
&= \frac{1}{11}\begin{bmatrix} -7 & 3\end{bmatrix}
\end{aligned}$$

13 a $2x + 3y = 4$

$-x + 4y = 9$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 2 & 3\\ -1 & 4\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix} = \begin{bmatrix} 4\\ 9\end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3\\ -1 & 4\end{bmatrix}, X = \begin{bmatrix} x\\ y\end{bmatrix}, K = \begin{bmatrix} 4\\ 9\end{bmatrix}$$

$$\begin{aligned}
AX &= K\\
A^{-1}AX &= A^{-1}K\\
X &= A^{-1}K
\end{aligned}$$

$$\begin{aligned}\det(A) &= 2\times 4-3\times -1\\ &= 8+3\\ &= 11\end{aligned}$$

$$A^{-1}=\frac{1}{11}\begin{bmatrix}4 & -3\\ 1 & 2\end{bmatrix}$$

$$\begin{aligned}X &= A^{-1}K\\ &= \frac{1}{11}\begin{bmatrix}4 & -3\\ 1 & 2\end{bmatrix}\begin{bmatrix}4\\ 9\end{bmatrix}\\ &= \frac{1}{11}\begin{bmatrix}4\times 4-3\times 9\\ 1\times 4+2\times 9\end{bmatrix}\\ &= \frac{1}{11}\begin{bmatrix}-11\\ 22\end{bmatrix}\\ &= \begin{bmatrix}-1\\ 2\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-1\\ 2\end{bmatrix}$$

$x=-1,\ y=2$

b $4x+5y=-6$

$2x-3y=8$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix}4 & 5\\ 2 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-6\\ 8\end{bmatrix}$$

$$A=\begin{bmatrix}4 & 5\\ 2 & -3\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix},\ K=\begin{bmatrix}-6\\ 8\end{bmatrix}$$

$$\begin{aligned}AX &= K\\ A^{-1}AX &= A^{-1}K\\ X &= A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A) &= 4\times -3-5\times 2\\ &= -12-10\\ &= -22\end{aligned}$$

$$\begin{aligned}A^{-1} &= -\frac{1}{22}\begin{bmatrix}-3 & -5\\ -2 & 4\end{bmatrix}\\ &= \frac{1}{22}\begin{bmatrix}3 & 5\\ 2 & -4\end{bmatrix}\end{aligned}$$

$$\begin{aligned}X &= A^{-1}K\\ &= \frac{1}{22}\begin{bmatrix}3 & 5\\ 2 & -4\end{bmatrix}\begin{bmatrix}-6\\ 8\end{bmatrix}\\ &= \frac{1}{22}\begin{bmatrix}3\times -6+5\times 8\\ 2\times -6-4\times 8\end{bmatrix}\\ &= \frac{1}{22}\begin{bmatrix}22\\ -44\end{bmatrix}\\ &= \begin{bmatrix}1\\ -2\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ -2\end{bmatrix}$$

$x=1,\ y=-2$

c $x-2y=8$

$5x+4y=-2$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix}1 & -2\\ 5 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}8\\ -2\end{bmatrix}$$

$$A=\begin{bmatrix}1 & -2\\ 5 & 4\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix},\ K=\begin{bmatrix}8\\ -2\end{bmatrix}$$

$$\begin{aligned}AX &= K\\ A^{-1}AX &= A^{-1}K\\ X &= A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A) &= 1\times 4-(-2\times 5)\\ &= 4+10\\ &= 14\end{aligned}$$

$$A^{-1}=\frac{1}{14}\begin{bmatrix}4 & 2\\ -5 & 1\end{bmatrix}$$

$$\begin{aligned}X &= A^{-1}K\\ &= \frac{1}{14}\begin{bmatrix}4 & 2\\ -5 & 1\end{bmatrix}\begin{bmatrix}8\\ -2\end{bmatrix}\\ &= \frac{1}{14}\begin{bmatrix}4\times 8+2\times -2\\ -5\times 8+1\times -2\end{bmatrix}\\ &= \frac{1}{14}\begin{bmatrix}28\\ -42\end{bmatrix}\\ &= \begin{bmatrix}2\\ -3\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ -3\end{bmatrix}$$

$x=2,\ y=-3$

d $-2x+7y+3=0$

$3x+y+7=0$

Write the equations in the correct format.

$-2x+7y=-3$

$3x+y=-7$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix}-2 & 7\\ 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-3\\ -7\end{bmatrix}$$

$$A=\begin{bmatrix}-2 & 7\\ 3 & 1\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix},\ K=\begin{bmatrix}-3\\ -7\end{bmatrix}$$

$$\begin{aligned}AX &= K\\ A^{-1}AX &= A^{-1}K\\ X &= A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A) &= -2\times 1-7\times 3\\ &= -2-21\\ &= -23\end{aligned}$$

$$A^{-1}=-\frac{1}{23}\begin{bmatrix}1 & -7\\ -3 & -2\end{bmatrix}$$

$$\begin{aligned}X &= A^{-1}K\\ &= -\frac{1}{23}\begin{bmatrix}1 & -7\\ -3 & -2\end{bmatrix}\begin{bmatrix}-3\\ -7\end{bmatrix}\\ &= -\frac{1}{23}\begin{bmatrix}1\times -3-7\times -7\\ -3\times -3-2\times -7\end{bmatrix}\\ &= -\frac{1}{23}\begin{bmatrix}46\\ 23\end{bmatrix}\\ &= \begin{bmatrix}-2\\ -1\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-2\\ -1\end{bmatrix}$$

$x=-2,\ y=-1$

14 a $3x+4y=6$

$2x+3y=5$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}6\\ 5\end{bmatrix}$$

$$A=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix},\ K=\begin{bmatrix}6\\ 5\end{bmatrix}$$

$$\begin{aligned}AX &= K\\ A^{-1}AX &= A^{-1}K\\ X &= A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A) &= 3\times 3-4\times 2\\ &= 9-8\\ &= 1\end{aligned}$$

$$A^{-1}=\begin{bmatrix} 3 & -4\\ -2 & 3\end{bmatrix}$$

$$\begin{aligned}X &= A^{-1}K\\ &= \begin{bmatrix} 3 & -4\\ -2 & 3\end{bmatrix}\begin{bmatrix} 6\\ 5\end{bmatrix}\\ &= \begin{bmatrix} 3\times 6-4\times 5\\ -2\times 6+3\times 5\end{bmatrix}\\ &= \begin{bmatrix} -2\\ 3\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} -2\\ 3\end{bmatrix}$$

$x=-2,\ y=3$

b $x+4y=5$

$3x-y=-11$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 1 & 4\\ 3 & -1\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 5\\ -11\end{bmatrix}$$

$$A=\begin{bmatrix} 1 & 4\\ 3 & -1\end{bmatrix},\ X=\begin{bmatrix} x\\ y\end{bmatrix},\ K=\begin{bmatrix} 5\\ -11\end{bmatrix}$$

$$\begin{aligned}AX &= K\\ A^{-1}AX &= A^{-1}K\\ X &= A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A) &= 1\times -1-4\times 3\\ &= -1-12\\ &= -13\end{aligned}$$

$$\begin{aligned}A^{-1} &= -\frac{1}{13}\begin{bmatrix} -1 & -4\\ -3 & 1\end{bmatrix}\\ &= \frac{1}{13}\begin{bmatrix} 1 & 4\\ 3 & -1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}X &= A^{-1}K\\ &= \frac{1}{13}\begin{bmatrix} 1 & 4\\ 3 & -1\end{bmatrix}\begin{bmatrix} 5\\ -11\end{bmatrix}\\ &= \frac{1}{13}\begin{bmatrix} 1\times 5+4\times -11\\ 3\times 5-1\times -11\end{bmatrix}\\ &= \frac{1}{13}\begin{bmatrix} -39\\ 26\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} -3\\ 2\end{bmatrix}$$

$x=-3,\ y=2$

c $-4x+3y=13$

$2x-y=5$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} -4 & 3\\ 2 & -1\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 13\\ 5\end{bmatrix}$$

$$A=\begin{bmatrix} -4 & 3\\ 2 & -1\end{bmatrix},\ X=\begin{bmatrix} x\\ y\end{bmatrix},\ K=\begin{bmatrix} 13\\ 5\end{bmatrix}$$

$$\begin{aligned}AX &= K\\ A^{-1}AX &= A^{-1}K\\ X &= A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A) &= -4\times -1-3\times 2\\ &= 4-6\\ &= -2\end{aligned}$$

$$\begin{aligned}A^{-1} &= -\frac{1}{2}\begin{bmatrix} -1 & -3\\ -2 & -4\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix} 1 & 3\\ 2 & 4\end{bmatrix}\end{aligned}$$

$$\begin{aligned}X &= A^{-1}K\\ &= \frac{1}{2}\begin{bmatrix} 1 & 3\\ 2 & 4\end{bmatrix}\begin{bmatrix} 13\\ 5\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix} 1\times 13+3\times 5\\ 2\times 13+4\times 5\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix} 28\\ 46\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 14\\ 23\end{bmatrix}$$

$x=14,\ y=23$

d $-2x+5y=15$

$3x-2y=16$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} -2 & 5\\ 3 & -2\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 15\\ 16\end{bmatrix}$$

$$A=\begin{bmatrix} -2 & 5\\ 3 & -2\end{bmatrix},\ X=\begin{bmatrix} x\\ y\end{bmatrix},\ K=\begin{bmatrix} 15\\ 16\end{bmatrix}$$

$$\begin{aligned}AX &= K\\ A^{-1}AX &= A^{-1}K\\ X &= A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A) &= -2\times -2-5\times 3\\ &= 4-15\\ &= -11\end{aligned}$$

$$\begin{aligned}A^{-1} &= -\frac{1}{11}\begin{bmatrix} -2 & -5\\ -3 & -2\end{bmatrix}\\ &= \frac{1}{11}\begin{bmatrix} 2 & 5\\ 3 & 2\end{bmatrix}\end{aligned}$$

$$\begin{aligned}X &= A^{-1}K\\ &= \frac{1}{11}\begin{bmatrix} 2 & 5\\ 3 & 2\end{bmatrix}\begin{bmatrix} 15\\ 16\end{bmatrix}\\ &= \frac{1}{11}\begin{bmatrix} 2\times 15+5\times 16\\ 3\times 15+2\times 16\end{bmatrix}\\ &= \frac{1}{11}\begin{bmatrix} 110\\ 77\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 10\\ 7\end{bmatrix}$$

$x=10,\ y=7$

15 a i Sub in the x, y point $(12, 6)$

$$\frac{12}{a}+\frac{6}{b}=1$$

Sub in the x, y point $(8, 3)$

$$\frac{8}{a}+\frac{3}{b}=1$$

Therefore the two equations are:

$$\frac{12}{a}+\frac{6}{b}=1$$

$$\frac{8}{a}+\frac{3}{b}=1$$

ii Writing the equations as a matrix equation gives:

$$\begin{bmatrix}12 & 6\\ 8 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{a}\\ \frac{1}{b}\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$A=\begin{bmatrix}12 & 6\\ 8 & 3\end{bmatrix}, X=\begin{bmatrix}\frac{1}{a}\\ \frac{1}{b}\end{bmatrix}, K=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\begin{aligned}AX&=K\\ A^{-1}AX&=A^{-1}K\\ X&=A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=12\times3-6\times8\\ &=36-48\\ &=-12\end{aligned}$$

$$\begin{aligned}A^{-1}&=-\frac{1}{12}\begin{bmatrix}3 & -6\\ -8 & 12\end{bmatrix}\\ &=\frac{1}{12}\begin{bmatrix}-3 & 6\\ 8 & -12\end{bmatrix}\end{aligned}$$

$$\begin{aligned}X&=A^{-1}K\\ &=\frac{1}{12}\begin{bmatrix}-3 & 6\\ 8 & -12\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\\ &=\frac{1}{12}\begin{bmatrix}-3\times1+6\times1\\ 8\times1-12\times1\end{bmatrix}\\ &=\frac{1}{12}\begin{bmatrix}3\\ -4\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}\frac{1}{a}\\ \frac{1}{b}\end{bmatrix}=\begin{bmatrix}\frac{3}{12}\\ \frac{-4}{12}\end{bmatrix}$$

Taking the equations out of the matrices gives:

$$\begin{aligned}\frac{1}{a}&=\frac{3}{12}\\ a&=\frac{12}{3}\\ a&=4\end{aligned}$$

$$\begin{aligned}\frac{1}{b}&=\frac{-4}{12}\\ b&=\frac{12}{-4}\\ b&=-3\end{aligned}$$

$\therefore a=4,\ b=-3$

b i Sub in the x, y point $(4,5)$

$$\frac{4}{a}+\frac{5}{b}=1$$

Sub in the x, y point $(-4,-15)$

$$\frac{-4}{a}-\frac{15}{b}=1$$

Therefore the two equations are:

$$\begin{aligned}\frac{4}{a}+\frac{5}{b}&=1\\ -\frac{4}{a}-\frac{15}{b}&=1\end{aligned}$$

ii Writing the equations as a matrix equation gives:

$$\begin{bmatrix}4 & 5\\ -4 & -15\end{bmatrix}\begin{bmatrix}\frac{1}{a}\\ \frac{1}{b}\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$A=\begin{bmatrix}4 & 5\\ -4 & -15\end{bmatrix}, X=\begin{bmatrix}\frac{1}{a}\\ \frac{1}{b}\end{bmatrix}, K=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\begin{aligned}AX&=K\\ A^{-1}AX&=A^{-1}K\\ X&=A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=4\times-15-5\times-4\\ &=-60+20\\ &=-40\end{aligned}$$

$$\begin{aligned}A^{-1}&=-\frac{1}{40}\begin{bmatrix}-15 & -5\\ 4 & 4\end{bmatrix}\\ &=\frac{1}{40}\begin{bmatrix}15 & 5\\ -4 & -4\end{bmatrix}\end{aligned}$$

$$\begin{aligned}X&=A^{-1}K\\ &=\frac{1}{40}\begin{bmatrix}15 & 5\\ -4 & -4\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\\ &=\frac{1}{40}\begin{bmatrix}15\times1+5\times1\\ -4\times1-4\times1\end{bmatrix}\\ &=\frac{1}{40}\begin{bmatrix}20\\ -8\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}\frac{1}{a}\\ \frac{1}{b}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{-1}{5}\end{bmatrix}$$

Taking the equations out of the matrices gives:

$$\begin{aligned}\frac{1}{a}&=\frac{1}{2}\\ a&=\frac{2}{1}\\ a&=2\end{aligned}$$

$$\begin{aligned}\frac{1}{b}&=\frac{-1}{5}\\ b&=-\frac{5}{1}\\ b&=-5\end{aligned}$$

$\therefore a=2,\ b=-5$

16 a i

$$\begin{aligned}x-3y&=k\\ -2x+6y&=6\end{aligned}$$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix}1 & -3\\ -2 & 6\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}k\\ 6\end{bmatrix}$$

$$A=\begin{bmatrix}1 & -3\\ -2 & 6\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}, K=\begin{bmatrix}k\\ 6\end{bmatrix}$$

If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\begin{aligned}\det(A)&=1\times6-(-3\times-2)\\ &=6-6\\ &=0\end{aligned}$$

Determinant of A is constant at 0.
Therefore look for when the lines are parallel.

$$\begin{aligned} x-3y&=k && (1)\\ -2x+6y&=6 && (2) \end{aligned}$$

$$(2)\times -\frac{1}{2} \Rightarrow x-3y=-3$$

The lines are parallel when the constant is different.

$$\begin{aligned} (2)\times -\frac{1}{2} &\neq (1)\\ k &\neq -3 \end{aligned}$$

Therefore there is no solution (lines are parallel) when $k \neq -3$.

ii For an infinite number of solutions, $\det(A)=0$ and the lines are the same.
Determinant is constant at 0.
Therefore look for when the lines are the same.

$$\begin{aligned} x-3y&=k && (1)\\ -2x+6y&=6 && (2) \end{aligned}$$

$$(2)\times -\frac{1}{2} \Rightarrow x-3y=-3$$

The lines are the same when the constant is the same.

$$\begin{aligned} (2)\times -\frac{1}{2} &= (1)\\ k &= -3 \end{aligned}$$

Therefore, there is an infinite number of solutions when $k=-3$.

b i $2x-5y=4$
$-4x+10y=k$
Writing the equations as a matrix equation gives.

$$\begin{bmatrix} 2 & -5 \\ -4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 4 \\ k \end{bmatrix}$$

$$A=\begin{bmatrix} 2 & -5 \\ -4 & 10 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, K=\begin{bmatrix} 4 \\ k \end{bmatrix}$$

If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\begin{aligned} \det(A)&=2\times 10-(-5\times -4)\\ &=20-20\\ &=0 \end{aligned}$$

Determinant of A is constant at 0.
Therefore look for when the lines are parallel.

$$\begin{aligned} 2x-5y&=4 && (1)\\ -4x+10y&=k && (2) \end{aligned}$$

$$(1)\times -2 \Rightarrow -4x+10y=-8$$

The lines are parallel when the constant is different.

$$\begin{aligned} (1)\times -2 &\neq (1)\\ k &\neq -8 \end{aligned}$$

Therefore there is no solution (lines are parallel) when $k \neq -8$.

ii For an infinite number of solutions, $\det(A)=0$ and the lines are the same.
Determinant is constant at 0.
Therefore look for when the lines are the same.

$$\begin{aligned} 2x-5y&=4 && (1)\\ -4x+10y&=k && (2) \end{aligned}$$

$$(1)\times -2 \Rightarrow -4x+10y=-8$$

The lines are the same when the constant is the same.

$$\begin{aligned} (1)\times -2 &= (1)\\ k &= -8 \end{aligned}$$

Therefore, there is an infinite number of solutions when $k=-8$.

c i $3x-5y=k$
$-6x+10y=10$
Writing the equations as a matrix equation gives.

$$\begin{bmatrix} 3 & -5 \\ -6 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} k \\ 10 \end{bmatrix}$$

$$A=\begin{bmatrix} 3 & -5 \\ -6 & 10 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, K=\begin{bmatrix} k \\ 10 \end{bmatrix}$$

If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\begin{aligned} \det(A)&=3\times 10-(-5\times -6)\\ &=30-30\\ &=0 \end{aligned}$$

Determinant of A is constant at 0.
Therefore look for when the lines are parallel.

$$\begin{aligned} 3x-5y&=k && (1)\\ -6x+10y&=10 && (2) \end{aligned}$$

$$(2)\times -\frac{1}{2} \Rightarrow 3x-5y=-5$$

The lines are parallel when the constant is different.

$$\begin{aligned} (2)\times -\frac{1}{2} &\neq (1)\\ k &\neq -5 \end{aligned}$$

Therefore there is no solution (lines are parallel) when $k \neq -5$.

ii For an infinite number of solutions, $\det(A)=0$ and the lines are the same.
Determinant is constant at 0.
Therefore look for when the lines are the same.

$$\begin{aligned} 3x-5y&=k && (1)\\ -6x+10y&=10 && (2) \end{aligned}$$

$$(2)\times -\frac{1}{2} \Rightarrow 3x-5y=-5$$

The lines are the same when the constant is the same.

$$\begin{aligned} (2)\times -\frac{1}{2} &= (1)\\ k &= -5 \end{aligned}$$

Therefore, there is an infinite number of solutions when $k=-5$.

d i $4x-6y=8$
$-2x+3y=k$
Writing the equations as a matrix equation gives.

$$\begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 8 \\ k \end{bmatrix}$$

$$A=\begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, K=\begin{bmatrix} 8 \\ k \end{bmatrix}$$

If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\begin{aligned} \det(A)&=4\times 3-(-6\times -2)\\ &=12-12\\ &=0 \end{aligned}$$

Determinant of A is constant at 0.

Therefore look for when the lines are parallel.

$$\begin{aligned} 4x-6y&=8 \quad (1)\\ -2x+3y&=k \quad (2)\end{aligned}$$

$$(1)\times -\frac{1}{2} \Rightarrow -2x+3y=-4$$

The lines are parallel when the constant is different.

$$(1)\times -\frac{1}{2} \neq (2)$$

$$k \neq -4$$

Therefore there is no solution (lines are parallel) when $k \neq -4$.

ii For an infinite number of solutions, $\det(A)=0$ and the lines are the same.

Determinant is constant at 0.

Therefore look for when the lines are the same.

$$\begin{aligned} 4x-6y&=8 \quad (1)\\ -2x+3y&=k \quad (2)\end{aligned}$$

$$(1)\times -\frac{1}{2} \Rightarrow -2x+3y=-4$$

The lines are the same when the constant is the same.

$$(1)\times -\frac{1}{2} = (2)$$

$$k=-4$$

Therefore, there is an infinite number of solutions when $k=-4$.

17 a

$$\begin{aligned} x-2y&=3 \quad (1)\\ -2x+4y&=-6 \quad (2)\end{aligned}$$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 1 & -2\\ -2 & 4\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 3\\ -6\end{bmatrix}$$

$$A=\begin{bmatrix} 1 & -2\\ -2 & 4\end{bmatrix}, X=\begin{bmatrix} x\\ y\end{bmatrix}, K=\begin{bmatrix} 3\\ -6\end{bmatrix}$$

$$\begin{aligned} AX&=K\\ A^{-1}AX&=A^{-1}K\\ X&=A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=1\times 4-(-2\times -2)\\ &=4-4\\ &=0\end{aligned}$$

Since the $\det(A)=0$ there is no unique solution for these simultaneous equations.

Since (2) is a multiple of (1) as $(2)=-2\times(1)$ the lines are the same and therefore there are infinitely many solutions.

In general let $y=t$ and then substitute for y into equation (1) and solve for x:

$$\begin{aligned} y&=t\\ x-2t&=3\\ x&=2t+3\end{aligned}$$

Therefore, infinite solutions are of the form $(2t+3,\ t),\ t\in R$

b

$$\begin{aligned} 2x-3y&=5 \quad (1)\\ -4x+6y&=-11 \quad (2)\end{aligned}$$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 2 & -3\\ -4 & 6\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 5\\ -11\end{bmatrix}$$

$$A=\begin{bmatrix} 2 & -3\\ -4 & 6\end{bmatrix}, X=\begin{bmatrix} x\\ y\end{bmatrix}, K=\begin{bmatrix} 5\\ -11\end{bmatrix}$$

$$\begin{aligned} AX&=K\\ A^{-1}AX&=A^{-1}K\\ X&=A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=2\times 6-(-3\times -4)\\ &=12-12\\ &=0\end{aligned}$$

Since the $\det(A)=0$ there is no unique solution for these simultaneous equations.

Since (2) is a not a multiple of (1) the lines are parallel and therefore there is no solution.

c

$$\begin{aligned} 2x-y&=4 \quad (1)\\ -4x+2y&=-7 \quad (2)\end{aligned}$$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 2 & -1\\ -4 & 2\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 4\\ -7\end{bmatrix}$$

$$A=\begin{bmatrix} 2 & -1\\ -4 & 2\end{bmatrix}, X=\begin{bmatrix} x\\ y\end{bmatrix}, K=\begin{bmatrix} 4\\ -7\end{bmatrix}$$

$$\begin{aligned} AX&=K\\ A^{-1}AX&=A^{-1}K\\ X&=A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=2\times 2-(-1\times -4)\\ &=4-4\\ &=0\end{aligned}$$

Since the $\det(A)=0$ there is no unique solution for these simultaneous equations.

Since (2) is a not a multiple of (1) the lines are parallel and therefore there is no solution.

d

$$\begin{aligned} 3x-4y&=5 \quad (1)\\ -6x+8y&=-10 \quad (2)\end{aligned}$$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 3 & -4\\ -6 & 8\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 5\\ -10\end{bmatrix}$$

$$A=\begin{bmatrix} 3 & -4\\ -6 & 8\end{bmatrix}, X=\begin{bmatrix} x\\ y\end{bmatrix}, K=\begin{bmatrix} 5\\ -10\end{bmatrix}$$

$$\begin{aligned} AX&=K\\ A^{-1}AX&=A^{-1}K\\ X&=A^{-1}K\end{aligned}$$

$$\begin{aligned}\det(A)&=3\times 8-(-4\times -6)\\ &=24-24\\ &=0\end{aligned}$$

Since the $\det(A)=0$ there is no unique solution for these simultaneous equations.

Since (2) is a multiple of (1) as $(2)=-2\times(1)$ the lines are the same and therefore there are infinitely many solutions.

In general let $y=t$ and then substitute for y into equation (1) and solve for x:

$$\begin{aligned} y&=t\\ 3x-4t&=5\\ 3x&=4t+5\\ \frac{3x}{3}&=\frac{4t+5}{3}\\ x&=\frac{4t+5}{3}\end{aligned}$$

Therefore, infinite solutions are of the form $\left(\frac{4t+5}{3},\ t\right),\ t\in R$

18 a i $(k-2)x-2y=k-1$

$-4x+ky=-6$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix} k-2 & -2 \\ -4 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} k-1 \\ -6 \end{bmatrix}$$

$$A=\begin{bmatrix} k-2 & -2 \\ -4 & k \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, K=\begin{bmatrix} k-1 \\ -6 \end{bmatrix}$$

For a unique solution, $\det(A)\neq 0$

$$\det(A)\neq 0$$
$$\begin{vmatrix} k-2 & -2 \\ -4 & k \end{vmatrix}\neq 0$$
$$(k-2)\times k-(-2\times -4)\neq 0$$
$$k(k-2)-8\neq 0$$
$$k^2-2k-8\neq 0$$
$$(k+2)(k-4)\neq 0$$
$$k\neq -2,4$$
$$k\in R\setminus\{-2,4\}$$

ii If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\det(A)=0$$
$$(k+2)(k-4)=0$$
$$k=-2,4$$
$$k=-2$$
$$(1)\quad -4x-2y=-3$$
$$(2)\quad -4x-2y=-6$$
$$(2)\neq(1)$$

Therefore there is no solution (lines are parallel) when $k=-2$.

iii For an infinite number of solutions, $\det(A)=0$ and the lines are the same.

$$\det(A)=0$$
$$(k+3)(k-2)=0$$
$$k=-3,2$$
$$k=4$$
$$(1)\quad 2x-2y=3$$
$$(2)\quad -4x+4y=-6$$
$$(2)\times-\frac{1}{2}\Rightarrow 2x-2y=3$$
$$=(1)$$

Therefore, there is an infinite number of solutions when $k=4$.

b i $(k+1)x+5y=4$

$6x+5ky=k+6$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix} k+1 & 5 \\ 6 & 5k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 4 \\ k+6 \end{bmatrix}$$

$$A=\begin{bmatrix} k+1 & 5 \\ 6 & 5k \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, K=\begin{bmatrix} 4 \\ k+6 \end{bmatrix}$$

For a unique solution, $\det(A)\neq 0$

$$\det(A)\neq 0$$
$$\begin{vmatrix} k+1 & 5 \\ 6 & 5k \end{vmatrix}\neq 0$$
$$(k+1)\times 5k-5\times 6\neq 0$$
$$5k(k+1)-30\neq 0$$
$$5k^2+5k-30\neq 0$$
$$5(k^2+k-6)\neq 0$$
$$(k+3)(k-2)\neq 0$$
$$k\neq -3,2$$
$$k\in R\setminus\{-3,2\}$$

ii If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\det(A)=0$$
$$(k+3)(k-2)=0$$
$$k=-3,2$$

$$k=-3$$

(1) $\quad -2x+5y=4$

(2) $\quad 6x-15y=3$

$$(2)\times-\frac{1}{3}\Rightarrow 2x-5y=-1$$
$$\neq(1)$$

Therefore there is no solution (lines are parallel) when $k=-3$.

iii For an infinite number of solutions, $\det(A)=0$ and the lines are the same.

$$\det(A)=0$$
$$(k+3)(k-2)=0$$
$$k=-3,2$$
$$k=2$$

(1) $\quad 3x+5y=4$

(2) $\quad 6x+10y=8$

$$(2)\times\frac{1}{2}\Rightarrow 3x+5y=4$$
$$=(1)$$

Therefore, there is an infinite number of solutions when $k=2$.

c i $(k-1)x-3y=k+2$

$-4x+2ky=-10$

Writing the equations as a matrix equation gives.

$$\begin{bmatrix} k-1 & -3 \\ -4 & 2k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} k+2 \\ -10 \end{bmatrix}$$

$$A=\begin{bmatrix} k-1 & -3 \\ -4 & 2k \end{bmatrix},\ X=\begin{bmatrix} x \\ y \end{bmatrix},\ K=\begin{bmatrix} k+2 \\ -10 \end{bmatrix}$$

For a unique solution, $\det(A)\neq 0$

$$\det(A)\neq 0$$
$$\begin{vmatrix} k-1 & -3 \\ -4 & 2k \end{vmatrix}\neq 0$$
$$(k-1)\times 2k-(-3\times -4)\neq 0$$
$$2k(k-1)-12\neq 0$$
$$2k^2-2k-12\neq 0$$
$$2(k^2-k-6)\neq 0$$
$$(k+2)(k-3)\neq 0$$
$$k\neq -2,3$$
$$k\in R\setminus\{-2,3\}$$

ii If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\det(A)=0$$
$$(k+2)(k-3)=0$$
$$k=-2,3$$
$$k=-2$$

(1) $\quad -3x-3y=0$

$\quad x+y=0$

(2) $\quad -4x-4y=-10$

$$x+y=\frac{-5}{2}$$

Therefore there is no solution (lines are parallel) when $k=-2$.

iii For an infinite number of solutions, $\det(A)=0$ and the lines are the same.

$$\det(A)=0$$
$$(k+2)(k-3)=0$$
$$k=-2,3$$
$$k=3$$

(1) $\quad 2x-3y=5$

(2) $\quad -4x+6y=-10$

$$(2)\times-\frac{1}{2}\Rightarrow 2x-3y=5$$
$$=(1)$$

Therefore, there is an infinite number of solutions when $k=3$.

d i $2x-(k-2)y=6$

$(k-5)x-2y=k-3$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 2 & -(k-2) \\ k-5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 6 \\ k-3 \end{bmatrix}$$

$$A=\begin{bmatrix} 2 & -(k-2) \\ k-5 & -2 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, K=\begin{bmatrix} 6 \\ k-3 \end{bmatrix}$$

For a unique solution, $\det(A)\neq 0$

$$\det(A)\neq 0$$

$$\begin{vmatrix} 2 & -(k-2) \\ k-5 & -2 \end{vmatrix}\neq 0$$

$$2\times -2-(-(k-2)(k-5))\neq 0$$

$$-4+(k-2)(k-5)\neq 0$$

$$k^2-7k+10-4\neq 0$$

$$k^2-7k+6\neq 0$$

$$(k-1)(k-6)\neq 0$$

$$k\neq 1,6$$

$$k\in R\setminus\{1,6\}$$

ii If there is no solution, $\det(A)=0$ and the lines are parallel.

$$\det(A)=0$$

$$(k-1)(k-6)=0$$

$$k=1,6$$

$$k=1$$

$$(1)\quad 2x+y=6$$

$$(2)\ -4x-2y=-2$$

$$(2)\times -\frac{1}{2}\Rightarrow 2x+y=1$$

$$\neq (1)$$

Therefore there is no solution (lines are parallel) when $k=1$.

iii For an infinite number of solutions, $\det(A)=0$ and the lines are the same.

$$\det(A)=0$$

$$(k-1)(k-6)=0$$

$$k=1,6$$

$$k=6$$

$$(1)\quad 2x-4y=6$$

$$(2)\quad x-2y=3$$

$$(2)\times 2\Rightarrow 2x-4y=6$$

$$=(1)$$

Therefore, there is an infinite number of solutions when $k=6$.

19 a i $-2x+3y=p$

$qx-6y=7$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} -2 & 3 \\ q & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} p \\ 7 \end{bmatrix}$$

$$A=\begin{bmatrix} -2 & 3 \\ q & -6 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, K=\begin{bmatrix} p \\ 7 \end{bmatrix}$$

For a unique solution, $\det(A)\neq 0$

$$\det(A)\neq 0$$

$$\begin{vmatrix} -2 & 3 \\ q & -6 \end{vmatrix}\neq 0$$

$$-2\times -6-3\times q\neq 0$$

$$12-3q\neq 0$$

$$-3q\neq -12$$

$$\frac{-3q}{-3}\neq\frac{-12}{-3}$$

$$q\neq 4$$

There is no restriction on p, so for a unique solution $q\neq 4,\ p\in R$.

ii If there is no solution, $\det(A) = 0$ and the lines are parallel.
$\det(A) = 0$
$q = 4$
Lines are parallel so $(1) \neq (2)$
Sub in $q = 4$
$-2x + 3y = p \qquad (1)$
$4x - 6y = 7 \qquad (2)$

$(2) \times -\frac{1}{2} \Rightarrow -2x + 3y = -\frac{7}{2}$
So $(1) \neq (2)$ when $p \neq -\frac{7}{2}$
Therefore there is no solution when $q = 4,\ p \neq -\frac{7}{2}$.

iii If there are infinitely many solutions, $\det(A) = 0$ and the lines are the same.
$\det(A) = 0$
$q = 4$
The lines are the same so $(1) = (2)$
Sub in $q = 4$
$-2x + 3y = p \qquad (1)$
$4x - 6y = 7 \qquad (2)$

$(2) \times -\frac{1}{2} \Rightarrow -2x + 3y = -\frac{7}{2}$
So $(1) = (2)$ when $p = -\frac{7}{2}$
Therefore there are infinitely many solutions when $q = 4,\ p = -\frac{7}{2}$.

b i $4x - 2y = q$
$3x + py = 10$
Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 4 & -2 \\ 3 & p \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q \\ 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -2 \\ 3 & p \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ K = \begin{bmatrix} q \\ 10 \end{bmatrix}$$

For a unique solution, $\det(A) \neq 0$

$$\begin{aligned} \det(A) &\neq 0 \\ \begin{vmatrix} 4 & -2 \\ 3 & p \end{vmatrix} &\neq 0 \\ 4 \times p - (-2 \times 3) &\neq 0 \\ 4p + 6 &\neq 0 \\ 4p &\neq -6 \\ \frac{4p}{4} &\neq \frac{-6}{4} \\ p &\neq -\frac{3}{2} \end{aligned}$$

There is no restriction on q, so for a unique solution $p \neq -\frac{3}{2},\ q \in R$.

ii If there is no solution, $\det(A) = 0$ and the lines are parallel.
$\det(A) = 0$
$p = -\frac{3}{2}$
The lines are parallel so $(1) \neq (2)$
Sub in $p = -\frac{3}{2}$
$4x - 2y = q \qquad (1)$
$3x - \frac{3}{2}y = 10 \qquad (2)$

$(2) \times \frac{4}{3} \Rightarrow 4x - 2y = \frac{40}{3}$
So $(1) \neq (2)$ when $q \neq \frac{40}{3}$
Therefore there is no solution when $p = -\frac{3}{2},\ p \neq \frac{40}{3}$.

iii If there are infinitely many solutions, $\det(A) = 0$ and the lines are the same.
$\det(A) = 0$
$p = -\frac{3}{2}$
The lines are the same so $(1) = (2)$
Sub in $p = -\frac{3}{2}$
$4x - 2y = q \qquad (1)$
$3x - \frac{3}{2}y = 10 \qquad (2)$

$(2) \times \frac{4}{3} \Rightarrow 4x - 2y = \frac{40}{3}$
So $(1) = (2)$ when $q = \frac{40}{3}$
Therefore there are infinitely many solutions when $p = -\frac{3}{2},\ q = \frac{40}{3}$.

c i $3x - py = 6$
$7x - 2y = q$
Writing the equations as a matrix equation gives:

$$\begin{bmatrix} 3 & -p \\ 7 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ q \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -p \\ 7 & -2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ K = \begin{bmatrix} 6 \\ q \end{bmatrix}$$

For a unique solution, $\det(A) \neq 0$

$$\begin{aligned} \det(A) &\neq 0 \\ \begin{vmatrix} 3 & -p \\ 7 & -2 \end{vmatrix} &\neq 0 \\ 3 \times -2 - (-p \times 7) &\neq 0 \\ -6 + 7p &\neq 0 \\ 7p &\neq 6 \\ \frac{7p}{7} &\neq \frac{6}{7} \\ p &\neq \frac{6}{7} \end{aligned}$$

There is no restriction on q, so for a unique solution $p \neq \frac{6}{7},\ q \in R$.

ii If there is no solution, $\det(A) = 0$ and the lines are parallel.
$\det(A) = 0$
$p = \frac{6}{7}$
The lines are parallel so $(1) \neq (2)$
Sub in $p = \frac{6}{7}$

$3x-\frac{6}{7}y=6 \quad (1)$

$7x-2y=q \quad (2)$

$(1)\times\frac{7}{3}\Rightarrow 7x-2y=14$

So $(1)\neq(2)$ when $q\neq 14$

Therefore there is no solution when $p=\frac{6}{7}$, $p\neq 14$.

iii If there are infinitely many solutions, $\det(A)=0$ and the lines are the same.

$\det(A)=0$

$p=\frac{6}{7}$

The lines are the same so $(1)=(2)$

Sub in $p=\frac{6}{7}$

$3x-\frac{6}{7}y=6 \quad (1)$

$7x-2y=q \quad (2)$

$(1)\times\frac{7}{3}\Rightarrow 7x-2y=14$

So $(1)=(2)$ when $q=14$

Therefore there are infinitely many solutions when $p=\frac{6}{7}$, $q=14$.

d i $px-y=3$

$-3x+2y=q$

Writing the equations as a matrix equation gives:

$$\begin{bmatrix} p & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 3 \\ q \end{bmatrix}$$

$$A=\begin{bmatrix} p & -1 \\ -3 & 2 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, K=\begin{bmatrix} 3 \\ q \end{bmatrix}$$

For a unique solution, $\det(A)\neq 0$

$$\det(A)\neq 0$$

$$\begin{vmatrix} p & -1 \\ -3 & 2 \end{vmatrix}\neq 0$$

$$p\times 2-(-1\times -3)\neq 0$$

$$2p-3\neq 0$$

$$2p\neq 3$$

$$\frac{2p}{2}\neq\frac{3}{2}$$

$$p\neq\frac{3}{2}$$

There is no restriction on q, so for a unique solution $p\neq\frac{3}{2}$, $q\in R$.

ii If there is no solution, $\det(A)=0$ and the lines are parallel.

$\det(A)=0$

$p=\frac{3}{2}$

The lines are parallel so $(1)\neq(2)$

Sub in $p=\frac{3}{2}$

$\frac{3}{2}x-y=3 \quad (1)$

$-3x+2y=q \quad (2)$

$(1)\times -2\Rightarrow -3x+2y=-6$

So $(1)\neq(2)$ when $q\neq -6$

Therefore there is no solution when $p=\frac{3}{2}$, $p\neq -6$.

iii If there are infinitely many solutions, $\det(A)=0$ and the lines are the same.

$\det(A)=0$

$p=\frac{3}{2}$

The lines are the same so $(1)=(2)$

Sub in $p=\frac{3}{2}$

$\frac{3}{2}x-y=3 \quad (1)$

$-3x+2y=q \quad (2)$

$(1)\times -2\Rightarrow -3x+2y=-6$

So $(1)=(2)$ when $q=-6$

Therefore there are infinitely many solutions when $p=\frac{3}{2}$, $q=-6$.

20 a $AX=C$

$$A^{-1}AX=A^{-1}C$$

$$X=A^{-1}C$$

$$A^{-1}=\frac{1}{1\times 4-(-2\times 5)}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}$$

$$=\frac{1}{14}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}$$

$$X=A^{-1}C$$

$$=\frac{1}{14}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ -19 \end{bmatrix}$$

$$=\frac{1}{14}\begin{bmatrix} 4\times -5+2\times -19 \\ -5\times -5+1\times -19 \end{bmatrix}$$

$$=\frac{1}{14}\begin{bmatrix} -58 \\ 6 \end{bmatrix}$$

$$=\frac{2}{14}\begin{bmatrix} -29 \\ 3 \end{bmatrix}$$

$$=\frac{1}{7}\begin{bmatrix} -29 \\ 3 \end{bmatrix}$$

b $XA=B$

$$XAA^{-1}=BA^{-1}$$

$$X=BA^{-1}$$

$$A^{-1}=\frac{1}{1\times 4-(-2\times 5)}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}$$

$$=\frac{1}{14}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}$$

$$X=BA^{-1}$$

$$=\begin{bmatrix} 3 & 1 \\ -7 & 2 \end{bmatrix}\times\frac{1}{14}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}$$

$$=\frac{1}{14}\begin{bmatrix} 3 & 1 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}$$

$$=\frac{1}{14}\begin{bmatrix} 3\times 4+1\times -5 & 3\times 2+1\times 1 \\ -7\times 4+2\times -5 & -7\times 2+2\times 1 \end{bmatrix}$$

$$=\frac{1}{14}\begin{bmatrix} 7 & 7 \\ -38 & -12 \end{bmatrix}$$

c

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{1\times 4-(-2\times 5)}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{14}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -7 & 2 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 4\times 3+2\times -7 & 4\times 1+2\times 2 \\ -5\times 3+1\times -7 & -5\times 1+1\times 2 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} -2 & 8 \\ -22 & -3 \end{bmatrix} \end{aligned}$$

d

$$\begin{aligned} XA &= D \\ XAA^{-1} &= DA^{-1} \\ X &= DA^{-1} \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{1\times 4-(-2\times 5)}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= DA^{-1} \\ &= \begin{bmatrix} 7 & 14 \end{bmatrix}\times\frac{1}{14}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 7 & 14 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 7\times 4+14\times -5 & 7\times 2+14\times 1 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} -42 & 28 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \end{bmatrix} \end{aligned}$$

21 a

$$\begin{aligned} AX &= C \\ A^{-1}AX &= A^{-1}C \\ X &= A^{-1}C \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{2\times 4-3\times -1}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= A^{-1}C \\ &= \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 14 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 4\times 5-3\times 14 \\ 1\times 5+2\times 14 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} -22 \\ 33 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 3 \end{bmatrix} \end{aligned}$$

b

$$\begin{aligned} XA &= B \\ XAA^{-1} &= BA^{-1} \\ X &= BA^{-1} \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{2\times 4-3\times -1}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= BA^{-1} \\ &= \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\times\frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 4\times 4+5\times 1 & 4\times -3+5\times 2 \\ 2\times 4-3\times 1 & 2\times -3+-3\times 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 21 & -2 \\ 5 & -12 \end{bmatrix} \end{aligned}$$

c

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{2\times 4-3\times -1}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 4\times 4-3\times 2 & 4\times 5-3\times -3 \\ 1\times 4+2\times 2 & 1\times 5+2\times -3 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 10 & 29 \\ 8 & -1 \end{bmatrix} \end{aligned}$$

d

$$\begin{aligned} XA &= D \\ XAA^{-1} &= DA^{-1} \\ X &= DA^{-1} \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{2\times 4-3\times -1}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= DA^{-1} \\ &= \begin{bmatrix} 1 & -2 \end{bmatrix}\times\frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 1\times 4-2\times 1 & 1\times -3-2\times 2 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 2 & -7 \end{bmatrix} \end{aligned}$$

22 a

$$\begin{aligned} AX &= C \\ A^{-1}AX &= A^{-1}C \\ X &= A^{-1}C \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{-2\times 5-3\times 4}\begin{bmatrix} 5 & -3 \\ -1 & -2 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} 5 & -3 \\ -4 & -2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= A^{-1}C \\ &= \frac{1}{22}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -5\times 3+3\times 1 \\ 4\times 3+2\times 1 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -12 \\ 14 \end{bmatrix} \\ &= \frac{2}{22}\begin{bmatrix} -6 \\ 7 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} -6 \\ 7 \end{bmatrix} \end{aligned}$$

b

$$\begin{aligned} XA &= B \\ XAA^{-1} &= BA^{-1} \\ X &= BA^{-1} \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{-2\times 5-3\times 4}\begin{bmatrix} 5 & -3 \\ -1 & -2 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} 5 & -3 \\ -4 & -2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= BA^{-1} \\ &= \begin{bmatrix} 2 & 19 \\ 12 & -7 \end{bmatrix}\times\frac{1}{22}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} 2 & 19 \\ 12 & -7 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} 2\times -5+19\times 4 & 2\times 3+19\times 2 \\ 12\times -5-7\times 4 & 12\times 3-7\times 2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} 66 & 44 \\ -88 & 22 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix} \end{aligned}$$

c

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{-2\times 5-3\times 4}\begin{bmatrix} 5 & -3 \\ -1 & -2 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} 5 & -3 \\ -4 & -2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{22}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 19 \\ 12 & -7 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -5\times 2+3\times 12 & -5\times 19+3\times -7 \\ 4\times 2+2\times 12 & 4\times 19+2\times -7 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} 26 & -116 \\ 32 & 62 \end{bmatrix} \\ &= \frac{2}{22}\begin{bmatrix} 13 & -58 \\ 16 & 31 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 13 & -58 \\ 16 & 31 \end{bmatrix} \end{aligned}$$

d

$$\begin{aligned} XA &= D \\ XAA^{-1} &= DA^{-1} \\ X &= DA^{-1} \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{-2\times 5-3\times 4}\begin{bmatrix} 5 & -3 \\ -1 & -2 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} 5 & -3 \\ -4 & -2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= DA^{-1} \\ &= \begin{bmatrix} -1 & 3 \end{bmatrix}\times\frac{1}{22}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -1 & 3 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} -1\times -5+3\times 4 & -1\times 3+3\times 2 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} 17 & 3 \end{bmatrix} \end{aligned}$$

23 a

$$P = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

$$\begin{aligned} P^{-1} &= \frac{1}{a\times d-0\times 0}\begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} \\ &= \frac{1}{ad}\begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1}{a} & 0 \\ 0 & \dfrac{1}{d} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} PP^{-1} &= \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\times\frac{1}{ad}\begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} \\ &= \frac{1}{ad}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} \\ &= \frac{1}{ad}\begin{bmatrix} a\times d+0\times 0 & a\times 0+0\times a \\ 0\times d+d\times 0 & 0\times 0+d\times a \end{bmatrix} \\ &= \frac{1}{ad}\begin{bmatrix} ad & 0 \\ 0 & ad \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

$$\begin{aligned}
P^{-1}P &= \frac{1}{ad}\begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \\
&= \frac{1}{ad}\begin{bmatrix} d\times a+0\times 0 & d\times 0+0\times d \\ 0\times a+a\times 0 & 0\times 0+a\times d \end{bmatrix} \\
&= \frac{1}{ad}\begin{bmatrix} ad & 0 \\ 0 & ad \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

$\therefore PP^{-1} = P^{-1}P = I$

b $Q = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$

$$\begin{aligned}
Q^{-1} &= \frac{1}{0\times 0-b\times c}\begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} \\
&= \frac{1}{-bc}\begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
QQ^{-1} &= \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\times\frac{1}{bc}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} 0\times 0+b\times c & 0\times b+b\times 0 \\ c\times 0+0\times c & c\times b+0\times 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

$$\begin{aligned}
Q^{-1}Q &= \frac{1}{bc}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} 0\times 0+b\times c & 0\times b+b\times 0 \\ c\times 0+0\times c & c\times b+0\times 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

$\therefore QQ^{-1} = Q^{-1}Q = I$

24 a $R = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$

$$\begin{aligned}
R^{-1} &= \frac{1}{a\times 0-b\times c}\begin{bmatrix} 0 & -b \\ -c & a \end{bmatrix} \\
&= -\frac{1}{bc}\begin{bmatrix} 0 & -b \\ -c & a \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} 0 & b \\ c & -a \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{c} \\ \frac{1}{b} & \frac{-a}{bc} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
RR^{-1} &= \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\times\frac{1}{bc}\begin{bmatrix} 0 & b \\ c & -a \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ c & -a \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} a\times 0+b\times c & a\times b+b\times -a \\ c\times 0+0\times c & c\times b+0\times -a \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

$$\begin{aligned}
R^{-1}R &= \frac{1}{bc}\begin{bmatrix} 0 & b \\ c & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} 0\times a+b\times c & 0\times b+b\times 0 \\ c\times a-a\times c & c\times b-a\times 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

$\therefore RR^{-1} = R^{-1}R = I$

b $S = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$

$$\begin{aligned}
S^{-1} &= \frac{1}{0\times d-b\times c}\begin{bmatrix} d & -b \\ -c & 0 \end{bmatrix} \\
&= \frac{1}{-bc}\begin{bmatrix} d & -b \\ -c & 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} -d & b \\ c & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
SS^{-1} &= \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}\times\frac{1}{bc}\begin{bmatrix} -d & b \\ c & 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}\begin{bmatrix} -d & b \\ c & 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} 0\times -d+b\times c & 0\times b+b\times 0 \\ c\times -d+d\times c & c\times b+d\times 0 \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

$$\begin{aligned}
S^{-1}S &= \frac{1}{bc}\begin{bmatrix} -d & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} -d\times 0+b\times c & -d\times b+b\times d \\ c\times 0+0\times c & c\times b+0\times d \end{bmatrix} \\
&= \frac{1}{bc}\begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

$\therefore SS^{-1} = S^{-1}S = I$

c $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{aligned}
AA^{-1}&=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\times\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\\
&=\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\\
&=\frac{1}{ad-bc}\begin{bmatrix} a\times d+b\times -c & a\times -b+b\times a \\ c\times d+d\times -c & c\times -b+d\times a \end{bmatrix}\\
&=\frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}\\
&=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\\
&=I
\end{aligned}$$

$$\begin{aligned}
A^{-1}A&=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\\
&=\frac{1}{ad-bc}\begin{bmatrix} d\times a-b\times c & d\times b-b\times d \\ -c\times a+a\times c & -c\times b+a\times d \end{bmatrix}\\
&=\frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}\\
&=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\\
&=I
\end{aligned}$$

$\therefore AA^{-1}=A^{-1}A=I$

Exercise 7.6 — Translations

1 Sub the x, y point $(1,2)$ into the matrix equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}+\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1+3 \\ 2-2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Therefore the image point is $(4,0)$.

2 Sub the x, y point $(3,-4)$ into the matrix equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 3 \\ -4 \end{bmatrix}+\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 3-1 \\ -4+2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Therefore the image point is $(2,-2)$.

3 $P'=P+T$

$T=P'-P$

$P=\Delta ABC=\begin{bmatrix} 0 & 2 & -3 \\ 0 & 3 & 4 \end{bmatrix}$

$P'=\Delta A'B'C'=\begin{bmatrix} 1 & 3 & -2 \\ -2 & 1 & 2 \end{bmatrix}$

$$T=\begin{bmatrix} 1 & 3 & -2 \\ -2 & 1 & 2 \end{bmatrix}-\begin{bmatrix} 0 & 2 & -3 \\ 0 & 3 & 4 \end{bmatrix}$$

$$T=\begin{bmatrix} 1-0 & 3-2 & -2+3 \\ -2-0 & 1-3 & 2-4 \end{bmatrix}$$

$$T=\begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}$$

4 $P'=P+T$

$T=P'-P$

$P=\Delta ABC=\begin{bmatrix} 3 & 2 & -2 \\ 0 & 4 & -5 \end{bmatrix}$

$P'=\Delta A'B'C'=\begin{bmatrix} 4 & 3 & -1 \\ 2 & 6 & -3 \end{bmatrix}$

$$T=\begin{bmatrix} 4 & 3 & -1 \\ 2 & 6 & -3 \end{bmatrix}-\begin{bmatrix} 3 & 2 & -2 \\ 0 & 4 & -5 \end{bmatrix}$$

$$T=\begin{bmatrix} 4-3 & 3-2 & -1+2 \\ 2-0 & 6-4 & -3+5 \end{bmatrix}$$

$$T=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

5 $\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} x+3 \\ y-2 \end{bmatrix}$$

$x'=x+3$

$y'=y-2$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$x'=x+3$

$x=x'-3$

$y'=y-2$

$y=y'+2$

Substitute the image equations for the two coordinates into the original equation, $y=x-1$

$$\begin{aligned}
y&=x-1\\
y'+2&=x'-3-1\\
y'&=x'-4-2\\
y'&=x'-6
\end{aligned}$$

Therefore the image equation is $y=x-6$.

6 $\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} x-2 \\ y+1 \end{bmatrix}$$

$x'=x-2$

$y'=y+1$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$x' = x - 2$
$x = x' + 2$

$y' = y + 1$
$y = y' - 1$

Substitute the image equations for the two coordinates into the original equation, $y = x + 3$

$$\begin{aligned} y &= x + 3 \\ y' - 1 &= x' + 2 + 3 \\ y' &= x' + 5 + 1 \\ y' &= x' + 6 \end{aligned}$$

Therefore the image equation is $y = x + 6$.

7 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2 \\ y - 1 \end{bmatrix}$

$x' = x + 2$
$y' = y - 1$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$x' = x + 2$
$x = x' - 2$

$y' = y - 1$
$y = y' + 1$

Substitute the image equations for the two coordinates into the original equation, $y = x^2$

$$\begin{aligned} y &= x^2 \\ y' + 1 &= (x' - 2)^2 \\ y' &= (x' - 2)^2 - 1 \end{aligned}$$

Therefore the image equation is $y = (x - 2)^2 - 1$.

8 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - 3 \\ y \end{bmatrix}$

$x' = x - 3$
$y' = y$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$x' = x - 3$
$x = x' + 3$

$y' = y$
$y = y'$

Substitute the image equations for the two coordinates into the original equation, $y = x^2 + 1$

$$\begin{aligned} y &= x^2 + 1 \\ y' &= (x' + 3)^2 + 1 \end{aligned}$$

Therefore the image equation is $y = (x + 3)^2 + 1$.

9 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2+1 \\ 5+2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(3, 7)$.

10 Sub the x, y point $(-1, 0)$ into the matrix equation:

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1-5 \\ 0+2 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ 2 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(-6, 2)$.

11 Put the equations into a matrix equation.

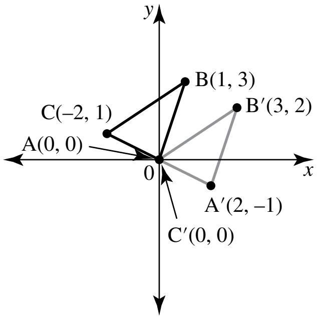
$x' = x + 2$
$y' = y + 1$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2 \\ y + 1 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

12 a $\Delta ABC = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix}$ gives the three points of the triangle, point $A(0,0)$, point $B(1,3)$ and point $C(-2,1)$

$\Delta A'B'C' = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ gives the three points of the triangle, point $A'(2,-1)$, point $B'(3,2)$ and point $C'(0,0)$



b $\Delta A'B'C' = \Delta ABC + T$

$$\begin{aligned} T &= \Delta A'B'C' - \Delta ABC \\ &= \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2-0 & 3-1 & 0+2 \\ -1-0 & 2-3 & 0-1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix} \end{aligned}$$

13 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x-1 \\ y+3 \end{bmatrix}$

$x' = x - 1$

$y' = y + 3$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$x' = x - 1$

$x = x' + 1$

$y' = y + 3$

$y = y' - 3$

Substitute the image equations for the two coordinates into the original equation, $y = x - 3$

$y = x - 3$

$y' - 3 = x' + 1 - 3$

$y' = x' + 1 - 3 + 3$

$y' = x' + 1$

Therefore the image equation is $y = x + 1$

14 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+7 \\ y-4 \end{bmatrix}$

$x' = x + 7$

$y' = y - 4$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$x' = x + 7$

$x = x' - 7$

$y' = y - 4$

$y = y' + 4$

Substitute the image equations for the two coordinates into the original equation, $y = x^2 - 2$

$y = x^2 - 2$

$y' + 4 = (x' - 7)^2 - 2$

$y' = (x' - 7)^2 - 2 - 4$

$y' = (x' - 7)^2 - 6$

Therefore the image equation is $y = (x - 7)^2 - 6$

15 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$

$x' = x + a$

$y' = y + b$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = x + a$

$x = x' - a$

$y' = y + b$

$y = y' - b$

Substitute the image equations for the two coordinates into the original equation, $y = x$

$y = x$

$y' - b = x' - a$

$y' = (x' - a) + b \quad \Leftrightarrow \quad y = x + 2$

$\therefore a = 0, b = 2$

$T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Or, you can notice from the equations that the image equation $y = x + 2$ has been translated 2 units in the positive y-direction.

This translation corresponds to the matrix, $T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

16 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$

$x' = x + a$

$y' = y + b$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = x + a$

$x = x' - a$

$y' = y + b$

$y = y' - b$

Substitute the image equations for the two coordinates into the original equation, $y = x$

$y = x^2$

$y' - b = (x' - a)^2$

$y' = (x' - a)^2 + b \quad \Leftrightarrow \quad y = (x - 7)^2 + 3$

$\therefore a = 7, b = 3$

$T = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

Or, you can notice from the equations that the image equation $y = (x - 7)^2 + 3$ has been translated 7 units in the positive x-direction and 3 units in the positive y-direction.

This translation corresponds to the matrix $T = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

17 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$

$x' = x + a$

$y' = y + b$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = x + a$

$x = x' - a$

$y = y + b$

$y = y' - b$

Substitute the image equations for the two coordinates into the original equation, $y = x^2$

$$y = x^2$$
$$y' - b = (x' - a)^2$$
$$y' = (x' - a)^2 + b$$
$$= x'^2 - 2xa + a^2 + b \quad \Leftrightarrow \quad y = x^2 - 4x + 10$$

Equating x coefficients:

$$-2a = -4$$
$$\frac{-2a}{-2} = \frac{-4}{2}$$
$$a = 2$$

Equating constants and substituting $a = 2$:

$$a^2 + b = 10$$
$$2^2 + b = 10$$
$$4 + b = 10$$
$$b = 10 - 4$$
$$b = 6$$
$$\therefore a = 2, b = 6$$

$$T = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Or, you can rearrange the image equation to obtain the turning point form.

$$y = x^2 - 4x + 10$$
$$= (x^2 - 4x + 4) - 4 + 10$$
$$= (x - 2)^2 = 6$$

From the turning point form you can notice that the image equation $y = (x - 2)^2 + 6$ has been translated 2 units in the positive x-direction and 6 units in the positive y-direction.

This translation corresponds to the matrix $T = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

Therefore the translation equation is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

18 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \end{bmatrix}$$
$$x' = x + a$$
$$y' = y + b$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$x' = x + a$$
$$x = x' - a$$

$$y' = y + b$$
$$y = y' - b$$

Substitute the image equations for the two coordinates into the original equation, $x^2 + y^2 = 9$

$$x^2 + y^2 = 9$$
$$(x' - a)^2 + (y' - b)^2 = 9 \quad \Leftrightarrow \quad (x - 1)^2 + y^2 = 9$$

$$\therefore a = 1, b = 0$$

$$T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Or, you can notice that the image equation $(x - 1)^2 + y^2 = 9$ has been translated 1 unit in the positive x-direction.

This translation corresponds to the matrix $T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Therefore the translation equation is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

19 The image equation $y = (x - a)^2 + b$ has occurred from translating $y = x^2$, a units in the positive x-direction and b units in the positive y-direction.

This translation corresponds to the matrix $T = \begin{bmatrix} a \\ b \end{bmatrix}$.

Therefore the translation equation is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

20 The image equation $(x - a)^2 + y^2 = r^2$ has occurred from translating $x^2 + y^2 = r^2$, a units in the positive x-direction.

This translation corresponds to the matrix $T = \begin{bmatrix} a \\ 0 \end{bmatrix}$.

Therefore the translation equation is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}$$

Exercise 7.7 — Reflections

1 The matrix equation for a reflection in the x-axis is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Sub the x, y point $(-3, -1)$ into the matrix equation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times -3 + 0 \times -1 \\ 0 \times -3 - 1 \times -1 \end{bmatrix}$$
$$= \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Therefore the image point is $(-3, 1)$.

2 The matrix equation for a reflection in the y-axis is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Sub the x, y point $(5, -2)$ into the matrix equation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \times 5 + 0 \times -2 \\ 0 \times 5 + 1 \times -2 \end{bmatrix}$$
$$= \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

Therefore the image point is $(-5, -2)$.

3 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \times x + 0 \times y \\ 0 \times x + 1 \times y \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$
$$x' = -x$$
$$y' = y$$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$$\begin{aligned} x' &= -x \\ x &= -x' \end{aligned}$$

$$\begin{aligned} y' &= y \\ y &= y' \end{aligned}$$

Substitute the image equations for the two coordinates into the original equation, $y=(x-2)^2$.

$$\begin{aligned} y &= (x-2)^2 \\ y' &= (-x'-2)^2 \\ y' &= (-(x'+2))^2 \\ y' &= (x'+2)^2 \end{aligned}$$

Therefore the image equation is $y=(x+2)^2$.

4 $\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1\times x+0\times y \\ 0\times x-1\times y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} x \\ -y \end{bmatrix}$$

$$\begin{aligned} x' &= x \\ y' &= -y \end{aligned}$$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$$\begin{aligned} x' &= x \\ x &= x' \end{aligned}$$

$$\begin{aligned} y' &= -y \\ y &= -y' \end{aligned}$$

Substitute the image equations for the two coordinates into the original equation, $y=x^2+1$.

$$\begin{aligned} y &= x^2+1 \\ -y' &= x'^2+1 \\ y' &= -(x'^2+1) \\ y' &= -x'^2-1 \end{aligned}$$

Therefore the image equation is $y=-x^2-1$.

5 The matrix equation for a reflection in line with equation $y=x$ is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Sub the x,y point $(-2,5)$ into the matrix equation.

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 0\times -2+1\times 5 \\ 1\times -2+0\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -2 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(5,-2)$.

6 The matrix equation for a reflection in line with equation $y=x$ is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Sub the x,y point $(8,-3)$ into the matrix equation.

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 0\times 8+1\times -3 \\ 1\times 8+0\times -3 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 8 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(-3,8)$.

7 The matrix equation for a reflection in line with equation $x=-3$ is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

Sub the x,y point $(2,-1)$ into the matrix equation.

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -6 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}+\begin{bmatrix} -6 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1\times 2+0\times -1 \\ 0\times 2+1\times -1 \end{bmatrix}+\begin{bmatrix} -6 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ -1 \end{bmatrix}+\begin{bmatrix} -6 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -8 \\ -1 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(-8,-1)$.

8 The matrix equation for a reflection in line with equation $y=2$ is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Sub the x,y point $(-4,3)$ into the matrix equation.

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 0 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix}+\begin{bmatrix} 0 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1\times -4+0\times 3 \\ 0\times -4-1\times 3 \end{bmatrix}+\begin{bmatrix} 0 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -3 \end{bmatrix}+\begin{bmatrix} 0 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(-4,1)$.

9 **a** $\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 1\times -1+0\times 5 \\ 0\times -1-1\times 5 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -5 \end{bmatrix} \end{aligned}$

The image point is $(-1,-5)$.

b $\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -1\times -1+0\times 5 \\ 0\times -1+1\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} \end{aligned}$

The image point is $(1,5)$.

10 a A reflection in M_x is the same as a reflection in the x-axis.

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 8+0\times -4 \\ 0\times 8-1\times -4 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 4 \end{bmatrix}\end{aligned}$$

The image point is $(8,4)$.

b A reflection in M_y is the same as a reflection in the y-axis.

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -1\times 8+0\times -4 \\ 0\times 8+1\times -4 \end{bmatrix} \\ &= \begin{bmatrix} -8 \\ -4 \end{bmatrix}\end{aligned}$$

The image point is $(-8,-4)$.

11

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 9 \\ -6 \end{bmatrix} \\ &= \begin{bmatrix} 0\times 9+1\times -6 \\ 1\times 9+0\times -6 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ 9 \end{bmatrix}\end{aligned}$$

The image point is $(-6,9)$.

12

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0\times 0+1\times -1 \\ 1\times 0+0\times -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}\end{aligned}$$

The image point is $(-1,0)$.

13 a

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2\times 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}+\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1\times -2+0\times 3 \\ 0\times -2+1\times 3 \end{bmatrix}+\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 3 \end{bmatrix}+\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2+2 \\ 3+0 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 3 \end{bmatrix}\end{aligned}$$

The image point is $(4,3)$.

b

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 0 \\ 2\times -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 0 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}+\begin{bmatrix} 0 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} 1\times -2+0\times 3 \\ 0\times -2-1\times 3 \end{bmatrix}+\begin{bmatrix} 0 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ -3 \end{bmatrix}+\begin{bmatrix} 0 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -2+0 \\ -3-4 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ -7 \end{bmatrix}\end{aligned}$$

The image point is $(-2,-7)$.

14 a

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2\times -4 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -8 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix}+\begin{bmatrix} -8 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1\times 7+0\times -1 \\ 0\times 7+1\times -1 \end{bmatrix}+\begin{bmatrix} -8 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -7 \\ -1 \end{bmatrix}+\begin{bmatrix} -8 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -7-8 \\ -1+0 \end{bmatrix} \\ &= \begin{bmatrix} -15 \\ -1 \end{bmatrix}\end{aligned}$$

The image point is $(-15,-1)$.

b

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 0 \\ 2\times 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 0 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix}+\begin{bmatrix} 0 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 7+0\times -1 \\ 0\times 7-1\times -1 \end{bmatrix}+\begin{bmatrix} 0 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 1 \end{bmatrix}+\begin{bmatrix} 0 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 7+0 \\ 1+6 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 7 \end{bmatrix}\end{aligned}$$

The image point is $(7,7)$.

15
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2\times 2 \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1\times x + 0\times y \\ 0\times x - 1\times y \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} x+0 \\ -y+4 \end{bmatrix}$$
$$\begin{bmatrix} x \\ -y+4 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

Taking the equations out of the matrices gives:

$x = -3$

$-y + 4 = -5$

$-y = -9$

$y = 9$

Therefore, the original point has coordinates $P(-3, 9)$.

16
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2\times -1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1\times x + 0\times y \\ 0\times x + 1\times y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -x-2 \\ y+0 \end{bmatrix}$$
$$\begin{bmatrix} -x-2 \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Taking the equations out of the matrices gives:

$-x - 2 = -2$

$-x = 0$

$x = 0$

$y = 1$

Therefore, the original point has coordinates $P(0, 1)$.

17 a
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} 1\times x + 0\times y \\ 0\times x - 1\times y \end{bmatrix}$$
$$= \begin{bmatrix} x \\ -y \end{bmatrix}$$

$x' = x$

$y' = -y$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = x$

$x = x'$

$y' = -y$

$y = -y'$

Substitute the image equations for the two coordinates into the original equation, $y = -x + 3$

$y = -x + 3$

$-y' = -x' + 3$

$y' = -(-x' + 3)$

$y' = x' - 3$

Therefore the image equation is: $y = x - 3$.

b
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} -1\times x + 0\times y \\ 0\times x + 1\times y \end{bmatrix}$$
$$= \begin{bmatrix} -x \\ y \end{bmatrix}$$

$x' = -x$

$y' = y$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = -x$

$x = -x'$

$y' = y$

$y = y'$

Substitute the image equations for the two coordinates into the original equation, $y = -x + 3$

$y = -x + 3$

$y' = -(-x') + 3$

$y' = x' + 3$

Therefore the image equation is: $y = x + 3$.

c
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2\times 4 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1\times x + 0\times y \\ 0\times x + 1\times y \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -x+8 \\ y \end{bmatrix}$$

$x' = -x + 8$

$y' = y$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = -x + 8$

$-x = x' - 8$

$x = -x' + 8$

$y' = y$

$y = y'$

Substitute the image equations for the two coordinates into the original equation, $y = -x + 3$

$$y = -x + 3$$
$$y' = -(-x' + 8) + 3$$
$$y' = x' - 8 + 3$$
$$y' = x' - 5$$

Therefore the image equation is: $y = x - 5$.

18 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} 1 \times x + 0 \times y \\ 0 \times x - 1 \times y \end{bmatrix}$$
$$= \begin{bmatrix} x \\ -y \end{bmatrix}$$
$$x' = x$$
$$y' = -y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = x$
$x = x'$

$y' = -y$
$y = -y'$

Substitute the image equations for the two coordinates into the original equation $y = x^2 + 2x + 1$

$$y = x^2 + 2x + 1$$
$$-y' = x'^2 + 2x' + 1$$
$$y' = -(x'^2 + 2x' + 1)$$
$$y' = -x'^2 - 2x' - 1$$

Therefore the image equation is: $y = -x^2 - 2x - 1$ or $y = -\left(x^2 + 2x + 1\right)$

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} -1 \times x + 0 \times y \\ 0 \times x + 1 \times y \end{bmatrix}$$
$$= \begin{bmatrix} -x \\ y \end{bmatrix}$$
$$x' = -x$$
$$y' = y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = -x$
$x = -x'$

$y' = y$
$y = y'$

Substitute the image equations for the two coordinates into the original equation, $y = x^2 + 2x + 1$

$$y = x^2 + 2x + 1$$
$$y' = (-x')^2 + 2(-x') + 1$$
$$y' = x'^2 - 2x' + 1$$

Therefore the image equation is: $y = x^2 - 2x + 1$.

c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} 0 \times x + 1 \times y \\ 1 \times x + 0 \times y \end{bmatrix}$$
$$= \begin{bmatrix} y \\ x \end{bmatrix}$$
$$x' = y$$
$$y' = x$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$x' = y$
$y = x'$

$y' = x$
$x = y'$

Substitute the image equations for the two coordinates into the original equation, $y = x^2 + 2x + 1$

$$y = x^2 + 2x + 1$$
$$x' = y'^2 + 2y' + 1$$
$$x' = (y' + 1)^2$$
$$y' + 1 = \pm\sqrt{x'}$$
$$y' = \pm\sqrt{x'} - 1$$

Therefore the image equation is: $x = y^2 + 2y + 1$ or $y = \pm\sqrt{x} - 1$.

19 Reflection in the x-axis: $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflection in the line $y = x$: $T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Therefore the translation matrix is:

$$T = T_2T_1$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \times 1 + 1 \times 0 & 0 \times 0 + 1 \times -1 \\ 1 \times 1 + 0 \times 0 & 1 \times 0 + 0 \times -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore the matrix equation is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Sub in the x, y point $(2, -1)$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \times 2 - 1 \times -1 \\ 1 \times 2 + 0 \times -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore the image point is $(1, 2)$.

20 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 4 & 4 & 2 & 0 \\ 5 & 4 & 1 & 0 & 3 \end{bmatrix}$

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 & 2 & 0 \\ 5 & 4 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1\times2+0\times5 & -1\times4+0\times4 & -1\times4+0\times1 & -1\times2+0\times0 & -1\times0+0\times3 \\ 0\times2+1\times5 & 0\times2+1\times5 & 0\times4+1\times1 & 0\times2+1\times0 & 0\times0+1\times3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -4 & -4 & -2 & 0 \\ 5 & 5 & 1 & 0 & 3 \end{bmatrix} \end{aligned}$$

Therefore the coordinates of the vertices of the image of the pentagon are: $A'(-2,5)$, $B'(-4,4)$, $C'(-4,1)$, $D'(-2,0)$, $E'(0,3)$

Exercise 7.8 — Dilations

1 The matrix equation for a dilation factor of 3 from the x-axis is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Sub the x, y point $(2,-1)$ into the matrix equation

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1\times2+0\times-1 \\ 0\times2+3\times-1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(2,-3)$.

2 The matrix equation for a dilation factor of 2 from the y-axis is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Sub the x, y point $(-1,4)$ into the matrix equation

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 2\times-1+0\times4 \\ 0\times-1+1\times4 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 4 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(-2,4)$.

3 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3\times x+0\times y \\ 0\times x+1\times y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3x \\ y \end{bmatrix}$$

$$x' = 3x$$

$$y' = y$$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$$x' = 3x$$
$$x = \frac{x'}{3}$$

$$y' = y$$
$$y = y'$$

Substitute the image equations for the two coordinates into the original equation, $y = x^2$

$$y = x^2$$
$$y' = \left(\frac{x'}{3}\right)^2$$
$$y' = \frac{x'^2}{9}$$

Therefore the image equation is $y = \frac{x^2}{9}$.

4 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1\times x + 0\times y \\ 0\times x + \frac{1}{2}\times y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ \frac{y}{2} \end{bmatrix}$$

$$x' = x$$
$$y' = \frac{y}{2}$$

Rearrange the image equations for the two coordinates to make x and y the subjects.

$$x' = x$$
$$x = x'$$

$$y' = \frac{y}{2}$$
$$y = 2y'$$

Substitute the image equations for the two coordinates into the original equation, $y = x^2$

$$y = x^2$$
$$2y' = x'^2$$
$$y' = \frac{x'^2}{2}$$

Therefore the image equation is $y = \frac{x^2}{2}$.

5 The matrix equation for a dilation factor of 1.5 in the x-direction and 3 in the y-direction is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Sub the x, y points $A(0,0)$, $B(3,0)$, $C(3,4)$, $D(0,4)$ into the matrix equation

Point A:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2}\times 0 + 0\times 0 \\ 0\times 0 + 3\times 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Point B:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2}\times 3 + 0\times 0 \\ 0\times 3 + 3\times 0 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{9}{2} \\ 0 \end{bmatrix}$$

Point C:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2}\times 3 + 0\times 4 \\ 0\times 3 + 3\times 4 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{9}{2} \\ 12 \end{bmatrix}$$

Point D:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2}\times 0 + 0\times 4 \\ 0\times 0 + 3\times 4 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 12 \end{bmatrix}$$

Therefore she should relocate the fence posts to points $(0,0), \left(\frac{9}{2},0\right), (0,12), \left(\frac{9}{2},12\right)$.

6 The matrix equation for a dilation factor of 2 in the x-direction and 2 in the y-direction is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Sub the x, y points $A(2,1)$, $B(4,1)$, $C(3,2)$, $D(1,2)$ into the matrix equation

Point A:

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 2+0\times 1 \\ 0\times 2+2\times 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 2 \end{bmatrix}\end{aligned}$$

Point B:

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 4+0\times 1 \\ 0\times 4+2\times 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 2 \end{bmatrix}\end{aligned}$$

Point C:

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 3+0\times 2 \\ 0\times 3+2\times 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 4 \end{bmatrix}\end{aligned}$$

Point D:

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 1+0\times 2 \\ 0\times 1+2\times 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix}\end{aligned}$$

Therefore he should relocate the corners to points $(4,2), (8,2), (6,4), (2,4)$.

7

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 3\times 2+0\times -5 \\ 0\times 2+1\times -5 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -5 \end{bmatrix}\end{aligned}$$

Therefore the image point is $(6,-5)$.

8

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1\times -1+0\times 4 \\ 0\times -1+2\times 4 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 8 \end{bmatrix}\end{aligned}$$

Therefore the image point is $(-1,8)$.

9 Dilated 3 times wider means that the man has undergone a dilation of 3 parallel to the x-axis.
Therefore the matrix equation for this dilation is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

10 **a**

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3\times -1+0\times 3 \\ 0\times -1+2\times 3 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 6 \end{bmatrix}\end{aligned}$$

Therefore the image point is $(-3,6)$.

b The transformation matrix $T = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ corresponds to a dilation of 3 parallel to the x-axis (since all the x values are multiplied by 3) and a dilation of 2 parallel to the y-axis (since all the y values are multiplied by 2).

11

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 1\times x+0\times y \\ 0\times x+2\times y \end{bmatrix} \\ &= \begin{bmatrix} x \\ 2y \end{bmatrix}\end{aligned}$$

$$x' = x$$
$$y' = 2y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$x' = x$$
$$x = x'$$

$$y' = 2y$$
$$y = \frac{y'}{2}$$

Substitute the image equations for the two coordinates into the original equation, $2y + x = 3$

$$\begin{aligned}2y + x &= 3 \\ 2\left(\frac{y'}{2}\right) + x' &= 3 \\ y' + x' &= 3\end{aligned}$$

Therefore the image equation is: $x + y = 3$.

12

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 3\times x+0\times y \\ 0\times x+1\times y \end{bmatrix} \\ &= \begin{bmatrix} 3x \\ y \end{bmatrix}\end{aligned}$$

$$x' = 3x$$
$$y' = y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$x' = 3x$$
$$x = \frac{x'}{3}$$

$$y' = y$$
$$y = y'$$

Substitute the image equations for the two coordinates into the original equation, $y = x^2 - 1$

$$y = x^2 - 1$$
$$y' = \left(\frac{x'}{3}\right)^2 - 1$$

Therefore the image equation is: $y = \left(\frac{x}{3}\right)^2 - 1$ or $y = \frac{1}{9}x^2 - 1$.

13 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} 2 \times x + 0 \times y \\ 0 \times x + 1 \times y \end{bmatrix}$$
$$= \begin{bmatrix} 2x \\ y \end{bmatrix}$$
$$x' = 2x$$
$$y' = y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$x' = 2x$$
$$x = \frac{x'}{2}$$

$$y' = y$$
$$y = y'$$

Substitute the image equations for the two coordinates into the original equation, $y = \frac{1}{x+1}$

$$y = \frac{1}{x+1}$$
$$y' = \frac{1}{\frac{x'}{2} + 1} \times \frac{2}{2}$$
$$y' = \frac{2}{2\left(\frac{x'}{2} + 1\right)}$$
$$y' = \frac{2}{x' + 2}$$

Therefore the image equation is: $y = \frac{2}{x+2}$.

14 **a** The transformation matrix $T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ corresponds to a dilation of 2 parallel to the x-axis (since all the x values are multiplied by 2) and a dilation of 3 parallel to the y-axis (since all the y values are multiplied by 3).

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} 2 \times x + 0 \times y \\ 0 \times x + 3 \times y \end{bmatrix}$$
$$= \begin{bmatrix} 2x \\ 3y \end{bmatrix}$$
$$x' = 2x$$
$$y' = 3y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$x' = 2x$$
$$x = \frac{x'}{2}$$

$$y' = 3y$$
$$y = \frac{y'}{3}$$

Substitute the image equations for the two coordinates into the original equation, $y = 2\sqrt{x}$

$$y = 2\sqrt{x}$$
$$\frac{y'}{3} = 2\sqrt{\frac{x'}{2}}$$
$$\frac{y'}{3} = \sqrt{4} \times \sqrt{\frac{x'}{2}}$$
$$\frac{y'}{3} = \sqrt{4 \times \frac{x'}{2}}$$
$$\frac{y'}{3} = \sqrt{2x'}$$
$$y' = 3\sqrt{2x'}$$

Therefore the image equation is: $y = 3\sqrt{2x}$.

15 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} 2 \times x + 0 \times y \\ 0 \times x + 1 \times y \end{bmatrix}$$
$$= \begin{bmatrix} 2x \\ y \end{bmatrix}$$
$$x' = 2x$$
$$y' = y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$x' = 2x$$
$$x = \frac{x'}{2}$$

$$y' = y$$
$$y = y'$$

Substitute the image equations for the two coordinates into the original equation, $x^2 + y^2 = 4$

$$x^2 + y^2 = 4$$
$$\left(\frac{x'}{2}\right)^2 + y'^2 = 4$$
$$\frac{x'^2}{4} + y'^2 = 4$$

Therefore the image equation is: $\frac{x^2}{4} + y^2 = 4$.

16 **a** Sub in the x', y' point $A'(-3, 0)$ and the x, y point $A(-2, 0)$ (which comes from the first column of the 2×3 matrix)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} k \times -2 + 0 \times 0 \\ 0 \times -2 + 2 \times 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2k \\ 0 \end{bmatrix}$$

Taking the equation out of the matrix and solving for x:

$$\begin{aligned} -2k &= -3 \\ \frac{-2k}{-2} &= \frac{-3}{-2} \\ k &= \frac{3}{2} \end{aligned}$$

b $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & -1 & -3 \\ 0 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}\times -2 + 0\times 0 & \frac{3}{2}\times -1 + 0\times 3 & \frac{3}{2}\times -3 + 0\times 2 \\ 0\times -2 + 2\times 0 & 0\times -1 + 2\times 3 & 0\times -3 + 2\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & \frac{-3}{2} & \frac{-9}{2} \\ 0 & 6 & 4 \end{bmatrix}$$

Therefore the coordinates of the image vertices are $A'(-3,0)$, $B'\left(-\frac{3}{2},6\right)$, $C'\left(-\frac{9}{2},4\right)$.

17 $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k\times x + 0\times y \\ 0\times x + 1\times y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$$

$$\begin{aligned} x' &= kx \\ y' &= y \end{aligned}$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$\begin{aligned} x' &= kx \\ x &= \frac{x'}{k} \end{aligned}$$

$$\begin{aligned} y' &= y \\ y &= y' \end{aligned}$$

Substitute the image equations for the two coordinates into the original equation, $y = \frac{1}{x^2}$

$$\begin{aligned} y &= \frac{1}{x^2} \\ y' &= \frac{1}{\left(\frac{x'}{k}\right)^2} \\ y' &= \frac{1}{\frac{x'^2}{k^2}} \quad \Leftrightarrow \quad y = \frac{1}{3x^2} \end{aligned}$$

$$\begin{aligned} \frac{1}{k^2} &= 3 \\ k^2 &= \frac{1}{3} \\ k &= \pm\sqrt{\frac{1}{3}} \\ &= \pm\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} \\ &= \pm\frac{\sqrt{3}}{3} \end{aligned}$$

18 a $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 3\times x + 0\times y \\ 0\times x + 1\times y \end{bmatrix}$$

$$= \begin{bmatrix} 3x \\ y \end{bmatrix}$$

$$\begin{aligned} x' &= 3x \\ y' &= y \end{aligned}$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$\begin{aligned} x' &= 3x \\ x &= \frac{x'}{3} \end{aligned}$$

$$\begin{aligned} y' &= y \\ y &= y' \end{aligned}$$

Substitute the image equations for the two coordinates into the original equation, $x + 2y = 2$

$$\begin{aligned} x + 2y &= 2 \\ \frac{x'}{3} + 2y' &= 2 \\ 2y' &= -\frac{x'}{3} + 2 \\ y' &= \frac{1}{2}\left(-\frac{x'}{3} + 2\right) \\ y' &= -\frac{x'}{6} + 1 \end{aligned}$$

Therefore the image equation is: $y = -\frac{x}{6} + 1.$

b An invariant point is one that does not change when the graph is transformed. Therefore the invariant point will be when $y' = y$

Sub $y = -\frac{x}{6} + 1$ into $x + 2y = 2$

$$\begin{aligned} x + 2\left(-\frac{x}{6} + 1\right) &= 2 \\ x - \frac{2x}{6} + 2 &= 2 \\ x - \frac{x}{3} &= 0 \\ \frac{2x}{3} &= 0 \\ x &= 0 \end{aligned}$$

Sub $x = 0$ into $y = -\frac{x}{6} + 1$

$$\begin{aligned} y &= -\frac{1}{6}(0) + 1 \\ y &= 1 \end{aligned}$$

Therefore the invariant point is $(0,1)$.

Exercise 7.9 — Combinations of transformations

1 T_1 = reflection in the y-axis

$$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

T_2 = dilation factor of 3 from the x-axis

$$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

The order of multiplying matrices is important. Since T_1 occurs first, it needs to be multiplied to X first in the matrix equation $X' = TX$. Therefore the order of the transformations is opposite to the ordered the matrices are multiplied together to obtain T.

$$\begin{aligned} T &= T_2 T_1 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1\times -1 + 0\times 0 & 1\times 0 + 0\times 1 \\ 0\times -1 + 3\times 0 & 0\times 0 + 3\times 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \end{aligned}$$

2 T_1 = reflection in the line $y = x$

$$T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

T_2 = dilation factor of 2 from both the x-axis and y-axis

$$T_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The order of multiplying matrices is important. Since T_1 occurs first, it needs to be multiplied to X first in the matrix equation $X' = TX$. Therefore the order of the transformations is opposite to the ordered the matrices are multiplied together to obtain T.

$$\begin{aligned} T &= T_2 T_1 \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 0 + 0\times 1 & 2\times 1 + 0\times 0 \\ 0\times 0 + 2\times 1 & 0\times 1 + 2\times 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \end{aligned}$$

3 **a** $T = \begin{bmatrix} 3 & 2 \\ 5 & 8 \end{bmatrix}, X' = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

$$\begin{aligned} X' &= TX \\ TX &= X' \\ T^{-1}TX &= T^{-1}X' \\ X &= T^{-1}X' \end{aligned}$$

$$\begin{aligned} T^{-1} &= \frac{1}{3\times 8 - 2\times 5} \begin{bmatrix} 8 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 8 & -2 \\ -5 & 3 \end{bmatrix} \end{aligned}$$

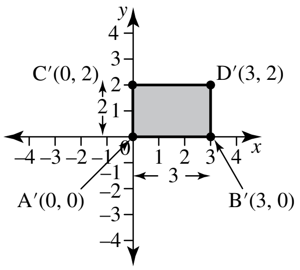
$$\begin{aligned} X &= T^{-1}X' \\ &= \frac{1}{14} \begin{bmatrix} 8 & -2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 8\times 0 - 2\times 0 & 8\times 3 - 2\times 0 & 8\times 0 - 2\times 2 & 8\times 3 - 2\times 2 \\ -5\times 0 + 3\times 0 & -5\times 3 + 3\times 0 & -5\times 0 + 3\times 2 & -5\times 3 + 3\times 2 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 0 & 24 & -4 & 20 \\ 0 & -15 & 6 & -9 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{24}{14} & \frac{-4}{14} & \frac{20}{14} \\ 0 & \frac{-15}{14} & \frac{6}{14} & \frac{-9}{14} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{12}{7} & \frac{-2}{7} & \frac{10}{7} \\ 0 & \frac{-15}{14} & \frac{3}{7} & \frac{-9}{14} \end{bmatrix} \end{aligned}$$

Therefore the vertices of the square $ABCD$ are $A(0,0)$, $B\left(\frac{12}{7}, \frac{-15}{14}\right)$, $C\left(\frac{-2}{7}, \frac{3}{7}\right)$, $D\left(\frac{10}{7}, \frac{-9}{14}\right)$

b Scale factor for the area increase by the transformation, T is given by $|\det(T)|$.

Therefore: $Area(A'B'C'D') = |\det(T)| \times Area(ABCD)$

To find the area of $A'B'C'D'$ plot the vertices of the and use the formula $A = l \times w$



$$\begin{aligned} A &= 3\times 2 \\ &= 6\,units^2 \end{aligned}$$

$$\begin{aligned} \text{Area}(A'B'C'D') &= |3\times 8 - 2\times 5| \times Area(ABCD) \\ 6 &= |14| \times Area(ABCD) \\ 6 &= 14\times Area(ABCD) \\ 14\times Area(ABCD) &= 6 \\ Area(ABCD) &= \frac{6}{14} \\ &= \frac{3}{7}\text{ units}^2 \end{aligned}$$

4 **a** Reflection in the y-axis: $M_x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Then a translation of 4 units in the positive direction of the x-axis: $T = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Therefore the matrix equation is:

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1\times x + 0\times y \\ 0\times x + 1\times y \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -x+4 \\ y \end{bmatrix} \end{aligned}$$

Therefore the image point is $(-x+4, y)$.

b Translation of 4 units in the positive direction of the x-axis: $T = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Then a reflection in the y-axis: $M_x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x+4 \\ y \end{bmatrix} \\ &= \begin{bmatrix} -1\times(x+4) + 0\times y \\ 0\times(x+4) + 1\times y \end{bmatrix} \\ &= \begin{bmatrix} -(x+4) \\ y \end{bmatrix} \\ &= \begin{bmatrix} -x-4 \\ y \end{bmatrix} \end{aligned}$$

Therefore the image point is $(-x-4, y)$.

5 The matrix equation for a translation of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, followed by a reflection in the x-axis is:

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x+2 \\ y-1 \end{bmatrix} \\ &= \begin{bmatrix} 1\times(x+2) + 0\times(y-1) \\ 0\times(x+2) - 1\times(y-1) \end{bmatrix} \\ &= \begin{bmatrix} x+2 \\ -(y-1) \end{bmatrix} \\ &= \begin{bmatrix} x+2 \\ -y+1 \end{bmatrix} \end{aligned}$$

Therefore the image point is $(x+2, -y+1)$.

6 For $T = T_2T_1$, the transformation of T_1 occurs first followed by the transformation of T_2

Therefore the transformation matrix $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ corresponds with:

First the matrix, $T_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ which represents a dilation factor of 2 parallel to both axes

Then the matrix $T_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which represents a rotation of 90° about the origin since: $\begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix}$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore this transformation matrix corresponds to a dilation factor of 2 parallel to both axes followed by a rotation of 90° about the origin.

7 Scale factor for the area increase by the transformation, T is given by $|\det(T)|$.

Therefore: $Area(\Delta A'B'C') = |\det(T)| \times Area(\Delta ABC)$

$$\begin{aligned} Area(\Delta A'B'C') &= |3\times 2 - (-1\times 1)| \times 10 \\ &= |6+1| \times 10 \\ &= |7| \times 10 \\ &= 7\times 10 \\ &= 70\text{ units}^2 \end{aligned}$$

8 **a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

For $T = T_2T_1$, the transformation of T_1 occurs first followed by the transformation of T_2

Therefore the transformation matrix $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ corresponds with:

First the matrix, $T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which represents a reflection in the line $y = x$

Then the matrix $T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ which represents a reflection in the x-axis.

Therefore this transformation corresponds to a reflection in the line $y = x$ followed by a reflection in the x-axis.

b $$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\\ &= \begin{bmatrix} 1\times 0+0\times 1 & 1\times 1+0\times 0 \\ 0\times 0-1\times 1 & 0\times 1-1\times 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\\ &= \begin{bmatrix} 0\times x+1\times y \\ -1\times x+0\times y \end{bmatrix}\\ &= \begin{bmatrix} y \\ -x \end{bmatrix}\end{aligned}$$

$$x' = y$$
$$y' = -x$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$\begin{aligned} x' &= y\\ y &= x' \end{aligned}$$

$$\begin{aligned} y' &= -x\\ x &= -y' \end{aligned}$$

Substitute the image equations for the two coordinates into the original equation, $2x-3y=12$

$$\begin{aligned} 2x-3y &= 12\\ 2(-y')-3(x') &= 12\\ -2y'-3x' &= 12\\ 3x'+2y' &= -12 \end{aligned}$$

Therefore the image equation is: $3x+2y=-12$ or $2y=-3x-12$.

9 a $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

First the matrix, $T=\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ corresponds to a dilation of factor 2 parallel to the x-axis and a reflection in the x-axis. Then the matrix $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ which corresponds to a translation of 1 units in the positive x-direction and 2 units in the positive y-direction.

Therefore this transformation corresponds to a dilation of factor 2 parallel to the y-axis and a reflection in the x-axis followed by a translation of 1 units in the positive x-direction and 2 units in the positive y-direction.

b $$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\\ &= \begin{bmatrix} 2\times x+0\times y \\ 0\times x-1\times y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\\ &= \begin{bmatrix} 2x \\ -y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\\ &= \begin{bmatrix} 2x+1 \\ -y+2 \end{bmatrix}\end{aligned}$$

$$x' = 2x+1$$
$$y' = -y+2$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$\begin{aligned} x' &= 2x+1\\ 2x &= x'-1\\ x &= \frac{x'-1}{2} \end{aligned}$$

$$\begin{aligned} y' &= -y+2\\ -y &= y'-2\\ y &= -y'+2 \end{aligned}$$

Substitute the image equations for the two coordinates into the original equation, $y=2x^2-1$

$$\begin{aligned} y &= 2x^2-1\\ -y'+2 &= 2\left(\frac{x'-1}{2}\right)^2-1\\ -y'+2 &= \frac{2(x'-1)^2}{4}-1\\ -y' &= \frac{(x'-1)^2}{2}-1-2\\ -y' &= \frac{(x'-1)^2}{2}-3\\ y' &= -\frac{(x'-1)^2}{2}+3 \end{aligned}$$

Therefore the image equation is: $y=-\frac{(x-1)^2}{2}+3$ or $y=-\frac{1}{2}(x-1)^2+3$.

10 Reflection in the x-axis: $T_1=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Dilation factor of 2 parallel to both the x and y-axis.

$$T_2=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\\ &= \begin{bmatrix} 2\times 1+0\times 0 & 2\times 0+0\times -1 \\ 0\times 1+2\times 0 & 0\times 0+2\times -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\\ &= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\\ &= \begin{bmatrix} 2\times x+0\times y \\ 0\times x-2\times y \end{bmatrix}\\ &= \begin{bmatrix} 2x \\ -2y \end{bmatrix}\end{aligned}$$

$$x' = 2x$$
$$y' = -2y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$\begin{aligned} x' &= 2x\\ x &= \frac{x'}{2} \end{aligned}$$

$$\begin{aligned} y' &= -2y\\ y &= -\frac{y'}{2} \end{aligned}$$

Substitute the image equations for the two coordinates into the original equation, $y=x^2$

$$y = x^2$$

$$-\frac{y'}{2} = \left(\frac{x'}{2}\right)^2$$

$$-\frac{y'}{2} = \frac{x'^2}{4}$$

$$y' = -2 \times \frac{x'^2}{4}$$

$$y' = -\frac{x'^2}{2}$$

Therefore the image equation is: $y = -\dfrac{x^2}{2}$.

11 Reflection in the y-axis: $T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Dilation factor of 2 parallel to both the x and y-axis.

$$T_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 3\times -1 + 0\times 0 & 3\times 0 + 0\times 1 \\ 0\times -1 + 1\times 0 & 0\times 0 + 1\times 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -3\times x + 0\times y \\ 0\times x + 1\times y \end{bmatrix}$$

$$= \begin{bmatrix} -3x \\ y \end{bmatrix}$$

$$x' = -3x$$

$$y' = y$$

Rearrange the image equations for the two coordinates to make x and y the subjects

$$x' = -3x$$

$$x = -\frac{x'}{3}$$

$$y' = y$$

$$y = y'$$

Substitute the image equations for the two coordinates into the original equation, $y = \sqrt{x}$

$$y = \sqrt{x}$$

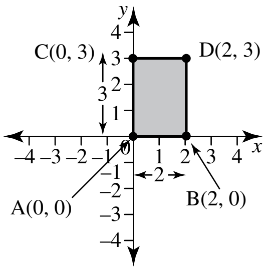
$$y' = \sqrt{-\frac{x'}{3}}$$

Therefore the image equation is: $y = \sqrt{-\dfrac{x}{3}}$.

12 Scale factor for the area increase by the transformation, T is given by $|\det(T)|$.

Therefore: $Area(\Delta A'B'C'D') = |\det(T)| \times Area(\Delta ABCD)$

To find the area of $ABCD$ plot the vertices of the and use the formula $A = l \times w$



$$Area(ABCD) = 2 \times 3$$

$$= 6\, units^2$$

Now use this and the determinant of the transformation matrix $T = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ to find the area of the transformed rectangle.

$$Area(\Delta A'B'C'D') = |3\times 2 - (-1\times 1)| \times 6$$

$$= |2\times 2 - (-1\times 1)| \times 6$$

$$= |4 + 1| \times 6$$

$$= 5 \times 6$$

$$= 30 \text{ units}^2$$

13 $D_k = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

$$D_k^2 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$= \begin{bmatrix} k\times k + 0\times 0 & k\times 0 + 0\times k \\ 0\times k + k\times 0 & 0\times 0 + k\times k \end{bmatrix}$$

$$= \begin{bmatrix} k^2 & 0 \\ 0 & k^2 \end{bmatrix}$$

Therefore D_k^2 gives a dilation factor of k^2 parallel to both axes.

14 Transformation A: Reflection in the y-axis followed by reflection in the line $y = x$:

$$T_A = T_2 T_1$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\times -1 + 1\times 0 & 0\times 0 + 1\times 1 \\ 1\times -1 + 0\times 0 & 1\times 0 + 0\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Transformation B: Reflection in the line $y = x$ followed by a reflection in the y-axis:

$$T_B = T_2 T_1$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1\times 0 + 0\times 1 & -1\times 1 + 0\times 0 \\ 0\times 0 + 1\times 1 & 0\times 1 + 1\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore the double transformation A is not the same as transformation B.

Topic 8 — Probability

Exercise 8.2 — Probability review

1 a $\xi = \{1,2,3,4,5,6,7,8\}$

As $\Pr(6) = \frac{9}{16}$ then the probability of not obtaining 6 is $1 - \frac{9}{16} = \frac{7}{16}$.

Since each of the numbers 1, 2, 3, 4, 5, 7, and 8 are equiprobable, the probability of each number is $\frac{7}{16} \times \frac{1}{7} = \frac{1}{16}$.

Hence, $\Pr(1) = \frac{1}{16}$.

b $A = \{2,3,5,7\}$

So, $\Pr(A) = \Pr(2) + \Pr(3) + \Pr(5) + \Pr(7)$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$
$$= \frac{1}{4}$$

Using rule for complimentary events,

$$\Pr(A') = 1 - \Pr(A)$$
$$= 1 - \frac{1}{4}$$
$$= \frac{3}{4}$$

2 a i $\Pr(G \text{ or } R) = \Pr(G) + \Pr(R)$

$$= \frac{9}{20} + \frac{6}{20}$$
$$= \frac{3}{4}$$

ii Using rule for complimentary events,

$$\Pr(R') = 1 - \Pr(R)$$
$$= 1 - \frac{6}{20}$$
$$= \frac{7}{10}$$

iii Using rule for complimentary events,

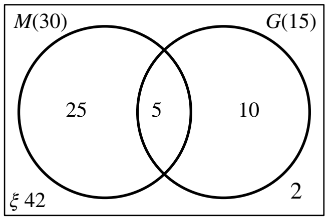
$$\Pr((G \text{ or } R)') = 1 - \Pr(G \text{ or } R)$$
$$= 1 - \frac{3}{4}$$
$$= \frac{1}{4}$$

b Let n be the number of additional red balls.

$$\frac{n(R)}{n(total)} = \Pr(R)$$
$$\frac{6+n}{20+n} = \frac{1}{2}$$
$$6 + n = \frac{1}{2} \times (20 + n)$$
$$6 + n = 10 + \frac{n}{2}$$
$$\frac{n}{2} = 4$$
$$n = 8$$

Therefore an additional 8 red balls must be added.

3 a Given $n(\xi) = 42$, $n(M) = 30$, $n(G) = 15$ and $n(G \cap M') = 10$



b $\Pr(M \cap G') = \frac{n(M \cap G')}{n(\xi)}$

$$\therefore \Pr(M \cap G') = \frac{25}{42}$$

The probability that the randomly chosen student studies Mathematical Methods but not Geography is $\frac{25}{42}$.

c $\Pr(M' \cap G') = \frac{n(M' \cap G')}{n(\xi)}$

$$\therefore \Pr(M' \cap G') = \frac{2}{42}$$
$$= \frac{1}{21}$$

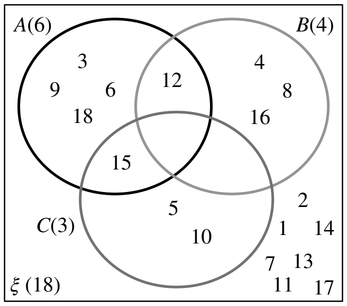
The probability that the randomly chosen student studies neither Mathematical Methods nor Geography is $\frac{1}{21}$.

d $\Pr(M \cap G') + \Pr(G \cap M') = \frac{25}{42} + \frac{10}{42}$

$$= \frac{35}{42} = \frac{5}{6}$$

The probability that the randomly chosen student studies only one of the subjects Mathematical Methods or Geography is $\frac{35}{42} = \frac{5}{6}$.

4 a Given $\xi = \{1,2,3,...,18\}$, $A = \{3,6,9,12,15,18\}$, $B = \{4,8,12,16\}$ and $C = \{5,10,15\}$



b Mutually exclusive means that the events cannot occur simultaneously, i.e. intersection = 0.
From Venn diagram, the events that are mutually exclusive are B and C.

c From Venn diagram,

$$\Pr(A) = \frac{n(A)}{n(\xi)}$$
$$= \frac{6}{18}$$
$$= \frac{1}{3}$$

d **i** $\Pr(A \cup C) = \dfrac{n(A \cup C)}{n(\xi)}$

$$= \frac{8}{18}$$
$$= \frac{4}{9}$$

ii $\Pr(A \cap B') = \dfrac{n(A \cap B')}{n(\xi)}$

$$= \frac{5}{18}$$

iii From Venn diagram,

$$\Pr((A \cup B \cup C)') = \frac{n((A \cup B \cup C)')}{n(\xi)}$$
$$= \frac{7}{18}$$

5 **a** Given $\Pr(A) = 0.65$, $\Pr(B) = 0.5$, $\Pr(A' \cap B') = 0.2$ and also $\Pr(\xi) = 1$.

	B	B'	
A			0.65
A'		0.2	
	0.5		1

$\Pr(A') = 1 - 0.65 = 0.35$ and $\Pr(B') = 1 - 0.5 = 0.5$

	B	B'	
A			0.65
A'		0.2	0.35
	0.5	0.5	1

For the second row, $0.15 + 0.2 = 0.4$
For the second column, $0.3 + 0.2 = 0.5$
For the first row, $0.35 + 0.3 = 0.65$

	B	B'	
A	0.35	0.3	0.65
A'	0.15	0.2	0.35
	0.5	0.5	1

b Using the addition formula,
$\Pr(B' \cup A) = \Pr(B') + \Pr(A) - \Pr(B' \cap A)$.
From the probability table, $\Pr(B' \cap A) = 0.3$.
$\therefore \Pr(B' \cup A) = 0.5 + 0.65 - 0.3$
$\therefore \Pr(B' \cup A) = 0.85$

6 **a** Given $\Pr(A') = 0.42$, $\Pr(B) = 0.55$, and also $\Pr(\xi) = 1$.

	B	B'	
A			
A'			0.42
	0.55		1

$\Pr(A) = 1 - 0.42 = 0.58$ and $\Pr(B') = 1 - 0.55 = 0.45$
Using the addition formula,
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$\therefore \Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$
Given $\Pr(A \cup B) = 0.75$,
$\therefore \Pr(A \cap B) = 0.58 + 0.55 - 0.75$
$\therefore \Pr(A \cap B) = 0.38$

	B	B'	
A	0.38		0.58
A'			0.42
	0.55	0.45	1

For the first row, $0.38 + 0.2 = 0.58$
For the first column, $0.38 + 0.17 = 0.55$
For the second row, $0.17 + 0.25 = 0.42$

	B	B'	
A	0.38	0.2	0.58
A'	0.17	0.25	0.42
	0.55	0.45	1

b From the probability table, $\Pr(A' \cap B') = 0.25$.

c From the table,

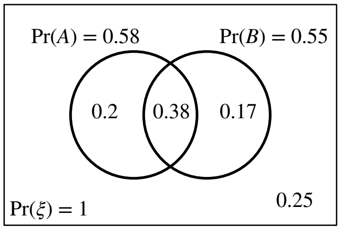
$\Pr(A \cup B)' = 0.25$
$\Pr(A' \cap B') = 0.25$
$\Rightarrow \Pr(A \cup B)' = \Pr(A' \cap B')$

d From the probability table, $\Pr(A \cap B) = 0.38$.

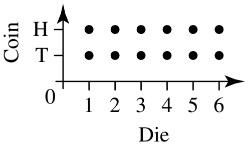
$$1 - \Pr(A' \cup B') = 1 - [\Pr(A') + \Pr(B') - \Pr(A' \cap B')]$$
$$= 1 - 0.42 - 0.45 + 0.25$$
$$= 0.38$$

So, $\Pr(A \cap B) = 1 - \Pr(A' \cup B')$.

e



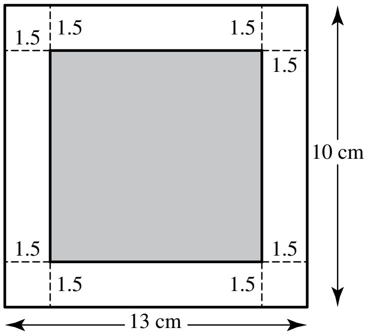
7 **a**



b There are 12 equally likely outcomes in the sample space. Only one of these outcomes is a 6 on the die and a head on the coin.
The probability of obtaining a 6 on the die and a head on the coin is $\frac{1}{12}$.

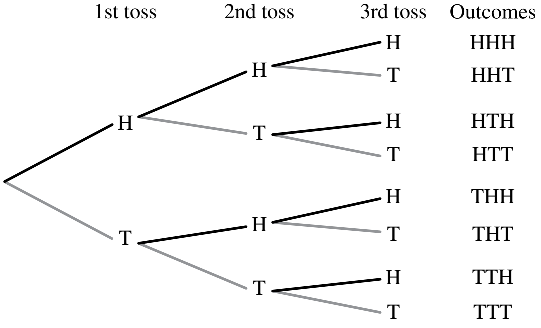
c This event occurs for the three events $2T$, $4T$ and $6T$.
So the probability of the event of obtaining an even number together with a tail on the coin is $\frac{3}{12} = \frac{1}{4}$.

d To land completely inside the rectangle, the area in which the centre of the coin may land is a rectangle of edge length $13 - 2 \times 1.5 = 10$ cm and edge width $10 - 2 \times 1.5 = 7$.



Area that coin could land in for coin to be completely inside area is $10 \times 7 = 70$ cm^2.
Total area that the coin could land in is $13 \times 10 = 130$ cm^2.
Therefore, the probability the coin lands completely inside the cardboard area is $\frac{70}{130} = \frac{7}{13}$.

8 a



b Obtaining at least one head is the complimentary event of obtaining no heads.
Therefore, using the rule for complimentary events, the probability of obtaining at least one head is
$1 - \Pr(TTT) = 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{7}{8}$.

c
$$\begin{aligned}\Pr(HH) + \Pr(TT) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{2}{8} \\ &= \frac{1}{4}\end{aligned}$$

9 a
$$\begin{aligned}\Pr(G) &= \frac{5}{5+6+4+8} \\ &= \frac{5}{23}\end{aligned}$$

b Using the addition formula:
$$\begin{aligned}\Pr(O \; or \; B) &= \Pr(A \cup B) \\ &= \Pr(O) + \Pr(B) - \Pr(A \cap B) \\ &= \frac{4}{23} + \frac{8}{23} - 0 \\ &= \frac{12}{23}\end{aligned}$$

c Using the rule for complimentary events:
$$\begin{aligned}P(B') &= 1 - \Pr(B) \\ &= 1 - \frac{8}{23} \\ &= \frac{15}{23}\end{aligned}$$

d There are no black counters so therefore the probability of selecting a black counter is zero.
$$\begin{aligned}\Pr(Bl) &= \frac{0}{5+6+4+8} \\ &= 0\end{aligned}$$

10 a
$$\begin{aligned}\Pr(1^{st}) &= \frac{10}{2000} \\ &= \frac{1}{200}\end{aligned}$$

b For Josephine to just win the second prize Josephine she must have not won first prize. Then since one ticket was already drawn for first prize, the solution space would decrease by 1 to 1999.
$$\begin{aligned}\Pr(2^{nd}) &= \Pr(1^{st}{}') \times \Pr(2^{nd}) \\ &= (1 - \Pr(1^{st})) \times \frac{10}{2000-1} \\ &= \left(1 - \frac{1}{200}\right) \times \frac{10}{1999} \\ &= \frac{199}{200} \times \frac{10}{1999} \\ &= \frac{199}{39\,980}\end{aligned}$$

c If Josephine wins the first prize then the solution space is reduced by 1 and her number of tickets is reduced by 1 for her to win the second prize as well. Then the solution space and her number of tickets is further reduced by 1 for her to also win the third prize.
$$\begin{aligned}\Pr(all \; three) &= \frac{10}{2000} \times \frac{9}{2000-1} \times \frac{8}{2000-2} \\ &= \frac{1}{200} \times \frac{9}{1999} \times \frac{8}{1998} \\ &= \frac{1}{11\,094\,450}\end{aligned}$$

11 a Using the rule for complementary events:
$$\begin{aligned}\Pr(G') &= 1 - \Pr(G) \\ &= 1 - 0 \\ &= 1\end{aligned}$$
Since, there are no green cards in a standard pack of 52 playing cards the probability of not drawing a green is 1.

b
$$\begin{aligned}\Pr(R) &= \frac{26}{52} \\ &= \frac{1}{2}\end{aligned}$$

c
$$\begin{aligned}\Pr(H) &= \frac{13}{52} \\ &= \frac{1}{4}\end{aligned}$$

d Using the addition formula:
$$\begin{aligned}\Pr(10 \; or \; R) &= \Pr(10 \cup R) \\ &= \Pr(10) + \Pr(red) - \Pr(10 \cap red) \\ &= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} \\ &= \frac{28}{52} \\ &= \frac{7}{13}\end{aligned}$$

e Using the rule for complimentary events:
$$\begin{aligned}\Pr(A') &= 1 - \Pr(A) \\ &= \frac{52-4}{52} \\ &= \frac{48}{52} \\ &= \frac{12}{13}\end{aligned}$$

12 a

1st Roll

2nd Roll	**1**	**2**	**3**	**4**	**5**	**6**
1	2	2	3	4	5	6
2	2	4	3	4	5	6
3	3	3	6	4	5	6
4	4	4	4	8	5	6
5	5	5	5	5	10	6
6	6	6	6	6	6	12

Since the size of the sample space is 36 and 5 appears 8 times in the table:

$$\begin{aligned}\Pr(5) &= \frac{8}{36}\\ &= \frac{2}{9}\end{aligned}$$

b Since the size of the sample space is 36 and 10 only occurs once.

$$\Pr(10) = \frac{1}{36}$$

c Count how many times a number greater than 5 appears in the table (i.e. numbers 6,8,10,12) and divide by the total amount of numbers in the table.

$$\begin{aligned}\Pr(x > 5) &= \Pr(x = 6) + \Pr(x = 8) + \Pr(x = 10) + \Pr(x = 12)\\ &= \frac{14}{36}\\ &= \frac{7}{18}\end{aligned}$$

d Can see from the table above that 7 does not appear in the table.
Therefore $\Pr(7) = 0$

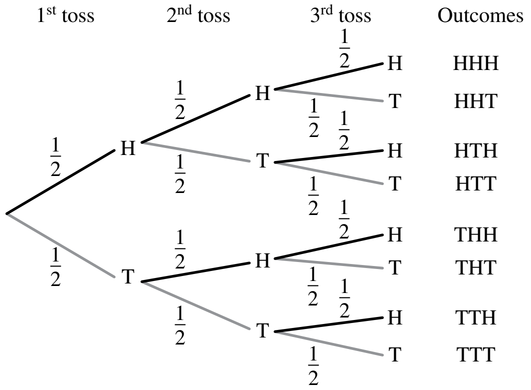
e Using the addition formula:

$$\begin{aligned}\Pr(2\ digits\ or\ x > 6) &= \Pr(2\ digits \cup x > 6)\\ &= \Pr(2\ digits) + \Pr(x > 6) - \Pr(2\ digits \cap x > 6)\\ &= (\Pr(10) + \Pr(12)) + \Pr(x > 6) - (\Pr(10) + \Pr(12))\\ &= \frac{1}{36} + \frac{1}{36} + \frac{3}{36} - \left(\frac{1}{36} + \frac{1}{36}\right)\\ &= \frac{3}{36}\\ &= \frac{1}{12}\end{aligned}$$

f Using the rule for complimentary events:

$$\begin{aligned}\Pr(9') &= 1 - \Pr(9)\\ &= 1 - 0\\ &= 1\end{aligned}$$

13 a



$$\begin{aligned}\Pr(2H \text{ and } 1T) &= \Pr(HHT) + \Pr(HTH) + \Pr(THH)\\ &= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\\ &= \frac{3}{8}\end{aligned}$$

b

$$\begin{aligned}\Pr(3H\ or\ 3T) &= \Pr(HHH) + \Pr(TTT)\\ &= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\\ &= \frac{1}{8} + \frac{1}{8}\\ &= \frac{2}{8}\\ &= \frac{1}{4}\end{aligned}$$

c $\Pr(H\ on\ first\ toss) = \Pr(HHH) + \Pr(HHT) + \Pr(HTH) + \Pr(HTT)$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$
$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2}$$

Or, you can see straight away from the tree diagram and the outcomes that half of the outcomes start with H. (and the tree diagram splits into two after the first toss)

d $\Pr(H \geq 1) = 1 - \Pr(no\ H)$

$$= 1 - \Pr(TTT)$$
$$= 1 - \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$
$$= 1 - \frac{1}{8}$$
$$= \frac{7}{8}$$

e $\Pr(T \leq 1) = \Pr(1T) + \Pr(0T)$

$$= \Pr(HHT) + \Pr(HTH) + \Pr(THH) + \Pr(HHH)$$
$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$
$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2}$$

14 a Total possible outcomes is 16:

RR, RG, RB, RY
GG, GR, GB, GY
YY, YR, YG, YB
BB, BR, BG, BY

$\Pr(same\ colour) = \Pr(RR) + \Pr(GG) + \Pr(YY) + \Pr(BB)$

$$= \frac{4}{16}$$
$$= \frac{1}{4}$$

b $\Pr(R \text{ and } Y) = \Pr(RY) + \Pr(YR)$

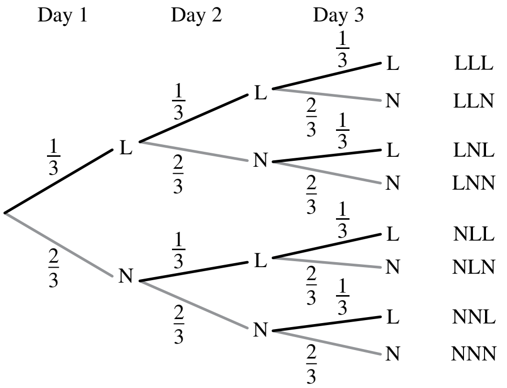
$$= \frac{2}{16}$$
$$= \frac{1}{8}$$

c Using the rule for complementary events:

$\Pr(G') = 1 - \Pr(G)$

$$= 1 - \left(\Pr(RG) + \Pr(GG) + \Pr(GR) + \Pr(GB) + \Pr(GY) + \Pr(YG) + \Pr(BG)\right)$$
$$= 1 - \frac{7}{16}$$
$$= \frac{9}{16}$$

15 a



$$\begin{aligned}\Pr(\textit{late on } 1 \textit{ day}) &= \Pr(LNN)+\Pr(NLN)+\Pr(NNL)\\ &= \left(\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}\right)+\left(\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}\right)+\left(\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}\right)\\ &= \frac{4}{27}+\frac{4}{27}+\frac{4}{27}\\ &= \frac{12}{27}\\ &= \frac{4}{9}\end{aligned}$$

b

$$\begin{aligned}\Pr(\textit{late at least } 2 \textit{ days}) &= \Pr(LLL)+\Pr(LLN)+\Pr(LNL)+\Pr(NLL)\\ &= \left(\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\right)+\left(\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}\right)+\left(\frac{1}{3}\times\frac{2}{3}\times\frac{1}{3}\right)+\left(\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}\right)\\ &= \frac{1}{27}+\frac{2}{27}+\frac{2}{27}+\frac{2}{27}\\ &= \frac{7}{27}\end{aligned}$$

c

$$\begin{aligned}\Pr(\textit{on time on last day}) &= \Pr(LLN)+\Pr(LNN)+\Pr(NLN)+\Pr(NNN)\\ &= \frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}+\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\\ &= \frac{2}{27}+\frac{4}{27}+\frac{4}{27}+\frac{8}{27}\\ &= \frac{18}{27}\\ &= \frac{2}{3}\end{aligned}$$

d

$$\begin{aligned}\Pr(\textit{on time on all days}) &= \Pr(NNN)\\ &= \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\\ &= \frac{8}{27}\end{aligned}$$

16 a

$$\begin{aligned}\Pr(L) &= \frac{178}{800}\\ &= \frac{89}{400}\end{aligned}$$

b

$$\begin{aligned}\Pr(\textit{not employed}) &= \Pr(\textit{unemployed})\\ &= \frac{53}{800}\end{aligned}$$

c Using the rule for complimentary events:

$$\begin{aligned}\Pr(E') &= 1-\Pr(E)\\ &= 1-\frac{128}{800}\\ &= 1-\frac{4}{25}\\ &= \frac{21}{25}\end{aligned}$$

d $\Pr(T\ or\ L) = \Pr(T) + \Pr(L)$

$$= \frac{261}{800} + \frac{178}{800}$$

$$= \frac{439}{800}$$

17 a Count the number of times 1 head was obtained and divide by the total number of trials, 40.

$\Pr(one\ \ female) = \Pr(1)$

$$= \frac{14}{40}$$

$$= \frac{7}{20}$$

b $\Pr(more\ than\ one\ \ female) = \Pr(2) + \Pr(3)$

$$= \frac{10}{40} + \frac{6}{40}$$

$$= \frac{16}{40}$$

$$= \frac{2}{5}$$

18 a $\Pr(F) = \dfrac{26+20}{73+26+81+20}$

$$= \frac{46}{200}$$

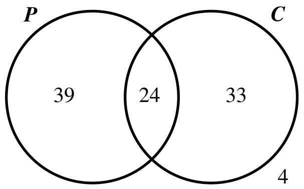
$$= \frac{23}{100}$$

b $\Pr(F \text{ and } P) = \dfrac{81}{200}$

19 a To find the number of students who study both chemistry and physics:

$63 + 57 + 4 = 124$ But there are only 100 students asked, so therefore $124 - 100 = 24$

Therefore, 24 students study both chemistry and physics.



$\Pr(P\ or\ C\ but\ not\ both) = \dfrac{39}{100} + \dfrac{33}{100}$

$$= \frac{72}{100}$$

$$= \frac{18}{25}$$

b $\Pr(both) = \Pr(P \cap C)$

$$= \frac{24}{100}$$

$$= \frac{6}{25}$$

c Let n = number of student who are likely to study both physics and chemistry.

$\Pr(both) = \Pr(P \cap C)$

$$= \frac{24}{100}$$

$$= \frac{6}{25}$$

$$n = \frac{6}{25} \times 1200$$

$$= 6 \times 48$$

$$= 288$$

Therefore there are 288 students expected to study both physics and chemistry.

d $\Pr(\mathit{both\ students\ study\ both}) = \Pr(P \cap C) \times \Pr(P \cap C)$

$$= \frac{24}{100} \times \frac{24}{100}$$
$$= \frac{6}{25} \times \frac{6}{25}$$
$$= \frac{36}{625}$$
$$= 0.0576$$

e Let A = the first student

And let B = the second student

$\Pr(\mathit{each\ student\ studies\ just\ one}) = \Pr(A \to P \cup C) \times \Pr(B \to P \cup C)$

$$= (\Pr(A \to P) + \Pr(A \to C) - \Pr(A \to P \cap C)) \times (\Pr(B \to P) + \Pr(B \to C) - \Pr(B \to P \cap C))$$
$$= \left(\frac{39}{100} + \frac{33}{100} - \frac{24}{100}\right) \times \left(\frac{39}{100} + \frac{33}{100} - \frac{24}{100}\right)$$
$$= \frac{48}{100} \times \frac{48}{100}$$
$$= \frac{144}{625}$$
$$= 0.2304$$

f $\Pr(\mathit{one\ student\ studies\ neither}) = \Pr(A\ \mathit{student\ neither\ or}\ B\ \mathit{studies\ neither})$

$$= \Pr((A \to N) \cup (B \to N))$$
$$= \Pr(A \to N) + \Pr(B \to N) - \Pr((A \to N) \cap (B \to N))$$
$$= \frac{4}{100} + \frac{4}{100} - \left(\frac{4}{100} \times \frac{4}{100}\right)$$
$$= \frac{8}{100} - \frac{1}{625}$$
$$= \frac{49}{625}$$
$$= 0.0784$$

20 **a** 5 teams: A, B, C, D, E

Therefore the games are:

$A \times B, A \times B, A \times C, A \times C, A \times D, A \times D, A \times E, A \times E,$
$B \times C, B \times C, B \times D, B \times D, B \times E, B \times E,$
$C \times D, C \times D, C \times E, C \times E,$
$D \times E, D \times E$

So, in total there must be 20 games for every team to play each other team twice.

b If there are n teams in the competition then there must be $n(n-1)$ games played. Each team plays $n-1$ games so therefore $n(n-1)$ gives the total number of games. (Usually when working this out need to divide the result by two since each game would be counted twice, eg $A \times B$ and $B \times A$, however since every team plays the same team twice this can be ignored for this situation.)

c Therefore each team must play each other team twice. Since there are $16-1=15$ teams to play, each team must play $15 \times 2 = 30$ games.

d $n = 16$

$$\mathit{total\ no.\ of\ games} = n(n-1)$$
$$= 16(16-1)$$
$$= 16 \times 15$$
$$= 240$$

Therefore there are a total of 240 games played if there are 16 teams in the competition.

Exercise 8.3 — Conditional probability

1 **a** From reading the table,

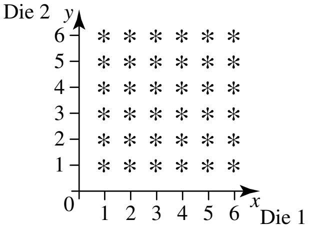
$$\Pr(B' \cap C') = \frac{n(B' \cap C')}{n(\xi)}$$
$$= \frac{20}{100}$$
$$= \frac{1}{5}$$

b $$\Pr(B'\mid C')=\frac{n(B'\cap C')}{n(C)}=\frac{20}{36}=\frac{5}{9}$$

c $$\Pr(C\mid B)=\frac{n(C\cap B)}{n(B)}=\frac{28}{44}=\frac{7}{11}$$

d $$\Pr(B)=\frac{n(B)}{n(\xi)}=\frac{44}{100}=\frac{11}{25}$$

2 a



Let A be the event of obtaining a sum of 8.
$\therefore A=\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$. There are 36 possible outcomes.

$$\Pr(A)=\frac{n(A)}{n(\xi)}=\frac{5}{36}$$

b Let B be the event of obtaining two numbers that are the same. $\therefore B=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$.

$$\Pr(A\mid B')=\frac{n(A\cap B')}{n(B')}=\frac{4}{30}=\frac{2}{15}$$

c Obtaining the sum of 8 but the numbers are not the same is $A\cap B'$.

$$\Pr(A\cap B')=\frac{n(A\cap B')}{n(\xi)}=\frac{4}{36}=\frac{1}{9}$$

d $$\Pr(B'\mid A)=\frac{n(B'\cap A)}{n(A)}=\frac{4}{5}$$

3 a Given $\Pr(A')=0.6$, $\Pr(B\mid A)=0.3$ and $\Pr(B)=0.5$
For complementary events,
$\Pr(A)=1-\Pr(A')$
$\therefore \Pr(A)=1-0.6$
$\therefore \Pr(A)=0.4$

$$\Pr(B\mid A)=\frac{\Pr(B\cap A)}{\Pr(A)}$$
$$\therefore 0.3=\frac{\Pr(B\cap A)}{0.4}$$
$\therefore \Pr(B\cap A)=0.3\times 0.4$
$\therefore \Pr(A\cap B)=\Pr(B\cap A)=0.12$

$$\Pr(A\mid B)=\frac{\Pr(A\cap B)}{\Pr(B)}=\frac{0.12}{0.5}=\frac{6}{25}$$

b Let G be the event a green ribbon is chosen.

$$\Pr(G)=\frac{n(G)}{n(\xi)}$$

For the first pick, $n(G)=8$ and $n(\xi)=12$, therefore $\Pr(G)=\frac{8}{12}$.

For the second pick, if one green ribbon has been removed, $n(G)=7$ and $n(\xi)=11$, therefore $\Pr(G)=\frac{7}{11}$.

For the third pick, if two green ribbons have been removed, $n(G)=6$ and $n(\xi)=10$, therefore $\Pr(G)=\frac{6}{10}$.

$$\Pr(G\cap G\cap G)=\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}=\frac{14}{55}$$

4 a Given $\Pr(A)=0.61$, $\Pr(B)=0.56$ and $\Pr(A\cup B)=0.81$,
$\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$
$\therefore \Pr(A\cap B)=\Pr(A)+\Pr(B)-\Pr(A\cup B)$
$=0.61+0.56-0.81$
$=0.36$

$$\Pr(A\mid B)=\frac{\Pr(A\cap B)}{\Pr(B)}=\frac{0.36}{0.56}=\frac{9}{14}$$

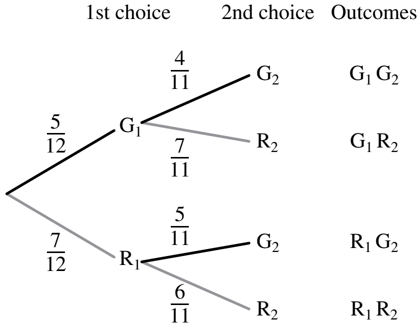
b $$\Pr(A\mid A\cup B)=\frac{\Pr(A\cap(A\cup B))}{\Pr(A\cup B)}=\frac{\Pr(A)}{\Pr(A\cup B)}=\frac{0.61}{0.81}=\frac{61}{81}$$

c $$\Pr(A\mid A'\cup B)=\frac{\Pr(A\cap(A'\cup B))}{\Pr(A'\cup B)}=\frac{\Pr(A\cap B)}{\Pr(A'\cup B)}=\frac{0.36}{0.75}=\frac{13}{25}$$

5 **a** If a red jube has already been chosen first, there remains in the box 6 red jubes and 5 green jubes.

$$\therefore \Pr(G_2 \mid R_1) = \frac{5}{11}$$

b



c $\Pr(G_1 \cap R_2) = \Pr(G_1) \times \Pr(R_2 \mid G_1)$

$$= \frac{5}{12} \times \frac{7}{11}$$

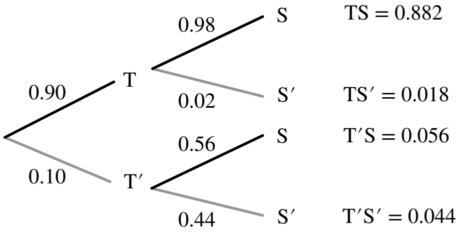
$$= \frac{35}{132}$$

d $\Pr((G_1 \cap G_2) \cup (R_1 \cap R_2)) = \Pr(G_1 \cap G_2) + \Pr(R_1 \cap R_2)$

$$= \frac{5}{12} \times \frac{4}{11} + \frac{7}{12} \times \frac{6}{11}$$

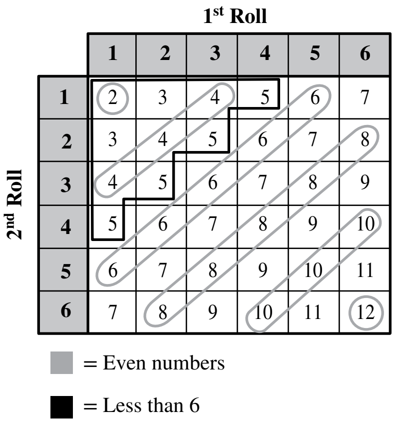
$$= \frac{31}{66}$$

6 **a** Let T = the bus being on time and T′ = the bus being late.
Let S = Rodney gets to school on time and S′ = Rodney gets to school late



b Pr(Rodney will arrive to school on time) = 0.882 + 0.056 = 0.938

7 Make a table for the sum of the two numbers rolled:



1st Roll

2nd Roll	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

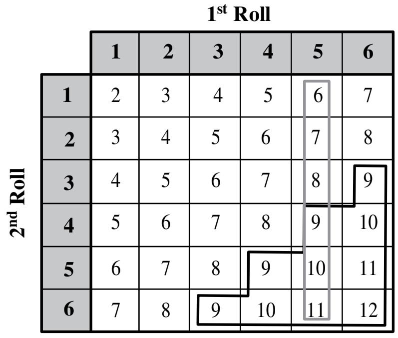
▨ = Even numbers

■ = Less than 6

Let x = the sum of the two numbers

$$\begin{aligned}\Pr(x<6\mid x\text{ is even})&=\frac{\Pr(x<6\cap x\text{ is even})}{\Pr(x\text{ is even})}\\&=\frac{\Pr(x=2)+\Pr(x=4)}{\Pr(x=2)+\Pr(x=4)+\Pr(x=6)+\Pr(x=8)+\Pr(x=10)+\Pr(x=12)}\\&=\frac{\frac{4}{36}}{\frac{18}{36}}\\&=\frac{4}{36}\times\frac{36}{18}\\&=\frac{4}{18}\\&=\frac{2}{9}\end{aligned}$$

8 Make a table for the sum of the two numbers rolled:



1st Roll

2nd Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

– = Sum greater than 8 – = 5 on First dice

Let x = the sum of the two numbers

$$\begin{aligned}\Pr(x>8\mid 5\text{ on first roll})&=\frac{\Pr(x>8\cap 5\text{ on first roll})}{\Pr(5\text{ on first roll})}\\&=\frac{\Pr(5,4)+\Pr(5,5)+\Pr(5,6)}{\Pr(5,1)+\Pr(5,2)+\Pr(5,3)+\Pr(5,4)+\Pr(5,5)+\Pr(5,6)}\\&=\frac{\frac{3}{36}}{\frac{6}{36}}\\&=\frac{3}{36}\times\frac{36}{6}\\&=\frac{3}{6}\\&=\frac{1}{2}\end{aligned}$$

9 **a** $\Pr(A\cap B)=\Pr(A)+\Pr(B)-\Pr(A\cup B)$

$$\begin{aligned}&=0.7+0.3-0.8\\&=1-0.8\\&=0.2\end{aligned}$$

b $\Pr(A\mid B)=\dfrac{\Pr(A\cap B)}{\Pr(B)}$

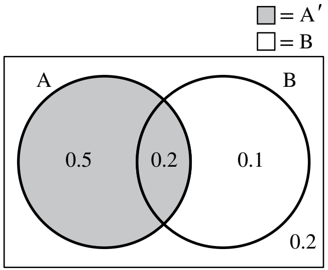
$$\begin{aligned}&=\frac{0.2}{0.3}\\&=\frac{2}{3}\end{aligned}$$

c $\Pr(B\mid A)=\dfrac{\Pr(B\cap A)}{\Pr(A)}$

$$\begin{aligned}&=\frac{\Pr(A\cap B)}{\Pr(A)}\\&=\frac{0.2}{0.7}\\&=\frac{2}{7}\end{aligned}$$

d Using the rule for complimentary events:

$$\begin{aligned}\Pr(B') &= 1 - \Pr(B)\\ &= 1 - 0.3\\ &= 0.7\end{aligned}$$



Pr(A ∩ B′) = 0.5

$$\begin{aligned}\Pr(A \mid B') &= \frac{\Pr(A \cap B')}{\Pr(B')}\\ &= \frac{0.5}{0.7}\\ &= \frac{5}{7}\end{aligned}$$

10 a Using the addition formula:

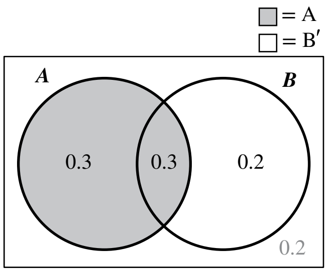
$$\begin{aligned}\Pr(A \cap B) &= \Pr(A) + \Pr(B) - \Pr(A \cup B)\\ &= 0.6 + 0.5 - 0.8\\ &= 1.1 - 0.8\\ &= 0.3\end{aligned}$$

b
$$\begin{aligned}\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\\ &= \frac{0.3}{0.5}\\ &= \frac{3}{5}\end{aligned}$$

c
$$\begin{aligned}\Pr(B \mid A) &= \frac{\Pr(B \cap A)}{\Pr(A)}\\ &= \frac{\Pr(A \cap B)}{\Pr(A)}\\ &= \frac{0.3}{0.6}\\ &= \frac{3}{6}\\ &= \frac{1}{2}\end{aligned}$$

d Using the rule for complimentary events:

$$\begin{aligned}\Pr(B') &= 1 - \Pr(B)\\ &= 1 - 0.5\\ &= 0.5\end{aligned}$$



Pr(A ∩ B′) = 0.3

$$\begin{aligned}\Pr(A \mid B') &= \frac{\Pr(A \cap B')}{\Pr(B')}\\ &= \frac{0.3}{0.5}\\ &= \frac{3}{5}\end{aligned}$$

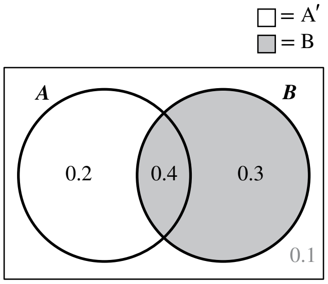
11 a Using the addition formula:

$$\begin{aligned}\Pr(A \cap B) &= \Pr(A) + \Pr(B) - \Pr(A \cup B)\\ \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= 0.6 + 0.7 - 0.4\\ &= 1.3 - 0.4\\ &= 0.9\end{aligned}$$

b
$$\begin{aligned}\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\\ &= \frac{0.4}{0.7}\\ &= \frac{4}{7}\end{aligned}$$

c Using the rule for complimentary events:

$$\begin{aligned}\Pr(A') &= 1 - \Pr(A)\\ &= 1 - 0.6\\ &= 0.4\end{aligned}$$

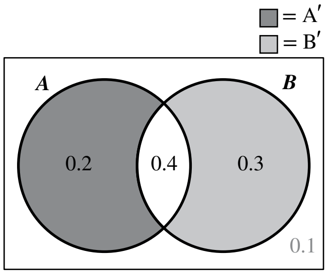


Pr(B ∩ A′) = 0.3

$$\begin{aligned}\Pr(B \mid A') &= \frac{\Pr(B \cap A')}{\Pr(A')}\\ &= \frac{0.3}{0.4}\\ &= \frac{3}{4}\end{aligned}$$

d Using the rule for complimentary events:

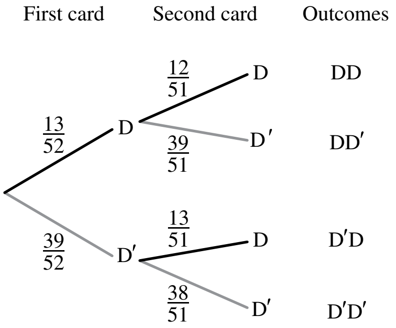
$$\begin{aligned}\Pr(B') &= 1 - \Pr(B)\\ &= 1 - 0.7\\ &= 0.3\end{aligned}$$



Pr(A′ ∩ B′) = 0.1

$$\begin{aligned}\Pr(A' \mid B') &= \frac{\Pr(A' \cap B')}{\Pr(B')}\\ &= \frac{0.1}{0.3}\\ &= \frac{1}{3}\end{aligned}$$

12 a



If one diamond is selected then the number of cards and the number of diamonds decreases by one for the second card.

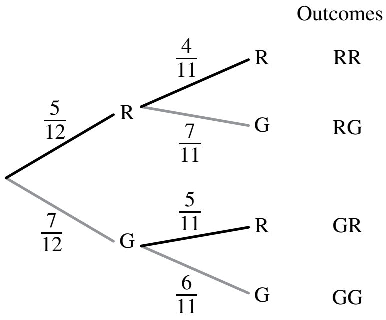
$$\begin{aligned}\Pr(DD)&=\frac{13}{52}\times\frac{12}{51}\\&=\frac{1}{17}\end{aligned}$$

b
$$\begin{aligned}\Pr(\textit{at least one diamond})&=\Pr(D\geq 1)\\&=1-\Pr(D=0)\\&=1-\Pr(D'D')\\&=1-\left(\frac{39}{52}\times\frac{38}{51}\right)\\&=1-\frac{19}{34}\\&=\frac{15}{34}\end{aligned}$$

c
$$\begin{aligned}\Pr(DD\mid D\geq 1)&=\frac{\Pr(DD\cap D\geq 1)}{\Pr(D\geq 1)}\\&=\frac{\Pr(DD)}{\Pr(D\geq 1)}\\&=\frac{\frac{1}{17}}{\frac{15}{34}}\\&=\frac{1}{17}\times\frac{34}{15}\\&=\frac{2}{15}\end{aligned}$$

d
$$\begin{aligned}\Pr(DD\mid \textit{first card } D)&=\frac{\Pr(DD\cap \textit{first card } D)}{\Pr(\textit{first card } D)}\\&=\frac{\Pr(DD)}{\Pr(DD)+\Pr(DD')}\\&=\frac{\frac{1}{17}}{\frac{1}{17}+\left(\frac{13}{52}\times\frac{39}{51}\right)}\\&=\frac{\frac{1}{17}}{\frac{1}{17}+\frac{13}{68}}\\&=\frac{\frac{1}{17}}{\frac{17}{68}}\\&=\frac{1}{17}\times\frac{68}{17}\\&=\frac{68}{289}\\&=\frac{4}{17}\end{aligned}$$

13 a



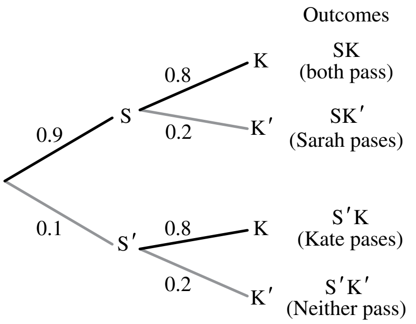
$$\begin{aligned}\Pr(GG)&=\frac{7}{12}\times\frac{6}{11}\\&=\frac{42}{132}\\&=\frac{7}{22}\end{aligned}$$

b
$$\begin{aligned}\Pr(G\geq 1)&=1-\Pr(RR)\\&=1-\left(\frac{5}{12}\times\frac{4}{11}\right)\\&=1-\frac{20}{132}\\&=\frac{112}{132}\\&=\frac{28}{33}\end{aligned}$$

c
$$\begin{aligned}\Pr(GG\mid G\geq 1)&=\frac{\Pr(GG\cap G\geq 1)}{\Pr(G\geq 1)}\\&=\frac{\Pr(GG)}{G\geq 1}\\&=\frac{\frac{7}{22}}{\frac{28}{33}}\\&=\frac{7}{22}\times\frac{33}{28}\\&=\frac{1}{2}\times\frac{3}{4}\\&=\frac{3}{8}\end{aligned}$$

d
$$\begin{aligned}\Pr(G\textit{ first}\mid \textit{different colours})&=\frac{\Pr(G\textit{ first}\cap \textit{different colours})}{\Pr(\textit{different colours})}\\&=\frac{\Pr(GR)}{\Pr(GR)+\Pr(RG)}\\&=\frac{\frac{7}{12}\times\frac{5}{11}}{\left(\frac{7}{12}\times\frac{5}{11}\right)+\left(\frac{5}{12}\times\frac{7}{11}\right)}\\&=\frac{\frac{35}{132}}{\frac{35}{132}+\frac{35}{132}}\\&=\frac{\frac{35}{132}}{2\left(\frac{35}{132}\right)}\\&=\frac{1}{2}\end{aligned}$$

14 a



$$\begin{aligned}\Pr(\textit{both pass}) &= 0.9\times 0.8\\ &= 0.72\end{aligned}$$

b
$$\begin{aligned}\Pr(\textit{at least one passes}) &= 1-\Pr(\textit{neither pass})\\ &= 1-0.1\times 0.2\\ &= 1-0.02\\ &= 0.98\end{aligned}$$

c
$$\begin{aligned}\Pr(\textit{only one passes} \mid \textit{S passes}) &= \frac{\Pr(\textit{only one passes} \cap \textit{S passes})}{\Pr(\textit{S passes})}\\ &= \frac{\Pr(SK')}{\Pr(SK')+\Pr(SK)}\\ &= \frac{0.9\times 0.2}{0.9\times 0.2+0.9\times 0.8}\\ &= \frac{0.18}{0.18+0.72}\\ &= \frac{0.18}{0.9}\\ &= \frac{18}{90}\\ &= \frac{1}{5} \text{ or } 0.2\end{aligned}$$

15 a
$$\begin{aligned}\Pr(S') &= \Pr(FS')+\Pr(MS')\\ &= \frac{224}{500}+\frac{203}{500}\\ &= \frac{427}{500}\end{aligned}$$

b
$$\begin{aligned}\Pr(M) &= \Pr(MS)+\Pr(MS')\\ &= \frac{32}{500}+\frac{203}{500}\\ &= \frac{235}{500}\\ &= \frac{47}{100}\end{aligned}$$

c
$$\begin{aligned}\Pr(F \mid S') &= \frac{\Pr(F\cap S')}{\Pr(S')}\\ &= \frac{\frac{224}{500}}{\frac{427}{500}}\\ &= \frac{224}{500}\times\frac{500}{427}\\ &= \frac{224}{427}\\ &= \frac{32}{61}\end{aligned}$$

16 a
$$\begin{aligned}\Pr(O) &= \Pr(OH)+\Pr(OH')\\ &= \frac{82}{1000}+\frac{185}{1000}\\ &= \frac{267}{1000}\end{aligned}$$

b
$$\begin{aligned}\Pr(H) &= \Pr(OH)+\Pr(O'H)\\ &= \frac{82}{1000}+\frac{175}{1000}\\ &= \frac{257}{1000}\end{aligned}$$

c
$$\begin{aligned}\Pr(H \mid O) &= \frac{\Pr(H\cap O)}{\Pr(O)}\\ &= \frac{\frac{82}{1000}}{\frac{267}{1000}}\\ &= \frac{82}{1000}\times\frac{1000}{267}\\ &= \frac{82}{267}\end{aligned}$$

d Using rule for complimentary events:

$$\begin{aligned}\Pr(H') &= 1-\frac{257}{1000}\\ &= \frac{743}{1000}\end{aligned}$$

$$\begin{aligned}\Pr(O \mid H') &= \frac{\Pr(O\cap H')}{\Pr(H')}\\ &= \frac{\frac{185}{1000}}{\frac{743}{1000}}\\ &= \frac{185}{1000}\times\frac{1000}{743}\\ &= \frac{185}{743}\end{aligned}$$

17 a
$$\begin{aligned}\Pr(R) &= \Pr(RA)+\Pr(RB)\\ &= \frac{25}{400}+\frac{47}{400}\\ &= \frac{72}{400}\\ &= \frac{9}{50}\end{aligned}$$

b
$$\begin{aligned}\Pr(A) &= \Pr(RA)+\Pr(R'A)\\ &= \frac{25}{400}+\frac{143}{400}\\ &= \frac{168}{400}\\ &= \frac{21}{50}\end{aligned}$$

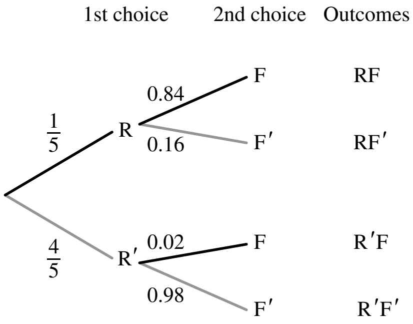
c Using the rule for complimentary events:

$$\begin{aligned}\Pr(B) &= 1\Pr(B')\\ &= 1-\Pr(A)\\ &= 1-\frac{21}{50}\\ &= \frac{29}{50}\end{aligned}$$

$$\begin{aligned}\Pr(R \mid B) &= \frac{\Pr(R\cap B)}{\Pr(B)}\\ &= \frac{\frac{47}{400}}{\frac{29}{50}}\\ &= \frac{47}{400}\times\frac{50}{29}\\ &= \frac{47}{232}\end{aligned}$$

d $$\begin{aligned}\Pr(R' \mid A) &= \frac{\Pr(R' \cap A)}{\Pr(A)} \\ &= \frac{\frac{143}{400}}{\frac{21}{50}} \\ &= \frac{143}{400} \times \frac{50}{21} \\ &= \frac{143}{168}\end{aligned}$$

18 a



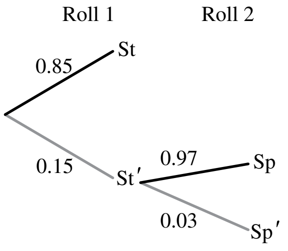
$$\begin{aligned}\Pr(RF) &= 0.2 \times 0.84 \\ &= 0.168\end{aligned}$$

b $$\begin{aligned}\Pr(F \mid R) &= \frac{\Pr(F \cap R)}{\Pr(R)} \\ &= \frac{\Pr(RF)}{\Pr(RF) + \Pr(RF')} \\ &= \frac{0.2 \times 0.84}{0.2 \times 0.84 + 0.2 \times 0.16} \\ &= \frac{0.168}{0.2} \\ &= 0.84\end{aligned}$$

c Incy Wincy makes it to the top so therefore he does not fall. Therefore we need to find the probability it is raining given he does not fall.

$$\begin{aligned}\Pr(R \mid F') &= \frac{\Pr(R \cap F')}{\Pr(F')} \\ &= \frac{\Pr(R \cap F')}{\Pr(RF') + \Pr(R'F')} \\ &= \frac{0.2 \times 0.16}{0.2 \times 0.16 + 0.8 \times 0.98} \\ &= \frac{0.032}{0.032 + 0.784} \\ &= \frac{0.032}{0.816} \\ &= \frac{2}{51} \\ &= 0.0392\end{aligned}$$

19 a



The tree diagram for each frame is the same since Richard's probability for obtaining a strike and spare remains the same throughout the club championship. Therefore frame 2 is not dependent on the previous frame, frame 1.

$$\begin{aligned}\Pr(\textit{both strikes}) &= 0.85 \times 0.85 \\ &= 0.7225\end{aligned}$$

b $$\begin{aligned}\Pr(\textit{all down}) &= 1 - \Pr((\textit{all down})') \\ &= 1 - (\Pr(St'Sp') \times \Pr(St'Sp')) \\ &= 1 - ((0.15 \times 0.03) \times (0.15 \times 0.03)) \\ &= 0.999\,98\end{aligned}$$

c $$\begin{aligned}\Pr(1^{st}\, St \mid 2^{nd}\, St) &= \frac{\Pr(1^{st}\, St \cap 2^{nd}\, St)}{\Pr(2^{nd}\, St)} \\ &= \frac{\Pr(StSt)}{\Pr(2^{nd}\, St)} \\ &= \frac{0.85 \times 0.85}{1 \times 0.85} \\ &= \frac{0.85 \times 0.85}{1 \times 0.85} \\ &= 0.85\end{aligned}$$

Exercise 8.4 — Independence

1 a $\xi = \{\mathrm{HH, HT, TH, TT}\}$.
$A = \{\mathrm{TH, TT}\}$
$B = \{\mathrm{HT, TH}\}$
$C = \{\mathrm{HH, HT, TH}\}$

b A and B are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$\Pr(A) = \frac{2}{4}$ and $\Pr(B) = \frac{2}{4}$.

Since $A \cap B = \{TH\}$, $\Pr(A \cap B) = \frac{1}{4}$.

Substitute values into the formula $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$$\text{LHS} = \frac{1}{4}$$

$$\begin{aligned}\text{RHS} &= \frac{2}{4} \times \frac{2}{4} \\ &= \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{4}\end{aligned}$$

Since LHS = RHS, the events A and B are independent.

c B and C are independent if $\Pr(B \cap C) = \Pr(B)\Pr(C)$.

$\Pr(B) = \frac{2}{4}$ and $\Pr(C) = \frac{3}{4}$.

Since $B \cap C = \{HT, TH\}$, $\Pr(B \cap C) = \frac{1}{2}$.

Substitute values into the formula $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$$\text{LHS} = \frac{1}{2}$$

$$\begin{aligned}\text{RHS} &= \frac{2}{4} \times \frac{3}{4} \\ &= \frac{1}{2} \times \frac{3}{4} \\ &= \frac{3}{8}\end{aligned}$$

Since LHS ≠ RHS, the events B and C are not independent.

d $\Pr(B \cup A) = \Pr(B) + \Pr(A) - \Pr(B \cap A)$

Since B and A are independent, $\Pr(B \cap A) = \Pr(B)\Pr(A)$

$\therefore \Pr(B \cup A) = \Pr(B) + \Pr(A) - \Pr(B) \times \Pr(A)$

$\therefore \Pr(B \cup A) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2}$

$\therefore \Pr(B \cup A) = \frac{3}{4}$

2 a A is the event the same number is obtained on each die, $\therefore A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$.
B is the event the sum of the numbers on each die exceeds 8, $\therefore B = \{(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}$.
A and B are mutually exclusive if $n(A \cap B) = 0$.
Since $A \cap B = \{(5,5),(6,6)\}$, A and B are not mutually exclusive.

b A and B are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.
$\Pr(A) = \frac{6}{36}$ and $\Pr(B) = \frac{10}{36}$.
Since $A \cap B = \{(5,5),(6,6)\}$, $\Pr(A \cap B) = \frac{2}{36}$.
Substitute values into the formula $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$$\text{LHS} = \frac{2}{36}$$

$$\begin{aligned}\text{RHS} &= \frac{6}{36} \times \frac{10}{36}\\ &= \frac{1}{6} \times \frac{5}{18}\\ &= \frac{5}{108}\end{aligned}$$

Since LHS ≠ RHS, the events A and B are not independent.

c i C is the event the sum of the two numbers equals 8, $\therefore C = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.
B and C are mutually exclusive if $n(B \cap C) = 0$.
Since there are no outcomes common to B and C, they are mutually exclusive.
Note that B is the event the sum of the numbers on each die *exceeds* 8 and C is the event the sum of the two numbers *equals* 8. Hence they cannot occur simultaneously, so they must be mutually exclusive.

ii B and C are independent if $\Pr(B \cap C) = \Pr(B)\Pr(C)$.
From (i), $\Pr(B \cap C) = 0$.
Therefore, assuming that neither $\Pr(B)$ nor $\Pr(C)$ are zero, $\Pr(B \cap C) \neq \Pr(B)\Pr(C)$
Since LHS ≠ RHS, the events A and B are not independent.
Note that mutually exclusive events cannot be independent, and vice versa.

3 a Let A be the event that Ava sticks to the diet, B be the event that Bambi sticks to the diet and C be the event that Chi sticks to the diet.
$\Pr(A) = 0.4$, $\Pr(B) = 0.9$ and $\Pr(C) = 0.6$
Since the events are independent,

$$\begin{aligned}\Pr(A \cap B \cap C) &= \Pr(A) \times \Pr(B) \times \Pr(C)\\ &= 0.4 \times 0.9 \times 0.6\\ &= 0.216\end{aligned}$$

The probability all three stick to the diet is 0.216.

b

$$\begin{aligned}\Pr(A \cap B' \cap C) &= \Pr(A) \times \Pr(B') \times \Pr(C)\\ &= 0.4 \times (1 - 0.9) \times 0.6\\ &= 0.4 \times 0.1 \times 0.6\\ &= 0.024\end{aligned}$$

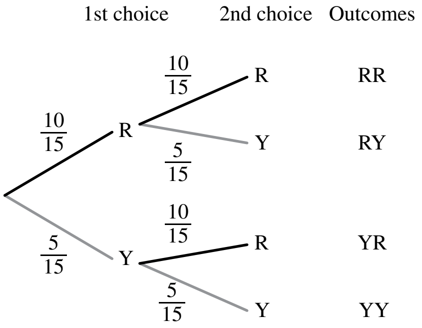
The probability only Ava and Chi stick to the diet is 0.024.

c Using the rule for complimentary events,

$$\begin{aligned}\Pr(\text{at least one does not stick to the diet}) &= 1 - \Pr(\text{all stick to the diet})\\ &= 1 - \Pr(A \cap B \cap C)\\ &= 1 - 0.4 \times 0.9 \times 0.6\\ &= 1 - 0.216\\ &= 0.784\end{aligned}$$

The probability that at least one person does not stick to the diet is 0.784.

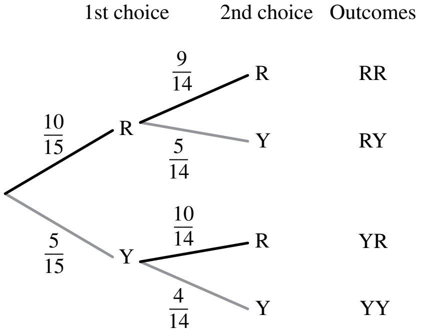
4 a



Let A be the event that one block of each colour is obtained (*with* replacement). $\therefore A = \{RY, YR\}$.

$$\begin{aligned}\Pr(A) &= \Pr(RY) + \Pr(YR)\\ &= \frac{10}{15}\times\frac{5}{15}+\frac{5}{15}\times\frac{10}{15}\\ &= \frac{2}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{3}\\ &= \frac{4}{9}\end{aligned}$$

b



Let B be the event that one block of each colour is obtained (*without* replacement). $\therefore B = \{RY, YR\}$.

$$\begin{aligned}\Pr(B) &= \Pr(RY) + \Pr(YR)\\ &= \frac{10}{15}\times\frac{5}{14}+\frac{5}{15}\times\frac{10}{14}\\ &= \frac{50}{210}+\frac{50}{210}\\ &= \frac{10}{21}\end{aligned}$$

c Let C be the event that three blocks of the same colour are obtained (*with* replacement). $\therefore C = \{RRR, YYY\}$.

$$\begin{aligned}\Pr(C) &= \Pr(RRR) + \Pr(YYY)\\ &= \frac{10}{15}\times\frac{10}{15}\times\frac{10}{15}+\frac{5}{15}\times\frac{5}{15}\times\frac{5}{15}\\ &= \frac{1000}{3375}+\frac{125}{3375}\\ &= \frac{1}{3}\end{aligned}$$

5 If two events A and B are independent:
$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$$\begin{aligned}\text{LHS} &= \Pr(A\cap B)\\ &= \Pr(A)+\Pr(B)-\Pr(A\cup B)\\ &= 0.7+0.8-0.94\\ &= 1.5-0.94\\ &= 0.56\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \Pr(A)\times\Pr(B)\\ &= 0.7\times 0.8\\ &= 0.56\end{aligned}$$

$$\text{LHS} = \text{RHS}$$

Therefore since $\Pr(A\cap B)=\Pr(A)\times\Pr(B)$, the events A and B are independent.

6 If two events A and B are independent:
$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$$\begin{aligned}\text{LHS} &= \Pr(A\cap B)\\ &= \Pr(A)+\Pr(B)-\Pr(A\cup B)\\ &= 0.75+0.64-0.91\\ &= 1.39-0.91\\ &= 0.48\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \Pr(A)\times\Pr(B)\\ &= 0.75\times 0.64\\ &= 0.48\end{aligned}$$

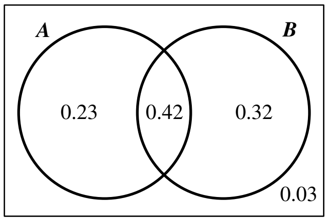
$$\text{LHS} = \text{RHS}$$

Therefore since $\Pr(A\cap B)=\Pr(A)\times\Pr(B)$, the events A and B are independent.

7

$$\begin{aligned}\Pr(A) &= 0.65\\ \Pr(B) &= 0.74\\ \Pr(\text{at least one}) &= 0.97\end{aligned}$$

Since at least one of the cars is used 97% of the time then none of the cars are used 3% of the time. Therefore when forming a Venn diagram for this situation 0.03 lies outside circles and the rest of the 0.97 is distributed with A, B and $A \cap B$.



$$\begin{aligned}\Pr(A)+\Pr(B) &= 0.65+0.74\\ &= 1.39\end{aligned}$$

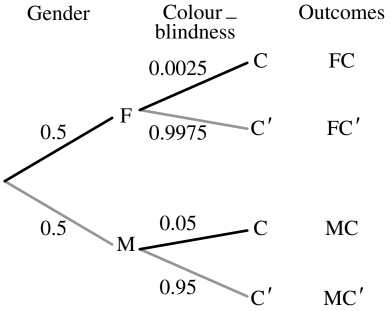
$$\begin{aligned}\Pr(A\cap B) &= 1.39-0.97\\ &= 0.42\end{aligned}$$

$$\begin{aligned}\Pr(A)\times\Pr(B) &= 0.65\times 0.74\\ &= 0.481\end{aligned}$$

$$\Pr(A\cap B) \neq \Pr(A)\times\Pr(B)$$

Therefore since $\Pr(A\cap B)\neq\Pr(A)\times\Pr(B)$, the cars A and B are not used independently.

8 **a**



$$\begin{aligned}\Pr(FC) &= 0.5\times 0.0025\\ &= \frac{1}{800}\\ &= 0.001\,25\end{aligned}$$

b $$\begin{aligned}\Pr(C \mid F) &= \frac{\Pr(C \cap F)}{\Pr(F)}\\ &= \frac{\Pr(F \cap C)}{\Pr(F)}\\ &= \frac{\frac{1}{800}}{\frac{1}{2}}\\ &= \frac{1}{800} \times \frac{2}{1}\\ &= \frac{1}{400}\\ &= 0.0025\end{aligned}$$

c $$\begin{aligned}\Pr(M \mid C) &= \frac{\Pr(M \cap C)}{\Pr(C)}\\ &= \frac{\Pr(M \cap C)}{\Pr(MC) + \Pr(FC)}\\ &= \frac{0.5 \times 0.05}{0.5 \times 0.05 + 0.5 \times 0.0025}\\ &= \frac{0.025}{0.025 + 0.001\,25}\\ &= \frac{0.025}{0.026\,25}\\ &= \frac{20}{21}\\ &= 0.9524\end{aligned}$$

d $$\begin{aligned}\Pr(2 \times MC) &= (0.5 \times 0.05) \times (0.5 \times 0.05)\\ &= \frac{1}{40} \times \frac{1}{40}\\ &= \frac{1}{1600}\\ &= 0.000\,625\end{aligned}$$

e $$\begin{aligned}\Pr(1C \mid M \text{ and } F) &= \frac{\Pr(1C \cap M \text{ and } F)}{\Pr(M \text{ and } F)}\\ &= \frac{\Pr(MC) \times \Pr(FC') + \Pr(MC') \times \Pr(FC)}{\Pr(M \text{ and } F)}\\ &= \frac{0.5 \times 0.05 \times 0.5 \times 0.9975 + 0.5 \times 0.95 \times 0.5 \times 0.0025}{0.5 \times 0.5}\\ &= \frac{209}{4000}\\ &= 0.052\,25\end{aligned}$$

9 **a** $$\begin{aligned}\Pr(x < 25) &= \frac{8 + 30 + 7}{200}\\ &= \frac{45}{200}\\ &= \frac{9}{40}\end{aligned}$$

b $$\begin{aligned}\Pr(\textit{at least one violation}) &= 1 - \Pr(\textit{zero violations})\\ &= 1 - \frac{8 + 47 + 45 + 20}{200}\\ &= 1 - \frac{120}{200}\\ &= \frac{80}{200}\\ &= \frac{2}{5}\end{aligned}$$

c $$\begin{aligned}\Pr(\textit{only one violation} \mid \textit{at least one violation}) &= \frac{\Pr(\textit{only one violation} \cap \textit{at least one violation})}{\Pr(\textit{at least one violation})}\\ &= \frac{\Pr(\textit{only one violation})}{\Pr(\textit{at least one violation})}\\ &= \frac{\frac{30+15+18+5}{200}}{\frac{2}{5}}\\ &= \frac{68}{200} \times \frac{5}{2}\\ &= \frac{17}{20}\end{aligned}$$

d Note: If the person is 38 they are in the 25–45 age group range.
Therefore look in the 25–45 row and 0 violations column.

$$\Pr(38 \text{ and } \textit{no violations}) = \frac{47}{200}$$

e $$\begin{aligned}\Pr(x < 25 \mid 2\ \textit{violations}) &= \frac{\Pr(x < 25 \cap 2\ \textit{violations})}{\Pr(2\ \textit{violations})}\\ &= \frac{\frac{7}{200}}{\frac{7+2+3}{200}}\\ &= \frac{7}{200} \times \frac{200}{12}\\ &= \frac{7}{12}\end{aligned}$$

10 **a** Let A = has the disease
Let B = positive test result

	A	A′	
B	23	7	30
B′	4	66	70
	27	73	100

$$\Pr(A') = \frac{73}{100} \text{ or } 0.73$$

b $$\Pr(B \cap A') = \frac{7}{100}$$

c $$\begin{aligned}\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\\ &= \frac{\frac{23}{100}}{\frac{30}{100}}\\ &= \frac{23}{100} \times \frac{100}{30}\\ &= \frac{23}{30}\end{aligned}$$

d $$\begin{aligned}\Pr(A' \mid B') &= \frac{\Pr(A' \cap B')}{\Pr(B')}\\ &= \frac{\frac{66}{100}}{\frac{70}{100}}\\ &= \frac{66}{100} \times \frac{100}{70}\\ &= \frac{66}{70}\end{aligned}$$

11 **a** Use the formula for conditional probability and substitute in the given values to find $\Pr(A \cap B)$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\frac{4}{5} = \frac{\Pr(A \cap B)}{\frac{2}{3}}$$

$$\Pr(A \cap B) = \frac{4}{5} \times \frac{2}{3}$$

$$\Pr(A \cap B) = \frac{8}{15}$$

Then, since A and B are independent:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

$$\frac{8}{15} = \Pr(A) \times \frac{2}{3}$$

$$\frac{2}{3} \times \Pr(A) = \frac{8}{15}$$

$$\begin{aligned}\Pr(A) &= \frac{3}{2} \times \frac{8}{15}\\ &= \frac{4}{5}\end{aligned}$$

Therefore $\Pr(A) = \frac{4}{5}$.

b $\Pr(B \mid A) = \dfrac{\Pr(B \cap A)}{\Pr(A)}$

Then, since A and B are independent and $\Pr(A) = \frac{4}{5}$ (from question 7a.)

$$\begin{aligned}\Pr(B \cap A) &= \Pr(A \cap B)\\ &= \Pr(A) \times \Pr(B)\\ &= \frac{4}{5} \times \frac{2}{3}\\ &= \frac{8}{15}\end{aligned}$$

$$\begin{aligned}\Pr(B \mid A) &= \frac{\Pr(B \cap A)}{\Pr(A)}\\ &= \frac{\frac{8}{15}}{\frac{4}{5}}\\ &= \frac{8}{15} \times \frac{5}{4}\\ &= \frac{2}{3}\end{aligned}$$

Therefore $\Pr(B \mid A) = \frac{2}{3}$.

c Using the formula for conditional probability:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\frac{4}{5} = \frac{\Pr(A \cap B)}{\frac{2}{3}}$$

$$\Pr(A \cap B) = \frac{4}{5} \times \frac{2}{3}$$

$$\Pr(A \cap B) = \frac{8}{15}$$

d Using the addition formula:

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= \frac{4}{5} + \frac{2}{3} - \frac{8}{15}\\ &= \frac{12}{15} + \frac{10}{15} - \frac{8}{15}\\ &= \frac{22}{15} - \frac{8}{15}\\ &= \frac{14}{15}\end{aligned}$$

12 **a** Using the formula for conditional probability:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\frac{1}{3} = \frac{\Pr(A \cap B)}{\frac{3}{5}}$$

$$\frac{1}{3} = \Pr(A \cap B) \times \frac{5}{3}$$

$$\Pr(A \cap B) = \frac{1}{3} \times \frac{3}{5}$$

$$\Pr(A \cap B) = \frac{1}{5}$$

b Using the addition formula:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\begin{aligned}\Pr(A) &= \Pr(A \cup B) - \Pr(B) + \Pr(A \cap B)\\ &= \frac{23}{30} - \frac{3}{5} + \frac{1}{5}\\ &= \frac{23}{30} - \frac{2}{5}\\ &= \frac{23}{30} - \frac{12}{30}\\ &= \frac{11}{30}\end{aligned}$$

c If two events A and B are independent:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

$$\text{LHS} = \frac{1}{5}$$

$$\begin{aligned}\text{RHS} &= \Pr(A) \times \Pr(B)\\ &= \frac{11}{30} \times \frac{3}{5}\\ &= \frac{11}{50}\end{aligned}$$

$$\text{LHS} \neq \text{RHS}$$

Therefore since $\Pr(A \cap B) \neq \Pr(A) \times \Pr(B)$, the events A and B are not independent.

13 Given events A and B are independent,

$$\begin{aligned}\Pr(A \mid B') &= \frac{\Pr(A \cap B')}{\Pr(B')} = 0.6\\ &= \frac{\Pr(A) \times \Pr(B')}{\Pr(B')} = 0.6\\ &= \Pr(A) = 0.6\end{aligned}$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$0.8 = \Pr(A) + \Pr(B) - \Pr(A) \times \Pr(B)$$

$$0.8 = 0.6 + x - 0.6 \times x$$

$$0.5 = x$$

$$\Rightarrow \Pr(B) = 0.5$$

14 Let Y = age 15 to 30
Let T = TV
If two events Y and T are independent:
$\Pr(Y \cap T) = \Pr(Y) \times \Pr(T)$

$$\text{LHS} = \frac{95}{600} = \frac{19}{120}$$

$$\begin{aligned}\text{RHS} &= \Pr(Y) \times \Pr(T)\\ &= \frac{290}{600} \times \frac{270}{600}\\ &= \frac{26}{60} \times \frac{9}{20}\\ &= \frac{87}{400}\end{aligned}$$

$$\text{LHS} \neq \text{RHS}$$

Therefore since $\Pr(Y \cap T) \neq \Pr(Y) \times \Pr(T)$, the events P and T are not independent.

15 If two events P and T are independent:
$\Pr(P \cap T) = \Pr(P) \times \Pr(T)$

$$\text{LHS} = \frac{34}{100} = \frac{17}{50}$$

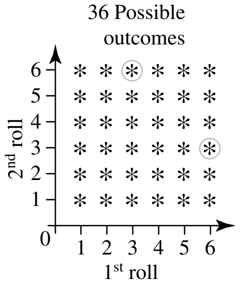
$$\begin{aligned}\text{RHS} &= \Pr(P) \times \Pr(T)\\ &= \frac{58}{100} \times \frac{50}{100}\\ &= \frac{29}{50} \times \frac{1}{2}\\ &= \frac{29}{100}\end{aligned}$$

$$\text{LHS} \neq \text{RHS}$$

Therefore since $\Pr(P \cap T) \neq \Pr(P) \times \Pr(T)$, the events P and T are not independent.

16 **a** $\Pr(A) = \frac{1}{6}$

$\Pr(B) = \frac{1}{6}$



$$\Pr(A \cap B) = \frac{2}{36} = \frac{1}{18}$$

If two events A and B are independent:

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$$\text{LHS} = \frac{1}{18}$$

$$\begin{aligned}\text{RHS} &= \Pr(A) \times \Pr(B)\\ &= \frac{1}{6} \times \frac{1}{6}\\ &= \frac{1}{36}\end{aligned}$$

$$\text{LHS} \neq \text{RHS}$$

Therefore since $\Pr(A \cap B) \neq \Pr(A) \times \Pr(B)$, the events A and B are not independent.

b Similarly, B and C are not independent

c Similarly, B and C are not independent

Exercise 8.5 — Counting techniques

1 **a** There are five choices for the first digit, leaving four choices for the second digit, three choices for the third digit and then two choices for the fourth digit.

5	4	3	2

Using the multiplication principle, there are $5 \times 4 \times 3 \times 2 = 120$ possible four digit numbers that could be formed.

b At least three digit numbers means either three digit, four digit or five digit numbers are to be counted.
For three digit numbers:

5	4	3

For four digit numbers:

5	4	3	2

For five digit numbers:

5	4	3	2	1

There are $5 \times 4 \times 3 = 60$ three digit numbers, $5 \times 4 \times 3 \times 2 = 120$ four digit numbers and $5 \times 4 \times 3 \times 2 \times 1 = 120$ five digit numbers. Using the addition principle there are $60 + 120 + 120 = 300$ possible three, four or five digit numbers.

c For the number to be even its last digit must be even, so the number must end in 6. This means there is one choice for the last digit.

				1

Once the last digit has been formed, there are four choices for the first digit then three choices for the second digit, two choices for the third digit and one choice for the fourth digit.

4	3	2	1	1

Using the multiplication principle, there are $4 \times 3 \times 2 \times 1 \times 1 = 24$ even five digit numbers possible.

d The sample space is the set of five digit numbers. From part b, $n(\xi) = 120$.
Let A be the event the five digit number is even. From part c, $n(A) = 24$.

$$P(A)=\frac{n(A)}{n(\xi)}$$
$$=\frac{24}{120}$$
$$=\frac{1}{5}$$

The probability the five digit number is even is $\frac{1}{5}$.

2 a There are 26 letters in the English alphabet and 10 digits from 0 to 9. Repetition of letters and digits is allowed.

26	26	10	10	10	26

Using the multiplication principle, there are $26\times26\times10\times10\times10\times26=17\,576\,000$ possible number plates that could be formed.

b If the letter X is used exactly once, there will be one choice for that position, and 25 choices for the other two positions where letters are used.

For an X in the first position:

1	25	10	10	10	25

For an X in the second position:

25	1	10	10	10	25

For an X in the sixth position:

25	25	10	10	10	1

Using the multiplication principle, there are $1\times25\times10\times10\times10\times25=625\,000$ number plates that use the letter X exactly once (in the first position), $25\times1\times10\times10\times10\times25=625\,000$ (in the second position) and $25\times25\times10\times10\times10\times1=625\,000$ (in the sixth position). Using the addition principle there are $625\,000+625\,000+625\,000=1\,875\,000$ possible number plates that use the letter X exactly once.

c Using the multiplication principle, there are $26\times1\times10\times1\times1\times26=6760$ possible number plates that could be formed.

$$\text{Pr(first 2 letters identical)}=\frac{6760}{17\,576\,000}$$
$$=\frac{1}{2704}$$

3 a Six people can arrange in a straight line in $6!$ ways.

Since $6!=6\times5!$ and $5!=120$,

$$6!=6\times120$$
$$=720$$

There are 720 ways in which the students can form the queue.

b For circular arrangements, 6 people can be arranged in $(6-1)!=5!$ ways.

Since $5!=120$, there are 120 different arrangements in which the six students may be seated.

c There are three prizes. Each prize can be awarded to any one of the six students.

6	6	6

The total number of ways the prizes can be awarded is $6\times6\times6$.

$\therefore n(\xi)=6\times6\times6$

Let A be the event that the same student receives all three prizes. There are six choices for that student.

$\therefore n(A)=6$

$$\Pr(A)=\frac{n(A)}{n(\xi)}$$
$$=\frac{6}{6\times6\times6}$$
$$=\frac{1}{36}$$

The probability that one students receives all three prizes is $\frac{1}{36}$.

4 a For circular arrangements, 4 people can be arranged in $(4-1)!=3!$ ways.

Since $3!=6$, there are 6 different arrangements in which the four boys may be seated.

b There are 4 possible seats each girl can sit in. For circular arrangements, 4 people can be arranged in $(4-1)!=3!$ ways.

Since $3!=6$, there are 6 different arrangements in which the four girls may be seated.

So, 24 ways in total.

5 a Treat the letters, Q and U, as one unit.

Now there are eight groups to arrange:

(QU), E, A, T, I, O, N, S.

These arrange in $8!$ ways.

The unit (QU) can internally re-arrange in $2!$ ways.

Hence, the total number of arrangements

$$=8!\times2!$$
$$=8\times7\times6\times5!$$
$$=336\times120$$
$$=40\,320$$

b The number of arrangements with the letters Q and U separated is equal to the total number of arrangements minus the number of arrangements with the vowels together.

The nine letters of the word EQUATIONS can be arranged in $9!=362\,880$ ways.

From part a, there are 40 320 arrangements with the letters together.

Therefore, there are $362\,880-40\,320=322\,560$ arrangements in which the two letters are separated.

c The word SIMULTANEOUS contains 12 letters of which there are 2 S's and 2 U's.

The number of arrangements of the word SIMULTANEOUS is $\frac{12!}{2!\times2!}=119\,750\,400$.

d As there are 119 750 400 total arrangements of the word SIMULTANEOUS, $n(\xi)=119\,750\,400$ or $\frac{12!}{2!\times2!}$.

For the letters U to be together, treat these two letters as one unit. This creates eleven groups (UU), S, I, M, L, T, A, N, E, O, S of which two are identical S's.

The eleven groups arrange in $\frac{11!}{2!}$ ways. As the unit (UU) contains two identical letters, there are no distinct internal re-arrangements of this unit that need to be taken into account.

Hence the number of elements in the event is $\frac{11!}{2!}$.

The probability that the U's are together

$$
\begin{aligned}
&= \frac{\text{number of arrangements with the U's together}}{\text{total number of arrangements}}\\
&= \frac{11!}{2!} \div \frac{12!}{2! \times 2!}\\
&= \frac{11!}{2!} \times \frac{2! \times 2!}{12 \times 11!}\\
&= \frac{2}{12}\\
&= \frac{1}{6}
\end{aligned}
$$

6 a The words 'PARALLEL LINES' contain 13 letters of which there are 2 A's, 4 L's and 2 E's.

The number of arrangements in a row of the words PARALLEL LINES equals $\frac{13!}{2! \times 4! \times 2!}$.

$$
\begin{aligned}
&\frac{13!}{2! \times 4! \times 2!}\\
&= \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{2! \times 4! \times 2!}\\
&= 64\,864\,800
\end{aligned}
$$

There are 64 864 800 arrangements in a row.

b For circular arrangements, the 13 letters of which there are 2 A's, 4 L's and 2 E's, can be arranged in $\frac{(13-1)!}{2! \times 4! \times 2!}$ ways.

$$
\begin{aligned}
&\frac{(13-1)!}{2! \times 4! \times 2!}\\
&= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{2! \times 4! \times 2!}\\
&= 4\,989\,600
\end{aligned}
$$

There are 4 989 600 arrangements in a circle.

c There are five vowels in the words PARALLEL LINES.
Treat these letters, A, A, E, I and E as one unit.
Now there are nine groups to arrange:
(AAEIE), P, R, L, L, L, L, N, S.
These arrange in 9! ways.
The unit (AAEIE) can internally re-arrange in 5! ways.
Hence, the total number of arrangements

$$
\begin{aligned}
&= 9! \times 5!\\
&= 9 \times 8 \times 7 \times 6 \times 5!\\
&= 3024 \times 120\\
&= 362\,880
\end{aligned}
$$

7 a There are 14 students in total from whom 5 students are to be chosen. This can be done in ${}^{14}C_5$ ways.

$$
\begin{aligned}
{}^{14}C_5 &= \frac{14!}{5! \times (14-5)!}\\
&= \frac{14!}{5! \times 9!}\\
&= \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9!}{5! \times 9!}\\
&= \frac{14 \times 13 \times 12 \times 11 \times 10}{120}\\
&= 2002
\end{aligned}
$$

There are 2002 possible committees.

b The 2 boys can be chosen from the 6 boys available in ${}^6C_2$ ways.
The 3 girls can be chosen from the 8 girls available in ${}^8C_3$ ways.
The total number of committees which contain two boys and three girls is
${}^6C_2 \times {}^8C_3$.

$$
\begin{aligned}
{}^6C_2 \times {}^8C_3 &= \frac{6!}{2! \times 4!} \times \frac{8!}{3! \times 5!}\\
&= \frac{6 \times 5 \times 4!}{2! \times 4!} \times \frac{8 \times 7 \times 6 \times 5!}{3! \times 5!}\\
&= 15 \times 56\\
&= 840
\end{aligned}
$$

There are 840 committees possible with the given restriction.

c As there are six boys available, at least four boys means either four or five boys.
The committees of five students which satisfy this restriction have either 4 boys and 1 girl or they have 5 boys and no girls.
4 boys and 1 girl are chosen in ${}^6C_4 \times {}^8C_1$ ways.
5 boys and no girls are chosen in ${}^6C_5 \times {}^8C_0$ ways.
The number of committees with at least four boys is ${}^6C_4 \times {}^8C_1 + {}^6C_5 \times {}^8C_0$.

$$
\begin{aligned}
{}^6C_4 \times {}^8C_1 + {}^6C_5 \times {}^8C_0 &= 15 \times 8 + 6 \times 1\\
&= 126
\end{aligned}
$$

There are 126 committees with at least four boys.

d The total number of committees of five students is ${}^{14}C_5 = 2002$ from part a.
Each committee must have five students. If neither the oldest nor youngest student are placed on the committee, then 5 students need to be selected from the remaining 12 students to form the committee of five. This can be done in ${}^{12}C_5$ ways.
Let A be the event that neither the oldest nor youngest are on the committee.

$$
\begin{aligned}
\Pr(A) &= \frac{n(A)}{n(\xi)}\\
&= \frac{{}^{12}C_5}{{}^{14}C_5}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Pr(A) &= \frac{12!}{5! \times 7!} \div \frac{14!}{5! \times 9!}\\
&= \frac{12!}{5! \times 7!} \times \frac{5! \times 9!}{14!}\\
&= \frac{1}{1} \times \frac{9 \times 8}{14 \times 13}\\
&= \frac{72}{182}\\
&= \frac{36}{91}
\end{aligned}
$$

The probability of the committee containing neither the youngest nor oldest student is $\frac{36}{91}$.

8 There are 17 players in total from whom 11 players are to be chosen. This can be done in ${}^{17}C_{11}$ ways.

$$
\begin{aligned}
{}^{17}C_{11} &= \frac{17!}{11! \times (17-11)!}\\
&= \frac{17!}{11! \times 6!}\\
&= \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11!}{11! \times 6!}\\
&= \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12}{720}\\
&= 12\,376
\end{aligned}
$$

There are 12 376 possible teams.

The 1 wicketkeeper can be chosen from the 3 wicketkeepers available in ${}^{3}C_1$ ways.
The 4 bowlers can be chosen from the 6 bowlers available in ${}^{6}C_4$ ways.
The 6 batsmen can be chosen from the 8 batsmen available in ${}^{8}C_6$ ways.
The total number of teams which contain one wicketkeeper, four bowlers and six batsmen is
${}^{3}C_1 \times {}^{6}C_4 \times {}^{8}C_6$.

$$\begin{aligned} {}^{3}C_1 \times {}^{6}C_4 \times {}^{8}C_6 &= \frac{3!}{1!\times 2!} \times \frac{6!}{4!\times 2!} \times \frac{8!}{6!\times 2!} \\ &= \frac{3\times 2!}{2!} \times \frac{6\times 5\times 4!}{4!\times 2!} \times \frac{8\times 7\times 6}{6!\times 2!} \\ &= 3\times 15\times 28 \end{aligned}$$

There are 1260 teams possible with the given restriction.
Let A be the event that the team chosen consists of one wicketkeeper, four bowlers and six batsmen.

$$\begin{aligned} \Pr(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{1260}{12\,367} \\ &= \frac{45}{442} \end{aligned}$$

The probability that the team chosen consists of one wicketkeeper, four bowlers and six batsmen is $\frac{45}{442}$.

9 **a** One bib can be selected from ten bibs in ${}^{10}C_1 = 10$ ways.
One body suit can be selected from twelve body suits in ${}^{12}C_1 = 12$ ways.
Using the multiplication principle, there are $10\times 12 = 120$ different combinations of bib and body suit she can wear.

b For the first leg of her trip, Christine has 2 options, the motorway or the highway.
For the second leg of her trip, Christine has 3 options of routes through the suburban streets.
Using the multiplication principle, there are $2\times 3 = 6$ different routes she can take. Therefore, if she wishes to take a different route to work each day, she will be able to take a different route on 6 days before she must use a route already travelled.

c For the first characteristic, Abdul has 2 options, manual or automatic.
For the second characteristic, Abdul has 5 options of exterior colour.
For the third characteristic, Abdul has 2 options, leather or vinyl seats.
For the fourth characteristic, Abdul has 3 options of interior colour.
For the fifth characteristic, Abdul has 2 options, seat heating or notself parking.
For the sixth characteristic, Abdul has 2 options, self parking or not.
Using the multiplication principle, there are $2\times 5\times 2\times 3\times 2\times 2 = 240$ different combinations of Peugeot Abdul can choose from.

d Using the multiplication principle, there are $3\times 2\times 7\times 5 = 210$ different combinations of clothes possible.

e Using the multiplication principle, there are $6\times 52 = 312$ different starting combinations.

f Using the multiplication principle, there are $3\times 6\times 12 = 216$ different trips possible.

10 **a** There are 26 letters in the English alphabet and 10 digits from 0 to 9. Repetition of letters and digits is allowed.

26	26	10	10	26

Using the multiplication principle, there are $26\times 26\times 10\times 110\times 26 = 1\,757\,600$ possible number plates that could be formed.

b There are 7 different letters, from which 5 letter words are formed. Repetitions are allowed.

7	7	7	7	7

Using the multiplication principle, there are $7\times 7\times 7\times 7\times 7 = 16\,807$ possible words that could be formed.

c For each single roll there are 6 possible outcomes.
Using the multiplication principle, for a die rolled three times there are $6\times 6\times 6 = 216$ possible outcomes.

d There are 6 different digits, from which 3-digit numbers are formed. Repetitions are allowed.

6	6	6

Using the multiplication principle, there are $6\times 6\times 6 = 216$ possible 3-digit numbers that could be formed.

e Three rooms can be selected from the four available in ${}^{4}C_3 = 4$ ways.
The three selected rooms can be arranged among the three friends in $3! = 3\times 2\times 1 = 6$ ways.
Using the multiplication principle, there are $4\times 6 = 24$ possible ways can the rooms be allocated.

11 **a** If there are no restrictions, eight people can be arranged in a row in $8! = 40\,320$ ways.

b If the boys and girls are to alternate, assuming the row begins with a boy, there are 4 choices of boy for the first position, 4 choices of girl for the second position, 3 choices of boy for the third position, 3 choices of girl for the fourth position and so on.

4	4	3	3	2	2	1	1

If the row begins with a girl, since there are an equal number of boys and girls, the box table will be identical.
Using the multiplication principle, there are $4\times 4\times 3\times 3\times 2\times 2\times 1\times 1 = 576$ ways that 4 boys and 4 girls can be arranged in a row, if a boy is first. By the same rule there will be 576 ways if a girl goes first.
Therefore, using the addition principle, there is a total of $576 + 576 = 1152$ ways that 4 boys and 4 girls can be arranged in a row if the boys and girls are to alternate.

c If the end seats must be occupied by a girl, there are 4 choices of girl for the first position and 3 choices of girl for the last position. Hence, for the remaining seats there are a remainder of 4 boys and 2 girls = 6 people left.

4	6	5	4	3	2	1	3

Using the multiplication principle, there are $4\times 6\times 5\times 4\times 3\times 2\times 1\times 3 = 8640$ ways that 4 boys and 4 girls can be arranged in a row, if the end seats must be occupied by a girl.

d If the brother and sister *must not* sit together, this is the complementary event of the brother and sister *must* sit together. Find the number of ways for the latter event first.
Treat the brother and sister as one unit. Now there are 7 groups, which will arrange in 7! ways.
The unit (brother and sister) can internally rearrange in 2! ways.

Hence number of arrangements

$= 7! \times 2!$

$= 5040 \times 2$

$= 10\,080$

From a), total number of ways 8 people can be arranged is 40 320.

Number of ways if brother and sister must not sit together

$= 40\,320 - 10\,080$

$= 30\,240$

e If the girls must sit together, treat the girls as one unit. Now there are 5 groups, which will arrange in 5! ways. The unit (the girls) can internally rearrange in 4! ways.

Hence number of arrangements

$= 5! \times 4!$

$= 120 \times 24$

$= 2880$

12 a There are ten digits. There are no repetitions are allowed and the number cannot start with 0. Therefore there are 9 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit.

9	9	8

Using the multiplication principle, there are $9 \times 9 \times 8 = 648$ possible three-digit numbers that could be formed.

b For the number to be even its last digit must be even, so the number must end in a 0, 2, 4, 6 or 8. This means there are 5 choices for the last digit.

Once the last digit has been formed, if the last digit was 0, there are 9 choices for the first digit then 8 choices for the second digit.

9	8	1

Therefore there are $9 \times 8 \times 1 = 72$ numbers possible.

If the last digit was not 0, there are 8 choices for the first digit then 8 choices for the second digit (because the first digit cannot be 0).

8	8	4

Therefore there are $8 \times 8 \times 4 = 256$ numbers possible.

Using the addition principle, there are $72 + 256 = 328$ even 3-digit numbers that could be formed.

c For the number to be less than 400 its first digit must be less than 4, so the number must start in a 1, 2 or 3 (remembering the number cannot start with 0). This means there are 3 choices for the first digit.

3		

Once the first digit has been formed, there are 9 choices for the second digit then 8 choices for the last digit.

3	9	8

Using the multiplication principle, there are $3 \times 9 \times 8 = 216$ even 3-digit numbers that could be formed.

d For the number to made up of odd digits only, each digit may only be a 1, 3, 5, 7 or a 9. Therefore there are 5 choices for the first digit, 4 choices for the second digit and 3 choices for the third digit.

5	4	3

Using the multiplication principle, there are $5 \times 4 \times 3 = 60$ possible three-digit numbers that could be formed.

13 a For circular arrangements, 9 people can be arranged in $(9-1)! = 8! = 40\,320$ ways.

b If the men can only be seated in pairs, treat each pair as one unit. Now there are 6 groups, which will arrange in $(6-1)! = 5!$ ways.

The 6 men can be arranged into 3 pairs in $\dfrac{{}^6C_2 \times {}^4C_2 \times {}^2C_2}{3!} = 15$ ways.

The units (the pairs) can each internally rearrange in 2! ways.

Hence number of arrangements

$= 5! \times 15 \times 2!$

$= 120 \times 30$

$= 3600$

14 a Using the rule for number of arrangements, some of which are identical, number of arrangements of mugs

$= \dfrac{12!}{4! \times 3! \times 5!}$

$= 27\,720$

b If the 12 mugs from part a are to be arranged in 2 rows of 6 and the green ones must be on the front row,

$\left(\dfrac{6!}{4! \times 2!} + \dfrac{6!}{3! \times 3!}\right) \times \dfrac{6!}{5! \times 1!}$

$= 210$

15 a The word *bananas* contains 7 letters of which there are 3 *a*s and 2 *n*s.

Number of words that can be formed, given all letters are used

$= \dfrac{7!}{3! \times 2!}$

$= 420$

b A 4-letter word is to be formed including at least one *a*. The complementary event of this is forming a word with no *a*s. Notice that there are only 4 other letters in the word *bananas* which are not an *a*. These are *b*, *n*, *n* and *s*. Note the two identical *n*s.

Number of 4-letter words with no *a*s

$= \dfrac{4!}{2!}$

$= 12$

From part a, total number of arrangements is 420.

Therefore number of 4-letter words with at least one *a*

$= 420 - 12$

$= 408$

c To form a 4-letter word using all different letters, exclude any repeated letter. Hence the available letters are *b*, *a*, *n* and *s*.

Number of words that can be formed $= 4! = 24$

16 a Number of ways 7 men can be selected from a group of 15 men

$= {}^{15}C_7$

$= \dfrac{15!}{7!(15-7)!}$

$= 6435$

b Number of 5-card hands that can be dealt from a standard pack of 52 cards

$= {}^{52}C_5$

$= \dfrac{52!}{5!(52-5)!}$

$= 2\,598\,960$

c If a 5-card hand is to contain all 4 aces, this leaves $52 - 4 = 48$ choices for the remaining 1 card. Therefore number of 5-card hands that contain all 4 aces, that can be dealt from a standard pack of 52 cards = 48.

d Number of ways 3 prime numbers be selected from the set containing the first 10 prime numbers

$$\begin{aligned} &= {}^{10}C_3 \\ &= \frac{10!}{3!(10-3)!} \\ &= 120 \end{aligned}$$

17 **a** There are 18 people in total from whom 8 people are to be chosen. This can be done in ${}^{18}C_8$ ways.

$$\begin{aligned} {}^{18}C_8 &= \frac{18!}{8! \times (18-8)!} \\ &= \frac{18!}{8! \times 10!} \\ &= 43\,758 \end{aligned}$$

There are 43 758 possible committees.

b The 5 men can be chosen from the 8 men available in ${}^8C_5$ ways.
The 3 women can be chosen from the 10 women available in ${}^{10}C_3$ ways.
The total number of committees which contain 5 men and 3 women is ${}^8C_5 \times {}^{10}C_3$.

$$\begin{aligned} {}^8C_5 \times {}^{10}C_3 &= \frac{8!}{5! \times 3!} \times \frac{10!}{3! \times 7!} \\ &= \frac{8 \times 7 \times 6}{3!} \times \frac{10 \times 9 \times 8}{3!} \\ &= 56 \times 120 \\ &= 6720 \end{aligned}$$

There are 6720 committees possible with the given restriction.

c As there are 8 men available, at least 6 men means either 6, 7 or 8 men.
The panels of 8 people which satisfy this restriction have either 6 men and 2 women, 7 men and 1 woman, or they have 8 men and no women.
6 men and 2 women are chosen in ${}^8C_6 \times {}^{10}C_2$ ways.
7 men and 1 woman are chosen in ${}^8C_7 \times {}^{10}C_1$ ways.
8 men and 0 women are chosen in ${}^8C_8 \times {}^{10}C_0$ ways.

The number of committees with at least 6 men is ${}^8C_6 \times {}^{10}C_2 + {}^8C_7 \times {}^{10}C_1 + {}^8C_8 \times {}^{10}C_0$.

$$\begin{aligned} {}^6C_4 \times {}^8C_1 + {}^6C_5 \times {}^8C_0^8C_6 \times {}^{10}C_2 + {}^8C_7 \times {}^{10}C_1 + {}^8C_8 \times {}^{10}C_0 &= 1260 + 80 + 1 \\ &= 1341 \end{aligned}$$

There are 1341 committees with at least 6 men.

d The total number of panels of 8 people is ${}^{18}C_8 = 43\,758$ from part a.
If two particular men cannot both be included, this is the complementary event of the two men both being included. In this case, the other 6 panel members need to be selected from the remaining 16 people to form the panel of 8. This can be done in ${}^{16}C_6$ ways.

$$\begin{aligned} {}^{16}C_6 &= \frac{16!}{6! \times (16-6)!} \\ &= \frac{16!}{6! \times 10!} \\ &= 8008 \end{aligned}$$

Therefore the number of panels that can be formed if two particular men cannot both be included = 43 758 – 8008 = 35 750.

e If a particular man and woman *must* be included on the panel, then the other 6 panel members need to be selected from the remaining 16 people to form the panel of 8. This can be done in ${}^{16}C_6$ ways.

$$\begin{aligned} {}^{16}C_6 &= \frac{16!}{6! \times (16-6)!} \\ &= \frac{16!}{6! \times 10!} \\ &= 8008 \end{aligned}$$

There are 8008 committees possible with the given restriction.

18 **a** There are 12 different numbers in *S*.
The total number of subsets of *S* is equal to the number of ways 0 numbers can be selected from *S* plus the number of ways 1 number can be selected from *S* plus the number of ways 2 numbers can be selected from *S*, and so on.
Hence,
Number of subsets of *S*

$= {}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 + {}^{12}C_4 + {}^{12}C_5 + {}^{12}C_6 + {}^{12}C_7 + {}^{12}C_8 + {}^{12}C_9 + {}^{12}C_{10} + {}^{12}C_{11}{}^{12}C_{12}$
$= 4096$

b To determine the number of subsets whose elements are all even numbers, only regard the even numbers in the subset, i.e. $S_{even} = \{2, 4, 8, 10, 14\}$. There are 5 different numbers in S_{even}.
Hence,
Number of subsets of S_{even}
$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$
$= 31$

Note that the empty set found by ${}^5C_0$ is not included since an empty set does not fit the condition.
Therefore there are 31 subsets whose elements are all even numbers.

c The total number of subsets of S is 4096 from part a.
Let C be the event that a subset selected at random will contain only even numbers.

$$\begin{aligned}\Pr(C) &= \frac{n(C)}{n(\xi)}\\ &= \frac{31}{4096}\end{aligned}$$

The probability that a subset selected at random will contain only even numbers is $\frac{31}{4096}$.

d The total number of subsets of S is 4096 from part a.
The number of subsets containing at least 3 elements will be the total number of subsets less those containing 0, 1 or 2 elements.
Hence,
Number of subsets containing at least 3 elements
$= 4096 - \left({}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2\right)$
$= 4017$

Let D be the event that a subset selected at random will contain at least 3 elements

$$\begin{aligned}\Pr(D) &= \frac{n(D)}{n(\xi)}\\ &= \frac{4017}{4096}\end{aligned}$$

The probability that a subset selected at random will contain only even numbers is $\frac{4017}{4096}$.

e The total number of subsets of S is 4096 from part a.
To determine the number of subsets whose elements are all prime numbers, only regard the prime numbers in the subset, i.e. $S_{prime} = \{2, 3, 5, 7, 11, 13, 17\}$. There are 7 different numbers in S_{prime}.
Hence,
Number of subsets of S_{prime} that contain exactly 3 elements
$= {}^7C_3$
$= 35$

Therefore there are 35 subsets of S that contain exactly 3 elements, all of which are prime numbers.
Let E be the event that a subset selected at random will contain exactly 3 elements, all of which are prime numbers.

$$\begin{aligned}\Pr(E) &= \frac{n(E)}{n(\xi)}\\ &= \frac{35}{4096}\end{aligned}$$

The probability that a subset selected at random will contain exactly 3 elements, all of which are prime numbers, is $\frac{35}{4096}$.

19 a There are 56 players in total from whom 7 members are to be selected. This can be done in ${}^{56}C_7$ ways.

$$\begin{aligned}{}^{56}C_7 &= \frac{56!}{7!\times(56-7)!}\\ &= \frac{56!}{7!\times 49!}\\ &= \frac{56\times 55\times 54\times 53\times 52\times 51\times 50}{7!}\\ &= 231\,917\,400\end{aligned}$$

There are 231 917 400 possible teams.
The 7 members can be selected from the 17 squash players available in ${}^{17}C_7 = 19\,448$ ways.
The total number of committees which contain 7 squash players is 19 448.
Let A be the event that the committee contains 7 squash players.

$$\begin{aligned}\Pr(A) &= \frac{n(A)}{n(\xi)}\\ &= \frac{19\,448}{231\,917\,400}\\ &= \frac{1}{11\,925}\end{aligned}$$

The probability the committee consists of 7 squash players is $\frac{1}{11\,925}$.

b As there are 21 tennis players available, at least 5 tennis players means either 5, 6 or 7 tennis players. The committees of 7 members which satisfy this restriction have the other members chosen from the remaining 35 squash or badminton players.

5 tennis players and 2 squash or badminton players are chosen in ${}^{21}C_5 \times {}^{35}C_2$ ways.

6 tennis players and 1 squash or badminton player are chosen in ${}^{21}C_6 \times {}^{35}C_1$ ways.

7 tennis players and no squash or badminton players are chosen in ${}^{21}C_7 \times {}^{35}C_0$ ways.

The number of committees with at least four boys is ${}^{21}C_5 \times {}^{35}C_2 + {}^{21}C_6 \times {}^{35}C_1 + {}^{21}C_7 \times {}^{35}C_0$.

$$\begin{aligned}{}^{21}C_5 \times {}^{35}C_2 + {}^{21}C_6 \times {}^{35}C_1 + {}^{21}C_7 \times {}^{35}C_0 &= 12\,107\,655 + 1\,899\,240 + 116\,280\\ &= 14\,123\,175\end{aligned}$$

There are 14 123 175 committees with at least 5 tennis players.

Let B be the event that the committee contains at least 5 tennis players.

$$\begin{aligned}\Pr(B) &= \frac{n(B)}{n(\xi)}\\ &= \frac{14\,123\,175}{231\,917\,400}\\ &= \frac{19}{312}\end{aligned}$$

The probability the committee contains 7 squash players is $\frac{19}{312}$.

c The event of the committee containing at least one representative from each sport is the complementary event of the committee containing no representative from any one or two of the three sports.

In this case, if the committee contains no tennis players, there are 35 remaining players from which 7 members must be chosen. Hence number of combinations = ${}^{35}C_7 = 6\,724\,520$.

If the committee contains no squash players, there are 39 remaining players from which 7 members must be chosen. Hence number of combinations = ${}^{39}C_7 = 15\,380\,937$.

If the committee contains no badminton players, there are 38 remaining players from which 7 members must be chosen. Hence number of combinations = ${}^{38}C_7 = 12\,620\,256$.

The cases where the committee contains no representative from two of the three sports must also be taken into account. This is the same as the committee containing only players from one sport.

Number of combinations of committee containing only players from one sport

$$\begin{aligned}&= {}^{21}C_7 + {}^{17}C_7 + {}^{18}C_7\\ &= 167\,552\end{aligned}$$

Therefore total number of committees that contain at least one representative from each sport

$$\begin{aligned}&= 231\,917\,400 - (6\,724\,520 + 15\,380\,937 + 12\,620\,256 + 167\,552)\\ &= 197\,024\,135\end{aligned}$$

Let C be the event that the committee contains at least one representative from each sport.

$$\begin{aligned}\Pr(C) &= \frac{n(C)}{n(\xi)}\\ &= \frac{197\,024\,135}{231\,917\,400}\\ &= \frac{210\,721}{248\,040}\end{aligned}$$

The probability the committee contains at least one representative from each sport is $\frac{210\,721}{248\,040} (\simeq 0.85)$.

d To find probability that the committee contains exactly 3 badminton players, given that it contains at least 1 badminton player, first find number of committees which contain at least 1 badminton player.

If the committee contains no badminton players, there are 38 remaining players from which 7 members must be chosen. Hence number of combinations = ${}^{38}C_7 = 12\,620\,256$.

Therefore, number of committees which contain at least 1 badminton player

$$\begin{aligned}&= 231\,917\,400 - 12\,620\,256\\ &= 219\,297\,144\end{aligned}$$

For a committee that contains exactly 3 badminton players, the 3 badminton players and 4 tennis or squash players are chosen in ${}^{18}C_3 \times {}^{38}C_4$ ways.

$$\begin{aligned}{}^{18}C_3 \times {}^{38}C_4 &= 816 + 73\,815\\ &= 74\,631\end{aligned}$$

There are 74 631 committees with exactly 3 badminton players.

Let D_1 be the event that the committee contains exactly 3 badminton players.

Let D_2 be the event that the committee contains at least 1 badminton player.

$$\begin{aligned}\Pr(D_1 \mid D_2) &= \frac{n(D_1 \cap D_2)}{n(D_2)}\\ &= \frac{74\,631}{219\,297\,144}\\ &= \frac{24\,877}{73\,099\,048}\end{aligned}$$

The probability the committee contains exactly 3 badminton players, given that it contains at least 1 badminton player is $\frac{24\,877}{73\,099\,048} \simeq 0.2747$.

20 **a** First find the total number possible 3-digit numbers. There are ten digits. There are repetitions are allowed, but the number cannot start with 0. Therefore there are 9 choices for the first digit, 10 choices for the second digit and 10 choices for the third digit.

9	10	10

Using the multiplication principle, there are $9 \times 10 \times 10 = 900$ possible three-digit numbers that could be formed.

4 3-digit numbers can be selected from 900 3-digit numbers in ${}^{900}C_4 = 27\,155\,621\,025$ ways.

There are 143 3-digit prime numbers.

4 3-digit prime numbers can be selected from 143 3-digit prime numbers in ${}^{143}C_4 = 16\,701\,685$ ways.

Let A be the event that the 4 numbers selected are all primes.

$$\begin{aligned}\Pr(A) &= \frac{n(A)}{n(\xi)}\\ &= \frac{16\,701\,685}{27\,155\,621\,025}\\ &= \frac{256\,949}{417\,778\,785}\end{aligned}$$

The probability the that the 4 numbers selected are all primes is $\frac{256\,949}{417\,778\,785} \simeq 0.0006$.

b There are ten digits. There is one repetition required and the number cannot start with 0. Therefore, for example, if the first and second digits are the same, there are 9 choices for the first digit, 1 choice for the second digit and 9 choices for the third digit (not 10 choices, because this digit cannot be the same as the repeated digit).

9	1	9

If the first and third digits are the same:

9	9	1

If the second and third digits are the same:

9	9	1

Using the multiplication and addition principles, there are $9 \times 1 \times 9 + 9 \times 9 \times 1 + 9 \times 9 \times 1 = 243$ possible three-digit numbers with a single repeated digit that could be formed.

4 3-digit numbers with a single repeated digit can be selected from 243 3-digit numbers with a single repeated digit in ${}^{243}C_4 = 141\,722\,460$ ways.

Let B be the event that the 4 numbers selected all have a single repeated digit.

$$\begin{aligned}\Pr(B) &= \frac{n(B)}{n(\xi)}\\ &= \frac{141\,722\,460}{27\,155\,621\,025}\\ &= \frac{3\,149\,388}{603\,458\,245}\end{aligned}$$

The probability the that the 4 numbers selected all have a single repeated digit is $\frac{3\,149\,388}{603\,458\,245} \simeq 0.0052$.

c There are 22 3-digit perfect squares.

4 3-digit perfect squares can be selected from 22 3-digit perfect squares in ${}^{22}C_4 = 7315$ ways.

Let C be the event that the 4 numbers selected are all perfect squares.

$$\begin{aligned}\Pr(C) &= \frac{n(C)}{n(\xi)}\\ &= \frac{7315}{27\,155\,621\,025}\\ &= \frac{1463}{5\,431\,124\,205}\end{aligned}$$

The probability the that the 4 numbers selected are all perfect squares is $\frac{1463}{5\,431\,124\,205} \simeq 2.63 \times 10^8$.

d There are ten digits. There is no repetition allowed and the number cannot start with 0.

Therefore there are 9 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit.

9	9	8

Using the multiplication principle, there are $9 \times 9 \times 8 = 648$ 3-digit numbers with no repeated digits.

4 3-digit numbers, with no repeated digits, can be selected from 648 3-digit numbers with no repeated digits in ${}^{648}C_4 = 7\,278\,808\,230$ ways.

Let E be the event that there are no repeated digits in any of the numbers.

$$\begin{aligned}\Pr(E) &= \frac{n(E)}{n(\xi)}\\ &= \frac{7\,278\,808\,230}{27\,155\,621\,025}\\ &= \frac{161\,751\,294}{603\,458\,245}\end{aligned}$$

The probability the that the 4 numbers selected are all perfect squares is $\frac{161\,751\,294}{603\,458\,245} \simeq 0.268$.

e To find the probability that the 4 numbers lie between 300 and 400, given that the 4 numbers are greater than 200, first find how many 3-digit numbers are greater than 200.

There are 799 3-digit numbers greater than 200.

4 3-digit numbers can be selected from 799 3-digit numbers in ${}^{799}C_4 = 16\,854\,265\,001$ ways.

There are 101 3-digit numbers that lie between 300 and 400.

4 3-digit numbers can be selected from 101 3-digit numbers in ${}^{101}C_4 = 4\,082\,925$ ways.
Let F_1 be the event that the 4 numbers lie between 300 and 400.
Let F_2 be the event that the 4 numbers are greater than 200.

$$\begin{aligned}\Pr(F_1 \mid F_2) &= \frac{n(F_1 \cap F_2)}{n(F_2)}\\ &= \frac{4\,082\,925}{16\,854\,265\,001}\\ &= \frac{583\,275}{2\,407\,752\,143}\end{aligned}$$

The probability the 4 numbers lie between 300 and 400, given that the 4 numbers are greater than 200 is $\frac{583\,275}{2\,407\,752\,143} \simeq 0.000\,242\,2$.

Exercise 8.6 — Binomial coefficients and Pascal's triangle

1 a For $(a+b)^4$, $n=4,\ a=a,\ b=b$
Using the rule for binomial expansion,

$$\begin{aligned}(a+b)^4 &= \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4\\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

b For $(2+x)^4$, $n=4,\ a=2,\ b=x$
Using the rule for binomial expansion,

$$\begin{aligned}(2+x)^4 &= \binom{4}{0}2^4 + \binom{4}{1}2^3x + \binom{4}{2}2^2x^2 + \binom{4}{3}2x^3 + \binom{4}{4}x^4\\ &= 16 + 32x + 24x^2 + 8x^3 + x^4\end{aligned}$$

c For $(t-2)^3$, $n=3,\ a=t,\ b=-2$
Using the rule for binomial expansion,

$$\begin{aligned}(t-2)^3 &= \binom{3}{0}t^3 + \binom{3}{1}t^2(-2) + \binom{3}{2}t(-2)^2 + \binom{3}{3}(-2)^3\\ &= t^3 - 6t^2 + 12t - 8\end{aligned}$$

2 a For $(m+3b)^2$, $n=2,\ a=m,\ b=3b$
Using the rule for binomial expansion,

$$\begin{aligned}(m+3b)^2 &= \binom{2}{0}m^2 + \binom{2}{1}(m)(3b) + \binom{2}{2}(3b)^2\\ &= m^2 + 6bm + 9b^2\end{aligned}$$

b For $(2d-x)^4$, $n=4,\ a=2d,\ b=-x$
Using the rule for binomial expansion,

$$\begin{aligned}(2d-x)^4 &= \binom{4}{0}(2d)^4 + \binom{4}{1}(2d)^3(-x) + \binom{4}{2}(2d)^2(-x)^2 + \binom{4}{3}(2d)(-x)^3 + \binom{4}{4}(-x)^4\\ &= 16d^4 - 32d^3x + 24d^2x^2 - 8dx^3 + x^4\end{aligned}$$

c For $\left(h+\frac{2}{h}\right)^3$, $n=3,\ a=h,\ b=\frac{2}{h}$
Using the rule for binomial expansion,

$$\begin{aligned}\left(h+\frac{2}{h}\right)^3 &= \binom{3}{0}h^3 + \binom{3}{1}h^2\left(\frac{2}{h}\right) + \binom{3}{2}h\left(\frac{2}{h}\right)^2 + \binom{3}{3}\left(\frac{2}{h}\right)^3\\ &= h^3 + 3h^2 \times \frac{2}{h} + 3h \times \frac{4}{h^2} + \frac{8}{h^3}\\ &= h^3 + 6h + \frac{12}{h} + \frac{8}{h^3}\end{aligned}$$

3 a
$$\begin{aligned}\left(x+\frac{1}{2x}\right)^4 &= \sum_{r=0}^{4}\binom{4}{r}x^{4-r}\left(\frac{1}{2x}\right)^r\\ &= \sum_{r=0}^{4}\binom{4}{r}2^{-r}x^{4-r}x^{-r}\\ &= \sum_{r=0}^{4}\binom{4}{r}2^{-r}x^{4-2r}\end{aligned}$$

For independent term of x,

$$4-2r=0$$
$$\to r=2$$
$$\begin{aligned}\binom{4}{r}2^{-r}x^{4-2r}&=\binom{4}{2}2^{-2}\\&=6\times\frac{1}{4}\\&=\frac{3}{2}\end{aligned}$$

The 5th term, $\frac{3}{2}$, is independent of x.

b $$\begin{aligned}\left(2m-\frac{1}{3m}\right)^6&=\sum_{r=0}^{6}\binom{6}{r}(2m)^{6-r}\left(\frac{1}{3m}\right)^r\\&=\sum_{r=0}^{6}\binom{6}{r}2^{6-r}m^{6-r}3^{-r}m^{-r}\\&=\sum_{r=0}^{4}\binom{4}{r}2^{6-r}3^{-r}m^{6-2r}\end{aligned}$$

For the term in m^2, we need to make the power of m equal to 2.

$$6-2r=2$$
$$\to r=2$$
$$\begin{aligned}\binom{6}{r}2^{6-r}3^{-r}m^{6-2r}&=\binom{6}{2}2^{4}3^{-2}m^{2}\\&=15\times16\times\frac{1}{9}\times m^2\\&=\frac{80}{3}m^2\end{aligned}$$

4 a $$\begin{aligned}(p+3q)^7&=\sum_{r=0}^{7}\binom{7}{r}p^{7-r}(3q)^r\\&=\sum_{r=0}^{7}\binom{7}{r}3^r p^{7-r}q^r\end{aligned}$$

For the 5th term, $r=4$.

$$\begin{aligned}\binom{7}{r}3^r p^{7-r}q^r&=\binom{7}{4}3^4p^{7-4}q^4\\&=35\times81p^3q^4\\&=2835p^3q^4\end{aligned}$$

Therefore the 5th term is $2835p^3q^4$.

b $$\begin{aligned}(x^2+ky)^5&=\sum_{r=0}^{5}\binom{5}{r}x^{5-r}(ky)^r\\&=\sum_{r=0}^{5}\binom{5}{r}k^rx^{5-r}y^r\end{aligned}$$

For the 4th term, $r=3$.

$$\frac{5}{4}x^4y^3=\binom{5}{3}k^3x^{5-3}y^3$$
$$\frac{5}{4}x^4y^3=10k^3x^4y^3$$
$$\frac{5}{4}=10k^3$$
$$\frac{5}{4\times10}=k^3$$
$$k=\sqrt[3]{\frac{1}{8}}=\frac{1}{2}$$

5 **a** For 4th coefficient in the 7th row of Pascal's triangle, $n = 7$ and $r = 3$
4th coefficient in the 7th row is ${}^{7}\mathrm{C}_{3}$.

b
$$\begin{aligned}\binom{9}{6} &= \frac{9!}{(9-6)!6!}\\ &= \frac{9\times8\times7}{3!}\\ &= \frac{504}{6}\\ &= 84\end{aligned}$$
$${}^{9}C_{6} = 84$$

c
$$\begin{aligned}\binom{18}{12} &= \frac{18!}{(18-12)!12!}\\ &= \frac{18!}{6!12!}\\ &= \frac{18!}{12!6!}\\ &= \frac{18!}{(18-6)!6!}\\ &= \binom{18}{6}\end{aligned}$$

d
$$\binom{15}{7}+\binom{15}{6}=\binom{16}{7}$$

Evaluate LHS:

$$\binom{15}{7}+\binom{15}{6}=\binom{16}{7}$$

$$\begin{aligned}\binom{15}{7}+\binom{15}{6} &= \frac{15!}{(15-7)!7!}+\frac{15!}{(15-6)!6!}\\ &= \frac{15!}{8!7!}+\frac{15!}{9!6!}\\ &= \frac{15\times14\times13\times12\times11\times10\times9}{7!}+\frac{15\times14\times13\times12\times11\times10}{6!}\\ &= 6435+5005\\ &= 11\,440\end{aligned}$$

Evaluate RHS:

$$\begin{aligned}\binom{16}{7} &= \frac{16!}{(16-7)!7!}\\ &= \frac{16!}{9!7!}\\ &= \frac{16\times15\times14\times13\times12\times11\times10}{7!}\\ &= 11\,440\end{aligned}$$

LHS = RHS

6 **a**
$$\begin{aligned}\binom{22}{8} &= \frac{22!}{(22-8)!8!}\\ &= \frac{22!}{14!8!}\\ &= \frac{22!}{8!14!}\\ &= \frac{22!}{(22-14)!14!}\\ &= \binom{22}{8}\end{aligned}$$

b
$$\binom{19}{15}+\binom{19}{14}={}^{19}\mathrm{C}_{15}+{}^{19}\mathrm{C}_{14}$$

c Given ${}^{n+1}\mathrm{C}_{r-1}+{}^{n+1}\mathrm{C}_{r}={}^{n+2}\mathrm{C}$,
Use the formula to express LHS

$$\begin{aligned}{}^{n+1}C_{r-1} + {}^{n+1}C_r &= \frac{(n+1)!}{(n+1-(r-1))!(r-1)!} + \frac{(n+1)!}{(n+1-r)!r!} \\ &= (n+1)!\left[\frac{1}{(n-r+2)!(r-1)!} + \frac{1}{(n+1-r)!r!}\right]\end{aligned}$$

7 **a** For $(x+y)^3$, $n=3$, $a=x$, $b=y$.

$(x+y)^3$

$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} x^3 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} x^2 y + \begin{pmatrix} 3 \\ 2 \end{pmatrix} xy^2 + \begin{pmatrix} 3 \\ 3 \end{pmatrix} y^3$

$= x^3 + 3x^2 y + 3xy^2 + y^3$

b For $(a+2)^4$, $n=4$, $a=a$, $b=2$.

$(a+2)^4$

$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} a^4 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} a^3(2) + \begin{pmatrix} 4 \\ 2 \end{pmatrix} a^2(2)^2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} a(2)^3 + \begin{pmatrix} 4 \\ 4 \end{pmatrix} (2)^4$

$= a^4 + 8a^3 + 24a^2 + 32a + 16$

c For $(m-3)^4$, $n=4$, $a=m$, $b=-3$.

$(m-3)^4$

$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} m^4 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} m^3(-3) + \begin{pmatrix} 4 \\ 2 \end{pmatrix} m^2(-3)^2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} m(-3)^3 + \begin{pmatrix} 4 \\ 4 \end{pmatrix} (-3)^4$

$= m^4 - 12m^3 + 54m^2 - 108m + 81$

d For $(2-x)^5$, $n=5$, $a=2$, $b=-x$.

$(2-x)^5$

$= \begin{pmatrix} 5 \\ 0 \end{pmatrix} (2)^5 + \begin{pmatrix} 5 \\ 1 \end{pmatrix} (2)^4(-x) + \begin{pmatrix} 5 \\ 2 \end{pmatrix} (2)^3(-x)^2 + \begin{pmatrix} 5 \\ 3 \end{pmatrix} (2)^2(-x)^3 + \begin{pmatrix} 5 \\ 4 \end{pmatrix} (2)(-x)^4 + \begin{pmatrix} 5 \\ 5 \end{pmatrix} (-x)^5$

$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

8 **a** For $\left(1-\frac{2}{x}\right)^3$, $n=3$, $a=1$, $b=-\frac{2}{x}$.

$\left(1-\frac{2}{x}\right)^3$

$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} (1)^3 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} (1)^2\left(-\frac{2}{x}\right) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} (1)\left(-\frac{2}{x}\right)^2 + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \left(-\frac{2}{x}\right)^3$

$= 1 - \frac{6}{x} + \frac{12}{x^2} - \frac{8}{x^3}$

b For $\left(1+\frac{p}{q}\right)^4$, $n=4$, $a=1$, $b=\frac{p}{q}$.

$\left(1+\frac{p}{q}\right)^4$

$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} (1)^4 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} (1)^3\left(\frac{p}{q}\right) + \begin{pmatrix} 4 \\ 2 \end{pmatrix} (1)^2\left(\frac{p}{q}\right)^2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} (1)\left(\frac{p}{q}\right)^3 + \begin{pmatrix} 4 \\ 4 \end{pmatrix} \left(\frac{p}{q}\right)^4$

$= 1 + \frac{4p}{q} + \frac{6p^2}{q^2} + \frac{4p^3}{q^3} + \frac{p^4}{q^4}$

c For $\left(3-\frac{m}{2}\right)^4$, $n=4$, $a=3$, $b=-\frac{m}{2}$.

$\left(3-\frac{m}{2}\right)^4$

$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} (3)^4 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} (3)^3\left(-\frac{m}{2}\right) + \begin{pmatrix} 4 \\ 2 \end{pmatrix} (3)^2\left(-\frac{m}{2}\right)^2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} (3)\left(-\frac{m}{2}\right)^3 + \begin{pmatrix} 4 \\ 4 \end{pmatrix} \left(-\frac{m}{2}\right)^4$

$= 81 - 54m + \frac{27}{2}m^2 - \frac{3}{2}m^3 + \frac{1}{16}m^4$

d For $\left(2x-\frac{1}{x}\right)^3$, $n=5$, $a=2x$, $b=-\frac{1}{x}$.

$\left(2x-\frac{1}{x}\right)^3$

$$=\binom{3}{0}(2x)^3\left(-\frac{1}{x}\right)^0+\binom{3}{1}(2x)^2\left(-\frac{1}{x}\right)^1+\binom{3}{2}(2x)\left(-\frac{1}{x}\right)^2+\binom{3}{3}\left(-\frac{1}{x}\right)^3$$
$$=8x^3-12x+\frac{6}{x}-\frac{1}{x^3}$$

9 **a** $(2w-3)^5=\sum_{r=0}^{5}\binom{5}{r}(2w)^{5-r}(-3)^r$

$$=\sum_{r=0}^{5}\binom{5}{r}2^{5-r}(-3)^r w^{5-r}$$

For the 3$^{\text{rd}}$ term, $r=2$.

$$\binom{5}{r}2^{5-r}(-3)^r w^{5-r}=\binom{5}{2}2^{5-2}(-3)^2 w^{5-2}$$
$$=80\times 9w^3$$
$$=720w^3$$

Therefore the 3$^{\text{rd}}$ term is $720w^3$.

b $\left(3-\frac{1}{b}\right)^7=\sum_{r=0}^{7}\binom{7}{r}3^{7-r}\left(-\frac{1}{b}\right)^r$

$$=\sum_{r=0}^{7}(-1)^r\binom{7}{r}3^{7-r}\frac{1}{b^r}$$

For the 5th term, $r=4$.

$$(-1)^r\binom{7}{r}3^{7-r}\frac{1}{b^r}=(-1)^4\binom{7}{4}3^{7-4}\frac{1}{b^4}$$
$$=35\times 27\times\frac{1}{b^4}$$
$$=\frac{945}{b^4}$$

Therefore the 5$^{\text{th}}$ term is $\frac{945}{b^4}$.

c $\left(y-\frac{3}{y}\right)^4=\sum_{r=0}^{4}\binom{4}{r}y^{4-r}\left(-\frac{3}{y}\right)^r$

$$=\sum_{r=0}^{4}(-1)^r\binom{4}{r}3^r y^{4-2r}$$

For the constant term, $4-2r=0$.

$\rightarrow r=2$

$$(-1)^r\binom{4}{r}3^r y^{4-2r}=(-1)^2\binom{4}{2}3^2 y^{4-4}$$
$$=1\times 6\times 9$$
$$=54$$

Therefore the constant term is 54.

10 **a** $(2b+3d)^5=\sum_{r=0}^{5}\binom{5}{r}(2b)^{5-r}(3d)^r$

$$=\sum_{r=0}^{5}\binom{5}{r}2^{5-r}3^r b^{5-r}d^r$$

For the 6$^{\text{th}}$ term, $r=5$.

$$\binom{5}{r}2^{5-r}3^r b^{5-r}d^r=\binom{5}{5}2^{5-5}3^5 b^{5-5}d^5$$
$$=1\times 1\times 243\times 1\times d^5$$
$$=243d^5$$

Therefore the 6$^{\text{th}}$ term is $243d^5$.

b $(3x-5y)^5=\sum_{r=0}^{5}\binom{5}{r}(3x)^{5-r}(-5y)^r$

$$=\sum_{r=0}^{5}\binom{5}{r}3^{5-r}(-5)^r x^{5-r}y^r$$

For the coefficient of the term x^2y^3, $5-r=2$.

$$\begin{aligned}&\rightarrow r=3\\&\binom{5}{r}3^{5-r}(-5)^r x^{5-r}y^r=\binom{5}{2}3^{5-3}(-5)^3x^{5-3}y^3\\&\qquad=10\times9\times-125\times x^2\times y^3\\&\qquad=-11\,250x^2y^3\end{aligned}$$

Therefore the coefficient of the term x^2y^3 is $-11\,250$.

11 Given $\left(2-\sqrt{5}\right)^4=a+b\sqrt{5}$,

For $\left(2-\sqrt{5}\right)^4$, $n=4$, $a=2$, $b=-\sqrt{5}$.

$$\begin{aligned}\left(2-\sqrt{5}\right)^4&=\binom{4}{0}2^4+\binom{4}{1}2^3\left(-\sqrt{5}\right)+\binom{4}{2}2^2\left(-\sqrt{5}\right)^2+\binom{4}{3}2\left(-\sqrt{5}\right)^3+\binom{4}{4}\left(-\sqrt{5}\right)^4\\&=16-32\sqrt{5}+120-40\sqrt{5}+25\\&=161-72\sqrt{5}\\&\therefore a=161\text{ and }b=-72\end{aligned}$$

12 $1-6m+15m^2-20m^3+15m^4-6m^5+m^6=(a-m)^6$

$$\begin{aligned}(a-m)^6&=\binom{6}{0}a^6+\binom{6}{1}a^5(-m)+\binom{6}{2}a^4(-m)^2+\binom{6}{3}a^3(-m)^3+\binom{6}{4}a^2(-m)^4+\binom{6}{5}a(-m)^5+\binom{6}{6}(-m)^6\\&=a^6-6a^5m+15a^4m^2-20a^3m^3+15a^2m^4+6am^5+m^6\end{aligned}$$

$\therefore a^6-6a^5m+15a^4m^2-20a^3m^3+15a^2m^4+6am^5+m^6=1-6m+15m^2-20m^3+15m^4-6m^5+m^6$

Equating equivalent terms (first term):

$$\begin{aligned}a^6&=1\\a&=\pm1\end{aligned}$$

Check if positive or negative by equating equivalent terms (second term):

$$\begin{aligned}-6a^5m&=-6m\\a^5&=1\\a&=1\end{aligned}$$

$\therefore 1-6m+15m^2-20m^3+15m^4-6m^5+m^6=(1-m)^6$

13 $(1-x)^4-4(1-x)^3+6(1-x)^2-4(1-x)+1$

Expand the first term:

$$\begin{aligned}(1-x)^4&=\binom{4}{0}1^4+\binom{4}{1}1^3(-x)+\binom{4}{2}1^2(-x)^2+\binom{4}{3}1(-x)^3+\binom{4}{4}(-x)^4\\&=1-4x+6x^2-4x^3+x^4\end{aligned}$$

Expand the second term:

$$\begin{aligned}-4(1-x)^3&=-4\left(\binom{3}{0}1^3+\binom{3}{1}1^2(-x)+\binom{3}{2}1(-x)^2+\binom{3}{3}(-x)^3\right)\\&=-4\left(1-3x+3x^2-x^3\right)\\&=-4+12x-12x^2+4x^3\end{aligned}$$

Expand the third term:

$$\begin{aligned}6(1-x)^2&=6\left(\binom{2}{0}1^2+\binom{2}{1}1(-x)+\binom{2}{2}(-x)^2\right)\\&=6\left(1-2x+x^2\right)\\&=6-12x+6x^2\end{aligned}$$

Expand the fourth term:

$-4(1-x)=-4+4x$

The fifth term does not need to be expanded.

Now add up all the expanded terms:

$$\begin{aligned}&(1-x)^4-4(1-x)^3+6(1-x)^2-4(1-x)+1\\&=1-4x+6x^2-4x^3+x^4-4+12x-12x^2+4x^3+6-12x+6x^2-4+4x+1\\&=x^4+\cancel{\left(-4x^3+4x^3\right)}+\cancel{\left(6x^2-12x^2+6x^2\right)}+\cancel{(-4x+12x-12x+4x)}+\cancel{(1-4+6-4+1)}\\&=x^4\end{aligned}$$

14 $\left(2-\dfrac{1}{y}\right)^3(1+2y)^5$

Expanding with a CAS calculator gives

$$\left(2-\frac{1}{y}\right)^3(1+2y)^5=256y^5+256y^4-128y^3-192y^2+\frac{8}{y}-\frac{4}{y^2}-\frac{1}{y^3}+48$$

The term independent of y is 48.

15 $$\left(3m^2+\frac{k}{m}\right)^6=\sum_{r=0}^{6}\binom{6}{r}\left(3m^2\right)^{6-r}\left(\frac{k}{m}\right)^r$$
$$=\sum_{r=0}^{6}\binom{6}{r}3^{6-r}k^rm^{2(6-r)}m^{-r}$$
$$=\sum_{r=0}^{6}\binom{6}{r}3^{6-r}k^rm^{12-3r}$$

For the term independent of m, $12-3r=0$.

$\rightarrow r=4$

$$\binom{6}{r}3^{6-r}k^rm^{12-3r}=\binom{6}{4}3^{6-4}k^4m^{12-3\times4}$$
$$=15\times9\times k^4\times1$$
$$=135k^4$$

Given the independent term is 2160,

$$135k^4=2160$$
$$k^4=16$$
$$k=\pm2$$

16 $$(ax+2y)^5=\sum_{r=0}^{5}\binom{5}{r}(ax)^{5-r}(2y)^r$$
$$=\sum_{r=0}^{5}\binom{5}{r}a^{5-r}2^rx^{5-r}y^r$$

For the 4th term, $r=3$.

$$\binom{5}{r}a^{5-r}2^rx^{5-r}y^r=\binom{5}{3}a^{5-3}2^3x^{5-3}y^3$$
$$=10\times a^2\times8\times x^2\times y^3$$
$$=80a^2x^2y^3$$

Therefore the coefficient of the 5th term is $80a^2$.

For the 5th term, $r=4$.

$$\binom{5}{r}a^{5-r}2^rx^{5-r}y^r=\binom{5}{4}a^{5-4}2^4x^{5-4}y^4$$
$$=5\times a\times16\times x\times y^4$$
$$=80axy^4$$

Therefore the coefficient of the 5th term is $80a$.

The ratio of coefficients is 3:1.

$$\frac{80a^2}{3}=\frac{80a}{1}$$
$$80a^2=240a$$
$$80a^2-240a=0$$
$$80a(a-3)=0$$
$$a=0,3$$

$\therefore a=3$ (Assuming that a is non-zero)

17 $$(1+kx)^n=\sum_{r=0}^{n}\binom{n}{r}1^{n-r}(kx)^r$$
$$=\sum_{r=0}^{n}\binom{n}{r}1^{n-r}k^rx^r$$

For the 1st term, $r=0$.

$$\binom{n}{r}1^{n-r}k^rx^r=\binom{n}{0}1^{n-0}k^0x^0$$
$$=\binom{n}{0}1^n$$
$$=1^n$$

The first term in the expansion is 1.

$1^n = 1$

This is unhelpful because 1 raised to any power is still 1.

For the 2nd term, $r = 1$.

$$\binom{n}{r}1^{n-r}k^r x^r = \binom{n}{1}1^{n-1}k^1 x^1$$
$$= \binom{n}{1}1^{n-1}kx$$

The second term in the expansion is $2x$.

$$\binom{n}{1}1^{n-1}kx = 2x$$
$$n \times 1 \times kx = 2x$$
$$nk = 2$$
$$k = \frac{2}{n} \qquad \rightarrow [1]$$

For the 3rd term, $r = 2$.

$$\binom{n}{r}1^{n-r}k^r x^r = \binom{n}{2}1^{n-2}k^2 x^2$$
$$= \frac{n!}{2!(n-2)!} \times 1 \times k^2 \times x^2$$
$$= \frac{n \times (n-1) \times \cancel{(n-2)!}}{2!\,\cancel{(n-2)!}}k^2 x^2$$
$$= \frac{n \times (n-1)}{2}k^2 x^2$$

The third term in the expansion is $\frac{3}{2}x^2$.

$$\frac{n \times (n-1)}{2}k^2 x^2 = \frac{3}{2}x^2$$
$$n \times (n-1) \times k^2 = 3$$
$$k^2\left(n^2 - n\right) = 3 \qquad \rightarrow [2]$$

Sub EQ [1] into EQ [2]:

$$\left(\frac{2}{n}\right)^2\left(n^2 - n\right) = 3$$
$$\frac{4}{n^2}\left(n^2 - n\right) = 3$$
$$4 - \frac{4}{n} = 3$$
$$1 = \frac{4}{n}$$
$$n = 4$$

Sub $n = 4$ into EQ [1]:

$$k = \frac{2}{4}$$
$$k = \frac{1}{2}$$

$\therefore n = 4$ and $k = \frac{1}{2}$

18 a 125 970, 77 520, 38 700

19 b Given $\left(3 + \sqrt{2}\right)^9 = a + b\sqrt{2}$,

For $\left(3 + \sqrt{2}\right)^9$, $n = 9$, $a = 3$, $b = \sqrt{2}$.

$$\left(3+\sqrt{2}\right)^9 = \binom{9}{0}3^9 + \binom{9}{1}3^8\left(\sqrt{2}\right) + \binom{9}{2}3^7\left(\sqrt{2}\right)^2 + \binom{9}{3}3^6\left(\sqrt{2}\right)^3 + \binom{9}{4}3^5\left(\sqrt{2}\right)^4$$
$$+ \binom{9}{5}3^4\left(\sqrt{2}\right)^5 + \binom{9}{6}3^3\left(\sqrt{2}\right)^6 + \binom{9}{7}3^2\left(\sqrt{2}\right)^7 + \binom{9}{8}3\left(\sqrt{2}\right)^8 + \binom{9}{9}\left(\sqrt{2}\right)^9$$
$$= 19\,683 + 59\,049\sqrt{2} + 157\,464 + 122\,472\sqrt{2} + 122\,472 + 40\,824\sqrt{2} + 18\,144 + 2592\sqrt{2} + 432 + 16\sqrt{2}$$
$$= 318\,195 + 224\,953\sqrt{2}$$

$\therefore a = 318\,195$ and $b = 224\,953$

c $\left(1+\frac{x}{2}\right)^n=\sum_{r=0}^{n}\binom{n}{r}1^{n-r}\left(\frac{x}{2}\right)^r$

$$=\sum_{r=0}^{n}\binom{n}{r}\frac{1}{2^r}x^r$$

For the term in x^3, $r=3$.

$$\begin{aligned}\binom{n}{r}\frac{1}{2^r}x^r&=\binom{n}{3}\frac{1}{2^3}x^3\\&=\frac{n!}{3!(n-3)!}\times\frac{1}{8}x^3\\&=\frac{n\times(n-1)\times(n-2)\times\cancel{(n-3)!}}{3!\,\cancel{(n-3)!}}\times\frac{1}{8}x^3\\&=\frac{n\times(n-1)\times(n-2)}{48}x^3\end{aligned}$$

Given that the coefficient of x^3 is 70,

$$\begin{aligned}\frac{n(n-1)(n-2)}{48}&=70\\n(n-1)(n-2)&=3360\\(n^2-n)(n-2)&=3360\\n^3-2n^2-n^2+2n&=3360\\n^3-2n^2-n^2+2n-3360&=0\\n^3-3n^2+2n-3360&=0\\(n-16)(n^2+13n+210)&=0\end{aligned}$$

$\therefore n=16$

Topic 9 — Trigonometric functions 1

Exercise 9.2 — Trigonometric ratios

1 **a** $\sin(50°) = \frac{h}{10}$

$\therefore h = 10\sin(50°)$

$\therefore h \simeq 7.66$

b Recognising the "3,4,5" Pythagorean triad gives

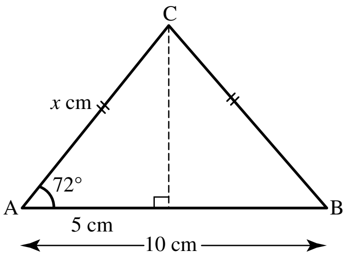
$\tan(a°) = \frac{5}{2}$

$\therefore a° = \tan^{-1}(2.5)$

$\therefore a° \simeq 68.20$

Hence, $a \simeq 68.20$.

2 Divide the isosceles triangle in half to create a right angled triangle.



$\cos(72°) = \frac{5}{x}$

$\therefore x = \frac{5}{\cos(72°)}$

$\therefore x \simeq 16.18$

The equal sides are approximately 16.18 cm in length.
The equal angles $\angle CBA = \angle CAB = 72°$ and $\angle ACB = 180° - 2\times 72° = 36°$

3 Let x metres be the required distance.

$\cos(45°) = \frac{x}{4}$

$\therefore x = 4\cos(45°)$

$= 4\times\frac{\sqrt{2}}{2}$

$= 2\sqrt{2}$

The foot of the ladder is $2\sqrt{2}$ metres from the fence.

4 $\frac{\cos(30°)\sin(45°)}{\tan(45°)+\tan(60°)}$

$= \frac{\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}}{1+\sqrt{3}}$

$= \frac{\sqrt{6}}{4\left(\sqrt{3}+1\right)}$

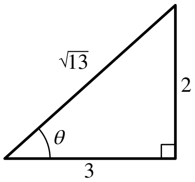
Rationalising the denominator,

$= \frac{\sqrt{6}}{4\left(\sqrt{3}+1\right)}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$

$= \frac{\sqrt{6}\left(\sqrt{3}-1\right)}{4(3-1)}$

$= \frac{3\sqrt{2}-\sqrt{6}}{8}$

5 **a** If $\tan(a°) = \frac{2}{3}$, using Pythagoras' theorem a triangle with sides 2, 3 and hypotenuse $\sqrt{13}$ gives $\cos(a°) = \frac{3}{\sqrt{13}}$.



b Let the horizontal run be x cm.

$\cos(a°) = \frac{x}{26}$

$\therefore x = 26\cos(a°)$

$= 26\times\frac{3}{\sqrt{13}}$

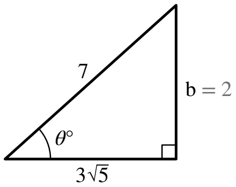
$= \frac{26\times 3\sqrt{13}}{13}$

$= 6\sqrt{13}$

Therefore the horizontal run is $6\sqrt{13}$ cm.

6 $\cos(\theta°) = \frac{3\sqrt{5}}{7}$

Construct a right angled triangle with hypotenuse 7 and adjacent side to θ of $3\sqrt{5}$.



Let opposite side be b.
Using Pythagoras' theorem.

$b^2 + \left(3\sqrt{5}\right)^2 = 7^2$

$\therefore b^2 + 45 = 49$

$\therefore b^2 = 4$

$\therefore b = 2$

Hence, $\sin(\theta°) = \frac{2}{7}$ and $\tan(\theta°) = \frac{2}{3\sqrt{5}}$ or $\frac{2\sqrt{5}}{15}$.

7 $a = 10, b = 6\sqrt{2}, c = 2\sqrt{13}$ cm and $C = 45°$.

Area is $A = \frac{1}{2}ab\sin(C)$

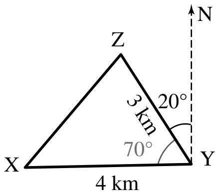
$\therefore A = \frac{1}{2}\times 10\times 6\sqrt{2}\times\sin(45°)$

$= 30\sqrt{2}\times\frac{\sqrt{2}}{2}$

$\therefore A = 30$

The area is 30 sq cm.

8



In triangle XYZ, the angle of 70° is included between the sides XY and YZ of lengths 4 and 3 km respectively.

The area of triangle XYZ is $A = \frac{1}{2}\times 3\times 4\times\sin(70°)$.

$\therefore A = 6\times\sin(70°)$

$\therefore A \simeq 5.64$

Correct to two decimal places, the horses can graze over an area of 5.64 square km.

9 a $$\frac{\sin(30^\circ)\cos(45^\circ)}{\tan(60^\circ)}$$

$$=\frac{1}{2}\times\frac{\sqrt{2}}{2}\div\frac{\sqrt{3}}{1}$$

$$=\frac{\sqrt{2}}{4}\times\frac{1}{\sqrt{3}}$$

$$=\frac{\sqrt{2}}{4\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\frac{\sqrt{6}}{12}$$

b $$\frac{\tan(45^\circ)+\cos(60^\circ)}{\sin(60^\circ)-\sin(45^\circ)}$$

$$=\frac{\left(1+\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\right)}$$

$$=\frac{3}{2}\div\frac{\sqrt{3}-\sqrt{2}}{2}$$

$$=\frac{3}{2}\times\frac{2}{\sqrt{3}-\sqrt{2}}$$

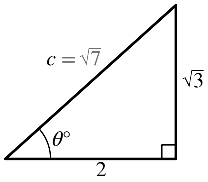
$$=\frac{3}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

$$=\frac{3\left(\sqrt{3}+\sqrt{2}\right)}{3-2}$$

$$=3\sqrt{3}+3\sqrt{2}$$

10 a i $\tan(\theta^\circ)=\dfrac{\sqrt{3}}{2}$

Draw a right angled triangle containing the angle θ° and label the sides opposite and adjacent to the angle in the ratio $\sqrt{3}$ to 2.



Using Pythagoras' theorem,

$$c^2=2^2+\left(\sqrt{3}\right)^2$$

$$\therefore c^2=4+3$$

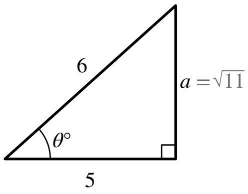
$$\therefore c=\sqrt{7}\quad (c>0)$$

$\sin(\theta^\circ)$ is the ratio of the opposite side to the hypotenuse.

$$\sin(\theta^\circ)=\frac{\sqrt{3}}{\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}$$

$$\therefore \sin(\theta^\circ)=\frac{\sqrt{21}}{7}$$

ii $\cos(\theta^\circ)=\dfrac{5}{6}$



Using Pythagoras'

$$a^2+5^2=6^2$$

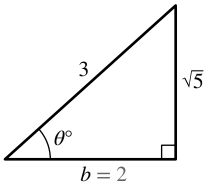
$$\therefore a^2+25=36$$

$$\therefore a^2=11$$

$$\therefore a=\sqrt{11}$$

Hence, $\tan(\theta^\circ)=\dfrac{\sqrt{11}}{5}$.

iii $\sin(\theta^\circ)=\dfrac{\sqrt{5}}{3}$



Using Pythagoras'

$$b^2+\left(\sqrt{5}\right)^2=3^2$$

$$\therefore b^2+5=9$$

$$\therefore b^2=4$$

$$\therefore b=2$$

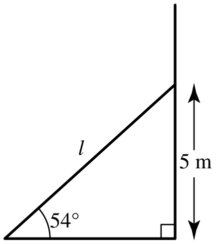
Hence, $\cos(\theta^\circ)=\dfrac{2}{3}$.

b Since $\sin(\theta^\circ)=\dfrac{3}{5}$, the ratio of the opposite side to the hypotenuse is 3 : 5. The set of numbers '3, 4, 5' are a Pythagorean triple so the sides of a triangle for which $\sin(\theta^\circ)=\dfrac{3}{5}$ must be in the ratio 3 : 4 : 5.

Since the longest side is the hypotenuse, for the given triangle, its hypotenuse is 60 cm. This is a factor of 12 times 5, so the other sides of the triangle must be $3\times12=36$ cm and $4\times12=48$ cm.

The shortest side is 36 cm.

11 a Let the length of the ladder be l metres.



$$\sin(54^\circ)=\frac{5}{l}$$

$$\therefore l=\frac{5}{\sin(54^\circ)}$$

$$\therefore l\simeq 6.1803$$

The ladder is 6.18 metres in length.

b Let the initial distance of the ladder from the pole be x meters.

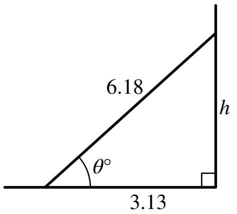
$$\tan(54^\circ)=\frac{5}{x}$$

$$\therefore x=\frac{5}{\tan(54^\circ)}$$

$$\therefore x\simeq 3.6327$$

Initially the ladder is 3.6327 metres from the pole. Moving the ladder 0.5 metres closer to the pole reduces this distance to 3.1327 metres from the pole.

Let the new inclination to the ground be θ° and the new height the ladder reaches up the pole be h metres.



$$\cos(\theta°) = \frac{3.1327}{6.1803}$$
$$\therefore \theta° = \cos^{-1}\left(\frac{3.1327}{6.1803}\right)$$
$$\therefore \theta° \simeq 59.54°$$

The new inclination to the ground is 59.5°.

$$\tan(59.54°) = \frac{h}{3.1327}$$
$$\therefore h = 3.1327\tan(59.54°)$$
$$\therefore h \simeq 5.33$$

The new height the ladder reaches up the pole is 5.3 metres.

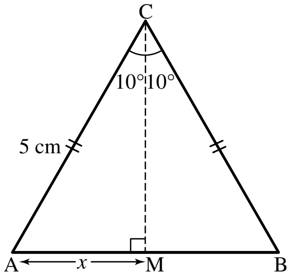
12 a i In the isosceles triangle, the 20° angle is included between the two equal sides of 5 cm.

The area of the triangle is

$$A = \frac{1}{2}\times 5\times 5\times \sin(20°)$$
$$= 12.5\times \sin(20°)$$
$$\therefore A = 4.275$$

The area is 4.275 sq cm correct to three decimal places.

ii Divide the isosceles triangle into two right angled triangles by joining C to the midpoint M of the side AB.



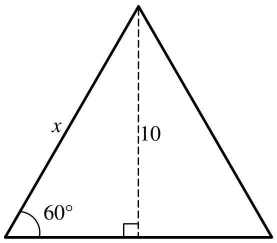
CM bisects the angle ACB and the side AB.

Let AM have length x metres so AB has length $2x$ metres.

$$\sin(10°) = \frac{x}{5}$$
$$\therefore x = 5\sin(10°)$$

The third side, AB has length $10\sin(10°) \simeq 1.736$ cm.

b The angles in an equilateral triangle are each 60° and the sides are equal in length. Let the length of a side be x cm.



$$\sin(60°) = \frac{10}{x}$$
$$\therefore \frac{\sqrt{3}}{2} = \frac{10}{x}$$
$$\therefore \sqrt{3}x = 20$$
$$\therefore x = \frac{20}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$\therefore x = \frac{20\sqrt{3}}{3}$$

The perimeter is $3x = 20\sqrt{3}$ cm.

The base and height of the triangle are known so its area is:

$$A = \frac{1}{2}bh$$
$$\therefore A = \frac{1}{2}\times\frac{20\sqrt{3}}{3}\times 10$$
$$\therefore A = \frac{100\sqrt{3}}{3}$$

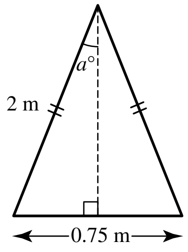
Area is $\frac{100\sqrt{3}}{3}$ sq cm.

c In triangle ABC $a = 4\sqrt{2}, b = 6$ cm and $C = 30°$.

$$A = \frac{1}{2}ab\sin(C)$$
$$= \frac{1}{2}\times 4\sqrt{2}\times 6\times \sin(30°)$$
$$= 12\sqrt{2}\times\frac{1}{2}$$
$$= 6\sqrt{2}$$

Area is $6\sqrt{2}$ sq cm.

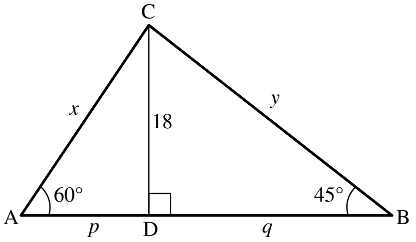
13 Let the angle between the legs be $2a°$.



$$\sin(a°) = \frac{0.75\div 2}{2}$$
$$\therefore \sin(a°) = \frac{0.375}{2}$$
$$\therefore a° = \sin^{-1}(0.1875)$$
$$\therefore 2a° = 2\sin^{-1}(0.1875)$$
$$\simeq 21.6°$$

The angle between the legs of the ladder is approximately 21.6°.

14



Consider triangle ACD.

$$\tan(60°)=\frac{18}{p}$$
$$\therefore p=\frac{18}{\tan(60°)}$$
$$\therefore p=\frac{18}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$\therefore p=6\sqrt{3}$$

Also,

$$\cos(60°)=\frac{p}{x}$$
$$\therefore \cos(60°)=\frac{6\sqrt{3}}{x}$$
$$\therefore x=\frac{6\sqrt{3}}{\cos(60°)}$$
$$\therefore x=6\sqrt{3}\div\frac{1}{2}$$
$$\therefore x=12\sqrt{3}$$

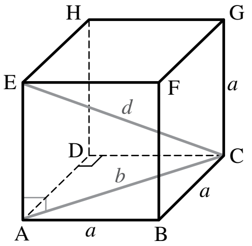
Consider triangle BCD.
Since $\angle BCD=45°$ (angle sum of a triangle is $180°$) then triangle BCD is isosceles.
$\therefore q=18$

Also,

$$\cos(45°)=\frac{q}{y}$$
$$\therefore \cos(45°)=\frac{18}{y}$$
$$\therefore y=\frac{18}{\cos(45°)}$$
$$\therefore y=18\div\frac{1}{\sqrt{2}}$$
$$\therefore y=18\sqrt{2}$$

The sides have lengths: $AC=12\sqrt{3}$ cm, $BC=18\sqrt{2}$ cm and $AB=\left(6\sqrt{3}+18\right)$ cm.

15 Consider the cube ABCDEFGH of edge a units.



a First calculate AC, a diagonal of the base.
Triangle ADC is right angled at D. Let AC have length b units.
Using Pythagoras' theorem

$$b^2=a^2+a^2$$
$$=2a^2$$
$$\therefore b=\sqrt{2}a$$

Now the diagonal of the cube, EC, can be calculated from the right angled triangle EAC.
Let EC have length c units.
Using Pythagoras' theorem

$$c^2=a^2+b^2$$
$$=a^2+2a^2$$
$$=3a^2$$
$$\therefore c=\sqrt{3}a$$

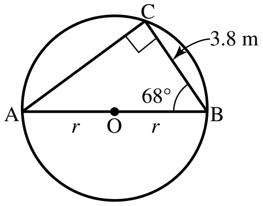
The diagonal of the cube is $\sqrt{3}a$ units in length.

b The inclination of the diagonal to the horizontal is the angle ECA. Let this angle be $\theta°$.
In triangle EAC,

$$\sin(\theta°)=\frac{\text{Opposite}}{\text{Hypotenuse}}=\frac{a}{\sqrt{3}a}$$
$$\therefore \sin(\theta°)=\frac{1}{\sqrt{3}}$$
$$\therefore \theta°=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$\therefore \theta°\simeq 35.26°$$

The diagonal is inclined at 35.26° to the horizontal.

16 a

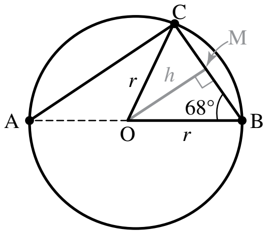


The angle in the semicircle is right angled so $\angle ACB=90°$.
If the radius is r cm then AB has length $2r$ cm.

$$\cos(68°)=\frac{3.8}{2r}$$
$$\therefore r=\frac{1.9}{\cos(68°)}$$
$$\therefore r\simeq 5.072$$

The radius is 5.07 cm.

b



The shortest distance is the perpendicular distance OM where O is the centre of the circle and M the midpoint of the side CB of the isosceles triangle OCB.
$MB=\frac{1}{2}\times 3.8=1.9$ cm.
Let the shortest distance be h cm.
In triangle OMB,

$$\tan(68°)=\frac{h}{1.9}$$
$$\therefore h=1.9\tan(68°)$$
$$\therefore h\simeq 4.703$$

The distance is 4.7 cm.

17 a Refer to the diagram given in the question.
In triangle ABC,

$$\sin(a°)=\frac{BC}{m}$$
$$\therefore BC=m\sin(a°)....(1)$$

Since DB is parallel to CA, the co-interior angles are supplementary.

$$\therefore \angle BDC+\angle ACD=180°$$
$$\therefore \angle BDC+90°=180°$$
$$\therefore \angle BDC=90°$$

Also, $\angle ACB=90°-a°$ (angle sum of triangle ABC)

$$\therefore \angle BCD=90°-(90°-a°)$$
$$\therefore \angle BCD=a°$$

In right angled triangle BCD,

$$\sin(a°) = \frac{n}{BC}$$

$$\therefore BC = \frac{n}{\sin(a°)} \text{....(2)}$$

Equating the two expressions for BC,

$$m\sin(a°) = \frac{n}{\sin(a°)}$$

$$\therefore m\sin(a°)\sin(a°) = n$$

$$\therefore n = m[\sin(a°)]^2$$

$$\therefore n = m\sin^2(a°)$$

as required.

b As alternate angles formed by parallel lines are equal, $\angle EBA = \angle BAC$.

Given $\angle EBA = 60°$, then $a° = 60°$.

In triangle BCD, $\angle BCD = a° = 60°$ and $CD = 4\sqrt{3}$ units.

$$\therefore \tan(60°) = \frac{n}{4\sqrt{3}}$$

$$\therefore n = 4\sqrt{3}\tan(60°)$$

$$= 4\sqrt{3} \times \sqrt{3}$$

$$\therefore n = 12$$

Substitute $n = 12$ in $n = m\sin^2(a°)$

$$\therefore 12 = m\sin^2(60°)$$

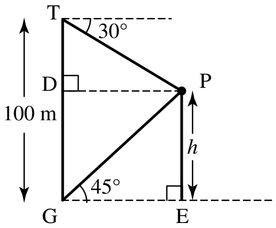
$$\therefore m\left(\frac{\sqrt{3}}{2}\right)^2 = 12$$

$$\therefore \frac{3m}{4} = 12$$

$$\therefore m = 16$$

Answer: $a = 60,\ m = 16,\ n = 12$

18 Let TG be the tower of height 100 metres and PE be the tower of height h metres.



Construct PD parallel to the ground level

The angle of elevation $\angle EGP = 45°$. Hence triangle EGP is an isosceles right angled triangle with $GE = EP = h$ metres.

Triangle GDP is also isosceles with $GD = DP = h$ metres.

In triangle TDP, $TD = (100 - h)$ metres, $DP = h$ metres, $\angle DPT = 30°$ and $\angle TDP = 90°$.

$$\therefore \tan(30°) = \frac{100 - h}{h}$$

$$\therefore \frac{1}{\sqrt{3}} = \frac{100 - h}{h}$$

$$\therefore h = \sqrt{3}(100 - h)$$

$$\therefore h = 100\sqrt{3} - \sqrt{3}h$$

$$\therefore \sqrt{3}h + h = 100\sqrt{3}$$

$$\therefore h\left(\sqrt{3} + 1\right) = 100\sqrt{3}$$

$$\therefore h = \frac{100\sqrt{3}}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

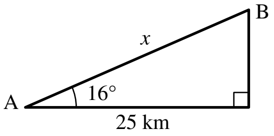
$$= \frac{100\sqrt{3}\left(\sqrt{3} - 1\right)}{2}$$

$$\therefore h = 50\sqrt{3}\left(\sqrt{3} - 1\right)$$

$$\therefore h = 150 - 50\sqrt{3}$$

Correct to one decimal place, the height of the second tower is 63.4 metres and this tower is 63.4 metres horizontally from the lookout tower.

19



Let the actual distance between A and B be x km.

$$\cos(16°) = \frac{25}{x}$$

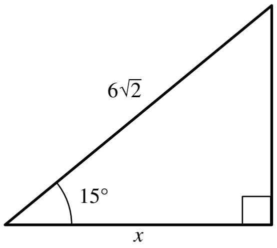
$$\therefore x = \frac{25}{\cos(16°)}$$

To evaluate, ensure calculator is on Deg mode and Decimal, not Standard. The trigonometric functions are obtained from the keyboard → mth → TRIG.

$\therefore x \simeq 26.007$

The actual distance between A and B is 26.007 km.

20



Let the horizontal distance required be x metres.

$$\cos(15°) = \frac{x}{6\sqrt{2}}$$

$$\therefore x = 6\sqrt{2}\cos(15°)$$

To evaluate this expression exactly, place the calculator on Standard, not Decimal.

$\therefore x = 3\left(\sqrt{3} + 1\right)$

The horizontal distance of the foot of the beam from the fence is $3\left(\sqrt{3} + 1\right)$ metres.

Exercise 9.3 — Circular measure

1 **a** $60° = 60 \times \frac{\pi}{180}$

$$= \frac{\pi}{3}$$

b $\frac{3\pi^c}{4} = \frac{3\pi}{4} \times \frac{180°}{\pi}$

$$= 135°$$

c $\frac{\pi}{6} = \frac{\cancel{\pi}}{6} \times \frac{180°}{\cancel{\pi}}$

$$= 30°$$

$$\therefore \tan\left(\frac{\pi}{6}\right) = \tan(30°)$$

$$= \frac{\sqrt{3}}{3}$$

2 $145°12' = 145.2°$

$$145.2° = 145.2 \times \frac{\pi}{180}$$

$$= \frac{726}{5} \times \frac{\pi}{180}$$

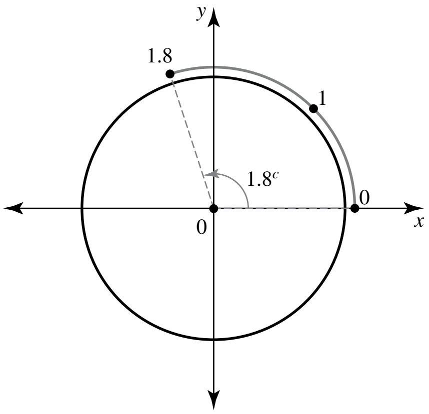
$$= \frac{121\pi}{150}$$

$$\simeq 2.53$$

3 **a** 1.8^c

$$\begin{aligned}1.8^c &= 1.8 \times \frac{180^\circ}{\pi}\\ &\simeq 103.1^\circ\end{aligned}$$

b



4 Every rotation of 2π maps numbers to same positions. These rotations may be anticlockwise or clockwise.

a the numbers $-4\pi, -2\pi, 0, 2\pi, 4\pi$ are all mapped to the same positions on the circumference of the unit circle.

b $-1-4\pi, -1-2\pi, -1, -1+2\pi, -1+4\pi$ are mapped to the same positions.

5 **a** $l = r\theta$, $r = 8$, $\theta = 75 \times \frac{\pi}{180}$

$$\begin{aligned}l &= 8 \times 75 \times \frac{\pi}{180}\\ &= 8 \times \frac{15\pi}{36}\\ &= \frac{30\pi}{9}\end{aligned}$$

arc length is $\frac{30\pi}{9} \simeq 10.47$ cm correct to two decimal places.

b Calculate the angle at the centre
$l = r\theta$, where $l = 12\pi$, $r = 10$
$12\pi = 10\theta$
$\therefore \theta = 1.2\pi$
This angle is in radian measure so it needs to be converted to degrees.
In degrees,

$$\begin{aligned}1.2\pi &= 1.2\pi \times \frac{180^\circ}{\pi}\\ &= 216^\circ\end{aligned}$$

Therefore the angle at the centre of the circle subtended by the arc is 216°.

6 **a** The angle has no degree sign so it must be assumed to be in radians. Ensure the calculator is on Rad mode.
$\tan(1.2) = \tan(1.2^c)$
$\therefore \tan(1.2) = 2.572$ correct to three decimal places.

b Using degree mode on the calculator,
$\tan(1.2^\circ) = 0.021$ correct to three decimal places.

7 **a** To convert degrees to radians, multiply by $\frac{\pi}{180}$.

$$30^\circ = \cancel{30}^1 \times \frac{\pi}{\cancel{180}_6} = \frac{\pi}{6} \qquad 45^\circ = \cancel{45}^1 \times \frac{\pi}{\cancel{180}_4} = \frac{\pi}{4} \qquad 60^\circ = \cancel{60}^1 \times \frac{\pi}{\cancel{180}_3} = \frac{\pi}{3}$$

Degrees	30°	45°	60°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

b $0^\circ = 0^c$, $180^\circ = \pi^c$
$\therefore 90^\circ = \frac{1}{2}\pi^c = \frac{\pi}{2}$ and $360^\circ = 2 \times \pi^c = 2\pi$.

$$\begin{aligned}270^\circ &= 180^\circ + 90^\circ\\ &= \pi + \frac{\pi}{2}\\ &= \frac{3\pi}{2}\end{aligned}$$

Degrees	0°	90°	180°	270°	360°
Radians	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

8 To convert radian to degree measure, multiply by $\frac{180}{\pi}$.

a $\frac{\pi^c}{5}$

$$\begin{aligned}&= \frac{\pi}{5} \times \frac{180^\circ}{\pi}\\ &= 36^\circ\end{aligned}$$

b $\frac{2\pi^c}{3}$

$$\begin{aligned}&= \frac{2\pi}{3} \times \frac{180^\circ}{\pi}\\ &= 2 \times 60^\circ\\ &= 120^\circ\end{aligned}$$

c $\frac{5\pi}{12}$

$$\begin{aligned}&= \frac{5\pi}{12} \times \frac{180^\circ}{\pi}\\ &= 5 \times 15^\circ\\ &= 75^\circ\end{aligned}$$

d $\frac{11\pi}{6}$

$$\begin{aligned}&= \frac{11\pi}{6} \times \frac{180^\circ}{\pi}\\ &= 11 \times 30^\circ\\ &= 330^\circ\end{aligned}$$

e $\frac{7\pi}{9}$

$$\begin{aligned}&= \frac{7\pi}{9} \times \frac{180^\circ}{\pi}\\ &= 7 \times 20^\circ\\ &= 140^\circ\end{aligned}$$

f $\frac{9\pi}{2}$

$$\begin{aligned}&= \frac{9\pi}{2} \times \frac{180^\circ}{\pi}\\ &= 9 \times 90^\circ\\ &= 810^\circ\end{aligned}$$

9 **a** 40°

$$\begin{aligned}&= 40 \times \frac{\pi}{180}\\ &= \frac{2\pi}{9}\end{aligned}$$

b 150°

$$\begin{aligned}&= 150 \times \frac{\pi}{180}\\ &= \frac{15\pi}{18}\\ &= \frac{5\pi}{6}\end{aligned}$$

c 225°

$$\begin{aligned}&= 225 \times \frac{\pi}{180}\\ &= \frac{45\pi}{36}\\ &= \frac{5\pi}{4}\end{aligned}$$

d 300°

$$\begin{aligned}&= 300\times\frac{\pi}{180}\\&=\frac{30\pi}{18}\\&=\frac{5\pi}{3}\end{aligned}$$

e 315°

$$\begin{aligned}&= 315\times\frac{\pi}{180}\\&=\frac{63\pi}{36}\\&=\frac{7\pi}{4}\end{aligned}$$

f 720°

$$\begin{aligned}&= 720\times\frac{\pi}{180}\\&=\frac{72\pi}{18}\\&=\frac{8\pi}{2}\\&=4\pi\end{aligned}$$

or, $720° = 2\times 360° = 2\times 2\pi = 4\pi$

10 **a** **i** 3°

$$\begin{aligned}&= 3\times\frac{\pi}{180}\\&=\frac{\pi}{60}\\&\simeq 0.052\end{aligned}$$

ii $112°15' = 112.25°$

$$\begin{aligned}&= 112.25\times\frac{\pi}{180}\\&\simeq 1.959\end{aligned}$$

iii 215.36°

$$\begin{aligned}&= 215.36\times\frac{\pi}{180}\\&\simeq 3.759\end{aligned}$$

b **i** 3^c

$$\begin{aligned}&= 3\times\frac{180°}{\pi}\\&\simeq 171.887°\end{aligned}$$

ii 2.3π

$$\begin{aligned}&= 2.3\pi\times\frac{180°}{\pi}\\&=2.3\times 180°\\&=414°\end{aligned}$$

c $\left\{1.5^c,\ 50°,\ \frac{\pi}{7}^c\right\}$

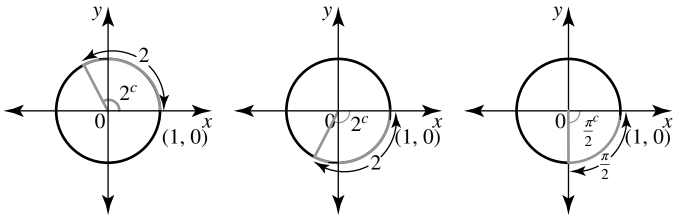
Converting radians to degrees,

$$\begin{aligned}1.5^c &= 1.5\times\frac{180°}{\pi}\\&\simeq 85.9°\end{aligned}$$

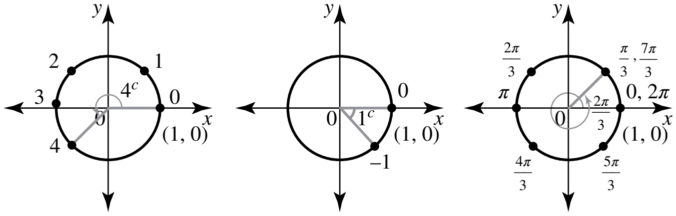
$$\begin{aligned}\frac{\pi}{7}^c &= \frac{\pi}{7}\times\frac{180°}{\pi}\\&=\frac{180°}{7}\\&\simeq 25.7°\end{aligned}$$

In order from smallest to largest, $\left\{\frac{\pi}{7}^c,\ 50°,\ 1.5^c\right\}$.

11 a



b



12 a $l = r\theta,\ r = 12,\ \theta = 150 \times \dfrac{\pi}{180}$

$$\begin{aligned}\therefore l &= 12 \times \frac{15\pi}{18}\\ &= 12 \times \frac{5\pi}{6}\\ &= 10\pi\end{aligned}$$

Arc length is 10π cm.

b If the angle at the circumference is $\dfrac{2\pi^c}{9}$, then the angle at the centre is $\dfrac{4\pi^c}{9}$.

$l = r\theta,\ r = \pi,\ \theta = \dfrac{4\pi}{9}$

$$\begin{aligned}\therefore l &= \pi \times \frac{4\pi}{9}\\ &= \frac{4\pi^2}{9}\end{aligned}$$

Arc length is $\dfrac{4\pi^2}{9}$ cm.

c As the chord or the minor arc, subtends an angle of 60° at the centre, the major arc subtends an angle of $(360° - 60°) = 300°$ at the centre.

$l = r\theta,\ r = 3,\ \theta = 300 \times \dfrac{\pi}{180}$

$$\begin{aligned}\therefore l &= 3 \times \frac{30\pi}{18}\\ &= 3 \times \frac{5\pi}{3}\\ &= 5\pi\end{aligned}$$

Arc length is 5π cm.

13 a The rope length is the radius of 2.5 metres; the arc length of 75 cm is 0.75 metres.

$l = r\theta,\ l = 0.75,\ r = 2.5$

$$\begin{aligned}\therefore 0.75 &= 2.5 \times \theta\\ \therefore \theta &= \frac{0.75}{2.5}\\ \therefore \theta &= 0.3\end{aligned}$$

Convert the radians to degrees.

$$\begin{aligned}0.3^c &= 0.3 \times \frac{180°}{\pi}\\ &= \frac{54°}{\pi}\\ &\simeq 17.2°\end{aligned}$$

The ball swings through an angle of 17.2°.

b Speed is 2 m/s, so in 5 seconds, the point travels a distance of 10 metres around the circumference of the wheel.
The radius of the wheel is 3 metres

$l = r\theta,\ r = 3,\ l = 10$

$$\begin{aligned}\therefore 10 &= 3\theta\\ \therefore \theta &= \frac{10}{3}\end{aligned}$$

In degrees, $\dfrac{10}{3}$ radians is

$$\left(\frac{10}{3}\times\frac{180}{\pi}\right)^\circ$$
$$=\frac{600^\circ}{\pi}$$
$$\simeq 191^\circ$$

The angle of rotation is 191° or $\frac{10^c}{3}$.

c Every 60 minutes, the minute hand rotates through 360°. In 1 minute it rotates through 6°.
In the 5 minutes between 9 : 45 am and 9 : 50 am, the minute hand will rotate through 30°.
Arc length:

$$l=r\theta,\ \ r=11,\ \ \theta=30\times\frac{\pi}{180}$$
$$\therefore l=11\times\frac{\pi}{6}$$
$$=\frac{11\pi}{6}$$
$$\simeq 5.76$$

The arc length is 5.76 mm.

d The angle at the centre is twice the angle at the circumference, so the arc subtends an angle of $2\times22.5^\circ=45^\circ$ at the centre of the circle.
The arc length and angle are known so the radius can be calculated.

$l=r\theta$, where $l=4,\theta=45\times\frac{\pi}{180}$, so $\theta=\frac{\pi}{4}$

$$4=r\times\frac{\pi}{4}$$
$$\therefore r=\frac{16}{\pi}$$

Area of circle: $A=\pi r^2$

$$A=\pi\times\left(\frac{16}{\pi}\right)^2$$
$$=\frac{256}{\pi}$$
$$\simeq 81.5$$

Correct to 1 decimal place, the area is 81.5 sq cm.

14 a i $\tan(1)=\tan(1^c)=1.557$

ii $\cos\left(\frac{2\pi}{7}\right)=\cos\left(\frac{2\pi^c}{7}\right)=0.623$

iii $\sin(1.46^\circ)=0.025$

b As $\frac{\pi}{6}=30^\circ,\frac{\pi}{4}=45^\circ,\frac{\pi}{3}=60^\circ$ then $\sin\left(\frac{\pi}{6}\right)=\sin(30^\circ)$ and so on.

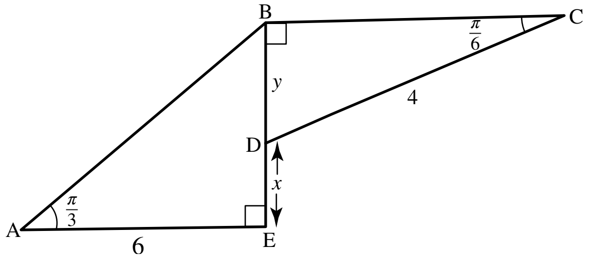
θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

15 a In triangle ABC, $a=2\sqrt{2},c=2,B=\frac{\pi}{4}$. Two sides and the included angle are known.

$$A_\Delta=\frac{1}{2}ac\sin B$$
$$=\frac{1}{2}\times2\sqrt{2}\times2\times\sin\left(\frac{\pi}{4}\right)$$
$$=2\sqrt{2}\times\frac{\sqrt{2}}{2}$$
$$=2$$

The area of the triangle is 2 square units.

b



In triangle BDC,

$$\sin\left(\frac{\pi}{6}\right)=\frac{y}{4}$$
$$\therefore y=4\sin\left(\frac{\pi}{6}\right)$$
$$=4\times\frac{1}{2}$$
$$=2$$

In triangle AEB,

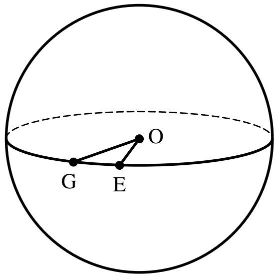
$$\tan\left(\frac{\pi}{3}\right)=\frac{y+x}{6}$$
$$=\frac{2+x}{6}$$
$$\therefore 2+x=6\tan\left(\frac{\pi}{3}\right)$$
$$\therefore x=6\times\sqrt{3}-2$$
$$\therefore x=6\sqrt{3}-2$$

16 a Let the radius of the earth be R km.
The arc BK of length 1490 km, is an arc of a circle, centre O, radius R km, with the arc subtending an angle of 13.4° at the centre O.

$$l=r\theta,\ r=R,\ \theta=13.4\times\frac{\pi}{180},l=1490$$
$$\therefore 1490=R\times\frac{13.4\pi}{180}$$
$$\therefore R=\frac{1490\times180}{13.4\pi}$$
$$\therefore R\simeq 6371$$

The radius of the Earth is estimated to be 6371 km.

b



Let G be the position of the Galapogas islands and E the position of Eucador; O is the centre of the earth.

600 nautical miles is equal to $600\times1.85=1110$ km. The arc GE is 1110 km and the angle GOE is the difference in longitudes of 10°.

$$l=r\theta,\ r=R,\theta=10\times\frac{\pi}{180},l=1110$$

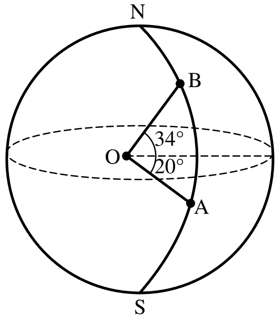
$$\therefore 1110=R\times\frac{\pi}{18}$$

$$\therefore R=\frac{1110\times18}{\pi}$$

$$\therefore R\simeq 6360$$

The radius of the Earth is estimated to be 6360 km.

c



As A is south of the equator and B is north of the equator, the angle the arc AB subtends at the centre is the sum of the latitudes, $(20+34)=54°$.

$$l=r\theta,\ r=6370,\theta=54\times\frac{\pi}{180}$$

$$\therefore l=6370\times\frac{6\pi}{20}$$

$$\therefore l=6370\times0.3\pi$$

$$\therefore l=1911\pi$$

$$\therefore l\simeq 6004$$

The distance between A and B measured along the meridian is 6004 km.

17 In the main menu in Rad mode and Standard mode, enter 135. Tap from the keyboard → Mth → Trig → *o* to obtain 135° and press EXE.
This converts the degrees to radians, giving $\frac{3\pi}{4}$.

$$\therefore 135°=\frac{3\pi}{4}$$

18 In the main menu in Deg mode and Standard mode, enter 5. Tap from the keyboard → Mth → Trig → *r* to obtain 5^r and press EXE.
This converts the radians to degrees, giving $\frac{900}{\pi}$. By switching to Decimal mode, and pressing EXE, this gives 286.4788976.

$$\therefore 5^c=\frac{900°}{\pi}\simeq 286.4789°$$

Exercise 9.4 — Unit circle definitions

1 a To reach the boundary between quadrants 1 and 2 a rotation of $\frac{\pi}{2}$ would be needed from the point (1,0). Therefore the trigonometric point is $P\left[\frac{\pi}{2}\right]$.

b

120°	−400° $=-360°-40°$	$\frac{4\pi}{3}=1\frac{1}{3}\pi$	$\frac{\pi}{4}$
quadrant 2	quadrant 4	quadrant 3	quadrant 1

c rotating clockwise $(360-120)°=240°$ gives $Q[-240°]$; one full revolution of 360° plus another 120° gives $R[480°]$.

2 An anticlockwise rotation of $\frac{3\pi}{2}$ or a clockwise rotation of $\frac{\pi}{2}$ would reach the boundary between quadrants 3 and 4. The point could be $P\left[\frac{3\pi}{2}\right]$ or $P\left[-\frac{\pi}{2}\right]$.

3 a $P\left[\frac{\pi}{6}\right],\theta=\frac{\pi}{6}$

Cartesian co-ordinates:

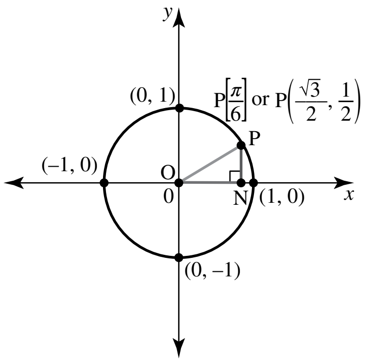
$$x=\cos(\theta)\qquad y=\sin(\theta)$$

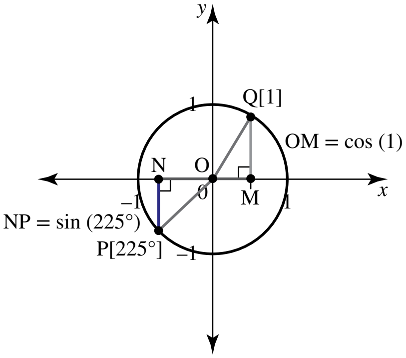
$$=\cos\left(\frac{\pi}{6}\right)\ \text{and}\ =\sin\left(\frac{\pi}{6}\right)$$

$$=\frac{\sqrt3}{2}\qquad =\frac{1}{2}$$

Therefore P is the point $\left(\frac{\sqrt3}{2},\frac{1}{2}\right)$.

b





c The point $P\left[-\frac{\pi}{2}\right]$ is the point $(0,-1)$.

$$x=\cos(\theta),\theta=-\frac{\pi}{2},x=0,\ \therefore\cos\left(-\frac{\pi}{2}\right)=0$$

$$y=\sin(\theta),\theta=-\frac{\pi}{2},y=-1,\ \therefore\sin\left(-\frac{\pi}{2}\right)=-1$$

d $f(\theta)=\sin(\theta)$

$\therefore f(0)=\sin(0)$

The trigonometric point [0] has Cartesian co-ordinates (1,0). Its *y* co-ordinate gives the value of sin(0).

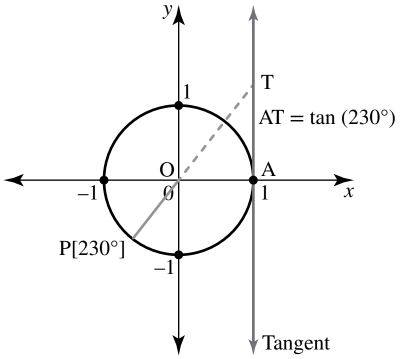
$\therefore \sin(0)=0$

$\therefore f(0)=0$

4 a Since $\sin(\theta)$ is the *y* co-ordinate, $\sin(\theta)$ is positive in the first and second quadrants.

b $\cos(\theta)$ is the *x* co-ordinate, so $\cos(\theta)$ is positive in the first and fourth quadrants.

5 a $\tan(230°) \simeq 1.192$



b $P[2\pi]$ is the point $(1, 0)$.

$$\tan(\theta) = \frac{y}{x}, \theta = 2\pi, x = 1, y = 0$$

$$\therefore \tan(2\pi) = \frac{0}{1}$$

$$\therefore \tan(2\pi) = 0$$

6 $\left\{\tan(-3\pi), \tan\left(\frac{5\pi}{2}\right), \tan(-90°), \tan\left(\frac{3\pi}{4}\right), \tan(780°)\right\}$

a Neither $\tan\left(\frac{5\pi}{2}\right)$ nor $\tan(-90°)$ are defined since the ray forming each is parallel to the vertical tangent.

b $\tan\left(\frac{3\pi}{4}\right)$ will be negative since $\frac{3\pi}{4}$ lies in the second quadrant so the extended ray forming it would intersect the tangent below the x axis.

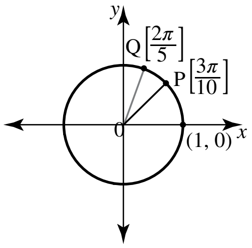
7 a Since $585° = 360° + 225°$ then the end ray of $585°$ lies in the same quadrant as that of $225°$, which is quadrant 3.

b $\frac{11\pi}{12} = \frac{12\pi}{12} - \frac{\pi}{12} = \pi - \frac{\pi}{12}$ so it lies in quadrant 2.

c $-18\pi = 9 \times (-2\pi)$ so it lies on the boundary between quadrants 1 and 4.

d $\frac{7\pi}{4} = \frac{8\pi}{4} - \frac{\pi}{4} = 2\pi - \frac{\pi}{4}$ so it lies in quadrant 4.

8 a As $\frac{\pi}{2} = \frac{5\pi}{10}$, and $\frac{3\pi}{10} < \frac{\pi}{2}$ and $\frac{2\pi}{5} = \frac{4\pi}{10} < \frac{\pi}{2}$, the points P and Q lie in quadrant 1.



b $\angle QOP = \frac{4\pi}{10} - \frac{3\pi}{10} = \frac{\pi^c}{10}$

c Using a clockwise rotation from $(1, 0)$ to reach point Q would require a rotation of $-\frac{3\pi}{2} - \frac{\pi}{10} = -\frac{16\pi}{10}$. Q could be described as the trigonometric point $\left[-\frac{8\pi}{5}\right]$.

To reach point P a further rotation from Q of $-\frac{\pi}{10}$ would be required, making the rotation from $(1, 0)$ to reach point P of $-\frac{16\pi}{10} - \frac{\pi}{10} = -\frac{17\pi}{10}$. P could be described as the trigonometric point $\left[-\frac{17\pi}{10}\right]$.

Other answers can be formed by adding multiples of -2π.

d Rotating anticlockwise a complete revolution of 2π and then a further $\frac{3\pi}{10}$ would reach point P. Since $2\pi + \frac{3\pi}{10} = \frac{23\pi}{10}$, P could be described as the trigonometric point $\left[\frac{23\pi}{10}\right]$.

Similarly, $2\pi + \frac{2\pi}{5} = \frac{12\pi}{5}$ so Q could be described as the trigonometric point $\left[\frac{12\pi}{5}\right]$.

Other answers can be formed by adding multiples of 2π.

9 a $P\left[\frac{\pi}{4}\right]$

$x = \cos\theta$ and $y = \sin\theta$ where $\theta = \frac{\pi}{4}$.

$$\therefore x = \cos\left(\frac{\pi}{4}\right) \quad \text{and} \quad y = \sin\left(\frac{\pi}{4}\right)$$

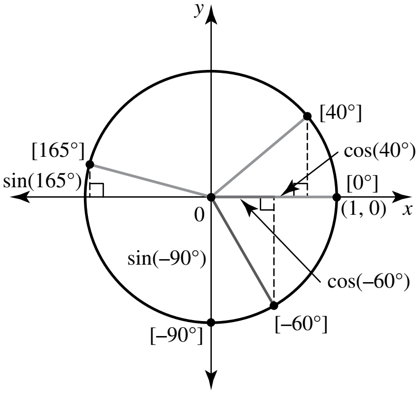
$$= \frac{\sqrt{2}}{2} \qquad\qquad = \frac{\sqrt{2}}{2}$$

P has Cartesian co-ordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

b The point $P(0, -1)$ lies on the boundary between quadrants 3 and 4.

Therefore, P could be the trigonometric point $\left[\frac{3\pi}{2}\right]$ or the trigonometric point $\left[-\frac{\pi}{2}\right]$.

10



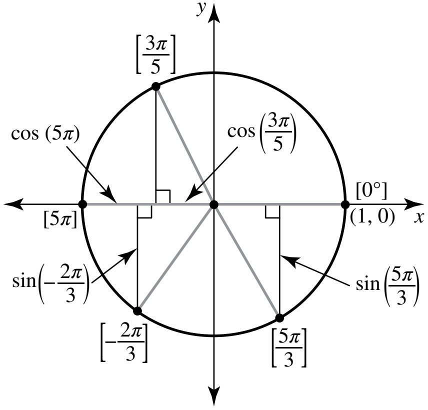
a cos(40°) is the x co-ordinate of the trigonometric point [40°] which lies in the first quadrant.

b sin(165°) is the y co-ordinate of the trigonometric point [165°] which lies in the second quadrant.

c cos(−60°) is the x co-ordinate of the trigonometric point [−60°] which lies in the fourth quadrant.

d sin(−90°) is the y co-ordinate of the trigonometric point [−90°] which lies on the boundary between the third and fourth quadrants.

11



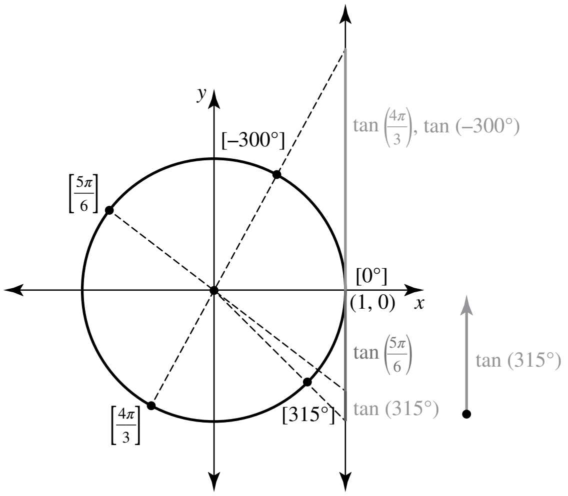
a $\sin\left(\frac{5\pi}{3}\right)$ is the y co-ordinate of the trigonometric point $\left[\frac{5\pi}{3}\right]$ which lies in the fourth quadrant.

b $\cos\left(\frac{3\pi}{5}\right)$ is the x co-ordinate of the trigonometric point $\left[\frac{3\pi}{5}\right]$ which lies in the second quadrant.

c $\cos(5\pi)$ is the x co-ordinate of the trigonometric point $[5\pi]$ which lies on the boundary between the second and third quadrants.

d $\sin\left(-\frac{2\pi}{3}\right)$ is the y co-ordinate of the trigonometric point $\left[-\frac{2\pi}{3}\right]$ which lies in the third quadrant.

12



The tangent to the unit circle at the point $(1,0)$ is drawn.

a tan(315°) is the length of the intercept cut off on the tangent by the extended radius through the trigonometric point [315°] in the fourth quadrant.

b $\tan\left(\frac{5\pi}{6}\right)$ is the length of the intercept cut off on the tangent by the extended radius from the trigonometric point $\left[\frac{5\pi}{6}\right]$ in the second quadrant.

c $\tan\left(\frac{4\pi}{3}\right)$ is the length of the intercept cut off on the tangent by the extended radius from the trigonometric point $\left[\frac{4\pi}{3}\right]$ in the third quadrant.

d tan(–300°) is the length of the intercept cut off on the tangent by the extended radius through the trigonometric point [–300°] in the first quadrant. This is the same value as $\tan\left(\frac{4\pi}{3}\right)$.

13 a $f(t) = \sin(t)$

Let $t = 2$

$$\begin{aligned}\therefore f(2) &= \sin(2)\\ &= \sin\left(2^c\right)\\ &= 0.91\end{aligned}$$

b $g(u) = \cos(u)$

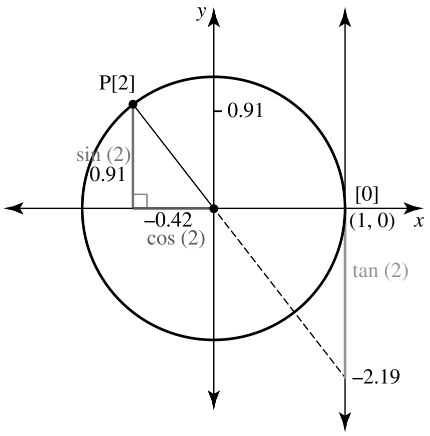
Let $u = 2$

$$\begin{aligned}\therefore g(2) &= \cos(2)\\ &= -0.42\end{aligned}$$

c $h(\theta) = \tan(\theta)$

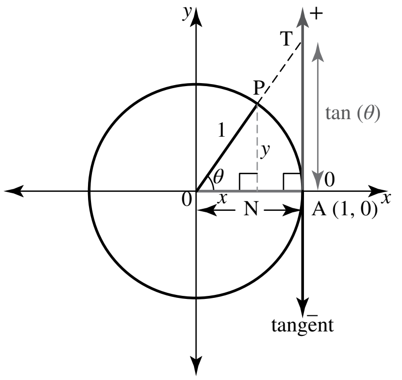
$$\begin{aligned}\therefore h(2) &= \tan(2)\\ &= -2.19\end{aligned}$$

14 a i



ii P has the Cartesian co-ordinates $(-0.42, 0.91)$.

b



The arc length AP:

$l = r\theta,\ r = 1$

$\therefore l = \theta$

From the diagram, a comparison of the lengths of PN, arc AP and TA shows

$PN < \text{arc } AP < TA$

$\therefore \sin\theta < \theta < \tan\theta$

15 a $P[\theta]$ is the Cartesian point $P(-0.8, 0.6)$.

Since $x < 0, y > 0$ the point lies in quadrant 2.

Check the point lies on a unit circle by showing $x^2 + y^2 = 1$.

$$\begin{aligned}x^2 + y^2 &= (-0.8)^2 + (0.6)^2\\ &= 0.64 + 0.36\\ &= 1\end{aligned}$$

$\sin(\theta) = y$ co-ordinate of P

$\therefore \sin(\theta) = 0.6$

$\cos(\theta) = x$ co-ordinate of P

$\therefore \cos(\theta) = -0.8$

$$\tan(\theta) = \frac{y}{x}$$

$$\begin{aligned}\therefore \tan(\theta) &= \frac{0.6}{-0.8}\\ &= -\frac{3}{4}\end{aligned}$$

b $Q[\theta]$ is the trigonometric point $Q\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Since $x > 0, y < 0$, Q lies in quadrant 4.

Check Q lies on a unit circle.

$$\begin{aligned} x^2+y^2 &= \left(\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{\sqrt{2}}{2}\right)^2 \\ &= \frac{2}{4}+\frac{2}{4} \\ &= 1 \end{aligned}$$

$\sin(\theta) = y = -\dfrac{\sqrt{2}}{2}$

$\cos(\theta) = x = \dfrac{\sqrt{2}}{2}$

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= -1 \end{aligned}$$

c $R[\theta]$ is the trigonometric point $R\left(\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)$.

Since $x > 0, y > 0$, R lies in quadrant 1.
Check R lies on a unit circle.

$$\begin{aligned} x^2+y^2 &= \left(\frac{2}{\sqrt{5}}\right)^2+\left(\frac{1}{\sqrt{5}}\right)^2 \\ &= \frac{4}{5}+\frac{1}{5} \\ &= 1 \end{aligned}$$

$\sin(\theta) = \dfrac{1}{\sqrt{5}}$

$\cos(\theta) = \dfrac{2}{\sqrt{5}}$

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \frac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} \\ &= \frac{1}{2} \end{aligned}$$

d $S[\theta]$ is the Cartesian point $S(0,1)$. It lies on the boundary between the first and second quadrants.
$\sin(\theta) = y = 1$
$\cos(\theta) = x = 0$
$\tan(\theta) = \dfrac{y}{x}$

$\therefore \tan(\theta) = \dfrac{1}{0}$ which is undefined.

16 a $\cos(0)$ is the x co-ordinate of the trigonometric point $[0]$ which has Cartesian co-ordinates $(1, 0)$.
$\therefore \cos(0) = 1$

b $\sin\left(\dfrac{\pi}{2}\right)$ is the y co-ordinate of the trigonometric point $\left[\dfrac{\pi}{2}\right]$ which has Cartesian co-ordinates $(0, 1)$.

$\therefore \sin\left(\dfrac{\pi}{2}\right) = 1$

c $\tan(\pi)$ is the ratio of the y co-ordinate to the x co-ordinate of the trigonometric point $[\pi]$ which has Cartesian co-ordinates $(-1, 0)$.

$$\begin{aligned} \therefore \tan(\pi) &= \frac{y}{x} \\ &= \frac{0}{-1} \\ &= 0 \end{aligned}$$

d $\cos\left(\dfrac{3\pi}{2}\right)$ is the x co-ordinate of the trigonometric point $\left[\dfrac{3\pi}{2}\right]$ which has Cartesian co-ordinates $(0, -1)$.

$\therefore \cos\left(\dfrac{3\pi}{2}\right) = 0$

e $\sin(2\pi)$ is the y co-ordinate of the trigonometric point $[2\pi]$ which has Cartesian co-ordinates $(1, 0)$.
$\therefore \sin(2\pi) = 0$

f $\cos\left(\dfrac{17\pi}{2}\right) + \tan(-11\pi) + \sin\left(\dfrac{11\pi}{2}\right)$

$\cos\left(\dfrac{17\pi}{2}\right)$ is the x co-ordinate of the trigonometric point $\left[\dfrac{17\pi}{2}\right]$.

Since $\dfrac{17\pi}{2} = 8\pi + \dfrac{\pi}{2}$, the trigonometric point has the same position as $\left[\dfrac{\pi}{2}\right]$ which has cartesian co-ordinates $(0, 1)$.

$\therefore \cos\left(\dfrac{17\pi}{2}\right) = 0$

$\tan(-11\pi)$ is the ratio of the y co-ordinate to the x co-ordinate of the trigonometric point $[-11\pi]$.
Since $-11\pi = -10\pi - \pi$, the trigonometric point has the same position as $[-\pi]$ which has cartesian co-ordinates $(-1, 0)$.

$$\begin{aligned} \therefore \tan(-11\pi) &= \frac{y}{x} \\ &= \frac{0}{-1} \\ &= 0 \end{aligned}$$

$\sin\left(\dfrac{11\pi}{2}\right)$ is the y co-ordinate of the trigonometric point $\left[\dfrac{11\pi}{2}\right]$.

Since $\dfrac{11\pi}{2} = 4\pi + \dfrac{3\pi}{2}$, the trigonometric point has the same position as $\left[\dfrac{3\pi}{2}\right]$ which has cartesian co-ordinates $(0, -1)$.

$\therefore \sin\left(\dfrac{11\pi}{2}\right) = -1.$

Hence,

$$\begin{aligned} &\cos\left(\frac{17\pi}{2}\right) + \tan(-11\pi) + \sin\left(\frac{11\pi}{2}\right) \\ &= 0 + 0 - 1 \\ &= -1 \end{aligned}$$

17 a $\cos^2\left(\dfrac{7\pi}{6}\right) + \sin^2\left(\dfrac{7\pi}{6}\right)$

In the Main menu using TRIG in the mth Keyboard key in $(\cos(7\pi/6))\wedge 2 + (\sin(7\pi/6))\wedge 2$, with the calculator set on Rad and Standard modes.
The value is 1.

b The value of $\cos(7\pi/6) + \sin(7\pi/6)$ is $\dfrac{-\sqrt{3}}{2} - \dfrac{1}{2}$.

c $(\sin(7/6))\wedge 2 + (\cos(7/6))\wedge 2$

$= \left(\sin\left(\dfrac{7}{6}\right)\right)^2 + \left(\cos\left(\dfrac{7}{6}\right)\right)^2$

Switch to Decimal mode to obtain the value 1.

d $\sin^2(60°)+\cos^2(60°)$

In Standard and Deg modes the value is 1. (In fact, it will be the value 1 even if Rad mode is used).

e $\sin^2(t)+\cos^2(t)=1$

$\sin(t)=y$ co-ordinate of trigonometric point $[t]$ and $\cos(t)=x$ co-ordinate of point $[t]$.

Point $[t]$ lies on the unit circle $x^2+y^2=1$

$$\therefore (\cos(t))^2+(\sin(t))^2=1$$
$$\therefore \cos^2(t)+\sin^2(t)=1$$
$$\therefore \sin^2(t)+\cos^2(t)=1$$

18 a $P\left[\frac{7\pi}{4}\right]$ has Cartesian co-ordinates of $x=\cos\left(\frac{7\pi}{4}\right)$ and $y=\sin\left(\frac{7\pi}{4}\right)$.

Evaluating on Standard and Rad modes, $x=\frac{\sqrt{2}}{2}$ and $y=\frac{-\sqrt{2}}{2}$.

P is the Cartesian point $\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$.

$Q\left[\frac{\pi}{4}\right]$ has Cartesian co-ordinates of $x=\cos\left(\frac{\pi}{4}\right)$ and $y=\sin\left(\frac{\pi}{4}\right)$.

Evaluating on Standard and Rad modes, $x=\frac{\sqrt{2}}{2}$ and $y=\frac{\sqrt{2}}{2}$.

Q is the Cartesian point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

Point Q lies in the first quadrant and point P lies in the fourth quadrant. The two points are symmetric about the x axis with P the reflection of Q in the x axis and vice versa.

b $R\left[\frac{4\pi}{5}\right]$ has Cartesian co-ordinates

$$x=\cos\left(\frac{4\pi}{5}\right)=\frac{-\left(\sqrt{5}+1\right)}{4} \quad \text{and } y=\sin\left(\frac{4\pi}{5}\right)=\frac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}$$

R is the Cartesian point $\left(\frac{-\left(\sqrt{5}+1\right)}{4},\frac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}\right)$.

$S\left[\frac{\pi}{5}\right]$ has Cartesian co-ordinates

$$x=\cos\left(\frac{\pi}{5}\right)=\frac{\sqrt{5}+1}{4} \quad \text{and } y=\sin\left(\frac{\pi}{5}\right)=\frac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}$$

S is the Cartesian point $\left(\frac{\sqrt{5}+1}{4},\frac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}\right)$.

R and S have the same y values but their x values are opposite in sign. R lies in the first quadrant and S lies in the second quadrant. The two points are symmetric about the y axis with S the reflection of R in the y axis and vice versa.

c i From part **a**, $\sin\left(\frac{7\pi}{4}\right)=-\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

$\therefore \sin\left(\frac{7\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)$.

Also from part **a**, $\cos\left(\frac{7\pi}{4}\right)=\frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

$\therefore \cos\left(\frac{7\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)$.

For the tangent values: $\tan\left(\frac{7\pi}{4}\right)=-1$ and $\tan\left(\frac{\pi}{4}\right)=1$

$\therefore \tan\left(\frac{7\pi}{4}\right)=-\tan\left(\frac{\pi}{4}\right)$.

ii From part **b**, $\sin\left(\frac{4\pi}{5}\right)=\frac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}$ and $\sin\left(\frac{\pi}{5}\right)=\frac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}$

$\therefore \sin\left(\frac{4\pi}{5}\right)=\sin\left(\frac{\pi}{5}\right)$.

Also from part **a**, $\cos\left(\frac{4\pi}{5}\right)=-\frac{\sqrt{5}+1}{4}$ and $\cos\left(\frac{\pi}{5}\right)=\frac{\sqrt{5}+1}{4}$

$\therefore \cos\left(\frac{4\pi}{5}\right)=-\cos\left(\frac{\pi}{5}\right)$.

For the tangent values: $\tan\left(\frac{4\pi}{5}\right)=-\sqrt{-2\sqrt{5}+5}$ and $\tan\left(\frac{\pi}{5}\right)=\sqrt{-2\sqrt{5}+5}$

$\therefore \tan\left(\frac{4\pi}{5}\right)=-\tan\left(\frac{\pi}{5}\right)$.

Exercise 9.5 — Symmetry properties

1 a $\cos(\theta)$ is negative and $\tan(\theta)$ is positive in the third quadrant.

b
$$\begin{aligned} f(t)&=\tan(t)\\ \therefore f(4\pi)&=\tan(4\pi)\\ &=0 \end{aligned}$$

2
$$\begin{aligned} f(t)&=\sin(\pi t)\\ \therefore f(2.5)&=\sin(2.5\pi)\\ &=\sin\left(\frac{5\pi}{2}\right)\\ &=1 \end{aligned}$$

3 a Third quadrant, base $\frac{\pi}{3}$ since $\frac{4\pi}{3}=\pi+\frac{\pi}{3}$

$$\begin{aligned} &\sin\left(\frac{4\pi}{3}\right)\\ &=-\sin\left(\frac{\pi}{3}\right)\\ &=-\frac{\sqrt{3}}{2} \end{aligned}$$

b
$$\begin{aligned} &\tan\left(\frac{5\pi}{6}\right)\\ &=\tan\left(\pi-\frac{\pi}{6}\right)\\ &=-\tan\left(\frac{\pi}{6}\right)\\ &=-\frac{\sqrt{3}}{3} \end{aligned}$$

c cos(−30°), fourth quadrant, base 30°

$$\begin{aligned}&\cos(-30^\circ)\\&=\cos(30^\circ)\\&=\frac{\sqrt{3}}{2}\end{aligned}$$

4 $-\frac{5\pi}{4}$ lies in second quadrant with base $\frac{\pi}{4}$ since $-\frac{5\pi}{4}=-\pi-\frac{\pi}{4}$. Only sine is positive in the second quadrant.

$$\begin{array}{lll}\sin\left(-\frac{5\pi}{4}\right) \quad\text{and} & \cos\left(-\frac{5\pi}{4}\right) \quad\text{and} & \tan\left(-\frac{5\pi}{4}\right)\\ =\sin\left(\frac{\pi}{4}\right) & =-\cos\left(\frac{\pi}{4}\right) & =-\tan\left(\frac{\pi}{4}\right)\\ =\frac{\sqrt{2}}{2} & =-\frac{\sqrt{2}}{2} & =-1\end{array}$$

5 a [75°]. Symmetric points are found from $180^\circ \pm 75^\circ, 360^\circ \pm 75^\circ$.

Therefore, second quadrant [105°], third quadrant [255°], fourth quadrant [285°].

Cosine is positive in the fourth quadrant so $\cos(285^\circ)=\cos(75^\circ)$. (The first and fourth quadrant points have the same x co-ordinate). The trigonometric point is [285°].

b $\frac{6\pi}{7}=\pi-\frac{\pi}{7}$

$$\begin{aligned}&\therefore \tan\left(\frac{6\pi}{7}\right)\\&=\tan\left(\pi-\frac{\pi}{7}\right)\\&=-\tan\left(\frac{\pi}{7}\right)\end{aligned}$$

c

$$\begin{aligned}\sin(\theta)&=0.8\\ \sin(\pi-\theta)&=\sin(\theta)\\&=0.8\\ \sin(2\pi-\theta)&=-\sin(\theta)\\&=-0.8\end{aligned}$$

d i

$$\begin{aligned}\cos\left(\frac{5\pi}{4}\right)&=\cos\left(\pi+\frac{\pi}{4}\right)\\&=-\cos\left(\frac{\pi}{4}\right)\\&=-\frac{\sqrt{2}}{2}\end{aligned}$$

ii

$$\begin{aligned}\sin\left(\frac{25\pi}{6}\right)&=\sin\left(4\pi+\frac{\pi}{6}\right)\\&=\sin\left(2\pi+\frac{\pi}{6}\right).\\&=\sin\left(\frac{\pi}{6}\right)\\&=\frac{1}{2}\end{aligned}$$

6 $\cos(\theta)=p$

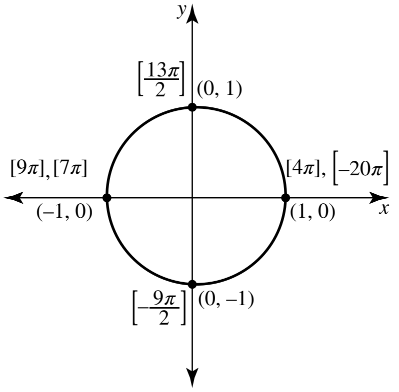
a fourth quadrant symmetry property

$$\begin{aligned}\cos(-\theta)&=\cos(\theta)\\&=p\end{aligned}$$

b third quadrant symmetry property

$$\begin{aligned}\cos(5\pi+\theta)&=\cos(\pi+\theta)\\&=-\cos(\theta)\\&=-p\end{aligned}$$

7



a $\cos(4\pi)$ is the x co-ordinate of the Cartesian point $(1,0)$.

$\therefore \cos(4\pi)=1$

b $\tan(9\pi)=\frac{y}{x}$ for the point $(-1,0)$

$\therefore \tan(9\pi)=\frac{0}{-1}=0$

c $\sin(7\pi)$ is the y co-ordinate of the Cartesian point $(-1,0)$.

$\therefore \sin(7\pi)=0$

d $\sin\left(\frac{13\pi}{2}\right)$ is the y co-ordinate of the Cartesian point $(0,1)$.

$\therefore \sin\left(\frac{13\pi}{2}\right)=1$

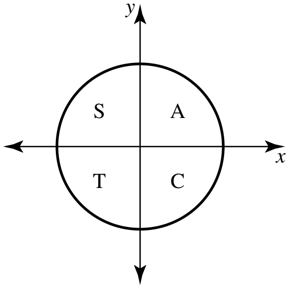
e $\cos\left(-\frac{9\pi}{2}\right)$ is the x co-ordinate of the Cartesian point $(0,-1)$.

$\therefore \cos\left(-\frac{9\pi}{2}\right)=0$

f $\tan(-20\pi)=\frac{y}{x}$ for the point $(1,0)$.

$\therefore \tan(-20\pi)=\frac{0}{1}=0$

8 Consider the ‘CAST’ diagram.



a $\cos(\theta)>0, \sin(\theta)<0$ in quadrant 4.

b $\tan(\theta)>0, \cos(\theta)>0$ in quadrant 1.

c $\sin(\theta)>0, \cos(\theta)<0$ in quadrant 2.

d $\cos(\theta)=0$ at the points on the unit circle which have $x=0$. This occurs at the boundary between quadrants 1 and 2, and at the boundary between quadrants 3 and 4.

e $\cos(\theta)=0, \sin(\theta)>0$ when $x=0, y>0$. This occurs at the boundary between quadrants 1 and 2.

f $\sin(\theta)=0, \cos(\theta)<0$ when $y=0, x<0$. This occurs at the boundary between quadrants 2 and 3.

9 a For θ in the first quadrant, symmetric points can be calculated as $\pi \pm \theta$ and $2\pi-\theta$.

Points symmetric to $\frac{\pi}{3}$ are:

second quadrant $\pi-\frac{\pi}{3}=\frac{2\pi}{3}$; third quadrant $\pi+\frac{\pi}{3}=\frac{4\pi}{3}$ and fourth quadrant $2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$.

b Points symmetric to $\frac{\pi}{6}$ are:
second quadrant $\pi-\frac{\pi}{6}=\frac{5\pi}{6}$; third quadrant $\pi+\frac{\pi}{6}=\frac{7\pi}{6}$
and fourth quadrant $2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$.

c Points symmetric to $\frac{\pi}{4}$ are:
second quadrant $\pi-\frac{\pi}{4}=\frac{3\pi}{4}$; third quadrant $\pi+\frac{\pi}{4}=\frac{5\pi}{4}$
and fourth quadrant $2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$.

d Points symmetric to $\frac{\pi}{5}$ are:
second quadrant $\pi-\frac{\pi}{5}=\frac{4\pi}{5}$; third quadrant $\pi+\frac{\pi}{5}=\frac{6\pi}{5}$
and fourth quadrant $2\pi-\frac{\pi}{5}=\frac{9\pi}{5}$.

e Points symmetric to $\frac{3\pi}{8}$ are:
second quadrant $\pi-\frac{3\pi}{8}=\frac{5\pi}{8}$; third quadrant $\pi+\frac{3\pi}{8}=\frac{11\pi}{8}$
and fourth quadrant $2\pi-\frac{3\pi}{8}=\frac{13\pi}{8}$.

f Points symmetric to 1 are:
second quadrant $\pi-1$; third quadrant $\pi+1$ and fourth quadrant $2\pi-1$.

10 a cos(120°) cosine is negative in the second quadrant and $120°=180°-60°$.
$=-\cos(60°)$
$=-\frac{1}{2}$

b $\tan(225°)$ tangent is positive in the third quadrant and $225°=180°+45°$.
$=\tan(45°)$
$=1$

c sin(330°) sine is negative in the fourth quadrant and $330°=360°-30°$.
$=-\sin(30°)$
$=-\frac{1}{2}$

d tan(−60°) tangent is negative in the fourth quadrant
$=-\tan(60°)$
$=-\sqrt{3}$

e cos(−315°) cosine is positive in the first quadrant
$=\cos(45°)$
$=\frac{\sqrt{2}}{2}$

f sin(510°)
$=\sin(360°+150°)$
$=\sin(150°)$
$=\sin(30°)$
$=\frac{1}{2}$

11 a $\sin\left(\frac{3\pi}{4}\right)$ sine is positive in the second quadrant and
$\frac{3\pi}{4}=\pi-\frac{\pi}{4}$
$=\sin\left(\frac{\pi}{4}\right)$
$=\frac{\sqrt{2}}{2}$

b $\tan\left(\frac{2\pi}{3}\right)$ tangent is negative in the second quadrant and
$\frac{2\pi}{3}=\pi-\frac{\pi}{3}$.
$=-\tan\left(\frac{\pi}{3}\right)$
$=-\sqrt{3}$

c $\cos\left(\frac{5\pi}{6}\right)$ cosine is negative in the second quadrant and
$\frac{5\pi}{6}=\pi-\frac{\pi}{6}$.
$=-\cos\left(\frac{\pi}{6}\right)$
$=-\frac{\sqrt{3}}{2}$

d $\cos\left(\frac{4\pi}{3}\right)$ cosine is negative in the third quadrant and
$\frac{4\pi}{3}=\pi+\frac{\pi}{3}$
$=-\cos\left(\frac{\pi}{3}\right)$
$=-\frac{1}{2}$

e $\tan\left(\frac{7\pi}{6}\right)$ tangent is positive in the third quadrant and
$\frac{7\pi}{6}=\pi+\frac{\pi}{6}$
$=\tan\left(\frac{\pi}{6}\right)$
$=\frac{\sqrt{3}}{3}$

f $\sin\left(\frac{11\pi}{6}\right)$ sine is negative in the fourth quadrant and
$\frac{11\pi}{6}=2\pi-\frac{\pi}{6}$
$=-\sin\left(\frac{\pi}{6}\right)$
$=-\frac{1}{2}$

12 a $\cos\left(-\frac{\pi}{4}\right)$
$=\cos\left(\frac{\pi}{4}\right)$
$=\frac{\sqrt{2}}{2}$

b $\sin\left(-\frac{\pi}{3}\right)$
$=-\sin\left(\frac{\pi}{3}\right)$
$=-\frac{\sqrt{3}}{2}$

c $\tan\left(-\frac{5\pi}{6}\right)$ tangent is positive in the third quadrant and
$-\frac{5\pi}{6}=-\pi+\frac{\pi}{6}$
$=\tan\left(\frac{\pi}{6}\right)$
$=\frac{\sqrt{3}}{3}$

d $\sin\left(\frac{8\pi}{3}\right)$

$$\begin{aligned}&=\sin\left(2\pi+\frac{2\pi}{3}\right)\\&=\sin\left(\frac{2\pi}{3}\right)\\&=\sin\left(\frac{\pi}{3}\right)\\&=\frac{\sqrt{3}}{2}\end{aligned}$$

e $\cos\left(\frac{9\pi}{4}\right)$

$$\begin{aligned}&=\cos\left(2\pi+\frac{\pi}{4}\right)\\&=\cos\left(\frac{\pi}{4}\right)\\&=\frac{\sqrt{2}}{2}\end{aligned}$$

f $\tan\left(\frac{23\pi}{6}\right)$

$$\begin{aligned}&=\tan\left(4\pi-\frac{\pi}{6}\right)\\&=-\tan\left(\frac{\pi}{6}\right)\\&=-\frac{\sqrt{3}}{3}\end{aligned}$$

13 a $\cos\left(\frac{7\pi}{6}\right)+\cos\left(\frac{2\pi}{3}\right)$

$$\begin{aligned}&=-\cos\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{3}\right)\\&=-\frac{\sqrt{3}}{2}-\frac{1}{2}\\&=\frac{-\left(\sqrt{3}+1\right)}{2}\end{aligned}$$

b $2\sin\left(\frac{7\pi}{4}\right)+4\sin\left(\frac{5\pi}{6}\right)$

$$\begin{aligned}&=2\times-\sin\left(\frac{\pi}{4}\right)+4\times\sin\left(\frac{\pi}{6}\right)\\&=-2\times\frac{\sqrt{2}}{2}+4\times\frac{1}{2}\\&=-\sqrt{2}+2\end{aligned}$$

c $\sqrt{3}\tan\left(\frac{5\pi}{4}\right)-\tan\left(\frac{5\pi}{3}\right)$

$$\begin{aligned}&=\sqrt{3}\times\tan\left(\frac{\pi}{4}\right)-\left[-\tan\left(\frac{\pi}{3}\right)\right]\\&=\sqrt{3}\times1+\sqrt{3}\\&=2\sqrt{3}\end{aligned}$$

d $\sin\left(\frac{8\pi}{9}\right)+\sin\left(\frac{10\pi}{9}\right)$

$$\begin{aligned}&=\sin\left(\pi-\frac{\pi}{9}\right)+\sin\left(\pi+\frac{\pi}{9}\right)\\&=\sin\left(\frac{\pi}{9}\right)-\sin\left(\frac{\pi}{9}\right)\\&=0\end{aligned}$$

e $2\cos^2\left(-\frac{5\pi}{4}\right)-1$

$$\begin{aligned}&=2\left[\cos\left(-\frac{5\pi}{4}\right)\right]^2-1\\&=2\left[\cos\left(-\pi-\frac{\pi}{4}\right)\right]^2-1\\&=2\left[-\cos\left(\frac{\pi}{4}\right)\right]^2-1\\&=2\left(-\frac{\sqrt{2}}{2}\right)^2-1\\&=2\times\frac{2}{4}-1\\&=0\end{aligned}$$

f $\dfrac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

$$\begin{aligned}&=\frac{\tan\left(4\pi+\frac{\pi}{4}\right)\cos(-6\pi-\pi)}{\sin\left(-2\pi+\frac{\pi}{6}\right)}\\&=\frac{\tan\left(\frac{\pi}{4}\right)\cos(-\pi)}{\sin\left(\frac{\pi}{6}\right)}\\&=\frac{1\times-1}{\frac{1}{2}}\\&=-2\end{aligned}$$

14 Given $\cos(\theta)=0.91,\ \sin(t)=0.43$ and $\tan(x)=0.47$.

a $\cos(\pi+\theta)=-\cos(\theta)$
$\quad=-0.91$

b $\sin(\pi-t)=\sin(t)$
$\quad=0.43$

c $\tan(2\pi-x)=-\tan(x)$
$\quad=-0.47$

d $\cos(-\theta)=\cos(\theta)$
$\quad=0.91$

e $\sin(-t)=-\sin(t)$
$\quad=-0.43$

f $\tan(2\pi+x)=\tan(x)$
$\quad=0.47$

15 Given $\sin(\theta)=p$

a $\sin(2\pi-\theta)=-\sin(\theta)$
$\quad=-p$

b $\sin(3\pi-\theta)=\sin(\theta)$
$\quad=p$

c $\sin(-\pi+\theta)=-\sin(\theta)$
$\quad=-p$

d $\sin(\theta+4\pi)=\sin(\theta)$
$\quad=p$

16 a Statement: $\sin^2\left(\frac{5\pi}{4}\right)+\cos^2\left(\frac{5\pi}{4}\right)=1$

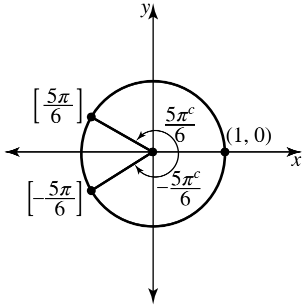
$\sin\left(\frac{5\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{5\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)$

$=-\frac{\sqrt{2}}{2}$ $\qquad=-\frac{\sqrt{2}}{2}$

Substitute these values into the left side of the statement.

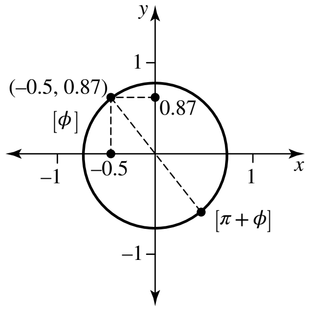
$$\begin{aligned}\text{LHS} &= \left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 \\ &= \frac{2}{4} + \frac{2}{4} \\ &= 1 \\ &= \text{RHS}\end{aligned}$$

b Plot the trigonometric points $\left[\frac{5\pi}{6}\right]$ and $\left[-\frac{5\pi}{6}\right]$ on a unit circle diagram.



The two points are symmetric relative to the x axis since $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$ and $-\frac{5\pi}{6} = -\pi + \frac{\pi}{6}$. Their x values are the same so $\cos\left(\frac{5\pi}{6}\right) = \cos\left(-\frac{5\pi}{6}\right)$.

c The point $[\phi]$ is the Cartesian point $(-0.5, 0.87)$.



Comparing y co-ordinates of $[\phi]$ and $[\pi+\phi]$,

$$\begin{aligned}\sin(\pi+\phi) &= -\sin(\phi) \\ &= -0.87\end{aligned}$$

Comparing x co-ordinates of $[\phi]$ and $[\pi+\phi]$,

$$\begin{aligned}\cos(\pi+\phi) &= -\cos(\phi) \\ &= -(-0.5) \\ &= 0.5\end{aligned}$$

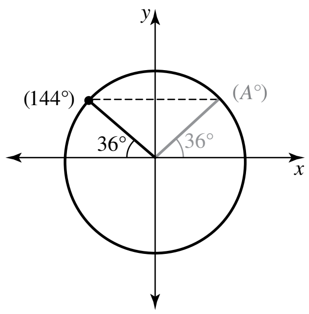
$$\begin{aligned}\tan(\pi+\phi) &= \frac{\sin(\pi+\phi)}{\cos(\pi+\phi)} \\ &= \frac{-0.87}{0.5} \\ &= -1.74\end{aligned}$$

d $\sin(-\pi+t) + \sin(-3\pi-t) + \sin(t+6\pi)$

$$\begin{aligned}&= -\sin(t) + \sin(t) + \sin(t) \\ &= \sin(t)\end{aligned}$$

e If $\sin(A°) = \sin(144°)$ then the points $[A°]$ and $[144°]$ have the same y co-ordinates.

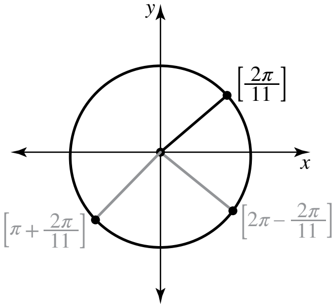
The point $[144°]$ is in the second quadrant.



As $144° = 180° - 36°$, the point $[36°]$ has the same y value as the point $[144°]$. Hence one value for $A°$ is $A° = 36°$. Another value could be $-360° + 36° = -324°$.

Other values are possible, including other values for $[144°]$ such as $[-216°]$ so $A° = -216°$.

f If $\sin(B) = -\sin\left(\frac{2\pi}{11}\right)$, then the y value of the point $[B]$ has the opposite sign to that of the point $\left[\frac{2\pi}{11}\right]$. Since $\left[\frac{2\pi}{11}\right]$ lies in the first quadrant, its y value is positive, so the y value of the point $[B]$ is negative.



The point $[B]$ is in either the third or the fourth quadrant. Possible values are:

$$B = \pi + \frac{2\pi}{11} \quad \text{and} \quad B = 2\pi - \frac{2\pi}{11}.$$
$$= \frac{13\pi}{11} \qquad\qquad = \frac{20\pi}{11}$$

For a third value, $B = -\frac{2\pi}{11}$. However, there are many answers possible for $B = -\frac{2\pi}{11}$.

17 a Since $\pi \simeq 3.142$ and $\frac{3\pi}{2} \simeq 4.712$, $\pi < 4.2 < \frac{3\pi}{2}$. The point $P[4.2]$ lies in quadrant 3.

b The Cartesian co-ordinates of P are:

$$x = \cos(4.2) \quad \text{and} \quad y = \sin(4.2)$$
$$= -0.49 \qquad\qquad = -0.87$$

The Cartesian co-ordinates are $(-0.49, -0.87)$.

c Let $[\theta]$ be the point in the first quadrant that is symmetric to P.

As P is in the third quadrant,

$$\begin{aligned}&4.2 = \pi + \theta \\ &\therefore \theta = 4.2 - \pi \\ &\therefore \theta \simeq 1.0584\end{aligned}$$

The symmetric point in the second quadrant is given by

$$\begin{aligned}\pi - \theta &= \pi - (4.2 - \pi) \\ &= 2\pi - 4.2 \\ &\simeq 2.0832\end{aligned}$$

The symmetric point in the fourth quadrant is given by

$$\begin{aligned}2\pi - \theta &= 2\pi - (4.2 - \pi) \\ &= 3\pi - 4.2 \\ &\simeq 5.2248\end{aligned}$$

18 $Q[\theta]$ where $\tan(\theta) = 5$.

a Since $\tan(\theta) > 0$, the point Q could lie in wither the first or the third quadrants.

b For quadrant 1,

$$\begin{aligned}\theta &= \tan^{-1}(5) \\ &\simeq 1.3734\end{aligned}$$

For quadrant 3,

$$\begin{aligned}\theta &= \pi + \tan^{-1}(5) \\ &\simeq 4.5150\end{aligned}$$

c For quadrant 1, with $\theta = \tan^{-1}(5)$,

$$\begin{aligned}&x = \cos(\theta) \\ &\therefore x = \cos\left(\tan^{-1}(5)\right)\end{aligned}$$

Evaluate this in Standard and Rad modes to obtain $x = \frac{\sqrt{26}}{26}$.

$y = \sin(\theta)$

$\therefore y = \sin\left(\tan^{-1}(5)\right)$

$\therefore y = \frac{5\sqrt{26}}{26}$

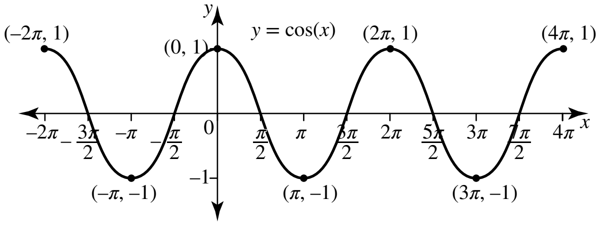
The first quadrant point has Cartesian co-ordinates $\left(\frac{\sqrt{26}}{26}, \frac{5\sqrt{26}}{26}\right)$.

For the point in the third quadrant, $x < 0$ and $y < 0$, so the point must have Cartesian co-ordinates $\left(-\frac{\sqrt{26}}{26}, -\frac{5\sqrt{26}}{26}\right)$.

For this third quadrant point $Q[\theta]$, $\cos(\theta) = -\frac{\sqrt{26}}{26}$ and $\sin(\theta) = -\frac{5\sqrt{26}}{26}$.

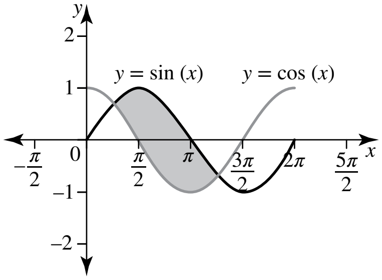
Exercise 9.6 — The graphs of the sine and cosine functions

1 To sketch $y = \cos(x)$ over the domain $[-2\pi, 4\pi]$, sketch the basic graph on the domain $[0, 2\pi]$ and continue the pattern.



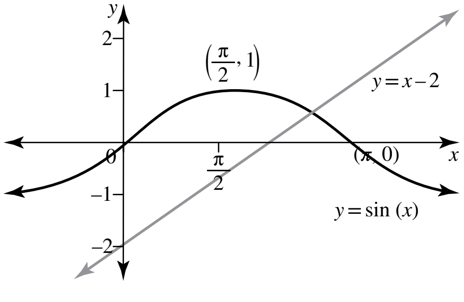
The graph shows three cycles of the cosine function.

2 Over the domain $[0, 2\pi]$ one cycle of each of the graphs of $y = \cos(x)$ and $y = \sin(x)$ is required.



The graphs intersect at the points A and B. The region below the sine graph and above the cosine graph between these points is the required region $\{(x, y) : \sin(x) \ge \cos(x), x \in [0, 2\pi]\}$.

3 a The graph of $y = \sin(x)$ has x intercepts at the origin and at $(\pi, 0) \simeq (-3.14, 0)$ as well others. The line $y = x - 2$ has intercepts with the axes at $(0, -2)$ and $(2, 0)$.



There is only one point of intersection of the two graphs. Therefore, the equation $\sin(x) = x - 2$ has only one solution.

b From the graph, the point of intersection lies in the interval between $x = 2$ and $x = 3$ (answers could vary).

c Let $f(x) = \sin(x) - x + 2$

From the graph the sine graph is higher than the line at $x = 2$, so $f(2) > 0$.

However, the sine graph lies below the line at $x = 3$, so $f(3) < 0$.

Midpoint of $[2, 3]$ is $x = 2.5$.

$$\begin{aligned} f(2.5) &= \sin(2.5) - 2.5 + 2 \\ &= \sin(2.5^c) - 0.5 \\ &= 0.098.. \\ &> 0 \end{aligned}$$

The root lies in the interval $[2.5, 3]$.

Midpoint of $[2.5, 3]$ is $x = 2.75$

$$\begin{aligned} f(2.75) &= \sin(2.75) - 2.75 + 2 \\ &= -0.368.. \\ &< 0 \end{aligned}$$

The root lies in the interval $[2.5, 2.75]$,

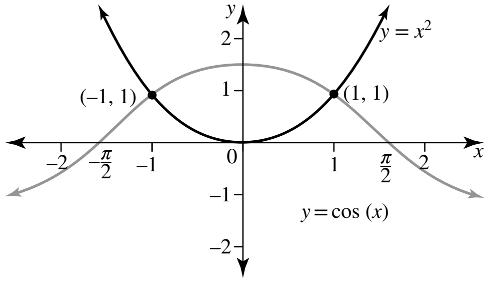
The midpoint of $[2.5, 2.75]$ gives an estimate of the root.

An estimate is $x = 2.625$.

4 a The number of solutions to the equation $\cos(x) - x^2 = 0$ is determined by the number of intersections of the graphs of $y = \cos(x)$ and $y = x^2$. At the intersection of these graphs, $\cos(x) = x^2$ and therefore $\cos(x) - x^2 = 0$.

b The graph of $y = x^2$ contains the points $(0, 0), (\pm 1, 1)$.

The graph of $y = \cos(x)$ contains the point $(0, 1)$ and among its x intercepts are $\left(\pm\frac{\pi}{2}, 0\right) \simeq (\pm 1.57, 0)$.



As the graphs intersect twice, there are two roots to the equation $\cos(x) - x^2 = 0$.

c If the graph of $y = x^2$ is vertically translated upwards by one unit, then it will meet the cosine graph at the point $(0, 1)$. Therefore if $k = 1$, the equation $\cos(x) = x^2 + k$ will have exactly one solution.

5 a $\sin(1.8°)$

Converting the angle to radian measure,

$1.8° = 1.8 \times \frac{\pi^c}{180}$

$\therefore 1.8° = \frac{\pi^c}{100}$

$\sin(1.8°) = \sin\left(\frac{\pi^c}{100}\right)$.

For small x, $\sin(x) \simeq x$ and as $\frac{\pi}{100}$ is small then $\sin\left(\frac{\pi^c}{100}\right) \simeq \frac{\pi}{100}$.

$\therefore \sin(1.8°) \simeq 0.01\pi$.

From a calculator, $\sin(1.8°) \simeq 0.03141$ and $0.01\pi \simeq 0.03142$, so the two values are the same correct to four decimal places.

b $\cos(x) - 10.5x^2 = 0$

Let $f(x) = \cos(x) - 10.5x^2$

$$\begin{aligned} f(0) &= \cos(0) - 10.5(0)^2 \\ &= 1 \\ &> 0 \end{aligned}$$

$$\begin{aligned} f(0.4) &= \cos(0.4) - 10.5(0.4)^2 \\ &= -0.7589.. \\ &< 0 \end{aligned}$$

Therefore, the equation $\cos(x) - 10.5x^2 = 0$ has a root for which $0 \le x \le 0.4$.

As the root is small, let $\cos(x) = 1 - 0.5x^2$.

$$\begin{aligned} \therefore 1 - 0.5x^2 - 10.5x^2 &= 0 \\ \therefore 1 &= 11x^2 \\ \therefore x^2 &= \frac{1}{11} \\ \therefore x &= \pm\frac{1}{\sqrt{11}} \end{aligned}$$

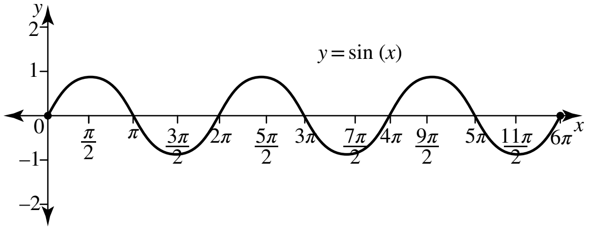
The positive root is $x = \frac{1}{\sqrt{11}} = 0.3015$, correct to four decimal places.

6 a As 0.5 is small, the quadratic approximation $\cos(x) \simeq 1 - 0.5x^2$ can be used to calculate the value of $\cos(0.5)$

$$\begin{aligned} \cos(0.5) &\simeq 1 - 0.5(0.5)^2 \\ &= 1 - 0.125 \\ &= 0.875 \end{aligned}$$

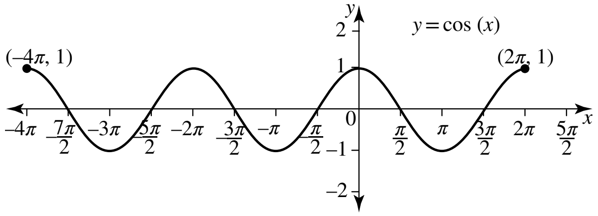
b As 5 is not a small number close to zero, the quadratic approximation for $\cos(5)$ is not applicable.

7 a The graph of $y = \sin(x)$, $0 \le x \le 6\pi$ covers three cycles. Sketch one cycle over $[0, 2\pi]$ and extend the pattern.



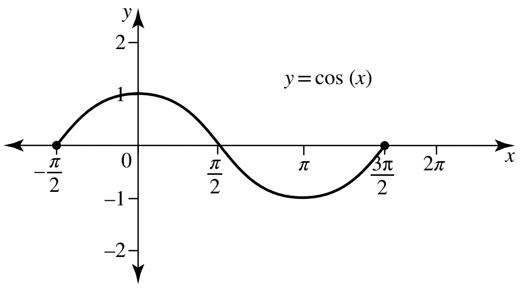
b $y = \cos(x)$, $-4\pi \le x \le 2\pi$

Sketch one cycle of the cosine graph over $[0, 2\pi]$ and extend the pattern to the left of the origin for two cycles.



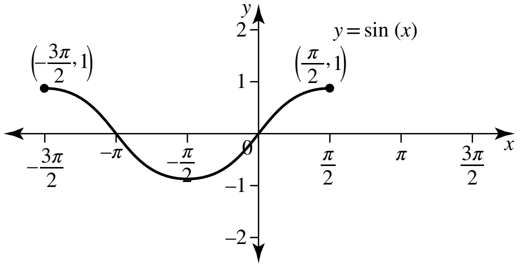
c $y = \cos(x)$, $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Draw part of the cycle of the basic cosine graph but stop it at $x = \frac{3\pi}{2}$. Extend the graph back to its equilibrium position at $x = -\frac{\pi}{2}$.



d $y = \sin(x)$, $-\frac{3\pi}{2} \le x \le \frac{\pi}{2}$

One cycle is required starting at maximum point at $x = -\frac{3\pi}{2}$ and finishing at maximum point at $x = \frac{\pi}{2}$.



8 a $f: [-4\pi, 0] \to R, f(x) = \sin(x)$

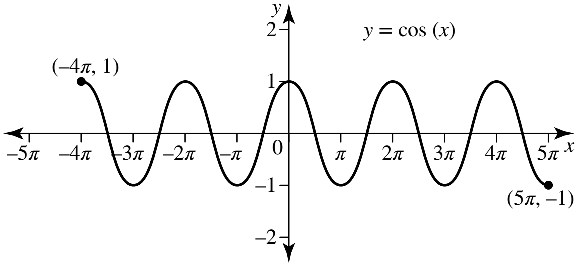
In every cycle of the sine graph, there is one maximum turning point. The domain $[-4\pi, 0]$ covers two cycles. Therefore, the function has 2 maximum turning points.

b $f: [0, 14\pi] \to R, f(x) = \cos(x)$

In every cycle of the cosine graph, there is one minimum turning point. The domain $[0, 14\pi]$ covers seven cycles. Therefore, the function has 7 minimum turning points.

c $y = \cos(x)$, $-4\pi \le x \le 5\pi$

Continue the cosine pattern to cover the domain $[-4\pi, 5\pi]$.



A cycle covers an interval of 2π, so there are $4\frac{1}{2}$ cycles over the given domain.

9 a $y = \cos(x)$, $0 \le x \le \frac{7\pi}{2}$

Over one period of 2π, the cosine graph has two x intercepts. The domain interval $\left[0, \frac{7\pi}{2}\right]$ covers $1\frac{3}{4}$ cycles, ending at equilibrium which is the x axis. So the graph has $2 + 2 = 4$ x intercepts.

b Over one period of 2π, the sine graph has three x intercepts but as the graph shape starts and stops at the equilibrium position, the x intercept at the end of one cycle is also the x intercept for the start of the next cycle.

Over the domain $[-2\pi, 4\pi]$ there will be three cycles with $3 + 2 + 2 = 7$ x intercepts.

c $y = \sin(x)$, $0 \le x \le 20\pi$

Over the domain $[0, 20\pi]$ there will be ten cycles with $3 + 9 \times 2 = 21$ x intercepts.

d $y = \cos(x)$, $\pi \le x \le 4\pi$

Over the domain $[\pi, 4\pi]$ there will be $1\frac{1}{2}$ cycles with $2 + 1 = 3$ x intercepts.

10 a $f:[0,a]\rightarrow R, f(x)=\cos(x)$

As the domain starts at $x=0$, and every interval of length 2π has two intersections with the x axis, for 10 such intersections the graph must cover between $4\frac{3}{4}$ and 5 cycles. The smallest value for a is $a=4\frac{3}{4}\pi=\dfrac{19\pi}{4}$.

b $f:[b,5\pi]\rightarrow R, f(x)=\sin(x)$

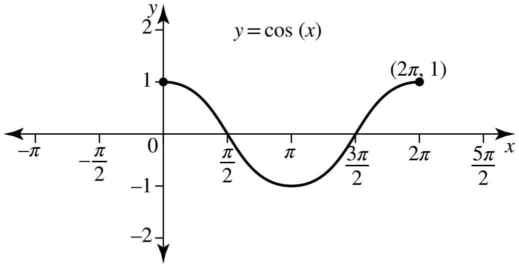
The sine graph has two turning points per cycle, one a maximum and one a minimum. The distance between a maximum and a minimum is π, half the period. Over the interval $[0,5\pi]$ there would be $2+2+1=5$ turning points. As there needs to be 7 turning points and the graph must start from equilibrium since $f(b)=0$, then one full cycle in the negative direction is required. Therefore $b=-2\pi$ and the function has domain $[-2\pi,5\pi]$,

c $f:[-c,c]\rightarrow R, f(x)=\sin(x)$

The graph covers 2.5 periods between $[-c,c]$ so between $[0,c]$ the graph covers 1.25 periods of 2π. This means

$$\begin{aligned}c&=1.25\times 2\pi\\&=2.5\pi\\&=\frac{5\pi}{2}\end{aligned}$$

11 a $y=\cos(x),\ 0\le x\le 2\pi$



$\cos(x)<0$ when the graph lies below the x axis. Hence, $\cos(x)<0$ for $\dfrac{\pi}{2}<x<\dfrac{3\pi}{2}$.

b If x is in either the second or third quadrants, then $\dfrac{\pi}{2}<x<\dfrac{3\pi}{2}$ and $y=\cos(x)<0$.

In the first quadrant, $0<x<\dfrac{\pi}{2}$ and the graph lies above the x axis showing $\cos(x)>0$; and in the fourth quadrant $\dfrac{3\pi}{2}<x<2\pi$ and $\cos(x)>0$.

Cosine is negative in the second and third quadrants and positive in the first and fourth quadrants. The graph is illustrating what the ‘CAST’ diagram said about the sign of cosine.

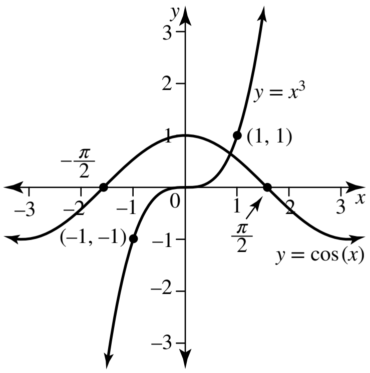
12 a i $\cos(x)-x^3=0$

$\therefore \cos(x)=x^3$

Sketch $y=\cos(x)$ and $y=x^3$ to determine the number of intersections.

$y=x^3$ has stationary point of inflection at the origin and passes through points $(-1,-1)$ and $(1,1)$.

$y=\cos(x)$ passes through $(0,1)$ and $\left(\pm\dfrac{\pi}{2},0\right)\simeq(\pm 1.57,0)$.



As there is one point of intersection of the graphs, there is one solution to the equation $\cos(x)-x^3=0$.

ii $4\cos(x)-x=0$

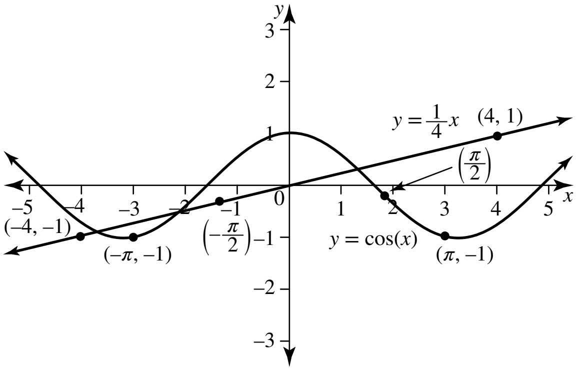
$\therefore 4\cos(x)=x$

$\therefore \cos(x)=\dfrac{1}{4}x$

Sketch $y=\cos(x)$ and $y=\dfrac{1}{4}x$ to determine the number of intersections.

The cosine graph has minimum turning points at $(-\pi,-1)$ and $(\pi,-1)$

$y=\dfrac{1}{4}x$ contains the points $(0,0)$ and $(\pm 4,\pm 1)$.



There are three points of intersection so there are three solutions to the equation.

iii $\sin(x)-x^2+2x-1=0$

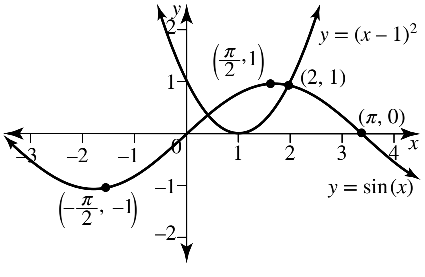
$\therefore \sin(x)=x^2-2x+1$

$\therefore \sin(x)=(x-1)^2$

Sketch $y=\sin(x)$ and $y=(x-1)^2$ to determine the number of intersections.

$y=(x-1)^2$ has a minimum turning point at $(1,0)$ and contains the points $(0,1)$ and $(2,1)$.

$y=\sin(x)$ has x intercepts at the origin and $(\pi,0)\simeq(3.14,0)$. There is a maximum turning point at $\left(\dfrac{\pi}{2},1\right)\simeq(1.57,1)$.



The graphs intersect twice so there are two solutions to the equation.

b The equation $\cos(x) - x^3 = 0$ has one solution which lies between $x = 0$ and $x = 1$.
Let $f(x) = \cos(x) - x^3$.
Looking at the graph in part **a i**, the cosine graph lies above the cubic graph at $x = 0$ and below it at $x = 1$.
$\therefore f(0) > 0$ and $\therefore f(1) < 0$.
Midpoint of $[0,1]$ is $x = 0.5$ and this is the first estimate.

$$\begin{aligned} f(0.5) &= \cos(0.5) - (0.5)^3 \\ &= 0.75.. \\ &> 0 \end{aligned}$$

The solution lies in the interval $[0.5,1]$.
The second estimate is $x = 0.75$

$$\begin{aligned} f(0.75) &= \cos(0.75) - (0.75)^3 \\ &= 0.3.. \\ &> 0 \end{aligned}$$

The solution lies in the interval $[0.75,1]$.
The third estimate is $x = 0.875$.

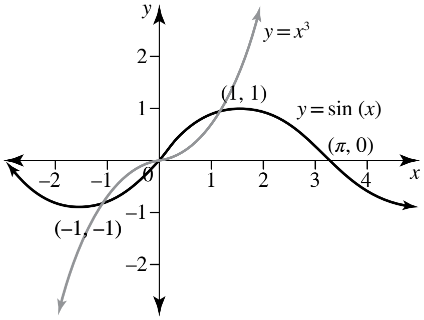
c $\sin(x) + ax^2 - 1 = 0$

$$\therefore \sin(x) = 1 - ax^2$$

i The graphs of $y = \sin(x)$ and the family of parabolas $y = 1 - ax^2, a \neq 0$ will always intersect if the parabolas are concave down. However, as the turning point of each of the parabolas is $(0,1)$, if the parabolas are concave up they will not intersect the sine curve.
No solutions if $-a < 0 \Rightarrow a > 0$.

ii If $a = 0$, the graphs of $y = \sin(x)$ and the horizontal line $y = 1$ must meet at every maximum turning point of the sine graph. Hence, there would be infinite solutions.

13 **a** Graphing $y = \sin(x)$ and $y = x^3$ shows they have 3 points of intersection, one of them being at the origin.



$x = 0$ is an exact solution of the equation $\sin(x) = x^3$.

b Let $f(x) = \sin(x) - x^3$
Near the point of intersection for which $x \in [0,1]$, the sine graph lies above the cubic graph before the point of intersection is reached and below the cubic graph after the point of intersection has been passed.
The sign of the function f changes sign from positive to negative.
The midpoint of $[0,1]$ is $x = 0.5$

$$\begin{aligned} f(0.5) &= \sin(0.5) - (0.5)^3 \\ &= 0.35.. \\ &> 0 \end{aligned}$$

So the solution lies in the interval $[0.5,1]$.

Continuing the procedure:

Midpoint	Value of $f(x)$ at midpoint	New interval
		$[0.5,1]$
$x = 0.75$	$f(0.75) = 0.25.. > 0$	$[0.75,1]$
$x = 0.875$	$f(0.875) = 0.09.. > 0$	$[0.875,1]$
$x = 0.9375$	$f(0.9375) = -0.017.. < 0$	$[0.875, 0.9375]$
$x = 0.90625$	$f(0.90625) = 0.04.. > 0$	$[0.90625, 0.9375]$

To one decimal point accuracy, the solution is $x = 0.9$.

c The graphs have a symmetry about the origin, so the other solution must be $x = -0.9$.

14 $\sin(x) \simeq x$ for small values of x.

a
$$\begin{aligned} 1° &= \frac{\pi^c}{180} \\ \sin(1°) &= \sin\left(\frac{\pi^c}{180}\right) \\ &= \sin\left(\frac{\pi}{180}\right) \\ &\simeq \frac{\pi}{180} \end{aligned}$$

b $\sin\left(\frac{\pi}{9}\right) \simeq \frac{\pi}{9}$

c
$$\begin{aligned} -2° &= -2 \times \frac{\pi^c}{180} \\ \therefore \sin(-2°) &= \sin\left(-\frac{\pi}{90}\right) \\ &\simeq -\frac{\pi}{90} \end{aligned}$$

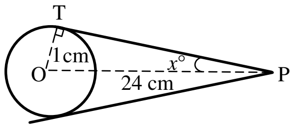
d Using the approximation, $\sin\left(\frac{\pi}{6}\right) \simeq \frac{\pi}{6}$.

$\therefore \sin\left(\frac{\pi}{6}\right) \simeq 0.5236$ correct to four decimal places.

The exact value of $\sin\left(\frac{\pi}{6}\right)$ is $\frac{1}{2} = 0.5$.

The discrepancy is due to the approximation $\sin(x) \simeq x$ becoming less accurate as the size of x increases.

e



The tangent PT is perpendicular to the radius OP.
Let $\theta = 2x$
In the right angled triangle OPT, $OP = 24$ cm, $OT = 1$ and $\angle OPT = x^{o}$.

$$\sin(x°) = \frac{1}{24}$$

Converting the angle measure to radians,

$$\sin\left(\frac{\pi x}{180}\right) = \frac{1}{24}$$

Using the linear approximation, $\sin\left(\frac{\pi x}{180}\right) \simeq \frac{\pi x}{180}$

$$\begin{aligned} \therefore \frac{\pi x}{180} &= \frac{1}{24} \\ \therefore x &= \frac{1}{24} \times \frac{180}{\pi} \\ \therefore x &= \frac{15}{2\pi} \end{aligned}$$

Hence $\theta = 2 \times \frac{15}{2\pi}$

$\therefore \theta = \dfrac{15}{\pi}$

As a decimal, $\dfrac{15}{\pi} = 4.77$, so the angle between the lines of sight is approximately 4.77°.

15 a $\cos\left(-\dfrac{1}{4}\right) \simeq 1 - \dfrac{1}{2}\left(-\dfrac{1}{4}\right)^2$

$\therefore \cos\left(-\dfrac{1}{4}\right) \simeq 1 - \dfrac{1}{2} \times \dfrac{1}{16}$

$= 1 - \dfrac{1}{32}$

$\therefore \cos\left(-\dfrac{1}{4}\right) \simeq \dfrac{31}{32}$

b $\cos(x) + 5x - 2 = 0$

Substitute $1 - \dfrac{1}{2}x^2$ for $\cos(x)$

$\therefore 1 - \dfrac{1}{2}x^2 + 5x - 2 = 0$

$\therefore 2 - x^2 + 10x - 4 = 0$

$\therefore x^2 - 10x + 2 = 0$

Completing the square,

$x^2 - 10x = -2$

$\therefore x^2 - 10x + 25 = -2 + 25$

$\therefore (x-5)^2 = 23$

$\therefore x - 5 = \pm\sqrt{23}$

$\therefore x = 5 \pm \sqrt{23}$

Since $\sqrt{16} < \sqrt{23} < \sqrt{25}$, $4 < \sqrt{23} < 5$

$\therefore 9 < 5 + \sqrt{23} < 10$ and $0 < 5 - \sqrt{23} < 1$

As $x = 5 + \sqrt{23}$ is not a value close to zero, reject $x = 5 + \sqrt{23}$

$\therefore x = 5 - \sqrt{23}$

16 a Let $f(x) = 4x\sin(x) - 1$

$f(0) = -1$

< 0

$f(0.6) = 4(0.6)\sin(0.6) - 1$

$= 0.355..$

> 0

Therefore there is a solution to the equation for which $0 \le x \le 0.6$.

Let $\sin(x) = x$

$\therefore 4x(x) - 1 = 0$

$\therefore 4x^2 = 1$

$\therefore x^2 = \dfrac{1}{4}$

$\therefore x = \pm\dfrac{1}{2}$

An estimate of the solution for which $0 \le x \le 0.6$ is $x = 0.5$.

b Rearranging the equation,

$4x\sin(x) - 1 = 0$

$\therefore 4x\sin(x) = 1$

$\therefore \sin(x) = \dfrac{1}{4x}$

The intersection of the graph of $y = \dfrac{1}{4x}$ together with the graph of $y = \sin(x)$ determine the number of solutions.

The function for which $y = \dfrac{1}{4x}$ is a hyperbola with the x axis as its horizontal asymptote. For positive values of x, as $x \to \infty, y \to 0$, so the hyperbola will intersect with the sine graph in an infinite number of points.

c The linear approximation only applies for small values of x around zero. The other positive solutions will all be greater than 1.

d Consider one cycle of $y = \sin(x)$ together with $y = \dfrac{1}{4x}$ for $x > 0$.

The sine graph remains above the x axis for $0 < x < \pi$ and the hyperbola remains above the x axis for $x > 0$.

If x is very close to zero, the hyperbola has a large y value and the sine graph has y value close to zero. The hyperbola is above the sine graph.

If $x = \dfrac{\pi}{2}$, the hyperbola has a y value, $y = \dfrac{1}{2\pi} \simeq 0.16$, while the sine graph has $y = 1$. The hyperbola is now below the sine graph so it has crossed the sine graph between $0 < x < \dfrac{\pi}{2}$. As the hyperbola is approaching the x axis it must intersect the sine graph again between $\dfrac{\pi}{2} < x < \pi$.

For $\pi < x < 2\pi$, the sine graph is below the x axis so the hyperbola will not intersect it.

A similar analysis can be made for $x < 0$ with the hyperbola intersecting the sine graph when it lies below the x axis.

Hence the hyperbola will intersect the sine graph twice over a period.

Over the domain $[-4\pi, 4\pi]$ the sine function covers 4 periods, so the hyperbola will intersect the sine curve $4 \times 2 = 8$ times.

17 a i $\sin(x) = 1 - x^2$ when $x = -1.4096$ or $x = 0.6367$ using equation solver.

ii Using the linear approximation, $x = 1 - x^2$. This equation has solutions $x = 0.618$ or $x = -1.618$.

For the solution closer to zero, the linear approximation gives $x = 0.618$ compared to $x = 0.6367$ from CAS. They agree at one decimal place accuracy.

b Comparing the values of $\sin(x)$ and x for $0 \le x \le 0.7$, some of the values are shown in the table.

x	0	0.1	0.5	0.6	0.7
$\sin(x)$	0	0.9983	0.47943	0.56464	0.64422

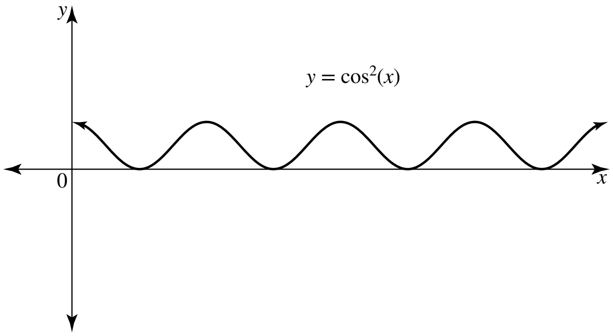
At $x = 0$, $\sin(x) = x$. The values of each are close until $x = 0.7$ where they no longer agree to one decimal place.

The approximation seems good for $0 \le x \le 0.5$ and reasonable for $x = 0.6$.

As $\sin(-x) = -\sin(x)$, a similar degree of closeness is obtained for negative values.

The approximation is 'reasonable' for $-0.6 \le x \le 0.6$.

18 $y = \cos^2(x)$ for $x \in [0, 4\pi]$



Where $\cos(x) < 0$, its value squared will be positive. The period of the cosine graph is 2π so the period of the graph of $y = \cos^2(x)$ is π.

Topic 10 — Trigonometric functions 2

Exercise 10.2 — Trigonometric equations

1 **a** $\sin(x)=\frac{1}{2},\ 0\le x\le 2\pi$

Quadrants 1 and 2, base $\frac{\pi}{6}$

$\therefore x=\frac{\pi}{6},\pi-\frac{\pi}{6}$

$\therefore x=\frac{\pi}{6},\frac{5\pi}{6}$

b $\sqrt{3}-2\cos(x)=0,\ 0\le x\le 2\pi$

Rearranging the equation gives $\cos(x)=\frac{\sqrt{3}}{2}$

Quadrants 1 and 4, base $\frac{\pi}{6}$

$\therefore x=\frac{\pi}{6},2\pi-\frac{\pi}{6}$

$\therefore x=\frac{\pi}{6},\frac{11\pi}{6}$

c $4+4\tan(x)=0,\ -2\pi\le x\le 2\pi$

Rearranging the equation gives $\tan(x)=-1$

Quadrants 2 and 4, base $\frac{\pi}{4}$, one positive and one negative rotation

$\therefore x=\pi-\frac{\pi}{4},2\pi-\frac{\pi}{4}$ or $-\frac{\pi}{4},-\pi-\frac{\pi}{4}$

$\therefore x=\frac{3\pi}{4},\frac{7\pi}{4},-\frac{\pi}{4},-\frac{5\pi}{4}$

2 $\cos(\theta°)=-\frac{1}{2},\ -180°\le\theta°\le 540°$

a Solutions lie in quadrants 2 and 3.

The negative rotation 0° to −180° picks up one solution in quadrant 3 while the positive rotation 0° to 180° picks up one solution in quadrant 2.

Therefore there are 2 solutions

b Base is 60°.

$\therefore \theta°=-180°+60°$ or $180°-60°$

$\therefore \theta°=-120°$ or $120°$

3 **a** $1-\sin(x)=0,\ -4\pi\le x\le 4\pi$

$\therefore \sin(x)=1$

Boundary of first and second quadrants where $\sin\left(\frac{\pi}{2}\right)=1$

For two positive and two negative rotations,

$\therefore x=\frac{\pi}{2},2\pi+\frac{\pi}{2}$ or $-\frac{3\pi}{2},-2\pi-\frac{3\pi}{2}$

$\therefore x=\frac{\pi}{2},\frac{5\pi}{2},-\frac{3\pi}{2},-\frac{7\pi}{2}$

b $\tan(x)=0.75,\ 0\le x\le 4\pi$

Quadrants 1 and 3. Base is $\tan^{-1}(0.75)=0.644$ using radian mode on calculator

Two complete positive rotations give

$x=0.644,\pi+0.644$ or $2\pi+0.644,3\pi+0.644$

$=0.64,3.79,6.93,10.07$

c $4\cos(x°)+1=0,\ -180°\le x°\le 180°$

$\therefore \cos(x°)=-\frac{1}{4}$

Quadrants 2 and 3, base $\cos^{-1}\left(\frac{1}{4}\right)=75.52°$ using degree mode

$\therefore x°=-180°+75.52°$ or $180°-75.52°$

$\therefore x°=\pm 104.5°$

Hence, $x°=\pm 104.5$

4 $f:[0,2]\to R, f(x)=\cos(\pi x)$

a $f(0)=\cos(0)$

$=1$

b $f(x)=0$

$\therefore \cos(\pi x)=0$

Boundary value between first and second quadrants and between third and fourth quadrants

$\therefore (\pi x)=\frac{\pi}{2}$ or $\frac{3\pi}{2}$

Dividing by π gives $x=\frac{1}{2},\frac{3}{2}$.

Solution set is $\left\{\frac{1}{2},\frac{3}{2}\right\}$.

5 **a** $\sqrt{3}\sin(x)=3\cos(x)$

$\therefore \frac{\sqrt{3}\sin(x)}{\cos(x)}=3$

$\therefore \sqrt{3}\tan(x)=3$

$\therefore \tan(x)=\frac{3}{\sqrt{3}}$

$\therefore \tan(x)=\sqrt{3}$

Quadrants 1 and 3, base $\frac{\pi}{3}$

$\therefore x=\frac{\pi}{3},\pi+\frac{\pi}{3}$

$\therefore x=\frac{\pi}{3},\frac{4\pi}{3}$

b $\sin^2(x)-5\sin(x)+4=0$

Let $a=\sin(x)$

$\therefore a^2-5a+4=0$

Solving this quadratic equation gives

$(a-4)(a-1)=0$

$\therefore a=4$ or $a=1$

Hence $\sin(x)=4$ or $\sin(x)=1$

Reject $\sin(x)=4$ since $-1\le\sin(x)\le 1$

Therefore $\sin(x)=1$

Boundary between first and second quadrants

$\therefore x=\frac{\pi}{2}$

6 $\cos^2(x)=\frac{3}{4},\ 0\le x\le 2\pi$

Taking square roots of both sides gives

$\cos(x)=\pm\sqrt{\frac{3}{4}}$

$\therefore \cos(x)=\pm\frac{\sqrt{3}}{2}$

There is a solution in each quadrant since cos can be positive or negative. Base is $\frac{\pi}{6}$

$\therefore x=\frac{\pi}{6},\pi-\frac{\pi}{6},\pi+\frac{\pi}{6},2\pi-\frac{\pi}{6}$

$\therefore x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

7 a $\sin(2x) = \frac{1}{\sqrt{2}}, 0 \le x \le 2\pi$

Since $0 \le x \le 2\pi$, $0 \le 2x \le 4\pi$

Sine is positive in quadrants 1 and 2, base is $\frac{\pi}{4}$

$\therefore 2x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi - \frac{\pi}{4}$

$\therefore 2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

$\therefore x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

b $\cos\left(2x + \frac{\pi}{6}\right) = 0,\ 0 \le x \le \frac{3\pi}{2}$

Let $\theta = 2x + \frac{\pi}{6}$

Since $0 \le x \le \frac{3\pi}{2}$ then $\frac{\pi}{6} \le \theta \le 3\pi + \frac{\pi}{6}$

$\therefore \cos(\theta) = 0,\ \frac{\pi}{6} \le \theta \le \frac{19\pi}{6}$

Boundary value at points $(0,-1)$ and $(0,1)$ since $\cos\left(\frac{\pi}{2}\right) = 0 = \cos\left(\frac{3\pi}{2}\right)$.

As $\frac{\pi}{6} \le \theta \le \frac{19\pi}{6}$, solutions for θ are:

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\therefore 2x + \frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\therefore 2x = \frac{2\pi}{6}, \frac{8\pi}{6}, \frac{14\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$

8 $\sin\left(\frac{x}{2}\right) = \sqrt{3}\cos\left(\frac{x}{2}\right), 0 \le x \le 2\pi$

$\therefore \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \sqrt{3}$

$\therefore \tan\left(\frac{x}{2}\right) = \sqrt{3}$

As $0 \le x \le 2\pi$ then $0 \le \frac{x}{2} \le \pi$

tan is positive in quadrants 1 and 3. Base is $\frac{\pi}{3}$.

Since $0 \le \frac{x}{2} \le \pi$ only the first quadrant value is reached

$\therefore \frac{x}{2} = \frac{\pi}{3}$

$\therefore x = \frac{2\pi}{3}$

9 $0 \le x \le 2\pi$

a $\cos(x) = \frac{1}{\sqrt{2}}$

cosine is positive in quadrants 1 and 4; since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, base for the solutions is $\frac{\pi}{4}$.

$\therefore x = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{7\pi}{4}$

b $\sin(x) = -\frac{1}{\sqrt{2}}$

sine is negative in quadrants 3 and 4; since $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, base for the solutions is $\frac{\pi}{4}$.

$\therefore x = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{5\pi}{4}, \frac{7\pi}{4}$

c $\tan(x) = -\frac{1}{\sqrt{3}}$

tangent is negative in quadrants 2 and 4; since $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, base for the solutions is $\frac{\pi}{6}$.

$\therefore x = \pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore x = \frac{5\pi}{6}, \frac{11\pi}{6}$

d $2\sqrt{3}\cos(x) + 3 = 0$

$\therefore 2\sqrt{3}\cos(x) = -3$

$\therefore \cos(x) = -\frac{3}{2\sqrt{3}}$

$\therefore \cos(x) = -\frac{\sqrt{3}}{2}$

Quadrants 2 and 3, base $\frac{\pi}{6}$

$\therefore x = \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}$

$\therefore x = \frac{5\pi}{6}, \frac{7\pi}{6}$

e $4 - 8\sin(x) = 0$

$\therefore 4 = 8\sin(x)$

$\therefore \sin(x) = \frac{4}{8}$

$\therefore \sin(x) = \frac{1}{2}$

Quadrants 1 and 2, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

f $2\sqrt{2}\tan(x) = \sqrt{24}$

$\therefore 2\sqrt{2}\tan(x) = 2\sqrt{6}$

$\therefore \tan(x) = \frac{\sqrt{6}}{\sqrt{2}}$

$\therefore \tan(x) = \sqrt{3}$

Quadrants 1 and 3, base $\frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$

10 $0° \le a° \le 360°$

a $\sqrt{3} + 2\sin(a°) = 0$

$\therefore 2\sin(a°) = -\sqrt{3}$

$\therefore \sin(a°) = -\frac{\sqrt{3}}{2}$

Quadrants 3 and 4, base 60°

$\therefore a° = 180° + 60°, 360° - 60°$

$\therefore a° = 240°, 300°$

b $\tan(a°) = 1$

Quadrants 1 and 3, base 45°

$\therefore a° = 45°, 180° + 45°$

$\therefore a° = 45°, 225°$

c $6 + 8\cos(a°) = 2$

$\therefore 8\cos(a°) = -4$

$\therefore \cos(a°) = -\frac{4}{8}$

$\therefore \cos(a°) = -\frac{1}{2}$

Quadrants 2 and 3, base 60°

$\therefore a° = 180° - 60°, 180° + 60°$

$\therefore a° = 120°, 240°$

d $4(2 + \sin(a°)) = 11 - 2\sin(a°)$

$\therefore 8 + 4\sin(a°) = 11 - 2\sin(a°)$

$\therefore 6\sin(a°) = 3$

$\therefore \sin(a°) = \frac{3}{6}$

$\therefore \sin(a°) = \frac{1}{2}$

Quadrants 1 and 2, base 30°

$\therefore a° = 30°, 180° - 30°$

$\therefore a° = 30°, 150°$

11 $t \in [-\pi, 4\pi]$. Solutions are generated by two complete anticlockwise rotations and half a clockwise rotation.

a $\tan(t) = 0$ at the boundary points $(1,0), (-1,0)$ since $\frac{y}{x} = 0$ for these points.

$\therefore t = 0, \pi, 2\pi, 3\pi, 4\pi$ or $t = -\pi$

b $\cos(t) = 0$ at the boundary points $(0,1), (0,-1)$ since $x = 0$ for these points.

$\therefore t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ or $t = -\frac{\pi}{2}$

c $\sin(t) = -1$ at the boundary point $(0,-1)$ since $y = -1$ at this point.

$\therefore t = \frac{3\pi}{2}, \frac{7\pi}{2}$ or $t = -\frac{\pi}{2}$

d $\cos(t) = 1$ at the boundary point $(1,0)$ since $x = 1$ at this point.

$\therefore t = 0, 2\pi, 4\pi$

e $\sin(t) = 1$ at the boundary point $(0,1)$ since $y = 1$ at this point.

$\therefore t = \frac{\pi}{2}, \frac{5\pi}{2}$

f $\tan(t) = 1$. This is not a boundary value.

Quadrants 1 and 3, base $\frac{\pi}{4}$

$\therefore t = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}$ or $t = -\pi + \frac{\pi}{4}$

$\therefore t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$ or $t = -\frac{3\pi}{4}$

12 **a** $2 + 3\cos(\theta) = 0, \quad 0 \le \theta \le 2\pi$

$\therefore \cos(\theta) = -\frac{2}{3}$

Quadrants 2 and 3, base is $\cos^{-1}\left(\frac{2}{3}\right) = 0.841.$

$\therefore \theta = \pi - 0.841, \pi + 0.841$

$\therefore \theta = 2.30, 3.98$

b $\tan(\theta) = \frac{1}{\sqrt{2}}, \quad -2\pi \le \theta \le 3\pi$

Quadrants 1 and 3, base is $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 0.615$. Solutions are generated by one and a half anticlockwise rotations and one clockwise rotation.

$\therefore \theta = 0.615, \pi + 0.615, 2\pi + 0.615$ or $\theta = -\pi + 0.615, -2\pi + 0.615$

$\therefore \theta = 0.62, 3.76, 6.90$ or $\theta = -2.53, -5.67$

c $5\sin(\theta°) + 4 = 0, \quad -270° \le \theta° \le 270°$

$\therefore \sin(\theta°) = -\frac{4}{5}$

Quadrants 3 and 4, base in degrees is $\sin^{-1}\left(\frac{4}{5}\right) = 53.130°$,

Solutions are generated by rotating $0° \to 270°$ anticlockwise and $0° \to -270°$ clockwise.

$\therefore \theta° = 180° + 53.13°$ or $\theta° = -53.13°, -180° + 53.13°$

$\therefore \theta° = 233.13°$ or $\theta° = -53.13°, -126.87°$

$\therefore \theta = -126.87, -53.13, 233.13$

d $\cos^2(\theta°) = 0.04, \quad 0° \le \theta° \le 360°$

$\therefore \cos(\theta°) = \pm\sqrt{0.04}$

$\therefore \cos(\theta°) = \pm 0.2$

Quadrants 1 and 4 and quadrants 2 and 3. Base, in degrees, is $\cos^{-1}(0.2) = 78.463°$.

$\therefore \theta° = 78.463°, 180° - 78.463°, 180° + 78.463°, 360° - 78.463°$

$\therefore \theta° = 78.46°, 101.54°, 258.46°, 281.54°$

$\therefore \theta = 78.46, 101.54, 258.46, 281.54$

13 **a** $4\sin(a) + 3 = 5, \quad -2\pi < a < 0$

$\therefore 4\sin(a) = 2$

$\therefore \sin(a) = \frac{1}{2}$

Quadrants 1 and 2, base $\frac{\pi}{6}$. Solutions are generated by a clockwise rotation.

$\therefore a = -\pi - \frac{\pi}{6}, -2\pi + \frac{\pi}{6}$

$\therefore a = -\frac{7\pi}{6}, -\frac{11\pi}{6}$

b $6\tan(b) - 1 = 11, \quad -\frac{\pi}{2} < b < 0$

$\therefore 6\tan(b) = 12$

$\therefore \tan(b) = 2$

Quadrants 1 and 3, base is $\tan^{-1}(2)$.

However, $b \in \left(-\frac{\pi}{2}, 0\right)$ and as there is no solution in quadrant 4, there is no solution to the equation.

c $8\cos(c) - 7 = 1, \quad -\frac{9\pi}{2} < c < 0$

$\therefore 8\cos(c) = 8$

$\therefore \cos(c) = 1$

Boundary value solution at the point $(1,0)$. Solutions are generated by a clockwise rotation of 2.5 revolutions.

$\therefore c = 0, -2\pi, -4\pi$. However, $-\frac{9\pi}{2} < c < 0$ so $c = 0$ is not a solution.

$\therefore c = -4\pi, -2\pi$.

d $\frac{9}{\tan(d)} - 9 = 0, \quad 0 < d \le \frac{5\pi}{12}$

$\therefore \frac{9}{\tan(d)} = 9$

$\therefore 9 = 9\tan(d)$

$\therefore \tan(d) = 1$

Quadrants 1 and 3, base is $\frac{\pi}{4}$.

As $\frac{\pi}{4} = \frac{3\pi}{12}$ and $0 < d \le \frac{5\pi}{12}$, there is only one solution $d = \frac{\pi}{4}$.

e $2\cos(e) = 1, \quad -\frac{\pi}{6} \le e \le \frac{13\pi}{6}$

$\therefore \cos(e) = \frac{1}{2}$

Quadrants 1 and 4, base is $\frac{\pi}{3}$.

Positive solutions:

$e = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}$. However, $2\pi + \frac{\pi}{3} = \frac{7\pi}{3} = \frac{14\pi}{6}$, so reject this value.

$\therefore e = \frac{\pi}{3}, \frac{5\pi}{3}$

Negative solutions:

$e = -\frac{\pi}{3}$. However, $-\frac{\pi}{3} = -\frac{2\pi}{6}$, so reject this value.

Answer is $e = \frac{\pi}{3}, \frac{5\pi}{3}$.

f $\sin(f°) = -\cos(150°), \quad -360° \le f° \le 360°$

Evaluate $\cos(150°)$.

$$\begin{aligned}\cos(150°) &= -\cos(30°)\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

The equation becomes

$\therefore \sin(f°) = -\left(-\frac{\sqrt{3}}{2}\right)$

$\therefore \sin(f°) = \frac{\sqrt{3}}{2}$

Quadrants 1 and 2, base is 60°, Solutions are generated by one anticlockwise rotation and one clockwise rotation.

$\therefore f° = 60°, 180° - 60°$ or $f° = -180° - 60°, -360° + 60°$

$\therefore f° = 60°, 120°, -240°, -300°$

$\therefore f = -300, -240, 120, 60.$

14 a $\cos(x) = -0.3, \quad -\frac{3\pi}{2} \le x \le \frac{5\pi}{2}$

Solutions lie in quadrants 2 and 3 and are generated by rotations anticlockwise from 0 to $\frac{5\pi}{2}$ and clockwise from 0 to $-\frac{3\pi}{2}$. The anticlockwise rotation passes through each of quadrants 2 and 3 once, generating two positive solutions. The clockwise rotation also passes though each of quadrants 2 and 3 once to generate two negative solutions. The number of solutions is 4.

b $\sin(x) = 0.2, \quad 0 \le x \le 2\pi$

Quadrants 1 and 2, base $\sin^{-1}(0.2)$.

$\therefore x = \sin^{-1}(0.2), \pi - \sin^{-1}(0.2)$

The sum of the two solutions is

$\sin^{-1}(0.2) + \left(\pi - \sin^{-1}(0.2)\right) = \pi.$

c $\tan(x) = c, \quad 0 \le x \le 3\pi$

Given that $x = 0.4$ is one solution, then $\tan(0.4) = c$.

As 0.4 lies in the first quadrant, $c > 0$.

Solutions to the equation for which tangent is positive must lie in quadrants 1 and 3 with base 0.4.

$\therefore x = 0.4, \pi + 0.4, 2\pi + 0.4$

The other solutions are $x = \pi + 0.4, 2\pi + 0.4$.

d i $y = -3x$ is the equation of a line with gradient −3.

Let $\theta°$ be the angle of inclination of the line to the positive direction of the x axis.

This angle is the solution to the equation $\tan(\theta°) = -3, \ 0° < \theta° < 180°$.

ii $y = \sqrt{3}x$ is the equation of a line with gradient $\sqrt{3}$.

Its angle of inclination with the positive direction of the x axis is the solution to the equation $\tan(\theta°) = \sqrt{3}, \ 0° < \theta° < 180°$.

e i $\tan(\theta°) = -3, \ 0° < \theta° < 180°$

As tangent is negative the solution is in quadrant 2. The base, in degrees, is $\tan^{-1}(3)$.

$\therefore \theta° = 180° - \tan^{-1}(3)$

ii $\tan(\theta°) = \sqrt{3}, \ 0° < \theta° < 180°$

As tangent is positive the solution is in quadrant 1. The base is 60°.

$\therefore \theta° = 60°$

f $\tan(\theta) = m, \ 0° < \theta < 180°$

If $m > 0$, the solution is in quadrant 1 and the base is $\tan^{-1}(m)$.

If $m < 0$, the solution is in quadrant 2. The base is $\tan^{-1}(|m|)$.

The second quadrant solution is $\theta = 180° - \tan^{-1}(|m|)$. This is the rule used in earlier calculations.

15 a $f: [0, 2\pi] \to R, f(x) = a\sin(x)$

$f\left(\frac{\pi}{6}\right) = 4$

$$\begin{aligned}\therefore a\sin\left(\frac{\pi}{6}\right) &= 4\\ \therefore a \times \frac{1}{2} &= 4\\ \therefore a &= 8\end{aligned}$$

b $f(x) = 8\sin(x)$

i $f(x) = 3$

$$\begin{aligned}\therefore 8\sin(x) &= 3\\ \therefore \sin(x) &= \frac{3}{8}\end{aligned}$$

Quadrants 1 and 2, base $\sin^{-1}\left(\frac{3}{8}\right) = 0.384$

$\therefore x = 0.384, \pi - 0.384$

$\therefore x = 0.38, 2.76$

ii $f(x) = 8$

$$\begin{aligned}\therefore 8\sin(x) &= 8\\ \therefore \sin(x) &= 1\end{aligned}$$

Boundary value at point $(0, 1)$

$\therefore x = \frac{\pi}{2}$

Correct to two decimal places, $x = 1.57$.

iii $f(x) = 10$

$$\begin{aligned}\therefore 8\sin(x) &= 10\\ \therefore \sin(x) &= \frac{10}{8}\\ \therefore \sin(x) &= \frac{5}{4} > 1\end{aligned}$$

Since $-1 \le \sin(x) \le 1$, there is no solution.

16 $0 \le x \le 2\pi$

a $\sin(x) = \sqrt{3}\cos(x)$

$\therefore \dfrac{\sin(x)}{\cos(x)} = \sqrt{3}$

$\therefore \tan(x) = \sqrt{3}$

Quadrants 1 and 3, base $\dfrac{\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}$

b $\sin(x) = -\dfrac{\cos(x)}{\sqrt{3}}$

$\therefore \dfrac{\sin(x)}{\cos(x)} = -\dfrac{1}{\sqrt{3}}$

$\therefore \tan(x) = -\dfrac{1}{\sqrt{3}}$

Quadrants 2 and 4, base $\dfrac{\pi}{6}$

$\therefore x = \pi - \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

$\therefore x = \dfrac{5\pi}{6}, \dfrac{11\pi}{6}$

c $\sin(2x) + \cos(2x) = 0$

$\therefore \sin(2x) = -\cos(2x)$

$\therefore \dfrac{\sin(2x)}{\cos(2x)} = -1$

$\therefore \tan(2x) = -1$

Quadrants 2 and 4, base $\dfrac{\pi}{4}$. Since $0 \le x \le 2\pi$, then $0 \le 2x \le 4\pi$.

$\therefore 2x = \pi - \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}, 3\pi - \dfrac{\pi}{4}, 4\pi - \dfrac{\pi}{4}$

$\therefore 2x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \dfrac{15\pi}{4}$

$\therefore x = \dfrac{3\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}, \dfrac{15\pi}{8}$

d $\dfrac{3\sin(x)}{8} = \dfrac{\cos(x)}{2}$

$\therefore \sin(x) = \dfrac{\cos(x)}{2} \times \dfrac{8}{3}$

$\therefore \sin(x) = \dfrac{4\cos(x)}{3}$

$\therefore \dfrac{\sin(x)}{\cos(x)} = \dfrac{4}{3}$

$\therefore \tan(x) = \dfrac{4}{3}$

Quadrants 1 and 3, base $\tan^{-1}\left(\dfrac{4}{3}\right) = 0.927$

$\therefore x = 0.927, \pi + 0.927$

$\therefore x = 0.93, 4.07$

e $\sin^2(x) = \cos^2(x)$

$\therefore \dfrac{\sin^2(x)}{\cos^2(x)} = 1$

$\therefore \left(\dfrac{\sin(x)}{\cos(x)}\right)^2 = 1$

$\therefore (\tan(x))^2 = 1$

$\therefore \tan(x) = \pm 1$

Quadrants 1, 2, 3 and 4, base $\dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

f $\cos(x)(\cos(x) - \sin(x)) = 0$

Using the null factor law,

$\cos(x) = 0$ or $\cos(x) - \sin(x) = 0$

If $\cos(x) = 0$, boundary solutions occur at points $(0, 1)$ and $(0, -1)$.

$\therefore x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

If $\cos(x) - \sin(x) = 0$ then

$\cos(x) = \sin(x)$

$\therefore 1 = \dfrac{\sin(x)}{\cos(x)}$

$\therefore \tan(x) = 1$

Quadrants 1 and 3, base $\dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

Answers are $x = \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}$

17 $0 \le x \le 2\pi$

a $\sin^2(x) = \dfrac{1}{2}$

$\therefore \sin(x) = \pm\dfrac{1}{\sqrt{2}}$

Quadrants 1, 2, 3 and 4, base $\dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

b $2\cos^2(x) + 3\cos(x) = 0$

$\cos(x)$ is a common factor

$\therefore \cos(x)[2\cos(x) + 3] = 0$

$\therefore \cos(x) = 0$ or $2\cos(x) + 3 = 0$

$\therefore \cos(x) = 0$ or $\cos(x) = -\dfrac{3}{2}$

Reject $\cos(x) = -\dfrac{3}{2}$ since $-1 \le \cos(x) \le 1$

$\therefore \cos(x) = 0$

Boundary solutions occur at points $(0, 1)$ and $(0, -1)$.

$\therefore x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

c $2\sin^2(x) - \sin(x) - 1 = 0$

Let $a = \sin(x)$

$\therefore 2a^2 - a - 1 = 0$

$\therefore (2a + 1)(a - 1) = 0$

$\therefore a = -\dfrac{1}{2}, a = 1$

$\therefore \sin(x) = -\dfrac{1}{2}$ or $\sin(x) = 1$

For $\sin(x) = -\dfrac{1}{2}$, solutions lie in quadrants 3 and 4 with base $\dfrac{\pi}{6}$

$\therefore x = \pi + \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

$\therefore x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

For $\sin(x)=1$, boundary solution occurs at $(0,1)$

$\therefore x=\frac{\pi}{2}$

Answers are $x=\frac{\pi}{2},\frac{7\pi}{6},\frac{11\pi}{6}$.

d $\tan^2(x)+2\tan(x)-3=0$

Let $a=\tan(x)$

$\therefore a^2+2a-3=0$

$\therefore (a+3)(a-1)=0$

$\therefore a=-3, a=1$

$\therefore \tan(x)=-3$ or $\tan(x)=1$

For $\tan(x)=-3$, solutions lie in quadrants 2 and 4 with base $\tan^{-1}(3)=1.25$.

$\therefore x=\pi-1.25, 2\pi-1.25$

$\therefore x=1.89, 5.03$

For $\tan(x)=1$: quadrants 1 and 3, base $\frac{\pi}{4}$

$\therefore x=\frac{\pi}{4},\pi+\frac{\pi}{4}$

$\therefore x=\frac{\pi}{4},\frac{5\pi}{4}$

Answers are $x=\frac{\pi}{4},1.89,\frac{5\pi}{4},5.03$

e $\sin^2(x)+2\sin(x)+1=0$

$\therefore (\sin(x)+1)^2=0$

$\therefore \sin(x)+1=0$

$\therefore \sin(x)=-1$

Boundary solution at $(0,-1)$

$\therefore x=\frac{3\pi}{2}$

f $\cos^2(x)-9=0$

$\therefore (\cos(x)-3)(\cos(x)+3)=0$

$\therefore \cos(x)=3$ or $\cos(x)=-3$

There are no solutions possible as $\cos(x)\in[-1,1]$.

18 $0\le\theta\le 2\pi$

a $\sqrt{3}\tan(3\theta)+1=0$

$\therefore \tan(3\theta)=-\frac{1}{\sqrt{3}},\quad 0\le 3\theta\le 6\pi$

Quadrants 2 and 4, base $\frac{\pi}{6}$

$\therefore 3\theta=\pi-\frac{\pi}{6},2\pi-\frac{\pi}{6},3\pi-\frac{\pi}{6},4\pi-\frac{\pi}{6},5\pi-\frac{\pi}{6},6\pi-\frac{\pi}{6}$

$\therefore 3\theta=\frac{5\pi}{6},\frac{11\pi}{6},\frac{17\pi}{6},\frac{23\pi}{6},\frac{29\pi}{6},\frac{35\pi}{6}$

$\therefore \theta=\frac{5\pi}{18},\frac{11\pi}{18},\frac{17\pi}{18},\frac{23\pi}{18},\frac{29\pi}{18},\frac{35\pi}{18}$

b $2\sqrt{3}\sin\left(\frac{3\theta}{2}\right)-3=0$

$\therefore 2\sqrt{3}\sin\left(\frac{3\theta}{2}\right)=3$

$\therefore \sin\left(\frac{3\theta}{2}\right)=\frac{3}{2\sqrt{3}}$

$\therefore \sin\left(\frac{3\theta}{2}\right)=\frac{\sqrt{3}}{2},\quad 0\le\frac{3\theta}{2}\le 3\pi$

Quadrants 1 and 2, base $\frac{\pi}{3}$

$\therefore \frac{3\theta}{2}=\frac{\pi}{3},\frac{2\pi}{3},\frac{7\pi}{3},\frac{8\pi}{3}$

$\therefore 3\theta=\frac{2\pi}{3},\frac{4\pi}{3},\frac{14\pi}{3},\frac{16\pi}{3}$

$\therefore \theta=\frac{2\pi}{9},\frac{4\pi}{9},\frac{14\pi}{9},\frac{16\pi}{9}$

c $4\cos^2(-\theta)=2$

As $\cos(-\theta)=\cos(\theta)$, the equation becomes $4\cos^2(\theta)=2$

$\therefore \cos^2(\theta)=\frac{1}{2}$

$\therefore \cos(\theta)=\pm\frac{1}{\sqrt{2}}$

Quadrants 1, 2, 3 and 4, base $\frac{\pi}{4}$

$\therefore x=\frac{\pi}{4},\pi-\frac{\pi}{4},\pi+\frac{\pi}{4},2\pi-\frac{\pi}{4}$

$\therefore x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

d $\sin\left(2\theta+\frac{\pi}{4}\right)=0$

Boundary solutions at $(1,0)$ and $(-1,0)$.

As $0\le\theta\le 2\pi$, then $0\le 2\theta\le 4\pi$

$\therefore 0+\frac{\pi}{4}\le 2\theta+\frac{\pi}{4}\le 4\pi+\frac{\pi}{4}$

$\therefore \frac{\pi}{4}\le 2\theta+\frac{\pi}{4}\le 4\pi+\frac{\pi}{4}$

Solutions in this interval are

$2\theta+\frac{\pi}{4}=\pi,2\pi,3\pi,4\pi$

$\therefore 2\theta=\pi-\frac{\pi}{4},2\pi-\frac{\pi}{4},3\pi-\frac{\pi}{4},4\pi-\frac{\pi}{4}$

$=\frac{3\pi}{4},\frac{7\pi}{4},\frac{11\pi}{4},\frac{15\pi}{4}$

$\therefore \theta=\frac{3\pi}{8},\frac{7\pi}{8},\frac{11\pi}{8},\frac{15\pi}{8}$

19 **a** $5\cos(x)=2,\ 0\le x\le 2\pi$

i For the exact solutions, use the calculator in Standard and rad modes.

In the main menu enter the equation as $5\cos(x)=2|0\le x\le 2\pi$. Highlight $5\cos(x)=2$ and drop it into Equation/Inequality.

The solution given is

$x=\cos^{-1}\left(\frac{2}{5}\right), x=-\cos^{-1}\left(\frac{2}{5}\right)+2\pi$

ii For decimal solutions, set the calculator to Decimal and Rad modes and re-calculate. This gives, correct to two decimal places, $x=1.16, x=5.12$.

b Enter $5\sin(2x+3)=4|0\le x\le 2\pi$ with the calculator set on Decimal and Rad modes. Highlight $5\sin(2x+3)=4$ and drop in Equation/Inequality to obtain $x=2.11, x=2.75, x=5.25, x=5.89$.

c With the calculator on Standard and Rad modes, enter the equation as $\tan(x)=1$ without a domain condition. Highlight and drop into Equation/Inequality to obtain the general solution as $x=\pi constn(1)+\frac{\pi}{4}$,

This means $x=n\pi+\frac{\pi}{4}$ where the constant $n\in Z$.

For the first three positive solutions, set n equal to $0,1,2$.

Let $n=0$,

$\therefore x=0\pi+\frac{\pi}{4}=\frac{\pi}{4}$

Let $n=1$,

$\therefore x=1\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

Let $n=2$,

$\therefore x=2\pi+\frac{\pi}{4}=\frac{9\pi}{4}$

The first three positive solutions are $x=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4}$.

20 a Set the calculator to Standard and Deg modes and enter sin(75).

The value given, $\frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$, is the exact value of $\sin(75^{\circ})$.

b $\sin(A^{\circ})=\frac{\sqrt{6}+\sqrt{2}}{4},\ -270\le A\le 270$

As the question says 'hence solve', the result in part a must be used in solving the equation.

Expressing $\sqrt{6}+\sqrt{2}=\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)$, the equation becomes

$$\sin(A^{\circ})=\frac{\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)}{4}$$

From part **a**, $A^{\circ}=75^{\circ}$ is a solution and forms the base for other solutions.

Solutions lie in quadrants 1 and 2 since $\sin(A^{\circ})>0$.

$\therefore A^{\circ}=75^{\circ},180^{\circ}-75^{\circ}$ or $A^{\circ}=-180^{\circ}-75^{\circ}$

$\therefore A^{\circ}=75^{\circ},105^{\circ}$ or -255°

$\therefore A^{\circ}=-255^{\circ},75^{\circ},105^{\circ}$

Exercise 10.3 — Transformations of sine and cosine graphs

1 a $y=2\sin(x)+1,\ 0\le x\le 2\pi$

Period 2π, amplitude 1, equilibrium position $y=1$, oscillates between $[-1,3]$.

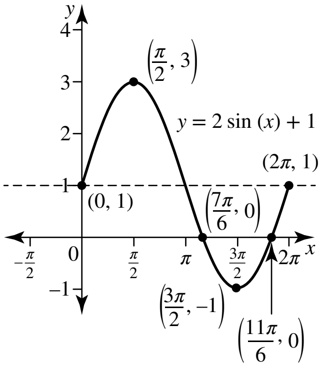
x intercepts: $2\sin(x)+1=0$

$\therefore \sin(x)=-\frac{1}{2}$

Quadrants 3 and 4, base $\frac{\pi}{6}$

$\therefore x=\pi+\frac{\pi}{6},2\pi-\frac{\pi}{6}$

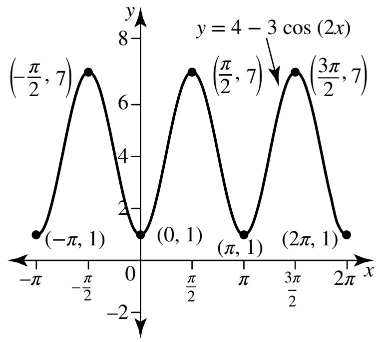
$\therefore x=\frac{7\pi}{6},\frac{11\pi}{6}$



b $y=4-3\cos(2x),\ -\pi\le x\le 2\pi$

$\therefore y=-3\cos(2x)+4$

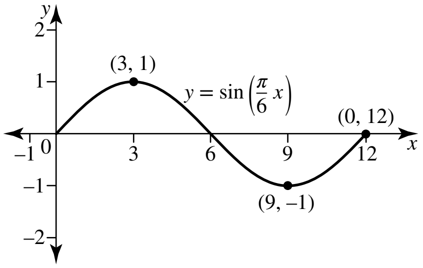
Period $\frac{2\pi}{2}=\pi$, amplitude 3, inverted, equilibrium $y=4$, range $[4-3,4+3]=[1,7]$, no x intercepts. Domain $[-\pi,2\pi]$.



2 $f:[0,12]\to R, f(x)=\sin\left(\frac{\pi x}{6}\right)$

Period $\frac{2\pi}{\frac{\pi}{6}}=12$, amplitude 1, equilibrium $y=0$, range $[-1,1]$, domain $[0,12]$.

Scale on x axis $x=0,3,6,9,12$ give the five key points



3 a $y=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right),\ 0\le x\le 2\pi$

Period 2π, amplitude $\sqrt{2}$, phase shift $-\frac{\pi}{4}$ from $y=\sqrt{2}\sin(x)$, range $\left[-\sqrt{2},\sqrt{2}\right]$.

translated points:

$(0,0)\to\left(-\frac{\pi}{4},0\right),\left(\frac{\pi}{2},\sqrt{2}\right)\to\left(\frac{\pi}{4},\sqrt{2}\right),(\pi,0)\to\left(\frac{3\pi}{4},0\right),$

$\left(\frac{3\pi}{2},-\sqrt{2}\right)\to\left(\frac{5\pi}{4},-\sqrt{2}\right),(2\pi,0)\to\left(\frac{7\pi}{4},0\right).$

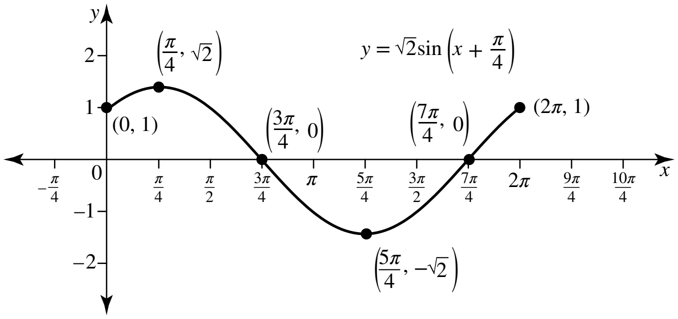
End points: $x=0$,

$$\begin{aligned}y&=\sqrt{2}\sin\left(\frac{\pi}{4}\right)\\&=\sqrt{2}\times\frac{1}{\sqrt{2}}\\&=1\end{aligned}$$

$x=2\pi$,

$$\begin{aligned}y&=\sqrt{2}\sin\left(2\pi+\frac{\pi}{4}\right)\\&=\sqrt{2}\sin\left(\frac{\pi}{4}\right)\\&=1\end{aligned}$$

End points $(0,1),(2\pi,1)$



b $y = -3\cos\left(2x + \frac{\pi}{4}\right) + 1$

Express in transformation form

$y = -3\cos\left(2\left(x + \frac{\pi}{8}\right)\right) + 1$

Period $\frac{2\pi}{2} = \pi$, amplitude 3, phase shift factor $-\frac{\pi}{8}$, equilibrium $y = 1$ so oscillates between $y = 1 \pm 3$. Therefore range is $[-2, 4]$.

4 $y = \sin\left(2x - \frac{\pi}{3}\right),\ 0 \le x \le \pi$

$\therefore y = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$

period π, amplitude 1, phase $\frac{\pi}{6}$, domain $[0, \pi]$.

End points: $x = 0, y = \sin\left(-\frac{\pi}{3}\right)$

$\therefore y = -\sin\left(\frac{\pi}{3}\right)$ point $\left(0, -\frac{\sqrt{3}}{2}\right)$

$= -\frac{\sqrt{3}}{2}$

$x = \pi, y = \sin\left(2\pi - \frac{\pi}{3}\right)$

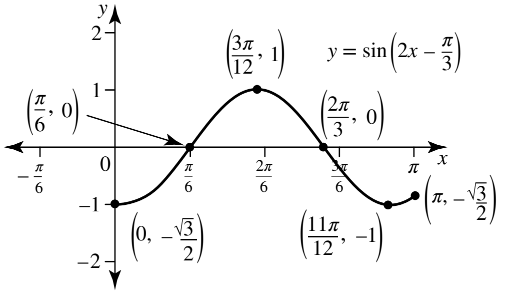
$\therefore y = \sin\left(2\pi - \frac{\pi}{3}\right)$ point $\left(\pi, -\frac{\sqrt{3}}{2}\right)$

$= -\sin\left(\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}$

Sketch $y = \sin(2x)$ and horizontally translate the 5 key points $\frac{\pi}{6}$ to the right.

Translated points: $(0,0) \to \left(\frac{\pi}{6}, 0\right), \left(\frac{\pi}{4}, 1\right) \to \left(\frac{5\pi}{12}, 1\right), \left(\frac{\pi}{2}, 0\right) \to \left(\frac{2\pi}{3}, 0\right), \left(\frac{3\pi}{4}, -1\right) \to \left(\frac{11\pi}{12}, -1\right), (\pi, 0) \to \left(\frac{7\pi}{6}, 0\right)$



5 Consider the graph as an inverted sine graph with amplitude 2. Its equation could be $y = -2\sin(nx)$.

Period is 3π, so $\frac{2\pi}{n} = 3\pi$

Therefore $n = \frac{2}{3}$ and a possible equation is $y = -2\sin\left(\frac{2x}{3}\right)$.

Another possibility is to consider the graph as a sine graph that has been translated horizontally 1.5π units to the right. Its equation could be

$$y = 2\sin\left(\frac{2}{3}\left(x - \frac{3\pi}{2}\right)\right)$$
$$= 2\sin\left(\frac{2x}{3} - \pi\right)$$

As a cosine graph, a possible equation could be

$$y = -2\cos\left(\frac{2}{3}\left(x - \frac{3\pi}{4}\right)\right)$$
$$= -2\cos\left(\frac{2x}{3} - \frac{\pi}{2}\right)$$

Other answers are possible.

6 Range is $[2,8]$ so equilibrium is $y = 5$ and amplitude is 3.
Period is π.
Let equation be $y = a\sin(nx) + k$ with $a = 3, k = 5$.
$\therefore y = 3\sin(nx) + 5$
Period: $\frac{2\pi}{n} = \pi \Rightarrow n = 2$
Therefore a possible equation is $y = 3\sin(2x) + 5$.

7 **a** $y = 6\cos(2x)$ has amplitude 6 and period $\frac{2\pi}{2} = \pi$.

b $y = -7\cos\left(\frac{x}{2}\right)$ has amplitude 7 and period $\frac{2\pi}{\frac{1}{2}} = 4\pi$.

c $y = -\frac{3}{5}\sin\left(\frac{3x}{5}\right)$ has amplitude $\frac{3}{5}$.
Its period is
$$2\pi \div \frac{3}{5}$$
$$= 2\pi \times \frac{5}{3}$$
$$= \frac{10\pi}{3}$$

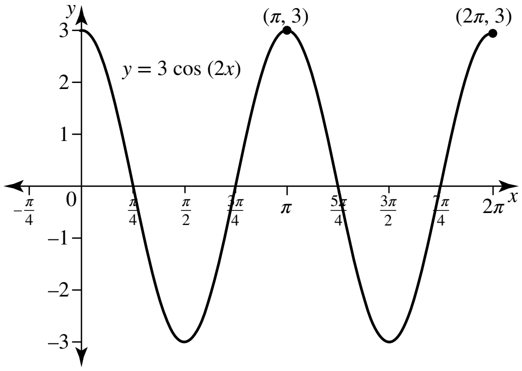
d $y = \sin\left(\frac{6\pi x}{7}\right)$ has amplitude 1.
Its period is
$$2\pi \div \frac{6\pi}{7}$$
$$= 2\pi \times \frac{7}{6\pi}$$
$$= \frac{7}{3}$$

e The graph has an amplitude of 2 and a period of 4π.

f Given points $\left(-\frac{7\pi}{4}, -4\right), \left(\frac{5\pi}{4}, 4\right)$ are minimum and maximum points respectively, so the graph has amplitude 4.
The graph completes $1\frac{1}{2}$ cycles between the given points.
Therefore, $1\frac{1}{2}$ cycles covers an interval of
$$\frac{5\pi}{4} - \left(-\frac{7\pi}{4}\right) = \frac{12\pi}{4} = 3\pi.$$
Therefore, one cycle covers an interval of $\frac{2}{3} \times 3\pi = 2\pi$.
The period is 2π.

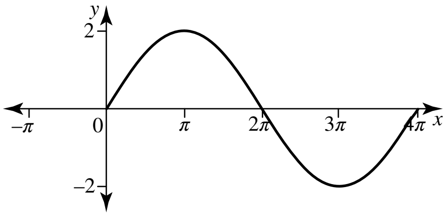
8 **a** $y = 3\cos(2x),\ 0 \le x \le 2\pi$

period $\frac{2\pi}{2} = \pi$, amplitude 3.



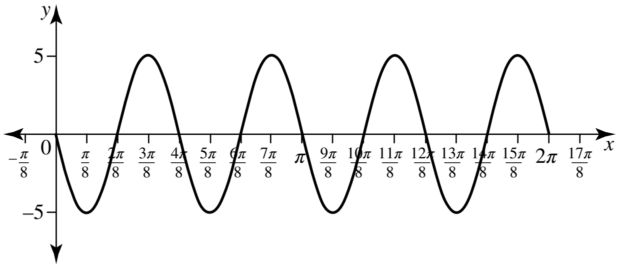
b $y = 2\sin\left(\frac{1}{2}x\right),\ 0 \le x \le 4\pi$

Period $\frac{2\pi}{\frac{1}{2}} = 4\pi$, amplitude 2



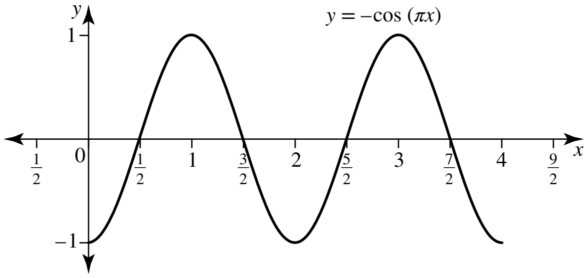
c $y = -5\sin(4x),\ 0 \le x \le 2\pi$

Period $\frac{2\pi}{4} = \frac{\pi}{2}$, amplitude 5, inverted



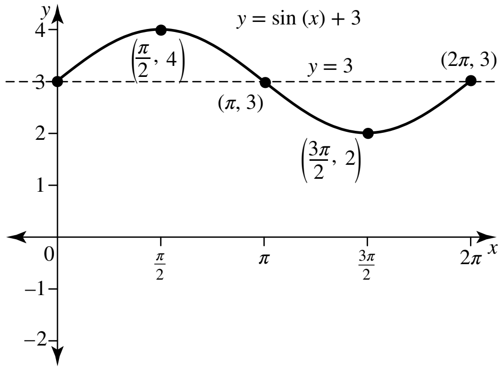
d $y = -\cos(\pi x),\ 0 \le x \le 4$

Period $\frac{2\pi}{\pi} = 2$, amplitude 1, inverted



9 a $y = \sin(x) + 3,\ 0 \le x \le 2\pi$

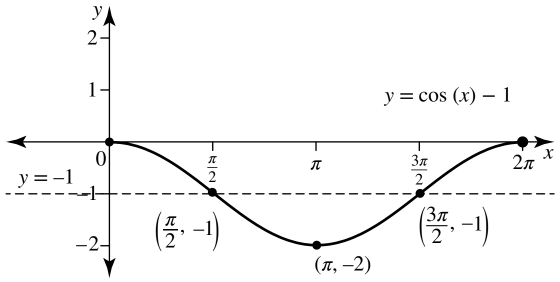
Period 2π, amplitude 1, equilibrium or mean position $y = 0$.



Range is $[2, 4]$.

b $y = \cos(x) - 1,\ 0 \le x \le 2\pi$

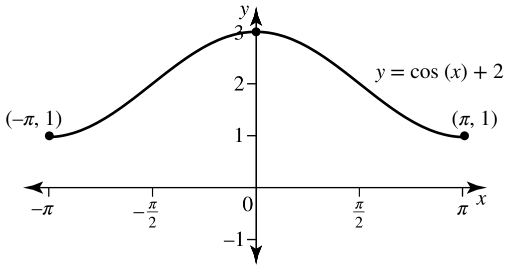
Period 2π, amplitude 1, equilibrium or mean position $y = -1$.



Range is $[-2, 0]$.

c $y = \cos(x) + 2,\ -\pi \le x \le \pi$

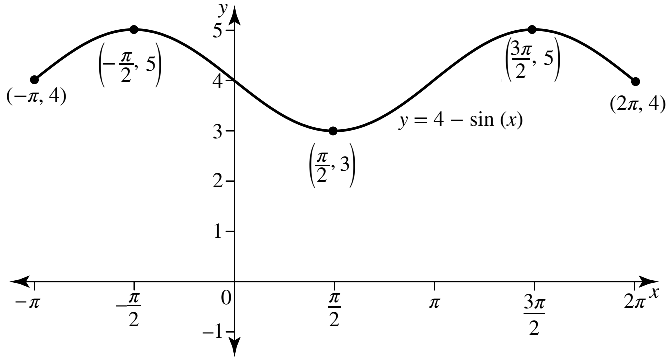
Period 2π, amplitude 1, equilibrium $y = 2$, range $[2-1, 2+1] = [1, 3]$



d $y = 4 - \sin(x),\ -\pi \le x \le 2\pi$

$\therefore y = -\sin(x) + 4$

Period 2π, amplitude 1, inverted, equilibrium $y = 4$, range $[4-1, 4+1] = [3, 5]$.



10 a $f: R \to R, f(x) = 3 + 2\sin(5x)$

The function has amplitude 2 and equilibrium position $y = 3$.

Its range is $[3-2, 3+2] = [1, 5]$.

b $f: [0, 2\pi] \to R, f(x) = 10\cos(2x) - 4$

The function has amplitude 10 and equilibrium position $y = -4$.

As its period is $\frac{2\pi}{2} = \pi$, the function will cover its maximal range over the interval $x \in [0, 2\pi]$.

Its range is $[-4-10, -4+10] = [-14, 6]$.

Its minimum value is -14.

Alternatively, replacing $\cos(2x)$ by its minimum value of -1,

$$\begin{aligned} f_{\min}(x) &= 10 \times (-1) - 4 \\ &= -14 \end{aligned}$$

c $f: [0, 2\pi] \to R, f(x) = 56 - 12\sin(x)$

If $\sin(x)$ is replaced by its most negative value, the greatest value of the function will occur.

$$\begin{aligned} f_{\max}(x) &= 56 - 12 \times (-1) \\ &= 56 + 12 \\ &= 68 \end{aligned}$$

The maximum occurs when $\sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$.

Alternatively, the range of the function is $[56-12, 56+12] = [44, 68]$, so the maximum value is 68.

When $f(x) = 68$,

$$\begin{aligned} 56 - 12\sin(x) &= 68 \\ \therefore -12 &= 12\sin(x) \\ \therefore \sin(x) &= -1 \\ \therefore x &= \frac{3\pi}{2} \end{aligned}$$

d i $\sin(x) \to 3 + 2\sin(5x)$ under the transformations:

Dilation of factor $\frac{1}{5}$ from the y axis and dilation of factor 2 from the x axis followed by a vertical translation of 3 units upwards.

ii $\cos(x) \to 10\cos(2x) - 4$ under the transformations:

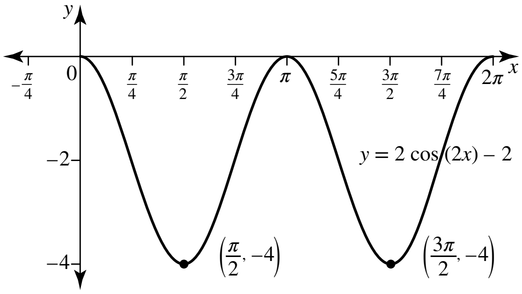
Dilation of factor $\frac{1}{2}$ from the y axis and dilation of factor 10 from the x axis, followed by a vertical translation of 4 units downwards.

iii $\sin(x) \to 56 - 12\sin(x)$ under the transformations:

Dilation of factor 12 from the x axis, reflection in the x axis, vertical translation of 56 units upwards.

11 a $y = 2\cos(2x) - 2,\ 0 \le x \le 2\pi$

Amplitude 2, period $\frac{2\pi}{2} = \pi$, equilibrium $y = -2$, range $[-2-2, -2+2] = [-4, 0]$.



b $y=2\sin(x)+\sqrt{3},\ 0\le x\le 2\pi$

Period 2π, amplitude 2, equilibrium $y=\sqrt{3}$, Range $\left[\sqrt{3}-2,\sqrt{3}+2\right]$.

Since $\sqrt{3}-2<0$ there will be x intercepts.

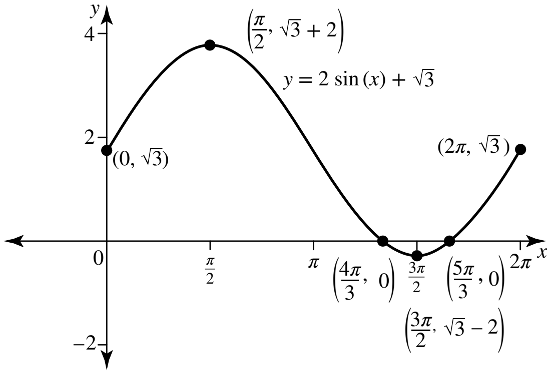
Let $y=0$

$\therefore 2\sin(x)+\sqrt{3}=0$

$$\therefore \sin(x)=-\frac{\sqrt{3}}{2}$$

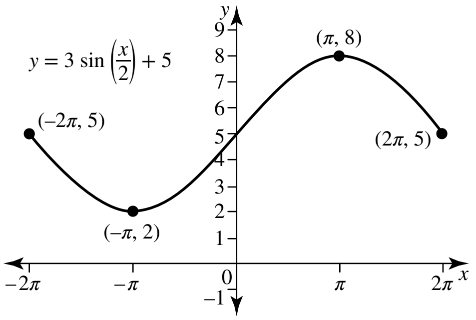
$$\therefore x=\pi+\frac{\pi}{3},2\pi-\frac{\pi}{3}$$

$$\therefore x=\frac{4\pi}{3},\frac{5\pi}{3}$$



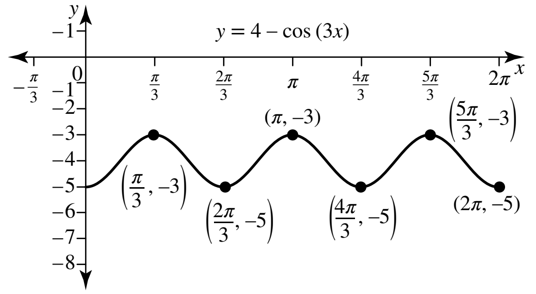
c $y=3\sin\left(\frac{x}{2}\right)+5,\ -2\pi\le x\le 2\pi$

Amplitude 3, period $\frac{2\pi}{\frac{1}{2}}=4\pi$, equilibrium $y=5$, Range $[5-3,5+3]=[2,8]$, no x intercepts.



d $y=-4-\cos(3x),\ 0\le x\le 2\pi$

Amplitude 1, inverted, period $\frac{2\pi}{3}$, equilibrium $y=-4$, Range $[-4-1,-4+1]=[-5,-3]$, no x intercepts.



e $y = 1 - 2\sin(2x),\ -\pi \le x \le 2\pi$

Amplitude 2, inverted, period $\frac{2\pi}{2} = \pi$, equilibrium $y = 1$, Range $[1-2, 1+2] = [-1, 3]$.

x intercepts: Let $y = 0$

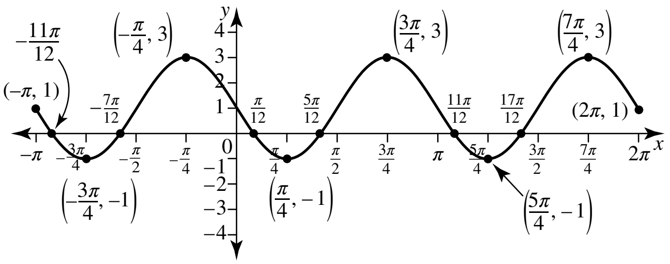
$\therefore 1 - 2\sin(2x) = 0$

$\therefore \sin(2x) = \frac{1}{2},\ -2\pi \le 2x \le 4\pi$

$\therefore 2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$ or $x = -\pi - \frac{\pi}{6}, -2\pi + \frac{\pi}{6}$

$\therefore 2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{-7\pi}{6}, \frac{-11\pi}{6}$

$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, -\frac{7\pi}{12}, -\frac{11\pi}{12}$



f $y = 2[1 - 3\cos(x°)],\ 0 \le x \le 360$

$\therefore y = 2 - 6\cos(x°)$

Amplitude 6, inverted, period 360°, equilibrium $y = 2$, Range $[2-6, 2+6] = [-4, 8]$.

x intercepts: Let $y = 0$

$\therefore 2 - 6\cos(x°) = 0$

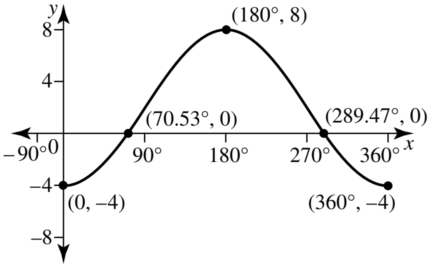
$\therefore \cos(x°) = \frac{2}{6}$

$\therefore \cos(x°) = \frac{1}{3}$

Base, in degrees is $\cos^{-1}\left(\frac{1}{3}\right) = 70.53°$

$\therefore x° = 70.53°, 360° - 70.53°$

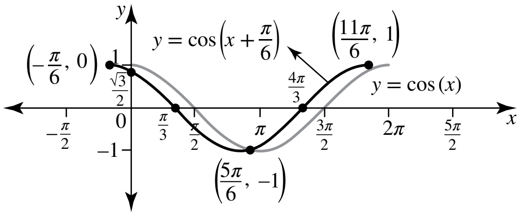
$\therefore x° = 70.53°, 289.47°$



12 a i The graph of $y = \cos\left(x + \frac{\pi}{6}\right)$ is obtained from $y = \cos(x)$ by a horizontal translation of $\frac{\pi}{6}$ to the left. Subtract $\frac{\pi}{6}$ from the x co-ordinates of all key points:

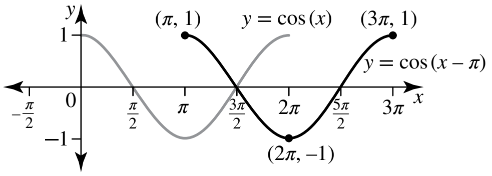
$(0,1) \to \left(-\frac{\pi}{6}, 1\right), \left(\frac{\pi}{2}, 0\right) \to \left(\frac{\pi}{3}, 0\right), (\pi, -1) \to \left(\frac{5\pi}{6}, -1\right), \left(\frac{3\pi}{2}, 0\right) \to \left(\frac{4\pi}{3}, 0\right)$ and $(2\pi, 1) \to \left(\frac{11\pi}{6}, 1\right)$.

$y = \cos\left(x + \frac{\pi}{6}\right)$ has y intercept $\left(0, \cos\left(\frac{\pi}{6}\right)\right) = \left(0, \frac{\sqrt{3}}{2}\right)$.



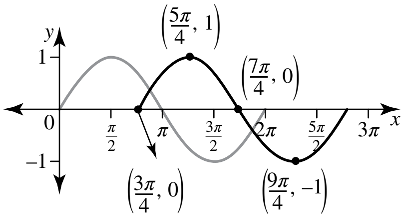
ii $y=\cos(x-\pi)$ is obtained from $y=\cos(x)$ by a horizontal translation of π to the right. Add π to the x co-ordinates of all key points.

$(0,1)\rightarrow(\pi,1),\left(\frac{\pi}{2},0\right)\rightarrow\left(\frac{3\pi}{2},0\right),(\pi,-1)\rightarrow(2\pi,-1),\left(\frac{3\pi}{2},0\right)\rightarrow\left(\frac{5\pi}{2},0\right)$ and $(2\pi,1)\rightarrow(3\pi,1)$.



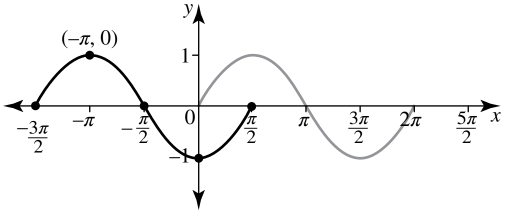
b **i** The graph of $y=\sin\left(x-\frac{3\pi}{4}\right)$ is obtained from $y=\sin(x)$ by a horizontal translation of $\frac{3\pi}{4}$ to the right. Add $\frac{3\pi}{4}$ to the x co-ordinates of all key points.

$(0,0)\rightarrow\left(\frac{3\pi}{4},0\right),\left(\frac{\pi}{2},1\right)\rightarrow\left(\frac{5\pi}{4},1\right),(\pi,0)\rightarrow\left(\frac{7\pi}{4},0\right),\left(\frac{3\pi}{2},-1\right)\rightarrow\left(\frac{9\pi}{4},-1\right)$ and $(2\pi,0)\rightarrow\left(\frac{11\pi}{4},0\right)$.



ii The graph of $y=\sin\left(x+\frac{3\pi}{2}\right)$ is obtained from $y=\sin(x)$ by a horizontal translation of $\frac{3\pi}{2}$ to the left. Subtract $\frac{3\pi}{2}$ from the x co-ordinates of all key points.

$(0,0)\rightarrow\left(-\frac{3\pi}{2},0\right),\left(\frac{\pi}{2},1\right)\rightarrow(-\pi,1),(\pi,0)\rightarrow\left(-\frac{\pi}{2},0\right),\left(\frac{3\pi}{2},-1\right)\rightarrow(0,-1)$ and $(2\pi,0)\rightarrow\left(\frac{\pi}{2},0\right)$.



13 **a** $y=2\sin\left(x-\frac{\pi}{4}\right),\ 0\le x\le 2\pi$

Amplitude 2, period 2π, range $[-2,2]$, horizontal translation $\frac{\pi}{4}$ to right.

Endpoints: When $x=0$,

$$\begin{aligned}y&=2\sin\left(-\frac{\pi}{4}\right)\\&=-2\sin\left(\frac{\pi}{4}\right)\\&=-2\times\frac{\sqrt{2}}{2}\\&=-\sqrt{2}\end{aligned}$$

$\left(0,-\sqrt{2}\right)$

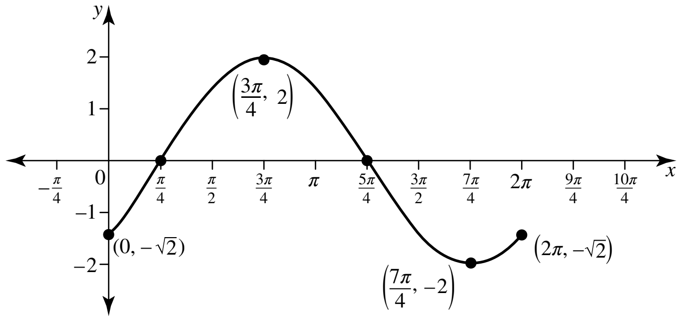
When $x=2\pi$,

$$\begin{aligned}y&=2\sin\left(2\pi-\frac{\pi}{4}\right)\\&=-2\sin\left(\frac{\pi}{4}\right)\\&=-\sqrt{2}\end{aligned}$$

$\left(2\pi,-\sqrt{2}\right)$

For $y=2\sin(x)\to y=2\sin\left(x-\frac{\pi}{4}\right)$, key points are:

$$(0,0)\to\left(\frac{\pi}{4},0\right),\left(\frac{\pi}{2},2\right)\to\left(\frac{3\pi}{4},2\right),(\pi,0)\to\left(\frac{5\pi}{4},0\right),\left(\frac{3\pi}{2},-2\right)\to\left(\frac{7\pi}{4},-2\right)$$



b $y=-4\sin\left(x+\frac{\pi}{6}\right),\ 0\le x\le 2\pi$

Amplitude 4, inverted, period 2π, range $[-4,4]$, horizontal translation $\frac{\pi}{6}$ left.
Endpoints: When $x=0$,

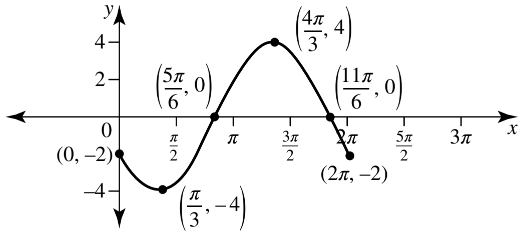
$$\begin{aligned}y&=-4\sin\left(\frac{\pi}{6}\right)\\&=-4\times\frac{1}{2}\\&=-2\end{aligned}$$

$(0,-2)$
When $x=2\pi$,

$$\begin{aligned}y&=-4\sin\left(2\pi+\frac{\pi}{6}\right)\\&=-4\sin\left(\frac{\pi}{6}\right)\\&=-2\end{aligned}$$

$(2\pi,-2)$
For $y=-4\sin(x)\to y=-4\sin\left(x+\frac{\pi}{6}\right)$, key points are:

$$\left(\frac{\pi}{2},-4\right)\to\left(\frac{\pi}{3},-4\right),(\pi,0)\to\left(\frac{5\pi}{6},0\right),\left(\frac{3\pi}{2},4\right)\to\left(\frac{4\pi}{3},4\right),(2\pi,0)\to\left(\frac{11\pi}{6},0\right)$$



c $y=\cos\left(2\left(x+\frac{\pi}{3}\right)\right),\ 0\le x\le 2\pi$

Amplitude 1, period $\frac{2\pi}{2}=\pi$, range $[-1,1]$, horizontal translation $\frac{\pi}{3}$ to left.

Endpoints: When $x=0$,

$$\begin{aligned}y&=\cos\left(\frac{2\pi}{3}\right)\\&=-\cos\left(\frac{\pi}{3}\right)\\&=-\frac{1}{2}\end{aligned}$$

$\left(0,-\frac{1}{2}\right)$

When $x = 2\pi$,

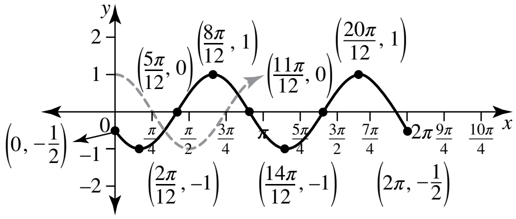
$$y = \cos\left(4\pi + \frac{2\pi}{3}\right)$$
$$= \cos\left(\frac{2\pi}{3}\right)$$
$$= -\frac{1}{2}$$

$\left(2\pi, -\frac{1}{2}\right)$

Sketch a cycle of $y = \cos(2x)$ and translate the points.

$$\cancel{\left(\frac{\pi}{4},0\right)\rightarrow\left(-\frac{\pi}{12}\right.},0\Big),\left(\frac{\pi}{2},-1\right)\rightarrow\left(\frac{2\pi}{12},-1\right),\left(\frac{3\pi}{4},0\right)\rightarrow\left(\frac{5\pi}{12},0\right),(\pi,1)\rightarrow\left(\frac{8\pi}{12},1\right).$$

Then continue the pattern for the points $\left(\frac{11\pi}{12},0\right),\left(\frac{14\pi}{12},-1\right),\left(\frac{17\pi}{12},0\right),\left(\frac{20\pi}{12},1\right),\left(\frac{23\pi}{12},0\right)$.



d $y = \cos\left(2x - \frac{\pi}{2}\right),\ 0 \le x \le 2\pi$

$$\therefore y = \cos\left(2\left(x - \frac{\pi}{4}\right)\right)$$

Amplitude 1, period $\frac{2\pi}{2} = \pi$, range $[-1,1]$, horizontal translation $\frac{\pi}{4}$ to right.

Endpoints: When $x = 0$,

$$y = \cos\left(-\frac{\pi}{2}\right)$$
$$= 0$$

$(0,0)$

When $x = 2\pi$,

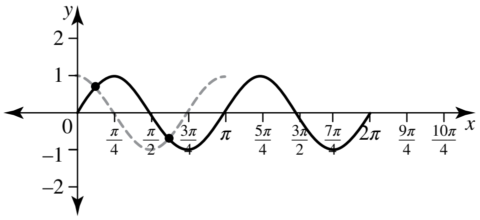
$$y = \cos\left(4\pi - \frac{\pi}{2}\right)$$
$$= \cos\left(\frac{\pi}{2}\right)$$
$$= 0$$

$(2\pi, 0)$

Sketch a cycle of $y = \cos(2x)$ and translate the points.

$$(0,1)\rightarrow\left(\frac{\pi}{4},1\right),\left(\frac{\pi}{4},0\right)\rightarrow\left(\frac{\pi}{2},0\right),\left(\frac{\pi}{2},-1\right)\rightarrow\left(\frac{3\pi}{4},-1\right),\left(\frac{3\pi}{4},0\right)\rightarrow(\pi,0)$$

Then continue the pattern for the points $\left(\frac{5\pi}{4},1\right),\left(\frac{3\pi}{2},0\right),\left(\frac{7\pi}{4},-1\right),(2\pi,0)$.



e $y = \cos\left(x + \frac{\pi}{2}\right) + 2,\ 0 \le x \le 2\pi$

Amplitude 1, period 2π, equilibrium $y = 2$, range $[2-1, 2+1] = [1,3]$, horizontal translation $\frac{\pi}{2}$ to left.

Endpoints: When $x = 0$,

$$\begin{aligned}y &= \cos\left(\frac{\pi}{2}\right)+2\\ &= 0+2\\ &= 2\end{aligned}$$

$(0,2)$

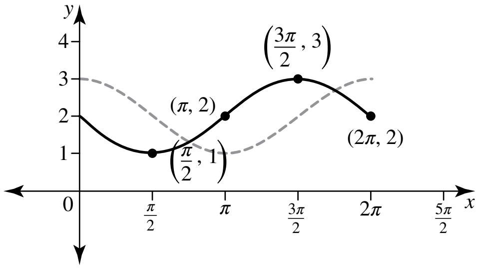
When $x = 2\pi$,

$$\begin{aligned}y &= \cos\left(2\pi+\frac{\pi}{2}\right)+2\\ &= \cos\left(\frac{\pi}{2}\right)+2\\ &= 2\end{aligned}$$

$(2\pi,2)$

Sketch a cycle of $y=\cos(x)+2$ and translate the points.

$\left(\frac{\pi}{2},2\right)\rightarrow(0,2),(\pi,1)\rightarrow\left(\frac{\pi}{2},1\right),\left(\frac{3\pi}{2},2\right)\rightarrow(\pi,2),(2\pi,3)\rightarrow\left(\frac{3\pi}{2},3\right)$



f $y=3-3\sin(2x-4\pi),\ 0\le x\le 2\pi$

$\therefore y=-3\sin(2(x-2\pi))+3$

Amplitude 3, inverted, period $\frac{2\pi}{2}=\pi$, equilibrium $y=3$, range $[3-3,3+3]=[0,6]$, horizontal shift 2π to right.

Endpoints: When $x=0$,

$$\begin{aligned}y &= -3\sin(-4\pi)+3\\ &= 0+3\\ &= 3\end{aligned}$$

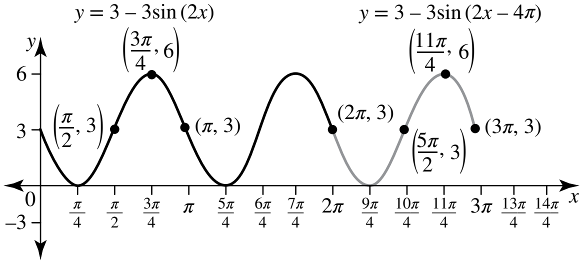
$(0,3)$

When $x=2\pi$,

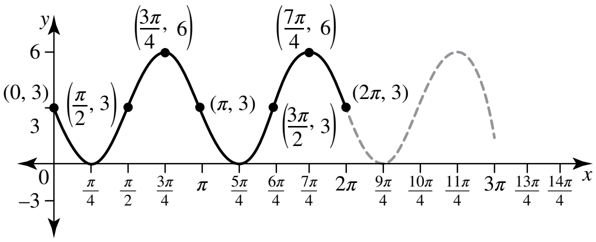
$$\begin{aligned}y &= -3\sin(4\pi-2\pi)+3\\ &= -3\sin(2\pi)+3\\ &= 0+3\\ &= 3\end{aligned}$$

$(2\pi,3)$

Sketch a cycle of $y=-3\sin(2x)+3$ and translate the points 2π units to the right.



The graph obtained does not lie on the correct domain. Extend the pattern of key points back to fit the domain.



The graph could have been obtained using symmetry properties:

$$\begin{aligned} y &= 3 - 3\sin(2x - 4\pi) \\ &= 3 - 3\sin(-(4\pi - 2x)) \\ &= 3 + 3\sin(4\pi - 2x) \\ &= 3 + 3\sin(2\pi - 2x) \\ &= 3 - 3\sin(2x) \end{aligned}$$

14 **a** Consider the given graph as an inverted sine graph with equilibrium position $y = 0$. Let the equation be $y = -a\sin(nx)$.

The maximum point $(3\pi, 3)$ shows the amplitude is 3.

$\therefore y = -3\sin(nx)$

Between the points $(0,0)$ and $(6\pi, 0)$, the graph completes $1\frac{1}{2}$ cycles. Therefore, its period is $\frac{2}{3} \times 6\pi = 4\pi$.

$\therefore \frac{2\pi}{n} = 4\pi$

$\therefore \frac{2\pi}{4\pi} = n$

$\therefore n = \frac{1}{2}$

A possible equation is $y = -3\sin\left(\frac{x}{2}\right)$.

b Consider the given graph as a cosine graph with equilibrium position $y = 0$. Let the equation be $y = a\cos(nx)$.

The minimum point $\left(\frac{\pi}{3}, -4\right)$ shows the amplitude is 4.

$\therefore y = 4\cos(nx)$

Between the points $(0,4)$ and $\left(\frac{\pi}{3}, -4\right)$ the graph completes $\frac{1}{2}$ a cycle. Therefore, its period is $2 \times \frac{\pi}{3} = \frac{2\pi}{3}$.

$\therefore \frac{2\pi}{n} = \frac{2\pi}{3}$

$\therefore n = 3$

A possible equation is $y = 4\cos(3x)$.

c The range of the graph is $[2,10]$.

Hence, its equilibrium position is $y = \frac{2+10}{2} = 6$ and its amplitude is $10 - 6 = 4$.

Consider the graph as an inverted cosine with equation $y = -a\cos(nx) + k$.

$\therefore y = -4\cos(nx) + 6$

The period of the graph is 2π

$\therefore \frac{2\pi}{n} = 2\pi$

$\therefore n = 1$

A possible equation is $y = -4\cos(x) + 6$.

d Consider the given graph as a sine graph with a horizontal translation of $\frac{\pi}{4}$ to the right and equilibrium position $y = 0$.

Let the equation be $y = a\sin(n(x - h))$.

$\therefore y = a\sin\left(n\left(x - \frac{\pi}{4}\right)\right)$

Amplitude is $2 \Rightarrow y = 2\sin\left(n\left(x - \frac{\pi}{4}\right)\right)$.

Between the x intercepts $\left(\frac{\pi}{4}, 0\right)$ and $\left(\frac{9\pi}{4}, 0\right)$, the graph completes one cycle. its period is 2π and therefore $n = 1$.

A possible equation is $y = 2\sin\left(x - \frac{\pi}{4}\right)$.

e The graph in part d could be considered to be a cosine graph that is horizontally translated $\frac{3\pi}{4}$ to the right. The period 2π and amplitude 2 are unaltered.

An alternative equation could be $y = 2\cos\left(x - \frac{3\pi}{4}\right)$.

f $y = \cos(-x)$

$\therefore y = \cos(x)$ is an alternative.

$y = \sin(-x)$

$\therefore y = -\sin(x)$ is an alternative.

15 $f(x) = a\sin(bx) + c$

a Since $f(x) = f\left(x + \frac{2\pi}{3}\right)$, it is a periodic function with period $\frac{2\pi}{3}$.

b Constants a, b and c are positive.

Period $\frac{2\pi}{3}$

$\therefore \frac{2\pi}{b} = \frac{2\pi}{3}$

$\therefore b = 3$

Range $[5,9]$ so equilibrium position is $y = \frac{5+9}{2} = 7$

$\therefore c = 7$

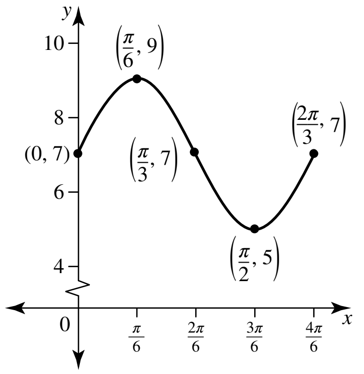
Amplitude is $9 - 7 = 2$.

Since $a > 0$, $a = 2$.

Answer is $a = 2,\ b = 3,\ c = 7$.

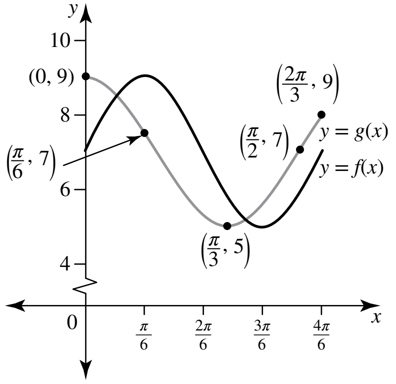
c The rule for the function is $f(x) = 2\sin(3x) + 7$.

For one cycle, the domain $D = \left[0, \frac{2\pi}{3}\right]$.



d $g(x) = a\cos(bx) + c,\ x \in D$

$\therefore g(x) = 2\cos(3x) + 7,\ x \in \left[0, \frac{2\pi}{3}\right]$.



e At intersection,

$2\sin(3x)+7=2\cos(3x)+7,\ x\in\left[0,\frac{2\pi}{3}\right]$

$\therefore 2\sin(3x)=2\cos(3x)$

$\therefore \sin(3x)=\cos(3x)$

$\therefore \frac{\sin(3x)}{\cos(3x)}=1$

$\therefore \tan(3x)=1$

Quadrants 1 and 3, base $\frac{\pi}{4}$. The diagram shows there are 2 points of intersection.

$\therefore 3x=\frac{\pi}{4},\pi+\frac{\pi}{4}$

$\therefore 3x=\frac{\pi}{4},\frac{5\pi}{4}$

$\therefore x=\frac{\pi}{12},\frac{5\pi}{12}$

If $x=\frac{\pi}{12}$,

$$\begin{aligned}f\left(\frac{\pi}{12}\right)&=2\sin\left(\frac{\pi}{4}\right)+7\\&=2\times\frac{\sqrt{2}}{2}+7\\&=\sqrt{2}+7\end{aligned}$$

If $x=\frac{5\pi}{12}$,

$$\begin{aligned}f\left(\frac{5\pi}{12}\right)&=2\sin\left(\frac{5\pi}{4}\right)+7\\&=-2\sin\left(\frac{\pi}{4}\right)+7\\&=-2\times\frac{\sqrt{2}}{2}+7\\&=-\sqrt{2}+7\end{aligned}$$

The points of intersection are $\left(\frac{\pi}{12},7+\sqrt{2}\right),\left(\frac{5\pi}{12},7-\sqrt{2}\right)$.

f From the diagram, $f(x)\ge g(x)$ for $\frac{\pi}{12}\le x\le\frac{5\pi}{12}$.

16 a Under the vertical translation of 3 units up followed by the reflection in the x axis followed by a dilation of factor 3 from the y axis:

$y=\sin(x)$

$\rightarrow y=\sin(x)+3$

$\rightarrow y=-[\sin(x)+3]=-\sin(x)-3$

$\rightarrow y=-\sin\left(\frac{x}{3}\right)-3$

The final image has the equation $y=-\sin\left(\frac{x}{3}\right)-3$.

The range of the final image is $[-3-1,-3+1]=[-4,-2]$. As the range of $y=\sin(x)$ is $[-1,1]$, the two graphs do not intersect.

b Under the linear transformation,

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}1&0\\0&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$\therefore x'=x$

$y'=-2y$

$\therefore x=x'$ and $y=\frac{y'}{-2}$

$y=\sin(x)\rightarrow\frac{y'}{-2}=\sin(x')$

The image has the equation $\frac{y}{-2}=\sin(x)$

$\therefore y=-2\sin(x)$

There has been a reflection in the x axis and a dilation of factor 2 from the x axis.

c $f:[0,4]\rightarrow R, f(x)=a-20\sin\left(\frac{\pi x}{3}\right)$

i Given $f(4)=10\left(\sqrt{3}+1\right)$.

$\therefore a-20\sin\left(\frac{4\pi}{3}\right)=10\left(\sqrt{3}+1\right)$

$\therefore a-20\left(-\sin\left(\frac{\pi}{3}\right)\right)=10\left(\sqrt{3}+1\right)$

$\therefore a+20\times\frac{\sqrt{3}}{2}=10\sqrt{3}+10$

$\therefore a+10\sqrt{3}=10\sqrt{3}+10$

$\therefore a=10$

ii $f(x)=10-20\sin\left(\frac{\pi x}{3}\right)$

Let $f(x)=0$.

$\therefore 10-20\sin\left(\frac{\pi x}{3}\right)=0$

$\therefore 10=20\sin\left(\frac{\pi x}{3}\right)$

$\therefore \sin\left(\frac{\pi x}{3}\right)=\frac{1}{2}$

Quadrants 1 and 2, base $\frac{\pi}{6}$.

As $x\in[0,4]$, then $\frac{\pi x}{3}\in\left[0,\frac{4\pi}{3}\right]$

$\therefore \frac{\pi x}{3}=\frac{\pi}{6},\pi-\frac{\pi}{6}$

$=\frac{\pi}{6},\frac{5\pi}{6}$

$\therefore \pi x=\frac{3\pi}{6},\frac{15\pi}{6}$

$\therefore x=\frac{1}{2},\frac{5}{2}$

iii $f(x)=10-20\sin\left(\frac{\pi x}{3}\right),\ 0\le x\le 4$

Amplitude 20, inverted, equilibrium $y=10$.

period $\frac{2\pi}{\frac{\pi}{3}}$

$=2\pi\times\frac{3}{\pi}$

$=6$

Domain is $[0,4]$ so not a full cycle of the graph since the period is 6.

Endpoints: Let $x=0$

$f(0)=10-20\sin(0)$

$=10$

$(0,10)$

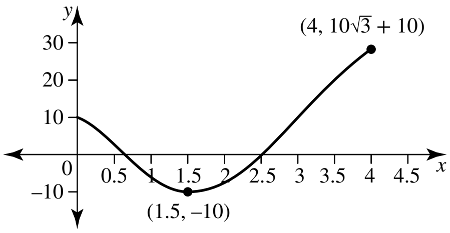
If $x=4$, then $f(4)=10\sqrt{3}+10$ (given), so endpoint is $\left(4,10\sqrt{3}+10\right)$.

x intercepts occur when $x=\frac{1}{2},\frac{5}{2}$.

Therefore the graph will reach its minimum point at $x=\frac{\frac{1}{2}+\frac{5}{2}}{2}=\frac{3}{2}$.

The minimum point is $\left(\frac{3}{2},10-20\right)=\left(\frac{3}{2},-10\right)$. Its maximum point is its endpoint $\left(4,10\sqrt{3}+10\right)$.

Range is $\left[-10,10\sqrt{3}+10\right]$.



iv $y=\cos(x), x\in\left[-\frac{\pi}{2},\frac{5\pi}{6}\right]\rightarrow y=10-20\sin\left(\frac{\pi x}{3}\right), x\in[0,4].$

Cosine and sine are out of phase by $\frac{\pi}{2}$, so $y=\cos(x)\rightarrow y=\sin(x)$ under a horizontal translation $\frac{\pi}{2}$ to the right and $x\in\left[-\frac{\pi}{2},\frac{5\pi}{6}\right]\rightarrow x\in\left[0,\frac{4\pi}{3}\right]$.

Now consider $y=\sin(x)\rightarrow y=-20\sin\left(\frac{\pi}{3}x\right)+10$.

This occurs for dilations of factor 20 from the x axis and factor $\frac{3}{\pi}$ from the y axis, reflection in the x axis and vertical translation up 10 units.

For these transformations,

$x\in\left[0,\frac{4\pi}{3}\right]\rightarrow x\in\left[0,\frac{4\pi}{3}\times\frac{3}{\pi}\right]=[0,4].$

The sequence of transformations for $y=\cos(x)\rightarrow y=10-20\sin\left(\frac{\pi x}{3}\right)$ is:

Horizontal translation $\frac{\pi}{2}$ to the right, dilations of factor 20 from the x axis and factor $\frac{3}{\pi}$ from the y axis, reflection in the x axis and vertical translation up 10 units.

17 a Sketch $y=4\sin(2x-4)$ in the Graph&Tab menu on Rad mode. Either enter the equation as $y1=4\sin(2x-4)|0\le x\le 2\pi$ or setting the window with $x_{\min}=0, x_{\max}=2\pi$.

There are 4 intercepts with the x axis. Tap Analysis → G-Solve → Root to obtain $x=0.43, 2.00, 3.57, 5.14$.

b $T:R^2\rightarrow R^2, T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Under the transformation, $x'=y$ and $y'=x$.

$y=4\sin(2x-4)\rightarrow x=4\sin(2y-4)$.

This is the inverse of $y=4\sin(2x-4)$.

c Its domain is the range of $y=4\sin(2x-4)$ and its range is the domain of $y=4\sin(2x-4)$.

Domain is $[-4,4]$, range $[0,2\pi]$.

Its y intercepts are the x intercepts of $y=4\sin(2x-4)$.

The graph has four y intercepts with values $y=0.43, 2.00, 3.57, 5.14$.

18 $y=\cos\left(x+k\frac{\pi}{2}\right), k\in Z.$

Since the period of the graph of $y=\cos(x)$ is 2π, $\cos(x)=\cos(x+2\pi)=\cos(x+4\pi)=......$

This means there are only four cases to be investigated: $k=0, k=1, k=2, k=3$.

Type 1: $k=0$

If $k=0$, then $y=\cos(x)$ is the graph obtained. This will be the same as the graph for any of $k=....-4,0,4,8,12,.....$

Type 2: $k=1$

If $k=1$, then $y=\cos\left(x+\frac{\pi}{2}\right)$ is the graph. This is also the graph of $y=-\sin(x)$. The same graph is obtained for any of $k=...,-3,1,5,9,....$

Type 3: $k=2$

If $k=2$, then $y=\cos(x+\pi)$ is the graph. This is also the graph of $y=-\cos(x)$. The same graph is obtained for any of $k=....-2,2,6,10,.....$

Type 4: $k=3$

If $k=3$, then $y=\cos\left(x+\frac{3\pi}{2}\right)$ is the graph. This is also the graph of $y=\sin(x)$. The same graph is obtained for any of $k=....-1,3,7,11,....$

In summary, the four types of graphs possible are:

For $k=4n, n\in Z$, $y=\cos(x)$; for $k=4n+1, n\in Z$, $y=-\sin(x)$; for $k=4n+2, n\in Z$, $y=-\cos(x)$; and for $k=4n+3, n\in Z$, $y=\sin(x)$.

Exercise 10.4 — Applications of sine and cosine functions

1 a $T=19-3\sin\left(\frac{\pi}{12}t\right)$

At midnight, $t=0$

Therefore, at midnight, $T=19-3\sin(0)\Rightarrow T=19$.

The temperature was 19° at midnight.

b Temperature will be a maximum when $\sin\left(\frac{\pi}{12}t\right)=-1$

$\therefore T_{\max}=19-3\times(-1)$

$\therefore T_{\max}=22$

maximum temperature is 22°.

Maximum occurs when $\sin\left(\frac{\pi}{12}t\right)=-1$

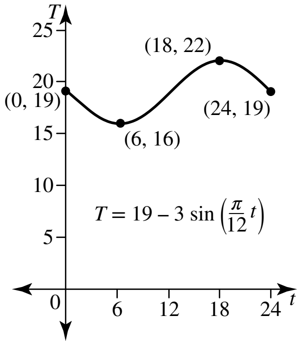
$\therefore \frac{\pi}{12}t=\frac{3\pi}{2}$

$\therefore t=18$

temperature reaches its maximum of 22° at 6 pm.

c Since the amplitude is 3 and the equilibrium occurs at $T=19$. the range of temperature is given by 19 ± 3 degrees. Therefore the temperature varied over the interval 16° to 22°.

d period $2\pi\div\frac{\pi}{12}=24$ hours



e For the temperature to be below k for 3 hours, the interval must lie between $t=6-\frac{3}{2}$ and $t=6+\frac{3}{2}$, that is, $t=4.5$ to $t=7.5$.

When $t=4.5$,

$T=19-3\sin\left(\frac{\pi}{12}\times\frac{9}{2}\right)$

$=19-3\sin\left(\frac{3\pi}{8}\right)$

$\simeq 16.2$

Therefore, $k=16.2$.

2 Period is 12 hours, range $[20,36]$, amplitude is $\frac{36-20}{2}=8$ so equilibrium is $T=28$.

The equation is $T=-8\cos(nt)+28$, where $\frac{2\pi}{n}=12$. Therefore, $n=\frac{\pi}{6}$.

The equations is $T=28-8\cos\left(\frac{\pi}{6}t\right)$.

Let $T=30$

$\therefore 30=28-8\cos\left(\frac{\pi}{6}t\right)$

$\therefore \cos\left(\frac{\pi}{6}t\right)=-\frac{1}{4}$

second and third quadrants and base is $\cos^{-1}\left(\frac{1}{4}\right)=1.32$.

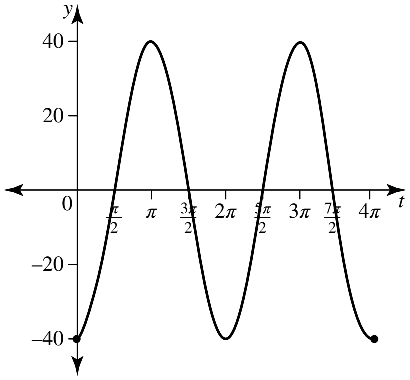
$\frac{\pi}{6}t=\pi-1.32$ or $\pi+1.32$

$\therefore t=3.48$ or $t=8.52$

Since time is measured from 8 am, the temperature exceeds 30° between 11:29 am and 4:31 pm.

3 **a** $y=-40\cos(t)$

Period 2π, amplitude 40, inverted, range $[-40,40]$, domain for two cycles $[0,4\pi]$.



b The greatest distance below rest, or equilibrium position, is 40 cm.

c The yo-yo is at its rest, or equilibrium position when $y=0$.

The times are $t=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}$ in seconds.

d Let $y=20$

$\therefore 20=-40\cos(t)$

$\therefore \cos(t)=-\frac{1}{2}$

For the first positive solution,

$t=\pi-\frac{\pi}{3}$

$\therefore t=\frac{2\pi}{3}$

The yo-yo first reaches the height after $\frac{2\pi}{3}\simeq 2.1$ seconds.

4 **a** Since the range is $[0,10]$, the equilibrium position is $I=5$ and the amplitude is 5.

Let the equation be $I=a\sin(nt)+k$

$\therefore I=5\sin(nt)+5$

Period is 4 weeks.

$\therefore \frac{2\pi}{n}=4$

$\therefore n=\frac{2\pi}{4}$

$\therefore n=\frac{\pi}{2}$

The equation is $I=5\sin\left(\frac{\pi}{2}t\right)+5$.

b Let $I=6$

$\therefore 6=5\sin\left(\frac{\pi}{2}t\right)+5$

$\therefore 1=5\sin\left(\frac{\pi}{2}t\right)$

$\therefore \sin\left(\frac{\pi}{2}t\right)=\frac{1}{5}$

Quadrants 1 and 2, base is $\sin^{-1}\left(\frac{1}{5}\right)\simeq 0.20$.

In one cycle, solutions are:

$\therefore \frac{\pi}{2}t=0.20,\pi-0.20$

$\therefore t=\frac{2}{\pi}\times 0.20,\frac{2}{\pi}\times(\pi-0.20)$

$\therefore t=0.128,1.872$

From the graph, $I\geq 6$ for $0.128\leq t\leq 1.872$.

The length of the interval is $1.872-0.128=1.744$ weeks.

The percentage of the 4 week cycle is $\frac{1.744}{4}\times 100=43.6\%$.

For 44% of the four week cycle, the person experiences a high level of happiness.

5 **a** $p=a\cos(nt)+b$

Period is 2

$\therefore \frac{2\pi}{n}=2$

$\therefore n=\pi$

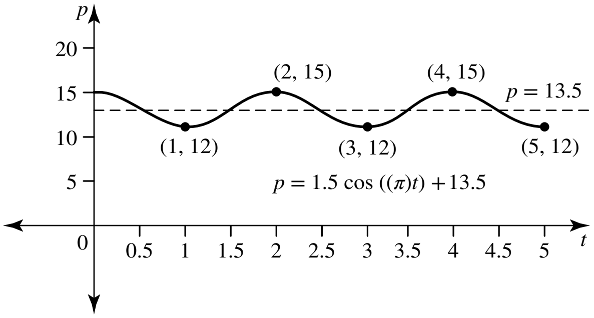
Range is $[12,15]$ so equilibrium is $p=\frac{12+15}{2}=13.5$.

$\therefore b=13.5$

Amplitude is $15-13.5=1.5$, so $a=1.5$ since the graph starts at its peak.

Answers are $a=1.5,\ n=\pi,\ b=13.5$.

b The equation is $p=1.5\cos(\pi t)+13.5$



From the graph, the share price at the end of the five weeks is 12 cents.

c Let $p=12.75$

$\therefore 12.75=1.5\cos(\pi t)+13.5$

$\therefore -0.75=1.5\cos(\pi t)$

$\therefore \cos(\pi t)=-\frac{0.75}{1.5}$

$\therefore \cos(\pi t)=-\frac{1}{2}$

Quadrants 2 and 3, base $\frac{\pi}{3}$. The share price will be falling in quadrant 2 and rising in quadrant 3. Therefore a quadrant 3 solution is required.

$\therefore \pi t=\pi+\frac{\pi}{3}$

$\therefore \pi t=\frac{4\pi}{3}$

$\therefore t=\frac{4}{3}$

In a cycle where $t \in [0,2]$, John buys at $t = \frac{4}{3}$ and sells at $t = 2$. The time between buying and selling is $2 - \frac{4}{3} = \frac{2}{3}$ of a week. If a week of 7 days is assumed, then it will be $\frac{2}{3} \times 7 \simeq 5$ days before John sells the shares.

d Buying cost

$= 10{,}000 \times \$0.1275 + 0.01 \times (10{,}000 \times \$0.1275)$

$= \$1275 + \$12.75$

$= \$1287.75$

Selling 10,000 shares at \$0.15 gives revenue of \$1500.
Brokerage cost is \$15, so selling revenue is \$1485.
Profit made is $\$1485 - \$1287.75 = \$197.25$.

6 $h = 4\sin\left(\frac{\pi(t-2)}{6}\right)$

a At 1 am, $t = 1$

$\therefore h = 4\sin\left(\frac{\pi(-1)}{6}\right)$

$= -4\sin\left(\frac{\pi}{6}\right)$

$= -4 \times \frac{1}{2}$

$\therefore h = -2$

The tide is 2 metres below mean sea level at 1 am.

b Since the mean position is $h = 0$ and the amplitude is 4, the high tide level is 4 metres above mean sea level.

High tide occurs when $\sin\left(\frac{\pi(t-2)}{6}\right) = 1$

$\therefore \frac{\pi(t-2)}{6} = \frac{\pi}{2}$

$\therefore \frac{t-2}{6} = \frac{1}{2}$

$\therefore t - 2 = 3$

$\therefore t = 5$

High tide first occurs 5 hours after midnight, that is, at 5 am.

c There is half a period between high tide and the following low tide.
Period, in hours,

$= 2\pi \div \frac{\pi}{6}$

$= 2\pi \times \frac{6}{\pi}$

$= 12$

Therefore there is an interval of 6 hours between high tide and the following low tide.

d $h = 4\sin\left(\frac{\pi(t-2)}{6}\right)$

Period 12, amplitude 4, horizontal translation 2 to the right.
Domain $[0,12]$, range $[-4,4]$
Endpoints: Let $t = 0$,

$\therefore h = 4\sin\left(\frac{\pi(-2)}{6}\right)$

$\therefore h = 4\sin\left(-\frac{\pi}{3}\right)$

$= -4\sin\left(\frac{\pi}{3}\right)$

$= -4 \times \frac{\sqrt{3}}{2}$

$\therefore h = -2\sqrt{3}$

$\left(0, -2\sqrt{3}\right)$

Let $t = 12$,

$\therefore h = 4\sin\left(\frac{\pi(10)}{6}\right)$

$\therefore h = 4\sin\left(\frac{5\pi}{3}\right)$

$= -4\sin\left(\frac{\pi}{3}\right)$

$\therefore h = -2\sqrt{3}$

$\left(12, -2\sqrt{3}\right)$

t intercepts: Let $h = 0$

$\therefore 4\sin\left(\frac{\pi(t-2)}{6}\right) = 0$

$\therefore \sin\left(\frac{\pi(t-2)}{6}\right) = 0$

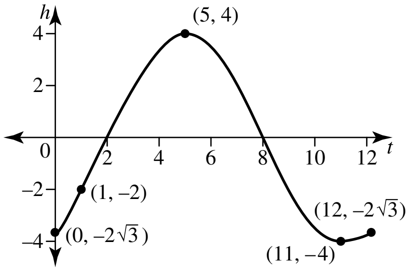
$\therefore \frac{\pi(t-2)}{6} = 0, \pi, 2\pi$

$\therefore \frac{t-2}{6} = 0, 1, 2$

$\therefore t - 2 = 0, 6, 12$

$\therefore t = 2, 8, \cancel{14}$

As high tide is at $(5,4)$, six hours later the minimum point is $(11,-4)$.
The point $(1,-2)$ is also known to lie on the graph.



e At 2 pm, $t = 14$.

$\therefore h = 4\sin\left(\frac{\pi(12)}{6}\right)$

$= 4\sin(2\pi)$

$= 0$

The tide is predicted to be at mean sea level.

f At 11:30 am, $t = 11.5$

$\therefore h = 4\sin\left(\frac{\pi(9.5)}{6}\right)$

$\simeq -3.86$

At low tide, $h = -4$.
Therefore the tide at 11:30 am is 0.14 metres higher than low tide.

7 **a** $h = a\sin\left(\frac{\pi}{5}x\right) + b$

Refer to the diagram given in the question.
The equilibrium position is $h = 4.5$, so $b = 4.5$.
The amplitude is $7 - 4.5 = 2.5$ so $a = 2.5$

The equation is $h = 2.5\sin\left(\frac{\pi}{5}x\right) + 4.5$.

b The base is one period in length.
Period is

$2\pi \div \frac{\pi}{5}$

$= 2\pi \times \frac{5}{\pi}$

$= 10$

The length of the base is 10 cm.

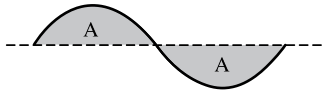
c The highest point on the curve occurs at a quarter of the cycle.

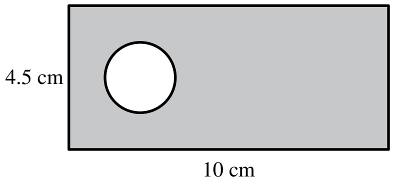
Hence, the highest point is $\left(\frac{10}{4}, 7\right) = (2.5, 7)$.

The x co-ordinate of the centre of the circle is therefore $x = 2.5$.
The y co-ordinate of the centre of the circle is
$y = \frac{2+4.5}{2} = 3.25$.
The centre of the circle is the point $(2.5, 3.25)$.

d The radius of the circle is $\frac{4.5-2}{2} = 1.25$.

Due to the symmetry of the sine curve, the area above the equilibrium position is equal to the area below the equilibrium position. The total area under the sine curve is that of a rectangle with dimensions 10 cm by 4.5 cm.





The required area is the rectangular area less the area of the circle.
Area, in sq cm, is $10 \times 4.5 - \pi(1.25)^2 \simeq 40.1$.
The shaded area is 40.1 sq cm.

8 $T = 30 - \cos\left(\frac{\pi}{12}t\right)$.

a Amplitude is 1, equilibrium is $T = 30$, so the range of the temperature in the incubator is $[29, 31]$, units being °C.

b As the graph is inverted, the cosine function will reach its maximum value after $\frac{1}{2}$ of its period.

The period, in minutes, is

$2\pi \div \frac{\pi}{12}$

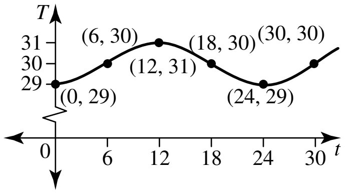
$= 2\pi \times \frac{12}{\pi}$

$= 24$

The maximum temperature is reached after $\frac{1}{2} \times 24 = 12$ minutes.

c $T = 30 - \cos\left(\frac{\pi}{12}t\right),\ t \in (0, 30)$

As the period is 24 minutes, the graph has $1\frac{1}{4}$ cycles.



d In 30 minutes, $1\frac{1}{4}$ cycles are completed so in 60 minutes, $2\frac{1}{2}$ cycles are completed.

e In 1 hour, $2\frac{1}{2}$ cycles are completed; in 2 hours, 5 cycles are completed and the temperature is 29°. Thus in 2.5 hours, $5 + 1\frac{1}{4} = 6\frac{1}{4}$ cycles will be completed and the temperature will be 30° as shown by the graph in part **c**.

f The graph can be considered to be a sine function with a horizontal translation of 6 to the right. A possible equation is $T = \sin\left(\frac{\pi}{12}(t-6)\right) + 30$.

9 $T = 19 + 6\sin\left(\frac{\pi t}{6}\right)$ with t the time in hours since 10 am.

a **i** As for any sine function, $-1 \le \sin\left(\frac{\pi t}{6}\right) \le 1$.

$$\begin{aligned}\therefore T_{\max} &= 19 + 6 \times 1\\ &= 25\end{aligned}$$

The maximum temperature is 25°.

The maximum temperature occurs when $\sin\left(\frac{\pi t}{6}\right) = 1$

$\therefore \frac{\pi t}{6} = \frac{\pi}{2}$

$\therefore t = 3$

The maximum temperature occurs at 1 pm.

ii The minimum temperature occurs when $\sin\left(\frac{\pi t}{6}\right) = -1$.

$\therefore \frac{\pi t}{6} = \frac{3\pi}{2}$

$\therefore t = 9$

$$\begin{aligned}T_{\min} &= 19 + 6 \times (-1)\\ &= 13°\end{aligned}$$

The minimum temperature of 13° occurs at 7 pm.

b **i** At 11:30 am, $t = 1.5$

$\therefore T = 19 + 6\sin\left(\frac{1.5\pi}{6}\right)$

$\therefore T = 19 + 6\sin\left(\frac{\pi}{4}\right)$

$= 19 + 6 \times \frac{\sqrt{2}}{2}$

$= 19 + 3\sqrt{2}$

$\therefore T \simeq 23.2$

The temperature at 11:30 am is 23.2°.

ii At 7:30 pm, $t = 9.5$

$\therefore T = 19 + 6\sin\left(\frac{9.5\pi}{6}\right)$

$\therefore T = 19 + 6\sin\left(\frac{19\pi}{12}\right)$

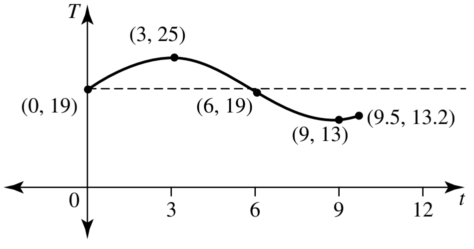
$\therefore T \simeq 13.2$

The temperature at 7:30 pm is 13.2°.

c $T = 19 + 6\sin\left(\frac{\pi t}{6}\right),\ t \in [0, 9.5]$.

Amplitude 6, equilibrium $T = 19$.

Period is $2\pi \div \frac{\pi}{6} = 12$, so for the domain specified the graph will not cover a full cycle.
Right endpoint is $(9.5, 13.2)$, maximum point $(3, 25)$, minimum point $(9, 13)$.
Left endpoint: Let $t = 0$
$\therefore T = 19 + 6\sin(0)$
$\therefore T = 19$
$(0, 19)$



d Let $T = 24$

$$\therefore 24 = 19 + 6\sin\left(\frac{\pi t}{6}\right)$$
$$\therefore 5 = 6\sin\left(\frac{\pi t}{6}\right)$$
$$\therefore \sin\left(\frac{\pi t}{6}\right) = \frac{5}{6}$$

Quadrants 1 and 2, base $\sin^{-1}\left(\frac{5}{6}\right) \simeq 0.99$

$$\therefore \frac{\pi t}{6} = 0.99, \pi - 0.99$$
$$\therefore t = \frac{6}{\pi} \times 0.99, \frac{6}{\pi} \times (\pi - 0.99)$$
$$\therefore t = 1.88, 4.12$$

The air conditioner is switched on at $t = 1.88$ and switched off 2.24 hours later at $t = 4.12$.

e From the graph in part **c**, the coldest two hour period is between $t = 7.5$ and $t = 9.5$.
When $t = 7.5$,

$$\therefore T = 19 + 6\sin\left(\frac{7.5\pi}{6}\right)$$
$$\therefore T = 19 + 6\sin\left(\frac{15\pi}{12}\right)$$
$$\therefore T = 19 + 6\sin\left(\frac{5\pi}{4}\right)$$
$$\therefore T = 19 - 6\sin\left(\frac{\pi}{4}\right)$$
$$\therefore T = 19 - 6 \times \frac{\sqrt{2}}{2}$$
$$\therefore T = 19 - 3\sqrt{2}$$

The heating is switched on at 5 : 30 pm when the temperature is $\left(19 - 3\sqrt{2}\right)^\circ$ or approximately 14.8°.

10 $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$

a When $t = 0$,
$h = 10 - 8.5\cos(0)$
$= 10 - 8.5 \times 1$
$= 1.5$
Initially the carriage is 1.5 metres above the ground.

b After 1 minute, $t = 60$,
$h = 10 - 8.5\cos(\pi)$
$= 10 - 8.5 \times (-1)$
$= 18.5$
After 1 minute the carriage is 18.5 metres above the ground.

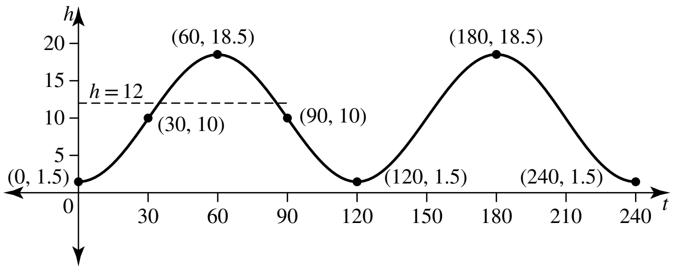
c Period is time to complete one revolution.

$$2\pi \div \frac{\pi}{60} = 2\pi \times \frac{60}{\pi}$$
$$= 120$$

The period is 120 seconds, or 2 minutes.
In 4 minutes two revolutions will be completed.

d $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right), t \in [0, 240]$

Amplitude 8.5, inverted, equilibrium $h = 10$, range $[1.5, 18.5]$, period 120, 2 cycles.



e Let $h = 12$

$$\therefore 12 = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$$
$$\therefore 2 = 8.5\cos\left(\frac{\pi}{60}t\right)$$
$$\therefore \cos\left(\frac{\pi}{60}t\right) = \frac{2}{8.5}$$
$$\therefore \cos\left(\frac{\pi}{60}t\right) = \frac{4}{17}$$

Quadrants 2 and 3, base $\cos^{-1}\left(\frac{4}{17}\right)$

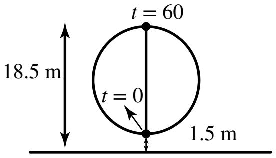
$$\therefore \frac{\pi}{60}t = \pi - \cos^{-1}\left(\frac{4}{17}\right), \pi + \cos^{-1}\left(\frac{4}{17}\right)$$
$$\therefore t = \frac{60}{\pi}\left[\pi - \cos^{-1}\left(\frac{4}{17}\right)\right], \frac{60}{\pi}\left[\pi + \cos^{-1}\left(\frac{4}{17}\right)\right]$$

The graph shows that the carriage is higher than 12 metres above the ground for the time interval between these two values for t.
The time, in seconds is

$$\frac{60}{\pi}\left[\pi + \cos^{-1}\left(\frac{4}{17}\right)\right] - \frac{60}{\pi}\left[\pi - \cos^{-1}\left(\frac{4}{17}\right)\right]$$
$$= 2 \times \frac{60}{\pi} \times \cos^{-1}\left(\frac{4}{17}\right)$$
$$\simeq 51$$

The time is 51 seconds.

f The highest height above the ground is 18.5 metres and the lowest height is 1.5 metres.



The diameter of the circle is $18.5 - 1.5 = 17$ metres.
Therefore, the length of a radial spoke is 8.5 metres.

11 $p = 3\sin(n\pi t) + 5$

a p measures the distance of the water from the sunbather. From the equation, amplitude is 3 and equilibrium is $p = 5$, so the range of values for p are $p \in [2, 8]$.
The closest distance the water reaches to the sunbather is 2 metres.

b In 1 hour, or 60 minutes, 40 cycles of the sine function are completed.

Therefore one cycle is completed in $\frac{60}{40}=\frac{3}{2}$ minutes.

The period is $\frac{3}{2}$ minutes.

$$\therefore \frac{2\pi}{n\pi}=\frac{3}{2}$$
$$\therefore \frac{2}{n}=\frac{3}{2}$$
$$\therefore n=\frac{4}{3}$$

c Second model has equation $p=a\sin(4\pi t)+5$.

Its range, assuming $a>0$, is $[5-a,5+a]$.

As the water just reaches the sunbather, $5-a=0$.

Therefore, $a=5$.

The period is $\frac{2\pi}{4\pi}=\frac{1}{2}$ minute, so in 30 minutes the function completes 60 cycles, reaching P once every cycle.

The water reaches the sunbather 60 times in half an hour.

d For the first model $p=3\sin(n\pi t)+5$ where $n=\frac{4}{3}$, one cycle was completed in $\frac{3}{2}$ minutes.

In 1 minute, $\frac{2}{3}$ of a cycle would be completed so the number of waves per minute is $\frac{2}{3}$.

For the second model $p=a\sin(4\pi t)+5$, one cycle is completed in $\frac{1}{2}$ minute so two cycles are completed in 1 minute. The number of waves per minute is 2.

Therefore, the second model, $p=a\sin(4\pi t)+5$, has the greater number of waves per minute.

12 $x=a\sin(bt)$

a Given information that the amplitude is 20 cm and that the can initially moves downwards from the mean position of $x=0$.

$\therefore a=-20$

The time interval between the lowest point and the following highest point is half a period.

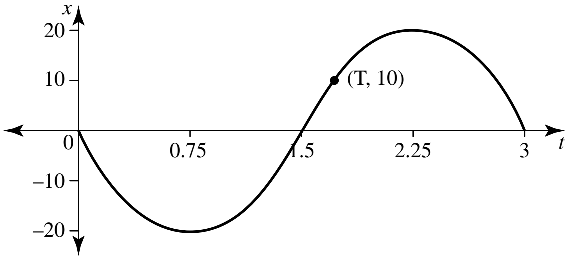
Therefore, the period is $2\times1.5=3$ seconds.

$$\therefore \frac{2\pi}{b}=3$$
$$\therefore b=\frac{2\pi}{3}$$

The equation for the vertical displacement is

$x=-20\sin\left(\frac{2\pi}{3}t\right)$.

b



c The amplitude is 20, so require the displacement to be 10.

Let $x=10$

$$\therefore 10=-20\sin\left(\frac{2\pi}{3}t\right)$$
$$\therefore \sin\left(\frac{2\pi}{3}t\right)=-\frac{1}{2}$$

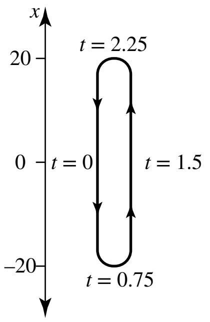
For the shortest time require the solution in quadrant 3.

Base is $\frac{\pi}{6}$

$$\therefore \frac{2\pi}{3}t=\pi+\frac{\pi}{6}$$
$$\therefore \frac{2\pi}{3}t=\frac{7\pi}{6}$$
$$\therefore t=\frac{7\pi}{6}\times\frac{3}{2\pi}$$
$$\therefore t=\frac{7}{4}$$
$$\therefore T=\frac{7}{4}$$

The shortest time is $\frac{7}{4}$ seconds.

d The can falls 20 centimetres, rises 20 centimetres to equilibrium, rises 20 centimetres to reach its greatest displacement and then falls 20 centimetres back to equilibrium. The total distance moved is 80 centimetres.



13 $I=4\sin(t)-3\cos(t)$

a Sketch the graph in the Graph&Tab menu over $[0,4\pi]$ and use the Analysis tools to obtain the co-ordinates of the first maximum point as (2.214...,5). The y co-ordinate is the value of the intensity.

The maximum intensity is 5 units.

b Tap Analysis → G-Solve → Root to obtain the first value of t for which $I=0$ as $t=0.64$.

c The amplitude of the graph is 5 and its period is 2π. This is confirmed by observations of the graph. To consider the graph as that of a sine function, the horizontal translation would be 0.64 units to the right.

$\therefore I=5\sin(t-0.64)$ is the same curve as $I=4\sin(t)-3\cos(t)$.

d In the form $I=a\cos(t+b)$, the horizontal translation would be 2.214....To two decimal places, $I=5\cos(t-2.21)$ is the same curve as $I=4\sin(t)-3\cos(t)$.

14 $y=x+4+4\cos(6x),\ 0\le x\le 4\pi$

a Counting the number of peaks on the given graph, including the ones at the two ends, gives 13 teeth.

b Sketch the graph in Graph&Tab menu and obtain the x co-ordinates of each peak. Test to see how far apart each is.

x co-ordinate of maximum points	**Difference between x values of successive maximums.**
$x_1=1.0541$	
	$x_2-x_1=1.0472$
$x_2=2.1013$	
	$x_3-x_2=1.0472$
$x_3=3.1485$	
	$x_4-x_3=1.0472$
$x_4=4.1957$	

The x values of the maximum points appear to be the same distance of 1.0472 cm apart.

The $\cos(6x)$ term has period $\frac{2\pi}{6}=\frac{\pi}{3}=1.0472$.

Successive peaks are $\frac{\pi}{3}$ cm apart.

c The greatest width is the y value of the 13th peak. This can be calculated in exact form from the equation.

Let $x=4\pi$

$$\begin{aligned}\therefore y&=4\pi+4+4\cos(24\pi)\\&=4\pi+4+4\times 1\\&=4\pi+8\end{aligned}$$

The greatest width of the saw is $(4\pi+8)$ cm.

As a decimal value of 20.566 cm, the greatest width could be calculated by tapping Analysis→ G-Solve → y-Cal for $x=4\pi$.

d Since $-1\le\cos(6x)\le 1$ then,

$-4\le 4\cos(6x)\le 4$

$\therefore -4+(x+4)\le 4\cos(6x)+(x+4)\le 4+(x+4)$

$\therefore x\le y\le x+8$

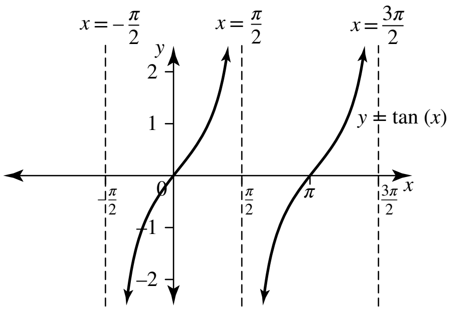
The line $y=x+8$ will touch the teeth. This can be confirmed by sketching this line with that of $y=x+4+4\cos(6x),\ 0\le x\le 4\pi$.

Exercise 10.5 — The tangent function

1 $y=\tan(x),\ x\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$

Asymptotes: $x=-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2}$

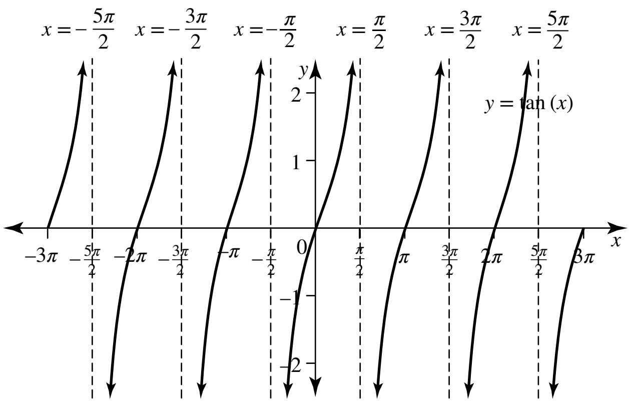
x intercepts are midway between the asymptotes at the origin and $(\pi,0)$.



2 $y=\tan(x),\ x\in[-3\pi,3\pi]$

Asymptotes: $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2}$

x intercepts midway between successive pairs of asymptotes: $(\pm\pi,0),(\pm 2\pi,0),(\pm 3\pi,0)$ and $(0,0)$.

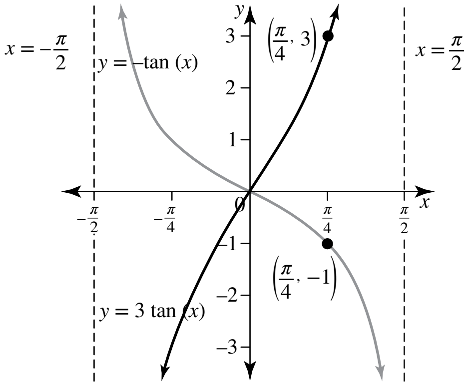


Domain of the graph is $[-3\pi,3\pi]\setminus\left\{\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2}\right\}$ and range is R.

3 **a** When $x=\frac{\pi}{4}$,

$$\begin{aligned}y&=-\tan(x) & y&=3\tan(x)\\&=-\tan\left(\frac{\pi}{4}\right)\quad\text{and} & &=3\tan\left(\frac{\pi}{4}\right)\\&=-1 & &=3\end{aligned}$$

Both graphs have asymptotes with equations $x=\pm\frac{\pi}{2}$, period of π and contain the point $(0,0)$.



b $y=\tan(x)+\sqrt{3},\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Vertical translation of $\sqrt{3}$ upwards does not affect the asymptotes at $x=\pm\frac{\pi}{2}$.

x intercept: Let $y=0$

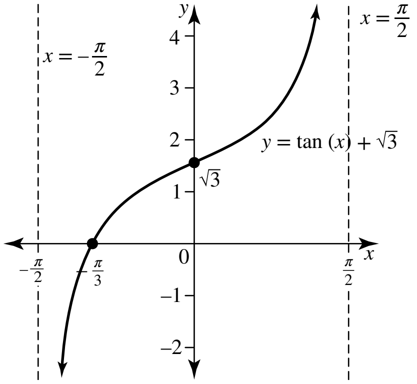
$\therefore \tan(x)+\sqrt{3}=0$

$\therefore \tan(x)=-\sqrt{3},\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\therefore x=-\frac{\pi}{3}$

$\left(-\frac{\pi}{3},0\right)$ is the intercept.

When $x=0$, $y=\sqrt{3}$.



4 $y=\tan(x)+1$ is obtained from $y=\tan(x)$ by a vertical translation upwards of 1 unit. For both graphs, the equations of the asymptotes are $x=\frac{\pi}{2},x=\frac{3\pi}{2}$ and the period is π.

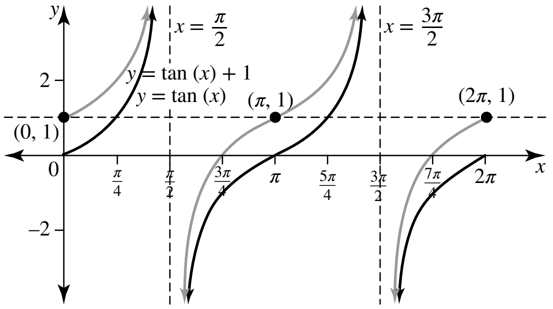
x intercepts for $y=\tan(x)$ occur at $x=0,\pi,2\pi$.

x intercepts for $y=\tan(x)+1$ occur when $\tan(x)+1=0$

$\therefore \tan(x)=-1$

$\therefore x=\pi-\frac{\pi}{4},2\pi-\frac{\pi}{4}$

$\therefore x=\frac{3\pi}{4},\frac{7\pi}{4}$



5 $y = \tan(3x)$ has period $\frac{\pi}{3}$.

Asymptotes occur when $3x = \frac{\pi}{2}, \frac{3\pi}{2}$

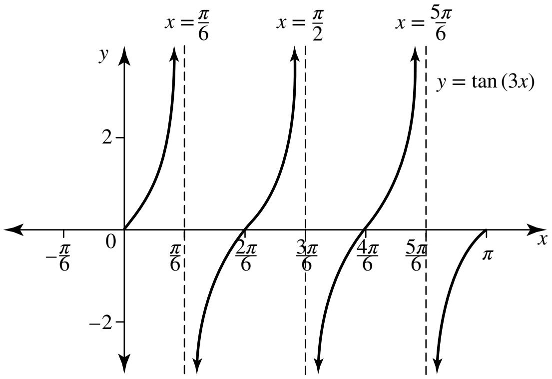
$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}$

Adding a period gives another asymptote at $x = \frac{\pi}{2} + \frac{\pi}{3}$ (Others are outside the given domain).

$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ are the equations of the asymptotes.

x intercepts lie midway between the asymptotes at $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$

Scale the horizontal axis in multiples of $\frac{\pi}{6}$.



6 The period of $y = -2\tan\left(\frac{x}{2}\right)$ is $\pi \div \frac{1}{2} = 2\pi$, There will only be one asymptote since the domain allows only one period.

asymptote when $\frac{x}{2} = \frac{\pi}{2}$ so therefore $x = \pi$.

Graph is inverted.

Point to illustrate the dilation factor of 2:

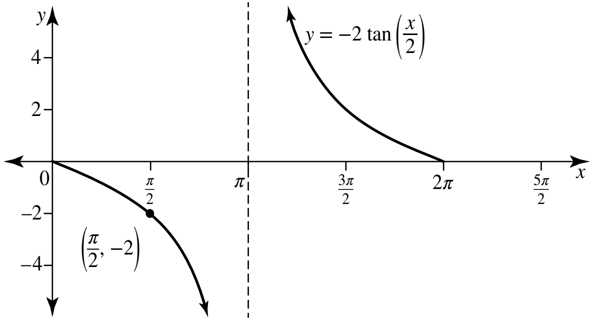
When $x = \frac{\pi}{2}$,

$$y = -2\tan\left(\frac{\pi}{4}\right)$$
$$= -2$$

Point $\left(\frac{\pi}{2}, -2\right)$

x intercepts: If $x = 0$ or $2\pi, y = 0$.

Scale the horizontal axis in multiples of $\frac{\pi}{2}$.



7 a $y = \tan\left(x - \frac{\pi}{4}\right)$ has period π and a horizontal translation of $\frac{\pi}{4}$ units to the right.

x intercepts, using the translation, become

$x = 0 + \frac{\pi}{4}, \pi + \frac{\pi}{4}, x = 2\pi + \frac{\pi}{4}$.

Within the domain, the x intercepts occur at $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

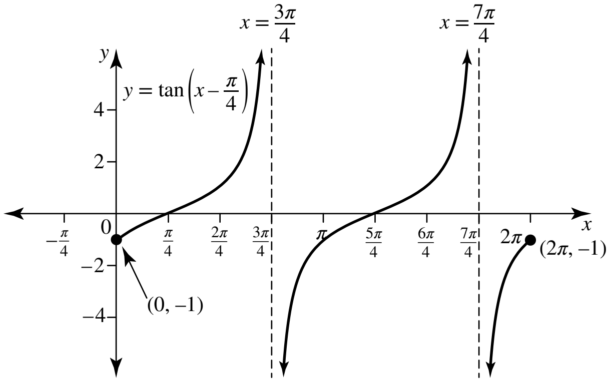
Asymptotes when $x - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}$

Therefore the equations of the asymptotes are $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

End points: When $x = 0$, When $x = 2\pi$,

$$y = \tan\left(-\frac{\pi}{4}\right) \qquad y = \tan\left(2\pi - \frac{\pi}{4}\right)$$
$$= -1 \qquad = -1$$

Endpoints are $(0, -1), (2\pi, -1)$.



b i $y = \tan(x) \to y = \tan\left(\frac{4\pi x}{3} + \frac{\pi}{3}\right)$

$$y = \tan\left(\frac{4\pi x}{3} + \frac{\pi}{3}\right)$$
$$\therefore y = \tan\left(\frac{4\pi}{3}\left(x + \frac{1}{4}\right)\right)$$

Dilation of factor $\frac{3}{4\pi}$ from the y axis followed by a horizontal translation of $\frac{1}{4}$ to the left.

ii Period $= \frac{\pi}{n}$, $n = \frac{4\pi}{3}$

$$= \pi \div \frac{4\pi}{3}$$
$$= \pi \times \frac{3}{4\pi}$$
$$= \frac{3}{4}$$

Period is $\frac{3}{4}$.

8 $y = \tan\left(2x + \frac{\pi}{4}\right) \Rightarrow y = \tan\left(2\left(x + \frac{\pi}{8}\right)\right)$

period $\frac{\pi}{2}$, horizontal translation $\frac{\pi}{8}$ to the left.

endpoints : $x = 0, y = \tan\left(\frac{\pi}{4}\right) = 1 \Rightarrow (0, 1)$ and

$x = \pi, y = \tan\left(2\pi + \frac{\pi}{4}\right) = 1 \Rightarrow (\pi, 1)$

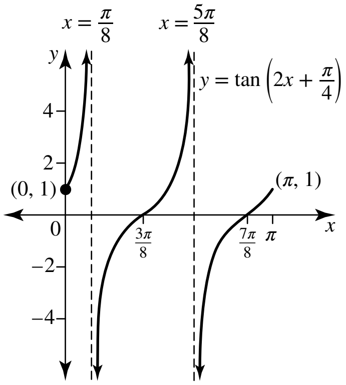
Asymptotes when $2x + \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}$

$$\therefore 2x = \frac{\pi}{4}, \frac{5\pi}{4}$$
$$\therefore x = \frac{\pi}{8}, \frac{5\pi}{8}$$

Other asymptotes are outside the domain.
x intercepts occur halfway between the asymptotes and one period after that, so

$x=\frac{3\pi}{8},\frac{3\pi}{8}+\frac{\pi}{2}\Rightarrow x=\frac{3\pi}{8},\frac{7\pi}{8}$

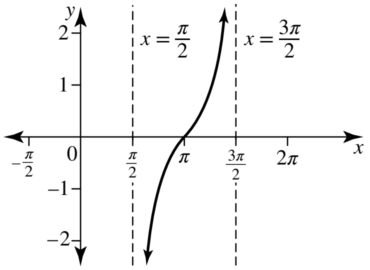
Scale the horizontal axis in units of $\frac{\pi}{8}$.



9 a $y=\tan(x),\ x\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

Asymptotes: $x=\frac{\pi}{2},\frac{3\pi}{2}$

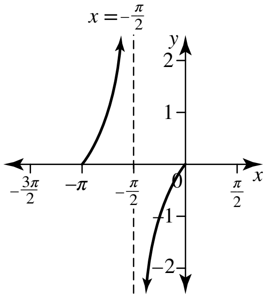
x intercept : $(\pi,0)$



b $y=\tan(x),\ x\in[-\pi,0]$

Asymptote: $x=-\frac{\pi}{2}$

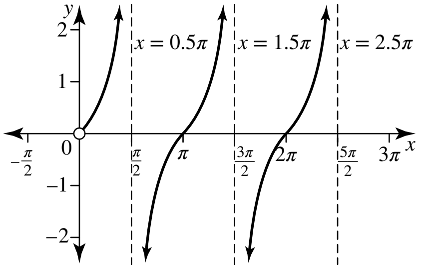
x intercepts and end points : $(-\pi,0),(0,0)$



c $y=\tan(x),\ x\in\left(0,\frac{5\pi}{2}\right)$

Asymptotes: $x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}$

x intercepts: $(\pi,0),(2\pi,0)$, open point at $(0,0)$.



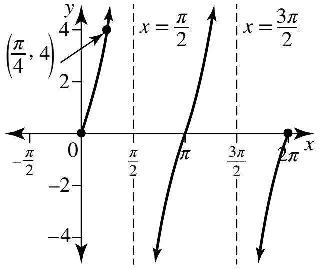
10 a $y=4\tan(x),\ x\in[0,2\pi]$

Period π, asymptotes $x=\frac{\pi}{2},x=\frac{3\pi}{2}$

Point: Let $x=\frac{\pi}{4}$

$$\begin{aligned}\therefore y&=4\tan\left(\frac{\pi}{4}\right)\\&=4\times1\\&=4\end{aligned}$$

$\left(\frac{\pi}{4},4\right)$



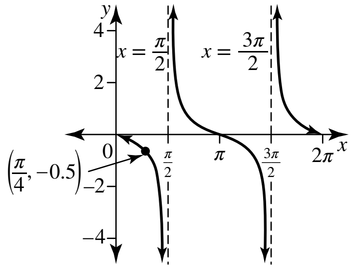
b $y=-0.5\tan(x),\ x\in[0,2\pi]$

Period π, asymptotes $x=\frac{\pi}{2},x=\frac{3\pi}{2}$, graph is inverted

Point: Let $x=\frac{\pi}{4}$

$$\begin{aligned}\therefore y&=-0.5\tan\left(\frac{\pi}{4}\right)\\&=-0.5\times1\\&=-0.5\end{aligned}$$

$\left(\frac{\pi}{4},-0.5\right)$



c $y=-\tan(x),-\frac{\pi}{2}<x<\frac{\pi}{2}$

Period π, asymptotes $x=\pm\frac{\pi}{2}$, x intercept $(0,0)$, reflected in x axis.

$y=1-\tan(x),-\frac{\pi}{2}<x<\frac{\pi}{2}$

Vertical translation up 1 unit from $y=-\tan(x)$.

Period π, asymptotes $x=\pm\frac{\pi}{2}$, passes through $(0,1)$.

x intercept: Let $y=0$

$\therefore 0=1-\tan(x)$

$\therefore \tan(x)=1$

$\therefore x=\frac{\pi}{4}$

$\left(\frac{\pi}{4},0\right)$

$y=-1-\tan(x),-\frac{\pi}{2}<x<\frac{\pi}{2}$

Vertical translation down 1 unit from $y=-\tan(x)$.

Period π, asymptotes $x=\pm\frac{\pi}{2}$, passes through $(0,-1)$.

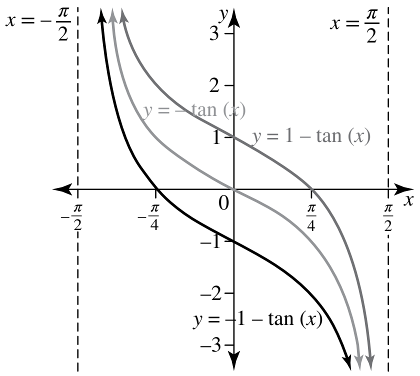
x intercept: Let $y=0$

$\therefore 0 = -1 - \tan(x)$

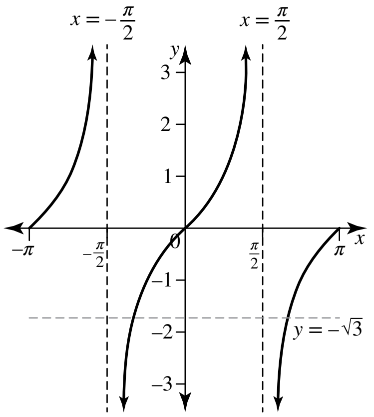
$\therefore \tan(x) = -1$

$\therefore x = -\dfrac{\pi}{4}$

$\left(-\dfrac{\pi}{4}, 0\right)$



11 a $y = \tan(x),\ \pi \le x \le \pi$



b Let $y = -\sqrt{3}$

$\therefore \tan(x) = -\sqrt{3},\ -\pi \le x \le \pi$

Solutions lie in quadrant 2 with a positive rotation and quadrant 4 with a negative rotation. Base $\dfrac{\pi}{3}$.

$\therefore x = -\dfrac{\pi}{3}, \pi - \dfrac{\pi}{3}$

$\therefore x = -\dfrac{\pi}{3}, \dfrac{2\pi}{3}$

c $\tan(x) + \sqrt{3} < 0$

$\therefore \tan(x) < -\sqrt{3}$

Reading from the graph, $-\dfrac{\pi}{2} < x < -\dfrac{\pi}{3}$ or $\dfrac{\pi}{2} < x < \dfrac{2\pi}{3}$.

12 a $y = \tan(6x)$

Period is $\dfrac{\pi}{6}$.

b $y = 5\tan\left(\dfrac{x}{4}\right)$

Period is $\pi \div \dfrac{1}{4} = 4\pi$.

c $y = -2\tan\left(\dfrac{3x}{2}\right) + 5$

Period is $\pi \div \dfrac{3}{2} = \dfrac{2\pi}{3}$

d $y = \tan(\pi x)$

Period is $\dfrac{\pi}{\pi} = 1$.

13 a $y = \tan(4x),\ 0 \le x \le \pi$

Period is $\dfrac{\pi}{4}$.

An asymptote occurs when $4x = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{8}$ and others are a period apart.

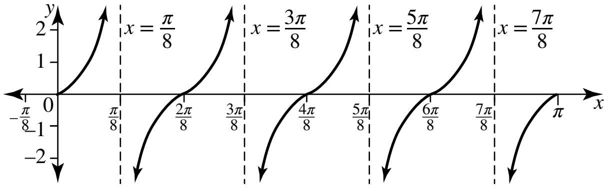
For $0 \le x \le \pi$, the equations of the asymptotes are:

$x = \dfrac{\pi}{8}, x = \dfrac{3\pi}{8}, x = \dfrac{5\pi}{8}, x = \dfrac{7\pi}{8}$.

x intercepts occur midway between the asymptotes at

$x = 0, x = \dfrac{2\pi}{8}, x = \dfrac{4\pi}{8}, x = \dfrac{6\pi}{8}, x = \pi$.

$\therefore x = 0, x = \dfrac{\pi}{4}, x = \dfrac{\pi}{2}, x = \dfrac{3\pi}{4}, x = \pi$



b $y = 2\tan(2x),\ 0 \le x \le \pi$

Period $\dfrac{\pi}{2}$

Asymptotes: When $2x = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{4}$

$x = \dfrac{\pi}{4}, x = \dfrac{3\pi}{4}$ are the equations of the asymptotes.

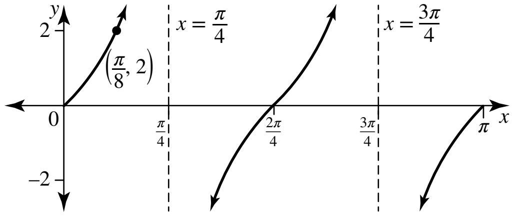
x intercepts at $x = 0, x = \dfrac{\pi}{2}, x = \pi$

Point: Let $x = \dfrac{\pi}{8}$

$\therefore y = 2\tan\left(\dfrac{\pi}{4}\right)$

$= 2$

$\left(\dfrac{\pi}{8}, 2\right)$



c $y = -\tan\left(\dfrac{x}{3}\right),\ 0 \le x \le \pi$

Period is $\pi \div \dfrac{1}{3} = 3\pi$, graph is inverted.

Asymptotes: When $\dfrac{x}{3} = \dfrac{\pi}{2} \Rightarrow x = \dfrac{3\pi}{2}$ which is outside the interval allowed.

Subtracting a period, $x = -\dfrac{3\pi}{2}$, which is also outside the interval allowed.

There are no asymptotes for $0 \le x \le \pi$.

x intercepts: Let $y = 0$

$\therefore 0 = -\tan\left(\dfrac{x}{3}\right)$

$\therefore \tan\left(\dfrac{x}{3}\right) = 0$

$\therefore \dfrac{x}{3} = -\pi, 0, \pi...$

$\therefore x = 0$

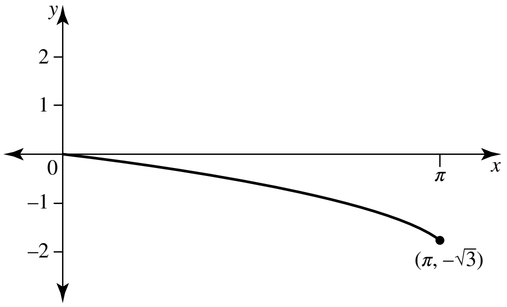
For $0 \le x \le \pi$, there is only one x intercept at $(0,0)$.

Endpoint: Let $x = \pi$

$\therefore y = -\tan\left(\dfrac{\pi}{3}\right)$

$= -\sqrt{3}$

$\left(\pi, -\sqrt{3}\right)$



d $y = 3\tan(x) + 2,\ 0 \le x \le \pi$

Period is π.

Asymptote: $x = \dfrac{\pi}{2}$

y intercept is $(0,2)$ since there is a vertical translation 2 units upwards.

x intercepts: Let $y = 0$

$\therefore 3\tan(x) + 2 = 0$

$\therefore \tan(x) = -\dfrac{2}{3}$

As $x \in [0, \pi]$, solution is in the second quadrant.

$\therefore x = \pi - \tan^{-1}\left(\dfrac{2}{3}\right)$

$\therefore x \simeq 2.55$

Point: Let $x = \dfrac{\pi}{4}$

$\therefore y = 3\tan\left(\dfrac{\pi}{4}\right) + 2$

$\therefore y = 5$

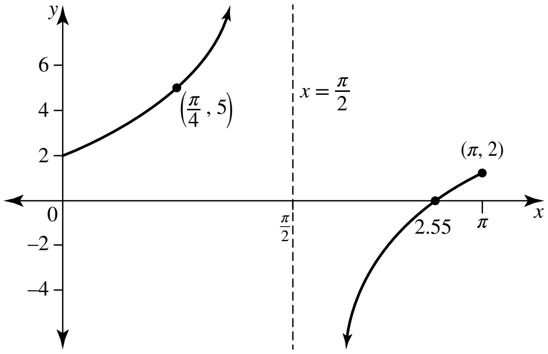
$\left(\dfrac{\pi}{4}, 5\right)$

Endpoint: Let $x = \pi$

$\therefore y = 3\tan(\pi) + 2$

$\therefore y = 2$

$(\pi, 2)$



e $y = 3(1 - \tan(x)),\ 0 \le x \le \pi$

$\therefore y = 3 - 3\tan(x)$

Period is π, asymptote equation $x = \dfrac{\pi}{2}$.

Graph is reflected in x axis and vertical translation 3 upwards, so y intercept is $(0,3)$.

x intercepts: Let $y = 0$

$\therefore 0 = 3 - 3\tan(x)$

$\therefore \tan(x) = 1$

$\therefore x = \dfrac{\pi}{4}$

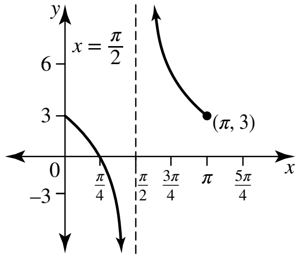
$\left(\dfrac{\pi}{4}, 0\right)$

Endpoint: Let $x = \pi$

$\therefore y = 3 - 3\tan(\pi)$

$\therefore y = 3$

$(\pi, 3)$



f $y = \tan(3x) - 1,\ 0 \le x \le \pi$

Period is $\dfrac{\pi}{3}$.

Asymptotes: When $3x = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{6}$

For $0 \le x \le \pi$, the equations of the asymptotes are:

$x = \dfrac{\pi}{6}, x = \dfrac{3\pi}{6} = \dfrac{\pi}{2}, x = \dfrac{5\pi}{6}$.

Vertical translation of 1 unit downwards so the y intercept is $(0, -1)$.

x intercepts: Let $y = 0$

$\therefore \tan(3x) - 1 = 0$

$\therefore \tan(3x) = 1,\ 0 \le 3x \le 3\pi$

$\therefore 3x = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi + \dfrac{\pi}{4}$

$= \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}$

$\therefore x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{3\pi}{4}$

$\left(\dfrac{\pi}{4}, 0\right)$

Endpoint: Let $x = \pi$

$\therefore y = 3 - 3\tan(\pi)$

$\therefore y = 3$

$(\pi, 3)$

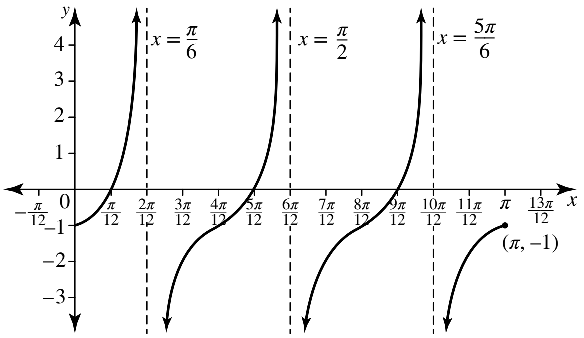
$\therefore \tan(3x) - 1 = 0$

$\therefore \tan(3x) = 1,\ 0 \le 3x \le 3\pi$

$\therefore 3x = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi + \dfrac{\pi}{4}$

$= \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}$

$\therefore x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{3\pi}{4}$



14 $y = \tan(bx),\ 0 \le x \le \frac{3\pi}{2}$

a **i** There are 4 cycles shown.

ii If 4 cycles are completed over an interval of $\frac{3\pi}{2}$ units, then one cycle is completed over an interval of length $\frac{3\pi}{8}$ units.

Therefore, the period of the graph is $\frac{3\pi}{8}$.

iii From the equation, the period is $\frac{\pi}{b}$.

$$\therefore \frac{\pi}{b} = \frac{3\pi}{8}$$
$$\therefore \frac{1}{b} = \frac{3}{8}$$
$$\therefore b = \frac{8}{3}$$

iv An asymptote occurs when $bx = \frac{\pi}{2}$

$$\therefore \frac{8}{3}x = \frac{\pi}{2}$$
$$\therefore x = \frac{\pi}{2} \times \frac{3}{8}$$
$$\therefore x = \frac{3\pi}{16}$$

Other asymptotes are a period apart. Since the period is $\frac{3\pi}{8} = \frac{6\pi}{16}$, the equations of the asymptotes are $x = \frac{3\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16}, \frac{21\pi}{16}$.

b $f(x) = a - \tan(cx)$

The function is undefined at its asymptote positions.

There is an asymptote when $cx = -\frac{\pi}{2} \Rightarrow x = -\frac{\pi}{2c}$. This is the least negative value for which the function is undefined.

$$\therefore -\frac{\pi}{2c} = -\frac{6}{5}$$
$$\therefore 5\pi = 12c$$
$$\therefore c = \frac{5\pi}{12}$$
$$\therefore f(x) = a - \tan\left(\frac{5\pi}{12}x\right)$$

Given $f(-12) = -2$,

$$a - \tan\left(\frac{5\pi}{12} \times -12\right) = -2$$
$$\therefore a - \tan(-5\pi) = -2$$
$$\therefore a - 0 = -2$$
$$\therefore a = -2$$

Answer: $a = -2,\ c = \frac{5\pi}{12}$.

15 a $y = \tan\left(x - \frac{\pi}{6}\right),\ 0 \le x \le 2\pi$

Period π, horizontal translation $\frac{\pi}{6}$ to right.

Asymptotes: $x - \frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}$

x intercepts: $x - \frac{\pi}{6} = 0, \pi, 2\pi$

$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}, \cancel{\frac{13\pi}{6}}$ (reject as outside the domain interval)

Endpoints: Let $x = 0$

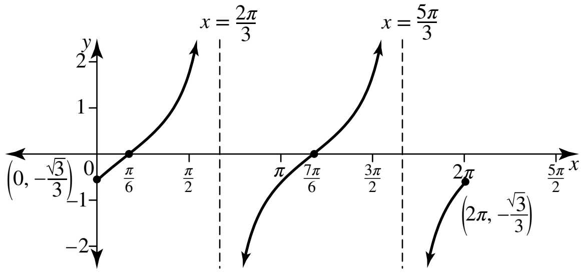
$$\therefore y = \tan\left(-\frac{\pi}{6}\right)$$
$$= -\tan\left(\frac{\pi}{6}\right)$$
$$= -\frac{\sqrt{3}}{3}$$

$\left(0, -\frac{\sqrt{3}}{3}\right)$

Let $x = 2\pi$

$$\therefore y = \tan\left(2\pi - \frac{\pi}{6}\right)$$
$$= -\tan\left(\frac{\pi}{6}\right)$$
$$= -\frac{\sqrt{3}}{3}$$

$\left(2\pi, -\frac{\sqrt{3}}{3}\right)$



b $y = \tan\left(x + \frac{\pi}{3}\right),\ 0 \le x \le 2\pi$

Period π, horizontal translation $\frac{\pi}{3}$ to left.

Asymptotes: $x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}$

x intercepts: $x + \frac{\pi}{3} = 0, \pi, 2\pi$

$\therefore x = -\cancel{\frac{\pi}{3}}, \frac{2\pi}{3}, \frac{5\pi}{3}$

Endpoints: Let $x = 0$

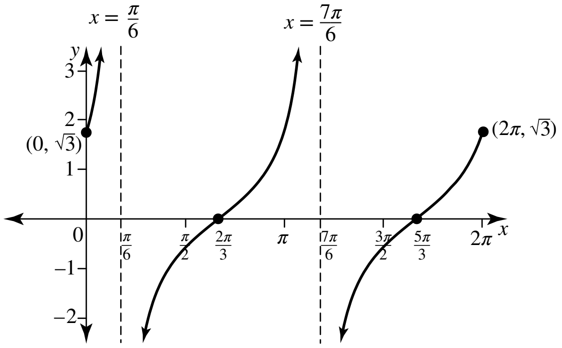
$$\therefore y = \tan\left(\frac{\pi}{3}\right)$$
$$= \sqrt{3}$$

$\left(0, \sqrt{3}\right)$

Let $x = 2\pi$

$$\begin{aligned}\therefore y &= \tan\left(2\pi+\frac{\pi}{3}\right)\\ &= \tan\left(\frac{\pi}{3}\right)\\ &= \sqrt{3}\end{aligned}$$

$\left(2\pi,\sqrt{3}\right)$



c $y=2\tan\left(x+\frac{\pi}{2}\right),\ 0\le x\le 2\pi$

Period π, horizontal translation $\frac{\pi}{2}$ to left.

Asymptotes: $x+\frac{\pi}{2}=\frac{\pi}{2},\frac{3\pi}{2}$

$\therefore x=0,\pi$ and adding another period $x=2\pi$ is also an asymptote.

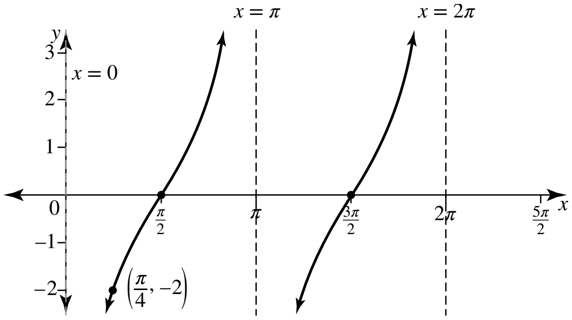
x intercepts: $x+\frac{\pi}{2}=0,\pi,2\pi$

$\therefore x=\cancel{-\frac{\pi}{2}},\frac{\pi}{2},\frac{3\pi}{2}$

Point: Let $x=\frac{\pi}{4}$

$$\begin{aligned}\therefore y &= 2\tan\left(\frac{3\pi}{4}\right)\\ &= -2\tan\left(\frac{\pi}{4}\right)\\ &= -2\end{aligned}$$

$\left(\frac{\pi}{4},-2\right)$



d $y=-\tan\left(2x-\frac{\pi}{2}\right),\ 0\le x\le 2\pi$

$\therefore y=-\tan\left(2\left(x-\frac{\pi}{4}\right)\right)$

Period is $\frac{\pi}{2}$, graph is reflected in the x axis and there is a horizontal translation $\frac{\pi}{4}$ to right.

Asymptotes: $2x-\frac{\pi}{2}=\frac{\pi}{2},\frac{3\pi}{2}$

$\therefore 2x=\pi,2\pi$

$\therefore x=\frac{\pi}{2},\pi$

Subtracting and adding the period, $x=0,\frac{3\pi}{2},2\pi$ are also asymptotes.

Asymptote equations are $x=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}2\pi$.

x intercepts: $2x-\frac{\pi}{2}=0,\pi,2\pi$

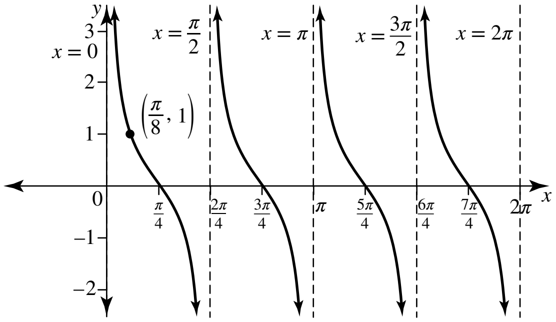
$$\begin{aligned}\therefore 2x &= \frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}\\ x &= \frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4}\end{aligned}$$

Adding the period, $x=\frac{7\pi}{4}$ will also be an x intercept.

Point: Let $x=\frac{\pi}{8}$

$$\begin{aligned}\therefore y &= -\tan\left(\frac{\pi}{4}-\frac{\pi}{2}\right)\\ &= -\tan\left(-\frac{\pi}{4}\right)\\ &= \tan\left(\frac{\pi}{4}\right)\\ &= 1\end{aligned}$$

$\left(\frac{\pi}{8},1\right)$



16 $f:[-3,6]\setminus D\rightarrow R, f(x)=\tan(ax)$

a $f(0)=\tan(0)=0$

b Since p is the smallest positive value for which $f(x)=f(x+p)$, then p is the period of the function. Given $p=3$, then $\frac{\pi}{a}=3$. Therefore, $a=\frac{\pi}{3}$.

c The function rule is $f(x)=\tan\left(\frac{\pi}{3}x\right)$.

The function is not defined at its asymptotes. The asymptotes occur when

$\frac{\pi}{3}x=\frac{\pi}{2},\frac{3\pi}{2}$

$\therefore x=\frac{3}{2},\frac{9}{2}$

Subtracting the period, another asymptotes occurs when $x=-\frac{3}{2}$.

The function is not defined for $x=-\frac{3}{2},\frac{3}{2},\frac{9}{2}$ so $D=\left\{-\frac{3}{2},\frac{3}{2},\frac{9}{2}\right\}$.

d Period and asymptotes are known.

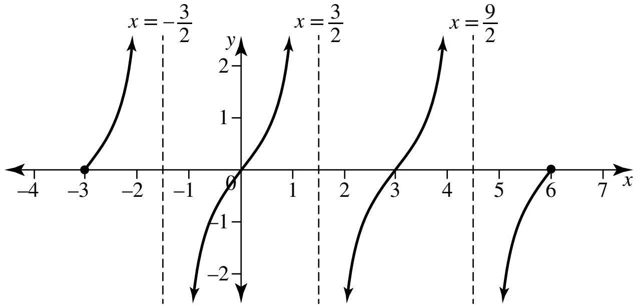
x intercepts: Let $y=0$

$\therefore \tan\left(\frac{\pi}{3}x\right)=0$

As $-3\le x\le 6$, then $-\pi\le\frac{\pi}{3}x\le 2\pi$.

$\therefore \frac{\pi}{3}x=-\pi,0,\pi,2\pi$

$\therefore x=-3,0,3,6$



e $f(x)=1$

$\therefore \tan\left(\frac{\pi}{3}x\right)=1,-\pi\le\frac{\pi}{3}x\le 2\pi$

Quadrants 1 and 3, base $\frac{\pi}{4}$

$\therefore \frac{\pi}{3}x=-\pi+\frac{\pi}{4},\frac{\pi}{4},\pi+\frac{\pi}{4}$

$\therefore \frac{\pi}{3}x=-\frac{3\pi}{4},\frac{\pi}{4},\frac{5\pi}{4}$

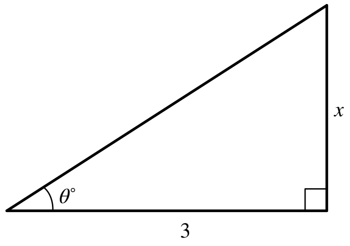
$\therefore x=-\frac{9}{4},\frac{3}{4},\frac{15}{4}$

f The graph of $y=f(x-1)$ is a horizontal translation 1 unit to the right of the graph of $y=f(x)$. Therefore, the asymptotes for $y=f(x-1)$ are:

$x=-\frac{3}{2}+1,\frac{3}{2}+1,\frac{9}{2}+1$

$\therefore x=-\frac{1}{2},\frac{5}{2},\frac{11}{2}$

17 a



$\tan(\theta°)=\frac{x}{3}$

$\therefore x=3\tan(\theta°)$

b $x=3\tan\left(\theta^{\text{o}}\right),\ 0^{\text{o}}\le\theta^{\text{o}}\le 70^{\text{o}}$

Endpoints: Let $\theta=0$

$\therefore x=3\tan(0°)$

$\therefore x=0$

$(0,0)$

Let $\theta=70$

$\therefore x=3\tan(70°)$

$\therefore x\simeq 8.24$

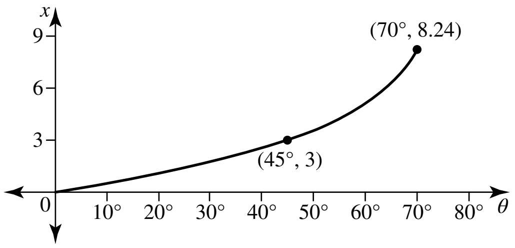
$(70°,8.24)$

Point: Let $\theta=45$

$\therefore x=3\tan(45°)$

$\therefore x=3$

$(45°,3)$



c As the angle $\theta°$ increases, the value of x increases, so the distance the ladder reaches up the wall increases.

d If $\theta°=90°$, the ladder is vertical and therefore parallel to the wall; it cannot reach the wall. The graph has an asymptote when $\theta°=90°$.

18 $h=12\tan\left(\frac{\pi}{144}t\right),\ 0\le t\le 48$

a When $t=24$,

$$\begin{aligned}h&=12\tan\left(\frac{\pi}{144}\times 24\right)\\&=12\tan\left(\frac{\pi}{6}\right)\\&=12\times\frac{\sqrt{3}}{3}\\&=4\sqrt{3}\end{aligned}$$

The wave is $4\sqrt{3}\simeq 6.93$ metres high.

b When $h=12$,

$$12=12\tan\left(\frac{\pi}{144}t\right)$$

$\therefore \tan\left(\frac{\pi}{144}t\right)=1$

$\therefore \frac{\pi}{144}t=\frac{\pi}{4}$

$\therefore t=\frac{\pi}{4}\times\frac{144}{\pi}$

$\therefore t=36$

The wave reaches a height of 12 metres after 36 minutes.

c When $t=48$,

$$\begin{aligned}h&=12\tan\left(\frac{\pi}{144}\times 48\right)\\&=12\tan\left(\frac{\pi}{3}\right)\\&=12\times\sqrt{3}\\&=12\sqrt{3}\end{aligned}$$

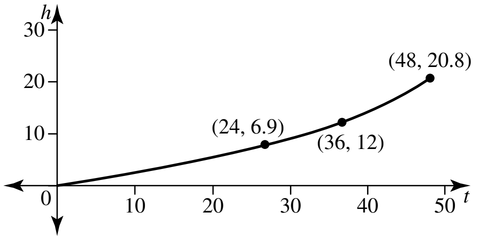
The peak height is $12\sqrt{3}\simeq 20.78$ metres. This is lower than the peak height of 40.5 metres reached by the Japanese tsunami by approximately 19.72 metres.

d Graph of $h=12\tan\left(\frac{\pi}{144}t\right),\ 0\le t\le 48$

Period is

$$\begin{aligned}&\pi\div\frac{\pi}{144}\\&=\pi\times\frac{144}{\pi}\\&=144\end{aligned}$$

Asymptote at $t = 72$ lies outside the domain specified.
Endpoints are $(0, 0)$ and $\left(48, 12\sqrt{3}\right)$.



19 a Graph $y = \tan(2x) - 3,\ 0 \le x \le 2\pi$ in the Graph&Tab menu with the calculator on Standard and Rad modes.
The gaps in the graph indicate there are four asymptotes, but their equations will need to be deduced.

Asymptotes occur when $2x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{4}$ and multiples of the period thereafter. The period is $\frac{\pi}{2}$, so the equations of the asymptotes are $x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$.

By checking that y-Cal is undefined when $x = \frac{\pi}{4}$, these asymptotes can be confirmed and then the vertical lines $x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$ can be added to the graph.

b A similar approach is required in order to graph $y = \tan(2x - 3)$.
The graph indicates there are four asymptotes for $0 \le x \le 2\pi$. The period is $\frac{\pi}{2} \simeq 1.57$.

An asymptote occurs when

$$2x - 3 = \frac{\pi}{2}$$
$$\therefore 2x = \frac{\pi}{2} + 3$$
$$\therefore x = \frac{\pi}{4} + \frac{3}{2}$$
$$\therefore x \quad 2.3$$

Adding and subtracting the period generates the other asymptotes.
The equations of the asymptotes are, to one decimal place, $x = 0.7, 2.3, 3.9, 5.4$

20 $y = \tan\left(\frac{\pi}{4} - x\right)$ and $y = \frac{1}{(6 + k)x}$ for $x \in \left[0, \frac{\pi}{4}\right]$

a If $k = 0$, $y = \frac{1}{6x}$.

Graphing this hyperbola and the tangent graph in Graph&Tab produces two graphs which lie close together so it is not possible with conviction to tell if they intersect by eye. Tapping Analysis → G-Solve → Intersect, says 'Not found', so there are no points of intersection.

b i For there to be intersections, the value of k will be only slightly more than $k = 0$.
One method is to use a trial and error method.

If $k = 0.1$, there are no intersections of $y = \frac{1}{(6 + 0.1)x}$ with the tangent graph.

If $k = 0.2$, there are two intersections of $y = \frac{1}{(6 + 0.2)x}$ with the tangent graph.

$0.1 < k < 0.2$

If $k = 0.15$, there are two intersections of $y = \frac{1}{(6 + 0.15)x}$ with the tangent graph.
$\therefore 0.1 < k < 0.15$
Continuing to test gives $k = 0.13$ as the first value, to two decimal places, for which there are two intersections.

ii If $k = 0.12$ there are no intersections but if $k = 0.13$ there are two intersections.
For one intersection, $0.12 < k < 0.13$.

If $k = 0.125$, there are no intersections.

$0.125 < k < 0.13$

Continued testing yields no intersection for $k = 0.1276$ and two intersections for $k = 0.1277$.

$\therefore 0.1276 < k < 0.1277$

Hence the first value of k, to three decimal places, for which there is one intersection would be $k = 0.128$. (This is the value to three decimal places, not the true value).

Exercise 10.6 — Trigonometric relationships

1 a $2 - 2\sin^2(\theta)$

$$= 2\left(1 - \sin^2(\theta)\right)$$
$$= 2\left(\cos^2(\theta)\right)$$
$$= 2\cos^2(\theta)$$

b $\frac{\sin^3(A) + \sin(A)\cos^2(A)}{\cos^3(A) + \cos(A)\sin^2(A)} = \tan(A)$

$$\text{LHS} = \frac{\sin^3(A) + \sin(A)\cos^2(A)}{\cos^3(A) + \cos(A)\sin^2(A)}$$
$$= \frac{\sin(A)\left(\sin^2(A) + \cos^2(A)\right)}{\cos(A)\left(\cos^2(A) + \sin^2(A)\right)}$$
$$= \frac{\sin(A)(1)}{\cos(A)(1)}$$
$$= \frac{\sin(A)}{\cos(A)}$$
$$= \tan(A)$$
$$= \text{RHS}$$

2 $\tan(u) + \frac{1}{\tan(u)}$

Replace $\tan(u)$ by $\frac{\sin(u)}{\cos(u)}$

$$\therefore \tan(u) + \frac{1}{\tan(u)} = \frac{\sin(u)}{\cos(u)} + \frac{\cos(u)}{\sin(u)}$$
$$= \frac{\sin(u) \times \sin(u)}{\cos(u)\sin(u)} + \frac{\cos(u) \times \cos(u)}{\sin(u)\cos(u)}$$
$$= \frac{\sin^2(u) + \cos^2(u)}{\cos(u)\sin(u)}$$
$$= \frac{1}{\cos(u)\sin(u)}$$

3 $\cos(x) = -\dfrac{2}{7},\ \dfrac{\pi}{2} \le x \le \pi$ second quadrant

First quadrant triangle has hypotenuse 7 and adjacent side 2.

Opposite side is $\sqrt{7^2 - 2^2} = \sqrt{45}$.

In second quadrant, $\sin(x) = \dfrac{3\sqrt{5}}{7}$ and $\tan(x) = -\dfrac{3\sqrt{5}}{2}$.

4 $\tan(x) = -3,\ \pi \le x \le 2\pi$. Since tan is negative, fourth quadrant is required.
First quadrant triangle has opposite side 3and adjacent side 1.
The hypotenuse is $\sqrt{3^2 + 1^2} = \sqrt{10}$.

In fourth quadrant, $\sin(x) = -\dfrac{3}{\sqrt{10}}$ and $\cos(x) = \dfrac{1}{\sqrt{10}}$.

5 **a** $\cos\left(\dfrac{3\pi}{2} - \theta\right)$ third quadrant so cos is negative. Base is $\dfrac{\pi}{2} - \theta$.
Therefore,

$$\begin{aligned}\cos\left(\frac{3\pi}{2} - \theta\right) &= -\cos\left(\frac{\pi}{2} - \theta\right)\\ &= -\sin(\theta)\end{aligned}$$

b Use the complement

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{2} - \frac{\pi}{12}\right)\\ &= \cos\left(\frac{5\pi}{12}\right)\end{aligned}$$

Therefore, $t = \dfrac{5\pi}{12}$.

6 $\sin\left(\dfrac{\pi}{2} + \theta\right) + \sin(\pi + \theta)$

$\sin\left(\dfrac{\pi}{2} + \theta\right) = \sin\left(\dfrac{\pi}{2} - \theta\right)$ using symmetry property of second quadrant
$= \cos(\theta)$ using complementary property

$\sin(\pi + \theta) = -\sin(\theta)$ using symmetry property of third quadrant.

Therefore, $\sin\left(\dfrac{\pi}{2} + \theta\right) + \sin(\pi + \theta) = \cos(\theta) - \sin(\theta)$.

7 **a** $4 - 4\cos^2(\theta)$

$$\begin{aligned}&= 4\left(1 - \cos^2(\theta)\right)\\ &= 4\sin^2(\theta)\end{aligned}$$

b $\dfrac{2\sin(\alpha)}{\cos(\alpha)}$

$$\begin{aligned}&= 2 \times \frac{\sin(\alpha)}{\cos(\alpha)}\\ &= 2\tan(\alpha)\end{aligned}$$

c $8\cos^2(\beta) + 8\sin^2(\beta)$

$$\begin{aligned}&= 8\left[\cos^2(\beta) + \sin^2(\beta)\right]\\ &= 8 \times 1\\ &= 8\end{aligned}$$

d $\dfrac{\sin^2(\theta)}{\cos^2(\theta)}$

$$\begin{aligned}&= \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2\\ &= (\tan(\theta))^2\\ &= \tan^2(\theta)\end{aligned}$$

e $(1 - \sin(A))(1 + \sin(A))$

Expand the difference of two squares

$$\begin{aligned}&= 1^2 - (\sin(A))^2\\ &= 1 - \sin^2(A)\\ &= \cos^2(A)\end{aligned}$$

f $\sin^4(\theta) + 2\sin^2(\theta)\cos^2(\theta) + \cos^4(\theta)$

$$\begin{aligned}&= \left(\sin^2(\theta) + \cos^2(\theta)\right)^2\\ &= (1)^2\\ &= 1\end{aligned}$$

8 **a** $\tan^2(\theta) + 1 = \dfrac{1}{\cos^2(\theta)}$

$$\begin{aligned}\text{LHS} &= \tan^2(\theta) + 1\\ &= \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 + 1\\ &= \frac{\sin^2(\theta)}{\cos^2(\theta)} + 1\\ &= \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)}\\ &= \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)}\\ &= \frac{1}{\cos^2(\theta)}\\ &= \text{RHS}\end{aligned}$$

b $\cos^3(\theta) + \cos(\theta)\sin^2(\theta) = \cos(\theta)$

$$\begin{aligned}\text{LHS} &= \cos^3(\theta) + \cos(\theta)\sin^2(\theta)\\ &= \cos(\theta)\left[\cos^2(\theta) + \sin^2(\theta)\right]\\ &= \cos(\theta) \times 1\\ &= \cos(\theta)\\ &= \text{RHS}\end{aligned}$$

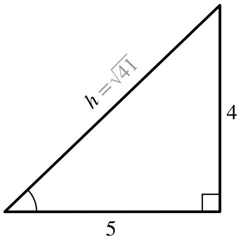
c $\dfrac{1}{1 - \cos(\theta)} + \dfrac{1}{1 + \cos(\theta)} = \dfrac{2}{\sin^2(\theta)}$

$$\begin{aligned}\text{LHS} &= \frac{1}{1 - \cos(\theta)} + \frac{1}{1 + \cos(\theta)}\\ &= \frac{1 + \cos(\theta) + 1 - \cos(\theta)}{(1 - \cos(\theta))(1 + \cos(\theta))}\\ &= \frac{2}{1 - \cos^2(\theta)}\\ &= \frac{2}{\sin^2(\theta)}\\ &= \text{RHS}\end{aligned}$$

d $(\sin(\theta)+\cos(\theta))^2+(\sin(\theta)-\cos(\theta))^2=2$

$$\begin{aligned}\text{LHS}&=(\sin(\theta)+\cos(\theta))^2+(\sin(\theta)-\cos(\theta))^2\\&=\sin^2(\theta)+\cancel{2\sin(\theta)\cos(\theta)}+\cos^2(\theta)+\sin^2(\theta)-\cancel{2\sin(\theta)\cos(\theta)}+\cos^2(\theta)\\&=2(\sin^2(\theta)+\cos^2(\theta))\\&=2\\&=\text{RHS}\end{aligned}$$

9 a $\tan(x)=-\frac{4}{5},\ \frac{3\pi}{2}\le x\le 2\pi$



In quadrant 1,

$$\begin{aligned}h^2&=4^2+5^2\\&=41\end{aligned}$$

$\therefore h=\sqrt{41}$

In quadrant 4,

$\sin(x)=-\frac{4}{\sqrt{41}},\ \cos(x)=\frac{5}{\sqrt{41}}.$

b $\sin(x)=\frac{\sqrt{3}}{2},\ \frac{\pi}{2}\le x\le\frac{3\pi}{2}$

Quadrant 2

As this is an exact value, $x=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$. (Alternatively, use the method of part a).

i $\cos(x)$

$$\begin{aligned}&=\cos\left(\frac{2\pi}{3}\right)\\&=-\cos\left(\frac{\pi}{3}\right)\\&=-\frac{1}{2}\end{aligned}$$

ii $1+\tan^2(x)$

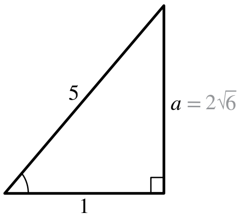
$$\begin{aligned}\tan(x)&=\tan\left(\frac{2\pi}{3}\right)\\&=-\tan\left(\frac{\pi}{3}\right)\\&=-\sqrt{3}\end{aligned}$$

$\therefore 1+\tan^2(x)=1+3=4$

c $\cos(x)=-0.2,\ \pi\le x\le\frac{3\pi}{2}$

$\therefore\cos(x)=-\frac{1}{5}$

Quadrant 3



In quadrant 1,

$a^2 + 1^2 = 5^2$

$\therefore a^2 = 24$

$\therefore a = 2\sqrt{6}$

i $\tan(x) = 2\sqrt{6}$

ii $1 - \sin^2(x)$

$= 1 - (\sin(x))^2$

$= 1 - \left(-\dfrac{2\sqrt{6}}{5}\right)^2$

$= 1 - \dfrac{24}{25}$

$= \dfrac{1}{25}$

iii $\sin(x)\tan(x)$

$= -\dfrac{2\sqrt{6}}{5} \times 2\sqrt{6}$

$= -\dfrac{24}{5}$

10 a i $\cos(x) = \dfrac{2\sqrt{3}}{5},\ 0 \le x \le \dfrac{\pi}{2}$

$\sin^2(x) + \cos^2(x) = 1$

$\therefore \sin^2(x) + \left(\dfrac{2\sqrt{3}}{5}\right)^2 = 1$

$\therefore \sin^2(x) + \dfrac{12}{25} = 1$

$\therefore \sin^2(x) = \dfrac{13}{25}$

As x lies in the first quadrant, $\sin(x) > 0$

$\therefore \sin(x) = \dfrac{\sqrt{13}}{5}$

ii $\cos(x) = \dfrac{2\sqrt{3}}{5},\ \dfrac{3\pi}{2} \le x \le 2\pi$

As x lies in the fourth quadrant, $\sin(x) < 0$

$\therefore \sin(x) = -\dfrac{\sqrt{13}}{5}$

b $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

For x in the first quadrant,

$\tan(x) = \dfrac{\sqrt{13}}{5} \div \dfrac{2\sqrt{3}}{5}$

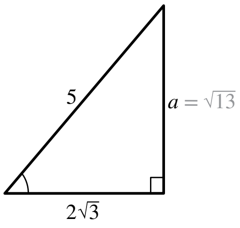
$= \dfrac{\sqrt{13}}{5} \times \dfrac{5}{2\sqrt{3}}$

$= \dfrac{\sqrt{13}}{2\sqrt{3}}$

$= \dfrac{\sqrt{39}}{6}$

For x in the fourth quadrant, $\tan(x) = -\dfrac{\sqrt{39}}{6}$.

c



In quadrant 1,

$a^2 + \left(2\sqrt{3}\right)^2 = 5^2$

$\therefore a^2 = 25 - 12$

$\therefore a = \sqrt{13}$

$\therefore \sin(x) = \dfrac{\sqrt{13}}{5}, \tan(x) = \dfrac{\sqrt{13}}{2\sqrt{3}} = \dfrac{\sqrt{39}}{6}$. This agrees with the answers in part **a**.

11 a $\cos(a°) = \sin(27°)$

$\therefore \cos(a°) = \cos(90° - 27°)$

$\therefore \cos(a°) = \cos(63°)$

$\therefore a = 63$

b $\sin(b) = \cos\left(\dfrac{\pi}{9}\right)$

$\therefore \sin(b) = \sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{9}\right)$

$\therefore \sin(b) = \sin\left(\dfrac{7\pi}{18}\right)$

$\therefore b = \dfrac{7\pi}{18}$

c $\sin(c) = \cos(0.35)$

$\therefore \sin(c) = \sin\left(\dfrac{\pi}{2} - 0.35\right)$

$\therefore c = \dfrac{\pi}{2} - 0.35$

$\therefore c = 1.22$

d $\cos(d) = \sin\left(\dfrac{\pi}{4}\right)$

$\therefore \cos(d) = \cos\left(\dfrac{\pi}{2} - \dfrac{\pi}{4}\right)$

$\therefore d = \dfrac{\pi}{4}$

or,

$\cos(d) = \sin\left(\dfrac{\pi}{4}\right)$

$\therefore \cos(d) = \cos\left(\dfrac{3\pi}{2} + \dfrac{\pi}{4}\right)$

$\therefore d = \dfrac{7\pi}{4}$

12 a $\cos\left(\dfrac{3\pi}{2} + \theta\right)$ fourth quadrant.

$= \cos\left(\dfrac{\pi}{2} - \theta\right)$

$= \sin(\theta)$

b $\cos\left(\dfrac{3\pi}{2} - \theta\right)$ third quadrant

$= -\cos\left(\dfrac{\pi}{2} - \theta\right)$

$= -\sin(\theta)$

c $\sin\left(\dfrac{\pi}{2} + \theta\right)$ second quadrant

$= \sin\left(\dfrac{\pi}{2} - \theta\right)$

$= \cos(\theta)$

d $\sin\left(\frac{5\pi}{2}-\theta\right)$

$=\sin\left(2\pi+\left(\frac{\pi}{2}-\theta\right)\right)$

$=\sin\left(\frac{\pi}{2}-\theta\right)$

$=\cos(\theta)$

13 $\sin(\theta)=0.8,\ 0\le\theta\le\frac{\pi}{2}$

a $\sin(\pi+\theta)$

$=-\sin(\theta)$

$=-0.8$

b $\cos\left(\frac{\pi}{2}-\theta\right)$

$=\sin(\theta)$

$=0.8$

c $\cos(\theta)$

Since $\cos^2(\theta)+\sin^2(\theta)=1$,

$\cos^2(\theta)=1-\sin^2(\theta)$

$=1-(0.8)^2$

$=1-0.64$

$=0.36$

As $0\le\theta\le\frac{\pi}{2}$ then $\cos(\theta)>0$

$\therefore\cos(\theta)=\sqrt{0.36}$

$\therefore\cos(\theta)=0.6$

d $\tan(\pi+\theta)$

$=\tan(\theta)$

$=\frac{\sin(\theta)}{\cos(\theta)}$

$=\frac{0.8}{0.6}$

$=\frac{4}{3}$

14 a $\frac{\cos(90°-a°)}{\cos(a°)}$

$=\frac{\sin(a°)}{\cos(a°)}$

$=\tan(a°)$

b $1-\sin^2\left(\frac{\pi}{2}-\theta\right)$

$=1-\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]^2$

$=1-[\cos(\theta)]^2$

$=1-\cos^2(\theta)$

$=\sin^2(\theta)$

c $\cos(\pi-x)+\sin\left(x+\frac{\pi}{2}\right)$

$=-\cos(x)+\sin\left(\frac{\pi}{2}+x\right)$

$=-\cos(x)+\cos(x)$

$=0$

d $\cos\left(x-\frac{\pi}{2}\right)=\cos\left(-\left(\frac{\pi}{2}-x\right)\right)$

Using the symmetry property that $\cos(-\theta)=\cos(\theta)$,

$\cos\left(-\left(\frac{\pi}{2}-x\right)\right)=\cos\left(\frac{\pi}{2}-x\right)$

$=\sin(x)$

$\therefore\cos\left(x-\frac{\pi}{2}\right)=\sin(x)$

This means that the graphs of $y=\cos(x)$ and $y=\sin(x)$ are out of phase by $\frac{\pi}{2}$ so a horizontal translation of $\frac{\pi}{2}$ to the right will transform the cosine graph into the sine graph.

15 a $\cos^2(x)+3\sin(x)-1=0,\ 0\le x\le 2\pi$

As $\cos^2(x)=1-\sin^2(x)$, substitute this in the equation

$\therefore 1-\sin^2(x)+3\sin(x)-1=0$

$\therefore -\sin^2(x)+3\sin(x)=0$

$\therefore \sin(x)[-\sin(x)+3]=0$

$\therefore \sin(x)=0$ or $\sin(x)=3$

Reject $\sin(x)=3$ since $\sin(x)\in[-1,1]$

$\therefore \sin(x)=0$ Boundary solutions at $(1,0)$ and $(-1,0)$

$\therefore x=0,\pi,2\pi$

b $3\cos^2(x)+6\sin(x)+3\sin^2(x)=0,\ 0\le x\le 2\pi$

$\therefore 3\left[\cos^2(x)+\sin^2(x)\right]+6\sin(x)=0$

$\therefore 3[1]+6\sin(x)=0$

$\therefore 6\sin(x)=-3$

$\therefore \sin(x)=-\frac{1}{2}$

Quadrants 3 and 4, base $\frac{\pi}{6}$

$\therefore x=\pi+\frac{\pi}{6},2\pi-\frac{\pi}{6}$

$\therefore x=\frac{7\pi}{6},\frac{11\pi}{6}$

c $\frac{1-\cos(x)}{1+\sin(x)}\times\frac{1+\cos(x)}{1-\sin(x)}=3,\ 0\le x\le 2\pi$

$\therefore \frac{(1-\cos(x))(1+\cos(x))}{(1+\sin(x))(1-\sin(x))}=3$

$\therefore \frac{1-\cos^2(x)}{1-\sin^2(x)}=3$

$\therefore \frac{\sin^2(x)}{\cos^2(x)}=3$

$\therefore \tan^2(x)=3$

$\therefore \tan(x)=\pm\sqrt{3}$

Quadrants $1,2,3$ and 4, base $\frac{\pi}{3}$

$\therefore x=\frac{\pi}{3},\pi-\frac{\pi}{3},\pi+\frac{\pi}{3},2\pi-\frac{\pi}{3}$

$\therefore x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$

d $\sin^2(x)=\cos(x)+1$

$\therefore 1-\cos^2(x)=\cos(x)+1$

$\therefore -\cos^2(x)=\cos(x)$

$\therefore \cos(x)+\cos^2(x)=0$

$\therefore \cos(x)(1+\cos(x))=0$

$\therefore \cos(x)=0$ or $\cos(x)=-1$

$\therefore x=\frac{\pi}{2},\frac{3\pi}{2}$ or $x=\pi$

$\therefore x=\frac{\pi}{2},\pi,\frac{3\pi}{2}$

16 **a** $1-\cos^2(40°)=\cos^2(50°)$

$$\begin{aligned}\text{LHS}&=1-\cos^2(40°)\\&=\sin^2(40°)\\&=[\sin(40°)]^2\\&=[\cos(50°)]^2\\&=\cos^2(50°)\\&=\text{RHS}\end{aligned}$$

b $\dfrac{\cos^3(x)-\sin^3(x)}{\cos(x)-\sin(x)}=1+\sin(x)\cos(x)$

Recall that the factors of the difference of two cubes are $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$.

$$\begin{aligned}\text{LHS}&=\frac{\cancel{[\cos(x)-\sin(x)]}\left[\cos^2(x)+\cos(x)\sin(x)+\sin^2(x)\right]}{\cancel{\cos(x)-\sin(x)}}\\&=\cos^2(x)+\cos(x)\sin(x)+\sin^2(x)\\&=\left(\cos^2(x)+\sin^2(x)\right)+\cos(x)\sin(x)\\&=1+\sin(x)\cos(x)\\&=\text{RHS}\end{aligned}$$

c $\dfrac{\sin^3(\theta)+\cos\left(\dfrac{\pi}{2}-\theta\right)\cos^2(\theta)}{\cos(\theta)}=\tan(\theta)$

$$\begin{aligned}\text{LHS}&=\frac{\sin^3(\theta)+\cos\left(\dfrac{\pi}{2}-\theta\right)\cos^2(\theta)}{\cos(\theta)}\\&=\frac{\sin^3(\theta)+\sin(\theta)\cos^2(\theta)}{\cos(\theta)}\\&=\frac{\sin(\theta)\left[\sin^2(\theta)+\cos^2(\theta)\right]}{\cos(\theta)}\\&=\frac{\sin(\theta)\times 1}{\cos(\theta)}\\&=\frac{\sin(\theta)}{\cos(\theta)}\\&=\tan(\theta)\\&=\text{RHS}\end{aligned}$$

d $\cos(2\pi-A)\cos(2\pi+A)-\cos\left(\dfrac{3\pi}{2}-A\right)\cos\left(\dfrac{3\pi}{2}+A\right)=1$

$$\begin{aligned}\text{LHS}&=\cos(2\pi-A)\cos(2\pi+A)-\cos\left(\frac{3\pi}{2}-A\right)\cos\left(\frac{3\pi}{2}+A\right)\\&=\cos(A)\cos(A)-(-\sin(A))(\sin(A))\\&=\cos^2(A)+\sin^2(A)\\&=1\\&=\text{RHS}\end{aligned}$$

17 **a** Enter $\sin(5.5\pi-x)\div\cos\left(x-\dfrac{\pi}{2}\right)$ in the Main Menu in Standard and Rad modes. Highlight and tap Interactive → Transformation → simplify.
This gives the result that
$\sin(5.5\pi-x)\div\cos\left(x-\dfrac{\pi}{2}\right)=\dfrac{1}{\tan(x)}$.

b Using the same method as in part **a**,
$\sin(3x)\cos(2x)-2\cos^2(x)\sin(3x)=-\sin(3x)$.

18 **a** $\cos\left(\sin^{-1}\left(\dfrac{3}{5}\right)\right)=\dfrac{4}{5}$.

If $\sin(\theta)=\dfrac{3}{5}$ then $\theta=\sin^{-1}\left(\dfrac{3}{5}\right)$.

$\therefore\cos\left(\sin^{-1}\left(\dfrac{3}{5}\right)\right)=\cos(\theta)$ where $\sin(\theta)=\dfrac{3}{5}$.

b Given $\cos(x)=-\dfrac{\sqrt{3}}{3}$ then $x=\cos^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$.

$\tan(x)=\tan\left(\cos^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)\right)$ and entering this into the Main Menu gives the result $-\sqrt{2}$.

Similarly, $\sin(x)=\sin\left(\cos^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)\right)=\dfrac{\sqrt{6}}{3}$.

Since $\sin(x)>0$ and $\tan(x)<0$, x lies in the second quadrant. Hence $x=\cos^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$ has been assumed to be in the second quadrant.

Topic 11 — Exponential functions

Exercise 11.2 — Indices as exponents

1 a $$\begin{aligned}\frac{2^{1-n}\times 8^{1+2n}}{16^{1-n}}&=\frac{2^{1-n}\times 2^{3(1+2n)}}{2^{4(1-n)}}\\&=\frac{2^{4+5n}}{2^{4-4n}}\\&=2^{9n}\end{aligned}$$

b $$\begin{aligned}\left(9a^3b^{-4}\right)^{\frac{1}{2}}\times 2\left(a^{\frac{1}{2}}b^{-2}\right)^{-2}&=3a^{\frac{3}{2}}b^{-2}\times 2\times a^{-1}b^4\\&=6a^{\frac{1}{2}}b^2\end{aligned}$$

c $$\begin{aligned}27^{-\frac{2}{3}}+\left(\frac{49}{81}\right)^{\frac{1}{2}}&=\frac{1}{27^{\frac{2}{3}}}+\left(\frac{49}{81}\right)^{\frac{1}{2}}\\&=\frac{1}{\left(\sqrt[3]{27}\right)^2}+\frac{\sqrt{49}}{\sqrt{81}}\\&=\frac{1}{(3)^2}+\frac{7}{9}\\&=\frac{1}{9}+\frac{7}{9}\\&=\frac{8}{9}\end{aligned}$$

2 $$\begin{aligned}\frac{20p^5}{m^3q^{-2}}\div\frac{5\left(p^2q^{-3}\right)^2}{-4m^{-1}}&=\frac{20p^5}{m^3q^{-2}}\times\frac{-4m^{-1}}{5p^4q^{-6}}\\&=\frac{{}^{4}\cancel{20}p^5q^2}{m^3}\times\frac{-4q^6}{\cancel{5}p^4m}\\&=\frac{-16p^5q^8}{m^4p^4}\\&=\frac{-16pq^8}{m^4}\end{aligned}$$

3 $$\begin{aligned}\frac{2^{5x-3}\times 8^{9-2x}}{4^x}&=1\\\frac{2^{5x-3}\times 2^{3(9-2x)}}{2^{2x}}&=1\\\frac{2^{24-x}}{2^{2x}}&=1\\2^{24-3x}&=1\\2^{24-3x}&=2^0\end{aligned}$$

Equating indices:

$$\begin{aligned}24-3x&=0\\\therefore x&=8\end{aligned}$$

4 a $2\times 5^x+5^x<75$

Adding:

$$\begin{aligned}3\times 5^x&<75\\5^x&<\frac{75}{3}\\5^x&<25\\5^x&<5^2\end{aligned}$$

Hence the indices give $x<2$.

b $$\begin{aligned}\left(\frac{1}{9}\right)^{2x-3}&>\left(\frac{1}{9}\right)^{7-x}\\9^{-(2x-3)}&>9^{-(7-x)}\\9^{-2x+3}&>9^{-7+x}\\-2x+3&>-7+x\\10&>3x\\x&<\frac{10}{3}\end{aligned}$$

5 $30\times 10^{2x}+17\times 10^x-2=0$

Let $a=10^x$.

$$\begin{aligned}&30a^2+17a-2=0\\&(10a-1)(3a+2)=0\\&a=\frac{1}{10},\ a=-\frac{2}{3}\end{aligned}$$

$\therefore 10^x=\frac{1}{10}$ or $10^x=-\frac{2}{3}$ for which there is no real solution

$$\begin{aligned}10^x&=\frac{1}{10}\\10^x&=10^{-1}\\x&=-1\end{aligned}$$

6 $2^x-48\times 2^{-x}=13$

$\therefore 2^x-\frac{48}{2^x}=13$

Let $a=2^x$.

$$\begin{aligned}a-\frac{48}{a}&=13\\a^2-48&=13a\\a^2-13a-48&=0\\(a-16)(a+3)&=0\\a&=16,\ a=-3\end{aligned}$$

$\therefore 2^x=16,\ 2^x=-3$

Reject $2^x=-3$ since there are no real solutions.

$$\begin{aligned}2^x&=16\\2^x&=2^4\\x&=4\end{aligned}$$

7 a i $1\,409\,000=1.409\times 10^6$ and it contains 4 significant figures.

ii $0.000\,130\,6=1.306\times 10^{-4}$ and it contains 4 significant figures.

b i $3.04\times 10^5=304\,000$

ii $5.803\times 10^{-2}=0.058\,03$

8 $$\begin{aligned}\left(4\times 10^6\right)^2\times(5\times 10^{-3})&=16\times 10^{12}\times 5\times 10^{-3}\\&=16\times 5\times 10^{12}\times 10^{-3}\\&=80\times 10^9\\&=8.0\times 10^1\times 10^9\\&=8\times 10^{10}\end{aligned}$$

9 a i $$\begin{aligned}\sqrt{a^3b^4}&=\left(a^3b^4\right)^{\frac{1}{2}}\\&=a^{\frac{3}{2}}b^2\end{aligned}$$

ii
$$\begin{aligned}\sqrt{\frac{a^5}{b^{-4}}}\times\sqrt[3]{a^2b}&=\left(\frac{a^5}{b^{-4}}\right)^{\frac{1}{2}}\times\left(a^2b\right)^{\frac{1}{3}}\\&=\frac{a^{\frac{5}{2}}}{b^{-2}}\times a^{\frac{2}{3}}b^{\frac{1}{3}}\\&=a^{\frac{5}{2}+\frac{2}{3}}b^{\frac{1}{3}+2}\\&=a^{\frac{19}{6}}b^{\frac{7}{3}}\end{aligned}$$

b **i**
$$\begin{aligned}a^{\frac{1}{2}}\div b^{\frac{3}{2}}&=\sqrt{a}\div\sqrt{b^3}\\&=\sqrt{\frac{a}{b^3}}\end{aligned}$$

ii
$$\begin{aligned}2^{\frac{5}{2}}&=\sqrt{2^5}\\&=\sqrt{32}\end{aligned}$$

iii
$$\begin{aligned}3^{-\frac{2}{5}}&=\sqrt[5]{3^{-2}}\\&=\sqrt[5]{\frac{1}{3^2}}\\&=\sqrt[5]{\frac{1}{9}}\end{aligned}$$

10 **a**
$$\begin{aligned}4^{\frac{3}{2}}&=\sqrt{4^3}\\&=\sqrt{64}\\&=8\end{aligned}$$

b
$$\begin{aligned}3^{-1}+5^0-2^2\times9^{-\frac{1}{2}}&=\frac{1}{3}+1-4\times\frac{1}{\sqrt{9}}\\&=\frac{4}{3}-\frac{4}{3}\\&=0\end{aligned}$$

c
$$\begin{aligned}2^3\times\left(\frac{4}{9}\right)^{-\frac{1}{2}}\div\left(6\times\left(3^{-2}\right)^2\right)&=8\times\sqrt{\frac{9}{4}}\div\left(6\times3^{-4}\right)\\&=8\times\frac{3}{2}\div\frac{6}{3^4}\\&=12\times\frac{81}{6}\\&=162\end{aligned}$$

d
$$\begin{aligned}\frac{15\times5^{\frac{3}{2}}}{125^{\frac{1}{2}}-20^{\frac{1}{2}}}&=\frac{15\times\sqrt{5^3}}{\sqrt{125}-\sqrt{20}}\\&=\frac{15\times5\sqrt{5}}{5\sqrt{5}-2\sqrt{5}}\\&=\frac{{}^{5}\cancel{15}\times5\cancel{\sqrt{5}}}{\cancel{3}\cancel{\sqrt{5}}}\\&=25\end{aligned}$$

11 **a**
$$\begin{aligned}\frac{3\left(x^2y^{-2}\right)^3}{\left(3x^4y^2\right)^{-1}}&=\frac{3x^6y^{-6}}{3^{-1}x^{-4}y^{-2}}\\&=3^2x^{10}y^{-4}\\&=\frac{9x^{10}}{y^4}\end{aligned}$$

b
$$\begin{aligned}\frac{2a^{\frac{2}{3}}b^{-3}}{3a^{\frac{1}{3}}b^{-1}}\times\frac{3^2\times2\times(ab)^2}{\left(-8a^2\right)^2b^2}&=\frac{2a^{\frac{1}{3}}b^{-2}}{3}\times\frac{18a^2b^2}{64a^4b^2}\\&=\frac{2a^{\frac{1}{3}}}{3b^2}\times\frac{9}{32a^2}\\&=\frac{a^{\frac{1}{3}}}{b^2}\times\frac{3}{16a^2}\\&=\frac{3}{16a^{\frac{5}{3}}b^2}\end{aligned}$$

c
$$\begin{aligned}\frac{\left(2mn^{-2}\right)^{-2}}{m^{-1}n}\div\frac{10n^4m^{-1}}{3\left(m^2n\right)^{\frac{3}{2}}}&=\frac{2^{-2}m^{-2}n^4}{m^{-1}n}\times\frac{3m^3n^{\frac{3}{2}}}{10n^4m^{-1}}\\&=\frac{n^3}{2^2m}\times\frac{3m^4}{10n^{\frac{5}{2}}}\\&=\frac{3n^3m^4}{40mn^{\frac{5}{2}}}\\&=\frac{3m^3n^{\frac{1}{2}}}{40}\end{aligned}$$

d
$$\begin{aligned}\frac{4m^2n^{-2}\times-2\left(m^2n^{\frac{3}{2}}\right)^2}{\left(-3m^3n^{-2}\right)^2}&=\frac{4m^2n^{-2}\times-2m^4n^3}{9m^6n^{-4}}\\&=\frac{-8m^6n}{9m^6n^{-4}}\\&=\frac{-8n^5}{9}\end{aligned}$$

e
$$\begin{aligned}\frac{m^{-1}-n^{-1}}{m^2-n^2}&=\left(\frac{1}{m}-\frac{1}{n}\right)\div\left(m^2-n^2\right)\\&=\left(\frac{n-m}{mn}\right)\times\frac{1}{m^2-n^2}\\&=\frac{n-m}{mn}\times\frac{1}{(m-n)(m+n)}\\&=\frac{-(m-n)}{mn}\times\frac{1}{(m-n)(m+n)}\\&=\frac{-1}{mn(m+n)}\end{aligned}$$

f
$$\begin{aligned}\sqrt{4x-1}-2x(4x-1)^{-\frac{1}{2}}&=(4x-1)^{\frac{1}{2}}-\frac{2x}{(4x-1)^{\frac{1}{2}}}\\&=\frac{(4x-1)^{\frac{1}{2}}(4x-1)^{\frac{1}{2}}-2x}{(4x-1)^{\frac{1}{2}}}\\&=\frac{(4x-1)-2x}{(4x-1)^{\frac{1}{2}}}\\&=\frac{2x-1}{(4x-1)^{\frac{1}{2}}}\end{aligned}$$

12 **a**
$$\begin{aligned}\frac{32\times4^{3x}}{16^x}&=\frac{2^5\times\left(2^2\right)^{3x}}{\left(2^4\right)^x}\\&=\frac{2^5\times2^{6x}}{2^{4x}}\\&=\frac{2^{5+6x}}{2^{4x}}\\&=2^{5+6x-4x}\\&=2^{5+2x}\end{aligned}$$

b
$$\begin{aligned}\frac{3^{1+n}\times81^{n-2}}{243^n}&=\frac{3^{1+n}\times\left(3^4\right)^{n-2}}{\left(3^5\right)^n}\\&=\frac{3^{1+n}\times3^{4n-8}}{3^{5n}}\\&=\frac{3^{5n-7}}{3^{5n}}\\&=3^{-7}\end{aligned}$$

c
$$\begin{aligned}
0.001\times\sqrt[3]{10}\times100^{\frac{5}{2}}\times(0.1)^{-\frac{2}{3}} &= \frac{1}{1000}\times10^{\frac{1}{3}}\times\left(10^2\right)^{\frac{5}{2}}\times\left(\frac{1}{10}\right)^{-\frac{2}{3}}\\
&= \frac{1}{10^3}\times10^{\frac{1}{3}}\times10^5\times\left(10^{-1}\right)^{-\frac{2}{3}}\\
&= 10^{-3}\times10^{\frac{1}{3}}\times10^5\times10^{\frac{2}{3}}\\
&= 10^{-3+\frac{1}{3}+5+\frac{2}{3}}\\
&= 10^{\frac{-45+5+75+10}{15}}\\
&= 10^{\frac{45}{15}}\\
&= 10^3
\end{aligned}$$

d
$$\begin{aligned}
\frac{5^{n+1}-5^n}{4} &= \frac{5^n\times5^1-5^n}{4}\\
&= \frac{5^n(5-1)}{4}\\
&= 5^n
\end{aligned}$$

13 a
$$\begin{aligned}
2^{2x}\times8^{2-x}\times16^{-\frac{3x}{2}} &= \frac{2}{4^x}\\
\therefore\ 2^{2x}\times(2^3)^{2-x}\times\left(2^4\right)^{-\frac{3x}{2}} &= \frac{2}{(2^2)^x}\\
\therefore\ 2^{2x}\times2^{6-3x}\times2^{-6x} &= \frac{2^1}{2^{2x}}\\
\therefore\ 2^{-7x+6} &= 2^{1-2x}
\end{aligned}$$

Equating indices,

$$\begin{aligned}
-7x+6 &= 1-2x\\
\therefore 5 &= 5x\\
\therefore x &= 1
\end{aligned}$$

b
$$\begin{aligned}
25^{3x-3} &\le 125^{4+x}\\
\therefore\left(5^2\right)^{3x-3} &\le \left(5^3\right)^{4+x}\\
\therefore 5^{6x-6} &\le 5^{12+3x}
\end{aligned}$$

As the base is greater than 1,

$$\begin{aligned}
6x-6 &\le 12+3x\\
\therefore 3x &\le 18\\
\therefore x &\le 6
\end{aligned}$$

c
$$\begin{aligned}
9^x\div27^{1-x} &= \sqrt{3}\\
\therefore\left(3^2\right)^x\div\left(3^3\right)^{1-x} &= 3^{\frac{1}{2}}\\
\therefore 3^{2x}\div3^{3-3x} &= 3^{\frac{1}{2}}\\
\therefore 3^{5x-3} &= 3^{\frac{1}{2}}\\
\therefore 5x-3 &= \frac{1}{2}\\
\therefore 5x &= \frac{7}{2}\\
\therefore x &= \frac{7}{10}
\end{aligned}$$

d
$$\begin{aligned}
\left(\frac{2}{3}\right)^{3-2x} &> \left(\frac{27}{8}\right)^{-\frac{1}{3}}\times\frac{1}{\sqrt{2\frac{1}{4}}}\\
\therefore\left(\frac{2}{3}\right)^{3-2x} &> \left(\frac{8}{27}\right)^{\frac{1}{3}}\times\frac{1}{\sqrt{\frac{9}{4}}}\\
\therefore\left(\frac{2}{3}\right)^{3-2x} &> \sqrt[3]{\frac{8}{27}}\times\frac{1}{\frac{3}{2}}\\
\therefore\left(\frac{2}{3}\right)^{3-2x} &> \frac{2}{3}\times\frac{2}{3}\\
\therefore\left(\frac{2}{3}\right)^{3-2x} &> \left(\frac{2}{3}\right)^2
\end{aligned}$$

As the base is less than 1,

$$\begin{aligned}
3-2x &< 2\\
\therefore -2x &< -1\\
\therefore x &> \frac{1}{2}
\end{aligned}$$

e
$$\begin{aligned}
4^{5x}+4^{5x} &= \frac{8}{2^{4x-5}}\\
\therefore 2\times4^{5x} &= \frac{2^3}{2^{4x-5}}\\
\therefore 2\times\left(2^2\right)^{5x} &= 2^{8-4x}\\
\therefore 2\times2^{10x} &= 2^{8-4x}\\
\therefore 2^{1+10x} &= 2^{8-4x}\\
\therefore 1+10x &= 8-4x\\
\therefore 14x &= 7\\
\therefore x &= \frac{7}{14}\\
\therefore x &= \frac{1}{2}
\end{aligned}$$

f
$$\begin{aligned}
5^{\frac{2x}{3}}\times5^{\frac{3x}{2}} &= 25^{x+4}\\
\therefore 5^{\frac{2x}{3}+\frac{3x}{2}} &= \left(5^2\right)^{x+4}\\
\therefore 5^{\frac{4x+9x}{6}} &= 5^{2x+8}\\
\therefore\frac{13x}{6} &= 2x+8\\
\therefore 13x &= 12x+48\\
\therefore x &= 48
\end{aligned}$$

14 a $3^{2x}-10\times3^x+9=0$

Let $a=3^x$

$$\begin{aligned}
&\therefore a^2-10a+9=0\\
&\therefore (a-1)(a-9)=0\\
&\therefore a=1 \text{ or } a=9\\
&\therefore 3^x=1 \text{ or } 3^x=9\\
&\therefore x=0 \text{ or } x=2
\end{aligned}$$

b $24\times2^{2x}+61\times2^x=2^3$

Let $a=2^x$

$$\begin{aligned}
&\therefore 24a^2+61a=8\\
&\therefore 24a^2+61a-8=0\\
&\therefore (8a-1)(3a+8)=0\\
&\therefore a=\frac{1}{8} \text{ or } a=-\frac{8}{3}\\
&\therefore 2^x=\frac{1}{8} \text{ or } 2^x=-\frac{8}{3}
\end{aligned}$$

Reject $2^x=-\frac{8}{3}$ since $2^x>0$

$$\begin{aligned}
&\therefore 2^x=\frac{1}{8}\\
&\therefore 2^x=2^{-3}\\
&\therefore x=-3
\end{aligned}$$

c
$$\begin{aligned}
&25^x+5^{2+x}-150=0\\
&\therefore 5^{2x}+5^2\times5^x-150=0
\end{aligned}$$

Let $a=5^x$

$$\begin{aligned}
&\therefore a^2+25a-150=0\\
&\therefore (a-5)(a+30)=0\\
&\therefore a=5 \text{ or } a=-30\\
&\therefore 5^x=5 \text{ or } 5^x=-30\\
&\therefore 5^x=5,\ (5^x>0)\\
&\therefore x=1
\end{aligned}$$

d $\left(2^x+2^{-x}\right)^2=4$

$\left(2^x+2^{-x}\right)^2=4$

$\therefore 2^x+2^{-x}=\pm\sqrt{4}$

$\therefore 2^x+2^{-x}=\pm 2$

However, $2^x+2^{-x}>0$

$\therefore 2^x+2^{-x}=2$

$\therefore 2^x+\dfrac{1}{2^x}=2$

Let $a=2^x$

$\therefore a+\dfrac{1}{a}=2$

$\therefore a^2+1=2a$

$\therefore a^2-2a+1=0$

$\therefore (a-1)^2=0$

$\therefore a=1$

$\therefore 2^x=1$

$\therefore x=0$

e $10^x-10^{2-x}=99$

$\therefore 10^x-\dfrac{10^2}{10^x}=99$

Let $a=10^x$

$\therefore a-\dfrac{100}{a}=99$

$\therefore a^2-100=99a$

$\therefore a^2-99a-100=0$

$\therefore (a-100)(a+1)=0$

$\therefore a=100$ or $a=-1$

$\therefore 10^x=100$ or $10^x=-1$

$\therefore 10^x=100 \quad (10^x>0)$

$\therefore 10^x=10^2$

$\therefore x=2$

f $2^{3x}+3\times 2^{2x-1}-2^x=0$

$\therefore 2^{3x}+3\times\dfrac{2^{2x}}{2^1}-2^x=0$

Let $a=2^x$

$\therefore a^3+3\times\dfrac{a^2}{2}-a=0$

$\therefore 2a^3+3a^2-2a=0$

$\therefore a(2a^2+3a-2)=0$

$\therefore a(2a-1)(a+2)=0$

$\therefore a=0$ or $a=\dfrac{1}{2}$ or $a=-2$

$\therefore 2^x=0$ or $2^x=\dfrac{1}{2}$ or $2^x=-2$

$\therefore 2^x=\dfrac{1}{2} \; (2^x>0)$

$\therefore 2^x=2^{-1}$

$\therefore x=-1$

15 a i $-0.000\,000\,050\,6=-5.06\times10^{-8}$

ii Diameter is $2\times6370=12\,740$ km.

In scientific notation, the diameter is 1.274×10^4 km.

iii $3.2\times10^4\times5\times10^{-2}$

$=(3.2\times5)\times\left(10^4\times10^{-2}\right)$

$=16\times10^2$

$=1.6\times10^3$

iv 16,878.7 km is equal to $1.687\,87\times10^4$ km.

b i $6.3\times10^{-4}+6.3\times10^4=0.000\,63+63\,000$

$=63\,000.000\,63$

ii $\left(1.44\times10^6\right)^{\frac{1}{2}}=(1.44)^{\frac{1}{2}}\times\left(10^6\right)^{\frac{1}{2}}$

$=\sqrt{1.44}\times10^3$

$=1.2\times10^3$

$=1200$

c i $60\,589$

$=6.0589\times10^4$

$\simeq 6.1\times10^4$

$\therefore 60\,589\simeq 61\,000$

Correct to 2 significant figures, 61 000 people attended the match.

ii $1.994\times10^{-2}\simeq 2.0\times10^{-2}$

The probability, correct to 2 significant figures, is 0.020.

iii $-0.006\,34$

$=-6.34\times10^{-3}$

$\simeq -6.3\times10^{-3}$

Correct to 2 significant figures, $x=-0.0063$.

iv $26{,}597{,}696$

$=2.659\,769\,6\times10^7$

$\simeq 2.7\times10^7$

Correct to 2 significant figures, the distance flown is 27 000 000 km.

16 Given $x=3^{\frac{1}{3}}+3^{-\frac{1}{3}}$, show that $x^3-3x=\dfrac{10}{3}$.

Let $x=a+\dfrac{1}{a}$ where $a=3^{\frac{1}{3}}$.

$x^3-3x=\left(a+\dfrac{1}{a}\right)^3-3\left(a+\dfrac{1}{a}\right)$

$=a^3+3a^2\times\dfrac{1}{a}+3a\times\dfrac{1}{a^2}+\dfrac{1}{a^3}-3a-\dfrac{3}{a}$

$=a^3+3a+\dfrac{3}{a}+\dfrac{1}{a^3}-3a-\dfrac{3}{a}$

$=a^3+\dfrac{1}{a^3}$

Substitute back that $a=3^{\frac{1}{3}}$.

$\therefore x^3-3x=\left(3^{\frac{1}{3}}\right)^3+\dfrac{1}{\left(3^{\frac{1}{3}}\right)^3}$

$\therefore x^3-3x=3+\dfrac{1}{3}$

$\therefore x^3-3x=\dfrac{10}{3}$

as required.

17 a Consider the system of equations:

$5^{2x-y}=\dfrac{1}{125}$.....(1)

$10^{2y-6x}=0.01$....(2)

From equation (1),

$5^{2x-y}=5^{-3}$

$\therefore 2x-y=-3$....(3)

From equation (2),

$10^{2y-6x}=\dfrac{1}{100}$

$\therefore 10^{2y-6x}=10^{-2}$

$\therefore 2y-6x=-2$

$\therefore -3x+y=-1....(4)$

Consider the simultaneous equations (3) and (4)

$2x-y=-3....(3)$

$-3x+y=-1....(4)$

Add equations (3) and (4)

$\therefore -x=-4$

$\therefore x=4$

Substitute $x=4$ in equation (4)

$\therefore -12+y=-1$

$\therefore y=11$

Answer: $x=4, y=11$

b Consider the system of equations

$a\times 2^{k-1}=40....(1)$

$a\times 2^{2k-2}=10....(2)$

Divide equation (1) by equation (2)

$\therefore \dfrac{\cancel{a}\times 2^{k-1}}{\cancel{a}\times 2^{2k-2}}=\dfrac{40}{10}$

$\therefore 2^{k-1-2k+2}=4$

$\therefore 2^{-k+1}=2^2$

$\therefore -k+1=2$

$\therefore k=-1$

Substitute $k=-1$ in equation (1)

$\therefore a\times 2^{-2}=40$

$\therefore a\times \dfrac{1}{4}=40$

$\therefore a=160$

Answer: $a=160,\ k=-1$

18 $\left(\dfrac{2x^2}{3a}\right)^{n-1}\div\left(\dfrac{3x}{a}\right)^{n+1}=\left(\dfrac{x}{4}\right)^3$

$\therefore \dfrac{2^{n-1}x^{2n-2}}{3^{n-1}a^{n-1}}\times\dfrac{a^{n+1}}{3^{n+1}x^{n+1}}=\dfrac{x^3}{64}$

$\therefore \dfrac{2^{n-1}x^{2n-2-n-1}a^{n+1-n+1}}{3^{n-1+n+1}}=\dfrac{x^3}{64}$

$\therefore \dfrac{2^{n-1}x^{n-3}a^2}{3^{2n}}=\dfrac{x^3}{64}$

$\therefore \dfrac{2^{n-1}a^2}{3^{2n}}x^{n-3}=\dfrac{1}{64}x^3$

For the equality to hold, the powers of x must be equal.

$\therefore n-3=3$

$\therefore n=6$

And, for the equality to hold, the coefficients of x^3 must be equal.

$\therefore \dfrac{2^{n-1}a^2}{3^{2n}}=\dfrac{1}{64}$

Substitute $n=6$

$\therefore \dfrac{2^5a^2}{3^{12}}=\dfrac{1}{2^6}$

$\therefore a^2=\dfrac{3^{12}}{2^5}\times\dfrac{1}{2^6}$

$\therefore a^2=\dfrac{3^{12}}{2^{11}}$

$\therefore a=\pm\dfrac{3^6}{2^{\frac{11}{2}}}$

$\therefore a=\pm\left(3^6\times 2^{-\frac{11}{2}}\right)$

Answer: $a=\pm\left(3^6\times 2^{-\frac{11}{2}}\right), n=6$

19 a In the Main window in Decimal mode, enter $5^{-4.3}$ to obtain $9.872541804E-4$, or use the exponential template x available in the 2 D keyboard.

Correct to 4 significant figures, $5^{-4.3}=9.873\times 10^{-4}$.

b $22.9\div 1.3E2$. The 'E' is found on the mth Keyboard. The value returned is 0.1762 correct to 4 significant figures. The notation $1.3E2$ stands for 1.3×10^2.

c $5.04\times 10^{-6}\div\left(3\times 10^9\right)$ could be entered as $5.04E-6\div 3E9$ (if wished) to return the answer $1.68E-15$. In standard form, $5.04\times 10^{-6}\div\left(3\times 10^9\right)=1.68\times 10^{-15}$.

d $5.04\times 10^{-6}\div\left(3.2\times 10^{4.2}\right)$. The E notation is for standard form where the power of 10 is always an integer, so it cannot be used for $10^{4.2}$ since 4.2 is not an integer.

$5.04\times 10^{-6}\div\left(3.2\times 10^{4.2}\right)=9.937578176E-11$

In standard form correct to 4 significant figures, $5.04\times 10^{-6}\div\left(3.2\times 10^{4.2}\right)=9.938\times 10^{-11}$.

20 a $\dfrac{x^2y^{-2}}{2x^{\frac{1}{3}}\sqrt{y^5}}$

In the 2 D keyboard the fraction template and the square root template and if wished, the exponential template could be used to enter the expression.

Highlight and tap Interactive → Transformation →

Simplify to obtain $\dfrac{x^{\frac{5}{3}}}{2y^2\sqrt{y^5}}$.

Simplifying by hand, this becomes $\dfrac{x^{\frac{5}{3}}}{2y^2y^{\frac{5}{2}}}=\dfrac{x^{\frac{5}{3}}}{2y^{\frac{9}{2}}}$.

Had the original expression been entered as $\dfrac{x^2y^{-2}}{2x^{\frac{1}{3}}y^{\frac{5}{2}}}$ then the answer would have been obtained immediately.

b i $5^x\times 25^{2x}=\dfrac{1}{5}$

With the calculator on Standard mode, solve the equation using Equation/Inequality. The answer is $x=-\dfrac{1}{5}$.

ii $5^x\times 25^{2x}=0.25$

With the calculator on Decimal mode, solve the equation using Equation/Inequality. The answer, to 4 significant figures, is $x=-0.1723$.

Exercise 11.3 — Indices as logarithms

1 a $5^4=625\Rightarrow 4=\log_5(625)$

b $\log_{36}(6)=\dfrac{1}{2}\Rightarrow 6=36^{\frac{1}{2}}$

c $10^x=8.52$

$x=\log_{10}(8.52)$

≈ 0.93

d $\log_3(x)=-1$

$x=3^{-1}$

$x=\dfrac{1}{3}$

2 a $\log_e(5)\approx 1.609$; index statement is $5=e^{1.609}$.

b $10^{3.5}\approx 3162$; log statement is $3.5=\log_{10}(3162)$

3 **a**
$$\begin{aligned}\log_{12}(3)+\log_{12}(4)&=\log_{12}(3\times 4)\\&=\log_{12}(12)\\&=1\end{aligned}$$

b
$$\begin{aligned}\log_2(192)-\log_2(12)&=\log_2\left(\frac{192}{12}\right)\\&=\log_2(16)\\&=\log_2\left(2^4\right)\\&=4\log_2(2)\\&=4\end{aligned}$$

c
$$\begin{aligned}\log_3(3a^3)-2\log_3\left(a^{\frac{3}{2}}\right)&=\log_3(3)+\log_3\left(a^3\right)-2\times\frac{3}{2}\log_3(a)\\&=1+3\log_3(a)-3\log_3(a)\\&=1\end{aligned}$$

d
$$\begin{aligned}\frac{\log_a(8)}{\log_a(4)}&=\frac{\log_a\left(2^3\right)}{\log_a\left(2^2\right)}\\&=\frac{3\cancel{\log_a(2)}}{2\cancel{\log_a(2)}}\\&=\frac{3}{2}\end{aligned}$$

4 $\log_a(2)=0.3;\ \log_a(5)=0.7$

a
$$\begin{aligned}\log_a(0.5)&=\log_a\left(\frac{1}{2}\right)\\&=\log_a\left(2^{-1}\right)\\&=-\log_a(2)\\&=-0.3\end{aligned}$$

b
$$\begin{aligned}\log_a(2.5)&=\log_a\left(\frac{5}{2}\right)\\&=\log_a(5)-\log_a(2)\\&=0.7-0.3\\&=0.4\end{aligned}$$

c
$$\begin{aligned}\log_a(20)&=\log_a\left(2^2\times 5\right)\\&=\log_a\left(2^2\right)+\log_a(5)\\&=2\log_a(2)+\log_a(5)\\&=2\times 0.3+0.7\\&=1.3\end{aligned}$$

5 **a** $7^x=15\Rightarrow x=\log_7(15)$

$$\log_7(15)=\frac{\log_{10}(15)}{\log_{10}(7)}$$

$$\therefore x\approx 1.392$$

b $3^{2x+5}=4^x$

Take logs base 10 of both sides:

$$\begin{aligned}\log\left(3^{2x+5}\right)&=\log\left(4^x\right)\\(2x+5)\log(3)&=x\log(4)\\2x\log(3)+5\log(3)&=x\log(4)\\5\log(3)&=x\log(4)-2x\log(3)\\5\log(3)&=x\left(\log(4)-2\log(3)\right)\\x&=\frac{5\log(3)}{\log(4)-2\log(3)}\end{aligned}$$

Correct to 3 decimal places, $x=-6.774$.

6
$$\begin{aligned}\log_2(3)-\log_2(2)&=\log_2(x)+\log_2(5)\\\log_2\left(\frac{3}{2}\right)&=\log_2(5x)\\\frac{3}{2}&=5x\\x&=\frac{3}{10}\end{aligned}$$

7
$$\begin{aligned}\log_3(x)+\log_3(2x+1)&=1\\\log_3\left(x(2x+1)\right)&=1\\\log_3\left(2x^2+x\right)&=1\\2x^2+x&=3^1\\2x^2+x&=3\\2x^2+x-3&=0\\(2x+3)(x-1)&=0\\x&=-1.5,\ x=1\end{aligned}$$

Check in $\log_3(x)+\log_3(2x+1)=1$.

If $x=-1.5$, $\text{LHS}=\log_3(-1.5)+\log_3(-2)$ is not admissible; therefore reject $x=-1.5$.

If $x=1$,
$$\begin{aligned}\text{LHS}&=\log_3(1)+\log_3(3)\\&=\log_3(3)\\&=1\\&=\text{RHS}\end{aligned}$$

Therefore $x=1$.

8
$$\begin{aligned}\log_6(x)-\log_6(x-1)&=2\\\log_6\left(\frac{x}{x-1}\right)&=2\\\frac{x}{x-1}&=6^2\\\frac{x}{x-1}&=36\\x&=36(x-1)\\35x&=36\\x&=\frac{36}{35}\end{aligned}$$

Check in $\log_6(x)-\log_6(x-1)=2$.

$$\begin{aligned}\text{LHS}&=\log_6\left(\frac{36}{35}\right)-\log_6\left(\frac{1}{5}\right)\\&=\log_6\left(\frac{36}{35}\div\frac{1}{5}\right)\\&=\log_6(36)\\&=2\\&=\text{RHS}\end{aligned}$$

Therefore $x=\frac{36}{35}$.

9 **a** **i** $2^5=32\Leftrightarrow 5=\log_2(32)$

ii $4^{\frac{1}{3}}=8\Leftrightarrow\frac{3}{2}=\log_4(8)$

iii $10^{-3}=0.001\Leftrightarrow -3=\log_{10}(0.001)$

b **i** $\log_2(16)=4\Leftrightarrow 2^4=16$

ii $\log_9(3)=\frac{1}{2}\Leftrightarrow 9^{\frac{1}{2}}=3$

iii $\log_{10}(0.1)=-1\Leftrightarrow 10^{-1}=0.1$

10 **a** $x=\log_2\left(\frac{1}{8}\right)$

$$\therefore 2^x=\frac{1}{8}$$

$$\therefore 2^x=2^{-3}$$

$$\therefore x=-3$$

b $\log_{25}(x)=-0.5$

$$\therefore 25^{-0.5}=x$$

$$\therefore x=\frac{1}{\sqrt{25}}$$

$$\therefore x=\frac{1}{5}$$

c $10^{2x} = 4$

$$\begin{aligned}\therefore 2x &= \log_{10}(4)\\ \therefore x &= \frac{1}{2}\log_{10}(4)\\ \therefore x &\simeq 0.30\end{aligned}$$

d $3 = e^{-x}$

$$\begin{aligned}\therefore -x &= \log_e(3)\\ \therefore x &= -\log_e(3)\\ \therefore x &\simeq -1.10\end{aligned}$$

e $\log_x(125) = 3$

$$\begin{aligned}\therefore x^3 &= 125\\ \therefore x &= 5\end{aligned}$$

f $\log_x(25) = -2$

$$\begin{aligned}\therefore x^{-2} &= 25\\ \therefore \frac{1}{x^2} &= 25\\ \therefore x^2 &= \frac{1}{25}\\ \therefore x &= \pm\frac{1}{5}\end{aligned}$$

However, the base of the logarithm is an element of the set $R^+ \setminus \{1\}$, so reject $x = -\frac{1}{5}$.

$$\therefore x = \frac{1}{5}$$

11 **a**
$$\begin{aligned}\log_9(3) + \log_9(27) &= \log_9(3 \times 27)\\ &= \log_9(81)\\ &= \log_9\left(9^2\right)\\ &= 2\log_9(9)\\ &= 2 \times 1\\ &= 2\end{aligned}$$

b
$$\begin{aligned}\log_9(3) - \log_9(27) &= \log_9(3 \div 27)\\ &= \log_9\left(\frac{1}{9}\right)\\ &= \log_9\left(9^{-1}\right)\\ &= -\log_9(9)\\ &= -1\end{aligned}$$

c
$$\begin{aligned}2\log_2(4) + \log_2(6) - \log_2(12) &= \log_2\left(4^2\right) + \log_2(6) - \log_2(12)\\ &= \log_2\left(\frac{16 \times 6}{12}\right)\\ &= \log_2(8)\\ &= \log_2\left(2^3\right)\\ &= 3\log_2(2)\\ &= 3\end{aligned}$$

d
$$\begin{aligned}\log_5\left(\log_3(3)\right) &= \log_5(1)\\ &= 0\end{aligned}$$

e
$$\begin{aligned}\log_{11}\left(\frac{7}{3}\right) + 2\log_{11}\left(\frac{1}{3}\right) - \log_{11}\left(\frac{11}{3}\right) - \log_{11}\left(\frac{7}{9}\right) &= \left[\log_{11}(7) - \log_{11}(3)\right] + 2\left[\log_{11}(1) - \log_{11}(3)\right] - \left[\log_{11}(11) - \log_{11}(3)\right] - \left[\log_{11}(7) - \log_{11}(9)\right]\\ &= \cancel{\log_{11}(7)} - \log_{11}(3) + 2\left[0 - \log_{11}(3)\right] - \left[1 - \log_{11}(3)\right] - \cancel{\log_{11}(7)} + \log_{11}(9)\\ &= -\cancel{\log_{11}(3)} - 2\log_{11}(3) - 1 + \cancel{\log_{11}(3)} + \log_{11}\left(3^2\right)\\ &= -2\log_{11}(3) - 1 + 2\log_{11}(3)\\ &= -1\end{aligned}$$

f
$$\begin{aligned}\log_2\left(\frac{\sqrt{x} \times x^{\frac{3}{2}}}{x^2}\right) &= \log_2\left(\frac{x^{\frac{1}{2}} \times x^{\frac{3}{2}}}{x^2}\right)\\ &= \log_2\left(\frac{x^2}{x^2}\right)\\ &= \log_2(1)\\ &= 0\end{aligned}$$

12 a
$$\begin{aligned}\frac{\log_a(9)}{\log_a\left(\sqrt[5]{3}\right)} &= \frac{\log_a\left(3^2\right)}{\log_a\left(3^{\frac{1}{5}}\right)}\\ &= \frac{2\log_a(3)}{\frac{1}{5}\log_a(3)}\\ &= 2 \div \frac{1}{5}\\ &= 10\end{aligned}$$

b
$$\begin{aligned}1-\log_4\left(3^n+3^{n+1}\right) &= \log_4(4)-\log_4\left(3^n(1+3)\right)\\ &= \log_4\left(\frac{4}{3^n(4)}\right)\\ &= \log_4\left(\frac{1}{3^n}\right)\\ &= \log_4\left(3^{-n}\right)\\ &= -n\log_4(3)\end{aligned}$$

c $\log_8(16)\times\log_{16}(8)$

Let $x=\log_8(16)$

$\therefore 8^x = 16....(1)$

Let $y = \log_{16}(8)$

$\therefore 16^y = 8....(2)$

Substitute $16 = 8^x$ in equation (2)

$\therefore \left(8^x\right)^y = 8$

$\therefore 8^{xy} = 8^1$

$\therefore xy = 1$

Hence, $\log_8(16)\times\log_{16}(8) = 1.$

d
$$\begin{aligned}\log_{10}(1000)+\log_{0.1}\left(\frac{1}{1000}\right) &= \log_{10}\left(10^3\right)+\log_{0.1}\left(0.1^3\right)\\ &= 3\log_{10}(10)+3\log_{0.1}(0.1)\\ &= 3+3\\ &= 6\end{aligned}$$

13 a
$$\begin{aligned}\log_2(10) &= \frac{\log_{10}(10)}{\log_{10}(2)}\\ &= \frac{1}{\log_{10}(2)}\end{aligned}$$

b i $11^x = 18$

$\therefore x = \log_{11}(18)$ is the exact solution.

Changing the base to 10,

$$\begin{aligned}\log_{11}(18) &= \frac{\log_{10}(18)}{\log_{10}(11)}\\ &\simeq 1.205\end{aligned}$$

$\therefore x \simeq 1.205$

ii $5^{-x} = 8$

$\therefore -x = \log_5(8)$

$\therefore x = -\log_5(8)$

Hence,

$$x = -\frac{\log_{10}(8)}{\log_{10}(5)}$$

$\therefore x \simeq -1.292$

iii $7^{2x} = 3$

$\therefore 2x = \log_7(3)$

$\therefore x = \frac{1}{2}\log_7(3)$

Hence,

$$x = \frac{1}{2}\times\frac{\log_{10}(3)}{\log_{10}(7)}$$

$\therefore x \simeq 0.2823$

c i $3^x \le 10$

$\therefore x \le \log_3(10)$

$\therefore x \le \frac{\log_{10}(10)}{\log_{10}(3)}$

$\therefore x \le \frac{1}{\log_{10}(3)}$

$\therefore x \le 2.096$

ii $5^{-x} > 0.4$

$\therefore -x > \log_5(0.4)$

$\therefore x < -\log_5(0.4)$

$\therefore x < -\frac{\log_{10}(0.4)}{\log_{10}(5)}$

$\therefore x < \frac{1}{\log_{10}(3)}$

$\therefore x < 0.5693$

d i $2^{\log_5(x)} = 8$

$\therefore 2^{\log_5(x)} = 2^3$

$\therefore \log_5(x) = 3$

$\therefore 5^3 = x$

$\therefore x = 125$

ii $2^{\log_2(x)} = 7$

$\therefore \log_2(x) = \log_2(7)$

$\therefore x = 7$

14 a $7^{1-2x} = 4$

$\therefore 1-2x = \log_7(4)$

$\therefore 1-\log_7(4) = 2x$

$\therefore x = \frac{1}{2}\left(1-\log_7(4)\right)$

$\therefore x = \frac{1}{2}\left(1-\frac{\log_{10}(4)}{\log_{10}(7)}\right)$

$\therefore x \simeq 0.14$

b $10^{-x} = 5^{x-1}$

Take base 10 logarithms of both sides

$\therefore \log_{10}\left(10^{-x}\right) = \log_{10}\left(5^{x-1}\right)$

$\therefore -x\log_{10}(10) = (x-1)\log_{10}(5)$

$\therefore -x = x\log_{10}(5)-\log_{10}(5)$

$\therefore \log_{10}(5) = x\log_{10}(5)+x$

$\therefore \log_{10}(5) = x\left[\log_{10}(5)+1\right]$

$\therefore x = \frac{\log_{10}(5)}{\log_{10}(5)+1}$

$\therefore x \simeq 0.41$

c $5^{2x-9} = 3^{7-x}$

$\therefore \log_{10}\left(5^{2x-9}\right) = \log_{10}\left(3^{7-x}\right)$

$\therefore (2x-9)\log_{10}(5) = (7-x)\log_{10}(3)$

$\therefore 2x\log(5)-9\log(5) = 7\log(3)-x\log(3)$

$\therefore 2x\log(5)+x\log_{10}(3) = 7\log(3)+9\log(5)$

$\therefore x\left[2\log(5)+\log(3)\right] = 7\log(3)+9\log(5)$

$\therefore x = \frac{7\log(3)+9\log(5)}{2\log(5)+\log(3)}$

$\therefore x \simeq 5.14$

d $10^{3x+5} = 6^{2-3x}$

$$\begin{aligned}
\therefore \log_{10}\left(10^{3x+5}\right) &= \log_{10}\left(6^{2-3x}\right)\\
\therefore (3x+5)\log(10) &= (2-3x)\log(6)\\
\therefore 3x+5 &= 2\log(6) - 3x\log(6)\\
\therefore 3x + 3x\log(6) &= 2\log(6) - 5\\
\therefore 3x\left(1+\log(6)\right) &= 2\log(6) - 5\\
\therefore x &= \frac{2\log(6)-5}{3\left(1+\log(6)\right)}\\
\therefore x &\simeq -0.65
\end{aligned}$$

e $0.25^{4x} = 0.8^{2-0.5x}$

$$\begin{aligned}
\therefore \log_{10}\left(0.25^{4x}\right) &= \log_{10}\left(0.8^{2-0.5x}\right)\\
\therefore 4x\log(0.25) &= (2-0.5x)\log(0.8)\\
\therefore 4x\log(0.25) &= 2\log(0.8) - 0.5x\log(0.8)\\
\therefore 4x\log(0.25) + 0.5x\log(0.8) &= 2\log(0.8)\\
\therefore x\left(4\log(0.25) + 0.5\log(0.8)\right) &= 2\log(0.8)\\
\therefore x &= \frac{2\log(0.8)}{4\log(0.25)+0.5\log(0.8)}\\
\therefore x &\simeq 0.08
\end{aligned}$$

f $4^{x+1} \times 3^{1-x} = 5^{x}$

$$\begin{aligned}
\therefore \log_{10}\left(4^{x+1}\times 3^{1-x}\right) &= \log_{10}\left(5^{x}\right)\\
\therefore \log\left(4^{x+1}\right) + \log\left(3^{1-x}\right) &= \log_{10}\left(5^{x}\right)\\
\therefore (x+1)\log(4) + (1-x)\log(3) &= x\log(5)\\
\therefore x\log(4) + \log(4) + \log(3) - x\log(3) &= x\log(5)\\
\therefore \log(4) + \log(3) &= x\log(5) - x\log(4) + x\log(3)\\
\therefore x\left(\log(5) - \log(4) + \log(3)\right) &= \log(4) + \log(3)\\
\therefore x &= \frac{\log(12)}{\log\left(\frac{5}{4}\times 3\right)}\\
\therefore x &= \frac{\log(12)}{\log\left(\frac{15}{4}\right)}\\
\therefore x &\simeq 1.88
\end{aligned}$$

15 a $\log_2(2x+1) + \log_2(2x-1) = 3\log_2(3)$

$$\begin{aligned}
\therefore \log_2\left((2x+1)(2x-1)\right) &= \log_2\left(3^3\right)\\
\therefore \log_2\left(4x^2-1\right) &= \log_2\left(3^3\right)\\
\therefore 4x^2 - 1 &= 27\\
\therefore 4x^2 &= 28\\
\therefore x^2 &= 7\\
\therefore x &= \pm\sqrt{7}
\end{aligned}$$

Reject $x = -\sqrt{7}$ since it creates logarithms of negative numbers in the original equation

$\therefore x = \sqrt{7}$

b $\log_3(2x) + \log_3(4) = \log_3(x+12) - \log_3(2)$

$$\begin{aligned}
\therefore \log_3(2x\times 4) &= \log_3\left(\frac{x+12}{2}\right)\\
\therefore \log_3(8x) &= \log_3\left(\frac{x+12}{2}\right)\\
\therefore 8x &= \frac{x+12}{2}\\
\therefore 16x &= x+12\\
\therefore 15x &= 12\\
\therefore x &= \frac{12}{15}\\
\therefore x &= \frac{4}{5}
\end{aligned}$$

c $\log_2(2x+12)-\log_2(3x)=4$

$$\begin{aligned}\therefore \log_2\left(\frac{2x+12}{3x}\right)&=4\\ \therefore 2^4&=\frac{2x+12}{3x}\\ \therefore 16\times 3x&=2x+12\\ \therefore 46x&=12\\ \therefore x&=\frac{12}{46}\\ \therefore x&=\frac{6}{23}\end{aligned}$$

d $\log_2(x)+\log_2(2-2x)=-1$

$$\begin{aligned}\therefore \log_2(x(2-2x))&=-1\\ \therefore \log_2\left(2x-2x^2\right)&=-1\\ \therefore 2^{-1}&=2x-2x^2\\ \therefore \frac{1}{2}&=2x-2x^2\\ \therefore 1&=4x-4x^2\\ \therefore 4x^2-4x+1&=0\\ \therefore (2x-1)^2&=0\\ \therefore x&=\frac{1}{2}\end{aligned}$$

e $\left(\log_{10}(x)+3\right)\left(2\log_4(x)-3\right)=0$

Using the null factor law,

$\log_{10}(x)+3=0$ or $2\log_4(x)-3=0$

$\therefore \log_{10}(x)=-3$ or $\log_4(x)=\frac{3}{2}$

$\therefore x=10^{-3}$ or $x=4^{\frac{3}{2}}$

$\therefore x=0.001$ or $x=8$

f $2\log_3(x)-1=\log_3(2x-3)$

$$\begin{aligned}\therefore \log_3\left(x^2\right)-\log_3(2x-3)&=1\\ \therefore \log_3\left(\frac{x^2}{2x-3}\right)&=1\\ \therefore \frac{x^2}{2x-3}&=3^1\\ \therefore x^2&=6x-9\\ \therefore x^2-6x+9&=0\\ \therefore (x-3)^2&=0\\ \therefore x&=3\end{aligned}$$

16 Given $\log_a(3)=p$ and $\log_a(5)=q$.

a $\log_a(15)=\log_a(3\times 5)$

$$\begin{aligned}&=\log_a(3)+\log_a(5)\\ &=p+q\end{aligned}$$

b $\log_a(125)=\log_a\left(5^3\right)$

$$\begin{aligned}&=3\log_a(5)\\ &=3q\end{aligned}$$

c $\log_a(45)=\log_a(9\times 5)$

$$\begin{aligned}&=\log_a\left(3^2\times 5\right)\\ &=\log_a\left(3^2\right)+\log_a(5)\\ &=2\log_a(3)+\log_a(5)\\ &=2p+q\end{aligned}$$

d $\log_a(0.6)=\log_a\left(\frac{3}{5}\right)$

$$\begin{aligned}&=\log_a(3)-\log_a(5)\\ &=p-q\end{aligned}$$

e $\log_a\left(\frac{25}{81}\right)=\log_a(25)-\log_a(81)$

$$\begin{aligned}&=\log_a\left(5^2\right)-\log_a\left(3^4\right)\\ &=2\log_a(5)-4\log_a(3)\\ &=2q-4p\end{aligned}$$

f $\log_a\left(\sqrt{5}\right)\times\log_a\left(\sqrt{27}\right)=\log_a\left(5^{\frac{1}{2}}\right)\times\log_a\left(\sqrt{3^3}\right)$

$$\begin{aligned}&=\log_a\left(5^{\frac{1}{2}}\right)\times\log_a\left(3^{\frac{3}{2}}\right)\\ &=\frac{1}{2}\log_a(5)\times\frac{3}{2}\log_a(3)\\ &=\frac{3}{4}\log_a(5)\times\log_a(3)\\ &=\frac{3}{4}qp\end{aligned}$$

17 **a** $\log_{10}(y)=\log_{10}(x)+2$

$$\begin{aligned}\therefore \log_{10}(y)-\log_{10}(x)&=2\\ \therefore \log_{10}\left(\frac{y}{x}\right)&=2\\ \therefore 10^2&=\frac{y}{x}\\ \therefore y&=100x\end{aligned}$$

b $\log_2\left(x^2\sqrt{y}\right)=x$

$$\begin{aligned}&\therefore 2^x=x^2\sqrt{y}\\ &\therefore \sqrt{y}=\frac{2^x}{x^2}\\ &\therefore y=\left(\frac{2^x}{x^2}\right)^2\\ &\therefore y=\frac{2^{2x}}{x^4}\\ &\therefore y=2^{2x}\times x^{-4}\end{aligned}$$

c $2\log_2\left(\frac{y}{2}\right)=6x-2$

$$\begin{aligned}\therefore \log_2\left(\frac{y}{2}\right)&=3x-1\\ \therefore 2^{3x-1}&=\frac{y}{2}\\ \therefore y&=2^{3x-1}\times 2\\ \therefore y&=2^{3x}\end{aligned}$$

d $x=10^{y-2}$

$$\begin{aligned}&\therefore y-2=\log_{10}(x)\\ &\therefore y=\log_{10}(x)+2\end{aligned}$$

e $\log_{10}\left(10^{3xy}\right)=3$

$$\begin{aligned}\therefore 3xy\log_{10}(10)&=3\\ \therefore 3xy&=3\\ \therefore y&=\frac{1}{x}\end{aligned}$$

f $10^{3\log_{10}(y)}=xy$

$$\begin{aligned}&\therefore \log_{10}\left(y^3\right)=\log_{10}(xy)\\ &\therefore y^3=xy\\ &\therefore y^3-xy=0\\ &\therefore y\left(y^2-x\right)=0\end{aligned}$$

$\therefore y=0$ (reject) or $y^2=x$

Both x and y must be positive

$\therefore y=\sqrt{x},\ x>0$

18 a $2^{2x} - 14 \times 2^x + 45 = 0$

Let $a = 2^x$

$\therefore a^2 - 14a + 45 = 0$

$\therefore (a-5)(a-9) = 0$

$\therefore a = 5$ or $a = 9$

$\therefore 2^x = 5$ or $2^x = 9$

$\therefore x = \log_2(5)$ or $x = \log_2(9)$

b $5^{-x} - 5^x = 4$

$\therefore \frac{1}{5^x} - 5^x = 4$

Let $a = 5^x$

$\therefore \frac{1}{a} - a = 4$

$\therefore 1 - a^2 = 4a$

$\therefore a^2 + 4a - 1 = 0$

Completing the square,

$\left(a^2 + 4a + 4\right) - 4 - 1 = 0$

$\therefore (a+2)^2 = 5$

$\therefore a + 2 = \pm\sqrt{5}$

$\therefore a = \sqrt{5} - 2$ or $a = -\sqrt{5} - 2$

$\therefore 5^x = \sqrt{5} - 2$ or $5^x = -\sqrt{5} - 2$

As $5^x > 0$, reject $5^x = -\sqrt{5} - 2$

$\therefore 5^x = \sqrt{5} - 2$

$\therefore x = \log_5\left(\sqrt{5} - 2\right)$

c $9^{2x} - 3^{1+2x} + 2 = 0$

$\therefore 9^{2x} - 3^1 \times 3^{2x} + 2 = 0$

$\therefore 9^{2x} - 3 \times 9^x + 2 = 0$

Let $a = 9^x$

$\therefore a^2 - 3a + 2 = 0$

$\therefore (a-1)(a-2) = 0$

$\therefore a = 1$ or $a = 2$

$\therefore 9^x = 1$ or $9^x = 2$

$\therefore x = 0$ or $x = \log_9(2)$

d $\log_a\left(x^3\right) + \log_a\left(x^2\right) - 4\log_a(2) = \log_a(x)$

$\therefore \log_a\left(x^3\right) + \log_a\left(x^2\right) - \log_a(x) = 4\log_a(2)$

$\therefore \log_a\left(\frac{x^3 \times x^2}{x}\right) = \log_a\left(2^4\right)$

$\therefore \log_a\left(x^4\right) = \log_a\left(2^4\right)$

$\therefore 4\log_a(x) = 4\log_a(2)$

$\therefore \log_a(x) = \log_a(2)$

$\therefore x = 2$

e $\left(\log_2(x)\right)^2 - \log_2\left(x^2\right) = 8$

$\therefore \left(\log_2(x)\right)^2 - 2\log_2(x) = 8$

Let $a = \log_2(x)$

$\therefore (a)^2 - 2a = 8$

$\therefore a^2 - 2a - 8 = 0$

$\therefore (a+2)(a-4) = 0$

$\therefore a = -2$ or $a = 4$

$\therefore \log_2(x) = -2$ or $\log_2(x) = 4$

$\therefore x = 2^{-2}$ or $x = 2^4$

$\therefore x = \frac{1}{4}$ or $x = 16$

f $\frac{\log_{10}\left(x^3\right)}{\log_{10}(x+1)} = \log_{10}(x)$

$\therefore \log_{10}\left(x^3\right) = \log_{10}(x) \times \log_{10}(x+1)$

$\therefore 3\log_{10}(x) = \log_{10}(x) \times \log_{10}(x+1)$

$\therefore 3\log_{10}(x) - \log_{10}(x) \times \log_{10}(x+1) = 0$

$\therefore \log_{10}(x)\left[3 - \log_{10}(x+1)\right] = 0$

$\therefore \log_{10}(x) = 0$ or $3 - \log_{10}(x+1) = 0$

$\therefore \log_{10}(x) = 0$ or $\log_{10}(x+1) = 3$

$\therefore x = 10^0$ or $x + 1 = 10^3$

$\therefore x = 1$ or $x + 1 = 1000$

$\therefore x = 1$ or $x = 999$

19 a $12^x = 50$

$\therefore x = \log_{12}(50)$

From the main menu in Decimal mode select the logarithm template $\log_w X$, from the 2 D Keyboard. Type in the base 12 and the numeral 50 to obtain $\log_{12}(50) = 1.574$, correct to 4 significant figures.

$\therefore x = 1.574$

b $\log(5x) + \log(x+5) = 1$

The base 10 logarithm is found in the Math keyboard. Use the calculator on Standard mode and solve the equation by highlighting it and dropping into Equation/Inequality to obtain the solution $x = \frac{\sqrt{33} - 5}{2}$.

20 a In Standard mode, using the logarithm templates enter $\log_{10}(5) + \log_5(10)$ and press the EXE key to obtain $\frac{\ln(5) + \ln(2)}{\ln(5)} + \log(5)$.

$\ln(5)$ means $\log_e(5)$ and $\log(5) = \log_5(10)$.

The calculator has used the change of base law to express $\log_5(10)$ in terms of the base e logarithm.

$$\begin{aligned}\log_5(10) &= \frac{\log_e(10)}{\log_e(5)} \\ &= \frac{\log_e(5 \times 2)}{\log_e(5)} \\ &= \frac{\log_e(5) + \log_e(2)}{\log_e(5)} \\ &= \frac{\ln(5) + \ln(2)}{\ln(5)}\end{aligned}$$

b $\log_y(x) \times \log_x(y)$

Enter using the 2 D template.

The answer given is 1.

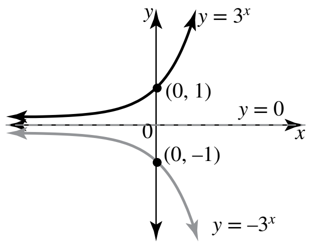
$\therefore \log_y(x) \times \log_x(y) = 1$

This can be explained by expressing $\log_y(x) \times \log_x(y)$ in terms of base 10 logarithms.

$$\begin{aligned}\log_y(x) \times \log_x(y) &= \frac{\log(x)}{\log(y)} \times \frac{\log(y)}{\log(x)} \\ &= 1\end{aligned}$$

Exercise 11.4 — Graphs of exponential functions

1 a



For $y = 3^x$, range is R^+ and for $y = -3^x$, the range is R^-. Asymptote is $y = 0$ for both graphs.

b Graph has 'decay' shape, so $y = a^{-x}$. Point $(-1, 3)$ lies on the graph.

$\therefore 3 = a^1$

$\therefore a = 3$

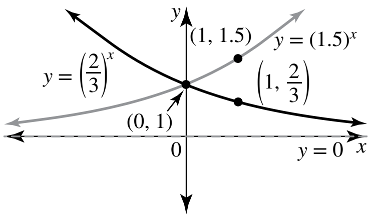
Equation is $y = 3^{-x}$.

2 $y = (1.5)^x$ $\qquad$ $y = \left(\frac{2}{3}\right)^x$

asymptote: $y = 0$ $\qquad$ asymptote: $y = 0$

y-intercept: If $x = 0, y = 1$ $\qquad$ y-intercept: If $x = 0, y = 1$

point: $x = 1, y = 1.5$ $\qquad$ point: $x = 1, y = \frac{2}{3}$



Note that $1.5 = \frac{3}{2}$, $y = (1.5)^x = \left(\frac{3}{2}\right)^x$ and since $\left(\frac{2}{3}\right) = \left(\frac{3}{2}\right)^{-1}$,

$y = \left(\frac{2}{3}\right)^x = \left(\frac{3}{2}\right)^{-x}$

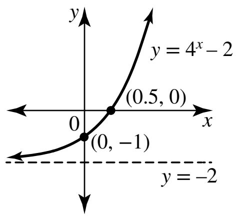
3 a $y = 4^x - 2$

asymptote: $y = -2$

y-intercept $x = 0, y = 1 - 2 \Rightarrow (0, -1)$

x-intercept: when $y = 0$,

$$\begin{aligned} 4^x - 2 &= 0 \\ 4^x &= 2 \\ 4^x &= 4^{\frac{1}{2}} \\ x &= \frac{1}{2} \end{aligned}$$



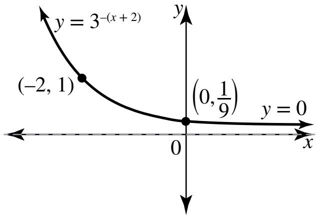
Range is $(-2, \infty)$.

b $y = 3^{-(x+2)}$

asymptote: $y = 0$

y-intercept $x = 0, y = 3^{-2} \Rightarrow \left(0, \frac{1}{9}\right)$

point: when $x = -2, y = 1$



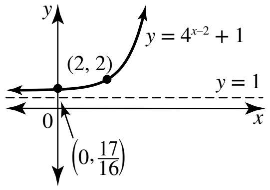
Range is R^+.

4 $y = 4^{x-2} + 1$

asymptote: $y = 1$

y-intercept $x = 0, y = 4^{-2} + 1 \Rightarrow \left(0, 1\frac{1}{16}\right)$

point: when $x = 2, y = 2$



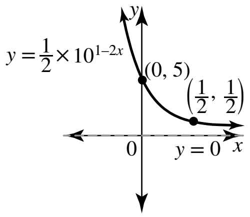
Range is $(1, \infty)$.

5 a $y = \frac{1}{2} \times 10^{1-2x}$

asymptote: $y = 0$

y-intercept: $x = 0, y = \frac{1}{2} \times 10 \Rightarrow (0, 5)$

point: when $x = \frac{1}{2}, y = \frac{1}{2}$



Range is R^+.

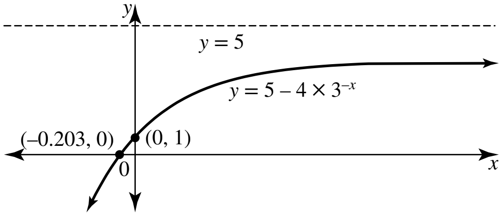
b $y = 5 - 4 \times 3^{-x}$

asymptote: $y = 5$

y-intercept: $x = 0, y = 5 - 4 \Rightarrow (0, 1)$

x-intercept: when $y = 0$,

$$\begin{aligned} 5 - 4 \times 3^{-x} &= 0 \\ 4 \times 3^{-x} &= 5 \\ 3^{-x} &= \frac{5}{4} \\ -x &= \log_3\left(\frac{5}{4}\right) \\ -x &= \frac{\log_{10}\left(\frac{5}{4}\right)}{\log_{10}(3)} \\ x &\approx -0.203 \end{aligned}$$



Range is $(-\infty, 5)$.

6 $y = a.3^x + b$

Asymptote is at $y = 2$, so $b = 2$.

$y = a.3^x + 2$

Graph passes through the origin.

$\Rightarrow 0 = a + 2$

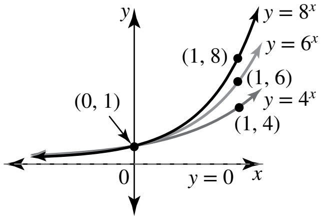
$\therefore a = -2$

Therefore the equation is $y = -2 \times 3^x + 2$ and $a = -2, b = 2$.

7 a i $y = 4^x, y = 6^x$ and $y = 8^x$.

Each graph passes through the point $(0,1)$ and the line $y = 0$ is the asymptote for each graph.

If $x = 1$, the graphs of $y = 4^x, y = 6^x$ and $y = 8^x$ pass through $(1,4), (1,6)$ and $(1,8)$ respectively.

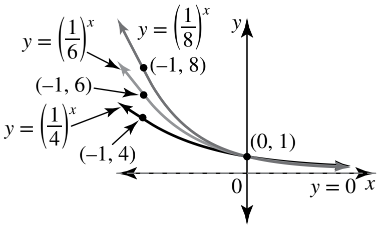


ii For $x > 0$, as the base increases, the steepness of the graph increases.

b i $y = \left(\frac{1}{4}\right)^x, y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$.

Each graph passes through the point $(0,1)$ and the line $y = 0$ is the asymptote for each graph.

If $x = -1$, the graphs of $y = \left(\frac{1}{4}\right)^x, y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$ pass through $(-1,4), (-1,6)$ and $(-1,8)$ respectively.

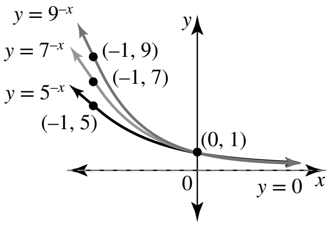


ii The rules for the graphs can be expressed as $y = 4^{-x}, y = 6^{-x}$ and $y = 8^{-x}$.

8 a i $y = 5^{-x}, y = 7^{-x}$ and $y = 9^{-x}$.

Each graph passes through the point $(0,1)$ and the line $y = 0$ is the asymptote for each graph.

If $x = -1$, the graphs of $y = 5^{-x}, y = 7^{-x}$ and $y = 9^{-x}$ pass through $(-1,5), (-1,7)$ and $(-1,9)$ respectively.

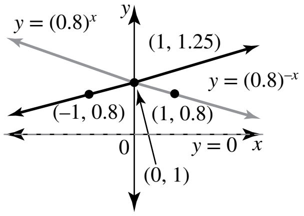


ii As the base increases, the decrease of the graph is more steep for $x < 0$.

b i $y = (0.8)^x, y = (1.25)^x$ and $y = (0.8)^{-x}$

Each graph passes through the point $(0,1)$ and the line $y = 0$ is the asymptote for each graph.

The point $(1, 0.8)$ lies on $y = (0.8)^x$, the point $(1, 1.25)$ lies on $y = (1.25)^x$ and the point $(-1, 0.8)$ lies on $y = (0.8)^{-x}$.



ii The graphs of $y = (0.8)^{-x}$ and $y = (1.25)^x$ are the same and the graph of $y = (0.8)^{-x}$ is the reflection in the y axis of the graph of $y = (0.8)^x$. This is because:

$y = (0.8)^{-x}$

$= \left(\frac{4}{5}\right)^{-x}$

$= \left(\frac{5}{4}\right)^x$

$y = (1.25)^x$

$= \left(\frac{5}{4}\right)^x$

$y = (0.8)^x$

$= \left(\frac{4}{5}\right)^x$ or $\left(\frac{5}{4}\right)^{-x}$

9 a $y = 5^{-x} + 1$

Asymptote: $y = 1$

y intercept: Let $x = 0$

$\therefore y = 5^0 + 1$

$\therefore y = 2$

$(0,2)$

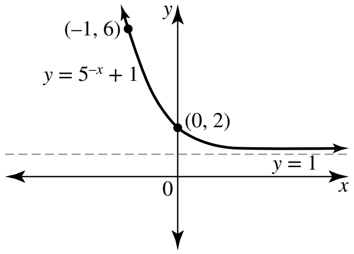
No x intercept.

Point: Let $x = -1$

$\therefore y = 5^1 + 1$

$\therefore y = 6$

$(-1,6)$



b $y = 1 - 4^x$

$\therefore y = -4^x + 1$

Asymptote: $y = 1$

y intercept: Let $x = 0$

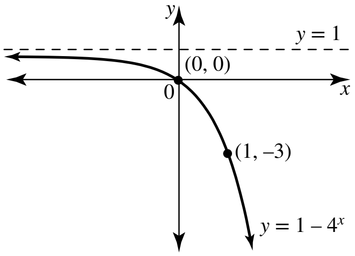
$\therefore y = -4^0 + 1$

$\therefore y = 0$

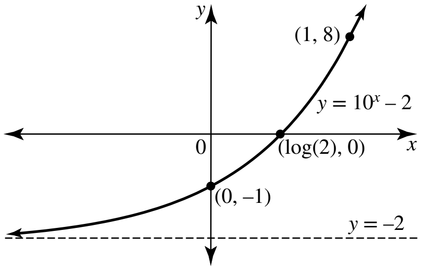
$(0,0)$

Point: Let $x = 1$

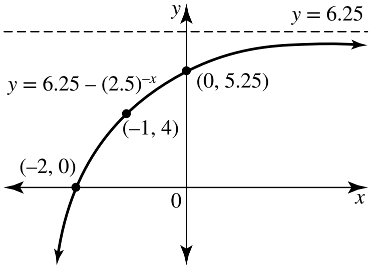
$\therefore y=-4^1+1$
$\therefore y=-3$
$(1,-3)$



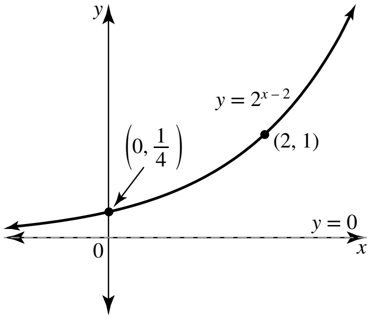
c $y=10^x-2$
Asymptote: $y=-2$
y intercept: Let $x=0$
$\therefore y=10^0-2$
$\therefore y=-1$
$(0,-1)$
x intercept: Let $y=0$
$\therefore 0=10^x-2$
$\therefore 10^x=2$
$\therefore x=\log_{10}(2)$
$\left(\log_{10}(2),0\right)$



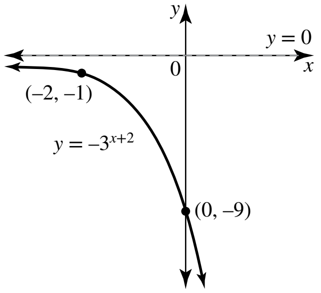
d $y=6.25-(2.5)^{-x}$
Asymptote: $y=6.25$
y intercept: Let $x=0$
$\therefore y=6.25-1$
$\therefore y=5.25$
$(0,\ 5.25)$
x intercept: Let $y=0$
$\therefore 0=6.25-(2.5)^{-x}$
$\therefore (2.5)^{-x}=6.25$
$\therefore (2.5)^{-x}=(2.5)^2$
$\therefore -x=2$
$\therefore x=-2$
$(-2,0)$



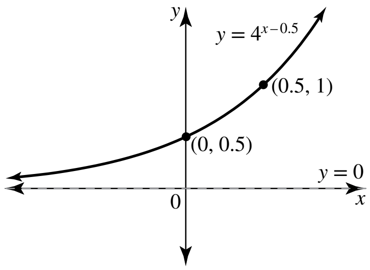
10 a $y=2^{x-2}$
Asymptote: $y=0$
y intercept: Let $x=0$
$\therefore y=2^{-2}$
$\therefore y=\frac{1}{4}$
$\left(0,\frac{1}{4}\right)$
Point: Let $x=2$
$\therefore y=2^0$
$\therefore y=1$
$(2,1)$



b $y=-3^{x+2}$
Asymptote: $y=0$
y intercept: Let $x=0$
$\therefore y=-3^2$
$\therefore y=-9$
$(0,-9)$
Point: Let $x=-2$
$\therefore y=-3^0$
$\therefore y=-1$
$(-2,-1)$



c $y=4^{x-0.5}$
Asymptote: $y=0$
y intercept: Let $x=0$
$\therefore y=4^{-0.5}$
$\therefore y=\frac{1}{\sqrt{4}}$
$\therefore y=\frac{1}{2}$
$\left(0,\frac{1}{2}\right)$
Point: Let $x=0.5$
$\therefore y=4^0$
$\therefore y=1$
$(0.5,1)$



d $y = 7^{1-x}$

Asymptote: $y = 0$

y intercept: Let $x = 0$

$\therefore y = 7^1$

$\therefore y = 7$

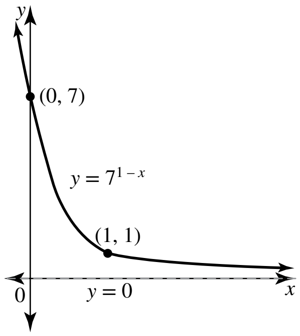
$(0,7)$

Point: Let $x = 1$

$\therefore y = 7^0$

$\therefore y = 1$

$(1,1)$



11 a $y = 3\times 2^x$

Asymptote: $y = 0$

y intercept: Let $x = 0$

$\therefore y = 3\times 1$

$\therefore y = 3$

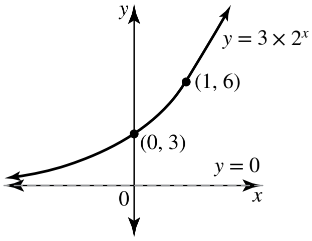
$(0,3)$

Point: Let $x = 1$

$\therefore y = 3\times 2^1$

$\therefore y = 6$

$(1,6)$



b $y = 2^{\frac{3x}{4}}$

Asymptote: $y = 0$

y intercept: Let $x = 0$

$\therefore y = 2^0$

$\therefore y = 1$

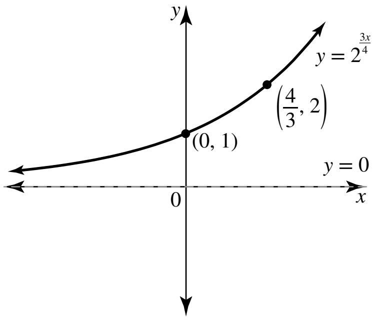
$(0,1)$

Point: Let $x = \frac{4}{3}$

$\therefore y = 2^1$

$\therefore y = 2$

$\left(\frac{4}{3},2\right)$



c $y = -3\times 2^{-3x}$

Asymptote: $y = 0$

y intercept: Let $x = 0$

$\therefore y = -3\times 2^0$

$\therefore y = -3$

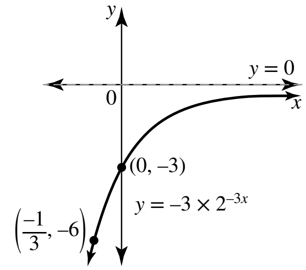
$(0,-3)$

Point: Let $x = -\frac{1}{3}$

$\therefore y = -3\times 2^1$

$\therefore y = -6$

$\left(-\frac{1}{3},-6\right)$



d $y = 1.5\times 10^{-\frac{x}{2}}$

Asymptote: $y = 0$

y intercept: Let $x = 0$

$\therefore y = 1.5\times 10^0$

$\therefore y = 1.5$

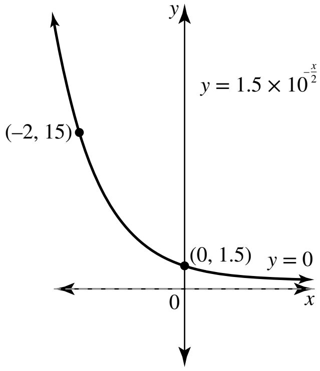
$(0,1.5)$

Point: Let $x = -2$

$\therefore y = 1.5\times 10^1$

$\therefore y = 15$

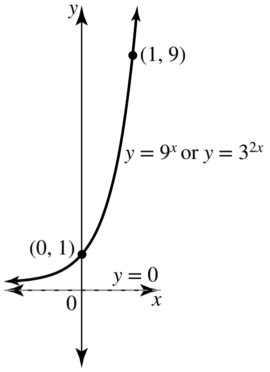
$(-2,15)$



12 a $y = 3^{2x}$ and $y = 9^x$ are identical equations since:

$$
\begin{aligned}
y &= 9^x \\
&= \left(3^2\right)^x \\
&= 3^{2x}
\end{aligned}
$$

The graph has an asymptote when $y=0$ and passes through the points $(0,1)$ and $(1,9)$.



b **i** $y=2\times 4^{0.5x}$

$\therefore y=2\times\left(2^2\right)^{0.5x}$

$\therefore y=2\times 2^x$

$\therefore y=2^{1+x}$

$\therefore y=2^{x+1}$

ii $y=2^{x+1}$

Asymptote: $y=0$

y intercept: Let $x=0$

$\therefore y=2^1$

$\therefore y=2$

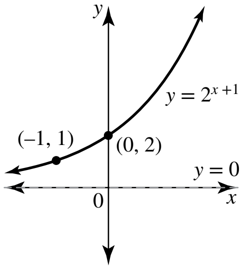
$(0,2)$

Point: Let $x=-1$

$\therefore y=2^0$

$\therefore y=1$

$(-1,1)$



13 **a** $y=a.10^x+b$

The asymptote equation is $y=3$, so $b=3$.

$\therefore y=a.10^x+3$

Substitute the point $(0,5)$

$\therefore 5=a.10^0+3$

$\therefore 5=a+3$

$\therefore a=2$

The rule for the graph is $y=2\times 10^x+3$.

b $y=a.3^{kx}$

Point $(1,36)\Rightarrow 36=a.3^k....(1)$

Point $(0,4)\Rightarrow 4=a.3^0\Rightarrow a=4$

Substitute $a=4$ in equation (1)

$\therefore 36=4\times 3^k$

$\therefore 3^k=9$

$\therefore k=2$

Hence the rule is $y=4\times 3^{2x}$ and the asymptote equation is $y=0$.

c $y=a-2\times 3^{b-x}$

Asymptote $y=6\Rightarrow a=6$

$\therefore y=6-2\times 3^{b-x}$

Substitute the point $(0,0)$

$\therefore 0=6-2\times 3^b$

$\therefore 2\times 3^b=6$

$\therefore 3^b=3$

$\therefore b=1$

The rule is $y=6-2\times 3^{1-x}$.

d $y=6-2\times 3^{1-x}$

$\therefore y=6-2\times 3^1\times 3^{-x}$

$\therefore y=6-6\times 3^{-x}$

14 **a** $y=2\times 10^{2x}-20$

Asymptote: $y=-20$

y intercept: Let $x=0$

$\therefore y=2\times 10^0-20$

$\therefore y=-18$

$(0,-18)$

Range $(-20,\infty)$

x intercept: Let $y=0$

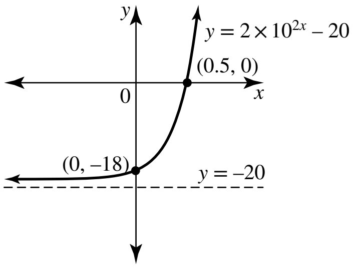
$\therefore 0=2\times 10^{2x}-20$

$\therefore 10^{2x}=10$

$\therefore 2x=1$

$\therefore x=\frac{1}{2}$

$(0.5,0)$



b $y=5\times 2^{1-x}-1$

Asymptote: $y=-1$

y intercept: Let $x=0$

$\therefore y=5\times 2^1-1$

$\therefore y=9$

$(0,9)$

Range $(-1,\infty)$

x intercept: Let $y=0$

$\therefore 0=5\times 2^{1-x}-1$

$\therefore 5\times 2^{1-x}=1$

$\therefore 2^{1-x}=\frac{1}{5}$

$\therefore \log\left(2^{1-x}\right)=\log(0.2)$

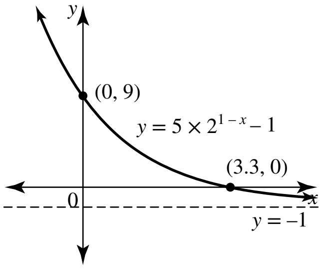
$\therefore (1-x)\log(2)=\log(0.2)$

$\therefore 1-x=\frac{\log(0.2)}{\log(2)}$

$\therefore 1-\frac{\log(0.2)}{\log(2)}=x$

$\therefore x\simeq 3.3$

$(3.3,0)$



c $y=3-2\left(\frac{2}{3}\right)^x$

Asymptote: $y=3$

y intercept: Let $x=0$

$\therefore y=3-2\times1$

$\therefore y=1$

$(0,1)$

Range $(-\infty,3)$

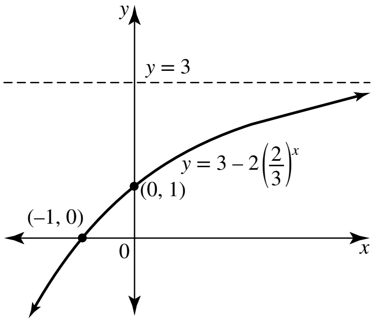
x intercept: Let $y=0$

$$\therefore 0=3-2\left(\frac{2}{3}\right)^x$$

$$\therefore \left(\frac{2}{3}\right)^x=\frac{3}{2}$$

$$\therefore x=-1$$

$(-1,0)$



d $y=2(3.5)^{x+1}-7$

Asymptote: $y=-7$

y intercept: Let $x=0$

$\therefore y=2\times(3.5)^1-7$

$\therefore y=0$

$(0,0)$

Range $(-7,\infty)$

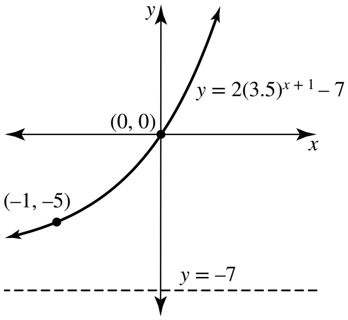
x intercept: $(0,0)$

Point: Let $x=-1$

$\therefore y=2\times1-7$

$\therefore y=-5$

$(-1,-5)$



e $y=8-4\times5^{2x-1}$

Asymptote: $y=8$

y intercept: Let $x=0$

$\therefore y=8-4\times5^{-1}$

$\therefore y=8-\frac{4}{5}$

$\therefore y=\frac{36}{5}$

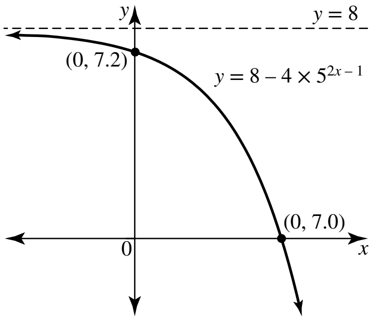
$(0,7.2)$

Range $(-\infty,8)$

x intercept: Let $y=0$

$$\begin{aligned}\therefore 0&=8-4\times5^{2x-1}\\ \therefore 4\times5^{2x-1}&=8\\ \therefore 5^{2x-1}&=2\\ \therefore \log\left(5^{2x-1}\right)&=\log(2)\\ \therefore (2x-1)\log(5)&=\log(2)\\ \therefore 2x\log(5)-\log(5)&=\log(2)\\ \therefore 2x\log(5)&=\log(2)+\log(5)\\ &=\log(10)\\ &=1\\ \therefore x&=\frac{1}{2\log(5)}\\ \therefore x&\simeq0.7\end{aligned}$$

$(0.7,0)$



f $y=-2\times10^{3x-1}-4$

Asymptote: $y=-4$

y intercept: Let $x=0$

$\therefore y=-2\times10^{-1}-4$

$\therefore y=-0.2-4$

$\quad=-4.2$

$(0,-4.2)$

Range $(-\infty,-4)$

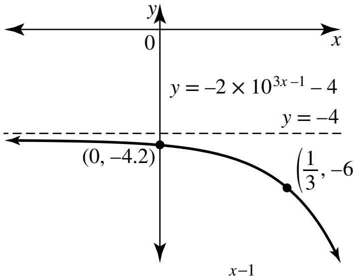
No x intercept

Point: Let $x=\frac{1}{3}$

$\therefore y=-2\times1-4$

$\therefore y=-6$

$\left(\frac{1}{3},-6\right)$



15 $f: R \to R, f(x) = 3 - 6 \times 2^{\frac{x-1}{2}}$

a **i** $f(1) = 3 - 6 \times 2^0$

$\therefore f(1) = -3$

ii $f(0) = 3 - 6 \times 2^{-\frac{1}{2}}$

$$= -3 - \frac{6}{\sqrt{2}}$$

$$= -3 - \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 3 - 3\sqrt{2}$$

b **i** $f(x) = -9$

$$\therefore 3 - 6 \times 2^{\frac{x-1}{2}} = -9$$

$$\therefore 12 = 6 \times 2^{\frac{x-1}{2}}$$

$$\therefore 2^{\frac{x-1}{2}} = 2$$

$$\therefore \frac{x-1}{2} = 1$$

$$\therefore x - 1 = 2$$

$$\therefore x = 3$$

ii $f(x) = 0$

$$\therefore 3 - 6 \times 2^{\frac{x-1}{2}} = 0$$

$$\therefore 3 = 6 \times 2^{\frac{x-1}{2}}$$

$$\therefore 2^{\frac{x-1}{2}} = \frac{1}{2}$$

$$\therefore \frac{x-1}{2} = -1$$

$$\therefore x - 1 = -2$$

$$\therefore x = -1$$

iii $f(x) = 9$

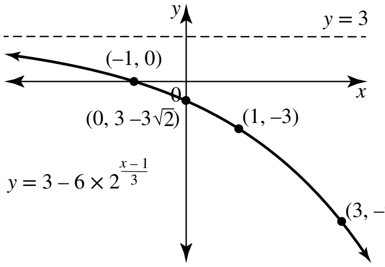
$$\therefore 3 - 6 \times 2^{\frac{x-1}{2}} = 9$$

$$\therefore -6 = 6 \times 2^{\frac{x-1}{2}}$$

$$\therefore 2^{\frac{x-1}{2}} = -1$$

As $2^{\frac{x-1}{2}} > 0$, there are no values of x for which $f(x) = 9$.

c From previous calculations, the points $(1,-3), \left(0, 3 - 3\sqrt{2}\right), (3,-9)$ and $(-1,0)$ lie on the function's graph. The equation of the asymptote is $y = 3$; the range is $(-\infty, 3)$.



d Let $f(x) = 1$

$$\therefore -1 = 3 - 6 \times 2^{\frac{x-1}{2}}$$

$$\therefore 6 \times 2^{\frac{x-1}{2}} = 4$$

$$\therefore 2^{\frac{x-1}{2}} = \frac{2}{3}$$

$$\therefore \log\left(2^{\frac{x-1}{2}}\right) = \log\left(\frac{2}{3}\right)$$

$$\therefore \frac{x-1}{2}\log(2) = \log\left(\frac{2}{3}\right)$$

$$\therefore x - 1 = \frac{2\log\left(\frac{2}{3}\right)}{\log(2)}$$

$$\therefore x = \frac{2\log\left(\frac{2}{3}\right)}{\log(2)} + 1$$

$$\therefore x \simeq -0.17$$

Using the graph, $f(x) \geq -1$ for $x \leq -0.17$.

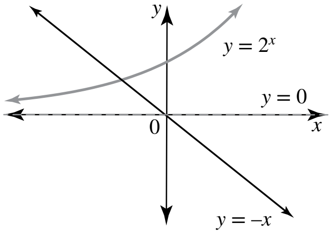
16 **a** Let $y_1 = 2^x$ and $y_2 = -x$.

When $x = 0$, $y_1 = 1$ and $y_2 = 0$; $y_1 > y_2$

When $x = -\frac{1}{2}$, $y_1 = \frac{1}{\sqrt{2}} \simeq 0.7$ and $y_2 = 0.5$; $y_1 > y_2$

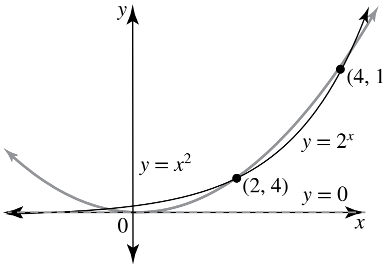
When $x = -1$, $y_1 = \frac{1}{2} = 0.5$ and $y_2 = 1$; $y_1 < y_2$

Therefore, $y_1 = y_2$ for some $x \in (-1, -0.5)$.



There is one point of intersection for which $x \in (-1, -0.5)$.

b $y = 2^x$ and $y = x^2$ both contain the points $(2,4)$ and $(4,16)$.



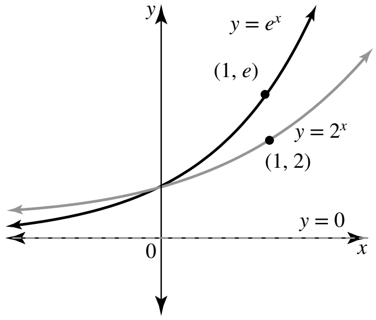
From the diagram there is a point of intersection for $x \in (-1, 0)$, so there are three points of intersection.

c $y = e^x$ and $y = 2^x$ both contain the point $(0,1)$.

As $e > 2$, the graph of $y = e^x$ should be steeper than that of $y = 2^x$, for $x > 0$.

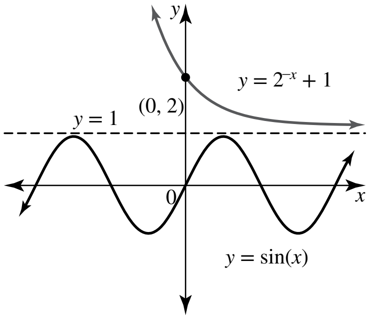
$y = e^x$ contains the point $(1, e)$, approximately $(1, 2.7)$ and $y = 2^x$ contains the point $(1, 2)$.

There will only one point of intersection, as confirmed by the diagram.



d $y = 2^{-x} + 1$ and $y = \sin(x)$
Consider $y = 2^{-x} + 1$:

Asymptote is $y = 1$ and y intercept is $(0, 2)$, so the range is $(1, \infty)$.
As the range of $y = \sin(x)$ is $[-1, 1]$, there will not be any intersections of the two graphs.
This is confirmed by the diagram.

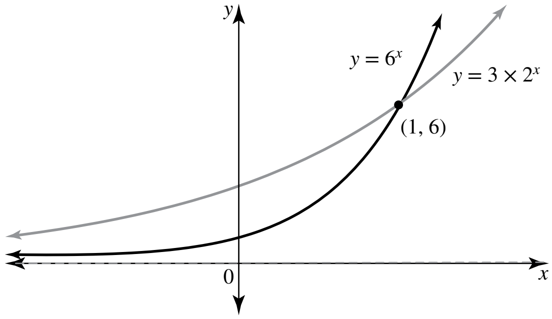


e $y = 3 \times 2^x$ and $y = 6^x$.
At intersection, $3 \times 2^x = 6^x$,

$$\begin{aligned}\therefore 3 &= \frac{6^x}{2^x}\\ \therefore 3 &= \left(\frac{6}{2}\right)^x\\ \therefore 3 &= 3^x\\ \therefore x &= 1\end{aligned}$$

There is one point of intersection $(1, 6)$.

Both $y = 3 \times 2^x$ and $y = 6^x$ have asymptote $y = 0$ and y intercepts of $(0, 3)$ and $(0, 1)$ respectively.



f $y = 2^{2x-1}$ and $y = \frac{1}{2} \times 16^{\frac{x}{2}}$

Consider $y = \frac{1}{2} \times 16^{\frac{x}{2}}$. This can be expressed as

$$\begin{aligned}y &= \frac{1}{2} \times \left(2^4\right)^{\frac{x}{2}}\\ &= 2^{-1} \times 2^{2x}\\ &= 2^{2x-1}\end{aligned}$$

The two curves are identical and therefore have an infinite number of intersections. The co-ordinates of the points of intersection are of the form $\left(t, 2^{2t-1}\right), t \in R$.

17 Sketch the graphs of $y = 2^x$ and $y = x^2$ and use Analysis → G-Solve → Intersect to obtain the co-ordinates of the points of intersection as $(-0.77, 0.59), (2, 4)$ and $(4, 16)$.

18 $y_1 = 33 - 2(11)^x$ and $y_2 = 33 - 2(11)^{x+1}$.
From the equations it can be seen that both have an asymptote $y = 33$ and that if $y_1 = f(x)$ then $y_2 = f(x + 1)$. Hence y_2 is a horizontal translation of y_1 one unit to the left.
Graph the functions and use Analysis → G-Solve → either Root or y-intercept or x-cal, to calculate the x intercept or y intercept or x value when $y = 10$, respectively.
The values obtained for y_1 are $(1.17, 0)$, $(0, 31)$ and $(1.0185, 10)$.
The values obtained for y_2 are $(0.17, 0)$, $(0, 11)$ and $(0.0185, 10)$.

Exercise 11.5 — Applications of exponential functions

1 $Q(t) = Q_0 \times 1.7^{-kt}$

a When $t = 0$, $Q(0) = Q_0 \times 1 = Q_0$, so Q_0 is the initial amount of the substance.

b $Q(300) = \frac{1}{2}Q_0$

$$\begin{aligned}\frac{1}{2}Q_0 &= Q_0 \times 1.7^{-300k}\\ 0.5 &= 1.7^{-300k}\\ \log(0.5) &= -300k \log(1.7)\\ k &= -\frac{\log(0.5)}{300 \log(1.7)}\\ k &\approx 0.004\end{aligned}$$

c $Q_0 = 250$, $Q = 250 \times 1.7^{-0.004t}$
When $t = 10$,

$$\begin{aligned}Q &= 250 \times 1.7^{-0.04}\\ &\approx 244.7\end{aligned}$$

The amount which has decayed is:
$250 - 244.7 = 5.3$
Therefore, 5.3 kg have decayed.

2 $D = 42 \times 2^{\frac{t}{16}}$

a When $t = 0$, $D = 42$. The initial average number of daily emails was 42 emails per day.

b When $D = 84$:

$$\begin{aligned}84 &= 42 \times 2^{\frac{t}{16}}\\ \frac{84}{42} &= 2^{\frac{t}{16}}\\ 2 &= 2^{\frac{t}{16}}\\ 1 &= \frac{t}{16}\\ t &= 16\end{aligned}$$

After 16 weeks the average number of daily emails is predicted to double.

3 Gradient $m = \frac{2}{0.8}$
The equation of the line is in the form $Y = mX + c$ where $m = 2.5$, $c = 2$ and $Y = \log(y)$, $X = \log(x)$.

Therefore,

$$\log(y) = 2.5\log(x) + 2$$
$$\log(y) - \log\left(x^{2.5}\right) = 2$$
$$\log\left(\frac{y}{x^{2.5}}\right) = 2$$
$$\frac{y}{x^{2.5}} = 10^2$$
$$y = 100x^{2.5}$$

4 Gradient of line through $(0,0)$ and $(1,0.3)$ is 0.3.

The equation of the line is of the form $Y = mX$ where $m = 0.3, Y = \log(y), X = x$.

$$\log(y) = 0.3x$$
$$y = 10^{0.3x}$$

5 $V = V_0 \times 2^{-kt}$

a When $t = 0$, $V = V_0$ so V_0 is the purchase price.

When $t = 5$, $V = \frac{1}{2}V_0$

$$\therefore \frac{1}{2}V_0 = V_0 \times 2^{-5k}$$
$$\therefore \frac{1}{2} = 2^{-5k}$$
$$\therefore 2^{-1} = 2^{-5k}$$
$$\therefore -1 = -5k$$
$$\therefore k = \frac{1}{5} \text{ or } 0.2$$

b The model is $V = V_0 \times 2^{-0.2t}$

When 75% of the purchase price is lost, 25% remains.

Let $V = 0.25V_0$

$$\therefore 0.25V_0 = V_0 \times 2^{-0.2t}$$
$$\therefore \frac{1}{4} = 2^{-0.2t}$$
$$\therefore 2^{-2} = 2^{-0.2t}$$
$$\therefore -2 = -0.2t$$
$$\therefore t = \frac{2}{0.2}$$
$$\therefore t = 10$$

It takes 10 years for the value of the car to lose 75% of its purchase price.

6 $N = 30 \times 2^{0.072t}$

a When $t = 0$, $N = 30$ so there were initially 30 drosophilae.

b When $t = 5$,

$$N = 30 \times 2^{0.072 \times 5}$$
$$= 30 \times 2^{0.36}$$
$$\simeq 38.503$$

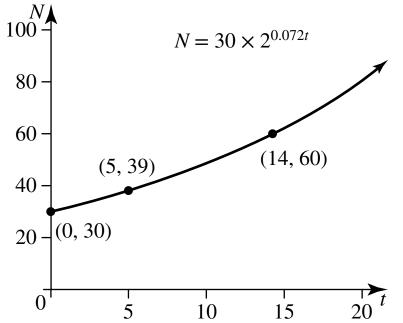
After 5 days there were approximately 39 drosophilae.

c When $N = 60$,

$$60 = 30 \times 2^{0.072t}$$
$$\therefore 2 = 2^{0.072t}$$
$$\therefore 1 = 0.072t$$
$$\therefore t = \frac{1}{0.072}$$
$$\therefore t \simeq 13.9$$

The population doubles after 14 days.

d



e Let $N = 100$

$$\therefore 100 = 30 \times 2^{0.072t}$$
$$\therefore \frac{10}{3} = 2^{0.072t}$$
$$\therefore \log\left(\frac{10}{3}\right) = \log\left(2^{0.072t}\right)$$
$$\therefore \log\left(\frac{10}{3}\right) = 0.072t\log(2)$$
$$\therefore t = \frac{\log\left(\frac{10}{3}\right)}{0.072\log(2)}$$
$$\therefore t \simeq 24.12$$

Using the graph, for N to exceed 100, then $t > 24.12$.

The population first exceeds 100 after 25 days.

7 $A = P\left(1 + \frac{r}{n}\right)^{nt}$

a **i** $P = 2000$, $r = 0.03$ and $n = 12$

$$\therefore A = 2000\left(1 + \frac{0.03}{12}\right)^{12t}$$
$$\therefore A = 2000(1 + 0.0025)^{12t}$$
$$\therefore A = 2000(1.0025)^{12t}$$

ii For 6 months, $t = \frac{1}{2}$

$$A = 2000(1.0025)^6$$
$$\simeq 2030.19$$

The investment is worth \$2030.19 after 6 months.

iii Let $A = 2500$

$$\therefore 2500 = 2000(1.0025)^{12t}$$
$$\therefore 1.0025^{12t} = \frac{2500}{2000}$$
$$\therefore 1.0025^{12t} = 1.25$$
$$\therefore \log\left(1.0025^{12t}\right) = \log(1.25)$$
$$\therefore 12t\log(1.0025) = \log(1.25)$$
$$\therefore t = \frac{\log(1.25)}{12\log(1.0025)}$$
$$\therefore t \simeq 7.45$$

It takes 7.45 years for the investment to reach \$2500.

b Let $A = 2500$ and $t = 4$

$$\therefore 2500 = 2000\left(1 + \frac{r}{12}\right)^{48}$$
$$\therefore \left(1 + \frac{r}{12}\right)^{48} = \frac{2500}{2000}$$
$$\therefore \left(1 + \frac{r}{12}\right)^{48} = 1.25$$

$$\therefore \left(1+\frac{r}{12}\right)=\sqrt[48]{1.25}$$

$$\therefore \frac{r}{12}=1.25^{\frac{1}{48}}-1$$

$$\therefore r=12\left(1.25^{\frac{1}{48}}-1\right)$$

$$\therefore r\simeq 0.056$$

The interest rate would need to be 5.6% p.a. to achieve the goal.

8 $T=85\times 3^{-0.008t}$

a When $t=0$, $T=85$ so the initial temperature is 85 degrees.

Let $t=10$

$\therefore T=85\times 3^{-0.08}$

$\therefore T\simeq 77.85$

After 10 minutes the temperature is approximately 78 degrees, so the coffee has cooled by 7 degrees.

b Let $T=65$

$$\therefore 65=85\times 3^{-0.008t}$$

$$\therefore 3^{-0.008t}=\frac{65}{85}$$

$$=\frac{13}{17}$$

$$\therefore \log\left(3^{-0.008t}\right)=\log\left(\frac{13}{17}\right)$$

$$\therefore -0.008t\log(3)=\log\left(\frac{13}{17}\right)$$

$$\therefore t=\frac{\log\left(\frac{13}{17}\right)}{-0.008\log(3)}$$

$$\therefore t\simeq 30.5$$

It takes just over half an hour to cool to 65 degrees.

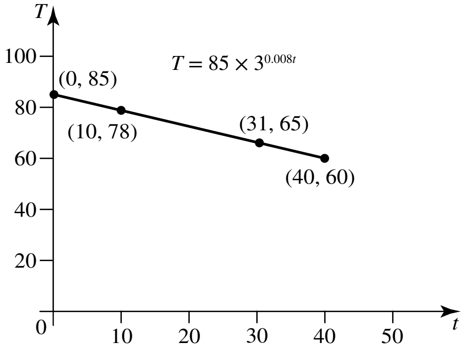
c The graph of $T=85\times 3^{-0.008t}$ passes through the three points $(0,85),(10,78),(31,65)$.

When $t=40$,

$T=85\times 3^{-0.008\times 40}$

$=85\times 3^{-0.32}$

$\simeq 59.8$



d The asymptote for the graph of $T=85\times 3^{-0.008t}$ is $T=0$. This model therefore predicts that the temperature will approach zero degrees. This makes the model unrealistic, particularly in Brisbane!

9 $T=a\times 3^{-0.13t}+25$

a When $t=0, T=95$

$\therefore 95=a\times 1+25$

$\therefore a=70$

b The model is $T=70\times 3^{-0.13t}+25$

Let $t=2$

$\therefore T=70\times 3^{-0.13\times 2}+25$

$=70\times 3^{-0.26}+25$

$\therefore T\simeq 77.6$

After 2 minutes, the pie has cooled to 77.6 degrees.

c Let $T=65$

$$\therefore 65=70\times 3^{-0.13t}+25$$

$$\therefore 40=70\times 3^{-0.13t}$$

$$\therefore 3^{-0.13t}=\frac{4}{7}$$

$$\therefore \log\left(3^{-0.13t}\right)=\log\left(\frac{4}{7}\right)$$

$$\therefore -0.13t\log(3)=\log\left(\frac{4}{7}\right)$$

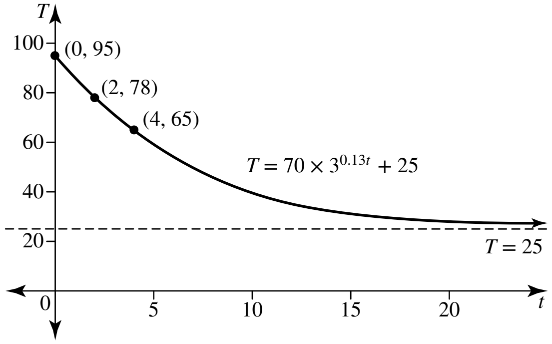
$$\therefore t=\frac{\log\left(\frac{4}{7}\right)}{-0.13\log(3)}$$

$$\therefore t\simeq 3.9$$

It takes 4 minutes for the pie to cool to 65 degrees.

d The graph of $T=70\times 3^{-0.13t}+25$ contains the points $(0,95),(2,77.6),(3.9,65)$.

Its asymptote is $T=25$. This means that the model predicts the temperature of the pie will approach 25 degrees. In the long term, the temperature of the pie will not fall below 25 degrees.



10 $P=P_0\times 10^{-kh}$

a When $h=0$, $P=P_0$, so P_0 is the barometric pressure at sea level.

For Mt Everest, $P=\frac{1}{3}P_0$ and $h=8.848$ (measured in km).

$$\therefore \frac{1}{3}P_0=P_0\times 10^{-8.848k}$$

$$\therefore \frac{1}{3}=10^{-8.848k}$$

$$\therefore \log\left(\frac{1}{3}\right)=-8.848k$$

$$\therefore k=\log\left(\frac{1}{3}\right)\div(-8.848)$$

$$\therefore k\simeq 0.054$$

b Model is $P=P_0\times 10^{-0.054h}$

Mt Kilimanjaro: $h=5.895,\ P=48.68$

$\therefore 48.68=P_0\times 10^{-0.054\times 5.895}$

$\therefore P_0=48.68\times 10^{0.054\times 5.895}$

$\therefore P_0\simeq 101.317$

c Use the model now as $P=101.317\times 10^{-0.054h}$.

For Mont Blanc, $h=4.810$

$\therefore P=101.317\times 10^{-0.054\times 4.810}$

$\therefore P\simeq 55.71$

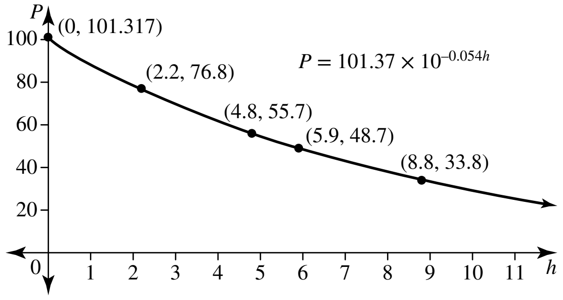
For Mt Kosciuszko, $h = 2.228$

$\therefore P = 101.317 \times 10^{-0.054\times 2.228}$

$\therefore P \simeq 76.80$

The atmospheric pressure is 55.71 kilopascals at the summit of Mont Blanc and 76.80 kilopascals at the summit of Mt Kosciuszko.

d



11 a $D = D_0 \times 10^{kt}$

In 1991: $t = 15, D = 15 \Rightarrow 15 = D_0 \times 10^{15k}$.....(1)

In 1994: $t = 18, D = 75 \Rightarrow 75 = D_0 \times 10^{18k}$.....(2)

b Divide equation (2) by equation (1)

$$\therefore \frac{75}{15} = \frac{D_0 \times 10^{18k}}{D_0 \times 10^{15k}}$$

$$\therefore 5 = 10^{3k}$$

$$\therefore 3k = \log(5)$$

$$\therefore k = \frac{1}{3}\log(5)$$

Substitute $k = \frac{1}{3}\log(5)$ in equation (1)

$$\therefore 15 = D_0 \times 10^{15\times\frac{1}{3}\log(5)}$$

$$\therefore 15 = D_0 \times 10^{5\log(5)}$$

$$\therefore 15 = D_0 \times 10^{\log(5^5)}$$

$$\therefore 15 = D_0 \times 5^5$$

$$\therefore D_0 = \frac{15}{5^5}$$

$$\therefore D_0 = 3 \times 5 \times 5^{-5}$$

$$\therefore D_0 = 3 \times 5^{-4}$$

Correct to three decimal places, $k = \frac{1}{3}\log(5) = 0.233$ and $D_0 = 3 \times 5^{-4} = 0.005$.

c The model is $D = 0.005 \times 10^{0.233t}$

1996: Let $t = 20$

$\therefore D = 0.005 \times 10^{0.233\times 20}$

$\therefore D \simeq 228.54$

In 1996, the density was 229 birds per square kilometre.

d New model is given by $D = 230 \times 10^{-\frac{t}{3}} + b$. This model has an asymptote when $D = b$.

When $t = 4, D = 40$

$$\therefore 40 = 30 \times 10^{-\frac{4}{3}} + b$$

$$\therefore b = 40 - 30 \times 10^{-\frac{4}{3}}$$

$$\therefore b \simeq 38.6$$

The asymptote is approximately $D = 39$ so the density cannot be expected to fall below 39 birds per square kilometre.

12 $C = C_0 \times \left(\frac{1}{2}\right)^{kt}$

a Half life of 5730 years means that $C = \frac{1}{2}C_0$ when $t = 5730$.

$$\therefore \frac{1}{2}C_0 = C_0 \times \left(\frac{1}{2}\right)^{5730k}$$

$$\therefore \frac{1}{2} = \left(\frac{1}{2}\right)^{5730k}$$

$$\therefore 1 = 5730k$$

$$\therefore k = \frac{1}{5730}$$

b The model is $C = C_0 \times \left(\frac{1}{2}\right)^{\frac{t}{5730}}$

Let $C = 0.83C_0$

$$\therefore 0.83C_0 = C_0 \times \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\therefore 0.83 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\therefore \log(0.83) = \log\left(\left(\frac{1}{2}\right)^{\frac{t}{5730}}\right)$$

$$\therefore \frac{t}{5730}\log\left(\frac{1}{2}\right) = \log(0.83)$$

$$\therefore t = \frac{5730\log(0.83)}{\log(0.5)}$$

$$\therefore t \simeq 1540$$

The bones are estimated to be 1540 years old.

13 a i Let $Y = \log_{10}(y)$ and $X = \log_{10}(x)$

The equation of the line is $Y = mX + c$

The gradient is $m = \frac{rise}{run} = \frac{1}{2}$ and the Y value of the Y intercept is $c = -1$.

$$\therefore Y = \frac{1}{2}X - 1$$

Hence, $\log(y) = \frac{1}{2}\log(x) - 1$

$$\therefore \log(y) - \frac{1}{2}\log(x) = -1$$

$$\therefore \log(y) - \log\left(x^{\frac{1}{2}}\right) = -1$$

$$\therefore \log\left(\frac{y}{\sqrt{x}}\right) = -1$$

$$\therefore \frac{y}{\sqrt{x}} = 10^{-1}$$

$$\therefore y = \frac{\sqrt{x}}{10} \text{ or } y = 0.1\sqrt{x}$$

ii Let $Y = \log_2(y)$

The equation of the line is $Y = mx + c$

The gradient is $m = \frac{rise}{run} = \frac{-1}{4}$ and the Y value of the Y intercept is $c = 0$.

$$\therefore Y = -\frac{1}{4}x$$

Hence, $\log_2(y) = -\frac{x}{4}$

$$\therefore y = 2^{-\frac{x}{4}}$$

b **i** $pH = -\log\left(\left[H^+\right]\right)$

Bleach: $\left[H^+\right] = 10^{-13}$

$\therefore pH = -\log\left(10^{-13}\right)$

$\therefore pH = 13\log(10)$

$\therefore pH = 13$

Water: $\left[H^+\right] = 10^{-7}$

$\therefore pH = -\log\left(10^{-7}\right)$

$\therefore pH = 7$

ii Lemon juice: $pH = 2$

$\therefore 2 = -\log\left(\left[H^+\right]\right)$

$\therefore -2 = \log\left(\left[H^+\right]\right)$

$\therefore \left[H^+\right] = 10^{-2}$

$\therefore \left[H^+\right] = 0.01$

The concentration of hydrogen ions in lemon juice is 1×10^{-2}.

Milk: $pH = 6$

$\therefore 6 = -\log\left(\left[H^+\right]\right)$

$\therefore -6 = \log\left(\left[H^+\right]\right)$

$\therefore \left[H^+\right] = 10^{-6}$

$\therefore \left[H^+\right] = 0.000\,001$

The concentration of hydrogen ions in milk is 1×10^{-6}.

iii The acidity is due to the hydrogen ions. Since $10^{-2} = 10^4 \times 10^{-6}$, the acidity of lemon juice is four times that of milk.

iv There is an increase of 4 in the *pH* of milk from that of lemon juice and the difference in their concentration of hydrogen ions decreases by a factor of 10^{-4}. For every one unit of increase of *pH*, the concentration of hydrogen ions decreases by a factor of 10.

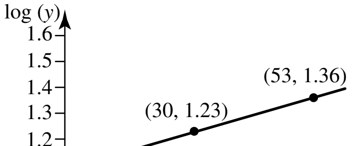
Hence for every unit of increase in *pH*, the solution becomes less acidic by a factor of 10.

14 **a** $\log(13.9) = 1.14, \log(17.1) = 1.23$ and $\log(22.9) = 1.36$.

The third row of the table contains these values.

b

x	15	30	53
$Y = \log(y)$	1.14	1.23	1.36



c The equation is of the form $Y = mx + c$

Using the points (15,1.14) and (30,1.23),

$$m = \frac{1.23 - 1.14}{30 - 15}$$

$$= \frac{0.09}{15}$$

$$= 0.006$$

$\therefore Y = 0.006x + c$

Substitute the point (30,1.23)

$\therefore 1.23 = 0.006 \times 30 + c$

$\therefore c = 1.23 - 0.18$

$\therefore c = 1.05$

$\therefore Y = 0.006x + 1.05$

d Replace Y by $\log(y)$

$\therefore \log(y) = 0.006x + 1.05$

$\therefore y = 10^{0.006x+1.05}$

$\therefore y = 10^{0.006x} \times 10^{1.05}$

$\therefore y = 10^{0.006x} \times 11.22$

$\therefore y = 11.22 \times 10^{0.006x}$

e For 1960, $x = 0$ and therefore $y = 11.22$.

Let $y = 2 \times 11.22$

$\therefore 2 \times 11.22 = 11.22 \times 10^{0.006x}$

$\therefore 2 = 10^{0.006x}$

$\therefore 0.006x = \log(2)$

$\therefore x = \dfrac{\log(2)}{0.006}$

$\therefore x \simeq 50.2$

The population doubles in just over 50 years.

f For the year 2030, $x = 70$

$\therefore y = 11.22 \times 10^{0.006\times70}$

$\therefore y = 11.22 \times 10^{0.42}$

$\therefore y \simeq 29.51$

The model predicts the population to be around 29.5 million by the year 2030 so this model supports the claim that the population will exceed 28 million.

15 **a** **i** Select Statistics from the Menu list. In List 1 enter the x values and enter the y values in List 2. Tap Calc.

For part **i**, select abExponential Reg and tap OK.

This gives the rule for the data as:

$y = ab^x$

$a = 4.003\,324\,7$

$b = 1.379\,721\,3$

with $r^2 = 0.999\,964$.

The rule is approximately $y = 4 \times 1.38^x$.

Tap the Histogram window to view the data points and the curve with the rule $y = 4 \times 1.38^x$.

ii From Calc, select Logarithmic Reg.

This gives the rule for the data as:

$y = a + b\ln(x)$

$a = 4.313\,698\,3$

$b = 6.216\,020\,8$

with $r^2 = 0.895\,887\,7$

The rule is approximately $y = 4.3 + 6.2\log_e(x)$.

The graph should automatically appear.

b From the graph of the logarithmic function it can be seen that fewer data points lie on the graph than is the case for the graph of the exponential function. As well, the closer the r^2 value is to 1, the better the fit so again the exponential model is the better.

16 $P(t) = (200t + 16) \times 2.7^{-t}$

a **i** When $t = 0$,

$P(0) = (16) \times 1$

$= 16$

Stephan's pain level is 16.

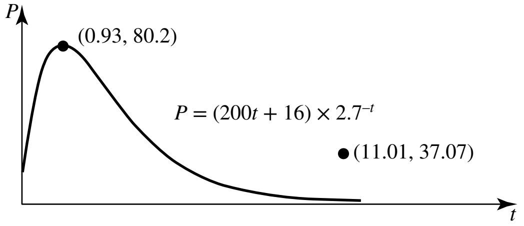
ii 15 seconds equals $\frac{15}{60}=\frac{1}{4}$ minutes.

When $t=\frac{1}{4}$,

$$P\left(\frac{1}{4}\right)=\left(200\times\frac{1}{4}+16\right)\times 2.7^{-\frac{1}{4}}$$
$$\simeq 51.5$$

After 15 seconds, Stephan's pain level is 51.5.

b The graph obtained should be similar to that shown.



i Use the Analysis tools to obtain the maximum turning point as $(0.926794, 80.202012)$.

Hence, the maximum pain measure is approximately 80.20.

ii It takes approximately 0.93 minutes or 55.61 seconds for the injection to start to reduce the pain.

iii Using y-Cal, when $x=5, y=7.0806786$ and when $x=10, y=0.097915$.

Hence, the pain level is 7.08 after 5 minutes and 0.10 after 10 minutes.

c The least pain level occurred at the end of the 10 minute interval, so the effectiveness of the injection was greatest after 10 minutes.

d **i** $P(t)=(100(t-10)+a)\times 2.7^{-(t-10)}$

$P(10)$ is known to equal 0.10.

$\therefore 0.10=(100(10-10)+a)\times 2.7^{-(0)}$

$\therefore 0.10=a\times 1$

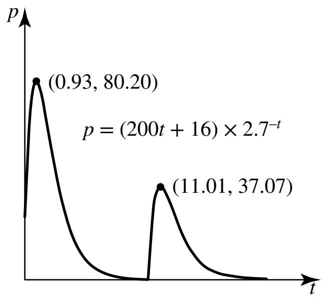
$\therefore a=0.10$

ii Enter the hybrid function as

$y1=(200x+16)\times 2.7^{-x} \mid 0\le x\le 10$

$y2=(100(x-10)+0.10)\times 2.7^{-(x-10)} \mid 10\le x\le 20$

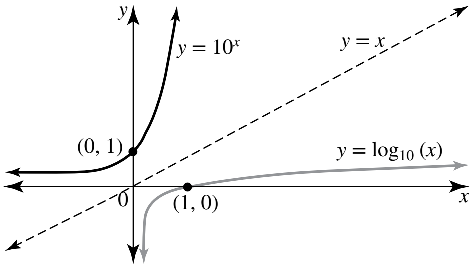
The graph obtained should be a similar shape to that shown. Use the Analysis tools to obtain the co-ordinates of the second maximum as approximately $(11.01, 37.07)$.



Exercise 11.6 — Inverses of exponential functions

1 **a** $y=\log_{10}(x)$

The inverse is $x=\log_{10}(y)$, which is the exponential function $y=10^x$.



b The points $(10,1),(100,2),(1000,3)$ lie on the logarithm graph.

With $m=1000, n=100$, the logarithm law is

$$\log_{10}(1000)-\log_{10}(100)=\log_{10}\left(\frac{1000}{100}\right)$$

$$\therefore \log_{10}(1000)-\log_{10}(100)=\log_{10}(10)$$

This means the difference between the y co-ordinates of the points $(1000,3),(100,2)$ should equal the y co-ordinate of the point $(10,1)$.

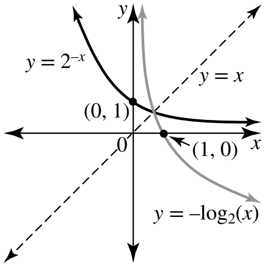
This does hold since $3-2=1$.

2 $y=2^{-x}$

Its inverse is $x=2^{-y}$ which rearranges to

$-y=\log_2(x)$

$\therefore y=-\log_2(x)$



3 $f:R\to R, f(x)=4-2^{3x}$

a Domain of inverse is range of f.

f: asymptote at $y=4$, y intercept $(0,3)$. Therefore range is $(-\infty,4)$.

Domain of inverse is $(-\infty,4)$.

b Let $y=f(x)$

f: $y=4-2^{3x}$

inverse: $x=4-2^{3y}$

$\therefore 2^{3y}=4-x$

$\therefore 3y=\log_2(4-x)$

$\therefore y=\frac{1}{3}\log_2(4-x)$

$\therefore f^{-1}(x)=\frac{1}{3}\log_2(4-x)$

As a mapping, $f^{-1}:(-\infty,4)\to R, f^{-1}(x)=\frac{1}{3}\log_2(4-x)$

c x intercept: $4-2^{3x}=0$

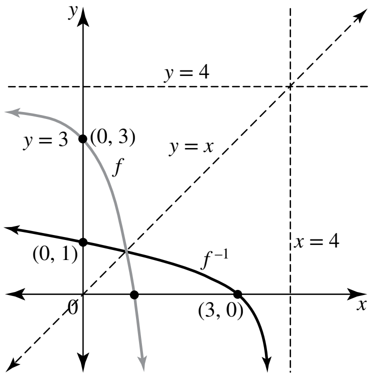
$\therefore 4=2^{3x}$

$\therefore 2^2=2^{3x}$

$\therefore 3x=2$

$\therefore x=\frac{2}{3}$

inverse: asymptote at $x=4$, x intercept $(3,0)$, y intercept $\left(0,\frac{2}{3}\right)$, domain $(-\infty,4)$.



4 $y=4+2\log_2(x)$

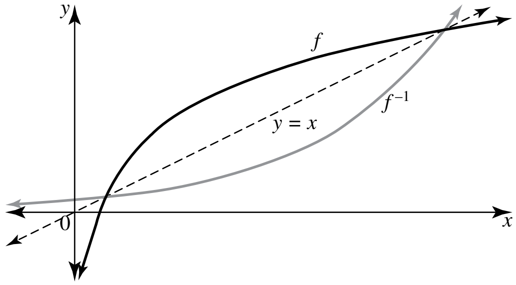
a inverse: $x=4+2\log_2(y)$

$$\therefore \log_2(y)=\frac{x-4}{2}$$

$$\therefore y=2^{\left(\frac{x-4}{2}\right)}$$

b Inverse: asymptote $y=0$, y intercept $\left(0,\frac{1}{4}\right)$, point $(4,1)$.

This means for the logarithm graph, asymptote $x=0$, x intercept $\left(\frac{1}{4},0\right)$, point $(1,4)$.



c The graphs intersect twice.

5 **a** $\log_6\left(2^{2x}\times 9^x\right)=\log_6\left(2^{2x}\times 3^{2x}\right)$

$$=\log_6\left((2\times 3)^{2x}\right)$$

$$=\log_6\left(6^{2x}\right)$$

$$=2x$$

b $2^{-3\log_2(10)}=2^{\log_2(10)^{-3}}$

$$=(10)^{-3}$$

$$=\frac{1}{1000}$$

$$=0.001$$

6 $5^{x\log_5(2)-\log_5(3)}$

$5^{x\log_5(2)-\log_5(3)}=5^{\log_5(2^x)-\log_5(3)}$

$$=5^{\log_5\left(\frac{2^x}{3}\right)}$$

$$=\frac{2^x}{3}$$

$$=\frac{1}{3}\times 2^x$$

7 **a** $y=\log_{10}(x-1)$

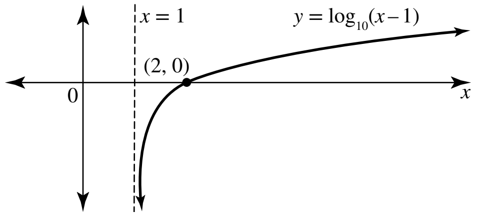
Horizontal translation 1 unit to the right gives the asymptote of $x=1$ and domain $(1,\infty)$. There will not be a y intercept.

x intercept: When $y=0$,

$\log_{10}(x-1)=0$

$\therefore x-1=10^0$

$\therefore x=2$



b $y=\log_5(x)-1$

Vertical translation of 1 unit downwards. This does not affect the asymptote or the domain.

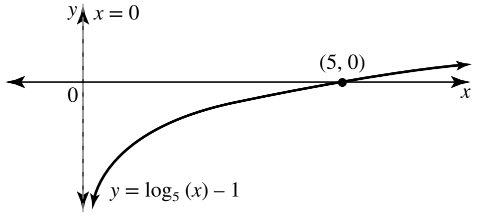
asymptote: $x=0$, no y intercept, domain: R^+

x intercept: $\log_5(x)-1=0$

$\therefore \log_5(x)=1$

$\therefore x=5^1$

$\therefore x=5$



c **i** Asymptote occurs when $x+b=0\Rightarrow x=-b$. The graph shows the asymptote is $x=-1$

and so $-b=-1\Rightarrow b=1$

Alternatively, as the asymptote is $x=-1$ there is a horizontal translation of 1 unit to the left so $b=1$.

ii The domain of the inverse is the range of the given graph, so domain is R.

The range of the inverse is the domain of the given graph, so range is $(-1,\infty)$.

The rule for the inverse:

function: $y=-\log_2(x+1)$

inverse: $x=-\log_2(y+1)$

$\therefore -x=\log_2(y+1)$

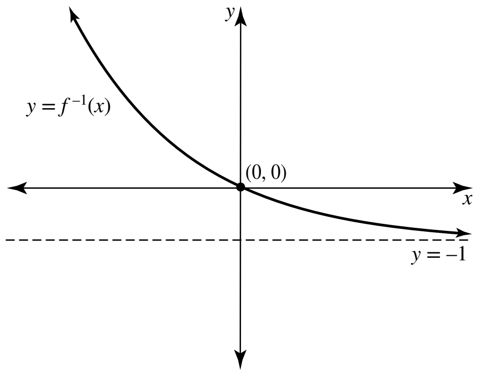
$\therefore 2^{-x}=y+1$

$\therefore y=2^{-x}-1$

The rule for the inverse function is given by $f^{-1}(x)=2^{-x}-1$.

iii Deduce the features of the inverse from the given graph:

asymptote $x=-1\Rightarrow y=-1$, point $(0,0)\Rightarrow(0,0)$, point $(1,-1)\Rightarrow(-1,1)$



8 $f: R^+ \to R, f(x) = 1 - \log_4(x)$

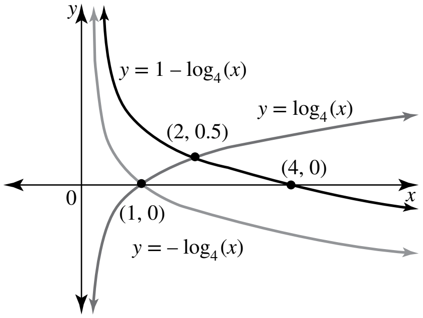
$f(x) = -\log_4(x) + 1$ is reflected in the x axis and then has a vertical translation of 1 unit up. This does not affect the vertical asymptote at $x = 0$.

x intercept: When $y = 0,\ 1 - \log_4(x) = 0$

$\therefore 1 = \log_4(x)$

$\therefore x = 4^1$

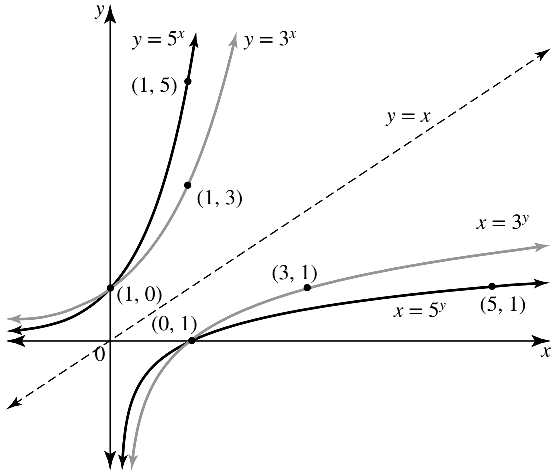
$x = 4$



9 a $y = 3^x$ has an asymptote $y = 0$ and passes through the points $(0,1)$ and $(1,3)$. Its inverse will have an asymptote $x = 0$ and pass through the points $(1,0)$ and $(3,1)$.

$y = 5^x$ has an asymptote $y = 0$ and passes through the points $(0,1)$ and $(1,5)$. Its inverse will have an asymptote $x = 0$ and pass through the points $(1,0)$ and $(5,1)$.

The inverse is a reflection of the exponential graph in the line $y = x$.

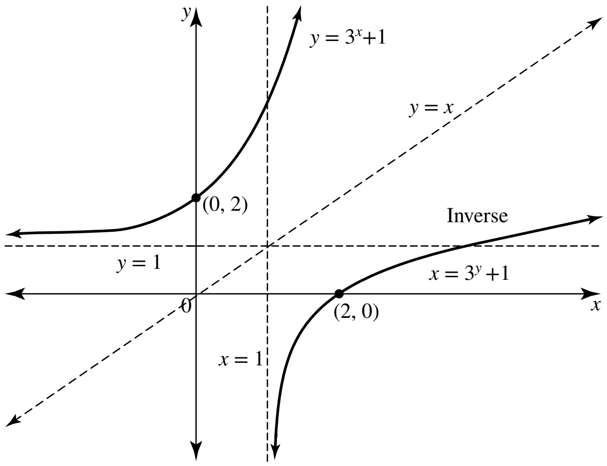


b The inverse of $y = 3^x$ is $x = 3^y$. Therefore the inverse is $y = \log_3(x)$.

The inverse of $y = 5^x$ is $x = 5^y$. Therefore the inverse is $y = \log_5(x)$.

c The larger the base of the logarithmic function, the more slowly its graph increases for $x > 1$.

d $y = 3^x + 1$ is a vertical translation up one unit of $y = 3^x$. The asymptote is $y = 1$ and it passes through $(0,2)$. Its inverse has asymptote $x = 1$ and passes through $(2,0)$.

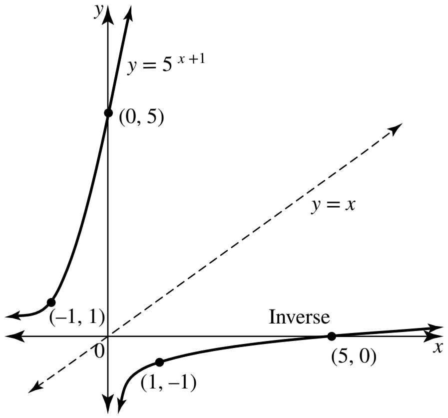


e The inverse of $y = 3^x + 1$ has the equation $x = 3^y + 1$.

$\therefore x - 1 = 3^y$

$\therefore y = \log_3(x - 1)$

f $y = 5^{x+1}$ is a horizontal translation one unit to the left of $y = 5^x$. Its asymptote is $y = 0$ and it passes through $(-1,1)$ and $(0,5)$. Its inverse has asymptote is $x = 0$ and it passes through $(1,-1)$ and $(5,0)$.



Rule for the inverse:

$x = 5^{y+1}$

$\therefore \log_5(x) = y + 1$

$\therefore y = \log_5(x) - 1$

10 a $y = 2^{-\frac{x}{3}}$

Asymptote: $y = 0$

y intercept: Let $x = 0$,

$\therefore y = 1\ (0,1)$.

Point: Let $x = -3$

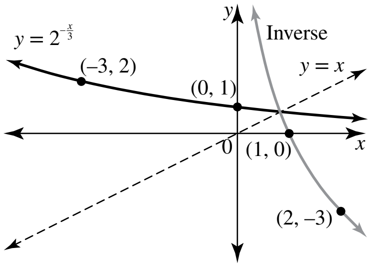
$\therefore y = 2\ (-3,2)$

Inverse: Asymptote $x = 0$, x intercept $(1,0)$, point $(2,-3)$.

Rule for the inverse: $x = 2^{-\frac{y}{3}}$

$\therefore \log_2(x) = -\frac{y}{3}$

$\therefore y = -3\log_2(x)$



b $y=\frac{1}{2}\times 8^{x}-1$

Asymptote: $y=-1$

y intercept: Let $x=0$,

$\therefore y=\frac{1}{2}-1$

$\therefore y=-\frac{1}{2}$.

$\left(0,-\frac{1}{2}\right)$

x intercept: Let $y=0$

$\therefore 0=\frac{1}{2}\times 8^{x}-1$

$\therefore 8^{x}=2$

$\therefore 2^{3x}=2^{1}$

$\therefore 3x=1$

$\therefore x=\frac{1}{3}$

$\left(\frac{1}{3},0\right)$

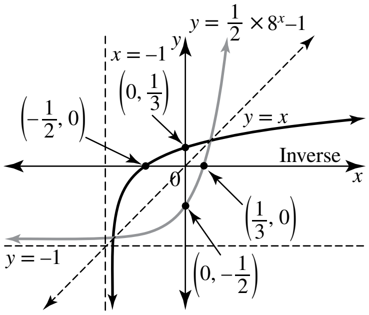
Inverse: Asymptote $x=-1$, x intercept $\left(-\frac{1}{2},0\right)$, y intercept $\left(0,\frac{1}{3}\right)$.

Rule for the inverse: $x=\frac{1}{2}\times 8^{y}-1$

$\therefore \frac{1}{2}\times 8^{y}=x+1$

$\therefore 8^{y}=2x+2$

$\therefore y=\log_{8}(2x+2)$



c $y=2-4^{x}$

Asymptote: $y=2$

y intercept: Let $x=0$,

$\therefore y=2-1$

$\therefore y=1$.

$(0,1)$

x intercept: Let $y=0$

$\therefore 0=2-4^{x}$

$\therefore 4^{x}=2$

$\therefore 2^{2x}=2^{1}$

$\therefore 2x=1$

$\therefore x=\frac{1}{2}$

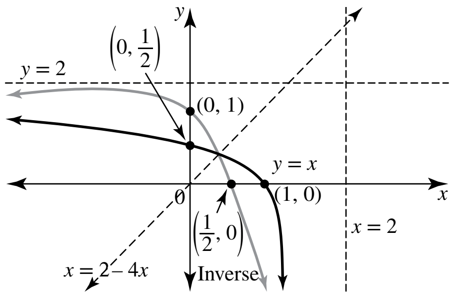
$\left(\frac{1}{2},0\right)$

Inverse: Asymptote $x=2$, x intercept $(1,0)$, y intercept $\left(0,\frac{1}{2}\right)$.

Rule for the inverse: $x=2-4^{y}$

$\therefore 4^{y}=2-x$

$\therefore y=\log_{4}(2-x)$



d $y=3^{x+1}+3$

Asymptote: $y=3$

y intercept: Let $x=0$,

$\therefore y=3+3$

$\therefore y=6$.

$(0,6)$

no x intercept

Point: Let $x=-1$

$\therefore y=1+3$

$\therefore y=4$

$(-1,4)$

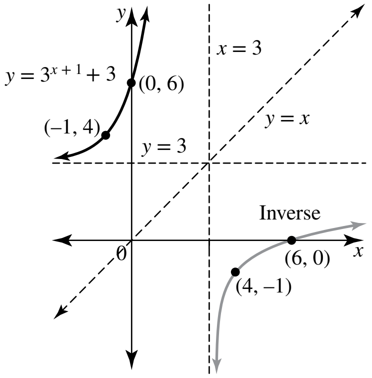
Inverse: Asymptote $x=3$, x intercept $(6,0)$, no y intercept, point $(4,-1)$.

Rule for the inverse: $x=3^{y+1}+3$

$\therefore 3^{y+1}=x-3$

$\therefore y+1=\log_{3}(x-3)$

$\therefore y=\log_{3}(x-3)-1$



e $y=-10^{-2x}$

Asymptote: $y=0$

no x intercept

y intercept: Let $x=0$,

$\therefore y=-1$

$(0,-1)$

Point: Let $x=-\frac{1}{2}$

$\therefore y=-10$

$\left(-\frac{1}{2},-10\right)$

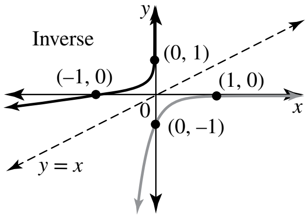
Inverse: Asymptote $x=0$, x intercept $(-1,0)$, no y intercept, point $\left(-10,-\frac{1}{2}\right)$.

Rule for the inverse: $x=-10^{-2y}$

$\therefore 10^{-2y}=-x$

$\therefore -2y=\log_{10}(-x)$

$\therefore y=-\frac{1}{2}\log_{10}(-x)$



f $y = 2^{1-x}$

Asymptote: $y = 0$

no x intercept

y intercept: Let $x = 0$,

$\therefore y = 2$

$(0,2)$

Point: Let $x = 1$

$\therefore y = 1$

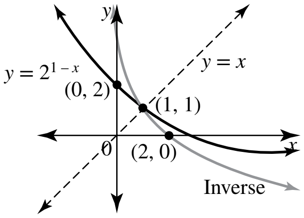
$(1,1)$

Inverse: Asymptote $x = 0$, x intercept $(2,0)$, no y intercept, point $(1,1)$.

Rule for the inverse: $x = 2^{1-y}$

$\therefore 1 - y = \log_2(x)$

$\therefore y = 1 - \log_2(x)$



11 $f: R \to R, f(x) = 8 - 2 \times 3^{2x}$

a $r_{f^{-1}} = d_f = R$.

The function has an asymptote at $y = 8$ and a y intercept $(0,6)$.

Therefore, $r_f = d_{f^{-1}} = (-\infty, 8)$.

b $f(0) = 6$

c As the point $(0,6)$ is on f, the point $(6,0)$ is on f^{-1}. This is the x intercept of the graph of f^{-1}.

d Function f: $y = 8 - 2 \times 3^{2x}$

Inverse function f^{-1}: $x = 8 - 2 \times 3^{2y}$

$\therefore 2 \times 3^{2y} = 8 - x$

$\therefore 3^{2y} = \dfrac{8-x}{2}$

$\therefore 2y = \log_3\left(\dfrac{8-x}{2}\right)$

$\therefore y = \dfrac{1}{2}\log_3\left(\dfrac{8-x}{2}\right)$

The rule for f^{-1} is $f^{-1}(x) = \dfrac{1}{2}\log_3\left(\dfrac{8-x}{2}\right)$.

As a mapping, the inverse function is

$f^{-1}: (-\infty, 8) \to R, f^{-1}(x) = \dfrac{1}{2}\log_3\left(\dfrac{8-x}{2}\right)$.

12 a $y = 2 \times (1.5)^{2-x}$

Let $y = 2$

$\therefore 2 = 2 \times (1.5)^{2-x}$

$\therefore (1.5)^{2-x} = 1$

$\therefore 2 - x = 0$

$\therefore x = 2$

b Let $x = 0$

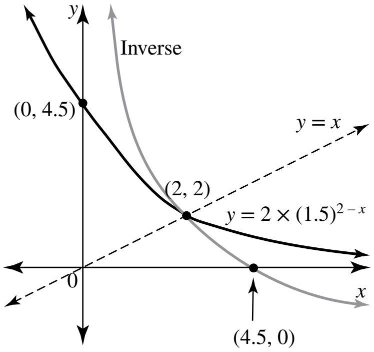
$\therefore y = 2 \times (1.5)^2$

$\therefore y = 2 \times 2.25$

$\therefore y = 4.5$

c & d For the graph of $y = 2 \times (1.5)^{2-x}$, asymptote is $y = 0$ and the graph passes through the points $(2,2)$ and $(0,4.5)$.

For its inverse, asymptote is $x = 0$ and the graph passes through the points $(2,2)$ and $(4.5,0)$.



e Rule for the inverse: $x = 2 \times (1.5)^{2-y}$

$\therefore \dfrac{x}{2} = (1.5)^{2-y}$

$\therefore 2 - y = \log_{1.5}\left(\dfrac{x}{2}\right)$

$\therefore y = 2 - \log_{1.5}\left(\dfrac{x}{2}\right)$

f The solution to the equation $2 \times (1.5)^{2-x} = 2 - \log_{1.5}\left(\dfrac{x}{2}\right)$ is the x co-ordinate of the point where the graphs of $y = 2 \times (1.5)^{2-x}$ and $y = 2 - \log_{1.5}\left(\dfrac{x}{2}\right)$ intersect.

Therefore, the solution is $x = 2$.

13 $g: (-1, \infty) \to R, g(x) = -3\log_5(x+1)$

a $r_{g^{-1}} = d_g = (-1, \infty)$

b $g(0) = -3\log_5(1)$

$= -3 \times 0$

$= 0$

c The point $(0,0)$ is on the graph of g and therefore on the graph of g^{-1}. This is the point where the graph of g^{-1} cuts the x axis.

d Function g: $y = -3\log_5(x+1)$

inverse function g^{-1}: $x = -3\log_5(y+1)$

$\therefore \log_5(y+1) = -\dfrac{x}{3}$

$\therefore y + 1 = 5^{-\frac{x}{3}}$

$\therefore y = 5^{-\frac{x}{3}} - 1$

The rule for g^{-1} is $g^{-1}(x) = 5^{-\frac{x}{3}} - 1$.

The maximal domain of an exponential function is R so as a mapping, the inverse function is

$g^{-1}: R \to R, g^{-1}(x) = 5^{-\frac{x}{3}} - 1$

e Consider $y = 5^{-\frac{x}{3}} - 1$

Asymptote: $y = -1$

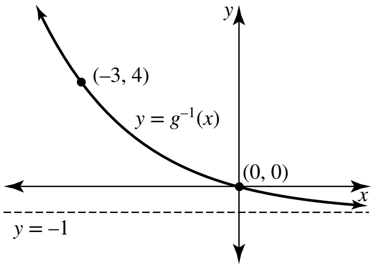
Axes intercept: $(0,0)$

Point: Let $x = -3$

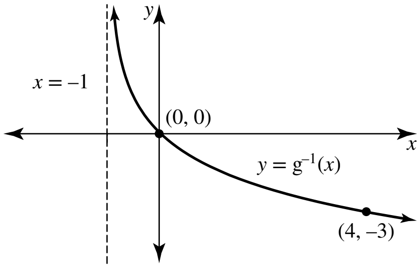
$\therefore y = 5 - 1$

$= 4$

$(-3,4)$



f The graph of $y = g(x)$ has an asymptote $x = -1$ and contains the points $(0,0)$ and $(4,-3)$ since it is the inverse of the graph in part **e**.



14 a i $3^{\log_3(8)} = 8$

ii $$\begin{aligned}10^{\log_{10}(2)+\log_{10}(3)} &= 10^{\log_{10}(2\times 3)}\\ &= 10^{\log_{10}(6)}\\ &= 6\end{aligned}$$

iii $$\begin{aligned}5^{-\log_5(2)} &= 5^{\log_5\left(2^{-1}\right)}\\ &= 2^{-1}\\ &= \frac{1}{2}\end{aligned}$$

iv $$\begin{aligned}6^{\frac{1}{2}\log_6(25)} &= 6^{\log_6\left(25^{\frac{1}{2}}\right)}\\ &= 25^{\frac{1}{2}}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

b i $3^{\log_3(x)} = x$

ii $$\begin{aligned}2^{3\log_2(x)} &= 2^{\log_2\left(x^3\right)}\\ &= x^3\end{aligned}$$

iii $$\begin{aligned}\log_2\left(2^x\right) + \log_3\left(9^x\right) &= \log_2\left(2^x\right) + \log_3\left(3^{2x}\right)\\ &= x + 2x\\ &= 3x\end{aligned}$$

iv $$\begin{aligned}\log_6\left(\frac{6^{x+1} - 6^x}{5}\right) &= \log_6\left(\frac{6^x(6^1 - 1)}{5}\right)\\ &= \log_6\left(\frac{6^x(5)}{5}\right)\\ &= \log_6\left(6^x\right)\\ &= x\end{aligned}$$

15 a $y = 2^{ax+b} + c$

From the diagram, the asymptote is $y = -4$, so $c = -4$.

$\therefore y = 2^{ax+b} - 4$

Substitute the known points on the curve.

$$\begin{aligned}(0,-2) &\Rightarrow -2 = 2^b - 4\\ &\therefore 2^b = 2\\ &\therefore b = 1\\ &\therefore y = 2^{ax+1} - 4\\ \left(\frac{1}{2},0\right) &\Rightarrow 0 = 2^{\frac{1}{2}a+1} - 4\\ &\therefore 2^{\frac{1}{2}a+1} = 4\\ &\therefore 2^{\frac{1}{2}a+1} = 2^2\\ &\therefore \frac{1}{2}a + 1 = 2\\ &\therefore a = 2\end{aligned}$$

The rule for the function is $y = 2^{2x+1} - 4$.

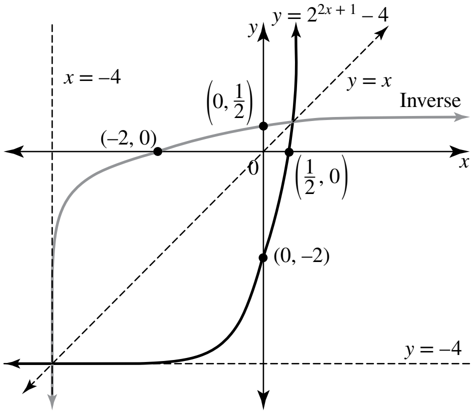
b Inverse function rule:

$$\begin{aligned}x &= 2^{2y+1} - 4\\ \therefore 2^{2y+1} &= x + 4\\ \therefore 2y + 1 &= \log_2(x+4)\\ \therefore 2y &= \log_2(x+4) - 1\\ \therefore y &= \frac{1}{2}\log_2(x+4) - \frac{1}{2}\end{aligned}$$

c Since $y = -4$ is an asymptote for the exponential function, $x = -4$ is the asymptote for the inverse function.

The points $\left(\frac{1}{2},0\right)$ and $(0,-2)$ are on the exponential function so the y intercept of the inverse is $\left(0,\frac{1}{2}\right)$ and the x intercept is $(-2,0)$.

d As the exponential graph intersects the line $y = x$ twice, the graph of the inverse must intersect the exponential at these two points, giving two points of intersection between the graphs.



e $y = 2^{2x+1} - 4$

Substitute the point $\left(\log_2(3), k\right)$

$$\begin{aligned}\therefore k &= 2^{2\log_2(3)+1} - 4\\ \therefore k &= 2^{2\log_2(3)} \times 2^1 - 4\\ \therefore k &= 2^{\log_2\left(3^2\right)} \times 2^1 - 4\\ \therefore k &= 3^2 \times 2 - 4\\ \therefore k &= 14\end{aligned}$$

f Inverse has equation $y = \frac{1}{2}\log_2(x+4) - \frac{1}{2}$

Substitute the point $\left(14, \log_2(3)\right)$

$\text{LHS} = \log_2(3)$

$\text{RHS} = \frac{1}{2}\log_2(14+4) - \frac{1}{2}$

$= \frac{1}{2}\log_2(18) - \frac{1}{2}$

$= \frac{1}{2}\log_2\left(3^2 \times 2\right) - \frac{1}{2}$

$= \frac{1}{2}\left[\log_2\left(3^2\right) + \log_2(2) - 1\right]$

$= \frac{1}{2}\left[2\log_2(3) + 1 - 1\right]$

$= \log_2(3)$

$= \text{LHS}$

The point $(14, \log_2(3))$ lies on the inverse function.

16 a $y = \log_5(x) - 2$

Domain: $x > 0 \Rightarrow$ domain is R^+

Asymptote: $x = 0$

No y intercept

Vertical translation 2 units down

Range: R

x intercept: Let $y = 0$

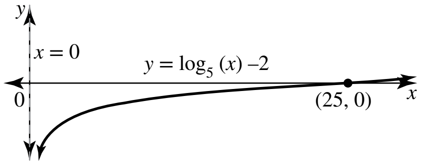
$\therefore 0 = \log_5(x) - 2$

$\therefore \log_5(x) = 2$

$\therefore x = 5^2$

$\therefore x = 25$

$(25, 0)$



b $y = \log_5(x-2)$

Domain: $x - 2 > 0 \Rightarrow x > 2$

domain is $(2, \infty)$

Asymptote: $x = 2$

No y intercept

Horizontal translation 2 units right

Range: R

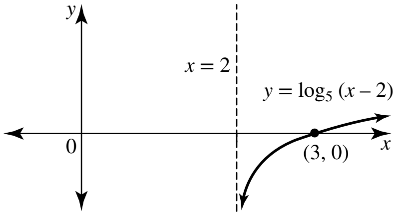
x intercept: Let $y = 0$

$\therefore 0 = \log_5(x-2)$

$\therefore x - 2 = 5^0$

$\therefore x = 3$

$(3, 0)$



c $y = \log_{10}(x) + 1$

Domain: $x > 0 \Rightarrow$ domain is R^+

Asymptote: $x = 0$

No y intercept

Vertical translation 1 unit up

Range: R

x intercept: Let $y = 0$

$\therefore 0 = \log_{10}(x) + 1$

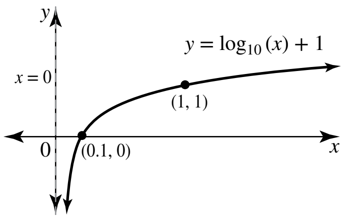
$\therefore \log_{10}(x) = -1$

$\therefore x = 10^{-1}$

$\therefore x = 0.1$

$(0.1, 0)$

Point: If $x = 1$, then $y = 1$. $(1,1)$ is on the graph.



d $y = \log_{10}(x+1)$

Domain: $x + 1 > 0 \Rightarrow x > -1$

domain is $(-1, \infty)$

Asymptote: $x = -1$

y intercept: Let $x = 0$

$\therefore y = \log_{10}(1)$

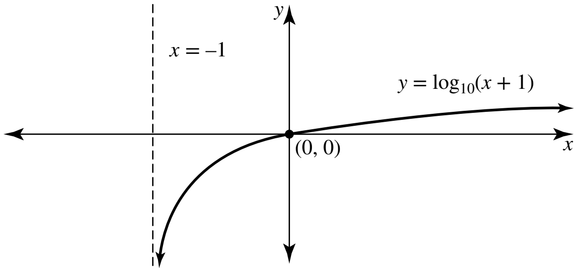
$\therefore y = 0$

$(0, 0)$

This is x intercept also.

Horizontal translation 1 unit left

Range: R



e $y = \log_3(4-x)$

Domain: $4 - x > 0$

$\therefore 4 > x$

$\therefore x < 4$

domain is $(-\infty, 4)$

Asymptote: $x = 4$

y intercept: Let $x = 0$

$\therefore y = \log_3(4)$

$(0, \log_3(4))$

Reflection in y axis, then horizontal translation 4 units right

Range: R

x intercept: Let $y = 0$

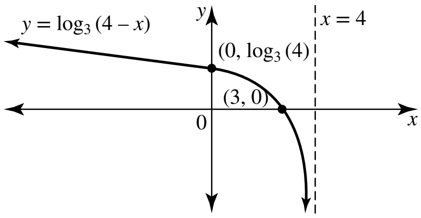
$\therefore 0 = \log_3(4-x)$

$\therefore 4 - x = 3^0$

$\therefore 4 - x = 1$

$\therefore x = 3$

$(3, 0)$



f $y = -\log_2(x+4)$

Domain: $x + 4 > 0 \Rightarrow x > -4$

domain is $(-4, \infty)$

Asymptote: $x = -4$

y intercept: Let $x = 0$

$\therefore y = -\log_2(4)$

$\therefore y = -\log_2\left(2^2\right)$

$\therefore y = -2$

$(0, -2)$

Reflection in x axis, then horizontal translation 4 units left

Range: R

x intercept: Let $y = 0$

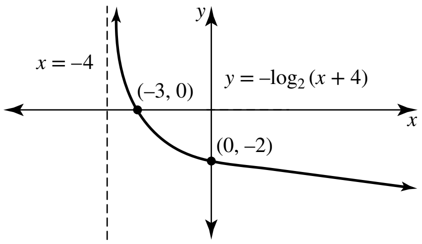
$\therefore 0 = \log_2(x+4)$

$\therefore x + 4 = 2^0$

$\therefore x + 4 = 1$

$\therefore x = -3$

$(-3, 0)$



17 a $y = -2\log_2(2-x)$

$\therefore y = -2\log_2(-(x-2))$

$y = \log_2(x) \rightarrow y = -2\log_2(-(x-2))$ under a dilation factor 2 from the x axis, a reflection in the x axis and a reflection in the y axis followed by a horizontal translation 2 units to the right.

b $y = \log_2(2x)$

$\therefore y = \log_2(x) + \log_2(2)$

$\therefore y = \log_2(x) + 1$

Therefore $y = \log_2(x) \rightarrow y = \log_2(x) + 1$ under a vertical translation of 1 unit upwards.

c $y = \log_2(x)$

Changing the base to 10,

$$y = \frac{\log_{10}(x)}{\log_{10}(2)}$$
$$= \frac{1}{\log_{10}(2)} \times \log_{10}(x)$$

So $y = \log_{10}(x) \rightarrow y = \log_2(x)$ is equivalent to

$y = \log_{10}(x) \rightarrow y = \dfrac{1}{\log_{10}(2)} \times \log_{10}(x)$.

The transformation required is a dilation of factor $\dfrac{1}{\log_{10}(2)}$ from the x axis.

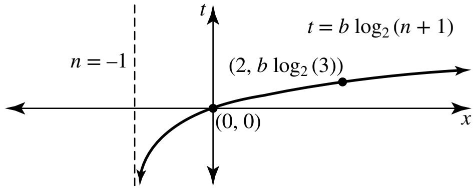
d $t = b\log_2(n+1)$, $n \geq 2$ and $b > 0$.

Asymptote $n = -1$ is outside the domain $[2, \infty)$.

Endpoint: When $n = 2$, $t = b\log_2(3)$, Point $\left(2, b\log_2(3)\right)$

Point not on the graph but helpful for sketching the curve is $(0,0)$ since when $n = 0$, $t = b\log_2(1) = 0$.

Although $n \in N$, the graph is drawn as a continuous one.



When the number of choices $n = 2$, the time $t_2 = b\log_2(3)$.

Double the number of choices from 2 to 4

When $n = 4$,the time $t_4 = b\log_2(5)$.

Test if $t_4 = 2t_2$,

$2t_2 = 2b\log_2(3)$

$= b\log_2\left(3^2\right)$

$= b\log_2(9)$

$\therefore t_4 < 2t_2$

Thus, doubling the number of choices does not double the decision time. The time does increase but by less than double.

18 a $y = a\log_7(bx)$

Substitute the given points.

Point $(2,0) \Rightarrow 0 = a\log_7(2b)$

$\therefore 0 = \log_7(2b)$

$\therefore 2b = 7^0$

$\therefore 2b = 1$

$\therefore b = \dfrac{1}{2}$

The equation is now $y = a\log_7\left(\dfrac{x}{2}\right)$.

Point $(14,14) \Rightarrow 14 = a\log_7(7)$

$\therefore 14 = a \times 1$

$\therefore a = 14$

The equation is $y = 14\log_7\left(\dfrac{x}{2}\right)$.

b i $y = a\log_3(x) + b$

Point $(1,4) \Rightarrow 4 = a\log_3(1) + b$

$\therefore 4 = a \times 0 + b$

$\therefore b = 4$

Hence, $y = a\log_3(x) + 4$

Point $\left(\dfrac{1}{3}, 8\right) \Rightarrow 8 = a\log_3\left(\dfrac{1}{3}\right) + 4$

$\therefore 4 = a\log_3\left(3^{-1}\right)$

$\therefore 4 = -a\log_3(3)$

$\therefore 4 = -a$

$\therefore a = -4$

The equation is $y = -4\log_3(x) + 4$.

ii Let the inverse cut the y axis at the point $(0, k)$. Then the function $y = -4\log_3(x) + 4$ cuts the x axis at $(k, 0)$.

$\therefore 0 = -4\log_3(k) + 4$

$\therefore 4\log_3(k) = 4$

$\therefore \log_3(k) = 1$

$\therefore k = 3^1$

$\therefore k = 3$

Therefore, the inverse function would cut the y axis at the point $(0,3)$.

c i $y = a\log_2(x-b) + c$

From the diagram the asymptote is $x = -2$ and from the equation the asymptote is $x = b$.

Therefore, $b = -2$

$\therefore y = a\log_2(x+2) + c$

Substitute the x and y intercepts shown on the diagram.

$$(-1.5,0) \Rightarrow 0 = a\log_2(-1.5+2)+c$$
$$\therefore 0 = a\log_2(0.5)+c$$
$$\therefore 0 = a\log_2\left(2^{-1}\right)+c$$
$$\therefore 0 = -a\log_2(2)+c$$
$$\therefore 0 = -a+c$$
$$\therefore a = c....(1)$$
$$(0,-2) \Rightarrow -2 = a\log_2(2)+c$$
$$\therefore -2 = a+c....(2)$$

Substitute equation (1) in equation (2)

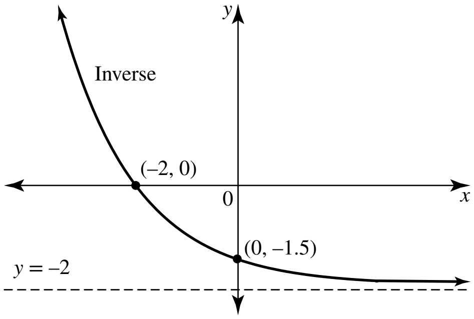
$$\therefore -2 = c+c$$
$$\therefore c = -1$$
$$\therefore a = -1$$

The equation of the graph is $y = -\log_2(x+2)-1$.

ii The graph of the inverse function has an asymptote $y=-2$, and axes intercepts $(-2,0)$ and $(0,-1.5)$.

Its rule is $x = -\log_2(y+2)-1$

$$\therefore \log_2(y+2) = -x-1$$
$$\therefore y+2 = 2^{-x-1}$$
$$\therefore y = 2^{-(x+1)}-2$$



d i $d_f = \left(-\frac{9}{4},\infty\right); d_g = (-\infty, 20)$

ii $x = -\frac{9}{4}, x = 20$

iii $f:(-2,0),(0,2); g:(10,0),\left(0,\frac{1}{2}\right)$

iv Sketch using CAS technology.

19 The inverse of $y = 2\times 3^{\frac{2-x}{2}}$ can be obtained by solving $x = 2\times 3^{\frac{2-y}{2}}$ for y in Equation/Inequality. This gives

$$y = -\frac{2\ln(x)}{\ln(3)} + \frac{2\ln(2)}{\ln(3)} + 2.$$

Enter the equations of both functions in the Graph&Tab editor and tap Analysis → G-Solve → Intersect to obtain three points of intersection. To 4 significant figures these points have co-ordinates $(0.4712, 4.632), (2,2)$ and $(4.632, 0.4712)$.

20 a i There is no logarithm law for expanding $\log_2(x+4)$. There is a logarithm law for $\log_2(x)+\log_2(4)$, It is equal to $\log_2(4x)$, not $\log_2(x+4)$.

The graphs will not be the same.

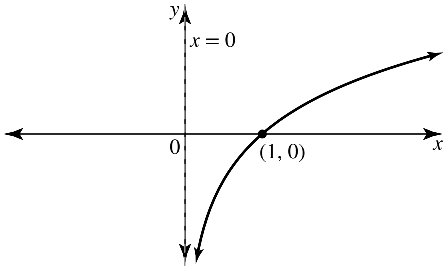
ii Use the Analysis tools to find that the graphs intersect at the point where $x = 1.\dot{3}$. For this value of x the graphs have the same value.

Algebraically, let $\log_2(x+4) = \log_2(x)+\log_2(4)$

$$\therefore \log_2(x+4) = \log_2(4x)$$
$$\therefore x+4 = 4x$$
$$\therefore 3x = 4$$
$$\therefore x = \frac{4}{3}$$

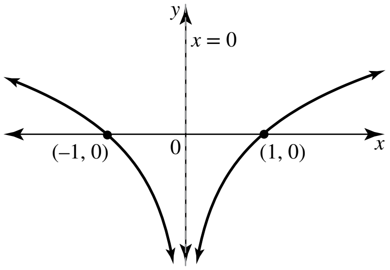
As $1.\dot{3} = \frac{4}{3}$, the calculator value supports the algebraic finding.

b The graph obtained for $y = 2\log_3(x)$ should be similar to that shown in the diagram.



The domain is R^+, the range is R and the correspondence is one-to-one.

c The graph obtained for $y = \log_2\left(x^2\right)$ should be similar to that shown in the diagram.



The domain is $R\setminus\{0\}$, the range is R and the correspondence is many-to-one.

For the domain of $y = \log_2\left(x^2\right)$, $x^2 > 0$. This is true for all real numbers excluding 0.

d The right hand branch of the graph of $y = \log_2\left(x^2\right)$ is identical to that of the graph of $y = 2\log_3(x)$.

For $x > 0$, $\log_2\left(x^2\right) = 2\log(x)$.

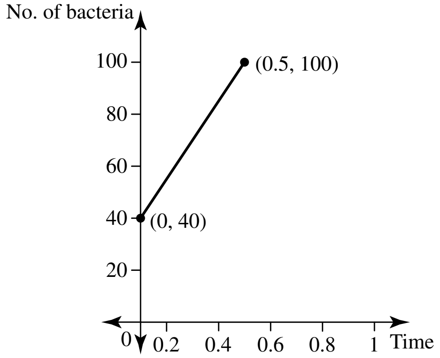
If $x < 0$, only the domain of $y = \log_2\left(x^2\right)$ includes these values.

The logarithm law $\log_a\left(m^p\right) = p\log_a(m)$ holds for any $m > 0$, so this has not been contradicted by the graphs in parts **b** and **c** : they are identical graphs for $x > 0$.

Topic 12 — Introduction to differential calculus

Exercise 12.2 — Rates of change

1 a Grows steadily so it's a linear relationship.



Rate of growth of bacteraia equals the gradient of the line.

$$\begin{aligned}\text{Rate} &= \frac{100-40}{0.5-0}\\ &= \frac{60}{0.5}\\ &= 120\end{aligned}$$

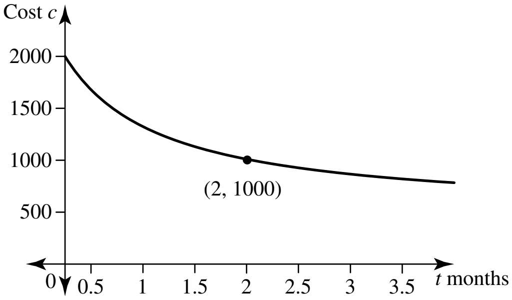
Therefore the bacteria are growing at 120 bacteria per hour.

b Cost of department is given by $c = \frac{4000}{t+2}$.

Hyperbola shape on domain where $t \geq 0$

$t = 0 \Rightarrow c = 2000$

$t = 2 \Rightarrow c = 1000$



Average rate of change of costs

$$\begin{aligned}&= \frac{\text{change in cost}}{\text{change in time}}\\ &= \frac{2000-1000}{0-2}\\ &= -500\end{aligned}$$

Therefore the costs have decreased at an average rate of \$500 per month over the first two months.

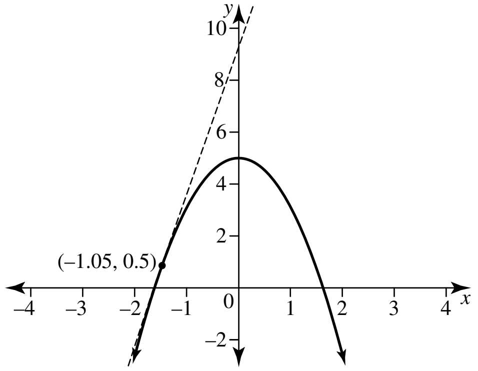
2 $f(x) = x^2 + 3,\ x \in [1,4]$

Average rate of change

$$\begin{aligned}&= \frac{f(4)-f(1)}{4-1}\\ &= \frac{19-4}{3}\\ &= 5\end{aligned}$$

3 Answers will vary.

a



Point Q $(-1.5, 0.5)$ lies on tangent and another point is estimated to be $(0, 9.5)$.

Gradient of tangent:

$$\begin{aligned}m &= \frac{9.5-0.5}{0-(-1.5)}\\ &= \frac{9}{1.5}\\ &= 6\end{aligned}$$

Therefore the gradient of the curve at point Q is estimated to be 6.

b Choose a point with $x = -1.4$ close to Q.

y-coordinate: $5 - 2(-1.4)^2 = 1.08$

Points $(-1.4, 1.08), (-1.5, 0.5)$

$$\begin{aligned}m &= \frac{1.08-0.5}{-1.4-(-1.5)}\\ &= \frac{0.58}{0.1}\\ &= 5.8\end{aligned}$$

Therefore the gradient of the curve at point Q is estimated to be 5.8.

4 a $x = t^3 + 4t^2 + 3t$ over the interval $[0, 2]$

$t = 0 \Rightarrow x = 0$

$t = 2 \Rightarrow x = 30$

Average rate of change

$$\begin{aligned}&= \frac{30-0}{2-0}\\ &= 15\end{aligned}$$

Therefore, average speed over the interval $t \in [0, 2]$ is 15 m/s.

Over the interval $[0.9, 1.1]$,

$t = 0.9 \Rightarrow x = (0.9)^3 + 4(0.9)^2 + 3(0.9)$

$\therefore t = 0.9,\ x = 6.669$

$t = 1.1 \Rightarrow x = (1.1)^3 + 4(1.1)^2 + 3(1.1)$

$\therefore t = 1.1,\ x = 9.471$

Average rate of change

$$\begin{aligned}&= \frac{9.471-6.669}{1.1-0.9}\\ &= \frac{2.802}{0.2}\\ &= 14.01\end{aligned}$$

Therefore, average speed over the interval is $t \in [0.9, 1.1]$ is 14.01 m/s.

b The speed at $t = 1$ is better estimated using the interval $[0.9, 1.1]$ because these endpoints give two closer points to $t = 1$ than do the endpoints of the interval $[0, 2]$.

Therefore the better estimate for the instantaneous speed at $t = 1$ is 14.01 m/s.

5 Refer to the diagram given in the question.

a Student II maintained the same level of interest for the semester so for this student there was a zero rate of change in interest level.

b The level of interest of Student I steadily increased during the semester, For this student there was a constant positive rate of change in interest level.

c The level of interest of Student III initially increased slowly but as the semester progressed this student's interest level grew more quickly, reaching a high level.

After an initially brief growth in interest, the level of interest of Student IV started to decline. However this loss of interest eventually slowed and thereafter the student's interest level grew again and quickly reached a high level.

d Answers will vary.

6 a 20 minutes is $\frac{1}{3}$ of an hour. In this time, \$200 is received.

The plumber's hourly rate of pay was \$600 per hour.

b 15 minutes is $\frac{1}{4}$ of an hour. In this time, \$180 is received.

The surgeon's hourly rate of pay was \$720 per hour.

c In 52 hours, \$1820 was received.

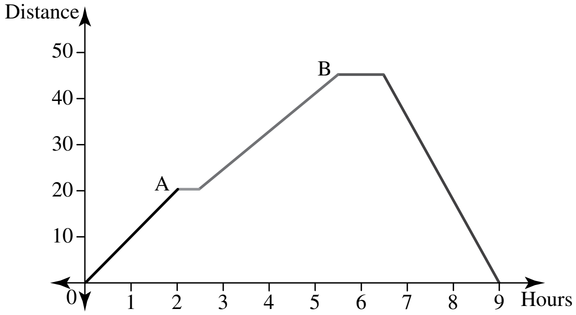
The teacher's hourly rate of pay was $\$\left(\frac{1820}{52}\right) = \$35$ per hour.

7 a 1 hour 20 minutes is $1\frac{1}{3}$ hours.

$$\begin{aligned} \text{speed} &= \frac{\text{distance}}{\text{time}} \\ &= \frac{48}{1\frac{1}{3}} \\ &= 48 \times \frac{3}{4} \\ &= 36 \end{aligned}$$

The speed of the car is 36 km/h.

b i



ii From O to A, took the cyclist 2 hours, rest period was 0.5 hours, lunch break was 1 hour and from B back to O took 2.5 hours. As the entire journey took 9 hours, the time it took for the cyclist to ride from A to B is $9 - (2 + 0.5 + 1 + 2.5) = 3$ hours.

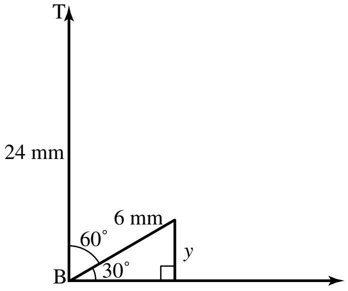
The cyclist travelled from O to A at a constant speed of 10 km/h, so A is 20 km from O. As B is 45 km from O, the distance between A and B is 25 km.

Thus, the constant speed the cyclist rode at between A and B is $\frac{25}{3} = 8\frac{1}{3}$ km/h.

iii The constant speed from B back to O is $\frac{45}{2.5} = 18$ km/h.

iv The total distance ridden in 9 hours is $2 \times 45 = 90$ km, giving the cyclist an average speed of $\frac{90}{9} = 10$ km/h.

8 a The shoot grows at 2 mm/week. In 3 weeks it will have grown to length 6 mm.

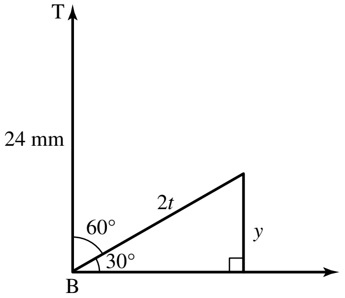


Let y be the vertical height of the tip of the shoot above the base B.

$$\begin{aligned} \sin(30°) &= \frac{y}{6} \\ \therefore y &= 6\sin(30°) \\ \therefore y &= 6 \times \frac{1}{2} \\ \therefore y &= 3 \end{aligned}$$

T is 24 mm vertically above B. Therefore, the tip of the shoot is 21 mm vertically below T.

b After t weeks, the shoot growing at 2 mm/week, will be $2t$ mm in length.



$$\begin{aligned} \sin(30°) &= \frac{y}{22} \\ \therefore y &= 2t\sin(30°) \\ \therefore y &= 2t \times \frac{1}{2} \\ \therefore y &= t \end{aligned}$$

The tip of the shoot is $(24 - t)$ mm below T.

c When he tip of the shoot is at the same height as T, $(24 - t) = 0$. This will occur after 24 weeks.

9 a There is a 9 month interval between the months of November and August of the following year. Over this interval, the Australian dollar rose in price by $(0.83 - 0.67) = 0.16$ euros.

The average rate of change of the dollar over this time period is $\frac{0.16}{9} \simeq 0.018$ euros per month.

b Over 3 years the value of the investment rose from \$1000 to \$1150 at a steady rate.

The rate of change is $\frac{150}{3} = 50$ dollars per year.

The percentage rate of interest: $\frac{50}{1000}\times 100 = 5$

Therefore the investment earns 5% per annum interest, and $r = 5$.

10 a $f(x) = 2x - x^2,\ x \in [-2, 6]$

$$\begin{aligned} f(-2) &= 2(-2) - (-2)^2 \\ &= -8 \\ f(6) &= 2(6) - (6)^2 \\ &= -24 \end{aligned}$$

Average rate of change:

$$\begin{aligned} \frac{f(6) - f(-2)}{6 - (-2)} &= \frac{-24 - (-8)}{8} \\ &= \frac{-16}{8} \\ &= -2 \end{aligned}$$

b $f(x) = 2 + 3x,\ x \in [12, 16]$

$$\begin{aligned} f(12) &= 2 + 3(12) \\ &= 38 \\ f(16) &= 2 + 3(16) \\ &= 50 \end{aligned}$$

Average rate of change:

$$\begin{aligned} \frac{f(16) - f(12)}{16 - (12)} &= \frac{50 - 38}{4} \\ &= 3 \end{aligned}$$

(which is the gradient of the line).

c $f(t) = t^2 + 3t - 1,\ t \in [1, 3]$

$$\begin{aligned} f(1) &= 1 + 3 - 1 \\ &= 3 \\ f(3) &= 9 + 9 - 1 \\ &= 17 \end{aligned}$$

Average rate of change:

$$\begin{aligned} \frac{f(3) - f(1)}{3 - 1} &= \frac{17 - 3}{2} \\ &= 7 \end{aligned}$$

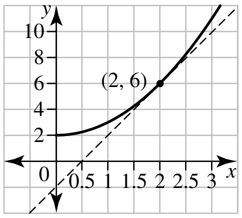
d $f(t) = t^3 - t,\ t \in [-1, 1]$

$$\begin{aligned} f(-1) &= (-1)^3 - (-1) \\ &= -1 + 1 \\ &= 0 \\ f(1) &= 1 - 1 \\ &= 0 \end{aligned}$$

Average rate of change:

$$\begin{aligned} \frac{f(1) - f(-1)}{1 - (-1)} &= \frac{0 - 0}{2} \\ &= 0 \end{aligned}$$

11 a Construct a tangent line to the curve at the point $(2, 6)$ and estimate a second point that lies on the tangent line.

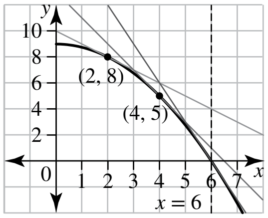


Answers will vary but suppose a second point is $(0, -2)$.

Gradient of tangent is $\frac{6 - (-2)}{2 - 0} = 4$.

An estimate of the gradient of the curve at the point on the curve where $x = 2$ is 4.

b Construct tangent lines to the curve at the points $(2, 8)$ and $(4, 5)$ and estimate a second point that lies on each of these lines.



The tangent at $(2, 8)$ passes through $(0, 10)$ (answers will vary).

Its gradient is $\frac{8 - 10}{2 - 0} = -1$.

An estimate of the gradient of the curve at the point on the curve where $x = 2$ is -1.

The tangent at $(4, 5)$ passes through $(6.5, 0)$ (answers will vary).

Its gradient is $\frac{5 - 0}{4 - 6.5} = -\frac{5}{2.5} = -2$.

An estimate of the gradient of the curve at the point on the curve where $x = 4$ is -2.

c The average rate of change over the interval $x \in [4, 6]$ is the gradient of the line joining the endpoints $(2, 8)$ and $(4, 5)$.

Average rate of change is $\frac{5 - 8}{4 - 2} = -\frac{3}{2}$.

d The gradient of the tangent at $(6, 0)$ is required to calculate the angle at which the curve cuts the x axis. Construct a tangent line to the curve at the point $(6, 0)$ and estimate a second point that lies on the tangent line. (see the diagram in part **b**).

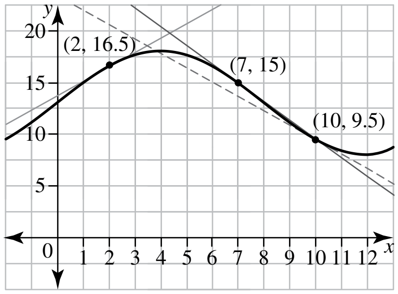
The tangent at $(6, 0)$ passes through $(3, 9)$ (answers will vary).

Its gradient is $\frac{9 - 0}{3 - 6} = -3$.

$$\begin{aligned} &\therefore \tan\theta = -3 \\ &\quad \therefore \theta = 180° - \tan^{-1}(3) \\ &\quad \therefore \theta \simeq 108° \end{aligned}$$

The curve cuts the x axis at an angle of approximately 108°.

12 Construct a tangent line at each of the given points and obtain the co-ordinates of a second point that lies on each tangent line. Answers will vary for parts **a**, **c** and **d**.



a i Two points estimated to be on tangent are $(2, 16.5)$ and $(4.5, 20)$.

$$\begin{aligned} m &= \frac{20 - 16.5}{4.5 - 2} \\ &= \frac{3.5}{2.5} \\ &\simeq 1.4 \end{aligned}$$

At 2 pm the temperature is estimated to be changing at 1.4°C per hour.

ii The tangent at the maximum turning point where $t = 4$ is horizontal so there is zero rate of change of temperature
at 4 pm.

iii Two points estimated to be on tangent are $(7,15)$ and $(12.5,5)$.

$$\begin{aligned} m &= \frac{5-15}{12.5-7} \\ &= \frac{-10}{5.5} \\ &\simeq -1.8 \end{aligned}$$

At 7 pm the temperature is estimated to be changing at −1.8°C per hour.

iv Two points estimated to be on tangent are $(10,9.5)$ and $(6,15)$.

$$\begin{aligned} m &= \frac{9.5-15}{10-6} \\ &= \frac{-5.5}{4} \\ &\simeq -1.4 \end{aligned}$$

At 10 pm the temperature is estimated to be changing at −1.4°C per hour.

b Judging where the tangent to the curve would be steepest with a negative gradient, suggests the temperature is decreasing most rapidly at around 8 pm. Answers may vary slightly.

c At 1 pm, the temperature is approximately 15° and at 9 : 30 pm the temperature is approximately 10°.

The average rate of change of temperature over this time interval is

$$\begin{aligned} &\frac{10-15}{9.5-1} \\ &= \frac{5}{8.5} \\ &\simeq 0.59 \end{aligned}$$

The average rate of change of the temperature between 1 pm and 9 : 30 pm is 0.59°C / hour.

13 a $d = \dfrac{200t}{t+1}, t \ge 0$

When $t = 0$, $d = 0$ and when $t = 4$, $d = \dfrac{800}{5} = 160$.

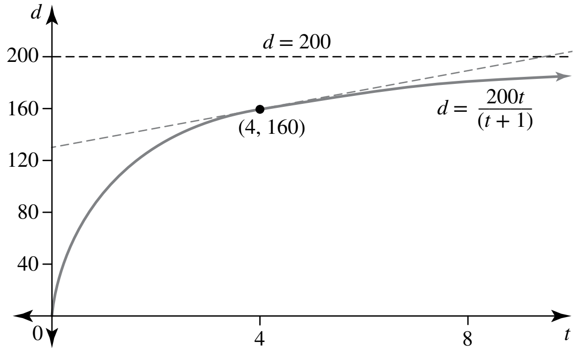
Average speed is $\dfrac{\text{distance travelled}}{\text{time taken}}$

Therefore, the average speed is $\dfrac{160}{4} = 40$ m/hour.

b $d = \dfrac{200t}{t+1}$

$$\begin{aligned} \therefore d &= \frac{200(t+1)-200}{t+1} \\ \therefore d &= \frac{200(t+1)}{t+1} - \frac{200}{t+1} \\ \therefore d &= 200 - \frac{200}{t+1} \end{aligned}$$

Asymptote $d = 200$ and $t = -1$ (outside the domain)
Right endpoint $(0,0)$ and point $(4,160)$ is on graph.



The boat travels away from the jetty quickly but its speed slows to almost stationary as it nears a distance of 200 metres from the jetty. It does not travel further beyond this distance.

c Draw the tangent at $(4,160)$. Another point on this tangent is approximately $(9,200)$.

Its gradient is

$$\begin{aligned} m &= \frac{200-160}{9-4} \\ &= \frac{40}{5} \\ &= 8 \end{aligned}$$

The instantaneous speed of the boat 4 hours after leaving the jetty is approximately 8 m/hour.

d $d = \dfrac{200t}{t+1}$

Let $t = 4.1$

$$\begin{aligned} d &= \frac{200 \times 4.1}{4.1+5} \\ &= \frac{820}{5.1} \\ &\simeq 160.78 \end{aligned}$$

Gradient of line segment joining $(4.1,160.78)$ and $(4,160)$ is

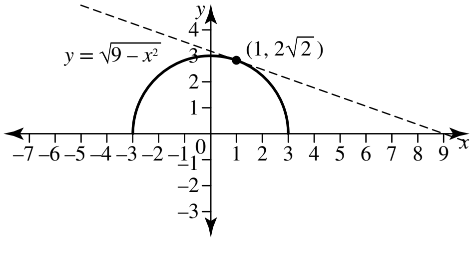
$$\begin{aligned} m &= \frac{160.78-160}{4.1-4} \\ &= \frac{0.78}{0.1} \\ &\simeq 7.8 \end{aligned}$$

An estimate of the speed of the boat is 7.8 m/hour.

14 a $y = \sqrt{9-x^2}$

Semicircle centre $(0,0)$, radius 3, domain $[-3,3]$, range $[0,3]$.

When $x = 1$, $y = \sqrt{8}$. Point is $\left(1, 2\sqrt{2}\right)$.



b When $x = 1.5$,

$$\begin{aligned} y &= \sqrt{9-2.25} \\ &= \sqrt{6.75} \\ &= \sqrt{\frac{27}{4}} \\ &= \frac{3\sqrt{3}}{2} \end{aligned}$$

Points $\left(1, 2\sqrt{2}\right)$ and $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

Gradient

$$m = \frac{\left(\frac{3\sqrt{3}}{2} - 2\sqrt{2}\right)}{\left(\frac{3}{2} - 1\right)}$$
$$= 3\sqrt{3} - 4\sqrt{2}$$
$$\simeq -0.46$$

This gradient is an estimate of the gradient of the semicircle at the point where $x = 1$.

c The tangent to the curve at point $\left(1, 2\sqrt{2}\right)$ is drawn on the diagram in part **a**. A second point on the tangent is estimated to be $(9, 0)$.

$$m = \frac{0 - 2\sqrt{2}}{9 - 1}$$
$$= -\frac{\sqrt{2}}{4}$$
$$\simeq -0.35$$

The gradient of the curve at $x = 1$ is estimated to be -0.35. (Answers will vary).

d The radius from $(0, 0)$ to $\left(1, 2\sqrt{2}\right)$ is perpendicular to the tangent.

$$m_{radius} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}.$$

For perpendicular lines, $m_1 m_2 = -1$

$$\therefore m_{\text{tangent}} = -\frac{1}{2\sqrt{2}}$$
$$= -\frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= -\frac{\sqrt{2}}{4}$$

The exact value of the gradient of the semicircle at the point where $x = 1$ is $-\frac{\sqrt{2}}{4}$.

Since $-\frac{\sqrt{2}}{4} \simeq -0.35$, the estimate in part **c** was accurate while that given in part **b** was less accurate. However, the comparisons depend on the answers obtained for parts **b** and **c**.

15 Enter the equation $y = 4 - 3x^2$ in the Graph&Tab editor. Tap Analysis → Sketch → Tangent and enter the x value -2. The tangent is sketched and its equation is given as $y = 12x + 16$. As the gradient of the tangent is 12, the gradient of $y = 4 - 3x^2$ at $x = -2$ is 12.

16 Sketch $y = 0.5x^2$ in Graph&Tab menu then sketch the tangents to the curve at any three x values following the same steps as in question **11**. From the equation of each tangent the gradient is obtained.

For example, choosing $x = 2.5$, tangent is $y = 2.5x - 3.125$, so the gradient of the curve at $x = 2.5$ is 2.5. Choosing $x = -3$, tangent is $y = -3x - 4.5$, so the gradient of the curve at $x = -3$ is -3.

For each point chosen, the gradient will be the same value as the x co-ordinate.

Exercise 12.3 — Gradients of secants

1 $y = 2x^2 - x$

a When $x = -1$, $y = 3 \Rightarrow (-1, 3)$

When $x = -1 + h$,

$$y = 2(-1 + h)^2 - (-1 + h)$$
$$= 2 - 4h + 2h^2 + 1 - h$$
$$= 2h^2 - 5h + 3$$
$$\Rightarrow (-1 + h, 2h^2 - 5h + 3)$$

Gradient of secant

$$= \frac{(2h^2 - 5h + 3) - (3)}{(-1 + h) - (-1)}$$
$$= \frac{2h^2 - 5h}{h}$$
$$= \frac{h(2h - 5)}{h}$$
$$= 2h - 5, \ h \neq 0$$

b If $h = 0.01$, the gradient of the secant is $0.02 - 5 = -4.98$. Therefore an estimate for the gradient of the tangent is -4.98.

c As $h \to 0$, $(2h - 5) \to -5$

$$\therefore \lim_{h \to 0}(2h - 5) = -5$$

Therefore the gradient of the tangent at $x = -1$ is -5.

2 $y = \frac{1}{3}x^3 - x^2 + x + 5$

a When $x = 1$, $y = 5\tfrac{1}{3} \Rightarrow \left(1, \frac{16}{3}\right)$

When $x = 1 - h$,

$$y = \frac{1}{3}(1 - h)^3 - (1 - h)^2 + (1 - h) + 5$$
$$= \frac{1}{3}(1 - 3h + 3h^2 - h^3) - (1 - 2h + h^2) + 1 - h + 5$$
$$= \frac{1}{3} - h + h^2 - \frac{1}{3}h^3 - 1 + 2h - h^2 + 1 - h + 5$$
$$= -\frac{1}{3}h^3 + 5\tfrac{1}{3}$$
$$\Rightarrow \left(1 - h, \frac{-h^3 + 16}{3}\right)$$

Gradient of secant

$$= \frac{\left({}^{(-h^3+16)}\!/_{3}\right) - {}^{16}\!/_{3}}{(1 - h) - 1}$$
$$= \frac{{}^{-h^3}\!/_{3}}{-h}$$
$$= \frac{h^2}{3}, \ h \neq 0$$

b As $h \to 0$, $\frac{h^2}{3} \to 0$

$$\therefore \lim_{h \to 0}\left(\frac{h^2}{3}\right) = 0$$

Therefore the gradient of the tangent at $x = 1$ is 0.

3 **a** $y = 2x^2 + 1$

When $x = 1$, $y = 3$ and when $x = 1.5$. $y = 2 \times 2.25 + 1 = 5.5$

Secant passes through $(1, 3)$ and $(1.5, 5.5)$

$$\begin{aligned} m_{\text{secant}} &= \frac{5.5-3}{1.5-1} \\ &= \frac{2.5}{0.5} \\ &= 5 \end{aligned}$$

b $f(x) = 3 + x - x^3$

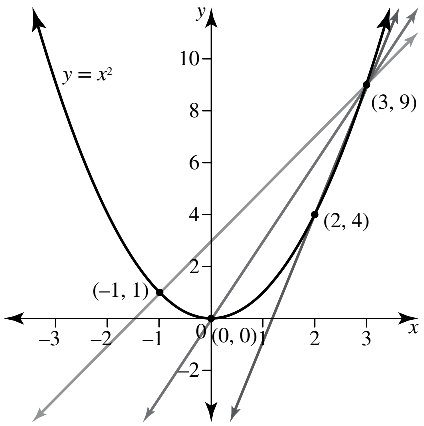
$f(-2) = 3 - 2 - (-8) = 9$

$f(-2.1) = 3 - 2.1 - (-2.1)^3 = 10.161$

Secant passes through $(-2, 9)$ and $(-2.1, 10.161)$

$$\begin{aligned} m_{\text{secant}} &= \frac{10.161-9}{-2.1+2} \\ &= -\frac{1.161}{0.1} \\ &= -11.61 \end{aligned}$$

4 a $y = x^2$



b The secant through $(2, 4)$ and $(3, 9)$ is the best of the three for approximating the tangent to the parabola at $(3, 9)$. To improve on this approximation, use a secant through $(3, 9)$ and a point closer to $(3, 9)$ than the point $(2, 4)$.

5 a $y = \frac{x^4}{4}$

When $x = 3.9$,

$$y = \frac{(3.9)^4}{4}$$

$\therefore y = 57.836025$

When $x = 4$,

$$\begin{aligned} y &= \frac{4^4}{4} \\ &= 64 \end{aligned}$$

Chord joins the points $(3.9, 57.836\,025)$ and $(4, 64)$.

$$\begin{aligned} m_{chord} &= \frac{64 - 57.836\,025}{4 - 3.9} \\ &= \frac{6.1639\,75}{0.1} \\ &= 61.639\,75 \end{aligned}$$

b A whole number estimate for the gradient of the tangent at $x = 4$ is 62.

c To improve on this estimate, use an endpoint of the chord which is closer to $(4, 64)$ than $(3.9, 57.836025)$.

6 a $y = 4 - x^2$

Let $x = 2 + h$

$$\begin{aligned} y &= 4 - (2+h)^2 \\ &= 4 - \left(4 + 4h + h^2\right) \\ &= -4h - h^2 \end{aligned}$$

A is the point $\left(2+h, -4h-h^2\right)$.

b C $(2, 0)$

$$\begin{aligned} m_{AC} &= \frac{-4h-h^2-0}{2+h-2} \\ &= \frac{-h(4+h)}{h} \\ &= -(4+h),\ h \neq 0 \\ &= -4 - h \end{aligned}$$

c $m_{AC} = -5$

$\therefore -4 - h = -5$

$\therefore h = 1$

For $h = 1$, the co-ordinates of point A $\left(2+h, -4h-h^2\right)$ become $(3, -5)$.

d If A has co-ordinates $(2.1, -0.41)$, then

$2 + h = 2.1$

$\therefore h = 0.1$

Checking y:

$$\begin{aligned} -4h - h^2 &= -0.4 - 0.01 \\ &= -0.41 \end{aligned}$$

as required.

With $h = 0.1$, then, $m_{AC} = -(4+h)$ becomes $m_{AC} = -4.1$.

e Let $x = 2 - h$

$$\begin{aligned} y &= 4 - (2-h)^2 \\ &= 4 - \left(4 - 4h + h^2\right) \\ &= 4h - h^2 \end{aligned}$$

B is the point $\left(2-h, 4h-h^2\right)$.

$$\begin{aligned} m_{BC} &= \frac{4h-h^2-0}{2-h-2} \\ &= \frac{h(4-h)}{-h} \\ &= -(4-h),\ h \neq 0 \\ &= h - 4 \end{aligned}$$

Substitute $h = 0.1$

$\therefore m_{BC} = -3.9$

f A $\left(2+h, -4h-h^2\right)$ and B $\left(2-h, 4h-h^2\right)$

$$\begin{aligned} m_{AB} &= \frac{\left(4h-h^2\right)-\left(-4h-h^2\right)}{(2-h)-(2+h)} \\ &= \frac{8h}{-2h} \\ &= -4, \quad h \neq 0 \end{aligned}$$

7 a $y = x^3 + x$

Let $x = 3 + h$

$$\begin{aligned} y &= (3+h)^3 + 3 + h \\ &= 3^3 + 3 \times 3^2 \times h + 3 \times 3 \times h^2 + h^3 + 3 + h \\ &= 27 + 27h + 9h^2 + h^3 + 3 + h \\ &= h^3 + 9h^2 + 28h + 30 \end{aligned}$$

The y co-ordinate of D is $h^3 + 9h^2 + 28h + 30$.

b Let $x = 3$, then $y = 27 + 3 = 30$.

D $\left(3+h, h^3+9h^2+28h+30\right)$ and $(3, 30)$

$$\begin{aligned} m &= \frac{\left(h^3+9h^2+28h+30\right)-30}{(3+h)-3} \\ &= \frac{h^3+9h^2+28h}{h} \\ &= \frac{h\left(h^2+9h+28\right)}{h} \\ &= h^2 + 9h + 28, \quad h \neq 0 \end{aligned}$$

c When $h = 0.001$,

$$\begin{aligned} h^2+9h+28 &= (0.001)^2+9(0.001)+28 \\ &= 28.009\,001 \end{aligned}$$

The gradient is 28.009 001

d The answer to part **c** suggests that the gradient of the tangent to the curve $y = x^3 + x$ at the point where $x = 3$ is 28.

8 a $y = \dfrac{1}{x}$

When $x = 1, y = 1$ so M is the point $(1,1)$.

When $x = 1+h, y = \dfrac{1}{1+h}$, so N is the point $\left(1+h, \dfrac{1}{1+h}\right)$.

b Gradient of the secant through M and N is

$$\begin{aligned} m &= \frac{\left(\dfrac{1}{1+h}\right)-1}{(1+h)-1} \\ &= \left(\frac{1}{1+h}-1\right) \div h \\ &= \frac{1-(1+h)}{1+h} \times \frac{1}{h} \\ &= \frac{-h}{1+h} \times \frac{1}{h} \\ &= \frac{-1}{1+h},\ h \neq 0 \end{aligned}$$

c As $h \to 0,\ \dfrac{-1}{1+h} \to \dfrac{-1}{1+0} = -1$

The gradient approaches the value -1.

d Hence, the gradient of the tangent to $y = \dfrac{1}{x}$ at M is -1.

9 a $y = 1+3x-x^2$

When $x = 1, y = 3$

When $x = 1-h$,

$$\begin{aligned} y &= 1+3(1-h)-(1-h)^2 \\ &= 1+3-3h-\left(1-2h+h^2\right) \\ &= 3-h-h^2 \end{aligned}$$

Gradient of secant through the points $(1,3)$ and $\left(1-h, 3-h-h^2\right)$ is

$$\begin{aligned} m_{\text{secant}} &= \frac{\left(3-h-h^2\right)-3}{(1-h)-1} \\ &= \frac{-h-h^2}{-h} \\ &= \frac{-h(1+h)}{-h} \\ &= 1+h,\ h \neq 0 \end{aligned}$$

b Gradient of tangent at $x = 1$ is $\lim\limits_{h \to 0}(1+h)$.

c As $h \to 0, 1+h \to 1+0 = 1$

Therefore, the gradient of the tangent is 1.

10 a $y = (x-2)(x+1)(x-3)$

When $x = 0, y = (-2)(1)(-3) = 6$

When $x = h, y = (h-2)(h+1)(h-3)$

$$\begin{aligned} \therefore y &= (h-2)\left(h^2-2h-3\right) \\ &= \left(h, h^3-4h^2+h+6\right) \end{aligned}$$

The gradient of the secant through the points $(0,6)$ and $\left(h, h^3-4h^2+h+6\right)$ is

$$\begin{aligned} m_{\text{secant}} &= \frac{\left(h^3-4h^2+h+6\right)-6}{h-0} \\ &= \frac{h^3-4h^2+h}{h} \\ &= \frac{h\left(h^2-4h+1\right)}{h} \\ &= h^2-4h+1,\ h \neq 0 \end{aligned}$$

as required.

b Hence, the gradient of the tangent to the curve at $x = 0$ is $\lim\limits_{h \to 0}\left(h^2-4h+1\right)$.

c As $h \to 0,\ h^2-4h+1 \to 0^2-4\times 0+1 = 1$

The gradient of the tangent is 1.

11 a $m_{\text{secant}} = 3+h$. This expression cannot be simplified further.

$\lim\limits_{h \to 0}(3+h) = 3$

b $m_{\text{secant}} = \dfrac{8h-2h^2}{h}$

$$\begin{aligned} \therefore m_{\text{secant}} &= \frac{h(8-2h)}{h} \\ &= 8-2h,\ h \neq 0 \end{aligned}$$

$$\begin{aligned} &\lim_{h \to 0}\left(\frac{8h-2h^2}{h}\right) \\ &= \lim_{h \to 0}(8-2h) \\ &= 8 \end{aligned}$$

c $m_{\text{secant}} = \dfrac{(2+h)^3-8}{h}$

$$\begin{aligned} \therefore m_{\text{secant}} &= \frac{(8+12h+6h^2+h^3)-8}{h} \\ &= \frac{12h+6h^2+h^3}{h} \\ &= \frac{h\left(12+6h+h^2\right)}{h} \\ &= 12+6h+h^2,\ h \neq 0 \end{aligned}$$

$$\begin{aligned} &\lim_{h \to 0}\left(\frac{(2+h)^3-8}{h}\right) \\ &= \lim_{h \to 0}\left(12+6h+h^2\right) \\ &= 12 \end{aligned}$$

d $m_{\text{secant}} = \dfrac{(1+h)^4-1}{h}$

$$\begin{aligned} \therefore m_{\text{secant}} &= \frac{(1+4h+6h^2+4h^3+h^4)-1}{h} \\ &= \frac{4h+6h^2+4h^3+h^4}{h} \\ &= \frac{h\left(4+6h+4h^2+h^3\right)}{h} \\ &= 4+6h+4h^2+h^3,\ h \neq 0 \end{aligned}$$

$$\begin{aligned} &\lim_{h \to 0}\left(\frac{(1+h)^4-1}{h}\right) \\ &= \lim_{h \to 0}\left(4+6h+4h^2+h^3\right) \\ &= 4 \end{aligned}$$

e $m_{\text{secant}} = \dfrac{\dfrac{1}{(4+h)} - \dfrac{1}{4}}{h}$

$$\begin{aligned}\therefore m_{\text{secant}} &= \left(\frac{1}{(4+h)} - \frac{1}{4}\right) \div h \\ &= \frac{4-(4+h)}{(4+h)4} \times \frac{1}{h} \\ &= \frac{-h}{4(4+h)} \times \frac{1}{h} \\ &= \frac{-1}{4(4+h)},\ h \neq 0\end{aligned}$$

$$\lim_{h\to 0}\left(\frac{-1}{4(4+h)}\right)$$
$$= \frac{-1}{4(4)}$$
$$= \frac{-1}{16}$$

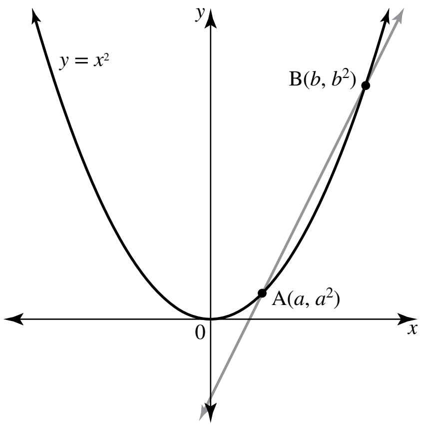
f $m_{\text{secant}} = \dfrac{h}{\sqrt{1+h}-1}$

$$\begin{aligned}\therefore m_{\text{secant}} &= \frac{h}{\sqrt{1+h}-1} \times \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} \\ &= \frac{h\left(\sqrt{1+h}+1\right)}{\left(\sqrt{1+h}\right)^2 - 1^2} \\ &= \frac{h\left(\sqrt{1+h}+1\right)}{1+h-1} \\ &= \frac{h\left(\sqrt{1+h}+1\right)}{h} \\ &= \sqrt{1+h}+1,\ h \neq 0\end{aligned}$$

$$\lim_{h\to 0}\left(\frac{h}{\sqrt{1+h}-1}\right)$$
$$= \lim_{h\to 0}\left(\sqrt{1+h}+1\right)$$
$$= \sqrt{1}+1$$
$$= 2$$

12 a $f(x) = x^2$

$f(a) = a^2$ and $f(b) = b^2$, so A is the point $\left(a, a^2\right)$ and B is the point $\left(b, b^2\right)$.



b Gradient of secant through points A and B is

$$\begin{aligned}m_{AB} &= \frac{b^2 - a^2}{b-a} \\ &= \frac{(b-a)(b+a)}{b-a} \\ &= a+b,\ \ a \neq b\end{aligned}$$

c i As $b \to a$, the secant AB $\to$ tangent at A.

Therefore, the gradient of the tangent at A is $\lim\limits_{b\to a}(b+a) = 2a$.

ii As $a \to b$, the secant AB $\to$ tangent at B.

Therefore, the gradient of the tangent at B is $\lim\limits_{a\to b}(b+a) = 2b$.

13 a $\lim\limits_{h\to 0}\left(\dfrac{(h+2)^2(h+1)-4}{h}\right)$

The limit template is in the 2*D* CALC section of the mth keyboard and *h* is in the abc section of the keyboard.

$$\lim_{h\to 0}\left(\frac{(h+2)^2(h+1)-4}{h}\right) = 8$$

b $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}+3\right) = 3$. The symbol ∞ is in the same row as π in the mth keyboard.

c
$$\lim_{x\to\infty}\left(\frac{2x+1}{3x-2}\right) = \frac{2}{3}$$

14 a $f(x) = \sqrt{x}$

$f(9) = \sqrt{9} = 3$ and $f(9+h) = \sqrt{9+h}$.

The gradient of the secant through the points $(9, 3)$ and $\left(9+h, \sqrt{9+h}\right)$ is

$$m_{\text{secant}} = \frac{\sqrt{9+h}-3}{9+h-9}$$
$$m_{\text{secant}} = \frac{\sqrt{9+h}-3}{h}$$

b Gradient of the tangent at the point $(9, 3)$ is

$$\lim_{h\to 0}\left(\frac{\sqrt{9+h}-3}{h}\right).$$

c Using the 2*D* CALC mth keyboard, $\lim\limits_{h\to 0}\left(\dfrac{\sqrt{9+h}-3}{h}\right) = \dfrac{1}{6}$.

The gradient of the tangent is $\dfrac{1}{6}$.

Exercise 12.4 — The derivative function

1 $f(x) = 2x^2 + x + 1$

a $f(x+h) = 2(x+h)^2 + (x+h) + 1$

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{2(x+h)^2 + (x+h) + 1 - (2x^2 + x + 1)}{h} \\ &= \lim_{h\to 0}\frac{2x^2 + 4xh + 2h^2 + x + h + 1 - 2x^2 - x - 1}{h} \\ &= \lim_{h\to 0}\frac{4xh + 2h^2 + h}{h} \\ &= \lim_{h\to 0}\frac{h(4x+2h+1)}{h} \\ &= \lim_{h\to 0}(4x+2h+1) \\ &= 4x+1\end{aligned}$$

b Gradient at $x = -1$ is the value of $f'(-1)$.

$$\begin{aligned}f'(-1) &= 4(-1)+1 \\ &= -3\end{aligned}$$

c Instantaneous rate of change is the gradient of the tangent i.e. the gradient of the curve.

$$\begin{aligned}\text{Rate} &= f'(0)\\ &= 1\end{aligned}$$

2 $f(x) = x^3$

$f(x+h) = (x+h)^3$

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}\\ &= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}\\ &= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}\\ &= \lim_{h\to 0}(3x^2+3xh+h^2)\\ &= 3x^2\end{aligned}$$

3 $y = x^2 + x$

$\therefore y+\delta y = (x+\delta x)^2 + (x+\delta x)$

$\therefore \delta y = (x+\delta x)^2 + (x+\delta x) - (x^2+x)$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x\to 0}\frac{\delta y}{\delta x}\\ &= \lim_{\delta x\to 0}\frac{(x+\delta x)^2+(x+\delta x)-(x^2+x)}{\delta x}\\ &= \lim_{\delta x\to 0}\frac{x^2+2x\delta x+(\delta x)^2+x+\delta x-x^2-x}{\delta x}\\ &= \lim_{\delta x\to 0}\frac{2x\delta x+(\delta x)^2+\delta x}{\delta x}\\ &= \lim_{\delta x\to 0}\frac{\delta x(2x+\delta x+1)}{\delta x}\\ &= \lim_{\delta x\to 0}(2x+\delta x+1)\\ &= 2x+1\end{aligned}$$

When $x = 1$, $\frac{dy}{dx} = 3$, so the gradient of the tangent to the curve is 3.

4 $f(x) = (3-x)(x+1)$

$$\begin{aligned}f(2+\delta x) &= \big(3-(2+\delta x)\big)\big((2+\delta x)+1\big) \quad\text{and}\quad f(2) = (3-2)(2+1)\\ &= (1-\delta x)(3+\delta x) \qquad\qquad\qquad\qquad\quad = 3\end{aligned}$$

$$\begin{aligned}\lim_{\delta x\to 0}\frac{f(2+\delta x)-f(2)}{\delta x} &= \lim_{\delta x\to 0}\frac{(1-\delta x)(3+\delta x)-3}{\delta x}\\ &= \lim_{\delta x\to 0}\frac{3+\delta x-3\delta x-(\delta x)^2-3}{\delta x}\\ &= \lim_{\delta x\to 0}\frac{-2\delta x-(\delta x)^2}{\delta x}\\ &= \lim_{\delta x\to 0}\frac{\delta x(-2-\delta x)}{\delta x}\\ &= \lim_{\delta x\to 0}(-2-\delta x)\\ &= -2\end{aligned}$$

This means the gradient of the curve at the point where $x = 2$ is –2.

5 Let $f(x)=5x+3x^4$

then $f(x+h)=5(x+h)+3(x+h)^4$.

$$
\begin{aligned}
f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to0}\frac{5(x+h)+3(x+h)^4-(5x+3x^4)}{h}\\
&=\lim_{h\to0}\frac{5x+5h+3(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-(5x+3x^4)}{h}\\
&=\lim_{h\to0}\frac{5x+5h+3x^4+12x^3h+18x^2h^2+12xh^3+3h^4-5x-3x^4}{h}\\
&=\lim_{h\to0}\frac{5h+12x^3h+18x^2h^2+12xh^3+3h^4}{h}\\
&=\lim_{h\to0}\frac{h(5+12x^3+18x^2h+12xh^2+3h^3)}{h}\\
&=\lim_{h\to0}(5+12x^3+18x^2h+12xh^2+3h^3)\\
&=5+12x^3
\end{aligned}
$$

Therefore, the derivative of $5x+3x^4$ with respect to x is $5+12x^3$.

6 Let $f(x)=ax^5+c$

$$
\begin{aligned}
f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to0}\frac{a(x+h)^5+c-(ax^5+c)}{h}\\
&=\lim_{h\to0}\frac{a(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)+c-ax^5-c}{h}\\
&=\lim_{h\to0}\frac{a(5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)}{h}\\
&=\lim_{h\to0}\frac{ah(5x^4+10x^3h+10x^2h^2+5xh^3+h^4)}{h}\\
&=\lim_{h\to0}\ a(5x^4+10x^3h+10x^2h^2+5xh^3+h^4)\\
&=5ax^4
\end{aligned}
$$

Therefore if $y=ax^5+c$, $\dfrac{dy}{dx}=5ax^4$.

7 **a** $f(x)=x^2$

$f(x+h)=(x+h)^2$

The difference quotient $\dfrac{f(x+h)-f(x)}{h}$ becomes

$$
\begin{aligned}
&\frac{(x+h)^2-x^2}{h}\\
&=\frac{x^2+2xh+h^2-x^2}{h}\\
&=\frac{2xh+h^2}{h}\\
&=\frac{h(2x+h)}{h}\\
&=2x+h,\ h\neq0
\end{aligned}
$$

b
$$
\begin{aligned}
f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to0}(2x+h)\\
&=2x
\end{aligned}
$$

c At (3,9) the value of x is 3

$\therefore f'(3)=2\times3$

$\therefore f'(3)=6$

The gradient of the tangent at the point $(3,9)$ is 6.

d A near neighbouring point to (3,9) is the point for which $x=3+h$ and $y=f(3+h)$.

The gradient of the secant through these two points is

$$
\begin{aligned}
m_{\text{secant}}&=\frac{f(3+h)-9}{3+h-3}\\
&=\frac{f(3+h)-f(3)}{h}
\end{aligned}
$$

since $9 = f(3)$ for this function $f(x) = x^2$.

The gradient of the tangent at $(3,9)$ is the limiting value the secant's gradient approaches as the two points become closer together.

Gradient of tangent at $(3,9)$ is $\lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$.

Evaluating this limit,

$$\begin{aligned}&\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}\\&=\lim_{h\to 0}\frac{(3+h)^2-9}{h}\\&=\lim_{h\to 0}\frac{9+6h+h^2-9}{h}\\&=\lim_{h\to 0}\frac{6h+h^2}{h}\\&=\lim_{h\to 0}\frac{h(6+h)}{h}\\&=\lim_{h\to 0}(6+h)\\&=6\\&=f'(3)\end{aligned}$$

8 a P $(1,5)$ and Q $\left(1+\delta x, 3(1+\delta x)^2+2\right)$ are on $y = 3x^2+2$.

The average rate of change of $y = 3x^2+2$ between P and Q

$$\begin{aligned}&=\frac{3(1+\delta x)^2+2-5}{1+\delta x-1}\\&=\frac{3\left(1+2\delta x+(\delta x)^2\right)-3}{\delta x}\\&=\frac{3+6\delta x+3(\delta x)^2-3}{\delta x}\\&=\frac{6\delta x+3(\delta x)^2}{\delta x}\\&=\frac{\delta x(6+3\delta x)}{\delta x}\\&=6+3\delta x,\ \delta x\neq 0\end{aligned}$$

b 6

c i When $x = 2, y = 14$ so S is the point $(2,14)$.

When $x = 2+\delta x$, $y = 3(2+\delta x)^2+2$ so T is the point $\left(2+\delta x, 3(2+\delta x)^2+2\right)$.

ii Gradient of secant through points S and T is

$$\begin{aligned}m_{\text{secant}}&=\frac{3(2+\delta x)^2+2-14}{2+\delta x-2}\\&=\frac{3\left(4+4\delta x+(\delta x)^2\right)-12}{\delta x}\\&=\frac{12+12\delta x+3(\delta x)^2-12}{\delta x}\\&=\frac{12\delta x+3(\delta x)^2}{\delta x}\\&=\frac{\delta x(12+3\delta x)}{\delta x}\\&=12+3\delta x,\ \delta x\neq 0\end{aligned}$$

iii Gradient of tangent at S = $\lim\limits_{\delta x\to 0}(12+3\delta x) = 12$.

9 a $\dfrac{f(x+h)-f(x)}{h}$ is the difference quotient representing the gradient of the secant through the points $(x, f(x))$ and $(x+h, f(x+h))$ on the curve $y = f(x)$.

b $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ is the gradient of the tangent to the curve $y = f(x)$ at the point $(x, f(x))$.

c $\lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$ is the gradient of the tangent to the curve $y = f(x)$ at the point $(3, f(3))$.

d $\dfrac{f(3)-f(3-h)}{h}$ is the gradient of the secant through the points $(3, f(3))$ and $(3-h, f(3-h))$ on the curve $y = f(x)$.

10 a $f(x) = 3x-2x^2$

$f(x+h) = 3(x+h)-2(x+h)^2$

By definition,

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to 0}\frac{\left[3(x+h)-2(x+h)^2\right]-\left[3x-2x^2\right]}{h}\\&=\lim_{h\to 0}\frac{3x+3h-2(x^2+2xh+h^2)-3x+2x^2}{h}\\&=\lim_{h\to 0}\frac{3h-4xh-2h^2}{h}\\&=\lim_{h\to 0}\frac{h(3-4x-2h)}{h}\\&=\lim_{h\to 0}(3-4x-2h)\\&=3-4x\end{aligned}$$

b The gradient of the tangent at $(0,0)$ is $f'(0) = 3$.

c $y = 3x-2x^2$

x intercepts: Let $y = 0$

$\therefore 3x-2x^2 = 0$

$\therefore x(3-2x) = 0$

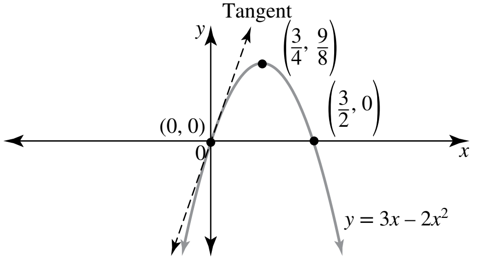
$\therefore x = 0, x = \dfrac{3}{2}$

Turning point: $x = \dfrac{0+\frac{3}{2}}{2} = \dfrac{3}{4}$

$$\begin{aligned}y&=3\times\frac{3}{4}-2\times\left(\frac{3}{4}\right)^2\\&=\frac{9}{4}-\frac{9}{8}\\&=\frac{9}{8}\end{aligned}$$

Maximum turning point is $\left(\dfrac{3}{4},\dfrac{9}{8}\right)$.

The tangent at $(0,0)$ can be drawn by eye. Alternatively, it has gradient 3 and passes through the origin so its equation is $y = 3x$.



11 **a** $f(x)=8x^2+2$

$$\begin{aligned}f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{\left[8(x+h)^2+2\right]-\left[8x^2+2\right]}{h}\\&=\lim_{h\to0}\frac{8x^2+16xh+2h^2+2-8x^2-2}{h}\\&=\lim_{h\to0}\frac{16xh+2h^2}{h}\\&=\lim_{h\to0}\frac{h(16x+2h)}{h}\\&=\lim_{h\to0}(16x+2h)\\&=16x\end{aligned}$$

b $f(x)=\frac{1}{2}x^2-4x-1$

$$\begin{aligned}f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{\left[\frac{1}{2}(x+h)^2-4(x+h)-1\right]-\left[\frac{1}{2}x^2-4x-1\right]}{h}\\&=\lim_{h\to0}\frac{\frac{1}{2}x^2+xh+\frac{1}{2}h^2-4x-4h-1-\frac{1}{2}x^2+4x+1}{h}\\&=\lim_{h\to0}\frac{xh+\frac{1}{2}h^2-4h}{h}\\&=\lim_{h\to0}\frac{h(x+\frac{1}{2}h-4)}{h}\\&=\lim_{h\to0}(x+\tfrac{1}{2}h-4)\\&=x-4\end{aligned}$$

c $f(x)=6-2x$

$$\begin{aligned}f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{[6-2(x+h)]-[6-2x]}{h}\\&=\lim_{h\to0}\frac{6-2x-2h-6+2x}{h}\\&=\lim_{h\to0}\frac{-2h}{h}\\&=\lim_{h\to0}(-2)\\&=-2\end{aligned}$$

d $f(x)=5$

$$\begin{aligned}f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{[5]-[5]}{h}\\&=\lim_{h\to0}\left(\frac{0}{h}\right)\\&=\lim_{h\to0}(0)\\&=0\end{aligned}$$

e $f(x)=x^3-6x^2+2x$

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{\left[(x+h)^3-6(x+h)^2+2(x+h)\right]-\left[x^3-6x^2+2x\right]}{h}\\
&=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-6x^2-12xh-6h^2+2x+2h-x^3+6x^2-2x}{h}\\
&=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-12xh-6h^2+2h}{h}\\
&=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2-12x-6h+2)}{h}\\
&=\lim_{h\to 0}(3x^2+3xh+h^2-12x-6h+2)\\
&=3x^2-12x+2
\end{aligned}$$

f $f(x)=2+x^6$ Binomial theorem or Pascal's triangle will need to be used in the working.

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{\left[2+(x+h)^6\right]-\left[2+x^6\right]}{h}\\
&=\lim_{h\to 0}\frac{2+x^6+6x^5h+15x^4h^2+20x^3h^3+15x^2h^4+6xh^5+h^6-2-x^6}{h}\\
&=\lim_{h\to 0}\frac{6x^5h+15x^4h^2+20x^3h^3+15x^2h^4+6xh^5+h^6}{h}\\
&=\lim_{h\to 0}\frac{h(6x^5+15x^4h+20x^3h^2+15x^2h^3+6xh^4+h^5)}{h}\\
&=\lim_{h\to 0}(6x^5+15x^4h+20x^3h^2+15x^2h^3+6xh^4+h^5)\\
&=6x^5
\end{aligned}$$

12 a $y=4-x^2$

$$\begin{aligned}
\frac{dy}{dx}&=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{-(x+\delta x)^2+x^2}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{-x^2-2x(\delta x)-(\delta x)^2+x^2}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{-2x(\delta x)-(\delta x)^2}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{\delta x(-2x-\delta x)}{\delta x}\\
&=\lim_{\delta x\to 0}(-2x-\delta x)\\
&=-2x
\end{aligned}$$

b $y=x^2+4x$

$y+\delta y=(x+\delta x)^2+4(x+\delta x)$

$\therefore \delta y=(x+\delta x)^2+4(x+\delta x)-(x^2+4x)$

$$\begin{aligned}
\frac{dy}{dx}&=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{(x+\delta x)^2+4(x+\delta x)-(x^2+4x)}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{x^2+2x(\delta x)+(\delta x)^2+4x+4(\delta x)-x^2-4x}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{2x(\delta x)+(\delta x)^2+4(\delta x)}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{\delta x(2x+\delta x+4)}{\delta x}\\
&=\lim_{\delta x\to 0}(2x+\delta x+4)\\
&=2x+4
\end{aligned}$$

c $y=\frac{x^3}{3}$

$$\begin{aligned}
y+\delta y&=\frac{(x+\delta x)^3}{3}\\
\therefore \delta y&=\frac{(x+\delta x)^3}{3}-\frac{x^3}{3}\\
\frac{dy}{dx}&=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{\frac{(x+\delta x)^3}{3}-\frac{x^3}{3}}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{\frac{1}{3}\left(x^3+3x^2(\delta x)+3x(\delta x)^2+(\delta x)^3-x^3\right)}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{3x^2(\delta x)+3x(\delta x)^2+(\delta x)^3}{3\delta x}\\
&=\lim_{\delta x\to 0}\frac{\delta x(3x^2+3x\delta x+(\delta x)^2)}{3\delta x}\\
&=\lim_{\delta x\to 0}\frac{(3x^2+3x\delta x+(\delta x)^2)}{3}\\
&=\frac{3x^2}{3}\\
&=x^2
\end{aligned}$$

d $y=x(x+1)=x^2+x$

$$\begin{aligned}
y+\delta y&=(x+\delta x)^2+(x+\delta x)\\
\therefore \delta y&=(x+\delta x)^2+(x+\delta x)-(x^2+x)\\
\frac{dy}{dx}&=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{(x+\delta x)^2+(x+\delta x)-(x^2+x)}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{x^2+2x(\delta x)+(\delta x)^2+x+\delta x-x^2-x}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{2x(\delta x)+(\delta x)^2+\delta x}{\delta x}\\
&=\lim_{\delta x\to 0}\frac{\delta x(2x+\delta x+1)}{\delta x}\\
&=\lim_{\delta x\to 0}(2x+\delta x+1)\\
&=2x+1
\end{aligned}$$

13 a $f(x)=(x+5)^2$

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{(x+h+5)^2-(x+5)^2}{h}\\
&=\lim_{h\to 0}\frac{[(x+h+5)-(x+5)][(x+h+5)+(x+5)]}{h}\\
&=\lim_{h\to 0}\frac{[h][2x+h+10]}{h}\\
&=\lim_{h\to 0}(2x+h+10)\\
&=2x+10
\end{aligned}$$

b $f'(-5)=2\times-5+10=0$

The gradient of the tangent at the point where $x=-5$ is zero. The tangent at this point is horizontal.

c At the y intercept, $x=0$.

$f'(0)=10$

The gradient of the tangent at the y intercept is 10.

d The instantaneous rate of change is measured by the gradient of the tangent.

At $(-2,9)$, $f'(-2)=-4+10=6$.

The instantaneous rate of change of the function at this point is 6.

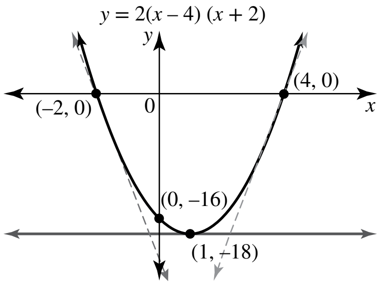
14 a $f(x)=2(x-4)(x+2)$

$f(0)=2(-4)(2)=-16$ so $(0,-16)$ is y intercept.

x intercepts occur at $x=4, x=-2$.

Turning point: $x=\frac{-2+4}{2}=1$, and $f(1)=2(-3)(3)=-18$.

Minimum turning point $(1,-18)$.



b $f(x)=2(x^2-2x-8)=2x^2-4x-16$

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{\left[2(x+h)^2-4(x+h)-16\right]-\left[2x^2-4x-16\right]}{h}\\
&=\lim_{h\to 0}\frac{2x^2+4xh+2h^2-4x-4h-16-2x^2+4x+16}{h}\\
&=\lim_{h\to 0}\frac{4xh+2h^2-4h}{h}\\
&=\lim_{h\to 0}\frac{h(4x+2h-4)}{h}\\
&=\lim_{h\to 0}(4x+2h-4)\\
&=4x-4
\end{aligned}$$

c $f'(1)=0$. The tangent at the turning point $(1,-18)$ is horizontal.

d The gradients of the tangents at the x intercepts are $f'(-2)=-12$ and $f'(4)=12$.

The tangent lines from parts **c** and **d** are shown on the diagram in part **a**.

15 a $f(x)=ax^2+bx+c$

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{\left[a(x+h)^2+b(x+h)+c\right]-\left[ax^2+bx+c\right]}{h}\\
&=\lim_{h\to 0}\frac{ax^2+2axh+ah^2+bx+bh+c-ax^2-bx-c}{h}\\
&=\lim_{h\to 0}\frac{2axh+ah^2+bh}{h}\\
&=\lim_{h\to 0}\frac{h(2ax+ah+b)}{h}\\
&=\lim_{h\to 0}(2ax+ah+b)\\
&=2ax+b
\end{aligned}$$

b The polynomial function $f:R\to R, f(x)=ax^2+bx+c$. Its derivative function is $f':R\to R, f'(x)=2ax+b$.

c For $f(x)=3x+4x+2$, $a=3, b=4, c=2$

$$\begin{aligned}
f'(x)&=2ax+b\\
\therefore f'(x)&=6x+4
\end{aligned}$$

16 a i $(2+h)^3-8=(2+h)^3-2^3$

$$\begin{aligned}&=((2+h)-2)\left((2+h)^2+(2+h)2+2^2\right)\\&=(h)(4+4h+h^2+4+2h+4)\\&=h(h^2+6h+12)\end{aligned}$$

ii $g(x)=x^3$

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

Hence, $g'(2)=\lim\limits_{h\to 0}\dfrac{g(2+h)-g(2)}{h}$

$$\therefore g'(2)=\lim_{h\to 0}\frac{(2+h)^3-8}{h}$$

iii Using the factorisation from part **i**,

$$\begin{aligned}g'(2)&=\lim_{h\to 0}\frac{h(h^2+6h+12)}{h}\\&=\lim_{h\to 0}(h^2+6h+12)\\&=12\end{aligned}$$

b i Let $f(x)=\dfrac{1}{x}, x\neq 0$

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to 0}\left[\left(\frac{1}{x+h}-\frac{1}{x}\right)\div h\right]\\&=\lim_{h\to 0}\left(\frac{x-x-h}{x(x+h)}\times\frac{1}{h}\right)\\&=\lim_{h\to 0}\frac{-h}{x(x+h)h}\\&=\lim_{h\to 0}\frac{-1}{x(x+h)}\\&=\frac{-1}{x^2}\end{aligned}$$

ii Let $f(x)=\dfrac{1}{x^2}, x\neq 0$

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to 0}\left[\left(\frac{1}{(x+h)^2}-\frac{1}{x^2}\right)\div h\right]\\&=\lim_{h\to 0}\left(\frac{x^2-(x+h)^2}{x^2(x+h)^2}\times\frac{1}{h}\right)\\&=\lim_{h\to 0}\left(\frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}\times\frac{1}{h}\right)\\&=\lim_{h\to 0}\frac{-2xh-h^2}{x^2(x+h)^2h}\\&=\lim_{h\to 0}\frac{h(-2x-h)}{x^2(x+h)^2h}\\&=\lim_{h\to 0}\frac{(-2x-h)}{x^2(x+h)^2}\\&=\frac{-2x}{x^2(x)^2}\\&=\frac{-2}{x^3}\end{aligned}$$

iii Let $f(x)=\sqrt{x}, x>0$

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\\&=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\&=\lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\&=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\&=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\&=\lim_{h\to 0}\frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}\\&=\frac{1}{\sqrt{x}+\sqrt{x}}\\&=\frac{1}{2\sqrt{x}}\end{aligned}$$

17 In the Main menu select Define from the Interactive options and complete

Func name: f
Variable/s: x
Expression: x^2

Use the limit template from 2 D CALC in the mth keyboard to enter $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$. The result is $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=2x$.

18 a Clear all variables then repeat the procedure in question **11** to define $f(x)=ax^3+bx^2+cx+d$ and to calculate the required limit.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=3ax^2+2bx+c$$

b Since $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$, $f'(x)=3ax^2+2bx+c$.

The degree of $f(x)=ax^3+bx^2+cx+d$ is three and the degree of its derivative is two.

Exercise 12.5 — Differentiation of polynomials by rule

1 a $\dfrac{d}{dx}\left(5x^8+\dfrac{1}{2}x^{12}\right)=40x^7+6x^{11}$

b Let $y=2t^3+4t^2-7t+12$

$$\frac{dy}{dt}=6t^2+8t-7$$

c
$$\begin{aligned}f(x)&=(2x+1)(3x-4)\\&=6x^2-5x-4\\f'(x)&=12x-5\end{aligned}$$

d
$$\begin{aligned}y&=\frac{4x^3-x^5}{2x^2}\\&=\frac{4x^3}{2x^2}-\frac{x^5}{2x^2}\\&=2x-\frac{x^3}{2}\\\frac{dy}{dx}&=2-\frac{3x^2}{2}\end{aligned}$$

2 $z=\frac{1}{420}(3x^5-3.5x^4+x^3-2x^2+12x-99)$

$$\begin{aligned}\frac{dz}{dx}&=\frac{1}{420}\times\frac{d}{dx}(3x^5-3.5x^4+x^3-2x^2+12x-99)\\&=\frac{1}{420}(15x^4-14x^3+3x^2-4x+12)\end{aligned}$$

3 $y=\frac{2x^3}{3}-x^2+3x-1$

a $\frac{dy}{dx}=2x^2-2x+3$

At the point $(6,125)$,

$$\begin{aligned}\frac{dy}{dx}&=2(6)^2-2(6)+3\\&=63\end{aligned}$$

Therefore, the rate of change is 63.

b $y=\frac{2x^3}{3}-x^2+3x-1$ over $x\in[0,6]$

When $x=0, y=-1\Rightarrow(0,-1)$ and when $x=6, y=125\Rightarrow(6,125)$

Average rate of change is $\frac{125-(-1)}{6-0}=21.$

c Gradient 3, so $\frac{dy}{dx}=3$

$$\begin{aligned}2x^2-2x+3&=3\\2x^2-2x&=0\\2x(x-1)&=0\\x&=0, x=1\end{aligned}$$

Substitute into equation of curve:

Points are $x=0, y=-1\Rightarrow(0,-1)$ and

$x=1, y=\frac{2}{3}-1+3-1\Rightarrow\left(1,\frac{5}{3}\right)$

4 $f(x)=(x-1)(x+2)$

$=x^2+x-2$

$f'(x)=2x+1$

The line $3x+3y=4$ has gradient -1.

Therefore, $f'(x)=-1$

$\therefore 2x+1=-1$

$\therefore x=-1$

When $x=-1$,

$f(x)=(-1-1)(-1+2)$

$=-2$

Therefore the point has coordinates $(-1,-2)$.

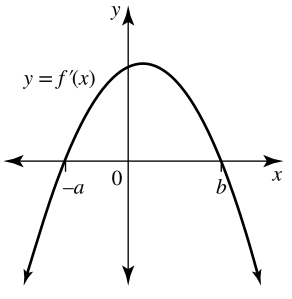
5 Turning points when $x=-a$ and $x=b\Rightarrow x=-a$ and $x=b$ are the x-intercepts of the gradient graph.

If $x<-a$, gradient is negative.

If $-a<x<b$, gradient is positive.

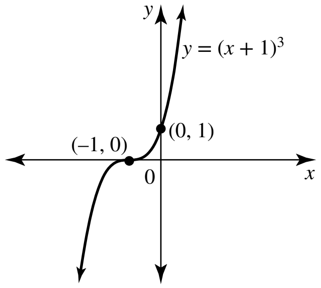
If $x>b$, gradient is negative.

This is a cubic graph, so gradient graph is a parabola with axis of symmetry $x=\frac{-a+b}{2}$.

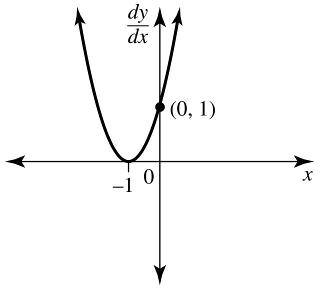


6 $y=(x+1)^3$

Stationary point of inflection at $(-1,0)$; y-intercept $(0,1)$



Gradient is positive everywhere except at $x=-1$ where the gradient is 0. Gradient graph is a parabola with a turning point at its x-intercept at $x=-1$.



7 **a** $f(x)=x^7$

$\therefore f'(x)=7x^{7-1}$

$\therefore f'(x)=7x^6$

b $y=3-2x^3$

$\therefore\frac{dy}{dx}=0-2\times3x^{3-1}$

$\therefore\frac{dy}{dx}=-6x^2$

c $\frac{d}{dx}\left(8x^2+6x-4\right)=16x+6$

d $f(x)=\frac{1}{6}x^3-\frac{1}{2}x^2+x-\frac{3}{4}$

$\therefore D_x(f)=\frac{1}{6}\times3x^2-\frac{1}{2}\times2x+1$

$\therefore D_x(f)=\frac{1}{2}x^2-x+1$

e $\frac{d}{du}\left(u^3-1.5u^2\right)=3u^2-3u$

f $z=4\left(1+t-3t^4\right)$

$\therefore\frac{dz}{dt}=4\left(1-12t^3\right)$

8 **a** Let $y=(2x+7)(8-x)$

Expand first

$\therefore y=-2x^2+9x+56$

Then differentiate

$\therefore\frac{dy}{dx}=-4x+9$

b Let $y=5x(3x+4)^2$

$\therefore y=5x(9x^2+24x+16)$

$\therefore y=45x^3+120x^2+80x$

$\frac{dy}{dx}=135x^2+240x+80$

c Let $y=(x-2)^4$

$\therefore y=x^4-4x^3(2)+6x^2(2)^2-4x(2)^3-(2)^4$

$\therefore y=x^4-8x^3+24x^2-32x+16$

$\frac{dy}{dx}=4x^3-24x^2+48x-32$

d Let $y = 250\left(3x + 5x^2 - 17x^3\right)$

$\therefore \frac{dy}{dx} = 250\left(3 + 10x - 51x^2\right)$

e Let $y = \frac{3x^2 + 10x}{x}$

$\therefore y = \frac{3x^2}{x} + \frac{10x}{x}$

$\therefore y = 3x + 10, \quad x \neq 0$

$\frac{dy}{dx} = 3$

f Let $y = \frac{4x^3 - 3x^2 + 10x}{2x}$

$\therefore y = \frac{4x^3}{2x} - \frac{3x^2}{2x} + \frac{10x}{2x}$

$\therefore y = 2x^2 - \frac{3}{2}x + 5, \quad x \neq 0$

$\frac{dy}{dx} = 4x - \frac{3}{2}$

9 a $y = 3x^2 - x + 4$

$\therefore \frac{dy}{dx} = 6x - 1$

When $x = \frac{1}{2}$,

$\frac{dy}{dx} = 6 \times \frac{1}{2} - 1$

$= 2$

The gradient of the tangent at the point where $x = \frac{1}{2}$ is 2.

b $f(x) = \frac{1}{6}x^3 + 2x^2 - 4x + 9$

$\therefore f'(x) = \frac{1}{6} \times 3x^2 + 4x - 4$

$\therefore f'(x) = \frac{1}{2}x^2 + 4x - 4$

At $(0,9)$, $f'(0) = -4$.

The gradient of the tangent at the point $(0,9)$ is -4.

c $y = (x+6)^2 - 2$ has turning point $(-6,-2)$.

Expanding the equation,

$y = x^2 + 12x + 36$

$\therefore \frac{dy}{dx} = 2x + 12$

At the point $(-6,-2)$,

$\frac{dy}{dx} = 2 \times -6 + 12$

$= 0$

The gradient of the tangent at the turning point is zero.

d $y = 2 - (x-3)^3$ has a stationary point of inflection at $(3,2)$.

Expending the equation,

$y = 2 - \left(x^3 - 3x^2(3) + 3x(3)^2 - (3)^3\right)$

$= 2 - x^3 + 9x^2 - 27x + 27$

$= 29 - x^3 + 9x^2 - 27x$

$\frac{dy}{dx} = -3x^2 + 18x - 27$

At the point $(3,2)$,

$\frac{dy}{dx} = -3(3)^2 + 18(3) - 27$

$= -27 + 54 - 27$

$= 0$

The gradient of the tangent at the stationary point of inflection is zero.

10 a $f(x) = 5x - \frac{3}{4}x^2$

$\therefore f'(x) = 5 - \frac{3}{4} \times 2x$

$\therefore f'(x) = 5 - \frac{3}{2}x$

$f'(-6) = 5 - \frac{3}{2} \times (-6)$

$= 5 + 9$

$= 14$

b $f'(0) = 5$

c $f'(x) = 0$

$\therefore 5 - \frac{3}{2}x = 0$

$\therefore 5 = \frac{3}{2}x$

$\therefore 10 = 3x$

$\therefore x = \frac{10}{3}$

The solution set is $\left\{\frac{10}{3}\right\}$.

d $f'(x) > 0$

$\therefore 5 - \frac{3}{2}x > 0$

$\therefore -\frac{3}{2}x > -5$

$\therefore -3x > -10$

$\therefore x < \frac{10}{3}$

The solution set is $\left\{x : x < \frac{10}{3}\right\}$.

e x intercepts: Let $f(x) = 0$

$\therefore 5x - \frac{3}{4}x^2 = 0$

$\therefore 20x - 3x^2 = 0$

$\therefore x(20 - 3x) = 0$

$\therefore x = 0, x = \frac{20}{3}$

Gradient at $x = 0$ is $f'(0) = 5$.

Gradient at $x = \frac{20}{3}$:

$f'\left(\frac{20}{3}\right) = 5 - \frac{3}{2} \times \frac{20}{3}$

$= 5 - 10$

$= -5$

The gradients of the curve at the points where it cuts the x axis are -5 and 5.

f If the gradient of the tangent is 11, then $f'(x) = 11$.

$\therefore 5 - \frac{3}{2}x = 11$

$\therefore -\frac{3}{2}x = 6$

$\therefore -3x = 12$

$\therefore x = -4$

Substitute $x = -4$ into the equation of the curve.

$f(-4) = 5 \times (-4) - \frac{3}{4} \times (-4)^2$

$= -20 - 12$

$= -32$

At the point $(-4,-32)$, the gradient of the tangent is 11.

11 a $f(x)=x^2-2x-3$

$\therefore f'(x)=2x-2$

Let $f'(x)=0$

$\therefore 2x-2=0$

$\therefore x=1$

$f(1)=1-2-3$

$\therefore f(1)=-4$

The gradient is zero at the point $(1,-4)$.

b The line $y=5-4x$ has gradient -4.

As the tangent is parallel to this line, the tangent has gradient -4.

Let $f'(x)=-4$

$\therefore 2x-2=-4$

$\therefore x=-1$

$f(-1)=1+2-3$

$\therefore f(-1)=0$

The tangent at the point $(-1,0)$ is parallel to the line $y=5-4x$.

c The line $x+y=7$ has gradient -1.

As the tangent is perpendicular to this line, the tangent has gradient 1.

Let $f'(x)=1$

$2x-2=1$

$\therefore x=\frac{3}{2}$

$f\left(\frac{3}{2}\right)=\frac{9}{4}-3-3$

$\therefore f\left(\frac{3}{2}\right)=-\frac{15}{4}$

The tangent at the point $\left(\frac{3}{2},-\frac{15}{4}\right)$ is perpendicular to the line $x+y=7$.

d At the point $(5,12)$,

$f'(5)=10-2$

$=8$

Require the point where $f'(x)=4$

$\therefore 2x-2=4$

$\therefore x=3$

$f(3)=9-6-3$

$\therefore f(3)=0$

The slope of the tangent at the point $(3,0)$ is half that of the tangent at the point $(5,12)$.

12 a $N=0.5\left(t^2+1\right)^2+2.5$

The instantaneous rate at which the population is changing is $\frac{dN}{dt}$.

$N=0.5\left(t^4+2t^2+1\right)+2.5$

$\therefore N=0.5t^4+t^2+3$

$\frac{dN}{dt}=2t^3+2t$

When $t=1$, $\frac{dN}{dt}=2+2=4$. The population is growing at 4 ants per hour when $t=1$.

When $t=2$,

$\frac{dN}{dt}=2\times 8+2\times 2$

$\therefore \frac{dN}{dt}=20$

The population is growing at 20 ants per hour when $t=2$.

b When $t=0$, $N=3$ and when $t=2$, $N=0.5\times 16+4+3=15$.

Average rate of change between $(0,3)$ and $(2,15)$ is

$\frac{15-3}{2-0}=6$ ants/hour.

c Let $\frac{dN}{dt}=60$

$\therefore 2t^3+2t=60$

$\therefore t^3+t-30=0$

Let $P(t)=t^3+t-30$

$P(3)=27+3-30=0$

$\therefore (t-3)$ is a factor

$\therefore t^3+t-30=(t-3)\left(t^2+3t+10\right)$

Hence, $t^3+t-30=0 \Rightarrow (t-3)\left(t^2+3t+10\right)=0$.

$\therefore t=3$ or $t^2+3t+10=0$

Consider $t^2+3t+10=0$

$\Delta=9-4\times 1\times 10$

$=-31$

<0

No real solutions.

$\therefore t=3$

The population of ants is increasing at 60 ants per hour 3 hours after the pot of honey is spilt.

13 a $y=4x^2+kx-5$

$\frac{dy}{dx}=8x+k$

Given that $\frac{dy}{dx}=4$ when $x=-2$.

$\therefore 4=8\times -2+k$

$\therefore k=20$

b $y=ax^2+bx+c$

$\frac{dy}{dx}=2ax+b$

At the turning point, the tangent is horizontal so $\frac{dy}{dx}=0$.

$\therefore 2ax+b=0$

$\therefore 2ax=-b$

$\therefore x=-\frac{b}{2a}$

c i $f(x)=x^3+9x^2+30x+c$

$\therefore f'(x)=3x^2+18x+30$

$=3\left(x^2+6x+10\right)$

$=3\left[\left(x^2+6x+9\right)-9+10\right]$

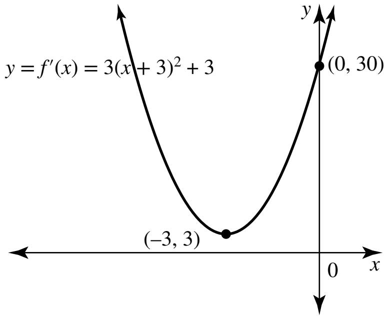
$=3\left[(x+3)^2+1\right]$

$=3(x+3)^2+3$

Since $(x+3)^2\geq 0$ then $f'(x)\geq 3$ and is therefore always positive.

ii Let $y=f'(x)$.

$y=3(x+3)^2+3$ is a parabola with minimum turning point $(-3,3)$ and y intercept $(0,30)$ since $f'(0)=30$.



d $r = at^2 + bt$

$\frac{dr}{dt} = 2at + b$

When $t = 1, \frac{dr}{dt} = 6$

$\therefore 6 = 2a + b....(1)$

When $t = 3, \frac{dr}{dt} = 14$

$\therefore 14 = 6a + b....(2)$

Equation (2) subtract equation (1)

$8 = 4a$

$\therefore a = 2$

Substitute $a = 2$ in equation (1)

$\therefore 6 = 4 + b$

$\therefore b = 2$

Hence, $a = 2, b = 2$.

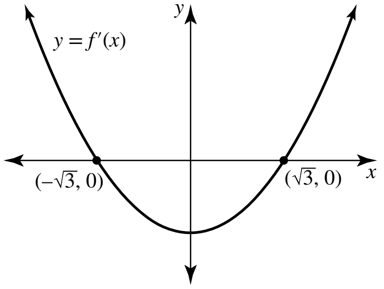
14 a Refer to the graph given in the question.

i $f'(x) = 0$ at the turning points. These occur at $x = \pm\sqrt{3}$.
The solution set is $\{\pm\sqrt{3}\}$.

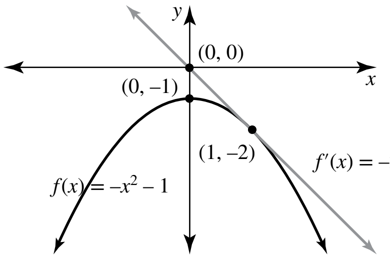
ii $f'(x) < 0$ where the tangent to the curve has a negative gradient. This occurs between the two turning points.
The solution set is $\{x : -\sqrt{3} < x < \sqrt{3}\}$.

iii $f'(x) > 0$ where the tangent to the curve has a positive gradient.
The solution set is $\{x : x < -\sqrt{3}\} \cup \{x : x > \sqrt{3}\}$.

b The graph of $y = f'(x)$ will have x intercepts at $x = \pm\sqrt{3}$.
The graph lies below the x axis for $\{x : -\sqrt{3} < x < \sqrt{3}\}$ and the graph lies above the x axis for $\{x : x < -\sqrt{3}\} \cup \{x : x > \sqrt{3}\}$.
A possible graph is shown.



15 a $f(x) = -x^2 - 1$. Parabola with maximum turning point at $(0, -1)$.
$f'(x) = -2x$. Straight line through the origin and the point $(1, -2)$.



b $f(x) = x^3 - x^2$

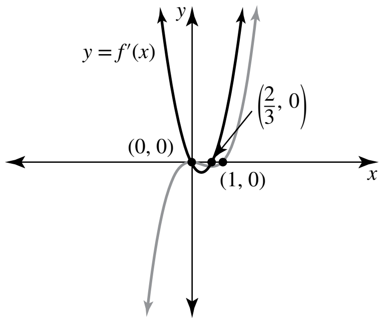
$\therefore f(x) = x^2(x - 1)$

Cubic graph touches the x axis at $(0, 0)$ and cuts the x axis at $(1, 0)$.

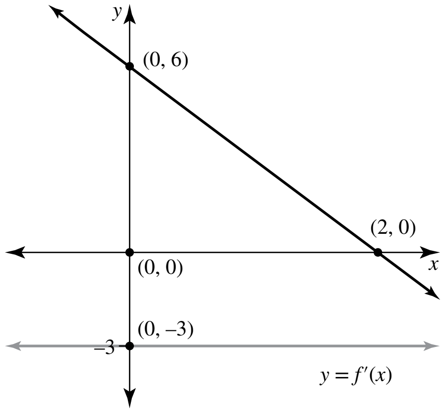
$f'(x) = 3x^2 - 2x$

$\therefore f'(x) = x(3x - 2)$

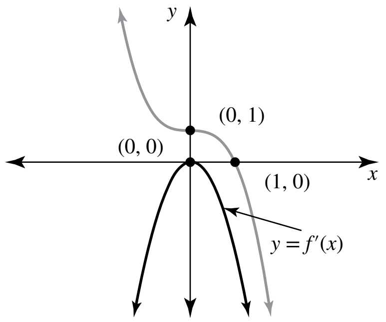
Quadratic graph cuts the x axis at $(0, 0)$ and $\left(\frac{2}{3}, 0\right)$.



c $f(x) = 6 - 3x$ is a straight line through $(0, 6)$ and $(2, 0)$ with gradient -3.
$f'(x) = -3$ is a horizontal line through $(0, -3)$.

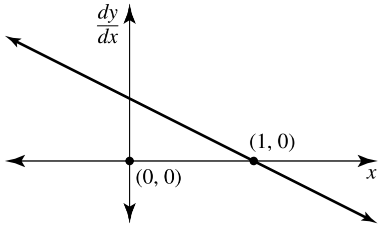


d $f(x) = 1 - x^3$. Cubic graph with stationary point of inflection at $(0, 1)$ and cutting x axis at $(1, 0)$.
$f'(x) = -3x^2$. Parabola with maximum turning point $(0, 0)$.

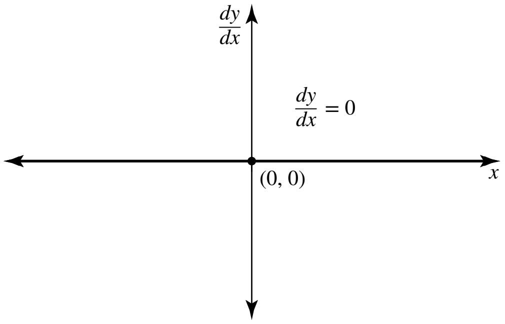


16 Refer to the graphs given in the question.

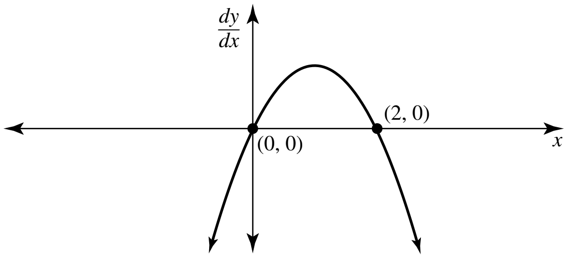
a Turning point at $(1, 4) \Rightarrow$ gradient graph has an x intercept when $x = 1$.
For $x < 1$, given graph has a positive gradient $\Rightarrow$ gradient graph lies above x axis.
For $x > 1$, given graph has a negative gradient $\Rightarrow$ gradient graph lies below x axis.
Given graph is degree 2 so gradient graph is degree 1 and the graph is linear.



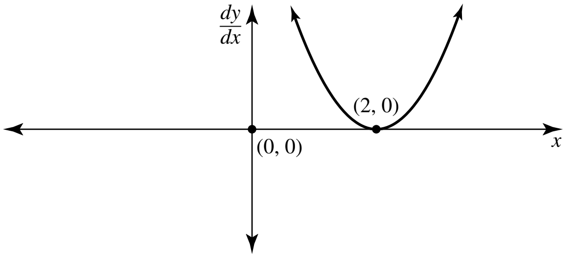
b The given graph is horizontal so its gradient is zero.



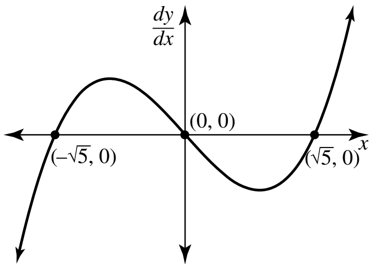
c The given graph has turning points at $(0,-2)$ and $(2,2)$. The gradient graph will have x intercepts at $x = 0, x = 2$.
For $x < 0$ the given graph has a negative gradient ⇒ the gradient graph lies below the x axis.
For $0 < x < 2$ the given graph has a positive gradient ⇒ the gradient graph lies above the x axis.
For $x > 2$ the given graph has a negative gradient ⇒ the gradient graph lies below the x axis.
The given graph has degree 3⇒ the gradient graph has degree 2.



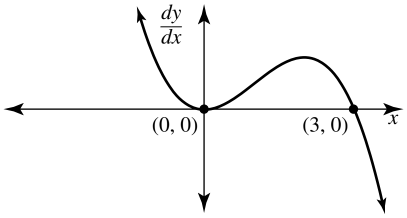
d The given graph has a stationary point of inflection at $(2,3)$. The gradient graph has an x intercept at $(2,0)$.
For $x < 2$ the given graph has a positive gradient ⇒ the gradient graph lies above the x axis.
For $x > 2$ the given graph has a positive gradient ⇒ the gradient graph lies above the x axis.
This means that the gradient graph must have a turning point at its x intercept $(2,0)$.



e The given graph has turning points at $\left(-\sqrt{5},-5\right),(0,3)$ and $\left(\sqrt{5},-5\right)$. The gradient graph will have x intercepts at $x = -\sqrt{5}, x = 0, x = \sqrt{5}$.
For $x < -\sqrt{5}$ the given graph has a negative gradient ⇒ the gradient graph lies below the x axis.
For $-\sqrt{5} < x < 0$ the given graph has a positive gradient ⇒ the gradient graph lies above the x axis.
For $0 < x < \sqrt{5}$ the given graph has a negative gradient ⇒ the gradient graph lies below the x axis.
For $x > \sqrt{5}$ the given graph has a positive gradient ⇒ the gradient graph lies above the x axis.
The given graph has degree 4⇒ the gradient graph has degree 3.



f The given graph has a stationary point of inflection at $(0,0)$ and a turning point at $(3,3)$. The gradient graph has x intercepts at $(0,0)$ and $(3,0)$.
For $x < 0$ the given graph has a positive gradient ⇒ the gradient graph lies above the x axis.
For $0 < x < 3$ the given graph has a positive gradient ⇒ the gradient graph lies above the x axis.
This means that the gradient graph must have a turning point at $(0,0)$.
For $x > 3$ the given graph has a negative gradient ⇒ the gradient graph lies below the x axis.
The given graph has degree 4 ⇒ the gradient graph has degree 3.



17 a The derivative template is in the 2 D CALC mth keyboard.

$$\frac{d}{dx}\left(x^3-2x\right)^{10} = 10\left(x^3-2x\right)^9\left(3x^2-2\right)$$

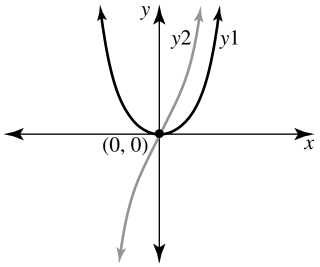
b Enter the function to be differentiated by tapping Interactive → Calculation → diff.

Tap Derivative at value
Expression: $\left(x^5-5x^8+2\right)^2$
Variable: x
Order: 1
Value: 1 (for the value of x at which the derivative is to be evaluated)
Press OK
The answer obtained is 140.

18 a $y = x^6 + 2x^2$

$\therefore \frac{dy}{dx} = 6x^5 + 4x$

In Graph&Tab enter:
$y1 = x^6 + 2x^2$
$y2 = 6x^5 + 4x$
and sketch. The graphs should be similar to that shown.



At a stationary point of inflection, the tangent to the curve is horizontal. To test whether the origin is a stationary point of inflection on the graph $y2$, use the Analysis tools to form the tangent to the curve at the origin. The tangent has a gradient of 4, not zero, and therefore the origin is not a stationary point of inflection on the derivative graph.

b If $\frac{dy}{dx} = y$ for $y = x^6 + 2x^2$, then intersections of the gradient graph and $y = x^6 + 2x^2$ will give the solutions to the equation $6x^5 + 4x = x^6 + 2x^2$.

c From the diagram it can be seen that the graph of $y = x^6 + 2x^2$ and its gradient graph intersect at the origin. In the main menu, solving $x^6 + 2x^2 = 6x^5 + 4x$ gives a second non zero solution of $x = 5.9938$.

d The graphs of $y = x^6 + 2x^2$ and its derivative will be parallel when the gradients of their tangents are the same.
For $y = x^6 + 2x^2$, the gradient of the tangent is $6x^5 + 4x$.
For $y = 6x^5 + 4x$, the gradient of its tangent is $30x^4 + 4$.
Solve $6x^5 + 4x = 30x^4 + 4$ to obtain $x = 4.9957$.
To the nearest integer, the graphs are parallel at the points where $x = 5$.

Topic 13 — Differentiation and applications

Exercise 13.2 — Limits, continuity and differentiability

1 a $\lim\limits_{x\to -5}(8-3x)=8+15$

$=23$

b $\lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right)$

$=\lim\limits_{x\to 3}\left(\dfrac{(x-3)(x+3)}{x-3}\right)$

$=\lim\limits_{x\to 3}(x+3)$

$=6$

c $\lim\limits_{x\to -3}\left(\dfrac{1}{x+3}\right)$ does not exist.

2 a $\lim\limits_{x\to 2}\left(\dfrac{x^3-8}{x-2}\right)$

$=\lim\limits_{x\to 2}\left(\dfrac{(x-2)(x^2+2x+4)}{x-2}\right)$

$=\lim\limits_{x\to 2}(x^2+2x+4)$

$=4+4+4$

$=12$

b $\lim\limits_{x\to 2}\left(\dfrac{1}{x+3}\right)$

$=\dfrac{1}{2+3}$

$=\dfrac{1}{5}$

3 a i $\lim\limits_{x\to 1}f(x)$

Limit from the left of $x=1$: $\lim\limits_{x\to 1}(x^2)=1$

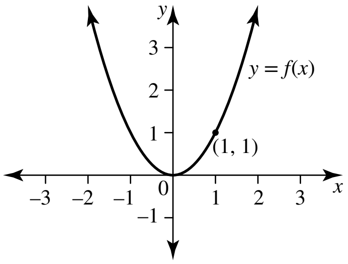
Limit from the right of $x=1$: $\lim\limits_{x\to 1}(2x-1)=1$

Since $L^-=L^+$, $\lim\limits_{x\to 1}f(x)=1$.

ii $f(1)=1$

The function is continuous at $x=1$ since $f(1)$ and $\lim\limits_{x\to 1}f(x)$ exist and $f(1)=\lim\limits_{x\to 1}f(x)$.

$$f(x)=\begin{cases}x^2, & x<1\\ 1, & x=1\\ 2x-1, & x>1\end{cases}$$



b i $g(x)=\dfrac{x^2-x}{x}$

Denominator cannot be zero, so maximal domain is $R\setminus\{0\}$.

ii $\lim\limits_{x\to 0}g(x)=\lim\limits_{x\to 0}\dfrac{x(x-1)}{x}$

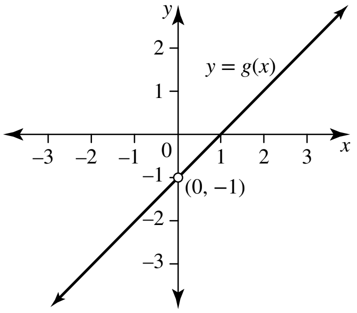
$=\lim\limits_{x\to 0}(x-1)$

$=-1$

iii The function is not continuous at $x=0$ because $g(0)$ is not defined.

$g(x)=x-1,\ \ x\neq 0$

iv Its graph will be the same as $y=x-1$ with a hole at $(0,-1)$.



4 $f(x)=\begin{cases}x^2-4, & x<0\\ 4-x^2, & x\geq 0\end{cases}$

$f(0)=4-0^2$

$=4$

$\lim\limits_{x\to 0}f(x)$:

Limit from the left:	Limit from the right:
$L^-=\lim\limits_{x\to 0}(x^2-4)$	$L^+=\lim\limits_{x\to 0}(4-x^2)$
$=-4$	$=4$

Since $L^-\neq L^+$, $\lim\limits_{x\to 0}f(x)$ does not exist.

Hence the function is not continuous at $x=0$. The branches of the function are polynomials so they are continuous except at their endpoint at $x=0$. Therefore the function is continuous for the domain $R\setminus\{0\}$.

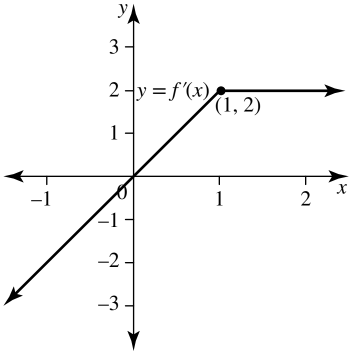
5 a As the function is continuous, test the derivative from the left and right of $x=1$.

Derivative from the left:	Derivative from the right:
$f(x)=x^2$	$f(x)=2x-1$
$\therefore f'(x)=2x$	$\therefore f'(x)=2$
$\therefore f'(1)=2$	$\therefore f'(1)=2$

Since the derivatives from each side are equal, the function is differentiable at $x=1$.

b $f'(x)=\begin{cases}2x, & x\leq 1\\ 2, & x>1\end{cases}$; domain is R.

The graph of $y=f'(x)$ is continuous.



6 $f(x)=\begin{cases} ax^2, & x \le 2 \\ 4x+b, & x > 2 \end{cases}$

To be continuous at $x=2$, the two branches must have the same endpoint when $x=2$.

Therefore, $a(2)^2=4(2)+b$

$\therefore 4a-b=8.....(1)$

To be smoothly continuous, the derivatives of the two branches must be equal when $x=2$.

From the left:	From the right:
$f(x)=ax^2$	$f(x)=4x+b$
$\therefore f'(x)=2ax$	$\therefore f'(x)=4$
$\therefore f'(2)=4a$	$\therefore f'(2)=4$

Hence, $4a=4$, giving $a=1$

Substitute into equation (1)

$4-b=8$

$\therefore b=-4$

Answer: $a=1, b=-4$

7 **a** $\lim\limits_{x\to 3}(6x-1)=6\times 3-1=17$

b $$\begin{aligned}&\lim_{x\to 3}\frac{2x^2-6x}{x-3}\\ &=\lim_{x\to 3}\frac{2x(x-3)}{x-3}\\ &=\lim_{x\to 3}(2x)\\ &=2\times 3\\ &=6\end{aligned}$$

c $$\begin{aligned}&\lim_{x\to 1}\frac{2x^2+3x-5}{x^2-1}\\ &=\lim_{x\to 1}\frac{(2x+5)(x-1)}{(x+1)(x-1)}\\ &=\lim_{x\to 1}\frac{2x+5}{x+1}\\ &=\frac{7}{2}\end{aligned}$$

d $$\begin{aligned}&\lim_{x\to 0}\frac{3x-5}{2x-1}\\ &=\frac{3(0)-5}{2(0)-1}\\ &=5\end{aligned}$$

e $$\begin{aligned}&\lim_{x\to -4}\frac{64+x^3}{x+4}\\ &=\lim_{x\to -4}\frac{(4+x)\left(16-4x+x^2\right)}{x+4}\\ &=\lim_{x\to -4}\left(16-4x+x^2\right)\\ &=16+16+16\\ &=48\end{aligned}$$

f $$\begin{aligned}&\lim_{x\to \infty}\frac{x+1}{x}\\ &=\lim_{x\to \infty}\left(\frac{x}{x}+\frac{1}{x}\right)\\ &=\lim_{x\to \infty}\left(1+\frac{1}{x}\right)\\ &=1+0\\ &=1\end{aligned}$$

(The hyperbola graph $y=1+\dfrac{1}{x}$ has an asymptote $y=1$, so as $x\to\infty, y\to 1$).

8 **a** $f(x)=\begin{cases} x^2, & x \le 2 \\ -2x, & x > 2 \end{cases}$

For $\lim\limits_{x\to 2} f(x)$ to exist, the limit from the left of $x=2$ must equal the limit from the right of $x=2$.

$$\begin{aligned}L^- &= \lim_{x\to 2^-} f(x) & L^+ &= \lim_{x\to 2^+} f(x)\\ &=\lim_{x\to 2}\left(x^2\right) & &=\lim_{x\to 2}(-2x)\\ &=4 & &=-4\end{aligned}$$

Since $L^- \ne L^+$, $\lim\limits_{x\to 2} f(x)$ does not exist.

b $f(x)=\begin{cases} (x-2)^2, & x < 2 \\ x-2, & x \ge 2 \end{cases}$

$$\begin{aligned}L^- &= \lim_{x\to 2^-} f(x) & L^+ &= \lim_{x\to 2^+} f(x)\\ &=\lim_{x\to 2}\left((x-2)^2\right) & &=\lim_{x\to 2}(x-2)\\ &=0 & &=0\end{aligned}$$

Since $L^-=L^+=0$, $\lim\limits_{x\to 2} f(x)=0.$

c $f(x)=\begin{cases} -x, & x < 2 \\ 0, & x=2 \\ x-4, & x > 2 \end{cases}$

$$\begin{aligned}L^- &= \lim_{x\to 2^-} f(x) & L^+ &= \lim_{x\to 2^+} f(x)\\ &=\lim_{x\to 2}(-x) & &=\lim_{x\to 2}(x-4)\\ &=-2 & &=-2\end{aligned}$$

Since $L^-=L^+=-2$, $\lim\limits_{x\to 2} f(x)=-2.$

d $f(x)=\dfrac{x^2-4}{x-2}$

$$\begin{aligned}&\lim_{x\to 2} f(x)\\ &=\lim_{x\to 2}\frac{\left(x^2-4\right)}{x-2}\\ &=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}\\ &=\lim_{x\to 2}(x+2)\\ &=4\end{aligned}$$

9 **a** Since $\lim\limits_{x\to 2} f(x)$ does not exist for the function in Q2a, the function is not continuous at $x=2$.

b For the function in Q2b, $\lim\limits_{x\to 2} f(x)=0.$

The value of the function at $x=2$ is $f(2)=2-2=0.$

$\therefore \lim\limits_{x\to 2} f(x)=f(2).$

The function is continuous at $x=2$.

c For the function in Q2c, $\lim\limits_{x\to 2} f(x)=-2.$

The value of the function at $x=2$ is $f(2)=0.$

$\therefore \lim\limits_{x\to 2} f(x)\ne f(2).$

The function is not continuous at $x=2$.

d For the function in Q2d, $\lim\limits_{x\to 2} f(x)=4.$

However, the value of the function at $x=2$ is not defined.

The function is not continuous at $x=2$.

10 **a** $f(x)=x^2+5x+2$ is a polynomial function so it is continuous over R.

b $g(x)=\dfrac{4}{x+2}$ is not defined when $x=-2$ so it is not continuous for all $x\in R$.

c $h(x)=\begin{cases} x^2+5x+2, & x<0 \\ 5x+2, & x>0 \end{cases}$ is not defined when $x=0$ so it is not continuous for all $x\in R$.

d $k(x)=\begin{cases} x^2+5x+2, & x<0 \\ \dfrac{4}{x+2}, & x\geq 0 \end{cases}$

The branch of the function for $x<0$ is a polynomial x^2+5x+2 so it is continuous for all $x<0$.

The branch of the function for $x\geq 0$ is $\dfrac{4}{x+2}$. Although this is not defined for $x=-2$, the value $x=-2$ is not in the domain for which this rule applies. So, for $x>0$ it is continuous.

Now we must test whether the function is continuous at $x=0$.

$$k(0)=\frac{4}{2}=2$$

$$\lim_{x\to 0}\left(x^2+5x+2\right)=2$$

$$\lim_{x\to 0}\left(\frac{4}{x+2}\right)=2$$

$$\therefore \lim_{x\to 0}k(x)=2=k(0)$$

The function is continuous at $x=0$.

Hence the function is continuous over R.

11 $y=\begin{cases} x+a, & x<1 \\ 4-x, & x\geq 1 \end{cases}$

Both branches are linear polynomials so they are each continuous. For the function to be continuous the two branches need to join at $x=1$.

When $x=1$ then $y=4-1=3$.

For continuity at $x=1$, $L^-=\lim\limits_{x\to 1}(x+a)=3$.

$$\therefore 1+a=3$$

$$\therefore a=2$$

12 $y=\begin{cases} ax+b, & x<-1 \\ 5, & -1\leq x\leq 2 \\ 2bx+a, & x>2 \end{cases}$

For continuity, $\lim\limits_{x\to -1}(ax+b)=5$ and $\lim\limits_{x\to 2}(2bx+a)=5$.

Hence,

$-a+b=5....(1)$

$4b+a=5....(2)$

Add the two equations together to eliminate a

$\therefore 5b=10$

$\therefore b=2$

Substitute $b=2$ in equation (1)

$\therefore -a+2=5$

$\therefore a=-3$

The function will be continuous if $a=-3$, $b=2$.

13 Refer to the diagram given in the question.

a There are breaks in the curve at x_3 and x_5, so the function is not continuous when $x=x_3$ and $x=x_5$. The left hand limit does not equal the right hand limit at each place.

b The function cannot be differentiated at a point of discontinuity so it is not differentiable at x_3 and x_5.

The function must be smoothly continuous to be differentiable. There are sharp points at x_1, x_2 and x_4 where the gradient from immediately left of the point does not equal the gradient immediately to the right. The function cannot be differentiated at these vales of x.

Overall, the function is not differentiable when $x=x_1, x_2, x_3, x_4, x_5$.

14 $f(x)=\begin{cases} 3-2x, & x<0 \\ x^2+3, & x\geq 0 \end{cases}$

a First test continuity at $x=0$.

$$L^-=\lim_{x\to 0}(3-2x)=3$$

$$L^+=\lim_{x\to 0}(x^2+3)=3$$

$$f(0)=0^2+3=3$$

$$\therefore \lim_{x\to 0}f(x)=f(0)$$

The function is continuous at $x=0$.

Derivative from the left of $x=0$:

$f(x)=3-2x$

$\therefore f'(x)=-2$

$\therefore f'(0)=-2$

Derivative from the right of $x=0$:

$f(x)=x^2+3$

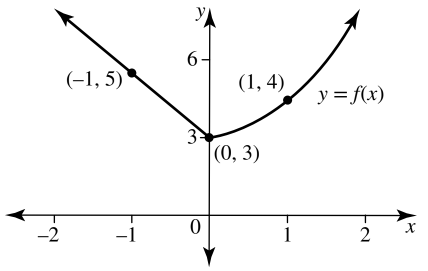
$\therefore f'(x)=2x$

$\therefore f'(0)=0$

Derivative from the left does not equal the derivative from the right.

The function is not differentiable at $x=0$.

b $f(0)=3, f(-1)=3+2=5$ and $f(1)=1^2+3=4$

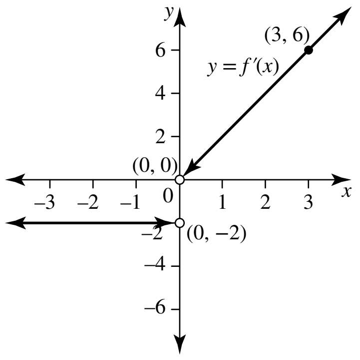


As the function is not differentiable at $x=0$, it is not smoothly continuous at $x=0$. Hence, the two branches do not join smoothly.

c $f'(x)=\begin{cases} -2, & x<0 \\ 2x, & x>0 \end{cases}$. The domain of the derivative is $R\setminus\{0\}$.

d $f'(3)=2\times 3=6$.

e



f $f(x)=\begin{cases} 3-2x, & x<0 \\ x^2-2x+3, & x\geq 0 \end{cases}$

part **a**: Test continuity at $x=0$.

$$L^-=\lim_{x\to 0}(3-2x)=3$$

$$L^+=\lim_{x\to 0}(x^2-2x+3)=3$$

$$f(0)=0^2-2\times 0+3=3$$

$$\therefore \lim_{x\to 0}f(x)=f(0)$$

The function is continuous at $x = 0$.
Derivative from the left of $x = 0$:

$f(x) = 3 - 2x$

$\therefore f'(x) = -2$

$\therefore f'(0) = -2$

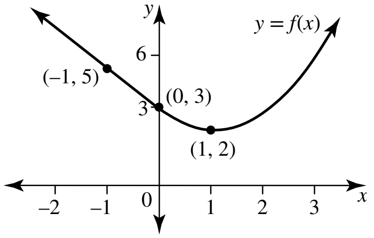
Derivative from the right of $x = 0$:

$f(x) = x^2 - 2x + 3$

$\therefore f'(x) = 2x - 2$

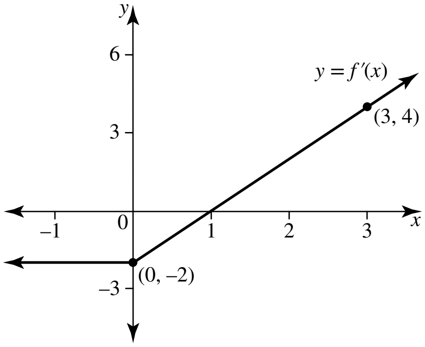
$\therefore f'(0) = -2$

Derivative from the left equals the derivative from the right.
The function is differentiable at $x = 0$.
part **b**: Left branch: $f(-1) = 3 + 2 = 5 \Rightarrow (-1, 5)$ is on the left linear branch.
Right branch: Minimum turning point at $(1, 2)$ since $y = x^2 - 2x + 3 \Rightarrow y = (x-1)^2 + 2$



As the function is differentiable at $x = 0$, it is smoothly continuous at $x = 0$. Hence, the two branches join smoothly.

part **c**: $f'(x) = \begin{cases} -2, & x < 0 \\ 2x - 2, & x \geq 0 \end{cases}$. The domain of the derivative is R.
part **d**: $f'(3) = 2 \times 3 - 2 = 4$.
part **e**:



15 **a** $f(x) = \begin{cases} 4x^2 - 5x + 2, & x \leq 1 \\ -x^3 + 3x^2, & x > 1 \end{cases}$

First test if continuous at $x = 1$.

$$f(1) = 4 - 5 + 2$$
$$= 1$$
$$L^+ = \lim_{x \to 1}\left(-x^3 + 3x^2\right)$$
$$= -1 + 3$$
$$= 2$$

The function is not continuous at $x = 1$ and therefore it is not differentiable at $x = 1$.

b $f'(x) = \begin{cases} 8x - 5, & x < 1 \\ -3x^2 + 6x, & x > 1 \end{cases}$

c $f'(x) = 0$
For $x < 1$, let $8x - 5 = 0$

$\therefore x = \frac{5}{8}$

For $x > 1$, let $-3x^2 + 6x = 0$

$\therefore -3x(x - 2) = 0$

$\therefore x = 0$ or $x = 2$

However, $x > 1$ so the only admissible value is $x = 2$.

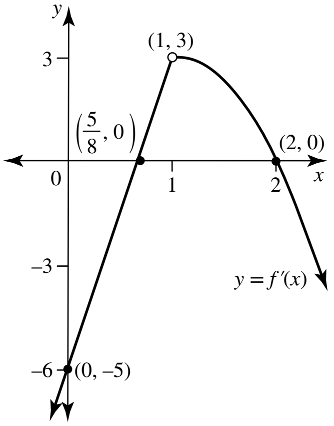
$f'(x) = 0$ when $x = \frac{5}{8}$ or $x = 2$

d $f'(x) = \begin{cases} 8x - 5, & x < 1 \\ -3x^2 + 6x, & x > 1 \\ \text{not defined}, & x = 1 \end{cases}$

Consider the linear left branch of the gradient function:
$y = 8x - 5$ passes through $(0, -5)$ and $\left(\frac{5}{8}, 0\right)$.
The endpoint $(1, 3)$ is open.
Consider the quadratic right branch of the gradient function:
$y = -3x^2 + 6x$ passes through $(2, 0)$.

$$y = -3\left[\left(x^2 - 2x + 1\right) - 1\right]$$
$$= -3(x-1)^2 + 3$$

Maximum turning point $(1, 3)$ is an open endpoint.



e Since $f'(x) = 0$ when $x = \frac{5}{8}$ or $x = 2$, at these points on $y = f(x)$ the gradient of the tangent is zero.

$$f\left(\frac{5}{8}\right) = 4 \times \left(\frac{5}{8}\right)^2 - 5 \times \frac{5}{8} + 2$$
$$= \frac{25}{16} - \frac{25}{8} + 2$$
$$= \frac{-25}{16} + \frac{32}{16}$$
$$= \frac{7}{16}$$
$$f(2) = -8 + 12$$
$$= 4$$

At the points $\left(\frac{5}{8}, \frac{7}{16}\right)$ and $(2, 4)$, the gradient of the tangent is zero.

f Consider the quadratic left branch of the function:
$y = 4x^2 - 5x + 2$ passes through $(0, 2)$.
Its minimum turning point is $\left(\frac{5}{8}, \frac{7}{16}\right)$ since the tangent there has zero gradient.
Its closed endpoint is $(1, 1)$.

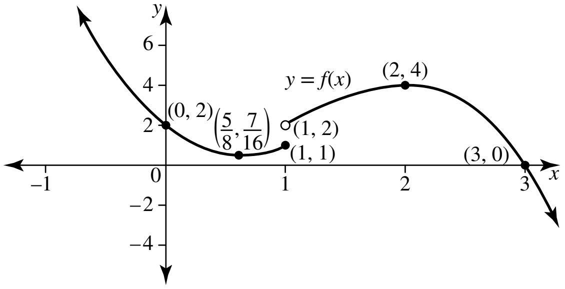
Consider the cubic right branch of the function:

$y = -x^3 + 3x^2$

$\therefore y = -x^2(x-3)$

Graph passes through $(3, 0)$ and has an open endpoint at $(1, 2)$.

There is a turning point at $(2, 4)$ since the gradient of the tangent is zero at that point.



16 $y = \begin{cases} ax^2 + b, & x \le 1 \\ 4x, & 1 < x < 2 \\ cx^2 + d, & x \ge 2 \end{cases}$

For continuity at the join of the two branches around $x = 1$,

$a \times 1^2 + b = 4 \times 1$

$\therefore a + b = 4....(1)$

For continuity at the join of the two branches around $x = 2$,

$4 \times 2 = c \times 2^2 + d$

$\therefore 8 = 4c + d....(2)$

For the joins to be smooth, the derivatives either side of each join must be equal.

For $x = 1$, $2ax = 4$ when $x = 1$

$\therefore 2a = 4$

$\therefore a = 2$

For $x = 2$, $4 = 2cx$ when $x = 2$

$\therefore 4 = 4c$

$\therefore c = 1$

Substitute $a = 2$ in equation (1)

$\therefore 2 + b = 4$

$\therefore b = 2$

Substitute $c = 1$ in equation (2)

$\therefore 8 = 4 + d$

$\therefore d = 4$

For the function to be differentiable over R,

$a = 2,\ b = 2,\ c = 1,\ d = 4.$

17 Using the limit template in the 2 DCALC mth keyboard,

a $\lim\limits_{x \to 1} \dfrac{x^{25} - 1}{x - 1} = 25$

b $\lim\limits_{x \to 1} \dfrac{x^n - 1}{x - 1} = n$

18 $f(x) = \begin{cases} x^2, & x < 2 \\ 2^x, & x \ge 2 \end{cases}$

a
$$L^- = \lim_{x \to 2}(x^2) = 4$$
$$L^+ = \lim_{x \to 2}(2^x) = 4$$
$$f(2) = 2^2 = 4$$
$$\therefore \lim_{x \to 2} f(x) = 2 = f(2)$$

hence, the function is continuous when $x = 2$.

b The derivative from the left of $x = 2$ is

$$f'(2^-) = \lim_{h \to 0} \frac{f(2) - f(2-h)}{2 - (2-h)}$$
$$= \lim_{h \to 0} \frac{4 - (2-h)^2}{h}$$

c The derivative from the right of $x = 2$ is

$$f'(2^+) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{2 + h - 2}$$
$$= \lim_{h \to 0} \frac{2^{2+h} - 4}{h}$$

d Using the limit template, $f'(2^-) = 4$ and $f'(2^+) = 4\ln(2)$. Since $4 \neq 4\log_e(2)$, the derivative from the left does not equal the derivative from the right. Hence, the function is not differentiable at $x = 2$.

Exercise 13.3 — Derivatives of power functions

1 a $y = \dfrac{4 - 3x + 7x^4}{x^4}$

$$= \frac{4}{x^4} - \frac{3x}{x^4} + \frac{7x^4}{x^4}$$
$$= 4x^{-4} - 3x^{-3} + 7$$
$$\frac{dy}{dx} = -16x^{-5} + 9x^{-4}$$
$$= -\frac{16}{x^5} + \frac{9}{x^4}$$

The derivative has x terms in its denominator so its domain is $R \backslash \{0\}$.

b i $f(x) = 4\sqrt{x} + \sqrt{2x}$

$$= 4\sqrt{x} + \sqrt{2}\sqrt{x}$$
$$= 4x^{\frac{1}{2}} + \sqrt{2}x^{\frac{1}{2}}$$
$$f'(x) = 2x^{-\frac{1}{2}} + \frac{\sqrt{2}}{2}x^{-\frac{1}{2}}$$
$$= \frac{2}{\sqrt{x}} + \frac{\sqrt{2}}{2\sqrt{x}}$$

The derivative has $\sqrt{x}$ terms in its denominator so its domain is R^+.

ii At $x = 1$, gradient is $f'(1)$

$$f'(1) = \frac{2}{1} + \frac{\sqrt{2}}{2}$$
$$= \frac{4 + \sqrt{2}}{2}$$

2 $\dfrac{d}{dx}\left(6x^{\frac{2}{3}} + \dfrac{1}{\sqrt{x}}\right)$

$$= \frac{d}{dx}(6x^{\frac{2}{3}} + x^{-\frac{1}{2}})$$
$$= 4x^{-\frac{1}{3}} - \frac{1}{2}x^{-\frac{3}{2}}$$
$$= \frac{4}{x^{\frac{1}{3}}} - \frac{1}{2x^{\frac{3}{2}}}$$

3 a Let $y = 4x^{-1} + 5x^{-2}$

$$\frac{dy}{dx} = 4 \times (-1)x^{-1-1} + 5 \times (-2)x^{-2-1}$$
$$= -4x^{-2} - 10x^{-3}$$

b Let $y = 4x^{\frac{1}{2}} - 3x^{\frac{2}{3}}$

$$\frac{dy}{dx} = 4 \times \frac{1}{2}x^{\frac{1}{2}-1} - 3 \times \frac{2}{3}x^{\frac{2}{3}-1}$$
$$= 2x^{-\frac{1}{2}} - 2x^{-\frac{1}{3}}$$

c Let $y=2+8x^{-\frac{1}{2}}$

$$\frac{dy}{dx}=8\times\frac{-1}{2}x^{-\frac{1}{2}-1}$$
$$=-4x^{-\frac{3}{2}}$$

d Let $y=0.5x^{1.8}-6x^{3.1}$

$$\frac{dy}{dx}=0.5\times1.8x^{1.8-1}-6\times3.1x^{3.1-1}$$
$$=0.9x^{0.8}-18.6x^{2.1}$$

4 a $y=1+\frac{1}{x}+\frac{1}{x^2}$

$$\therefore y=1+x^{-1}+x^{-2}$$
$$\frac{dy}{dx}=-1x^{-2}-2x^{-3}$$
$$=-\frac{1}{x^2}-\frac{2}{x^3}$$

b $y=5\sqrt{x}$

$$\therefore y=5x^{\frac{1}{2}}$$
$$\frac{dy}{dx}=5\times\frac{1}{2}x^{-\frac{1}{2}}$$
$$=\frac{5}{2x^{\frac{1}{2}}}$$
$$=\frac{5}{2\sqrt{x}}$$

c $y=3x-\sqrt[3]{x}$

$$\therefore y=3x-x^{\frac{1}{3}}$$
$$\frac{dy}{dx}=3-\frac{1}{3}x^{-\frac{2}{3}}$$
$$=3-\frac{1}{3x^{\frac{2}{3}}}$$

d $y=x\sqrt{x}$

$$\therefore y=x\times x^{\frac{1}{2}}$$
$$\therefore y=x^{\frac{3}{2}}$$
$$\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}$$
$$=\frac{3\sqrt{x}}{2}$$

5 a $f(x)=\frac{3x^2+5x-9}{3x^2}$

$$\therefore f(x)=\frac{3x^2}{3x^2}+\frac{5x}{3x^2}-\frac{9}{3x^2}$$
$$\therefore f(x)=1+\frac{5}{3}x^{-1}-3x^{-2}$$
$$f'(x)=-\frac{5}{3}x^{-2}+6x^{-3}$$
$$=-\frac{5}{3x^2}+\frac{6}{x^3}$$

b $f(x)=\left(\frac{x}{5}+\frac{5}{x}\right)^2$

$$\therefore f(x)=\frac{x^2}{25}+2\times\frac{x}{5}\times\frac{5}{x}+\frac{25}{x^2}$$
$$=\frac{1}{25}x^2+2+25x^{-2}$$
$$f'(x)=\frac{2}{25}x-50x^{-3}$$
$$=\frac{2x}{25}-\frac{50}{x^3}$$

c $f(x)=\sqrt[5]{x^2}+\sqrt{5x}+\frac{1}{\sqrt{x}}$

$$\therefore f(x)=\left(x^2\right)^{\frac{1}{5}}+\sqrt{5}\times\sqrt{x}+\frac{1}{x^{\frac{1}{2}}}$$
$$\therefore f(x)=x^{\frac{2}{5}}+\sqrt{5}x^{\frac{1}{2}}+x^{-\frac{1}{2}}$$
$$f'(x)=\frac{2}{5}x^{-\frac{3}{5}}+\sqrt{5}\times\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$$
$$=\frac{2}{5x^{\frac{3}{5}}}+\frac{\sqrt{5}}{2\sqrt{x}}-\frac{1}{2x^{\frac{3}{2}}}$$

d $f(x)=2x^{\frac{3}{4}}\left(4+x-3x^2\right)$

$$\therefore f(x)=8x^{\frac{3}{4}}+2x^{\frac{7}{4}}-6x^{\frac{11}{4}}$$
$$f'(x)=8\times\frac{3}{4}x^{-\frac{1}{4}}+2\times\frac{7}{4}x^{\frac{3}{4}}-6\times\frac{11}{4}x^{\frac{7}{4}}$$
$$=6x^{-\frac{1}{4}}+\frac{7}{2}x^{\frac{3}{4}}-\frac{33}{2}x^{\frac{7}{4}}$$
$$=\frac{6}{x^{\frac{1}{4}}}+\frac{7x^{\frac{3}{4}}}{2}-\frac{33}{2}x^{\frac{7}{4}}$$

6 $f:[0,\infty)\to R, f(x)=4-\sqrt{x}$

a $f(x)=4-\sqrt{x}$

$$\therefore f(x)=4-x^{\frac{1}{2}}$$
$$\therefore f'(x)=-\frac{1}{2}x^{-\frac{1}{2}}$$
$$\therefore f'(x)=-\frac{1}{2\sqrt{x}}$$

The domain of the function f is $[0,\infty)$. However, as $f'(0)$ is not defined, the domain of the derivative function is $(0,\infty)$. The derivative function f' is $f':(0,\infty)\to R, f'(x)=-\frac{1}{2\sqrt{x}}$.

b $y=f(x)$ has an x intercept when $y=0$.

$$\therefore 4-\sqrt{x}=0$$
$$\therefore \sqrt{x}=4$$
$$\therefore x=16$$
$$f'(16)=-\frac{1}{2\sqrt{16}}$$
$$\therefore f'(16)=-\frac{1}{8}$$

The gradient of the graph of $y=f(x)$ is $-\frac{1}{8}$ at its x intercept.

c Let $x = 0.0001$

$$f'(0.0001) = -\frac{1}{2\sqrt{0.0001}}$$
$$= -\frac{1}{2\times 0.01}$$
$$= -50$$

The gradient of the tangent at $x = 0.0001$ is -50.

Let $x = 10^{-10}$

$$f'\left(10^{-10}\right) = -\frac{1}{2\sqrt{10^{-10}}}$$
$$= -\frac{1}{2\times 10^{-5}}$$
$$= -\frac{10^5}{2}$$
$$= -50\,000$$

The gradient of the tangent at $x = 10^{-10}$ is $-50\,000$.

d As $x \to 0, f'(x) \to -\infty$.

e Since $f'(x) \to -\infty$ as $x \to 0$, the tangent to the curve $y = f(x)$ at $(0,4)$ is undefined.

7 $y = 1 - \frac{3}{x}$

a Let $y = 0$

$$\therefore 1 - \frac{3}{x} = 0$$
$$\therefore 1 = \frac{3}{x}$$
$$\therefore x = 3$$

The x intercept is $(3,0)$.

Gradient: $y = 1 - 3x^{-1}$

$$\therefore \frac{dy}{dx} = 3x^{-2}$$
$$= \frac{3}{x^2}$$

At $(3,0)$, $\frac{dy}{dx} = \frac{3}{9} = \frac{1}{3}$.

The gradient of the tangent is $\frac{1}{3}$ so $g = \frac{1}{3}$.

b Let $\frac{dy}{dx} = \frac{1}{3}$

$$\therefore \frac{3}{x^2} = \frac{1}{3}$$
$$\therefore 9 = x^2$$
$$\therefore x = \pm 3$$

When $x = -3$, $y = 1 - \frac{3}{-3} = 2$.

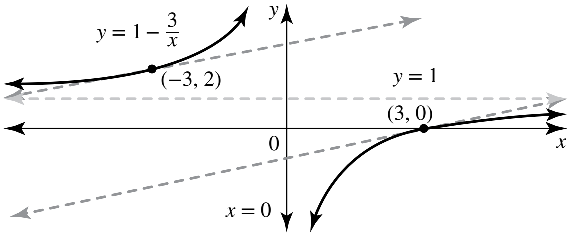
The other point where the gradient is $\frac{1}{3}$ is at $(-3,2)$

c $y = 1 - \frac{3}{x}$

Asymptotes: $x = 0, y = 1$

x intercept: $(3,0)$

Point: $(-3,2)$



d Let $x = 10$

$$\frac{dy}{dx} = \frac{3}{10^2}$$
$$= 3\times 10^{-2}$$

Let $x = 10^3$

$$\frac{dy}{dx} = \frac{3}{\left(10^3\right)^2}$$
$$= \frac{3}{10^6}$$
$$= 3\times 10^{-6}$$

As $x \to \infty, \frac{dy}{dx} \to 0$. The tangent to the graph becomes closer to being horizontal and approaches the horizontal asymptote as $x \to \infty$.

8 a i $f: R\setminus\{0\} \to R, f(x) = x - \frac{1}{x}$. The domain of the function is $R\setminus\{0\}$.

ii $f(x) = x - \frac{1}{x}$

$$\therefore f(x) = x - x^{-1}$$
$$f'(x) = 1 + x^{-2}$$
$$= 1 + \frac{1}{x^2}$$

The gradient function has domain $R\setminus\{0\}$ and rule $f'(x) = 1 + \frac{1}{x^2}$.

iii At the point $(1,0)$, $f'(1) = 1 + \frac{1}{1^2} = 2$. The gradient of the tangent at the point $(1,0)$ is 2.

iv Let $f'(x) = 5$

$$\therefore 1 + \frac{1}{x^2} = 5$$
$$\therefore \frac{1}{x^2} = 4$$
$$\therefore x^2 = \frac{1}{4}$$
$$\therefore x = \pm\frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{\frac{1}{2}}$$
$$= \frac{1}{2} - 2$$
$$= -\frac{3}{2}$$

$$f\left(-\frac{1}{2}\right) = -\frac{1}{2} + 2$$
$$= \frac{3}{2}$$

The tangents at the points $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and $\left(-\frac{1}{2}, \frac{3}{2}\right)$ have a gradient of 5.

b i $f(x) = x^2 + \frac{2}{x}$

$$\therefore f(x) = x^2 + 2x^{-1}$$
$$f'(x) = 2x - 2x^{-2}$$
$$= 2x - \frac{2}{x^2}$$
$$\therefore f'(2) = 4 - \frac{2}{4}$$
$$\therefore f'(2) = 3.5$$

ii Let $f'(x)=0$

$$\begin{aligned}
&\therefore 2x-\frac{2}{x^2}=0\\
&\therefore 2x=\frac{2}{x^2}\\
&\therefore x^3=1\\
&\therefore x=1
\end{aligned}$$

$f(1)=1+2=3$

At the point $(1,3)$, $f'(x)=0$.

iii Let $f'(x)=-4$

$$\begin{aligned}
&\therefore 2x-\frac{2}{x^2}=-4\\
&\therefore 2x^3-2=-4x^2\\
&\therefore x^3+2x^2-1=0
\end{aligned}$$

Let $P(x)=x^3+2x^2-1$

$P(-1)=-1+2-1=0$

$\therefore (x+1)$ is a factor

$$\begin{aligned}
\therefore x^3+2x^2-1&=(x+1)\left(x^2+x-1\right)=0\\
\therefore x&=-1 \text{ or } x^2+x-1=0\\
\therefore x&=-1 \text{ or } x=\frac{-1\pm\sqrt{1+4}}{2}\\
\therefore x&=-1 \text{ or } x=\frac{-1\pm\sqrt{5}}{2}
\end{aligned}$$

9 $f(x)=\begin{cases}(2-x)^2, & x<4\\ 2+\sqrt{x}, & x\ge 4\end{cases}$

a

$$\begin{array}{ll}
L^-=\lim\limits_{x\to 4^-} f(x) & L^+=\lim\limits_{x\to 4^+} f(x)\\
=\lim\limits_{x\to 4}(2-x)^2 & =\lim\limits_{x\to 4}(2+\sqrt{x})\\
=(-2)^2 & =2+\sqrt{4}\\
=4 & =4
\end{array}$$

Since $L^-=L^+=4$, $\lim\limits_{x\to 4} f(x)=4$.

b $f(4)=2+\sqrt{4}=4$. Since $\lim\limits_{x\to 4} f(x)=f(4)$, the function is continuous at $x=4$.

c Test whether the derivative from the left of 4 is equal to the derivative from the right of 4.

Derivative from the left:

$$\begin{aligned}
f(x)&=(2-x)^2\\
&=4-4x+x^2\\
\therefore f'(x)&=-4+2x\\
\therefore f'(4^-)&=-4+2\times 4\\
\therefore f'(4^-)&=4
\end{aligned}$$

Derivative from the right:

$$\begin{aligned}
f(x)&=2+\sqrt{x}\\
&=2+x^{\frac{1}{2}}\\
\therefore f'(x)&=\frac{1}{2}x^{-\frac{1}{2}}\\
\therefore f'(x)&=\frac{1}{2\sqrt{x}}\\
\therefore f'(4^+)&=\frac{1}{2\times\sqrt{4}}\\
\therefore f'(4^+)&=\frac{1}{4}
\end{aligned}$$

Since $f'(4^-)\ne f'(4^+)$, the function is not differentiable at $x=4$.

d $f'(x)=\begin{cases}-4+2x, & x<4\\ \dfrac{1}{2\sqrt{x}}, & x>4\end{cases}$

e $f'(0)=-4+2\times 0=-4$

f $f'(x)<0$

Since $\frac{1}{2\sqrt{x}}>0$ then solution can only come from considering $-4+2x<0$.

$$\begin{aligned}
&\therefore 2x<4\\
&\therefore x<2
\end{aligned}$$

Therefore $f'(x)<0$ when $x<2$.

10 $h=0.5+\sqrt{t}$

a Let $t=0$

$\therefore h=0.5$

The tree was 0.5 metres when first planted.

b $h=0.5+t^{\frac{1}{2}}$

$$\begin{aligned}
\therefore \frac{dh}{dt}&=\frac{1}{2}t^{-\frac{1}{2}}\\
\therefore \frac{dh}{dt}&=\frac{1}{2\sqrt{t}}
\end{aligned}$$

When $t=4$,

$$\begin{aligned}
\frac{dh}{dt}&=\frac{1}{2\sqrt{4}}\\
&=\frac{1}{4}
\end{aligned}$$

After 4 years, the height is growing at 0.25 metres per year.

c Let $h=3$

$$\begin{aligned}
\therefore 0.5+\sqrt{t}&=3\\
\therefore \sqrt{t}&=2.5\\
\therefore t&=6.25
\end{aligned}$$

The tree is 3 metres tall, $6\frac{1}{4}$ years after it was planted.

d $t=0, h=0.5$ and $t=6.25, h=3$

Average rate of growth

$$\begin{aligned}
&=\frac{3-0.5}{6.25-0}\\
&=\frac{2.5}{6.25}\\
&=\frac{1}{2.5}\\
&=0.4
\end{aligned}$$

The average rate of growth is 0.4 metres per year.

11 a

$$\begin{aligned}
&(x+h)^3-x^3\\
&=\left[(x+h)-x\right]\left[(x+h)^2+x(x+h)+x^2\right]\\
&=h\left(x^2+2xh+h^2+x^2+xh+x^2\right)\\
&=h\left(3x^2+3xh+h^2\right)
\end{aligned}$$

b $f(x)=\frac{1}{x^3}$

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{\frac{1}{(x+h)^3}-\frac{1}{x^3}}{h}\\
&=\lim_{h\to 0}\left(\frac{1}{(x+h)^3}-\frac{1}{x^3}\right)\div h\\
&=\lim_{h\to 0}\left(\frac{x^3-(x+h)^3}{(x+h)^3x^3}\right)\times\frac{1}{h}\\
&=\lim_{h\to 0}\frac{-\left((x+h)^3-x^3\right)}{(x+h)^3x^3h}
\end{aligned}$$

Using the result from part **a**,

$$f'(x)=\lim_{h\to 0}\frac{-h\left(3x^2+3xh+h^2\right)}{(x+h)^3x^3h}$$
$$=\lim_{h\to 0}\frac{-\left(3x^2+3xh+h^2\right)}{(x+h)^3x^3}$$
$$=\frac{-\left(3x^2\right)}{(x)^3x^3}$$
$$=-\frac{3x^2}{x^6}$$
$$=-\frac{3}{x^4}$$

c $f(x)=x^{-3}$

$\therefore f'(x)=-3x^{-4}$

$\therefore f'(x)=-\dfrac{3}{x^4}$

d **i** $\dfrac{d}{dx}\left(x^3+\dfrac{1}{x^3}\right)$

$$=\frac{d}{dx}\left(x^3+x^{-3}\right)$$
$$=3x^2-3x^{-4}$$
$$=3x^2-\frac{3}{x^4}$$

ii $\dfrac{d}{dx}\left(x-\dfrac{1}{x}\right)^3$

$$=\frac{d}{dx}\left(x^3-3x^2\times\frac{1}{x}+3x\times\frac{1}{x^2}-\frac{1}{x^3}\right)$$
$$=\frac{d}{dx}\left(x^3-3x+\frac{3}{x}-\frac{1}{x^3}\right)$$
$$=\frac{d}{dx}\left(x^3-3x+3x^{-1}-x^{-3}\right)$$
$$=3x^2-3-3x^{-2}+3x^{-4}$$
$$=3x^2-3-\frac{3}{x^2}+\frac{3}{x^4}$$

12 **a** $V=\dfrac{60t+2}{3t}, t>0$

$\therefore V=\dfrac{60t}{3t}+\dfrac{2}{3t}$

$\therefore V=20+\dfrac{2}{3}t^{-1}$

$$\frac{dV}{dt}=-\frac{2}{3}t^{-2}$$
$$=-\frac{2}{3t^2}$$

Since $t^2>0$, $-\dfrac{2}{3t^2}<0$. Hence $\dfrac{dV}{dt}<0$.

b Let $t=2$

$$\frac{dV}{dt}=-\frac{2}{3(2)^2}$$
$$=-\frac{1}{6}$$

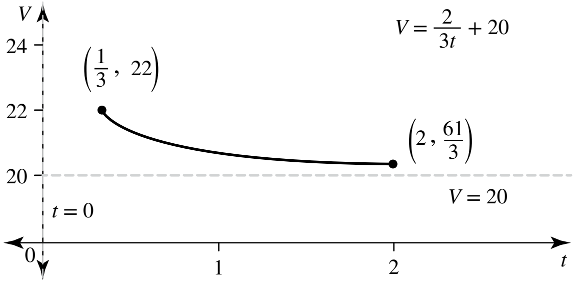
The water is evaporating at $\dfrac{1}{6}$ mL per hour.

c $V=\dfrac{2}{3t}+20,\ t\in\left[\dfrac{1}{3},2\right]$

First quadrant branch of hyperbola with asymptotes $t=0,\ V=20$.

Endpoints: When $t=\dfrac{1}{3}$, $V=2+20=22$. Point $\left(\dfrac{1}{3},22\right)$.

When $t=2$, $V=\dfrac{1}{3}+20=\dfrac{61}{3}$. Point $\left(2,\dfrac{61}{3}\right)$.



d Gradient of chord with endpoints $\left(\dfrac{1}{3},22\right)$ and $\left(2,\dfrac{61}{3}\right)$

$$=\frac{\frac{61}{3}-22}{2-\frac{1}{3}}$$
$$=\left(\frac{61}{3}-\frac{66}{3}\right)\div\left(\frac{6}{3}-\frac{1}{3}\right)$$
$$=-\frac{5}{3}\times\frac{3}{5}$$
$$=-1$$

This value measures the average rate of evaporation over the interval $t\in\left[\dfrac{1}{3},2\right]$.

13 In the main menu, tap Interactive → Calculation → diff and tap 'Derivative at value'.

Enter

Expression: $4-\dfrac{1}{3x-2}$

Variable: x

Order: 1

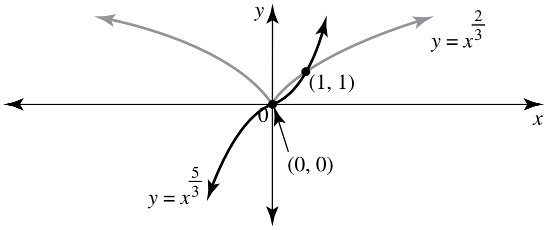
Value: $\dfrac{1}{3}$

and tap OK.

The gradient of the tangent is 3.

14 **a** The graphs of $y=x^{\frac{2}{3}}$ and $y=x^{\frac{5}{3}}$ will both pass through the points $(0,0)$ and $(1,1)$.

The graphs obtained should be similar to that shown in the diagram.



b Using the Analysis tools in Graph&Tab, the slopes of the tangents at $(1,1)$ can be deduced from the equation of the tangent.

At $(1,1)$, $y=x^{\frac{2}{3}}$ has gradient $\dfrac{2}{3}$ and $y=x^{\frac{5}{3}}$ has gradient $\dfrac{5}{3}$, making the graph of $y=x^{\frac{5}{3}}$ steeper than that of $y=x^{\frac{2}{3}}$ at the point $(1,1)$.

At $(0,0)$, the gradient of $y=x^{\frac{5}{3}}$ is zero. The gradient of $y=x^{\frac{2}{3}}$ is undefined. The tangent to $y=x^{\frac{2}{3}}$ is vertical while the tangent to $y=x^{\frac{5}{3}}$ is horizontal.

Exercise 13.4 — Coordinate geometry applications of differentiation

1 $y=5x-\frac{1}{3}x^3$

Point: $x=3$,

$$y=15-9$$
$$=6$$

Point is $(3,6)$.

Gradient: $\frac{dy}{dx}=5-x^2$

When $x=3$, $\frac{dy}{dx}=-4$, so gradient is -4.

Equation of tangent:

$$y-6=-4(x-3)$$
$$y=-4x+18$$

2 $y=\frac{4}{x^2}+3$

Gradient: as tangent is parallel to $y=-8x$, gradient is -8.

Point:

$$y=4x^{-2}+3$$
$$\frac{dy}{dx}=-8x^{-3}$$
$$=\frac{-8}{x^3}$$

As tangent has gradient -8,

$$\frac{-8}{x^3}=-8$$
$$x^3=1$$
$$x=1$$

Substitute $x=1$ into equation of curve.

$$y=\frac{4}{(1)^2}+3$$
$$=7$$

Point is $(1,7)$.

Equation of tangent:

$$y-7=-8(x-1)$$
$$y=-8x+15$$

3 **a** $y=x^2$ so $\frac{dy}{dx}=2x$

When $x=-1$, $\frac{dy}{dx}=-2$

The line perpendicular to tangent has a gradient of $\frac{1}{2}$ and a point $(-1,1)$.

Its equation is:

$$y-1=\frac{1}{2}(x+1)$$
$$y=\frac{1}{2}x+\frac{3}{2}$$

At Q,

$$x^2=\frac{1}{2}x+\frac{3}{2}$$
$$2x^2-x-3=0$$
$$(2x-3)(x+1)=0$$
$$x=\frac{3}{2},x=-1$$

At Q, $x=\frac{3}{2}$ and therefore $y=\frac{9}{4}$, so Q is the point $\left(\frac{3}{2},\frac{9}{4}\right)$.

b The line perpendicular to the tangent has a gradient of $\frac{1}{2}$.

Therefore, the required acute angle satisfies $\tan\theta=\frac{1}{2}$.

$$\theta=\tan^{-1}\left(\frac{1}{2}\right)$$
$$\approx 26.6°$$

To one decimal place, the angle of inclination with the x-axis is 26.6°.

4 $y=\frac{1}{x}$ so $\frac{dy}{dx}=-\frac{1}{x^2}$

When $x=\frac{1}{2}$, $y=2$, $\frac{dy}{dx}=-4$,

equation of tangent is:

$$y-2=-4\left(x-\frac{1}{2}\right)$$
$$y=-4x+4$$

A is point $(1,0)$ and B is point $(0,4)$.

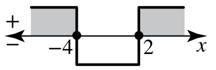
Therefore midpoint of AB is $\left(\frac{1}{2},2\right)$.

5 **a** $f(x)=\frac{1}{3}x^3+x^2-8x+6$

$$f'(x)=x^2+2x-8$$

Function is increasing when $f'(x)>0$, so:

$$x^2+2x-8>0$$
$$(x+4)(x-2)>0$$
$$x<-4 \text{ or } x>2$$



Interval over which the function is increasing is $(-\infty,-4)\cup(2,\infty)$.

b **i** $y=ax^2+4x+5$

Point: $x=1$, $y=a+9\Rightarrow(1,a+9)$

Gradient: $\frac{dy}{dx}=2ax+4$

When $x=1$, $\frac{dy}{dx}=2a+4$

Equation of tangent:

$$y-(a+9)=(2a+4)(x-1)$$
$$y=(2a+4)x-2a-4+a+9$$
$$=(2a+4)x+5-a$$

ii If the function is decreasing, the gradient of the tangent is negative.

Hence, $2a+4<0$

$\therefore a<-2$

6 $f(x)=10-\frac{2}{5x}$

$$f(x)=10-\frac{2}{5}x^{-1}$$
$$f'(x)=\frac{2}{5}x^{-2}$$
$$=\frac{2}{5x^2}$$

The function is increasing if $f'(x)>0$.

As $\frac{2}{5x^2}$ is positive for $x\neq 0$ then the function increases at all points in its domain.

The function is increasing for $x\in R\setminus\{0\}$.

7 $\frac{1}{3}x^3+7x-3=0, x=2$

Let $f(x)=\frac{1}{3}x^3+7x-3$

$=> f'(x)=x^2+7$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$\therefore x=x_0-\frac{f(x_n)}{f'(x_0)}$$

Let $x_0=2$

$$f(2)=\frac{1}{3}(2)^3+7(2)-3$$
$$=\frac{8}{3}+11$$
$$=\frac{41}{3}$$

$$f'(2)=2^2+7$$
$$=11$$

$$x_1=2-\frac{\frac{41}{3}}{11}$$
$$=\frac{25}{33}$$

$$f\left(\frac{25}{33}\right)=\frac{1}{3}\left(\frac{25}{33}\right)^3+7\left(\frac{25}{33}\right)-3$$
$$=2.45$$

$$f'\left(\frac{25}{33}\right)=\left(\frac{25}{33}\right)^2+7$$
$$=7.57$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$
$$=\frac{25}{33}-\frac{2.45}{7.57}$$
$$=0.43$$

Similarly, x_3 = 0.4249 (correct to 4 decimal places).

8 a Let $f(x) = -x^3 + x^2 - 3x + 5$.

when $x \le 1, f(x) > 0$

when $x \ge 2, f(x) < 0$

Since the polynomial changes sign between $x = 1$ and $x = 2$, there is a root which lies between $x = 1$ and $x = 2$.

b Let $f(x)=-x^3+x^2-3x+5$

$=>f'(x)=-3x^2+2x-3$

Let $x_0=1$

$$f(1)=-(1)^3+(1)^2-3(1)+5$$
$$=2$$

$$f'(1)=-3(1)^2+2(1)-3$$
$$=-4$$

$$=> x_1=1-\frac{2}{-4}$$
$$=1.5$$

$$f(1.5)=-(1.5)^3+(1.5)^2-3(1.5)+5$$
$$=-0.625$$

$$f'(1.5)=-3(1.5)^2+2(1.5)-3$$
$$=-6.75$$

$$=> x_2=1.5-\frac{-0.63}{-6.75}$$
$$=1.41$$

$$f(1.41)=-(1.41)^3+(1.41)^2-3(1.41)+5$$
$$=-0.0451$$

$$f'(1.41)=-3(1.41)^2+2(1.41)-3$$
$$=-6.1443$$

$$=> x_3=1.41-\frac{-0.0451}{-6.1443}$$
$$=1.4026$$

9 a $y=2x^2-7x+3$

$\frac{dy}{dx}=4x-7$

At $(0,3)$,

$\frac{dy}{dx}=4\times 0-7$

$=-7$

Equation of tangent:

$y-y_1=m(x-x_1),\ m=-7,(x_1,y_1)=(0,3)$

$\therefore y-3=-7x$

$\therefore y=-7x+3$

b $y=5-8x-3x^2$

Point: $(-1,10)$

Gradient: $\frac{dy}{dx}=-8-6x$

At $(-1,10)$, $\frac{dy}{dx}=-8-6\times -1=-2$

Equation of tangent: $y-10=-2(x+1)$

$\therefore y=-2x+8$

c $y=\frac{1}{2}x^3$

Point: $(2,4)$

Gradient: $\frac{dy}{dx}=\frac{3}{2}x^2$

At $(2,4)$, $\frac{dy}{dx}=\frac{3}{2}\times 2^2=6$

Equation of tangent: $y-4=6(x-2)$

$\therefore y=6x-8$

d $y=\frac{1}{3}x^3-2x^2+3x+5$

Point: $(3,5)$

Gradient: $\frac{dy}{dx}=x^2-4x+3$

At $(3,5)$, $\frac{dy}{dx}=3^2-4\times 3+3=0$ so tangent is horizontal.

Equation of tangent: $y=5$

e $y=\frac{6}{x}+9 \Rightarrow y=6x^{-1}+9$

Point: $\left(-\frac{1}{2},-3\right)$

Gradient: $\frac{dy}{dx}=-6x^{-2}$

$\therefore \frac{dy}{dx}=-\frac{6}{x^2}$

At $\left(-\frac{1}{2},-3\right)$,

$$\frac{dy}{dx}=-6\div\left(-\frac{1}{2}\right)^2$$
$$=-6\times 4$$
$$=-24$$

Equation of tangent: $y+3=-24\left(x+\frac{1}{2}\right)$

$\therefore y=-24x-15$

f $y = 38 - 2x^{\frac{3}{4}}$

Point: $(81, -16)$

Gradient: $\frac{dy}{dx} = -2 \times \frac{3}{4} x^{-\frac{1}{4}}$

$\therefore \frac{dy}{dx} = -\frac{3}{2x^{\frac{1}{4}}}$

At $(81, -16)$,

$$\frac{dy}{dx} = -\frac{3}{2(81)^{\frac{1}{4}}}$$
$$= -\frac{3}{2 \times 3}$$
$$= -\frac{1}{2}$$

Equation of tangent: $y + 16 = -\frac{1}{2}(x - 81)$

$\therefore 2y + 32 = -x + 81$

$\therefore 2y + x = 49$

10 a $y = 4\sqrt{x}$

Point: When $x = 4$, $y = 4\sqrt{4} = 8$

Point is $(4, 8)$.

Gradient: $y = 4x^{\frac{1}{2}}$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}}$$
$$= \frac{2}{\sqrt{x}}$$

At $(4, 8)$, $\frac{dy}{dx} = \frac{2}{\sqrt{4}} = 1$

Equation of tangent: $y - 8 = 1(x - 4)$

$\therefore y = x + 4$

b $y = \sqrt{2x}$

Point: When $x = 2$, $y = \sqrt{4} = 2$

Point is $(2, 2)$.

Gradient: $y = \sqrt{2}x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{\sqrt{2}}{2} x^{-\frac{1}{2}}$$
$$= \frac{\sqrt{2}}{2\sqrt{x}}$$

At $(2, 2)$, $\frac{dy}{dx} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$

Gradient of line perpendicular to the tangent is -2

Equation of this line: $y - 2 = -2(x - 2)$

$\therefore y = -2x + 6$

c $y = 6\sqrt{x}$

$\therefore y = 6x^{\frac{1}{2}}$

$\frac{dy}{dx} = 3x^{-\frac{1}{2}}$

$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{x}}$

The tangent has gradient $m = \tan(60°)$

$\therefore m = \sqrt{3}$

Let $\frac{dy}{dx} = \sqrt{3}$

$\therefore \frac{3}{\sqrt{x}} = \sqrt{3}$

$\therefore \frac{3}{\sqrt{3}} = \sqrt{x}$

$\therefore x = \frac{9}{3}$

$\therefore x = 3$

When $x = 3$, $y = 6\sqrt{3}$

Therefore the tangent is drawn to the curve at the point $(3, 6\sqrt{3})$, It has gradient $\sqrt{3}$.

The equation of the tangent is

$y - 6\sqrt{3} = \sqrt{3}(x - 3)$

$\therefore y = \sqrt{3}x - 3\sqrt{3} + 6\sqrt{3}$

$\therefore y = \sqrt{3}x + 3\sqrt{3}$

11 $y = x^2 - 6x + 3$

a As tangent is parallel to $y = 4x - 2$, the gradient of the tangent is 4.

$\frac{dy}{dx} = 2x - 6$

Let $\frac{dy}{dx} = 4$

$\therefore 2x - 6 = 4$

$\therefore 2x = 10$

$\therefore x = 5$

When $x = 5$, $y = 25 - 30 + 3 = -2$

The point is $(5, -2)$.

The equation of the tangent is

$y + 2 = 4(x - 5)$

$\therefore y = 4x - 22$

b If tangent is parallel to the x axis, its gradient is zero.

Let $\frac{dy}{dx} = 0$

$\therefore 2x - 6 = 0$

$\therefore x = 3$

When $x = 3$, $y = 9 - 18 + 3 = -6$

The tangent is the horizontal line through the point $(3, -6)$.

Its equation is $y = -6$.

c The line $6y + 3x - 1 = 0$ has gradient $-\frac{3}{6} = -\frac{1}{2}$.

The tangent is perpendicular to this line so the gradient of the tangent is 2.

Let $\frac{dy}{dx} = 2$

$\therefore 2x - 6 = 2$

$\therefore 2x = 8$

$\therefore x = 4$

When $x = 4$, $y = 16 - 24 + 3 = -5$

Point is $(4, -5)$.

Equation of tangent is

$y + 5 = 2(x - 4)$

$\therefore y = 2x - 13$

12 a $y = x(4 - x)$

The x intercepts are $(0, 0)$ and $(4, 0)$.

$y = 4x - x^2$

$\therefore \frac{dy}{dx} = 4 - 2x$

At $(0, 0)$, $\frac{dy}{dx} = 4$ and at $(4, 0)$, $\frac{dy}{dx} = -4$.

Equation of tangent at $(0,0)$ is $y = 4x$.

Equation of tangent at $(4,0)$ is $y = -4(x-4)$

$\therefore y = -4x + 16$

At the intersection of the tangent lines $y = 4x$ and $y = -4x + 16$,

$4x = -4x + 16$

$\therefore 8x = 16$

$\therefore x = 2$

Substitute $x = 2$ in $y = 4x$

$\therefore y = 8$

The tangents intersect at $(2,8)$.

b $y = (x-a)(x-b)$

x intercepts are $(a,0)$ and $(b,0)$.

$y = x^2 - ax - bx + ab$

$\therefore \frac{dy}{dx} = 2x - a - b$

At $(a,0)$,

$\frac{dy}{dx} = 2a - a - b$

$= a - b$

The equation of the tangent is $y = (a-b)(x-a)$.

At $(b,0)$,

$\frac{dy}{dx} = 2b - a - b$

$= b - a$

The equation of the tangent is $y = (b-a)(x-b)$.

At the intersection of the tangents,

$(a-b)(x-a) = (b-a)(x-b)$.

For two tangents to exist, $a \neq b$.

Dividing by $(a-b)$ gives

$x - a = -(x-b)$

$\therefore x - a = -x + b$

$\therefore 2x = a + b$

$\therefore x = \frac{a+b}{2}$

The axis of symmetry of a parabola lies midway between its x intercepts.

Therefore, the equation of the axis of symmetry is $x = \frac{a+b}{2}$, which is the same as the x co-ordinate of the point of intersection of the tangents drawn at the x intercepts.

Therefore the tangents intersect at some point on the parabola's axis of symmetry.

c For the curve $y = x(4-x)$, the point A has co-ordinates $(4,0)$.

From part **a**, the equation of the tangent to the curve at A is $y = -4x + 16$.

The line through A perpendicular to the tangent has gradient $\frac{1}{4}$, and its equation is

$y = \frac{1}{4}(x-4)$

$\therefore y = \frac{1}{4}x - 1$

This line intersects $y = x(4-x)$ when $\frac{1}{4}x - 1 = x(4-x)$.

$\therefore x - 4 = 4x(4-x)$

$\therefore x - 4 = 16x - 4x^2$

$\therefore 4x^2 - 15x - 4 = 0$

$\therefore (4x+1)(x-4) = 0$

$\therefore x = -\frac{1}{4}$ or $x = 4$

$x = 4$ is point A, so $x = -\frac{1}{4}$ gives the x co-ordinate of the point where $y = \frac{1}{4}x - 1$ again meets the parabola.

When $x = -\frac{1}{4}$, $y = -\frac{1}{16} - 1 = -\frac{17}{16}$

The point is $\left(-\frac{1}{4}, -\frac{17}{16}\right)$.

The gradient of the tangent to $y = x(4-x)$ at the point $\left(-\frac{1}{4}, -\frac{17}{16}\right)$ is

$\frac{dy}{dx} = 4 - 2x$

$= 4 - 2 \times -\frac{1}{4}$

$= \frac{9}{2}$

The equation of the tangent is $y + \frac{17}{16} = \frac{9}{2}\left(x + \frac{1}{4}\right)$

$\therefore 16y + 17 = 72\left(x + \frac{1}{4}\right)$

$\therefore 16y + 17 = 72x + 18$

$\therefore 16y - 72x = 1$

$16y = 72x + 1$

$y = \frac{72}{16}x + \frac{1}{16}$

$y = \frac{9}{2}x + \frac{1}{16}$

13 a $f(x) = 3 - 7x + 4x^2$

$f'(x) = -7 + 8x$

The function is decreasing when $f'(x) < 0$

$\therefore -7 + 8x < 0$

$\therefore 8x < 7$

$\therefore x < \frac{7}{8}$

The function is decreasing over the interval $x \in \left(-\infty, \frac{7}{8}\right)$.

b $y = -12x + x^3$

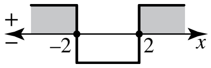
$\frac{dy}{dx} = -12 + 3x^2$

The function is increasing when $\frac{dy}{dx} > 0$

$\therefore -12 + 3x^2 > 0$

$\therefore x^2 - 4 > 0$

$\therefore (x+2)(x-2) > 0$



$\therefore x < -2$ or $x > 2$

The function is increasing over the interval $x \in (-\infty, -2) \cup (2, \infty)$.

c **i** $y = x^3 + 3x + 5$

$\frac{dy}{dx} = 3x^2 + 3$

Since $x^2 \geq 0$, $3x^2 + 3 \geq 3$

$\therefore \frac{dy}{dx} > 0$ for any value of $x \in R$.

Therefore, any tangent will have a positive gradient.

ii When $x = -1$, $y = -1 - 3 + 5 = 1$.

At the point $(-1,1)$, $\frac{dy}{dx} = 3 + 3 = 6$

The equation of the tangent is
$y-1=6(x+1)$
$\therefore y=6x+7$

d **i** $y=\dfrac{2-x^2}{x}, x\neq 0$
$\therefore y=\dfrac{2}{x}-\dfrac{x^2}{x}$
$\therefore y=2x^{-1}-x$
$\dfrac{dy}{dx}=-2x^{-2}-1$
$\therefore \dfrac{dy}{dx}=-\dfrac{2}{x^2}-1$

For all values of $x\in R\setminus\{0\}$, $\dfrac{dy}{dx}<0$. Therefore, any tangent will have a negative gradient.

ii When $x=-1$, $y=\dfrac{2-1}{-1}=-1$.
At the point $(-1,-1)$, $\dfrac{dy}{dx}=-\dfrac{2}{1}-1=-3$.
The equation of the tangent is
$y+1=-3(x+1)$
$\therefore y=-3x-4$

14 **a** $x^3+x-4=0$
Let $f(x)=x^3+x-4$
$\therefore f'(x)=3x^2+1$
Let $x_0=1$
$\therefore x_1=1-\dfrac{f(1)}{f'(1)}$
$=1-\dfrac{(1^3+1-4)}{(3\times 1^2+1)}$
$\therefore x_1=1.5$
Continuing the iteration using
$x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}$
$=Ans-\dfrac{(Ans^3+Ans-4)}{(3\times Ans^2+1)}$
gives the following values:
$x_1=1.387096774$
$x_2=1.378838948$
$x_3=1.378796701$
$x_4=1.3787967$
Correct to 4 decimal places, the solution is $x=1.3788$.

b $x^3-8x-10=0$
i Let $f(x)=x^3-8x-10$
$f(1)=1-8-10$
$=-17$
<0
$f(2)=8-16-10$
$=-18$
<0
$f(3)=27-24-10$
$=-7$
<0
$f(4)=64-32-10$
$=22$
>0

There is a sign change between $x=3$ and $x=4$ so the solution to the equation lies between these integers.

ii Let $x_0=3$ be a first estimate to the solution of the equation.
$x_1=3-\dfrac{f(3)}{f'(3)}$ where $f(x)=x^3-8x-10$ and
$f'(x)=3x^2-8$
Applying the iteration yields the values
$x_1=3.368421053$
$x_2=3.319585666$
$x_3=3.318628582$
$x_4=3.318628218$
Correct to 4 decimal places, the solution to the equation is $x=3.3186$.

c $x^3+5x^2=10$
Rewrite the equation as $x^3+5x^2-10=0$ and define $f(x)=x^3+5x^2-10$.
$f'(x)=3x^2+10x$
Let $x_0=-2$.
$x_1=-2-\dfrac{f(-2)}{f'(-2)}$
Applying the iteration yields the values
$x_1=-2.0625$
$x_2=-1.745094386$
$x_3=-1.755636788$
$x_4=-1.755640076$
Correct to 4 decimal places, the solution to the equation is $x=-1.7556$.

d If $x^3=16$ then $x=\sqrt[3]{16}$.
The equation $x^3-16=0$ has the exact solution $x=\sqrt[3]{16}$.
To apply the Newton-Raphson method, let $f(x)=x^3-16$.
$f'(x)=3x^2$.
As $2^3=8$ and $3^3=27$, $2<\sqrt[3]{16}<3$.
Let $x_0=2$
$x_1=2-\dfrac{f(2)}{f'(2)}$
Applying the iteration yields the values
$x_1=2.6666666...$
$x_2=2.5277777...$
$x_3=2.519866987$
$x_4=2.5198421$
$x_5=2.5198421$
Correct to 4 decimal places, $\sqrt[3]{16}=2.5198$.

e $x^2-8x+9=0$
Completing the square
$(x^2-8x+16)-16+9=0$
$\therefore (x-4)^2=7$
$\therefore x-4=\pm\sqrt{7}$
$\therefore x=4\pm\sqrt{7}$

Consider the larger root $4+\sqrt{7}$.
Since $4<7<9$, then $2<\sqrt{7}<3$, so $6<4+\sqrt{7}<7$
Now use the Newton-Raphson method to find the larger root.
Let $f(x)=x^2-8x+9$ and $x_0=7$.
$f'(x)=2x-8$
$x_1=7-\dfrac{f(7)}{f'(7)}$
Applying the iteration yields the values
$x_1=6.6666666...$
$x_2=6.6458333...$
$x_3=6.645751311$
The root is $x=6.6458$ correct to four decimal places.
$\therefore 4+\sqrt{7}=6.6458$
$\therefore \sqrt{7}=2.6458$

f $y = x^4 - 2x^3 - 5x^2 + 9x$

i $\dfrac{dy}{dx} = 4x^3 - 6x^2 - 10x + 9$

At $x = 0$, $\dfrac{dy}{dx} = 9 > 0$ so the quartic function is increasing.

At $x = 1$,

$$\begin{aligned}\frac{dy}{dx} &= 4 - 6 - 10 + 9 \\ &= -3 \\ &< 0\end{aligned}$$

The quartic function is decreasing at $x = 1$.

ii For $x \in [0,1]$, the quartic function changes from increasing to decreasing when $4x^3 - 6x^2 - 10x + 9 = 0$.

To solve this equation using Newton-Raphson method, let $f(x) = 4x^3 - 6x^2 - 10x + 9$.

$f'(x) = 12x^2 - 12x - 10$

Let $x_0 = 0$

$x_1 = 0 - \dfrac{f(0)}{f'(0)}$

Applying the iteration yields the values

$x_1 = 0.9$

$x_2 = 0.7345487365$

$x_3 = 0.7347267115$

The root is $x = 0.735$ correct to three decimal places.

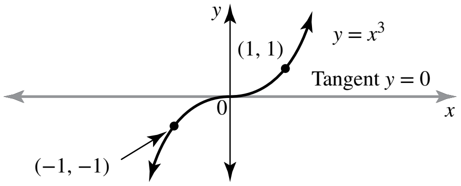
Therefore, at $x = 0.735$, the quartic function changes from increasing to decreasing.

15 $f : R \to R, f(x) = x^3$

a $f'(x) = 3x^2$

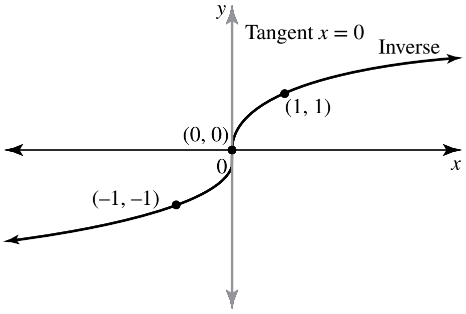
At $(0,0)$, $f'(0) = 0$ so the tangent is the horizontal line $y = 0$.

At $(0,0)$ there is a stationary point of inflection on the curve $f(x) = x^3$.



b **i** The graph of the inverse function is the reflection in $y = x$ of the graph of $y = f(x)$.

The inverse will pass through the points $(-1,-1)$, $(0,0)$ and $(1,1)$.



ii The tangent $y = 0$ will reflect in $y = x$ to become $x = 0$.

The equation of the tangent is $x = 0$.

c The function has rule $y = x^3$.

The inverse has rule $x = y^3$

$\therefore y = \sqrt[3]{x}$ or $y = x^{\frac{1}{3}}$

Hence, $f^{-1}(x) = x^{\frac{1}{3}}$.

d $\dfrac{d}{dx}\left(f^{-1}(x)\right) = \dfrac{1}{3}x^{-\frac{2}{3}}$

$\therefore \dfrac{d}{dx}\left(f^{-1}(x)\right) = \dfrac{1}{3x^{\frac{2}{3}}}$.

The derivative is undefined when $x = 0$. This agrees with the fact that the tangent is vertical.

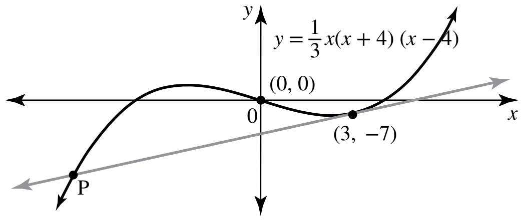
16 **a** $y = \dfrac{1}{3}x(x+4)(x-4)$

x intercepts occur at $x = 0$, $x = -4$ and $x = 4$(all cuts).

Point: Let $x = 3$

$$\begin{aligned}\therefore y &= \frac{1}{3} \times 3 \times 7 \times -1 \\ &= -7\end{aligned}$$

$(3,-7)$



b Point: $(3,-7)$

Gradient:

$$\begin{aligned}y &= \frac{1}{3}x(x^2 - 16) \\ &= \frac{1}{3}x^3 - \frac{16}{3}x\end{aligned}$$

$\therefore \dfrac{dy}{dx} = x^2 - \dfrac{16}{3}$

At $(3,-7)$,

$$\begin{aligned}\frac{dy}{dx} &= 9 - \frac{16}{3} \\ &= \frac{11}{3}\end{aligned}$$

Equation of tangent:

$y + 7 = \dfrac{11}{3}(x - 3)$

$\therefore y = \dfrac{11}{3}x - 11 - 7$

$\therefore y = \dfrac{11}{3}x - 18$

c **i** Tangent meets the curve again when

$$\frac{1}{3}x^3 - \frac{16}{3}x = \frac{11}{3}x - 18$$

$$\therefore x^3 - 16x = 11x - 54$$

$\therefore x^3 - 27x + 54 = 0$

ii As the tangent line touches the cubic graph at $(3,-7)$, $x = 3$ is a solution of the equation and this solution has multiplicity 2.

$\therefore (x-3)^2$ is a factor of the equation.

Since $(x-3)^2 = x^2 - 6x + 9$, then

$x^3 - 27x + 54 = (x^2 - 6x + 9)(x + 6)$

The equation becomes $(x-3)^2(x+6) = 0$ with solutions $x = 3, x = -6$.

At P, the tangent cuts the cubic graph, at P, $x = -6$.

When $x = -6$,

$$y = \frac{1}{3} \times -6 \times -2 \times -10$$

$\therefore y = -40$

P has co-ordinates $(-6,-40)$.

d $\frac{dy}{dx}=x^2-\frac{16}{3}$

When $x=-4$,

$\frac{dy}{dx}=16-\frac{16}{3}$

$=\frac{32}{3}$

When $x=4$,

$\frac{dy}{dx}=16-\frac{16}{3}$

$=\frac{32}{3}$

The tangents to the curve at $x=\pm4$ are parallel since they have the same gradient.

e **i** $y=x(x+a)(x-a)$

$\therefore y=x\left(x^2-a^2\right)$

$\therefore y=x^3-a^2x$

$\frac{dy}{dx}=3x^2-a^2$

At $x=\pm a$,

$\frac{dy}{dx}=3\times(\pm a)^2-a^2$

$=3a^2-a^2$

$=2a^2$

The tangents have the same gradients and therefore the tangents are parallel.

ii Equation of tangent at $(-a,0)$

$y=2a^2(x+a)$

$\therefore y=2a^2x+2a^3....(1)$

Equation of tangent at $(a,0)$

$y=2a^2(x-a)$

$\therefore y=2a^2x-2a^3....(2)$

Equation of tangent at $(0,0)$:

At $(0,0)$, $\frac{dy}{dx}=-a^2$

Therefore, the tangent has equation $y=-a^2x....(3)$

Intersection of tangents (1) and (3):

$2a^2x+2a^3=-a^2x$

$\therefore 3a^2x+2a^3=0$

$\therefore 3a^2x=-2a^3$

$\therefore x=-\frac{2a^3}{3a^2}$

$\therefore x=-\frac{2a}{3}$

Substitute $x=-\frac{2a}{3}$ in equation (3)

$\therefore y=-a^2\times-\frac{2a}{3}$

$\therefore y=\frac{2a^3}{3}$

Point of intersection is $\left(-\frac{2a}{3},\frac{2a^3}{3}\right)$

Intersection of tangents (2) and (3):

$2a^2x-2a^3=-a^2x$

$\therefore 3a^2x-2a^3=0$

$\therefore 3a^2x=2a^3$

$\therefore x=\frac{2a^3}{3a^2}$

$\therefore x=\frac{2a}{3}$

Substitute $x=\frac{2a}{3}$ in equation (3)

$\therefore y=-a^2\times\frac{2a}{3}$

$\therefore y=-\frac{2a^3}{3}$

Point of intersection is $\left(\frac{2a}{3},-\frac{2a^3}{3}\right)$.

17 $y=-\frac{4}{x}-1$

a $\therefore y=-4x^{-1}-1$

$\frac{dy}{dx}=4x^{-2}$

$\therefore \frac{dy}{dx}=\frac{4}{x^2}$

The gradient of the tangent is $m=\tan\left(45^o\right)$.

$\therefore m=1$

$\therefore \frac{dy}{dx}=1$

$\therefore \frac{4}{x^2}=1$

$\therefore x^2=4$

$\therefore x=\pm2$

When $x=2$, $y=-\frac{4}{2}-1=-3$ and when $x=-2$, $y=\frac{4}{2}-1=1$

Equation of tangent at $(2,-3)$

$y+3=1(x-2)$

$\therefore y=x-5$

Equation of tangent at $(-2,1)$

$y-1=1(x+2)$

$\therefore y=x+3$

b The line $2y+8x=5$ has gradient $m_1=-4$.

The gradient of the tangent perpendicular to this line is $m_2=\frac{1}{4}$

$\therefore \frac{dy}{dx}=\frac{1}{4}$

$\therefore \frac{4}{x^2}=\frac{1}{4}$

$\therefore 16=x^2$

$\therefore x=\pm4$

When $x=4$, $y=-\frac{4}{4}-1=-2$ and when $x=-4$, $y=\frac{4}{4}-1=0$

Equation of tangent at $(4,-2)$

$y+2=\frac{1}{4}(x-4)$

$\therefore 4y+8=x-4$

$\therefore 4y-x+12=0$

Equation of tangent at $(-4,0)$

$y=\frac{1}{4}(x+4)$

$\therefore 4y=x+4$

$\therefore 4y-x-4=0$

c At intersection of $y=x^2+2x-8$ and $y=-\frac{4}{x}-1$,

$x^2+2x-8=-\frac{4}{x}-1$

$\therefore x^3+2x^2-8x=-4-x$

$\therefore x^3+2x^2-7x+4=0$

Let $P(x)=x^3+2x^2-7x+4$

$P(1)=1+2-7+4=0$

$\therefore (x-1)$ is a factor

$\therefore x^3+2x^2-7x+4=(x-1)\left(x^2+3x-4\right)$
$=(x-1)(x+4)(x-1)$

$\therefore x^3+2x^2-7x+4=(x-1)^2(x+4)$

At intersection, $(x-1)^2(x+4)=0$

$\therefore x=1, x=-4$

As $x=1$ has multiplicity 2 and $x=-4$ has multiplicity 1, the parabola touches the hyperbola at $x=1$ and cuts the hyperbola at $x=-4$.

When $x=1$, $y=-\frac{4}{1}-1=-5$

Gradient of tangent to $y=-\frac{4}{x}-1$ at the point $(1,-5)$ is

$\frac{dy}{dx}=\frac{4}{x}$
$=\frac{4}{1}$
$=4$

Equation of the tangent is

$y+5=4(x-1)$

$\therefore y=4x-9$

Features of graph of $y=-\frac{4}{x}-1$:

Asymptotes $x=0, y=-1$

From previous working it is known the points $(2,-3),(-2,1),(4,-2),(-4,0)$ and $(1,-5)$ lie on the hyperbola.

Features of the graph of $y=x^2+2x-8$: $(0,-8)$ is y intercept

The equation can be expressed as

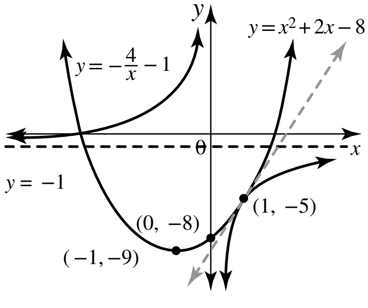
$y=\left(x^2+2x+1\right)-1-8$

$\therefore y=(x+1)^2-9$

Minimum turning point $(-1,-9)$

The equation can also be expressed as $y=(x+4)(x-2)$ so the x intercepts are $(-4,0)$ and $(2,0)$.

The parabola and hyperbola have a common tangent at $(1,-5)$ and the graphs also intersect at $(-4,0)$.



18 $C=\{(x,y): y=x^2+ax+3,\ a\in R\}$

a $\frac{dy}{dx}=2x+a$

When $x=-a$, $\frac{dy}{dx}=-2a+a=-a$

When $x=-a$, $y=a^2-a^2+3=3$

Equation of tangent at point $(-a,3)$ with gradient $-a$ is

$y-3=-a(x+a)$

$\therefore y=-ax+3-a^2$

b The tangent has y intercept when $x=0$

$\therefore y=3-a^2$

Given the tangent cuts the y axis at $y=-6$, then

$-6=3-a^2$

$\therefore a^2=9$

$\therefore a=\pm 3$

Also given that the curve $y=x^2+ax+3$ is increasing at $x=-a$ then its gradient at this point must be positive.

$\therefore \frac{dy}{dx}>0$

$\therefore -a>0$

$\therefore a<0$

Hence, $a=-3$.

c The family of curves are decreasing where $\frac{dy}{dx}<0$

$\therefore 2x+a<0$

$\therefore 2x<-a$

$\therefore x<-\frac{a}{2}$

For the domain $\left(-\infty,-\frac{a}{2}\right)$, all the curves in the family C are decreasing functions.

d Consider the case $a\neq 0$.

From part **a**, the tangent at $(-a,3)$ has gradient $-a$. The line perpendicular to the tangent has gradient $\frac{1}{a}$.

The equation of the line through $(-a,3)$ with gradient $\frac{1}{a}$ is

$y-3=\frac{1}{a}(x+a)$

$\therefore y-3=\frac{x}{a}+1$

$\therefore y=\frac{x}{a}+4$

When $x=0$, $y=4$ so all such lines pass through the point $(0,4)$ for $a\in R\setminus\{0\}$.

Consider the case $a=0$:

The curve in the family C for which $a=0$ is $y=x^2+3$

The tangent to this curve at $(0,3)$ is horizontal and has the equation $y=3$. The line perpendicular to $y=3$ through the point $(0,3)$ is the vertical line with equation $x=0$. This line, $\{(x,y): x=0\}$ does contain the point $(0,4)$.

Therefore, the statement will hold if $a=0$.

19 From the main menu select Interactive → Calculation → tanline and complete the dialogue box as

Expression: $(2x+1)^3$

Variable: x

Point: -4.5

Tap OK to obtain $y=384x+1216$ as the equation of the tangent to $y=(2x+1)^3$ at the point where $x=-4.5$.

(Other methods are possible).

20 $y=x^3+2x^2-4x-2$

Sketch the graph in Graph&Tab. Tap Analysis → Sketch → Tangent and enter the x value of 0. This sketches the tangent $y=-4x-2$ for which the gradient of the curve at the point of contact $x_c=0, y_c=-2$ is $\frac{dy}{dx}=-4$.

Tap Analysis → GSolve → Intersect to obtain $(-2,6)$ as the point where the tangent meets the curve again.

To sketch the tangent to the curve $y=x^3+2x^2-4x-2$ at the point $(-2,6)$, tap Analysis → Sketch → Tangent and enter the x value of -2.

The tangent is a horizontal line with equation $y=6$.

Exercise 13.5 — Curve sketching

1 a $f(x)=x^3+x^2-x+4$

$f'(x)=3x^2+2x-1$

At stationary points, $f'(x)=0$:

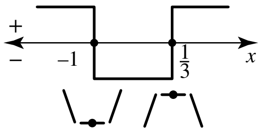
$$3x^2+2x-1=0$$
$$(3x-1)(x+1)=0$$
$$x=\frac{1}{3}, x=-1$$

When $x=\frac{1}{3}$,

$$y=\frac{1}{27}+\frac{1}{9}-\frac{1}{3}+4$$
$$=\frac{-5}{27}+4$$
$$=3\frac{22}{27}$$

When $x=-1$, $y=5$.

For type of stationary points, draw a sign diagram of $f'(x)$:



$(-1,5)$ is a maximum turning point and $\left(\frac{1}{3},\frac{103}{27}\right)$ is a minimum turning point.

b $y=ax^2+bx+c$

$$(0,5)\Rightarrow 5=c$$
$$\therefore y=ax^2+bx+5$$
$$(2,-14)\Rightarrow -14=4a+2b+5$$
$$\therefore 4a+2b=-19.......(1)$$
$$\frac{dy}{dx}=2ax+b$$

Since $(2,-14)$ is a stationary point, $4a+b=0.......(2)$

$(1)-(2)$

$$b=-19$$
$$\therefore a=\frac{19}{4}$$

Hence, $a=\frac{19}{4}, b=-19, c=5$

c $f'(x)=\frac{1}{5}-\frac{1}{x}$

Testing the sign of the gradient near the stationary point $(5,2)$:

x	4	5	6
$f'(x)$	$\frac{1}{5}-\frac{1}{4}=-\frac{1}{20}$	0	$\frac{1}{5}-\frac{1}{6}=\frac{1}{30}$
slope of tangent	╲	—	╱

Point $(5,2)$ is a minimum turning point.

2 $y=x^3+ax^2+bx-11$

a $\frac{dy}{dx}=3x^2+2ax+b$

At $x=2$ and $x=4$, $\frac{dy}{dx}=0$

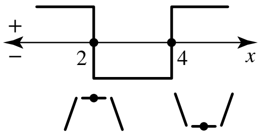
$$x=2\Rightarrow 3(2)^2+2a(2)+b=0$$
$$\therefore 12+4a+b=0.......(1)$$
$$x=4\Rightarrow 3(4)^2+2a(4)+b=0$$
$$\therefore 48+8a+b=0.......(2)$$

$(2)-(1)$

$$36+4a=0$$
$$\therefore a=-9$$
$$\therefore b=24$$

b Stationary points at $x=2$ and $x=4$ mean $(x-2)(x-4)$ must be factors of the gradient function.

$$y=x^3-9x^2+24x-11$$
$$\frac{dy}{dx}=3x^2-18x+24$$
$$=3(x-2)(x-4)$$



When $x=2$,

$$y=(2)^3-9(2)^2+24(2)-11$$
$$=9$$

When $x=4$,

$$y=(4)^3-9(4)^2+24(4)-11$$
$$=5$$

Therefore $(2,9)$ is a maximum turning point and $(4,5)$ is a minimum turning point.

3 $y=2x^2-x^3$

$$=x^2(2-x)$$

x-intercepts at $x=2, x=0$

stationary points: when $\frac{dy}{dx}=0$

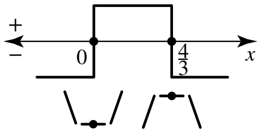
$$\frac{dy}{dx}=4x-3x^2$$
$$=x(4-3x)$$
$$\frac{dy}{dx}=0\Rightarrow x=0, x=\frac{4}{3}$$

When $x=0, y=0\Rightarrow(0,0)$

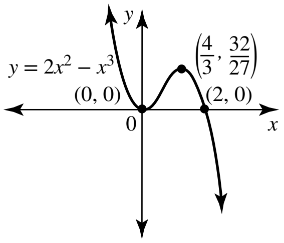
When $x=\frac{4}{3}$,

$$y=2\times\frac{16}{9}-\frac{64}{27}$$
$$=\frac{32}{27}\quad\Rightarrow\left(\frac{4}{3},\frac{32}{27}\right)$$

Sign diagram of $\frac{dy}{dx}$:



Therefore $(0,0)$ is a minimum turning point and $\left(\frac{4}{3},\frac{32}{27}\right)$ is a maximum turning point.



4 $y=x^4+2x^3-2x-1$

y-intercept: $(0,-1)$

x-intercepts: let $y=0$

$$x^4+2x^3-2x-1=0$$
$$(x^4-1)+(2x^3-2x)=0$$
$$(x^2-1)(x^2+1)+2x(x^2-1)=0$$
$$(x^2-1)(x^2+1+2x)=0$$
$$(x-1)(x+1)(x+1)^2=0$$
$$(x-1)(x+1)^3=0$$
$$x=1, x=-1$$

stationary points: $\frac{dy}{dx}=0$

$$\frac{dy}{dx}=4x^3+6x^2-2$$

$\therefore 4x^3+6x^2-2=0$

If $x=-1$, $\frac{dy}{dx}=0 \Rightarrow (x+1)$ is a factor

$4x^3+6x^2-2$

$=2(2x^3+3x^2-1)$

$=2(x+1)(2x^2+x-1)$

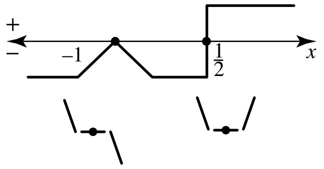
$=2(x+1)(2x-1)(x+1)$

$=2(x+1)^2(2x-1)$

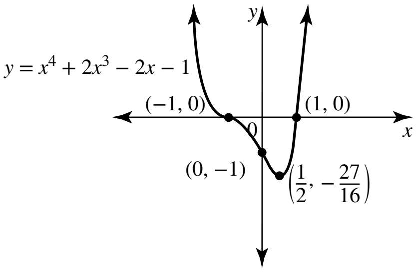
$\therefore x=-1, x=\frac{1}{2}$

When $x=-1$, $y=0$; when $x=\frac{1}{2}$, $y=-\frac{27}{16}$

For type of stationary point, draw a sign diagram of $\frac{dy}{dx}=4(x+1)^2(2x-1)$:



Therefore $(-1,0)$ is a stationary point of inflection and $\left(\frac{1}{2},\frac{-27}{16}\right)$ is a minimum turning point.



5 a $y=\frac{1}{16}x^2+\frac{1}{x}$

Endpoints: when $x=\frac{1}{4}, y=\frac{1}{256}+4 \Rightarrow \left(\frac{1}{4},\frac{1025}{256}\right)$

When $x=4, y=1+\frac{1}{4} \Rightarrow \left(4,\frac{5}{4}\right)$

b Stationary points:

$y=\frac{1}{16}x^2+x^{-1}$

$\frac{dy}{dx}=\frac{1}{8}x-\frac{1}{x^2}$

At a stationary point, $\frac{dy}{dx}=0$, so:

$\frac{1}{8}x-\frac{1}{x^2}=0$

$\frac{1}{8}x=\frac{1}{x^2}$

$x^3=8$

$x=2$

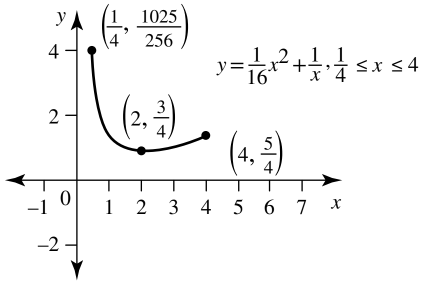
When $x=2$, $y=\frac{1}{4}+\frac{1}{2} \Rightarrow \left(2,\frac{3}{4}\right)$ is a stationary point

For type of stationary point, test the slope of the curve at $x=1, x=3$.

x	1	2	3
$\frac{dy}{dx}$	$-\frac{7}{8}$	0	$\frac{19}{72}$
Slope	\	—	/

Therefore $\left(2,\frac{3}{4}\right)$ is a minimum turning point.

c



d The global maximum occurs at left endpoint and equals $\frac{1025}{256}$.

The global minimum value occurs at the local minimum turning point and equals $\frac{3}{4}$.

6 $f(x)=4x^3-12x$

Endpoint: when $x=\sqrt{3}, f(x)=12\sqrt{3}-12\sqrt{3} \Rightarrow \left(\sqrt{3},0\right)$

Stationary points: $f'(x)=12x^2-12$

At stationary point, $f'(x)=0$, so:

$12x^2-12=0$

$12(x^2-1)=0$

$12(x+1)(x-1)=0$

$x=-1, x=1$

This is a positive cubic graph, so the first turning point is a maximum and the second is a minimum.

When $x=-1$, $f(x)=8 \Rightarrow (-1,8)$ is a maximum turning point.

When $x=1$, $f(x)=-8 \Rightarrow (1,-8)$ is a minimum turning point.

Domain: $(-\infty,\sqrt{3}]$ so as $x \to -\infty, y \to -\infty$

As global extrema occur either at a local turning point or at endpoint, there is no global minimum and there is a global maximum of 8 at the local maximum turning point.

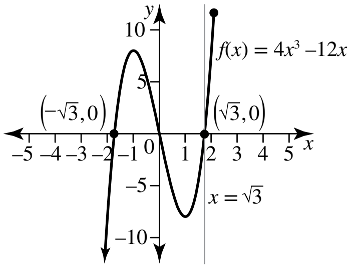
This is confirmed by sketching the graph:

x-intercepts: $4x^3-12x=0$

$4x(x^2-3)=0$

$4x(x+\sqrt{3})(x-\sqrt{3})=0$

$x=0, x=\pm\sqrt{3}$



7 a i $y=x^2-8x+10$

$\frac{dy}{dx}=2x-8$

At stationary point, $\frac{dy}{dx}=0$

$\therefore 2x-8=0$

$\therefore x=4$

When $x = 4$, $y = 16 - 32 + 10 = -6$
The stationary point is $(4, -6)$.

ii $y = -5x^2 + 6x - 12$

$\frac{dy}{dx} = -10x + 6$

At stationary point, $\frac{dy}{dx} = 0$

$\therefore -10x + 6 = 0$

$\therefore x = 0.6$

When $x = 0.6$, $y = -5 \times 0.36 + 6 \times 0.6 - 12 = -10.2$
The stationary point is $(0.6, -10.2)$.

b **i** $y = ax^2 + bx$

$\frac{dy}{dx} = 2ax + b$

$(4, -8)$ is a stationary point

$\therefore 2a(4) + b = 0$

$\therefore 8a + b = 0....(1)$

$(4, -8)$ lies on the curve

$\therefore -8 = a(4)^2 + b(4)$

$\therefore 16a + 4b = -8$

$\therefore 4a + b = -2....(2)$

Subtract equation (2) from equation (1)

$\therefore 4a = 2$

$\therefore a = \frac{1}{2}$

Substitute $a = \frac{1}{2}$ in equation (1)

$\therefore 4 + b = 0$

$\therefore b = -4$

Answer is $a = \frac{1}{2}, b = -4$

ii Equation of curve is $y = \frac{1}{2}x^2 - 4x$

Minimum turning point is $(4, -8)$.
x intercepts: Let $y = 0$

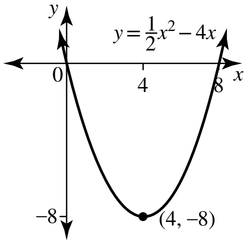
$\therefore \frac{1}{2}x^2 - 4x = 0$

$\therefore x^2 - 8x = 0$

$\therefore x(x - 8) = 0$

$\therefore x = 0, x = 8$

$(0, 0), (8, 0)$



8 $f(x) = x^3 + 3x^2 + 8$

a $f'(x) = 3x^2 + 6x$

$f'(-2) = 3 \times 4 + 6 \times -2$

$\therefore f'(-2) = 0$

The function has a stationary point when $x = -2$.
$f(-2) = -8 + 12 + 8 = 12$
Therefore, $(-2, 12)$ is a stationary point of the function.

b Test the slope of the tangent to the curve around $x = -2$.

x	−3	−2	−1
$f'(x)$	9	0	−3
Slope	/	—	\

The slope of the tangent shows the point $(-2, 12)$ is a maximum turning point.

c Let $f'(x) = 0$

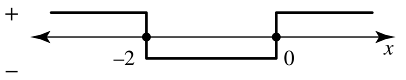
$\therefore 3x^2 + 6x = 0$

$\therefore 3x(x + 2) = 0$

$\therefore x = 0, x = -2$

Since $f(0) = 8$, the other stationary point is $(0, 8)$.

d Sign diagram of $f'(x) = 3x(x + 2)$



The sign of the gradient changes from negative to positive about $x = 0$, indicating $(0, 8)$ is a minimum turning point.

9 **a** $y = \frac{1}{3}x^3 + x^2 - 3x - 1$

$\frac{dy}{dx} = x^2 + 2x - 3$

At stationary points, $\frac{dy}{dx} = 0$

$\therefore x^2 + 2x - 3 = 0$

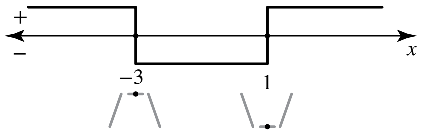
$\therefore (x + 3)(x - 1) = 0$

$\therefore x = -3, x = 1$

When $x = -3$, $y = \frac{1}{3} \times -27 + 9 + 9 - 1 = 8$ and when $x = 1$,

$y = \frac{1}{3} + 1 - 3 - 1 = -\frac{8}{3}$.

Sign of the derivative:



Hence, $(-3, 8)$ is a maximum turning point and $\left(1, -\frac{8}{3}\right)$ is a minimum turning point.

b $y = -x^3 + 6x^2 - 12x + 8$

$\frac{dy}{dx} = -3x^2 + 12x - 12$

At stationary points, $\frac{dy}{dx} = 0$

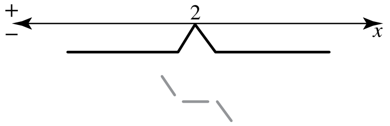
$\therefore -3x^2 + 12x - 12 = 0$

$\therefore -3\left(x^2 - 4x + 4\right) = 0$

$\therefore -3(x - 2)^2 = 0$

$\therefore x = 2$

When $x = 2$, $y = -8 + 24 - 24 + 8 = 0$
Sign of the derivative:



Hence, $(2, 0)$ is a stationary point of inflection.

c $y=\frac{23}{6}x(x-3)(x+3)$

$\therefore y=\frac{23}{6}x(x^2-9)$

$\therefore y=\frac{23}{6}\left(x^3-9x\right)$

$\frac{dy}{dx}=\frac{23}{6}\left(3x^2-9\right)$

$\therefore \frac{dy}{dx}=\frac{23}{2}\left(x^2-3\right)$

At stationary points, $\frac{dy}{dx}=0$

$\therefore \frac{23}{2}(x^2-3)=0$

$\therefore x^2=3$

$\therefore x=\pm\sqrt{3}$

Substitute $x=\sqrt{3}$ in $y=\frac{23}{6}\left(x^3-9x\right)$

$\therefore y=\frac{23}{6}\left(3\sqrt{3}-9\sqrt{3}\right)$

$=\frac{23}{6}\times -6\sqrt{3}$

$=-23\sqrt{3}$

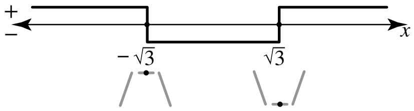
When $x=-\sqrt{3}$,

$y=\frac{23}{6}\left(-3\sqrt{3}+9\sqrt{3}\right)$

$=\frac{23}{6}\times 6\sqrt{3}$

$=23\sqrt{3}$

Sign of the derivative



$\left(-\sqrt{3},23\sqrt{3}\right)$ is a maximum turning point and $\left(\sqrt{3},-23\sqrt{3}\right)$ is a minimum turning point.

d $y=4x^3+5x^2+7x-10$

$\frac{dy}{dx}=12x^2+10x+7$

At stationary points, $\frac{dy}{dx}=0$

$\therefore 12x^2+10x+7=0$

$\Delta=100-4\times12\times7$

$=100-336$

<0

Since $\Delta<0$ there are no real solutions to $\frac{dy}{dx}=0.$

Therefore, there are no stationary points.

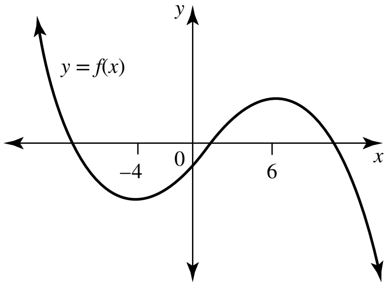
10 a $f'(-4)=0$ and $f'(6)=0 \Rightarrow$ there are stationary points at $x=-4$ and $x=6$.

As $f'(x)<0$ for $x<-4$ and $f'(x)>0$ for $-4<x<6$, the slope of the curve changes from negative to positive about $x=-4$. Therefore there is a minimum turning point when $x=-4$.

As $f'(x)>0$ for $-4<x<6$ and $f'(x)<0$ for $x>6$, the slope of the curve changes from positive to negative about $x=6$. Therefore there is a maximum turning point when $x=6$.

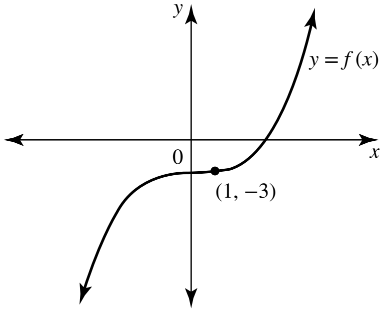
The y co-ordinates of these turning points are not known.

A possible graph for $y=f(x)$ is shown.



b $f'(1)=0$ and $f(1)=-3 \Rightarrow$ there is a stationary point at $(1,-3)$. Since $f'(x)>0$ for both $x<1$ and $x>1$, the point $(1,-3)$ is a stationary point of inflection.

A possible graph for $y=f(x)$ is shown.



11 a $y=x^3+bx^2+cx-26$

Point $(2,-54)$ lies on the curve.

$\therefore -54=8+4b+2c-26$

$\therefore -36=4b+2c$

$\therefore 2b+c=-18....(1)$

$\frac{dy}{dx}=3x^2+2bx+c$

As $(2,-54)$ is a stationary point, $\frac{dy}{dx}=0$ at $(2,-54)$.

$\therefore 12+4b+c=0$

$\therefore 4b+c=-12....(2)$

Subtract equation (1) from equation (2)

$\therefore 2b=6$

$\therefore b=3$

Substitute $b=3$ in equation (1)

$\therefore 6+c=-18$

$\therefore c=-24$

Hence, $b=3,\ c=-24$.

b $y=x^3+3x^2-24x-26$ and $\frac{dy}{dx}=3x^2+6x-24$

Let $\frac{dy}{dx}=0$

$\therefore 3x^2+6x-24=0$

$\therefore 3\left(x^2+2x-8\right)=0$

$\therefore 3(x+4)(x-2)=0$

$\therefore x=-4, x=2$

When $x=-4$, $y=-64+48+96-26=54$

The other stationary point is $(-4,54)$.

c $y=x^3+3x^2-24x-26$ has y intercept $(0,-26)$.

x intercepts: Let $y=0$

$\therefore x^3+3x^2-24x-26=0$

Let $P(x)=x^3+3x^2-24x-26$

$P(-1)=-1+3+24-26=0$

$\therefore (x+1)$ is a factor

$\therefore x^3+3x^2-24x-26=(x+1)(x^2+2x-26)$

$\therefore (x+1)(x^2+2x-26)=0$

$\therefore x=-1$ or $x^2+2x-26=0$

$\therefore x=-1$ or $\left(x^2+2x+1\right)-1-26=0$

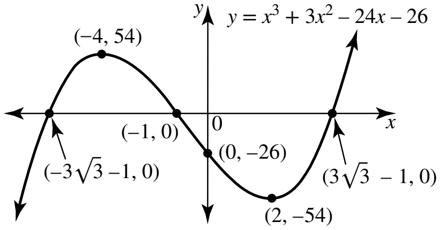
$\therefore x=-1$ or $(x+1)^2=27$

$\therefore x=-1$ or $x+1=\pm3\sqrt{3}$

$\therefore x=-1$ or $x=\pm3\sqrt{3}-1$

$(-1,0),\left(-3\sqrt{3}-1,0\right),\left(3\sqrt{3}-1,0\right)$

d



12 a $f: R\to R, f(x)=2x^3+6x^2$

x intercepts: Let $f(x)=0$

$\therefore 2x^3+6x^2=0$

$\therefore 2x^2(x+3)=0$

$\therefore x=0, x=-3$

Graph touches x axis at $(0,0)$ and cuts the x axis at $(-3,0)$.

Stationary points: $f'(x)=0$

$\therefore 6x^2+12x=0$

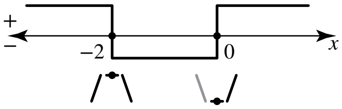
$\therefore 6x(x+2)=0$

$\therefore x=0, x=-2$

When $x=-2$, $y=-16+24=8$

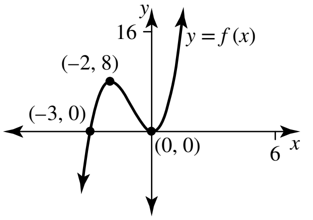
Stationary points at $(0,0)$ and $(-2,8)$.

Sign of derivative:



Therefore, $(-2,8)$ is a maximum turning point and $(0,0)$ is a minimum turning point.

Graph of $y=f(x)$:



b $g: R\to R, g(x)=-x^3+4x^2+3x-12$

y intercept is $(0,-12)$.

x intercepts when $-x^3+4x^2+3x-12=0$

$\therefore -x^2(x-4)+3(x-4)=0$

$\therefore (x-4)(-x^2+3)=0$

$\therefore x=4$ or $x^2=3$

$\therefore x=4$ or $x=\pm\sqrt{3}$

$(4,0),\left(-\sqrt{3},0\right),\left(\sqrt{3},0\right)$

Stationary points when $g'(x)=0$

$\therefore -3x^2+8x+3=0$

$\therefore (-3x-1)(x-3)=0$

$\therefore x=-\frac{1}{3}$ or $x=3$

$g\left(-\frac{1}{3}\right)=\frac{1}{27}+\frac{4}{9}-1-12$

$=\frac{13}{27}-13$

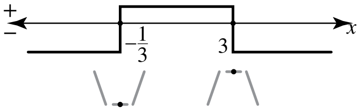
$=-12\frac{14}{27}$

$g(3)=-27+36+9-12$

$=6$

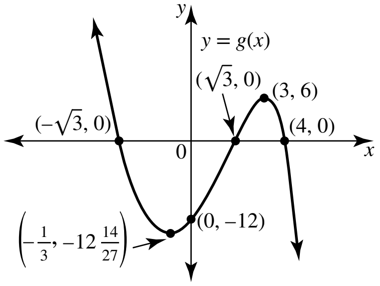
Stationary points are $\left(-\frac{1}{3},-12\frac{14}{27}\right)$ and $(3,6)$.

Sign of derivative:



Therefore, $\left(-\frac{1}{3},-12\frac{14}{27}\right)$ is a minimum turning point and $(3,6)$ is a maximum turning point.

Graph of $y=g(x)$:



c $h: R\to R, h(x)=9x^3-117x+108$

$(0,108)$ is the y intercept.

x intercepts occur when $h(x)=9x^3-117x+108=0$.

$h(1)=9-117+108=0$

$\therefore (x-1)$ is a factor

$\therefore x^3-117x+108=(x-1)\left(9x^2+9x-108\right)=0$

$\therefore (x-1)9\left(x^2+x-12\right)=0$

$\therefore 9(x-1)(x+4)(x-3)=0$

$\therefore x=1, x=-4, x=3$

$(-4,0),(1,0),(3,0)$

Stationary points when $h'(x)=0$

$\therefore 27x^2-117=0$

$\therefore x^2=\frac{117}{27}$

$\therefore x^2=\frac{39}{9}$

$\therefore x=\pm\frac{\sqrt{39}}{3}$

$h\left(-\frac{\sqrt{39}}{3}\right)=-9\times\frac{39\sqrt{39}}{27}+\frac{117\sqrt{39}}{3}+108$

$=-13\sqrt{39}+39\sqrt{39}+108$

$=26\sqrt{39}+108$

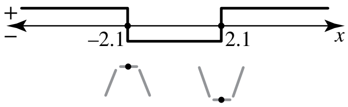
$h\left(\frac{\sqrt{39}}{3}\right)=9\times\frac{39\sqrt{39}}{27}-\frac{117\sqrt{39}}{3}+108$

$=13\sqrt{39}-39\sqrt{39}+108$

$=-26\sqrt{39}+108$

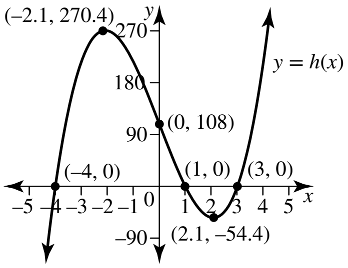
Stationary points are $\left(-\frac{\sqrt{39}}{3}, 26\sqrt{39}+108\right)$ and $\left(\frac{\sqrt{39}}{3}, 108-26\sqrt{39}\right)$. Approximately these are $(-2.1, 270.4)$ and $(2.1, 54.4)$.

Sign of derivative:



Therefore, $(-2.1, 270.4)$ is a maximum turning point and $(2.1, -54.4)$ is a minimum turning point.

Graph of $y = h(x)$:



d $p : [-1, 1] \to R, p(x) = x^3 + 2x$

$$p(x) = x^3 + 2x$$
$$= x\left(x^2 + 2\right)$$

Since $x^2 + 2 \neq 0$, there is only one x intercept at $(0, 0)$.

Endpoints: $p(-1) = -1 - 2 = -3$ $(-1, -3)$ is closed left endpoint.

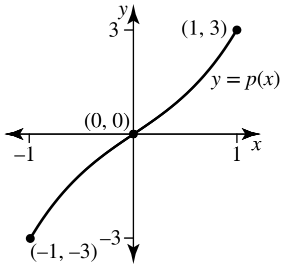
$p(1) = 1 + 2 = 3$ $(1, 3)$ is closed right endpoint.

Stationary points occur when $p'(x) = 0$

$$\therefore 3x^2 + 2 = 0$$
$$\therefore x^2 = -\frac{2}{3}$$

As there are no real solutions to this equation, there are no stationary points.

$p'(x) > 0$ so the graph of $y = p(x)$ is increasing.



e $\{(x, y) : y = x^4 - 6x^2 + 8\}$

$(0, 8)$ is y intercept.

x intercepts occur when $x^4 - 6x^2 + 8 = 0$

$$\therefore \left(x^2 - 4\right)\left(x^2 - 2\right) = 0$$
$$\therefore (x+2)(x-2)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right) = 0$$
$$\therefore x = -2, x = 2, x = -\sqrt{2}, x = \sqrt{2}$$

$\left(\pm\sqrt{2}, 0\right), (\pm 2, 0)$

Stationary points occur when $\frac{dy}{dx} = 0$

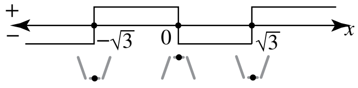
$$\therefore 4x^3 - 12x = 0$$
$$\therefore 4x\left(x^2 - 3\right) = 0$$
$$\therefore 4x\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right) = 0$$
$$\therefore x = 0, x = -\sqrt{3}, x = \sqrt{3}$$

When $x = -\sqrt{3}$, $y = 9 - 18 + 8 = -1$

When $x = \sqrt{3}$, $y = 9 - 18 + 8 = -1$

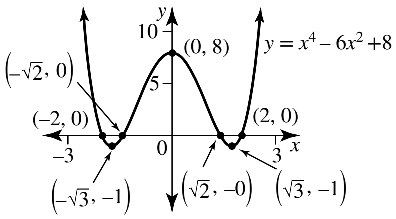
Stationary points are $(0, 8)$, $\left(-\sqrt{3}, -1\right)$ and $\left(\sqrt{3}, -1\right)$

Sign of the derivative:



Therefore, $\left(-\sqrt{3}, -1\right)$ is a minimum turning point, $(0, 8)$ is a maximum turning point and $\left(\sqrt{3}, -1\right)$ is a minimum turning point.

The graph:



f $\{(x, y) : y = 2x(x+1)^3\}$

Graph will cut x axis at $(0, 0)$ and saddle cut x axis at $(-1, 0)$, so point $(-1, 0)$ is a stationary point of inflection.

Expanding the function's equation,

$$y = 2x\left(x^3 + 3x^2 + 3x + 1\right)$$
$$\therefore y = 2x^4 + 6x^3 + 6x^2 + 2x$$

Other stationary points: Let $\frac{dy}{dx} = 0$

$$\therefore 8x^3 + 18x^2 + 12x + 2 = 0$$
$$\therefore 4x^3 + 9x^2 + 6x + 1 = 0$$

Since $(-1, 0)$ is a stationary point of inflection, $x = -1$ is a solution and therefore $(x+1)$ is a factor.

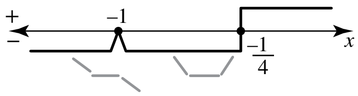
$$\therefore (x+1)\left(4x^2 + 5x + 1\right) = 0$$
$$\therefore (x+1)(4x+1)(x+1) = 0$$
$$\therefore (x+1)^2(4x+1) = 0$$
$$\therefore x = -1, x = -\frac{1}{4}$$

When $x = -\frac{1}{4}$,

$$y = 2 \times -\frac{1}{4}\left(-\frac{1}{4} + 1\right)^3$$
$$= -\frac{1}{2} \times \frac{27}{64}$$
$$= -\frac{27}{128}$$

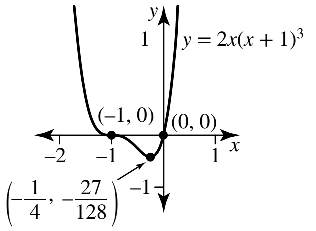
The stationary points are $\left(-\frac{1}{4}, -\frac{27}{128}\right)$ and $(-1, 0)$.

Sign of $\frac{dy}{dx} = 2(x+1)^2(4x+1)$



The point $(-1, 0)$ is a stationary point of inflection and the point $\left(-\frac{1}{4}, -\frac{27}{128}\right)$ is a minimum turning point.

The graph:



13 a The greatest number of turning points a cubic function can have is 2 and the least number is 0.

b $y = 3x^3 + 6x^2 + 4x + 6$

$\frac{dy}{dx} = 9x^2 + 12x + 4$

Stationary points occur when $\frac{dy}{dx} = 0$.

$\therefore 9x^2 + 12x + 4 = 0$

$\therefore (3x + 2)^2 = 0$

$\therefore x = -\frac{3}{2}$

There is only one stationary point.

As $\frac{dy}{dx} = (3x + 2)^2$, then $\frac{dy}{dx} > 0$ for $x \in R \setminus \left\{-\frac{3}{2}\right\}$. The stationary point is a stationary point of inflection.

c $y = 3x^3 + 6x^2 + kx + 6$

$\frac{dy}{dx} = 9x^2 + 12x + k$

Stationary points occur when $\frac{dy}{dx} = 0$.

For the function to have no stationary points, the quadratic equation $9x^2 + 12x + k = 0$ will have no real solutions. Therefore, its discriminant must be negative.

$\Delta = 144 - 36k$

$\therefore \Delta < 0 \Rightarrow 144 - 36k < 0$

$\therefore 144 < 36k$

$\therefore k > \frac{144}{36}$

$\therefore k > 4$

d For a cubic function with a positive coefficient of x^3, as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$. It is not possible for $x \to \infty, y \to \infty$ if there is exactly one stationary point which is a maximum turning point.

For there to be exactly one stationary point the point must be a stationary point of inflection.

e The gradient function has degree 2.

Suppose a cubic function has one stationary point of inflection at $x = a$ and one maximum turning point at $x = b$. Then $(x - a)^2$ and $(x - b)$ must be factors of the gradient function. However, this would make the gradient function's degree 3, which is not possible.

Therefore, it is not possible for a cubic function to have both a stationary point of inflection and a maximum turning point.

f $y = xa^2 - x^3$

$\frac{dy}{dx} = a^2 - 3x^2$

At stationary points, $a^2 - 3x^2 = 0$

$\therefore a^2 = 3x^2$

$\therefore x^2 = \frac{a^2}{3}$

$\therefore x = \pm\frac{a}{\sqrt{3}}$

When $x = -\frac{a}{\sqrt{3}}$,

$$\begin{aligned} y &= -\frac{a}{\sqrt{3}} \times a^2 + \frac{a^3}{3\sqrt{3}} \\ &= -\frac{3a^3}{3\sqrt{3}} + \frac{a^3}{3\sqrt{3}} \\ &= -\frac{2a^3}{3\sqrt{3}} \end{aligned}$$

When $x = \frac{a}{\sqrt{3}}$,

$$\begin{aligned} y &= \frac{a}{\sqrt{3}} \times a^2 - \frac{a^3}{3\sqrt{3}} \\ &= \frac{3a^3}{3\sqrt{3}} - \frac{a^3}{3\sqrt{3}} \\ &= \frac{2a^3}{3\sqrt{3}} \end{aligned}$$

The stationary points are $\left(-\frac{a}{\sqrt{3}}, -\frac{2a^3}{3\sqrt{3}}\right)$ and $\left(\frac{a}{\sqrt{3}}, \frac{2a^3}{3\sqrt{3}}\right)$.

Let A be $\left(-\frac{a}{\sqrt{3}}, -\frac{2a^3}{3\sqrt{3}}\right)$, B be $\left(\frac{a}{\sqrt{3}}, \frac{2a^3}{3\sqrt{3}}\right)$ and C be the point $(0, 0)$.

The line through A and B will pass through C if the three points are collinear.

$$\begin{aligned} m_{AC} &= \frac{\left(\frac{2a^3}{3\sqrt{3}} - 0\right)}{\left(\frac{a}{\sqrt{3}} - 0\right)} \\ &= \frac{2a^3}{3\sqrt{3}} \div \frac{a}{\sqrt{3}} \\ &= \frac{2a^3}{3\sqrt{3}} \times \frac{\sqrt{3}}{a} \\ &= \frac{2a^2}{3} \end{aligned}$$

$$\begin{aligned} m_{BC} &= \frac{\left(-\frac{2a^3}{3\sqrt{3}} - 0\right)}{\left(-\frac{a}{\sqrt{3}} - 0\right)} \\ &= -\frac{2a^3}{3\sqrt{3}} \div -\frac{a}{\sqrt{3}} \\ &= \frac{2a^3}{3\sqrt{3}} \times -\frac{\sqrt{3}}{a} \\ &= \frac{2a^2}{3} \end{aligned}$$

Since $m_{AC} = m_{BC}$ and point C is common, the three points A, B and C are collinear.

Therefore, the line joining the turning points passes through the origin. The equation of the line is $y = \frac{2a^2}{3}x$.

14 a $y = 4x^2 - 2x + 3, \quad -1 \le x \le 1$

Endpoints: When $x = -1$, $y = 4 + 2 + 3 = 9$. Left endpoint $(-1, 9)$.

When $x = 1$, $y = 4 - 2 + 3 = 5$. Right endpoint $(1, 5)$.

y intercept is $(0, 3)$

Turning point: $\frac{dy}{dx} = 8x - 2$

At turning point, $8x - 2 = 0$

$\therefore x = \frac{1}{4}$

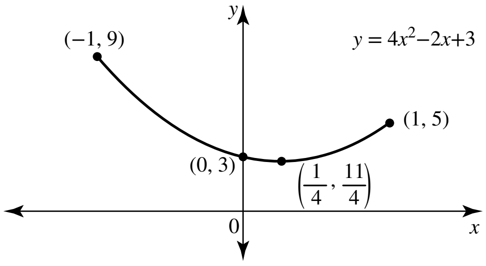
$\therefore y = 4 \times \frac{1}{16} - 2 \times \frac{1}{4} + 3$

$\therefore y = \frac{1}{4} - \frac{1}{2} + 3$

$\therefore y = \frac{11}{4}$

Minimum turning point $\left(\frac{1}{4}, \frac{11}{4}\right)$

no x intercepts.



Local minimum and global minimum value is $\frac{11}{4}$.
Global maximum value is 9.
There is no local maximum.

b $y = x^3 + 2x^2, \ -3 \le x \le 3$

Endpoints: When $x = -3$, $y = -27 + 18 = -9$. Left endpoint $(-3, -9)$.
When $x = 3$, $y = 27 + 18 = 45$. Right endpoint $(3, 45)$.

Stationary points: $\frac{dy}{dx} = 3x^2 + 4x$

At stationary points, $3x^2 + 4x = 0$

$\therefore x(3x + 4) = 0$

$\therefore x = 0, x = -\frac{4}{3}$

When $x = 0$, $y = 0$

When $x = -\frac{4}{3}$,

$$\begin{aligned} y &= -\frac{64}{27} + 2 \times \frac{16}{9} \\ &= -\frac{64}{27} + \frac{32}{9} \\ &= \frac{-64 + 96}{27} \\ &= \frac{32}{27} \end{aligned}$$

Stationary points are $\left(-\frac{4}{3}, \frac{32}{27}\right)$ and $(0, 0)$.

As the function is a positive cubic, $\left(-\frac{4}{3}, \frac{32}{27}\right)$ is a maximum turning point and $(0, 0)$ is a minimum turning point.

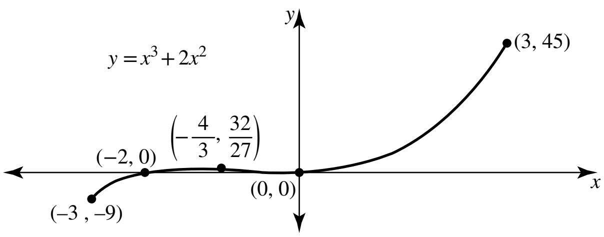
x intercepts occur when $y = 0$

$\therefore x^3 + 2x^2 = 0$

$\therefore x^2(x + 2) = 0$

$\therefore x = 0, x = -2$

$(0, 0), (-2, 0)$



Local maximum value is $\frac{32}{27}$.
Local minimum value is 0.
Global maximum value is 45.
Global minimum value is −9.

c $y = 3 - 2x^3, \ x \le 1$

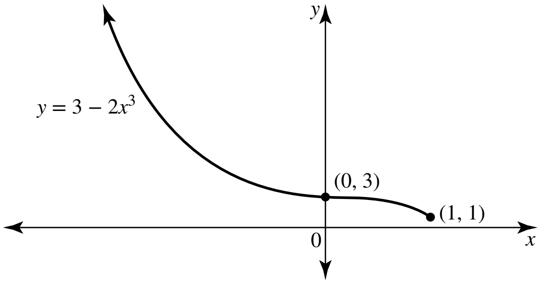
Endpoint: When $x = 1$, $y = 1$. Right endpoint $(1, 1)$.
y intercept is $(0, 3)$
Stationary point of inflection at $(0, 3)$.
x intercept: Let $y = 0$

$\therefore 3 - 2x^3 = 0$

$2x^3 = 3$

$\therefore x^3 = \frac{3}{2}$

Since $x^3 > 1$, then $x > 1$, so there is no x intercept in the given domain $(-\infty, 1]$.



No local maximum or minimum.
No global maximum.
Global minimum value is 1.

d $f: [0, \infty) \to R, f(x) = x^3 + 6x^2 + 3x - 10$

Let $y = f(x)$
Endpoint: When $x = 0$, $y = -10$. Left endpoint and y intercept $(0, -10)$.
Stationary points: $f'(x) = 3x^2 + 12x + 3$
At stationary points, $3x^2 + 12x + 3 = 0$

$\therefore 3\left(x^2 + 4x + 1\right) = 0$

$\therefore \left(x^2 + 4x + 4\right) - 4 + 1 = 0$

$\therefore (x + 2)^2 = 3$

$\therefore x = -2 \pm \sqrt{3}$

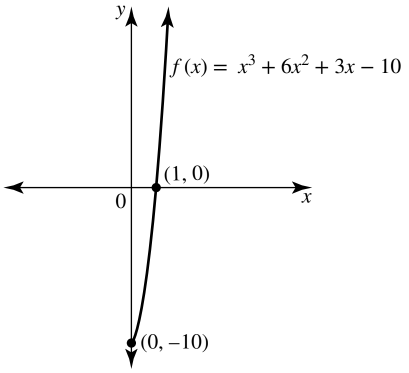
$-2 - \sqrt{3} < 0$ and $-2 + \sqrt{3} < 0$ so there are no stationary points in the given domain $[0, \infty)$.
x intercepts occur where $f(x) = x^3 + 6x^2 + 3x - 10 = 0$
$f(1) = 1 + 6 + 3 - 10 = 0$
$\therefore (x - 1)$ is a factor

$\therefore x^3 + 6x^2 + 3x - 10 = (x - 1)\left(x^2 + 7x + 10\right) = 0$

$\therefore (x - 1)(x + 2)(x + 5) = 0$

$\therefore x = 1, x = -2, x = -5$

Only $x = 1$ in the domain, so there is one x intercept at $(1, 0)$.



No local maximum or minimum.
No global maximum.
Global minimum value is −10.

15 $f(x)=2\sqrt{x}+\frac{1}{x},\ 0.25\le x\le 5$

a A is the left endpoint ⇒ $x=0.25$

$$\begin{aligned}f(0.25)&=2\sqrt{0.25}+\frac{1}{0.25}\\&=2\times 0.5+4\\&=5\end{aligned}$$

A is the point $(0.25,5)$.
C is the right endpoint ⇒ $x=5$

$$f(5)=2\sqrt{5}+\frac{1}{5}$$

C is the point $\left(5,2\sqrt{5}+0.2\right)$.
B is the stationary point so $f'(x)=0$ at B.

$$f(x)=2x^{\frac{1}{2}}+x^{-1}$$

$$\therefore f'(x)=x^{-\frac{1}{2}}-x^{-2}$$

$$\therefore f'(x)=\frac{1}{\sqrt{x}}-\frac{1}{x^2}$$

At B, $\frac{1}{\sqrt{x}}-\frac{1}{x^2}=0$

$$\therefore \frac{1}{\sqrt{x}}=\frac{1}{x^2}$$

$$\therefore \frac{x^2}{x^{\frac{1}{2}}}=1$$

$$\therefore x^{\frac{3}{2}}=1$$

$$\therefore x=1^{\frac{2}{3}}$$

$$\therefore x=1$$

$$f(1)=2+1=3$$

B is the point $(1,3)$.

b A $(0.25,5)$ and C $\left(5,2\sqrt{5}+0.2\right)$.
$2\sqrt{5}+0.2\simeq 4.67<5$
The global maximum occurs at point A.

c The global maximum value is 5.
The global minimum occurs at B. The global minimum value is 3.

16 **a** $y=ax^3+bx^2+cx+d$
Since $(0,0)$ lies on the curve, $d=0$.

$$\therefore y=ax^3+bx^2+cx$$

$$\frac{dy}{dx}=3ax^2+2bx+c$$

The tangent at $(0,0)$ has gradient $m=\tan(135°)$

$$\therefore m=-\tan(45°)$$

$$\therefore m=-1$$

At $(0,0)$, $\frac{dy}{dx}=-1$

$$\therefore -1=c$$

Hence, $c=-1, d=0$ and $y=ax^3+bx^2-x$.

b The point $(2,-2)$ lies on the curve.

$$\therefore -2=8a+4b-2$$

$$\therefore 8a+4b=0$$

$$\therefore b=-2a....(1)$$

The point $(2,-2)$ is a stationary point so $\frac{dy}{dx}=0$ at this point.

$$\frac{dy}{dx}=3ax^2+2bx-1$$

$$\therefore 12a+4b-1=0....(2)$$

Substitute $b=-2a$ in equation (2)

$$\therefore 12a-8a-1=0$$

$$\therefore 4a=1$$

$$\therefore a=\frac{1}{4}$$

Substitute $a=\frac{1}{4}$ in equation (1)

$$\therefore b=-\frac{1}{2}$$

Hence, $a=\frac{1}{4}, b=-\frac{1}{2}$ and $y=\frac{1}{4}x^3-\frac{1}{2}x^2-x$.

c $\frac{dy}{dx}=\frac{3}{4}x^2-x-1$

At stationary points, $\frac{3}{4}x^2-x-1=0$

$$\therefore 3x^2-4x-4=0$$

$$\therefore (3x+2)(x-2)=0$$

$$\therefore x=-\frac{2}{3}, x=2$$

The second stationary point occurs at $x=-\frac{2}{3}$.

When $x=-\frac{2}{3}$,

$$\begin{aligned}y&=\frac{1}{4}\times -\frac{8}{27}-\frac{1}{2}\times\frac{4}{9}+\frac{2}{3}\\&=-\frac{2}{27}-\frac{2}{9}+\frac{2}{3}\\&=\frac{-2-6+18}{27}\\&=\frac{10}{27}\end{aligned}$$

The other stationary point is $\left(-\frac{2}{3},\frac{10}{27}\right)$.

d Since $(2,-2)$ is a stationary point, the tangent to the curve at this point is the horizontal line $y=-2$.

This line intersects $y=\frac{1}{4}x^3-\frac{1}{2}x^2-x$ when $\frac{1}{4}x^3-\frac{1}{2}x^2-x=-2$.

$$\therefore x^3-2x^2-4x=-8$$

$$\therefore x^3-2x^2-4x+8=0$$

$$\therefore x^2(x-2)-4(x-2)=0$$

$$\therefore (x-2)\left(x^2-4\right)=0$$

$$\therefore (x-2)(x-2)(x+2)=0$$

$$\therefore (x-2)^2(x+2)=0$$

$\therefore x=2$(touch) or $x=-2$(cut)
The tangent meets the curve again when $x=-2$
When $x=-2$, $y=-\frac{1}{4}\times 8-\frac{1}{2}\times 4+2=-2$
The tangent meets the curve again at the point $(-2,-2)$.

17 **a** One approach is to Define $f(x)=-0.625x^3+7.5x^2-20x$ in the main menu. Then in the Graph&Tab menu, enter $y1=f(x)$. Graph the function and then select Max from the Analysis → G-Solve options to obtain the maximum turning point as $(6.31,15.40)$. The minimum turning point of $(1.69,-15.40)$ is obtained by selecting Min from the Analysis → G-Solve options.

b To sketch the derivative function, enter $y2=\frac{d}{dx}(f(x))$ and sketch. The maximum turning point $(4,10)$ is obtained by selecting Max from the Analysis → G-Solve options.

c The gradient reaches its greatest positive value when $x=4$. This means the point on $y=f(x)$ where $x=4$ will be the point at which the curve is steepest.

Evaluate in the Main menu to find that $f(4)=0$. Hence the gradient of $y=f(x)$ has its greatest positive gradient at the point $(4,0)$.

18 a $y=\frac{1}{96}(x+2)^3(x-3)(x-4)^2$.

Consider the x intercepts and their multiplicity.

$x=-2$ has multiplicity 3 so there is a stationary point of inflection at $(-2,0)$ as the graph saddle cuts the x axis.

$x=3$ has multiplicity 1 so the graph cuts the x axis at $(3,0)$.

$x=4$ has multiplicity 2 so there is a turning point at $(4,0)$ as the graph touches the x axis.

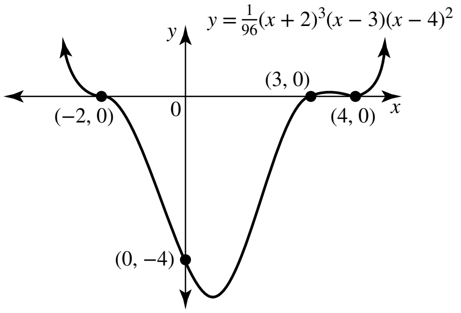
y intercept: Let $x=0$

$$\therefore y=\frac{1}{96}(2)^3(-3)(-4)^2$$

$$\therefore y=-4$$

$(0,-4)$

The polynomial function has degree 6 and a positive leading coefficient. The graph starts above the x axis.

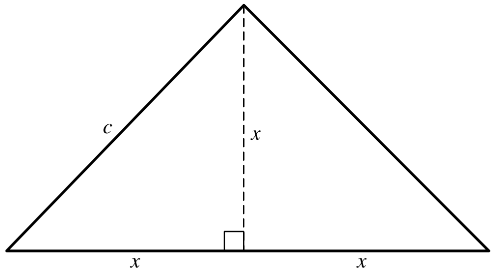


There are 4 stationary points: one stationary point of inflection, two minimum turning points and one maximum turning point.

b There is no global maximum. The global minimum can be found by tapping Analysis → G-Solve → fMin. It occurs at the minimum turning point $(0.79,-5.15)$. The global minimum value is -5.15 correct to two decimal places.

Exercise 13.6 — Optimisation problems

1 a



Consider the isosceles triangle to find the lengths of its sloping sides.

Using Pythagoras' theorem, $c^2=x^2+x^2$, so the sloping sides have lengths $\sqrt{2}x$ cm.

Since the perimeter of the figure is 150 cm,

$$2x+2y+2\sqrt{2}x=150$$

$$\therefore y=75-x-\sqrt{2}x$$

The area of the figure is the sum of the areas of the rectangle and the triangle, base $2x$, height x.

$$\begin{aligned}A&=2xy+\frac{1}{2}(2x)x\\&=2x(75-x-\sqrt{2}x)+x^2\\&=150x-2x^2-2\sqrt{2}x^2+x^2\\&=150x-\left(1+2\sqrt{2}\right)x^2\end{aligned}$$

b To find the maximum area, let $\frac{dA}{dx}=0$.

$$150-2\left(1+2\sqrt{2}\right)x=0$$

$$\begin{aligned}x&=\frac{75}{1+2\sqrt{2}}\\&\approx 19.6\end{aligned}$$

As the area function is a concave down parabola, the area is greatest when $x\approx 19.6$.

Substitute $x\approx 19.6$ to find y.

$$\begin{aligned}y&=75-\left(1+\sqrt{2}\right)x\\&=75-\left(1+\sqrt{2}\right)\times\frac{75}{1+2\sqrt{2}}\\&\approx 27.7\end{aligned}$$

Width: $2x=39.2$; total height: $y+x=47.3$

Therefore figure has width of 39.2 cm and total height of 47.3 cm for greatest area.

c If the width of the figure cannot exceed 30 cm, $x\le 15$ so the maximum area at $x\approx 19.6$ is not in the domain. Hence the maximum area will occur when $x=15$ and therefore $y=75-15\left(1+\sqrt{2}\right)\approx 38.8$.

The dimensions of the figure for maximum area are now width 30 cm and height 53.8 cm.

2 a Box has length $=(20-2x)$ cm, width $=(12-2x)$ cm and height $=x$ cm.

Therefore the volume, V cm^3, is $V=x(20-2x)(12-2x)$.

$$\therefore V=240x-64x^2+4x^3$$

b Greatest volume occurs when $\frac{dV}{dx}=0$.

$$240-128x+12x^2=0$$

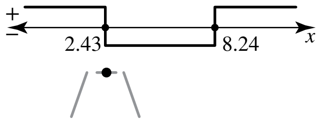
$$4(3x^2-32x+60)=0$$

Using the formulas for solving quadratic equations gives:

$$x=\frac{32\pm\sqrt{(32)^2-4(3)(60)}}{6}$$

$x\approx 2.43$ or $x\approx 8.24$

Sign diagram of $\frac{dV}{dx}$:

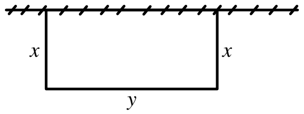


Maximum volume occurs for $x=2.43$.

(Alternatively, consider the domain which would require $x\in[0,6]$ and the shape of the volume function's graph.)

Therefore, the box with length 15.14 cm, width 7.14 cm and height 2.43 cm has the greatest volume of 32 cm^3, to the nearest whole number.

3 a Let width be x metres and length y metres.



There is an amount of 40 metres of fencing available.

$$\therefore 2x+y=40$$

$$\therefore y=40-2x$$

Area, A sq m, of the rectangular garden is

$$A=xy$$

$$\therefore A=x(40-2x)$$

$$\therefore A=40x-2x^2$$

b $\frac{dA}{dx} = 40 - 4x$

At the maximum area, $\frac{dA}{dx} = 0$

$\therefore 40 - 4x = 0$

$\therefore x = 10$

When $x = 10$, $y = 40 - 20 = 20$.

The dimensions for maximum area are width 10 metres and length 20 metres.

c The maximum area is 200 sq m.

d The area function is restricted to $A = 40x - 2x^2,\ 5 \le x \le 7$ so the maximum turning point occurs outside the domain $[5, 7]$. In this case, the greatest area will occur at an endpoint of the domain.

$A(5) = 200 - 50 = 150$

$A(7) = 280 - 98 = 182$

The greatest area that can be enclosed is 182 sq m.

4 $C = n^3 - 10n^2 - 32n + 400,\ 5 \le n \le 10$

$\frac{dC}{dn} = 3n^2 - 20n - 32$

At stationary points, $\frac{dC}{dn} = 0$

$\therefore 3n^2 - 20n - 32 = 0$

$\therefore (3n + 4)(n - 8) = 0$

$\therefore n = -\frac{4}{3}$(reject) or $n = 8$.

Test the slope of the function around $n = 8$ to determine the nature of the stationary point.

n	7	8	9
$\frac{dC}{dn}$	$(25)(-1) = -25$	0	$(33)(1) = 33$
Slope of tangent	negative	zero	positive

There is a minimum turning point at $n = 8$.

As the cost function is a cubic polynomial, $n = 8$ will be the value in the domain $[5,10]$ for which the cost is minimised.

Therefore, 8 people should be employed in order to minimise the cost.

5 **a** $y = 0.0001x^2\left(625 - x^2\right)$

$\therefore y = 0.0625x^2 - 0.0001x^4$

$\frac{dy}{dx} = 0.1250x - 0.0004x^3$

At greatest height, $\frac{dy}{dx} = 0$

$\therefore 0.1250x - 0.0004x^3 = 0$

$\therefore x\left(0.1250 - 0.0004x^2\right) = 0$

$\therefore x = 0$ (reject) or $0.1250 - 0.0004x^2 = 0$

$\therefore x^2 = \frac{0.1250}{0.0004}$

$\therefore x^2 = \frac{1250}{4}$

$\therefore x = \pm\frac{\sqrt{1250}}{2}$

$\therefore x = \pm\frac{25\sqrt{2}}{2}$

Reject the negative value

$\therefore x = \frac{25\sqrt{2}}{2} \simeq 17.68$

Test the slope of the curve either side of this value

x	17	$\frac{25\sqrt{2}}{2}$	18
$\frac{dy}{dx}$	$0.125 \times 17 - 0.0004 \times 17^3$ $\simeq 0.16$	0	$0.125 \times 18 - 0.0004 \times 1^3$ $\simeq 0.08$
slope	positive	zero	negative

There is a maximum turning point at $x = \frac{25\sqrt{2}}{2}$

When $x = \frac{25\sqrt{2}}{2}$, $x^2 = \frac{625 \times 2}{4} = \frac{625}{2}$

$\therefore y = 0.0001 \times \frac{625}{2}\left(625 - \frac{625}{2}\right)$

$\therefore y \simeq 9.77$

The greatest height the ball reaches is 9.77 metres above the ground.

b When the ball strikes the ground, $y = 0$

$\therefore 0.001x^2\left(625 - x^2\right) = 0$

$\therefore x = 0$ or $x^2 = 625$

$\therefore x = 0, x = 25, x = -25$

Only $x = 25$ is a practical solution.

Therefore the ball travels 25 metres horizontally before it strikes the ground.

6 **a** TSA of cylinder is the sum of the areas of the two circular ends together with the curved surface area.

$\therefore 200 = 2\pi r^2 + 2\pi rh$

$\therefore \pi r^2 + \pi rh = 100$

$\therefore \pi rh = 100 - \pi r^2$

$\therefore h = \frac{100 - \pi r^2}{\pi r}$

b The formula for the volume of a cylinder is $V = \pi r^2 h$

$\therefore V = \left(\cancel{\pi r^2}\right) \times \frac{100 - \pi r^2}{\cancel{\pi r}}$

$\therefore V = r\left(100 - \pi r^2\right)$

$\therefore V = 100r - \pi r^3$

c $\frac{dV}{dr} = 100 - 3\pi r^2$

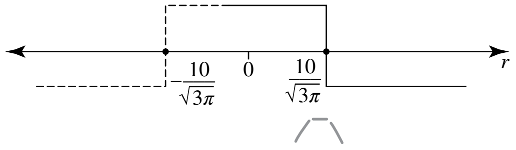
At maximum volume, $\frac{dV}{dr} = 0$

$\therefore 100 - 3\pi r^2 = 0$

$\therefore r^2 = \frac{100}{3\pi}$

$\therefore r = \frac{10}{\sqrt{3\pi}}$ (reject negative square root)

Sign of $\frac{dV}{dr}$ is that of a concave down parabola with zeros at $r = \pm\frac{10}{\sqrt{3\pi}}$



There is a maximum turning point at $r = \frac{10}{\sqrt{3\pi}}$.

When $r = \frac{10}{\sqrt{3\pi}}$,

$$h=\left(100-\pi\times\frac{100}{3\pi}\right)\div\pi\times\frac{10}{\sqrt{3\pi}}$$
$$=\left(100-\frac{100}{3}\right)\div\frac{10\sqrt{\pi}}{\sqrt{3}}$$
$$=\frac{200}{3}\times\frac{\sqrt{3}}{10\sqrt{\pi}}$$
$$=\frac{20\sqrt{3}}{3\sqrt{\pi}}$$
$$=\frac{20}{\sqrt{3\pi}}$$
$$=2\times\frac{10}{\sqrt{3\pi}}$$
$$=2r$$

For maximum volume, the height is equal to twice the radius, or, the height is equal to the diameter, of the base of the circular cylinder.

d $r=\frac{10}{\sqrt{3\pi}}\simeq 3.26$ is the value where the maximum volume occurs.

For the interval $2\le r\le 4$, there are no other stationary points. Hence the minimum volume must occur at one of the endpoints $r=2$ or $r=4$.

$V=100r-\pi r^3$

$V(2)=200-8\pi\simeq 174.88$

$V(4)=400-64\pi\simeq 198.94$

For the given restriction, the minimum volume is 175 cubic cm, correct to the nearest integer.

7 Refer to the diagram given in the question.

a The volume is required as a function of h only, so it is necessary to express r in terms of h.

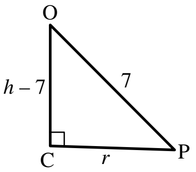
In the diagram, $OV=OP=7$ as theses lengths are radii of the sphere.

$CV=CO+OV$

$\therefore h=CO+7$

$\therefore CO=h-7$

Consider the right angle triangle COP.



Using Pythagoras' theorem,

$(h-7)^2+r^2=7^2$

$\therefore r^2=49-(h-7)^2$

$\therefore r^2=49-\left(h^2-14h+49\right)$

$\therefore r^2=14h-h^2$

The volume of the cone is $V=\frac{1}{3}\pi r^2h$

$\therefore V=\frac{1}{3}\pi\left(14h-h^2\right)h$

$\therefore V=\frac{1}{3}\pi\left(14h^2-h^3\right)$

b $\frac{dV}{dh}=\frac{1}{3}\pi\left(28h-3h^2\right)$

$\therefore\frac{dV}{dh}=\frac{1}{3}\pi h(28-3h)$

At the maximum volume, $\frac{dV}{dh}=0$

$\therefore\frac{1}{3}\pi h(28-3h)=0$

$\therefore h=0\text{(reject) or } h=\frac{28}{3}$

Test the slope of the function about $h=\frac{28}{3}=9\frac{1}{3}$

h	9	$\frac{28}{3}$	10
$\frac{dV}{dh}$	$\frac{1}{3}\pi\times9\times(1)=3\pi$	0	$\frac{1}{3}\pi\times10\times(-2)=-\frac{20\pi}{3}$
slope	positive	zero	negative

There is a maximum turning point when $h=\frac{28}{3}$.

The volume is greatest when $h=\frac{28}{3}$.

Substitute $h=\frac{28}{3}$ in $r^2=49-(h-7)^2$

$\therefore r^2=49-\left(\frac{28}{3}-\frac{21}{3}\right)^2$

$\therefore r^2=49-\frac{49}{9}$

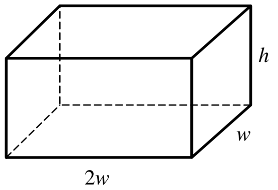
$\therefore r^2=\frac{8}{9}\times 49$

$\therefore r=\frac{2\sqrt{2}}{3}\times 7$ (reject negative square root)

$\therefore r=\frac{14\sqrt{2}}{3}$

When the height measure, $h=\frac{28}{3}$ and the base radius measure $r=\frac{14\sqrt{2}}{3}$, the volume of the cone is greatest.

8 a The rectangular prism has length $2w$ cm, width w cm and height h cm.



Surface area is the sum of the areas of the six rectangular faces.

$SA=2\times(2w\times w)+2\times(2w\times h)+2\times(w\times h)$

$=4w^2+4wh+2wh$

$=4w^2+6wh$

b Give the surface area is 300 cm^2, then

$4w^2+6wh=300$

$\therefore 2w^2+3wh=150$

$\therefore 3wh=150-2w^2$

$\therefore h=\frac{150-2w^2}{3w}$

The volume, V, of the prism is

$V=2w\times w\times h$

$\therefore V=2w^2h$

$\therefore V=2w^2\left(\frac{150-2w^2}{3w}\right)$

$\therefore V=\frac{2}{3}w\left(150-2w^2\right)$

$\therefore V=100w-\frac{4}{3}w^3$

$\frac{dV}{dw} = 100 - 4w^2$

At maximum volume, $\frac{dV}{dw} = 0$

$\therefore 100 - 4w^2 = 0$

$\therefore w^2 = 25$

$\therefore w = 5$ (reject negative square root)

To justify the nature of the turning point, consider the slope of the tangent either side of $w = 5$.

w	4	5	6
$\frac{dV}{dw}$	$100 - 64 = 36$	0	$100 - 144 = -44$
slope	positive	zero	negative

There is a maximum turning point when $w = 5$.

$$\begin{aligned}\therefore V_{\max} &= 100 \times 5 - \frac{4}{3} \times 125\\ &= \frac{1500 - 500}{3}\\ &= \frac{1000}{3}\end{aligned}$$

The maximum volume is $\frac{1000}{3}$ cubic cm.

c The maximum volume occurs when $w = 5$. Substitute $w = 5$ in $h = \frac{150 - 2w^2}{3w}$,

$\therefore h = \frac{150 - 50}{15}$

$\therefore h = \frac{20}{3}$

The dimensions of the rectangular prism with greatest volume are length 10 cm, width 5 cm and height $\frac{20}{3}$ cm.

d Since $5 \in [2, 6]$, the greatest volume is $\frac{1000}{3}$ cubic cm.

There is no other turning point in the interval $w \in [2, 6]$.

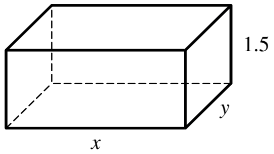
Test the value of $V = 100w - \frac{4}{3}w^3$ at the endpoints $w = 2$ and $w = 6$ to determine the minimum value over the domain $[2, 6]$.

$$\begin{aligned}V(2) &= 200 - \frac{32}{3}\\ &= \frac{568}{3}\\ V(6) &= 600 - 4 \times 72\\ &= 312\\ &= \frac{936}{3}\end{aligned}$$

The minimum volume is $\frac{568}{3}$ cubic cm.

In cubic cm, the range of values for the volume over the restricted domain is $\left[\frac{568}{3}, \frac{1000}{3}\right]$.

9



The volume of the bin is 12 cubic metres.

$\therefore 12 = 1.5xy$

$\therefore y = \frac{12}{1.5x}$

$\therefore y = \frac{8}{x}$

Let the cost in dollars be C.

$$\begin{aligned}C &= 10 \times (2 \times 1.5y + 2 \times 1.5x) + 25 \times (xy)\\ &= 30y + 30x + 25xy\end{aligned}$$

Substitute $y = \frac{8}{x}$

$\therefore C = \frac{240}{x} + 30x + 200$

For the minimum cost, $\frac{dC}{dx} = 0$

$C = 240x^{-1} + 30x + 200$

$\frac{dC}{dx} = -240x^{-2} + 30$

$\therefore \frac{-240}{x^2} + 30 = 0$

$\therefore 30 = \frac{240}{x^2}$

$\therefore x^2 = 8$

$\therefore x = 2\sqrt{2}$

(reject negative square root)

To justify $x = 2\sqrt{2}$ gives the minimum cost, consider the slope diagram for $x = \sqrt{4} < x = \sqrt{8} < x = \sqrt{9}$.

x	2	$2\sqrt{2}$	3
$\frac{dC}{dx}$	$-\frac{240}{4} + 30 = -30$	0	$-\frac{240}{9} + 30 = \frac{10}{3}$
slope	negative	zero	positive

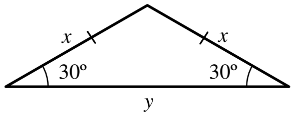
There is a minimum turning point when $x = 2\sqrt{2}$.

$$\begin{aligned}C_{\min} &= \frac{240}{2\sqrt{2}} + 60\sqrt{2} + 200\\ &= 60\sqrt{2} + 60\sqrt{2} + 200\\ &= 120\sqrt{2} + 200\end{aligned}$$

$\therefore C_{\min} \simeq 370$

The cost of the cheapest bin is \$370.

10 **a**



Area of a triangle is given by the formula $A_\Delta = \frac{1}{2}ab\sin C$,

$\therefore A = \frac{1}{2}xy\sin(30°)$

$\therefore A = \frac{1}{2}xy \times \frac{1}{2}$

$\therefore A = \frac{1}{4}xy$

b The perimeter P of the triangle is $P = 2x + y$

Given $A = 15$,

$15 = \frac{1}{4}xy$

$\therefore y = \frac{60}{x}$

Hence, $P = 2x + \frac{60}{x}$

$\therefore P = 2x + 60x^{-1}$

$\therefore \frac{dP}{dx} = 2 - \frac{60}{x^2}$

At minimum, $\frac{dP}{dx} = 0$

$\therefore 2 - \frac{60}{x^2} = 0$

$\therefore 2 = \frac{60}{x^2}$

$\therefore x^2 = 30$

$\therefore x = \sqrt{30}$

When $x = \sqrt{30}$,

$y = \frac{60}{\sqrt{30}}$

$= \frac{60\sqrt{30}}{30}$

$= 2\sqrt{30}$

Testing slope around $x = \sqrt{30}$ for
$x = \sqrt{25} < x = \sqrt{30} < x = \sqrt{36}$

x	5	$\sqrt{30}$	6
$\frac{dP}{dx}$	$2 - \frac{60}{25} = -\frac{2}{5}$	0	$2 - \frac{60}{36} = \frac{1}{3}$
slope	negative	zero	positive

There is a minimum turning point when $x = \sqrt{30}$.
The perimeter is least for $x = \sqrt{30}$ and $y = 2\sqrt{30}$.

11 Refer to the diagram given in the question.

a Let the perimeter be P metres.

$P = r + l + r$

$= 2r + l$

Since the arc length $l = r\theta$, then $P = 2r + r\theta$.
Given $P = 8$

$\therefore 2r + r\theta = 8$

$\therefore r\theta = 8 - 2r$

$\therefore \theta = \frac{8-2r}{r}$

b The formula for the area of a sector is $A = \frac{1}{2}r^2\theta$

$\therefore A = \frac{1}{2}r^2 \times \frac{8-2r}{r}$

$\therefore A = \frac{1}{2}r \times 2(4-r)$

$\therefore A = r(4-r)$

$\therefore A = 4r - r^2$

c $\frac{dA}{d\theta} = 4 - 2r$

At maximum, $\frac{dA}{d\theta} = 0$

$\therefore 4 - 2r = 0$

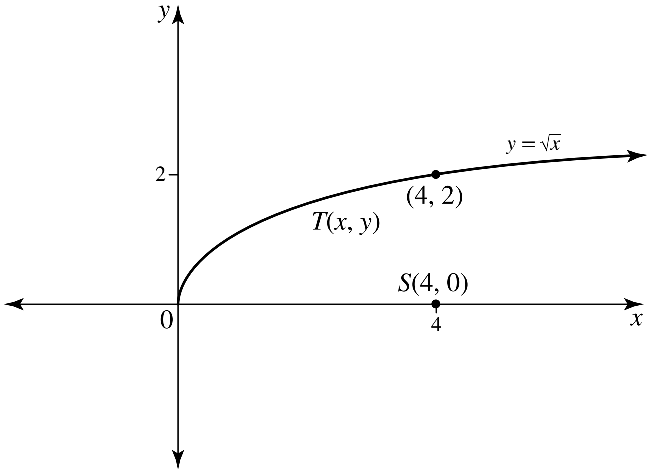
$\therefore r = 2$

As the area function is a concave down parabola, the stationary point must be a maximum.

When $r = 2$, $\theta = \frac{8-4}{2} = 2$.

For greatest area, the value of θ is 2 radians.

12 a $y = \sqrt{x}$



The point on $y = \sqrt{x}$ for which $x = 4$ is $(4, 2)$. The distance between the points $(4, 0)$ and $(4, 2)$ is 2 units.
The tram route is 2 km directly north of Shirley's position.

b Let T (x, y) be any point on the tram route $y = \sqrt{x}$.

The distance TS is $\sqrt{(x-4)^2 + (y-0)^2}$.
The function W is the square of this distance.

$\therefore W = \left(\sqrt{(x-4)^2 + y^2}\right)^2$

$\therefore W = (x-4)^2 + y^2$

Since T lies on $y = \sqrt{x}$,

$W = (x-4)^2 + \left(\sqrt{x}\right)^2$

$\therefore W = (x-4)^2 + x$

$\therefore W = x^2 - 8x + 16 + x$

$\therefore W = x^2 - 7x + 16$

c As W is a concave up quadratic function, it has a minimum turning point when $\frac{dW}{dx} = 0$.

$\frac{dW}{dx} = 2x - 7$

$\therefore 2x - 7 = 0$

$\therefore x = \frac{7}{2}$

W is minimised when $x = \frac{7}{2}$.

d When $x = \frac{7}{2}$, $y = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$.

The point T $\left(\frac{7}{2}, \frac{\sqrt{14}}{2}\right)$ is the closest point on the tram route to Shirley.

e When $x = \frac{7}{2}$,

$W = \frac{49}{4} - \frac{49}{2} + 16$

$= -\frac{49}{4} + 16$

$= \frac{15}{4}$

Since the least value of W is $\frac{15}{4}$, and W is the square of the distance TS,

The distance TS is $\sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$ km.

Shirley's speed is 5 km/h so the time it takes her to walk from S to T is $\frac{\sqrt{15}}{2} \div 5 \simeq 0.387$ hours. Multiply the hours by 60 to obtain the time in minutes.
To the nearest minute it takes Shirley 23 minutes to walk straight to the point.

f $y=\sqrt{x} \Rightarrow y = x^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2\sqrt{x}}$$

At T $\left(\frac{7}{2},\frac{\sqrt{14}}{2}\right)$,

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\frac{7}{2}}}$$

$$=\frac{1}{\sqrt{2}\sqrt{7}}$$

$$=\frac{1}{\sqrt{14}}$$

The gradient of the tangent to the curve at T is $m_1=\frac{1}{\sqrt{14}}$.

The gradient of the line through T $\left(\frac{7}{2},\frac{\sqrt{14}}{2}\right)$ and S $(4,0)$ is

$$m_2=\left(0-\frac{\sqrt{14}}{2}\right)\div\left(4-\frac{7}{2}\right)$$

$$=-\frac{\sqrt{14}}{2}\div\frac{1}{2}$$

$$=-\frac{\sqrt{14}}{2}\times\frac{2}{1}$$

$$=-\sqrt{14}$$

$$m_1\times m_2=\frac{1}{\sqrt{14}}\times-\sqrt{14}$$

$$\therefore m_1m_2=-1$$

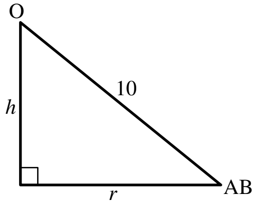
The line ST and the tangent at T are perpendicular, so Shirley's calculation is correct.

13 a Refer to the diagram given in the question.
The circumference of the circular base of the cone is the length of the major arc AB. The arc AB is part of the circle with radius 10 cm.

$$\therefore 2\pi r = 10\theta$$

$$\therefore r=\frac{5\theta}{\pi}$$

b Let the height of the cone be h cm.



Using Pythagoras' theorem,

$$h^2+r^2=10^2$$

$$\therefore h^2=100-r^2$$

$$\therefore h^2=100-\left(\frac{5\theta}{\pi}\right)^2$$

$$\therefore h=\sqrt{100-\frac{25\theta^2}{\pi^2}}$$

(negative square root not applicable).

c $V=\frac{1}{3}\pi r^2 h$

$$\therefore V=\frac{1}{3}\pi r^2\sqrt{100-\frac{25\theta^2}{\pi^2}}$$

$$\therefore V=\frac{1}{3}\pi\times\frac{25\theta^2}{\pi^2}\sqrt{100-\frac{25\theta^2}{\pi^2}}$$

This can be simplified using CAS, but the question says 'show' so we continue to simplify the volume expression ourselves.

$$V=\frac{25\theta^2}{3\pi}\sqrt{\frac{100\pi^2}{\pi^2}-\frac{25\theta^2}{\pi^2}}$$

$$=\frac{25\theta^2}{3\pi}\sqrt{\frac{100\pi^2-25\theta^2}{\pi^2}}$$

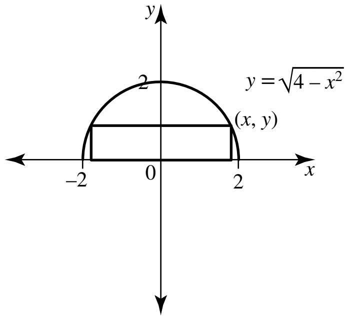
$$=\frac{25\theta^2}{3\pi}\times\frac{\sqrt{25\left(4\pi^2-\theta^2\right)}}{\sqrt{\pi^2}}$$

$$=\frac{25\theta^2}{3\pi}\times\frac{5\sqrt{4\pi^2-\theta^2}}{\pi}$$

$$=\frac{125\theta^2}{3\pi^2}\sqrt{4\pi^2-\theta^2}$$

d Solve $\frac{d}{d\theta}\left(\frac{125\theta^2}{3\pi^2}\sqrt{4\pi^2-\theta^2}\right)=0$ to obtain the value of θ, in radians, for which the volume of the cone is greatest.
Enter the expression in the main menu, highlight and tap Interactive → Equation/Inequality → Solve to obtain the solutions $\theta=0,\theta=-\frac{2\sqrt{6}\pi}{3},\theta=\frac{2\sqrt{6}\pi}{3}$. Of these the only applicable solution is $\theta=\frac{2\sqrt{6}\pi}{3}$.
To convert the radian value to degrees, multiply by $\frac{180}{\pi}$ to obtain $\theta°=120\sqrt{6}$. Switch from Standard to Decimal mode to obtain $\theta°=294°$, to the nearest degree.

14 $y=\sqrt{4-x^2}$



Inscribe a rectangle under the semicircle. Let the vertex of the rectangle which lies on the semicircle in the first quadrant be the point (x, y), with $x>0$ and $y>0$.
The length of the rectangle is $2x$ and the height is y.
The area of the rectangle $A=2xy$.
Since $y=\sqrt{4-x^2}$, $A=2x\sqrt{4-x^2}$.

Using CAS, $\frac{dA}{dx}=\frac{-\left(4x^2-8\right)}{\sqrt{-x^2+4}}$

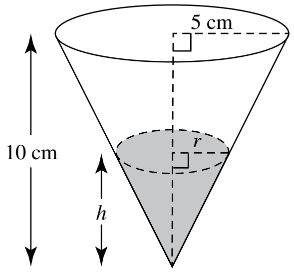
The greatest area occurs when $\frac{dA}{dx}=0$

$$\therefore\frac{-\left(4x^2-8\right)}{\sqrt{-x^2+4}}=0$$

Solving this in Equation/Inequality, gives $x = -\sqrt{2}, x = \sqrt{2}$.
Since $x > 0$, $x = \sqrt{2}$.
Substituting this value in $A = 2x\sqrt{4 - x^2}$ gives $A_{\max} = 4$.
The area of the largest rectangle is 4 square units.

Exercise 13.7 — Rates of change and kinematics

1 a



Using similar triangles,

$$\frac{r}{h} = \frac{5}{10}$$

$$\therefore r = \frac{1}{2}h$$

b Volume of water is $V = \frac{1}{3}\pi r^2 h$.

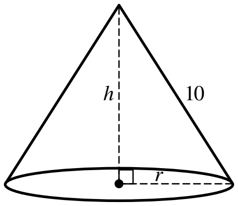
$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$
$$= \frac{1}{12}\pi h^3$$

c Rate: $\frac{dV}{dh} = \frac{1}{4}\pi h^2$

When $h = 3$, $\frac{dV}{dh} = \frac{9\pi}{4}$

With respect to its depth, the volume is changing at the rate of $\frac{9\pi}{4}$ cm^3/cm when depth is 3 cm.

2 a



Using Pythagoras' theorem:

$h^2 + r^2 = 10^2$

$\therefore r^2 = 100 - h^2$

Since $r > 0$, $r = \sqrt{100 - h^2}$

b Volume of cone is $V = \frac{1}{3}\pi r^2 h$.

$$V = \frac{1}{3}\pi(100 - h^2)h$$
$$= \frac{1}{3}\pi(100h - h^3)$$

c Rate: $\frac{dV}{dh} = \frac{1}{3}\pi(100 - 3h^2)$

When $h = 6$, $\frac{dV}{dh} = -\frac{8\pi}{3}$

With respect to its height, the volume is decreasing at the rate of $\frac{8\pi}{3}$ cm^3/cm when the height is 6 cm.

3 $x = 3t^2 - 6t,\ t \geq 0$

a When $t = 0$, $x = 0$ so initially the particle is at the origin.

b Velocity: $v = \frac{dx}{dt}$

$\therefore v = 6t - 6$

The initial velocity is –6m/s. The particle starts to move to the left.

c When $v = 0$, $6t - 6 = 0$

Therefore, $t = 1$ and when $t = 1$, $x = -3$.

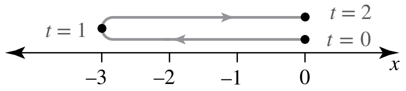
The particle is momentarily at rest after 1 second at the position 3 m to the left of the origin.

d When $t = 2$,

$x = 3(4) - 6(2)$

$= 0$

Therefore the particle is at the origin after 2 seconds.



The path of the particle shows the distance travelled is 3 m to the left and 3 m to the right, giving a total distance of 6 m travelled in the first 2 seconds.

e Average speed $= \frac{\text{distance}}{\text{time}}$

Therefore average speed $= \frac{6}{2} = 3$ m/s

f Average velocity $= \frac{x(2) - x(0)}{2 - 0} = 0$ m/s

4 $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1$

a $x(0) = 1$ so particle is initially 1 metre to the right of the origin.

$v(t) = x'(t)$

$= -t^2 + 2t + 8$

$\therefore v(0) = 8$ so initial velocity is 8 m/s.

b Particle changes its direction of motion when velocity is zero.

$-t^2 + 2t + 8 = 0$

$-(t - 4)(t + 2) = 0$

$t = 4, t = -2$

Since $t \geq 0$, velocity is zero when $t = 4$.

$$x(4) = -\frac{64}{3} + 16 + 32 + 1$$
$$= 17\frac{2}{3}$$

Distance travelled is $17\frac{2}{3} - 1 = 16\frac{2}{3}$ metres.

c $a = v'(t)$

$\therefore a(t) = -2t + 2$

When $t = 4$, $a(4) = -6$ so the acceleration is –6 m/s^2.

5 a The area of a circle is $A = \pi r^2$.

$\therefore \frac{dA}{dr} = 2\pi r$

When $r = 0.2$, $\frac{dA}{dr} = 0.4\pi$.

The area is changing with respect to its radius at 0.4π m^2/m at the instant its radius is 0.2 metres.

b Let the length of the edge of the cube be x mm.

The surface area is the sum of the area of its six square faces.

$\therefore A = 6x^2$

The rate at which the surface area changes with respect to its side length is $\frac{dA}{dx}=12x$.

When $x=8$, $\frac{dA}{dx}=96$.

The surface area is changing at 96 mm^2/mm.

c **i** $N=\frac{125}{t}, t>0$

$\therefore N=125t^{-1}$

$\therefore \frac{dN}{dt}=-125t^{-2}$

$\therefore \frac{dN}{dt}=-\frac{125}{t^2}$

When $t=5$, $\frac{dN}{dt}=-\frac{125}{25}=-5$

The rabbit population is decreasing at 5 rabbits per month after 5 months.

ii For $t\in[1,5]$, the average rate of change of the population is

$$\frac{N(5)-N(1)}{5-1}$$
$$=\frac{25-125}{4}$$
$$=-25$$

The average rate of change of the population over the given interval is –25 rabbits per month.

6 **a** **i** An equilateral triangle has angles of 60°.

The area of a triangle is $A=\frac{1}{2}ab\sin C$

$\therefore A=\frac{1}{2}\times x\times x\times \sin(60°)$

$=\frac{1}{2}x^2\times\frac{\sqrt{3}}{2}$

$\therefore A=\frac{\sqrt{3}}{4}x^2$

ii $\frac{dA}{dx}=\frac{\sqrt{3}}{2}x$

When $x=2$, $\frac{dA}{dx}=\sqrt{3}$.

The area is changing at $\sqrt{3}$ sq cm/cm.

iii When $A=64\sqrt{3}$,

$\frac{\sqrt{3}}{4}x^2=64\sqrt{3}$

$\therefore x^2=64\sqrt{3}\times\frac{4}{\sqrt{3}}$

$\therefore x^2=256$

$\therefore x=16$

(negative square root is not applicable)

When $x=16$, $\frac{dA}{dx}=8\sqrt{3}$.

The area is changing at $8\sqrt{3}$ sq cm/cm.

b Let the rectangle have length l and width w.

Its area, $A=lw$

$\therefore lw=50$

$\therefore w=\frac{50}{l}$

The perimeter of the rectangle is $P=2l+2w$

$\therefore P=2l+\frac{100}{l}$

$\therefore P=2l+100l^{-1}$

$\frac{dP}{dl}=2-100l^{-2}$

$\therefore \frac{dP}{dl}=2-\frac{100}{l^2}$

When $l=10$, $\frac{dP}{dl}=2-\frac{100}{100}=1$

The perimeter is changing at 1 cm/cm.

7 **a** **i** Refer to the diagram given in the question. The two right angled triangles are similar because they are equiangular.

$\therefore \frac{1.5}{9}=\frac{s}{s+x}$

$\therefore \frac{1}{6}=\frac{s}{s+x}$

$\therefore s+x=6s$

$\therefore 5s=x$

$\therefore s=\frac{x}{5}$

ii $\frac{ds}{dx}=\frac{1}{5}$.

This measures the rate of change of the length of the shadow with respect to the distance of the person from the foot of the pole. It shows the length of the shadow changes at a constant rate of $\frac{1}{5}$ m/m.

b The volume of a cylinder is $V=\pi r^2h$ or $V=Ah$ where A is the area of its circular base.
As the oil leaks out the depth of water decreases but the area of the circular base does not change.
When the oil drum is full, $V=0.25$ and $h=0.75$.

$\therefore 0.25=A\times 0.75$

$\therefore A=\frac{0.25}{0.75}$

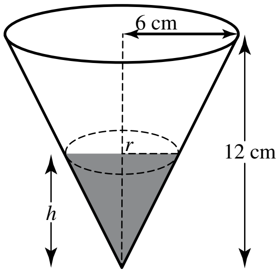
$\therefore A=\frac{1}{3}$

Hence, the volume is $V=\frac{1}{3}h$.

The rate of decrease of the volume with respect to the depth of oil is $\frac{dV}{dh}=\frac{1}{3}$.

Therefore, the volume is decreasing at $\frac{1}{3}$ m^3/m.

c **i** Consider the volume of water in the cone when the depth is h cm and the radius of the surface is r cm.



Using similar triangles,

$\frac{r}{h}=\frac{6}{12}$

$\therefore r=\frac{1}{2}h$

The volume of water is $V=\frac{1}{3}\pi r^2 h$

$$\therefore V=\frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

$$\therefore V=\frac{1}{12}\pi h^3$$

The rate of change of the volume with respect to the depth of water is $\frac{dV}{dh}=\frac{1}{4}\pi h^2$.

The height of the cone is 12 cm.

When $h=\frac{1}{2}\times 12=6$,

$$\begin{aligned}\frac{dV}{dh}&=\frac{1}{4}\pi\times 36\\&=9\pi\end{aligned}$$

The volume is changing at 9π cm^3/cm.

ii The depth when the container is one third full is required.

When the cone is full, $h=12, r=6$

$$\begin{aligned}V_{full}&=\frac{1}{3}\pi r^2 h\\&=\frac{1}{3}\pi\times 36\times 12\\&=144\pi\end{aligned}$$

At depth h, $V=\frac{1}{12}\pi h^3$

Let $V=\frac{1}{3}\times 144\pi=48\pi$

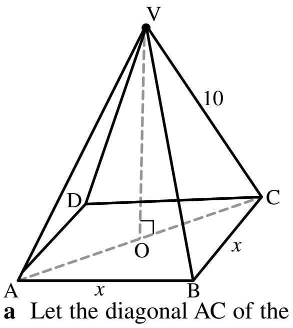
$$\begin{aligned}\therefore \frac{1}{12}\pi h^3&=48\pi\\ \therefore h^3&=48\times 12\\ \therefore h^3&=576\\ \therefore h&=\sqrt[3]{576}\end{aligned}$$

When $h=\sqrt[3]{576}$,

$$\begin{aligned}\frac{dV}{dh}&=\frac{1}{4}\pi h^2\\&=\frac{1}{4}\pi\times\left(\sqrt[3]{576}\right)^2\\&=\frac{1}{4}\pi\times(576)^{\frac{2}{3}}\\&=\frac{1}{4}\pi\times(64\times 9)^{\frac{2}{3}}\\&=\frac{1}{4}\pi\times 16\times 9^{\frac{2}{3}}\\&=4\pi\times 9^{\frac{2}{3}}\\&=4\times 3^{\frac{4}{3}}\pi\end{aligned}$$

The rate of change of the volume is $4\times 3^{\frac{4}{3}}\pi$ cm^3/cm.

8



a Let the diagonal AC of the square base be of length l metres.

Using Pythagoras' theorem in the right angled triangle ABC,

$$\begin{aligned}l^2&=x^2+x^2\\ \therefore l^2&=2x^2\\ \therefore l&=\sqrt{2}x\end{aligned}$$

(negative square root is not applicable)

The base diagonal is of length $\sqrt{2}x$ metres.

b Since AC has length $\sqrt{2}x$ metres, OC length $\frac{1}{2}\times\sqrt{2}x$ metres.

Consider the right angled triangle OCV where $OC=\frac{\sqrt{2}x}{2}, OV=h$ and $VC=10$.

Using Pythagoras' theorem,

$$\begin{aligned}h^2+\left(\frac{\sqrt{2}x}{2}\right)^2&=10^2\\ \therefore h^2+\frac{x^2}{2}&=100\\ \therefore 2h^2+x^2&=200\\ \therefore x^2&=200-2h^2\end{aligned}$$

The volume of air in the tent is $V=\frac{1}{3}Ah$

$$\therefore V=\frac{1}{3}\times\left(x^2\right)\times h$$

Substitute $x^2=200-2h^2$

$$\therefore V=\frac{1}{3}\times\left(200-2h^2\right)\times h$$

$$\therefore V=\frac{1}{3}\left(200h-2h^3\right)$$

c $\frac{dV}{dh}=\frac{1}{3}\left(200-6h^2\right)$

When $h=2\sqrt{3}$,

$$\begin{aligned}\frac{dV}{dh}&=\frac{1}{3}(200-6\times 4\times 3)\\&=\frac{128}{3}\\&=42\tfrac{2}{3}\end{aligned}$$

The volume is changing at the rate $42\frac{2}{3}$ m^3/m.

d Let $\frac{dV}{dh}=0$

$$\begin{aligned}\therefore \frac{1}{3}\left(200-6h^2\right)&=0\\ \therefore 200-6h^2&=0\\ \therefore h^2&=\frac{200}{6}\\ \therefore h^2&=\frac{100}{3}\\ \therefore h&=\frac{10}{\sqrt{3}}\end{aligned}$$

(negative square root not applicable)

The greatest volume occurs when the rate of change of the volume is instantaneously zero. Hence, the value $h=\frac{10}{\sqrt{3}}$ is the height of the tent for it to hold the greatest volume of air.

9 $x=5t-10,\ t\geq 0$

a Let $t=0$

$\therefore x=-10$

Initially, the particle is 10 cm to the left of the fixed origin.

Let $t=3$

$\therefore x=15-10=5$

After 3 seconds, the particle is 5 cm to the right of the origin.

b The distance between the positions $x=-10$ and $x=5$ is 15 cm.

c The velocity is the rate of change of the displacement,

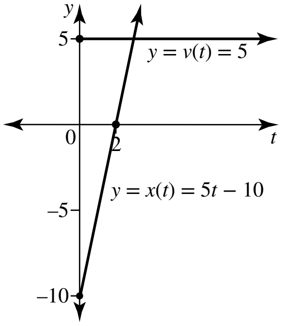
$$v=\frac{dx}{dt}$$

$\therefore v = \frac{d}{dt}(5t - 10)$

$\therefore v = 5$

The particle moves with a constant velocity of 5 cm/s.

d $x = 5t - 10,\ t \geq 0$



The velocity graph is the gradient graph of the displacement graph.

10 $x = 6t - t^2,\ t \geq 0$

a Velocity: $v = \frac{dx}{dt} = 6 - 2t$

Acceleration: $a = \frac{dv}{dt} = -2$

b Displacement-time graph: $x = 6t - t^2$

$\therefore x = t(6 - t)$

t intercepts occur at $t = 0, t = 6$

Therefore, the turning point occurs at $t = 3$.

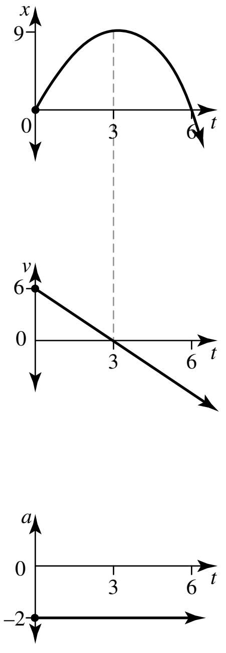
When $t = 3$, $x = 9$, so the maximum turning point is $(3, 9)$.

Velocity-time graph: $v = 6 - 2t$

Points $(0, 6)$ and $(3, 0)$.

Acceleration-time graph: $a = -2$

Horizontal line with endpoint $(0, -2)$.



The displacement-time graph is quadratic with maximum turning point when $t = 3$; the velocity-time graph is linear with $v = 0$ at the t intercept of $t = 3$; the acceleration-time graph is a horizontal line since the acceleration is constant. The acceleration is the gradient of the velocity graph.

c The velocity is zero when $t = 3$. At this time, the displacement graph is at its maximum turning point $(3, 9)$. The velocity is zero after 3 seconds when the value of x is 9. The displacement is 9 metres to the right of the origin.

d The displacement graph has a positive gradient for $0 \leq t < 3$. Over this same interval the velocity graph lies above the horizontal axis so the velocity is positive.

11 $x(t) = 3t^2 - 24t - 27,\ t \geq 0$

a $x(2) = 12 - 48 - 27 = -63$

The particle is a distance of 63 metres from O.

b $v = x'(t)$

$\therefore v(t) = 6t - 24$

$v(2) = 12 - 24$

$= -12$

The speed is 12 m/s.

c Average velocity is the average rate of change of displacement.

$= \frac{x(2) - x(0)}{2 - 0}$

$= \frac{-63 - (-27)}{2}$

$= -18$

The average velocity over the first two seconds of motion is −18 m/s.

d Let $x = 0$

$\therefore 3t^2 - 24t - 27 = 0$

$\therefore t^2 - 8t - 9 = 0$

$\therefore (t - 9)(t + 1) = 0$

$\therefore t = 9$ or $t = -1$ (reject)

$\therefore t = 9$

$v(9) = 54 - 24$

$= 30$

The particle returns to the origin after 9 seconds with a velocity of 30 m/s.

e The particle's position after 6 seconds is $x(6) = 108 - 144 - 27 = -63$.

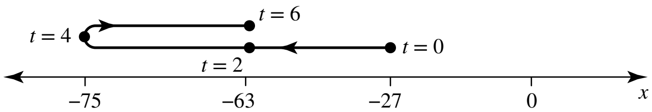
From the earlier working it is known that $x(0) = -27$, $x(2) = -63$ and $x(9) = 0$. Initially the particle moved to the left but at some time changed direction.

To find the time when the motion changed direction, let $v = 0$.

$\therefore 6t - 24 = 0$

$\therefore t = 4$

The position of the particle when it changes direction at $t = 4$ is $x(4) = 48 - 96 - 27 = -75$.



The distance travelled in the first 6 seconds of motion $= (48 + 12) = 60$ metres.

f Average speed $= \frac{\text{distance travelled}}{\text{time taken}}$

$= \frac{60}{6}$

$= 10$

The average speed was 10 m/s.

12 $x = \frac{1}{3}t^3 - t^2,\ t \geq 0$

a When $t = 0$, $x = 0$.

Velocity: $v = \frac{dx}{dt}$

$\therefore v = t^2 - 2t$

When $t = 0$, $v = 0$.

Therefore, the particle starts at the origin from rest.

b Let $v = 0$

$\therefore t^2 - 2t = 0$

$\therefore t(t-2) = 0$

$\therefore t = 0, t = 2$

The particle is next at rest at $t = 2$.

Its position when $t = 2$ is $x = \frac{8}{3} - 4 = -\frac{4}{3}$.

The particle is next at rest after 2 seconds when it is $\frac{4}{3}$ cm to the left of the origin.

c Let $x = 0$

$\therefore \frac{1}{3}t^3 - t^2 = 0$

$\therefore t^3 - 3t^2 = 0$

$\therefore t^2(t-3) = 0$

$\therefore t = 0, t = 3$

The particle returns to the origin after 3 seconds.

d When $t = 3$, $v = 9 - 6 = 3$.

The particle's speed when it returns to the origin is 3 cm/s.

Acceleration $a = \frac{dv}{dt}$

$\therefore a = 2t - 2$

When $t = 3$, $a = 4$.

The acceleration of the particle when it returns to the origin is 4 cm/s^2.

e Displacement-time graph $x = \frac{1}{3}t^3 - t^2$

Cubic graph with t intercepts at $t = 0, t = 3$ and minimum turning point at $\left(2, -\frac{4}{3}\right)$ since its derivative is zero when $t = 2$.

Velocity-time graph $v = t^2 - 2t$

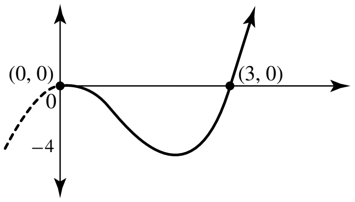
Quadratic graph with t intercepts at $t = 0, t = 2$.

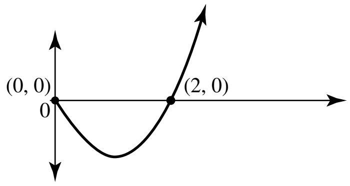
Turning point when $t = 1$ and therefore $v = -1$. Minimum turning point $(1, -1)$.

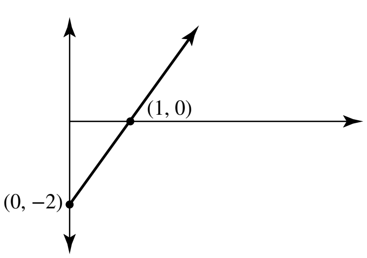
Acceleration-time graph $a = 2t - 2$

Linear graph containing points $(0, -2)$ and $(1, 0)$.

When $t = 2, a = 2$, so point $(2, 2)$ is also on the graph.







At $t = 2$, the displacement graph reaches its most negative displacement and the velocity is zero. The acceleration is positive with value 2 cm/s^2. This represents the instant when the particle ceases to move to the left and starts to move to towards the right.

f When $t = 1$ the velocity reaches its most negative value. The particle is moving to the left since its displacement is negative but at $t = 1$ the velocity starts to reduce to less negative values. At that instant the acceleration is zero as it changes from negative to positive acceleration. The overall effect is to start to slow the particle down until the point $t = 2$ where it will change its direction of motion and begin to move to the right under positive acceleration.

13 $h = 40t - 5t^2$

a The rate of change of the height of the ball is given by $\frac{dh}{dt}$.

$\frac{dh}{dt} = 40 - 10t$

When $t = 2$, $\frac{dh}{dt} = 20$.

The height is changing at 20 m/s.

b The velocity is the rate of change of height.

$\therefore v = \frac{dh}{dt} = 40 - 10t$

When $t = 3$, $v = 10$.

The vertical velocity upwards is 10 m/s.

c Let $v = -10$

$\therefore 40 - 10t = -10$

$\therefore 50 = 10t$

$\therefore t = 5$

After 5 seconds, the velocity of the ball is –10 m/s. The negative sign indicates the ball is travelling vertically downwards towards the ground.

d Let $v = 0$

$\therefore 40 - 10t = 0$

$\therefore t = 4$

The velocity is zero after 4 seconds.

e The greatest height occurs when $\frac{dh}{dt} = v = 0$. Hence, the greatest height occurs when $t = 4$.

When $t = 4$, $h = 160 - 80 = 80$.

The greatest height the ball reaches is 80 metres above the ground.

f When the ball reaches the ground, $h = 0$.

$\therefore 40t - 5t^2 = 0$

$\therefore 5t(8 - t) = 0$

$\therefore t = 0, t = 8$

The ball returns to the ground after 8 seconds.

When $t = 8$, $v = 40 - 80 = -40$.

The ball strikes the ground with speed 40 m/s.

14 $x_P(t) = t^3 - 12t^2 + 45t - 34$

a The particle is stationary when its velocity is zero.

$v_P = x'(t)$

$= 3t^2 - 24t + 45$

Let $v_P = 0$

$\therefore 3t^2 - 24t + 45 = 0$

$\therefore t^2 - 8t + 15 = 0$

$\therefore (t-3)(t-5) = 0$

$\therefore t = 3, t = 5$

The particle P is instantaneously stationary after 3 seconds and after 5 seconds.

b If $v < 0$, then $(t-3)(t-5) < 0$

Therefore, $v < 0$ when $3 < t < 5$.
The velocity is negative for the time interval $t \in (3, 5)$.

c $a_P = v'(t)$
$\therefore a_P = 6t - 24$
If $a_P < 0$ then $6t - 24 < 0$
$\therefore t < 4$
The acceleration is negative for the time interval $t \in [0, 4)$.

d $x_Q(t) = -12t^2 + 54t - 44$
$v_Q(t) = -24t + 54.$
P and Q have the same velocities when $v_P = v_Q$.
$\therefore 3t^2 - 24t + 45 = -24t + 54$
$\therefore 3t^2 = 9$
$\therefore t^2 = 3$
$\therefore t = \sqrt{3}$
(negative square root not applicable)
Particles P and Q are travelling with the same velocities after $\sqrt{3}$ seconds.

e P and Q have the same displacements when $x_P = x_Q$.
$\therefore t^3 - 12t^2 + 45t - 34 = -12t^2 + 54t - 44$
$\therefore t^3 - 9t + 10 = 0$
By inspection, $t = 2$ is a solution and therefore $(t-2)$ is a factor
$\therefore t^3 - 9t + 10 = (t-2)\left(t^2 + 2t - 5\right) = 0$
$\therefore t = 2$ or $t^2 + 2t - 5 = 0$
Consider $t^2 + 2t - 5 = 0$
Completing the square,
$\left(t^2 + 2t + 1\right) - 1 - 5 = 0$
$\therefore (t+1)^2 = 6$
$\therefore t = \pm\sqrt{6} - 1$
However, $t = -\sqrt{6} - 1 < 0$ so reject this solution.
P and Q have the same displacements when $t = 2$ and $t = \sqrt{6} - 1$, that is their displacements are equal after $\left(\sqrt{6} - 1\right)$ seconds and after 2 seconds.

15 $N = 200 - \dfrac{140}{t+1}, t \geq 0$

a Let $N = 172$
$\therefore 200 - \dfrac{140}{t+1} = 172$

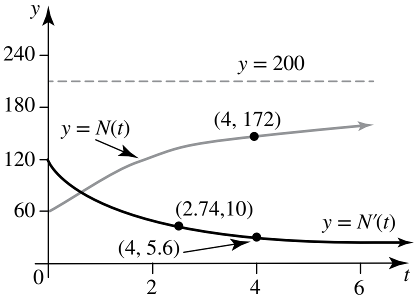
Solving in Equation/Inequality gives $t = 4$.

It took 4 years for the butterfly population to reach 172.
The rate of growth of the population at this time can be evaluated as $\left.\dfrac{dN}{dt}\right|_{t=4}$ using the 2 D CALC and OPTN mth keyboard.
The rate of growth of the population is $\dfrac{28}{5} = 5.6$ butterflies per year.

b Let $\dfrac{dN}{dt} = 10$
Solving this equation in Equation/Inequality gives $t = -\sqrt{14} - 1, t = \sqrt{14} - 1.$
As $t \geq 0$, the solution is only $t = \sqrt{14} - 1$. Converting this value to decimal form, $t \simeq 2.74$.
After 2.74 years, the growth rate is 10 butterflies per year.

c In the Graph&Tab editor enter
$y1 = 200 - 140/(x+1)$
$y2 = \dfrac{d}{dx}(200 - 140/(x+1))$

The graphs obtained should be similar to those shown.



As $t \to \infty$, $N \to 200$ and as $t \to \infty$, $\dfrac{dN}{dt} \to 0.$

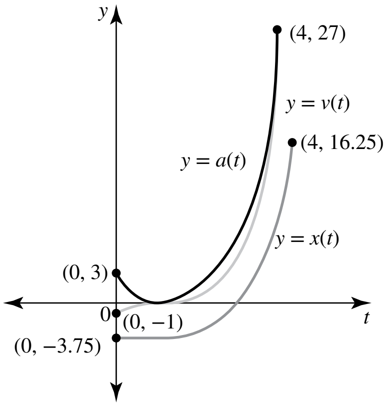
16 $x = 0.25t^4 - t^3 + 1.5t^2 - t - 3.75$

a $x(0) = -3.75$ and $x(4)$ evaluates to give $x(4) = 16.25$.
We need to check whether or not there has been a change of direction between these two positions.
$v = \dfrac{dx}{dt}$
$\therefore v = t^3 - 3t^2 + 3t - 1$
$\therefore v = (t-1)^3$
The velocity is zero when $t = 1$, so the direction of motion does change between $t = 0$ and $t = 4$.
$x(1) = -4$
The particle travels $x(0) = -3.75 \to x(1) = -4 \to$ origin $\to$ $x(4) = 16.25.$
The distance travelled is $(0.25 + 4 + 16.25) = 20.5$ metres.

b Let $x = 0$
Solve $0.25t^4 - t^3 + 1.5t^2 - t - 3.75 = 0$ to obtain $t = -1, t = 3.$
Rejecting $t = -1$, the particle is at the origin after 3 seconds.

c $a = \dfrac{dv}{dt}$
$\therefore a = 3t^2 - 6t + 3$
$= 3\left(t^2 - 2t + 1\right)$
$\therefore a = 3(t-1)^2$
$\therefore a \geq 0$
Therefore, the acceleration is never negative.

d The graphs obtained should be similar to those shown.



e Both the cubic velocity graph and the quadratic acceleration graph are positive for all values $t > 1$ and as t increases beyond $t = 1$, both the velocity and the acceleration continue to increase. The particle is at the origin at $t = 3$. Since its velocity and acceleration are both positive the particle continues moving to the right with increasing velocity due to the increasing acceleration and so the particle can never return to the origin.

Topic 14 — Anti-differentiation and introduction to integral calculus

Exercise 14.2 — Anti-derivatives

1 a $\frac{dy}{dx} = 12x^5$

$$y = 12 \times \frac{1}{6}x^6$$
$$y = 2x^6$$

b An anti-derivative of $4x^2 + 2x - 5$ equals $\frac{4}{3}x^3 + \frac{2x^2}{2} - 5x$.

Therefore, an anti-derivative could be $\frac{4}{3}x^3 + x^2 - 5x$.

c $f'(x) = (x-2)(3x+8)$

$$= 3x^2 + 8x - 6x - 16$$
$$= 3x^2 + 2x - 16$$
$$f(x) = x^3 + x^2 - 16x + c$$

2 $\frac{dy}{dx} = \frac{2x^3 - 3x^2}{x}, x \neq 0$

$$= \frac{2x^3}{x} - \frac{3x^2}{x}$$
$$= 2x^2 - 3x$$
$$y = \frac{2x^3}{3} - \frac{3x^2}{2} + c$$

3 a $\int \left(-7x^4 + 3x^2 - 6x\right) dx$

$$= -7 \times \frac{x^5}{5} + x^3 - 6 \times \frac{x^2}{2} + c$$
$$= -\frac{7x^5}{5} + x^3 - 3x^2 + c$$

b $\int \frac{5x^8 + 3x^3}{4x^2} dx$

$$= \int \left(\frac{5x^8}{4x^2} + \frac{3x^3}{4x^2}\right) dx$$
$$= \int \left(\frac{5}{4}x^6 + \frac{3}{4}x\right) dx$$
$$= \frac{5}{4} \times \frac{x^7}{7} + \frac{3}{4} \times \frac{x^2}{2} + c$$
$$= \frac{5}{28}x^7 + \frac{3}{8}x^2 + c$$

c $f(x) = \frac{2}{x^4}$

$$= 2x^{-4}$$
$$F(x) = 2 \times \frac{x^{-3}}{-3} + c$$
$$F(x) = -\frac{2}{3x^3} + c$$

d $\int \left(x + \frac{1}{x}\right)^2 dx$

$$= \int \left(x^2 + 2 + \frac{1}{x^2}\right) dx$$
$$= \int \left(x^2 + 2 + x^{-2}\right) dx$$
$$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} + c$$
$$= \frac{x^3}{3} + 2x - \frac{1}{x} + c$$

4 a $f(x) = \frac{1}{2\sqrt{x}}$

$$= \frac{1}{2} \times x^{-\frac{1}{2}}$$
$$F(x) = \frac{1}{2} \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \frac{1}{2} \times \frac{2}{1} \times x^{\frac{1}{2}} + c$$
$$= x^{\frac{1}{2}} + c$$
$$= \sqrt{x} + c$$

b $\int 2(t^2 + 2)\, dt$

$$= \int \left(2t^2 + 4\right) dt$$
$$= \frac{2t^3}{3} + 4t + c$$
$$2\int (t^2 + 2)\, dt$$
$$= 2\left(\frac{t^3}{3} + 2t\right) + c$$
$$= \frac{2t^3}{3} + 4t + c$$

Therefore, $\int 2\left(t^2 + 2\right) dt = 2\int \left(t^2 + 2\right) dt$.

5 a $\frac{dy}{dx} = 5x^9$

$$\therefore y = \frac{5x^{9+1}}{10} + c$$
$$\therefore y = \frac{1}{2}x^{10} + c$$

b $\frac{dy}{dx} = -3 + 4x^7$

$$\therefore y = -3x + \frac{4x^8}{8} + c$$
$$\therefore y = -3x + \frac{1}{2}x^8 + c$$

c $\frac{dy}{dx}=2\left(x^2-6x+7\right)$

$\therefore \frac{dy}{dx}=2x^2-12x+14$

$\therefore y=\frac{2x^3}{3}-\frac{12x^2}{2}+14x+c$

$\therefore y=\frac{2}{3}x^3-6x^2+14x+c$

d $\frac{dy}{dx}=(8-x)(2x+5)$

$=16x+40-2x^2-5x$

$\therefore \frac{dy}{dx}=-2x^2+11x+40$

$\therefore y=-\frac{2x^3}{3}+\frac{11x^2}{2}+40x+c$

$\therefore y=-\frac{2}{3}x^3+\frac{11}{2}x^2+40x+c$

6 a $f'(x)=\frac{1}{2}x^5+\frac{7}{3}x^6$

$\therefore f(x)=\frac{1}{2}\times\frac{x^6}{6}+\frac{7}{3}\times\frac{x^7}{7}+c$

$\therefore f(x)=\frac{1}{12}x^6+\frac{1}{3}x^7+c$

b $f'(x)=\frac{4x^6}{3}+5$

$\therefore f(x)=\frac{4}{3}\times\frac{x^7}{7}+5x+c$

$\therefore f(x)=\frac{4}{21}x^7+5x+c$

c $f'(x)=\frac{4x^4-6x^8}{x^2}$

$\therefore f'(x)=\frac{4x^4}{x^2}-\frac{6x^8}{x^2}$

$\therefore f'(x)=4x^2-6x^6$

$\therefore f(x)=\frac{4x^3}{3}-\frac{6x^7}{7}+c$

$\therefore f(x)=\frac{4}{3}x^3-\frac{6}{7}x^7+c$

d $f'(x)=\left(3-2x^2\right)^2$

$\therefore f'(x)=9-12x^2+4x^4$

$\therefore f(x)=9x-\frac{12x^3}{3}+\frac{4x^5}{5}+c$

$\therefore f(x)=9x-4x^3+\frac{4}{5}x^5+c$

7 a $\int\frac{3x^8}{5}dx$

$=\frac{3}{5}\times\frac{x^9}{9}+c$

$=\frac{1}{15}x^9+c$

b $\int 2\,dx=2x+c$

c $4\int\left(20x-5x^7\right)dx$

$=4\left[\frac{20x^2}{2}-\frac{5x^8}{8}\right]+c$

$=4\left[10x^2-\frac{5}{8}x^8\right]+c$

$=40x^2-\frac{5}{2}x^8+c$

d $\int\frac{1}{100}\left(9+6x^2-5.5x^{10}\right)dx$

$=\frac{1}{100}\int\left(9+6x^2-5.5x^{10}\right)dx$

$=\frac{1}{100}\left[9x+\frac{6x^3}{3}-\frac{5.5x^{11}}{11}\right]+c$

$=\frac{1}{100}\left[9x+2x^3-0.5x^{11}\right]+c$

8 a Let $f(x)=2ax+b$

The primitive function is $F(x)$ where

$F(x)=\frac{2ax^2}{2}+bx+c$

$\therefore F(x)=ax^2+bx+c$

b Let $f(x)=0.05x^{99}$

$\therefore F(x)=\frac{0.05x^{100}}{100}+c$

$\therefore F(x)=0.0005x^{100}+c$

c Let $f(x)=(2x+1)^3$

$\therefore f(x)=(2x)^3+3(2x)^2(1)+3(2x)(1)^2+(1)^3$

$\therefore f(x)=8x^3+12x^2+6x+1$

$\therefore F(x)=\frac{8x^4}{4}+\frac{12x^3}{3}+\frac{6x^2}{2}+x+c$

$\therefore F(x)=2x^4+4x^3+3x^2+x+c$

An anti-derivative could be $2x^4+4x^3+3x^2+x$.

d Let $f(x)=7-x\left(5x^3-4x-8\right)$

$\therefore f(x)=7-5x^4+4x^2+8x$

$\therefore F(x)=7x-\frac{5x^5}{5}+\frac{4x^3}{3}+\frac{8x^2}{2}+c$

$\therefore F(x)=7x-x^5+\frac{4}{3}x^3+4x^2+c$

9 a $f(x)=\frac{3x^2-2}{4}$

$\therefore f(x)=\frac{1}{4}\left(3x^2-2\right)$

$\therefore F(x)=\frac{1}{4}\left(\frac{3x^3}{3}-2x\right)+c$

$\therefore F(x)=\frac{1}{4}\left(x^3-2x\right)+c$

b $f(x)=\frac{3x}{4}+\frac{2(1-x)}{3}$

$\therefore f(x)=\frac{3x}{4}+\frac{2}{3}-\frac{2x}{3}$

$=\frac{x}{12}+\frac{2}{3}$

$\therefore F(x)=\frac{1}{12}\times\frac{x^2}{2}+\frac{2}{3}x+c$

$\therefore F(x)=\frac{1}{24}x^2+\frac{2}{3}x+c$

c $f(x)=0.25\left(1+5x^{14}\right)$

$\therefore f(x)=\frac{1}{4}+\frac{5}{4}x^{14}$

$\therefore F(x)=\frac{1}{4}x+\frac{5}{4}\times\frac{x^{15}}{15}+c$

$\therefore F(x)=\frac{1}{4}x+\frac{1}{12}x^{15}+c$

d $f(x)=\dfrac{12\left(x^5\right)^2-(4x)^2}{3x^2}$

$$\therefore f(x)=\frac{12x^{10}-16x^2}{3x^2}$$
$$=\frac{12x^{10}}{3x^2}-\frac{16x^2}{3x^2}$$
$$\therefore f(x)=4x^8-\frac{16}{3}$$
$$\therefore F(x)=\frac{4x^9}{9}-\frac{16}{3}x+c$$

10 a $\dfrac{dy}{dx}=x^{\frac{3}{2}}$

$$\therefore y=\frac{x^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)}+c$$
$$=\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+c$$
$$\therefore y=\frac{2}{5}x^{\frac{5}{2}}+c$$

b $\dfrac{dy}{dx}=x^{-\frac{3}{2}}$

$$\therefore y=\frac{x^{-\frac{3}{2}+1}}{\left(-\frac{3}{2}+1\right)}+c$$
$$=\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}+c$$
$$\therefore y=-2x^{-\frac{1}{2}}+c$$

11 a $f'(x)=\dfrac{5}{x^2}$

$$\therefore f'(x)=5x^{-2}$$
$$\therefore f(x)=\frac{5x^{-2+1}}{(-2+1)}+c$$
$$=\frac{5x^{-1}}{-1}+c$$
$$\therefore f(x)=-\frac{5}{x}+c$$

b $f'(x)=6x^3+6x^{-3}$

$$\therefore f(x)=\frac{6x^4}{4}+\frac{6x^{-2}}{-2}+c$$
$$\therefore f(x)=\frac{3}{2}x^4-3x^{-2}+c$$

12 a $\displaystyle\int\frac{2x^5+7x^3-5}{x^2}dx$

$$=\int\left(\frac{2x^5}{x^2}+\frac{7x^3}{x^2}-\frac{5}{x^2}\right)dx$$
$$=\int\left(2x^3+7x-5x^{-2}\right)dx$$
$$=\frac{2x^4}{4}+\frac{7x^2}{2}-\frac{5x^{-1}}{-1}+c$$
$$=\frac{x^4}{2}+\frac{7x^2}{2}+\frac{5}{x}+c$$

b $\displaystyle\int\left(2\sqrt{x}+1\right)^2dx$

$$=\int\left(\left(2\sqrt{x}\right)^2+4\sqrt{x}+1\right)dx$$
$$=\int\left(4x+4x^{\frac{1}{2}}+1\right)dx$$
$$=\frac{4x^2}{2}+\frac{4x^{\frac{3}{2}}}{\frac{3}{2}}+x+c$$
$$=2x^2+\frac{2}{3}\times4x^{\frac{3}{2}}+x+c$$
$$=2x^2+\frac{8}{3}x^{\frac{3}{2}}+x+c$$

c $\displaystyle\int x^{\frac{1}{5}}\left(x^2-13x^{\frac{1}{10}}+12\right)dx$

$$=\int\left(x^{\frac{11}{5}}-13x^{\frac{3}{10}}+12x^{\frac{1}{5}}\right)dx$$
$$=\frac{x^{\frac{16}{5}}}{\frac{16}{5}}-\frac{13x^{\frac{13}{10}}}{\frac{13}{10}}+\frac{12x^{\frac{6}{5}}}{\frac{6}{5}}+c$$
$$=\frac{5}{16}x^{\frac{16}{5}}-\frac{10}{13}\times13x^{\frac{13}{10}}+\frac{5}{6}\times12x^{\frac{6}{5}}+c$$
$$=\frac{5}{16}x^{\frac{16}{5}}-10x^{\frac{13}{10}}+10x^{\frac{6}{5}}+c$$

d $\displaystyle\int\frac{(x+4)(x-2)}{2x^4}dx$

$$=\int\frac{x^2+2x-8}{2x^4}dx$$
$$=\int\left(\frac{x^2}{2x^4}+\frac{2x}{2x^4}-\frac{8}{2x^4}\right)dx$$
$$=\int\left(\frac{1}{2}x^{-2}+x^{-3}-4x^{-4}\right)dx$$
$$=\frac{1}{2}\times\frac{x^{-1}}{-1}+\frac{x^{-2}}{-2}-\frac{4x^{-3}}{-3}+c$$
$$=-\frac{1}{2x}-\frac{1}{2x^2}+\frac{4}{3x^3}+c$$

13 a $f(x)=(2ax)^2+b^3$

$$\therefore f(x)=4a^2x^2+b^3$$
$$\therefore F(x)=4a^2\times\frac{x^3}{3}+b^3x+c$$
$$\therefore F(x)=\frac{4a^2x^3}{3}+b^3x+c$$

b $f(x)=\sqrt{3x^4}+\sqrt{3}a^2$

$$\therefore f(x)=\sqrt{3}x^2+\sqrt{3}a^2$$
$$\therefore F(x)=\sqrt{3}\times\frac{x^3}{3}+\sqrt{3}a^2x+c$$
$$\therefore F(x)=\frac{\sqrt{3}x^3}{3}+\sqrt{3}a^2x+c$$

14 a Let $f(x)=(x-1)(x+4)(x+1)$

$$\therefore f(x)=(x^2-1)(x+4)$$
$$\therefore f(x)=x^3+4x^2-x-4$$

The primitive is $F(x)=\dfrac{x^4}{4}+\dfrac{4x^3}{3}-\dfrac{x^2}{2}-4x+c.$

b The anti-derivative of $\left(\frac{5}{x}-\frac{x}{5}\right)^2$ is

$$\int\left(\frac{5}{x}-\frac{x}{5}\right)^2 dx$$
$$=\int\left(\frac{25}{x^2}-2\times\frac{5}{x}\times\frac{x}{5}+\frac{x^2}{25}\right)dx$$
$$=\int\left(25x^{-2}-2+\frac{1}{25}x^2\right)dx$$
$$=25\times\frac{x^{-1}}{-1}-2x+\frac{1}{25}\times\frac{x^3}{3}+c$$
$$=-\frac{25}{x}-2x+\frac{1}{75}x^3+c$$

c Let $f(x)=\frac{x^p}{x^q}$

$\therefore f(x)=x^{p-q}$

The anti-derivative is $F(x)=\frac{x^{p-q+1}}{p-q+1}+c$ provided $p-q+1\neq 0$

An anti-derivative could be $\frac{1}{p-q+1}x^{p-q+1}$ provided $p\neq q-1$.

d Let $f(x)=\frac{\sqrt[3]{x}-\sqrt[3]{x^5}}{x}$

$$\therefore f(x)=\frac{x^{\frac{1}{3}}}{x}-\frac{x^{\frac{5}{3}}}{x}$$
$$\therefore f(x)=x^{-\frac{2}{3}}-x^{\frac{2}{3}}$$

The anti-derivative is

$$F(x)=\frac{x^{\frac{1}{3}}}{\frac{1}{3}}-\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+c$$
$$=3x^{\frac{1}{3}}-\frac{3}{5}x^{\frac{5}{3}}+c$$

An anti-derivative could be $3x^{\frac{1}{3}}-\frac{3}{5}x^{\frac{5}{3}}$.

e $\frac{d}{dx}\left(\int(4x+7)\,dx\right)$

$$=\frac{d}{dx}\left(2x^2+7x+c\right)$$
$$=4x+7$$

15 An anti-derivative of x^3 could be obtained using the template in the 2D CALC mth keyboard.

$\int x^3\,dx=\frac{x^4}{4}$. For the family of anti-derivatives, the constant of integration needs to be inserted by oneself.

Another method in the main menu is to tap Interactive → Calculation → $\int$

Complete the dialogue box as follows:
Indefinite integral
Expression: x^3
Variable: x

Then tap OK to obtain $\frac{x^4}{4}$. The constant of integration is again not given.

Either of these methods is the most straightforward way to obtain an anti-derivative.

However, there is a third method whereby the constant of integration will be given. Calculating the anti-derivative of x^3 is equivalent to calculating y if $\frac{dy}{dx}=x^3$.

The equation $\frac{dy}{dx}=x^3$ is called a differential equation and it can be solved by going to Equation/Inequality and selecting dSolve.

Complete the dialogue box as follows:
No condition
Equation: $y'=x^3$. To form y' use the ‘ in the 2D CALC mth keyboard
Inde var: x
Depe var: y
and tap OK to obtain $y=\frac{x^4}{4}+\text{const}(1)$. The constant of integration is const(1).

16 Choosing to use the template method and inserting the constant of integration afterwards:

a $\int x^{\frac{3}{2}}\,dx=\frac{2x^{\frac{5}{2}}}{5}$, so $\int x^{\frac{3}{2}}\,dx=\frac{2x^{\frac{5}{2}}}{5}+c$.

b $\frac{dy}{dx}=\left(x^4+1\right)^2$

$$\therefore y=\int\left(x^4+1\right)^2 dx$$
$$\therefore y=\frac{x^9}{9}+\frac{2x^5}{5}+x+c$$

c $f(t)=100\left(t-\frac{5}{t^2}\right)$

$$\therefore F(t)=\int 100\left(t-\frac{5}{t^2}\right)dt$$
$$\therefore F(t)=50t^2+\frac{500}{t}+c$$

d $\int\left(y^5-y^{-5}\right)dy=\frac{2y^{10}+3}{12y^4}+c$

e $\int\sqrt{4u+5}du=\frac{(4u+5)^{\frac{3}{2}}}{6}+c$

f Answers will vary.

Exercise 14.3 — Anti-derivative functions and graphs

1 a $\frac{dy}{dx}=ax-6$

Stationary point at $(-1,10)$

$$\therefore \left.\frac{dy}{dx}\right|_{x=-1}=0$$
$$a(-1)-6=0$$
$$a=-6$$
$$\frac{dy}{dx}=-6x-6$$
$$\therefore y=-3x^2-6x+c$$

Substitute $(-1,10)$:

$$\therefore 10=-3(-1)^2-6(-1)+c$$
$$\therefore 10=-3+6+c$$
$$\therefore c=7$$

The equation of the curve is $y=-3x^2-6x+7$.

b $f'(x) = \dfrac{2x^2+9}{2x^2}$

$= \dfrac{2x^2}{2x^2} + \dfrac{9}{2x^2}$

$= 1 + \dfrac{9}{2}x^{-2}$

Therefore $f(x) = x - \dfrac{9}{2x} + c$.

Since $f(3) = 0$:

$0 = 3 - \dfrac{9}{2 \times 3} + c$

$c = -\dfrac{3}{2}$

Therefore $f(x) = x - \dfrac{9}{2x} - \dfrac{3}{2}$.

$f(-1) = -1 + \dfrac{9}{2} - \dfrac{3}{2}$

$= -1 + 3$

$= 2$

2 $\dfrac{dy}{dx} = 2\sqrt{x}$

$= 2x^{\frac{1}{2}}$

$y = 2 \times \dfrac{2}{3}x^{\frac{3}{2}} + c$

$= \dfrac{4}{3}x^{\frac{3}{2}} + c$

$y = 10$ when $x = 4$, so:

$10 = \dfrac{4}{3} \times 4^{\frac{3}{2}} + c$

$10 = \dfrac{4}{3} \times 8 + c$

$10 = \dfrac{32}{3} + c$

$c = -\dfrac{2}{3}$

Hence, $y = \dfrac{4}{3}x^{\frac{3}{2}} - \dfrac{2}{3}$.

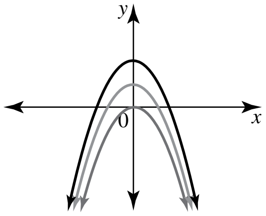
When $x = 1$:

$y = \dfrac{4}{3} - \dfrac{2}{3}$

$y = \dfrac{2}{3}$

3 a At $x = 0$ there is a stationary point on the graph of $y = F(x)$. Since f changes sign from positive to zero to negative, the point will be a maximum turning point.

b Three possible graphs with a maximum turning point at $x = 0$ are shown.



c The rule for the graph is of the form $y = ax$.

Substitute the point $(1, -2)$:

$-2 = a(1)$

$a = -2$

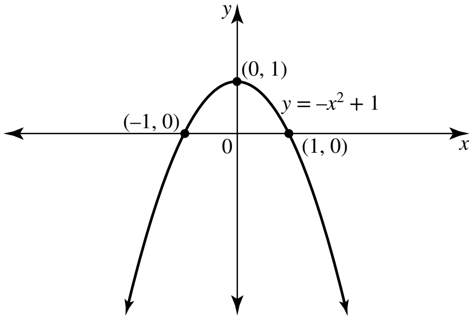
Hence, $f(x) = -2x$.

d $F(x) = -x^2 + c$

$F(0) = 1 \Rightarrow 1 = c$

$\therefore F(x) = -x^2 + 1$

Maximum turning point at $(0,1)$ and x-intercepts at $(\pm 1, 0)$.



4 The graph has a gradient of $\dfrac{1}{2}$ and a vertical axis intercept at $(0,2)$.

Its equation is $\dfrac{dy}{dx} = \dfrac{1}{2}x + 2$.

Therefore, the equation of the curve must be $y = \dfrac{1}{4}x^2 + 2x + c$.

Since $(-2,3)$ lies on the curve,

$3 = \dfrac{1}{4} \times 4 + 2 \times (-2) + c$

$\therefore c = 6$

Thus the equation of the curve is $y = \dfrac{1}{4}x^2 + 2x + 6$.

From the gradient graph, the turning point of the curve must occur at $x = -4$.

$y = \dfrac{1}{4} \times 16 - 8 + 6$

$\therefore y = 2$

The minimum turning point has coordinates $(-4, 2)$.

5 $f'(x) = -3x^2 + 4$

$\therefore f(x) = -x^3 + 4x + c$

Since the point $(-1,2)$ lies on the function, $f(-1) = 2$.

$\therefore -(-1)^3 + 4(-1) + c = 2$

$\therefore 1 - 4 + c = 2$

$\therefore c = 5$

The equation is $f(x) = -x^3 + 4x + 5$

6 a $\dfrac{dy}{dx} = \dfrac{2x}{5} - 3$

$\therefore y = \dfrac{x^2}{5} - 3x + c$

Substitute the point $(5,0)$

$\therefore 0 = 5 - 15 + c$

$\therefore c = 10$

The equation is $y = \dfrac{x^2}{5} - 3x + 10$.

b Let $y = 0$

$\therefore \dfrac{x^2}{5} - 3x + 10 = 0$

$\therefore x^2 - 15x + 50 = 0$

$\therefore (x-5)(x-10) = 0$

$\therefore x = 5, x = 10$

The x intercepts are $(5,0)$ and $(10,0)$.

7 a $\dfrac{dy}{dx} \infty x$

$\therefore \dfrac{dy}{dx} = kx$

Given that $\frac{dy}{dx}=-3$ at the point $(2,5)$

$\therefore -3=k\times 2$

$\therefore k=-\frac{3}{2}$

The constant of proportionality is $-\frac{3}{2}$.

b $\frac{dy}{dx}=-\frac{3}{2}x$

$\therefore y=-\frac{3}{2}\times\frac{x^2}{2}+c$

$\therefore y=-\frac{3}{4}x^2+c$

Substitute the point $(2,5)$

$\therefore 5=-\frac{3}{4}\times 4+c$

$\therefore c=8$

The equation is $y=-\frac{3}{4}x^2+8$

8 $f(x)=\frac{(4-x)(5-x)}{10x^4}$

$=\frac{20-9x+x^2}{10x^4}$

$=\frac{20}{10x^4}-\frac{9x}{10x^4}+\frac{x^2}{10x^4}$

$=\frac{2}{x^4}-\frac{9}{10x^3}+\frac{1}{10x^2}$

$\therefore f(x)=2x^{-4}-\frac{9}{10}x^{-3}+\frac{1}{10}x^{-2}$

The primitive function is

$F(x)=\frac{2x^{-3}}{-3}-\frac{9}{10}\times\frac{x^{-2}}{-2}+\frac{1}{10}\times\frac{x^{-1}}{-1}+c$

$=-\frac{2}{3x^3}+\frac{9}{20x^2}-\frac{1}{10x}+c$

Substitute $(1,-1)$

$\therefore -1=-\frac{2}{3}+\frac{9}{20}-\frac{1}{10}+c$

$\therefore -60=-40+27-6+60c$

$\therefore 60c=-41$

$\therefore c=-\frac{41}{60}$

The equation of the primitive function is

$F(x)=-\frac{2}{3x^3}+\frac{9}{20x^2}-\frac{1}{10x}-\frac{41}{60}$.

9 a $\frac{dy}{dx}=2x(3-x)$

$\therefore \frac{dy}{dx}=6x-2x^2$

$\therefore y=3x^2-\frac{2x^3}{3}+c$

$x=3, y=0\Rightarrow 0=27-18+c$

$\therefore c=-9$

$\therefore y=3x^2-\frac{2x^3}{3}-9$

Let $x=0$,

$\therefore y=-9$

b $\frac{dz}{dx}=(10-x)^2$

$\therefore \frac{dz}{dx}=100-20x+x^2$

$\therefore z=100x-10x^2+\frac{x^3}{3}+c$

$x=10, z=200\Rightarrow 200=1000-1000+\frac{1000}{3}+c$

$\therefore c=200-\frac{1000}{3}$

$\therefore c=-\frac{400}{3}$

$\therefore z=100x-10x^2+\frac{x^3}{3}-\frac{400}{3}$

Let $x=4$

$z=400-160+\frac{64}{3}-\frac{400}{3}$

$=240-\frac{336}{3}$

$=240-112$

$=128$

c $\frac{dA}{dt}=\frac{4}{\sqrt{t}}$

$\therefore \frac{dA}{dt}=4t^{-\frac{1}{2}}$

$\therefore A=4\times\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c$

$\therefore A=8\sqrt{t}+c$

$t=16, A=40\Rightarrow 40=8\times 4+c$

$\therefore c=8$

$\therefore A=8\sqrt{t}+8$

Let $t=64$,

$A=8\times 8+8$

$=72$

d $\frac{dx}{dy}=-\frac{3}{y^4}$

$\therefore \frac{dx}{dy}=-3y^{-4}$

$\therefore x=\frac{-3y^{-3}}{-3}+c$

$\therefore x=\frac{1}{y^3}+c$

$y=1, x=12\Rightarrow 12=1+c$

$\therefore c=11$

$\therefore x=\frac{1}{y^3}+11$

Let $x=75$

$\therefore 75=\frac{1}{y^3}+11$

$\therefore 64=\frac{1}{y^3}$

$\therefore y^3=\frac{1}{64}$

$\therefore y=\frac{1}{4}$

10 a $\frac{dy}{dx}=a-x^{\frac{2}{3}}$

Since the point (8, 32) is a stationary point, $\frac{dy}{dx}=0$ at (8, 32).

$\therefore 0=a-8^{\frac{2}{3}}$
$\therefore 0=a-4$
$\therefore a=4$

b $\frac{dy}{dx}=4-x^{\frac{2}{3}}$

$\therefore y=4x-\frac{3x^{\frac{5}{3}}}{5}+c$

Substitute the point (8, 32)

$\therefore 32=32-\frac{3\times 8^{\frac{5}{3}}}{5}+c$

$\therefore 0=-\frac{3\times 2^5}{5}+c$

$\therefore c=\frac{96}{5}$

The equation of the curve is $y=4x-\frac{3x^{\frac{5}{3}}}{5}+\frac{96}{5}$.

c Let $x=1$

$\frac{dy}{dx}=4-1=3$ so gradient of tangent is $m=3$

Point of contact: $x=1$

$$\begin{aligned}\therefore y&=4-\frac{3}{5}+\frac{96}{5}\\&=4+\frac{93}{5}\\&=\frac{113}{5}\end{aligned}$$

Point is $\left(1,\frac{113}{5}\right)$

Equation of tangent:

$y-\frac{113}{5}=3(x-1)$
$\therefore 5y-113=15x-15$
$\therefore 5y-15x=98$

11 a $f'(x)=\frac{a}{x^2}$

$\therefore f'(x)=ax^{-2}$

$\therefore f(x)=\frac{ax^{-1}}{-1}+c$

$\therefore f(x)=-\frac{a}{x}+c$

The horizontal asymptote of $y=f(x)$ has equation $y=c$. Since the given information gave the equation as $y=a$, then $c=a$.

Hence, $f(x)=-\frac{a}{x}+a$

The curve passes through (2, 3)

$\therefore 3=-\frac{a}{2}+a$

$\therefore 3=\frac{a}{2}$

$\therefore a=6$

b The equation of the curve is $f(x)=-\frac{6}{x}+6$.

c Let $f(x)=0$

$\therefore -\frac{6}{x}+6=0$

$\therefore 6=\frac{6}{x}$

$\therefore x=1$

The graph cuts the x axis at (1, 0).
The gradient at this point is

$f'(x)=\frac{6}{x^2}$

$\therefore f'(1)=6$

The angle at which the curve cuts the x axis is such that $\tan\theta=6$

$\therefore \theta=\tan^{-1}(6)$
$\therefore \theta\simeq 81°$

The angle is approximately 81°.

12 a The x intercepts of the given gradient graph show that $\frac{dy}{dx}=0$ when $x=-2$ and when $x=2$. Hence, the curve with this gradient function has stationary points where $x=-2$ and $x=2$.

Consider how the gradient changes about each point to determine the nature of the stationary points.

About $x=-2$:

For $x<-2$, the value of the gradient is negative as its graph lies below the x axis.

For $-2<x<2$, the value of the gradient is positive as its graph lies above the x axis.

Therefore, the curve with this gradient function has a minimum turning point where $x=-2$.

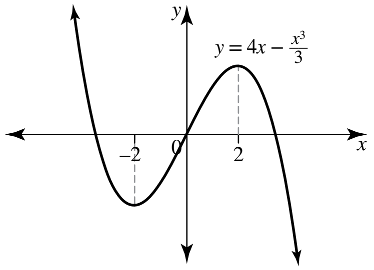
About $x=2$:

For $-2<x<2$, the value of the gradient is positive

For $x>2$, the value of the gradient is negative as its graph lies below the x axis.

Therefore, the curve with this gradient function has a maximum turning point where $x=2$.

b The y co-ordinates of the turning points are not known. A possible graph is shown.



c The gradient graph is a parabola with a maximum turning point at (0, 4).

Let its equation be $\frac{dy}{dx}=ax^2+4$

Substitute the point (2, 0)

$\therefore 0=4a+4$
$\therefore a=-1$

The rule for the gradient graph is $\frac{dy}{dx}=-x^2+4$

$\therefore y=-\frac{x^3}{3}+4x+c$

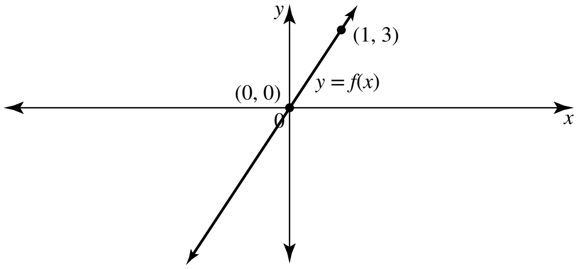
The family of curves with this gradient function are those given by $y=-\frac{x^3}{3}+4x+c$.

d For the curve containing the point $(3,0)$,
$0=-9+12+c$
$\therefore c=-3$

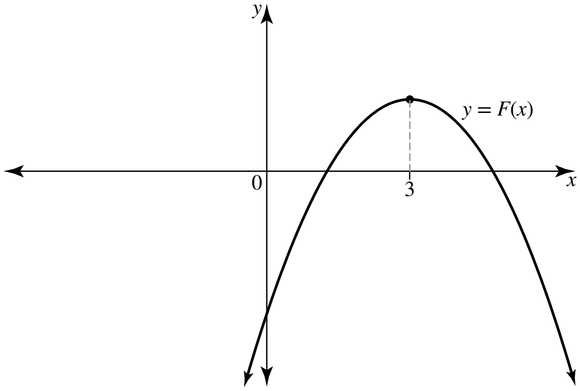
The equation of this particular curve is $y=-\dfrac{x^3}{3}+4x-3$.

At $(3,0)$, $\dfrac{dy}{dx}=-9+4=-5$, so the slope of the curve then is -5.

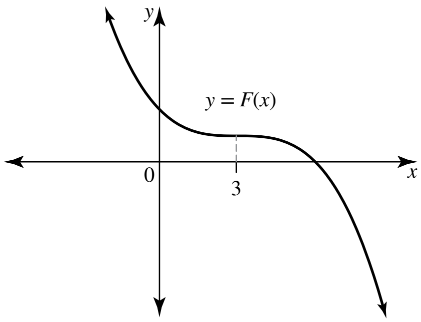
13 a The equation of the given graph is $f(x)=3$. Hence, the anti-derivative graph has equation of the form $F(x)=3x+c$. Any linear graph with gradient 3 is suitable. One such possibility is shown.



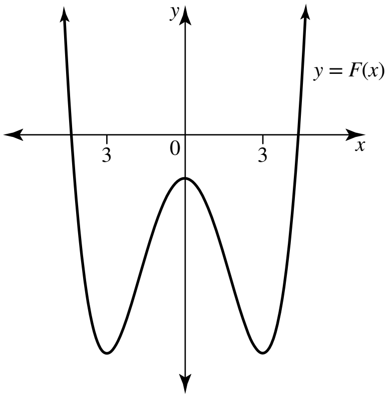
b Since $f(3)=0$ and $f(x)>0$ for $x<3$ and $f(x)<0$ for $x>3$, the graph of $y=F(x)$ will have a maximum turning point where $x=3$.
$y=f(x)$ is linear so $y=F(x)$ is quadratic.
Any concave down parabola with a maximum turning point where $x=3$ is suitable. An example is shown.



c Since $f(3)=0$ and $f(x)<0$ for $x<3$ and $f(x)<0$ for $x>3$, the graph of $y=F(x)$ will have a stationary point of inflection where $x=3$.
$y=f(x)$ is quadratic so $y=F(x)$ is cubic.
Any cubic graph with a stationary point of inflection where $x=3$ and negative gradient elsewhere is suitable. An example is shown.



d Since $f(-3)=0$, $f(0)=0$ and $f(3)=0$, the graph of $y=F(x)$ has stationary points where $x=-3$, $x=0$ and $x=3$.
About $x=-3$:
For $x<-3$, the value of the gradient is negative as its graph lies below the x axis.
For $-3<x<0$, the value of the gradient is positive as its graph lies above the x axis.
Therefore, the curve with this gradient function has a minimum turning point where $x=-3$.
About $x=0$:
For $-3<x<0$ the value of the gradient is positive
For $0<x<3$, the value of the gradient is negative as its graph lies below the x axis.
Therefore, the curve with this gradient function has a maximum turning point where $x=0$.
About $x=3$:
For $0<x<3$, the value of the gradient is negative
For $x>3$, the value of the gradient is positive as its graph lies above the x axis.
Therefore, the curve with this gradient function has a minimum turning point where $x=3$.
$y=f(x)$ is a cubic graph so $y=F(x)$ is a quartic graph.
An example of a possible graph for $y=F(x)$ is shown.



14 a Let the equation of the cubic function with stationary point of inflection $(0,18)$ be $f(x)=ax^3+18$.
Substitute the point $(1,9)$
$\therefore 9=a+18$
$\therefore a=-9$
Hence, $f(x)=-9x^3+18$.

$$\int f(x)\,dx=\int\left(-9x^3+18\right)dx$$
$$=-\frac{9x^4}{4}+18x+c$$

b i Let the anti-derivative function be $y=-\dfrac{9x^4}{4}+18x+c$.

Since the function contains the point $(0,0)$, $c=0$.

Its equation is $y=-\dfrac{9x^4}{4}+18x$.

ii Let $y=0$

$$\therefore -\frac{9x^4}{4}+18x=0$$
$$\therefore -\frac{x^4}{4}+2x=0$$
$$\therefore -x^4+8x=0$$
$$\therefore x\left(-x^3+8\right)=0$$
$$\therefore x=0 \text{ or } x^3=8$$
$$\therefore x=0 \text{ or } x=2$$

The x intercept other than the origin is $(2, 0)$.

The gradient of this curve is $f(x) = -9x^3 + 18$.

$f(0) = 18$ (positive slope) and $f(2) = -72 + 18 = -54$ (negative slope)

The graph is steeper at $(2, 0)$ than at $(0, 0)$.

iii At stationary points, the gradient function $f(x) = 0$.

$$\therefore -9x^3 + 18 = 0$$
$$\therefore x^3 = 2$$
$$\therefore x = \sqrt[3]{2}$$

From the graph of $y = f(x)$ given in the question, if $x < \sqrt[3]{2}$, the gradient is positive since $y = f(x)$ lies above the x axis and if $x > \sqrt[3]{2}$, the gradient is negative since $y = f(x)$ lies below the x axis.

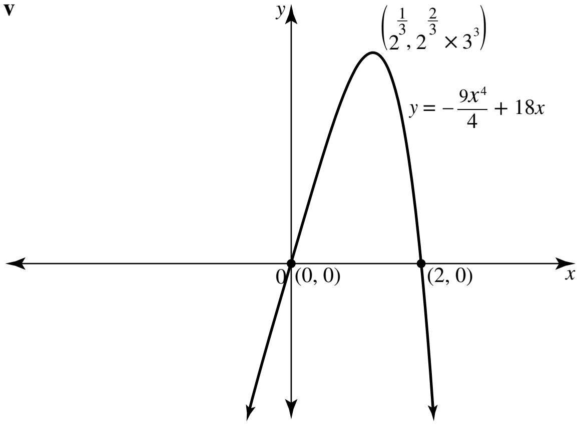
The anti-derivative function has a maximum turning point when $x = \sqrt[3]{2}$.

iv When $x = \sqrt[3]{2} = 2^{\frac{1}{3}}$,

$$\begin{aligned}
y &= -\frac{9}{4} \times \left(2^{\frac{1}{3}}\right)^4 + 18 \times 2^{\frac{1}{3}} \\
&= -\frac{3^2}{2^2} \times 2^{\frac{4}{3}} + 3^2 \times 2^1 \times 2^{\frac{1}{3}} \\
&= -3^2 \times 2^{\frac{4}{3}-2} + 3^2 \times 2^{1+\frac{1}{3}} \\
&= -3^2 \times 2^{-\frac{2}{3}} + 3^2 \times 2^{\frac{4}{3}} \\
&= 3^2 \times 2^{-\frac{2}{3}} \left[-1 + 2^{\frac{6}{3}}\right] \\
&= 3^2 \times 2^{-\frac{2}{3}} \left[-1 + 2^2\right] \\
&= 3^2 \times 2^{-\frac{2}{3}} \times 3 \\
&= 2^{-\frac{2}{3}} \times 3^3
\end{aligned}$$

The y co-ordinate of the turning point is $2^p \times 3^q$ where $p = -\frac{2}{3}$ and $q = 3$.

v

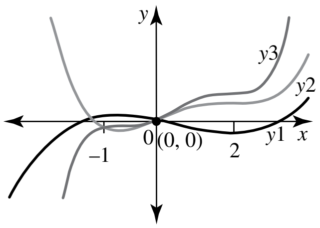


15 The shape of the graphs should be similar to those shown.

a $y1 = \int (x-2)(x+1)\,dx$

b $y2 = \int (x-2)^2(x+1)\,dx$

c $y3 = \int (x-2)^2(x+1)^2\,dx$



d The factor $(x-2)$ has a zero of $x = 2$ and the factor $(x+1)$ has a zero of $x = -1$.

The graph of $y1$ has turning points where $x = 2$ and $x = -1$; the graph of $y2$ has a stationary point of inflection at $x = 2$ and a turning point at $x = -1$; the graph of $y3$ has stationary points of inflection at $x = 2$ and at $x = -1$.

For factors of multiplicity 1, the anti-derivative graphs have turning points at the zero of each factor.

For factors of multiplicity 2, the anti-derivative graphs have stationary points of inflection at the zero of each factor.

16 Define $f(x) = x^2 - 6x$ by using the Interactive sub-menu.

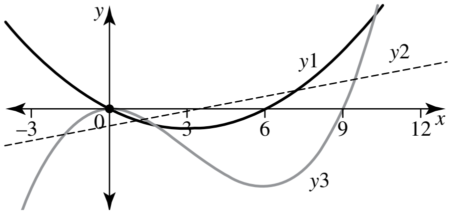
In Graph&Tab, enter and graph

$y1 = f(x)$

$y2 = \frac{d}{dx}\left(f(x)\right)$

$y3 = \int f(x)\,dx$

The shape of the graphs should be similar to those shown.



Connections between $y1$ and $y2$:

$y1$ is quadratic with x intercepts at $x = 0$ and $x = 6$. It has a minimum turning point when $x = 3$.

$y2$ is linear with x intercept at $x = 3$. Its sign changes from negative to zero to positive as it cuts through the x intercept. This shows that $y1$ has a minimum turning point when $x = 3$.

The x intercept of $y = f'(x)$ is connected to the turning point of $y = f(x)$; the degree of $f'(x)$ is one less than the degree of $f(x)$.

Connections between $y1$ and $y3$:

$y1$ is the gradient graph of $y3$. The x intercepts of $y1$ give the x co-ordinates of the stationary points of $y3$.

The sign of $y1$ changes from positive to zero to negative at $x = 0$, showing that $y3$ has a maximum turning point when $x = 0$.

The sign of $y1$ changes from negative to zero to positive at $x = 6$, showing that $y3$ has a minimum turning point when $x = 6$.

The turning points of $y = \int f(x)\,dx$ are connected to the x intercepts of $y = f(x)$; the degree of $\int f(x)\,dx$ is one more than the degree of $f(x)$.

(Note that the maximum turning point of $y3$ occurs at $x = 0$ but it is not necessarily at $(0, 0)$. The CAS calculator has used zero for the constant of integration).

Exercise 14.4 — Applications of anti-differentiation

1 a $v=3t^2-10t-8$

To obtain displacement, anti-differentiate velocity.

$\therefore x=t^3-5t^2-8t+c$

When $t=0,\ x=40$

$\therefore 40=c$

$\therefore x=t^3-5t^2-8t+40$

When the particle is at the origin, $x=0$.

$$t^3-5t^2-8t+40=0$$
$$t^2(t-5)-8(t-5)=0$$
$$(t-5)(t^2-8)=0$$
$$(t-5)(t-\sqrt{8})(t+\sqrt{8})=0$$
$$t=5,\pm\sqrt{8}$$

Since $t\geq 0,\ t=5,\sqrt{8}$.

The first time the particle is at the origin is when $t=\sqrt{8}$. Therefore, the particle is first at the origin after $2\sqrt{2}$ seconds.

b Acceleration is the rate of change of velocity.

$a=6t-10$.

When velocity is 0:

$$3t^2-10t-8=0$$
$$(3t+2)(t-4)=0$$
$$t=-\frac{2}{3},t=4$$

Since $t\geq 0,\ t=4$.

Therefore,

$a=6\times 4-10$

$=14$

The acceleration is 14 m/s^2 when the velocity is 0.

2 $v=\dfrac{1}{t^2}+2$

Acceleration: differentiate the velocity

$v=t^{-2}+2$

$a=-2t^{-3}$

$=-\dfrac{2}{t^3}$

Displacement: anti-differentiate the velocity

$v=t^{-2}+2$

$x=\dfrac{t^{-1}}{-1}+2t+c$

$=-\dfrac{1}{t}+2t+c$

When $t=1,\ x=-1$

$\therefore -1=-1+2+c$

$\therefore c=-2$

$\therefore x=-\dfrac{1}{t}+2t-2$

3 $a=8-18t$

Anti-differentiate to obtain velocity.

$v=8t-9t^2+c_1$

$v=10$ when $t=0\Rightarrow c_1=10$

$\therefore v=8t-9t^2+10$

When $t=1,\ v=9$, so the velocity is 9 m/s.

Displacement:

$v=8t-9t^2+10$

$\therefore x=4t^2-3t^3+10t+c_2$

When $t=0,x=-2\ \Rightarrow c_2=-2$

$\therefore x=4t^2-3t^3+10t-2$

When $t=1,\ x=9$, so the displacement is 9 metres. The position of the particle is 9 metres to the right of the origin.

4 $a=9.8$

$\therefore v=9.8t+c_1$

$v=0,t=0\Rightarrow c_1=0$

$\therefore v=9.8t$

$\therefore x=4.9t^2+c_2$

$x=0,t=0\Rightarrow c_2=0$

$\therefore x=4.9t^2$

When $t=5$,

$x=4.9\times 25$

$=122.5$

Therefore the displacement is 122.5 metres.

5 $\dfrac{dV}{dt}=-0.25$

a Anti-differentiation gives $V=-0.25t+c$

When $t=0,V=4.5\pi\Rightarrow c=4.5\pi$

$\therefore V=-0.25t+4.5\pi$

b When $V=0,\ t=\dfrac{4.5\pi}{0.25}$

Therefore, it takes approximately 56.5 seconds for the ice block to melt.

6 $h'(t)=0.2t$

$\therefore h(t)=0.1t^2+c$

When $t=0,h=50\Rightarrow c=50$

$\therefore h(t)=0.1t^2+50$

After one year or 12 months:

$h(12)=0.1\times 144+50$

$=64.4$

The rubber plant reached a height of 64.4 cm after one year.

7 a $v=8t^2-20t-12,\ t\geq 0$

Since $x=\int v\,dt$,

$x=\dfrac{8t^3}{3}-10t^2-12t+c$

Given $x=54$ when $t=0$,

$54=c$

$\therefore x=\dfrac{8t^3}{3}-10t^2-12t+54$

b When $t=1$,

$x=\dfrac{8}{3}-10-12+54$

$=\dfrac{8}{3}+32$

$=34\dfrac{2}{3}$

This position is $\left(54-34\dfrac{2}{3}\right)=19\dfrac{1}{3}$ metres from its initial position.

c Let $v = 0$

$\therefore 8t^2 - 20t - 12 = 0$

$\therefore 2t^2 - 5t - 3 = 0$

$\therefore (2t + 1)(t - 3) = 0$

$\therefore t = -\frac{1}{2}$, or $t = 3$

Reject the negative value since $t \geq 0$

$\therefore t = 3$

When $t = 3$,

$x = 72 - 90 - 36 + 54$

$= 0$

When the velocity is zero, the particle is at the origin.

8 $v = -3t^3,\ t \geq 0$

Given that $v = -24$ when $x = 1$.

Let $v = -24$

$\therefore -24 = -3t^3$

$\therefore t^3 = 8$

$\therefore t = 2$

Hence, when $t = 2$, $x = 1$.

Since $x = \int v\, dt$, then $x = \frac{-3t^4}{4} + c$.

Substitute $t = 2$, $x = 1$.

$\therefore 1 = -12 + c$

$\therefore c = 13$

$\therefore x = -\frac{3t^4}{4} + 13$

Let $t = 0$

$\therefore x = 13$

Initially the particle is 13 metres to the right of the origin.

9 **a** $v = 6 - 6t,\ t \geq 0$

Acceleration: $a = \frac{dv}{dt}$

$\therefore a = -6$

The particle moves with a constant acceleration of -6 m/s^2.

b Let $v = 0$

$\therefore 6 - 6t = 0$

$\therefore t = 1$

Position: $x = 6t - 3t^2 + c$

When $t = 0$, $x = 9$

$\therefore 9 = c$

$\therefore x = 6t - 3t^2 + 9$

Let $t = 1$

$x = 6 - 3 + 9$

$= 12$

The velocity is zero after 1 second when the particle is 12 metres to the right of the origin.

c Let $x = 0$

$\therefore 6t - 3t^2 + 9 = 0$

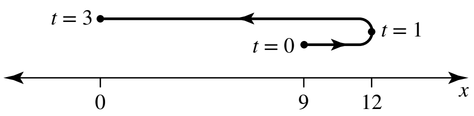
$\therefore t^2 - 2t - 3 = 0$

$\therefore (t - 3)(t + 1) = 0$

$\therefore t = 3$ or $t = -1$ reject since $t \geq 0$

$\therefore t = 3$

The particle reaches the origin after 3 seconds.



The distance travelled is $(3 + 12) = 15$ metres.

d The average speed in travelling 15 metres in 3 seconds is equal to $\frac{15}{3} = 5$ m/s

e The average velocity is the average rate of change of displacement.

$\frac{x(3) - x(0)}{3 - 0} = \frac{0 - 9}{3}$

$= -3$

The average velocity is -3 m/s.

10 **a** $v(t) = 3(t - 2)(t - 4),\ t \geq 0$

$v(0) = 3(-2)(-4)$

$= 24$

The initial velocity is 24 m/s.

Acceleration: $a = v'(t)$

$v(t) = 3(t^2 - 6t + 8)$

$\therefore v(t) = 3t^2 - 18t + 24$

$\therefore a = 6t - 18$

$a(0) = -18$, so the initial acceleration is -18 m/s^2.

b $v(t) = 3t^2 - 18t + 24$

$\therefore x(t) = t^3 - 9t^2 + 24t + c$

Since $x = 0$ when $t = 0$, $c = 0$

$\therefore x(t) = t^3 - 9t^2 + 24t$

c $x(5) = 125 - 225 + 120$

$\therefore x(5) = 20$

d Since $v(t) = 3(t - 2)(t - 4)$, the velocity is zero when $t = 2$ and when $t = 4$. The particle will therefore change its direction of motion twice during the first 5 seconds of motion.

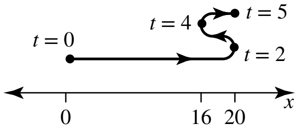
$x(0) = 0$ and $x(5) = 20$.

$x(2) = 8 - 36 + 48$

$= 20$

$x(4) = 64 - 144 + 96$

$= 16$



The distance travelled is $20 + 4 + 4 = 28$ metres.

11 **a** Acceleration: $a = 8 + 6t$

Velocity: $v = \int a \ \ dt$

$\therefore v = 8t + 3t^2 + c_1$

Given $v = 3$ when $t = 1$,

$\therefore 3 = 8 + 3 + c_1$

$\therefore c_1 = -8$

The velocity is $v = 3t^2 + 8t - 8$.

b Displacement: $x = \int v\, dt$

$\therefore x = t^3 + 4t^2 - 8t + c_2$

Given $x = 2$ when $t = 1$,

$\therefore 2 = 1 + 4 - 8 + c_2$

$\therefore c_2 = 5$

The displacement is $x = t^3 + 4t^2 - 8t + 5$.

12 **a** Acceleration, $a = -10$

Velocity: $v = -10t + c_1$

Given $v = 20$ when $t = 0$

$\therefore 20 = c_1$

$\therefore v = -10t + 20$

Displacement: $x = -5t^2 + 20t + c_2$

Given $x = 0$ when $t = 0$

$\therefore 0 = c_2$

$\therefore x = -5t^2 + 20t$

Let $v = 0$

$\therefore -10t + 20 = 0$

$\therefore t = 2$

When $t = 2$,

$x = -20 + 40$

$= 20$

The velocity is zero after 2 seconds when its position is 20 metres from the origin in the positive direction.

b The object starts from the origin.

Let $x = 0$

$\therefore -5t^2 + 20t = 0$

$\therefore -5t(t-4) = 0$

$\therefore t = 0, t = 4$

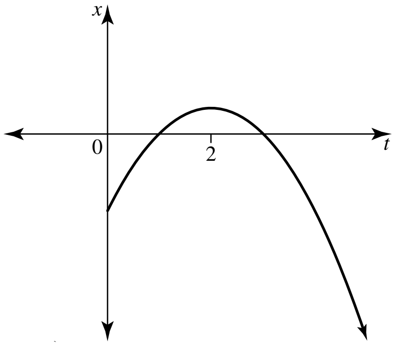
The object returns to its starting point after 4 seconds.

13 a The given velocity graph is the gradient graph for the displacement. The displacement graph is an anti-derivative graph.

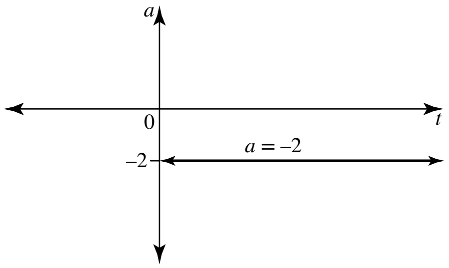
Since $v = 0$ when $t = 2$, there is a stationary point at $t = 2$ on the displacement graph. For increasing t, the velocity changes its sign from positive to zero to negative as it passes through $(2, 0)$. This means there is a maximum turning point at $t = 2$ on the displacement graph.

The velocity graph is linear so the displacement graph is quadratic.

An example of a possible displacement graph is shown.



b The acceleration graph is the gradient of the velocity graph. The gradient of the given graph is −2, so the acceleration graph is a the horizontal line $a = -2, t \geq 0$.



c The given velocity graph has gradient $m = -2$ and its vertical intercept is $(0, 4)$.

Its equation is $v = -2t + 4$.

$\therefore x = -t^2 + 4t + c$

When $t = 0$, $x = -4$

$\therefore -4 = c$

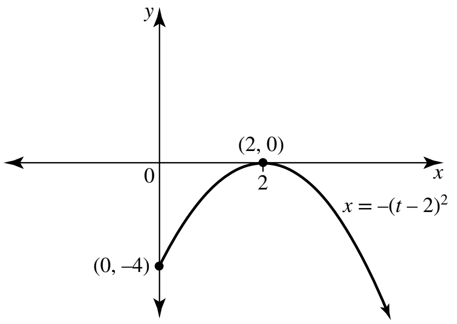
$\therefore x = -t^2 + 4t - 4$

Factorising the expression for the displacement,

$x = -\left(t^2 - 4t + 4\right)$

$= -(t-2)^2$

d There is a maximum turning point at $(2, 0)$.



14 a $\dfrac{dA}{dt} = -18t$

Anti-differentiate with respect to t

$\therefore A = -9t^2 + c$

When $t = 0$, $A = 90$

$\therefore 90 = c$

$\therefore A = -9t^2 + 90$

b Let $A = 0$

$\therefore -9t^2 + 90 = 0$

$\therefore t^2 = 10$

$\therefore t = \sqrt{10}$

(negative square root is not applicable).

Since $\sqrt{9} < \sqrt{10} < \sqrt{16}$, then $3 < \sqrt{10} < 4$. Therefore, after 4 whole days, the weed will be completely removed.

c The values of t lie in the interval $\left[0, \sqrt{10}\right]$, so the model is valid for $0 \leq t \leq \sqrt{10}$.

15 a $A'(t) = 4 - t$

$A'(0) = 4 > 0$, so the area increases initially.

Let the rate of change be zero.

$A'(t) = 0$

$\therefore 4 - t = 0$

$\therefore t = 4$

After 4 days the area stops increasing.

b $A(t) = 4t - \dfrac{t^2}{2} + c$

When $t = 0$, $A = 0$

$\therefore 0 = c$

$\therefore A(t) = 4t - \dfrac{t^2}{2}$

The area of the puddle is zero when $4t - \dfrac{t^2}{2} = 0$.

$\therefore 8t - t^2 = 0$

$\therefore t(8-t) = 0$

$\therefore t = 0,\ t = 8$

After 8 days, the puddle has dried out.

Therefore the domain is $[0, 8]$.

c Greatest area occurs when $A'(t) = 0$.

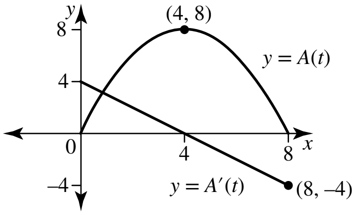
From part a, this occurs when $t = 4$

$A(4) = 16 - 8$

$= 8$

The greatest area of the puddle is 8 sq m.

d The domain for each graph is $[0,8]$.
$y = A'(t) = 4 - t$ is linear with endpoints $(0,4)$ and $(8,-4)$. It also contains the point $(4,0)$.
$y = A(t) = 4t - \frac{t^2}{2}$ is quadratic with maximum turning point $(4,8)$ and endpoints $(0,0)$ and $(8,0)$.



16 a $\frac{dm}{dt} = k\sqrt{t}$

When $t = 4$, $\frac{dm}{dt} = 300$

$\therefore 300 = 2k$

$\therefore k = 150$

b $\frac{dm}{dt} = 150t^{\frac{1}{2}}$

$\therefore m = 150 \times \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$

$\therefore m = 150 \times \frac{2}{3}t^{\frac{3}{2}} + c$

$\therefore m = 100t^{\frac{3}{2}} + c$

When $t = 0$, $m = 20$

$\therefore 20 = 100 \times 0 + c$

$\therefore c = 20$

$\therefore m = 100t^{\frac{3}{2}} + 20$

c Let $m = 6420$

$\therefore 6420 = 100t^{\frac{3}{2}} + 20$

$\therefore 100t^{\frac{3}{2}} = 6400$

$\therefore t^{\frac{3}{2}} = 64$

$\therefore (t^{\frac{3}{2}})^{\frac{2}{3}} = (64)^{\frac{2}{3}}$

$\therefore t = \left(\sqrt[3]{64}\right)^2$

$\therefore t = 16$

It takes 16 days for the population to reach 6420 microbes.

17 a $v = (2t+1)^4$

Acceleration $a = \frac{dv}{dt}$

Using the template in mth keyboard,

$a = \frac{d}{dt}\left((2t+1)^4\right)$

$= 8(2t+1)^3$

b Displacement: Using the template,

$x = \int (2t+1)^4\, dt$

$= \frac{(2t+1)^5}{10}$

Hence, $x = \frac{(2t+1)^5}{10} + c$

When $t = 0$, $x = 4.2$

$\therefore 4.2 = \frac{1}{10} + c$

$\therefore c = 4.1$

$\therefore x = \frac{(2t+1)^4}{10} + 4.1$

c Let $x = 8.4$

$\therefore 8.4 = \frac{(2t+1)^4}{10} + 4.1$

Solve in Equation/Inequality in Decimal mode to obtain $t = 0.561$.

When $t = 0.561$,

$v = (2 \times 0.561 + 1)^4$

$= 20.266$

Correct to two decimal places, the time is 0.56 seconds and velocity is 20.27 m/s (assuming time unit is seconds).

18 $a = \frac{1}{(t+1)^3}$

$v = \int \frac{1}{(t+1)^3}\, dt$

$= \frac{-1}{2(t+1)^2}$

$\therefore v = \frac{-1}{2(t+1)^2} + c_1$

Given $v = 0$ when $t = 0$.

$0 = \frac{-1}{2} + c_1$

$\therefore c_1 = \frac{1}{2}$

$\therefore v = \frac{-1}{2(t+1)^2} + \frac{1}{2}$

$x = \int \left(\frac{-1}{2(t+1)^2} + \frac{1}{2}\right) dt$

$= \frac{t}{2} + \frac{1}{2(t+1)}$

$\therefore x = \frac{t}{2} + \frac{1}{2(t+1)} + c_2$

Given $x = 0$ when $t = 0$.

$0 = \frac{1}{2} + c_2$

$\therefore c_2 = -\frac{1}{2}$

$\therefore x = \frac{t}{2} + \frac{1}{2(t+1)} - \frac{1}{2}$

Exercise 14.5 — The definite integral

1 $\int_0^3 (3x^2 - 2x)\, dx$

$= \left[x^3 - x^2\right]_0^3$

$= \left(3^3 - 3^2\right) - \left(0^3 - 0^2\right)$

$= 27 - 9 - 0$

$= 18$

2 a $\int_{-2}^{2}(x-2)(x+2)\,dx$

$$=\int_{-2}^{2}(x^2-4)\,dx$$
$$=\left[\frac{x^3}{3}-4x\right]_{-2}^{2}$$
$$=\left(\frac{8}{3}-8\right)-\left(\frac{-8}{3}-(-8)\right)$$
$$=\frac{8}{3}-8+\frac{8}{3}-8$$
$$=\frac{16}{3}-16$$
$$=-\frac{32}{3}$$

b $\int_{-1}^{1}x^3dx$

$$=\left[\frac{x^4}{4}\right]_{-1}^{1}$$
$$=\left(\frac{1}{4}\right)-\left(\frac{1}{4}\right)$$
$$=0$$

3 a Base is 4 units and height is 3 units.

$$A=\frac{1}{2}bh$$
$$=\frac{1}{2}\times 4\times 3$$
$$=6$$

Area is 6 square units.

b $\int_{0}^{4}0.75x\,dx$

c $\int_{0}^{4}0.75x\;\;dx$

$$=\left[0.75\times\frac{x^2}{2}\right]_{0}^{4}$$
$$=0.75\times 8-0$$
$$=6$$

Therefore the area is 6 square units.

4 a $\int_{1}^{3}\left(16-x^2\right)dx$

b $\int_{1}^{3}\left(16-x^2\right)dx$

$$=\left[16x-\frac{x^3}{3}\right]_{1}^{3}$$
$$=(48-9)-\left(16-\frac{1}{3}\right)$$
$$=39-15\frac{2}{3}$$
$$=23\frac{1}{3}$$

Therefore, the area is $23\frac{1}{3}$ square units.

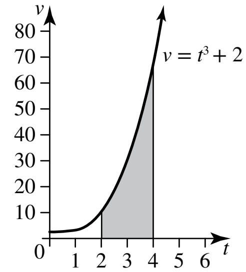
5 a $v=t^3+2$

When $t=1,\ v=3$ so velocity is 3 m/s.

$$a=\frac{dv}{dt}$$
$$\therefore a=3t^2$$

When $t=1,\ a=3$ so acceleration is 3 m/s^2.

b i



ii Measure of the distance equals $\int_{2}^{4}(t^3+2)\,dt$.

$$\int_{2}^{4}(t^3+2)\,dt$$
$$=\left[\frac{t^4}{4}+2t\right]_{2}^{4}$$
$$=\left(\frac{4^4}{4}+2(4)\right)-\left(\frac{2^4}{4}+2(2)\right)$$
$$=(64+8)-(4+4)$$
$$=64$$

Distance travelled is 64 metres.

6 a $v=3t^2-2t+5$

The velocity is a quadratic function in t.

The discriminant of this quadratic is:

$$\Delta=b^2-4ac,\ a=3, b=-2, c=5$$
$$=4-60$$
$$=-56$$

Since $\Delta<0$ and $a>0$, the velocity is always positive.

b i Distance measure is given by $\int_{0}^{2}(3t^2-2t+5)\,dt$.

$$\int_{0}^{2}(3t^2-2t+5)\,dt$$
$$=\left[t^3-t^2+5t\right]_{0}^{2}$$
$$=8-4+10$$
$$=14$$

Therefore, the distance travelled is 14 metres.

ii $v=3t^2-2t+5$

Anti-differentiate with respect to t:

$x=t^3-t^2+5t+c$

When $t=0$, $x=c$ and when $t=2, x=8-4+10+c=14+c$.

The distance travelled is:

$$x(1)-x(0)$$
$$=(14+c)-(c)$$
$$=14$$

Therefore, the distance travelled is 14 metres.

7 a $\int_{-2}^{2}5x^4\,dx$

$$=\left[x^5\right]_{-2}^{2}$$
$$=(32)-(-32)$$
$$=64$$

b $\int_{0}^{2}\left(7-2x^3\right)dx$

$$=\left[7x-\frac{x^4}{2}\right]_{0}^{2}$$
$$=(14-8)-(0)$$
$$=6$$

c $\int_1^3 \left(6x^2+5x-1\right)dx$

$$\begin{aligned}
&= \left[2x^3+\frac{5x^2}{2}-x\right]_1^3\\
&= \left(54+\frac{45}{2}-3\right)-\left(2+\frac{5}{2}-1\right)\\
&= \left(51+\frac{45}{2}\right)-\left(\frac{7}{2}\right)\\
&= 51+\frac{45}{2}-\frac{7}{2}\\
&= 51+19\\
&= 70
\end{aligned}$$

d $\int_{-3}^0 12x^2(x-1)\,dx$

$$\begin{aligned}
&= \int_{-3}^0 \left(12x^3-12x^2\right)dx\\
&= \left[3x^4-4x^3\right]_{-3}^0\\
&= (0)-(243+108)\\
&= -351
\end{aligned}$$

e $\int_{-4}^{-2} (x+4)^2\,dx$

$$\begin{aligned}
&= \int_{-4}^{-2} \left(x^2+8x+16\right)dx\\
&= \left[\frac{x^3}{3}+4x^2+16x\right]_{-4}^{-2}\\
&= \left(-\frac{8}{3}+16-32\right)-\left(-\frac{64}{3}+64-64\right)\\
&= -\frac{8}{3}-16+\frac{64}{3}\\
&= \frac{56}{3}-\frac{48}{3}\\
&= \frac{8}{3}
\end{aligned}$$

f $\int_{-\frac{1}{2}}^{\frac{1}{2}} (x+1)\left(x^2-x\right)dx$

$$\begin{aligned}
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(x^3-x\right)dx\\
&= \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-\frac{1}{2}}^{\frac{1}{2}}\\
&= \left(\frac{1}{64}-\frac{1}{8}\right)-\left(\frac{1}{64}-\frac{1}{8}\right)\\
&= 0
\end{aligned}$$

8 a $\int_1^2 (20x+15)\,dx = 5\int_1^2 (4x+3)\,dx$

$$\begin{aligned}
\text{LHS} &= \int_1^2 (20x+15)\,dx\\
&= \left[10x^2+15x\right]_1^2\\
&= (40+30)-(10+15)\\
&= 70-25\\
&= 45
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= 5\int_1^2 (4x+3)\,dx\\
&= 5\left[2x^2+3x\right]_1^2\\
&= 5\{(8+6)-(2+3)\}\\
&= 5\{14-5\}\\
&= 5\times 9\\
&= 45
\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

b $\int_{-1}^2 (x^2+2)\,dx = \int_{-1}^2 x^2\,dx + \int_{-1}^2 2\,dx$

$$\begin{aligned}
\text{LHS} &= \int_{-1}^2 (x^2+2)\,dx\\
&= \left[\frac{x^3}{3}+2x\right]_{-1}^2\\
&= \left(\frac{8}{3}+4\right)-\left(-\frac{1}{3}-2\right)\\
&= \left(\frac{20}{3}\right)-\left(-\frac{7}{3}\right)\\
&= \frac{20}{3}+\frac{7}{3}\\
&= 9
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \int_{-1}^2 x^2\,dx + \int_{-1}^2 2\,dx\\
&= \left[\frac{x^3}{3}\right]_{-1}^2 + [2x]_{-1}^2\\
&= \left[\left(\frac{8}{3}\right)-\left(-\frac{1}{3}\right)\right]+[(4)-(-2)]\\
&= (3)+(6)\\
&= 9
\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

c $\int_1^3 3x^2\,dx = \int_1^3 t^2\,dt$

$$\begin{aligned}
\text{LHS} &= \int_1^3 3x^2\,dx\\
&= \left[x^3\right]_1^3\\
&= (27)-(1)\\
&= 26
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \int_1^3 3t^2\,dt\\
&= \left[t^3\right]_1^3\\
&= (27)-(1)\\
&= 26
\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

d $\int_a^a 3x^2\,dx = 0$

$$\begin{aligned}
\text{LHS} &= \int_a^a 3x^2\,dx\\
&= \left[x^3\right]_a^a\\
&= \left(a^3\right)-\left(a^3\right)\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

e $\int_b^a 3x^2\,dx = -\int_a^b 3x^2\,dx$

$$\begin{aligned}
\text{LHS} &= \int_b^a 3x^2\,dx\\
&= \left[x^3\right]_b^a\\
&= \left(a^3\right)-\left(b^3\right)\\
&= a^3-b^3
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= -\int_a^b 3x^2\,dx\\
&= -\left[x^3\right]_a^b\\
&= -\left(b^3-a^3\right)\\
&= a^3-b^3
\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

f $\int_{-a}^{a} dx = 2a$

$\text{LHS} = \int_{-a}^{a} dx$

$$\begin{aligned}
&= \int_{-a}^{a} 1 \; dx \\
&= [x]_{-a}^{a} \\
&= (a) - (-a) \\
&= 2a \\
&= \text{RHS}
\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

9 **a** $\int_{1}^{2} \frac{1}{x^2} dx$

$$\begin{aligned}
&= \int_{1}^{2} x^{-2} \, dx \\
&= \left[\frac{x^{-1}}{-1} \right]_1^2 \\
&= \left[-\frac{1}{x} \right]_1^2 \\
&= \left(-\frac{1}{2} \right) - (-1) \\
&= -\frac{1}{2} + 1 \\
&= \frac{1}{2}
\end{aligned}$$

b $\int_{0}^{9} \sqrt{x} \, dx$

$$\begin{aligned}
&= \int_{0}^{9} x^{\frac{1}{2}} \, dx \\
&= \left[\frac{2x^{\frac{3}{2}}}{3} \right]_0^9 \\
&= \left(\frac{2}{3} \times 9^{\frac{3}{2}} \right) - (0) \\
&= \frac{2}{3} \times 27 \\
&= 18
\end{aligned}$$

c $\int_{1}^{4} 5x^{\frac{3}{2}} \, dx$

$$\begin{aligned}
&= \left[5 \times \frac{2x^{\frac{5}{2}}}{5} \right]_1^4 \\
&= \left[2x^{\frac{5}{2}} \right]_1^4 \\
&= \left(2 \times 4^{\frac{5}{2}} \right) - (2) \\
&= 2 \times 32 - 2 \\
&= 62
\end{aligned}$$

d $\int_{1}^{2} \frac{x^6 + 2x^3 + 4}{x^5} dx$

$$\begin{aligned}
&= \int_{1}^{2} \left(\frac{x^6}{x^5} + \frac{2x^3}{x^5} + \frac{4}{x^5} \right) dx \\
&= \int_{1}^{2} \left(x + 2x^{-2} + 4x^{-5} \right) dx \\
&= \left[\frac{x^2}{2} + \frac{2x^{-1}}{-1} + \frac{4x^{-4}}{-4} \right]_1^2 \\
&= \left[\frac{x^2}{2} - \frac{2}{x} - \frac{1}{x^4} \right]_1^2 \\
&= \left(2 - 1 - \frac{1}{16} \right) - \left(\frac{1}{2} - 2 - 1 \right) \\
&= \frac{15}{16} + \frac{5}{2} \\
&= \frac{55}{16}
\end{aligned}$$

e $\int_{3}^{5} 2\sqrt{t} \left(\sqrt{t} + 3\left(\sqrt{t}\right)^3 \right) dt$

$$\begin{aligned}
&= \int_{3}^{5} \left(2\left(\sqrt{t}\right)^2 + 6\left(\sqrt{t}\right)^4 \right) dt \\
&= \int_{3}^{5} \left(2t + 6t^2 \right) dt \\
&= \left[t^2 + 2t^3 \right]_3^5 \\
&= (25 + 250) - (9 + 54) \\
&= 275 - 63 \\
&= 212
\end{aligned}$$

f $\int_{-2}^{-1} \left(2 - \frac{4}{u^3} \right) du$

$$\begin{aligned}
&= \int_{-2}^{-1} \left(2 - 4u^{-3} \right) du \\
&= \left[2u - \frac{4u^{-2}}{-2} \right]_{-2}^{-1} \\
&= \left[2u + \frac{2}{u^2} \right]_{-2}^{-1} \\
&= (-2 + 2) - \left(-4 + \frac{1}{2} \right) \\
&= (0) - \left(-\frac{7}{2} \right) \\
&= \frac{7}{2}
\end{aligned}$$

10 **a** $\int_{4}^{n} 3 \, dx = 9$

$\therefore [3x]_4^n = 9$

$\therefore 3n - 12 = 9$

$\therefore 3n = 21$

$\therefore n = 7$

b $\int_0^p \sqrt{x}\,dx = 18$

$\therefore \int_0^p x^{\frac{1}{2}}\,dx = 18$

$\therefore \left[\dfrac{2x^{\frac{3}{2}}}{3}\right]_0^p = 18$

$\therefore \dfrac{2}{3}p^{\frac{3}{2}} - 0 = 18$

$\therefore \dfrac{2}{3}p^{\frac{3}{2}} = 18$

$\therefore 2p^{\frac{3}{2}} = 54$

$\therefore p^{\frac{3}{2}} = 27$

$\therefore \left(p^{\frac{3}{2}}\right)^{\frac{2}{3}} = 27^{\frac{2}{3}}$

$\therefore p = 9$

11 a The area measure is given by $\int_{-2}^{2}(3x+6)\,dx$.

Integration method to calculate the area:

$$\begin{aligned} A &= \int_{-2}^{2}(3x+6)\,dx \\ &= \left[\frac{3x^2}{2}+6x\right]_{-2}^{2} \\ &= (6+12)-(6-12) \\ &= 18+6 \\ &= 24 \end{aligned}$$

The area is 24 sq units.

The triangular area can also be calculated with the formula $A = \frac{1}{2}bh$.

The base, $b = 4$.

Height: When $x = 2$,

$$\begin{aligned} y &= 3\times 2+6 \\ &= 12 \end{aligned}$$

$\therefore h = 12$

$$\begin{aligned} A &= \frac{1}{2}\times 4\times 12 \\ &= 24 \end{aligned}$$

The area is 24 sq units.

b The area measure is given by $\int_{-6}^{-2}(1)\,dx$.

Integration method to calculate the area:

$$\begin{aligned} A &= \int_{-6}^{-2}(1)\,dx \\ &= [x]_{-6}^{-2} \\ &= (-2)-(-6) \\ &= 4 \end{aligned}$$

The area is 4 sq units.

The rectangular area can also be calculated from $A = lw,\ l = 4, w = 1$

$$\begin{aligned} \therefore A &= 4\times 1 \\ &= 4 \end{aligned}$$

The area is 4 sq units.

c The area measure is given by $\int_{-2}^{0}(4-3x)\,dx$.

Integration method to calculate the area:

$$\begin{aligned} A &= \int_{-2}^{0}(4-3x)\,dx \\ &= \left[4x-\frac{3x^2}{2}\right]_{-2}^{0} \\ &= (0)-(-8-6) \\ &= 14 \end{aligned}$$

The area is 14 sq units.

The trapezoidal area can also be calculated from $A = \frac{1}{2}h(a+b)$.

For the line $y = 4-3x$, when $x = -2$, $y = 10$ and when $x = 0$, $y = 4$.

$A = \frac{1}{2}h(a+b)$, $h = 2, a = 10, b = 4$

$$\begin{aligned} \therefore A &= \frac{1}{2}\times 2\times(10+4) \\ &= 14 \end{aligned}$$

The area is 14 sq units.

12 a i The area measure is given by $\int_{-1}^{1}\left(x^2+1\right)dx$

ii
$$\begin{aligned} A &= \int_{-1}^{1}\left(x^2+1\right)dx \\ &= \left[\frac{x^3}{3}+x\right]_{-1}^{1} \\ &= \left(\frac{1}{3}+1\right)-\left(-\frac{1}{3}-1\right) \\ &= \frac{4}{3}-\left(-\frac{4}{3}\right) \\ &= \frac{8}{3} \end{aligned}$$

The area is $\frac{8}{3}$ sq units.

b i The area measure is given by $\int_{-2}^{1}\left(1-x^3\right)dx$

ii
$$\begin{aligned} A &= \int_{-2}^{1}\left(1-x^3\right)dx \\ &= \left[x-\frac{x^4}{4}\right]_{-2}^{1} \\ &= \left(1-\frac{1}{4}\right)-(-2-4) \\ &= \frac{3}{4}-(-6) \\ &= 6\frac{3}{4} \end{aligned}$$

The area is $6\frac{3}{4}$ sq units.

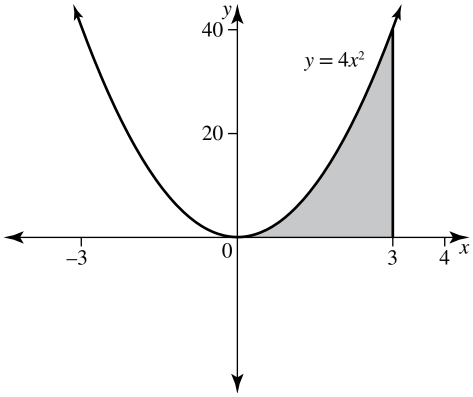
c i The area measure is given by $\int_{0.5}^{1.5}(1+x)(3-x)\,dx$

ii $A=\int_{0.5}^{1.5}(1+x)(3-x)\,dx$

$$\begin{aligned}
&=\int_{0.5}^{1.5}\left(3+2x-x^2\right)dx\\
&=\left[3x+x^2-\frac{x^3}{3}\right]_{0.5}^{1.5}\\
&=(4.5+2.25-1.125)-\left(1.5+0.25-\frac{0.125}{3}\right)\\
&=(5.625)-\left(1.75-\frac{1}{24}\right)\\
&=5.625-1.75+\frac{1}{24}\\
&=3.875+\frac{1}{24}\\
&=3+\frac{7}{8}+\frac{1}{24}\\
&=3\frac{11}{12}
\end{aligned}$$

The area is $3\frac{11}{12}$ sq units.

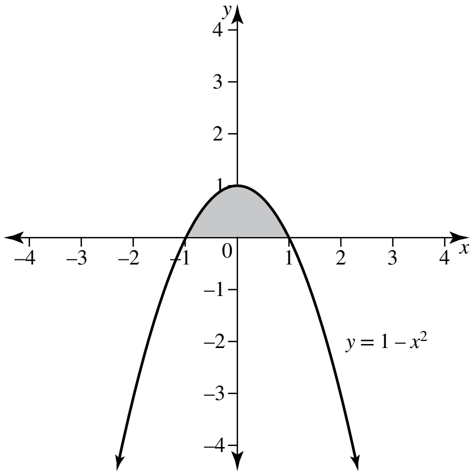
13 a The area represented by $\int_0^3 4x^2\,dx$ is the area enclosed by the graph of $y=4x^2$, the x axis and $x=0$, $x=3$.



$$\begin{aligned}
A&=\int_0^3 4x^2\,dx\\
&=\left[\frac{4x^3}{3}\right]_0^3\\
&=(36)-(0)\\
&=36
\end{aligned}$$

The area is 36 sq units.

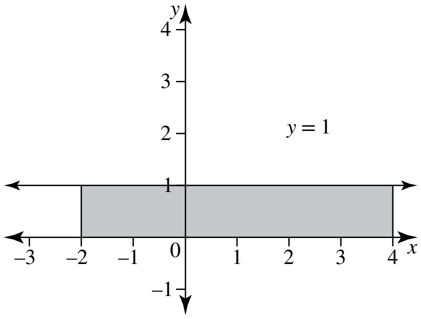
b The area represented by $\int_{-1}^{1}\left(1-x^2\right)dx$ is the area bounded by the graph of $y=1-x^2$, the x axis and $x=-1$, $x=1$.



$$\begin{aligned}
A&=\int_{-1}^{1}\left(1-x^2\right)dx\\
&=\left[x-\frac{x^3}{3}\right]_{-1}^{1}\\
&=\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\\
&=\left(\frac{2}{3}\right)-\left(-\frac{2}{3}\right)\\
&=\frac{4}{3}
\end{aligned}$$

The area is $\frac{4}{3}$ sq units.

c The area represented by $\int_{-2}^{4}dx$ is the rectangular area enclosed by the line $y=1$, the x axis and $x=-2$, $x=4$.



The area is $6\times1=6$ sq units.

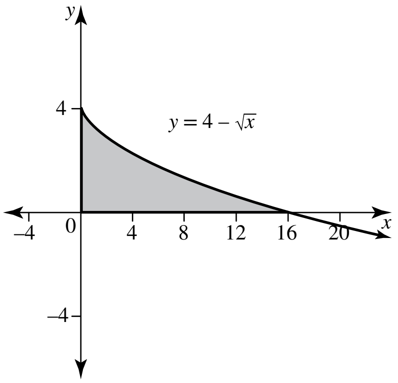
14 a Features of the graph of $y=f(x)$, for $f(x)=4-\sqrt{x}$.

y intercept and endpoint: $(0,4)$

x intercept: Let $y=0$

$$\begin{aligned}
&\therefore 4-\sqrt{x}=0\\
&\therefore 4=\sqrt{x}\\
&\therefore x=16
\end{aligned}$$

$(16,0)$



b The area bounded by the curve and the co-ordinate axes is shaded.

c The area measure is $\int_0^{16}\left(4-\sqrt{x}\right)dx$.

d $\int_0^{16}\left(4-\sqrt{x}\right)dx$

$$=\int_0^{16}\left(4-x^{\frac{1}{2}}\right)dx$$

$$=\left[4x-\frac{2x^{\frac{3}{2}}}{3}\right]_0^{16}$$

$$=\left(64-\frac{2}{3}\times 16^{\frac{3}{2}}\right)-(0)$$

$$=64-\frac{2}{3}\times 64$$

$$=\frac{1}{3}\times 64$$

$$=\frac{64}{3}$$

The area is $\frac{64}{3}$ sq units.

15 a $v=10t-5t^2,\ t\geq 0$

Displacement: $x=\int v\,dt$

$$x=5t^2-\frac{5t^3}{3}+c$$

When $t=0,\ x=0$

$\therefore c=0$

$$\therefore x=5t^2-\frac{5t^3}{3}$$

b Let $v=0$

$\therefore 10t-5t^2=0$

$\therefore 5t(2-t)=0$

$\therefore t=0$ or $t=2$

The particle is next at rest after 2 seconds.

When $t=2$,

$$x=5\times 4-5\times\frac{8}{3}$$

$$=20-\frac{40}{3}$$

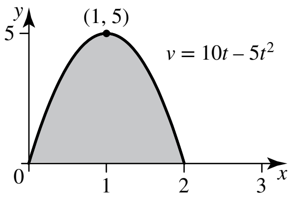
$$=\frac{20}{3}$$

The distance the particle travels by the time it is next at rest is $\frac{20}{3}$ units.

c $v=10t-5t^2$ for $t\in[0,2]$

The concave down parabolic graph has endpoints $(0,0)$ and $(2,0)$.

Its turning point occurs when $t=1$ and therefore $v=5$. The point $(1,5)$ is a maximum turning point.



d The area measure is given by $\int_0^2\left(10t-5t^2\right)dt$.

$$\int_0^2\left(10t-5t^2\right)dt$$

$$=\left[5t^2-\frac{5t^3}{3}\right]_0^2$$

$$=\left(20-\frac{5}{3}\times 8\right)-(0)$$

$$=20-\frac{40}{3}$$

$$=\frac{20}{3}$$

The area is $\frac{20}{3}$ sq units.

e The area under the velocity-time graph measures the distance travelled in the first 2 seconds.

16 a Refer to the velocity-time graph given in the question. The greatest speed occurs at the global maximum point $\left(4,\frac{22}{3}\right)$. Thus, the greatest speed is $\frac{22}{3}$ km/h.

The least speed occurs at the global minimum point $(0,2)$. The least speed is 2 km/h.

b Acceleration is the rate of change of the velocity and is measured by the gradient of the tangent to the velocity graph. If the acceleration is zero, the tangent is horizontal. Therefore, the acceleration is zero at the turning points. The acceleration is zero after 1 hour, after 2 hours and after 4 hours.

c $v=2+8t-7t^2+\frac{7t^3}{3}-\frac{t^4}{4}$.

The distance the athlete runs is the area under the velocity-time graph.

$$x=\int_0^5 v\,dt$$

$$=\int_0^5\left(2+8t-7t^2+\frac{7t^3}{3}-\frac{t^4}{4}\right)dt$$

$$=\left[2t+4t^2-\frac{7t^3}{3}+\frac{7}{3}\times\frac{t^4}{4}-\frac{1}{4}\times\frac{t^5}{5}\right]_0^5$$

$$=\left[2t+4t^2-\frac{7t^3}{3}+\frac{7t^4}{12}-\frac{t^5}{20}\right]_0^5$$

$$=\left(10+100-\frac{7\times 125}{3}+\frac{7\times 625}{12}-\frac{625\times 5}{20}\right)-(0)$$

$$=110+125\times\left(-\frac{7}{3}+\frac{7\times 5}{12}-\frac{5}{4}\right)$$

$$=110+125\times\left(\frac{-28+35-15}{12}\right)$$

$$=110+125\times\frac{-\cancel{8}^{2}}{\cancel{12}_{3}}$$

$$=110-\frac{250}{3}$$

$$=\frac{80}{3}$$

The distance the athlete ran during the five hour training period is $\frac{80}{3}=26\frac{2}{3}$ km.

17 **a** The definite integral template is the one used for indefinite integrals. For the definite integral the terminals are entered on the template.

$\int_{-3}^{4}\left(2-3x+x^2\right)dx=\frac{203}{6}$

b $\int_{4}^{-3}\left(2-3x+x^2\right)dx=-\frac{203}{6}$

c Interchanging the terminals changes the sign of the integral.

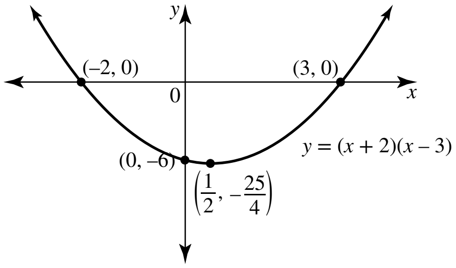
18 **a** $\int_{-2}^{3}(x+2)(x-3)\,dx=-\frac{125}{6}$

b The graph of $y=(x+2)(x-3)$ is a concave up parabola which cuts x axis at $x=-2$ and $x=3$. The y intercept is $(0,-6)$.

Turning point occurs when $x=\frac{-2+3}{2}=\frac{1}{2}$ and

$y=\frac{5}{2}\times\frac{-5}{2}=-\frac{25}{4}$.

Minimum turning point $\left(\frac{1}{2},-\frac{25}{4}\right)$.



c The area enclosed by the graph and the x axis lies completely below the x axis.

The area $A=-\int_{-2}^{3}(x+2)(x-3)\,dx$

$=-\left(-\frac{125}{6}\right)$

$=\frac{125}{6}$

The area is $\frac{125}{6}$ sq units.

d The definite integral measures the signed area. As the area lies under the the x axis, the signed area is negative and therefore the definite integral is negative.
The actual area and the value of the definite integral differ in sign.
A possible integral which would give the actual area is obtained by interchanging the terminals so that

$A=\int_{3}^{-2}(x+2)(x-3)\,dx$.

Another possible integral is to write

$A=-\int_{-2}^{3}(x+2)(x-3)\,dx$ as

$A=\int_{-2}^{3}(x+2)(3-x)\,dx$.