# MATHSQUEST 11

## MATHEMATICAL METHODS

VCE UNITS 1 AND 2

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VCE UNITS 1 AND 2


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# Contents

# Introduction

At Jacaranda, we are deeply committed to the ideal that learning brings life-changing benefits to all students. By continuing to provide resources for Mathematics of exceptional and proven quality, we ensure that all VCE students have the best opportunity to excel and to realise their full potential.

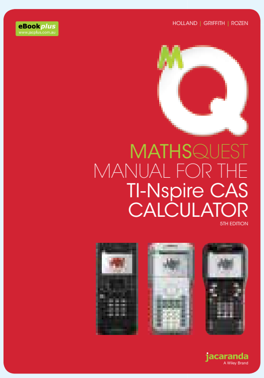
*Maths Quest 11 Mathematical Methods VCE Units 1 and 2* comprehensively covers the requirements of the revised Study Design 2016–2018.

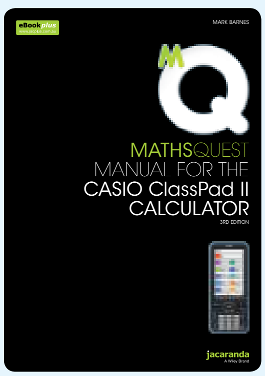
## Features of the new *Maths Quest* series

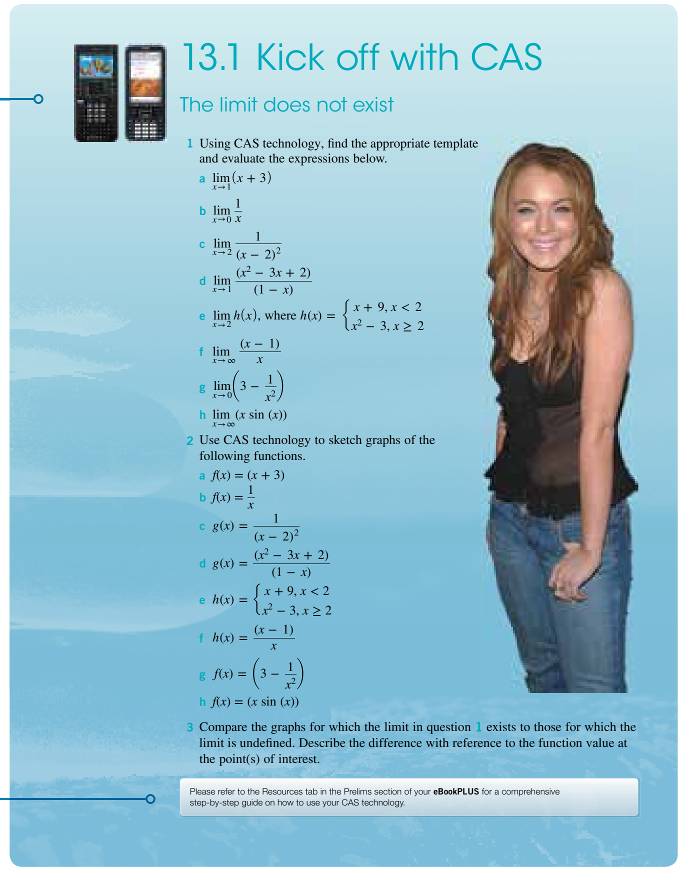
### CAS technology

Each topic opens with an engaging **Kick off with CAS** activity designed to stimulate students' interest and curiosity and to highlight the important applications of CAS technology in developing deep understanding of the mathematical concepts presented.

For up-to-date, step-by-step instructions on how to use CAS technology, we have provided the *Manual for the TI-Nspire CAS calculator* and the *Manual for the Casio ClassPad II* in the Prelims section of the eBook.



### 13.1 Kick off with CAS

The limit does not exist

1 Using CAS technology, find the appropriate template and evaluate the expressions below.

a $\lim_{x \to 1}(x + 3)$

b $\lim_{x \to 0}\frac{1}{x}$

c $\lim_{x \to 2}\frac{1}{(x-2)^2}$

d $\lim_{x \to 1}\frac{(x^2 - 3x + 2)}{(1 - x)}$

e $\lim_{x \to 2} h(x)$, where $h(x) = \begin{cases} x + 9, x < 2 \\ x^2 - 3, x \ge 2 \end{cases}$

f $\lim_{x \to \infty}\frac{(x-1)}{x}$

g $\lim_{x \to 0}\left(3 - \frac{1}{x^2}\right)$

h $\lim_{x \to \infty}(x \sin(x))$

2 Use CAS technology to sketch graphs of the following functions.

a $f(x) = (x + 3)$

b $f(x) = \frac{1}{x}$

c $g(x) = \frac{1}{(x-2)^2}$

d $g(x) = \frac{(x^2 - 3x + 2)}{(1 - x)}$

e $h(x) = \begin{cases} x + 9, x < 2 \\ x^2 - 3, x \ge 2 \end{cases}$

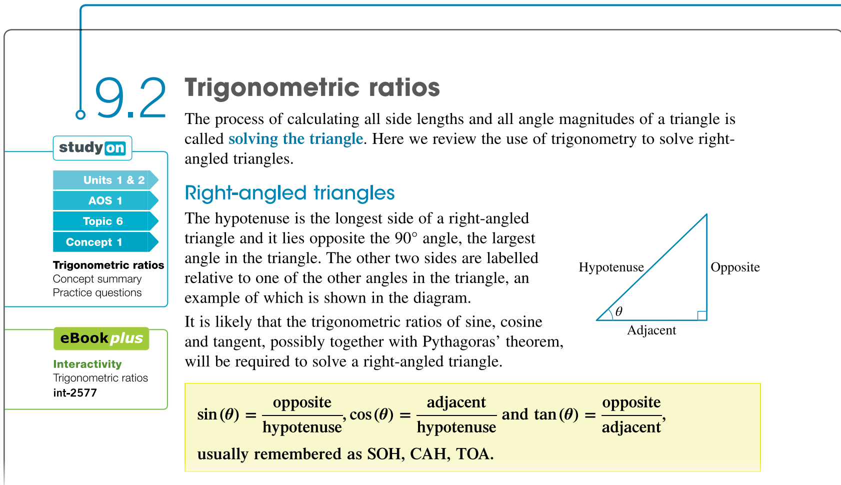
f $h(x) = \frac{(x-1)}{x}$

g $f(x) = \left(3 - \frac{1}{x^2}\right)$

h $f(x) = (x \sin(x))$

3 Compare the graphs for which the limit in question 1 exists to those for which the limit is undefined. Describe the difference with reference to the function value at the point(s) of interest.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.



### 9.2 Trigonometric ratios

studyon: Units 1 & 2 | AOS 1 | Topic 6 | Concept 1

**Trigonometric ratios**
Concept summary
Practice questions

eBookplus
**Interactivity**
Trigonometric ratios
**int-2577**

The process of calculating all side lengths and all angle magnitudes of a triangle is called **solving the triangle**. Here we review the use of trigonometry to solve right-angled triangles.

#### Right-angled triangles

The hypotenuse is the longest side of a right-angled triangle and it lies opposite the 90° angle, the largest angle in the triangle. The other two sides are labelled relative to one of the other angles in the triangle, an example of which is shown in the diagram.

Hypotenuse, Opposite, Adjacent, $\theta$

It is likely that the trigonometric ratios of sine, cosine and tangent, possibly together with Pythagoras' theorem, will be required to solve a right-angled triangle.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}, \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \text{ and } \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}},$$

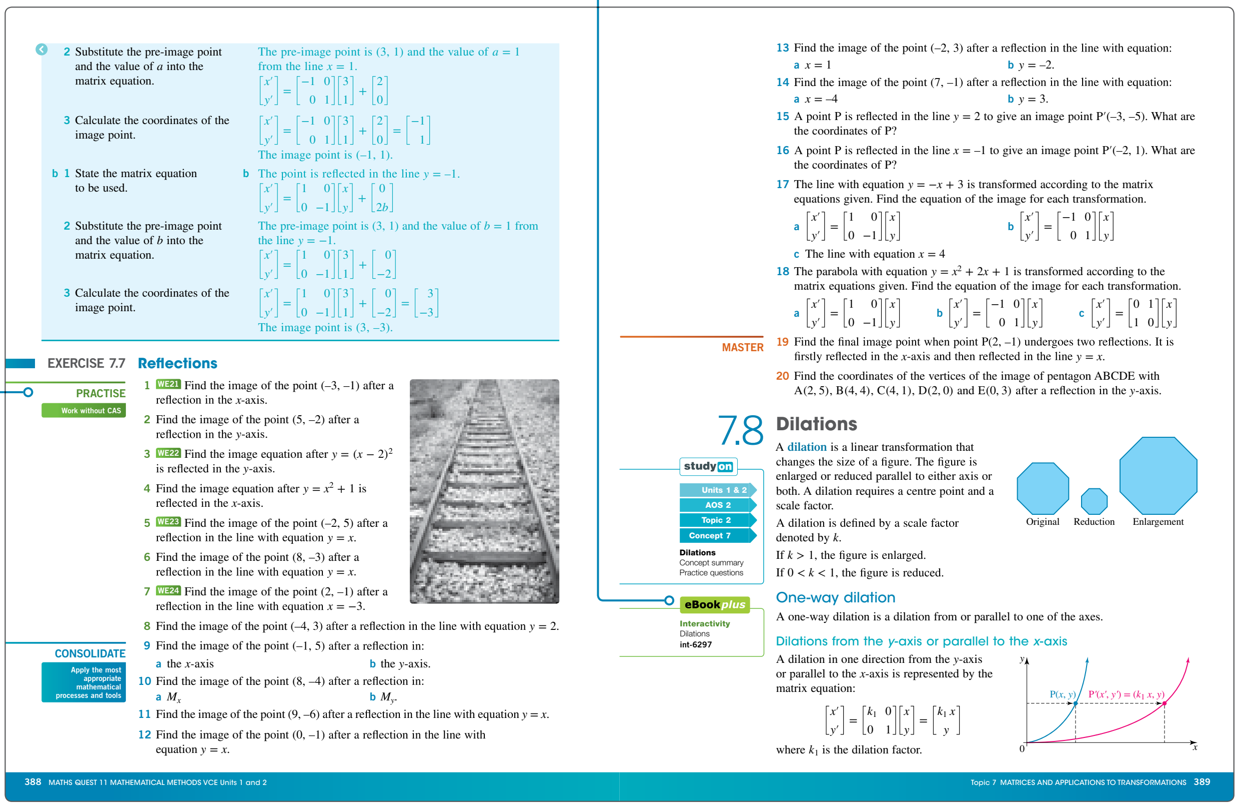
usually remembered as SOH, CAH, TOA.

### studyON links

Link to **studyON**, an interactive and highly visual study, revision and exam practice tool for instant feedback and on-demand progress reports.

## Interactivities

Many **NEW** interactivities in the resources tab of the eBookPLUS bring difficult concepts to life to engage and excite and to consolidate understanding.




## Graded questions

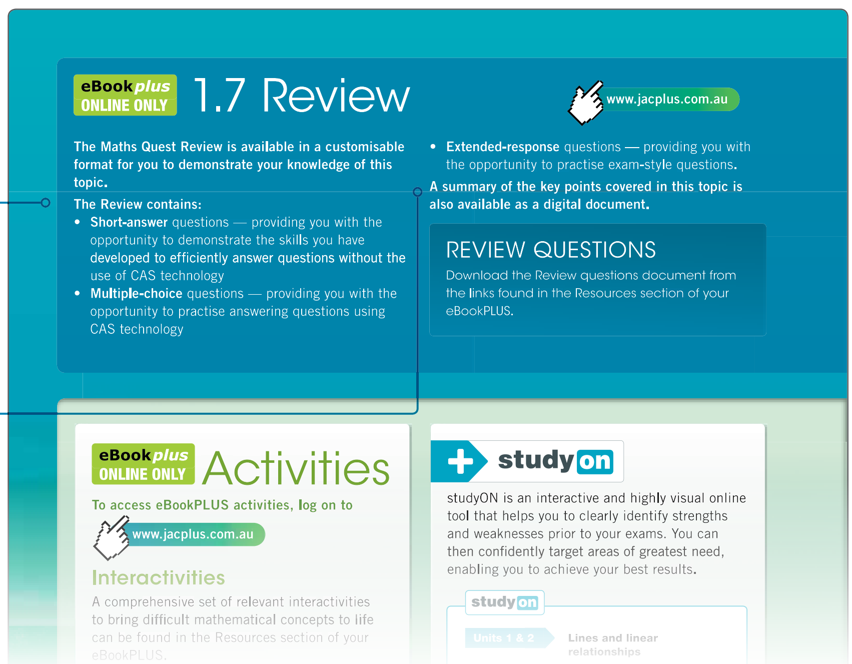
A wide variety of questions at **Practise**, **Consolidate** and **Master** levels allow students to build, apply and extend their knowledge independently and progressively.

## Review

Each topic concludes with a customisable **Review**, available in the resources tab of the **eBookPLUS**, giving students the opportunity to revise key concepts covered throughout the topic. A variety of typical question types is available including short-answer, multiple-choice and extended response.

## Summary

A comprehensive and fully customisable topic summary is available in the resources tab of the **eBookPLUS**, enabling students to add study notes and key information relevant to their personal study needs.



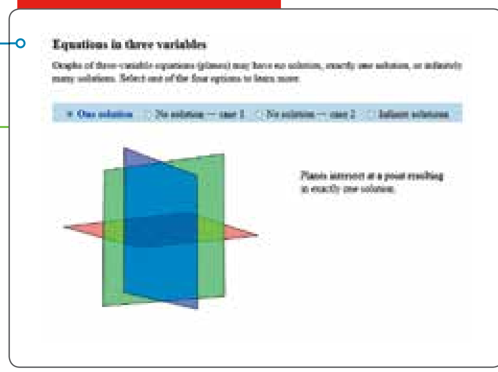
## eBook*plus*

The **eBookPLUS** is available for students and teachers and contains:

- the full text online in HTML format, including PDFs of all topics
- the *Manual for the TI-Nspire CAS calculator* for step-by-step instructions
- the *Manual for the Casio ClassPad II calculator* for step-by-step instructions
- **interactivities** to bring concepts to life
- topic reviews in a customisable format
- topic summaries in a customisable format
- links to **studyON**.

**Interactivity**

**Interactivities** bring concepts to life



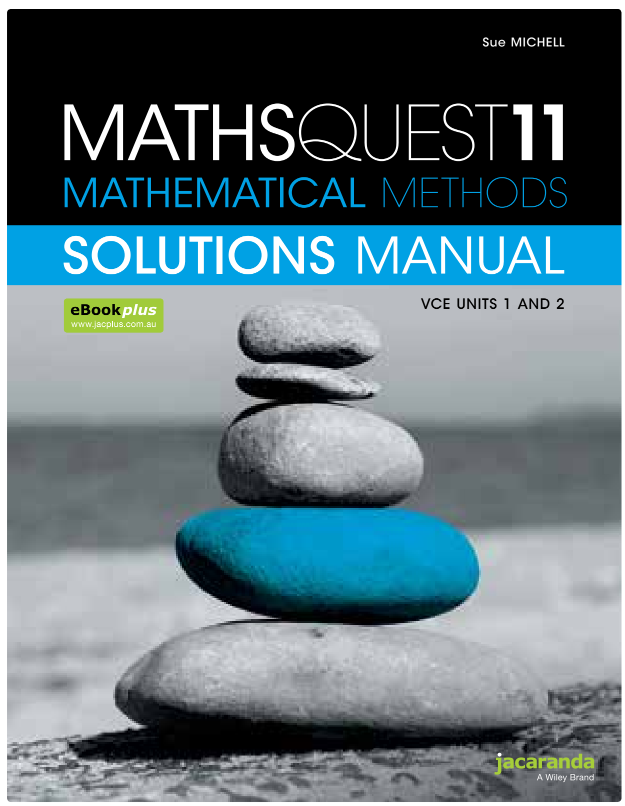

## eGuide*plus*

The **eGuidePLUS** is available for teachers and contains:

- the full **eBookPLUS**
- a Work Program to assist with planning and preparation
- School-assessed Coursework — Application task and Modelling and Problem-solving tasks, including fully worked solutions
- two tests per topic with fully worked solutions.

## *Maths Quest 11 Mathematical Methods Solutions Manual VCE Units 1 and 2*

Available to students and teachers to purchase separately, the Solutions Manual provides fully worked solutions to every question in the corresponding student text. The Solutions Manual is designed to encourage student independence and to model best practice. Teachers will benefit by saving preparation and correction time.



# About eBookPLUS and studyON

Access your online Jacaranda resources anywhere, anytime, from any device in three easy steps:

**STEP 1** Go to **www.jacplus.com.au** and create a user account.
**STEP 2** Enter your registration code.
**STEP 3** Instant access!

**eBookplus + studyon**

**eBookPLUS is an electronic version of the textbook, together with a targeted range of supporting multimedia resources.**

**eBookPLUS features:**

- **eBook** — the entire textbook in electronic format
- **Digital documents** designed for easy customisation and editing
- **Interactivities** to reinforce and enhance students' learning
- **eLessons** — engaging video clips and supporting material
- **Weblinks** to relevant support material on the internet

**eGuidePLUS features assessment and curriculum material to support teachers.**

**studyON is an interactive and highly visual online study, revision and exam practice tool designed to help students and teachers maximise exam results.**

**studyON features:**

- **Concept summary screens** provide concise explanations of key concepts, with relevant examples.
- **Access 1000+ past VCAA questions** or custom-authored practice questions at a concept, topic or entire course level, and receive immediate feedback.
- **Sit past VCAA exams** (Units 3 & 4) or **topic tests** (Units 1 & 2) in exam-like situations.
- **Video animations and interactivities** demonstrate concepts to provide a deep understanding (Units 3 & 4 only).
- **All results and performance in practice and sit questions** are tracked to a concept level to pinpoint strengths and weaknesses.

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- **Contact** John Wiley & Sons Australia, Ltd.
  **Email:** support@jacplus.com.au
  **Phone:** 1800 JAC PLUS (1800 522 7587)

**Minimum requirements**

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

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# 1

# Lines and linear relationships

# 1.1 Kick off with CAS

## Graphing straight lines with restrictions

1 Using CAS technology, graph the following straight lines.

a $y = 2x$

b $y = \frac{1}{2}x - 3$

c $y = -x$

d $y = -2x + 5$

2 The graphs of the linear relationships in question 1 stretch indefinitely in either direction. They are only limited by the boundaries of the screen. Using CAS technology, find the appropriate symbols and graph the following straight lines.

a $y = 2x | 0 \le x \le 4$

b $y = \frac{1}{2}x - 3 | -1 \le x \le 3$

c $y = -x | x \le 2$

d $y = -2x + 5 | x \ge 0$

3 Using CAS technology, determine the endpoints of the straight lines.

4 The image below is that of the famous Tower Bridge across the River Thames in London. Using the Cartesian plane and your knowledge of straight lines and parabolas, create a model of the image of the Tower Bridge with CAS technology.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 1.2 Linearly related variables, linear equations and inequations

studyon

Units 1 & 2

AOS 1

Topic 1

Concept 1

**Linearly related variables**
Concept summary
Practice questions

For linearly related variables $x$ and $y$, the rule connecting these variables is of the form $y = a + bx$. Plotting ordered pairs $(x, y)$ which satisfy this relationship will create a straight-line graph.

## Variables in direct proportion

eBook*plus*

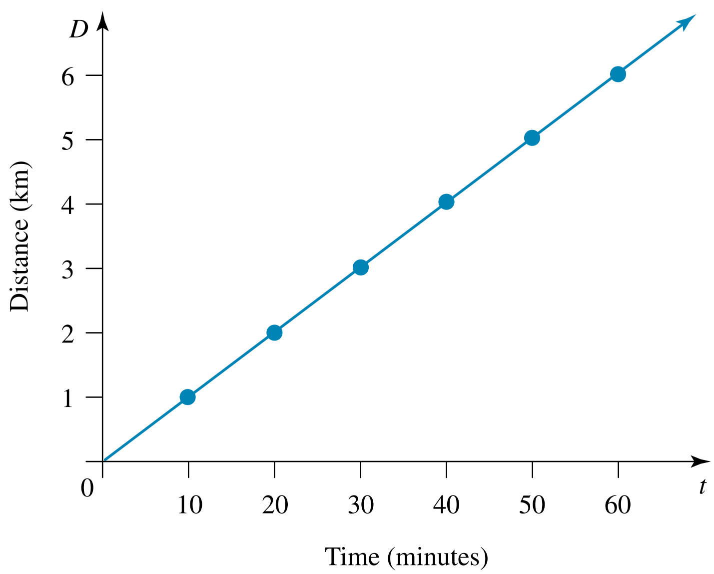
**Interactivity**
Proportionality scaling
**int-2548**

Two quantities which vary directly, or are in **direct proportion** or **direct variation**, have a linear relationship. For example, the distance travelled by a person jogging at a steady rate of 6 km/h (or 0.1 km/minute) would depend on the time spent jogging. The distance is in direct proportion to the time.

| Time (minutes) | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|
| Distance (kilometres) | 1 | 2 | 3 | 4 | 5 | 6 |

Doubling the time from 20 to 40 minutes causes the distance jogged to also double from 2 to 4 kilometres; trebling the time trebles the distance; halving the time halves the distance.

In this relationship, time is the **independent** or **explanatory variable** and distance the **dependent** or **response variable**. Plotting the points from the table on a distance versus time graph and joining these points gives a straight line through the origin. Here, time is the independent variable plotted along the horizontal axis, and distance is the dependent variable plotted along the vertical axis.




The linear relationship between these variables is described by the rule $D = 0.1t$, where $t$ is the time in minutes and $D$ the distance in kilometres.

**For any variables $x$ and $y$:**

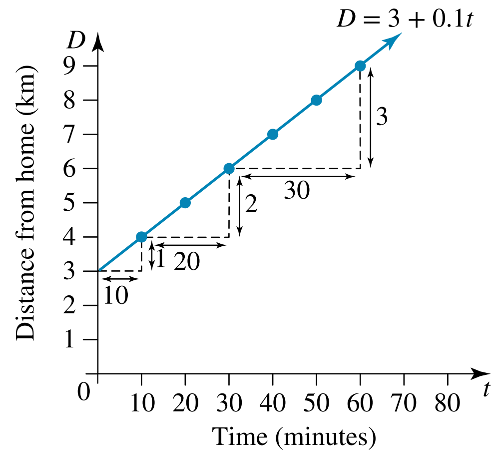
- **If $y$ is directly proportional to $x$, then $y = kx$, where $k$ is called the constant of proportionality.**
- **The graph of $y$ against $x$ is a straight line through the origin.**

### Other linearly related variables

eBook*plus*

**Interactivity**
Dynamic graph plotter
**int-2547**

Not all linear graphs pass through the origin, so not all linearly related variables are in direct proportion. However, intervals of change between the variables are in direct proportion.




If the jogger started from a point 3 km away from home, then the distance from home $D$ kilometres after jogging for $t$ minutes at the steady rate of 6 km/h (0.1 km/minute) would be given by $D = 3 + 0.1t$.

Although $D$ is not directly proportional to $t$, if $t$ changes by 10 minutes, then $D$ changes by 1 km; if $t$ changes by 20 minutes, $D$ changes by 2 km; if $t$ changes by 30 minutes, then $D$ changes by 3 km; and so the changes in $D$ are directly proportional to the changes in $t$. This occurs because the slope of the line is constant throughout.

The linear relationship $D = 3 + 0.1t$ can be described as the sum of two parts, one of which is a constant and the other of which is directly proportional to time. This is called a **part variation**.

**WORKED EXAMPLE 1**

**The circumference of a circle is directly proportional to its radius. If the circumference is $16\pi$ cm when the radius is 8 cm, calculate:**

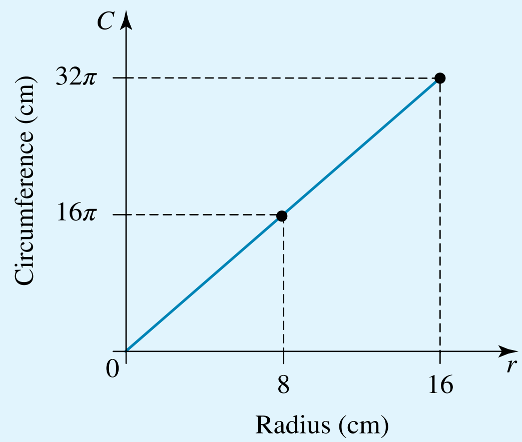
**a** **the constant of proportionality**

**b** **the radius of the circle when the circumference is doubled and sketch the graph of circumference versus radius.**

| THINK | WRITE |
|---|---|
| **a** 1 Define the two variables. | **a** $C$ = the length of the circumference in centimetres<br>$r$ = the length of the radius in centimetres |
| 2 Write the direct variation statement using $k$ for the proportionality constant. | $C = kr$ |
| 3 Substitute the given values and solve for $k$. | $C = 16\pi$ when $r = 8$<br>$16\pi = k(8)$<br>$k = \frac{16\pi}{8}$<br>$\therefore \quad k = 2\pi$ |
| **b** 1 State the rule connecting $C$ and $r$. | **b** $C = 2\pi r$ |
| 2 Substitute the new value for $C$ and solve for $r$. | Doubling the circumference $\Rightarrow C = 32\pi$<br>$32\pi = 2\pi r$<br>$\therefore r = 16$ |
| 3 Express the answer in context.<br>*Note:* Doubling the circumference has doubled the radius. | The radius is 16 cm. |

4 Sketch the graph.

$r$ is the independent variable and $C$ the dependent variable.




## Linear equations

Throughout this subject, the ability to solve equations and inequations of varying complexity will be required. Here we revise the underlying skills in solving linear equations, noting that a **linear equation** involves a variable which has an index or power of one. For example, $2x + 3 = 7$ is a linear equation in the variable $x$ whereas $2x^2 + 3 = 7$ is not (it is a quadratic equation); and $2x + 3$ alone is an algebraic expression.

**WORKED EXAMPLE 2**

Solve for $x$:

a $4(3 + 2x) = 22 + 3x$

b $\dfrac{2x + 3}{5} - \dfrac{1 - 4x}{7} = x.$

| THINK | WRITE |
|---|---|
| a Expand the brackets and collect all terms in $x$ together on one side in order to solve the equation. | a $4(3 + 2x) = 22 + 3x$<br>$12 + 8x = 22 + 3x$<br>$8x - 3x = 22 - 12$<br>$5x = 10$<br>$x = \frac{10}{5}$<br>$\therefore x = 2$ |
| b 1 Express fractions with a common denominator and simplify. | b $\frac{2x + 3}{5} - \frac{1 - 4x}{7} = x$<br>$\frac{7(2x + 3) - 5(1 - 4x)}{35} = x$<br>$\frac{14x + 21 - 5 + 20x}{35} = x$<br>$\frac{34x + 16}{35} = x$ |
| 2 Remove the fraction by multiplying both sides by the common denominator and solve for $x$. | $34x + 16 = 35x$<br>$16 = 35x - 34x$<br>$\therefore x = 16$ |

### Literal linear equations

**Literal equations** contain pronumerals rather than known numbers. The solution to a literal equation in $x$ usually expresses $x$ as a combination of these pronumerals rather than as a specific numerical value. Although the method of solution of linear literal equations is similar to those previously used, it is worth checking to see if answers may be simplified using algebraic skills such as factorisation.

**WORKED EXAMPLE 3** Solve for $x$: $\dfrac{x+a}{b} = \dfrac{b-x}{a}$.

| THINK | WRITE |
| --- | --- |
| 1 Each fraction can be placed on the common denominator and then each side multiplied by that term.<br>*Note:* Since there is only one fraction on each side of this equation, a quick way to do this is to 'cross-multiply'. | $\dfrac{x+a}{b} = \dfrac{b-x}{a}$<br>$a(x+a) = b(b-x)$<br>$ax + a^2 = b^2 - bx$ |
| 2 Collect all the terms in $x$ together and take out $x$ as the common factor. | $ax + bx = b^2 - a^2$<br>$x(a+b) = b^2 - a^2$ |
| 3 Divide by the coefficient of $x$ to obtain an expression for $x$. | $\therefore \quad x = \dfrac{b^2 - a^2}{a+b}$ |
| 4 Simplify the expression, if possible. | The numerator can be factorised as a difference of two squares.<br>$x = \dfrac{b^2 - a^2}{a+b}$<br>$= \dfrac{(b-a)(b+a)}{a+b}$ |
| 5 Cancel the common factor to give the solution in its simplest form. | $x = \dfrac{(b-a)\cancel{(b+a)}}{\cancel{a+b}}$<br>$\therefore x = b - a$ |

## Linear inequations

An **inequation** contains one of the order symbols:

$<$ (less than), $\leq$ (less than or equal to), $>$ (greater than), $\geq$ (greater than or equal to). The order symbols are referred to as inequality symbols.

Linear inequations are solved in a similar way to linear equations; however, care must be taken when multiplying or dividing by a negative number.

**Reverse the order symbol when multiplying or dividing by a negative number.**

To illustrate this, consider the inequality statement that $-6 < 15$. If this inequality is divided by $-3$ then the statement must become $2 > -5$, so the order symbol has been reversed.

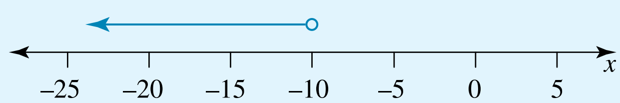
The solutions to linear inequations are sets of values satisfying an inequality, unlike linear equations where the solutions are unique.

When illustrating inequalities on a number line:

- an open circle, ○, is used when the endpoint is not included ($<$ or $>$)
- a closed circle, ●, is used when the endpoint is included ($\leq$ or $\geq$)

**WORKED EXAMPLE 4**

**Calculate the values for $x$ for which $5 - \frac{4x}{5} > 13$ and show this set of values on a number line.**

| THINK | WRITE |
|---|---|
| 1 Subtract 5 from both sides of the inequation. *Note:* Subtracting a number does not affect the inequality symbol. | $5 - \frac{4x}{5} > 13$<br>$-\frac{4x}{5} > 13 - 5$<br>$-\frac{4x}{5} > 8$ |
| 2 Multiply both sides by 5. *Note:* Multiplying by a positive number does not affect the inequality symbol. | $-4x > 8 \times 5$<br>$-4x > 40$ |
| 3 Divide both sides by $-4$. *Note:* Dividing by a negative number does require the symbol to be reversed. | $x < \frac{40}{-4}$<br>$\therefore \quad x < -10$ |
| 4 Illustrate this set of values on a number line. | <br>The number line has an open end at $x = -10$ since this value is not included in the set of solutions, $x < -10$. |



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**Interactivity**
Solving systems of equations
**int-2549**

## Systems of 2 × 2 simultaneous linear equations

To find the value of one variable, one equation is needed. However, to find the values of two variables, two equations are required. These two equations form a 2 × 2 **system of equations**.

The usual algebraic approach for finding the values of two variables, $x$ and $y$, which satisfy two linear equations **simultaneously** is by using either a ***substitution*** method or an ***elimination*** method.

### Substitution method

In the substitution method, one equation is used to express one variable in terms of the other and this expression is substituted in place of that variable in the second equation. This creates an equation with just one variable which can then be solved.

### Elimination method

In the elimination method the two equations are combined in such a way as to eliminate one variable, leaving an equation with just one variable which can then be solved.

**WORKED EXAMPLE 5**

**a** Use the substitution method to solve the following system of simultaneous equations for $x$ and $y$.

$$y = 3x - 1$$
$$x - 3y = 11$$

**b** Use the elimination method to solve the following system of simultaneous equations for $x$ and $y$.

$$8x + 3y = -13$$
$$5x + 4y = -6$$

| THINK | WRITE |
|---|---|
| **a 1** Write the two equations and number them for ease of reference. | **a** $y = 3x - 1$........(1)<br>$x - 3y = 11$......(2) |
| **2** Since the first equation already has $y$ in terms of $x$, it can be substituted for $y$ in the second equation. | Substitute equation (1) into equation (2).<br>$x - 3(3x - 1) = 11$ |
| **3** Solve for $x$. | $x - 9x + 3 = 11$<br>$-8x = 11 - 3$<br>$-8x = 8$<br>$\therefore x = -1$ |
| **4** Substitute the value of $x$ into one of the original equations and calculate the $y$-value. | Substitute $x = -1$ into equation (1).<br>$y = 3(-1) - 1$<br>$\therefore y = -4$ |
| **5** State the answer. | The solution is $x = -1$ and $y = -4$. |
| **6** It is a good idea to check that these values satisfy both equations by substituting the values for $x$ and $y$ into the other equation. | Check: In equation (2), if $x = -1$ and $y = -4$, then:<br>$x - 3y = (-1) - 3 \times (-4)$<br>$= -1 + 12$<br>$= 11$<br>True |
| **b 1** Write the two equations and number them for ease of reference. | **b** $8x + 3y = -13$.....(1)<br>$5x + 4y = -6$.......(2) |
| **2** Choose whether to eliminate $x$ or $y$ and adjust the coefficient of the chosen variable in each equation.<br>*Note:* Neither equation readily enables one variable to be expressed simply in terms of the other, so the elimination method is the more appropriate method here. | Eliminate $y$.<br>Multiply equation (1) by 4:<br>$32x + 12y = -52$.....(3)<br>Multiply equation (2) by 3:<br>$15x + 12y = -18$.....(4) |
| **3** Eliminate $y$ and solve for $x$.<br>*Note:* 'Same Signs Subtract' can be remembered as the SSS rule. | Equation (3) − equation (4)<br>$32x - 15x = -52 - (-18)$<br>$17x = -34$<br>$\therefore x = -2$ |

4 Substitute the value of $x$ into one of the original equations and calculate the $y$-value.

*Note:* Alternatively, start again but this time eliminate $x$.

Substitute $x = -2$ into equation (2)

$$5(-2) + 4y = -6$$
$$-10 + 4y = -6$$
$$4y = -6 + 10$$
$$4y = 4$$
$$\therefore \quad y = 1$$

5 State the answer.

The solution is $x = -2$, $y = 1$.

## Solving problems using equations

The symbolism of algebra enables statements of problems to be expressed very concisely and succinctly. This is part of its power and beauty. Mathematicians may construct sets of equations to model relationships between various variables and, by solving these, contribute to our collective understanding of real-world phenomena.

**In setting up equations:**

- **define the symbols used for the variables, specifying units where appropriate**
- **ensure any units used are consistent**
- **express answers in the context of the problem.**

WORKED EXAMPLE 6

**The organisers of an annual student fundraising event for charity know there will be fixed costs of \$120 plus an estimated cost of 60 cents per student for incidental costs on the day of the fundraiser. The entry fee to the fundraising event is set at \$5.**

**a** **Form an algebraic model for the profit the event can expect to make.**

**b** **What is the least number of students who must attend the event to avoid the organisers making a loss?**

**A proposal is made to lower the student entry fee by allowing staff to attend the event.**

**c** **If 30 students and 5 staff attend, the revenue gained on entry fees is \$145 whereas if 55 students and 15 staff attend the revenue is \$312.50. How much is the proposed entry fee for a student and how much is it for a member of staff?**

THINK

a 1 Define the variables.

WRITE

a Let $n$ = the number of students attending the event
$P$ = the profit made in dollars

2 Form expressions for the cost and the revenue, ensuring units are consistent.

Profit depends on costs and revenues.
Revenue (\$) = $5n$
Costs (\$) = $120 + 0.60n$

3 Form the expression for the profit to define the algebraic model.

Profit = revenue − costs
Hence,
$P = 5n - (120 + 0.60n)$
$\therefore P = 4.4n - 120$ gives the linear model for the profit.

**b 1** Impose the condition required to avoid a loss and calculate the consequent restriction on $n$.

**b** A loss is made if $P < 0$. To avoid making a loss, the organisers require $P \geq 0$.

$$\begin{aligned} 4.4n - 120 &\geq 0 \\ 4.4n &\geq 120 \\ n &\geq \frac{120}{4.4} \\ \therefore \quad n &\geq 27.2727... \end{aligned}$$

**2** Express the answer in the context of the question.

To avoid making a loss, at least 28 students need to attend the event.

**c 1** The problem is now posed in terms of other variables. Define these and express the given information using a pair of simultaneous equations.

**c** Let the entry fees be:
$s$ dollars for students and
$t$ dollars for staff.

$$30s + 5t = 145............(1)$$

$$55s + 15t = 312.5..........(2)$$

**2** Solve this system of simultaneous equations by elimination.

Eliminate $t$.
Multiply equation (1) by 3.

$$90s + 15t = 435.......(3)$$

$$55s + 15t = 312.5....(2)$$

Subtract equation (2) from equation (3).

$$\begin{aligned} 90s - 55s &= 435 - 312.5 \\ 35s &= 122.5 \\ \therefore s &= \frac{122.5}{35} \\ \therefore s &= 3.5 \end{aligned}$$

Substitute $s = 3.5$ into equation (1).

$$\begin{aligned} 30(3.5) + 5t &= 145 \\ 105 + 5t &= 145 \\ 5t &= 40 \\ \therefore \quad t &= 8 \end{aligned}$$

**3** Write the answer in context.

The student entry fee is \$3.50 and the staff entry fee is \$8.

## EXERCISE 1.2 Linearly related variables, linear equations and inequations

**PRACTISE**

Work without CAS
Questions 1–6

1 WE1 The volume of a cone of fixed base radius is directly proportional to its height. If the volume is $96\pi$ cm$^3$ when the height is 6 cm, calculate:

**a** the constant of proportionality

**b** the height of this cone when its volume is halved and sketch the graph of volume versus height.

2 The cost per day of running a children's crèche is the sum of two quantities, one of which is the constant fixed cost of \$210 and the other of which is directly proportional to the number of children attending the crèche. If the cost is \$330 on a day when 20 children attend:

a form the linear relationship between the cost per day, $C$ dollars, and $n$, the number of children attending the crèche that day.

b What will the cost be on a day when 25 children attend?

3 WE2 Solve for $x$:

a $3(5x - 1) = 4x - 14$

b $\dfrac{4 - x}{3} + \dfrac{3x - 2}{4} = 5$

4 Solve for $x$: $\dfrac{7(x - 3)}{8} + \dfrac{3(2x + 5)}{4} = \dfrac{3x}{2} + 1$

5 WE3 Solve for $x$: $\dfrac{d - x}{a} = \dfrac{a - x}{d}$

6 Solve for $x$: $b(x + c) = a(x - c) + 2bc$

7 WE4 Calculate the values for $x$ for which $7 - \dfrac{3x}{8} \leq -2$ and show this set of values on a number line.

8 Solve for $x$: $4(2 + 3x) > 8 - 3(2x + 1)$

9 a WE5 Use the substitution method to solve the following system of simultaneous equations for $x$ and $y$.

$$x = 2y + 5$$
$$4x - 3y = 25$$

b Use the elimination method to solve the following system of simultaneous equations for $x$ and $y$.

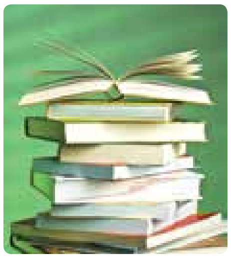
$$5x + 9y = -38$$
$$-3x + 2y = 8$$

10 Solve for $x$ and $y$:

a $2x - y = 7$
$7x - 5y = 42$

b $ax - by = a$
$bx + ay = b$

11 WE6 Although the organisers of a secondhand book sale are allowed free use of the local Scouts Hall for their fete, they must contribute \$100 towards heating and lighting costs and in addition donate 20$c$ from the sale of each book to the Scouts Association. The books are intended to be sold for \$2.50 each.



a Form an algebraic model for the profit the book sale can expect to make.

b What is the least number of books that must be sold to ensure the organisers make a profit?

A proposal is made to sell the hardcover books at a different price to the paperback books.

**c** If 40 hardback books and 65 paperbacks are sold, the revenue gained from the sales would be \$330 whereas the sale of 30 hardback books and 110 paperbacks would bring revenue of \$370. How much are the sale prices of hardback books and of paperback books under this proposal?

**12** A cyclist travels at an average speed of 16 km/h along a road. Fifteen minutes after the cyclist sets out, a motorcyclist starting from the same place travels along the same road at an average speed of 48 km/h. How many minutes will the cyclist have travelled before being overtaken by the motorcyclist?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**13** Form the linear equation connecting the two variables and then use the equation to answer the question in each of the following:

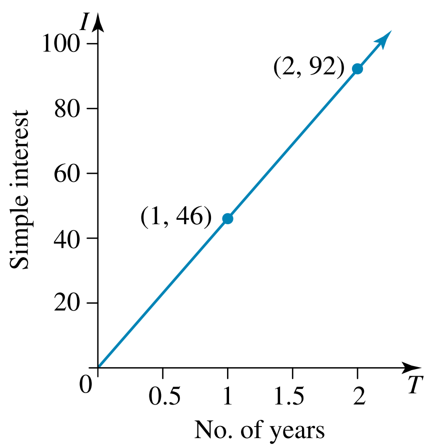
**a** When an elastic string is extended $x$ units beyond its natural length, the tension $T$ in the string is directly proportional to the extension. When the string is extended by 0.2 metres the tension is 0.7 newtons. What is the tension in the string after it has been extended a further 0.1 metre?

**b** The cost $C$ of purchasing petrol from a petrol station is directly proportional to the number of litres $l$ poured into the car's tank. If it costs \$52.49 to fill a tank with 36.2 litres, how many litres of petrol can be purchased for \$43.50?

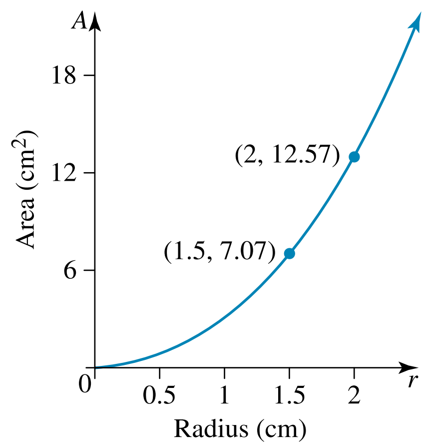
**c** A ball is thrown vertically upwards with a velocity $v$ of 12 m/s, which then decreases by an amount proportional to $t$, the number of seconds the ball moves upwards. After 0.5 seconds the ball has slowed to a velocity of 7.1 m/s. How many seconds after it is thrown upwards will the ball start to fall back to the ground?

**14** From the following diagrams **a** to **c**, select those graphs which show two linearly related variables. For these, describe the relationship and determine the linear rule between the variables.

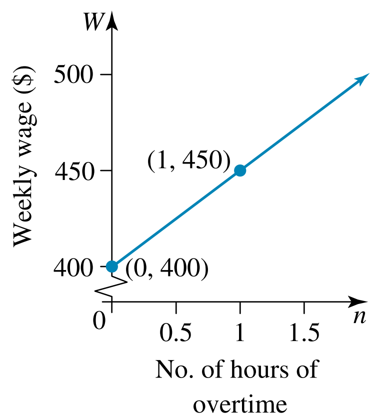
**a** The simple interest ($I$) earned on an investment of \$100 versus the number of years ($T$) of investment



**b** The area ($A$) of a circle, in cm$^2$, versus the length of the radius ($r$), in cm




**c** The weekly wage earned ($W$) versus the number of hours ($n$) of overtime work




**15** Solve the following linear equations for $x$.

**a** $7(2x - 3) = 5(3 + 2x)$

**b** $\frac{4x}{5} - 9 = 7$

**c** $4 - 2(x - 6) = \frac{2x}{3}$

**d** $\frac{3x + 5}{9} = \frac{4 - 2x}{5}$

**e** $\frac{x + 2}{3} + \frac{x}{2} - \frac{x + 1}{4} = 1$

**f** $\frac{7x}{5} - \frac{3x}{10} = 2\left(x + \frac{9}{2}\right)$

**16** Solve the following literal equations for $x$, expressing solutions in simplest form.

**a** $ax + b = c$

**b** $a(x - b) = bx$

**c** $a^2x + a^2 = ab + abx$

**d** $\frac{x}{a} + \frac{x}{b} = a + b$

**e** $\frac{bx - a}{c} = \frac{cx + a}{b}$

**f** $\frac{x + a}{b} - 2 = \frac{x - b}{a}$

**17** Solve the following inequations for $x$.

**a** $3x - 5 \le -8$

**b** $4(x - 6) + 3(2 - 2x) < 0$

**c** $1 - \frac{2x}{3} \ge -11$

**d** $\frac{5x}{6} - \frac{4 - x}{2} > 2$

**e** $8x + 7(1 - 4x) \le 7x - 3(x + 3)$

**f** $\frac{2}{3}(x - 6) - \frac{3}{2}(x + 4) > 1 + x$

**18** Solve the following simultaneous equations for $x$ and $y$.

**a** $y = 5x - 1$
$x + 2y = 9$

**b** $3x + 5y = 4$
$8x + 2y = -12$

**c** $x = 5 + \frac{y}{2}$
$-4x - 3y = 35$

**d** $8x + 3y = 8$
$-2x + 11y = \frac{35}{6}$

**e** $\frac{x}{2} + \frac{y}{3} = 8$
$\frac{x}{3} + \frac{y}{2} = 7$

**f** $ax + by = 4ab$
$ax - by = 2ab$

**19** Solve the following problems by first forming linear equations.

**a** The sum of two consecutive even numbers is nine times their difference. What are these two numbers?

**b** Three is subtracted from a certain number and the result is then multiplied by 4 to give 72. What is the number?

**c** The sum of three consecutive numbers is the same as the sum of 36 and one quarter of the smallest number. What are the three numbers?

**d** The length of a rectangle is 12 cm greater than twice its width. If the perimeter of the rectangle is 48 cm, calculate its length and width.

**e** The ratio of the length to the width to the height of a rectangular prism is 2:1:3 and the sum of the lengths of all its edges is 360 cm. Calculate the height of this rectangular prism.

**20 a** A car travelling at 60 km/h takes $t$ hours to go from A to B. If the speed of the car is reduced by 10 km/h, the time to go from A to B is increased by half an hour. Calculate the value of $t$ and find the distance between A and B.

**b** A cyclist travelling at an average speed of $u$ km/h takes $t$ hours to travel from A to B. On reaching B, the cyclist rests for $0.1t$ hours before returning from B to A along the same route. If the cyclist reaches A $2t$ hours after first setting out, find an expression for the average speed of the cyclist on the return journey.

**21 a** When 4 adults and 5 children attend a pantomime, the total cost of the tickets is \$160 whereas the cost of the tickets for 3 adults and 7 children is \$159. How much is an adult ticket and how much does a child's ticket cost?

**b** A householder's electricity bill consists of a fixed payment together with an amount proportional to the number of units used. When the number of units used was 1428 the total bill was \$235.90 and when the number of units was 2240, the bill was \$353.64. How much will the householder's bill be if 3050 units are used?

**c** The temperature measured in degrees Celsius, C, is linearly related to the temperature measured in degrees Fahrenheit (F). Water boils at 100 °C or 212 °F and freezes at 0 °C or 32 °F. Find the linear equation that enables conversion of degrees Fahrenheit to degrees Celsius.

**22** A cricket social club committee booked four tables at a local restaurant for a casual lunch for their members. The organisers were unsure in advance how many people were attending but on the day three or four people sat at each table.

**a** Construct an inequality to describe the number of people from the cricket club who actually attended the lunch.

**b** The lunch consisted of two courses for a fixed amount per person. If the total bill the cricket social club had to pay was exactly $418.50, how many people from the cricket club attended the lunch and what was the fixed charge per person?

**c** Included in the lunch cost was complimentary tea or coffee. If the number of tea drinkers was half the number of coffee drinkers and each person had either a coffee or a tea, how many of the cricket club members drank coffee and how many drank tea?

MASTER

**23** Use CAS technology to solve:

**a** $3(5x - 2) + 5(3x - 2) = 8(x - 2)$

**b** $\frac{2x - 1}{5} + \frac{3 - 2x}{4} < \frac{3}{20}$

**c** $4x - 3y = 23$

$7x + 4y = 31$

**d** $3(x + 2) = 2y$

$7x - 6y = 146$

**24 a** It is thought that the chirping rate of cicadas is linearly related to the temperature. Use the given data to construct a linear model of the chirping rate $C$ and temperature $T$.

| Temperature (°C) | Chirping rate (chirps/minute) |
|---|---|
| 21° | 113 |
| 27° | 173 |

**b** Solve for $x$ and $y$:

$ax + bcy = c$

$bx + acy = c.$

# 1.3 Systems of 3 × 3 simultaneous linear equations

studyon

Units 1 & 2
AOS 1
Topic 1
Concept 2

**Systems of simultaneous linear equations**
Concept summary
Practice questions

The yield from a crop of zucchini plants appears to be linearly related to the amount of sunshine and the amount of rainfall the crop receives. This could be expressed as a linear equation $z = a + bx + cy$ using the variables $z$ for the number of zucchini harvested, $x$ for the amount of sunshine and $y$ for the amount of rain. Such linear relationships have three variables, $x$, $y$ and $z$, so their graph would require a sketch in 3-dimensional space (something for future mathematical studies).

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**Interactivity**
Solving equations in 3 variables
**int-2550**

However, situations may arise in our course in which we need to solve a system of three simultaneous linear equations in three unknowns. It will be sufficient to restrict our attention to solving systems that have unique solutions (although this is not always the case for such systems).

## The manual method for solving 3 equations in 3 variables

- **Reduce the 3 × 3 system to a 2 × 2 system by eliminating the same variable from two different pairs of equations.**
- **Solve the 2 × 2 system of simultaneous equations using an elimination or substitution method to obtain the values of these two variables.**
- **Substitute these values for two of the variables into an original equation to obtain the value of the third variable.**

Where one equation allows a variable to easily be expressed in terms of the other two variables, substituting this expression into both of the other equations would create the 2 × 2 system of simultaneous equations.

*Note:* A CAS calculator or other technology may be used to solve 3 × 3 systems of equations.

WORKED EXAMPLE 7

**Solve for $x$, $y$ and $z$ in the following 3 × 3 system of simultaneous equations:**

$$3x - 4y + 5z = 10$$
$$2x + y - 3z = -7$$
$$5x + y - 2z = -9$$

| THINK | WRITE |
|---|---|
| 1 Number the equations for ease of reference. | $3x - 4y + 5z = 10$ ....(1)<br>$2x + y - 3z = -7$ ....(2)<br>$5x + y - 2z = -9$ ....(3) |
| 2 Decide which variable to eliminate and choose a pair of equations to carry out this elimination. Number the resultant equation. | Eliminate $y$ from equations (1) and (2).<br>Multiply equation (2) by 4.<br>$8x + 4y - 12z = -28$ ....(4)<br>$3x - 4y + 5z = 10$...... (1)<br>Add equations (4) and (1).<br>$\therefore 11x - 7z = -18$ ....(5) |
| 3 Choose a second pair of equations to eliminate the same variable. Number the resultant equation. | Eliminate $y$ from equations (3) and (2).<br>$5x + y - 2z = -9$ ....(3)<br>$2x + y - 3z = -7$ ....(2)<br>Subtract equation (2) from equation (3).<br>$\therefore 3x + z = -2$....(6) |
| 4 The original 3 × 3 system has been reduced to a 2 × 2 system of simultaneous equations in $x$ and $z$.<br>Write these equations. | $11x - 7z = -18$ ....(5)<br>$3x + z = -2$......(6) |

**5** Solve the equations to find $x$ and $z$.

*Note:* Either of the elimination or substitution methods could be used.

Looking at the coefficients of $z$ in each of the two equations, we shall choose to eliminate $z$.

Eliminate $z$.

Add equation (5) and 7 times equation (6).

$$11x + 7 \times 3x = -18 + 7 \times (-2)$$
$$32x = -32$$
$$x = -\frac{32}{32}$$
$$\therefore \quad x = -1$$

Substitute $x = -1$ into equation (6).

$$-3 + z = -2$$
$$\therefore \quad z = 1$$

**6** Finally, to obtain $y$, substitute the values of $x$ and $z$ into one of the original equations.

*Note:* For ease of calculation, it is usual to select the equation which looks the simplest.

Substitute $x = -1$, $z = 1$ into equation (2).

$$2(-1) + y - 3(1) = -7$$
$$-2 + y - 3 = -7$$
$$y - 5 = -7$$
$$\therefore \quad y = -2$$

**7** State the answer.

The solution is $x = -1$, $y = -2$ and $z = 1$.

**8** It is a good idea to check that the solution satisfies the other two equations by substituting the values and showing that LHS (left-hand side) equals RHS (right-hand side).

Check: In equation (1) substitute $x = -1$, $y = -2$ and $z = 1$.

$$\begin{aligned}\text{LHS} &= 3 \times (-1) - 4 \times (-2) + 5 \times (1) \\ &= -3 + 8 + 5 \\ &= 10 \\ &= \text{RHS}\end{aligned}$$

In equation (3) substitute $x = -1$, $y = -2$ and $z = 1$.

$$\begin{aligned}\text{LHS} &= 5 \times (-1) + (-2) - 2 \times (1) \\ &= -5 - 2 - 2 \\ &= -9 \\ &= \text{RHS}\end{aligned}$$

## EXERCISE 1.3 Systems of 3 × 3 simultaneous linear equations

**PRACTISE**

Work without CAS

**1** WE7 Solve for $x$, $y$ and $z$ in the following 3 × 3 system of simultaneous equations:

$$5x - 2y + z = 3$$
$$3x + y + 3z = 5$$
$$6x + y - 4z = 62.$$

**2** Solve for $x$, $y$ and $z$ in the following 3 × 3 system of simultaneous equations:

$$z = 12 - x + 4y$$
$$z = 4 + 5x + 3y$$
$$z = 5 - 12x - 5y.$$

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

In questions **3**–**8**, solve the $3 \times 3$ systems of simultaneous equations for $x$, $y$ and $z$.

**3** $2x + 3y - z = 3$

$5x + y + z = 15$

$4x - 6y + z = 6$

**4** $x - 2y + z = -1$

$x + 4y + 3z = 9$

$x - 7y - z = -9$

**5** $2x - y + z = -19$

$3x + y + 9z = -1$

$4x + 3y - 5z = -5$

**6** $2x + 3y - 4z = -29$

$-5x - 2y + 4z = 40$

$7x + 5y + z = 21$

**7** $3x - 2y + z = 8$

$3x + 6y + z = 32$

$3x + 4y - 5z = 14$

**8** $y = 3x - 5$

$\frac{x - z}{2} - y + 10 = 0$

$9x + 2y + z = 0$

In questions **9**–**12**, set up a system of simultaneous equations that could be used to solve the problems.

**9** At a recent art exhibition the total entry cost for a group of 3 adults, 2 concession holders and 3 children came to \$96; 2 adult, 1 concession and 6 child tickets cost \$100; and 1 adult, 4 concession and 1 child ticket cost \$72. Calculate the cost of an adult ticket, a concession ticket and a child ticket.

**10** Agnes, Bjork and Chi are part-time outsource workers for the manufacturing industry. When Agnes works 2 hours, Bjork 3 hours and Chi 4 hours, their combined earnings total \$194. If Agnes works 4 hours, Bjork 2 hours and Chi 3 hours, their total earnings are \$191; and if Agnes works 2 hours, Bjork 5 hours and Chi 2 hours their combined earnings total \$180. Calculate the hourly rate of pay for each person.

**11** A nutritionist at a zoo needs to produce a food compound in which the concentration of fats is 6.8 kg unsaturated fats, 3.1 kg saturated fats and 1.4 kg trans fats. The food compound is formed from three supplements whose concentration of fats per kg is shown in the following table:

| | Unsaturated fat | Saturated fat | Trans fat |
|---|---|---|---|
| Supplement X | 6% | 3% | 1% |
| Supplement Y | 10% | 4% | 2% |
| Supplement Z | 8% | 4% | 3% |

How many kilograms of each supplement must be used in order to create the food compound?

12 A student buys a sandwich at lunchtime from the school canteen for \$4.20 and pays the exact amount using 50 cent coins, 20 cent coins and 10 cent coins. If the number of 20 cent coins is the same as half the 10 cent coins plus four times the number of 50 cent coins and the student pays the cashier with 22 coins in total, how many coins of each type did the student use?

MASTER

13 a Use CAS technology to obtain the values of $x$, $y$ and $z$ for the system of equations:

$$2x + 6y + 5z = 2$$
$$5x - 10y - 8z = 20.8$$
$$7x + 4y + 10z = 1$$

b Use CAS technology to obtain the values of $x$, $y$, $z$ and $w$ for the $4 \times 4$ system of equations:

$$x - y + 4z - 2w = 8$$
$$3x + 2y - 2z + 10w = 67$$
$$2x + 8y + 18z + w = -14$$
$$8x - 7y - 80z + 7w = 117$$

c Use CAS technology to obtain the values of $x_1$, $x_2$, $x_3$ and $x_4$ for the $4 \times 4$ system of equations:

$$4x_1 + 2x_2 + 3x_3 + 6x_4 = -13$$
$$12x_1 - 11x_2 - 7.5x_3 + 9x_4 = 16.5$$
$$x_1 + 18x_3 - 12x_4 = 8$$
$$-3x_1 + 12x_2 - x_3 + 10x_4 = -41$$

14 The yield of zucchini, $z$ kg/hectare, over a period of $x$ hours of sunshine and $y$ mm of rainfall is shown in the following table.

| $z$ | $x$ | $y$ |
|---|---|---|
| 23 | 30 | 320 |
| 28 | 50 | 360 |
| 30 | 40 | 400 |

a Form a linear model in the form $z = a + bx + cy$ for this data.

b How many kilograms per hectare of zucchini would the model predict for 40 hours of sunshine and only 200 mm of rain?

**Earlier than the Europeans, possibly around 250 BC, the work of Chinese mathematicians in *Chui-chang suan-shu* (*Nine chapters on the Mathematical Art*), gave the solution to simultaneous linear equations in 3 unknowns.**

# 1.4 Linear graphs and their equations

Intervals of change between linearly related variables are in direct proportion. This means there is a steady, or constant, rate of change of the dependent variable with respect to the independent variable.

**study on**

Units 1 & 2
AOS 1
Topic 2
Concept 3

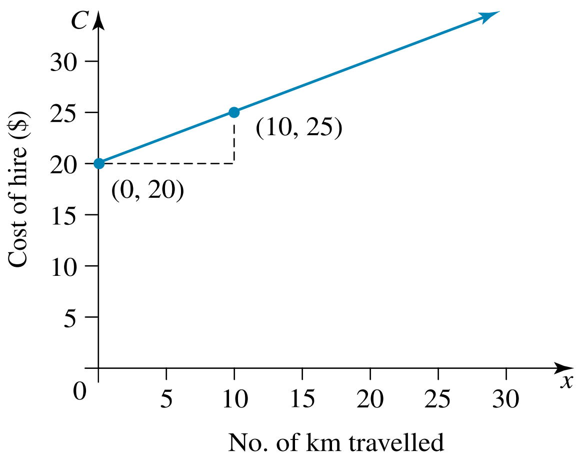
**Linear graphs and their equations**
Concept summary
Practice questions

## Steady rates of change and linear rules

If the cost of hiring a bicycle is \$20 plus an additional \$0.50 per kilometre of travel then the cost increases steadily from \$20 by 50 cents for every kilometre travelled. The rate of change of the cost is \$0.50 per kilometre. Thus, if the bicycle is hired for a journey of 10 km, then the cost would be $\$(20 + 10 \times 0.50)$ which equals \$25.

The relationship between the cost and the distance variables could be illustrated by plotting the ordered pairs (0, 20) and (10, 25) and joining these points with a straight line which extends beyond the point (10, 25).

| Number of km travelled | 0 | 10 |
|---|---|---|
| Cost in dollars | 20 | 25 |




The rate of increase of the cost remains constant between any two points on the line.

$$\text{For this graph, the rate of change of the cost} = \frac{\text{change in cost}}{\text{change in km travelled}}$$

$$= \frac{25 - 20}{10 - 0}$$

$$= 0.50$$

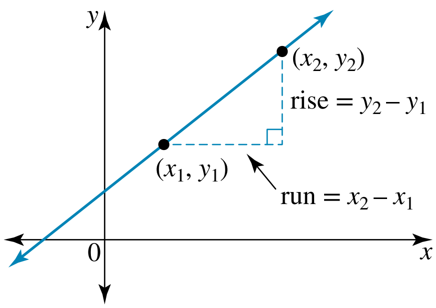
The linear rule for the hire cost, $C$ dollars, is $C = 20 + 0.50x$, where $x$ is the number of kilometres travelled. The rate of change of the cost appears in this rule as the coefficient of the independent variable $x$.

## Gradient of a line

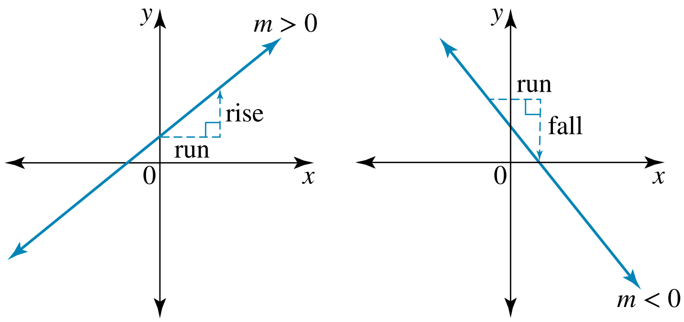
The **gradient**, or **slope** measures the rate of change of the dependent variable with respect to the independent variable. Along a line, there is a steady rate of change so the gradient is constant. It measures the steepness of the line as the ratio $\frac{\text{rise}}{\text{run}}$.

The gradient of a line, usually denoted by the pronumeral $m$, can be calculated by choosing any two points that are known to lie on the line. If a line contains the two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ then the gradient of the line is

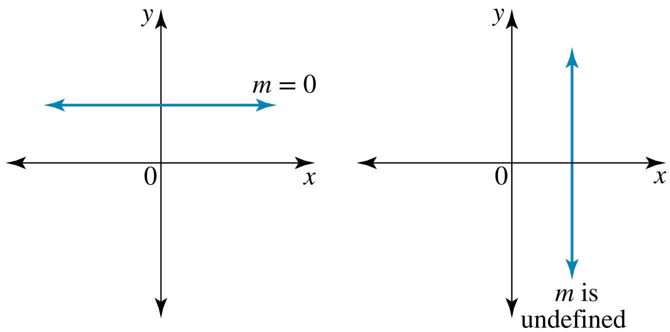
$$m = \frac{\textbf{rise}}{\textbf{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$




An oblique line has either a positive or a negative gradient, depending on whether the relationship between $x$ and $y$ is **increasing** or **decreasing**. If it is increasing, as the run increases, the rise increases so $y_2 - y_1 > 0$ and $m > 0$; if the relationship is decreasing, as the run increases, the rise decreases or falls so $y_2 - y_1 < 0$ and $m < 0$.




Horizontal lines have a gradient of zero since the rise between any two points is zero; and the gradient of a vertical line is undefined since the run between any two points is zero.



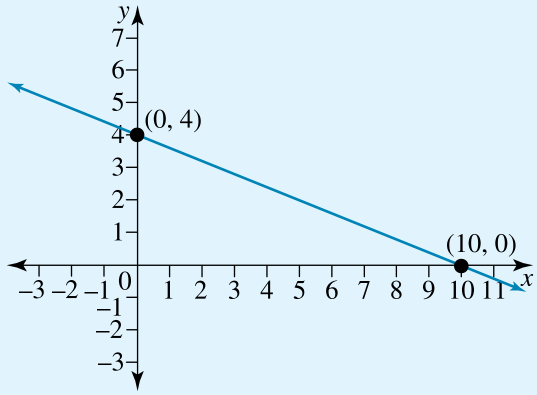

All vertical lines are parallel, and all horizontal lines are **parallel**. For other lines, if they have the same gradient then they have the same steepness, so they must be parallel to each other.

The gradient appears in the linear rule which connects the two variables $x$ and $y$.

If the rule or equation is expressed in the form $y = ax + b$ then the gradient is the coefficient of $x$ in this equation. Hence, for the line described by $y = 20 + 0.50x$ (or $y = 0.50x + 20$), the gradient is 0.50.

WORKED EXAMPLE 8

**Calculate the gradient of the given line.**




**THINK**

1 Examine the diagram to locate two known points on the line and state their coordinates.

**WRITE**

The intercepts with the coordinate axes are shown.
Given points: (0, 4), (10, 0)

2 Apply the gradient formula using one point as $(x_1, y_1)$ and the other as $(x_2, y_2)$.

Let $(x_1, y_1) = (0, 4)$ and $(x_2, y_2) = (10, 0)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{0 - 4}{10 - 0}$$

$$= -\frac{4}{10}$$

$$\therefore m = -0.4$$

3 State the answer.

The gradient of the given line is −0.4.

4 An alternative method would be to calculate the rise and the run from the diagram.

Run = 10, and rise (fall) = −4

$$m = \frac{\text{rise}}{\text{run}}$$

$$\therefore m = \frac{-4}{10} = -0.4$$

## Sketching lines and half planes from their linear rules

While many straight lines may be drawn through a single point, only one line can be drawn through two fixed points. Two points are said to determine the line. The coordinates of every point on a line must satisfy the rule or equation which describes the line.

### Sketching oblique lines

While any two points whose coordinates satisfy the equation of a line can be used to sketch it, the two points which are usually chosen are the $x$- and $y$-intercepts. Should the line pass through the origin then the coordinates of the $x$- and $y$-intercepts are both (0, 0), which gives one point. A second point would then need to be identified. The coordinates of a second point may be obtained by substituting a value for $x$ or $y$ into the rule or equation that describes the line, or alternatively the gradient could be used.

## Sketching horizontal and vertical lines

The $y$- or $x$-axis intercept is used to sketch a horizontal or vertical line respectively. These intercepts are evident from their equations.

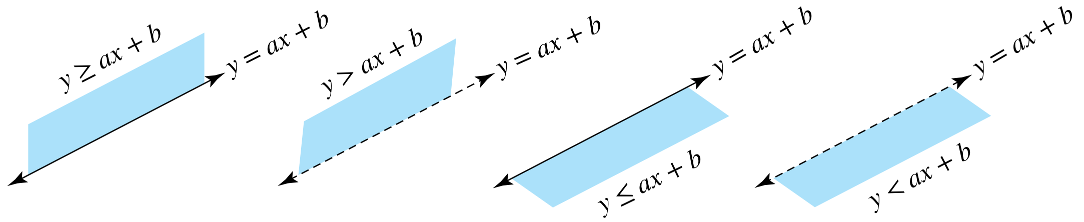
The equation $y = c$, where $c$ is a constant, represents a horizontal line with a $y$-axis intercept at $(0, c)$. The equation $x = d$, where $d$ is a constant, represents a vertical line with $x$-axis intercept of $(d, 0)$.

## Sketching half planes

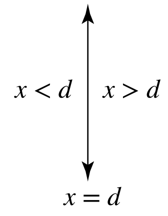
**Half planes** are the regions which lie on either side of a line, with the line referred to as the boundary line of the region.

If a line is described by a linear equation such as $y = ax + b$, then the linear inequations $y < ax + b$ or $y > ax + b$ or $y \leq ax + b$ or $y \geq ax + b$ describe half planes. The boundary line is included in a closed region for which $y \leq ax + b$ or $y \geq ax + b$, and not included in an open region for which $y < ax + b$ or $y > ax + b$.

Any non-vertical line divides the Cartesian plane into a region above the line and a region below the line. Since the $y$-coordinate measures vertical distance, the half plane, or region, above the line consists of all the points which have $y$-values greater than each of those of the corresponding points on the line while the half plane, or region, below the line has the points with $y$-values smaller than those on the line.




A vertical line also divides the Cartesian plane into two half planes, one of which consists of the points with $x$-coordinates smaller than that of the boundary line and the other with $x$-coordinates greater than that of the boundary line.




**To sketch a half plane,**
- **sketch the boundary line, drawing it either as an open or closed line depending on the inequality sign of the region.**
- **shade the appropriate side of the boundary line. (We shall always shade the required region, not the unrequired region, which is an alternative convention some texts adopt.)**

When sketching half planes, it may be helpful to test a point not on the boundary line to determine whether that point satisfies the inequation. If it does, then shade the side of the line containing that point; if not, shade the side of the line which does not contain the point. The origin $(0, 0)$ is usually a convenient point to use.

**WORKED EXAMPLE 9**

**Sketch the set of points for which:**

**a** $y = -\frac{3x}{2}$

**b** $2x - y = 6$. Hence, sketch the region described by $2x - y < 6$.

**c** $x = 4$

**d** $y \geq -2$.

**THINK** | **WRITE**

**a 1** Calculate the $y$-intercept.

**a** $y = -\frac{3x}{2}$

When $x = 0$, $y = 0$.

$\Rightarrow$ (0, 0) is both the $x$- and the $y$-intercept. The line must pass through the origin.

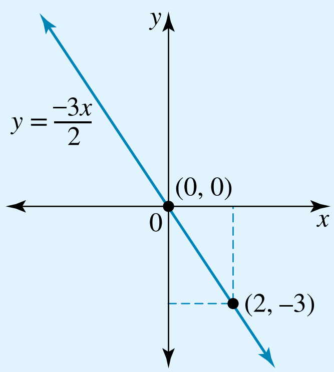
**2** A second point is needed. Substitute another value for $x$ in the equation of the line.

Point: let $x = 2$.

$$y = -\frac{3 \times (2)}{2}$$
$$= -3$$

$\Rightarrow$ (2, −3) is a point on the line.

**3** Plot the two points and sketch the line.




**b 1** Calculate the $y$-intercept.

**b** $2x - y = 6$

$y$-intercept: let $x = 0$,

$$2 \times 0 - y = 6$$
$$-y = 6$$
$$\therefore \quad y = -6$$

$\Rightarrow$ (0, −6) is the $y$-intercept.

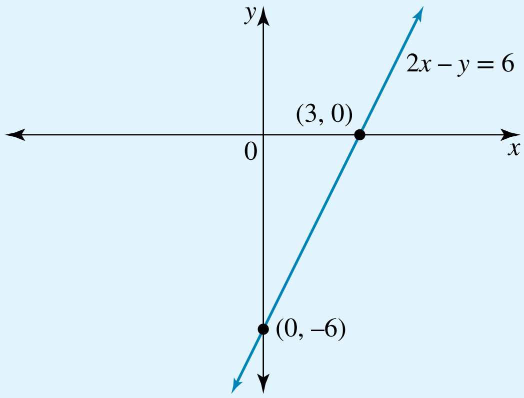
**2** Calculate the $x$-intercept.

$x$-intercept: let $y = 0$,

$$2x - 0 = 6$$
$$2x = 6$$
$$\therefore \quad x = 3$$

$\Rightarrow$ (3, 0) is the $x$-intercept.

**3** Plot the two points and draw a straight line through them. Label the points and the line appropriately.




**4** Notice the inequation indicates a half plane, or region on one side of the line $2x - y = 6$ is required.

Rearrange the inequation to make $y$ the subject.

$2x - y < 6$

$-y < -2x + 6$

$\therefore y > 2x - 6$

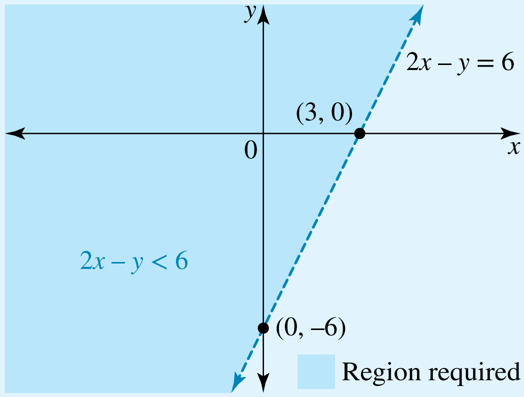
*Note:* The inequality sign is reversed by the division by $-1$.

**5** Determine whether the region lies above or below the boundary line.

Region: $y > 2x - 6$
Hence, the boundary line is $y = 2x - 6$.
Therefore the required region is the open half plane above the boundary line.

**6** Sketch the boundary line and shade the required region.

$y = 2x - 6$ is the same as $2x - y = 6$.
From part **b**, this boundary line has axis intercepts at $(0, -6)$ and $(3, 0)$. It is drawn as an open or non-solid line because of the $>$ sign in the inequation describing the region.
The required region is shaded on the diagram.




**7** An alternative method is to test a point such as the origin to see if its coordinates satisfy the inequation.

Substitute $x = 0$, $y = 0$ into each side of the inequation.

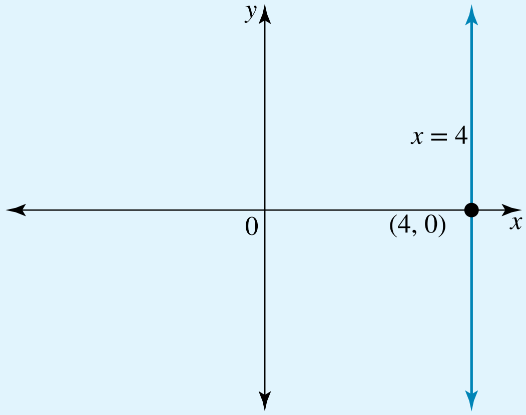
$$\begin{aligned} \text{LHS} &= 2x - y \\ &= 2(0) - (0) \\ &= 0 \end{aligned}$$

RHS $= 6$
Since $0 < 6$, the origin does satisfy the inequation.
Therefore the side of the line $2x - y = 6$ containing the origin should be shaded.

**c** $x = 4$ is the equation of a vertical line. Sketch this line.

**d** $x = 4$

$x$-intercept $(4, 0)$




**d 1** Determine the region that is required.

**e** Region: $y \geq -2$

Boundary line: $y = -2$

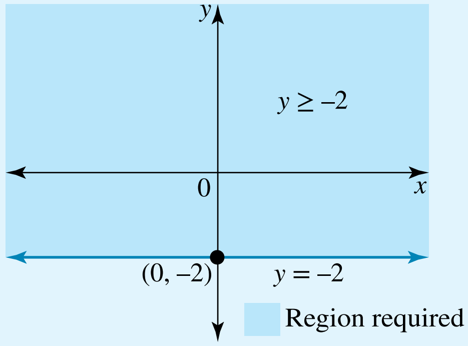
Therefore the required region is the closed half plane above, and including, the boundary line.

**2** Sketch the boundary line and shade the required region.

The boundary line is a horizontal line with $y$-intercept $(0, -2)$.

It is drawn as a closed, or solid, line because of the $\geq$ sign in the inequation describing the region.

The required region is shaded on the diagram.




**3** The alternative method of testing the origin is included as a check.

Substitute $x = 0$, $y = 0$ into each side of the inequation.

$$\text{LHS} = y$$
$$= 0$$
$$\text{RHS} = -2$$

Since $0 > -2$, the origin does satisfy the inequation.

Therefore, the side of the line $y = -2$ which contains the origin should be shaded.

## Forming equations of lines

**eBook*plus***

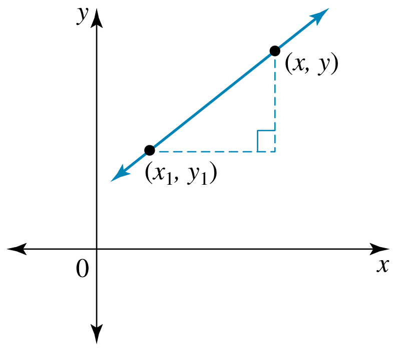
**Interactivity**
Point–gradient and gradient–$y$-intercept equations
**int-2551**

Regardless of whether the line is oblique, horizontal or vertical, two pieces of information are required in order to form its equation. The type of information usually determines the method used to form the equation of the line. The forms of the equation of an oblique line that are most frequently used are:

- the **point–gradient form**
- the **gradient–$y$-intercept form**.

## Point–gradient form of the equation of a line

Given the gradient $m$ and a point $(x_1, y_1)$ on the line, the equation of the line can be formed as follows:




For any point $(x, y)$ on the line with gradient $m$:

$$m = \frac{y - y_1}{x - x_1}$$

$$\therefore y - y_1 = m(x - x_1)$$

**A line with gradient $m$ and passing through the point $(x_1, y_1)$ has the equation:**

$$\boldsymbol{y - y_1 = m(x - x_1)}$$

This equation is known as the point–gradient form of the equation of a line.

## Gradient–*y*-intercept form

A variation on the point–gradient form is obtained if the known point is the $y$-intercept.

If a line with gradient $m$ cuts the $y$-axis at the point $(0, c)$, then using $(0, c)$ for $(x_1, y_1)$,

$$y - y_1 = m(x - x_1)$$

$$\therefore y - c = m(x - 0)$$

$$\therefore \quad y = mx + c$$

The equation of a line in the form $y = mx + c$ features the gradient as the coefficient of $x$ and the $y$-intercept as $c$. In this context it is common to refer to $c$ as the $y$-intercept, with the understanding that the actual coordinates of the $y$-intercept are $(0, c)$.

**A line with gradient $m$ and $y$-intercept $c$ has the equation:**

$$\boldsymbol{y = mx + c}$$

This equation is known as gradient–$y$-intercept form.

### General form of the equation

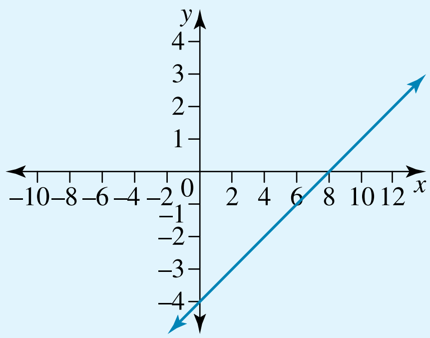
The general form of the equation of a line can be written as $ax + by + c = 0$. Any equivalent form of this equation is an acceptable form for the equation of a line. For instance, the equation $3x + y - 2 = 0$ is equally as well expressed in equivalent forms which include $y = -3x + 2$, $y = 2 - 3x$, $y + 3x = 2$, $x + \frac{y}{3} = \frac{2}{3}$. Of course, it is important to be able to recognise that the examples given are all equivalent.

WORKED EXAMPLE 10

**a** Form the equation of the line with gradient 4, passing through the point (3, −7).

**b** Form the equation of the line passing through the points (5, 9) and (12, 0).

**c** For the line shown, determine its equation.




**d** Obtain the gradient and the coordinates of the $y$-intercept of the line with the equation $3x - 8y + 5 = 0$.

**THINK**

**a** 1 State the given information.

2 Write the point–gradient form of the equation.

3 Substitute the given information and simplify to obtain the equation.

**b** 1 State the given information.

2 Use the two points to calculate the gradient.

3 Write the point–gradient form of the equation.

4 The point–gradient equation can be used with either of the points. Substitute one of the points and simplify to obtain the equation.

5 The equation could be expressed without fractions. Although this is optional, it looks more elegant.

**WRITE**

**a** The gradient and a point are given.

$m = 4, (x_1, y_1) = (3, -7)$

$$y - y_1 = m(x - x_1)$$

$$y - (-7) = 4(x - 3)$$

$$y + 7 = 4x - 12$$

$$\therefore y = 4x - 19$$

**b** Two points are given.

Let $(x_1, y_1) = (5, 9)$ and $(x_2, y_2) = (12, 0)$

$$m = \frac{0 - 9}{12 - 5} = -\frac{9}{7}$$

$$y - y_1 = m(x - x_1)$$

Let (12, 0) be the given point $(x_1, y_1)$ in this equation.

$$y - 0 = -\frac{9}{7}(x - 12)$$

$$y = -\frac{9}{7}(x - 12)$$

$$7y = -9(x - 12)$$

$$7y = -9x + 108$$

$$\therefore 7y + 9x = 108$$

6 Had the point (5, 9) been used, the same answer would have been obtained.

Check:

$$y - 9 = -\tfrac{9}{7}(x - 5)$$
$$7(y - 9) = -9(x - 5)$$
$$7y - 63 = -9x + 45$$
$$\therefore 7y + 9x = 108$$

Same equation as before.

**c** 1 Calculate the gradient from the graph (or use the coordinates of the $y$-intercept and the $x$-intercept points).

**c** $m = \dfrac{\text{rise}}{\text{run}}$

$\therefore m = \dfrac{4}{8}$

$\therefore m = \dfrac{1}{2}$

2 One of the points given is the $y$-intercept. State $m$ and $c$.

$m = \dfrac{1}{2}, \; c = -4$

3 Use the gradient–$y$-intercept form to obtain the required equation.

$y = mx + c$

$\therefore y = \dfrac{1}{2}x - 4$

**d** 1 Express the equation in the form $y = mx + c$.

**d** $3x - 8y + 5 = 0$

$\therefore \quad 3x + 5 = 8y$

$\therefore \quad y = \dfrac{3x}{8} + \dfrac{5}{8}$

2 State $m$ and $c$.

$m = \dfrac{3}{8}, \; c = \dfrac{5}{8}$

3 Express the answer in the required form.

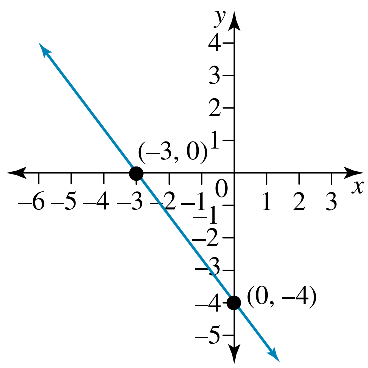
Gradient is $\dfrac{3}{8}$ and $y$-intercept is $\left(0, \dfrac{5}{8}\right)$.

## EXERCISE 1.4 Linear graphs and their equations

**PRACTISE**

Work without CAS

1 WE8 Calculate the gradient of the given line.




2 Show that the line passing through the points $(a, b)$ and $(-b, -a)$ is parallel to the line joining the points $(-c, d)$ and $(-d, c)$.

3 WE9 Sketch the set of points for which:

a $y = 4x$

b $3x + 2y = 6$. Hence sketch the region described by $3x + 2y > 6$

c $x < 3$

d $y = 2$

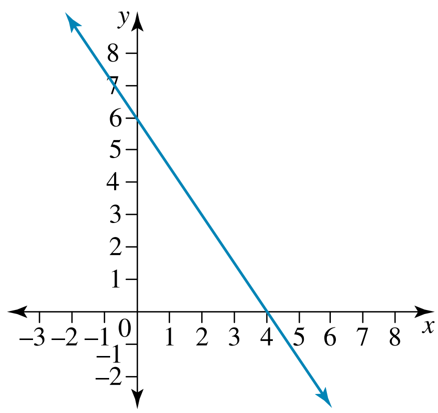
4 Determine whether the point $(-2.2, 17.6)$ lies on, above or below the line with equation $y = 4.4 - 10x$.

5 a WE10 Form the equation of the line with gradient $-2$, passing through the point $(-8, 3)$.

b Form the equation of the line passing through the points $(4, -1)$ and $(-3, 1)$.

c Determine the equation of the line shown.

d Obtain the gradient and the coordinates of the $y$-intercept of the line with equation $6y - 5x - 18 = 0$.




6 What is the equation of the line parallel to the $x$-axis that passes through the point $(2, 10)$?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 Calculate the gradient of the line joining the following points.

a $(-3, 8)$ and $(-7, 18)$

b $(0, -4)$ and $(12, 56)$

c $(-2, -5)$ and $(10, -5)$

d $(3, -3)$ and $(3, 15)$

8 a If $y$ is directly proportional to $x$, and $y = 52.5$ when $x = 15$:

i find the rule for $y$ in terms of $x$

ii sketch the graph of $y$ versus $x$ and state its gradient.

b If the cost, $C$ dollars, of hiring a rowing boat is \$30 plus \$1.50 per hour, or part thereof:

i find the rule for $C$ in terms of the hire time in hours $t$

ii sketch the graph of $C$ versus $t$ and state its gradient.

9 Sketch the lines with the following equations.

a $y = 3x + 8$

b $4y - x + 4 = 0$

c $6x + 5y = 30$

d $3y - 5x = 0$

e $\frac{x}{2} - \frac{3y}{4} = 6$

f $y = -\frac{6x}{7}$

10 Sketch the half planes described by the following inequations.

a $y \leq 3 - 3x$

b $4x + y > 12$

c $5x - 2y \leq 8$

d $x > -4$

e $y \leq 0$

f $y \leq x$

11 Find the equations of the straight lines that are determined by the following information.

a Gradient $-5$, passing through the point $(7, 2)$

b Gradient $\frac{2}{3}$, passing through the point $(-4, -6)$

c Gradient $-1\frac{3}{4}$, passing through the point $(0, -9)$

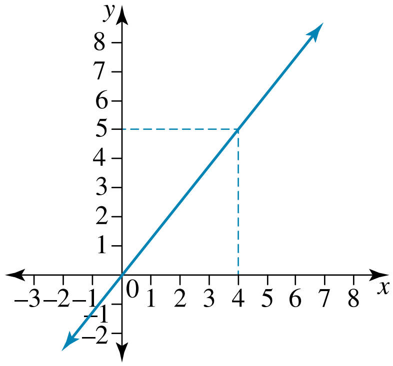
d Gradient $-0.8$, passing through the point $(0.5, -0.2)$

e Passing through the points $(-1, 8)$ and $(-4, -2)$

f Passing through the points $(0, 10)$ and $(10, -10)$

**12 a** Form the equations of the given graphs.

**i**

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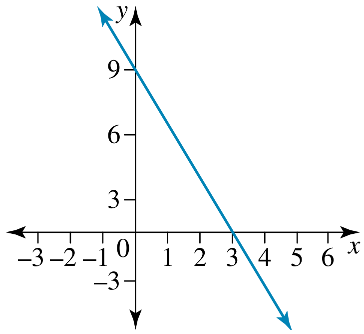
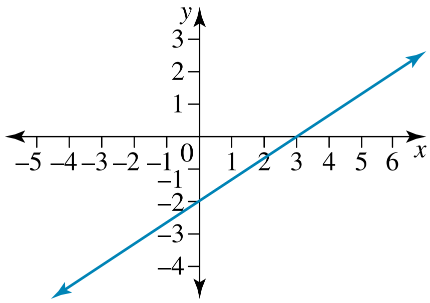
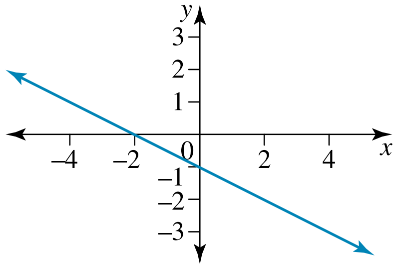



**iii**




**iv**




**b** A line contains the points $(-12, 8)$ and $(-12, -1)$. Form its equation and sketch its graph.

**c** Find the equation of the line passing through $(10, -8)$ which is parallel to the line $y = 2$, and sketch its graph.

**13** Find the gradient and the $y$-intercept of the lines with the following equations.

**a** $4x + 5y = 20$

**b** $\dfrac{2x}{3} - \dfrac{y}{4} = -5$

**c** $x - 6y + 9 = 0$

**d** $2y - 3 = 0$

**14 a** Find the value of $a$ if the point $(2a, 2 - a)$ lies on the line given by $5y = -3x + 4$.

**b** Does the point $(-22, 13)$ lie on, above or below the line with equation $7y - 3x = 25$?

**c** Form the equation of the line containing the points $(p, q)$ and $(-p, -q)$.

**15** A girl's pulse rate measured 180 beats per minute immediately following an exercise session, and thereafter decreased by an amount proportional to the time that had elapsed since she finished exercising.

**a** Construct a rule relating her pulse rate, $p$ beats per minute and $t$, the time in minutes since she stopped the exercise.

**b** If the girl's pulse rate decreases at 10 beats per minute, how long does it take for her pulse rate to return to its normal rest rate of 60 beats per minute?

**c** Express $p$ in terms of $t$ and sketch the $t$–$p$ graph over an appropriate interval.

**d** What is the gradient of the graph?

16 A family of parallel lines has the equation $3x - 2y = k$ where $k$ is a real number.

a What is the gradient of each member of this family of lines?
b Show that all lines in the family contain the point $(k, k)$.
c Which member of the family contains the point $(-3, -8)$?
d Which member of the family has a $y$-intercept of 2?
e Sketch the lines found in part c and part d on the same set of axes.
f Describe the closed region between these two lines using inequations.

MASTER

17 Sketch with CAS technology the lines with the following equations.

a $2y - 4x = -11$ b $x = 5$ c $y = -3$

18 Use CAS technology to sketch the regions defined by the following.

a $2y - 4x > -11$ b $x \leq 5$ and $y \geq 1$

# 1.5 Intersections of lines and their applications

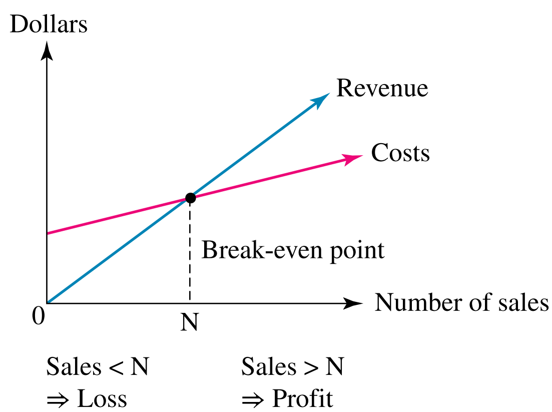
studyon

Units 1 & 2
AOS 1
Topic 1
Concept 5

**Coordinate geometry graphical representation**
Concept summary
Practice questions

Graphs offer visual communication and allow comparisons between models to be made. If a revenue model and cost model are graphed together, it is possible to see when a profit is made.

A profit will be made only when the revenue graph lies above the cost graph. At the point of intersection of the two graphs, the break-even point, the revenue equals the costs. Before this point a loss occurs since the revenue graph lies below the cost graph.




The point of intersection of the two graphs becomes a point of interest as the number of sales needed to make a profit can be deduced from it. However, this key feature cannot often be obtained with precision by reading from the graph.

## Intersections of lines

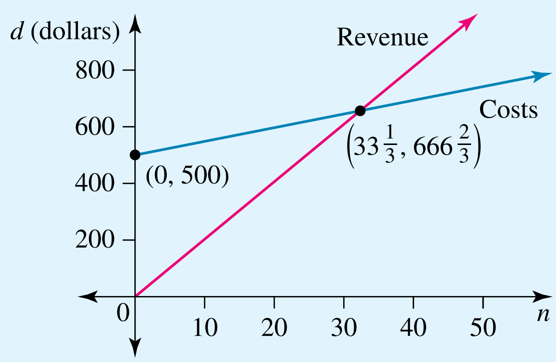
Two lines with different gradients will always intersect at a point. Since this point must lie on both lines, its coordinates can reliably be found algebraically using simultaneous equations.

WORKED EXAMPLE 11

**The model for the revenue in dollars, $d$, from the sale of $n$ items is $d_R = 20n$ and the cost of manufacture of the $n$ items is modelled by $d_C = 500 + 5n$.**

**a** **Find the coordinates of the point of intersection of the graphs of these two models and sketch the graphs on the same set of axes.**

**b** **Obtain the least value of $n$ for a profit to be made.**

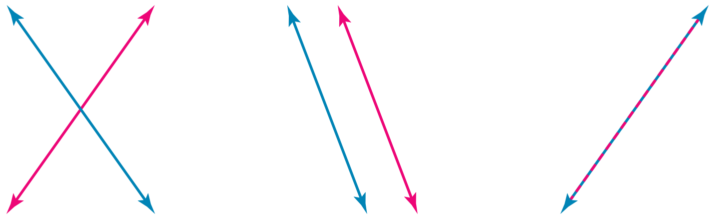
| THINK | WRITE |
|---|---|
| **a 1** Write the equation at the point of intersection. | **a** At the intersection, or break-even point: $d_R = d_C$ <br> $20n = 500 + 5n$ |
| **2** Solve to find $n$. | $15n = 500$ <br> $n = \frac{100}{3}$ <br> $\therefore \quad n = 33\frac{1}{3}$ |
| **3** Calculate the $d$ coordinate. | When $n = \frac{100}{3}$, <br> $d = 20 \times \frac{100}{3}$ <br> $\therefore d = 666\frac{2}{3}$ |
| **4** State the coordinates of the point of intersection. | The point of intersection is $\left(33\frac{1}{3}, 666\frac{2}{3}\right)$. |
| **5** Both graphs contain the intersection point. Find one other point on each graph. | Points: Let $n = 0$, <br> $\therefore d_R = 0$ and $d_C = 500$ <br> The revenue graph contains the points $(0, 0)$, $\left(33\frac{1}{3}, 666\frac{2}{3}\right)$. <br> The cost graph contains the points $(0, 500)$, $\left(33\frac{1}{3}, 666\frac{2}{3}\right)$. |
| **6** Sketch the graphs. |  |
| **b 1** State the condition for a profit to be made. | **b** For a profit, $d_R > d_C$. <br> From the graph $d_R > d_C$ when $n > 33\frac{1}{3}$. |
| **2** Answer the question. | Therefore, at least 34 items need to be sold for a profit to be made. |

## The number of solutions to systems of 2 × 2 linear simultaneous equations

**eBook*plus***

**Interactivity**
Intersecting, parallel and identical lines
**int-2552**

Since a linear equation represents a straight line graph, by considering the possible intersections of two lines, three possible outcomes arise when solving a system of simultaneous equations.



Case 1
$m_1 \neq m_2$

Case 2
$m_1 = m_2$
(parallel)

Case 3
$m_1 = m_2,\ c_1 = c_2$
(identical)

Case 1: Unique solution to the system

The equations represent two lines which intersect at a single point.

Case 2: No solution to the system

The equations represent parallel lines.

Case 3: Infinitely many solutions

The equations are equivalent and represent the same line. Every point on the line is a solution.

**If the system of equations is rearranged to be of the form**

$$y = m_1x + c_1$$

$$y = m_2x + c_2$$

**then:**
- **unique solution if $m_1 \neq m_2$**
- **no solution if $m_1 = m_2$ and $c_1 \neq c_2$**
- **infinitely many solutions if $m_1 = m_2$ and $c_1 = c_2$.**

**WORKED EXAMPLE 12**

Find the value of $m$ so that the system of equations

$$mx - y = 2$$

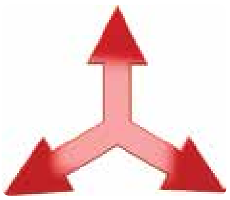
$$3x + 4y = 12$$

has no solutions.

| THINK | WRITE |
|---|---|
| 1 Rearrange both equations to the $y = mx + c$ form. | $mx - y = 2 \Rightarrow y = mx - 2$<br>and<br>$3x + 4y = 12 \Rightarrow y = -\frac{3}{4}x + 3$ |
| 2 State the gradients of the lines the equations represent. | Gradients are $m$ and $-\frac{3}{4}$. |
| 3 State the condition for the system of equations to have no solution, and calculate $m$. | For the system of equations to have no solution, the lines have to be parallel, but have different $y$-intercepts.<br>For the lines to be parallel, the two gradients have to be equal.<br>$\therefore m = -\frac{3}{4}$ |

| | |
|---|---|
| 4 The possibility of the equations being equivalent has to be checked. | Substitute $m = -\frac{3}{4}$ into the $y = mx + c$ forms of the equations.<br>$y = -\frac{3}{4}x - 2$ and $y = -\frac{3}{4}x + 3$ represent parallel lines since they have the same gradients and different $y$-intercepts. |
| 5 State the answer. | Therefore if $m = -\frac{3}{4}$, the system will have no solution. |

## Concurrent lines



Three or more lines that intersect at a common point are said to be **concurrent**. Their point of intersection is known as the point of concurrency.

To show that three lines are concurrent, the simplest method is to find the point of intersection of two of the lines and then check whether that point lies on the third line.

**WORKED EXAMPLE 13**

**Show that the three lines with equations $5x + 3y = 1$, $4x + 7y = 10$ and $2x - y = -4$ are concurrent.**

| THINK | WRITE |
|---|---|
| 1 Select the pair of equations to solve simultaneously. | Consider<br>$4x + 7y = 10$ ...... (1)<br>$2x - y = -4$ ....... (2) |
| 2 Solve this system of equations. | Eliminate $x$.<br>Multiply equation (2) by 2 and subtract it from equation (1).<br>$7y - 2 \times (-y) = 10 - 2 \times (-4)$<br>$7y + 2y = 10 + 8$<br>$9y = 18$<br>$\therefore \quad y = 2$<br>Substitute $y = 2$ into equation (2).<br>$2x - 2 = -4$<br>$2x = -2$<br>$\therefore \quad x = -1$<br>Lines (1) and (2) intersect at $(-1, 2)$. |
| 3 Test whether the values for $x$ and $y$ satisfy the third equation. | Substitute $x = -1, y = 2$ into $5x + 3y = 1$.<br>LHS $= 5 \times (-1) + 3 \times (2)$<br>$= 1$<br>$=$ RHS<br>Therefore $(-1, 2)$ lies on $5x + 3y = 1$. |
| 4 Write a conclusion. | Since $(-1, 2)$ lies on all three lines, the three lines are concurrent. The point $(-1, 2)$ is their point of concurrency. |

## EXERCISE 1.5 Intersections of lines and their applications

**PRACTISE**

Work without CAS

1 WE11 If the model for the revenue in dollars, $d$, from the sale of $n$ items is $d_R = 25n$ and the cost of manufacture of the $n$ items is modelled by $d_C = 260 + 12n$:

a find the coordinates of the point of intersection of the graphs of these two models and sketch the graphs on the same set of axes

b obtain the least value of $n$ for a profit to be made.

2 Use simultaneous equations to find the coordinates of the point of intersection of the lines with equations $3x - 2y = 15$ and $x + 4y = 54$.

3 WE12 Find the value of $m$ so that the system of equations

$$2mx + 3y = 2m$$
$$4x + y = 5$$

has no solutions.

4 For what values of $a$ and $b$ will the system of equations

$$ax + y = b$$
$$3x - 2y = 4$$

have infinitely many solutions?

5 WE13 Show that the three lines with equations $2x + 3y = 0$, $x - 8y = 19$ and $9x + 5y = 17$ are concurrent.

6 Find the value of $a$ so that the three lines defined by $x + 4y = 13$, $5x - 4y = 17$ and $-3x + ay = 5$ are concurrent.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 Find the coordinates of the point of intersection of each of the following pairs of lines.

a $4x - 3y = 13$ and $2y - 6x = -7$

b $y = \frac{3x}{4} - 9$ and $x + 5y + 7 = 0$

c $y = -5$ and $x = 7$

8 The line passing through the point $(4, -8)$ with gradient $-2$ intersects the line with gradient 3 and $y$-intercept 5 at the point Q. Find the coordinates of Q.

9 A triangle is bounded by the lines with equations $x = 3$, $y = 6$ and $y = -3x$.

a Find the coordinates of its vertices.

b Calculate its area in square units.

10 For what value of $p$ will the lines $2x + 3y = 23$ and $7x + py = 8$ not intersect?

11 a Express the lines given by $px + 5y = q$ and $3x - qy = 5q$, $(q \neq 0)$, in the $y = mx + c$ form.

b Hence, determine the values of $p$ and $q$ so the system of equations

$$px + 5y = q$$
$$3x - qy = 5q$$

will have infinitely many solutions.

c What relationship must exist between $p$ and $q$ so the lines $px + 5y = q$ and $3x - qy = 5q$ will intersect?

12 a Show that the following three lines are concurrent and state their point of concurrency:

$3x - y + 3 = 0$, $5x + 2y + 16 = 0$ and $9x - 5y + 3 = 0$

b Determine the values of $d$ so that the three lines $x + 4y = 9$, $3x - 2y = -1$ and $4x + 3y = d$ are not concurrent.

13 The daily cost of hiring a bicycle from the Pedal On company is \$10 plus 75 cents per kilometre whereas from the Bikes R Gr8 company the cost is a flat rate of \$20 with unlimited kilometres.

a State the linear equations that model the costs of hiring the bicycles from each company.

b On one set of axes, sketch the graphs showing the cost versus the number of kilometres ridden, for each company.

c After how many kilometres are the costs equal?

d Shay wishes to hire a bicycle for the day. How can Shay decide from which of the two companies to hire the bicycle?

14 A tram starting from rest travels for part of its journey at a speed, in km/h, which is directly proportional to its time, in minutes, of travel. After 2 minutes the speed of the tram is 10 km/h.

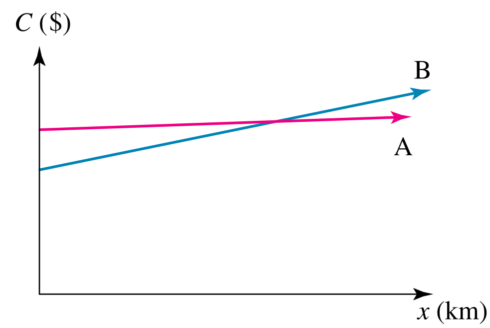

Travelling along the same route over the same period of time, the speed of a bus is partly constant and partly varies as the time of travel. Two minutes after the tram starts, the bus is travelling at 18 km/h; three minutes after that its speed has increased to 27 km/h.

a Assuming neither the tram nor the bus stop for passengers during this time interval, construct the linear models for the speed of the tram and the speed of the bus over this time period.

b Sketch each model on the same set of axes over a 5-minute time period.

c How long does it take for the tram's speed to be faster than that of the bus?

15 The graph shows cost, $C$, in dollars, versus distance $x$, in kilometres, for two different car rental companies A and B.

The cost models for each company are $C = 300 + 0.05x$ and $C = 250 + 0.25x$.

**a** Match each cost model to each company.
**b** Explain what the gradient of each graph represents.
**c** Construct a linear rule in terms of $x$ for $y = C_A - C_B$, the difference in cost between Company A and Company B.
**d** Sketch the graph of $y = C_A - C_B$ showing the intercepts with the coordinate axes.
**e** Use the graph in part **d** to determine the number of kilometres when:
  **i** the costs of each company are the same
  **ii** the costs of Company A are cheaper than those of Company B.

**16** The position of a boat at sea is measured as $x$ km east and $y$ km north of a lookout taken to be the origin $(0, 0)$. Initially, at 6 am, the boat is 2 km due north of the lookout and after 1 hour, its position is 6 km east and 3 km north of the lookout.

**a** Write down the coordinates of the two positions of the boat and, assuming the boat travels in a straight line, form the equation of its path.

**b** The boat continues to sail on this linear path and at some time $t$ hours after 6 am, its distance east of the lookout is $6t$ km. At that time, show that its position north of the lookout is $(t + 2)$ km.
**c** Determine the coordinates of the position of the boat at 9.30 am.
**d** The positions east and north of the lighthouse of a second boat, a large fishing trawler, sailing along a linear path are given by $x = \frac{4t - 1}{3}$ and $y = t$ respectively, where $t$ is the time in hours since 6 am. Find the coordinates of the positions of the trawler at 6 am and 7 am and hence (or otherwise) find the Cartesian equation of its linear path.
**e** Show that the paths of the boat and the trawler contain a common point and give the coordinates of this point.
**f** Sketch the paths of the boats on the same axes and explain whether the boat and the trawler collide.

MASTER

**17** Use the graphing facility on CAS technology to obtain the point of intersection of the pair of lines $y = \frac{17 + 9x}{5}$ and $y = 8 - \frac{3x}{2}$, to 2 decimal places.

**18** At time $t$ a particle $P_1$ moving on a straight line has coordinates given by $x = t, y = 3 + 2t$, while at the same time a second particle $P_2$ moving along another straight line has coordinates given by $x = t + 1, y = 4t - 1$.

**a** Use CAS technology to sketch their paths simultaneously and so determine whether the particles collide.
**b** What are the coordinates of the common point on the paths?

# 1.6 Coordinate geometry of the straight line

studyon

Units 1 & 2
AOS 1
Topic 1
Concept 6

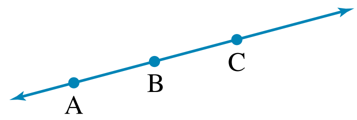
**Coordinate geometry of the straight line**
Concept summary
Practice questions

The gradient of a line has several applications. It determines if lines are parallel or perpendicular, it determines the angle a line makes with the horizontal and it determines if three or more points lie on the same line.

## Collinearity

Three or more points that lie on the same line are said to be **collinear**.

If $m_{AB} = m_{BC}$ then the line through the points A and B is parallel to the line through the points B and C; but, since the point B is common to both AB and BC, the three points A, B and C must lie on the same line.




Alternatively, the equation of the line through two of the points can be used to test if the third point also lies on that line.

**WORKED EXAMPLE 14**

**Show that the points A(−5, −3), B(−1, 7) and C(1, 12) are collinear.**

| THINK | WRITE |
|---|---|
| 1 Select two of the points. | For the points A and B, let $(x_1, y_1)$ be $(-5, -3)$ and $(x_2, y_2)$ be $(-1, 7)$. |
| 2 Calculate the gradient of AB. | $m = \frac{y_2 - y_1}{x_2 - x_1}$ <br> $= \frac{7 - (-3)}{-1 - (-5)}$ <br> $= \frac{10}{4}$ <br> $\therefore m_{AB} = \frac{5}{2}$ |
| 3 Select another pair of points containing a common point with the interval AB. | For the points B and C, let $(x_1, y_1)$ be $(-1, 7)$ and $(x_2, y_2)$ be $(1, 12)$. |
| 4 Calculate the gradient of BC. <br> *Note:* The interval AC could equally as well have been chosen. | $m = \frac{y_2 - y_1}{x_2 - x_1}$ <br> $m = \frac{12 - 7}{1 - (-1)}$ <br> $\therefore m_{BC} = \frac{5}{2}$ |
| 5 Compare the gradients to determine collinearity. | Since $m_{AB} = m_{BC}$ and the point B is common, the three points lie on the same line so they are collinear. |

## Angle of inclination of a line to the horizontal

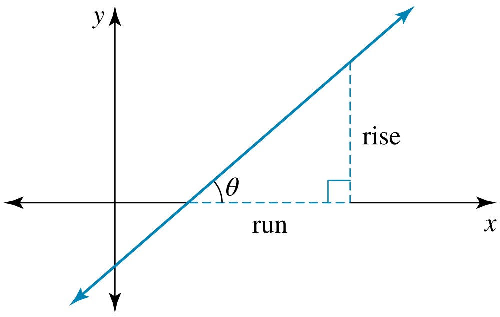
If $\theta$ is the angle a line makes with the positive direction of the $x$-axis (or other horizontal line), and $m$ is the gradient of the line, then a relationship between the gradient and this angle can be formed using trigonometry.

In the right-angled triangle,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{\text{rise}}{\text{run}}$$

$$\therefore \tan\theta = m$$



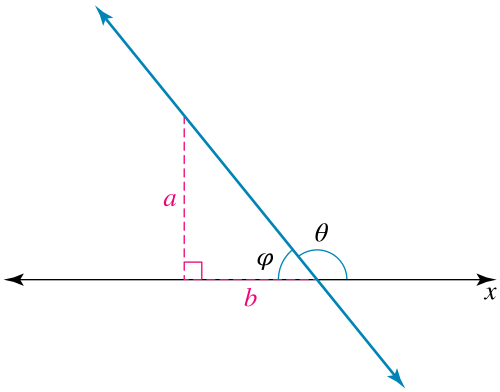

Hence, the gradient of the line is given by

$$\boldsymbol{m = \tan\theta}$$

If the line is vertical, $\theta = 90°$ and if the line is horizontal, $\theta = 0°$.

For oblique lines, the angle is either acute ($0° < \theta < 90°$) or obtuse ($90° < \theta < 180°$) and a calculator is used to find $\theta$.

In the obtuse-angle case, $\theta = 180° - \varphi$ where $\varphi$ is acute. If the gradient of the line is $m = -\frac{a}{b}$ then, using trigonometry, $\tan\varphi = \frac{a}{b}$ and therefore $\theta = 180° - \tan^{-1}\left(\frac{a}{b}\right)$.




Note that $\frac{a}{b}$ is the positive part of $m$. The mathematical term for this positive part is the absolute value and it can be written as $|m| = \frac{a}{b}$, or alternatively, just remember to use the positive part of the gradient when calculating the obtuse **angle of inclination**. For example, if $m = -3$, then $|m| = 3$ and the obtuse angle is calculated from $\theta = 180° - \tan^{-1}(3)$.

**If $m > 0$ then $\theta$ is an acute angle, so $\theta = \tan^{-1}(m)$**
**If $m < 0$ then $\theta$ is an obtuse angle, so $\theta = 180° - \tan^{-1}(|m|)$**

WORKED EXAMPLE 15

**a** Calculate, correct to 2 decimal places, the angle made with the positive direction of the $x$-axis by the line which passes through the points $(-1, 2)$ and $(2, 8)$.

**b** Calculate the angle of inclination with the horizontal made by a line which has a gradient of $-0.6$.

**c** Obtain the equation of the line which passes through the point $(5, 3)$ at an angle of 45° to the horizontal.

| THINK | WRITE |
| --- | --- |
| **a** 1 Calculate the gradient of the line through the given points. | **a** Points $(-1, 2)$ and $(2, 8)$<br>$m = \frac{8-2}{2-(-1)}$<br>$= \frac{6}{3}$<br>$= 2$ |

| | | |
|---|---|---|
| | **2** Write down the relationship between the angle and the gradient. | $\tan\theta = m$<br>$\therefore \tan\theta = 2$ |
| | **3** Find $\theta$ correct to 2 decimal places using a calculator. | $\theta = \tan^{-1}(2)$<br>$\therefore \theta \simeq 63.43°$ |
| | **4** Answer in context. | Therefore the required angle is 63.43°. |
| **b** | **1** State the relationship between the angle and the gradient given. | **b** The gradient −0.6 is given.<br>$\tan\theta = m$<br>$\therefore \tan\theta = -0.6$ |
| | **2** Write down the method for finding the obtuse angle. | The required angle is obtuse since the gradient is negative.<br>$\theta = 180° - \tan^{-1}(\lvert m \rvert)$<br>Since $m = -0.6$, $\lvert m \rvert = 0.6$<br>$\theta = 180° - \tan^{-1}(0.6)$ |
| | **3** Calculate the angle. | $\therefore \theta \simeq 149.04°$ |
| | **4** Answer in context. | Therefore the required angle is 149°, to the nearest degree. |
| **c** | **1** State the relationship between the gradient and the angle. | **c** $m = \tan\theta$ |
| | **2** Substitute the given angle for $\theta$ and calculate the gradient. | $m = \tan(45°)$<br>$\therefore m = 1$ |
| | **3** Use the point–gradient form to obtain the equation of the line. | $y - y_1 = m(x - x_1)$, $m = 1$, $(x_1, y_1) = (5, 3)$<br>$y - 3 = 1(x - 5)$<br>$\therefore \quad y = x - 2$ |
| | **4** State the answer. | The equation of the line is $y = x - 2$. |

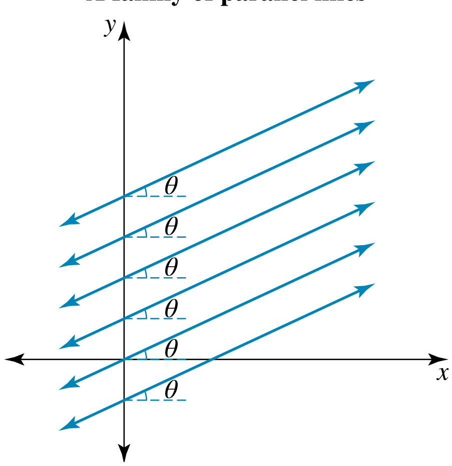
## Parallel lines

Parallel lines have the same gradient but different $y$-intercepts. A set of parallel lines is called a family of lines since each line has a common feature, namely the same gradient.

If two lines with gradients $m_1$ and $m_2$ are parallel, then $m_1 = m_2$.

Each line would be inclined at the same angle to the horizontal.

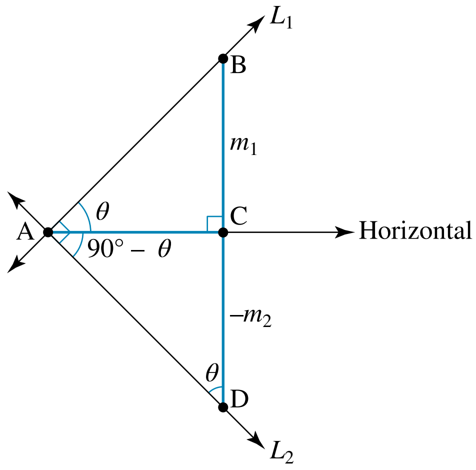
**A family of parallel lines**



## Perpendicular lines

A pair of lines are perpendicular to each other when the angle between them is 90°. For a pair of oblique lines, one must have a positive gradient and the other a negative gradient.

To find the relationship between these gradients consider two **perpendicular lines** $L_1$ and $L_2$ with gradients $m_1$ and $m_2$ respectively. Suppose $m_1 > 0$, and $m_2 < 0$ and that $L_1$ is inclined at an angle $\theta$ to the horizontal.




The diagram shows the line $L_1$ passing through the points A and B and the line $L_2$ passing through the points A and D with the angle *BAD* a right angle.

Taking AC as 1 unit, the sides in the diagram are labelled with their lengths. The side CB has length $m_1$. Since lengths must be positive, the side CD is labelled as $-m_2$ since $m_2 < 0$.

From the triangle ABC in the diagram, $\tan\theta = \frac{m_1}{1} = m_1$ and from the triangle ACD in the diagram, $\tan\theta = \frac{1}{-m_2}$.

Hence,

$$m_1 = \frac{1}{-m_2}$$

$$\therefore\ m_1 m_2 = -1$$

This is the relationship between the gradients of perpendicular lines.

$$\boldsymbol{m_1 m_2 = -1 \quad \text{or} \quad m_2 = -\frac{1}{m_1}}$$

- If two lines with gradients $m_1$ and $m_2$ are perpendicular, then the product of their gradients is $-1$. One gradient is the negative reciprocal of the other.
- It follows that if $m_1 m_2 = -1$ then the two lines are perpendicular. This can be used to test for perpendicularity.

**WORKED EXAMPLE 16**

**a** State the gradient of a line which is

**i** parallel to $2y - 5x = 4$ **ii** perpendicular to $2y - 5x = 4$.

**b** Show that the lines $y = 4x$ and $y = -0.25x$ are perpendicular.

**c** Determine the equation of the line through the point $(1, 1)$ perpendicular to the line $y = -3x - 9$.

| THINK | WRITE |
| --- | --- |
| **a i 1** Rearrange the equation of the given line to express it in the $y = mx + c$ form. | **a i** $2y - 5x = 4$<br>$2y = 5x + 4$<br>$y = \frac{5x}{2} + \frac{4}{2}$<br>$\therefore \quad y = \frac{5x}{2} + 2$ |
| **2** Give the gradient of the given line. | The given line has $m = \frac{5}{2}$. |
| **3** State the gradient of a line parallel to the given line. | The gradient of a parallel line will be the same as that of the given line.<br>Therefore a line parallel to $2y - 5x = 4$ has a gradient of $\frac{5}{2}$. |
| **ii 1** State the gradient of a perpendicular line to the given line. | **ii** The gradient of a perpendicular line will be the negative reciprocal of the gradient of the given line.<br>If $m_1 = \frac{5}{2}$ then $m_2 = -\frac{2}{5}$.<br>Therefore a line perpendicular to $2y - 5x = 4$ has a gradient of $-\frac{2}{5}$. |
| **b 1** Write down the gradients of each line. | **b** Lines: $y = 4x$ and $y = -0.25x$<br>Gradients: $m_1 = 4,\ m_2 = -0.25$ |
| **2** Test the product of the gradients. | $m_1 m_2 = 4 \times -0.25$<br>$\therefore m_1 m_2 = -1$<br>Therefore the lines are perpendicular. |
| **c 1** State the gradient of the given line and calculate the gradient of the perpendicular line. | **c** For $y = -3x - 9$, its gradient is $m_1 = -3$.<br>The perpendicular line has gradient $m_2 = -\frac{1}{m_1}$.<br>Therefore the perpendicular line has gradient $\frac{1}{3}$. |
| **2** Use the point–gradient form to obtain the equation of the required line. | $m = \frac{1}{3}$ and $(x_1, y_1) = (1, 1)$<br>$y - y_1 = m(x - x_1)$<br>$\therefore \quad y - 1 = \frac{1}{3}(x - 1)$<br>$\therefore 3(y - 1) = 1(x - 1)$<br>$\therefore \quad 3y - 3 = x - 1$<br>$\therefore \quad 3y - x = 2$ |
| **3** State the answer. | The required line has equation $3y - x = 2$. |

## Line segments

**eBookplus**

**Interactivity**
Midpoint and perpendicular bisection
**int-2553**

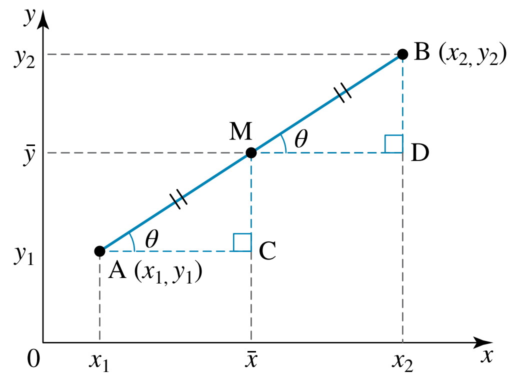
In theory, lines are never-ending and of infinite length; however, sections of lines having two endpoints have finite lengths. These sections are called **line segments**. For simplicity, the notation AB will be used both as the name of the line segment with endpoints A and B and, in context, as the length of that line segment.

## The coordinates of the midpoint of a line segment

The point of bisection of a line segment is its **midpoint**. This point is **equidistant** from the endpoints of the interval.

Let the endpoints of the line segment be $A(x_1, y_1)$ and $B(x_2, y_2)$.

Let the midpoint of AB be the point $M(\bar{x}, \bar{y})$.
Since M bisects AB, AM = MB and the triangles ACM and MDB are congruent (identical).




Equating the 'runs':

$$
\begin{aligned}
AC &= MD \\
\bar{x} - x_1 &= x_2 - \bar{x} \\
2\bar{x} &= x_1 + x_2 \\
\bar{x} &= \frac{x_1 + x_2}{2}
\end{aligned}
$$

and equating the 'rises':

$$
\begin{aligned}
CM &= DB \\
\bar{y} - y_1 &= y_2 - \bar{y} \\
2\bar{y} &= y_1 + y_2 \\
\bar{y} &= \frac{y_1 + y_2}{2}
\end{aligned}
$$

Hence the coordinates of the midpoint of a line segment are found by averaging the coordinates of the endpoints.

**The coordinates of the midpoint of a line segment with endpoints $(x_1, y_1)$ and $(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.**

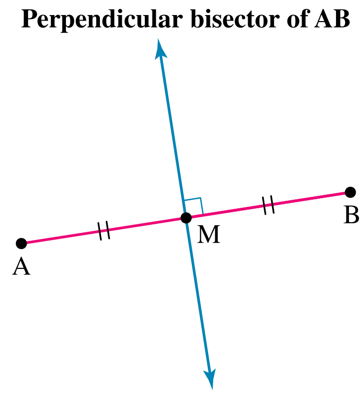
**WORKED EXAMPLE 17**

**Calculate the coordinates of the midpoint of the line segment joining the points (−3, 5) and (7, −8).**

| THINK | WRITE | |
|---|---|---|
| 1 Average the $x$- and the $y$-coordinates of the endpoints. | $\bar{x} = \frac{x_1 + x_2}{2}$<br>$= \frac{(-3) + 7}{2}$<br>$= \frac{4}{2}$<br>$= 2$ | $\bar{y} = \frac{y_1 + y_2}{2}$<br>$= \frac{5 + (-8)}{2}$<br>$= \frac{-3}{2}$<br>$= -1.5$ |
| 2 Write the coordinates as an ordered pair. | Therefore the midpoint is (2, −1.5). | |

## The perpendicular bisector of a line segment

The line which passes through the midpoint of a line segment and at right angles to the line segment is called the **perpendicular bisector** of the line segment.




To find the equation of the perpendicular bisector:

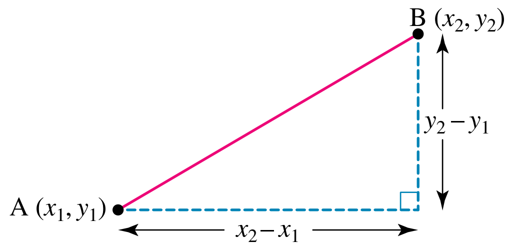
- its gradient is $-\frac{1}{m_{AB}}$ since it is perpendicular to the line segment AB
- the midpoint of the line segment AB also lies on the perpendicular bisector.

**WORKED EXAMPLE 18** Determine the equation of the perpendicular bisector of the line segment joining the points A(6, 3) and B(−8, 5).

| THINK | WRITE |
|---|---|
| 1 Calculate the gradient of the line segment. | A(6, 3) and B(−8, 5)<br>$m = \frac{5-3}{-8-6}$<br>$= \frac{2}{-14}$<br>$= -\frac{1}{7}$ |
| 2 Obtain the gradient of a line perpendicular to the line segment. | Since $m_{AB} = -\frac{1}{7}$, the gradient of a line perpendicular to AB is $m_{\perp} = 7$. |
| 3 Calculate the coordinates of the midpoint of the line segment. | Midpoint of AB:<br>$\bar{x} = \frac{6+(-8)}{2} = \frac{-2}{2} = -1$<br>$\bar{y} = \frac{3+5}{2} = \frac{8}{2} = 4$<br>Midpoint is (−1, 4). |
| 4 Form the equation of the perpendicular bisector using the point–gradient equation. | Point (−1, 4); gradient $m = 7$<br>$y - y_1 = m(x - x_1)$<br>$y - 4 = 7(x + 1)$<br>$y - 4 = 7x + 7$<br>$\therefore \quad y = 7x + 11$ |
| 5 State the answer. | The equation of the perpendicular bisector is $y = 7x + 11$. |

## The length of a line segment

The **length of a line segment** is the distance between its endpoints. For a line segment AB with endpoints A$(x_1, y_1)$ and B$(x_2, y_2)$, the run $x_2 - x_1$ measures the distance between $x_1$ and $x_2$ and the rise $y_2 - y_1$ measures the distance between $y_1$ and $y_2$.




Using Pythagoras' theorem, $(\text{AB})^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

**The length of the line segment is AB $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.**

This could be expressed as the distance-between-two-points formula:
$d(\text{A, B}) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where $d(\text{A, B})$ is a symbol for the distance between the points A and B.

WORKED EXAMPLE 19

**Calculate the length of the line segment joining the points A(−2, −5) and B(1, 3).**

| THINK | WRITE |
|---|---|
| 1 Write the distance formula. | $d(\text{A, B}) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ |
| 2 Substitute the coordinates of the two points. *Note:* It does not matter which point is labelled $(x_1, y_1)$ and which $(x_2, y_2)$. | A(−2, −5) and B(1, 3) Let A be $(x_1, y_1)$ and B be $(x_2, y_2)$. $d(\text{A, B}) = \sqrt{(1 - (-2))^2 + (3 - (-5))^2}$ $= \sqrt{(3)^2 + (8)^2}$ $= \sqrt{9 + 64}$ $= \sqrt{73}$ |
| 3 State the answer. By choice, both the exact surd value and its approximate value to 2 decimal places have been given. *Note:* Always re-read the question to see if the degree of accuracy is specified. | Therefore the length of AB is $\sqrt{73} \simeq 8.54$ units. |

## EXERCISE 1.6 Coordinate geometry of the straight line

**PRACTISE**

Work without CAS
Questions 1–2

1 WE14 Show that the points A(−3, −12), B(0, 3) and C(4, 23) are collinear.

2 For the points P(−6, −8), Q(6, 4) and R(−32, 34), find the equation of the line through P and Q and hence determine if the three points are collinear.

3 a WE15 Calculate, correct to 2 decimal places, the angle made with the positive direction of the $x$-axis by the line which passes through the points $(1, -8)$ and $(5, -2)$.

b Calculate the angle of inclination with the horizontal made by a line which has a gradient of $-2$.

c Obtain the equation of the line which passes through the point $(2, 7)$ at an angle of $135^{\circ}$ to the horizontal.

4 Calculate the angle of inclination with the horizontal made by each of the lines whose gradients are 5 and 4 respectively, and hence find the magnitude of the acute angle between these two lines.

5 a WE16 State the gradient of a line which is:

i parallel to $3y - 6x = 1$

ii perpendicular to $3y - 6x = 1$.

b Show that the lines $y = x$ and $y = -x$ are perpendicular.

c Determine the equation of the line through the point $(1, 1)$ perpendicular to the line $y = 5x + 10$.

6 Find the coordinates of the $x$-intercept of the line which passes through the point $(8, -2)$, and is parallel to the line $2y - 4x = 7$.

7 WE17 Calculate the coordinates of the midpoint of the line segment joining the points $(12, 5)$ and $(-9, -1)$.

8 If the midpoint of PQ has coordinates $(3, 0)$ and Q is the point $(-10, 10)$, find the coordinates of point P.

9 WE18 Determine the equation of the perpendicular bisector of the line segment joining the points $A(-4, 4)$ and $B(-3, 10)$.

10 Given that the line $ax + by = c$ is the perpendicular bisector of the line segment CD where C is the point $(-2, -5)$ and D is the point $(2, 5)$, find the smallest non-negative values possible for the integers $a$, $b$ and $c$.

11 WE19 Calculate the length of the line segment joining the points $(6, -8)$ and $(-4, -5)$.

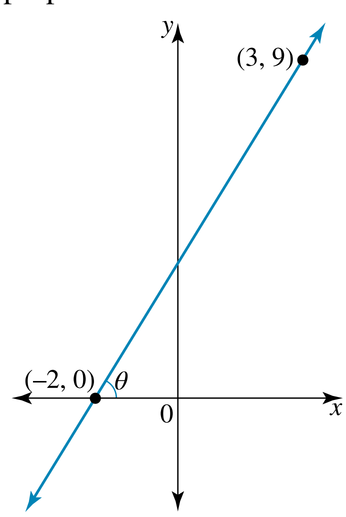
12 Calculate the distance between the point $(3, 10)$ and the midpoint of the line segment AB where A is the point $(-1, 1)$ and B is the point $(6, -1)$. Give the answer correct to 2 decimal places.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

13 Calculate the magnitude of the angle the following lines make with the positive direction of the $x$-axis, expressing your answer correct to 2 decimal places where appropriate.

a




b The line that cuts the $x$-axis at $x = 4$ and the $y$-axis at $y = 3$

c The line that is parallel to the $y$ axis and passes through the point $(1, 5)$

d The line with gradient $-7$

**14** Determine the equation of the line, in the form $ax + by = c$, which:

**a** passes through the point $(0, 6)$ and is parallel to the line $7y - 5x = 0$

**b** passes through the point $\left(-2, \frac{4}{5}\right)$ and is parallel to the line $3y + 4x = 2$

**c** passes through the point $\left(-\frac{3}{4}, 1\right)$ and is perpendicular to the line $2x - 3y + 7 = 0$

**d** passes through the point $(0, 0)$ and is perpendicular to the line $3x - y = 2$

**e** passes through the point $(-6, 12)$ making an angle of $\tan^{-1}(1.5)$ with the horizontal

**f** passes through the point of intersection of the lines $2x - 3y = 18$ and $5x + y = 11$, and is perpendicular to the line $y = 8$.

**15** Given the points A$(-7, 2)$ and B$(-13, 10)$, obtain:

**a** the distance between the points A and B

**b** the coordinates of the midpoint of the line segment AB

**c** the equation of the perpendicular bisector of AB

**d** the coordinates of the point where the perpendicular bisector meets the line $3x + 4y = 24$.

**16** A line $L$ cuts the $x$-axis at the point A where $x = 4$, and is inclined at an angle of 123.69° to the positive direction of the $x$-axis.

**a** Form the equation of the line $L$ specifying its gradient to 1 decimal place.

**b** Form the equation of a second line, $K$, which passes through the same point A at right angles to the line $L$.

**c** What is the distance between the $y$-intercepts of $K$ and $L$?

**17** **a** Determine whether the points A$(-4, 13)$, B$(7, -9)$ and C$(12, -19)$ are collinear.

**b** Explain whether or not the points A$(-15, -95)$, B$(12, 40)$ and C$(20, 75)$ may be joined to form a triangle.

**c** Given the points A$(3, 0)$, B$(9, 4)$, C$(5, 6)$ and D$(-1, 2)$, show that AC and BD bisect each other.

**d** Given the points P$(-2, -3)$, Q$(2, 5)$, R$(6, 9)$ and S$(2, 1)$, show that PQRS is a parallelogram. Is PQRS a rectangle?

**18** Triangle CDE has vertices C$(-8, 5)$, D$(2, 4)$ and E$(0.4, 0.8)$.

**a** Calculate its perimeter to the nearest whole number.

**b** Show that the magnitude of angle CED is 90°.

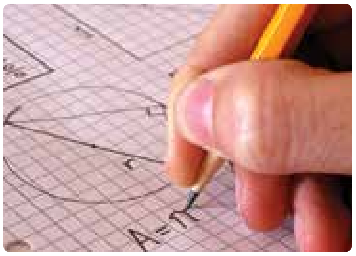
**c** Find the coordinates of M, the midpoint of its hypotenuse.

**d** Show that M is equidistant from each of the vertices of the triangle.

**19** A circle has its centre at $(4, 8)$ and one end of the diameter at $(-2, -2)$.

**a** Specify the coordinates of the other end of the diameter.

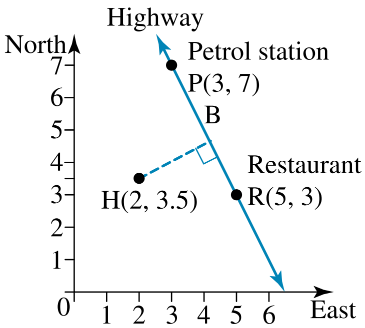
**b** Calculate the area of the circle as a multiple of $\pi$.



20 a Find the value of $a$ so that the line $ax - 7y = 8$ is:
i parallel to the line $3y + 6x = 7$
ii perpendicular to the line $3y + 6x = 7$.
b Find the value of $b$ if the three points $(3, b)$, $(4, 2b)$ and $(8, 5 - b)$ are collinear.
c Find the value of $c$ if the line through the points $(2c, -c)$ and $(c, -c - 2)$ makes an angle of 45° with the horizontal.
d Find the value of $d$ so the line containing the points $(d + 1, d - 1)$ and $(4, 8)$ is:
i parallel to the line which cuts the $x$-axis at $x = 7$ and the $y$-axis at $y = -2$
ii parallel to the $x$-axis
iii perpendicular to the $x$-axis.
e If the distance between the points $(p, 8)$ and $(0, -4)$ is 13 units, find two possible values for $p$.
f The angle between the two lines with gradients $-1.25$ and $0.8$ respectively has the magnitude $\alpha°$. Calculate the value of $\alpha$.

21 Two friends planning to spend some time bushwalking argue over which one of them should carry a rather heavy rucksack containing food and first aid items. Neither is keen so they agree to each throw a small coin towards the base of a tree and the person whose coin lands the greater distance from the tree will have to carry the rucksack. Taking the tree as the origin, and the distances in centimetres east and north of the origin as $(x, y)$ coordinates, Anna's coin lands at $(-2.3, 1.5)$ and Liam's coin lands at $(1.7, 2.1)$. Who carries the rucksack, Anna or Liam? Support your answer with a mathematical argument.

22 The diagram shows a main highway through a country town. The section of this highway running between a petrol station at P and a restaurant at R can be considered a straight line. Relative to a fixed origin, the coordinates of the petrol station and restaurant are P(3, 7) and R(5, 3) respectively. Distances are measured in kilometres.




a How far apart are the petrol station and restaurant? Answer to 1 decimal place.
b Form the equation of the straight line PR.

Ada is running late for her waitressing job at the restaurant. She is still at home at the point H(2, 3.5).
There is no direct route to the restaurant from her home, but there is a bicycle track that goes straight to the nearest point B on the highway from her home. Ada decides to ride her bike to point B and then to travel along the highway from B to the restaurant.


c Form the equation of the line through H perpendicular to PR.

**d** Hence find the coordinates of B, the closest point on the highway from her home.

**e** If Ada's average speed is 10 km/h, how long, to the nearest minute, does it take her to reach the restaurant from her home?

**MASTER**

For questions **23a** and **24a**, use the geometry facility on CAS technology to draw a triangle.

**23** **a** Construct the perpendicular bisectors of each of the three sides of the triangle. What do you notice? Repeat this procedure using other triangles. Does your observation appear to hold for these triangles?

**b** For the triangle formed by joining the points O(0, 0), A(6, 0), B(4, 4), find the point of intersection of the perpendicular bisectors of each side. Check your answer algebraically.

**24** **a** Construct the line segments joining each vertex to the midpoint of the opposite side (these are called medians). What do you notice? Repeat this procedure using other triangles. Does your observation appear to hold for these triangles?

**b** For the triangle formed by joining the points O(0, 0), A(6, 0) and B(4, 4), find the point of intersection of the medians drawn to each side. Check your answer algebraically.

**René Descartes, seventeenth century French mathematician and philosopher, was one of the first to combine algebra and geometry together as coordinate geometry in his work *La Géométrie*.**

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# 1.7 Review

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- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

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studyon

Units 1 & 2

**Lines and linear relationships**

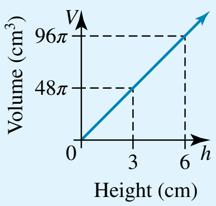
**Sit topic test**

# 1 Answers

## EXERCISE 1.2

**1 a** $k = 16\pi$

**b** Height 3 cm;




**2 a** $C = 210 + 6n$

**b** \$360

**3 a** $x = -1$

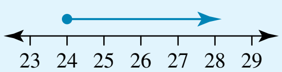
**b** $x = 10$

**4** $x = -\frac{1}{7}$

**5** $x = a + d$

**6** $x = c$

**7** $x \geq 24$




**8** $x > -\frac{1}{6}$

**9 a** $x = 7, y = 1$

**b** $x = -4, y = -2$

**10 a** $x = -\frac{7}{3}, y = -\frac{35}{3}$

**b** $x = 1, y = 0$

**11 a** Profit for sale of $n$ books is $P = 2.3n - 100$

**b** 44 books

**c** Hardcovers \$5; paperbacks \$2

**12** 22.5 minutes

**13 a** $T = 3.5x$; tension is 1.05 newton

**b** $C = 1.45l$, 30 litres

**c** $v = 12 - 9.8t$; approximately 1.22 seconds

**14 a** Interest is directly proportional to number of years invested, $I = 46T$

**b** Not linearly related

**c** Wage is a fixed amount plus an amount proportional to the number of hours of overtime; $W = 400 + 50n$

**15 a** $x = 9$ **b** $x = 20$

**c** $x = 6$ **d** $x = \frac{1}{3}$

**e** $x = 1$ **f** $x = -10$

**16 a** $x = \dfrac{c - b}{a}$ **b** $x = \dfrac{ab}{a - b}$

**c** $x = -1$ **d** $x = ab$

**e** $x = \dfrac{a}{b - c}$ **f** $x = b - a$

**17 a** $x \leq -1$ **b** $x > -9$

**c** $x \leq 18$ **d** $x > 3$

**e** $x \geq \frac{2}{3}$ **f** $x < -6$

**18 a** $x = 1, y = 4$

**b** $x = -2, y = 2$

**c** $x = -\frac{1}{2}, y = -11$

**d** $x = \frac{3}{4}, y = \frac{2}{3}$

**e** $x = 12, y = 6$

**f** $x = 3b, y = a$

**19 a** 8 and 10

**b** 21

**c** 12, 13 and 14

**d** Length is 20 cm; width is 4 cm

**e** Height is 45 cm

**20 a** $t = 2.5$; distance is 150 km

**b** $\dfrac{10u}{9}$

**21 a** Adult ticket costs \$25; child ticket costs \$12

**b** \$471.09

**c** $C = \frac{5}{9}(F - 32)$

**22 a** $12 \leq n \leq 16$ where the whole number $n$ is the number of people

**b** 15 people attended at a cost of \$27.90 per person

**c** 10 drank coffee, 5 drank tea

**23 a** $x = 0$

**b** $x > 4$

**c** $x = 5, y = -1$

**d** $x = -82, y = -120$

**24 a** $C = 10T - 97$

**b** $x = \dfrac{c}{a + b}, y = \dfrac{1}{a + b}$

## EXERCISE 1.3

**1** $x = 5, y = 8, z = -6$

**2** $x = 1, y = -2, z = 3$

**3** $x = 2, y = 1, z = 4$

**4** $x = 4, y = 2, z = -1$

**5** $x = -6, y = 8, z = 1$

**6** $x = -2 \;\; y = 5 \;\; z = 10$

**7** $x = 4, y = 3, z = 2$

**8** $x = -2, y = -11, z = 40$

**9** Adult ticket \$14; concession ticket \$12; child ticket \$10

**10** Agnes \$20 per hour; Bjork \$18 per hour; Chi \$25 per hour

11 The food compound requires 50 kg of supplement X, 30 kg of supplement Y and 10 kg of supplement Z.

12 Two 50-cent coins, twelve 20-cent coins and eight 10-cent coins

13 a $x = 3, y = 1.5, z = -2.6$

b $x = 10, y = -6, z = 0.5, w = 5$

c $x_1 = -2, x_2 = -4, x_3 = \frac{2}{3}, x_4 = \frac{1}{6}$

14 a $z = -4 + 0.1x + 0.075y$

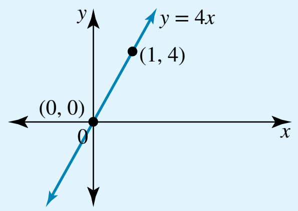
b 15 kg/hectare

## EXERCISE 1.4

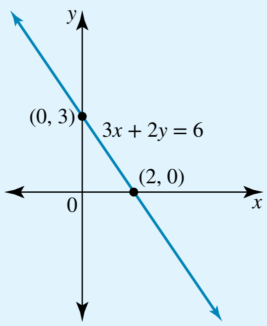
1 $m = -\frac{4}{3}$

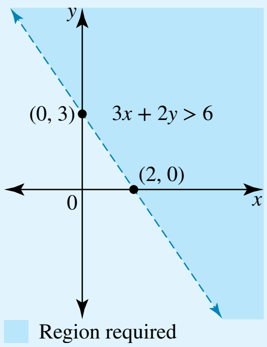
2 Show both gradients equal 1

3 a

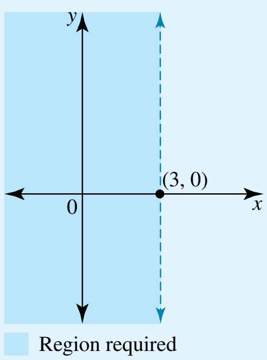



b

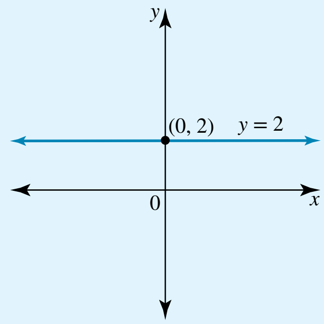



c




d




4 $y < 4.4 - 10x$
Point lies below the line.

5 a $y = -2x - 13$

b $7y + 2x = 1$

c $y = -1.5x + 6$

d $m = \frac{5}{6}$; $(0, 3)$

6 $y = 10$

7 a $m = -2.5$

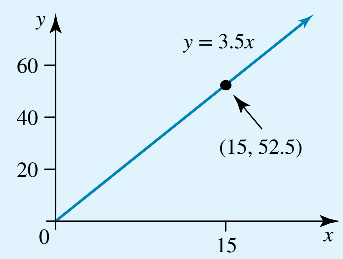
b $m = 5$

c $m = 0$

d Undefined

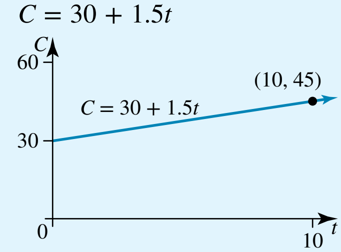
8 a i $y = 3.5x$

ii

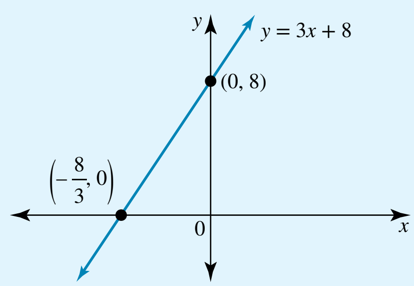



b i $C = 30 + 1.5t$

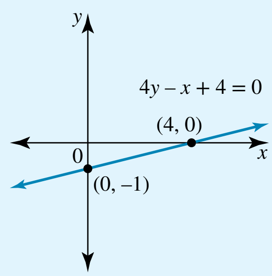
ii



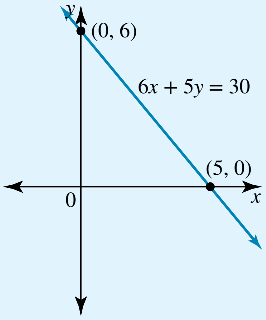

9 a



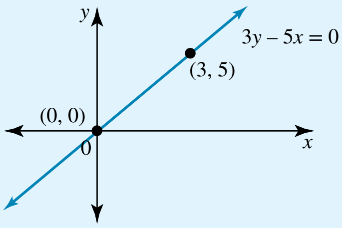

b



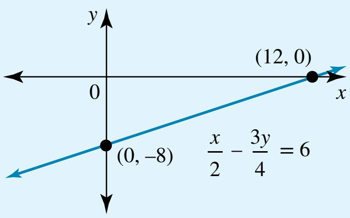
**c**



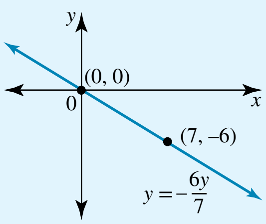

**d**



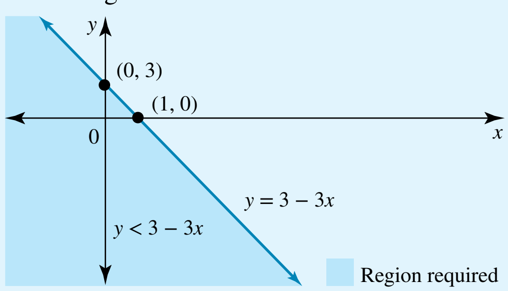

**e**



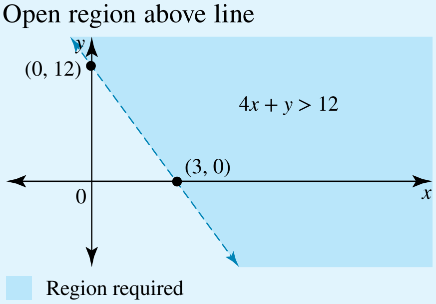

**f**



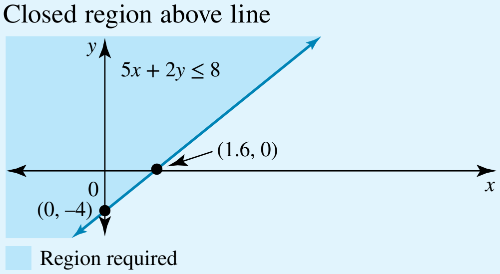

**10 a** Closed region below line



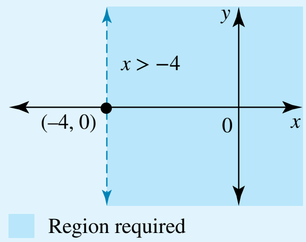

**b** Open region above line



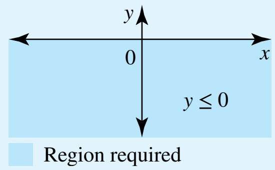

**c** Closed region above line



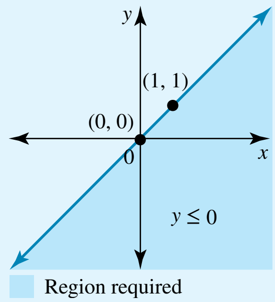

**d** Open region to right of vertical line




**e** Closed region below $x$-axis




**f** Closed region below line




**11 a** $y = -5x + 37$ **b** $3y - 2x + 10 = 0$

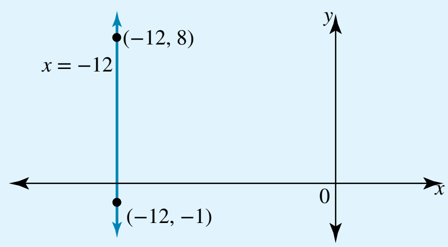
**c** $4y + 7x + 36 = 0$ **d** $y = -0.8x + 0.2$

**e** $3y - 10x = 34$ **f** $y = -2x + 10$

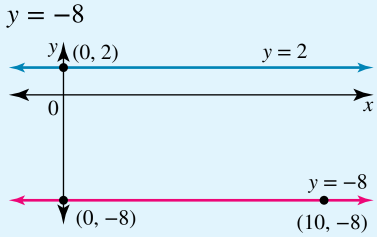
**12 a** **i** $y = \dfrac{5x}{4}$ **ii** $y = -3x + 9$

**iii** $y = \frac{2}{3}x - 2$ **iv** $y = -\frac{1}{2}x - 1$

**b** $x = -12$




**c** $y = -8$




**13 a** $m = -\dfrac{4}{5}, c = 4$ **b** $m = \dfrac{8}{3}, c = 20$

**c** $m = \dfrac{1}{6}, c = \dfrac{3}{2}$ **d** $m = 0, c = \dfrac{3}{2}$

**14 a** $a = -6$

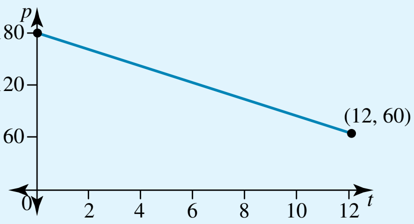
**b** Above the line

**c** $y = \dfrac{q}{p}x$

**15 a** $p = 180 - kt$

**b** 12 minutes

**c** $p = 180 - 10t$




**d** Gradient $= -10$

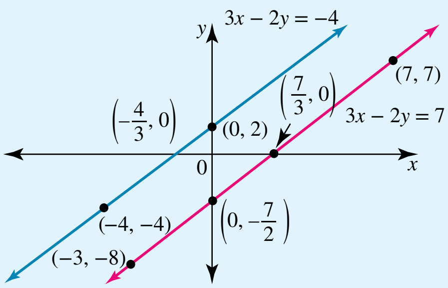
**16 a** $m = \frac{3}{2}$

**b** Proof required — check with your teacher

**c** $3x - 2y = 7$

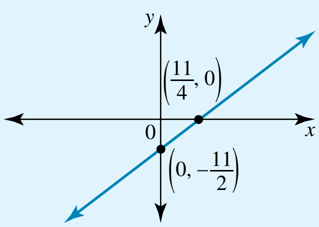
**d** $3x - 2y = -4$

**e**

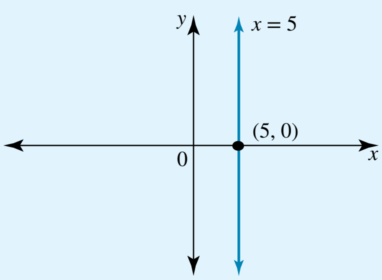



**f** $3x - 2y \le 7$ and $3x - 2y \ge -4$, that is $-4 \le 3x - 2y \le 7$

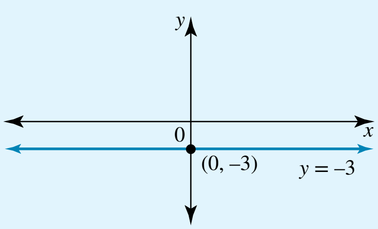
**17 a**



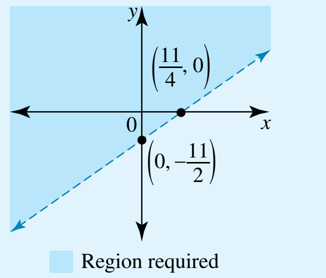

**b**



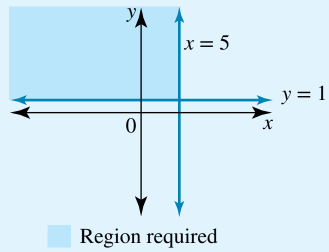

**c**




**18 a**

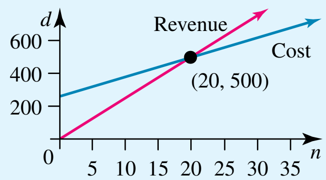



**b**




## EXERCISE 1.5

**1 a** (20, 500)




**b** At least 21 items

**2** (12, 10.5)

**3** $m = 6$

**4** $a = -1.5, b = -2$

**5** Concurrent; point of concurrency at $(3, -2)$

**6** $a = 10$

**7 a** $(-0.5, -5)$ **b** $(8, -3)$ **c** $(7, -5)$

**8** $(-1, 2)$

**9 a** $(-2, 6), (3, 6), (3, -9)$

**b** 37.5 square units

**10** $p = 10.5$

**11 a** $y = -\frac{px}{5} + \frac{q}{5}, \; y = \frac{3x}{q} - 5$

**b** $p = 0.6, \; q = -25$

**c** $pq \neq -15$

**12 a** $(-2, -3)$

**b** Any real number except for $d = 10$

**13 a** $C_1 = 10 + 0.75x$, $C_2 = 20$ where $C$ is the cost and $x$ the distance

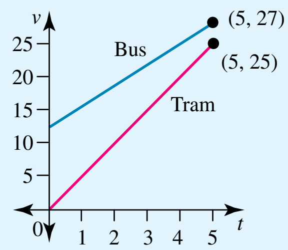
**b** Graphs required

**c** $13\frac{1}{3}$ km

**d** If the distance is less than $13\frac{1}{3}$ km, Pedal On is cheaper; if the distance exceeds $13\frac{1}{3}$ km, Bikes R Gr8 is cheaper.

**14 a** $s_T = 5t,\ s_B = 12 + 3t$

**b**



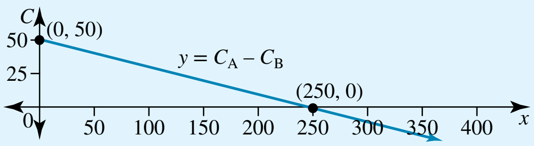

**c** After 6 minutes the tram has the faster speed.

**15 a** $C_A = 300 + 0.05x,\ C_B = 250 + 0.25x$

**b** Cost per kilometre of travel

**c** $y = 50 - 0.2x$

**d**




**e** **i** 250 km

**ii** More than 250 km

**16 a** 6 am (0, 2); 7 am (6, 3); $y = \dfrac{x}{6} + 2$

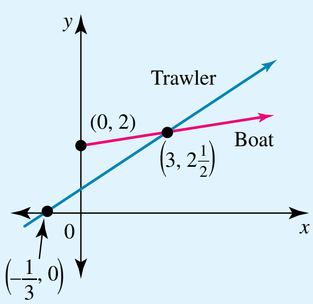
**b** Proof required — check with your teacher

**c** (21, 5.5)

**d** 6 am $\left(-\frac{1}{3}, 0\right)$; 7 am (1, 1); $y = \dfrac{3x}{4} + \dfrac{1}{4}$

**e** (3, 2.5)

**f** No collision since boat is at common point at 6.30 am and trawler at 8.30 am




**17** (1.39, 5.91)

**18 a** No collision

**b** (4, 11)

## EXERCISE 1.6

**1** Proof required — check with your teacher

**2** $y = x - 2$; not collinear

**3 a** 56.31°

**b** 116.57°

**c** $y = -x + 9$

**4** 78.69°, 75.96°, 2.73°

**5 a** **i** $m = 2$ **ii** $m = -0.5$

**b** $m_1 m_2 = -1$

**c** $5y + x = 6$

**6** (9, 0)

**7** (1.5, 2)

**8** P(16, −10)

**9** $2x + 12y = 77$

**10** $a = 2, b = 5, c = 0$

**11** $\sqrt{109} \approx 10.44$

**12** 10.01

**13 a** 143.13° **b** 90° **c** 98.13°

**14 a** $-5x + 7y = 42$ **b** $20x + 15y = -28$

**c** $12x + 8y = -1$ **d** $x + 3y = 0$

**e** $3x - 2y = -42$ **f** $x = 3$

**15 a** 10 units **b** (−10, 6)

**c** $4y - 3x = 54$ **d** $\left(-5, 9\frac{3}{4}\right)$

**16 a** $y = -1.5x + 6$

**b** $2x - 3y = 8$

**c** $\frac{26}{3}$ units

**17 a** Collinear

**b** Not collinear so triangle can be formed

**c** (4, 3) is midpoint of both AC and BD.

**d** $m_{PQ} = m_{SR} = 2,\ m_{PS} = m_{QR} = 1$
PQRS is a parallelogram as opposite sides are parallel.
Adjacent sides are not perpendicular, so PQRS is not a rectangle.

**18 a** 23 units

**b** Proof required — check with your teacher

**c** (−3, 4.5)

**d** $ME = MC = MD = \sqrt{25.25}$

**19 a** (10, 18) **b** $136\pi$ square units

**20 a** **i** $a = -14$ **ii** $a = 3.5$

**b** $b = \dfrac{5}{7}$

**c** $c = 2$

**d** **i** $d = 11.4$ **ii** $d = 9$ **iii** $d = 3$

**e** $p = \pm 5$

**f** $\alpha = 90$

**21** Anna

**22 a** 4.5 km **b** $y = -2x + 13$

**c** $y = 0.5x + 2.5$ **d** (4.2, 4.6)

**e** 26 minutes

**23 a** Perpendicular bisectors are concurrent.

**b** (3, 1)

**24 a** Medians are concurrent.

**b** $\left(\frac{10}{3}, \frac{4}{3}\right)$

# 2

# Algebraic foundations

# 2.1 Kick off with CAS

## Playing lotto

1 Using CAS technology, calculate the following products.

a $3 \times 2 \times 1$

b $5 \times 4 \times 3 \times 2 \times 1$

c $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

d $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

2 Using CAS technology, find the symbol ! and evaluate the following:

a 3!

b 5!

c 7!

d 10!

This symbol is called factorial.

3 Compare the answers to questions 1 and 2.

4 Using the factorial symbol or another method, in how many ways can:

a one number be arranged

b two numbers be arranged

c three numbers be arranged

d six numbers be arranged

e nine numbers be arranged?

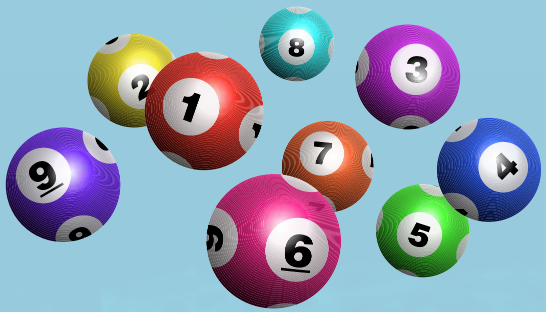
5 Using the factorial symbol or another method, answer the following.

a How many 2-digit numbers can be formed from 6 different numbers?

b How many 3-digit numbers can be formed from 5 different numbers?

c How many 4-digit numbers can be formed from 10 different numbers?

6 In the game of lotto, how many different combinations of 6 numbers can be chosen from 45 numbers?



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 2.2 Algebraic skills

**studyon**

Units 1 & 2
AOS 2
Topic 1
Concept 1

**Algebraic skills**
Concept summary
Practice questions

This chapter covers some of the algebraic skills required as the foundation to learning and understanding of Mathematical Methods. Some basic algebraic techniques will be revised and some new techniques will be introduced.

## Review of factorisation and expansion

### Expansion

The **Distributive Law** $a(b + c) = ab + ac$ is fundamental in expanding to remove brackets.

Some simple **expansions** include:

$$(a + b)(c + d) = ac + ad + bc + bd$$
$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

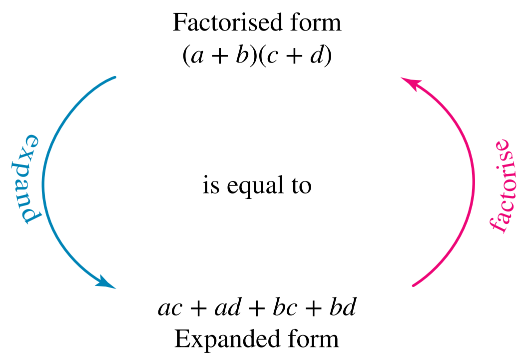
**WORKED EXAMPLE 1** **Expand $2(4x - 3)^2 - (x - 2)(x + 2) + (x + 5)(2x - 1)$ and state the coefficient of the term in $x$.**

| THINK | WRITE |
|---|---|
| 1 Expand each pair of brackets.<br>*Note:* The first term contains a perfect square, the second a difference of two squares and the third a quadratic trinomial. | $2(4x - 3)^2 - (x - 2)(x + 2) + (x + 5)(2x - 1)$<br>$= 2(16x^2 - 24x + 9) - (x^2 - 4) + (2x^2 - x + 10x - 5)$ |
| 2 Expand fully, taking care with signs. | $= 32x^2 - 48x + 18 - x^2 + 4 + 2x^2 + 9x - 5$ |
| 3 Collect like terms together. | $= 33x^2 - 39x + 17$ |
| 4 State the answer.<br>*Note:* Read the question again to ensure the answer given is as requested. | The expansion gives $33x^2 - 39x + 17$ and the coefficient of $x$ is $-39$. |

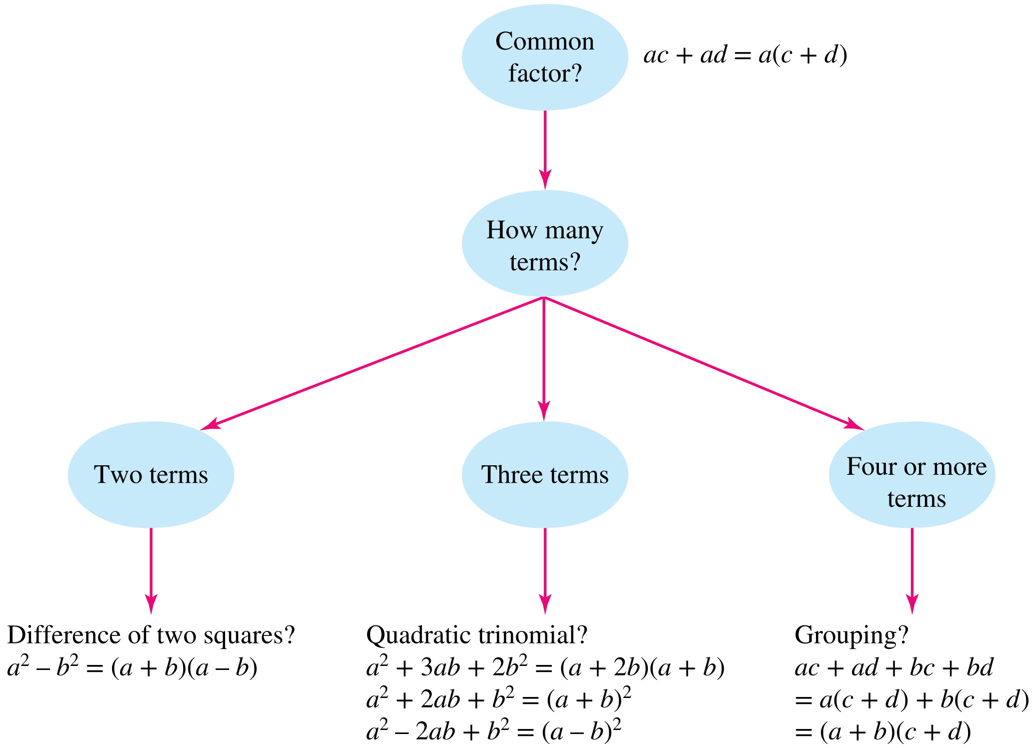
### Factorisation

Some simple **factors** include:

- common factor
- difference of two perfect squares
- perfect squares and factors of other quadratic trinomials.



A systematic approach to factorising is displayed in the following diagram.




Grouping terms commonly referred to as grouping '2 and 2' and grouping '3 and 1' depending on the number of terms grouped together, are often used to create factors. For example, as the first three terms of $a^2 + 2ab + b^2 - c^2$ are a perfect square, grouping '3 and 1' would create a difference of two squares expression, allowing the whole expression to be factorised.

$$a^2 + 2ab + b^2 - c^2$$
$$= (a^2 + 2ab + b^2) - c^2$$
$$= (a + b)^2 - c^2$$

This factorises to give $(a + b - c)(a + b + c)$.

**WORKED EXAMPLE 2**

**Factorise:**

**a** $2x^3 + 5x^2y - 12y^2x$ **b** $4y^2 - x^2 + 10x - 25$

**c** $7(x + 1)^2 - 8(x + 1) + 1$ **using the substitution** $a = (x + 1)$.

| THINK | WRITE |
|---|---|
| **a 1** Take out the common factor. | **a** $2x^3 + 5x^2y - 12y^2x$<br>$= x(2x^2 + 5xy - 12y^2)$ |
| **2** Factorise the quadratic trinomial. | $= x(2x - 3y)(x + 4y)$ |
| **b 1** The last three terms of the expression can be grouped together to form a perfect square. | **b** $4y^2 - x^2 + 10x - 25$<br>$= 4y^2 - (x^2 - 10x + 25)$ |

2. Use the grouping '3 and 1' technique to create a difference of two squares. $= 4y^2 - (x-5)^2$
$= (2y)^2 - (x-5)^2$

3. Factorise the difference of two squares. $= [2y - (x-5)][2y + (x-5)]$

4. Remove the inner brackets to obtain the answer. $= (2y - x + 5)(2y + x - 5)$

**c** 1. Substitute $a = (x+1)$ to form a quadratic trinomial in $a$.
**c** $7(x+1)^2 - 8(x+1) + 1 = 7a^2 - 8a + 1$ where $a = (x+1)$

2. Factorise the quadratic trinomial. $= (7a - 1)(a - 1)$

3. Substitute $(x+1)$ back in place of $a$ and simplify. $= (7(x+1) - 1)((x+1) - 1)$

4. Remove the inner brackets and simplify to obtain the answer.
$= (7x + 7 - 1)(x + 1 - 1)$
$= (7x + 6)(x)$
$= x(7x + 6)$

## Factorising sums and differences of two perfect cubes

Check the following by hand or by using a CAS technology.

Expanding $(a+b)(a^2 - ab + b^2)$ gives $a^3 + b^3$, the sum of two cubes; and expanding $(a-b)(a^2 + ab + b^2)$ gives $a^3 - b^3$, the difference of two cubes.

Hence the factors of the sum and difference of two cubes are:

$$\mathbf{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$
$$\mathbf{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$$

**WORKED EXAMPLE 3**

**Factorise:**

**a** $x^3 - 27$ **b** $2x^3 + 16$.

| THINK | WRITE |
|---|---|
| **a** 1. Express $x^3 - 27$ as a difference of two cubes. | **a** $x^3 - 27 = x^3 - 3^3$ |
| 2. Apply the factorisation rule for the difference of two cubes. | Using $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ with $a = x,\ b = 3$, $x^3 - 3^3 = (x-3)(x^2 + 3x + 3^2)$ |
| 3. State the answer. | $\therefore x^3 - 27 = (x-3)(x^2 + 3x + 9)$ |
| **b** 1. Take out the common factor. | **b** $2x^3 + 16$ $= 2(x^3 + 8)$ |
| 2. Express $x^3 + 8$ as a sum of two cubes. | $= 2(x^3 + 2^3)$ |
| 3. Apply the factorisation rule for the sum of two cubes. | Using $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = x,\ b = 2$, $x^3 + 2^3 = (x+2)(x^2 - 2x + 2^2)$ $2(x^3 + 2^3) = 2(x+2)(x^2 - 2x + 2^2)$ |
| 4. State the answer. | $\therefore 2x^3 + 16 = 2(x+2)(x^2 - 2x + 4)$ |

## Algebraic fractions

Factorisation techniques may be used in the simplification of algebraic fractions under the arithmetic operations of multiplication, division, addition and subtraction.

### Multiplication and division of algebraic fractions

An algebraic fraction can be simplified by cancelling any common factor between its numerator and its denominator. For example:

$$\frac{ab + ac}{ad} = \frac{\cancel{a}(b + c)}{\cancel{a}d}$$
$$= \frac{b + c}{d}$$

For the product of algebraic fractions, once numerators and denominators have been factorised, any common factors can then be cancelled. The remaining numerator terms are usually left in factors, as are any remaining denominator terms. For example:

$$\frac{\cancel{a}(b + c)}{\cancel{a}\cancel{d}} \times \frac{\cancel{d}(a + c)}{b} = \frac{(b + c)(a + c)}{b}$$

Note that $b$ is not a common factor of the numerator so it cannot be cancelled with the $b$ in the denominator.

As in arithmetic, to divide by an algebraic fraction, multiply by its reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

**WORKED EXAMPLE 4**

**Simplify:**

**a** $\dfrac{x^2 - 2x}{x^2 - 5x + 6}$  **b** $\dfrac{x^4 - 1}{x - 3} \div \dfrac{1 + x^2}{3 - x}$

| THINK | WRITE |
|---|---|
| **a 1** Factorise both the numerator and the denominator. *Note:* The numerator has a common factor; the denominator is a quadratic trinomial. | **a** $\dfrac{x^2 - 2x}{x^2 - 5x + 6} = \dfrac{x(x - 2)}{(x - 3)(x - 2)}$ |
| **2** Cancel the common factor in the numerator and denominator. | $= \dfrac{x\cancel{(x - 2)}}{(x - 3)\cancel{(x - 2)}}$ |
| **3** Write the fraction in its simplest form. | $= \dfrac{x}{x - 3}$ No further cancellation is possible. |
| **b 1** Change the division into multiplication by replacing the divisor by its reciprocal. | **b** $\dfrac{x^4 - 1}{x - 3} \div \dfrac{1 + x^2}{3 - x} = \dfrac{x^4 - 1}{x - 3} \times \dfrac{3 - x}{1 + x^2}$ |

2 Factorise where possible.

*Note:* The aim is to create common factors of both the numerator and denominator. For this reason, write $(3 - x)$ as $-(x - 3)$.

Since $x^4 - 1 = (x^2)^2 - 1^2$
$= (x^2 - 1)(x^2 + 1)$

then:

$$\frac{x^4 - 1}{x - 3} \times \frac{3 - x}{1 + x^2}$$
$$= \frac{(x^2 - 1)(x^2 + 1)}{x - 3} \times \frac{-(x - 3)}{1 + x^2}$$
$$= \frac{(x^2 - 1)\cancel{(x^2 + 1)}}{\cancel{x - 3}} \times \frac{-\cancel{(x - 3)}}{\cancel{1 + x^2}}$$

3 Cancel the two sets of common factors of the numerator and denominator.

$$= \frac{(x^2 - 1)}{1} \times \frac{-1}{1}$$

4 Multiply the remaining terms in the numerator together and the remaining terms in the denominator together.

$$= \frac{-(x^2 - 1)}{1}$$

5 State the answer.

*Note:* The answer could be expressed in different forms, including as a product of linear factors, but this is not necessary as it does not lead to any further simplification.

$$= -(x^2 - 1)$$
$$= 1 - x^2$$

## Addition and subtraction of algebraic fractions

Factorisation and expansion techniques are often required when adding or subtracting algebraic fractions.

- Denominators should be factorised in order to select the lowest common denominator.
- Express each fraction with this lowest common denominator.
- Simplify by expanding the terms in the numerator and collect any like terms together.

**WORKED EXAMPLE 5** Simplify $\frac{2}{3x + 3} - \frac{1}{x - 2} + \frac{x}{x^2 - x - 2}$.

**THINK**

1 Factorise each denominator.

**WRITE**

$$\frac{2}{3x + 3} - \frac{1}{x - 2} + \frac{x}{x^2 - x - 2}$$
$$= \frac{2}{3(x + 1)} - \frac{1}{(x - 2)} + \frac{x}{(x + 1)(x - 2)}$$

2 Select the lowest common denominator and express each fraction with this as its denominator.

$$= \frac{2 \times (x - 2)}{3(x + 1)(x - 2)} - \frac{1 \times 3(x + 1)}{3(x + 1)(x - 2)} + \frac{x \times 3}{3(x + 1)(x - 2)}$$

3 Combine the fractions into one fraction. $= \frac{2(x-2) - 3(x+1) + 3x}{3(x+1)(x-2)}$

4 Expand the terms in the numerator. $= \frac{2x - 4 - 3x - 3 + 3x}{3(x+1)(x-2)}$

*Note:* It is not necessary to expand the denominator terms.

5 Collect like terms in the numerator and state the answer. $= \frac{2x-7}{3(x+1)(x-2)}$

*Note:* Since there are no common factors between the numerator and the denominator, the fraction is in its simplest form.

## EXERCISE 2.2 Algebraic skills

**PRACTISE**

Work without CAS

1 WE1 Expand $3(2x+1)^2 + (7x+11)(7x-11) - (3x+4)(2x-1)$ and state the coefficient of the term in $x$.

2 Expand $(2+3x)(x+6)(3x-2)(6-x)$.

3 WE2 Factorise:

a $4x^3 - 8x^2y - 12y^2x$

b $9y^2 - x^2 - 8x - 16$

c $4(x-3)^2 - 3(x-3) - 22$ using the substitution $a = (x-3)$.

4 Factorise $x^2 - 6x + 9 - xy + 3y$.

5 WE3 Factorise the following.

a $x^3 - 125$

b $3 + 3x^3$

6 Factorise $2y^4 + 2y(x-y)^3$.

7 WE4 Simplify:

a $\frac{x^2+4x}{x^2+2x-8}$

b $\frac{x^4-64}{5-x} \div \frac{x^2+8}{x-5}$.

8 Simplify $\frac{x^3-125}{x^2-25} \times \frac{5}{x^3+5x^2+25x}$.

9 WE5 Simplify $\frac{6}{5x-25} + \frac{1}{x-1} - \frac{2x}{x^2-6x+5}$.

10 Simplify $\left(\frac{4}{x+1} - \frac{3}{(x+1)^2}\right) \div \frac{16x^2-1}{x^2+2x+1}$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

This exercise should be attempted by hand rather than by using CAS technology.

11 Expand each of the following expressions.

a $(2x+3)^2$

b $4a(b-3a)(b+3a)$

c $10 - (c+2)(4c-5)$

d $(5-7y)^2$

e $(3m^3+4n)(3m^3-4n)$

f $(x+1)^3$

12 Expand the following.

a $(g+12+h)^2$

b $(2p+7q)^2(7q-2p)$

c $(x+10)(5+2x)(10-x)(2x-5)$

13 Expand and simplify, and state the coefficient of the term in $x$.

a $2(2x - 3)(x - 2) + (x + 5)(2x - 1)$

b $(2 + 3x)(4 - 6x - 5x^2) - (x - 6)(x + 6)$

c $(4x + 7)(4x - 7)(1 - x)$

d $(x + 1 - 2y)(x + 1 + 2y) + (x - 1)^2$

e $(3 - 2x)(2x + 9) - 3(5x - 1)(4 - x)$

f $x^2 + x - 4(x^2 + x - 4)$

14 Factorise each of the following expressions.

a $x^2 + 7x - 60$

b $4a^2 - 64$

c $2bc + 2b + 1 + c$

d $15x + 27 - 2x^2$

e $1 - 9(1 - m)^2$

f $8x^2 - 48xy + 72y^2$

15 Fully factorise the following.

a $x^3 + 2x^2 - 25x - 50$

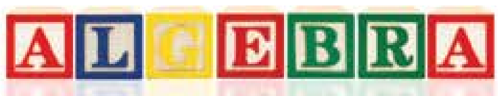
b $100p^3 - 81pq^2$

c $4n^2 + 4n + 1 - 4p^2$

d $49(m + 2n)^2 - 81(2m - n)^2$

e $13(a - 1) + 52(1 - a)^3$

f $a^2 - b^2 - a + b + (a + b - 1)^2$




16 Use a substitution method to factorise the following.

a $(x + 5)^2 + (x + 5) - 56$

b $2(x + 3)^2 - 7(x + 3) - 9$

c $70(x + y)^2 - y(x + y) - 6y^2$

d $x^4 - 8x^2 - 9$

e $9(p - q)^2 + 12(p^2 - q^2) + 4(p + q)^2$

f $a^2\left(a + \frac{1}{a}\right)^2 - 4a^2\left(a + \frac{1}{a}\right) + 4a^2$

17 Factorise the following.

a $x^3 - 8$

b $x^3 + 1000$

c $1 - x^3$

d $27x^3 + 64y^3$

e $x^4 - 125x$

f $(x - 1)^3 + 216$

18 Fully factorise the following.

a $24x^3 - 81y^3$

b $8x^4y^4 + xy$

c $125(x + 2)^3 + 64(x - 5)^3$

d $2(x - y)^3 - 54(2x + y)^3$

e $a^5 - a^3b^2 + a^2b^3 - b^5$

f $x^6 - y^6$

19 Simplify the following.

a $\dfrac{3x^2 - 7x - 20}{25 - 9x^2}$

b $\dfrac{x^3 + 4x^2 - 9x - 36}{x^2 + x - 12}$

c $\dfrac{(x + h)^3 - x^3}{h}$

d $\dfrac{2x^2}{9x^3 + 3x^2} \times \dfrac{1 - 9x^2}{18x^2 - 12x + 2}$

e $\dfrac{m^3 - 2m^2n}{m^3 + n^3} \div \dfrac{m^2 - 4n^2}{m^2 + 3mn + 2n^2}$

f $\dfrac{1 - x^3}{1 + x^3} \times \dfrac{1 - x^2}{1 + x^2} \div \dfrac{1 + x + x^2}{1 - x + x^2}$

**20** Simplify the following expressions.

**a** $\dfrac{4}{x^2+1}+\dfrac{4}{x-x^2}$

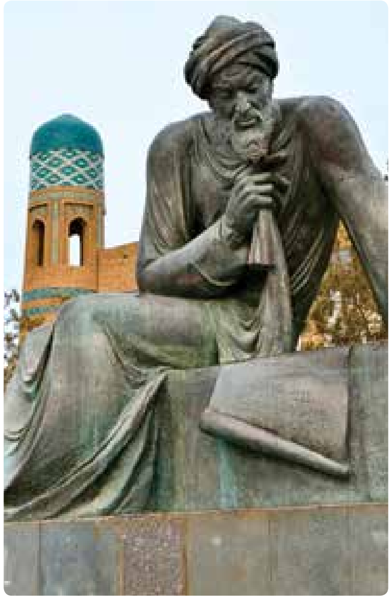
**b** $\dfrac{4}{x^2-4}-\dfrac{3}{x+2}+\dfrac{5}{x-2}$

**c** $\dfrac{5}{x+6}+\dfrac{4}{5-x}+\dfrac{3}{x^2+x-30}$

**d** $\dfrac{1}{4y^2-36y+81}+\dfrac{2}{4y^2-81}-\dfrac{1}{2y^2-9y}$

**e** $\dfrac{1}{p-q}-\dfrac{p}{p^2-q^2}-\dfrac{q^3}{p^4-q^4}$

**f** $(a+6b)\div\left(\dfrac{7}{a^2-3ab+2b^2}-\dfrac{5}{a^2-ab-2b^2}\right)$



**The word *algebra* is of Arabic origin. It is derived from *al-jabr*, and was developed by the mathematician Muhammad ibn Musa al-Khwarizmi (c.780–850). The word *algorithm* is derived from his name.**

MASTER

**21** **a** Expand $(x+5)(2-x)(3x+7)$.

**b** Factorise $27(x-2)^3+64(x+2)^3$.

**22** **a** Simplify $\dfrac{3}{x-1}+\dfrac{8}{x+8}$.

**b** Use CAS technology to complete some of the questions in Exercise 2.2.

# 2.3 Pascal's triangle and binomial expansions

## Expansions of perfect cubes

studyon

Units 1 & 2
AOS 2
Topic 1
Concept 2

**Pascal's triangle and binomial expansions**
Concept summary
Practice questions

The **perfect square** $(a+b)^2$ may be expanded quickly by the rule $(a+b)^2=a^2+2ab+b^2$. The **perfect cube** $(a+b)^3$ can also be expanded by a rule. This rule is derived by expressing $(a+b)^3$ as the product of repeated factors and expanding.

$$\begin{aligned}(a+b)^3&=(a+b)(a+b)(a+b)\\&=(a+b)(a+b)^2\\&=(a+b)(a^2+2ab+b^2)\\&=a^3+2a^2b+ab^2+ba^2+2ab^2+b^3\\&=a^3+3a^2b+3ab^2+b^3\end{aligned}$$

Therefore, the rules for expanding a perfect cube are:

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

### Features of the rule for expanding perfect cubes

- The powers of the first term, $a$, decrease as the powers of the second term, $b$, increase.
- The coefficients of each term in the expansion of $(a+b)^3$ are 1, 3, 3, 1.
- The coefficients of each term in the expansion of $(a-b)^3$ are 1, −3, 3, −1.
- The signs alternate + − + − in the expansion of $(a-b)^3$.

**WORKED EXAMPLE 6**

Expand $(2x - 5)^3$.

*Note:* It is appropriate to use CAS technology to perform expansions such as this.

**THINK**

**WRITE**

1 Use the rule for expanding a perfect cube.

$(2x - 5)^3$

Using $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$, let $2x = a$ and $5 = b$.

$(2x - 5)^3$

$= (2x)^3 - 3(2x)^2(5) + 3(2x)(5)^2 - (5)^3$

2 Simplify each term.

$= 8x^3 - 3 \times 4x^2 \times 5 + 3 \times 2x \times 25 - 125$

$= 8x^3 - 60x^2 + 150x - 125$

3 State the answer.

$\therefore (2x - 5)^3 = 8x^3 - 60x^2 + 150x - 125$

4 An alternative approach to using the rule would be to write the expression in the form $(a + b)^3$.

$(2x - 5)^3 = (2x + (-5))^3$

Using $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, let $2x = a$ and $-5 = b$.

$(2x - 5)^3 = (2x + (-5))^3$

$= (2x)^3 + 3(2x)^2(-5) + 3(2x)(-5)^2 + (-5)^3$

$= 8x^3 - 60x^2 + 150x - 125$

## Pascal's triangle

**eBook*plus***

**Interactivity**
Pascal's triangle
**int-2554**

Although known to Chinese mathematicians many centuries earlier, the following pattern is named after the seventeenth century French mathematician Blaise Pascal. **Pascal's triangle** contains many fascinating patterns. Each row from row 1 onwards begins and ends with '1'. Each other number along a row is formed by adding the two terms to its left and right from the preceding row.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Row 0 | | | | | 1 | | | | |
| Row 1 | | | | 1 | | 1 | | | |
| Row 2 | | | 1 | | 2 | | 1 | | |
| Row 3 | | 1 | | 3 | | 3 | | 1 | |
| Row 4 | 1 | | 4 | | 6 | | 4 | | 1 |

The numbers in each row are called **binomial coefficients**.

The numbers 1, 2, 1 in row 2 are the coefficients of the terms in the expansion of $(a + b)^2$.

$$(a + b)^2 = 1a^2 + 2ab + 1b^2$$

The numbers 1, 3, 3, 1 in row 3 are the coefficients of the terms in the expansion of $(a + b)^3$.

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

Each row of Pascal's triangle contains the coefficients in the expansion of a power of $(a + b)$. Such expansions are called binomial expansions because of the two terms $a$ and $b$ in the brackets.

Row $n$ contains the coefficients in the binomial expansion $(a + b)^n$.

To expand $(a + b)^4$ we would use the binomial coefficients, 1, 4, 6, 4, 1, from row 4 to obtain:

$$\begin{aligned}(a + b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

Notice that the powers of $a$ decrease by 1 as the powers of $b$ increase by 1, with the sum of the powers of $a$ and $b$ always totalling 4 for each term in the expansion of $(a + b)^4$.

For the expansion of $(a - b)^4$ the signs would alternate:

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

By extending Pascal's triangle, higher powers of such binomial expressions can be expanded.

**WORKED EXAMPLE 7**

**Form the rule for the expansion of $(a - b)^5$ and hence expand $(2x - 1)^5$.**

***Note:*** **It is appropriate to use CAS technology to perform expansions such as this.**

| THINK | WRITE |
| --- | --- |
| 1 Choose the row in Pascal's triangle which contains the required binomial coefficients. | For $(a - b)^5$, the power of the binomial is 5. Therefore the binomial coefficients are in row 5. The binomial coefficients are: 1, 5, 10, 10, 5, 1 |
| 2 Write down the required binomial expansion. | Alternate the signs: $(a - b)^5 = a^5 - 5a^4b + 10a^3b^5 - 10a^2b^3 + 5ab^4 - b^5$ |
| 3 State the values to substitute in place of $a$ and $b$. | To expand $(2x - 1)^5$, replace $a$ by $2x$ and $b$ by 1. |
| 4 Write down the expansion. | $(2x - 1)^5$ $= (2x)^5 - 5(2x)^4(1) + 10(2x)^3(1)^2 - 10(2x)^2(1)^3 + 5(2x)(1)^4 - (1)^5$ |
| 5 Evaluate the coefficients and state the answer. | $= 32x^5 - 5 \times 16x^4 + 10 \times 8x^3 - 10 \times 4x^2 + 10x - 1$ $= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$ $\therefore (2x - 1)^5 = 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$ |

## EXERCISE 2.3 Pascal's triangle and binomial expansions

**PRACTISE**

Work without CAS

1 WE6 Expand $(3x - 2)^3$.

2 Expand $\left(\frac{a}{3} + b^2\right)^3$ and give the coefficient of $a^2b^2$.

3 WE7 Form the rule for the expansion of $(a - b)^6$ and hence expand $(2x - 1)^6$.

4 Expand $(3x + 2y)^4$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

5 Expand the following.

a $(3x + 1)^3$

b $(1 - 2x)^3$

c $(5x + 2y)^3$

d $\left(\frac{x}{2} - \frac{y}{3}\right)^3$

6 Select the correct statement(s).

A $(x + 2)^3 = x^3 + 6x^2 + 12x + 8$

B $(x + 2)^3 = x^3 + 2^3$

C $(x + 2)^3 = (x + 2)(x^2 - 2x + 4)$

D $(x + 2)^3 = (x + 2)(x^2 + 2x + 4)$

E $(x + 2)^3 = x^3 + 3x^2 + 3x + 8$

7 Find the coefficient of $x^2$ in the following expressions.

a $(x + 1)^3 - 3x(x + 2)^2$

b $3x^2(x + 5)(x - 5) + 4(5x - 3)^3$

c $(x - 1)(x + 2)(x - 3) - (x - 1)^3$

d $(2x^2 - 3)^3 + 2(4 - x^2)^3$

8 a Write down the numbers in row 7 of Pascal's triangle.

b Identify which row of Pascal's triangle contains the binomial coefficients: 1, 9, 36, 84, 126, 126, 84, 36, 9, 1.

c Row 0 contains 1 term, row 1 contains 2 terms. How is the number of terms related to the row number of Pascal's triangle?

9 Copy and complete the following table by making use of Pascal's triangle.

| Binomial power | Expansion | Number of terms in the expansion | Sum of indices in each term |
|---|---|---|---|
| $(x + a)^2$ | | | |
| $(x + a)^3$ | | | |
| $(x + a)^4$ | | | |
| $(x + a)^5$ | | | |

10 Expand the following using the binomial coefficients from Pascal's triangle.

a $(x + 4)^5$

b $(x - 4)^5$

c $(xy + 2)^5$

d $(3x - 5y)^4$

e $(3 - x^2)^4$

f $(1 + x)^6 - (1 - x)^6$

11 a Expand and simplify $[(x - 1) + y]^4$.

b Find the term independent of $x$ in the expansion of $\left(\frac{x}{2} + \frac{2}{x}\right)^6$.

c If the coefficient of $x^2y^2$ in the expansion of $(x + ay)^4$ is 3 times the coefficient of $x^2y^3$ in the expansion of $(ax^2 - y)^4$, find the value of $a$.

d Find the coefficient of $x$ in the expansion of $(1 + 2x)(1 - x)^5$.

**12 a** The sum of the binomial coefficients in row 2 is $1 + 2 + 1 = 4$. Form the sum of the binomial coefficients in each of rows 3 to 5.

**b** Create a formula for the sum of the binomial coefficients of row $n$.

**c** Expand $(1 + x)^4$.

**d** In the expansion in part **c**, let $x = 1$ and comment on the result.

**e** Using a suitably chosen value for $x$ evaluate $1.1^4$ using the expansion in part **c**.

**13 a** Expand $(x + 1)^5 - (x + 1)^4$ and hence show that $(x + 1)^5 - (x + 1)^4 = x(x + 1)^4$.

**b** Prove $(x + 1)^{n+1} - (x + 1)^n = x(x + 1)^n$.

**14** A section of Pascal's triangle is shown. Determine the values of $a$, $b$ and $c$.

| | | | | |
|---|---|---|---|---|
| | 45 | | $a$ | |
| $b$ | | 165 | | 330 |
| | 220 | | $c$ | |

MASTER

**15** Pascal's triangle can be written as:

| | | | | | |
|---|---|---|---|---|---|
| 1 | | | | | |
| 1 | 1 | | | | |
| 1 | 2 | 1 | | | |
| 1 | 3 | 3 | 1 | | |
| 1 | 4 | 6 | 4 | 1 | |
| 1 | 5 | 10 | 10 | 5 | 1 |

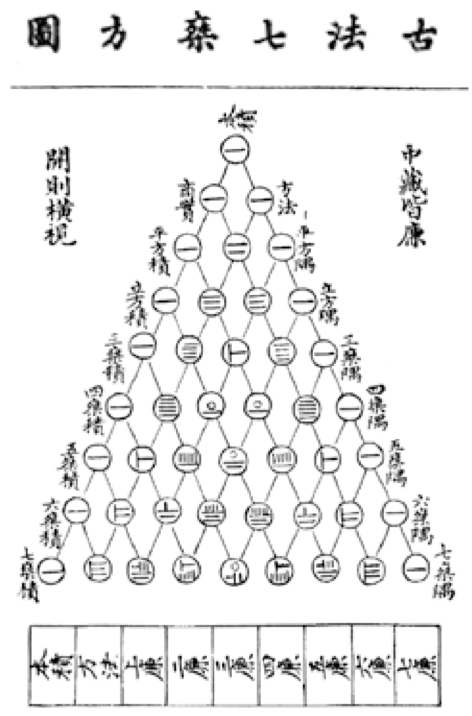
**a** Describe the pattern in the second column.

**b** What would be the sixth entry in the third column?

**c** Describe the pattern of the terms in the third column by forming a rule for the $n$th entry.

**d** What would be the rule for finding the $n$th entry of the fourth column?

**16** Expand $(1 + x + x^2)^4$ and hence, using a suitably chosen value for $x$, evaluate $0.91^4$.



**The Yang Hui (Pascal's) triangle as depicted in 1303 in a work by the Chinese mathematician Chu Shih-chieh**

# 2.4 The binomial theorem

*Note:* The binomial theorem is not part of the Study Design but is included here to enhance understanding.

studyon

Units 1 & 2
AOS 2
Topic 1
Concept 3

**The binomial theorem**
Concept summary
Practice questions

Pascal's triangle is useful for expanding small powers of binomial terms. However, to obtain the coefficients required for expansions of higher powers, the triangle needs to be extensively extended. The binomial theorem provides the way around this limitation by providing a rule for the expansion of $(x + y)^n$. Before this theorem can be presented, some notation needs to be introduced.

## Factorial notation

In this and later chapters, calculations such as $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ will be encountered. Such expressions can be written in shorthand as 7! and are read as '7 factorial'. There is a factorial key on most calculators, but it is advisable to remember some small factorials by heart.

### Definition

eBook*plus*

**Interactivity**
The binomial theorem
**int-2555**

$n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$ for any natural number $n$.

It is also necessary to define $0! = 1$.

7! is equal to 5040. It can also be expressed in terms of other factorials such as:

$$\begin{aligned} 7! &= 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) \\ &= 7 \times 6! \end{aligned} \quad \text{or} \quad \begin{aligned} 7! &= 7 \times 6 \times (5 \times 4 \times 3 \times 2 \times 1) \\ &= 7 \times 6 \times 5! \end{aligned}$$

This is useful when working with fractions containing factorials. For example:

$$\begin{aligned} \frac{7!}{6!} &= \frac{7 \times \cancel{6!}}{\cancel{6!}} \\ &= 7 \end{aligned} \quad \text{or} \quad \begin{aligned} \frac{5!}{7!} &= \frac{\cancel{5!}}{7 \times 6 \times \cancel{5!}} \\ &= \frac{1}{42} \end{aligned}$$

By writing the larger factorial in terms of the smaller factorial, the fractions were simplified.

Factorial notation is just an abbreviation so factorials cannot be combined arithmetically. For example, $3! - 2! \neq 1!$. This is verified by evaluating $3! - 2!$.

$$\begin{aligned} 3! - 2! &= 3 \times 2 \times 1 - 2 \times 1 \\ &= 6 - 2 \\ &= 4 \\ &\neq 1 \end{aligned}$$

**WORKED EXAMPLE 8**

Evaluate $5! - 3! + \frac{50!}{49!}$

| THINK | WRITE |
|---|---|
| 1 Expand the two smaller factorials. | $5! - 3! + \frac{50!}{49!}$ <br> $= 5 \times 4 \times 3 \times 2 \times 1 - 3 \times 2 \times 1 + \frac{50!}{49!}$ |

**2** To simplify the fraction, write the larger factorial in terms of the smaller factorial.

$= 5 \times 4 \times 3 \times 2 \times 1 - 3 \times 2 \times 1 + \dfrac{50 \times 49!}{49!}$

**3** Calculate the answer.

$= 120 - 6 + \dfrac{50 \times \cancel{49!}}{\cancel{49!}}$

$= 120 - 6 + 50$

$= 164$

## Formula for binomial coefficients

Each of the terms in the rows of Pascal's triangle can be expressed using factorial notation. For example, row 3 contains the coefficients 1, 3, 3, 1.

These can be written as $\dfrac{3!}{0! \times 3!}, \dfrac{3!}{1! \times 2!}, \dfrac{3!}{2! \times 1!}, \dfrac{3!}{3! \times 0!}$.
(Remember that 0! was defined to equal 1.)

The coefficients in row 5 (1, 5, 10, 10, 5, 1) can be written as:

$$\frac{5!}{0! \times 5!}, \frac{5!}{1! \times 4!}, \frac{5!}{2! \times 3!}, \frac{5!}{3! \times 2!}, \frac{5!}{4! \times 1!}, \frac{5!}{5! \times 0!}$$

The third term of row 4 would equal $\dfrac{4!}{2! \times 2!}$ and so on.

The $(r + 1)$th term of row $n$ would equal $\dfrac{n!}{r! \times (n - r)!}$. This is normally written using the notations ${}^nC_r$ or $\dbinom{n}{r}$.

These expressions for the binomial coefficients are referred to as **combinatoric coefficients**. They occur frequently in other branches of mathematics including probability theory. Blaise Pascal is regarded as the 'father of probability' and it could be argued he is best remembered for his work in this field.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}^nC_r$$, **where $r \le n$ and $r$, $n$ are non-negative whole numbers.**

## Pascal's triangle with combinatoric coefficients

Pascal's triangle can now be expressed using this notation:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Row 0 | | | | | $\binom{0}{0}$ | | | | |
| Row 1 | | | | $\binom{1}{0}$ | | $\binom{1}{1}$ | | | |
| Row 2 | | | $\binom{2}{0}$ | | $\binom{2}{1}$ | | $\binom{2}{2}$ | | |
| Row 3 | | $\binom{3}{0}$ | | $\binom{3}{1}$ | | $\binom{3}{2}$ | | $\binom{3}{3}$ | |
| Row 4 | $\binom{4}{0}$ | | $\binom{4}{1}$ | | $\binom{4}{2}$ | | $\binom{4}{3}$ | | $\binom{4}{4}$ |

Binomial expansions can be expressed using this notation for each of the binomial coefficients.

The expansion $(a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$.

Note the following patterns:

- $\binom{n}{0} = 1 = \binom{n}{n}$ (the start and end of each row of Pascal's triangle)
- $\binom{n}{1} = n = \binom{n}{n-1}$ (the second from the start and the second from the end of each row) and $\binom{n}{r} = \binom{n}{n-r}$.

While most calculators have a ${}^nC_r$ key to assist with the evaluation of the coefficients, the formula for $\binom{n}{r}$ or ${}^nC_r$ should be known. Some values of $\binom{n}{r}$ can be committed to memory.

**WORKED EXAMPLE 9** Evaluate $\binom{8}{3}$.

| THINK | WRITE |
|---|---|
| 1 Apply the formula. | $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$<br>Let $n = 8$ and $r = 3$.<br>$\binom{8}{3} = \dfrac{8!}{3!(8-3)!}$<br>$= \dfrac{8!}{3!5!}$ |
| 2 Write the largest factorial in terms of the next largest factorial and simplify. | $= \dfrac{8 \times 7 \times 6 \times \cancel{5!}}{3!\cancel{5!}}$<br>$= \dfrac{8 \times 7 \times 6}{3!}$ |
| 3 Calculate the answer. | $= \dfrac{8 \times 7 \times \cancel{6}}{\cancel{3 \times 2} \times 1}$<br>$= 8 \times 7$<br>$= 56$ |

## Binomial theorem

The binomial coefficients in row $n$ of Pascal's triangle can be expressed as $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots \binom{n}{n}$ and hence the expansion of $(x + y)^n$ can be formed.

The binomial theorem gives the rule for the expansion of $(x + y)^n$ as:

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n}y^n$$

Since $\binom{n}{0} = 1 = \binom{n}{n}$ this formula becomes:

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + y^n$$

### Features of the binomial theorem formula for the expansion of $(x + y)^n$

- There are $(n + 1)$ terms.
- In each successive term the powers of $x$ decrease by 1 as the powers of $y$ increase by 1.
- For each term, the powers of $x$ and $y$ add up to $n$.
- For the expansion of $(x - y)^n$ the signs alternate $+ - + - +$ ... with every even term assigned the $-$ sign and every odd term assigned the $+$ sign.

**WORKED EXAMPLE 10** **Use the binomial theorem to expand $(3x + 2)^4$.**

| THINK | WRITE |
|---|---|
| 1 Write out the expansion using the binomial theorem. *Note:* There should be 5 terms in the expansion. | $(3x + 2)^4$ <br> $= (3x)^4 + \binom{4}{1}(3x)^3(2) + \binom{4}{2}(3x)^2(2)^2 + \binom{4}{3}(3x)(2)^3 + (2)^4$ |
| 2 Evaluate the binomial coefficients. | $= (3x)^4 + 4 \times (3x)^3(2) + 6 \times (3x)^2(2)^2 + 4 \times (3x)(2)^3 + (2)^4$ |
| 3 Complete the calculations and state the answer. | $= 81x^4 + 4 \times 27x^3 \times 2 + 6 \times 9x^2 \times 4 + 4 \times 3x \times 8 + 16$ <br> $\therefore (3x + 2)^4 = 81x^4 + 216x^3 + 216x^2 + 96x + 16$ |

## Using the binomial theorem

The binomial theorem is very useful for expanding $(x + y)^n$. However, for powers $n \geq 7$ the calculations can become quite tedious. If a particular term is of interest then, rather than expand the expression completely to obtain the desired term, an alternative option is to form an expression for the general term of the expansion.

### The general term of the binomial theorem

Consider the terms of the expansion:

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + y^n$$

Term 1: $t_1 = \binom{n}{0}x^n y^0$

Term 2: $t_2 = \binom{n}{1}x^{n-1}y^1$

Term 3: $t_3 = \binom{n}{2}x^{n-2}y^2$

Following the pattern gives:

Term $(r + 1)$: $t_{r+1} = \binom{n}{r} x^{n-r} y^r$

For the expansion of $(x + y)^n$, the general term is $t_{r+1} = \binom{n}{r} x^{n-r} y^r$.

For the expansion of $(x - y)^n$, the general term could be expressed as

$t_{r+1} = \binom{n}{r} x^{n-r} (-y)^r$.

The general-term formula enables a particular term to be evaluated without the need to carry out the full expansion. As there are $(n + 1)$ terms in the expansion, if the middle term is sought there will be two middle terms if $n$ is odd and one middle term if $n$ is even.

**WORKED EXAMPLE 11**

**Find the fifth term in the expansion of $\left(\frac{x}{2} - \frac{y}{3}\right)^9$.**

| THINK | WRITE |
|---|---|
| 1 State the general term formula of the expansion. | $\left(\frac{x}{2} - \frac{y}{3}\right)^9$<br>The $(r + 1)$th term is $t_{r+1} = \binom{n}{r}\left(\frac{x}{2}\right)^{n-r}\left(-\frac{y}{3}\right)^r$.<br>Since the power of the binomial is 9, $n = 9$.<br>$\therefore t_{r+1} = \binom{9}{r}\left(\frac{x}{2}\right)^{9-r}\left(-\frac{y}{3}\right)^r$ |
| 2 Choose the value of $r$ for the required term. | For the fifth term, $t_5$:<br>$r + 1 = 5$<br>$r = 4$<br>$t_5 = \binom{9}{4}\left(\frac{x}{2}\right)^{9-4}\left(-\frac{y}{3}\right)^4$<br>$= \binom{9}{4}\left(\frac{x}{2}\right)^5\left(-\frac{y}{3}\right)^4$ |
| 3 Evaluate to obtain the required term. | $= 126 \times \frac{x^5}{32} \times \frac{y^4}{8}$<br>$= \frac{7x^5y^4}{144}$ |

## Identifying a term in the binomial expansion

The general term can also be used to determine which term has a specified property such as the term independent of $x$ or the term containing a particular power of $x$.

**WORKED EXAMPLE 12** Identify which term in the expansion of $(8 - 3x^2)^{12}$ would contain $x^8$ and express the coefficient of $x^8$ as a product of its prime factors.

| THINK | WRITE |
|---|---|
| 1 Write down the general term for this expansion. | $(8 - 3x^2)^{12}$<br>The general term: $t_{r+1} = \binom{12}{r}(8)^{12-r}(-3x^2)^r$ |
| 2 Rearrange the expression for the general term by grouping the numerical parts together and the algebraic parts together. | $t_{r+1} = \binom{12}{r}(8)^{12-r}(-3)^r(x^2)^r$<br>$= \binom{12}{r}(8)^{12-r}(-3)^r x^{2r}$ |
| 3 Find the value of $r$ required to form the given power of $x$. | For $x^8$, $2r = 8$ so $r = 4$. |
| 4 Identify which term is required. | Hence it is the fifth term which contains $x^8$. |
| 5 Obtain an expression for this term. | With $r = 4$,<br>$t_5 = \binom{12}{4}(8)^{12-4}(-3)^4 x^8$<br>$= \binom{12}{4}(8)^8(-3)^4 x^8$ |
| 6 State the required coefficient. | The coefficient of $x^8$ is $\binom{12}{4}(8)^8(-3)^4$. |
| 7 Express the coefficient in terms of prime numbers. | $\binom{12}{4}(8)^8(-3)^4$<br>$= \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} \times (2^3)^8 \times 3^4$<br>$= 11 \times 5 \times 9 \times 2^{24} \times 3^4$<br>$= 11 \times 5 \times 3^2 \times 2^{24} \times 3^4$<br>$= 11 \times 5 \times 3^6 \times 2^{24}$ |
| 8 State the answer. | Therefore the coefficient of $x^8$ is $11 \times 5 \times 3^6 \times 2^{24}$. |

## EXERCISE 2.4 The binomial theorem

**PRACTISE**

Work without CAS

1 WE8 Evaluate $6! + 4! - \frac{10!}{9!}$.

2 Simplify $\frac{n!}{(n-2)!}$.

3 WE9 Evaluate $\binom{7}{4}$.

4 Find an algebraic expression for $\binom{n}{2}$ and use this to evaluate $\binom{21}{2}$.

5 WE10 Use the binomial theorem to expand $(2x + 3)^5$.

6 Use the binomial theorem to expand $(x - 2)^7$.

7 WE11 Find the fourth term in the expansion of $\left(\frac{x}{3} - \frac{y}{2}\right)^7$.

8 Find the middle term in the expansion of $\left(x^2 + \frac{y}{2}\right)^{10}$.

9 WE12 Identify which term in the expansion of $(4 + 3x^3)^8$ would contain $x^{15}$ and express the coefficient of $x^{15}$ as a product of its prime factors.

10 Find the term independent of $x$ in the expansion of $\left(x + \frac{2}{x}\right)^6$.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

11 Evaluate the following.

a $6!$
b $4! + 2!$
c $7 \times 6 \times 5!$
d $\frac{6!}{3!}$
e $10! - 9!$
f $(4! + 3!)^2$

12 Evaluate the following.

a $\frac{26!}{24!}$
b $\frac{42!}{43!}$
c $\frac{49!}{50!} \div \frac{69!}{70!}$
d $\frac{11! + 10!}{11! - 10!}$

13 Simplify the following.

a $(n + 1) \times n!$
b $(n - 1)(n - 2)(n - 3)!$
c $\frac{n!}{(n - 3)!}$
d $\frac{(n - 1)!}{(n + 1)!}$
e $\frac{(n - 1)!}{n!} - \frac{(n + 1)!}{(n + 2)!}$
f $\frac{n^3 - n^2 - 2n}{(n + 1)!} \times \frac{(n - 2)!}{n - 2}$

14 Evaluate the following.

a $\binom{5}{2}$
b $\binom{5}{3}$
c $\binom{12}{12}$
d ${}^{20}C_3$
e $\binom{7}{0}$
f $\binom{13}{10}$

15 Simplify the following.

a $\binom{n}{3}$
b $\binom{n}{n - 3}$
c $\binom{n + 3}{n}$
d $\binom{2n + 1}{2n - 1}$
e $\binom{n}{2} + \binom{n}{3}$
f $\binom{n + 1}{3}$

16 Expand the following.

a $(x + 1)^5$
b $(2 - x)^5$
c $(2x + 3y)^6$
d $\left(\frac{x}{2} + 2\right)^7$
e $\left(x - \frac{1}{x}\right)^8$
f $(x^2 + 1)^{10}$

**17** Obtain each of the following terms.

**a** The fourth term in the expansion of $(5x + 2)^6$

**b** The tenth term in the expansion of $(1 + 2x)^{12}$

**c** The sixth term in the expansion of $(2x + 3)^{10}$

**d** The third term in the expansion of $(3x^2 - 1)^6$

**e** The middle term(s) in the expansion of $(x - 5)^6$

**f** The middle term(s) in the expansion of $(x + 2y)^7$

**18 a** Specify the term which contains $x^4$ in the expansion of $(x + 3)^{12}$.

**b** Obtain the coefficient of $x^6$ in the expansion of $(1 - 2x^2)^9$.

**c** Express the coefficient of $x^5$ in the expansion of $(3 + 4x)^{11}$ as a product of its prime factors.

**d** Calculate the coefficient of $x^2$ in the expansion of $\left(\frac{x}{2} - \frac{2}{x}\right)^8$.

**e** Find the term independent of $x$ in the expansion of $\left(x^2 + \frac{1}{x^3}\right)^{10}$.

**f** Find the term independent of $x$ in the expansion of $\left(x + \frac{1}{x}\right)^6\left(x - \frac{1}{x}\right)^6$.

**19 a** Determine the value of $a$ so that the coefficients of the fourth and the fifth terms in the expansion of $(1 + ax)^{10}$ are equal.

**b** If the coefficient of $x^2$ in the expansion of $(1 + x + x^2)^4$ is equal to the coefficient of $x$ in the expansion of $(1 + x)^n$, find the value of $n$.

**20 a** Use the expansion of $(1 + x)^{10}$ with suitably chosen $x$ to show that $2^{10} = \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \ldots + \binom{10}{10}$ and interpret this result for Pascal's triangle.

**b** Show that $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$ and interpret this result for Pascal's triangle.

**MASTER**

**21** Evaluate the following using CAS technology.

**a** 15!

**b** $\binom{15}{10}$

**22 a** Solve for $n$: $\binom{n}{2} = 1770$

**b** Solve for $r$: $\binom{12}{r} = 220$

**'I'm very well acquainted, too, with matters mathematical;**
**I understand equations, both the simple and quadratical;**
**About Binomial Theorem I am teeming with a lot o' news,**
**With many cheerful facts about the square of the hypotenuse!'**

***Source:*** Verse 2 of 'I am the very model of a modern major general' from *Pirates of Penzance* by Gilbert and Sullivan.

# 2.5 Sets of real numbers

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AOS 2

Topic 1

Concept 4

**Real numbers**
Concept summary
Practice questions

eBook*plus*

**Interactivity**
Sets
int-2556

The concept of numbers in counting and the introduction of symbols for numbers marked the beginning of major intellectual development in the minds of the early humans. Every civilisation appears to have developed a system for counting using written or spoken symbols for a few, or more, numbers. Over time, technologies were devised to assist in counting and computational techniques, and from these counting machines the computer was developed.

Over the course of history, different categories of numbers have evolved which collectively form the real number system. Real numbers are all the numbers which are positive or zero or negative. Before further describing and classifying the real number system, a review of some mathematical notation is given.

## Set notation

A **set** is a collection of objects, these objects being referred to as the **elements** of the set. The elements may be listed as, for example, the set $A = \{1, 2, 3, 4, 5\}$ and the set $B = \{1, 3, 5\}$.

The statement $2 \in A$ means 2 is an element of set $A$, and the statement $2 \notin B$ means 2 does not belong to, or is not an element of, set $B$.

Since every element in set $B = \{1, 3, 5\}$ is also an element of set $A = \{1, 2, 3, 4, 5\}$, $B$ is a **subset** of set $A$. This is written as $B \subset A$. However we would write $A \not\subset B$ since $A$ is not a subset of $B$.

The **union** of the sets $A$ and $B$ contains the elements which are either in $A$ or in $B$ or in both. Element should not be counted twice.

The **intersection** of the sets $A$ and $B$ contains the elements which must be in both $A$ and $B$. This is written as $A \cap B$ and would be the same as the set $B$ for this example.

The **exclusion** notation $A \backslash B$ excludes, or removes, any element of $B$ from $A$. This leaves a set with the elements $\{2, 4\}$.

Sets may be given a description as, for example, set $C = \{x : 1 < x < 10\}$. The set $C$ is read as '$C$ is the set of numbers $x$ such that $x$ is between 1 and 10'.

The set of numbers not in set $C$ is called the **complement** of $C$ and given the symbol $C'$. The description of this set could be written as $C' = \{x : x \leq 1 \text{ or } x \geq 10\}$.

A set and its complement cannot intersect. This is written as $C \cap C' = \varnothing$ where $\varnothing$ is a symbol for 'empty set'. Such sets are called **disjoint** sets.

There will be ongoing use of set notation throughout the coming chapters.

## Classification of numbers

While counting numbers are sufficient to solve equations such as $2 + x = 3$, they are not sufficient to solve, for example, $3 + x = 2$ where negative numbers are needed, nor $3x = 2$ where fractions are needed.

The following sequence of subsets of the real number system, while logical, does not necessarily reflect the historical order in which the real number system was established. For example, fractions were established long before the existence of negative numbers was accepted.

Natural numbers are the positive whole numbers or counting numbers. The **set of natural numbers** is $N = \{1, 2, 3, \ldots\}$.

The positive and negative whole numbers, together with the number zero, are called integers. The **set of integers** is $Z = \{\ldots -2, -1, 0, 1, 2, 3, \ldots\}$. The symbol $Z$ is derived from the German word 'zahl' for number.

Rational numbers are those which can be expressed as quotients in the form $\frac{p}{q}$, where $q \neq 0$, and $p$ and $q$ are integers which have no common factors other than 1. The symbol for the **set of rational numbers** is $Q$ (for quotients). Rational numbers include finite and recurring decimals as well as fractions and integers. For example:

$$1\tfrac{1}{8} = \frac{9}{8},\ 0.75 = \frac{75}{100} = \frac{3}{4},\ 0.\overline{3} = 0.3333\ldots = \frac{1}{3},\ \text{and } 5 = \frac{5}{1} \text{ are rational.}$$

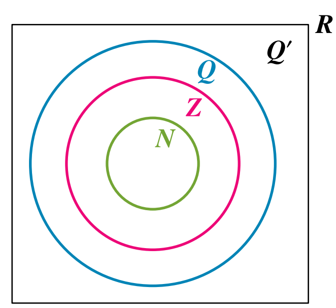
Natural numbers and integers are subsets of the set of rational numbers with $N \subset Z \subset Q$. Irrational numbers are numbers which are not rational; they cannot be expressed in fraction form as the ratio of two integers. Irrational numbers include numbers such as $\sqrt{2}$ and $\pi$. The **set of irrational** numbers is denoted by the symbol $Q'$ using the complement symbol $'$ for 'not'. $Q \cap Q' = \varnothing$ as the rational and irrational sets do not intersect.

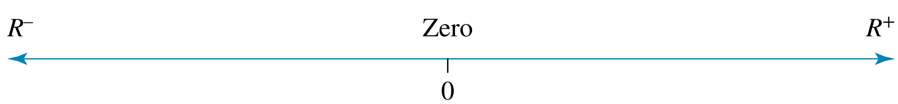
The irrational numbers are further classified into the algebraic irrationals and the non-algebraic ones known as **transcendental numbers**. Algebraic irrationals are those which, like rational numbers, can be solutions to an equation with integer coefficients, while transcendental numbers cannot. For example, $\pi$ is transcendental while $\sqrt{2}$ is algebraic since it is a solution of the equation $x^2 - 2 = 0$.

The union of the set of rational and irrational numbers forms the **set of real numbers** $R$.

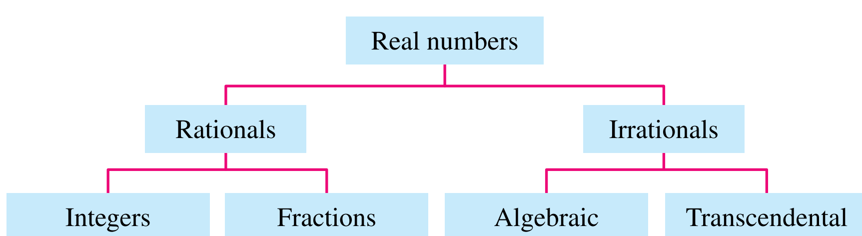
Hence $R = Q \cup Q'$. This is displayed in the diagram showing the subsets of the real numbers.

The set of all real numbers forms a number line continuum on which all of the positive or zero or negative numbers are placed. Hence $R = R^- \cup \{0\} \cup R^+$.




The sets which formed the building blocks of the real number system have been defined, enabling the real number system to be viewed as the following hierarchy.



## Expressions and symbols that do not represent real numbers

It is important to recognise that the following are not numbers.

- The symbol for infinity $\infty$ may suggest this is a number but that is not so. We can speak of numbers getting larger and larger and approaching infinity, but infinity is a concept, not an actual number.
- Any expression of the form $\frac{a}{0}$ does not represent a number since division by zero is not possible. If $a = 0$, the expression $\frac{0}{0}$ is said to be indeterminate. It is not defined as a number.

To illustrate the second point, consider $\frac{3}{0}$.

Suppose 3 divided by 0 is possible and results in a number we shall call $n$.

$$\frac{3}{0} = n$$

$$3 = 0 \times n$$

$$\therefore 3 = 0$$

The conclusion is nonsensical so $\frac{3}{0}$ is not defined.

However, if we try the same process for zero divided by zero, we obtain:

$$\frac{0}{0} = n$$

$$\therefore 0 = 0 \times n$$

$$\therefore 0 = 0$$

While the conclusion holds, it is not possible to determine a value for $n$, so $\frac{0}{0}$ is indeterminate.

It is beyond the Mathematical Methods course, but there are numbers that are not elements of the set of real numbers. For example, the square roots of negative numbers, such as $\sqrt{-1}$, are unreal, but these square roots are numbers. They belong to the set of complex numbers. These numbers are very important in higher levels of mathematics.

**WORKED EXAMPLE 13**

**a** **Classify each of the following numbers as an element of a subset of the real numbers.**

i $-\frac{3}{5}$

ii $\sqrt{7}$

iii $6 - 2 \times 3$

iv $\sqrt{9}$

**b** **Which of the following are correct statements?**

i $5 \in Z$

ii $Z \subset N$

iii $R^- \cup R^+ = R$

| THINK | WRITE |
| --- | --- |
| **a** i Fractions are rational numbers. | **a** i $-\frac{3}{5} \in Q$ |
| ii Surds are irrational numbers. | ii $\sqrt{7} \in Q'$ |

**iii** Evaluate the number using the correct order of operations.

**iii** $6 - 2 \times 3$
$= 6 - 6$
$= 0$
$\therefore (6 - 2 \times 3) \in Z$.
An alternative answer is $(6 - 2 \times 3) \in Q$.

**iv** Evaluate the square root.

**iv** $\sqrt{9} = \pm 3$
$\therefore \sqrt{9} \in Z$

**b i** $Z$ is the set of integers.

**b i** $5 \in Z$ is a correct statement since 5 is an integer.

**ii** $N$ is the set of natural numbers.

**ii** $Z \subset N$ is incorrect since $N \subset Z$.

**iii** This is the union of $R^-$, the set of negative real numbers, and $R^+$, the set of positive real numbers.

**iii** $R^- \cup R^+ = R$ is incorrect since $R$ includes the number zero which is neither positive nor negative.

## Interval notation

**Interval notation** provides an alternative and often convenient way of describing certain sets of numbers.

### Closed interval

$[a, b] = \{x : a \le x \le b\}$ is the set of real numbers that lie between $a$ and $b$, including the endpoints, $a$ and $b$. The inclusion of the endpoints is indicated by the use of the square brackets [ ].

This is illustrated on a number line using closed circles at the endpoints.


### Open interval

$(a, b) = \{x : a < x < b\}$ is the set of real numbers that lie between $a$ and $b$, not including the endpoints $a$ and $b$. The exclusion of the endpoints is indicated by the use of the round brackets ( ).

This is illustrated on a number line using open circles at the endpoints.


### Half-open intervals

Half-open intervals have only one endpoint included.

$[a, b) = \{x : a \le x < b\}$


$(a, b] = \{x : a < x \le b\}$

Interval notation can be used for infinite intervals using the symbol for infinity with an open end. For example, the set of real numbers, $R$, is the same as the interval $(-\infty, \infty)$.

**WORKED EXAMPLE 14**

**a** **Illustrate the following on a number line and express in alternative notation.**

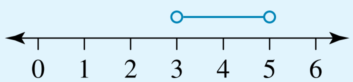
**i** $(-2, 2]$

**ii** $\{x : x \geq 1\}$

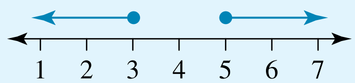
**iii** $\{1, 2, 3, 4\}$.

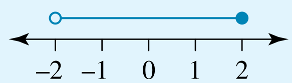
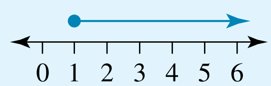
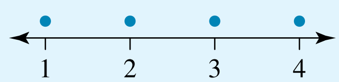
**b** **Use interval notation to describe the sets of numbers shown on the following number lines.**

**i**




**ii**




| THINK | WRITE |
|---|---|
| **a i 1** Describe the given interval. *Note:* The round bracket indicates 'not included' and a square bracket indicates 'included'. | **a i** $(-2, 2]$ is the interval representing the set of numbers between $-2$ and 2, closed at 2, open at $-2$.  |
| **2** Write the set in alternative notation. | An alternative notation for the set is $(-2, 2] = \{x : -2 < x \leq 2\}$. |
| **ii 1** Describe the given set. | **ii** $\{x : x \geq 1\}$ is the set of all numbers greater than or equal to 1. This is an infinite interval which has no right-hand endpoint.  |
| **2** Write the set in alternative notation. | An alternative notation is $\{x : x \geq 1\} = [1, \infty)$. |
| **iii 1** Describe the given set. *Note:* This set does not contain all numbers between the beginning and end of an interval. | **iii** $\{1, 2, 3, 4\}$ is a set of discrete elements.  |
| **2** Write the set in alternative notation. | Alternative notations could be $\{1, 2, 3, 4\} = \{x : 1 \leq x \leq 4,\ x \in N\}$, or $\{1, 2, 3, 4\} = [1, 4] \cap N$. |
| **b i** Describe the set using interval notation with appropriate brackets. | **b i** The set of numbers lie between 3 and 5, with both endpoints excluded. The set is described as $(3, 5)$. |
| **ii 1** Describe the set as the union of the two disjoint intervals. | **ii** The left branch is $(-\infty, 3]$ and the right branch is $[5, \infty)$. The set of numbers is the union of these two. It can be described as $(-\infty, 3] \cup [5, \infty)$. |
| **2** Describe the same set by considering the interval that has been excluded from $R$. | Alternatively, the diagram can be interpreted as showing what remains after the set $(3, 5)$ is excluded from the set $R$. An alternative description is $R\backslash(3, 5)$. |

## EXERCISE 2.5 Sets of real numbers

**PRACTISE**

Work without CAS

1 **WE13** a Classify each of the following numbers as an element of a subset of the real numbers.

i $\frac{6}{11}$ ii $\sqrt{27}$ iii $(6-2)\times 3$ iv $\sqrt{0.25}$

b Which of the following are correct statements?

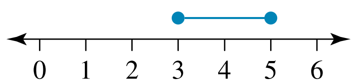
i $17 \in N$ ii $Q \subset N$ iii $Q \cup Q' = R$

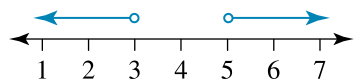
2 For what value(s) of $x$ would $\dfrac{x-5}{(x+1)(x-3)}$ be undefined?

3 **WE14** a Illustrate the following on a number line and express in alternative notation.

i $[-2, 2)$ ii $\{x : x < -1\}$ iii $\{-2, -1, 0, 1, 2\}$

b Use interval notation to describe the sets of numbers shown on the following number lines.

i 


ii 


4 Write $R\backslash\{x : 1 < x \le 4\}$ as the union of two sets expressed in interval notation.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

5 Which of the following does not represent a real number?

A $\sqrt[3]{-4}$ B $\sqrt{0}$ C $\dfrac{6\times 2 - 3\times 4}{7}$

D $5\pi^2 + 3$ E $\dfrac{(8-4)\times 2}{8 - 4\times 2}$

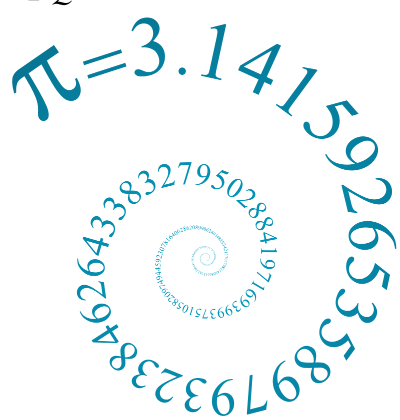
6 Explain why each of the following statements is false and then rewrite it as a correct statement.

a $\sqrt{16+25} \in Q$ b $\left(\frac{4}{9} - 1\right) \in Z$

c $R^+ = \{x : x \ge 0\}$ d $\sqrt{2.25} \in Q'$

7 Select the irrational numbers from the following set of numbers.

$\{\sqrt{11}, \frac{2}{11}, 11^{11}, 11\pi, \sqrt{121}, 2^{\pi}\}$




8 State whether the following are true or false.

a $R^- \subset R$ b $N \subset R^+$

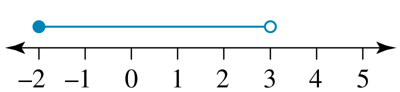
c $Z \cup N = R$ d $Q \cap Z = Z$

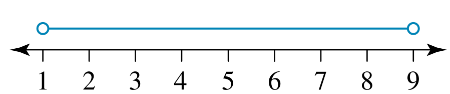
e $Q' \cup Z = R \setminus Q$ f $Z \setminus N = Z^-$

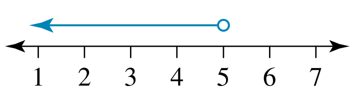
9 Determine any values of $x$ for which the following would be undefined.

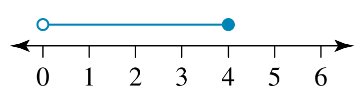
a $\dfrac{1}{x+5}$ b $\dfrac{x+2}{x-2}$ c $\dfrac{x+8}{(2x+3)(5-x)}$ d $\dfrac{4}{x^2-4x}$

10 Use interval notation to describe the intervals shown on the following number lines.

a 


b 


c 

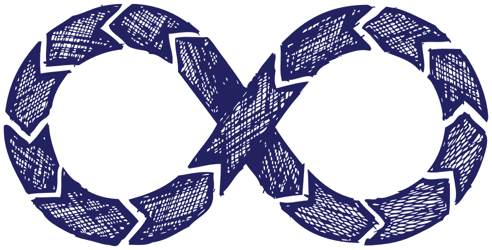

d 

11 Express the following in interval notation.

a $\{x : 4 < x \leq 8\}$
b $\{x : x > -3\}$
c $\{x : x \leq 0\}$
d $\{x : -2 \leq x \leq 0\}$

12 Show the following intervals on a number line.

a $[-5, 5)$
b $(4, \infty)$
c $[-3, 7]$
d $(-3, 7]$
e $(-\infty, 3]$
f $(-\infty, \infty)$



13 Illustrate the following on a number line.

a $R \backslash [-2, 2]$
b $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$
c $[-4, 2) \cap (0, 4)$
d $[-4, 2) \cup (0, 4)$
e $\{-1, 0, 1\}$
f $R \backslash \{0\}$

14 Use an alternative form of notation to describe the following sets.

a $\{x : 2 < x < 6, x \in Z\}$
b $R \backslash (-1, 5]$
c $R^-$
d $(-\infty, -4) \cup [2, \infty)$

MASTER

15 Determine which of the following are rational and which are irrational numbers.

a $\sqrt{7225}$
b $\sqrt{75600}$
c 0.234234234...

16 The ancient Egyptians devised the formula $A = \frac{64}{81}d^2$ for calculating the area $A$ of a circle of diameter $d$. Use this formula to find a rational approximation for $\pi$ and evaluate it to 9 decimal places. Is it a better approximation than $\frac{22}{7}$?

## 2.6 Surds

studyon

Units 1 & 2
AOS 2
Topic 1
Concept 5

**Surds**
Concept summary
Practice questions

A **surd** is an $n$th root, $\sqrt[n]{x}$. Surds are irrational numbers, and cannot be expressed in the quotient form $\frac{p}{q}$. Hence, surds have neither a finite nor a recurring decimal form. Any decimal value obtained from a calculator is just an approximation.

All surds have **radical** signs, such as square roots or cube roots, but not all numbers with radical signs are surds. For surds, the roots cannot be evaluated exactly. Hence, $\sqrt{26}$ is a surd. $\sqrt{25}$ is not a surd since 25 is a perfect square, $\sqrt{25} = 5$, which is rational.

### Ordering surds

Surds are real numbers and therefore have a position on the number line. To estimate the position of $\sqrt{6}$, we can place it between two rational numbers by placing 6 between its closest perfect squares.

$$4 < 6 < 9$$
$$\sqrt{4} < \sqrt{6} < \sqrt{9}$$
$$\therefore 2 < \sqrt{6} < 3$$

So $\sqrt{6}$ lies between 2 and 3, closer to 2, since 6 lies closer to 4 than to 9. Checking with a calculator, $\sqrt{6} \simeq 2.4495$. Note that the symbol $\sqrt{\ }$ always gives a positive number, so the negative surd $-\sqrt{6}$ would lie on the number line between $-3$ and $-2$ at the approximate position $-2.4495$.

To order the sizes of two surds such as $3\sqrt{5}$ and $5\sqrt{3}$, express each as an entire surd.

$$\begin{aligned} 3\sqrt{5} &= \sqrt{9} \times \sqrt{5} \\ &= \sqrt{9 \times 5} \\ &= \sqrt{45} \end{aligned} \qquad \text{and} \qquad \begin{aligned} 5\sqrt{3} &= \sqrt{25} \times \sqrt{3} \\ &= \sqrt{25 \times 3} \\ &= \sqrt{75} \end{aligned}$$

Since $\sqrt{45} < \sqrt{75}$ it follows that $3\sqrt{5} < 5\sqrt{3}$.

## Surds in simplest form

**study on**

Units 1 & 2
AOS 2
Topic 1
Concept 6

**Simplifying surds**
Concept summary
Practice questions

Surds are said to be in simplest form when the number under the square root sign contains no perfect square factors. This means that $3\sqrt{5}$ is the simplest form of $\sqrt{45}$ and $5\sqrt{3}$ is the simplest form of $\sqrt{75}$.

If the radical sign is a cube root then the simplest form has no perfect cube factors under the cube root.

To express $\sqrt{128}$ in its simplest form, find perfect square factors of 128.

$$\begin{aligned} \sqrt{128} &= \sqrt{64 \times 2} \\ &= \sqrt{64} \times \sqrt{2} \\ &= 8\sqrt{2} \\ \therefore \sqrt{128} &= 8\sqrt{2} \end{aligned}$$

**WORKED EXAMPLE 15**

**a** Express $\{6\sqrt{2},\ 4\sqrt{3},\ 2\sqrt{5},\ 7\}$ with its elements in increasing order.
**b** Express in simplest form

**i** $\sqrt{56}$ **ii** $2\sqrt{252a^2b}$ assuming $a > 0$.

| THINK | WRITE |
|---|---|
| **a 1** Express each number entirely as a square root. | **a** $\{6\sqrt{2},\ 4\sqrt{3},\ 2\sqrt{5},\ 7\}$<br>$6\sqrt{2} = \sqrt{36} \times \sqrt{2} = \sqrt{72}$,<br>$4\sqrt{3} = \sqrt{16} \times \sqrt{3} = \sqrt{48}$,<br>$2\sqrt{5} = \sqrt{4} \times \sqrt{5} = \sqrt{20}$<br>and $7 = \sqrt{49}$. |
| **2** Order the terms. | $\sqrt{20} < \sqrt{48} < \sqrt{49} < \sqrt{72}$<br>In increasing order, the set of numbers is $\{2\sqrt{5}, 4\sqrt{3}, 7, 6\sqrt{2}\}$. |

| | |
|---|---|
| **b** **i** Find a perfect square factor of the number under the square root sign. | **b** **i** $\sqrt{56} = \sqrt{4 \times 14}$<br>$= \sqrt{4} \times \sqrt{14}$<br>$= 2\sqrt{14}$ |
| **ii** **1** Find any perfect square factors of the number under the square root sign. | **ii** $2\sqrt{252a^2b}$<br>$= 2\sqrt{4 \times 9 \times 7a^2b}$ |
| **2** Express the square root terms as products and simplify where possible. | $= 2 \times \sqrt{4} \times \sqrt{9} \times \sqrt{7} \times \sqrt{a^2} \times \sqrt{b}$<br>$= 2 \times 2 \times 3 \times \sqrt{7} \times a \times \sqrt{b}$<br>$= 12a\sqrt{7b}$ |
| **3** Try to find the largest perfect square factor for greater efficiency. | Alternatively, recognising that 252 is $36 \times 7$,<br>$2\sqrt{252a^2b}$<br>$= 2\sqrt{36 \times 7a^2b}$<br>$= 2 \times \sqrt{36} \times \sqrt{7} \times \sqrt{a^2} \times \sqrt{b}$<br>$= 2 \times 6 \times \sqrt{7} \times a \times \sqrt{b}$<br>$= 12a\sqrt{7b}$ |

## Operations with surds

As surds are real numbers, they obey the usual laws for addition and subtraction of like terms and the laws of multiplication and division.

- Addition and subtraction

$$a\sqrt{c} + b\sqrt{c} = (a + b)\sqrt{c}$$
$$a\sqrt{c} - b\sqrt{c} = (a - b)\sqrt{c}$$

Surdic expressions such as $\sqrt{2} + \sqrt{3}$ cannot be expressed in any simpler form as $\sqrt{2}$ and $\sqrt{3}$ are 'unlike' surds. Like surds have the same number under the square root sign. Expressing surds in simplest form enables any like surds to be recognised.

- Multiplication and division

$$\sqrt{c} \times \sqrt{d} = \sqrt{(cd)}$$
$$a\sqrt{c} \times b\sqrt{d} = (ab)\sqrt{(cd)}$$
$$\frac{\sqrt{c}}{\sqrt{d}} = \sqrt{\frac{c}{d}}$$

Note that $(\sqrt{c})^2 = c$ because $(\sqrt{c})^2 = \sqrt{c} \times \sqrt{c} = \sqrt{c^2} = c$.

WORKED EXAMPLE 16

**Simplify the following.**

a $3\sqrt{5}+7\sqrt{2}+6\sqrt{5}-3\sqrt{2}$

b $3\sqrt{98}-4\sqrt{72}+2\sqrt{125}$

c $4\sqrt{3}\times 6\sqrt{15}$

| THINK | WRITE |
|---|---|
| a Collect like surds together and simplify. | a $3\sqrt{5}+7\sqrt{2}+6\sqrt{5}-3\sqrt{2}$<br>$=3\sqrt{5}+6\sqrt{5}+7\sqrt{2}-3\sqrt{2}$<br>$=9\sqrt{5}+4\sqrt{2}$ |
| b 1 Write each surd in simplest form. | b $3\sqrt{98}-4\sqrt{72}+2\sqrt{125}$<br>$=3\sqrt{49\times 2}-4\sqrt{36\times 2}+2\sqrt{25\times 5}$<br>$=3\times 7\sqrt{2}-4\times 6\sqrt{2}+2\times 5\sqrt{5}$<br>$=21\sqrt{2}-24\sqrt{2}+10\sqrt{5}$ |
| 2 Collect like surds together. | $=(21\sqrt{2}-24\sqrt{2})+10\sqrt{5}$<br>$=-3\sqrt{2}+10\sqrt{5}$ |
| c 1 Multiply the rational numbers together and multiply the surds together. | c $4\sqrt{3}\times 6\sqrt{15}$<br>$=(4\times 6)\sqrt{(3\times 15)}$<br>$=24\sqrt{45}$ |
| 2 Write the surd in its simplest form. | $=24\sqrt{9\times 5}$<br>$=24\times 3\sqrt{5}$<br>$=72\sqrt{5}$ |

## Expansions

Expansions of brackets containing surds are carried out using the Distributive Law in the same way as algebraic expansions.

$$\sqrt{a}(\sqrt{b}+\sqrt{c})=\sqrt{ab}+\sqrt{ac}$$
$$(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{d})=\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$$

The perfect squares formula for binomial expansions involving surds becomes: $(\sqrt{a}\pm\sqrt{b})^2=a\pm 2\sqrt{ab}+b$, since:

$$\begin{aligned}(\sqrt{a}\pm\sqrt{b})^2&=(\sqrt{a})^2\pm 2\sqrt{a}\sqrt{b}+(\sqrt{b})^2\\&=a\pm 2\sqrt{ab}+b\end{aligned}$$

The difference of two squares formula becomes $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$ since:

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b$$

The binomial theorem can be used to expand higher powers of binomial expressions containing surds.

WORKED EXAMPLE 17

Expand and simplify the following.

a $3\sqrt{2}(4\sqrt{2} - 5\sqrt{6})$

b $(2\sqrt{3} - 5\sqrt{2})(4\sqrt{5} + \sqrt{14})$

c $(3\sqrt{3} + 2\sqrt{5})^2$

d $(\sqrt{7} + 3\sqrt{2})(\sqrt{7} - 3\sqrt{2})$

| THINK | WRITE |
|---|---|
| a Use the Distributive Law to expand, then simplify each term. | a $3\sqrt{2}(4\sqrt{2} - 5\sqrt{6})$<br>$= 12\sqrt{4} - 15\sqrt{12}$<br>$= 12 \times 2 - 15\sqrt{4 \times 3}$<br>$= 24 - 15 \times 2\sqrt{3}$<br>$= 24 - 30\sqrt{3}$ |
| b 1 Expand as for algebraic terms. | b $(2\sqrt{3} - 5\sqrt{2})(4\sqrt{5} + \sqrt{14})$<br>$= 8\sqrt{15} + 2\sqrt{42} - 20\sqrt{10} - 5\sqrt{28}$ |
| 2 Simplify where possible. | $= 8\sqrt{15} + 2\sqrt{42} - 20\sqrt{10} - 5\sqrt{4 \times 7}$<br>$= 8\sqrt{15} + 2\sqrt{42} - 20\sqrt{10} - 5 \times 2\sqrt{7}$<br>$= 8\sqrt{15} + 2\sqrt{42} - 20\sqrt{10} - 10\sqrt{7}$<br>There are no like surds so no further simplification is possible. |
| c 1 Use the rule for expanding a perfect square. | c $(3\sqrt{3} + 2\sqrt{5})^2$<br>$= (3\sqrt{3})^2 + 2(3\sqrt{3})(2\sqrt{5}) + (2\sqrt{5})^2$ |
| 2 Simplify each term remembering that $(\sqrt{a})^2 = a$ and collect any like terms together. | $= 9 \times 3 + 2 \times 3 \times 2\sqrt{3 \times 5} + 4 \times 5$<br>$= 27 + 12\sqrt{15} + 20$<br>$= 47 + 12\sqrt{15}$ |
| d Use the rule for expanding a difference of two squares. | d $(\sqrt{7} + 3\sqrt{2})(\sqrt{7} - 3\sqrt{2})$<br>$= (\sqrt{7})^2 - (3\sqrt{2})^2$<br>$= 7 - 9 \times 2$<br>$= 7 - 18$<br>$= -11$ |

## Rationalising denominators

It is usually desirable to express any fraction whose denominator contains surds as a fraction with a denominator containing only a rational number. This does not necessarily mean the rational denominator form of the fraction is simpler, but it can provide a form which allows for easier manipulation and it can enable like surds to be recognised in a surdic expression.

The process of obtaining a rational number on the denominator is called rationalising the denominator. There are different methods for rationalising denominators, depending on how many terms there are in the denominator.

## Rationalising monomial denominators

Consider $\frac{a}{b\sqrt{c}}$, where $a, b, c \in Q$. This fraction has a monomial denominator since its denominator contains the one term, $b\sqrt{c}$.

In order to rationalise the denominator of this fraction, we use the fact that

$\sqrt{c} \times \sqrt{c} = c$, a rational number.

Multiply both the numerator and the denominator by $\sqrt{c}$. As this is equivalent to multiplying by 1, the value of the fraction is not altered.

$$\begin{aligned}\frac{a}{b\sqrt{c}} &= \frac{a}{b\sqrt{c}} \times \frac{\sqrt{c}}{\sqrt{c}} \\ &= \frac{a\sqrt{c}}{b(\sqrt{c} \times \sqrt{c})} \\ &= \frac{a\sqrt{c}}{bc}\end{aligned}$$

By this process $\frac{a}{b\sqrt{c}} = \frac{a\sqrt{c}}{bc}$ and the denominator, $bc$, is now rational.

Once the denominator has been rationalised, it may be possible to simplify the expression if, for example, any common factor exists between the rationals in the numerator and denominator.

## Rationalising binomial denominators

$\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called **conjugate surds**. Multiplying a pair of conjugate surds always results in a rational number since $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$. This fact is used to rationalise binomial denominators.

Consider $\frac{1}{\sqrt{a} + \sqrt{b}}$ where $a, b \in Q$. This fraction has a binomial denominator since its denominator is the addition of two terms.

To rationalise the denominator, multiply both the numerator and the denominator by $\sqrt{a} - \sqrt{b}$, the conjugate of the surd in the denominator. This is equivalent to multiplying by 1, so the value of the fraction is unaltered; however, it creates a difference of two squares on the denominator.

$$\begin{aligned}\frac{1}{\sqrt{a} + \sqrt{b}} &= \frac{1}{\sqrt{a} + \sqrt{b}} \times \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} \\ &= \frac{\sqrt{a} - \sqrt{b}}{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})} \\ &= \frac{\sqrt{a} - \sqrt{b}}{a - b}\end{aligned}$$

By this process we have $\frac{1}{\sqrt{a} + \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{a - b}$ where the denominator, $a - b$, is a rational number.

WORKED EXAMPLE 18

**a** Express the following with a rational denominator.

**i** $\dfrac{12}{5\sqrt{3}}$ **ii** $\dfrac{5\sqrt{3} - 3\sqrt{10}}{4\sqrt{5}}$

**b** Simplify $5\sqrt{2} - \dfrac{4}{\sqrt{2}} + 3\sqrt{18}$.

**c** Express $\dfrac{6}{5\sqrt{3} - 3\sqrt{2}}$ with a rational denominator.

**d** Given $p = 3\sqrt{2} - 1$, calculate $\dfrac{1}{p^2 - 1}$, expressing the answer with a rational denominator.

| THINK | WRITE |
|---|---|
| **a i 1** The denominator is monomial. Multiply both numerator and denominator by the surd part of the monomial term. | **a i** $\dfrac{12}{5\sqrt{3}} = \dfrac{12}{5\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ |
| **2** Multiply the numerator terms together and multiply the denominator terms together. | $= \dfrac{12\sqrt{3}}{5 \times 3} = \dfrac{12\sqrt{3}}{15}$ |
| **3** Cancel the common factor between the numerator and denominator. | $= \dfrac{{}^{4}\cancel{12}\sqrt{3}}{\cancel{15}_{5}} = \dfrac{4\sqrt{3}}{5}$ |
| **ii 1** The denominator is monomial. Multiply both numerator and denominator by the surd part of the monomial term. | **ii** $\dfrac{5\sqrt{3} - 3\sqrt{10}}{4\sqrt{5}} = \dfrac{(5\sqrt{3} - 3\sqrt{10})}{4\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{\sqrt{5}(5\sqrt{3} - 3\sqrt{10})}{4 \times 5} = \dfrac{5\sqrt{15} - 3\sqrt{50}}{20}$ |
| **2** Simplify the surds, where possible. | $= \dfrac{5\sqrt{15} - 3 \times 5\sqrt{2}}{20} = \dfrac{5\sqrt{15} - 15\sqrt{2}}{20}$ |
| **3** Take out the common factor in the numerator since it can be cancelled as a factor of the denominator. | $= \dfrac{5(\sqrt{15} - 3\sqrt{2})}{20} = \dfrac{\sqrt{15} - 3\sqrt{2}}{4}$ |

**b** Rationalise any denominators containing surds and simplify all terms in order to identify any like surds that can be collected together.

**b** $5\sqrt{2} - \dfrac{4}{\sqrt{2}} + 3\sqrt{18}$

$= 5\sqrt{2} - \dfrac{4}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} + 3 \times 3\sqrt{2}$

$= 5\sqrt{2} - \dfrac{4\sqrt{2}}{2} + 9\sqrt{2}$

$= 5\sqrt{2} - 2\sqrt{2} + 9\sqrt{2}$

$= 12\sqrt{2}$

**c 1** The denominator is binomial. Multiply both numerator and denominator by the conjugate of the binomial surd contained in the denominator.

**c** The conjugate of $5\sqrt{3} - 3\sqrt{2}$ is $5\sqrt{3} + 3\sqrt{2}$

$\dfrac{6}{5\sqrt{3} - 3\sqrt{2}}$

$= \dfrac{6}{5\sqrt{3} - 3\sqrt{2}} \times \dfrac{5\sqrt{3} + 3\sqrt{2}}{5\sqrt{3} + 3\sqrt{2}}$

$= \dfrac{6(5\sqrt{3} + 3\sqrt{2})}{(5\sqrt{3} - 3\sqrt{2})(5\sqrt{3} + 3\sqrt{2})}$

**2** Expand the difference of two squares in the denominator.

*Note:* This expansion should always result in a rational number.

$= \dfrac{6(5\sqrt{3} + 3\sqrt{2})}{(5\sqrt{3})^2 - (3\sqrt{2})^2}$

$= \dfrac{6(5\sqrt{3} + 3\sqrt{2})}{25 \times 3 - 9 \times 2}$

$= \dfrac{6(5\sqrt{3} + 3\sqrt{2})}{57}$

**3** Cancel the common factor between the numerator and the denominator.

*Note:* The numerator could be expanded but there is no further simplification to gain by doing so.

$= \dfrac{{}^{2}\cancel{6}(5\sqrt{3} + 3\sqrt{2})}{\cancel{57}_{19}}$

$= \dfrac{2(5\sqrt{3} + 3\sqrt{2})}{19}$ or $\dfrac{10\sqrt{3} + 6\sqrt{2}}{19}$

**d 1** Substitute the given value and simplify.

**d** Given $p = 3\sqrt{2} - 1$,

$\dfrac{1}{p^2 - 1}$

$= \dfrac{1}{(3\sqrt{2} - 1)^2 - 1}$

$= \dfrac{1}{(9 \times 2 - 6\sqrt{2} + 1) - 1}$

$= \dfrac{1}{18 - 6\sqrt{2}}$

**2** Factorise the denominator so that the binomial surd is simpler.

$= \dfrac{1}{6(3 - \sqrt{2})}$

3 Multiply numerator and denominator by the conjugate of the binomial surd contained in the denominator.

$$= \frac{1}{6(3-\sqrt{2})} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

$$= \frac{3+\sqrt{2}}{6(3-\sqrt{2})(3+\sqrt{2})}$$

4 Expand the difference of two squares and simplify.

$$= \frac{3+\sqrt{2}}{6((3)^2-(\sqrt{2})^2)}$$

$$= \frac{3+\sqrt{2}}{6(9-2)}$$

$$= \frac{3+\sqrt{2}}{6 \times 7}$$

$$= \frac{3+\sqrt{2}}{42}$$

## EXERCISE 2.6 Surds

**PRACTISE**

Work without CAS

1 WE15 a Express $\{3\sqrt{3}, 4\sqrt{5}, 5\sqrt{2}, 5\}$ with its elements in increasing order.

b Express the following in simplest form.

i $\sqrt{84}$

ii $2\sqrt{108ab^2}$ assuming $b > 0$

2 Express $\sqrt[3]{384}$ in simplest form.

3 WE16 Simplify the following.

a $\sqrt{5} - 4\sqrt{7} - 7\sqrt{5} + 3\sqrt{7}$

b $3\sqrt{48} - 4\sqrt{27} + 3\sqrt{32}$

c $3\sqrt{5} \times 7\sqrt{15}$

4 Simplify $\frac{1}{2}\sqrt{12} - \frac{1}{5}\sqrt{80} + \frac{\sqrt{10}}{\sqrt{2}} + \sqrt{243} + 5$.

5 WE17 Expand and simplify the following.

a $2\sqrt{3}(4\sqrt{15} + 5\sqrt{3})$

b $(\sqrt{3} - 8\sqrt{2})(5\sqrt{5} - 2\sqrt{21})$

c $(4\sqrt{3} - 5\sqrt{2})^2$

d $(3\sqrt{5} - 2\sqrt{11})(3\sqrt{5} + 2\sqrt{11})$

6 a Expand $(\sqrt{2} + 1)^3$.

b If $(\sqrt{2} + \sqrt{6})^2 - 2\sqrt{3}(\sqrt{2} + \sqrt{6})(\sqrt{2} - \sqrt{6}) = a + b\sqrt{3}$, $a, b \in N$, find the values of $a$ and $b$.

7 WE18 a Express the following with a rational denominator.

i $\frac{6}{7\sqrt{2}}$

ii $\frac{3\sqrt{5} + 7\sqrt{15}}{2\sqrt{3}}$

b Simplify $14\sqrt{6} + \frac{12}{\sqrt{6}} - 5\sqrt{24}$.

c Express $\frac{10}{4\sqrt{3} + 3\sqrt{2}}$ with a rational denominator.

d Given $p = 4\sqrt{3} + 1$, calculate $\frac{1}{p^2 - 1}$, expressing the answer with a rational denominator.

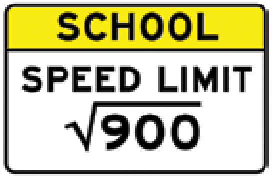
8 a Simplify $\dfrac{2\sqrt{3}-1}{\sqrt{3}+1}-\dfrac{\sqrt{3}}{\sqrt{3}+2}$ by first rationalising each denominator.

b Show that $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}+\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ is rational by first placing each fraction on a common denominator.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 Select the surds from the following set of real numbers.




$$\left\{\sqrt{8}, \sqrt{900}, \sqrt{\frac{4}{9}}, \sqrt{1.44}, \sqrt{10^3}, \pi, \sqrt[3]{27}, \sqrt[3]{36}\right\}$$

10 Express the following as entire surds.

a $4\sqrt{5}$
b $2\sqrt[3]{6}$
c $\dfrac{9\sqrt{7}}{4}$
d $\dfrac{3}{\sqrt{3}}$
e $ab\sqrt{c}$
f $m\sqrt[3]{n}$

11 Express each of the following in simplified form.

a $\sqrt{75}$
b $5\sqrt{48}$
c $\sqrt{2000}$
d $3\sqrt{288}$
e $2\sqrt{72}$
f $\sqrt[3]{54}$

12 Simplify the following.

a $3\sqrt{7}+8\sqrt{3}+12\sqrt{7}-9\sqrt{3}$
b $10\sqrt{2}-12\sqrt{6}+4\sqrt{6}-8\sqrt{2}$
c $3\sqrt{50}-\sqrt{18}$
d $8\sqrt{45}+2\sqrt{125}$
e $\sqrt{6}+7\sqrt{5}+4\sqrt{24}-8\sqrt{20}$
f $2\sqrt{12}-7\sqrt{243}+\frac{1}{2}\sqrt{8}-\frac{2}{3}\sqrt{162}$

13 Carry out the following operations and express answers in simplest form.

a $4\sqrt{5}\times 2\sqrt{7}$
b $-10\sqrt{6}\times -8\sqrt{10}$
c $3\sqrt{8}\times 2\sqrt{5}$
d $\sqrt{18}\times\sqrt{72}$
e $\dfrac{4\sqrt{27}\times\sqrt{147}}{2\sqrt{3}}$
f $5\sqrt{2}\times\sqrt{3}\times 4\sqrt{5}\times\dfrac{\sqrt{6}}{6}+3\sqrt{2}\times 7\sqrt{10}$

14 Expand and simplify the following.

a $\sqrt{2}(3\sqrt{5}-7\sqrt{6})$
b $5\sqrt{3}(7-3\sqrt{3}+2\sqrt{6})$
c $2\sqrt{10}-3\sqrt{6}(3\sqrt{15}+2\sqrt{6})$
d $(2\sqrt{3}+\sqrt{5})(3\sqrt{2}+4\sqrt{7})$
e $(5\sqrt{2}-3\sqrt{6})(2\sqrt{3}+3\sqrt{10})$
f $(\sqrt[3]{x}-\sqrt[3]{y})((\sqrt[3]{x})^2+\sqrt[3]{xy}+(\sqrt[3]{y})^2)$

15 Expand the following.

a $(2\sqrt{2}+3)^2$
b $(3\sqrt{6}-2\sqrt{3})^2$
c $(\sqrt{7}-\sqrt{5})^3$
d $(2\sqrt{5}+\sqrt{3})(2\sqrt{5}-\sqrt{3})$
e $(10\sqrt{2}-3\sqrt{5})(10\sqrt{2}+3\sqrt{5})$
f $(\sqrt{3}+\sqrt{2}+1)(\sqrt{3}+\sqrt{2}-1)$

16 Express the following in simplest form with rational denominators.

a $\dfrac{3\sqrt{2}}{4\sqrt{3}}$
b $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{2}}$
c $\dfrac{\sqrt{12}-3\sqrt{2}}{2\sqrt{18}}$
d $\dfrac{1}{\sqrt{6}+\sqrt{2}}$
e $\dfrac{2\sqrt{10}+1}{5-\sqrt{10}}$
f $\dfrac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

17 Express the following as a single fraction in simplest form.

a $4\sqrt{5} - 2\sqrt{6} + \frac{3}{\sqrt{6}} - \frac{10}{3\sqrt{5}}$

b $\sqrt{2}(2\sqrt{10} + 9\sqrt{8}) - \frac{\sqrt{5}}{\sqrt{5} - 2}$

c $\frac{3}{2\sqrt{3}(\sqrt{2} + \sqrt{3})^2} + \frac{2}{\sqrt{3}}$

d $\frac{2\sqrt{3} - \sqrt{2}}{2\sqrt{3} + \sqrt{2}} + \frac{2\sqrt{3} + \sqrt{2}}{2\sqrt{3} - \sqrt{2}}$

e $\frac{\sqrt{2}}{16 - 4\sqrt{7}} - \frac{\sqrt{14} + 2\sqrt{2}}{9\sqrt{7}}$

f $\frac{(2 - \sqrt{3})^2}{2 + \sqrt{3}} + \frac{2\sqrt{3}}{4 - 3\sqrt{2}}$

18 a If $x = 2\sqrt{3} - \sqrt{10}$, calculate the value of the following.

i $x + \frac{1}{x}$

ii $x^2 - 4\sqrt{3}x$

b If $y = \frac{\sqrt{7} + 2}{\sqrt{7} - 2}$, calculate the value of the following.

i $y - \frac{1}{y}$

ii $\frac{1}{y^2 - 1}$

c Determine the values of $m$ and $n$ for which each of the following is a correct statement.

i $\frac{1}{\sqrt{7} + \sqrt{3}} - \frac{1}{\sqrt{7} - \sqrt{3}} = m\sqrt{7} + n\sqrt{3}$

ii $(2 + \sqrt{3})^4 - \frac{7\sqrt{3}}{(2 + \sqrt{3})^2} = m + \sqrt{n}$

d The real numbers $x_1$ and $x_2$ are a pair of conjugates. If $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$:

i state $x_2$

ii calculate the sum $x_1 + x_2$

iii calculate the product $x_1 x_2$.

**MASTER**

19 A triangle has vertices at the points A$(\sqrt{2}, -1)$, B$(\sqrt{5}, \sqrt{10})$ and C$(\sqrt{10}, \sqrt{5})$.

a Calculate the lengths of each side of the triangle in simplest surd form.

b An approximation attributable to the Babylonians is that $\sqrt{a^2 \pm b} = a \pm \frac{b}{2a}$.

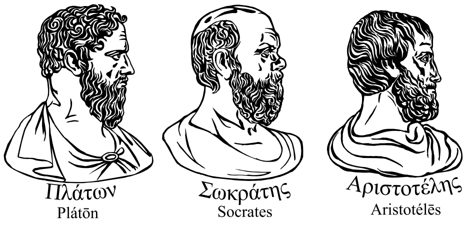
Use this formula to calculate approximate values for the lengths of each side of the triangle.

c Calculate from the surd form the length of the longest side to 1 decimal place.

20 A rectangular lawn has dimensions $(\sqrt{6} + \sqrt{3} + 1)$ m by $(\sqrt{3} + 2)$ m. Hew agrees to mow the lawn for the householder.

a Calculate the exact area of the lawn.

b If the householder received change of \$23.35 from \$50, what was the cost per square metre that Hew charged for mowing the lawn?




**Aristotle was probably the first to prove $\sqrt{2}$ was what we call irrational and what he called incommensurable. Plato, an ancient Greek philosopher, claimed his teacher Theodorus of Cyrene, building on Aristotle's approach, was the first to prove the irrationality of the non-perfect squares from 3 to 17. The work of Theodorus no longer exists.**

# eBookplus ONLINE ONLY 2.7 Review

www.jacplus.com.au

**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

## REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

# eBookplus ONLINE ONLY Activities

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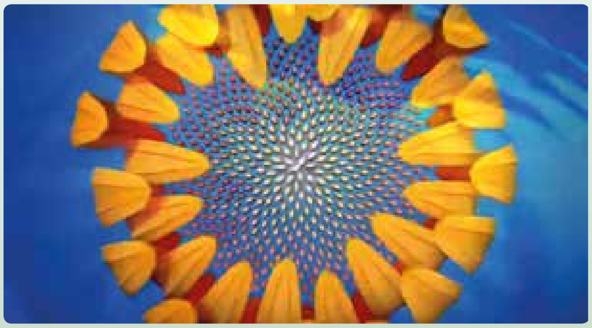
## Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.



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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

**studyon**

**Units 1 & 2** **Algebraic foundations**

**Sit topic test**

# 2 Answers

## EXERCISE 2.2

1 The expansion gives $55x^2 + 7x - 114$. The coefficient of $x$ is 7.

2 $-9x^4 + 328x^2 - 144$

3 a $4x(x - 3y)(x + y)$

b $(3y - x - 4)(3y + x + 4)$

c $(4x - 23)(x - 1)$

4 $(x - 3)(x - 3 - y)$

5 a $(x - 5)(x^2 + 5x + 25)$

b $3(1 + x)(1 - x + x^2)$

6 $2xy(x^2 - 3xy + 3y^2)$

7 a $\frac{x}{x - 2}$

b $8 - x^2$

8 $\frac{5}{x(x + 5)}$

9 $\frac{x - 31}{5(x - 5)(x - 1)}$

10 $\frac{1}{4x - 1}$

11 a $4x^2 + 12x + 9$

b $4ab^2 - 36a^3$

c $20 - 3c - 4c^2$

d $25 - 70y + 49y^2$

e $9m^6 - 16n^2$

f $x^3 + 3x^2 + 3x + 1$

12 a $g^2 + h^2 + 2gh + 24g + 24h + 144$

b $343q^3 + 98q^2p - 28qp^2 - 8p^3$

c $-4x^4 + 425x^2 - 2500$

13 a $6x^2 - 5x + 7; -5$

b $44 - 29x^2 - 15x^3; 0$

c $-16x^3 + 16x^2 + 49x - 49; 49$

d $2x^2 - 4y^2 + 2; 0$

e $39 - 75x + 11x^2; -75$

f $-3x^2 - 3x + 16; -3$

14 a $(x + 12)(x - 5)$

b $4(a - 4)(a + 4)$

c $(c + 1)(2b + 1)$

d $(3 + 2x)(9 - x)$

e $(3m - 2)(4 - 3m)$

f $8(x - 3y)^2$

15 a $(x + 2)(x - 5)(x + 5)$

b $p(10p - 9q)(10p + 9q)$

c $(2n + 1 - 2p)(2n + 1 + 2p)$

d $5(n + 5m)(23n - 11m)$

e $13(a - 1)(3 - 2a)(2a - 1)$

f $(a + b - 1)(2a - 1)$

16 a $(x + 13)(x - 2)$

b $(2x - 3)(x + 4)$

c $(7x + 9y)(10x + 7y)$

d $(x - 3)(x + 3)(x^2 + 1)$

e $(5p - q)^2$

f $(a - 1)^4$

17 a $(x - 2)(x^2 + 2x + 4)$

b $(x + 10)(x^2 - 10x + 100)$

c $(1 - x)(1 + x + x^2)$

d $(3x + 4y)(9x^2 - 12xy + 16y^2)$

e $x(x - 5)(x^2 + 5x + 25)$

f $(x + 5)(x^2 - 8x + 43)$

18 a $3(2x - 3y)(4x^2 + 6xy + 9y^2)$

b $xy(2xy + 1)(4x^2y^2 - 2xy + 1)$

c $7(9x - 10)(3x^2 + 100)$

d $-2(5x + 4y)(43x^2 + 31xy + 7y^2)$

e $(a - b)(a + b)^2(a^2 - ab + b^2)$

f $(x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$

19 a $\frac{x - 4}{5 - 3x}$

b $x + 3$

c $3x^2 + 3xh + h^2$

d $\frac{1}{3(1 - 3x)}$

e $\frac{m^2}{m^2 - mn + n^2}$

f $\frac{(1 - x)^2}{1 + x^2}$

20 a $\frac{4(x + 1)}{x(1 - x)(x^2 + 1)}$

b $\frac{2x + 20}{x^2 - 4}$

c $\frac{x - 46}{(x + 6)(x - 5)}$

d $\frac{2y^2 - 9y + 81}{y(2y - 9)^2(2y + 9)}$

e $\frac{p^2q}{p^4 - q^4}$

f $\frac{1}{2}(a - 2b)(a - b)(a + b)$

**21 a** $-3x^3 - 16x^2 + 9x + 70$

**b** $(13x^2 + 28x + 148)(7x + 2)$

**22 a** $\dfrac{11x + 16}{(x + 8)(x - 1)}$

**b** CAS use up to student

## EXERCISE 2.3

**1** $27x^3 - 54x^2 + 36x - 8$

**2** $\dfrac{a^3}{27} + \dfrac{a^2b^2}{3} + ab^4 + b^6$. The coefficient of $a^2b^2$ is $\dfrac{1}{3}$.

**3** $(a - b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

$(2x - 1)^6 = 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

**4** $(3x + 2y)^4 = 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$

**5 a** $(3x + 1)^3 = 27x^3 + 27x^2 + 9x + 1$

**b** $(1 - 2x)^3 = 1 - 6x + 12x^2 - 8x^3$

**c** $(5x + 2y)^3 = 125x^3 + 150x^2y + 60xy^2 + 8y^3$

**d** $\left(\dfrac{x}{2} - \dfrac{y}{3}\right)^3 = \dfrac{x^3}{8} - \dfrac{x^2y}{4} + \dfrac{xy^2}{6} - \dfrac{y^3}{27}$

**6** A

**7 a** $-9$

**b** $-975$

**c** 1

**d** $-42$

**8 a** 1, 7, 21, 35, 35, 21, 7, 1

**b** Row 9

**c** Row $n$ has $(n + 1)$ terms.

**9** *

**10 a** $(x + 4)^5 = x^5 + 20x^4 + 160x^3 + 640x^2 + 1280x + 1024$

**b** $(x - 4)^5 = x^5 - 20x^4 + 160x^3 - 640x^2 + 1280x - 1024$

**c** $(xy + 2)^5 = x^5y^5 + 10x^4y^4 + 40x^3y^3 + 80x^2y^2 + 80xy + 32$

**d** $(3x - 5y)^4 = 81x^4 - 540x^3y + 1350x^2y^2 - 1500xy^3 + 625y^4$

**e** $(3 - x^2)^4 = 81 - 108x^2 + 54x^4 - 12x^6 + x^8$

**f** $(1 + x)^6 - (1 - x)^6 = 12x + 40x^3 + 12x^5$

**11 a** $x^4 + y^4 - 4x^3 - 4y^3 + 6x^2 + 6y^2 - 4x - 4y + 4x^3y + 4xy^3 - 12x^2y - 12xy^2 + 6x^2y^2 + 12xy + 1$

**b** 20

**c** $a = -2$

**d** $-3$

**12 a** Row 3 sum is 8, row 4 sum is 16, row 5 sum is 32

**b** Sum of terms in row $n$ is $2^n$.

**c** $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

**d** $2^4 = 1 + 4 + 6 + 4 + 1 = 16$ illustrates the rule in part **b**.

**e** Let $x = 0.1$, $1.1^4 = 1.4641$

**13 a** $(x + 1)^5 - (x + 1)^4 = x^5 + 4x^4 + 6x^3 + 4x^2 + x$; proof required – check with your teacher

**b** Proof required – check with your teacher

**14** $a = 120$, $b = 55$, $c = 495$

**15 a** The consecutive natural numbers 1, 2, 3, 4, … where each number is one more than the preceding number.

**b** 21

**c** $\dfrac{n(n + 1)}{2}$

**d** $\dfrac{n(n + 1)(n + 2)}{6}$

**16** $x^8 + 4x^7 + 10x^6 + 16x^5 + 19x^4 + 16x^3 + 10x^2 + 4x + 1$; $0.91^4 = 0.68574961$

## EXERCISE 2.4

**1** 734

**2** $n(n - 1)$

**3** 35

**4** $\dbinom{n}{2} = \dfrac{n(n - 1)}{2}$ and $\dbinom{21}{2} = 210$

**5** $32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$

**6** $x^7 - 14x^6 + 84x^5 - 280x^4 + 560x^3 - 672x^2 + 448x - 128$

**7** $-\dfrac{35}{648}x^4y^3$

**8** $\dfrac{63x^{10}y^5}{8}$

*9

| Binomial power | Expansion | Number of terms in the expansion | Sum of indices in each term |
|---|---|---|---|
| $(x + a)^2$ | $x^2 + 2xa + a^2$ | 3 | 2 |
| $(x + a)^3$ | $x^3 + 3x^2a + 3xa^2 + a^3$ | 4 | 3 |
| $(x + a)^4$ | $x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$ | 5 | 4 |
| $(x + a)^5$ | $x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$ | 6 | 5 |

9 Sixth term; the coefficient of $x^{15}$ is $2^9 \times 3^5 \times 7$.

10 $t_4 = 160$

11 a 720

b 26

c 5040

d 120

e 3265920

f 900

12 a 650

b $\frac{1}{43}$

c $\frac{7}{5}$

d $\frac{6}{5}$

13 a $(n+1)!$

b $(n-1)!$

c $n(n-1)(n-2)$

d $\frac{1}{n(n+1)}$

e $\frac{2}{n(n+2)}$

f $\frac{1}{n-1}$

14 a 10

b 10

c 1

d 1140

e 1

f 286

15 a $\frac{n(n-1)(n-2)}{6}$

b $\frac{n(n-1)(n-2)}{6}$

c $\frac{(n+3)(n+2)(n+1)}{6}$

d $n(2n+1)$

e $\frac{n(n-1)(n+1)}{6}$

f $\frac{n(n-1)(n+1)}{6}$

16 a $(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

b $(2-x)^5 = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

c $(2x+3y)^6 = 64x^6 + 576x^5y + 2160x^4y^2 + 4320x^3y^3 + 4860x^2y^4 + 2916xy^5 + 729y^6$

d $\left(\frac{x}{2}+2\right)^7 = \frac{x^7}{128} + \frac{7x^6}{32} + \frac{21x^5}{8} + \frac{35x^4}{2} + 70x^3 + 168x^2 + 224x + 128$

e $\left(x - \frac{1}{x}\right)^8 = x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - \frac{56}{x^2} + \frac{28}{x^4} - \frac{8}{x^6} + \frac{1}{x^8}$

f $(x^2+1)^{10} = x^{20} + 10x^{18} + 45x^{16} + 120x^{14} + 210x^{12} + 252x^{10} + 210x^8 + 120x^6 + 45x^4 + 10x^2 + 1$

17 a $t_4 = 20000x^3$

b $t_{10} = 112640x^9$

c $t_6 = 1959552x^5$

d $t_3 = 1215x^8$

e $t_4 = -2500x^3$

f $t_4 = 280x^4y^3,\ t_5 = 560x^3y^4$

18 a $t_9 = 3247695x^4$

b $-672$

c $11 \times 7 \times 3^7 \times 2^{11}$

d $-14$

e 210

f $-20$

19 a $a = \frac{4}{7}$

b $n = 10$

20 a Proof required. Sum of row 10 is $2^{10}$.

b Proof required. Each term formed by adding the two terms to its left and right from the preceding row.

21 a 1 307 674 368 000

b 3003

22 a $n = 60$

b $r = 3$ or $r = 9$

## EXERCISE 2.5

1 a i $\frac{6}{11} \in Q$

ii $\sqrt{27} \in Q'$

iii $12 \in N$ or $12 \in Z$ or $12 \in Q$

iv $0.5 \in Q$

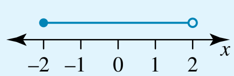
b i True

ii False

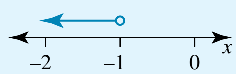
iii True

2 $x = -1$ or $x = 3$

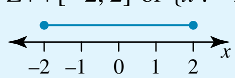
3 a i $\{x : -2 \le x < 2\}$




ii $(-\infty, -1)$




iii $Z \cap [-2, 2]$ or $\{x : -2 \le x \le 2, x \in Z\}$




b i $[3, 5]$

ii $R \setminus (3, 5)$ or $(-\infty, 3) \cup (5, \infty)$

4 $(-\infty, 1] \cup (4, \infty)$

5 E since $\frac{8}{0}$ is not defined

6 a $\sqrt{41} \in Q'$

b $-\frac{5}{9} \in Q$

c $R^+ = \{x : x > 0\}$

d $\sqrt{2.25} \in Q$

7 $\sqrt{11}, 11\pi, 2^\pi$

8 a True

b True

c False

d True

e False

f False

9 a $-5$

b 2

c $-\frac{3}{2}, 5$

d 0, 4

10 a $[-2, 3)$

b $(1, 9)$

c $(-\infty, 5)$

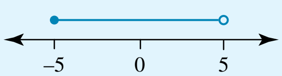
d $(0, 4]$

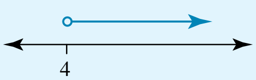
11 a $(4, 8]$

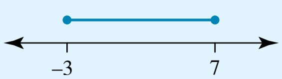
b $(-3, \infty)$

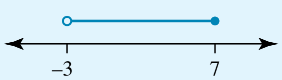
c $(-\infty, 0]$

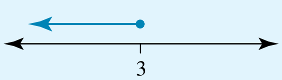
d $[-2, 0]$

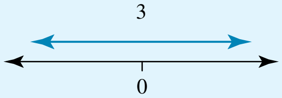
12 a 

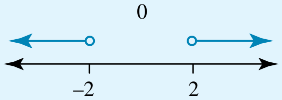

b 

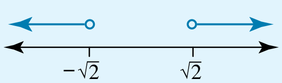

c 

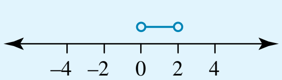

d 

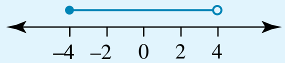

e 

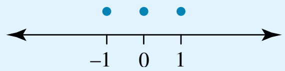

f 

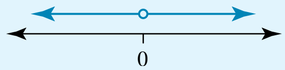

13 a 


b 


c 


d 


e 


f 


14 a $\{3, 4, 5\}$

b $(-\infty, -1] \cup (5, \infty)$

c $(-\infty, 0)$

d $R \setminus [-4, 2)$

15 a Rational

b Irrational

c Rational

16 $\pi \approx \frac{256}{81}$, $\frac{256}{81} = 3.160493827$ to 9 decimal places. $\frac{22}{7}$ is a better approximation.

## EXERCISE 2.6

1 a $\{5, 3\sqrt{3}, 5\sqrt{2}, 4\sqrt{5}\}$

b i $2\sqrt{21}$

ii $12b\sqrt{3a}$

2 $4\sqrt[3]{6}$

3 a $-6\sqrt{5} - \sqrt{7}$

b $12\sqrt{2}$

c $105\sqrt{3}$

4 $10\sqrt{3} + \frac{1}{5}\sqrt{5} + 5$

5 a $24\sqrt{5} + 30$

b $5\sqrt{15} - 6\sqrt{7} - 40\sqrt{10} + 16\sqrt{42}$

c $98 - 40\sqrt{6}$

d 1

6 a $7 + 5\sqrt{2}$

b $a = 8, b = 12$

7 a i $\frac{3\sqrt{2}}{7}$

ii $\frac{\sqrt{15} + 7\sqrt{5}}{2}$

b $6\sqrt{6}$

c $\frac{4\sqrt{3} - 3\sqrt{2}}{3}$

d $\frac{6 - \sqrt{3}}{264}$

8 a $\frac{13 - 7\sqrt{3}}{2}$

b 4, which is rational

9 $\sqrt{8}, \sqrt{10^3}, \sqrt[3]{36}$ are surds.

10 a $\sqrt{80}$

b $\sqrt[3]{48}$

c $\sqrt{\frac{567}{16}}$

d $\sqrt{3}$

e $\sqrt{a^2b^2c}$

f $\sqrt[3]{m^3n}$

11 a $5\sqrt{3}$

b $20\sqrt{3}$

c $20\sqrt{5}$

d $36\sqrt{2}$

e $12\sqrt{2}$

f $3\sqrt[3]{2}$

12 a $15\sqrt{7} - \sqrt{3}$

b $2\sqrt{2} - 8\sqrt{6}$

c $12\sqrt{2}$

d $34\sqrt{5}$

e $9\sqrt{6} - 9\sqrt{5}$

f $-59\sqrt{3} - 5\sqrt{2}$

13 a $8\sqrt{35}$

b $160\sqrt{15}$

c $12\sqrt{10}$

d 36

e $42\sqrt{3}$

f $62\sqrt{5}$

14 a $3\sqrt{10} - 14\sqrt{3}$

b $35\sqrt{3} + 30\sqrt{2} - 45$

c $-25\sqrt{10} - 36$

d $6\sqrt{6} + 8\sqrt{21} + 3\sqrt{10} + 4\sqrt{35}$

e $10\sqrt{6} + 30\sqrt{5} - 18\sqrt{2} - 18\sqrt{15}$

f $x - y$

15 a $17 + 12\sqrt{2}$

b $66 - 36\sqrt{2}$

c $22\sqrt{7} - 26\sqrt{5}$

d 17

e 155

f $4 + 2\sqrt{6}$

16 a $\frac{\sqrt{6}}{4}$

b $\frac{\sqrt{10} + 2}{2}$

c $\frac{\sqrt{6} - 3}{6}$

d $\frac{\sqrt{6} - \sqrt{2}}{4}$

e $\frac{25 + 11\sqrt{10}}{15}$

f $5 - \sqrt{6}$

17 a $\frac{20\sqrt{5} - 9\sqrt{6}}{6}$

b $2\sqrt{5} + 31$

c $\frac{19\sqrt{3} - 18\sqrt{2}}{6}$

d $\frac{14}{5}$

e $-\frac{\sqrt{14}}{252}$

f $26 - 19\sqrt{3} - 3\sqrt{6}$

18 a i $\frac{6\sqrt{3} - \sqrt{10}}{2}$

ii $-2$

b i $\frac{8\sqrt{7}}{3}$

ii $\frac{11\sqrt{7} - 28}{56}$

c i $m = 0, n = -\frac{1}{2}$

ii $m = 181, n = 147$

d i $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

ii $x_1 + x_2 = \frac{-b}{a}$

iii $x_1x_2 = \frac{c}{a}$

19 a $AB = 3\sqrt{2}, BC = \sqrt{30 - 20\sqrt{2}}, AC = \sqrt{18 - 2\sqrt{5}}$

b $AB \approx 4.5, BC \approx 1.333, AC \approx 3.6875$

c $AB \approx 4.2$ units

20 a $2\sqrt{6} + 3\sqrt{3} + 3\sqrt{2} + 5\ \text{m}^2$

b \$1.38 per square metre

# 3

# Quadratic relationships

# 3.1 Kick off with CAS

## Parabolic transformations

1 Using CAS technology sketch the following parabolas.

a $y = x^2$ b $y = -x^2$ c $y = -3x^2$ d $y = \frac{1}{2}x^2$ e $y = -\frac{2}{5}x^2$

2 Using CAS technology, enter $y = ax^2$ into the function entry line and use a slider to change the values of $a$.

3 Complete the following sentences.

a When sketching a quadratic function, a negative sign in front of the $x^2$ term ________ the graph of $y = x^2$.

b When sketching a quadratic function $y = ax^2$ for values of $a < -1$ and $a > 1$, the graph of $y = x^2$ becomes ___________.

c When sketching a quadratic function $y = ax^2$ for values of $-1 < a < 1, a \neq 0$, the graph of $y = x^2$ becomes ___________.

4 Using CAS technology sketch the following parabolas.

a $y = x^2$ b $y = (x + 1)^2$ c $y = -(x - 2)^2$

d $y = x^2 - 1$ e $y = -x^2 + 2$ f $y = 3 - x^2$

5 Using CAS technology, enter $y = (x - h)^2$ into the function entry line and use a slider to change the values of $h$.

6 Using CAS technology, enter $y = x^2 + c$ into the function entry line and use a slider to change the values of $c$.

7 Complete the following sentences.

a When sketching a quadratic function $y = (x - h)^2$, the graph of $y = x^2$ is _________.

b When sketching a quadratic function $y = x^2 + c$, the graph of $y = x^2$ is _________.

8 Use CAS technology and your answers to questions 1–7 to determine the equation that could model the shape of the Sydney Harbour Bridge. If the technology permits, upload a photo of the bridge to make this easier.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 3.2 Quadratic equations with rational roots

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Units 1 & 2
AOS 1
Topic 2
Concept 1

**Quadratic equations with rational roots**
Concept summary
Practice questions

Expressions such as $2x^2 + 3x + 4$ and $x^2 - 9$ are in quadratic form. Quadratic expansions always have an $x^2$ term as the term containing the highest power of the variable $x$. An expression such as $x^3 + 2x^2 + 3x + 4$ is not quadratic; it is called a cubic due to the presence of the $x^3$ term.

## Quadratic equations and the Null Factor Law

The general quadratic equation can be written as $ax^2 + bx + c = 0$, where $a, b, c$ are real constants and $a \neq 0$. If the quadratic expression on the left-hand side of this equation can be factorised, the solutions to the quadratic equation may be obtained using the **Null Factor Law**.

The Null Factor Law states that, for any $a$ and $b$, if the product $ab = 0$ then $a = 0$ or $b = 0$ or both $a$ and $b = 0$.

Applying the Null Factor Law to a quadratic equation expressed in the factorised form as $(x - \alpha)(x - \beta) = 0$, would mean that

$$(x - \alpha) = 0 \text{ or } (x - \beta) = 0$$
$$\therefore x = \alpha \text{ or } x = \beta$$

To apply the Null Factor Law, one side of the equation must be zero and the other side must be in factorised form.

### Roots, zeros and factors

eBookplus

**Interactivity**
Roots, zeros and factors
**int-2557**

The solutions of an equation are also called the **roots** of the equation or the **zeros** of the quadratic expression. This terminology applies to all algebraic and not just quadratic equations. The quadratic equation $(x - 1)(x - 2) = 0$ has roots $x = 1,\ x = 2$. These solutions are the zeros of the quadratic expression $(x - 1)(x - 2)$ since substituting either of $x = 1,\ x = 2$ in the quadratic expression $(x - 1)(x - 2)$ makes the expression equal zero.

As a converse of the Null Factor Law it follows that if the roots of a quadratic equation, or the zeros of a quadratic, are $x = \alpha$ and $x = \beta$, then $(x - \alpha)$ and $(x - \beta)$ are linear factors of the quadratic. The quadratic would be of the form $(x - \alpha)(x - \beta)$ or any multiple of this form, $a(x - \alpha)(x - \beta)$.

WORKED EXAMPLE 1

**a** Solve the equation $5x^2 - 18x = 8$.

**b** Given that $x = 2$ and $x = -2$ are zeros of a quadratic, form its linear factors and expand the product of these factors.

| THINK | WRITE |
|---|---|
| **a** 1 Rearrange the equation to make one side of the equation equal zero. | **a** $5x^2 - 18x = 8$<br>Rearrange:<br>$5x^2 - 18x - 8 = 0$ |
| 2 Factorise the quadratic trinomial. | $(5x + 2)(x - 4) = 0$ |
| 3 Apply the Null Factor Law. | $5x + 2 = 0$ or $x - 4 = 0$ |

| | |
|---|---|
| **4** Solve these linear equations for $x$. | $5x = -2$ or $x = 4$<br>$x = -\frac{2}{5}$ or $x = 4$ |
| **b 1** Use the converse of the Null Factor Law. | **b** Since $x = 2$ is a zero, then $(x - 2)$ is a linear factor, and since $x = -2$ is a zero, then $(x - (-2)) = (x + 2)$ is a linear factor.<br>Therefore the quadratic has the linear factors $(x - 2)$ and $(x + 2)$. |
| **2** Expand the product of the two linear factors. | The product $= (x - 2)(x + 2)$<br>Expanding, $(x - 2)(x + 2) = x^2 - 4$<br>The quadratic has the form $x^2 - 4$ or any multiple of this form $a(x^2 - 4)$. |

## Using the perfect square form of a quadratic

**eBook*plus***

**Interactivity**
The perfect square
**int-2558**

As an alternative to solving a quadratic equation by using the Null Factor Law, if the quadratic is a perfect square, solutions to the equation can be found by taking square roots of both sides of the equation. A simple illustration is

| using square root method | | using Null Factor Law method |
|---|---|---|
| $x^2 = 9$ | or | $x^2 = 9$ |
| $x = \pm\sqrt{9}$ | | $x^2 - 9 = 0$ |
| $= \pm 3$ | | $(x - 3)(x + 3) = 0$ |
| | | $x = \pm 3$ |

If the square root method is used, both the positive and negative square roots must be considered.

WORKED EXAMPLE 2

**Solve the equation $(2x + 3)^2 - 25 = 0$.**

| THINK | WRITE |
|---|---|
| **1** Rearrange so that each side of the equation contains a perfect square. | $(2x + 3)^2 - 25 = 0$<br>$(2x + 3)^2 = 25$ |
| **2** Take the square roots of both sides. | $2x + 3 = \pm 5$ |
| **3** Separate the linear equations and solve. | $2x + 3 = 5$ or $2x + 3 = -5$<br>$2x = 2$ $\quad$ $2x = -8$<br>$x = 1$ or $x = -4$ |
| **4** An alternative method uses the Null Factor Law. | Alternatively:<br>$(2x + 3)^2 - 25 = 0$<br>Factorise:<br>$((2x + 3) - 5)((2x + 3) + 5) = 0$<br>$(2x - 2)(2x + 8) = 0$<br>$2x = 2$ or $2x = -8$<br>$\therefore x = 1$ or $x = -4$ |

## Equations which reduce to quadratic form

Substitution techniques can be applied to the solution of equations such as those of the form $ax^4 + bx^2 + c = 0$. Once reduced to quadratic form, progress with the solution can be made.

The equation $ax^4 + bx^2 + c = 0$ can be expressed in the form $a(x^2)^2 + bx^2 + c = 0$. Letting $u = x^2$, this becomes $au^2 + bu + c = 0$, a quadratic equation in variable $u$.

By solving the quadratic equation for $u$, then substituting back $x^2$ for $u$, any possible solutions for $x$ can be obtained. Since $x^2$ cannot be negative, it would be necessary to reject negative $u$ values since $x^2 = u$ would have no real solutions.

The quadratic form may be achieved from substitutions other than $u = x^2$, depending on the form of the original equation. The choice of symbol for the substitution is at the discretion of the solver. The symbol $u$ does not have to be used; a commonly chosen symbol is $a$. However, if the original equation involves variable $x$, do not use $x$ for the substitution symbol.

**WORKED EXAMPLE 3**

**Solve the equation $4x^4 - 35x^2 - 9 = 0$.**

| THINK | WRITE |
|---|---|
| 1 Use an appropriate substitution to reduce the equation to quadratic form. | $4x^4 - 35x^2 - 9 = 0$<br>Let $a = x^2$<br>$4a^2 - 35a - 9 = 0$ |
| 2 Solve for $a$ using factorisation. | $(4a + 1)(a - 9) = 0$<br>$\therefore a = -\frac{1}{4}$ or $a = 9$ |
| 3 Substitute back, replacing $a$ by $x^2$. | $x^2 = -\frac{1}{4}$ or $x^2 = 9$ |
| 4 Since $x^2$ cannot be negative, any negative value of $a$ needs to be rejected. | Reject $x^2 = -\frac{1}{4}$ since there are no real solutions. |
| 5 Solve the remaining equation for $x$ | $x^2 = 9$<br>$x = \pm\sqrt{9}$<br>$x = \pm 3$ |

## EXERCISE 3.2 Quadratic equations with rational roots

**PRACTISE**

Work without CAS

1 WE1 a Solve the equation $10x^2 + 23x = 21$.

b Given that $x = -5$ and $x = 0$ are zeros of a quadratic, form its linear factors and expand the product of these factors.

2 Find the roots of the equation $32x^2 - 96x + 72 = 0$.

3 WE2 Solve the equation $(5x - 1)^2 - 16 = 0$.

4 Solve the equation $(px + q)^2 = r^2$ for $x$ in terms of $p, q$ and $r$, $r > 0$.

5 WE3 Solve the equation $9x^4 + 17x^2 - 2 = 0$.

6 Solve the equation $\left(x + \frac{1}{x}\right)^2 - 4\left(x + \frac{1}{x}\right) + 4 = 0$.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

For questions **7** and **8**, solve each of the given equations using the Null Factor Law.

**7** **a** $3x(5-x)=0$ **b** $(3-x)(7x-1)=0$
**c** $(x+8)^2=0$ **d** $2(x+4)(6+x)=0$

**8** **a** $6x^2+5x+1=0$ **b** $12x^2-7x=10$
**c** $49=14x-x^2$ **d** $5x+25-30x^2=0$
**e** $44+44x^2=250x$
**f** The use of the symbol $x$ for the variable is a conventional notation, although not obligatory. The Babylonians, who were the first to solve quadratic equations, just used words equivalent to 'length', 'breadth', 'area' for example, for the unknown quantity, and ignored their different dimensions. Write the following statement in contemporary form in terms of $x$ and hence obtain the required quantity.

Obtain the side of a square if the 'area' less the 'side' is 870.

(The name first given to an unknown was 'shay', meaning 'thing', and it appeared in the work of al-Khwarizmi. De Nemore was the first European mathematician to use a symbol for an unknown. For reasons not understood, he used the symbol *abc* as the unknown.)

For questions **9** and **10**, express each equation in quadratic form and hence solve the equations for $x$.

**9** **a** $x(x-7)=8$ **b** $4x(3x-16)=3(4x-33)$
**c** $(x+4)^2+2x=0$ **d** $(2x+5)(2x-5)+25=2x$

**10** **a** $2-3x=\dfrac{1}{3x}$ **b** $\dfrac{4x+5}{x+125}=\dfrac{5}{x}$
**c** $7x-\dfrac{2}{x}+\dfrac{11}{5}=0$ **d** $\dfrac{12}{x+1}-\dfrac{14}{x-2}=19$

**11** Obtain the solutions to the following equations.
**a** $x^2=121$ **b** $9x^2=16$
**c** $(x-5)^2=1$ **d** $(5-2x)^2-49=0$
**e** $2(3x-1)^2-8=0$ **f** $(x^2+1)^2=100$

**12** Use a substitution technique to solve the following equations.
**a** $(3x+4)^2+9(3x+4)-10=0$ **b** $2(1+2x)^2+9(1+2x)=18$
**c** $x^4-29x^2+100=0$ **d** $2x^4=31x^2+16$
**e** $36x^2=\dfrac{9}{x^2}-77$ **f** $(x^2+4x)^2+7(x^2+4x)+12=0$

**13** Obtain the solutions to the following equations.
**a** $x^4=81$ **b** $(9x^2-16)^2=20(9x^2-16)$
**c** $\left(x-\dfrac{2}{x}\right)^2-2\left(x-\dfrac{2}{x}\right)+1=0$ **d** $2\left(1+\dfrac{3}{x}\right)^2+5\left(1+\dfrac{3}{x}\right)+3=0$

**14** Express the value of $x$ in terms of the positive real numbers $a$ and $b$.
**a** $(x-2b)(x+3a)=0$ **b** $2x^2-13ax+15a^2=0$
**c** $(x-b)^4-5(x-b)^2+4=0$ **d** $(x-a-b)^2=4b^2$
**e** $(x+a)^2-3b(x+a)+2b^2=0$ **f** $ab\left(x+\dfrac{a}{b}\right)\left(x+\dfrac{b}{a}\right)=(a+b)^2x$

15 Consider the quadratic equation $(x-\alpha)(x-\beta)=0$.

a If the roots of the equation are $x=1$ and $x=7$, form the equation.
b If the roots of the equation are $x=-5$ and $x=4$, form the equation.
c If the roots of the equation are $x=0$ and $x=10$, form the equation.
d If the only root of the equation is $x=2$, form the equation.

16 a If the zeros of the quadratic expression $4x^2+bx+c$ are $x=-4$ and $x=\frac{3}{4}$, find the values of the integer constants $b$ and $c$.

b Express the roots of $px^2+(p+q)x+q=0$ in terms of $p$ and $q$ for $p, q \in Q, p \neq 0$ and hence solve $p(x-1)^2+(p+q)(x-1)+q=0$.

MASTER

In questions 17 and 18, use CAS technology to solve the equations.

17 $60x^2+113x-63=0$

18 $4x(x-7)+8(x-3)^2=x-26$

# 3.3 Quadratics over *R*

When $x^2-4$ is expressed as $(x-2)(x+2)$ it has been factorised over $Q$, as both of the zeros are rational numbers. However, over $Q$, the quadratic expression $x^2-3$ cannot be factorised into linear factors. Surds need to be permitted for such an expression to be factorised.

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Units 1 & 2
AOS 1
Topic 2
Concept 2

**Quadratics over *R***
Concept summary
Practice questions

## Factorisation over *R*

The quadratic $x^2-3$ can be expressed as the difference of two squares $x^2-3=x^2-(\sqrt{3})^2$ using surds. This can be factorised over $R$ because it allows the factors to contain surds.

$$\begin{aligned} x^2-3 &= x^2-(\sqrt{3})^2 \\ &= (x-\sqrt{3})(x+\sqrt{3}) \end{aligned}$$

If a quadratic can be expressed as the difference of two squares, then it can be factorised over $R$. To express a quadratic trinomial as a difference of two squares a technique called '**completing the square**' is used.

### 'Completing the square' technique

eBook*plus*

**Interactivity**
Completing the square
**int-2559**

Expressions of the form $x^2 \pm px + \left(\frac{p}{2}\right)^2 = \left(x \pm \frac{p}{2}\right)^2$ are perfect squares. For example, $x^2+4x+4=(x+2)^2$.

To illustrate the 'completing the square' technique, consider the quadratic trinomial $x^2+4x+1$.

If 4 is added to the first two terms $x^2+4x$ then this will form a perfect square $x^2+4x+4$. However, 4 must also be subtracted in order not to alter the value of the expression.

$$x^2+4x+1 = x^2+4x+4-4+1$$

Grouping the first three terms together to form the perfect square and evaluating the last two terms,

$$= (x^2 + 4x + 4) - 4 + 1$$
$$= (x + 2)^2 - 3$$

By writing this difference of two squares form using surds, factors over $R$ can be found.

$$= (x + 2)^2 - (\sqrt{3})^2$$
$$= (x + 2 - \sqrt{3})(x + 2 + \sqrt{3})$$

Thus $\quad x^2 + 4x + 1 = (x + 2 - \sqrt{3})(x + 2 + \sqrt{3})$.

'Completing the square' is the method used to factorise **monic** quadratics over $R$. A monic quadratic is one for which the coefficient of $x^2$ equals 1.

For a monic quadratic, to complete the square, add and then subtract the square of half the coefficient of $x$. This squaring will always produce a positive number regardless of the sign of the coefficient of $x$.

$$x^2 \pm px = \left[x^2 \pm px + \left(\frac{p}{2}\right)^2\right] - \left(\frac{p}{2}\right)^2$$
$$= \left(x \pm \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$$

To complete the square on $ax^2 + bx + c$, the quadratic should first be written as $a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$ and the technique applied to the monic quadratic in the bracket. The common factor $a$ is carried down through all the steps in the working.

**WORKED EXAMPLE 4**

**Factorise the following over $R$.**

**a** $x^2 - 14x - 3$ **b** $2x^2 + 7x + 4$ **c** $4x^2 - 11$

| THINK | WRITE |
|---|---|
| **a 1** Add and subtract the square of half the coefficient of $x$. *Note:* The negative sign of the coefficient of $x$ becomes positive when squared. | **a** $x^2 - 14x - 3$<br>$= x^2 - 14x + 7^2 - 7^2 - 3$ |
| **2** Group the first three terms together to form a perfect square and evaluate the last two terms. | $= (x^2 - 14x + 49) - 49 - 3$<br>$= (x - 7)^2 - 52$ |
| **3** Factorise the difference of two squares expression. | $= (x - 7)^2 - (\sqrt{52})^2$<br>$= (x - 7 - \sqrt{52})(x - 7 + \sqrt{52})$ |
| **4** Express any surds in their simplest form. | $= (x - 7 - 2\sqrt{13})(x - 7 + 2\sqrt{13})$ |
| **5** State the answer. | Therefore:<br>$x^2 - 14x - 3 = (x - 7 - 2\sqrt{13})(x - 7 + 2\sqrt{13})$ |

**b** **1** First create a monic quadratic by taking the coefficient of $x^2$ out as a common factor. This may create fractions.

**2** Add and subtract the square of half the coefficient of $x$ for the monic quadratic expression.

**3** Within the bracket, group the first three terms together and evaluate the remaining terms.

**4** Factorise the difference of two squares that has been formed.

**5** State the answer.

**b** $2x^2 + 7x + 4$

$$= 2\left(x^2 + \frac{7}{2}x + 2\right)$$

$$= 2\left(x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 + 2\right)$$

$$= 2\left[\left(x^2 + \frac{7}{2}x + \frac{49}{16}\right) - \frac{49}{16} + 2\right]$$

$$= 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{49}{16} + 2\right]$$

$$= 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{49}{16} + \frac{32}{16}\right]$$

$$= 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{17}{16}\right]$$

$$= 2\left[\left(x + \frac{7}{4}\right) - \sqrt{\frac{17}{16}}\right]\left[\left(x + \frac{7}{4}\right) + \sqrt{\frac{17}{16}}\right]$$

$$= 2\left(x + \frac{7}{4} - \frac{\sqrt{17}}{4}\right)\left(x + \frac{7}{4} + \frac{\sqrt{17}}{4}\right)$$

$2x^2 + 7x + 4$

$$= 2\left(x + \frac{7}{4} - \frac{\sqrt{17}}{4}\right)\left(x + \frac{7}{4} + \frac{\sqrt{17}}{4}\right)$$

$$= 2\left(x + \frac{7 - \sqrt{17}}{4}\right)\left(x + \frac{7 + \sqrt{17}}{4}\right)$$

**c** The quadratic is a difference of two squares. Factorise it.

**c** $4x^2 - 11$

$$= (2x)^2 - (\sqrt{11})^2$$

$$= (2x - \sqrt{11})(2x + \sqrt{11})$$

## The discriminant

**eBook*plus***

**Interactivity**
Discriminant
**int-2560**

Some quadratics factorise over $Q$ and others factorise only over $R$. There are also some quadratics which cannot be factorised over $R$ at all. This happens when the 'completing the square' technique does not create a difference of two squares but instead leads to a sum of two squares. In this case no further factorisation is possible over $R$. For example, completing the square on $x^2 - 2x + 6$ would give:

$$x^2 - 2x + 6 = (x^2 - 2x + 1) - 1 + 6$$
$$= (x - 1)^2 + 5$$

As this is the sum of two squares, it cannot be factorised over $R$.

Completing the square can be a tedious process when fractions are involved so it can be useful to be able to determine in advance whether a quadratic factorises over $Q$ or over $R$, or does not factorise over $R$. Evaluating what is called the **discriminant** will allow these three possibilities to be discriminated between. In order to define the discriminant, we need to complete the square on the general quadratic trinomial $ax^2 + bx + c$.

$$
\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)\\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right)\\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]\\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}\right]\\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]
\end{aligned}
$$

The sign of the term $b^2 - 4ac$ will determine whether a difference of two squares or a sum of two squares has been formed. If this term is positive, a difference of two squares is formed, but if the term is negative then a sum of two squares is formed. This term, $b^2 - 4ac$, is called the discriminant of the quadratic. It is denoted by the Greek letter delta, $\Delta$.

$$\Delta = b^2 = 4ac$$

- If $\Delta < 0$ the quadratic has no real factors.
- If $\Delta \geq 0$ the quadratic has two real factors. The two factors are distinct (different) if $\Delta > 0$ and the two factors are identical if $\Delta = 0$.

For a quadratic $ax^2 + bx + c$ with real factors and $a, b, c \in Q$:

- If $\Delta$ is a perfect square, the factors are rational; the quadratic factorises over $Q$.
- If $\Delta > 0$ but not a perfect square, the factors contain surds; the quadratic factorises over $R$. Completing the square will be required if $b \neq 0$.
- If $\Delta = 0$, the quadratic is a perfect square.

**WORKED EXAMPLE 5**

**For each of the following quadratics, calculate the discriminant and hence state the number and type of factors and whether the 'completing the square' method would be needed to obtain the factors.**

**a** $2x^2 + 15x + 13$
**b** $5x^2 - 6x + 9$
**c** $-3x^2 + 3x + 8$
**d** $\frac{81}{4}x^2 - 12x + \frac{16}{9}$

| THINK | WRITE |
|---|---|
| **a 1** State the values of $a, b$ and $c$ needed to calculate the discriminant. | **a** $2x^2 + 15x + 13,\ a = 2, b = 15, c = 13$ |
| **2** State the formula for the discriminant. | $\Delta = b^2 - 4ac$ |
| **3** Substitute the values of $a, b$ and $c$. | $\therefore \Delta = (15)^2 - 4 \times (2) \times (13)$ <br> $= 225 - 104$ <br> $= 121$ |

4 Interpret the value of the discriminant.

Since $\Delta > 0$ and is a perfect square, the quadratic has two rational factors.
Completing the square is not essential as the quadratic factorises over $Q$.
Check: $2x^2 + 15x + 13 = (2x + 13)(x + 1)$

**b** 1 State $a, b, c$ and calculate the discriminant.

**b** $5x^2 - 6x + 9,\ a = 5, b = -6, c = 9$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-6)^2 - 4 \times (5) \times (9) \\ &= 36 - 180 \\ &= -144\end{aligned}$$

2 Interpret the value of the discriminant.

Since $\Delta < 0$, the quadratic has no real factors.

**c** 1 State $a, b, c$ and calculate the discriminant.

**c** $-3x^2 + 3x + 8,\ a = -3, b = 3, c = 8$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (3)^2 - 4 \times (-3) \times (8) \\ &= 9 + 96 \\ &= 105\end{aligned}$$

2 Interpret the value of the discriminant.

Since $\Delta > 0$ but is not a perfect square, there are two real factors. The quadratic factorises over $R$, so completing the square would be needed to obtain the factors.

**d** 1 State $a, b, c$ and calculate the discriminant.

**d** $\frac{81}{4}x^2 - 12x + \frac{16}{9}$

$a = \frac{81}{4}, b = -12, c = \frac{16}{9}$

$$\Delta = b^2 - 4ac$$

$$\begin{aligned}\therefore \Delta &= (-12)^2 - 4 \times \frac{81}{4} \times \frac{16}{9} \\ &= 144 - 144 \\ &= 0\end{aligned}$$

2 Interpret the value of the discriminant.

Since $\Delta = 0$, there are two identical rational factors. The quadratic is a perfect square. It factorises over $Q$, so completing the square is not essential.

Check: $\frac{81}{4}x^2 - 12x + \frac{16}{9} = \left(\frac{9}{2}x - \frac{4}{3}\right)^2$

## Quadratic equations with real roots

The choices of method to consider for solving the quadratic equation $ax^2 + bx + c = 0$ are:

- factorise over $Q$ and use the Null Factor Law
- factorise over $R$ by completing the square and use the Null Factor Law
- use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

## The quadratic formula

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**Interactivity**
The quadratic formula
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The **quadratic formula** is used for solving quadratic equations and is obtained by completing the square on the left-hand side of the equation $ax^2 + bx + c = 0$.

Using completing the square it has been shown that

$ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right)$.

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) &= 0 \\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

**The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.**

Often the coefficients in the quadratic equation make the use of the formula less tedious than completing the square. Although the formula can also be used to solve a quadratic equation which factorises over $Q$, factorisation is usually simpler, making it the preferred method.

**WORKED EXAMPLE 6**

**Use the quadratic formula to solve the equation $x(9 - 5x) = 3$.**

| THINK | WRITE |
|---|---|
| 1 The equation needs to be expressed in the general quadratic form $ax^2 + bx + c = 0$. | $x(9 - 5x) = 3$<br>$9x - 5x^2 = 3$<br>$5x^2 - 9x + 3 = 0$ |
| 2 State the values of $a$, $b$ and $c$. | $a = 5, b = -9, c = 3$ |
| 3 State the formula for solving a quadratic equation. | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| 4 Substitute the $a$, $b$, $c$ values and evaluate. | $= \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times (5) \times (3)}}{2 \times (5)}$<br>$= \frac{9 \pm \sqrt{81 - 60}}{10}$<br>$= \frac{9 \pm \sqrt{21}}{10}$ |

| | |
|---|---|
| **5** Express the roots in simplest surd form and state the answers. *Note:* If the question asked for answers correct to 2 decimal places, use your calculator to find approximate answers of $x \simeq 0.44$ and $x \simeq 1.36$. Otherwise, do *not* approximate answers. | The solutions are: $x = \frac{9 - \sqrt{21}}{10},\ x = \frac{9 + \sqrt{21}}{10}$ |

## The role of the discriminant in quadratic equations

The type of factors determines the type of solutions to an equation, so it is no surprise that the discriminant determines the number and type of solutions as well as the number and type of factors.

The formula for the solution to the quadratic equation $ax^2 + bx + c = 0$ can be expressed as $x = \frac{-b \pm \sqrt{\Delta}}{2a}$, where the discriminant $\Delta = b^2 - 4ac$.

- If $\Delta < 0$, there are no real solutions to the equation.
- If $\Delta = 0$, there is one real solution (or two equal solutions) to the equation.
- If $\Delta > 0$, there are two distinct real solutions to the equation.

For $a, b, c \in Q$:

- If $\Delta$ is a perfect square, the roots are rational.
- If $\Delta$ is not a perfect square, the roots are irrational.

WORKED EXAMPLE 7

**a** **Use the discriminant to determine the number and type of roots to the equation $15x^2 + 8x - 5 = 0$.**

**b** **Find the values of $k$ so the equation $x^2 + kx - k + 8 = 0$ will have one real solution and check the answer.**

| THINK | WRITE |
|---|---|
| **a 1** Identify the values of $a, b, c$ from the general $ax^2 + bx + c = 0$ form. | **a** $15x^2 + 8x - 5 = 0,\ a = 15, b = 8, c = -5$ |
| **2** State the formula for the discriminant. | $\Delta = b^2 - 4ac$ |
| **3** Substitute the values of $a, b, c$ and evaluate. | $= (8)^2 - 4 \times (15) \times (-5)$<br>$= 64 + 300$<br>$= 364$ |
| **4** Interpret the result. | Since the discriminant is positive but not a perfect square, the equation has two irrational roots. |
| **b 1** Express the equation in general form and identify the values of $a, b$ and $c$. | **b** $x^2 + kx - k + 8 = 0$<br>$\therefore\ x^2 + kx + (-k + 8) = 0$<br>$a = 1, b = k, c = (-k + 8)$ |

| | |
|---|---|
| **2** Substitute the values of $a, b, c$ and obtain an algebraic expression for the discriminant. | $\Delta = b^2 - 4ac$<br>$= (k)^2 - 4 \times (1) \times (-k + 8)$<br>$= k^2 + 4k - 32$ |
| **3** State the condition on the discriminant for the equation to have one solution. | For one solution, $\Delta = 0$ |
| **4** Solve for $k$. | $k^2 + 4k - 32 = 0$<br>$(k + 8)(k - 4) = 0$<br>$k = -8,\ k = 4$ |
| **5** Check the solutions of the equation for each value of $k$. | If $k = -8$, the original equation becomes:<br>$x^2 - 8x + 16 = 0$<br>$(x - 4)^2 = 0$<br>$\therefore \quad x = 4$<br>This equation has one solution.<br>If $k = 4$, the original equation becomes:<br>$x^2 + 4x + 4 = 0$<br>$(x + 2)^2 = 0$<br>$\therefore \quad x = -2$<br>This equation has one solution. |

## Quadratic equations with rational and irrational coefficients

From the converse of the Null Factor Law, if the roots of a quadratic equation are $x = \alpha$ and $x = \beta$, then the factorised form of the quadratic equation would be $a(x - \alpha)(x - \beta) = 0$, $a \in R$. The roots could be rational or irrational. If both roots are rational, expanding the factors will always give a quadratic with rational coefficients. However, if the roots are irrational not all of the coefficients of the quadratic may be rational.

For example, if the roots of a quadratic equation are $x = \sqrt{2}$ and $x = 3\sqrt{2}$, then the factorised form of the equation is $\left(x - \sqrt{2}\right)\left(x - 3\sqrt{2}\right) = 0$. This expands to $x^2 - 4\sqrt{2}x + 6 = 0$, where the coefficient of $x$ is irrational.

However, if the roots were a pair of conjugate surds such as $x = \sqrt{2}$ and $x = -\sqrt{2}$ then the equation is $\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) = 0$. This expands to $x^2 - 2 = 0$, where all coefficients are rational.

The formula gives the roots of the general equation $ax^2 + bx + c = 0,\ a, b, c \in Q$ as $x_1 = \dfrac{-b - \sqrt{\Delta}}{2a}$ and $x_2 = \dfrac{-b + \sqrt{\Delta}}{2a}$. If $\Delta > 0$ but not a perfect square, these roots are irrational and $x_1 = \dfrac{-b}{2a} - \dfrac{\sqrt{\Delta}}{2a}$, $x_2 = \dfrac{-b}{2a} + \dfrac{\sqrt{\Delta}}{2a}$ are a pair of conjugate surds.

- If a quadratic equation with rational coefficients has irrational roots, the roots must occur in conjugate surd pairs.
- The conjugate surd pairs are of the form $m - n\sqrt{p}$ and $m + n\sqrt{p}$ with $m, n, p \in Q$.

WORKED EXAMPLE 8

**a** Use 'completing the square' to solve the quadratic equation $x^2 + 8\sqrt{2}x - 17 = 0$. Are the solutions a pair of conjugate surds?

**b** One root of the quadratic equation with rational coefficients $x^2 + bx + c = 0, b, c \in Q$ is $x = 6 + \sqrt{3}$. State the other root and calculate the values of $b$ and $c$.

| THINK | WRITE |
|---|---|
| **a 1** Add and also subtract the square of half the coefficient of $x$. | **a** $x^2 + 8\sqrt{2}x - 17 = 0$<br>$x^2 + 8\sqrt{2}x + (4\sqrt{2})^2 - (4\sqrt{2})^2 - 17 = 0$ |
| **2** Group the first three terms of the left-hand side of the equation together to form a perfect square and evaluate the last two terms. | $(x^2 + 8\sqrt{2}x + (4\sqrt{2})^2) - 32 - 17 = 0$<br>$(x + 4\sqrt{2})^2 - 49 = 0$ |
| **3** Rearrange so that each side of the equation contains a perfect square. | $(x + 4\sqrt{2})^2 = 49$ |
| **4** Take the square roots of both sides. | $x + 4\sqrt{2} = \pm\sqrt{49}$<br>$x + 4\sqrt{2} = \pm 7$ |
| **5** Solve for $x$. | $x = \pm 7 - 4\sqrt{2}$<br>$\therefore\ x = 7 - 4\sqrt{2},\ x = -7 - 4\sqrt{2}$ |
| **6** Explain whether or not the solutions are conjugate surds. | The solutions are not a pair of conjugate surds. The conjugate of $7 - 4\sqrt{2}$ is $7 + 4\sqrt{2}$, not $-7 - 4\sqrt{2}$. |
| **b 1** State the conjugate surd root. | **b** $x^2 + bx + c = 0,\ b, c \in Q$<br>Since the equation has rational coefficients, the roots occur in conjugate surd pairs. Therefore, the other root is $x = 6 - \sqrt{3}$. |
| **2** Form the two linear factors of the equation in factorised form. | $x = 6 - \sqrt{3} \Rightarrow (x - (6 - \sqrt{3}))$ is a factor and<br>$x = 6 + \sqrt{3} \Rightarrow (x - (6 + \sqrt{3}))$ is a factor. |
| **3** Write the equation. | The equation is<br>$(x - (6 + \sqrt{3}))(x - (6 - \sqrt{3})) = 0$ |
| **4** Expand as a difference of two squares. | $\therefore\ ((x - 6) + \sqrt{3})((x - 6) - \sqrt{3}) = 0$<br>$(x - 6)^2 - (\sqrt{3})^2 = 0$ |
| **5** Express in the form $x^2 + bx + c = 0$. | $x^2 - 12x + 36 - 3 = 0$<br>$x^2 - 12x + 33 = 0$ |
| **6** State the values of $b$ and $c$. | $b = -12,\ c = 33$ |

## Equations of the form $\sqrt{x} = ax + b$

Equations of the form $\sqrt{x} = ax + b$ could be written as $\sqrt{x} = a(\sqrt{x})^2 + b$ and reduced to a quadratic equation $u = au^2 + b$ by the substitution $u = \sqrt{x}$. Any negative solution for $u$ would need to be rejected as $\sqrt{x} \geq 0$.

An alternative technique to obtain quadratic form is discussed here.

By squaring both sides of the equation $\sqrt{x} = ax + b$ the quadratic equation $x = (ax + b)^2$ is formed, with no substitution required.

However, since the same quadratic equation would be obtained by squaring $\sqrt{x} = -(ax + b)$, the squaring process may produce extraneous 'solutions' — ones that do not satisfy the original equation. It is always necessary to verify the solutions by testing whether they satisfy the original equation.

**WORKED EXAMPLE 9**

**Solve the equation $3 + 2\sqrt{x} = x$ for $x$.**

| THINK | WRITE |
| --- | --- |
| 1 Isolate the surd term on one side of the equation. | $3 + 2\sqrt{x} = x$<br>$2\sqrt{x} = x - 3$ |
| 2 Square both sides to remove the surd term. | $(2\sqrt{x})^2 = (x - 3)^2$<br>$4x = (x - 3)^2$ |
| 3 Expand to form the quadratic equation and solve this equation. | $4x = x^2 - 6x + 9$<br>$x^2 - 10x + 9 = 0$<br>$(x - 9)(x - 1) = 0$<br>$\therefore x = 9$ or $x = 1$ |
| 4 Check $x = 9$ using the original equation. | Substitute $x = 9$ into $3 + 2\sqrt{x} = x$.<br>$\text{LHS} = 3 + 2\sqrt{x}$ $= 3 + 2\sqrt{9}$ $= 9$; $\text{RHS} = x$ $= 9$ $= \text{LHS}$<br>Therefore accept $x = 9$. |
| 5 Check $x = 1$ using the original equation. | Substitute $x = 1$ in $3 + 2\sqrt{x} = x$<br>$\text{LHS} = 3 + 2\sqrt{x}$ $= 3 + 2\sqrt{1}$ $= 5$; $\text{RHS} = x$ $= 1$ $\neq \text{LHS}$<br>Therefore reject $x = 1$. |
| 6 State the answer. | Answer is $x = 9$. |

## EXERCISE 3.3 Quadratics over *R*

**PRACTISE**

Work without CAS

1 WE4 Factorise the following over $R$.

a $x^2 - 10x - 7$ b $3x^2 + 7x + 3$ c $5x^2 - 9$

2 Use the 'completing the square' method to factorise $-3x^2 + 8x - 5$ and check the answer by using another method of factorisation.

3 WE5 For each of the following quadratics, calculate the discriminant and hence state the number and type of factors and whether the 'completing the square' method would be needed to obtain the factors.

a $4x^2 + 5x + 10$ b $169x^2 - 78x + 9$ c $-3x^2 + 11x - 10$ d $\frac{1}{3}x^2 - \frac{8}{3}x + 2$

**4** Factorise the following where possible.

**a** $3(x-8)^2-6$ **b** $(xy-7)^2+9$

**5** **WE6** Use the quadratic formula to solve the equation $(2x+1)(x+5)-1=0$.

**6** Solve the equation $3(2x+1)^4-16(2x+1)^2-35=0$ for $x \in R$.

**7** **WE7** **a** Use the discriminant to determine the number and type of roots to the equation $0.2x^2-2.5x+10=0$.

**b** Find the values of $k$ so the equation $kx^2-(k+3)x+k=0$ will have one real solution.

**8** Show that the equation $mx^2+(m-4)x=4$ will always have real roots for any real value of $m$.

**9** **WE8** **a** Complete the square to solve the equation $x^2-20\sqrt{5}x+100=0$.

**b** One root of the quadratic equation with rational coefficients $x^2+bx+c=0,\ b, c \in Q$ is $x=1-\sqrt{2}$. State the other root and calculate the values of $b$ and $c$.

**10** Solve the equation $\sqrt{2}x^2+4\sqrt{3}x-8\sqrt{2}=0$, expressing solutions in simplest surd form.

**11** **WE9** Solve the equation $4x-3\sqrt{x}=1$ for $x$.

**12** Use the substitution $u=\sqrt{x}$ to solve $4x-3\sqrt{x}=1$.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

**13** Complete the following statements about perfect squares.

**a** $x^2+10x+\ldots=(x+\ldots)^2$ **b** $x^2-7x+\ldots=(x-\ldots)^2$

**c** $x^2+x+\ldots=(x+\ldots)^2$ **d** $x^2-\frac{4}{5}x+\ldots=(x-\ldots)^2$

**14** Factorise the following over $R$, where possible.

**a** $x^2-12$ **b** $x^2-12x+4$ **c** $x^2+9x-3$

**d** $2x^2+5x+1$ **e** $3x^2+4x+3$ **f** $1+40x-5x^2$

**15** For each of the following, calculate the discriminant and hence state the number and type of linear factors.

**a** $5x^2+9x-2$ **b** $12x^2-3x+1$

**c** $121x^2+110x+25$ **d** $x^2+10x+23$

**16** **a** Factorise the difference of two cubes, $x^3-8$, and explain why there is only one linear factor over $R$.

**b** Form linear factors from the following information and expand the product of these factors to obtain a quadratic expression.

**i** The zeros of a quadratic are $x=\sqrt{2}$ and $x=-\sqrt{2}$.

**ii** The zeros of a quadratic are $x=-4+\sqrt{2}$ and $x=-4-\sqrt{2}$.

**17** Solve the following quadratic equations, giving the answers in simplest exact form.

**a** $9x^2-3x-4=0$

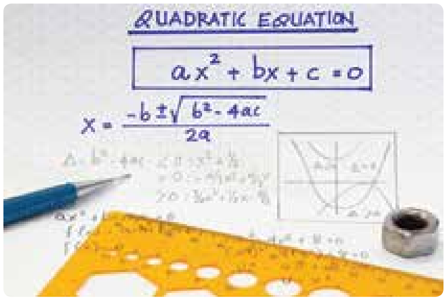
**b** $5x(4-x)=12$

**c** $(x-10)^2=20$

**d** $x^2+6x-3=0$

**e** $56x^2+51x-27=0$

**f** $5x-x(7+2x)=(x+5)(2x-1)$



18 Without actually solving the equations, determine the number and the nature of the roots of the following equations.

a $-5x^2 - 8x + 9 = 0$
b $4x^2 + 3x - 7 = 0$
c $4x^2 + x + 2 = 0$
d $28x - 4 - 49x^2 = 0$
e $4x^2 + 25 = 0$
f $3\sqrt{2}x^2 + 5x + \sqrt{2} = 0$

19 Use an appropriate substitution to reduce the following equations to quadratic form and hence obtain all solutions over $R$.

a $(x^2 - 3)^2 - 4(x^2 - 3) + 4 = 0$
b $5x^4 - 39x^2 - 8 = 0$
c $x^2(x^2 - 12) + 11 = 0$
d $\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) - 3 = 0$
e $(x^2 - 7x - 8)^2 = 3(x^2 - 7x - 8)$
f $3\left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) - 2 = 0$ given that $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$

20 a Find the values of $m$ so the equation $x^2 + (m + 2)x - m + 5 = 0$ has one root.
b Find the values of $m$ so the equation $(m + 2)x^2 - 2mx + 4 = 0$ has one root.
c Find the values of $p$ so the equation $3x^2 + 4x - 2(p - 1) = 0$ has no real roots.
d Show that the equation $kx^2 - 4x - k = 0$ always has two solutions for $k \in R\backslash\{0\}$.
e Show that for $p, q \in Q$, the equation $px^2 + (p + q)x + q = 0$ always has rational roots.

21 a Solve the following for $x$, expressing solutions in simplest surd form with rational denominators.
i $x^2 + 6\sqrt{2}x + 18 = 0$
ii $2\sqrt{5}x^2 - 3\sqrt{10}x + \sqrt{5} = 0$
iii $\sqrt{3}x^2 - (2\sqrt{2} - \sqrt{3})x - \sqrt{2} = 0$
b One root of the quadratic equation with rational coefficients, $x^2 + bx + c = 0$, $b, c \in Q$ is $x = \frac{-1 + \sqrt{5}}{2}$.
i State the other root.
ii Form the equation and calculate the values of $b$ and $c$.
c The roots of the quadratic equation with real coefficients, $x^2 + bx + c = 0,\ b, c \in R$, are $x = 4\sqrt{3} \pm 5\sqrt{6}$. Form the equation and calculate the values of $b$ and $c$.

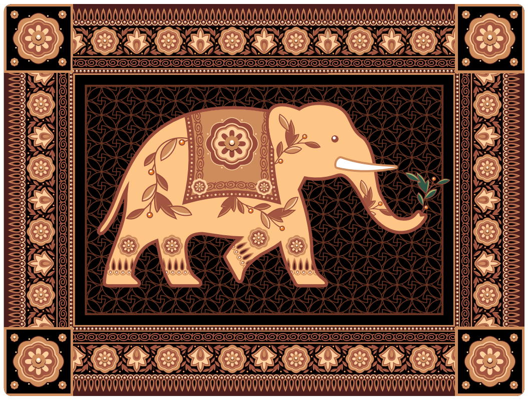
22 a Solve the equation $2\sqrt{x} = 8 - x$ by the following two methods:
i squaring both sides of the equation
ii using a suitable substitution to reduce to quadratic form.
b Solve $1 + \sqrt{x + 1} = 2x$.

MASTER

23 Use CAS technology to write down the factors over $R$ of $12x^2 + 4x - 9$.

24 a Show that the solutions of the simultaneous equations $x + y = p$ and $xy = q$ are the solutions of the quadratic equation $x^2 + q = px$.
b Write down the solutions for $x$ and $y$ in part a using CAS technology.

**The seventh century Hindu mathematician Brahmagupta may have been the first to solve quadratic equations involving roots that were negative or irrational, and he was reputedly the first person to use zero as a number. In the twelfth century another Indian mathematician, Bhaskara, refined much of Brahmagupta's work and contributed significantly to algebraic analysis.**



# 3.4 Applications of quadratic equations

Quadratic equations may occur in problem solving and as mathematical models. In formulating a problem, variables should be defined and it is important to check whether mathematical solutions are feasible in the context of the problem.

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Concept 3

**Applications of quadratic equations**
Concept summary
Practice questions

## Quadratically related variables

The formula for the area, $A$, of a circle in terms of its radius, $r$, is $A = \pi r^2$. This is of the form $A = kr^2$ as $\pi$ is a constant. The area varies directly as the square of its radius with the constant of proportionality $k = \pi$. This is a quadratic relationship between $A$ and $r$.

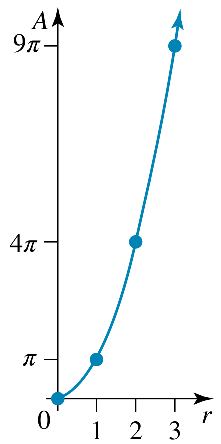
**WORKED EXAMPLE 10**

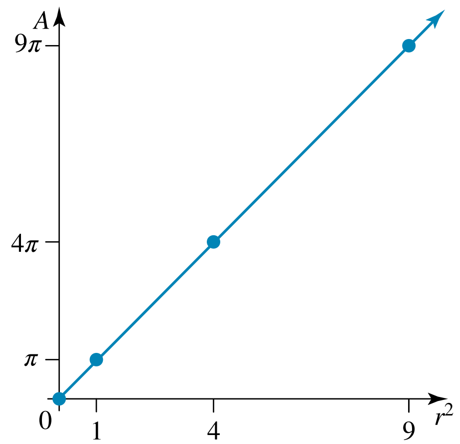
**The owner of a gift shop imported a certain number of paperweights for \$900 and was pleased when all except 4 were sold for \$10 more than what each paperweight had cost the owner to import. From the sale of the paperweights the gift shop owner received a total of \$1400. How many paperweights were imported?**

| THINK | WRITE |
|---|---|
| 1 Define the key variable. | Let $x$ be the number of paperweights imported. |
| 2 Find an expression for the cost of importing each paperweight. | The total cost of importing $x$ paperweights is \$900. Therefore the cost of each paperweight is $\left(\frac{900}{x}\right)$ dollars. |
| 3 Find an expression for the selling price of each paperweight and identify how many are sold. | The number of paperweights sold is $(x - 4)$ and each is sold for $\left(\frac{900}{x} + 10\right)$ dollars. |
| 4 Create the equation showing how the sales revenue of \$1400 is formed. | $\left(\frac{900}{x} + 10\right) \times (x - 4) = 1400$ |

| | |
|---|---|
| **5** Now the equation has been formulated, solve it. | Expand:<br>$900 - \frac{3600}{x} + 10x - 40 = 1400$<br>$-\frac{3600}{x} + 10x = 540$<br>$-3600 + 10x^2 = 540x$<br>$10x^2 - 540x - 3600 = 0$<br>$x^2 - 54x - 360 = 0$<br>$(x - 60)(x + 6) = 0$<br>$\therefore\ x = 60,\ x = -6$ |
| **6** Check the feasibility of the mathematical solutions. | Reject $x = -6$ since $x$ must be a positive whole number. |
| **7** Write the answer in context. | Therefore 60 paperweights were imported by the gift shop owner. |

| $r$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $A$ | 0 | $\pi$ | $4\pi$ | $9\pi$ |




Plotting the graph of $A$ against $r$ gives a curve which is part of a **parabola**. However, if $A$ was plotted against $r^2$ then a straight-line graph containing the origin would be obtained.

For any variables $x$ and $y$, if $y$ is directly proportional to $x^2$, then $y = kx^2$ where $k$ is the constant of proportionality.

Other quadratically related variables would include, for example, those where $y$ was the sum of two parts, one part of which was constant and the other part of which was in direct proportion to $x^2$ so that $y = c + kx^2$.

If, however, $y$ was the sum of two parts, one part of which varied as $x$ and another as $x^2$, then $y = k_1x + k_2x^2$ with different constants of proportionality required for each part.

The quadratic relation $y = c + k_1x + k_2x^2$ shows $y$ as the sum of three parts, one part constant, one part varying as $x$ and one part varying as $x^2$.

**WORKED EXAMPLE 11**

**The volume of a cone of fixed height is directly proportional to the square of the radius of its base. When the radius is 3 cm, the volume is $30\pi$ cm$^3$. Calculate the radius when the volume is $480\pi$ cm$^3$.**

| THINK | WRITE |
|---|---|
| 1 Write the variation equation, defining the symbols used. | $V = kr^2$ where $V$ is the volume of a cone of fixed height and radius $r$. $k$ is the constant of proportionality. |
| 2 Use the given information to find $k$. | $r = 3, V = 30\pi \Rightarrow 30\pi = 9k$ <br> $\therefore \quad k = \frac{30\pi}{9}$ <br> $= \frac{10\pi}{3}$ |
| 3 Write the rule connecting $V$ and $r$. | $V = \frac{10\pi}{3}r^2$ |
| 4 Substitute $V = 480\pi$ and find $r$. | $480\pi = \frac{10\pi}{3}r^2$ <br> $10\pi r^2 = 480\pi \times 3$ <br> $r^2 = \frac{480\pi \times 3}{10\pi}$ <br> $r^2 = 144$ <br> $r = \pm\sqrt{144}$ <br> $r = \pm 12$ |
| 5 Check the feasibility of the mathematical solutions. | Reject $r = -12$ since $r$ must be positive. <br> $\therefore r = 12$ |
| 6 Write the answer in context. | The radius of the cone is 12 cm. |

## EXERCISE 3.4 Applications of quadratic equations

**PRACTISE**

Work without CAS

1 WE10 The owner of a fish shop bought $x$ kilograms of salmon for \$400 from the wholesale market. At the end of the day all except for 2 kg of the fish were sold at a price per kg which was \$10 more than what the owner paid at the market. From the sale of the fish, a total of \$540 was made. How many kilograms of salmon did the fish-shop owner buy at the market?

2 The product of two consecutive even natural numbers is 440. What are the numbers?

**3** WE11 The surface area of a sphere is directly proportional to the square of its radius. When the radius is 5 cm, the area is $100\pi$ cm$^2$. Calculate the radius when the area is $360\pi$ cm$^2$.

**4** The cost of hiring a chainsaw is \$10 plus an amount that is proportional to the square of the number of hours for which the chainsaw is hired. If it costs \$32.50 to hire the chainsaw for 3 hours, find, to the nearest half hour, the length of time for which the chainsaw was hired if the cost of hire was \$60.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**5** **a** The area of an equilateral triangle varies directly as the square of its side length. A triangle of side length $2\sqrt{3}$ cm has an area of $3\sqrt{3}$ cm$^2$. Calculate the side length if the area is $12\sqrt{3}$ cm$^2$.

**b** The distance a particle falls from rest is in direct proportion to the square of the time of fall. What is the effect on the distance fallen if the time of fall is doubled?

**c** The number of calories of heat produced in a wire in a given time varies as the square of the voltage. If the voltage is reduced by 20%, what is the effect on the number of calories of heat produced?

**6** The sum of the first $n$ whole numbers is equal to the sum of two parts, one of which varies as $n$ and the other varies as $n^2$.

**a** Using $k_1$ and $k_2$ as the constants of proportionality for each part, write down an expression for the sum $S$ in terms of $n$.

**b** By calculating the sum of the first 4 whole numbers and the sum of the first 5 whole numbers, find the values of $k_1$ and $k_2$.

**c** If the sum of the first $n$ whole numbers is equal to 1275, what is the value of $n$?

**7** Given $y$ is the sum of three parts, one part constant, one part varying as $x$ and one part varying as $x^2$ and that $y = 2$ when $x = 0$, $y = 9$ when $x = 1$ and $y = 24$ when $x = 2$, find:

**a** the rule connecting $x$ and $y$

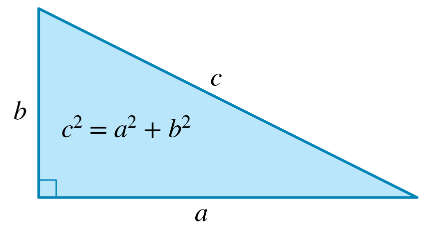
**b** the positive value of $x$ when $y = 117$.

**8** The cost of producing $x$ hundred litres of olive oil is $20 + 5x$ dollars. If the revenue from the sale of $x$ hundred litres of the oil is $1.5x^2$ dollars, calculate to the nearest litre, the number of litres that must be sold to make a profit of \$800.

9 The sum of the squares of two consecutive natural numbers plus the square of their sums is 662. What are the numbers?

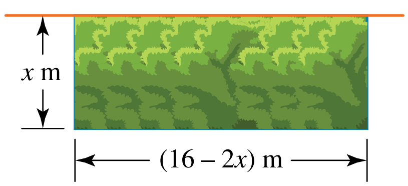
10 The ratio of the height of a triangle to the length of its base is $\sqrt{2} : 1$. If the area of the triangle is $\sqrt{32}$ cm$^2$, calculate the length of its base and of its height.

11 The hypotenuse of a right angled-triangle is $(3x + 3)$ cm and the other two sides are $3x$ cm and $(x - 3)$ cm. Determine the value of $x$ and calculate the perimeter of the triangle.




12 A photograph, 17 cm by 13 cm, is placed in a rectangular frame. If the border around the photograph is of uniform width and has an area of 260 cm$^2$, measured to the nearest cm$^2$, what are the dimensions of the frame measured to the nearest cm?

13 A gardener has 16 metres of edging to place around three sides of a rectangular garden bed, the fourth side of which is bounded by the backyard fence.




a If the width of the garden bed is $x$ metres, explain why its length is $(16 - 2x)$ metres.

b If the area of the rectangular garden is $k$ square metres, show that $2x^2 - 16x + k = 0$.

c Find the discriminant and hence find the values of $k$ for which this equation will have

i no solutions ii one solution iii two solutions.

d What is the largest area the garden bed can be and what are its dimensions in this case?

e The gardener decides the area of the garden bed is to be 15 square metres. Given that the gardener would also prefer to use as much of the backyard fence as possible as a boundary to the garden bed, calculate the dimensions of the rectangle in this case, correct to 1 decimal place.

14 A young collector of fantasy cards buys a parcel of the cards in a lucky dip at a fete for \$10 but finds on opening the parcel that only two are of interest. Keeping those two cards aside, the collector decides to resell the remaining cards to an unsuspecting friend for \$1 per card more than the original cost, thereby making a nice profit of \$6. How many cards did the collector's friend receive?

**MASTER**

Use the following information in questions **15** and **16**: the formula for the total surface area $A$ of a cone of base radius $r$ and slant height $l$ is $A = \pi r^2 + \pi rl$.

**15** Find, correct to 3 decimal places, the radius of the base of a cone with slant height 5 metres and total surface area 20 m$^2$.

**16** For any cone which has a surface area of 20 m$^2$, find $r$ in terms of $l$ and use this expression to check the answer to question **15**.

# 3.5 Graphs of quadratic polynomials

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Concept 4

**Graphs of quadratic polynomials**
Concept summary
Practice questions

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A quadratic **polynomial** is an algebraic expression of the form $ax^2 + bx + c$ where each power of the variable $x$ is a positive whole number, with the highest power of $x$ being 2. It is called a second-degree polynomial, whereas a linear polynomial of the form $ax + b$ is a first degree polynomial since the highest power of $x$ is 1.

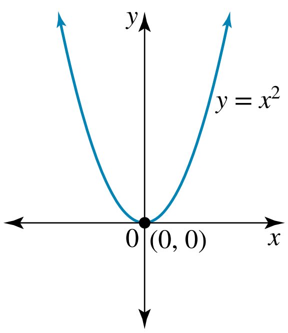
The graph of a quadratic polynomial is called a **parabola**.

## The graph of $y = x^2$ and transformations

The simplest parabola has the equation $y = x^2$.

Key features of the graph of $y = x^2$:

- it is symmetrical about the $y$-axis
- the **axis of symmetry** has the equation $x = 0$
- the graph is **concave up** (opens upwards)
- it has a minimum turning point, or vertex, at the point $(0, 0)$.

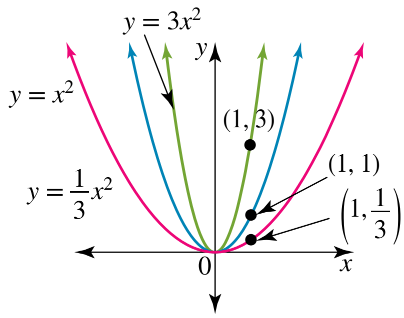



### Making the graph wider or narrower

The graphs of $y = ax^2$ for $a = \frac{1}{3}$, 1 and 3 are drawn on the same set of axes.

Comparison of the graphs of $y = x^2$, $y = 3x^2$ and $y = \frac{1}{3}x^2$ shows that the graph of $y = ax^2$ will be:

- narrower than the graph of $y = x^2$ if $a > 1$
- wider than the graph of $y = x^2$ if $0 < a < 1$.

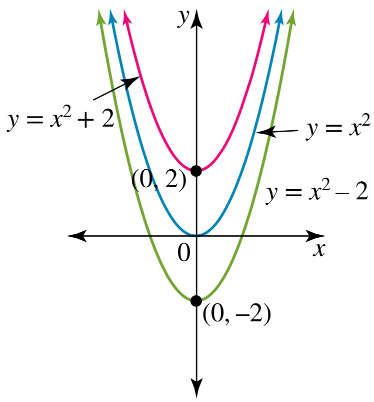



The coefficient of $x^2$, $a$, is called the **dilation factor**. It measures the amount of stretching or compression from the $x$-axis.

For $y = ax^2$, the graph of $y = x^2$ has been dilated by a factor of $a$ from the $x$-axis or by a factor of $a$ parallel to the $y$-axis.

### Translating the graph up or down

The graphs of $y = x^2 + k$ for $k = -2, 0$ and $2$ are drawn on the same set of axes.



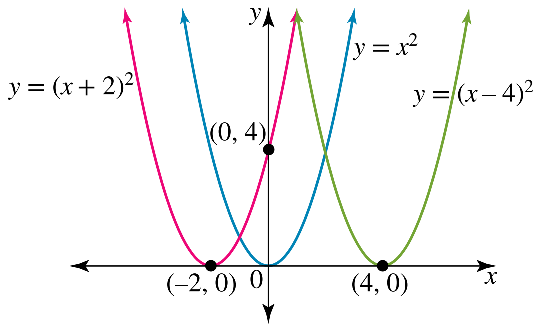

Comparison of the graphs of $y = x^2$, $y = x^2 + 2$ and $y = x^2 - 2$ shows that the graph of $y = x^2 + k$ will:

- have a turning point at $(0, k)$
- move the graph of $y = x^2$ vertically upwards by $k$ units if $k > 0$
- move the graph of $y = x^2$ vertically downwards by $k$ units if $k < 0$.

The value of $k$ gives the vertical translation.
For the graph of $y = x^2 + k$, the graph of $y = x^2$ has been translated by $k$ units from the $x$-axis.

### Translating the graph left or right

The graphs of $y = (x - h)^2$ for $h = -2, 1$ and $4$ are drawn on the same set of axes.



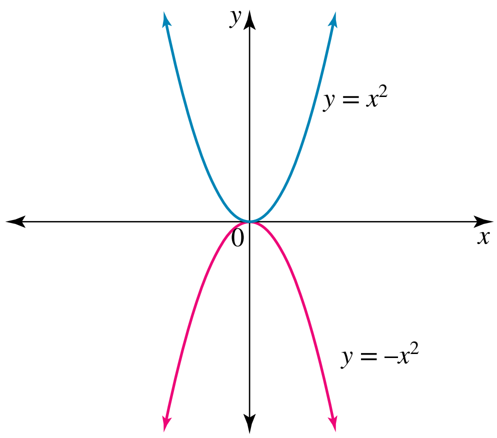

Comparison of the graphs of $y = x^2$, $y = (x + 2)^2$ and $y = (x - 4)^2$ shows that the graph of $y = (x - h)^2$ will:

- have a turning point at $(h, 0)$
- move the graph of $y = x^2$ horizontally to the right by $h$ units if $h > 0$
- move the graph of $y = x^2$ horizontally to the left by $h$ units if $h < 0$.

The value of $h$ gives the horizontal translation. For the graph of $y = (x - h)^2$, the graph of $y = x^2$ has been translated by $h$ units from the $y$-axis.

### Reflecting the graph in the x-axis

The graph of $y = -x^2$ is obtained by reflecting the graph of $y = x^2$ in the $x$-axis.




Key features of the graph of $y = -x^2$:

- it is symmetrical about the $y$-axis
- the axis of symmetry has the equation $x = 0$
- the graph is **concave down** (opens downwards)
- it has a maximum turning point, or vertex, at the point $(0, 0)$.

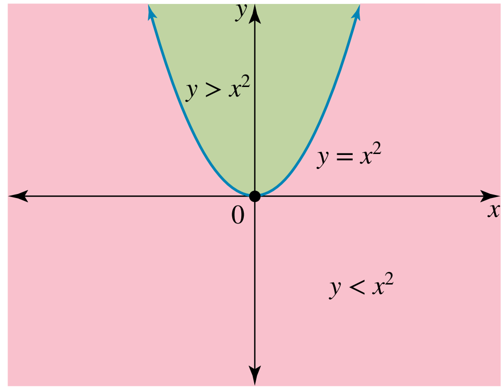
A negative coefficient of $x^2$ indicates the graph of a parabola is concave down.

### Regions above and below the graph of a parabola

A parabola divides the Cartesian plane into a region above the parabola and a region below the parabola. The regions that lie above and below the parabola $y = x^2$ are illustrated in the diagram.

The regions are described in a similar manner to the half planes that lie above or below a straight line.

- The region where the points have larger $y$-coordinates than the turning point, or any other point, on the parabola $y = x^2$ is described by the inequation $y > x^2$.
- The region where the points have smaller $y$-coordinates than the turning point, or any other point, on the parabola is described by the inequation $y < x^2$.




If the region is closed, the points on the boundary parabola are included in the region. If the region is open, the points on the boundary parabola are not included in the region.
A point can be tested to confirm which side of the parabola to shade.

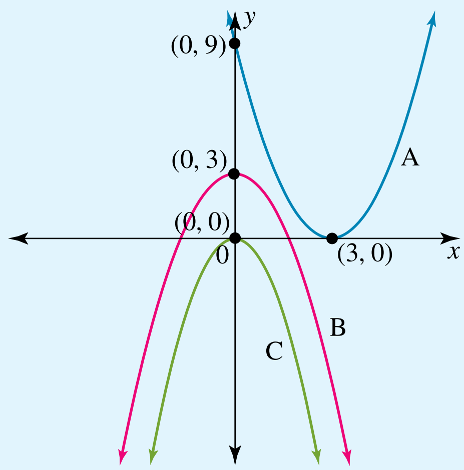
**WORKED EXAMPLE 12**

**Match the graphs of the parabolas A, B, C with the following equations.**

**a** $y = -x^2 + 3$

**b** $y = -3x^2$

**c** $y = (x - 3)^2$




| THINK | WRITE |
| --- | --- |
| **1** Compare graph A with the basic graph $y = x^2$ to identify the transformations. | Graph A opens upwards and has been moved horizontally to the right. Graph A matches with equation **c** $y = (x - 3)^2$. |
| **2** Compare graph B with the basic graph $y = x^2$ to identify the transformations. | Graph B opens downwards and has been moved vertically upwards. Graph B matches with equation **a** $y = -x^2 + 3$. |
| **3** Check graph C for transformations. | Graph C opens downwards. It is narrower than both graphs *A* and *B*. Graph C matches with equation **b** $y = -3x^2$. |

## Sketching parabolas from their equations

The key points required when sketching a parabola are:

- the turning point
- the $y$-intercept
- any $x$-intercepts.

The axis of symmetry is also a key feature of the graph.

The equation of a parabola allows this information to be obtained but in differing ways, depending on the form of the equation.

We shall consider three forms for the equation of a parabola:

- general form
- turning point form
- $x$-intercept form.

## The general, or polynomial form, $y = ax^2 + bx + c$

If $a > 0$ then the parabola is concave up and has a minimum turning point.

If $a < 0$ then the parabola is concave down and has a maximum turning point.

The dilation factor $a$, $a > 0$, determines the width of the parabola. The dilation factor is always a positive number (so it could be expressed as $|a|$).

The methods to determine the key features of the graph are as follows.

- Substitute $x = 0$ to obtain the $y$-intercept (the $y$-intercept is obvious from the equation).
- Substitute $y = 0$ and solve the quadratic equation $ax^2 + bx + c = 0$ to obtain the $x$-intercepts. There may be 0, 1 or 2 $x$-intercepts, as determined by the discriminant.
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$.
  This is because the formula for solving $ax^2 + bx + c = 0$ gives $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ as the $x$-intercepts and these are symmetrical about their midpoint $x = -\frac{b}{2a}$.
- the turning point lies on the axis of symmetry so its $x$-coordinate is $x = -\frac{b}{2a}$. Substitute this value into the parabola's equation to calculate the $y$-coordinate of the turning point.

**WORKED EXAMPLE 13** **Sketch the graph of $y = \frac{1}{2}x^2 - x - 4$ and label the key points with their coordinates.**

| THINK | WRITE |
| --- | --- |
| 1 Write down the $y$-intercept. | $y = \frac{1}{2}x^2 - x - 4$<br>$y$-intercept: if $x = 0$ then $y = -4 \Rightarrow (0, -4)$ |
| 2 Obtain any $x$-intercepts. | $x$-intercepts: let $y = 0$<br>$\frac{1}{2}x^2 - x - 4 = 0$<br>$x^2 - 2x - 8 = 0$<br>$(x + 2)(x - 4) = 0$<br>$\therefore \quad x = -2, 4$<br>$\Rightarrow (-2, 0), (4, 0)$ |

3 Find the equation of the axis of symmetry.

Axis of symmetry formula $x = -\frac{b}{2a}$, $a = \frac{1}{2}$, $b = -1$

$$x = -\frac{-1}{\left(2 \times \frac{1}{2}\right)}$$
$$= 1$$

4 Find the coordinates of the turning point.

Turning point: when $x = 1$,

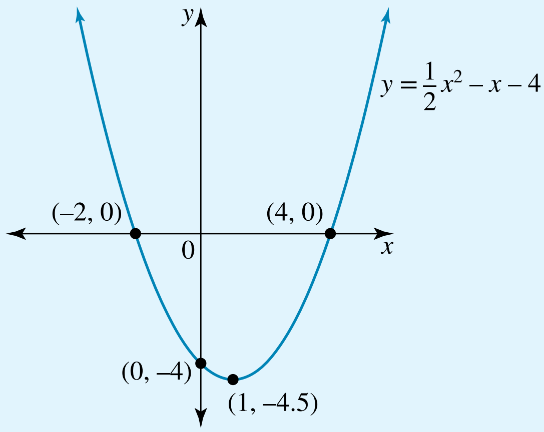
$$y = \tfrac{1}{2} - 1 - 4$$
$$= -4\tfrac{1}{2}$$

$\Rightarrow \left(1, -4\tfrac{1}{2}\right)$ is the turning point.

5 Identify the type of turning point.

Since $a > 0$, the turning point is a minimum turning point.

6 Sketch the graph using the information obtained in previous steps. Label the key points with their coordinates.




## Turning point form, $y = a(x - h)^2 + k$

Since $h$ represents the horizontal translation and $k$ the vertical translation, this form of the equation readily provides the coordinates of the turning point.

- The turning point has coordinates $(h, k)$.
  If $a > 0$, the turning point is a minimum and if $a < 0$ it will be a maximum. Depending on the nature of the turning point the $y$-coordinate of the turning point gives the minimum or maximum value of the quadratic.
- Find the $y$-intercept by substituting $x = 0$.
- Find the $x$-intercepts by substituting $y = 0$ and solving the equation $a(x - h)^2 + k = 0$. However, before attempting to find $x$-intercepts, consider the type of turning point and its $y$-coordinate as this will indicate whether there are any $x$-intercepts.

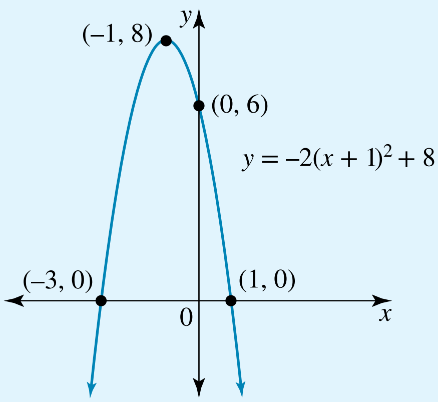
The general form of the equation of a parabola can be converted to turning point form by the use of the completing the square technique: by expanding turning point form, the general form would be obtained.

**WORKED EXAMPLE 14**

**a** Sketch the graph of $y = -2(x + 1)^2 + 8$ and label the key points with their coordinates.

**b** **i** Express $y = 3x^2 - 12x + 18$ in the form $y = a(x - h)^2 + k$ and hence state the coordinates of its vertex.

**ii** Sketch its graph.

| THINK | WRITE |
|---|---|
| **a 1** Obtain the coordinates and the type of turning point from the given equation. *Note:* The $x$-coordinate of the turning point could also be obtained by letting $(x + 1) = 0$ and solving this for $x$. | **a** $y = -2(x + 1)^2 + 8$<br>$\therefore\ y = -2(x - (-1))^2 + 8$<br>Maximum turning point at $(-1, 8)$ |
| **2** Calculate the $y$-intercept. | Let $x = 0$<br>$\therefore\ y = -2(1)^2 + 8$<br>$= 6$<br>$\Rightarrow (0, 6)$ |
| **3** Calculate any $x$-intercepts. *Note:* The graph is concave down with maximum $y$-value of 8, so there will be $x$-intercepts. | $x$-intercepts: let $y = 0$<br>$0 = -2(x + 1)^2 + 8$<br>$2(x + 1)^2 = 8$<br>$(x + 1)^2 = 4$<br>$(x + 1) = \pm\sqrt{4}$<br>$x = \pm 2 - 1$<br>$x = -3, 1$<br>$\Rightarrow (-3, 0), (1, 0)$ |
| **4** Sketch the graph, remembering to label the key points with their coordinates. | <br> |
| **b i 1** Apply the completing the square technique to the general form of the equation. | **b** $y = 3x^2 - 12x + 18$<br>$= 3(x^2 - 4x + 6)$<br>$= 3((x^2 - 4x + (2)^2) - (2)^2 + 6)$<br>$= 3((x - 2)^2 + 2)$ |

2 Expand to obtain the form $y = a(x - h)^2 + k$.

$= 3(x - 2)^2 + 6$

$\therefore y = 3(x - 2)^2 + 6$

3 State the coordinates of the vertex (turning point).

The vertex is (2, 6).

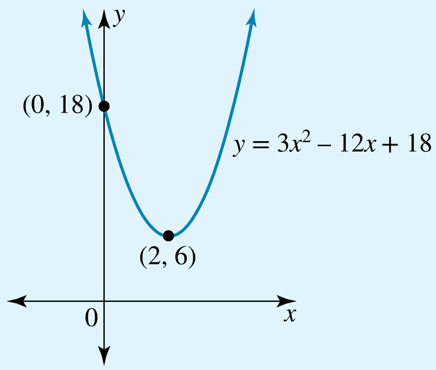
ii 1 Obtain the $y$-intercept from the general form.

$y$-intercept is (0, 18).

2 Will the graph have $x$-intercepts?

Since the graph is concave up with minimum $y$-value of 6, there are no $x$-intercepts.

3 Sketch the graph.




## Factorised, or $x$-intercept, form $y = a(x - x_1)(x - x_2)$

This form of the equation readily provides the $x$-intercepts.

- The $x$-intercepts occur at $x = x_1$ and $x = x_2$.
- The axis of symmetry lies halfway between the $x$-intercepts and its equation, $x = \dfrac{x_1 + x_2}{2}$, gives the $x$-coordinate of the turning point.
- The turning point is obtained by substituting $x = \dfrac{x_1 + x_2}{2}$ into the equation of the parabola and calculating the $y$-coordinate.
- The $y$-intercept is obtained by substituting $x = 0$.

If the linear factors are distinct, the graph cuts through the $x$-axis at each $x$-intercept.

If the linear factors are identical making the quadratic a perfect square, the graph touches the $x$-axis at its turning point.

**WORKED EXAMPLE 15**

**Sketch the graph of $y = -\dfrac{1}{2}(x + 5)(x - 1)$.**

**THINK**

1 Identify the $x$-intercepts.

**WRITE**

$y = -\dfrac{1}{2}(x + 5)(x - 1)$

$x$-intercepts: let $y = 0$

$\dfrac{1}{2}(x + 5)(x - 1) = 0$

$x + 5 = 0$ or $x - 1 = 0$

$x = -5$ or $x = 1$

$x$-intercepts are $(-5, 0)$, $(1, 0)$.

**2** Calculate the equation of the axis of symmetry.

Axis of symmetry has equation

$$x = \frac{-5 + 1}{2}$$

$\therefore x = -2$

**3** Obtain the coordinates of the turning point.

Turning point: substitute $x = -2$ in to the equation

$$\begin{aligned} y &= -\frac{1}{2}(x + 5)(x - 1) \\ &= -\frac{1}{2}(3)(-3) \\ &= \frac{9}{2} \end{aligned}$$

Turning point is $\left(-2, \frac{9}{2}\right)$.

**4** Calculate the $y$-intercept.

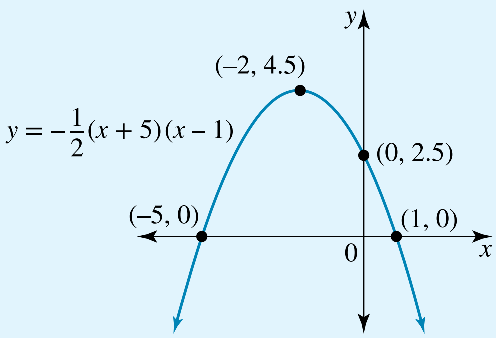
$y = -\frac{1}{2}(x + 5)(x - 1)$

$y$-intercept: let $x = 0$,

$$\begin{aligned} y &= -\frac{1}{2}(5)(-1) \\ &= \frac{5}{2} \end{aligned}$$

$y$-intercept is $\left(0, \frac{5}{2}\right)$.

**5** Sketch the graph.



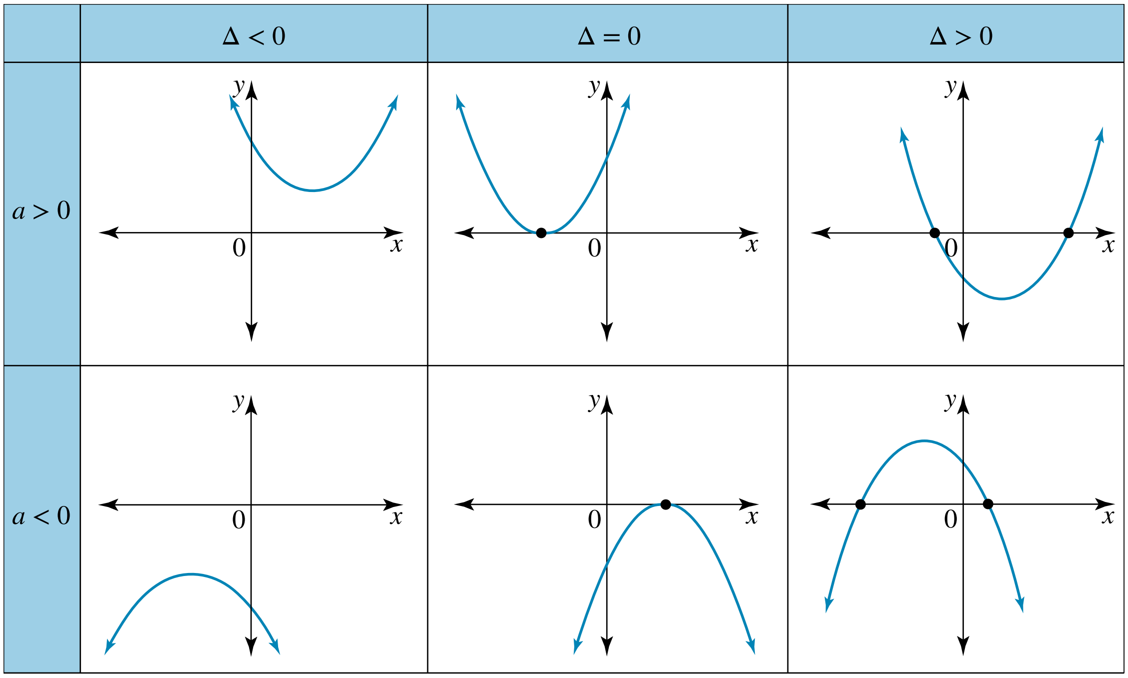

## The discriminant and the $x$-intercepts

The zeros of the quadratic expression $ax^2 + bx + c$, the roots of the quadratic equation $ax^2 + bx + c = 0$ and the $x$-intercepts of the graph of a parabola with rule $y = ax^2 + bx + c$ all have the same $x$-values; and the discriminant determines the type and number of these values.

- If $\Delta > 0$, there are two $x$-intercepts. The graph *cuts* through the $x$-axis at two different places.
- If $\Delta = 0$, there is one $x$-intercept. The graph *touches* the $x$-axis at its turning point.
- If $\Delta < 0$, there are no $x$-intercepts. The graph does not intersect the $x$-axis and lies entirely above or entirely below the $x$-axis, depending on its concavity.

If $a > 0$, $\Delta < 0$, the graph lies entirely above the $x$-axis and every point on it has a positive $y$-coordinate. $ax^2 + bx + c$ is called **positive definite** in this case.

If $a < 0$, $\Delta < 0$, the graph lies entirely below the $x$-axis and every point on it has a negative $y$-coordinate. $ax^2 + bx + c$ is called **negative definite** in this case.




When $\Delta \geq 0$ and for $a, b, c \in Q$, the $x$ intercepts are rational if $\Delta$ is a perfect square and irrational if $\Delta$ is not a perfect square.

**WORKED EXAMPLE 16**

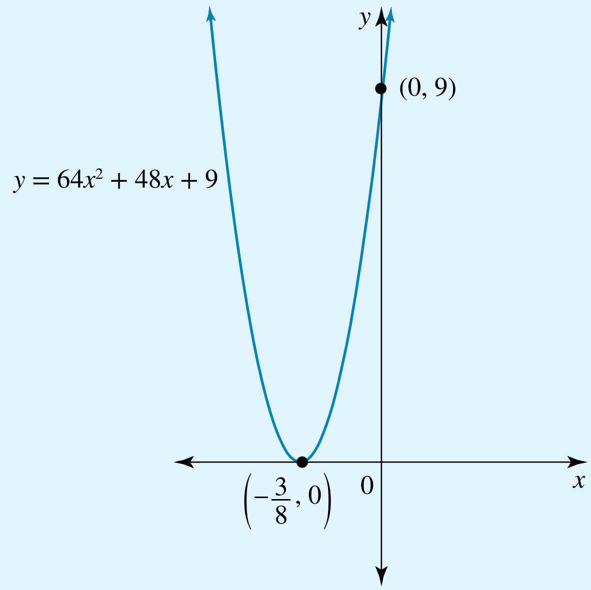
**Use the discriminant to:**

**a** **determine the number and type of $x$-intercepts of the graph defined by $y = 64x^2 + 48x + 9$**

**b** **sketch the graph of $y = 64x^2 + 48x + 9$.**

| THINK | WRITE |
|---|---|
| **a** **1** State the $a, b, c$ values and evaluate the discriminant. | **a** $y = 64x^2 + 48x + 9,\ a = 64,\ b = 48,\ c = 9$<br>$\Delta = b^2 - 4ac$<br>$= (48)^2 - 4 \times (64) \times (9)$<br>$= 2304 - 2304$<br>$= 0$ |
| **2** Interpret the result. | Since the discriminant is zero, the graph has one rational $x$-intercept. |
| **b** **1** Interpret the implication of a zero discriminant for the factors. | **b** The quadratic must be a perfect square.<br>$y = 64x^2 + 48x + 9$<br>$= (8x + 3)^2$ |
| **2** Identify the key points. | $x$-intercept: let $y = 0$.<br>$8x + 3 = 0$<br>$x = -\frac{3}{8}$<br>Therefore $\left(-\frac{3}{8}, 0\right)$ is both the $x$-intercept and the turning point.<br>$y$-intercept: let $x = 0$ in $y = 64x^2 + 48x + 9$.<br>$\therefore y = 9$<br>Therefore $(0, 9)$ is the $y$-intercept. |

**3** Sketch the graph.




## EXERCISE 3.5 Graphs of quadratic polynomials

**PRACTISE**

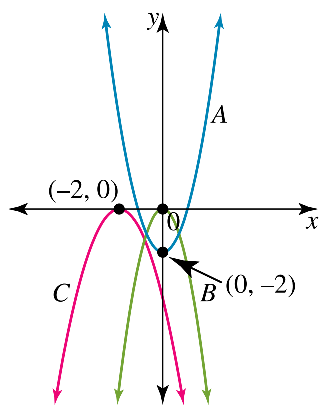
Work without CAS

1 WE12 Match the graphs of the parabolas $A$, $B$ and $C$ with the following equations.

i $y = x^2 - 2$

ii $y = -2x^2$

iii $y = -(x + 2)^2$




2 Using the graph of $y = -2x^2$ identified in question 1, shade the region described by $y \le -2x^2$.

3 WE13 Sketch the graph of $y = \frac{1}{3}x^2 + x - 6$ and label the key points with their coordinates.

4 Sketch the region given by $y > \frac{1}{3}x^2 + x - 6$.

5 WE14 a Sketch the graph of $y = -2(x + 1)^2 + 8$ and label the key points with their coordinates.

b i Express $y = -x^2 + 10x - 30$ in the form $y = a(x - h)^2 + k$ and hence state the coordinates of its vertex.

ii Sketch its graph.

6 State the nature and the coordinates of the turning point for each of the following parabolas.

a $y = 4 - 3x^2$

b $y = (4 - 3x)^2$

7 WE15 Sketch the graph of $y = 2x(4 - x)$.

8 Sketch the graph of $y = (2 + x)^2$.

9 WE16 Use the discriminant to:

a determine the number and type of $x$-intercepts of the graph defined by $y = 42x - 18x^2$

b sketch the graph of $y = 42x - 18x^2$.

**10** Show that $7x^2 - 4x + 9$ is positive definite.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

**11** Sketch the following parabolas on the same set of axes.

a $y = 2x^2$

b $y = -2x^2$

c $y = 0.5x^2$

d $y = -0.5x^2$

e $y = (2x)^2$

f $y = \left(-\frac{x}{2}\right)^2$

For each of the parabolas in questions **12** to **15**, give the coordinates of:

i the turning point   ii the $y$-intercept   iii any $x$-intercepts.

iv Sketch the graph.

**12** a $y = x^2 - 9$   b $y = (x - 9)^2$   c $y = 6 - 3x^2$

d $y = -3(x + 1)^2$   e $y = \frac{1}{4}(1 - 2x)^2$   f $y = -0.25(1 + 2x)^2$

**13** a $y = x^2 + 6x - 7$   b $y = 3x^2 - 6x - 7$   c $y = 5 + 4x - 3x^2$

d $y = 2x^2 - x - 4$   e $y = -2x^2 + 3x - 4$   f $y = 10 - 2x^2 + 8x$

**14** a $y = (x - 5)^2 + 2$   b $y = 2(x + 1)^2 - 2$   c $y = -2(x - 3)^2 - 6$

d $y = -(x - 4)^2 + 1$   e $y + 2 = \dfrac{(x + 4)^2}{2}$   f $9y = 1 - \frac{1}{3}(2x - 1)^2$

**15** a $y = -5x(x + 6)$   b $y = (4x - 1)(x + 2)$   c $y = -2(1 + x)(2 - x)$

d $y = (2x + 1)(2 - 3x)$   e $y = 0.8x(10x - 27)$   f $y = (3x + 1)^2$

**16** Use the discriminant to determine the number and type of intercepts each of the following graphs makes with the $x$-axis.

a $y = 9x^2 + 17x - 12$   b $y = -5x^2 + 20x - 21$

c $y = -3x^2 - 30x - 75$   d $y = 0.02x^2 + 0.5x + 2$

**17** a i Express $2x^2 - 12x + 9$ in the form $a(x + b)^2 + c$.

ii Hence state the coordinates of the turning point of the graph of $y = 2x^2 - 12x + 9$.

iii What is the minimum value of the polynomial $2x^2 - 12x + 9$?

b i Express $-x^2 - 18x + 5$ in the form $a(x + b)^2 + c$.

ii Hence state the coordinates of the turning point of the graph of $y = -x^2 - 18x + 5$.

iii What is the maximum value of the polynomial $-x^2 - 18x + 5$?

**18** Shade the regions described by the following inequations.

a $y \geq x^2 - 6x$   b $y \leq 8 - 2x^2$   c $y < x^2 + 4x + 4$

d $y > 3(x + 2)^2 + 6$   e $y \leq (3 - x)(3 + x)$   f $y < 7x + 14x^2$

**19** For what values of $k$ does the graph of $y = 5x^2 + 10x - k$ have:

i one $x$-intercept

ii two $x$-intercepts

iii no $x$-intercepts?

20 a For what values of $m$ is $mx^2 - 2x + 4$ positive definite?
b i Show that there is no real value of $p$ for which $px^2 + 3x - 9$ is positive definite.
ii If $p = 3$, find the equation of the axis of symmetry of the graph of $y = px^2 + 3x - 9$.
c i For what values of $t$ does the turning point of $y = 2x^2 - 3tx + 12$ lie on the $x$-axis?
ii For what values of $t$ will the equation of the axis of symmetry of $y = 2x^2 - 3tx + 12$ be $x = 3t^2$?

MASTER

21 Use CAS technology to give the exact coordinates of:
i the turning point
ii any $x$-intercepts
for the parabolas defined by:
a $y = 12x^2 - 18x + 5$
b $y = -8x^2 + 9x + 12$.

22 Use CAS technology to sketch the graph of $y = -2.1x^2 + 52x + 10$, giving coordinates of key points correct to 2 decimal places, where appropriate.

# 3.6 Determining the rule of a quadratic polynomial from a graph

studyon

Units 1 & 2
AOS 1
Topic 2
Concept 5

**The rule of a quadratic polynomial from a graph**
Concept summary
Practice questions

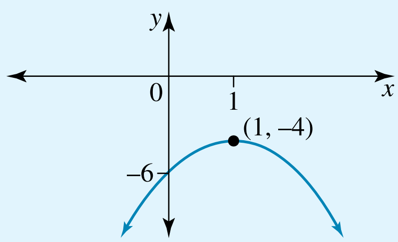
Whether the equation of the graph of a quadratic polynomial is expressed in $y = ax^2 + bx + c$ form, $y = a(x - h)^2 + k$ form or $y = a(x - x_1)(x - x_2)$ form, each equation contains 3 unknowns. Hence, 3 pieces of information are needed to fully determine the equation. This means that exactly one parabola can be drawn through 3 non-collinear points.

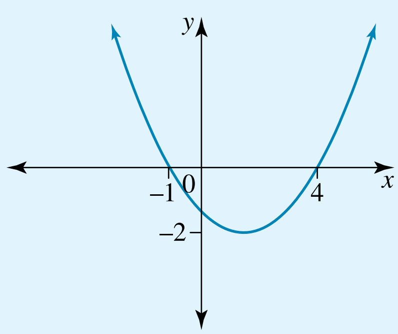
If the information given includes the turning point or the intercepts with the axes, one form of the equation may be preferable over another.

**As a guide:**
- **if the turning point is given, use the $y = a(x - h)^2 + k$ form**
- **if the $x$-intercepts are given, use the $y = a(x - x_1)(x - x_2)$ form**
- **if 3 points on the graph are given, use the $y = ax^2 + bx + c$ form.**

WORKED EXAMPLE 17

**Determine the rules for the following parabolas.**

a



b


| THINK | WRITE |
| --- | --- |
| **a** **1** Consider the given information to choose the form of the equation for the graph. | **a** Let the equation be $y = a(x - h)^2 + k$.<br>Turning point $(1, -4)$<br>$\therefore y = a(x - 1)^2 - 4$ |
| **2** Determine the value of $a$. | Substitute the given point $(0, -6)$.<br>$-6 = a(0 - 1)^2 - 4$<br>$-6 = a - 4$<br>$\therefore \quad a = -2$ |
| **3** Is the sign of $a$ appropriate? | Check: graph is concave down so $a < 0$. |
| **4** Write the rule for the graph.<br>*Note:* Check if the question specifies whether the rule needs to be expanded into general form. | The equation of the parabola is<br>$y = -2(x - 1)^2 - 4$. |
| **b** **1** Consider the given information to choose the form of the equation for the graph. | **b** Let the equation be $y = a(x - x_1)(x - x_2)$.<br>Given $x_1 = -1, x_2 = 4$<br>$\therefore y = a(x + 1)(x - 4)$ |
| **2** Determine the value of $a$. | Substitute the third given point $(0, -2)$.<br>$-2 = a(0 + 1)(0 - 4)$<br>$-2 = a(1)(-4)$<br>$-2 = -4a$<br>$a = \frac{-2}{-4}$<br>$= \frac{1}{2}$ |
| **3** Is the sign of $a$ appropriate? | Check: graph is concave up so $a > 0$. |
| **4** Write the rule for the graph. | The equation of the parabola is<br>$y = \frac{1}{2}(x + 1)(x - 4)$. |

## Using simultaneous equations

In Worked example 17b three points were available, but because two of them were key points, the $x$-intercepts, we chose to form the rule using the $y = a(x - x_1)(x - x_2)$ form. If the points were not key points, then simultaneous equations need to be created using the coordinates given.

**WORKED EXAMPLE 18** **Determine the equation of the parabola that passes through the points $(1, -4)$, $(-1, 10)$ and $(3, -2)$.**

| THINK | WRITE |
| --- | --- |
| **1** Consider the given information to choose the form of the equation for the graph. | Let $y = ax^2 + bx + c$. |

**2** Substitute the first point to form the an equation in $a$, $b$ and $c$.

First point
$(1, -4) \Rightarrow -4 = a(1)^2 + b(1) + c$
$\therefore -4 = a + b + c$.......(1)

**3** Substitute the second point to form a second equation in $a$, $b$ and $c$.

Second point
$(-1, 10) \Rightarrow 10 = a(-1)^2 + b(-1) + c$
$\therefore 10 = a - b + c$.......(2)

**4** Substitute the third point to form a third equation in $a$, $b$ and $c$.

Third point
$(3, -2) \Rightarrow -2 = a(3)^2 + b(3) + c$
$\therefore -2 = 9a + 3b + c$.......(3)

**5** Write the equations as a system of $3 \times 3$ simultaneous equations.

$-4 = a + b + c$..........(1)
$10 = a - b + c$..........(2)
$-2 = 9a + 3b + c$.......(3)

**6** Solve the system of simultaneous equations.

*Note:* CAS technology may be used to find the solutions.

Eliminate $c$ from equations (1) and (2).
Equation (2) − equation (1)

$14 = -2b$

$b = -7$

Eliminate $c$ from equations (1) and (3).
Equation (3) − equation (1)

$2 = 8a + 2b$....(4)

Substitute $b = -7$ in to equation (4).

$2 = 8a - 14$

$16 = 8a$

$a = 2$

Substitute $a = 2$, $b = -7$ in to equation (1).

$-4 = 2 - 7 + c$

$c = 1$

**7** State the equation.

The equation of the parabola is
$y = 2x^2 - 7x + 1$.

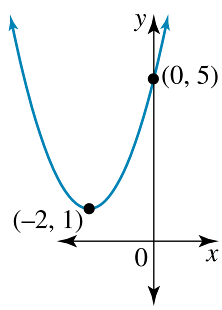
## EXERCISE 3.6 Determining the rule of a quadratic polynomial from a graph

**PRACTISE**

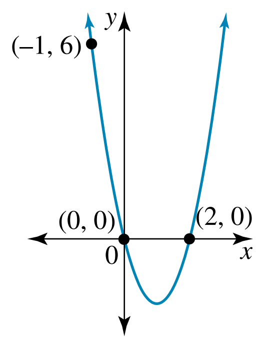
Work without CAS

**1** WE17 Determine the rules for each of the following parabolas.

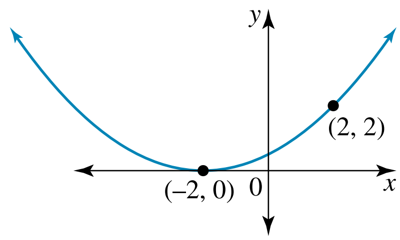
**a**




**b**



2 Form the rule, expressing it in expanded polynomial form, for the following parabola.



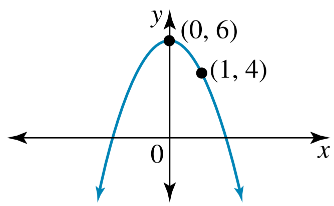

3 **WE18** Determine the equation of the parabola which passes through the points $(-1, -7)$, $(2, -10)$ and $(4, -32)$.

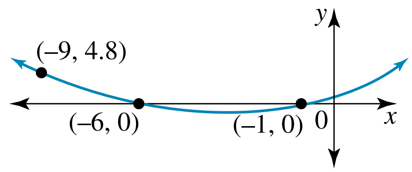
4 Use simultaneous equations to form the equation of the parabola through the points $(0, -2)$, $(-1, 0)$ and $(4, 0)$, and show this equation agrees with that found for the same parabola in Worked example 17b.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Determine the equation of each of the parabolas shown in the diagrams.

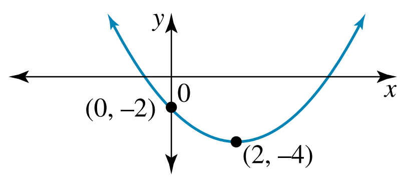
a



b



6 For the graph of the parabola shown:

a determine its rule

b calculate the length of the intercept cut off by the graph on the $x$-axis.




7 A parabola contains the three points $A(-1, 10), B(1, 0), C(2, 4)$.

a Determine its equation.

b Find the coordinates of its intercepts with the coordinate axes.

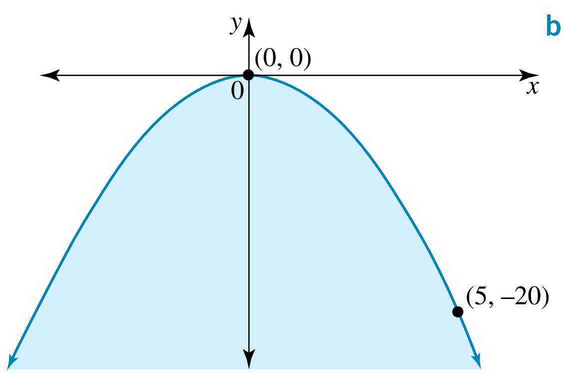
c Find the coordinates of its vertex.

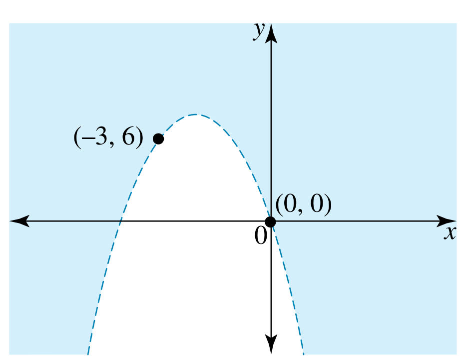
d Sketch the graph showing the points A, B, C.

8 a Give equations for three possible members of the family of parabolas which have $x$-intercepts of $(-3, 0)$ and $(5, 0)$.

b One member of this family of parabolas has a $y$-intercept of $(0, 45)$. Find its equation and its vertex.

9 Form the rule for the regions shaded in the diagrams.

a



b


**10 a** A parabola has the same shape as $y = 3x^2$ but its vertex is at $(-1, -5)$. Write its equation, and express it in $y = ax^2 + bx + c$ form.

**b** The parabola with equation $y = (x - 3)^2$ is translated 8 units to the left. What is the equation of its image?

**11** The axis of symmetry of a parabola has the equation $x = 4$. If the points $(0, 6)$ and $(6, 0)$ lie on the parabola, form its equation, expressing it in the $y = ax^2 + bx + c$ form.

**12** A parabola has the equation $y = (ax + b)(x + c)$. When $x = 5$, its graph cuts the $x$-axis and when $y = -10$ the graph cuts the $y$-axis.

**a** Show that $y = ax^2 + (2 - 5a)x - 10$.

**b** Express the discriminant in terms of $a$.

**c** If the discriminant is equal to 4, find the equation of the parabola and the coordinates of its other $x$-intercept.

**13 a** The graph of a parabola touches the $x$-axis at $x = -4$ and passes through the point $(2, 9)$. Determine its equation.

**b** A second parabola touches the $x$-axis at $x = p$ and passes through the points $(2, 9)$ and $(0, 36)$. Show there are two possible values for $p$ and for each, form the equation of the parabola and sketch each on the same axes.

**14** A concave up parabola with vertex V contains the points A$(-1, 15)$ and B$(5, 15)$. If the distance VB is $\sqrt{90}$ units:

**a** calculate the coordinates of V

**b** hence form the equation of the parabola.

**c** Show that the straight line through V and B passes through the origin.

**d** Calculate the area of the triangle VAB.

MASTER

**15** Use CAS technology to find the equation of the parabola through the points $(3.5, 62.45)$, $(5, 125)$ and $(6.2, 188.648)$.

**16** A parabola has the equation $y = ax^2 + b$. If the points $(20.5, 47.595)$ and $(42, 20.72)$ lie on its curve:

**a** determine the values of $a$ and $b$

**b** calculate the $x$-intercepts correct to 3 decimal places.

# 3.7 Quadratic inequations

If $ab > 0$, this could mean $a > 0$ and $b > 0$ or it could mean $a < 0$ and $b < 0$.

So, solving a quadratic inequation involving the product of factors is not as straightforward as solving a linear inequation. To assist in the solution of a quadratic inequation, either the graph or its **sign diagram** is a useful reference.

studyon

Units 1 & 2
AOS 1
Topic 2
Concept 6

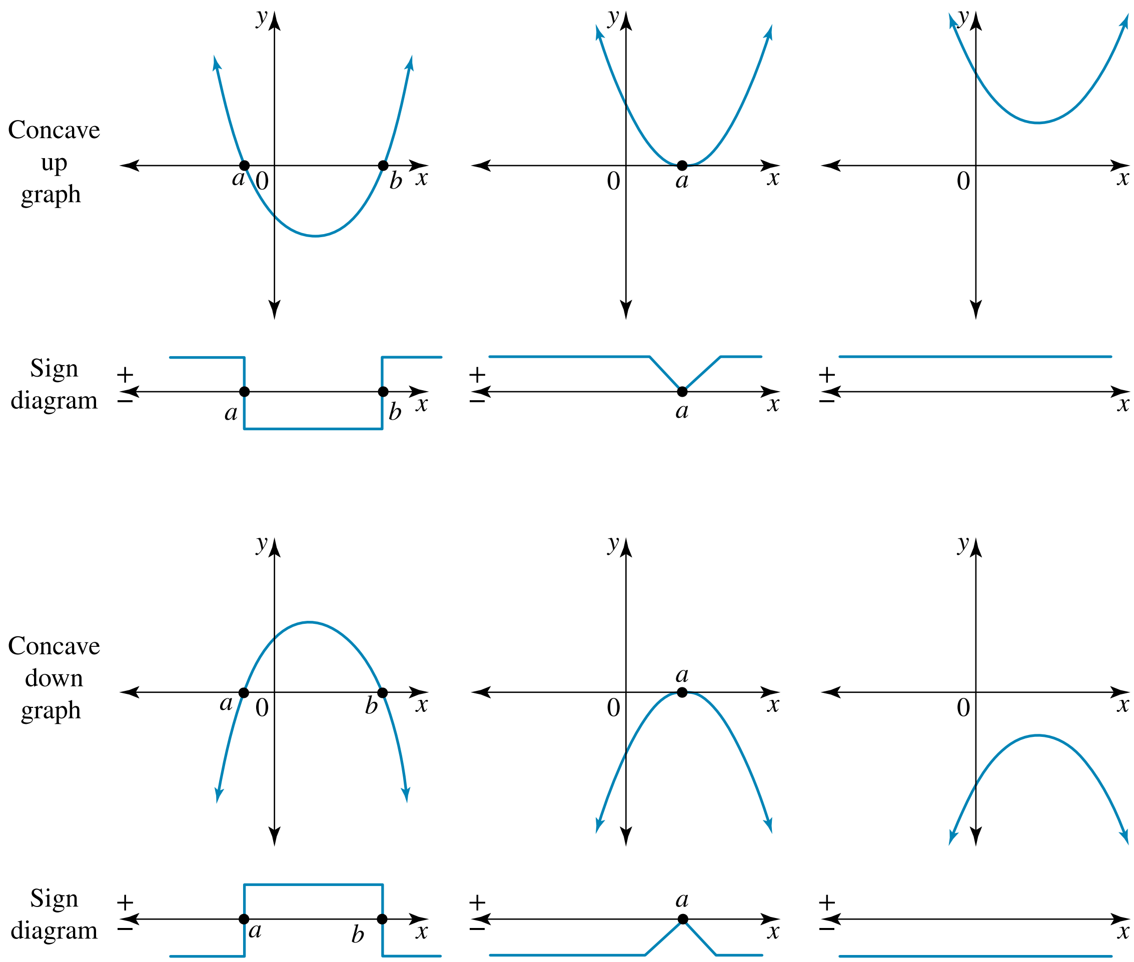
**Quadratic inequations**
Concept summary
Practice questions

## Sign diagrams of quadratics

A sign diagram is like a 'squashed' graph with only the $x$-axis showing. The sign diagram indicates the values of $x$ where the graph of a quadratic polynomial is above, on or below the $x$-axis. It shows the $x$-values for which $ax^2 + bx + c > 0$, the

$x$-values for which $ax^2 + bx + c = 0$ and the $x$-values for which $ax^2 + bx + c < 0$. A graph of the quadratic shows the same information and could be used, but usually the sign diagram is simpler to draw when solving a quadratic inequation. Unlike a graph, scaling and turning points that do not lie on the $x$-axis are not important in a sign diagram.

The three types of graphs with either 2, 1 or 0 $x$-intercepts are shown together with their matching sign diagrams for concave up and concave down parabolas.




**To draw a sign diagram of $ax^2 + bx + c$:**

- **find the zeros of the quadratic expression by solving $ax^2 + bx + c = 0$**
- **start the sign diagram below the axis if $a < 0$ and above the axis if $a > 0$**
- **either touch or cut through the axis at each zero depending whether the zero is a repeated one or not.**

Repeated zeros are said to have **multiplicity** 2, while non-repeated ones have multiplicity 1.

**WORKED EXAMPLE 19** Draw the sign diagram of $(4 - x)(2x - 3)$.

**THINK**

1 Find the zeros of the quadratic.

**WRITE**

Zeros occur when $(4 - x)(2x - 3) = 0$.

$(4 - x) = 0$ or $(2x - 3) = 0$

$x = 4$ or $x = 1.5$

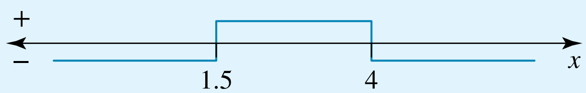
2 Draw the $x$-axis and mark the zeros in the correct order.


3 Consider the coefficient of $x^2$ to determine the concavity.

Multiplying the $x$ terms from each bracket of $(4 - x)(2x - 3)$ gives $-2x^2$.
So, concave down shape

4 Draw the sign diagram.

Sign diagram starts below the $x$-axis and cuts the axis at each zero.




## Solving quadratic inequations

To solve a quadratic inequation:

- rearrange the terms in the inequation, if necessary, so that one side of the inequation is 0 (similar to solving a quadratic equation)
- calculate the zeros of the quadratic expression and draw the sign diagram of this quadratic
- read from the sign diagram the set of values of $x$ which satisfy the inequation.

**WORKED EXAMPLE 20**

**a** Solve the quadratic inequation $(4 - x)(2x - 3) > 0$ using the sign diagram from Worked example 19 and check the solution using a selected value for $x$.

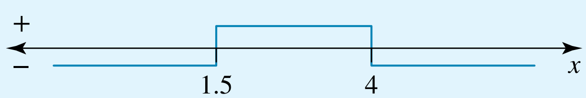
**b** Find $\{x: x^2 \geq 3x + 10\}$.

**THINK**

a 1 Copy the sign diagram from Worked example 19.

**WRITE**

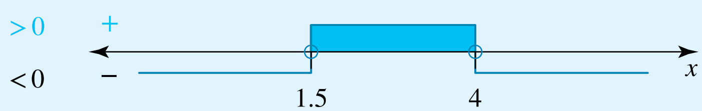
a




2 Highlight the part of the sign diagram which shows the values required by the inequation.

$(4 - x)(2x - 3) > 0$

Positive values of the quadratic lie above the $x$-axis.




3 State the interval in which the required solutions lie.

*Note:* The interval is open because the original inequation has an open inequality sign.

Therefore the solution is $1.5 < x < 4$.

4 Choose an $x$-value that lies in the solution interval and check it satisfies the inequation.

Let $x = 2$.
Substitute in LHS of $(4 - x)(2x - 3) > 0$.

$$\begin{aligned}(4 - 2)(2 \times 2 - 3) &= 2 \times 1 \\ &= 2 \\ &> 0\end{aligned}$$

Therefore $x = 2$ lies in the solution set.

**b** 1 Rearrange the inequation to make one side 0.

**b** $x^2 \geq 3x + 10$

$x^2 - 3x - 10 \geq 0$

2 Calculate the zeros of the quadratic.

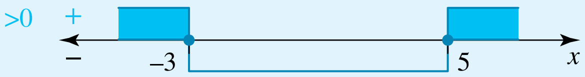
Let $x^2 - 3x - 10 = 0$

$(x - 5)(x + 3) = 0$

$x = 5$, or $x = -3$

3 Draw the sign diagram and highlight the interval(s) with the required sign.

$x^2 - 3x - 10 \geq 0$
Quadratic is concave up, values above or on the $x$-axis required.




4 State the intervals required.

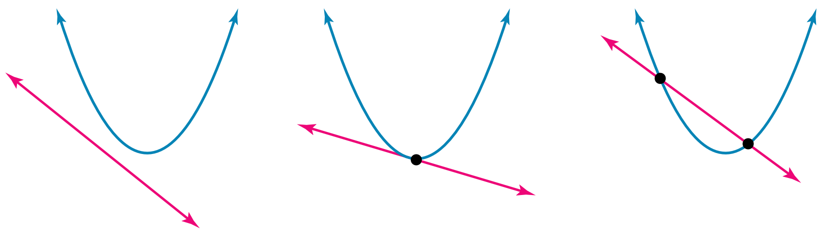
$\therefore (x - 5)(x + 3) \geq 0$ when $x \leq -3$ or $x \geq 5$

5 State the answer in set notation.

Answer is: $\{x: x \leq -3\} \cup \{x: x \geq 5\}$

## Intersections of lines and parabolas

The possible number of points of intersection between a straight line and a parabola will be either 0, 1 or 2 points.



- If there is no point of intersection, the line makes no contact with the parabola.
- If there is 1 point of intersection, a non-vertical line is a **tangent line** to the parabola, touching the parabola at that one point of contact.
- If there are 2 points of intersection, the line cuts through the parabola at these points.

Simultaneous equations can be used to find any points of intersection and the discriminant can be used to predict the number of solutions. To solve a pair of linear-quadratic simultaneous equations, usually the method of substitution from the linear into the quadratic equation is used.

**WORKED EXAMPLE 21**

**a** **Calculate the coordinates of the points of intersection of the parabola $y = x^2 - 3x - 4$ and the line $y - x = 1$.**

**b** **How many points of intersection will there be between the graphs of $y = 2x - 5$ and $y = 2x^2 + 5x + 6$?**

| THINK | WRITE |
|---|---|
| **a** 1 Set up the simultaneous equations. | **a** $y = x^2 - 3x - 4.....(1)$<br>$y - x = 1.....(2)$ |
| 2 Substitute from the linear equation into the quadratic equation. | From equation (2), $y = x + 1$.<br>Substitute this into equation (1).<br>$x + 1 = x^2 - 3x - 4$<br>$x^2 - 4x - 5 = 0$ |
| 3 Solve the newly created quadratic equation for the $x$ coordinates of the points of intersection of the line and parabola. | $x^2 - 4x - 5 = 0$<br>$(x + 1)(x - 5) = 0$<br>$x = -1$ or $x = 5$ |
| 4 Find the matching $y$ coordinates using the simpler linear equation. | In equation (2):<br>when $x = -1$, $y = 0$<br>when $x = 5$, $y = 6$. |
| 5 State the coordinates of the points of intersection. | The points of intersection are $(-1, 0)$ and $(5, 6)$. |
| **b** 1 Set up the simultaneous equations. | **b** $y = 2x - 5.....(1)$<br>$y = 2x^2 + 5x + 6.....(2)$ |
| 2 Create the quadratic equation from which any solutions are generated. | Substitute equation (1) in equation (2).<br>$2x - 5 = 2x^2 + 5x + 6$<br>$2x^2 + 3x + 11 = 0$ |
| 3 The discriminant of this quadratic equation determines the number of solutions. | $\Delta = b^2 - 4ac$, $a = 2, b = 3, c = 11$<br>$= (3)^2 - 4 \times (2) \times (11)$<br>$= -79$<br>$\therefore \Delta < 0$<br>There are no points of intersection between the two graphs. |

## Quadratic inequations in discriminant analysis

The need to solve a quadratic inequation as part of the analysis of a problem can occur in a number of situations, an example of which arises when a discriminant is itself a quadratic polynomial in some variable.

**WORKED EXAMPLE 22**

**For what values of $m$ will there be at least one intersection between the line $y = mx + 5$ and the parabola $y = x^2 - 8x + 14$?**

**THINK** / **WRITE**

1 Set up the simultaneous equations and form the quadratic equation from which any solutions are generated.

$y = mx + 5$.....(1)
$y = x^2 - 8x + 14$.....(2)
Substitute from equation (1) into equation (2).

$mx + 5 = x^2 - 8x + 14$

$x^2 - 8x - mx + 9 = 0$

$x^2 - (8 + m)x + 9 = 0$

2 Obtain an algebraic expression for the discriminant of this equation.

$\Delta = b^2 - 4ac,\ a = 1, b = -(8 + m), c = 9$

$= (-(8 + m))^2 - 4(1)(9)$

$= (8 + m)^2 - 36$

3 For at least one intersection, $\Delta \geq 0$. Impose this condition on the discriminant to set up a quadratic inequation.

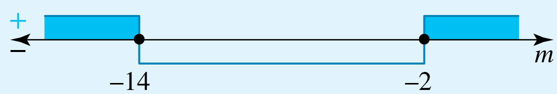
$\Delta \geq 0$ for one or two intersections.
$\therefore (8 + m)^2 - 36 \geq 0$

4 Solve this inequation for $m$ by finding the zeros and using a sign diagram of the discriminant.

$((8 + m) - 6)((8 + m) + 6) \geq 0$

$(2 + m)(14 + m) \geq 0$

Zeros are $m = -2, m = -14$.
The sign diagram of $(2 + m)(14 + m)$ will be that of a concave up quadratic.




$\therefore m \leq -14$ or $m \geq -2$

5 State the answer.

If $m \leq -14$ or $m \geq -2$ there will be at least one intersection between the line and the parabola.

## EXERCISE 3.7 Quadratic inequations

**PRACTISE**

Work without CAS

1 WE19 Draw the sign diagram of $(x + 3)(x - 4)$.

2 Draw the sign diagram of $81x^2 - 18x + 1$.

3 WE20 a Solve the quadratic inequation $(4 - x)(2x - 3) \leq 0$ using the sign diagram from Worked example 19 and check the solution using a selected value for $x$.

b Find $\{x: 6x^2 < x + 2\}$.

4 Solve $6x < x^2 + 9$.

5 WE21 a Calculate the coordinates of the points of intersection of the parabola $y = x^2 + 3x - 10$ and the line $y + x = 2$.

b How many points of intersection will there be between the graphs of $y = 6x + 1$ and $y = -x^2 + 9x - 5$?

6 Show that the line $y = 4x$ is a tangent to the parabola $y = x^2 + 4$ and sketch the line and parabola on the same diagram, labelling the coordinates of the point of contact.

7 **WE22** For what values of $m$ will there be at least one intersection between the line $y = mx - 7$ and the parabola $y = 3x^2 + 6x + 5$?

8 For what values of $k$ will there be no intersection between the line $y = kx + 9$ and the parabola $y = x^2 + 14$?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 Solve the following quadratic inequations.

a $x^2 + 8x - 48 \leq 0$
b $-x^2 + 3x + 4 \leq 0$
c $3(3 - x) < 2x^2$
d $(x + 5)^2 < 9$
e $9x < x^2$
f $5(x - 2) \geq 4(x - 2)^2$

10 Find the following sets.

a $\{x: 36 - 12x + x^2 > 0\}$
b $\{x: 6x^2 - 12x + 6 \leq 0\}$
c $\{x: -8x^2 + 2x < 0\}$
d $\{x: x(1 + 10x) \leq 21\}$

11 Solve each of the following pairs of simultaneous equations.

a $y = 5x + 2$
$y = x^2 - 4$

b $4x + y = 3$
$y = x^2 + 3x - 5$

c $2y + x - 4 = 0$
$y = (x - 3)^2 + 4$

d $\dfrac{x}{3} + \dfrac{y}{5} = 1$
$x^2 - y + 5 = 0$

12 Obtain the coordinates of the point(s) of intersection of:

a the line $y = 2x + 5$ and the parabola $y = -5x^2 + 10x + 2$
b the line $y = -5x - 13$ and the parabola $y = 2x^2 + 3x - 5$
c the line $y = 10$ and the parabola $y = (5 - x)(6 + x)$
d the line $19x - y = 46$ and the parabola $y = 3x^2 - 5x + 2$.

13 Use a discriminant to determine the number of intersections of:

a the line $y = 4 - 2x$ and the parabola $y = 3x^2 + 8$
b the line $y = 2x + 1$ and the parabola $y = -x^2 - x + 2$
c the line $y = 0$ and the parabola $y = -2x^2 + 3x - 2$.

14 Determine the values of $k$ so that the line $y = (k - 2)x + k$ and the parabola $y = x^2 - 5x$ will have:

a no intersections
b one point of intersection
c two points of intersection.

15 a For what values of $p$ will the equation $px^2 - 2px + 4 = 0$ have real roots?

b For what real values of $t$ will the line $y = tx + 1$ not intersect the parabola $y = 2x^2 + 5x + 11$?

c For what values of $n$ will the line $y = x$ be a tangent to the parabola $y = 9x^2 + nx + 1$?

d Obtain the solutions to the inequation $x^4 - x^2 < 12$.

16 Consider the line $2y - 3x = 6$ and the parabola $y = x^2$.

a Calculate the coordinates of their points of intersection, correct to 2 decimal places.
b Sketch the line and the parabola on the same diagram.
c Use inequations to describe the region enclosed between the two graphs.
d Calculate the $y$-intercept of the line parallel to $2y - 3x = 6$ which is a tangent to the parabola $y = x^2$.

17 a The equation $x^2 - 5x + 4 = 0$ gives the $x$-coordinates of the points of intersection of the parabola $y = x^2$ and a straight line. What is the equation of this line?
b The equation $3x^2 + 9x - 2 = 0$ gives the $x$-coordinates of the points of intersection of a parabola and the straight line $y = 3x + 1$. What could be the equation of the parabola?

18 a Sketch the parabolas $y = (x + 2)^2$ and $y = 4 - x^2$ on the one diagram and hence determine their points of intersection.
b i Show that the parabolas $y = (x + 2)^2$ and $y = k - x^2$ have one point of intersection if $k = 2$.
ii Sketch $y = (x + 2)^2$ and $y = 2 - x^2$ on the one diagram, labelling their common point C with its coordinates.
iii The line $y = ax + b$ is the tangent to both curves at point C. Find its equation.

MASTER

19 Use CAS technology to find the solutions to:
a $19 - 3x - 5x^2 < 0$
b $6x^2 + 15x \leq 10$.

20 a Using CAS technology, draw on the one diagram the graphs of $y = 2x^2 - 10x$ with the family of lines $y = -4x + a$, for $a = -6, -4, -2, 0$ and find the points of intersection for each of these values of $a$.
b Use CAS technology to solve the equation $2x^2 - 10x = -4x + a$ to obtain $x$ in terms of $a$.
c Hence obtain the value of $a$ for which $y = -4x + a$ is a tangent to $y = 2x^2 - 10x$ and give the coordinates of the point of contact in this case.

# 3.8 Quadratic models and applications

Quadratic polynomials can be used to model a number of situations such as the motion of a falling object and the time of flight of a projectile. They can be used to model the shape of physical objects such as bridges, and they can also occur in economic models of cost and revenue.

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Units 1 & 2
AOS 1
Topic 2
Concept 7

**Quadratic models**
Concept summary
Practice questions

## Maximum and minimum values

The greatest or least value of the quadratic model is often of interest.

A quadratic reaches its maximum or minimum value at its turning point. The $y$-coordinate of the turning point represents the maximum or minimum value, depending on the nature of the turning point.

If $a < 0,\ a(x - h)^2 + k \leq k$ so the maximum value of the quadratic is $k$.

If $a > 0,\ a(x - h)^2 + k \geq k$ so the minimum value of the quadratic is $k$.

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Projectiles
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**WORKED EXAMPLE 23**

**A stone is thrown vertically into the air so that its height $h$ metres above the ground after $t$ seconds is given by $h = 1.5 + 5t - 0.5t^2$.**

**a** What is the greatest height the stone reaches?

**b** After how many seconds does the stone reach its greatest height?

**c** When is the stone 6 metres above the ground? Why are there two times?

**d** Sketch the graph and give the time to return to the ground to 1 decimal place.

| THINK | WRITE |
|---|---|
| **a 1** The turning point is required. Calculate the coordinates of the turning point and state its type. | **a** $h = 1.5 + 5t - 0.5t^2$<br>$a = -0.5, b = 5, c = 1.5$<br>Turning point:<br>Axis of symmetry has equation $t = -\frac{b}{2a}$.<br>$t = -\frac{5}{2 \times (-0.5)}$<br>$= 5$<br>When $t = 5$,<br>$h = 1.5 + 5(5) - 0.5(5)^2$<br>$= 14$<br>Turning point is $(5, 14)$. This is a maximum turning point as $a < 0$. |
| **2** State the answer.<br>*Note:* The turning point is in the form $(t, h)$ as $t$ is the independent variable and $h$ the dependent variable. The greatest height is the $h$-coordinate. | Therefore the greatest height the stone reaches is 14 metres above the ground. |
| **b** The required time is the $t$-coordinate of the turning point. | **b** The stone reaches its greatest height after 5 seconds. |
| **c 1** Substitute the given height and solve for $t$. | **c** $h = 1.5 + 5t - 0.5t^2$<br>When $h = 6$, $6 = 1.5 + 5t - 0.5t^2$<br>$0.5t^2 - 5t + 4.5 = 0$<br>$t^2 - 10t + 9 = 0$<br>$(t - 1)(t - 9) = 0$<br>$\therefore t = 1$ or $t = 9$ |
| **2** Interpret the answer. | Therefore the first time the stone is 6 metres above the ground is 1 second after it has been thrown into the air and is rising upwards. It is again 6 metres above the ground after 9 seconds when it is falling down. |

**d** **1** Calculate the time the stone returns to the ground.

**d** Returns to ground when $h = 0$

$0 = 1.5 + 5t - 0.5t^2$

$t^2 - 10t - 3 = 0$

$t^2 - 10t = 3$

$t^2 - 10t + 25 = 3 + 25$

$(t - 5)^2 = 28$

$t = 5 \pm \sqrt{28}$

$t \simeq 10.3$ (reject negative value)

The stone reaches the ground after 10.3 seconds.

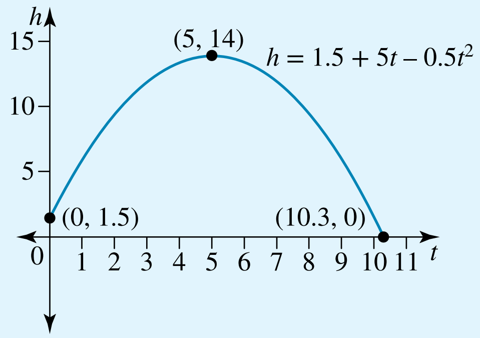
**2** Sketch the graph, from its initial height to when the stone hits the ground. Label the axes appropriately.

When $t = 0, h = 1.5$ so stone is thrown from a height of 1.5 metres.

Initial point: (0, 1.5)

Maximum turning point: (5, 14)

Endpoint: (10.3, 0)




## EXERCISE 3.8 Quadratic models and applications

**PRACTISE**

Work without CAS

**1** WE23 A missile is fired vertically into the air from the top of a cliff so that its height $h$ metres above the ground after $t$ seconds is given by $h = 100 + 38t - \frac{19}{12}t^2$.

**a** What is the greatest height the missile reaches?

**b** After how many seconds does the missile reach its greatest height?

**c** Sketch the graph and give the time to return to the ground to 1 decimal place.

**2** A gardener has 30 metres of edging to enclose a rectangular area using the back fence as one edge.

**a** Show the area function is $A = 30x - 2x^2$ where $A$ square metres is the area of the garden bed of width $x$ metres.

**b** Calculate the dimensions of the garden bed for maximum area.

**c** What is the maximum area that can be enclosed?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**3** A child throws a ball vertically upwards so that after $t$ seconds its height $h$ metres above the ground is given by $10h = 16t + 4 - 9t^2$.

**a** How long does it take for the ball to reach the ground?

**b** Will the ball strike the foliage overhanging from a tree if the foliage is 1.6 metres vertically above the ball's point of projection?

**c** What is the greatest height the ball could reach?

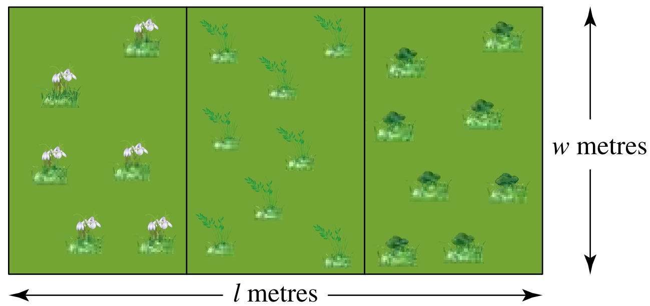
**4** In a game of volleyball a player serves a 'sky-ball' serve from the back of a playing court of length 18 metres. The path of the ball can be considered to be part of the parabola $y = 1.2 + 2.2x - 0.2x^2$ where $x$ (metres) is the horizontal distance travelled by the ball from where it was hit and $y$ (metres) is the vertical height the ball reaches.

**a** Use completing the square technique to express the equation in the form $y = a(x - b)^2 + c$.

**b** How high does the volleyball reach?

**c** The net is 2.43 metres high and is placed in the centre of the playing court. Show that the ball clears the net and calculate by how much.

**5** Georgie has a large rectangular garden area with dimensions $l$ metres by $w$ metres which she wishes to divide into three sections so she can grow different vegetables. She plans to put a watering system along the perimeter of each section. This will require a total of 120 metres of hosing.




**a** Show the total area of the three sections, $A$ m$^2$ is given by $A = 60w - 2w^2$ and hence calculate the dimensions when the total area is a maximum.

**b** Using the maximum total area, Georgie decides she wants the areas of the three sections to be in the ratio 1 : 2 : 3. What is the length of hosing for the watering system that is required for each section?

**6** The number of bacteria in a slowly growing culture at time $t$ hours after 8.00 am is given by $N = 100 + 46t + 2t^2$.

**a** How long does it take for the initial number of bacteria to double?

**b** How many bacteria are present at 1.00 pm?

**c** At 1.00 pm a virus is introduced that initially starts to destroy the bacteria so that $t$ hours after 1.00 pm the number of bacteria is given by $N = 380 - 180t + 30t^2$. What is the minimum number the population of bacteria reaches and at what time does this occur?

**7** Let $z = 5x^2 + 4xy + 6y^2$. Given $x + y = 2$, find the minimum value of $z$ and the values of $x$ and $y$ for which $z$ is minimum.

**8** A piece of wire of length 20 cm is cut into two sections, and each is used to form a square. The sum of the areas of these two squares is $S$ cm$^2$.

**a** If one square has a side length of 4 cm, calculate the value of $S$.

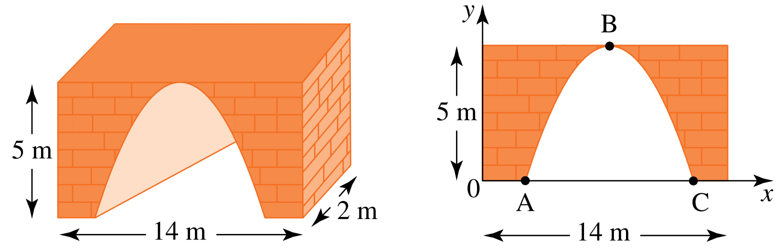
**b** If one square has a side length of $x$ cm, express $S$ in terms of $x$ and hence determine how the wire should be cut for the sum of the areas to be a minimum.

**9** The cost $C$ dollars of manufacturing $n$ dining tables is the sum of three parts. One part represents the fixed overhead costs $c$, another represents the cost of raw materials and is directly proportional to $n$ and the third part represents the labour costs which are directly proportional to the square of $n$.

**a** If 5 tables cost \$195 to manufacture, 8 tables cost \$420 to manufacture and 10 tables cost \$620 to manufacture, find the relationship between $C$ and $n$.

**b** Find the maximum number of dining tables that can be manufactured if costs are not to exceed $1000.

**10** The arch of a bridge over a small creek is parabolic in shape with its feet evenly spaced from the ends of the bridge. Relative to the coordinate axes, the points A, B and C lie on the parabola.




**a** If AC = 8 metres, write down the coordinates of the points A, B and C.
**b** Determine the equation of the parabola containing points A, B and C.
**c** Following heavy rainfall the creek floods and overflows its bank, causing the water level to reach 1.5 metres above AC. What is the width of the water level to 1 decimal place?

**11** The daily cost $C$ dollars of producing $x$ kg of plant fertiliser for use in market gardens is $C = 15 + 10x$. The manufacturer decides that the fertiliser will be sold for $v$ dollars per kg where $v = 50 - x$.

Find an expression for the profit in terms of $x$ and hence find the price per kilogram that should be charged for maximum daily profit.

**12** **a** If the sum of two numbers is 16, find the numbers for which:
**i** their product is greatest **ii** the sum of their squares is least.
**b** If the sum of two non-zero numbers is $k$:
**i** express their greatest product in terms of $k$
**ii** are there any values of $k$ for which the sum of the squares of the numbers and their product are equal? If so, state the values; if not, explain why.

**MASTER**

**13** Meteorology records for the heights of tides above mean sea level in Tuvalu predict the tide levels shown in the following table.

| Time of day | Height of tide (in metres) |
|---|---|
| 10.15 am | 1.05 |
| 4.21 pm | 3.26 |
| 10.30 pm | 0.94 |

**a** Use CAS technology to find the equation of a quadratic model which fits these three data points in the form $h = at^2 + bt + c$ where $h$ is the height in metres of the tide $t$ hours after midnight. Express the coefficients to 2 decimal places.
**b** Find the greatest height of the tide above sea level and the time of day it is predicted to occur.

**14** A piece of wire of length 20 cm is cut into two sections, one a square and the other a circle. The sum of the areas of the square and the circle is $S$ cm$^2$.

**a** If the square has a side length of $x$ cm, express $S$ in terms of $x$.
**b** Graph the $S$–$x$ relationship and hence calculate the lengths of the two sections of the wire for $S$ to be a minimum. Give the answer to 1 decimal place.
**c** Use the graph to find the value of $x$ for which $S$ is a maximum.

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# 3.9 Review

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- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

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Units 1 & 2 **Quadratic relationships**

**Sit topic test**

# 3 Answers

## EXERCISE 3.2

1 a $x = -3, 0.7$

b $(x + 5)x = x^2 + 5x$

2 $x = 1.5$

3 $x = -0.6,\ 1$

4 $x = \frac{r - q}{p},\ x = -\frac{(r + q)}{p}$

5 $x = \pm\frac{1}{3}$

6 $x = 1$

7 a $0, 5$

b $\frac{1}{7}, 3$

c $-8$

d $-6, -4$

8 a $-\frac{1}{2}, -\frac{1}{3}$

b $-\frac{2}{3}, \frac{5}{4}$

c $7$

d $-\frac{5}{6}, 1$

e $\frac{2}{11}, \frac{11}{2}$

f Side of square is 30 units.

9 a $-1, 8$

b $\frac{11}{6}, \frac{9}{2}$

c $-8, -2$

d $0, \frac{1}{2}$

10 a $\frac{1}{3}$

b $\pm\frac{25}{2}$

c $-\frac{5}{7}, \frac{2}{5}$

d $0, \frac{17}{19}$

11 a $\pm 11$

b $\pm\frac{4}{3}$

c $4, 6$

d $-1, 6$

e $-\frac{1}{3}, 1$

f $\pm 3$

12 a $-\frac{14}{3}, -1$

b $-\frac{7}{2}, \frac{1}{4}$

c $\pm 2, \pm 5$

d $\pm 4$

e $\pm\frac{1}{3}$

f $-3, -2, -1$

13 a $\pm 3$

b $\pm\frac{4}{3}, \pm 2$

c $-1, 2$

d $-\frac{3}{2}, -\frac{6}{5}$

14 a $x = -3a, x = 2b$

b $x = \frac{3a}{2}, x = 5a$

c $x = b - 1, x = b + 1, x = b - 2, x = b + 2$

d $x = a - b, x = a + 3b$

e $x = b - a, x = 2b - a$

f $x = 1$

15 a $(x - 1)(x - 7) = 0$

b $(x + 5)(x - 4) = 0$

c $x(x - 10) = 0$

d $(x - 2)^2 = 0$

16 a $b = 13, c = -12$

b Roots are $-\frac{q}{p}, -1$, solutions $x = \frac{p - q}{p}, 0$

17 $x = -\frac{7}{3}, x = \frac{9}{20}$

18 $x = \frac{7}{4}, x = \frac{14}{3}$

## EXERCISE 3.3

1 a $(x - 5 - 4\sqrt{2})(x - 5 + 4\sqrt{2})$

b $3\left(x + \frac{7 - \sqrt{13}}{6}\right)\left(x + \frac{7 + \sqrt{13}}{6}\right)$

c $(\sqrt{5}x - 3)(\sqrt{5}x + 3)$

2 $(-3x + 5)(x - 1)$

3 a $\Delta = -135$, no real factors

b $\Delta = 0$, two identical rational factors

c $\Delta = 1$, two rational factors

d $\Delta = \frac{40}{9}$, two real factors, completing the square needed to obtain the factors

4 a $3(x - 8 - \sqrt{2})(x - 8 + \sqrt{2})$

b No real factors

5 $x = \frac{-11 \pm \sqrt{89}}{4}$

6 $x = \frac{-1 \pm \sqrt{7}}{2}$

7 a There are no real roots

b $k = -1,\ k = 3$

8 $\Delta = (m + 4)^2 \Rightarrow \Delta \geq 0$

9 a $x = 10\sqrt{5} \pm 20$

b $x = 1 + \sqrt{2}, b = -2, c = -1$

10 $x = -\sqrt{6} \pm \sqrt{14}$

11 $x = 1$

12 $x = 1$

13 a $x^2 + 10x + 25 = (x + 5)^2$

b $x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$

c $x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$

d $x^2 - \frac{4}{5}x + \frac{4}{25} = \left(x - \frac{2}{5}\right)^2$

14 a $(x - 2\sqrt{3})(x + 2\sqrt{3})$

b $(x - 6 - 4\sqrt{2})(x - 6 + 4\sqrt{2})$

c $\left(x + \dfrac{9 - \sqrt{93}}{2}\right)\left(x + \dfrac{9 + \sqrt{93}}{2}\right)$

d $2\left(x + \dfrac{5 - \sqrt{17}}{4}\right)\left(x + \dfrac{5 + \sqrt{17}}{4}\right)$

e $3\left(\left(x + \frac{2}{3}\right)^2 + \frac{5}{9}\right)$ no linear factors over $R$

f $-5\left(x - 4 - \dfrac{9\sqrt{5}}{5}\right)\left(x - 4 + \dfrac{9\sqrt{5}}{5}\right)$

15 a $\Delta = 121$, 2 rational factors

b $\Delta = -39$, no real factors

c $\Delta = 0$, 1 repeated rational factor

d $\Delta = 8$, 2 irrational factors

16 a $(x - 2)(x^2 + 2x + 4)$, quadratic factor has a negative discriminant

b i $(x - \sqrt{2})(x + \sqrt{2}) = x^2 - 2$

ii $(x + 4 - \sqrt{2})(x + 4 + \sqrt{2}) = x^2 + 8x + 14$

17 a $\dfrac{1 \pm \sqrt{17}}{6}$

b $\dfrac{10 \pm 2\sqrt{10}}{5}$

c $10 \pm 2\sqrt{5}$

d $-3 \pm 2\sqrt{3}$

e $-\frac{9}{7}, \frac{3}{8}$

f $\dfrac{-11 \pm \sqrt{201}}{8}$

18 a 2 irrational roots

b 2 rational roots

c no real roots

d 1 rational root

e no real roots

f 2 irrational roots

19 a $\pm\sqrt{5}$

b $\pm 2\sqrt{2}$

c $\pm\sqrt{11}, \pm 1$

d $\dfrac{-3 \pm \sqrt{5}}{2}$

e $-1, 8, \dfrac{7 \pm \sqrt{93}}{2}$

f $-1$

20 a $m = -4 \pm 4\sqrt{2}$

b $m = 2 \pm 2\sqrt{3}$

c $p < \frac{1}{3}$

d $\Delta > 0$

e $\Delta$ is a perfect square.

21 a i $x = -3\sqrt{2}$

ii $x = \dfrac{3\sqrt{2} \pm \sqrt{10}}{4}$

iii $x = \dfrac{-3 + 2\sqrt{6} \pm \sqrt{33}}{6}$

b i $x = \dfrac{-1 - \sqrt{5}}{2}$

ii $x^2 + x - 1 = 0, b = 1, c = -1$

c $x^2 - 8\sqrt{3}x - 102 = 0, b = -8\sqrt{3}, c = -102$

22 a i 4

ii 4

b $x = \frac{5}{4}$

23 $12\left(x + \dfrac{\sqrt{7}}{3} + \dfrac{1}{6}\right)\left(x - \dfrac{\sqrt{7}}{3} + \dfrac{1}{6}\right)$

24 a Proof required – check with your teacher

b $(x, y) = \left(\dfrac{p \pm \sqrt{p^2 - 4q}}{2}, \dfrac{p \pm \sqrt{p^2 - 4q}}{2}\right)$

## EXERCISE 3.4

1 20 kg

2 20 and 22

3 $3\sqrt{10}$ cm

4 $4\frac{1}{2}$ hours

5 a $4\sqrt{3}$ cm

b Distance is quadrupled.

c Heat is reduced by 36%.

6 a $S = k_1 n + k_2 n^2$

b $k_1 = 0.5 = k_2$

c $n = 50$

7 a $y = 2 + 3x + 4x^2$ b $x = 5$

8 2511 litres

9 10, 11

10 Base $2\sqrt{2}$ cm; height 4 cm

11 $x = 24$; perimeter = 168 cm

12 Length 24 cm; width 20 cm

13 a Proof required – check with your teacher

b Proof required – check with your teacher

c $\Delta = 256 - 8k$

i $k > 32$

ii $k = 32$

iii $0 < k < 32$

d 32 m$^2$; width 4 m, length 8 m

e Width 1.1 metres and length 13.8 metres

14 8 cards

**15** 1.052 metres

**16** $r = \dfrac{-\left(\pi l - \sqrt{\pi^2 l^2 + 80\pi}\right)}{2\pi}$

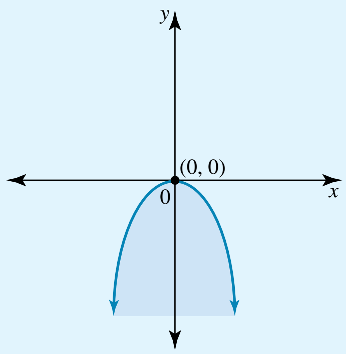
## EXERCISE 3.5

**1 A** i $y = x^2 - 2$

**B** ii $y = -2x^2$

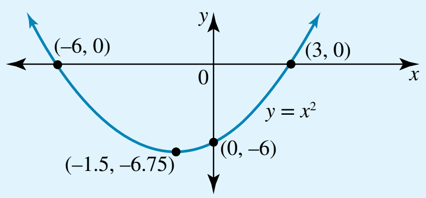
**C** iii $y = -(x + 2)^2$

**2**

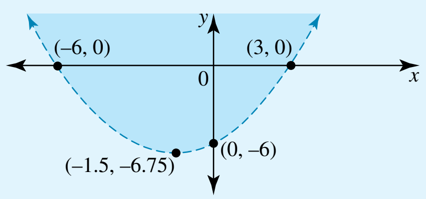



Shade the region below the parabola, including the curve.

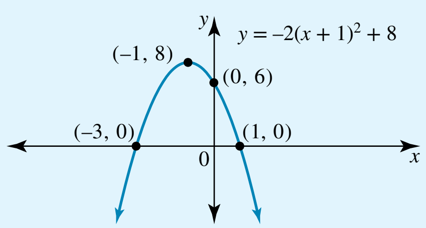
**3** Axis intercepts $(-6, 0)$, $(3, 0)$, $(0, -6)$; minimum turning point $(-1.5, -6.75)$




**4** Shade the region above the graph not including the curve.

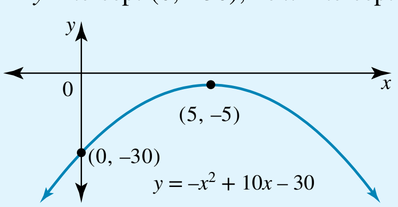



**5 a** Maximum turning point $(-1, 8)$; axis intercepts $(0, 6)$, $(-3, 0)$, $(1, 0)$




**b** i $y = -(x - 5)^2 - 5$
Vertex is $(5, -5)$, a maximum turning point

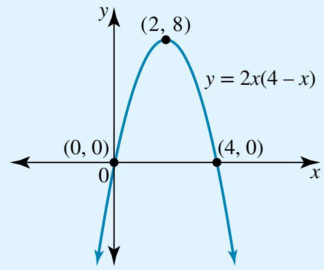
ii $y$-intercept $(0, -30)$, no $x$-intercepts



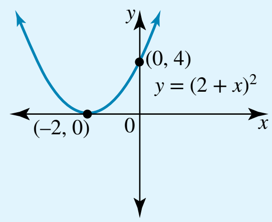

**6 a** Maximum turning point at $(0, 4)$

**b** Minimum turning point at $\left(\frac{4}{3}, 0\right)$

**7** Axis intercepts $(0, 0)$, $(4, 0)$; maximum turning point $(2, 8)$

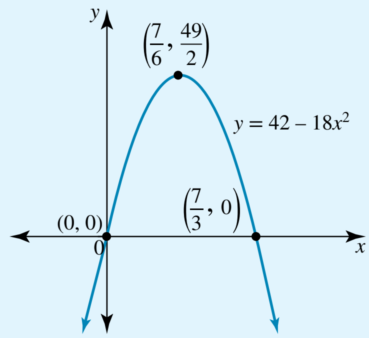



**8** Minimum turning point and $x$-intercept $(-2, 0)$, $y$-intercept $(0, 4)$



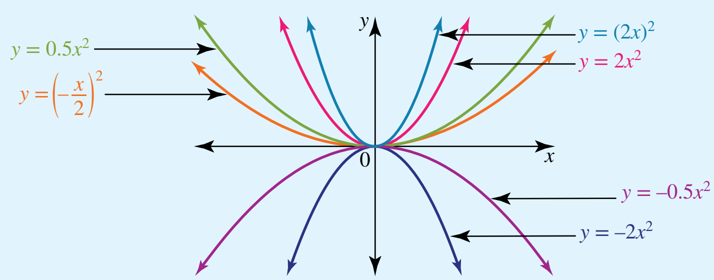

**9 a** 2 rational $x$-intercepts

**b** Axis intercepts $(0, 0)$, $\left(\frac{7}{3}, 0\right)$; maximum turning point $\left(\frac{7}{6}, \frac{49}{2}\right)$




**10** $\Delta < 0$ and $a > 0$

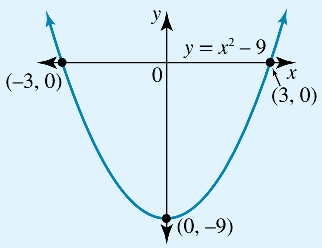
**11**



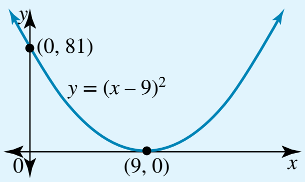

**12**

| | Turning point | $y$-intercept | $x$-intercepts |
|---|---|---|---|
| **a** | $(0, -9)$ | $(0, -9)$ | $(\pm 3, 0)$ |
| **b** | $(9, 0)$ | $(0, 81)$ | $(9, 0)$ |
| **c** | $(0, 6)$ | $(0, 6)$ | $(\pm\sqrt{2}, 0)$ |
| **d** | $(-1, 0)$ | $(0, -3)$ | $(-1, 0)$ |
| **e** | $\left(\frac{1}{2}, 0\right)$ | $\left(0, \frac{1}{4}\right)$ | $\left(\frac{1}{2}, 0\right)$ |
| **f** | $\left(-\frac{1}{2}, 0\right)$ | $\left(0, -\frac{1}{4}\right)$ | $\left(-\frac{1}{2}, 0\right)$ |

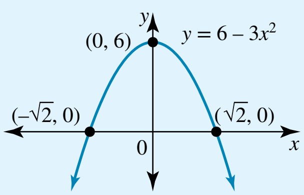
**a**



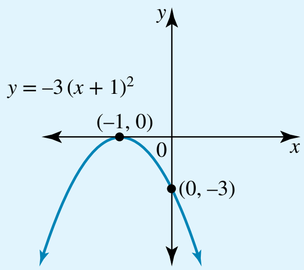

**b**



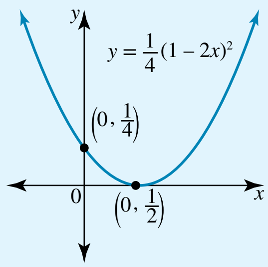

**c**



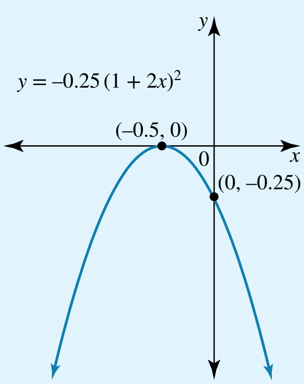

**d**




**e**



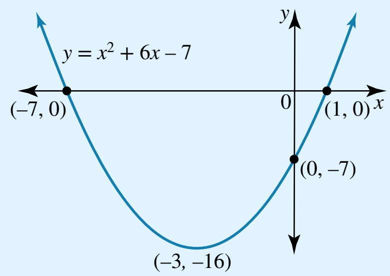

**f**



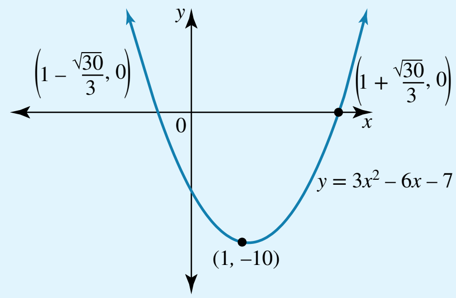

**13**

| | Turning point | *y*-intercept | *x*-intercepts |
|---|---|---|---|
| **a** | $(-3, -16)$ | $(0, -7)$ | $(-7, 0), (1, 0)$ |
| **b** | $(1, -10)$ | $(0, -7)$ | $\left(1 \pm \frac{\sqrt{30}}{3}, 0\right)$ |
| **c** | $\left(\frac{2}{3}, \frac{19}{3}\right)$ | $(0, 5)$ | $\left(\frac{2 \pm \sqrt{19}}{3}, 0\right)$ |
| **d** | $\left(\frac{1}{4}, -\frac{33}{8}\right)$ | $(0, -4)$ | $\left(\frac{1 \pm \sqrt{33}}{4}, 0\right)$ |
| **e** | $\left(\frac{3}{4}, -\frac{23}{8}\right)$ | $(0, -4)$ | none |
| **f** | $(2, 18)$ | $(0, 10)$ | $(-1, 0), (5, 0)$ |

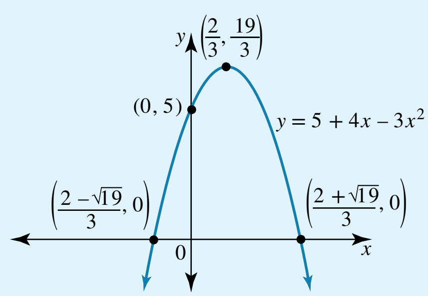
**a**



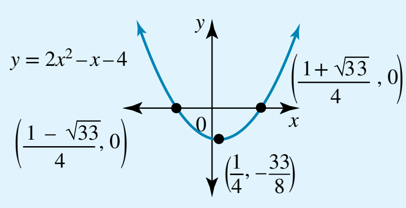

**b**



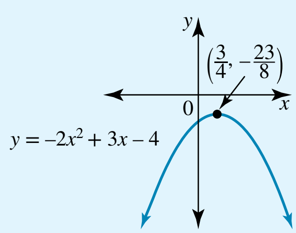

**c**



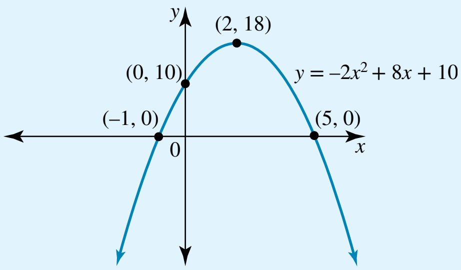

**d**




**e**



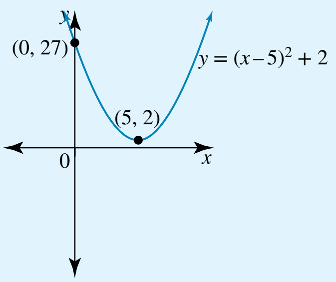
**f**



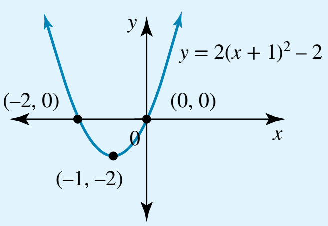

**14**

| | Turning point | $y$-intercept | $x$-intercepts |
|---|---|---|---|
| **a** | (5, 2) | (0, 27) | none |
| **b** | (−1, −2) | (0, 0) | (0, 0), (−2, 0) |
| **c** | (3, −6) | (0, −24) | none |
| **d** | (4, 1) | (0, −15) | (3, 0), (5, 0) |
| **e** | (−4, −2) | (0, 6) | (−6, 0), (−2, 0) |
| **f** | $\left(\frac{1}{2}, \frac{1}{9}\right)$ | $\left(0, \frac{2}{27}\right)$ | $\left(\frac{1 \pm \sqrt{3}}{2}, 0\right)$ |

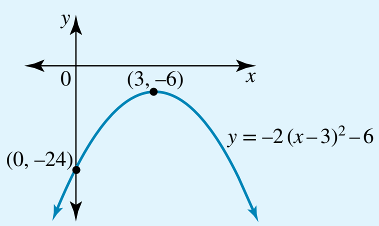
**a**



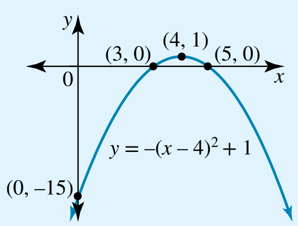

**b**



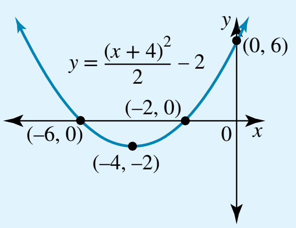

**c**



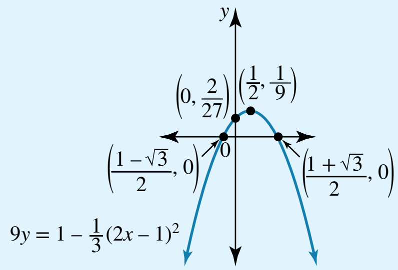

**d**




**e**



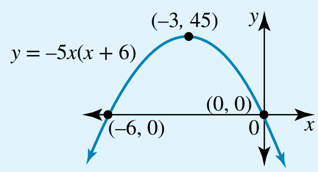

**f**



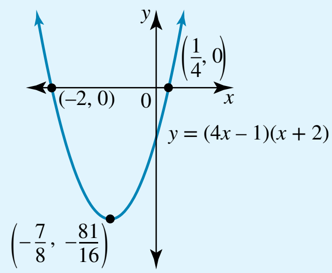

**15**

| | Turning point | $y$-intercept | $x$-intercepts |
|---|---|---|---|
| **a** | (−3, 45) | (0, 0) | (0, 0), (−6, 0) |
| **b** | $\left(-\frac{7}{8}, -\frac{81}{16}\right)$ | (0, −2) | (−2, 0), (0.25, 0) |
| **c** | (0.5, −4.5) | (0, −4) | (−1, 0), (2, 0) |
| **d** | $\left(\frac{1}{12}, \frac{49}{24}\right)$ | (0, 2) | $\left(-\frac{1}{2}, 0\right), \left(\frac{2}{3}, 0\right)$ |
| **e** | (1.35, −14.58) | (0, 0) | (0, 0), (2.7, 0) |
| **f** | $\left(-\frac{1}{3}, 0\right)$ | (0, 1) | $\left(-\frac{1}{3}, 0\right)$ |

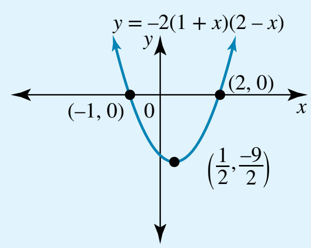
**a**



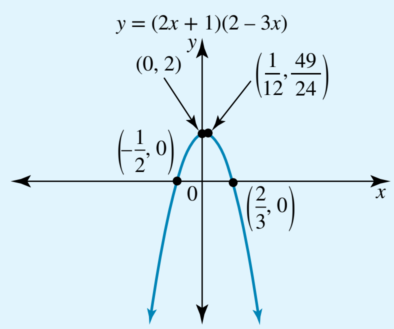

**b**



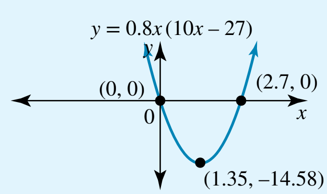

**c**

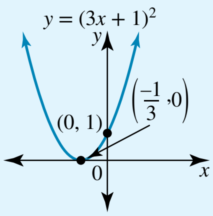



**d**




**e**



f



**16 a** 2 irrational $x$-intercepts

**b** No $x$-intercepts

**c** 1 rational $x$-intercept

**d** 2 rational $x$-intercepts

**17 a** i $2(x-3)^2 - 9$

ii $(3, -9)$

iii $-9$

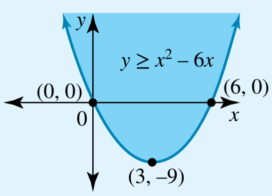
**b** i $-(x+9)^2 + 86$

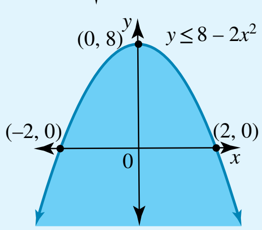
ii $(-9, 86)$

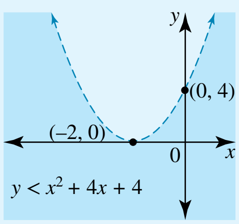
iii $86$

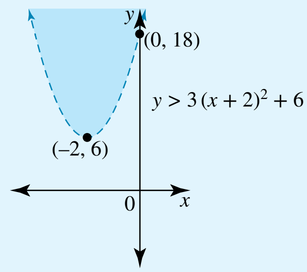
**18**

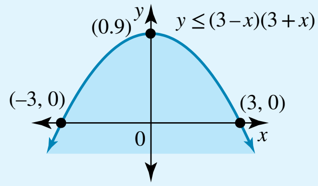
| | Turning point | $x$-intercepts | $y$-intercept | Region |
|---|---|---|---|---|
| a | $(3, -9)$ | $(0, 0), (6, 0)$ | $(0, 0)$ | Shade above closed curve |
| b | $(0, 8)$ | $(\pm 2, 0)$ | $(0, 8)$ | Shade below closed curve |
| c | $(-2, 0)$ | $(-2, 0)$ | $(0, 4)$ | Shade below open curve |
| d | $(-2, 6)$ | none | $(0, 18)$ | Shade above open curve |
| e | $(0, 9)$ | $(\pm 3, 0)$ | $(0, 9)$ | Shade below closed curve |
| f | $\left(-\frac{1}{4}, -\frac{7}{8}\right)$ | $(0, 0)$, $\left(-\frac{1}{2}, 0\right)$ | $(0, 0)$ | Shade below open curve |

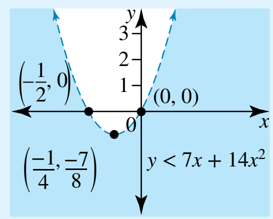
a



b



c



d



e



f



**19** i $k = -5$

ii $k > -5$

iii $k < -5$

**20 a** $m > \frac{1}{4}$

**b** i Proof required – check with your teacher

ii $x = -\frac{1}{2}$

**c** i $t = \pm\frac{4\sqrt{6}}{3}$

ii $t = 0, 0.25$

**21 a** i $\left(\frac{3}{4}, -\frac{7}{4}\right)$

ii $\left(\frac{9 \pm \sqrt{21}}{12}, 0\right)$

**b** i $\left(\frac{9}{16}, \frac{465}{32}\right)$

ii $\left(\frac{9 \pm \sqrt{465}}{16}, 0\right)$

**22** Graph using CAS technology

| Turning point | $x$-intercept | $y$-intercept |
|---|---|---|
| (12.38, 331.90) | (–0.19, 0), (24.95, 0) | (0, 10) |

## EXERCISE 3.6

**1 a** $y = (x+2)^2 + 1$

**b** $y = 2x(x-2)$

**2** $y = \frac{1}{8}x^2 + \frac{1}{2}x + \frac{1}{2}$

**3** $y = -2x^2 + x - 4$

**4** $y = \frac{1}{2}x^2 - \frac{3}{2}x - 2$ which is the same as $y = \frac{1}{2}(x+1)(x-4)$

**5 a** $y = -2x^2 + 6$

**b** $y = 0.2(x+6)(x+1)$

**6 a** $y = \frac{1}{2}(x-2)^2 - 4$

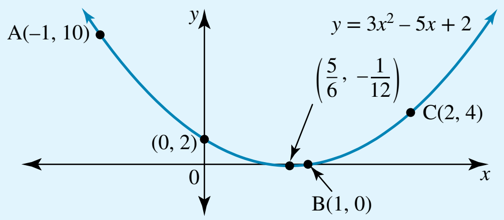
**b** $4\sqrt{2}$ units

**7 a** $y = 3x^2 - 5x + 2$

**b** $\left(\frac{2}{3}, 0\right), (1, 0)$

**c** $\left(\frac{5}{6}, -\frac{1}{12}\right)$

**d**




**8 a** Answers will vary but of the form $y = a(x+3)(x-5)$

**b** $y = -3(x+3)(x-5)$, vertex $(1, 48)$

**9 a** $y \le -0.8x^2$

**b** $y > -2x^2 - 8x$

**10 a** $y = 3(x+1)^2 - 5 = 3x^2 + 6x - 2$

**b** $y = (x+5)^2$

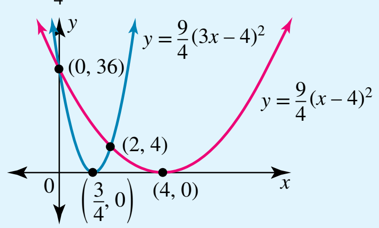
**11** $y = \frac{1}{2}x^2 - 4x + 6$

**12 a** Proof required – check with your teacher

**b** $\Delta = (2 + 5a)^2$

**c** $y = -\frac{4}{5}x^2 + 6x - 10, \left(\frac{5}{2}, 0\right)$

**13 a** $y = \frac{1}{4}(x+4)^2$




**b** $p = \frac{4}{3} \Rightarrow y = \frac{9}{4}(3x-4)^2$

$p = 4 \Rightarrow y = \frac{9}{4}(x-4)^2$

**14 a** V is point $(2, 6)$

**b** $y = (x-2)^2 + 6$

**c** Equation of VB is $y = 3x$ which passes through the origin

**d** 27 square units

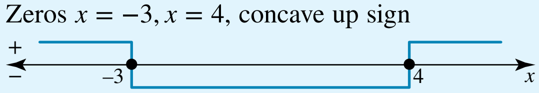
**15** $y = 4.2x^2 + 6x - 10$

**16 a** $a = -0.02, b = 56$

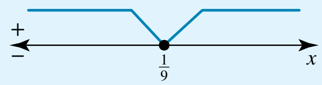
**b** $(\pm 52.915, 0)$

## EXERCISE 3.7

**1** Zeros $x = -3, x = 4$, concave up sign




**2** Zero $x = \frac{1}{9}$ multiplicity 2 concave up sign diagram touching at the zero required




**3 a** $x \le \frac{3}{2}$ or $x \ge 4$

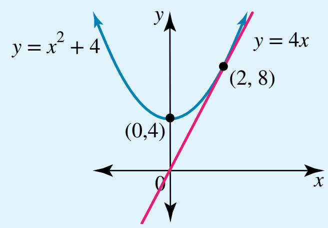
**b** $\left\{x : -\frac{1}{2} < x < \frac{2}{3}\right\}$

**4** $x \in R \backslash \{3\}$

**5 a** $(-6, 8)$ and $(2, 0)$

**b** No intersections

**6** $(2, 8)$;




**7** $m \le -6$ or $m \ge 18$

**8** $-2\sqrt{5} < k < 2\sqrt{5}$

**9 a** $-12 \le x \le 4$

**b** $x \le -1$ or $x \ge 4$

**c** $x < -3$ or $x > \frac{3}{2}$

**d** $-8 < x < -2$

**e** $x < 0$ or $x > 9$

**f** $2 \le x \le \frac{13}{4}$

**10 a** $R \backslash \{6\}$

**b** $\{1\}$

**c** $\{x : x < 0\} \cup \left\{x : x > \frac{1}{4}\right\}$

**d** $\left\{x : -\frac{3}{2} \le x \le \frac{7}{5}\right\}$

**11 a** $x = 6, y = 32$ or $x = -1, y = -3$

**b** $x = -8, y = 35$ or $x = 1, y = -1$

**c** No solution

**d** $x = 0, y = 5$ or $x = -\frac{5}{3}, y = \frac{70}{9}$

**12 a** $\left(\frac{3}{5}, \frac{31}{5}\right), (1, 7)$

**b** $(-2, -3)$

**c** $(-5, 10), (4, 10)$

**d** $(4, 30)$

**13 a** No intersections

**b** 2 intersections

**c** No intersections

**14 a** $-9 < k < -1$

**b** $k = -9, k = -1$

**c** $k < -9$ or $k > -1$

**15 a** $p \le 0$ or $p \ge 4$

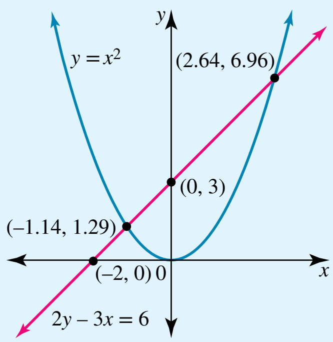
**b** $5 - 4\sqrt{5} < t < 5 + 4\sqrt{5}$

**c** $n = -5, n = 7$

**d** $-2 < x < 2$

**16 a** $(-1.14, 1.29), (2.64, 6.96)$

**b**



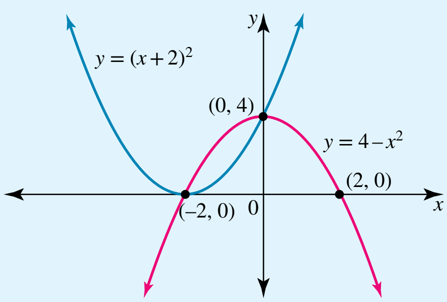

**c** $2y - 3x \le 6$ and $y \ge x^2$

**d** $\left(0, -\frac{9}{16}\right)$

**17 a** $y = 5x - 4$

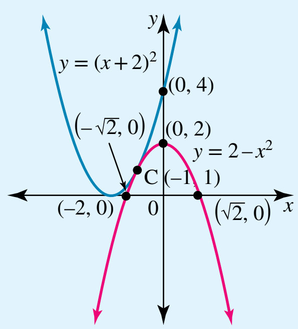
**b** $y = \frac{5}{3} - x^2$ Other answers possible

**18 a** $(-2, 0), (0, 4)$




**b** **i** Proof required — check with your teacher

**ii** C(−1, 1);




**iii** $y = 2x + 3$

**19 a** $x < -\frac{1}{10}(\sqrt{389} + 3)$ or $x > \frac{1}{10}(\sqrt{389} - 3)$

**b** $-\frac{1}{12}(\sqrt{465} + 15) \le x \le \frac{1}{12}(\sqrt{465} - 15)$

**20 a** Use CAS technology to sketch the graph. $a = -6$ no intersections; $a = -4$ points $(1, -8), (2, -12)$; $a = -2$ points $(0.38, -3.53), (2.62, -12, 47)$; $a = 0$ points $(0, 0), (3, -12)$

**b** $x = -\frac{1}{2}(\sqrt{2a + 9} - 3)$ or $x = \frac{1}{2}(\sqrt{2a + 9} + 3)$

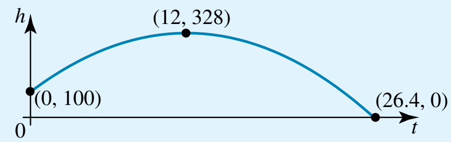
**c** $a = -4.5$, point $(1.5, -10.5)$

## EXERCISE 3.8

**1 a** 328 metres

**b** 12 seconds

**c** Reaches ground after 26.4 seconds;




**2 a** Proof required – check with your teacher

**b** Width 7.5 metres; length 15 metres

**c** 112.5 square metres

**3 a** 2 seconds

**b** Does not strike foliage

**c** $\frac{10}{9}$ metres

**4 a** $y = -0.2(x - 5.5)^2 + 7.25$

**b** 7.25 metres

**c** 2.37 metres

**5 a** Proof required; length 30 metres, width 15 metres

**b** 40 metres, 35 metres, 45 metres

**6 a** 2 hours

**b** 380

**c** 110 bacteria at 4 pm

**7** $z = \frac{104}{7}$ when $x = \frac{8}{7}, y = \frac{6}{7}$

**8 a** 17

**b** $S = 2x^2 - 10x + 25$, two pieces of 10 cm

**9 a** $C = 20 + 10n + 5n^2$

**b** 13

**10 a** $A(3, 0), B(7, 5), C(11, 0)$

**b** $y = -\frac{5}{16}(x - 7)^2 + 5$

**c** 6.7 metres

**11** Profit $= -x^2 + 40x - 15$, \$20 per kg

**12 a** **i** Both numbers are 8.

**ii** Both numbers are 8.

**b** **i** $\frac{k^2}{4}$

**ii** No values possible

**13 a** $h = -0.06t^2 + 1.97t - 12.78$

**b** 3.39 metres above mean sea level at 4.25 pm

**14 a** $S = x^2 + \frac{(10 - 2x)^2}{\pi}$

**b** 11.2 cm and 8.8 cm

**c** $x = 0$

# 4

# Cubic polynomials

# 4.1 Kick off with CAS

## Cubic transformations

**1** Using CAS technology, sketch the following cubic functions.

**a** $y = x^3$ **b** $y = -x^3$ **c** $y = -3x^3$ **d** $y = \frac{1}{2}x^3$ **e** $y = -\frac{2}{5}x^3$

**2** Using CAS technology, enter $y = ax^3$ into the function entry line and use a slider to change the values of $a$.

**3** Complete the following sentences.

**a** When sketching a cubic function, a negative sign in front of the $x^3$ term ________ the graph of $y = x^3$.

**b** When sketching a cubic function, $y = ax^3$, for values of $a < -1$ and $a > 1$, the graph of $y = x^3$ becomes ___________.

**c** When sketching a cubic function, $y = ax^3$, for values of $-1 < a < 1$, $a \neq 0$, the graph of $y = x^3$ becomes ___________.

**4** Using CAS technology, sketch the following functions.

**a** $y = x^3$ **b** $y = (x + 1)^3$ **c** $y = -(x - 2)^3$

**d** $y = x^3 - 1$ **e** $y = -x^3 + 2$ **f** $y = 3 - x^3$

**5** Using CAS technology, enter $y = (x - h)^3$ into the function entry line and use a slider to change the values of $h$.

**6** Using CAS technology, enter $y = x^3 + k$ into the function entry line and use a slider to change the values of $k$.

**7** Complete the following sentences.

**a** When sketching a cubic function, $y = (x - h)^3$, the graph of $y = x^3$ is _________.

**b** When sketching a cubic function, $y = x^3 + k$, the graph of $y = x^3$ is _________.

**8** Use CAS technology and your answers to questions 1–7 to determine the equation that could model the shape of the Bridge of Peace in Georgia. If the technology permits, upload a photo of the bridge to make this easier.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 4.2 Polynomials

A **polynomial** is an algebraic expression in which the power of the variable is a positive whole number. For example, $3x^2 + 5x - 1$ is a quadratic polynomial in the variable $x$ but $\frac{3}{x^2} + 5x - 1$ i.e. $3x^{-2} + 5x - 1$ is not a polynomial because of the $3x^{-2}$ term. Similarly, $\sqrt{3}x + 5$ is a linear polynomial but $3\sqrt{x} + 5$ i.e. $3x^{\frac{1}{2}} + 5$ is not a polynomial because the power of the variable $x$ is not a whole number. Note that the coefficients of $x$ can be positive or negative integers, rational or irrational real numbers.

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Units 1 & 2
AOS 1
Topic 3
Concept 1

**Polynomials**
Concept summary
Practice questions

## Classification of polynomials

- The **degree of a polynomial** is the highest power of the variable. For example, linear polynomials have degree 1, quadratic polynomials have degree 2 and cubic polynomials have degree 3.
- The **leading term** is the term containing the highest power of the variable.
- If the coefficient of the leading term is 1 then the polynomial is said to be monic.
- The **constant term** is the term that does not contain the variable.

**A polynomial of degree $n$ has the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $n \in N$ and the coefficients $a_n, a_{n-1}, \ldots a_1, a_0 \in R$. The leading term is $a_n x^n$ and the constant term is $a_0$.**

WORKED EXAMPLE 1

**Select the polynomials from the following list of algebraic expressions and for these polynomials, state the degree, the coefficient of the leading term, the constant term and the type of coefficients.**

**A** $5x^3 + 2x^2 - 3x + 4$ **B** $5x - x^3 + \frac{x^4}{2}$ **C** $4x^5 + 2x^2 + 7x^{-3} + 8$

| THINK | WRITE |
|---|---|
| 1 Check the powers of the variable $x$ in each algebraic expression. | A and B are polynomials since all the powers of $x$ are positive integers. C is not a polynomial due to the $7x^{-3}$ term. |
| 2 For polynomial A, state the degree, the coefficient of the leading term and the constant term. | Polynomial A: the leading term of $5x^3 + 2x^2 - 3x + 4$ is $5x^3$. Therefore, the degree is 3 and the coefficient of the leading term is 5. The constant term is 4. |
| 3 Classify the coefficients of polynomial A as elements of a subset of $R$. | The coefficients in polynomial A are integers. Therefore, A is a polynomial over Z. |
| 4 For polynomial B, state the degree, the coefficient of the leading term and the constant term. | Polynomial B: the leading term of $5x - x^3 + \frac{x^4}{2}$ is $\frac{x^4}{2}$. Therefore, the degree is 4 and the coefficient of the leading term is $\frac{1}{2}$. The constant term is 0. |
| 5 Classify the coefficients of polynomial B as elements of a subset of $R$. | The coefficients in polynomial B are rational numbers. Therefore, B is a polynomial over Q. |

## Polynomial notation

- The polynomial in variable $x$ is often referred to as $P(x)$.
- The value of the polynomial $P(x)$ when $x = a$ is written as $P(a)$.
- $P(a)$ is evaluated by substituting $a$ in place of $x$ in the $P(x)$ expression.

**WORKED EXAMPLE 2**

**a** If $P(x) = 5x^3 + 2x^2 - 3x + 4$ calculate $P(-1)$.

**b** If $P(x) = ax^2 - 2x + 7$ and $P(4) = 31$, obtain the value of $a$.

| THINK | WRITE |
|---|---|
| **a** Substitute $-1$ in place of $x$ and evaluate. | **a** $P(x) = 5x^3 + 2x^2 - 3x + 4$<br>$P(-1) = 5(-1)^3 + 2(-1)^2 - 3(-1) + 4$<br>$= -5 + 2 + 3 + 4$<br>$= 4$ |
| **b 1** Find an expression for $P(4)$ by substituting 4 in place of $x$, and then simplify. | **b** $P(x) = ax^2 - 2x + 7$<br>$P(4) = a(4)^2 - 2(4) + 7$<br>$= 16a - 1$ |
| **2** Equate the expression for $P(4)$ with 31. | $P(4) = 31$<br>$\Rightarrow 16a - 1 = 31$ |
| **3** Solve for $a$. | $16a = 32$<br>$a = 2$ |

## Identity of polynomials

If two polynomials are **identically equal** then the coefficients of like terms are equal. **Equating coefficients** means that if $ax^2 + bx + c \equiv 2x^2 + 5x + 7$ then $a = 2$, $b = 5$ and $c = 7$. The identically equal symbol '$\equiv$' means the equality holds for all values of $x$. For convenience, however, we shall replace this symbol with the equality symbol '$=$' in working steps.

**WORKED EXAMPLE 3**

Calculate the values of $a$, $b$ and $c$ so that $x(x - 7) \equiv a(x - 1)^2 + b(x - 1) + c$.

| THINK | WRITE |
|---|---|
| **1** Expand each bracket and express both sides of the equality in expanded polynomial form. | $x(x - 7) \equiv a(x - 1)^2 + b(x - 1) + c$<br>$\therefore x^2 - 7x = a(x^2 - 2x + 1) + bx - b + c$<br>$\therefore x^2 - 7x = ax^2 + (-2a + b)x + (a - b + c)$ |
| **2** Equate the coefficients of like terms. | Equate the coefficients.<br>$x^2 : 1 = a$ ..................(1)<br>$x : -7 = -2a + b$ ......(2)<br>Constant: $0 = a - b + c$ ....(3) |

**3** Solve the system of simultaneous equations.

Since $a = 1$, substitute $a = 1$ into equation (2).

$-7 = -2(1) + b$

$b = -5$

Substitute $a = 1$ and $b = -5$ into equation (3).

$0 = 1 - (-5) + c$

$c = -6$

**4** State the answer.

$\therefore a = 1, b = -5, c = -6$

## Operations on polynomials

The addition, subtraction and multiplication of two or more polynomials results in another polynomial. For example, if $P(x) = x^2$ and $Q(x) = x^3 + x^2 - 1$, then $P(x) + Q(x) = x^3 + 2x^2 - 1$, a polynomial of degree 3; $P(x) - Q(x) = -x^3 + 1$, a polynomial of degree 3; and $P(x)Q(x) = x^5 + x^4 - x^2$, a polynomial of degree 5.

**WORKED EXAMPLE 4**

**Given $P(x) = 3x^3 + 4x^2 + 2x + m$ and $Q(x) = 2x^2 + kx - 5$, find the values of $m$ and $k$ for which $2P(x) - 3Q(x) = 6x^3 + 2x^2 + 25x - 25$.**

**THINK**

**1** Form a polynomial expression for $2P(x) - 3Q(x)$ by collecting like terms together.

**WRITE**

$2P(x) - 3Q(x)$

$= 2(3x^3 + 4x^2 + 2x + m) - 3(2x^2 + kx - 5)$

$= 6x^3 + 2x^2 + (4 - 3k)x + (2m + 15)$

**2** Equate the two expressions for $2P(x) - 3Q(x)$.

Hence, $6x^3 + 2x^2 + (4 - 3k)x + (2m + 15)$
$= 6x^3 + 2x^2 + 25x - 25$

**3** Calculate the values of $m$ and $k$.

Equate the coefficients of $x$.

$4 - 3k = 25$

$k = -7$

Equate the constant terms.

$2m + 15 = -25$

$m = -20$

**4** State the answer.

Therefore $m = -20,\ k = -7$

## Division of polynomials

There are several methods for performing the **division of polynomials** and CAS technology computes the division readily. Here, two 'by-hand' methods will be shown.

### The inspection method for division

The division of one polynomial by another polynomial of equal or lesser degree can be carried out by expressing the numerator in terms of the denominator.

To divide $(x+3)$ by $(x-1)$, or to find $\frac{x+3}{x-1}$, write the numerator $x+3$ as $(x-1)+1+3=(x-1)+4$.

$$\frac{x+3}{x-1}=\frac{(x-1)+4}{x-1}$$

This expression can then be split into the sum of **partial fractions** as:

$$\begin{aligned}\frac{x+3}{x-1}&=\frac{(x-1)+4}{x-1}\\&=\frac{x-1}{x-1}+\frac{4}{x-1}\\&=1+\frac{4}{x-1}\end{aligned}$$

**The division is in the form: $\frac{\text{dividend}}{\text{divisor}}=\text{quotient}+\frac{\text{remainder}}{\text{divisor}}$**

In the language of division, when the dividend $(x+3)$ is divided by the divisor $(x-1)$ it gives a quotient of 1 and a remainder of 4. Note that from the division statement $\frac{x+3}{x-1}=1+\frac{4}{x-1}$ we can write $x+3=1\times(x-1)+4$.

This is similar to the division of integers. For example, 7 divided by 2 gives a quotient of 3 and a remainder of 1.

$$\frac{7}{2}=3+\frac{1}{2}$$

$$\therefore 7=3\times 2+1$$

This inspection process of division can be extended, with practice, to division involving non-linear polynomials. It could be used to show that $\frac{x^2+4x+1}{x-1}=\frac{x(x-1)+5(x-1)+6}{x-1}$ and therefore $\frac{x^2+4x+1}{x-1}=x+5+\frac{6}{x-1}$.

This result can be verified by checking that $x^2+4x+1=(x+5)(x-1)+6$.

**WORKED EXAMPLE 5**

**a** Calculate the quotient and the remainder when $(x+7)$ is divided by $(x+5)$.

**b** Use the inspection method to find $\frac{3x-4}{x+2}$.

| THINK | WRITE |
|---|---|
| **a 1** Write the division of the two polynomials as a fraction. | **a** $\frac{x+7}{x+5}=\frac{(x+5)-5+7}{x+5}$ |
| **2** Write the numerator in terms of the denominator. | $=\frac{(x+5)+2}{x+5}$ |

3 Split into partial fractions.

$$= \frac{(x + 5)}{x + 5} + \frac{2}{x + 5}$$

4 Simplify.

$$= 1 + \frac{2}{x + 5}$$

5 State the answer.

The quotient is 1 and the remainder is 2.

b 1 Express the numerator in terms of the denominator.

b The denominator is $(x + 2)$.
Since $3(x + 2) = 3x + 6$, the numerator is
$3x - 4 = 3(x + 2) - 6 - 4$
$\therefore 3x - 4 = 3(x + 2) - 10$

2 Split the given fraction into its partial fractions.

$$\frac{3x - 4}{x + 2} = \frac{3(x + 2) - 10}{x + 2}$$
$$= \frac{3(x + 2)}{(x + 2)} - \frac{10}{x + 2}$$

3 Simplify and state the answer.

$$= 3 - \frac{10}{x + 2}$$
$$\therefore \frac{3x - 4}{x + 2} = 3 - \frac{10}{x + 2}$$

## Algorithm for long division of polynomials

eBook*plus*

**Interactivity**
Division of polynomials
**int-2564**

The inspection method of division is very efficient, particularly when the division involves only linear polynomials. However, it is also possible to use the long-division **algorithm** to divide polynomials.

**The steps in the long-division algorithm are:**

**1. Divide the leading term of the divisor into the leading term of the dividend.**

**2. Multiply the divisor by this quotient.**

**3. Subtract the product from the dividend to form a remainder of lower degree.**

**4. Repeat this process until the degree of the remainder is lower than that of the divisor.**

To illustrate this process, consider $(x^2 + 4x + 1)$ divided by $(x - 1)$. This is written as:

$$x - 1\overline{)x^2 + 4x + 1}$$

Step 1. The leading term of the divisor $(x - 1)$ is $x$; the leading term of the dividend $(x^2 + 4x + 1)$ is $x^2$. Dividing $x$ into $x^2$, we get $\frac{x^2}{x} = x$. We write this quotient $x$ on top of the long-division symbol.

$$\begin{array}{r} x \phantom{+ 4x + 1} \\ x - 1\overline{)x^2 + 4x + 1} \end{array}$$

Step 2. The divisor $(x - 1)$ is multiplied by the quotient $x$ to give $x(x - 1) = x^2 - x$. This product is written underneath the terms of $(x^2 + 4x + 1)$; like terms are placed in the same columns.

$$\begin{array}{r}
x \\
x - 1\overline{)x^2 + 4x + 1} \\
x^2 - x
\end{array}$$

Step 3. $x^2 - x$ is subtracted from $(x^2 + 4x + 1)$. This cancels out the $x^2$ leading term to give $x^2 + 4x + 1 - (x^2 - x) = 5x + 1$.

$$\begin{array}{r}
x \\
x - 1\overline{)x^2 + 4x + 1} \\
\underline{-(x^2 - x)} \\
5x + 1
\end{array}$$

The division statement, so far, would be $\frac{x^2 + 4x + 1}{x - 1} = x + \frac{5x + 1}{x - 1}$. This is incomplete since the remainder $(5x + 1)$ is not of a smaller degree than the divisor $(x - 1)$. The steps in the algorithm must be repeated with the same divisor $(x - 1)$ but with $(5x + 1)$ as the new dividend.

Continue the process.

Step 4. Divide the leading term of the divisor $(x - 1)$ into the leading term of $(5x + 1)$; this gives $\frac{5x}{x} = 5$. Write this as $+5$ on the top of the long-division symbol.

$$\begin{array}{r}
x + 5 \\
x - 1\overline{)x^2 + 4x + 1} \\
\underline{-(x^2 - x)} \\
5x + 1
\end{array}$$

Step 5. Multiply $(x - 1)$ by 5 and write the result underneath the terms of $(5x + 1)$.

$$\begin{array}{r}
x + 5 \\
x - 1\overline{)x^2 + 4x + 1} \\
\underline{-(x^2 - x)} \\
5x + 1 \\
5x - 5
\end{array}$$

Step 6. Subtract $(5x - 5)$ from $(5x + 1)$.

$$\begin{array}{rl}
x + 5 & \leftarrow \text{Quotient} \\
x - 1\overline{)x^2 + 4x + 1} & \\
\underline{-(x^2 - x)} & \\
5x + 1 & \\
\underline{-(5x - 5)} & \\
\underline{6} & \leftarrow \text{Remainder}
\end{array}$$

The remainder is of lower degree than the divisor so no further division is possible and we have reached the end of the process.

Thus: $\frac{x^2 + 4x + 1}{x - 1} = x + 5 + \frac{6}{x - 1}$

This method can be chosen instead of the inspection method, or if the inspection method becomes harder to use.

WORKED EXAMPLE 6

a Given $P(x) = 4x^3 + 6x^2 - 5x + 9$, use the long-division method to divide $P(x)$ by $(x + 3)$ and state the quotient and the remainder.

b Use the long-division method to calculate the remainder when $\left(3x^3 + \frac{5}{3}x\right)$ is divided by $(5 + 3x)$.

| THINK | WRITE |
| --- | --- |
| a 1 Set up the long division. | a $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$ |
| 2 The first stage of the division is to divide the leading term of the divisor into the leading term of the dividend. | $\begin{array}{r} 4x^2 \phantom{+ 6x^2 - 5x + 9} \\ x + 3\overline{)4x^3 + 6x^2 - 5x + 9} \end{array}$ |
| 3 The second stage of the division is to multiply the result of the first stage by the divisor. Write this product placing like terms in the same columns. | $\begin{array}{r} 4x^2 \phantom{- 5x + 9} \\ x + 3\overline{)4x^3 + 6x^2 - 5x + 9} \\ 4x^3 + 12x^2 \phantom{- 5x + 9} \end{array}$ |
| 4 The third stage of the division is to subtract the result of the second stage from the dividend. This will yield an expression of lower degree than the original dividend. | $\begin{array}{r} 4x^2 \phantom{- 5x + 9} \\ x + 3\overline{)4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{- 5x + 9} \\ \hline - 6x^2 - 5x + 9 \end{array}$ |
| 5 The algorithm needs to be repeated. Divide the leading term of the divisor into the leading term of the newly formed dividend. | $\begin{array}{r} 4x^2 - 6x \phantom{+ 9} \\ x + 3\overline{)4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{- 5x + 9} \\ \hline - 6x^2 - 5x + 9 \end{array}$ |
| 6 Multiply the result by the divisor and write this product keeping like terms in the same columns. | $\begin{array}{r} 4x^2 - 6x \phantom{+ 9} \\ x + 3\overline{)4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{- 5x + 9} \\ \hline - 6x^2 - 5x + 9 \\ - 6x^2 - 18x \phantom{+ 9} \end{array}$ |
| 7 Subtract to yield an expression of lower degree. *Note:* The degree of the expression obtained is still not less than the degree of the divisor so the algorithm will need to be repeated again. | $\begin{array}{r} 4x^2 - 6x \phantom{+ 9} \\ x + 3\overline{)4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{- 5x + 9} \\ \hline - 6x^2 - 5x + 9 \\ -(- 6x^2 - 18x) \phantom{+ 9} \\ \hline 13x + 9 \end{array}$ |

**8** Divide the leading term of the divisor into the dividend obtained in the previous step.

$$
\begin{array}{r}
4x^2 - 6x + 13 \\
x + 3\,\overline{)\,4x^3 + 6x^2 - 5x + 9} \\
-(4x^3 + 12x^2) \qquad\qquad \\
\hline
-6x^2 - 5x + 9 \\
-(-6x^2 - 18x) \quad \\
\hline
13x + 9
\end{array}
$$

**9** Multiply the result by the divisor and write this product keeping like terms in the same columns.

$$
\begin{array}{r}
4x^2 - 6x + 13 \\
x + 3\,\overline{)\,4x^3 + 6x^2 - 5x + 9} \\
(4x^3 + 12x^2) \qquad\qquad \\
\hline
-6x^2 - 5x + 9 \\
-(-6x^2 - 18x) \quad \\
\hline
13x + 9 \\
13x + 39
\end{array}
$$

**10** Subtract to yield an expression of lower degree.

*Note:* The term reached is a constant so its degree is less than that of the divisor. The division is complete.

$$
\begin{array}{r}
4x^2 - 6x + 13 \\
x + 3\,\overline{)\,4x^3 + 6x^2 - 5x + 9} \\
-(4x^3 + 12x^2) \qquad\qquad \\
\hline
-6x^2 - 5x + 9 \\
-(-6x^2 - 18x) \quad \\
\hline
13x + 9 \\
-(13x + 39) \\
\hline
-30 \\
\hline
\end{array}
$$

**11** State the answer.

$$\frac{4x^3 + 6x^2 - 5x + 9}{x + 3} = 4x^2 - 6x + 13 - \frac{30}{x + 3}$$

The quotient is $4x^2 - 6x + 13$ and the remainder is $-30$.

**b** **1** Set up the division, expressing both the divisor and the dividend in decreasing powers of $x$. This creates the columns for like terms.

**b** $3x^3 + \frac{5}{3}x = 3x^3 + 0x^2 + \frac{5}{3}x + 0$

$5 + 3x = 3x + 5$

$$3x + 5\,\overline{)\,3x^3 + 0x^2 + \tfrac{5}{3}x + 0}$$

**2** Divide the leading term of the divisor into the leading term of the dividend, multiply this result by the divisor and then subtract this product from the dividend.

$$
\begin{array}{r}
x^2 \qquad\qquad\qquad \\
3x + 5\,\overline{)\,3x^3 + 0x^2 + \tfrac{5}{3}x + 0} \\
-(3x^3 + 5x^2) \qquad\quad \\
\hline
-5x^2 + \tfrac{5}{3}x + 0
\end{array}
$$

3 Repeat the three steps of the algorithm using the dividend created by the first application of the algorithm.

$$\begin{array}{r} x^2 - \frac{5}{3}x \\ 3x + 5 \overline{)3x^3 + 0x^2 + \frac{5}{3}x + 0} \\ -(3x^3 + 5x^2) \\ \hline - 5x^2 + \frac{5}{3}x + 0 \\ -\left(- 5x^2 - \frac{25}{3}x\right) \\ \hline 10x + 0 \end{array}$$

4 Repeat the algorithm using the dividend created by the second application of the algorithm.

$$\begin{array}{r} x^2 - \frac{5}{3}x + \frac{10}{3} \\ 3x + 5 \overline{)3x^3 + 0x^2 + \frac{5}{3}x + 0} \\ -(3x^3 + 5x^2) \\ \hline - 5x^2 + \frac{5}{3}x + 0 \\ -\left(- 5x^2 - \frac{25}{3}x\right) \\ \hline 10x + 0 \\ -\left(10x + \frac{50}{3}\right) \\ \hline - \frac{50}{3} \\ \hline \end{array}$$

5 State the answer.

$$\frac{3x^3 + \frac{5}{3}x}{3x + 5} = x^2 - \frac{5}{3}x + \frac{10}{3} + \frac{-\frac{50}{3}}{3x + 5}$$

$$= x^2 - \frac{5}{3}x + \frac{10}{3} - \frac{50}{3(3x + 5)}$$

The remainder is $-\frac{50}{3}$.

## EXERCISE 4.2 Polynomials

**PRACTISE**

Work without CAS

1 WE1 Select the polynomials from the following list of algebraic expressions and state their degree, the coefficient of the leading term, the constant term and the type of coefficients.

A $30x + 4x^5 - 2x^3 + 12$ B $\frac{3x^2}{5} - \frac{2}{x} + 1$ C $5.6 + 4x - 0.2x^2$

2 Write down a monic polynomial over $R$ in the variable $y$ for which the degree is 7, the coefficient of the $y^2$ term is $-\sqrt{2}$, the constant term is 4 and the polynomial contains four terms.

3 WE2 a If $P(x) = 7x^3 - 8x^2 - 4x - 1$ calculate $P(2)$.

b If $P(x) = 2x^2 + kx + 12$ and $P(-3) = 0$, find $k$.

4 If $P(x) = -2x^3 + 9x + m$ and $P(1) = 2P(-1)$, find $m$.

5 WE3 Calculate the values of $a$, $b$ and $c$ so that $(2x + 1)(x - 5) \equiv a(x + 1)^2 + b(x + 1) + c$.

6 Express $(x + 2)^3$ in the form $px^2(x + 1) + qx(x + 2) + r(x + 3) + t$.

7 WE4 Given $P(x) = 4x^3 - px^2 + 8$ and $Q(x) = 3x^2 + qx - 7$, find the values of $p$ and $q$ for which $P(x) + 2Q(x) = 4x^3 + x^2 - 8x - 6$.

8 If $P(x) = x^4 + 3x^2 - 7x + 2$ and $Q(x) = x^3 + x + 1$, expand the product $P(x)Q(x)$ and state its degree.

9 WE5 a Calculate the quotient and the remainder when $(x - 12)$ is divided by $(x + 3)$.

b Use the inspection method to find $\dfrac{4x + 7}{2x + 1}$.

10 Calculate the values of $a$, $b$ and $c$ if $3x^2 - 6x + 5 \equiv ax(x + 2) + b(x + 2) + c$ and hence express $\dfrac{3x^2 - 6x + 5}{x + 2}$ in the form $P(x) + \dfrac{d}{x + 2}$ where $P(x)$ is a polynomial and $d \in R$.

11 WE6 a Given $P(x) = 2x^3 - 5x^2 + 8x + 6$, divide $P(x)$ by $(x - 2)$ and state the quotient and the remainder.

b Use the long-division method to calculate the remainder when $(x^3 + 10)$ is divided by $(1 - 2x)$.

12 Obtain the quotient and remainder when $(x^4 - 3x^3 + 6x^2 - 7x + 3)$ is divided by $(x - 1)^2$.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

13 Consider the following list of algebraic expressions.

A $3x^5 + 7x^4 - \dfrac{x^3}{6} + x^2 - 8x + 12$

B $9 - 5x^4 + 7x^2 - \sqrt{5}x + x^3$

C $\sqrt{4x^5} - \sqrt{5}x^3 + \sqrt{3}x - 1$

D $2x^2(4x - 9x^2)$

E $\dfrac{x^6}{10} - \dfrac{2x^8}{7} + \dfrac{5}{3x^2} - \dfrac{7x}{5} + \dfrac{4}{9}$

F $(4x^2 + 3 + 7x^3)^2$

a Select the polynomials from the list and for each of these polynomials state:
   i its degree
   ii the type of coefficients
   iii the leading term
   iv the constant term.

b Give a reason why each of the remaining expressions is not a polynomial.

14 Given $P(x) = 2x^3 + 3x^2 + x - 6$, evaluate the following.

a $P(3)$
b $P(-2)$
c $P(1)$
d $P(0)$
e $P\left(-\frac{1}{2}\right)$
f $P(0.1)$

15 If $P(x) = x^2 - 7x + 2$, obtain expressions for the following.

a $P(a) - P(-a)$
b $P(1 + h)$
c $P(x + h) - P(x)$

16 a If $P(x) = 4x^3 + kx^2 - 10x - 4$ and $P(1) = 15$, obtain the value of $k$.

b If $Q(x) = ax^2 - 12x + 7$ and $Q(-2) = -5$, obtain the value of $a$.

c If $P(x) = x^3 - 6x^2 + nx + 2$ and $P(2) = 3P(-1)$, obtain the value of $n$.

d If $Q(x) = -x^2 + bx + c$ and $Q(0) = 5$ and $Q(5) = 0$, obtain the values of $b$ and $c$.

17 a If $3x^2 + 4x - 7 \equiv a(x + 1)^2 + b(x + 1) + c$ calculate $a$, $b$ and $c$.

b If $x^3 + mx^2 + nx + p \equiv (x - 2)(x + 3)(x - 4)$ calculate $m$, $n$ and $p$.

c If $x^2 - 14x + 8 \equiv a(x - b)^2 + c$ calculate $a$, $b$ and $c$ and hence express $x^2 - 14x + 8$ in the form $a(x - b)^2 + c$.

d Express $4x^3 + 2x^2 - 7x + 1$ in the form $ax^2(x + 1) + bx(x + 1) + c(x + 1) + d$.

**18 a** If $P(x) = 2x^2 - 7x - 11$ and $Q(x) = 3x^4 + 2x^2 + 1$, find, expressing the terms in descending powers of $x$:

i $Q(x) - P(x)$ ii $3P(x) + 2Q(x)$ iii $P(x)Q(x)$

**b** If $P(x)$ is a polynomial of degree $m$ and $Q(x)$ is a polynomial of degree $n$ where $m > n$, state the degree of:

i $P(x) + Q(x)$ ii $P(x) - Q(x)$ iii $P(x)Q(x)$

**19 a** Determine the values of $a$, $b$, $p$ and $q$ if $P(x) = x^3 - 3x^2 + px - 2$, $Q(x) = ax^3 + bx^2 + 3x - 2a$ and $2P(x) - Q(x) = 5(x^3 - x^2 + x + q)$.

**b** i Express $4x^4 + 12x^3 + 13x^2 + 6x + 1$ in the form $(ax^2 + bx + c)^2$ where $a > 0$.

ii Hence state a square root of $4x^4 + 12x^3 + 13x^2 + 6x + 1$.

**20** $P(x) = x^4 + kx^2 + n^2$, $Q(x) = x^2 + mx + n$ and the product $P(x)Q(x) = x^6 - 5x^5 - 7x^4 + 65x^3 - 42x^2 - 180x + 216$.

**a** Calculate $k$, $m$ and $n$.

**b** Obtain the linear factors of $P(x)Q(x) = x^6 - 5x^5 - 7x^4 + 65x^3 - 42x^2 - 180x + 216$.

**21** Specify the quotient and the remainder when:

**a** $(x + 7)$ is divided by $(x - 2)$

**b** $(8x + 5)$ is divided by $(2x + 1)$

**c** $(x^2 + 6x - 17)$ is divided by $(x - 1)$

**d** $(2x^2 - 8x + 3)$ is divided by $(x + 2)$

**e** $(x^3 + 2x^2 - 3x + 5)$ is divided by $(x - 3)$

**f** $(x^3 - 8x^2 + 9x - 2)$ is divided by $(x - 1)$.

**22** Perform the following divisions.

**a** $(8x^3 + 6x^2 - 5x + 15)$ divided by $(1 + 2x)$

**b** $(4x^3 + x + 5)$ divided by $(2x - 3)$

**c** $(x^3 + 6x^2 + 6x - 12) \div (x + 6)$

**d** $(2 + x^3) \div (x + 1)$

**e** $\dfrac{x^4 + x^3 - x^2 + 2x + 5}{x^2 - 1}$

**f** $\dfrac{x(7 - 2x^2)}{(x + 2)(x - 3)}$

MASTER

**23 a** Use CAS technology to divide $(4x^3 - 7x^2 + 5x + 2)$ by $(2x + 3)$.

**b** State the remainder and the quotient.

**c** Evaluate the dividend if $x = -\frac{3}{2}$.

**d** Evaluate the divisor if $x = -\frac{3}{2}$.

**24 a** Define $P(x) = 3x^3 + 6x^2 - 8x - 10$ and $Q(x) = -x^3 + ax - 6$.

**b** Evaluate $P(-4) + P(3) - P\left(\dfrac{2}{3}\right)$.

**c** Give an algebraic expression for $P(2n) + 24Q(n)$.

**d** Obtain the value of $a$ so that $Q(-2) = -16$.

# 4.3 The remainder and factor theorems

studyon

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**The remainder and factor theorems**
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The remainder and factor theorems
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The remainder obtained when dividing $P(x)$ by the linear divisor $(x - a)$ is of interest because if the remainder is zero, then the divisor must be a linear factor of the polynomial. To pursue this interest we need to be able to calculate the remainder quickly without the need to do a lengthy division.

## The remainder theorem

The actual division, as we know, will result in a quotient and a remainder. This is expressed in the division statement $\frac{P(x)}{x-a} = \text{quotient} + \frac{\text{remainder}}{x-a}$.

Since $(x - a)$ is linear, the remainder will be some constant term independent of $x$.

From the division statement it follows that:

$$P(x) = (x - a) \times \text{quotient} + \text{remainder}$$

If we let $x = a$, this makes $(x - a)$ equal to zero and the statement becomes:

$$P(a) = 0 \times \text{quotient} + \text{remainder}$$

Therefore:

$$P(a) = \text{remainder}$$

This result is known as the **remainder theorem**.

> **If a polynomial $P(x)$ is divided by $(x - a)$ then the remainder is $P(a)$.**

Note that:

- If $P(x)$ is divided by $(x + a)$ then the remainder would be $P(-a)$ since replacing $x$ by $-a$ would make the $(x + a)$ term equal zero.
- If $P(x)$ is divided by $(ax + b)$ then the remainder would be $P\left(-\frac{b}{a}\right)$ since replacing $x$ by $-\frac{b}{a}$ would make the $(ax + b)$ term equal zero.

**WORKED EXAMPLE 7**

Find the remainder when $P(x) = x^3 - 3x^2 - 2x + 9$ is divided by:

**a** $x - 2$ **b** $2x + 1$.

| THINK | WRITE |
|---|---|
| **a 1** What value of $x$ will make the divisor zero? | **a** $(x - 2) = 0 \Rightarrow x = 2$ |
| **2** Write an expression for the remainder. | $P(x) = x^3 - 3x^2 - 2x + 9$<br>Remainder is $P(2)$. |
| **3** Evaluate to obtain the remainder. | $P(2) = (2)^3 - 3(2)^2 - 2(2) + 9$<br>$= 1$<br>The remainder is 1. |
| **b 1** Find the value of $x$ which makes the divisor zero. | **b** $(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}$ |

2 Write an expression for the remainder and evaluate it.

Remainder is $P\left(-\frac{1}{2}\right)$.

$$P\left(-\tfrac{1}{2}\right) = \left(-\tfrac{1}{2}\right)^3 - 3\left(-\tfrac{1}{2}\right)^2 - 2\left(-\tfrac{1}{2}\right) + 9$$
$$= -\tfrac{1}{8} - \tfrac{3}{4} + 1 + 9$$
$$= 9\tfrac{1}{8}$$

Remainder is $9\frac{1}{8}$.

## The factor theorem

We know 4 is a factor of 12 because it divides 12 exactly, leaving no remainder. Similarly, if the division of a polynomial $P(x)$ by $(x - a)$ leaves no remainder, then the divisor $(x - a)$ must be a factor of the polynomial $P(x)$.

$$P(x) = (x - a) \times \text{quotient} + \text{remainder}$$

If the remainder is zero, then $P(x) = (x - a) \times$ quotient.

Therefore $(x - a)$ is a factor of $P(x)$.

This is known as the **factor theorem**.

**If $P(x)$ is a polynomial and $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.**

Conversely, if $(x - a)$ is a factor of a polynomial $P(x)$ then $P(a) = 0$.

$a$ is a **zero** of the polynomial.

It also follows from the remainder theorem that if $P\left(-\dfrac{b}{a}\right) = 0$, then $(ax + b)$ is a factor of $P(x)$ and $-\dfrac{b}{a}$ is a zero of the polynomial.

**WORKED EXAMPLE 8**

**a** Show that $(x + 3)$ is a factor of $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$.

**b** Determine the polynomial $P(x) = ax^3 + bx + 2$ which leaves a remainder of $-9$ when divided by $(x - 1)$ and is exactly divisible by $(x + 2)$.

**THINK**

**a** 1 State how the remainder can be calculated when $Q(x)$ is divided by the given linear expression.

**WRITE**

**a** $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$
When $Q(x)$ is divided by $(x + 3)$, the remainder equals $Q(-3)$.

2 Evaluate the remainder.

$$Q(-3) = 4(-3)^4 + 4(-3)^3 - 25(-3)^2 - (-3) + 6$$
$$= 324 - 108 - 225 + 3 + 6$$
$$= 0$$

3 It is important to explain in the answer why the given linear expression is a factor.

Since $Q(-3) = 0$, $(x + 3)$ is a factor of $Q(x)$.

**b** **1** Express the given information in terms of the remainders.

**b** $P(x) = ax^3 + bx + 2$
Dividing by $(x - 1)$ leaves a remainder of $-9$.
$\Rightarrow P(1) = -9$
Dividing by $(x + 2)$ leaves a remainder of 0.
$\Rightarrow P(-2) = 0$

**2** Set up a pair of simultaneous equations in $a$ and $b$.

$P(1) = a + b + 2$
$a + b + 2 = -9$
$\therefore \quad a + b = -11$..............................(1)
$P(-2) = -8a - 2b + 2$
$-8a - 2b + 2 = 0$
$\therefore \quad 4a + b = 1$............................(2)

**3** Solve the simultaneous equations.

$a + b = -11$.....(1)
$4a + b = 1$..........(2)
Equation (2) – equation (1):
$3a = 12$
$a = 4$
Substitute $a = 4$ into equation (1).
$4 + b = -11$
$b = -15$

**4** Write the answer.

$\therefore P(x) = 4x^3 - 15x + 2$

## Factorising polynomials

When factorising a cubic or higher-degree polynomial, the first step should be to check if any of the standard methods for factorising can be used. In particular, look for a common factor, then look to see if a grouping technique can produce either a common linear factor or a difference of two squares. If the standard techniques do not work then the remainder and factor theorems can be used to factorise, since the zeros of a polynomial enable linear factors to be formed.

Cubic polynomials may have up to three zeros and therefore up to three linear factors. For example, a cubic polynomial $P(x)$ for which it is known that $P(1) = 0$, $P(2) = 0$ and $P(-4) = 0$, has 3 zeros: $x = 1$, $x = 2$ and $x = -4$. From these, its three linear factors $(x - 1)$, $(x - 2)$ and $(x + 4)$ are formed.

Integer zeros of a polynomial may be found through a trial-and-error process where factors of the polynomial's constant term are tested systematically. For the polynomial $P(x) = x^3 + x^2 - 10x + 8$, the constant term is 8 so the possibilities to test are $1, -1, 2, -2, 4, -4, 8$ and $-8$. This is a special case of what is known as the **rational root theorem**.

**The rational solutions to the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$, where the coefficients are integers and $a_n$ and $a_0$ are non-zero, will have solutions $x = \frac{p}{q}$ (in simplest form), where $p$ is a factor of $a_0$ and $q$ is a factor of $a_n$.**

In practice, not all of the zeros need to be, nor necessarily can be, found through trial and error. For a cubic polynomial it is sufficient to find one zero by trial and error and form its corresponding linear factor using the factor theorem. Dividing this linear factor into the cubic polynomial gives a quadratic quotient and zero remainder, so the quadratic quotient is also a factor. The standard techniques for factorising quadratics can then be applied.

For the division step, long division could be used; however, it is more efficient to use a division method based on equating coefficients. With practice, this can usually be done by inspection. To illustrate, $P(x) = x^3 + x^2 - 10x + 8$ has a zero of $x = 1$ since $P(1) = 0$. Therefore $(x - 1)$ is a linear factor and $P(x) = (x - 1)(ax^2 + bx + c)$.

Note that the $x^3$ term of $(x - 1)(ax^2 + bx + c)$ can only be formed by the product of the $x$ term in the first bracket with the $x^2$ term in the second bracket; likewise, the constant term of $(x - 1)(ax^2 + bx + c)$ can only be formed by the product of the constant terms in the first and second brackets.

The coefficients of the quadratic factor are found by equating coefficients of like terms in $x^3 + x^2 - 10x + 8 = (x - 1)(ax^2 + bx + c)$.

For $x^3$: $1 = a$

For constants: $8 = -c \Rightarrow c = -8$

This gives $x^3 + x^2 - 10x + 8 = (x - 1)(x^2 + bx - 8)$ which can usually be written down immediately.

For the right-hand expression $(x - 1)(x^2 + bx - 8)$, the coefficient of $x^2$ is formed after a little more thought. An $x^2$ term can be formed by the product of the $x$ term in the first bracket with the $x$ term in the second bracket and also by the product of the constant term in the first bracket with the $x^2$ term in the second bracket.

$$x^3 + x^2 - 10x + 8 = (x - 1)(x^2 + bx - 8)$$

Equating coefficients of $x^2$: $1 = b - 1$

$$\therefore b = 2$$

If preferred, the coefficients of $x$ could be equated or used as check.

It follows that:

$$\begin{aligned} P(x) &= (x - 1)(x^2 + 2x - 8) \\ &= (x - 1)(x - 2)(x + 4) \end{aligned}$$

**WORKED EXAMPLE 9**

**a** Factorise $P(x) = x^3 - 2x^2 - 5x + 6$.

**b** Given that $(x + 1)$ and $(5 - 2x)$ are factors of $P(x) = -4x^3 + 4x^2 + 13x + 5$, completely factorise $P(x)$.

| THINK | WRITE |
|---|---|
| **a 1** The polynomial does not factorise by a grouping technique so a zero needs to be found. The factors of the constant term are potential zeros. | **a** $P(x) = x^3 - 2x^2 - 5x + 6$<br>The factors of 6 are $\pm1, \pm2, \pm3$ and $\pm6$. |
| **2** Use the remainder theorem to test systematically until a zero is obtained. Then use the factor theorem to state the corresponding linear factor. | $P(1) = 1 - 2 - 5 + 6$<br>$= 0$<br>$\therefore (x - 1)$ is a factor. |

**3** Express the polynomial in terms of a product of the linear factor and a general quadratic factor.

$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(ax^2 + bx + c)$

**4** State the values of $a$ and $c$.

For the coefficient of $x^3$ to be 1, $a = 1$.
For the constant term to be 6, $c = -6$.
$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 + bx - 6)$

**5** Calculate the value of $b$.

Equating the coefficients of $x^2$ gives:

$$x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 + bx - 6)$$

$-2 = b - 1$

$b = -1$

$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$

**6** Factorise the quadratic factor so the polynomial is fully factorised into its linear factors.

Hence,

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 5x + 6 \\ &= (x - 1)(x^2 - x - 6) \\ &= (x - 1)(x - 3)(x + 2) \end{aligned}$$

**b 1** Multiply the two given linear factors to form the quadratic factor.

**b** $P(x) = -4x^3 + 4x^2 + 13x + 5$
Since $(x + 1)$ and $(5 - 2x)$ are factors, then $(x + 1)(5 - 2x) = -2x^2 + 3x + 5$ is a quadratic factor.

**2** Express the polynomial as a product of the quadratic factor and a general linear factor.

The remaining factor is linear.

$$\begin{aligned} \therefore P(x) &= (x + 1)(5 - 2x)(ax + b) \\ &= (-2x^2 + 3x + 5)(ax + b) \end{aligned}$$

**3** Find $a$ and $b$.

$-4x^3 + 4x^2 + 13x + 5 = (-2x^2 + 3x + 5)(ax + b)$
Equating coefficients of $x^3$ gives:

$-4 = -2a$

$\therefore a = 2$

Equating constants gives:

$5 = 5b$

$\therefore b = 1$

**4** State the answer.

$$\begin{aligned} -4x^3 + 4x^2 + 13x + 5 &= (-2x^2 + 3x + 5)(2x + 1) \\ &= (x + 1)(5 - 2x)(2x + 1) \\ \therefore P(x) &= (x + 1)(5 - 2x)(2x + 1) \end{aligned}$$

## Polynomial equations

If a polynomial is expressed in factorised form, then the polynomial equation can be solved using the Null Factor Law.

$$(x - a)(x - b)(x - c) = 0$$
$$\therefore (x - a) = 0,\ (x - b) = 0,\ (x - c) = 0$$
$$\therefore x = a,\ x = b \text{ or } x = c$$

$x = a$, $x = b$ and $x = c$ are called the roots or the solutions to the equation $P(x) = 0$.

The factor theorem may be required to express the polynomial in factorised form.

**WORKED EXAMPLE 10**

**Solve the equation $3x^3 + 4x^2 = 17x + 6$.**

| THINK | WRITE |
| --- | --- |
| 1 Rearrange the equation so one side is zero. | $3x^3 + 4x^2 = 17x + 6$<br>$3x^3 + 4x^2 - 17x - 6 = 0$ |
| 2 Since the polynomial does not factorise by grouping techniques, use the remainder theorem to find a zero and the factor theorem to form the corresponding linear factor.<br>*Note:* It is simpler to test for integer zeros first. | Let $P(x) = 3x^3 + 4x^2 - 17x - 6$.<br>Test factors of the constant term:<br>$P(1) \neq 0$<br>$P(-1) \neq 0$<br>$P(2) = 3(2)^3 + 4(2)^2 - 17(2) - 6$<br>$= 24 + 16 - 34 - 6$<br>$= 0$<br>Therefore $(x - 2)$ is a factor. |
| 3 Express the polynomial as a product of the linear factor and a general quadratic factor. | $3x^3 + 4x^2 - 17x - 6 = (x - 2)(ax^2 + bx + c)$ |
| 4 Find and substitute the values of $a$ and $c$. | $\therefore 3x^3 + 4x^2 - 17x - 6 = (x - 2)(3x^2 + bx + 3)$ |
| 5 Calculate $b$. | Equate the coefficients of $x^2$:<br>$4 = b - 6$<br>$b = 10$ |
| 6 Completely factorise the polynomial. | $3x^3 + 4x^2 - 17x - 6 = (x - 2)(3x^2 + 10x + 3)$<br>$= (x - 2)(3x + 1)(x + 3)$ |
| 7 Solve the equation. | The equation $3x^3 + 4x^2 - 17x - 6 = 0$ becomes:<br>$(x - 2)(3x + 1)(x + 3) = 0$<br>$x - 2 = 0,\ 3x + 1 = 0,\ x + 1 = 0$<br>$x = 2,\ x = -\frac{1}{3},\ x = -3$ |

## EXERCISE 4.3 The remainder and factor theorems

**PRACTISE**

Work without CAS

1 WE7 Find the remainder when $P(x) = x^3 + 4x^2 - 3x + 5$ is divided by:

a $x + 2$

b $2x - 1$.

2 If $x^3 - kx^2 + 4x + 8$ leaves a remainder of 29 when it is divided by $(x - 3)$, find the value of $k$.

3 WE8 a Show that $(x - 2)$ is a factor of $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$.

b Determine the polynomial $P(x) = 3x^3 + ax^2 + bx - 2$ which leaves a remainder of $-22$ when divided by $(x + 1)$ and is exactly divisible by $(x - 1)$.

4 Given $(2x + a)$ is a factor of $12x^2 - 4x + a$, obtain the value(s) of $a$.

5 WE9 a Factorise $P(x) = x^3 + 3x^2 - 13x - 15$.

b Given that $(x + 1)$ and $(3x + 2)$ are factors of $P(x) = 12x^3 + 41x^2 + 43x + 14$, completely factorise $P(x)$.

6 Given the zeros of the polynomial $P(x) = 12x^3 + 8x^2 - 3x - 2$ are not integers, use the rational root theorem to calculate one zero and hence find the three linear factors of the polynomial.

7 WE10 Solve the equation $6x^3 + 13x^2 = 2 - x$.

8 Solve for $x$, $2x^4 + 3x^3 - 8x^2 - 12x = 0$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 Calculate the remainder without actual division when:

a $x^3 - 4x^2 - 5x + 3$ is divided by $(x - 1)$

b $6x^3 + 7x^2 + x + 2$ is divided by $(x + 1)$

c $-2x^3 + 2x^2 - x - 1$ is divided by $(x - 4)$

d $x^3 + x^2 + x - 10$ is divided by $(2x + 1)$

e $27x^3 - 9x^2 - 9x + 2$ is divided by $(3x - 2)$

f $4x^4 - 5x^3 + 2x^2 - 7x + 8$ is divided by $(x - 2)$.

10 a When $P(x) = x^3 - 2x^2 + ax + 7$ is divided by $(x + 2)$, the remainder is 11. Find the value of $a$.

b If $P(x) = 4 - x^2 + 5x^3 - bx^4$ is exactly divisible by $(x - 1)$, find the value of $b$.

c If $2x^3 + cx^2 + 5x + 8$ has a remainder of 6 when divided by $(2x - 1)$, find the value of $c$.

d Given that each of $x^3 + 3x^2 - 4x + d$ and $x^4 - 9x^2 - 7$ have the same remainder when divided by $(x + 3)$, find the value of $d$.

11 a Calculate the values of $a$ and $b$ for which $Q(x) = ax^3 + 4x^2 + bx + 1$ leaves a remainder of 39 when divided by $(x - 2)$, given $(x + 1)$ is a factor of $Q(x)$.

b Dividing $P(x) = \frac{1}{3}x^3 + mx^2 + nx + 2$ by either $(x - 3)$ or $(x + 3)$ results in the same remainder. If that remainder is three times the remainder left when $P(x)$ is divided by $(x - 1)$, determine the values of $m$ and $n$.

12 a A monic polynomial of degree 3 in $x$ has zeros of 5, 9 and –2. Express this polynomial in:

i factorised form

ii expanded form.

b A polynomial of degree 3 has a leading term with coefficient $-2$ and zeros of $-4$, $-1$ and $\frac{1}{2}$. Express this polynomial in:

i factorised form

ii expanded form.

13 a Given $(x - 4)$ is a factor of $P(x) = x^3 - x^2 - 10x - 8$, fully factorise $P(x)$.

b Given $(x + 12)$ is a factor of $P(x) = 3x^3 + 40x^2 + 49x + 12$, fully factorise $P(x)$.

c Given $(5x + 1)$ is a factor of $P(x) = 20x^3 + 44x^2 + 23x + 3$, fully factorise $P(x)$.

**d** Given $(4x - 3)$ is a factor of $P(x) = -16x^3 + 12x^2 + 100x - 75$, fully factorise $P(x)$.

**e** Given $(8x - 11)$ and $(x - 3)$ are factors of $P(x) = -8x^3 + 59x^2 - 138x + 99$, fully factorise $P(x)$.

**f** Given $(3x - 5)$ is a factor of $P(x) = 9x^3 - 75x^2 + 175x - 125$, fully factorise $P(x)$.

**14** Factorise the following:

**a** $x^3 + 5x^2 + 2x - 8$

**b** $x^3 + 10x^2 + 31x + 30$

**c** $2x^3 - 13x^2 + 13x + 10$

**d** $-18x^3 + 9x^2 + 23x - 4$

**e** $x^3 - 7x + 6$

**f** $x^3 + x^2 - 49x - 49$

**15 a** The polynomial $24x^3 + 34x^2 + x - 5$ has three zeros, none of which are integers. Calculate the three zeros and express the polynomial as the product of its three linear factors.

**b** The polynomial $P(x) = 8x^3 + mx^2 + 13x + 5$ has a zero of $\frac{5}{2}$.

**i** State a linear factor of the polynomial.

**ii** Fully factorise the polynomial.

**iii** Calculate the value of $m$.

**c** **i** Factorise the polynomials $P(x) = x^3 - 12x^2 + 48x - 64$ and $Q(x) = x^3 - 64$.

**ii** Hence, show that $\dfrac{P(x)}{Q(x)} = 1 - \dfrac{12x}{x^2 + 4x + 16}$.

**d** A cubic polynomial $P(x) = x^3 + bx^2 + cx + d$ has integer coefficients and $P(0) = 9$. Two of its linear factors are $(x - \sqrt{3})$ and $(x + \sqrt{3})$. Calculate the third linear factor and obtain the values of $b$, $c$ and $d$.

**16** Solve the following equations for $x$.

**a** $(x + 4)(x - 3)(x + 5) = 0$

**b** $2(x - 7)(3x + 5)(x - 9) = 0$

**c** $x^3 - 13x^2 + 34x + 48 = 0$

**d** $2x^3 + 7x^2 = 9$

**e** $3x^2(3x + 1) = 4(2x + 1)$

**f** $8x^4 + 158x^3 - 46x^2 - 120x = 0$

**17 a** Show that $(x - 2)$ is a factor of $P(x) = x^3 + 6x^2 - 7x - 18$ and hence fully factorise $P(x)$ over $R$.

**b** Show that $(3x - 1)$ is the only real linear factor of $3x^3 + 5x^2 + 10x - 4$.

**c** Show that $(2x^2 - 11x + 5)$ is a factor of $2x^3 - 21x^2 + 60x - 25$ and hence calculate the roots of the equation $2x^3 - 21x^2 + 60x - 25 = 0$.

**18 a** If $(x^2 - 4)$ divides $P(x) = 5x^3 + kx^2 - 20x - 36$ exactly, fully factorise $P(x)$ and hence obtain the value of $k$.

**b** If $x = a$ is a solution to the equation $ax^2 - 5ax + 4(2a - 1) = 0$, find possible values for $a$.

**c** The polynomials $P(x) = x^3 + ax^2 + bx - 3$ and $Q(x) = x^3 + bx^2 + 3ax - 9$ have a common factor of $(x + a)$. Calculate $a$ and $b$ and fully factorise each polynomial.

**d** $(x + a)^2$ is a repeated linear factor of the polynomial $P(x) = x^3 + px^2 + 15x + a^2$. Show there are two possible polynomials satisfying this information and, for each, calculate the values of $x$ which give the roots of the equation $x^3 + px^2 + 15x + a^2 = 0$.

**MASTER**

**19** Specify the remainder when $(9 + 19x - 2x^2 - 7x^3)$ is divided by $(x - \sqrt{2} + 1)$.

**20** Solve the equation $10x^3 - 5x^2 + 21x + 12 = 0$ expressing the values of $x$ to 4 decimal places.

# 4.4 Graphs of cubic polynomials

The graph of the general cubic polynomial has an equation of the form $y = ax^3 + bx^2 + cx + d$, where $a$, $b$, $c$ and $d$ are real constants and $a \neq 0$. Since a cubic polynomial may have up to three linear factors, its graph may have up to three $x$-intercepts. The shape of its graph is affected by the number of $x$-intercepts.

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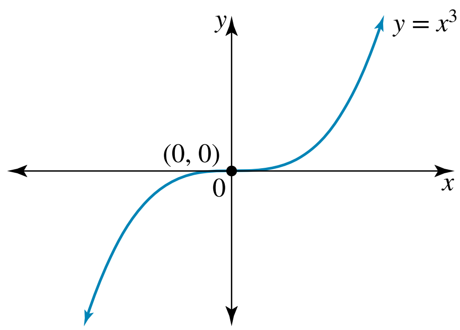
## The graph of $y = x^3$ and transformations

The graph of the simplest cubic polynomial has the equation $y = x^3$.




The 'maxi–min' point at the origin is sometimes referred to as a 'saddle point'. Formally, it is called a **stationary point of inflection** (or inflexion as a variation of spelling). It is a key feature of this cubic graph.

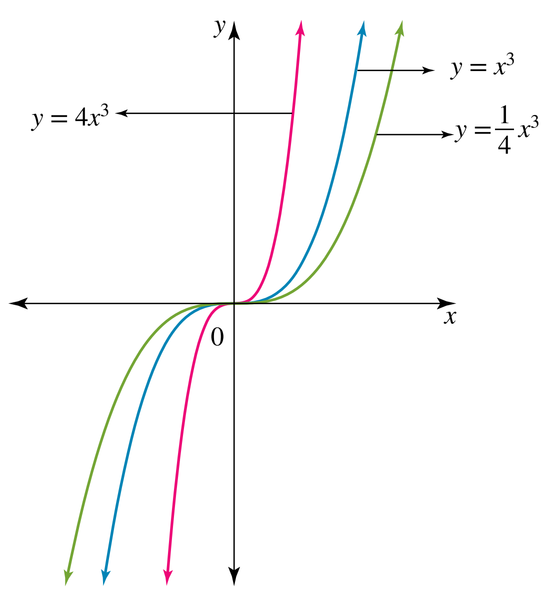
Key features of the graph of $y = x^3$:

- (0, 0) is a stationary point of inflection.
- The shape of the graph changes from concave down to concave up at the stationary point of inflection.
- There is only one $x$-intercept.
- As the values of $x$ become very large positive, the behaviour of the graph shows its $y$-values become increasingly large positive also. This means that as $x \to \infty$, $y \to \infty$. This is read as 'as $x$ approaches infinity, $y$ approaches infinity'.
- As the values of $x$ become very large negative, the behaviour of the graph shows its $y$-values become increasingly large negative. This means that as $x \to -\infty$, $y \to -\infty$.
- The graph starts from below the $x$-axis and increases as $x$ increases.

Once the basic shape is known, the graph can be dilated, reflected and translated in much the same way as the parabola $y = x^2$.

### Dilation

The graph of $y = 4x^3$ will be narrower than the graph of $y = x^3$ due to the dilation factor of 4 from the $x$-axis. Similarly, the graph of $y = \frac{1}{4}x^3$ will be wider than the graph of $y = x^3$ due to the dilation factor of $\frac{1}{4}$ from the $x$-axis.

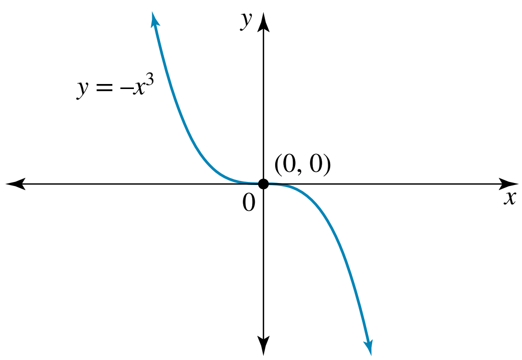


## Reflection

The graph of $y = -x^3$ is the reflection of the graph of $y = x^3$ in the $x$-axis.

For the graph of $y = -x^3$ note that:

- as $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$
- the graph starts from above the $x$-axis and decreases as $x$ increases
- at $(0, 0)$, the stationary point of inflection, the graph changes from concave up to concave down.

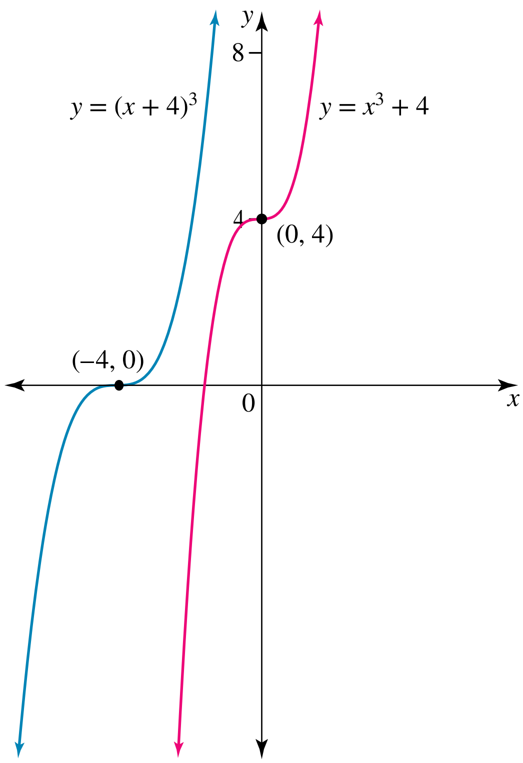



## Translation

The graph of $y = x^3 + 4$ is obtained when the graph of $y = x^3$ is translated vertically upwards by 4 units. The stationary point of inflection is at the point $(0, 4)$.

The graph of $y = (x + 4)^3$ is obtained when the graph of $y = x^3$ is translated horizontally 4 units to the left. The stationary point of inflection is at the point $(-4, 0)$.

The transformations from the basic parabola $y = x^2$ are recognisable from the equation $y = a(x - h)^2 + k$, and the equation of the graph of $y = x^3$ can be transformed to a similar form.




**The key features of the graph of $y = a(x - h)^3 + k$ are:**

- **stationary point of inflection at $(h, k)$**
- **change of concavity at the stationary point of inflection**
- **if $a > 0$, the graph starts below the $x$-axis and increases, like $y = x^3$**
- **if $a < 0$, the graph starts above the $x$-axis and decreases, like $y = -x^3$**
- **the one $x$-intercept is found by solving $a(x - h)^3 + k = 0$**
- **the $y$-intercept is found by substituting $x = 0$.**

WORKED EXAMPLE 11

Sketch:

**a** $y = (x + 1)^3 + 8$

**b** $y = 6 - \frac{1}{2}(x - 2)^3$.

| THINK | WRITE |
|---|---|
| **a 1** State the point of inflection. | **a** $y = (x + 1)^3 + 8$<br>Point of inflection is $(-1, 8)$. |

2 Calculate the $y$-intercept.

$y$-intercept: let $x = 0$

$y = (1)^3 + 8$

$= 9$

$\Rightarrow (0, 9)$

3 Calculate the $x$-intercept.

$x$-intercept: let $y = 0$

$(x + 1)^3 + 8 = 0$

$(x + 1)^3 = -8$

Take the cube root of both sides:

$x + 1 = \sqrt[3]{-8}$

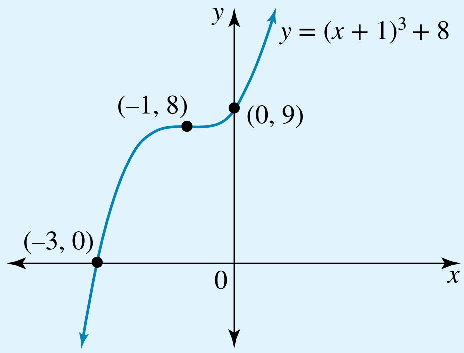
$x + 1 = -2$

$x = -3$

$\Rightarrow (-3, 0)$

4 Sketch the graph. Label the key points and ensure the graph changes concavity at the point of inflection.

The coefficient of $x^3$ is positive so the graph starts below the $x$-axis and increases.




**b** 1 Rearrange the equation to the $y = a(x - h)^3 + k$ form and state the point of inflection.

**b** $y = 6 - \frac{1}{2}(x - 2)^3$

$= -\frac{1}{2}(x - 2)^3 + 6$

Point of inflection: $(2, 6)$

2 Calculate the $y$-intercept.

$y$-intercept: let $x = 0$

$y = -\dfrac{1}{2}(-2)^3 + 6$

$= 10$

$\Rightarrow (0, 10)$

3 Calculate the $x$-intercept.

*Note:* A decimal approximation helps locate the point.

$x$-intercept: let $y = 0$

$-\frac{1}{2}(x - 2)^3 + 6 = 0$

$\frac{1}{2}(x - 2)^3 = 6$

$(x - 2)^3 = 12$

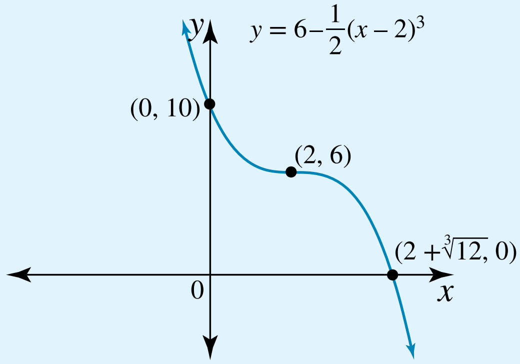
$x - 2 = \sqrt[3]{12}$

$x = 2 + \sqrt[3]{12}$

$\Rightarrow (2 + \sqrt[3]{12}, 0) \approx (4.3, 0)$

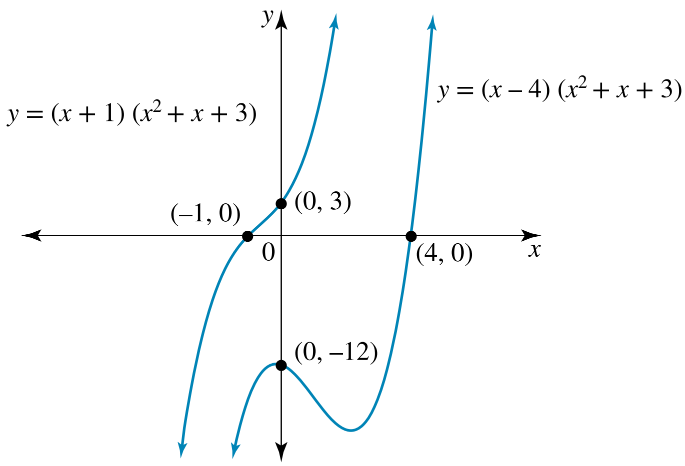
4 Sketch the graph showing all key features.

$a < 0$ so the graph starts above the $x$-axis and decreases.




## Cubic graphs with one $x$-intercept but no stationary point of inflection

There are cubic graphs which have one $x$-intercept but no stationary point of inflection. The equations of such cubic graphs cannot be expressed in the form $y = a(x - h)^3 + k$. Their equations can be expressed as the product of a linear factor and a quadratic factor which is **irreducible**, meaning the quadratic has no real factors.



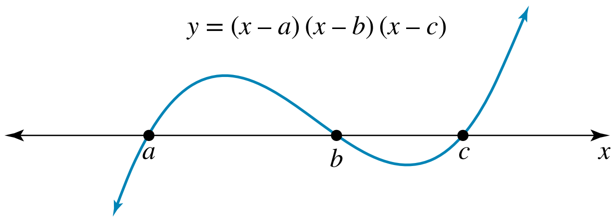

Technology is often required to sketch such graphs. Two examples, $y = (x + 1)(x^2 + x + 3)$ and $y = (x - 4)(x^2 + x + 3)$, are shown in the diagram. Each has a linear factor and the discriminant of the quadratic factor $x^2 + x + 3$ is negative; this means it cannot be further factorised over $R$.

Both graphs maintain the **long-term behaviour** exhibited by all cubics with a positive leading-term coefficient; that is, as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$. Every cubic polynomial must have at least one linear factor in order to maintain this long-term behaviour.

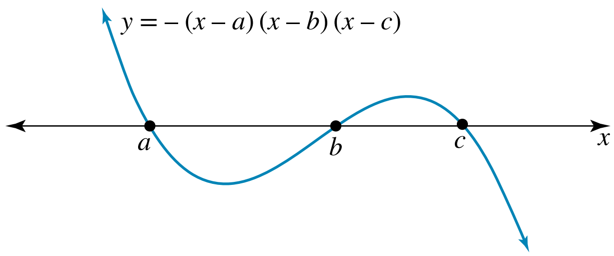
## Cubic graphs with three $x$-intercepts

For the graph of a cubic polynomial to have three $x$-intercepts, the polynomial must have three distinct linear factors. This is the case when the cubic polynomial expressed as the product of a linear factor and a quadratic factor is such that the quadratic factor has two distinct linear factors.

This means that the graph of a monic cubic with an equation of the form $y = (x - a)(x - b)(x - c)$ where $a, b, c \in R$ and $a < b < c$ will have the shape of the graph shown.



If the graph is reflected in the $x$-axis, its equation is of the form $y = -(x - a)(x - b)(x - c)$ and the shape of its graph satisfies the long-term behaviour that as $x \to \pm\infty$, $y \to \mp\infty$.



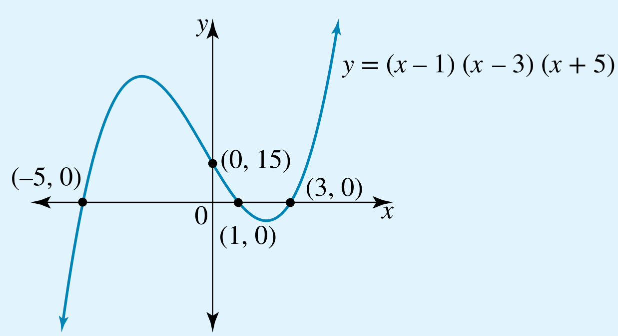

It is important to note the graph is not a quadratic so the maximum and minimum turning points do not lie halfway between the $x$-intercepts. In a later chapter we will learn how to locate these points without using technology.

To sketch the graph, it is usually sufficient to identify the $x$- and $y$-intercepts and to ensure the shape of the graph satisfies the long-term behaviour requirement determined by the sign of the leading term.

**WORKED EXAMPLE 12**

**Sketch the following without attempting to locate turning points.**

**a** $y = (x - 1)(x - 3)(x + 5)$ **b** $y = (x + 1)(2x - 5)(6 - x)$

| THINK | WRITE |
|---|---|
| **a 1** Calculate the $x$-intercepts. | **a** $y = (x - 1)(x - 3)(x + 5)$<br>$x$-intercepts: let $y = 0$<br>$(x - 1)(x - 3)(x + 5) = 0$<br>$x = 1,\ x = 3,\ x = -5$<br>$\Rightarrow (-5, 0), (1, 0), (3, 0)$ are the $x$-intercepts. |
| **2** Calculate the $y$-intercept. | $y$-intercept: let $x = 0$<br>$y = (-1)(-3)(5)$<br>$= 15$<br>$\Rightarrow (0, 15)$ is the $y$-intercept. |
| **3** Determine the shape of the graph. | Multiplying together the terms in $x$ from each bracket gives $x^3$, so its coefficient is positive.<br>The shape is of a positive cubic. |
| **4** Sketch the graph. |  |
| **b 1** Calculate the $x$-intercepts. | **b** $y = (x + 1)(2x - 5)(6 - x)$<br>$x$-intercepts: let $y = 0$<br>$(x + 1)(2x - 5)(6 - x) = 0$<br>$x + 1 = 0,\ 2x - 5 = 0,\ 6 - x = 0$<br>$x = -1,\ x = 2.5,\ x = 6$<br>$\Rightarrow (-1, 0), (2.5, 0), (6, 0)$ are the $x$-intercepts. |

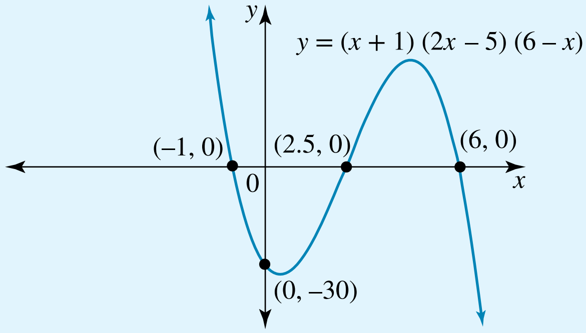
2 Calculate the $y$-intercept.

$y$-intercept: let $x = 0$
$y = (1)(-5)(6)$
$= -30$
$\Rightarrow (0, -30)$ is the $y$-intercept.

3 Determine the shape of the graph.

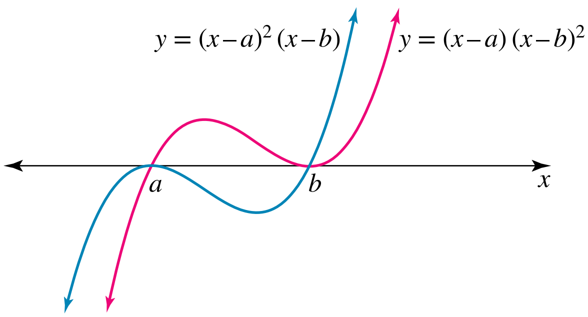
Multiplying the terms in $x$ from each bracket gives $(x) \times (2x) \times (-x) = -2x^3$ so the shape is of a negative cubic.

4 Sketch the graph.




## Cubic graphs with two $x$-intercepts

If a cubic has two $x$-intercepts, one at $x = a$ and one at $x = b$, then in order to satisfy the long-term behaviour required of any cubic, the graph either touches the $x$-axis at $x = a$ and turns, or it touches the $x$-axis at $x = b$ and turns. One of the $x$-intercepts must be a turning point.




Thinking of the cubic polynomial as the product of a linear and a quadratic factor, for its graph to have two instead of three $x$-intercepts, the quadratic factor must have two identical factors. Either the factors of the cubic are $(x - a)(x - a)(x - b) = (x - a)^2(x - b)$ or the factors are $(x - a)(x - b)(x - b) = (x - a)(x - b)^2$. The repeated factor identifies the $x$-intercept which is the turning point. The repeated factor is said to be of multiplicity 2 and the single factor of multiplicity 1.

The graph of a cubic polynomial with equation of the form $y = (x - a)^2(x - b)$ has a turning point on the $x$-axis at $(a, 0)$ and a second $x$-intercept at $(b, 0)$. The graph is said to *touch* the $x$-axis at $x = a$ and *cut* it at $x = b$.

Although the turning point on the $x$-axis must be identified when sketching the graph, there will be a second turning point that cannot yet be located without technology.

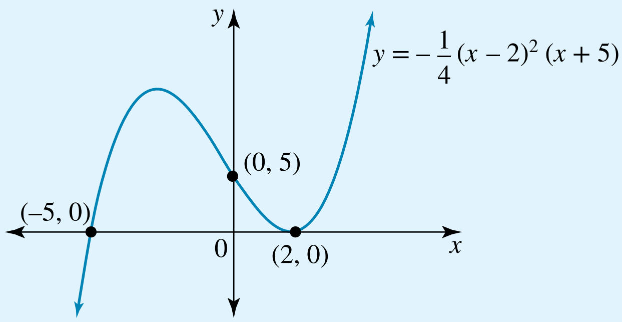
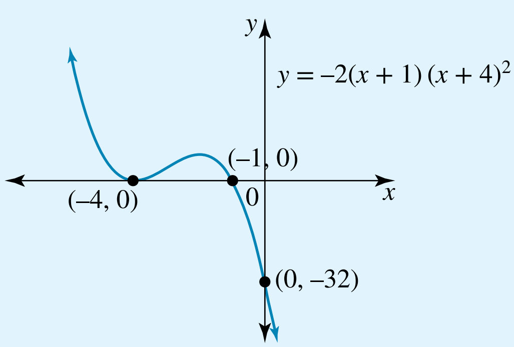
Note that a cubic graph whose equation has a repeated factor of multiplicity 3, such as $y = (x - h)^3$, would have only one $x$-intercept as this is a special case of $y = a(x - h)^3 + k$ with $k = 0$. The graph would cut the $x$-axis at its stationary point of inflection $(h, 0)$.

**WORKED EXAMPLE 13**

Sketch the graphs of:

**a** $y = \frac{1}{4}(x-2)^2(x+5)$

**b** $y = -2(x+1)(x+4)^2$

| THINK | WRITE |
|---|---|
| **a 1** Calculate the $x$-intercepts and interpret the multiplicity of each factor. | **a** $y = \frac{1}{4}(x-2)^2(x+5)$<br>$x$-intercepts: let $y = 0$<br>$\frac{1}{4}(x-2)^2(x+5) = 0$<br>$\therefore x = 2$ (touch), $x = -5$ (cut)<br>$x$-intercept at $(-5, 0)$ and turning-point $x$-intercept at $(2, 0)$ |
| **2** Calculate the $y$-intercept. | $y$-intercept: let $x = 0$<br>$y = \frac{1}{4}(-2)^2(5)$<br>$= 5$<br>$\Rightarrow (0, 5)$ |
| **3** Sketch the graph. | <br> |
| **b 1** Calculate the $x$-intercepts and interpret the multiplicity of each factor. | **b** $y = -2(x+1)(x+4)^2$<br>$x$-intercepts: let $y = 0$<br>$-2(x+1)(x+4)^2 = 0$<br>$(x+1)(x+4)^2 = 0$<br>$\therefore x = -1$ (cut), $x = -4$ (touch)<br>$x$-intercept at $(-1, 0)$ and turning-point $x$-intercept at $(-4, 0)$ |
| **2** Calculate the $y$-intercept. | $y$-intercept: let $x = 0$<br>$y = -2(1)(4)^2$<br>$= -32$<br>$y$-intercept at $(0, -32)$ |
| **3** Sketch the graph. | <br> |

## Cubic graphs in the general form $y = ax^3 + bx^2 + cx + d$

If the cubic polynomial with equation $y = ax^3 + bx^2 + cx + d$ can be factorised, then the shape of its graph and its key features can be determined. Standard factorisation techniques such as grouping terms together may be sufficient, or the factor theorem may be required in order to obtain the factors.

The sign of $a$, the coefficient of $x^3$, determines the long-term behaviour the graph exhibits. For $a > 0$ as $x \to \pm\infty$, $y \to \pm\infty$; for $a < 0$ as $x \to \pm\infty$, $y \to \mp\infty$.

The value of $d$ determines the $y$-intercept.

The factors determine the $x$-intercepts and the multiplicity of each factor will determine how the graph intersects the $x$-axis.

Every cubic graph must have at least one $x$-intercept and hence the polynomial must have at least one linear factor. Considering a cubic as the product of a linear and a quadratic factor, it is the quadratic factor which determines whether there is more than one $x$-intercept.

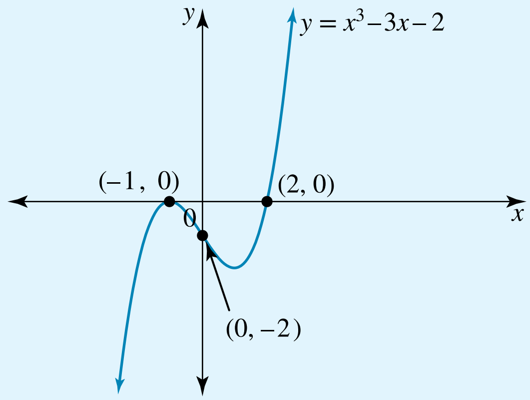
Graphs which have only one $x$-intercept may be of the form $y = a(x - h)^3 + k$ where the stationary point of inflection is a major feature. Recognition of this equation from its expanded form would require the expansion of a perfect cube to be recognised, since $a(x^3 - 3x^2h + 3xh^2 - h^3) + k = a(x - h)^3 + k$. However, as previously noted, not all graphs with only one $x$-intercept have a stationary point of inflection.

**WORKED EXAMPLE 14**

**Sketch the graph of $y = x^3 - 3x - 2$, without attempting to obtain any turning points that do not lie on the coordinate axes.**

| THINK | WRITE |
|---|---|
| 1 Obtain the $y$-intercept first since it is simpler to obtain from the expanded form. | $y = x^3 - 3x - 2$<br>$y$-intercept: $(0, -2)$ |
| 2 Factorisation will be needed in order to obtain the $x$-intercepts. | $x$-intercepts: let $y = 0$<br>$x^3 - 3x - 2 = 0$ |
| 3 The polynomial does not factorise by grouping so the factor theorem needs to be used. | Let $P(x) = x^3 - 3x - 2$<br>$P(1) \neq 0$<br>$P(-1) = -1 + 3 - 2 = 0$<br>$\therefore (x + 1)$ is a factor<br>$x^3 - 3x - 2 = (x + 1)(x^2 + bx - 2)$<br>$= (x + 1)(x^2 - x - 2)$<br>$= (x + 1)(x - 2)(x + 1)$<br>$= (x + 1)^2(x - 2)$<br>$\therefore x^3 - 3x - 2 = 0$<br>$\Rightarrow (x + 1)^2(x - 2) = 0$<br>$\therefore x = -1, 2$ |
| 4 What is the nature of these $x$-intercepts? | $y = P(x) = (x + 1)^2(x - 2)$<br>$x = -1$ (touch) and $x = 2$ (cut)<br>Turning point at $(-1, 0)$ |

5 Sketch the graph.




EXERCISE 4.4

## Graphs of cubic polynomials

PRACTISE

Work without CAS

1 WE11 Sketch the graphs of these polynomials.

a $y = (x - 1)^3 - 8$

b $y = 1 - \frac{1}{36}(x + 6)^3$

2 State the coordinates of the point of inflection and sketch the graph of the following.

a $y = \left(\frac{x}{2} - 3\right)^3$

b $y = 2x^3 - 2$

3 WE12 Sketch the following, without attempting to locate turning points.

a $y = (x + 1)(x + 6)(x - 4)$

b $y = (x - 4)(2x + 1)(6 - x)$

4 Sketch $y = 3x(x^2 - 4)$.

5 WE13 Sketch the graphs of these polynomials.

a $y = \frac{1}{9}(x - 3)^2(x + 6)$

b $y = -2(x - 1)(x + 2)^2$

6 Sketch $y = 0.1x(10 - x)^2$ and hence shade the region for which $y \le 0.1x(10 - x)^2$.

7 WE14 Sketch the graph of $y = x^3 - 3x^2 - 10x + 24$ without attempting to obtain any turning points that do not lie on the coordinate axes.

8 a Sketch the graph of $y = -x^3 + 3x^2 + 10x - 30$ without attempting to obtain any turning points that do not lie on the coordinate axes.

b Determine the coordinates of the stationary point of inflection of the graph with equation $y = x^3 + 3x^2 + 3x + 2$ and sketch the graph.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 a Sketch and clearly label the graphs of $y = x^3$, $y = 3x^3$, $y = x^3 + 3$ and $y = (x + 3)^3$ on the one set of axes.

b Sketch and clearly label the graphs of $y = -x^3$, $y = -3x^3$, $y = -x^3 + 3$ and $y = -(x + 3)^3$ on the one set of axes.

10 Sketch the graphs of the following, identifying all key points.

a $y = (x + 4)^3 - 27$

b $y = 2(x - 1)^3 + 10$

c $y = 27 + 2(x - 3)^3$

d $y = 16 - 2(x + 2)^3$

e $y = -\frac{3}{4}(3x + 4)^3$

f $y = 9 + \frac{x^3}{3}$

11 Sketch the graphs of the following, without attempting to locate any turning points that do not lie on the coordinate axes.

a $y = (x - 2)(x + 1)(x + 4)$
b $y = -0.5x(x + 8)(x - 5)$
c $y = (x + 3)(x - 1)(4 - x)$
d $y = \dfrac{1}{4}(2 - x)(6 - x)(4 + x)$
e $y = 0.1(2x - 7)(x - 10)(4x + 1)$
f $y = 2\left(\dfrac{x}{2} - 1\right)\left(\dfrac{3x}{4} + 2\right)\left(x - \dfrac{5}{8}\right)$

12 Sketch the graphs of the following, without attempting to locate any turning points that do not lie on the coordinate axes.

a $y = -(x + 4)^2(x - 2)$
b $y = 2(x + 3)(x - 3)^2$
c $y = (x + 3)^2(4 - x)$
d $y = \frac{1}{4}(2 - x)^2(x - 12)$
e $y = 3x(2x + 3)^2$
f $y = -0.25x^2(2 - 5x)$

13 Sketch the graphs of the following, showing any intercepts with the coordinate axes and any stationary point of inflection.

a $y = (x + 3)^3$
b $y = (x + 3)^2(2x - 1)$
c $y = (x + 3)(2x - 1)(5 - x)$
d $2(y - 1) = (1 - 2x)^3$
e $4y = x(4x - 1)^2$
f $y = -\frac{1}{2}(2 - 3x)(3x + 2)(3x - 2)$

14 Factorise, if possible, and then sketch the graphs of the cubic polynomials with equations given by:

a $y = 9x^2 - 2x^3$
b $y = 9x^3 - 4x$
c $y = 9x^2 - 3x^3 + x - 3$
d $y = 9x(x^2 + 4x + 3)$
e $y = 9x^3 + 27x^2 + 27x + 9$
f $y = -9x^3 - 9x^2 + 9x + 9$

15 Sketch, without attempting to locate any turning points that do not lie on the coordinate axes.

a $y = 2x^3 - 3x^2 - 17x - 12$
b $y = 6 - 55x + 57x^2 - 8x^3$
c $y = x^3 - 17x + 4$
d $y = 6x^3 - 13x^2 - 59x - 18$
e $y = -5x^3 - 7x^2 + 10x + 14$
f $y = -\dfrac{1}{2}x^3 + 14x - 24$

16 Consider $P(x) = 30x^3 + kx^2 + 1$.

a Given $(3x - 1)$ is a factor, find the value of $k$.
b Hence express $P(x)$ as the product of its linear factors.
c State the values of $x$ for which $P(x) = 0$.
d Sketch the graph of $y = P(x)$.
e Does the point $(-1, -40)$ lie on the graph of $y = P(x)$? Justify your answer.
f On your graph shade the region for which $y \geq P(x)$.

17 a Express $-\frac{1}{2}x^3 + 6x^2 - 24x + 38$ in the form $a(x - b)^3 + c$.
b Hence sketch the graph of $y = -\frac{1}{2}x^3 + 6x^2 - 24x + 38$.

18 Consider $y = x^3 - 5x^2 + 11x - 7$.

a Show that the graph of $y = x^3 - 5x^2 + 11x - 7$ has only one $x$-intercept.
b Show that $y = x^3 - 5x^2 + 11x - 7$ cannot be expressed in the form $y = a(x - b)^3 + c$.
c Describe the behaviour of the graph as $x \to \infty$.
d Given the graph of $y = x^3 - 5x^2 + 11x - 7$ has no turning points, draw a sketch of the graph.

MASTER

19 a Sketch, locating turning points, the graph of $y = x^3 + 4x^2 - 44x - 96$.
b Show that the turning points are not placed symmetrically in the interval between the adjoining $x$-intercepts.

20 Sketch, locating intercepts with the coordinate axes and any turning points. Express values to 1 decimal place where appropriate.
a $y = 10x^3 - 20x^2 - 10x - 19$
b $y = -x^3 + 5x^2 - 11x + 7$
c $y = 9x^3 - 70x^2 + 25x + 500$

## 4.5 Equations of cubic polynomials

studyon

Units 1 & 2
AOS 1
Topic 3
Concept 4

**Equations of cubic polynomials**
Concept summary
Practice questions

eBook*plus*

**Interactivity**
Graph plotter:
1, 2, 3 intercepts
**int-2567**

The equation $y = ax^3 + bx^2 + cx + d$ contains four unknown coefficients that need to be specified, so four pieces of information are required to determine the equation of a cubic graph, unless the equation is written in turning point form when 3 pieces of information are required.

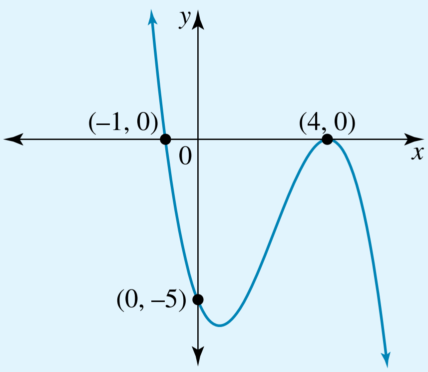
**As a guide:**
- **If there is a stationary point of inflection given, use the $y = a(x - h)^3 + k$ form.**
- **If the $x$-intercepts are given, use the $y = a(x - x_1)(x - x_2)(x - x_3)$ form, or the repeated factor form $y = a(x - x_1)^2(x - x_2)$ if there is a turning point at one of the $x$-intercepts.**
- **If four points on the graph are given, use the $y = ax^3 + bx^2 + cx + d$ form.**

WORKED EXAMPLE 15

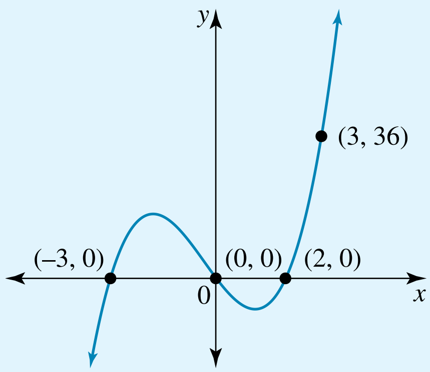
Determine the equation for each of the following graphs.

a The graph of a cubic polynomial which has a stationary point of inflection at the point $(-7, 4)$ and an $x$-intercept at $(1, 0)$.

b




c




| THINK | WRITE |
|---|---|
| a 1 Consider the given information and choose the form of the equation to be used. | a Stationary point of inflection is given.<br>Let $y = a(x - h)^3 + k$<br>Point of inflection is $(-7, 4)$.<br>$\therefore y = a(x + 7)^3 + 4$ |

**2** Calculate the value of $a$.

*Note:* The coordinates of the given points show the $y$-values decrease as the $x$-values increase, so a negative value for $a$ is expected.

Substitute the given $x$-intercept point $(1, 0)$.

$$0 = a(8)^3 + 4$$
$$(8)^3 a = -4$$
$$a = \frac{-4}{8 \times 64}$$
$$a = -\frac{1}{128}$$

**3** Write the equation of the graph.

The equation is $y = -\frac{1}{128}(x + 7)^3 + 4$.

**b 1** Consider the given information and choose the form of the equation to be used.

**b** Two $x$-intercepts are given.
One shows a turning point at $x = 4$ and the other a cut at $x = -1$.
Let the equation be $y = a(x + 1)(x - 4)^2$.

**2** Calculate the value of $a$.

Substitute the given $y$-intercept point $(0, -5)$.

$$-5 = a(1)(-4)^2$$
$$-5 = a(16)$$
$$a = -\frac{5}{16}$$

**3** Write the equation of the graph.

The equation is $y = -\frac{5}{16}(x + 1)(x - 4)^2$.

**c 1** Consider the given information and choose the form of the equation to be used.

**c** Three $x$-intercepts are given.
Let the equation be

$$y = a(x + 3)(x - 0)(x - 2)$$
$$\therefore y = ax(x + 3)(x - 2)$$

**2** Calculate the value of $a$.

Substitute the given point $(3, 36)$.

$$36 = a(3)(6)(1)$$
$$36 = 18a$$
$$a = 2$$

**3** Write the equation of the graph.

The equation is $y = 2x(x + 3)(x - 2)$.

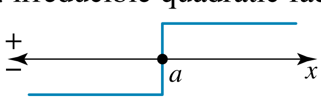
## Cubic inequations

**eBook*plus***

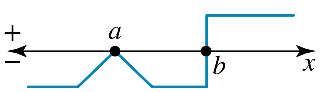
**Interactivity**
Cubic inequations
**int-2568**

The sign diagram for a cubic polynomial can be deduced from the shape of its graph and its $x$-intercepts. The values of the zeros of the polynomial are those of the $x$-intercepts. For a cubic polynomial with a positive coefficient of $x^3$, the sign diagram starts from below the $x$-axis. For examples below, assume $a < b < c$.

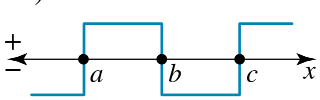
One zero $(x - a)^3$ or $(x - a) \times$ irreducible quadratic factor




Two zeros $(x - a)^2(x - b)$




Three zeros $(x - a)(x - b)(x - c)$



At a zero of multiplicity 2, the sign diagram touches the $x$-axis. For a zero of odd multiplicity, either multiplicity 1 or multiplicity 3, the sign diagram cuts the $x$-axis. This 'cut and touch' nature applies if the coefficient of $x^3$ is negative; however, the sign diagram would start from above the $x$-axis in that case.

**To solve a cubic inequation:**

- **Rearrange the terms in the inequation, if necessary, so that one side of the inequation is 0.**
- **Factorise the cubic expression and calculate its zeros.**
- **Draw the sign diagram, or the graph, of the cubic.**
- **Read from the sign diagram the set of values of $x$ which satisfy the inequation.**

An exception applies to inequations of forms such as $a(x-h)^3+k>0$. These inequations are solved in a similar way to solving linear inequations without the need for a sign diagram or a graph. Note the similarity between the sign diagram of the cubic polynomial with one zero and the sign diagram of a linear polynomial.

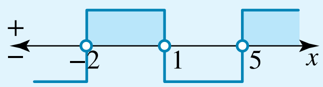
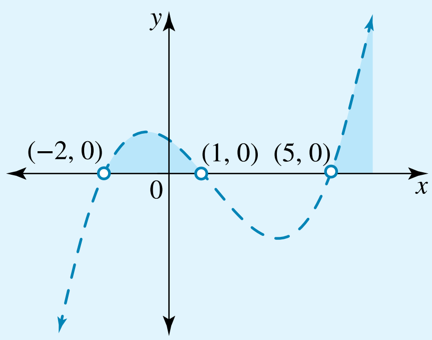
WORKED EXAMPLE 16

**Solve the inequations.**

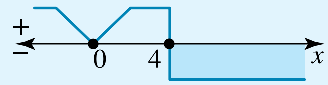
**a** $(x+2)(x-1)(x-5)>0$

**b** $\{x : 4x^2 \leq x^3\}$

**c** $(x-2)^3-1>0$

| THINK | WRITE |
|---|---|
| **a 1** Read the zeros from each factor. | **a** $(x+2)(x-1)(x-5)>0$<br>Zeros: $x=-2,\ x=1,\ x=5$ |
| **2** Consider the leading-term coefficient to draw the sign diagram. This is a positive cubic. |  |
| **3** State the intervals in which the required solutions lie. | $-2<x<1$ or $x>5$ |
| **4** Show the solution on a graph. |  |
| **b 1** Rearrange so one side is 0. | **b** $4x^2 \leq x^3$<br>$4x^2-x^3 \leq 0$<br>$\therefore x^2(4-x) \leq 0$ |
| **2** Factorise to calculate the zeros. | Let $x^2(4-x)=0$<br>$x^2=0$ or $4-x=0$<br>$x=0$ or $x=4$<br>Zeros: $x=0$ (multiplicity 2), $x=4$ |

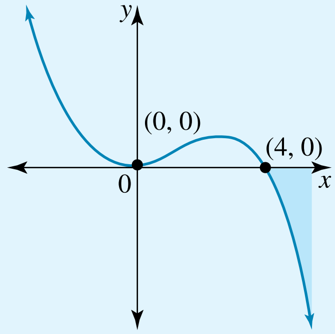
3 Consider the leading-term coefficient to draw the sign diagram. $4x^2 - x^3$ is a negative cubic.




4 State the answer from the sign diagram, using set notation, since the question is in set notation.

$\{x : x \geq 4\} \cup \{0\}$

5 Show the solution on a graph.




**c** 1 Solve for $x$.

**c** $(x - 2)^3 - 1 > 0$

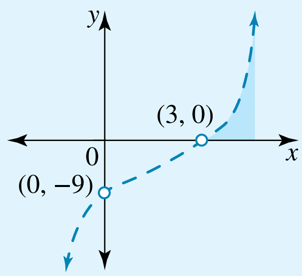
$(x - 2)^3 > 1$

$(x - 2) > \sqrt[3]{1}$

$x - 2 > 1$

$x > 3$

2 Show the solution on a graph.




## Intersections of cubic graphs with linear and quadratic graphs

If $P(x)$ is a cubic polynomial and $Q(x)$ is either a linear or a quadratic polynomial, then the intersection of the graphs of $y = P(x)$ and $y = Q(x)$ occurs when $P(x) = Q(x)$. Hence the $x$-coordinates of the points of intersection are the roots of the equation $P(x) - Q(x) = 0$. This is a cubic equation since $P(x) - Q(x)$ is a polynomial of degree 3.

WORKED EXAMPLE 17

**Sketch the graphs of $y = x(x - 1)(x + 1)$ and $y = x$ and calculate the coordinates of the points of intersection. Hence state the values of $x$ for which $x > x(x - 1)(x + 1)$.**

**THINK**

1 Sketch the graphs.

**WRITE**

$y = x(x - 1)(x + 1)$

This is a positive cubic.

$x$-intercepts: let $y = 0$

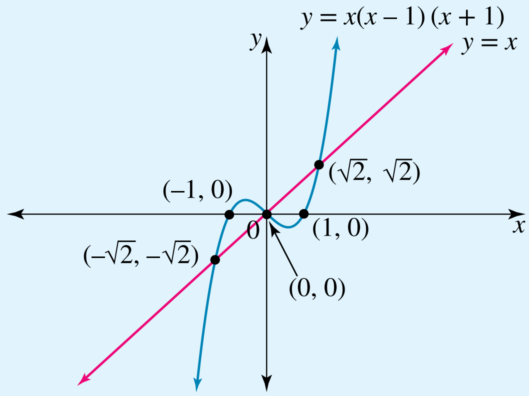
$x(x - 1)(x + 1) = 0$

$x = 0,\ x = \pm 1$

$(-1, 0), (0, 0), (1, 0)$ are the three $x$-intercepts.

$y$-intercept is $(0, 0)$.

Line: $y = x$ passes through $(0, 0)$ and $(1, 1)$.



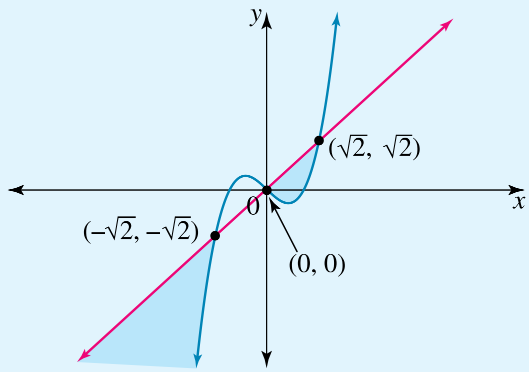

**2** Calculate the coordinates of the points of intersections.

At intersection:

$$\begin{aligned} x(x-1)(x+1) &= x \\ x(x^2-1)-x &= 0 \\ x^3-2x &= 0 \\ x(x^2-2) &= 0 \\ x &= 0,\ x^2 = 2 \\ x &= 0,\ x = \pm\sqrt{2} \end{aligned}$$

Substituting these $x$-values in the equation of the line $y = x$, the points of intersection are $(0, 0), (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$.

**3** Shade the regions of the graph where $x > x(x-1)(x+1)$.




**4** Use the diagram to state the intervals for which the given inequation holds.

$x > x(x-1)(x+1)$ for

$\{x: x < \sqrt{2}\} \cup \{x: 0 < x - \sqrt{2}\}$

## EXERCISE 4.5 Equations of cubic polynomials

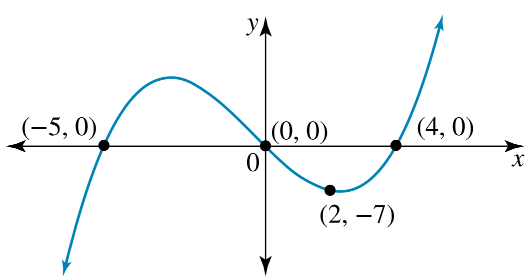
**PRACTISE**

Work without CAS

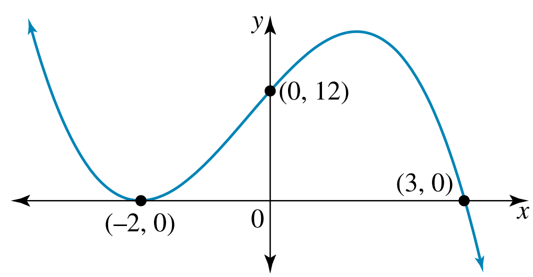
**1** WE15 Determine the equation of each of the following graphs.

**a** The graph of a cubic polynomial which has a stationary point of inflection at the point $(3, -7)$ and an $x$-intercept at $(10, 0)$.

**b**




**c**



2 Use simultaneous equations to determine the equation of the cubic graph containing the points $(0, 3)$, $(1, 4)$, $(-1, 8)$, $(-2, 7)$.

3 **WE16** Solve the inequations.

a $(x-2)^2(6-x) > 0$ b $\{x : 4x \le x^3\}$ c $2(x+4)^3 - 16 < 0$

4 Calculate $\{x : 3x^3 + 7 > 7x^2 + 3x\}$.

5 **WE17** Sketch the graphs of $y = (x+2)(x-1)^2$ and $y = -3x$ and calculate the coordinates of the points of intersection. Hence state the values of $x$ for which $-3x < (x+2)(x-1)^2$.

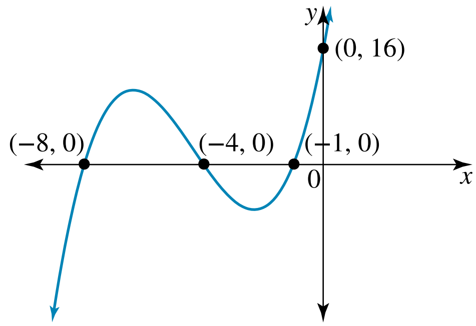
6 Calculate the coordinates of the points of intersection of $y = 4 - x^2$ and $y = 4x - x^3$ and then sketch the graphs on the same axes.

## CONSOLIDATE

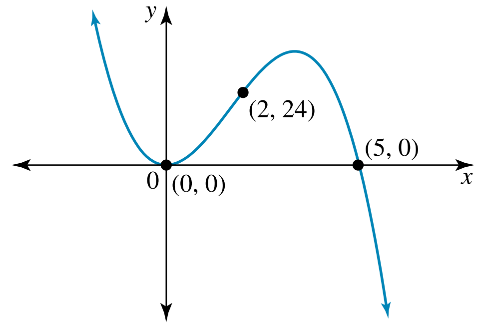
Apply the most appropriate mathematical processes and tools

7 Determine the equation for each of the following graphs of cubic polynomials.

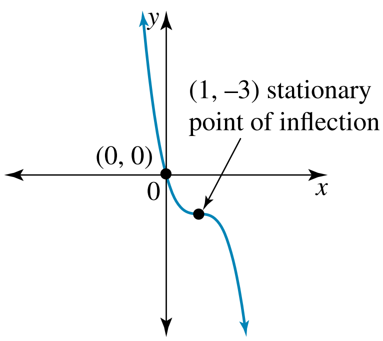
a



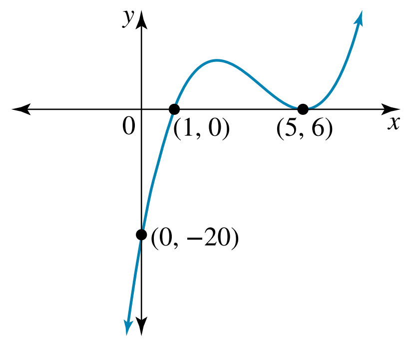

b




c




d



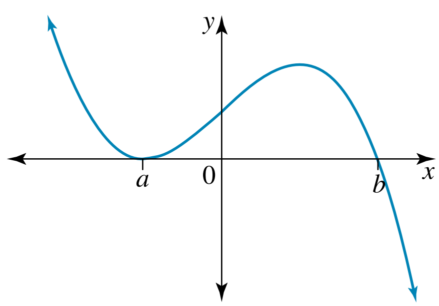

8 a Give the equation of the graph which has the same shape as $y = -2x^3$ and a point of inflection at $(-6, -7)$.

b Calculate the $y$-intercept of the graph which is created by translating the graph of $y = x^3$ two units to the right and four units down.

c A cubic graph has a stationary point of inflection at $(-5, 2)$ and a $y$-intercept of $(0, -23)$. Calculate its exact $x$-intercept.

d A curve has the equation $y = ax^3 + b$ and contains the points $(1, 3)$ and $(-2, 39)$. Calculate the coordinates of its stationary point of inflection.

9 The graph of $y = P(x)$ shown is the reflection of a monic cubic polynomial in the $x$-axis. The graph touches the $x$-axis at $x = a$, $a < 0$ and cuts it at $x = b$, $b > 0$.




a Form an expression for the equation of the graph.

b Use the graph to find $\{x : P(x) \ge 0\}$.

c How far horizontally to the left would the graph need to be moved so that both of its $x$-intercepts are negative?

d How far horizontally to the right would the graph need to be moved so that both of its $x$-intercepts are positive?

**10** A graph of a cubic polynomial with equation $y = x^3 + ax^2 + bx + 9$ has a turning point at $(3, 0)$.

**a** State the factor of the equation with greatest multiplicity.
**b** Determine the other $x$-intercept.
**c** Calculate the values of $a$ and $b$.

**11** Solve the cubic inequations.

**a** $(x - 2)(x + 1)(x + 9) \geq 0$
**b** $x^2 - 5x^3 < 0$
**c** $8(x - 2)^3 - 1 > 0$
**d** $x^3 + x \leq 2x^2$
**e** $5x^3 + 6x^2 - 20x - 24 < 0$
**f** $2(x + 1) - 8(x + 1)^3 < 0$

**12** Find the coordinates of the points of intersection of the following.

**a** $y = 2x^3$ and $y = x^2$
**b** $y = 2x^3$ and $y = x - 1$
**c** Illustrate the answers to parts **a** and **b** with a graph.
**d** Solve the inequation $2x^3 - x^2 \leq 0$ algebraically and explain how you could use your graph from part **c** to solve this inequation.

**13** **a** The number of solutions to the equation $x^3 + 2x - 5 = 0$ can be found by determining the number of intersections of the graphs of $y = x^3$ and a straight line. What is the equation of this line and how many solutions does $x^3 + 2x - 5 = 0$ have?
**b** Use a graph of a cubic and a linear polynomial to determine the number of solutions to the equation $x^3 + 3x^2 - 4x = 0$.
**c** Use a graph of a cubic and a quadratic polynomial to determine the number of solutions to the equation $x^3 + 3x^2 - 4x = 0$.
**d** Solve the equation $x^3 + 3x^2 - 4x = 0$.

**14** The graph of a polynomial of degree 3 cuts the $x$-axis at $x = 1$ and at $x = 2$. It cuts the $y$-axis at $y = 12$.

**a** Explain why this is insufficient information to completely determine the equation of the polynomial.
**b** Show that this information identifies a family of cubic polynomials with equation $y = ax^3 + (6 - 3a)x^2 + (2a - 18)x + 12$.
**c** On the same graph, sketch the two curves in the family for which $a = 1$ and $a = -1$.
**d** Determine the equation of the curve for which the coefficient of $x^2$ is 15. Specify the $x$-intercepts and sketch this curve.

**15** The graph with equation $y = (x + a)^3 + b$ passes through the three points $(0, 0), (1, 7), (2, 26)$.

**a** Use this information to determine the values of $a$ and $b$.
**b** Find the points of intersection of the graph with the line $y = x$.
**c** Sketch both graphs in part **b** on the same axes.
**d** Hence, with the aid of the graphs, find $\{x : x^3 + 3x^2 + 2x > 0\}$.

**16** **a** Show the line $y = 3x + 2$ is a tangent to the curve $y = x^3$ at the point $(-1, -1)$.
**b** What are the coordinates of the point where the line cuts the curve?
**c** Sketch the curve and its tangent on the same axes.
**d** Investigate for what values of $m$ will the line $y = mx + 2$ have one, two or three intersections with the curve $y = x^3$.

MASTER

17 Give the equation of the cubic graph containing the points $(-2, 53)$, $(-1, -6)$, $(2, 33)$, $(4, -121)$.

18 a Write down, to 2 decimal places, the coordinates of the points of intersection of $y = (x + 1)^3$ and $y = 4x + 3$.

b Form the cubic equation $ax^3 + bx^2 + cx + d = 0$ for which the $x$-coordinates of the points of intersection obtained in part a are the solution.

c What feature of the graph of $y = ax^3 + bx^2 + cx + d$ would the $x$-coordinates of these points of intersection be?

**Omar Khayyam (1050–1123) is not only a brilliant poet, but also an extraordinary mathematician, and is noted for linking algebra with geometry by solving cubic equations as the intersection of two curves.**

# 4.6 Cubic models and applications

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Units 1 & 2
AOS 1
Topic 3
Concept 5

**Cubic models and applications**
Concept summary
Practice questions

Practical situations which use cubic polynomials as models are likely to require a restriction of the possible values the variable may take. This is called a **domain** restriction. The domain is the set of possible values of the variable that the polynomial may take. We shall look more closely at domains in later chapters.

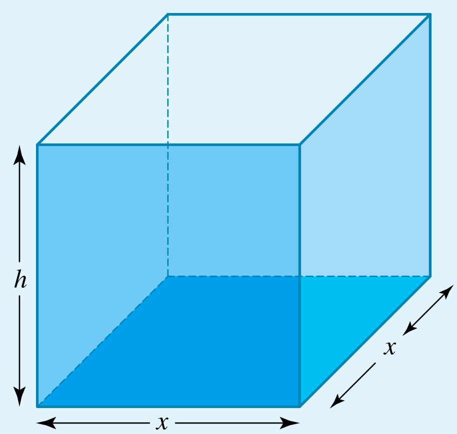
The polynomial model should be expressed in terms of one variable.

Applications of cubic models where a maximum or minimum value of the model is sought will require identification of turning point coordinates. In a later chapter we will see how this is done. For now, obtaining turning points may require the use of graphing or CAS technology.

WORKED EXAMPLE 18

**A rectangular storage container is designed to have an open top and a square base.**

**The base has side length $x$ cm and the height of the container is $h$ cm. The sum of its dimensions (the sum of the length, width and height) is 48 cm.**




a **Express $h$ in terms of $x$.**

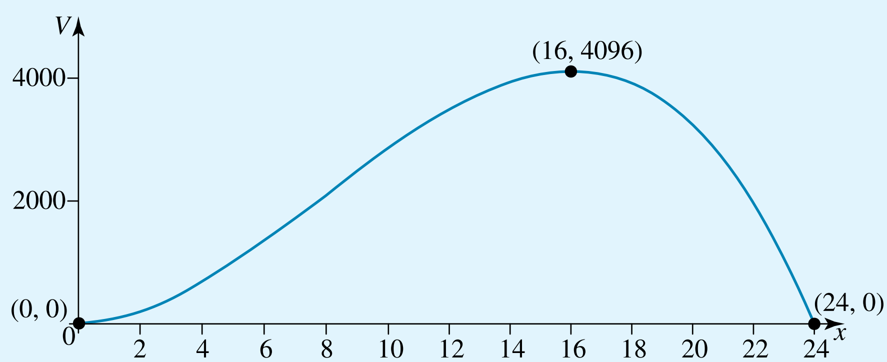
b **Show that the volume $V$ cm$^3$ of the container is given by $V = 48x^2 - 2x^3$.**

c **State any restrictions on the values $x$ can take.**

**d** **Sketch the graph of $V$ against $x$ for appropriate values of $x$, given its maximum turning point has coordinates (16, 4096).**

**e** **Calculate the dimensions of the container with the greatest possible volume.**

| THINK | WRITE |
|---|---|
| **a** Write the given information as an equation connecting the two variables. | **a** Sum of dimensions is 48 cm.<br>$x + x + h = 48$<br>$h = 48 - 2x$ |
| **b** Use the result from part **a** to express the volume in terms of one variable and prove the required statement. | **b** The formula for volume of a cuboid is<br>$V = lwh$<br>$\therefore V = x^2 h$<br>Substitute $h = 48 - 2x$.<br>$V = x^2(48 - 2x)$<br>$\therefore V = 48x^2 - 2x^3$, as required |
| **c** State the restrictions.<br>*Note:* It could be argued that the restriction is $0 < x < 24$ because when $x = 0$ or $x = 48$ there is no storage container, but we are adopting the closed convention. | **c** Length cannot be negative, so $x \geq 0$.<br>Height cannot be negative, so $h \geq 0$.<br>$48 - 2x \geq 0$<br>$-2x \geq -48$<br>$\therefore x \leq 24$<br>Hence the restriction is $0 \leq x \leq 24$. |
| **d** Draw the cubic graph but only show the section of the graph for which the restriction applies. Label the axes with the appropriate symbols and label the given turning point. | **d** $V = 48x^2 - 2x^3$<br>$= 2x^2(24 - x)$<br>$x$-intercepts: let $V = 0$<br>$x^2 = 0$ or $24 - x = 0$<br>$\therefore x = 0$ (touch), $x = 24$ (cut)<br>(0, 0), (24, 0) are the $x$-intercepts.<br>This is a negative cubic.<br>Maximum turning point (16, 4096)<br>Draw the section for which $0 \leq x \leq 24$. |

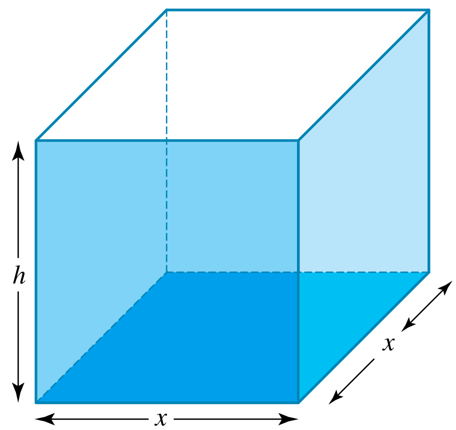


| | |
|---|---|
| **e 1** Calculate the required dimensions. *Note:* The maximum turning point $(x, V)$ gives the maximum value of $V$ and the value of $x$ when this maximum occurs. | **e** The maximum turning point is $(16, 4096)$. This means the greatest volume is 4096 cm$^3$. It occurs when $x = 16$. $\therefore h = 48 - 2(16) \Rightarrow h = 16$ Dimensions: length = 16 cm, width = 16 cm, height = 16 cm |
| **2** State the answer. | The container has the greatest volume when it is a cube of edge 16 cm. |

## EXERCISE 4.6 Cubic models and applications

**PRACTISE**

**1** WE18 A rectangular storage container is designed to have an open top and a square base.




The base has side length $x$ metres and the height of the container is $h$ metres. The total length of its 12 edges is 6 metres.

**a** Express $h$ in terms of $x$.

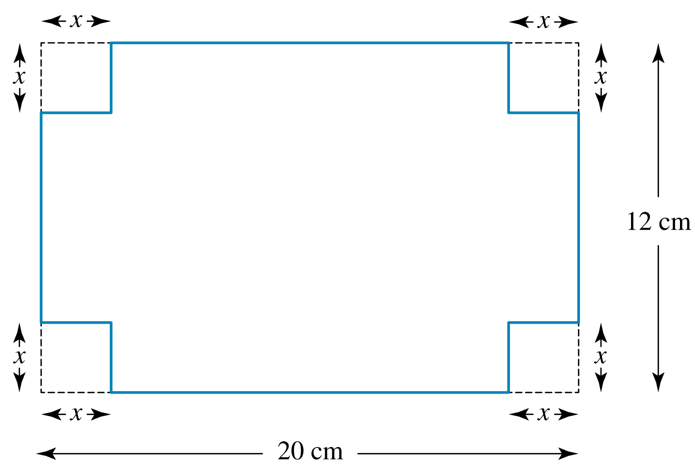
**b** Show that the volume $V$ m$^3$ of the container is given by $V = 1.5x^2 - 2x^3$.

**c** State any restrictions on the values $x$ can take.

**d** Sketch the graph of $V$ against $x$ for appropriate values of $x$, given its maximum turning point has coordinates $(0.5, 0.125)$.

**e** Calculate the dimensions of the container with the greatest possible volume.

**2** A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 20 cm by 12 cm by cutting equal squares of side length $x$ cm out of the four corners and folding the flaps up.




The box has length $l$ cm, width $w$ cm and volume $V$ cm$^3$.

**a** Express $l$ and $w$ in terms of $x$ and hence express $V$ in terms of $x$.

**b** State any restrictions on the values of $x$.

c Sketch the graph of $V$ against $x$ for appropriate values of $x$, given the unrestricted graph would have turning points at $x = 2.43$ and $x = 8.24$.

d Calculate the length and width of the box with maximum volume and give this maximum volume to the nearest whole number.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 The cost $C$ dollars for an artist to produce $x$ sculptures by contract is given by $C = x^3 + 100x + 2000$. Each sculpture is sold for \$500 and as the artist only makes the sculptures by order, every sculpture produced will be paid for. However, too few sales will result in a loss to the artist.

a Show the artist makes a loss if only 5 sculptures are produced and a profit if 6 sculptures are produced.

b Show that the profit, $P$ dollars, from the sale of $x$ sculptures is given by $P = -x^3 + 400x - 2000$.

c What will happen to the profit if a large number of sculptures are produced? Why does this effect occur?

d Calculate the profit (or loss) from the sale of:

i 16 sculptures ii 17 sculptures.

e Use the above information to sketch the graph of the profit $P$ for $0 \le x \le 20$. Place its intersection with the $x$-axis between two consecutive integers but don't attempt to obtain its actual $x$-intercepts.

f In order to guarantee a profit is made, how many sculptures should the artist produce?

4 The number of bacteria in a slow-growing culture at time $t$ hours after 9 am is given by $N = 54 + 23t + t^3$.

a What is the initial number of bacteria at 9 am?

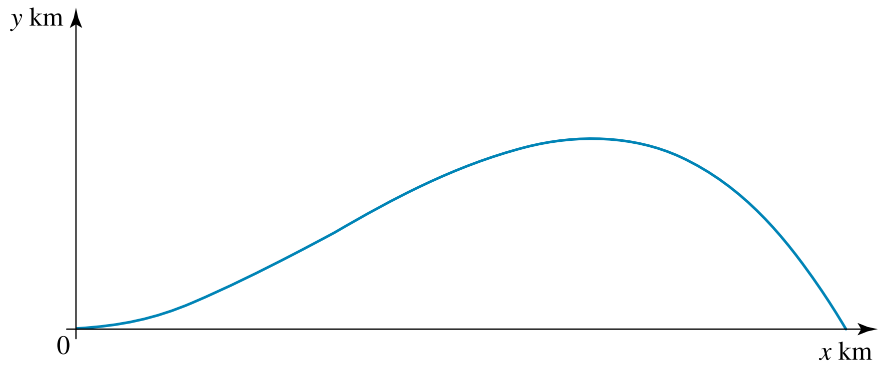
b How long does it take for the initial number of bacteria to double?

c How many bacteria are there by 1 pm?

d Once the number of bacteria reaches 750, the experiment is stopped. At what time of the day does this happen?

5 Engineers are planning to build an underground tunnel through a city to ease traffic congestion.

The cross-section of their plan is bounded by the curve shown.

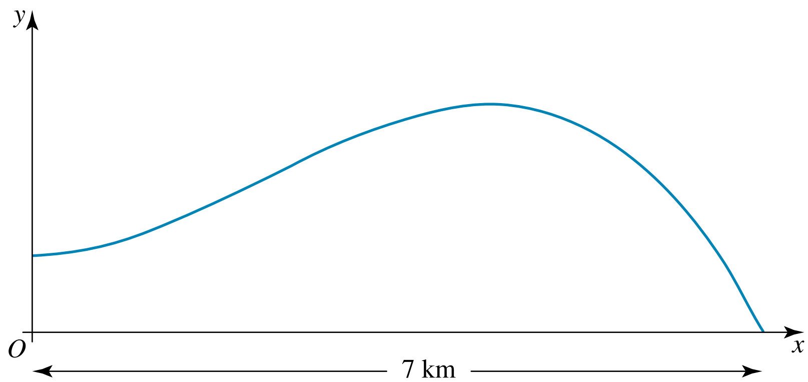


The equation of the bounding curve is $y = ax^2(x - b)$ and all measurements are in kilometres.

It is planned that the greatest breadth of the bounding curve will be 6 km and the greatest height will be 1 km above this level at a point 4 km from the origin.

a Determine the equation of the bounding curve.

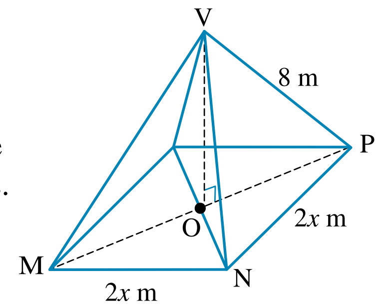
b If the greatest breadth of the curve was extended to 7 km, what would be the greatest height of the curve above this new lowest level?




6 Find the smallest positive integer and the largest negative integer for which the difference between the square of 5 more than this number and the cube of 1 more than the number exceeds 22.

7 A tent used by a group of bushwalkers is in the shape of a square-based right pyramid with a slant height of 8 metres.

For the figure shown, let OV, the height of the tent, be $h$ metres and the edge of the square base be $2x$ metres.




a Use Pythagoras' theorem to express the length of the diagonal of the square base of the tent in terms of $x$.

b Use Pythagoras' theorem to show $2x^2 = 64 - h^2$.

c The volume $V$ of a pyramid is found using the formula $V = \frac{1}{3}Ah$ where $A$ is the area of the base of the pyramid. Use this formula to show that the volume of space contained within the bushwalkers' tent is given by $V = \frac{1}{3}(128h - 2h^3)$.

d i If the height of the tent is 3 metres, what is the volume?

ii What values for the height does this mathematical model allow?

e Sketch the graph of $V = \dfrac{1}{3}(128h - 2h^3)$ for appropriate values of $h$ and estimate the height for which the volume is greatest.

f The greatest volume is found to occur when the height is half the length of the base. Use this information to calculate the height which gives the greatest volume and compare this value with your estimate from your graph in part **e**.

8 A cylindrical storage container is designed so that it is open at the top and has a surface area of $400\pi$ cm$^2$. Its height is $h$ cm and its radius is $r$ cm.

a Show that $h = \dfrac{400 - r^2}{2r}$.

b Show that the volume $V$ cm$^3$ the container can hold is given by $V = 200\pi r - \frac{1}{2}\pi r^3$.

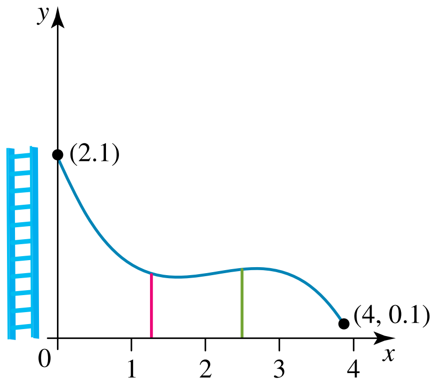
c State any restrictions on the values $r$ can take.

d Sketch the graph of $V$ against $r$ for appropriate values of $r$.

e Find the radius and height of the container if the volume is $396\pi$ cm$^3$.

f State the maximum possible volume to the nearest cm$^3$ if the maximum turning point on the graph of $y = 200\pi x - \frac{1}{2}\pi x^3$ has an $x$-coordinate of $\frac{20}{\sqrt{3}}$.

9 A new playground slide for children is to be constructed at a local park. At the foot of the slide the children climb a vertical ladder to reach the start of the slide. The slide must start at a height of 2.1 metres above the ground and end at a point 0.1 metres above the ground and 4 metres horizontally from its foot. A model for the slide is $h = ax^3 + bx^2 + cx + d$ where $h$ metres is the height of the slide above ground level at a horizontal distance of $x$ metres from its foot. The foot is at the origin.




The ladder supports the slide at one end and the slide also requires two vertical struts as support. One strut of length 1 metre is placed at a point 1.25 metres horizontally from the foot of the slide and the other is placed at a point 1.5 metres horizontally from the end of the slide and is of length 1.1 metres.

a Give the coordinates of 4 points which lie on the cubic graph of the slide.

b State the value of $d$ in the equation of the slide.

c Form a system of 3 simultaneous equations, the solutions to which give the coefficients $a, b, c$ in the equation of the slide.

d The equation of the slide can be shown to be $y = -0.164x^3 + x^2 - 1.872x + 2.1$. Use this equation to calculate the length of a third strut thought necessary at $x = 3.5$. Give your answer to 2 decimal places.

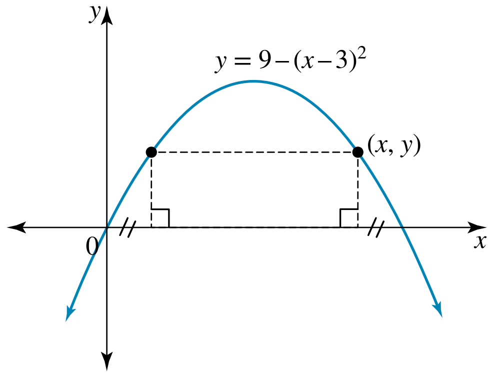
10 Since 1988, the world record times for the men's 100-m sprint can be roughly approximated by the cubic model $T(t) = -0.00005(t - 6)^3 + 9.85$ where $T$ is the time in seconds and $t$ is the number of years since 1988.

a In 1991 the world record was 9.86 seconds and in 2008 the record was 9.72 seconds. Compare these times with those predicted by the cubic model.

**b** Sketch the graph of $T$ versus $t$ from 1988 to 2008.

**c** What does the model predict for 2016? Is the model likely to be a good predictor beyond 2016?

**11** A rectangle is inscribed under the parabola $y = 9 - (x - 3)^2$ so that two of its corners lie on the parabola and the other two lie on the $x$-axis at equal distances from the intercepts the parabola makes with the $x$-axis.




**a** Calculate the $x$-intercepts of the parabola.

**b** Express the length and width of the rectangle in terms of $x$.

**c** Hence show that the area of the rectangle is given by $A = -2x^3 + 18x^2 - 36x$.

**d** For what values of $x$ is this a valid model of the area?

**e** Calculate the value(s) of $x$ for which $A = 16$.

**12** A pathway through the countryside passes through 5 scenic points. Relative to a fixed origin, these points have coordinates A$(-3, 0)$, B$(-\sqrt{3}, -12\sqrt{3})$, C$(\sqrt{3}, 12\sqrt{3})$, D$(3, 0)$; the fifth scenic point is the origin, O$(0, 0)$. The two-dimensional shape of the path is a cubic polynomial.

**a** State the maximum number of turning points and $x$-intercepts that a cubic graph can have.

**b** Determine the equation of the pathway through the 5 scenic points.

**c** Sketch the path, given that points B and C are turning points of the cubic polynomial graph.

**d** It is proposed that another pathway be created to link B and C by a direct route. Show that if a straight-line path connecting B and C is created, it will pass through O and give the equation of this line.

**e** An alternative plan is to link B and C by a cubic path which has a stationary point of inflection at O. Determine the equation of this path.

**MASTER**

Use CAS technology to answer the following questions.

**13 a** Consider question 7e.

Use the calculator to sketch the graph of $V = \frac{1}{3}(128h - 2h^3)$ and state the height for which the volume is greatest, correct to 2 decimal places.

**b** Consider question 7f.

Show that the greatest volume occurs when the height is half the length of the base.

**c** Consider question 8f.

Use technology to obtain the maximum volume and calculate the corresponding dimensions of the cylindrical storage container, expressed to 1 decimal place.

**14 a** Consider question 9c.

Use technology to obtain the equation of the slide.

**b** Consider question 11.

Calculate, to 3 decimal places, the length and width of the rectangle which has the greatest area.

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# 4.7 Review

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**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

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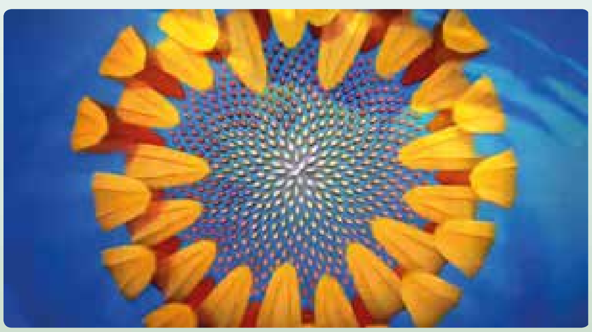

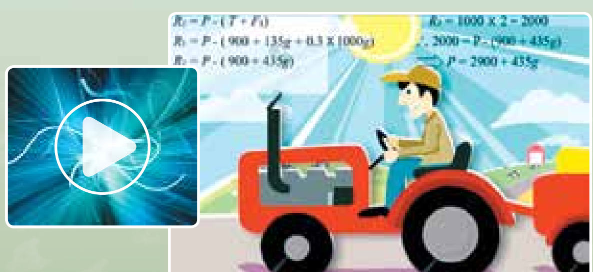


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Units 1 & 2 | **Cubic polynomials**

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# 4 Answers

## EXERCISE 4.2

1 A: Degree 5; leading coefficient 4; constant term 12; coefficients $\in Z$

C: Degree 2; leading coefficient −0.2; constant term 5.6; coefficients $\in Q$

2 $y^7 + 2y^5 - \sqrt{2}y^2 + 4$ (answers may vary)

3 a 15 b 10

4 21

5 $a = 2; b = -13; c = 6$

6 $(x+2)^3 = x^2(x+1) + 5x(x+2) + 2(x+3) + 2$

7 $p = 5; q = -4$

8 $x^7 + 4x^5 - 6x^4 + 5x^3 - 4x^2 - 5x + 2$; degree 7

9 a $\frac{x-12}{x+3} = 1 - \frac{15}{x+3}$; quotient is 1; remainder is −15.

b $\frac{4x+7}{2x+1} = 2 + \frac{5}{2x+1}$

10 $a = 3,\ b = -12,\ c = 29$;
$\frac{3x^2 - 6x + 5}{x+2} = 3x - 12 + \frac{29}{x+2}$;
$P(x) = 3x - 12,\ d = 29$

11 a $\frac{2x^3 - 5x^2 + 8x + 6}{x-2} = 2x^2 - x + 6 + \frac{18}{x-2}$; quotient is $2x^2 - x + 6$; remainder is 18.

b $\frac{x^3+10}{1-2x} = -\frac{1}{2}x^2 - \frac{1}{4}x - \frac{1}{8} + \frac{81}{8(1-2x)}$;

remainder is $\frac{81}{8}$.

12 Quotient is $x^2 - x + 3$; remainder is 0.

13 a A, B, D, F are polynomials.

| | Degree | Type of coefficient | Leading term | Constant term |
|---|---|---|---|---|
| **A** | 5 | Q | $3x^5$ | 12 |
| **B** | 4 | R | $-5x^4$ | 9 |
| **D** | 4 | Z | $-18x^4$ | 0 |
| **F** | 6 | N | $49x^6$ | 9 |

b C is not a polynomial due to $\sqrt{4x^5} = 2x^{\frac{5}{2}}$ term.

E is not a polynomial due to $\frac{5}{3x^2} = \frac{5}{3}x^{-2}$ term.

14 a 78 b −12
c 0 d −6
e −6 f −5.868

15 a $-14a$
b $h^2 - 5h - 4$
c $2xh + h^2 - 7h$

16 a $k = 25$ b $a = -9$
c $n = -\frac{1}{5}$ d $b = 4; c = 5$

17 a $a = 3; b = -2; c = -8$
b $m = -3; n = -10; p = 24$
c $a = 1; b = 7; c = -41; x^2 - 14x + 8 = (x-7)^2 - 41$
d $4x^3 + 2x^2 - 7x + 1$
$= 4x^2(x+1) - 2x(x+1) - 5(x+1) + 6$

18 a i $3x^4 + 7x + 12$
ii $6x^4 + 10x^2 - 21x - 31$
iii $6x^6 - 21x^5 - 29x^4 - 14x^3 - 20x^2 - 7x - 11$
b i $m$ ii $m$
iii $m + n$

19 a $a = -3; b = -1; p = 4; q = -2$
b i $4x^4 + 12x^3 + 13x^2 + 6x + 1 = (2x^2 + 3x + 1)^2$
ii $2x^2 + 3x + 1$

20 a $k = -13; m = -5; n = 6$
b $(x-3)^2(x-2)^2(x+3)(x+2)$

21

| | Quotient | Remainder |
|---|---|---|
| a | 1 | 9 |
| b | 4 | 1 |
| c | $x + 7$ | −10 |
| d | $2x - 12$ | 27 |
| e | $x^2 + 5x + 12$ | 41 |
| f | $x^2 - 7x + 2$ | 0 |

22 a $4x^2 + x - 3 + \frac{18}{1+2x}$
b $2x^2 + 3x + 5 + \frac{20}{2x-3}$
c $x^2 + 6 - \frac{48}{x+6}$
d $x^2 - x + 1 + \frac{1}{x+1}$
e $x^2 + x + \frac{3x+5}{x^2-1}$
f $-2x - 2 - \frac{7x+12}{(x+2)(x-3)}$

23 a $2x^2 - \frac{13}{2}x - \frac{139}{4(2x+3)} + \frac{49}{4}$
b Remainder is $-\frac{139}{4}$; quotient is $2x^2 - \frac{13}{2}x + \frac{49}{4}$.
c Dividend equals $-\frac{139}{4}$.
d Divisor equals 0.

**24 a** Define using CAS technology.

**b** $\frac{349}{9}$

**c** $24n^2 + 24an - 16n - 154$

**d** 9

## EXERCISE 4.3

**1 a** The remainder is 19.

**b** The remainder is $\frac{37}{8}$.

**2** $k = 2$

**3 a** Proof required to show $Q(2) = 0$

**b** $a = -9$; $b = 8$; $P(x) = 3x^3 - 9x^2 + 8x - 2$

**4** $a = 0$; $a = -1$

**5 a** $P(-1) = 0 \Rightarrow (x + 1)$ is a factor; $P(x) = (x + 1)(x + 5)(x - 3)$

**b** $P(x) = (x + 1)(3x + 2)(4x + 7)$

**6** Linear factors are $(2x - 1)$, $(2x + 1)$ and $(3x + 2)$.

**7** $x = -2, \frac{1}{3}, -\frac{1}{2}$

**8** $x = 0, x = -\frac{3}{2}, x = 2, x = -2$

**9 a** $-5$ **b** 2

**c** $-101$ **d** $-10\frac{3}{8}$

**e** 0 **f** 26

**10 a** $-10$ **b** 8

**c** $-19$ **d** $-19$

**11 a** $a = 2$; $b = 3$ **b** $m = -\frac{2}{3}$; $n = -3$

**12 a i** $(x - 5)(x - 9)(x + 2)$

**ii** $x^3 - 12x^2 + 17x + 90$

**b i** $(x + 4)(x + 1)(1 - 2x)$

**ii** $-2x^3 - 9x^2 - 3x + 4$

**13 a** $(x - 4)(x + 1)(x + 2)$

**b** $(x + 12)(3x + 1)(x + 1)$

**c** $(5x + 1)(2x + 3)(2x + 1)$

**d** $(4x - 3)(5 - 2x)(5 + 2x)$

**e** $-(x - 3)^2(8x - 11)$

**f** $(3x - 5)^2(x - 5)$

**14 a** $(x - 1)(x + 2)(x + 4)$

**b** $(x + 2)(x + 3)(x + 5)$

**c** $(x - 2)(2x + 1)(x - 5)$

**d** $(x + 1)(3x - 4)(1 - 6x)$

**e** $(x - 1)(x + 3)(x - 2)$

**f** $(x + 1)(x - 7)(x + 7)$

**15 a** $(2x + 1)(3x - 1)(4x + 5)$

**b i** $(2x - 5)$

**ii** $(2x - 5)(4x + 1)(x - 1)$

**iii** $m = -26$

**c i** $P(x) = (x - 4)^3$; $Q(x) = (x - 4)(x^2 + 4x + 16)$

**ii** Proof required — check with your teacher

**d** Third factor is $(x - 3)$; $b = c = -3$; $d = 9$

**16 a** $-4, 3, -5$ **b** $7, -\frac{5}{3}, 9$

**c** $-1, 6, 8$ **d** $1, -\frac{3}{2}, -3$

**e** $1, -\frac{2}{3}$ **f** $0, 1, -\frac{3}{4}, -20$

**17 a** Proof required; $(x - 2)(x + 4 - \sqrt{7})(x + 4 + \sqrt{7})$

**b** Proof required — check with your teacher

**c** $(2x - 1)(x - 5)$; equation has roots $x = \frac{1}{2}$, $x = 5$

**18 a** $(x - 2)(x + 2)(5x + 9)$; $k = 9$

**b** $a = 1$; $a = 2$

**c** $a = -3$; $b = 1$; $P(x) = (x - 3)(x^2 + 1)$, $Q(x) = (x - 3)(x + 3)(x + 1)$

**d** $x^3 + 7x^2 + 15x + 9 : x = -3,\ x = -1$; $x^3 - 9x^2 + 15x + 25 : x = 5,\ x = -1$

**19** $-12\sqrt{2} + 33$

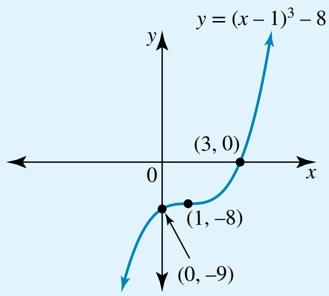
**20** $-0.4696$

## EXERCISE 4.4

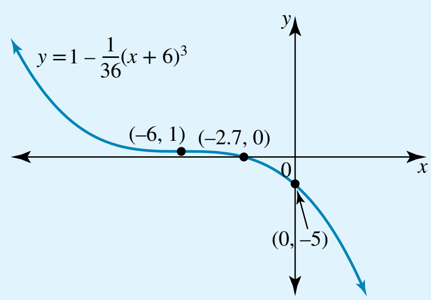
**1**

| | Inflection point | y-intercept | x-intercept |
|---|---|---|---|
| **a** | (1, −8) | (0, −9) | (3, 0) |
| **b** | (−6, 1) | (0, −5) | (−2.7, 0) approx. |

**a**



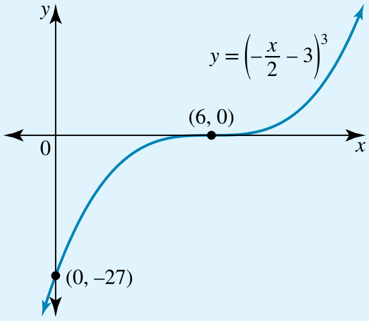

**b**



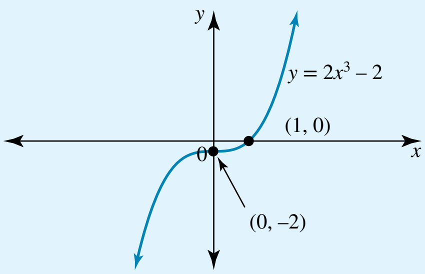
**2**

| | Inflection point | $y$-intercept | $x$-intercept |
|---|---|---|---|
| **a** | (6, 0) | (0, −27) | (6, 0) |
| **b** | (0, −2) | (0, −2) | (1, 0) |

**a**



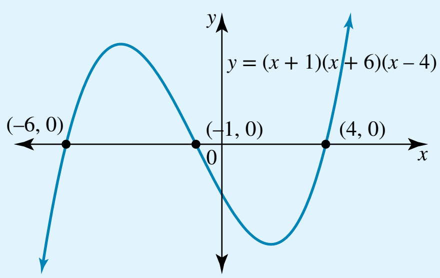

**b**



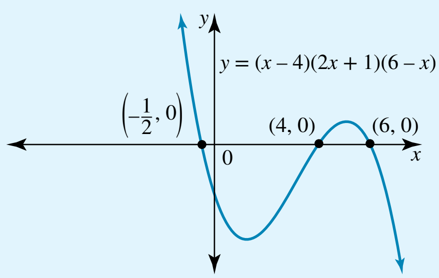

**3**

| | $y$-intercept | $x$-intercepts |
|---|---|---|
| **a** | (0, −24) | (−6, 0), (−1, 0), (4, 0) |
| **b** | (0, −24) | $\left(-\frac{1}{2}, 0\right)$, (2, 0), (4, 0) |

**a**

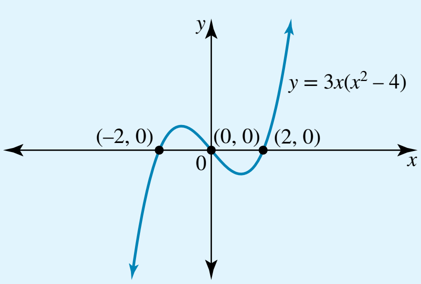



**b**




**4**

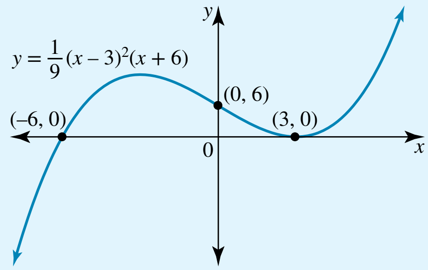
| $y$-intercept | $x$-intercepts |
|---|---|
| (0, 0) | (−2, 0), (0, 0), (2, 0) |



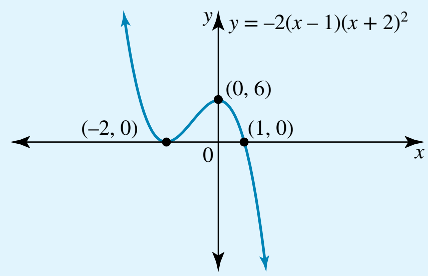

**5**

| | $y$-intercept | $x$-intercepts |
|---|---|---|
| **a** | (0, 6) | (−6, 0) and (3, 0) which is a turning point |
| **b** | (0, 8) | (−2, 0) is a turning point and (1, 0) |

**a**

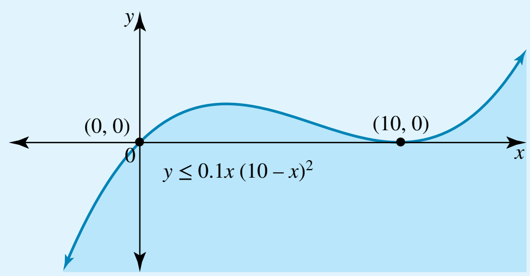



**b**



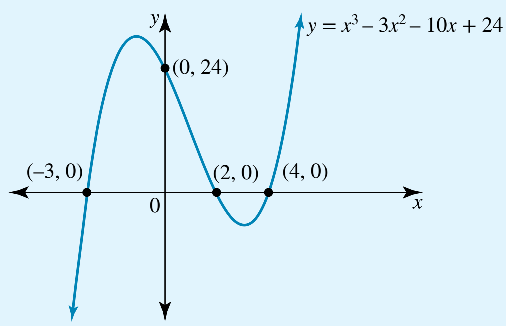

**6**

| $y$-intercept | $x$-intercept |
|---|---|
| (0, 0) | (0, 0) and (10, 0) is a turning point |



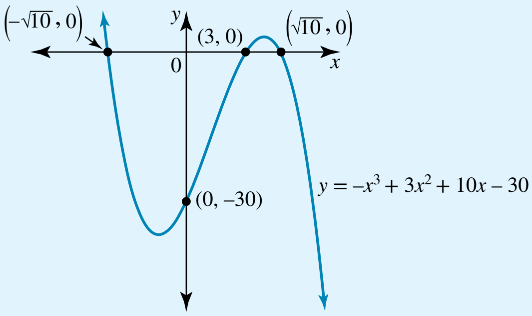

**7** $x^3 - 3x^2 - 10x + 24 = (x - 2)(x - 4)(x + 3)$

| $y$-intercept | $x$-intercepts |
|---|---|
| (0, 24) | (−3, 0), (2, 0), (4, 0) |

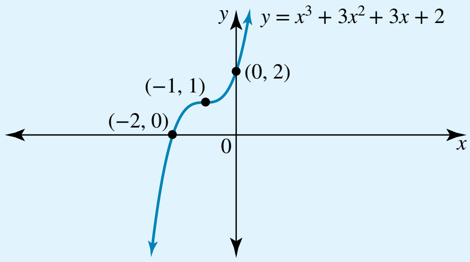



**8 a** $-x^3 + 3x^2 + 10x - 30 = -(x - 3)\left(x - \sqrt{10}\right)\left(x + \sqrt{10}\right)$

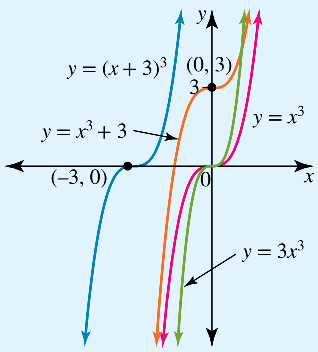
| $y$-intercept | $x$-intercepts |
|---|---|
| (0, −30) | $\left(-\sqrt{10}, 0\right)$, (3, 0), $\left(\sqrt{10}, 0\right)$ |



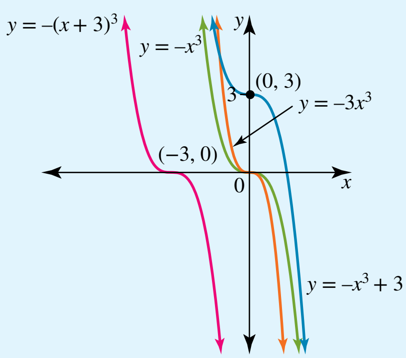

**b** Stationary point of inflection $(-1, 1)$; $y$-intercept $(0, 2)$; $x$-intercept $(-2, 0)$




**9 a**



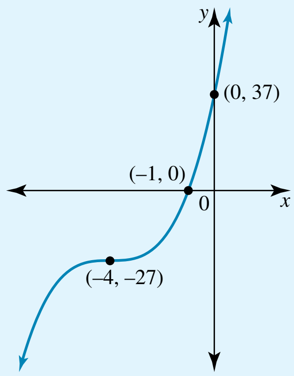

**b**



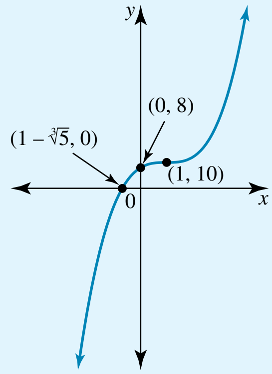

**10**

| | Inflection point | $y$-intercept | $x$-intercept |
|---|---|---|---|
| **a** | $(-4, -27)$ | $(0, 37)$ | $(-1, 0)$ |
| **b** | $(1, 10)$ | $(0, 8)$ | $(-0.7, 0)$ approx. |
| **c** | $(3, 27)$ | $(0, -27)$ | $(0.6, 0)$ approx. |
| **d** | $(-2, 16)$ | $(0, 0)$ | $(0, 0)$ |
| **e** | $\left(-\frac{4}{3}, 0\right)$ | $(0, -48)$ | $\left(-\frac{4}{3}, 0\right)$ |
| **f** | $(0, 9)$ | $(0, 9)$ | $(-3, 0)$ |

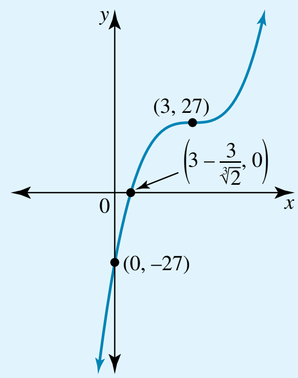
**a**



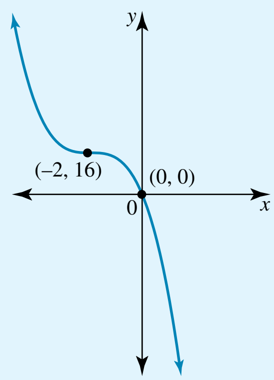

**b**



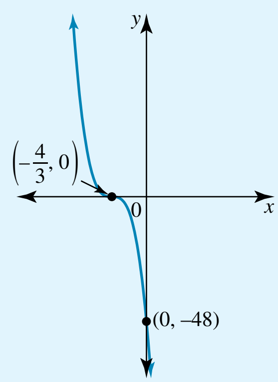

**c**



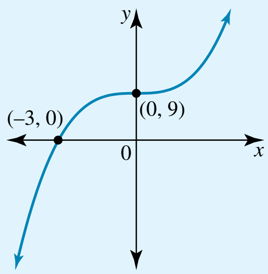

**d**




**e**



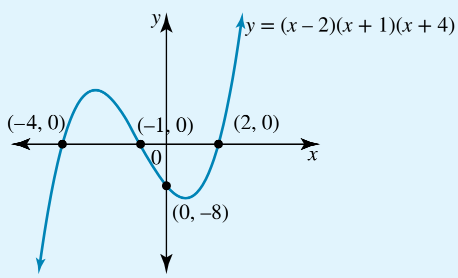
f



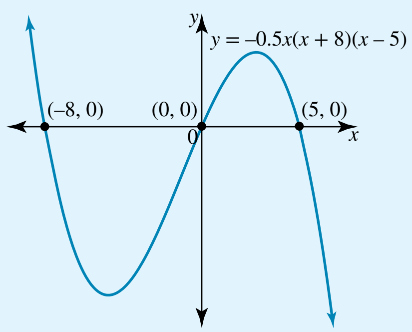

11

| | y-intercept | x-intercepts |
|---|---|---|
| a | (0, −8) | (−4, 0), (−1, 0), (2, 0) |
| b | (0, 0) | (−8, 0), (0, 0), (5, 0) |
| c | (0, −12) | (−3, 0), (1, 0), (4, 0) |
| d | (0, 12) | (−4, 0), (2, 0), (6, 0) |
| e | (0, 7) | $\left(-\frac{1}{4}, 0\right), \left(\frac{7}{2}, 0\right)$, (10, 0) |
| f | $\left(0, \frac{5}{2}\right)$ | $\left(-\frac{8}{3}, 0\right), \left(\frac{5}{8}, 0\right)$, (2, 0) |

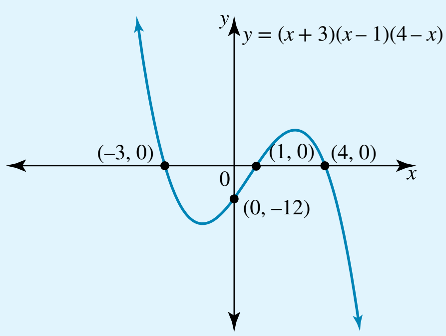
a



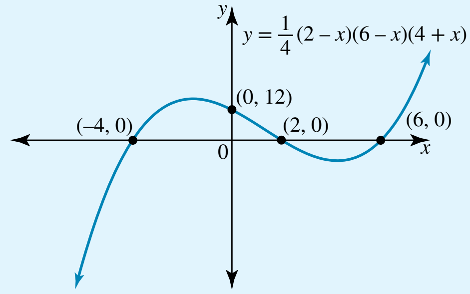

b



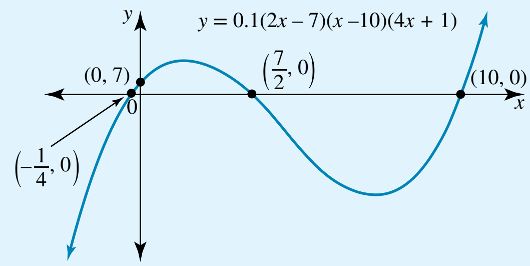

c



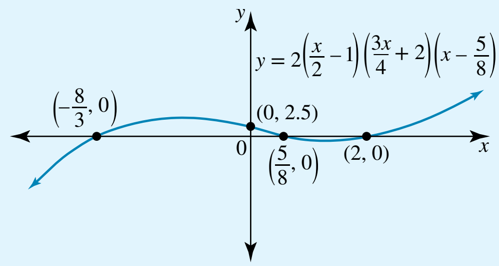

d




e



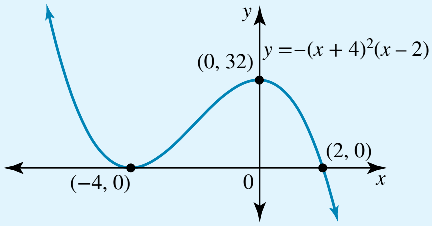

f



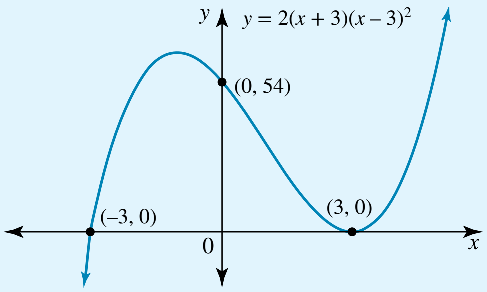

12

| | y-intercept | x-intercepts |
|---|---|---|
| a | (0, 32) | (−4, 0) is a turning point;<br>(2, 0) is a cut |
| b | (0, 54) | (3, 0) is a turning point;<br>(−3, 0) is a cut |
| c | (0, 36) | (−3, 0) is a turning point;<br>(4, 0) is a cut |
| d | (0, −12) | (2, 0) is a turning point;<br>(12, 0) is a cut |
| e | (0, 0) | $\left(-\frac{3}{2}, 0\right)$ is a turning point;<br>(0, 0) is a cut |
| f | (0, 0) | (0, 0) is a turning point;<br>(0.4, 0) is a cut |

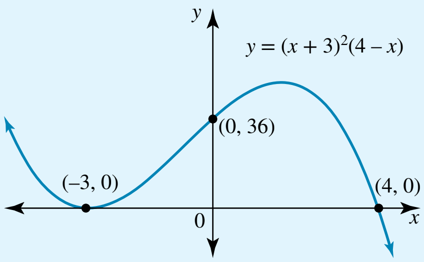
**a**



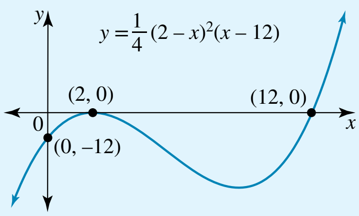

**b**



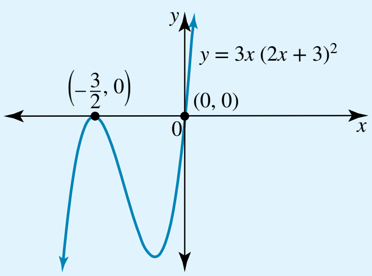

**c**



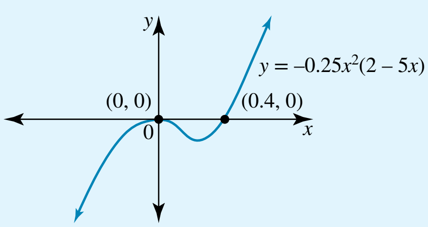

**d**




**e**



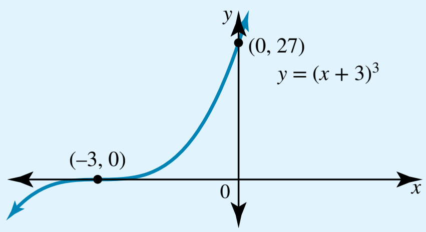

**f**



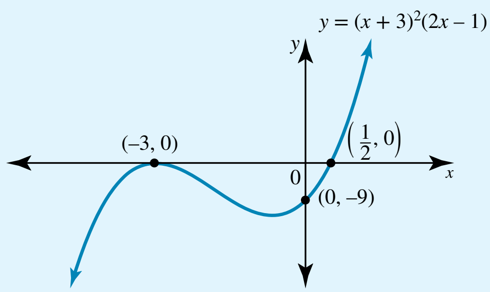

**13**

| | Stationary point of inflection | $y$-intercept | $x$-intercepts |
|---|---|---|---|
| **a** | $(-3, 0)$ | $(0, 27)$ | $(-3, 0)$ |
| **b** | none | $(0, -9)$ | $(-3, 0)$ is a turning point; $\left(\frac{1}{2}, 0\right)$ is a cut |
| **c** | none | $(0, -15)$ | $(-3, 0)$, $\left(\frac{1}{2}, 0\right)$, $(5, 0)$ |
| **d** | $\left(\frac{1}{2}, 1\right)$ | $\left(0, \frac{3}{2}\right)$ | $(1.1, 0)$ approx. |
| **e** | none | $(0, 0)$ | $\left(\frac{1}{4}, 0\right)$ is a turning point; $(0, 0)$ is a cut |
| **f** | none | $(0, 4)$ | $\left(\frac{2}{3}, 0\right)$ is a turning point; $\left(-\frac{2}{3}, 0\right)$ is a cut |

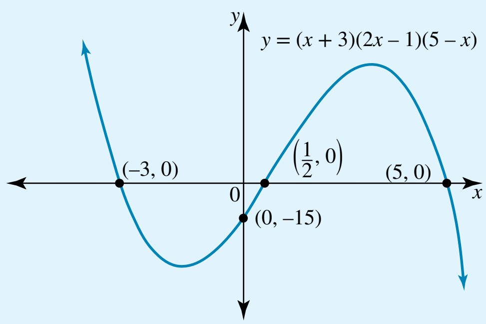
**a**



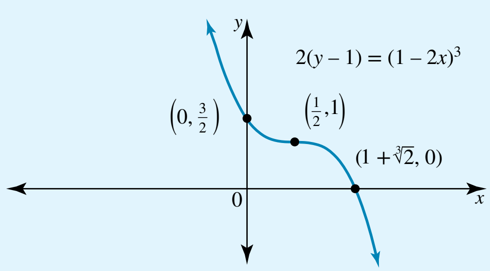

**b**



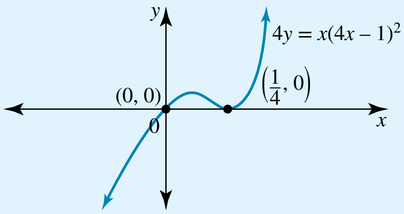

**c**



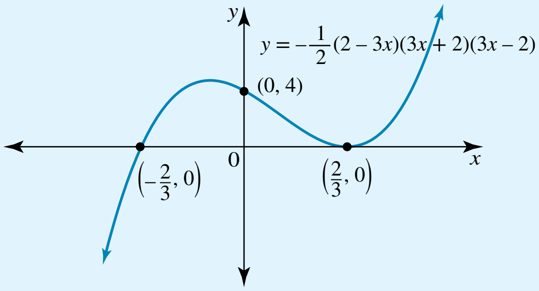

**d**



**e**

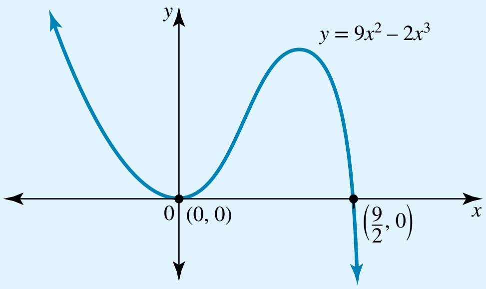



**f**

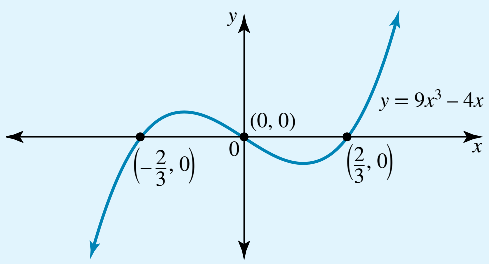



**14** See table at foot of page.*

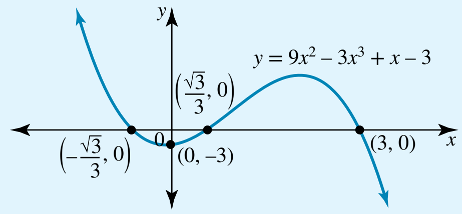
**a**



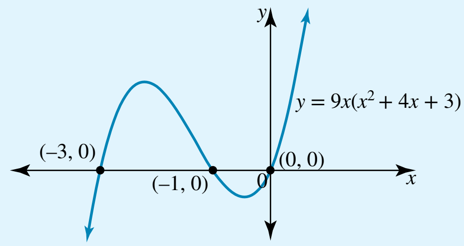

**b**



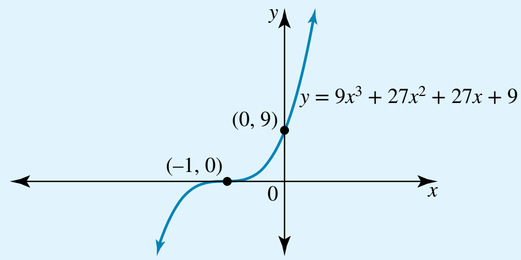

**c**



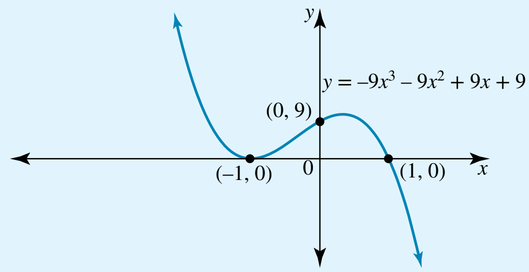

**d**




**e**




**f**

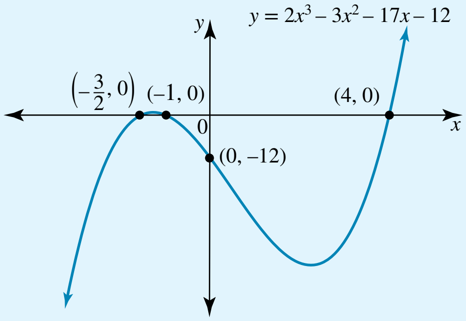



*14

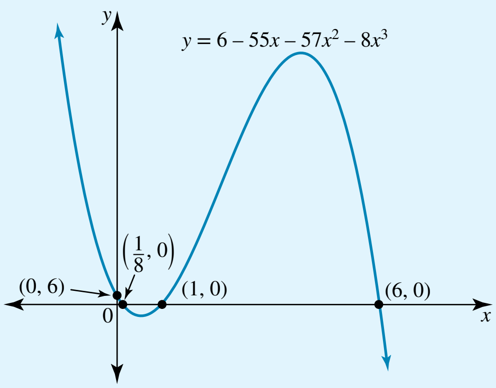
| | Factorised form | Stationary point of inflection | $y$-intercept | $x$-intercepts |
|---|---|---|---|---|
| a | $y = x^2(9 - 2x)$ | none | (0, 0) | (0, 0) is a turning point; $\left(\frac{9}{2}, 0\right)$ is a cut |
| b | $y = x(3x - 2)(3x + 2)$ | none | (0, 0) | $\left(-\frac{2}{3}, 0\right)$, (0, 0), $\left(\frac{2}{3}, 0\right)$ |
| c | $y = -3(x - 3)\left(x - \frac{\sqrt{3}}{3}\right)\left(x + \frac{\sqrt{3}}{3}\right)$ | none | (0, −3) | $\left(-\frac{\sqrt{3}}{3}, 0\right)$, $\left(\frac{\sqrt{3}}{3}, 0\right)$, (3, 0) |
| d | $y = 9x(x + 1)(x + 3)$ | none | (0, 0) | (−3, 0), (−1, 0), (0, 0) |
| e | $y = 9(x + 1)^3$ | (−1, 0) | (0, 9) | (−1, 0) |
| f | $y = -9(x + 1)^2(x - 1)$ | none | (0, 9) | (−1, 0) is a turning point; (1, 0) is a cut |

**15** See table at foot of page.*

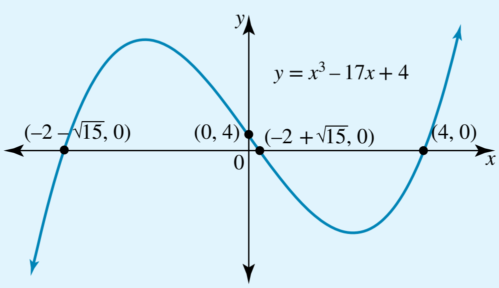
**a**



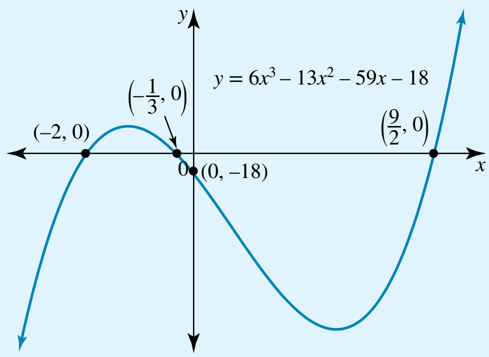

**b**



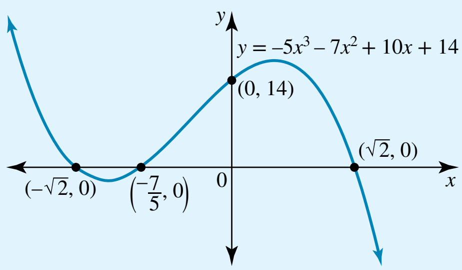

**c**



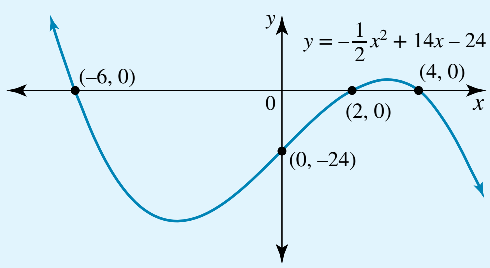

**d**




**e**




**f**




*15

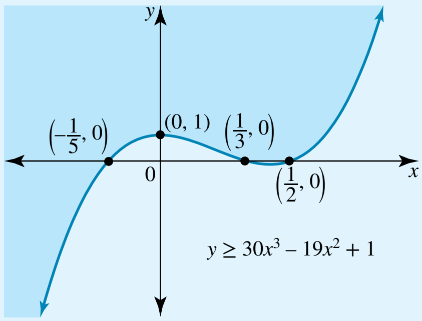
| | Factorised form | y-intercept | x-intercepts |
|---|---|---|---|
| a | $y = (x + 1)(2x + 3)(x - 4)$ | (0, −12) | $\left(-\frac{3}{2}, 0\right)$, (−1, 0), (4, 0) |
| b | $y = -(x - 1)(8x - 1)(x - 6)$ | (0, 6) | $\left(\frac{1}{8}, 0\right)$, (1, 0), (6, 0) |
| c | $y = (x - 4)(x + 2 - \sqrt{5})(x + 2 + \sqrt{5})$ | (0, 4) | $(-2 - \sqrt{5}, 0)$, $(-2 + \sqrt{5}, 0)$, (4, 0) |
| d | $y = (x + 2)(3x + 1)(2x - 9)$ | (0, −18) | (−2, 0), $\left(-\frac{1}{3}, 0\right)$, $\left(\frac{9}{2}, 0\right)$ |
| e | $y = (5x + 7)(\sqrt{2} - x)(\sqrt{2} + x)$ | (0, 14) | $(-\sqrt{2}, 0)$, $\left(-\frac{7}{2}, 0\right)$, $(\sqrt{2}, 0)$ |
| f | $y = -\frac{1}{2}(x - 2)(x + 6)(x - 4)$ | (0, −24) | (−6, 0), (2, 0), (4, 0) |

**16 a** $k = -19$

**b** $P(x) = (3x - 1)(5x + 1)(2x - 1)$

**c** $x = -\frac{1}{5}, \frac{1}{3}, \frac{1}{2}$

**d** $y$-intercept $(0, 1)$; $x$-intercepts $\left(-\frac{1}{5}, 0\right), \left(\frac{1}{3}, 0\right), \left(\frac{1}{2}, 0\right)$

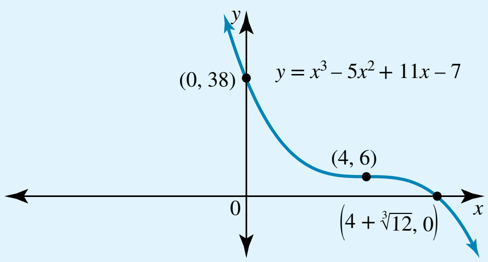



**e** No

**f** Shade the closed region above the graph of $y = P(x)$.

**17 a** $-\frac{1}{2}(x - 4)^3 + 6$

**b**



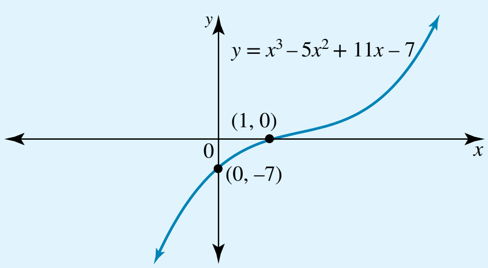

Stationary point of inflection $(4, 6)$; $y$-intercept $(0, 38)$; $x$-intercept approximately $(6.3, 0)$

**18 a** Proof required — check with your teacher
$y = (x - 1)(x^2 - 4x + 7)$

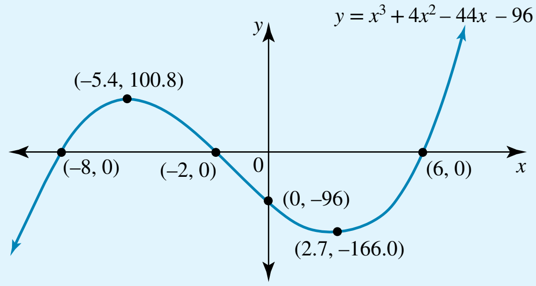
**b** Proof required — check with your teacher

**c** $y \to \infty$

**d**



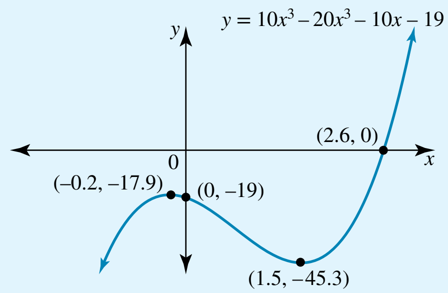

**19 a** Maximum turning point $(-5.4, 100.8)$; minimum turning point $(2.7, -166.0)$; $y$-intercept $(0, -96)$; $x$-intercepts $(-8, 0), (-2, 0), (6, 0)$



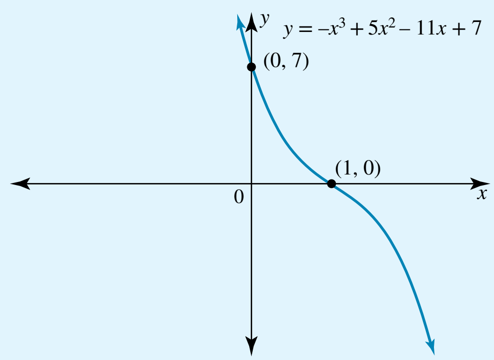

**b** Proof required — check with your teacher

**20** See table at foot of page.*

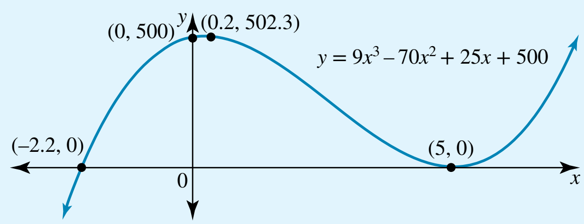
**a**




**b**




**c**




## EXERCISE 4.5

**1 a** $y = \frac{1}{49}(x - 3)^3 - 7$

**b** $y = 0.25x(x + 5)(x - 4)$

**c** $y = -(x + 2)^2(x - 3)$

**2** $y = 2x^3 + 3x^2 - 4x + 3$

**3 a** $x < 6, x \neq 2$

**b** $\{x : -2 \leq x \leq 0\} \cup \{x : x \geq 2\}$

**c** $x < -2$

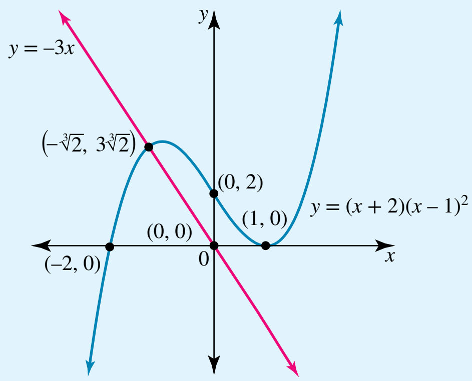
*20

| | Maximum turning point | Minimum turning point | $y$-intercept | $x$-intercept |
|---|---|---|---|---|
| a | $(-0.2, -17.9)$ | $(1.5, -45.3)$ | $(0, -19)$ | $(2.6, 0)$ |
| b | none | none | $(0, 7)$ | $(1, 0)$ |
| c | $(0.2, 502.3)$ | $(5, 0)$ | $(0, 500)$ | $(-2.2, 0), (5, 0)$ |

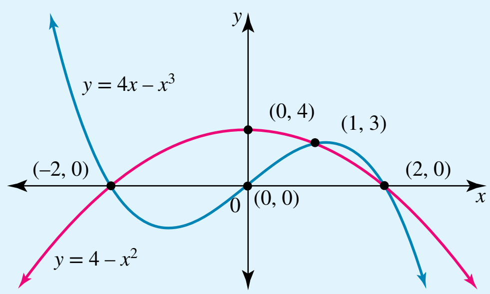
**4** $\{x : -1 < x < 1\} \cup \left\{x : x > \frac{7}{3}\right\}$

**5** Point of intersection is $(-\sqrt[3]{2}, 3\sqrt[3]{2})$.

When $x > -\sqrt[3]{2}$, $-3x < (x + 2)(x - 1)^2$




**6** Points of intersection are $(1, 3)$, $(2, 0)$, $(-2, 0)$.




**7 a** $y = \frac{1}{2}(x + 8)(x + 4)(x + 1)$

**b** $y = -2x^2(x - 5)$

**c** $y = -3(x - 1)^3 - 3$

**d** $y = \frac{4}{5}(x - 1)(x - 5)^2$

**8 a** $y = -2(x + 6)^3 - 7$

**b** $(0, -12)$

**c** $\left(\sqrt[3]{10} - 5, 0\right)$

**d** $(0, 7)$

**9 a** $y = -(x - a)^2(x - b)$

**b** $\{x : x \leq b\}$

**c** More than $b$ units to the left

**d** More than $-a$ units to the right

**10 a** $(x - 3)^2$

**b** $(-1, 0)$

**c** $a = -5, b = 3$

**11 a** $-9 \leq x \leq -1$ or $x \geq 2$

**b** $x > \frac{1}{5}$

**c** $x > 2.5$

**d** $x \leq 0$ or $x = 1$

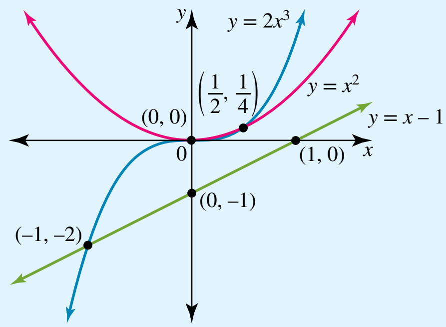
**e** $x < -2$ or $-\frac{6}{5} < x < 2$

**f** $-\frac{3}{2} < x < -1$ or $x > -\frac{1}{2}$

**12 a** $(0, 0), \left(\frac{1}{2}, \frac{1}{4}\right)$

**b** $(-1, -2)$

**c**




**d** $x \leq 0.5$

**13 a** $y = -2x + 5$; 1 solution

**b** $y = x^3 + 3x^2, y = 4x$; 3 solutions

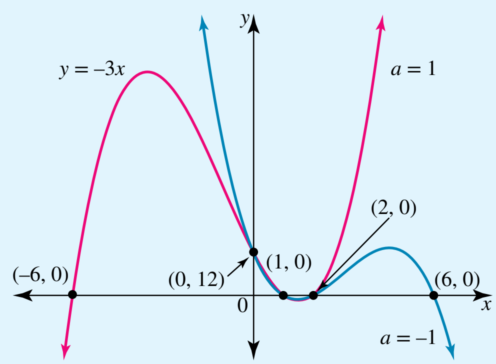
**c** There are 3 solutions. One method is to use $y = x^3, y = -3x^2 + 4x$.

**d** $x = -4, 0, 1$

**14 a** Fewer than 4 pieces of information are given.

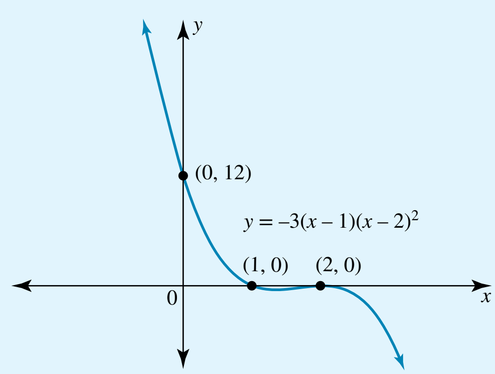
**b** Proof required — check with your teacher

**c**




If $a = 1$, graph contains the points $(1, 0)$, $(2, 0)$, $(0, 12)$, $(-6, 0)$; if $a = -1$, the points are $(1, 0)$, $(2, 0)$, $(0, 12)$, $(6, 0)$.

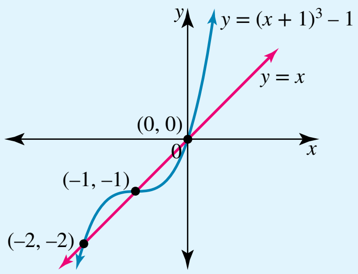
**d** Equation is $y = -3x^3 + 15x^2 - 24x + 12$ or $y = -3(x - 1)(x - 2)^2$; $x$-intercepts are $(1, 0)$, $(2, 0)$; $(2, 0)$ is a maximum turning point.



**15 a** $a = 1; b = -1$

**b** $(-2, -2), (-1, -1), (0, 0)$

**c**

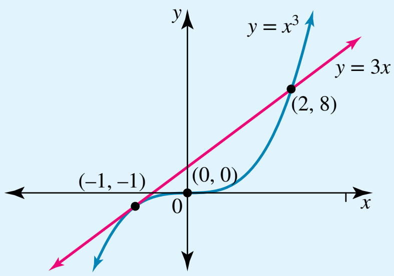



**d** $\{x : -2 < x < -1\} \cup \{x : x > 0\}$

**16 a** Proof required — check with your teacher

**b** $(2, 8)$

**c**




**d** One intersection if $m < 3$; two intersections if $m = 3$; three intersections if $m > 3$

**17** $y = -6x^3 + 12x^2 + 19x - 5$

**18 a** $(-3.11, -9.46), (-0.75, 0.02), (0.86, 6.44)$

**b** $x^3 + 3x^2 - x - 2 = 0$

**c** $x$-intercepts

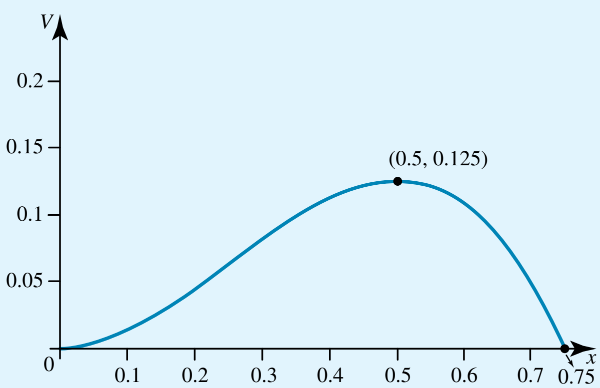
## EXERCISE 4.6

**1 a** $h = \dfrac{3 - 4x}{2}$

**b** Proof required — check with your teacher

**c** $0 \le x \le \frac{3}{4}$

**d**



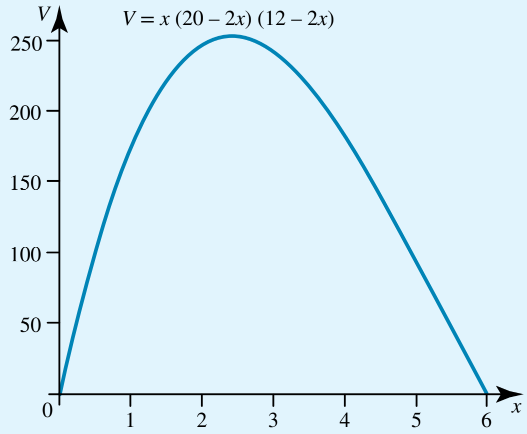

$x$-intercepts at $x = 0$ (touch), $x = 0.75$ (cut); shape of a negative cubic

**e** Cube of edge 0.5 m

**2 a** $l = 20 - 2x$; $w = 12 - 2x$; $V = (20 - 2x)(12 - 2x)x$

**b** $0 \le x \le 6$

**c**




$x$-intercepts at $x = 10, x = 6, x = 0$ but since $0 \le x \le 6$, the graph won't reach $x = 10$; shape is of a positive cubic.

**d** Length 15.14 cm; width 7.14 cm; height 2.43 cm; greatest volume 263 $cm^3$

**3 a** Loss of \$125; profit of \$184

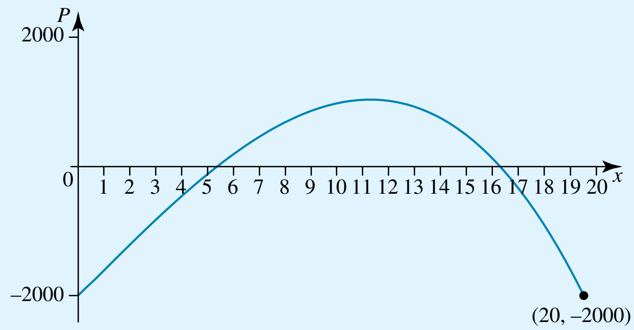
**b** Proof required — check with your teacher

**c** Too many and the costs outweigh the revenue from the sales. A negative cubic tends to $-\infty$ as $x$ becomes very large.

**d** **i** Profit \$304

**ii** Loss \$113

**e**




$x$-intercepts lie between 5 and 6 and between 16 and 17.

**f** Between 6 and 16

**4 a** 54

**b** 2 hours

**c** 210

**d** 5 pm

**5 a** $y = -\dfrac{1}{32}x^2(x - 6)$

**b** $\dfrac{81}{32}$ km

**6** $-4, 1$

**7 a** $2\sqrt{2}x$

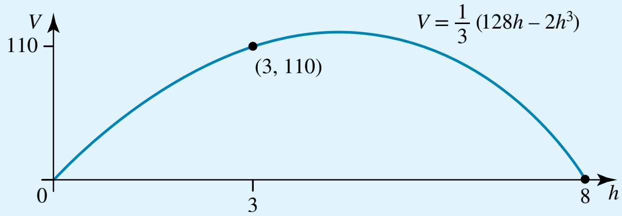
**b** Proof required — check with your teacher

**c** Proof required — check with your teacher

**d** **i** 110 $m^3$

**ii** Mathematically $0 \le h \le 8$

**e**




Max volume when $h$ is between 4 and 5 (estimates will vary).

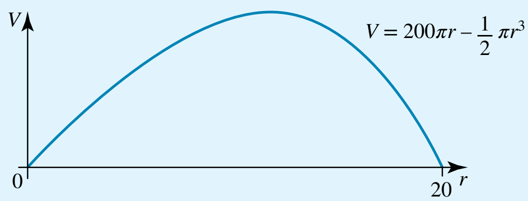
**f** Height $\frac{8}{\sqrt{3}} \approx 4.6$ m

**8 a** Proof required — check with your teacher

**b** Proof required — check with your teacher

**c** $0 \le r \le 20$

**d**




**e** Radius 2 cm, height 99 cm or radius 18.9 cm, height 1.1 cm

**f** 4837 $cm^3$

**9 a** (0, 2.1), (1.25, 1), (2.5, 1.1), (4, 0.1)

**b** $d = 2.1$

**c** $125a + 100b + 80c = -70.4$

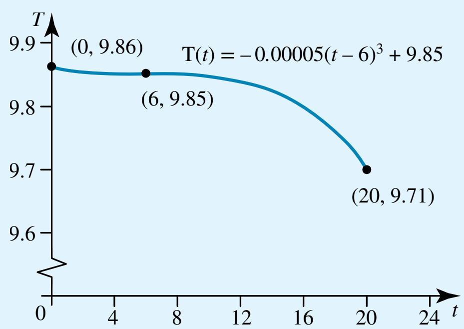
$125a + 50b + 20c = -8$

$64a + 16b + 4c = -2$

**d** 0.77 m

**10 a** $T(3) = 9.85, T(20) = 9.71$

**b**




**c** $T(28) = 9.32$; unlikely, but not totally impossible. Model is probably not a good predictor.

**11 a** $x = 0, x = 6$

**b** Length $2x - 6$; width $6x - x^2$

**c** Proof required — check with your teacher

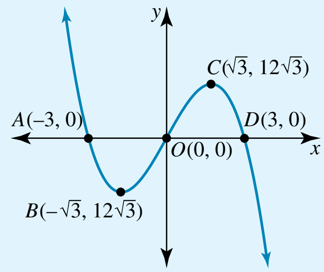
**d** $3 \le x \le 6$

**e** $x = 4, x = \frac{5 + \sqrt{33}}{2}$

**12 a** 3 $x$-intercepts; 2 turning points

**b** $y = -2x(x^2 - 9)$

**c**




**d** $y = 12x$

**e** $y = 4x^3$

**13 a** 4.62 m

**b** Proof required — check with your teacher

**c** Greatest volume of 4836.8 $cm^3$; base radius 11.5 cm; height 11.5 cm

**14 a** $y = -0.164x^3 + x^2 - 1.872x + 2.1$

**b** Length 3.464 units; width 6.000 units

# 5

# Higher-degree polynomials

# 5.1 Kick off with CAS

## Quartic transformations

1 Using CAS technology, sketch the following quartic functions.

a $y = x^4$ b $y = -x^4$ c $y = -3x^4$

d $y = \frac{1}{2}x^4$ e $y = -\frac{2}{5}x^4$

2 Using CAS technology, enter $y = ax^4$ into the function entry line and use a slider to change the values of $a$.

3 Complete the following sentences.

a When sketching a quartic function, a negative sign in front of the $x^4$ term ___________ the graph of $y = x^4$.

b When sketching a quartic function, $y = ax^4$, for values of $a < -1$ and $a > 1$, the graph of $y = x^4$ becomes ___________.

c When sketching a quartic function, $y = ax^4$, for values of $-1 < a < 1$, $a \neq 0$, the graph of $y = x^4$ becomes ___________.

4 Using CAS technology, sketch the following functions.

a $y = x^4$ b $y = (x + 1)^4$ c $y = -(x - 2)^4$

d $y = x^4 - 1$ e $y = -x^4 + 2$ f $y = 3 - x^4$

5 Using CAS technology, enter $y = (x - h)^4$ into the function entry line and use a slider to change the value of $h$.

6 Using CAS technology, enter $y = x^4 + k$ into the function entry line and use a slider to change the value of $k$.

7 Complete the following sentences.

a When sketching a quadratic function, $y = (x - h)^4$, the graph of $y = x^4$ is ___________.

b When sketching a quadratic function, $y = x^4 + k$, the graph of $y = x^4$ is ___________.

8 Use CAS technology and your answers to questions 1–7 to determine the equation that could model the shape of the Gateway Arch in St Louis. If the technology permits, upload a photo of the bridge to make this easier.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 5.2 Quartic polynomials

A quartic polynomial is a polynomial of degree 4 and is of the form $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$ and $a, b, c, d, e \in R$.

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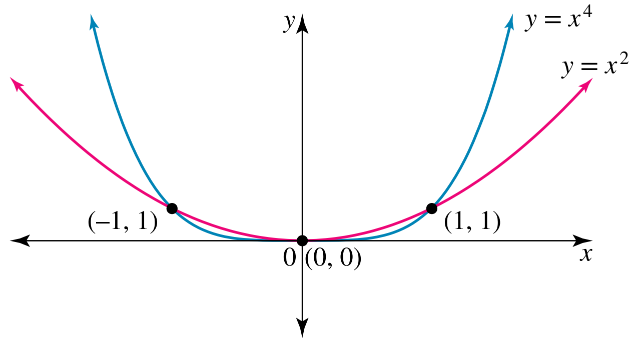
Graph plotter: Polynomials of higher degree
**int-2569**

## Graphs of quartic polynomials of the form $y = a(x - h)^4 + c$

The simplest quartic polynomial graph has the equation $y = x^4$. As both negative and positive numbers raised to an even power, in this case 4, will be positive, the long-term behaviour of the graph of $y = x^4$ must be that as $x \to -\infty$ or as $x \to \infty$, then $y \to \infty$.




The graph of $y = x^4$ is similar to that of the parabola $y = x^2$. Both graphs are concave up with a minimum turning point at (0, 0) and both contain the points (−1, 1) and (1, 1). However, for the intervals where $x < -1$ and $x > 1$, the graph of $y = x^4$ lies above the parabola. This is because $x^4 > x^2$ for these intervals. Likewise, the graph of $y = x^4$ lies below that of the parabola for the intervals $-1 < x < 0$ and $0 < x < 1$, since $x^4 < x^2$ for these intervals.

Despite these differences, the two graphs are of sufficient similarity to enable us to obtain the key features of graphs of quartic polynomials of the form $y = a(x - h)^4 + k$ in much the same manner as for quadratics of the form $y = a(x - h)^2 + k$.

Under a dilation of $a$ units, a horizontal translation of $h$ units and a vertical translation of $k$ units, the graph of $y = x^4$ is transformed to that of $y = a(x - h)^2 + k$.

**The graph of $y = a(x - h)^4 + k$ has the following features.**
- **A turning point with coordinates $(h, k)$.**
  **If $a > 0$, the turning point is a minimum and if $a < 0$ it is a maximum.**
- **Axis of symmetry with equation $x = h$.**
- **Zero, one or two $x$-intercepts.**
  **These are obtained as the solution to the equation $a(x - h)^4 + k = 0$.**

WORKED EXAMPLE 1

Sketch the graph of:

a $y = \frac{1}{4}(x + 3)^4 - 4$ b $y = -(3x - 1)^4 - 7$

| THINK | WRITE |
| --- | --- |
| **a** 1 State the coordinates and type of turning point. | **a** $y = \frac{1}{4}(x + 3)^4 - 4$<br>Turning point is (−3, −4).<br>As $a = \frac{1}{4}$, $a > 0$, so the turning point is a minimum. |

**2** Calculate the $y$-intercept.

$y$-intercept: let $x = 0$

$y = \frac{1}{4}(3)^4 - 4$

$= \frac{81}{4} - \frac{16}{4}$

$= \frac{65}{4}$

$y$-intercept: $\left(0, \frac{65}{4}\right)$

**3** Determine whether there will be any $x$-intercepts.

As the $y$-coordinate of the minimum turning point is negative, the concave up graph must pass through the $x$-axis.

**4** Calculate the $x$-intercepts.

*Note:* $\pm$ is needed in taking the fourth root of each side.

$x$-intercepts: let $y = 0$

$\frac{1}{4}(x + 3)^4 - 4 = 0$

$\frac{1}{4}(x + 3)^4 = 4$

$\therefore (x + 3)^4 = 16$

Take the fourth root of both sides.

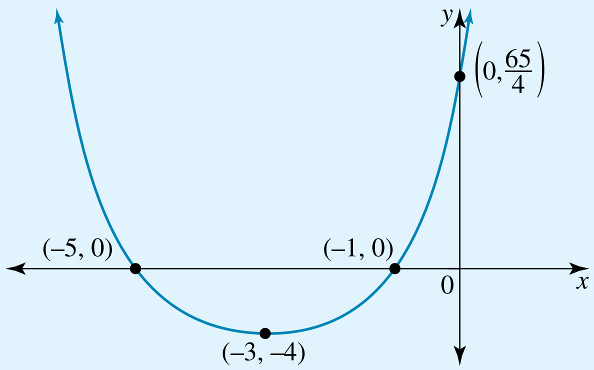
$(x + 3) = \pm\sqrt[4]{16}$

$x + 3 = \pm 2$

$\therefore x = -5$ or $x = -1$

$x$-intercepts: $(-5, 0)$ and $(-1, 0)$

**5** Sketch the graph.




**b 1** Express the equation in the form $y = a(x - h)^4 + k$.

**b** $y = -(3x - 1)^4 - 7$

$= -\left(3\left(x - \frac{1}{3}\right)\right)^4 - 7$

$= -81\left(x - \frac{1}{3}\right)^4 - 7$

**2** State the coordinates of the turning point and its type.

The graph has a maximum turning point at $\left(\frac{1}{3}, -7\right)$.

**3** Calculate the $y$-intercept.

$y$-intercept: let $x = 0$ in the original form

$y = -(3x - 1)^4 - 7$

$= -(-1)^4 - 7$

$= -(1) - 7$

$= -8$

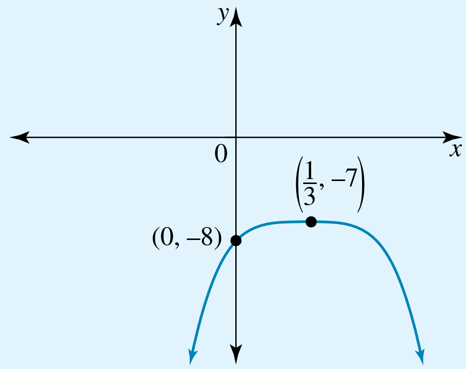
$y$-intercept: $(0, -8)$

**4** Determine whether there will be any $x$-intercepts.

As the $y$-coordinate of the maximum turning point is negative, the concave down graph will not pass through the $x$-axis.

**5** Sketch the graph.

The graph is symmetric about its axis of symmetry, $x = \frac{1}{3}$.



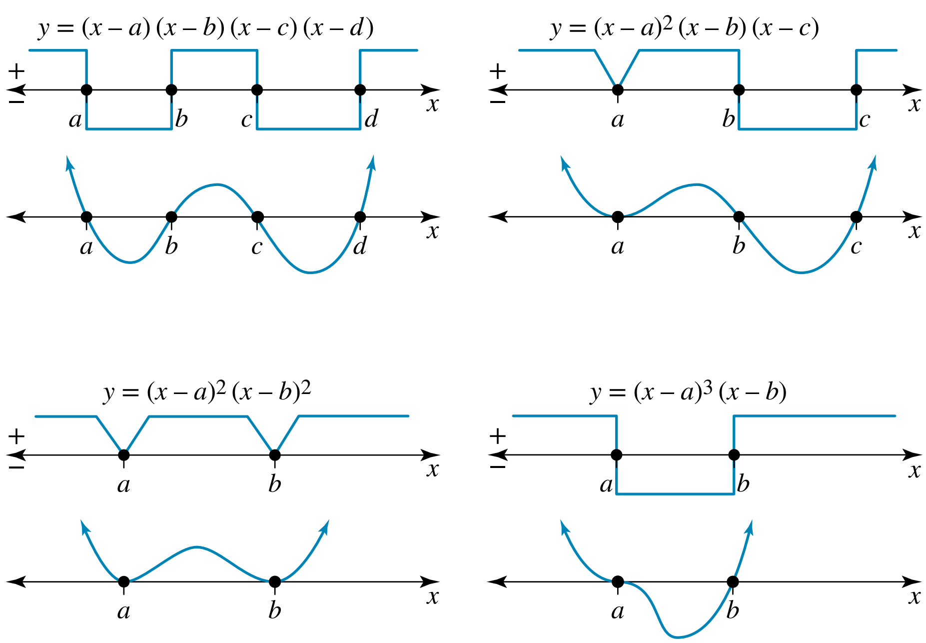

## Quartic polynomials which can be expressed as the product of linear factors

Not all quartic polynomials have linear factors. However, the graphs of those which can be expressed as the product of linear factors can be readily sketched by analysing these factors.

**A quartic polynomial may have up to 4 linear factors since it is of fourth degree. The possible combinations of these linear factors are:**

- **four distinct linear factors $y = (x - a)(x - b)(x - c)(x - d)$**
- **one repeated linear factor $y = (x - a)^2(x - b)(x - c)$**
- **two repeated linear factors $y = (x - a)^2(x - b)^2$**
- **one factor of multiplicity 3$y = (x - a)^3(x - b)$**
- **one factor of multiplicity 4$y = (x - a)^4$. This case in which the graph has a minimum turning point at $(a, 0)$ has already been considered.**

Given the long-term behaviour of a quartic polynomial whereby $y \to \infty$ as $x \to \pm\infty$ for a positive coefficient of the term in $x^4$, the sign diagrams and accompanying shape of the graphs must be of the form shown in the diagrams.

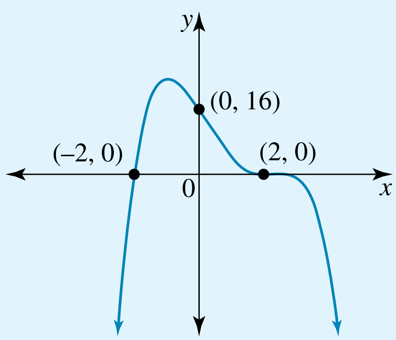


For a negative coefficient of $x^4$, $y \to -\infty$ as $x \to \pm\infty$, so the sign diagrams and graphs are inverted.

**The single factor identifies an $x$-intercept where the graph cuts the axis; a repeated factor identifies an $x$-intercept which is a turning point; and the factor of multiplicity 3 identifies an $x$-intercept which is a stationary point of inflection.**

WORKED EXAMPLE 2

Sketch the graph of $y = (x + 2)(2 - x)^3$.

| THINK | WRITE |
|---|---|
| 1 Calculate the $x$-intercepts. | $y = (x + 2)(2 - x)^3$<br>$x$-intercepts: let $y = 0$<br>$(x + 2)(2 - x)^3 = 0$<br>$\therefore x + 2 = 0$ or $(2 - x)^3 = 0$<br>$\therefore x = -2$ or $x = 2$<br>$x$-intercepts: $(-2, 0)$ and $(2, 0)$ |
| 2 Interpret the nature of the graph at each $x$-intercept. | Due to the multiplicity of each factor, at $x = -2$ the graph cuts the $x$-axis and at $x = 2$ it saddle-cuts the $x$-axis. The point $(2, 0)$ is a stationary point of inflection. |
| 3 Calculate the $y$-intercept. | $y$-intercept: let $x = 0$<br>$y = (2)(2)^3$<br>$= 16$<br>$y$-intercept: $(0, 16)$ |
| 4 Determine the sign of the coefficient of the leading term and identify the long-term behaviour of the graph. | Leading term is $(x)(-x)^3 = -x^4$. The coefficient of the leading term is negative, so as $x \to \pm\infty$ then $y \to -\infty$. This means the sketch of the graph must start and finish below the $x$-axis. |
| 5 Sketch the graph. |  |

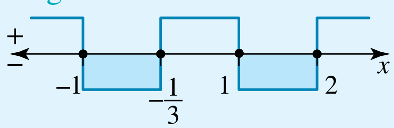
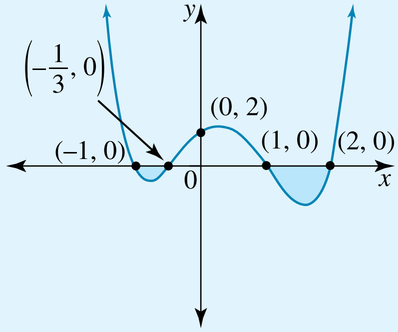


## Equations and inequations

Factorisation techniques may enable a quartic polynomial $P(x)$ given in its general form to be rewritten as the product of linear factors. Its graph $y = P(x)$ can then be readily sketched from this form and the equation $P(x) = 0$ solved using the Null Factor Law. With the aid of a sign diagram, or a graph, an inequation such as $P(x) \leq 0$ can be solved.

The factor theorem may be one method employed if no simpler method can be found. Once one zero and its corresponding factor are obtained using the factor theorem, division of the quartic polynomial by this factor would produce a cubic quotient which in turn could be factorised with further use of the factor theorem. Alternatively, if two zeros can be found from trial and error, then division of the quartic by the product of their corresponding factors would produce a quadratic quotient.

**WORKED EXAMPLE 3** **Factorise $P(x) = 3x^4 - 5x^3 - 5x^2 + 5x + 2$ and hence solve the inequation $3x^4 - 5x^3 - 5x^2 + 5x + 2 \leq 0$.**

| THINK | WRITE/DRAW |
|---|---|
| 1 Use the factor theorem to obtain one linear factor of the polynomial. | $P(x) = 3x^4 - 5x^3 - 5x^2 + 5x + 2$<br>$P(1) = 3 - 5 - 5 + 5 + 2$<br>$= 0$<br>$\therefore (x - 1)$ is a factor. |
| 2 Use the factor theorem to obtain a second linear factor. | $P(-1) = 3 + 5 - 5 - 5 + 2$<br>$= 0$<br>$\therefore (x + 1)$ is a factor. |
| 3 State a quadratic factor of the polynomial. | Hence, $(x - 1)(x + 1) = x^2 - 1$ is a quadratic factor of $P(x)$. |
| 4 Divide the known quadratic factor into the polynomial. | $3x^4 - 5x^3 - 5x^2 + 5x + 2 = (x^2 - 1)(ax^2 + bx + c)$<br>$= (x^2 - 1)(3x^2 + bx - 2)$<br>Equating coefficients of $x^3$: $-5 = b$<br>$\therefore 3x^4 - 5x^3 - 5x^2 + 5x + 2 = (x^2 - 1)(3x^2 - 5x - 2)$ |
| 5 Completely factorise the polynomial. | $P(x) = (x^2 - 1)(3x^2 - 5x - 2)$<br>$= (x^2 - 1)(3x + 1)(x - 2)$<br>$= (x - 1)(x + 1)(3x + 1)(x - 2)$ |
| 6 State the zeros of the polynomial. | The zeros of the polynomial are:<br>$x = 1, x = -1, x = -\frac{1}{3}, x = 2$ |
| 7 Draw the sign diagram. | The leading term has a positive coefficient so the sign diagram is:<br><br> |
| 8 Sketch the graph. | <br> |
| 9 State the solution to the inequation. | $P(x) \leq 0$<br>$\therefore -1 \leq x \leq -\frac{1}{3}$ or $1 \leq x \leq 2$ |

## EXERCISE 5.2 Quartic polynomials

**PRACTISE**

Work without CAS

1 WE1 Sketch the following graphs.

a $y = (x - 2)^4 - 1$

b $y = -(2x + 1)^4$

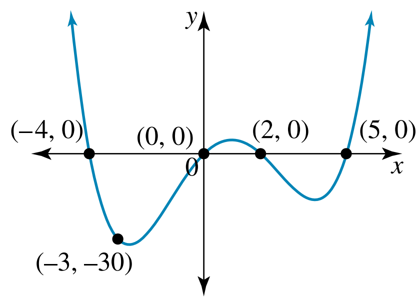
2 A graph with the equation $y = a(x - b)^4 + c$ has a maximum turning point at $(-2, 4)$ and cuts the $y$-axis at $y = 0$.

a Determine its equation.

b Sketch the graph and so determine $\{x : a(x - b)^4 + c > 0\}$.

3 WE2 Sketch the graph of $y = (x + 2)^2(2 - x)^2$.

4 Give a suitable equation for the graph of the quartic polynomial shown.




5 WE3 Factorise $P(x) = x^4 + 5x^3 - 6x^2 - 32x + 32$ and hence solve the inequation $x^4 + 5x^3 - 6x^2 - 32x + 32 > 0$.

6 Solve the equation $(x + 2)^4 - 13(x + 2)^2 - 48 = 0$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 a On the same set of axes, sketch the graphs of $y = x^4$, $y = 2x^4$ and $y = \frac{1}{2}x^4$. Label the points for which $x = -1, 0$ and $1$.

b On the same set of axes, sketch the graphs of $y = x^4$, $y = -x^4$, $y = -2x^4$ and $y = (-2x)^4$. Label the points for which $x = -1, 0$ and $1$.

c On the same set of axes, sketch the graphs of $y = x^4$, $y = -(x + 1)^4$ and $y = (1 - x)^4$. Label the points for which $x = -1, 0$ and $1$.

d On the same set of axes, sketch the graphs of $y = x^4$, $y = x^4 + 2$ and $y = -x^4 - 1$. Label the points for which $x = -1, 0$ and $1$.

8 Sketch the following graphs, identifying the coordinates of the turning point and any point of intersection with the coordinate axes.

a $y = (x - 1)^4 - 16$

b $y = \frac{1}{9}(x + 3)^4 + 12$

c $y = 250 - 0.4(x + 5)^4$

d $y = -\left(6(x - 2)^4 + 11\right)$

e $y = \frac{1}{8}(5x - 3)^4 - 2$

f $y = 1 - \left(\frac{2 - 7x}{3}\right)^4$

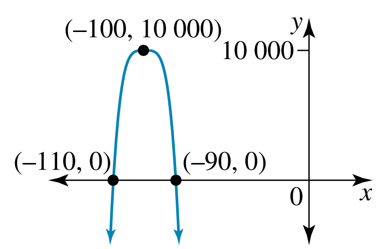
9 Determine a posssible equation for each of the following.

a A quartic graph with the same shape as $y = \frac{2}{3}x^4$ but whose turning point has the coordinates $(-9, -10)$.

b The curve with the equation $y = a(x + b)^4 + c$ which has a minimum turning point at $(-3, -8)$ and passes through the point $(-4, -2)$.

c A curve has the equation $y = (ax + b)^4$ where $a > 0$ and $b < 0$. The points $(0, 16)$ and $(2, 256)$ lie on the graph.

d The graph shown has the equation $y = a(x - h)^4 + k$.



10 The curve with equation $y = ax^4 + k$ passes through the points $(-1, 1)$ and $\left(\frac{1}{2}, \frac{3}{8}\right)$.

**a** Determine the values of $a$ and $k$.

**b** State the coordinates of the turning point and its nature.

**c** Give the equation of the axis of symmetry.

**d** Sketch the curve.

11 The graph of $y = a(x + b)^4 + c$ passes through the points $(-2, 3)$ and $(4, 3)$.

**a** State the equation of its axis of symmetry.

**b** Given the greatest $y$-value the graph reaches is 10, state the coordinates of the turning point of the graph.

**c** Determine the equation of the graph.

**d** Calculate the coordinates of the point of intersection with the $y$-axis.

**e** Calculate the exact value(s) of any intercepts the graph makes with the $x$-axis.

**f** Sketch the graph.

12 Sketch the following quartic polynomials without attempting to locate any turning points that do not lie on the coordinate axes.

**a** $y = (x + 8)(x + 3)(x - 4)(x - 10)$

**b** $y = -\frac{1}{100}(x + 3)(x - 2)(2x - 15)(3x - 10)$

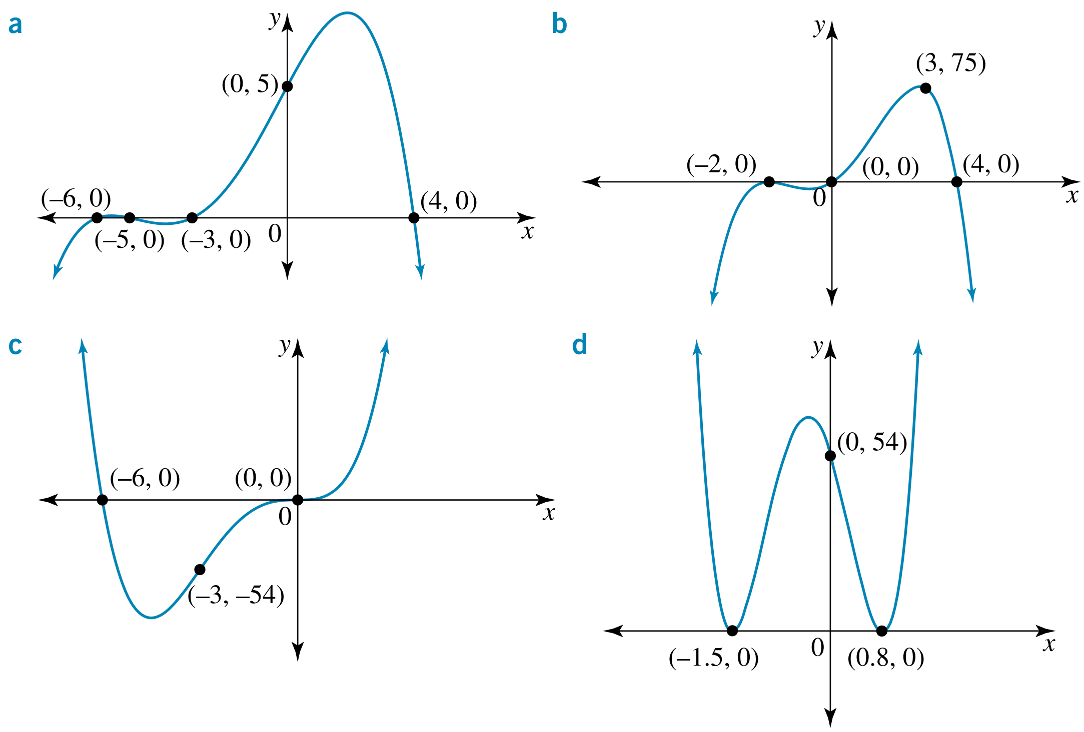
**c** $y = -2(x + 7)(x - 1)^2(2x - 5)$

**d** $y = \frac{2}{3}x^2(4x - 15)^2$

**e** $y = 3(1 + x)^3(4 - x)$

**f** $y = (3x + 10)(3x - 10)^3$

13 For each of the following quartic graphs, form a possible equation.




14 Solve the following equations and inequations.

**a** $(x + 3)(2x - 1)(4 - x)(2x - 11) < 0$

**b** $9x^4 - 49x^2 = 0$

**c** $300x^4 + 200x^3 + 28x^2 \leq 0$

**d** $-3x^4 + 20x^3 + 10x^2 - 20x - 7 = 0$

**e** $x^4 + x^3 - 8x = 8$

**f** $20(2x - 1)^4 - 8(1 - 2x)^3 \geq 0$

15 a Factorise $-x^4 + 18x^2 - 81$.
b Sketch the graph of $y = -x^4 + 18x^2 - 81$.
c Use the graph to obtain $\{x : -x^4 + 18x^2 - 81 > 0\}$.
d State the solution set for $\{x : x^4 - 18x^2 + 81 > 0\}$.

16 The graphs of $y = x^4$ and $y = 2x^3$ intersect at the origin and at a point P.
a Calculate the coordinates of the point P.
b The parabola $y = ax^2$ and the straight line $y = mx$ pass through the origin and the point P. Determine the values of $a$ and $m$.
c Using the values obtained for $a$ and $m$ in part b, sketch the graphs of $y = x^4$, $y = 2x^3$, $y = ax^2$ and $y = mx$ on the same set of axes.
d i At what points would the graphs of $y = nx^3$ and $y = x^4$ intersect?
ii If each of the four curves $y = x^4$, $y = nx^3$, $y = ax^2$ and $y = mx$ intersect at the same two points, express $a$ and $m$ in terms of $n$.

MASTER

17 Sketch the graph of $y = x^4 - x^3 - 12x^2 - 4x + 4$, locating turning points and intersections with the coordinate axes. Express coordinates to 2 decimal places where appropriate.

18 a Sketch the graph of $y = x^4 - 7x - 8$, locating any turning points and intersections with the coordinate axes.
b Hence express $x^4 - 7x - 8$ as the product of a linear and a cubic polynomial with rational coefficients.

# 5.3 Families of polynomials

studyon

Units 1 & 2
AOS 1
Topic 4
Concept 2

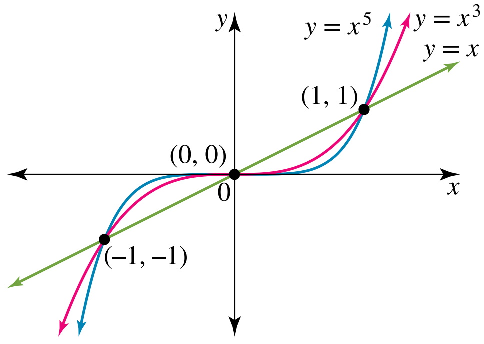
**Families of polynomials**
Concept summary
Pratice questions

Of the polynomials, the simplest is the linear polynomial. In some ways it is the exception, because its graph is a straight line, whereas the graphs of all other polynomials are curves. Nevertheless, the graphs of linear and all other polynomials of odd degree do display some similarities. Likewise, the graphs of polynomials of even degree also display similarities with each other. In considering families of polynomials it is therefore helpful to separate them into two categories: those with an odd degree and those with an even degree.

## Graphs of $y = x^n$, where $n \in N$ and $n$ is odd

Consider the shapes of the graphs of some odd-degree polynomials with simple equations. Once the basic shape is established, we can deduce the effect of transformations on these graphs.

Comparison of the graphs of $y = x$, $y = x^3$ and $y = x^5$ shows a number of similarities.




- Each graph exhibits the same long-term behaviour that as $x \to \pm\infty$, $y \to \pm\infty$.
- Each graph passes through the points $(-1, -1)$, $(0, 0)$, $(1, 1)$.
- With the exception of $y = x$, the other two graphs have a stationary point of inflection at $(0, 0)$ and essentially similar shapes.
- The larger the power, the narrower the graph for $x > 1$ and for $x < -1$.

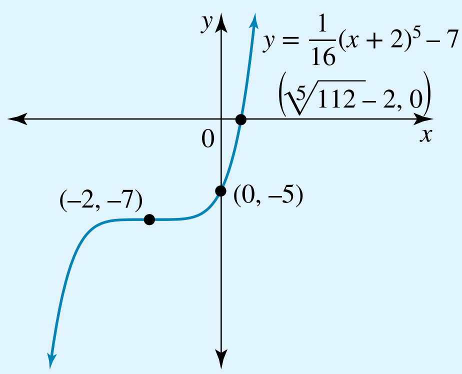
Thus, if $n$ is an odd positive integer, $n \neq 1$, the graph of $y = x^n$ will have a stationary point of inflection at $(0, 0)$ and essentially resemble the shape of $y = x^3$.

Under a sequence of transformations whereby the graph of $y = x^n$, $n \in N \setminus \{1\}$ is dilated by factor $a$ from the $x$-axis and translated horizontally $h$ units right and vertically $k$ units up, the equation of the transformed graph takes the form $y = a(x - h)^n + k$.

**The key features of the graphs of the family of odd-degree polynomials, with the equation $y = a(x - h)^n + k$, $n \in N \setminus \{1\}$, and $n$ is odd, are as follows:**
- **There is a stationary point of inflection at $(h, k)$.**
- **If $a > 0$, then as $x \to \pm\infty, y \to \pm\infty$.**
- **If $a < 0$, then as $x \to \pm\infty, y \to \mp\infty$.**
- **There is one $x$-intercept which is calculated by solving $a(x - h)^n + k = 0$.**
- **There is one $y$-intercept which is calculated by substituting $x = 0$.**

WORKED EXAMPLE 4

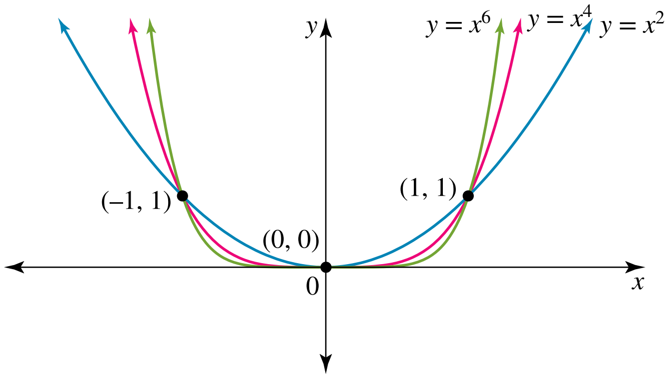
Sketch the graph of $y = \frac{1}{16}(x + 2)^5 - 7$.

| THINK | WRITE |
|---|---|
| 1 State the coordinates of the stationary point of inflection. | $y = \frac{1}{16}(x + 2)^5 - 7$<br>Stationary point of inflection at $(-2, -7)$ |
| 2 Calculate the $y$-intercept. | $y$-intercept: let $x = 0$<br>$y = \frac{1}{16}(2)^5 - 7$<br>$= \frac{32}{16} - 7$<br>$= -5$<br>$y$-intercept: $(0, -5)$ |
| 3 Calculate the $x$-intercept. | $x$-intercept: let $y = 0$<br>$\frac{1}{16}(x + 2)^5 - 7 = 0$<br>$\frac{1}{16}(x + 2)^5 = 7$<br>$\therefore (x + 2)^5 = 112$<br>Take the fifth root of both sides:<br>$x + 2 = \sqrt[5]{112}$<br>$x = \sqrt[5]{112} - 2$<br>$x$-intercept: approximately $(0.6, 0)$ |
| 4 Sketch the graph. |  |

## Graphs of $y = x^n$, where $n \in N$ and $n$ is even

Comparison of the graphs of $y = x^2$, $y = x^4$ and $y = x^6$ shows a number of similarities.

- Each graph exhibits the same long-term behaviour that as $x \to \pm\infty, y \to \infty$.
- Each graph passes through the points $(-1, 1), (0, 0), (1, 1)$.
- All graphs have a minimum turning point at $(0, 0)$ and essentially similar shapes.
- The larger the power, the narrower the graph for $x > 1$ and for $x < -1$.




Thus, if $n$ is an even positive integer, the graph of $y = x^n$ will have a minimum turning point at $(0, 0)$ and essentially resemble the shape of $y = x^2$.

**Under a sequence of transformations, the equation takes the form $y = a(x - h)^n + k$. If $n$ is an even positive integer, the key features of the graphs of the family of even-degree polynomials with the equation $y = a(x - h)^n + k$ are as follows:**

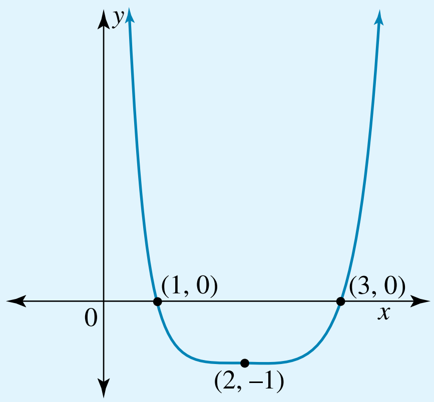
- **There is a turning point, or vertex, at $(h, k)$.**
- **For $a > 0$, the turning point is a minimum; for $a < 0$ it is a maximum.**
- **For $a > 0$, as $x \to \pm\infty, y \to \infty$.**
- **For $a < 0$, as $x \to \pm\infty, y \to -\infty$.**
- **The axis of symmetry has the equation $x = h$.**
- **There may be 0, 1 or 2 $x$-intercepts.**
- **There is one $y$-intercept.**

WORKED EXAMPLE 5

Sketch the graph of $y = (x - 2)^6 - 1$.

| THINK | WRITE |
| --- | --- |
| 1 State the type of turning point and its coordinates. | $y = (x - 2)^6 - 1$<br>As $a > 0$, minimum turning point at $(2, -1)$ |
| 2 Calculate the $y$-intercept. | $y$-intercept: let $x = 0$<br>$y = (-2)^6 - 1$<br>$= 63$<br>$y$-intercept: $(0, 63)$ |
| 3 Calculate the $x$-intercepts if they exist.<br>*Note:* $\pm$ is needed in taking an even root of each side. | $x$-intercepts: let $y = 0$<br>$(x - 2)^6 - 1 = 0$<br>$(x - 2)^6 = 1$<br>$x - 2 = \pm\sqrt[6]{1}$<br>$x - 2 = \pm 1$<br>$\therefore x = 1, x = 3$<br>$x$-intercepts: $(1, 0)$ and $(3, 0)$ |

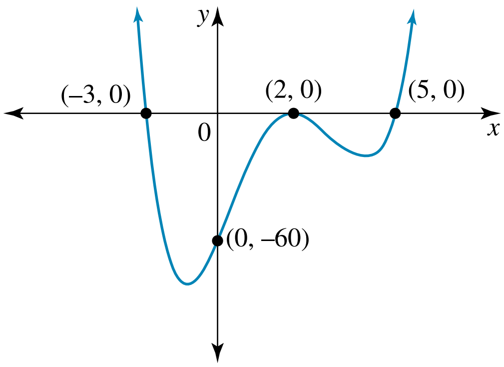
4 Sketch the graph.




## Families of polynomials which can be expressed as the product of linear factors

The factors of a polynomial determine the $x$-intercepts of the graph. From the multiplicity of each linear factor, the behaviour of a graph at its $x$-intercepts can be determined. For example, if you look at the equation $y = (x + 3)(x - 2)^2(x - 5)$, the graph of this quartic polynomial is predicted to:

- cut the $x$-axis at $x = -3$
- touch the $x$-axis at $x = 2$
- cut the $x$-axis at $x = 5$.




Its graph confirms this prediction.

This interpretation can be extended to any polynomial expressed in factorised form.

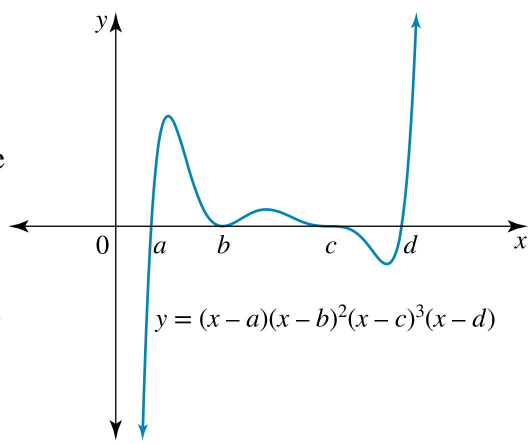
### Effect of multiplicity of zeros and linear factors

**The graph of $y = (x - a)(x - b)^2(x - c)^3(x - d)$ would:**

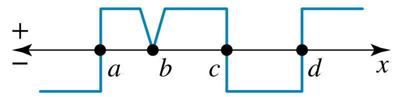
- **cut the $x$-axis at $x = a$ (single zero from factor of multiplicity 1)**
- **touch the $x$-axis at $x = b$ (double zero from factor of multiplicity 2)**
- **saddle-cut the $x$-axis at $x = c$ (triple zero from factor of multiplicity 3)**
- **cut the $x$-axis at $x = d$ (single zero from factor of multiplicity 1).**

At a 'touch' $x$-intercept, there is a turning point; at a 'saddle-cut' $x$-intercept there is a stationary point of inflection.

This polynomial has degree 7 since $x \times x^2 \times x^3 \times x = x^7$. As the coefficient of the leading term is positive, the graph follows the long-term behaviour of an odd-degree polynomial. It must initially start below the $x$-axis and it must end above the $x$-axis, since as $x \to \pm\infty, y \to \pm\infty$. A possible graph of $y = (x - a)(x - b)^2(x - c)^3(x - d)$ is shown in the diagram.

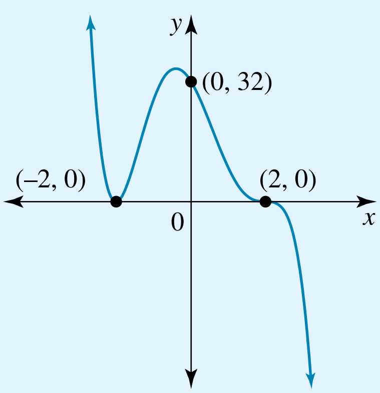


Sign diagrams or graphs can be drawn using the nature of the zeros to solve inequations. On a graph, there is a significant difference in shape between the way the graph crosses the $x$-axis at a 'saddle-cut' $x$-intercept compared with a 'cut' $x$-intercept. However, on a sign diagram, a 'saddle cut' is not treated differently to a 'cut'since the sign diagram is simply showing the graph crosses through the $x$-axis at these points. The sign diagram for the graph drawn for $y = (x - a)(x - b)^2(x - c)^3(x - d)$ is:



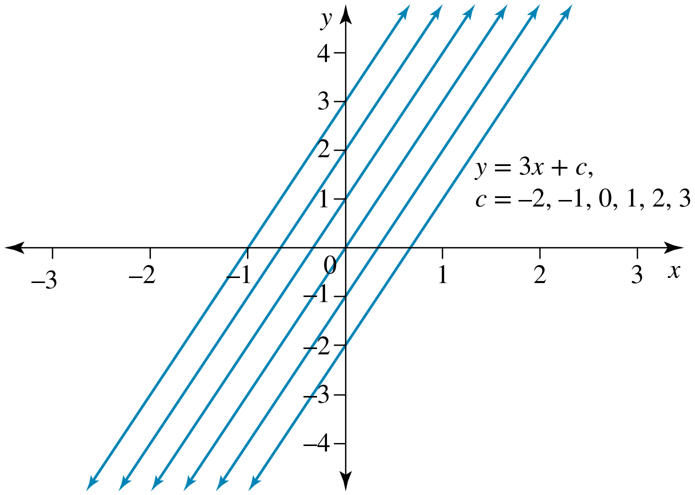

**WORKED EXAMPLE 6** Sketch the graph of $y = (x + 2)^2(2 - x)^3$.

| THINK | WRITE |
|---|---|
| 1 Calculate the $x$-intercepts. | $y = (x + 2)^2(2 - x)^3$<br>$x$-intercepts: let $y = 0$<br>$(x + 2)^2(2 - x)^3 = 0$<br>$\therefore x + 2 = 0$ or $2 - x = 0$<br>$x = -2$ (touch), $x = 2$ (saddle cut)<br>$x$-intercepts: $(-2, 0)$ and $(2, 0)$ |
| 2 State the nature of the graph at each $x$-intercept. | Due to the multiplicity of each factor, at $(-2, 0)$ the graph has a turning point and at $(2, 0)$ there is a stationary point of inflection. |
| 3 Calculate the $y$-intercept. | $y$-intercept: let $x = 0$<br>$y = (2)^2(2)^3$<br>$= 32$<br>$y$-intercept: $(0, 32)$ |
| 4 Identify the degree of the polynomial. | Leading term is $(x)^2(-x)^3 = -x^5$, so the polynomial is of degree 5. |
| 5 Sketch the graph. | As the coefficient of the leading term is negative, the graph starts above the $x$-axis. |



## Other families of polynomials

Various sets of polynomials which share a common feature or features may be considered a family. Often these families are described by a common equation which contains one or more constants that can be varied in value. Such a varying constant is called a **parameter**. An example is the set of linear polynomials, each with a gradient of 3 but with a differing $y$-intercept. This is the family of parallel lines defined by the equation $y = 3x + c,\ c \in R$, some members of which are shown in the diagram.




This set of lines is generated by allowing the parameter $c$ to take the values $-2, -1$, 0, 1, 2 and 3. As could be anticipated, these values of $c$ are the $y$-intercepts of each line.

**WORKED EXAMPLE 7**

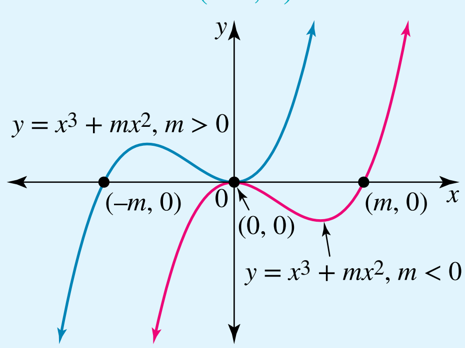
**Consider the family of cubic polynomials defined by the equation $y = x^3 + mx^2$ where the parameter $m$ is a real non-zero constant.**

**a Calculate the $x$-intercepts and express them in terms of $m$, where appropriate.**

**b Draw a sketch of the shape of the curve for positive and negative values of $m$ and comment on the behaviour of the graph at the origin in each case.**

**c Determine the equation of the member of the family that contains the point (7, 49).**

| THINK | WRITE |
|---|---|
| **a** Calculate the $x$-intercepts. | **a** $y = x^3 + mx^2$<br>$x$-intercepts: let $y = 0$<br>$x^3 + mx^2 = 0$<br>$x^2(x + m) = 0$<br>$\therefore x = 0$ or $x = -m$<br>$x$-intercepts: $(0, 0)$ and $(-m, 0)$ |
| **b 1** Describe the behaviour of the curve at each $x$-intercept. | **b** Due to the multiplicity of the factor there is a turning point at $(0, 0)$. At $(-m, 0)$ the graph cuts the $x$-axis. |
| **2** Sketch the shape of the graph, keeping in mind whether the parameter is positive or negative. | If $m < 0$ then $(-m, 0)$ lies to the right of the origin.<br>If $m > 0$ then $(-m, 0)$ lies to the left of the origin. |



| | |
|---|---|
| **3** Comment on the behaviour of the graph at the origin. | The graph has a maximum turning point at the origin if $m$ is negative. If $m$ is positive there is a minimum turning point at the origin. |
| **c 1** Use the given point to determine the value of the parameter. | **c** $y = x^3 + mx^2$ Substitute the point $(7, 49)$. $49 = 7^3 + m \times 7^2$ $49 = 49(7 + m)$ $1 = 7 + m$ $\therefore m = -6$ |
| **2** State the equation of the required curve. | If $m = -6$ then the member of the family which passes through the point $(7, 49)$ has the equation $y = x^3 - 6x$. |

## EXERCISE 5.3 Families of polynomials

**PRACTISE**

Work without CAS

1 **WE4** Sketch the graph of $y = 32 - (x - 1)^5$.

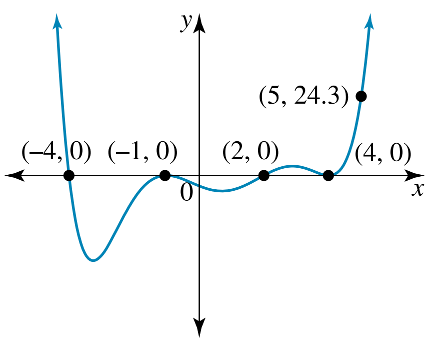
2 On the same set of axes sketch the graphs of $y = x^7$ and $y = x$, labelling the points of intersection with their coordinates. Hence state $\{x : x^7 \leq x\}$.

3 **WE5** Sketch the graph of $y = -(2x + 1)^6 - 6$.

4 A graph with equation $y = a(x - b)^8 + c$ has a minimum turning point at $(-1, -12)$ and passes through the origin.
   - **a** Determine its equation.
   - **b** State the equation of its axis of symmetry.
   - **c** What is its other $x$-intercept?

5 **WE6** Sketch the graph of $y = (2 + x)^3(2 - x)^2$.

6 The graph of a polynomial is shown.
   - **a** State the degree of the polynomial.
   - **b** Given the point $(5, 24.3)$ lies on the graph, form its equation.




7 **WE7** Consider the family of quartic polynomials defined by the equation $y = x^4 - mx^3$ where the parameter $m$ is a real non-zero constant.
   - **a** Calculate the $x$-intercepts and express them in terms of $m$ where appropriate.
   - **b** Draw a sketch of the shape of the curve for positive and negative values of $m$ and comment on the behaviour of the graph at the origin in each case.
   - **c** Determine the equation of the member of the family that contains the point $(-1, -16)$.

8 Consider the family of quadratic polynomials defined by $y = a(x - 3)^2 + 5 - 4a,\ a \in R \backslash \{0\}$.
   - **a** Show that every member of this family passes through the point $(1, 5)$.
   - **b** For what value(s) of $a$ will the turning point of the parabola lie on the $x$-axis?
   - **c** For what value(s) of $a$ will the parabola have no $x$-intercepts?

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

**9** **a** On the same set of axes, sketch the graphs of $y = x^6$, $y = -x^6$, $y = x$ and $y = -x$.

**b** On the same set of axes, sketch the graphs of $y = x^5$, $y = 2x^5$ and $y = \frac{1}{2}x^5$.

**10** **a** **i** On the same set of axes, sketch the graphs of $y = -x^{11}$ and $y = -x$.

**ii** Hence state the solution set to $\{x : x^{11} > x\}$.

**b** **i** Sketch the graphs of $y = x^6$ and $y = x^7$.

**ii** Hence or otherwise, solve the equation $x^6 = x^7$.

**c** **i** On the same set of axes sketch the graphs of $y = (x + 1)^4 + 1$ and $y = (x + 1)^5 + 1$.

**ii** Hence state the coordinates of the points of intersection of the two graphs.

**11** **a** State the coordinates of the turning point and specify its nature for each of the following:

**i** $y = \frac{1}{16}(x - 4)^{10} + 3$ **ii** $y = -\frac{1}{125}\left(\frac{3x}{2} - 5\right)^6$

**b** State the coordinates of the stationary point of inflection for each of the following:

**i** $y = \frac{x^5 + 27}{54}$ **ii** $y = 16 - (2x + 1)^7$

**12** Sketch the shape of the following, identifying any intercepts with the coordinate axes.

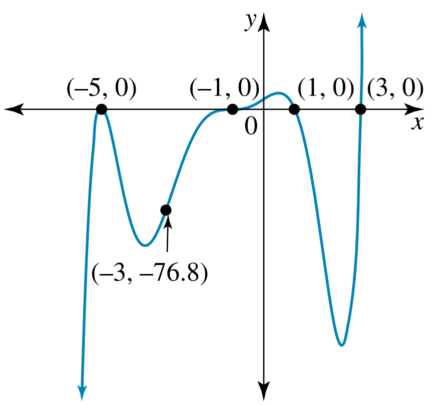
**a** $y = (x + 3)^2(x + 1)(x - 2)^2$ **b** $y = \frac{1}{4}(x + 2)^3(8 - x)$

**c** $y = -x^3(x^2 - 1)$ **d** $y = -x^2(x - 1)^2(x - 3)^2$

**13** **a** The curve belonging to the family of polynomials for which $y = a(x + b)^5 + c$ has a stationary point of inflection at $(-1, 7)$ and passes through the point $(-2, -21)$. Determine the equation of the curve.

**b** The graph of a monic polynomial of degree 4 has a turning point at $(-2, 0)$ and one of its other $x$-intercepts occurs at $(4, 0)$. Its $y$-intercept is $(0, 48)$. Determine a posssible equation for the polynomial and identify the coordinates of its other $x$-intercept.

**c** Give a possible equation for the graph of the polynomial shown.




**d** The curve has the equation $y = (ax + b)^4$. When expanded, the coefficients of the terms in $x$ and $x^2$ are equal and the coefficient of $x^3$ is 1536. Determine the equation of this curve.

**14** **a** Solve the following equations:

**i** $9x^3 - 36x^5 = 0$ **ii** $x^6 - 2x^3 + 1 = 0$

**iii** $x^5 + x^4 - x^3 - x^2 - 2x - 2 = 0$

**b** Solve the following inequations:

**i** $x(x + 4)^3(x - 3)^2 < 0$ **ii** $\frac{1}{81}(9 - 10x)^5 - 3 \geq 0$

**iii** $(x - 2)^4 \leq 81$

**15** **a** **i** Using a parameter, form the equation of the family of lines which pass through the point $(2, 3)$.

**ii** Use the equation from part **i** to find which line in this family also passes through the origin.

**b** **i** What is the point which is common to the family of parabolas defined by the equation $y = ax^2 + bx,\ a \neq 0$?

**ii** Express the equation of those parabolas, in the form $y = ax^2 + bx,\ x \neq 0$, which pass through the point $(2, 6)$ in terms of the one parameter, $a$, only.

**c** Calculate the $x$-intercepts of the graphs of the family of polynomials defined by the equation $y = mx^2 - x^4,\ m > 0$ and draw a sketch of the shape of the graph.

**16** Consider the two families of polynomials for which $y = k$ and $y = x^2 + bx + 10$, where $k$ and $b$ are real constants.

**a** Describe a feature of each family that is shared by all members of that family.

**b** Sketch the graphs of $y = x^2 + bx + 10$ for $b = -7, 0$ and $7$.

**c** For what values of $k$ will members of the family $y = k$ intersect $y = x^2 + 7x + 10$?

**i** Once only **ii** Twice **iii** Never

**d** If $k = 7$, then for what values of $b$ will $y = k$ intersect $y = x^2 + bx + 10$?

**i** Once only **ii** Twice **iii** Never

**17** Consider the curves for which $y = (x - a)^3(x + a)^2$, where $a$ is a positive real constant.

**a** Identify the intercepts with the axes in terms of $a$ and comment on their nature.

**b** Draw a sketch of the shape of the curve.

**c** In how many places will the line $y = -x$ intersect these curves?

**d** For what value(s) of $a$ will the line $y = -x$ intersect the curve at the point $\left(\frac{a}{2}, -\frac{a}{2}\right)$?

**18** Consider the family of cubic polynomials for which $y = ax^3 + (3 - 2a)x^2 + (3a + 1)x - 4 - 2a$ where $a \in R\backslash\{0\}$.

**a** Show that the point $(1, 0)$ is common to all this family.

**b** For the member of the family which passes through the origin, form its equation and sketch its graph.

**c** A member of the family passes through the point $(-1, -10)$. Show that its graph has exactly one $x$-intercept.

**d** By calculating the coordinates of the point of intersection of the polynomials with equations $y = ax^3 + (3 - 2a)x^2 + (3a + 1)x - 4 - 2a$ and $y = (a - 1)x^3 - 2ax^2 + (3a - 2)x - 2a - 5$, show that for all values of $a$ the point of intesection will always lie on a vertical line.

MASTER

**19** Use CAS technology to sketch the family of quartic polynomials for which $y = x^4 + ax^2 + 4,\ a \in R$, for $a = -6, -4, -2, 0, 2, 4, 6$, and determine the values of $a$ for which the polynomial equation $x^4 + ax^2 + 4 = 0$ will have:

**a** four roots **b** two roots **c** no real roots.

**20** A parabola has an axis of symmetry with equation $x = 1$. If the parabola intersects the graph of $y = \frac{1}{18}(x + 2)^6 - 2$ at both the turning point and the $y$-intercept of this curve, determine the equation of the parabola and sketch each curve on the same set of axes.

# 5.4 Numerical approximations to roots of polynomial equations

studyon

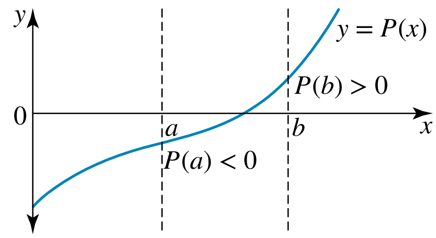
Units 1 & 2
AOS 1
Topic 4
Concept 3

**Numerical approximations to roots of polynomial equations**
Concept summary
Pratice questions

For any polynomial $P(x)$ the values of the $x$-axis intercepts of the graph of $y = P(x)$ are the roots of the polynomial equation $P(x) = 0$. These roots can always be obtained if the polynomial is linear or quadratic, or if the polynomial can be expressed as a product of linear factors. However, there are many polynomial equations that cannot be solved by an algebraic method. In such cases, if an approximate value of a root can be estimated, then this value can be improved upon by a method of **iteration**. An iterative procedure is one which is repeated by using the values obtained from one stage to calculate the value of the next stage and so on.

## Existence of roots

**For a polynomial $P(x)$, if $P(a)$ and $P(b)$ are of opposite signs, then there is at least one value of $x \in (a, b)$ where $P(x) = 0$.**




For example, in the diagram shown, $P(a) < 0$ and $P(b) > 0$. The graph cuts the $x$-axis at a point for which $a < x < b$.

This means that the equation $P(x) = 0$ has a root which lies in the interval $(a, b)$. This gives an estimate of the root. Often the values of $a$ and $b$ are integers and these may be found through trial and error. Alternatively, if the polynomial graph has been sketched, it may be possible to obtain their values from the graph. Ideally, the values of $a$ and $b$ are not too far apart in order to avoid, if possible, there being more than one $x$-intercept of the graph, or one root of the polynomial equation, that lies between them.

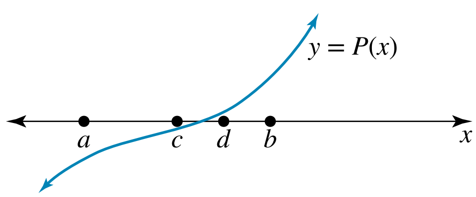
## The method of bisection

Either of the values of $a$ and $b$ for which $P(a)$ and $P(b)$ are of opposite sign provides an estimate for one of the roots of the equation $P(x) = 0$. The **method of bisection** is a procedure for improving this estimate by halving the interval in which the root is known to lie.

Let $c$ be the midpoint of the interval $[a, b]$ so $c = \frac{1}{2}(a + b)$.

The value $x = c$ becomes an estimate of the root.

By testing the sign of $P(c)$ it can be determined whether the root lies in $(a, c]$ or in $[c, b)$.




In the diagram shown, $P(a) < 0$ and $P(c) < 0$ so the root does not lie between $a$ and $c$. It lies between $c$ and $b$ since $P(c) < 0$ and $P(b) > 0$.

The midpoint $d$ of the interval $[c, b]$ can then be calculated. The value of $d$ may be an acceptable approximation to the root. If not, the accuracy of the approximation can be further improved by testing which of $[c, d]$ and $[d, b]$ contains the root and then halving that interval and so on. The use of some form of technology helps considerably with the calculations as it can take many iterations to achieve an estimate that has a high degree of accuracy.

Any other roots of the polynomial equation may be estimated by the same method once an interval in which each root lies has been established.

**WORKED EXAMPLE 8**

**Consider the cubic polynomial $P(x) = x^3 - 3x^2 + 7x - 4$.**

**a** **Show the equation $x^3 - 3x^2 + 7x - 4 = 0$ has a root which lies between $x = 0$ and $x = 1$.**

**b** **State a first estimate of the root.**

**c** **Carry out two iterations of the method of bisection by hand to obtain two further estimates of this root.**

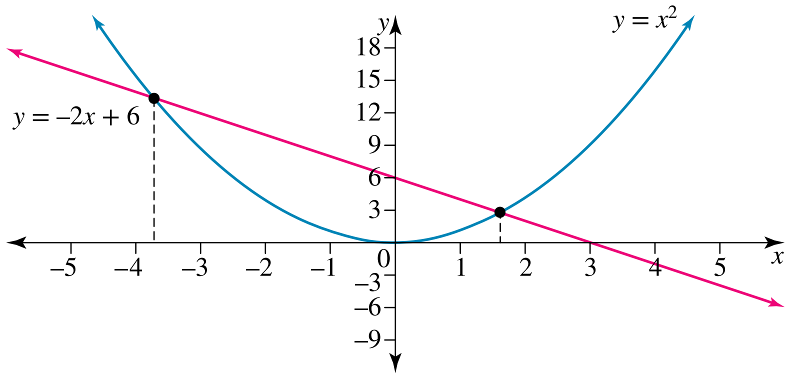
**d** **Continue the iteration using a calculator until the error in using this estimate as the root of the equation is less than 0.05.**

| THINK | WRITE |
|---|---|
| **a 1** Calculate the value of the polynomial at each of the given values. | **a** $P(x) = x^3 - 3x^2 + 7x - 4$<br>$P(0) = -4$<br>$P(1) = 1 - 3 + 7 - 4$<br>$= 1$ |
| **2** Interpret the values obtained. | As $P(0) < 0$ and $P(1) > 0$, there is a value of $x$ which lies between $x = 0$ and $x = 1$ where $P(x) = 0$.<br>Hence the equation $x^3 - 3x^2 + 7x - 4 = 0$ has a root which lies between $x = 0$ and $x = 1$. |
| **b** By comparing the values calculated select the one which is closer to the root. | **b** $P(0) = -4$ and $P(1) = 1$, so the root is closer to $x = 1$ than to $x = 0$. A first estimate of the root of the equation is $x = 1$. |
| **c 1** Calculate the midpoint of the first interval. | **c** The midpoint of the interval between $x = 0$ and $x = 1$ is $x = \frac{1}{2}$. |
| **2** State the second estimate. | $x = 0.5$ is a second estimate. |
| **3** Determine in which half interval the root lies. | $P\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{3}{4} + \frac{7}{2} - 4$<br>$= -\frac{9}{8}$<br>$< 0$<br>As $P(0) < 0$ and $P(1) > 0$, the root lies between $x = \frac{1}{2}$ and $x = 1$. |
| **4** Calculate the midpoint of the second interval. | The midpoint of the interval between $x = \frac{1}{2}$ and $x = 1$ is:<br>$x = \frac{1}{2}\left(\frac{1}{2} + 1\right)$<br>$= \frac{3}{4}$<br>$= 0.75$ |
| **5** State the third estimate. | $x = 0.75$ is a third estimate. |
| **d 1** Continue the calculations using a calculator. | **d** $P(0.75) = -0.015625$<br>As $-0.05 < P(0.75) < 0.05$, this estimate provides a solution to the equation with an error that's less than 0.05. |
| **2** State the estimate of the root.<br>*Note:* The value is still an estimate, not the exact value. | Hence $x = 0.75$ is a good estimate of the root of the equation which lies between $x = 0$ and $x = 1$. |

## Using the intersections of two graphs to estimate solutions to equations

Consider the quadratic equation $x^2 + 2x - 6 = 0$. Although it can be solved algebraically to give $x = \pm\sqrt{7} - 1$, we shall use it to illustrate another non-algebraic method for solving equations. If the equation is rearranged to $x^2 = -2x + 6$, then any solutions to the equation are the $x$-coordinates of any points of intersection of the parabola $y = x^2$ and the straight line $y = -2x + 6$.

Both of these polynomial graphs are relatively simple graphs to draw. The line can be drawn accurately using its intercepts with the coordinate axes, and the parabola can be drawn with reasonable accuracy by plotting some points that lie on it. The diagram of their graphs shows there are two points of intersection and hence that the equation $x^2 + 2x - 6 = 0$ has two roots.



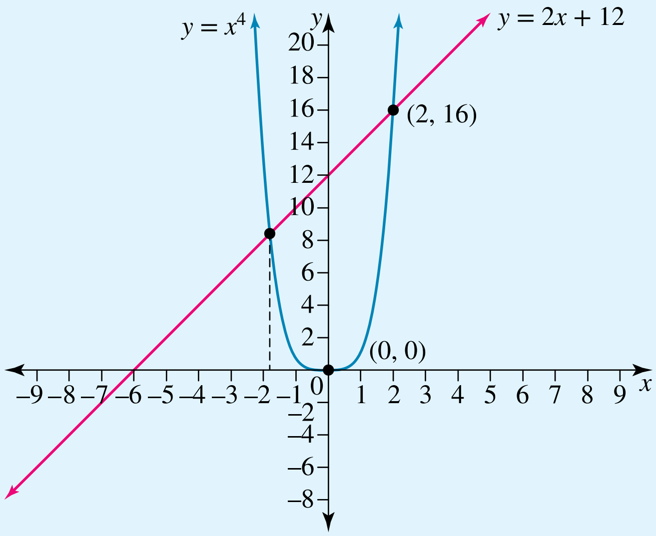

Estimates of the roots can be read from the graph. One root is approximately $x = -3.6$ and the other is approximately $x = 1.6$. (This agrees with the actual solutions which, to 3 decimal places, are $x = -3.646$ and $x = 1.646$).

Alternatively, we can confidently say that one root lies in the interval $[-4, -3]$ and the other in the interval $[1, 2]$ and by applying the method of bisection the roots could be obtained to a greater accuracy than that of the estimates that were read from the graph.

**To use the graphical method to solve the polynomial equation $H(x) = 0$:**

- **Rearrange the equation into the form $P(x) = Q(x)$ where each of the polynomials $P(x)$ and $Q(x)$ have graphs that can be drawn quite simply and accurately.**
- **Sketch the graphs of $y = P(x)$ and $y = Q(x)$ with care.**
- **The number of intersections determines the number of solutions to the equation $H(x) = 0$.**
- **The $x$-coordinates of the points of intersection are the solutions to the equation.**
- **Estimate these $x$-coordinates by reading off the graph.**
- **Alternatively, an interval in which the $x$-coordinates lie can be determined from the graph and the method of bisection applied to improve the approximation.**

**WORKED EXAMPLE 9** **Use a graphical method to estimate any solutions to the equation $x^4 - 2x - 12 = 0$.**

| THINK | WRITE |
| --- | --- |
| 1 Rearrange the equation so that it is expressed in terms of two familiar polynomials. | $x^4 - 2x - 12 = 0$<br>$x^4 = 2x + 12$ |
| 2 State the equations of the two polynomial graphs to be drawn. | The solutions to the equation are the $x$-coordinates of the points of intersection of the graphs of $y = x^4$ and $y = 2x + 12$. |
| 3 Determine any information which will assist you to sketch each graph with some accuracy. | The straight line $y = 2x + 12$ has a $y$-intercept at $(0, 12)$ and an $x$-intercept at $(-6, 0)$.<br>The quartic graph $y = x^4$ has a minimum turning point at $(0, 0)$ and contains the points $(\pm1, 1)$ and $(\pm2, 16)$. |
| 4 Carefully sketch each graph on the same set of axes. |  |
| 5 State the number of solutions to the original equation given. | As there are two points of intersection, the equation $x^4 - 2x - 12 = 0$ has two solutions. |
| 6 Use the graph to obtain the solutions. | From the graph it is clear that one point of intersection is at $(2, 16)$, so $x = 2$ is an exact solution of the equation.<br>An estimate of the $x$-coordinate of the other point of intersection is approximately $-1.7$, so $x = -1.7$ is an approximate solution to the equation. |



## Estimating coordinates of turning points

If the linear factors of a polynomial are known, sketching the graph of the polynomial is a relatively easy task to undertake. Turning points, other than those which lie on the $x$-axis, have largely been ignored, or, at best, allowed to occur where our pen and 'empathy' for the polynomial have placed them. Promises of rectifying this later when calculus is studied have been made. While this remains the case, we will address this unfinished aspect of our graph-sketching by considering a numerical method of systematic trial and error to locate the approximate position of a turning point.

For any polynomial with zeros at $x = a$ and $x = b$, its graph will have at least one turning point between $x = a$ and $x = b$.

To illustrate, consider the graph of $y = (x + 2)(x - 1)(x - 4)$. The factors show there are three $x$-intercepts: one at $x = -2$, a second one at $x = 1$ and a third at $x = 4$.

There would be a turning point between $x = -2$ and $x = 1$, and a second turning point between $x = 1$ and $x = 4$. The first turning point must be a maximum and the second one must be a minimum in order to satisfy the long-term behaviour requirements of a positive cubic polynomial.

The interval in which the $x$-coordinate of the maximum turning point lies can be narrowed using a table of values.

| $x$ | −2 | −1.5 | −1 | −0.5 | 0 | 0.5 | 1 |
|---|---|---|---|---|---|---|---|
| $y$ | 0 | 6.875 | 10 | 10.125 | 8 | 4.375 | 0 |

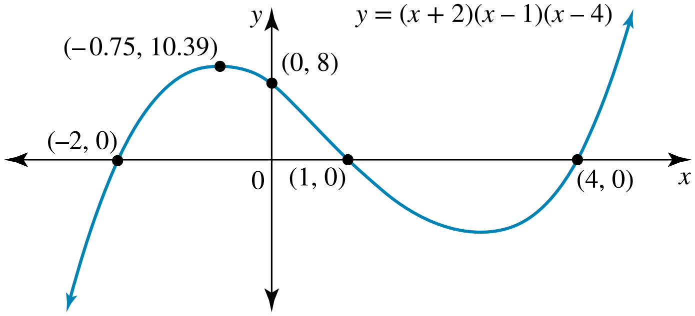
As a first approximation, the maximum turning point lies near the point $(-0.5, 10.125)$. Zooming in further around $x = -0.5$ gives greater accuracy.

| $x$ | −0.75 | −0.5 | −0.25 |
|---|---|---|---|
| $y$ | 10.390625 | 10.125 | 9.2989 |

An improved estimate is that the maximum turning point lies near the point $(-0.75, 10.39)$. The process could continue by zooming in around $x = -0.75$ if greater accuracy is desired.

An approximate position of the minimum turning point could be estimated by the same numerical method of systematic trial and error.

The shape of this positive cubic with a $y$-intercept at $(0, 8)$ could then be sketched.




## An alternative approach

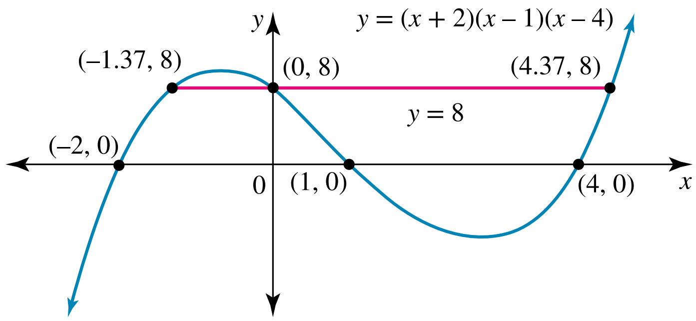
For any polynomial $P(x)$, if $P(a) = P(b)$ then its graph will have at least one turning point between $x = a$ and $x = b$. This means for the cubic polynomial shown in the previous diagram, the maximum turning point must lie between the $x$-values for which $y = 8$ (the $y$-intercept value).

Substitute $y = 8$ into $y = (x + 2)(x - 1)(x - 4)$:

$$x^3 - 3x^2 - 6x + 8 = 8$$
$$x^3 - 3x^2 - 6x = 0$$
$$x(x^2 - 3x - 6) = 0$$
$$x = 0 \text{ or } x^2 - 3x - 6 = 0$$

As the cubic graph must have a maximum turning point, the quadratic equation must have a solution. Solving it would give the negative solution as $x = -1.37$. Rather than test values between $x = -2$ and $x = 1$ as we have previously, the starting interval for testing values could be narrowed to between $x = -1.37$ and $x = 0$.

The positive solution $x = 4.37$ indicates that the minimum turning point lies between $x = 0$ and $x = 4.37$. In this case the interval between the two positive $x$-intercepts provides a narrower and therefore better interval to zoom into.




WORKED EXAMPLE 10

**a** State an interval in which the $x$-coordinate of the minimum turning point on the graph of $y = x(x - 2)(x + 3)$ must lie.

**b** Use a numerical method to zoom in on this interval and hence estimate the position of the minimum turning point of the graph, with the $x$-coordinate correct to 1 decimal place.

THINK

WRITE

**a 1** State the values of the $x$-intercepts in increasing order.

**a** $y = x(x - 2)(x + 3)$
The $x$-intercepts occur when $x = 0, x = 2, -3$.
In increasing order they are $x = -3, x = 0, x = 2$.

**2** Determine the pair of $x$-intercepts between which the required turning point lies.

The graph is a positive cubic so the first turning point is a maximum and the second is a minimum.
The minimum turning point must lie between $x = 0$ and $x = 2$.

**b 1** Construct a table of values which zooms in on the interval containing the required turning point.

**b** Values of the polynomial calculated over the interval $[0, 2]$ are tabulated.

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| $y$ | 0 | −2.625 | −4 | −3.375 | 0 |

**2** State an estimate of the position of the turning point.

The turning point is near $(1, -4)$.

**3** Zoom in further to obtain a second estimate.

Zooming in around $x = 1$ gives the table of values:

| $x$ | 0.9 | 1 | 1.1 | 1.2 |
|---|---|---|---|---|
| $y$ | −3.861 | −4 | −4.059 | −4.032 |

**4** State the approximate position.

The minimum turning point is approximately $(1.1, -4.059)$.

## EXERCISE 5.4 Numerical approximations to roots of polynomial equations

**PRACTISE**

1 WE8 Consider the cubic polynomial $P(x) = x^3 + 3x^2 - 7x - 4$.

a Show the equation $x^3 + 3x^2 - 7x - 4 = 0$ has a root which lies between $x = 1$ and $x = 2$.

b State a first estimate of the root.

c Carry out two iterations of the method of bisection by hand to obtain two further estimates of this root.

d Continue the iteration using a calculator until the error in using this estimate as the root of the equation is less than 0.05.

2 The graph of $y = x^4 - 2x - 12$ has two $x$-intercepts.

a Construct a table of values for this polynomial rule for $x = -3, -2, -1, 0, 1, 2, 3$.

b Hence state an exact solution to the equation $x^4 - 2x - 12 = 0$.

c State an interval within which the other root of the equation lies and use the method of bisection to obtain an estimate of this root correct to 1 decimal place.

3 WE9 Use a graphical method to estimate any solutions to the equation $x^4 + 3x - 4 = 0$.

4 Use a graphical method to estimate any solutions to the equation $x^3 - 6x + 4 = 0$.

5 WE10 a State an interval in which the $x$-coordinate of the maximum turning point on the graph of $y = -x(x + 2)(x - 3)$ must lie.

b Use a numerical method to zoom in on this interval and hence estimate the position of the maximum turning point of the graph with the $x$-coordinate correct to 1 decimal place.

6 Consider the cubic polynomial $y = 2x^3 - x^2 - 15x + 9$.

a State the $y$-intercept.

b What other points on the graph have the same $y$-coordinate as the $y$-intercept?

c Between which two $x$-values does the maximum turning point lie?

d Use a numerical method to zoom in on this interval and hence estimate the position of the maximum turning point of the graph, with the $x$-coordinate correct to 1 decimal place.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 For each of the following polynomials, show that there is a zero of each in the interval $[a, b]$.

a $P(x) = x^2 - 12x + 1, a = 10, \ b = 12$

b $P(x) = -2x^3 + 8x + 3, a = -2, \ b = -1$

c $P(x) = x^4 + 9x^3 - 2x + 1, a = -2, \ b = 1$

d $P(x) = x^5 - 4x^3 + 2, a = 0, \ b = 1$

8 The following polynomial equations are formed using the polynomials in question 7. Use the method of bisection to obtain two narrower intervals in which the root lies and hence give an estimate of the root which lies in the interval $[a, b]$.

a $x^2 - 12x + 1 = 0, a = 10, \ b = 12$

b $-2x^3 + 8x + 3 = 0, a = -2, \ b = -1$

c $x^4 + 9x^3 - 2x + 1 = 0, a = -2, \ b = 1$

d $x^5 - 4x^3 + 2 = 0, a = 0, \ b = 1$

9 The quadratic equation $5x^2 - 26x + 24 = 0$ has a root in the interval for which $1 \leq x \leq 2$.

a Use the method of bisection to obtain this root correct to 1 decimal place.

b What is the other root of this equation?

c Comment on the efficiency of the method of bisection.

10 Consider the polynomial defined by the rule $y = x^4 - 3$.

a Complete the table of values for the polynomial.

| $x$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | | | | | |

b Hence, state an interval in which $\sqrt[4]{3}$ lies.

c Use the method of bisection to show that $\sqrt[4]{3} = 1.32$ to 2 decimal places.

11 Consider the polynomial equation $P(x) = x^3 + 5x - 2 = 0$.

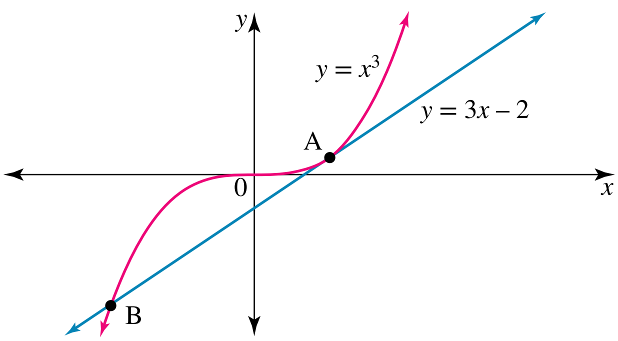
a Determine an interval $[a, b]$, $a, b \in Z$ in which there is a root of this equation.

b Use the method of bisection to obtain this root with an error less than 0.1.

c State the equations of two graphs, the intersections of which would give the number of solutions to the equation.

d Sketch the two graphs and hence state the number of solutions to the equation $P(x) = x^3 + 5x - 2 = 0$. Does the diagram support the answer obtained in part b?

12 The diagram shows that the line $y = 3x - 2$ is a tangent to the curve $y = x^3$ at a point A and that the line intersects the curve again at a point B.




a Form the polynomial equation $P(x) = 0$ for which the $x$-coordinates of the points A and B are solutions.

b Describe the number and multiplicity of the linear factors of the polynomial specified in part a.

c Use an algebraic method to calculate the exact roots of the polynomial equation specified in part a and hence state the coordinates of the points A and B.

d Using the graph, state how many solutions there are to the equation $x^3 - 3x + 1 = 0$.

13 Use a numerical systematic trial and error process to estimate the position of the following turning points. Express the $x$-coordinate correct to 1 decimal place.

a The maximum turning point of $y = (x + 4)(x - 2)(x - 6)$

b The minimum turning point of $y = x(2x + 5)(2x + 1)$

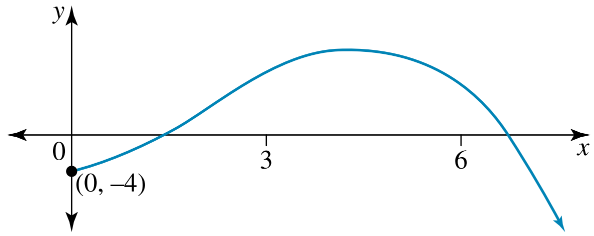
c The maximum and minimum turning points of $y = x^2 - x^4$

14 For the following polynomials, $P(0) = d$. Solve the equations $P(x) = d$ and hence state intervals in which the turning points of the graphs of $y = P(x)$ lie.

a $P(x) = x^3 - 3x^2 - 4x + 9$
b $P(x) = x^3 - 12x + 18$
c $P(x) = -2x^3 + 10x^2 - 8x + 1$
d $P(x) = x^3 + x^2 + 7$

15 The weekly profit $y$, in tens of dollars, from the sale of $10x$ containers of whey protein sold by a health food business is given by $y = -x^3 + 7x^2 - 3x - 4,\ x \geq 0$.

The graph of the profit is shown in the diagram.




a Show that the business first started to make a profit when the number of containers sold was between 10 and 20.
b Use the method of bisection to construct two further intervals for the value of $x$ required for the business to first start making a profit.
c Hence, use a numerical systematic trial-and-error method to calculate the number of containers that need to be sold for a profit to be made.
d Use the graph to state an interval in which the greatest profit lies.
e Use a numerical systematic trial and error process to estimate the number of containers that need to be sold for greatest profit, and state the greatest profit to the nearest dollar.
f As the containers are large, storage costs can lower profits. State an estimate from the graph of the number of containers beyond which no profit is made and improve upon this value with a method of your choice.

16 a For a polynomial $P(x)$ it is found that the product $P(x_1)P(x_2) < 0$. Explain what conclusion can be made about any roots of the equation $P(x) = 0$.

b Consider the polynomial $P(x) = 6x^3 - 11x^2 - 4x - 15$.
  i Show that $P(2)P(3) < 0$.
  ii Use the method of bisection to obtain a root of the equation $6x^3 - 11x^2 - 4x - 15 = 0$.
  iii Show that there is only one root to the equation $6x^3 - 11x^2 - 4x - 15 = 0$.
  iv Determine the number of points of intersection of the graph of $y = 6x^3 - 11x^2 - 4x - 15$ and the graph of $y = -15$. What information does this provide about the graph of $y = 6x^3 - 11x^2 - 4x - 15$?
  v Use the information gathered to draw a sketch of $y = 6x^3 - 11x^2 - 4x - 15$.

MASTER

17 Consider the cubic polynomial $y = -x^3 - 4$ and the family of lines $y = ax$.

a Sketch $y = -x^3 - 4$ and the family of lines $y = ax$.
b What is the largest integer value of $a$ for which the equation $x^3 + ax + 4 = 0$ has three roots?
c If $a > 0$, how many solutions are there to $x^3 + ax + 4 = 0$?
d Determine the root(s) of the equation $x^3 + x + 4 = 0$ using the method of bisection. Express the answer correct to 2 decimal places.

18 A rectangular sheet of cardboard measures 18 cm by 14 cm. Four equal squares of side length $x$ cm are cut from the corners and the sides are folded to form an open rectangular box in which to place some clothing.


a Express the volume of the box in terms of $x$.

b State an interval within which lies the value of $x$ for which the volume is greatest.

c Use the spreadsheet option on a CAS calculator to systematically test values in order to determine, to 3 decimal places, the side length of the square needed for the volume of the box to be greatest.

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# 5.5 Review

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**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

## REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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# Activities

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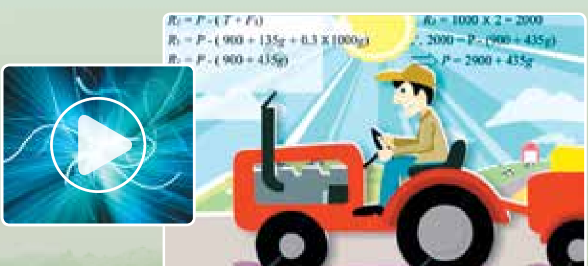
## Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.



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**studyon**

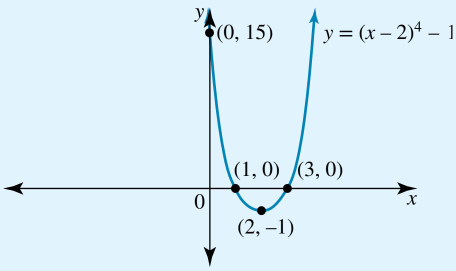
**Units 1 & 2** — **Higher-degree polynomials**

**Sit topic test**

# 5 Answers

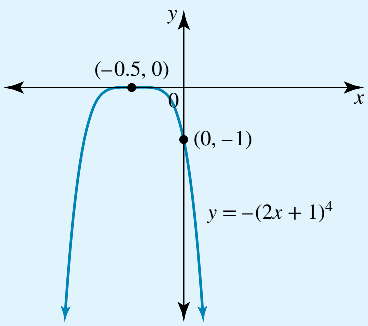
## EXERCISE 5.2

**1 a**




Minimum turning point $(2, -1)$; $y$-intercept $(0, 15)$; $x$-intercepts $(1, 0)$, $(3, 0)$

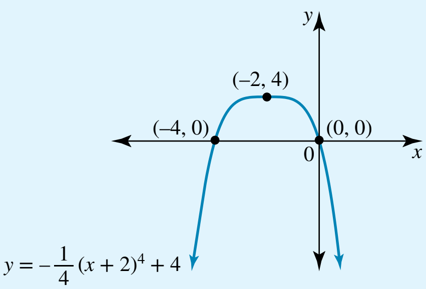
**b**




$x$-intercept and maximum turning point $\left(-\frac{1}{2}, 0\right)$; $y$-intercept $(0, -1)$

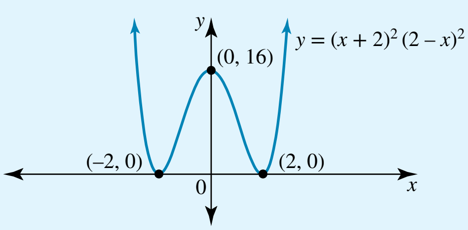
**2 a** $y = -\frac{1}{4}(x + 2)^4 + 4$

**b**




$x$-intercepts $(-4, 0)$, $(0, 0)$;
$\left\{x : -\frac{1}{4}(x + 2)^4 + 4 > 0\right\} = \{x : -4 < x < 0\}$

**3**



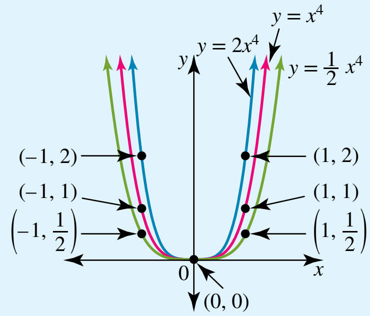

$x$-intercepts $(-2, 0)$ and $(2, 0)$ are turning points; $y$-intercept $(0, 16)$

**4** $y = \frac{1}{4}x(x + 4)(x - 2)(x - 5)$

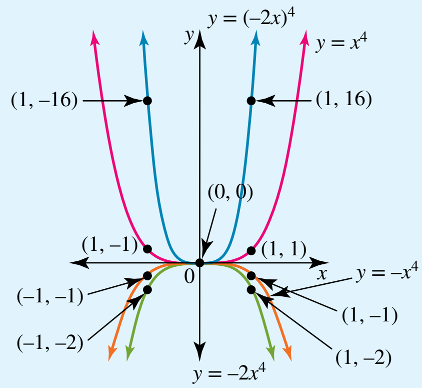
**5** $P(x) = (x - 1)(x - 2)(x + 4)^2$, $x < -4$, or $-4 < x < 1$ or $x > 2$

**6** $x = -6$ or $x = 2$

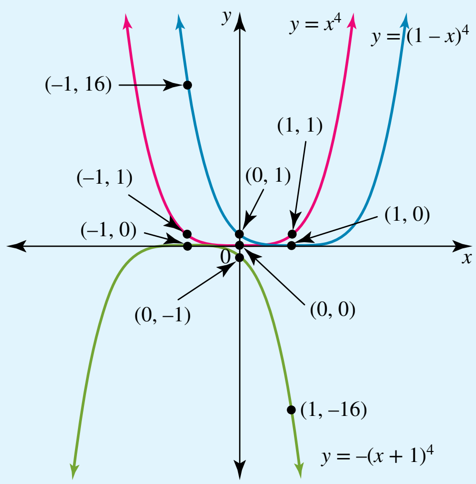
**7 a**



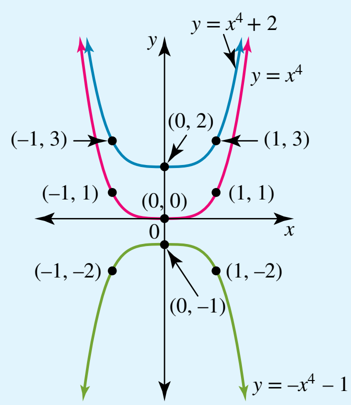

**b**




**c**

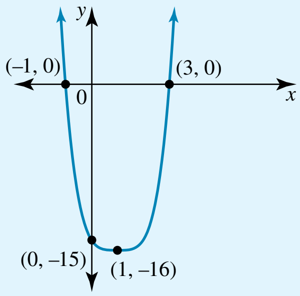


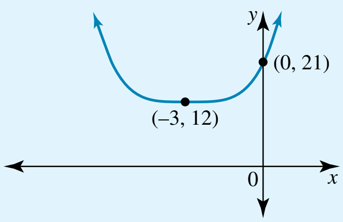

**d**

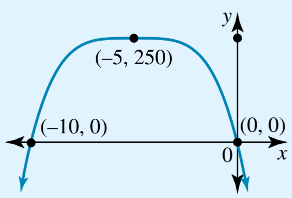


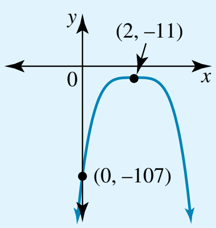
8

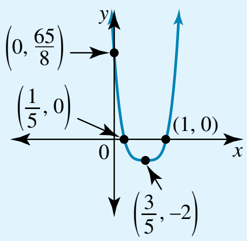
| | Turning point | $y$-intercept | $x$-intercepts |
|---|---|---|---|
| a | $(1, -16)$ minimum | $(0, -15)$ | $(-1, 0), (3, 0)$ |
| b | $(-3, 12)$ minimum | $(0, 21)$ | none |
| c | $(-5, 250)$ maximum | $(0, 0)$ | $(-10, 0), (0, 0)$ |
| d | $(2, -11)$ maximum | $(0, -107)$ | none |
| e | $\left(\frac{3}{5}, -2\right)$ minimum | $\left(0, \frac{65}{8}\right)$ | $\left(\frac{1}{5}, 0\right), (1, 0)$ |
| f | $\left(\frac{2}{7}, 1\right)$ maximum | $\left(0, \frac{65}{81}\right)$ | $\left(-\frac{1}{7}, 0\right), \left(\frac{5}{7}, 0\right)$ |

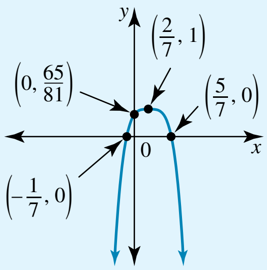
a



b



c



d



e



f



9 a $y = \frac{2}{3}(x + 9)^4 - 10$

b $y = 6(x + 3)^4 - 8$

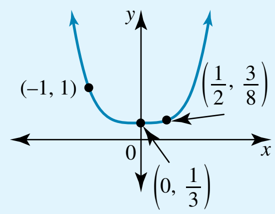
c $y = (3x - 2)^4$

d $y = -(x + 100)^4 + 10\,000$

10 a $a = \frac{2}{3}, k = \frac{1}{3}$

b Minimum turning point $\left(0, \frac{1}{3}\right)$

c $x = 0$

d


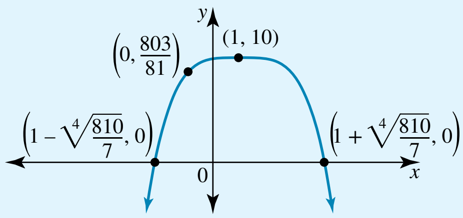

11 a $x = 1$

b $(1, 10)$

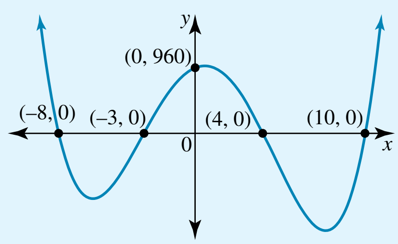
c $y = -\frac{7}{81}(x - 1)^4 + 10$

d $\left(0, \frac{803}{81}\right)$

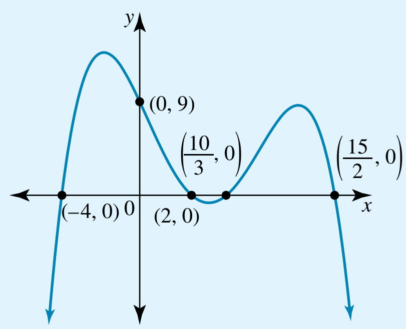
e $\left(1 - \sqrt[4]{\frac{810}{7}}, 0\right), \left(1 + \sqrt[4]{\frac{810}{7}}, 0\right)$

f


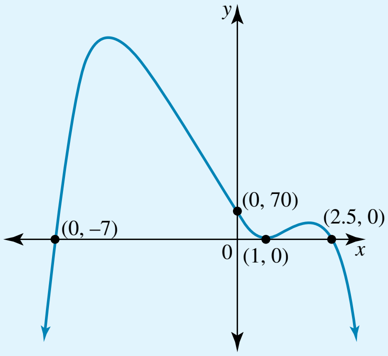

12 a $x$-intercepts $(-8, 0), (-3, 0), (4, 0), (10, 0)$; $y$-intercept $(0, 960)$



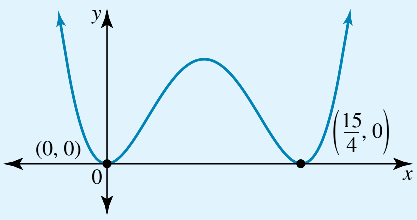

b $x$-intercepts $(-3, 0), (2, 0), \left(\frac{15}{2}, 0\right), \left(\frac{10}{3}, 0\right)$; $y$-intercept $(0, 9)$



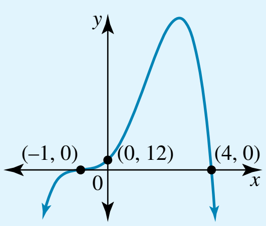
**c** $x$-intercepts $(-7, 0)$, $(2.5, 0)$; turning point $(1, 0)$; $y$-intercept $(0, 70)$



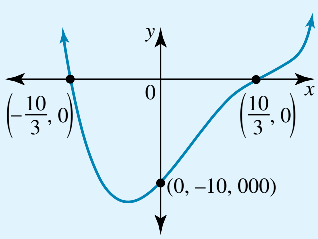

**d** $x$-intercepts and turning points $(0, 0)$, $\left(\frac{15}{4}, 0\right)$




**e** $x$-intercept and stationary point of inflection $(-1, 0)$; $x$-intercept $(4, 0)$; $y$-intercept $(0, 12)$




**f** $x$-intercept and stationary point of inflection $\left(\frac{10}{3}, 0\right)$; $x$-intercept $\left(-\frac{10}{3}, 0\right)$; $y$-intercept $(0, -10\,000)$




**13 a** $y = -\frac{1}{72}(x + 6)(x + 5)(x + 3)(x - 4)$

**b** $y = -x(x - 4)(x + 2)^2$

**c** $y = \frac{2}{3}x^3(x + 6)$

**d** $y = \frac{3}{8}(2x + 3)^2(5x - 4)^2$

**14 a** $x < -3$ or $0.5 < x < 4$ or $x > 5.5$

**b** $x = -\frac{7}{3}, x = 0, x = \frac{7}{3}$

**c** $-\frac{7}{15} \le x \le -\frac{1}{5}$ or $x = 0$

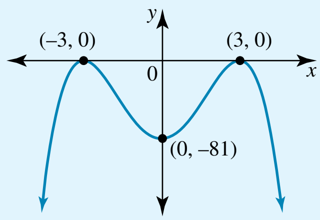
**d** $x = -1, x = -\frac{1}{3}, x = 1, x = 7$

**e** $x = -1, x = 2$

**f** $x \le \frac{3}{10}$ or $x \ge \frac{1}{2}$

**15 a** $-(x - 3)^2(x + 3)^2$

**b**




$x$-intercepts and maximum turning points $(-3, 0)$, $(3, 0)$; $y$-intercept and minimum turning point $(0, -81)$

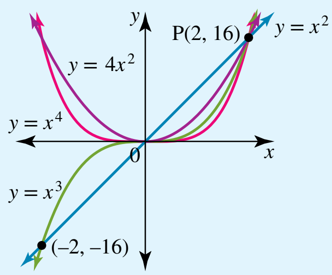
**c** $\{x: -x^4 + 18x^2 - 81 > 0\} = \varnothing$

**d** $R\backslash\{-3, 3\}$

**16 a** $(2, 16)$

**b** $a = 4$, $m = 8$

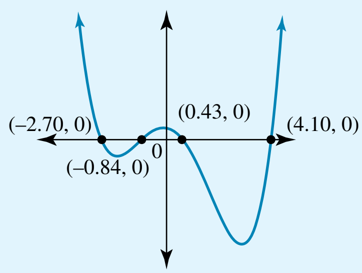
**c**




**d i** $(0, 0)$ and $(n, n^4)$

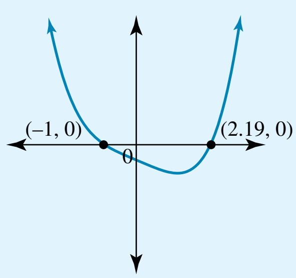
**ii** $a = n^2$, $m = n^3$

**17**




$x$-intercepts are $(-2.70, 0)$, $(-0.84, 0)$, $(0.43, 0)$, $(4.10, 0)$; minimum turning points $(-2, -12)$, $(2.92, -62.19)$; maximum turning point $(-0.17, 4.34)$

**18 a**

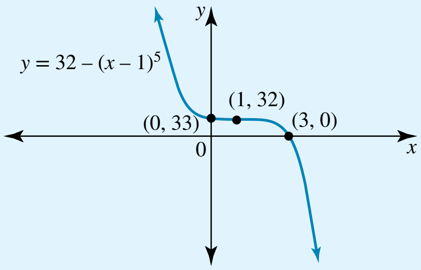



Minimum turning point $(1.21, -14.33)$; $x$-intercepts $(-1, 0)$, $(2.19, 0)$

**b** $x^4 - 7x - 8 = (x + 1)(x^3 - x^2 + x - 8)$

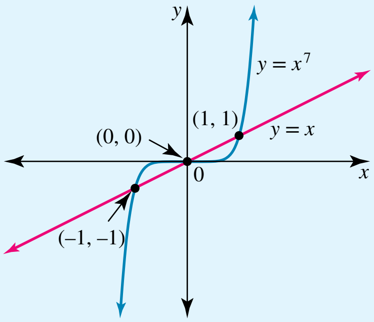
## EXERCISE 5.3

**1**



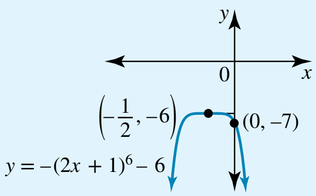

Stationary point of inflection $(1, -32)$; $y$-intercept $(0, 33)$, $x$-intercept $(3, 0)$

**2**




Points of intersection $(-1, -1), (0, 0), (1, 1)$
$\{x: x^7 \leq x\} = \{x: x \leq -1\} \cup \{x: 0 \leq x \leq 1\}$

**3**



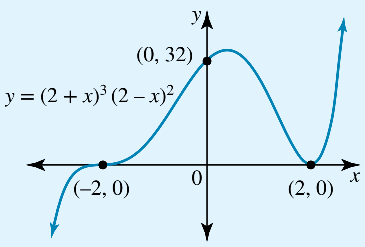

Maximum turning point $\left(-\frac{1}{2}, -6\right)$; $y$-intercept $(0, -7)$; no $x$-intercepts

**4 a** $y = 12(x + 1)^8 - 12$

**b** $x = -1$

**c** $(-2, 0)$

**5**



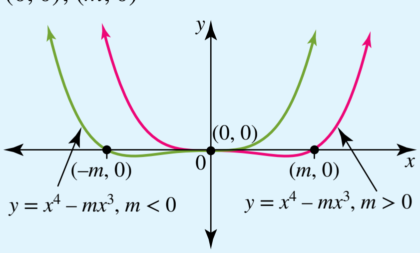

$x$-intercepts: $(-2, 0)$ is a point of inflection and $(2, 0)$ is a turning point; $y$-intercept $(0, 32)$

**6 a** Degree 6

**b** $y = 0.025(x + 4)(x + 1)^2(x - 2)(x - 4)^2$

**7 a** $(0, 0), (m, 0)$

**b**




There is a stationary point of inflection at the origin.

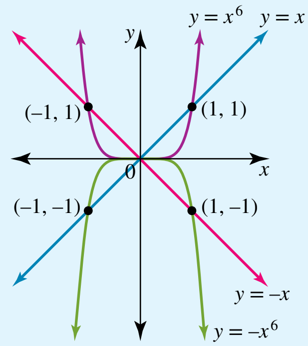
**c** $y = x^4 + 17x^3$

**8 a** Proof required — check with your teacher

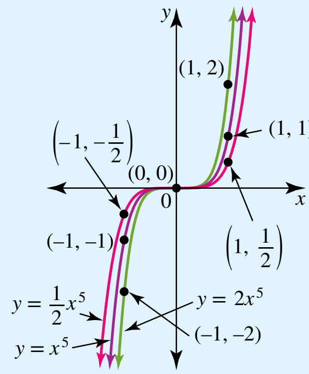
**b** $a = \frac{5}{4}$

**c** $0 < a < \frac{5}{4}$

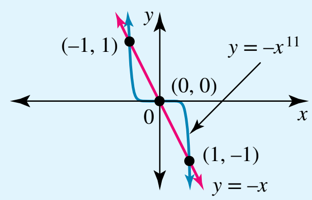
**9 a**




**b**

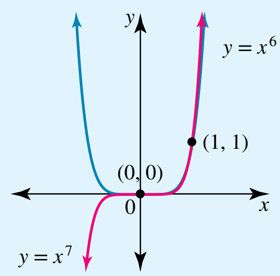



**10 a i**



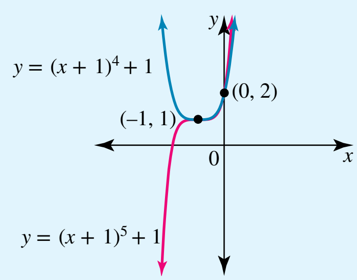

**ii** $\{x: -1 < x < 0\} \cup \{x: x > 1\}$

**b i**




**ii** $x = 0, x = 1$

**c i**




**ii** $(-1, 1), (0, 2)$

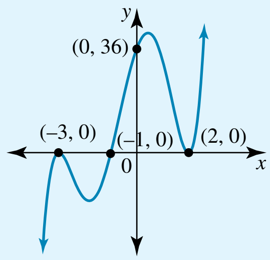
**11 a** i Minimum turning point $(4, 3)$

ii Maximum turning point $\left(\frac{10}{3}, 0\right)$

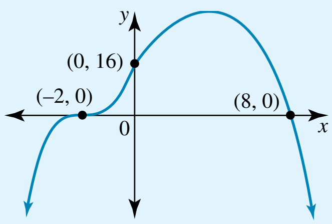
**b** i $\left(0, \frac{1}{2}\right)$

ii $\left(-\frac{1}{2}, 16\right)$

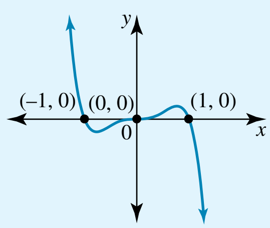
**12 a** $x$-intercept and maximum turning point $(-3, 0)$; $x$-intercept $(-1, 0)$; $x$-intercept and minimum turning point $(2, 0)$; $y$-intercept $(0, 36)$



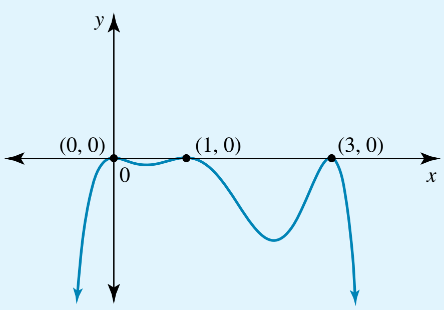

**b** $x$-intercept and stationary point of inflection $(-2, 0)$ and $x$-intercept $(8, 0)$; $y$-intercept $(0, 16)$




**c** $x$-intercepts $(-1, 0)$, $(1, 0)$; stationary point of inflection $(0, 0)$




**d** $(0, 0)$, $(1, 0)$, $(3, 0)$ are all maximum turning points.




**13 a** $y = 28(x + 1)^5 + 7$

**b** $y = (x + 2)^2(x - 4)(x - 3)$, $(3, 0)$

**c** $y = 0.1(x + 5)^2(x + 1)^3(x - 1)(x - 3)$

**d** $y = 16(2x + 3)^4$

**14 a** i $x = -\frac{1}{2}, 0, \frac{1}{2}$

ii $x = 1$

iii $x = -\sqrt{2}, -1, \sqrt{2}$

**b** i $-4 < x < 0$

ii $x \leq \frac{3}{5}$

iii $-1 \leq x \leq 5$

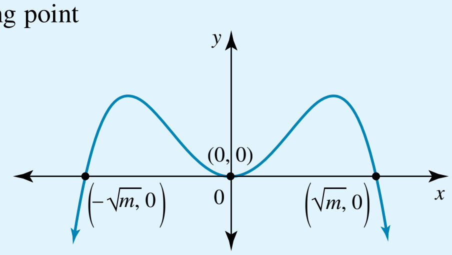
**15 a** i $y = mx - 2m + 3$

ii $y = \frac{3}{2}x$

**b** i The origin

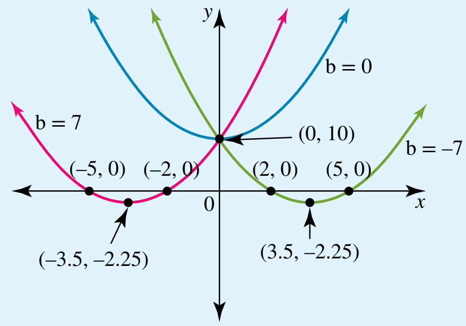
ii $y = ax^2 + (3 - 2a)x$, $a \neq 0$.

**c** $(-\sqrt{m}, 0)$, $(\sqrt{m}, 0)$ and $(0, 0)$ which is also a turning point




**16 a** The set of horizontal lines and the set of concave up parabolas with $y$-intercept $(0, 10)$

**b**




**c** i $k = -2.25$

ii $k > -2.25$

iii $k < -2.25$

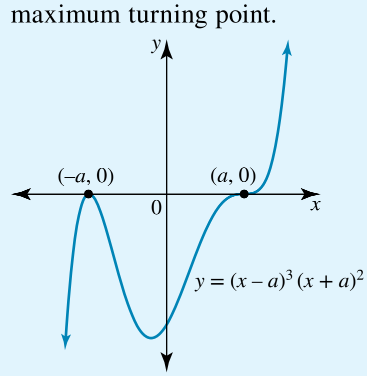
**d** i $b = \pm 2\sqrt{3}$

ii $b < -2\sqrt{3}$ or $b > 2\sqrt{3}$

iii $-2\sqrt{3} < b < 2\sqrt{3}$

**17 a** $(a, 0)$ stationary point of inflection and $(-a, 0)$ maximum turning point.

**b**

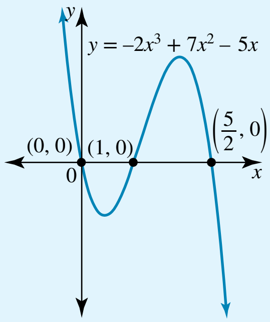



**c** Once

**d** $a = \dfrac{2\sqrt{3}}{3}$

**18 a** Proof required — check with your teacher

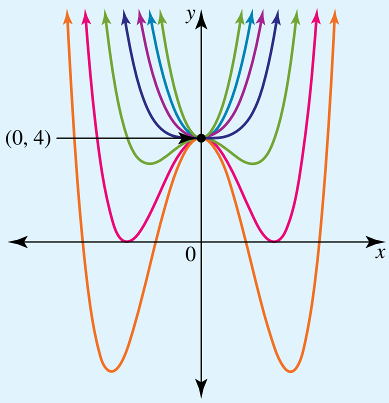
**b** $a = -2$; $y = -2x^3 + 7x^2 - 5x$; $x$-intercepts at origin, $(1, 0)$ and $\left(\frac{5}{2}, 0\right)$




**c** $a = 1$; $y = x^3 + x^2 + 4x - 6$; proof required

**d** Point of intersection $(-1, -8a - 2)$ lies on the vertical line $x = -1$.

**19**

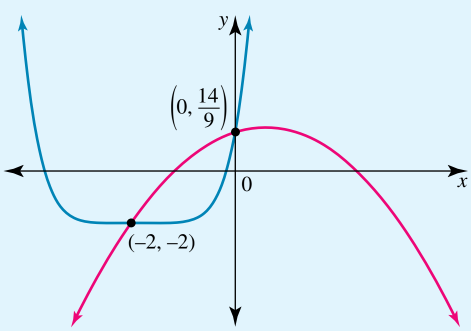



**a** $a < -4$

**b** $a = -4$

**c** $a > -4$

**20** $y = -\frac{4}{9}(x - 1)^2 + 2$




## EXERCISE 5.4

**1 a** $P(1) < 0, P(2) > 0$

**b** $x = 2$

**c** $x = 1.5$; $x = 1.75$

**d** $x = 1.875$

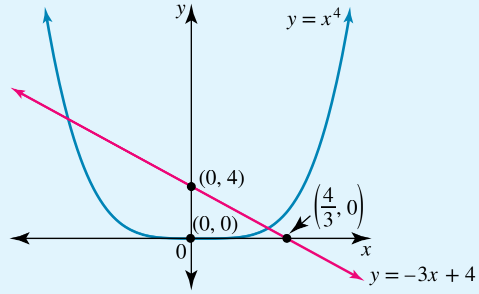
**2 a**

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | $75$ | $8$ | $-9$ | $-12$ | $-13$ | $0$ | $63$ |

**b** $x = 2$

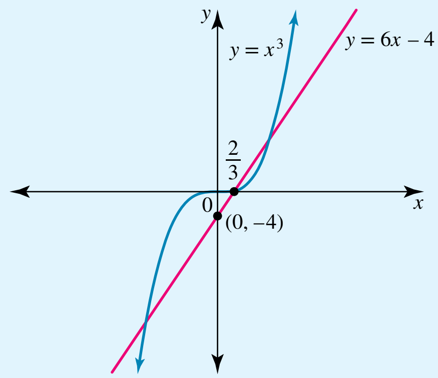
**c** $[-2, -1]$; $x = -1.7$

**3**




$x = -1.75$ (estimate); $x = 1$ (exact)

**4**




Exact solution $x = 2$; approximate solutions $x = -2.7$ and $x = 0.7$

**5 a** $x \in [0, 3]$

**b** $(1.8, 8.208)$

**6 a** $(0, 9)$

**b** $\left(-\frac{5}{2}, 9\right)$ and $(3, 9)$

**c** Between $x = -\frac{5}{2}$ and $x = 0$

**d** $(-1.4, 22.552)$

**7 a** $P(10) = -19, P(12) = 1$

**b** $P(-2) = 3, P(-1) = -3$

**c** $P(-2) = -51, P(1) = 9$

**d** $P(0) = 2, P(1) = -1$

**8 a** $[11, 12]$, $[11.5, 12]$; $x = 11.75$

**b** $[-2, -1.5]$, $[-2, -1.75]$; $x = -1.875$

**c** $[-2, -0.5]$, $[-1.25, -0.5]$; $x = -0.875$

**d** $[0.5, 1]$, $[0.75, 1]$; $x = 0.875$

**9 a** $x = 1.2$

**b** $x = 4$

**c** Method of bisection very slowly converges towards the solution.

**10 a**

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $13$ | $-2$ | $-3$ | $-2$ | $13$ |

**b** $x \in [1, 2]$

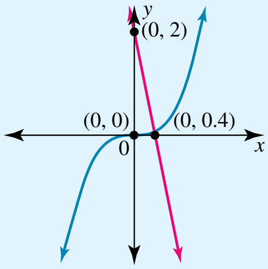
**c** Proof required — check with your teacher

**11 a** $[0, 1]$

**b** $x = 0.375$

**c** $y = x^3$ and $y = -5x + 2$ (or other)

**d**




One root close to $x = 0.375$

**12 a** $x^3 - 3x + 2 = 0$

**b** Two factors, one of multiplicity 2, one of multiplicity 1

**c** $x = -2, x = 1$, A(1, 1), B(−2, −8)

**d** Three solutions

**13 a** (−1.6, 65.664)

**b** (−0.2, −0.552)

**c** Maximum turning points approximately (−0.7, 0.2499) and (0.7, 0.2499); minimum turning point exactly (0, 0)

**14 a** $x \in [-1, 0]$ and $x \in [0, 4]$

**b** $x \in [-2\sqrt{3}, 0]$ and $x \in [0, 2\sqrt{3}]$

**c** $x \in [0, 1]$ and $x \in [1, 4]$

**d** $x \in [-1, 0]$ and at the point (0, 7)

**15 a** $y = 0$ for $x \in [1, 2]$

**b** $x \in [1, 1.5]$; $x \in [1, 1.25]$

**c** 12 containers

**d** $x \in [3, 6]$

**e** 44 containers; $331

**f** 65 or more containers

**16 a** There is at least one root of the equation $P(x) = 0$ that lies between $x_1$ and $x_2$.

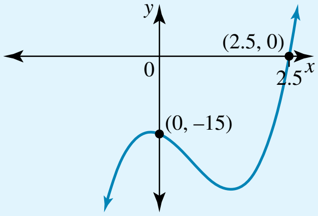
**b i** $P(2)P(3) = -684$

**ii** $x = 2.5$

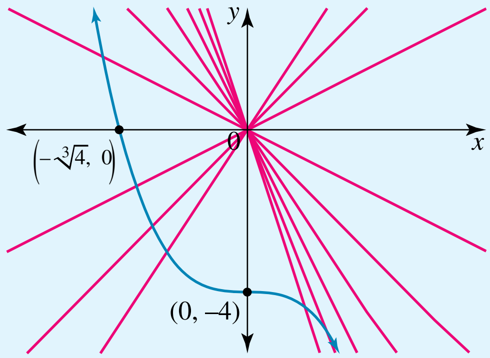
**iii** Proof required — check with your teacher

**iv** 3 points of intersection; both turning points lie below the $x$-axis.

**v**




**17 a**




**b** $a = -5$

**c** One

**d** $x = -1.38$

**18 a** $V = x(18 - 2x)(14 - 2x)$

**b** Between $x = 0$ and $x = 7$

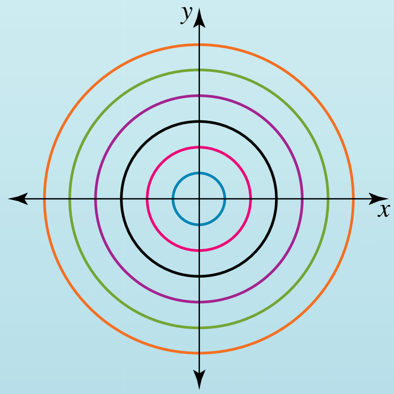
**c** 2.605 cm

# 6

# Functions and relations

# 6.1 Kick off with CAS

## Circles




1 Sketch the following circles using CAS technology.

a $x^2 + y^2 = 1$ b $x^2 + y^2 = 2^2$

c $x^2 + y^2 = 3^2$ d $x^2 + y^2 = 4^2$

e $x^2 + y^2 = 5^2$ f $x^2 + y^2 = 6^2$

2 Using CAS technology, enter $x^2 + y^2 = r^2$ into the equation entry line and use a slider to change the values of $r$.

3 Complete the following sentences.

a When sketching a circle, what is the effect of changing the value of $r$?

b When sketching a circle, can the value of $r$ be negative? Explain your answer.

4 Sketch the following circles using CAS technology.

a $(x - 1)^2 + y^2 = 2^2$ b $(x - 2)^2 + y^2 = 2^2$ c $(x - 3)^2 + y^2 = 2^2$

d $(x + 1)^2 + y^2 = 2^2$ e $(x + 2)^2 + y^2 = 2^2$ f $(x + 3)^2 + y^2 = 2^2$

5 Using CAS technology, enter $(x - h)^2 + y^2 = 2^2$ into the equation entry line and use a slider to change the values of $h$.

What effect does changing the value of $h$ have on the graph of the circle?

6 Sketch the following circles using CAS technology.

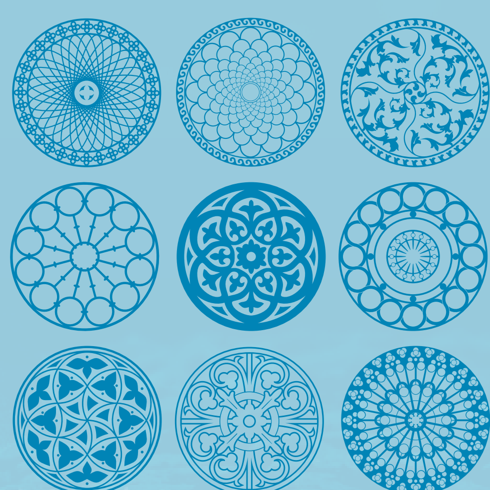
a $x^2 + (y - 1)^2 = 2^2$ b $x^2 + (y - 2)^2 = 2^2$ c $x^2 + (y - 3)^2 = 2^2$

d $x^2 + (y + 1)^2 = 2^2$ e $x^2 + (y + 2)^2 = 2^2$ f $x^2 + (y + 3)^2 = 2^2$

7 Using CAS technology, enter $x^2 + (y - k)^2 = 2^2$ into the equation entry line and use a slider to change the values of $k$.

What effect does changing the value of $k$ have on the graph of the circle?

8 Using CAS technology and your answers to questions 1–7, create a pattern using circles.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 6.2 Functions and relations

studyon

Units 1 & 2
AOS 1
Topic 5
Concept 1

**Functions and relations**
Concept summary
Practice questions

The mathematical language, notation and functionality concepts that will be introduced in this chapter will continue to be used throughout all subsequent chapters. Within this chapter, functionality concepts will be illustrated using the now-familiar polynomial relationships. They will also be applied to some new relationships.

## Relations

A mathematical **relation** is any set of ordered pairs.

The ordered pairs may be listed or described by a rule or presented as a graph. Examples of relations could include $A = \{(-2, 4), (1, 5), (3, 4)\}$ where the ordered pairs have been listed, $B = \{(x, y) : y = 2x\}$ where the ordered pairs are described by a linear equation, and $C = \{(x, y) : y \leq 2x\}$ where the ordered pairs are described by a linear inequation. These relations could be presented visually by being graphed on coordinate axes. The graph of $A$ would consist of three points, the graph of $B$ would be a straight line and the graph of $C$ would be a closed half-plane.

## Domain and range

For a set of ordered pairs $(x, y)$, the **domain** is the set of all the $x$-values of the ordered pairs and the **range** is the set of all the $y$-values of the ordered pairs.

For $A = \{(-2, 4), (1, 5), (3, 4)\}$, the domain is $\{-2, 1, 3\}$ and the range is $\{4, 5\}$. For both $B = \{(x, y) : y = 2x\}$ and $C = \{(x, y) : y \leq 2x\}$, the domain is $R$ and the range is $R$.

The graph of any polynomial relation normally has a domain of $R$. For some practical situations, restrictions have been placed on the values of the variables in some polynomial models. In these cases the polynomial relation has been defined on a **restricted domain**. A restricted domain usually affects the range.

Set notation or interval notation should be used for domains and ranges.

## Functions

A **function** is a set of ordered pairs in which every $x$-value is paired to a unique $y$-value.

The relations $A = \{(-2, 4), (1, 5), (3, 4)\}$ and $B = \{(x, y) : y = 2x\}$ are functions, but not every relation is a function. The relation $C = \{(x, y) : y \leq 2x\}$ is not a function as it contains many points such as (3, 1) and (3, 2) which are in the region defined by $y \leq 2x$ but which have the same $x$-coordinate. No two ordered pairs of a function can have the same $x$-coordinate.

## Vertical line test

Functions can be recognised from their graphs by the **vertical line test**. Any vertical line which cuts the graph of a function will do so exactly once. If the vertical line cuts the graph at more than one place, the graph is not that of a function.

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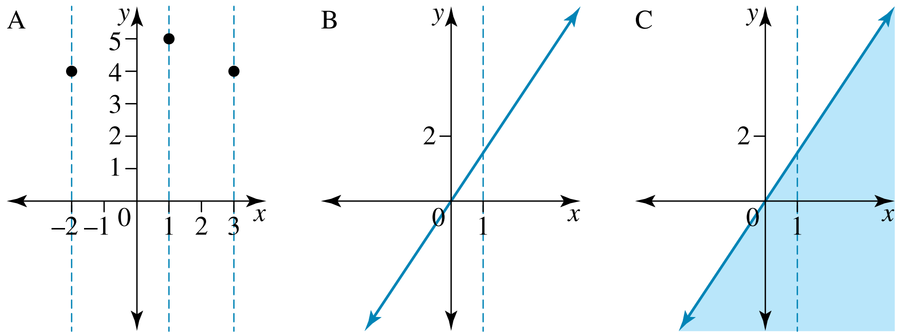
**Interactivity**
Vertical and horizontal line test
**int-2570**




a vertical line cuts once — a vertical line cuts once — a vertical line cuts many times

In the diagram, a vertical line only cuts the graphs of $A$ and $B$ in one place; it cuts the graph of $C$ in many places. $A$ and $B$ are therefore functions; $C$ is not a function.

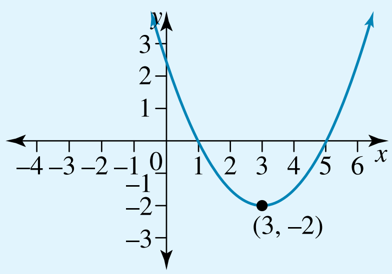
$B$ is a linear polynomial function. The graph of any polynomial can be cut by a vertical line exactly once: every polynomial relation is a function.

**WORKED EXAMPLE 1**

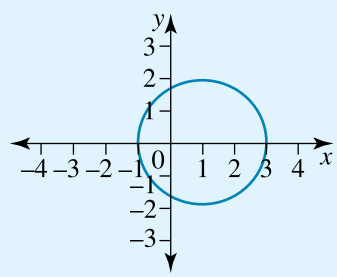
**For each of the following, state the domain and range, and whether the relation is a function or not.**

**a** $\{(1, 4), (2, 0), (2, 3), (5, -1)\}$

**b**




**c**



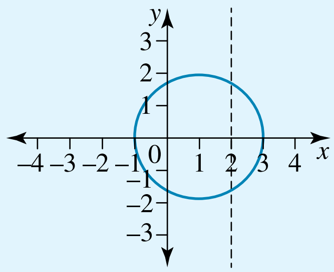
**d** $\{(x, y) : y = 4 - x^3\}$

| THINK | WRITE |
|---|---|
| **a 1** State the domain. | **a** $\{(1, 4), (2, 0), (2, 3), (5, -1)\}$ The domain is the set of $x$-values: $\{1, 2, 5\}$. |
| **2** State the range. | The range is the set of $y$-values: $\{-1, 0, 3, 4\}$. |
| **3** Are there any ordered pairs which have the same $x$-coordinate? | The relation is not a function since there are two different points with the same $x$-coordinate: (2, 0) and (2, 3). |
| **b 1** State the domain. | **b** Reading from left to right horizontally in the direction of the $x$-axis, the graph uses every possible $x$-value. The domain is $(-\infty, \infty)$ or $R$. |
| **2** State the range. | Reading from bottom to top vertically in the direction of the $y$-axis, the graph's $y$-values start at $-2$ and increase from there. The range is $[-2, \infty)$ or $\{y : y \geq -2\}$. |

3 Use the vertical line test. | This is a function since any vertical line cuts the graph exactly once.

**c** 1 State the domain and range. | **c** The domain is $[-1, 3]$; the range is $[-2, 2]$.

2 Use the vertical line test. | This is not a function as a vertical line can cut the graph more than once.




**d** 1 State the domain. | **d** $y = 4 - x^3$
This is the equation of a polynomial so its domain is $R$.

2 State the range. | It is the equation of a cubic polynomial with a negative coefficient of its leading term, so as $x \to \pm\infty, y \to \mp\infty$. The range is $R$.

3 Is the relation a function? | This is a function, because all polynomial relations are functions.

## Type of correspondence

Functions and relations can be classified according to the **correspondence** between the coordinates of their ordered pairs. There are four possible types:

- **one-to-one correspondence** where each $x$-value is paired to a unique $y$-value, such as $\{(1, 2), (3, 4)\}$
- **many-to-one correspondence** where more than one $x$-value may be paired to the same $y$-value, such as $\{(1, 2), (3, 2)\}$
- **one-to-many correspondence** where each $x$-value may be paired to more than one $y$-value, such as $\{(1, 2), (1, 4)\}$
- **many-to-many correspondence** where more than one $x$-value may be paired to more than one $y$-value, such as $\{(1, 2), (1, 4), (3, 4)\}$.

Here 'many' means more than one.

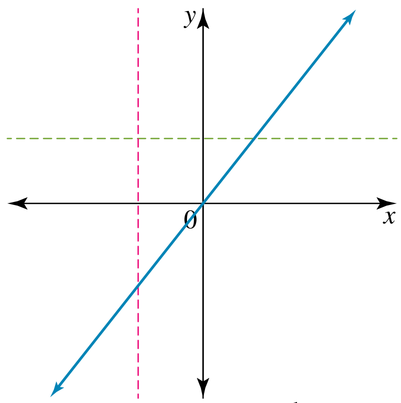
The graph of a function is recognised by the vertical line test and its type of correspondence is determined by the **horizontal line test**.

**A function has a:**

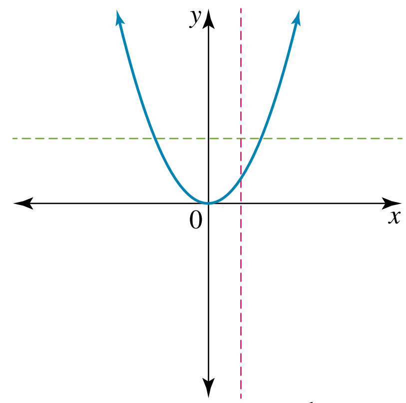
- **one-to-one correspondence if a horizontal line cuts its graph exactly once**
- **many-to-one correspondence if a horizontal line cuts its graph more than once.**

**A relation which is not a function has a:**

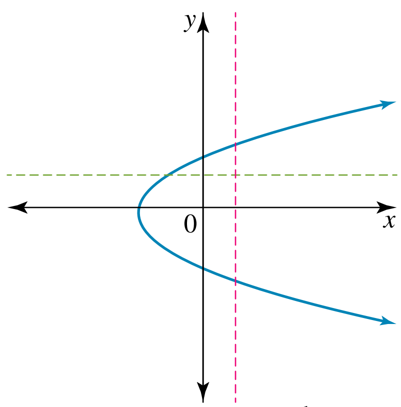
- **one-to-many correspondence if a horizontal line cuts its graph exactly once**
- **many-to-many correspondence if a horizontal line cuts its graph more than once.**



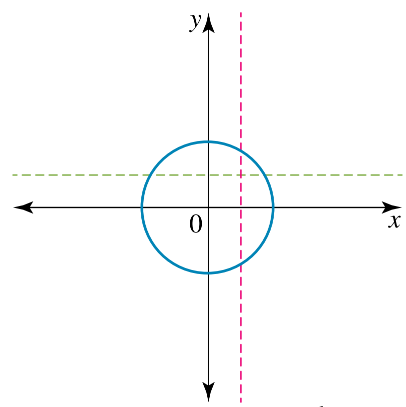

one-to-one correspondence
function




many-to-one correspondence
function




one-to-many correspondence
not a function



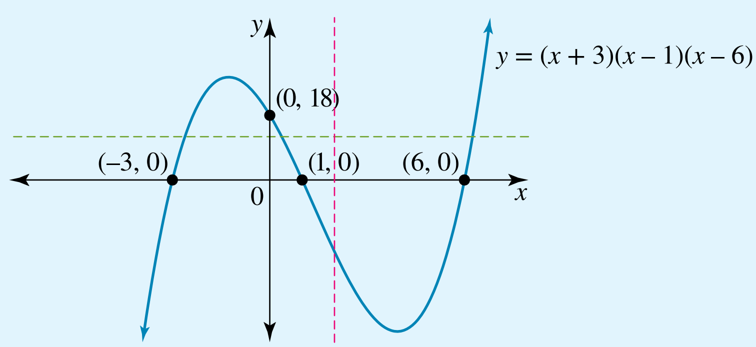

many-to-many correspondence
not a function

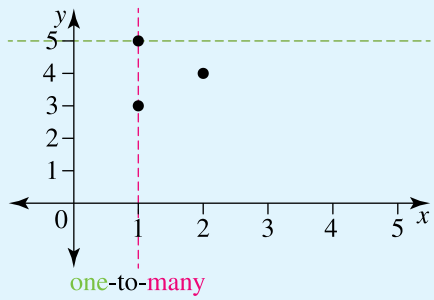
Graphically, the type of correspondence is determined by the number of intersections of a horizontal line (one or many) to the number of intersections of a vertical line (one or many) with the graph. Functions have either a one-to-one or many-to-one correspondence, since their graphs must pass the vertical line test.

**WORKED EXAMPLE 2**

**Identify the type of correspondence and state whether each relation is a function or not.**

**a** $\{(x, y) : y = (x + 3)(x - 1)(x - 6)\}$ **b** $\{(1, 3), (2, 4), (1, 5)\}$

| THINK | WRITE |
| --- | --- |
| **a 1** Draw the graph. | **a** $y = (x + 3)(x - 1)(x - 6)$<br>$x$-intercepts: $(-3, 0), (1, 0), (6, 0)$<br>$y$-intercept: $(0, 18)$<br>The graph is a positive cubic.<br> |

| | |
|---|---|
| **2** Use the horizontal line test and the vertical line test to determine the type of correspondence. | A horizontal line cuts the graph in more than one place. A vertical line cuts the graph exactly once. This is a many-to-one correspondence. |
| **3** State whether the relation is a function. | $y = (x + 3)(x - 1)(x - 6)$ is the equation of a polynomial function with a many-to-one correspondence. |
| **b 1** Look to see if there are points with the same $x$- or $y$-coordinates. | **b** $\{(1, 3), (2, 4), (1, 5)\}$ $x = 1$ is paired to both $y = 3$ and $y = 5$. The relation has a one-to-many correspondence. It is not a function. |
| **2** Alternatively, or as a check, plot the points and use the horizontal and vertical line tests. |  A horizontal line cuts the graph exactly once. A vertical line cuts the graph in more than one place. This is a one-to-many correspondence. |



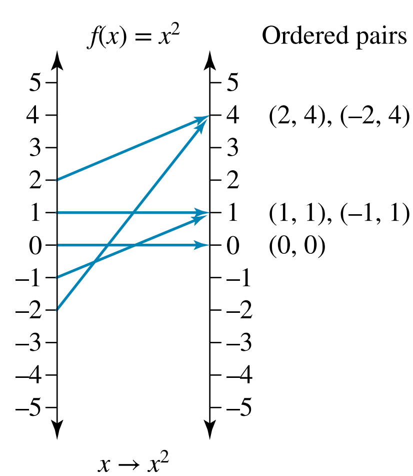
## Function notation

The rule for a function such as $y = x^2$ will often be written as $f(x) = x^2$. This is read as '$f$ of $x$ equals $x^2$'. We shall also refer to a function as $y = f(x)$, particularly when graphing a function as the set of ordered pairs $(x, y)$ with $x$ as the independent or explanatory variable and $y$ as the dependent or response variable.

$f(x)$ is called the **image** of $x$ under the function **mapping**, which means that if, for example, $x = 2$ then $f(2)$ is the $y$-value that $x = 2$ is paired with (mapped to), according to the function rule. We have used $P(x)$ notation for polynomial functions.

For $f(x) = x^2$, $f(2) = 2^2 = 4$. The image of 2 under the mapping $f$ is 4; the ordered pair $(2, 4)$ lies on the graph of $y = f(x)$; 2 is mapped to 4 under $f$: these are all variations of the mathematical language that could be used for this function.

The ordered pairs that form the function with rule $f(x) = x^2$ could be illustrated on a mapping diagram. The mapping diagram shown uses two number lines, one for the $x$-values and one for the $y$-values, but there are varied ways to show mapping diagrams.





Under the mapping, every $x$-value in the domain is mapped to its square, $x \to x^2$. The range is the set of the images, or corresponding $y$-values, of each $x$-value in the domain. For this example, the polynomial function has a domain of $R$ and a range of $[0, \infty)$, since squared numbers are not negative. Not all of the real numbers on the $y$ number line are elements of the range in this example. The set of all the available $y$-values, whether used in the mapping or not, is called the **codomain**. Only the set of those $y$-values which are used for the mapping form the range. For this example, the codomain is $R$ and the range is a subset of the codomain since $[0, \infty) \subset R$.

The mapping diagram also illustrates the many-to-one correspondence of the function defined by $y = x^2$.

### Formal mapping notation

The mapping $x \to x^2$ is written formally as:

| $f$ : | $R$ | $\to$ | $R$, | $f(x) = x^2$ |
|---|---|---|---|---|
| ↓ | ↓ | | ↓ | ↓ |
| name of function | domain of $f$ | | codomain | rule for, or equation of, $f$ |

The domain of the function must always be specified when writing functions formally.

We will always use $R$ as the codomain. Mappings will be written as $f : D \to R$, where $D$ is the domain. Usually a graph of the function is required in order to determine its range.

Note that $f$ is a symbol for the name of the function or mapping, whereas $f(x)$ is an element of the range of the function: $f(x)$ gives the image of $x$ under the mapping $f$. While $f$ is the commonly used symbol for a function, other symbols may be used.

WORKED EXAMPLE 3

**Consider $f : R \to R$, $f(x) = a + bx$, where $f(1) = 4$ and $f(-1) = 6$.**

**a** **Calculate the values of $a$ and $b$ and state the function rule.**

**b** **Evaluate $f(0)$.**

**c** **Calculate the value of $x$ for which $f(x) = 0$.**

**d** **Find the image of $-5$.**

**e** **Write the mapping for a function $g$ which has the same rule as $f$ but a domain restricted to $R^+$.**

| THINK | WRITE |
|---|---|
| **a** 1 Use the given information to set up a system of simultaneous equations. | **a** $f(x) = a + bx$<br>$f(1) = 4 \Rightarrow 4 = a + b \times 1$<br>$\therefore a + b = 4$......... (1)<br>$f(-1) = 6 \Rightarrow 6 = a + b \times -1$<br>$\therefore a - b = 6$........ (2) |
| 2 Solve the system of simultaneous equations to obtain the values of $a$ and $b$. | Equation (1) + equation (2)<br>$2a = 10$<br>$a = 5$<br>Substitute $a = 5$ into equation (1)<br>$\therefore b = -1$ |

3 State the answer. $a = 5,\ b = -1$
$f(x) = 5 - x$

b Substitute the given value of $x$.

b $f(x) = 5 - x$
$f(0) = 5 - 0$
$= 5$

c Substitute the rule for $f(x)$ and solve the equation for $x$.

c $f(x) = 0$
$5 - x = 0$
$\therefore x = 5$

d Write the expression for the image and then evaluate it.

d The image of $-5$ is $f(-5)$.
$f(x) = 5 - x$
$f(-5) = 5 - (-5)$
$= 10$
The image is 10.

e Change the name of the function and change the domain.

e $g : R^+ \to R, g(x) = 5 - x$

## EXERCISE 6.2 Functions and relations

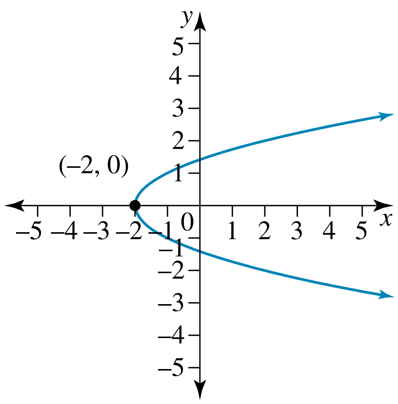
**PRACTISE**

Work without CAS

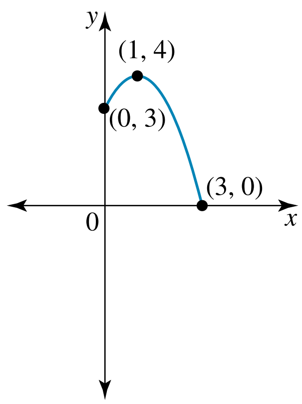
1 WE1 For each of the following, state the domain and range and whether the relation is a function or not.

a $\{(4, 4), (3, 0), (2, 3), (0, -1)\}$

b




c



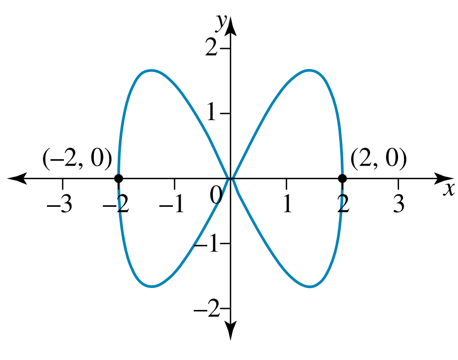

d $\{(x, y) : y = 4 - x^2\}$

2 Sketch the graph of $y = -4x, x \in [-1, 3)$ and state its domain and range.

3 WE2 Identify the type of correspondence and state whether each relation is a function or not:

a $\{(x, y) : y = 8(x + 1)^3 - 1\}$

b



4 a Sketch the graph of $y = (x - 2)^2$, stating its domain, range and type of correspondence.
b Restrict the domain of the function defined by $y = (x - 2)^2$ so that it will be a one-to-one and increasing function.

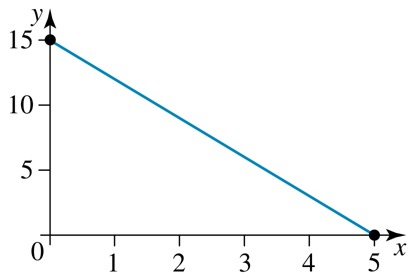
5 WE3 Consider $f: R \to R,\ f(x) = ax + b$, where $f(2) = 7$ and $f(3) = 9$.
a Calculate the values of $a$ and $b$ and state the function rule.
b Evaluate $f(0)$.
c Calculate the value of $x$ for which $f(x) = 0$.
d Find the image of $-3$.
e Write the mapping for a function $g$ which has the same rule as $f$ but a domain restricted to $(-\infty, 0]$.

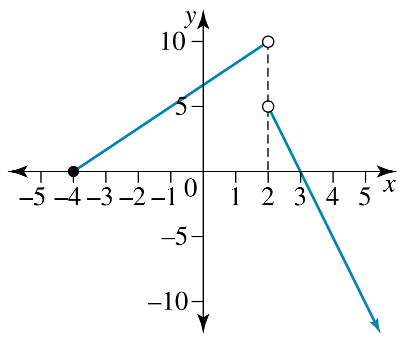
6 Express $y = x^2 - 6x + 10,\ 0 \le x < 7$ in mapping notation and state its domain and range.

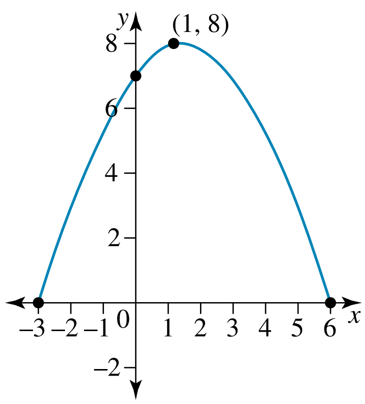
## CONSOLIDATE

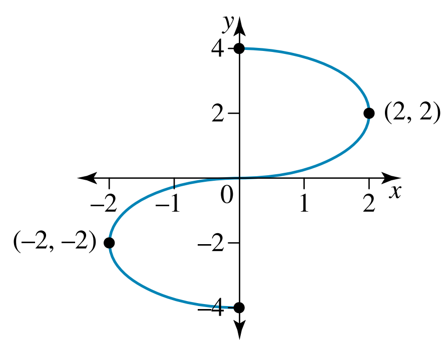
Apply the most appropriate mathematical processes and tools

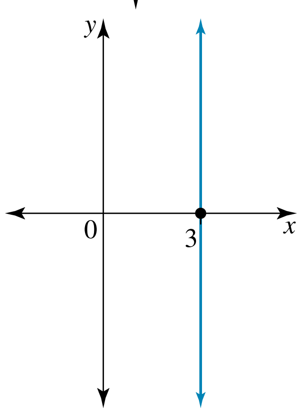
7 State the domain and range for each of the following relations.

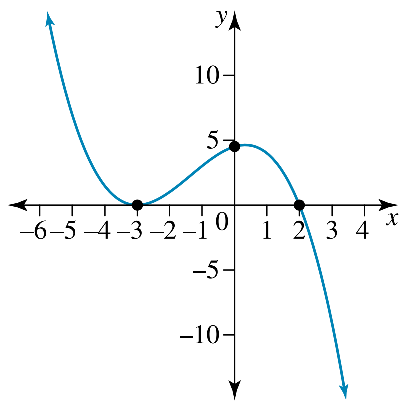
a


b


c



d



e


f


8 Consider each of the graphs in question 7.
a State the type of correspondence of each relation.
b Identify any of the relations which are not functions.

**9** For each of the following relations, state the domain, range, type of correspondence and whether the relation is a function.

**a** $\{(-11, 2), (-3, 8), (-1, 0), (5, 2)\}$

**b** $\{(20, 6), (20, 20), (50, 10), (60, 10)\}$

**c** $\{(-14, -7), (0, 0), (0, 2), (14, 7)\}$

**d** $\{(x, y) : y = 2(x - 16)^3 + 13\}$

**e** $\{(x, y) : y = 4 - (x - 6)^2\}$

**f** $\{(x, y) : y = 3x^2(x - 5)^2\}$

**10** Given $A = \{(1, 2), (2, 3), (3, 2), (k, 3)\}$, state the possible values for $k$ and the type of correspondence, if the following apply.

**a** $A$ is not a function.

**b** $A$ is a function.

**c** Draw a mapping diagram for the relation in part **a** using the chosen $k$ values.

**d** Draw a mapping diagram for the function in part **b** using the chosen $k$ values.

**11** Sketch each of the functions defined by the following rules and state the domain and range of each.

**a** $f(x) = 4x + 2,\ x \in (-1, 1]$

**b** $g(x) = 4x(x + 2),\ x \geq -2$

**c** $h(x) = 4 - x^3,\ x \in R^+$

**d** $y = 4,\ x \in R\backslash[-2, 2]$

**12** If $f(x) = x^2 + 2x - 3$, calculate the following.

**a** **i** $f(-2)$ **ii** $f(9)$

**b** **i** $f(2a)$ **ii** $f(1 - a)$

**c** $f(x + h) - f(x)$

**d** $\{x : f(x) > 0\}$

**e** The values of $x$ for which $f(x) = 12$

**f** The values of $x$ for which $f(x) = 1 - x$

**13** Consider $f : R \to R, f(x) = x^3 - x^2$.

**a** Find the image of 2.

**b** Sketch the graph of $y = f(x)$.

**c** State the domain and range of the function $f$.

**d** What is the type of correspondence?

**e** Give a restricted domain so that $f$ is one-to-one and increasing.

**f** Calculate $\{x : f(x) = 4\}$.

**14** Select the functions from the following list, express them in function mapping notation and state their ranges.

**a** $y = x^2, x \in Z^+$

**b** $2x + 3y = 6$

**c** $y = \pm x$

**d** $\{(x, 5), x \in R\}$

**e** $\{(-1, y), y \in R\}$

**f** $y = -x^5, x \in R^+$

**15** Consider the functions $f$ and $g$ where $f(x) = a + bx + cx^2$ and $g(x) = f(x - 1)$.

**a** Given $f(-2) = 0, f(5) = 0$ and $f(2) = 3$, determine the rule for the function $f$.

**b** Express the rule for $g$ as a polynomial in $x$.

**c** Calculate any values of $x$ for which $f(x) = g(x)$.

**d** On the same axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ and describe the relationship between the two graphs.

**16** **a** A toy car moves along a horizontal straight line so that at time $t$ its position $x$ from a child at a fixed origin is given by the function $x(t) = 4 + 5t$. For the interval $t \in [0, 5]$, state the domain and range of this function and calculate how far the toy car travels in this interval.

b A hat is thrown vertically into the air and at time $t$ seconds its height above the ground is given by the function $h(t) = 10t - 5t^2$. Calculate how long it takes the hat to return to the ground and hence state the domain and range of this function.

c For part of its growth over a two-week period, the length of a leaf at time $t$ weeks is given by the function $l(t) = 0.5 + 0.2t^3, 0 \le t \le 2$.

i State the domain and determine the range of this function.

ii Calculate how long it takes for the leaf to reach half the length that it reaches by the end of the time period.

MASTER

17 Sketch $\{(x, y) : y^2 = x^2 + 1\}$ using CAS technology and hence state the domain and range, and determine if this relation is a function or not.

18 Define $f(x) = x^3 + lx^2 + mx + n$. Given $f(3) = -25, f(5) = 49, f(7) = 243$, answer the following questions.

a Calculate the constants $l$, $m$ and $n$ and hence state the rule for $f(x)$.

b What is the image of 1.2?

c Calculate, correct to 3 decimal places, the value of $x$ such that $f(x) = 20$.

d Given $x \ge 0$, express the function as a mapping and state its range.

# 6.3 The circle

The **circle** is an example of a relation with a many-to-many correspondence. A circle is not a function.

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Units 1 & 2

AOS 1

Topic 5

Concept 2

**The circle**

Concept summary

Practice questions

## Equation of a circle

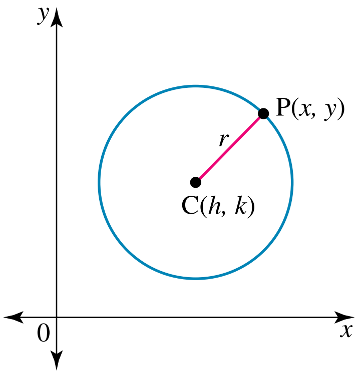
To obtain the equation of a circle, consider a circle of radius $r$ and centre at the point $C(h, k)$.

Let $P(x, y)$ be any point on the circumference. CP, of length $r$, is the radius of the circle.

Using the formula for the distance between two points:

$$\sqrt{(x - h)^2 + (y - k)^2} = CP = r$$

$$(x - h)^2 + (y - k)^2 = r^2$$




**The equation of a circle with centre $(h, k)$ and radius $r$ is:**

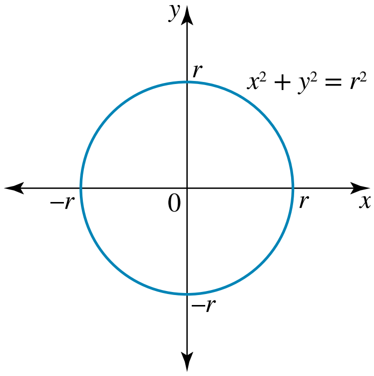
$$(x - h)^2 + (y - k)^2 = r^2$$

The endpoints of the horizontal diameter have coordinates $(h - r, k)$ and $(h + r, k)$; the endpoints of the vertical diameter are $(h, k - r)$ and $(h, k + r)$. These points, together with the centre point, are usually used to sketch the circle. The intercepts with the coordinate axes are not always calculated.

The domain and range are obtained from the endpoints of the horizontal and vertical diameters.

The circle with the centre $(h, k)$ and radius $r$ has domain $[h - r, h + r]$ and range $[k - r, k + r]$.

If the centre is at $(0, 0)$, then the circle has equation $x^2 + y^2 = r^2$, with domain $[-r, r]$ and range $[-r, r]$.




## General form of the equation of a circle

The general form of the equation of a circle is the expanded form of $(x - h)^2 + (y - k)^2 = r^2$. Expanding gives $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$. This is equivalent to $x^2 + y^2 + ax + by + c = 0$ where $a = -2h, b = -2k, c = h^2 + k^2 - r^2$, and shows that three pieces of information are needed to calculate $a$, $b$ and $c$ in order to determine the equation.

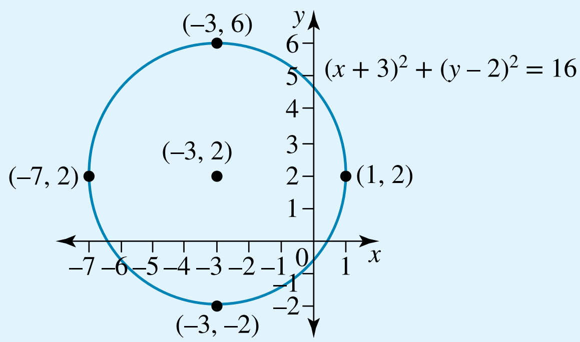
The general form is converted into the standard centre–radius form by completing the square both on the $x$ terms and on the $y$ terms.

**WORKED EXAMPLE 4**

**a** **State the domain and range of the circle with equation $(x + 3)^2 + (y - 2)^2 = 16$ and sketch the graph.**

**b** **Find the centre, radius, domain and range of the circle with equation $2x^2 + 2y^2 + 12x - 4y + 3 = 0$.**

| THINK | WRITE |
|---|---|
| **a** 1 Identify the centre and radius from the equation of the circle. | **a** $(x + 3)^2 + (y - 2)^2 = 16$<br>Centre $(-3, 2)$; radius $\sqrt{16} = 4$ |
| 2 Use the $x$-coordinates of the endpoints of the horizontal diameter to state the domain. | Domain:<br>$x \in [h - r, h + r]$<br>$x \in [-3 - 4, -3 + 4]$<br>Therefore, the domain is $[-7, 1]$. |
| 3 Use the $y$ coordinates of the endpoints of the vertical diameter to state the range. | Range:<br>$y \in [k - r, k + r]$<br>$y \in [2 - 4, 2 + 4]$<br>Therefore, the range is $[-2, 6]$. |
| 4 Sketch the circle using the endpoints of the domain and range, and the centre. | Circle has centre $(-3, 2)$ and contains the points $(-7, 2), (1, 2), (-3, -2), (-3, 6)$. |



**b** **1** Express the equation in the form where the coefficients of $x^2$ and $y^2$ are both 1.

**b** $2x^2 + 2y^2 + 12x - 4y + 3 = 0$

Divide both sides by 2.

$$\therefore x^2 + y^2 + 6x - 2y + \frac{3}{2} = 0$$

**2** Group the terms in $x$ together and the terms in $y$ together, and complete the squares.

$$x^2 + 6x + y^2 - 2y = -\frac{3}{2}$$

$$(x^2 + 6x + 9) - 9 + (y^2 - 2y + 1) - 1 = -\frac{3}{2}$$

$$(x + 3)^2 + (y - 1)^2 = -\frac{3}{2} + 9 + 1$$

$$(x + 3)^2 + (y - 1)^2 = \frac{17}{2}$$

**3** State the centre and radius.

Centre $(-3, 1)$; radius $\sqrt{\frac{17}{2}} = \frac{\sqrt{34}}{2}$

**4** State the domain and range.

Domain $\left[-3 - \frac{\sqrt{34}}{2}, -3 + \frac{\sqrt{34}}{2}\right]$

Range $\left[1 - \frac{\sqrt{34}}{2}, 1 + \frac{\sqrt{34}}{2}\right]$

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**Interactivity**
Graph plotter: Circles, semicircles and regions
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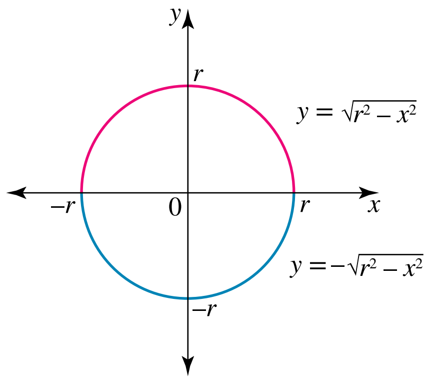
## Semicircles

The equation of the circle $x^2 + y^2 = r^2$ can be rearranged to make $y$ the subject.

$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$



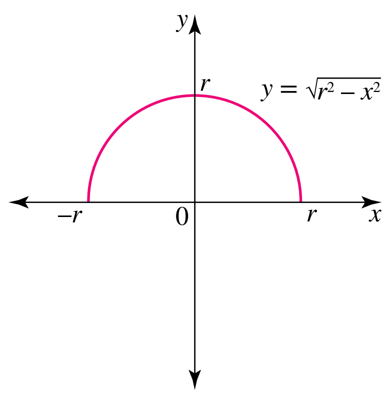

The equation of the circle can be expressed as $y = \pm\sqrt{r^2 - x^2}$. This form of the equation indicates two semicircle functions which together make up the whole circle.

For $y = +\sqrt{r^2 - x^2}$, the $y$-coordinates must be positive (or zero) so this is the equation of the **semicircle** which lies above the $x$-axis.

For $y = -\sqrt{r^2 - x^2}$, the $y$-coordinates must be negative (or zero) so this is the equation of the semicircle which lies below the $x$-axis.

## The semicircle $y = \sqrt{r^2 - x^2}$

The semicircle with equation $y = \sqrt{r^2 - x^2}$ is a function with a many-to-one correspondence. It is the top half of the circle, with centre $(0, 0)$, radius $r$, domain $[-r, r]$ and range $[0, r]$.

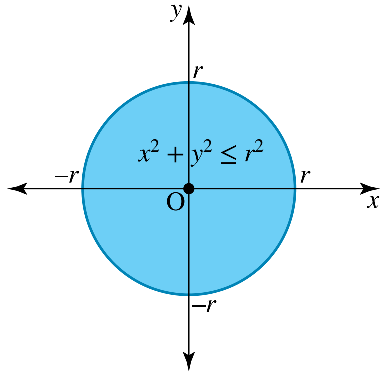



The domain can be deduced algebraically since $\sqrt{r^2 - x^2}$ is only real if $r^2 - x^2 \geq 0$. From this the domain requirement $-r \leq x \leq r$ can be obtained.

For the circle with centre $(h, k)$ and radius $r$, rearranging its equation $(x - h)^2 + (y - k)^2 = r^2$ gives the equation of the top, or upper, semicircle as $y = \sqrt{r^2 - (x - h)^2} + k$.

## Regions

The region defined by $x^2 + y^2 \le r^2$ could be determined by testing if $(0, 0)$ satisfies the inequation. Substitution gives the true statement that $0 \le r^2$. The region is therefore the closed interior of the circle and includes its circumference. The region on or outside the circle is defined by the inequation $x^2 + y^2 \ge r^2$.

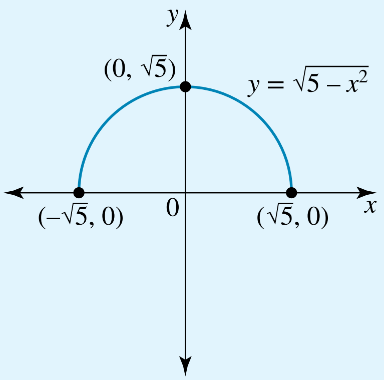



**WORKED EXAMPLE 5**

**a** Sketch the graph of $y = \sqrt{5 - x^2}$ and state the domain and range.

**b** Sketch $\{(x, y) : 4x^2 + 4y^2 < 1\}$.

**c** For the circle with equation $4x^2 + 4y^2 = 1$, give the equation of its lower semicircle and state its domain and range.

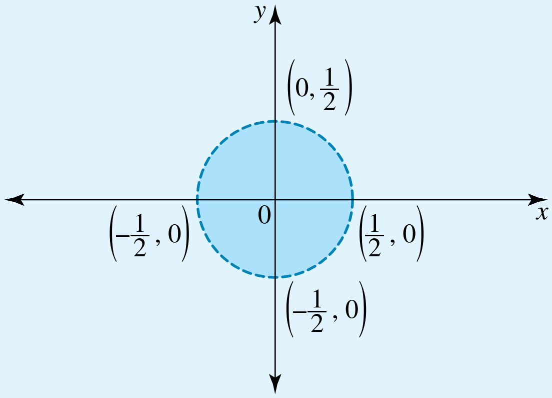
| THINK | WRITE |
|---|---|
| **a 1** State the centre and radius of the circle this semicircle is part of. | **a** $y = \sqrt{5 - x^2}$ is the equation of a semicircle in the form of $y = \sqrt{r^2 - x^2}$.<br>Centre: $(0, 0)$<br>Radius: $r^2 = 5 \Rightarrow r = \sqrt{5}$ since $r$ cannot be negative. |
| **2** Sketch the graph. | This is an upper semicircle.<br> |
| **3** Read from the graph its domain and range. | Domain $[-\sqrt{5}, \sqrt{5}]$; range $[0, \sqrt{5}]$ |
| **b 1** State the equation of the circle which is a boundary of this region. | **b** $4x^2 + 4y^2 < 1$ represents the open region inside the circle $4x^2 + 4y^2 = 1$. |
| **2** Express the equation in the form $x^2 + y^2 = r^2$. | $4x^2 + 4y^2 = 1$<br>$x^2 + y^2 = \frac{1}{4}$ |

**3** State the centre and radius of the circle.

Centre $(0, 0)$; radius $r = \sqrt{\frac{1}{4}} = \frac{1}{2}$

**4** Sketch the circle with an open boundary and shade the required region.

An open region inside the circle but not including the circumference is required.




**c** **1** Rearrange the equation of the circle to make $y$ the subject and state the equation of the lower semicircle.

**c** $4x^2 + 4y^2 = 1$

Rearrange:

$$4y^2 = 1 - 4x^2$$

$$y^2 = \frac{1 - 4x^2}{4}$$

$$y = \pm\sqrt{\frac{1}{4} - x^2}$$

Therefore the lower semicircle has the equation

$$y = -\sqrt{\frac{1}{4} - x^2}.$$

**2** Alternatively, or as a check, use the centre and radius already identified in part **b**.

Check: using part **b**, the centre of the circle has coordinates $(0, 0)$ and $r = \frac{1}{2}$. The lower semicircle has equation $y = -\sqrt{r^2 - x^2}$ with $r = \frac{1}{2}$.

$$\therefore y = -\sqrt{\frac{1}{4} - x^2}$$

or:

$$y = -\sqrt{\frac{1 - 4x^2}{4}}$$

$$= -\frac{\sqrt{1 - 4x^2}}{2}$$

$$= -\frac{1}{2}\sqrt{1 - 4x^2}$$

**3** State the domain and range.

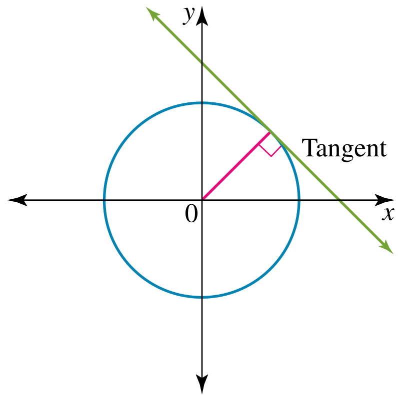
The domain is $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This is the lower semicircle, so the range is $\left[-\frac{1}{2}, 0\right]$.

**eBook*plus***

**Interactivity**
Graph plotter:
Tangents to a circle
**int-2572**

## Tangents to circles

A line and a circle can intersect in 2, 1 or 0 places. The coordinates of any points of intersection are found using simultaneous equations. If there are 2 points of intersection, the line is called a **secant**; a line segment joining these points is called a **chord**.




If there is exactly 1 point of intersection, the line is a **tangent** touching the circle at that point of contact. This tangent line is perpendicular to the radius drawn to the point of contact.

The gradient of the tangent and the gradient of the radius drawn to the point of contact must satisfy the relationship $m_{\text{tangent}} \times m_{\text{radius}} = -1$. Other coordinate geometry formulae may be required to determine the equation of a tangent or to calculate the length of a segment of the tangent.

**WORKED EXAMPLE 6**

**For the circle with equation $(x-1)^2 + y^2 = 5$, determine:**

**a the equation of the tangent at the point (2, 2) on the circumference of the circle**

**b the length of the tangent drawn from the point (−4, −5) to its point of contact with the circle, R.**

**c the number of intersections the line $y + 3x + 4 = 0$ makes with the circle.**

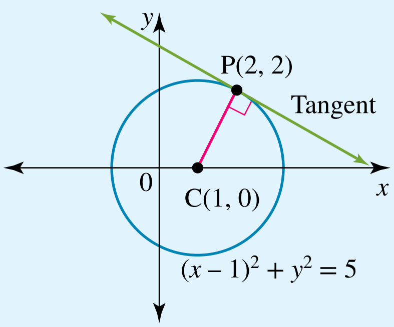
**THINK**

**a 1** State the centre and radius of the circle.

**WRITE**

**a** $(x-1)^2 + y^2 = 5$
Centre (1, 0)
Radius: $r^2 = 5 \Rightarrow r = \sqrt{5}$

**2** Draw a sketch of the circle and the tangent at the given point.




**3** Calculate the gradient of the tangent.

Let C be the centre (1, 0) and P the point (2, 2).
The tangent is perpendicular to the radius CP.

$$m_{CP} = \frac{2-0}{2-1}$$
$$= 2$$

As $m_{\text{tangent}} \times m_{CP} = -1$, $m_{\text{tangent}} = -\frac{1}{2}$.

**4** Form the equation of the tangent line.

Equation of the tangent line:

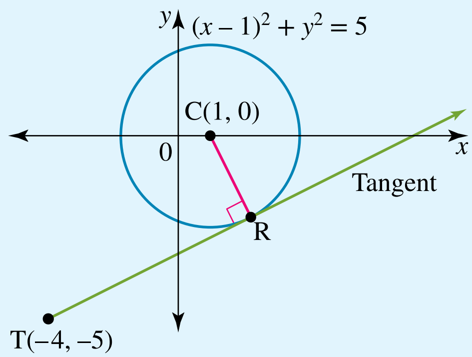
$y - y_1 = m(x - x_1)$, $m = -\frac{1}{2}$, $(x_1, y_1) = (2, 2)$

$$y - 2 = -\frac{1}{2}(x-2)$$
$$y = -\frac{1}{2}x + 3$$

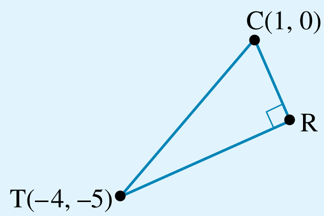
**b** **1** Sketch the circle and a tangent drawn to the circle from the given point.

*Note:* Two tangents can be drawn to the circle from a given point but the required length will be the same in either case.

**b**




Let T be the point $(-4, -5)$ and R the point of contact with the circle. The centre C is the point $(1, 0)$.
The tangent is perpendicular to the radius CR, so the triangle CRT is right-angled.




**2** Calculate the distance between the centre and the given external point.

Using the formula for the distance between two points:

$$\begin{aligned} TC &= \sqrt{(-4-1)^2 + (-5-0)^2} \\ &= \sqrt{25+25} \\ &= \sqrt{50} \end{aligned}$$

**3** Calculate the required length of the tangent.

$r = \sqrt{5} \Rightarrow CR = \sqrt{5}$
Let $TR = t$, $t > 0$
Using Pythagoras' theorem:

$$\begin{aligned} (\sqrt{50})^2 &= (\sqrt{5})^2 + t^2 \\ \therefore 50 &= 5 + t^2 \\ \therefore t^2 &= 45 \\ \therefore t &= 3\sqrt{5} \end{aligned}$$

**4** State the answer.

The length of the tangent from the external point to its point of contact with the circle is $3\sqrt{5}$ units.

**c** **1** Set up a system of simultaneous equations.

**c** $y + 3x + 4 = 0$........(1)
$(x-1)^2 + y^2 = 5$....(2)

**2** Use the substitution method to form the quadratic equation which determines the number of solutions.

From equation (1), $y = -3x - 4$.
Substitute this into equation (2).

$$\begin{aligned} (x-1)^2 + (-3x-4)^2 &= 5 \\ \therefore x^2 - 2x + 1 + 9x^2 + 24x + 16 &= 5 \\ \therefore 10x^2 + 22x + 12 &= 0 \\ \therefore 5x^2 + 11x + 6 &= 0 \end{aligned}$$

| | |
|---|---|
| **3** Calculate the discriminant. | $\Delta = b^2 - 4ac \quad a = 5,\ b = 11,\ c = 6$<br>$= (11)^2 - 4 \times 5 \times 6$<br>$= 121 - 120$<br>$= 1$ |
| **4** State the answer. | Since $\Delta > 0$, there are two points of intersection. |

## EXERCISE 6.3 The circle

**PRACTISE**

Work without CAS

**1** WE4 **a** State the domain and range of the circle with equation $(x - 1)^2 + (y + 3)^2 = 9$ and sketch the graph.

**b** Find the centre, radius, domain and range of the circle with equation $x^2 + y^2 + 2x + 8y = 0$.

**2** A circle with centre $(-5, 0)$ passes through the point $(2, 3)$. Determine its equation and express it in general form.

**3** WE5 **a** Sketch the graph of $y = -\sqrt{2 - x^2}$, stating the domain and range.

**b** Sketch $\{(x, y) : 4x^2 + 4y^2 > 25\}$.

**c** For the circle with equation $4x^2 + 4y^2 = 25$, give the equation of its upper semicircle and state its domain and range.

**4** A semicircle has the equation $y = 2 + \sqrt{8 - 4x - x^2}$.

**a** Identify the equation of the circle of which it is part.

**b** State the domain and range of the semicircle.

**5** WE6 For the circle with equation $(x + 2)^2 + (y - 1)^2 = 10$, determine:

**a** the equation of the tangent at the point $(-3, -2)$ on the circumference of the circle

**b** the length of the tangent drawn from the point $(6, 0)$ to its point of contact with the circle

**c** the number of intersections the line $y + 2x - 5 = 0$ makes with the circle.

**6** Determine the value(s) of $m$ so that $y = mx - 3$ is a tangent to the circle $x^2 + y^2 = 4$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**7** Sketch the following circles and state the centre, radius, domain and range of each.

**a** $x^2 + (y - 1)^2 = 1$

**b** $(x + 2)^2 + (y + 4)^2 = 9$

**c** $16x^2 + 16y^2 = 81$

**d** $x^2 + y^2 - 6x + 2y + 6 = 0$

**e** $16x^2 + 16y^2 - 16x - 16y + 7 = 0$

**f** $(2x + 6)^2 + (6 - 2y)^2 = 4$

**8** Form the equations of the following circles from the given information.

**a** Centre $(-8, 9)$; radius 6

**b** Centre $(7, 0)$; radius $2\sqrt{2}$

**c** Centre $(1, 6)$ and containing the point $(-5, -4)$

**d** Endpoints of a diameter are $\left(-\frac{4}{3}, 2\right)$ and $\left(\frac{4}{3}, 2\right)$.

9 Sketch the following relations, state the domain and range, and express the functions, if any, as mappings.

a $\{(x, y) : y = \sqrt{36 - x^2}\}$

b $\{(x, y) : y = -\sqrt{0.25 - x^2}\}$

c $\{(x, y) : y = \sqrt{1 - x^2} + 3\}$

d $\{(x, y) : x^2 + y^2 < 5\}$

e $\{(x, y) : (x - 2)^2 + (y - 4)^2 \geq 4\}$

f $\{(x, y) : y \leq \sqrt{4 - x^2}\}$

10 a Does $(3, -3)$ lie on, inside or outside the circle $x^2 + y^2 - 3x + 3y + 3 = 0$?

b Find the two values of $a$ so that the point $(a, 2)$ lies on the circle $x^2 + y^2 + 8x - 3y + 2 = 0$, and identify the equation of the semicircle (of the given circle) on which these points lie.

11 a Calculate the coordinates of the points of intersection of the line $y = 2x$ and the circle $(x - 2)^2 + (y - 2)^2 = 1$.

b Calculate the coordinates of the points of intersection of $y = 7 - x$ with the circle $x^2 + y^2 = 49$. On a diagram, sketch the region $\{(x, y) : y \geq 7 - x\} \cap \{(x, y) : x^2 + y^2 \leq 49\}$.

12 a Show that the line $y + 2x = 5$ is a tangent to the circle $x^2 + y^2 = 5$ and calculate the coordinates of the point of contact.

b Determine the value of $k$ so the line $y = kx + 2$ and the circle $x^2 + y^2 + 5x - 4y + 10 = 0$ have:

i one intersection

ii two intersections

iii no intersections.

13 Consider the circle defined by $x^2 + y^2 - 6x + 4y - 12 = 0$.

a Specify its centre and radius.

b i Show that the point $(-1, 1)$ lies on the circle.

ii Find the equation of the tangent drawn to the circle at the point $(-1, 1)$.

c i Show that the point $(3, 3)$ lies on the circle.

ii What is the equation of the tangent at the point $(3, 3)$?

d The tangent drawn from the point T(4, 10) meets the circle at a point R. Calculate the length of TR.

e Deduce the length of the tangent drawn from the point $(8, -7)$ to where it meets the circle.

f Obtain the coordinates of the point of intersection of the tangent in part **b** with the tangent in part **e**.

14 Consider the circle with equation $x^2 + y^2 - 2x - 4y - 20 = 0$.

a Calculate the exact length of the intercept, or chord, cut off on the $x$-axis by the circle.

b Using clearly explained mathematical analysis, calculate the exact distance of the centre of the circle from the chord joining the points $(5, -1)$ and $(4, 6)$.

15 A circle passes through the three points $(1, 0)$, $(0, 2)$ and $(0, 8)$. The general equation of the circle is $x^2 + y^2 + ax + by + c = 0$.

a Calculate the values of $a, b$ and $c$.

b Determine the coordinates of the centre and the length of the radius.

c Sketch the circle labelling all intercepts with the coordinate axes with their coordinates.

d Calculate the shortest distance from the origin to the circle, giving your answer correct to 2 decimal places.

e What is the greatest distance from the origin to the circle? Express the answer correct to 2 decimal places.

16 a Sketch the circle $x^2 + y^2 = 16$ and two other circles that also pass through both of the points $(0, 4)$ and $(4, 0)$. Why is it possible for several circles to pass through these two points?

b Find the equation of the circle for which the points $(0, 4)$ and $(4, 0)$ are the endpoints of a diameter and show that this circle passes through the origin.

c Relative to a fixed origin O on the edge of a circular lake, the circumference of the circular shoreline passes through O and the points 4 km due north of O and 4 km due east of O, so that the equation of its circumference is that of the circle obtained in part b.

Two friends, Sam and Rufus, have a competition to see who can first reach kiosk K which lies at the intersection of the line $y = x$ with the circle.

i Find the coordinates of K.

ii Sam decides to swim from O directly to K along a straight line while Rufus claims it would be faster to walk around the lake from O to K. Given that Sam swims at 2.5 km/h while Rufus walks at 4 km/h, who reaches the kiosk first and by how many minutes?

MASTER

17 Use the conic screen on a CAS calculator, or other technology, to sketch the circle $x^2 + y^2 + 4x - 7y + 2 = 0$, locating the intercepts with the coordinate axes to 3 decimal places and stating the centre and radius.

18 Sketch $(x - 6)^2 + (y + 4)^2 = 66$ and determine the endpoints of the domain and range to 3 decimal places. Why does CAS say the gradient of the tangent is undefined at the endpoints of the domain?

## 6.4 The rectangular hyperbola and the truncus

studyon

Units 1 & 2
AOS 1
Topic 5
Concept 3

**The rectangular hyperbola and the truncus**
Concept summary
Practice questions

The family of functions with rules $y = x^n, n \in N$ are the now familiar polynomial functions. If $n$ is a negative number, however, these functions cannot be polynomials. Here we consider the functions with rule $y = x^n$ where $n = -1$ and $n = -2$.

### The graph of $y = \frac{1}{x}$

With $n = -1$, the rule $y = x^{-1}$ can also be written as $y = \frac{1}{x}$. This is the rule for a **rational function** called a **hyperbola**. Two things can be immediately observed from the rule:

- $x = 0$ must be excluded from the domain, since division by zero is not defined.
- $y = 0$ must be excluded from the range, since there is no number whose reciprocal is zero.

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**Interactivity**
Graph plotter: The hyperbola
**int-2573**

The lines $x = 0$ and $y = 0$ are **asymptotes**. An asymptote is a line the graph will approach but never reach. As these two asymptotes $x = 0$ and $y = 0$ are a pair of perpendicular lines, the hyperbola is known as a **rectangular hyperbola**. The asymptotes are a key feature of the graph of a hyperbola.

Completing a table of values can give us a 'feel' for this graph.

| $x$ | $-10$ | $-4$ | $-2$ | $-1$ | $-\frac{1}{2}$ | $-\frac{1}{4}$ | $-\frac{1}{10}$ | 0 | $\frac{1}{10}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | $-\frac{1}{10}$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-1$ | $-2$ | $-4$ | $-10$ | no value possible | 10 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{10}$ |

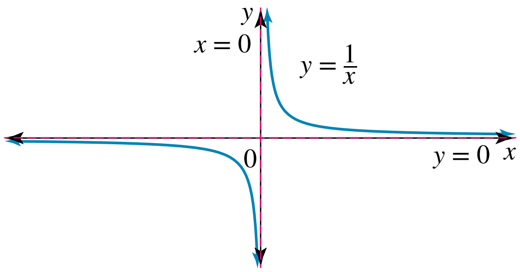
The values in the table illustrate that as $x \to \infty, y \to 0$ and as $x \to -\infty, y \to 0$.

The table also illustrates that as $x \to 0$, either $y \to -\infty$ or $y \to \infty$.

These observations describe the asymptotic behaviour of the graph.

The graph of the basic hyperbola $y = \frac{1}{x}$ is shown.

Key features:




- Vertical asymptote has equation $x = 0$ (the $y$-axis).
- Horizontal asymptote has equation $y = 0$ (the $x$-axis).
- Domain is $R\backslash\{0\}$.
- Range is $R\backslash\{0\}$.
- As $x \to \infty, y \to 0$ from above and as $x \to -\infty, y \to 0$ from below. This can be written as: as $x \to \infty, y \to 0^+$ and as $x \to -\infty, y \to 0^-$.
- As $x \to 0^-, y \to -\infty$ and as $x \to 0^+, y \to \infty$.
- The graph is of a function with a one-to-one correspondence.
- The graph has two branches separated by the asymptotes.
- As the two branches do not join at $x = 0$, the function is said to be **discontinuous** at $x = 0$.
- The graph lies in **quadrants** 1 and 3 as defined by the asymptotes.

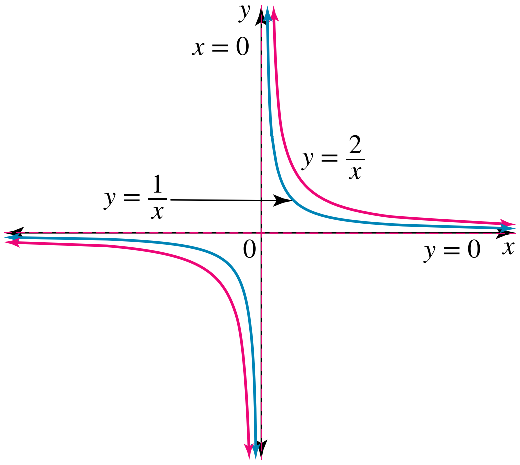
The asymptotes divide the Cartesian plane into four areas or quadrants. The quadrants formed by the asymptotes are numbered 1 to 4 anticlockwise.

With the basic shape of the hyperbola established, transformations of the graph of $y = \frac{1}{x}$ can be studied.

## Dilation from the *x*-axis

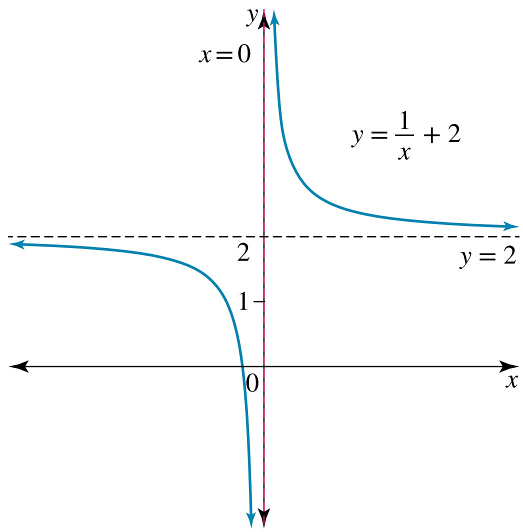
The effect of a dilation factor on the graph can be illustrated by comparing $y = \frac{1}{x}$ and $y = \frac{2}{x}$.

For $x = 1$ the point $(1, 1)$ lies on $y = \frac{1}{x}$ whereas the point $(1, 2)$ lies on $y = \frac{2}{x}$. The dilation effect on $y = \frac{2}{x}$ is to move the graph further out from the $x$-axis. The graph has a dilation factor of 2 from the $x$-axis.



### Vertical translation

The graph of $y = \frac{1}{x} + 2$ illustrates the effect of a vertical translation of 2 units upwards.

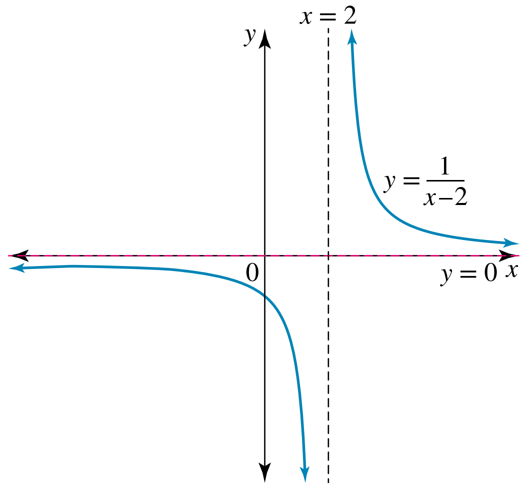



Key features:

- The horizontal asymptote has equation $y = 2$. This means that as $x \to \pm\infty, y \to 2$.
- The vertical asymptote is unaffected and remains $x = 0$.
- Domain is $R \backslash \{0\}$.
- Range is $R \backslash \{2\}$.

### Horizontal translation

For $y = \frac{1}{x - 2}$ as the denominator cannot be zero, $x - 2 \neq 0 \Rightarrow x \neq 2$. The domain must exclude $x = 2$, so the line $x = 2$ is the vertical asymptote.



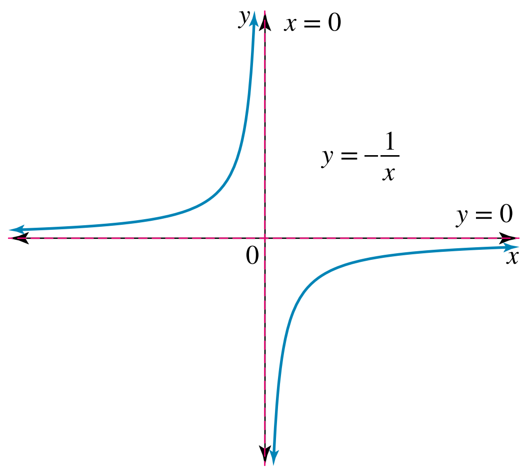

Key features:

- Vertical asymptote has equation $x = 2$.
- Horizontal asymptote is unaffected by the horizontal translation and still has the equation $y = 0$.
- Domain is $R \backslash \{2\}$.
- Range is $R \backslash \{0\}$.

The graph of $y = \frac{1}{x - 2}$ demonstrates the same effect that we have seen with other graphs that are translated 2 units to the right.

### Reflection in the $x$-axis

The graph of $y = -\frac{1}{x}$ illustrates the effect of inverting the graph by reflecting $y = \frac{1}{x}$ in the $x$-axis.




The graph of $y = -\frac{1}{x}$ lies in quadrants 2 and 4 as defined by the asymptotes.

## General equation of a hyperbola

The equation $y = \frac{a}{x-h} + k$ is that of a hyperbola with the following key features.

- Vertical asymptote has the equation $x = h$.
- Horizontal asymptote has the equation $y = k$.
- Domain is $R\backslash\{h\}$.
- Range is $R\backslash\{k\}$.
- Asymptotic behaviour: as $x \to \pm\infty, y \to k$ and as $x \to h, y \to \pm\infty$.
- There are two branches to the graph and the graph is discontinuous at $x = h$.
- If $a > 0$ the graph lies in the asymptote-formed quadrants 1 and 3.
- If $a < 0$ the graph lies in the asymptote-formed quadrants 2 and 4.
- $|a|$ gives the dilation factor from the $x$-axis.

If the equation of the hyperbola is in the form $y = \frac{a}{bx+c} + k$, then the vertical asymptote can be identified by finding the $x$-value for which the denominator term $bx + c = 0$. The horizontal asymptote is $y = k$ because as $x \to \pm\infty$, $\frac{a}{bx+c} \to 0$ and therefore $y \to k$.

**WORKED EXAMPLE 7**

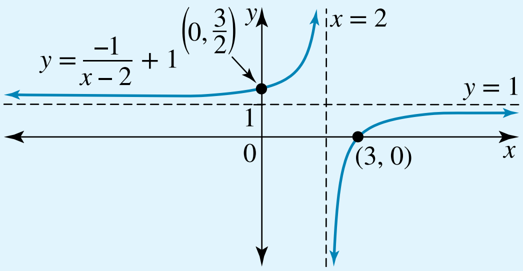
**Sketch the graphs of the following functions, stating the domain and range.**

**a** $y = \frac{-1}{x-2} + 1$ **b** $y = \frac{4}{x} - 2$

| THINK | WRITE |
|---|---|
| **a 1** State the equations of the asymptotes. | **a** $y = \frac{-1}{x-2} + 1$<br>Vertical asymptote occurs when $x - 2 = 0$.<br>Vertical asymptote: $x = 2$<br>Horizontal asymptote occurs as $x \to \pm\infty$.<br>Horizontal asymptote: $y = 1$ |
| **2** Calculate the coordinates of any axis intercepts. | $y$-intercept: let $x = 0$<br>$y = \frac{-1}{-2} + 1$<br>$= \frac{3}{2}$<br>$\left(0, \frac{3}{2}\right)$ is the $y$-intercept.<br>$x$-intercept: let $y = 0$<br>$\frac{-1}{x-2} + 1 = 0$<br>$\therefore \frac{1}{x-2} = 1$<br>$\therefore x - 2 = 1$<br>$\therefore x = 3$<br>$(3, 0)$ is the $x$-intercept. |

**3** Sketch the graph ensuring it approaches but does not touch the asymptotes.

*Note:* Check the graph has the shape anticipated.




Since $a < 0$ the graph lies in the asymptote-formed quadrants 2 and 4, as expected.

**4** State the domain and range.

The domain is $R\backslash\{2\}$; the range is $R\backslash\{1\}$.

**b** **1** State the equations of the asymptotes.

**b** $y = \frac{4}{x} - 2$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

**2** Calculate the coordinates of any axes intercepts.

No $y$-intercept since the $y$-axis is the vertical asymptote

$x$-intercept: let $y = 0$

$$0 = \frac{4}{x} - 2$$

$$\therefore \frac{4}{x} = 2$$

$$\therefore 4 = 2x$$

$$\therefore x = 2$$

$(2, 0)$ is the $x$-intercept.

**3** Anticipating the position of the graph, calculate the coordinates of a point so each branch of the graph will have a known point.

Graph lies in asymptote-formed quadrants 1 and 3. A point to the left of the vertical asymptote is required.

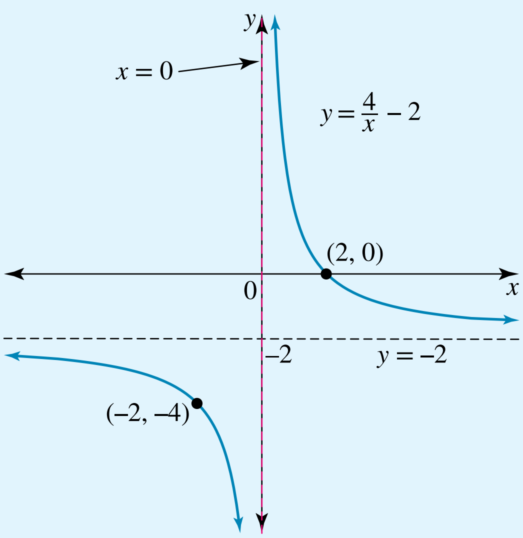
Point: let $x = -2$

$$y = \frac{4}{-2} - 2$$

$$= -4$$

$(-2, -4)$ is a point.

**4** Sketch the graph.




**5** State the domain and range.

Domain $R\backslash\{0\}$; range $R\backslash\{-2\}$

## Proper rational functions

The equation of the hyperbola $y = \frac{a}{x - h} + k$ is expressed in **proper rational function** form. This means the rational term, $\frac{a}{x - h}$, has a denominator of a higher degree than that of the numerator. For example, $y = \frac{x - 1}{x - 2}$ is not expressed as a proper rational function: the numerator has the same degree as the denominator. Using division, it can be converted to the proper form $y = \frac{1}{x - 2} + 1$ which is recognisable as a hyperbola and from which the asymptotes can be obtained.

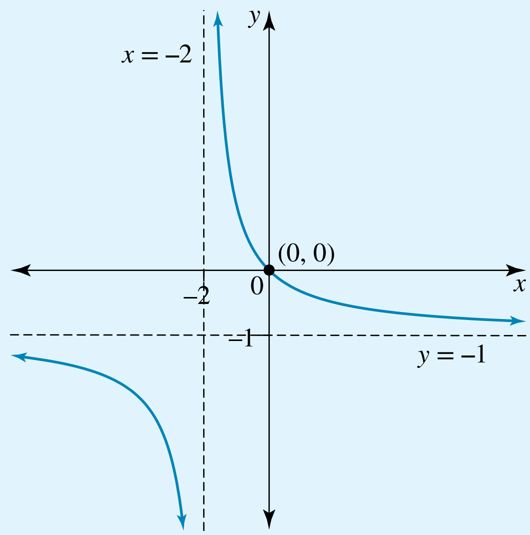
## Forming the equation

From the equation $y = \frac{a}{x - h} + k$ it can be seen that three pieces of information will be needed to form the equation of a hyperbola. These are usually the equations of the asymptotes and the coordinates of a point on the graph.

WORKED EXAMPLE 8

**a** **Identify the asymptotes of the hyperbola with equation $y = \frac{2x - 3}{5 - 2x}$.**

**b** **Form the equation of the hyperbola shown.**




| THINK | WRITE |
|---|---|
| **a** 1 The equation is in improper form so reduce it to proper form using division.<br>*Note:* The long-division algorithm could also have been used to reduce the function to proper form. | **a** $y = \frac{2x - 3}{5 - 2x}$<br>$= \frac{-1(5 - 2x) + 2}{5 - 2x}$<br>$= -1 + \frac{2}{5 - 2x}$<br>$y = \frac{2}{5 - 2x} - 1$ is the proper rational function form. |
| 2 State the equations of the asymptotes. | Vertical asymptote when $5 - 2x = 0$<br>$\therefore x = \frac{5}{2}$<br>Horizontal asymptote is $y = -1$. |

**b** **1** Substitute the equations of the asymptotes shown on the graph into the general equation of a hyperbola.

**b** Let equation of the graph be $y = \dfrac{a}{x - h} + k$.

From the graph, asymptotes have equations $x = -2, y = -1$

$\therefore y = \dfrac{a}{x + 2} - 1$

**2** Use a known point on the graph to determine the remaining unknown constant.

Point (0, 0) lies on the graph.

$0 = \dfrac{a}{2} - 1$

$\therefore 1 = \dfrac{a}{2}$

$\therefore a = 2$

**3** State the equation of the hyperbola.

The equation is $y = \dfrac{2}{x + 2} - 1$.

## Inverse proportion

The hyperbola is also known as the **inverse proportion** graph. To illustrate this, consider the time taken to travel a fixed distance of 60 km.

The time to travel a fixed distance depends on the speed of travel. For a distance of 60 km, the times taken for some different speeds are shown in the table.

| **Speed, *v* (km/h)** | 10 | 15 | 20 | 30 |
|---|---|---|---|---|
| **Time, *t* (hours)** | 6 | 4 | 3 | 2 |

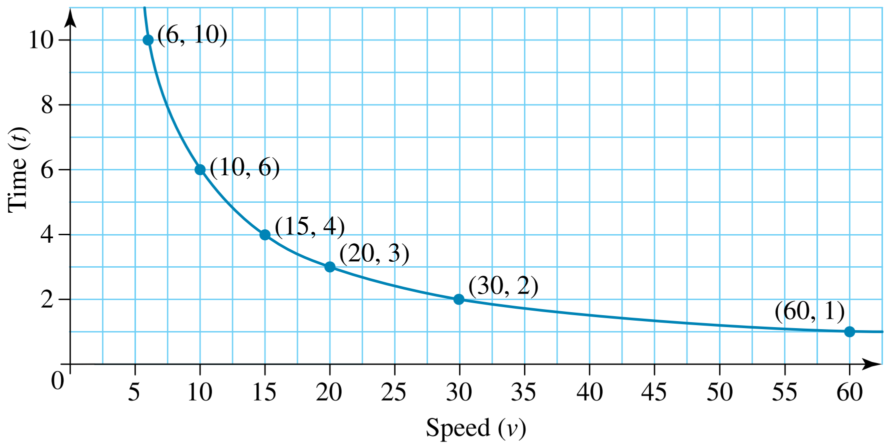
As the speed increases, the time will decrease; as the speed decreases, the time will increase. The time is inversely proportional to the speed, or the time varies inversely as the speed.

From the table:

$$t \times v = 60$$

$$\therefore t = \frac{60}{v}$$

This is the equation of a hyperbola where $v$ is the independent variable and $t$ the dependent variable.



The graph of time versus speed exhibits the asymptotic behaviour typical of a hyperbola. As the speed increases, becoming faster and faster, the time decreases, becoming smaller and smaller. In other words, as $v \to \infty$, $t \to 0$, but $t$ can never reach zero nor can speed reach infinity. Further, as $v \to 0$, $t \to \infty$. Only one branch of the hyperbola is given since neither time nor speed can be negative.

**In general, the following rules apply.**

- **'$y$ is inversely proportional to $x$' is written as $y \propto \frac{1}{x}$.**
- **If $y$ is inversely proportional to $x$, then $y = \frac{k}{x}$ where $k$ is the constant of proportionality.**
- **This relationship can also be expressed as $xy = k$ so if the product of two variables is constant, the variables are in inverse proportion.**

If $y = \frac{k}{x}$, then it could also be said that $y$ is directly proportional to $\frac{1}{x}$; the graph of $y$ against $\frac{1}{x}$ is linear. The graph of $y$ against $x$ is a hyperbola.

Functions of variables may be in inverse proportion. For example, the strength of a radio signal $I$ varies inversely as the square of the distance $d$ from the transmitter, so $I = \frac{k}{d^2}$. The graph of $I$ against $d$ is not a hyperbola; it is called a **truncus** and is from the family $y = x^n, n = -2$.

**WORKED EXAMPLE 9**

**Boyle's Law says that if the temperature of a given mass of gas remains constant, its volume $V$ is inversely proportional to the pressure $P$.**

**A container of volume 100 $cm^3$ is filled with a gas under a pressure of 75 cm of mercury.**

**a** **Find the relationship between the volume and pressure.**

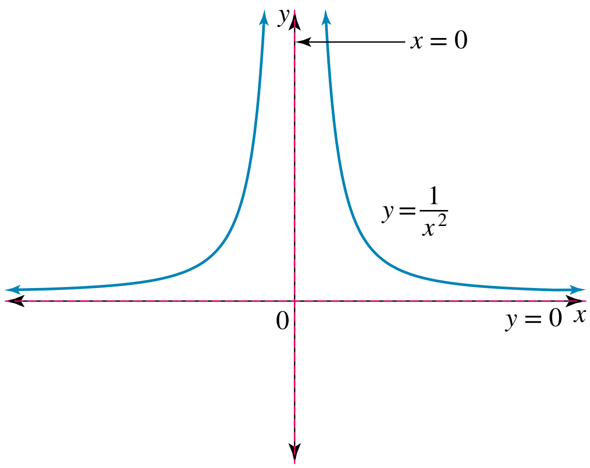
**b** **The container is connected by a hose to an empty container of volume 50 $cm^3$. Find the pressure in the two containers.**

| THINK | WRITE |
|---|---|
| **a** **1** Write the rule for the inverse proportion relation. | **a** $V \propto \frac{1}{P}$ <br> $\therefore V = \frac{k}{P}$ |
| **2** Use the given data to find $k$ and hence the rule. | Substitute $V = 50,\ P = 75$. <br> $50 = \frac{k}{75}$ <br> $k = 50 \times 75$ <br> $= 3750$ <br> Hence $V = \frac{3750}{P}$. |

| | |
|---|---|
| **b** **1** State the total volume. | **b** The two containers are connected and can be thought of as one. Therefore, the combined volume is $100 + 50 = 150$ cm$^3$. |
| **2** Calculate the pressure for this volume. | $V = \frac{3750}{P}$<br>When $V = 150$,<br>$150 = \frac{3750}{P}$<br>$P = \frac{3750}{150}$<br>$= 25$ |
| **3** State the answer. | The gas in the containers is under a pressure of 25 cm of mercury. |

## The graph of the truncus $y = \frac{1}{x^2}$

With $n = -2$, the rule $y = x^{-2}$ is also written as $y = \frac{1}{x^2}$. The rational function with this rule is called a truncus. Its graph shares similarities with the graph of the hyperbola $y = \frac{1}{x}$ as the rule shows that $x = 0$ and $y = 0$ are vertical and horizontal asymptotes, respectively.

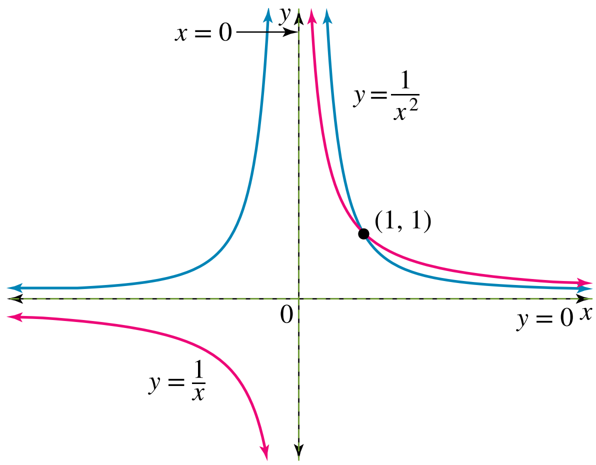



A major difference between the two curves is that for a truncus, whether $x < 0$ or $x > 0$, the $y$-values must be positive. This means the graph of the truncus $y = \frac{1}{x^2}$ will have two branches, one in quadrant 1 and the other in quadrant 2. The quadrants are formed by the asymptote positions.

Key features:

- Vertical asymptote has equation $x = 0$ (the $y$-axis).
- Horizontal asymptote has equation $y = 0$ (the $x$-axis).
- Domain is $R \backslash \{0\}$.
- Range is $R^+$.
- As $x \to \pm\infty, y \to 0^+$.
- As $x \to 0, y \to \infty$.
- The graph is of a function with a many-to-one correspondence.
- The graph has two symmetric branches separated by the asymptotes.
- The graph lies in quadrants 1 and 2 as defined by the asymptotes.
- The function is discontinuous at $x = 0$.

The branches of the truncus $y = \frac{1}{x^2}$ approach the horizontal asymptote more rapidly than those of the hyperbola $y = \frac{1}{x}$ because $\frac{1}{x^2} < \frac{1}{x}$, for $x > 1$. This can be seen when the graphs of the hyperbola and truncus are drawn on the same diagram.




Under a dilation of factor $a$ and translations of $h$ units horizontally and $k$ units vertically, $y = \frac{1}{x^2}$ becomes $y = \frac{a}{(x-h)^2} + k$.

From this general form, $y = \frac{a}{(x-h)^2} + k$, the key features can be identified:

- There is a vertical asymptote with the equation $x = h$.
- There is a horizontal asymptote with the equation $y = k$.
- If $a > 0$ the graph lies in the asymptote-formed quadrants 1 and 2.
- If $a < 0$ the graph lies in the asymptote-formed quadrants 3 and 4.
- Domain is $R \backslash \{h\}$.
- If $a > 0$, range is $(k, \infty)$; if $a < 0$, range is $(-\infty, k)$.

**WORKED EXAMPLE 10**

**Sketch the graphs of the following functions, stating the domain and range.**

**a** $y = \frac{1}{(x+1)^2} + 2$

**b** $y = 1 - \frac{4}{(x-2)^2}$

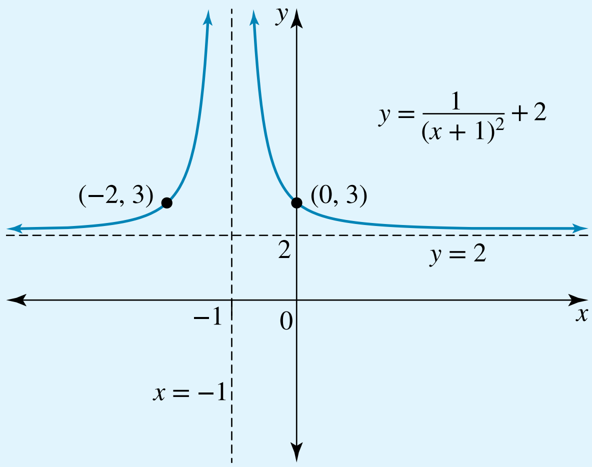
| THINK | WRITE |
|---|---|
| **a 1** State the equations of the asymptotes. | **a** $y = \frac{1}{(x+1)^2} + 2$<br>Vertical asymptote occurs when $(x+1)^2 = 0$.<br>$\therefore x + 1 = 0$<br>$\therefore x = -1$<br>Vertical asymptote: $x = -1$<br>Horizontal asymptote: $y = 2$ |
| **2** Calculate the coordinates of any axis intercepts. | $y$-intercept: let $x = 0$<br>$y = \frac{1}{(1)^2} + 2$<br>$= 3$<br>$(0, 3)$ is the $y$-intercept.<br>Since $a > 0$ the graph lies above the horizontal asymptote, $y = 2$.<br>Hence, there is no $x$-intercept. |

**3** Sketch the graph ensuring that each branch has at least one known point.

Let $x = -2$.

$$y = \frac{1}{(-1)^2} + 2$$
$$= 3$$

$(-2, 3)$ lies on the graph.




**4** State the domain and range.

Domain is $R\backslash\{-1\}$; range is $(2, \infty)$.

**b 1** State the equations of the asymptotes.

**b** $y = 1 - \frac{4}{(x-2)^2}$

$$= \frac{-4}{(x-2)^2} + 1$$

Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 1$

**2** Calculate the coordinates of any axis intercepts.

*Note:* The symmetry of the graph about its vertical asymptote may enable key points to be identified.

$y$-intercept: let $x = 0$

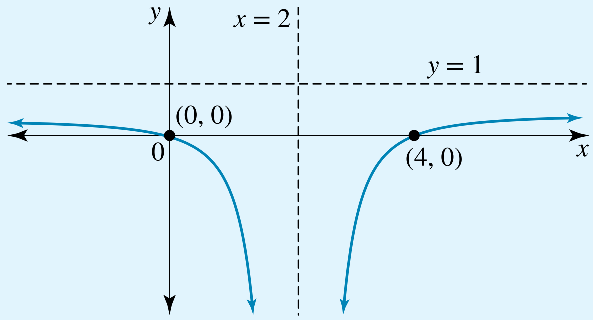
$$y = \frac{-4}{(-2)^2} + 1$$
$$= 0$$

The graph passes through the origin $(0, 0)$.
$x$-intercepts: let $y = 0$

$$\frac{-4}{(x-2)^2} + 1 = 0$$
$$\frac{4}{(x-2)^2} = 1$$
$$(x-2)^2 = 4$$
$$x - 2 = \pm 2$$
$$x = 0,\ x = 4$$

$(4, 0)$ is also an $x$-intercept.

3 Sketch the graph ensuring that each branch has at least one known point.




4 State the domain and range.

Domain $R\backslash\{2\}$; range $(-\infty, 1)$

## Modelling with the hyperbola and truncus

As for inverse proportionality, practical applications involving hyperbola or truncus models may need domain restrictions. Unlike many polynomial models, neither the hyperbola nor the truncus has maximum and minimum turning points, so the asymptotes are often where the interest will lie. The horizontal asymptote is often of particular interest as it represents the limiting value of the model.

**WORKED EXAMPLE 11**

**A relocation plan to reduce the number of bats in a public garden is formed and $t$ months after the plan is introduced the number of bats $N$ in the garden is thought to be modelled by $N = 250 + \dfrac{30}{t+1}$.**

**a How many bats were removed from the garden in the first 9 months of the relocation plan?**

**b Sketch the graph of the bat population over time using the given model and state its domain and range.**

**c What is the maximum number of bats that will be relocated according to this model?**

**THINK**

a Find the number of bats at the start of the plan and the number after 9 months and calculate the difference.

**WRITE**

a $N = 250 + \dfrac{30}{t+1}$

When $t = 0$, $N = 250 + \dfrac{30}{1}$.

Therefore there were 280 bats when the plan was introduced.

When $t = 9$, $N = 250 + \dfrac{30}{10}$.

Therefore 9 months later there were 253 bats.
Hence, over the first 9 months, 27 bats were removed.

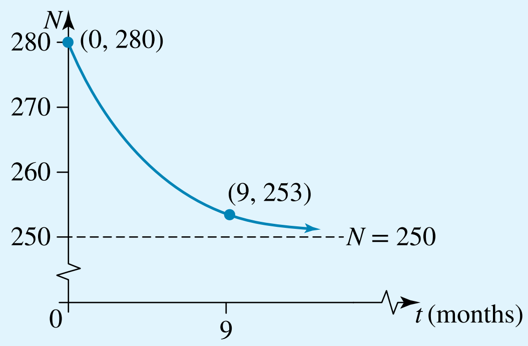
**b 1** Identify the asymptotes and other key features which are appropriate for the restriction $t \geq 0$.

**b** $N = 250 + \dfrac{30}{t+1}, t \geq 0$

Vertical asymptote $t = -1$ (not applicable)
Horizontal asymptote $N = 250$
Initial point is $(0, 280)$.

**2** Sketch the part of the graph of the hyperbola that is applicable and label axes appropriately.

*Note:* The vertical scale is broken in order to better display the graph.




**3** State the domain and range for this model.

Domain $\{t : t \geq 0\}$
Range $(250, 280]$

**c 1** Interpret the meaning of the horizontal asymptote.

**c** The horizontal asymptote shows that as $t \to \infty, N \to 250$. This means $N = 250$ gives the limiting population of the bats.

**2** State the answer.

Since the population of bats cannot fall below 250 and there were 280 initially, the maximum number of bats that can be relocated is 30.

## EXERCISE 6.4 The rectangular hyperbola and the truncus

**PRACTISE**

Work without CAS

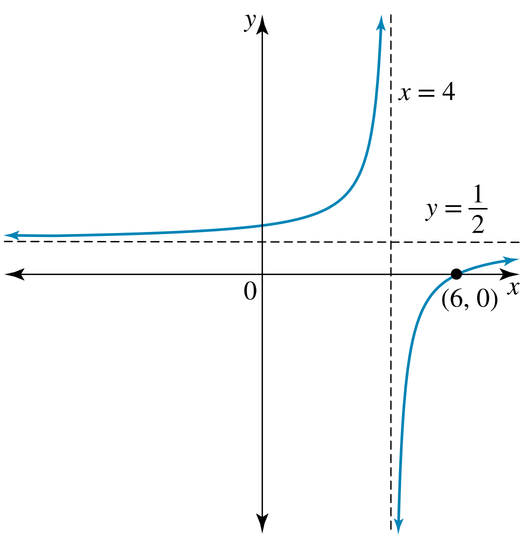
**1** WE7 Sketch the graphs of the following functions, stating the domain and range.

**a** $y = \dfrac{2}{x+2} - 1$ **b** $y = \dfrac{-1}{x-2}$

**2** For $f : R\backslash\left\{\dfrac{1}{2}\right\} \to R, f(x) = 4 - \dfrac{3}{1-2x}$, state the equations of the asymptotes and the asymptote-formed quadrants in which the graph would lie.

**3** WE8 **a** Identify the asymptotes of the hyperbola with equation $y = \dfrac{6x}{3x+2}$.

**b** Form the equation of the hyperbola shown.



4 Find the domain and range of the hyperbola with equation $xy - 4y + 1 = 0$.

5 WE9 From Ohm's Law, the electrical resistance, $R$ ohms, of a metal conductor in which the voltage is constant is inversely proportional to the current, $I$ amperes. When the current is 0.6 amperes, the resistance is 400 ohms.

a Find the relationship between the resistance and current.
b If the current is increased by 20%, what is the resistance?

6 Select the table of data which shows $y \propto \frac{1}{x}$, complete the table and state the rule for $y$ in terms of $x$.

a

| $x$ | 0.5 | 1 | 2 | 4 | | 8 |
|---|---|---|---|---|---|---|
| $y$ | 20.5 | 11 | 7 | 6.5 | 6.4 | |

b

| $x$ | 0.5 | 1 | 2 | 4 | | 8 |
|---|---|---|---|---|---|---|
| $y$ | 14.4 | 7.2 | 3.6 | 1.8 | 6.4 | |

7 WE10 Sketch the graphs of the following functions, stating the domain and range.

a $y = \frac{1}{(x-3)^2} - 1$

b $y = \frac{-8}{(x+2)^2} - 4$

8 Determine the equations of the asymptotes and state the domain and range of the truncus with equation $y = \frac{3}{2(1-5x)^2}$.

9 WE11 The number $P$ of cattle owned by a farmer at a time $t$ years after purchase is modelled by $P = 30 + \frac{100}{2+t}$.

a By how many cattle is the herd reduced after the first 2 years?
b Sketch the graph of the number of cattle over time using the given model and state its domain and range.
c What is the minimum number the herd of cattle is expected to reach according to this model?

10 The height $h$ metres of a hot air balloon above ground level $t$ minutes after take-off is given by $h = 25 - \frac{100}{(t+2)^2}$, $t \geq 0$.

a How long, to the nearest second, does it take the balloon to reach an altitude of 12.5 metres above ground level?
b What is its limiting altitude?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

11 State the equations of the asymptotes of the following hyperbolas.

a $y = \frac{1}{x+5} + 2$

b $y = \frac{8}{x} - 3$

c $y = \frac{-3}{4x}$

d $y = \frac{-3}{14+x} - \frac{3}{4}$

12 Sketch the graph of the following functions, stating the domain and range.

a $y = \frac{1}{x+1} - 3$

b $y = 4 - \frac{3}{x-3}$

c $y = -\frac{5}{3+x}$

d $y = -\left(1 + \frac{5}{2-x}\right)$

13 a If $\frac{11-3x}{4-x} = a - \frac{b}{4-x}$ calculate the values of $a$ and $b$.

b Hence, sketch the graph of $y = \frac{11-3x}{4-x}$.

c For what values of $x$ is $\frac{11-3x}{4-x} > 0$?

14 Express in the form $y = \frac{a}{bx+c} + d$ and state the equations of the asymptotes for each of the following.

a $y = \frac{x}{4x+1}$

b $(x-4)(y+2) = 4$

c $y = \frac{1+2x}{x}$

d $2xy + 3y + 2 = 0$

15 Identify the equations of the asymptotes and sketch the graphs of the following, stating the domain and range.

a $y = \frac{12}{(x-2)^2} + 5$

b $y = \frac{-24}{(x+2)^2} + 6$

c $y = 7 - \frac{1}{7x^2}$

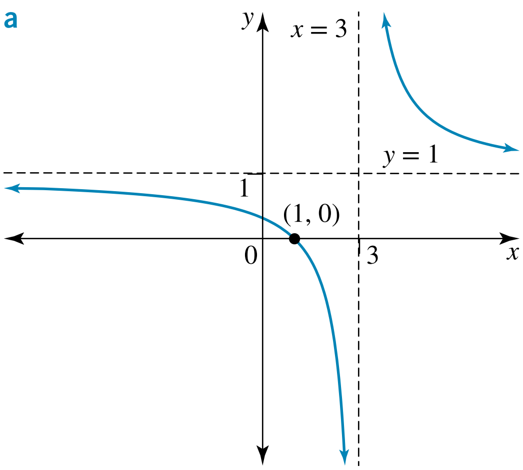
d $y = \frac{4}{(2x-1)^2}$

e $y = -2 - \frac{1}{(2-x)^2}$

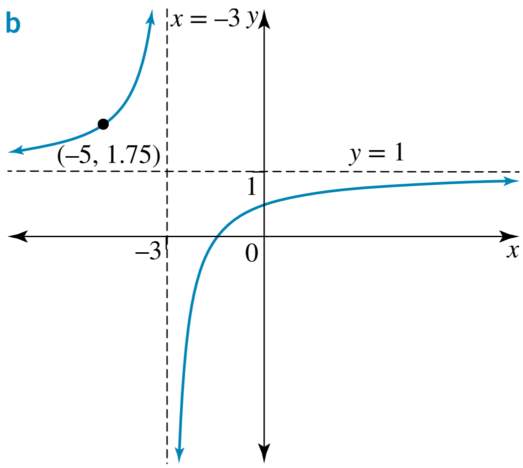
f $y = \frac{x^2+2}{x^2}$

16 The diagrams in parts a to d are either of a hyperbola or a truncus. Form the equations of each graph.

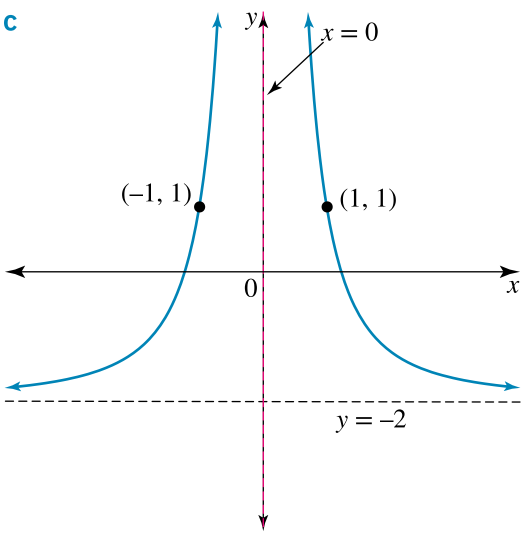
a



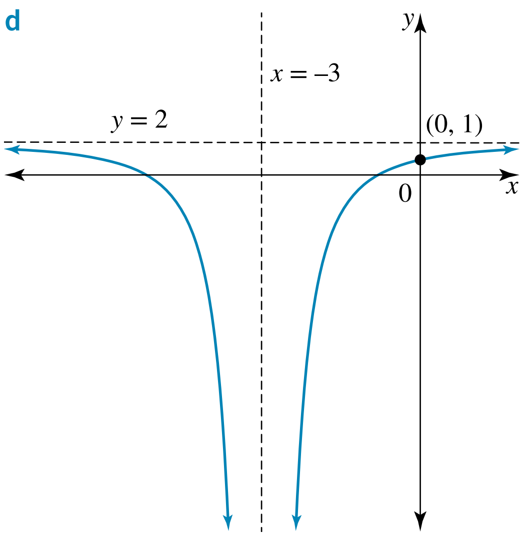

b




c




d



**e** The graph of a truncus has the same shape as $y = \frac{4}{x^2}$ but its vertical asymptote has the equation $x = -2$. Give its rule and write the function in mapping notation.

**f** A hyperbola is undefined when $x = \frac{1}{4}$. As $x \to -\infty$ its graph approaches the line $y = -\frac{1}{2}$ from below. The graph cuts the $x$-axis where $x = 1$.

**i** Determine the equation of the hyperbola and express it in the form $y = \frac{ax + b}{cx + d}$ where $a, b, c, d \in Z$.

**ii** Write the function in mapping notation.

**17** The strength of a radio signal $I$ is given by $I = \frac{k}{d^2}$, where $d$ is the distance from the transmitter.

**a** Draw a sketch of the shape of the graph of $I$ against $d$.

**b** Describe the effect on the strength of the signal when the distance from the transmitter is doubled.

**18** The time $t$ taken to travel a fixed distance of 180 km is given by $t = \frac{k}{v}$ where $v$ is the speed of travel.

**a** What is the constant of proportionality, $k$?

**b** Sketch a graph to show the nature of the relationship between the time and the speed.

**c** What speed needs to be maintained if the entire journey is to be completed in $2\frac{1}{4}$ hours?

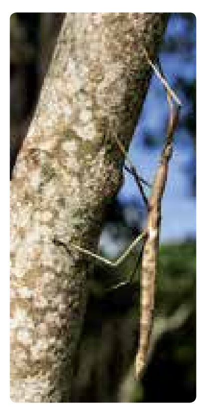
**19 a** Calculate the coordinates of the point(s) of intersection of $xy = 2$ and $y = \frac{x^2}{4}$.

**b** On the same axes sketch the graphs of $xy = 2$ and $y = \frac{x^2}{4}$.

**c** Calculate the coordinates of the point(s) of intersection of $y = \frac{4}{x^2}$ and $y = \frac{x^2}{4}$ and sketch the graphs of $y = \frac{4}{x^2}$ and $y = \frac{x^2}{4}$ on the same set of axes.

**d** Express the coordinates of the point(s) of intersection of $y = \frac{a}{x^2}$ and $y = \frac{x^2}{a}$ in terms of $a$, for $a > 0$.

**20** In an effort to protect a rare species of stick insect, 20 of the species were captured and relocated to a small island where there were few predators. After 2 years the population size grew to 240 stick insects.



A model for the size $N$ of the stick insect population after $t$ years on the island is thought to be defined by the function $N: R^+ \cup \{0\} \to R,\ N(t) = \frac{at + b}{t + 2}$.

**a** Calculate the values of $a$ and $b$.

**b** After what length of time, to the nearest month, did the stick insect population reach 400?

**c** Show that $N(t + 1) - N(t) = \frac{880}{(t + 2)(t + 3)}$.

d Hence, or otherwise, find the increase in the stick insect population during the 12th year and compare this with the increase during the 14th year. What is happening to the growth in population?

e When would the model predict the number of stick insects reaches 500?

f How large can the stick insect population grow?

MASTER

21 Use CAS technology to sketch $y = \dfrac{x+1}{x+2}$ together with its asymptotes and use the graphing screen to obtain:

a the number of intersections of $y = x$ with $y = \dfrac{x+1}{x+2}$

b the values of $k$ for which $y = x + k$ intersects $y = \dfrac{x+1}{x+2}$ once, twice or not at all.

22 a Sketch the graphs of $xy = 1$ and $x^2 - y^2 = 2$ using the conic screen or other technology and give the equations of their asymptotes.

b These hyperbolas are the same but they are sketched on different orientations of the axes. Suggest a way to transform one graph into the other.

# 6.5 The relation $y^2 = x$

Here we shall consider a relation with a one-to-many correspondence. This relation is not a function.

studyon

Units 1 & 2
AOS 1
Topic 5
Concept 4

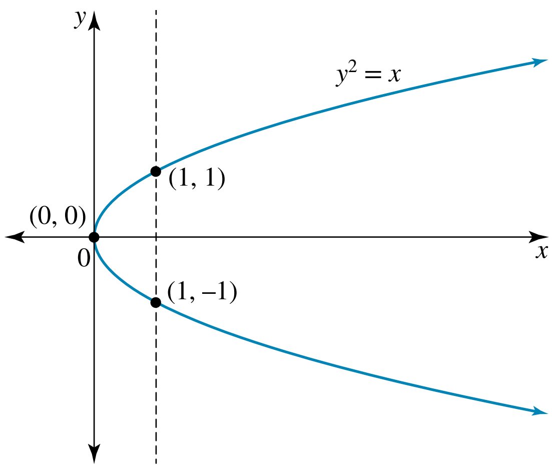
**The relation $y^2 = x$**
Concept summary
Practice questions

eBook*plus*

**Interactivity**
Graph plotter: $y^2 = x$
**int-2574**

## The relation $y^2 = x$

The relation $y^2 = x$ cannot be a function since, for example, $x = 1$ is paired with both $y = 1$ and $y = -1$; the graph of $y^2 = x$ therefore fails the vertical line test for a function.

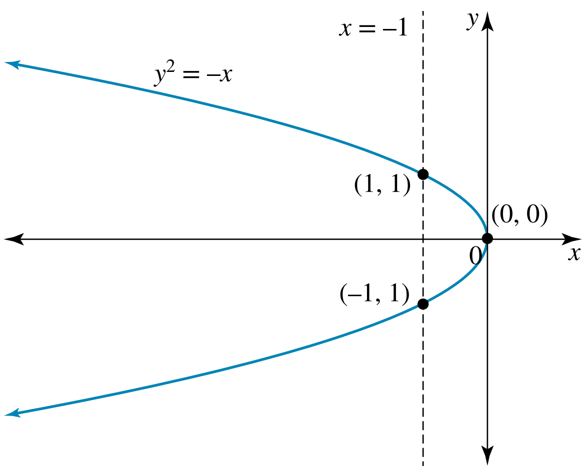



The shape of the graph of $y^2 = x$ could be described as a **sideways parabola** opening to the right, like the reflector in a car's headlight.

Key features of the graph of $y^2 = x$ are:

- domain $R^+ \cup \{0\}$ with the graph opening to the right
- range $R$
- turning point, usually called a vertex, at $(0, 0)$
- axis of symmetry is horizontal with equation $y = 0$ (the $x$-axis)
- one-to-many correspondence.

The graph of $y^2 = -x$ will open to the left, with domain $R^- \cup \{0\}$.



## Transformations of the graph of $y^2 = x$

Horizontal and vertical translations are identifiable from the coordinates of the vertex, just as they are for translations of the parabolic function $y = x^2$; the analysis of the curve is very similar to that applied to the parabolic function.

From the equation $(y - k)^2 = a(x - h)$ we can deduce:

- The vertex has coordinates $(h, k)$, due to the horizontal and vertical translations $h$ and $k$ respectively.
- The axis of symmetry has equation $y = k$.
- If $a > 0$, the graph opens to the right; if $a < 0$, it opens to the left.
- There is always one $x$-intercept obtained by substituting $y = 0$.
- There may be two, one or no $y$-intercepts, determined by substituting $x = 0$ and solving the resulting quadratic equation for $y$.

By considering the sign of $a$ and the position of the vertex, it is possible to deduce whether or not there will be a $y$-intercept. If there is no $y$-intercept, this consideration can avoid wasted effort in attempting to solve a quadratic equation for which there are no real solutions.

If the equation of the graph is not given in the vertex form $(y - k)^2 = a(x - h)$, completing the square on the $y$ terms may be necessary to transform the equation into this form.

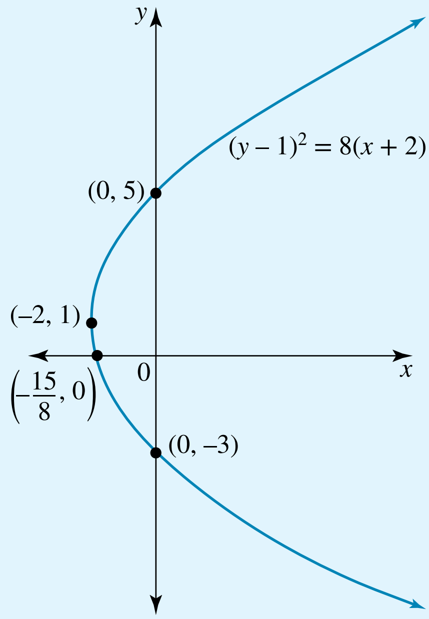
**WORKED EXAMPLE 12**

**For each of the following relations, state the coordinates of the vertex and sketch the graph stating its domain and range.**

**a** $(y - 1)^2 = 8(x + 2)$ **b** $y^2 = 6 - 3x$

| THINK | WRITE |
|---|---|
| **a** 1 State the coordinates of the vertex. | **a** $(y - k)^2 = a(x - h)$ has vertex $(h, k)$<br>$(y - 1)^2 = 8(x + 2)$ has vertex $(-2, 1)$ |
| 2 Calculate any intercepts with the axes. | $x$-intercept: let $y = 0$<br>$(-1)^2 = 8(x + 2)$<br>$8x = -15$<br>$\therefore x = -\frac{15}{8}$<br>$x$-intercept $\left(-\frac{15}{8}, 0\right)$<br>$y$-intercepts: let $x = 0$<br>$(y - 1)^2 = 16$<br>$y - 1 = \pm 4$<br>$\therefore y = -3$ or $y = 5$<br>$y$-intercepts $(0, -3), (0, 5)$ |

3 Sketch the graph showing the key features and state the domain and range.




Domain $[-2, \infty)$ and range $R$.

**b** 1 Express the equation in the form $(y-k)^2 = a(x-h)$ and state the vertex.

**b** $y^2 = 6 - 3x$

$= -3(x-2)$

Vertex is $(2, 0)$.

2 Calculate any intercepts with the axes.

$x$-intercept is the vertex $(2, 0)$.

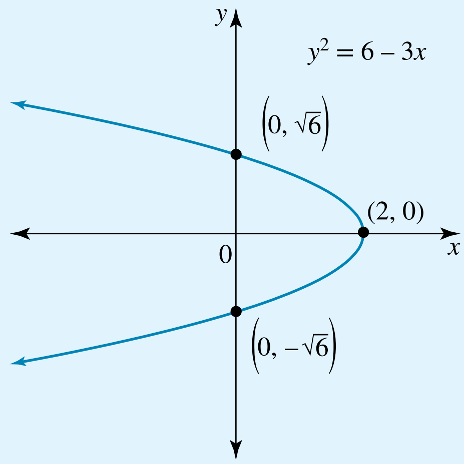
$y$-intercepts: in $y^2 = 6 - 3x$, let $x = 0$

$y^2 = 6$

$\therefore y = \pm\sqrt{6}$

$y$-intercepts $(0, \pm\sqrt{6})$

3 Sketch the graph and state the domain and range.




Domain $(-\infty, 2]$; range $R$.

## Determining the rule for the sideways parabola

Since the most common form given for the equation of the sideways parabola is the vertex form $(y-k)^2 = a(x-h)$, once the coordinates of the vertex are known, a second point can be used to obtain the value of $a$. Other sets of three pieces of information and analysis could also determine the equation, including that the axis of symmetry lies midway between the $y$-intercepts.

**WORKED EXAMPLE 13**

**a** Determine the equation of the relation with rule $(y-k)^2 = a(x-h)$ and vertex $(3, 5)$ which passes through the point $(5, 3)$.

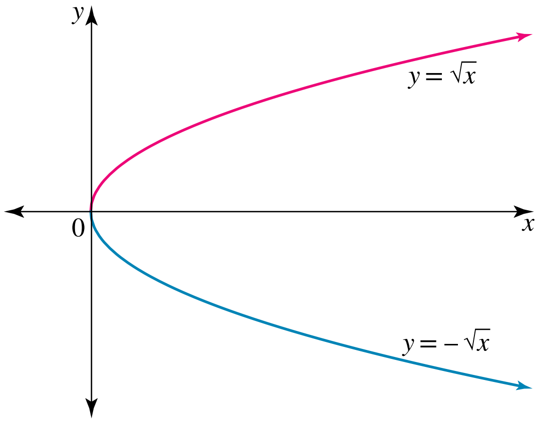
**b** Determine the equation of the sideways parabola which contains the three points $(0, 0)$, $(0, -4)$, $(3, 2)$.

| THINK | WRITE |
|---|---|
| **a 1** Substitute the coordinates of the vertex into the general form of the equation. | **a** $(y-k)^2 = a(x-h)$<br>Vertex $(3, 5)$<br>$\Rightarrow (y-5)^2 = a(x-3)$ |
| **2** Use the given point on the graph to determine the remaining unknown constant. | Point $(5, 3)$ is on the curve.<br>$\Rightarrow (3-5)^2 = a(5-3)$<br>$4 = 2a$<br>$\therefore a = 2$ |
| **3** State the equation. | The equation is $(y-5)^2 = 2(x-3)$. |
| **b 1** Calculate the equation of the axis of symmetry.<br>*Note:* An alternative approach would be to set up a system of 3 simultaneous equations using the coordinates of the 3 given points. | **b** Two of the given points, $(0, 0)$ and $(0, -4)$, lie on the $y$-axis, so the axis of symmetry lies midway between these two points.<br>Equation of axis of symmetry is:<br>$y = \frac{y_1 + y_2}{2}$<br>$= \frac{0 + (-4)}{2}$<br>$= -2$<br>$y = -2$ is the equation of the axis of symmetry. |
| **2** Substitute the equation of the axis of symmetry into the general equation of a sideways parabola. | Let the equation be $(y-k)^2 = a(x-h)$.<br>Axis of symmetry $y = -2$<br>$\therefore (y+2)^2 = a(x-h)$ |
| **3** Use the third point and one of the $y$-intercepts to form a system of two simultaneous equations. | Substitute the point $(0, 0)$.<br>$(2)^2 = a(-h)$<br>$4 = -ah$<br>Substitute the point $(3, 2)$.<br>$(2+2)^2 = a(3-h)$<br>$16 = a(3-h)$<br>$= 3a - ah$ |
| **4** Solve the simultaneous equations to obtain $a$ and $h$. | $4 = -ah$ ............. (1)<br>$16 = 3a - ah$ ...... (2)<br>Equation (2) − equation (1)<br>$12 = 3a$<br>$a = 4$<br>Equation (1) $\Rightarrow h = -1$ |
| **5** State the answer. | The equation of the sideways parabola is $(y+2)^2 = 4(x+1)$. |

## The square root function

The **square root function** is formed as part of the relation $y^2 = x$ in much the same way as the semicircle forms part of a circle.

Although $y^2 = x$ is not a function, it is made up of two branches, the upper branch and the lower branch, each of which is a function. Since $y^2 = x \Rightarrow y = \pm\sqrt{x}$, the upper branch has the equation $y = \sqrt{x}$ and the lower branch has the equation $y = -\sqrt{x}$.




## The graph of $y = \sqrt{x}$

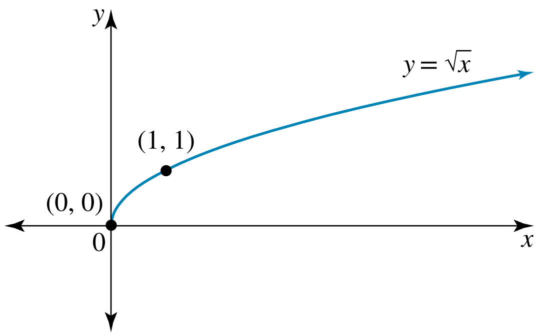
The square root function has the rule $y = x^n,\ n = \frac{1}{2}$. It is not a polynomial function.

Since $x^{\frac{1}{2}} = \sqrt{x}$, the square root function is defined by the equation $y = \sqrt{x}$ or $y = x^{\frac{1}{2}}$.

The $y$-values of this function must be such that $y \geq 0$. No term under a square root symbol can be negative so this function also requires that $x \geq 0$. The graph of the square root function is shown in the diagram.

Key features of the graph of $y = \sqrt{x}$ are:

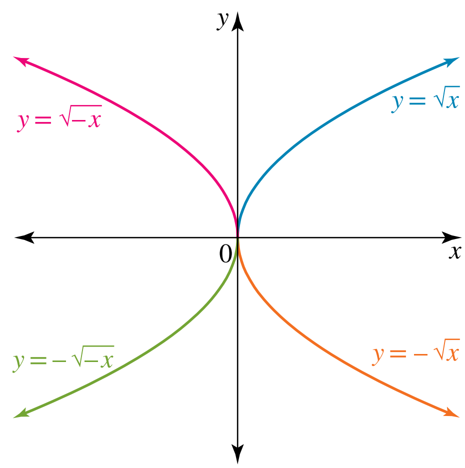
- endpoint $(0, 0)$
- as $x \to \infty, y \to \infty$
- defined only for $x \geq 0$ so the domain is $R^+ \cup \{0\}$
- $y$-values cannot be negative so the range is $R^+ \cup \{0\}$
- one-to-one correspondence
- upper half of the sideways parabola $y^2 = x$




### Variations of the basic graph

The term under the square root cannot be negative so $y = \sqrt{-x}$ has a domain requiring $-x \geq 0 \Rightarrow x \in (-\infty, 0]$. This graph can be obtained by reflecting the graph of $y = \sqrt{x}$ in the $y$-axis. The four variations in position of the basic graph are shown in the diagram.

The diagram could also be interpreted as displaying the graphs of the two relations $y^2 = x$ (on the right of the $y$-axis) and $y^2 = -x$ (on the left of the $y$-axis). The endpoints of the square root functions are the vertices of these sideways parabolas.




## Transformations of the square root function

The features of the graph of $y = a\sqrt{x - h} + k$ are:

- endpoint $(h, k)$
- if $a > 0$, the endpoint is the minimum point

- if $a < 0$, the endpoint is the maximum point
- either one or no $x$-intercepts
- either one or no $y$-intercepts
- domain $[h, \infty)$ since $x - h \geq 0 \Rightarrow x \geq h$
- range is $[k, \infty)$ if $a > 0$ or $(-\infty, k]$ if $a < 0$

The graph of $y = a\sqrt{-(x - h)} + k$ also has its endpoint at $(h, k)$ but its domain is $(-\infty, h]$.

**WORKED EXAMPLE 14**

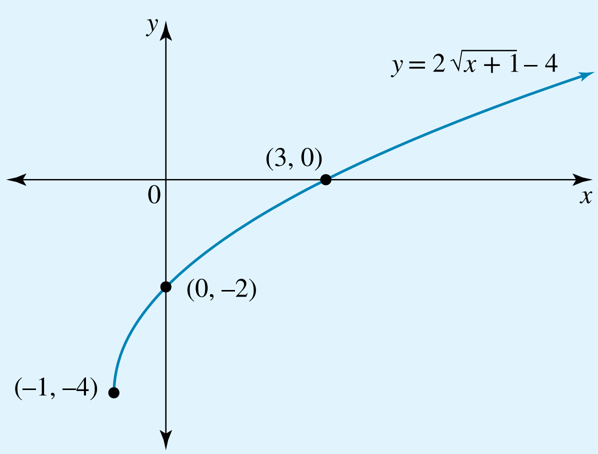
**a** **Sketch $y = 2\sqrt{x + 1} - 4$, stating its domain and range.**

**b** **For the function $f(x) = \sqrt{2 - x}$:**

**i** **state its domain**

**ii** **sketch the graph of $y = f(x)$**

**iii** **form the equation of the sideways parabola of which it is part.**

| THINK | WRITE |
|---|---|
| **a** **1** State the coordinates of the endpoint. | **a** $y = 2\sqrt{x + 1} - 4$<br>Endpoint $(-1, -4)$ |
| **2** Calculate any intercepts with the coordinate axes. | $y$-intercept: let $x = 0$<br>$y = 2\sqrt{1} - 4$<br>$= -2$<br>$\Rightarrow (0, -2)$<br>$x$-intercept: let $y = 0$<br>$2\sqrt{x + 1} - 4 = 0$<br>$\therefore \sqrt{x + 1} = 2$<br>Square both sides:<br>$x + 1 = 4$<br>$\therefore x = 3$<br>$\Rightarrow (3, 0)$ |
| **3** Sketch the graph and state its domain and range. | <br>Domain $[-1, \infty)$; range $[-4, \infty)$ |

**b i** State the requirement on the expression under the square root and then state the domain.

**b i** $f(x) = \sqrt{2 - x}$

Domain: require $2 - x \geq 0$

$\therefore -x \geq -2$

$\therefore x \leq 2$

Therefore the domain is $(-\infty, 2]$.

**ii** Identify the key points of the graph and sketch.

**ii** $f(x) = \sqrt{2 - x}$

$y = \sqrt{2 - x}$

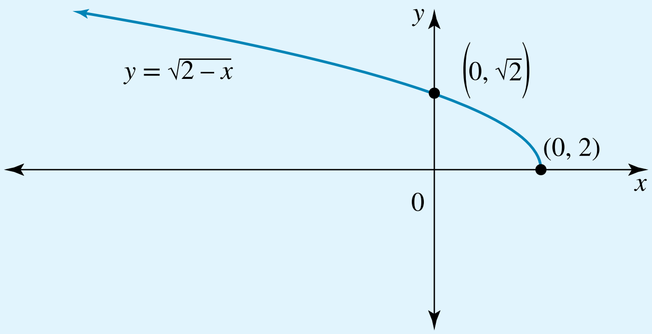
$= \sqrt{-(x - 2)}$

Endpoint $(2, 0)$ is also the $x$-intercept.

$y$-intercept: let $x = 0$ in $y = \sqrt{2 - x}$

$y = \sqrt{2}$

$y$-intercept is $(0, \sqrt{2})$.




**iii** Form the equation of the sideways parabola.

*Note:* The required equation cannot be written using $f(x)$ notation since the sideways parabola is not a function.

**iii** Square root function: $y = \sqrt{2 - x}$

Square both sides:

$y^2 = 2 - x$

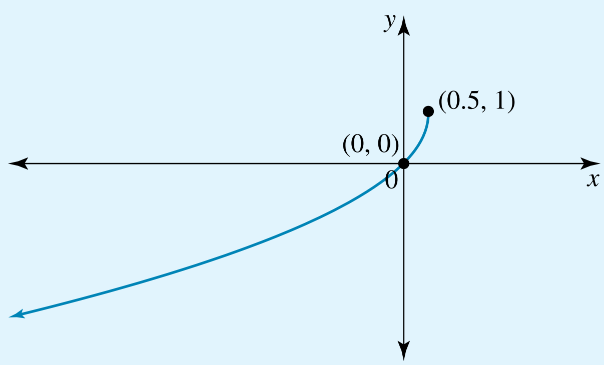
The equation of the sideways parabola is $y^2 = 2 - x$ or $y^2 = -(x - 2)$ or $y = \pm\sqrt{2 - x}$.

## Determining the equation of a square root function

If a diagram is given, the direction of the graph will indicate whether to use the equation in the form of $y = a\sqrt{x - h} + k$ or of $y = a\sqrt{-(x - h)} + k$. If a diagram is not given, the domain or a rough sketch of the given information may clarify which form of the equation to use.

WORKED EXAMPLE 15

**Form a possible equation for the square root function shown.**



| THINK | WRITE |
| --- | --- |
| 1 Note the direction of the graph to decide which form of the equation to use. | Graph opens to the left with $x \leq 0.5$.<br>Let equation be $y = a\sqrt{-(x-h)} + k$. |
| 2 Substitute the coordinates of the endpoint into the equation. | Endpoint $(0.5, 1)$<br>$y = a\sqrt{-(x-0.5)} + 1$<br>$= a\sqrt{-x+0.5} + 1$ |
| 3 Use a second point on the graph to determine the value of $a$. | $(0, 0)$ lies on the graph.<br>$0 = a\sqrt{0.5} + 1$<br>$-1 = a \times \frac{1}{\sqrt{2}}$<br>$a = -\sqrt{2}$ |
| 4 State the equation of the graph. | The equation is $y = -\sqrt{2}\sqrt{-x+0.5} + 1$. |
| 5 Express the equation in a simplified form. | $y = -\sqrt{2}\sqrt{-x+\frac{1}{2}} + 1$<br>$= -\sqrt{2}\sqrt{\frac{-2x+1}{2}} + 1$<br>$= -\sqrt{2}\frac{\sqrt{-2x+1}}{\sqrt{2}} + 1$<br>$= -\sqrt{-2x+1} + 1$<br>Therefore the equation is $y = 1 - \sqrt{1-2x}$. |

## EXERCISE 6.5 The relation $y^2 = x$

**PRACTISE**

Work without CAS

1 WE12 For each of the following relations state the coordinates of the vertex and sketch the graph, stating its domain and range.

a $(y+3)^2 = 4(x-1)$

b $(y-3)^2 = -9x$

2 Express the relation given by $y^2 + 8y - 3x + 20 = 0$ in the form $(y-k)^2 = a(x-h)$ and hence state the coordinates of its vertex and the equation of its axis of symmetry.

3 WE13 a Determine the equation of the relation with rule $(y-k)^2 = a(x-h)$ which passes through the point $(-10, 0)$ and has a vertex at $(4, -7)$.

b Determine the equation of the sideways parabola which contains the points $(0, 0)$, $(0, 6)$ and $(9, -3)$.

4 A parabola touches the $y$-axis at $y = 3$ and cuts the $x$-axis at $x = 2$. Explain whether this parabola is a function or not and form its equation.

5 WE14 a Sketch $y = \sqrt{x-1} - 3$, stating its domain and range.

b For the function $f(x) = -\sqrt{2x+4}$:

i state its domain

ii sketch the graph of $y = f(x)$

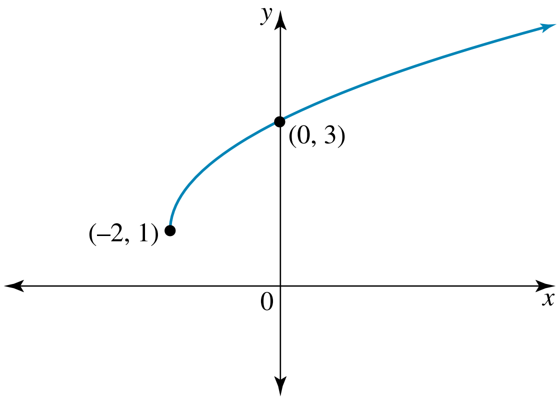
iii form the equation of the sideways parabola of which it is part.

6 Consider the relation $S = \{(x, y) : (y + 2)^2 = 9(x - 1)\}$.

a Find the coordinates of its vertex and $x$-intercept and hence sketch its graph.

b Form the equation of the square root function which forms the lower half of $S$.

7 WE15 Form a possible equation for the square root function shown.




8 The relation $(y - a)^2 = b(x - c)$ has a vertex at $(2, 5)$ and cuts the $x$-axis at $x = -10.5$. Determine the values of $a$, $b$ and $c$, and hence state the equation of the relation and its domain and range.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 On the same diagram, sketch the graphs of $y^2 = x$, $y^2 = 4x$ and $y^2 = \frac{1}{4}x$ and comment on the effect of the change of the coefficient of the $x$-term.

10 Sketch the following, labelling the coordinates of the vertex and any axis intercepts.

a $(y + 1)^2 = 3x$

b $9y^2 = x + 1$

c $(y + 2)^2 = 8(x - 3)$

d $(y - 4)^2 = 2x + 1$

11 Sketch the following, stating the coordinates of the vertex and the exact coordinates of any intercept with the axes.

a $y^2 = -2x$

b $(y + 1)^2 = -2(x - 4)$

c $(6 - y)^2 = -8 - 2x$

d $x = -(2y - 6)^2$

12 Express the following equations in the form $(y - k)^2 = a(x - h)$ and hence state the coordinates of the vertex and the domain.

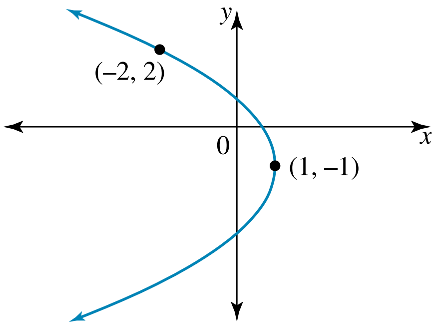
a $y^2 + 16y - 5x + 74 = 0$

b $y^2 - 3y + 13x - 1 = 0$

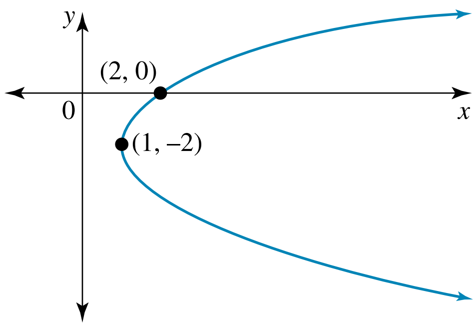
c $(5 + 2y)^2 = 8 - 4x$

d $(5 - y)(1 + y) + 5(x - 1) = 0$

13 a Form the equation of the graph of the parabola relation shown.




b Give a possible equation for the graph shown.

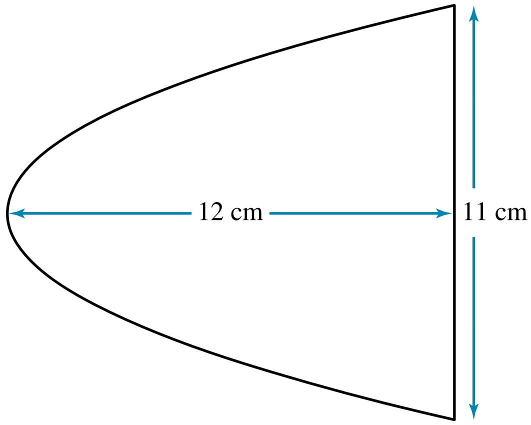


**c** A curve in the shape of a sideways parabola touches the $y$-axis and passes through the points $(1, 12)$ and $(1, -4)$.

  **i** State the equation of its axis of symmetry.

  **ii** Determine the equation of the curve.

**d** The reflector in a car's headlight has a parabolic shape.




Placing the coordinate axes with the origin at the vertex of the parabola, form the equation of the parabola relative to these axes.

**14** Consider the relation $S = \{(x, y) : (y - 2)^2 = 1 - x\}$.

**a** Express the equation of the relation $S$ with $y$ as the subject.

**b** Define, as mappings, the two functions $f$ and $g$ which together form the relation $S$.

**c** Sketch, on the same diagram, the graphs of $y = f(x)$ and $y = g(x)$.

**d** Give the image of $-8$ for each function.

**15** Sketch the following relations and state their domains and ranges.

**a** $y = \sqrt{x + 3} - 2$

**b** $y = 5 - \sqrt{5x}$

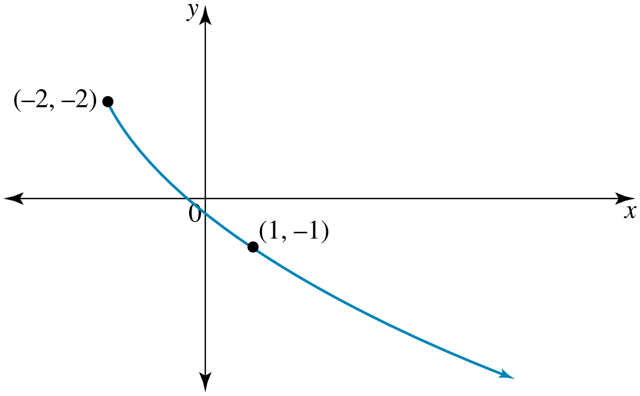
**c** $y = 2\sqrt{9 - x} + 4$

**d** $y = \sqrt{49 - 7x}$

**e** $y = 2 \pm \sqrt{x + 4}$

**f** $y + 1 + \sqrt{-2x + 3} = 0$

**16 a** Determine the equation for the graph shown, given it represents a square root function with endpoint $(-2, -2)$.




**b** Form the equation of the square root function with endpoint $(4, -1)$ and containing the point $(0, 9)$. At what point does this function cut the $x$-axis?

**c** Give the equation of the function which has the same shape as $y = \sqrt{-x}$ and an endpoint with coordinates $(4, -4)$.

**d** A small object falls to the ground from a vertical height of $h$ metres. The time, $t$ seconds, is proportional to the square root of the height. If it takes 2 seconds for the object to reach the ground from a height of 19.6 metres, obtain the rule for $t$ in terms of $h$.

**17** A function is defined as $f: [0, \infty) \to R, f(x) = \sqrt{mx} + n$, $f(1) = 1$ and $f(4) = 4$.

**a** Determine the values of $m$ and $n$.

**b** Calculate the value of $x$ for which $f(x) = 0$.

**c** Sketch the graph of $y = f(x)$ together with the graph of $y = x$ and state $\{x : f(x) > x\}$.

**d** State the equation of the sideways parabola of which the function $f$ is one of its branches.

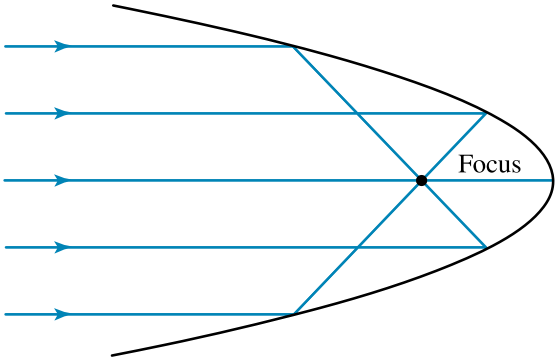
**18** Consider the curve with equation $y^2 = -8x$.

**a** State the domain of the curve and show that the point $P(-3, 2\sqrt{6})$ lies on the curve. Identify which branch of the curve it lies on.

**b** Show that both the vertex V and the point $P(-3, 2\sqrt{6})$ are at positions which are equidistant from the point $F(-2, 0)$ and the vertical line $D$ with equation $x = 2$.

**c** Q is a point on the other branch of the curve to P, where $x = a, a < 0$. Express the coordinates of Q in terms of $a$ and show that Q is also equidistant from the point $F(-2, 0)$ and the vertical line $D$ with equation $x = 2$.

**d** A property of a parabola is that rays travelling parallel to its axis of symmetry are all reflected through a point called the focus. A radio telescope is designed on this principle so that signals received from outer space will be concentrated at its focus.




Consider the equation $y^2 = -8x$ as a two-dimensional model of a telescope dish. Its focus is the point $F(-2, 0)$. A signal, travelling parallel to the axis of symmetry, strikes the dish at the point $P(-3, 2\sqrt{6})$ and is reflected through the focus F, striking the curve at point Q where $x = a, a < 0$. Calculate the value of $a$.

MASTER

**19** Consider the functions with rules $f(x) = 3\sqrt{x + 1} + 2$ and $g(x) = \sqrt{4 - x^2} + 2$.

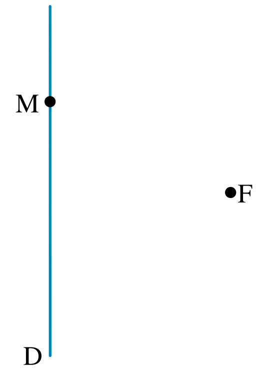
**a** State their domains and what types of function they are.

**b** Obtain the coordinates of any points of intersection of the graphs of $y = f(x)$ and $y = g(x)$, expressing the coordinates to 1 decimal place.

**c** The function $f$ is a branch of a relation $A$ with the same domain as $f$; the function $g$ is a branch of a relation $B$ with the same domain as $g$. Form the rules for the two relations $A$ and $B$.

**d** Give the coordinates of the points of intersection of the two relations $A$ and $B$ obtained in part **c**, to 1 decimal place.

**20** A parabola can be defined as the path traced out by the set of points which lie in positions that are equidistant from a fixed point and a fixed line. Using Cabri or other geometry software, construct a vertical line and label it D; select a point to the right of the line D and label it F.



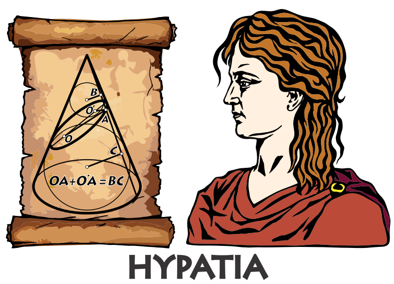

Construct the following:

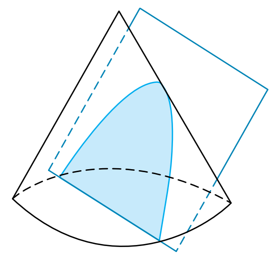
- a point M on the line *D*
- the perpendicular line to *D* through M
- the perpendicular bisector of MF
- the intersection point P of these last two lines
- the locus of P as M moves along line *D*.

**a** What shape is the locus path?

**b** Erase the locus and create, then measure, the line segments *FP* and *PM*. What do you observe about the measurements? Move F and M to test whether your observation continues to hold. What conclusion can you form?

**Hypatia, who was born in Alexandria around 350 BC, is the first recorded female mathematician of note. Amongst her mathematical interests were the curves, known as the conic sections, which arise from a plane cutting a cone. One of these conics is the parabola. When a plane parallel to the slant height of a right cone cuts the cone, the cross-section obtained is a parabola.**



# 6.6 Other functions and relations

studyon
Units 1 & 2
AOS 1
Topic 5
Concept 5
**Other functions and relations**
Concept summary
Practice questions

In this section we will use the functions we have learnt about so far to shed insight into and explore some connections between other functions and relations.

## Maximal domains

When we write the function for which $f(x) = x^2$ we imply that this function has domain $R$. This is called the **maximal** or **implied** domain of the function. The knowledge acquired about the domains of polynomial, hyperbola, truncus, semicircle and square root functions can now be applied to identify the maximal domains of algebraic functions with which we are not familiar.

We know, for example, that the maximal domain of any polynomial function is $R$; the domain of the square root function $y = \sqrt{x}$ is $R^+ \cup \{0\}$ since the expression under the square root symbol cannot be negative; and the domain of the hyperbola $y = \frac{1}{x}$ is $R \backslash \{0\}$ since the denominator cannot be zero.

**The maximal domain of any function must exclude:**
- **any value of $x$ for which the denominator would become zero**
- **any value of $x$ which would create a negative number or expression under a square root sign.**

Hence, to obtain maximal domains, the following conditions would have to be satisfied:

$$y = \frac{g(x)}{f(x)} \Rightarrow f(x) \neq 0$$
$$y = \sqrt{f(x)} \Rightarrow f(x) \geq 0$$
$$y = \frac{g(x)}{\sqrt{f(x)}} \Rightarrow f(x) > 0$$

Numerators can be zero. For $y = \frac{g(x)}{f(x)}$, if $g(a) = 0$ and $f(a) = 2$, for example, the value of $\frac{g(a)}{f(a)} = \frac{0}{2} = 0$, which is defined.

At a value of $x$ for which its denominator would be zero, a function will be discontinuous and its graph will have a vertical asymptote. As noted, this value of $x$ is excluded from the function's domain.

Where a function is composed of the sum or difference of two functions, its domain must be the set over which all of its parts are defined.

**The maximal domain of $y = f(x) \pm g(x)$ is $d_f \cap d_g$ where $d_f$ and $d_g$ are the domains of $f$ and $g$ respectively.**

WORKED EXAMPLE 16

Identify the maximal domains of the rational functions with the following rules.

**a** $y = \frac{1}{x^2 - 9}$ **b** $y = \frac{x - 4}{x^2 + 9}$ **c** $y = \frac{1}{\sqrt{x^3 - 8}}$

| THINK | WRITE |
|---|---|
| **a 1** Determine any values of $x$ for which the denominator would become zero and exclude these values from the domain. | **a** $y = \frac{1}{x^2 - 9}$<br>If the denominator $x^2 - 9 = 0$, then:<br>$(x + 3)(x - 3) = 0$<br>$\therefore x = -3, x = 3$ |
| **2** State the maximal domain. | The values $x = -3,\ x = 3$ must be excluded from the domain.<br>Therefore, the maximal domain is $R\backslash\{\pm 3\}$. |

| | |
|---|---|
| **b 1** Determine any values of $x$ for which the denominator would become zero. | **b** $y = \frac{x-4}{x^2+9}$<br>If the denominator is zero:<br>$x^2 + 9 = 0$<br>$x^2 = -9$<br>There is no real solution.<br>As the denominator $x^2 + 9$ is the sum of two squares, it can never be zero. |
| **2** State the maximal domain. | The denominator cannot be zero and both the numerator and denominator are polynomials, so they are always defined for any $x$-value.<br>Therefore the maximal domain of the function is $R$. |
| **c 1** State the condition the expression contained under the square root sign must satisfy. | **c** $y = \frac{1}{\sqrt{x^3-8}}$<br>The term $\sqrt{x^3-8}$ requires $x^3 - 8 \geq 0$.<br>However, this term cannot be allowed to be zero since it is in the denominator.<br>Hence, $x^3 - 8 > 0$. |
| **2** Solve the inequation and state the maximal domain. | $x^3 > 8$<br>$\therefore x > 2$<br>The maximal domain is $(2, \infty)$. |

## Inverse relations and functions

The relation $A = \{(-1, 4), (0, 3), (1, 5)\}$ is formed by the mapping:

$$-1 \rightarrow 4$$
$$0 \rightarrow 3$$
$$1 \rightarrow 5$$

The **inverse** relation is formed by 'undoing' the mapping where:

$$4 \rightarrow -1$$
$$3 \rightarrow 0$$
$$5 \rightarrow 1$$

The inverse of $A$ is the relation $\{(4, -1), (3, 0), (5, 1)\}$.

The $x$- and $y$-coordinates of the points in relation $A$ have been interchanged in its inverse. This causes the domains and ranges to be interchanged also.

Domain of $A = \{-1, 0, 1\}$ = range of its inverse

Range of $A = \{3, 4, 5\}$ = domain of its inverse

- **For any relation, the inverse is obtained by interchanging the $x$- and $y$-coordinates of the ordered pairs.**
- **Domains and ranges are interchanged between a pair of inverse relations.**

## The equation of the inverse

The effect of the operation 'multiply by 2' is undone by the operation 'divide by 2'; the effect of the operation 'subtract 2' is undone by the operation 'add 2', and so on. These are examples of inverse operations and we use these 'undoing' operations to solve equations. The 'undoing' operations can also be used to deduce the equation of the inverse of a given relation.

The 'multiply by 2' rule is that of the relation $y = 2x$, so its inverse with the 'divide by 2' rule is $y = \frac{x}{2}$.

Algebraically, interchanging $x$ and $y$ in the equation $y = 2x$ gives $x = 2y$. Rearranging this to make $y$ the subject gives $y = \frac{x}{2}$, the equation of its inverse.

**To obtain the equation of the inverse:**

- **Interchange $x$ and $y$ in the equation of the original relation or function and rearrange to make $y$ the subject.**

In some cases, the domain will need to be included when stating the equation of the inverse. For example, to find the equation of the inverse of the function $y = \sqrt{x}$, interchanging coordinates gives $x = \sqrt{y}$. Expressing $x = \sqrt{y}$ with $y$ as the subject gives $y = x^2$. This rule is not unexpected since 'square root' and 'squaring' are inverse operations. However, as the range of the function $y = \sqrt{x}$ is $[0, \infty)$, this must be the domain of its inverse. Hence, the equation of the inverse of $y = \sqrt{x}$ is $y = x^2$, with the restriction that $x \geq 0$.

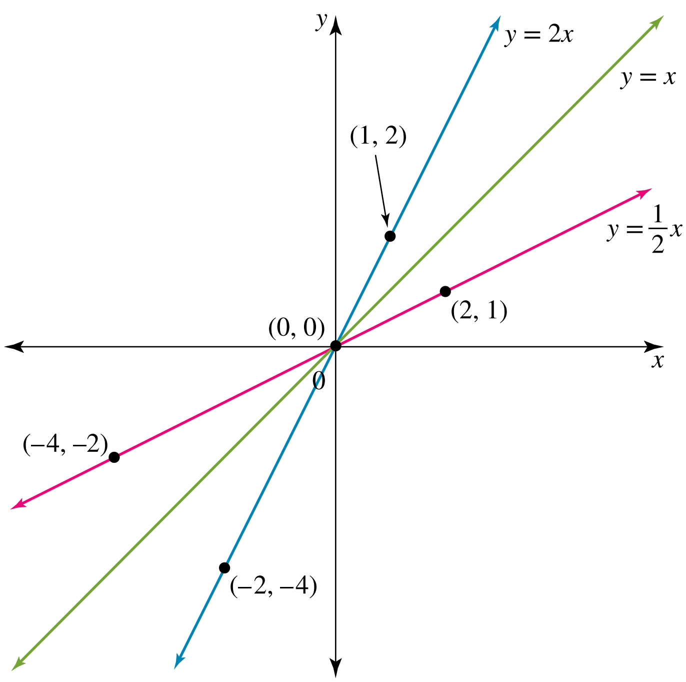
### Graphs of inverse relations

The graphs of the pair of inverses, $y = 2x$ and $y = \frac{x}{2}$ show that their graphs are symmetric about the line $y = x$. The coordinates of symmetric points relative to $y = x$ are interchanged on each graph.

Both graphs contain the point (0, 0) since interchanging coordinates of this point does not change the point.

- Given the graph of a function or another relation, reflection of this graph in the line $y = x$ will create the graph of its inverse.

The coordinates of known points, such as the axes intercepts, are interchanged by this reflection. If the graphs intersect, they will do so on the line $y = x$ since interchanging the coordinates of any point on $y = x$ would not cause any alteration to the coordinates.



**eBook*plus***

**Interactivity**
Graph plotter: Inverse functions
**int-2575**

## Notation for inverse functions

Neither the original relation nor its inverse have to be functions.

However, when both a relation and its inverse are functions, there is a special symbol for the name of the inverse.

> **If the inverse of a function $f$ is itself a function, then the inverse function is denoted by $f^{-1}$.**

The equation of the inverse of the square root function $f(x) = \sqrt{x}$ can be written as $f^{-1}(x) = x^2,\ x \geq 0$.

In mapping notation, if $f : [0, \infty) \to R,\ f(x) = \sqrt{x}$, then the inverse function is $f^{-1} : [0, \infty) \to R,\ f^{-1}(x) = x^2$.

The domain of $f^{-1}$ equals the range of $f$ and the range of $f^{-1}$ equals the domain of $f$; that is, $d_{f^{-1}} = r_f$ and $r_{f^{-1}} = d_f$.

Note that $f^{-1}$ is a function notation and cannot be used for relations which are not functions. Note also that the inverse function $f^{-1}$ and the reciprocal function $\frac{1}{f}$ represent different functions: $f^{-1} \neq \frac{1}{f}$.

**WORKED EXAMPLE 17**

**Consider the linear function $f : [0, 3) \to R,\ f(x) = 2x - 2$.**

**a** **State the domain and determine the range of $f$.**

**b** **State the domain and range of $f^{-1}$, the inverse of $f$.**

**c** **Form the rule for the inverse and express the inverse function in mapping notation.**

**d** **Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.**

| THINK | WRITE |
|---|---|
| **a** State the domain and use its endpoints to determine the range. | **a** $f: [0, 3) \to R, f(x) = 2x - 2$<br>Domain $[0, 3)$<br>Endpoints: when $x = 0,\ f(0) = -2$ (closed); when $x = 3,\ f(3) = 4$ (open)<br>Therefore the range is $[-2, 4)$. |
| **b** State the domain and range of the inverse. | **b** $d_{f^{-1}} = r_f$<br>$= [-2, 4)$<br>$r_{f^{-1}} = d_f$<br>$= [0, 3)$<br>The inverse has domain $[-2, 4)$ and range $[0, 3)$. |
| **c** **1** Interchange $x$- and $y$-coordinates to form the equation of the inverse function. | **c** $f(x) = 2x - 2$<br>Let $y = 2x - 2$<br>Function: $y = 2x - 2$<br>Inverse: $x = 2y - 2$ |

2 Rearrange the equation to make $y$ the subject.

$$x + 2 = 2y$$

$$\therefore y = \frac{x+2}{2}$$

Therefore the rule for the inverse is $f^{-1}(x) = \frac{x+2}{2}$.

3 Express the inverse function as a mapping.

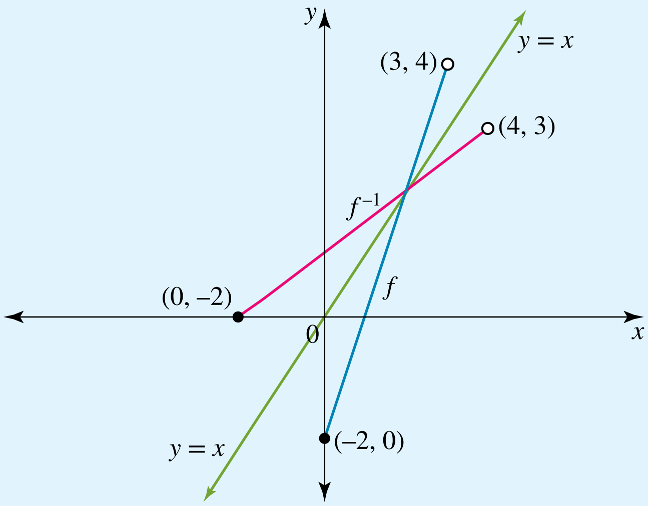
The inverse function is:

$f^{-1}: [-2, 4) \to R, f^{-1}(x) = \frac{x+2}{2}$

d Sketch each line using its endpoints and include the line $y = x$ as part of the graph.

*Note:* The graph of $f^{-1}$ could be deduced from the graph of $f$ by interchanging coordinates of the endpoints.

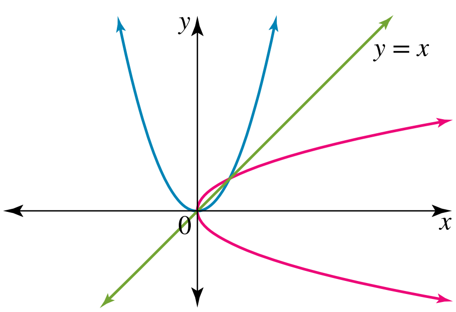
d




$f$ and $f^{-1}$ intersect on the line $y = x$.

## Condition for the inverse function $f^{-1}$ to exist

The inverse of the parabola with equation $y = x^2$ can be obtained algebraically by interchanging $x$- and $y$-coordinates to give $x = y^2$ as the equation of its inverse.




Graphically, the inverse of the parabola is obtained by reflecting the graph of the parabola in the line $y = x$. This not only illustrates that the sideways parabola, $y^2 = x$, is the inverse of the parabola $y = x^2$, but it also illustrates that although the parabola is a function, its inverse is not a function.

The reflection has interchanged the types of correspondence, as well as the domains and ranges. The many-to-one function $y = x^2$ has an inverse with a one-to-many correspondence and therefore its inverse is not a function. This has an important implication for functions.

**For $f^{-1}$ to exist, $f$ must be a one-to-one function.**

To ensure the inverse exists as a function, the domain of the original function may need to be restricted in order to ensure its correspondence is one-to-one.

WORKED EXAMPLE 18

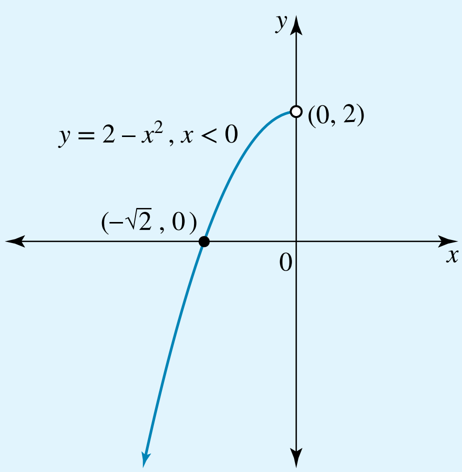
**Consider the quadratic function defined by $y = 2 - x^2$.**

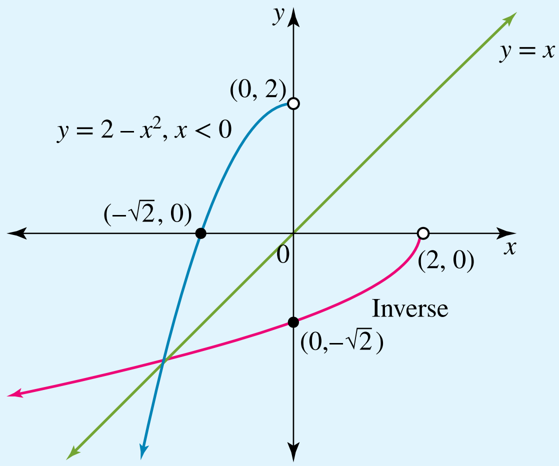
**a** **Form the rule for its inverse and explain why the inverse is not a function.**

**b** **Sketch the graph of $y = 2 - x^2$, $x \in R^-$ and use this to sketch its inverse on the same diagram.**

**c** **Form the equation of the inverse of $y = 2 - x^2$, $x \in R^-$.**

**d** **At what point do the two graphs intersect?**

| THINK | WRITE |
|---|---|
| **a** 1 Interchange $x$- and $y$-coordinates to form the rule for its inverse. | **a** $y = 2 - x^2$<br>Inverse: $x = 2 - y^2$<br>$\therefore y^2 = 2 - x$ is the rule for the inverse. |
| 2 Explain why the inverse is not a function. | The quadratic function is many-to-one so its inverse has a one-to-many correspondence. Therefore the inverse is not a function. |
| **b** 1 Sketch the graph of the function for the restricted domain. | **b** $y = 2 - x^2$, $x \in R^-$<br>Domain: $R^-$<br>$y$-intercept: $(0, 2)$ is open since $x \in R^-$.<br>$x$-intercept: let $y = 0$<br>$2 - x^2 = 0$<br>$\therefore x^2 = 2$<br>$\therefore x = \pm\sqrt{2}$<br>$\Rightarrow x = -\sqrt{2}$ since $x \in R^-$<br>$x$-intercept $\left(-\sqrt{2}, 0\right)$<br>Turning point $(0, 2)$<br> |
| 2 Deduce the key features of the inverse and sketch its graph and the line $y = x$ on the same diagram as the graph of the function. | For the inverse, $(2, 0)$ is an open point on the $x$-axis and $\left(0, -\sqrt{2}\right)$ is the $y$-intercept. Its graph is the reflection of the graph of $y = 2 - x^2$, $x \in R^-$ in the line $y = x$. |




**c** Use the range of the inverse to help deduce its equation.

**c** From part **a**, the inverse of $y = 2 - x^2$ is:

$$y^2 = 2 - x$$
$$\therefore y = \pm\sqrt{2 - x}$$

The range of the inverse must be $R^-$, so the branch with the negative square root is required.
Therefore the equation of the inverse is $y = -\sqrt{2 - x}$.

**d** Choose two of the three equations that contain the required point and solve this system of simultaneous equations.

**d** Point of intersection lies on $y = x$, $y = 2 - x^2$ and $y = -\sqrt{2 - x}$.
Using the first two equations, at intersection:
$x = 2 - x^2, x \in R^-$

$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2, x = 1$$

Reject $x = 1$ since $x \in R^-$.
Therefore $(-2, -2)$ is the point of intersection.

## The inverse of $y = x^3$

The cubic function $y = x^3$ has a one-to-one correspondence, and both its domain and range are $R$. Its inverse will also be a function with both the domain and range of $R$.

The rule for the inverse is obtained by interchanging the $x$- and $y$-coordinates.

Function: $y = x^3$

Inverse function: $x = y^3$

$$\therefore y = \sqrt[3]{x}$$

The inverse of the cubic function $y = x^3$ is the **cube root function** $f: R \to R, f(x) = \sqrt[3]{x}$.

In index form $y = \sqrt[3]{x}$ is written as $y = x^{\frac{1}{3}}$, just as the square root function $y = \sqrt{x}$ can be expressed as $y = x^{\frac{1}{2}}$. Both $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{2}}$ belong to a category of functions known as power functions of the form $y = x^{\frac{p}{q}}, p, q \in N$.

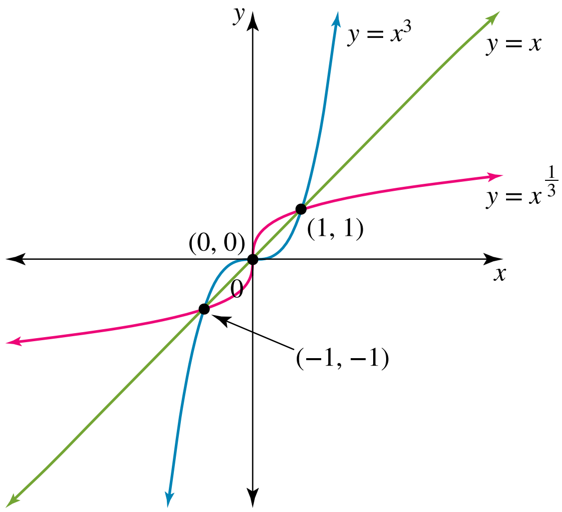
### The graph of $y = x^n, n = \frac{1}{3}$

The graph of $y = x^{\frac{1}{3}}$ is obtained by reflecting the graph of $y = x^3$ in the line $y = x$.

Note that the points $(-1, -1)$, $(0, 0)$, $(1, 1)$ lie on $y = x^3$ so the same three points must lie on $y = x^{\frac{1}{3}}$.

Key features:

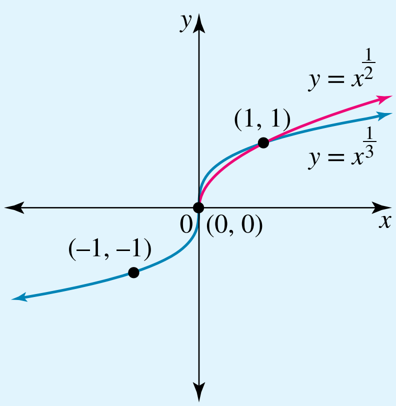
- Point of inflection at $(0, 0)$
- As $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$
- Domain $R$ and range $R$




Unlike the square root function, the cube root function has domain $R$ since cube roots of negative numbers can be calculated.

**WORKED EXAMPLE 19**

**On the same set of axes sketch the graphs of $y = x^n$ for $n = \frac{1}{2}$ and $n = \frac{1}{3}$ and hence state $\left\{x : x^{\frac{1}{3}} > x^{\frac{1}{2}}\right\}$.**

| THINK | WRITE |
|---|---|
| **1** State the features of the first graph. | The function $y = x^n$ for $n = \frac{1}{2}$ is the square root function, $y = x^{\frac{1}{2}}$ or $y = \sqrt{x}$.<br>Endpoint: $(0, 0)$<br>Point: the point $(1, 1)$ lies on the graph.<br>Domain requires $x \geq 0$. |
| **2** State features of the second graph. | The function $y = x^n$ for $n = \frac{1}{3}$ is the cube root function, $y = x^{\frac{1}{3}}$ or $y = \sqrt[3]{x}$.<br>Point of inflection: $(0, 0)$<br>Point: the point $(1, 1)$ lies on the graph.<br>Domain allows $x \in R$. |
| **3** Sketch the graphs on the same set of axes. |  |
| **4** State the answer to the inequation. | For $x > 1$ the square root values are larger than the cube root values.<br>For $0 < x < 1$ the cube root values are the larger.<br>$\left\{x : x^{\frac{1}{3}} > x^{\frac{1}{2}}\right\}$ is $\{x : 0 < x < 1\}$ |

## Hybrid functions

*Note:* Although hybrid functions are not part of the Unit 1 and 2 course, they are included here in preparation for Units 3 and 4.

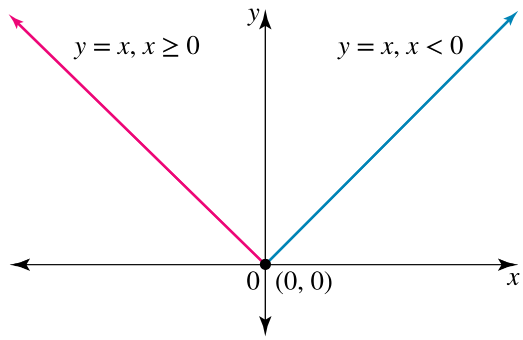
A **hybrid function** is one in which the rule may take a different form over different sections of its domain. An example of a simple hybrid function is one defined by the rule:

$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Graphing this function would give a line with positive gradient to the right of the $y$-axis and a line with negative gradient to the left of the $y$-axis.

This hybrid function is continuous at $x = 0$ since both of its branches join, but that may not be the case for all hybrid functions. If the branches do not join, the function is not continuous for that value of $x$: it is discontinuous at that point of its domain.

Sketching a hybrid function is like sketching a set of functions with restricted domains all on the same graph. Each branch of the hybrid rule is valid only for part of the domain and, if the branches do not join, it is important to indicate which endpoints are open and which are closed.




As for any function, each $x$-value can only be paired to exactly one $y$-value in a hybrid function. To calculate the corresponding $y$-value for a given value of $x$, the choice of which branch of the hybrid rule to use depends on which section of the domain the $x$-value belongs to.

**WORKED EXAMPLE 20**

**Consider the function:**

$$f(x) = \begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$$

**a** **Evaluate**

**i** $f(-2)$ **ii** $f(1)$ **iii** $f(2)$.

**b** **Sketch the graph of $y = f(x)$ and state the domain and range.**

**c** **State any value of $x$ for which the function is not continuous.**

**THINK**

**a** Decide for each $x$-value which section of the domain it is in and calculate its image using the branch of the hybrid function's rule applicable to that section of the domain.

**WRITE**

**a** $f(x) = \begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$

**i** $f(-2)$: Since $x = -2$ lies in the domain section $x < 1$, use the rule $f(x) = x^2$.

$f(-2) = (-2)^2$

$\therefore f(-2) = 4$

**ii** $f(1)$: Since $x = 1$ lies in the domain section $x \geq 1$, use the rule $f(x) = -x$.

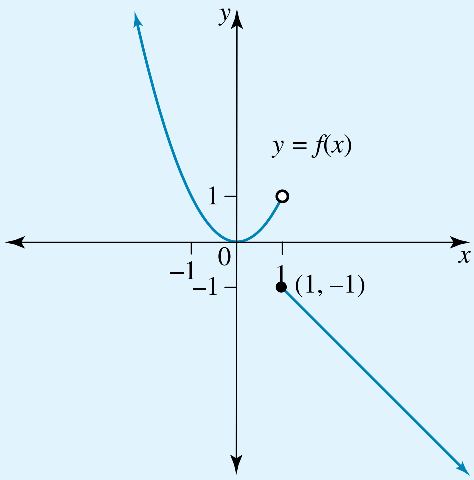
$\therefore f(1) = -1$

**iii** $f(2)$: Since $x = 2$ lies in the domain section $x \geq 1$, use the rule $f(x) = -x$.

$\therefore f(2) = -2$

**b 1** Sketch each branch over its restricted domain to form the graph of the hybrid function.

**b** Sketch $y = x^2, x < 1$
Parabola, turning point $(0, 0)$ open endpoint $(1, 1)$
Sketch $y = -x, x \geq 1$
Line, closed endpoint $(1, -1)$
Point $x = 2 \Rightarrow (2, -2)$




**2** State the domain and range.

The domain is $R$.
The range is $(-\infty, -1] \cup [0, \infty)$.

**c** State any value of $x$ where the branches of the graph do not join.

**c** The function is not continuous at $x = 1$ because there is a break in the graph.

## EXERCISE 6.6 Other functions and relations

**PRACTISE**

Work without CAS

**1** WE16 Identify the maximal domains of the rational functions with the following rules.

**a** $y = \dfrac{1}{16 - x^2}$ **b** $y = \dfrac{2 - x}{x^2 + 3}$ **c** $y = \dfrac{1}{\sqrt{x^3 + 1}}$

**2** Give the implied domains of the functions defined by the following rules.

**a** $f(x) = \dfrac{1}{x^2 + 5x + 4}$ **b** $g(x) = \sqrt{x + 3}$ **c** $h(x) = f(x) + g(x)$

**3** WE17 Consider the linear function $f : (-\infty, 2] \to R, f(x) = 6 - 3x$.

**a** State the domain and determine the range of $f$.
**b** State the domain and range of $f^{-1}$, the inverse of $f$.
**c** Form the rule for the inverse and express the inverse function in mapping notation.
**d** Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

**4** Consider the semicircle with equation $y = \sqrt{4 - x^2}$.

**a** Sketch the semicircle and its inverse on the same set of axes; state the type of correspondence for each graph.
**b** Is the inverse a function?

**5** WE18 Consider the quadratic function $y = (x + 1)^2$ defined on its maximal domain.

**a** Form the rule for its inverse and explain why the inverse is not a function.
**b** Sketch the graph of $y = (x + 1)^2, x \in [-1, \infty)$ and use this to sketch its inverse on the same diagram.

c Form the equation of the inverse of $y = (x + 1)^2, x \in [-1, \infty)$.
d At what point do the two graphs intersect?

6 Given $f: [a, \infty) \to R, f(x) = x^2 - 4x + 9$, find the smallest value of $a$ so that $f^{-1}$ exists.

7 **WE19** On the same set of axes sketch the graphs of $y = x^n$ for $n = 1$ and $n = \frac{1}{3}$; hence, state $\{x : x^{\frac{1}{3}} > x\}$.

8 a Calculate the coordinates of the points of intersection of the two functions $y = x^n, n = -1$ and $n = \frac{1}{3}$.
b Sketch the graphs of $y = x^n, n = -1$ and $n = \frac{1}{3}$ on the same set of axes.

9 **WE20** Consider the function for which $f(x) = \begin{cases} x^3, & x < 1 \\ 2, & x \geq 1 \end{cases}$
a Evaluate the following.
i $f(-2)$ ii $f(1)$ iii $f(2)$
b Sketch the graph of $y = f(x)$ and state the domain and range.
c State any value of $x$ for which the function is not continuous.

10 Consider the function defined by $f(x) = \begin{cases} 4x + a, & x < 1 \\ \frac{2}{x}, & 1 \leq x \leq 4 \end{cases}$
a Determine the value of $a$ so the function will be continuous at $x = 1$.
b Explain whether the function is continuous at $x = 0$.

**CONSOLIDATE**
Apply the most appropriate mathematical processes and tools

11 Give the maximal domains of the functions with the following rules.
a $y = \sqrt{x^2 - 4}$ b $y = \frac{2x}{x^2 - 4}$ c $y = \frac{1}{\sqrt{4 - x}}$
d $y = \sqrt{x} + \sqrt{2 - x}$ e $y = 4x^6 + 6x^{-1} + 2x^{\frac{1}{2}}$ f $y = \frac{\sqrt{x^2 + 2}}{x^2 + 8}$

12 Find the rule for the inverse of each of the following.
a $4x - 8y = 1$ b $y = -\frac{2}{3}x - 4$ c $y^2 = 4x$
d $y = 4x^2$ e $x^2 + (y - 3)^2 = 1$ f $y = \sqrt{2x + 1}$

13 Consider the hyperbola function with the rule $f(x) = \frac{1}{x - 2}$ defined on its maximal domain.
a State the maximal domain and the equations of the asymptotes of the function.
b Obtain the rule for its inverse.
c State the equations of the asymptotes of the inverse.
d For any hyperbola with equation $y = \frac{1}{x - a} + b$, what would the equations of the asymptotes of its inverse be? Write down the equation of the inverse.

14 Consider $f: [-2, 4) \to R, f(x) = 4 - 2x$.
a What are the domain and range of $f^{-1}$?
b Obtain the rule for $f^{-1}$ and write $f^{-1}$ as a mapping.
c Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.
d For what value of $x$ does $f(x) = f^{-1}(x)$?

15 **a** Sketch the graphs of the following by reflecting the given relation in the line $y = x$.

**i** $y = 1$ and its inverse

**ii** $y = \sqrt{-x}$ and its inverse

**iii** $y = \dfrac{1}{x^2}$ and its inverse

**iv** $y = x^3 - 1$ and its inverse

**v** $y = 1 - \sqrt{1 - x^2}$ and its inverse

**vi** $(y - 2)^2 = 1 - x$ and its inverse

**b** For those relations in part **a** that are functions, state which of them have inverses that are also functions and give the inverses' rules.

16 Sketch the graphs of each of the following hybrid functions and state their domain and range.

**a** $y = \begin{cases} 2, & x \le 0 \\ 1 + x^2, & x > 0 \end{cases}$

**b** $y = \begin{cases} x^3, & x < 1 \\ x, & x \ge 1 \end{cases}$

**c** $y = \begin{cases} -2x, & x < -1 \\ 2, & -1 \le x \le 1 \\ 2x, & x > 1 \end{cases}$

**d** $y = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ \dfrac{1}{2 - x}, & x > 2 \end{cases}$

**e** $y = \begin{cases} x^{\frac{1}{3}}, & x \ge 0 \\ -x^{-2}, & x < 0 \end{cases}$

**f** $y = \begin{cases} x^3, & x < -1 \\ \sqrt[3]{x}, & -1 \le x \le 1 \\ 3x, & x > 1 \end{cases}$

17 Consider $f : R \to R, f(x) = \begin{cases} -x - 1, & x < -1 \\ \sqrt{1 - x^2}, & -1 \le x \le 1 \\ x + 1, & x > 1 \end{cases}$

**a** Calculate the value of:

**i** $f(0)$

**ii** $f(3)$

**iii** $f(-2)$

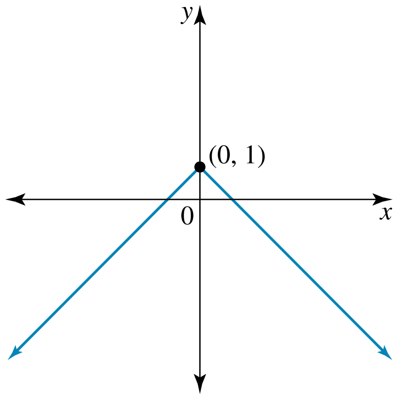
**iv** $f(1)$

**b** Show the function is not continuous at $x = 1$.

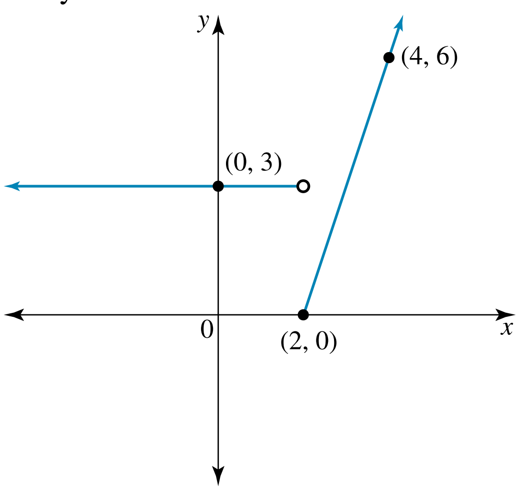
**c** Sketch the graph of $y = f(x)$ and state the type of correspondence.

**d** Determine the value of $a$ such that $f(a) = a$.

18 **a** Form a rule for the graph of the hybrid function.




**b** Form the rule for the graph of the hybrid function.



c Determine the values of $a$ and $b$ so that the function with the rule

$$f(x) = \begin{cases} a, & x \in (-\infty, -3] \\ x + 2, & x \in (-3, 3) \\ b, & x \in [3, \infty) \end{cases}$$

is continuous for $x \in R$; for these values, sketch the graph of $y = f(x)$.

d In an effort to reduce the time her children spend in the shower, a mother introduced a penalty scheme with fines to be paid from the children's pocket money according to the following: If someone spends more than 5 minutes in the shower, the fine in dollars is equal to the shower time in minutes; if someone spends up to and including 5 minutes in the shower, there is no fine. If someone chooses not to shower at all, there is a fine of \$2 because that child won't be nice to be near.

Defining appropriate symbols, express the penalty scheme as a mathematical rule in hybrid form and sketch the graph which represents it.

19 Consider the function $g : D \to R, g(x) = x^2 + 8x - 9$.

a Give two possible domains $D$ for which $g^{-1}$ exists.
b If the domain of $g$ is chosen to be $R^+$, form $g^{-1}$ and state its range.
c i For this domain $R^+$, sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes.
 ii Calculate the exact coordinates of any point(s) of intersection of the two graphs.
d What number is its own image under the mapping $g^{-1}$?

20 A function is defined by the rule $f(x) = \begin{cases} \sqrt{x-4} + 2, & x > 4 \\ 2, & 1 \le x \le 4 \\ \sqrt{1-x} + 2, & x < 1 \end{cases}$

a Calculate the values of $f(0)$ and $f(5)$.
b Sketch $y = f(x)$ and state its domain and range.
c For what values of $x$ does $f(x) = 8$?
d Will the inverse of this function also be a function? Draw a sketch of the graph of the inverse on a new set of axes.

MASTER

21 a Use CAS technology to find algebraically the rule for the inverse of $y = x^2 + 5x - 2$.
b Use CAS technology to draw the graph of $y = x^2 + 5x - 2$ and its inverse and find, to 2 decimal places, the coordinates of the points of intersection of $y = x^2 + 5x - 2$ with its inverse.

22 Consider the hybrid function with the rule $y = \begin{cases} \sqrt{2 + x^2}, & x \le -2 \\ x\sqrt{2 + x}, & -2 < x < 2 \\ \dfrac{1}{\sqrt{2 + x}}, & x \ge 2 \end{cases}$

a Identify any points of discontinuity.
b Graph the function and give its exact range.

# 6.7 Transformations of functions

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**Transformations of functions**
Concept summary
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Graph plotter: Transformations of functions
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Once the basic shape of a function is known, its features can be identified after various **transformations** have been applied to it simply by interpreting the transformed equation of the image. For example, we recognise the equation $y = (x - 1)^2 + 3$ to be a parabola formed by translating the basic parabola $y = x^2$ horizontally 1 unit to the right and vertically 3 units up. In exactly the same way, we recognise $y = \frac{1}{x-1} + 3$ to be a hyperbola formed by translating the basic hyperbola $y = \frac{1}{x}$ horizontally 1 unit to the right and vertically 3 units up. The only difference is the basic shape.

In this section we extend such interpretation of the transformed equation to consider the image of a general function $y = f(x)$ under a sequence of transformations.

## Horizontal and vertical translations of *y* = *f*(*x*)

Translations parallel to the *x*- and *y*-axis move graphs horizontally to the left or right and vertically up or down, respectively.

Under a horizontal translation of *h* units to the right, the following effect is seen:

$$y = x^2 \rightarrow y = (x - h)^2;$$
$$y = \frac{1}{x} \rightarrow y = \frac{1}{x - h};$$
$$y = \sqrt{x} \rightarrow y = \sqrt{x - h};$$

and so, for any function, $y = f(x) \rightarrow y = f(x - h)$.

Under a vertical translation of *k* units upwards:

$$y = x^2 \rightarrow y = x^2 + k;$$
$$y = \frac{1}{x} \rightarrow y = \frac{1}{x} + k;$$
$$y = \sqrt{x} \rightarrow y = \sqrt{x} + k;$$

and so, for any function, $y = f(x) \rightarrow y = f(x) + k$

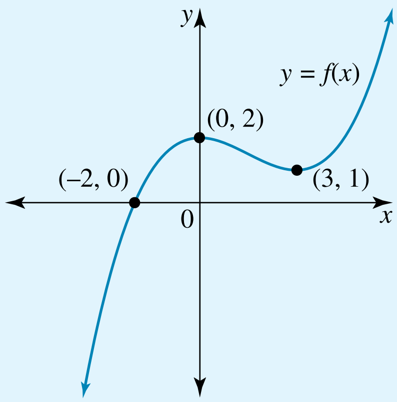
**Thus, for any function:**

- $y = f(x - h)$ **is the image of** $y = f(x)$ **under a horizontal translation of** $h$ **units to the right.**
- $y = f(x) + k$ **is the image of** $y = f(x)$ **under a vertical translation of** $k$ **units upwards.**
- **Under the combined transformations of** $h$ **units parallel to the** $x$**-axis and** $k$ **units parallel to the** $y$**-axis,** $y = f(x) \rightarrow y = f(x - h) + k$**.**

WORKED EXAMPLE 21

The diagram shows the graph of $y = f(x)$ passing through points $(-2, 0)$, $(0, 2)$, $(3, 1)$.

Sketch the graph of $y = f(x + 1)$ using the images of these three points.




THINK

1 Identify the transformation required.

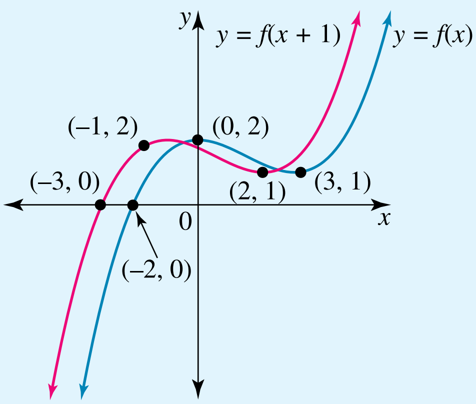
WRITE

$y = f(x + 1)$
This is a horizontal translation 1 unit to the left of the graph of $y = f(x)$.

2 Find the image of each key point.

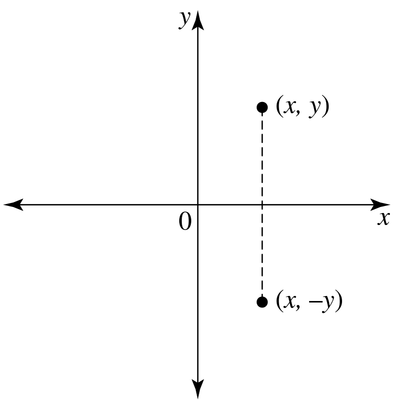
Under this transformation:
$(-2, 0) \rightarrow (-3, 0)$
$(0, 2) \rightarrow (-1, 2)$
$(3, 1) \rightarrow (2, 1)$

3 Sketch the image.

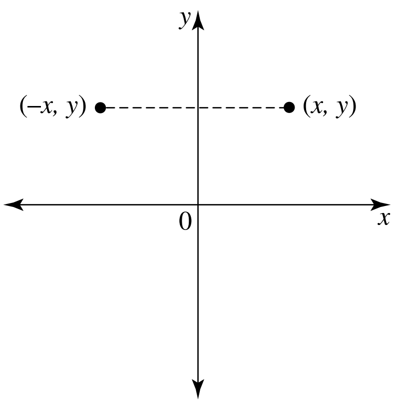



## Reflections in the coordinate axes

The point $(x, y)$ becomes $(x, -y)$ when reflected in the $x$-axis and $(-x, y)$ when reflected in the $y$-axis.




Reflection in the $x$-axis




Reflection in the $y$-axis

Reflecting the graph of $y = \sqrt{x}$ in the $x$-axis gives the graph of $y = -\sqrt{x}$, so under a reflection in the $x$-axis, $y = f(x) \rightarrow y = -f(x)$.

Reflecting the graph of $y = \sqrt{x}$ in the $y$-axis gives the graph of $y = \sqrt{-x}$, so under a reflection in the $y$-axis, $y = f(x) \rightarrow y = f(-x)$.

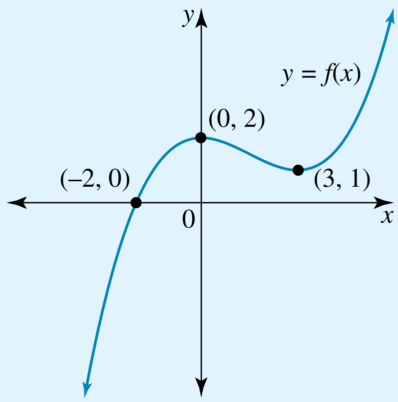
**For any function:**

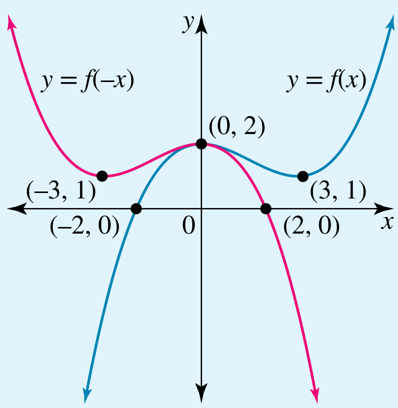
- **$y = -f(x)$ is the image of $y = f(x)$ under a reflection in the $x$-axis**
- **$y = f(-x)$ is the image of $y = f(x)$ under a reflection in the $y$-axis**

WORKED EXAMPLE 22

**Consider again the graph in Worked example 21.**

**Sketch the graph of $y = f(-x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.**




| THINK | WRITE |
|---|---|
| 1 Identify the transformation required. | $y = f(-x)$<br>This is a reflection in the $y$-axis of the graph of $y = f(x)$. |
| 2 Find the image of each key point. | Under this transformation, $(x, y) \rightarrow (-x, y)$<br>$(-2, 0) \rightarrow (2, 0)$<br>$(0, 2) \rightarrow (0, 2)$<br>$(3, 1) \rightarrow (-3, 1)$ |
| 3 Sketch the image. |  |



## Dilations from the coordinate axes

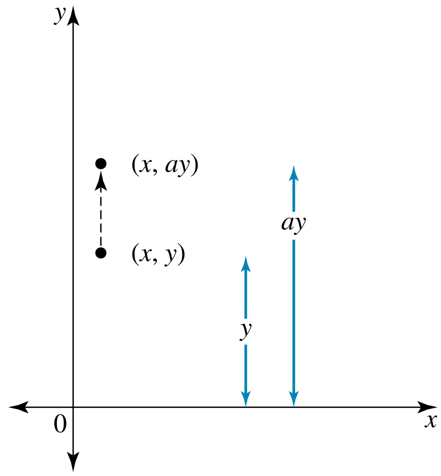
A dilation from an axis either stretches or compresses a graph from that axis, depending on whether the dilation factor is greater than 1 or between 0 and 1, respectively.

## Dilation from the $x$-axis by factor $a$

A dilation from the $x$-axis acts parallel to the $y$-axis, or in the $y$-direction.

The point $(x, y) \to (x, ay)$ when dilated by a factor $a$ from the $x$-axis.

A dilation of factor $a$ from the $x$-axis transforms $y = x^2$ to $y = ax^2$ and, generalising, under a dilation of factor $a$ from the $x$-axis, $y = f(x) \to y = af(x)$.



Dilation of factor $a$, $(a > 1)$, from the $x$-axis

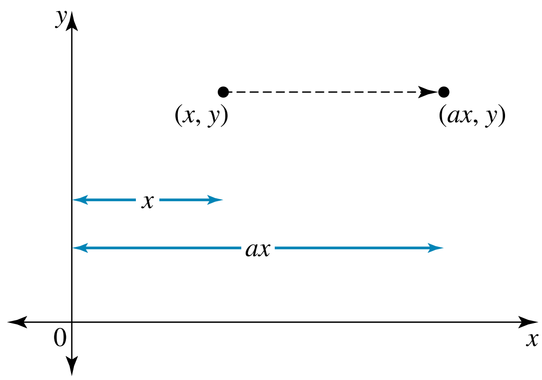
**For any function:**

- **$y = af(x)$ is the image of $y = f(x)$ under a dilation of factor $a$ from the $x$-axis, parallel to the $y$-axis.**

## Dilation from the $y$-axis by factor $a$

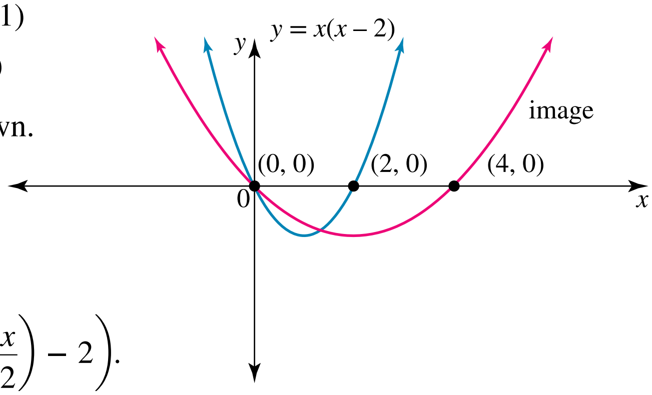
A dilation from the $y$-axis acts parallel to the $x$-axis, or in the $x$-direction.

The point $(x, y) \to (ax, y)$ when dilated by a factor $a$ from the $y$-axis. To see the effect of this dilation, consider the graph of $y = x(x - 2)$ under a dilation of factor 2 from the $y$-axis. Choosing the key points, under this dilation:




$$(0, 0) \to (0, 0)$$
$$(1, -1) \to (2, -1)$$
$$(2, 0) \to (4, 0)$$

and the transformed graph is as shown.




The equation of the image of $y = x(x - 2)$ can be found by fitting the points to a quadratic equation. Its equation is

$$y = (0.5x)(0.5x - 2) \Rightarrow y = \left(\frac{x}{2}\right)\left(\left(\frac{x}{2}\right) - 2\right).$$

This illustrates that dilating $y = f(x)$ by a factor $a$ from the $y$-axis gives the image $y = f\left(\frac{x}{a}\right)$.

**For any function:**

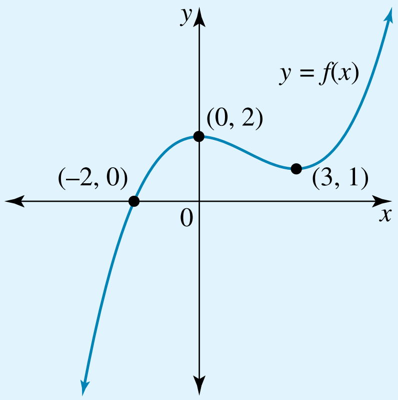
- **$y = f(ax)$ is the image of $y = f(x)$ under a dilation of factor $\frac{1}{a}$ from the $y$-axis, parallel to the $x$-axis.**
- **$y = f\left(\frac{x}{a}\right)$ is the image of $y = f(x)$ under a dilation of factor $a$ from the $y$-axis, parallel to the $x$-axis.**

Since $y = af(x)$ could be written as $\frac{y}{a} = f(x)$, and $y = f(ax)$ as $y = f\left(\frac{x}{\frac{1}{a}}\right)$, dilations from the coordinate axes can be summarised as follows.

- Under a dilation of factor $a$ from the $x$-axis (in the $y$-direction), $y = f(x) \rightarrow \frac{y}{a} = f(x)$.
- Under a dilation of factor $a$ from the $y$-axis (in the $x$-direction), $y = f(x) \rightarrow y = f\left(\frac{x}{a}\right)$.

**WORKED EXAMPLE 23**

For the graph given in Worked examples 21 and 22, sketch the graph of $y = f(2x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.




**THINK**

**WRITE**

1 Identify the transformation.

$y = f(2x) \Rightarrow y = f\left(\frac{x}{\frac{1}{2}}\right)$

The transformation is a dilation from the $y$-axis of factor $\frac{1}{2}$.

This dilation acts in the $x$-direction.

2 Find the image of each key point.

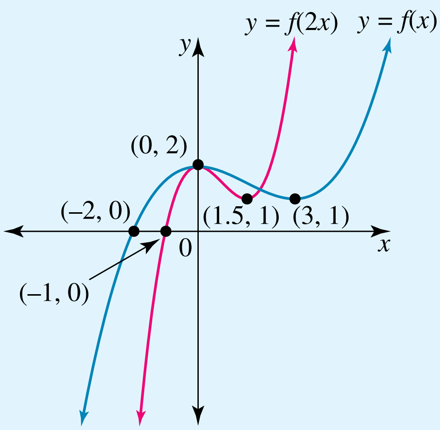
Under this dilation $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$

$(-2, 0) \rightarrow (-1, 0)$

$(0, 2) \rightarrow (0, 2)$

$(3, 1) \rightarrow (1.5, 1)$

3 Sketch the image.



## Combinations of transformations

**The graph of $y = af(n(x - h)) + k$ is the graph of $y = f(x)$ under a set of transformations which are identified as follows.**

- **$a$ gives the dilation factor $|a|$ from the $x$-axis, parallel to the $y$-axis.**
- **If $a < 0$, there is a reflection in the $x$-axis.**
- **$n$ gives the dilation factor $\frac{1}{|n|}$ from the $y$-axis, parallel to the $x$-axis.**
- **If $n < 0$, there is a reflection in the $y$-axis.**
- **$h$ gives the horizontal translation parallel to the $x$-axis.**
- **$k$ gives the vertical translation parallel to the $y$-axis.**

When applying transformations to $y = f(x)$ to form the graph of $y = af(n(x - h)) + k$, the order of operations can be important, so any dilation or reflection should be applied before any translation.

It is quite possible that more than one order or more than one set of transformations may achieve the same image. For example, $y = 4x^2$ could be considered a dilation of $y = x^2$ by factor 4 from the $x$-axis or, as $y = (2x)^2$, it's also a dilation of $y = x^2$ by a factor $\frac{1}{2}$ from the $y$-axis.

WORKED EXAMPLE 24

**a** Describe the transformations applied to the graph of $y = f(x)$ to obtain $y = 4 - 2f(3x + 2)$.

**b** Describe the transformations applied to the graph of $y = \sqrt[3]{x}$ to obtain $y = \sqrt[3]{6 - 2x}$.

| THINK | WRITE |
|---|---|
| **a 1** Express the image equation in the summary form. | **a** $y = 4 - 2f(3x + 2)$<br>$= -2f\left(3\left(x + \frac{2}{3}\right)\right) + 4$ |
| **2** State the values of $a, n, h, k$ from the summary form. | $y = af(n(x - h)) + k$<br>$a = -2,\ n = 3,\ h = -\frac{2}{3},\ k = 4$ |
| **3** Interpret the transformations, leaving the translations to last. | Dilation of factor 2 from the $x$-axis, followed by a reflection in the $x$-axis; then, a dilation of factor $\frac{1}{3}$ from the $y$-axis; then, a horizontal translation $\frac{2}{3}$ units to the left; finally, a vertical translation upwards of 4 units |
| **b 1** Express the image equation in the summary form. | **b** $y = \sqrt[3]{6 - 2x}$<br>$= \sqrt[3]{-2(x - 3)}$ |
| **2** Identify the transformations in the correct order. | Dilation of factor $\frac{1}{2}$ from the $y$-axis, followed by a reflection in the $y$-axis; then, a horizontal translation 3 units to the right |

## EXERCISE 6.7 Transformations of functions

**PRACTISE**

Work without CAS

1 WE21 For the graph of $y = f(x)$ given in Worked example 21, sketch the graph of $y = f(x) - 2$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.

2 For the graph of $y = f(x)$ given in Worked example 21, sketch the graph of $y = f(x - 2) + 1$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.

3 WE22 Consider again the graph given in Worked examples 21 and 22. Sketch the graph of $y = -f(x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.

4 a The parabola with equation $y = (x - 1)^2$ is reflected in the $x$-axis followed by a vertical translation upwards of 3 units. What is the equation of its final image?

b Obtain the equation of the image if the order of the transformations in part **a** was reversed. Is the image the same as that in part **a**?

5 WE23 For the graph in Worked example 23, sketch the graph of $y = f\left(\frac{x}{2}\right)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.

6 For the graph of Worked example 23, sketch the graph of $y = \frac{1}{2}f(x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.

7 WE24 a Describe the transformations applied to the graph of $y = f(x)$ to obtain $y = 4f\left(\frac{x}{2} - 1\right) + 3$.

b Describe the transformations applied to the graph of $y = \sqrt{x}$ to obtain $y = \sqrt{3 - \frac{x}{4}}$.

8 a The graph of $y = \frac{1}{x}$ undergoes two transformations in the order: dilation of factor $\frac{1}{2}$ from the $y$-axis, followed by a horizontal translation of 3 units to the left. What is the equation of its image?

b Describe the sequence of transformations that need to be applied to the image to undo the effect of the transformations and revert to the graph of $y = \frac{1}{x}$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 Identify the transformations that would be applied to the graph of $y = x^2$ to obtain each of the following graphs.

a $y = 3x^2$ b $y = -x^2$ c $y = x^2 + 5$ d $y = (x + 5)^2$

10 Describe the transformations that have been applied to the graph of $y = x^3$ to obtain each of the following graphs.

a $y = \left(\frac{x}{3}\right)^3$ b $y = (2x)^3 + 1$

c $y = (x - 4)^3 - 4$ d $y = (1 + 2x)^3$

11 Give the equation of the image of **i** $y = \sqrt{x}$ and **ii** $y = x^4$ if their graphs are:

a dilated by a factor 2 from the $x$-axis

b dilated by a factor 2 from the $y$-axis

c reflected in the $x$-axis and then translated 2 units vertically upwards

d translated 2 units vertically upwards and then reflected in the $x$-axis

**e** reflected in the $y$-axis and then translated 2 units to the right

**f** translated 2 units to the right and then reflected in the $y$-axis.

**12** Give the coordinates of the image of the point $(3, -4)$ if it is:

**a** translated 2 units to the left and 4 units down

**b** reflected in the $y$-axis and then reflected in the $x$-axis

**c** dilated by a factor $\frac{1}{5}$ from the $x$-axis parallel to the $y$-axis

**d** dilated by a factor $\frac{1}{5}$ from the $y$-axis parallel to the $x$-axis

**e** reflected in the line $y = x$

**f** rotated 90° anticlockwise about the origin.

**13** **a** **i** Give the equation of the image of $y = \frac{1}{x}$ after the two transformations are applied in the order given: dilation by a factor 3 from the $y$-axis, then reflection in the $y$-axis.

**ii** Reverse the order of the transformations and give the equation of the image.

**b** **i** Give the equation of the image of $y = \frac{1}{x^2}$ after the two transformations are applied in the order given: dilation by a factor 3 from the $x$-axis, then vertical translation 6 units up.

**ii** Reverse the order of the transformations and give the equation of the image.

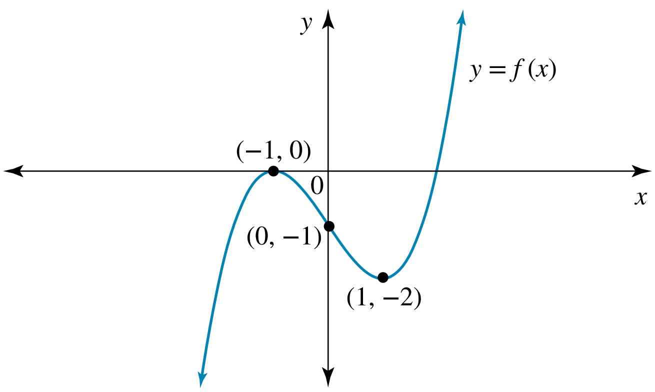
**c** Describe the transformations applied to $y = \frac{1}{x}$ if its image has the equation

$$y = -\frac{1}{2x + 2} + 1.$$

**d** If $f(x) = \frac{1}{x^2}$, give the equations of the asymptotes of $y = -2f(x + 1)$.

**14** The graph of $y = f(x)$ is shown.

On separate diagrams sketch the graphs of the following.




**a** $y = f(x - 1)$

**b** $y = -f(x)$

**c** $y = 2f(x)$

**d** $y = f(-x)$

**e** $y = f\left(\frac{x}{2}\right)$

**f** $y = f(x) + 2$

**15** Describe the transformations applied to $y = f(x)$ if its image is:

**a** $y = 2f(x + 3)$

**b** $y = 6f(x - 2) + 1$

**c** $y = f(2x + 2)$

**d** $y = f(-x + 3)$

**e** $y = 1 - f(4x)$

**f** $y = \frac{1}{9}f\left(\frac{x - 3}{9}\right)$.

**16** Form the equation of the image after the given functions have been subjected to the set of transformations in the order specified.

**a** $y = \frac{1}{x^2}$ undergoes a dilation of factor $\frac{1}{3}$ from the $x$-axis followed by a horizontal translation of 3 units to the left.

b $y = x^5$ undergoes a vertical translation of 3 units down followed by reflection in the $x$-axis.

c $y = \frac{1}{x}$ undergoes a reflection in the $y$-axis followed by a horizontal translation of 1 unit to the right.

d $y = \sqrt[3]{x}$ undergoes a horizontal translation of 1 unit to the right followed by a dilation of factor 0.5 from the $y$-axis.

e $y = (x + 9)(x + 3)(x - 1)$ undergoes a horizontal translation of 6 units to the right followed by a reflection in the $x$-axis.

f $y = x^2(x + 2)(x - 2)$ undergoes a dilation of factor 2 from both the $x$- and $y$-axis.

**17** a The function $g : R \to R, g(x) = x^2 - 4$ is reflected in the $y$-axis. Describe its image.

b Show that the image of the function $f : R \to R, f(x) = x^{\frac{1}{3}}$ when it is reflected in the $y$-axis is the same as when it is reflected in the $x$-axis.

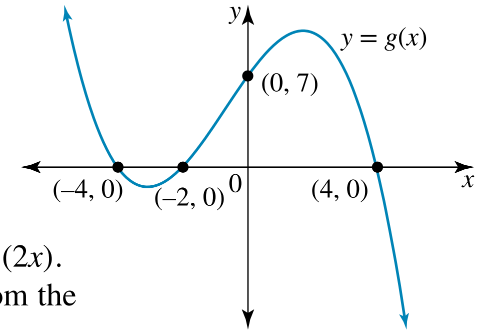
c The function $h : [-3, 3] \to R, h(x) = -\sqrt{9 - x^2}$ is reflected in the $x$-axis. Describe its image. What single transformation when applied to the image would return the curve back to its original position?

d The graph of $y = (x - 2)^2 + 5$ is reflected in both the $x$- and $y$-axis. What is the nature, and the coordinates, of the turning point of its image?

e The graph of a relation is shifted vertically down 2 units, then reflected in the $y$-axis. If the equation of its image is $y^2 = (x - 3)$, undo the transformations (that is, form the inverse transformations) to obtain the equation of the original graph.

f A curve $y = f(x)$ is dilated by a factor 2 from the $x$-axis, then vertically translated 1 unit up, then reflected in the $x$-axis. After these three transformations have been applied, the equation of its image is $y = 6(x - 2)^3 - 1$. Determine the equation of $y = f(x)$.

**18** The graph of the function $y = g(x)$ is given.




a Sketch the graph of $y = -g(2x)$.

b Sketch the graph of $y = g(2 - x)$.

c For what values of $h$ will all the $x$-intercepts of the graph of $y = g(x + h)$ be negative?

d Give a possible equation for the graph of $y = g(x)$ and hence find an expression for $g(2x)$.

e Reduce the graph of $y = x^3 (x - 4)^2 + 7$ from the graph of $y = x^3(x - 4)^2$

MASTER

**19** On the same screen, graph $y_1 = x^2 + 5x - 6$, $y_2 = (2x)^2 + 5(2x) - 6$ and $y_3 = \left(\frac{x}{2}\right)^2 + 5\left(\frac{x}{2}\right) - 6$ and compare the graphs. Which are parabolas?

**20** For the function defined by $f(x) = \sqrt{16 - x^2}$, sketch on the same screen $y_1 = f(x)$, $y_2 = f(2x)$, $y_3 = 2f(x)$ and $y_4 = f\left(\frac{x}{2}\right)$ and compare the graphs. Which are semicircles?

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# 6.8 Review

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**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

## REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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# Activities

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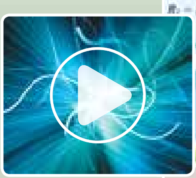
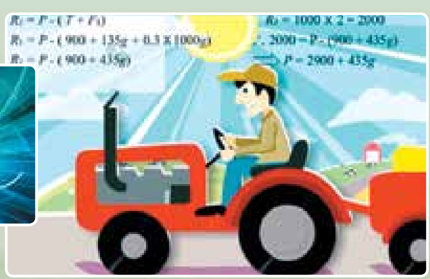
## Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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Units 1 & 2 — **Functions and relations**

**Sit topic test**

# 6 Answers

## EXERCISE 6.2

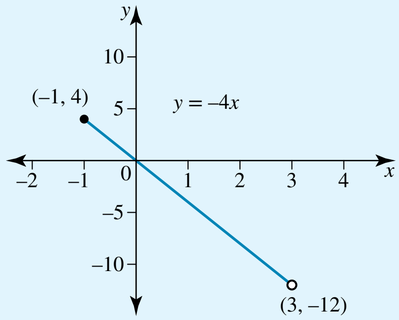
**1 a** Domain $\{0, 2, 3, 4\}$; range $\{-1, 0, 3, 4\}$; a function

**b** Domain $[-2, \infty)$; range $R$; not a function

**c** Domain $[0, 3]$; range $[0, 4]$; a function

**d** Maximal domain $R$; range $(-\infty, 4]$; a function

**2**

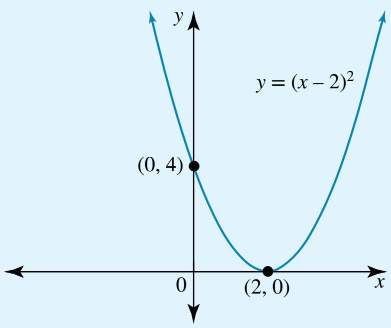



Domain is $[-1, 3)$; range is $(-12, 4]$.

**3 a** One-to-one correspondence; a function

**b** Many-to-many correspondence; not a function

**4 a**




Domain $R$; range $R^+ \cup \{0\}$; many-to-one correspondence

**b** An answer is $[2, \infty)$.

**5 a** $a = 2,\ b = 3;\ f(x) = 2x + 3$

**b** $f(0) = 3$

**c** $x = -1.5$

**d** $f(-3) = -3$

**e** $g : (-\infty, 0] \to R,\ g(x) = 2x + 3$

**6** $f : [0, 7) \to R, f(x) = x^2 - 6x + 10$; domain is $[0, 7)$; range is $[1, 17)$.

**7 a** Domain $[0, 5]$; range $[0, 15]$

**b** Domain $[-4, 2) \cup (2, \infty)$; range $(-\infty, 10)$

**c** Domain $[-3, 6]$; range $[0, 8]$

**d** Domain $[-2, 2]$; range $[-4, 4]$

**e** Domain $\{3\}$; range $R$

**f** Domain $R$; range $R$

**8 a** Relation **a** is one-to-one; **b** is many-to-one; **c** is many-to-one; **d** is one-to-many; **e** is one-to-many; **f** is many-to-one

**b** Relations **d** and **e** are not functions.

**9 a** Domain $\{-11, -3, -1, 5\}$; range $\{0, 2, 8\}$; many-to-one function

**b** Domain $\{20, 50, 60\}$; range $\{6, 10, 20\}$; many-to-many: not a function

**c** Domain $\{-14, 0, 14\}$; range $\{-7, 0, 2, 7\}$; one-to-many: not a function

**d** Domain $R$; range $R$; one-to-one function

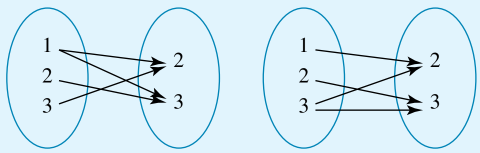
**e** Domain $R$; range $(-\infty, 4]$; many-to-one function

**f** Domain $R$; range $[0, \infty)$; many-to-one function

**10 a** $k = 1$ or $k = 3$; many-to-many

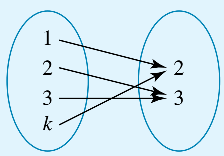
**b** $k \in R \backslash \{1, 3\}$; many-to-one

**c**

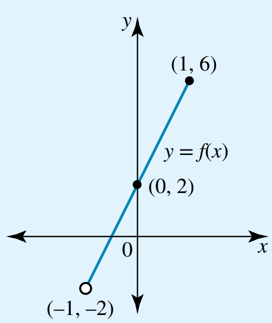



$k = 1$ $\qquad$ $k = 3$

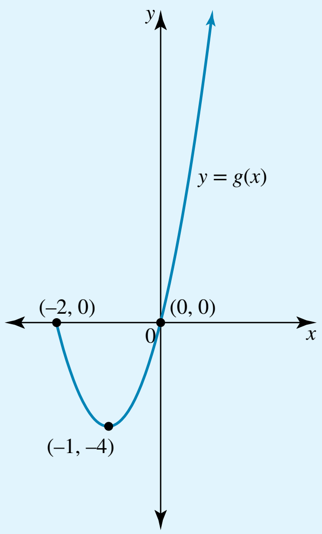
**d**



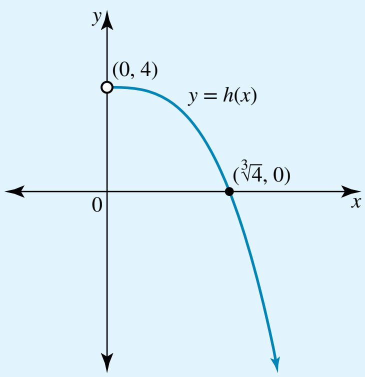

**11 a** Line; endpoints $(-1, -2)$(open), $(1, 6)$(closed); domain $(-1, 1]$; range $(-2, 6]$



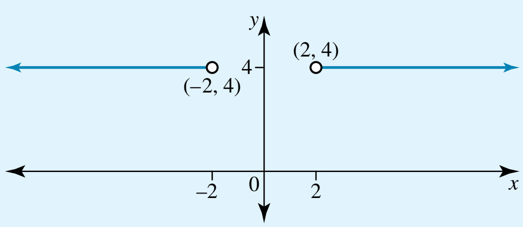
**b** Parabola; endpoint $(-2, 0)$(closed); minimum turning point $(-1, -4)$; $y$-intercept $(0, 0)$; domain $[-2, \infty)$; range $[-4, \infty)$




**c** Cubic with stationary point of inflection; open endpoint $(0, 4)$; $x$-intercept $(\sqrt[3]{4}, 0)$; domain $R^+$; range $(-\infty, 4)$




**d** Two sections of a horizontal line; open endpoints at $(\pm 2, 4)$; domain $R\backslash[-2, 2]$; range $\{4\}$




**12 a** i $-3$

ii $96$

**b** i $f(2a) = 4a^2 + 4a - 3$

ii $f(1 - a) = a^2 - 4a$

**c** $2xh + h^2 + 2h$

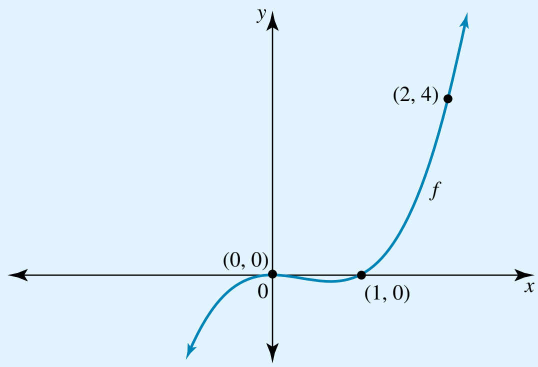
**d** $\{x : x < -3\} \cup \{x : x > 1\}$

**e** $x = -5, x = 3$

**f** $x = -4, x = 1$

**13 a** 4

**b** $x$-intercepts: $(0, 0)$ at turning point, $(1, 0)$




**c** Domain $R$; range $R$

**d** Many-to-one

**e** An answer is $R^-$.

**f** $\{2\}$

**14 a** Function; $f : Z^+ \to R, f(x) = x^2$; domain $Z^+$; range $\{1, 4, 9, 16, \ldots\}$

**b** Function; $f : R \to R, f(x) = \dfrac{6 - 2x}{3}$; domain $R$; range $R$

**c** Not a function

**d** Function; $f : R \to R, f(x) = 5$; domain $R$; range $\{5\}$

**e** Not a function

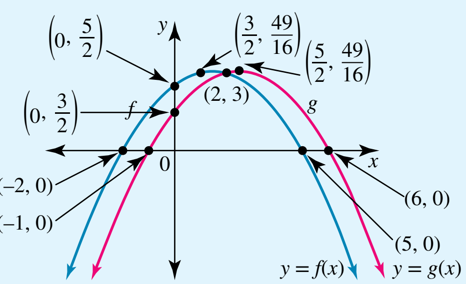
**f** Function; $f : R^+ \to R, f(x) = -x^5$; domain $R^+$; range $R^-$

**15 a** $f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$

**b** $g(x) = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$

**c** $x = 2$

**d**




Graphs intersect at $(2, 3)$.

$y = f(x)$: maximum turning point $\left(\frac{3}{2}, \frac{49}{16}\right)$;

$x$-intercepts $(-2, 0)$, $(5, 0)$; $y$-intercept $\left(0, \frac{5}{2}\right)$

$y = g(x)$: maximum turning point $\left(\frac{5}{2}, \frac{49}{16}\right)$;

$x$-intercepts $(-1, 0)$, $(6, 0)$; $y$-intercept $\left(0, \frac{3}{2}\right)$

$y = g(x)$ has the same shape as $y = f(x)$ but it has been translated 1 unit to the right.

**16 a** Domain $\{t : 0 \le t \le 5\}$; range $\{x : 4 \le x \le 29\}$; the distance travelled is 25 units.

**b** 2 seconds; domain $[0, 2]$; range $[0, 5]$

**c** i Domain $[0, 2]$; range $[0.5, 2.1]$

ii Approximately 1.4 weeks

**17** Not a function; domain $R$; range $R\backslash(-1, 1)$

**18 a** $l = 0$; $m = -12$; $n = -16$; $f(x) = x^3 - 12x - 16$

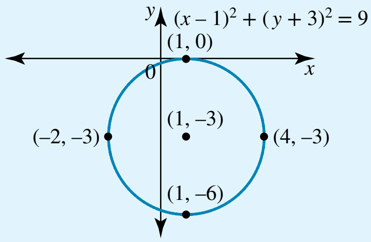
**b** $-28.672$

**c** $4.477$

**d** $f: R^+ \cup \{0\} \rightarrow R, f(x) = x^3 - 12x - 16$; domain $R^+ \cup \{0\}$; range $[-32, \infty)$

## EXERCISE 6.3

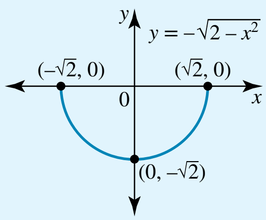
**1 a** Centre $(1, -3)$; radius 3; domain $[-2, 4]$; range $[-6, 0]$



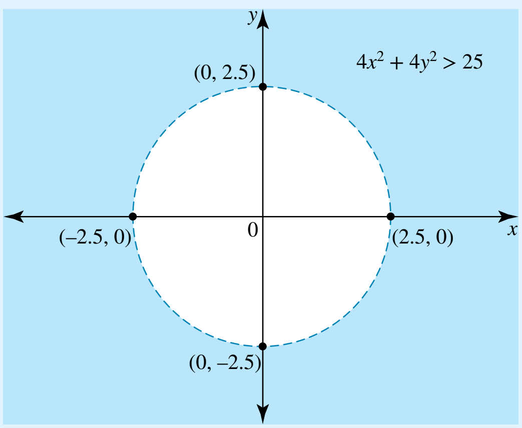

**b** Centre $(-1, -4)$; radius $\sqrt{17}$; domain $[-1 - \sqrt{17}, -1 + \sqrt{17}]$; range $[-4 - \sqrt{17}, -4 + \sqrt{17}]$

**2** $x^2 + y^2 + 10x - 33 = 0$

**3 a** Centre $(0, 0)$; radius $\sqrt{2}$; domain $[-\sqrt{2}, \sqrt{2}]$; range $[-\sqrt{2}, 0]$




**b** Open region outside the circle with centre $(0, 0)$ and radius $\frac{5}{2}$




**c** Upper semicircle $y = \sqrt{\frac{25}{4} - x^2}$; domain $\left[-\frac{5}{2}, \frac{5}{2}\right]$; range $\left[0, \frac{5}{2}\right]$

**4 a** $(x + 2)^2 + (y - 2)^2 = 12$

**b** Domain $[-2 - 2\sqrt{3}, -2 + 2\sqrt{3}]$; range $[2, 2 + 2\sqrt{3}]$

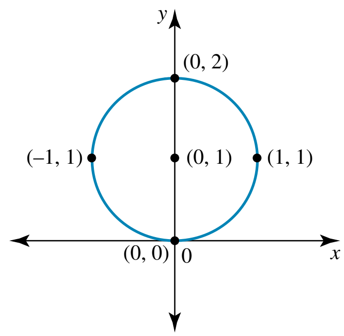
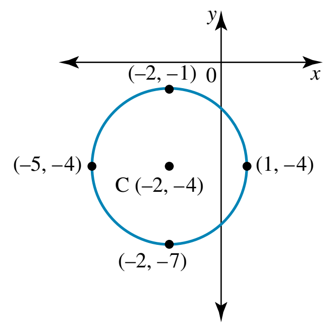
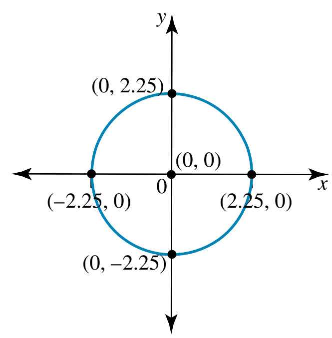
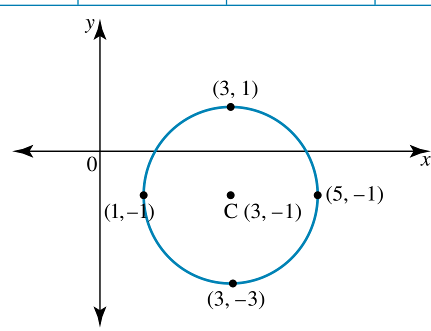
**5 a** $y = -\frac{1}{3}x - 3$

**b** $\sqrt{55}$ units

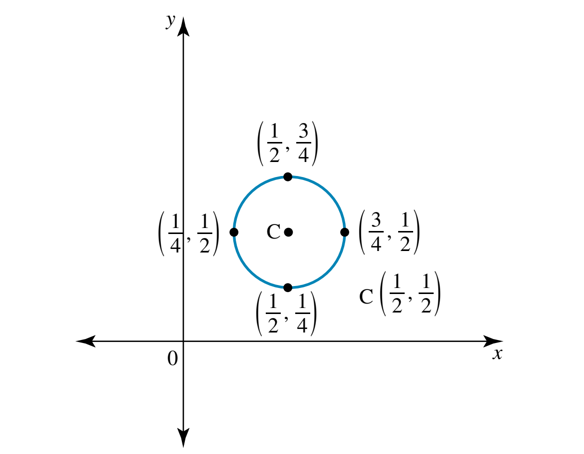
**c** No intersections

**6** $m = \pm\frac{\sqrt{5}}{2}$

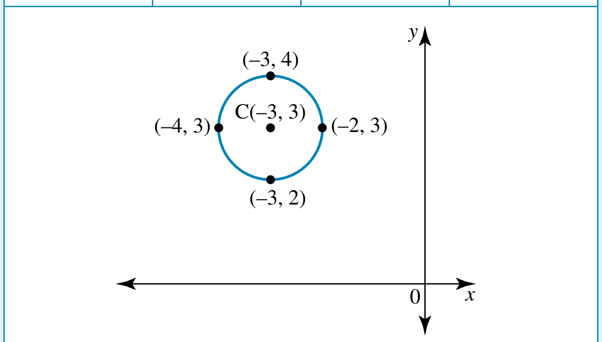
**7**

<table>
<tr><th></th><th>Centre</th><th>Radius</th><th>Domain</th><th>Range</th></tr>
<tr><td>a</td><td>$(0, 1)$</td><td>1</td><td>$[-1, 1]$</td><td>$[0, 2]$</td></tr>
<tr><td></td><td colspan="4">
</td></tr>
<tr><td>b</td><td>$(-2, -4)$</td><td>3</td><td>$[-5, 1]$</td><td>$[-7, -1]$</td></tr>
<tr><td></td><td colspan="4">
</td></tr>
<tr><td>c</td><td>$(0, 0)$</td><td>$\frac{9}{4}$</td><td>$\left[-\frac{9}{4}, \frac{9}{4}\right]$</td><td>$\left[-\frac{9}{4}, \frac{9}{4}\right]$</td></tr>
<tr><td></td><td colspan="4">
</td></tr>
<tr><td>d</td><td>$(3, -1)$</td><td>2</td><td>$[1, 5]$</td><td>$[-3, 1]$</td></tr>
<tr><td></td><td colspan="4">
</td></tr>
</table>

| | Centre | Radius | Domain | Range |
|---|---|---|---|---|
| e | $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $\frac{1}{4}$ | $\left[\frac{1}{4}, \frac{3}{4}\right]$ | $\left[\frac{1}{4}, \frac{3}{4}\right]$ |




| | Centre | Radius | Domain | Range |
|---|---|---|---|---|
| f | (−3, 3) | 1 | [−4, −2] | [2, 4] |




**8** a $(x + 8)^2 + (y - 9)^2 = 36$

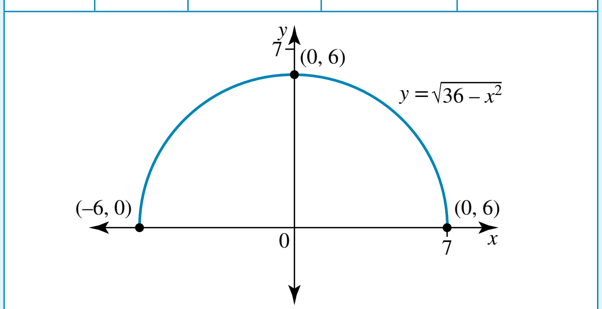
b $(x - 7)^2 + y^2 = 8$

c $(x - 1)^2 + (y - 6)^2 = 136$

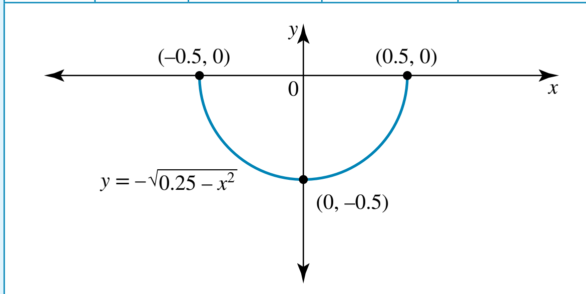
d $9x^2 + 9(y - 2)^2 = 16$

**9**

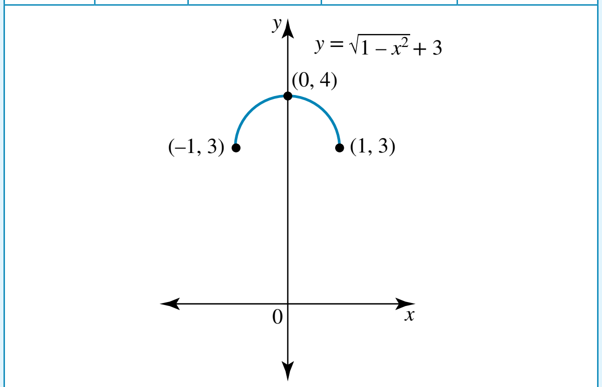
| | Centre | Radius | Domain | Range | Description |
|---|---|---|---|---|---|
| a | (0, 0) | 6 | [−6, 6] | [0, 6] | Upper semicircle |



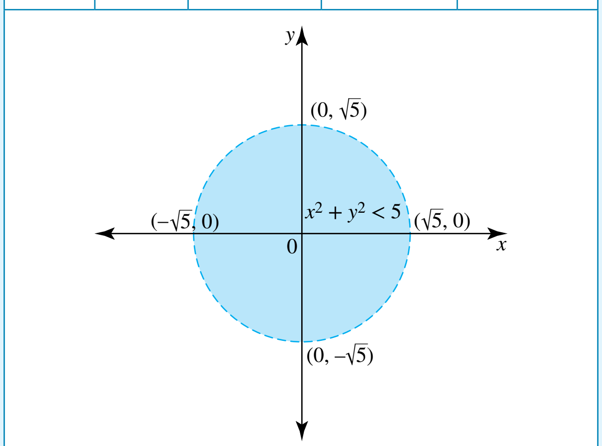

| | Centre | Radius | Domain | Range | Description |
|---|---|---|---|---|---|
| b | (0, 0) | 0.5 | [−0.5, 0.5] | [−0.5, 0] | Lower semicircle |




| | Centre | Radius | Domain | Range | Description |
|---|---|---|---|---|---|
| c | (0, 3) | 1 | [−1, 1] | [3, 4] | Upper semicircle |

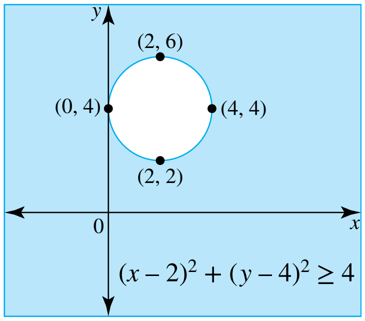



| | Centre | Radius | Domain | Range | Description |
|---|---|---|---|---|---|
| d | (0, 0) | $\sqrt{5}$ | $(-\sqrt{5}, \sqrt{5})$ | $(-\sqrt{5}, \sqrt{5})$ | Open region inside circle |

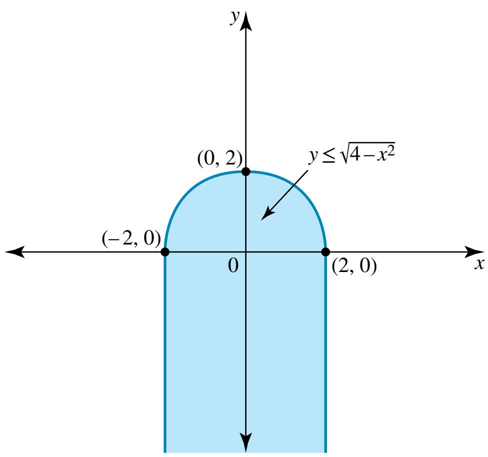



(*Continued*)

| | Centre | Radius | Domain | Range | Description |
|---|---|---|---|---|---|
| e | (2, 4) | 2 | $R$ | $R$ | Closed region outside circle |




| | Centre | Radius | Domain | Range | Description |
|---|---|---|---|---|---|
| f | (0, 0) | 2 | [−2, 2] | (−∞, 2] | Closed region below semicircle and between $x = -2$ and $x = 2$ |




**a**, **b** and **c** are functions.

**a** $f : [-6, 6] \to R, f(x) = \sqrt{36 - x^2}$

**b** $g : [-0.5, 0.5] \to R, g(x) = -\sqrt{0.25 - x^2}$

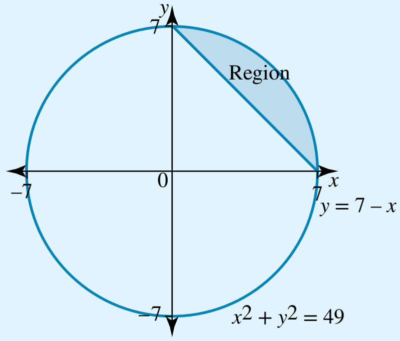
**c** $h : [-1, 1] \to R, h(x) = \sqrt{1 - x^2} + 3$

**10 a** Outside

**b** $a = 0, -8$; $y = 0.5\sqrt{1 - 32x - 4x^2} + 1.5$

**11 a** (1, 2), (1.4, 2.8)

**b** (0, 7), (7, 0); region is inside circle and above the line (boundaries included)




**12 a** (2, 1)

**b i** $k = \pm\frac{1}{2\sqrt{6}} = \pm\frac{\sqrt{6}}{12}$

**ii** $-\frac{1}{2\sqrt{6}} < k < \frac{1}{2\sqrt{6}}$ or $\frac{-\sqrt{6}}{12} < k < \frac{\sqrt{6}}{12}$

**iii** $k < -\frac{1}{2\sqrt{6}}$ or $k > \frac{1}{2\sqrt{6}}$ or $k < \frac{-\sqrt{6}}{12}$ or $k > \frac{\sqrt{6}}{12}$

**13 a** Centre (3, −2); radius 5

**b i** Proof required — check with your teacher

**ii** $3y - 4x = 7$

**c i** Proof required — check with your teacher

**ii** $y = 3$

**d** $2\sqrt{30}$ units

**e** 5 units

**f** Either (8, 13) or (−7, −7)

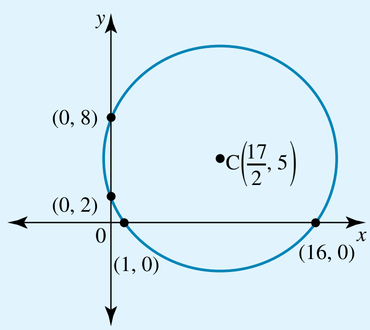
**14 a** $2\sqrt{21}$

**b** $\frac{5\sqrt{2}}{2}$

**15 a** $a = -17,\ b = -10,\ c = 16$

**b** Centre $\left(\frac{17}{2}, 5\right)$; radius $\frac{5\sqrt{13}}{2}$

**c** $x$-intercepts (1, 0), (16, 0); $y$-intercepts (0, 2), (0, 8)




**d** 0.85 units

**e** 18.88 units

**16 a** Three points are required to determine a circle but only two are given.

**b** $(x-2)^2+(y-2)^2=8$; proof required

**c** **i** $(4, 4)$

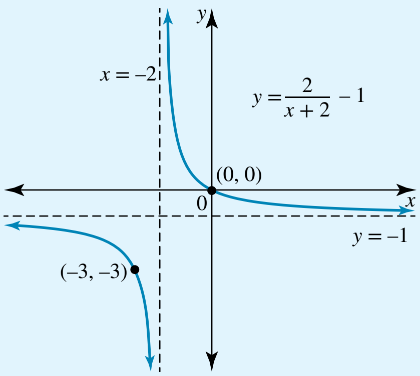
**ii** Rufus by approximately 2 minutes

**17** Centre $(-2, 3.5)$; radius 3.775; $x$-intercepts at $x=-3.414$, $x=-0.586$; $y$-intercepts at $y=6.702$, $y=0.298$

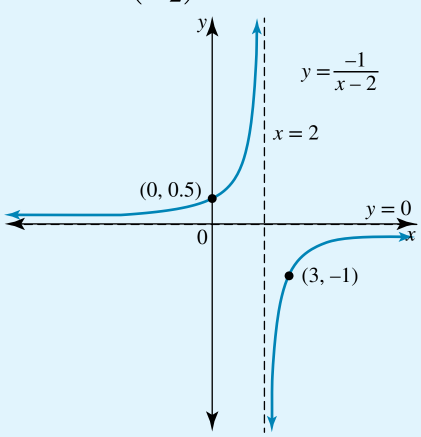
**18** Domain $[-2.124, 14.124]$; range $[-12.124, 4.124]$; tangent is vertical.

## EXERCISE 6.4

**1 a** Vertical asymptote $x=-2$; horizontal asymptote $y=-1$; $y$-intercept $(0, 0)$; domain $R\backslash\{-2\}$; range $R\backslash\{-1\}$




**b** Vertical asymptote $x=2$; horizontal asymptote $y=0$; $y$-intercept $\left(0, \frac{1}{2}\right)$; domain $R\backslash\{2\}$; range $R\backslash\{0\}$




**2** Vertical asymptote $x=\frac{1}{2}$; horizontal asymptote $y=4$; graph lies in quadrants 1 and 3 (quadrants as defined by the asymptotes)

**3 a** Vertical asymptote $x=-\frac{2}{3}$; horizontal asymptote $y=2$

**b** $y=\dfrac{-1}{x-4}+\dfrac{1}{2}$

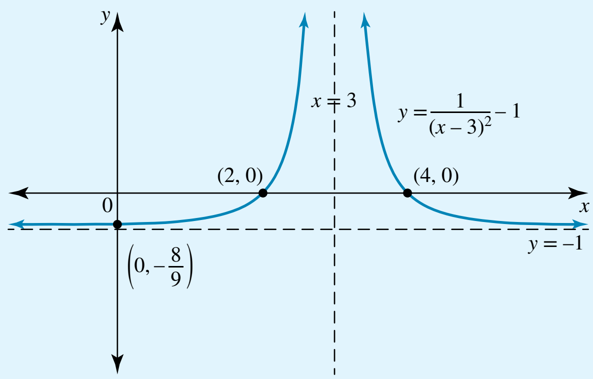
**4** Domain $R\backslash\{4\}$; range $R\backslash\{0\}$

**5 a** $R=\dfrac{240}{I}$

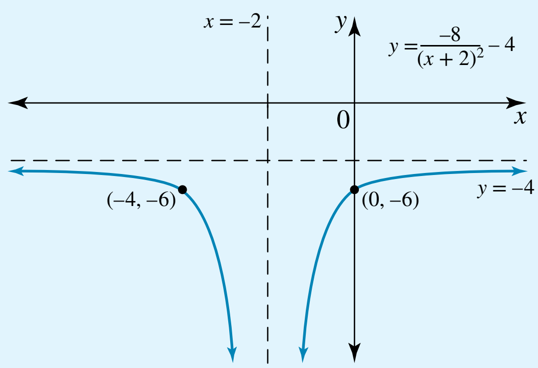
**b** $333\frac{1}{3}$ ohm

**6** Table **b**; $y=\dfrac{7.2}{x}$, $y=6.4, x=1.125$; $x=8, y=0.9$

**7 a** Domain $R\backslash\{3\}$; range $(-1, \infty)$



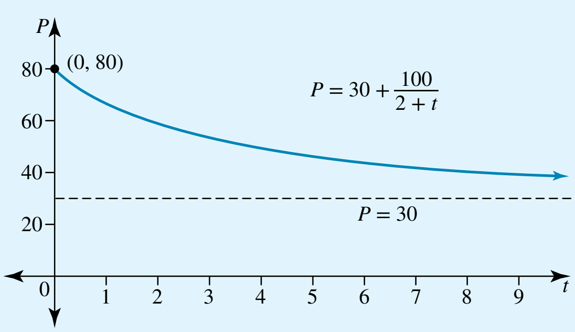

**b** Domain $R\backslash\{-2\}$; range $(-\infty, -4)$




**8** Asymptotes $x=\frac{1}{5}$, $y=0$; domain $R\backslash\left\{\frac{1}{5}\right\}$; range $(0, \infty)$

**9 a** Reduced by 25 cattle

**b** Domain $\{t : t \geq 0\}$; range $(30, 80]$




**c** The number of cattle will never go below 30.

**10 a** 50 seconds

**b** 25 metres above the ground

**11 a** $x = -5, y = 2$

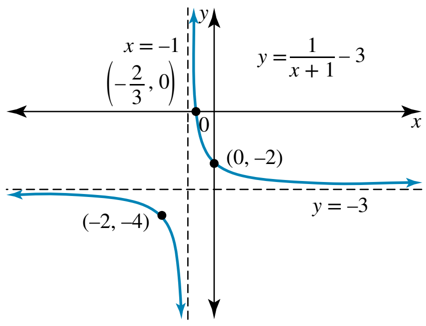
**b** $x = 0, y = -3$

**c** $x = 0, y = 0$

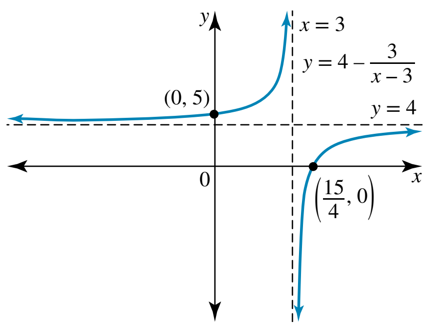
**d** $x = -14, y = -\frac{3}{4}$

**12**

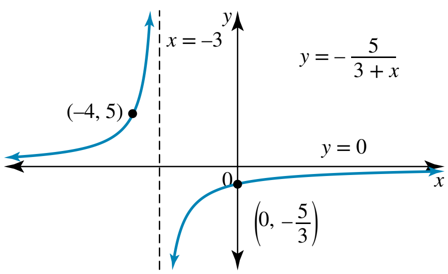
| | Asymptotes | $y$-intercept | $x$-intercept | Domain | Range | Point |
|---|---|---|---|---|---|---|
| **a** | $x = -1, y = -3$ | $(0, -2)$ | $\left(-\frac{2}{3}, 0\right)$ | $R\backslash\{-1\}$ | $R\backslash\{-3\}$ | $(-2, -4)$ |



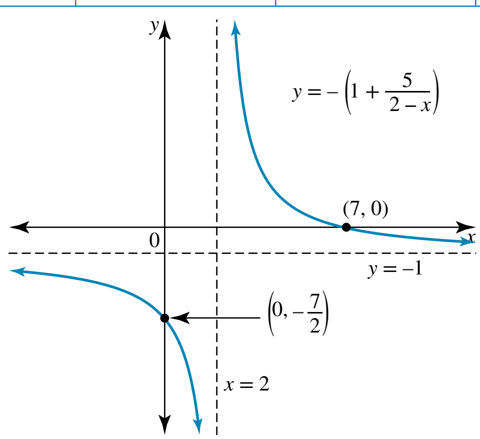

| | Asymptotes | $y$-intercept | $x$-intercept | Domain | Range | Point |
|---|---|---|---|---|---|---|
| **b** | $x = 3, y = 4$ | $(0, 5)$ | $\left(\frac{15}{4}, 0\right)$ | $R\backslash\{3\}$ | $R\backslash\{4\}$ | |




| | Asymptotes | $y$-intercept | $x$-intercept | Domain | Range | Point |
|---|---|---|---|---|---|---|
| **c** | $x = -3, y = 0$ | $\left(0, -\frac{5}{3}\right)$ | none | $R\backslash\{-3\}$ | $R\backslash\{0\}$ | $(-4, 5)$ |

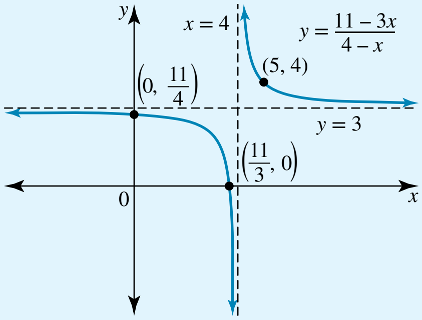



| | Asymptotes | $y$-intercept | $x$-intercept | Domain | Range | Point |
|---|---|---|---|---|---|---|
| **d** | $x = 2, y = -1$ | $\left(0, -\frac{7}{2}\right)$ | $(7, 0)$ | $R\backslash\{2\}$ | $R\backslash\{-1\}$ | |



**13 a** $a = 3; b = 1$

**b** Asymptotes $x = 4, y = 3$; $y$-intercept $\left(0, \frac{11}{4}\right)$; $x$-intercept $\left(\frac{11}{3}, 0\right)$; point $(5, 4)$




**c** $x < \frac{11}{3}$ or $x > 4$

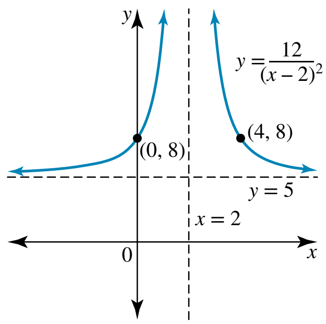
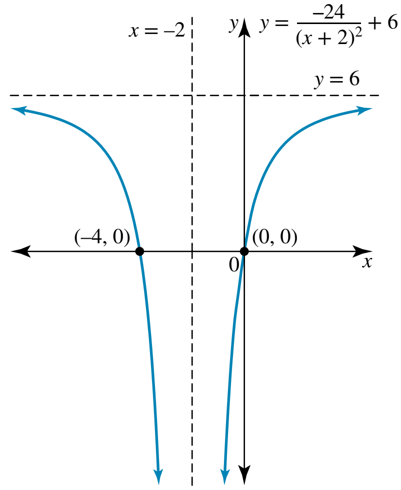
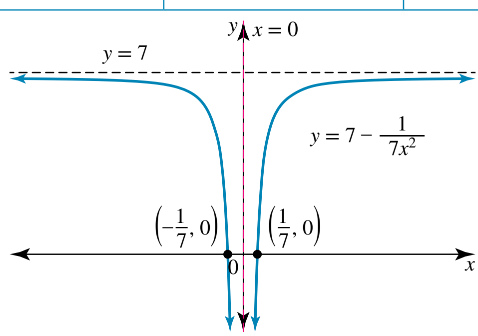
**14 a** $y = \dfrac{-1}{16x + 4} + \dfrac{1}{4},\ x = -\dfrac{1}{4}, y = \dfrac{1}{4}$

**b** $y = \dfrac{4}{x - 4} - 2,\ x = 4, y = -2$

**c** $y = \dfrac{1}{x} + 2,\ x = 0, y = 2$

**d** $y = \dfrac{-2}{2x + 3},\ x = -\dfrac{3}{2}, y = 0$

**15**

| | Asymptotes | $y$-intercept | $x$-intercept | Domain | Range |
|---|---|---|---|---|---|
| **a** | $x = 2, y = 5$ | $(0, 8)$ | none | $R\backslash\{2\}$ | $(5, \infty)$ |
| |  | | | | |
| **b** | $x = -2, y = 6$ | $(0, 0)$ | $(0, 0)$ | $R\backslash\{-2\}$ | $(-\infty, 6)$ |
| |  | | | | |
| **c** | $x = 0, y = 7$ | none | $\left(\pm\frac{1}{7}, 0\right)$ | $R\backslash\{0\}$ | $(-\infty, 7)$ |
| |  | | | | |



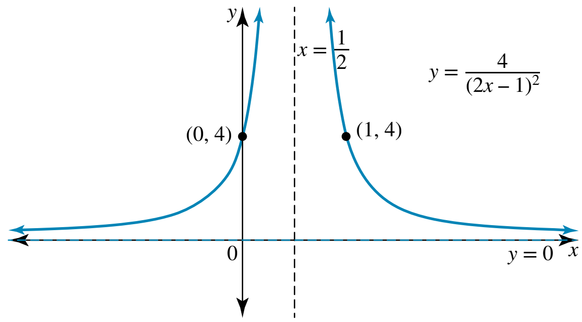




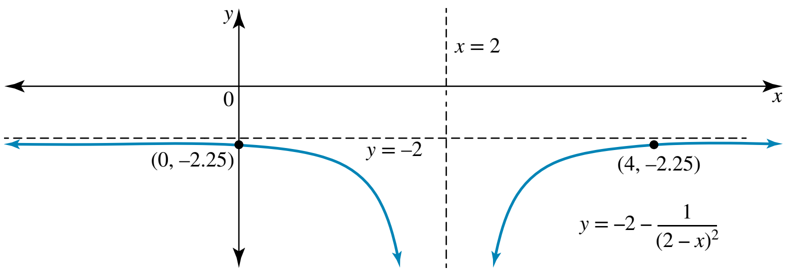
(*Continued*)

**15**

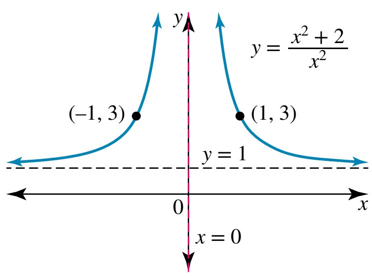
| | Asymptotes | y-intercept | x-intercept | Domain | Range |
|---|---|---|---|---|---|
| **d** | $x=\frac{1}{2}, y=0$ | $(0, 4)$ | none | $R\backslash\left\{\frac{1}{2}\right\}$ | $R^+$ |




| | Asymptotes | y-intercept | x-intercept | Domain | Range |
|---|---|---|---|---|---|
| **e** | $x = 2, y = -2$ | $\left(0, -\frac{9}{4}\right)$ | none | $R\backslash\{2\}$ | $(-\infty, -2)$ |




| | Asymptotes | y-intercept | x-intercept | Domain | Range |
|---|---|---|---|---|---|
| **f** | $x = 0, y = 1$ | none | none | $R\backslash\{0\}$ | $(1, \infty)$ |




**16 a** $y = \dfrac{2}{x-3} + 1$

**b** $y = \dfrac{-1.5}{x+3} + 1$

**c** $y = \dfrac{3}{x^2} - 2$

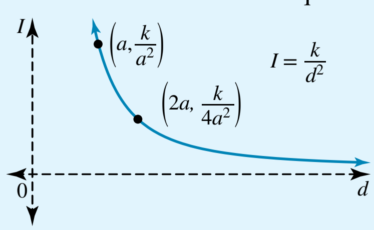
**d** $y = 2 - \dfrac{9}{(x+3)^2}$

**e** $y = \dfrac{4}{(x+2)^2}, f: R\backslash\{-2\} \to R, f(x) = \dfrac{4}{(x+2)^2}$

**f** **i** $y = \dfrac{-2x+2}{4x-1}$

**ii** $f: R\backslash\left\{\dfrac{1}{4}\right\} \to R, f(x) = \dfrac{-2x+2}{4x-1}$

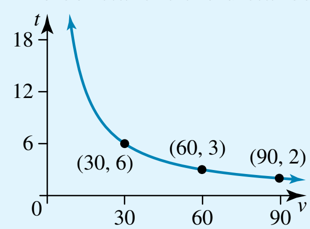
**17 a** One branch of a truncus required




**b** Intensity reduced by 75%

**18 a** The constant is the distance; k = 180

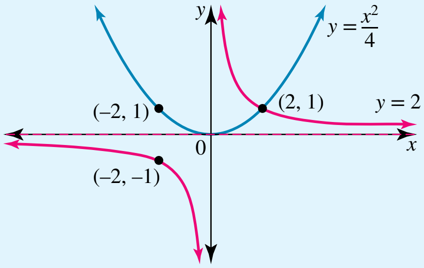
**b**



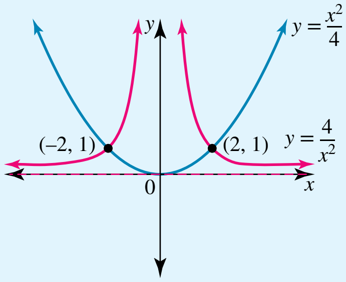

**c** 80 km/h

19 a (2, 1)

b




c (−2, 1), (2, 1)




d $(-\sqrt{a}, 1)$, $(\sqrt{a}, 1)$

20 a $a = 460$; $b = 40$

b 12 years 8 months

c Proof required — check with your teacher

d Increases by approximately 4 insects in 12th year and 3 in 14th year; growth is slowing.

e Never reaches 500 insects

f Cannot be larger than 460

21 a Two intersections

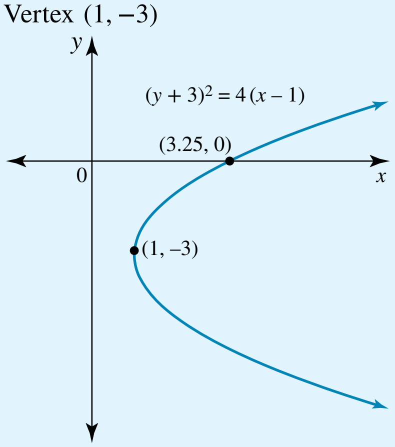
b One intersection if $k = 1, 5$; two intersections if $k < 1$ or $k > 5$; no intersections if $1 < k < 5$

22 a Asymptotes $x = 0, y = 0$ and $y = -x, y = x$

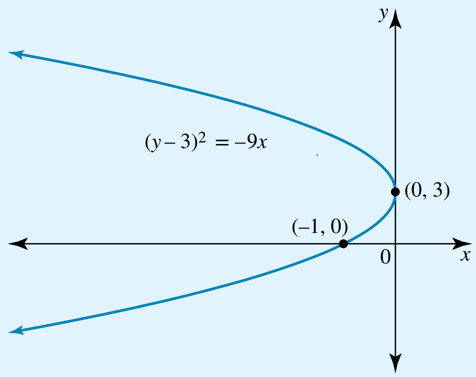
b Rotate axes on $xy = 1$ graph anticlockwise by 45°

## EXERCISE 6.5

1 a Vertex (1, −3)




b Vertex (0, 3)



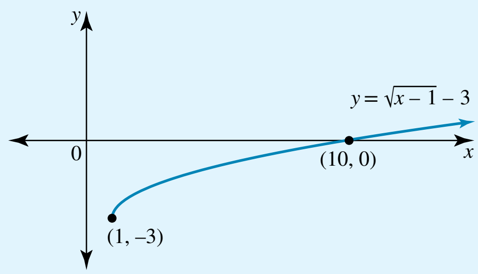

2 $(y + 4)^2 = 3\left(x - \frac{4}{3}\right)$; vertex $\left(\frac{4}{3}, -4\right)$; axis of symmetry $y = -4$

3 a $(y + 7)^2 = -\frac{7}{2}(x - 4)$

b $(y - 3)^2 = 3(x + 3)$

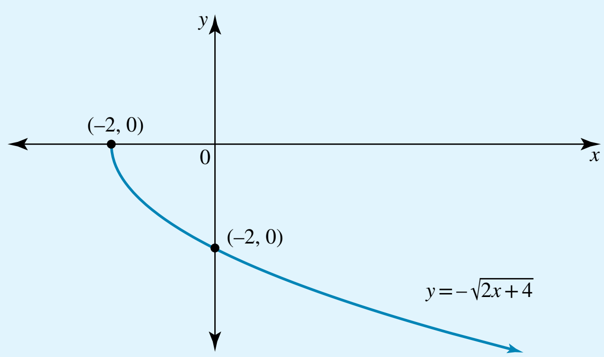
4 Sideways parabola fails vertical line test for functions; $(y - 3)^2 = \frac{9}{2}x$

5 a Domain $[1, \infty)$; range $[-3, \infty)$



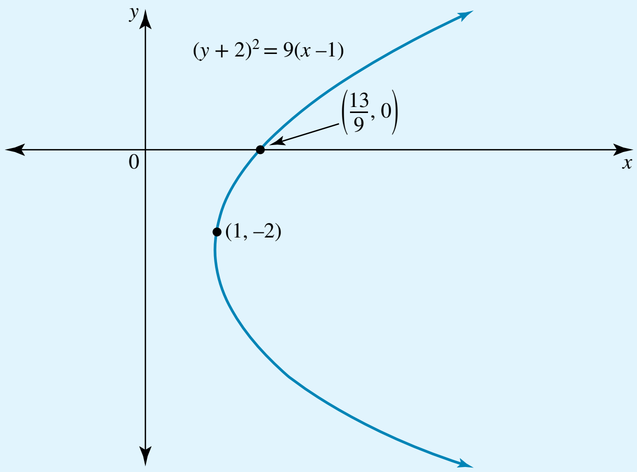

b i $[-2, \infty)$

ii




iii $y^2 = 2(x + 2)$

**6 a** Vertex $(1, -2)$; $x$-intercept $\left(\frac{13}{9}, 0\right)$

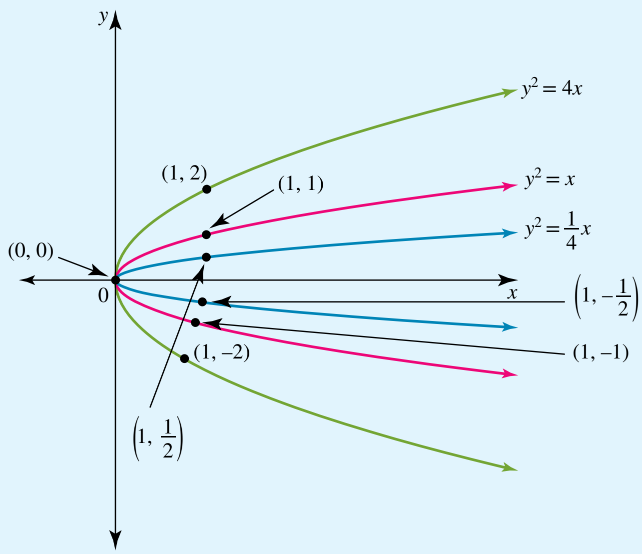



**b** $y = -\sqrt{9(x-1)} - 2$

**7** $y = \sqrt{2}\sqrt{x+2} + 1 \Rightarrow y = \sqrt{2(x+2)} + 1$

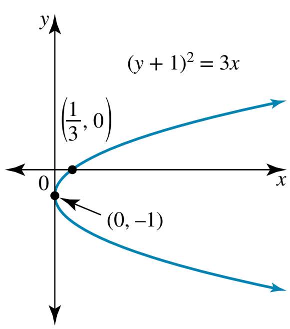
**8** $a = 5, b = -2, c = 2$; $(y-5)^2 = -2(x-2)$; domain $(-\infty, 2]$; range $R$

**9** Increasing the coefficient of $x$ makes the graph wider (in the $y$ direction) or more open.

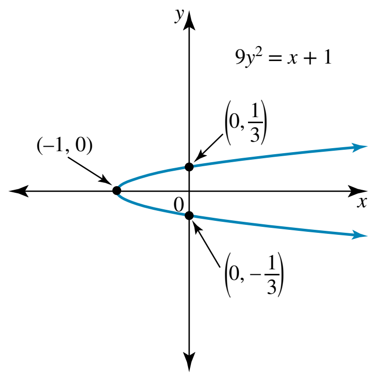



**10**

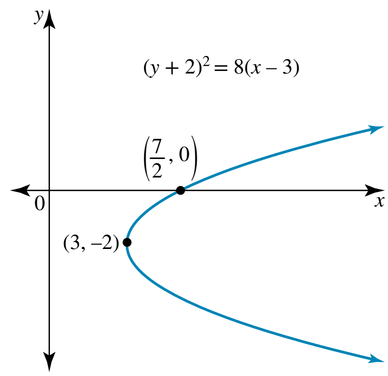
| | Vertex | $x$-intercept | $y$-intercepts |
|---|---|---|---|
| **a** | $(0, -1)$ | $\left(\frac{1}{3}, 0\right)$ | $(0, -1)$ |



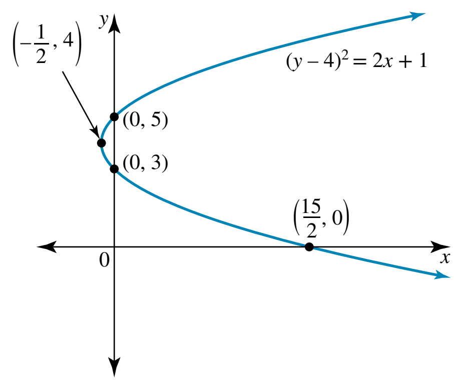

| | | | |
|---|---|---|---|
| **b** | $(-1, 0)$ | $(-1, 0)$ | $\left(0, \pm\frac{1}{3}\right)$ |



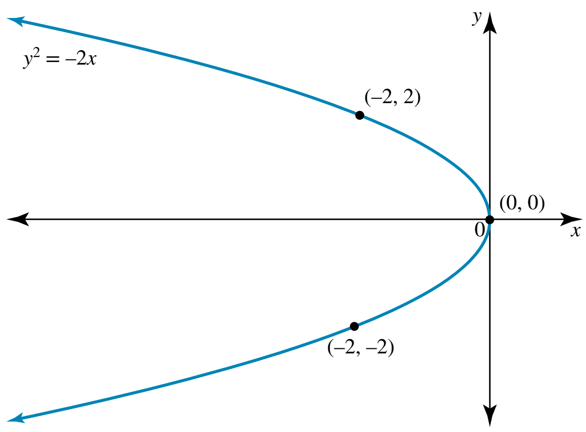
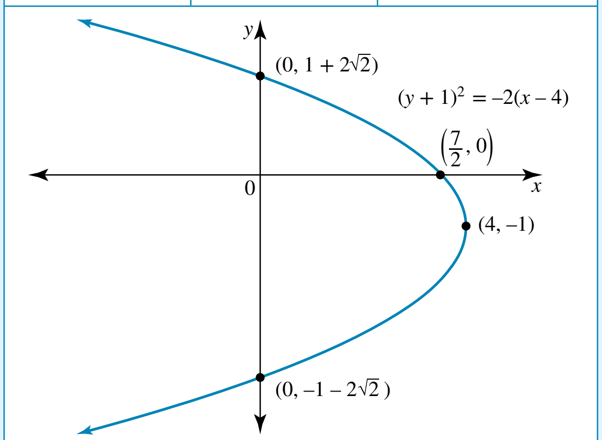
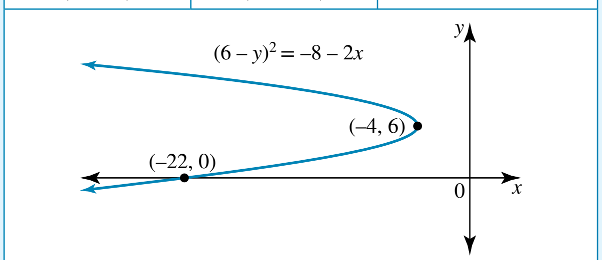
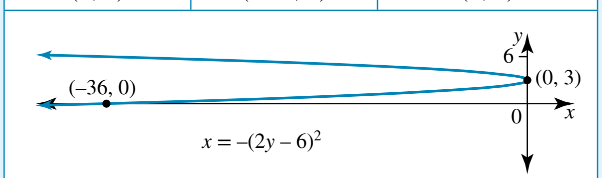

| | | | |
|---|---|---|---|
| **c** | $(3, -2)$ | $\left(\frac{7}{2}, 0\right)$ | none |




| | | | |
|---|---|---|---|
| **d** | $\left(-\frac{1}{2}, 4\right)$ | $\left(\frac{15}{2}, 0\right)$ | $(0, 3), (0, 5)$ |



**11**

| | Vertex | $x$-intercept | $y$-intercepts |
|---|---|---|---|
| **a** | (0, 0) | (0, 0) | (0, 0) |
| |   | | |
| **b** | (4, −1) | $\left(\frac{7}{2}, 0\right)$ | $(0, -1 \pm 2\sqrt{2})$ |
| |   | | |
| **c** | (−4, 6) | (−22, 0) | none |
| |   | | |
| **d** | (0, 3) | (−36, 0) | (0, 3) |
| |   | | |

**12 a** $(y + 8)^2 = 5(x - 2)$; vertex $(2, -8)$; domain $[2, \infty)$

**b** $\left(y - \frac{3}{2}\right)^2 = -13\left(x - \frac{1}{4}\right)$; vertex $\left(\frac{1}{4}, \frac{3}{2}\right)$; domain $\left(-\infty, \frac{1}{4}\right]$

**c** $\left(y + \frac{5}{2}\right)^2 = -(x - 2)$; vertex $\left(2, -\frac{5}{2}\right)$; domain $(-\infty, 2]$

**d** $(y - 2)^2 = 5\left(x + \frac{4}{5}\right)$; vertex $\left(-\frac{4}{5}, 2\right)$; domain $\left[-\frac{4}{5}, \infty\right)$

**13 a** $(y + 1)^2 = -3(x - 1)$

**b** $(y + 2)^2 = 4(x - 1)$

**c** **i** $y = 4$

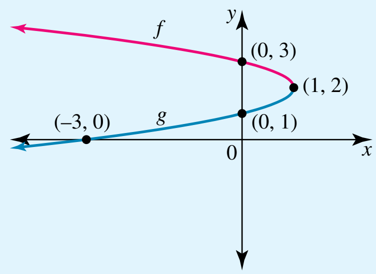
**ii** $(y - 4)^2 = 64x$

**d** $y^2 = \frac{121}{48}x$

**14 a** $y = 2 \pm \sqrt{1 - x}$

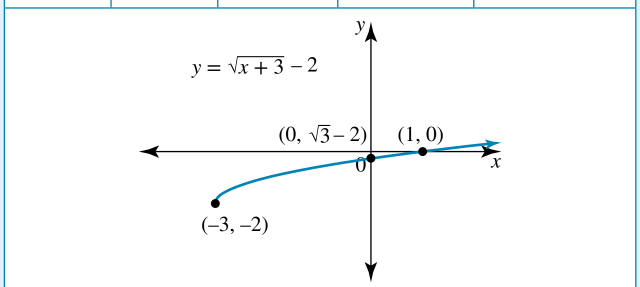
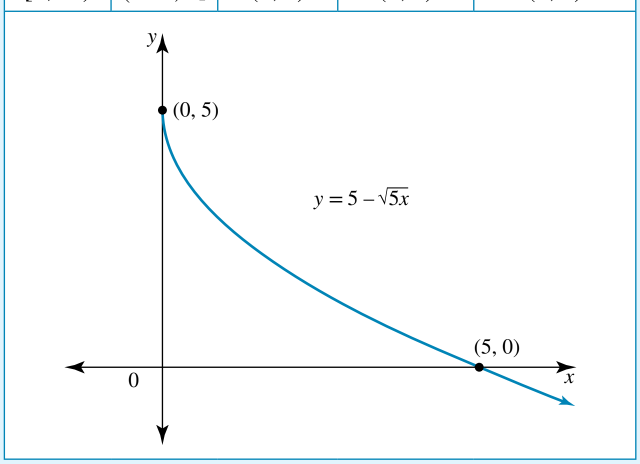
**b** $f:(-\infty, 1] \rightarrow R, f(x) = 2 + \sqrt{1 - x}$,
$g:(-\infty, 1] \rightarrow R, g(x) = 2 - \sqrt{1 - x}$

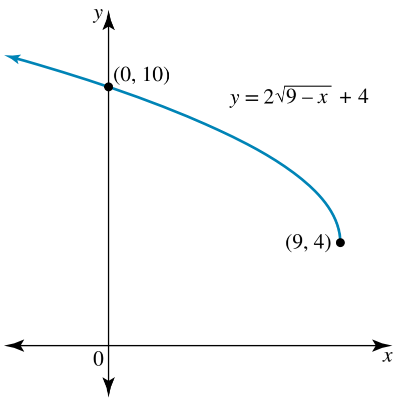
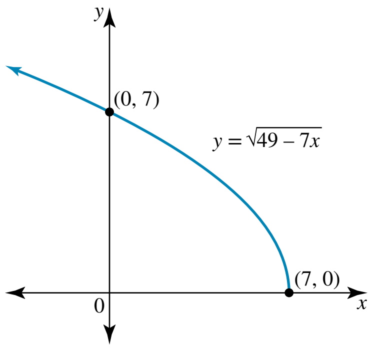
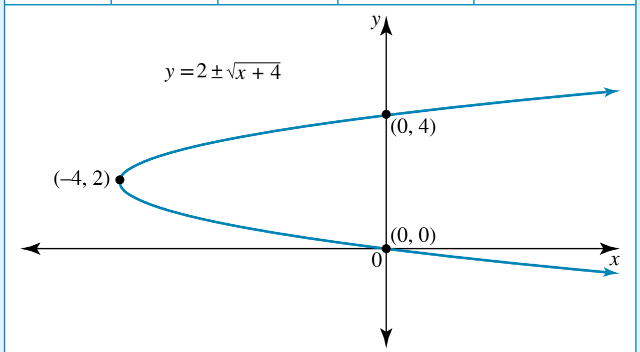
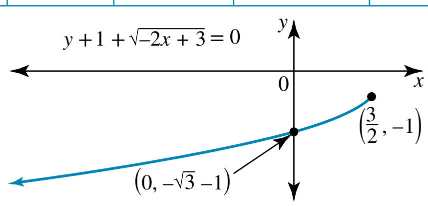
**c**




**d** Image is 5 under $f$ and −1 under $g$.

**15**

| | Domain | Range | Endpoint | $x$-intercept | $y$-intercept |
|---|---|---|---|---|---|
| **a** | $[-3, \infty)$ | $[-2, \infty)$ | (−3, −2) | (1, 0) | $(0, \sqrt{3} - 2)$ |
| |   | | | | |
| **b** | $[0, \infty)$ | $(-\infty, 5]$ | (0, 5) | (5, 0) | (0, 5) |
| |   | | | | |

| | Domain | Range | Endpoint | $x$-intercept | $y$-intercept |
|---|---|---|---|---|---|
| **c** | $(-\infty, 9]$ | $[4, \infty)$ | $(9, 4)$ | none | $(0, 10)$ |
|  | | | | | |
| **d** | $(-\infty, 7]$ | $[0, \infty)$ | $(7, 0)$ | $(7, 0)$ | $(0, 7)$ |
|  | | | | | |
| **e** | $[-4, \infty)$ | $R$ | vertex $(-4, 2)$ | $(0, 0)$ | $(0, 0)$, $(0, 4)$ |
|  | | | | | |
| **f** | $\left(-\infty, \frac{3}{2}\right]$ | $(-\infty, -1]$ | $\left(\frac{3}{2}, -1\right)$ | none | $(0, -\sqrt{3} - 1)$ |
|  | | | | | |

**16 a** $y = \sqrt{\frac{x+2}{3}} - 2$

**b** $y = 5\sqrt{4 - x} - 1$; $\left(\frac{99}{25}, 0\right)$

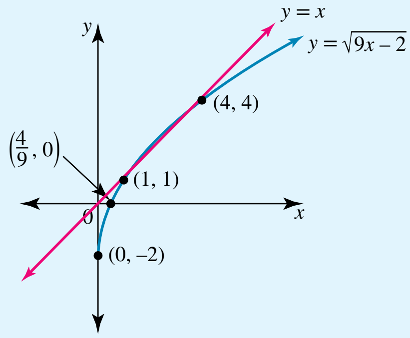
**c** $y = \sqrt{4 - x} - 4$

**d** $t = \sqrt{\frac{h}{4.9}}$

**17 a** $m = 9$; $n = -2$

**b** $x = \frac{4}{9}$

**c** $\{x : 1 < x < 4\}$




**d** $(y + 2)^2 = 9x$

**18 a** Domain $(-\infty, 0]$; P lies on upper branch

**b** V is 2 units from both F and line $D$ and P is 5 units from F and line $D$.

**c** $(a, -\sqrt{-8a})$; Q is $2 - a$ units from F and line $D$

**d** $a = -\frac{4}{3}$

**19 a** $f$ is a square root function with domain $[-1, \infty)$; $g$ is a semicircle function with domain $[-2, 2]$

**b** $(-0.6, 3.9)$

**c** $A$ has the rule $(y - 2)^2 = 9(x + 1)$; $B$ has the rule $x^2 + (y - 2)^2 = 4$.

**d** $(-0.6, 3.9)$, $(-0.6, 0.1)$

**20 a** A sideways parabola opening to the right

**b** Distances are equal. Any point on the parabola is equidistant from the point F and the line $D$.

## EXERCISE 6.6

**1 a** $R\backslash\{\pm 4\}$

**b** $R$

**c** $(-1, \infty)$

**2 a** $R\backslash\{-4, -1\}$

**b** $[-3, \infty)$

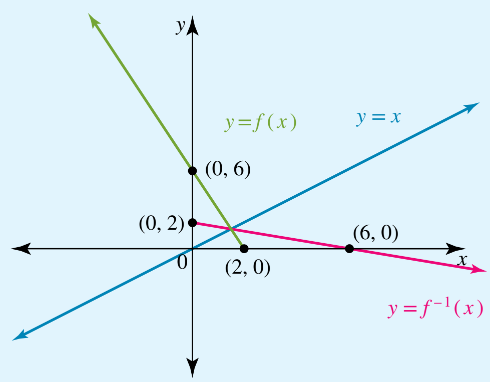
**c** $[-3, \infty)\backslash\{-1\}$

**3 a** Domain $(-\infty, 2]$; range $R^+ \cup \{0\}$

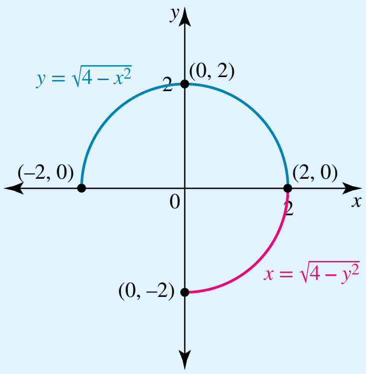
**b** Domain $R^+ \cup \{0\}$; range $(-\infty, 2]$

**c** $f^{-1} : R^+ \cup \{0\} \rightarrow R$, $f^{-1}(x) = \frac{6 - x}{3}$

**d**



**4 a**

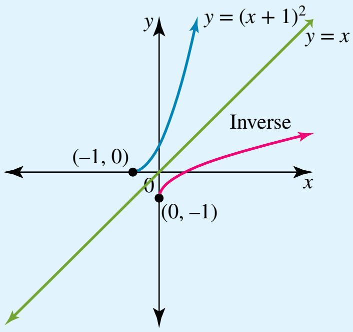



Upper semicircle has many-to-one correspondence; semicircle to the right of the $y$-axis has one-to-many correspondence.

**b** Not a function

**5 a** $(y + 1)^2 = x$ or $y = \pm\sqrt{x} - 1$; one-to-many correspondence so not a function

**b**

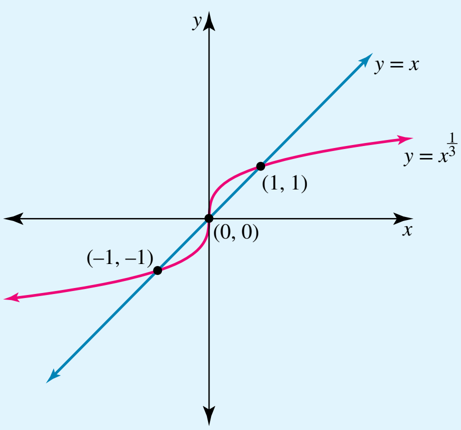



**c** $y = \sqrt{x} - 1$

**d** No intersection

**6** $a = 2$

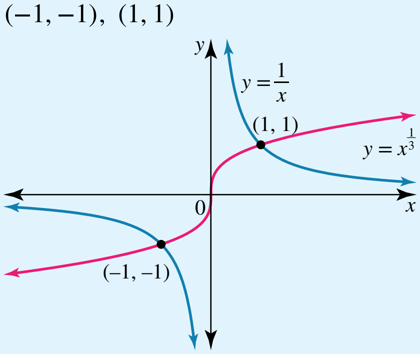
**7**




$\{x : x < -1\} \cup \{x : 0 < x < 1\}$

**8 a** $(-1, -1)$, $(1, 1)$

**b**

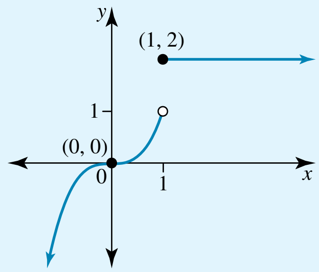



**9 a i** $f(-2) = -8$

**ii** $f(1) = 2$

**iii** $f(2) = 2$

**b** Domain $R$; range $(-\infty, 1) \cup \{2\}$




**c** Not continuous at $x = 1$

**10 a** $a = -2$

**b** Continuous

**11 a** $R\backslash(-2, 2)$

**b** $R\backslash\{\pm 2\}$

**c** $(-\infty, 4)$

**d** $[0, 2]$

**e** $R^+$

**f** $R$

**12 a** $4y - 8x = 1$

**b** $y = -\dfrac{3x}{2} - 6$

**c** $y = \dfrac{x^2}{4}$

**d** $y^2 = \dfrac{x}{4}\left(\text{or } y = \pm\dfrac{\sqrt{x}}{2}\right)$

**e** $(x - 3)^2 + y^2 = 1$

**f** $y = \dfrac{x^2 - 1}{2}, \; x \geq 0$

**13 a** Maximal domain is $R\backslash\{2\}$; asymptote equations $x = 2, y = 0$

**b** $f^{-1}(x) = \dfrac{1}{x} + 2$

**c** $x = 0, \; y = 2$

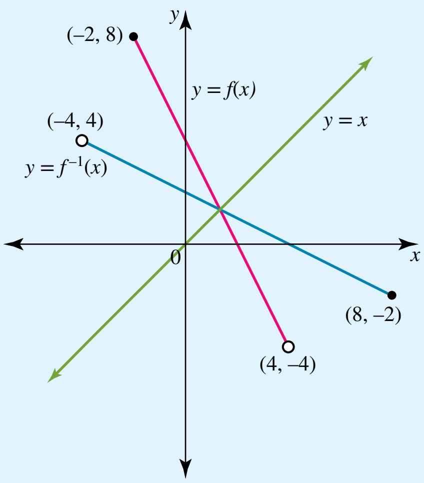
**d** $x = b; y = a; y = \dfrac{1}{x - b} + a$

**14 a** Domain $(-4, 8]$; range $[-2, 4)$

**b** $f^{-1}(x) = \dfrac{4 - x}{2}, \; -4 < x \leq 8;$

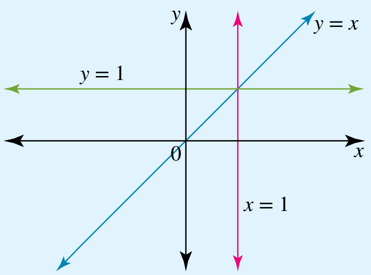
$f^{-1} : (-4, 8] \rightarrow R, f^{-1}(x) = \dfrac{4 - x}{2}$

**c**

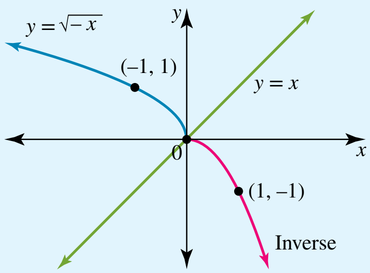



**d** $x = \dfrac{4}{3}$

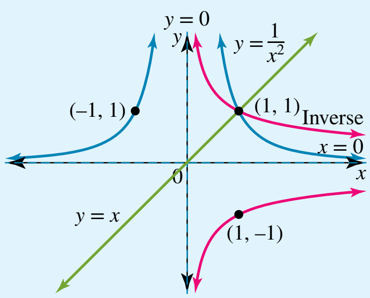
**15 a** **i**



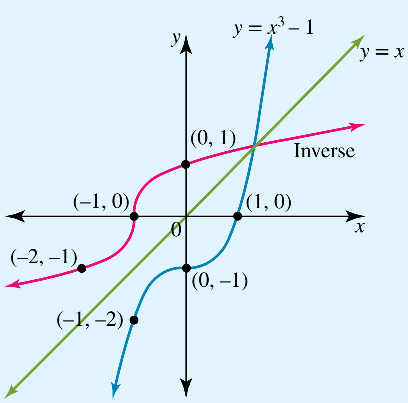

**ii**



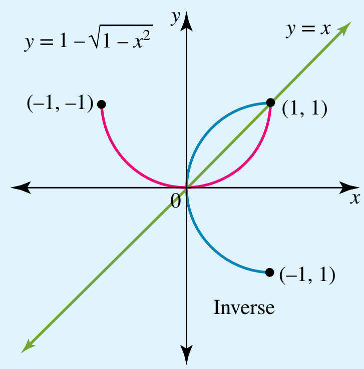

**iii**



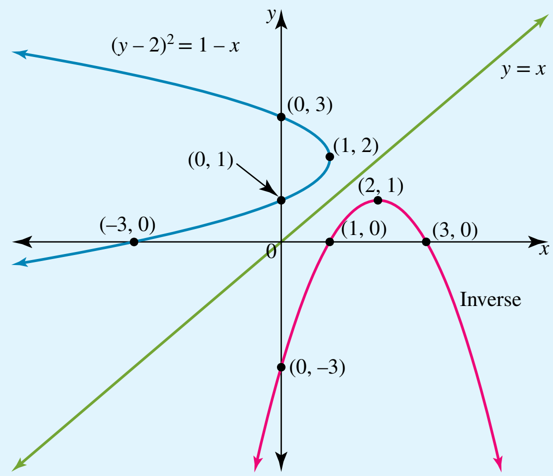

**iv**




**v**

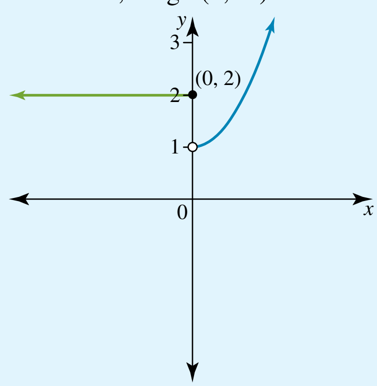



**vi**

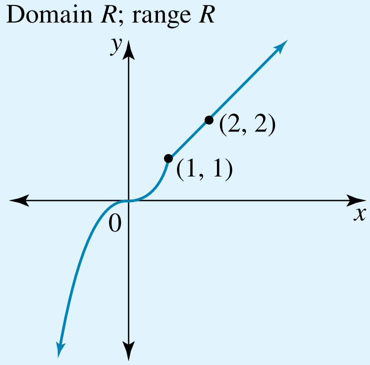



**b** Functions in **ii** and **iv** have inverses which are also functions; $y = -x^2,\ x \geq 0,\ y = \sqrt[3]{x+1}$

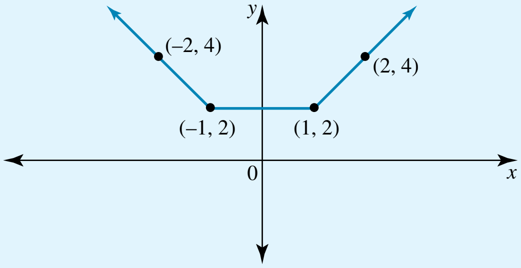
**16 a** Domain $R$; range $(1, \infty)$



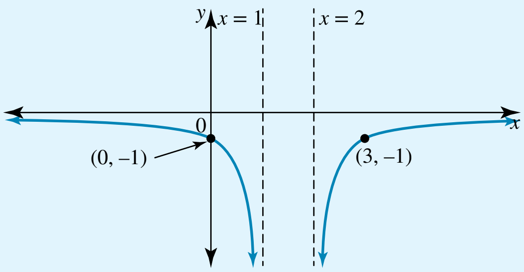

**b** Domain $R$; range $R$



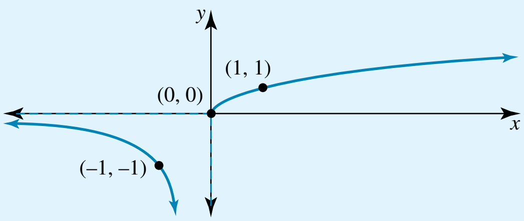
**c** Domain $R$; range $[2, \infty)$



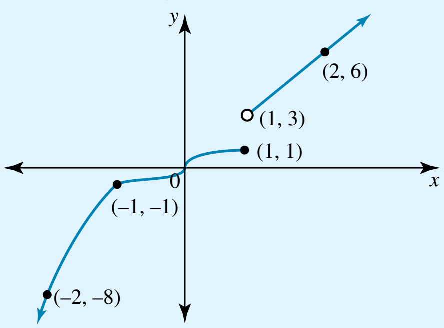

**d** Domain $R\backslash[1, 2]$; range $R^-$




**e** Domain $R$; range $R$




**f** Domain $R$; range $R\backslash(1, 3]$




**17 a** **i** $f(0) = 1$

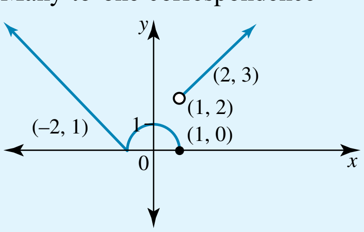
**ii** $f(3) = 4$

**iii** $f(-2) = 1$

**iv** $f(1) = 0$

**b** Proof required — check with your teacher

**c** Many-to-one correspondence

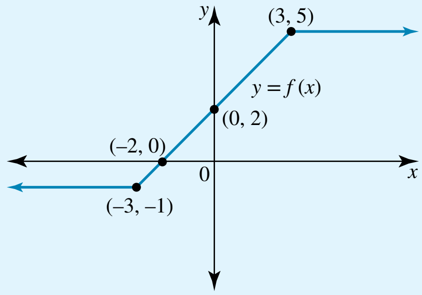



**d** $a = \dfrac{\sqrt{2}}{2}$

**18 a** $y = \begin{cases} x + 1, & x \le 0 \\ -x + 1, & x > 0 \end{cases}$

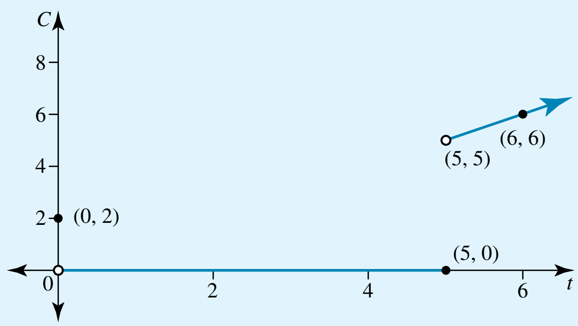
**b** $y = \begin{cases} 3, & x < 2 \\ 3x - 6, & x \ge 2 \end{cases}$

**c** $a = -1, b = 5$




**d** With cost $C$ dollars and time $t$ minutes:

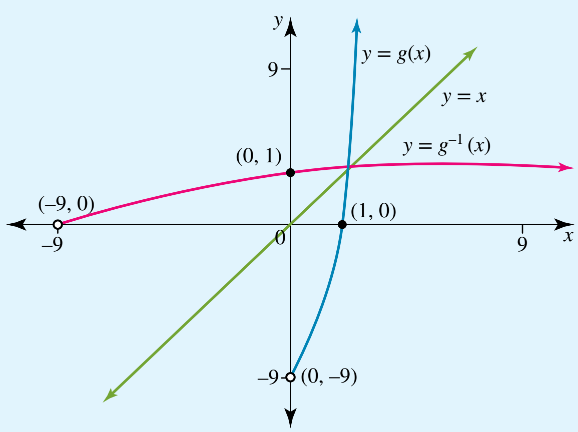
$$C = \begin{cases} 2, & t = 0 \\ 0, & 0 < t \le 5 \\ t, & t > 5 \end{cases}$$




**19 a** Either $D = [-4, \infty)$ or $D = (-\infty, -4]$
(Other answers are possible.)

**b** $g^{-1}: (-9, \infty) \to R, g^{-1}(x) = \sqrt{x + 25} - 4$; range $R^+$

**c** **i**

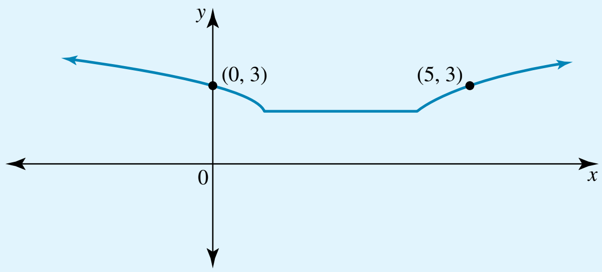



**ii** $\left(\dfrac{-7 + \sqrt{85}}{2}, \dfrac{-7 + \sqrt{85}}{2}\right)$

**d** $\dfrac{-7 + \sqrt{85}}{2}$

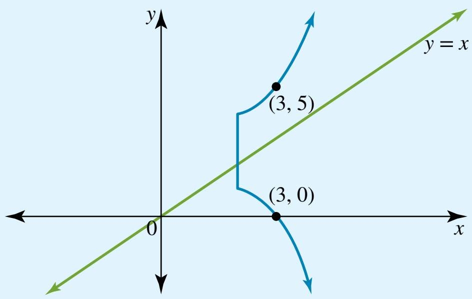
**20 a** $f(0) = 3 = f(5)$

**b** Domain $R$; range $[2, \infty)$




**c** $x = -35, x = 40$

**d** Inverse is not a function.




**21 a** $y = 0.5\left((4x + 33)^{0.5} - 5\right), y = -0.5\left((4x + 33)^{0.5} + 5\right)$

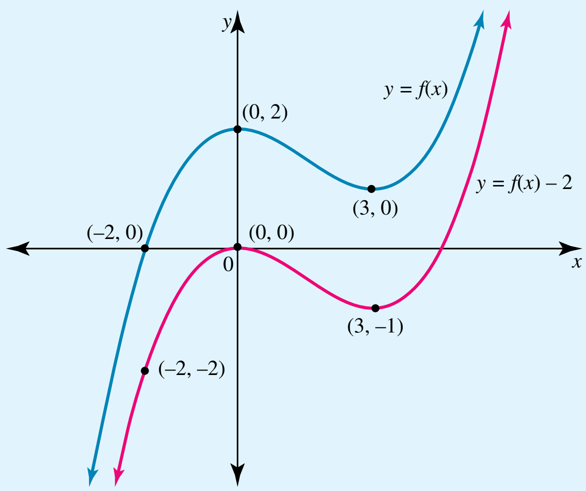
**b** $(-4.45, -4.45), (0.45, 0.45), (-0.76, -5.24), (-5.24, -0.76)$

**22 a** $x = -2,\ x = 2$

**b** Range $\left[-\dfrac{4\sqrt{6}}{9}, \infty\right)$

## EXERCISE 6.7

**1**

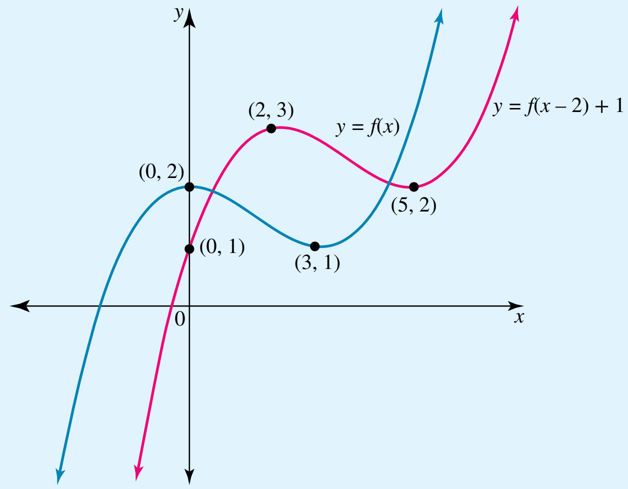



$(-2, 0) \to (-2, -2)$

$(0, 2) \to (0, 0)$

$(3, 1) \to (3, -1)$

**2**

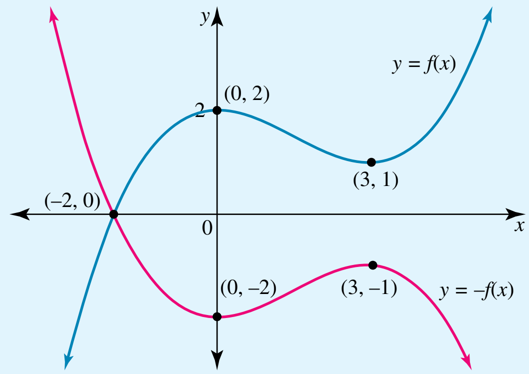



$(-2, 0) \to (0, 1)$

$(0, 2) \to (2, 3)$

$(3, 1) \to (5, 2)$

**3**




$(-2, 0) \to (-2, 0)$

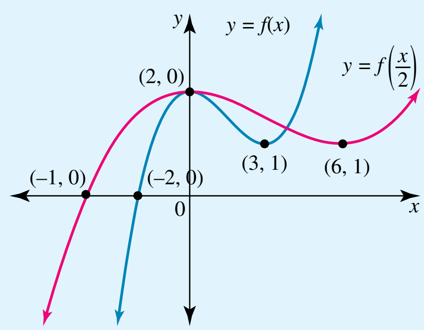
$(0, 2) \to (0, -2)$

$(3, 1) \to (3, -1)$

**4 a** $y = -(x - 1)^2 + 3$

**b** $y = -(x - 1)^2 - 3$; a different image

**5**

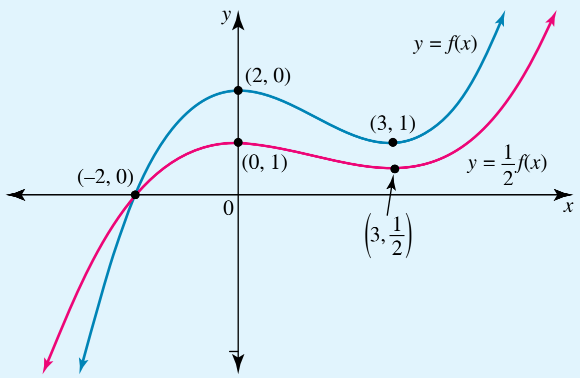



$(-2, 0) \to (-4, 0)$

$(0, 2) \to (0, 2)$

$(3, 1) \to (6, 1)$

**6**




$(-2, 0) \rightarrow (-2, 0)$

$(0, 2) \rightarrow (0, 1)$

$(3, 1) \rightarrow \left(3, \frac{1}{2}\right)$

**7 a** Dilation of factor 4 from the $x$-axis; dilation of factor 2 from the $y$-axis; horizontal translation 2 units to the right; vertical translation 3 units upwards

**b** Either reflection in $y$-axis, dilation of factor 4 from the $y$-axis followed by horizontal translation 12 units to the right, or reflection in $y$-axis, dilation of factor $\frac{1}{2}$ from the $x$-axis followed by horizontal translation 12 units to the right.

**8 a** $y = \dfrac{1}{2(x + 3)}$

**b** Horizontal translation 3 units to the right followed by dilation of factor 2 from the $y$-axis.

**9 a** Dilation factor 3 from $x$-axis

**b** Reflection in $x$-axis

**c** Translation up 5

**d** Translation 5 to the left

**10 a** Dilation factor 3 from $y$-axis

**b** Dilation factor $\frac{1}{2}$ from $y$-axis and translation 1 up

**c** Horizontal translation 4 to the right; vertical translation 4 down

**d** Dilation factor $\frac{1}{2}$ from $y$-axis; translation $\frac{1}{2}$ left; or dilation factor 8 from $x$-axis, translation $\frac{1}{2}$ left

**11 i a** $y = 2\sqrt{x}$

**b** $y = \sqrt{\dfrac{x}{2}}$

**c** $y = -\sqrt{x} + 2$

**d** $y = -\sqrt{x} - 2$

**e** $y = \sqrt{2 - x}$

**f** $y = \sqrt{-x - 2}$

**ii a** $y = 2x^4$

**b** $y = \frac{1}{16}x^4$

**c** $y = -x^4 + 2$

**d** $y = -x^4 - 2$

**e** $y = (x - 2)^4$

**f** $y = (x + 2)^4$

**12 a** $(1, -8)$

**b** $(-3, 4)$

**c** $(3, -0.8)$

**d** $(0.6, -4)$

**e** $(-4, 3)$

**f** $(4, 3)$

**13 a i** $y = -\dfrac{3}{x}$

**ii** $y = -\dfrac{3}{x}$

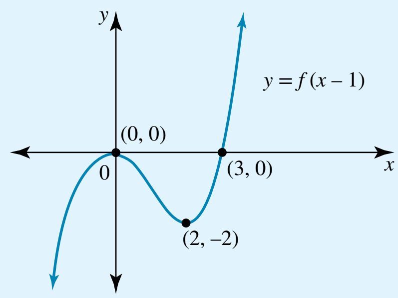
**b i** $y = \dfrac{3}{x^2} + 6$

**ii** $y = \dfrac{3}{x^2} + 18$

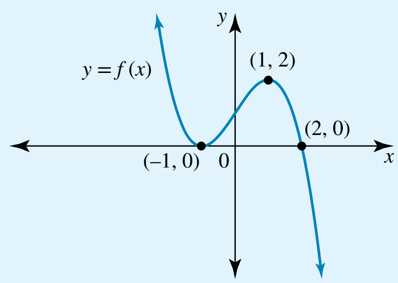
**c** Reflection in $x$-axis; dilation factor $\frac{1}{2}$ from $x$-axis (or factor $\frac{1}{2}$ from $y$-axis); translation 1 unit left and 1 unit up

**d** $x = -1, y = 0$

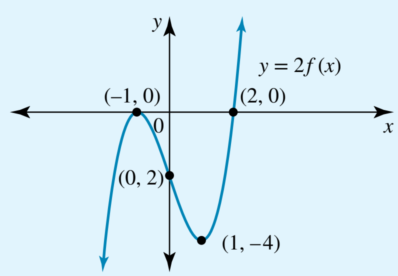
**14 a**



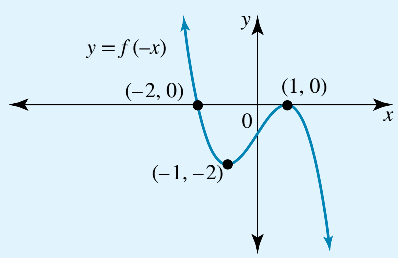

**b**



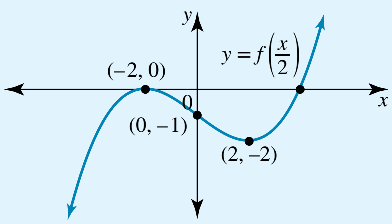

**c**



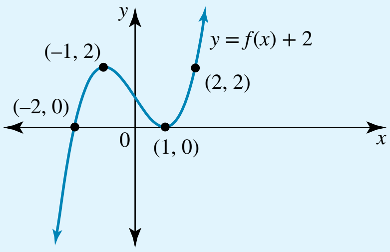

**d**



e




f




**15 a** Dilation factor 2 from $x$-axis; translation 3 left

**b** Dilation factor 6 from $x$-axis; translation 2 right, 1 up

**c** Dilation factor 0.5 from $y$-axis; translation 1 left

**d** Reflection in $y$-axis; translation 3 right

**e** Reflection in $x$-axis; dilation factor 0.25 from $y$-axis; translation 1 up

**f** Dilation factor $\frac{1}{9}$ from $x$-axis; dilation factor 9 from $y$-axis; translation 3 right

**16 a** $y = \dfrac{1}{3(x+3)^2}$

**b** $y = -x^5 + 3$

**c** $y = \dfrac{1}{1-x}$

**d** $y = \sqrt[3]{2x-1}$

**e** $y = -(x+3)(x-3)(x-7)$

**f** $y = \frac{1}{8}x^2(x+4)(x-4)$

**17 a** Image is the same parabola $g$.

**b** Image is $y = -x^{\frac{1}{3}}$ for either reflection.

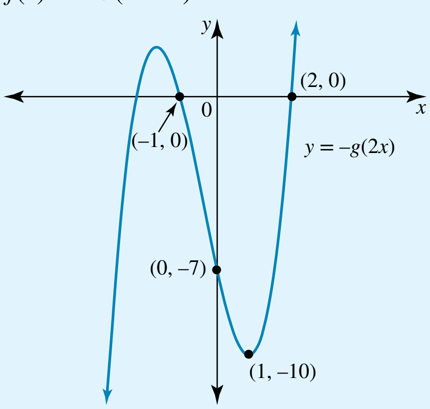
**c** Upper semicircle; reflect in the $x$-axis again

**d** Maximum turning point $(-2, -5)$

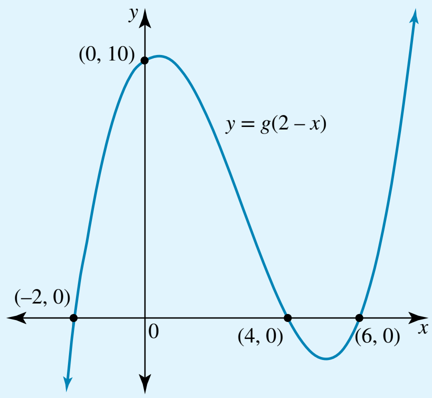
**e** $(y-2)^2 = -(x+3)$

**f** $f(x) = -3(x-2)^3$

**18 a**




**b**

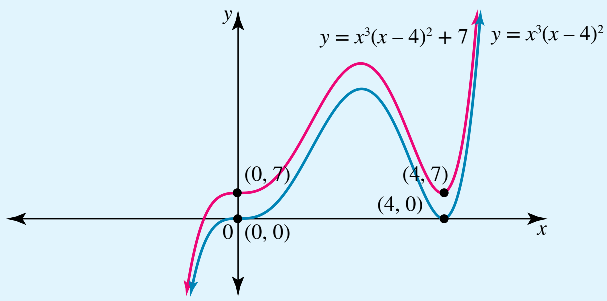



**c** $h > 4$

**d** $g(x) = -\frac{7}{32}(x+4)(x+2)(x-4)$;

$g(2x) = -\frac{7}{4}(x+2)(x+1)(x-2)$

**e**




**19** $y_2$ is a dilation of $y_1$ factor $\frac{1}{2}$ from $y$-axis; $y_3$ is a dilation of $y_1$ factor 2 from $y$-axis. All are parabolas.

**20** $y_2$ is a dilation of $y_1$ factor $\frac{1}{2}$ from $y$-axis; $y_3$ is a dilation of $y_1$ factor 2 from $x$-axis; $y_4$ is a dilation of $y_1$ factor 2 from $y$-axis. Only $y_1$ is a semicircle.

# 7

# Matrices and applications to transformations

# 7.1 Kick off with CAS

## Matrices

**1** Using CAS technology, create the following matrices.

**a** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $1\begin{bmatrix} 1 & 5 & 3 & -1 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 4 & 7 & -2 \\ 5 & 1 & 3 & 1 \\ -8 & 3 & 2 & 6 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 6 \\ -2 & 8 \\ 9 & 1 \\ 4 & -5 \\ 2 & 7 \end{bmatrix}$

**2** Using CAS technology, calculate each of the following.

**a** $2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $6\begin{bmatrix} 1 & 5 & 3 & -1 & 2 \end{bmatrix}$ **c** $\frac{1}{3}\begin{bmatrix} 1 & 4 & 7 & -2 \\ 5 & 1 & 3 & 1 \\ -8 & 3 & 2 & 6 \end{bmatrix}$ **d** $x\begin{bmatrix} 1 & 6 \\ -2 & 8 \\ 9 & 1 \\ 4 & -5 \\ 2 & 7 \end{bmatrix}$

**3** What do you notice about your answers to question **2**?

**4** Using CAS technology, define the following matrices.

$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 1 & -2 \end{bmatrix}$, and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**5** Calculate each of the following.

**a** $5A$ **b** $-2B$ **c** $2A + 3B$ **d** $\det A$ **e** $B^{-1}$ **f** $BI$

**6** Examine the answers to question **5**. What do you notice?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 7.2 Addition, subtraction and scalar multiplication of matrices

## Introduction to matrices

studyon

Units 1 & 2
AOS 2
Topic 2
Concept 1

**Addition, subtraction and scalar multiplication of matrices**
Concept summary
Practice questions

The table shows the final medal tally for the top four countries at the 2012 London Olympic Games.

| Country | Gold | Silver | Bronze |
|---|---|---|---|
| United States of America | 46 | 29 | 29 |
| People's Republic of China | 38 | 27 | 22 |
| Great Britain | 29 | 17 | 19 |
| Russian Federation | 24 | 25 | 33 |

This information can be presented in a matrix, without the country names, and without the headings for gold, silver and bronze:

$$\begin{bmatrix} 46 & 29 & 29 \\ 38 & 27 & 22 \\ 29 & 17 & 19 \\ 24 & 25 & 33 \end{bmatrix}$$

The data is presented in a rectangular array and is called a **matrix**. It conveys information such as that the second country won 38 gold, 27 silver and 22 bronze medals. This matrix has four rows and three columns. The numbers in the matrix, in this case representing the number of medals won, are called **elements of the matrix**.

### Matrices

In general we enclose a matrix in square brackets and usually use capital letters to denote it. The size or **order of a matrix** is important, and is determined by the number of rows and the number of columns, strictly in that order.

Consider the following set of matrices:

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$$

$A$ is a $2 \times 2$ matrix: it has two rows and two columns.

$$B = \begin{bmatrix} 2 & -5 & -3 \\ -4 & 2 & -5 \\ 1 & 3 & 4 \end{bmatrix}$$

$B$ is a $3 \times 3$ matrix: it has three rows and three columns. When the numbers of rows and columns in a matrix are equal, it is called a **square matrix**.

$$C = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$C$ is a $3 \times 1$ matrix as it has three rows and one column. If a matrix has only one column, it is also called a **column** or **vector matrix**.

$$D = \begin{bmatrix} 3 & -2 \end{bmatrix}$$

$D$ is a $1 \times 2$ matrix as it has one row and two columns. If a matrix has only one row it is also called a **row matrix**.

$$E = \begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}$$

$E$ is a $3 \times 2$ matrix as it has three rows and two columns.

$$F = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}$$

$F$ is a $2 \times 3$ matrix as it has two rows and three columns.

Each element in a matrix can also be identified by its position in the matrix. We use subscripts to identify the row and column number. For example, in matrix $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, the element 3 is in the first row and first column, so $a_{11} = 3$. The element 5 is in the first row and second column, so $a_{12} = 5$. The element 4 is in the second row and first column, so $a_{21} = 4$. Finally, the element 7 is in the second row and second column, so $a_{22} = 7$.

In general, we can write a $2 \times 2$ matrix as:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A general matrix of order $m \times n$ can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Here, $a_{14}$ denotes the element in the first row and fourth column, $a_{43}$ denotes the element in the fourth row and third column, and $a_{ij}$ denotes the element in the $i$th row and $j$th column.

## Operations on matrices

Operations include addition, subtraction and multiplication of two matrices. Note that we cannot divide matrices.

### Equality of matrices

Two matrices are equal if and only if they have the same size or order and each of the corresponding elements are equal.

For example, if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, then $x = 1,\ y = -2$ and $z = 3$; if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, then $a = 3,\ b = 5,\ c = 4$ and $d = 7$.

### Addition of matrices

Only two matrices of the same size or order can be added together. To add two matrices, we add the elements in the corresponding positions. For example, if $P = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 6 & 3 \end{bmatrix}$, then:

$$P + Q = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2+4 & 3-2 \\ -1+6 & 5+3 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 5 & 8 \end{bmatrix}$$

If two matrices cannot be added together (if the sum does not exist), we say that the two matrices are not conformable for addition.

### Scalar multiplication of matrices

To multiply any matrix (of any size or order) by a **scalar**, we multiply every element in the matrix by the scalar.

If $P = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ then:

$$\begin{aligned} 2P &= 2\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 2 & 2\times 3 \\ 2\times -1 & 2\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 \\ -2 & 10 \end{bmatrix} \end{aligned}$$

### Subtraction of matrices

Only two matrices of the same size or order can be subtracted. To subtract two matrices, we subtract the elements in the corresponding positions. For example, if $P = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 6 & 3 \end{bmatrix}$, then:

$$P - Q = P + (-Q) = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2-4 & 3+2 \\ -1-6 & 5-3 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -7 & 2 \end{bmatrix}$$

If two matrices cannot be subtracted (if the difference does not exist), we say that the matrices are not conformable for subtraction. Note that none of the matrices defined by $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -5 & -3 \\ -4 & 2 & -5 \\ 1 & 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -2 \end{bmatrix}$, $E = \begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}$,

$F = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}$ can be added or subtracted from one another, as they are all of different orders.

**WORKED EXAMPLE 1** **At a football match one food outlet sold 280 pies, 210 hotdogs and 310 boxes of chips. Another food outlet sold 300 pies, 220 hotdogs and 290 boxes of chips. Represent this data as a 1 × 3 matrix, and find the total number of pies, hotdogs and chips sold by these two outlets.**

| THINK | WRITE |
|---|---|
| 1 Use a 1 × 3 matrix to represent the number of pies, hotdogs and chips sold. | [pies  hotdogs  chips] |
| 2 Write down the matrix for the sales from the first outlet. | $S_1 = \begin{bmatrix} 280 & 210 & 310 \end{bmatrix}$ |
| 3 Write down the matrix for the sales from the second outlet. | $S_2 = \begin{bmatrix} 300 & 220 & 290 \end{bmatrix}$ |
| 4 Find the sum of these two matrices. | $S_1 + S_2 = \begin{bmatrix} 280+300 & 210+220 & 310+290 \end{bmatrix}$ <br> $= \begin{bmatrix} 580 & 430 & 600 \end{bmatrix}$ |

**WORKED EXAMPLE 2** **Given the matrices $A = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $B = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ y \end{bmatrix}$ find the values of $x$ and $y$ if $xA + 2B = C$.**

| THINK | WRITE |
|---|---|
| 1 Substitute for the given matrices. | $xA + 2B = C$ <br> $x\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 2\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$ |
| 2 Apply the rules for scalar multiplication. | $\begin{bmatrix} 3x \\ -5x \end{bmatrix} + \begin{bmatrix} -10 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$ |
| 3 Apply the rules for addition of matrices. | $\begin{bmatrix} 3x-10 \\ -5x+8 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$ |
| 4 Apply the rules for equality of matrices. | $3x - 10 = 2$ <br> $-5x + 8 = y$ |
| 5 Solve the first equation for $x$. | $3x = 12$ <br> $x = 4$ |
| 6 Substitute for $x$ into the second equation and solve this equation for $y$. | $-5x + 8 = y$ <br> $y = 8 - 20$ <br> $= -12$ |

## Special matrices

### The zero matrix

The $2 \times 2$ **null matrix** or **zero matrix** $O$, with all elements equal to zero, is given by $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

When matrices $A + B = O$, matrix $B$ is the **additive inverse** of $A$. So, $B = O - A$.

### The identity matrix

The $2 \times 2$ **identity matrix** $I$ is defined by $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This matrix has ones down the leading diagonal and zeros on the other diagonal.

**WORKED EXAMPLE 3**

Given the matrices $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$, find the matrix $X$ if:

**a** $X = 2A - 3B$ **b** $A + X = O$ **c** $X = B + 2A - 3I$

| THINK | WRITE |
|---|---|
| **a 1** Substitute for the given matrices. | **a** $X = 2A - 3B$ <br> $= 2\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} - 3\begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$ |
| **2** Apply the rules for scalar multiplication. | $= \begin{bmatrix} 6 & 10 \\ 8 & 14 \end{bmatrix} - \begin{bmatrix} -15 & -9 \\ 9 & 12 \end{bmatrix}$ |
| **3** Apply the rules for subtraction of matrices. | $= \begin{bmatrix} 21 & 19 \\ -1 & 2 \end{bmatrix}$ |
| **b 1** Transpose the equation to make $X$ the subject. | **b** $A + X = O$ <br> $X = O - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ |
| **2** State the final answer. | $= \begin{bmatrix} -3 & -5 \\ -4 & -7 \end{bmatrix}$ |
| **c 1** Substitute for the given matrices. | **c** $X = B + 2A - 3I$ <br> $= \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| **2** Apply the rules for scalar multiplication. | $= \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 10 \\ 8 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ |
| **3** Apply the rules for addition and subtraction of matrices. | $= \begin{bmatrix} -5 + 6 - 3 & -3 + 10 + 0 \\ 3 + 8 - 0 & 4 + 14 - 3 \end{bmatrix}$ |
| **4** State the final answer. | $= \begin{bmatrix} -2 & 7 \\ 11 & 15 \end{bmatrix}$ |

## EXERCISE 7.2 Addition, subtraction and scalar multiplication of matrices

**PRACTISE**

Work without CAS

1 **WE1** At football matches, commentators often quote player statistics. In one particular game, the top ranked player on the ground had 25 kicks, 8 marks and 10 handballs. The second ranked player on the same team on the ground had 20 kicks, 6 marks and 8 handballs, while the third ranked player on the same team on the ground had 18 kicks, 5 marks and 7 handballs. Represent this data as a $1 \times 3$ matrix, and find the total number of kicks, marks and handballs by these three players from the same team.

2 At the end of a doubles tennis match, one player had 2 aces, 3 double faults, 25 forehand winners and 10 backhand winners, while his partner had 4 aces, 5 double faults, 28 forehand winners and 7 backhand winners. Represent this data as a $2 \times 2$ matrix and find the total number of aces, double faults, forehand and backhand winners for these players.

3 **WE2** Given the matrices $A = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ y \end{bmatrix}$ find the values of $x$ and $y$ if $xA + 2B = C$.

4 Given the matrices $A = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 \\ y \\ z \end{bmatrix}$ find the values of $x$, $y$ and $z$ if $xA - 2B = C$.

5 **WE3** If $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$ find matrix $X$ given the following.

a $X = 3A - 2B$

b $2A + X = O$

c $X = B - 3A + 2I$

6 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$ find the values of $a$, $b$, $c$ and $d$ given the following.

a $A + 2I - 2B = O$

b $3I + 4B - 2A = O$

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 If $A = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ find matrix $C$ given the following.

a $C = A + B$

b $A + C = B$

c $3A + 2C = 4B$

8 If $A = \begin{bmatrix} 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \end{bmatrix}$ find matrix $C$ given the following.

a $C = A + B$

b $A + C = B$

c $3A + 2C = 4B$

9 Consider these matrices: $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$

a Find the following matrices.

i $B + C$

ii $A + B$

b Verify the Associative Law for matrix addition: $A + (B + C) = (A + B) + C$.

10 If $A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ find matrix $C$ given the following.

a $3A = C - 2B$

b $C + 3A - 2B = O$

c $2C + 3A - 2B = O$

**11** Given the matrices $A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find matrix $C$ if the following apply.

**a** $3A + C - 2B + 4I = O$ **b** $4A - C + 3B - 2I = O$

**12** If $A = \begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix}$ and $B = \begin{bmatrix} 2 & y \\ y & -3 \end{bmatrix}$ find the values of $x$ and $y$ given the following.

**a** $A + B = \begin{bmatrix} 7 & 4 \\ 9 & 2 \end{bmatrix}$ **b** $A - B = \begin{bmatrix} 2 & -9 \\ -4 & 7 \end{bmatrix}$ **c** $B - A = \begin{bmatrix} -1 & 1 \\ -4 & -6 \end{bmatrix}$

**13** If $D = \begin{bmatrix} 1 & 4 \\ -3 & 2 \\ 2 & 5 \end{bmatrix}$ and $E = \begin{bmatrix} 2 & -2 \\ -1 & 5 \\ 3 & -3 \end{bmatrix}$ find matrix $C$ given the following.

**a** $C = D + E$ **b** $D + C = E$ **c** $3D + 2C = 4E$

**14** If $D = \begin{bmatrix} 1 & 4 & 5 \\ -3 & 2 & -2 \end{bmatrix}$ and $E = \begin{bmatrix} 2 & -2 & 4 \\ 1 & 4 & -3 \end{bmatrix}$ find the matrix $C$ given the following.

**a** $C = D + E$ **b** $D + C = E$ **c** $3D + 2C = 4E$

Given $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, answer questions **15** and **16**.

**15 a** Write down the values of $a_{11}$, $a_{12}$, $a_{21}$ and $a_{22}$.

**b** Find the $2 \times 2$ matrix $A$ if $a_{11} = 3$, $a_{12} = -2$, $a_{21} = -3$ and $a_{22} = 5$.

**16 a** Find the $2 \times 2$ matrix $A$ whose elements are $a_{ij} = 2i - j$ for $j \neq i$ and $a_{ij} = ij$ for $j = i$.

**b** Find the $2 \times 2$ matrix $A$ whose elements are $a_{ij} = i + j$ for $i < j$, $a_{ij} = i - j + 1$ for $i > j$ and $a_{ij} = i + j + 1$ for $i = j$.

MASTER

**17** The **trace of a matrix** $A$ denoted by $\text{tr}(A)$ is equal to the sum of leading diagonal elements. For $2 \times 2$ matrices, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\text{tr}(A) = a_{11} + a_{22}$. Consider the following matrices: $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$

**a** Find the following.

**i** $\text{tr}(A)$ **ii** $\text{tr}(B)$ **iii** $\text{tr}(C)$

**b** Is $\text{tr}(A + B + C) = \text{tr}(A) + \text{tr}(B) + \text{tr}(C)$?

**c** Is $\text{tr}(2A + 3B - 4C) = 2\text{tr}(A) + 3\text{tr}(B) - 4\text{tr}(C)$?

**18** If $A = \begin{bmatrix} 12 & 10 & 4 \\ 8 & 6 & 8 \\ 14 & 12 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 15 \\ 2 \\ 4 \end{bmatrix}$, use your calculator to calculate $A \times B$.

What type of matrix is $AB$?

# 7.3 Matrix multiplication

## Multiplying matrices

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Units 1 & 2
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Topic 2
Concept 2

**Matrix multiplication**
Concept summary
Practice questions

At the end of an AFL football match between Sydney and Melbourne the scores were as shown.


This information is represented in a matrix as:

$$\begin{array}{r} \\ \text{Sydney} \\ \text{Melbourne} \end{array} \begin{array}{c} \text{Goals} \quad \text{Behinds} \\ \begin{bmatrix} 12 & 15 \\ 9 & 10 \end{bmatrix} \end{array}$$

One goal in AFL football is worth 6 points and one behind is worth 1 point.

This information is represented in a matrix as:

$$\begin{array}{r} \text{Goals} \\ \text{Behinds} \end{array} \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

To get the total points scored by both teams the matrices are multiplied.

$$\begin{bmatrix} 12 & 15 \\ 9 & 10 \end{bmatrix} \times \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \times 6 + 15 \times 1 \\ 9 \times 6 + 10 \times 1 \end{bmatrix} = \begin{bmatrix} 87 \\ 64 \end{bmatrix}$$

This is an example of multiplying a $2 \times 2$ matrix by a $2 \times 1$ matrix to obtain a $2 \times 1$ matrix.

## Multiplying matrices in general

Two matrices $A$ and $B$ may be multiplied together to form the product $AB$ when the number of columns in $A$ is equal to the number of rows in $B$. Such matrices are said to be conformable with respect to multiplication. If $A$ is of order $m \times n$ and $B$ is of order $n \times p$, then the product has order $m \times p$. The number of columns in the first matrix must be equal to the number of rows in the second matrix. The product is obtained by multiplying each element in each row of the first matrix by the corresponding elements of each column in the second matrix.

In general, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} d \\ e \end{bmatrix}$ then $AB = \begin{bmatrix} ad + be \\ cd + de \end{bmatrix}$.

For $2 \times 2$ matrices if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ then:

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

In general $AB \neq BA$.

**WORKED EXAMPLE 4**

Given the matrices $A = \begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix}$ and $X = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, find the following matrices.

**a** $AX$ **b** $XA$

| THINK | WRITE |
|---|---|
| **a 1** Substitute for the given matrices. | **a** $AX = \begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ |
| **2** Apply the rules for matrix multiplication. Since $A$ is a $2 \times 2$ matrix and $X$ is a $2 \times 1$ matrix, the product $AX$ will be a $2 \times 1$ matrix. | $AX = \begin{bmatrix} 6 \times 3 + 5 \times -2 \\ 4 \times 3 + 7 \times -2 \end{bmatrix}$ |
| **3** Simplify and give the final result. | $AX = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$ |
| **b** Apply the rules for matrix multiplication. Since $X$ is a $2 \times 1$ matrix and $A$ is a $2 \times 2$ matrix, the product $XA$ does not exist because the number of columns of the first matrix is not equal to the number or rows in the second matrix. | **b** $XA$ does not exist. |

**WORKED EXAMPLE 5**

Given the matrices $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$ find the following matrices.

**a** $AB$ **b** $BA$ **c** $B^2$

| THINK | WRITE |
|---|---|
| **a 1** Substitute for the given matrices. Since $A$ and $B$ are both $2 \times 2$ matrices, the product $AB$ will also be a $2 \times 2$ matrix. | **a** $AB = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$ |
| **2** Apply the rules for matrix multiplication. | $= \begin{bmatrix} 3 \times -5 + 5 \times 3 & 3 \times -3 + 5 \times 4 \\ 4 \times -5 + 7 \times 3 & 4 \times -3 + 7 \times 4 \end{bmatrix}$ |
| **3** Simplify and give the final result. | $= \begin{bmatrix} 0 & 11 \\ 1 & 16 \end{bmatrix}$ |
| **b 1** Substitute for the given matrices. Since both $A$ and $B$ are $2 \times 2$ matrices, the product $BA$ will also be a $2 \times 2$ matrix. | **b** $BA = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ |
| **2** Apply the rules for matrix multiplication. | $= \begin{bmatrix} -5 \times 3 + -3 \times 4 & -5 \times 5 + -3 \times 7 \\ 3 \times 3 + 4 \times 4 & 3 \times 5 + 4 \times 7 \end{bmatrix}$ |
| **3** Simplify and give the final result. | $= \begin{bmatrix} -27 & -46 \\ 25 & 43 \end{bmatrix}$ |
| **c 1** $B^2 = B \times B$. Write the matrices. | **c** $B^2 = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$ |

**2** Since $B$ is a $2 \times 2$ matrix, $B^2$ will also be a $2 \times 2$ matrix. Apply the rules for matrix multiplication.

$$= \begin{bmatrix} -5 \times -5 + -3 \times 3 & -5 \times -3 + -3 \times 4 \\ 3 \times -5 + 4 \times 3 & 3 \times -3 + 4 \times 4 \end{bmatrix}$$

**3** Simplify and give the final result.

$$= \begin{bmatrix} 16 & 3 \\ -3 & 7 \end{bmatrix}$$

These last two examples show that matrix multiplication in general is not commutative: $AB \neq BA$, although there are exceptions. It is also possible that one product exists and the other simply does not exist, and that the products may have different orders. Note that squaring a matrix (when defined) is not the square of each individual element.

**WORKED EXAMPLE 6**

**Given the matrices** $E = \begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}$ **and** $F = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}$**, find the following matrices.**

**a** $EF$ **b** $FE$

| THINK | WRITE |
|---|---|
| **a 1** Substitute for the given matrices. | **a** $EF = \begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}$ |
| **2** Apply the rules for matrix multiplication. Since $E$ is a $3 \times 2$ matrix and $F$ is a $2 \times 3$ matrix, the product $EF$ will be a $3 \times 3$ matrix. | $= \begin{bmatrix} 3 \times 2 + 5 \times 4 & 3 \times 3 + 5 \times -5 & 3 \times 4 + 5 \times -2 \\ -4 \times 2 + 2 \times 4 & -4 \times 3 + 2 \times -5 & -4 \times 4 + 2 \times -2 \\ -1 \times 2 + 3 \times 4 & -1 \times 3 + 3 \times -5 & -1 \times 4 + 3 \times -2 \end{bmatrix}$ |
| **3** Simplify and give the final result. | $= \begin{bmatrix} 26 & -16 & 2 \\ 0 & -22 & -20 \\ 10 & -18 & -10 \end{bmatrix}$ |
| **b 1** Substitute for the given matrices. | **b** $FE = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}$ |
| **2** Apply the rules for matrix multiplication. Since $F$ is a $2 \times 3$ matrix and $E$ is a $3 \times 2$ matrix, the product $EF$ will be a $2 \times 2$ matrix. | $= \begin{bmatrix} 2 \times 3 + 3 \times -4 + 4 \times -1 & 2 \times 5 + 3 \times 2 + 4 \times 3 \\ 4 \times 3 + -5 \times -4 + 2 \times -1 & 4 \times 5 + -5 \times 2 + -2 \times 3 \end{bmatrix}$ |
| **3** Simplify and give the final result. | $= \begin{bmatrix} -10 & 28 \\ 34 & 4 \end{bmatrix}$ |

# EXERCISE 7.3 Matrix multiplication

**PRACTISE**

Work without CAS

1 WE4 Given the matrices $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, find the following matrices.

a $AX$ b $XA$

2 Given the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$ find the following matrices.

a $AX$ b $XA$

3 WE5 Given the matrices $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$ find the following matrices.

a $AB$ b $BA$ c $A^2$

4 Given the matrices $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find the following matrices.

a $AO$ b $OA$ c $AI$ d $IA$

What do you observe from this example?

5 WE6 Given the matrices $D = \begin{bmatrix} 2 & -1 \\ -3 & 5 \\ -1 & -4 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 5 \end{bmatrix}$ find the following matrices.

a $DE$ b $ED$

6 Given the matrices $C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & -2 \end{bmatrix}$ find the following matrices.

a $CD$ b $DC$

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 a Given the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ verify the following.

i $AI = IA = A$ ii $AO = OA = O$

b Given the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ verify the following.

i $AI = IA = A$ ii $AO = OA = O$

8 a Given the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ verify that $(I - A)(I + A) = I - A^2$.

b If $A = \begin{bmatrix} 3 & 0 \\ -4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}$ show that $AB = O$ where $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Does $BA = O$?

c If $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}$ show that $AB = O$ where $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Does $BA = O$?

**9** Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$.

**a** Verify the Distributive Law: $A(B + C) = AB + AC$.
**b** Verify the Associative Law for Multiplication: $A(BC) = (AB)C$.
**c** Is $(A + B)^2 = A^2 + 2AB + B^2$? Explain.
**d** Show that $(A + B)^2 = A^2 + AB + BA + B^2$.

**10** If $A = \begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix}$ and $B = \begin{bmatrix} 2 & x \\ x & -3 \end{bmatrix}$ find the value of $x$ given the following.

**a** $AB = \begin{bmatrix} 3 & 18 \\ 13 & 3 \end{bmatrix}$
**b** $BA = \begin{bmatrix} -16 & 10 \\ 10 & 24 \end{bmatrix}$
**c** $A^2 = \begin{bmatrix} -2 & 12 \\ -8 & -2 \end{bmatrix}$
**d** $B^2 = \begin{bmatrix} 8 & -2 \\ -2 & 13 \end{bmatrix}$

**11** Given the matrices $A = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$ find, if possible, each of the following matrices.

**a** $A + B$
**b** $A + C$
**c** $B + C$
**d** $AB$
**e** $BA$
**f** $AC$
**g** $CA$
**h** $BC$
**i** $CB$
**j** $ABC$
**k** $CBA$
**l** $CAB$

**12** Given $D = \begin{bmatrix} 1 & 4 \\ -3 & 2 \\ 2 & 5 \end{bmatrix}$ and $E = \begin{bmatrix} 2 & -2 & 4 \\ 1 & 4 & -3 \end{bmatrix}$ find the following matrices.

**a** $DE$
**b** $ED$
**c** $E + D$
**d** $D^2$

**13** Given $D = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -4 \end{bmatrix}$ and $E = \begin{bmatrix} 6 & 2 \\ 3 & -1 \\ 3 & -3 \end{bmatrix}$ find the following matrices.

**a** $DE$
**b** $ED$
**c** $E + D$
**d** $D^2$

**14** **a** If $P = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ find the matrices $P^2$, $P^3$, $P^4$ and deduce the matrix $P^n$.

**b** If $Q = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ find the matrices $Q^2$, $Q^3$, $Q^4$ and deduce the matrix $Q^n$.

**c** If $R = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ find the matrices $R^2$, $R^3$, $R^4$ and deduce the matrix $R^n$.

**d** If $S = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ find the matrices $S^2$, $S^3$, $S^4$, and deduce the matrices $S^8$ and $S^9$.

**15** If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ evaluate the matrix $A^2 - 6A + 11I$.

**16** **a** If $B = \begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ evaluate the matrix $B^2 - B - 2I$.

**b** If $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ evaluate the matrix $C^2 - 5C + 14I$.

MASTER

**17** If $D = \begin{bmatrix} d & -4 \\ -2 & 8 \end{bmatrix}$ evaluate the matrix $D^2 - 9D$.

**18** The trace of a matrix $A$ denoted by tr$(A)$ is equal to the sum of leading diagonal elements. For $2 \times 2$ matrices if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\text{tr}(A) = a_{11} + a_{22}$.

Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$.

**a** Find the following.

**i** $\text{tr}(AB)$ **ii** $\text{tr}(BA)$ **iii** $\text{tr}(A)\text{tr}(B)$

**b** Is $\text{tr}(ABC) = \text{tr}(A)\text{tr}(B)\text{tr}(C)$?

# 7.4 Determinants and inverses of 2 × 2 matrices

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Units 1 & 2
AOS 2
Topic 2
Concept 3

**Determinants and inverses of 2 × 2 matrices**
Concept summary
Practice questions

## Determinant of a 2 × 2 matrix

Associated with a square matrix is a single number called the **determinant of a matrix**. For $2 \times 2$ matrices, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of the matrix $A$ is denoted by $\det(A)$ or often given the symbol $\Delta$. The determinant is represented not by the square brackets that we use for matrices, but by straight lines; that is, $\det(A) = \Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. To evaluate the determinant, multiply the elements in the leading diagonal and subtract the product of the elements in the other diagonal: $\det(A) = \Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

**WORKED EXAMPLE 7**

Find the determinant of the matrix $F = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$.

| THINK | WRITE |
|---|---|
| 1 Apply the definition and multiply the elements in the leading diagonal. Subtract the product of the elements in the other diagonal. | $F = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$<br>$\det(F) = 3 \times 7 - 4 \times 5$<br>$= 21 - 20$ |
| 2 State the value of the determinant. | $\det(F) = 1$ |

## Inverses of 2 × 2 matrices

The multiplicative $2 \times 2$ identity matrix $I$, defined by $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, has the property that for a $2 \times 2$ non-zero matrix $A$, $AI = IA = A$.

When any square matrix is multiplied by its multiplicative inverse, the identity matrix $I$ is obtained. The multiplicative inverse is called the inverse matrix and is denoted by $A^{-1}$. Note that $A^{-1} \neq \frac{1}{A}$ as division of matrices is not defined; furthermore, $A A^{-1} = A^{-1} A = I$.

Consider the products of $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$:

$$A A^{-1} = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 \times 7 + 5 \times -4 & 3 \times -5 + 5 \times 3 \\ 4 \times 7 + 7 \times -4 & 4 \times -5 + 7 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} A = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 7 \times 3 + -5 \times 4 & 7 \times 5 + -5 \times 7 \\ -4 \times 3 + 3 \times 4 & -4 \times 5 + 3 \times 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now for the matrix $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ the determinant $\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} = 1$.

$A^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$ is obtained from the matrix $A$ by swapping the elements on the leading diagonal, and placing a negative sign on the other two elements. These results are true in general for $2 \times 2$ matrices, but we must also account for the value of a non-unit determinant. To find the inverse of a $2 \times 2$ matrix, the value of the determinant is calculated first; then, provided that the determinant is non-zero, we divide by the determinant, then swap the elements on the leading diagonal and place a negative sign on the other two elements.

In general if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the inverse matrix $A^{-1}$ is given by

$A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. We can show that $A A^{-1} = A^{-1} A = I$.

**WORKED EXAMPLE 8** Find the inverse of the matrix $P = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ and verify that $P P^{-1} = P^{-1} P = I$.

| THINK | WRITE |
| --- | --- |
| 1 Calculate the determinant. If $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\lvert P \rvert = ad - bc$. | $\lvert P \rvert = \begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} = 2 \times 5 - -1 \times 3 = 10 + 3 = 13$ |
| 2 To find the inverse of matrix $P$, apply the rule $P^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. | $P^{-1} = \dfrac{1}{13}\begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix}$ |
| 3 Substitute and evaluate $P P^{-1}$. | $P P^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \times \dfrac{1}{13}\begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix}$ |
| 4 Apply the rules for scalar multiplication and multiplication of matrices. | $= \dfrac{1}{13}\begin{bmatrix} 2 \times 5 + 3 \times 1 & 2 \times -3 + 3 \times 2 \\ -1 \times 5 + 5 \times 1 & -1 \times -3 + 5 \times 2 \end{bmatrix}$ |

5 Simplify the matrix product to show that $PP^{-1} = I$.

$$= \frac{1}{13}\begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6 Substitute and evaluate $P^{-1}P$.

$$P^{-1}P = \frac{1}{13}\begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

7 Use the rules for scalar multiplication and multiplication of matrices.

$$= \frac{1}{13}\begin{bmatrix} 5 \times 2 + -3 \times -1 & 5 \times 3 + -3 \times 5 \\ 1 \times 2 + 2 \times -1 & 1 \times 3 + 2 \times 5 \end{bmatrix}$$

8 Simplify the matrix product to show that $P^{-1}P = I$.

$$= \frac{1}{13}\begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### Singular matrices

If a matrix has a zero determinant then the inverse matrix simply does not exist, and the original matrix is termed a **singular matrix**.

**WORKED EXAMPLE 9**

**Show that the matrix $\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}$ is singular.**

**THINK**

1 Evaluate the determinant.

**WRITE**

$$\begin{vmatrix} -3 & 2 \\ 6 & -4 \end{vmatrix} = (-3 \times -4) - (6 \times 2)$$
$$= 12 - 12$$
$$= 0$$

2 Since the determinant is zero, the matrix $\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}$ is singular.

$$\begin{vmatrix} -3 & 2 \\ 6 & -4 \end{vmatrix} = 0$$

**WORKED EXAMPLE 10**

**If $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, express the determinant of the matrix $A - kI$ in the form $pk^2 + qk + r$ and evaluate the matrix $pA^2 + qA + rI$.**

**THINK**

1 Substitute to find the matrix $A - kI$. Apply the rules for scalar multiplication and subtraction of matrices.

**WRITE**

$$A - kI = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2-k & 4 \\ 3 & 5-k \end{bmatrix}$$

2 Evaluate the determinant of the matrix $A - kI$.

$$\det(A - kI) = \begin{vmatrix} -2-k & 4 \\ 3 & 5-k \end{vmatrix}$$
$$= (-2-k)(5-k) - 3 \times 4$$

3 Simplify the determinant of the matrix $A - kI$.

$$= -(2+k)(5-k) - 1$$
$$= -(10 + 3k - k^2) - 12$$
$$= k^2 - 3k - 22$$

| | |
|---|---|
| **4** State the values of $p$, $q$ and $r$. | $p = 1; q = -3; r = -22$ |
| **5** Determine the matrix $A^2$. | $A^2 = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix}$ |
| **6** Substitute for $p$, $q$ and $r$ and evaluate the matrix $A^2 - 3A - 22I$. | $A^2 - 3A - 22I = \begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix} - 3\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} - 22\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| **7** Simplify by applying the rules for scalar multiplication of matrices. | $= \begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix} - \begin{bmatrix} -6 & 12 \\ 9 & 15 \end{bmatrix} - \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$ |
| **8** Simplify and apply the rules for addition and subtraction of matrices. | $\therefore A^2 - 3A - 22I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ |

## EXERCISE 7.4 Determinants and inverses of 2 × 2 matrices

**PRACTISE**

Work without CAS

**1** WE7 Find the determinant of the matrix $G = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$.

**2** The matrix $\begin{bmatrix} x & 5 \\ 3 & x+2 \end{bmatrix}$ has a determinant equal to 9. Find the values of $x$.

**3** WE8 Find the inverse of the matrix $A = \begin{bmatrix} 4 & -2 \\ 5 & 6 \end{bmatrix}$ and verify that $A\,A^{-1} = A^{-1}\,A = I$.

**4** The inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} p & 3 \\ 3 & q \end{bmatrix}$. Find the values of $p$ and $q$.

**5** WE9 Show that the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 10 \end{bmatrix}$ is singular.

**6** Find the value of $x$ if the matrix $\begin{bmatrix} x & 4 \\ 3 & x+4 \end{bmatrix}$ is singular.

**7** WE10 If $A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ express the determinant of the matrix $A - kI$ in the form $pk^2 + qk + r$ and evaluate the matrix $pA^2 + qA + rI$.

**8** If $A = \begin{bmatrix} 4 & -8 \\ -3 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find the value of $k$ for which the determinant of the matrix $A - kI$ is equal to zero.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**9** Consider the matrix $P = \begin{bmatrix} 6 & -2 \\ 4 & 2 \end{bmatrix}$.

**a** Find the following.

**i** $\det(P)$ **ii** $P^{-1}$

**b** Verify that $P\,P^{-1} = P^{-1}\,P = I$.

**c** Find the following.

**i** $\det(P^{-1})$ **ii** $\det(P)\det(P^{-1})$

**10** Find the inverse matrix of each of the following matrices.

**a** $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & -3 \\ 2 & -1 \end{bmatrix}$

The following matrices refer to questions **11** and **12**.

$$A = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

**11** **a** Find $\det(A)$, $\det(B)$ and $\det(C)$.

**b** Is $\det(AB) = \det(A)\det(B)$?

**c** Verify that $\det(ABC) = \det(A)\det(B)\det(C)$.

**12** **a** Find the matrices $A^{-1}$, $B^{-1}$, $C^{-1}$.

**b** Is $(AB)^{-1} = A^{-1}B^{-1}$?

**c** Is $(AB)^{-1} = B^{-1}A^{-1}$?

**d** Is $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$?

**13** Find the value of $x$ for each of the following.

**a** $\begin{vmatrix} x & -3 \\ 4 & 2 \end{vmatrix} = 6$ **b** $\begin{vmatrix} x & x \\ 8 & 2 \end{vmatrix} = 12$ **c** $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = 4$ **d** $\begin{vmatrix} \frac{1}{x} & x \\ -2 & 3 \end{vmatrix} = 7$

**14** Find the values of $x$ if each of the following are singular matrices.

**a** $\begin{bmatrix} x & -3 \\ 4 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} x & \frac{1}{x} \\ 8 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} x & 3 \\ 4 & x \end{bmatrix}$ **d** $\begin{bmatrix} x+1 & -3 \\ -2 & x \end{bmatrix}$

**15** Given $A = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$ find, if possible, the following matrices.

**a** $(AB)^{-1}$ **b** $A^{-1}$ **c** $B^{-1}$

**d** $C^{-1}$ **e** $(ABC)^{-1}$

**16** If $A = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ express the determinant of the matrix $A - kI$, $k \in R$ in the form $pk^2 + qk + r$, and evaluate the matrix $pA^2 + qA + rI$.

**17** If $B = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ express the determinant of the matrix $B - kI$, $k \in R$ in the form $pk^2 + qk + r$, and evaluate the matrix $pB^2 + qB + rI$.

**18** Consider the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**a** Find the values of $k$, for which the determinant of the matrix $A - kI = 0$.

**b** Find the matrix $P^{-1}AP$.

MASTER

**19** Consider the matrices $B = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**a** Find the values of $k$ for which the determinant of the matrix $B - kI = 0$.

**b** Find the matrix $Q^{-1}BQ$.

**20** Let $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

**a** Use a CAS technology, or otherwise, to find the following matrices.

**i** $R\left(\frac{\pi}{2}\right)$ **ii** $R\left(\frac{\pi}{6}\right)$ **iii** $R\left(\frac{\pi}{3}\right)$ **iv** $R^2$ CAS **v** $R^{-1}$

**b** Show that $R\left(\frac{\pi}{6}\right)R\left(\frac{\pi}{3}\right) = R\left(\frac{\pi}{2}\right)$.

**c** Show that $R(\alpha)R(\beta) = R(\alpha + \beta)$.

# 7.5 Matrix equations and solving 2 × 2 linear simultaneous equations

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Units 1 & 2

AOS 2

Topic 2

Concept 4

**Matrix equations and solving 2 × 2 linear simultaneous equations**
Concept summary
Practice questions

## Systems of 2 × 2 simultaneous linear equations

Consider the two linear equations $ax + by = e$ and $cx + dy = f$.

These equations written in matrix form are:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

This is the matrix equation $AX = K$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $K = \begin{bmatrix} e \\ f \end{bmatrix}$.

Recall that $A^{-1} = \dfrac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $\Delta = ad - bc$, and that this matrix has the property that $A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

To solve the matrix equation $AX = K$, pre-multiply both sides of the equation $AX = K$ by $A^{-1}$. Recall that the order of multiplying matrices is important.

$$\begin{aligned} A^{-1}AX &= A^{-1}K, \\ \text{since } A^{-1}A &= I \\ IX &= A^{-1}K \quad \text{and} \quad IX = X \\ X &= A^{-1}K \end{aligned}$$

$$\Rightarrow X = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}$$

**WORKED EXAMPLE 11**

**Solve for $x$ and $y$ using inverse matrices.**

$4x + 5y = 6$

$3x + 2y = 8$

| THINK | WRITE |
|---|---|
| 1 First rewrite the two equations as a matrix equation. | $\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ |
| 2 Write down the matrices $A$, $X$ and $K$. | $A = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $K = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ |
| 3 Find the determinant of the matrix $A$. | $\Delta = \begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} = 4 \times 2 - 3 \times 5 = -7$ |
| 4 Find the inverse matrix $A^{-1}$, and apply the rules for scalar multiplication to simplify this inverse. | $A^{-1} = \dfrac{1}{-7}\begin{bmatrix} 2 & -5 \\ -3 & 4 \end{bmatrix} = \dfrac{1}{7}\begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$ |

| | |
|---|---|
| **5** The unknown matrix $X$ satisfies the equation $X = A^{-1}K$. Write the equation in matrix form. | $X = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$ |
| **6** Apply the rules for matrix multiplication. The product is a 2 × 1 matrix. | $= \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 \times 6 + 5 \times 8 \\ 3 \times 6 + -4 \times 8 \end{bmatrix}$ |
| **7** Apply the rules for scalar multiplication, and the rules for equality of matrices. | $= \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 28 \\ -14 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ |
| **8** State the final answer. | $x = 4$ and $y = -2$ |

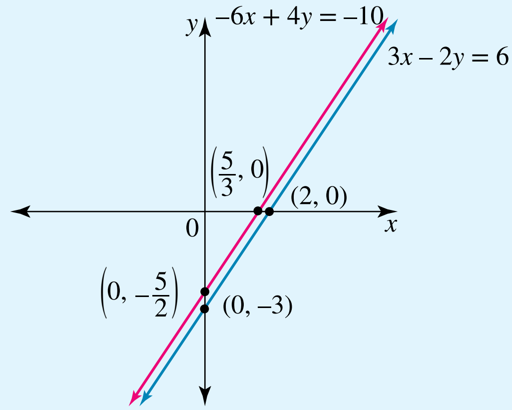
**WORKED EXAMPLE 12**

**Solve the following simultaneous linear equations for $x$ and $y$.**

$$3x - 2y = 6$$
$$-6x + 4y = -10$$

| THINK | WRITE |
|---|---|
| **1** First write the two equations as matrix equations. | $\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix}$ |
| **2** Write down the matrices $A$, $X$ and $K$. | $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $K = \begin{bmatrix} 6 \\ -10 \end{bmatrix}$ |
| **3** Find the determinant of the matrix $A$. | $\Delta = \begin{vmatrix} 3 & -2 \\ -6 & 4 \end{vmatrix} = (3 \times 4) - (-6 \times -2) = 0$ |
| **4** The inverse matrix $A^{-1}$ does not exist. This method cannot be used to solve the simultaneous equations. | The matrix $A$ is singular. |
| **5** Apply the method of elimination by first numbering each equation. | (1) $3x - 2y = 6$<br>(2) $-6x + 4y = -10$ |
| **6** To eliminate $x$, multiply equation (1) by 2 and add this to equation (2). | (1) × 2 $6x - 4y = 12$<br>(2) $-6x + 4y = -10$ |
| **7** In eliminating $x$, we have also eliminated $y$ and obtained a contradiction. | $\Rightarrow\ 0 = 2$ ? |
| **8** Apply another method to solving simultaneous equations: the graphical method. Since both equations represent straight lines, determine the $x$- and $y$-intercepts. | Line (1) $3x - 2y = 6$ crosses the $x$-axis at $(2, 0)$ and the $y$-axis at $(0, -3)$.<br>Line (2) $-6x + 4y = -10$ crosses the $x$-axis at $\left(\frac{5}{3}, 0\right)$ and the $y$-axis at $\left(0, -\frac{5}{2}\right)$. |

9 Sketch the graphs. Note that the two lines are parallel and therefore have no points of intersection.



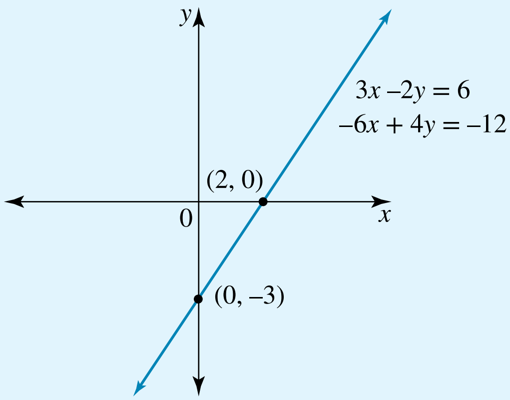

10 State the final answer. — There is no solution.

**WORKED EXAMPLE 13** **Solve the following linear simultaneous equations for $x$ and $y$.**

$$3x - 2y = 6$$
$$-6x + 4y = -12$$

| THINK | WRITE |
|---|---|
| 1 First write the two equations as matrix equations. | $\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$ |
| 2 Write down the matrices $A$, $X$ and $K$. | $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $K = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$ |
| 3 Find the determinant of the matrix $A$. | $\Delta = \begin{vmatrix} 3 & -2 \\ -6 & 4 \end{vmatrix} = (3 \times 4) - (-6 \times -2) = 0$ |
| 4 The inverse matrix $A^{-1}$ does not exist. This method cannot be used to solve the simultaneous equations. | The matrix $A$ is singular. |
| 5 Apply another method of solving simultaneous equations: elimination. Number the equations. | (1) $3x - 2y = 6$<br>(2) $-6x + 4y = -12$ |
| 6 To eliminate $x$, multiply equation (1) by 2 and add this to equation (2). | (1) × 2 $6x - 4y = 12$<br>(2) $-6x + 4y = -12$ |
| 7 In eliminating $x$, we have also eliminated $y$; however, we have obtained a true consistent equation. | $\Rightarrow 0 = 0$ |
| 8 Apply another method of solving simultaneous equations: the graphical method. Determine the $x$- and $y$-intercepts. | $3x - 2y = 6$ (line 1) crosses the $x$-axis at $(2, 0)$ and the $y$-axis at $(0, -3)$.<br>$-6x + 4y = -12$ (line 2) is actually the same line, since $(2) = -2 \times (1)$. |

9 Sketch the graphs. Note that since the lines overlap, there is an infinite number of points of intersection.




Since $3x - 2y = 6 \quad \Rightarrow \quad x = \dfrac{6 + 2y}{3}$

If $y = 0,\ x = 2 \quad (2, 0)$

If $y = 1,\ x = \dfrac{8}{3} \quad \left(\dfrac{8}{3}, 1\right)$

If $y = 2,\ x = \dfrac{10}{3} \quad \left(\dfrac{10}{3}, 2\right)$

If $y = 3\ x = 4 \quad (4, 3)$

In general, let $y = t$ so that $x = \dfrac{6 + 3t}{3}$.

As a coordinate: $\left(\dfrac{6 + 2t}{3}, t\right)$

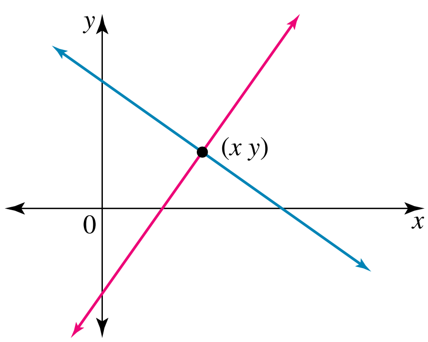
10 State the final answer.

There is an infinite number of solutions of the form $\left(2 + \dfrac{2t}{3}, t\right)$ where $t \in R$.

## Geometrical interpretation of solutions

Consider the simultaneous linear equations $ax + by = e$ and $cx + dy = f$ and the determinant:

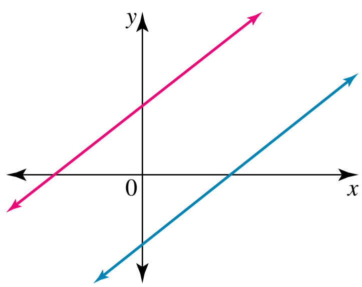
$$\det(A) = \Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$




If the determinant is non-zero ($\Delta \neq 0$) then these two equations are consistent; graphically, the two lines have different gradients and therefore they intersect at a unique point.

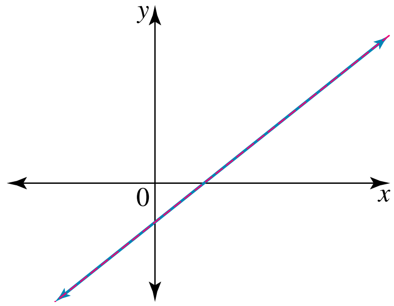
If the determinant is zero ($\Delta = 0$) then there are two possibilities.

- The only certainty is that there is not a unique solution.
- If the lines are parallel, the equations are inconsistent and there is a contradiction; this indicates that there is no solution. That is, graphically the two lines have the same gradient but different $y$-intercepts.



- If the lines are simply multiples of one another, the equations are consistent and dependent. This indicates that there is an infinite number of solutions.

That is, they have the same gradient and the same $y$-intercept (they overlap).




**WORKED EXAMPLE 14**

**Find the values of $k$ for which the equations $kx - 3y = k - 1$ and $10x - (k + 1)y = 8$ have:**

**a a unique solution**

**b no solution**

**c an infinite number of solutions.**

**(You are not required to find the solution set.)**

| THINK | WRITE |
|---|---|
| **a 1** First write the two equations as matrix equations. | **a** $\begin{bmatrix} k & -3 \\ 10 & -(k+1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k-1 \\ 8 \end{bmatrix}$ |
| **2** Write out the determinant, as it is the key to answering this question. | $\Delta = \begin{vmatrix} k & -3 \\ 10 & -(k+1) \end{vmatrix}$ |
| **3** Evaluate the determinant in terms of $k$. | $\Delta = -k(k+1) + 30$ <br> $= -k^2 - k + 30$ <br> $= -(k^2 + k - 30)$ |
| **4** If the solution is unique then $\Delta \neq 0$; that is, there is a unique solution when $k \neq -6$ and $k \neq 5$. <br> Now investigate these two cases. | $\Delta = -(k+6)(k-5)$ <br> There is a unique solution when $k \neq -6$ and $k \neq 5$, or $k \in R\backslash\{-6, 5\}$. |
| **5** Substitute $k = -6$ into the two equations. | (1) $-6x - 3y = -7 \Rightarrow 2x + y = \frac{7}{3}$ <br> (2) $10x + 5y = 8 \quad \Rightarrow 2x + y = \frac{8}{5}$ |
| **b 1** If there is no solution then the two equations represent parallel lines with different $y$-intercepts. Interpret the answer. | **b** When $k = -6$ there is no solution. |
| **2** Substitute $k = 5$ into the two equations. | (1) $5x - 3y = 4$ <br> (2) $10x - 6y = 8$ |
| **c** If there are infinite solutions, the two equations are multiples of one another. Interpret the answer. | **c** When $k = 5$ there is an infinite number of solutions. |

## Matrix equations

In matrix algebra, matrices are generally not commutative. The order in which matrices are multiplied is important. Consider the matrix equations $AX = B$ and $XA = B$, where all the matrices $A$, $B$ and $X$ are $2 \times 2$ matrices, the matrices $A$ and $B$ are given, and in each case the unknown matrix $X$ needs to be found.

If $AX = B$, to find the matrix $X$ both sides of the equation must be pre-multiplied by $A^{-1}$, the inverse of matrix $A$.

$\Rightarrow A^{-1}AX = A^{-1}B$ since $A^{-1}A = I$

$IX = A^{-1}B$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; recall that this $2 \times 2$ identity matrix satisfies $IX = X$ so that if $AX = B$ then $X = A^{-1}B$.

If $XA = B$, to find the matrix $X$, post-multiply both sides of the equation by $A^{-1}$.

$\Rightarrow XAA^{-1} = BA^{-1}$ since $AA^{-1} = I$

$XI = BA^{-1}$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; recall that this $2 \times 2$ identity matrix satisfies $XI = X$ so that if $XA = B$ then $X = BA^{-1}$.

**WORKED EXAMPLE 15**

**Given the matrices $A = \begin{bmatrix} 3 & -5 \\ 4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$ find matrix $X$, if:**

**a** $AX = B$

**b** $XA = B$.

| THINK | WRITE |
|---|---|
| **a 1** Evaluate the determinant of the matrix $A$. | **a** $\det(A) = \begin{vmatrix} 3 & -5 \\ 4 & -6 \end{vmatrix} = 3 \times -6 - 4 \times -5 = 2$ |
| **2** Find the inverse matrix $A^{-1}$. | $A^{-1} = \frac{1}{2}\begin{bmatrix} -6 & 5 \\ -4 & 3 \end{bmatrix}$ |
| **3** If $AX = B$, pre-multiply both sides by the inverse matrix $A^{-1}$; then $X = A^{-1}B$. | $X = \frac{1}{2}\begin{bmatrix} -6 & 5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$ |
| **4** $X$ is a $2 \times 2$ matrix. Apply the rules for multiplying matrices. | $= \frac{1}{2}\begin{bmatrix} -6 \times -5 + 5 \times 3 & -6 \times -3 + 5 \times 4 \\ -4 \times -5 + 3 \times 3 & -4 \times -3 + 3 \times 4 \end{bmatrix}$ |
| **5** State the answer. | $= \frac{1}{2}\begin{bmatrix} 45 & 38 \\ 29 & 24 \end{bmatrix}$ |
| **b 1** If $XA = B$, post-multiply both sides by the inverse matrix $A^{-1}$ so that $X = BA^{-1}$. | **b** $X = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -6 & 5 \\ -4 & 3 \end{bmatrix}$ <br> $= \frac{1}{2}\begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -6 & 5 \\ -4 & 3 \end{bmatrix}$ |
| **2** $X$ is a $2 \times 2$ matrix. Apply the rules to multiply the matrices. | $= \frac{1}{2}\begin{bmatrix} -5 \times -6 + -3 \times -4 & -5 \times 5 + -3 \times 3 \\ 3 \times -6 + 4 \times -4 & 3 \times 5 + 4 \times 3 \end{bmatrix}$ |
| **3** State the answer. | $= \frac{1}{2}\begin{bmatrix} 42 & -34 \\ -34 & 27 \end{bmatrix}$ |

**WORKED EXAMPLE 16** Given the matrices $A = \begin{bmatrix} 3 & -4 \\ 5 & -6 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & -2 \end{bmatrix} = [3\ \ -2]$ find matrix $X$ if:

**a** $AX = C$

**b** $XA = D$.

| THINK | WRITE |
|---|---|
| **a 1** Evaluate the determinant of the matrix $A$. | **a** $\det(A) = \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 3 \times -6 - 5 \times -4 = 2$ |
| **2** Find the inverse matrix $A^{-1}$. | $A^{-1} = \frac{1}{2}\begin{bmatrix} -6 & 4 \\ -5 & 3 \end{bmatrix}$ |
| **3** If $AX = C$ then pre-multiply both sides by the inverse matrix $A^{-1}$. | $AX = C$<br>$A^{-1}AX = A^{-1}C$ |
| **4** Substitute for the given matrices. | $IX = X = A^{-1}C$<br>$X = \frac{1}{2}\begin{bmatrix} -6 & 4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ |
| **5** $X$ is a $2 \times 1$ matrix. Apply the rules to multiply the matrices. | $= \frac{1}{2}\begin{bmatrix} -6 \times -1 + 4 \times 2 \\ -5 \times -1 + 3 \times 2 \end{bmatrix}$ |
| **6** State the answer. | $= \frac{1}{2}\begin{bmatrix} 14 \\ 11 \end{bmatrix}$ |
| **b 1** If $XA = D$, post-multiply both sides by the inverse matrix $A^{-1}$. | **b** $XA = D$<br>$XAA^{-1} = DA^{-1}$ |
| **2** Substitute for the given matrices. | $XI = X = DA^{-1}$<br>$X = \begin{bmatrix} 3 & -2 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -6 & 4 \\ -5 & 3 \end{bmatrix}$ |
| **3** $X$ is a $1 \times 2$ matrix. Apply the rules to multiply the matrices. | $= \frac{1}{2}\begin{bmatrix} 3 \times -6 + -2 \times -5 & 3 \times 4 + -2 \times 3 \end{bmatrix}$<br>$= \frac{1}{2}\begin{bmatrix} -8 & 6 \end{bmatrix}$ |
| **4** State the answer. | $= \begin{bmatrix} -4 & 3 \end{bmatrix}$ |

## EXERCISE 7.5 Matrix equations and solving 2 × 2 linear simultaneous equations

**PRACTISE**

Work without CAS

1 **WE11** Solve for $x$ and $y$ using inverse matrices.

$$3x - 4y = 23$$
$$5x + 2y = 21$$

2 Solve for $x$ and $y$ using inverse matrices.

$$2x + 5y = -7$$
$$3x - 2y = 18$$

3 **WE12** Solve the following linear simultaneous equations for $x$ and $y$.

$$4x - 3y = 12$$
$$-8x + 6y = -18$$

4 Find the value of $k$, if the following linear simultaneous equations for $x$ and $y$ have no solution.

$$5x - 4y = 20$$
$$kx + 2y = -8$$

5 WE13 Solve the following linear simultaneous equations for $x$ and $y$.

$$4x - 3y = 12$$
$$-8x + 6y = -24$$

6 Find the value of $k$ if the following linear simultaneous equations for $x$ and $y$ have an infinite number of solutions.

$$5x - 4y = 20$$
$$kx + 2y = -10$$

7 WE14 Find the values of $k$ for which the system of equations $(k + 1)x - 2y = 2k$ and $-6x + 2ky = -8$ has:

a a unique solution

b no solution

c an infinite number of solutions.

(You are not required to find the solution set.)

8 Find the values of $k$ for which the system of equations $2x + (k - 1)y = 4$ and $kx + 6y = k + 4$ has:

a a unique solution

b no solution

c an infinite number of solutions.

(You are not required to find the solution set.)

9 WE15 If $A = \begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix}$ find matrix $X$ given the following.

a $AX = B$

b $XA = B$

10 If $P = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & -1 \\ -3 & 6 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ find matrix $X$ given the following.

a $XP - Q = O$

b $PX - Q = O$

11 WE16 If $A = \begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}$, $C = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & -5 \end{bmatrix}$ find matrix $X$ given the following.

a $AX = C$

b $XA = D$

12 If $B = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 3 \end{bmatrix}$ find matrix $X$ given the following.

a $BX = C$

b $XB = D$

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

13 Solve each of the following simultaneous linear equations using inverse matrices.

a $2x + 3y = 4$
$-x + 4y = 9$

b $4x + 5y = -6$
$2x - 3y = 8$

c $x - 2y = 8$
$5x + 4y = -2$

d $-2x + 7y + 3 = 0$
$3x + y + 7 = 0$

14 Solve each of the following simultaneous linear equations using inverse matrices.

a $3x + 4y = 6$
$2x + 3y = 5$

b $x + 4y = 5$
$3x - y = -11$

**c** $-4x + 3y = 13$
$2x - y = 5$

**d** $-2x + 5y = 15$
$3x - 2y = 16$

**15** **a** The line $\frac{x}{a} + \frac{y}{b} = 1$ passes through points $(12, 6)$ and $(8, 3)$.

**i** Write down two simultaneous equations that can be used to solve for $a$ and $b$.
**ii** Using inverse matrices, find the values of $a$ and $b$.

**b** The line $\frac{x}{a} + \frac{y}{b} = 1$ passes through points $(4, 5)$ and $(-4, -15)$.

**i** Write down two simultaneous equations that can be used to solve for $a$ and $b$.
**ii** Using inverse matrices, find the values of $a$ and $b$.

**16** Find the values of $k$ for which the following simultaneous linear equations have:

**i** no solution
**ii** an infinite number of solutions.

**a** $x - 3y = k$
$-2x + 6y = 6$

**b** $2x - 5y = 4$
$-4x + 10y = k$

**c** $3x - 5y = k$
$-6x + 10y = 10$

**d** $4x - 6y = 8$
$-2x + 3y = k$

**17** Show that each of the following does not have a unique solution. Describe the solution set and solve if possible.

**a** $x - 2y = 3$
$-2x + 4y = -6$

**b** $2x - 3y = 5$
$-4x + 6y = -11$

**c** $2x - y = 4$
$-4x + 2y = -7$

**d** $3x - 4y = 5$
$-6x + 8y = -10$

**18** Find the values of $k$ for which the system of equations has:

**i** a unique solution
**ii** no solution
**iii** an infinite number of solutions.

(You are not required to find the solution set.)

**a** $(k - 2)x - 2y = k - 1$
$-4x + ky = -6$

**b** $(k + 1)x + 5y = 4$
$6x + 5ky = k + 6$

**c** $(k - 1)x - 3y = k + 2$
$-4x + 2ky = -10$

**d** $2x - (k - 2)y = 6$
$(k - 5)x - 2y = k - 3$

**19** Find the values of $p$ and $q$ for which the system of equations has:

**i** a unique solution
**ii** no solution
**iii** an infinite number of solutions.

(You are not required to find the solution set.)

**a** $-2x + 3y = p$
$qx - 6y = 7$

**b** $4x - 2y = q$
$3x + py = 10$

**c** $3x - py = 6$
$7x - 2y = q$

**d** $px - y = 3$
$-3x + 2y = q$

20 Consider the matrices $A = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -7 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -5 \\ -19 \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 14 \end{bmatrix}$.

Find the matrix $X$ in each of the following cases.

a $AX = C$ b $XA = B$ c $AX = B$ d $XA = D$

21 Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -2 \end{bmatrix}$.

Find the matrix $X$ in each of the following cases.

a $AX = C$ b $XA = B$ c $AX = B$ d $XA = D$

22 Consider matrices $A = \begin{bmatrix} -2 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 19 \\ 12 & -7 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 3 \end{bmatrix}$. Find the matrix $X$ in each of the following cases.

a $AX = C$ b $XA = B$ c $AX = B$ d $XA = D$

MASTER

23 $a$, $b$, $c$ and $d$ are all non-zero real numbers.

a If $P = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ find $P^{-1}$ and verify that $PP^{-1} = P^{-1}P = I$.

b If $Q = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ find $Q^{-1}$ and verify that $QQ^{-1} = Q^{-1}Q = I$.

24 $a$, $b$, $c$ and $d$ are all non-zero real numbers.

a If $R = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ find $R^{-1}$ and verify that $RR^{-1} = R^{-1}R = I$.

b If $S = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ find $S^{-1}$ and verify that $SS^{-1} = S^{-1}S = I$.

c If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ write down $A^{-1}$ and verify that $AA^{-1} = A^{-1}A = I$.

# 7.6 Translations

## Introduction to transformations

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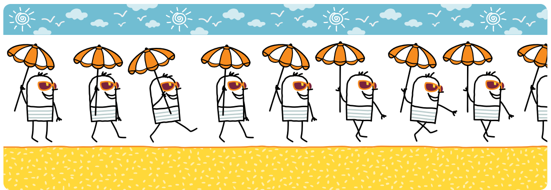
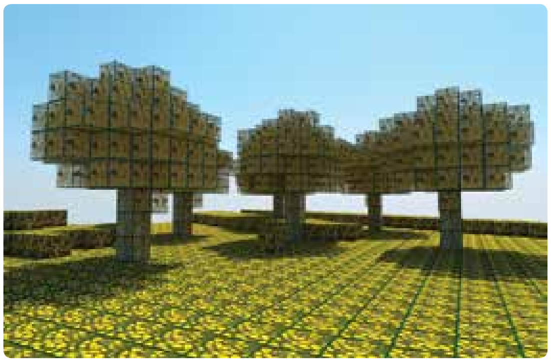
Computer engineers perform matrix transformations in the computer animation used in movies and video games. The animation models use matrices to describe the locations of specific points in images. Transformations are added to images to make them look more realistic and interesting.





## Matrix transformations

A **transformation** is a function which maps the points of a set $X$, called the **pre-image**, onto a set of points $Y$, called the **image**, or onto itself.

A transformation is a change of position of points, lines, curves or shapes in a plane, or a change in shape due to an enlargement or reduction by a scale factor.

Each point of the plane is transformed or mapped onto another point.

The transformation, $T$, is written as:

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix}$$

which means $T$ maps the points of the original or the pre-image point $(x, y)$ onto a new position point known as the image point $(x', y')$.

Any transformation that can be represented by a $2 \times 2$ matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is called a **linear transformation**. The origin never moves under a linear transformation.

An **invariant point** or **fixed point** is a point of the domain of the function which is mapped onto itself after a transformation. The pre-image point is the same as the image point.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

An invariant point of a transformation is a point which is unchanged by the transformation.

For example, a reflection in the line $y = x$ leaves every point on the line $y = x$ unchanged.

The transformations which will be studied in this chapter are translations, reflections, rotations and dilations.

When two or more transformations are applied to a function, the general order of transformations is dilations, reflections and translations (DRT). However, with the use of matrices, as long as the matrix calculations are completed in the order of the transformations, the correct equation or final result will be obtained.

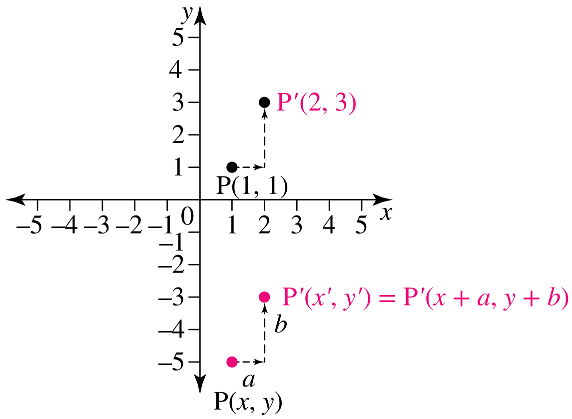
## Translations

A **translation** is a transformation of a figure where each point in the plane is moved a given distance in a horizontal or vertical direction. It is when a figure is moved from one location to another without changing size, shape or orientation.

Consider a marching band marching in perfect formation. As the leader of the marching band moves from a position $P(x, y)$ $a$ steps across and $b$ steps up to a new position $P'(x', y')$, all members of the band will also move to a new position $P'(x + a, y + b)$. Their new position could be defined as $P'(x', y') = P'(x + a, y + b)$ where $a$ represents the horizontal translation and $b$ represents the vertical translation.

If the leader of the marching band moves from position P(1, 1) to a new position P′(2, 3), which is across 1 and up 2 steps, all members of the marching band will also move the same distance and in the same direction. Their new position could be defined as:

$$P(x, y) \rightarrow P'(x + 1, y + 2)$$




The matrix transformation for a translation can be given by

$$\begin{matrix} P' & & P & & T \end{matrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

where $a$ represents the horizontal translation and $b$ is the vertical translation.

The matrix $\begin{bmatrix} a \\ b \end{bmatrix}$ is called the translation matrix and is denoted by $T$. The translation matrix maps the point $P(x, y)$ onto the point $P'(x', y')$, giving the image point $(x', y') = (x + a, y + b)$.

**WORKED EXAMPLE 17**

**A cyclist in a bicycle race needs to move from the front position at (0, 0) across 2 positions to the left so that the other cyclists can pass. Write the translation matrix and find the cyclist's new position.**

| THINK | WRITE |
|---|---|
| 1 Write down the translation matrix, $T$, using the information given. | The cyclist moves across to the left by 2 units. Translating 2 units to the left means each $x$-coordinate decreases by 2. $T = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ |
| 2 Apply the matrix transformation for a translation equation. | $\begin{matrix} P' & P & T \end{matrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ |
| 3 Substitute the pre-image point into the matrix equation. | The pre-image point is (0,0). $\begin{matrix} P' & P & T \end{matrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ |

4 State the cyclist's new position by calculating the coordinates of the image point from the matrix equation.

$$\overset{P'}{\begin{bmatrix} x' \\ y' \end{bmatrix}} = \overset{P}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \overset{T}{\begin{bmatrix} -2 \\ 0 \end{bmatrix}} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

The new position is (–2, 0).

## Translations of a shape

Matrix addition can be used to find the coordinates of a translated shape when a shape is moved or translated from one location to another on the coordinate plane without changing its size or orientation.

Consider the triangle ABC with coordinates A(–1, 3), B(0, 2) and C(–2, 1). It is to be moved 3 units to the right and 1 unit down. To find the coordinates of the vertices of the translated $\Delta A'B'C'$, we can use matrix addition.

First, the coordinates of the triangle $\Delta ABC$ can be written as a **coordinate matrix**. The coordinates of the vertices of a figure are arranged as columns in the matrix.

$$\Delta ABC = \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

Secondly, translating the triangle 3 units to the right means each $x$-coordinate increases by 3.

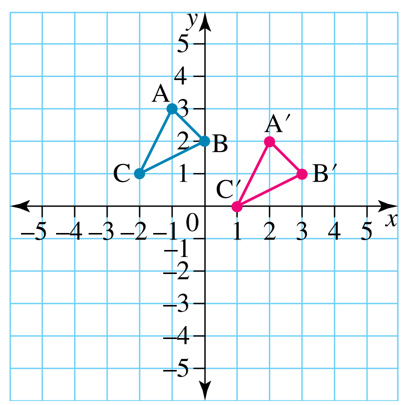
Translating the triangle 1 unit down means that each $y$-coordinate decreases by 1.

The translation matrix that will do this is $\begin{bmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix}$.

Finally, to find the coordinates of the vertices of the translated triangle $\Delta A'B'C'$ add the translation matrix to the coordinate matrix.

$$\overset{A \quad B \quad C}{\begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}} + \begin{bmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix} = \overset{A' \quad B' \quad C'}{\begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}}$$

The coordinates of the vertices of the translated triangle $\Delta A'B'C' = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ are $A'(2, 2)$, $B'(3, 1)$ and $C'(1, 0)$.




**WORKED EXAMPLE 18**

**Find the translation matrix if $\Delta ABC$ with coordinates A(–1, 3), B(0, 2) and C(–2, 1) is translated to $\Delta A'B'C'$ with coordinates $A'(2, 4)$, $B'(3, 3)$ and $C'(1, 2)$.**

| THINK | WRITE |
|---|---|
| 1 Write the coordinates of $\Delta ABC$ as a coordinate matrix. | The coordinates of the vertices of a figure are arranged as columns in the matrix. $\Delta ABC = \overset{A \quad B \quad C}{\begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}}$ |

2 Write the coordinates of the vertices of the translated triangle $\Delta A'B'C'$ as a coordinate matrix.

$$\Delta A'B'C' = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix} \quad (\text{columns } A', B', C')$$

3 Calculate the translation matrix by using the matrix equation:

$P' = P + T$

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix} + T$$

$$T = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

4 Translating the triangle 3 units to the right means that each $x$-coordinate increases by 3.

Translating the triangle 1 unit up means that each $y$-coordinate increases by 1.

The translation matrix is:

$$T = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

## Translations of a curve

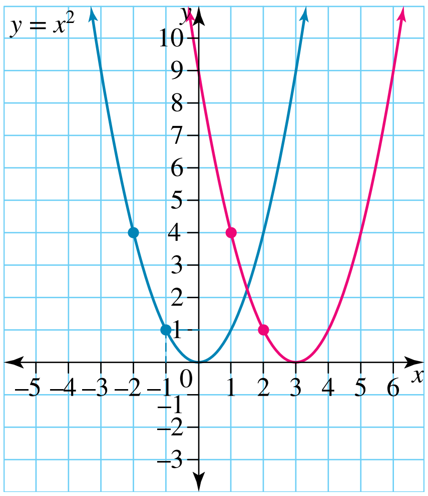
A translation of a curve maps every original point $(x, y)$ of the curve onto a new unique and distinct image point $(x', y')$.

Consider the parabola with the equation $y = x^2$.

If the parabola is translated 3 units in the positive direction of the $x$-axis (right), what is the image equation and what happens to the coordinates?

As seen from the table of values below, each coordinate $(x, y)$ has a new coordinate pair or image point $(x + 3, y)$.

| $x$ | $y$ | $(x, y)$ | $x' = x + 3$ | $y' = y$ | $(x', y')$ |
|---|---|---|---|---|---|
| –3 | 9 | (–3, 9) | –3 + 3 | 9 | (0, 9) |
| –2 | 4 | (–2, 4) | –2 + 3 | 4 | (1, 4) |
| –1 | 1 | (–1, 1) | –1 + 3 | 1 | (2, 1) |
| 0 | 0 | (0, 0) | 0 + 3 | 0 | (3, 0) |
| 1 | 1 | (1, 1) | 1 + 3 | 1 | (4, 1) |
| 2 | 4 | (2, 4) | 2 + 3 | 4 | (5, 4) |
| 3 | 9 | (3, 9) | 3 + 3 | 9 | (6, 9) |




The matrix equation for the transformation for any point on the curve $y = x^2$ can be written as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

The image equations for the two coordinates are $x' = x + 3$ and $y' = y$.

Rearranging the image equations to make the pre-image coordinates the subject, we get $x' = x + 3 \leftrightarrow x = x' - 3$ and $y = y'$ (no change).

To find the image equation, substitute the image expressions into the pre-image equation.

$$y = x^2$$

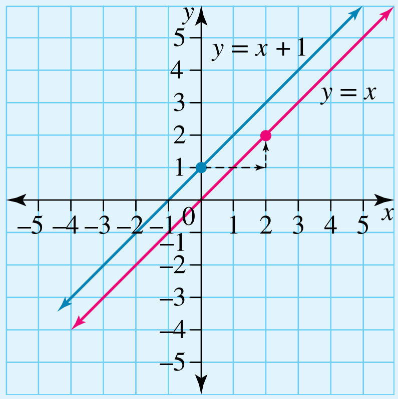
$$y = y' \qquad x = x' - 3$$

$$\therefore y' = (x' - 3)^2$$

The image equation is $y = (x - 3)^2$.

WORKED EXAMPLE 19

**Determine the image equation when the line with equation $y = x + 1$ is transformed by the translation matrix $T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.**

| THINK | WRITE |
|---|---|
| **1** State the matrix equation for the transformation given. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ |
| **2** State the image equations for the two coordinates. | $x' = x + 2$ and $y' = y + 1$ |
| **3** Rearrange the equations to make the pre-image coordinates $x$ and $y$ the subjects. | $x = x' - 2$ and $y = y' - 1$ |
| **4** Substitute the image equations into the pre-image equation to find the image equation. | $y = x + 1$<br>$y' - 1 = (x' - 2) + 1$<br>$y' = x'$<br>The image equation is $y = x$. |
| **5** Graph the image and pre-image equation to verify the translation. |  |

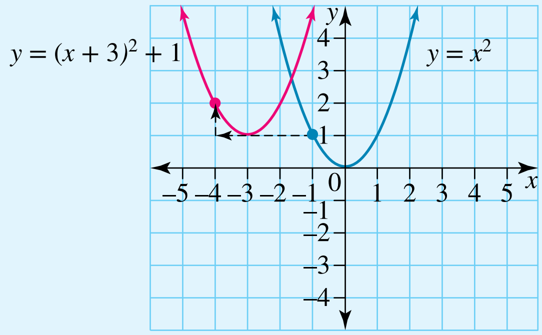


WORKED EXAMPLE 20

**Determine the image equation when the parabola with equation $y = x^2$ is transformed by the translation matrix $T = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.**

| THINK | WRITE |
|---|---|
| **1** State the matrix equation for the transformation given. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ |
| **2** State the image equations for the two coordinates. | $x' = x - 3$ and $y' = y + 1$ |
| **3** Rearrange the equations to make the pre-image coordinates $x$ and $y$ the subjects. | $x = x' + 3$ and $y = y' - 1$ |
| **4** Substitute the image expressions into the pre-image equation to find the image equation. | $y = x^2$<br>$y' - 1 = (x' + 3)^2$<br>$y' = (x' + 3)^2 + 1$<br>The image equation is $y = (x + 3)^2 + 1$. |

5 Graph the image and pre-image equation to verify the translation.




## EXERCISE 7.6 Translations

**PRACTISE**

Work without CAS

1 WE17 Find the image point for the pre-image point (1, 2) using the matrix equation for translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

2 Find the image point for the pre-image point (3, –4) using the matrix equation for translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

3 WE18 Find the translation matrix if ΔABC with coordinates A(0, 0), B(2, 3) and C(–3, 4) is translated to ΔA′B′C′ with coordinates A′(1, –2), B′(3, 1) and C′(–2, 2).

4 Find the translation matrix if ΔABC with coordinates A(3, 0), B(2, 4) and C(–2, –5) is translated to ΔA′B′C′ with coordinates A′(4, 2), B′(3, 6) and C′(–1, –3).

5 WE19 Determine the image equation when the line with equation $y = x - 1$ is transformed by the translation matrix $T = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

6 Determine the image equation when the line with equation $y = x + 3$ is transformed by the translation matrix $T = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

7 WE20 Determine the image equation when the parabola with equation $y = x^2$ is transformed by the translation matrix $T = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

8 Determine the image equation when the parabola with equation $y = x^2 + 1$ is transformed by the translation matrix $T = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 A chess player moves his knight 1 square to the right and 2 squares up from position (2, 5). Find the new position of the knight.

10 Find the image point for the pre-image point (–1, 0) using the matrix equation for the translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

11 The image points are given by $x' = x + 2$ and $y' = y + 1$. Express the transformation in matrix equation form.

12 a On a Cartesian plane, draw $\Delta ABC = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix}$ and $\Delta A'B'C' = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix}$.

b Find the translation matrix if $\Delta ABC = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix}$ is translated to $\Delta A'B'C' = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix}$.

13 Find the image equation when the line with equation $y = x - 3$ is transformed by the translation matrix $T = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

14 Find the image equation when the parabola with equation $y = x^2 - 2$ is transformed by the translation matrix $T = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$.

15 Find the translation matrix that maps the line with equation $y = x$ onto the line with equation $y = x + 2$.

16 Find the translation matrix that maps the parabola with equation $y = x^2$ onto the parabola with equation $y = (x - 7)^2 + 3$.

17 Write the translation equation that maps the parabola with equation $y = x^2$ onto the parabola with equation $y = x^2 - 4x + 10$.

18 Write the translation equation that maps the circle with equation $x^2 + y^2 = 9$ onto the circle with equation $(x - 1)^2 + y^2 = 9$.

MASTER

19 Write the translation equation that maps the parabola with equation $y = x^2$ onto the parabola with equation $y = (x - a)^2 + b$.

20 Write the translation equation that maps the circle with equation $x^2 + y^2 = r^2$ onto the circle with equation $(x - a)^2 + y^2 = r^2$.

# 7.7 Reflections

studyon

Units 1 & 2
AOS 2
Topic 2
Concept 6

**Reflections**
Concept summary
Practice questions

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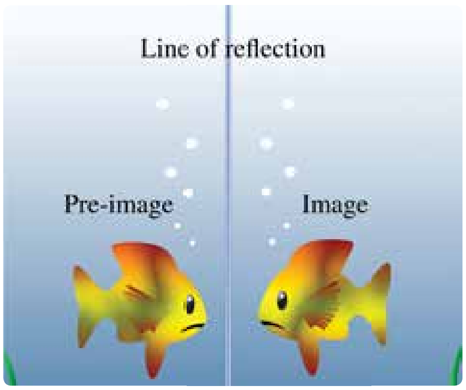
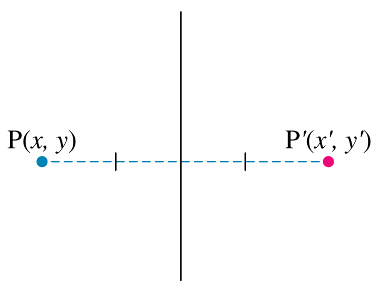
**Interactivity**
Reflections
**int-6295**

A **reflection** is a transformation defined by the line of reflection where the image point is a mirror image of the pre-image point.




Under a reflection, the image point P′ is a mirror image of the pre-image point P. The distances from the pre-image and image point to the reflection line are equal, with P and P′ on opposite sides of the reflection line. The line segment $PP'$ joining a point and its image is bisected perpendicularly to the reflection line. The reflection line or reflecting surface is called the mediator.

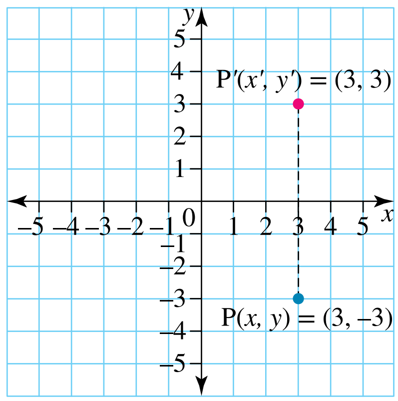


## Reflection in the *x*-axis

The reflection in the $x$-axis maps the point $P(x, y)$ onto the point $P'(x', y')$, giving the image point $(x', y') = (x, -y)$.

The matrix for reflection mapping in the $x$-axis is: $M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

In matrix form, the reflection for any point in the $x$-axis is: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

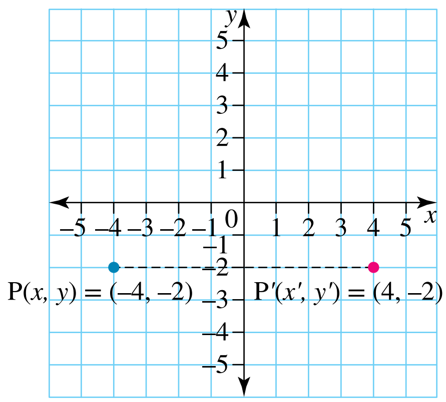



## Reflection in the *y*-axis

The reflection in the $y$-axis maps the point $P(x, y)$ onto the point $P'(x', y')$, giving the image point $(x', y') = (-x, y)$.

The matrix for reflection mapping in the $y$-axis is: $M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

In matrix form, the reflection for any point in the $y$-axis is: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$




**WORKED EXAMPLE 21**

**Find the image of the point (–2, 3) after a reflection in:**

**a** the $x$-axis **b** the $y$-axis.

| THINK | WRITE |
|---|---|
| **a 1** State the reflection matrix to be used. | **a** $M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| **2** Use the matrix equation for reflection in the $x$-axis. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ |
| **3** Substitute the pre-image point into the matrix equation. | The pre-image point is (–2, 3). $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ |
| **4** Calculate the coordinates of the image point. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ The image point is (–2, –3). |
| **b 1** State the reflection matrix to be used. | **b** $M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| **2** Use the matrix equation for reflection in the $y$-axis. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ |

**3** Substitute the pre-image point into the matrix equation.

The pre-image point is (–2, 3).

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

**4** Calculate the coordinates of the image point.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The image point is (2, 3).

**WORKED EXAMPLE 22** **Find the image equation after $y = (x + 1)^2$ is reflected in the $y$-axis.**

**THINK** / **WRITE**

**1** State the matrix equation for reflection in the $y$-axis.

$$M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

**2** Find the image coordinates.

$x' = -x$

$y' = y$

**3** Rearrange the equations to make the pre-image coordinates $x$ and $y$ the subjects.

$x = -x'$

$y = y'$

**4** Substitute the image expressions into the pre-image equation $y = (x + 1)^2$ to find the image equation.

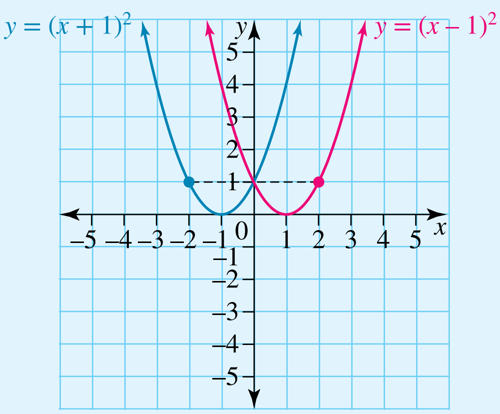
$y = (x + 1)^2$

$y' = (-x' + 1)^2$

$y' = (x' - 1)^2$

The image equation is $y = (x - 1)^2$.

**5** Graph the image and the pre-image equation to verify the reflection.




**6** Alternatively:

State the matrix equation for reflection in the $y$-axis.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

**7** Find the pre-image coordinates by using the inverse of the transformational matrix.

$X = T^{-1}x'$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

8 Multiply and simplify the matrix equation.

$x = -x'$
$y = y'$

9 Substitute the image expressions into the pre-image equation $y = (x+1)^2$ to find the image equation.

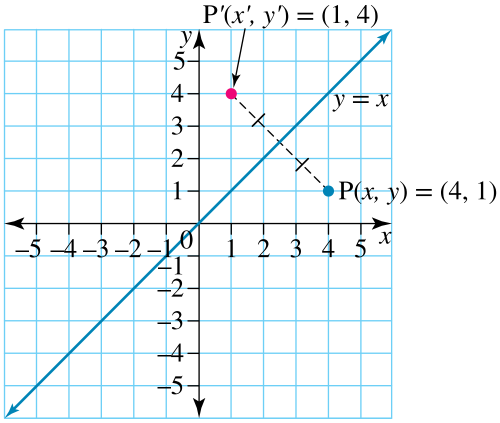
$y = (x+1)^2$
$y' = (-x'+1)^2$
$y' = (x'-1)^2$
The image equation is $y = (x-1)^2$.

## Reflection in the line with equation $y = x$

The reflection in the line $y = x$ maps the point $P(x, y)$ onto the point $P'(x', y')$, giving the image point $(x', y') = (y, x)$.

The matrix for reflection mapping in the line $y = x$ is $M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

In matrix form, the reflection for any point in the line $y = x$ is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.




WORKED EXAMPLE 23

**Find the image of the point (3, 1) after a reflection in the line with equation $y = x$.**

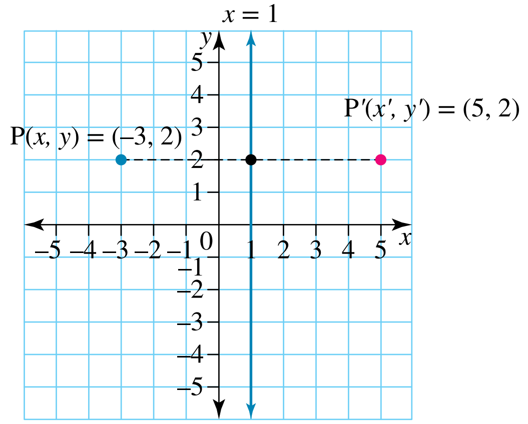
| THINK | WRITE |
|---|---|
| 1 State the reflection matrix to be used. | $M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| 2 Use the matrix equation for a reflection about the line with equation $y = x$. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ |
| 3 Substitute the pre-image point into the matrix equation. | The pre-image point is (3, 1). $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ |
| 4 Calculate the coordinates of the image point. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ The image point is (1, 3). |

## Reflection in a line parallel to either axis

To determine the image point $P'(x', y')$ from a reflection in a line parallel to either the $x$-axis or the $y$-axis, we need to consider the distance between the point $P(x, y)$ and the parallel line.

If we consider the distance from the $x$-coordinate of P to the vertical reflection line as $\overline{PD} = a - x$, to obtain the image $x$-coordinate we need to add the distance to the value of the mediator line, giving $x' = a + a - x = 2a - x$.

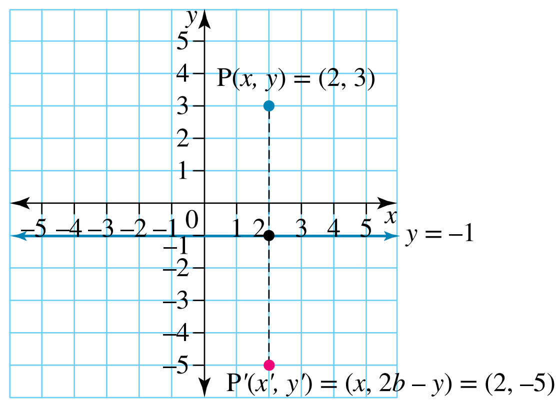
The reflection in the line $x = a$ maps the point $P(x, y)$ onto the point $P'(x', y')$, giving the image point $(x', y') = (2a - x, y)$.




In matrix form, the reflection for any point in the line $x = a$ is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2a \\ 0 \end{bmatrix}$$

The reflection in the line $y = b$ maps the point $P(x, y)$ onto the point $P'(x', y')$ giving the image point $(x', y') = (x, 2b - y)$.




In matrix form, the reflection for any point in the line $y = b$ is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2b \end{bmatrix}$$

A summary of the matrices for reflections are shown in the following table.

| Reflection in | Matrix | Matrix equation |
|---|---|---|
| $x$-axis | $M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ |
| $y$-axis | $M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ |
| line $y = x$ | $M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ |
| line $x = a$ | | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2a \\ 0 \end{bmatrix}$ |
| line $y = b$ | | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2b \end{bmatrix}$ |

**WORKED EXAMPLE 24**

**Find the image of the point (3, 1) after a reflection in the line with equation:**

**a** $x = 1$

**b** $y = -1$.

**THINK**

**a 1** State the matrix equation to be used.

**WRITE**

**a** The point is reflected in the line $x = 1$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2a \\ 0 \end{bmatrix}$$

| | |
|---|---|
| **2** Substitute the pre-image point and the value of $a$ into the matrix equation. | The pre-image point is (3, 1) and the value of $a = 1$ from the line $x = 1$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ |
| **3** Calculate the coordinates of the image point. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ The image point is (–1, 1). |
| **b 1** State the matrix equation to be used. | **b** The point is reflected in the line $y = -1$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2b \end{bmatrix}$ |
| **2** Substitute the pre-image point and the value of $b$ into the matrix equation. | The pre-image point is (3, 1) and the value of $b = 1$ from the line $y = -1$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ |
| **3** Calculate the coordinates of the image point. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ The image point is (3, –3). |

## EXERCISE 7.7 Reflections

**PRACTISE**

Work without CAS

1. WE21 Find the image of the point (–3, –1) after a reflection in the $x$-axis.
2. Find the image of the point (5, –2) after a reflection in the $y$-axis.
3. WE22 Find the image equation after $y = (x - 2)^2$ is reflected in the $y$-axis.
4. Find the image equation after $y = x^2 + 1$ is reflected in the $x$-axis.
5. WE23 Find the image of the point (–2, 5) after a reflection in the line with equation $y = x$.
6. Find the image of the point (8, –3) after a reflection in the line with equation $y = x$.
7. WE24 Find the image of the point (2, –1) after a reflection in the line with equation $x = -3$.

8. Find the image of the point (–4, 3) after a reflection in the line with equation $y = 2$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9. Find the image of the point (–1, 5) after a reflection in:
   a the $x$-axis
   b the $y$-axis.
10. Find the image of the point (8, –4) after a reflection in:
   a $M_x$
   b $M_y$.
11. Find the image of the point (9, –6) after a reflection in the line with equation $y = x$.
12. Find the image of the point (0, –1) after a reflection in the line with equation $y = x$.

13 Find the image of the point (–2, 3) after a reflection in the line with equation:

a $x = 1$ b $y = -2$.

14 Find the image of the point (7, –1) after a reflection in the line with equation:

a $x = -4$ b $y = 3$.

15 A point P is reflected in the line $y = 2$ to give an image point P′(–3, –5). What are the coordinates of P?

16 A point P is reflected in the line $x = -1$ to give an image point P′(–2, 1). What are the coordinates of P?

17 The line with equation $y = -x + 3$ is transformed according to the matrix equations given. Find the equation of the image for each transformation.

a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

c The line with equation $x = 4$

18 The parabola with equation $y = x^2 + 2x + 1$ is transformed according to the matrix equations given. Find the equation of the image for each transformation.

a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

MASTER

19 Find the final image point when point P(2, –1) undergoes two reflections. It is firstly reflected in the $x$-axis and then reflected in the line $y = x$.

20 Find the coordinates of the vertices of the image of pentagon ABCDE with A(2, 5), B(4, 4), C(4, 1), D(2, 0) and E(0, 3) after a reflection in the $y$-axis.

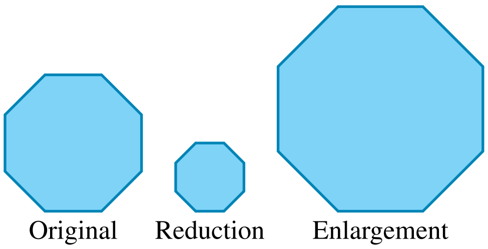
# 7.8 Dilations

studyon

Units 1 & 2
AOS 2
Topic 2
Concept 7

**Dilations**
Concept summary
Practice questions

A **dilation** is a linear transformation that changes the size of a figure. The figure is enlarged or reduced parallel to either axis or both. A dilation requires a centre point and a scale factor.




A dilation is defined by a scale factor denoted by $k$.

If $k > 1$, the figure is enlarged.

If $0 < k < 1$, the figure is reduced.

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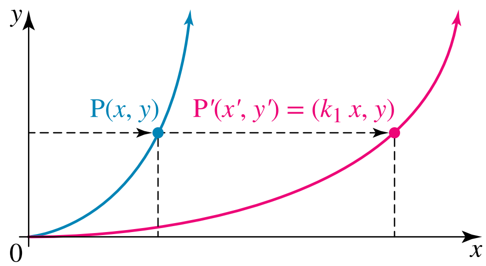
**Interactivity**
Dilations
**int-6297**

## One-way dilation

A one-way dilation is a dilation from or parallel to one of the axes.

### Dilations from the $y$-axis or parallel to the $x$-axis

A dilation in one direction from the $y$-axis or parallel to the $x$-axis is represented by the matrix equation:




$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 x \\ y \end{bmatrix}$$

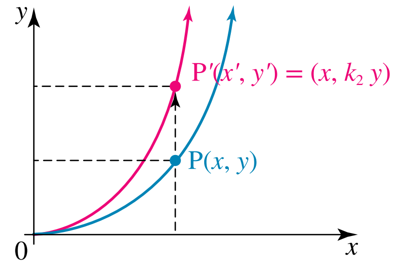
where $k_1$ is the dilation factor.

The points $(x, y)$ are transformed onto points with the same $y$-coordinate but with the $x$-coordinate $k_1$ times the distance from the $y$-axis that it was originally.

The point moves away from the $y$-axis in the direction of the $x$-axis by a factor of $k_1$. This determines the horizontal enlargement of the figure if $k_1 > 1$ or the horizontal compression if $0 < k_1 < 1$.

### Dilations from the $x$-axis or parallel to the $y$-axis

A dilation in one direction from the $x$-axis or parallel to the $y$-axis is represented by the matrix equation:




$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ k_2 y \end{bmatrix}$$

where $k_2$ is the dilation factor.

The point moves away from the $x$-axis in the direction of the $y$-axis by a factor of $k_2$. This determines the vertical enlargement of the figure if $k_2 > 1$ or if $0 < k_2 < 1$, the vertical compression.

**WORKED EXAMPLE 25** **Find the coordinates of the image of the point (3, –1) under a dilation of factor 2 from the $x$-axis.**

| THINK | WRITE |
|---|---|
| 1 State the dilation matrix to be used. | $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ |
| 2 Use the matrix equation for dilation by substituting the value of $k$. | The dilation factor is $k = 2$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ |
| 3 Substitute the pre-image point into the matrix equation. | The pre-image point is (3, –1). $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ |
| 4 Calculate the coordinates of the image point. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ The image point is (3, –2). |

**WORKED EXAMPLE 26** **Find the image equation when the parabola with equation $y = x^2$ is dilated by a factor of 2 from the $y$-axis.**

| THINK | WRITE |
|---|---|
| 1 State the matrix equation for dilation. | $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$ |
| 2 Find the equation of the image coordinates in terms of the pre-image coordinates. | $x' = 2x$ and $y' = y$ |

3 Rearrange the equations to make the pre-image coordinates $x$ and $y$ the subjects.

$x = \frac{x'}{2}$ and $y = y'$

4 Substitute the image values into the pre-image equation to find the image equation.

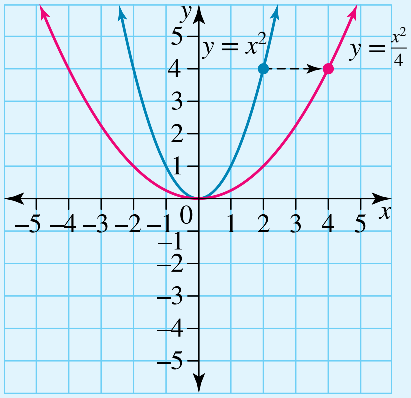
$$y = x^2$$

$$y' = \left(\frac{x'}{2}\right)^2$$

$$= \frac{(x\ )^2}{4}$$

The image equation is $y = \frac{x^2}{4}$.

5 Graph the image and the pre-image equation to verify the translation.



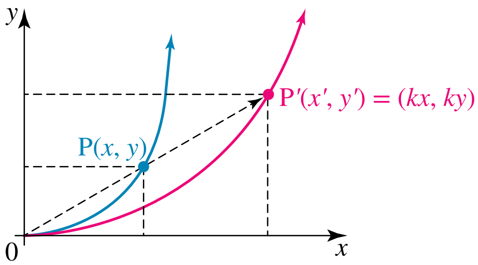

## Two-way dilations

A dilation parallel to both the $x$-axis and $y$-axis can be represented by the matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 x \\ k_2 y \end{bmatrix}$$

where $k_1$ and $k_2$ are the dilation factors in the $x$-axis and $y$-axis directions respectively.

- When $k_1 \neq k_2$ the object is skewed.
- When $k_1 = k_2 = k$ the size of the object is enlarged or reduced by the same factor, and the matrix equation is:




$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= kI\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= k\begin{bmatrix} x \\ y \end{bmatrix}$$

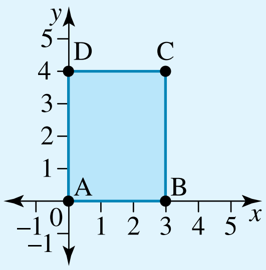
$$= \begin{bmatrix} kx \\ ky \end{bmatrix}$$ where $k$ is the dilation factor.

WORKED EXAMPLE 27

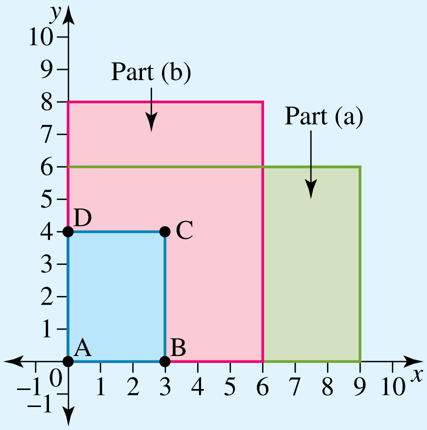
Jo has fenced a rectangular vegetable patch with fence posts at A(0, 0), B(3, 0), C(3, 4) and D(0, 4).

**a** She wants to increase the size of the vegetable patch by a dilation factor of 3 in the $x$-direction and a dilation factor of 1.5 in the $y$-direction. Where should Jo relocate the fence posts?

**b** Jo has noticed that the vegetable patch in part (a) is too long and can only increase the vegetable patch size by a dilation factor of 2 in both the $x$-direction and the $y$-direction. Where should she relocate the fence posts? Will this give her more area to plant vegetables? Explain.

| THINK | WRITE |
|---|---|
| **a 1** Draw a diagram to represent this situation. | **a**  |
| **2** State the coordinates of the vegetable patch as a coordinate matrix. | The coordinates of the vegetable patch ABCD can be written as a coordinate matrix. $\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$ |
| **3** State the dilation matrix. | $\begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix}$ |
| **4** Multiply the dilation matrix by the coordinate matrix to calculate the new fence post coordinates. | $\begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 9 & 9 & 0 \\ 0 & 0 & 6 & 6 \end{bmatrix}$ The new fence posts are located at A′(0, 0), B′(9, 0), C′(9, 6) and D′(0, 6). |
| **b 1** State the coordinates of the vegetable patch as a coordinate matrix. | **b** The coordinates of the vegetable patch ABCD can be written as a coordinate matrix: $\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$ |
| **2** State the dilation matrix. | The dilation matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. |
| **3** Calculate the new fence post coordinates A′B′C′D′ by multiplying the dilation matrix by the coordinate matrix. | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 & 0 \\ 0 & 0 & 8 & 8 \end{bmatrix}$ The new fence posts are located at A′(0, 0), B′(6, 0), C′(6, 8) and D′(0, 8). |

4 Draw a diagram of the original vegetable patch, and the two transformed vegetable patches on the same Cartesian plane.




5 Determine the area for each vegetable patch.

The vegetable patch size when dilated by a factor of 3 in the $x$-direction and a dilation factor of 1.5 in the $y$-direction gives an area of 54 units$^2$.
When dilated by a factor of 2 in both the $x$-direction and the $y$-direction, the vegetable patch has an area of 48 units$^2$.
The farmer will have less area to plant vegetables in the second option.

## EXERCISE 7.8 Dilations

**PRACTISE**

Work without CAS

1 WE25 Find the coordinates of the image of the point (2, –1) under a dilation of factor 3 from the $x$-axis.

2 Find the coordinates of the image of the point (–1, 4) under a dilation of factor 2 from the $y$-axis.

3 WE26 Find the image equation when the parabola with equation $y = x^2$ is dilated by a factor of 3 from the $y$-axis.

4 Find the image equation when the parabola with equation $y = x^2$ is dilated by a factor of $\frac{1}{2}$ from the $x$-axis.

5 WE27 A farmer has fenced a vegetable patch with fence posts at A(0, 0), B(3, 0), C(3, 4) and D(0, 4). She wants to increase the vegetable patch size by a dilation factor of 1.5 in the $x$-direction and a dilation factor of 3 in the $y$-direction. Where should she relocate the fence posts?

6 Jack wants to plant flowers on a flower patch with corners at A(2, 1), B(4, 1), C(3, 2) and D(1, 2). He wants to increase the flower patch size by a dilation factor of 2 in both the $x$-direction and the $y$-direction. Where should he relocate the new corners of the flower patch?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 Find the image of (2, –5) after a dilation of 3 parallel to the $x$-axis.

8 Find the image of (–1, 4) after a dilation of 2 parallel to the $y$-axis.

9 A man standing in front of a carnival mirror looks like he has been dilated 3 times wider. Write a matrix equation for this situation.

10 A transformation $T$ is given by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

a Find the image of the point A(–1, 3).

b Describe the transformation represented by $T$.

11 Find the image equation when the line with equation $2y + x = 3$ is dilated by $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

12 Find the image equation when the parabola with equation $y = x^2 - 1$ is dilated by $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

13 Find the image equation when the hyperbola with equation $y = \dfrac{1}{x+1}$ is dilated by a factor of 2 from the $y$-axis.

14 The equation $y = 2\sqrt{x}$ is transformed according to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

a What is the mapping produced in the matrix transformation?

b What is the image equation?

15 Find the image equation when the circle with equation $x^2 + y^2 = 4$ is transformed according to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

16 The coordinates of ΔABC can be written as a coordinate matrix $\begin{bmatrix} -2 & -1 & -3 \\ 0 & 3 & 2 \end{bmatrix}$.

It has undergone a transformation $T$ given by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

a Find the dilation factor, $k$, if the image coordinate point A′ is (–3, 0).

b Calculate the coordinates of the vertices of ΔA′B′C′.

MASTER

17 Find the factor of dilation when the graph of $y = \dfrac{1}{3x^2}$ is obtained by dilating the graph of $y = \dfrac{1}{x^2}$ from the $y$-axis.

18 a Find the image equation of $x + 2y = 2$ under the transformation dilation by a factor of 3 parallel to the $x$-axis.

b Is there an invariant point?

# 7.9 Combinations of transformations

A **combined transformation** is made up of two or more transformations.

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Topic 2

Concept 8

**Combinations of transformations**

Concept summary

Practice questions

## Double transformation matrices

If a linear transformation $T_1$ of a plane is followed by a second linear transformation $T_2$, then the results may be represented by a single transformation matrix $T$.

When transformation $T_1$ is applied to the point $P(x, y)$ it results in $P'(x', y')$.

When transformation $T_2$ is then applied to $P'(x', y')$ it results in $P''(x'', y'')$.

Summarising in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T_1\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = T_2\begin{bmatrix} x' \\ y' \end{bmatrix}$$

Substituting $T_1\begin{bmatrix} x \\ y \end{bmatrix}$ for $\begin{bmatrix} x' \\ y' \end{bmatrix}$ into $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = T_2\begin{bmatrix} x' \\ y' \end{bmatrix}$ results in $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = T_2T_1\begin{bmatrix} x \\ y \end{bmatrix}$.

To form the single transformation matrix $T$, the first transformation matrix $T_1$ must be pre-multiplied by the second transformation matrix $T_2$.

This is written as:

$$\boldsymbol{T = T_2\,T_1}$$

| Common transformation matrices used for combinations of transformations | |
|---|---|
| $M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | Reflection in the $x$-axis |
| $M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ | Reflection in the $y$-axis |
| $M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | Reflection in the line $y = x$ |
| $D_{k_1,1} = \begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}$ | Dilation in one direction parallel to the $x$-axis or from the $y$-axis |
| $D_{1,k_2} = \begin{bmatrix} 1 & 0 \\ 0 & k_2 \end{bmatrix}$ | Dilation in one direction parallel to the $y$-axis or from the $x$-axis |
| $D_{k_1,k_2} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ | Dilation parallel to both the $x$- and $y$-axis ($k_1$ and $k_2$ are the dilation factors) |

**WORKED EXAMPLE 28** **Determine the single transformation matrix $T$ that describes a reflection in the $x$-axis followed by a dilation of factor 3 from the $y$-axis.**

| THINK | WRITE |
|---|---|
| **1** Determine the transformation matrices being used. | $T_1$ = reflection in the $x$-axis<br>$T_1$: $M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$<br>$T_2$ = dilation of factor 3 from the $y$-axis<br>$T_2$: $D_{3,1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ |
| **2** State the combination of transformations matrix and simplify. | $T = T_2T_1$<br>$T = D_{3,1}M_x$<br>$T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$<br>$= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ |
| **3** State the single transformation matrix. | The single transformation matrix is:<br>$T = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ |

## EXERCISE 7.9 Combinations of transformations

**PRACTISE**

Work without CAS

1 WE28 Determine the single transformation matrix $T$ that describes a reflection in the $y$-axis followed by a dilation factor of 3 from the $x$-axis.

2 Determine the single transformation matrix $T$ that describes a reflection in the line $y = x$ followed by a dilation of factor 2 from both the $x$ and $y$-axis.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

3 A rectangle ABCD is transformed under the transformation matrix $T = \begin{bmatrix} 3 & 2 \\ 5 & 8 \end{bmatrix}$, to give vertices at A′(0, 0), B′(3, 0), C′(3, 2) and D′(0, 2).

a Find the vertices of the square ABCD.

b Calculate the area of the transformed figure ABCD.

4 a Find the image point of point $P'(x', y')$ when the point $P(x, y)$ undergoes a double transformation: a reflection in the $y$-axis followed by a translation of 4 units in the positive direction of the $x$-axis.

b Reverse the order of the pair of transformations in part a. Is the image different?

5 State the image of $(x, y)$ for a translation of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ followed by a reflection in the $x$-axis.

6 Describe fully a sequence of two geometrical transformations represented by $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

7 The triangle ABC is mapped by the transformation represented by $T = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$ onto the triangle A′B′C′. Given that the area of ABC is 10 units$^2$, find the area of A′B′C′.

8 a State the transformations that have undergone $T\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

b Determine the image of the curve with equation $2x - 3y = 12$.

9 a State the transformations that have undergone $T\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

b Determine the image of the curve with equation $y = 2x^2 - 1$.

10 Find the image equation of $y = x^2$ under a double transformation: a reflection in the $x$-axis followed by a dilation factor of 2 parallel to both the $x$- and $y$-axis.

11 Find the image equation of $y = \sqrt{x}$ under a double transformation: a reflection in the $y$-axis followed by a dilation of 3 parallel to the $x$-axis.

12 A rectangle ABCD with vertices at A(0, 0), B(2, 0), C(2, 3) and D(0, 3) is transformed under the transformation matrix $T = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Find the new area of the transformed rectangle.

**MASTER**

13 If $D_k$ denotes a dilation factor of $k$ parallel to both axes, what single dilation would be equivalent to $D_k^2$?

14 Check whether the transformation ‘a reflection in the $y$-axis followed by a reflection in the line $y = x$’ is the same as ‘a reflection in the line $y = x$ followed by a reflection in the $y$-axis’.

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# 7.10 Review

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**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

## REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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# Activities

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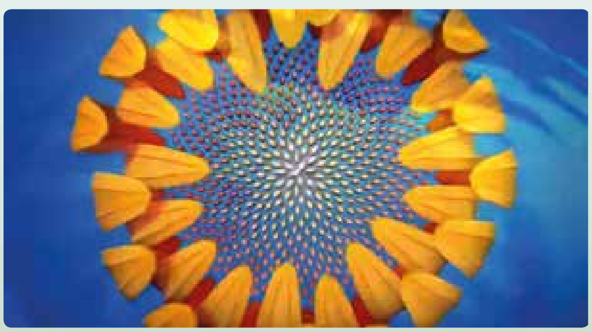
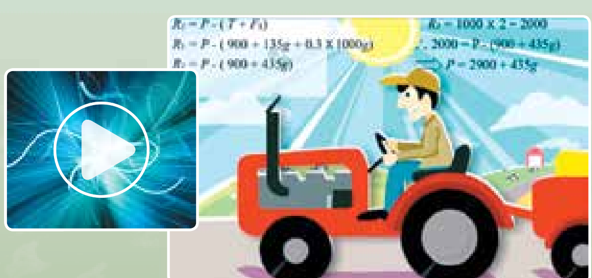
## Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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# 7 Answers

## EXERCISE 7.2

**1** $\begin{bmatrix} 63 & 19 & 25 \end{bmatrix}$

**2** $\begin{bmatrix} 6 & 8 \\ 53 & 17 \end{bmatrix}$

**3** $x = 2;\ y = 18$

**4** $x = 3;\ y = -16;\ z = 11$

**5 a** $\begin{bmatrix} -10 & 4 \\ 11 & 21 \end{bmatrix}$

**b** $\begin{bmatrix} 4 & -8 \\ -6 & -10 \end{bmatrix}$

**c** $\begin{bmatrix} 10 & -8 \\ -10 & -16 \end{bmatrix}$

**6 a** $a = 2;\ b = 8;\ c = -2;\ d = -8$

**b** $a = \frac{11}{2};\ b = 8;\ c = -2;\ d = -\frac{9}{2}$

**7 a** $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$

**b** $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

**c** $\frac{1}{2}\begin{bmatrix} 15 \\ 14 \end{bmatrix}$

**8 a** $\begin{bmatrix} 4 & -7 \end{bmatrix}$

**b** $\begin{bmatrix} 2 & -3 \end{bmatrix}$

**c** $\begin{bmatrix} \frac{9}{2} & -7 \end{bmatrix}$

**9 a i** $\begin{bmatrix} 5 & 3 \\ 7 & 1 \end{bmatrix}$

**ii** $\begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix}$

**b** $\begin{bmatrix} 7 & 6 \\ 6 & 5 \end{bmatrix}$

**10 a** $\begin{bmatrix} 11 & 8 \\ -3 & 16 \end{bmatrix}$

**b** $\begin{bmatrix} 5 & -16 \\ 15 & 4 \end{bmatrix}$

**c** $\frac{1}{2}\begin{bmatrix} 5 & -16 \\ 15 & 4 \end{bmatrix}$

**11 a** $\begin{bmatrix} 1 & -16 \\ 15 & 0 \end{bmatrix}$

**b** $\begin{bmatrix} 14 & 10 \\ -3 & 21 \end{bmatrix}$

**12 a** $x = 5;\ y = 7$

**b** $x = 4;\ y = 6$

**c** $x = 3;\ y = -2$

**13 a** $\begin{bmatrix} 3 & 2 \\ -4 & 7 \\ 5 & 2 \end{bmatrix}$

**b** $\begin{bmatrix} 1 & -6 \\ 2 & 3 \\ 1 & -8 \end{bmatrix}$

**c** $\frac{1}{2}\begin{bmatrix} 5 & -20 \\ 5 & 14 \\ 6 & -27 \end{bmatrix}$

**14 a** $\begin{bmatrix} 3 & 2 & 9 \\ -2 & 6 & -5 \end{bmatrix}$

**b** $\begin{bmatrix} 1 & -6 & -1 \\ 4 & 2 & -1 \end{bmatrix}$

**c** $\frac{1}{2}\begin{bmatrix} 5 & -20 & 1 \\ 13 & 10 & -6 \end{bmatrix}$

**15 a** $a_{11} = 2;\ a_{12} = 3;\ a_{21} = -1;\ a_{22} = 4$

**b** $\begin{bmatrix} 3 & -2 \\ -3 & 5 \end{bmatrix}$

**16 a** $\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$

**b** $\begin{bmatrix} 3 & 3 \\ 2 & 5 \end{bmatrix}$

**17 a i** 6

**ii** 9

**iii** 5

**b** Yes

**c** Yes

**18** A $3 \times 1$ matrix $\begin{bmatrix} 216 \\ 164 \\ 274 \end{bmatrix}$

## EXERCISE 7.3

**1 a** $\begin{bmatrix} 14 \\ 1 \end{bmatrix}$

**b** Does not exist

**2 a** $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$

**b** Does not exist

**3 a** $\begin{bmatrix} -8 & -20 \\ 1 & -3 \end{bmatrix}$

**b** $\begin{bmatrix} 8 & 28 \\ -7 & -19 \end{bmatrix}$

**c** $\begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix}$

**4 a** $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

**b** $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

**c** $\begin{bmatrix}-2 & 4\\3 & 5\end{bmatrix}$

**d** $\begin{bmatrix}-2 & 4\\3 & 5\end{bmatrix}$

**5 a** $\begin{bmatrix}0 & 8 & -11\\7 & -26 & 34\\-9 & 14 & -17\end{bmatrix}$

**b** $\begin{bmatrix}-1 & 21\\11 & -42\end{bmatrix}$

**6 a** $\begin{bmatrix}3 & -2\\-6 & 4\end{bmatrix}$

**b** $[7]$

**7 a i** $\begin{bmatrix}2 & 3\\-1 & 4\end{bmatrix}$

**ii** $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

**b i** $\begin{bmatrix}a & b\\c & d\end{bmatrix}$

**ii** $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

**8 a** $\begin{bmatrix}0 & -18\\6 & -12\end{bmatrix}$

**b** No; $\begin{bmatrix}0 & 0\\10 & 0\end{bmatrix}$

**c** No; $\begin{bmatrix}0 & 0\\ax+by & 0\end{bmatrix}$

**9 a** $\begin{bmatrix}31 & 9\\23 & 1\end{bmatrix}$

**b** $\begin{bmatrix}19 & -24\\-81 & -76\end{bmatrix}$

**c** LHS $= \begin{bmatrix}44 & 56\\7 & 9\end{bmatrix}$

RHS $= \begin{bmatrix}55 & 25\\4 & -2\end{bmatrix}$

No since $BA \neq AB$

**d** $\begin{bmatrix}44 & 56\\7 & 9\end{bmatrix}$

**10 a** $x = -3$

**b** $x = -4$

**c** $x = -2$

**d** $x = 2$

**11 a** Does not exist

**b** Does not exist

**c** Does not exist

**d** $\begin{bmatrix}-3 & 5\\6 & -10\end{bmatrix}$

**e** $[-13]$

**f** Does not exist

**g** $\begin{bmatrix}6\\13\end{bmatrix}$

**h** $[21 \quad -13]$

**i** Does not exist

**j** $\begin{bmatrix}-21 & 13\\42 & -26\end{bmatrix}$

**k** Does not exist

**l** $\begin{bmatrix}18 & -30\\39 & -65\end{bmatrix}$

**12 a** $\begin{bmatrix}6 & 14 & -8\\-4 & 14 & -18\\9 & 16 & -7\end{bmatrix}$

**b** $\begin{bmatrix}16 & 24\\-17 & -3\end{bmatrix}$

**c** Does not exist

**d** Does not exist

**13 a** $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

**b** $\begin{bmatrix}2 & -8 & 4\\5 & -20 & 10\\9 & -36 & 18\end{bmatrix}$

**c** Does not exist

**d** Does not exist

**14 a** $P^2 = \begin{bmatrix}1 & 0\\0 & 16\end{bmatrix}$; $P^3 = \begin{bmatrix}-1 & 0\\0 & 64\end{bmatrix}$; $P^4 = \begin{bmatrix}1 & 0\\0 & 256\end{bmatrix}$;

$P^n = \begin{bmatrix}(-1)^n & 0\\0 & 4^n\end{bmatrix}$

**b** $Q^2 = \begin{bmatrix}4 & 0\\0 & 9\end{bmatrix}$; $Q^3 = \begin{bmatrix}8 & 0\\0 & -27\end{bmatrix}$; $Q^4 = \begin{bmatrix}16 & 0\\0 & 81\end{bmatrix}$;

$Q^n = \begin{bmatrix}2^n & 0\\0 & (-3)^n\end{bmatrix}$

**c** $R^2 = \begin{bmatrix}1 & 0\\6 & 1\end{bmatrix}$; $R^3 = \begin{bmatrix}1 & 0\\9 & 1\end{bmatrix}$; $R^4 = \begin{bmatrix}1 & 0\\12 & 1\end{bmatrix}$; $R^n = \begin{bmatrix}1 & 0\\3n & 1\end{bmatrix}$

**d** $S^2 = \begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$; $S^3 = \begin{bmatrix}0 & 18\\12 & 0\end{bmatrix}$; $S^4 = \begin{bmatrix}36 & 0\\0 & 36\end{bmatrix}$;

$S^8 = \begin{bmatrix}1296 & 0\\0 & 1296\end{bmatrix}$; $S^9 = \begin{bmatrix}0 & 3888\\2592 & 0\end{bmatrix}$

**15** $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

**16 a** $\begin{bmatrix}-20 & 0\\0 & -20\end{bmatrix}$

**b** $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

**17** $\begin{bmatrix}d^2 - 9d + 8 & 4 - 4d\\2 - 2d & 0\end{bmatrix}$

**18 a i** 39

**ii** 39

**iii** 54

**b** No

## EXERCISE 7.4

1 $-22$

2 $x = 4, -6$

3 $\frac{1}{34}\begin{bmatrix} 6 & 2 \\ -5 & 4 \end{bmatrix}$

4 $p = -4;\ q = -2$

5 $\Delta = 0$

6 $x = -6, 2$

7 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8 $k = -2, 8$

9 a i 20

ii $\frac{1}{10}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

b Proof required — check with your teacher

c i $\frac{1}{20}$

ii 1

10 a $\frac{1}{4}\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$

b $\frac{1}{6}\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$

c $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

d $\frac{1}{6}\begin{bmatrix} -1 & 3 \\ -2 & 0 \end{bmatrix}$

11 a $-11;\ 2;\ 10$

b Yes

c Proof required — check with your teacher

12 a $A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & 3 \\ 1 & 2 \end{bmatrix} B^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$

$C^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$

b No

c Yes

d Yes

13 a $-3$

b $-2$

c $\pm 4$

d $\frac{1}{2}, 3$

14 a $-6$

b $\pm 2$

c $\pm 2\sqrt{3}$

d $-3, 2$

15 a Does not exist

b Does not exist

c Does not exist

d $\frac{1}{22}\begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix}$

e Does not exist

16 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

17 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

18 a $k = -1, 4$

b $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

19 a $k = -1, 2$

b $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

20 a i $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

ii $\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

iii $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

iv $\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$

v $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

b Proof required — check with your teacher

c Proof required — check with your teacher

## EXERCISE 7.5

1 $x = 5;\ y = -2$

2 $x = 4;\ y = -3$

3 No solution

4 $-\frac{5}{2}$

5 $\left(3 + \frac{3t}{4}, t\right) t \in R$

6 $-\frac{5}{2}$

7 a $k \in R\backslash\{-3,\ 2\}$

b $k = -3$

c $k = 2$

8 a $k \in R\backslash\{-3, 4\}$

b $k = -3$

c $k = 4$

9 a $\frac{1}{5}\begin{bmatrix} -13 & 1 \\ -7 & -6 \end{bmatrix}$

b $\frac{1}{5}\begin{bmatrix} 0 & -5 \\ 17 & -19 \end{bmatrix}$

10 a $\frac{1}{10}\begin{bmatrix} 11 & 3 \\ -30 & 0 \end{bmatrix}$

b $\frac{1}{10}\begin{bmatrix} 2 & 8 \\ -9 & 9 \end{bmatrix}$

**11 a** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

**b** $\frac{1}{2}\begin{bmatrix} -5 & -2 \end{bmatrix}$

**12 a** $\frac{1}{11}\begin{bmatrix} -2 \\ 7 \end{bmatrix}$

**b** $\frac{1}{11}\begin{bmatrix} -7 & 3 \end{bmatrix}$

**13 a** $x = -1, y = 2$

**b** $x = 1, y = -2$

**c** $x = 2, y = -3$

**d** $x = -2, y = -1$

**14 a** $x = -2, y = 3$

**b** $x = -3, y = 2$

**c** $x = 14, y = 23$

**d** $x = 10, y = 7$

**15 a** **i** $\frac{12}{a} + \frac{6}{b} = 1,\ \frac{8}{a} + \frac{3}{b} = 1$

**ii** $a = 4; b = -3$

**b** **i** $\frac{4}{a} + \frac{5}{b} = 1,\ -\frac{4}{a} - \frac{15}{b} = 1$

**ii** $a = 2; b = -5$

**16 a** **i** $k \neq -3$

**ii** $k = -3$

**b** **i** $k \neq -8$

**ii** $k = -8$

**c** **i** $k \neq -5$

**ii** $k = -5$

**d** **i** $k \neq -4$

**ii** $k = -4$

**17 a** $(2t + 3, t), t \in R$

**b** No solution

**c** No solution

**d** $\left(\frac{4t + 5}{3}, t\right)\ t \in R$

**18 a** **i** $k \in R\backslash\{-2, 4\}$

**ii** $k = -2$

**iii** $k = 4$

**b** **i** $k \in R\backslash\{-3, 2\}$

**ii** $k = -3$

**iii** $k = 2$

**c** **i** $k \in R\backslash\{-2, 3\}$

**ii** $k = -2$

**iii** $k = 3$

**d** **i** $k \in R\backslash\{1, 6\}$

**ii** $k = 1$

**iii** $k = 6$

**19 a** **i** $q \neq 4, p \in R$

**ii** $q = 4, p \neq -\frac{7}{2}$

**iii** $q = 4, p = -\frac{7}{2}$

**b** **i** $p \neq -\frac{3}{2}, q \in R$

**ii** $p = -\frac{3}{2}, q \neq \frac{40}{3}$

**iii** $p = -\frac{3}{2}, q = \frac{40}{3}$

**c** **i** $p \neq \frac{6}{7}, q \in R$

**ii** $p = \frac{6}{7}, q \neq 14$

**iii** $p = \frac{6}{7}, q = 14$

**d** **i** $p \neq \frac{3}{2}, q \in R$

**ii** $p = \frac{3}{2}, q \neq -6$

**iii** $p = \frac{3}{2}, q = -6$

**20 a** $\frac{1}{7}\begin{bmatrix} -29 \\ 3 \end{bmatrix}$

**b** $\frac{1}{14}\begin{bmatrix} 7 & 7 \\ -38 & -12 \end{bmatrix}$

**c** $\frac{1}{14}\begin{bmatrix} -2 & 8 \\ -22 & -3 \end{bmatrix}$

**d** $\begin{bmatrix} -3 & 2 \end{bmatrix}$

**21 a** $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

**b** $\frac{1}{11}\begin{bmatrix} 21 & -2 \\ 5 & -12 \end{bmatrix}$

**c** $\frac{1}{11}\begin{bmatrix} 10 & 29 \\ 8 & -1 \end{bmatrix}$

**d** $\frac{1}{11}\begin{bmatrix} 2 & -7 \end{bmatrix}$

**22 a** $\frac{1}{11}\begin{bmatrix} -6 \\ 7 \end{bmatrix}$

**b** $\begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix}$

**c** $\frac{1}{11}\begin{bmatrix} 13 & -58 \\ 16 & 31 \end{bmatrix}$

**d** $\frac{1}{22}\begin{bmatrix} 17 & 3 \end{bmatrix}$

**23 a** $P^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}$

**b** $Q^{-1} = \begin{bmatrix} 0 & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}$

24 a $R^{-1} = \begin{bmatrix} 0 & \frac{1}{c} \\ \frac{1}{b} & \frac{-a}{bc} \end{bmatrix}$

b $S^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}$

c $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

## EXERCISE 7.6

1 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ Image point is (4, 0)

2 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

3 $T = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}$

4 $T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

5 $y = x - 6$

6 $y = x + 6$

7 $y = (x - 2)^2 - 1$

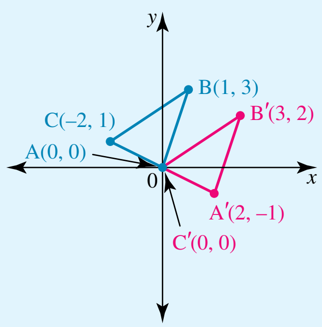
8 $y = (x + 3)^2 + 1$

9 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ Image point is (3, 7)

10 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$,(−6, 2)

11 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

12 a




b $T = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$

13 $y = x + 1$

14 $y = (x - 7)^2 - 6$

15 $T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$

16 $T = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

17 $T = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

18 $T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

19 $T = \begin{bmatrix} a \\ b \end{bmatrix}$

20 $T = \begin{bmatrix} a \\ 0 \end{bmatrix}$

## EXERCISE 7.7

1 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ Image point is (−3, 1)

2 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$ Image point is (−5, −2)

3 $y = (x + 2)^2$

4 $y = -x^2 - 1$

5 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, (5, −2)

6 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$, (−3, 8)

7 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \end{bmatrix}$, (−8, −1)

8 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$, (−4, 1)

9 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$, (−1, −5)

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, (1, 5)

10 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, (8, 4)

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$, (−8, −4)

11 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, (−6, 9)

12 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, (−1, 0)

13 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, (4, 3)

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$, (−2, −7)

**14 a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -15 \\ -1 \end{bmatrix}$, $(-15, -1)$

**b** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$, $(7, 7)$

**15** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$, $(-3, 9)$

**16** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $(0, -1)$

**17 a** $y = x - 3$

**b** $y = x + 3$

**c** $y = x - 5$

**18 a** $y = -(x^2 + 2x + 1)$

**b** $y = x^2 - 2x + 1$

**c** $x = y^2 + 2y + 1$

**19** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

**20** $A'(-2, 5)$; $B'(-4, 4)$; $C'(-4, 1)$; $D'(-2, 0)$; $E'(0, 3)$

## EXERCISE 7.8

**1** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $(2, -3)$

**2** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $(-2, 4)$

**3** $y = \left(\frac{x}{3}\right)^2 = \frac{1}{9}(x)^2$

**4** $y = \frac{1}{2}x^2$

**5** $(0, 0)$, $(4.5, 0)$, $(4.5, 12)$, $(0, 12)$

**6** $(4, 2)$, $(8, 2)$, $(6, 4)$, $(2, 4)$

**7** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $(6, -5)$

**8** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$, $(-1, 8)$

**9** $T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

**10 a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$, $(-3, 6)$

**b** A dilation of 3 parallel to the $x$-axis and a dilation of 2 parallel to the $y$-axis

**11** $x + y = 3$

**12** $y = \left(\frac{x}{3}\right)^2 - 1 = \frac{1}{9}x^2 - 1$

**13** $y = \dfrac{2}{x + 2}$

**14 a** A dilation of 2 from the $y$-axis and dilation of 3 from the $x$-axis

**b** $y = 3\sqrt{2x}$

**15** $\dfrac{x^2}{4} + y^2 = 4$

**16 a** $k = \frac{3}{2}$

**b** $\begin{bmatrix} -3 & -\frac{3}{2} & -\frac{9}{2} \\ 0 & 6 & 4 \end{bmatrix}$; $A'(-3, 0)$, $B'\left(-\frac{3}{2}, 6\right)$, $C'\left(-\frac{9}{2}, 4\right)$

**17** $k = \dfrac{1}{\sqrt{3}}$ or $\dfrac{\sqrt{3}}{3}$

**18 a** $y = -\frac{1}{6}x + 1$

**b** Invariant point is $(0, 1)$.

## EXERCISE 7.9

**1** $T = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

**2** $T = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

**3 a** $A(0, 0)$, $B\left(\frac{12}{7}, \frac{-15}{14}\right)$, $C\left(\frac{10}{7}, \frac{-9}{14}\right)$, $D\left(\frac{-2}{7}, \frac{3}{7}\right)$

**b** $\frac{3}{7}$ units$^2$

**4 a** $(-x + 4, y)$

**b** Yes: $(-x - 4, y)$

**5** $(x + 2, -y + 1)$

**6** Dilation of factor 2 to both axes followed by a rotation of 90° about the origin

**7** 70 units$^2$

**8 a** Reflection in line $y = x$ followed by a reflection in $x$-axis

**b** $2y = -3x - 12$

**9 a** Dilation of factor 2 from $y$-axis followed by a reflection in $x$-axis and translation of 1 unit across and 2 up

**b** $y = -\frac{1}{2}(x - 1)^2 + 3$

**10** $y = -\frac{1}{2}x^2$

**11** $y = \sqrt{-\frac{x}{3}}$

**12** 30 units$^2$

**13** $D_k^2$ gives a dilation factor of $k^2$ parallel to both axes.

**14** Not the same

# 8

# Probability

# 8.1 Kick off with CAS

## Probability

1 Using CAS technology, calculate each of the following.

a $4 \times 3 \times 2 \times 1$

b $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

c $11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

d $15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

2 Using CAS technology, calculate each of the following.

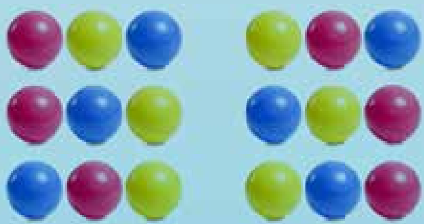
a $4!$ b $8!$ c $11!$ d $15!$

3 What do you notice about your answers to questions 1 and 2?

4 Consider the 3 balls shown.

If the order of the colours is important, in how many ways can the balls be arranged?

5 The answer to question 4 is displayed.



Use CAS technology to calculate the value of ${}^3P_3$.
How does the answer compare to the answer to question 4?

6 Using the same three balls, in how many ways can they be arranged if the order of the colours is not important?

7 Use CAS technology to calculate the value of ${}^3C_3$.
How does the answer compare to the answer to question 6?

8 Consider the macaroons in the picture.

In how many ways can 3 out of the 4 macaroons be arranged if:

a the order matters b the order doesn't matter?

9 Calculate ${}^4P_3$ and ${}^4C_3$. How do these answers relate to your answers to question 8?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 8.2 Probability review

The origins of probability as a branch of mathematics arose from the work of Blaise Pascal in relation to a game of chance in the seventeenth century. Today the study of mathematical probability and statistics has become enormously important. The science of quantum mechanics is based on probability; those who work in many fields including farmers, firefighters, environmentalists, urban planners, geneticists and medical researchers, make decisions based on probability models.

The language and the notation used in the theory of probability is that which appears in set theory. Some set notation has already been introduced in earlier topics.

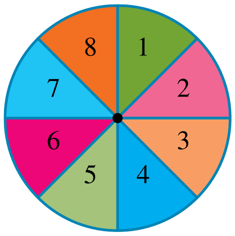
studyon

Units 1 & 2
AOS 4
Topic 1
Concept 1

**Probability review**
Concept summary
Practice questions

## Notation and fundamentals: outcomes, sample spaces and events

Consider the experiment or trial of spinning a wheel which is divided into eight equal sectors, with each sector marked with one of the numbers 1 to 8. If the wheels is unbiased, each of these numbers is equally likely to occur.




The **outcome** of each trial is one of the eight numbers.

The **sample space**, $\xi$, is the set of all possible outcomes: $\xi = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

An **event** is a particular set of outcomes which is a subset of the sample space. For example, if $M$ is the event of obtaining a number which is a multiple of 3, then $M = \{3, 6\}$. This set contains two outcomes. This is written in set notation as $n(M) = 2$.

The **probability** of an event is the long-term proportion, or relative frequency, of its occurrence.

For any event $A$, the probability of its occurrence is $\Pr(A) = \dfrac{n(A)}{n(\xi)}$.

Hence, for the event $M$:

$$\begin{aligned}\Pr(M) &= \frac{n(M)}{n(\xi)}\\ &= \frac{2}{8}\\ &= \frac{1}{4}\end{aligned}$$

This value does not mean that a multiple of 3 is obtained once in every four spins of the wheel. However, it does mean that after a very large number of spins of the wheel, the proportion of times that a multiple of 3 would be obtained approaches $\frac{1}{4}$. The closeness of this proportion to $\frac{1}{4}$ would improve in the long term as the number of spins is further increased.

**For any event $A$, $0 \le \Pr(A) \le 1$.**

- If $\Pr(A) = 0$ then it is not possible for $A$ to occur. For example, the chance that the spinner lands on a negative number would be zero.
- If $\Pr(A) = 1$ then the event $A$ is certain to occur. For example, it is 100% certain that the number the spinner lands on will be smaller than 9.

The probability of each outcome $\Pr(1) = \Pr(2) = \Pr(3) = \ldots = \Pr(8) = \frac{1}{8}$ for this spinning wheel. As each outcome is equally likely to occur, the outcomes are **equiprobable**. In other situations, some outcomes may be more likely than others.

For any sample space, $\Pr(\xi) = \frac{n(\xi)}{n(\xi)} = 1$ and the sum of the probabilities of each of the outcomes in any sample space must total 1.

### Complementary events

For the spinner example, the event that the number is not a multiple of 3 is the complement of the event $M$. The complementary event is written as $M'$ or as $\overline{M}$.

$$\begin{aligned}\Pr(M') &= 1 - \Pr(M)\\ &= 1 - \frac{1}{4}\\ &= \frac{3}{4}\end{aligned}$$

For any complementary events, $\Pr(A) + \Pr(A') = 1$ and therefore $\Pr(A') = 1 - \Pr(A)$.

**WORKED EXAMPLE 1**

**A spinning wheel is divided into eight sectors, each of which is marked with one of the numbers 1 to 8. This wheel is biased so that Pr(8) = 0.3, while the other numbers are equiprobable.**

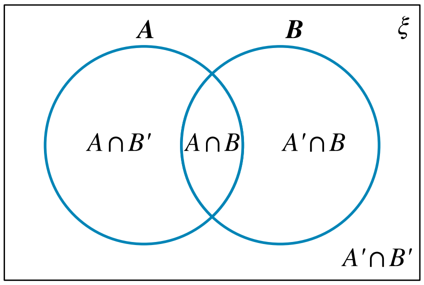
**a Calculate the probability of obtaining the number 4.**

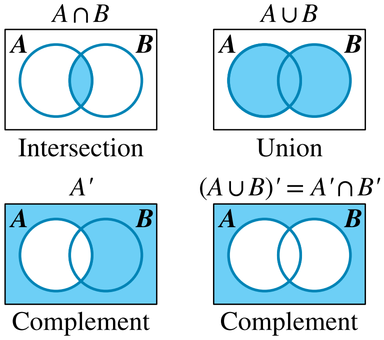
**b If $A$ is the event the number obtained is even, calculate $\Pr(A)$ and $\Pr(A')$.**

| THINK | WRITE |
|---|---|
| **a 1** State the complement of obtaining the number 8 and the probability of this. | **a** The sample space contains the numbers 1 to 8 so the complement of obtaining 8 is obtaining one of the numbers 1 to 7.<br>As $\Pr(8) = 0.3$ then the probability of not obtaining 8 is $1 - 0.3 = 0.7$. |
| **2** Calculate the required probability. | Since each of the numbers 1 to 7 are equiprobable, the probability of each number is $\frac{0.7}{7} = 0.1$.<br>Hence, $\Pr(4) = 0.1$. |
| **b 1** Identify the elements of the event. | **b** $A = \{2, 4, 6, 8\}$ |
| **2** Calculate the probability of the event. | $\begin{aligned}\Pr(A) &= \Pr(2 \text{ or } 4 \text{ or } 6 \text{ or } 8)\\ &= \Pr(2) + \Pr(4) + \Pr(6) + \Pr(8)\\ &= 0.1 + 0.1 + 0.1 + 0.3\\ &= 0.6\end{aligned}$ |
| **3** State the complementary probability. | $\begin{aligned}\Pr(A') &= 1 - \Pr(A)\\ &= 1 - 0.6\\ &= 0.4\end{aligned}$ |

## Venn diagrams

A **Venn diagram** can be useful for displaying the union and intersection of sets. Such a diagram may be helpful in displaying compound events in probability, as illustrated for the sets or events $A$ and $B$.




The information shown in the Venn diagram may be the actual outcomes for each event, or it may only show a number which represents the number of outcomes for each event. Alternatively, the Venn diagram may show the probability of each event. The total probability is 1; that is, $\Pr(\xi) = 1$.

### The addition formula

The number of elements contained in set $A$ is denoted by $n(A)$.

The Venn diagram illustrates that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Hence, dividing by the number of elements in the sample space gives:

$$\frac{n(A \cup B)}{n(\xi)} = \frac{n(A)}{n(\xi)} + \frac{n(B)}{n(\xi)} - \frac{n(A \cap B)}{n(\xi)}$$

$$\therefore \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

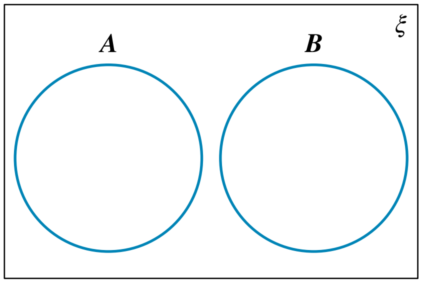
The result is known as the addition formula.

$$\mathbf{Pr}(A \cup B) = \mathbf{Pr}(A) + \mathbf{Pr}(B) - \mathbf{Pr}(A \cap B)$$

If the events A and B are **mutually exclusive** then they cannot occur simultaneously. For mutually exclusive events, $n(A \cap B) = 0$ and therefore $\Pr(A \cap B) = 0$.

The addition formula for mutually exclusive events becomes:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$




**WORKED EXAMPLE 2**

**From a survey of a group of 50 people it was found that in the past month 30 of the group had made a donation to a local charity, 25 had donated to an international charity and 20 had made donations to both local and international charities.**

**Let $L$ be the set of people donating to a local charity and $I$ the set of people donating to an international charity.**

**a** **Draw a Venn diagram to illustrate the results of this survey.**
**One person from the group is selected at random.**

**b** **Using appropriate notation, calculate the probability that this person donated to a local charity but not an international one.**

**c** What is the probability that this person did not make a donation to either type of charity?

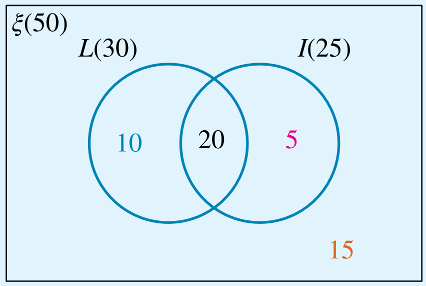
**d** Calculate the probability that this person donated to at least one of the two types of charity.

**THINK**

**a** Show the given information on a Venn diagram and complete the remaining sections using arithmetic.

**WRITE**

**a** Given: $n(\xi) = 50$, $n(L) = 30$, $n(I) = 25$ and $n(L \cap I) = 20$




**b 1** State the required probability using set notation.

**b** $\Pr(L \cap I') = \dfrac{n(L \cap I')}{n(\xi)}$

**2** Identify the value of the numerator from the Venn diagram and calculate the probability.

$\Pr(L \cap I') = \dfrac{10}{50}$

$= \dfrac{1}{5}$

**3** Express the answer in context.

The probability that the randomly chosen person donated to a local charity but not an international one is 0.2.

**c 1** State the required probability using set notation.

**c** $\Pr(L' \cap I') = \dfrac{n(L' \cap I')}{n(\xi)}$

**2** Identify the value of the numerator from the Venn diagram and calculate the probability.

$\Pr(L' \cap I') = \dfrac{15}{50}$

$= \dfrac{3}{10}$

**3** Express the answer in context.

The probability that the randomly chosen person did not donate is 0.3.

**d 1** State the required probability using set notation.

**d** $\Pr(L \cup I) = \dfrac{n(L \cup I)}{n(\xi)}$

**2** Identify the value of the numerator from the Venn diagram and calculate the probability.

$\Pr(L \cup I) = \dfrac{10 + 20 + 5}{50}$

$= \dfrac{35}{50}$

$= \dfrac{7}{10}$

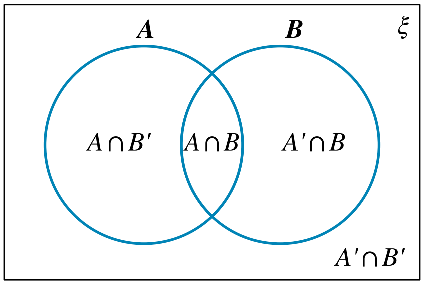
**3** Express the answer in context.

The probability that the randomly chosen person donated to at least one type of charity is 0.7.

## Probability tables

For situations involving two events, a probability table can provide an alternative to a Venn diagram. Consider the Venn diagram shown.

A **probability table** presents any known probabilities of the four compound events $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$ in rows and columns.




| | $B$ | $B'$ | |
|---|---|---|---|
| $A$ | $\Pr(A \cap B)$ | $\Pr(A \cap B')$ | $\Pr(A)$ |
| $A'$ | $\Pr(A' \cap B)$ | $\Pr(A' \cap B')$ | $\Pr(A')$ |
| | $\Pr(B)$ | $\Pr(B')$ | $\Pr(\xi) = 1$ |

This allows the table to be completed using arithmetic calculations since, for example, $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$ and $\Pr(B) = \Pr(A \cap B) + \Pr(A' \cap B)$. The probabilities of complementary events can be calculated using the formula $\Pr(A') = 1 - \Pr(A)$.

To obtain $\Pr(A \cup B)$, the addition formula $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ can be used.

Probability tables are also known as **Karnaugh maps**.

WORKED EXAMPLE 3

**$\Pr(A) = 0.4$, $\Pr(B) = 0.7$ and $\Pr(A \cap B) = 0.2$**

**a** **Construct a probability table for the events $A$ and $B$.**

**b** **Calculate $\Pr(A' \cup B)$.**

THINK

**a 1** Enter the given information in a probability table.

WRITE

**a** Given: $\Pr(A) = 0.4$, $\Pr(B) = 0.7$, $\Pr(A \cap B) = 0.2$ and also $\Pr(\xi) = 1$

| | $B$ | $B'$ | |
|---|---|---|---|
| $A$ | 0.2 | | 0.4 |
| $A'$ | | | |
| | 0.7 | | 1 |

**2** Add in the complementary probabilities.

$\Pr(A') = 1 - 0.4 = 0.6$ and $\Pr(B') = 1 - 0.7 = 0.3$

| | $B$ | $B'$ | |
|---|---|---|---|
| $A$ | 0.2 | | 0.4 |
| $A'$ | | | 0.6 |
| | 0.7 | 0.3 | 1 |

3 Complete the remaining sections using arithmetic.

For the first row, $0.2 + 0.2 = 0.4$
For the first column, $0.2 + 0.5 = 0.7$

| | $B$ | $B'$ | |
|---|---|---|---|
| $A$ | 0.2 | 0.2 | 0.4 |
| $A'$ | 0.5 | 0.1 | 0.6 |
| | 0.7 | 0.3 | 1 |

b 1 State the addition formula.

b $\Pr(A' \cup B) = \Pr(A') + \Pr(B) - \Pr(A' \cap B)$

2 Use the values in the probability table to carry out the calculation.

From the probability table, $\Pr(A' \cap B) = 0.5$.

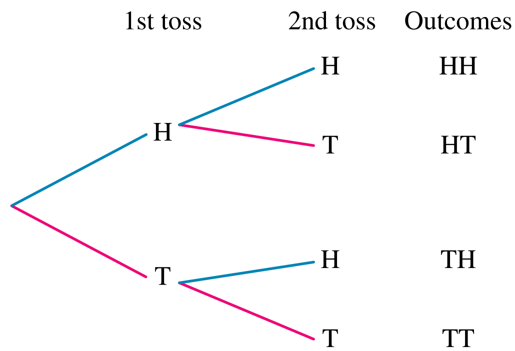
$$\therefore \Pr(A' \cup B) = 0.6 + 0.7 - 0.5$$
$$= 0.8$$

## Other ways to illustrate sample spaces

In experiments involving two tosses of a coin or two rolls of a die, the sample space can be illustrated using a simple tree diagram or a lattice diagram.

### Simple tree diagram

The outcome of each toss of the coin is either Heads (H) or Tails (T). For two tosses, the outcomes are illustrated by the tree diagram shown.

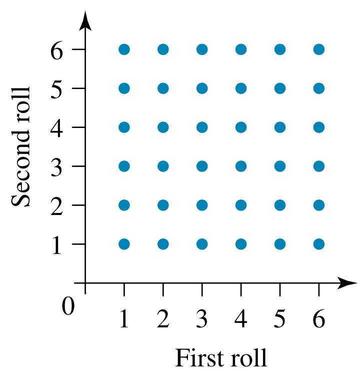



The sample space consists of the four equally likely outcomes HH, HT, TH and TT. This means the probability of obtaining two Heads in two tosses of a coin would be $\frac{1}{4}$.

The tree diagram could be extended to illustrate repeated tosses of the coin.

### Lattice diagram

For two rolls of a six-sided die, the outcomes are illustrated graphically using a grid known as a lattice diagram.




There are 36 points on the grid which represent the equally likely elements of the sample space. Each point can be described by a pair of coordinates, with the first coordinate giving the outcome from the first roll and the second coordinate giving the outcome from the second roll of the die. As there is only one point with coordinates $(6, 6)$, this means the probability that both rolls result in a 6 is $\frac{1}{36}$.

### Use of physical measurements in infinite sample spaces

When an arrow is fired at an archery target there would be an infinite number of points at which the arrow could land. The sample space can be represented by the area measure of the target, assuming the arrow hits it. The probability of such an arrow landing in a particular section of the target could then be calculated as the ratio of that area to the total area of the target.

WORKED EXAMPLE 4

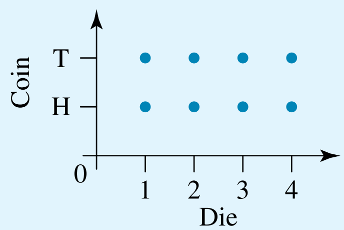
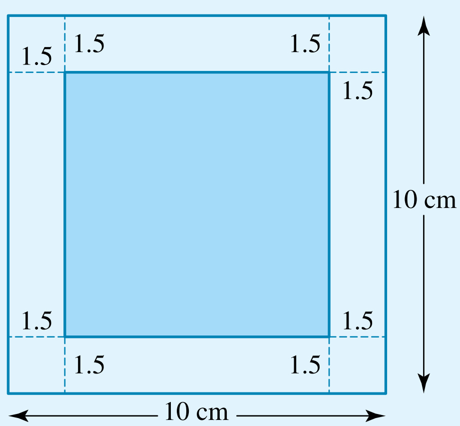
A four-sided die with faces labelled 1, 2, 3, 4 is rolled at the same time a coin is tossed.

**a** Draw a lattice diagram to represent the sample space.

**b** What is the probability of obtaining a 1 on the die and a Tail on the coin?

**c** What is the probability of the event of obtaining a number which is at least 3, together with a Head on the coin?

**d** The coin is thrown onto a square sheet of cardboard of edge 10 cm. The coin lands with its centre inside or on the boundary of the cardboard. Given the radius of the circular coin is 1.5 cm, calculate the probability the coin lands completely inside the area covered by the square piece of cardboard.

| THINK | WRITE |
|---|---|
| **a** Construct a lattice diagram to illustrate the sample space. | **a**  |
| **b** Calculate the required probability. | **b** There are 8 equally likely outcomes in the sample space. Only one of these outcomes is a 1 on the die and a Tail on the coin.<br>The probability of obtaining a 1 on the die and a Tail on the coin is $\frac{1}{8}$. |
| **c 1** Identify the number of outcomes which make up the event. | **c** The event of obtaining a number which is at least 3 together with a Head on the coin occurs for the two outcomes 3H and 4H. |
| **2** Calculate the probability. | The probability of obtaining a number which is at least 3 together with a Head is $\frac{2}{8} = \frac{1}{4}$. |
| **d 1** Draw a diagram to show the area in which the centre of the coin may land. | **d** For the coin to land inside the square its centre must be no less than 1.5 cm from each edge of the square.<br>The area in which the centre of the coin may land is a square of edge $10 - 2 \times 1.5 = 7$ cm.<br> |

2 Calculate the required probability.

The area that the centre of the coin could land in is the area of the cardboard which is $10 \times 10 = 100 \text{ cm}^2$.
The area that the centre of the coin must land in for the coin to be completely inside the area of the cardboard is $7 \times 7 = 49 \text{ cm}^2$.
Therefore, the probability the coin lands completely inside the cardboard area is $\frac{49}{100} = 0.49$.

## Simulations

To estimate the probability of success, it may be necessary to perform experiments or simulations. For example, the probability of obtaining a total of 11 when rolling two dice can be estimated by repeatedly rolling two dice and counting the number of times a total of 11 appears.

The results of a particular experiment or simulation can only give an estimate of the true probability. The more times the simulation is carried out, the better the estimate of the true probability.

For example, to simulate whether a baby is born male or female, a coin is flipped. If the coin lands Heads up, the baby is a boy. If the coin lands Tails up, the baby is a girl. The coin is flipped 100 times and returns 43 Heads and 57 Tails.

This simulation gives a probability of 0.43 of the baby being a boy and 0.57 for a girl. This closely resembles the theoretical probability of 0.5.

## EXERCISE 8.2 Probability review

**PRACTISE**

Work without CAS

1 WE1 A spinning wheel is divided into eight sectors, each of which is marked with one of the numbers 1 to 8. This wheel is biased so that $\Pr(6) = \frac{9}{16}$ while the other numbers are equiprobable.

a Calculate the probability of obtaining the number 1.

b If $A$ is the event that a prime number is obtained, calculate $\Pr(A)$ and $\Pr(A')$.

2 A bag contains 20 balls of which 9 are green and 6 are red. One ball is selected at random.

a What is the probability that this ball is:

i either green or red

ii not red

iii neither green nor red?

b How many additional red balls must be added to the original bag so that the probability that the chosen ball is red is 0.5?

3 WE2 From a group of 42 students it was found that 30 students studied Mathematical Methods and 15 studied Geography. Ten of the Geography students did not study Mathematical Methods.

Let $M$ be the set of students studying Mathematical Methods and let $G$ be the set of students studying Geography.

a Draw a Venn diagram to illustrate this situation.

One student from the group is selected at random.

b Using appropriate notation, calculate the probability that this student studies Mathematical Methods but not Geography.

c What is the probability that this student studies neither Mathematical Methods nor Geography?

d Calculate the probability that this student studies only one of Mathematical Methods or Geography.

4 From a set of 18 cards numbered 1, 2, 3, …, 18, one card is drawn at random.

Let $A$ be the event of obtaining a multiple of 3, $B$ be the event of obtaining a multiple of 4 and let $C$ be the event of obtaining a multiple of 5.

a List the elements of each event and then illustrate the three events as sets on a Venn diagram.

b Which events are mutually exclusive?

c State the value of $\Pr(A)$.

d Calculate the following.

i $\Pr(A \cup C)$ ii $\Pr(A \cap B')$ iii $\Pr((A \cup B \cup C)')$

5 WE3 Given $\Pr(A) = 0.65$, $\Pr(B) = 0.5$ and $\Pr(A' \cap B') = 0.2$:

a construct a probability table for the events $A$ and $B$

b calculate $\Pr(B' \cup A)$.

6 For two events $A$ and $B$ it is known that $\Pr(A \cup B) = 0.75$, $\Pr(A') = 0.42$ and $\Pr(B) = 0.55$.

a Form a probability table for these two events.

b State $\Pr(A' \cap B')$.

c Show that $\Pr(A \cup B)' = \Pr(A' \cap B')$.

d Show that $\Pr(A \cap B) = 1 - \Pr(A' \cup B')$.

e Draw a Venn diagram for the events $A$ and $B$.

7 WE4 A six-sided die with faces labelled 1, 2, 3, 4, 5, 6 is rolled at the same time that a coin is tossed.

a Draw a lattice diagram to represent the sample space.

b What is the probability of obtaining a 6 on the die and a Head on the coin?

c What is the probability of obtaining an even number together with a Tail on the coin?

d The coin is thrown onto a rectangular sheet of cardboard with dimensions 13 cm by 10 cm. The coin lands with its centre inside or on the boundary of the cardboard. Given the radius of the circular coin is 1.5 cm, calculate the probability that the coin lands completely inside the area covered by the rectangular piece of cardboard.

8 A coin is tossed three times.

a Draw a simple tree diagram to show the possible outcomes.

b What is the probability of obtaining at least one Head?

c Calculate the probability of obtaining either exactly two Heads or two Tails.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 A bag contains 5 green, 6 pink, 4 orange and 8 blue counters. A counter is selected at random. Find the probability that the counter is:

a green

b orange or blue

c not blue

d black.

**10** Tickets are drawn randomly from a barrel containing 200 tickets. There are 3 prizes to be won: first, second and third. Josephine has purchased 10 tickets. Assuming Josephine wins only one prize, what is the probability that she wins:

**a** the first prize **b** the second prize **c** all three prizes?

**11** A card is drawn randomly from a standard pack of 52 cards. Find the probability that the card is:

**a** not green **b** red **c** a heart
**d** a 10 or a red card **e** not an ace.

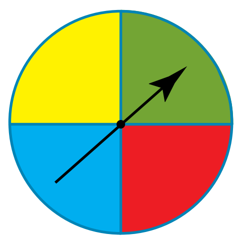
**12** Two unbiased dice are rolled and the larger of the two numbers is noted. If the two dice show the same number, then the sum of the two numbers is recorded. Use a table to show all the possible outcomes. Hence find the probability that the result is:

**a** 5
**b** 10
**c** a number greater than 5
**d** 7
**e** either a two-digit number or a number greater than 6
**f** not 9.

**13** A coin is tossed three times. Show the sample space on a tree diagram and hence find the probability of getting:

**a** 2 Heads and 1 Tail
**b** either 3 Heads or 3 Tails
**c** a Head on the first toss of the coin
**d** at least 1 Head
**e** no more than 1 Tail.

**14** A spinner is divided into 4 sections coloured red, blue, green and yellow. Each section is equally likely to occur.



The spinner is spun twice. List all the possible outcomes and hence find the probability of obtaining:

**a** the same colour
**b** a red and a yellow
**c** not a green.

**15** The 3.38 train to the city is late on average 1 day out of 3. Draw a probability tree to show the outcomes on three consecutive days. Hence find the probability that the bus is:

**a** late on 1 day
**b** late on at least 2 days
**c** on time on the last day
**d** on time on all 3 days.

**16** A survey was carried out to find the type of occupation of 800 adults in a small suburb. There were 128 executives, 180 professionals, 261 trades workers, 178 labourers and 53 unemployed people.

A person from this group is chosen at random. What is the probability that the person chosen is:

**a** a labourer **b** not employed **c** not an executive
**d** either a tradesperson or a labourer?

**17** The gender of babies in a set of triplets is simulated by flipping 3 coins. If a coin lands Tails up, the baby is a boy. If a coin lands Heads up, the baby is a girl. In the simulation, the trial is repeated 40 times and the following results show the number of Heads obtained in each trial:

0, 3, 2, 1, 1, 0, 1, 2, 1, 0, 1, 0, 2, 0, 1, 0, 1, 2, 3, 2, 1, 3, 0, 2, 1, 2, 0, 3, 1, 3, 0, 1, 0, 1, 3, 2, 2, 1, 2, 1.

**a** Calculate the probability that exactly one of the babies in a set of triplets is female.
**b** Calculate the probability that more than one of the babies in the set of triplets is female.

**18** Two hundred people applied to do their driving test in October. The results are shown below.

| | Passed | Failed |
|---|---|---|
| Male | 73 | 26 |
| Female | 81 | 20 |

**a** Find the probability that a person selected at random has failed the test.
**b** What is the probability that a person selected at random is a female who passed the test?

MASTER

**19** A sample of 100 first-year university science students were asked if they study physics or chemistry. It was found that 63 study physics, 57 study chemistry and 4 study neither.

A student is then selected at random. What is the probability that the student studies:

**a** either physics or chemistry but not both
**b** both physics and chemistry?

In total, there are 1200 first-year university science students.

**c** Estimate the number of students who are likely to study both physics and chemistry.

Two students are chosen at random from the total number of students. Find the probability that:

**d** both students study physics and chemistry
**e** each student studies just one of the two subjects
**f** one of the two students studies neither physics nor chemistry.

**20** In a table tennis competition, each team must play every other team twice.

**a** How many games must be played if there are 5 teams in the competition?
**b** How many games must be played if there are $n$ teams in the competition?

A regional competition consists of 16 teams, labelled A, B, C, …, N, O, P.

**c** How many games must each team play?
**d** What is the total number of games played?

# 8.3 Conditional probability

Some given information may reduce the number of elements in a sample space. For example, in two tosses of a coin, if it is known that at least one Head is obtained then this reduces the sample space from {HH, HT, TH, TT} to {HH, HT, TH}. This affects the probability of obtaining two Heads.

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**Conditional probability**
Concept summary
Practice questions

The probability of obtaining two Heads given at least one Head has occurred is called **conditional probability**. It is written as $\Pr(A|B)$ where $A$ is the event of obtaining two Heads and $B$ is the event of at least one Head. The event $B$ is the conditional event known to have occurred. Since event $B$ has occurred, the sample space has been reduced to three elements. This means $\Pr(A|B) = \frac{1}{3}$.

Who no information is given about what has occurred, the sample space contains four elements and the probability of obtaining two Heads is $\Pr(A) = \frac{1}{4}$.

**WORKED EXAMPLE 5**

**The table shows the results of a survey of 100 people aged between 16 and 29 about their preferred choice of food when eating at a café.**

| | Vegetarian ($V$) | Non-vegetarian ($V'$) | |
|---|---|---|---|
| Male ($M$) | 18 | 38 | 56 |
| Female ($F$) | 25 | 19 | 44 |
| | 43 | 57 | 100 |

**One person is selected at random from those surveyed. Identify the event and use the table to calculate:**

**a** $\Pr(M \cap V)$
**b** $\Pr(M|V)$
**c** $\Pr(V'|F)$
**d** $\Pr(V)$

| THINK | WRITE |
|---|---|
| **a 1** Describe the event $M \cap V$. | **a** The event $M \cap V$ is the event the selected person is both male and vegetarian. |
| **2** Calculate the probability. | $\Pr(M \cap V) = \frac{n(M \cap V)}{n(\xi)}$ <br> $= \frac{18}{100}$ <br> $\therefore \Pr(M \cap V) = 0.18$ |
| **b 1** Describe the conditional probability. | **b** The event $M|V$ is the event of a person being male given that the person is vegetarian. |
| **2** State the number of elements in the reduced sample space. | Since the person is known to be vegetarian, the sample space is reduced to $n(V) = 43$ people. |
| **3** Calculate the required probability. | Of the 43 vegetarians, 18 are male. <br> $\therefore \Pr(M|V) = \frac{n(M \cap V)}{n(V)}$ <br> $= \frac{18}{43}$ |
| **c 1** Identify the event. | **c** The event $V'|F$ is the event of a person being non-vegetarian given the person is female. |

| | |
|---|---|
| **2** State the number of elements in the reduced sample space. | Since the person is known to be female, the sample space is reduced to $n(F) = 44$ people. |
| **3** Calculate the probability. | Of the 44 females, 19 are non-vegetarian. $\therefore \Pr(V' \mid F) = \frac{n(V' \cap F)}{n(F)} = \frac{19}{44}$ |
| **d 1** State the event. | **d** The event $V$ is the event the selected person is vegetarian. |
| **2** Calculate the required probability. *Note:* This is not a conditional probability. | $\Pr(V) = \frac{n(V)}{n(\xi)} = \frac{43}{100}$ $\therefore \Pr(V) = 0.43$ |

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## Formula for conditional probability

Consider the events $A$ and $B$:

$$\Pr(A) = \frac{n(A)}{n(\xi)},\ \Pr(B) = \frac{n(B)}{n(\xi)} \text{ and } \Pr(A \cap B) = \frac{n(A \cap B)}{n(\xi)}$$

For the conditional probability, $\Pr(A \mid B)$, the sample space is reduced to $n(B)$.

$$\begin{aligned}\Pr(A \mid B) &= \frac{n(A \cap B)}{n(B)} \\ &= n(A \cap B) \div n(B) \\ &= \frac{n(A \cap B)}{n(\xi)} \div \frac{n(B)}{n(\xi)} \\ &= \Pr(A \cap B) \div \Pr(B) \\ &= \frac{\Pr(A \cap B)}{\Pr(B)}\end{aligned}$$

Hence, the conditional probability formula is:

$$\mathbf{Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}}$$

This formula illustrates that if the events $A$ and $B$ are mutually exclusive so that $\Pr(A \cap B) = 0$, then $\Pr(A \mid B) = 0$. That is, if $B$ occurs then it is impossible for $A$ to occur.

However, if $B$ is a subset of $A$ so that $\Pr(A \cap B) = \Pr(B)$, then $\Pr(A \mid B) = 1$. That is, if $B$ occurs, then it is certain that $A$ will occur.

## Multiplication of probabilities

Consider the conditional probability formula for $\Pr(B \mid A)$:

$$\Pr(B \mid A) = \frac{\Pr(B \cap A)}{\Pr(A)}$$

Since $B \cap A$ is the same as $A \cap B$, then $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$.

Rearranging, the formula for multiplication of probabilities is formed.

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$$

For example, the probability of obtaining first an aqua ($A$) and then a black ($B$) ball when selecting two balls without replacement from a bag containing 16 balls, 6 of which are aqua and 10 of which are black, would be $\Pr(A \cap B) = \Pr(A) \times \Pr(B|A) = \dfrac{6}{16} \times \dfrac{10}{15}$.

The multiplication formula can be extended. For example, the probability of obtaining 3 black balls when selecting three balls without replacement from the bag containing 16 balls, 10 of which are black, would be $\dfrac{10}{16} \times \dfrac{9}{15} \times \dfrac{8}{14}$.

**WORKED EXAMPLE 6**

**a** **If $\Pr(A) = 0.6$, $\Pr(A|B) = 0.6125$ and $\Pr(B') = 0.2$, calculate $\Pr(A \cap B)$ and $\Pr(B|A)$.**

**b** **Three girls each select one ribbon at random, one after the other, from a bag containing 8 green ribbons and 10 red ribbons. What is the probability that the first girl selects a green ribbon and both the other girls select a red ribbon?**

| THINK | WRITE |
|---|---|
| **a** 1 State the conditional probability formula for $\Pr(A|B)$. | **a** $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ |
| 2 Obtain the value of $\Pr(B)$. | For complementary events: $\Pr(B) = 1 - \Pr(B')$ $= 1 - 0.2$ $= 0.8$ |
| 3 Use the formula to calculate $\Pr(A \cap B)$. | $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ <br> $0.6125 = \dfrac{\Pr(A \cap B)}{0.8}$ <br> $\Pr(A \cap B) = 0.6125 \times 0.8$ $= 0.49$ |
| 4 State the conditional probability formula for $\Pr(B|A)$. | $\Pr(B|A) = \dfrac{\Pr(B \cap A)}{\Pr(A)}$ |
| 5 Calculate the required probability. | $\Pr(B \cap A) = \Pr(A \cap B)$ <br> $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$ $= \dfrac{0.49}{0.6}$ $= \dfrac{49}{60}$ |

**b** **1** Define the events.

**b** Let $G$ be the event a green ribbon is chosen and $R$ be the event a red ribbon is chosen.

**2** Describe the sample space.

There are 18 ribbons in the bag forming the elements of the sample space. Of these 18 ribbons, 8 are green and 10 are red.

**3** State the probability the first ribbon selected is green.

$\Pr(G) = \frac{8}{18}$

**4** Calculate the conditional probability the second ribbon is red by reducing the number of elements in the sample space.

Once a green ribbon has been chosen, there are 7 green and 10 red ribbons remaining, giving a total of 17 ribbons in the bag.

$\therefore \Pr(R|G) = \frac{10}{17}$

**5** Calculate the conditional probability the third ribbon is red by reducing the number of elements in the sample space.

Once a green and a red ribbon have been chosen, there are 7 green and 9 red ribbons remaining, giving a total of 16 ribbons in the bag.

$\therefore \Pr(R|G \cap R) = \frac{9}{16}$

**6** Calculate the required probability.

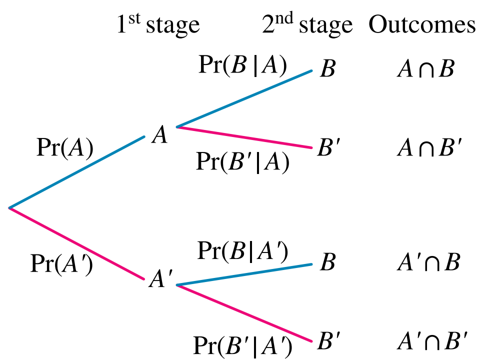
*Note:*

$\Pr(G \cap R \cap R) = \Pr(G) \times \Pr(R|G) \times \Pr(R|G \cap R)$

$$\Pr(G \cap R \cap R) = \frac{8}{18} \times \frac{10}{17} \times \frac{9}{16} = \frac{5}{34}$$

## Probability tree diagrams

The sample space of a two-stage trial where the outcomes of the second stage are dependent on the outcomes of the first stage can be illustrated with a **probability tree diagram**.




Each branch is labelled with its probability; conditional probabilities are required for the second-stage branches. Calculations are performed according to the addition and multiplication laws of probability.

The formula for multiplication of probabilities is applied by multiplying the probabilities that lie along the respective branches to calculate the probability of an outcome. For example, to obtain the probability of $A$ and $B$ occurring, we need to multiply the probabilities along the branches $A$ to $B$ since $\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$.

The addition formula for mutually exclusive events is applied by adding the results from separate outcome branches to calculate the union of any of the four outcomes. For example, to obtain the probability that $A$ occurs, add together the results from the two branches where $A$ occurs. This gives $\Pr(A) = \Pr(A)\Pr(B|A) + \Pr(A)\Pr(B'|A)$.

- Multiply along the branch
- Add the results from each complete branch

**WORKED EXAMPLE 7**

**A box of chocolates contains 6 soft-centre and 4 hard-centre chocolates. A chocolate is selected at random and once eaten, a second chocolate is chosen.**

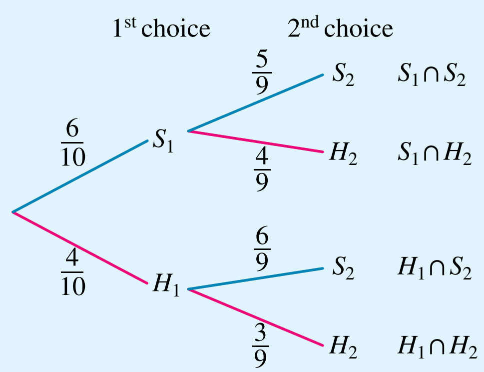
**Let $S_i$ be the event a soft-centre chocolate is chosen on the $i^{th}$ selection and $H_i$ be the event that a hard-centre chocolate is chosen on the $i^{th}$ selection, $i = 1, 2$.**

**a** **Deduce the value of $\Pr(H_2 | S_1)$.**

**b** **Construct a probability tree diagram to illustrate the possible outcomes.**

**c** **What is the probability that the first chocolate has a hard centre and the second a soft centre?**

**d** **Calculate the probability that either both chocolates have soft centres or both have hard centres.**

| THINK | WRITE |
|---|---|
| **a 1** Identify the meaning of $\Pr(H_2 \| S_1)$. | **a** $\Pr(H_2 \| S_1)$ is the probability that the second chocolate has a hard centre given that the first has a soft centre. |
| **2** State the required probability. | If a soft centre has been chosen first, there remain in the box 5 soft- and 4 hard-centre chocolates.<br>$\therefore \Pr(H_2 \| S_1) = \frac{4}{9}$ |
| **b** Construct the two-stage probability tree diagram. | **b**   |
| **c** Identify the appropriate branch and multiply along it to obtain the required probability.<br>*Note:* The multiplication law for probability is $\Pr(H_1 \cap S_2) = \Pr(H_1) \times \Pr(S_2 \| H_1)$. | **c** The required outcome is $H_1 \cap S_2$.<br>$\Pr(H_1 \cap S_2) = \frac{4}{10} \times \frac{6}{9}$<br>$= \frac{2}{5} \times \frac{2}{3}$<br>$= \frac{4}{15}$<br>The probability is $\frac{4}{15}$. |
| **d 1** Identify the required outcome. | **d** The probability that both chocolates have the same type of centre is $\Pr((S_1 \cap S_2) \cup (H_1 \cap H_2))$. |
| **2** Calculate the probabilities along each relevant branch. | $\Pr(S_1 \cap S_2) = \frac{6}{10} \times \frac{5}{9}$<br>$\Pr(H_1 \cap H_2) = \frac{4}{10} \times \frac{3}{9}$ |

3 Use the addition law for mutually exclusive events by adding the probabilities from the separate branches.

The probability that both chocolates have the same type of centre is:

$$\frac{6}{10}\times\frac{5}{9}+\frac{4}{10}\times\frac{3}{9}$$
$$=\frac{30}{90}+\frac{12}{90}$$
$$=\frac{42}{90}$$
$$=\frac{7}{15}$$

The probability both centres are the same type is $\frac{7}{15}$.

## EXERCISE 8.3 Conditional probability

**PRACTISE**

Work without CAS

1 WE5 The table shows the results of a survey of 100 people aged between 16 and 29 who were asked whether they rode a bike and what drink they preferred.

| | Drink containing caffeine ($C$) | Caffeine-free drink ($C'$) | |
|---|---|---|---|
| Bike rider ($B$) | 28 | 16 | 44 |
| Non–bike rider ($B'$) | 36 | 20 | 56 |
| | 64 | 36 | 100 |

One person is selected at random from those surveyed. Identify the event and use the table to calculate the following.

a $\Pr(B'\cap C')$ b $\Pr(B'|C')$ c $\Pr(C|B)$ d $\Pr(B)$

2 Two six-sided dice are rolled. Using appropriate symbols, calculate the probability that:

a the sum of 8 is obtained
b a sum of 8 is obtained given the numbers are not the same
c the sum of 8 is obtained but the numbers are not the same
d the numbers are not the same given the sum of 8 is obtained.

3 WE6 a If $\Pr(A') = 0.6$, $\Pr(B|A) = 0.3$ and $\Pr(B) = 0.5$, calculate $\Pr(A\cap B)$ and $\Pr(A|B)$.

b Three girls each select one ribbon at random, one after the other, from a bag containing 8 green ribbons and 4 red ribbons. What is the probability that all three girls select a green ribbon?

4 If $\Pr(A) = 0.61$, $\Pr(B) = 0.56$ and $\Pr(A\cup B) = 0.81$, calculate the following.

a $\Pr(A|B)$ b $\Pr(A|A\cap B)$ c $\Pr(A|A'\cap B)$

5 WE7 A box of jubes contains 5 green jubes and 7 red jubes. One jube is selected at random and once eaten, a second jube is chosen.

Let $G_i$ be the event a green jube is chosen on the $i^{\text{th}}$ selection and $R_i$ the event that a red jube is chosen on the $i^{\text{th}}$ selection, $i = 1, 2$.

a Deduce the value of $\Pr(G_2|R_1)$.
b Construct a probability tree diagram to illustrate the possible outcomes.

c What is the probability that the first jube is green and the second is red?
d Calculate the probability that either both jubes are green or both are red.

6 To get to school Rodney catches a bus and then walks the remaining distance.

If the bus is on time, Rodney has a 98% chance of arriving at school on time. However, if the bus is late, Rodney's chance of arriving at school on time is only 56%. On average the bus is on time 90% of the time.

a Draw a probability tree diagram to describe the given information, defining the symbols used.
b Calculate the probability that Rodney will arrive at school on time.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 Two unbiased dice are rolled and the sum of the topmost numbers is noted. Given that the sum is less than 6, find the probability that the sum is an even number.

8 Two unbiased dice are rolled. Find the probability that the sum is greater than 8, given that a 5 appears on the first die.

9 Given $\Pr(A) = 0.7, \Pr(B) = 0.3$ and $\Pr(A \cup B) = 0.8$, find the following.

a $\Pr(A \cap B)$ b $\Pr(A \mid B)$ c $\Pr(B \mid A)$ d $\Pr(A \mid B')$

10 Given $\Pr(A) = 0.6, \Pr(B) = 0.5$ and $\Pr(A \cup B) = 0.8$, find the following.

a $\Pr(A \cap B)$ b $\Pr(A \mid B)$ c $\Pr(B \mid A)$ d $\Pr(A \mid B')$

11 Given $\Pr(A) = 0.6,\ \Pr(B) = 0.7$ and $\Pr(A \cap B) = 0.4$, find the following.

a $\Pr(A \cup B)$ b $\Pr(A \mid B)$ c $\Pr(B \mid A')$ d $\Pr(A' \mid B')$

12 Two cards are drawn randomly from a standard pack of 52 cards. Find the probability that:

a both cards are diamonds
b at least 1 card is a diamond
c both cards are diamonds, given that at least one card is a diamond
d both cards are diamonds, given that the first card drawn is a diamond.

13 A bag contains 5 red marbles and 7 green marbles. Two marbles are drawn from the bag, one at a time, without replacement. Find the probability that:

a both marbles are green
b at least 1 marble is green
c both marbles are green given that at least 1 is green
d the first marble drawn is green given that the marbles are of different colours.

14 Sarah and Kate sit a Biology exam. The probability that Sarah passes the exam is 0.9 and the probability that Kate passes the exam is 0.8. Find the probability that:

a both Sarah and Kate pass the exam
b at least one of the two girls passes the exam
c only 1 girl passes the exam, given that Sarah passes.

15 In a survey designed to check the number of male and female smokers in a population, it was found that there were 32 male smokers, 41 female smokers, 224 female non-smokers and 203 male non-smokers.

A person is selected at random from this group of people. Find the probability that the person selected is:

a non-smoker
b male
c female, given that the person is a non-smoker.

16 In a sample of 1000 people, it is found that:

82 people are overweight and suffer from hypertension

185 are overweight but do not suffer from hypertension

175 are not overweight but suffer from hypertension

558 are not overweight and do not suffer from hypertension.

A person is selected at random from the sample. Find the probability that the person:

a is overweight
b suffers from hypertension
c suffers from hypertension given that he is overweight
d is overweight given that he does not suffer from hypertension.

17 A group of 400 people were tested for allergic reactions to two new medications. The results are shown in the table below:

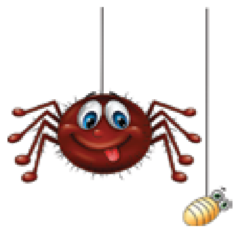
| | Allergic reaction | No allergic reaction |
|---|---|---|
| Medication A | 25 | 143 |
| Medication B | 47 | 185 |

If a person is selected at random from the group, calculate the following.

a The probability that the person suffers an allergic reaction
b The probability that the person was administered medication A
c Given there was an allergic reaction, the probability that medication B was administered
d Given the person was administered medication A, the probability that the person did not have an allergic reaction

MASTER

18 When Incy Wincy Spider climbs up the waterspout, the chances of his falling are affected by whether or not it is raining. When it is raining, the probability that he will fall is 0.84. When it is not raining, the probability that he will fall is 0.02. On average it rains 1 day in 5 around Incy Wincy Spider's spout.



Draw a probability tree to show the sample space and hence find the probability that:

a Incy Wincy will fall when it is raining
b Incy Wincy falls, given that it is raining
c it is raining given that Incy Wincy makes it to the top of the spout.

19 In tenpin bowling, a game is made up of 10 frames. Each frame represents one turn for the bowler. In each turn, a bowler is allowed up to 2 rolls of the ball to knock down the 10 pins. If the bowler knocks down all 10 pins with the first ball, this is called a strike. If it takes 2 rolls of the ball to knock the 10 pins, this is called a spare. Otherwise it is called an open frame.

On average, Richard hits a strike 85% of the time. If he needs a second roll of the ball then, on average, he will knock down the remaining pins 97% of the time. While training for the club championship, Richard plays a game.

Draw a tree diagram to represent the outcomes of the first 2 frames of his game.

Hence find the probability that:

a both are strikes
b all 10 pins are knocked down in both frames
c the first frame is a strike, given that the second is a strike.

# 8.4 Independence

If a coin is tossed twice, the chance of obtaining a Head on the coin on its second toss is unaffected by the result of the first toss. The probability of a Head on the second toss given a Head is obtained on the first toss is still $\frac{1}{2}$.

Events which have no effect on each other are called **independent events**. For such events, $\Pr(A|B) = \Pr(A)$. The given information does not affect the chance of event $A$ occurring.

Events which do affect each other are dependent events. For dependent events $\Pr(A|B) \neq \Pr(A)$ and the conditional probability formula is used to evaluate $\Pr(A|B)$.

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Units 1 & 2
AOS 4
Topic 1
Concept 3

**Independence**
Concept summary
Practice questions

## Test for mathematical independence

While it may be obvious that the chance of obtaining a Head on a coin on its second toss is unaffected by the result of the first toss, in more complex situations it can be difficult to intuitively judge whether events are independent or dependent. For such situations there is a test for mathematical independence that will determine the matter.

The multiplication formula states $\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$.

If the events $A$ and $B$ are independent then $\Pr(B|A) = \Pr(B)$.

Hence for independent events:

$$\mathbf{Pr}(A \cap B) = \mathbf{Pr}(A) \times \mathbf{Pr}(B)$$

This result is used to test whether events are mathematically independent or not.

WORKED EXAMPLE 8

**Consider the trial of tossing a coin twice.**

**Let $A$ be the event of at least one Tail, $B$ be the event of either two Heads or two Tails, and $C$ be the event that the first toss is a Head.**

**a** **List the sample space and the set of outcomes in each of $A$, $B$ and $C$.**

**b** **Test whether $A$ and $B$ are independent.**

**c** **Test whether $B$ and $C$ are independent.**

**d** **Use the addition formula to calculate $\Pr(B \cup C)$.**

| THINK | WRITE |
| --- | --- |
| **a** 1 List the elements of the sample space. | **a** The sample space is the set of equiprobable outcomes $\{HH, HT, TH, TT\}$. |
| 2 List the elements of $A$, $B$ and $C$. | $A = \{HT, TH, TT\}$<br>$B = \{HH, TT\}$<br>$C = \{HH, HT\}$ |
| **b** 1 State the test for independence. | **b** $A$ and $B$ are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$. |
| 2 Calculate the probabilities needed for the test for independence to be applied. | $\Pr(A) = \frac{3}{4}$ and $\Pr(B) = \frac{2}{4}$<br>Since $A \cap B = \{TT\}$, $\Pr(A \cap B) = \frac{1}{4}$. |

**3** Determine whether the events are independent.

Substitute values into the formula $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$\text{LHS} = \frac{1}{4}$

$\text{RHS} = \frac{3}{4} \times \frac{2}{4} = \frac{3}{8}$

Since LHS ≠ RHS, the events $A$ and $B$ are not independent.

**c 1** State the test for independence.

**c** $B$ and $C$ are independent if $\Pr(B \cap C) = \Pr(B)\Pr(C)$.

**2** Calculate the probabilities needed for the test for independence to be applied.

$\Pr(B) = \frac{2}{4}$ and $\Pr(C) = \frac{2}{4}$

Since $\Pr(B \cap C) = \{HH\}$, $\Pr(B \cap C) = \frac{1}{4}$.

**3** Determine whether the events are independent.

Substitute values in $\Pr(B \cap C) = \Pr(B)\Pr(C)$.

$\text{LHS} = \frac{1}{4}$

$\text{RHS} = \frac{2}{4} \times \frac{2}{4} = \frac{1}{4}$

Since LHS = RHS, the events $B$ and $C$ are independent.

**d 1** State the addition formula for $\Pr(B \cup C)$.

**d** $\Pr(B \cup C) = \Pr(B) + \Pr(C) - \Pr(B \cap C)$

**2** Replace $\Pr(B \cap C)$.

Since $B$ and $C$ are independent, $\Pr(B \cap C) = \Pr(B)\Pr(C)$.

$\therefore \Pr(B \cup C) = \Pr(B) + \Pr(C) - \Pr(B) \times \Pr(C)$

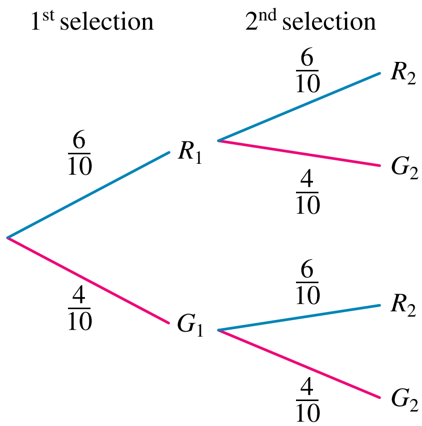
**3** Complete the calculation.

$\Pr(B \cup C) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$

## Independent trials

Consider choosing a ball from a bag containing 6 red and 4 green balls, noting its colour, returning the ball to the bag and then choosing a second ball. These trials are independent as the chance of obtaining a red or green ball is unaltered for each draw. This is an example of **sampling with replacement**.

The probability tree diagram is as shown.




The second stage branch outcomes are not dependent on the results of the first stage.

The probability that both balls are red is $\Pr(R_1R_2) = \frac{6}{10} \times \frac{6}{10}$.

As the events are independent:

$$\begin{aligned}\Pr(R_1 \cap R_2) &= \Pr(R_1) \times \Pr(R_2 \mid R_1)\\ &= \Pr(R_1) \times \Pr(R_2)\end{aligned}$$

However, if **sampling without replacement**, then $\Pr(R_1 \cap R_2) = \frac{6}{10} \times \frac{5}{9}$ since the events are not independent.

### Sequences of independent events

If the events $A$, $B$, $C$, … are independent, then
$\Pr(A \cap B \cap C \cap \ldots) = \Pr(A) \times \Pr(B) \times \Pr(C) \times \ldots$

**WORKED EXAMPLE 9**

**Three overweight people, Ari, Barry and Chris, commence a diet. The chances that each person sticks to the diet are 0.6, 0.8 and 0.7 respectively, independent of each other. What is the probability that:**

**a all three people stick to the diet**

**b only Chris sticks to the diet**

**c at least one of the three people sticks to the diet?**

| THINK | WRITE |
|---|---|
| **a 1** Define the independent events. | **a** Let $A$ be the event that Ari sticks to the diet, $B$ be the event that Barry sticks to the diet and $C$ be the event that Chris sticks to the diet.<br>$\Pr(A) = 0.6$, $\Pr(B) = 0.8$ and $\Pr(C) = 0.7$ |
| **2** State an expression for the required probability. | The probability that all three people stick to the diet is $\Pr(A \cap B \cap C)$. |
| **3** Calculate the required probability. | Since the events are independent,<br>$\Pr(A \cap B \cap C) = \Pr(A) \times \Pr(B) \times \Pr(C)$<br>$= 0.6 \times 0.8 \times 0.7$<br>$= 0.336$<br>The probability all three stick to the diet is 0.336. |
| **b 1** State an expression for the required probability. | **b** If only Chris sticks to the diet then neither Ari nor Barry do. The probability is $\Pr(A' \cap B' \cap C)$. |
| **2** Calculate the probability.<br>*Note:* For complementary events, $\Pr(A') = 1 - \Pr(A)$. | $\Pr(A' \cap B' \cap C) = \Pr(A') \times \Pr(B') \times \Pr(C)$<br>$= (1 - 0.6) \times (1 - 0.8) \times 0.7$<br>$= 0.4 \times 0.2 \times 0.7$<br>$= 0.056$<br>The probability only Chris sticks to the diet is 0.056. |
| **c 1** Express the required event in terms of its complementary event. | **c** The event that at least one of the three people sticks to the diet is the complement of the event that no one sticks to the diet. |
| **2** Calculate the required probability. | Pr(at least one sticks to the diet)<br>= 1 − Pr(no one sticks to the diet)<br>$= 1 - \Pr(A' \cap B' \cap C')$<br>$= 1 - 0.4 \times 0.2 \times 0.3$<br>$= 1 - 0.024$<br>$= 0.976$<br>The probability that at least one person sticks to the diet is 0.976. |

## EXERCISE 8.4 Independence

**PRACTICE**

Work without CAS

1 WE8 Consider the experiment of tossing a coin twice.

Let $A$ be the event the first toss is a Tail, $B$ the event of one Head and one Tail and $C$ be the event of no more than one Tail.

a List the sample space and the set of outcomes in each of $A$, $B$ and $C$.
b Test whether $A$ and $B$ are independent.
c Test whether $B$ and $C$ are independent.
d Use the addition formula to calculate $\Pr(B \cup A)$.

2 Two unbiased six-sided dice are rolled. Let $A$ be the event the same number is obtained on each die and $B$ be the event the sum of the numbers on each die exceeds 8.

a Are events $A$ and $B$ mutually exclusive?
b Are events $A$ and $B$ independent?
Justify your answers.
c If $C$ is the event the sum of the two numbers equals 8, determine whether $B$ and $C$ are:
i mutually exclusive
ii independent.

3 WE9 Three underweight people, Ava, Bambi and Chi, commence a carbohydrate diet. The chances that each person sticks to the diet are 0.4, 0.9 and 0.6 respectively, independent of each other. What is the probability that:

a all three people stick to the diet
b only Ava and Chi stick to the diet
c at least one of the three people does not stick to the diet?

4 A box of toy blocks contains 10 red blocks and 5 yellow blocks. A child draws out two blocks at random.

a Draw the tree diagram if the sampling is with replacement and calculate the probability that one block of each colour is obtained.
b Draw the tree diagram if the sampling is without replacement and calculate the probability that one block of each colour is obtained.
c If the child was to draw out three blocks, rather than two, calculate the probability of obtaining three blocks of the same colour if the sampling is with replacement.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

5 Two events $A$ and $B$ are such that $\Pr(A) = 0.7$, $\Pr(B) = 0.8$ and $\Pr(A \cup B) = 0.94$. Determine whether $A$ and $B$ are independent.

6 Events $A$ and $B$ are such that $\Pr(A) = 0.75$, $\Pr(B) = 0.64$ and $\Pr(A \cup B) = 0.91$. Determine whether events $A$ and $B$ are independent.

7 A family owns two cars, A and B. Car A is used 65% of the time, car B is used 74% of the time and at least one of the cars is used 97% of the time. Determine whether the two cars are used independently.

8 The probability that a male is colourblind is 0.05 while the probability that a female is colourblind is 0.0025.

If there is an equal number of males and females in a population, find the probability that a person selected at random from the population is:

a female and colourblind
b colourblind given that the person is female
c male given that the person is colourblind.

If two people are chosen at random, find the probability that:

**d** both are colourblind males

**e** one is colourblind given a male and a female are chosen.

**9** A survey of 200 people was carried out to determine the number of traffic violations committed by different age groups. The results are shown in the table below.

| Age group | Number of violations 0 | 1 | 2 |
|---|---|---|---|
| Under 25 | 8 | 30 | 7 |
| 25–45 | 47 | 15 | 2 |
| 45–65 | 45 | 18 | 3 |
| 65+ | 20 | 5 | 0 |

If one person is selected at random from the group, find the probability that:

**a** the person belongs to the under-25 age group

**b** the person has had at least one traffic violation

**c** given that the person has had at least 1 traffic violation, he or she has had only 1 violation

**d** the person is 38 and has had no traffic violations

**e** the person is under 25, given that he/she has had 2 traffic violations.

**10** In an attempt to determine the efficacy of a test used to detect a particular disease, 100 subjects, of which 27 had the disease, were tested. A positive result means the test detected the disease and a negative result means the test did not detect the disease. Only 23 of the 30 people who tested positive actually had the disease. Draw up a two-way table to show this information and hence find the probability that a subject selected at random:

**a** does not have the disease

**b** tested positive but did not have the disease

**c** had the disease given that the subject tested positive

**d** did not have the disease, given that the subject tested negative

**11** Events $A$ and $B$ are independent. If $\Pr(B) = \frac{2}{3}$ and $\Pr(A|B) = \frac{4}{5}$, find:

**a** $\Pr(A)$ **b** $\Pr(B|A)$ **c** $\Pr(A \cap B)$ **d** $\Pr(A \cup B)$.

**12** Events $A$ and $B$ are such that $\Pr(B) = \frac{3}{5}$, $\Pr(A|B) = \frac{1}{3}$ and $\Pr(A \cup B) = \frac{23}{30}$.

**a** Find $\Pr(A \cap B)$. **b** Find $\Pr(A)$.

**c** Determine whether events $A$ and $B$ are independent.

**13** Events $A$ and $B$ are independent such that $\Pr(A \cup B) = 0.8$ and $\Pr(A|B') = 0.6$. Find $\Pr(B)$.

**14** 600 people were surveyed about whether they watched movies on their TV or on their devices. They are classified according to age, with the following results.

| | Age 15 to 30 | Age 30 to 70 | |
|---|---|---|---|
| TV | 95 | 175 | 270 |
| Device | 195 | 135 | 330 |
| | 290 | 310 | 600 |

Based on these findings, is the device used to watch movies independent of a person's age?

MASTER

15 A popular fast-food restaurant has studied the customer service provided by a sample of 100 of its employees across Australia. They wanted to know if employees who were with the company longer received more positive feedback from customers than newer employees.

The results of their study are shown below.

| | Positive customer feedback ($P$) | Negative customer feedback ($P'$) | |
|---|---|---|---|
| Employed 2 years or more ($T$) | 34 | 16 | 50 |
| Employed fewer than 2 years ($T'$) | 22 | 26 | 50 |
| | 58 | 42 | 100 |

Are the events $P$ and $T$ independent? Justify your answer.

16 Roll two fair dice and record the number uppermost on each. Let $A$ be the event of rolling a 6 on one die, $B$ be the event of rolling a 3 on the other, and $C$ be the event of the product of the numbers on the dice being at least 20.

a Are the events $A$ and $B$ independent? Explain.
b Are the events $B$ and $C$ independent? Explain.
c Are the events $A$ and $C$ independent? Explain.

## 8.5 Counting techniques

When calculating the probability of an event $A$ from the fundamental rule $\Pr(A) = \frac{n(A)}{n(\xi)}$, the number of elements in both $A$ and the sample space need to be able to be counted. Here we shall consider two counting techniques, one where order is important and one where order is not important. Respectively, these are called **arrangements** and **selections** or, alternatively, **permutations** and **combinations**.

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AOS 4
Topic 1
Concept 3

**Counting techniques**
Concept summary
Practice questions

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Counting techniques
**int-6293**

### Arrangements or permutations

The arrangement AB is a different arrangement to BA.

If two of three people designated by A, B and C are to be placed in a line, the possible arrangements are AB, BA, AC, CA, BC, CB. There are 6 possible arrangements or permutations.

Rather than list the possible arrangements, the number of possibilities could be calculated as follows using a **box table**.

| 3 | 2 |
|---|---|

↑

There are 3 people who can occupy the left position.

Once that position is filled this leaves 2 people who can occupy the remaining position.

Multiplying these figures together gives the total number of $3 \times 2 = 6$ arrangements. This is an illustration of the **multiplication principle**.

### Multiplication principle

If there are $m$ ways of doing the first procedure and for each one of these there are $n$ ways of doing the second procedure, then there are $m \times n$ ways of doing the first **and** the second procedures. This can be extended.

Suppose either two or three of four people A, B, C, D are to be arranged in a line. The possible arrangements can be calculated as follows:

Arrange two of the four people:

| | |
|---|---|
| 4 | 3 |

This gives $4 \times 3 = 12$ possible arrangements using the multiplication principle.

Arrange three of the four people:

| | | |
|---|---|---|
| 4 | 3 | 2 |

This gives $4 \times 3 \times 2 = 24$ possible arrangements using the multiplication principle.

The total number of arrangements of either two or three from the four people is $12 + 24 = 36$. This illustrates the **addition principle**.

### Addition principle for mutually exclusive events

For mutually exclusive procedures, if there are $m$ ways of doing one procedure and $n$ ways of doing another procedure, then there are $m + n$ ways of doing one **or** the other procedure.

**AND** $\times$ (Multiplication principle)

**OR** $+$ (Addition principle)

WORKED EXAMPLE 10

**Consider the set of five digits {2, 6, 7, 8, 9}.**

**Assume no repetition of digits in any one number can occur.**

**a How many three-digit numbers can be formed from this set?**

**b How many numbers with at least four digits can be formed?**

**c How many five-digit odd numbers can be formed?**

**d One of the five-digit numbers is chosen at random. What is the probability it will be an odd number?**

| THINK | WRITE |
|---|---|
| a 1 Draw a box table with three divisions. | a There are five choices for the first digit, leaving four choices for the second digit and then three choices for the third digit.<br>\| 5 \| 4 \| 3 \| |
| 2 Calculate the answer. | Using the multiplication principle, there are $5 \times 4 \times 3 = 60$ possible three-digit numbers that could be formed. |
| b 1 Interpret the event described. | b At least four digits means either four-digit or five-digit numbers are to be counted. |

2 Draw the appropriate box tables.

For four-digit numbers:

| | | | |
|---|---|---|---|
| 5 | 4 | 3 | 2 |

For five-digit numbers:

| | | | | |
|---|---|---|---|---|
| 5 | 4 | 3 | 2 | 1 |

3 Calculate the answer.

There are $5 \times 4 \times 3 \times 2 = 120$ four-digit numbers and $5 \times 4 \times 3 \times 2 \times 1 = 120$ five-digit numbers. Using the addition principle there are $120 + 120 = 240$ possible four- or five-digit numbers.

**c** 1 Draw the box table showing the requirement imposed on the number.

**c** For the number to be odd its last digit must be odd, so the number must end in either 7 or 9. This means there are two choices for the last digit.

| | | | | |
|---|---|---|---|---|
| | | | | 2 |

2 Complete the box table.

Once the last digit has been formed, there are four choices for the first digit then three choices for the second digit, two choices for the third digit and one choice for the fourth digit.

| | | | | |
|---|---|---|---|---|
| 4 | 3 | 2 | 1 | 2 |

3 Calculate the answer.

Using the multiplication principle, there are $4 \times 3 \times 2 \times 1 \times 2 = 48$ odd five-digit numbers possible.

**d** 1 Define the sample space and state $n(\xi)$.

**d** The sample space is the set of five-digit numbers. From part **b**, $n(\xi) = 120$.

2 State the number of elements in the required event.

Let $A$ be the event the five-digit number is odd. From part **c**, $n(A) = 48$.

3 Calculate the probability.

*Note:* The last digit must be odd. Of the five possible last digits, two are odd. Hence the probability the number is odd is $\frac{2}{5}$.

$$P(A) = \frac{n(A)}{n(\xi)} = \frac{48}{120} = \frac{2}{5}$$

The probability the five-digit number is odd is $\frac{2}{5}$.

## Factorial notation

The number of ways that four people can be arranged in a row can be calculated using the box table shown.

| | | | |
|---|---|---|---|
| 4 | 3 | 2 | 1 |

Using the multiplication principle, the total number of arrangements is $4 \times 3 \times 2 \times 1 = 24$. This can be expressed using factorial notation as 4!.

**In general, the number of ways of arranging $n$ objects in a row is $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$.**

## Arrangements in a circle

Consider arranging the letters A, B and C.

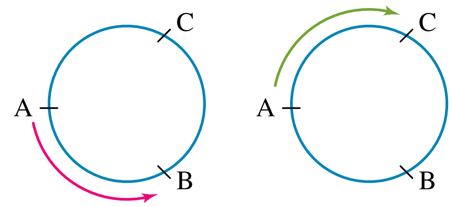
If the arrangements are in a row then there are $3! = 6$ arrangements.

In a row arrangement there is a first and a last position.

| | | |
|---|---|---|
| 3 | 2 | 1 |
| ↑ first | | ↑ last |

In a circular arrangement there is no first or last position; order is only created clockwise or anticlockwise from one letter once this letter is placed. This means ABC and BCA are the same circular arrangement, as the letters have the same anticlockwise order relative to A.

There are only two distinct circular arrangements of three letters: ABC or ACB as shown.




**$n!$ is the number of ways of arranging $n$ objects in a row.**

**$(n - 1)!$ is the number of ways of arranging $n$ objects in a circle.**

Three objects A, B and C arranged in a row in $3!$ ways; three objects arranged in a circle in $(3 - 1)! = 2!$ ways.

**WORKED EXAMPLE 11**

**A group of 7 students queue in a straight line at a canteen to buy a drink.**

**a** **In how many ways can the queue be formed?**

**b** **The students carry their drinks to a circular table. In how many different seating arrangements can the students sit around the table?**

**c** **This group of students have been shortlisted for the Mathematics, History and Art prizes.**

**What is the probability that one person in the group receives all three prizes?**

| THINK | WRITE |
|---|---|
| **a** 1 Use factorial notation to describe the number of arrangements. | **a** Seven people can arrange in a straight line in 7! ways. |
| 2 Calculate the answer. *Note:* A calculator could be used to evaluate the factorial. | Since $7! = 7 \times 6 \times 5!$ and $5! = 120$, $7! = 7 \times 6 \times 120$ $= 7 \times 720$ $= 5040$ There are 5040 ways in which the students can form the queue. |
| **b** 1 State the rule for circular arrangements. | **b** For circular arrangements, 7 people can be arranged in $(7 - 1)! = 6!$ ways. |

| | |
|---|---|
| **2** State the answer. | Since $6! = 720$, there are 720 different arrangements in which the 7 students may be seated. |
| **c 1** Form the number of elements in the sample space. | **c** There are three prizes. Each prize can be awarded to any one of the 7 students. <br> \| 7 \| 7 \| 7 \| <br> The total number of ways the prizes can be awarded is $7 \times 7 \times 7$. <br> $\therefore n(\xi) = 7 \times 7 \times 7$ |
| **2** Form the number of elements in the event under consideration. | Let $A$ be the event that the same student receives all three prizes. There are seven choices for that student. <br> $\therefore n(A) = 7$ |
| **3** Calculate the required probability. | $\Pr(A) = \dfrac{n(A)}{n(\xi)} = \dfrac{7}{7 \times 7 \times 7} = \dfrac{1}{49}$ <br> The probability that one student receives all three prizes is $\dfrac{1}{49}$. |

## Arrangements with objects grouped together

Where a group of objects are to be together, treat them as one unit in order to calculate the number of arrangements. Having done this, then allow for the number of internal rearrangements within the objects grouped together, and apply the multiplication principle.

For example, consider arranging the letters A, B, C and D with the restriction that ABC must be together.

Treating ABC as one unit would mean there are two objects to arrange: D and the unit (ABC).

Two objects arrange in 2! ways.

For each of these arrangements, the unit (ABC) can be internally arranged in 3! ways.

The multiplication principle then gives the total number of possible arrangements which satisfy the restriction is $2! \times 3!$ or $2 \times 6 = 12$ arrangements.

## Arrangements where some objects may be identical

The word SUM has three distinct letters. These letters can be arranged in 3! ways. The six arrangements can be listed as 3 pairs SUM and MUS, USM and UMS, SMU and MSU formed when the S and the M are interchanged.

Now consider the word MUM. Although this word also has three letters, two of the letters are identical. Interchanging the two M's will not create a new arrangement. SUM and MUS where the S and the M are interchanged are different, but MUM and MUM are the same.

There are only three arrangements: MUM, UMM and MMU.

The number of arrangements is $\dfrac{3!}{2!} = 3$.

This can be considered as cancelling out the rearrangements of the 2 identical letters from the total number of arrangements of 3 letters.

**The number of arrangements of $n$ objects, $p$ of which are of one type, $q$ of which are of another type, is $\dfrac{n!}{p!\,q!\ldots}$.**

WORKED EXAMPLE 12

**Consider the two words 'PARALLEL' and 'LINES'.**

**a** **How many arrangements of the letters of the word LINES have the vowels grouped together?**

**b** **How many arrangements of the letters of the word LINES have the vowels separated?**

**c** **How many arrangements of the letters of the word PARALLEL are possible?**

**d** **What is the probability that in a randomly chosen arrangement of the word PARALLEL, the letters A are together?**

| THINK | WRITE |
|---|---|
| **a** **1** Group the required letters together. | **a** There are two vowels in the word LINES. Treat these letters, I and E, as one unit. |
| **2** Arrange the unit of letters together with the remaining letters. | Now there are four groups to arrange: (IE), L, N, S. These arrange in 4! ways. |
| **3** Use the multiplication principle to allow for any internal rearrangements. | The unit (IE) can internally rearrange in 2! ways. Hence, the total number of arrangements is: $4! \times 2!$ $= 24 \times 2$ $= 48$ |
| **b** **1** State the method of approach to the problem. | **b** The number of arrangements with the vowels separated is equal to the total number of arrangements minus the number of arrangements with the vowels together. |
| **2** State the total number of arrangements. | The five letters of the word LINES can be arranged in $5! = 120$ ways. |
| **3** Calculate the answer. | From part **a**, there are 48 arrangements with the two vowels together. Therefore, there are $120 - 48 = 72$ arrangements in which the two vowels are separated. |
| **c** **1** Count the letters, stating any identical letters. | **c** The word PARALLEL contains 8 letters of which there are 2 A's and 3 L's. |

| | |
|---|---|
| **2** Using the rule $\frac{n!}{p!\,q!\,...}$ state the number of distinct arrangements. | There are $\frac{8!}{2! \times 3!}$ arrangements of the word PARALLEL. |
| **3** Calculate the answer. | $\frac{8!}{2! \times 3!} = \frac{8 \times 7 \times 6 \times 5 \times \not{4}^2 \times \not{3!}}{\not{2} \times \not{3!}}$ <br> $= 3360$ <br> There are 3360 arrangements. |
| **d 1** State the number of elements in the sample space. | **d** There are 3360 total arrangements of the word PARALLEL, so $n(\xi) = 3360$ or $\frac{8!}{2! \times 3!}$. |
| **2** Group the required letters together. | For the letters A to be together, treat these two letters as one unit. This creates seven groups (AA), P, R, L, L, E, L of which three are identical L's. |
| **3** Calculate the number of elements in the event. | The seven groups arrange in $\frac{7!}{3!}$ ways. As the unit (AA) contains two identical letters, there are no distinct internal rearrangements of this unit that need to be taken into account. <br> Hence $\frac{7!}{3!}$ is the number of elements in the event. |
| **4** Calculate the required probability. <br> *Note:* It helps to use factorial notation in the calculations. | The probability that the A's are together <br> $= \frac{\text{number of arrangements with the As together}}{\text{total number of arrangements}}$ <br> $= \frac{7!}{3!} \div \frac{8!}{2! \times 3!}$ <br> $= \frac{7!}{3!} \times \frac{2! \times 3!}{8 \times 7!}$ <br> $= \frac{2}{8}$ <br> $= \frac{1}{4}$ |

## Formula for permutations

The number of arrangements of $n$ objects taken $r$ at a time is shown in the box table.

| $n$ | $n-1$ | $n-2$ | .................................... | $n-r+1$ |
|---|---|---|---|---|
| ↑ | ↑ | ↑ | | ↑ |
| $1^{st}$ | $2^{nd}$ | $3^{rd}$ | ............................................ | $r^{th}$ object |

The number of arrangements equals $n(n-1)(n-2)\ldots(n-r+1)$.

This can be expressed using factorial notation as:

$$n(n-1)(n-2)\ldots(n-r+1)$$

$$= n(n-1)(n-2)\ldots(n-r+1) \times \frac{(n-r)(n-r-1) \times \ldots 2 \times 1}{(n-r)(n-r-1) \times \ldots 2 \times 1}$$

$$= \frac{n!}{(n-r)!}$$

The formula for the number of permutations or arrangements of $n$ objects taken $r$ at a time is ${}^{n}P_{r} = \frac{n!}{(n-r)!}$. Although we have preferred to use a box table, it is possible to count arrangements using this formula.

## Combinations or selections

Now we shall consider the counting technique for situations where order is unimportant. This is the situation where the selection AB is the same as the selection BA. For example the entry Alan and Bev is no different to the entry Bev and Alan as a pair of mixed doubles players in a tennis match: they are the same entry.

The number of combinations of $r$ objects from a total group of $n$ distinct objects is calculated by counting the number of arrangements of the objects $r$ at a time and then dividing that by the number of ways each group of these $r$ objects can rearrange between themselves. This is done in order to cancel out counting these as different selections.

The number of combinations is therefore $\frac{{}^{n}P_{r}}{r!} = \frac{n!}{(n-r)!r!}$.

The symbol for the number of ways of choosing $r$ objects from a total of $n$ objects is ${}^{n}C_{r}$ or $\binom{n}{r}$.

**The number of combinations of $r$ objects from a total of $n$ objects is ${}^{n}C_{r} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$, $0 \le r \le n$ where $r$ and $n$ are non-negative integers.**

The formula for ${}^{n}C_{r}$ is exactly that for the binomial coefficients used in the binomial theorem. These values are the combinatoric terms in Pascal's triangle as encountered in an earlier chapter.

Drawing on that knowledge, we have:

- ${}^{n}C_{0} = 1 = {}^{n}C_{n}$, there being only one way to choose none or all of the $n$ objects.
- ${}^{n}C_{1} = n$, there being $n$ ways to choose one object from a group of $n$ objects.
- ${}^{n}C_{r} = {}^{n}C_{n-r}$ since choosing $r$ objects must leave behind a group of $(n-r)$ objects and vice versa.

### Calculations

The formula is always used for calculations in selection problems. Most calculators have a ${}^{n}C_{r}$ key to assist with the evaluation when the figures become large.

Both the multiplication and addition principles apply and are used in the same way as for arrangements.

The calculation of probabilities from the rule $\Pr(A) = \dfrac{n(A)}{n(\xi)}$ requires that the same counting technique used for the numerator is also used for the denominator. We have seen for arrangements that it can assist calculations to express numerator and denominator in terms of factorials and then simplify. Similarly for selections, express the numerator and denominator in terms of the appropriate combinatoric coefficients and then carry out the calculations.

**WORKED EXAMPLE 13**

**A committee of 5 students is to be chosen from 7 boys and 4 girls.**

**a** **How many committees can be formed?**

**b** **How many of the committees contain exactly 2 boys and 3 girls?**

**c** **How many committees have at least 3 girls?**

**d** **What is the probability of the oldest and youngest students both being on the committee?**

**THINK**

**a** 1 As there is no restriction, choose the committee from the total number of students.

2 Use the formula ${}^nC_r = \dfrac{n!}{r! \times (n-r)!}$ to calculate the answer.

**b** 1 Select the committee to satisfy the given restriction.

2 Use the multiplication principle to form the total number of committees.

*Note:* The upper numbers on the combinatoric coefficients sum to the total available, $7 + 4 = 11$, while the lower numbers sum to the number that must be on the committee, $2 + 3 = 5$.

**WRITE**

**a** There are 11 students in total from whom 5 students are to be chosen. This can be done in ${}^{11}C_5$ ways.

$$
\begin{aligned}
{}^{11}C_5 &= \frac{11!}{5! \times (11-5)!} \\
&= \frac{11!}{5! \times 6!} \\
&= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{5! \times 6!} \\
&= \frac{11 \times \cancel{10}^{2} \times \cancel{9 \times 8}^{3} \times 7}{\cancel{5} \times \cancel{4 \times 3 \times 2} \times 1} \\
&= 462
\end{aligned}
$$

There are 462 possible committees.

**b** The 2 boys can be chosen from the 7 boys available in ${}^7C_2$ ways.
The 3 girls can be chosen from the 4 girls available in ${}^4C_3$ ways.

The total number of committees which contain two boys and three girls is ${}^7C_2 \times {}^4C_3$.

**3** Calculate the answer.

$$
\begin{aligned}
{}^{7}C_2 \times {}^{4}C_3 &= \frac{7!}{2! \times 5!} \times 4 \\
&= \frac{7 \times 6}{2!} \times 4 \\
&= 21 \times 4 \\
&= 84
\end{aligned}
$$

There are 84 committees possible with the given restriction.

**c 1** List the possible committees which satisfy the given restriction.

**c** As there are 4 girls available, at least 3 girls means either 3 or 4 girls.
The committees of 5 students which satisfy this restriction have either 3 girls and 2 boys, or they have 4 girls and 1 boy.

**2** Write the number of committees in terms of combinatoric coefficients.

3 girls and 2 boys are chosen in ${}^{4}C_3 \times {}^{7}C_2$ ways.
4 girls and 1 boy are chosen in ${}^{4}C_4 \times {}^{7}C_1$ ways.

**3** Use the addition principle to state the total number of committees.

The number of committees with at least three girls is ${}^{4}C_3 \times {}^{7}C_2 + {}^{4}C_4 \times {}^{7}C_1$.

**4** Calculate the answer.

$$
\begin{aligned}
{}^{4}C_3 \times {}^{7}C_2 + {}^{4}C_4 \times {}^{7}C_1 &= 84 + 1 \times 7 \\
&= 91
\end{aligned}
$$

There are 91 committees with at least 3 girls.

**d 1** State the number in the sample space.

**d** The total number of committees of 5 students is ${}^{11}C_5 = 462$ from part **a**.

**2** Form the number of ways the given event can occur.

Each committee must have 5 students. If the oldest and youngest students are placed on the committee, then 3 more students need to be selected from the remaining 9 students to form the committee of 5. This can be done in ${}^{9}C_3$ ways.

**3** State the probability in terms of combinatoric coefficients.

Let $A$ be the event the oldest and the youngest students are on the committee.

$$
\begin{aligned}
\Pr(A) &= \frac{n(A)}{n(\xi)} \\
&= \frac{{}^{9}C_3}{{}^{11}C_5}
\end{aligned}
$$

4 Calculate the answer.

$$\Pr(A) = \frac{9!}{3! \times 6!} \div \frac{11!}{5! \times 6!}$$
$$= \frac{9!}{3! \times 6!} \times \frac{5! \times 6!}{11!}$$
$$= \frac{1}{3!} \times \frac{5!}{11 \times 10}$$
$$= \frac{5 \times 4}{110}$$
$$= \frac{2}{11}$$

The probability of the committee containing the youngest and the oldest students is $\frac{2}{11}$.

## EXERCISE 8.5 Counting techniques

**PRACTISE**

Work without CAS

1 WE10 Consider the set of five digits $\{3, 5, 6, 7, 9\}$.
Assume no repetition of digits in any one number can occur.
- a How many four-digit numbers can be formed from this set?
- b How many numbers with at least three digits can be formed?
- c How many five-digit even numbers can be formed?
- d One of the five-digit numbers is chosen at random. What is the probability it will be an even number?

2 A car's number plate consists of two letters of the English alphabet followed by three of the digits 0 to 9, followed by one single letter. Repetition of letters and digits is allowed.
- a How many such number plates are possible?
- b How many of the number plates use the letter X exactly once?
- c What is the probability the first two letters are identical and all three numbers are the same, but the single letter differs from the other two?

3 WE11 A group of six students queue in a straight line to take out books from the library.
- a In how many ways can the queue be formed?
- b The students carry their books to a circular table. In how many different seating arrangements can the students sit around the table?
- c This group of students have been shortlisted for the Mathematics, History and Art prizes. What is the probability that one person in the group receives all three prizes?

4 A group of four boys seat themselves in a circle.
- a How many different seating arrangements of the boys are possible?
- b Four girls join the boys and seat themselves around the circle so that each girl is sat between two boys. In how many ways can the girls be seated?

5 WE12 Consider the words SIMULTANEOUS and EQUATIONS.
- a How many arrangements of the letters of the word EQUATIONS have the letters Q and U grouped together?
- b How many arrangements of the letters of the word EQUATIONS have the letters Q and U separated?

c How many arrangements of the letters of the word SIMULTANEOUS are possible?

d What is the probability that in a randomly chosen arrangement of the word SIMULTANEOUS, both the letters U are together?

6 In how many ways can the thirteen letters PARALLEL LINES be arranged:

a in a row

b in a circle

c in a row with the vowels together?

7 WE13 A committee of 5 students is to be chosen from 6 boys and 8 girls.

a How many committees can be formed?

b How many of the committees contain exactly 2 boys and 3 girls?

c How many committees have at least 4 boys?

d What is the probability of neither the oldest nor the youngest student being on the committee?

8 A cricket team of eleven players is to be selected from a list of 3 wicketkeepers, 6 bowlers and 8 batsmen. What is the probability the team chosen consists of one wicketkeeper, four bowlers and six batsmen?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 a Baby Amelie has 10 different bibs and 12 different body suits. How many different combinations of bib and body suit can she wear?

b On her way to work each morning, Christine has the option of taking the motorway or the highway. She then must travel through some suburban streets to get to work. She has the option of three different routes through the suburban streets. If Christine wishes to take a different route to work each day, on how many days will she be able to take a different route before she must use a route already travelled?

c When selecting his new car, Abdul has the option of a manual or an automatic. He is also offered a choice of 5 exterior colours, leather or vinyl seats, 3 interior colours and the options of individual seat heating and self-parking. How many different combinations of new car can Abdul choose from?


d Sarah has a choice of 3 hats, 2 pairs of sunglasses, 7 T-shirts and 5 pairs of shorts. When going out in the sun, she chooses one of each of these items to wear. How many different combinations are possible?

e In order to start a particular game, each player must roll an unbiased die, then select a card from a standard pack of 52. How many different starting combinations are possible?

f On a recent bushwalking trip, a group of friends had a choice of travelling by car, bus or train to the Blue Mountains. They decided to walk to one of the waterfalls, then a mountain-top scenic view. How many different trips were possible if there are 6 different waterfalls and 12 mountain-top scenic views?

10 a Registration plates on a vehicle consist of 2 letters followed by 2 digits followed by another letter. How many different number plates are possible if repetitions are allowed?

b How many five-letter words can be formed using the letters B, C, D, E, G, I and M if repetitions are allowed?

c A die is rolled three times. How many possible outcomes are there?

d How many three-digit numbers can be formed using the digits 2, 3, 4, 5, 6, 7 if repetitions are allowed?

e Three friends on holidays decide to stay at a hotel which has 4 rooms available. In how many ways can the rooms be allocated if there are no restrictions and each person has their own room?

11 Eight people, consisting of 4 boys and 4 girls, are to be arranged in a row. Find the number of ways this can be done if:

a there are no restrictions

b the boys and girls are to alternate

c the end seats must be occupied by a girl

d the brother and sister must not sit together

e the girls must sit together.

12 How many three-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 given that no repetitions are allowed, the number cannot start with 0, and the following condition holds:

a there are no other restrictions

b the number must be even

c the number must be less than 400

d the number is made up of odd digits only?

13 In how many ways can 6 men and 3 women be arranged at a circular table if:

a there are no restrictions

b the men can only be seated in pairs?

14 a A set of 12 mugs that are identical except for the colour are to be placed on a shelf. In how many ways can this be done if 4 of the mugs are blue, 3 are orange and 5 are green?

b In how many ways can the 12 mugs from part **a** be arranged in 2 rows of 6 if the green ones must be in the front row?

15 How many words can be formed from the letters of the word BANANAS given the following conditions?

a All the letters are used.

b A four-letter word is to be used including at least one A.

c A four-letter word using all different letters is to be used.

16 a In how many ways can 7 men be selected from a group of 15 men?

b How many five-card hands can be dealt from a standard pack of 52 cards?

c How many five-card hands that contain all 4 aces, can be dealt from a standard pack of 52 cards?

d In how many ways can 3 prime numbers be selected from the set containing the first 10 prime numbers?

17 A panel of 8 is to be selected from a group of 8 men and 10 women. Find how many panels can be formed if:

a there are no restrictions

b there are 5 men and 3 women on the panel

c there are at least 6 men on the panel

d two particular men cannot both be included

e a particular man and woman must both be included.

18 Consider the universal set $S = \{2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 15, 17\}$.

a How many subsets of $S$ are there?

b Determine the number of subsets whose elements are all even numbers.

**c** What is the probability that a subset selected at random will contain only even numbers?

**d** Find the probability that a subset selected at random will contain at least 3 elements.

**e** Find the probability that a subset selected at random will contain exactly 3 prime numbers.

MASTER

**19** A representative sport committee consisting of 7 members is to be formed from 21 tennis players, 17 squash players and 18 badminton players. Find the probability that the committee will contain:

**a** 7 squash players
**b** at least 5 tennis players
**c** at least one representative from each sport
**d** exactly 3 badminton players, given that it contains at least 1 badminton player.

**20** One three-digit number is selected at random from all the possible three-digit numbers. Find the probability that:

**a** the digits are all primes
**b** the number has just a single repeated digit
**c** the digits are perfect squares
**d** there are no repeated digits
**e** the number lies between 300 and 400, given that the number is greater than 200.

# 8.6 Binomial coefficients and Pascal's triangle

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**Binomial coefficients and Pascal's triangle**
Concept summary
Practice questions

## Binomial coefficients

In this section the link between counting techniques and the **binomial coefficients** will be explored.

Consider the following example.

A coin is biased in such a way that the probability of tossing a Head is 0.6. The coin is tossed three times. The outcomes and their probabilities are shown below.

$\Pr(TTT) = (0.4)^3$ or $\Pr(0H) = (0.4)^3$

$$\left.\begin{array}{l}\Pr(HTT) = (0.6)(0.4)^2\\ \Pr(THT) = (0.6)(0.4)^2\\ \Pr(TTH) = (0.6)(0.4)^2\end{array}\right\} \rightarrow \Pr(1H) = \binom{3}{1}(0.6)(0.4)^2 \text{ or } \Pr(1H) = 3(0.6)(0.4)^2$$

$$\left.\begin{array}{l}\Pr(HHT) = (0.6)^2(0.4)\\ \Pr(HTH) = (0.6)^2(0.4)\\ \Pr(THH) = (0.6)^2(0.4)\end{array}\right\} \rightarrow \Pr(2H) = \binom{3}{2}(0.6)^2(0.4) \text{ or } \Pr(2H) = 3(0.6)^2(0.4)$$

$\Pr(HHH) = 0.6^3 \rightarrow \Pr(3H) = 0.6^3$

Since this represents the sample space, the sum of these probabilities is 1.

So we have $\Pr(0H) + \Pr(1H) + \Pr(2H) + \Pr(3H) = 1$

$$\binom{3}{0}(0.4)^3 + \binom{3}{1}(0.4)^2(0.6) + \binom{3}{2}(0.4)(0.6)^2 + \binom{3}{3}0.6^3 = 1$$

$$\rightarrow 1 = (0.4)^3 + 3(0.4)^2(0.6) + 3(0.4)(0.6)^2 + 0.6^3$$

Note that $\binom{3}{2}$, which is the number of ways in which we can get 2 Heads from 3 tosses, is also the number of ordered selections of 3 objects where 2 are alike of one type (Heads) and 1 of another (Tails), as was noted previously.

Now compare the following expansion:

$$(q + p)^3 = q^3 + 3p^2q + 3pq^2 + p^3$$

We see that if we let $p = 0.6$ (the probability of getting a Head on our biased die) and $q = 0.4$ (the probability of not getting a Head), we have identical expressions. Hence:

$$\begin{aligned}(0.4 + 0.6)^3 &= (0.4)^3 + 3(0.4)^2(0.6) + 3(0.4)(0.6)^2 + 0.6^3 \\ &= \binom{3}{0}(0.4)^3 + \binom{3}{1}(0.4)^2(0.6) + \binom{3}{2}(0.4)(0.6)^2 + \binom{3}{3}0.6^3\end{aligned}$$

So the coefficients in the binomial expansion are equal to the number of ordered selections of 3 objects of just 2 types.

If we were to toss the coin $n$ times, with $p$ being the probability of a Head and $q$ being the probability of not Head (Tail), we have the following.

The probability of 0 Heads ($n$ Tails) in $n$ tosses is $\binom{n}{0}q^n$

The probability of 1 Head ($n - 1$ Tails) in $n$ tosses is $\binom{n}{1}q^{n-1}p$

The probability of 2 Heads ($n - 2$ Tails) in $n$ tosses is $\binom{n}{2}q^{n-2}p^2$

These probabilities are known as **binomial probabilities**.

In general, the probability of getting $r$ favourable and $n - r$ non-favourable outcomes, in $n$ repetitions of an experiment, is $\binom{n}{r}q^{n-r}p^r$, where $\binom{n}{r}$ is the number of ways of getting $r$ favourable outcomes.

So again equating the sum of all the probabilities in the sample space to the binomial expansion of $(q + p)^n$, we have:

$$(q + p)^n = \binom{n}{0}q^n + \binom{n}{1}q^{n-1}p + \binom{n}{2}q^{n-2}p^2 + \ldots + \binom{n}{r}q^{n-r}p^r + \ldots \binom{n}{n-1}qp^{n-1} + \binom{n}{n}p^n$$

Hence, generalising, in the binomial expansion of $(a + b)^n$, the coefficient of $a^{n-r}b^r$ is $\binom{n}{r}$, where $\binom{n}{r}$ is the number of ordered selections of $n$ objects, in which $r$ are alike of one type and $n - r$ are alike of another type.

We have:

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

This can be written using sigma notation as:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

We now have a simple formula for calculating binomial coefficients.

Important features to note in the expansion are:

1. The powers of $a$ decrease as the powers of $b$ increase.
2. The sum of the powers of $a$ and $b$ for each term in the expansion is equal to $n$.
3. As we would expect, $\binom{n}{0} = \binom{n}{n} = 1$ since there is only 1 way of selecting none, or of selecting all $n$ objects.
4. If we look at the special case of

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

then $\binom{n}{r}$, or ${}^nC_r$, is the coefficient of $x^r$.

**WORKED EXAMPLE 14**

**Write down the expansion of the following.**

**a** $(p+q)^4$ **b** $(x-y)^3$ **c** $(2m+5)^3$

| THINK | WRITE |
|---|---|
| **a 1** Determine the values of $n$, $a$ and $b$ in the expansion of $(a+b)^n$. | **a** $n = 4; a = p; b = q$ |
| **2** Use the formula to write the expansion and simplify. | $(p+q)^4 = \binom{4}{0}p^4 + \binom{4}{1}p^3q + \binom{4}{2}p^2q^2 + \binom{4}{3}pq^3 + \binom{4}{4}p^4$ <br> $= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + p^4$ |
| **b 1** Determine the values of $n$, $a$ and $b$ in the expansion of $(a+b)^n$. | **b** $n = 3; a = x; b = -y$ |
| **2** Use the formula to write the expansion and simplify. | $(x-y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2(-y) + \binom{3}{2}x(-y)^2 + \binom{3}{3}(-y)^3$ <br> $= x^3 - 3x^2y + 3xy^2 - y^3$ |
| **c 1** Determine the values of $n$, $a$ and $b$ in the expansion of $(a+b)^n$. | **c** $n = 3; a = 2m; b = 5$ |
| **2** Use the formula to write the expansion and simplify. | $(2m+5)^3 = \binom{3}{0}(2m)^3 + \binom{3}{1}(2m)^2(5) + \binom{3}{2}(2m)(5)^2 + \binom{3}{3}(5)^3$ <br> $= 8m^3 + 60m^2 + 150m + 125$ |

## Sigma notation

The expansion of $(1+x)^n$ can also be expressed in sigma notation as

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

**WORKED EXAMPLE 15**

Find each of the following in the expansion of $\left(3x^2 - \dfrac{1}{x}\right)^6$.

**a** The term independent of $x$ **b** The term in $x^{-6}$

| THINK | WRITE |
|---|---|
| **a 1** Express the expansion in sigma notation. | **a** $\left(3x^2 - \frac{1}{x}\right)^6 = \sum_{r=0}^{6} \binom{6}{r} (3x^2)^{6-r}\left(-\frac{1}{x}\right)^r$ |
| **2** Simplify by collecting powers of $x$. | $\left(3x^2 - \frac{1}{x}\right)^6 = \sum_{r=0}^{6} (-1)^r \binom{6}{r} 3^{6-r} x^{12-2r} x^{-r}$ <br> $= \sum_{r=0}^{6} (-1)^r \binom{6}{r} 3^{6-r} x^{12-3r}$ |
| **3** For the term independent of $x$, we need the power of $x$ to be 0. | $12 - 3r = 0$ <br> $r = 4$ |
| **4** Substitute the value of $r$ in the expression. | $(-1)^r \binom{6}{r} 3^{6-r} x^{12-3r} = (-1)^4 \binom{6}{4} 3^2$ <br> $= 135$ |
| **5** Answer the question. | The 5th term, 135, is independent of $x$. |
| **b 1** For the term in $x^{-6}$, we need to make the power of $x$ equal to $-6$. | **b** $12 - 3r = -6$ <br> $r = 6$ |
| **2** Substitute the value of $r$ in the expression. | $(-1)^r \binom{6}{r} 3^{6-r} x^{12-3r} = (-1)^6 \binom{6}{6} 3^0 x^{-6}$ <br> $= x^{-6}$ |
| **3** Answer the question. | The 7th term is $x^{-6}$. |

## Pascal's triangle

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**Interactivity**
Pascal's triangle
**int-2554**

We saw earlier that the coefficient of $p^{n-r}q^r$ in the binomial expansion of $(p + q)^n$ is $\binom{n}{r}$, the number of ordered selections of $n$ objects, in which $r$ are alike of one type and $n - r$ are alike of another type. These coefficients reappear in Pascal's triangle, shown in the diagram.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Compare the following expansions:

| | | |
|---|---|---|
| $(p+q)^0 = 1$ | Coefficient: 1 | $\binom{0}{0}$ |
| $(p+q)^1 = p + q$ | Coefficients: 1 1 | $\binom{1}{0}, \binom{1}{1}$ |
| $(p+q)^2 = p^2 + 2pq + q^2$ | Coefficients: 1 2 1 | $\binom{2}{0}, \binom{2}{1}, \binom{2}{2}$ |
| $(q+p)^3 = q^3 + 3p^2q + 3pq^2 + p^3$ | Coefficients: 1 3 3 1 | $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$ |
| $(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$ | Coefficients: 1 4 6 4 1 | $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$ |

The coefficients in the binomial expansion are equal to the numbers in Pascal's triangle.
Hence the triangle can be written using the $\binom{n}{r}$ or ${}^nC_r$ notation as follows:

$n = 0$: ${}^0C_0$ 1

$n = 1$: ${}^1C_0$ ${}^1C_1$ 1 1

$n = 2$: ${}^2C_0$ ${}^2C_1$ ${}^2C_2$ 1 2 1

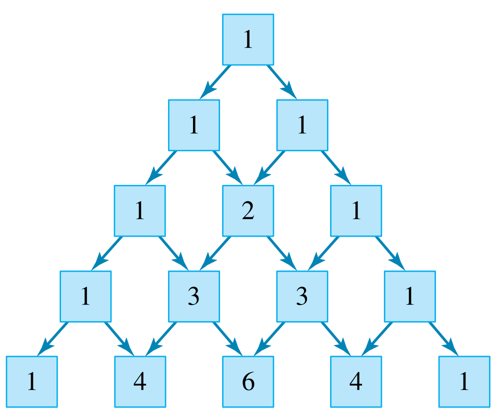
$n = 3$: ${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$ 1 3 3 1

$n = 4$: ${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$ 1 4 6 4 1

$n = 5$: ${}^5C_0$ ${}^5C_1$ ${}^5C_2$ ${}^5C_3$ ${}^5C_4$ ${}^5C_5$ 1 5 10 10 5 1

Note that the first and last number in each row is always 1.

Each coefficient is obtained by adding the two coefficients immediately above it.




This gives **Pascal's identity**:

$$\mathbf{{}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad \text{for } 0 < r < n}$$

WORKED EXAMPLE 16

**a** Write down the notation for the eighth coefficient in the 13th row of Pascal's triangle.

**b** Calculate the value of $\binom{7}{5}$.

**c** Verify that $\binom{16}{5} = \binom{16}{11}$.

**d** Show that ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$.

| THINK | WRITE |
|---|---|
| **a** 1 Write down the values of $n$ and $r$. Remember, numbering starts at 0. | **a** $n = 13$ and $r = 7$ |
| 2 Substitute into the notation. | 8th coefficient in the 13th row is ${}^{13}C_{7}$. |
| **b** 1 Use the formula. | **b** $\binom{7}{5} = \frac{7!}{(7-5)!5!} = \frac{7 \times 6}{2!} = 21$ |
| 2 Verify using a calculator. | ${}^{7}C_{5} = 21$ |
| **c** 1 Use the formula to express one side. | **c** $\binom{16}{11} = \frac{16!}{(16-11)!11!} = \frac{16!}{5!11!} = \frac{16!}{11!5!} = \binom{16}{11}$ |
| 2 Verify your answer by evaluating each side. | $\binom{16}{5} = 4368$ <br> $\binom{16}{11} = 4368$ |
| **d** 1 Use the formula to express the left-hand side. | **d** ${}^{n}C_{r} + {}^{n}C_{r+1} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r-1)!(r+1)!}$ |
| 2 Simplify the expression. | $= n!\left[\frac{1}{(n-r)!r!} + \frac{1}{(n-r-1)!(r+1)!}\right]$ <br> $= n!\left[\frac{r+1+n-r}{(n-r)!(r+1)!}\right]$ <br> $= n!\left[\frac{n+1}{(n-r)!(r+1)!}\right]$ <br> $= \frac{(n+1)!}{(n-r)!(r+1)!}$ <br> $= {}^{n+1}C_{r+1}$ |

## EXERCISE 8.6 Binomial coefficients and Pascal's triangle

**PRACTISE**

Work without CAS

1 WE14 Write down the expansion of each of the following.

a $(a+b)^4$ b $(2+x)^4$ c $(t-2)^3$

2 Write down the expansion of each of the following.

a $(m+3b)^2$ b $(2d-x)^4$ c $\left(h+\frac{2}{h}\right)^3$

3 WE15 a Find the term independent of $x$ in the expansion of $\left(x+\frac{1}{2x}\right)^4$.

b Find the term in $m^2$ in the expansion of $\left(2m-\frac{1}{3m}\right)^6$.

4 a Find the 5th term in the expansion of $(p+3q)^7$.

b Find the value of $k$ in the expansion of $(x^2+ky)^5$ if the 4th term is $\frac{5}{4}x^4y^3$.

5 WE16 a Write down the notation for the 4th coefficient in the 7th row of Pascal's triangle.

b Evaluate $\binom{9}{6}$.

c Show that $\binom{18}{12}=\binom{18}{6}$.

d Show that $\binom{15}{7}+\binom{15}{6}=\binom{16}{7}$.

6 a Verify that $\binom{22}{8}=\binom{22}{14}$.

b Express $\binom{19}{15}+\binom{19}{14}$ in simplified ${}^{n}C_{r}$ form.

c Show that ${}^{n+1}C_{r-1}+{}^{n+1}C_{r}={}^{n+2}C_{r}$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 Expand and simplify each of the following.

a $(x+y)^3$ b $(a+2)^4$ c $(m-3)^4$ d $(2-x)^5$

8 Expand and simplify each of the following.

a $\left(1-\frac{2}{x}\right)^3$ b $\left(1+\frac{p}{q}\right)^4$ c $\left(3-\frac{m}{2}\right)^4$ d $\left(2x-\frac{1}{x}\right)^3$

9 For each of the following find the term specified in the expansion.

a The 3rd term in $(2w-3)^5$

b The 5th term in $\left(3-\frac{1}{b}\right)^7$

c The constant term in $\left(y-\frac{3}{y}\right)^4$

10 For each of the following find the term specified in the expansion.

a The 6th term in $(2b+3d)^5$

b The coefficient of the term $x^2y^3$ in $(3x-5y)^5$

11 Find integers $a$ and $b$ such that $(2-\sqrt{5})^4=a+b\sqrt{5}$.

12 Simplify $1-6m+15m^2-20m^3+15m^4-6m^5+m^6$.

13 Simplify $(1 - x)^4 - 4(1 - x)^3 + 6(1 - x)^2 - 4(1 - x) + 1$.

14 Find the term independent of $y$ in the expansion of $\left(2 - \frac{1}{y}\right)^3 (1 + 2y)^5$.

15 Find the values of $k$ in the expansion of $\left(3m^2 + \frac{k}{m}\right)^6$, given that the term independent of $m$ is 2160.

16 The ratio of the coefficients of the 4th and 5th terms in the expansion of $(ax + 2y)^5$ is 3 : 1. Find the value of $a$.

MASTER

17 The first 3 terms in the expansion of $(1 + kx)^n$ are 1, $2x$ and $\frac{3}{2}x^2$. Find the values of $k$ and $n$.

18 a Three consecutive terms of Pascal's triangle are in the ratio 13 : 8 : 4. Find the three terms.

b Find the values of $a$ and $b$ given $(3 + \sqrt{2})^9 = a + b\sqrt{2}$.

c In the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, the coefficient of $x^3$ is 70. Find the value of $n$.

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# 8.7 Review

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**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

## REVIEW QUESTIONS

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## Activities

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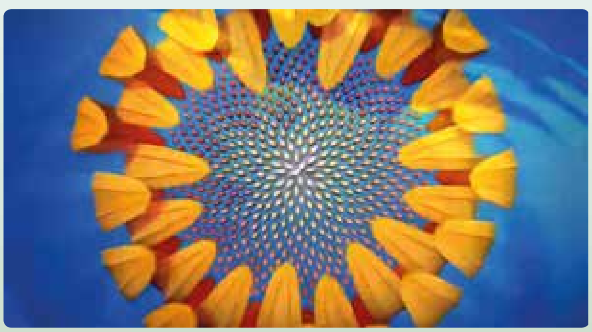
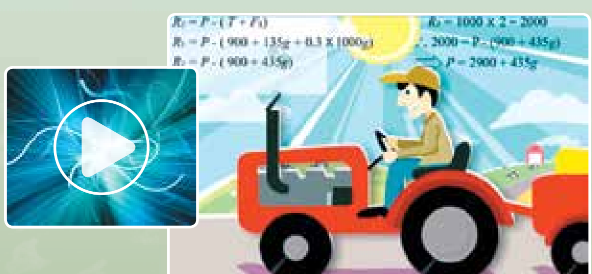
### Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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# 8 Answers

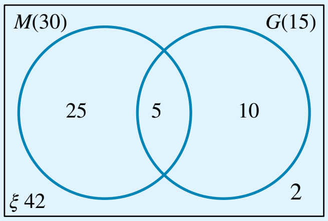
## EXERCISE 8.2

**1 a** $\frac{1}{16}$    **b** $\frac{3}{4}, \frac{1}{4}$

**2 a i** $\frac{3}{4}$    **ii** $\frac{7}{10}$    **iii** $\frac{1}{4}$

**b** 8 additional red balls

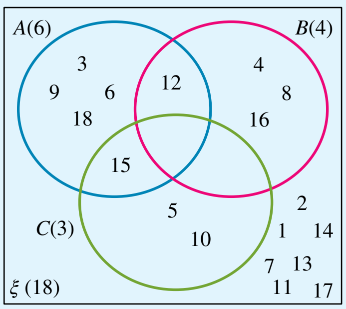
**3 a**




**b** $\frac{25}{42}$    **c** $\frac{1}{21}$

**d** $\frac{35}{42} = \frac{5}{6}$

**4 a** $\xi = \{1, 2, 3, ..., 18\}$, $A = \{3, 6, 9, 12, 15, 18\}$, $B = \{4, 8, 12, 16\}$ and $C = \{5, 10, 15\}$




**b** $B$ and $C$    **c** $\frac{1}{3}$

**d i** $\frac{4}{9}$    **ii** $\frac{5}{18}$    **iii** $\frac{7}{18}$

**5 a**

| | $B$ | $B'$ | |
|---|---|---|---|
| $A$ | 0.35 | 0.3 | 0.65 |
| $A'$ | 0.15 | 0.2 | 0.35 |
| | 0.5 | 0.5 | 1 |

**b** 0.85

**6 a**

| | $B$ | $B'$ | |
|---|---|---|---|
| $A$ | 0.38 | 0.2 | 0.58 |
| $A'$ | 0.17 | 0.25 | 0.42 |
| | 0.55 | 0.45 | 1 |

**b** 0.25

**c** $\Pr(A \cup B)' = 0.25$

$\Pr(A' \cap B') = 0.25$

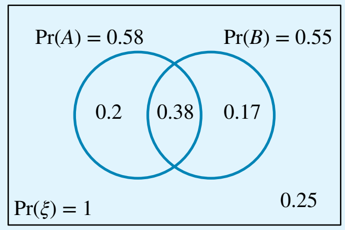
$\Rightarrow \Pr(A \cup B)' = \Pr(A' \cap B')$

**d** From the probability table, $\Pr(A \cap B) = 0.38$.

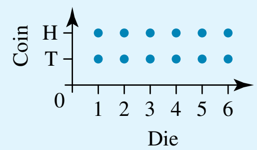
$$\begin{aligned} 1 - \Pr(A' \cup B') &= 1 - [\Pr(A') + \Pr(B') - \Pr(A' \cap B')] \\ &= 1 - 0.42 - 0.45 + 0.25 \\ &= 0.38 \end{aligned}$$

So, $\Pr(A \cap B) = 1 - \Pr(A' \cup B')$.

**e** $\text{pr}(\xi) = 1$

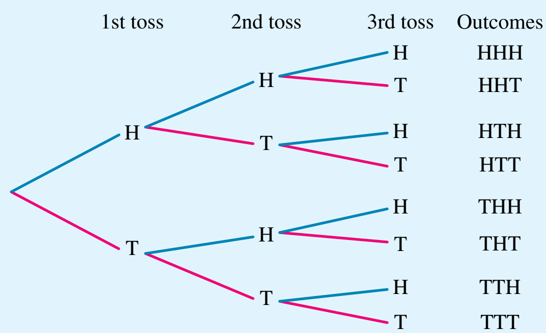



**7 a**




**b** $\frac{1}{12}$    **c** $\frac{1}{4}$    **d** $\frac{7}{13}$

**8 a**




**b** $\frac{7}{8}$    **c** $\frac{3}{4}$

**9 a** $\frac{5}{23}$    **b** $\frac{12}{23}$

**c** $\frac{15}{23}$    **d** 0

**10 a** $\frac{1}{200}$    **b** $\frac{199}{39\,980}$    **c** $\frac{1}{11\,094\,450}$

**11 a** 1    **b** $\frac{1}{2}$    **c** $\frac{1}{4}$

**d** $\frac{7}{13}$    **e** $\frac{12}{13}$

**12 a** $\frac{2}{9}$    **b** $\frac{1}{36}$    **c** $\frac{7}{18}$

**d** 0    **e** $\frac{1}{12}$    **f** 1

13 a $\frac{3}{8}$ b $\frac{1}{4}$ c $\frac{1}{2}$

d $\frac{7}{8}$ e $\frac{1}{2}$

14 a $\frac{1}{4}$ b $\frac{1}{8}$ c $\frac{9}{16}$

15 a $\frac{4}{9}$ b $\frac{7}{27}$

c $\frac{2}{3}$ d $\frac{8}{27}$

16 a $\frac{89}{400}$ b $\frac{53}{800}$

c $\frac{21}{25}$ d $\frac{439}{800}$

17 a $\frac{7}{20}$ b $\frac{2}{5}$

18 a $\frac{23}{100}$ b $\frac{81}{200}$

19 a $\frac{18}{25}$ b $\frac{6}{25}$ c 288 students

d 0.0576 e 0.2304 f 0.0784

20 a 20 games b $n(n-1)$

c 30 games d 240 games

## EXERCISE 8.3

1 a $\frac{1}{5}$ b $\frac{5}{9}$

c $\frac{7}{11}$ d $\frac{11}{25}$

2 a $\frac{5}{36}$ b $\frac{2}{15}$

c $\frac{1}{9}$ d $\frac{4}{5}$

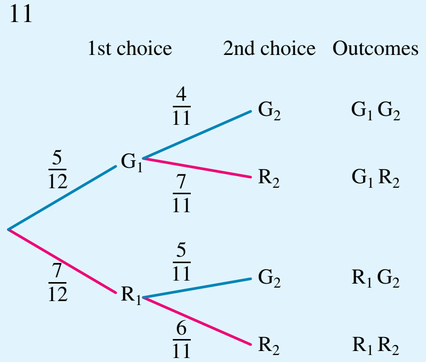
3 a $\Pr(A \cap B) = 0.12$; $\Pr(A|B) = \frac{6}{25}$

b $\frac{14}{55}$

4 a $\frac{9}{14}$ b $\frac{61}{81}$ c $\frac{13}{25}$

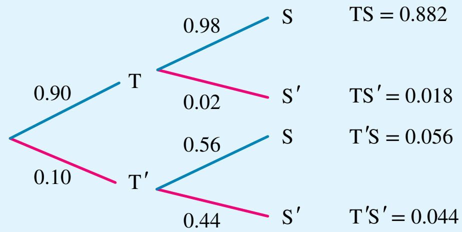
5 a $\frac{5}{11}$

b




c $\frac{35}{132}$ d $\frac{31}{66}$

6 a Let T = the bus being on time and T′ = the bus being late. Let S = Rodney gets to school on time and S′ = Rodney gets to school late.




b Pr(Rodney will arrive at school on time) = 0.882 + 0.056 = 0.938

7 $\frac{2}{9}$

8 $\frac{1}{2}$

9 a 0.2 b $\frac{3}{5}$

c $\frac{2}{7}$ d $\frac{5}{7}$

10 a 0.3 b $\frac{2}{3}$

c $\frac{1}{2}$ d $\frac{3}{5}$

11 a 0.9 b $\frac{4}{7}$

c $\frac{3}{4}$ d $\frac{1}{3}$

12 a $\frac{1}{17}$ b $\frac{15}{34}$

c $\frac{2}{15}$ d $\frac{4}{17}$

13 a $\frac{7}{22}$ b $\frac{28}{33}$

c $\frac{3}{8}$ d $\frac{1}{2}$

14 a 0.72 b 0.98 c 0.2

15 a $\frac{427}{500}$ b $\frac{47}{100}$ c $\frac{32}{61}$

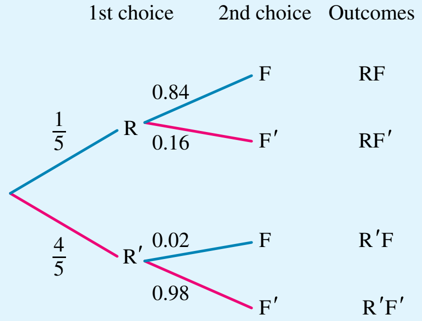
16 a $\frac{267}{1000}$ b $\frac{257}{1000}$

c $\frac{82}{267}$ d $\frac{185}{743}$

17 a $\frac{9}{50}$ b $\frac{21}{50}$

c $\frac{47}{232}$ d $\frac{43}{232}$

**18**




**a** 0.168 **b** 0.84 **c** 0.0392

**19 a** 0.7225 **b** 0.99998 **c** 0.85

## EXERCISE 8.4

**1 a** $\xi = \{HH, HT, TH, TT\}$; $A = \{TH, TT\}$; $B = \{HT, TH\}$; $C = \{HH, HT, TH\}$

**b** $A$ and $B$ are independent.

**c** $B$ and $C$ are not independent.

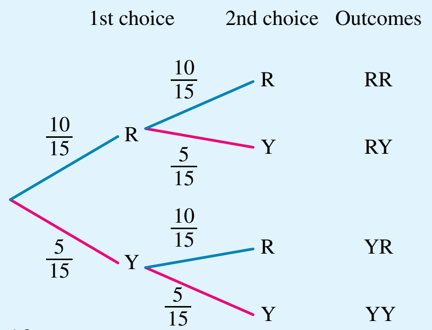
**d** $\frac{3}{4}$

**2 a** No **b** No

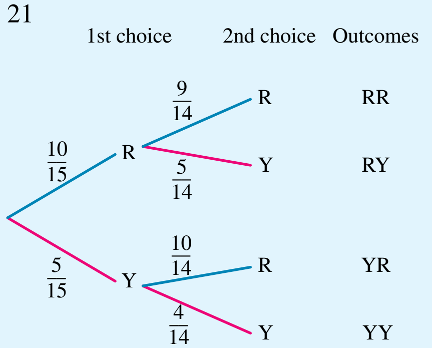
**c i** Yes **ii** No

**3 a** 0.216 **b** 0.024 **c** 0.784

**4 a** $\frac{4}{9}$




**b** $\frac{10}{21}$




**c** $\frac{1}{3}$

**5** Yes

**6** Yes

**7** No

**8 a** 0.001 25 **b** 0.0025 **c** 0.9524

**d** 0.000 625 **e** 0.052 25

**9 a** $\frac{9}{40}$ **b** $\frac{2}{5}$ **c** $\frac{17}{20}$

**d** $\frac{47}{200}$ **e** $\frac{7}{12}$

**10 a** $\frac{73}{100}$ **b** $\frac{7}{100}$

**c** $\frac{23}{30}$ **d** $\frac{66}{70}$

**11 a** $\frac{4}{5}$ **b** $\frac{2}{3}$

**c** $\frac{8}{15}$ **d** $\frac{14}{15}$

**12 a** $\frac{1}{5}$ **b** $\frac{11}{30}$ **c** No

**13** $\Pr(B) = 0.5$

**14** No; proof required

**15** No; proof required

**16 a** No; explanation required

**b** No; explanation required

**c** No; explanation required

## EXERCISE 8.5

**1 a** 120 **b** 300

**c** 24 **d** $\frac{1}{5}$

**2 a** 17 576 000 **b** 1 875 000 **c** $\frac{1}{2704}$

**3 a** 720 **b** 120 **c** $\frac{1}{36}$

**4 a** 6 **b** 24

**5 a** 80 640 **b** 28 2240

**c** 119 750 400 **d** $\frac{1}{6}$

**6 a** 64 864 800 **b** 4 989 600

**c** 453 600

**7 a** 2002 **b** 840

**c** 126 **d** $\frac{36}{91}$

**8** $\frac{45}{442}$

**9 a** 120 **b** 6 days **c** 240

**d** 210 **e** 312 **f** 216

**10 a** 1 757 600 **b** 16 807 **c** 216

**d** 216 **e** 24

**11 a** 40 320 **b** 1152 **c** 8640

**d** 30 240 **e** 2880

**12 a** 648 **b** 328

**c** 216 **d** 60

**13 a** 40 320 **b** 3600

**14 a** 27 720 **b** 210

**15 a** 420 **b** 408 **c** 24

**16 a** 6435 **b** 2 598 960

**c** 48 **d** 120

**17 a** 43 758 **b** 6720 **c** 1341

**d** 35 750 **e** 8008

**18 a** 4096 **b** 31 **c** $\frac{31}{4096}$

**d** $\frac{4017}{4096}$ **e** $\frac{35}{4096}$

**19 a** $\frac{1}{11\,925}$ **b** $\frac{19}{312}$

**c** $\frac{210\,721}{248\,040}(\approx 0.85)$ **d** $\frac{24\,877}{73\,099\,048} \approx 0.2747$

**20 a** $\frac{256\,949}{417\,778\,785}(\approx 0.0006)$

**b** $\frac{3\,149\,388}{603\,458\,245}(\approx 0.0052)$

**c** $\frac{1463}{5\,431\,124\,205}(\approx 2.63 \times 10^{-8})$

**d** $\frac{161\,751\,294}{603\,458\,245}(\approx 0.268)$

**e** $\frac{583\,275}{2\,407\,752\,143}(\approx 0.000\,242\,2)$

## EXERCISE 8.6

**1 a** $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

**b** $16 + 32x + 24x^2 + 8x^3 + x^4$

**c** $t^3 - 6t^2 + 12t - 8$

**2 a** $m^2 + 6bm + 9b^2$

**b** $16d^4 - 32d^3x + 24d^2x^2 - 8dx^3 + x^4$

**c** $h^3 + 6h + \frac{12}{h} + \frac{8}{h^3}$

**3 a** The third term, $\frac{3}{2}$ **b** $\frac{80}{3}m^2$

**4 a** $2835p^3q^4$ **b** $k = \frac{1}{2}$

**5 a** $^7C_3$ **b** 84

**c** Proof required **d** Proof required

**6 a** Proof required **b** $^{19}C_{15} + {}^{19}C_{14} = {}^{20}C_{15}$

**c** Proof required

**7 a** $x^3 + 3x^2y + 3xy^2 + y^3$

**b** $a^4 + 8a^3 + 24a^2 + 32a + 16$

**c** $m^4 - 12m^3 + 54m^2 - 108m + 81$

**d** $32x - 80 + 80x^2 - 40x^3 + 10x^4 - x^5$

**8 a** $1 - \frac{6}{x} + \frac{12}{x^2} - \frac{8}{x^3}$

**b** $1 + \frac{4p}{q} + \frac{6p^2}{q^2} + \frac{4p^3}{q^3} + \frac{p^4}{q^4}$

**c** $81 - 54m + \frac{27}{2}m^2 - \frac{3}{2}m^3 + \frac{1}{16}m^4$

**d** $8x^3 - 12x + \frac{6}{x} - \frac{1}{x^3}$

**9 a** $720w^3$ **b** $945b^{-4}$

**c** 54

**10 a** $243d^5$ **b** $-11250$

**11** $a = 161$ and $b = -72$

**12** $(1 - m)^6$

**13** $x^4$

**14** 48

**15** $\pm 2$

**16** 3

**17** $n = 4; k = \frac{1}{2}$

**18 a** 125 970, 77 520, 38 760

**b** $a = 318\,195$ and $b = 224\,953$

**c** $n = 16$

# 9

# Trigonometric functions 1

# 9.1 Kick off with CAS

## Trigonometric functions

1 Using CAS technology in radian mode, sketch the following trigonometric functions over $0 \le x \le 2\pi$.

a $y = \sin(x)$
b $y = 2\sin(x)$
c $y = -3\sin(x)$
d $y = 5\sin(x)$
e $y = -\frac{1}{2}\sin(x)$

2 Using CAS technology, enter $y = a\sin(x)$ into the function entry line and use a slider to change the value of $a$.

3 When sketching a trigonometric function, what is the effect of changing the value of $a$ in front of $\sin(x)$?

4 Using CAS technology in radian mode, sketch the following trigonometric functions over $0 \le x \le 2\pi$.

a $y = \cos(x)$
b $y = 2\cos(x)$
c $y = -3\cos(x)$
d $y = 5\cos(x)$
e $y = -\frac{1}{2}\cos(x)$

5 Using CAS technology, enter $y = a\cos(x)$ into the function entry line and use a slider to change the value of $a$.

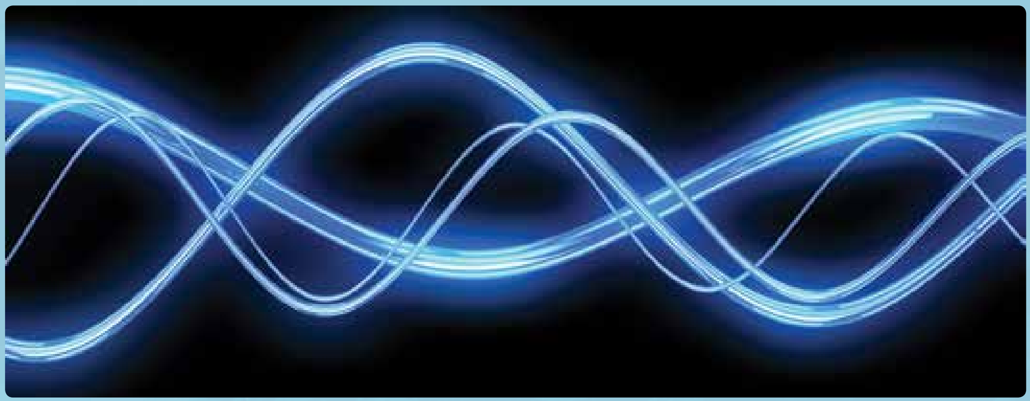
6 When sketching a trigonometric function, what is the effect of changing the value of $a$ in front of $\cos(x)$?

7 Using CAS technology in radian mode, sketch the following trigonometric functions over $0 \le x \le 2\pi$.

a $y = \sin(2x)$
b $y = \cos(4x)$
c $y = \sin(\pi x)$
d $y = \cos\left(\frac{x}{2}\right)$
e $y = \sin\left(\frac{x}{4}\right)$

8 Using CAS technology, enter $y = \cos(nx)$ into the function entry line and use a slider to change the value of $n$.

9 When sketching a trigonometric function, what is the effect of changing the value of $n$ in the equation?



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 9.2 Trigonometric ratios

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Units 1 & 2
AOS 1
Topic 6
Concept 1

**Trigonometric ratios**
Concept summary
Practice questions

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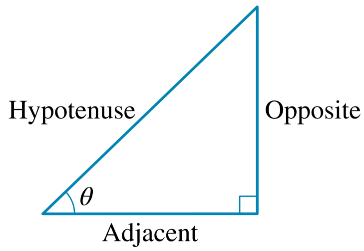
**Interactivity**
Trigonometric ratios
**int-2577**

The process of calculating all side lengths and all angle magnitudes of a triangle is called **solving the triangle**. Here we review the use of trigonometry to solve right-angled triangles.

## Right-angled triangles

The hypotenuse is the longest side of a right-angled triangle and it lies opposite the 90° angle, the largest angle in the triangle. The other two sides are labelled relative to one of the other angles in the triangle, an example of which is shown in the diagram.




It is likely that the trigonometric ratios of sine, cosine and tangent, possibly together with Pythagoras' theorem, will be required to solve a right-angled triangle.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}, \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \text{ and } \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}},$$

**usually remembered as SOH, CAH, TOA.**

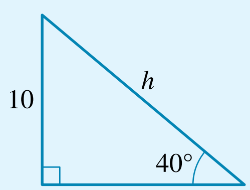
These ratios cannot be applied to triangles which do not have a right angle. However, isosceles and equilateral triangles can easily be divided into two right-angled triangles by dropping a perpendicular from the vertex between a pair of equal sides to the midpoint of the opposite side.

By and large, as first recommended by the French mathematician François Viète in the sixteenth century, decimal notation has been adopted for magnitudes of angles rather than the sexagesimal system of degrees and minutes; although, even today we still may see written, for example, either 15°24′ or 15.4° for the magnitude of an angle.

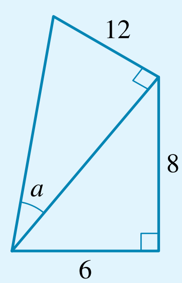
**WORKED EXAMPLE 1**

**Calculate, to 2 decimal places, the value of the pronumeral shown in each diagram.**

**a**




**b**




| THINK | WRITE |
| --- | --- |
| **a 1** Choose the appropriate trigonometric ratio. | **a** Relative to the angle, the sides marked are the opposite and the hypotenuse. $\sin(40°) = \frac{10}{h}$ |

**2** Rearrange to make the required side the subject and evaluate, checking the calculator is in degree mode.

$h = \dfrac{10}{\sin(40°)}$

$= 15.56$ to 2 decimal places

**b** **1** Obtain the hypotenuse length of the lower triangle.

**b** From Pythagoras' theorem the sides 6, 8, 10 form a Pythagorean triple, so the hypotenuse is 10.

**2** In the upper triangle choose the appropriate trigonometric ratio.

The opposite and adjacent sides to the angle $a°$ are now known.

$\tan(a) = \frac{12}{10}$

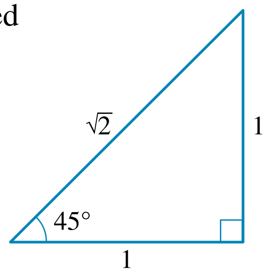
**3** Rearrange to make the required angle the subject and evaluate.

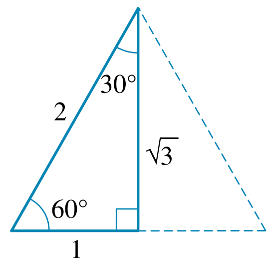
$\tan(a) = 1.2$

$\therefore a = \tan^{-1}(1.2)$

$= 50.19°$ to 2 decimal places

## Exact values for trigonometric ratios of 30°, 45°, 60°




By considering the isosceles right-angled triangle with equal sides of one unit, the trigonometric ratios for 45° can be obtained. Using Pythagoras' theorem, the hypotenuse of this triangle will be $\sqrt{1^2 + 1^2} = \sqrt{2}$ units.

The equilateral triangle with the side length of two units can be divided in half to form a right-angled triangle containing the 60° and the 30° angles. This right-angled triangle has a hypotenuse of 2 units and the side divided in half has length $\frac{1}{2} \times 2 = 1$ unit. The third side is found using Pythagoras' theorem: $\sqrt{2^2 - 1^2} = \sqrt{3}$ units.

The exact values for trigonometric ratios of 30°, 45°, 60° can be calculated from these triangles using SOH, CAH, TOA. Alternatively, these values can be displayed in a table and committed to memory.

| $\theta$ | 30° | 45° | 60° |
|---|---|---|---|
| $\sin(\theta)$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos(\theta)$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |
| $\tan(\theta)$ | $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ |

As a memory aid, notice the sine values in the table are in the order $\frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$. The cosine values reverse this order, while the tangent values are the sine values divided by the cosine values.

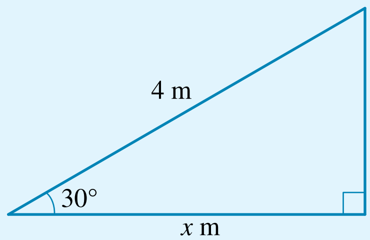
For other angles, a calculator, or other technology, is required. It is essential to set the calculator mode to degree in order to evaluate a trigonometric ratio involving angles in degree measure.

**WORKED EXAMPLE 2**

**A ladder of length 4 metres leans against a fence. If the ladder is inclined at 30° to the ground, how far exactly is the foot of the ladder from the fence?**

**THINK** | **WRITE**

1 Draw a diagram showing the given information.




Let the distance of the ladder from the fence be $x$ m.

2 Choose the appropriate trigonometric ratio.

Relative to the angle, the sides marked are the adjacent and the hypotenuse.

$\cos(30°) = \frac{x}{4}$

3 Calculate the required length using the exact value for the trigonometric ratio.

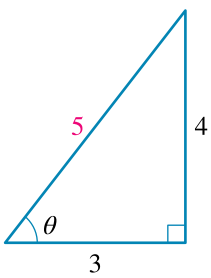
$$\begin{aligned} x &= 4\cos(30°) \\ &= 4 \times \frac{\sqrt{3}}{2} \\ &= 2\sqrt{3} \end{aligned}$$

4 State the answer.

The foot of the ladder is $2\sqrt{3}$ metres from the fence.

## Deducing one trigonometric ratio from another

Given the sine, cosine or tangent value of some unspecified angle, it is possible to obtain the exact value of the other trigonometric ratios of that angle using Pythagoras' theorem.




One common example is that given $\tan(\theta) = \frac{4}{3}$ it is possible to deduce that $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$ without evaluating $\theta$. The reason for this is that $\tan(\theta) = \frac{4}{3}$ means that the sides opposite and adjacent to the angle $\theta$ in a right-angled triangle would be in the ratio $4:3$.

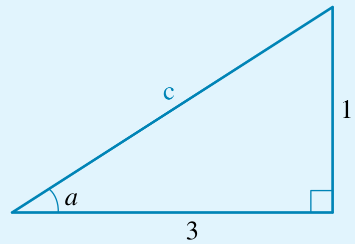
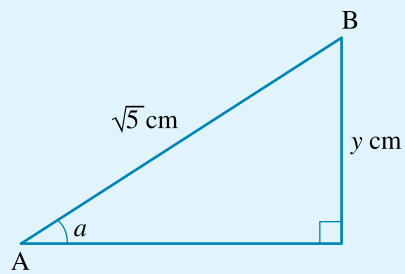
Labelling these sides 4 and 3 respectively and using Pythagoras' theorem (or recognising the Pythagorean triad '3,4,5') leads to the hypotenuse being 5 and hence the ratios $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$ are obtained.

**WORKED EXAMPLE 3**

**A line segment AB is inclined at $a$ degrees to the horizontal, where $\tan(a) = \frac{1}{3}$.**

**a Deduce the exact value of $\sin(a)$.**

**b Calculate the vertical height of B above the horizontal through A if the length of AB is $\sqrt{5}$ cm.**

| THINK | WRITE |
| --- | --- |
| **a** **1** Draw a right-angled triangle with two sides in the given ratio and calculate the third side. | **a** $\tan(a) = \frac{1}{3} \Rightarrow$ sides opposite and adjacent to angle $a$ are in the ratio 1:3.  Using Pythagoras' theorem: $c^2 = 1^2 + 3^2$ $\therefore c = \sqrt{10}$ |
| **2** State the required trigonometric ratio. | $\sin(a) = \dfrac{1}{\sqrt{10}}$ |
| **b** **1** Draw the diagram showing the given information. | **b** Let the vertical height be $y$ cm.  |
| **2** Choose the appropriate trigonometric ratio and calculate the required length. | $\sin(a) = \dfrac{y}{\sqrt{5}}$ $y = \sqrt{5}\sin(a)$ $= \sqrt{5} \times \dfrac{1}{\sqrt{10}}$ as $\sin(a) = \dfrac{1}{\sqrt{10}}$ $= \dfrac{1}{\sqrt{2}}$ or $\dfrac{\sqrt{2}}{2}$ |
| **3** State the answer. | The vertical height of B above the horizontal through A is $\dfrac{\sqrt{2}}{2}$ cm. |

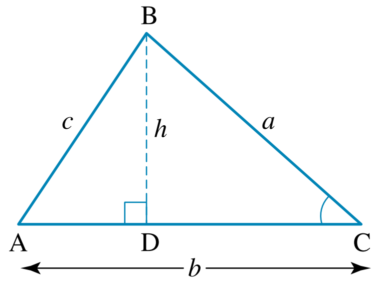
## Area of a triangle

> **The formula for calculating the area of a right-angled triangle is:**
>
> **Area = $\frac{1}{2}$(base) × (height)**

For a triangle that is not right-angled, if two sides and the angle included between these two sides are known, it is also possible to calculate the area of the triangle from that given information.

Consider the triangle ABC shown, where the convention of labelling the sides opposite the angles A, B and C with lower case letters $a$, $b$ and $c$ respectively has been adopted in the diagram.

In triangle ABC construct the perpendicular height, $h$, from B to a point D on AC. As this is not necessarily an isosceles triangle, D is not the midpoint of AC.




In the right-angled triangle BCD,
$\sin(C) = \frac{h}{a} \Rightarrow h = a\sin(C)$.

This means the height of triangle ABC is $a\sin(C)$ and its base is $b$.

The area of the triangle ABC can now be calculated.

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}(\text{base}) \times (\text{height})\\ &= \tfrac{1}{2}b \times a\sin(C)\\ &= \tfrac{1}{2}ab\sin(C)\end{aligned}$$

The formula for the area of the triangle ABC, $A_\Delta = \frac{1}{2}ab\sin(C)$, is expressed in terms of two of its sides and the angle included between them.

Alternatively, using the height as $c\sin(A)$ from the right-angled triangle ABD on the left of the diagram, the area formula becomes $A_\Delta = \frac{1}{2}bc\sin(A)$.

It can also be shown that the area is $A_\Delta = \frac{1}{2}ac\sin(B)$.

Hence, the area of a triangle is $\frac{1}{2}$ × (product of two sides) × (sine of the angle included between the two given sides).

**Area of a triangle: $A_\Delta = \frac{1}{2}ab\sin(C)$**

**WORKED EXAMPLE 4**

**Calculate the exact area of the triangle ABC for which $a = \sqrt{62}, b = 5\sqrt{2}, c = 6\sqrt{2}$ cm, and $A = 60°$.**

**THINK**

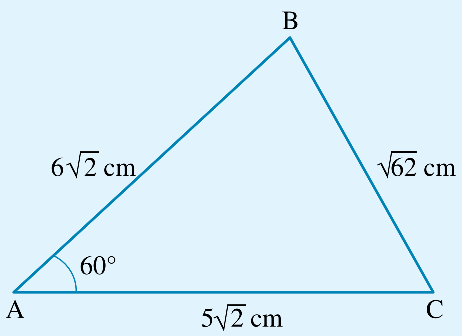
1 Draw a diagram showing the given information.

*Note:* The naming convention for labelling the angles and the sides opposite them with upper- and lower-case letters is commonly used.

2 State the two sides and the angle included between them.

3 State the appropriate area formula and substitute the known values.

**WRITE**




The given angle $A$ is included between the sides $b$ and $c$.

The area formula is:

$A_\Delta = \frac{1}{2}bc\sin(A),\ b = 5\sqrt{2}, c = 6\sqrt{2}, A = 60°$

$\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \sin(60°)$

4 Evaluate, using the exact value for the trigonometric ratio.

$\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \frac{\sqrt{3}}{2}$

$= \frac{1}{2} \times 30 \times 2 \times \frac{\sqrt{3}}{2}$

$= 15\sqrt{3}$

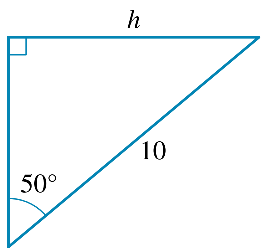
5 State the answer.

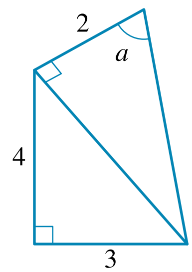
The area of the triangle is $15\sqrt{3}$ cm$^2$.

## EXERCISE 9.2 Trigonometric ratios

**PRACTISE**

1 **WE1** Calculate, to 2 decimal places, the value of the pronumeral shown in each diagram.

a



b



2 Triangle ACB is an isosceles triangle with equal sides CA and CB. If the third side AB has length of 10 cm and the angle CAB is 72°, solve this triangle by calculating the length of the equal sides and the magnitudes of the other two angles.

3 **WE2** A ladder of length 4 metres leans against a fence. If the ladder is inclined at 45° to the horizontal ground, how far exactly is the foot of the ladder from the fence?

4 Evaluate $\frac{\cos(30°)\sin(45°)}{\tan(45°)+\tan(60°)}$, expressing the answer in exact form with a rational denominator.

5 **WE3** A line segment AB is inclined at $a$ degrees to the horizontal, where $\tan(a) = \frac{2}{3}$.

a Deduce the exact value of $\cos(a)$.

b Calculate the run of AB along the horizontal through A if the length of AB is 26 cm.

6 For an acute angle $\theta$, $\cos(\theta) = \frac{3\sqrt{5}}{7}$. Calculate the exact values of $\sin(\theta)$ and $\tan(\theta)$.

7 **WE4** Calculate the exact area of the triangle ABC for which $a = 10$, $b = 6\sqrt{2}$, $c = 2\sqrt{13}$ cm and $C = 45°$.

8 Horses graze over a triangular area XYZ where Y is 4 km east of X and Z is 3 km from Y on a bearing of N 20° W. Over what area, correct to 2 decimal places, can the horses graze?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 a Evaluate $\frac{\sin(30°)\cos(45°)}{\tan(60°)}$, expressing the answer in exact form with a rational denominator.

b Evaluate $\frac{\tan(45°)+\cos(60°)}{\sin(60°)-\sin(45°)}$, expressing the answer in exact form with a rational denominator.

10 a For an acute angle $\theta$, obtain the following trigonometric ratios without evaluating $\theta$.

i Given $\tan(\theta) = \frac{\sqrt{3}}{2}$, form the exact value of $\sin(\theta)$.

ii Given $\cos(\theta) = \frac{5}{6}$, form the exact value of $\tan(\theta)$.

iii Given $\sin(\theta) = \frac{\sqrt{5}}{3}$, form the exact value of $\cos(\theta)$.

b A right-angled triangle contains an angle $\theta$ where $\sin(\theta) = \frac{3}{5}$. If the longest side of the triangle is 60 cm, calculate the exact length of the shortest side.

11 In order to check the electricity supply, a technician uses a ladder to reach the top of an electricity pole. The ladder reaches 5 metres up the pole and its inclination to the horizontal ground is 54°.

a Calculate the length of the ladder to 2 decimal places.

b If the foot of the ladder is moved 0.5 metres closer to the pole, calculate its new inclination to the ground and the new vertical height it reaches up the electricity pole, both to 1 decimal place.

12 a An isosceles triangle ABC has sides BC and AC of equal length 5 cm. If the angle enclosed between the equal sides is 20°, calculate:

i the area of the triangle to 3 decimal places

ii the length of the third side AB to 3 decimal places.

b An equilateral triangle has a vertical height of 10 cm. Calculate the exact perimeter and area of the triangle.

c Calculate the area of the triangle ABC if, using the naming convention, $a = 4\sqrt{2}$, $b = 6$ cm and $C = 30°$.

13 The two legs of a builder's ladder are of length 2 metres. The ladder is placed on horizontal ground with the distance between its two feet of 0.75 metres. Calculate the magnitude of the angle between the legs of the ladder.

14 Triangle ABC has angles such that ∠CAB = 60° and ∠ABC = 45°. The perpendicular distance from C to AB is 18 cm. Calculate the exact lengths of each of its sides.

15 A cube of edge length $a$ units rests on a horizontal table. Calculate:

a the length of the diagonal of the cube in terms of $a$

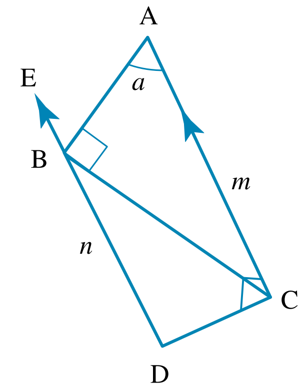
b the inclination of the diagonal to the horizontal, to 2 decimal places.

16 AB is a diameter of a circle and C is a point on the circumference of the circle such that ∠CBA = 68° and CB = 3.8 cm.

a Calculate the length of the radius of the circle to 2 decimal places.

b Calculate the shortest distance of CB from the centre of the circle, to 1 decimal place.

17 In the diagram, angles ABC and ACD are right angles and DE is parallel to CA. Angle BAC is $a$ degrees and the length measures of AC and BD are $m$ and $n$ respectively.




a Show that $n = m\sin^2(a)$, where $\sin^2(a)$ is the notation for the square of $\sin(a)$.
b If angle EBA is 60° and CD has length measure of $4\sqrt{3}$, calculate the values of $a$, $m$ and $n$.

18 A lookout tower is 100 metres in height. From the top of this tower, the angle of depression of the top of a second tower stood on the same level ground is 30°; from the bottom of the lookout tower, the angle of elevation to the top of the second tower is 45°. Calculate the height of the second tower and its horizontal distance from the lookout tower, expressing both measurements to 1 decimal place.

MASTER

19 The distances shown on a map are called projections. They give the horizontal distances between two places without taking into consideration the slope of the line connecting the two places. If a map gives the projection as 25 km between two points which actually lie on a slope of 16°, what is the true distance between the points?

20 A beam of length $6\sqrt{2}$ metres acts as one of the supports for a new fence. This beam is inclined at an angle of 15° to the horizontal. Calculate the exact horizontal distance of its foot from the fence.

# 9.3 Circular measure

Measurements of angles up to now have been given in degree measure. An alternative to degree measure is **radian measure**. This alternative can be more efficient for certain calculations that involve circles, and it is essential for the study of trigonometric functions.

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Units 1 & 2
AOS 1
Topic 6
Concept 2

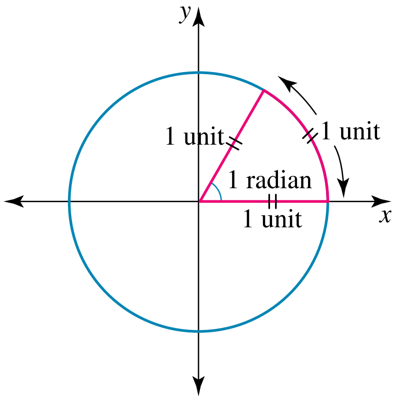
**Circular measure**
Concept summary
Practice questions

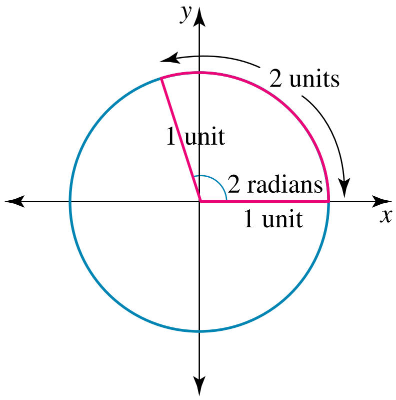
## Definition of radian measure

Radian measure is defined in relation to the length of an **arc** of a circle. An arc is a part of the circumference of a circle.

**One radian is the measure of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.**

In particular, an arc of 1 unit subtends an angle of one radian at the centre of a **unit circle**, a circle with radius 1 unit and, conventionally, a centre at the origin.



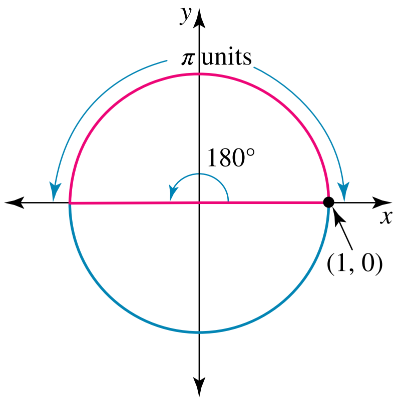

Doubling the arc length to 2 units doubles the angle to 2 radians. This illustrates the direct proportionality between the arc length and the angle in a circle of fixed radius.

The diagram suggests an angle of one radian will be a little less than 60° since the sector containing the angle has one 'edge' curved and therefore is not a true equilateral triangle.

The degree equivalent for 1 radian can be found by considering the angle subtended by an arc which is half the circumference. The circumference of a circle is given by $2\pi r$ so the circumference of a unit circle is $2\pi$.

In a unit circle, an arc of $\pi$ units subtends an angle of $\pi$ radians at the centre. But we know this angle to be 180°.

This gives the relationship between radian and degree measure.




**$\pi$ radians = 180°**

Hence, 1 radian equals $\frac{180°}{\pi}$, which is approximately 57.3°; 1° equals $\frac{\pi}{180}$ radians, which is approximately 0.0175 radians.

From these relationships it is possible to convert from radians to degrees and vice versa.

**To convert radians to degrees, multiply by $\frac{180}{\pi}$.**

**To convert degrees to radians, multiply by $\frac{\pi}{180}$.**

Radians are often expressed in terms of $\pi$, perhaps not surprisingly, since a radian is a circular measure and $\pi$ is so closely related to the circle.

## Notation

$\pi$ radian can be written as $\pi^c$, where $c$ stands for circular measure. However, linking radian measure with the length of an arc, a real number, has such importance that the symbol $c$ is usually omitted. Instead, the onus is on degree measure to always include the degree sign in order not to be mistaken for radian measure.

**WORKED EXAMPLE 5**

**a** Convert 30° to radian measure.

**b** Convert $\frac{4\pi^c}{3}$ to degree measure.

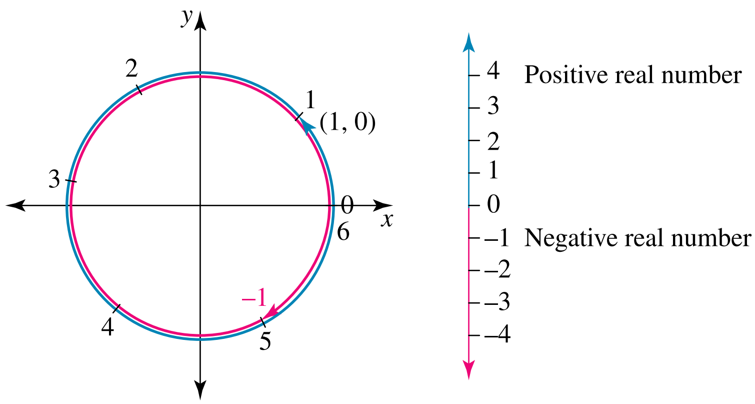
**c** Convert $\frac{\pi}{4}$ to degree measure and hence state the value of $\sin\left(\frac{\pi}{4}\right)$.

| THINK | WRITE |
|---|---|
| **a** Convert degrees to radians. | **a** To convert degrees to radians, multiply by $\frac{\pi}{180}$. <br> $30° = 30 \times \frac{\pi}{180}$ <br> $= \cancel{30} \times \frac{\pi}{\cancel{180}_6}$ <br> $= \frac{\pi}{6}$ |
| **b** Convert radians to degrees. *Note:* The degree sign must be used. | **b** To convert radians to degrees, multiply by $\frac{180}{\pi}$. <br> $\frac{4\pi^c}{3} = \left(\frac{4\pi}{3} \times \frac{180}{\pi}\right)^\circ$ <br> $= \left(\frac{4\cancel{\pi}}{3} \times \frac{180^{60}}{\cancel{\pi}}\right)^\circ$ <br> $= 240°$ |
| **c 1** Convert radians to degrees. | **c** $\frac{\pi}{4} = \frac{\cancel{\pi}}{4} \times \frac{180°}{\cancel{\pi}}$ <br> $= 45°$ |
| **2** Calculate the trigonometric value. | $\sin\left(\frac{\pi}{4}\right) = \sin(45°)$ <br> $= \frac{\sqrt{2}}{2}$ |

## Extended angle measure

By continuing to rotate around the circumference of the unit circle, larger angles are formed from arcs which are multiples of the circumference. For instance, an angle of $3\pi$ radians is formed from an arc of length $3\pi$ units created by one and a half revolutions of the unit circle: $3\pi = 2\pi + \pi$. This angle, in degrees, equals $360° + 180° = 540°$ and its endpoint on the circumference of the circle is in the same position as that of 180° or $\pi^c$; this is the case with any other angle which is a multiple of $2\pi$ added to $\pi^c$.

What is important here is that this process can continue indefinitely so that any real number, the arc length, can be associated with a radian measure. The real number line can be wrapped around the circumference of the unit circle so that the real number $\theta$ corresponds to the angle $\theta$ in radian measure. By convention, the positive reals wrap around the circumference anticlockwise while the negative reals wrap clockwise, with the number zero placed at the point $(1, 0)$ on the unit circle. The wrapping of the real number line around the circumference results in many numbers being placed in the same position on the unit circle's circumference.




**WORKED EXAMPLE 6**

**a** **Convert $-3^c$ to degree measure.**

**b** **Draw a unit circle diagram to show the position the real number $-3$ is mapped to when the real number line is wrapped around the circumference of the unit circle.**

**THINK**

**a** Convert radians to degrees.

**b** **1** State how the wrapping of the number line is made.

**2** Draw the unit circle diagram and mark the position of the number.

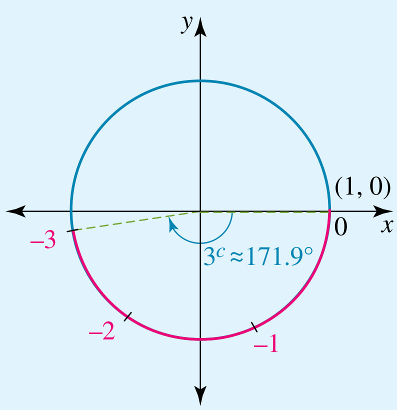
**WRITE**

**a** $-3^c = -\left(3 \times \frac{180}{\pi}\right)^\circ$

As the $\pi$ can't be cancelled, a calculator is used to evaluate.

$-3^c \approx -171.9^\circ$

**b** The number zero is placed at the point $(1, 0)$ and the negative number line is wrapped clockwise around the circumference of the unit circle through an angle of $171.9^\circ$ so that the number $-3$ is its endpoint.




## Using radians in calculations

From the definition of a radian, for any circle of radius $r$, an angle of $1^c$ is subtended at the centre of the circle by an arc of length $r$. So, if the angle at the centre of this circle is $\theta^c$, then the length of the arc subtending this angle must be $\theta \times r$.

This gives a formula for calculating the length of an arc.

$$l = r\theta$$

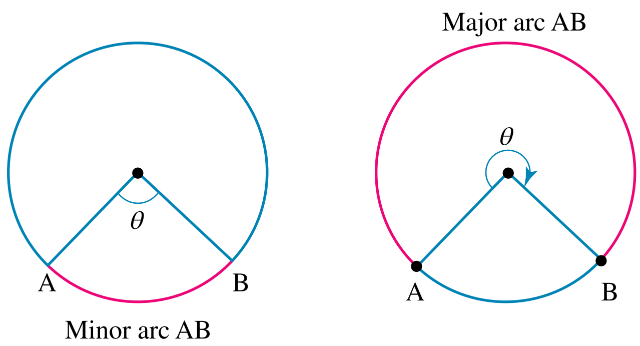
In the formula $l$ is the arc length and $\theta$ is the angle, in radians, subtended by the arc at the centre of the circle of radius $r$.

Any angles given in degree measure will need to be converted to radian measure to use this arc length formula.

Some calculations may require recall of the geometry properties of the angles in a circle, such as the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.

## Major and minor arcs

For a minor arc, $\theta < \pi$ and for a major arc $\theta > \pi$, with the sum of the minor and major arc angles totalling $2\pi$ if the major and minor arcs have their endpoints on the same chord.




To calculate the length of the major arc, the reflex angle with $\theta > \pi$ should be used in the arc length formula. Alternatively, the sum of the minor and major arc lengths gives the circumference of the circle, so the length of the major arc could be calculated as the difference between the lengths of the circumference and the minor arc.

## Trigonometric ratios of angles expressed in radians

Problems in trigonometry may be encountered where angles are given in radian mode and their sine, cosine or tangent value is required to solve the problem. A calculator, or other technology, can be set on radian or 'rad' mode and the required trigonometric ratio evaluated directly without the need to convert the angle to degrees.

Care must be taken to ensure the calculator is set to the appropriate degree or radian mode to match the measure in which the angle is expressed. Care is also needed with written presentation: if the angle is measured in degrees, the degree symbol must be given; if there is no degree sign then it is assumed the measurement is in radians.

**WORKED EXAMPLE 7**

**a** **An arc subtends an angle of 56° at the centre of a circle of radius 10 cm. Calculate the length of the arc to 2 decimal places.**

**b** **Calculate, in degrees, the magnitude of the angle that an arc of length $20\pi$ cm subtends at the centre of a circle of radius 15 cm.**

| THINK | WRITE |
|---|---|
| **a** **1** The angle is given in degrees so convert it to radian measure. | **a** $\theta^\circ = 56^\circ$<br>$\theta^c = 56 \times \frac{\pi}{180}$<br>$= \frac{14\pi}{45}$ |

2 Calculate the arc length.

$l = r\theta, r = 10, \theta = \dfrac{14\pi}{45}$

$l = 10 \times \dfrac{14\pi}{45}$

$= \dfrac{28\pi}{9}$

$\approx 9.77$

The arc length is 9.77 cm (to 2 decimal places).

b 1 Calculate the angle at the centre of the circle subtended by the arc.

b $l = r\theta, r = 15, l = 20\pi$

$15\theta = 20\pi$

$\therefore \theta = \dfrac{20\pi}{15}$

$= \dfrac{4\pi}{3}$

The angle is $\dfrac{4\pi}{3}$ radians.

2 Convert the angle from radians to degrees.

In degree measure:

$\theta^\circ = \dfrac{4\pi}{3} \times \dfrac{180^\circ}{\pi}$

$= 240^\circ$

The magnitude of the angle is 240°.

## EXERCISE 9.3 Circular measure

**PRACTISE**

1 WE5 a Convert 60° to radian measure.

b Convert $\dfrac{3\pi^c}{4}$ to degree measure.

c Convert $\dfrac{\pi}{6}$ to degree measure and hence state the value of $\tan\left(\dfrac{\pi}{6}\right)$.

2 Express 145°12' in radian measure, correct to 2 decimal places.

3 WE6 a Convert $1.8^c$ to degree measure.

b Draw a unit circle diagram to show the position the real number 1.8 is mapped to when the real number line is wrapped around the circumference of the unit circle.

4 The real number line is wrapped around the circumference of the unit circle. Give two positive and two negative real numbers which lie in the same position as the following numbers.

a 0 b −1

5 WE7 a An arc subtends an angle of 75° at the centre of a circle of radius 8 cm. Calculate the length of the arc, to 2 decimal places.

b Calculate, in degrees, the magnitude of the angle that an arc of length $12\pi$ cm subtends at the centre of a circle of radius 10 cm.

6 Evaluate the following to 3 decimal places.

a $\tan(1.2)$ b $\tan(1.2°)$

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 a Copy, complete and learn the following table by heart.

| Degrees | 30° | 45° | 60° |
|---|---|---|---|
| Radians | | | |

b Copy, complete and learn the following table by heart.

| Degrees | 0° | 90° | 180° | 270° | 360° |
|---|---|---|---|---|---|
| Radians | | | | | |

8 Convert the following to degrees.

a $\frac{\pi^c}{5}$ b $\frac{2\pi^c}{3}$ c $\frac{5\pi}{12}$

d $\frac{11\pi}{6}$ e $\frac{7\pi}{9}$ f $\frac{9\pi}{2}$

9 Convert the following to radian measure.

a 40° b 150° c 225°

d 300° e 315° f 720°

10 a Express in radian measure, to 3 decimal places.

i 3° ii 112°15′ iii 215.36°

b Express in degree measure to 3 decimal places.

i $3^c$ ii $2.3\pi$

c Rewrite $\left\{1.5^c, 50°, \frac{\pi^c}{7}\right\}$ with the magnitudes of the angles ordered from smallest to largest.

11 a For each of the following, draw a unit circle diagram to show the position of the angle and the arc which subtends the angle.

i An angle of 2 radians ii An angle of −2 radians

iii An angle of $-\frac{\pi}{2}$

b For each of the following, draw a unit circle diagram with the real number line wrapped around its circumference to show the position of the number and the associated angle subtended at the centre of the circle.

i The number 4 ii The number −1

iii The number $\frac{7\pi}{3}$

12 Calculate the exact lengths of the following arcs.

a The arc which subtends an angle of 150° at the centre of a circle of radius 12 cm

b The arc which subtends an angle of $\frac{2\pi^c}{9}$ at the circumference of a circle of radius $\pi$ cm

c The major arc with endpoints on a chord which subtends an angle of 60° at the centre of a circle of radius 3 cm

13 a A ball on the end of a rope of length 2.5 metres swings through an arc of 75 cm. Through what angle, to the nearest tenth of a degree, does the ball swing?

**b** A fixed point on the rim of a wheel of radius 3 metres rolls along horizontal ground at a speed of 2 m/s. After 5 seconds, calculate the angle the point has rotated through and express the answer in both degrees and radians.

**c** An analogue wristwatch has a minute hand of length 11 mm. Calculate, to 2 decimal places, the arc length the minute hand traverses between 9.45 am and 9.50 am.

**d** An arc of length 4 cm subtends an angle of 22.5° at the circumference of a circle. Calculate the area of the circle correct to 1 decimal place.

**14 a** Calculate the following to 3 decimal places.

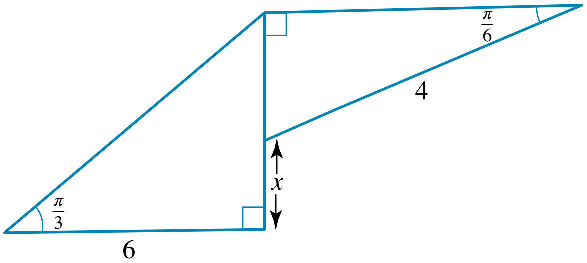
**i** $\tan(1^c)$ **ii** $\cos\left(\frac{2\pi}{7}\right)$ **iii** $\sin(1.46°)$

**b** Complete the following table with exact values.

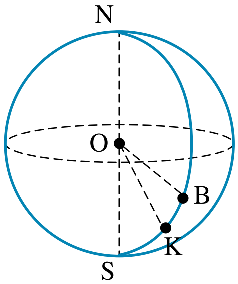
| $\theta$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|
| $\sin(\theta)$ | | | |
| $\cos(\theta)$ | | | |
| $\tan(\theta)$ | | | |

**15 a** Calculate the area of triangle ABC in which $a = 2\sqrt{2},\ c = 2,\ B = \frac{\pi}{4}$.

**b** Calculate the exact value of $x$ in the following diagram.




**16 a** The Western Australian towns of Broome(B) and Karonie(K) both lie on approximately the same longitude. Broome is approximately 1490 km due north of Karonie (the distance being measured along the meridian). When the sun is directly over Karonie, it is 13.4° south of Broome. Use this information to estimate the radius of the Earth.




This method dates back to Eratosthenes in 250 BC, although he certainly didn't use these Australian towns to calculate his results.

**b** A ship sailing due east along the equator from the Galapagos Islands to Ecuador travels a distance of 600 nautical miles. If the ship's longitude changes from 90° W to 80° W during this journey, estimate the radius of the Earth, given that 1 nautical mile is approximately 1.85 km.

**c** Taking the radius of the earth as 6370 km, calculate the distance, to the nearest kilometre, along the meridian between place A, located 20° S, 110° E, and place B, located 34° N, 110° E.

MASTER

**17** Convert 135° to radian measure using the 'mth TRIG' function on a CAS calculator.

**18** Convert 5 radians to degree measure using the 'mth TRIG' function, expressing the answer both as an exact value and as a value to 4 decimal places.

# 9.4 Unit circle definitions

With the introduction of radian measure, we encountered positive and negative angles of any size and then associated them with the wrapping of the real number line around the circumference of a unit circle. Before applying this wrapping to define the sine, cosine and tangent functions, we first consider the conventions for angle rotations and the positions of the endpoints of these rotations.

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AOS 1
Topic 6
Concept 3

**Unit circle definitions**
Concept summary
Practice questions

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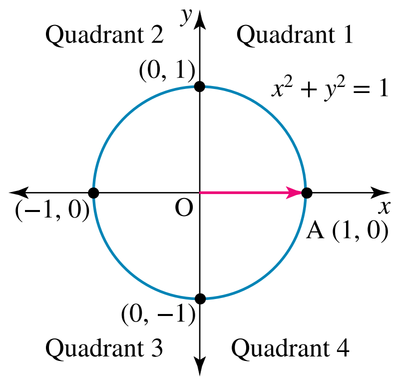
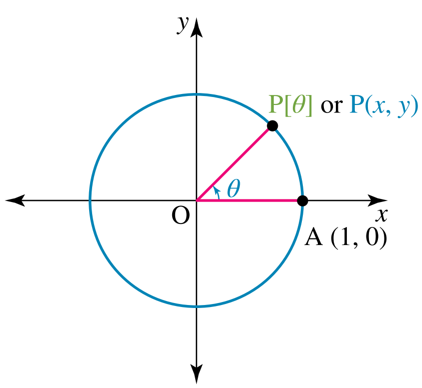
## Trigonometric points

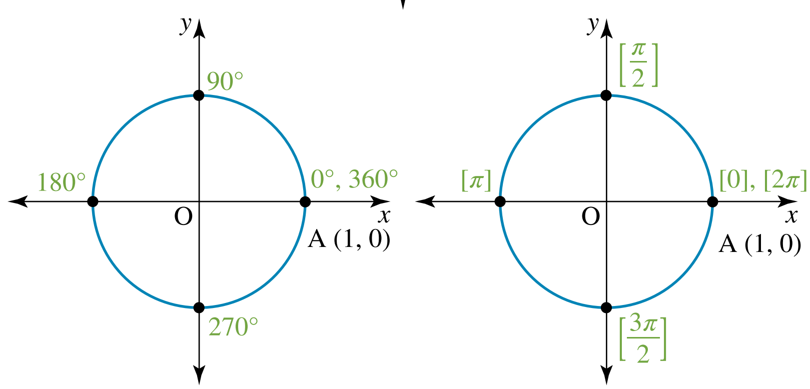
The unit circle has centre $(0, 0)$ and radius 1 unit. Its Cartesian equation is $x^2 + y^2 = 1$.




The coordinate axes divide the Cartesian plane into four quadrants. The points on the circle which lie on the boundaries between the quadrants are the endpoints of the horizontal and vertical diameters. These **boundary points** have coordinates $(-1, 0)$, $(1, 0)$ on the horizontal axis and $(0, -1)$, $(0, 1)$ on the vertical axis.

A rotation starts with an initial ray OA, where A is the point $(1, 0)$ and O $(0, 0)$. Angles are created by rotating the initial ray anticlockwise for positive angles and clockwise for negative angles. If the point on the circumference the ray reaches after a rotation of $\theta$ is P, then $\angle AOP = \theta$ and P is called the **trigonometric point** $[\theta]$. The angle of rotation $\theta$ may be measured in radian or degree measure. In radian measure, the value of $\theta$ corresponds to the length of the arc AP of the unit circle the rotation cuts off.




The point $P[\theta]$ has Cartesian coordinates $(x, y)$ where:

- $x > 0, y > 0$ if P is in quadrant 1, $0 < \theta < \dfrac{\pi}{2}$
- $x < 0, y > 0$ if P is in quadrant 2, $\dfrac{\pi}{2} < \theta < \pi$

- $x < 0, y < 0$ if P is in quadrant 3, $\pi < \theta < \frac{3\pi}{2}$
- $x > 0, y < 0$ if P is in quadrant 4, $\frac{3\pi}{2} < \theta < 2\pi$

Continued rotation, anticlockwise or clockwise, can be used to form other values for $\theta$ greater than $2\pi$, or values less than 0, respectively. No trigonometric point has a unique $\theta$ value.

The angle $\theta$ is said to lie in the quadrant in which its endpoint P lies.

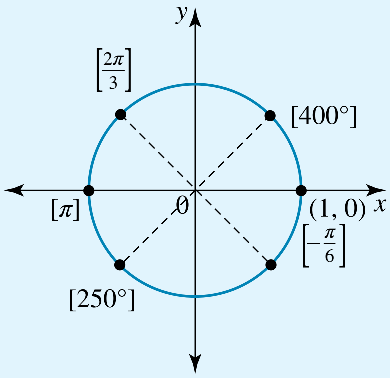
**WORKED EXAMPLE 8**

**a** Give a trigonometric value, using radian measure, of the point P on the unit circle which lies on the boundary between the quadrants 2 and 3.

**b** Identify the quadrants the following angles would lie in: $250°, 400°, \frac{2\pi^c}{3}, -\frac{\pi^c}{6}$.

**c** Give two other trigonometric points, Q and R, one with a negative angle and one with a positive angle respectively, which would have the same position as the point P[250°].

| THINK | WRITE |
| --- | --- |
| **a 1** State the Cartesian coordinates of the required point. | **a** The point $(-1, 0)$ lies on the boundary of quadrants 2 and 3. |
| **2** Give a trigonometric value of this point. *Note:* Other values are possible. | An anticlockwise rotation of 180° or $\pi^c$ from the point $(1, 0)$ would have its endpoint at $(-1, 0)$. The point P has the trigonometric value $[\pi]$. |
| **b 1** Explain how the quadrant is determined. | **b** For positive angles, rotate anticlockwise from $(1, 0)$; for negative angles rotate clockwise from $(1, 0)$. The position of the endpoint of the rotation determines the quadrant. |
| **2** Identify the quadrant the endpoint of the rotation would lie in for each of the given angles. | Rotating anticlockwise 250° from $(1, 0)$ ends in quadrant 3; rotating anticlockwise from $(1, 0)$ through 400° would end in quadrant 1; rotating anticlockwise from $(1, 0)$ by $\frac{2}{3}$ of $\pi$ would end in quadrant 2; rotating clockwise from $(1, 0)$ by $\frac{\pi}{6}$ would end in quadrant 4. |



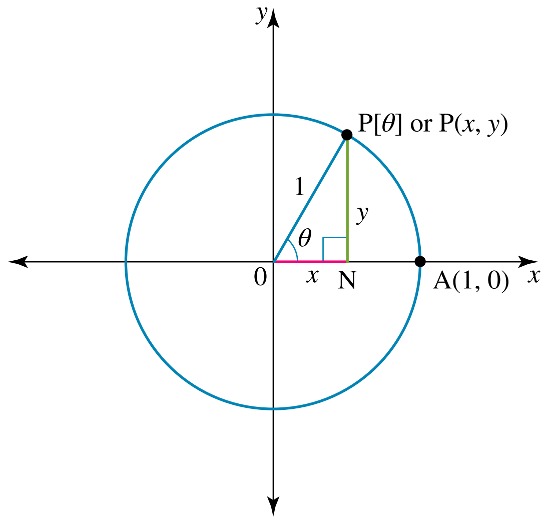

| | |
| --- | --- |
| **3** State the answer. | The angle 250° lies in quadrant 3, 400° in quadrant 1, $\frac{2\pi^c}{3}$ in quadrant 2, and $-\frac{\pi^c}{6}$ in quadrant 4. |

| | |
|---|---|
| **c** **1** Identify a possible trigonometric point Q. | **c** A rotation of 110° in the clockwise direction from (1, 0) would end in the same position as P[250°]. Therefore the trigonometric point could be Q[−110°]. |
| **2** Identify a possible trigonometric point R. | A full anticlockwise revolution of 360° plus another anticlockwise rotation of 250° would end in the same position as P[250°]. Therefore the trigonometric point could be R[610°]. |

## Unit circle definitions of the sine and cosine functions

Consider the unit circle and trigonometric point P[$\theta$] with Cartesian coordinates $(x, y)$ on its circumference. In the triangle ONP, $\angle NOP = \theta = \angle AOP$, ON $= x$ and NP $= y$.

As the triangle ONP is right-angled, $\cos(\theta) = \frac{x}{1} = x$ and $\sin(\theta) = \frac{y}{1} = y$. This enables the following definitions to be given.




For a rotation from the point $(1, 0)$ of any angle $\theta$ with endpoint P[$\theta$] on the unit circle:

**cos ($\theta$) is the $x$-coordinate of the trigonometric point P[$\theta$]**
**sin ($\theta$) is the $y$-coordinate of the trigonometric point P[$\theta$]**

The importance of these definitions is that they enable sine and cosine functions to be defined for any real number $\theta$. With $\theta$ measured in radians, the trigonometric point [$\theta$] also marks the position the real number $\theta$ is mapped to when the number line is wrapped around the circumference of the unit circle, with zero placed at the point $(1, 0)$. This relationship enables the sine or cosine of a real number $\theta$ to be evaluated as the sine or cosine of the angle of rotation of $\theta$ radians in a unit circle: $\sin(\theta) = \sin(\theta^c)$ and $\cos(\theta) = \cos(\theta^c)$.

The sine and cosine functions are
$f : R \to R, f(\theta) = \sin(\theta)$ and $f : R \to R, f(\theta) = \cos(\theta)$.

They are **trigonometric functions**, also referred to as **circular functions**. The use of parentheses in writing $\sin(\theta)$ or $\cos(\theta)$ emphasises their functionality.

The mapping has a many-to-one correspondence as many values of $\theta$ are mapped to the one trigonometric point. The functions have a **period** of $2\pi$ since rotations of $\theta$ and of $2\pi + \theta$ have the same endpoint on the circumference of the unit circle. The cosine and sine values repeat after each complete revolution around the unit circle.

For $f(\theta) = \sin(\theta)$, the image of a number such as 4 is $f(4) = \sin(4) = \sin(4^c)$. This is evaluated as the $y$-coordinate of the trigonometric point [4] on the unit circle.

The values of a function for which $f(t) = \cos(t)$, where $t$ is a real number, can be evaluated through the relation $\cos(t) = \cos(t^c)$ as $t$ will be mapped to the trigonometric point [$t$] on the unit circle.

The sine and cosine functions are periodic functions which have applications in contexts which may have nothing to do with angles, as we shall study in later chapters.

**WORKED EXAMPLE 9**

**a** **Calculate the Cartesian coordinates of the trigonometric point P$\left[\frac{\pi}{3}\right]$ and show the position of this point on a unit circle diagram.**

**b** **Illustrate $\cos(330°)$ and $\sin(2)$ on a unit circle diagram.**

**c** **Use the Cartesian coordinates of the trigonometric point $[\pi]$ to obtain the values of $\sin(\pi)$ and $\cos(\pi)$.**

**d** **If $f(\theta) = \cos(\theta)$, evaluate $f(0)$.**

**THINK**

**a** **1** State the value of $\theta$.

**2** Calculate the exact Cartesian coordinates.

*Note:* The exact values for sine and cosine of $\frac{\pi^c}{3}$, or 60°, need to be known.

**3** Show the position of the given point on a unit circle diagram.

**WRITE**

**a** $P\left[\frac{\pi}{3}\right]$

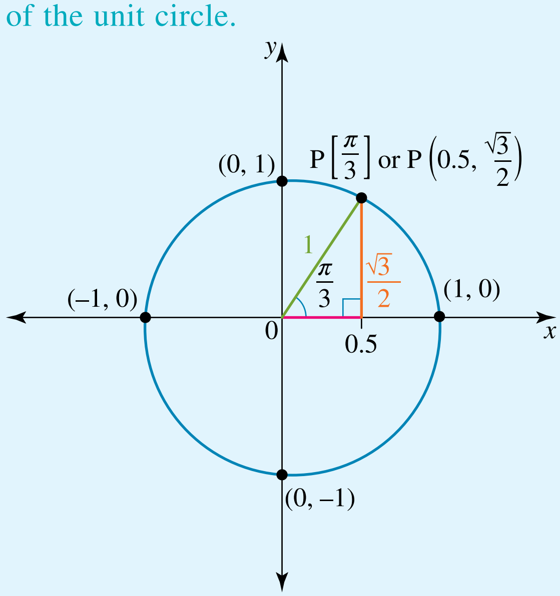
This is the trigonometric point with $\theta = \frac{\pi}{3}$.

The Cartesian coordinates are:

$$x = \cos(\theta) \qquad y = \sin(\theta)$$
$$= \cos\left(\frac{\pi}{3}\right) \qquad = \sin\left(\frac{\pi}{3}\right)$$
$$= \cos\left(\frac{\pi}{3}\right)^c \qquad = \sin\left(\frac{\pi}{3}\right)^c$$
$$= \cos(60^\circ) \qquad = \sin(60^\circ)$$
$$= \frac{1}{2} \qquad = \frac{\sqrt{3}}{2}$$

Therefore P has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

P$\left[\frac{\pi}{3}\right]$ or P$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ lies in quadrant 1 on the circumference of the unit circle.



**b 1** Identify the trigonometric point and which of its Cartesian coordinates gives the first value.

**b** $\cos(330°)$:
The value of $\cos(330°)$ is given by the $x$-coordinate of the trigonometric point $[330°]$.

**2** State the quadrant in which the trigonometric point lies.

The trigonometric point $[330°]$ lies in quadrant 4.

**3** Identify the trigonometric point and which of its Cartesian coordinates gives the second value.

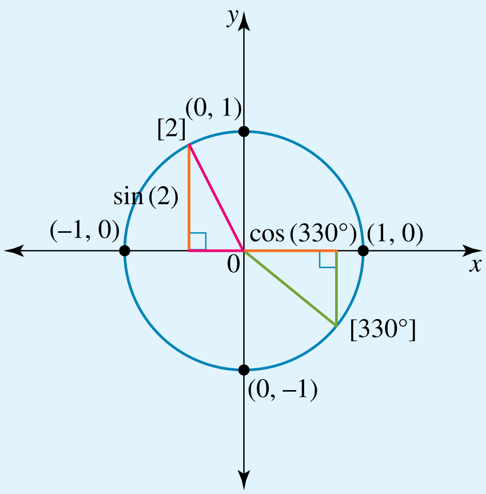
$\sin(2)$:
The value of $\sin(2)$ is given by the $y$-coordinate of the trigonometric point $[2]$.

**4** State the quadrant in which the trigonometric point lies.

As $\frac{\pi}{2} \approx 1.57 < 2 < \pi \approx 3.14$, the trigonometric point $[2]$ lies in quadrant 2.

**5** Draw a unit circle showing the two trigonometric points and construct the line segments which illustrate the $x$- and $y$-coordinates of each point.

For each of the points on the unit circle diagram, the horizontal line segment gives the $x$-coordinate and the vertical line segment gives the $y$-coordinate.




**6** Label the line segments which represent the appropriate coordinate for each point.

The value of $\cos(330°)$ is the length measure of the horizontal line segment.
The value of $\sin(2)$ is the length measure of the vertical line segment.
The line segments illustrating these values are highlighted in orange on the diagram.

**c 1** State the Cartesian coordinates of the given point.

**c** An anticlockwise rotation of $\pi$ from $(1, 0)$ gives the endpoint $(-1, 0)$.
The trigonometric point $[\pi]$ is the Cartesian point $(-1, 0)$.

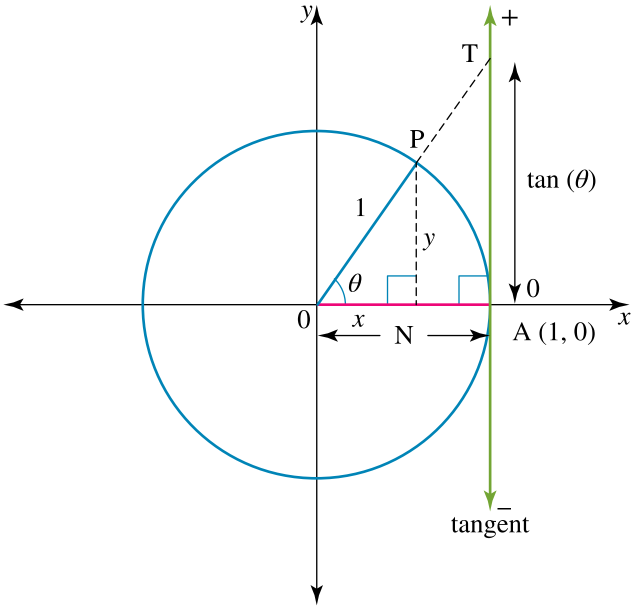
**2** State the required values.

The point $(-1, 0)$ has $x = -1, y = 0$.
Since $x = \cos(\theta)$,
$$\cos(\pi) = x = -1$$
Since $y = \sin(\theta)$,
$$\sin(\pi) = y = 0$$

| | |
|---|---|
| **d** **1** Substitute the given value in the function rule. | **d** $f(\theta) = \cos(\theta)$<br>$\therefore f(0) = \cos(0)$ |
| **2** Identify the trigonometric point and state its Cartesian coordinates. | The trigonometric point [0] has Cartesian coordinates $(1, 0)$. |
| **3** Evaluate the required value of the function. | The value of $\cos(0)$ is given by the $x$-coordinate of the point $(1, 0)$.<br>$\therefore \cos(0) = 1$<br>$\therefore \quad f(0) = 1$ |

## Unit circle definition of the tangent function

Consider again the unit circle with centre O(0, 0) containing the points A(1, 0) and the trigonometric point P[$\theta$] on its circumference. A tangent line to the circle is drawn at point A. The radius OP is extended to intersect the tangent line at point T.




**For any point P[$\theta$] on the unit circle, tan($\theta$) is defined as the length of the intercept AT that the extended ray OP cuts off on the tangent drawn to the unit circle at the point A(1, 0).**

Intercepts that lie above the $x$-axis give positive tangent values; intercepts that lie below the $x$-axis give negative tangent values.

Unlike the sine and cosine functions, there are values of $\theta$ for which $\tan(\theta)$ is undefined. These occur when OP is vertical and therefore parallel to the tangent line through A(1, 0); these two vertical lines cannot intersect no matter how far OP is extended. The values of $\tan\left(\frac{\pi}{2}\right)$ and $\tan\left(\frac{3\pi}{2}\right)$, for instance, are not defined.

The value of $\tan(\theta)$ can be calculated from the coordinates $(x, y)$ of the point P[$\theta$], provided the $x$-coordinate is not zero.

Using the ratio of sides of the similar triangles ONP and OAT:

$$\frac{AT}{OA} = \frac{NP}{ON}$$

$$\frac{\tan(\theta)}{1} = \frac{y}{x}$$

Hence:

$\tan(\theta) = \frac{y}{x},\ x \neq 0$, where $(x, y)$ are the coordinates of the trigonometric point P[$\theta$].

Since $x = \cos(\theta),\ y = \sin(\theta)$, this can be expressed as the relationship:

$$\boldsymbol{\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}}$$

## Domains and ranges of the trigonometric functions

The domain and range of the unit circle require $-1 \le x \le 1$ and $-1 \le y \le 1$ so $-1 \le \cos(\theta) \le 1$ and $-1 \le \sin(\theta) \le 1$.

Since $\theta$ can be any real number, this means that:

- the function $f$ where $f$ is either sine or cosine has domain $R$ and range $[-1, 1]$.

Unlike the sine and cosine functions, the domain of the tangent function is not the set of real numbers $R$ since $\tan(\theta)$ is not defined for any value of $\theta$ which is an odd multiple of $\frac{\pi}{2}$. Excluding these values, intercepts of any size may be cut off on the tangent line so $\tan(\theta) \in R$.

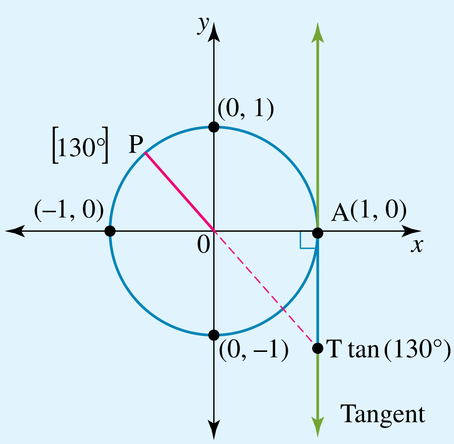
This means that the function $f$ where $f$ is tangent has domain $R\backslash\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\right\}$ and range $R$.

The domain of the tangent function can be written as $R\backslash\left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$ and the tangent function as the mapping $f: R\backslash\left\{(2n+1)\frac{\pi}{2}, n \in Z\right\} \to R, f(\theta) = \tan(\theta)$.

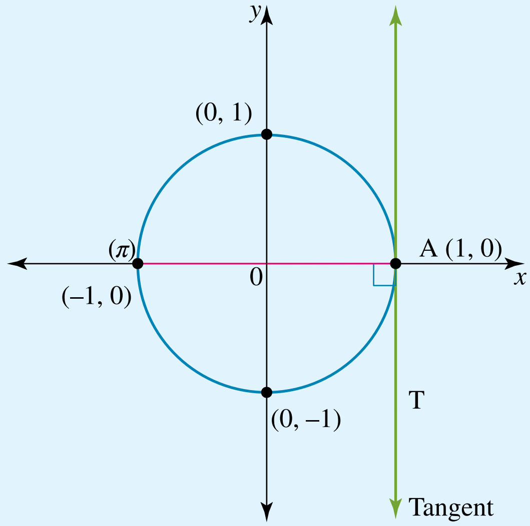
**WORKED EXAMPLE 10**

**a** **Illustrate $\tan(130°)$ on a unit circle diagram and use a calculator to evaluate $\tan(130°)$ to 3 decimal places.**

**b** **Use the Cartesian coordinates of the trigonometric point $P[\pi]$ to obtain the value of $\tan(\pi)$.**

| THINK | WRITE |
|---|---|
| **a 1** State the quadrant in which the angle lies. | **a** 130° lies in the second quadrant. |
| **2** Draw the unit circle with the tangent at the point A(1, 0). *Note:* The tangent line is always drawn at this point (1, 0). |  |
| **3** Extend PO until it reaches the tangent line. | Let T be the point where the extended radius PO intersects the tangent drawn at A. The intercept AT is tan(130°). |
| **4** State whether the required value is positive, zero or negative. | The intercept lies below the $x$-axis, which shows that tan(130°) is negative. |
| **5** Calculate the required value. | The value of $\tan(130°) = -1.192$, correct to 3 decimal places. |
| **b 1** Identify the trigonometric point and state its Cartesian coordinates. | **b** The trigonometric point $P[\pi]$ is the endpoint of a rotation of $\pi^c$ or 180°. It is the Cartesian point $P(-1, 0)$. |

| | |
|---|---|
| **2** Calculate the required value. | The point $(-1, 0)$ has $x = -1, y = 0$.<br>Since $\tan(\theta) = \frac{y}{x}$,<br>$\tan(\pi) = \frac{0}{-1}$<br>$= 0$ |
| **3** Check the answer using the unit circle diagram. | Check:<br>PO is horizontal and runs along the $x$-axis. Extending PO, it intersects the tangent at the point A. This means the intercept is 0, which means $\tan(\pi) = 0$. |




## EXERCISE 9.4 Unit circle definitions

**PRACTISE**

**1** WE8 **a** Give a trigonometric value, using radian measure, of the point P on the unit circle which lies on the boundary between the quadrants 1 and 2.

**b** Identify the quadrants the following angles would lie in: 120°, –400°, $\frac{4\pi^c}{3}$, $\frac{\pi^c}{4}$.

**c** Give two other trigonometric points, Q with a negative angle and R with a positive angle, which would have the same position as the point P[120°].

**2** Using a positive and a negative radian measure, state trigonometric values of the point on the unit circle which lies on the boundary between quadrants 3 and 4.

**3** WE9 **a** Calculate the Cartesian coordinates of the trigonometric point $P\left[\frac{\pi}{6}\right]$ and show the position of this point on a unit circle diagram.

**b** Illustrate $\sin(225°)$ and $\cos(1)$ on a unit circle diagram.

**c** Use the Cartesian coordinates of the trigonometric point $\left[-\frac{\pi}{2}\right]$ to obtain the values of $\sin\left(-\frac{\pi}{2}\right)$ and $\cos\left(-\frac{\pi}{2}\right)$.

**d** If $f(\theta) = \sin(\theta)$, evaluate $f(0)$.

**4** Identify the quadrants where:

**a** $\sin(\theta)$ is always positive

**b** $\cos(\theta)$ is always positive.

**5** WE10 **a** Illustrate $\tan(230°)$ on a unit circle diagram and use a calculator to evaluate $\tan(230°)$ to 3 decimal places.

**b** Use the Cartesian coordinates of the trigonometric point P[$2\pi$] to obtain the value of $\tan(2\pi)$.

**6** Consider $\left\{\tan(-3\pi), \tan\left(\frac{5\pi}{2}\right), \tan(-90°), \tan\left(\frac{3\pi}{4}\right), \tan(780°)\right\}$.

**a** Which elements in the set are not defined?
**b** Which elements have negative values?

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**7** Identify the quadrant in which each of the following lies.

**a** $585°$ **b** $\frac{11\pi}{12}$ **c** $-18\pi$ **d** $\frac{7\pi}{4}$

**8** Consider O, the centre of the unit circle, and the trigonometric points P$\left[\frac{3\pi}{10}\right]$ and Q$\left[\frac{2\pi}{5}\right]$ on its circumference.

**a** Sketch the unit circle showing these points.
**b** How many radians are contained in the angle ∠QOP?
**c** Express each of the trigonometric points P and Q with a negative $\theta$ value.
**d** Express each of the trigonometric points P and Q with a larger positive value for $\theta$ than the given values P$\left[\frac{3\pi}{10}\right]$ and Q$\left[\frac{2\pi}{5}\right]$.

**9 a** Calculate the Cartesian coordinates of the trigonometric point P$\left[\frac{\pi}{4}\right]$.

**b** Express the Cartesian point P(0, −1) as two different trigonometric points, one with a positive value for $\theta$ and one with a negative value for $\theta$.

**10** Illustrate the following on a unit circle diagram.

**a** $\cos(40°)$ **b** $\sin(165°)$ **c** $\cos(-60°)$ **d** $\sin(-90°)$

**11** Illustrate the following on a unit circle diagram.

**a** $\sin\left(\frac{5\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{5}\right)$ **c** $\cos(5\pi)$ **d** $\sin\left(-\frac{2\pi}{3}\right)$

**12** Illustrate the following on a unit circle diagram.

**a** $\tan(315°)$ **b** $\tan\left(\frac{5\pi}{6}\right)$ **c** $\tan\left(\frac{4\pi}{3}\right)$ **d** $\tan(-300°)$

**13 a** Given $f(t) = \sin(t)$, use a calculator to evaluate $f(2)$ to 2 decimal places.
**b** Given $g(u) = \cos(u)$, use a calculator to evaluate $g(2)$ to 2 decimal places.
**c** Given $h(\theta) = \tan(\theta)$, use a calculator to evaluate $h(2)$ to 2 decimal places.

**14 a i** On a unit circle diagram show the trigonometric point P[2] and the line segments $\sin(2)$, $\cos(2)$ and $\tan(2)$. Label them with their length measures expressed to 2 decimal places.
**ii** State the Cartesian coordinates of P to 2 decimal places.
**b** On a unit circle diagram show the trigonometric points A[0] and P[$\theta$] where $\theta$ is acute, and show the line segments $\sin(\theta)$ and $\tan(\theta)$. By comparing the lengths of the line segments with the length of the arc AP, explain why $\sin(\theta) < \theta < \tan(\theta)$ for acute $\theta$.

**15 a** The trigonometric point P[$\theta$] has Cartesian coordinates (−0.8, 0.6). State the quadrant in which P lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.
**b** The trigonometric point Q[$\theta$] has Cartesian coordinates $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. State the quadrant in which Q lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

**c** For the trigonometric point R[$\theta$] with Cartesian coordinates $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, state the quadrant in which R lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

**d** The Cartesian coordinates of the trigonometric point S[$\theta$] are (0, 1). Describe the position of S and state the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

**16** By locating the appropriate trigonometric point and its corresponding Cartesian coordinates, obtain the exact values of the following.

**a** $\cos(0)$ **b** $\sin\left(\frac{\pi}{2}\right)$ **c** $\tan(\pi)$ **d** $\cos\left(\frac{3\pi}{2}\right)$

**e** $\sin(2\pi)$ **f** $\cos\left(\frac{17\pi}{2}\right) + \tan(-11\pi) + \sin\left(\frac{11\pi}{2}\right)$

**MASTER**

**17** Use CAS technology to calculate the exact value of the following.

**a** $\cos^2\left(\frac{7\pi}{6}\right) + \sin^2\left(\frac{7\pi}{6}\right)$ **b** $\cos\left(\frac{7\pi}{6}\right) + \sin\left(\frac{7\pi}{6}\right)$

**c** $\sin^2\left(\frac{7}{6}\right) + \cos^2\left(\frac{7}{6}\right)$ **d** $\sin^2(76°) + \cos^2(76°)$

**e** $\sin^2(t) + \cos^2(t)$; explain the result with reference to the unit circle

**18** **a** Obtain the exact Cartesian coordinates of the trigonometric points $P\left[\frac{7\pi}{4}\right]$ and $Q\left[\frac{\pi}{4}\right]$, and describe the relative position of the points P and Q on the unit circle.

**b** Obtain the exact Cartesian coordinates of the trigonometric points $R\left[\frac{4\pi}{5}\right]$ and $S\left[\frac{\pi}{5}\right]$, and describe the relative position of these points on the unit circle.

**c** Give the exact sine, cosine and tangent values of:

**i** $\frac{7\pi}{4}$ and $\frac{\pi}{4}$, and compare the values **ii** $\frac{4\pi}{5}$ and $\frac{\pi}{5}$, and compare the values.

# 9.5 Symmetry properties

There are relationships between the coordinates and associated trigonometric values of trigonometric points placed in symmetric positions in each of the four quadrants. There will now be investigated.

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## The signs of the sine, cosine and tangent values in the four quadrants

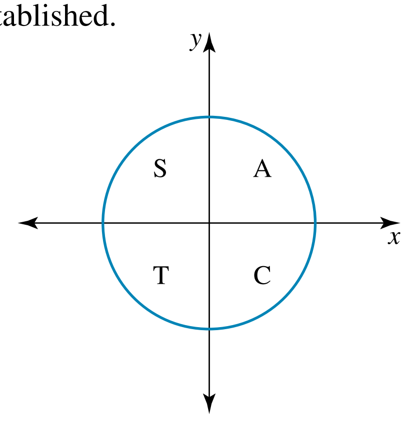
The definitions $\cos(\theta) = x$, $\sin(\theta) = y$, $\tan(\theta) = \frac{y}{x}$ where $(x, y)$ are the Cartesian coordinates of the trigonometric point [$\theta$] have been established.

If $\theta$ lies in the first quadrant, **A**ll of the trigonometric values will be positive, since $x > 0, y > 0$.

If $\theta$ lies in the second quadrant, only the **S**ine value will be positive, since $x < 0, y > 0$.

If $\theta$ lies in the third quadrant, only the **T**angent value will be positive, since $x < 0, y < 0$.

If $\theta$ lies in the fourth quadrant, only the **C**osine value will be positive, since $x > 0, y < 0$.




This is illustrated in the diagram shown.

There are several mnemonics for remembering the allocation of signs in this diagram: we shall use 'CAST' and refer to the diagram as the CAST diagram.

### The sine, cosine and tangent values at the boundaries of the quadrants

The points which do not lie within a quadrant are the four coordinate axes intercepts of the unit circle. These are called the boundary points. Since the Cartesian coordinates and the trigonometric positions of these points are known, the boundary values can be summarised by the following table.

| Boundary point | (1, 0) | (0, 1) | (−1, 0) | (0, −1) | (1, 0) |
|---|---|---|---|---|---|
| $\theta$ radians | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
| $\theta$ degrees | 0° | 90° | 180° | 270° | 360° |
| $\sin(\theta)$ | 0 | 1 | 0 | −1 | 0 |
| $\cos(\theta)$ | 1 | 0 | −1 | 0 | 1 |
| $\tan(\theta)$ | 0 | undefined | 0 | undefined | 0 |

Other values of $\theta$ could be used for the boundary points, including negative values.

WORKED EXAMPLE 11

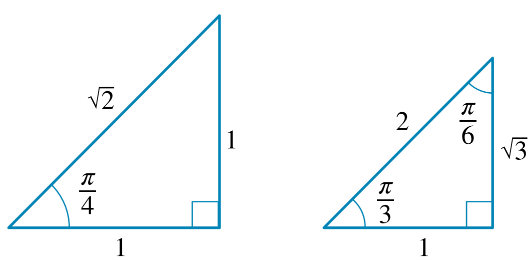
**a** **Identify the quadrant(s) where both $\cos(\theta)$ and $\sin(\theta)$ are negative.**

**b** **If $f(\theta) = \cos(\theta)$, evaluate $f(-6\pi)$.**

| THINK | WRITE |
|---|---|
| **a** Refer to the CAST diagram. | **a** $\cos(\theta) = x,\ \sin(\theta) = y$<br>The quadrant where both $x$ and $y$ are negative is quadrant 3. |
| **b** 1 Substitute the given value in the function rule. | **b** $f(\theta) = \cos(\theta)$<br>$\therefore f(-6\pi) = \cos(-6\pi)$ |
| 2 Identify the Cartesian coordinates of the trigonometric point. | A clockwise rotation of $6\pi$ from (1, 0) shows that the trigonometric point $[-6\pi]$ is the boundary point with coordinates (1, 0). |
| 3 Evaluate the required value of the function. | The $x$-coordinate of the boundary point gives the cosine value.<br>$\cos(-6\pi) = 1$<br>$\therefore f(-6\pi) = 1$ |

### Exact trigonometric values of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$

As the exact trigonometric ratios are known for angles of 30°, 45° and 60°, these give the trigonometric ratios for $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ respectively. A summary of these is given with the angles in each triangle expressed in radian measure. The values should be memorised.

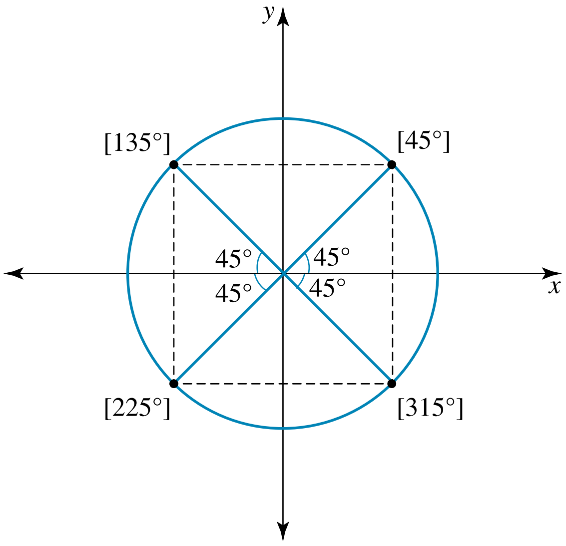


| $\theta$ | $\frac{\pi}{6}$ or 30° | $\frac{\pi}{4}$ or 45° | $\frac{\pi}{3}$ or 60° |
|---|---|---|---|
| $\sin(\theta)$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos(\theta)$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |
| $\tan(\theta)$ | $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ |

These values can be used to calculate the exact trigonometric values for other angles which lie in positions symmetric to these first-quadrant angles.

## Trigonometric points symmetric to [$\theta$] where $\theta \in \left\{30°, 45°, 60°, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\}$

The symmetrical points to [45°] are shown in the diagram.




Each radius of the circle drawn to each of the points makes an acute angle of 45° with either the positive or the negative $x$-axis. The symmetric points to [45°] are the endpoints of a rotation which is 45° short of, or 45° beyond, the horizontal $x$-axis. The calculations $180° - 45°$, $180° + 45°$ and $360° - 45°$ give the symmetric trigonometric points [135°], [225°] and [315°] respectively.

Comparisons between the coordinates of these trigonometric points with those of the first quadrant point [45°] enable the trigonometric values of these non-acute angles to be calculated from those of the acute angle 45°.

Consider the $y$-coordinate of each point.

As the $y$-coordinates of the trigonometric points [135°] and [45°] are the same, $\sin(135°) = \sin(45°)$. Similarly, the $y$-coordinates of the trigonometric points [225°] and [315°] are the same, but both are the negative of the $y$-coordinate of [45°]. Hence, $\sin(225°) = \sin(315°) = -\sin(45°)$. This gives the following exact sine values:

$$\sin(45°) = \frac{\sqrt{2}}{2}; \sin(135°) = \frac{\sqrt{2}}{2}; \sin(225°) = -\frac{\sqrt{2}}{2}; \sin(315°) = -\frac{\sqrt{2}}{2}$$

Now consider the $x$-coordinate of each point.

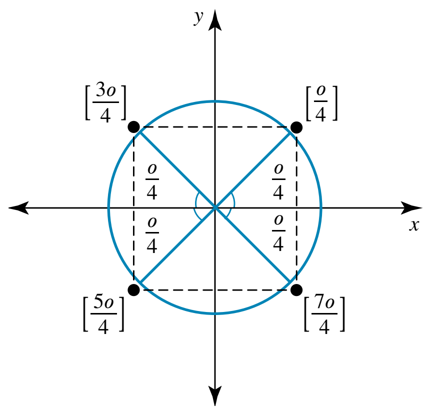
As the $x$-coordinates of the trigonometric points [315°] and [45°] are the same, $\cos(315°) = \cos(45°)$. Similarly, the $x$-coordinates of the trigonometric points [135°] and [225°] are the same but both are the negative of the $x$-coordinate of [45°]. Hence, $\cos(135°) = \cos(225°) = -\cos(45°)$. This gives the following exact cosine values:

$$\cos(45°) = \frac{\sqrt{2}}{2}; \cos(135°) = -\frac{\sqrt{2}}{2}; \cos(225°) = -\frac{\sqrt{2}}{2}; \cos(315°) = \frac{\sqrt{2}}{2}$$

Either by considering the intercepts cut off on the vertical tangent drawn at $(1, 0)$ or by using $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$, you will find that the corresponding relationships for the four points are $\tan(225°) = \tan(45°)$ and $\tan(135°) = \tan(315°) = -\tan(45°)$. Hence the exact tangent values are:

$$\tan(45°) = 1; \tan(135°) = -1; \tan(225°) = 1; \tan(315°) = -1$$

The relationships between the Cartesian coordinates of [45°] and each of [135°], [225°] and [315°] enable the trigonometric values of 135°, 225° and 315° to be calculated from those of 45°.



If, instead of degree measure, the radian measure of $\frac{\pi}{4}$ is used, the symmetric points to $\left[\frac{\pi}{4}\right]$ are the endpoints of rotations which lie $\frac{\pi}{4}$ short of, or $\frac{\pi}{4}$ beyond, the horizontal $x$-axis. The positions of the symmetric points are calculated as $\pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$, giving the symmetric trigonometric points $\left[\frac{3\pi}{4}\right], \left[\frac{5\pi}{4}\right], \left[\frac{7\pi}{4}\right]$ respectively.

By comparing the Cartesian coordinates of the symmetric points with those of the first quadrant point $\left[\frac{\pi}{4}\right]$, it is possible to obtain results such as the following selection:

Second quadrant

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Third quadrant

$$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

Fourth quadrant

$$\sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

A similar approach is used to generate symmetric points to the first quadrant points $\left[\frac{\pi}{6}\right]$ and $\left[\frac{\pi}{3}\right]$.

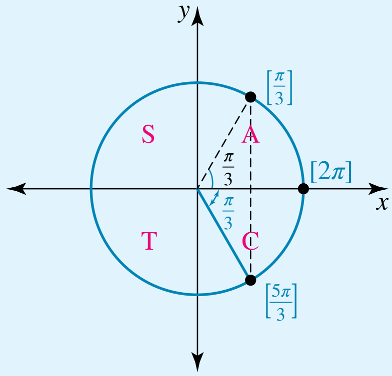
**WORKED EXAMPLE 12**

**Calculate the exact values of the following.**

**a** $\cos\left(\frac{5\pi}{3}\right)$ **b** $\sin\left(\frac{7\pi}{6}\right)$ **c** $\tan(-30°)$

| THINK | WRITE |
|---|---|
| **a 1** State the quadrant in which the trigonometric point lies. | **a** $\cos\left(\frac{5\pi}{3}\right)$<br>As $\frac{5\pi}{3} = \frac{5}{3}\pi = 1\frac{2}{3}\pi$,<br>the point $\left[\frac{5\pi}{3}\right]$ lies in quadrant 4. |

**2** Identify the first-quadrant symmetric point.




Since $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, the rotation of $\frac{5\pi}{3}$ stops short of the $x$-axis by $\frac{\pi}{3}$. The points $\left[\frac{\pi}{3}\right]$ and $\left[\frac{5\pi}{3}\right]$ are symmetric.

**3** Compare the coordinates of the symmetric points and obtain the required value.

*Note:* Check the +/− sign follows the CAST diagram rule.

The $x$-coordinates of the symmetric points are equal.

$$\cos\left(\frac{5\pi}{3}\right) = +\cos\left(\frac{\pi}{3}\right)$$
$$= \frac{1}{2}$$

Check: cosine is positive in quadrant 4.

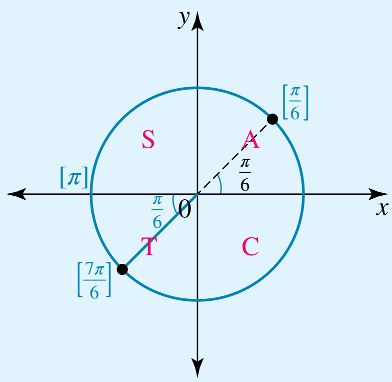
**b 1** State the quadrant in which the trigonometric point lies.

**b** $\sin\left(\frac{7\pi}{6}\right)$

$\frac{7\pi}{6} = 1\frac{1}{6}\pi$

The point lies in quadrant 3.

**2** Identify the first-quadrant symmetric point.




As $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$, the rotation of $\frac{7\pi}{6}$ goes beyond the $x$-axis by $\frac{\pi}{6}$. The points $\left[\frac{\pi}{6}\right]$ and $\left[\frac{7\pi}{6}\right]$ are symmetric.

3 Compare the coordinates of the symmetric points and obtain the required value.

The $y$-coordinate of $\left[\frac{7\pi}{6}\right]$ is the negative of that of $\left[\frac{\pi}{6}\right]$ in the first quadrant.

$$\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$$
$$= -\frac{1}{2}$$

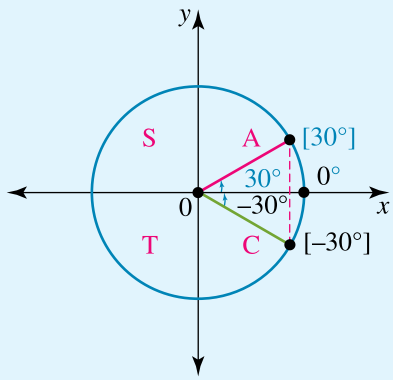
Check: sine is negative in quadrant 3.

**c** 1 State the quadrant in which the trigonometric point lies.

**c** $\tan(-30°)$

$[-30°]$ lies in quadrant 4.

2 Identify the first-quadrant symmetric point.




$-30°$ is a clockwise rotation of $30°$ from the horizontal so the symmetric point in the first quadrant is $[30°]$.

3 Compare the coordinates of the symmetric points and obtain the required value.

*Note:* Alternatively, consider the intercepts that would be cut off on the vertical tangent at $(1, 0)$.

The points $[30°]$ and $[-30°]$ have the same $x$-coordinates but opposite $y$-coordinates. The tangent value is negative in quadrant 4.

$$\tan(-30°) = -\tan(30°)$$
$$= -\frac{\sqrt{3}}{3}$$

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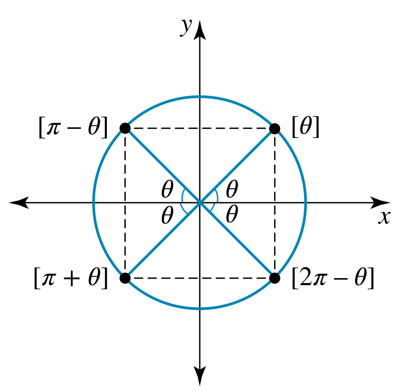
**Interactivity**
Symmetry points & quadrants
**int-2584**

## Symmetry properties

The **symmetry properties** give the relationships between the trigonometric values in quadrants 2, 3, 4 and that of the first quadrant value, called the base, with which they are symmetric. The symmetry properties are simply a generalisation of what was covered for the bases $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.

For any real number $\theta$ where $0 < \theta < \frac{\pi}{2}$, the trigonometric point $[\theta]$ lies in the first quadrant. The other quadrant values can be expressed in terms of the base $\theta$, since the symmetric values will either be $\theta$ short of, or $\theta$ beyond, the horizontal $x$-axis.




The symmetric points to $[\theta]$ are:

- second quadrant $[\pi - \theta]$
- third quadrant $[\pi + \theta]$
- fourth quadrant $[2\pi - \theta]$

Comparing the Cartesian coordinates with those of the first-quadrant base leads to the following general statements.

The symmetry properties for the second quadrant are:

$$\sin(\pi - \theta) = \sin(\theta)$$
$$\cos(\pi - \theta) = -\cos(\theta)$$
$$\tan(\pi - \theta) = -\tan(\theta)$$

The symmetry properties for the third quadrant are:

$$\sin(\pi + \theta) = -\sin(\theta)$$
$$\cos(\pi + \theta) = -\cos(\theta)$$
$$\tan(\pi + \theta) = \tan(\theta)$$

The symmetry properties for the fourth quadrant are:

$$\sin(2\pi - \theta) = -\sin(\theta)$$
$$\cos(2\pi - \theta) = \cos(\theta)$$
$$\tan(2\pi - \theta) = -\tan(\theta)$$

## Other forms for the symmetric points

The rotation assigned to a point is not unique. With clockwise rotations or repeated revolutions, other values are always possible. However, the symmetry properties apply no matter how the points are described.

The trigonometric point $[2\pi + \theta]$ would lie in the first quadrant where all ratios are positive. Hence:

$$\sin(2\pi + \theta) = \sin(\theta)$$
$$\cos(2\pi + \theta) = \cos(\theta)$$
$$\tan(2\pi + \theta) = \tan(\theta)$$

The trigonometric point $[-\theta]$ would lie in the fourth quadrant where only cosine is positive. Hence:

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$

For negative rotations, the points symmetric to $[\theta]$ could be given as:

- fourth quadrant $[-\theta]$
- third quadrant $[-\pi + \theta]$
- second quadrant $[-\pi - \theta]$
- first quadrant $[-2\pi + \theta]$

## Using symmetry properties to calculate values of trigonometric functions

Trigonometric values are either the same as, or the negative of, the associated trigonometric values of the first-quadrant base; the sign is determined by the CAST diagram.

The base involved is identified by noting the rotation needed to reach the $x$-axis and determining how far short of or how far beyond this the symmetric point is. It is important to emphasise that for the points to be symmetric this is always measured from the horizontal and not the vertical axis.

**To calculate a value of a trigonometric function, follow these steps.**
- **Locate the quadrant in which the trigonometric point lies**
- **Identify the first-quadrant base with which the trigonometric point is symmetric**
- **Compare the coordinates of the trigonometric point with the coordinates of the base point or use the CAST diagram rule to form the sign in the first instance**
- **Evaluate the required value exactly if there is a known exact value involving the base.**

With practice, the symmetry properties allow us to recognise, for example, that $\sin\left(\frac{8\pi}{7}\right) = -\sin\left(\frac{\pi}{7}\right)$ because $\frac{8\pi}{7} = \pi + \frac{\pi}{7}$ and sine is negative in the third quadrant. Recognition of the symmetry properties is very important and we should aim to be able to apply these quickly. For example, to evaluate $\cos\left(\frac{3\pi}{4}\right)$ think: 'Second quadrant; cosine is negative; base is $\frac{\pi}{4}$,' and write:

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$$
$$= -\frac{\sqrt{2}}{2}$$

WORKED EXAMPLE 13

**a** **Identify the symmetric points to [20°]. At which of these points is the tangent value the same as tan (20°)?**

**b** **Express $\sin\left(\frac{6\pi}{5}\right)$ in terms of a first-quadrant value.**

**c** **If $\cos(\theta) = 0.6$, give the values of $\cos(\pi - \theta)$ and $\cos(2\pi - \theta)$.**

**d** **Calculate the exact value of the following.**

**i** $\tan\left(\frac{7\pi}{6}\right)$ **ii** $\sin\left(\frac{11\pi}{3}\right)$

| THINK | WRITE |
|---|---|
| **a** **1** Calculate the symmetric points to the given point. | **a** Symmetric points to [20°] will be ±20° from the $x$-axis. The points are:<br>second quadrant [180° − 20°] = [160°]<br>third quadrant [180° + 20°] = [200°]<br>fourth quadrant [360° − 20°] = [340°] |
| **2** Identify the quadrant. | The point [20°] is in the first quadrant so tan (20°) is positive. As tangent is also positive in the third quadrant, tan (200°) = tan (20°). |
| **3** State the required point. | The tangent value at the trigonometric point [200°] has the same value as tan (20°). |

**b** **1** Express the trigonometric value in the appropriate quadrant form.

**b** $\frac{6\pi}{5}$ is in the third quadrant.

$$\sin\left(\frac{6\pi}{5}\right) = \sin\left(\pi + \frac{\pi}{5}\right)$$

**2** Apply the symmetry property for that quadrant.

$$\therefore \sin\left(\frac{6\pi}{5}\right) = -\sin\left(\frac{\pi}{5}\right)$$

**c** **1** Use the symmetry property for the appropriate quadrant.

**c** $(\pi - \theta)$ is second quadrant form.
$\therefore \cos(\pi - \theta) = -\cos(\theta)$

**2** State the answer.

Since $\cos(\theta) = 0.6$, $\cos(\pi - \theta) = -0.6$.

**3** Use the symmetry property for the appropriate quadrant.

$2\pi - \theta$ is fourth quadrant form.
$\therefore \cos(2\pi - \theta) = \cos(\theta)$

**4** State the answer.

Since $\cos(\theta) = 0.6$, $\cos(2\pi - \theta) = 0.6$.

**d** **i** **1** Express the trigonometric value in an appropriate quadrant form.

**d** **i** $$\tan\left(\frac{7\pi}{6}\right) = \tan\left(\pi + \frac{\pi}{6}\right)$$

**2** Apply the symmetry property and evaluate.

$$= \tan\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{3}$$
$$\therefore \tan\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

**ii** **1** Express the trigonometric value in an appropriate quadrant form.

**ii** $\frac{11\pi}{3}$ is in quadrant 4.

$$\sin\left(\frac{11\pi}{3}\right) = \sin\left(4\pi - \frac{\pi}{3}\right)$$

**2** Apply the symmetry property and evaluate.

$$= \sin\left(2\pi - \frac{\pi}{3}\right)$$
$$= -\sin\left(\frac{\pi}{3}\right)$$
$$= -\frac{\sqrt{3}}{2}$$
$$\therefore \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

## EXERCISE 9.5 Symmetry properties

### PRACTISE

1 WE11 **a** Identify the quadrant(s) where $\cos(\theta)$ is negative and $\tan(\theta)$ is positive.
**b** If $f(\theta) = \tan(\theta)$, evaluate $f(4\pi)$.

2 If $f(t) = \sin(\pi t)$, evaluate $f(2.5)$.

3 WE12 Calculate the exact values of the following.

a $\sin\left(\frac{4\pi}{3}\right)$ b $\tan\left(\frac{5\pi}{6}\right)$ c $\cos(-30°)$

4 Calculate the exact values of $\sin\left(-\frac{5\pi}{4}\right)$, $\cos\left(-\frac{5\pi}{4}\right)$ and $\tan\left(-\frac{5\pi}{4}\right)$.

5 WE13 a Identify the symmetric points to [75°]. At which of these points is the cosine value the same as $\cos(75°)$?

b Express $\tan\left(\frac{6\pi}{7}\right)$ in terms of a first quadrant value.

c If $\sin(\theta) = 0.8$, give the values of $\sin(\pi - \theta)$ and $\sin(2\pi - \theta)$.

d Calculate the exact value of the following.

i $\cos\left(\frac{5\pi}{4}\right)$ ii $\sin\left(\frac{25\pi}{6}\right)$

6 Given $\cos(\theta) = p$, express the following in terms of $p$.

a $\cos(-\theta)$ b $\cos(5\pi + \theta)$

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 Show the boundary points on a diagram and then state the value of the following.

a $\cos(4\pi)$ b $\tan(9\pi)$ c $\sin(7\pi)$

d $\sin\left(\frac{13\pi}{2}\right)$ e $\cos\left(-\frac{9\pi}{2}\right)$ f $\tan(-20\pi)$

8 Identify the quadrant(s), or boundaries, for which the following apply.

a $\cos(\theta) > 0, \sin(\theta) < 0$ b $\tan(\theta) > 0, \cos(\theta) > 0$

c $\sin(\theta) > 0, \cos(\theta) < 0$ d $\cos(\theta) = 0$

e $\cos(\theta) = 0, \sin(\theta) > 0$ f $\sin(\theta) = 0, \cos(\theta) < 0$

9 Determine positions for the points in quadrants 2, 3 and 4 which are symmetric to the trigonometric point [$\theta$] for which the value of $\theta$ is:

a $\frac{\pi}{3}$ b $\frac{\pi}{6}$ c $\frac{\pi}{4}$

d $\frac{\pi}{5}$ e $\frac{3\pi}{8}$ f 1

10 Calculate the exact values of the following.

a $\cos(120°)$ b $\tan(225°)$ c $\sin(330°)$

d $\tan(-60°)$ e $\cos(-315°)$ f $\sin(510°)$

11 Calculate the exact values of the following.

a $\sin\left(\frac{3\pi}{4}\right)$ b $\tan\left(\frac{2\pi}{3}\right)$ c $\cos\left(\frac{5\pi}{6}\right)$

d $\cos\left(\frac{4\pi}{3}\right)$ e $\tan\left(\frac{7\pi}{6}\right)$ f $\sin\left(\frac{11\pi}{6}\right)$

12 Calculate the exact values of the following.

a $\cos\left(-\frac{\pi}{4}\right)$ b $\sin\left(-\frac{\pi}{3}\right)$ c $\tan\left(-\frac{5\pi}{6}\right)$

d $\sin\left(\frac{8\pi}{3}\right)$ e $\cos\left(\frac{9\pi}{4}\right)$ f $\tan\left(\frac{23\pi}{6}\right)$

13 Calculate the exact values of the following.

a $\cos\left(\frac{7\pi}{6}\right) + \cos\left(\frac{2\pi}{3}\right)$

b $2\sin\left(\frac{7\pi}{4}\right) + 4\sin\left(\frac{5\pi}{6}\right)$

c $\sqrt{3}\tan\left(\frac{5\pi}{4}\right) - \tan\left(\frac{5\pi}{3}\right)$

d $\sin\left(\frac{8\pi}{9}\right) + \sin\left(\frac{10\pi}{9}\right)$

e $2\cos^2\left(-\frac{5\pi}{4}\right) - 1$

f $\dfrac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

14 Given $\cos(\theta) = 0.91$, $\sin(t) = 0.43$ and $\tan(x) = 0.47$, use the symmetry properties to obtain the values of the following.

a $\cos(\pi + \theta)$
b $\sin(\pi - t)$
c $\tan(2\pi - x)$
d $\cos(-\theta)$
e $\sin(-t)$
f $\tan(2\pi + x)$

15 If $\sin(\theta) = p$, express the following in terms of $p$.

a $\sin(2\pi - \theta)$
b $\sin(3\pi - \theta)$
c $\sin(-\pi + \theta)$
d $\sin(\theta + 4\pi)$

16 a Verify that $\sin^2\left(\frac{5\pi}{4}\right) + \cos^2\left(\frac{5\pi}{4}\right) = 1$.

b Explain, with the aid of a unit circle diagram, why $\cos(-\theta) = \cos(\theta)$ is true for $\theta = \frac{5\pi}{6}$.

c The point $[\phi]$ lies in the second quadrant and has Cartesian coordinates $(-0.5, 0.87)$. Show this on a diagram and give the values of $\sin(\pi + \phi)$, $\cos(\pi + \phi)$ and $\tan(\pi + \phi)$.

d Simplify $\sin(-\pi + t) + \sin(-3\pi - t) + \sin(t + 6\pi)$.

e Use the unit circle to give two values of an angle $A$ for which $\sin(A) = \sin(144°)$.

f With the aid of the unit circle, give three values of $B$ for which $\sin(B) = -\sin\left(\frac{2\pi}{11}\right)$.

MASTER

17 a Identify the quadrant in which the point P[4.2] lies.

b Calculate the Cartesian coordinates of point P[4.2] to 2 decimal places.

c Identify the trigonometric positions, to 4 decimal places, of the points in the other three quadrants which are symmetric to the point P[4.2].

18 Consider the point $Q[\theta]$, $\tan(\theta) = 5$.

a In which two quadrants could Q lie?

b Determine, to 4 decimal places, the value of $\theta$ for each of the two points.

c Calculate the exact sine and cosine values for each $\theta$ and state the exact Cartesian coordinates of each of the two points.

# 9.6 Graphs of the sine and cosine functions

As the two functions sine and cosine are closely related, we shall initially consider their graphs together.

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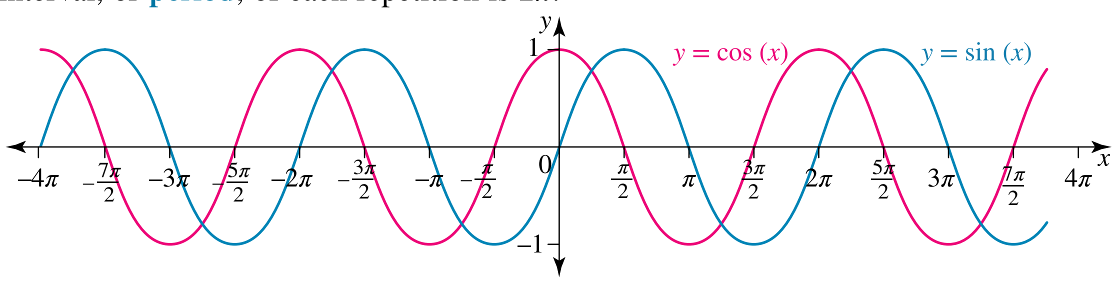
## The graphs of $y = \sin(x)$ and $y = \cos(x)$

The functions sine and cosine are both periodic and have many-to-one correspondences, which means values repeat after every revolution around the unit circle. This means both functions have a period of $2\pi$ since $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$.

The graphs of $y = \sin(x)$ and $y = \cos(x)$ can be plotted using the boundary values from continued rotations, clockwise and anticlockwise, around the unit circle.

| $x$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $\sin(x)$ | 0 | 1 | 0 | −1 | 0 |
| $\cos(x)$ | 1 | 0 | −1 | 0 | 1 |

The diagram shows four cycles of the graphs drawn on the domain $[-4\pi, 4\pi]$. The graphs continue to repeat their wavelike pattern over their maximal domain $R$; the interval, or **period**, of each repetition is $2\pi$.




The first observation that strikes us about these graphs is how remarkably similar they are: a horizontal translation of $\frac{\pi}{2}$ to the right will transform the graph of $y = \cos(x)$ into the graph of $y = \sin(x)$, while a horizontal translation of $\frac{\pi}{2}$ to the left transforms the graph of $y = \sin(x)$ into the graph of $y = \cos(x)$.

Recalling our knowledge of transformations of graphs, this observation can be expressed as:

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

The two functions are said to be '**out of phase**' by $\frac{\pi}{2}$ or to have a **phase difference** or **phase shift** of $\frac{\pi}{2}$.

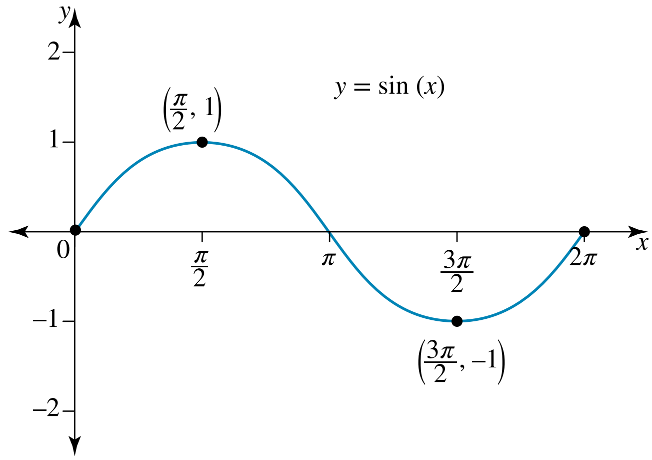
Both graphs oscillate up and down one unit from the $x$-axis. The $x$-axis is the **equilibrium** or **mean position** and the distance the graphs oscillate up and down from this mean position to a maximum or minimum point is called the **amplitude**.

The graphs keep repeating this cycle of oscillations up and down from the equilibrium position, with the amplitude measuring half the vertical distance between maximum and minimum points and the period measuring the horizontal distance between successive maximum points or between successive minimum points.

## One cycle of the graph of $y = \sin(x)$

The basic graph of $y = \sin(x)$ has the domain $[0, 2\pi]$, which restricts the graph to one cycle.

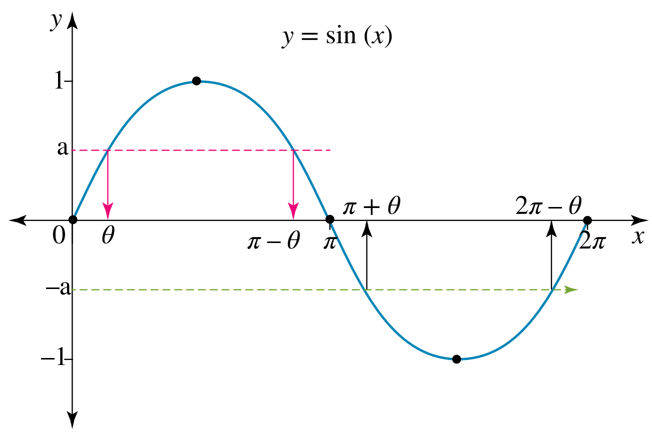
The graph of the function $f: [0, 2\pi] \to R,\ f(x) = \sin(x)$ is shown.




**Key features of the graph of $y = \sin(x)$:**

- **Equilibrium position is the $x$-axis, the line with equation $y = 0$.**
- **Amplitude is 1 unit.**
- **Period is $2\pi$ units.**
- **Domain is $[0, 2\pi]$.**
- **Range is $[-1, 1]$.**
- **The $x$-intercepts occur at $x = 0, \pi, 2\pi$.**
- **Type of correspondence is many-to-one.**

The graph lies above the $x$-axis for $x \in (0, \pi)$ and below for $x \in (\pi, 2\pi)$, which matches the quadrant signs of sine given in the CAST diagram. The symmetry properties of sine are displayed in its graph as $\sin(\pi - \theta) = \sin(\theta)$ and $\sin(\pi + \theta) = \sin(2\pi - \theta) = -\sin(\theta)$.



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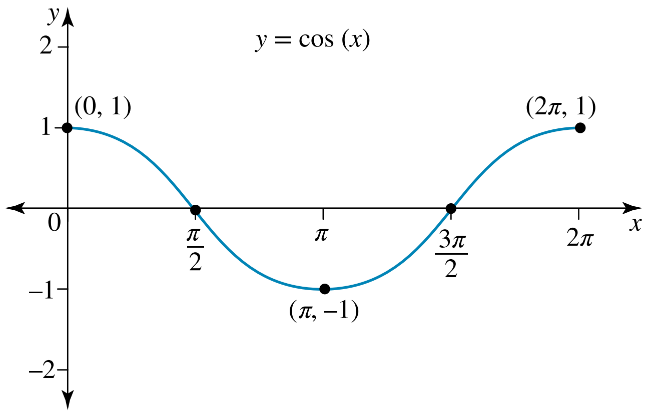



**Key features of the graph of $y = \cos(x)$:**

- **Equilibrium position is the $x$-axis, the line with equation $y = 0$.**
- **Amplitude is 1 unit.**
- **Period is $2\pi$ units.**
- **Domain is $[0, 2\pi]$.**
- **Range is $[-1, 1]$.**
- **The $x$-intercepts occur at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.**
- **Type of correspondence is many-to-one.**

The graph of $y = \cos(x)$ has the same amplitude, period, equilibrium (or mean) position, domain, range and type of correspondence as the graph of $y = \sin(x)$.

### Guide to sketching the graphs on extended domains

There is a pattern of 5 points to the shape of the basic sine and cosine graphs created by the division of the period into four equal intervals.

For $y = \sin(x)$: first point starts at the equilibrium; the second point at $\frac{1}{4}$ of the period, reaches up one amplitude to the maximum point; the third point, at $\frac{1}{2}$ of the period, is back at equilibrium; the fourth point at $\frac{3}{4}$ of the period goes down one amplitude to the minimum point; the fifth point at the end of the period interval returns back to equilibrium.

In other words:

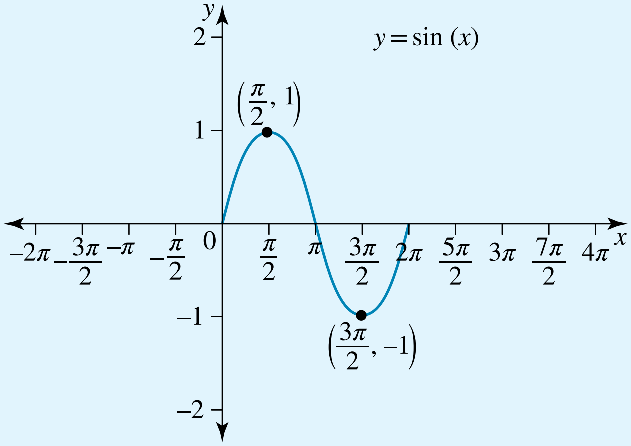
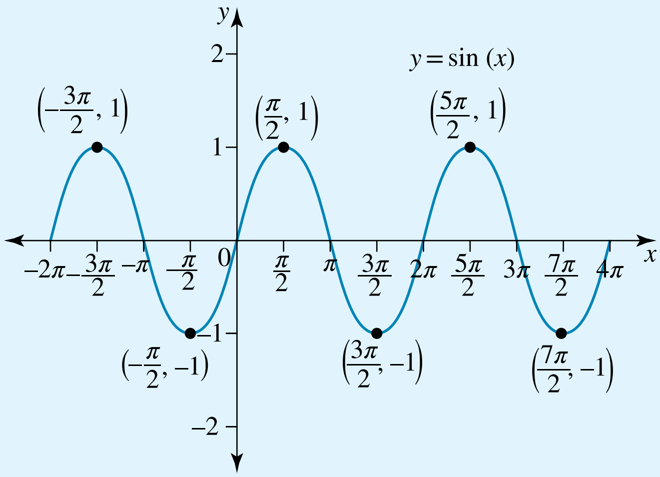
equilibrium → range maximum → equilibrium → range minimum → equilibrium

For $y = \cos(x)$: the pattern for one cycle is summarised as:

range maximum → equilibrium → range minimum → equilibrium → range maximum

This pattern only needs to be continued in order to sketch the graph of $y = \sin(x)$ or $y = \cos(x)$ on a domain other than $[0, 2\pi]$.

**WORKED EXAMPLE 14** Sketch the graph of $y = \sin(x)$ over the domain $[-2\pi, 4\pi]$ and state the number of cycles of the sine function drawn.

| THINK | WRITE |
|---|---|
| 1 Draw the graph of the function over $[0, 2\pi]$. | The basic graph of $y = \sin(x)$ over the domain $[0, 2\pi]$ is drawn.  |
| 2 Extend the pattern to cover the domain specified. | The pattern is extended for one cycle in the negative direction and one further cycle in the positive direction to cover the domain $[-2\pi, 3\pi]$.  |
| 3 State the number of cycles of the function that are shown in the graph. | Altogether, 3 cycles of the sine function have been drawn. |

## Graphical and numerical solutions to equations

The sine and cosine functions are **transcendental functions**, meaning they cannot be expressed as algebraic expressions in powers of $x$. There is no algebraic method of solution for an equation such as $\sin(x) = 1 - 2x$ because it contains a transcendental function and a linear polynomial function. However, whether solutions exist or not can usually be determined by graphing both functions to see if, or in how many places, they intersect. If a solution exists, an interval in which the root lies could be specified and the bisection method could refine this interval.

When sketching, care is needed with the scaling of the $x$-axis. The polynomial function is normally graphed using an integer scale whereas the trigonometric function normally uses multiples of $\pi$. It can be helpful to remember that $\pi \approx 3.14$, so $\frac{\pi}{2} \approx 1.57$ and so on.

WORKED EXAMPLE 15

Consider the equation $\sin(x) = 1 - 2x$.

**a** Sketch the graphs of $y = \sin(x)$ and $y = 1 - 2x$ on the same set of axes and explain why the equation $\sin(x) = 1 - 2x$ has only one root.

**b** Use the graph to give an interval in which the root of the equation $\sin(x) = 1 - 2x$ lies.

**c** Use the bisection method to create two narrower intervals for the root and hence give an estimate of its value.

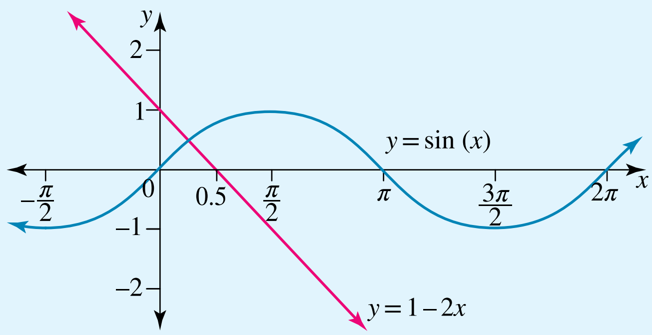
**THINK**

**a 1** Calculate the points needed to sketch the two graphs.

**WRITE**

**a** $y = \sin(x)$
One cycle of this graph has a domain $[0, 2\pi]$. The axis intercepts are $(0, 0)$, $(\pi, 0)$ and $(2\pi, 0)$.
$y = 1 - 2x$ has axis intercepts at $(0, 1)$ and $(0.5, 0)$.

**2** Sketch the graphs on the same set of axes.




**3** Give an explanation about the number of roots to the equation.

The two graphs intersect at one point only so the equation $\sin(x) = 1 - 2x$ has only one root.

**b** State an interval in which the root of the equation lies.

**b** From the graph it can be seen that the root lies between the origin and the $x$-intercept of the line. The interval in which the root lies is therefore $[0, 0.5]$.

**c 1** Define the function whose sign is to be tested in the bisection method procedure.

**c** $\sin(x) = 1 - 2x$
$\therefore \sin(x) - 1 + 2x = 0$
Let $f(x) = \sin(x) - 1 + 2x$

**2** Test the sign at the endpoints of the interval in which the root has been placed.

At $x = 0$, the sine graph lies below the line so $f(0) < 0$.
Check:

$$\begin{aligned} f(0) &= \sin(0) - 1 + 2(0) \\ &= -1 \\ &< 0 \end{aligned}$$

At $x = 0.5$, the sine graph lies above the line so $f(0.5) > 0$.
Check:

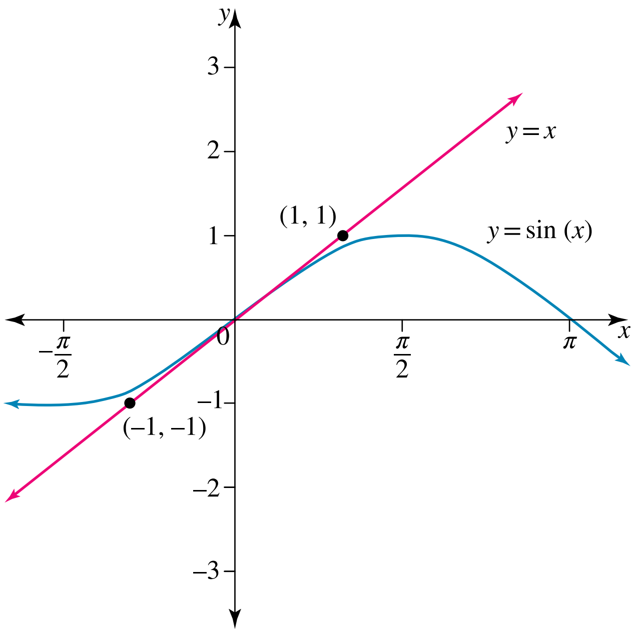
$$\begin{aligned} f(0.5) &= \sin(0.5) - 1 + 2(0.5) \\ &= \sin(0.5^c) - 1 + 1 \\ &= 0.48\ldots \\ &> 0 \end{aligned}$$

| | |
|---|---|
| **3** Create the first of the narrower intervals. | Midpoint of $[0, 0.5]$ is $x = 0.25$.<br>$f(0.25) = \sin(0.25) - 1 + 2(0.25)$<br>$= -0.25\ldots$<br>$< 0$<br>The root lies in the interval $[0.25, 0.5]$. |
| **4** Create the second interval. | Midpoint of $[0.25, 0.5]$ is $x = 0.375$.<br>$f(0.375) = \sin(0.375) - 1 + 2(0.375)$<br>$= 0.116\ldots$<br>$> 0$<br>The root lies in the interval $[0.25, 0.375]$. |
| **5** State an estimated value of the root of the equation. | The midpoint of $[0.25, 0.375]$ is an estimate of the root.<br>An estimate is $x = 0.3125$. |

## Approximations to the sine and cosine functions for values close to zero

Despite the sine and cosine functions having no algebraic form, for small values of $x$ it is possible to approximate them by simple polynomial functions for a domain close to zero.

Comparing the graphs of $y = \sin(x)$ and $y = x$ we can see the two graphs resemble each other for a domain around $x = 0$.

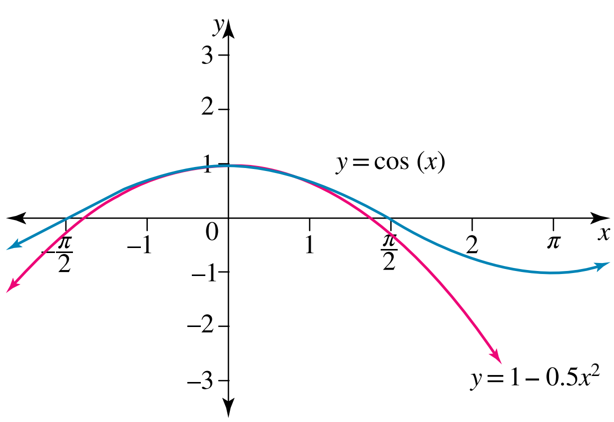



**For small $x$, $\sin(x) \approx x$.**

This offers another way to obtain an estimate of the root of the equation $\sin(x) = 1 - 2x$, as the graphs drawn in Worked Example 15 placed the root in the small interval $[0, 0.5]$.

Replacing $\sin(x)$ by $x$, the equation becomes $x = 1 - 2x$, the solution to which is $x = \frac{1}{3}$ or 0.333...
To 1-decimal-place accuracy, this value agrees with that obtained with greater effort using the bisection method.

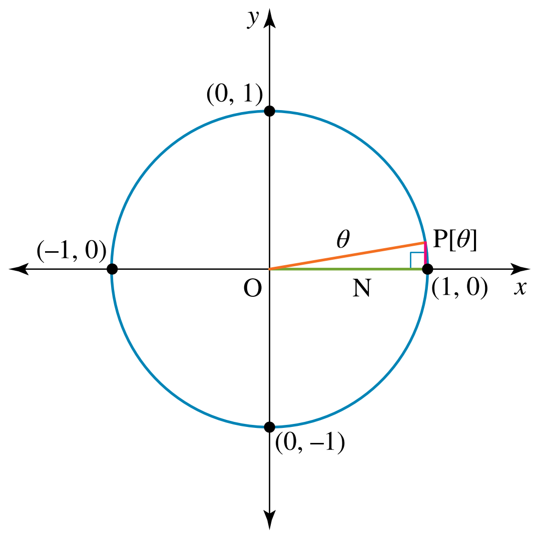
The graph of the cosine function around $x = 0$ suggests a quadratic polynomial could approximate $\cos(x)$ for a small section of its domain around $x = 0$.



Comparing the graphs of $y = \cos(x)$ and $y = 1 - 0.5x^2$ for small $x$, show the two graphs are close together for a small part of the domain around $x = 0$.

**For small $x$, $\cos(x) \approx 1 - 0.5x^2$**

Returning to the unit circle definitions of the sine and cosine functions, the line segments whose lengths give the values of sine and cosine are shown in the unit circle diagram for a small value of $\theta$, the angle of rotation.




The length PN $= \sin(\theta)$ and for small $\theta$, this length is approximately the same as the length of the arc which subtends the angle $\theta$ at the centre of the unit circle. The arc length is $r\theta = 1\theta = \theta$. Hence, for small $\theta$, $\sin(\theta) \approx \theta$.

The length ON $= \cos(\theta)$ and for small $\theta$, ON $\approx 1$, the length of the radius of the circle. Hence, for small $\theta$, $\cos(\theta) \approx 1$. As the polynomial $1 - 0.5x^2 \approx 1$ for small $x$, this remains consistent with the unit circle observation.

**WORKED EXAMPLE 16**

**a** **Use the linear approximation for $\sin(x)$ to evaluate $\sin(3°)$ and compare the accuracy of the approximation with the calculator value for $\sin(3°)$.**

**b** **Show there is a root to the equation $\cos(x) - 12x^2 = 0$ for which $0 \le x \le 0.4$.**

**c** **Use the quadratic approximation $1 - 0.5x^2$ for $\cos(x)$ to obtain an estimate of the root in part b, expressed to 4 decimal places.**

| THINK | WRITE |
|---|---|
| **a 1** Express the angle in radian mode. | **a** $3° = 3 \times \frac{\pi}{180}$<br>$\therefore 3° = \frac{\pi^c}{60}$ |
| **2** Use the approximation for the trigonometric function to estimate the value of the trigonometric ratio. | $\sin(x) \approx x$ for small values of $x$.<br>As $\frac{\pi}{60}$ is small,<br>$\sin\left(\frac{\pi^c}{60}\right) \approx \frac{\pi}{60}$<br>$\therefore \sin(3°) \approx \frac{\pi}{60}$ |
| **3** Compare the approximate value with that given by a calculator for the value of the trigonometric ratio. | From a calculator, $\sin(3°) = 0.05234$ and $\frac{\pi}{60} = 0.05236$ to 5 decimal places.<br>The two values would be the same when expressed correct to 3 decimal places. |

| | |
|---|---|
| **b** Show there is a root to the equation in the given interval. | **b** $\cos(x) - 12x^2 = 0$<br>Let $f(x) = \cos(x) - 12x^2$<br>$f(0) = \cos(0) - 12(0)$<br>$= 1$<br>$> 0$<br>$f(0.4) = \cos(0.4) - 12(0.4)^2$<br>$= -0.9989\ldots$<br>$< 0$<br>Therefore, there is a root of the equation for which $0 \le x \le 0.4$. |
| **c** Use the approximation given to obtain an estimate of the root in part **b**. | **c** Values of $x$ in the interval $0 \le x \le 0.4$ are small and for small $x$, $\cos(x) \approx 1 - 0.5x^2$.<br>Let $\cos(x) = 1 - 0.5x^2$<br>The equation $\cos(x) - 12x^2 = 0$ becomes:<br>$0 = 1 - 0.5x^2 - 12x^2$<br>$1 = 12.5x^2$<br>$2 = 25x^2$<br>$x^2 = \frac{2}{25}$<br>$x = \pm\frac{\sqrt{2}}{5}$<br>$= \pm 0.2828$<br>The root for which $x$ is positive is 0.2828 to 4 decimal places. |

## EXERCISE 9.6 Graphs of the sine and cosine functions

**PRACTISE**

**1** WE14 Sketch the graph of $y = \cos(x)$ over the domain $[-2\pi, 4\pi]$ and state the number of cycles of the cosine function drawn.

**2** On the same set of axes, sketch the graphs of $y = \cos(x)$ and $y = \sin(x)$ over the domain $[0, 2\pi]$ and shade the region $\{(x, y) : \sin(x) \ge \cos(x), x \in [0, 2\pi]\}$.

**3** WE15 Consider the equation $\sin(x) = x - 2$.

**a** Sketch the graphs of $y = \sin(x)$ and $y = x - 2$ on the same set of axes and explain why the equation $\sin(x) = x - 2$ has only one root.

**b** Use the graph to give an interval in which the root of the equation $\sin(x) = x - 2$ lies.

**c** Use the bisection method to create two narrower intervals for the root and hence give an estimate of its value.

**4** Consider the equation $\cos(x) - x^2 = 0$.

**a** Give the equations of the two graphs whose intersection determines the number of solutions to the equation.

**b** Sketch the two graphs and hence determine the number of roots of the equation $\cos(x) - x^2 = 0$.

**c** For what value of $k$ will the equation $\cos(x) = x^2 + k$ have exactly one solution?

**5** WE16 **a** Use the linear approximation for $\sin(x)$ to evaluate $\sin(1.8°)$ and compare the accuracy of the approximation with the calculator value for $\sin(1.8°)$.

**b** Show there is a root to the equation $\cos(x) - 10.5x^2 = 0$ for which $0 \le x \le 0.4$.

**c** Use the quadratic approximation $1 - 0.5x^2$ for $\cos(x)$ to obtain an estimate of the root in part **b**, expressed to 4 decimal places.

**6 a** Evaluate $\cos(0.5)$ using the quadratic polynomial approximation for $\cos(x)$ and compare the value with that given by a calculator.

**b** Explain why the quadratic approximation is not applicable for calculating the value of $\cos(5)$.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

**7** Sketch the graphs of $y = \sin(x)$ and $y = \cos(x)$ over the given domain interval.

**a** $y = \sin(x), 0 \le x \le 6\pi$

**b** $y = \cos(x), -4\pi \le x \le 2\pi$

**c** $y = \cos(x), -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

**d** $y = \sin(x), -\frac{3\pi}{2} \le x \le \frac{\pi}{2}$.

**8 a** State the number of maximum turning points on the graph of the function $f: [-4\pi, 0] \to R,\ f(x) = \sin(x)$.

**b** State the number of minimum turning points of the graph of the function $f: [0, 14\pi] \to R,\ f(x) = \cos(x)$.

**c** Sketch the graph of $y = \cos(x), -4\pi \le x \le 5\pi$ and state the number of cycles of the cosine function drawn.

**9** State the number of intersections that the graphs of the following make with the $x$-axis.

**a** $y = \cos(x), 0 \le x \le \frac{7\pi}{2}$

**b** $y = \sin(x), -2\pi \le x \le 4\pi$

**c** $y = \sin(x), 0 \le x \le 20\pi$

**d** $y = \cos(x), \pi \le x \le 4\pi$

**10 a** The graph of the function $f: [0, a] \to R, f(x) = \cos(x)$ has 10 intersections with the $x$-axis. What is the smallest value possible for $a$?

**b** The graph of the function $f: [b, 5\pi] \to R, f(x) = \sin(x)$ has 6 turning points. If $f(b) = 0$, what is the value of $b$?

**c** If the graph of the function $f: [-c, c] \to R,\ f(x) = \sin(x)$ covers 2.5 periods of the sine function, what must the value of $c$ be?

**11 a** Draw one cycle of the cosine graph over $[0, 2\pi]$ and give the values of $x$ in this interval for which $\cos(x) < 0$.

**b** Explain how the graph in part **a** illustrates what the CAST diagram says about the sign of the cosine function.

**12 a** Determine the number of solutions to each of the following equations by drawing the graphs of an appropriate pair of functions.

**i** $\cos(x) - x^3 = 0$

**ii** $4\cos(x) - x = 0$

**iii** $\sin(x) - x^2 + 2x - 1 = 0$

**b** For the equation in part **a** which has one solution, state an interval between two integers in which the solution lies and apply three iterations of the bisection method to obtain an estimate of the solution.

**c** **i** For what values of $a$ will the equation $\sin(x) + ax^2 - 1 = 0$ have no solution?

**ii** How many solutions does the equation $\sin(x) + ax^2 - 1 = 0$ have if $a = 0$?

**13** Consider the equation $\sin(x) = x^3$.

**a** Show the equation has 3 solutions and state the exact value of one of these solutions.

**b** One of the solutions lies in the interval $[0, 2]$. Use the bisection method to obtain this solution with an accuracy that is correct to 1 decimal place.

**c** What is the value of the third solution?

**14** Use the linear approximation for $\sin(x)$ to calculate the following.

**a** $\sin(1°)$ **b** $\sin\left(\frac{\pi}{9}\right)$ **c** $\sin(-2°)$

**d** $\sin\left(\frac{\pi}{6}\right)$; comment on the reason for the discrepancy with its exact value.

**e** A person undergoing a particular type of eye test is looking at a round circle of radius 1 cm on a screen. The person's eye is at a distance of 24 cm from the centre of the circle. The two lines of sight of the person act tangentially to the circle and enclose an angle of $\theta$ degrees. Use an approximation to calculate the value of $\theta$ in terms of $\pi$ and hence state the magnitude of the angle between the lines of sight, to 2 decimal places.

**15** **a** Use the quadratic approximation $\cos(x) \approx 1 - \frac{1}{2}x^2$ to express the value of $\cos\left(-\frac{1}{4}\right)$ as a rational number.

**b** The solution to the equation $\cos(x) + 5x - 2 = 0$ is small. Use the quadratic approximation $\cos(x) \approx 1 - \frac{1}{2}x^2$ to obtain the solution as an irrational number.

**16** Consider the equation $4x\sin(x) - 1 = 0$.

**a** Show the equation has a solution for which $0 \leq x \leq 0.6$ and use the linear polynomial approximation for $\sin(x)$ to estimate this solution.

**b** Specify the function whose intersection with $y = \sin(x)$ determines the number of solutions to the equation and hence explain why there would be other positive solutions to the equation.

**c** Why were these other solutions not obtained using the linear approximation for $\sin(x)$?

**d** Analyse the behaviour of the function specified in part **b** to determine how many solutions of $4x\sin(x) - 1 = 0$ lie in the interval $[-4\pi, 4\pi]$.

**MASTER**

**17** **a** **i** Use a CAS technology to obtain the two solutions to the equation $\sin(x) = 1 - x^2$.

**ii** For the solution closer to zero, compare its value with that obtained using the linear approximation for $\sin(x)$.

**b** Investigate over what interval the approximation $\sin(x) = x$ is reasonable.

**18** Sketch the graph of $y = \cos^2(x)$ for $x \in [0, 4\pi]$ and state its period.

# 9.7 Review

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- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
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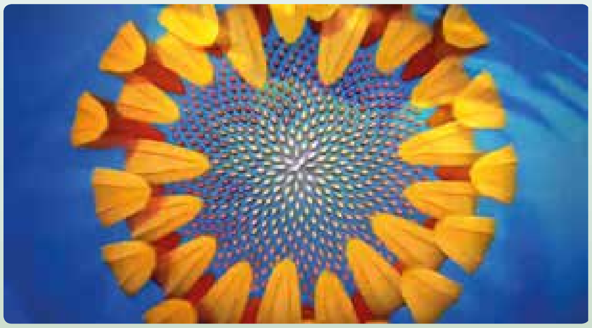

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# 9 Answers

## EXERCISE 9.2

**1 a** $h = 7.66$

**b** $a \approx 68.20$

**2** CA = CB = 16.18 cm;
$\angle CBA = \angle CAB = 72°$; $\angle ACB = 36°$

**3** $2\sqrt{2}$ metres

**4** $\dfrac{3\sqrt{2} - \sqrt{6}}{8}$

**5 a** $\cos(a) = \dfrac{3}{\sqrt{13}}$

**b** $6\sqrt{13}$ cm

**6** $\sin(\theta) = \dfrac{2}{7}$; $\tan(\theta) = \dfrac{2\sqrt{5}}{15}$

**7** 30 $cm^2$

**8** 5.64 $km^2$

**9 a** $\dfrac{\sqrt{6}}{12}$

**b** $3\sqrt{3} + 3\sqrt{2}$

**10 a i** $\dfrac{\sqrt{21}}{7}$

**ii** $\dfrac{\sqrt{11}}{5}$

**iii** $\dfrac{2}{3}$

**b** 36 cm

**11 a** 6.18 metres

**b** 59.5°; 5.3 metres

**12 a i** 4.275 $cm^2$

**ii** 1.736 cm

**b** $20\sqrt{3}$ cm; $\dfrac{100\sqrt{3}}{3}$ $cm^2$

**c** $6\sqrt{2}$ $cm^2$

**13** 21.6°

**14** $AC = 12\sqrt{3}$ cm; $BC = 18\sqrt{2}$ cm; $AB = 18 + 6\sqrt{3}$ cm

**15 a** $\sqrt{3}a$ units

**b** 35.26°

**16 a** 5.07 cm

**b** 4.7 cm

**17 a** Proof required — check with your teacher

**b** $a = 60$; $m = 16$; $n = 12$

**18** Height 63.4 metres; distance 63.4 metres

**19** 26.007 km

**20** $3(\sqrt{3} + 1)$ metres

## EXERCISE 9.3

**1 a** $\dfrac{\pi}{3}$

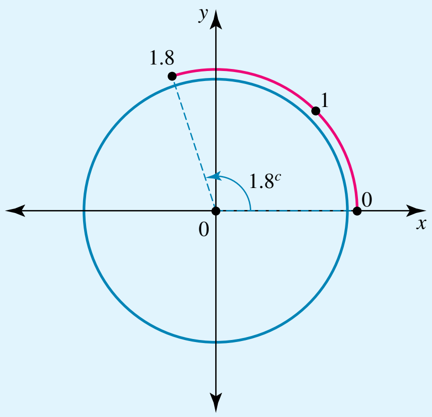
**b** 135°

**c** 30°; $\dfrac{\sqrt{3}}{3}$

**2** 2.53

**3 a** 103.1°

**b**




**4 a** $-4\pi, -2\pi, 0, 2\pi, 4\pi$

**b** $-1 - 4\pi, -1 - 2\pi, -1, -1 + 2\pi, -1 + 4\pi$

**5 a** 10.47 cm

**b** 216°

**6 a** 2.572

**b** 0.021

**7 a**

| Degrees | 30° | 45° | 60° |
|---|---|---|---|
| Radians | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |

**b**

| Degrees | 0° | 90° | 180° | 270° | 360° |
|---|---|---|---|---|---|
| Radians | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |

**8 a** 36° **b** 120° **c** 75°
**d** 330° **e** 140° **f** 810°

**9 a** $\dfrac{2\pi}{9}$ **b** $\dfrac{5\pi}{6}$ **c** $\dfrac{5\pi}{4}$
**d** $\dfrac{5\pi}{3}$ **e** $\dfrac{7\pi}{4}$ **f** $4\pi$

**10 a i** 0.052

**ii** 1.959

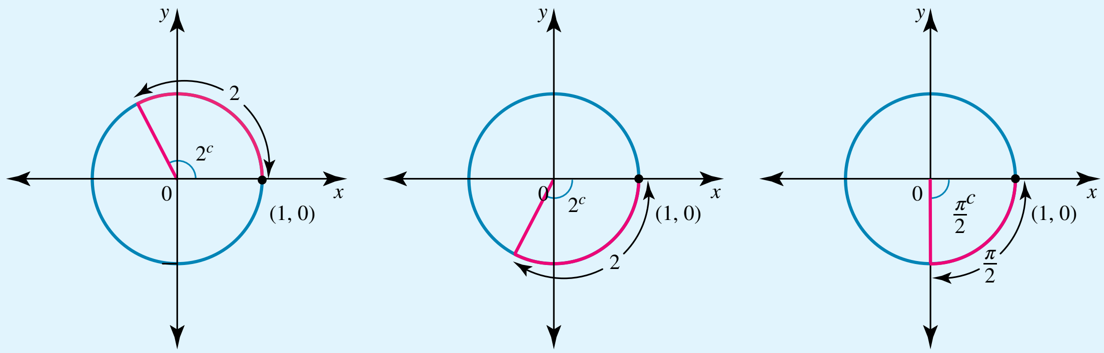
**iii** 3.759

**b i** 171.887°

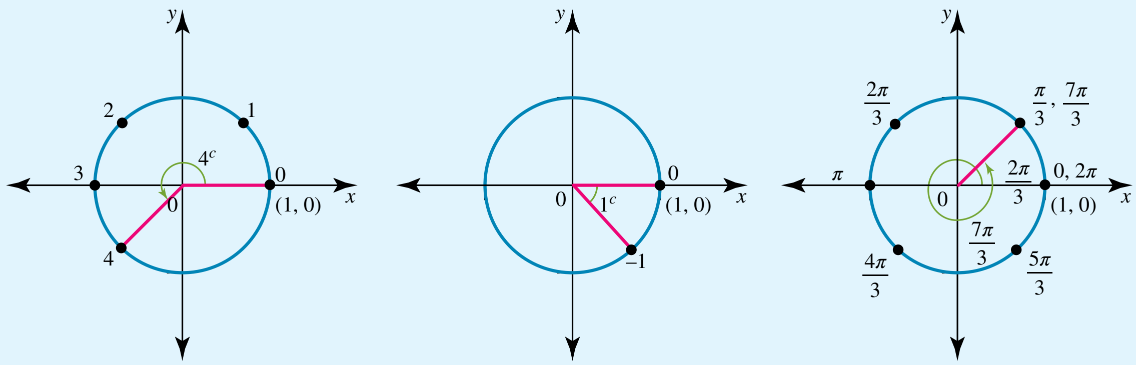
**ii** 414°

**c** $\left\{\dfrac{\pi^c}{7}, 50°, 1.5^c\right\}$

**11 a**




**b**




**12 a** $10\pi$ cm

**b** $\frac{4\pi^2}{9}$ cm

**c** $5\pi$ cm

**13 a** 17.2°

**b** 191° or $\frac{10^c}{3}$

**c** 5.76 mm

**d** 81.5 $cm^2$

**14 a** **i** 1.557

**ii** 0.623

**iii** 0.025

**b**

| $\theta$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|
| $\sin(\theta)$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos(\theta)$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |
| $\tan(\theta)$ | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ |

**15 a** 2 $units^2$

**b** $6\sqrt{3} - 2$

**16 a** 6371 km

**b** 6360 km

**c** 6004 km

**17** $\frac{3\pi}{4}$

**18** $\frac{900°}{\pi} \approx 286.4789°$

## EXERCISE 9.4

**1 a** $P\left[\frac{\pi}{2}\right]$

**b**

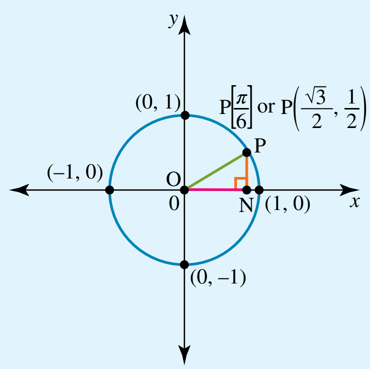
| | | | |
|---|---|---|---|
| 120° | −400° $= -360° - 40°$ | $\frac{4\pi}{3} = \pi + \frac{1}{3}\pi$ | $\frac{\pi}{4}$ |
| quadrant 2 | quadrant 4 | quadrant 3 | quadrant 1 |

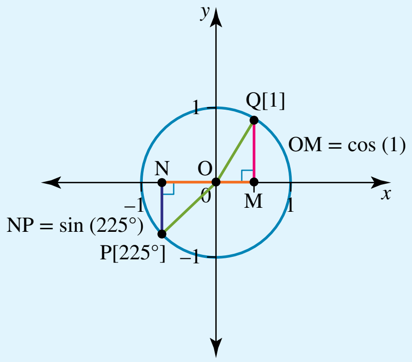
**c** Q[−240°]; R[480°]

**2** $P\left[\frac{3\pi}{2}\right]$ or $P\left[-\frac{\pi}{2}\right]$

**3 a** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

**b**





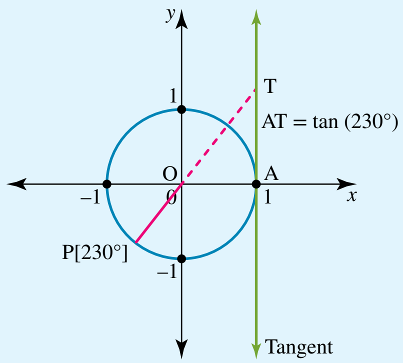

c $\cos\left(-\frac{\pi}{2}\right) = 0; \sin\left(-\frac{\pi}{2}\right) = -1$

d $f(0) = 0$

**4 a** First and second quadrants

b First and fourth quadrants

**5 a** $\tan(230°) \approx 1.192$




b $\tan(2\pi) = 0$

**6 a** $\tan\left(\frac{5\pi}{2}\right); \tan(-90°)$

b $\tan\left(\frac{3\pi}{4}\right)$

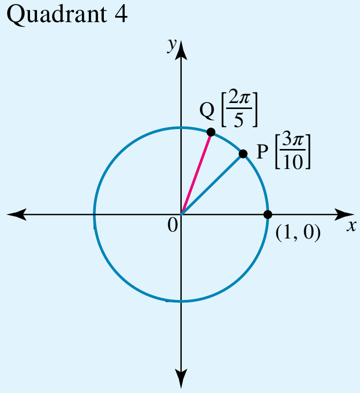
**7 a** Quadrant 3

b Quadrant 2

c Boundary of quadrant 1 and quadrant 4

d Quadrant 4

**8 a**




b $\frac{\pi}{10}$

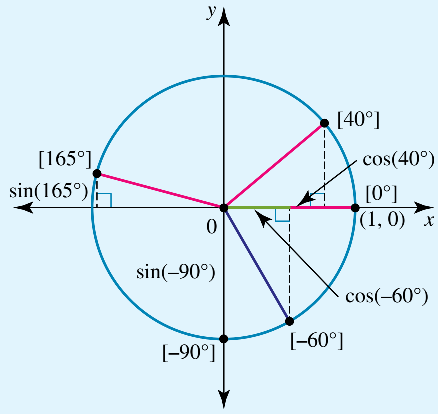
c $P\left[-\frac{17\pi}{10}\right]; Q\left[-\frac{8\pi}{5}\right]$ (other answers are possible)

d $P\left[\frac{23\pi}{10}\right]; Q\left[\frac{12\pi}{5}\right]$ (other answers are possible)

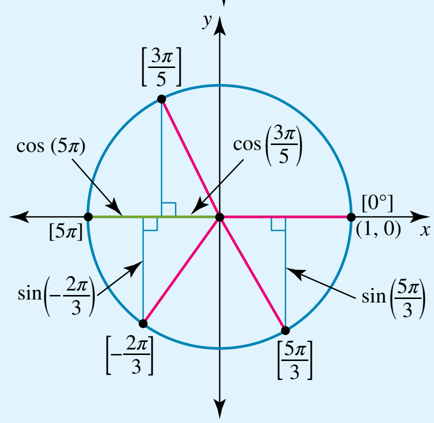
**9 a** $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

b $\left[\frac{3\pi}{2}\right]$ or $\left[-\frac{\pi}{2}\right]$

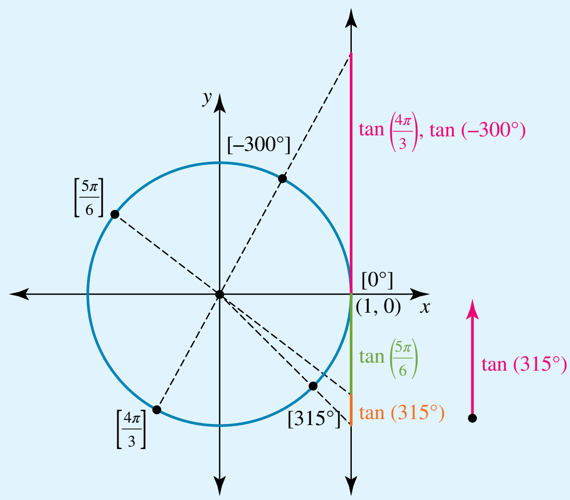
**10**




**11**




**12**

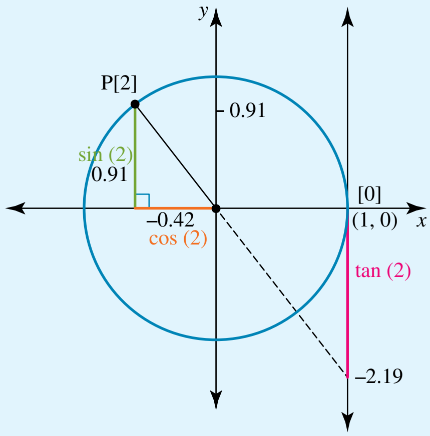



**13 a** 0.91

b −0.42

c −2.19

**14 a i**




**ii** $(-0.42, 0.91)$

**b** Proof required — check with your teacher

**15 a** Quadrant 2;
$\sin(\theta) = 0.6,\ \cos(\theta) = -0.8,\ \tan(\theta) = -0.75$

**b** Quadrant 4;
$\sin(\theta) = -\frac{\sqrt{2}}{2},\ \cos(\theta) = \frac{\sqrt{2}}{2},\ \tan(\theta) = -1$

**c** Quadrant 1;
$\sin(\theta) = \frac{1}{\sqrt{5}},\ \cos(\theta) = \frac{2}{\sqrt{5}},\ \tan(\theta) = \frac{1}{2}$

**d** Boundary between quadrant 1 and quadrant 2;
$\sin(\theta) = 1$, $\cos(\theta) = 0$, $\tan(\theta)$ undefined

**16 a** 1 **b** 1 **c** 0
**d** 0 **e** 0 **f** $-1$

**17 a** 1

**b** $-\frac{\sqrt{3}}{2} - \frac{1}{2}$

**c** 1

**d** 1

**e** 1; proof required — check with your teacher

**18 a** $P\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$ and $Q\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ are reflections in the $x$-axis.

**b** $R\left(\frac{-(\sqrt{5}+1)}{4}, \frac{\sqrt{2(-\sqrt{5}+5)}}{4}\right)$ and
$S\left(\frac{\sqrt{5}+1}{4}, \frac{\sqrt{2(-\sqrt{5}+5)}}{4}\right)$ are reflections in the $y$-axis.

**c i** $\sin\left(\frac{7\pi}{4}\right) = \frac{-\sqrt{2}}{2}, \cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(\frac{7\pi}{4}\right) = -1$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(\frac{\pi}{4}\right) = 1$

$\sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right),$

$\tan\left(\frac{7\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right)$

**ii** $\sin\left(\frac{4\pi}{5}\right) = \frac{\sqrt{2(-\sqrt{5}+5)}}{4},$

$\cos\left(\frac{4\pi}{5}\right) = -\frac{(\sqrt{5}+1)}{4}, \tan\left(\frac{4\pi}{5}\right) = -\sqrt{-2\sqrt{5}+5}$

$\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{2(-\sqrt{5}+5)}}{4}, \cos\left(\frac{\pi}{5}\right) = \frac{(\sqrt{5}+1)}{4},$

$\tan\left(\frac{\pi}{5}\right) = \sqrt{-2\sqrt{5}+5}$

$\sin\left(\frac{4\pi}{5}\right) = \sin\left(\frac{\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right) = -\cos\left(\frac{\pi}{5}\right),$

$\tan\left(\frac{4\pi}{5}\right) = -\tan\left(\frac{\pi}{5}\right)$

## EXERCISE 9.5

**1 a** Third quadrant

**b** 0

**2** 1

**3 a** $-\frac{\sqrt{3}}{2}$

**b** $-\frac{\sqrt{3}}{3}$

**c** $\frac{\sqrt{3}}{2}$

**4** $\sin\left(-\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}; \cos\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}; \tan\left(-\frac{5\pi}{4}\right) = -1$

**5 a** [105°]; [255°]; [285°]; $\cos(285°) = \cos(75°)$; [285°]

**b** $-\tan\left(\frac{\pi}{7}\right)$

**c** $\sin(\pi - \theta) = 0.8, \sin(2\pi - \theta) = -0.8$

**d i** $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

**ii** $\sin\left(\frac{25\pi}{6}\right) = \frac{1}{2}$

**6 a** $\cos(-\theta) = p$

**b** $\cos(5\pi + \theta) = -p$

**7 a** 1 **b** 0 **c** 0
**d** 1 **e** 0 **f** 0

**8 a** Fourth

**b** First

**c** Second

**d** Boundary between quadrants 1 and 2, and boundary between quadrants 3 and 4

**e** Boundary between quadrants 1 and 2

**f** Boundary between quadrants 2 and 3

9

| | Quadrant 2 | Quadrant 3 | Quadrant 4 |
|---|---|---|---|
| a | $\frac{2\pi}{3}$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ |
| b | $\frac{5\pi}{6}$ | $\frac{7\pi}{6}$ | $\frac{11\pi}{6}$ |
| c | $\frac{3\pi}{4}$ | $\frac{5\pi}{4}$ | $\frac{7\pi}{4}$ |
| d | $\frac{4\pi}{5}$ | $\frac{6\pi}{5}$ | $\frac{9\pi}{5}$ |
| e | $\frac{5\pi}{8}$ | $\frac{11\pi}{8}$ | $\frac{13\pi}{8}$ |
| f | $\pi - 1$ | $\pi + 1$ | $2\pi - 1$ |

10 a $-\frac{1}{2}$ b $1$ c $-\frac{1}{2}$

d $-\sqrt{3}$ e $\frac{\sqrt{2}}{2}$ f $\frac{1}{2}$

11 a $\frac{\sqrt{2}}{2}$ b $-\sqrt{3}$ c $-\frac{\sqrt{3}}{2}$

d $-\frac{1}{2}$ e $\frac{\sqrt{3}}{3}$ f $-\frac{1}{2}$

12 a $\frac{\sqrt{2}}{2}$ b $-\frac{\sqrt{3}}{2}$ c $\frac{\sqrt{3}}{3}$

d $\frac{\sqrt{3}}{2}$ e $\frac{\sqrt{2}}{2}$ f $-\frac{\sqrt{3}}{3}$

13 a $\frac{-\left(\sqrt{3}+1\right)}{2}$ b $2 - \sqrt{2}$ c $2\sqrt{3}$

d $0$ e $0$ f $-2$

14 a $-0.91$ b $0.43$ c $-0.47$

d $0.91$ e $-0.43$ f $0.47$

15 a $-p$ b $p$ c $-p$ d $p$

16 a Proof required — check with your teacher

b Same $x$-coordinates

c $\sin(\pi + \phi) = -0.87$, $\cos(\pi + \phi) = 0.5$, $\tan(\pi + \phi) = -1.74$

d $\sin(t)$

e $A = 36°$ or $-216°$ (other answers are possible)

f $B = \frac{13\pi}{11}$ or $\frac{20\pi}{11}$ or $-\frac{2\pi}{11}$ (other answers are possible)

17 a Third quadrant b $(-0.49, -0.87)$

c Quadrant 1, $\theta = 1.0584$; quadrant 2, $\theta = 2.0832$; quadrant 4, $\theta = 5.2248$

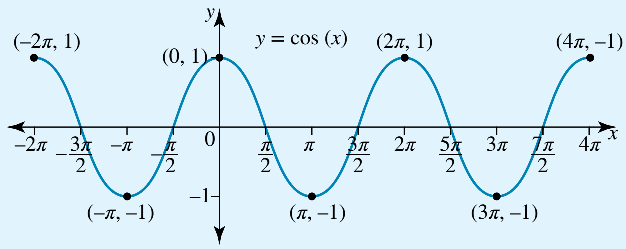
18 a Quadrant 1 or quadrant 3

b Quadrant 1, $\theta = 1.3734$; quadrant 3, $\theta = 4.5150$

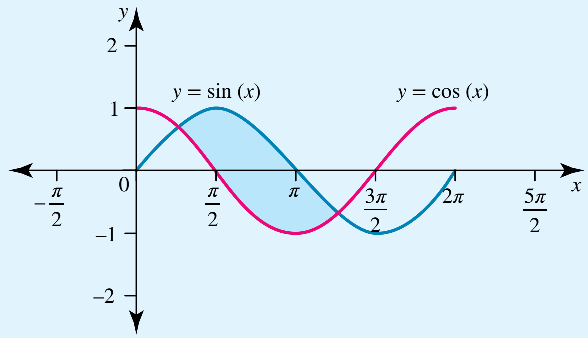
c Quadrant 1, $\cos(\theta) = \frac{\sqrt{26}}{26}$, $\sin(\theta) = \frac{5\sqrt{26}}{26}$, $\left(\frac{\sqrt{26}}{26}, \frac{5\sqrt{26}}{26}\right)$; or quadrant 3 $\cos(\theta) = -\frac{\sqrt{26}}{26}$, $\sin(\theta) = -\frac{5\sqrt{26}}{26}$, $\left(-\frac{\sqrt{26}}{26}, -\frac{5\sqrt{26}}{26}\right)$

## EXERCISE 9.6

1 Three cycles

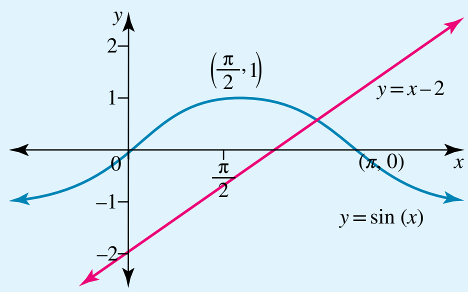



2




Region required lies below the sine graph and above the cosine graph between their points of intersection.

3 a



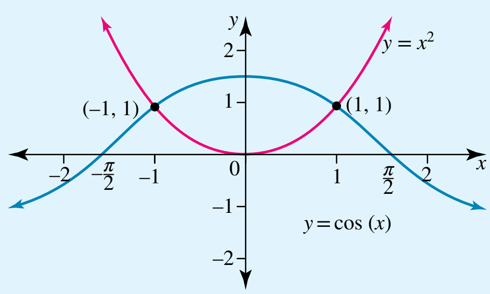

Only one point of intersection of the graphs

b Between $x = 2$ and $x = 3$

c $[2.5, 3]$; $[2.5, 2.75]$; $x = 2.625$

4 a $y = \cos(x)$ and $y = x^2$

b Two solutions




c $k = 1$

5 a $\sin(1.8°) \approx 0.01\pi$, accurate to 4 decimal places

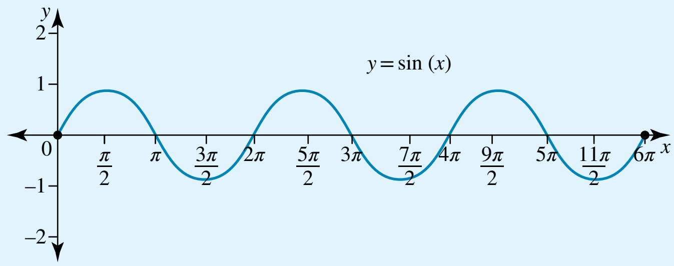
b Proof required — check with your teacher

c $x = 0.3015$

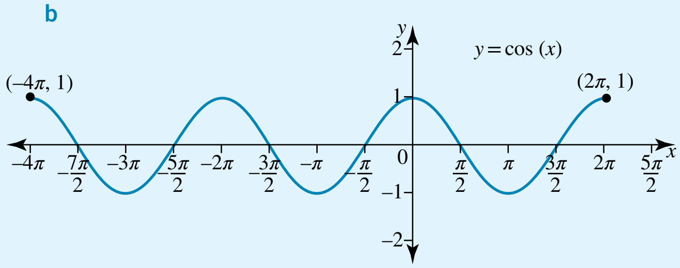
6 a $\cos(0.5) \approx 0.875$

b The approximation is only applicable for small numbers close to zero.

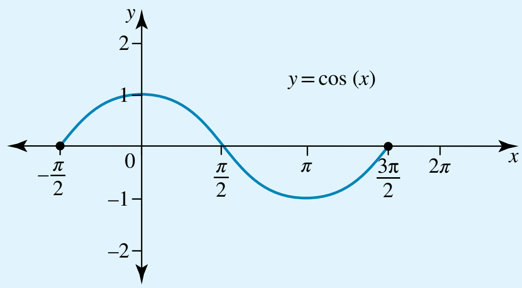
**7 a**



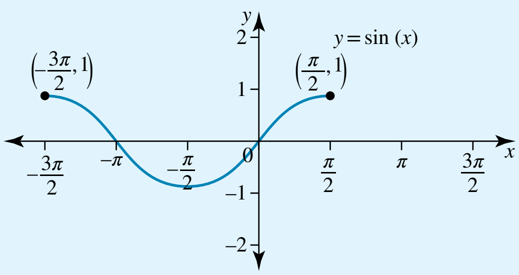

**b**




**c**



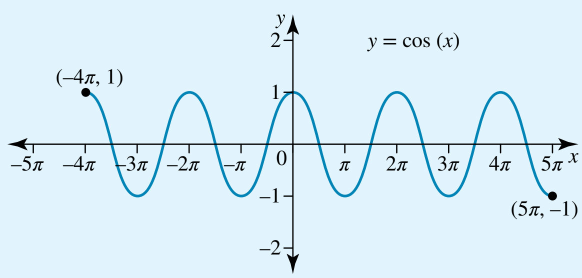

**d**




**8 a** 2 maximum turning points

**b** 7 minimum turning points

**c** $4\frac{1}{2}$ cycles



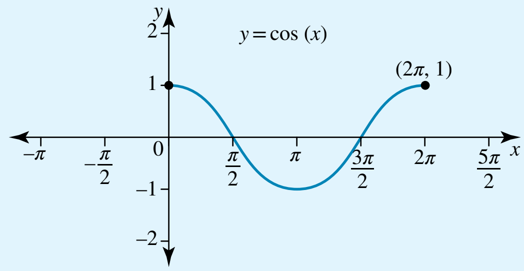

**9 a** 4 **b** 7 **c** 21 **d** 3

**10 a** $a = \frac{19\pi}{4}$

**b** $b = -2\pi$

**c** $c = \frac{5\pi}{2}$

**11 a** $\frac{\pi}{2} < x < \frac{3\pi}{2}$




**b** Explanation required

**12 a i** $y = \cos(x)$ and $y = x^3$; one solution

**ii** $y = \cos(x)$ and $y = \frac{1}{4}x$; three solutions

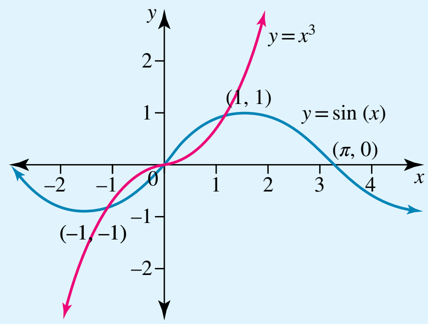
**iii** $y = \sin(x)$ and $y = (x - 1)^2$; two solutions

**b** $0 \le x \le 1$; $x = 0.875$

**c i** $a < 0$

**ii** Infinite solutions

**13 a** $x = 0$




**b** $x = 0.9$

**c** $x = -0.9$

**14 a** $\frac{\pi}{180}$

**b** $\frac{\pi}{9}$

**c** $-\frac{\pi}{90}$

**d** $\frac{\pi}{6}$; the approximation becomes less accurate as the size of $x$ increases.

**e** $\theta = \frac{15}{\pi}$; angle is 4.77°

**15 a** $\frac{31}{32}$

**b** $x = 5 - \sqrt{23}$

**16 a** Proof required; $x = 0.5$

**b** $y = \frac{1}{4x}$; explanation required

**c** Not small solutions

**d** 8

**17 a i** $x = -1.4096$ or $x = 0.6367$

**ii** $x = 0.618$

**b** $-0.6 \le x \le 0.6$

**18** $\pi$

# 10

# Trigonometric functions 2

# 10.1 Kick off with CAS

## Trigonometric functions

1 Using CAS technology in radian mode, sketch the following trigonometric functions over $-2\pi \le x \le 2\pi$.

a $y = \sin(x) + 2$
b $y = \sin(x) - 3$
c $y = \sin(x) - 5$
d $y = \sin(x) + 4$
e $y = \sin(x) - 1$

2 Using CAS technology, enter $y = \sin(x) + k$ into the function entry line and use a slider to change the value of $k$.

3 When sketching a trigonometric function, what is the effect of changing the value of $k$ in the equation?

4 Using CAS technology in radian mode, sketch the following trigonometric functions over $-2\pi \le x \le 2\pi$.

a $y = \cos(x) + 2$
b $y = \cos(x) - 3$
c $y = \cos(x) - \pi$
d $y = \cos(x) + 4$
e $y = \cos(x) - 1$

5 Using CAS technology, enter $y = \cos(x) + k$ into the function entry line and use a slider to change the value of $a$.

6 When sketching a trigonometric function, what is the effect of changing the value of $k$ in the equation?

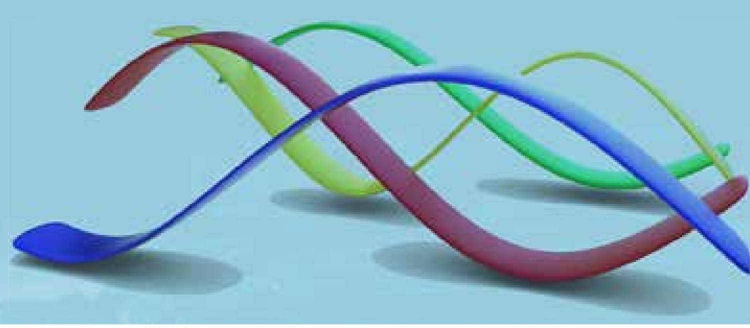
7 Using CAS technology in radian mode, sketch the following trigonometric functions over $-2\pi \le x \le 2\pi$.

a $y = \sin\left(x - \frac{\pi}{2}\right)$
b $y = \sin\left(x + \frac{\pi}{2}\right)$
c $y = \sin(x - \pi)$
d $y = \cos\left(x - \frac{\pi}{2}\right)$
e $y = \cos\left(x + \frac{\pi}{2}\right)$

8 Using CAS technology, enter $y = \sin(x - h)$ into the function entry line and use a slider to change the value of $h$.

9 When sketching a trigonometric function, what is the effect of changing the value of $h$ in the equation?

10 On the one set of axes, sketch the graphs of $y_1 = \sin(x)$ and $y_2 = 2\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 2$. Describe the transformations to get from $y_1$ to $y_2$.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 10.2 Trigonometric equations

studyon

Units 1 & 2
AOS 1
Topic 7
Concept 1

**Trigonometric equations**
Concept summary
Practice questions

The symmetry properties of trigonometric functions can be used to obtain solutions to equations of the form $f(x) = a$ where $f$ is sine, cosine or tangent. If $f$ is sine or cosine, then $-1 \leq a \leq 1$ and, if $f$ is tangent, then $a \in R$.

Once the appropriate base value of the first quadrant is known, symmetric points in any other quadrant can be obtained. However, there are many values, generated by both positive and negative rotations, which can form these symmetric quadrant points. Consequently, the solution of a trigonometric equation such as $\sin(x) = a,\ x \in R$ would have infinite solutions. We shall consider trigonometric equations in which a subset of $R$ is specified as the domain in order to have a finite number of solutions.

## Solving trigonometric equations on finite domains

To solve the basic type of equation $\sin(x) = a,\ 0 \leq x \leq 2\pi$:

- Identify the quadrants in which solutions lie from the sign of $a$.
  - If $a > 0$, $x$ must lie in quadrants 1 and 2 where sine is positive.
  - If $a < 0$, $x$ must be in quadrants 3 and 4 where sine is negative.
- Obtain the base value, or first-quadrant value, by solving $\sin(x) = a$ if $a > 0$ or ignoring the negative sign if $a < 0$ (to ensure the first-quadrant value is obtained).
  - This may require recognition of an exact value ratio or it may require the use of a calculator.
- Once obtained, use the base value to generate the values for the quadrants required from their symmetric forms.

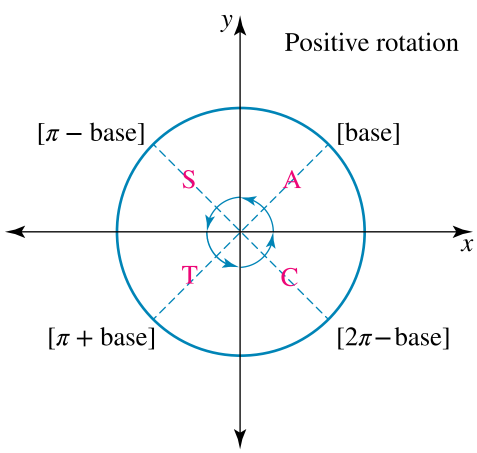
The basic equations $\cos(x) = a$ or $\tan(x) = a$, $0 \leq x \leq 2\pi$ are solved in a similar manner, with the sign of $a$ determining the quadrants in which solutions lie.

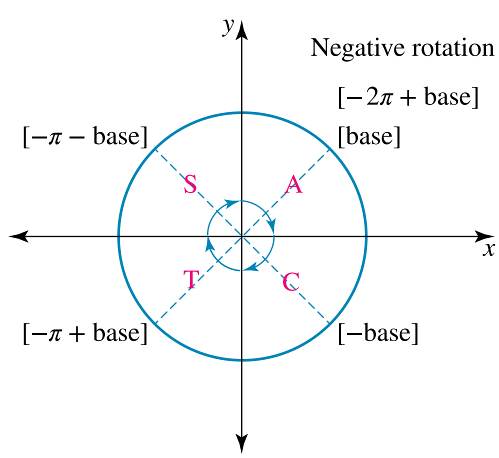
For $\cos(x) = a$: if $a > 0$, $x$ must lie in quadrants 1 and 4 where cosine is positive; if $a < 0$, $x$ must be in quadrants 2 and 3 where cosine is negative.

For $\tan(x) = a$: if $a > 0$, $x$ must lie in quadrants 1 and 3 where tangent is positive; if $a < 0$, $x$ must be in quadrants 2 and 4 where tangent is negative.

## Symmetric forms

For one positive and one negative rotation, the symmetric points to the first-quadrant base are shown in the diagrams.



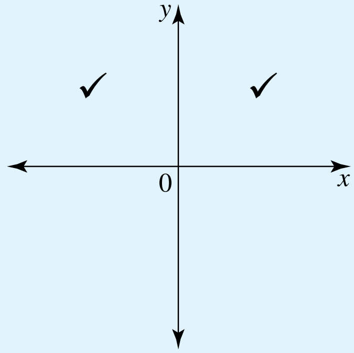
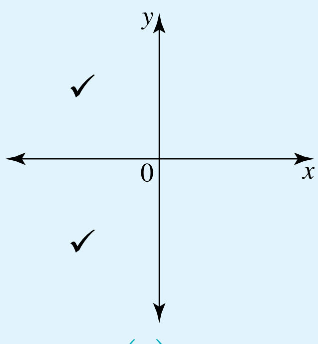
**WORKED EXAMPLE 1**

**Solve the following equations to obtain exact values for $x$.**

**a** $\sin(x) = \frac{\sqrt{3}}{2},\ 0 \le x \le 2\pi$

**b** $\sqrt{2}\cos(x) + 1 = 0,\ 0 \le x \le 2\pi$

**c** $\sqrt{3} - 3\tan(x) = 0,\ -2\pi \le x \le 2\pi$

| THINK | WRITE |
|---|---|
| **a** **1** Identify the quadrants in which the solutions lie. | **a** $\sin(x) = \frac{\sqrt{3}}{2},\ 0 \le x \le 2\pi$<br>Sine is positive in quadrants 1 and 2.<br> |
| **2** Use knowledge of exact values to state the first-quadrant base. | Base is $\frac{\pi}{3}$ since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. |
| **3** Generate the solutions using the appropriate quadrant forms. | Since $x \in [0, 2\pi]$ there will be two positive solutions, one from quadrant 1 and one from quadrant 2.<br>$\therefore x = \frac{\pi}{3}$ or $x = \pi - \frac{\pi}{3}$ |
| **4** Calculate the solutions from their quadrant forms. | $\therefore x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ |
| **b** **1** Rearrange the equation so the trigonometric function is isolated on one side. | **b** $\sqrt{2}\cos(x) + 1 = 0,\ 0 \le x \le 2\pi$<br>$\sqrt{2}\cos(x) = -1$<br>$\cos(x) = -\frac{1}{\sqrt{2}}$ |
| **2** Identify the quadrants in which the solutions lie. | Cosine is negative in quadrants 2 and 3.<br> |
| **3** Identify the base.<br>*Note:* The negative sign is ignored in identifying the base since the base is the first-quadrant value. | Since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, the base is $\frac{\pi}{4}$. |

4 Generate the solutions using the appropriate quadrant forms.

Since $x \in [0, 2\pi]$ there will be two positive solutions.

$\therefore x = \pi - \frac{\pi}{4}$ or $x = \pi + \frac{\pi}{4}$

5 Calculate the solutions from their quadrant forms.

$\therefore x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

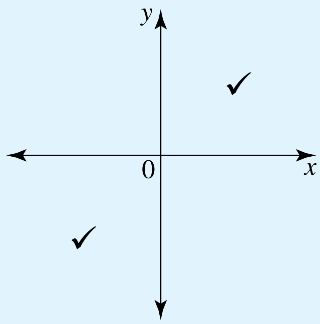
**c** 1 Rearrange the equation so the trigonometric function is isolated on one side.

**c** $\sqrt{3} - 3\tan(x) = 0, \ -2\pi \le x \le 2\pi$

$\therefore \tan(x) = \frac{\sqrt{3}}{3}$

2 Identify the quadrants in which the solutions lie.

Tangent is positive in quadrants 1 and 3.




3 Identify the base.

Base is $\frac{\pi}{6}$ since $\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$.

4 Generate the solutions using the appropriate quadrant forms.

Since $-2\pi \le x \le 2\pi$, there will be 4 solutions, two from a positive rotation and two from a negative rotation.

$x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$ or $x = -\pi + \frac{\pi}{6}, -2\pi + \frac{\pi}{6}$

5 Calculate the solutions from their quadrant forms.

$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}, -\frac{5\pi}{6}, -\frac{11\pi}{6}$

## Trigonometric equations with boundary value solutions

Recognition of exact trigonometric values allows us to identify the base for solving trigonometric equations to obtain exact solutions. However, there are also exact trigonometric values for boundary points. These need to be recognised should they appear in an equation. The simplest strategy to solve trigonometric equations involving boundary values is to use a unit circle diagram to generate the solutions. The domain for the equation determines the number of rotations required around the unit circle. It is not appropriate to consider quadrant forms to generate solutions, since boundary points lie between two quadrants.

### Using technology

When bases are not recognisable from exact values, calculators are needed to identify the base. Whether the calculator, or other technology, is set on radian mode or degree mode is determined by the given equation. For example, if $\sin(x) = -0.7, 0 \le x \le 2\pi$, the base is calculated as $\sin^{-1}(0.7)$ in radian mode.

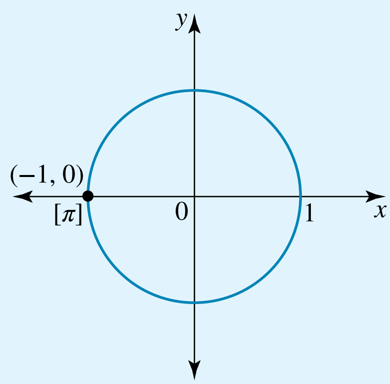
However for $\sin(x) = 0.7, 0° \le x \le 360°$, degree mode is used when calculating the base as $\sin^{-1}(0.7)$. The degree of accuracy required for the answer is usually specified in the question; if not, express answers rounded to 2 decimal places.

**WORKED EXAMPLE 2**

**a** Solve for $x$, $3\cos(x) + 3 = 0, \ -4\pi \le x \le 4\pi$.

**b** Solve for $x$ to 2 decimal places, $\sin(x) = -0.75, \ 0 \le x \le 4\pi$.

**c** Solve for $x$, to 1 decimal place, $\tan(x°) + 12.5 = 0, \ -180° \le x° \le 180°$.

| THINK | WRITE |
|---|---|
| **a 1** Express the equation with the trigonometric function as subject. | **a** $3\cos(x) + 3 = 0, \ -4\pi \le x \le 4\pi$<br>$\therefore \cos(x) = -1$ |
| **2** Identify any boundary points. | –1 is a boundary value since $\cos(\pi) = -1$.<br>The boundary point $[\pi]$ has Cartesian coordinates $(-1, 0)$. |
| **3** Use a unit circle to generate the solutions. | <br>As $-4\pi \le x \le 4\pi$, this means 2 anticlockwise revolutions and 2 clockwise revolutions around the circle are required, with each revolution generating one solution.<br>The solutions are:<br>$x = \pi, 3\pi$ and $x = -\pi, -3\pi$<br>$\therefore x = \pm\pi, \ \pm 3\pi$ |
| **b 1** Identify the quadrants in which the solutions lie. | **b** $\sin(x) = -0.75, \ 0 \le x \le 4\pi$<br>Sine is negative in quadrants 3 and 4. |
| **2** Calculate the base. | Base is $\sin^{-1}(0.75)$. Using radian mode, $\sin^{-1}(0.75) = 0.848$ to 3 decimal places. |
| **3** Generate the solutions using the appropriate quadrant forms. | Since $x \in [0, 4\pi]$ there will be four positive solutions from two anticlockwise rotations.<br>$x = \pi + 0.848, 2\pi - 0.848$ or $x = 3\pi + 0.848, 4\pi - 0.848$ |
| **4** Calculate the solutions to the required accuracy.<br>*Note:* If the base is left as $\sin^{-1}(0.75)$ then the solutions such as $x = \pi + \sin^{-1}(0.75)$ could be calculated on radian mode in one step. | $\therefore x = 3.99, 5.44, 10.27, 11.72$ (correct to 2 decimal places) |

**c 1** Identify the quadrants in which the solutions lie.

**c** $\tan(x^\circ) + 12.5 = 0,\ -180^\circ \le x^\circ \le 180^\circ$
$\tan(x^\circ) = -12.5$
Tangent is negative in quadrants 2 and 4.

**2** Calculate the base.

Base is $\tan^{-1}(12.5)$. Using degree mode, $\tan^{-1}(12.5) = 85.43^\circ$ to 2 decimal places.

**3** Generate the solutions using the appropriate quadrant forms.

Since $-180^\circ \le x^\circ \le 180^\circ$, a clockwise rotation of 180° gives one negative solution in quadrant 4 and an anticlockwise rotation of 180° gives one positive solution in quadrant 2.
$x^\circ = -85.43^\circ$ or $x^\circ = 180^\circ - 85.43^\circ$

**4** Calculate the solutions to the required accuracy.

$\therefore x = -85.4, 94.6$ (correct to 1 decimal place)

## Further types of trigonometric equations

Trigonometric equations may require algebraic techniques or the use of relationships between the functions before they can be reduced to the basic form $f(x) = a$, where $f$ is either sin, cos or tan.

- **Equations of the form $\sin(x) = a\cos(x)$ can be converted to $\tan(x) = a$ by dividing both sides of the equation by $\cos(x)$.**
- **Equations of the form $\sin^2(x) = a$ can be converted to $\sin(x) = \pm\sqrt{a}$ by taking the square roots of both sides of the equation.**
- **Equations of the form $\sin^2(x) + b\sin(x) + c = 0$ can be converted to standard quadratic equations by using the substitution $u = \sin(x)$.**

Since $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$, neither $\sin(x)$ nor $\cos(x)$ can have values greater than 1 or smaller than −1. This may have implications requiring the rejection of some steps when working with sine or cosine trigonometric equations. As $\tan(x) \in R$, there is no restriction on the values the tangent function can take.

WORKED EXAMPLE 3

**Solve for $x$, where $0 \le x \le 2\pi$.**

**a** $\sqrt{3}\sin(x) = \cos(x)$

**b** $\cos^2(x) + \cos(x) - 2 = 0$

THINK

**a 1** Reduce the equation to one trigonometric function.

WRITE

**a** $\sqrt{3}\sin(x) = \cos(x),\ 0 \le x \le 2\pi$
Divide both sides by $\cos(x)$.

$$\frac{\sqrt{3}\sin(x)}{\cos(x)} = 1$$

$$\therefore \sqrt{3}\tan(x) = 1$$

$$\therefore \tan(x) = \frac{1}{\sqrt{3}}$$

**2** Calculate the solutions.

Tangent is positive in quadrants 1 and 3.

Base is $\frac{\pi}{6}$.

$x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$

$= \frac{\pi}{6}, \frac{7\pi}{6}$

**b** **1** Use substitution to form a quadratic equation.

**b** $\cos^2(x) + \cos(x) - 2 = 0, \quad 0 \le x \le 2\pi$

Let $a = \cos(x)$.

The equation becomes $a^2 + a - 2 = 0$.

**2** Solve the quadratic equation.

$(a + 2)(a - 1) = 0$

$\therefore a = -2$ or $a = 1$

**3** Substitute back for the trigonometric function.

Since $a = \cos(x)$, $\cos(x) = -2$ or $\cos(x) = 1$.

Reject $\cos(x) = -2$ since $-1 \le \cos(x) \le 1$.

$\therefore \cos(x) = 1$

**4** Solve the remaining trigonometric equation.

$\cos(x) = 1, \quad 0 \le x \le 2\pi$

Boundary value since $\cos(0) = 1$

$\therefore x = 0, \quad 2\pi$

## Solving trigonometric equations which require a change of domain

Equations such as $\sin(2x) = 1,\ 0 \le x \le 2\pi$ can be expressed in the basic form by the substitution of $\theta = 2x$. The accompanying domain must be changed to be the domain for $\theta$. This requires the endpoints of the domain for $x$ to be multiplied by 2. Hence, $0 \le x \le 2\pi \Rightarrow 2 \times 0 \le 2x \le 2 \times 2\pi$ gives the domain requirement for $\theta$ as $0 \le \theta \le 4\pi$.

This allows the equation to be written as $\sin(\theta) = 1,\ 0 \le \theta \le 4\pi$.

This equation can then be solved to give $\theta = \frac{\pi}{2}, \frac{5\pi}{2}$.

Substituting back for $x$ gives $2x = \frac{\pi}{2}, \frac{5\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$. The solutions are in the domain specified for $x$.

The change of domain ensures all possible solutions are obtained.

However, in practice, it is quite common not to formally introduce the pronumeral substitution for equations such as $\sin(2x) = 1, 0 \le x \le 2\pi$.

With the domain change, the equation can be written as $\sin(2x) = 1, 0 \le 2x \le 4\pi$ and the equation solved for $x$ as follows:

$$\sin(2x) = 1, 0 \le 2x \le 4\pi$$

$$\therefore 2x = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$$

**WORKED EXAMPLE 4**

**a** Solve $\cos(3x) = -\frac{1}{2}$ for $x$, $0 \le x \le 2\pi$.

**b** Use substitution to solve the equation $\tan\left(2x - \frac{\pi}{4}\right) = -1$, $0 \le x \le \pi$.

| THINK | WRITE |
|---|---|
| **a 1** Change the domain to be that for the given multiple of the variable. | **a** $\cos(3x) = -\frac{1}{2}$, $0 \le x \le 2\pi$<br>Multiply the endpoints of the domain of $x$ by 3<br>$\therefore \cos(3x) = -\frac{1}{2}$, $0 \le 3x \le 6\pi$ |
| **2** Solve the equation for $3x$.<br>*Note:* Alternatively, substitute $\theta = 3x$ and solve for $\theta$. | Cosine is negative in quadrants 2 and 3.<br>Base is $\frac{\pi}{3}$.<br>As $3x \in [0, 6\pi]$, each of the three revolutions will generate 2 solutions, giving a total of 6 values for $3x$.<br>$3x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, 3\pi + \frac{\pi}{3}, 5\pi - \frac{\pi}{3}, 5\pi + \frac{\pi}{3}$<br>$= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}$ |
| **3** Calculate the solutions for $x$. | Divide each of the 6 values by 3 to obtain the solutions for $x$.<br>$x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$ |
| **b 1** State the substitution required to express the equation in basic form. | **b** $\tan\left(2x - \frac{\pi}{4}\right) = -1$, $0 \le x \le \pi$<br>Let $\theta = 2x - \frac{\pi}{4}$. |
| **2** Change the domain of the equation to that of the new variable. | For the domain change:<br>$0 \le x \le \pi$<br>$\therefore 0 \le 2x \le 2\pi$<br>$\therefore -\frac{\pi}{4} \le 2x - \frac{\pi}{4} \le 2\pi - \frac{\pi}{4}$<br>$\therefore -\frac{\pi}{4} \le \theta \le \frac{7\pi}{4}$ |
| **3** State the equation in terms of $\theta$. | The equation becomes $\tan(\theta) = -1, -\frac{\pi}{4} \le \theta \le \frac{7\pi}{4}$. |
| **4** Solve the equation for $\theta$. | Tangent is negative in quadrants 2 and 4.<br>Base is $\frac{\pi}{4}$.<br>$\theta = -\frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$<br>$= -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$ |
| **5** Substitute back in terms of $x$. | $\therefore 2x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$ |

6 Calculate the solutions for $x$.

Add $\frac{\pi}{4}$ to each value.

$\therefore 2x = 0, \pi, 2\pi$

Divide by 2.

$\therefore x = 0, \frac{\pi}{2}, \pi$

## EXERCISE 10.2 Trigonometric equations

### PRACTISE

**1** WE1 Solve the following equations to obtain exact values for $x$.

a $\sin(x) = \frac{1}{2},\ 0 \le x \le 2\pi$

b $\sqrt{3} - 2\cos(x) = 0,\ 0 \le x \le 2\pi$

c $4 + 4\tan(x) = 0,\ -2\pi \le x \le 2\pi$

**2** Consider the equation $\cos(\theta) = -\frac{1}{2},\ -180° \le \theta \le 180°$.

a How many solutions for $\theta$ does the equation have?

b Calculate the solutions of the equation.

**3** WE2 a Solve $1 - \sin(x) = 0, -4\pi \le x \le 4\pi$ for $x$.

b Solve $\tan(x) = 0.75, 0 \le x \le 4\pi$ for $x$, to 2 decimal places.

c Solve $4\cos(x°) + 1 = 0, -180° \le x° \le 180°$ for $x$, to 1 decimal place.

**4** Consider the function $f: [0, 2] \to R, f(x) = \cos(\pi x)$.

a Calculate $f(0)$.

b Obtain $\{x : f(x) = 0\}$.

**5** WE3 Solve the following for $x$, given $0 \le x \le 2\pi$.

a $\sqrt{3}\sin(x) = 3\cos(x)$

b $\sin^2(x) - 5\sin(x) + 4 = 0$

**6** Solve $\cos^2(x) = \frac{3}{4}, 0 \le x \le 2\pi$ for $x$.

**7** WE4 a Solve $\sin(2x) = \frac{1}{\sqrt{2}}, 0 \le x \le 2\pi$ for $x$.

b Use a substitution to solve the equation $\cos\left(2x + \frac{\pi}{6}\right) = 0, 0 \le x \le \frac{3\pi}{2}$.

**8** Solve $\sin\left(\frac{x}{2}\right) = \sqrt{3}\cos\left(\frac{x}{2}\right), 0 \le x \le 2\pi$ for $x$.

### CONSOLIDATE

Apply the most appropriate mathematical processes and tools

**9** Solve for $x$, given $0 \le x \le 2\pi$.

a $\cos(x) = \frac{1}{\sqrt{2}}$

b $\sin(x) = -\frac{1}{\sqrt{2}}$

c $\tan(x) = -\frac{1}{\sqrt{3}}$

d $2\sqrt{3}\cos(x) + 3 = 0$

e $4 - 8\sin(x) = 0$

f $2\sqrt{2}\tan(x) = \sqrt{24}$

**10** Solve for $a$, given $0° \le a \le 360°$.

a $\sqrt{3} + 2\sin(a) = 0$

b $\tan(a) = 1$

c $6 + 8\cos(a) = 2$

d $4(2 + \sin(a)) = 11 - 2\sin(a)$

**11** Obtain all values for $t$, $t \in [-\pi, 4\pi]$, for which:

a $\tan(t) = 0$

b $\cos(t) = 0$

c $\sin(t) = -1$

d $\cos(t) = 1$

e $\sin(t) = 1$

f $\tan(t) = 1$.

**12** Calculate the values of $\theta$, correct to 2 decimal places, which satisfy the following conditions.

a $2 + 3\cos(\theta) = 0, 0 \le \theta \le 2\pi$

b $\tan(\theta) = \frac{1}{\sqrt{2}}, -2\pi \le \theta \le 3\pi$

c $5\sin(\theta) + 4 = 0, -270° \le \theta \le 270°$

d $\cos^2(\theta) = 0.04, 0° \le \theta \le 360°$

**13** Solve the following equations, where possible, to obtain the values of the pronumerals.

**a** $4\sin(a) + 3 = 5, -2\pi < a < 0$

**b** $6\tan(b) - 1 = 11, -\frac{\pi}{2} < b < 0$

**c** $8\cos(c) - 7 = 1, -\frac{9\pi}{2} < c < 0$

**d** $\frac{9}{\tan(d)} - 9 = 0, 0 < d \leq \frac{5\pi}{12}$

**e** $2\cos(e) = 1, -\frac{\pi}{6} \leq e \leq \frac{13\pi}{6}$

**f** $\sin(f) = -\cos(150°), -360° \leq f \leq 360°$

**14 a** How many solutions are there to the equation $\cos(x) = -0.3, -\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$?

**b** Calculate the sum of the solutions to $\sin(x) = 0.2, 0 \leq x \leq 2\pi$.

**c** If $x = 0.4$ is a solution of the equation $\tan(x) = c, 0 \leq x \leq 3\pi$, obtain the other possible solutions.

**d** Set up a trigonometric equation, the solution of which gives the angle of inclination of the following two lines with the positive direction of the $x$-axis.

**i** $y = -3x$ **ii** $y = \sqrt{3}x$

**e** Give the exact angle in each case for part **d**.

**f** Earlier, $\theta$, the obtuse angle of inclination with the $x$-axis of a line with a negative gradient $m$, was calculated from the rule $\theta = 180° - \tan^{-1}(|m|)$ where $|m|$ gave the positive part of $m$. Explain this rule by reference to the solution of a trigonometric equation.

**15** Consider the function $f: [0, 2\pi] \rightarrow R, f(x) = a\sin(x)$.

**a** If $f\left(\frac{\pi}{6}\right) = 4$, calculate the value of $a$.

**b** Use the answer to part **a** to find, where possible, any values of $x$, to 2 decimal places, for which the following apply.

**i** $f(x) = 3$ **ii** $f(x) = 8$ **iii** $f(x) = 10$

**16** Solve for $x$, where $0 \leq x \leq 2\pi$.

**a** $\sin(x) = \sqrt{3}\cos(x)$

**b** $\sin(x) = -\frac{\cos(x)}{\sqrt{3}}$

**c** $\sin(2x) + \cos(2x) = 0$

**d** $\frac{3\sin(x)}{8} = \frac{\cos(x)}{2}$

**e** $\sin^2(x) = \cos^2(x)$

**f** $\cos(x)(\cos(x) - \sin(x)) = 0$

**17** Solve for $x$, where $0 \leq x \leq 2\pi$.

**a** $\sin^2(x) = \frac{1}{2}$

**b** $2\cos^2(x) + 3\cos(x) = 0$

**c** $2\sin^2(x) - \sin(x) - 1 = 0$

**d** $\tan^2(x) + 2\tan(x) - 3 = 0$

**e** $\sin^2(x) + 2\sin(x) + 1 = 0$

**f** $\cos^2(x) - 9 = 0$

**18** Solve for $\theta$, $0 \leq \theta \leq 2\pi$.

**a** $\sqrt{3}\tan(3\theta) + 1 = 0$

**b** $2\sqrt{3}\sin\left(\frac{3\theta}{2}\right) - 3 = 0$

**c** $4\cos^2(-\theta) = 2$

**d** $\sin\left(2\theta + \frac{\pi}{4}\right) = 0$

MASTER

19 a Use CAS technology to obtain the solutions to $5\cos(x) = 2, 0 \le x \le 2\pi$:
  i in exact form
  ii to 2 decimal places.
 b Solve $5\sin(2x + 3) = 4, 0 \le x \le 2\pi$ to 2 decimal places.
 c Obtain the general solution to $\tan(x) = 1, x \in R$ and use this to write down the first three positive solutions.

20 a Give the exact value of $\sin(75°)$.
 b Hence solve the equation $\sin(A) = \dfrac{\sqrt{6} + \sqrt{2}}{4}, -270° \le A \le 270°$ for $A$.

# 10.3 Transformations of sine and cosine graphs

studyon

Units 1 & 2
AOS 1
Topic 7
Concept 2

**Transformations of sine and cosine graphs**
Concept summary
Practice questions

The shapes of the graphs of the two functions, sine and cosine, are now familiar. Each graph has a wavelike pattern of period $2\pi$ and the pair are 'out of phase' with each other by $\frac{\pi}{2}$. We now consider the effect of transformations on the shape and on the properties of these basic sine and cosine graphs.

## Transformations of the sine and cosine graphs

The following transformations are applied to the graph of $y = f(x) \to y = af(n(x - h)) + k$:

- dilation of factor $a$ (assuming $a$ is positive) from the $x$-axis parallel to the $y$-axis
- reflection in the $x$-axis if $a < 0$
- dilation of factor $\frac{1}{n}, n > 0$ from the $y$-axis parallel to the $x$-axis
- horizontal translation of $h$ units to the right if $h > 0$ or to the left if $h < 0$
- vertical translation of $k$ units up if $k > 0$ or down if $k < 0$

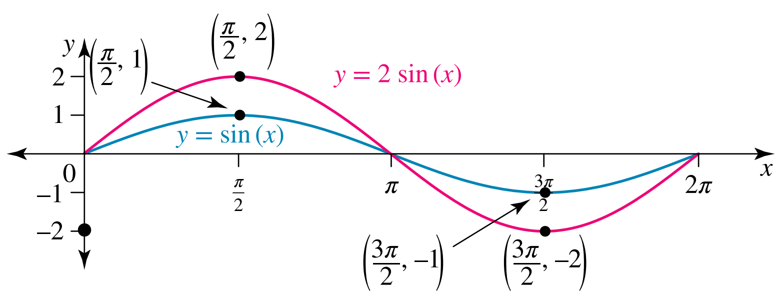
We can therefore infer that the graph of $y = a\sin(n(x - h)) + k$ is the image of the basic $y = \sin(x)$ graph after the same set of transformations are applied.

However, the period, amplitude and equilibrium, or mean, position of a trigonometric graph are such key features that we shall consider each transformation in order to interpret its effect on each of these features.

## Amplitude changes

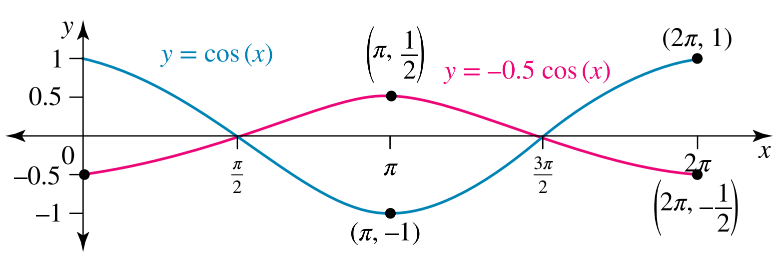
Consider the graphs of $y = \sin(x)$ and $y = 2\sin(x)$.

Comparison of the graph of $y = 2\sin(x)$ with the graph of $y = \sin(x)$ shows the dilation of factor 2 from the $x$-axis affects the amplitude, but neither the period nor the equilibrium position is altered.




Consider the graphs of $y = \cos(x)$ and $y = -\frac{1}{2}\cos(x)$.

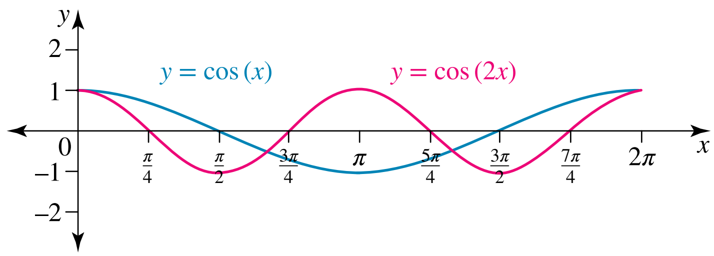
Comparison of the graph of $y = -\frac{1}{2}\cos(x)$ where



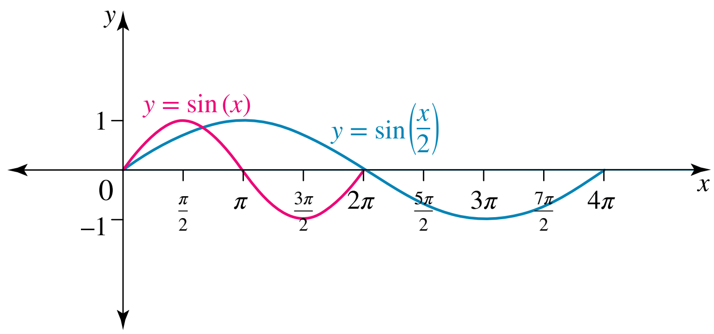
$a = -\frac{1}{2}$ with the graph of $y = \cos(x)$ shows the dilation factor affecting the amplitude is $\frac{1}{2}$ and the graph of $y = -\frac{1}{2}\cos(x)$ is reflected in the $x$-axis.

- **The graphs of $y = a\sin(x)$ and $y = a\cos(x)$ have amplitude $a$, if $a > 0$.**
- **If $a < 0$, the graph is reflected in the $x$-axis, (inverted) and the amplitude is the positive part of $a$ (or $|a|$).**

## Period changes



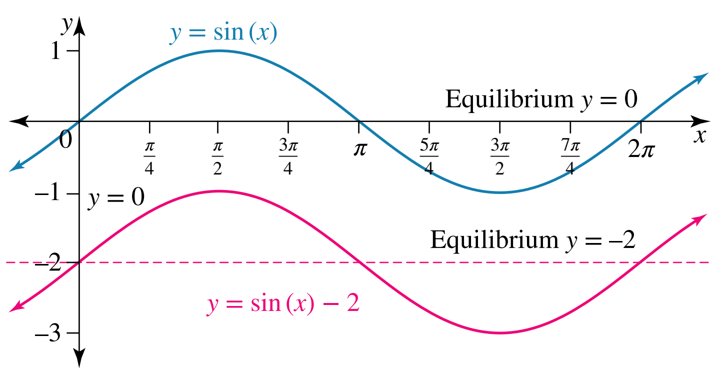

Comparison of the graph of $y = \cos(2x)$ with the graph of $y = \cos(x)$ shows the dilation factor of $\frac{1}{2}$ from the $y$-axis affects the period: it halves the period. The period of $y = \cos(x)$ is $2\pi$ while the period of $y = \cos(2x)$ is $\frac{1}{2}$ of $2\pi$; that is, $y = \cos(2x)$ has a period of $\frac{2\pi}{2} = \pi$. Neither the amplitude nor the equilibrium position has been altered.




Comparison of one cycle of the graph of $y = \sin\left(\frac{x}{2}\right)$ with one cycle of the graph of $y = \sin(x)$ shows the dilation factor of 2 from the $y$-axis doubles the period. The period of the graph of $y = \sin\left(\frac{x}{2}\right)$ is $\frac{2\pi}{\frac{1}{2}} = 2 \times 2\pi = 4\pi$.

- **The graphs of $y = \sin(nx)$ and $y = \cos(nx)$ have period $\frac{2\pi}{n}$, $n > 0$.**

## Equilibrium (or mean) position changes




Comparison of the graph of $y = \sin(x) - 2$ with the graph of $y = \sin(x)$ shows that vertical translation affects the equilibrium position. The graph of $y = \sin(x) - 2$ oscillates about the line $y = -2$, so its range is $[-3, -1]$. Neither the period nor the amplitude is affected.

- **The graphs of $y = \sin(x) + k$ and $y = \cos(x) + k$ both oscillate about the equilibrium (or mean) position $y = k$.**
- **The range of both graphs is $[k - 1, k + 1]$ since the amplitude is 1.**

## Summary of amplitude, period and equilibrium changes

**The graphs of $y = a\sin(nx) + k$ and $y = a\cos(nx) + k$ have:**

- **amplitude $a$ for $a > 0$; the graphs are reflected in the $x$-axis (inverted) if $a < 0$**
- **period $\frac{2\pi}{n}$ (for $n > 0$)**
- **equilibrium or mean position $y = k$**
- **range $[k - a, k + a]$.**

The oscillation about the equilibrium position of the graph of $y = a\sin(nx) + k$ always starts at the equilibrium with the pattern, for each period divided into quarters, of:

- equilibrium → range maximum → equilibrium → range minimum → equilibrium if $a > 0$, or: equilibrium → range minimum → equilibrium → range maximum → equilibrium if $a < 0$.

The oscillation about the equilibrium position of the graph of $y = a\cos(nx) + k$ either starts from its maximum or minimum point with the pattern:

- range maximum → equilibrium → range minimum → equilibrium → range maximum if $a > 0$, or: range minimum → equilibrium → range maximum → equilibrium → range minimum if $a < 0$.

When sketching the graphs, any intercepts with the $x$-axis are usually obtained by solving the trigonometric equation $a\sin(nx) + k = 0$ or $a\cos(nx) + k = 0$.

WORKED EXAMPLE 5

**Sketch the graphs of the following functions.**

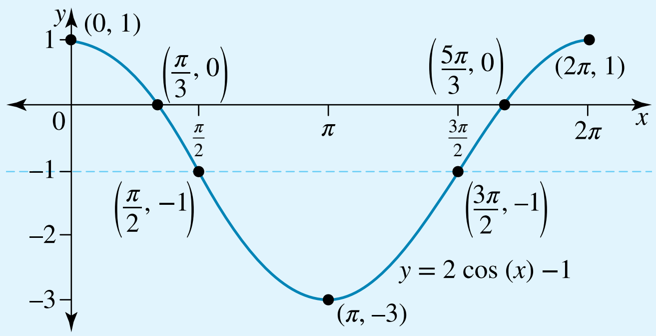
**a** $y = 2\cos(x) - 1, 0 \le x \le 2\pi$

**b** $y = 4 - 2\sin(3x), -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

| THINK | WRITE |
|---|---|
| **a 1** State the period, amplitude and equilibrium position by comparing the equation with $y = a\cos(nx) + k$. | **a** $y = 2\cos(x) - 1, 0 \le x \le 2\pi$<br>$a = 2,\ n = 1,\ k = -1$<br>Amplitude 2, period $2\pi$, equilibrium position $y = -1$ |
| **2** Determine the range and whether there will be $x$-intercepts. | The graph oscillates between $y = -1 - 2 = -3$, and $y = -1 + 2 = 1$, so it has range $[-3, 1]$.<br>It will have $x$-intercepts. |
| **3** Calculate the $x$-intercepts. | $x$-intercepts: let $y = 0$<br>$2\cos(x) - 1 = 0$<br>$\therefore \cos(x) = \frac{1}{2}$<br>Base $\frac{\pi}{3}$, quadrants 1 and 4<br>$x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$<br>$= \frac{\pi}{3}, \frac{5\pi}{3}$<br>$x$-intercepts are $\left(\frac{\pi}{3}, 0\right), \left(\frac{5\pi}{3}, 0\right)$. |

**4** Scale the axes by marking $\frac{1}{4}$-period intervals on the $x$-axis. Mark the equilibrium position and endpoints of the range on the $y$-axis. Then plot the graph using its pattern.

Period is $2\pi$ so the scale on the $x$-axis is in multiples of $\frac{\pi}{2}$. Since $a > 0$, graph starts at range maximum at its $y$-intercept $(0, 1)$.




**5** Label all key features of the graph including the maximum and minimum points.

The maximum points are $(0, 1)$ and $(2\pi, 1)$. The minimum point is $(\pi, -3)$.

**b 1** State the information the equation provides by comparing the equation with $y = a\sin(nx) + k$.

*Note:* The amplitude is always a positive value.

**b** $y = 4 - 2\sin(3x), -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

$y = -2\sin(3x) + 4$, domain $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$a = -2,\ n = 3,\ k = 4$

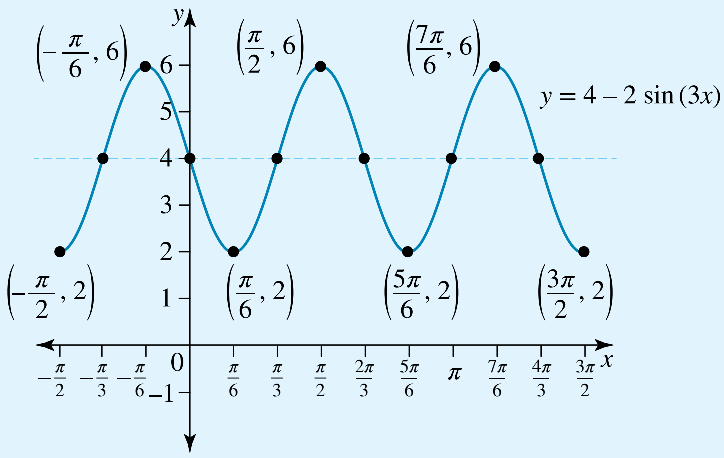
Amplitude 2; graph is inverted; period $\frac{2\pi}{3}$; equilibrium $y = 4$

**2** Determine the range and whether there will be $x$-intercepts.

The graph oscillates between $y = 4 - 2 = 2$ and $y = 4 + 2 = 6$, so its range is $[2, 6]$.
There are no $x$-intercepts.

**3** Scale the axes and extend the $\frac{1}{4}$-period intervals on the $x$-axis to cover the domain. Mark the equilibrium position and endpoints of the range on the $y$-axis. Then plot the graph using its pattern and continue the pattern over the given domain.

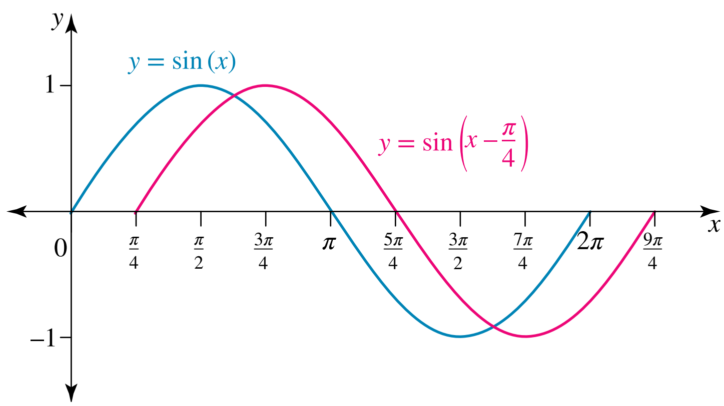
Dividing the period of $\frac{2\pi}{3}$ into four gives a horizontal scale of $\frac{\pi}{6}$. The first cycle of the graph starts at its equilibrium position at its $y$-intercept $(0, 4)$ and decreases as $a < 0$.



| | |
|---|---|
| 4 Label all key features of the graph including the maximum and minimum points. *Note:* Successive maximum points are one period apart, as are the successive minimum points. | Maximum points are $\left(-\frac{\pi}{6}, 6\right)$, $\left(\frac{\pi}{2}, 6\right)$ and $\left(\frac{7\pi}{6}, 6\right)$. Minimum points are $\left(-\frac{\pi}{2}, 2\right)$, $\left(\frac{\pi}{6}, 2\right)$, $\left(\frac{5\pi}{6}, 2\right)$ and $\left(\frac{3\pi}{2}, 2\right)$. |

## Phase changes

Horizontal translations of the sine and cosine graphs do not affect the period, amplitude or equilibrium, as one cycle of each of the graphs of $y = \sin(x)$ and $y = \sin\left(x - \frac{\pi}{4}\right)$ illustrate.




The horizontal translation causes the two graphs to be 'out of phase' by $\frac{\pi}{4}$.

- The graph of $y = \sin(x - h)$ has a phase shift of $h$ from the graph of $y = \sin(x)$.
- The graph of $y = \cos(x + h)$ has a phase shift of $-h$ from the graph of $y = \cos(x)$.

## The graphs of $y = a\sin(n(x - h)) + k$ and $y = a\cos(n(x - h)) + k$

**The features of the graphs of $y = a\sin(n(x - h)) + k$ and $y = a\cos(n(x - h)) + k$ are:**

- **period $\frac{2\pi}{n}$ $(n > 0)$**
- **amplitude $a$ (for $a > 0$), inverted if $a < 0$**
- **equilibrium at $y = k$ oscillating between $y = k \pm a$**
- **phase shift of $h$ from the graph of $y = a\sin(nx)$ or $y = a\cos(nx)$.**

Horizontal translation of the 5 key points that create the pattern for the graph of either $y = a\sin(nx)$ or $y = a\cos(nx)$ will enable one cycle of the graph with the phase shift to be sketched. This transformed graph may be extended to fit a given domain, with its rule used to calculate the coordinates of endpoints.

**WORKED EXAMPLE 6**

**a** Sketch the graph of $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right), 0 \le x \le 2\pi$.

**b** State the period, amplitude, range and phase shift factor for the graph of $y = -2\sin\left(4x + \frac{\pi}{3}\right) + 5$.

**THINK**

**a 1** Identify the key features of the graph from the given equation.

**2** Sketch one cycle of the graph without the horizontal translation.

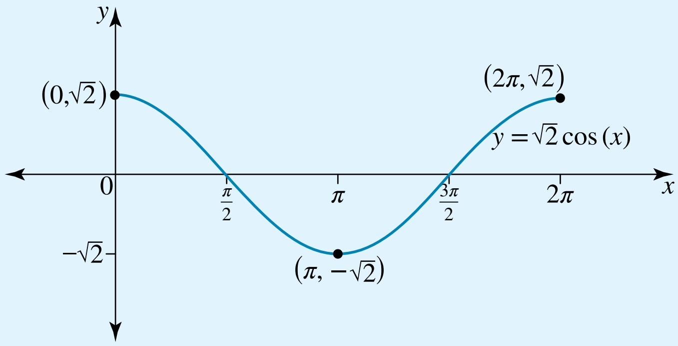
**3** Sketch the required graph using horizontal translation.

**WRITE**

**a** $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right), 0 \le x \le 2\pi$

Period $2\pi$; amplitude $\sqrt{2}$; equilibrium position $y = 0$; horizontal translation of $\frac{\pi}{4}$ to the right; domain $[0, 2\pi]$.

Sketching the graph of $y = \sqrt{2}\cos(x)$ using the pattern gives:




The key points of the graph of $y = \sqrt{2}\cos(x)$ become, under a horizontal translation of $\frac{\pi}{4}$ units to the right:

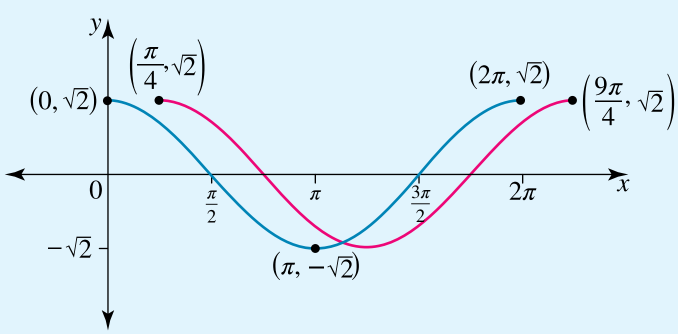
$(0, \sqrt{2}) \to \left(\frac{\pi}{4}, \sqrt{2}\right)$

$\left(\frac{\pi}{2}, 0\right) \to \left(\frac{3\pi}{4}, 0\right)$

$(\pi, -\sqrt{2}) \to \left(\frac{5\pi}{4}, -\sqrt{2}\right)$

$\left(\frac{3\pi}{2}, 0\right) \to \left(\frac{7\pi}{4}, 0\right)$

$(2\pi, \sqrt{2}) \to \left(\frac{9\pi}{4}, \sqrt{2}\right)$



4 Calculate the endpoints of the domain.

The translated graph is not on the required domain.
Endpoints for the domain $[0, 2\pi]$:

$$y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$$

When $x = 0$,

$$\begin{aligned} y &= \sqrt{2}\cos\left(-\frac{\pi}{4}\right) \\ &= \sqrt{2}\cos\left(\frac{\pi}{4}\right) \\ &= \sqrt{2} \times \frac{1}{\sqrt{2}} \\ &= 1 \end{aligned}$$

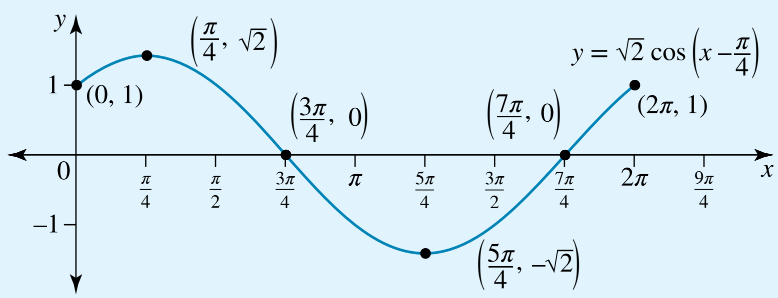
When $x = 2\pi$,

$$\begin{aligned} y &= \sqrt{2}\cos\left(2\pi - \frac{\pi}{4}\right) \\ &= \sqrt{2}\cos\left(\frac{\pi}{4}\right) \\ &= 1 \end{aligned}$$

Endpoints are $(0, 1)$ and $(2\pi, 1)$.

5 Sketch the graph on the required domain.

*Note:* As the graph covers one full cycle, the endpoints should have the same $y$-coordinates.




**b** 1 Express the equation in the form $y = a\sin(n(x - h)) + k$.

**b** $y = -2\sin\left(4x + \frac{\pi}{3}\right) + 5$

$$= -2\sin\left(4\left(x + \frac{\pi}{12}\right)\right) + 5$$

$a = -2, n = 4, h = -\frac{\pi}{12}, k = 5$

2 Calculate the required information.

Period is $\frac{2\pi}{n} = \frac{2\pi}{4}$, so the period is $\frac{\pi}{2}$. Amplitude is 2. (graph inverted)
Graph oscillates between $y = 5 - 2 = 3$ and $y = 5 + 2 = 7$, so the range is $[3, 7]$.

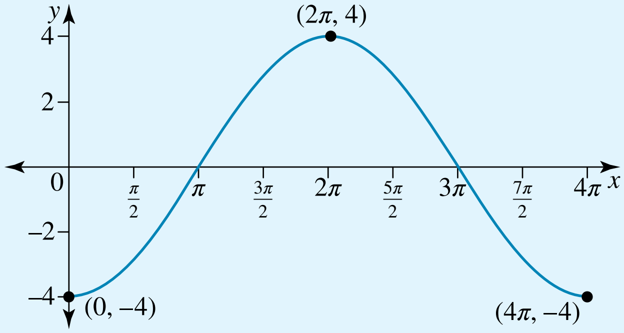
Phase shift factor from $y = -2\sin(4x)$ is $-\frac{\pi}{12}$.

## Forming the equation of a sine or cosine graph

The graph of $y = \sin\left(x + \frac{\pi}{2}\right)$ is the same as the graph of $y = \cos(x)$ since sine and cosine have a phase difference of $\frac{\pi}{2}$. This means that it is possible for the equation of the graph to be expressed in terms of either function. This is true for all sine and cosine graphs so their equations are not uniquely expressed. Given the choice, it is simpler to choose the form which does not require a phase shift.

WORKED EXAMPLE 7

**Determine two possible equations for the following graph.**




| THINK | WRITE |
|---|---|
| **1** Identify the key features of the given graph. | The graph has a period of $4\pi$, amplitude 4, and the equilibrium position is $y = 0$. |
| **2** Form a possible equation for the graph that does not involve any horizontal translation. | The graph could be an inverted cosine graph. A possible cosine equation for the graph could be $y = a\cos(nx) + k$ with $a = -4$ and $k = 0$<br>$\therefore y = -4\cos(nx)$<br>The period is $\frac{2\pi}{n}$.<br>From the diagram the period is $4\pi$.<br>$\frac{2\pi}{n} = 4\pi$<br>$\frac{2\pi}{4\pi} = n$<br>$n = \frac{1}{2}$<br>Therefore, a possible equation is $y = -4\cos\left(\frac{1}{2}x\right)$. |
| **3** Form a possible equation for the graph that does involve a horizontal translation.<br>*Note:* Other equations for the graph are possible by considering other phase shifts. | Alternatively, the graph could be a sine function that has been horizontally translated $\pi$ units to the right. This sine graph is not inverted so $a$ is positive.<br>A possible sine equation could be:<br>$y = a\sin(n(x - h)) + k$ with $a = 4$, $h = \pi$, $k = 0$<br>$\therefore y = 4\sin(n(x - \pi))$.<br>The graph has the same period of $4\pi$, so $n = \frac{1}{2}$.<br>Therefore, a possible equation is $y = 4\sin\left(\frac{1}{2}(x - \pi)\right)$. |

## EXERCISE 10.3 Transformations of sine and cosine graphs

### PRACTISE

1 **WE5** Sketch the graphs of the following functions.

a $y = 2\sin(x) + 1, 0 \le x \le 2\pi$

b $y = 4 - 3\cos(2x), -\pi \le x \le 2\pi$

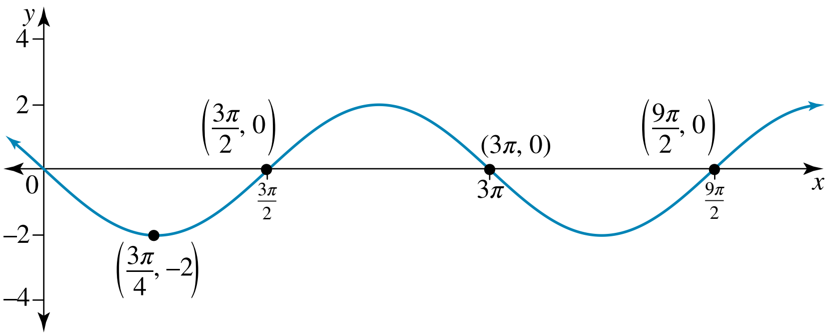
2 Sketch the graph of $y = f(x)$ for the function $f: [0, 12] \to R, f(x) = \sin\left(\frac{\pi x}{6}\right)$.

3 **WE6** a Sketch the graph of $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right), 0 \le x \le 2\pi$.

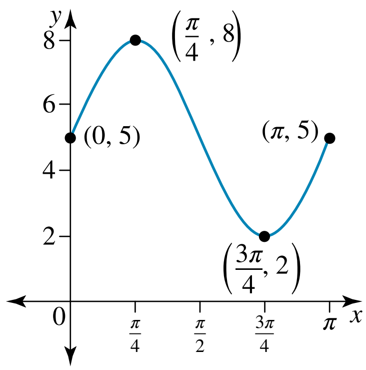
b State the period, amplitude, range and phase shift factor for the graph of $y = -3\cos\left(2x + \frac{\pi}{4}\right) + 1$.

4 Sketch the graph of $y = \sin\left(2x - \frac{\pi}{3}\right), 0 \le x \le \pi$.

5 **WE7** Determine two possible equations for the following graph.




6 Determine a possible equation for the following graph.




### CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 State the period and amplitude of the following.

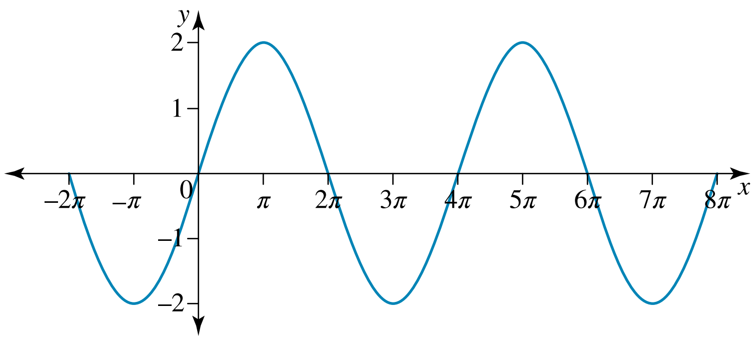
a $y = 6\cos(2x)$

b $y = -7\cos\left(\frac{x}{2}\right)$

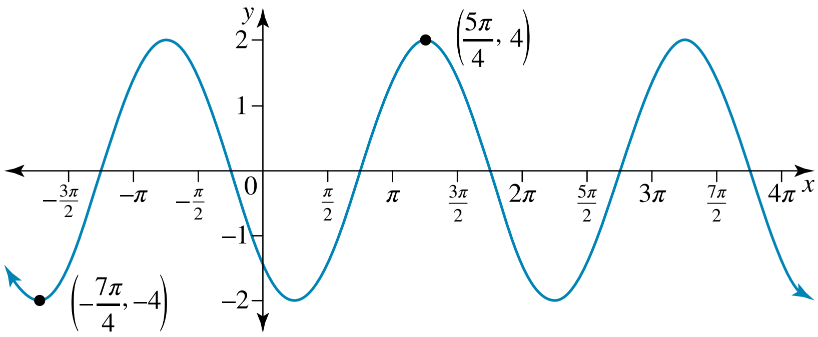
c $y = -\frac{3}{5}\sin\left(\frac{3x}{5}\right)$

d $y = \sin\left(\frac{6\pi x}{7}\right)$

**e**




**f**




**8** Sketch the following graphs over the given domains.

**a** $y = 3\cos(2x), 0 \le x \le 2\pi$ **b** $y = 2\sin\left(\frac{1}{2}x\right), 0 \le x \le 4\pi$

**c** $y = -5\sin(4x), 0 \le x \le 2\pi$ **d** $y = -\cos(\pi x), 0 \le x \le 4$

**9** Sketch each of the following over the domain specified and state the range.

**a** $y = \sin(x) + 3, 0 \le x \le 2\pi$ **b** $y = \cos(x) - 1, 0 \le x \le 2\pi$

**c** $y = \cos(x) + 2, -\pi \le x \le \pi$ **d** $y = 4 - \sin(x), -\pi \le x \le 2\pi$

**10 a** Give the range of $f: R \to R, f(x) = 3 + 2\sin(5x)$.

**b** What is the minimum value of the function $f: [0, 2\pi] \to R, f(x) = 10\cos(2x) - 4$?

**c** What is the maximum value of the function $f: [0, 2\pi] \to R, f(x) = 56 - 12\sin(x)$ and for what value of $x$ does the maximum occur?

**d** Describe the sequence of transformations that must be applied for the following.

**i** $\sin(x) \to 3 + 2\sin(5x)$ **ii** $\cos(x) \to 10\cos(2x) - 4$

**iii** $\sin(x) \to 56 - 12\sin(x)$

**11** Sketch the following graphs over the given domains and state the ranges of each.

**a** $y = 2\cos(2x) - 2, 0 \le x \le 2\pi$ **b** $y = 2\sin(x) + \sqrt{3}, 0 \le x \le 2\pi$

**c** $y = 3\sin\left(\frac{x}{2}\right) + 5, -2\pi \le x \le 2\pi$ **d** $y = -4 - \cos(3x), 0 \le x \le 2\pi$

**e** $y = 1 - 2\sin(2x), -\pi \le x \le 2\pi$ **f** $y = 2[1 - 3\cos(x)], 0° \le x \le 360°$

**12 a i** Sketch one cycle of each of $y = \cos(x)$ and $y = \cos\left(x + \frac{\pi}{6}\right)$ on the same axes.

**ii** Sketch one cycle of each of $y = \cos(x)$ and $y = \cos(x - \pi)$ on a second set of axes.

**b i** Sketch one cycle of each of $y = \sin(x)$ and $y = \sin\left(x - \frac{3\pi}{4}\right)$ on the same axes.

**ii** Sketch one cycle of each of $y = \sin(x)$ and $y = \sin\left(x + \frac{3\pi}{2}\right)$ on a second set of axes.

**13** Sketch the following graphs for $0 \le x \le 2\pi$.

**a** $y = 2\sin\left(x - \dfrac{\pi}{4}\right)$

**b** $y = -4\sin\left(x + \dfrac{\pi}{6}\right)$

**c** $y = \cos\left(2\left(x + \dfrac{\pi}{3}\right)\right)$

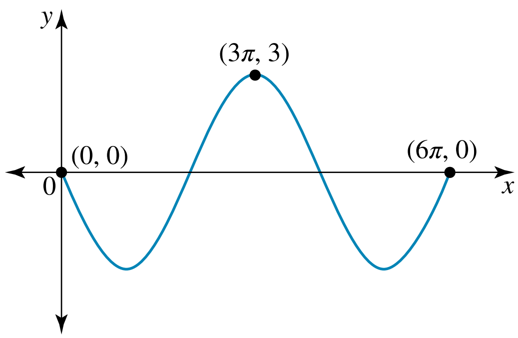
**d** $y = \cos\left(2x - \dfrac{\pi}{2}\right)$

**e** $y = \cos\left(x + \dfrac{\pi}{2}\right) + 2$

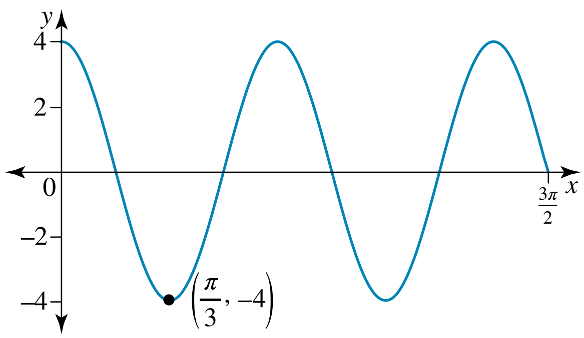
**f** $y = 3 - 3\sin(2x - 4\pi)$

**14** In parts **a–d**, obtain a possible equation for each of the given graphs.

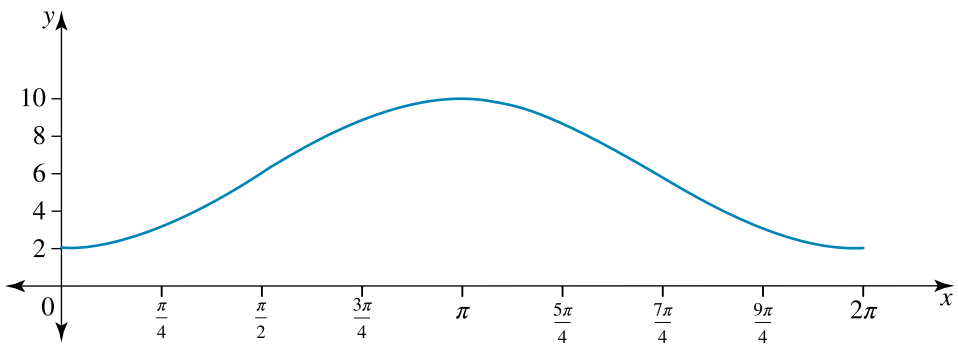
**a**



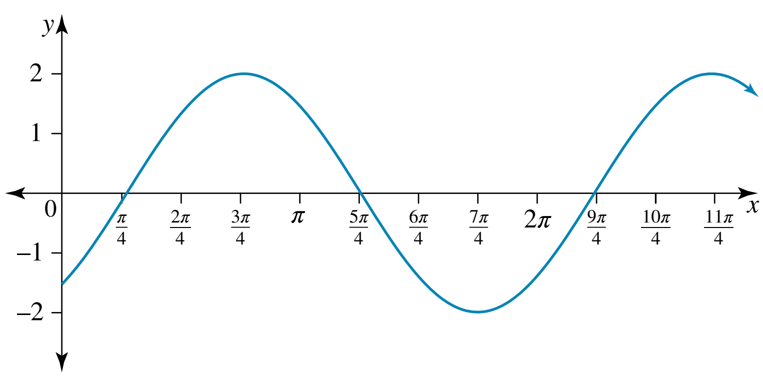

**b**




**c**




**d**




**e** Give an alternative equation for the graph shown in part **d**.

**f** Use the symmetry properties to give an alternative equation for $y = \cos(-x)$ and for $y = \sin(-x)$.

**15** A function has the rule $f(x) = a\sin(bx) + c$ and range of $[5, 9]$.

$f(x) = f\left(x + \frac{2\pi}{3}\right)$ and $\frac{2\pi}{3}$ is the smallest positive value for which this relationship holds.

**a** State the period of the function.

**b** Obtain possible values for the positive constants $a$, $b$ and $c$.

**c** Sketch one cycle of the graph of $y = f(x)$, stating its domain, $D$.

**d** A second function has the rule $g(x) = a\cos(bx) + c$ where $a$, $b$ and $c$ have the same values as those of $y = f(x)$. Sketch one cycle of the graph of $y = g(x)$, $x \in D$, on the same axes as the graph of $y = f(x)$.

**e** Obtain the coordinates of any points of intersection of the graphs of $y = f(x)$ and $y = g(x)$.

**f** Give the values of $x$ which are solutions to the inequation $f(x) \geq g(x)$ where $x \in D$.

**16 a** The graph of $y = \sin(x)$ is vertically translated upwards 3 units and is then reflected in the $x$-axis; this is followed by a dilation of factor 3 from the $y$-axis. Give the equation of its final image and determine if the graph of the image would intersect the graph of $y = \sin(x)$.

**b** Form the equation of the image of $y = \sin(x)$ under a linear transformation given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ and describe this transformation.

**c** Under the function $f\colon [0, 4] \to R, f(x) = a - 20\sin\left(\frac{\pi x}{3}\right)$ the image of 4 is $10(\sqrt{3} + 1)$.

**i** Determine the value of $a$.

**ii** For what value(s) of $x$ is $f(x) = 0$?

**iii** Sketch the graph of $y = f(x)$, stating its range.

**iv** Describe the sequence of transformations which would map $y = \cos(x) \to y = f(x)$, given that initially the domain of the cosine function is $\left[-\frac{\pi}{2}, \frac{5\pi}{6}\right]$.

MASTER

**17 a** Sketch the graph of $y = 4\sin(2x - 4), 0 \leq x \leq 2\pi$ using a CAS technology and give the values of any $x$-intercepts to 2 decimal places.

**b** Form the equation of the image of $y = 4\sin(2x - 4)$ under the transformation defined by $T: R^2 \to R^2, T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ and identify the relationship between $y = 4\sin(2x - 4)$ and its image under the transformation.

**c** Sketch the graph of the image using a CAS technology and state its domain, range and the values of any $y$-intercepts to 2 decimal places.

**18** Sketch and compare the graphs of the family defined by $y = \cos\left(x + k\frac{\pi}{2}\right), k \in Z$ and hence specify the $k$ values for which the graphs can be classified into four types.

# 10.4 Applications of sine and cosine functions

Phenomena that are cyclical in nature can often be modelled by a sine or cosine function.

Examples of periodic phenomena include sound waves, ocean tides and ovulation cycles. Trigonometric models may be able to approximate things like the movement in the value of the All Ordinaries Index of the stock market or fluctuations in temperature or the vibrations of violin strings about a mean position.

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## Maximum and minimum values

As $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$, the maximum value of both $\sin(x)$ and $\cos(x)$ is 1 and the minimum value of both functions is $-1$. This can be used to calculate, for example, the maximum value of $y = 2\sin(x) + 4$ by substituting 1 for $\sin(x)$:

$$y_{\max} = 2 \times 1 + 4 \Rightarrow y_{\max} = 6$$

The minimum value can be calculated as:

$$y_{\min} = 2 \times (-1) + 4 \Rightarrow y_{\min} = 2$$

To calculate the maximum value of $y = 5 - 3\cos(2x)$ the largest negative value of $\cos(2x)$ would be substituted for $\cos(2x)$. Thus:

$$y_{\max} = 5 - 3 \times (-1) \Rightarrow y_{\max} = 8$$

The minimum value can be calculated by substituting the largest positive value of $\cos(2x)$:

$$y_{\min} = 5 - 3 \times 1 \Rightarrow y_{\min} = 2$$

Alternatively, identifying the equilibrium position and amplitude enables the range to be calculated. For $y = 5 - 3\cos(2x)$, with amplitude 3 and equilibrium at $y = 5$, the range is calculated from $y = 5 - 3 = 2$ to $y = 5 + 3 = 8$, giving a range of $[2, 8]$. This also shows the maximum and minimum values.

WORKED EXAMPLE 8

**The temperature, $T$ °C, during one day in April is given by $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$ where $t$ is the time in hours after midnight.**

**a** **What was the temperature at midnight?**

**b** **What was the minimum temperature during the day and at what time did it occur?**

**c** **Over what interval did the temperature vary that day?**

**d** **State the period and sketch the graph of the temperature for $t \in [0, 24]$.**

**e** **If the temperature was below $k$ degrees for 2.4 hours, obtain the value of $k$ to 1 decimal place.**

| THINK | WRITE |
|---|---|
| **a** State the value of $t$ and use it to calculate the required temperature. | **a** At midnight, $t = 0$.<br>Substitute $t = 0$ into $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$.<br>$T = 17 - 4\sin(0)$<br>$= 17$<br>The temperature at midnight was 17°. |

**b 1** State the condition on the value of the trigonometric function for the minimum value of $T$ to occur.

**b** $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$

The minimum value occurs when

$\sin\left(\frac{\pi}{12}t\right) = 1.$

**2** Calculate the minimum temperature.

$$\begin{aligned} T_{\min} &= 17 - 4 \times 1 \\ &= 13 \end{aligned}$$

The minimum temperature was 13°.

**3** Calculate the time when the minimum temperature occurred.

The minimum temperature occurs when

$\sin\left(\frac{\pi}{12}t\right) = 1$

Solving this equation,

$$\begin{aligned} \frac{\pi}{12}t &= \frac{\pi}{2} \\ t &= 6 \end{aligned}$$

The minimum temperature occurred at 6 am.

**c** Use the equilibrium position and amplitude to calculate the range of temperatures.

**c** $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$

Amplitude 4, equilibrium $T = 17$

Range is $[17 - 4, 17 + 4] = [13, 21]$.

Therefore the temperature varied between 13°C and 21°C.

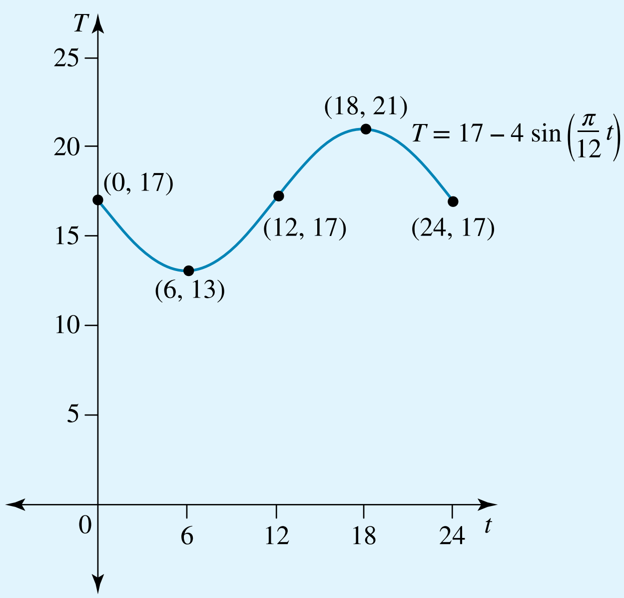
**d 1** Calculate the period.

**d** $$\begin{aligned} 2\pi \div \frac{\pi}{12} &= 2\pi \times \frac{12}{\pi} \\ &= 24 \end{aligned}$$

The period is 24 hours.

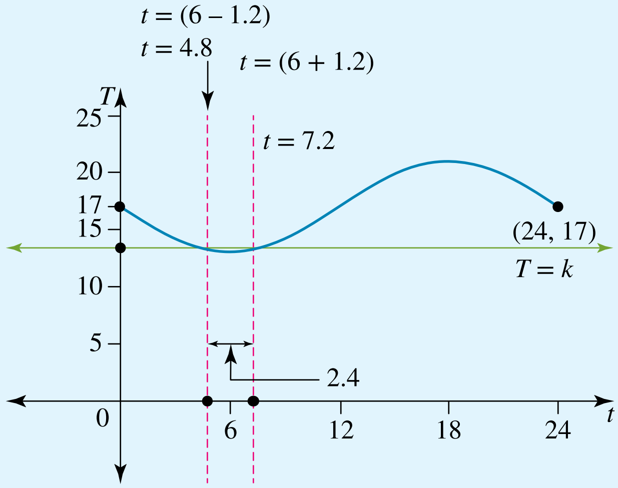
**2** Sketch the graph.

Dividing the period into quarters gives a horizontal scale of 6.



**e** **1** Use the symmetry of the curve to deduce the endpoints of the interval involved.

**e** $T < k$ for an interval of 2.4 hours.
For $T$ to be less than the value for a time interval of 2.4 hours, the 2.4-hour interval is symmetric about the minimum point.




The endpoints of the $t$ interval must each be $\frac{1}{2} \times 2.4$ from the minimum point where $t = 6$. The endpoints of the interval occur at $t = 6 \pm 1.2$.
$\therefore T = k$ when $t = 4.8$ or $7.2$.

**2** Calculate the required value.

Substituting either endpoint into the temperature model will give the value of $k$.
If $t = 4.8$:

$$\begin{aligned} T &= 17 - 4\sin\left(\frac{\pi}{12} \times 4.8\right) \\ &= 17 - 4\sin(0.4\pi) \\ &= 13.2 \end{aligned}$$

Therefore $k = 13.2$ and the temperature is below 13.2° for 2.4 hours.

## EXERCISE 10.4 Applications of sine and cosine functions

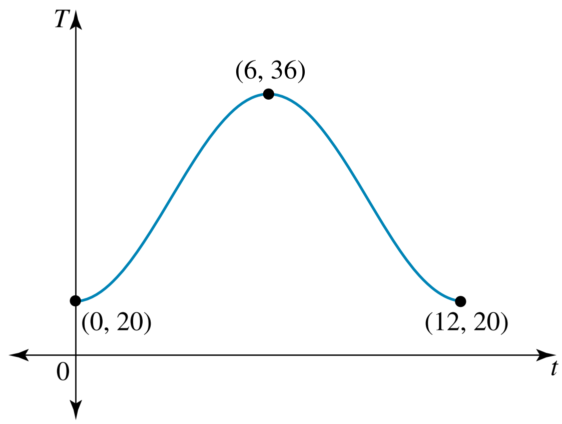
**PRACTISE**

**1** WE8 During one day in October the temperature $T$ °C is given by $T = 19 - 3\sin\left(\frac{\pi}{12}t\right)$ where $t$ is the time in hours after midnight.

**a** What was the temperature at midnight?
**b** What was the maximum temperature during the day and at what time did it occur?
**c** Over what interval did the temperature vary that day?
**d** State the period and sketch the graph of the temperature for $t \in [0, 24]$.
**e** If the temperature was below $k$ degrees for 3 hours find, to 1 decimal place, the value of $k$.

2 The temperature from 8 am to 10 pm on a day in February is shown.

If $T$ is the temperature in degrees Celsius $t$ hours from 8 am, form the equation of the temperature model and use this to calculate the times during the day when the temperature exceeded 30 degrees.




**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

3 A child plays with a yo-yo attached to the end of an elastic string. The yo-yo rises and falls about its rest position so that its vertical distance, $y$ cm, above its rest position at time $t$ seconds is given by $y = -40\cos(t)$.

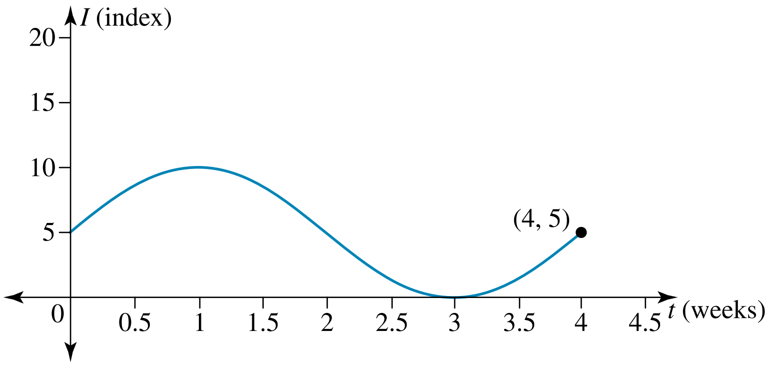
a Sketch the graph of $y = -40\cos(t)$ showing two complete cycles.

b What is the greatest distance the yo-yo falls below its rest position?

c At what times does the yo-yo return to its rest position during the two cycles?

d After how many seconds does the yo-yo first reach a height of 20 cm above its rest position?

4 Emotional ups and downs are measured by a wellbeing index which ranges from 0 to 10 in increasing levels of happiness. A graph of this index over a four-week cycle is shown.




a Express the relationship between the wellbeing index $I$ and the time $t$ in terms of a trigonometric equation.

b A person with a wellbeing index of 6 or higher is considered to experience a high level of happiness. For what percentage of the four-week cycle does the model predict this feeling would last?

**5** John is a keen amateur share trader who keeps careful records showing the fluctuation in prices of shares. One share in particular, Zentium, appears to have been following a sinusoidal shape with a period of two weeks over the last five weeks. Its share price has fluctuated between 12 and 15 cents, with its initial price at the start of the observations at its peak.

**a** Given the Zentium share price can be modelled by $p = a\cos(nt) + b$ where $p$ is the price in cents of the share $t$ weeks after the start of the recorded observations, determine the values of $a$, $n$ and $b$.

**b** Sketch the graph of the share price over the last five weeks and state the price of the shares at the end of the five weeks.

**c** When John decides to purchase the share, its price is 12.75 cents and rising. He plans to sell it immediately once it reaches 15 cents. According to the model, how many days will it be before he sells the share? Round your answer to the nearest day. Assume 7 days trading week applies.

**d** If John buys 10 000 shares at 12.75 cents, sells them at 15 cents and incurs brokerage costs of 1% when buying and selling, how much profit does he make?

**6** The height, $h$ metres, of the tide above mean sea level is given by $h = 4\sin\left(\dfrac{\pi(t-2)}{6}\right)$, where $t$ is the time in hours since midnight.

**a** How far below mean sea level was the tide at 1 am?

**b** State the high tide level and show that this first occurs at 5 am.

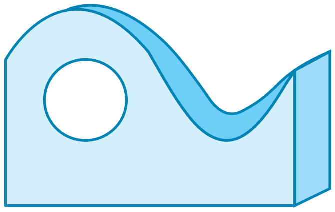
**c** How many hours are there between high tide and the following low tide?

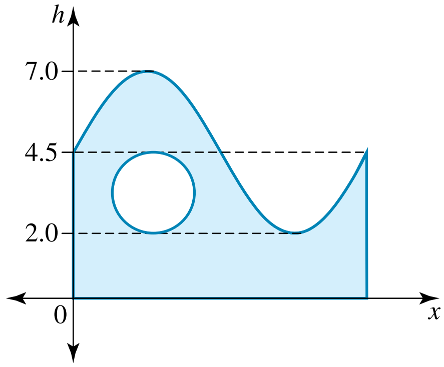
**d** Sketch the graph of $h$ versus $t$ for $t \in [0, 12]$.

**e** What is the height of the tide predicted to be at 2 pm?

**f** How much higher than low tide level is the tide at 11.30 am? Give the answer to 2 decimal places.

**7**






The diagram shows the cross-section of a sticky tape holder with a circular hole through its side. It is bounded by a curve with the equation $h = a\sin\left(\frac{\pi}{5}x\right) + b$, where $h$ cm is the height of the curve above the base of the holder at distance $x$ cm along the base.

**a** Use the measurements given on the diagram to state the values of $a$ and $b$ and hence write down the equation of the bounding curve.

**b** Calculate the length of the base of the holder.

**c** If the centre of the circular hole lies directly below the highest point of the curve, what are the coordinates of the centre?

**d** Using the symmetry of the curve, calculate the cross-sectional area shaded in the diagram, to 1 decimal place.

**8** In a laboratory, an organism is being grown in a test tube. To help increase the rate at which the organism grows, the test tube is placed in an incubator where the temperature $T$ °C varies according to the rule $T = 30 - \cos\left(\frac{\pi}{12}t\right)$, where $t$ is the time in minutes since the test tube has been placed in the incubator.

**a** State the range of temperature in the incubator.

**b** How many minutes after the test tube is placed in the incubator does the temperature reach its greatest value?

**c** Sketch the graph of the temperature against time for the first half hour.

**d** How many cycles are completed in 1 hour?

**e** If the organism is kept in the incubator for 2.5 hours, what is the temperature at the end of this time?

**f** Express the rule for the temperature in terms of a sine function.

**9** During a particular day in a Mediterranean city, the temperature inside an office building between 10 am and 7.30 pm fluctuates so that $t$ hours after 10 am, the temperature $T$ °C is given by $T = 19 + 6\sin\left(\frac{\pi t}{6}\right)$.

**a** **i** State the maximum temperature and the time it occurs.

**ii** State the minimum temperature and the time it occurs.

**b** **i** What is the temperature in the building at 11.30 am? Answer to 1 decimal place.

**ii** What is the temperature in the building at 7.30 pm? Answer to 1 decimal place.

**c** Sketch the graph of the temperature against time from 10 am and 7.30 pm.

**d** When the temperature reaches 24°, an air conditioner in the boardroom is switched on and it is switched off when the temperature in the rest of the building falls below 24°. For how long is the air conditioner on in the boardroom?

e The office workers who work the shift between 11.30 am and 7.30 pm complain that the temperature becomes too cool towards the end of their shift. If management agrees that heating can be used for the coldest two-hour period of their shift, at what time and at what temperature would the heating be switched on? Express the temperature in both exact form and to 1 decimal place.

10 The height above ground level of a particular carriage on a Ferris wheel is given by $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$ where $h$ is the height in metres above ground level after $t$ seconds.

a How far above the ground is the carriage initially?
b After one minute, how high will the carriage be?
c How many revolutions will the Ferris wheel complete in a four-minute time interval?
d Sketch the graph of $h$ against $t$ for the first four minutes.
e For how long, to the nearest second, in one revolution, is the carriage higher than 12 metres above the ground?
f The carriage is attached by strong wire radial spokes to the centre of the circular wheel. Calculate the length of a radial spoke.

11 A person sunbathing at a position P on a beachfront observes the waves wash onto the beach in such a way that after $t$ minutes, the distance $p$ metres of the end of the water's wave from P is given by $p = 3\sin(n\pi t) + 5$.

a What is the closest distance the water reaches to the sunbather at P?
b Over a one-hour interval, the sunbather counts 40 complete waves that have washed onto the beach. Calculate the value of $n$.
c At some time later in the day, the distance $p$ metres of the end of the water's wave from P is given by $p = a\sin(4\pi t) + 5$. If the water just reaches the sunbather who is still at P, deduce the value of $a$ and determine how many times in half an hour the water reaches the sunbather at P.
d In which of the two models of the wave motion, $p = 3\sin(n\pi t) + 5$ or $p = a\sin(4\pi t) + 5$, is the number of waves per minute greater?

12 A discarded drink can floating in the waters of a creek oscillates vertically up and down 20 cm about its equilibrium position. Its vertical displacement, $x$ metres, from its equilibrium position after $t$ seconds is given by $x = a\sin(bt)$. Initially the can moved vertically downwards from its mean position and it took 1.5 seconds to rise from its lowest point to its highest point.

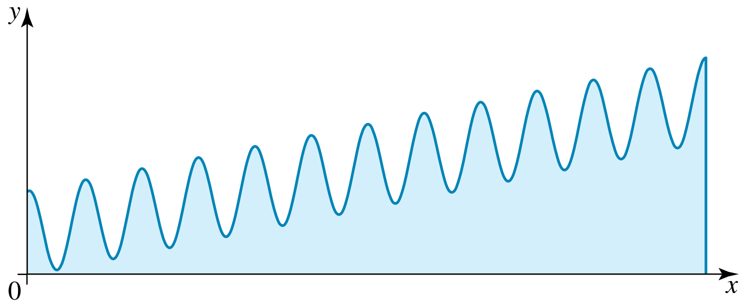
a Determine the values of $a$ and $b$ and state the equation for the vertical displacement.
b Sketch one cycle of the motion.
c Calculate the shortest time, $T$ seconds, for the can's displacement to be one half the value of its amplitude.
d What is the total distance the can moved in one cycle of the motion?

MASTER

13 The intensity, $I$, of sound emitted by a device is given by $I = 4\sin(t) - 3\cos(t)$ where $t$ is the number of hours the device has been operating.

a Use the $I - t$ graph to obtain the maximum intensity the device produces.
b State, to 2 decimal places, the first value of $t$ for which $I = 0$.
c Express the equation of the $I - t$ graph in the form $I = a\sin(t + b)$.
d Express the equation of the $I - t$ graph in the form $I = a\cos(t + b)$.

14 The teeth of a tree saw can be approximated by the function $y = x + 4 + 4\cos(6x)$, $0 \le x \le 4\pi$, where $y$ cm is the vertical height of the teeth at a horizontal distance $x$ cm from the end of the saw.




*Note:* This diagram is representative only.

Each peak of the graph represents one of the teeth.

a How many teeth does the saw have?
b Exactly how far apart are the successive peaks of the teeth?
c The horizontal length of the saw is $4\pi$ cm. What is the greatest width measurement of this saw?
d Give the equation of the linear function which will touch each of the teeth of the saw.

# 10.5 The tangent function

studyon

Units 1 & 2
AOS 1
Topic 7
Concept 4

**The tangent function**
Concept summary
Practice questions

The graph of $y = \tan(x)$ has a distinct shape quite different to that of the wave shape of the sine and cosine graphs. As values such as $\tan\left(\frac{\pi}{2}\right)$ and $\tan\left(\frac{3\pi}{2}\right)$ are undefined, a key feature of the graph of $y = \tan(x)$ is the presence of vertical asymptotes at odd multiples of $\frac{\pi}{2}$.

**The relationship $\tan(x) = \frac{\sin(x)}{\cos(x)}$ shows that:**
- **$\tan(x)$ will be undefined whenever $\cos(x) = 0$**
- **$\tan(x) = 0$ whenever $\sin(x) = 0$.**

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Graph plotter: Tangent
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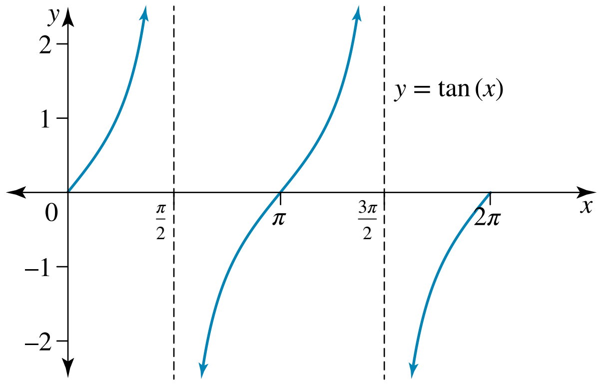
## The graph of $y = \tan(x)$

The diagram shows the graph of $y = \tan(x)$ over the domain $[0, 2\pi]$.

The key features of the graph of $y = \tan(x)$ are:

- Vertical asymptotes at $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$ for the domain $[0, 2\pi]$. For extended domains this pattern continues with asymptotes at $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$.
- Period $\pi$. Two cycles are completed over the domain $[0, 2\pi]$.

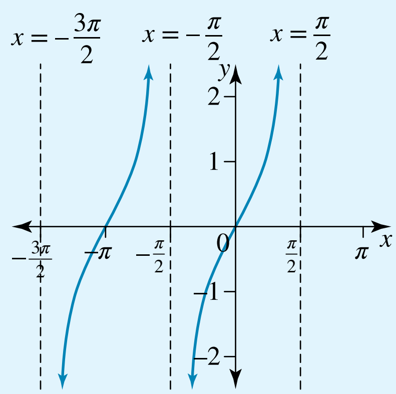
- Range is $R$.
- $x$-intercepts occur at $x = 0,\ \pi,\ 2\pi$ for the domain $[0, 2\pi]$. For extended domains this pattern continues with $x$-intercepts at $x = n\pi,\ n \in Z$.
- Mean position is $y = 0$.
- The asymptotes are one period apart.
- The $x$-intercepts are one period apart.




Unlike the sine and cosine graphs, 'amplitude' has no meaning for the tangent graph. As for any graph, the $x$-intercepts of the tangent graph are the solutions to the equation formed when $y = 0$.

WORKED EXAMPLE 9

**Sketch the graph of $y = \tan(x)$ for $x \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$.**

| THINK | WRITE |
|---|---|
| 1 State the equations of the asymptotes. | Period: the period of $y = \tan(x)$ is $\pi$. Asymptotes: The graph has an asymptote at $x = \frac{\pi}{2}$. As the asymptotes are a period apart, for the domain $\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$ there is an asymptote at $x = \frac{\pi}{2} - \pi \Rightarrow x = -\frac{\pi}{2}$ and another at $x = -\frac{\pi}{2} - \pi = -\frac{3\pi}{2}$. |
| 2 State where the graph cuts the $x$-axis. *Note:* The $x$-intercepts could be found by solving the equation $\tan(x) = 0,\ x \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$. | The $x$-intercepts occur midway between the asymptotes. $(-\pi, 0)$ and $(0, 0)$ are the $x$-intercepts. |
| 3 Sketch the graph. |  |



## Transformations of the graph of $y = \tan(x)$

The vertical asymptotes are major features of the graph of any tangent function and their positions may be affected by transformations of the graph of $y = \tan(x)$. This is not the case if the graph is reflected in either axis, dilated from the $x$-axis, or vertically translated.

The $x$-intercepts which lie midway between the asymptotes of $y = \tan(x)$ may not always maintain this symmetry under transformations of this graph.

## The graphs of $y = a\tan(x)$ and $y = \tan(x) + k$

First consider $y = a\tan(x)$.

A dilation of factor $a$ from the $x$-axis makes the graph of $y = a\tan(x)$ narrower than $y = \tan(x)$ if $a > 1$, or wider than $y = \tan(x)$ if $0 < a < 1$; if $a < 0$, the graph is reflected in the $x$-axis. The dilation does not affect the position of the vertical asymptotes. The $x$-intercepts will lie midway between successive pairs of asymptotes.

Now consider $y = \tan(x) + k$.

A vertical translation of $y = \tan(x)$ upwards by $k$ units if $k > 0$ or downwards by $k$ units if $k < 0$ does not affect the position of the vertical asymptotes. However, the position of the $x$-intercepts no longer lies midway between the asymptotes, since $y = 0$ is no longer the mean position. The $x$-intercepts are located by solving the equation $\tan(x) + k = 0$.

The period of both $y = a\tan(x)$ and $y = \tan(x) + k$ is $\pi$ since neither a dilation from the $x$-axis nor a vertical translation affects the period of the graph.

**WORKED EXAMPLE 10**

**a** On the same diagram, sketch the graphs of $y = \tan(x)$ and $y = -2\tan(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

**b** Sketch the graph of $y = \tan(x) - \sqrt{3}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

| THINK | WRITE |
| --- | --- |
| **a 1** State the key features of the graph of $y = \tan(x)$ for the given interval. | **a** $y = \tan(x)$<br>Period is $\pi$.<br>Asymptotes: $x = -\frac{\pi}{2}, x = \frac{\pi}{2}$<br>$x$-intercept: the $x$-intercept occurs midway between the asymptotes. Therefore $(0, 0)$ is the $x$-intercept and the $y$-intercept. |
| **2** Describe the transformations applied to $y = \tan(x)$ to obtain the second function. | $y = -2\tan(x)$<br>Dilation from the $x$-axis by a factor of 2 and reflection in the $x$-axis |
| **3** State the key features of the graph of the second function. | The transformations do not alter the period, $x$-intercept or asymptotes.<br>The period is $\pi$, the graph contains the point $(0, 0)$, and the asymptotes have equations $x = \pm\frac{\pi}{2}$. |
| **4** To illustrate the relative positions of the graphs, calculate the coordinates of another point on each graph. | $y = \tan(x)$<br>Let $x = \frac{\pi}{4}$:<br>$y = \tan\left(\frac{\pi}{4}\right)$<br>$= 1$ |

Point $\left(\frac{\pi}{4}, 1\right)$ lies on the graph of $y = \tan(x)$.

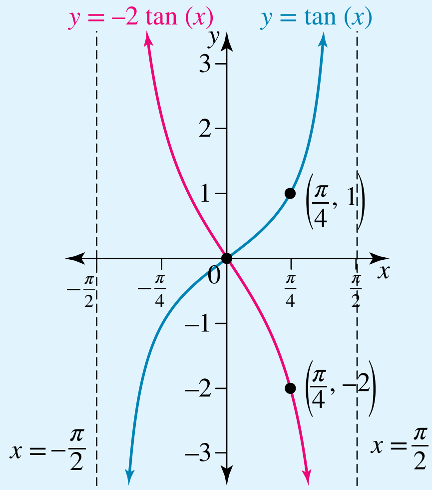
$y = -2\tan(x)$

Let $x = \frac{\pi}{4}$:

$$\begin{aligned} y &= -2\tan\left(\frac{\pi}{4}\right) \\ &= -2 \end{aligned}$$

Point $\left(\frac{\pi}{4}, -2\right)$ lies on the graph of $y = -2\tan(x)$.

**5** Sketch the two graphs.




**b 1** Describe the transformation.

**b** $y = \tan(x) - \sqrt{3}$

There is a vertical translation down of $\sqrt{3}$ units.

**2** Calculate any $x$-intercepts.

$x$-intercepts: let $y = 0$

$\tan(x) = \sqrt{3}, -\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore x = \frac{\pi}{3}$

$\left(\frac{\pi}{3}, 0\right)$ is the $x$-intercept.

**3** State any other features of the graph.

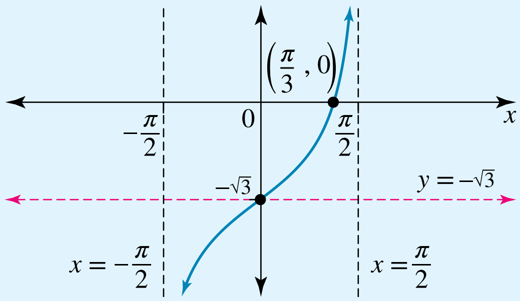
The transformation does not alter the period or asymptotes.

Period is $\pi$ and the asymptotes are $x = \pm\frac{\pi}{2}$.

Mean position is $y = -\sqrt{3}$.

$y$-intercept is $(0, -\sqrt{3})$.

**4** Sketch the graph.



## Period changes: the graph of $y = \tan(nx)$

The dilation factor of $\frac{1}{n}$ from the $y$-axis affects the period of $y = \tan(nx)$.
Assuming $n > 0$:

- $y = \tan(nx)$ has period $\frac{\pi}{n}$

Altering the period will alter the position of the asymptotes, although they will still remain one period apart. Since the graph of $y = \tan(x)$ has an asymptote at $x = \frac{\pi}{2}$, one way to obtain the equations of the asymptotes is to solve $nx = \frac{\pi}{2}$ to obtain one asymptote: other asymptotes can then be generated by adding or subtracting multiples of the period.

The $x$-intercepts of the graph of $y = \tan(nx)$ remain midway between successive pairs of asymptotes.

**WORKED EXAMPLE 11** **Sketch the graph of $y = \tan(4x), 0 \le x \le \pi$.**

| THINK | WRITE |
|---|---|
| 1 State the period by comparing the equation to $y = \tan(nx)$. | For $y = \tan(4x)$, $n = 4$<br>Period is $\frac{\pi}{n}$.<br>Therefore the period is $\frac{\pi}{4}$. |
| 2 Obtain the equation of an asymptote. | An asymptote occurs when:<br>$4x = \frac{\pi}{2}$<br>$\therefore x = \frac{\pi}{8}$ |
| 3 Calculate the equations of all the asymptotes in the given domain.<br>*Note:* Adding $\frac{\pi}{4}$ to $\frac{7\pi}{8}$ would give a value which exceeds the domain endpoint of $\pi$; subtracting $\frac{\pi}{4}$ from $\frac{\pi}{8}$ would also give a value which lies outside the domain. | The other asymptotes in the domain are formed by adding multiples of the period to $x = \frac{\pi}{8}$.<br>The other asymptotes in the domain $[0, \pi]$ occur at:<br>$x = \frac{\pi}{8} + \frac{\pi}{4}$<br>$= \frac{3\pi}{8}$<br>and:<br>$x = \frac{3\pi}{8} + \frac{\pi}{4}$<br>$= \frac{5\pi}{8}$ |

and:

$$x = \frac{5\pi}{8} + \frac{\pi}{4}$$

$$= \frac{7\pi}{8}$$

The equations of the asymptotes are $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

**4** Calculate the $x$-intercepts.

*Note:* The $x$-intercepts lie midway between each pair of asymptotes, one period apart.

$x$-intercepts: let $y = 0$

$$\tan(4x) = 0, 0 \le x \le \pi$$

$$\tan(4x) = 0, 0 \le 4x \le 4\pi$$

$$4x = 0, \pi, 2\pi, 3\pi, 4\pi$$

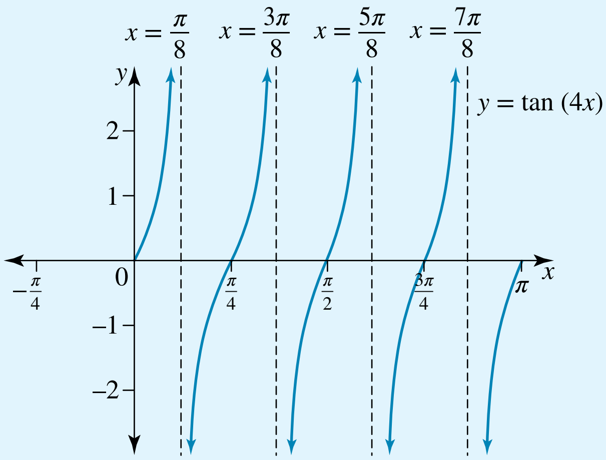
$$x = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}$$

$$= 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$$

$x$-intercepts are $(0, 0), \left(\frac{\pi}{4}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{4}, 0\right), (\pi, 0)$.

**5** Sketch the graph.

The period is $\frac{\pi}{4}$ so the horizontal axis is scaled in multiples of $\frac{1}{2}$ of $\frac{\pi}{4} = \frac{\pi}{8}$.




## Horizontal translation: the graph of $y = \tan(x - h)$

Horizontal translations will affect the position of the asymptotes. An asymptote can be established by solving $x - h = \frac{\pi}{2}$ and other asymptotes obtained by adding or subtracting multiples of the period.

The $x$-intercepts must again remain midway between pairs of asymptotes and this can be useful to determine their positions. Alternatively, their position can be calculated using the horizontal translation or by solving the equation formed when $y = 0$.

Coordinates of the endpoints of the domain should always be calculated and clearly labelled on the graph.

For the equation in the form $y = a\tan(nx + c)$, the equation should be expressed as $y = a\tan\left(n\left(x + \frac{c}{n}\right)\right) = a\tan(n(x + h))$ in order to recognise that $-\frac{c}{n}$ is the horizontal translation.

**WORKED EXAMPLE 12**

**a** Sketch the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$, $0 \le x \le 2\pi$.

**b i** Describe the transformations for $y = \tan(x) \to y = \tan\left(\frac{\pi x}{5} - \frac{\pi}{10}\right)$.

**ii** State the period of the function $y = \tan\left(\frac{\pi x}{5} - \frac{\pi}{10}\right)$.

| THINK | WRITE |
|---|---|
| **a 1** State the period and describe the transformation. | **a** $y = \tan\left(x + \frac{\pi}{4}\right), 0 \le x \le 2\pi$<br>The period is $\pi$.<br>There is a horizontal translation of $\frac{\pi}{4}$ units to the left. |
| **2** Calculate the equations of the asymptotes. | Asymptotes: an asymptote occurs when<br>$x + \frac{\pi}{4} = \frac{\pi}{2}$<br>$x = \frac{\pi}{2} - \frac{\pi}{4}$<br>$= \frac{\pi}{4}$<br>For the domain $[0, 2\pi]$ and period $\pi$, other asymptotes occur when<br>$x = \frac{\pi}{4} + \pi$<br>$= \frac{5\pi}{4}$<br>Adding another period would go beyond the domain.<br>The equations of the asymptotes are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. |
| **3** Calculate the $x$-intercepts.<br>*Note:* Solving the equation $\tan\left(x + \frac{\pi}{4}\right) = 0$ would also locate the $x$-intercepts. | $x$-intercepts: the $x$-intercept that is midway between the pair of asymptotes is $x = \frac{3\pi}{4}$. Adding a period gives another at $x = \frac{7\pi}{4}$.<br>The $x$-intercepts are $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$. |

**4** Determine the coordinates of the endpoints.

Endpoints: when $x = 0$,

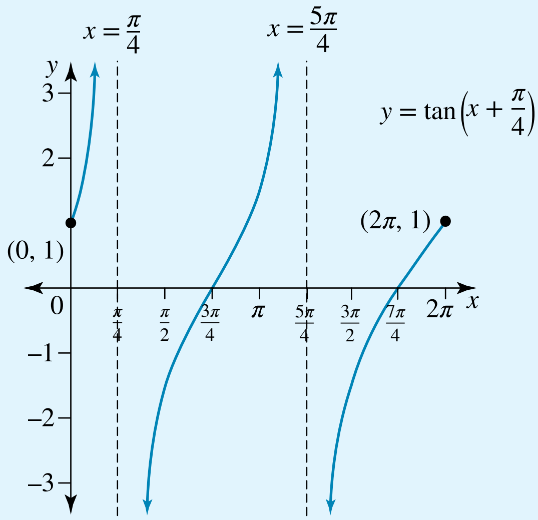
$$y = \tan\left(\frac{\pi}{4}\right)$$
$$= 1$$
$\Rightarrow (0, 1)$

When $x = 2\pi$,

$$y = \tan\left(2\pi + \frac{\pi}{4}\right)$$
$$= \tan\left(\frac{\pi}{4}\right)$$
$$= 1$$
$\Rightarrow (2\pi, 1)$

**5** Sketch the graph.

To allow for the horizontal translation, the horizontal scale is in multiples of $\frac{\pi}{4}$.




**b i 1** Rearrange the equation of the image into the form in which the transformations can be identified.

**b** $y = \tan(x) \to y = \tan\left(\frac{\pi x}{5} - \frac{\pi}{10}\right)$

$$y = \tan\left(\frac{\pi x}{5} - \frac{\pi}{10}\right)$$
$$= \tan\left(\frac{\pi}{5}\left(x - \frac{1}{2}\right)\right)$$

**2** Describe the sequence of transformations.

Dilation factor of $\frac{5}{\pi}$ from the $y$-axis followed by a horizontal translation of $\frac{1}{2}$ to the right.

**ii** State the period.

$$\text{Period} = \frac{\pi}{n}, n = \frac{\pi}{5}$$
$$= \pi \div \frac{\pi}{5}$$
$$= \pi \times \frac{5}{\pi}$$
$$= 5$$

Therefore the period is 5.

## EXERCISE 10.5 The tangent function

**PRACTISE**

1 WE9 Sketch the graph of $y = \tan(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

2 Sketch the graph of $y = \tan(x)$ over the interval for which $x \in [-3\pi, 3\pi]$ and state the domain and range of the graph.

3 WE10 **a** On the same diagram sketch the graphs of $y = -\tan(x)$ and $y = 3\tan(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

**b** Sketch the graph of $y = \tan(x) + \sqrt{3}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

4 On the same diagram sketch the graphs of $y = \tan(x)$ and $y = \tan(x) + 1$ for $x \in [0, 2\pi]$, showing any intercepts with the coordinate axes.

5 WE11 Sketch the graph of $y = \tan(3x),\ 0 \le x \le \pi$.

6 Sketch the graph of $y = -2\tan\left(\frac{x}{2}\right),\ 0 \le x \le 2\pi$.

7 WE12 **a** Sketch the graph of $y = \tan\left(x - \frac{\pi}{4}\right),\ 0 \le x \le 2\pi$.

**b** **i** Describe the transformations for $y = \tan(x) \rightarrow y = \tan\left(\frac{4\pi x}{3} + \frac{\pi}{3}\right)$.

**ii** State the period of the function $y = \tan\left(\frac{4\pi x}{3} + \frac{\pi}{3}\right)$.

8 Sketch the graph of $y = \tan\left(2x + \frac{\pi}{4}\right), 0 \le x \le \pi$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 Sketch the following over the given interval.

**a** $y = \tan(x),\ x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

**b** $y = \tan(x),\ x \in [-\pi, 0]$

**c** $y = \tan(x),\ x \in \left(0, \frac{5\pi}{2}\right)$

10 Sketch the following two graphs over the interval $x \in [0, 2\pi]$.

**a** $y = 4\tan(x)$

**b** $y = -0.5\tan(x)$

**c** On the same set of axes, sketch the graphs of $y = -\tan(x)$, $y = 1 - \tan(x)$ and $y = -1 - \tan(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

11 Consider the function defined by $y = \tan(x), -\pi \le x \le \pi$.

**a** Sketch the graph over the given interval.

**b** Obtain the $x$-coordinates of the points on the graph for which the $y$-coordinate is $-\sqrt{3}$.

**c** Use the answers to parts **a** and **b** to solve the inequation $\tan(x) + \sqrt{3} < 0$ for $x \in [-\pi, \pi]$.

**12** State the period of the following graphs:

**a** $y = \tan(6x)$

**b** $y = 5\tan\left(\dfrac{x}{4}\right)$

**c** $y = -2\tan\left(\dfrac{3x}{2}\right) + 5$

**d** $y = \tan(\pi x)$

**13** For $x \in [0, \pi]$, sketch the graphs of the following functions.

**a** $y = \tan(4x)$

**b** $y = 2\tan(2x)$

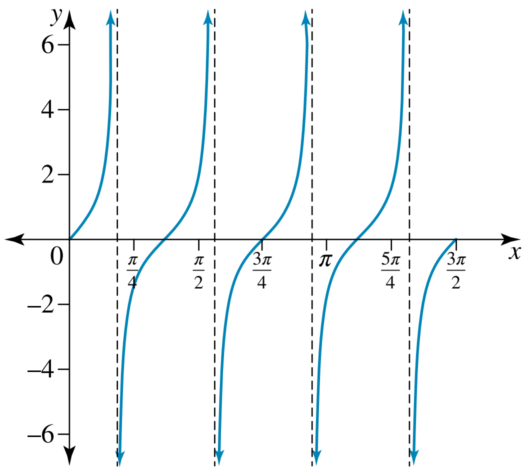
**c** $y = -\tan\left(\dfrac{x}{3}\right)$

**d** $y = 3\tan(x) + 2$

**e** $y = 3(1 - \tan(x))$

**f** $y = \tan(3x) - 1$

**14 a** The graph of $y = \tan(bx), 0 \le x \le \dfrac{3\pi}{2}$ is shown.




**i** State the number of cycles shown.

**ii** State the period.

**iii** Obtain the value of the positive constant $b$.

**iv** Give the equations of the four vertical asymptotes.

**b** Given the least negative value for which $f(x) = a - \tan(cx)$ is undefined is $x = -\frac{6}{5}$, and that $f(-12) = -2$, calculate the values of $a$ and $c$.

**15** Sketch the graphs of each of the following for $x \in [0, 2\pi]$.

**a** $y = \tan\left(x - \dfrac{\pi}{6}\right)$

**b** $y = \tan\left(x + \dfrac{\pi}{3}\right)$

**c** $y = 2\tan\left(x + \dfrac{\pi}{2}\right)$

**d** $y = -\tan\left(2x - \dfrac{\pi}{2}\right)$

**16** Consider the function $f: [-3, 6]\backslash D \to R, f(x) = \tan(ax)$.

**a** Evaluate $f(0)$.

**b** Given $p = 3$ is the smallest positive value of $p$ for which $f(x) = f(x + p)$, show that $a = \dfrac{\pi}{3}$.

**c** Obtain the values of $x$ for which the function is not defined and hence state the set $D$ excluded from the domain.

**d** Sketch the graph of $y = f(x)$.

**e** For what values of $x$ does $f(x) = 1$?

**f** State the equations of the asymptotes for the graph of $y = f(x - 1)$.

**17** A ladder is inclined at an angle $\theta$ to the horizontal ground, with its foot 3 metres from a vertical wall. The distance the ladder reaches up the wall is $x$ metres.

**a** Express $x$ in terms of $\theta$.
**b** Sketch the graph of $x$ against $\theta$ for $0° \le \theta \le 70°$.
**c** Describe how $x$ changes as the angle $\theta$ increases.
**d** What would happen if $\theta = 90°$?

**18** An underground earthquake triggers an ocean tsunami event. The height, $h$ metres, of the tsunami wave above mean sea level after $t$ minutes is modelled by $h = 12\tan\left(\frac{\pi}{144}t\right)$, $0 \le t \le 48$.

**a** How high was the wave after the first 24 minutes?
**b** After how many minutes did the height of the tsunami wave reach 12 metres?
**c** The severity of the tsunami started to ease after 48 minutes. What was the peak height the wave reached? How did this compare with the highest tsunami wave of 40.5 metres in the Japanese tsunami of 2011?
**d** Sketch the height against time for $0 \le t \le 48$.

MASTER

**19** **a** Sketch using CAS technology, the graph of $y = \tan(2x) - 3,\ 0 \le x \le 2\pi$ and obtain the equations of the asymptotes.
**b** Sketch using CAS technology, the graph of $y = \tan(2x - 3),\ 0 \le x \le 2\pi$ and obtain the equations of the asymptotes expressed to 1 decimal place.

**20** For the domain where $x \in \left[0, \frac{\pi}{4}\right]$, consider the graphs of $y = \tan\left(\frac{\pi}{4} - x\right)$ and $y = \frac{1}{(6 + k)x}$.

**a** If $k = 0$, how many points of intersection do the two graphs have?
**b** Use the calculator, or other technology, to explore different values of $k$.
  **i** Determine to 2 decimal places the first positive value of $k$ so that there will be two intersections.
  **ii** Determine to 3 decimal places the first positive value of $k$ so that there will be one intersection.

# 10.6 Trigonometric relationships

There are many properties and relationships that exist between the trigonometric functions. Two of these which have previously been established are the symmetry properties and the relationship $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Some others will now be considered.

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Units 1 & 2
AOS 1
Topic 7
Concept 5

**Trigonometric relationships**
Concept summary
Practice questions

## Pythagorean identity

Consider again the definitions of the sine and cosine functions. The trigonometric point P[$\theta$] lies on the circumference of the unit circle with equation $x^2 + y^2 = 1$. By definition, the Cartesian coordinates of P[$\theta$] are $x = \cos(\theta)$, $y = \sin(\theta)$. Substituting these coordinate values into the equation of the circle gives:

$$x^2 + y^2 = 1$$
$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$$
$$\cos^2(\theta) + \sin^2(\theta) = 1$$

This relationship, $\cos^2(\theta) + \sin^2(\theta) = 1$, is known as the **Pythagorean identity**. It is a true statement for any value of $\theta$.

The Pythagorean identity may be rearranged to give either $\cos^2(\theta) = 1 - \sin^2(\theta)$ or $\sin^2(\theta) = 1 - \cos^2(\theta)$.

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$\sin^2(\theta) = 1 - \cos^2(\theta)$$
$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

These identities should be committed to memory. Together with the application of standard algebraic techniques, they can enable other relationships to be verified.

WORKED EXAMPLE 13

**a** Simplify $2\sin^2(\theta) + 2\cos^2(\theta)$.

**b** Show that $\dfrac{3 - 3\cos^2(A)}{\sin(A)\cos(A)} = 3\tan(A)$.

| THINK | WRITE |
| --- | --- |
| **a 1** Factorise the given expression. | **a** $2\sin^2(\theta) + 2\cos^2(\theta)$ <br> $= 2(\sin^2(\theta) + \cos^2(\theta))$ |
| **2** Simplify by using the Pythagorean identity. | $= 2 \times 1$, as $\sin^2(\theta) + \cos^2(\theta) = 1$ <br> $= 2$ |
| **b 1** Simplify the expression on the left-hand side. | **b** $\dfrac{3 - 3\cos^2(A)}{\sin(A)\cos(A)} = 3\tan(A)$ <br> There is a common factor in the numerator of the LHS. <br> $\text{LHS} = \dfrac{3 - 3\cos^2(A)}{\sin(A)\cos(A)}$ <br> $= \dfrac{3(1 - \cos^2(A))}{\sin(A)\cos(A)}$ |
| **2** Apply a form of the Pythagorean identity. | Since $1 - \cos^2(A) = \sin^2(A)$, <br> $\text{LHS} = \dfrac{3\sin^2(A)}{\sin(A)\cos(A)}$ |

3 Complete the simplification to show the two sides are equal.

Cancelling sin (A) from numerator and denominator,

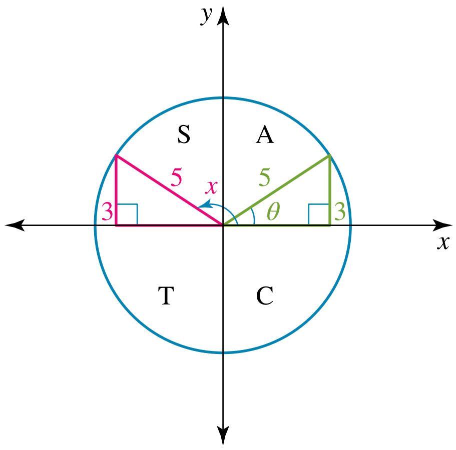
$$\text{LHS} = \frac{3\sin(A)}{\cos(A)}$$
$$= 3\tan(A)$$
$$= \text{RHS}$$

$$\therefore \frac{3 - 3\cos^2(A)}{\sin(A)\cos(A)} = 3\tan(A)$$

## Deducing trigonometric ratios for any real number

Given $\sin(\theta) = \frac{3}{5}$, where $\theta$ is acute, we have used the Pythagorean triple '3, 4, 5' to create a right-angled triangle from which to deduce $\cos(\theta) = \frac{4}{5}$ and $\tan(\theta) = \frac{3}{4}$, without actually evaluating $\theta$. This same method can be employed to find trigonometric ratios for any real-valued $\theta$ if one ratio is known. The additional step needed is to ensure the quadrant in which the triangle lies is taken into consideration and the $\pm$ signs allocated as appropriate to the trigonometric values for that quadrant.

For example, if $\sin(x) = \frac{3}{5}$, $\frac{\pi}{2} \le x \le \pi$, the triangle would lie in the second quadrant, giving $\cos(x) = -\frac{4}{5}$ and $\tan(x) = -\frac{3}{4}$.




In the diagram shown, the circle in which the triangles are drawn has a radius of 5 units. In practice, just the first quadrant triangle needs to be drawn as it, together with the appropriate sign from the CAST diagram, will determine the values for the other quadrants.

### An alternative approach

Alternatively, the Pythagorean identity could be used so that given $\sin(x) = \frac{3}{5}$, $\frac{\pi}{2} \le x \le \pi$ we can write $\cos^2(x) = 1 - \sin^2(x)$ and substitute the value of $\sin(x)$. This gives:

$$\cos^2(x) = 1 - \left(\frac{3}{5}\right)^2$$
$$= 1 - \frac{9}{25}$$
$$= \frac{16}{25}$$
$$\therefore \cos(x) = \pm\sqrt{\frac{16}{25}}$$

As $\cos(x) < 0$ for $\frac{\pi}{2} < x < \pi$, the negative square root is required.

$$\begin{aligned}\cos(x) &= -\sqrt{\frac{16}{25}}\\ &= -\frac{4}{5}\end{aligned}$$

The value of $\tan(x)$ can then be calculated from the identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

$$\begin{aligned}\tan(x) &= \frac{3}{5} \div -\frac{4}{5}\\ &= -\frac{3}{4}\end{aligned}$$

While both methods are appropriate and both are based on Pythagoras' theorem, the triangle method is often the quicker to apply.

**WORKED EXAMPLE 14** **Given $\cos(x) = -\frac{1}{3}$, $\pi \le x \le \frac{3\pi}{2}$, deduce the exact values of $\sin(x)$ and $\tan(x)$.**

**THINK**

1 Identify the quadrant required.

**WRITE**

$\cos(x) = -\frac{1}{3}$, $\pi \le x \le \frac{3\pi}{2}$

The sign of $\cos(x)$ indicates either the second or the third quadrant. The condition that $\pi \le x \le \frac{3\pi}{2}$ makes it the third, not the second quadrant.

Quadrant 3 is required.

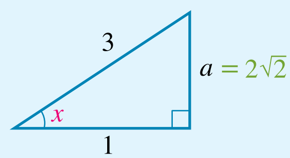
**THINK**

2 Draw a right-angled triangle as if for the first quadrant, label its sides in the ratio given and calculate the third side.

**WRITE**

First quadrant, $\cos(x) = \frac{1}{3}$.

The ratio adjacent : hypotenuse is 1 : 3.




Using Pythagoras's theorem:

$$\begin{aligned}a^2 + 1^2 &= 3^2\\ a &= \sqrt{8} \quad \text{(positive root required)}\\ a &= 2\sqrt{2}\end{aligned}$$

**THINK**

3 Calculate the required values.

*Note:* The alternative approach is to use the Pythagorean identity to obtain $\sin(x)$, and to then use $\tan(x) = \frac{\sin(x)}{\cos(x)}$ to obtain $\tan(x)$.

**WRITE**

From the triangle, the ratio opposite : hypotenuse is $2\sqrt{2} : 3$ and the ratio opposite : adjacent is $2\sqrt{2} : 1$.

In quadrant 3, sine is negative and tangent is positive. Allocating these signs gives $\sin(x) = -\frac{2\sqrt{2}}{3}$ and $\tan(x) = +\frac{2\sqrt{2}}{1}$.

For the third quadrant where $\cos(x) = -\frac{1}{3}$,

$\sin(x) = -\frac{2\sqrt{2}}{3}$, $\tan(x) = 2\sqrt{2}$.

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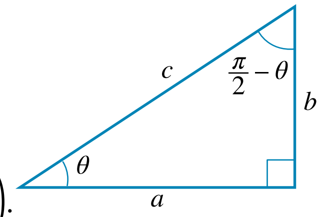
**Interactivity**
Complementary properties for sine and cosine
**int-2979**

## Complementary properties for sine and cosine

In geometry, complementary angles add to 90° or $\frac{\pi}{2}$ in radian measure, so $\theta$ and $\frac{\pi}{2}-\theta$ are called a complementary pair.




In a right-angled triangle, the trigonometric ratios give $\sin(\theta) = \frac{b}{c} = \cos\left(\frac{\pi}{2}-\theta\right)$ and $\cos(\theta) = \frac{a}{c} = \sin\left(\frac{\pi}{2}-\theta\right)$.

For example, $\sin(60°) = \cos(90° - 60°) = \cos(30°)$ and $\cos\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right)$.

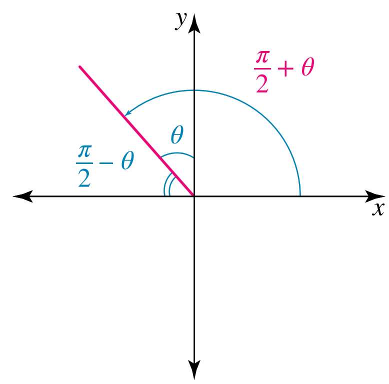
The **complementary properties** are:

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta) \quad \text{and} \quad \cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta)$$

The name ***co**sine* is derived from the ***co**mplement* of sine.

These properties can be extended to other quadrants with the aid of symmetry properties. To illustrate this, the steps in simplifying $\cos\left(\frac{\pi}{2}+\theta\right)$ would be as follows.

- Write in second quadrant form as $\cos\left(\pi - \left(\frac{\pi}{2}-\theta\right)\right)$
- from which the symmetry property gives $-\cos\left(\frac{\pi}{2}-\theta\right)$ and
- from which the complementary property gives $-\sin(\theta)$.




So $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin(\theta)$.

### Distinction between complementary and symmetry properties

Note that for complementary properties $\theta$ is measured from the vertical axis whereas the base needed for symmetry properties is always measured from the horizontal axis. Hence the three trigonometric points $\left[\frac{\pi}{2}+\theta\right], \left[\frac{3\pi}{2}-\theta\right], \left[\frac{3\pi}{2}+\theta\right]$ will each have the same base $\frac{\pi}{2}-\theta$, while the three trigonometric points $[\pi-\theta], [\pi+\theta], [2\pi-\theta]$ each have the base $\theta$.

$$\cos\left(\frac{\pi}{2}+\theta\right) = -\cos\left(\frac{\pi}{2}-\theta\right) \text{ using second quadrant symmetry property}$$
$$= -\sin(\theta) \text{ using complementary property}$$

With practice, seeing $\cos\left(\frac{\pi}{2}+\theta\right)$ should trigger the reaction that this is a complementary $\sin(\theta)$ with the second-quadrant negative sign, since the original cosine expression is negative in the second quadrant. This is much the same as

$\cos(\pi + \theta)$ triggering the reaction this is a symmetry $\cos(\theta)$ with the third-quadrant negative sign.

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta) \text{ complementary property;}$$

$$\cos(\pi + \theta) = -\cos(\theta) \text{ symmetry property}$$

(Cosine is negative in quadrant 2.)

**WORKED EXAMPLE 15**

**a** Simplify $\sin\left(\frac{3\pi}{2} + \theta\right)$.

**b** Give a value for $t$, if $\cos\left(\frac{2\pi}{5}\right) = \sin(t)$.

| THINK | WRITE |
|---|---|
| **a 1** Write the given expression in the appropriate quadrant form. | **a** $\sin\left(\frac{3\pi}{2} + \theta\right)$<br>Quadrant 4<br>$= \sin\left(2\pi - \left(\frac{\pi}{2} - \theta\right)\right)$ |
| **2** Apply the symmetry property. | $= -\sin\left(\frac{\pi}{2} - \theta\right)$ using symmetry property |
| **3** Apply the complementary property. | $= -\cos(\theta)$ using complementary property |
| **4** State the answer.<br>*Note:* Since $\theta$ is inclined to the vertical axis, this implies the complementary trigonometric function is required together with the appropriate sign from the CAST diagram and the answer could be written in one step. | $\therefore \sin\left(\frac{3\pi}{2} + \theta\right) = -\cos(\theta)$ |
| **b 1** Apply a complementary property to one of the given expressions. | **b** $\cos\left(\frac{2\pi}{5}\right)$<br>Quadrant 1<br>Using the complementary property,<br>$\cos\left(\frac{2\pi}{5}\right) = \sin\left(\frac{\pi}{2} - \frac{2\pi}{5}\right)$<br>$= \sin\left(\frac{\pi}{10}\right)$ |
| **2** State an answer.<br>*Note:* Other answers are possible using the symmetry properties of the sine function. | Since $\cos\left(\frac{2\pi}{5}\right) = \sin(t)$,<br>$\sin\left(\frac{\pi}{10}\right) = \sin(t)$<br>$\therefore t = \frac{\pi}{10}$ |

## EXERCISE 10.6 Trigonometric relationships

**PRACTISE**

1 WE13 a Simplify $2 - 2\sin^2(\theta)$.

b Show that $\dfrac{\sin^3(A) + \sin(A)\cos^2(A)}{\cos^3(A) + \cos(A)\sin^2(A)} = \tan(A)$.

2 Simplify $\tan(u) + \dfrac{1}{\tan(u)}$.

3 WE14 Given $\cos(x) = -\dfrac{2}{7}, \dfrac{\pi}{2} \le x \le \pi$, deduce the exact values of $\sin(x)$ and $\tan(x)$.

4 Given $\tan(x) = -3, \pi \le x \le 2\pi$, deduce the exact values of $\sin(x)$ and $\cos(x)$.

5 WE15 a Simplify $\cos\left(\dfrac{3\pi}{2} - \theta\right)$.

b Give a value for $t$, if $\sin\left(\dfrac{\pi}{12}\right) = \cos(t)$.

6 Simplify $\sin\left(\dfrac{\pi}{2} + \theta\right) + \sin(\pi + \theta)$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 Simplify each of the following expressions.

a $4 - 4\cos^2(\theta)$

b $\dfrac{2\sin(\alpha)}{\cos(\alpha)}$

c $8\cos^2(\beta) + 8\sin^2(\beta)$

d $\dfrac{\sin^2(\theta)}{\cos^2(\theta)}$

e $(1 - \sin(A))(1 + \sin(A))$

f $\sin^4(\theta) + 2\sin^2(\theta)\cos^2(\theta) + \cos^4(\theta)$

8 a Show that $\tan^2(\theta) + 1 = \dfrac{1}{\cos^2(\theta)}$

b Show that $\cos^3(\theta) + \cos(\theta)\sin^2(\theta) = \cos(\theta)$.

c Show that $\dfrac{1}{1 - \cos(\theta)} + \dfrac{1}{1 + \cos(\theta)} = \dfrac{2}{\sin^2(\theta)}$.

d Show that $(\sin(\theta) + \cos(\theta))^2 + (\sin(\theta) - \cos(\theta))^2 = 2$.

9 a Given $\tan(x) = -\dfrac{4}{5}, \dfrac{3\pi}{2} \le x \le 2\pi$, obtain the exact values of $\sin(x)$ and $\cos(x)$.

b Given $\sin(x) = \dfrac{\sqrt{3}}{2}, \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}$, obtain the exact values of:

i $\cos(x)$

ii $1 + \tan^2(x)$.

c Given $\cos(x) = -0.2, \pi \le x \le \dfrac{3\pi}{2}$, obtain the exact values of:

i $\tan(x)$

ii $1 - \sin^2(x)$

iii $\sin(x)\tan(x)$.

10 a Given $\cos(x) = \dfrac{2\sqrt{3}}{5}$, use the identity $\sin^2(x) + \cos^2(x) = 1$ to obtain $\sin(x)$ if:

i $x$ lies in the first quadrant

ii $x$ lies in the fourth quadrant.

b Calculate $\tan(x)$ for each of parts a i and ii using $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

c Check your answer to part b using the triangle method to obtain the values of $\sin(x)$ and $\tan(x)$ if $x$ lies in the first quadrant.

11 a Give a value for $a$ for which $\cos(a) = \sin(27°)$.

b Give a value for $b$ for which $\sin(b) = \cos\left(\frac{\pi}{9}\right)$.

c Give a value, to 2 decimal places, for $c$ so that $\sin(c) = \cos(0.35)$.

d Identify two values, between 0 and $2\pi$, for $d$ so that $\cos(d) = \sin\left(\frac{\pi}{4}\right)$.

12 Simplify the following expressions.

a $\cos\left(\frac{3\pi}{2} + \theta\right)$

b $\cos\left(\frac{3\pi}{2} - \theta\right)$

c $\sin\left(\frac{\pi}{2} + \theta\right)$

d $\sin\left(\frac{5\pi}{2} - \theta\right)$

13 Given $\sin(\theta) = 0.8$, $0 \le \theta \le \frac{\pi}{2}$, evaluate the following.

a $\sin(\pi + \theta)$

b $\cos\left(\frac{\pi}{2} - \theta\right)$

c $\cos(\theta)$

d $\tan(\pi + \theta)$

14 Simplify the expressions in parts a, b and c.

a $\dfrac{\cos(90° - a)}{\cos(a)}$

b $1 - \sin^2\left(\frac{\pi}{2} - \theta\right)$

c $\cos(\pi - x) + \sin\left(x + \frac{\pi}{2}\right)$

d Show that $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$ and explain what this means about the graphs of $y = \sin(x)$ and $y = \cos(x)$.

15 Solve the following equations for $x$, $0 \le x \le 2\pi$.

a $\cos^2(x) + 3\sin(x) - 1 = 0$

b $3\cos^2(x) + 6\sin(x) + 3\sin^2(x) = 0$

c $\dfrac{1 - \cos(x)}{1 + \sin(x)} \times \dfrac{1 + \cos(x)}{1 - \sin(x)} = 3$

d $\sin^2(x) = \cos(x) + 1$

16 Show that the following are true statements.

a $1 - \cos^2(40°) = \cos^2(50°)$

b $\dfrac{\cos^3(x) - \sin^3(x)}{\cos(x) - \sin(x)} = 1 + \sin(x)\cos(x)$

c $\dfrac{\sin^3(\theta) + \cos\left(\frac{\pi}{2} - \theta\right)\cos^2(\theta)}{\cos(\theta)} = \tan(\theta)$

d $\cos(2\pi - A)\cos(2\pi + A) - \cos\left(\frac{3\pi}{2} - A\right)\cos\left(\frac{3\pi}{2} + A\right) = 1$

MASTER

17 Use CAS technology to simplify the following.

a $\sin(5.5\pi - x) \div \cos\left(x - \frac{3\pi}{2}\right)$

b $\sin(3x)\cos(2x) - 2\cos^2(x)\sin(3x)$

18 a Evaluate $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$ on CAS technology and explain what has been found.

b Use a CAS method to obtain $\tan(x)$ and $\sin(x)$, given $\cos(x) = -\frac{\sqrt{3}}{3}$. What quadrant does CAS ‘assume’ in its calculation?

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# 10.7 Review

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**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

## REVIEW QUESTIONS

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# Activities

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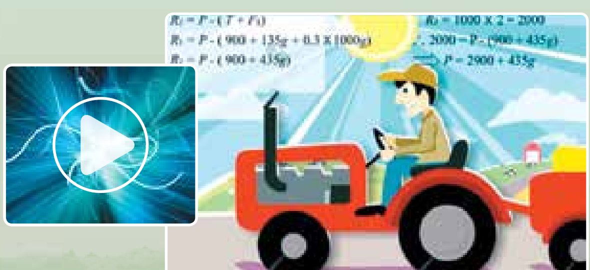
## Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.



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**Units 1 & 2** — **Trigonometric functions 2**

**Sit topic test**

# 10 Answers

## EXERCISE 10.2

**1 a** $\frac{\pi}{6}, \frac{5\pi}{6}$

**b** $\frac{\pi}{6}, \frac{11\pi}{6}$

**c** $-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

**2 a** 2 solutions

**b** $\pm 120°$

**3 a** $-\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$

**b** 0.64, 3.79, 6.93, 10.07

**c** $x° = \pm 104.5°$ so $x° = \pm 104.5$

**4 a** 1

**b** $\left\{\frac{1}{2}, \frac{3}{2}\right\}$

**5 a** $\frac{\pi}{3}, \frac{4\pi}{3}$

**b** $\frac{\pi}{2}$

**6** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

**7 a** $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

**b** $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$

**8** $x = \frac{2\pi}{3}$

**9 a** $\frac{\pi}{4}, \frac{7\pi}{4}$

**b** $\frac{5\pi}{4}, \frac{7\pi}{4}$

**c** $\frac{5\pi}{6}, \frac{11\pi}{6}$

**d** $\frac{5\pi}{6}, \frac{7\pi}{6}$

**e** $\frac{\pi}{6}, \frac{5\pi}{6}$

**f** $\frac{\pi}{3}, \frac{4\pi}{3}$

**10 a** 240°, 300°

**b** 45°, 225°

**c** 120°, 240°

**d** 30°, 150°

**11 a** $-\pi, 0, \pi, 2\pi, 3\pi, 4\pi$

**b** $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

**c** $-\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$

**d** $0, 2\pi, 4\pi$

**e** $\frac{\pi}{2}, \frac{5\pi}{2}$

**f** $-\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

**12 a** 2.30, 3.98

**b** −5.67, −2.53, 0.62, 3.76, 6.90

**c** −53.13, −126.87, 233.13

**d** 78.46, 101.54, 258.46, 281.54

**13 a** $-\frac{11\pi}{6}, -\frac{7\pi}{6}$

**b** No solution

**c** $-4\pi, -2\pi$

**d** $\frac{\pi}{4}$

**e** $\frac{\pi}{3}, \frac{5\pi}{3}$

**f** −300, −240, 60, 120

**14 a** Four solutions

**b** $\pi$

**c** $\pi + 0.4, 2\pi + 0.4$

**d** **i** $\tan(\theta) = -3,\ 0° < \theta < 180°$

**ii** $\tan(\theta) = \sqrt{3},\ 0° < \theta < 180°$

**e** **i** $\theta = 180° - \tan^{-1}(3)$

**ii** $\theta = 60°$

**f** $\theta$ is the second quadrant solution to $\tan(\theta) = m,\ m < 0.$

**15 a** $a = 8$

**b** **i** 0.38, 2.76

**ii** 1.57

**iii** No solution

**16 a** $\frac{\pi}{3}, \frac{4\pi}{3}$

**b** $\frac{5\pi}{6}, \frac{11\pi}{6}$

**c** $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

**d** 0.93, 4.07

**e** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

**f** $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}$

**17 a** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

**b** $\frac{\pi}{2}, \frac{3\pi}{2}$

**c** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

**d** $\frac{\pi}{4}, \frac{5\pi}{4}$, 1.89, 5.03

**e** $\frac{3\pi}{2}$

**f** No solution

**18 a** $\frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}, \frac{29\pi}{18}, \frac{35\pi}{18}$

**b** $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$

**c** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

**d** $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

**19 a** **i** $x = \cos^{-1}\left(\frac{2}{5}\right),\ x = -\cos^{-1}\left(\frac{2}{5}\right) + 2\pi$

**ii** $x = 1.16,\ x = 5.12$

**b** $x \approx 2.11, 2.75, 5.25, 5.89$

**c** $x = \pi\,\text{constn}(1) + \frac{\pi}{4}$ which means $x = n\pi + \frac{\pi}{4}, n \in Z$;

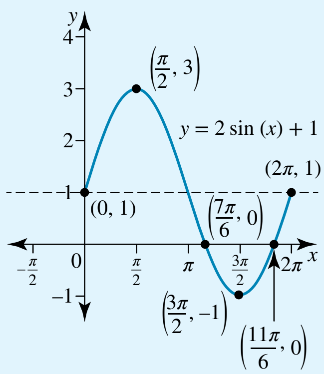
$n = 0, 1, 2 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

**20 a** $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$

**b** $-255°, 75°, 105°$

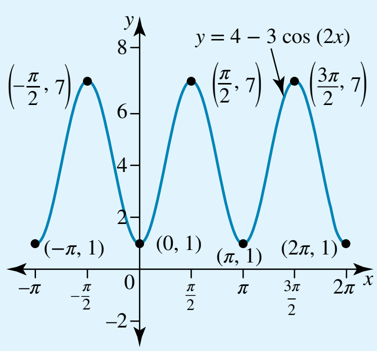
## EXERCISE 10.3

**1 a**



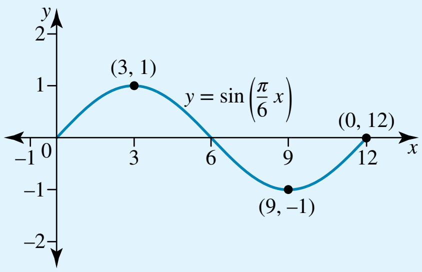

Period $2\pi$; amplitude 1; equilibrium position $y = 1$; range $[-1, 3]$; $x$-intercepts at $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

**b**



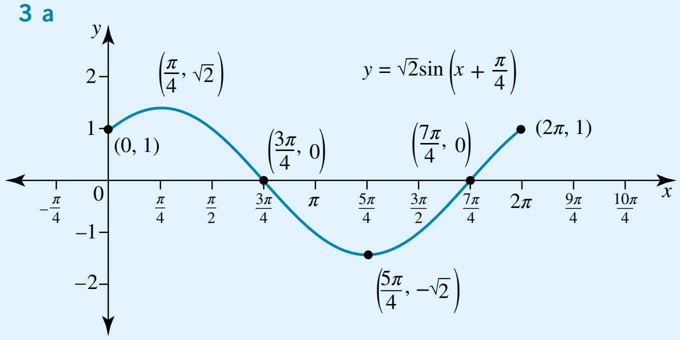

Period $\pi$; amplitude 3; inverted; equilibrium $y = 4$; range $[1, 7]$

**2**




Period 12; amplitude 1; equilibrium $y = 0$; range $[-1, 1]$; domain $[0, 12]$

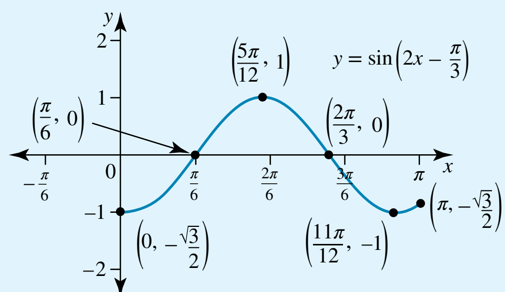
**3 a**




Period $2\pi$; amplitude $\sqrt{2}$; phase shift $-\frac{\pi}{4}$ from $y = \sqrt{2}\sin(x)$; endpoints $(0, 1), (2\pi, 1)$; range $[-\sqrt{2}, \sqrt{2}]$

**b** Period $\pi$; amplitude 3; phase shift factor $-\frac{\pi}{8}$; range $[-2, 4]$

**4**




Period $\pi$; amplitude 1; phase $\frac{\pi}{6}$; endpoints $\left(0, -\frac{\sqrt{3}}{2}\right)$, $\left(\pi, -\frac{\sqrt{3}}{2}\right)$

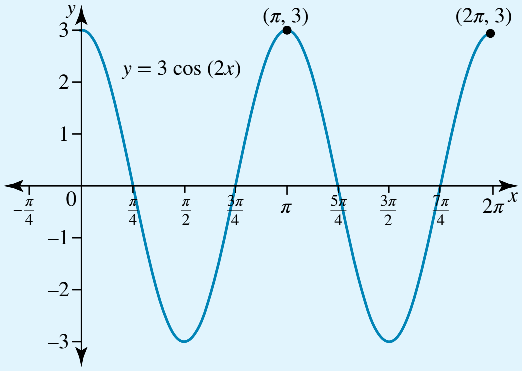
**5** $y = -2\sin\left(\frac{2x}{3}\right)$ or $y = -2\cos\left(\frac{2x}{3} - \frac{\pi}{2}\right)$ or $y = 2\sin\left(\frac{2x}{3} - \pi\right)$. Other answers are possible.

**6** $y = 3\sin(2x) + 5$ (other answers are possible).

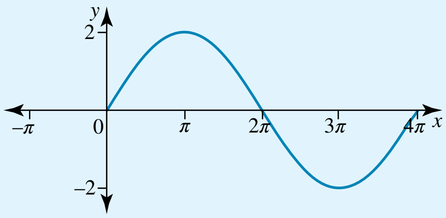
**7**

| | Amplitude | Period |
|---|---|---|
| **a** | 6 | $\pi$ |
| **b** | 7 | $4\pi$ |
| **c** | $\frac{3}{5}$ | $\frac{10\pi}{3}$ |
| **d** | 1 | $\frac{7}{3}$ |
| **e** | 2 | $4\pi$ |
| **f** | 4 | $2\pi$ |

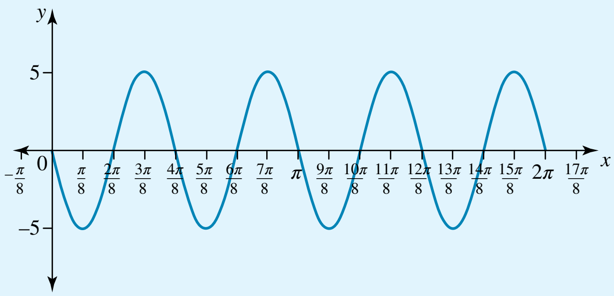
**8 a**



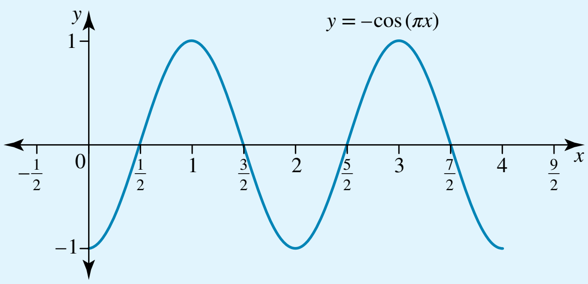

**b**



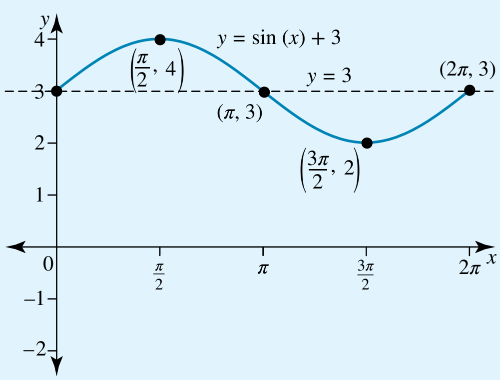
**c**



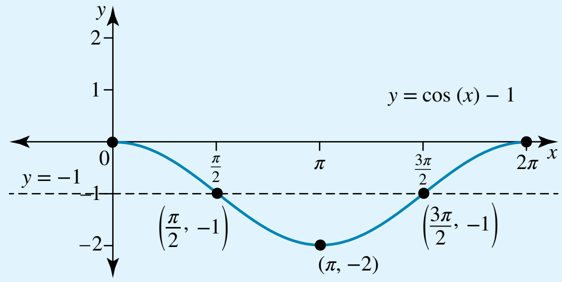
**d**



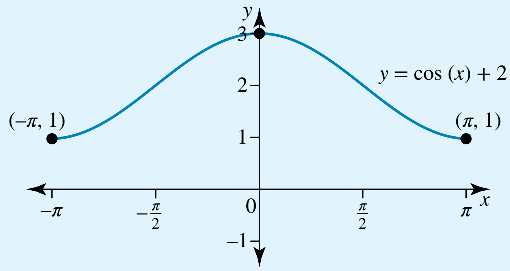

**9 a** [2, 4]



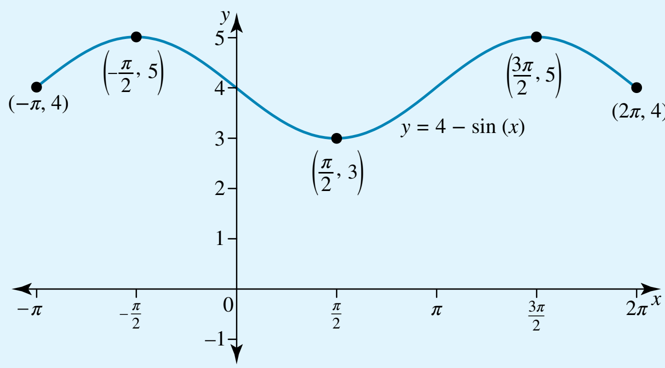

**b** [−2, 0]




**c** [1, 3]




**d** [3, 5]




**10 a** [1, 5]

**b** −14

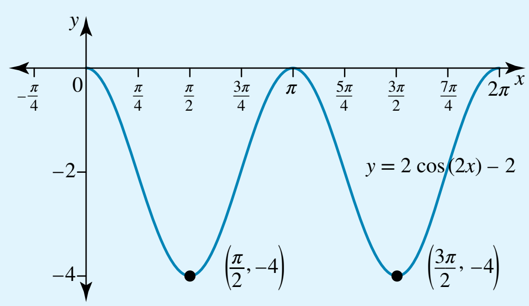
**c** 68; $x = \frac{3\pi}{2}$

**d** **i** Dilation factor $\frac{1}{5}$ from $y$-axis; dilation factor 2 from $x$-axis; vertical translation up 3

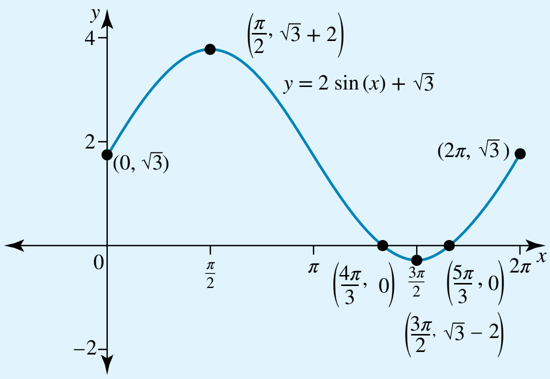
**ii** Dilation factor $\frac{1}{2}$ from $y$-axis; dilation factor 10 from $x$-axis; vertical translation down 4

**iii** Dilation factor 12 from $x$-axis; reflection in $x$-axis; vertical translation up 56

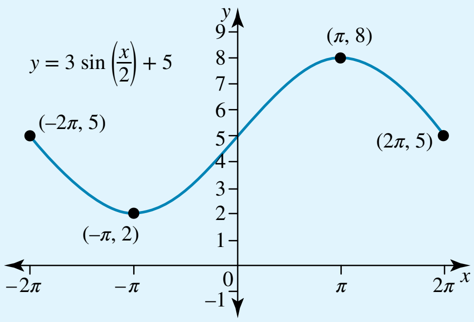
**11 a** Range [−4, 0]



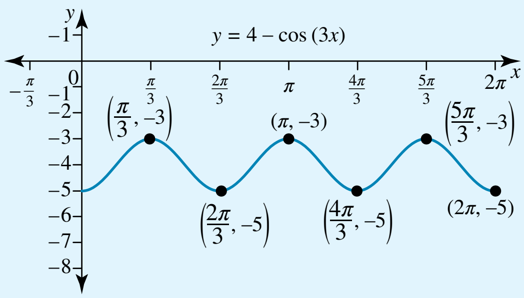
**b** Range $[\sqrt{3}-2, \sqrt{3}+2]$



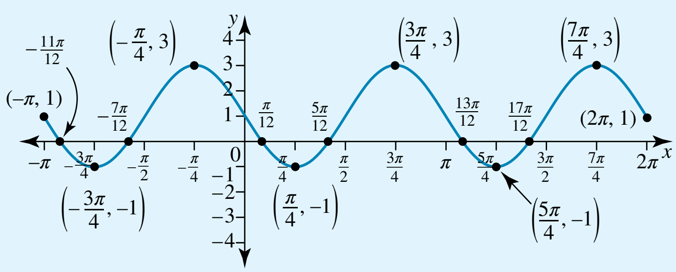

**c** Range [2, 8]



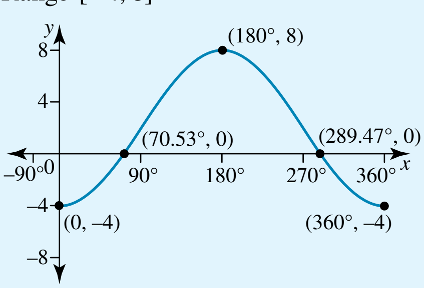

**d** Range [−5, −3]



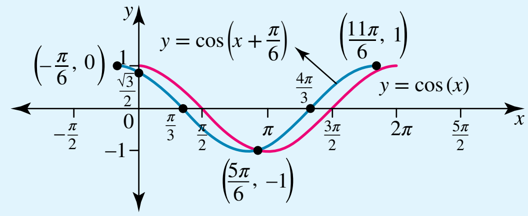

**e** Range [−1, 3]



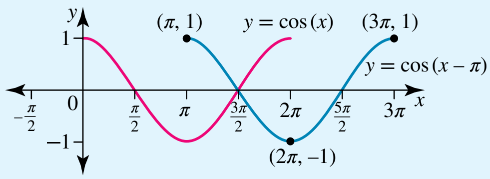

**f** Range [−4, 8]



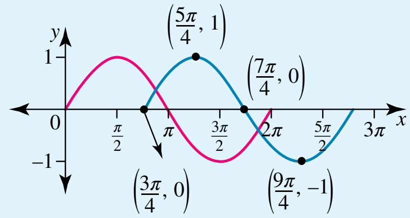

**12 a i**



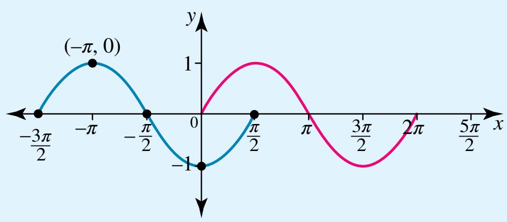

**ii**



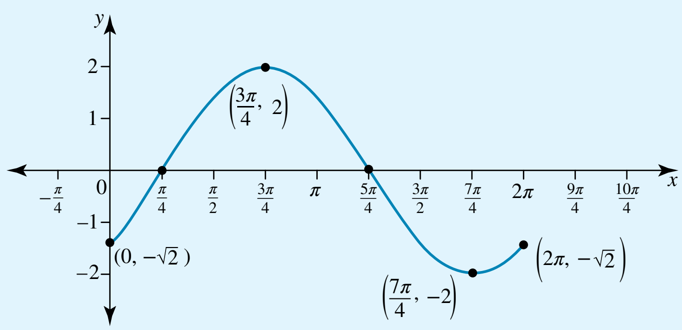

**b i**



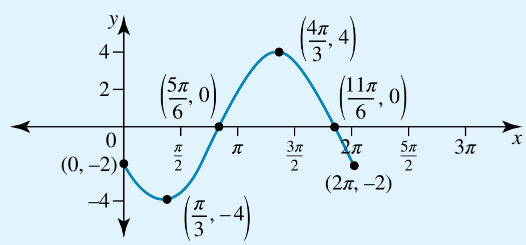

**ii**



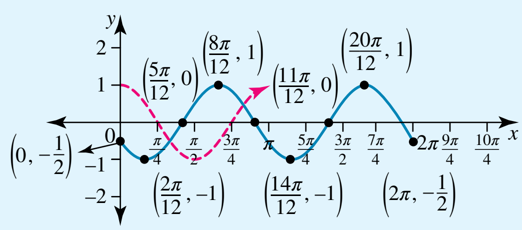

**13 a**



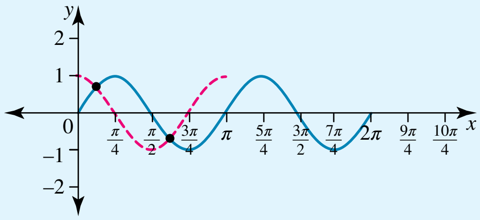

**b**



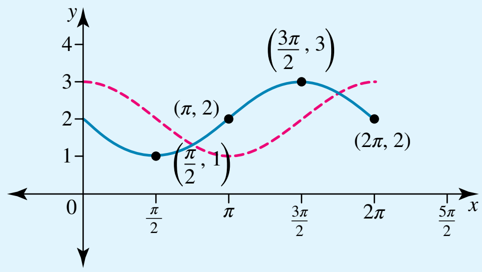

**c**



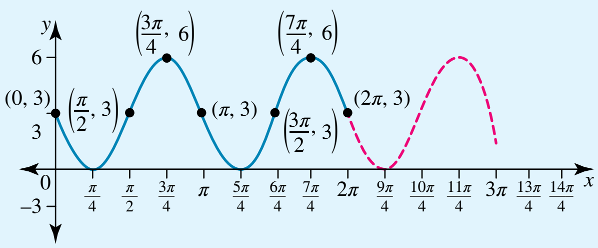
**d**



**e**




**f**




**14 a** $y = -3 \sin\left(\frac{x}{2}\right)$

**b** $y = 4 \cos(3x)$

**c** $y = -4 \cos(x) + 6$

**d** $y = 2 \sin\left(x - \frac{\pi}{4}\right)$

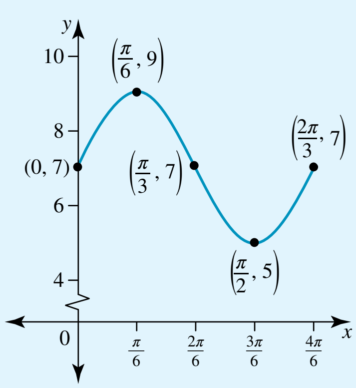
**e** $y = 2 \cos\left(x - \frac{3\pi}{4}\right)$

**f** $y = \cos(x)$; $y = -\sin(x)$

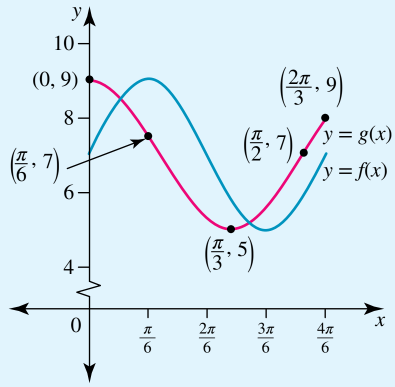
**15 a** $\frac{2\pi}{3}$

**b** $a = 2$; $b = 3$; $c = 7$

**c** $D = \left[0, \frac{2\pi}{3}\right]$




**d**




**e** $\left(\frac{\pi}{12}, 7 + \sqrt{2}\right), \left(\frac{5\pi}{12}, 7 - \sqrt{2}\right)$

**f** $\frac{\pi}{12} \le x \le \frac{5\pi}{12}$

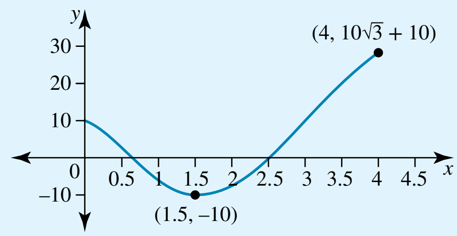
**16 a** $y = -\sin\left(\frac{x}{3}\right) - 3$; no intersections

**b** $y = -2\sin(x)$; reflection in the $x$-axis; dilation of factor 2 from the $x$-axis

**c** **i** $a = 10$

**ii** $x = 0.5, 2.5$

**iii** Range $[-10, 10\sqrt{3} + 10]$




**iv** Horizontal translation $\frac{\pi}{2}$ to right; dilation factor $\frac{3}{\pi}$ from $y$-axis; dilation factor 20 from $x$-axis; reflection in $x$-axis; vertical translation up 10

**17 a** 4$x$-intercepts: 0.43, 2.00, 3.57, 5.14

**b** $x = 4 \sin(2y - 4)$; the inverse

**c** Domain $[-4, 4]$; range $[0, 2\pi]$; four $y$-intercepts at 0.43, 2.00, 3.57, 5.14

**18** Four types of graphs:

$y = \cos(x),\ k = 4n, n \in Z$

$y = -\sin(x),\ k = 4n + 1, n \in Z$

$y = -\cos(x),\ k = 4n + 2, n \in Z$

$y = \sin(x),\ k = 4n + 3, n \in Z$

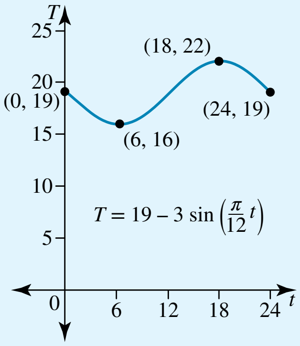
## EXERCISE 10.4

**1 a** 19°

**b** 22° at 6 pm

c Between 16° and 22°

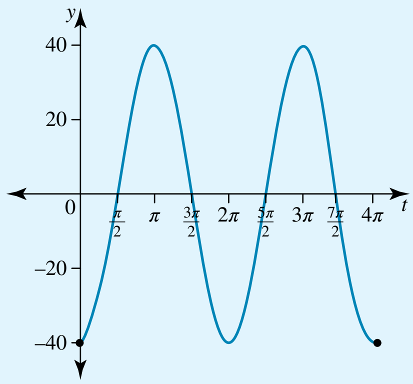
d Period is 24 hours.




e $k = 16.2$

**2** $T = 28 - 8\cos\left(\frac{\pi}{6}t\right)$; between 11.29 am and 4.31 pm

**3 a**




b 40 cm

c $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

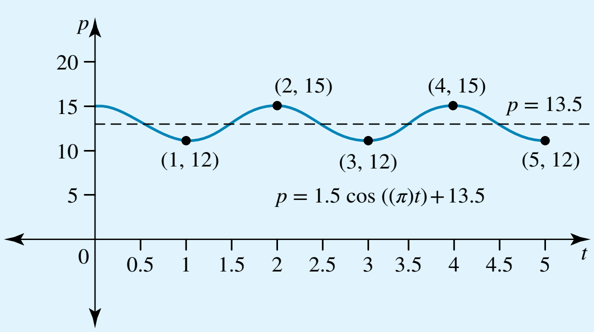
d $\frac{2\pi}{3}$ seconds ≈ 2.1 seconds

**4 a** $I = 5\sin\left(\frac{\pi}{2}t\right) + 5$

b 44%

**5 a** $a = 1.5; n = \pi; b = 13.5$

b 12 cents




c 5 days

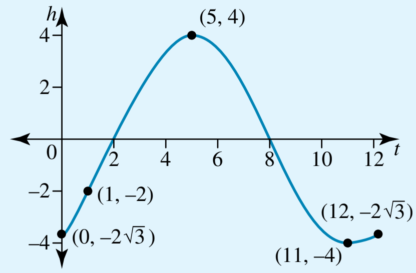
d $197.25

**6 a** 2 metres below mean sea level

b 4 metres above mean sea level; proof required

c 6 hours

d




e At mean sea level

f Risen by 0.14 metres

**7 a** $a = 2.5; b = 4.5; h = 2.5\sin\left(\frac{\pi}{5}x\right) + 4.5$

b 10 cm

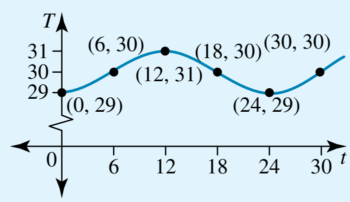
c $(2.5, 3.25)$

d $40.1\ \text{cm}^2$

**8 a** Between 29° and 31°

b 12 minutes

c




d 2.5 cycles

e 30 °C

f $T = \sin\left(\frac{\pi}{12}(t - 6)\right) + 30$

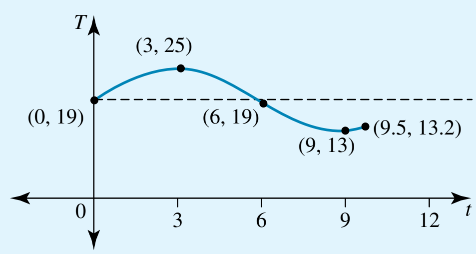
**9 a** i 25° at 1 pm

ii 13° at 7 pm

b i 23.2 °C

ii 13.2 °C

c




d 2.24 hours

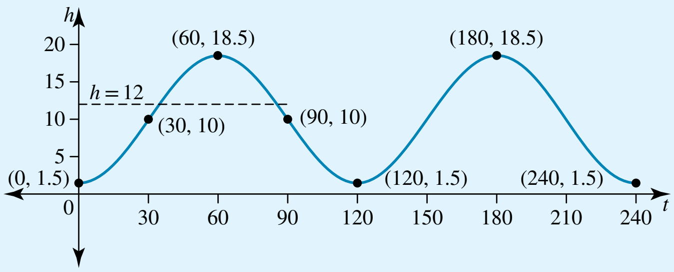
e $(19 - 3\sqrt{2})° \approx 14.8°$ at 5.30 pm

**10 a** 1.5 metres

b 18.5 metres

c 2

**d**




**e** 51 seconds

**f** 8.5 metres

**11 a** 2 metres

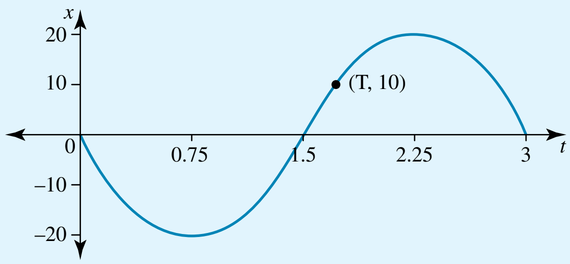
**b** $n = \frac{4}{3}$

**c** $a = 5$; 60 times

**d** $p = a \sin(4\pi t) + 5$

**12 a** $a = -20,\ b = \dfrac{2\pi}{3},\ x = -20 \sin\left(\dfrac{2\pi}{3}t\right)$

**b**




**c** $T = 1.75$

**d** 80 cm

**13 a** 5 units

**b** $t = 0.64$

**c** $I = 5 \sin(t - 0.64)$

**d** $I = 5 \cos(t - 2.21)$

**14 a** 13 (includes the single-sided teeth at the ends)

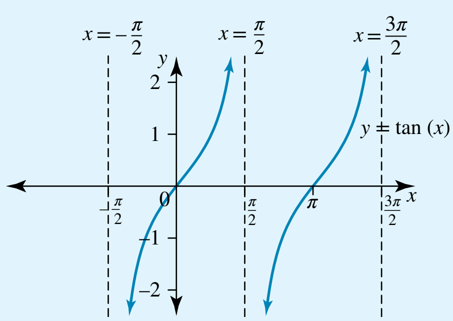
**b** $\dfrac{\pi}{3}$ cm

**c** $(4\pi + 8)$ cm

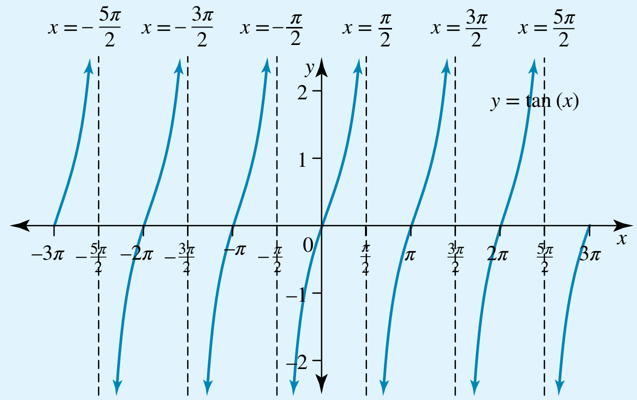
**d** $y = x + 8$

## EXERCISE 10.5

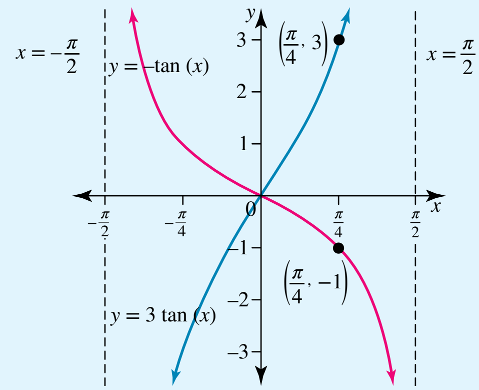
**1**



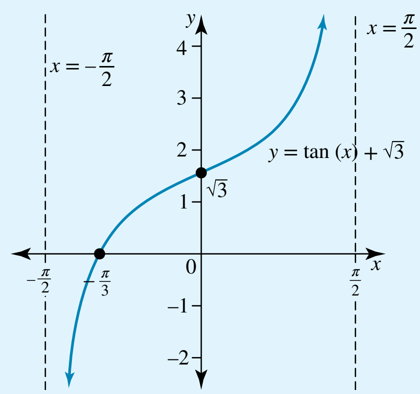

**2** Domain $[-3\pi, 3\pi] \backslash \left\{\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}\right\}$; range $R$



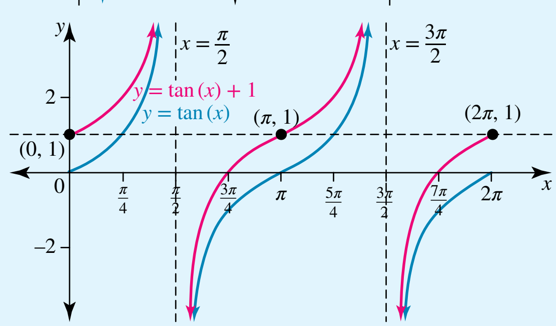

**3 a**



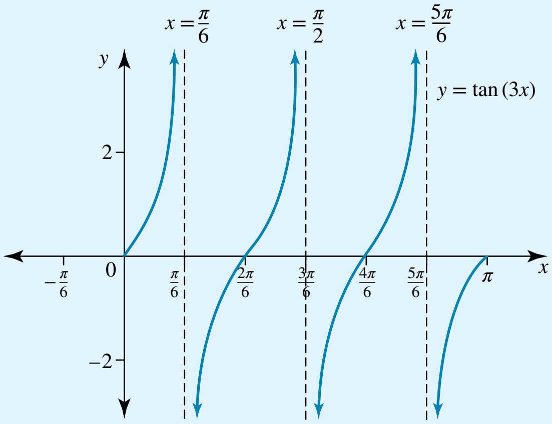

**b**



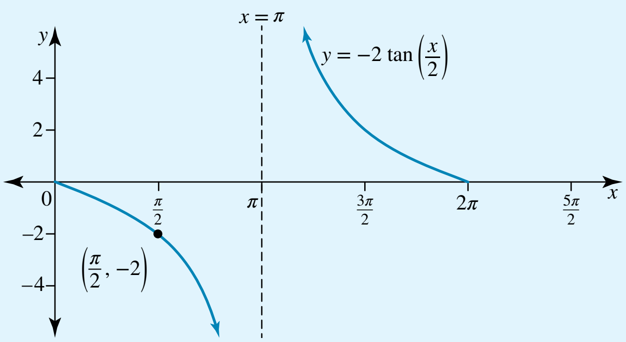

**4**



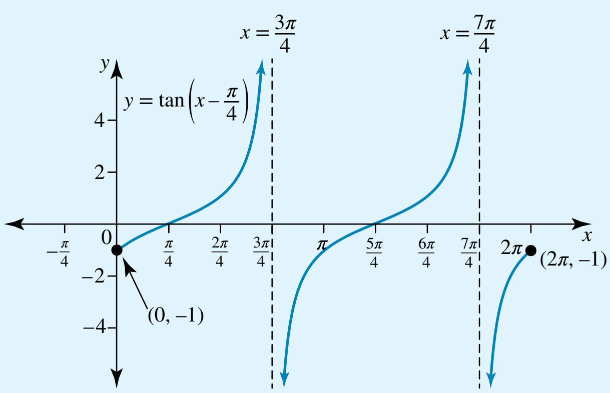
**5** Period $\frac{\pi}{3}$; asymptotes $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$; $x$-intercepts $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$




**6** Period $2\pi$; asymptote $x = \pi$; $x$-intercepts $x = 0, 2\pi$; point $\left(\frac{\pi}{2}, -2\right)$



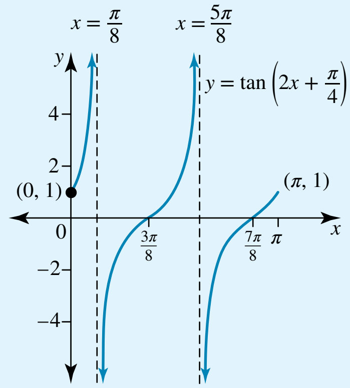

**7 a** Period $\pi$; endpoints $(0, -1)$, $(2\pi, -1)$; asymptotes $x = \frac{3\pi}{4}, \frac{7\pi}{4}$; $x$-intercepts $x = \frac{\pi}{4}, \frac{5\pi}{4}$



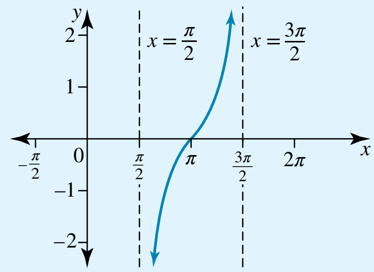

**b** **i** Dilation of factor $\frac{3}{4\pi}$ from the $y$-axis followed by a horizontal translation of $\frac{1}{4}$ to the left.

**ii** Period is $\frac{3}{4}$.

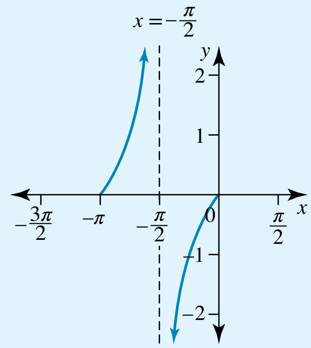
**8** Period $\frac{\pi}{2}$; endpoints $(0, 1)$, $(\pi, 1)$; asymptotes $x = \frac{\pi}{8}, \frac{5\pi}{8}$; $x$-intercepts $x = \frac{3\pi}{8}, \frac{7\pi}{8}$



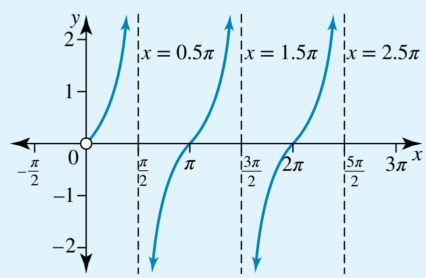

**9 a**



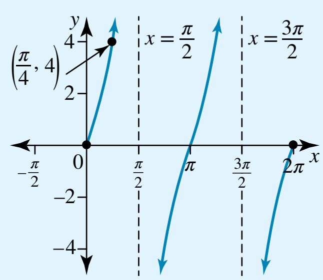

**b**



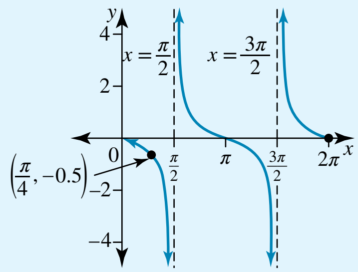

**c**



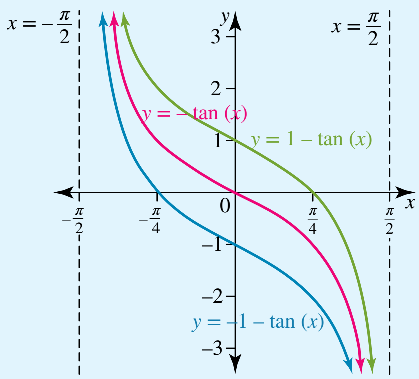

**10 a**



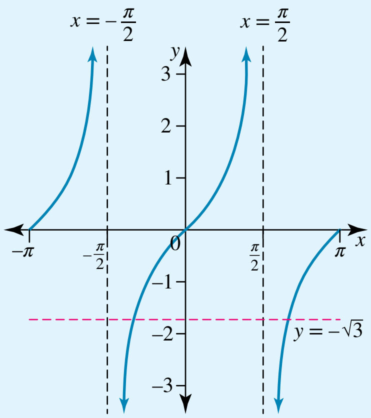
**b**




**c**




**11 a**



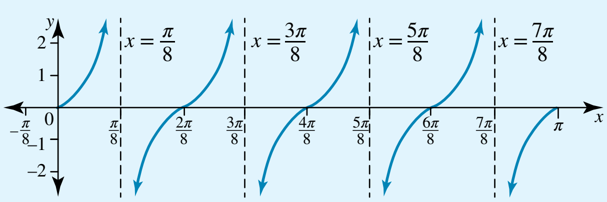

**b** $x = -\frac{\pi}{3}, \frac{2\pi}{3}$

**c** $x \in \left(-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

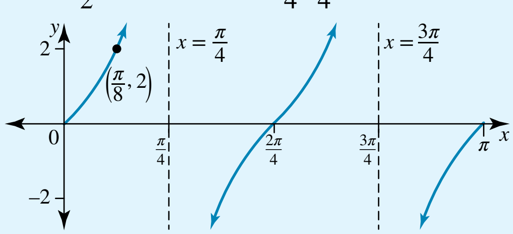
**12 a** $\frac{\pi}{6}$ **b** $4\pi$ **c** $\frac{2\pi}{3}$ **d** 1

**13 a** Period $\frac{\pi}{4}$; asymptotes $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$;

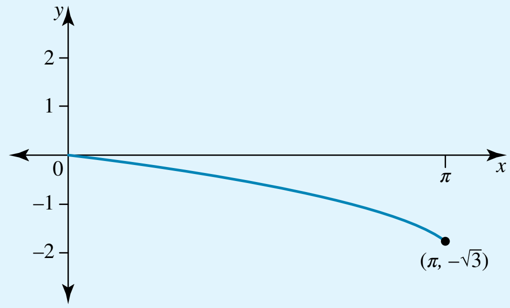
$x$-intercepts $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$



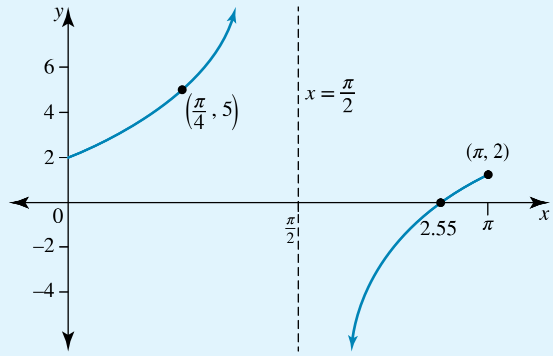

**b** Period $\frac{\pi}{2}$; asymptotes $x = \frac{\pi}{4}, \frac{3\pi}{4}$; $x$-intercepts $x = 0, \frac{\pi}{2}, \pi$



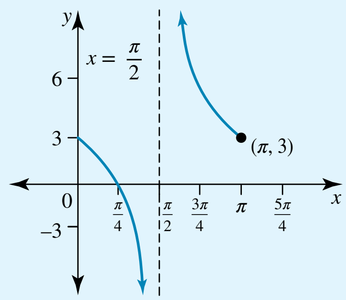

**c** Period $3\pi$; no asymptotes in the domain; $x$-intercept $x = 0$



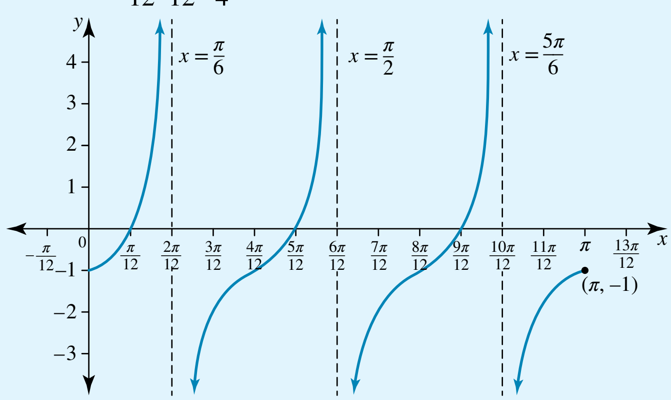

**d** Period $\pi$; asymptote $x = \frac{\pi}{2}$; $x$-intercept $x \approx 2.55$




**e** Period $\pi$; asymptote $x = \frac{\pi}{2}$; $x$-intercept $x = \frac{\pi}{4}$




**f** Period $\frac{\pi}{3}$; asymptotes $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$; $x$-intercepts $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

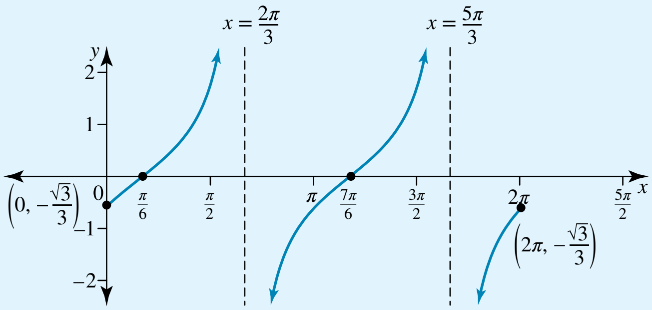



**14 a** **i** 4 **ii** $\frac{3\pi}{8}$

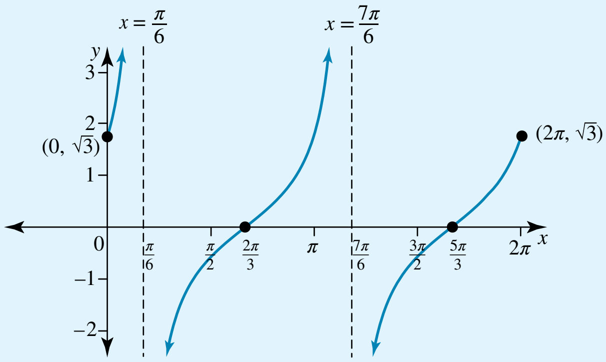
**iii** $b = \frac{8}{3}$ **iv** $x = \frac{3\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16}, \frac{21\pi}{16}$

**b** $a = -2, c = \frac{5\pi}{12}$

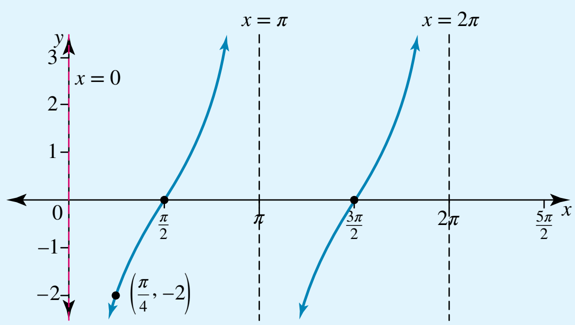
**15 a** Period $\pi$; asymptotes $x = \frac{2\pi}{3}, \frac{5\pi}{3}$; $x$-intercepts $x = \frac{\pi}{6}, \frac{7\pi}{6}$



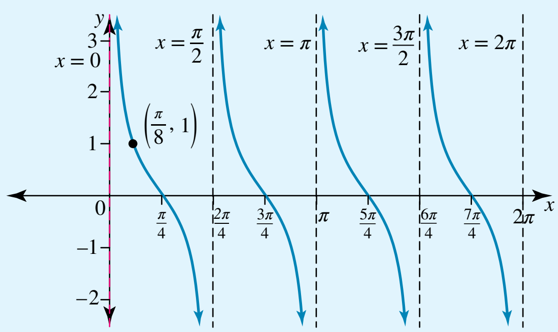

**b** Period $\pi$; asymptotes $x = \frac{\pi}{6}, \frac{7\pi}{6}$; $x$-intercepts $x = \frac{2\pi}{3}, \frac{5\pi}{3}$




**c** Period $\pi$; asymptotes $x = 0, \pi, 2\pi$; $x$-intercepts $x = \frac{\pi}{2}, \frac{3\pi}{2}$




**d** Period $\frac{\pi}{2}$; asymptotes $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$; $x$-intercepts $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

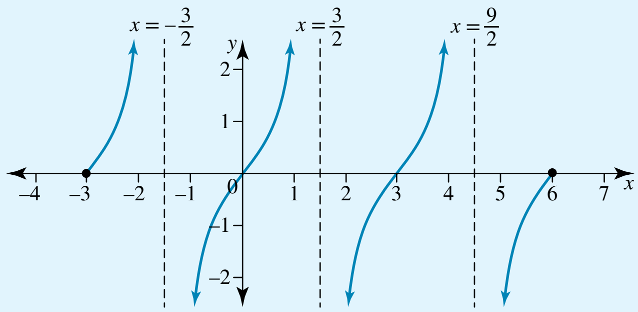



**16 a** 0

**b** Proof required — check with your teacher

**c** $D = \left\{-\frac{3}{2}, \frac{3}{2}, \frac{9}{2}\right\}$

**d** Period 3; asymptotes $x = -\frac{3}{2}, \frac{3}{2}, \frac{9}{2}$; $x$-intercepts $x = -3, 0, 3, 6$

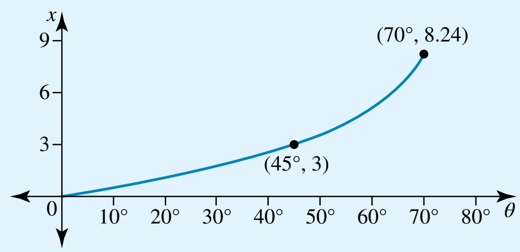



**e** $x = -\frac{9}{4}, \frac{3}{4}, \frac{15}{4}$

**f** $x = -\frac{1}{2}, \frac{5}{2}, \frac{11}{2}$

**17 a** $x = 3\tan(\theta)$

**b**




**c** As $\theta$ increases, $x$ increases

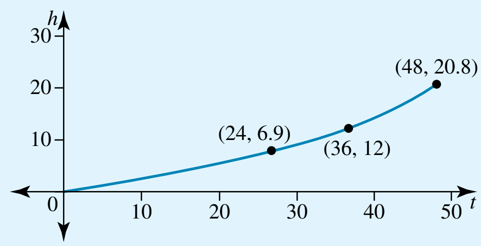
**d** Impossible for ladder to reach the wall since it is parallel to the wall

**18 a** $4\sqrt{3} \approx 6.9$ metres

**b** 36 minutes

**c** $12\sqrt{3} \approx 20.8$ metres, 19.7 metres lower than the Japanese tsunami

**d**




**19 a** $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

**b** $x = 0.7, 2.3, 3.9, 5.4$

**20 a** No intersections

**b i** $k = 0.13$

**ii** $k = 0.128$

## EXERCISE 10.6

**1 a** $2\cos^2(\theta)$

**b** Proof required — check with your teacher

**2** $\frac{1}{\cos(u)\sin(u)}$

**3** $\sin(x) = \frac{3\sqrt{5}}{7}$; $\tan(x) = -\frac{3\sqrt{5}}{2}$

**4** $\sin(x) = -\frac{3}{\sqrt{10}}$; $\cos(x) = \frac{1}{\sqrt{10}}$

**5 a** $-\sin(\theta)$

**b** $t = \frac{5\pi}{12}$

**6** $\cos(\theta) - \sin(\theta)$

**7 a** $4\sin^2(\theta)$

**b** $2\tan(\alpha)$

**c** 8

**d** $\tan^2(\theta)$

**e** $\cos^2(A)$

**f** 1

**8** Proofs required — check with your teacher

**9 a** $\sin(x) = -\frac{4}{\sqrt{41}}$; $\cos(x) = \frac{5}{\sqrt{41}}$

**b i** $-\frac{1}{2}$

**ii** 4

**c i** $2\sqrt{6}$

**ii** $\frac{1}{25}$

**iii** $-\frac{24}{5}$

**10 a i** $\sin(x) = \frac{\sqrt{13}}{5}$

**ii** $\sin(x) = -\frac{\sqrt{13}}{5}$

**b** $\frac{\sqrt{39}}{6}$; $-\frac{\sqrt{39}}{6}$

**c** Answers should agree.

**11 a** $a = 63$

**b** $b = \frac{7\pi}{18}$

**c** $c = 1.22$

**d** $d = \frac{\pi}{4}, \frac{7\pi}{4}$

**12 a** $\sin(\theta)$

**b** $-\sin(\theta)$

**c** $\cos(\theta)$

**d** $\cos(\theta)$

**13 a** $-0.8$

**b** 0.8

**c** $\frac{3}{5}$

**d** $\frac{4}{3}$

**14 a** $\tan(a)$

**b** $\sin^2(\theta)$

**c** 0

**d** Proof required; graphs are out of phase by $\frac{\pi}{2}$.

**15 a** $x = 0, \pi, 2\pi$

**b** $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

**c** $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

**d** $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

**16** Proofs required — check with your teacher

**17 a** $\frac{1}{\tan(x)}$

**b** $-\sin(3x)$

**18 a** $\frac{4}{5}$; CAS calculates $\cos(\theta)$ given $\sin(\theta) = \frac{3}{5}$.

**b** $\tan(x) = -\sqrt{2}$, $\sin(x) = \frac{\sqrt{6}}{3}$; second quadrant

# 11

# Exponential functions

# 11.1 Kick off with CAS

## Exponential functions

1 Using CAS technology, sketch the following exponential functions on the same set of axes.

a $y = 2^x$ b $y = 3^x$ c $y = 5^x$ d $y = 8^x$ e $y = \left(\frac{1}{2}\right)^x$

2 Using CAS technology, enter $y = a^x$ into the function entry line and use a slider to change the value of $a$.

3 When sketching an exponential function, what is the effect of changing the value of $a$ in the equation?

4 Using CAS technology, sketch the following exponential functions on the same set of axes.

a $y = 2^x$ b $y = 2^x + 2$ c $y = 2^x + 5$ d $y = 2^x - 3$ e $y = 2^x - 5$

5 Using CAS technology, enter $y = 2^x + k$ into the function entry line and use a slider to change the value of $k$.

6 When sketching an exponential function, what is the effect of changing the value of $k$ in the equation?

7 Using CAS technology, sketch the following exponential functions on the same set of axes.

a $y = 2^x$ b $y = 2^{x-2}$ c $y = 2^{x+3}$ d $y = 2^{x-5}$ e $y = 2^{x-8}$

8 Using CAS technology, enter $y = 2^{x-h}$ into the function entry line and use a slider to change the value of $h$.

9 When sketching an exponential function, what is the effect of changing the value of $h$ in the equation?

10 On the one set of axes, sketch the graphs of $y_1 = 2^x$ and $y_2 = -2^{x-2} + 2$. Describe the transformations to get from $y_1$ to $y_2$.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 11.2 Indices as exponents

## Index or exponential form

Units 1 & 2
AOS 1
Topic 8
Concept 1

**Indices as exponents**
Concept summary
Practice questions

When the number 8 is expressed as a power of 2, it is written as $8 = 2^3$. In this form, the base is 2 and the power (also known as index or **exponent**) is 3. The form of 8 expressed as $2^3$ is known both as its index form with index 3 and base 2, and its **exponential form** with exponent 3 and base 2. The words 'index' and 'exponent' are interchangeable.

For any positive number $n$ where $n = a^x$, the statement $n = a^x$ is called an index or exponential statement. For our study:

- the base is $a$ where $a \in R^+ \backslash \{1\}$
- the exponent, or index, is $x$ where $x \in R$
- the number $n$ is positive, so $a^x \in R^+$.

Index laws control the simplification of expressions which have the same base.

## Review of index laws

Recall the basic index laws:

$$\begin{aligned} a^m \times a^n &= a^{m+n} \\ a^m \div a^n &= a^{m-n} \\ (a^m)^n &= a^{mn} \end{aligned}$$

From these, it follows that:

$$a^0 = 1$$

$$a^{-n} = \frac{1}{a^n} \text{ and } \frac{1}{a^{-n}} = a^n$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

## Fractional indices

Since $\left(a^{\frac{1}{2}}\right)^2 = a$ and $(\sqrt{a})^2 = a$, then $a^{\frac{1}{2}} = \sqrt{a}$. Thus, surds such as $\sqrt{3}$ can be written in index form as $3^{\frac{1}{2}}$.

The symbols $\sqrt{\ }, \sqrt[3]{\ }, \ldots \sqrt[n]{\ }$ are radical signs. Any radical can be converted to and from a fractional index using the index law:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

A combination of index laws allows $a^{\frac{m}{n}}$ to be expressed as $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m$. It can also be expressed as $a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$.

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \text{ or } a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

WORKED EXAMPLE 1

a Express $\frac{3^{2n} \times 9^{1-n}}{81^{n-1}}$ as a power of 3.

b Simplify $\left(3a^2b^{\frac{5}{2}}\right)^2 \times 2(a^{-1}b^2)^{-2}$.

c Evaluate $8^{\frac{2}{3}} + \left(\frac{4}{9}\right)^{-\frac{1}{2}}$.

| THINK | WRITE |
|---|---|
| **a** 1 Express each term with the same base. | **a** $\frac{3^{2n} \times 9^{1-n}}{81^{n-1}} = \frac{3^{2n} \times (3^2)^{1-n}}{(3^4)^{n-1}}$ |
| 2 Apply an appropriate index law. *Note:* To raise a power to a power multiply the indices. | $= \frac{3^{2n} \times 3^{2(1-n)}}{3^{4(n-1)}}$ <br> $= \frac{3^{2n} \times 3^{2-2n}}{3^{4n-4}}$ |
| 3 Apply an appropriate index law. *Note:* To multiply numbers with the same base, add the indices. | $= \frac{3^{2n+2-2n}}{3^{4n-4}}$ <br> $= \frac{3^2}{3^{4n-4}}$ |
| 4 Apply an appropriate index law and state the answer. *Note:* To divide numbers with the same base, subtract the indices. | $= 3^{2-(4n-4)}$ <br> $= 3^{6-4n}$ <br> $\therefore \frac{3^{2n} \times 9^{1-n}}{81^{n-1}} = 3^{6-4n}$ |
| **b** 1 Use an index law to remove the brackets. | **b** $\left(3a^2b^{\frac{5}{2}}\right)^2 \times 2(a^{-1}b^2)^{-2}$ <br> $= 3^2a^4b^5 \times 2 \times a^2b^{-4}$ <br> $= 9a^4b^5 \times 2a^2b^{-4}$ |
| 2 Apply the index laws to terms with the same base to simplify the expression. | $= 18a^6b^1$ <br> $= 18a^6b$ |
| **c** 1 Express each term with a positive index. *Note:* There is no index law for the addition of numbers. | **c** $8^{\frac{2}{3}} + \left(\frac{4}{9}\right)^{-\frac{1}{2}} = 8^{\frac{2}{3}} + \left(\frac{9}{4}\right)^{\frac{1}{2}}$ |
| 2 Apply the index law for fractional indices. *Note:* The fractional indices could be interpreted in other ways including, for example, as $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}$. | $= \left(\sqrt[3]{8}\right)^2 + \frac{\sqrt{9}}{\sqrt{4}}$ |
| 3 Evaluate each term separately and calculate the answer required. | $= (2)^2 + \frac{3}{2}$ <br> $= 4 + 1\frac{1}{2}$ <br> $= 5\frac{1}{2}$ |

## Indicial equations

An **indicial equation** has the unknown variable as an exponent. In this section we shall consider indicial equations which have rational solutions.

## Method of equating indices

If index laws can be used to express both sides of an equation as single powers of the same base, then this allows indices to be equated. For example, if an equation can be simplified to the form $2^{3x} = 2^4$, then for the equality to hold, $3x = 4$. Solving this linear equation gives the solution to the indicial equation as $x = \frac{4}{3}$.

In the case of an inequation, a similar method is used. If $2^{3x} < 2^4$, for example, then $3x < 4$. Solving this linear inequation gives the solution to the indicial inequation as $x < \frac{4}{3}$.

> **To solve for the exponent $x$ in equations of the form $a^x = n$:**
> - **Express both sides as powers of the same base.**
> - **Equate the indices and solve the equation formed to obtain the solution to the indicial equation.**

Inequations are solved in a similar manner. However, you need to ensure the base $a$ is greater than 1 prior to expressing the indices with the corresponding order sign between them.

For example, you first need to write $\left(\frac{1}{2}\right)^x < \left(\frac{1}{2}\right)^3$ as $2^{-x} < 2^{-3}$ and then solve the linear inequation $-x < -3$ to obtain the solution $x > 3$.

**WORKED EXAMPLE 2**

Solve $5^{3x} \times 25^{4-2x} = \frac{1}{125}$ for $x$.

| THINK | WRITE |
|---|---|
| 1 Use the index laws to express the left-hand side of the equation as a power of a single base. | $5^{3x} \times 25^{4-2x} = \frac{1}{125}$<br>$5^{3x} \times 5^{2(4-2x)} = \frac{1}{125}$<br>$5^{3x+8-4x} = \frac{1}{125}$<br>$5^{8-x} = \frac{1}{125}$ |
| 2 Express the right-hand side as a power of the same base. | $5^{8-x} = \frac{1}{5^3}$<br>$5^{8-x} = 5^{-3}$ |
| 3 Equate indices and calculate the required value of $x$. | Equating indices,<br>$8 - x = -3$<br>$x = 11$ |

## Indicial equations which reduce to quadratic form

The technique of substitution to form a quadratic equation may be applicable to indicial equations.

To solve equations of the form $p \times a^{2x} + q \times a^x + r = 0$:

- Note that $a^{2x} = (a^x)^2$.
- Reduce the indicial equation to quadratic form by using a substitution for $a^x$.
- Solve the quadratic and then substitute back for $a^x$.
- Since $a^x$ must always be positive, solutions for $x$ can only be obtained for $a^x > 0$; reject any negative or zero values for $a^x$.

**WORKED EXAMPLE 3** **Solve $3^{2x} - 6 \times 3^x - 27 = 0$ for $x$.**

| THINK | WRITE |
| --- | --- |
| 1 Use a substitution technique to reduce the indicial equation to quadratic form. *Note:* The subtraction signs prevent the use of index laws to express the left-hand side as a power of a single base. | $3^{2x} - 6 \times 3^x - 27 = 0$<br>Let $a = 3^x$<br>$\therefore a^2 - 6a - 27 = 0$ |
| 2 Solve the quadratic equation. | $(a - 9)(a + 3) = 0$<br>$a = 9,\ a = -3$ |
| 3 Substitute back and solve for $x$. | Replace $a$ by $3^x$.<br>$\therefore 3^x = 9$ or $3^x = -3$ (reject negative value)<br>$3^x = 9$<br>$\therefore 3^x = 3^2$<br>$\therefore x = 2$ |

## Scientific notation (standard form)

Index notation provides a convenient way to express numbers which are either very large or very small. Writing a number as $a \times 10^b$ (the product of a number $a$ where $1 \le a < 10$ and a power of 10) is known as writing the number in **scientific notation** (or **standard form**). The age of the earth since the Big Bang is estimated to be $4.54 \times 10^9$ years, while the mass of a carbon atom is approximately $1.994 \times 10^{-23}$ grams. These numbers are written in scientific notation.

**To convert scientific notation back to a basic numeral:**

- **move the decimal point $b$ places to the right if the power of 10 has a positive index, in order to obtain the large number $a \times 10^b$ represents;**

  **or**
- **move the decimal point $b$ places to the left if the power of 10 has a negative index, in order to obtain the small number $a \times 10^{-b}$ represents. This is because multiplying by $10^{-b}$ is equivalent to dividing by $10^b$.**

### Significant figures

When a number is expressed in scientific notation as either $a \times 10^b$ or $a \times 10^{-b}$, the number of digits in $a$ determines the number of **significant figures** in the basic numeral. The age of the Earth is $4.54 \times 10^9$ years in scientific notation or 4 540 000 000 years to three significant figures. To one significant figure, the age would be 5 000 000 000 years.

WORKED EXAMPLE 4

a Express each of the following numerals in scientific notation and state the number of significant figures each numeral contains.

i 3 266 400 ii 0.009 876 03

b Express the following as basic numerals.

i $4.54 \times 10^9$ ii $1.037 \times 10^{-5}$

| THINK | WRITE |
|---|---|
| a i 1 Write the given number as a value between 1 and 10 multiplied by a power of 10. *Note:* The number is large so the power of 10 should be positive. | a i In scientific notation, $3\,266\,400 = 3.2664 \times 10^6$ |
| 2 Count the number of digits in the number $a$ in the scientific notation form $a \times 10^b$ and state the number of significant figures. | There are 5 significant figures in the number 3 266 400. |
| ii 1 Write the given number as a value between 1 and 10 multiplied by a power of 10. *Note:* The number is small so the power of 10 should be negative. | ii $0.009\,876\,03 = 9.876\,03 \times 10^{-3}$ |
| 2 Count the number of digits in the number $a$ in the scientific notation form and state the number of significant figures. | 0.009 876 03 has 6 significant figures. |
| b i 1 Perform the multiplication. *Note:* The power of 10 is positive, so a large number should be obtained. | b i $4.54 \times 10^9$ Move the decimal point 9 places to the right. $\therefore 4.54 \times 10^9 = 4\,540\,000\,000$ |
| ii 2 Perform the multiplication. *Note:* The power of 10 is negative, so a small number should be obtained. | ii $1.037 \times 10^{-5}$ Move the decimal point 5 places to the left. $\therefore 1.037 \times 10^{-5} = 0.00001037$ |

## EXERCISE 11.2 Indices as exponents

PRACTISE

Work without CAS

1 WE1 a Express $\dfrac{2^{1-n} \times 8^{1+2n}}{16^{1-n}}$ as a power of 2.

b Simplify $(9a^3b^{-4})^{\frac{1}{2}} \times 2\left(a^{\frac{1}{2}}b^{-2}\right)^{-2}$.

c Evaluate $27^{-\frac{2}{3}} + \left(\frac{49}{81}\right)^{\frac{1}{2}}$.

2 Simplify $\dfrac{20p^5}{m^3q^{-2}} \div \dfrac{5(p^2q^{-3})^2}{-4m^{-1}}$.

3 WE2 Solve $\dfrac{2^{5x-3} \times 8^{9-2x}}{4^x} = 1$ for $x$.

4 Solve the following inequations.

a $2 \times 5^x + 5^x < 75$

b $\left(\frac{1}{9}\right)^{2x-3} > \left(\frac{1}{9}\right)^{7-x}$

5 WE3 Solve $30 \times 10^{2x} + 17 \times 10^{x} - 2 = 0$ for $x$.

6 Solve $2^{x} - 48 \times 2^{-x} = 13$ for $x$.

7 WE4 a Express each of the following numbers in scientific notation, and state the number of significant figures each number contains.

i 1 409 000 ii 0.000 130 6

b Express the following as basic numerals.

i $3.04 \times 10^{5}$ ii $5.803 \times 10^{-2}$

8 Calculate $(4 \times 10^{6})^{2} \times (5 \times 10^{-3})$ without using a calculator.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 a Express the following in index (exponent) form.

i $\sqrt{a^3b^4}$ ii $\sqrt{\dfrac{a^5}{b^{-4}}} \times \sqrt[3]{a^2b}$

b Express the following in surd form.

i $a^{\frac{1}{2}} \div b^{\frac{3}{2}}$ ii $2^{\frac{5}{2}}$ iii $3^{-\frac{2}{5}}$

10 Evaluate without a calculator:

a $4^{\frac{3}{2}}$

b $3^{-1} + 5^{0} - 2^{2} \times 9^{-\frac{1}{2}}$

c $2^{3} \times \left(\dfrac{4}{9}\right)^{-\frac{1}{2}} \div (6 \times (3^{-2})^{2})$

d $\dfrac{15 \times 5^{\frac{3}{2}}}{125^{\frac{1}{2}} - 20^{\frac{1}{2}}}$

11 Simplify and express the answer with positive indices.

a $\dfrac{3(x^2y^{-2})^3}{(3x^4y^2)^{-1}}$

b $\dfrac{2a^{\frac{2}{3}}b^{-3}}{3a^{\frac{1}{3}}b^{-1}} \times \dfrac{3^2 \times 2 \times (ab)^2}{(-8a^2)^2b^2}$

c $\dfrac{(2mn^{-2})^{-2}}{m^{-1}n} \div \dfrac{10n^4m^{-1}}{3(m^2n)^{\frac{3}{2}}}$

d $\dfrac{4m^2n^{-2} \times -2\left(m^2n^{\frac{3}{2}}\right)^2}{(-3m^3n^{-2})^2}$

e $\dfrac{m^{-1} - n^{-1}}{m^2 - n^2}$

f $\sqrt{4x-1} - 2x(4x-1)^{-\frac{1}{2}}$

12 a Express $\dfrac{32 \times 4^{3x}}{16^{x}}$ as a power of 2.

b Express $\dfrac{3^{1+n} \times 81^{n-2}}{243^{n}}$ as a power of 3.

c Express $0.001 \times \sqrt[3]{10} \times 100^{\frac{5}{2}} \times (0.1)^{-\frac{2}{3}}$ as a power of 10.

d Express $\dfrac{5^{n+1} - 5^{n}}{4}$ as a power of 5.

13 Solve for $x$:

a $2^{2x} \times 8^{2-x} \times 16^{-\frac{3x}{2}} = \dfrac{2}{4^{x}}$

b $25^{3x-3} \leq 125^{4+x}$

c $9^{x} \div 27^{1-x} = \sqrt{3}$

d $\left(\dfrac{2}{3}\right)^{3-2x} > \left(\dfrac{27}{8}\right)^{-\frac{1}{3}} \times \dfrac{1}{\sqrt{2\frac{1}{4}}}$

e $4^{5x} + 4^{5x} = \dfrac{8}{2^{4x-5}}$

f $5^{\frac{2x}{3}} \times 5^{\frac{3x}{2}} = 25^{x+4}$

14 Use a suitable substitution to solve the following equations.

a $3^{2x} - 10 \times 3^x + 9 = 0$
b $24 \times 2^{2x} + 61 \times 2^x = 2^3$
c $25^x + 5^{2+x} - 150 = 0$
d $(2^x + 2^{-x})^2 = 4$
e $10^x - 10^{2-x} = 99$
f $2^{3x} + 3 \times 2^{2x-1} - 2^x = 0$

15 a Express in scientific notation:

i $-0.000\,000\,050\,6$
ii the diameter of the Earth, given its radius is 6370 km
iii $3.2 \times 10^4 \times 5 \times 10^{-2}$
iv the distance between Roland Garros and Kooyong tennis stadiums of 16 878.7 km.

b Express as a basic numeral:

i $6.3 \times 10^{-4} + 6.3 \times 10^4$
ii $\left(1.44 \times 10^6\right)^{\frac{1}{2}}$

c Express the following to 2 significant figures:

i 60 589 people attended a football match.
ii The probability of winning a competition is $1.994 \times 10^{-2}$.
iii The solution to an equation is $x = -0.006\,34$.
iv The distance flown per year by the Royal Flying Doctor Service is 26 597 696 km.

16 If $x = 3^{\frac{1}{3}} + 3^{-\frac{1}{3}}$, show that $x^3 - 3x = \frac{10}{3}$.

17 a Solve the pair of simultaneous equations for $x$ and $y$:

$$5^{2x-y} = \frac{1}{125}$$
$$10^{2y-6x} = 0.01$$

b Solve the pair of simultaneous equations for $a$ and $k$.

$$a \times 2^{k-1} = 40$$
$$a \times 2^{2k-2} = 10$$

18 If $\left(\frac{2x^2}{3a}\right)^{n-1} \div \left(\frac{3x}{a}\right)^{n+1} = \left(\frac{x}{4}\right)^3$, determine the values of the constants $a$ and $n$.

MASTER

19 Evaluate:

a $5^{-4.3}$ and express the answer in scientific notation to 4 significant figures
b $22.9 \div 1.3E2$ to 4 significant figures and explain what the $1.3E2$ notation means
c $5.04 \times 10^{-6} \div (3 \times 10^9)$, expressing the answer in standard form
d $5.04 \times 10^{-6} \div (3.2 \times 10^{4.2})$, expressing the answer in standard form to 4 significant figures.

20 a Simplify $\dfrac{x^2y^{-2}}{2x^{\frac{1}{3}}\sqrt{y^5}}$.

b Solve the equations:

i $5^x \times 25^{2x} = \frac{1}{5}$ to obtain $x$ exactly
ii $5^x \times 25^{2x} = 0.25$ to obtain $x$ to 4 significant figures.

# 11.3 Indices as logarithms

Not all solutions to indicial equations are rational. In order to obtain the solution to an equation such as $2^x = 5$, we need to learn about logarithms.

studyon

Units 1 & 2
AOS 1
Topic 8
Concept 2

**Indices as logarithms**
Concept summary
Practice questions

## Index-logarithm forms

A **logarithm** is also another name for an index.

The index statement, $n = a^x$, with base $a$ and index $x$, can be expressed with the index as the subject. This is called the logarithm statement and is written as $x = \log_a(n)$. The statement is read as '$x$ equals the log to base $a$ of $n$' (adopting the abbreviation of 'log' for logarithm).

> **The statements $n = a^x$ and $x = \log_a(n)$ are equivalent.**
> - $n = a^x \Leftrightarrow x = \log_a(n)$, **where the base** $a \in R^+\backslash\{1\}$, **the number** $n \in R^+$ **and the logarithm, or index,** $x \in R$.

Consider again the equation $2^x = 5$. The solution is obtained by converting this index statement to the logarithm statement.

$$2^x = 5$$
$$\therefore x = \log_2(5)$$

The number $\log_2(5)$ is irrational: the power of 2 which gives the number 5 is not rational. A decimal approximation for this logarithm can be obtained using a calculator. The exact solution to the indicial equation $2^x = 5$ is $x = \log_2(5)$; an approximate solution is $x \approx 2.3219$ to 5 significant figures.

Not all expressions containing logarithms are irrational. Solving the equation $2^x = 8$ by converting to logarithm form gives:

$$2^x = 8$$
$$\therefore x = \log_2(8)$$

In this case, the expression $\log_2(8)$ can be simplified. As the power of 2 which gives the number 8 is 3, the solution to the equation is $x = 3$; that is, $\log_2(8) = 3$.

### Use of a calculator

Calculators have two inbuilt logarithmic functions.

Base 10 logarithms are obtained from the LOG key. Thus $\log_{10}(2)$ is evaluated as $\log(2)$, giving the value of 0.3010 to 4 decimal places. Base 10 logarithms are called common logarithms and, in a previous era, they were commonly used to perform calculations from tables of values. They are also known as Briggsian logarithms in deference to the English mathematician Henry Briggs who first published their table of values in the seventeenth century.

Base $e$ logarithms are obtained from the LN key. Thus $\log_e(2)$ is evaluated as $\ln(2)$, giving the value 0.6931 to 4 decimal places. Base $e$ logarithms are called natural or Naperian logarithms, after their inventor John Napier, a seventeenth-century Scottish baron with mathematical interests. These logarithms occur extensively in calculus. The number $e$ itself is known as Euler's number and, like $\pi$, it is a transcendental irrational number that has great importance in higher mathematical studies, as you will begin to discover in Units 3 and 4 of Mathematical Methods.

CAS technology enables logarithms to bases other than 10 or $e$ to be evaluated.

**WORKED EXAMPLE 5**

a Express $3^4 = 81$ as a logarithm statement.

b Express $\log_7\left(\frac{1}{49}\right) = -2$ as an index statement.

c Solve the equation $10^x = 12.8$, expressing the exponent $x$ to 2 significant figures.

d Solve the equation $\log_5(x) = 2$ for $x$.

| THINK | WRITE |
|---|---|
| a 1 Identify the given form. | a $3^4 = 81$<br>The index form is given with base 3, index or logarithm 4 and number 81. |
| 2 Convert to the equivalent form. | Since $n = a^x \Leftrightarrow x = \log_a(n)$,<br>$81 = 3^4 \Rightarrow 4 = \log_3(81)$.<br>The logarithm statement is $4 = \log_3(81)$. |
| b 1 Identify the given form. | b $\log_7\left(\frac{1}{49}\right) = -2$<br>The logarithm form is given with base 7, number $\frac{1}{49}$ and logarithm, or index, of −2. |
| 2 Convert to the equivalent form. | Since $x = \log_a(n) \Leftrightarrow n = a^x$<br>$-2 = \log_7\left(\frac{1}{49}\right) \Rightarrow \frac{1}{49} = 7^{-2}$<br>The index statement is $\frac{1}{49} = 7^{-2}$. |
| c 1 Convert to the equivalent form. | c $10^x = 12.8$<br>$\therefore x = \log_{10}(12.8)$ |
| 2 Evaluate using a calculator and state the answer to the required accuracy.<br>*Note:* The base is 10 so use the LOG key on the calculator. | $\approx 1.1$ to 2 significant figures |
| d 1 Convert to the equivalent form. | d $\log_5(x) = 2$<br>$\therefore \quad x = 5^2$ |
| 2 Calculate the answer. | $= 25$ |

## Logarithm laws

Since logarithms are indices, unsurprisingly there is a set of laws which control simplification of logarithmic expressions with the same base.

**For any $a, m, n > 0$, $a \neq 1$, the laws are:**

1. $\log_a(1) = 0$
2. $\log_a(a) = 1$
3. $\log_a(m) + \log_a(n) = \log_a(mn)$
4. $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
5. $\log_a(m^p) = p\log_a(m)$.

Note that there is no logarithm law for either the product or quotient of logarithms or for expressions such as $\log_a(m \pm n)$.

## Proofs of the logarithm laws

1. Consider the index statement.

$$a^0 = 1$$
$$\therefore \log_a(1) = 0$$

2. Consider the index statement.

$$a^1 = a$$
$$\therefore \log_a(a) = 1$$

3. Let $x = \log_a(m)$ and $y = \log_a(n)$.

$$\therefore m = a^x \text{ and } n = a^y$$
$$mn = a^x \times a^y$$
$$= a^{x+y}$$

Convert to logarithm form:

$$x + y = \log_a(mn)$$

Substitute back for $x$ and $y$:

$$\log_a(m) + \log_a(n) = \log_a(mn)$$

4. With $x$ and $y$ as given in law 3:

$$\frac{m}{n} = \frac{a^x}{a^y}$$
$$= a^{x-y}$$

Converting to logarithm form, and then substituting back for $x$ and $y$ gives:

$$x - y = \log_a\left(\frac{m}{n}\right)$$

Substitute back for $x$ and $y$:

$$\therefore \log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$$

5. With $x$ as given in law 3:

$$x = \log_a(m)$$
$$\therefore m = a^x$$

Raise both sides to the power $p$:

$$\therefore (m)^p = (a^x)^p$$
$$\therefore m^p = a^{px}$$

Express as a logarithm statement with base $a$:

$$px = \log_a(m^p)$$

Substitute back for $x$:

$$\therefore p\log_a(m) = \log_a(m^p)$$

**WORKED EXAMPLE 6**

Simplify the following using the logarithm laws.

a $\log_{10}(5) + \log_{10}(2)$

b $\log_2(80) - \log_2(5)$

c $\log_3(2a^4) + 2\log_3\left(\sqrt{\dfrac{a}{2}}\right)$

d $\dfrac{\log_a\left(\frac{1}{4}\right)}{\log_a(2)}$

| THINK | WRITE |
|---|---|
| **a 1** Apply the appropriate logarithm law. | **a** $\log_{10}(5) + \log_{10}(2) = \log_{10}(5 \times 2)$ <br> $= \log_{10}(10)$ |
| **2** Simplify and state the answer. | $= 1$ since $\log_a(a) = 1$ |
| **b 1** Apply the appropriate logarithm law. | **b** $\log_2(80) - \log_2(5) = \log_2\left(\dfrac{80}{5}\right)$ <br> $= \log_2(16)$ |
| **2** Further simplify the logarithmic expression. | $= \log_2(2^4)$ <br> $= 4\log_2(2)$ |
| **3** Calculate the answer. | $= 4 \times 1$ <br> $= 4$ |
| **c 1** Apply a logarithm law to the second term. <br> *Note:* As with indices, there is often more than one way to approach the simplification of logarithms. | **c** $\log_3(2a^4) + 2\log_3\left(\sqrt{\dfrac{a}{2}}\right)$ <br> $= \log_2(2a^4) + 2\log_a\left(\left(\dfrac{a}{2}\right)^{\frac{1}{2}}\right)$ <br> $= \log_2(2a^4) + 2 \times \frac{1}{2}\log_a\left(\dfrac{a}{2}\right)$ <br> $= \log_2(2a^4) + \log_a\left(\dfrac{a}{2}\right)$ |
| **2** Apply an appropriate logarithm law to combine the two terms as one logarithmic expression. | $= \log_3\left(2a^4 \times \dfrac{a}{2}\right)$ <br> $= \log_3(a^5)$ |
| **3** State the answer. | $= 5\log_3(a)$ |
| **d 1** Express the numbers in the logarithm terms in index form. <br> *Note:* There is no law for division of logarithms. | **d** $\dfrac{\log_a\left(\frac{1}{4}\right)}{\log_a(2)} = \dfrac{\log_a(2^{-2})}{\log_a(2)}$ |
| **2** Simplify the numerator and denominator separately. | $= \dfrac{-2\log_a(2)}{\log_a(2)}$ |
| **3** Cancel the common factor in the numerator and denominator. | $= \dfrac{-2\cancel{\log_a(2)}}{\cancel{\log_a(2)}}$ <br> $= -2$ |

## Logarithms as operators

Just as both sides of an equation may be raised to a power and the equality still holds, taking logarithms of both sides of an equation maintains the equality.

If $m = n$, then it is true that $\log_a(m) = \log_a(n)$ and vice versa, provided the same base is used for the logarithms of each side.

This application of logarithms can provide an important tool when solving indicial equations.

Consider again the equation $2^x = 5$ where the solution was given as $x = \log_2(5)$.

Take base 10 logarithms of both sides of this equation.

$$2^x = 5$$

$$\log_{10}(2^x) = \log_{10}(5)$$

Using one of the logarithm laws, this becomes $x\log_{10}(2) = \log_{10}(5)$ from which the solution to the indicial equation is obtained as $x = \dfrac{\log_{10}(5)}{\log_{10}(2)}$. This form of the solution can be evaluated on a scientific calculator and is the prime reason for choosing base 10 logarithms in solving the indicial equation.

It also demonstrates that $\log_2(5) = \dfrac{\log_{10}(5)}{\log_{10}(2)}$, which is a particular example of another logarithm law called the *change of base law.*

### Change of base law for calculator use

The equation $a^x = p$ for which $x = \log_a(p)$ could be solved in a similar way to $2^x = 5$, giving the solution as $x = \dfrac{\log_{10}(p)}{\log_{10}(a)}$. Thus $\log_a(p) = \dfrac{\log_{10}(p)}{\log_{10}(a)}$. This form enables decimal approximations to logarithms to be calculated on scientific calculators.

The change of base law is the more general statement allowing base $a$ logarithms to be expressed in terms of any other base $b$ as $\log_a(p) = \dfrac{\log_b(p)}{\log_b(a)}$. This more general form shall be left until Units 3 and 4.

### Convention

There is a convention that if the base of a logarithm is not stated, this implies it is base 10. As it is on a calculator, $\log(n)$ represents $\log_{10}(n)$. When working with base 10 logarithms it can be convenient to adopt this convention.

**WORKED EXAMPLE 7**

**a** **State the exact solution to $5^x = 8$ and calculate its value to 3 decimal places.**

**b** **Calculate the exact value and the value to 3 decimal places of the solution to the equation $2^{1-x} = 6^x$.**

| THINK | WRITE |
|---|---|
| **a** **1** Convert to the equivalent form and state the exact solution. | **a** $5^x = 8$<br>$\therefore x = \log_5(8)$<br>The exact solution is $x = \log_5(8)$. |

2 Use the change of base law to express the answer in terms of base 10 logarithms.

Since

$$\log_a(p) = \frac{\log_{10}(p)}{\log_{10}(a)}$$

then

$$\log_5(8) = \frac{\log_{10}(8)}{\log_{10}(5)}$$

$$\therefore \quad x = \frac{\log_{10}(8)}{\log_{10}(5)}$$

3 Calculate the approximate value.

$\therefore x \approx 1.292$ to 3 decimal places.

b 1 Take base 10 logarithms of both sides.

*Note:* The convention is not to write the base 10.

b $2^{1-x} = 6^x$

Take logarithms to base 10 of both sides:

$\log(2^{1-x}) = \log(6^x)$

2 Apply the logarithm law so that $x$ terms are no longer exponents.

$(1 - x)\log(2) = x\log(6)$

3 Solve the linear equation in $x$.

*Note:* This is no different to solving any other linear equation of the form $a - bx = cx$ except the constants $a$, $b$, $c$ are expressed as logarithms.

Expand:

$\log(2) - x\log(2) = x\log(6)$

Collect $x$ terms together:

$$\begin{aligned}\log(2) &= x\log(6) + x\log(2)\\ &= x(\log(6) + \log(2))\end{aligned}$$

$$x = \frac{\log(2)}{\log(6) + \log(2)}$$

This is the exact solution.

4 Calculate the approximate value.

*Note:* Remember to place brackets around the denominator for the division.

$x \approx 0.279$ to 3 decimal places.

## Equations containing logarithms

While the emphasis in this topic is on exponential (indicial) relations for which some knowledge of logarithms is essential, it is important to know that logarithms contribute substantially to Mathematics. As such, some equations involving logarithms are included, allowing further consolidation of the laws which logarithms must satisfy.

Remembering the requirement that $x$ must be positive for $\log_a(x)$ to be real, it is advisable to check any solution to an equation involving logarithms. Any value of $x$ which when substituted back into the original equation creates a '$\log_a$(negative number)' term must be rejected as a solution. Otherwise, normal algebraic approaches together with logarithm laws are the techniques for solving such equations.

**WORKED EXAMPLE 8**

**Solve the equation $\log_6(x) + \log_6(x - 1) = 1$ for $x$.**

**THINK**

1 Apply the logarithm law which reduces the equation to one logarithm term.

**WRITE**

$$\begin{aligned}\log_6(x) + \log_6(x - 1) &= 1\\ \therefore \log_6(x(x - 1)) &= 1\\ \therefore \log_6(x^2 - x) &= 1\end{aligned}$$

| | |
|---|---|
| **2** Convert the logarithm form to its equivalent form. *Note:* An alternative method is to write $\log_6(x^2 - x) = \log_6(6)$ from which $x^2 - x = 6$ is obtained. | Converting from logarithm form to index form gives: $x^2 - x = 6^1$ $\therefore\ x^2 - x = 6$ |
| **3** Solve the quadratic equation. | $x^2 - x - 6 = 0$ $\therefore (x-3)(x+2) = 0$ $\therefore x = 3,\ x = -2$ |
| **4** Check the validity of both solutions in the original equation. | Check in $\log_6(x) + \log_6(x-1) = 1$ If $x = 3$, LHS $= \log_6(3) + \log_6(2)$ $= \log_6(6)$ $= 1$ $=$ RHS If $x = -2$, LHS $= \log_6(-2) + \log_6(-3)$ which is not admissible. Therefore reject $x = -2$. |
| **5** State the answer. | The solution is $x = 3$. |

## EXERCISE 11.3 Indices as logarithms

**PRACTISE**

Work without CAS Q1–4

**1** WE5 **a** Express $5^4 = 625$ as a logarithm statement.

**b** Express $\log_{36}(6) = \frac{1}{2}$ as an index statement.

**c** Solve the equation $10^x = 8.52$, expressing the exponent $x$ to 2 significant figures.

**d** Solve the equation $\log_3(x) = -1$ for $x$.

**2** **a** Evaluate $\log_e(5)$ to 4 significant figures and write the equivalent index statement.

**b** Evaluate $10^{3.5}$ to 4 significant figures and write the equivalent logarithm statement.

**3** WE6 Simplify using the logarithm laws:

**a** $\log_{12}(3) + \log_{12}(4)$

**b** $\log_2(192) - \log_2(12)$

**c** $\log_3(3a^3) - 2\log_3\left(a^{\frac{3}{2}}\right)$

**d** $\dfrac{\log_a(8)}{\log_a(4)}$

**4** Given $\log_a(2) = 0.3$ and $\log_a(5) = 0.7$, evaluate:

**a** $\log_a(0.5)$

**b** $\log_a(2.5)$

**c** $\log_a(20)$

**5** WE7 **a** State the exact solution to $7^x = 15$ and calculate its value to 3 decimal places.

**b** Calculate the exact value and the value to 3 decimal places of the solution to the equation $3^{2x+5} = 4^x$.

**6** If $\log_2(3) - \log_2(2) = \log_2(x) + \log_2(5)$, solve for $x$.

**7** WE8 Solve the equation $\log_3(x) + \log_3(2x+1) = 1$ for $x$.

**8** Solve the equation $\log_6(x) - \log_6(x-1) = 2$ for $x$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 a Express as a logarithm statement with the index as the subject:

i $2^5 = 32$ ii $4^{\frac{3}{2}} = 8$ iii $10^{-3} = 0.001$

b Express as an index statement:

i $\log_2(16) = 4$ ii $\log_9(3) = \frac{1}{2}$ iii $\log_{10}(0.1) = -1$

10 Rewrite each of the following in the equivalent index or logarithm form and hence calculate the value of $x$.

a $x = \log_2\left(\frac{1}{8}\right)$

b $\log_{25}(x) = -0.5$

c $10^{(2x)} = 4$. Express the answer to 2 decimal places.

d $3 = e^{-x}$. Express the answer to 2 decimal places.

e $\log_x(125) = 3$

f $\log_x(25) = -2$

11 Use the logarithm laws to evaluate the following.

a $\log_9(3) + \log_9(27)$ b $\log_9(3) - \log_9(27)$

c $2\log_2(4) + \log_2(6) - \log_2(12)$ d $\log_5(\log_3(3))$

e $\log_{11}\left(\frac{7}{3}\right) + 2\log_{11}\left(\frac{1}{3}\right) - \log_{11}\left(\frac{11}{3}\right) - \log_{11}\left(\frac{7}{9}\right)$

f $\log_2\left(\frac{\sqrt{x} \times x^{\frac{3}{2}}}{x^2}\right)$

12 Simplify the following.

a $\dfrac{\log_a(9)}{\log_a(\sqrt[5]{3})}$ b $1 - \log_4(3^n + 3^{n+1})$

c $\log_8(16) \times \log_{16}(8)$ d $\log_{10}(1000) + \log_{0.1}\left(\frac{1}{1000}\right)$

13 a Express $\log_2(10)$ in terms of $\log_{10}(2)$.

b State the exact solution and then give the approximate solution to 4 significant figures for each of the following indicial equations.

i $11^x = 18$ ii $5^{-x} = 8$ iii $7^{2x} = 3$

c Obtain the approximate solution to 4 significant figures for each of the following inequations.

i $3^x \le 10$ ii $5^{-x} > 0.4$

d Solve the following equations.

i $2^{\log_5(x)} = 8$ ii $2^{\log_2(x)} = 7$

14 Solve the indicial equations to obtain the value of $x$ to 2 decimal places.

a $7^{1-2x} = 4$ b $10^{-x} = 5^{x-1}$ c $5^{2x-9} = 3^{7-x}$

d $10^{3x+5} = 6^{2-3x}$ e $0.25^{4x} = 0.8^{2-0.5x}$ f $4^{x+1} \times 3^{1-x} = 5^x$

15 Solve the following for $x$.

a $\log_2(2x + 1) + \log_2(2x - 1) = 3\log_2(3)$

b $\log_3(2x) + \log_3(4) = \log_3(x + 12) - \log_3(2)$

c $\log_2(2x + 12) - \log_2(3x) = 4$

d $\log_2(x) + \log_2(2 - 2x) = -1$

e $(\log_{10}(x) + 3)(2\log_4(x) - 3) = 0$

f $2\log_3(x) - 1 = \log_3(2x - 3)$

**16** Given $\log_a(3) = p$ and $\log_a(5) = q$, express the following in terms of $p$ and $q$.

**a** $\log_a(15)$
**b** $\log_a(125)$
**c** $\log_a(45)$
**d** $\log_a(0.6)$
**e** $\log_a\left(\frac{25}{81}\right)$
**f** $\log_a(\sqrt{5}) \times \log_a(\sqrt{27})$

**17** Express $y$ in terms of $x$.

**a** $\log_{10}(y) = \log_{10}(x) + 2$
**b** $\log_2(x^2\sqrt{y}) = x$
**c** $2\log_2\left(\frac{y}{2}\right) = 6x - 2$
**d** $x = 10^{y-2}$
**e** $\log_{10}(10^{3xy}) = 3$
**f** $10^{3\log_{10}(y)} = xy$

**18** Solve the following equations, giving exact solutions.

**a** $2^{2x} - 14 \times 2^x + 45 = 0$
**b** $5^{-x} - 5^x = 4$
**c** $9^{2x} - 3^{1+2x} + 2 = 0$
**d** $\log_a(x^3) + \log_a(x^2) - 4\log_a(2) = \log_a(x)$
**e** $(\log_2(x))^2 - \log_2(x^2) = 8$
**f** $\dfrac{\log_{10}(x^3)}{\log_{10}(x + 1)} = \log_{10}(x)$

MASTER

**19 a** Give the solution to $12^x = 50$ to 4 significant figures.
**b** Give the exact solution to the equation $\log(5x) + \log(x + 5) = 1$.

**20 a** Evaluate $\log_{10}(5) + \log_5(10)$ with the calculator on 'Standard' mode and explain the answer obtained.
**b** Evaluate $\log_y(x) \times \log_x(y)$ and explain how the result is obtained.

# 11.4 Graphs of exponential functions

studyon

Units 1 & 2
AOS 1
Topic 8
Concept 3

**Graphs of exponential functions**
Concept summary
Practice questions

eBook*plus*

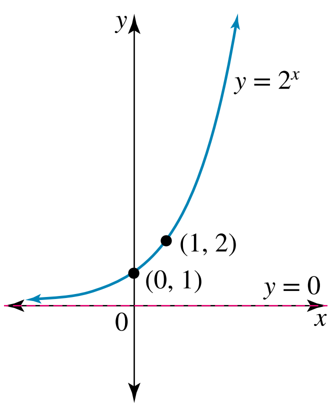
**Interactivity**
Exponential functions
**int-5959**

**Exponential functions** are functions of the form $f: R \to R, f(x) = a^x, a \in R^+\backslash\{1\}$. They provide mathematical models of exponential growth and exponential decay situations such as population increase and radioactive decay respectively.

## The graph of $y = a^x$ where $a > 1$

Before sketching such a graph, consider the table of values for the function with rule $y = 2^x$.

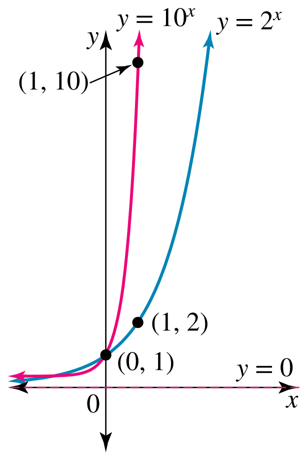
| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y = 2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ |




From the table it is evident that $2^x > 0$ for all values of $x$, and that as $x \to -\infty$, $2^x \to 0$. This means that the graph will have a horizontal asymptote with equation $y = 0$.

It is also evident that as $x \to \infty$, $2^x \to \infty$ with the values increasing rapidly.

Since these observations are true for any function $y = a^x$ where $a > 1$, the graph of $y = 2^x$ will be typical of the basic graph of any exponential with base larger than 1.



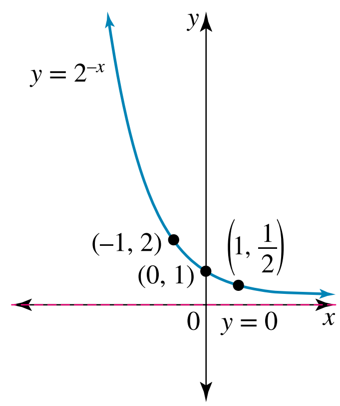

Key features of the graph of $y = 2^x$ and any such function $y = a^x$ where $a > 1$:

- horizontal asymptote with equation $y = 0$
- $y$-intercept is $(0, 1)$
- shape is of 'exponential growth'
- domain $R$
- range $R^+$
- one-to-one correspondence

For $y = 2^x$, the graph contains the point $(1, 2)$; for the graph of $y = a^x, a > 1$, the graph contains the point $(1, a)$, showing that as the base increases, the graph becomes steeper more quickly for values $x > 0$. This is illustrated by the graphs of $y = 2^x$ and $y = 10^x$, with the larger base giving the steeper graph.

## The graph of $y = a^x$ where $0 < a < 1$

An example of a function whose rule is in the form $y = a^x$ where $0 < a < 1$ is $y = \left(\frac{1}{2}\right)^x$. Since $\left(\frac{1}{2}\right)^x = 2^{-x}$, the rule for the graph of this exponential function $y = \left(\frac{1}{2}\right)^x$ where the base lies between 0 and 1 is identical to the rule $y = 2^{-x}$ where the base is greater than 1.

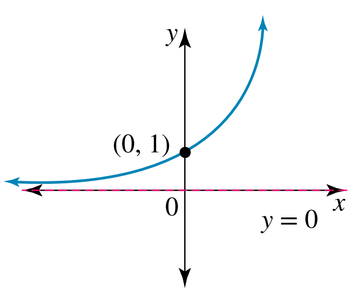



The graph of $y = 2^{-x}$ shown is typical of the graph of $y = a^{-x}$ where $a > 1$ and of the graph of $y = a^x$ where $0 < a < 1$.

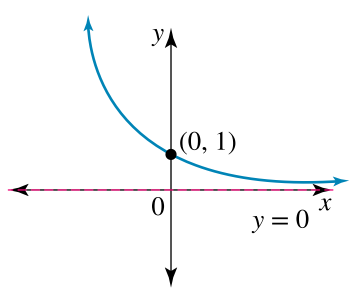
Key features of the graph of $y = 2^{-x}$ and any such function with rule expressed as either $y = a^x$ where $0 < a < 1$ or as $y = a^{-x}$ where $a > 1$:

- horizontal asymptote with equation $y = 0$
- $y$-intercept is $(0, 1)$
- shape is of 'exponential decay'
- domain $R$
- range $R^+$
- one-to-one correspondence
- reflection of $y = 2^x$ in the $y$-axis

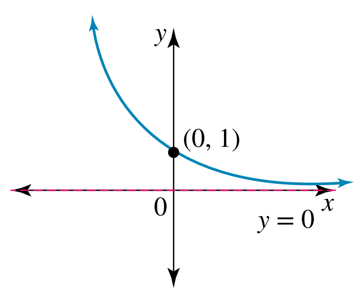
The basic shape of an exponential function is either one of 'growth' or 'decay'.




$y = a^x, a > 1$




$y = a^{-x}, a > 1$



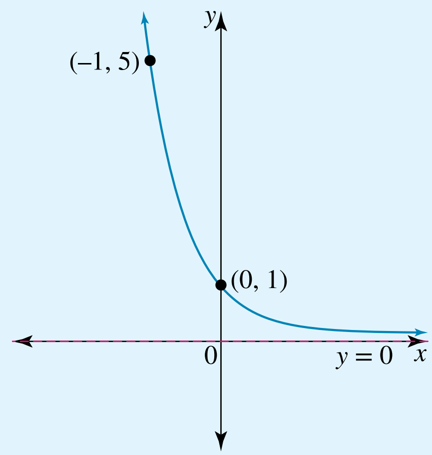

$y = a^x, 0 < a < 1$

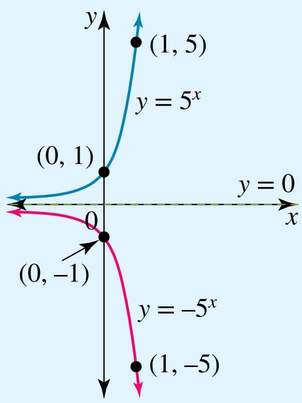
As with other functions, the graph of $y = -a^x$ will be inverted (reflected in the $x$-axis).

**WORKED EXAMPLE 9**

**a** On the same set of axes, sketch the graphs of $y = 5^x$ and $y = -5^x$, stating their ranges.

**b** Give a possible equation for the graph shown.




| THINK | WRITE |
|---|---|
| **a 1** Identify the asymptote of the first function. | **a** $y = 5^x$<br>The asymptote is the line with equation $y = 0$. |
| **2** Find the $y$-intercept. | $y$-intercept: when $x = 0$, $y = 1 \Rightarrow (0, 1)$ |
| **3** Calculate the coordinates of a second point. | Let $x = 1$.<br>$y = 5^1$<br>$= 5$<br>$\Rightarrow (1, 5)$ |
| **4** Use the relationship between the two functions to deduce the key features of the second function. | $y = -5^x$<br>This is the reflection of $y = 5^x$ in the $x$-axis.<br>The graph of $y = -5^x$ has the same asymptote as that of $y = 5^x$.<br>Equation of its asymptote is $y = 0$.<br>Its $y$-intercept is $(0, -1)$.<br>Point $(1, -5)$ lies on the graph. |
| **5** Sketch and label each graph. |  |
| **6** State the range of each graph. | The range of $y = 5^x$ is $R^+$ and the range of $y = -5^x$ is $R^-$. |
| **b 1** Use the shape of the graph to suggest a possible form for the rule. | **b** The graph has a 'decay' shape.<br>Let the equation be $y = a^{-x}$. |
| **2** Use a given point on the graph to calculate $a$. | The point $(-1, 5) \Rightarrow 5 = a^1$<br>$\therefore \quad a = 5$ |
| **3** State the equation of the graph. | The equation of the graph could be $y = 5^{-x}$.<br>The equation could also be expressed as $y = \left(\frac{1}{5}\right)^x$ or $y = 0.2^x$. |

## Translations of exponential graphs

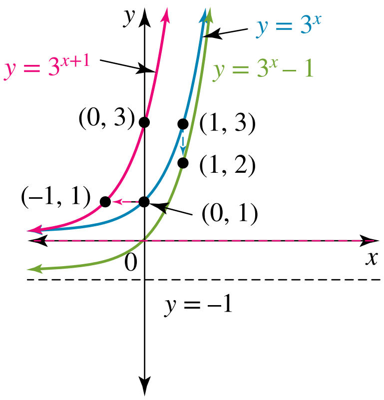
Once the basic exponential growth or exponential decay shapes are known, the graphs of exponential functions can be translated in similar ways to graphs of any other functions previously studied.

### The graph of $y = a^x + k$

Under a vertical translation the position of the asymptote will be altered to $y = k$. If $k < 0$, the graph will have $x$-axis intercepts which are found by solving the exponential equation $a^x + k = 0$.

### The graph of $y = a^{x-h}$

Under a horizontal translation the asymptote is unaffected. The point on the $y$-axis will no longer occur at $y = 1$. An additional point to the $y$-intercept that can be helpful to locate is the one where $x = h$, since $a^{x-h}$ will equal 1 when $x = h$.




A horizontal translation and a vertical translation of the graph of $y = 3^x$ are illustrated in the diagram by the graphs of $y = 3^{x+1}$ and $y = 3^x - 1$ respectively. Under the horizontal translation of 1 unit to the left, the point $(0, 1) \rightarrow (-1, 1)$; under the vertical translation of 1 unit down, the point $(1, 3) \rightarrow (1, 2)$.

**WORKED EXAMPLE 10**

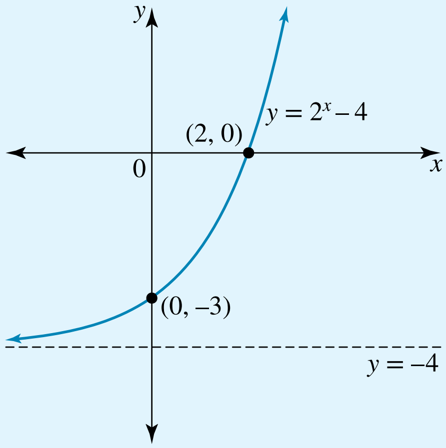
**Sketch the graphs of each of the following and state the range of each.**

**a** $y = 2^x - 4$  **b** $y = 10^{-(x+1)}$

| THINK | WRITE |
|---|---|
| **a 1** State the equation of the asymptote. | **a** $y = 2^x - 4$<br>The vertical translation 4 units down affects the asymptote.<br>The asymptote has the equation $y = -4$. |
| **2** Calculate the $y$-intercept. | $y$-intercept: let $x = 0$,<br>$y = 1 - 4$<br>$= -3$<br>$y$-intercept is $(0, -3)$. |
| **3** Calculate the $x$-intercept. | $x$-intercept: let $y = 0$,<br>$2^x - 4 = 0$<br>$\therefore 2^x = 4$<br>$\therefore 2^x = 2^2$<br>$\therefore x = 2$<br>$x$-intercept is $(2, 0)$. |

4 Sketch the graph and state the range.

A 'growth' shape is expected since the coefficient of $x$ is positive.




Range is $(-4, \infty)$.

**b** 1 Identify the key features from the given equation.

**b** $y = 10^{-(x+1)}$

Reflection in $y$-axis, horizontal translation 1 unit to the left. The asymptote will not be affected.

Asymptote: $y = 0$

There is no $x$-intercept.

$y$-intercept: let $x = 0$,

$$y = 10^{-1}$$
$$= \frac{1}{10}$$

$y$-intercept is $(0, 0.1)$.

2 Calculate the coordinates of a second point on the graph.

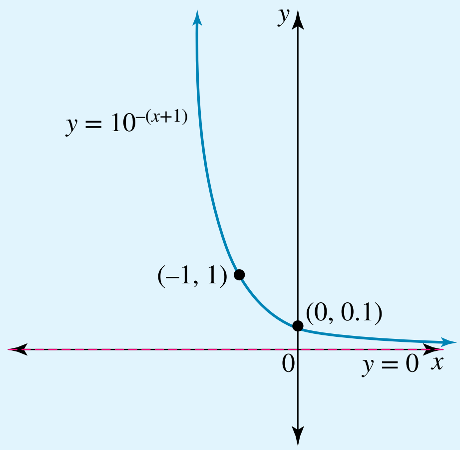
Let $x = -1$

$$y = 10^{0}$$
$$= 1$$

The point $(-1, 1)$ lies on the graph.

3 Sketch the graph and state the range.

A 'decay' shape is expected since the coefficient of $x$ is negative.



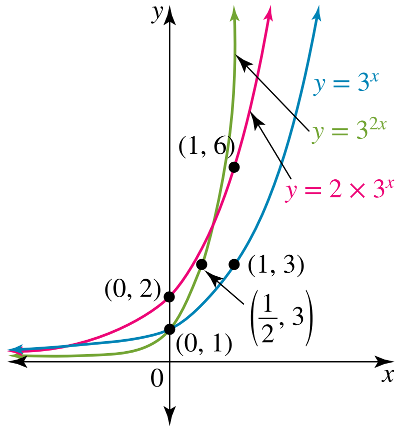

Range is $R^{+}$.

## Dilations

Exponential functions of the form $y = b \times a^x$ have been dilated by a factor $b$ $(b > 0)$ from the $x$-axis. This affects the $y$-intercept, but the asymptote remains at $y = 0$.

Exponential functions of the form $y = a^{nx}$ have been dilated by a factor $\frac{1}{n}$ $(n > 0)$ from the $y$-axis. This affects the steepness of the graph but does not affect either the $y$-intercept or the asymptote.

A dilation from the $x$-axis of factor 2 and a dilation from the $y$-axis of factor $\frac{1}{2}$ of the graph of $y = 3^x$ are illustrated in the diagram by the graphs of $y = 2 \times 3^x$ and $y = 3^{2x}$ respectively. Under the dilation from the $x$-axis of factor 2, the point $(1, 3) \rightarrow (1, 6)$; under the dilation from the $y$-axis of factor $\frac{1}{2}$, the point $(1, 3) \rightarrow \left(\frac{1}{2}, 3\right)$.




## Combinations of transformations

Exponential functions with equations of the form $y = b \times a^{n(x-h)} + k$ are derived from the basic graph of $y = a^x$ by applying a combination of transformations. The key features to identify in order to sketch the graphs of such exponential functions are:

- the asymptote
- the $y$-intercept
- the $x$-intercept, if there is one.

Another point that can be obtained simply could provide assurance about the shape. Always aim to show at least two points on the graph.

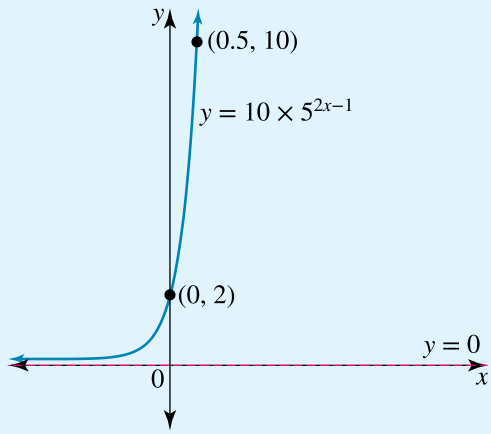
**WORKED EXAMPLE 11**

**Sketch the graphs of each of the following and state the range of each.**

**a** $y = 10 \times 5^{2x-1}$ **b** $y = 1 - 3 \times 2^{-x}$

| THINK | WRITE |
|---|---|
| **a 1** Identify the key features using the given equation. | **a** $y = 10 \times 5^{2x-1}$<br>asymptote: $y = 0$<br>no $x$-intercept<br>$y$-intercept: let $x = 0$<br>$y = 10 \times 5^{-1}$<br>$= 10 \times \frac{1}{5}$<br>$= 2$<br>$y$-intercept is $(0, 2)$. |
| **2** Calculate the coordinates of a second point. | Since the horizontal translation is $\frac{1}{2}$ to the right, let $x = \frac{1}{2}$.<br>$y = 10 \times 5^{2 \times \frac{1}{2} - 1}$<br>$= 10 \times 5^0$<br>$= 10 \times 1$<br>$= 10$<br>Point $\left(\frac{1}{2}, 10\right)$ lies on the graph. |

**3** Sketch the graph and state the range.




Range is $R^+$.

**b 1** Write the equation in the form $y = b \times a^{n(x-h)} + k$ and state the asymptote.

**b** $y = 1 - 3 \times 2^{-x}$

$\therefore y = -3 \times 2^{-x} + 1$

Asymptote: $y = 1$

**2** Calculate the $y$-intercept.

$y$-intercept: let $x = 0$

$$y = -3 \times 2^0 + 1$$
$$= -3 \times 1 + 1$$
$$= -2$$

$y$-intercept is $(0, -2)$.

**3** Calculate the $x$-intercept.

*Note:* As the point $(0, -2)$ lies below the asymptote and the graph must approach the asymptote, there will be an $x$-intercept.

$x$-intercept: let $y = 0$

$$0 = 1 - 3 \times 2^{-x}$$
$$2^{-x} = \frac{1}{3}$$

In logarithm form,

$$-x = \log_2\left(\frac{1}{3}\right)$$
$$= \log_2(3^{-1})$$
$$= -\log_2(3)$$
$$\therefore x = \log_2(3)$$

The exact $x$-intercept is $(\log_2(3), 0)$.

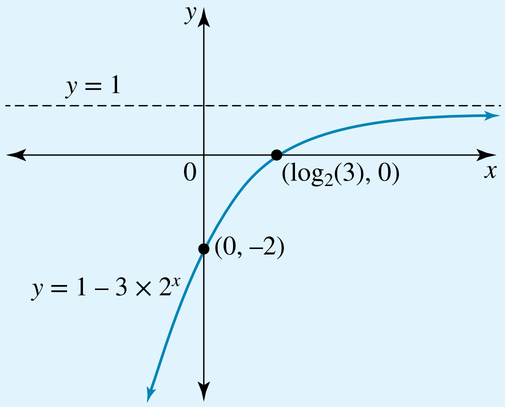
**4** Calculate an approximate value for the $x$-intercept to help determine its position on the graph.

$$x = \log_2(3)$$
$$= \frac{\log_{10}(3)}{\log_{10}(2)}$$
$$\approx 1.58$$

The $x$-intercept is approximately $(1.58, 0)$.

5 Sketch the graph and state the range.

*Note:* Label the $x$-intercept with its exact coordinates once the graph is drawn.




Range is $(-\infty, 1)$.

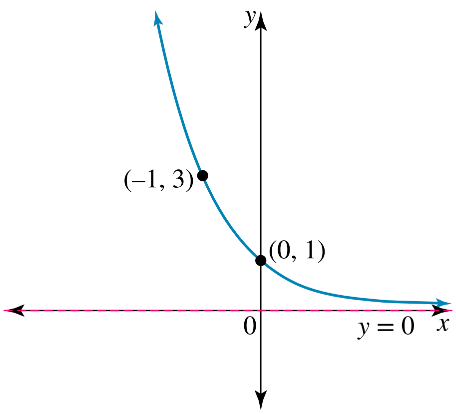
## EXERCISE 11.4 Graphs of exponential functions

**PRACTISE**

Work without CAS

1 WE9 **a** On the same set of axes, sketch the graphs of $y = 3^x$ and $y = -3^x$, stating their ranges.

**b** Give a possible equation for the graph shown.




2 Sketch the graphs of $y = (1.5)^x$ and $y = \left(\frac{2}{3}\right)^x$ on the same set of axes.

3 WE10 Sketch the graphs of each of the following and state the range of each.

**a** $y = 4^x - 2$

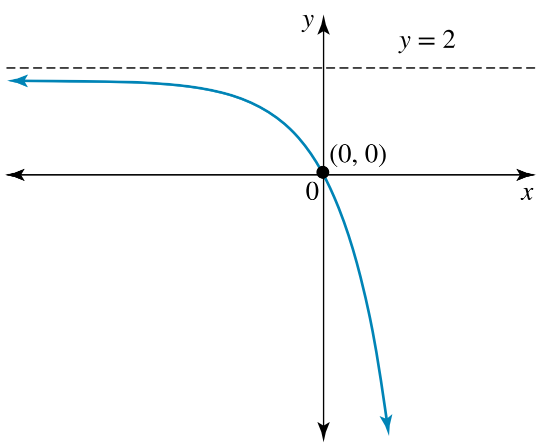
**b** $y = 3^{-(x+2)}$

4 Sketch the graph of $y = 4^{x-2} + 1$ and state its range.

5 WE11 Sketch the graphs of each of the following and state the range of each.

**a** $y = \frac{1}{2} \times 10^{1-2x}$ **b** $y = 5 - 4 \times 3^{-x}$

6 The graph shown has the equation $y = a.3^x + b$. Determine the values of $a$ and $b$.



**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**7 a i** Sketch, on the same set of axes, the graphs of $y = 4^x$, $y = 6^x$ and $y = 8^x$.

**ii** Describe the effect produced by increasing the base.

**b i** Sketch, on the same set of axes, the graphs of $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$.

**ii** Express each rule in a different form.

**8 a i** Sketch, on the same set of axes, the graphs of $y = 5^{-x}$, $y = 7^{-x}$ and $y = 9^{-x}$.

**ii** Describe the effect produced by increasing the base.

**b i** Sketch, on the same set of axes, the graphs of $y = (0.8)^x$, $y = (1.25)^x$ and $y = (0.8)^{-x}$.

**ii** Describe the relationships between the three graphs.

**9** Sketch each of the following graphs, showing the asymptote and labelling any intersections with the coordinate axes with their exact coordinates.

**a** $y = 5^{-x} + 1$
**b** $y = 1 - 4^x$
**c** $y = 10^x - 2$
**d** $y = 6.25 - (2.5)^{-x}$

**10** Sketch the graphs of:

**a** $y = 2^{x-2}$
**b** $y = -3^{x+2}$
**c** $y = 4^{x-0.5}$
**d** $y = 7^{1-x}$

**11** Sketch the graphs of:

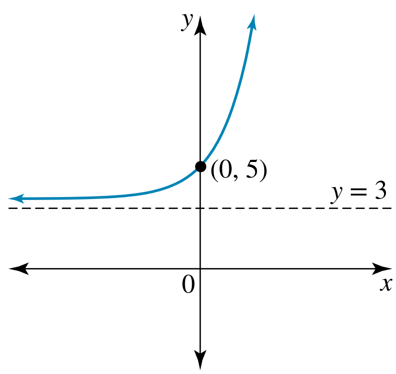
**a** $y = 3 \times 2^x$
**b** $y = 2^{\frac{3x}{4}}$
**c** $y = -3 \times 2^{-3x}$
**d** $y = 1.5 \times 10^{-\frac{x}{2}}$

**12 a** Sketch the graphs of $y = 3^{2x}$ and $y = 9^x$ and explain the result.

**b i** Use index laws to obtain another form of the rule for $y = 2 \times 4^{0.5x}$.

**ii** Hence or otherwise, sketch the graph of $y = 2 \times 4^{0.5x}$.

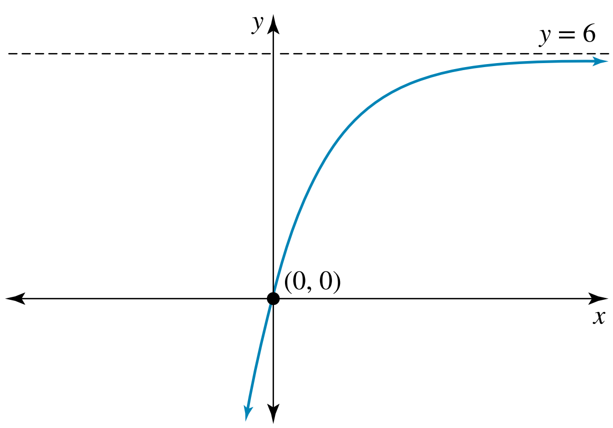
**13 a** Determine a possible rule for the given graph in the form $y = a.10^x + b$.




**b** The graph of an exponential function of the form $y = a.3^{kx}$ contains the points $(1, 36)$ and $(0, 4)$. Determine its rule and state the equation of its asymptote.

**c** For the graph shown, determine a possible rule in the form $y = a - 2 \times 3^{b-x}$.

**d** Express the equation given in part **c** in another form not involving a horizontal translation.



**14** Sketch the graphs of the following exponential functions and state their ranges. Where appropriate, any intersections with the coordinate axes should be given to 1 decimal place.

**a** $y = 2 \times 10^{2x} - 20$ **b** $y = 5 \times 2^{1-x} - 1$ **c** $y = 3 - 2\left(\frac{2}{3}\right)^x$

**d** $y = 2(3.5)^{x+1} - 7$ **e** $y = 8 - 4 \times 5^{2x-1}$ **f** $y = -2 \times 10^{3x-1} - 4$

**15** Consider the function $f: R \to R, f(x) = 3 - 6 \times 2^{\frac{x-1}{2}}$.

**a** Evaluate:

**i** $f(1)$

**ii** $f(0)$, expressing the answer in simplest surd form.

**b** For what value of $x$, if any, does:

**i** $f(x) = -9$

**ii** $f(x) = 0$

**iii** $f(x) = 9$?

**c** Sketch the graph of $y = f(x)$ and state its range.

**d** Solve the inequation $f(x) \geq -1$ correct to 2 significant figures.

**16** Use a graphical means to determine the number of intersections between:

**a** $y = 2^x$ and $y = -x$, specifying an interval in which the $x$-coordinate of any point of intersection lies.

**b** $y = 2^x$ and $y = x^2$

**c** $y = e^x$ and $y = 2^x$

**d** $y = 2^{-x} + 1$ and $y = \sin(x)$

**e** $y = 3 \times 2^x$ and $y = 6^x$, determining the coordinates of any points of intersection algebraically.

**f** $y = 2^{2x-1}$ and $y = \frac{1}{2} \times 16^{\frac{x}{2}}$, giving the coordinates of any points of intersection.

MASTER

**17** Obtain the coordinates of the points of intersection of $y = 2^x$ and $y = x^2$.

**18** Sketch the graphs of $y_1 = 33 - 2(11)^x$ and $y_2 = 33 - 2(11)^{x+1}$ and compare their asymptotes, $x$- and $y$-intercepts and the value of their $x$-coordinates when $y = 10$. What transformation maps $y_1$ to $y_2$?

# 11.5 Applications of exponential functions

study on
Units 1 & 2
AOS 1
Topic 8
Concept 4
**Applications of exponential functions**
Concept summary
Practice questions

The importance of exponential functions lies in the frequency with which they occur in models of phenomena involving growth and decay situations, in chemical and physical laws of nature and in higher-level mathematical analysis.

## Exponential growth and decay models

For time $t$, the exponential function defined by $y = b \times a^{nt}$ where $a > 1$ represents exponential growth over time if $n > 0$ and exponential decay over time if $n < 0$. The domain of this function would be restricted according to the way the independent time variable $t$ is defined. The rule $y = b \times a^{nt}$ may also be written as $y = b.a^{nt}$.

In some mathematical models such as population growth, the initial population may be represented by a symbol such as $N_0$. For an exponential decay model, the time it takes for 50% of the initial amount of the substance to decay is called its half-life.

**WORKED EXAMPLE 12**

**The decay of a radioactive substance is modelled by $Q(t) = Q_0 \times 2.7^{-kt}$ where $Q$ kg is the amount of the substance present at time $t$ years and $Q_0$ and $k$ are positive constants.**

**a** **Show that the constant $Q_0$ represents the initial amount of the substance.**

**b** **If the half-life of the radioactive substance is 100 years, calculate $k$ to one significant figure.**

**c** **If initially there was 25 kg of the radioactive substance, how many kilograms would decay in 10 years? Use the value of $k$ from part b in the calculations.**

| THINK | WRITE |
|---|---|
| **a 1** Calculate the initial amount. | **a** $Q(t) = Q_0 \times 2.7^{-kt}$<br>The initial amount is the value of $Q$ when $t = 0$.<br>Let $t = 0$:<br>$Q(0) = Q_0 \times 2.7^0$<br>$= Q_0$<br>Therefore $Q_0$ represents the initial amount of the substance. |
| **b 1** Form an equation in $k$ from the given information.<br>*Note:* It does not matter that the value of $Q_0$ is unknown since the $Q_0$ terms cancel. | **b** The half-life is the time it takes for 50% of the initial amount of the substance to decay.<br>Since the half-life is 100 years, when $t = 100$,<br>$Q(100) = 50\%$ of $Q_0$<br>$Q(100) = 0.50Q_0 \ldots\ldots (1)$<br>From the equation, $Q(t) = Q_0 \times 2.7^{-kt}$.<br>When $t = 100$, $Q(100) = Q_0 \times 2.7^{-k(100)}$<br>$\therefore Q(100) = Q_0 \times 2.7^{-100k} \ldots\ldots (2)$<br>Equate equations (1) and (2):<br>$0.50Q_0 = Q_0 \times 2.7^{-100k}$<br>Cancel $Q_0$ from each side:<br>$0.50 = 2.7^{-100k}$ |
| **2** Solve the exponential equation to obtain $k$ to the required accuracy. | Convert to the equivalent logarithm form.<br>$-100k = \log_{2.7}(0.5)$<br>$k = -\frac{1}{100}\log_{2.7}(0.5)$<br>$= -\frac{1}{100} \times \frac{\log_{10}(0.5)}{\log_{10}(2.7)}$<br>$\approx 0.007$ |
| **c 1** Use the values of the constants to state the actual rule for the exponential decay model. | **c** $Q_0 = 25,\ k = 0.007$<br>$\therefore Q(t) = 25 \times 2.7^{-0.007t}$ |
| **2** Calculate the amount of the substance present at the time given. | When $t = 10$,<br>$Q(10) = 25 \times 2.7^{-0.07}$<br>$\approx 23.32$ |

3 Calculate the amount that has decayed.

*Note:* Using a greater accuracy for the value of $k$ would give a slightly different answer for the amount decayed.

Since $25 - 23.32 = 1.68$, in 10 years approximately 1.68 kg will have decayed.

## Analysing data

One method for detecting if data has an exponential relationship can be carried out using logarithms. If the data is suspected of following an exponential rule such as $y = A \times 10^{kx}$, then the graph of $\log(y)$ against $x$ should be linear. The reasoning for this is as follows:

$$\begin{aligned} y &= A \times 10^{kx} \\ \therefore \frac{y}{A} &= 10^{kx} \\ \therefore \log\left(\frac{y}{A}\right) &= kx \\ \therefore \log(y) - \log(A) &= kx \\ \therefore \log(y) &= kx + \log(A) \end{aligned}$$

This equation can be written in the form $Y = kx + c$ where $Y = \log(y)$ and $c = \log(A)$.

The graph of $Y$ versus $x$ is a straight line with gradient $k$ and vertical axis $Y$-intercept $(0, \log(A))$.

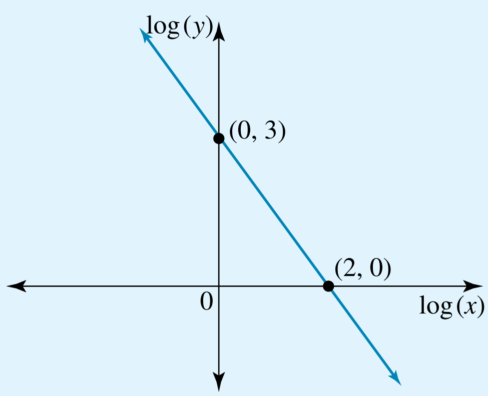
Such an analysis is called a **semi-log plot**. While experimental data is unlikely to give a perfect fit, the equation would describe the line of best fit for the data.

Logarithms can also be effective in determining a power law that connects variables. If the law connecting the variables is of the form $y = x^p$ then $\log(y) = p\log(x)$. Plotting $\log(y)$ values against $\log(x)$ values will give a straight line of gradient $p$ if the data does follow such a law. Such an analysis is called a **log-log plot**.

**WORKED EXAMPLE 13**

**For a set of data $\{(x, y)\}$, plotting $\log(y)$ versus $\log(x)$ gave the straight line shown in the diagram.**

**Form the equation of the graph and hence determine the rule connecting $y$ and $x$.**




**THINK**

1 State the gradient and the coordinates of the intercept with the vertical axis.

**WRITE**

$$\text{Gradient} = \frac{\text{rise}}{\text{run}} = \frac{-3}{2}$$

Intercept with vertical axis: $(0, 3)$

2 Form the equation of the line.

Let $Y = \log(y)$ and $X = \log(x)$.

The equation of the line is $Y = mX + c$ where $m = -\frac{3}{2}, c = 3$.

Therefore the equation of the line is $Y = -\frac{3}{2}X + 3$.

3 Express the equation in terms of the variables marked on the axes of the given graph.

The vertical axis is $\log(y)$ and the horizontal axis is $\log(x)$, so the equation of the graph is

$$\log(y) = -\frac{3}{2}\log(x) + 3.$$

$$\therefore \log(y) = -1.5\log(x) + 3$$

4 Collect the terms involving logarithms together and simplify to create a logarithm statement.

$$\log(y) + 1.5\log(x) = 3$$

$$\log(y) + \log(x^{1.5}) = 3$$

$$\log(yx^{1.5}) = 3$$

5 Express the equation with $y$ as the subject.

*Note:* Remember the base of the logarithm is 10.

$$\therefore \log_{10}(yx^{1.5}) = 3$$

$$yx^{1.5} = 10^3$$

$$y = 1000x^{-1.5}$$

## EXERCISE 11.5 Applications of exponential functions

**PRACTISE**

1 **WE12** The decay of a radioactive substance is modelled by $Q(t) = Q_0 \times 1.7^{-kt}$ where $Q$ is the amount of the substance present at time $t$ years and $Q_0$ and $k$ are positive constants.

a Show that the constant $Q_0$ represents the initial amount of the substance.

b If the half-life of the radioactive substance is 300 years, calculate $k$ to one significant figure.

c If initially there was 250 kg of the radioactive substance, how many kilograms would decay in 10 years? Use the value of $k$ from part **b** in the calculations.

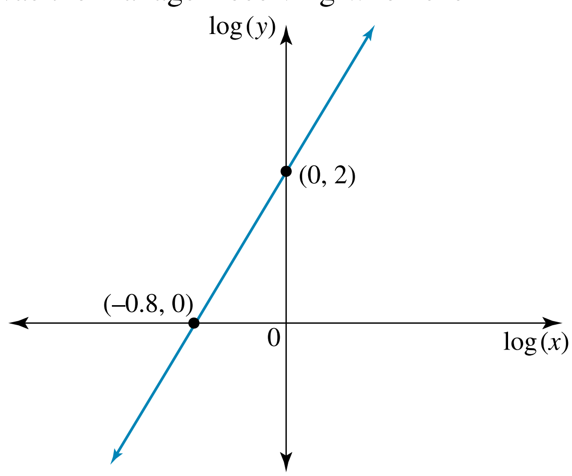
2 The manager of a small business is concerned about the amount of time she spends dealing with the growing number of emails she receives. The manager starts keeping records and finds the average number of emails received per day can be modelled by $D = 42 \times 2^{\frac{t}{16}}$ where $D$ is the average number of emails received per day $t$ weeks from the start of the records.

a How many daily emails on average was the manager receiving when she commenced her records?

b After how many weeks does the model predict that the average number of emails received per day will double?

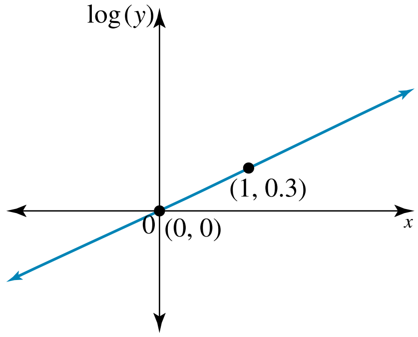
3 **WE13** For a set of data $\{(x, y)\}$, plotting $\log(y)$ versus $\log(x)$ gave the straight line shown in the diagram.

From the equation of the graph and hence determine the rule connecting $y$ and $x$.



4 For a set of data $\{(x, y)\}$, the semi-log plot of $\log(y)$ versus $x$ gave the straight line shown in the diagram.

Form the equation of the graph and hence determine an exponential rule connecting $y$ and $x$.




## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 The value $V$ of a new car depreciates so that its value after $t$ years is given by $V = V_0 \times 2^{-kt}$.

a If 50% of the purchase value is lost in 5 years, calculate $k$.

b How long does it take for the car to lose 75% of its purchase value?

6 The number of drosophilae (fruit flies), $N$, in a colony after $t$ days of observation is modelled by $N = 30 \times 2^{0.072t}$. Give whole-number answers to the following.

a How many drosophilae were present when the colony was initially observed?

b How many of the insects were present after 5 days?

c How many days does it take the population number to double from its initial value?

d Sketch a graph of $N$ versus $t$ to show how the population changes.

e After how many days will the population first exceed 100?

7 The value of an investment which earns compound interest can be calculated from the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ where $P$ is the initial investment, $r$ the interest rate per annum (yearly), $n$ the number of times per year the interest is compounded and $t$ the number of years of the investment.

An investor deposits \$2000 in an account where interest is compounded monthly.

a If the interest rate is 3% per annum:

i Show that the formula giving the value of the investment is $A = 2000(1.0025)^{12t}$.

ii Calculate how much the investment is worth after a 6-month period.

iii What time period would be needed for the value of the investment to reach \$2500?

b The investor would like the \$2000 to grow to \$2500 in a shorter time period. What would the interest rate, still compounded monthly, need to be for the goal to be achieved in 4 years?

8 A cup of coffee is left to cool on a kitchen table inside a Brisbane home. The temperature of the coffee $T$ (°C) after $t$ minutes is thought to be given by $T = 85 \times 3^{-0.008t}$.

a By how many degrees does the coffee cool in 10 minutes?

b How long does it take for the coffee to cool to 65 °C?

c Sketch a graph of the temperature of the coffee for $t \in [0, 40]$.

d By considering the temperature the model predicts the coffee will eventually cool to, explain why the model is not realistic in the long term.

9 The contents of a meat pie immediately after being heated in a microwave have a temperature of 95 °C. The pie is removed from the microwave and left to cool. A model for the temperature of the pie as it cools is given by $T = a \times 3^{-0.13t} + 25$ where $T$ is the temperature after $t$ minutes of cooling.

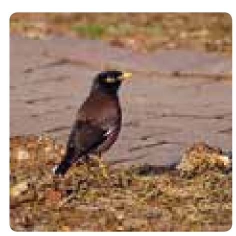
a Calculate the value of $a$.
b What is the temperature of the contents of the pie after being left to cool for 2 minutes?
c Determine how long, to the nearest minute, it will take for the contents of the meat pie to cool to 65 °C.
d Sketch the graph showing the temperature over time and state the temperature to which this model predicts the contents of the pie will eventually cool if left unattended.

10 The barometric pressure $P$, measured in kilopascals, at height $h$ above sea level, measured in kilometres, is given by $P = P_o \times 10^{-kh}$ where $P_o$ and $k$ are positive constants. The pressure at the top of Mount Everest is approximately one third that of the pressure at sea level.

a Given the height of Mount Everest is approximately 8848 metres, calculate the value of $k$ to 2 significant figures.
Use the value obtained for $k$ for the remainder of question 6.
b Mount Kilimanjaro has a height of approximately 5895 metres. If the atmospheric pressure at its summit is approximately 48.68 kilopascals, calculate the value of $P_0$ to 3 decimal places.
c Use the model to estimate the atmospheric pressure to 2 decimal places at the summit of Mont Blanc, 4810 metres, and of Mount Kosciuszko, 2228 metres in height.
d Draw a graph of the atmospheric pressure against height showing the readings for the four mountains from the above information.



11 The common Indian mynah bird was introduced into Australia in order to control insects affecting market gardens in Melbourne. It is now considered to be Australia's most important pest problem. In 1976, the species was introduced to an urban area in New South Wales. By 1991 the area averaged 15 birds per square kilometre and by 1994 the density reached an average of 75 birds per square kilometre.



A model for the increasing density of the mynah bird population is thought to be $D = D_0 \times 10^{kt}$ where $D$ is the average density of the bird per square kilometre $t$ years after 1976 and $D_0$ and $k$ are constants.

**a** Use the given information to set up a pair of simultaneous equations in $D$ and $t$.

**b** Solve these equations to show that $k = \frac{1}{3}\log(5)$ and $D_0 = 3 \times 5^{-4}$ and hence that $k \approx 0.233$ and $D_0 \approx 0.005$.

**c** A project was introduced in 1996 to curb the growth in numbers of these birds. What does the model predict was the average density of the mynah bird population at the time the project was introduced in the year 1996? Use $k \approx 0.233$ and $D_0 \approx 0.005$ and round the answer to the nearest whole number.

**d** Sometime after the project is successfully implemented, a different model for the average density of the bird population becomes applicable. This model is given by $D = 30 \times 10^{-\frac{t}{3}} + b$. Four years later, the average density is reduced to 40 birds per square kilometre. How much can the average density expect to be reduced?

**12** Carbon dating enables estimates of the age of fossils of once living organisms to be ascertained by comparing the amount of the radioactive isotope carbon-14 remaining in the fossil with the normal amount present in the living entity, which can be assumed to remain constant during the organism's life. It is known that carbon-14 decays with a half-life of approximately 5730 years according to an exponential model of the form $C = C_o \times \left(\frac{1}{2}\right)^{kt}$, where $C$ is the amount of the isotope remaining in the fossil $t$ years after death and $C_o$ is the normal amount of the isotope that would have been present when the organism was alive.

**a** Calculate the exact value of the positive constant $k$.

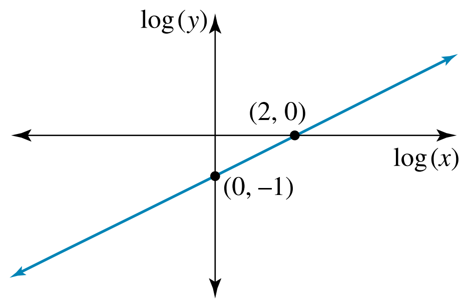
**b** The bones of an animal are unearthed during digging explorations by a mining company. The bones are found to contain 83% of the normal amount of the isotope carbon-14. Estimate how old the bones are.

**13 a** Obtain the equation of the given linear graphs and hence determine the relationship between $y$ and $x$.

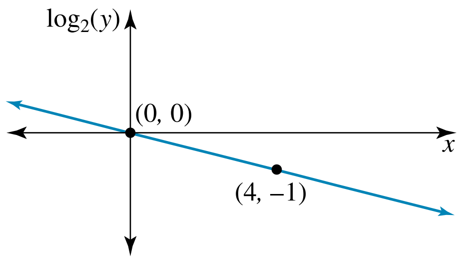
**i** The linear graph of $\log_{10}(y)$ against $\log_{10}(x)$ is shown.

**ii** The linear graph of $\log_2(y)$ against $x$ is shown.

i




ii




**b** The acidity of a solution is due to the presence of hydrogen ions. The concentration of these ions is measured by the pH scale calculated as $\text{pH} = -\log([H^+])$ where $[H^+]$ is the concentration of hydrogen ions.

**i** The concentration of hydrogen ions in bleach is $10^{-13}$ per mole and in pure water the concentration is $10^{-7}$ per mole. What are the pH readings for bleach and for pure water?

ii Lemon juice has a pH reading of 2 and milk has a pH reading of 6. Use scientific notation to express the concentration of hydrogen ions in each of lemon juice and milk and then write these concentrations as numerals.

iii Solutions with pH smaller than 7 are acidic and those with pH greater than 7 are alkaline. Pure water is neutral. How much more acidic is lemon juice than milk?

iv For each one unit of change in pH, explain the effect on the concentration of hydrogen ions and acidity of a solution.

14 The data shown in the table gives the population of Australia, in millions, in years since 1960.

| | 1975 | 1990 | 2013 |
|---|---|---|---|
| $x$ (years since 1960) | 15 | 30 | 53 |
| $y$ (population in millions) | 13.9 | 17.1 | 22.9 |
| $\log(y)$ | | | |

a Complete the third row of the table by evaluating the $\log(y)$ values to 2 decimal places.

b Plot $\log(y)$ against $x$ and construct a straight line to fit the points.

c Show that the equation of the line is approximately $Y = 0.006x + 1.05$ where $Y = \log(y)$.

d Use the equation of the line to show that the exponential rule between $y$ and $x$ is approximately $y = 11.22 \times 10^{0.006x}$.


e After how many years did the population double the 1960 population?

f It is said that the population of Australia is likely to exceed 28 million by the year 2030. Does this model support this claim?

MASTER

15 Experimental data yielded the following table of values:

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | 5.519 | 6.483 | 7.615 | 8.994 | 10.506 | 12.341 | 14.496 |

a Enter the data as lists in the Statistics menu and obtain the rule connecting the data by selecting the following from the Calc menu:

i Exponential Reg

ii Logarithmic Reg

b Graph the data on the calculator to confirm which rule better fits the data.

16 Following a fall from his bike, Stephan is feeling some shock but not, initially, a great deal of pain. However, his doctor gives him an injection for relief from the pain that he will start to feel once the shock of the accident wears off. The amount of pain Stephan feels over the next 10 minutes is modelled by the function

$P(t) = (200t + 16) \times 2.7^{-t}$, where $P$ is the measure of pain on a scale from 0 to 100 that Stephan feels $t$ minutes after receiving the injection.

**a** Give the measure of pain Stephan is feeling:
  **i** at the time the injection is administered
  **ii** 15 seconds later when his shock is wearing off but the injection has not reached its full effect.

**b** Use technology to draw the graph showing Stephan's pain level over the 10-minute interval and hence give, to 2 decimal places:
  **i** the maximum measure of pain he feels
  **ii** the number of seconds it takes for the injection to start lowering his pain level
  **iii** his pain levels after 5 minutes and after 10 minutes have elapsed.

**c** Over the 10-minute interval, when was the effectiveness of the injection greatest?

**d** At the end of the 10 minutes, Stephan receives a second injection modelled by $P(t) = (100(t - 10) + a) \times 2.7^{-(t-10)}$, $10 \le t \le 20$.
  **i** Determine the value of $a$.
  **ii** Sketch the pain measure over the time interval $t \in [0, 20]$ and label the maximum points with their coordinates.

# 11.6 Inverses of exponential functions

studyon

Units 1 & 2
AOS 1
Topic 8
Concept 5

**Inverses of exponential functions**
Concept summary
Practice questions

## The inverse of $y = a^x$, $a \in R^+ \backslash \{1\}$

The exponential function has a one-to-one correspondence so its inverse must also be a function. To form the inverse of $y = a^x$, interchange the $x$- and $y$-coordinates.

function: $y = a^x$ domain $R$, range $R^+$

inverse function: $x = a^y$ domain $R^+$, range $R$

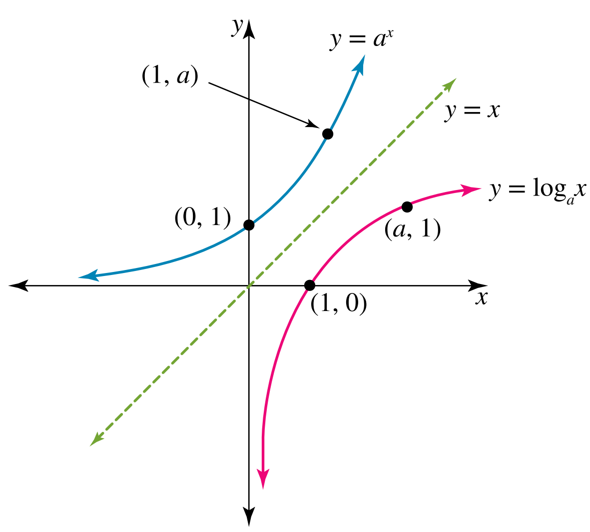
$$\therefore y = \log_a(x)$$

Therefore, the inverse of an exponential function is a logarithmic function: $y = \log_a(x)$ and $y = a^x$ are the rules for a pair of inverse functions. These are transcendental functions, not algebraic functions. However, they can be treated similarly to the inverse pairs of algebraic functions previously encountered. This means the graph of $y = \log_a(x)$ can be obtained by reflecting the graph of $y = a^x$ in the line $y = x$.

## The graph of $y = \log_a(x)$, for $a > 1$

The shape of the basic logarithmic graph with rule $y = \log_a(x)$, $a > 1$ is shown as the reflection in the line $y = x$ of the exponential graph with rule $y = a^x$, $a > 1$.

The key features of the graph of $y = \log_a(x)$ can be deduced from those of the exponential graph.



| $y = a^x$ | $y = \log_a(x)$ |
|---|---|
| horizontal asymptote with equation $y = 0$ | vertical asymptote with equation $x = 0$ |
| $x$-intercept $(1, 0)$ | $y$-intercept $(0, 1)$ |
| point $(1, a)$ lies on the graph | point $(a, 1)$ lies on the graph |
| range $R^+$ | domain $R^+$ |
| domain $R$ | range $R$ |
| one-to-one correspondence | one-to-one correspondence |

Note that logarithmic growth is much slower than exponential growth and also note that, unlike $a^x$ which is always positive, $\log_a(x) \begin{cases} > 0, & \text{if } x > 1 \\ = 0, & \text{if } x = 1 \\ < 0, & \text{if } 0 < x < 1 \end{cases}$

The logarithmic function is formally written as $f: R^+ \to R,\ f(x) = \log_a(x)$.

**WORKED EXAMPLE 14**

**a** **Form the exponential rule for the inverse of $y = \log_2(x)$ and hence deduce the graph of $y = \log_2(x)$ from the graph of the exponential.**

**b** **Given the points $(1, 2)$, $(2, 4)$ and $(3, 8)$ lie on the exponential graph in part a, explain how these points can be used to illustrate the logarithm law $\log_2(m) + \log_2(n) = \log_2(mn)$.**

**THINK**

**a** 1 Form the rule for the inverse by interchanging coordinates and then make $y$ the subject of the rule.

2 Sketch the exponential function.

**WRITE**

**a** $y = \log_2(x)$

Inverse: $x = \log_2(y)$

$\therefore y = 2^x$

$y = 2^x$

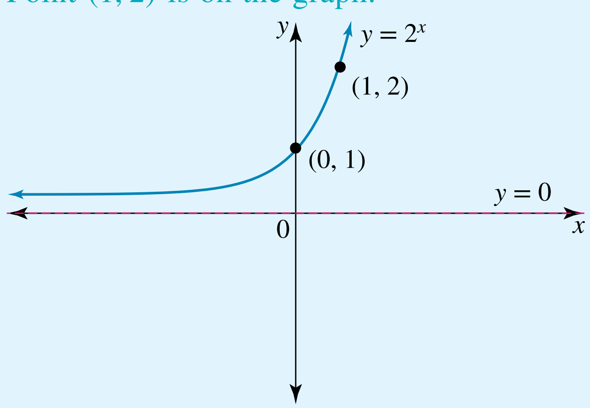
Asymptote: $y = 0$

$y$-intercept: $(0, 1)$

second point: let $x = 1$,

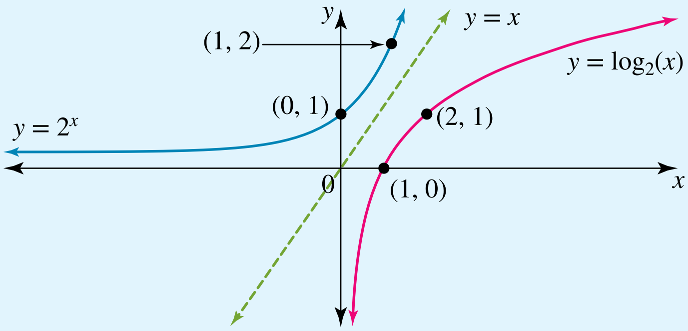
$\therefore y = 2$

Point $(1, 2)$ is on the graph.



3 Reflect the exponential graph in the line $y = x$ to form the required graph.

$y = \log_2(x)$ has:
asymptote: $x = 0$
$x$-intercept: $(1, 0)$
second point: $(2, 1)$




**b** 1 State the coordinates of the corresponding points on the logarithm graph.

**b** Given the points $(1, 2)$, $(2, 4)$ and $(3, 8)$ lie on the exponential graph, the points $(2, 1)$, $(4, 2)$ and $(8, 3)$ lie on the graph of $y = \log_2(x)$.

2 State the $x$- and $y$-values for each of the points on the logarithmic graph.

$y = \log_2(x)$
point $(2, 1)$: when $x = 2, y = 1$
point $(4, 2)$: when $x = 4, y = 2$
point $(8, 3)$: when $x = 8, y = 3$

3 Use the relationship between the $y$-coordinates to illustrate the logarithm law.

The sum of the $y$-coordinates of the points on $y = \log_2(x)$ when $x = 2$ and $x = 4$ equals the $y$-coordinate of the point on $y = \log_2(x)$ when $x = 8$, as $1 + 2 = 3$.

$\log_2(2) + \log_2(4) = \log_2(8)$

$\log_2(2) + \log_2(4) = \log_2(2 \times 4)$

This illustrates the logarithm law
$\log_2(m) + \log_2(n) = \log_2(mn)$
with $m = 2$ and $n = 4$.

## The inverse of exponential functions of the form $y = b \times a^{n(x-h)} + k$

The rule for the inverse of $y = b \times a^{n(x-h)} + k$ is calculated in the usual way by interchanging $x$- and $y$-coordinates to give $x = b \times a^{n(y-h)} + k$.

Expressing this equation as an index statement:

$$x - k = b \times a^{n(y-h)}$$

$$\frac{x-k}{b} = a^{n(y-h)}$$

Converting to the equivalent logarithm form:

$$\log_a\left(\frac{x-k}{b}\right) = n(y-h)$$

Rearranging to make $y$ the subject:

$$\frac{1}{n}\log_a\left(\frac{x-k}{b}\right) = y - h$$

$$y = \frac{1}{n}\log_a\left(\frac{x-k}{b}\right) + h$$

The inverse of any exponential function is a logarithmic function. Since the corresponding pair of graphs of these functions must be symmetric about the line $y = x$, this provides one approach for sketching the graph of any logarithmic function.

In this section, the graphs of logarithmic functions are obtained by deduction using the previously studied key features of exponential functions under a sequence of transformations. Should either the exponential or the logarithmic graph intersect the line $y = x$ then the other graph must also intersect that line at exactly the same point. Due to the transcendental nature of these functions, technology is usually required to obtain the coordinates of any such point of intersection.

**WORKED EXAMPLE 15**

**Consider the function $f: R \to R, f(x) = 5 \times 2^{-x} + 3$.**

**a What is the domain of its inverse?**

**b Form the rule for the inverse function and express the inverse function as a mapping.**

**c Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.**

| THINK | WRITE |
|---|---|
| **a 1** Determine the range of the given function. | **a** $f(x) = 5 \times 2^{-x} + 3$<br>Asymptote: $y = 3$<br>$y$-intercept: $(0, 8)$<br>This point lies above the asymptote, so the range of $f$ is $(3, \infty)$. |
| **2** State the domain of the inverse. | The domain of the inverse function is the range of the given function.<br>The domain of the inverse function is $(3, \infty)$. |
| **b 1** Form the rule for the inverse function by interchanging $x$- and $y$-coordinates and rearranging the equation obtained.<br>*Note:* Remember to express the rule for the inverse function $f^{-1}$ with $f^{-1}(x)$ in place of $y$ as its subject. | **b** Let $f(x) = y$.<br>Function: $y = 5 \times 2^{-x} + 3$<br>Inverse:<br>$x = 5 \times 2^{-y} + 3$<br>$\therefore x - 3 = 5 \times 2^{-y}$<br>$\therefore \frac{x-3}{5} = 2^{-y}$<br>Converting to logarithm form:<br>$-y = \log_2\left(\frac{x-3}{5}\right)$<br>$\therefore y = -\log_2\left(\frac{x-3}{5}\right)$<br>The inverse function has the rule $f^{-1}(x) = -\log_2\left(\frac{x-3}{5}\right)$. |

**2** Write the inverse function as a mapping.

The inverse function has domain $(3, \infty)$ and its rule is $f^{-1}(x) = -\log_2\left(\frac{x-3}{5}\right)$.

Hence, as a mapping:

$f^{-1}: (3, \infty) \to R, f^{-1}(x) = -\log_2\left(\frac{x-3}{5}\right)$

**c 1** Sketch the graph of the exponential function $f$ and use this to deduce the graph of the inverse function $f^{-1}$.

**c** $f(x) = 5 \times 2^{-x} + 3,\ f^{-1}(x) = -\log_2\left(\frac{x-3}{5}\right)$

The key features of $f$ determine the key features of $f^{-1}$.

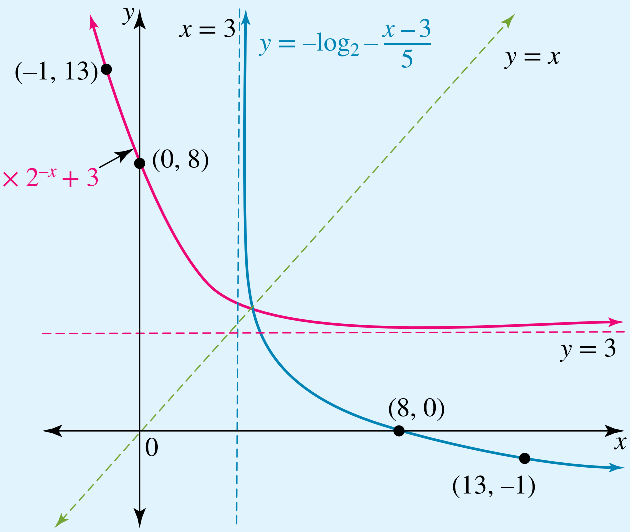
| | |
|---|---|
| $y = f(x)$ | $y = f^{-1}(x)$ |
| asymptote: $y = 3$ | asymptote: $x = 3$ |
| $y$-intercept $(0, 8)$ | $x$-intercept $(8, 0)$ |

second point on $y = f(x)$:

let $x = -1$

$$y = 5 \times 2^1 + 3$$
$$= 13$$

Point $(-1, 13)$ is on $y = f(x)$ and the point $(13, -1)$ is on $y = f^{-1}(x)$.




The graphs intersect on the line $y = x$.

## Relationships between the inverse pairs

As the exponential and logarithmic functions are a pair of inverses, each 'undoes' the effect of the other. From this it follows that:

$$\log_a(a^x) = x \text{ and } a^{\log_a(x)} = x$$

The first of these statements could also be explained using logarithm laws:

$$\log_a(a^x) = x\log_a(a)$$
$$= x \times 1$$
$$= x$$

The second statement can also be explained from the index-logarithm definition that $a^n = x \Leftrightarrow n = \log_a(x)$. Replacing $n$ by its logarithm form in the definition gives:

$$a^n = x$$
$$a^{\log_a(x)} = x$$

**WORKED EXAMPLE 16**

**a** Simplify $\log_{12}(2^{2x} \times 3^x)$ using the inverse relationship between exponentials and logarithms.

**b** Evaluate $10^{2\log_{10}(5)}$.

| THINK | WRITE |
|---|---|
| **a 1** Use index laws to simplify the product of powers given in the logarithm expression. | **a** $\log_{12}(2^{2x} \times 3^x)$<br>Consider the product $2^{2x} \times 3^x$.<br>$2^{2x} \times 3^x = (2^2)^x \times 3^x$<br>$= 4^x \times 3^x$<br>$= (4 \times 3)^x$<br>$= 12^x$ |
| **2** Simplify the given expression. | $\log_{12}(2^{2x} \times 3^x) = \log_{12}(12^x)$<br>$= x$ |
| **b 1** Apply a logarithm law to the term in the index. | **b** $10^{2\log_{10}(5)}$<br>$= 10^{\log_{10}(5)^2}$ |
| **2** Simplify the expression. | $= 10^{\log_{10}(25)}$<br>$= 25$<br>Therefore, $10^{2\log_{10}(5)} = 25$. |

## Transformations of logarithmic graphs

Knowledge of the transformations of graphs enables the graph of any logarithmic function to be obtained from the basic graph of $y = \log_a(x)$. This provides an alternative to sketching the graph as the inverse of that of an exponential function. Further, given the logarithmic graph, the exponential graph could be obtained as the inverse of the logarithmic graph.

The logarithmic graph under a combination of transformations will be studied in Units 3 and 4. In this section we shall consider the effect a single transformation has on the key features of the graph of $y = \log_a(x)$.

### Dilations

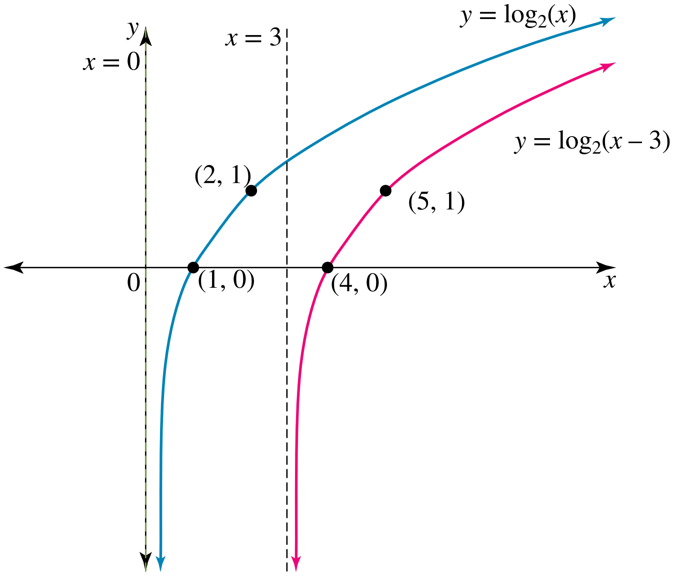
Dilations from either coordinate axis are recognisable from the equation of the logarithmic function: for example, $y = 2\log_a(x)$ and $y = \log_a\left(\frac{x}{2}\right)$ would give the images when $y = \log_a(x)$ undergoes a dilation of factor 2 from the $x$-axis and from the $y$-axis respectively. The asymptote at $x = 0$ would be unaffected by either dilation. The position of the $x$-intercept is affected by the dilation from the $y$-axis as $(1, 0) \rightarrow (2, 0)$. The dilation from the $x$-axis does not affect the $x$-intercept.

### Horizontal translation: the graph of $y = \log_a(x - h)$

The vertical asymptote will always be affected by a horizontal translation and this affects the domain of the logarithmic function. Under a horizontal translation of $h$ units to the right or left, the vertical asymptote at $x = 0$ must move $h$ units to the right or left respectively. Hence, horizontally translating the graph of $y = \log_a(x)$ by $h$ units to obtain the graph of $y = \log_a(x - h)$ produces the following changes to the key features:

- equation of asymptote: $x = 0 \rightarrow x = h$
- domain: $\{x : x > 0\} \rightarrow \{x : x > h\}$
- $x$-intercept: $(1, 0) \rightarrow (1 + h, 0)$

These changes are illustrated in the diagram by the graph of $y = \log_2(x)$ and its image, $y = \log_2(x - 3)$, after a horizontal translation of 3 units to the right.




The diagram shows that the domain of $y = \log_2(x - 3)$ is $(3, \infty)$. Its range is unaffected by the horizontal translation and remains $R$.

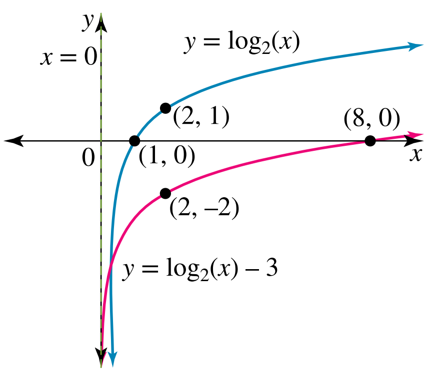
It is important to realise that the domain and the asymptote position can be calculated algebraically, since we only take logarithms of positive numbers. For example, the domain of $y = \log_2(x - 3)$ can be calculated by solving the inequation $x - 3 > 0 \Rightarrow x > 3$. This means that the domain is $(3, \infty)$ as the diagram shows. The equation of the asymptote of $y = \log_2(x - 3)$ can be calculated from the equation $x - 3 = 0 \Rightarrow x = 3$.

The function defined by $y = \log_a(nx + c)$ would have a vertical asymptote when $nx + c = 0$ and its domain can be calculated by solving $nx + c > 0$.

### Vertical translation: the graph of $y = \log_a(x) + k$

Under a vertical translation of $k$ units, neither the domain nor the position of the asymptote alters from that of $y = \log_a(x)$. The translated graph will have an $x$-intercept which can be obtained by solving the equation $\log_a(x) + k = 0$.

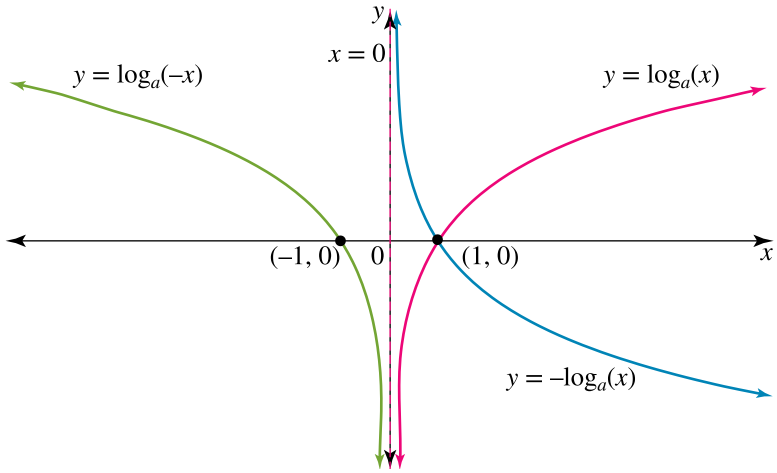
The graph of $y = \log_2(x) - 3$ is a vertical translation down by 3 units of the graph of $y = \log_2(x)$. Solving $\log_2(x) - 3 = 0$ gives $x = 2^3$ so the graph cuts the $x$-axis at $x = 8$, as illustrated.




### Reflections: the graphs of $y = -\log_a(x)$ and $y = \log_a(-x)$

The graph of $y = -\log_a(x)$ is obtained by inverting the graph of $y = \log_a(x)$; that is, by reflecting it in the $x$-axis.

The graph of $y = \log_a(-x)$ is obtained by reflecting the graph of $y = \log_a(x)$ in the $y$-axis. For $\log_a(-x)$ to be defined, $-x > 0$ so the graph has domain $\{x : x < 0\}$.




The relative positions of the graphs of $y = \log_a(x)$, $y = -\log_a(x)$ and $y = \log_a(-x)$ are illustrated in the diagram. The vertical asymptote at $x = 0$ is unaffected by either reflection.

**WORKED EXAMPLE 17**

**a** **Sketch the graph of $y = \log_2(x + 2)$ and state its domain.**

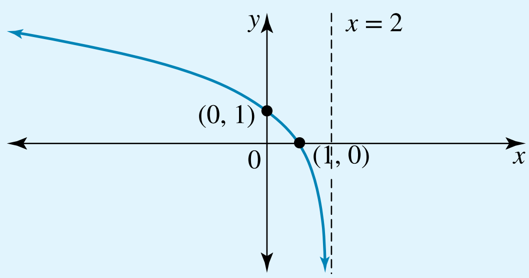
**b** **Sketch the graph of $y = \log_{10}(x) + 1$ and state its domain.**

**c** **The graph of the function for which $f(x) = \log_2(b - x)$ is shown below.**

**i** **Determine the value of $b$.**

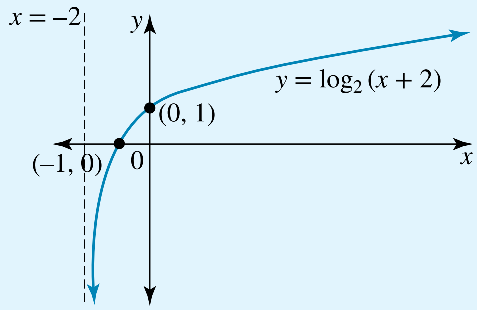
**ii** **State the domain and range of, and form the rule for, the inverse function.**

**iii** **Sketch the graph of $y = f^{-1}(x)$.**




| THINK | WRITE |
|---|---|
| **a 1** Identify the transformation involved. | **a** $y = \log_2(x + 2)$<br>Horizontal translation 2 units to the left |
| **2** Use the transformation to state the equation of the asymptote and the domain. | The vertical line $x = 0 \rightarrow$ the vertical line $x = -2$ under the horizontal translation.<br>The domain is $\{x : x > -2\}$. |
| **3** Calculate any intercepts with the coordinate axes.<br>*Note:* The domain indicates there will be an intercept with the $y$-axis as well as the $x$-axis. | $y$-intercept: when $x = 0$,<br>$y = \log_2(2)$<br>$= 1$<br>$y$-intercept $(0, 1)$<br>$x$-intercept: when $y = 0$,<br>$\log_2(x + 2) = 0$<br>$x + 2 = 2^0$<br>$x + 2 = 1$<br>$x = -1$<br>$x$-intercept $(-1, 0)$<br>Check: the point $(1, 0) \rightarrow (-1, 0)$ under the horizontal translation. |

**4** Sketch the graph.




**b** **1** Identify the transformation involved.

**b** $y = \log_{10}(x) + 1$
Vertical translation of 1 unit upwards

**2** State the equation of the asymptote and the domain.

The vertical transformation does not affect either the position of the asymptote or the domain.
Hence, the equation of the asymptote is $x = 0$.
The domain is $R^+$.

**3** Obtain any intercept with the coordinate axes.

Since the domain is $R^+$ there is no $y$-intercept.
$x$-intercept: when $y = 0$,

$$\begin{aligned}\log_{10}(x) + 1 &= 0\\ \log_{10}(x) &= -1\\ x &= 10^{-1}\\ &= \frac{1}{10} \text{ or } 0.1\end{aligned}$$

$x$-intercept is $(0.1, 0)$.

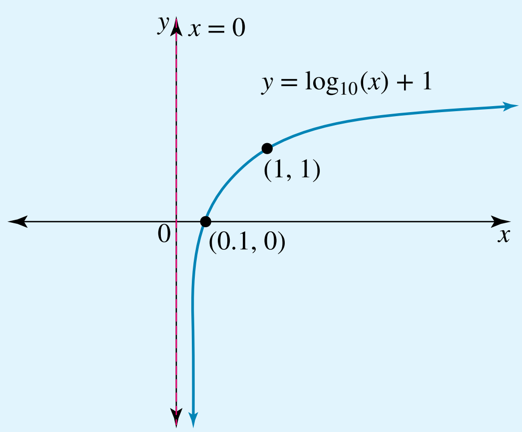
**4** Calculate the coordinates of a second point on the graph.

Point: let $x = 1$.

$$\begin{aligned}y &= \log_{10}(1) + 1\\ &= 0 + 1\\ &= 1\end{aligned}$$

The point $(1, 1)$ lies on the graph.
Check: the point $(1, 0) \rightarrow (1, 1)$ under the vertical translation.

**5** Sketch the graph.



**c i 1** State the equation of the asymptote shown in the graph and use this to calculate the value of $b$.

*Note:* The function rule can be rearranged to show the horizontal translation and a reflection in the $y$-axis.

$$f(x) = \log_2(b - x)$$
$$= \log_2(-(x - b))$$

The horizontal translation determines the position of the asymptote.

**c i** $f(x) = \log_2(b - x)$

From the diagram, the asymptote of the graph is $x = 2$.

From the function rule, the asymptote occurs when:

$b - x = 0$

$x = b$

Hence, $b = 2$.

**ii 1** Give the domain and range of the inverse function.

**ii** The given function has domain $(-\infty, 2)$ and range $R$. Therefore the inverse function has domain $R$ and range $(-\infty, 2)$.

**2** Form the rule for the inverse function by interchanging $x$- and $y$-coordinates and rearranging the equation obtained.

function $f : y = \log_2(2 - x)$

inverse $f^{-1} : x = \log_2(2 - y)$

$2^x = 2 - y$

$\therefore y = 2 - 2^x$

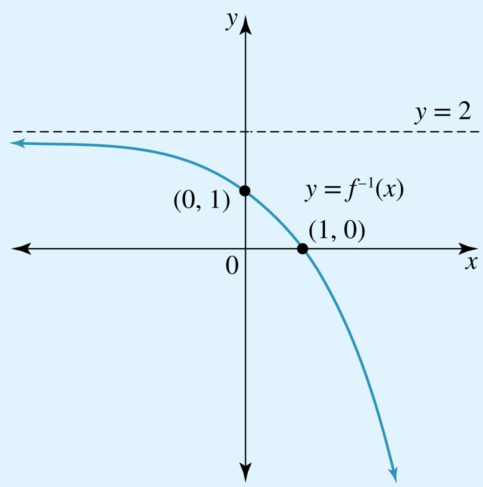
$\therefore f^{-1}(x) = 2 - 2^x$

**iii 1** Use the features of the logarithm graph to deduce the features of the exponential graph.

**iii** The key features of $f$ give those for $f^{-1}$.

| $y = f(x)$ | $y = f^{-1}(x)$ |
|---|---|
| asymptote: $x = 2$ | asymptote $y = 2$ |
| $x$-intercept $(1, 0)$ | $y$-intercept $(0, 1)$ |
| $y$-intercept $(0, 1)$ | $x$-intercept $(1, 0)$ |

**2** Sketch the graph of $y = f^{-1}(x)$.



## EXERCISE 11.6 Inverses of exponential functions

**PRACTISE**

1 **WE14** a Form the exponential rule for the inverse of $y = \log_{10}(x)$ and hence deduce the graph of $y = \log_{10}(x)$ from the graph of the exponential.

b Given the points (1, 10), (2, 100) and (3, 1000) lie on the exponential graph in part a, explain how these points can be used to illustrate the logarithm law $\log_{10}(m) - \log_{10}(n) = \log_{10}\left(\frac{m}{n}\right)$.

2 Sketch the graphs of $y = 2^{-x}$ and its inverse and form the rule for this inverse.

3 **WE15** Consider the function $f: R \to R, f(x) = 4 - 2^{3x}$.

a What is the domain of its inverse?

b Form the rule for the inverse function and express the inverse function as a mapping.

c Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

4 Consider the function defined by $y = 4 + 2\log_2(x)$.

a Form the rule for the inverse function.

b Sketch the graph of the inverse function and hence draw the graph of $y = 4 + 2\log_2(x)$ on the same set of axes.

c In how many places do the two graphs intersect?

5 **WE16** a Simplify $\log_6(2^{2x} \times 9^x)$ using the inverse relationship between exponentials and logarithms.

b Evaluate $2^{-3\log_2(10)}$.

6 Simplify $5^{x\log_5(2) - \log_5(3)}$.

7 **WE17** a Sketch the graph of $y = \log_{10}(x - 1)$ and state its domain.

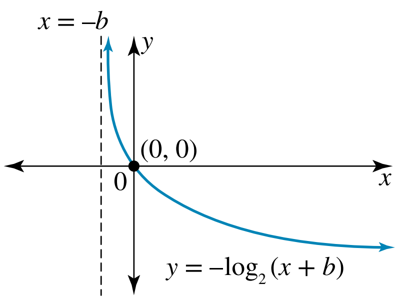
b Sketch the graph of $y = \log_5(x) - 1$ and state its domain.

c The graph of the function for which $f(x) = -\log_2(x + b)$ is shown.

i Determine the value of $b$.

ii State the domain and range of, and form the rule for, the inverse function.

iii Sketch the graph of $y = f^{-1}(x)$.




8 Sketch the graph of the function $f: R^+ \to R, f(x) = 1 - \log_4(x)$ by identifying the transformations involved.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

9 a On the same axes, sketch the graphs of $y = 3^x$ and $y = 5^x$ together with their inverses.

b State the rules for the inverses graphed in part a as logarithmic functions.

c Describe the effect of increasing the base on a logarithm graph.

d On a new set of axes, sketch $y = 3^x + 1$ and draw its inverse.

e Give the equation of the inverse of $y = 3^x + 1$.

f Sketch the graphs of $y = 5^{x+1}$ and its inverse on the same set of axes and give the rule for the inverse.

**10** Sketch the graphs of the following functions and their inverses and form the rule for the inverse.

**a** $y = 2^{-\frac{x}{3}}$ **b** $y = \frac{1}{2} \times 8^x - 1$ **c** $y = 2 - 4^x$

**d** $y = 3^{x+1} + 3$ **e** $y = -10^{-2x}$ **f** $y = 2^{1-x}$

**11** Given $f: R \to R, f(x) = 8 - 2 \times 3^{2x}$:

**a** Determine the domain and the range of the inverse function $f^{-1}$.
**b** Evaluate $f(0)$.
**c** At what point would the graph of $f^{-1}$ cut the $x$-axis?
**d** Obtain the rule for $f^{-1}$ and express $f^{-1}$ as a mapping.

**12** Consider the function defined by $y = 2 \times (1.5)^{2-x}$.

**a** For what value of $x$ does $y = 2$?
**b** For what value of $y$ does $x = 0$?
**c** Sketch the graph of $y = 2 \times (1.5)^{2-x}$ showing the key features.
**d** On the same set of axes sketch the graph of the inverse function.
**e** Form the rule for the inverse.
**f** Hence state the solution to the equation $2 \times (1.5)^{2-x} = 2 - \log_{1.5}\left(\frac{x}{2}\right)$.

**13** Given $g: (-1, \infty) \to R, g(x) = -3\log_5(x + 1)$:

**a** State the range of $g^{-1}$.
**b** Evaluate $g(0)$.
**c** At what point would the graph of $g^{-1}$ cut the $x$-axis?
**d** Obtain the rule for $g^{-1}$ and express $g^{-1}$ as a mapping.
**e** Sketch the graph of $y = g^{-1}(x)$.
**f** Use the graph in part **e** to deduce the graph of $y = g(x)$.

**14 a** Evaluate:

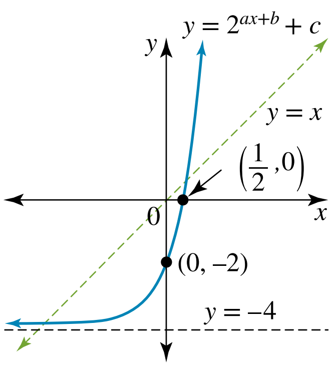
**i** $3^{\log_3(8)}$ **ii** $10^{\log_{10}(2)+\log_{10}(3)}$ **iii** $5^{-\log_5(2)}$ **iv** $6^{\frac{1}{2}\log_6(25)}$

**b** Simplify:

**i** $3^{\log_3(x)}$ **ii** $2^{3\log_2(x)}$

**iii** $\log_2(2^x) + \log_3(9^x)$ **iv** $\log_6\left(\frac{6^{x+1} - 6^x}{5}\right)$

**15** The diagram shows the graph of the exponential function $y = 2^{ax+b} + c$. The graph intersects the line $y = x$ twice and cuts the $x$-axis at $\left(\frac{1}{2}, 0\right)$ and the $y$-axis at $(0, -2)$.




**a** Form the rule for the exponential function.
**b** Form the rule for the inverse function.
**c** For the inverse, state the equation of its asymptote and the coordinates of the points where its graph would cut the $x$- and $y$-axes.
**d** Copy the diagram and sketch the graph of the inverse on the same diagram. How many points of intersection of the inverse and the exponential graphs are there?
**e** The point $(\log_2(3), k)$ lies on the exponential graph. Calculate the exact value of $k$.
**f** Using the equation for the inverse function, verify that the point $(k, \log_2(3))$ lies on the inverse graph.

**16** Sketch the graphs of the following transformations of the graph of $y = \log_a(x)$, stating the domain and range, equation of the asymptote and any points of intersection with the coordinate axes.

**a** $y = \log_5(x) - 2$
**b** $y = \log_5(x - 2)$
**c** $y = \log_{10}(x) + 1$
**d** $y = \log_3(x + 1)$
**e** $y = \log_3(4 - x)$
**f** $y = -\log_2(x + 4)$

**17 a** Describe the transformations which map $y = \log_2(x) \rightarrow y = -2\log_2(2 - x)$.

**b** Use a logarithm law to describe the vertical translation which maps $y = \log_2(x) \rightarrow y = \log_2(2x)$.

**c** Use the change of base law to express $y = \log_2(x)$ in terms of base 10 logarithms and hence describe the dilation which would map $y = \log_{10}(x) \rightarrow y = \log_2(x)$.

**d** Hick's Law arose from research into the time taken for a person to make a decision when faced with a number of possible choices. For $n$ equally probable options, the law is expressed as $t = b\log_2(n + 1)$ where $t$ is the time taken to choose an option, $b$ is a positive constant and $n \geq 2$. Draw a sketch of the time against the number of choices and show that doubling the number of options does not double the time to make the choice between them.

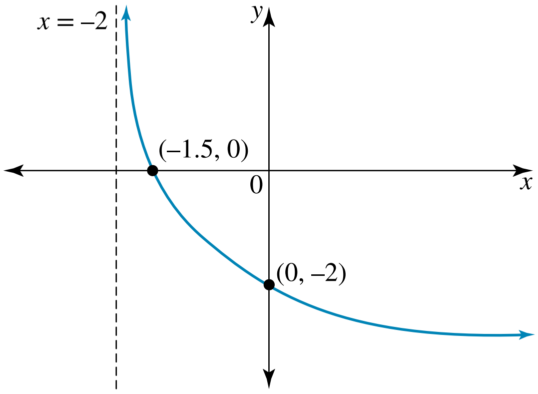
**18 a** The graph of the function with equation $y = a\log_7(bx)$ contains the points $(2, 0)$ and $(14, 14)$. Determine its equation.

**b** The graph of the function with equation $y = a\log_3(x) + b$ contains the points $\left(\frac{1}{3}, 8\right)$ and $(1, 4)$.

**i** Determine its equation.

**ii** Obtain the coordinates of the point where the graph of the inverse function would cut the $y$-axis.

**c i** For the graph illustrated in the diagram, determine a possible equation in the form $y = a\log_2(x - b) + c$.




**ii** Use the diagram to sketch the graph of the inverse and form the rule for the inverse.

**d** Consider the functions $f$ and $g$ for which $f(x) = \log_3(4x + 9)$ and $g(x) = \log_4(2 - 0.1x)$.

**i** Determine the maximal domain of each function.

**ii** State the equations of the asymptotes of the graphs of $y = f(x)$ and $y = g(x)$.

**iii** Calculate the coordinates of the points of intersection of each of the graphs with the coordinate axes.

**iv** Sketch the graphs of $y = f(x)$ and $y = g(x)$ on separate diagrams.

MASTER

19 Obtain the coordinates of any points of intersection of the graph of $y = 2 \times 3^{\frac{2-x}{2}}$ with its inverse. Express the values to 4 significant figures, where appropriate.

20 Consider the two functions with rules $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$.

a i Should the graphs of $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$ be the same graphs? Use CAS technology to sketch the graphs of $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$ to verify your answer.

ii Give any values of $x$ for which the graphs have the same value and justify algebraically.

b Sketch the graph of $y = 2\log_3(x)$, stating its domain, range and type of correspondence.

c Sketch the graph of $y = \log_3(x^2)$, stating its domain, range and type of correspondence.

d The graphs in parts b and c are not identical. Explain why this does not contradict the logarithm law $\log_a(m^p) = p\log_a(m)$.

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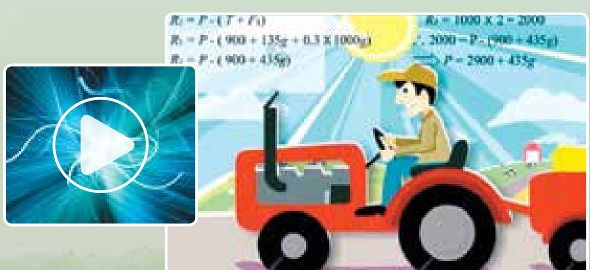

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**Units 1 & 2** **Exponential functions**

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# 11 Answers

## EXERCISE 11.2

**1 a** $2^{9n}$

**b** $6a^{\frac{1}{2}}b^2$

**c** $\frac{8}{9}$

**2** $\frac{-16pq^8}{m^4}$

**3** $x = 8$

**4 a** $x < 2$

**b** $x < \frac{10}{3}$

**5** $x = -1$

**6** $x = 4$

**7 a i** $1.409 \times 10^6$; 4 significant figures

**ii** $1.306 \times 10^{-4}$; 4 significant figures

**b i** 304 000

**ii** 0.058 03

**8** $8 \times 10^{10}$

**9 a i** $a^{\frac{3}{2}}b^2$

**ii** $a^{\frac{19}{6}}b^{\frac{7}{3}}$

**b i** $\sqrt{\frac{a}{b^3}}$

**ii** $\sqrt{32}$

**iii** $\sqrt[5]{\frac{1}{9}}$

**10 a** 8

**b** 0

**c** 162

**d** 25

**11 a** $\frac{9x^{10}}{y^4}$

**b** $\frac{3}{16a^{\frac{5}{3}}b^2}$

**c** $\frac{3m^3n^{\frac{1}{2}}}{40}$

**d** $\frac{-8n^5}{9}$

**e** $\frac{-1}{mn(m+n)}$

**f** $\frac{2x-1}{(4x-1)^{\frac{1}{2}}}$

**12 a** $2^{5+2x}$

**b** $3^{-7}$

**c** $10^3$

**d** $5^n$

**13 a** $x = 1$

**b** $x \le 6$

**c** $x = \frac{7}{10}$

**d** $x > \frac{1}{2}$

**e** $x = \frac{1}{2}$

**f** $x = 48$

**14 a** $x = 0,\ 2$

**b** $x = -3$

**c** $x = 1$

**d** $x = 0$

**e** $x = 2$

**f** $x = -1$

**15 a i** $-5.06 \times 10^{-8}$

**ii** $1.274 \times 10^4$ km

**iii** $1.6 \times 10^3$

**iv** $1.687\,87 \times 10^4$ km

**b i** 63 000.000 63

**ii** 1200

**c i** 61 000

**ii** 0.020

**iii** $x = -0.0063$

**iv** 27 million km

**16** Proof required — check with your teacher

**17 a** $x = 4,\ y = 11$

**b** $a = 160,\ k = -1$

**18** $a = \pm\left(3^6 \times 2^{-\frac{11}{2}}\right); n = 6$

**19 a** $9.873 \times 10^{-4}$

**b** $0.1762$; $1.3E2 = 1.3 \times 10^2$

**c** $1.68 \times 10^{-15}$

**d** $9.938 \times 10^{-11}$

**20 a** $\frac{x^{\frac{5}{3}}}{2y^2\sqrt{y^5}} = \frac{x^{\frac{5}{3}}}{2y^{\frac{9}{2}}}$

**b i** $-\frac{1}{5}$

**ii** $-0.1723$

## EXERCISE 11.3

**1 a** $4 = \log_5(625)$

**b** $6 = 36^{\frac{1}{2}}$

c $x \approx 0.93$

d $x = \frac{1}{3}$

2 a $\log_e(5) \approx 1.609; 5 = e^{1.609}$

b $10^{3.5} \approx 3162; 3.5 = \log_{10}(3162)$

3 a 1

b 4

c 1

d $\frac{3}{2}$

4 a −0.3

b 0.4

c 1.3

5 a $x = \log_7(15); x \approx 1.392$

b $x = \frac{5\log(3)}{\log(4) - 2\log(3)}; x = -6.774$

6 $x = \frac{3}{10}$

7 $x = 1$

8 $x = \frac{36}{35}$

9 a i $5 = \log_2(32)$

ii $\frac{3}{2} = \log_4(8)$

iii $-3 = \log_{10}(0.001)$

b i $2^4 = 16$

ii $9^{\frac{1}{2}} = 3$

iii $10^{-1} = 0.1$

10 a $2^x = \frac{1}{8}; x = -3$

b $25^{-0.5} = x; x = \frac{1}{5}$

c $2x = \log_{10}(4); x \approx 0.30$

d $-x = \log_e(3); x \approx -1.10$

e $x^3 = 125; x = 5$

f $x^{-2} = 25; x = \frac{1}{5}$

11 a 2

b −1

c 3

d 0

e −1

f 0

12 a 10

b $-n\log_4(3)$

c 1

d 6

13 a $\frac{1}{\log_{10}(2)}$

b i $\log_{11}(18) \approx 1.205$

ii $-\log_5(8) \approx -1.292$

iii $\frac{1}{2}\log_7(3) \approx 0.2823$

c i $x \le 2.096$

ii $x < 0.5693$

d i 125

ii 7

14 a 0.14

b 0.41

c 5.14

d −0.65

e 0.08

f 1.88

15 a $\sqrt{7}$

b $\frac{4}{5}$

c $\frac{6}{23}$

d $\frac{1}{2}$

e 0.001, 8

f 3

16 a $p + q$

b $3q$

c $2p + q$

d $p - q$

e $2q - 4p$

f $\frac{3}{4}pq$

17 a $y = 100x$

b $y = 2^{2x} \times x^{-4}$

c $y = 2^{3x}$

d $y = \log_{10}(x) + 2$

e $y = \frac{1}{x}$

f $y = \sqrt{x},\ x > 0$

18 a $\log_2(5),\ \log_2(9)$

b $\log_5(\sqrt{5} - 2)$

c $0, \log_9(2)$ $\left(\text{alternatively, } x = 0, \frac{1}{2}\log_3(2)\right)$

d 2

e $\frac{1}{4}$, 16

f 1, 999

19 a $x \approx 1.574$

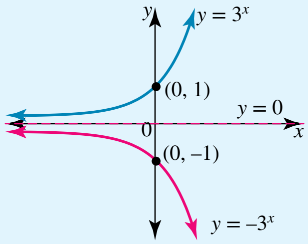
b $x = \frac{\sqrt{33} - 5}{2}$

20 a $\frac{\ln(5) + \ln(2)}{\ln(5)} + \log(5)$; proof required

b 1; proof required — check with your teacher

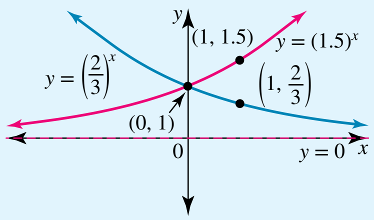
## EXERCISE 11.4

**1 a** For $y = 3^x$, range is $R^+$ and for $y = -3^x$, the range is $R^-$. Asymptote is $y = 0$ for both graphs.



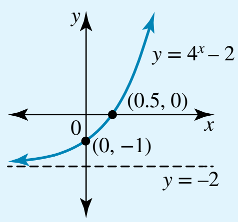

**b** $y = 3^{-x}$

**2**



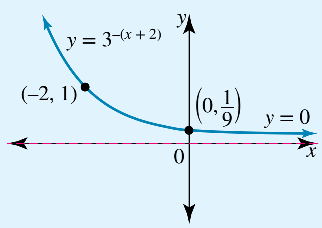

$y = \left(\frac{2}{3}\right)^x = \left(\frac{3}{2}\right)^{-x}$ and $y = (1.5)^x = \left(\frac{3}{2}\right)^x$

**3 a**



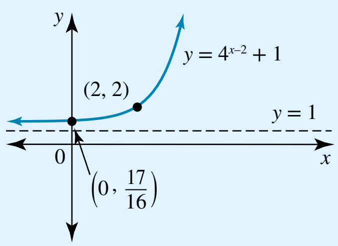

range $(-2, \infty)$

**b**



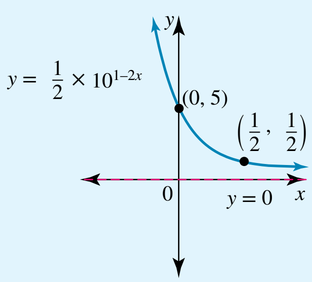

range $R^+$

**4**



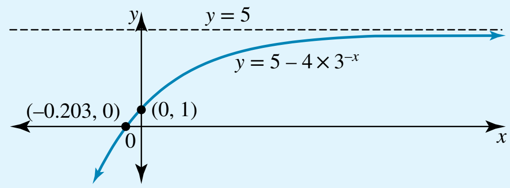

range $(1, \infty)$

**5 a**




range $R^+$

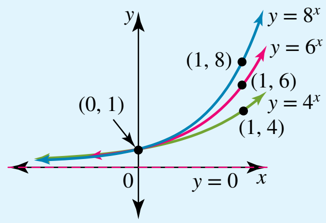
**b**




range $(-\infty, 5)$

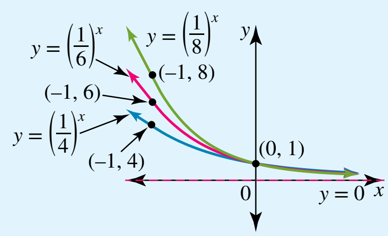
**6** $a = -2,\ b = 2$

**7 a i**



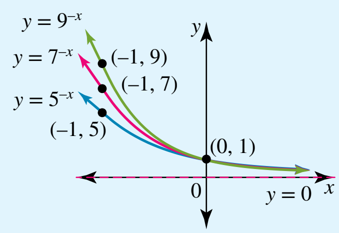

**ii** As the base increases, the graphs become steeper (for $x > 0$).

**b i**



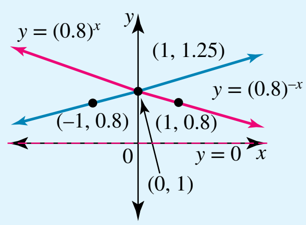

**ii** $y = 4^{-x};\ y = 6^{-x};\ y = 8^{-x}$

**8 a i**




**ii** As the base increases, the graphs decrease more steeply (for $x < 0$).

**b i**

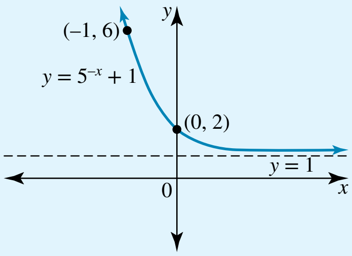



**ii** $y = (1.25)^x$ is the same as $y = 0.8^{-x}$. Reflecting these in the $y$-axis gives $y = 0.8^x$.

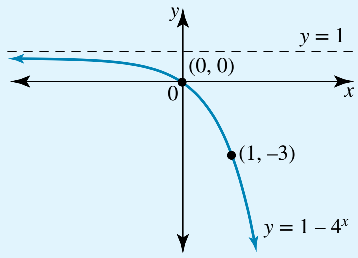
**9**

| | Asymptote | $y$-intercept | $x$-intercept | Point |
|---|---|---|---|---|
| **a** | $y = 1$ | $(0, 2)$ | none | $(-1, 6)$ |
| **b** | $y = 1$ | $(0, 0)$ | $(0, 0)$ | $(1, -3)$ |
| **c** | $y = -2$ | $(0, -1)$ | $(\log_{10}(2), 0)$ | $(1, 8)$ |
| **d** | $y = 6.25$ | $(0, 5.25)$ | $(-2, 0)$ | $(-1, 4)$ |

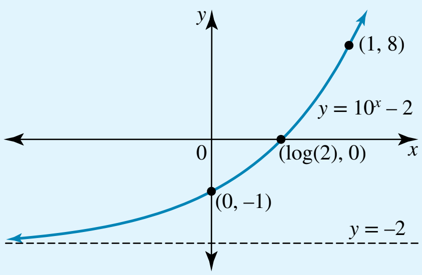
**a**



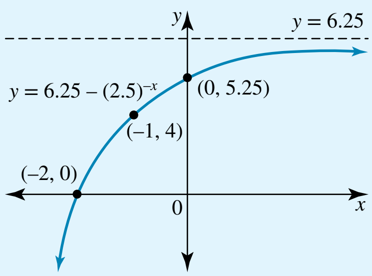

**b**




**c**

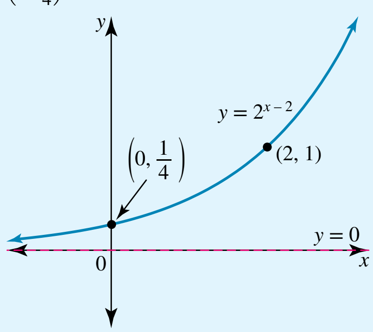



**d**

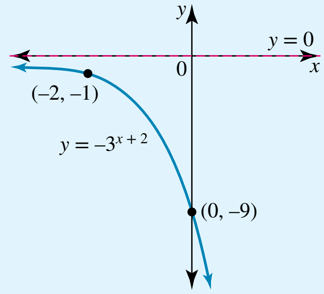



**10** Each graph has asymptote at $y = 0$.

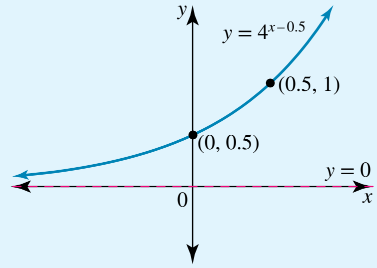
**a** $\left(0, \frac{1}{4}\right)$, $(2, 1)$



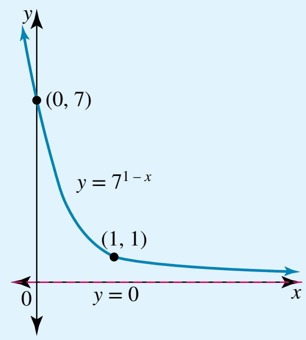

**b** $(0, -9)$, $(-2, -1)$




**c** $\left(0, \frac{1}{2}\right)$, $(0.5, 1)$

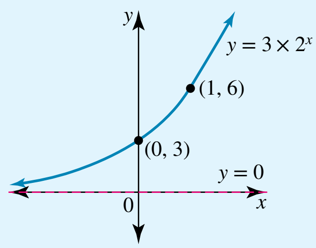



**d** $(0, 7)$, $(1, 1)$

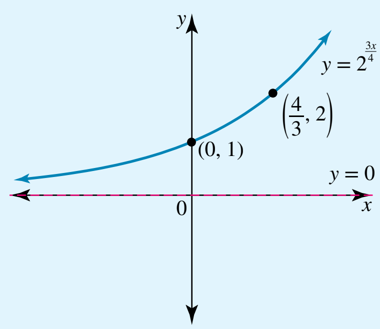



**11** Each graph has asymptote at $y = 0$.

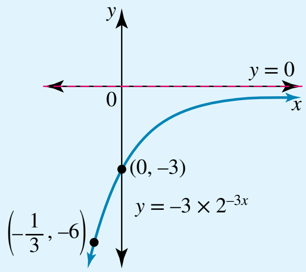
**a** $(0, 3)$, $(1, 6)$



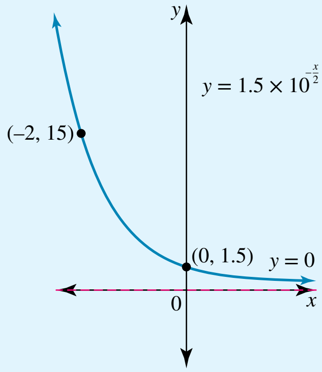

**b** $(0, 1)$, $\left(\frac{4}{3}, 2\right)$



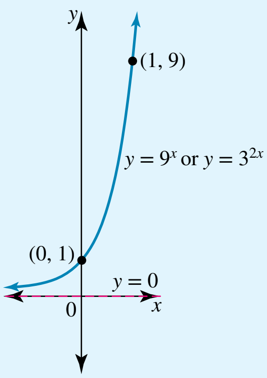
**c** $(0, -3), \left(-\frac{1}{3}, -6\right)$




**d** $(0, 1.5), (-2, 15)$

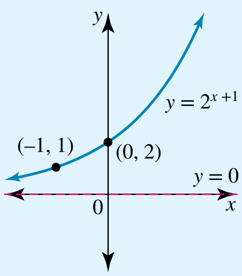



**12 a** Graphs are identical with asymptote $y = 0$; $y$-intercept $(0, 1)$; point $(1, 9)$




**b i** $y = 2^{x+1}$

**ii**




Asymptote $y = 0$; $y$-intercept $(0, 2)$; point $(-1, 1)$

**13 a** $y = 2 \times 10^x + 3$

**b** $y = 4 \times 3^{2x}$; asymptote $y = 0$

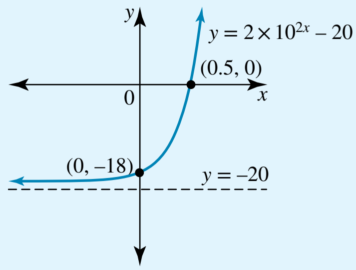
**c** $y = 6 - 2 \times 3^{1-x}$

**d** $y = 6 - 6 \times 3^{-x}$

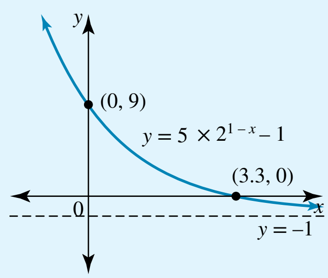
**14**

| | Asymptote | $y$-intercept | $x$-intercept | Range |
|---|---|---|---|---|
| **a** | $y = -20$ | $(0, -18)$ | $(0.5, 0)$ | $(-20, \infty)$ |
| **b** | $y = -1$ | $(0, 9)$ | $(3.3, 0)$ | $(-1, \infty)$ |
| **c** | $y = 3$ | $(0, 1)$ | $(-1, 0)$ | $(-\infty, 3)$ |
| **d** | $y = -7$ | $(0, 0)$ | $(0, 0)$ | $(-7, \infty)$ |
| **e** | $y = 8$ | $(0, 7.2)$ | $(0.7, 0)$ | $(-\infty, 8)$ |
| **f** | $y = -4$ | $(0, -4.2)$ | none | $(-\infty, -4)$ |

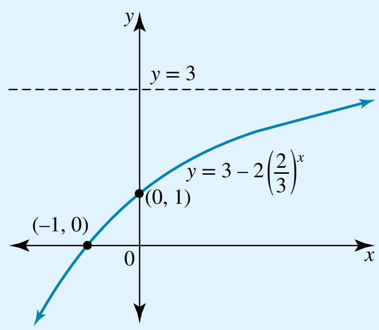
**a**



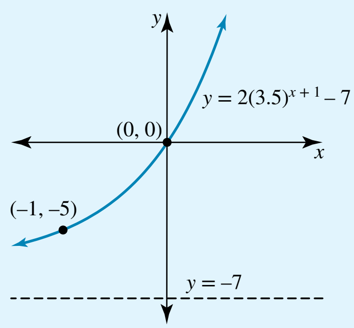

**b**



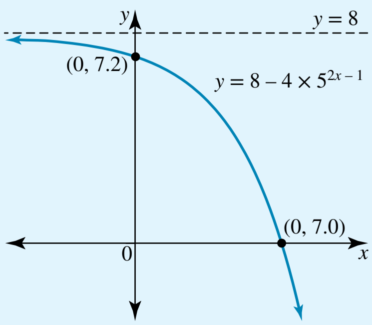

**c**



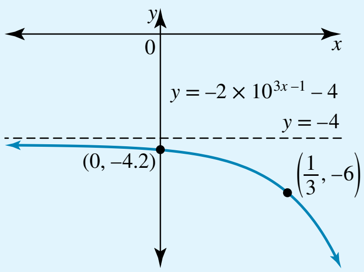

**d**



e




f

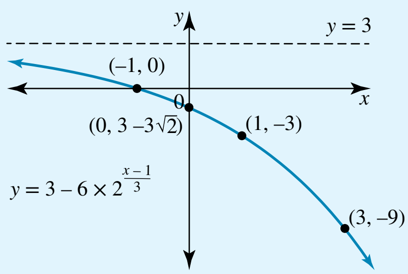



**15 a** i $-3$ ii $3 - 3\sqrt{2}$

**b** i $x = 3$ ii $x = -1$

iii Not possible

**c**




Asymptote $y = 3$; range $(-\infty, 3)$

**d** $x \leq -0.17$

**16 a** One intersection for which $x \in (-1, -0.5)$

**b** Three

**c** One

**d** None

**e** One point of intersection $(1, 6)$

**f** Infinite points of intersection $(t, 2^{2t-1}),\ t \in R$

**17** $(-0.77, 0.59), (2, 4), (4, 16)$

**18**

| | Asymptote | $y$-intercept | $x$-intercept | Point |
|---|---|---|---|---|
| $y_1$ | $y = 33$ | $(0, 31)$ | $(1.17, 0)$ | $(1.0185, 10)$ |
| $y_2$ | $y = 33$ | $(0, 11)$ | $(0.17, 0)$ | $(0.0185, 10)$ |

Horizontal translation is one unit to the left.

## EXERCISE 11.5

**1 a** Proof required — check with your teacher

**b** $k \approx 0.004$

**c** 5.3 kg

**2 a** 42 emails per day on average

**b** 16 weeks

**3** $\log(y) = 2.5\log(x) + 2$; $y = 100x^{2.5}$

**4** $\log(y) = 0.3x$, $y = 10^{0.3x}$

**5 a** $k = \frac{1}{5}$

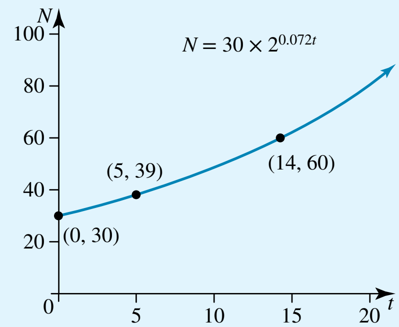
**b** 10 years

**6 a** 30 drosophilae

**b** 39 insects

**c** 14 days

**d**




**e** After 25 days

**7 a** i Proof required — check with your teacher

ii \$2030.19

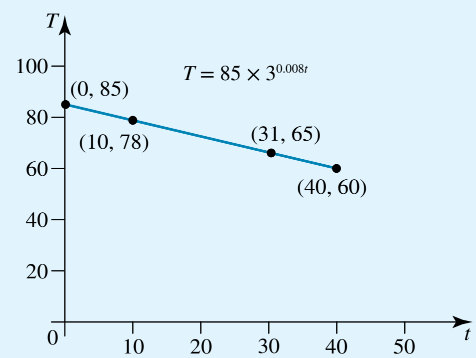
iii 7.45 years

**b** 5.6 %

**8 a** 7 °C

**b** Approximately $\frac{1}{2}$ hour

**c**



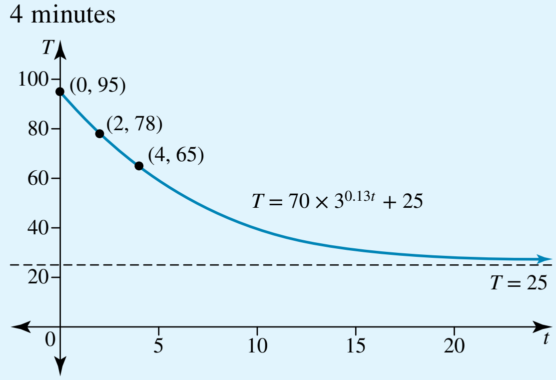

**d** Unrealistic for temperature to approach 0°

**9 a** $a = 70$

**b** 77.6 °C

**c** 4 minutes

**d**



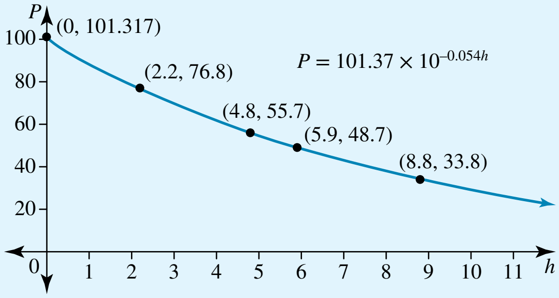

temperature approaches 25 °C

**10 a** $k \approx 0.054$

**b** $P_0 \approx 101.317$

**c** 55.71 kilopascals; 76.80 kilopascals

**d**




**11 a** $D_0 \times 10^{15k} = 15$ and $D_0 \times 10^{18k} = 75$

**b** Proof required — check with your teacher

**c** 229 birds per square kilometre

**d** Not lower than 39 birds per square kilometre

**12 a** $k = \frac{1}{5730}$

**b** 1540 years old

**13 a i** $y = 0.1\sqrt{x}$

**ii** $y = 2^{-\frac{x}{4}}$

**b i** Bleach $pH = 13$, water $pH = 7$

**ii** $1 \times 10^{-2} = 0.01,\ 1 \times 10^{-6} = 0.000001$

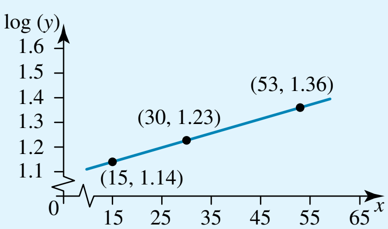
**iii** Lemon juice is $10^4$ times more acidic

**iv** An increase of one unit makes the solution 10 times less acidic; concentration of hydrogen ions decreases by a factor of 10.

**14 a**

| log(y) | 1.14 | 1.23 | 1.36 |
|---|---|---|---|

**b**




**c** Proof required — check with your teacher

**d** Proof required — check with your teacher

**e** Approximately 50 years

**f** Model supports the claim.

**15 a i** $y \approx 4 \times 1.38^x$

**ii** $y \approx 4.3 + 6.2\log_e(x)$

**b** Exponential model is the better model.

**16 a i** 16

**ii** 51.5

**b i** 80.20

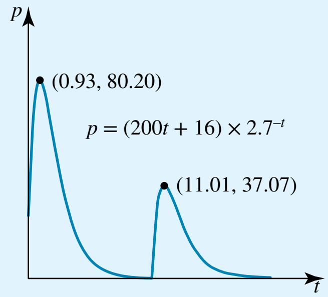
**ii** 55.61 seconds

**iii** 7.08; 0.10

**c** 10 minutes

**d i** $a = 0.10$

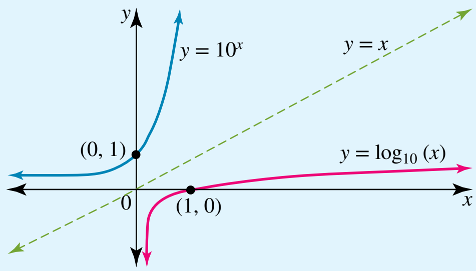
**ii**




Maximum turning points at $(0.93, 80.2)$, $(11.01, 37.07)$

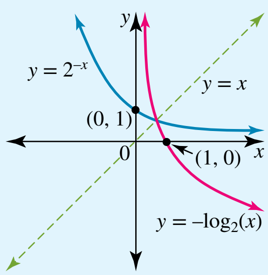
## EXERCISE 11.6

**1 a** $y = 10^x$




**b** $\log_{10}(1000) - \log_{10}(100) = \log_{10}(10)$ and $3 - 2 = 1$

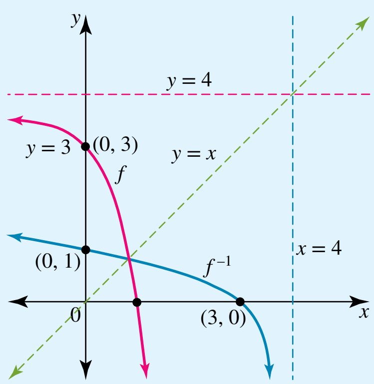
**2** Inverse is $y = -\log_2(x)$




**3 a** $(-\infty, 4)$

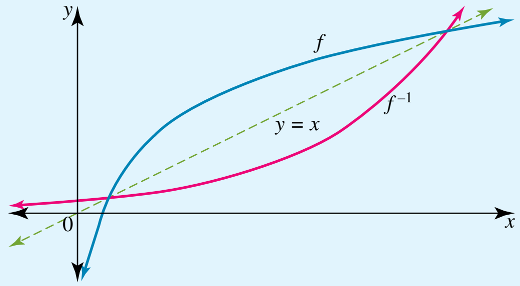
**b** Rule is $f^{-1}(x) = \frac{1}{3}\log_2(4 - x)$; mapping is $f^{-1}: (-\infty, 4) \to R,\ f^{-1}(x) = \frac{1}{3}\log_2(4 - x)$

**c**



**4 a** $y = 2^{\left(\frac{x-4}{2}\right)}$

**b**

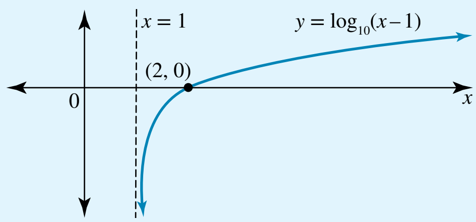



**c** Twice

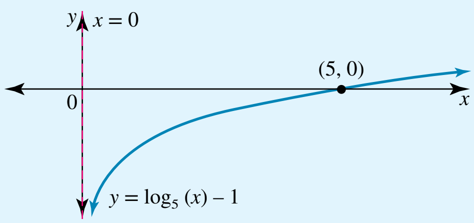
**5 a** $2x$ **b** 0.001

**6** $\frac{1}{3} \times 2^x$

**7 a** Asymptote $x = 1$; domain $(1, \infty)$; $x$-intercept $(2, 0)$;



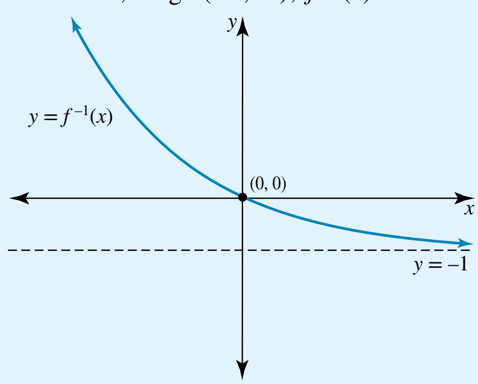

**b** Asymptote $x = 0$; domain $R^+$; $x$-intercept $(5, 0)$;




**c** **i** $b = 1$

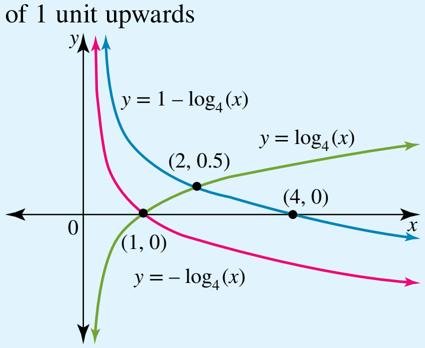
**ii** Domain $R$; range $(-1, \infty)$, $f^{-1}(x) = 2^{-x} - 1$

**iii**



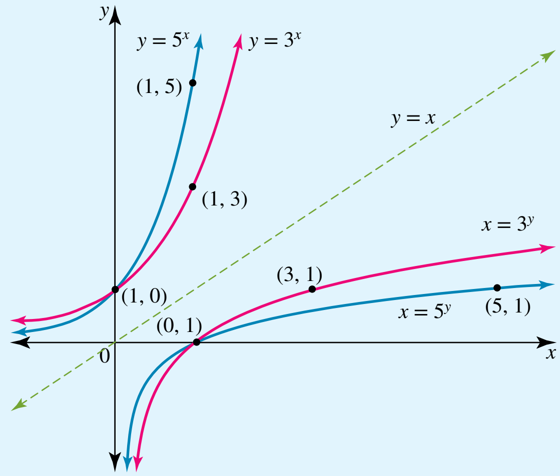

Asymptote $y = -1$, contains the origin

**8** Reflection in the $x$-axis followed by a vertical translation of 1 unit upwards




Asymptote $x = 0$, domain $R^+$, $x$-intercept $(4, 0)$

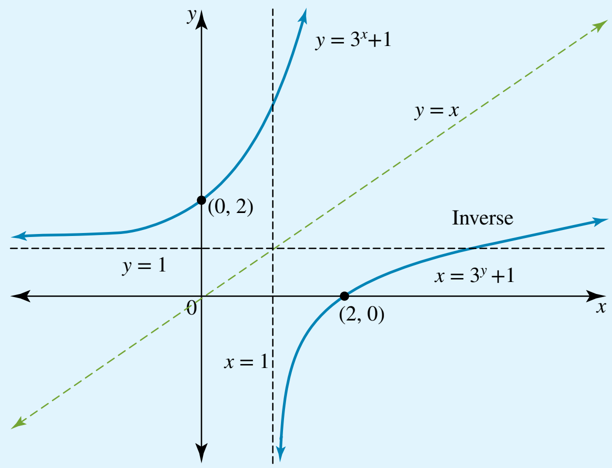
**9 a**




**b** $y = \log_3(x)$, $y = \log_5(x)$

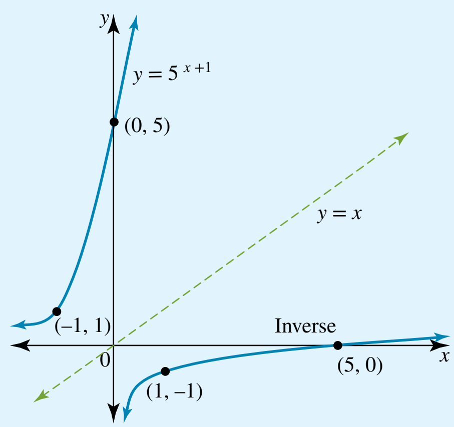
**c** The larger the base, the more slowly the graph increases for $x > 1$.

**d**



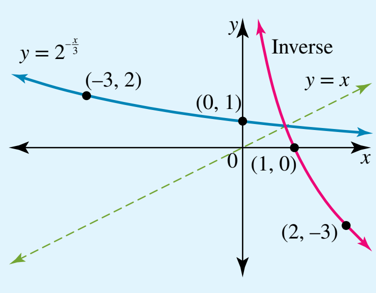

**e** $y = \log_3(x - 1)$

**f**

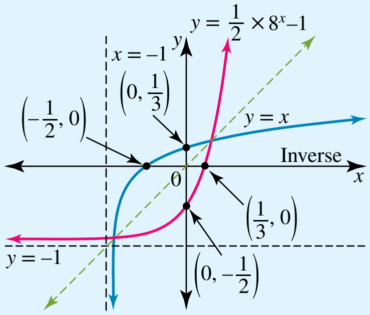



$y = \log_5(x) - 1$

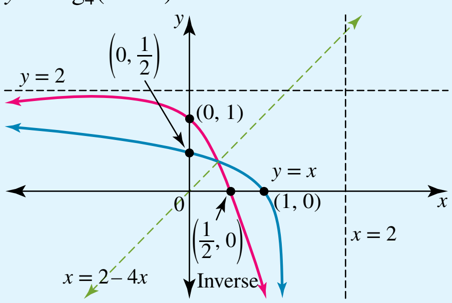
**10 a** $y = -3\log_2(x)$



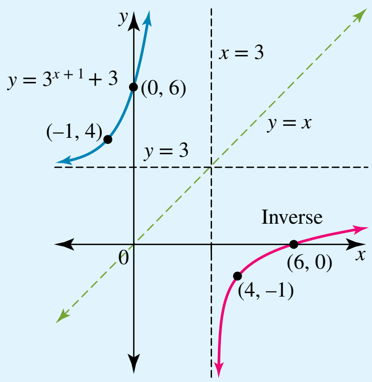
**b** $y = \log_8(2x + 2)$



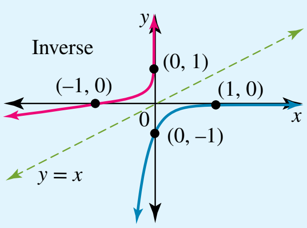

**c** $y = \log_4(2 - x)$



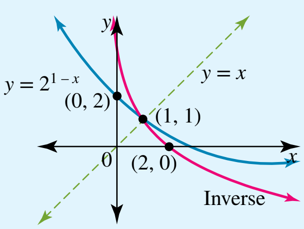

**d** $y = \log_3(x - 3) - 1$




**e** $y = -\frac{1}{2}\log_{10}(-x)$




**f** $y = 1 - \log_2(x)$




**11 a** Domain $(-\infty, 8)$; range $R$

**b** 6

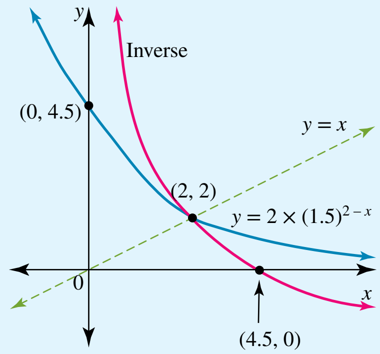
**c** $(6, 0)$

**d** Rule is $f^{-1}(x) = \frac{1}{2}\log_3\left(\frac{8 - x}{2}\right)$; mapping is $f^{-1}: (-\infty, 8) \to R, f^{-1}(x) = \frac{1}{2}\log_3\left(\frac{8 - x}{2}\right)$.

**12 a** $x = 2$

**b** $y = 4.5$

**c and d**




**e** $y = 2 - \log_{1.5}\left(\frac{x}{2}\right)$

**f** $x = 2$

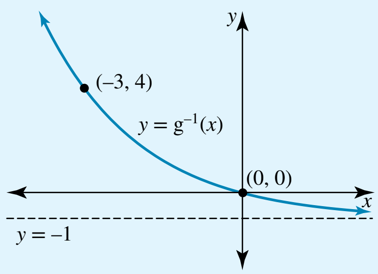
**13 a** Range $(-1, \infty)$

**b** 0

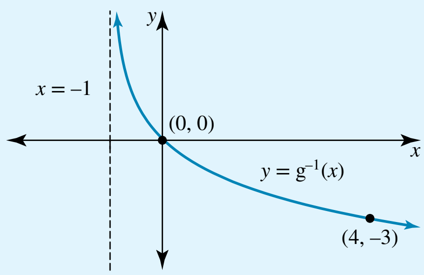
**c** $(0, 0)$

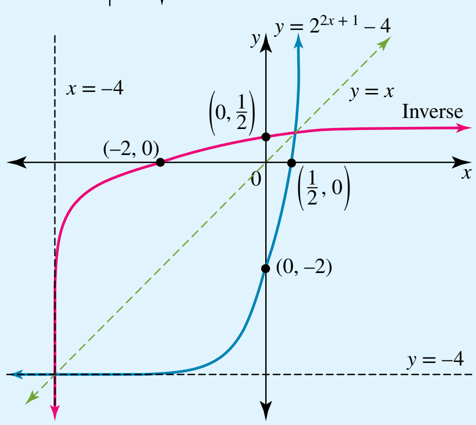
**d** $g^{-1}: R \to R, g^{-1}(x) = 5^{-\frac{x}{3}} - 1$

**e**




**f**




| | | | | |
|---|---|---|---|---|
| **14 a** | **i** 8 | **ii** 6 | **iii** 0.5 | **iv** 5 |
| **b** | **i** $x$ | **ii** $x^3$ | **iii** $3x$ | **iv** $x$ |

**15 a** $y = 2^{2x+1} - 4$

**b** $y = \frac{1}{2}\log_2(x + 4) - \frac{1}{2}$

c Asymptote: $x = -4$; intercepts: $(-2, 0)$, $(0, 0.5)$

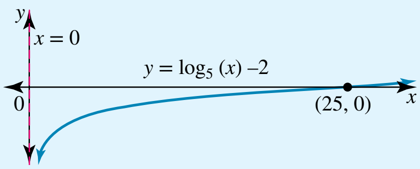
d Two points of intersection

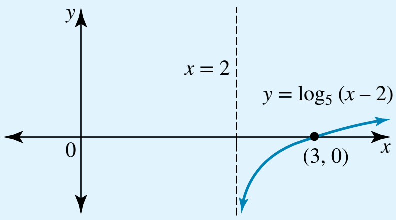
e $k = 14$

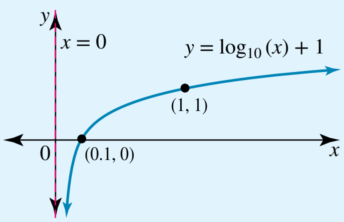
f Proof required — check with your teacher

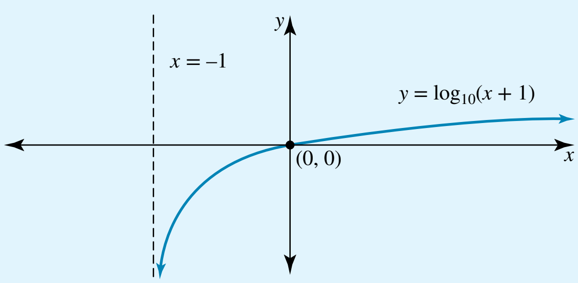
**16** Range is $R$ for each graph.

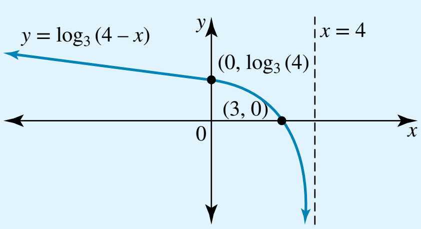
| | Domain | Asymptote equation | Axes intercepts |
|---|---|---|---|
| a | $R^+$ | $x = 0$ | $(25, 0)$ |
| b | $(2, \infty)$ | $x = 2$ | $(3, 0)$ |
| c | $R^+$ | $x = 0$ | $(0.1, 0)$ |
| d | $(-1, \infty)$ | $x = -1$ | $(0, 0)$ |
| e | $(-\infty, 4)$ | $x = 4$ | $(3, 0)$, $(0, \log_3(4))$ |
| f | $(-4, \infty)$ | $x = -4$ | $(-3, 0)$, $(0, -2)$ |

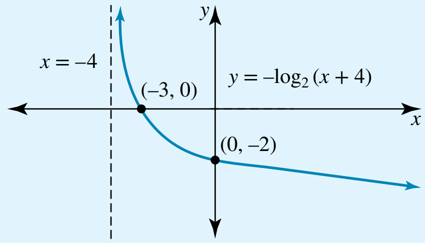
a



b



c



d



e


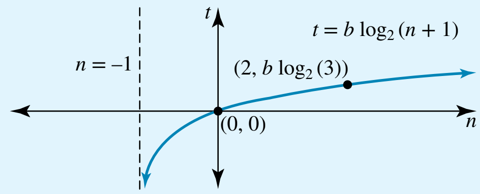

f



**17** a Dilation factor 2 from the $x$-axis; reflection in $x$-axis; reflection in $y$-axis; horizontal shift 2 units to right

b Vertical translation of 1 unit up

c Dilation of factor $\dfrac{1}{\log_{10}(2)}$ from the $x$-axis

Proof required — check with your teacher

d



Proof required — check with your teacher

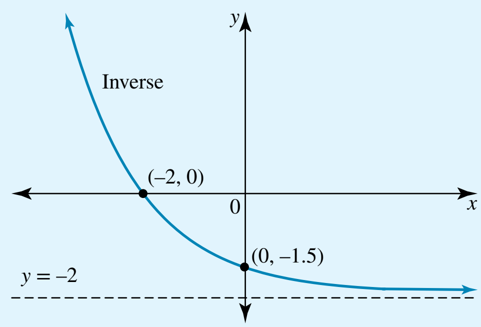
**18** a $y = 14\log_7\left(\dfrac{x}{2}\right)$

b i $y = -4\log_3(x) + 4$

ii $(0, 3)$

c i $y = -\log_2(x + 2) - 1$

ii $y = 2^{-(x+1)} - 2$




d i $d_f = \left(-\dfrac{9}{4}, \infty\right)$; $d_g = (-\infty, 20)$

ii $x = -\dfrac{9}{4}$, $x = 20$

iii $f : (-2, 0), (0, 2)$; $g : (10, 0), \left(0, \dfrac{1}{2}\right)$

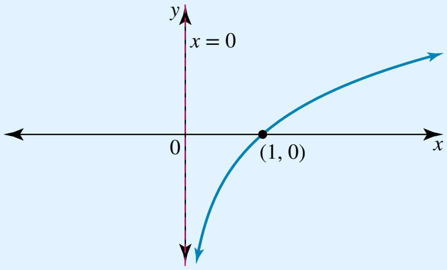
iv Sketch here graphs using a CAS technology

**19** $(0.4712, 4.632)$, $(2, 2)$, $(4.632, 0.4712)$

**20 a** **i** Graphs are not the same.

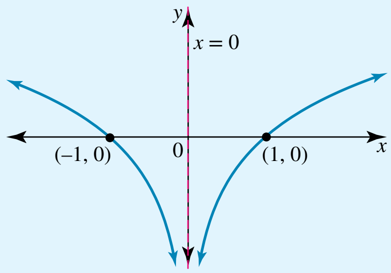
**ii** $x = \dfrac{4}{3}$

**b**




domain $R^+$; range $R$; one-to-one correspondence

**c**




domain $R\backslash\{0\}$; range $R$; many-to-one correspondence

**d** The law holds for $m > 0$ and it does hold since $\log_3(x^2) = 2\log_3(x), x > 0$.

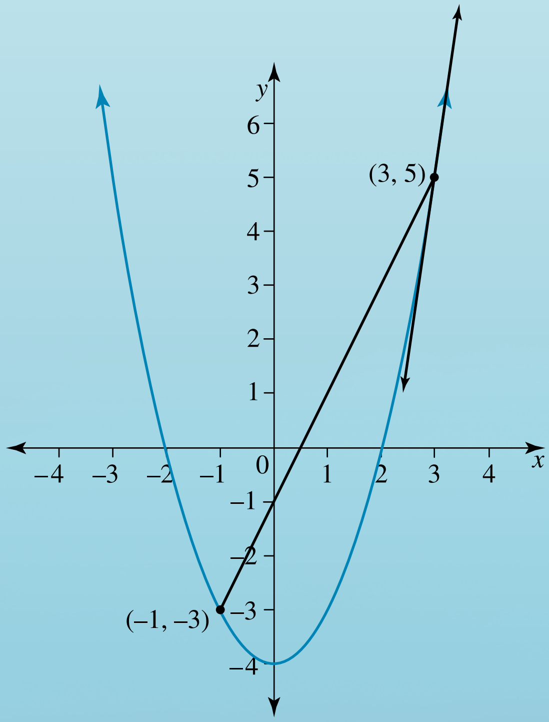
# 12

# Introduction to differential calculus

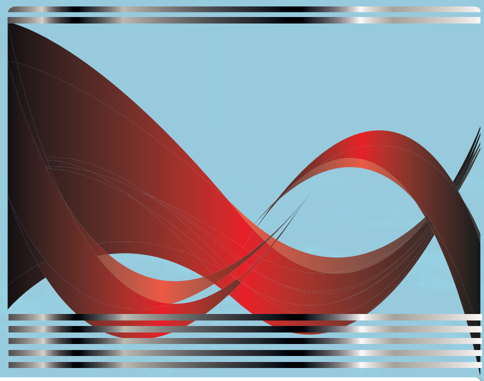
# 12.1 Kick off with CAS

## Differential calculus

1 Sketch the graph of $f(x) = x^2 - 4$ using CAS technology.
2 Place two points on the graph at $x = -1$ and $x = 3$.
3 Using CAS technology, calculate the gradient of the line segment joining the points at $x = -1$ and $x = 3$. This gives you the average rate of change of $f(x)$ between $x = -1$ and $x = 3$.
4 Draw a tangent to $f(x)$ at $x = 3$.




5 Using CAS technology, calculate the gradient of the tangent at $x = 3$. This represents the instantaneous rate of change of $f(x)$ between $x = -1$ and $x = 3$.
6 Determine the equation of the tangent using CAS technology.
7 Sketch the graph of $g(x) = x^3 - 2$ using CAS technology.
8 Place two points on the graph at $x = -1$ and $x = 3$.
9 Using CAS technology, calculate the gradient of the line segment joining the points at $x = -1$ and $x = 3$. This gives you the average rate of change of $g(x)$ between $x = -1$ and $x = 3$.
10 Draw a tangent to $g(x)$ at $x = 3$.
11 Using CAS technology, calculate the gradient of the tangent at $x = 3$. This represents the instantaneous rate of change of $g(x)$ between $x = -1$ and $x = 3$.
12 Using CAS technology, determine the equation of the tangent.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 12.2 Rates of change

studyon

Units 1 & 2
AOS 3
Topic 1
Concept 1

**Rates of change**
Concept summary
Practice questions

eBook*plus*

**interactivity**
Rates of change
**int-5960**

A rate of change measures the change in one variable relative to another. An oil spill may grow at the rate of 20 square metres per day; the number of subscribers to a newspaper may decrease at the rate of 50 people per month. In practical situations, the rate of change is often expressed with respect to time.

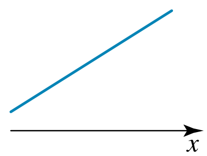
## Gradients as rates of change

A linear function has a straight-line graph whose gradient, or slope, is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

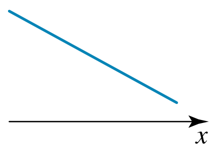
This gradient measures the rate at which the $y$-values change with respect to the change in the $x$-values. The gradient measures the rate of change of the function.

The rate of change of a linear function is a constant: either the function increases steadily if $m > 0$ or decreases steadily if $m < 0$. The function neither increases nor decreases if $m = 0$.

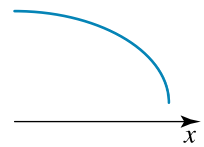
The rate of change of any function is measured by the gradient or slope of its graph. However, the gradient of a non-linear function is not constant; it depends at what point on the graph, or at what instant, it is to be measured.




This linear function increases steadily as $x$ increases.




This linear function decreases steadily as $x$ increases.



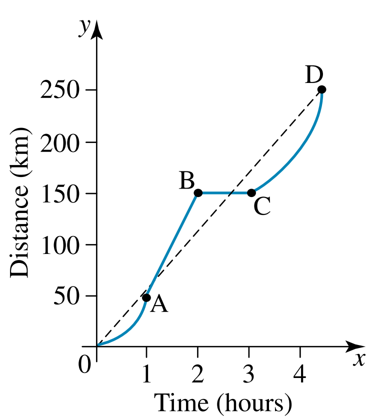

This non-linear function decreases as $x$ increases but not in a steady way.

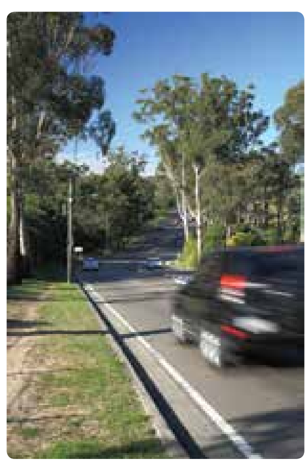
We know how to find the gradient of a linear function but learning how to measure the gradient of a non-linear function is the aim of this chapter.

## Average rate of change

A car journey of 240 km completed in 4 hours may not have been undertaken with a steady driving speed of 60 km/h. The speed of 60 km/h is the average speed for the journey found from the calculation: average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$. It gives the average rate of change of the distance with respect to the time.

The diagram shows a graph of the distance travelled by a motorist over a period of 4 hours.



A possible scenario for this journey is that slow progress was made for the first hour (O–A) due to city traffic congestion. Once on a freeway, good progress was made, with the car travelling at the constant speed limit on cruise control (A–B); then after a short break for the driver to rest (B–C), the journey was completed with the car's speed no longer constant (C–D).

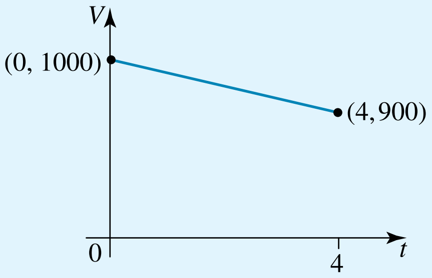
The average speed of the journey is the gradient of the line segment joining the origin (0, 0) and D(4, 240), the endpoints for the start and finish of the journey.

For any function $y = f(x)$, the **average rate of change** of the function over the interval $x \in [a, b]$ is calculated as $\dfrac{f(b) - f(a)}{b - a}$. This is the gradient of the line joining the endpoints of the interval.

**WORKED EXAMPLE 1**

**a** **A water tank contains 1000 litres of water. The volume decreases steadily and after 4 weeks there are 900 litres of water remaining in the tank. Show this information on a diagram and calculate the rate at which the volume is changing with respect to time.**

**b** **The profit made by a company in its first 4 years of operation can be modelled by the function $p$ for which $p(t) = \sqrt{t}$, where the profit is $p \times \$10\,000$ after $t$ years. Sketch the graph of the function over the time interval and calculate the average rate of change of the profit over the first 4 years.**

| THINK | WRITE |
|---|---|
| **a 1** Define the variables and state the information given about their values. | **a** Let volume be $V$ litres at time $t$ weeks. Given: when $t = 0$, $V = 1000$ and when $t = 4$, $V = 900$, then the points (0, 1000) and (4, 900) are the endpoints of the volume–time graph. |
| **2** Interpret the keyword 'steadily' and sketch the given information on a graph. | As the volume decreases steadily, the graph of $V$ against $t$ is linear and the rate of change is constant.  |
| **3** Calculate the rate of change of the volume with respect to time. | The rate of change of the volume is the gradient of the graph. It is measured by $\dfrac{\text{change in volume}}{\text{change in time}}$, so: |

$$\frac{\text{change in volume}}{\text{change in time}} = \frac{V(4) - V(0)}{4 - 0}$$

$$= \frac{900 - 1000}{4}$$

$$= -25$$

**4** Write the answer in context using appropriate units.

*Note:* The negative value for the rate represents the fact the volume is decreasing. The units for the rate are the volume unit per time unit.

The volume of water is decreasing at a rate of 25 litres/week.

**b 1** Sketch the graph of the function over the given time interval.

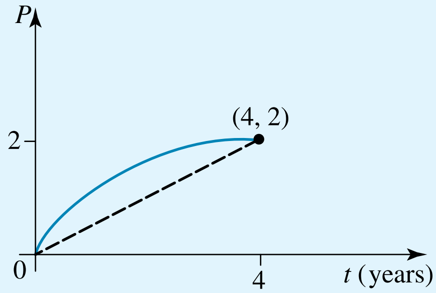
*Note:* The graph is not linear so the rate of change is not constant.

**b** $p(t) = \sqrt{t}$

When $t = 0, p = 0$

When $t = 4, p = 2$

(0, 0) and (4, 2) are the endpoints of the square root function graph.




**2** Calculate the average rate of the function over the time interval.

The average rate is measured by the gradient of the line joining the endpoints.

$$\text{Average rate of change of } p = \frac{p(4) - p(0)}{4 - 0}$$

$$= \frac{2 - 0}{4}$$

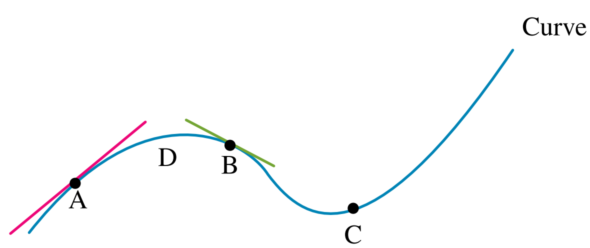
$$= 0.5$$

**3** Calculate the answer.

Since the profit is $p \times \$10\,000$, the profit is increasing at an average rate of \$5000 per year over this time period.

## Instantaneous rate of change

The gradient at any point on the graph of a function is given by the gradient of the tangent to the curve at that point. This gradient measures the **instantaneous rate of change**, or rate of change, of the function at that point or at that instant.




For the curve shown, the gradient of the tangent at point A is positive and the function is increasing; at point B the gradient of the tangent is negative and the function is decreasing. At point C, the curve is rising and if the tangent to the curve is drawn at C, its gradient would be positive. The curve is steeper at point A than at point C as the tangent at point A has a more positive gradient than the tangent at point C. The instantaneous rate of change of the function at A is greater than the instantaneous rate of change at C.

At the turning point D, the function is neither increasing nor decreasing and the tangent is horizontal, with a zero gradient. The function is said to be stationary at that instant.

- The rate of change of a function is measured by the gradient of the tangent to its curve.
- For non-linear functions, the rate of change is not constant and its value depends on the point, or instant, at which it is evaluated.

## Estimating the gradient of a curve

Two approximation methods that could be used to obtain an estimate of the gradient of a curve at a point P are:

Method 1: Draw the tangent line at P and obtain the coordinates of two points on this line in order to calculate its gradient.

Method 2: Calculate the average rate of change between the point P and another point on the curve which is close to the point P.

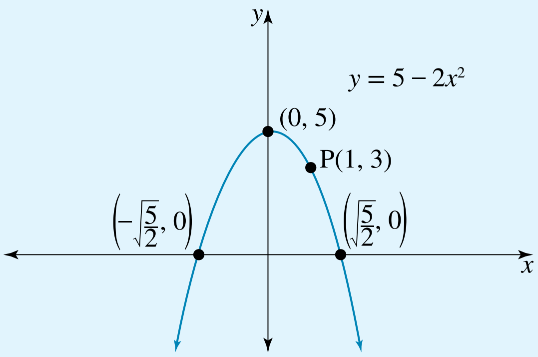
The accuracy of the estimate of the gradient obtained using either of these methods may have limitations. In method 1 the accuracy of the construction of the tangent line, especially when drawn 'by eye', and the accuracy with which the coordinates of the two points on the tangent can be obtained will affect the validity of the estimate of the gradient. Method 2 depends on knowing the equation of the curve for validity. The closeness of the two points also affects the accuracy of the estimate obtained using this method.

**WORKED EXAMPLE 2**

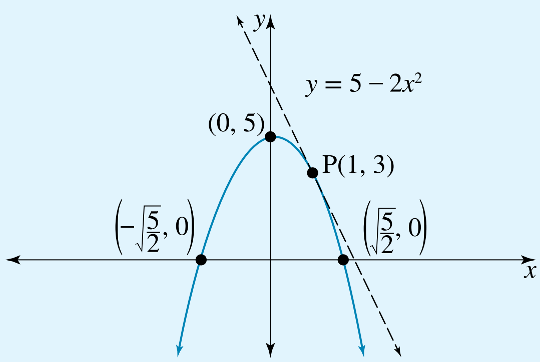
**For the curve with equation $y = 5 - 2x^2$ shown, estimate the gradient of the curve at the point P(1, 3) by:**

**a constructing a tangent at P and calculating its gradient**

**b choosing a point on the curve close to P and calculating the average rate of change between P and this point.**




| THINK | WRITE |
|---|---|
| **a** **1** Draw the tangent line which just touches the curve at P. | **a** The tangent at P is shown. |



| | |
|---|---|
| **2** Determine the coordinates of two points that lie on the tangent. *Note:* The second point may vary and its coordinates may lack accuracy. | The tangent contains point P(1, 3). A second point is estimated to be (0, 7), the $y$-intercept. |
| **3** Calculate the gradient using the two points. | $m = \dfrac{7-3}{0-1}$ <br> $= -4$ |
| **4** State the answer. *Note:* Answers may vary. | The gradient of the curve at point P is approximately equal to –4. |
| **b 1** Choose a value for $x$ close to the $x$-value at point P and use the equation of the curve to calculate the corresponding $y$-coordinate. *Note:* Other points could be chosen. | **b** A point close to P(1, 3) could be the point for which $x = 0.9$. <br> Equation of curve: $y = 5 - 2x^2$ <br> If $x = 0.9$, $y = 5 - 2(0.9)^2$ <br> $= 5 - 2 \times 0.81$ <br> $= 3.38$ <br> The point (0.9, 3.38) lies on the curve close to point P. |
| **2** Calculate the average rate of change between the two points. | Average rate of change $= \dfrac{3.38 - 3}{0.9 - 1}$ <br> $= -3.8$ |
| **3** State the answer. *Note:* Answers may vary. | The gradient of the curve at point P is approximately equal to –3.8. |

## EXERCISE 12.2 Rates of change

**PRACTISE**

**1** **WE1** **a** A Petri dish contains 40 bacteria which grow steadily so that after half an hour there are 100 bacteria in the dish. Show this information on a graph and calculate the rate at which the bacteria are growing with respect to time.

**b** After a review of its operations, part of a company's strategic plan is to improve efficiency in one of its administrative departments. The operating costs for this section of the company are modelled by the function $c$ for which $c(t) = \dfrac{4000}{t+2}$, where $t$ months after the strategic plan came into effect, the operating cost is \$$c$. Sketch the graph of the operating costs over time and calculate the average rate of change of the operating costs over the first 2 months.

**2** **WE2** Calculate the average rate of change of the function $f(x) = x^2 + 3$ over the interval for which $x \in [1, 4]$.

**3** For the curve with equation $y = 5 - 2x^2$ shown in Worked example 2, estimate the gradient of the curve at the point Q(−1.5, 0.5) by:

**a** constructing a tangent at Q and calculating its gradient

**b** choosing a point on the curve close to Q and calculating the average rate of change between Q and this point.

4 The distance travelled by a particle moving in a straight line can be expressed as a function of time. If $x$ metres is the distance travelled after $t$ seconds, and $x = t^3 + 4t^2 + 3t$:

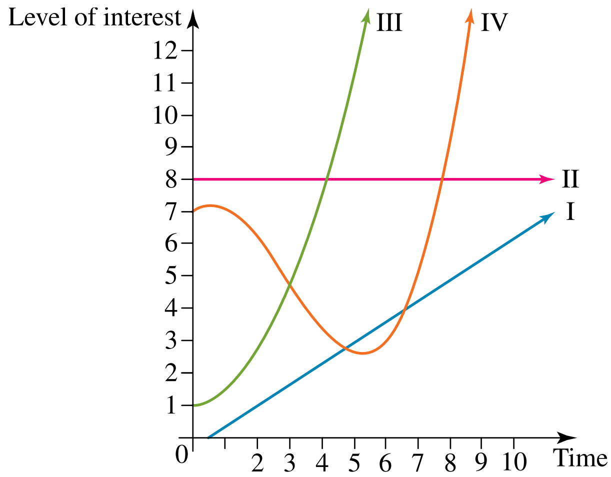
a Calculate the average rate of change of the distance $x$ over the time interval $t \in [0, 2]$ and over the time interval $t \in [0.9, 1.1]$.

b Which of the average speeds calculated in part a would be the better estimate for the instantaneous speed at $t = 1$?

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 The sketch graphs I–IV show the level of interest of four students in Maths Methods during semester 1.




a For which student was there a zero rate of change in interest level for the entire semester?

b For which student was there a constant positive rate of change in interest level for the entire semester?

c Describe the other two students' interest levels in Maths Methods during semester 1.

d Sketch a graph showing your own interest level in Maths Methods during semester 1.

6 Calculate the hourly rate of pay for each of the following people based on one event that may be atypical:

a a plumber who receives \$200 for a task that takes 20 minutes to complete

b a surgeon who is paid \$180 for a consultation lasting 15 minutes

c a teacher who receives \$1820 for working a 52-hour week.

7 a Calculate the constant speed of a car if in 1 hour 20 minutes it has driven a distance of 48 km.

b i Draw the distance–time graph for a cyclist riding along a straight road, returning to point O after 9 hours, which fits the following description:

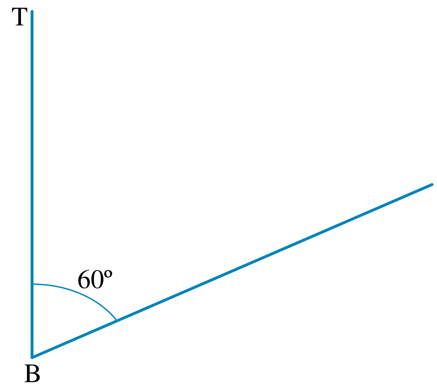
A cyclist travels from O to A in 2 hours at a constant speed of 10 km/h; she then has a rest break for 0.5 hours, after which she rides at constant speed to reach point B, which is 45 km from O. After a lunch break of 1 hour, the cyclist takes 2.5 hours to return to O at constant speed.

ii Calculate the time between the cyclist leaving A and arriving at B and hence calculate the constant speed with which the cyclist rode from A to B.

iii Calculate the constant speed with which the cyclist rode from B to return to O.

iv What was the cyclist's average speed calculated over the entire 9-hour journey?

8 The top T of the stem of a small plant is 24 mm directly above its base B. From the base B a small shoot grows at a rate of 2 mm/week at an angle of 60° to the stem BT. The stem of the plant does not grow further.




a After 3 weeks, find how far vertically the tip of the small shoot is below T.

b The shoot continues to grow at the constant rate of 2 mm/week. How far vertically is its tip below T after $t$ weeks?

c After how many weeks will the tip of the shoot be at the same height as T?

9 a One Australian dollar was worth 0.67 euros in November and by August of the following year one Australian dollar was worth 0.83 euros. What was the average rate of change per month in the exchange value of the Australian dollar with respect to the euro over this time period?

b \$1000 is invested in a term deposit for 3 years at a fixed rate of $r$% per annum interest. If the value of the investment is \$1150 at the end of the 3 years, calculate the value of $r$.

10 Calculate the average rate of change of the following functions over the given interval.

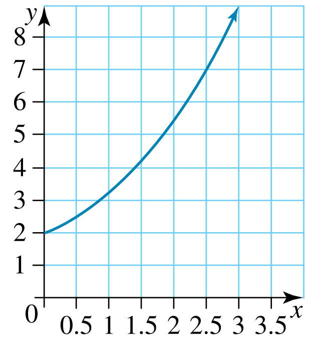
a $f(x) = 2x - x^2,\ x \in [-2, 6]$

b $f(x) = 2 + 3x,\ x \in [12, 16]$

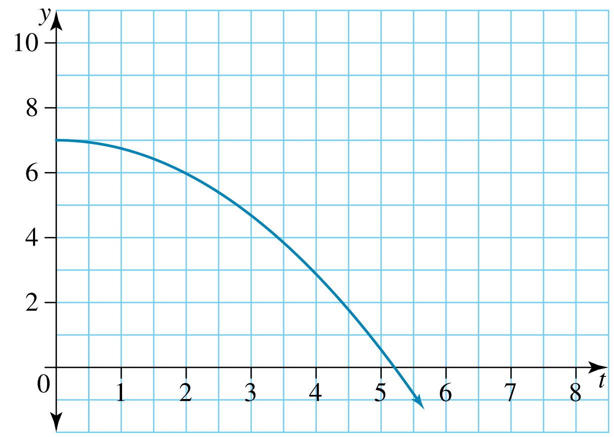
c $f(t) = t^2 + 3t - 1,\ t \in [1, 3]$

d $f(t) = t^3 - t,\ t \in [-1, 1]$

11 a Estimate the gradient at the point on the curve where $x = 2$.



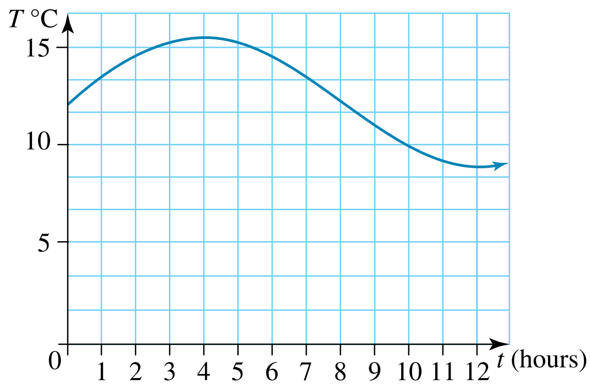

b Estimate the gradient at the points on the curve where $x = 2$ and $x = 4$.




c For the curve in part **b** calculate the average rate of change over the interval $x \in [4, 6]$.

d For the curve in part **b**, find an estimate, to the nearest degree, for the magnitude of the angle at which the tangent to the curve cuts the $x$-axis.

**12** The temperature $T$ °C over the 12-hour time interval ($t$) from midday to midnight is shown in the diagram.




**a** Use the graph and the tangent line method to estimate the rate at which the temperature is changing at:

**i** 2 pm

**ii** 4 pm

**iii** 7 pm

**iv** 10 pm.

**b** At what time does the temperature appear to be decreasing most rapidly?

**c** What is the average rate of change of temperature from 1 pm to 9.30 pm?

**13** A small boat sets out to sea away from a jetty. Its distance in metres from the jetty after $t$ hours is given by $d = \dfrac{200t}{t + 1}$, $t > 0$.

**a** Calculate the average speed of the boat over the first 4 hours.

**b** Sketch the graph of the distance from the jetty against time and give a short description of the motion of the boat.

**c** Draw a tangent to the curve at the point where $t = 4$ and hence find an approximate value for the instantaneous speed of the boat 4 hours after it leaves the jetty.

**d** Use an interval close to $t = 4$ to obtain another estimate of this speed.

**14** Consider the semicircle with equation $y = \sqrt{9 - x^2}$.

**a** Sketch the semicircle and mark the point on the curve where $x = 1$ with its coordinates.

**b** Calculate the gradient of the line joining the points on the curve for which $x = 1$ and $x = 1.5$, and hence estimate the gradient of the curve at $x = 1$.

**c** Draw a tangent to the semicircle at the point where $x = 1$ and use this to estimate the gradient of the curve at $x = 1$.

**d** Use coordinate geometry to find the exact value of the gradient of the semicircle at the point where $x = 1$. Compare the exact value with your answers in parts **b** and **c**.

**MASTER**

**15** Use CAS technology to sketch the curve $y = 4 - 3x^2$ and its tangent at the point where $x = -2$ and to state the equation of the tangent. Hence state the gradient of $y = 4 - 3x^2$ at $x = -2$.

**16** Choose three different points which lie on the graph of $y = 0.5x^2$ and find the gradient at these points. What pattern do you notice?

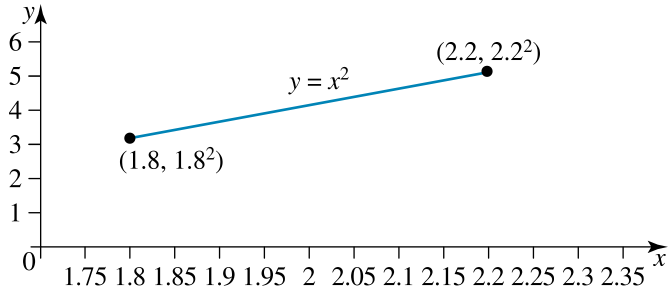
# 12.3 Gradients of secants

studyon

Units 1 & 2
AOS 3
Topic 1
Concept 2

**Gradients of secants**
Concept summary
Practice questions

The gradient of a curve can be estimated by finding the average rate of change between two nearby or near-neighbouring points. As the interval between these two points is reduced the curve becomes 'straighter', which is why the average rate starts to become a good estimate for the actual gradient. This method of zooming in on two close points is called the **near neighbour** method or the **straightening the curve** method.




The graph of the curve $y = x^2$ appears quite straight if viewed on the domain [1.8, 2.2] and straighter still for intervals with endpoints closer together.

If this section of the curve is treated as straight, its gradient is $\dfrac{(2.2)^2 - (1.8)^2}{2.2 - 1.8} = 4$.

This value should be a very good estimate of the gradient of the curve at the midpoint of the interval, $x = 2$. Zooming in closer would either boost confidence in this estimate or produce an even better estimate.

## Gradient of a secant

Straightening the curve gives a good approximation to the gradient of a curve. However, it is still an approximation, since the curve is not absolutely straight. The completely straight line through the endpoints of the interval formed by the two neighbouring points is called a **secant**, or a chord if just the line segment joining the endpoints is formed. The gradient of a secant, the line passing through two points on a curve, can be used to find the gradient of the tangent, the line that just touches the curve at one point.

To illustrate this, consider finding the gradient of the curve $y = x^2$ at the point A where $x = 3$ by forming a secant line through A and a nearby close-neighbour point B.

As A has an $x$-coordinate of 3, let B have an $x$-coordinate of $3 + h$, where $h$ represents the small difference between the $x$-coordinates of points A and B. It does not matter whether $h$ is positive or negative. What matters is that it is small, so that B is close to A.

The $y$-coordinate of A will be $y = (3)^2 = 9$.

The $y$-coordinate of B will be $y = (3 + h)^2$.

The gradient of the secant through the points A(3, 9) and B$(3 + h, (3 + h)^2)$ is:

$$\begin{aligned} m_{\text{secant}} &= \frac{(3+h)^2 - 9}{(3+h) - 3} \\ &= \frac{9 + 6h + h^2 - 9}{3 + h - 3} \\ &= \frac{6h + h^2}{h} \\ &= \frac{h(6+h)}{h} \\ &= 6 + h, \; h \neq 0 \end{aligned}$$

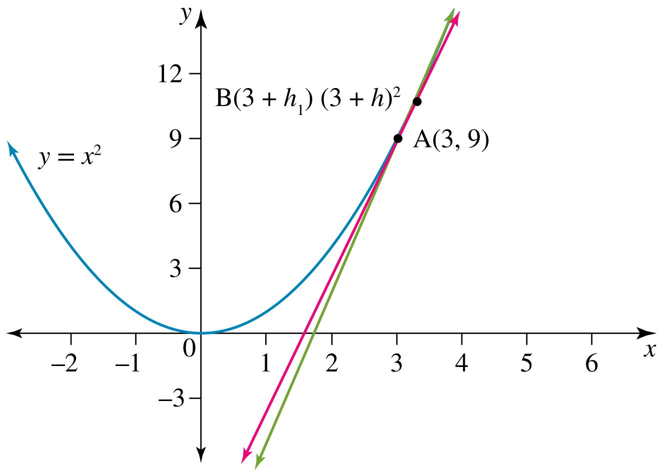
Thus, the gradient of the secant is $6 + h$.

By choosing small values for $h$, the gradient of the secant can be calculated to any desired accuracy.

| $h$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 |
|---|---|---|---|---|---|
| **Gradient $6+h$** | 6.1 | 6.01 | 6.001 | 6.0001 | 6.00001 |

As $h$ becomes smaller, the gradient appears to approach the value of 6.

Also, as $h$ becomes smaller, the point $B(3+h, (3+h)^2)$ moves closer and closer towards point $A(3, 9)$; the secant becomes closer and closer to the tangent at point A.




Therefore, as $h$ approaches 0, the gradient of the secant through A and B approaches the value of the gradient of the tangent at A; that is, as $h$ approaches 0, $(6+h)$ approaches 6.

This statement is written as: as $h \to 0,\ 6+h \to 6$.

Hence, the gradient of the tangent to the curve $y = x^2$ at $x = 3$ is the **limiting** value of the gradient of the secant as $h$ approaches 0.

This statement is written as: the gradient of the tangent is $\lim\limits_{h \to 0}(6+h) = 6$.

### Evaluating the limit expression

For two neighbouring points whose $x$-coordinates differ by $h$, the gradient of the secant will be an expression involving $h$. Algebraic techniques such as expansion and factorisation are used to simplify the expression, allowing any common factor of $h$ in the numerator and denominator to be cancelled, for $h \neq 0$. Once this simplified expression for the gradient of the secant has been obtained, then the limit expression as $h \to 0$ can be calculated simply by replacing $h$ by 0. It is essential that the expression is simplified first before replacing $h$ by 0.

**WORKED EXAMPLE 3**

**Consider the curve with equation $y = 3x - x^2$.**

**a** **Express the gradient of the secant through the points on the curve where $x = 2$ and $x = 2 + h$ in terms of $h$.**

**b** **Use $h = 0.01$ to obtain an estimate of the gradient of the tangent to the curve at $x = 2$.**

**c** **Deduce the gradient of the tangent to the curve at the point where $x = 2$.**

| THINK | WRITE |
|---|---|
| **a** 1 Form the coordinates of the two points on the secant. | **a** $y = 3x - x^2$<br>When $x = 2$, $y = 6 - 4$<br>$= 2$<br>Point $(2, 2)$<br>When $x = 2 + h$,<br>$y = 3(2 + h) - (2 + h)^2$<br>$= 6 + 3h - (4 + 4h + h^2)$<br>$= 6 + 3h - 4 - 4h - h^2$<br>$= 2 - h - h^2$<br>Point $(2 + h,\ 2 - h - h^2)$<br>The two neighbouring points are $(2, 2)$ and $(2 + h, 2 - h - h^2)$. |
| 2 Calculate the gradient of the secant line through the two points and simplify the expression obtained. | $m_{\text{secant}} = \dfrac{(2 - h - h^2) - (2)}{(2 + h) - 2}$<br>$= \dfrac{-h - h^2}{h}$<br>$= \dfrac{-h(1 + h)}{h}$<br>$= -(1 + h),\ h \neq 0$<br>Therefore, the gradient of the secant is $-(1 + h) = -h - 1$. |
| **b** Calculate an estimate of the gradient of the tangent. | **b** Since $h$ is small, the secant's gradient with $h = 0.01$ should be an estimate of the gradient of the tangent.<br>If $h = 0.01$, $m_{\text{secant}} = -0.01 - 1 = -1.01$.<br>An estimate of the gradient of the tangent at $x = 2$ is $-1.01$. |
| **c** Calculate the limiting value of the secant's gradient as $h$ approaches 0. | **c** As $h \to 0$, $(-h - 1) \to -1$<br>$\therefore \lim\limits_{h \to 0}(-h - 1) = -1$<br>$m_{\text{tangent}} = \lim\limits_{h \to 0}(-h - 1)$<br>$= -1$<br>The gradient of the tangent at the point $(2, 2)$ is $-1$. |

## EXERCISE 12.3 Gradients of secants

**PRACTISE**

Work without CAS

1 WE3 Consider the curve with equation $y = 2x^2 - x$.

a Express the gradient of the secant through the points on the curve where $x = -1$ and $x = -1 + h$ in terms of $h$.

b Use $h = 0.01$ to obtain an estimate of the gradient of the tangent to the curve at $x = -1$.

c Deduce the gradient of the tangent to the curve at the point where $x = -1$.

2 Consider the curve with equation $y = \frac{1}{3}x^3 - x^2 + x + 5$.

a Calculate the gradient of the chord joining the points on the curve where $x = 1$ and $x = 1 - h$.

b Hence deduce the gradient of the tangent at the point where $x = 1$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

3 a Calculate the gradient of the secant through the points where $x = 1$ and $x = 1.5$ on the curve with equation $y = 2x^2 + 1$.

b Calculate the gradient of the secant through the points where $x = -2$ and $x = -2.1$ on the graph of the function defined by $f(x) = 3 + x - x^3$.

4 a Sketch the curve $y = x^2$ and draw the secant through each of the following pairs of points: $(-1, 1)$ and $(3, 9)$; $(0, 0)$ and $(3, 9)$; $(2, 4)$ and $(3, 9)$.

b Of the three secants in part a, which one is the best approximation to the tangent at $(3, 9)$? How could this approximation be improved?

5 a Calculate the gradient of the chord joining the points where $x = 3.9$ and $x = 4$ on the curve with equation $y = \frac{x^4}{4}$.

b Hence state a whole-number estimate for the gradient of the tangent to the curve at the point where $x = 4$.

c How could the estimate be improved?

6 a Calculate the coordinates of the point A on the parabola $y = 4 - x^2$ for which $x = 2 + h$.

b Express the gradient of the chord joining the point A to the point C$(2, 0)$ in terms of $h$.

c If the gradient of the chord AC is $-5$, identify the value of $h$ and state the coordinates of A.

d If A is the point $(2.1, -0.41)$, identify the value of $h$ and evaluate the gradient of the chord AC.

e The point B for which $x = 2 - h$ also lies on the parabola $y = 4 - x^2$. Obtain an expression for the gradient of the chord BC and evaluate this gradient for $h = 0.1$.

f Calculate the gradient of the secant which passes through the points A and B.

7 The point D lies on the curve with equation $y = x^3 + x$.

a Obtain an expression for the $y$-coordinate of D given its $x$-coordinate is $3 + h$.

b Find the gradient of the secant passing through the point D and the point on the curve $y = x^3 + x$ where $x = 3$.

c Evaluate this gradient if $h = 0.001$.

d Suggest the value for the gradient of the tangent to the curve at $x = 3$.

8 a Obtain the coordinates of the points M and N on the hyperbola $y = \frac{1}{x}$ where $x = 1$ and $x = 1 + h$ respectively.

b Find, in terms of $h$, the gradient of the secant passing through the points M and N.

c As $h \to 0$, what value does this gradient approach?

d Deduce the gradient of the tangent to $y = \frac{1}{x}$ at the point M.

9 For the graph of $y = 1 + 3x - x^2$:

a Calculate the gradient of the secant passing through the points on the graph with $x$-coordinates 1 and $1 - h$.

b Write a limit expression for the gradient of the tangent to curve at the point where $x = 1$.

c Hence, state the gradient of the tangent to the curve at the point for which $x = 1$.

10 For the graph of $y = (x - 2)(x + 1)(x - 3)$:

a Show that the gradient of the secant passing through the points on the graph with $x$-coordinates 0 and $h$ is $h^2 - 4h + 1$ for $h \neq 0$.

b Hence, write down a limit expression for the gradient of the tangent to curve at $x = 0$.

c State the gradient of the tangent to the curve at the point for which $x = 0$.

11 Simplify the given secant gradient expression where possible for $h \neq 0$, and then evaluate the given limit.

a $m_{\text{secant}} = 3 + h$; $\lim\limits_{h \to 0}(3 + h)$

b $m_{\text{secant}} = \dfrac{8h - 2h^2}{h}$; $\lim\limits_{h \to 0}\left(\dfrac{8h - 2h^2}{h}\right)$

c $m_{\text{secant}} = \dfrac{(2 + h)^3 - 8}{h}$; $\lim\limits_{h \to 0}\left(\dfrac{(2 + h)^3 - 8}{h}\right)$

d $m_{\text{secant}} = \dfrac{(1 + h)^4 - 1}{h}$; $\lim\limits_{h \to 0}\left(\dfrac{(1 + h)^4 - 1}{h}\right)$

e $m_{\text{secant}} = \dfrac{\dfrac{1}{(4 + h)} - \dfrac{1}{4}}{h}$; $\lim\limits_{h \to 0}\left(\dfrac{-1}{4(4 + h)}\right)$

f $m_{\text{secant}} = \dfrac{h}{\sqrt{1 + h} - 1}$; $\lim\limits_{h \to 0}\left(\dfrac{h}{\sqrt{1 + h} - 1}\right)$

12 For the function with the rule $f(x) = x^2$:

a Draw a diagram showing the curve and the secant through the points A and B on the curve which have $x$-coordinates $a$ and $b$ respectively.

b Calculate the gradient of the secant through the points A and B in terms of $a$ and $b$.

c Hence, deduce the gradient of the tangent to curve at:

i the point A   ii the point B.

MASTER

13 Evaluate using CAS technology.

a $\lim\limits_{h \to 0}\left(\dfrac{(h + 2)^2(h + 1) - 4}{h}\right)$

b $\lim\limits_{x \to \infty}\left(\dfrac{1}{x} + 3\right)$

c $\lim\limits_{x \to \infty}\dfrac{2x + 1}{3x - 2}$

14 For the graph of the function defined by $f(x) = \sqrt{x}$:

a Calculate the gradient of the secant which passes through the points on the graph which have $x$-coordinates 9 and $9 + h$.

**b** Hence state the limit expression for the gradient of the tangent to the graph at the point for which $x = 9$.

**c** Calculate this limit to obtain the gradient of the graph at the point where $x = 9$.

# 12.4 The derivative function

studyon

Units 1 & 2
AOS 3
Topic 1
Concept 3

**The derivative function**
Concept summary
Practice questions

While the work of the seventeenth century mathematicians Sir Isaac Newton and Gottfried Wilhelm Leibniz created a bitter rivalry between them, it is now widely accepted that each independently invented calculus. Their geniuses bequeathed to the world the most profound of all mathematical inventions and one which underpins most advanced mathematics even today. In essence, it all started from the problem of finding the rate of change of a function at a particular instant: that is, of finding the gradient of a curve at any point.

The approach both Newton and Leibniz used is based on the concept of a limit: that if two points on a curve are sufficiently close, the tangent's gradient is the limiting value of the secant's gradient. The only difference in their approach was the form of notation each used.

Both mathematicians essentially found the gradient of a curve $y = f(x)$ at some general point $(x, f(x))$, by:

- taking a neighbouring point $(x + h,\ f(x + h))$ where $h$ represents a small change in the $x$-coordinates
- forming the **difference quotient** $\dfrac{f(x + h) - f(x)}{h}$, which is the gradient of the secant or the average rate of change of the function between the two points
- calculating the gradient of the curve as the limiting value of the difference quotient as $h \to 0$.

This method created a new function called the **gradient function** or the **derived function** or the **derivative function**, which is frequently just referred to as the derivative. For the function $f$, the symbol for the gradient function is $f'$, which is read as '$f$ dashed'. The rule for the derivative is $f'(x)$.

## Definition of the gradient or derivative function

For the function $y = f(x)$, its gradient or derivative function, $y = f'(x)$, is defined as the limiting value of the difference quotient as $h \to 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Once the derivative function is obtained, the gradient at a given point on the curve $y = f(x)$ is evaluated by substituting the $x$-coordinate of the given point in place of $x$ in the equation of the derivative function. The value of $f'(a)$ represents the rate of change of the function at the instant when $x = a$; it gives the gradient of the tangent to the curve $y = f(x)$ at the point on the curve where $x = a$.

The limit expression must be simplified in order to cancel the $h$ term from its denominator before the limit can be evaluated by substituting $h = 0$. As a limit is concerned with the value approached as $h \to 0$, there is no need to add the proviso $h \neq 0$ when cancelling the $h$ term from the denominator of the limit expression.

**WORKED EXAMPLE 4**

**Consider the function for which $f(x) = x^2 - 5x + 4$.**

**a** **Use the difference quotient definition to form the rule $f'(x)$ for the gradient function.**

**b** **Hence obtain the gradient of the graph of the function at the point (1, 0).**

**c** **Evaluate the instantaneous rate of change of the function when $x = 4$.**

| THINK | WRITE |
|---|---|
| **a 1** Form an expression for $f(x+h)$. | **a** $f(x) = x^2 - 5x + 4$<br>Replacing every $x$ in the function rule with $(x+h)$, we get $f(x+h) = (x+h)^2 - 5(x+h) + 4$. |
| **2** State the difference quotient definition. | The difference quotient is $\dfrac{f(x+h)-f(x)}{h}$. |
| **3** Substitute the expressions for the functions into the difference quotient. | $\therefore \dfrac{f(x+h)-f(x)}{h}$<br>$= \dfrac{[(x+h)^2 - 5(x+h) + 4] - [x^2 - 5x + 4]}{h}$ |
| **4** Expand the numerator and simplify.<br>*Note:* The terms in the numerator not containing $h$ have cancelled out. | $= \dfrac{x^2 + 2xh + h^2 - 5x - 5h + 4 - x^2 + 5x - 4}{h}$<br>$= \dfrac{2xh + h^2 - 5h}{h}$ |
| **5** State the definition of the gradient function. | By definition,<br>$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$ |
| **6** Substitute the expression for the difference equation and simplify.<br>*Note:* As the limit is yet to be evaluated, we must write $\lim\limits_{h \to 0}$ at each line of the simplification steps. | $f'(x) = \lim\limits_{h \to 0} \dfrac{2xh + h^2 - 5h}{h}$<br>$= \lim\limits_{h \to 0} \dfrac{h(2x + h - 5)}{h}$<br>$= \lim\limits_{h \to 0} (2x + h - 5)$ |
| **7** Evaluate the limit to calculate the rule for the gradient function. | Substitute $h = 0$ to calculate the limit.<br>$f'(x) = 2x + 0 - 5$<br>$= 2x - 5$<br>The rule for the gradient function is $f'(x) = 2x - 5$. |
| **b** Calculate the value of the gradient function at the given point. | **b** For the point $(1, 0)$, $x = 1$.<br>The gradient at this point is the value of $f'(1)$.<br>$f'(x) = 2x - 5$<br>$f'(1) = 2(1) - 5$<br>$f'(1) = -3$<br>Therefore the gradient at the point $(1, 0)$ is $-3$. |

| | |
|---|---|
| **c** Evaluate the instantaneous rate of change at the given point. | **c** The instantaneous rate of change at $x = 4$ is $f'(4)$.<br>$f'(x) = 2x - 5$<br>$f'(4) = 2(4) - 5$<br>$= 3$<br>The rate of change of the function at $x = 4$ is 3. |

## Other forms of notation

It was the eighteenth century mathematicians Joseph Lagrange and Augustin-Louis Cauchy who introduced the notation and the method to evaluate the limit, respectively, similar to that used in Worked example 4. Other forms of notation include replacing $h$ for the **small increment** or small change in $x$ by one of the symbols $\Delta x$ or $\delta x$, where $\Delta$ and $\delta$ are upper and lower case respectively of the Greek letter 'delta'.

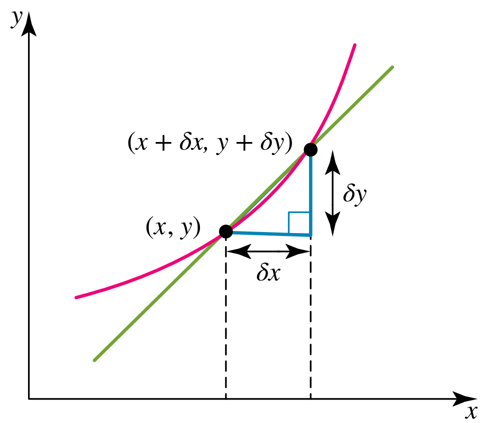
The gradient function could be expressed as $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ or as $f'(x) = \lim\limits_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$. Newton's notation for the derivative function is commonly used in higher-level studies of motion but this is beyond our scope. Leibniz's notation, however, is important to our current study.

## Leibniz's notation

Leibniz used the symbol $\delta x$ ('delta $x$') for a small change or small increment in $x$ and the symbol $\delta y$ for a small change or small increment in $y$. These are used as single symbols, not as products of two symbols.

He expressed the coordinates of the two neighbouring points as $(x, y)$ and $(x + \delta x, y + \delta y)$.

The gradient of the secant, or difference quotient, is $\frac{(y + \delta y) - y}{(x + \delta x) - x} = \frac{\delta y}{\delta x}$.




The gradient of the tangent at the point $(x, y)$ is therefore $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$.

Leibniz used the symbol $\frac{dy}{dx}$ for the rule of this gradient or derivative function.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

This is different notation but exactly the same thinking to that used previously in the chapter.

Note that $\frac{dy}{dx}$ (pronounced 'dee y by dee x') is a single symbol for the derivative; it is not a fraction. It represents the derivative of the dependent variable $y$ with respect to the independent variable $x$.

Importantly, both symbols $\frac{dy}{dx}$ and $f'(x)$ are used for the derivative of $y = f(x)$.

The gradient of the tangent at a given point is calculated by substituting the $x$-coordinate of the given point in place of $x$ in the expression for $\frac{dy}{dx}$.

$\left.\frac{dy}{dx}\right|_{x=a}$ is sometimes used as the notation for the evaluation of $\frac{dy}{dx}$ at $x = a$.

WORKED EXAMPLE 5

**Use Leibniz's notation to obtain the derivative of $y = x^2 + 1$ and hence calculate the gradient of the tangent to the curve $y = x^2 + 1$ at the point where $x = 1$.**

| THINK | WRITE |
|---|---|
| 1 Use the coordinates of the two neighbouring points to form an expression for $\delta y$. | Let the neighbouring points be $(x, y)$ and $(x + \delta x, y + \delta y)$ on the curve $y = x^2 + 1$.<br>For the point $(x, y)$, $y = x^2 + 1 \ldots (1)$<br>For the point $(x + \delta x, y + \delta y)$,<br>$y + \delta y = (x + \delta x)^2 + 1$<br>$\delta y = (x + \delta x)^2 + 1 - y$<br>Substitute $y = x^2 + 1$ from equation (1).<br>$\therefore \delta y = \left[(x + \delta x)^2 + 1\right] - \left[(x^2 + 1)\right]$ |
| 2 Form the difference quotient. | $\dfrac{\delta y}{\delta x} = \dfrac{[(x + \delta x)^2 + 1] - [x^2 + 1]}{\delta x}$ |
| 3 Expand and simplify the numerator.<br>*Note:* Write the square of $\delta x$ as $(\delta x)^2$. | $\dfrac{\delta y}{\delta x} = \dfrac{x^2 + 2x(\delta x) + (\delta x)^2 + 1 - x^2 - 1}{\delta x}$<br>$= \dfrac{2x(\delta x) + (\delta x)^2}{\delta x}$ |
| 4 State the definition of the derivative and substitute the expression for $\dfrac{\delta y}{\delta x}$. | $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$<br>$= \lim\limits_{\delta x \to 0} \dfrac{2x(\delta x) + (\delta x)^2}{\delta x}$ |
| 5 Evaluate the limit. | Factorise and cancel $\delta x$ from the denominator:<br>$\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta x(2x + \delta x)}{\delta x}$<br>$= \lim\limits_{\delta x \to 0} (2x + \delta x)$<br>$= 2x$ |
| 6 Use the derivative to obtain the gradient of the tangent to the curve at the given point. | When $x = 1$, $\dfrac{dy}{dx} = 2(1)$<br>$= 2$<br>The gradient of the tangent to $y = x^2 + 1$ at the point where $x = 1$ is 2. |

## Differentiation from first principles

The process of obtaining the gradient or derivative function is called **differentiation**. Using the limit definition from the difference quotient to obtain the derivative is called **differentiation from first principles**.

From first principles, the derivative is calculated from forms such as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \text{ or } f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \text{ or } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

The derivative with respect to $x$ of the function $y = f(x)$ is $\frac{dy}{dx} = f'(x)$.

Once the derivative is known, the gradient of the tangent to the curve $y = f(x)$ at a point where $x = a$ is calculated as $\left.\frac{dy}{dx}\right|_{x=a}$ or $f'(a)$. Alternatively, this gradient value could be calculated directly from first principles using $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

In Worked examples 4 and 5, second-degree polynomials were differentiated from first principles: in Worked example 4 with function notation and in Worked example 5 with Leibniz's notation. For higher-degree polynomials, the binomial theorem or Pascal's triangle may be required to expand terms to calculate the derivative from first principles.

**WORKED EXAMPLE 6** **Differentiate $3x - 2x^4$ with respect to $x$ using first principles.**

| THINK | WRITE |
|---|---|
| 1 Define $f(x)$ and form the expression for $f(x+h)$. *Note:* Leibniz's notation could be used instead of function notation. | Let $f(x) = 3x - 2x^4$ <br> $f(x+h) = 3(x+h) - 2(x+h)^4$ |
| 2 State the limit definition of the derivative. | $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ |
| 3 Substitute the expressions for $f(x)$ and $f(x+h)$. | $\therefore f'(x) = \lim\limits_{h \to 0} \frac{\left[3(x+h) - 2(x+h)^4\right] - \left[3x - 2x^4\right]}{h}$ |
| 4 Expand and simplify the numerator. *Note:* Either the binomial theorem or Pascal's triangle could be used to expand $(x+h)^4$. | $f'(x) = \lim\limits_{h \to 0} \frac{3(x+h) - 2(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - 3x + 2x^4}{h}$ <br> $= \lim\limits_{h \to 0} \frac{\cancel{3x} + 3h - \cancel{2x^4} - 8x^3h - 12x^2h^2 - 8xh^3 - 2h^4 - \cancel{3x} + \cancel{2x^4}}{h}$ <br> $= \lim\limits_{h \to 0} \frac{3h - 8x^3h - 12x^2h^2 - 8xh^3 - 2h^4}{h}$ |
| 5 Factorise, to allow $h$ to be cancelled from the denominator. | $f'(x) = \lim\limits_{h \to 0} \frac{h\left[3 - 8x^3 - 12x^2h - 8xh^2 - 2h^3\right]}{h}$ <br> $= \lim\limits_{h \to 0} (3 - 8x^3 - 12x^2h - 8xh^2 - 2h^3)$ |
| 6 Evaluate the limit by substituting 0 for $h$. | $f'(x) = 3 - 8x^3 - 12x^2(0) - 8x(0)^2 - 2(0)^3$ <br> $= 3 - 8x^3$ |
| 7 State the answer. | The derivative of $3x - 2x^4$ with respect to $x$ is $3 - 8x^3$. |

## EXERCISE 12.4 The derivative function

**PRACTISE**

Work without CAS

1 WE4 Consider the function $f(x) = 2x^2 + x + 1$.

a Use the difference quotient definition to form the rule $f'(x)$ for the gradient function.

b Hence obtain the gradient of the function at the point $(-1, 2)$.

c Evaluate the instantaneous rate of change of the function when $x = 0$.

2 Use the difference quotient definition to form the derivative of $f(x) = x^3$.

3 WE5 Use Leibniz's notation to obtain the derivative of $y = x^2 + x$ and hence calculate the gradient of the tangent to the curve $y = x^2 + x$ at the point where $x = 1$.

4 If $f(x) = (3 - x)(x + 1)$, evaluate $\lim\limits_{\delta x \to 0} \dfrac{f(2 + \delta x) - f(2)}{\delta x}$ and give a geometrical interpretation of what the answer represents.

5 WE6 Using first principles, differentiate $5x + 3x^4$ with respect to $x$.

6 Use first principles to obtain $\dfrac{dy}{dx}$ if $y = ax^5 + c,\ a, c \in R$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 Consider the function defined by $f(x) = x^2$.

a Form the difference quotient $\dfrac{f(x + h) - f(x)}{h}$.

b Use the difference quotient to find $f'(x)$, the rule for the gradient function.

c Hence state the gradient of the tangent at the point $(3, 9)$.

d Show that the gradient at the point $(3, 9)$ could be calculated by evaluating $\lim\limits_{h \to 0} \dfrac{f(3 + h) - f(3)}{h}$.

8 The points $P(1, 5)$ and $Q(1 + \delta x, 3(1 + \delta x)^2 + 2)$ lie on the curve $y = 3x^2 + 2$.

a Express the average rate of change of $y = 3x^2 + 2$ between P and Q in terms of $\delta x$.

b Calculate the gradient of the tangent to the curve at P.

c S and T are also points on the curve $y = 3x^2 + 2$, where $x = 2$ and $x = 2 + \delta x$ respectively.

i Give the coordinates of the points S and T.

ii Form an expression in terms of $\delta x$ for the gradient of the secant through S and T.

iii Calculate the gradient of the tangent to the curve at S.

9 Explain the geometric meaning of the following.

a $\dfrac{f(x + h) - f(x)}{h}$

b $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

c $\lim\limits_{h \to 0} \dfrac{f(3 + h) - f(3)}{h}$

d $\dfrac{f(3) - f(3 - h)}{h}$

10 Given $f(x) = 3x - 2x^2$:

a Find $f'(x)$ using the limit definition.

b Calculate the gradient of the tangent to the curve at $(0, 0)$.

c Sketch the curve $y = f(x)$ showing the tangent at $(0, 0)$.

Sir Isaac Newton
(1643–1727)

Gottfried Wilhelm
Leibniz
(1646–1716)

**11** Using first principles, differentiate the following with respect to $x$.

**a** $f(x) = 8x^2 + 2$
**b** $f(x) = \frac{1}{2}x^2 - 4x - 1$
**c** $f(x) = 6 - 2x$
**d** $f(x) = 5$
**e** $f(x) = x^3 - 6x^2 + 2x$
**f** $f(x) = 2 + x^6$

**12** Find $\frac{dy}{dx}$ using the definition $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ for the following functions.

**a** $y = 4 - x^2$
**b** $y = x^2 + 4x$
**c** $y = \frac{x^3}{3}$
**d** $y = x(x + 1)$

**13** Given $f(x) = (x + 5)^2$:

**a** find $f'(x)$ using first principles
**b** calculate $f'(-5)$ and explain its geometric meaning
**c** calculate the gradient of the tangent to the curve $y = f(x)$ at its $y$-intercept
**d** calculate the instantaneous rate of change of the function $y = f(x)$ at $(-2, 9)$.

**14** Given $f(x) = 2(x - 4)(x + 2)$:

**a** sketch the graph of $y = f(x)$ showing its key points
**b** find $f'(x)$ using first principles
**c** calculate $f'(1)$ and hence draw the tangent at $x = 1$ on the graph
**d** calculate the gradient of the tangent to the curve at each of its $x$-intercepts and draw these tangents on the graph.

**15** Consider the function $f: R \to R, f(x) = ax^2 + bx + c$ where $a$, $b$ and $c$ are constants.

**a** Using first principles, find $f'(x)$.
**b** Form the derivative function of $f$ and express it as a mapping.
**c** Hence, write down the derivative of $f(x) = 3x^2 + 4x + 2$.
**d** Compare the degree of the function $f$ with that of the function $f'$.

**16** **a** Consider the function $g: R \to R, g(x) = x^3$.

**i** Factorise $(2 + h)^3 - 8$ as a difference of two cubes.
**ii** Write down a limit expression for $g'(2)$.
**iii** Use the answers to parts **i** and **ii** to evaluate $g'(2)$.

**b** Find, from first principles, the derivative of:

**i** $\frac{1}{x}, x \neq 0$
**ii** $\frac{1}{x^2}, x \neq 0$
**iii** $\sqrt{x}, x > 0$

**MASTER**

**17** Define the function $f(x) = x^2$ and use CAS technology to calculate

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

**18** **a** Define the function $f(x) = ax^3 + bx^2 + cx + d$ and use CAS technology to calculate $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

**b** Compare the degree of $f(x) = ax^3 + bx^2 + cx + d$ with that of its derivative.

# 12.5 Differentiation of polynomials by rule

study on

Units 1 & 2
AOS 3
Topic 1
Concept 4

**Differentiation of polynomials by rule**
Concept summary
Practice questions

Differentiation by first principles using the limit definition is an important procedure, yet it can be quite lengthy and for non-polynomial functions it can be quite difficult. Examining the results obtained from differentiation of polynomials by first principles, some simple patterns appear. These allow some basic rules for differentiation to be established.

## Differentiation of polynomial functions

The results obtained from Worked examples 4, 5 and 6 and from Exercise 12.4 Practise questions are compiled into the following table:

| $f(x)$ | $f'(x)$ |
|---|---|
| $x^2 - 5x + 4$ | $2x - 5$ |
| $2x^2 + x + 1$ | $4x + 1$ |
| $x^3$ | $3x^2$ |
| $x^2 + 1$ | $2x$ |
| $x^2 + x$ | $2x + 1$ |
| $3x - 2x^4$ | $3 - 8x^3$ |
| $5x + 3x^4$ | $5 + 12x^3$ |
| $ax^5 + c$ | $5ax^4$ |

Close examination of these results suggests that the derivative of the polynomial function, $f(x) = x^n$, $n \in N$ is $f'(x) = nx^{n-1}$, a polynomial of one degree less. This observation can be proven to be correct by differentiating $x^n$ from first principles.

## Derivative of $x^n$, $n \in N$

Let $f(x) = x^n$, $n \in N$

Then $f(x + h) = (x + h)^n$

By definition, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$$\therefore f'(x) = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

To simplify this limit to the stage where it can be evaluated, the binomial theorem is used to expand $(x + h)^n$.

$$f'(x) = \lim_{h \to 0} \frac{\left(x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n\right) - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n}{h}$$

As expected, the remaining terms in the numerator have a common factor of $h$.

$$f'(x) = \lim_{h \to 0} \frac{h\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \ldots + h^{n-1}\right]}{h}$$

$$= \lim_{h \to 0}\left(\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \ldots + h^{n-1}\right)$$

Evaluating the limit by replacing $h$ by zero,

$$f'(x) = \binom{n}{1}x^{n-1}$$

Recall that the binomial coefficient $\binom{n}{1} = n$ leads to the result that $f'(x) = nx^{n-1}$.

This proves the original observation.

**If $f(x) = x^n$, $n \in N$**
**then $f'(x) = nx^{n-1}$**

It also follows that any coefficient of $x^n$ will be unaffected by the differentiation operation. If $f(x) = ax^n$ where the coefficient $a \in R$, then $f'(x) = anx^{n-1}$. However, if a function is itself a constant then its graph is a horizontal line with zero gradient. The derivative with respect to $x$ of such a function must be zero.

**For $f(x) = c$ where $c$ is a constant, $f'(x) = 0$**

Where the polynomial function consists of the sum or difference of a number of terms of different powers of $x$, the derivative is calculated as the sum or difference of the derivative of each of these terms.

**If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$**
**then $f'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + \ldots + a_1$**

The above results illustrate what are known as the **linearity properties** of the derivative:

- If $f(x) = g(x) \pm h(x)$ then $f'(x) = g'(x) \pm h'(x)$.
- If $f(x) = ag(x)$ where $a$ is a constant, then $f'(x) = ag'(x)$.

## Calculating the derivative

The linearity properties of the derivative enable the derivative of $f(x) = 3x^4$ to be calculated as $f'(x) = 3 \times 4x^3 \Rightarrow f'(x) = 12x^3$ and the derivative of $y = 3x^4 + x^2$ to be calculated as $\frac{dy}{dx} = 12x^3 + 2x$. This is differentiation by rule. Differentiation becomes an operation that can be applied to a polynomial function with relative ease in comparison with the process of differentiation from first principles. While the limit definition defines the derivative function, from now on we will only use first principles to calculate the derivative if specific instructions to use this method are given.

The notation for the derivative has several forms attributable to several mathematicians, some of whom have already been acknowledged. These forms include $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}(f(x))$, $\frac{df}{dx}$ or $D_x(f)$.

This means that 'the derivative of $3x^4$ equals $12x^3$' could be written as: $f'(x) = 12x^3$ or $\frac{dy}{dx} = 12x^3$ or $\frac{d}{dx}(3x^4) = 12x^3$ or $\frac{df}{dx} = 12x^3$ or $D_x(f) = 12x^3$.

For functions of variables other than $x$, such as for example $p = f(t)$, the derivative would be $\frac{dp}{dt} = f'(t)$.

**WORKED EXAMPLE 7**

**a** Calculate $\frac{d}{dx}(5x^3 - 8x^2)$.

**b** Differentiate $\frac{1}{15}x^5 - 7x + 9$ with respect to $x$.

**c** If $p(t) = (2t + 1)^2$, calculate $p'(t)$.

**d** If $y = \frac{x - 4x^5}{x}$, $x \neq 0$, calculate $\frac{dy}{dx}$.

| THINK | WRITE |
| --- | --- |
| **a** Apply the rules for differentiation of polynomials. | **a** This is a difference of two functions.<br>$\frac{d}{dx}(5x^3 - 8x^2)$<br>$= \frac{d}{dx}(5x^3) - \frac{d}{dx}(8x^2)$<br>$= 5 \times 3x^{3-1} - 8 \times 2x^{2-1}$<br>$= 15x^2 - 16x$ |
| **b** Choose which notation to use and then differentiate. | **b** Let $f(x) = \frac{1}{15}x^5 - 7x + 9$<br>$f'(x) = \frac{1}{15} \times 5x^{5-1} - 7 \times 1x^{1-1} + 0$<br>$= \frac{1}{3}x^4 - 7$ |
| **c** **1** Express the function in expanded polynomial form. | **c** $p(t) = (2t + 1)^2$<br>$= 4t^2 + 4t + 1$ |
| **2** Differentiate the function. | $\therefore p'(t) = 8t + 4$ |
| **d** **1** Express the function in partial fraction form and simplify.<br>*Note:* An alternative method is to factorise the numerator as $y = \frac{x(1 - 4x^4)}{x}$ and cancel to obtain $y = 1 - 4x^4$.<br>Another alternative is to use an index law to write $y = x^{-1}(x - 4x^5)$ and expand using index laws to obtain $y = 1 - 4x^4$. | **d** $y = \frac{x - 4x^5}{x}$, $x \neq 0$<br>$= \frac{x}{x} - \frac{4x^5}{x}$<br>$= 1 - 4x^4$ |
| **2** Calculate the derivative. | $y = 1 - 4x^4$<br>$\frac{dy}{dx} = -16x^3$ |

## Using the derivative

Differential calculus is concerned with the analysis of rates of change. At the start of this chapter we explored ways to estimate the instantaneous rate of change of a function. Now, at least for polynomial functions, we can calculate these rates of change using calculus. The instantaneous rate of change is obtained by evaluating the derivative at a particular point. An average rate, however, is the gradient of the chord joining the endpoints of an interval and its calculation does not involve the use of calculus.

**WORKED EXAMPLE 8**

**Consider the polynomial function with equation $y = \frac{x^3}{3} - \frac{2x^2}{3} + 7$.**

**a** **Calculate the rate of change of the function at the point (3, 10).**

**b** **Calculate the average rate of change of the function over the interval $x \in [0, 3]$.**

**c** **Obtain the coordinates of the point(s) on the graph of the function where the gradient is $-\frac{4}{9}$.**

| THINK | WRITE |
| --- | --- |
| **a** **1** Differentiate the function. | **a** $y = \frac{x^3}{3} - \frac{2x^2}{3} + 7$ <br> $\frac{dy}{dx} = \frac{3x^2}{3} - \frac{4x}{3}$ <br> $= x^2 - \frac{4x}{3}$ |
| **2** Calculate the rate of change at the given point. | At the point (3, 10), $x = 3$. <br> The rate of change is the value of $\frac{dy}{dx}$ when $x = 3$. <br> $\frac{dy}{dx} = (3)^2 - \frac{4(3)}{3}$ <br> $= 5$ <br> The rate of change of the function at (3, 10) is 5. |
| **b** **1** Calculate the endpoints of the interval. | **b** $y = \frac{x^3}{3} - \frac{2x^2}{3} + 7$ over $[0, 3]$ <br> When $x = 0, y = 7 \Rightarrow (0, 7)$ <br> When $x = 3, y = 10 \Rightarrow (3, 10)$ (given) |
| **2** Calculate the average rate of change over the given interval. | The gradient of the line joining the endpoints of the interval is: <br> $m = \frac{10 - 7}{3 - 0}$ <br> $= 1$ <br> Therefore, the average rate of change is 1. |
| **c** **1** Form an equation from the given information. | **c** When the gradient is $-\frac{4}{9}$, $\frac{dy}{dx} = -\frac{4}{9}$ <br> $\therefore x^2 - \frac{4x}{3} = -\frac{4}{9}$ |

**2** Solve the equation.

Multiply both sides by 9.

$$9x^2 - 12x = -4$$
$$9x^2 - 12x + 4 = 0$$
$$(3x - 2)^2 = 0$$
$$3x - 2 = 0$$
$$x = \frac{2}{3}$$

**3** Calculate the $y$-coordinate of the point.

$$y = \frac{1}{3}x^3 - \frac{2}{3}x^2 + 7$$

When $x = \frac{2}{3}$,

$$y = \frac{1}{3} \times \left(\frac{2}{3}\right)^3 - \frac{2}{3} \times \left(\frac{2}{3}\right)^2 + 7$$
$$= \frac{8}{81} - \frac{8}{27} + 7$$
$$= \frac{-16}{81} + 7$$
$$= 6\frac{65}{81}$$

At the point $\left(\frac{2}{3}, 6\frac{65}{81}\right)$ the gradient is $-\frac{4}{9}$.

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Graph of a derivative function
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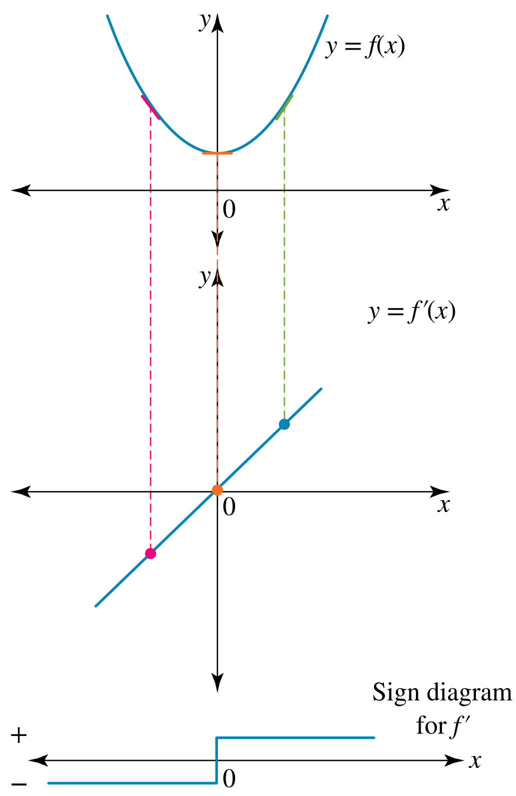
## The graph of the derivative function

The derivative of a polynomial function in $x$ is also a polynomial function but with one degree less than the original polynomial.

As a mapping, for $n \in N$,
if $f: R \rightarrow R, f(x) = x^n$ then
$f': R \rightarrow R, f'(x) = nx^{n-1}$.

The graphs of a function and its derivative function are interrelated. To illustrate this, consider the graphs of $y = f(x)$ and $y = f'(x)$ for the functions defined as $f: R \rightarrow R, f(x) = x^2 + 1$ and $f': R \rightarrow R, f'(x) = 2x$.



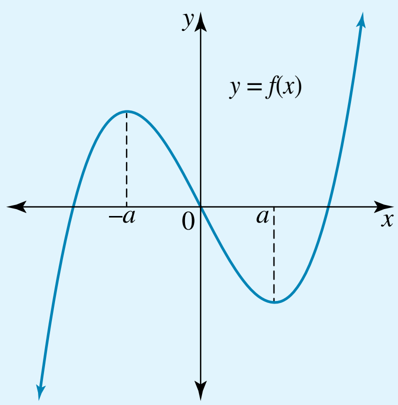
A comparison of the graphs of $y = f(x)$ and $y = f'(x)$ shows:

- The graph of $f$ is quadratic (degree 2) and the graph of $f'$ is linear (degree 1).
- The $x$-coordinate of the turning point of $f$ is the $x$-intercept of $f'$ (this is because the tangent at the turning point is horizontal so its gradient, $f'(x)$, is zero).
- Over the section of the domain where any tangent to the graph of $f$ would have a negative gradient, the graph of $f'$ lies below the $x$-axis.
- Over the section of the domain where any tangent to the graph of $f$ would have a positive gradient, the graph of $f'$ lies above the $x$-axis.
- The turning point on the graph of $f$ is a minimum and the graph of $f'$ cuts the $x$-axis from below to above; that is, from negative to positive $f'$ values.

These connections enable the graph of the derivative function to be deduced from the graph of a function.

**WORKED EXAMPLE 9**

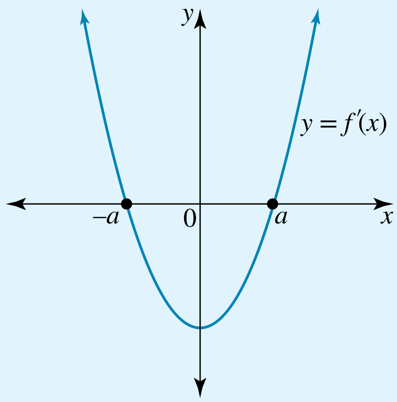
**The graph of a cubic polynomial function $y = f(x)$ is given. The turning points occur where $x = \pm a$. Sketch a possible graph for its gradient function.**




| THINK | WRITE |
|---|---|
| 1 Identify the values of the $x$-intercepts of the gradient function's graph. | The $x$-coordinates of the turning points of $y = f(x)$ give the $x$-intercepts of the gradient graph.<br>Turning points of $y = f(x)$ occur at $x = \pm a$<br>$\therefore f'(\pm a) = 0$<br>$\Rightarrow$ gradient graph has $x$-intercepts at $x = \pm a$. |
| 2 By considering the gradient of any tangent drawn to the given curve at sections on either side of the turning points, determine whether the corresponding gradient graph lies above or below its $x$-axis. | For $x < -a$, the gradient of the curve is positive<br>$\Rightarrow$ gradient graph lies above the $x$-axis.<br>For $-a < x < a$, the gradient of the curve is negative<br>$\Rightarrow$ gradient graph lies below the $x$-axis.<br>For $x > a$, the gradient of the curve is positive<br>$\Rightarrow$ gradient graph lies above the $x$-axis. |
| 3 State the degree of the gradient function. | The given function has degree 3 so the gradient function has degree 2. |
| 4 Identify any other key features of the gradient graph.<br>*Note:* It is often not possible to locate the $y$-coordinate of the turning point of the gradient graph without further information. | The gradient function is a quadratic.<br>Its axis of symmetry is the vertical line halfway between $x = \pm a$.<br>$\Rightarrow$ the gradient graph is symmetric about $x = 0$ ($y$-axis).<br>The $x$-coordinate of its turning point is $x = 0$. |

5 Sketch the shape of the gradient function.

*Note:* The $y$-axis scale is not known.




## EXERCISE 12.5 Differentiation of polynomials by rule

**PRACTISE**

Work without CAS

1 WE7 a Calculate $\frac{d}{dx}\left(5x^8 + \frac{1}{2}x^{12}\right)$.

b Differentiate $2t^3 + 4t^2 - 7t + 12$ with respect to $t$.

c If $f(x) = (2x + 1)(3x - 4)$, calculate $f'(x)$.

d If $y = \frac{4x^3 - x^5}{2x^2}$, calculate $\frac{dy}{dx}$.

2 If $z = \frac{1}{420}(3x^5 - 3.5x^4 + x^3 - 2x^2 + 12x - 99)$, calculate $\frac{dz}{dx}$.

3 WE8 Consider the polynomial function with equation $y = \frac{2x^3}{3} - x^2 + 3x - 1$.

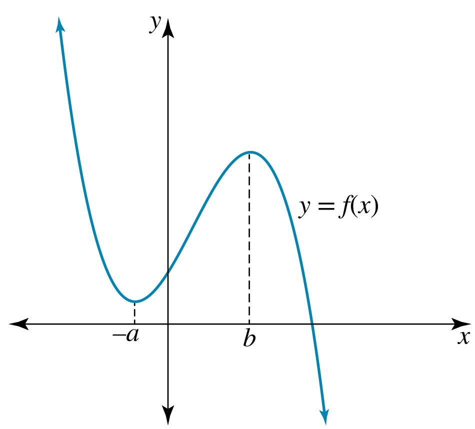
a Calculate the rate of change of the function at the point $(6, 125)$.

b Calculate the average rate of change of the function over the interval $x \in [0, 6]$.

c Obtain the coordinates of the point(s) on the graph of the function where the gradient is 3.

4 For the curve defined by $f(x) = (x - 1)(x + 2)$, calculate the coordinates of the point where the tangent is parallel to the line with equation $3x + 3y = 4$.

5 WE9 The graph of a cubic polynomial function $y = f(x)$ is given. The turning points occur when $x = -a$ and $x = b$. Sketch a possible graph for its gradient function.




6 Sketch the graph of $y = (x + 1)^3$ and hence sketch the graph of its gradient $\frac{dy}{dx}$ versus $x$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 a If $f(x) = 5x^7$, find $f'(x)$.

b If $y = 3 - 2x^3$, find $\frac{dy}{dx}$.

c Find $\frac{d}{dx}(8x^2 + 6x - 4)$.

d If $f(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + x - \frac{3}{4}$, find $D_x(f)$.

e Find $\frac{d}{du}(u^3 - 1.5u^2)$.

f If $z = 4(1 + t - 3t^4)$, find $\frac{dz}{dt}$.

8 Differentiate the following with respect to $x$:

a $(2x + 7)(8 - x)$

b $5x(3x + 4)^2$

c $(x - 2)^4$

d $250(3x + 5x^2 - 17x^3)$

e $\dfrac{3x^2 + 10x}{x}, \ x \neq 0$

f $\dfrac{4x^3 - 3x^2 + 10x}{2x}, \ x \neq 0$

9 a Calculate the gradient of the tangent to the curve $y = 3x^2 - x + 4$ at the point where $x = \dfrac{1}{2}$.

b Calculate the gradient of the tangent to the curve $f(x) = \dfrac{1}{6}x^3 + 2x^2 - 4x + 9$ at the point $(0, 9)$.

c Show that the gradient of the tangent to the curve $y = (x + 6)^2 - 2$ at its turning point is 0.

d Show that the gradient of the tangent to the curve $y = 2 - (x - 3)^3$ at its point of inflection is 0.

10 For the curve with equation $f(x) = 5x - \dfrac{3}{4}x^2$, calculate:

a $f'(-6)$

b $f'(0)$

c $\{x : f'(x) = 0\}$

d $\{x : f'(x) > 0\}$

e the gradient of the curve at the points where it cuts the $x$-axis

f the coordinates of the point where the tangent has a gradient of 11.

11 Determine the coordinates of the point(s) on the curve $f(x) = x^2 - 2x - 3$ where:

a the gradient is zero

b the tangent is parallel to the line $y = 5 - 4x$

c the tangent is perpendicular to the line $x + y = 7$

d the slope of the tangent is half that of the slope of the tangent at point $(5, 12)$.

12 The size of a population of ants in a kitchen cupboard at time $t$ hours after a pot of honey is spilt is given by $N = 0.5(t^2 + 1)^2 + 2.5$.

a Calculate the rate at which the population is growing at times $t = 1$ and $t = 2$.

b What is the average rate of growth over the first 2 hours?

c At what time is the population of ants increasing at 60 ants/hour?

13 a If the curve $y = 4x^2 + kx - 5$ has a gradient of 4 at the point where $x = -2$, determine the value of $k$.

b Use calculus to obtain the $x$-coordinate of the turning point of $y = ax^2 + bx + c$.

c i Show that the gradient of the curve $f(x) = x^3 + 9x^2 + 30x + c, \ c \in R$ is always positive.

ii Hence sketch the graph of the gradient function $y = f'(x)$ using the rule obtained for $f'(x)$.

**d** The radius $r$ metres of the circular area covered by an oil spill after $t$ days is $r = at^2 + bt$. After the first day the radius is growing at 6 metres/day and after 3 days the rate is 14 metres/day. Calculate $a$ and $b$.

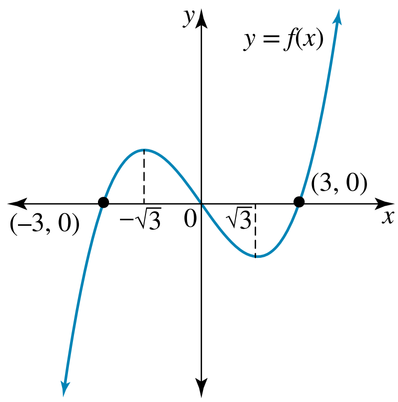
**14** **a** For the graph of $y = f(x)$ shown, state:

**i** $\{x : f'(x) = 0\}$

**ii** $\{x : f'(x) < 0\}$

**iii** $\{x : f'(x) > 0\}$.

**b** Sketch a possible graph for $y = f'(x)$.




**15** On the same set of axes, sketch the graphs of $y = f(x)$ and $y = f'(x)$, given:

**a** $f(x) = -x^2 - 1$

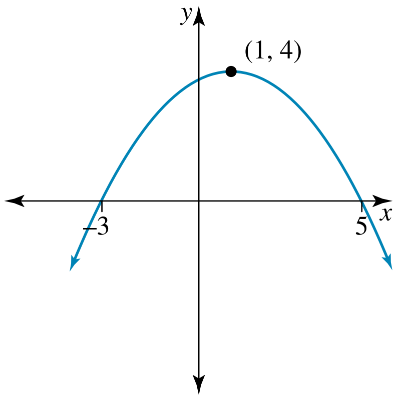
**b** $f(x) = x^3 - x^2$

**c** $f(x) = 6 - 3x$

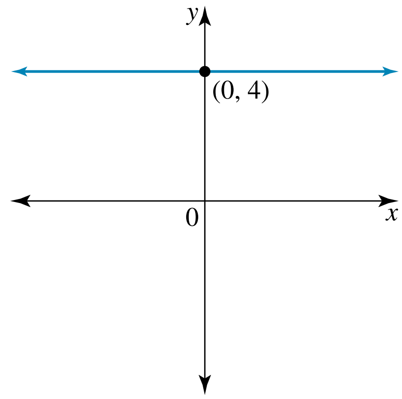
**d** $f(x) = 1 - x^3$

**16** Sketch a possible graph of $\dfrac{dy}{dx}$ versus $x$ for each of the following curves:

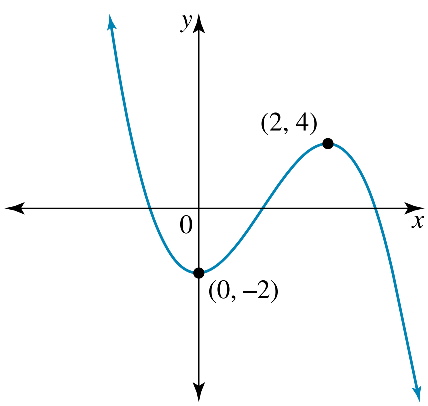
**a**



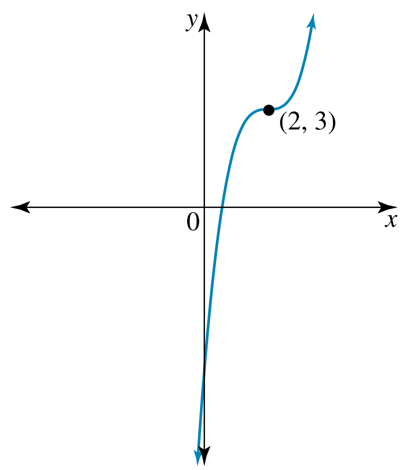

**b**



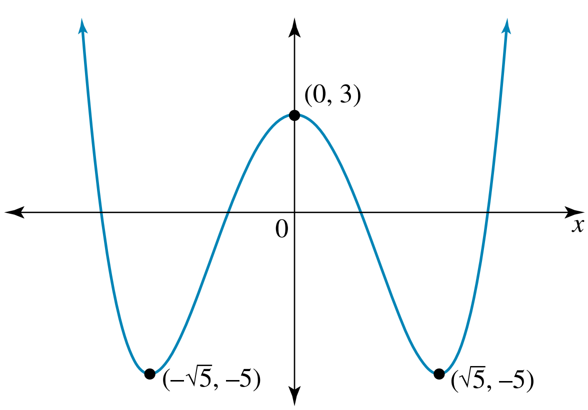

**c**



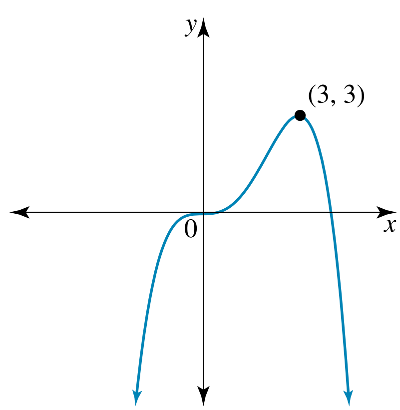

**d**




**e**




**f**



MASTER

**17 a** Use CAS technology to calculate $\frac{d}{dx}(x^3 - 2x)^{10}$.

**b** Give the value of the derivative of $(x^5 - 5x^8 + 2)^2$ when $x = 1$.

**18** Consider the function defined by $y = x^6 + 2x^2$.

**a** Sketch, using CAS technology the graphs of $y = x^6 + 2x^2$ and its derivative.

**b** Explain whether or not the derivative graph has a stationary point of inflection at the origin.

**c** Obtain, to 4 decimal places, any non-zero values of $x$ for which $\frac{dy}{dx} = y$, given $y = x^6 + 2x^2$.

**d** Form an equation the solution to which would give the $x$-values when the graphs of $y = x^6 + 2x^2$ and its derivative are parallel. Give the solution to the nearest integer.

# 12.6 Review

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**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

## REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

## Activities

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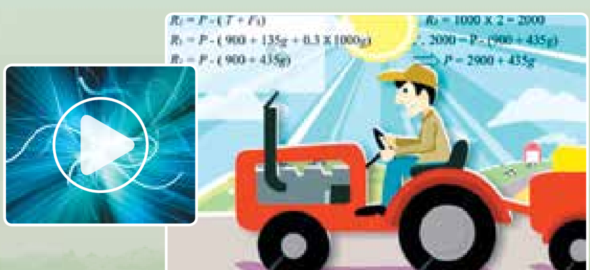
### Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.



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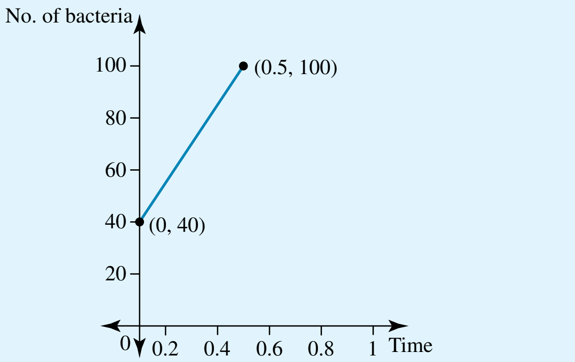
**Introduction to differential calculus**

**Sit topic test**

# 12 Answers

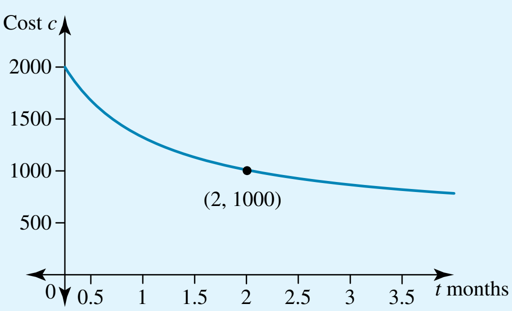
## EXERCISE 12.2

**1 a**




Linear graph with positive gradient; rate of growth is 120 bacteria per hour.

**b**




Hyperbola shape for $t \geq 0$; costs decreased at an average rate of \$500 per month over the first two months.

**2** Average rate of change is 5.

**3 a** Approximately 6

**b** Approximately 5.8

**4 a** Average speed over the interval $t \in [0, 2]$ is 15 m/s; average speed over the interval $t \in [0.9, 1.1]$ is 14.01 m/s.

**b** 14.01 m/s

**5 a** Student II

**b** Student I

**c** Student III's level of interest increased slowly initially but grew to a high level of interest; after a brief growth in interest, student IV started to lose interest but eventually this slowed and the student gained a high level of interest quickly afterwards.

**d** Answers will vary.

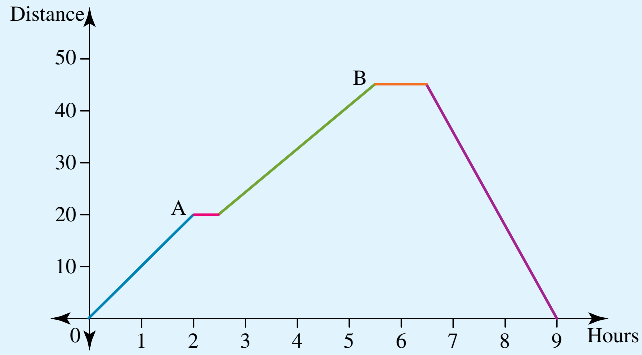
**6 a** \$600 per hour

**b** \$720 per hour

**c** \$35 per hour

**7 a** 36 km/h

**b i**




**ii** 3 hours; $8\frac{1}{3}$ km/h

**iii** 18 km/h

**iv** 10 km/h

**8 a** 21 mm

**b** $(24 - t)$ mm

**c** 24 weeks

**9 a** Growing at approximately 0.018 euros per month

**b** $r = 5$ (5% per annum)

**10 a** −2

**b** 3

**c** 7

**d** 0

**11 a** Answers will vary; approximately 4

**b** Answers will vary; approximately −1 and −2

**c** −1.5

**d** Answers will vary, approximately 108°

**12** Answers will vary. Approximately:

**a i** 1.4 °C/hour

**ii** 0

**iii** −0.8 °C/hour

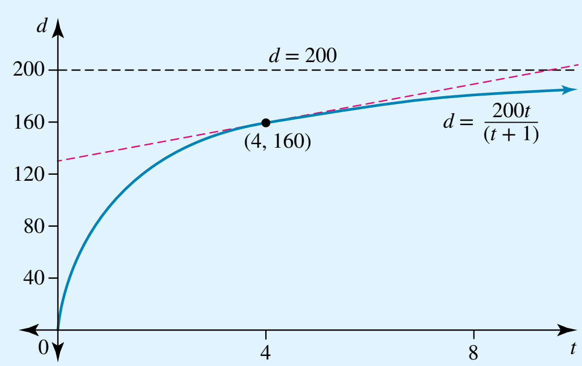
**iv** −1.4 °C

**b** 8 pm

**c** −0.59 °C/hour

**13 a** 40 m/h

**b**

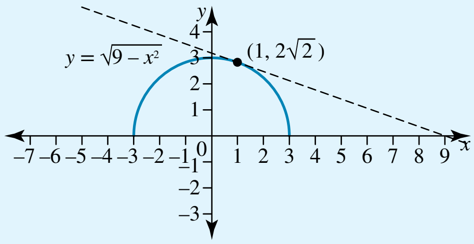


Boat initially travels away from the jetty quickly but slows almost to a stop as it nears the distance of 200 metres from the jetty.

c Answers will vary; approximately 8 m/h

d Answers will vary; for $t \in [4, 4.1]$ approximately 7.84 m/h

14 a




$(1, 2\sqrt{2})$

b $-0.46$

c Approximately $-0.35$

d Exactly $-\frac{\sqrt{2}}{4}$

15 $y = 12x + 16; m = 12$

16 The value of the gradient is the same as the value of the $x$-coordinate.

## EXERCISE 12.3

1 a $2h - 5,\ h \neq 0$

b $-4.98$

c $-5$

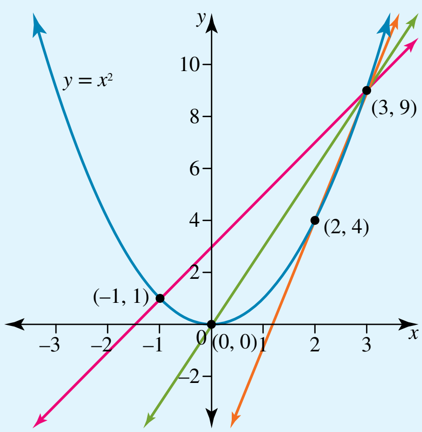
2 a $\frac{h^2}{3},\ h \neq 0$

b 0

3 a 5

b $-11.61$

4 a




b Best approximation is secant through $(2, 4)$ and $(3, 9)$; choose a closer point.

5 a 61.63975

b 62

c Use closer points to form the secant.

6 a $(2 + h, -4h - h^2)$

b $-4 - h,\ h \neq 0$

c $h = 1; (3, -5)$

d $h = 0.1; -4.1$

e $h - 4,\ h \neq 0, -0.39$

f $-4$

7 a The $y$-coordinate is $h^3 + 9h^2 + 28h + 30$.

b $h^2 + 9h + 28,\ h \neq 0$

c 28.009001

d 28

8 a $M(1, 1); N\left(1 + h, \frac{1}{1 + h}\right)$

b $\frac{-1}{1 + h},\ h \neq 0$

c $-1$

d $-1$

9 a $1 + h,\ h \neq 0$

b $\lim_{h \to 0}(1 + h)$

c 1

10 a Proof required — check with your teacher

b $\lim_{h \to 0}(h^2 - 4h + 1)$

c 1

11 a Cannot be simplified; 3

b $8 - 2h,\ h \neq 0; 8$

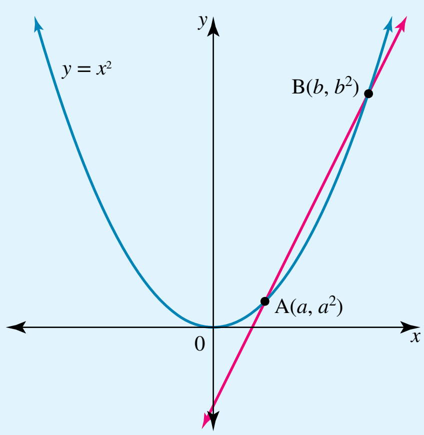
c $12 + 6h + h^2,\ h \neq 0; 12$

d $4 + 6h + 4h^2 + h^3,\ h \neq 0; 4$

e $\frac{-1}{4(4 + h)},\ h \neq 0; -\frac{1}{16}$

f $\sqrt{1 + h} + 1,\ h \neq 0; 2$

12 a




b $a + b,\ a \neq b$

c i $2a$

ii $2b$

13 a 8

b 3

c $\frac{2}{3}$

14 a $\frac{\sqrt{9 + h} - 3}{h}$

b $\lim_{h \to 0}\frac{\sqrt{9 + h} - 3}{h}$

c $\frac{1}{6}$

## EXERCISE 12.4

1 a $f'(x) = 4x + 1$

b $-3$

c 1

2 $f'(x) = 3x^2$

3 $\dfrac{dy}{dx} = 2x + 1$; gradient of the tangent to the curve at $x = 1$ is 3

4 $-2$; the gradient of the curve at $x = 2$ is $-2$

5 $5 + 12x^3$

6 $\dfrac{dy}{dx} = 5ax^4$

7 a $2x + h,\ h \neq 0$

b $f'(x) = 2x$

c 6

d 6

8 a $6 + 3\delta x,\ \delta x \neq 0$

b 6

c i S $(2, 14)$; T $(2 + \delta x, 3(2 + \delta x)^2 + 2)$

ii $12 + 3\delta x,\ \delta x \neq 0$

iii 12

9 a Gradient of secant through $(x, f(x))$ and $(x + h, f(x + h))$

b Gradient of tangent at $(x, f(x))$

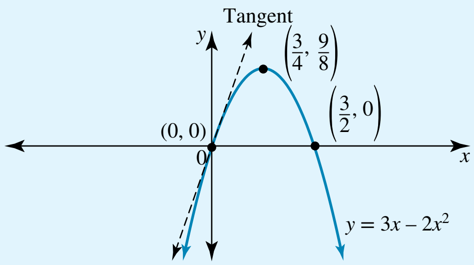
c Gradient of tangent at $(3, f(3))$

d Gradient of secant through $(3 - h, f(3 - h))$ and $(3, f(3))$

10 a $f'(x) = 3 - 4x$

b 3

c




11 a $f'(x) = 16x$

b $f'(x) = x - 4$

c $f'(x) = -2$

d $f'(x) = 0$

e $f'(x) = 3x^2 - 12x + 2$

f $f'(x) = 6x^5$

12 a $\dfrac{dy}{dx} = -2x$

b $\dfrac{dy}{dx} = 2x + 4$

c $\dfrac{dy}{dx} = x^2$

d $\dfrac{dy}{dx} = 2x + 1$

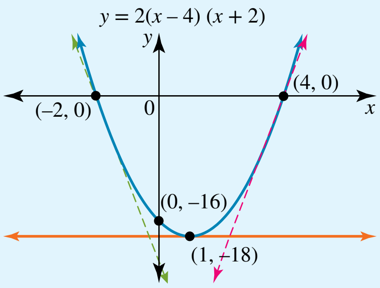
13 a $f'(x) = 2x + 10$

b 0; tangent at $x = -5$ has zero gradient

c 10

d 6

14 a




$x$-intercepts at $x = -2, x = 4$; $y$-intercept at $y = -16$; minimum turning point $(1, -18)$

b $f'(x) = 4x - 4$

c 0; tangent at turning point is horizontal

d $-12$; 12

15 a $f'(x) = 2ax + b$

b $f' : R \to R, f'(x) = 2ax + b$

c $f'(x) = 6x + 4$

d $f$ has degree 2; $f'$ has degree 1

16 a i $h(h^2 + 6h + 12)$

ii $g'(2) = \lim\limits_{h \to 0} \dfrac{(2 + h)^3 - 8}{h}$

iii 12

b i $\dfrac{-1}{x^2},\ x \neq 0$

ii $\dfrac{-2}{x^3},\ x \neq 0$

iii $\dfrac{1}{2\sqrt{x}},\ x > 0$

17 $2x$

18 a $3ax^2 + 2bx + c$

b $f$ has degree 3; $f'$ has degree 2

## EXERCISE 12.5

1 a $40x^7 + 6x^{11}$

b $6t^2 + 8t - 7$

c $12x - 5$

d $2 - \dfrac{3x^2}{2}$

2 $\dfrac{1}{420}(15x^4 - 14x^3 + 3x^2 - 4x + 12)$

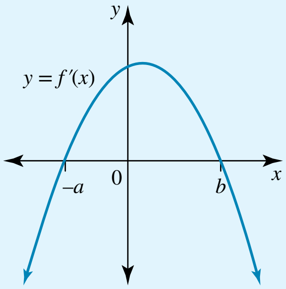
3 a 63

b 21

c $(0, -1), \left(1, \dfrac{5}{3}\right)$

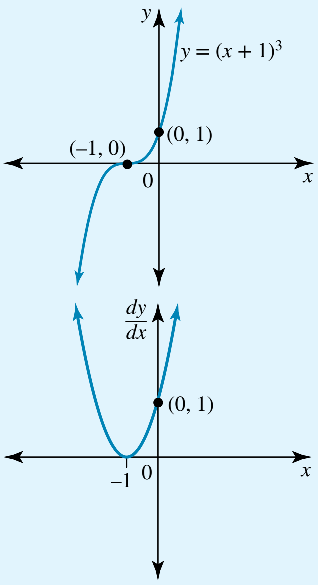
4 $(-1, -2)$

**5**




Concave down parabola with $x$-intercepts $x = -a$ and $x = b$

**6**




$y = (x + 1)^3$ has a stationary point of inflection at $(-1, 0)$, $y$-intercept $(0, 1)$; gradient graph is concave up parabola with turning point at its $x$-intercept, at $(-1, 0)$

**7 a** $f'(x) = 35x^6$

**b** $\frac{dy}{dx} = -6x^2$

**c** $16x + 6$

**d** $D_x(f) = \frac{1}{2}x^2 - x + 1$

**e** $3u^2 - 3u$

**f** $\frac{dz}{dt} = 4(1 - 12t^3)$

**8 a** $-4x + 9$

**b** $135x^2 + 240x + 80$

**c** $4x^3 - 24x^2 + 48x - 32$

**d** $250(3 + 10x - 51x^2)$

**e** 3

**f** $4x - \frac{3}{2}$

**9 a** 2

**b** −4

**c** Proof required — check with your teacher

**d** Proof required — check with your teacher

**10 a** 14

**b** 5

**c** $\left\{\frac{10}{3}\right\}$

**d** $\left\{x : x < \frac{10}{3}\right\}$

**e** 5 and −5

**f** (−4, −32)

**11 a** (1, −4)

**b** (−1, 0)

**c** $\left(\frac{3}{2}, -\frac{15}{4}\right)$

**d** (3, 0)

**12 a** 4 ants/hour when $t = 1$ and 20 ants/hour when $t = 2$

**b** 6 ants/hour

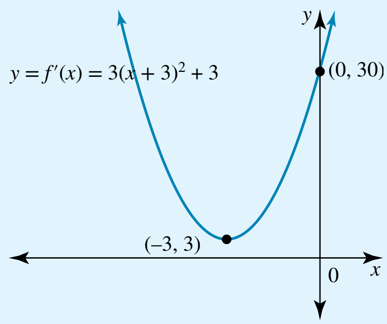
**c** 3 hours

**13 a** $k = 20$

**b** $x = -\frac{b}{2a}$

**c i** Proof required — check with your teacher

**ii**



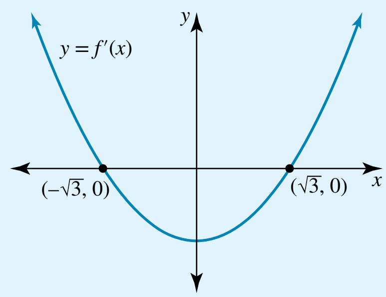

**d** $a = 2; b = 2$

**14 a i** $\{\pm\sqrt{3}\}$

**ii** $\{x : -\sqrt{3} < x < \sqrt{3}\}$

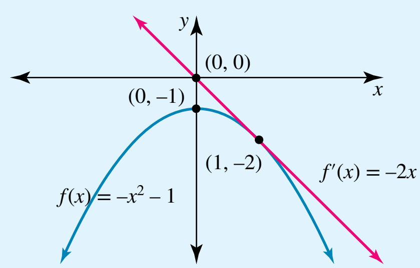
**iii** $\{x : x < -\sqrt{3}\} \cup \{x : x > \sqrt{3}\}$

**b**

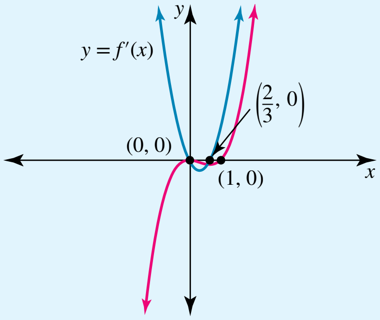



Concave up parabola with $x$-intercepts at $x = \pm\sqrt{3}$

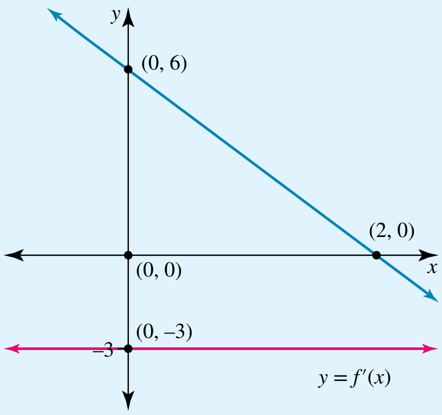
**15 a** $f'(x) = -2x$



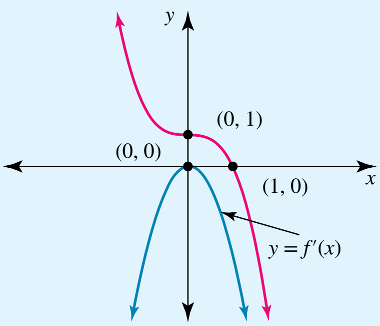
**b** $f'(x) = 3x^2 - 2x$



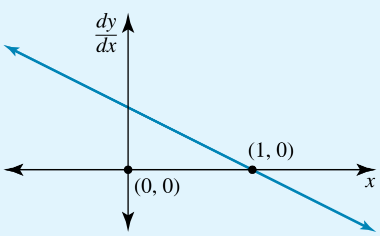

**c** $f'(x) = -3$



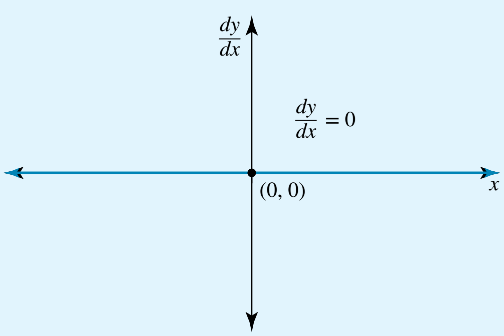

**d** $f'(x) = -3x^2$



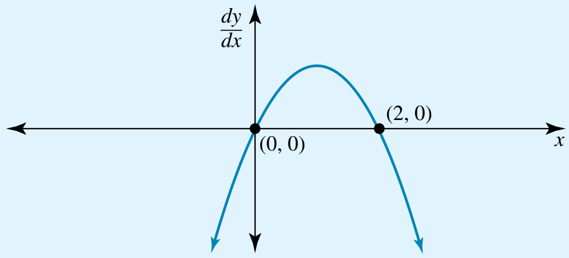

**16 a**



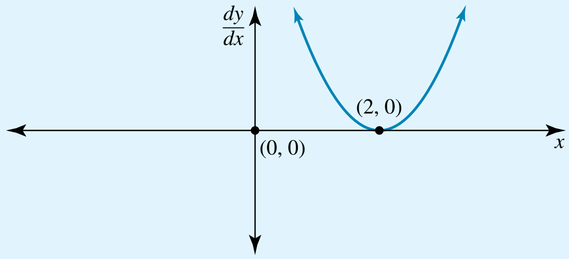

**b**



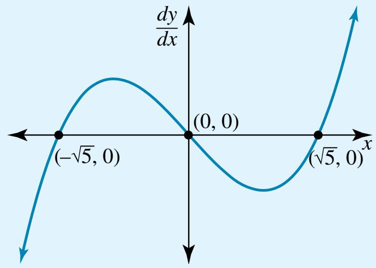

**c**



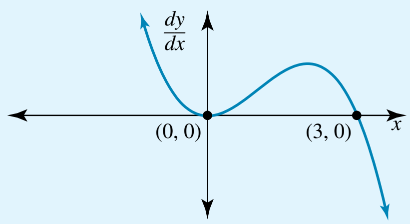

**d**




**e**



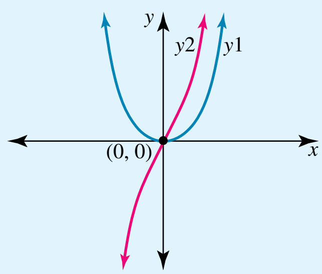

**f**




**17 a** $10(x^3 - 2x)^9(3x^2 - 2)$

**b** 140

**18 a**




**b** Gradient of derivative graph at origin is not 0.

**c** $x = 5.9938$

**d** $3x^5 - 15x^4 + 2x - 2 = 0$; $x = 5$

# 13

# Differentiation and applications

# 13.1 Kick off with CAS

## The limit does not exist

1 Using CAS technology, find the appropriate template and evaluate the expressions below.

a $\lim_{x \to 1} (x + 3)$

b $\lim_{x \to 0} \frac{1}{x}$

c $\lim_{x \to 2} \frac{1}{(x - 2)^2}$

d $\lim_{x \to 1} \frac{(x^2 - 3x + 2)}{(1 - x)}$

e $\lim_{x \to 2} h(x)$, where $h(x) = \begin{cases} x + 9, x < 2 \\ x^2 - 3, x \ge 2 \end{cases}$

f $\lim_{x \to \infty} \frac{(x - 1)}{x}$

g $\lim_{x \to 0} \left(3 - \frac{1}{x^2}\right)$

h $\lim_{x \to \infty} (x \sin(x))$

2 Use CAS technology to sketch graphs of the following functions.

a $f(x) = (x + 3)$

b $f(x) = \frac{1}{x}$

c $g(x) = \frac{1}{(x - 2)^2}$

d $g(x) = \frac{(x^2 - 3x + 2)}{(1 - x)}$

e $h(x) = \begin{cases} x + 9, x < 2 \\ x^2 - 3, x \ge 2 \end{cases}$

f $h(x) = \frac{(x - 1)}{x}$

g $f(x) = \left(3 - \frac{1}{x^2}\right)$

h $f(x) = (x \sin(x))$

3 Compare the graphs for which the limit in question 1 exists to those for which the limit is undefined. Describe the difference with reference to the function value at the point(s) of interest.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 13.2 Limits, continuity and differentiability

Having established the rules for differentiating polynomial functions in the previous topic, we now consider the calculation of derivatives of other types of functions. Differential calculus is founded on the concept of limits. To develop an understanding of when a function can and cannot be differentiated, we need to have a closer look at how limits can be calculated.

**study on**

Units 1 & 2
AOS 3
Topic 2
Concept 1, 2, 3

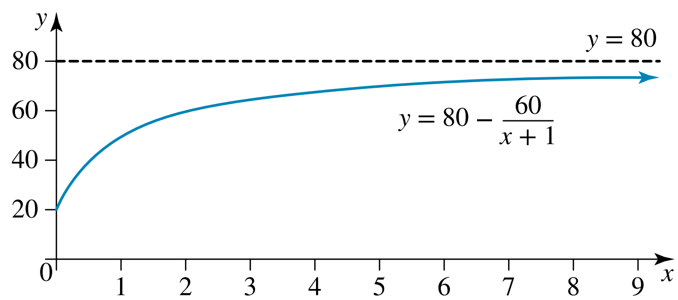
**Limits, continuity and differentiability**
Concept summary
Practice questions

## Limits

For any function $y = f(x)$, the statement $\lim\limits_{x \to a} f(x) = L$ means that as the $x$-values approach the value $a$, the $y$-values approach the value $L$; as $x \to a$, $f(x) \to L$.

A **limit** is only concerned with the behaviour of a function as it approaches a particular point, and not with its behaviour at the point. What happens to the function at $x = a$ is immaterial; indeed, $f(a)$ may not even exist.

It is not only in calculus that limits occur. We have encountered limits when we considered the long-term behaviour of a hyperbolic function as $x \to \infty$, thereby identifying its limiting behaviour. The graph of the hyperbola with the rule $y = 80 - \dfrac{60}{x+1}$ illustrates that as $x \to \infty$, $y \to 80$.




Although the graph never reaches the horizontal asymptote at $y = 80$, as the $x$-values increase indefinitely, the $y$-values come closer and closer to 80.

This is written as $\lim\limits_{x \to \infty} \left(80 - \dfrac{60}{x+1}\right) = 80$.

## Calculating limits algebraically

In differentiation from first principles, we encountered limits such as $\lim\limits_{h \to 0} \dfrac{2xh + h^2}{h}$. The value $h = 0$ could not be substituted immediately because the expression $\dfrac{2xh + h^2}{h}$ is not a *well-behaved* function at $h = 0$: substitution of $h = 0$ is not possible since division by zero is not possible.

Factorisation, $\dfrac{\cancel{h}(2x + h)}{\cancel{h}}$, was used to create an equivalent expression, $2x + h$, which was well behaved at $h = 0$ and therefore allowed for $h = 0$ to be substituted.

(The legitimacy of this process originally concerned some mathematicians until the nineteenth century when Karl Weierstrass and others added the rigour needed to quell any doubts, but this is beyond our course and our current needs.)

Karl Weierstrass (1815–1897)

**To evaluate the limit of a function as $x \to a$:**

- **If the function is well behaved, substitute $x = a$.**
- **If the function is not well behaved at $x = a$, simplify it to obtain a well-behaved function, then substitute $x = a$.**
- **If it is not possible to obtain a well-behaved function, then the limit does not exist.**

WORKED EXAMPLE 1

**Calculate, where possible, the following limits.**

a $\lim_{x \to 2}(4x + 3)$  b $\lim_{x \to -1}\left(\frac{x^2 - 1}{x + 1}\right)$  c $\lim_{x \to 1}\left(\frac{1}{x - 1}\right)$

| THINK | WRITE |
|---|---|
| a 1 State whether or not the function is well behaved at the value of $x$. | a $\lim_{x \to 2}(4x + 3)$<br>The function $f(x) = 4x + 3$ is well behaved at $x = 2$, since it is possible to substitute $x = 2$ into the equation. |
| 2 Calculate the limit. | Substitute $x = 2$.<br>$\lim_{x \to 2}(4x + 3) = 4 \times 2 + 3$<br>$= 11$ |
| b 1 State whether or not the function is well behaved at the value of $x$. | b $\lim_{x \to -1}\left(\frac{x^2 - 1}{x + 1}\right)$<br>The function $f(x) = \frac{x^2 - 1}{x + 1}$ is not well behaved at $x = -1$ since its denominator would be zero. |
| 2 Use algebra to simplify the function to create an equivalent limit statement. | $\lim_{x \to -1}\left(\frac{x^2 - 1}{x + 1}\right) = \lim_{x \to -1}\left(\frac{(x - 1)(x + 1)}{x + 1}\right)$<br>$= \lim_{x \to -1}(x - 1)$ |
| 3 Calculate the limit. | $= -1 - 1$<br>$= -2$<br>$\therefore \lim_{x \to -1}\left(\frac{x^2 - 1}{x + 1}\right) = -2$ |
| c Determine whether the limit exists. | c $\lim_{x \to 1}\left(\frac{1}{x - 1}\right)$<br>The function $f(x) = \frac{1}{x - 1}$ is not well behaved at $x = 1$ but it cannot be simplified any further.<br>Therefore, $\lim_{x \to 1}\left(\frac{1}{x - 1}\right)$ does not exist. |

## Left and right limits

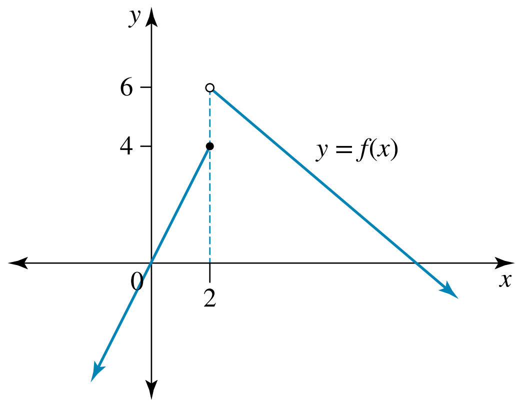
For $\lim_{x \to a} f(x)$ to exist, the limit of the function as $x$ approaches $a$ from values smaller than $a$ should be the same as its limit as $x$ approaches $a$ from values larger than $a$. These are called **left and right limits** and are written as follows:

$$L^- = \lim_{x \to a^-} f(x) \text{ and } L^+ = \lim_{x \to a^+} f(x)$$

The direction as $x$ approaches $a$ must not matter if the limit is to exist.

- If $L^- = L^+$, then $\lim\limits_{x \to a} f(x)$ exists and $\lim\limits_{x \to a} f(x) = L^- = L^+$.
- If $L^- \neq L^+$, then $\lim\limits_{x \to a} f(x)$ does not exist.

For the graph of the hybrid function shown, the limiting value of the function as $x \to 2$ depends on which of the two branches are considered.




The limit as $x \to 2$ from the left of $x = 2$ is:

$$L^- = \lim_{x \to 2^-} f(x) = 4$$

The limit as $x \to 2$ from the right of $x = 2$ is:

$$L^+ = \lim_{x \to 2^+} f(x) = 6$$

Since $L^- \neq L^+$, the limit of the function as $x \to 2$ does not exist.

## Limits and continuity

For a function to be **continuous** at $x = a$, there should be no break in its graph at $x = a$, so both its behaviour as it approaches $x = a$ and its behaviour at that point are important.

**If a function is continuous at a point where $x = a$, then:**

- **$f(a)$ exists**
- **$\lim\limits_{x \to a} f(x)$ exists**
- **$\lim\limits_{x \to a} f(x) = f(a)$.**

For the graph of the hybrid function previously considered, the two branches do not join. There is a break in the graph at $x = 2$ so the function is not continuous at $x = 2$.

However, to illustrate the definition of continuity, we can reason that although $f(2) = 4$ so the function is defined at $x = 2$, $\lim\limits_{x \to 2} f(x)$ does not exist since $L^- \neq L^+$ and therefore the function is not continuous at $x = 2$.

From a graph, places where a function is not continuous (the points of **discontinuity**) are readily identified: the graph may contain a vertical asymptote, or it may contain branches which do not join, or there may be a gap in the graph at some point. Without a graph, we may need to test the definition to decide if a function is continuous or not.

WORKED EXAMPLE 2

**a** A function is defined as $f(x) = \begin{cases} x, & x < 1 \\ 1, & x = 1 \\ x^2, & x > 1 \end{cases}$

**i** Calculate $\lim\limits_{x \to 1} f(x)$.

**ii** Determine whether the function is continuous at $x = 1$ and sketch the graph of $y = f(x)$ to illustrate the answer.

**b** For the function with rule $g(x) = \dfrac{x^2 + x}{x + 1}$:

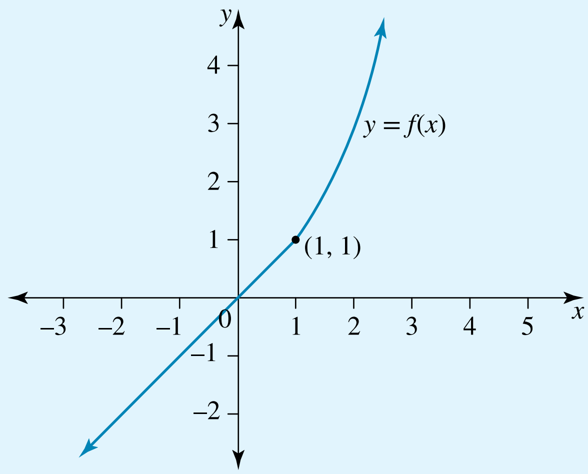
i state its maximal domain

ii calculate $\lim\limits_{x \to -1} g(x)$

iii explain why the function is not continuous at $x = -1$

iv sketch the graph of $y = g(x)$.

| THINK | WRITE/DRAW |
|---|---|
| **a i 1** Calculate the left and right limits of the function at the value of $x$. | **a i** $f(x) = \begin{cases} x, & x < 1 \\ 1, & x = 1 \\ x^2, & x > 1 \end{cases}$<br>Limit from the left of $x = 1$:<br>$L^- = \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1}(x) = 1$<br>Limit from the right of $x = 1$:<br>$L^+ = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1}(x^2) = 1$ |
| **2** State the answer. | Since $L^- = L^+$, $\lim\limits_{x \to 1} f(x) = 1$. |
| **ii 1** Identify the value of the function at the $x$-value under consideration. | **ii** From the given rule, $f(1) = 1$. |
| **2** Explain whether or not the function is continuous at the given $x$-value. | Both $f(1)$ and $\lim\limits_{x \to 1} f(x)$ exist, and since $f(1) = \lim\limits_{x \to 1} f(x)$, the function is continuous at $x = 1$. |
| **3** Sketch the hybrid function. | The branch of the graph for $x < 1$ is the straight line $y = x$. Two points to plot are $(0, 0)$ and $(1, 1)$.<br>The branch of the graph for $x > 1$ is the parabola $y = x^2$. Two points to plot are $(1, 1)$ and $(2, 4)$.<br>The two branches join at $(1, 1)$ so there is no break in the graph. |



**b i** Determine the maximal domain.

The function has a denominator $x + 1$ which would equal zero if $x = -1$. This value must be excluded from the domain of the function.

**b i** $g(x) = \dfrac{x^2 + x}{x + 1}$

The maximal domain is $R \setminus \{-1\}$.

**ii** Calculate the required limit of the function.

**ii**
$$\begin{aligned}\lim_{x \to -1} g(x) &= \lim_{x \to -1} \frac{x^2 + x}{x + 1} \\ &= \lim_{x \to -1} \frac{x(x + 1)}{x + 1} \\ &= \lim_{x \to -1} (x) \\ &= -1\end{aligned}$$
$$\therefore \lim_{x \to -1} g(x) = -1$$

**iii** Explain why the function is not continuous at the given point.

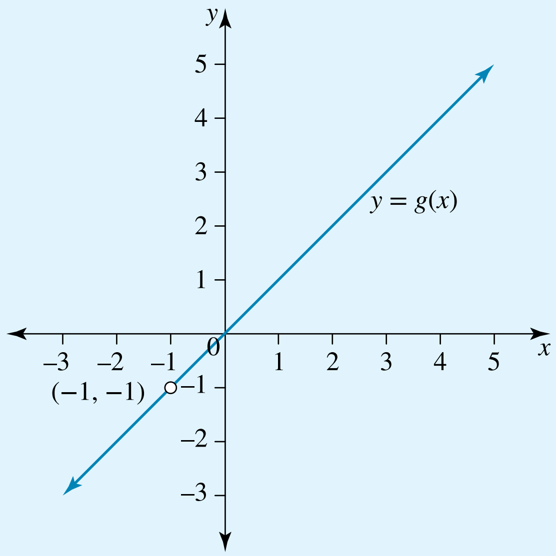
**iii** The domain of the function is $R \setminus \{-1\}$ so $g(-1)$ does not exist. Therefore, the function is not continuous at $x = -1$.

**iv 1** Identify the key features of the graph.

**iv**
$$\begin{aligned}g(x) &= \frac{x(x + 1)}{x + 1} \\ &= x, \quad x \neq -1\end{aligned}$$

Therefore the graph of $y = g(x)$ is the graph of $y = x$, $x \neq -1$.
The straight-line graph has a gap, or point of discontinuity, at $(-1, -1)$.

**2** Sketch the graph.

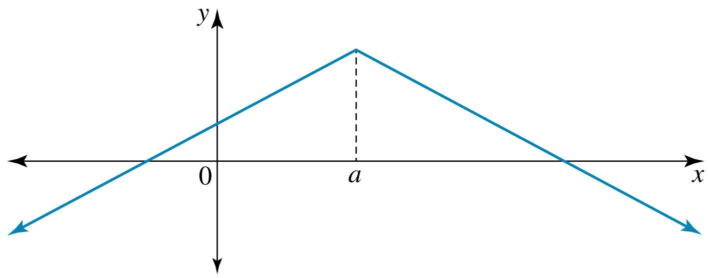



## Differentiability

While a function must be continuous at a point in order to find the gradient of its tangent, that condition is necessary but not sufficient. The derivative function has a limit definition, and as a limit exists only if the limit from the left equals the limit from the right, to test whether a continuous function is **differentiable** at a point $x = a$ in its domain, the derivative from the left of $x = a$ must equal the derivative from the right of $x = a$.

**A function $f$ is differentiable at $x = a$ if:**
- **$f$ is continuous at $x = a$ and**
- **$f'(a^-) = f'(a^+)$.**

This means that only **smoothly continuous functions** are differentiable at $x = a$: their graph contains no 'sharp' points at $x = a$. Polynomials are smoothly continuous functions but a hybrid function, for example, may not always join its ends smoothly. An example of a hybrid function which is continuous but not differentiable at $x = a$ is shown in the diagram below.




For this hybrid function, the line on the left of $a$ has a positive gradient but the line on the right of $a$ has a negative gradient; $\left.\frac{dy}{dx}\right|_{x=a^-} \neq \left.\frac{dy}{dx}\right|_{x=a^+}$. The 'sharp' point at $x = a$ is clearly visible. However, we cannot always rely on the graph to determine if a continuous function is differentiable because visually it may not always be possible to judge whether the join is smooth or not.

WORKED EXAMPLE 3

**The function defined as $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ is continuous at $x = 1$.**

**a** **Test whether the function is differentiable at $x = 1$.**

**b** **Give the rule for $f'(x)$ stating its domain and sketch the graph of $y = f'(x)$.**

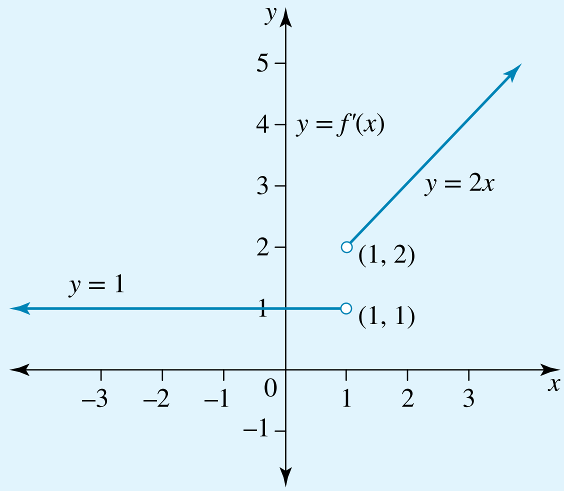
| THINK | WRITE/DRAW |
|---|---|
| **a** **1** Calculate the derivative from the left and the derivative from the right at the given value of $x$. | **a** $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$<br>Derivative from the left of $x = 1$:<br>$f(x) = x$<br>$f'(x) = 1$<br>$\therefore f'(1^-) = 1$<br>Derivative from the right of $x = 1$:<br>$f(x) = x^2$<br>$f'(x) = 2x$<br>$\therefore f'(1^+) = 2 \times 1$<br>$= 2$ |
| **2** State whether the function is differentiable at the given value of $x$. | Since the derivative from the left does not equal the derivative from the right, the function is not differentiable at $x = 1$. |

b 1 State the rule for the derivative in the form of a hybrid function.

b Each branch of this function is a polynomial so the derivative of the function is

$$f'(x) = \begin{cases} 1, & x < 1 \\ 2x, & x > 1 \end{cases} \text{ with domain } R\backslash\{1\}.$$

2 Sketch the graph.




## EXERCISE 13.2 Limits, continuity and differentiability

**PRACTISE**

Work without CAS

1 WE1 Calculate, where possible, the following limits.

a $\lim_{x \to -5} (8 - 3x)$

b $\lim_{x \to 3} \left( \dfrac{x^2 - 9}{x - 3} \right)$

c $\lim_{x \to -3} \left( \dfrac{1}{x + 3} \right)$

2 Calculate the following limits.

a $\lim_{x \to 2} \left( \dfrac{x^3 - 8}{x - 2} \right)$

b $\lim_{x \to 2} \left( \dfrac{1}{x + 3} \right)$

3 a WE2 A function is defined as $f(x) = \begin{cases} x^2, & x < 1 \\ 1, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$

i Calculate $\lim_{x \to 1} f(x)$.

ii Determine whether the function is continuous at $x = 1$ and sketch the graph of $y = f(x)$ to illustrate the answer.

b Consider the function with rule $g(x) = \dfrac{x^2 - x}{x}$.

i State its maximal domain.

ii Calculate $\lim_{x \to 0} g(x)$.

iii Explain why the function is not continuous at $x = 0$.

iv Sketch the graph of $y = g(x)$.

4 For the function defined by

$$f(x) = \begin{cases} x^2 - 4, & x < 0 \\ 4 - x^2, & x \geq 0 \end{cases}$$

calculate where possible $f(0)$ and $\lim_{x \to 0} f(x)$.

Hence, determine the domain over which the function is continuous.

5 WE3 The function defined as

$$f(x) = \begin{cases} x^2, & x \le 1 \\ 2x - 1, & x > 1 \end{cases}$$

is continuous at $x = 1$.

a Test whether the function is differentiable at $x = 1$.

b Give the rule for $f'(x)$ stating its domain and sketch the graph of $y = f'(x)$.

6 Determine the values of $a$ and $b$ so that

$$f(x) = \begin{cases} ax^2, & x \le 2 \\ 4x + b, & x > 2 \end{cases}$$

is smoothly continuous at $x = 2$.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Evaluate the following limits:

a $\lim\limits_{x \to 3} (6x - 1)$

b $\lim\limits_{x \to 3} \dfrac{2x^2 - 6x}{x - 3}$

c $\lim\limits_{x \to 1} \dfrac{2x^2 + 3x - 5}{x^2 - 1}$

d $\lim\limits_{x \to 0} \dfrac{3x - 5}{2x - 1}$

e $\lim\limits_{x \to -4} \dfrac{64 + x^3}{x + 4}$

f $\lim\limits_{x \to \infty} \dfrac{x + 1}{x}$

8 Calculate, if it exists, $\lim\limits_{x \to 2} f(x)$ for each of the following functions.

a $f(x) = \begin{cases} x^2, & x \le 2 \\ -2x, & x > 2 \end{cases}$

b $f(x) = \begin{cases} (x - 2)^2, & x < 2 \\ x - 2, & x \ge 2 \end{cases}$

c $f(x) = \begin{cases} -x, & x < 2 \\ 0, & x = 2 \\ x - 4, & x > 2 \end{cases}$

d $f(x) = \dfrac{x^2 - 4}{x - 2},\ x \ne 2$

9 For each of the functions in question 8, explain whether the function is continuous at $x = 2$.

10 Identify which of the following functions are continuous over $R$.

a $f(x) = x^2 + 5x + 2$

b $g(x) = \dfrac{4}{x + 2}$

c $h(x) = \begin{cases} x^2 + 5x + 2, & x < 0 \\ 5x + 2, & x > 0 \end{cases}$

d $k(x) = \begin{cases} x^2 + 5x + 2, & x < 0 \\ \dfrac{4}{x + 2}, & x \ge 0 \end{cases}$

11 Determine the value of $a$ so that

$$y = \begin{cases} x + a, & x < 1 \\ 4 - x, & x \ge 1 \end{cases}$$

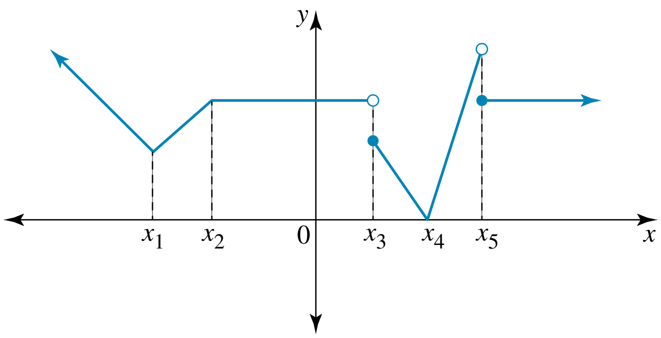
is a continuous function.

12 Determine the values of $a$ and $b$ so that

$$y = \begin{cases} ax + b, & x < -1 \\ 5, & -1 \le x \le 2 \\ 2bx + a, & x > 2 \end{cases}$$

is a continuous function.

**13**




The graph shown consists of a set of straight-line segments and rays.

**a** State any values of $x$ for which the function is not continuous.

**b** State any values of $x$ for which the function is not differentiable.

**14** Consider the function defined by the rule

$$f(x) = \begin{cases} 3 - 2x, & x < 0 \\ x^2 + 3, & x \geq 0 \end{cases}$$

**a** Determine whether the function is differentiable at $x = 0$.

**b** Sketch the graph of $y = f(x)$. Do the two branches join smoothly at $x = 0$?

**c** Form the rule for $f'(x)$ and state its domain.

**d** Calculate $f'(3)$.

**e** Sketch the graph of $y = f'(x)$.

**f** Repeat parts **a–e** if the rule for the function is altered to

$$f(x) = \begin{cases} 3 - 2x, & x < 0 \\ x^2 - 2x + 3, & x \geq 0 \end{cases}$$

**15 a** Explain why the function

$$f(x) = \begin{cases} 4x^2 - 5x + 2, & x \leq 1 \\ -x^3 + 3x^2, & x > 1 \end{cases}$$

is not differentiable at $x = 1$.

**b** Form the rule for $f'(x)$ for $x \in R\backslash\{1\}$.

**c** Calculate the values of $x$ for which $f'(x) = 0$.

**d** Sketch the graph of $y = f'(x)$ showing all key features.

**e** Calculate the coordinates of the points on the graph of $y = f(x)$ where the gradient of the tangent to the curve is zero.

**f** Sketch the graph of $y = f(x)$, locating any turning points and intercepts with the coordinate axes.

**16** Determine the values of $a$, $b$, $c$ and $d$ so that

$$y = \begin{cases} ax^2 + b, & x \leq 1 \\ 4x, & 1 < x < 2 \\ cx^2 + d, & x \geq 2 \end{cases}$$

is a differentiable function for $x \in R$.

**MASTER**

**17** Evaluate the following.

**a** $\lim\limits_{x \to 1} \dfrac{x^{25} - 1}{x - 1}$

**b** $\lim\limits_{x \to 1} \dfrac{x^n - 1}{x - 1}$

18 For the function $f(x) = \begin{cases} x^2, & x < 2 \\ 2^x, & x \geq 2 \end{cases}$

a show the function is continuous at $x = 2$
b write down the limit definition for the derivative from the left of $x = 2$ using a nearby point with $x = 2 - h$
c write down the limit definition for the derivative from the right of $x = 2$ using a nearby point with $x = 2 + h$
d evaluate these limits and hence determine whether the function is differentiable at $x = 2$.

# 13.3 Derivatives of power functions

A **power function** is of the form $y = x^n$ where $n$ is a rational number. The hyperbola $y = \frac{1}{x} = x^{-1}$ and the square root function $y = \sqrt{x} = x^{\frac{1}{2}}$ are examples of power functions that we have already studied.

**study on**
Units 1 & 2
AOS 3
Topic 2
Concept 4
**Derivatives of power functions**
Concept summary
Practice questions

## The derivative of $y = x^{-n}, n \in N$

$$\text{Let } f(x) = x^{-n},\ n \in N.$$
$$\text{Then } f(x) = \frac{1}{x^n}.$$

Since this function is undefined and therefore not continuous at $x = 0$, it cannot be differentiated at $x = 0$.

For $x \neq 0$, the derivative can be calculated using differentiation from first principles.

By definition, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\therefore f'(x) = \lim_{h \to 0} \frac{(x+h)^{-n} - x^{-n}}{h}$$
$$= \lim_{h \to 0} \frac{\dfrac{1}{(x+h)^n} - \dfrac{1}{x^n}}{h}$$
$$= \lim_{h \to 0} \left( \left( \frac{1}{(x+h)^n} - \frac{1}{x^n} \right) \div h \right)$$
$$= \lim_{h \to 0} \left( \frac{x^n - (x+h)^n}{(x+h)^n x^n} \times \frac{1}{h} \right)$$
$$= \lim_{h \to 0} \frac{x^n - (x+h)^n}{(x+h)^n x^n h}$$

To simplify the numerator, $(x+h)^n$ is expanded using the binomial theorem, as used in the proof of the rule for the derivative of a polynomial function.

$$f'(x) = \lim_{h \to 0} \frac{x^n - \left(x^n - \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \ldots + h^n\right)}{(x+h)^n x^n h}$$
$$= \lim_{h \to 0} \frac{-\binom{n}{1} x^{n-1} h - \binom{n}{2} x^{n-2} h^2 - \ldots - h^n}{(x+h)^n x^n h}$$
$$= \lim_{h \to 0} \frac{-n x^{n-1} h - \binom{n}{2} x^{n-2} h^2 - \ldots - h^n}{(x+h)^n x^n h}$$

$$= \lim_{h \to 0} \frac{-h\left(nx^{n-1} + \binom{n}{2}x^{n-2}h^2 - \ldots - h^{n-1}\right)}{(x+h)^n x^n h}$$

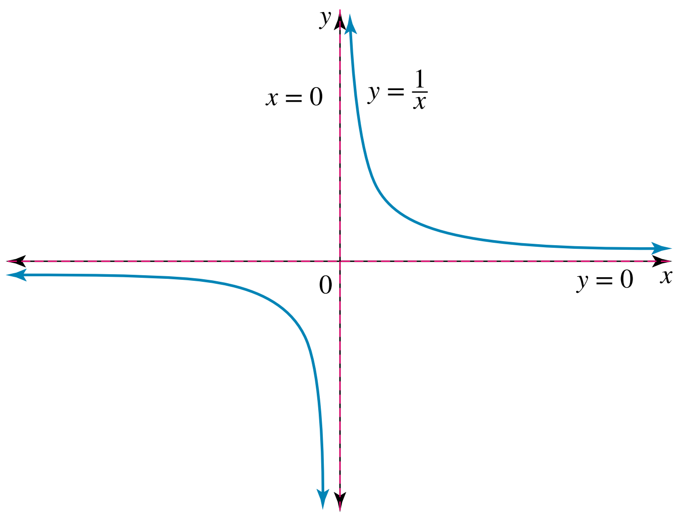
$$= \lim_{h \to 0} \frac{-\left(nx^{n-1} + \binom{n}{2}x^{n-2}h^2 - \ldots - h^{n-1}\right)}{(x+h)^n x^n}$$

The limit is calculated by substituting $h = 0$.

$$\begin{aligned} \therefore f'(x) &= \frac{-(nx^{n-1} + 0)}{(x+0)^n x^n} \\ &= \frac{-nx^{n-1}}{x^n x^n} \\ &= -nx^{n-1-2n} \\ &= -nx^{-n-1} \end{aligned}$$

For $x \neq 0$, if $f(x) = x^{-n}$ where $n \in N$, then $f'(x) = -nx^{-n-1}$.

This rule means the derivative of the hyperbola $y = \dfrac{1}{x}$ can be obtained in a very similar way to differentiating a polynomial function.




$$\begin{aligned} y &= \frac{1}{x} = x^{-1} \\ \frac{dy}{dx} &= (-1)x^{-1-1} \\ &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

Since this derivative is always negative, it tells us that the hyperbola $y = \dfrac{1}{x}$ is always decreasing, as the gradient of its graph is always negative. This is consistent with what we already know about this hyperbola.

## Derivative of $y = x^{\frac{1}{n}}$, $n \in N$

To differentiate the function $y = x^{\frac{1}{n}}$, $n \in N$, we use the result that

$$\frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)}$$

For $y = x^{\frac{1}{n}}$, $n \in N$, raising both sides to the power of $n$ gives:

$$\begin{aligned} y^n &= \left(x^{\frac{1}{n}}\right)^n \\ &= x \\ \therefore x &= y^n \end{aligned}$$

Since $x$ is expressed as a function of $y$,

$$\frac{dx}{dy} = ny^{n-1}$$

Using the result $\dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{dx}{dy}\right)}$,

$$\frac{dy}{dx} = \frac{1}{ny^{n-1}}$$

$$= \frac{1}{n}y^{1-n}$$

Substituting back $y = x^{\frac{1}{n}}$ gives

$$\frac{dy}{dx} = \frac{1}{n}\left(x^{\frac{1}{n}}\right)^{1-n}$$

$$\therefore \frac{dy}{dx} = \frac{1}{n}x^{\frac{1}{n}-1}$$

**If $y = x^{\frac{1}{n}}$, $n \in N$, then**

$$\frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)}$$

$$\frac{dy}{dx} = \frac{1}{n}x^{\frac{1}{n}-1}$$

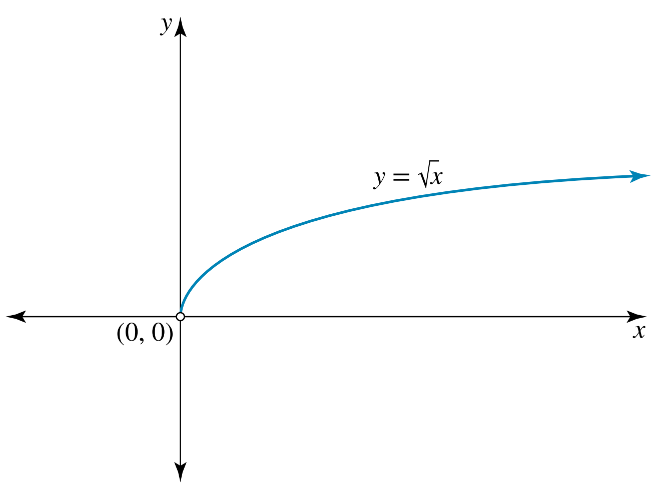
This rule means the derivative of the square root function $y = \sqrt{x}$ can also be obtained in a very similar way to differentiating a polynomial function.

In index form, $y = \sqrt{x}$ is written as $y = x^{\frac{1}{2}}$.
Its derivative is calculated as:

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$



This derivative is always positive, which tells us that the square root function $y = \sqrt{x}$ is an increasing function, as its gradient is always positive. This is consistent with our knowledge of the graph of $y = \sqrt{x}$.

Although the domain of the function $y = \sqrt{x}$ is $R^+ \cup \{0\}$, the domain of its derivative is $R^+$ as $\frac{1}{2\sqrt{x}}$ is undefined when $x = 0$.

## The derivative of any power function

The two previous types of power functions considered lead us to generalise that any power function can be differentiated using the same basic rule that was applied to polynomial functions. In fact, $y = x^n$ for any real number $n$ is differentiated using that same basic rule.

**For $n \in R$, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.**

The linearity properties of the derivative also hold for rational as well as polynomial functions.

$$\frac{d(ky)}{dx} = k\frac{dy}{dx},\ k \in R$$

$$\frac{d(f \pm g)}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$$

If $n \in N$, the domain of the derivative of $f(x) = x^n$, a polynomial function, will be $R$. However, if the power $n$ is a negative number, then $x = 0$, at least, must be excluded from the domains of both the function and its derivative function. This is illustrated by the hyperbola $y = \frac{1}{x}$ where the domain of the function and its derivative are both $R\backslash\{0\}$.

The derivative function will be undefined at any value of $x$ for which the tangent to the graph of the function is undefined.

Other restrictions on the domain of the derivative may arise when considering, for example, a function which is a sum of rational terms, or a hybrid function. Any function must be smoothly continuous for it to be differentiable.

Before the derivative can be calculated, index laws may be required to express power functions as a sum of terms where each $x$ term is placed in the numerator with a rational index. For example, $y = \frac{1}{x^p} + \sqrt[p]{x}$ would be expressed as $y = x^{-p} + x^{\frac{1}{p}}$ and then its derivative can be calculated using the rule for differentiation.

WORKED EXAMPLE 4

**a** If $y = \frac{2 + x^2 - 8x^5}{2x^3}$, calculate $\frac{dy}{dx}$, stating its domain.

**b** If $f(x) = 2\sqrt{3x}$:

**i** calculate $f'(x)$, stating its domain

**ii** calculate the gradient of the tangent at the point where $x = 3$.

| THINK | WRITE |
|---|---|
| **a** **1** Express the rational function as the sum of its partial fractions. | **a** $y = \dfrac{2 + x^2 - 8x^5}{2x^3}$ $= \dfrac{2}{2x^3} + \dfrac{x^2}{2x^3} - \dfrac{8x^5}{2x^3}$ $= \dfrac{1}{x^3} + \dfrac{1}{2x} - 4x^2$ |
| **2** Express each term as a power of $x$. | $\therefore y = x^{-3} + \frac{1}{2}x^{-1} - 4x^2$ |
| **3** Differentiate with respect to $x$. | $\dfrac{dy}{dx} = -3x^{-3-1} - \frac{1}{2}x^{-1-1} - 8x^{2-1}$ $= -3x^{-4} - \frac{1}{2}x^{-2} - 8x$ |
| **4** Express each term in $x$ with a positive index. | $\therefore \dfrac{dy}{dx} = \dfrac{-3}{x^4} - \dfrac{1}{2x^2} - 8x$ |
| **5** State the domain of the derivative. | There are terms in $x$ in the denominator, so $x \neq 0$. The domain of the derivative is $R\backslash\{0\}$. |
| **b** **i** **1** Express the term in $x$ with a rational index. | **b** **i** $f(x) = 2\sqrt{3x}$ $f(x) = 2\sqrt{3}\sqrt{x}$ $= 2\sqrt{3}x^{\frac{1}{2}}$ |
| **2** Calculate the derivative. | $f'(x) = 2\sqrt{3} \times \frac{1}{2}x^{\frac{1}{2}-1}$ $= \sqrt{3}x^{-\frac{1}{2}}$ |
| **3** Express $x$ with a positive index. | $f'(x) = \dfrac{\sqrt{3}}{\sqrt{x}}$ $= \sqrt{\dfrac{3}{x}}$ |
| **4** State the domain of the derivative. | Due to the square root term on the denominator, $x > 0$. The domain of the derivative is $R^+$. |
| **ii** Use the derivative to calculate the gradient at the given point. | **ii** Gradient at $x = 3$ is $f'(3)$. $f'(3) = \sqrt{\dfrac{3}{3}}$ $= \sqrt{1}$ $= 1$ Therefore the gradient is 1. |

## EXERCISE 13.3 Derivatives of power functions

**PRACTISE**

Work without CAS

1 a WE4 If $y = \dfrac{4 - 3x + 7x^4}{x^4}$, calculate $\dfrac{dy}{dx}$ and state its domain.

b If $f(x) = 4\sqrt{x} + \sqrt{2x}$,

i calculate $f'(x)$, stating its domain

ii calculate the gradient of the tangent at the point where $x = 1$.

2 Calculate $\dfrac{d}{dx}\left(6x^{\frac{2}{3}} + \dfrac{1}{\sqrt{x}}\right)$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

3 Differentiate the following with respect to $x$.

a $y = 4x^{-1} + 5x^{-2}$

b $y = 4x^{\frac{1}{2}} - 3x^{\frac{2}{3}}$

c $y = 2 + 8x^{-\frac{1}{2}}$

d $y = 0.5x^{1.8} - 6x^{3.1}$

4 Find $\dfrac{dy}{dx}$ for each of the following.

a $y = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

b $y = 5\sqrt{x}$

c $y = 3x - \sqrt[3]{x}$

d $y = x\sqrt{x}$

5 Calculate $f'(x)$, expressing the answer with positive indices, if:

a $f(x) = \dfrac{3x^2 + 5x - 9}{3x^2}$

b $f(x) = \left(\dfrac{x}{5} + \dfrac{5}{x}\right)^2$

c $f(x) = \sqrt[5]{x^2} + \sqrt{5x} + \dfrac{1}{\sqrt{x}}$

d $f(x) = 2x^{\frac{3}{4}}(4 + x - 3x^2)$

6 A function $f$ is defined as $f: [0, \infty) \to R, f(x) = 4 - \sqrt{x}$.

a Define the derivative function $f'(x)$.

b Obtain the gradient of the graph of $y = f(x)$ as the graph cuts the $x$-axis.

c Calculate the gradient of the graph of $y = f(x)$ when $x = 0.0001$ and when $x = 10^{-10}$.

d What happens to the gradient as $x \to 0$?

e Describe the tangent to the curve at the point $(0, 4)$.

7 Consider the hyperbola defined by $y = 1 - \dfrac{3}{x}$.

a At its $x$-intercept, the gradient of the tangent is $g$. Calculate the value of $g$.

b Obtain the coordinates of the other point where the gradient of the tangent is $g$.

c Sketch the hyperbola showing both tangents.

d Express the gradients of the hyperbola at the points where $x = 10$ and $x = 10^3$ in scientific notation; describe what is happening to the tangent to the curve as $x \to \infty$.

8 a Consider the function $f: R\backslash\{0\} \to R,\ f(x) = x - \dfrac{1}{x}$.

i State the domain of the function.

ii Form the rule for its gradient function, stating its domain.

iii Calculate the gradient of the tangent to the curve at the point $(1, 0)$.

iv Find the coordinates of the points on the curve where the tangent has a gradient of 5.

b Consider the function $f: R\backslash\{0\} \to R,\ f(x) = x^2 + \dfrac{2}{x}$.

i Evaluate $f'(2)$.

ii Determine the coordinates of the point for which $f'(x) = 0$.

iii Calculate the exact values of $x$ for which $f'(x) = -4$.

9 A function is defined by the rule

$$f(x) = \begin{cases} (2 - x)^2, & x < 4 \\ 2 + \sqrt{x}, & x \geq 4 \end{cases}$$

a Calculate, if it exists, $\lim\limits_{x \to 4} f(x)$.
b Explain whether the function is continuous at $x = 4$.
c Determine whether the function is differentiable at $x = 4$.
d Calculate the rule for $f'(x)$.
e Evaluate, if possible, $f'(0)$.
f Calculate the values of $x$ for which $f'(x) < 0$.

10 The height of a magnolia tree, in metres, is modelled by $h = 0.5 + \sqrt{t}$ where $h$ is the height $t$ years after the tree was planted.

a How tall was the tree when it was planted?
b At what rate is the tree growing 4 years after it was planted?
c When will the tree be 3 metres tall?
d What will be the average rate of growth of the tree over the time period from planting to a height of 3 metres?

11 a Factorise $(x + h)^3 - x^3$ as the difference of two cubes.
b Hence, use first principles to find $f'(x)$ if $f(x) = \dfrac{1}{x^3}$.
c Show that the answer in part b is obtained when $f(x) = \dfrac{1}{x^3}$ is differentiated by rule.
d Calculate the following.
i $\dfrac{d}{dx}\left(x^3 + \dfrac{1}{x^3}\right)$
ii $\dfrac{d}{dx}\left(x - \dfrac{1}{x}\right)^3$

12 On a warm day in a garden, water in a bird bath evaporates in such a way that the volume, $V$ mL, at time $t$ hours is given by

$$V = \frac{60t + 2}{3t}, \quad t > 0.$$

a Show that $\dfrac{dV}{dt} < 0$.
b At what rate is the water evaporating after 2 hours?
c Sketch the graph of $V = \dfrac{60t + 2}{3t}$ for $t \in \left[\dfrac{1}{3}, 2\right]$.
d Calculate the gradient of the chord joining the endpoints of the graph for $t \in \left[\dfrac{1}{3}, 2\right]$ and explain what the value of this gradient measures.

MASTER

13 Give the gradient of the tangent to the hyperbola $y = 4 - \frac{1}{3x - 2}$ at the point $\left(\frac{1}{3}, \frac{9}{2}\right)$.

14 a Sketch the graphs of $y = x^{\frac{2}{3}}$ and $y = x^{\frac{5}{3}}$ using CAS technology and find the coordinates of the points of intersection.

b Compare the gradients of the tangents to each curve at the points of intersection.

# 13.4 Coordinate geometry applications of differentiation

studyon

Units 1 & 2
AOS 3
Topic 2
Concept 5

**Coordinate geometry applications of differentiation**
Concept summary
Practice questions

eBookplus

**Interactivity**
Equations of tangents
**int-5962**

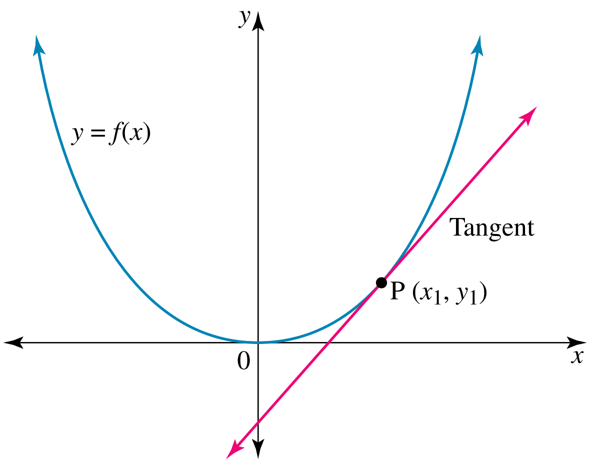
The derivative measures the gradient of the tangent to a curve at any point. This has several applications in coordinate geometry.

## Equations of tangents

A **tangent** is a straight line, so, to form its equation, its gradient and a point on the tangent are needed. The equation can then be formed using

$$y - y_1 = m(x - x_1).$$

For the tangent to a curve $y = f(x)$ at a point P, the gradient $m$ is found by evaluating the curve's derivative $f'(x)$ at P, the point of contact or point of tangency. The coordinates of P provide the point $(x_1, y_1)$ on the line.




WORKED EXAMPLE 5

**Form the equation of the tangent to the curve $y = 2x^3 + x^2$ at the point on the curve where $x = -2$.**

| THINK | WRITE |
| --- | --- |
| 1 Obtain the coordinates of the point of contact of the tangent with the curve by substituting $x = -2$ into the equation of the curve. | $y = 2x^3 + x^2$<br>When $x = -2$,<br>$y = 2(-2)^3 + (-2)^2$<br>$= -12$<br>The point of contact is $(-2, -12)$. |
| 2 Calculate the gradient of the tangent at the point of contact by finding the derivative of the function at the point $x = -2$. | $y = 2x^3 + x^2$<br>$\therefore \frac{dy}{dx} = 6x^2 + 2x$<br>When $x = -2$,<br>$\frac{dy}{dx} = 6(-2)^2 + 2(-2)$<br>$= 20$<br>The gradient of the tangent is 20. |

3 Form the equation of the tangent line.

Equation of the tangent:

$y - y_1 = m(x - x_1)$, $m = 20$, $(x_1, y_1) = (-2, -12)$

$\therefore y + 12 = 20(x + 2)$

$\therefore y = 20x + 28$

The equation of the tangent is $y = 20x + 28$.

## Coordinate geometry

Since the tangent is a straight line, all the results obtained for the coordinate geometry of a straight line apply to the tangent line. These include:

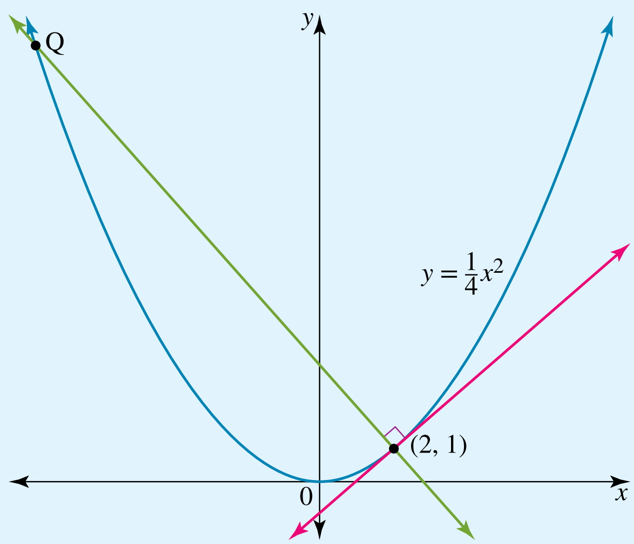
- the angle of inclination of the tangent to the horizontal can be calculated using $m = \tan(\theta)$
- tangents which are parallel have the same gradient
- the gradient of a line perpendicular to the tangent is found using $m_1 m_2 = -1$
- the gradient of a horizontal tangent is zero
- the gradient of a vertical tangent is undefined
- coordinates of points of intersections of tangents with other lines and curves can be found using simultaneous equations.

**WORKED EXAMPLE 6**

**At the point (2, 1) on the curve $y = \frac{1}{4}x^2$, a line is drawn perpendicular to the tangent to the curve at that point. This line meets the curve $y = \frac{1}{4}x^2$ again at the point Q.**

**a Calculate the coordinates of the point Q.**

**b Calculate the magnitude of the angle that the line passing through Q and the point (2, 1) makes with the positive direction of the $x$-axis.**




| THINK | WRITE |
|---|---|
| **a 1** Calculate the gradient of the tangent at the given point. | **a** $y = \frac{1}{4}x^2$<br>$\frac{dy}{dx} = \frac{1}{2}x$<br>At the point (2, 1),<br>$\frac{dy}{dx} = \frac{1}{2} \times 2 = 1$<br>The gradient of the tangent at the point (2, 1) is 1. |
| **2** Calculate the gradient of the line perpendicular to the tangent. | For perpendicular lines, $m_1 m_2 = -1$.<br>Since the gradient of the tangent is 1, the gradient of the line perpendicular to the tangent is $-1$. |

**3** Form the equation of the perpendicular line.

Equation of the line perpendicular to the tangent:

$y - y_1 = m(x - x_1),\ m = -1, (x_1, y_1) = (2, 1)$

$y - 1 = -(x - 2)$

$\therefore y = -x + 3$

**4** Use simultaneous equations to calculate the coordinates of Q.

Point Q lies on the line $y = -x + 3$ and the curve $y = \frac{1}{4}x^2$.

At Q,

$$\begin{aligned}\frac{1}{4}x^2 &= -x + 3\\ x^2 &= -4x + 12\\ x^2 + 4x - 12 &= 0\\ (x + 6)(x - 2) &= 0\\ \therefore x &= -6, x = 2\end{aligned}$$

$x = 2$ is the $x$-coordinate of the given point. Therefore, the $x$-coordinate of Q is $x = -6$.

Substitute $x = -6$ into $y = -x + 3$:

$$\begin{aligned}y &= -(-6) + 3\\ &= 9\end{aligned}$$

Point Q has the coordinates $(-6, 9)$.

**b** Calculate the angle of inclination required.

**b** For the angle of inclination, $m = \tan\theta$.

As the gradient of the line passing through Q and the point (2, 1) is $-1$, $\tan\theta = -1$.

Since the gradient is negative, the required angle is obtuse.

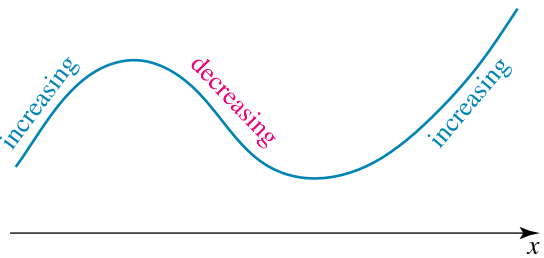
The second quadrant solution is

$$\begin{aligned}\theta &= 180° - \tan^{-1}(1)\\ &= 180° - 45°\\ &= 135°\end{aligned}$$

The angle made with the positive direction of the $x$-axis is 135°.

## Increasing and decreasing functions

If the tangents to a function's curve have a positive gradient over some domain interval then the function is increasing over that domain; if the gradients of the tangents are negative then the function is decreasing. From this it follows that for a function $y = f(x)$ which is differentiable over an interval:



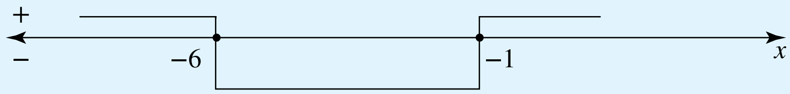

- if $f'(x) > 0$, then the function increases as $x$ increases over the interval
- if $f'(x) < 0$, then the function decreases as $x$ increases over the interval.

**WORKED EXAMPLE 7**

**a** Over what domain interval is the function $f(x) = \frac{1}{3}x^3 + \frac{7}{2}x^2 + 6x - 5$ decreasing?

**b** **i** Form the equation of the tangent to the curve $y = x^2 + ax + 4$ at the point where $x = -1$.

**ii** Hence, find the values of $a$ for which the function $y = x^2 + ax + 4$ is increasing at $x = -1$.

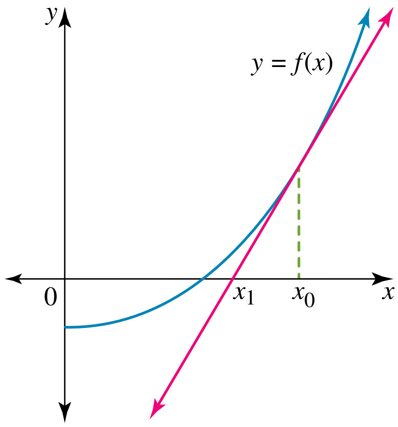
| THINK | WRITE/DRAW |
|---|---|
| **a** **1** Apply the condition for a function to be decreasing to the given function. | **a** For a decreasing function, $f'(x) < 0$ <br> $f(x) = \frac{1}{3}x^3 + \frac{7}{2}x^2 + 6x - 5$ <br> $f'(x) = x^2 + 7x + 6$ <br> $\therefore x^2 + 7x + 6 < 0$ |
| **2** Solve the inequation using a sign diagram of $f'(x)$. <br> *Note:* Alternatively, sketch the gradient function. | $\therefore (x + 6)(x + 1) < 0$ <br> Zeros are $x = -6, x = -1$ <br>   <br> $\therefore \ -6 < x < -1$ |
| **3** State the answer. | The function is decreasing over the interval $x \in (-6, \ -1)$. |
| **b i 1** Express the coordinates of the point of tangency in terms of $a$. | **b i** $y = x^2 + ax + 4$ <br> When $x = -1$ <br> $y = (-1)^2 + a(-1) + 4$ <br> $= 5 - a$ <br> The point is $(-1, \ 5 - a)$. |
| **2** Obtain the gradient of the tangent in terms of $a$. | Gradient: $\frac{dy}{dx} = 2x + a$ <br> At the point $(-1, 5 - a)$, $\frac{dy}{dx} = -2 + a$. <br> Therefore the gradient is $-2 + a$. |
| **3** Form the equation of the tangent. | Equation of tangent: <br> $y - y_1 = m(x - x_1), \ m = -2 + a, (x_1, y_1) = (-1, 5 - a)$ <br> $y - (5 - a) = (-2 + a)(x + 1)$ <br> $y = (a - 2)x - 2 + a + 5 - a$ <br> $y = (a - 2)x + 3$ |
| **ii** Apply the condition on the tangent for a function to be increasing, and calculate the values of $a$. | **ii** If a function is increasing, the tangent to its curve must have a positive gradient. <br> The gradient of the tangent at $x = -1$ is $a - 2$. <br> Therefore, the function $y = x^2 + ax + 4$ is increasing at $x = -1$ when: <br> $a - 2 > 0$ <br> $\therefore a > 2$ |

## Newton’s method

### Using tangents to obtain approximate solutions to equations

Isaac Newton and his assistant Joseph Raphson devised an iterative method for obtaining numerical approximations to the roots of equations of the form $f(x) = 0$. This method is known as **Newton’s method** or the **Newton–Raphson method**.

Consider a curve $y = f(x)$ in the region near its intersection with the $x$-axis. An increasing curve has been chosen. The solution to, or root of, the equation $f(x) = 0$ is given by the $x$-intercept of the graph of $y = f(x)$.




Let $x_0$ be a first estimate of the $x$-intercept, that is the first estimate of the root of the equation $f(x) = 0$.

The tangent to the curve $y = f(x)$ is drawn at the point on the curve where $x = x_0$. To obtain the equation of this tangent, its gradient and the coordinates of the point on the curve are needed.

The gradient of the tangent is $\dfrac{dy}{dx} = f'(x)$.

At the point $(x_0, f(x_0))$, the gradient is $f'(x_0)$.

The equation of the tangent is:

$$
\begin{aligned}
y - f(x_0) &= f'(x_0)(x - x_0) \\
\therefore y &= f(x_0) + f'(x_0)(x - x_0)
\end{aligned}
$$

The intercept, $x_1$, that this tangent makes with the $x$-axis is a better estimate of the required root. Its value is obtained from the equation of the tangent.

Let $y = 0$.

$$
\begin{aligned}
0 &= f(x_0) + f'(x_0)(x - x_0) \\
f'(x_0)(x - x_0) &= -f(x_0) \\
(x - x_0) &= -\frac{f(x_0)}{f'(x_0)} \\
x &= x_0 - \frac{f(x_0)}{f'(x_0)}
\end{aligned}
$$

Hence the improved estimate is $x_1 = x_0 - \dfrac{f(x_1)}{f'(x_0)}$.

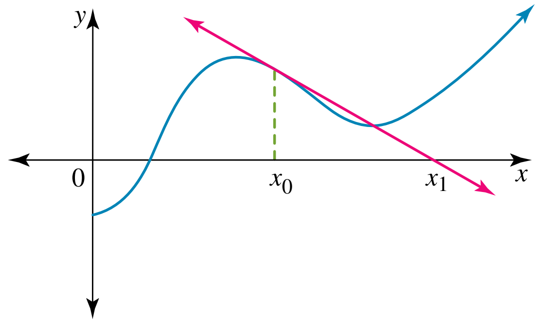
The tangent at the point on the curve where $x = x_1$ is then constructed and its $x$-intercept $x_2$ calculated from $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

The procedure continues to be repeated until the desired degree of accuracy to the root of the equation is obtained.

The formula for the iterative relation is generalised as

$$
\mathbf{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0, 1, 2\ldots}
$$

The first estimate needs to be reasonably close to the desired root for the procedure to require only a few iterations. It is also possible that a less appropriate choice of first estimate could lead to subsequent estimates going further away from the true value. An example of such a situation is shown.




**WORKED EXAMPLE 8**

**Use Newton's method to calculate the root of the equation $x^3 - 2x - 2 = 0$ that lies near $x = 2$. Express the answer correct to 4 decimal places.**

| THINK | WRITE |
| --- | --- |
| 1 Define $f(x)$ and state $f'(x)$. | Let $f(x) = x^3 - 2x - 2$.<br>Then $f'(x) = 3x^2 - 2$. |
| 2 State Newton's formula for calculating $x_1$ from $x_0$. | $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$<br>$\therefore x = x_0 - \dfrac{f(x_n)}{f'(x_0)}$ |
| 3 Use the value of $x_0$ to calculate the value of $x_1$. | Let $x_0 = 2$.<br>$f(2) = (2)^3 - 2(2) - 2$<br>$= 2$<br>$f'(2) = 3(2)^2 - 2$<br>$= 10$<br>$x_1 = 2 - \dfrac{2}{10} = 1.8$ |
| 4 Use the value of $x_1$ to calculate the value of $x_2$. | $f(1.8) = (1.8)^3 - 2(1.8) - 2$<br>$= 0.232$<br>$f'(1.8) = 3(1.8)^2 - 2$<br>$= 7.72$<br>$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$<br>$= 1.8 - \dfrac{0.232}{7.72}$<br>$\therefore x_2 \approx 1.769\,948\,187$ |
| 5 Use the *Ans* key to carry the value of $x_2$ in a calculator, and calculate the value of $x_3$.<br>*Note:* In practice, carrying the previous value as *Ans* can be commenced from the beginning. | With the value of $x_2$ as *Ans*:<br>$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$<br>$= Ans - \dfrac{f(Ans)}{f'(Ans)}$<br>$= Ans - \dfrac{(Ans^3 - 2 \times Ans - 2)}{(3 \times Ans^2 - 2)}$<br>$\therefore x_3 = 1.769\,292\,663$ |

| | |
|---|---|
| 6 Continue the procedure until the 4th decimal place remains unchanged. | $x_4 = Ans - \dfrac{f(Ans)}{f'(Ans)}$<br>$= 1.769\,292\,354$<br>Both $x_3$ and $x_4$ are the same value for the desired degree of accuracy. |
| 7 State the solution. | To 4 decimal places, the solution to $x^3 - 2x - 2 = 0$ is $x = 1.7693$. |

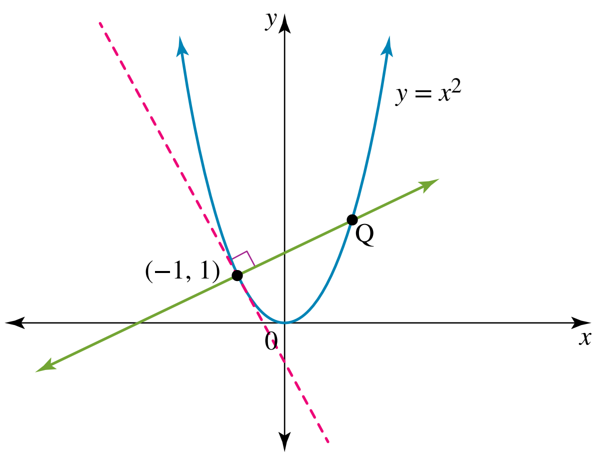
## EXERCISE 13.4 Coordinate geometry applications of differentiation

**PRACTISE**

1 WE5 Form the equation of the tangent to the curve $y = 5x - \frac{1}{3}x^3$ at the point on the curve where $x = 3$.

2 Form the equation of the tangent to the curve $y = \frac{4}{x^2} + 3$ at the point where the tangent is parallel to the line $y = -8x$.

3 WE6 At the point $(-1, 1)$ on the curve $y = x^2$ a line is drawn perpendicular to the tangent to the curve at that point. This line meets the curve $y = x^2$ again at the point Q.




a Calculate the coordinates of the point Q.

b Calculate the magnitude of the angle that the line through Q and the point $(-1, 1)$ makes with the positive direction of the $x$-axis.

4 The tangent to the curve $y = \frac{1}{x}$ at the point where $x = \frac{1}{2}$ meets the $x$-axis at A and the $y$-axis at B. Calculate the coordinates of the midpoint of the line segment AB.

5 a WE7 Over what domain interval is the function $f(x) = \frac{1}{3}x^3 + x^2 - 8x + 6$ increasing?

b i Form the equation of the tangent to the curve $y = ax^2 + 4x + 5$ at the point where $x = 1$.

ii Hence find the values of $a$ for which the function $y = ax^2 + 4x + 5$ is decreasing at $x = 1$.

6 Over what domain interval is the function $f(x) = 10 - \frac{2}{5x}$ increasing?

7 WE8 Use Newton's method to calculate the root of the equation $\frac{1}{3}x^3 + 7x - 3 = 0$ that lies near $x = 2$. Express the answer correct to 4 decimal places.

8 Consider the equation $-x^3 + x^2 - 3x + 5 = 0$.

a Show that there is a root which lies between $x = 1$ and $x = 2$.

b Use Newton's method to calculate the root to an accuracy of 4 decimal places.

## CONSOLIDATE

Apply the most appropriate mathematical processes and tools

**9** Form the equation of the tangent to the curve at the given point.

**a** $y = 2x^2 - 7x + 3$; $(0, 3)$

**b** $y = 5 - 8x - 3x^2$; $(-1, 10)$

**c** $y = \frac{1}{2}x^3$; $(2, 4)$

**d** $y = \frac{1}{3}x^3 - 2x^2 + 3x + 5$; $(3, 5)$

**e** $y = \frac{6}{x} + 9$; $\left(-\frac{1}{2}, -3\right)$

**f** $y = 38 - 2x^{\frac{3}{4}}$; $(81, -16)$

**10 a** Obtain the equation of the tangent to $y = 4\sqrt{x}$ at the point where $x = 4$.

**b** Calculate the equation of the line which is perpendicular to the tangent to $y = \sqrt{2x}$ at the point where $x = 2$.

**c** A tangent to the curve $y = 6\sqrt{x}$ is inclined at 60° to the horizontal. Determine its equation.

**11** Form the equation of the tangent to $y = x^2 - 6x + 3$ which is:

**a** parallel to the line $y = 4x - 2$

**b** parallel to the $x$-axis

**c** perpendicular to the line $6y + 3x - 1 = 0$.

**12 a** Calculate the coordinates of the point of intersection of the tangents drawn to the $x$-intercepts of the curve $y = x(4 - x)$.

**b** Show that the tangents drawn to the $x$-intercepts of the parabola $y = (x - a)(x - b)$ intersect on the parabola's axis of symmetry.

**c** Consider the tangent drawn to A, the positive $x$-intercept of the curve $y = x(4 - x)$. A line perpendicular to this tangent is drawn through point A. Determine the equation of the tangent to the curve $y = x(4 - x)$ at the second point where this line intersects this curve.

**13 a** Over what interval is the function $f(x) = 3 - 7x + 4x^2$ decreasing?

**b** Over what interval is the function $y = -12x + x^3$ increasing?

**c i** Show that any tangent to the curve $y = x^3 + 3x + 5$ will have a positive gradient.

**ii** Form the equation of the tangent at the point on the curve where $x = -1$.

**d i** Show that any tangent to the curve $y = \frac{2 - x^2}{x}, x \neq 0$ will have a negative gradient.

**ii** Form the equation of the tangent at the point on the curve where $x = -1$.

**14 a** Use Newton's method to obtain the root of the equation $x^3 + x - 4 = 0$ near $x = 1$. Express the answer to an accuracy of 4 decimal places.

**b** The equation $x^3 - 8x - 10 = 0$ has only one solution.

**i** Between which two integers does this solution lie?

**ii** Use Newton's method to obtain the solution to an accuracy of 4 decimal places.

**c** Obtain the root of the equation $x^3 + 5x^2 = 10$ near $x = -2$ using Newton's method, to an accuracy of 4 decimal places.

**d** Use a suitable cubic equation and Newton's method to calculate $\sqrt[3]{16}$ correct to 4 decimal places.

**e** Calculate the exact roots of the quadratic equation $x^2 - 8x + 9 = 0$ and hence, using Newton's method to obtain one of the roots, calculate the value of $\sqrt{7}$ to 4 decimal places.

f Consider the quartic function defined by $y = x^4 - 2x^3 - 5x^2 + 9x$.

i Show the function is increasing when $x = 0$ and decreasing when $x = 1$.

ii Use Newton's method to obtain the value of $x$ where the quartic function changes from increasing to decreasing in the interval $[0, 1]$. Express the value to an accuracy of 3 decimal places.

**15** Consider the function $f: R \to R,\ f(x) = x^3$.

a Find the equation of the tangent to $f$ at the point $(0, 0)$ and sketch the curve $y = f(x)$ showing this tangent.

b i Use the graph of $y = f(x)$ to sketch the graph of the inverse function $y = f^{-1}(x)$.

ii On your sketch of $y = f^{-1}(x)$, draw the tangent to the curve at $(0, 0)$. What is the equation of this tangent?

c Form the rule for the inverse function.

d Calculate $\frac{d}{dx}(f^{-1}(x))$ and explain what happens to this derivative if $x = 0$.

**16** Consider the curve with equation $y = \frac{1}{3}x(x + 4)(x - 4)$.

a Sketch the curve and draw a tangent to the curve at the point where $x = 3$.

b Form the equation of this tangent.

c i The tangent meets the curve again at a point P. Show that the $x$-coordinate of the point P satisfies the equation $x^3 - 27x + 54 = 0$.

ii Explain why $(x - 3)^2$ must be a factor of this equation and hence calculate the coordinates of P.

d Show that the tangents to the curve at the points where $x = \pm 4$ are parallel.

e i For $a \in R\backslash\{0\}$, show that the tangents to the curve $y = x(x + a)(x - a)$ at the points where $x = \pm a$ are parallel.

ii Calculate the coordinates, in terms of $a$, of the points of intersection of the tangent at $x = 0$ with each of the tangents at $x = -a$ and $x = a$.

**17** Determine the equations of the tangents to the curve $y = -\frac{4}{x} - 1$ at the point(s):

a where the tangent is inclined at 45° to the positive direction of the $x$-axis

b where the tangent is perpendicular to the line $2y + 8x = 5$

c where the parabola $y = x^2 + 2x - 8$ touches the curve $y = -\frac{4}{x} - 1$; draw a sketch of the two curves showing the common tangent.

**18** Consider the family of curves $C = \{(x, y): y = x^2 + ax + 3,\ a \in R\}$.

a Express the equation of the tangent to the curve $y = x^2 + ax + 3$ at the point where $x = -a$ in terms of $a$.

b If the tangent at $x = -a$ cuts the $y$-axis at $y = -6$ and the curve $y = x^2 + ax + 3$ is increasing at $x = -a$, calculate the value of $a$.

c Over what domain interval are all the curves in the family $C$ decreasing functions?

d Show that for $a \neq 0$, each line through $x = -a$ perpendicular to the tangents to the curves in $C$ at $x = -a$ will pass through the point $(0, 4)$. If $a = 0$, explain whether or not this would still hold.

MASTER

**19** Obtain the equation of the tangent to the curve $y = (2x + 1)^3$ at the point where $x = -4.5$.

**20** Sketch the curve $y = x^3 + 2x^2 - 4x - 2$ and its tangent at the point where $x = 0$, and hence find the coordinates of the second point where the tangent meets the curve. Give the equation of the tangent at this second point.

# 13.5 Curve sketching

At the points where a differentiable function is neither increasing nor decreasing, the function is **stationary** and its gradient is zero. Identifying such stationary points provides information which assists curve sketching.

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Topic 2
Concept 6, 7, 8

**Curve sketching**
Concept summary
Practice questions

eBook*plus*

**Interactivity**
Stationary points
**int-5963**

## Stationary points

At a **stationary point** on a curve $y = f(x)$, $f'(x) = 0$.

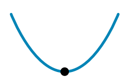
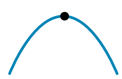
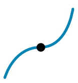
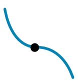
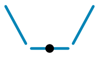
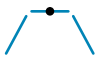
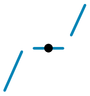
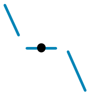
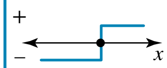
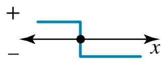
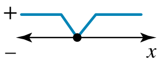
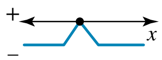
There are three types of stationary points:

- **(local) minimum** turning point
- **(local) maximum** turning point
- **stationary point of inflection**.

The word 'local' means the point is a minimum or a maximum in a particular locality or neighbourhood. Beyond this section of the graph there could be other points on the graph which are lower than the local minimum or higher than the local maximum. Our purpose for the time being is simply to identify the turning points and their nature, so we shall continue to refer to them just as minimum or maximum turning points.

## Determining the nature of a stationary point

At each of the three types of stationary points, $f'(x) = 0$. This means that the tangents to the curve at these points are horizontal. By examining the slope of the tangent to the curve immediately before and immediately after the stationary point, the nature or type of stationary point can be determined.

| | Minimum turning point | Maximum turning point | Stationary point of inflection |
|---|---|---|---|
| Stationary point |  |  |  or  |
| Slope of tangent |  |  |  or  |
| Sign diagram of gradient function $f'(x)$ |   |   |   or   |

For a minimum turning point, the behaviour of the function changes from decreasing just before the point, to stationary at the point, to increasing just after the point; the slope of the tangent changes from negative to zero to positive.

For a maximum turning point, the behaviour of the function changes from increasing just before the point, to stationary at the point, to decreasing just after the point; the slope of the tangent changes from positive to zero to negative.

For a stationary point of inflection, the behaviour of the function remains either increasing or decreasing before and after the point, and stationary at the point; the slope of the tangent is zero at the point but it does not change sign either side of the point.

To identify stationary points and their nature:

- establish where $f'(x) = 0$
- determine the nature by drawing the sign diagram of $f'(x)$ around the stationary point or, alternatively, by testing the slope of the tangent at selected points either side of, and in the neighbourhood of, the stationary point.

For polynomial functions, the sign diagram is usually the preferred method for determining the nature, or for justifying the anticipated nature, of the turning point, as the gradient function is also a polynomial. If the function is not a polynomial, then testing the slope of the tangent is usually the preferred method for determining the nature of a stationary point.

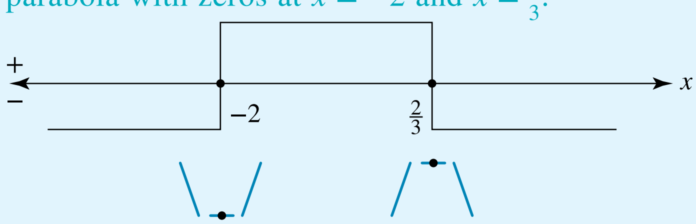
**WORKED EXAMPLE 9**

**a** **Determine the stationary points of $f(x) = 2 + 4x - 2x^2 - x^3$ and justify their nature.**

**b** **The curve $y = ax^2 + bx - 24$ has a stationary point at $(-1, -25)$. Calculate the values of $a$ and $b$.**

**c** **The point $(4, -1)$ is a stationary point on a curve $y = f(x)$ for which $x > 0$. Given that $f'(x) = \frac{1}{x} - \frac{1}{4}$, determine the nature of the stationary point.**

| THINK | WRITE/DRAW |
|---|---|
| **a 1** Calculate the $x$-coordinates of the stationary points. *Note:* Always include the reason why $f'(x) = 0$. | **a** $f(x) = 2 + 4x - 2x^2 - x^3$<br>$f'(x) = 4 - 4x - 3x^2$<br>At stationary points, $f'(x) = 0$, so:<br>$4 - 4x - 3x^2 = 0$<br>$(2 - 3x)(2 + x) = 0$<br>$x = \frac{2}{3}$ or $x = -2$ |
| **2** Calculate the corresponding $y$-coordinates. | When $x = \frac{2}{3}$,<br>$f(x) = 2 + 4\left(\frac{2}{3}\right) - 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$<br>$= \frac{94}{27}$<br>When $x = -2$,<br>$f(x) = 2 + 4(2) - 2(2)^2 - (2)^3$<br>$= -6$<br>Stationary points are $\left(\frac{2}{3}, \frac{94}{27}\right)$, $(-2, -6)$. |
| **3** To justify the nature of the stationary points, draw the sign diagram of $f'(x)$. *Note:* The shape of the cubic graph would suggest the nature of the stationary points. | Since $f'(x) = 4 - 4x - 3x^2$<br>$= (2 - 3x)(2 + x)$<br>the sign diagram of $f'(x)$ is that of a concave down parabola with zeros at $x = -2$ and $x = \frac{2}{3}$. |



**4** Identify the nature of each stationary point by examining the sign of the gradient before and after each point.

At $x = -2$, the gradient changes from negative to positive, so $(-2, -6)$ is a minimum turning point.
At $x = \frac{2}{3}$, the gradient changes from positive to negative, so $\left(\frac{2}{3}, \frac{94}{27}\right)$ is a maximum turning point.

**b 1** Use the coordinates of the given point to form an equation.
*Note:* As there are two unknowns to determine, two pieces of information are needed to form two equations in the two unknowns.

**b** $y = ax^2 + bx - 24$
Point $(-1, -25)$ lies on the curve.
$$-25 = a(-1)^2 + b(-1) - 24$$
$\therefore a - b = -1 \ldots.(1)$

**2** Use the other information given about the point to form a second equation.

Point $(-1, -25)$ is a stationary point, so $\frac{dy}{dx} = 0$ at this point.
$y = ax^2 + bx - 24$
$$\frac{dy}{dx} = 2ax + b$$
At $(-1, -25)$, $\frac{dy}{dx} = 2a(-1) + b$
$= -2a + b$
$\therefore -2a + b = 0 \ldots.(2)$

**3** Solve the simultaneous equations and state the answer.

$a - b = -1 \ldots.(1)$
$-2a + b = 0 \ldots.(2)$
Adding the equations,
$-a = -1$
$\therefore a = 1$
Substitute $a = 1$ into equation (2):
$-2 + b = 0$
$\therefore b = 2$
The values are $a = 1$ and $b = 2$.

**c 1** Test the sign of the gradient function at two selected points either side of the given stationary point.

**c** $f'(x) = \frac{1}{x} - \frac{1}{4}$

| $x$ | 3 | 4 | 5 |
|---|---|---|---|
| $f'(x)$ | $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ | 0 | $\frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$ |
| **Slope of tangent** | / | — | \ |

**2** State the nature of the stationary point.

As the gradient changes from positive to zero to negative, the point $(4, -1)$ is a maximum turning point.

## Curve sketching

To sketch the graph of any function $y = f(x)$:

- Obtain the $y$-intercept by evaluating $f(0)$.
- Obtain any $x$-intercepts by solving, if possible, $f(x) = 0$. This may require the use of factorisation techniques including the factor theorem.

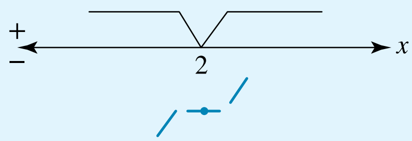
- Calculate the $x$-coordinates of the stationary points by solving $f'(x) = 0$ and use the equation of the curve to obtain the corresponding $y$-coordinates.
- Identify the nature of the stationary points.
- Calculate the coordinates of the endpoints of the domain, where appropriate.
- Identify any other key features of the graph, where appropriate.

**WORKED EXAMPLE 10** **Sketch the function $y = \frac{1}{2}x^3 - 3x^2 + 6x - 8$. Locate any intercepts with the coordinate axes and any stationary points, and justify their nature.**

| THINK | WRITE/DRAW |
|---|---|
| 1 State the $y$-intercept. | $y = \frac{1}{2}x^3 - 3x^2 + 6x - 8$<br>$y$-intercept: $(0, -8)$ |
| 2 Calculate any $x$-intercepts. | $x$-intercepts: when $y = 0$,<br>$\frac{1}{2}x^3 - 3x^2 + 6x - 8 = 0$<br>$x^3 - 6x^2 + 12x - 16 = 0$<br>Let $P(x) = x^3 - 6x^2 + 12x - 16$<br>$P(4) = 64 - 96 + 48 - 16$<br>$= 0$<br>$\therefore (x - 4)$ is a factor.<br>$x^3 - 6x^2 + 12x - 16 = (x - 4)(x^2 - 2x + 4)$<br>$\therefore x = 4$ or $x^2 - 2x + 4 = 0$<br>The discriminant of $x^2 - 2x + 4$ is $\Delta = 4 - 16 < 0$.<br>There is only one $x$-intercept: $(4, 0)$. |
| 3 Obtain the derivative in order to locate any stationary points. | Stationary points:<br>$y = \frac{1}{2}x^3 - 3x^2 + 6x - 8$<br>$\frac{dy}{dx} = \frac{3}{2}x^2 - 6x + 6$<br>At stationary points, $\frac{dy}{dx} = 0$, so:<br>$\frac{3}{2}x^2 - 6x + 6 = 0$<br>$\frac{3}{2}(x^2 - 4x + 4) = 0$<br>$\frac{3}{2}(x - 2)^2 = 0$<br>$x = 2$<br>Substitute $x = 2$ into the function's equation:<br>$y = \frac{1}{2} \times 8 - 3 \times 4 + 6 \times 2 - 8$<br>$= -4$<br>Stationary point is $(2, -4)$. |

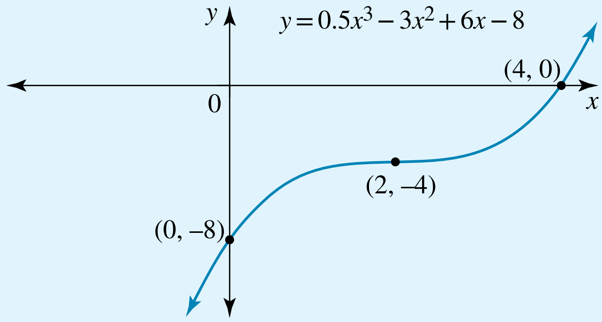
4 Identify the type of stationary point using the sign diagram of the gradient function.

Sign diagram of $\frac{dy}{dx} = \frac{3}{2}(x - 2)^2$:



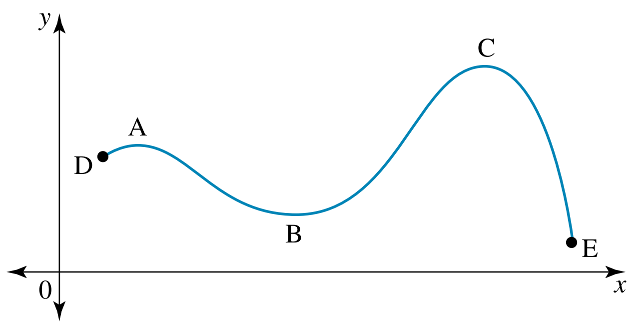
The point (2, −4) is a stationary point of inflection.

5 Sketch the curve showing the intercepts with the axes and the stationary point.




## Local and global maxima and minima

The diagram shows the graph of a function sketched over a domain with endpoints D and E.




There are three turning points: A and C are maximum turning points, and B is a minimum turning point.

The $y$-coordinate of point A is greater than those of its neighbours, so A is a local maximum point. At point C, not only is its $y$-coordinate greater than those of its neighbours, it is greater than that of any other point on the graph. For this reason, C is called the **global** or **absolute maximum** point.

The **global** or **absolute minimum** point is the point whose $y$-coordinate is smaller than any others on the graph. For this function, point E, an endpoint of the domain, is the global or absolute minimum point. Point B is a local minimum point; it is not the global minimum point.

Global maximums and global minimums may not exist for all functions. For example, a cubic function on its maximal domain may have one local maximum turning point and one local minimum turning point, but there is neither a global maximum nor a global minimum point since as $x \to \pm\infty$, $y \to \pm\infty$ (assuming a positive coefficient of $x^3$).

If a differentiable function has a global maximum or a global minimum value, then this will either occur at a turning point or at an endpoint of the domain. The $y$-coordinate of such a point gives the value of the global maximum or the global minimum.

### Definitions

- **A function $y = f(x)$ has a global maximum $f(a)$ if $f(a) \geq f(x)$ for all $x$-values in its domain.**
- **A function $y = f(x)$ has a global minimum $f(a)$ if $f(a) \leq f(x)$ for all $x$-values in its domain.**
- **A function $y = f(x)$ has a local maximum $f(x_0)$ if $f(x_0) \geq f(x)$ for all $x$-values in the neighbourhood of $x_0$.**
- **A function $y = f(x)$ has a local minimum $f(x_0)$ if $f(x_0) \leq f(x)$ for all $x$-values in the neighbourhood of $x_0$.**

WORKED EXAMPLE 11

**A function defined on a restricted domain has the rule $y = \frac{x}{2} + \frac{2}{x}$, $x \in \left[\frac{1}{4}, 4\right]$.**

a **Specify the coordinates of the endpoints of the domain.**

b **Obtain the coordinates of any stationary point and determine its nature.**

c **Sketch the graph of the function.**

d **State the global maximum and the global minimum values of the function, if they exist.**

| THINK | WRITE/DRAW |
|---|---|
| a Use the given domain to calculate the coordinates of the endpoints. | a $y = \frac{x}{2} + \frac{2}{x}$<br>For the domain, $\frac{1}{4} \leq x \leq 4$.<br>Substitute each of the end values of the domain in the function's rule.<br>Left endpoint:<br>When $x = \frac{1}{4}$,<br>$y = \frac{x}{2} + \frac{2}{x}$<br>$= \frac{1}{8} + 8$<br>$= 8\frac{1}{8}$<br>Right endpoint:<br>When $x = 4$,<br>$y = 2 + \frac{1}{2}$<br>$= 2\frac{1}{2}$<br>Endpoints are $\left(\frac{1}{4}, \frac{65}{8}\right)$, $\left(4, \frac{5}{2}\right)$. |

**b** **1** Calculate the derivative of the function.

**b**
$$y = \frac{x}{2} + \frac{2}{x}$$
$$= \frac{x}{2} + 2x^{-1}$$
$$\frac{dy}{dx} = \frac{1}{2} - 2x^{-2}$$
$$= \frac{1}{2} - \frac{2}{x^2}$$

**2** Calculate the coordinates of any stationary point.

At a stationary point, $\frac{dy}{dx} = 0$, so:

$$\frac{1}{2} - \frac{2}{x^2} = 0$$
$$\frac{1}{2} = \frac{2}{x^2}$$
$$x^2 = 4$$
$$x = \pm 2$$
$$x = 2,\ x \in \left[\frac{1}{4}, 4\right]$$

When $x = 2$, $y = \frac{2}{2} + \frac{2}{2}$

$= 2$

$(2, 2)$ is a stationary point.

**3** Test the gradient at two selected points either side of the stationary point.

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $\frac{dy}{dx}$ | $\frac{1}{2} - \frac{2}{1} = -\frac{3}{2}$ | 0 | $\frac{1}{2} - \frac{2}{9} = \frac{5}{18}$ |
| Slope | \ | — | / |

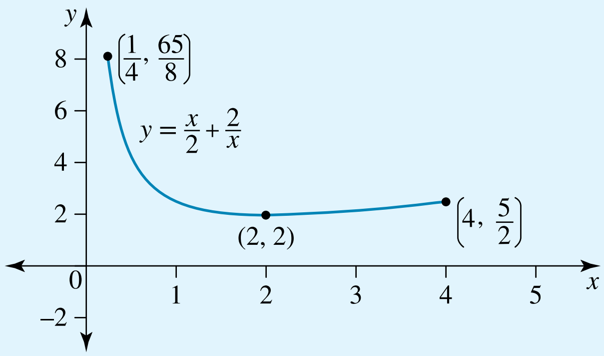
**4** State the nature of the stationary point.

The gradient changes from negative to zero to positive about the stationary point.
The point $(2, 2)$ is a minimum turning point.

**c** Calculate any intercepts with the coordinate axes.

**c** There is no $y$-intercept since $x = 0$ is not in the given domain, nor is $y = \frac{x}{2} + \frac{2}{x}$ defined at $x = 0$.
There is no $x$-intercept since the endpoints and the minimum turning point all have positive $y$-coordinates and there are no other turning points.

2 Sketch the graph using the three known points.




d Examine the graph and the $y$-coordinates to identify the global extrema.

d The function has a global maximum of $8\frac{1}{8}$ at the left endpoint and a global minimum, and local minimum, of 2 at its turning point.

## EXERCISE 13.5 Curve sketching

**PRACTISE**

1 a WE9 Determine the stationary points of $f(x) = x^3 + x^2 - x + 4$ and justify their nature.

b The curve $y = ax^2 + bx + c$ contains the point (0, 5) and has a stationary point at (2, −14). Calculate the values of $a$, $b$ and $c$.

c The point (5, 2) is a stationary point on a curve $y = f(x)$ for which $x > 0$. Given $f'(x) = \frac{1}{5} - \frac{1}{x}$, determine the nature of the stationary point.

2 The curve $y = x^3 + ax^2 + bx - 11$ has stationary points when $x = 2$ and $x = 4$.

a Calculate $a$ and $b$.

b Determine the coordinates of the stationary points and their nature.

3 WE10 Sketch the function $y = 2x^2 - x^3$. Locate any intercepts with the coordinate axes and any stationary points, and justify their nature.

4 Sketch the function $y = x^4 + 2x^3 - 2x - 1$. Locate any intercepts with the coordinate axes and any stationary points, and justify their nature.

5 WE11 A function defined on a restricted domain has the rule

$$y = \frac{1}{16}x^2 + \frac{1}{x}, x \in \left[\frac{1}{4}, 4\right]$$

a Specify the coordinates of the endpoints of the domain.

b Obtain the coordinates of any stationary point and determine its nature.

c Sketch the graph of the function.

d State the global maximum and global minimum values of the function, if they exist.

6 Find, if possible, the global maximum and minimum values of the function $f(x) = 4x^3 - 12x$ over the domain $\{x : x \leq \sqrt{3}\}$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

7 a Use calculus to identify the coordinates of the turning points of the following parabolas:

i $y = x^2 - 8x + 10$

ii $y = -5x^2 + 6x - 12$

b The point (4, −8) is a stationary point of the curve $y = ax^2 + bx$.

i Calculate the values of $a$ and $b$.

ii Sketch the curve.

8 Consider the function defined by $f(x) = x^3 + 3x^2 + 8$.

a Show that $(-2, 12)$ is a stationary point of the function.
b Use a slope diagram to determine the nature of this stationary point.
c Give the coordinates of the other stationary point.
d Use a sign diagram of the gradient function to justify the nature of the second stationary point.

9 Obtain any stationary points of the following curves and justify their nature using the sign of the derivative.

a $y = \frac{1}{3}x^3 + x^2 - 3x - 1$
b $y = -x^3 + 6x^2 - 12x + 8$
c $y = \frac{23}{6}x(x-3)(x+3)$
d $y = 4x^3 + 5x^2 + 7x - 10$

10 Sketch a possible graph of the function $y = f(x)$ for which:

a $f'(-4) = 0$, $f'(6) = 0$, $f'(x) < 0$ for $x < -4$, $f'(x) > 0$ for $-4 < x < 6$ and $f'(x) < 0$ for $x > 6$
b $f'(1) = 0$, $f'(x) > 0$ for $x \in R\backslash\{1\}$ and $f(1) = -3$.

11 The point $(2, -54)$ is a stationary point of the curve $y = x^3 + bx^2 + cx - 26$.

a Find the values of $b$ and $c$.
b Obtain the coordinates of the other stationary point.
c Identify where the curve intersects each of the coordinate axes.
d Sketch the curve and label all key points with their coordinates.

12 Sketch the graphs of each of the following functions and label any axis intercepts and any stationary points with their coordinates. Justify the nature of the stationary points.

a $f: R \to R, f(x) = 2x^3 + 6x^2$
b $g: R \to R, g(x) = -x^3 + 4x^2 + 3x - 12$
c $h: R \to R, h(x) = 9x^3 - 117x + 108$
d $p: [-1, 1] \to R, p(x) = x^3 + 2x$
e $\{(x, y) : y = x^4 - 6x^2 + 8\}$
f $\{(x, y) : y = 2x(x+1)^3\}$

13 a What is the greatest and least number of turning points a cubic function can have?

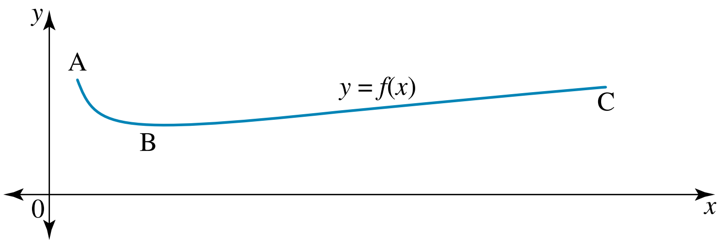
b Show that $y = 3x^3 + 6x^2 + 4x + 6$ has one stationary point and determine its nature.
c Determine the values of $k$ so the graph of $y = 3x^3 + 6x^2 + kx + 6$ will have no stationary points.
d If a cubic function has exactly one stationary point, explain why it is not possible for that stationary point to be a maximum turning point. What type of stationary point must it be?
e State the degree of the gradient function of a cubic function and use this to explain whether it is possible for the graph of a cubic function to have two stationary points: one a stationary point of inflection and the other a maximum turning point.
f Show that the line through the turning points of the cubic function $y = xa^2 - x^3$ must pass through the origin for any real positive constant $a$.

14 Sketch the graphs and state any local and global maximum or minimum values over the given domain for each of the following functions.

a $y = 4x^2 - 2x + 3,\ -1 \leq x \leq 1$
b $y = x^3 + 2x^2,\ -3 \leq x \leq 3$
c $y = 3 - 2x^3,\ x \leq 1$
d $f: [0, \infty) \to R,\ f(x) = x^3 + 6x^2 + 3x - 10$

15 The graph of $f(x) = 2\sqrt{x} + \frac{1}{x},\ 0.25 \leq x \leq 5$ is shown below.

a Determine the coordinates of the endpoints A and C and the stationary point B.
b At which point does the global maximum occur?
c State the global maximum and global minimum values.




16 Consider the function $y = ax^3 + bx^2 + cx + d$. The graph of this function passes through the origin at an angle of 135° with the positive direction of the $x$-axis.

a Obtain the values of $c$ and $d$.
b Calculate the values of $a$ and $b$ given that the point $(2, -2)$ is a stationary point.
c Find the coordinates of the second stationary point.
d Calculate the coordinates of the point where the tangent at $(2, -2)$ meets the curve again.

MASTER

17 a Give the coordinates and state the type of any stationary points on the graph of $f(x) = -0.625x^3 + 7.5x^2 - 20x$, expressing answers to 2 decimal places.
b Sketch $y = f'(x)$ and state the coordinates of its turning point.
c What does the behaviour of $y = f'(x)$ at its turning point tell us about the behaviour of $y = f(x)$ at the point with the same $x$-coordinate?

18 a Use your knowledge of polynomials to draw a sketch of the graph, showing the intercepts with the coordinate axes only, for the function $y = \frac{1}{96}(x + 2)^3(x - 3)(x - 4)^2$. How many stationary points does this function have?
b Use CAS technology to sketch $y = \frac{1}{96}(x + 2)^3(x - 3)(x - 4)^2$ and so determine, if they exist, the global extrema of the function.

# 13.6 Optimisation problems

studyon
Units 1 & 2
AOS 3
Topic 2
Concept 9
**Optimisation problems**
Concept summary
Practice questions

Optimisation problems involve determining the greatest possible, or least possible, value of some quantity, subject to certain conditions. These types of problems provide an important practical application of differential calculus and global extrema.

## Solving optimisation problems

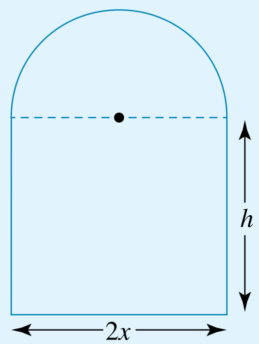
If the mathematical model of the quantity to be optimised is a function of more than one variable, then it is necessary to reduce it to a function of one variable before its derivative can be calculated. The given conditions may enable one variable to be expressed in terms of another. Techniques that may be required include the use of Pythagoras' theorem, trigonometry, similar triangles, or standard mensuration formulas.

To solve optimisation problems:

- Draw a diagram of the situation, where appropriate.
- Identify the quantity to be maximised or minimised and define the variables involved.
- Express the quantity to be optimised as a function of one variable.
- Find the stationary points of the function and determine their nature.
- Consider the domain of the function as there are likely to be restrictions in practical problems and, if appropriate, find the endpoints of the domain.
- The maximum or minimum value of the function will either occur at a stationary point or at an endpoint of the domain.

WORKED EXAMPLE 12

The new owner of an apartment wants to install a window in the shape of a rectangle surmounted by a semicircle in order to allow more light into the apartment.




The owner has 336 cm of wood for a surround to the window and wants to determine the dimensions, as shown in the diagram, that will allow as much light into the apartment as possible.

a Show that the area $A$, in $cm^2$, of the window is $A = 336x - \frac{1}{2}(4 + \pi)x^2$.

b Hence determine, to the nearest cm, the width and the height of the window for which the area is greatest.

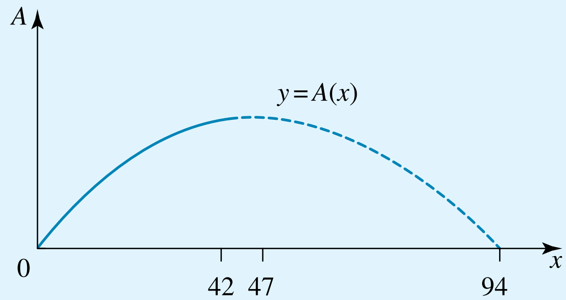
c Structural problems require that the width of the window should not exceed 84 cm. What should the new dimensions of the window be for maximum area?

| THINK | WRITE/DRAW |
| --- | --- |
| a 1 Form an expression for the total area.<br>*Note:* This expression involves more than one variable. | a The total area is the sum of the areas of the rectangle and semicircle.<br>Rectangle: length $2x$ cm, width $h$ cm<br>$\therefore A_{\text{rectangle}} = 2xh$<br>Semicircle: diameter $2x$ cm, radius $x$ cm<br>$\therefore A_{\text{semicircle}} = \frac{1}{2}\pi x^2$<br>The total area of the window is $A = 2xh + \frac{1}{2}\pi x^2$. |

| | |
|---|---|
| **2** Use the given condition to form an expression connecting the two variables. | The perimeter of the window is 336 cm.<br>The circumference of the semicircle is $\frac{1}{2}(2\pi x)$, so the perimeter of the shape is $h + 2x + h + \frac{1}{2}(2\pi x)$.<br>Hence, $2h + 2x + \pi x = 336$. |
| **3** Express one appropriately chosen variable in terms of the other. | The required expression for the area is in terms of $x$, so express $h$ in terms of $x$.<br>$2h = 336 - 2x - \pi x$<br>$h = \frac{1}{2}(336 - 2x - \pi x)$ |
| **4** Write the area as a function of one variable in the required form. | Substitute $h$ into the area function:<br>$A = x(2h) + \frac{1}{2}\pi x^2$<br>$= x(336 - 2x - \pi x) + \frac{1}{2}\pi x^2$<br>$= 336x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$<br>$= 336x - \left(2 + \frac{1}{2}\pi\right)x^2$<br>$\therefore A = 336x - \frac{1}{2}(4 + \pi)x^2$ as required. |
| **b 1** Determine where the stationary point occurs and justify its nature. | **b** At the stationary point, $\frac{dA}{dx} = 0$<br>$\frac{dA}{dx} = 336 - (4 + \pi)x$<br>$336 - (4 + \pi)x = 0$<br>$x = \frac{336}{4 + \pi}$<br>$\approx 47.05$<br>As the area function is a concave down quadratic, the stationary point at $x = \frac{336}{4 + \pi}$ is a maximum turning point. |
| **2** State the values of both variables. | When $x = \frac{336}{4 + \pi}$<br>$2h = 336 - 2 \times \left(\frac{336}{4 + \pi}\right) - \pi \times \left(\frac{336}{4 + \pi}\right)$<br>$h \approx 47.05$ |
| **3** Calculate the required dimensions and state the answer. | Width of the window is $2x \approx 94$ cm.<br>Total height of the window is $h + x \approx 94$ cm. Therefore the area of the window will be greatest if its width is 94 cm and its height is 94 cm. |
| **c 1** Give the restricted domain of the area function. | **c** If the width is not to exceed 84 cm, then:<br>$2x \leq 84$<br>$x \leq 42$<br>With the restriction, the domain of the area function is $[0, 42]$. |

2 Determine where the function is greatest.

As the stationary point occurs when $x = 47$, for the domain $[0, 42]$ there is no stationary point, so the greatest area must occur at an endpoint of the domain.




Maximum occurs when $x = 42$.

3 Calculate the required dimensions and state the answer.

When $x = 42$,

$$h = \frac{1}{2}(336 - 84 - 42\pi)$$
$$\approx 60$$

Width of window is $2x = 84$ cm.

Height of window is $h + x = 102$ cm.

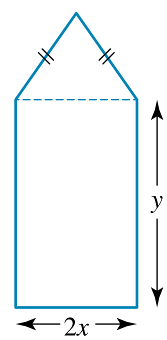
With the restriction, the area of the window will be greatest if its width is 84 cm and its height is 102 cm.

## EXERCISE 13.6 Optimisation problems

### PRACTISE

1 WE12 The owner of an apartment wants to create a stained glass feature in the shape of a rectangle surmounted by an isosceles triangle of height equal to half its base, next to the door opening on to a balcony.

The owner has 150 cm of plastic edging to place around the perimeter of the feature and wants to determine the dimensions of the figure with the greatest area.




a Show that the area $A$, in $\text{cm}^2$, of the stained glass figure is $A = 150x - (2\sqrt{2} + 1)x^2$.

b Hence determine, to 1 decimal place, the width and the height of the figure for which the area is greatest.

c Structural problems require that the width of the figure should not exceed 30 cm. What are the dimensions of the stained glass figure that has maximum area within this requirement?

2 A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 20 cm by 12 cm by cutting equal squares of side length $x$ cm out of the four corners and folding the flaps up.

a Express the volume as a function of $x$.

b Determine the dimensions of the box with greatest volume and give this maximum volume to the nearest whole number.

### CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 A rectangular vegetable garden patch uses part of a back fence as the length of one side. There are 40 metres of fencing available for enclosing the other three sides of the vegetable garden.

a Draw a diagram of the garden and express the area in terms of the width (the width being the length of the sides perpendicular to the back fence).
b Use calculus to obtain the dimensions of the garden for maximum area.
c State the maximum area.
d If the width is to be between 5 and 7 metres, calculate the greatest area of garden that can be enclosed with this restriction.

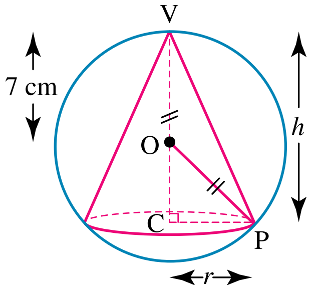
4 The cost in dollars of employing $n$ people per hour in a small distribution centre is modelled by $C = n^3 - 10n^2 - 32n + 400,\ 5 \le n \le 10$. Calculate the number of people who should be employed in order to minimise the cost and justify your answer.

5 A batsman opening the innings in a cricket match strikes the ball so that its height $y$ metres above the ground after it has travelled a horizontal distance $x$ metres is given by $y = 0.0001x^2(625 - x^2)$.

a Calculate, to 2 decimal places, the greatest height the ball reaches and justify the maximum nature.
b Determine how far the ball travels horizontally before it strikes the ground.

6 The total surface area of a closed cylinder is 200 cm$^2$. If the base radius is $r$ cm and the height $h$ cm:
a Express $h$ in terms of $r$.
b Show that the volume, $V$ cm$^3$, is $V = 100r - \pi r^3$.
c Hence show that for maximum volume the height must equal the diameter of the base.
d Calculate, to the nearest integer, the minimum volume if $2 \le r \le 4$.

7 A right circular cone is inscribed in a sphere of radius 7 cm. In the diagram shown, O is the centre of the sphere, C is the centre of the circular base of the cone, and V is the vertex of the cone.




The formula for the volume of a cone of height $h$ cm and base radius $r$ cm is $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$.
a Show that the volume $V$ cm$^3$ of the cone satisfies the relationship $V = \frac{1}{3}\pi(14h^2 - h^3)$.
b Hence, obtain the exact values of $r$ and $h$ for which the volume is greatest, justifying your answer.

8 A shop offers a gift-wrapping service where purchases are wrapped in the shape of closed rectangular prisms.

For one gift-wrapped purchase, the rectangular prism has length $2w$ cm, width $w$ cm, height $h$ cm and a surface area of 300 cm$^2$.
a Draw a diagram of the prism and write down an expression for its surface area.
b Express its volume as a function of $w$ and determine the maximum value of the volume.
c Give the exact dimensions of the prism with greatest volume.
d If $2 \le w \le 6$, determine the range of values for the volume.

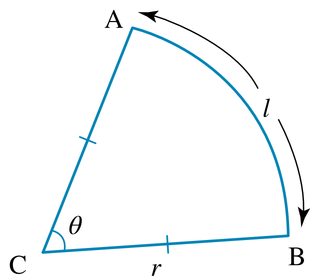
**9** An open rectangular storage bin is to have a volume of 12 $m^3$. The cost of the materials for its sides is \$10 per square metre and the material for the reinforced base costs \$25 per square metre. If the dimensions of the base are $x$ and $y$ metres and the bin has a height of 1.5 metres, find, with justification, the cost, to the nearest dollar, of the cheapest bin that can be formed under these conditions.

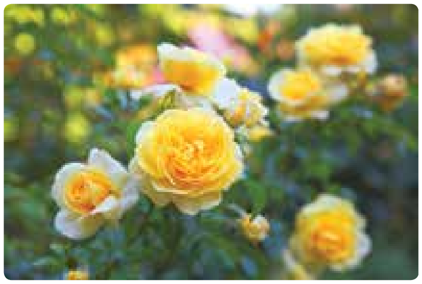
**10** An isosceles triangle with equal base angles of 30° has equal sides of length $x$ cm and a third side of length $y$ cm.

**a** Show the information on a diagram and form an expression in terms of $x$ and $y$ for the area of the triangle.

**b** If the area of the triangle is 15 $cm^2$, for what exact values of $x$ and $y$ will the perimeter of this triangle be smallest?

**11** A section of a rose garden is enclosed by edging to form the shape of a sector ABC of radius $r$ metres and arc length $l$ metres.



The perimeter of this section of the garden is 8 metres.

**a** If $\theta$ is the angle, in radian measure, subtended by the arc at C, express $\theta$ in terms of $r$.

**b** The formula for the area of a sector is $A_{\text{sector}} = \frac{1}{2}r^2\theta$. Express is the area of a sector in terms of $r$.

**c** Calculate the value of $\theta$ when the area is greatest.

**12** The city of Prague has an excellent transport system. Shirley is holidaying in Prague and has spent the day walking in the countryside. Relative to a fixed origin, with measurements in kilometres, Shirley is at the point S(4, 0). She intends to catch a tram back to her hotel in the heart of the city. Looking at her map, Shirley notices the tram route follows a path that could be modelled by the curve $y = \sqrt{x}$.

**a** Draw a diagram showing the tram route and Shirley's position, and calculate how far directly north of Shirley (in the direction of the $y$-axis) the tram route is.

Being a smart mathematician, Shirley realises she can calculate how far it is to the closest point on that tram route. She calculates a function $W$, the square of the distance from the point S(4, 0) to the point T($x$, $y$) on the curve $y = \sqrt{x}$.

**b** Write down an expression for the distance TS and hence show that $W = x^2 - 7x + 16$.

**c** Use calculus to obtain the value of $x$ for which $W$ is minimised.

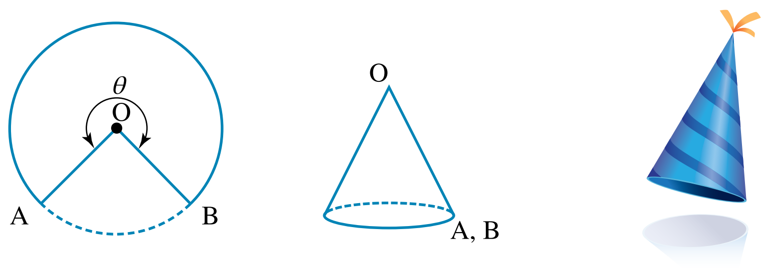
**d** Obtain the coordinates of T, the closest point on the tram route to Shirley.

**e** Calculate, to the nearest minute, the time it takes Shirley to walk to the closest point if she walks directly to it at 5 km/h.

**f** Once she is on the tram travelling back to her hotel, Shirley does a quick calculation on the back of an envelope and finds that the line ST she had walked along is perpendicular to the tangent to the curve at T. Show that Shirley's calculation is correct.

MASTER

**13** From a circle with centre O and radius of 10 cm, sector OAB is cut out, with $\theta$ the reflex angle AOB. A conical party hat is formed by joining the sides OA and OB together, with O as the vertex of the cone.




If the base radius of the cone is $r$ cm:

**a** Express $r$ in terms of $\theta$.

**b** Obtain an expression for the height of the cone in terms of $\theta$.

**c** Hence show that the volume of the cone in terms of $\theta$ is $V = \dfrac{125\theta^2}{3\pi^2}\sqrt{4\pi^2 - \theta^2}$.

(The formula for the volume of a cone is $V_{\text{cone}} = \dfrac{1}{3}\pi r^2 h$.)

**d** Use CAS technology to obtain the value of $\theta$, to the nearest degree, for which the volume of the cone is greatest.

**14** Calculate the area of the largest rectangle with its base on the $x$-axis that can be inscribed in the semicircle $y = \sqrt{4 - x^2}$.

# 13.7 Rates of change and kinematics

Calculus enables us to analyse the behaviour of a changing quantity. Many of the fields of interest in the biological, physical and social sciences involve the study of **rates of change**. In this section we shall consider the application of calculus to rates of change in general contexts and then as applied to the motion of a moving object.

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Units 1 & 2

AOS 3

Topic 2

Concept 10

**Rates of change and kinematics**
Concept summary
Practice questions

## Rates of change

The derivative of a function measures the instantaneous rate of change. For example, the derivative $\dfrac{dV}{dt}$ could be the rate of change of volume with respect to time, with possible units being litres per minute; the rate of change of volume with respect to height would be $\dfrac{dV}{dh}$ with possible units being litres per cm. To calculate these rates, $V$ would need to be expressed as a function of one independent variable: either time or height. Similar methods to those encountered in optimisation problems may be required to connect variables together in order to obtain this function of one variable.

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**Interactivity**
Kinematics
**int-5964**

**To solve rates of change problems:**

- **Draw a diagram of the situation, where appropriate.**
- **Identify the rate of change required and define the variables involved.**
- **Express the quantity which is changing as a function of one independent variable, the variable the rate is measured with respect to.**
- **Calculate the derivative which measures the rate of change.**
- **To obtain the rate at a given value or instant, substitute the given value into the derivative expression.**
- **A negative value for the rate of change means the quantity is decreasing; a positive value for the rate of change means the quantity is increasing.**

**WORKED EXAMPLE 13**

**A container in the shape of an inverted right cone of radius 2 cm and depth 5 cm is being filled with water. When the depth of water is *h* cm, the radius of the water level is *r* cm.**

**a** **Use similar triangles to express *r* in terms of *h*.**

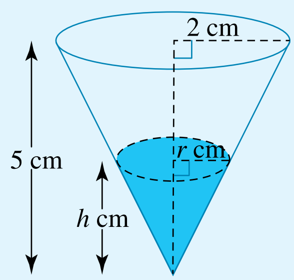
**b** **Express the volume of the water as a function of *h*.**

**c** **At what rate, with respect to the depth of water, is the volume of water changing, when its depth is 1 cm?**

**THINK** | **WRITE/DRAW**

**a 1** Draw a diagram of the situation.

**a**




**2** Obtain the required relationship between the variables.

Using similar triangles,

$$\frac{r}{h} = \frac{2}{5}$$

$$\therefore r = \frac{2h}{5}$$

**b** Express the function in the required form.

**b** The volume of a cone is $V_{cone} = \frac{1}{3}\pi r^2 h$.

Therefore, the volume of water is $V = \frac{1}{3}\pi r^2 h$.

Substitute $r = \frac{2h}{5}$:

$$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$$

$$= \frac{4\pi h^3}{75}$$

**c** **1** Calculate the derivative of the function.

**c** The derivative gives the rate of change at any depth.

$$\frac{dV}{dh} = \frac{4\pi}{75} \times 3h^2$$
$$= \frac{4\pi}{25}h^2$$

**2** Evaluate the derivative at the given value.

When $h = 1$,
$$\frac{dV}{dh} = \frac{4\pi}{25}$$
$$= 0.16\pi$$

**3** Write the answer in context, with the appropriate units.

At the instant the depth is 1 cm, the volume of water is increasing at the rate of $0.16\pi$ cm$^3$ per cm.

## Kinematics

Many quantities change over time so many rates measure that change with respect to time. Motion is one such quantity. The study of the motion of a particle without considering the causes of the motion is called **kinematics**. Analysing motion requires interpretation of the **displacement**, **velocity** and **acceleration**, and this analysis depends on calculus. For the purpose of our course, only motion in a straight line, also called **rectilinear motion**, will be considered.

## Displacement

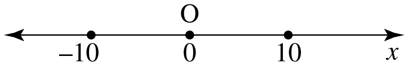
The displacement, $x$, gives the position of a particle by specifying both its distance and direction from a fixed origin.

Common units for displacement and distance are cm, m and km.

The commonly used conventions for motion along a horizontal straight line are:

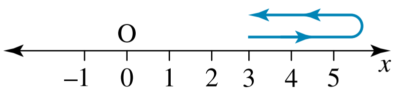
- if $x > 0$, the particle is to the right of the origin;
- if $x < 0$, the particle is to the left of the origin;
- if $x = 0$, the particle is at the origin.

For example, if $x = -10$, this means the particle is 10 units to the left of origin O.




Note that its distance from the origin of 10 units is the same as the distance from the origin of a particle with displacement $x = 10$.

Distance is not concerned with the direction of motion. This can have implications if there is a change of direction in a particle's motion. For example, suppose a particle initially 3 cm to the right of the origin travels 2 cm further to the right, then 2 cm to the left, to return to where it started.




Its change in displacement is zero, but the distance it has travelled is 4 cm.

## Velocity

**Velocity**, $v$, measures the rate of change of displacement, so $v = \frac{dx}{dt}$.

For a particle moving in a horizontal straight line, the sign of the velocity indicates that:

- if $v > 0$, the particle is moving to the right;
- if $v < 0$, the particle is moving to the left;
- if $v = 0$, the particle is stationary (instantaneously at rest).

Common units for velocity and speed include m/s or km/h.

As for distance, speed is not concerned with the direction in which the particle travels, and it is never negative. A velocity of $-10$ m/s means the particle is travelling at 10 m/s to the left. Its speed, however, is 10 m/s regardless of whether the particle is moving to the left or to the right; that is, the speed is 10 m/s for $v = \pm 10$ m/s.

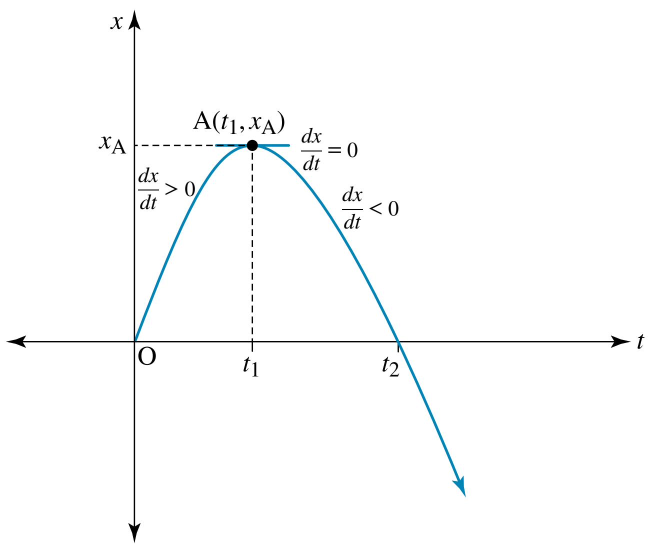
Since $v = \frac{dx}{dt}$, if the position or displacement $x$ of a particle is plotted against time $t$ to create the position–time graph $x = f(t)$, then the gradient of the tangent to its curve at any point represents the velocity of the particle at that point: $v = \frac{dx}{dt} = f'(t)$.

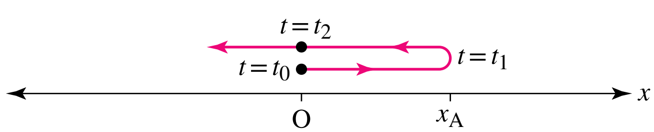
The position–time graph shows the displacement or position of a particle which starts at the origin and initially moves to the right as the gradient of the graph, the velocity, is positive.

At point A the tangent is horizontal and the velocity is zero, indicating the particle changes its direction of motion at that point.

The particle then starts to move to the left, which is indicated by the gradient of the graph, the velocity, having a negative sign. It returns to the origin and continues to move to the left, so its displacement becomes negative.

The same motion is also shown along the horizontal displacement line.




### Average velocity

**Average velocity** is the average rate of change of the displacement. It is measured by the gradient of the chord joining two points on the position–time graph and requires coordinate geometry to evaluate it, not calculus.

$$\textbf{Average velocity} = \frac{\textbf{change in displacement}}{\textbf{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$\textbf{Average speed} = \frac{\textbf{distance travelled}}{\textbf{time taken}}$$

## Acceleration

The **acceleration**, $a$, measures the rate of change of velocity, so $a = \frac{dv}{dt}$.

A common unit for acceleration is m/s$^2$.

Displacement, velocity and acceleration are linked by calculus. Differentiation enables the velocity function to be obtained from the displacement function; it also enables the acceleration function to be obtained from the velocity function.

$$x \rightarrow v \rightarrow a$$
$$x \rightarrow \frac{dx}{dt} \rightarrow \frac{dv}{dt}$$

### Average acceleration

The acceleration acts tangentially to the velocity–time graph, while **average acceleration** measures the gradient of the chord joining two points on the velocity–time graph.

$$\text{Average acceleration} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{v_2 - v_1}{t_2 - t_1}$$

**WORKED EXAMPLE 14**

**A particle moves in a straight line such that its displacement, $x$ metres, from a fixed origin at time $t$ seconds is modelled by $x = t^2 - 4t - 12, t \geq 0$.**

**a** **Identify its initial position.**

**b** **Obtain its velocity and hence state its initial velocity and describe its initial motion.**

**c** **At what time and position is the particle momentarily at rest?**

**d** **Show the particle is at the origin when $t = 6$ and calculate the distance it has travelled to reach the origin.**

**e** **Calculate the average speed over the first 6 seconds.**

**f** **Calculate the average velocity over the first 6 seconds.**

| THINK | WRITE/DRAW |
|---|---|
| **a** Calculate the value of $x$ when $t = 0$. | **a** $x = t^2 - 4t - 12, \ t \geq 0$<br>When $t = 0$, $x = -12$.<br>Initially the particle is 12 metres to the left of the origin. |
| **b 1** Calculate the rate of change required. | **b** $v = \frac{dx}{dt}$<br>$\therefore v = 2t - 4$ |
| **2** Calculate the value of $v$ at the given instant. | When $t = 0$, $v = -4$.<br>The initial velocity is $-4$ m/s. |
| **3** Describe the initial motion. | Since the initial velocity is negative, the particle starts to move to the left with an initial speed of 4 m/s. |

**c** **1** Calculate when the particle is momentarily at rest.

*Note:* This usually represents a change of direction of motion.

**c** The particle is momentarily at rest when its velocity is zero.

When $v = 0$,

$$2t - 4 = 0$$
$$\therefore t = 2$$

The particle is at rest after 2 seconds.

**2** Calculate where the particle is momentarily at rest.

The position of the particle when $t = 2$ is:

$$x = (2)^2 - 4(2) - 12$$
$$= -16$$

Therefore, the particle is momentarily at rest after 2 seconds at the position 16 metres to the left of the origin.

**d** **1** Calculate the position to show the particle is where stated at the given time.

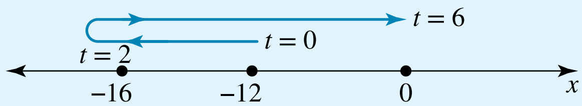
**d** When $t = 6$,

$$x = 36 - 24 - 12$$
$$= 0$$

Particle is at the origin when $t = 6$.

**2** Track the motion on a horizontal displacement line and calculate the required distance.

The motion of the particle for the first 6 seconds is shown.




Distances travelled are 4 metres to the left then 16 metres to the right.
The total distance travelled is the sum of the distances in each direction.
The particle has travelled a total distance of 20 metres.

**e** Calculate the value required.

**e** $$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$
$$= \frac{20}{6}$$
$$= 3\tfrac{1}{3}$$

The average speed is $3\frac{1}{3}$ m/s.

**f** Calculate the average rate of change required.

*Note:* As there is a change of direction, the average velocity will not be the same as the average speed.

**f** Average velocity is the average rate of change of displacement.

For the first 6 seconds,

$(t_1, x_1) = (0, -12)$, $(t_2, x_2) = (6, 0)$

$$\text{Average velocity} = \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{0 - (-12)}{6 - 0}$$
$$= 2$$

The average velocity is 2 m/s.

## EXERCISE 13.7 Rates of change and kinematics

**PRACTISE**

**1** WE13 A container in the shape of an inverted right cone of radius 5 cm and depth 10 cm is being filled with water. When the depth of water is $h$ cm, the radius of the water level is $r$ cm.

- **a** Use similar triangles to express $r$ in terms of $h$.
- **b** Express the volume of the water as a function of $h$.
- **c** At what rate, with respect to the depth of water, is the volume of water changing when its depth is 3 cm?

**2** A cone has a slant height of 10 cm. The diameter of its circular base is increased in such a way that the cone maintains its slant height of 10 cm while its perpendicular height decreases. When the base radius is $r$ cm, the perpendicular height of the cone is $h$ cm.

- **a** Use Pythagoras' theorem to express $r$ in terms of $h$.
- **b** Express the volume of the cone as a function of $h$.
- **c** What is the rate of change of the volume with respect to the perpendicular height when the height is 6 cm?

**3** WE14 A particle moves in a straight line such that its displacement, $x$ metres, from a fixed origin at time $t$ seconds is $x = 3t^2 - 6t,\ t \geq 0$.

- **a** Identify its initial position.
- **b** Obtain its velocity and hence state its initial velocity and describe its initial motion.
- **c** At what time and position is the particle momentarily at rest?
- **d** Show the particle is at the origin when $t = 2$ and calculate the distance it has travelled to reach the origin.
- **e** Calculate the average speed over the first 2 seconds.
- **f** Calculate the average velocity over the first 2 seconds.

**4** The position of a particle after $t$ seconds is given by $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1,\ t \geq 0$.

- **a** Find its initial position and initial velocity.
- **b** Calculate the distance travelled before it changes its direction of motion.
- **c** What is its acceleration at the instant it changes direction?

**CONSOLIDATE**

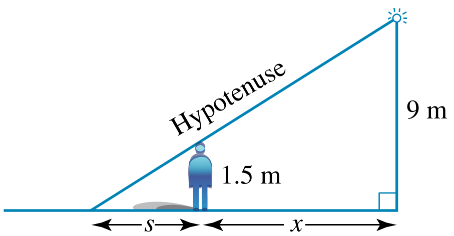
Apply the most appropriate mathematical processes and tools

**5** **a** As the area covered by an ink spill from a printer cartridge grows, it maintains a circular shape. Calculate the rate at which the area of the circle is changing with respect to its radius at the instant its radius is 0.2 metres.

- **b** An ice cube melts in such a way that it maintains its shape as a cube. Calculate the rate at which its surface area is changing with respect to its side length at the instant the side length is 8 mm.
- **c** The number of rabbits on a farm is modelled by $N = \frac{125}{t},\ t > 0$ where $N$ is the number of rabbits present after $t$ months.
  - **i** At what rate is the population of rabbits changing after 5 months?
  - **ii** Calculate the average rate of change of the population over the interval $t \in [1, 5]$.

6 a An equilateral triangle has side length $x$ cm.
 i Express its area in terms of $x$.
 ii At what rate, with respect to $x$, is the area changing when $x = 2$?
 iii At what rate, with respect to $x$, is the area changing when the area is $64\sqrt{3}$ cm$^2$?
b A rectangle has a fixed area of 50 cm$^2$. At what rate with respect to its length is its perimeter changing when the length is 10 cm?

7 a A streetlight pole is 9 metres above the horizontal ground level. When a person of height 1.5 metres is $x$ metres from the foot of the pole, the length of the person's shadow is $s$ metres.




 i Express $s$ in terms of $x$.
 ii Calculate $\frac{ds}{dx}$ and explain what this measures.
b Oil starts to leak from a small hole in the base of a cylindrical oil drum. The oil drum has a height of 0.75 metres and is initially full, with a volume of 0.25 m$^3$ of oil. At what rate is the volume decreasing with respect to the depth of the oil in the oil drum?

c A container in the shape of an inverted right circular cone is being filled with water. The cone has a height of 12 cm and a radius of 6 cm. At what rate is the volume of water changing with respect to the depth of water when:
 i the depth of water reaches half the height of the cone?
 ii the container is one-third full?

8 A tent in the shape of a square-based right pyramid has perpendicular height $h$ metres, side length of its square base $x$ metres and volume $\frac{1}{3}Ah$, where $A$ is the area of its base.
a Express the length of the diagonal of the square base in terms of $x$.
If the slant height of the pyramid is 10 metres:
b show that $x^2 = 200 - 2h^2$ and hence express the volume of air inside the tent in terms of $h$.
c Calculate the rate of change of the volume, with respect to height, when the height is $2\sqrt{3}$ metres.
d For what value of $h$ does the rate of change of the volume equal zero? What is significant about this value for $h$?

9 The position, $x$ cm, relative to a fixed origin of a particle moving in a straight line at time $t$ seconds is $x = 5t - 10$, $t \geq 0$.
a Give its initial position and its position after 3 seconds.
b Calculate the distance travelled in the first 3 seconds.
c Show the particle is moving with a constant velocity.
d Sketch the $x - t$ and $v - t$ graphs and explain their relationship.

10 Relative to a fixed origin, the position of a particle moving in a straight line at time $t$ seconds, $t \geq 0$, is given by $x = 6t - t^2$, where $x$ is the displacement in metres.
a Write down expressions for its velocity and its acceleration at time $t$.
b Sketch the three motion graphs showing displacement, velocity and acceleration versus time and describe their shapes.

**c** Use the graphs to find when the velocity is zero; find the value of $x$ at that time.

**d** Use the graphs to find when the displacement is increasing and what the sign of the velocity is for that time interval.

**11** A particle moves in a straight line so that at time $t$ seconds, its displacement, $x$ metres, from a fixed origin O is given by $x(t) = 3t^2 - 24t - 27,\ t \geq 0$.

**a** Calculate the distance the particle is from O after 2 seconds.

**b** At what speed is the particle travelling after 2 seconds?

**c** What was the average velocity of the particle over the first 2 seconds of motion?

**d** At what time, and with what velocity, does it reach O?

**e** Calculate the distance the particle travels in the first 6 seconds of motion.

**f** What was the average speed of the particle over the first 6 seconds of motion?

**12** The position $x$ cm relative to a fixed origin of a particle moving in a straight line at time $t$ seconds is $x = \frac{1}{3}t^3 - t^2,\ t \geq 0$.

**a** Show the particle starts at the origin from rest.

**b** At what time and at what position is the particle next at rest?

**c** When does the particle return to the origin?

**d** What are the particle's speed and acceleration when it returns to the origin?

**e** Sketch the three motion graphs: $x - t$, $v - t$ and $a - t$, and comment on their behaviour at $t = 2$.

**f** Describe the motion at $t = 1$.

**13** A ball is thrown vertically upwards into the air so that after $t$ seconds, its height $h$ metres above the ground is $h = 40t - 5t^2$.

**a** At what rate is its height changing after 2 seconds?

**b** Calculate its velocity when $t = 3$.

**c** At what time is its velocity $-10$ m/s and in what direction is the ball then travelling?

**d** When is its velocity zero?

**e** What is the greatest height the ball reaches?

**f** At what time and with what speed does the ball strike the ground?

**14** A particle P moving in a straight line has displacement, $x$ metres, from a fixed origin O of $x_P(t) = t^3 - 12t^2 + 45t - 34$ for time $t$ seconds.

**a** At what time(s) is the particle stationary?

**b** Over what time interval is the velocity negative?

**c** When is its acceleration negative?

A second particle Q also travels in a straight line with its position from O at time $t$ seconds given by $x_Q(t) = -12t^2 + 54t - 44$.

**d** At what time are P and Q travelling with the same velocity?

**e** At what times do P and Q have the same displacement from O?

**MASTER**

**15** A population of butterflies in an enclosure at a zoo is modelled by $N = 200 - \dfrac{140}{t + 1},\ t \geq 0$ where $N$ is the number of butterflies $t$ years after observations of the butterflies commenced.

**a** How long did it take for the butterfly population to reach 172 butterflies and at what rate was the population growing at that time?

**b** At what time was the growth rate 10 butterflies per year?

**c** Sketch the population versus time graph and the rate of growth versus time graph and explain what happens to each as $t \to \infty$.

**16** A particle moving in a straight line has position, $x$ m, from a fixed origin O of $x = 0.25t^4 - t^3 + 1.5t^2 - t - 3.75$ for time $t$ seconds.

**a** Calculate the distance it travels in the first 4 seconds of motion.

**b** At what time is the particle at O?

**c** Show that the acceleration is never negative.

**d** Sketch the three motion graphs showing displacement, velocity and acceleration versus time, for $0 \le t \le 4$.

**e** Explain, using the velocity and acceleration graphs, why the particle will not return to O at a later time.

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# 13.8 Review

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**The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.**

**The Review contains:**

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

**A summary of the key points covered in this topic is also available as a digital document.**

## REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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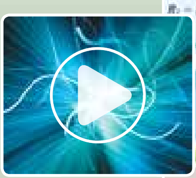
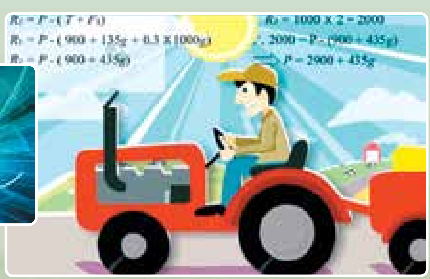
### Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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# 13 Answers

## EXERCISE 13.2

**1 a** 23

**b** 6

**c** Limit does not exist.

**2 a** 12

**b** $\frac{1}{5}$

**3 a i** 1

**ii** Continuous at $x = 1$ since both $f(1)$ and $\lim_{x \to 1} f(x)$ exist, and $f(1) = 1 = \lim_{x \to 1} f(x)$; graph required showing that the two branches join at $x = 1$.

**b i** $R \backslash \{0\}$

**ii** $-1$

**iii** $g(0)$ is not defined.

**iv** Graph required is $y = x - 1$ with an open circle at $(0, -1)$.

**4** $f(0) = 4$, $\lim_{x \to 0} f(x)$ does not exist; continuous for domain $R \setminus \{0\}$

**5 a** Differentiable at $x = 1$

**b** $f'(x) = \begin{cases} 2x, & x \le 1 \\ 2, & x > 1 \end{cases}$; domain is $R$; graph required showing that the two branches join.

**6** $a = 1, b = -4$

**7 a** 17

**b** 6

**c** 3.5

**d** 5

**e** 48

**f** 1

**8 a** Does not exist

**b** 0

**c** $-2$

**d** 4

**9** Only **b** is continuous at $x = 2$; explanations are required.

**10 a** Yes

**b** No

**c** No

**d** Yes

**11** $a = 2$

**12** $a = -3, b = 2$

**13 a** $x = x_3, x_5$

**b** $x = x_1, x_2, x_3, x_4, x_5$

**14 a** Not differentiable at $x = 0$

**b** Graph is required; branches do not join smoothly.

**c** $f'(x) = \begin{cases} -2, & x < 0 \\ 2x, & x > 0 \end{cases}$, domain $R \backslash \{0\}$

**d** 6

**e** Graph required; graph is not continuous at $x = 0$.

**f (a)** Differentiable at $x = 0$

**(b)** Graph required; branches join smoothly.

**(c)** $f'(x) = \begin{cases} -2, & x < 0 \\ 2x - 2, & x \ge 0 \end{cases}$ domain $R$

**(d)** 4

**(e)** Graph required; graph is continuous at $x = 0$.

**15 a** Not continuous at $x = 1$

**b** $f'(x) = \begin{cases} 8x - 5, & x < 1 \\ -3x^2 + 6x, & x > 1 \end{cases}$

**c** $x = \frac{5}{8}, x = 2$

**d** Graph required; point of discontinuity at $(1, 3)$; intercepts with axes at $(0, -5)$, $\left(\frac{5}{8}, 0\right)$, $(2, 0)$

**e** $\left(\frac{5}{8}, \frac{7}{16}\right)$ and $(2, 4)$

**f** Graph required; minimum turning point at $\left(\frac{5}{8}, \frac{7}{16}\right)$, maximum turning point at $(2, 4)$; intercepts with axes $(0, 2)$, $(3, 0)$; endpoints $(1, 1)$ closed, $(1, 2)$ open

**16** $a = 2, b = 2, c = 1, d = 4$

**17 a** 25

**b** $n$

**18 a** Proof is required. Check with your teachers

**b** $\lim_{h \to 0} \frac{4 - (2 - h)^2}{h}$

**c** $\lim_{h \to 0} \frac{2^{2+h} - 4}{h}$

**d** $4, 4 \ln 2$ respectively. Not differentiable at $x = 2$

## EXERCISE 13.3

**1 a** $\frac{dy}{dx} = -\frac{16}{x^5} + \frac{9}{x^4}$; domain $R \backslash \{0\}$

**b i** $f'(x) = \frac{2}{\sqrt{x}} + \frac{\sqrt{2}}{2\sqrt{x}}$; domain $R^+$

**ii** Gradient $= \frac{4 + \sqrt{2}}{2}$

**2** $\frac{4}{x^{\frac{1}{3}}} - \frac{1}{2x^{\frac{3}{2}}}$

**3 a** $\frac{dy}{dx} = -4x^{-2} - 10x^{-3}$

**b** $\frac{dy}{dx} = 2x^{-\frac{1}{2}} - 2x^{-\frac{1}{3}}$

**c** $\dfrac{dy}{dx} = -4x^{-\frac{3}{2}}$

**d** $\dfrac{dy}{dx} = 0.9x^{0.8} - 18.6x^{2.1}$

**4 a** $\dfrac{dy}{dx} = -\dfrac{1}{x^2} - \dfrac{2}{x^3}$

**b** $\dfrac{dy}{dx} = \dfrac{5}{2\sqrt{x}}$

**c** $\dfrac{dy}{dx} = 3 - \dfrac{1}{3x^{\frac{2}{3}}}$

**d** $\dfrac{dy}{dx} = \dfrac{3\sqrt{x}}{2}$

**5 a** $f'(x) = -\dfrac{5}{3x^2} + \dfrac{6}{x^3}$

**b** $f'(x) = \dfrac{2x}{25} - \dfrac{50}{x^3}$

**c** $f'(x) = \dfrac{2}{5x^{\frac{3}{5}}} + \dfrac{\sqrt{5}}{2\sqrt{x}} - \dfrac{1}{2x^{\frac{3}{2}}}$

**d** $f'(x) = \dfrac{6}{x^{\frac{1}{4}}} + \dfrac{7x^{\frac{3}{4}}}{2} - \dfrac{33x^{\frac{7}{4}}}{2}$

**6 a** $f':(0, \infty) \to R, f'(x) = -\dfrac{1}{2\sqrt{x}}$

**b** $-\dfrac{1}{8}$

**c** $-50, -50\,000$

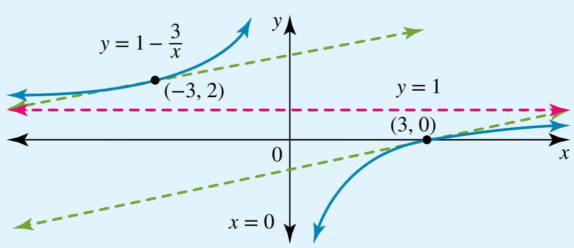
**d** $f'(x) \to -\infty$

**e** Tangent is undefined.

**7 a** $g = \frac{1}{3}$

**b** $(-3, 2)$

**c**




Asymptotes $x = 0, y = 1$

**d** $3 \times 10^{-2}$, $3 \times 10^{-6}$; tangent approaches the horizontal asymptote.

**8 a** **i** $R\backslash\{0\}$

**ii** $f'(x) = 1 + \dfrac{1}{x^2}$; domain $R\backslash\{0\}$

**iii** 2

**iv** $\left(-\dfrac{1}{2}, \dfrac{3}{2}\right), \left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$

**b** **i** 3.5

**ii** $(1, 3)$

**iii** $x = -1,\ x = \dfrac{-1 \pm \sqrt{5}}{2}$

**9 a** 4

**b** Continuous as $f(4) = \lim\limits_{x \to 4} f(x)$

**c** Not differentiable as $f'(4^-) \neq f'(4^+)$

**d** $f'(x) = \begin{cases} -4 + 2x, & x < 4 \\ \dfrac{1}{2\sqrt{x}}, & x > 4 \end{cases}$

**e** $-4$

**f** $x < 2$

**10 a** 0.5 metres

**b** 0.25 metres/year

**c** $6\frac{1}{4}$ years

**d** 0.4 metres/year

**11 a** $h(3x^2 + 3xh + h^2)$

**b** $f'(x) = -\dfrac{3}{x^4}$

**c** $f'(x) = -\dfrac{3}{x^4}$

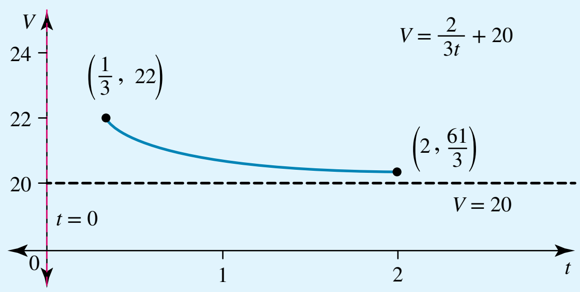
**d** **i** $3x^2 - \dfrac{3}{x^4}$

**ii** $3x^2 - 3 - \dfrac{3}{x^2} + \dfrac{3}{x^4}$

**12 a** $\dfrac{dV}{dt} = -\dfrac{2}{3t^2}$ which is $< 0$.

**b** $\frac{1}{6}$ mL/hour

**c**

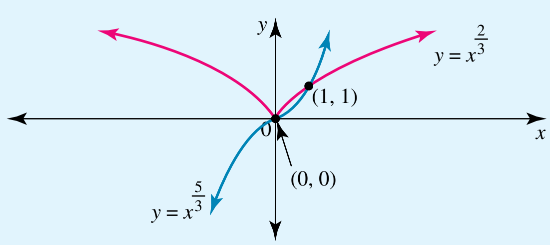



Part of hyperbola with asymptotes at $V = 20,\ t = 0$; endpoints $\left(\dfrac{1}{3}, 22\right), \left(2, \dfrac{61}{3}\right)$

**d** $-1$, average rate of evaporation

**13** 3

**14 a**




$(0, 0)$ and $(1, 1)$

**b** At $(1, 1)$, gradient of $y = x^{\frac{2}{3}}$ is $\dfrac{2}{3}$ and gradient of $y = x^{\frac{5}{3}}$ is $\dfrac{5}{3}$ so $y = x^{\frac{5}{3}}$ is steeper; at $(0, 0)$, gradient of $y = x^{\frac{2}{3}}$ is undefined and gradient of $y = x^{\frac{5}{3}}$ is zero.

## EXERCISE 13.4

1 $y = -4x + 18$

2 $y = -8x + 15$

3 a $\left(\frac{3}{2}, \frac{9}{4}\right)$  b $26.6°$

4 $\left(\frac{1}{2}, 2\right)$

5 a $(-\infty, -4) \cup (2, \infty)$

b i $y = (2a + 4)x + 5 - a$

ii $a < -2$

6 $x \in R\backslash\{0\}$

7 $x = 0.4249$

8 a Polynomial changes sign.

b $x = 1.4026$

9 a $y = -7x + 3$  b $y = -2x + 8$

c $y = 6x - 8$  d $y = 5$

e $y = -24x - 15$  f $2y + x = 49$

10 a $y = x + 4$

b $y = -2x + 6$

c $y = \sqrt{3}x + 3\sqrt{3}$

11 a $y = 4x - 22$

b $y = -6$

c $y = 2x - 13$

12 a $(2, 8)$

b Proof required — check with your teacher

c $y = \frac{9}{2}x + \frac{1}{16}$

13 a $x \in \left(-\infty, \frac{7}{8}\right)$

b $x \in (-\infty, -2) \cup (2, \infty)$

c i Proof required — check with your teacher

ii $y = 6x + 7$

d i Proof required — check with your teacher

ii $y = -3x - 4$

14 a $x = 1.3788$

b i Between $x = 3$ and $x = 4$

ii $x = 3.3186$

c $x = -1.7556$

d $\sqrt[3]{16} = 2.5198$

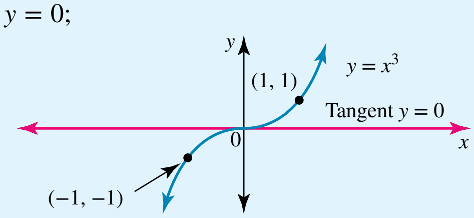
e $x = 4 \pm \sqrt{7}$

$\sqrt{7} \approx 2.6458$

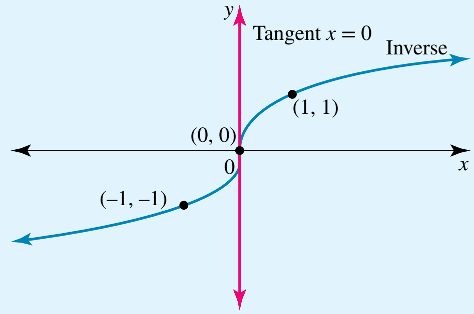
f i Gradient is positive at $x = 0$ and negative at $x = 1$.

ii $x = 0.735$

15 a $y = 0$;




b i

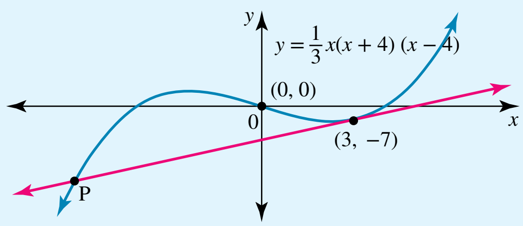



ii Tangent is the vertical line $x = 0$.

c $f^{-1}(x) = x^{\frac{1}{3}}$

d $\frac{1}{3x^{\frac{2}{3}}}$; undefined at $x = 0$; tangent is vertical.

16 a




b $y = \frac{11}{3}x - 18$

c i Proof required — check with your teacher

ii $(-6, -40)$

d Same gradient

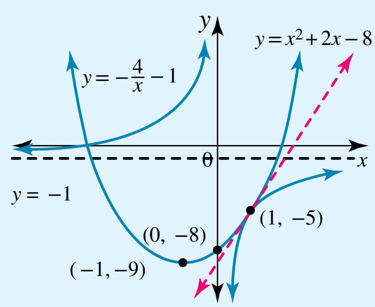
e i Same gradient

ii $\left(-\frac{2a}{3}, \frac{2a^3}{3}\right), \left(\frac{2a}{3}, -\frac{2a^3}{3}\right)$

17 a $y = x - 5,\ y = x + 3$

b $4y - x + 12 = 0,\ 4y - x - 4 = 0$

c $y = 4x - 9$;




18 a $y = -ax + 3 - a^2$

b $a = -3$

c $\left(-\infty, -\frac{a}{2}\right)$

d Proof required; true also when $a = 0$

19 $y = 384x + 1216$

20 $(-2, 6)$; $y = 6$

## EXERCISE 13.5

1 a $(-1, 5)$ is a maximum turning point; $\left(\frac{1}{3}, \frac{103}{27}\right)$ is a minimum turning point.

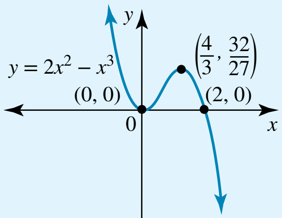
**b** $a = \frac{19}{4},\ b = -19,\ c = 5$

**c** Minimum turning point

**2 a** $a = -9,\ b = 24$

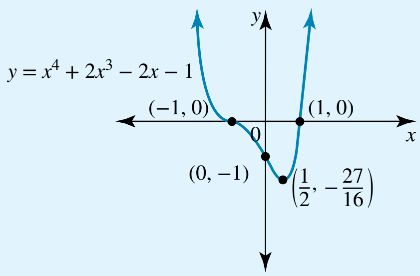
**b** $(2, 9)$ is a maximum turning point; $(4, 5)$ is a minimum turning point.

**3**




$x$-intercepts when $x = 2, x = 0$; $(0, 0)$ is a minimum turning point; $\left(\frac{4}{3}, \frac{32}{27}\right)$ is a maximum turning point.

**4**

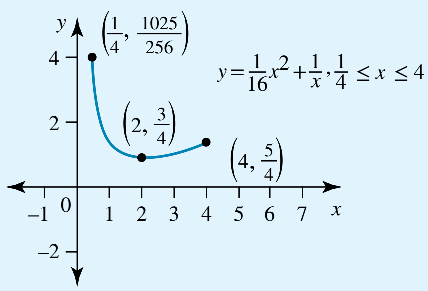



$(0, -1)$, $(\pm 1, 0)$ intercepts with axes; $(-1, 0)$ is a stationary point of inflection; $\left(\frac{1}{2}, -\frac{27}{16}\right)$ is a minimum turning point.

**5 a** $\left(\frac{1}{4}, \frac{1025}{256}\right)$, $\left(4, \frac{5}{4}\right)$

**b** $\left(2, \frac{3}{4}\right)$ is a minimum turning point.

**c**



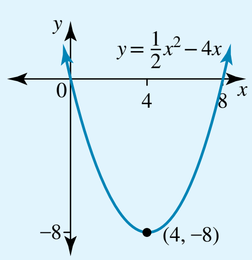

**d** Global maximum $\frac{1025}{256}$; global minimum $\frac{3}{4}$

**6** No global minimum; global maximum 8

**7 a** **i** $(4, -6)$ **ii** $\left(\frac{3}{5}, -\frac{51}{5}\right)$

**b** **i** $a = \frac{1}{2}, b = -4$

**ii**




**8 a** Proof required — check with your teacher

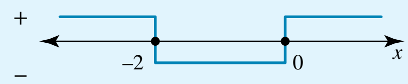
**b**

| $x$ | −3 | −2 | −1 |
|---|---|---|---|
| $f'(x)$ | 9 | 0 | −3 |
| **Slope** | / | — | \ |

Maximum turning point

**c** $(0, 8)$

**d**




Minimum turning point

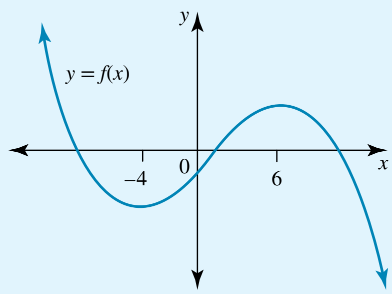
**9 a** $(-3, 8)$ is a maximum turning point; $\left(1, -\frac{8}{3}\right)$ is a minimum turning point.

**b** $(2, 0)$ is a stationary point of inflection.

**c** $(-\sqrt{3}, 23\sqrt{3})$ is a maximum turning point; $(\sqrt{3}, -23\sqrt{3})$ is a minimum turning point.

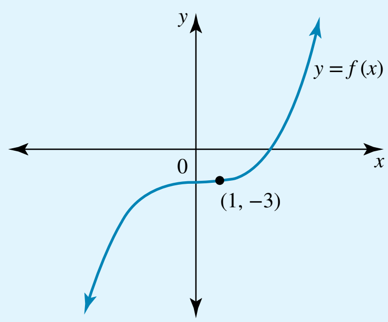
**d** There are no stationary points.

**10 a**




Negative cubic with minimum turning point where $x = -4$ and maximum turning point where $x = 6$

**b**



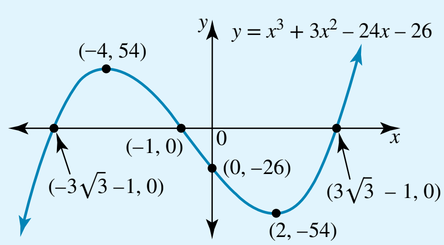

Positive cubic with stationary point of inflection at $(1, -3)$

**11 a** $b = 3, c = -24$

**b** $(-4, 54)$

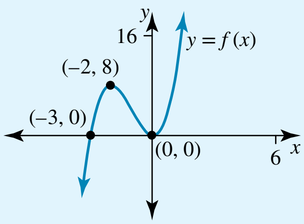
**c** $(0, -26)$, $(-1 \pm 3\sqrt{3}, 0)$, $(-1, 0)$

**d**



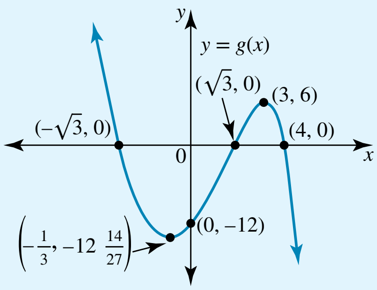

Key points are $(-1 - 3\sqrt{3}, 0)$, $(-4, 54)$, $(-1, 0)$, $(0, -26)$ $(2, -54)$, $(-1 + 3\sqrt{3}, 0)$.

**12 a**




Minimum turning point $(0, 0)$; maximum turning point $(-2, 8)$; intercept $(-3, 0)$

**b**




Minimum turning point $\left(-\frac{1}{3}, -12\,\frac{14}{27}\right)$; maximum turning point $(3, 6)$; intercepts $(\pm\sqrt{3}, 0)$, $(4, 0)$, $(0, -12)$

**c** Minimum turning point $\left(\frac{\sqrt{39}}{3}, 108 - 26\sqrt{39}\right) \approx (2.1, -54.4)$; maximum turning point $\left(-\frac{\sqrt{39}}{3}, 108 + 26\sqrt{39}\right) \approx (-2.1, 270.4)$; intercepts $(1, 0)$, $(3, 0)$, $(-4, 0)$, $(0, 108)$

**d** Endpoints $(-1, -3)$, $(1, 3)$; intercept $(0, 0)$; no stationary points

**e** Minimum turning points $(\pm\sqrt{3}, -1)$; maximum turning point and $y$-intercept $(0, 8)$; $x$-intercepts $(\pm\sqrt{2}, 0)$, $(\pm 2, 0)$

**f** Minimum turning point $\left(-\frac{1}{4}, -\frac{27}{128}\right)$; stationary point of inflection and $x$-intercept $(-1, 0)$; intercept $(0, 0)$

**13 a** 2, 0

**b** Proof required; stationary point of inflection

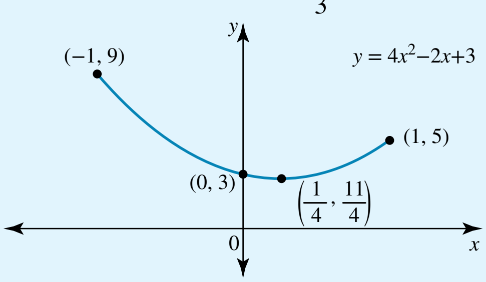
**c** $k > 4$

**d** Explanation required; stationary point of inflection

**e** Degree 2; not possible; explanation required

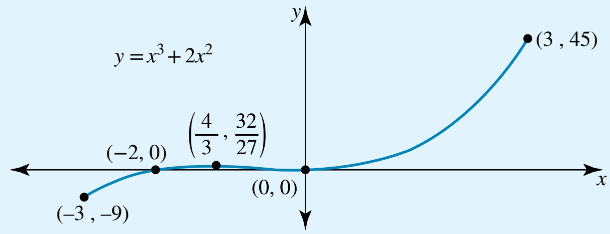
**f** Origin lies on the line $y = \frac{2a^2x}{3}$.

**14 a**



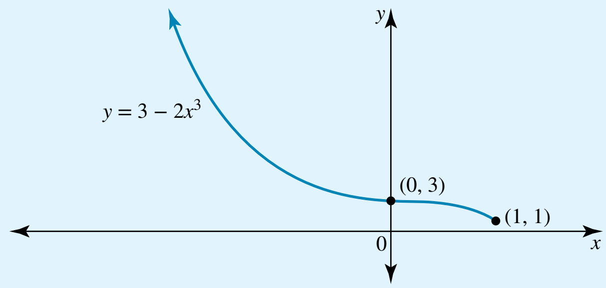

Local and global minimum $\frac{11}{4}$; no local maximum; global maximum 9

**b**



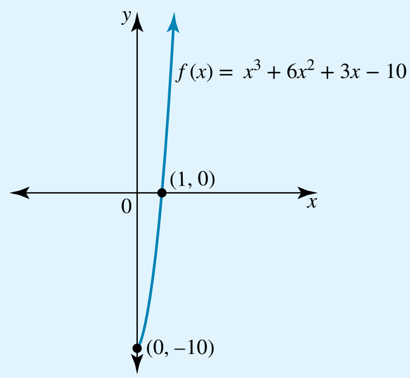

Local maximum $\frac{32}{27}$; local minimum 0; global maximum 45; global minimum −9

**c**




No local or global maximum; no local minimum; global minimum 1

**d**




No local or global maximum; no local minimum; global minimum −10

**15 a** A $(0.25, 5)$, B $(1, 3)$, C $(5, 2\sqrt{5} + 0.2)$

**b** A

**c** 5, 3

**16 a** $c = -1,\ d = 0$

**b** $a = \frac{1}{4},\ b = -\frac{1}{2}$

**c** $\left(-\frac{2}{3}, \frac{10}{27}\right)$

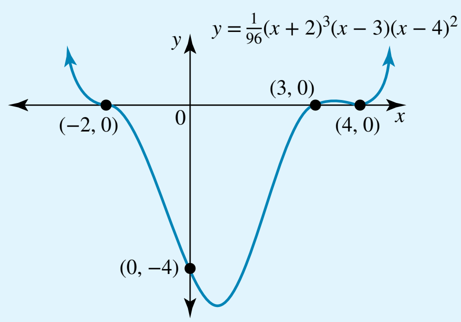
**d** $(-2, -2)$

**17 a** Maximum turning point $(6.31, 15.40)$; minimum turning point $(1.69, -15.40)$

**b** $(4, 10)$

**c** Gradient of the curve is greatest at the point $(4, 0)$.

**18 a**




4 stationary points

**b** Global minimum −5.15; no global maximum

## EXERCISE 13.6

1 a Proof required — check with your teacher

b Width 39.2 cm and height 47.3 cm

c Width 30 cm and height 53.8 cm

2 a $V = 240x - 64x^2 + 4x^3$

b Length 15.14 cm; width 7.14 cm; height 2.43 cm; volume 32 $\text{cm}^3$

3 a $A = 40x - 2x^2$

b Width 10 metres; length 20 metres

c 200 $\text{m}^2$

d 182 $\text{m}^2$

4 8 people

5 a 9.77 metres

b 25 metres

6 a $h = \dfrac{100 - \pi r^2}{\pi r}$

b Proof required — check with your teacher

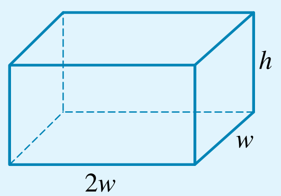
c Proof required — check with your teacher

d 175 $\text{cm}^3$

7 a Proof required — check with your teacher

b $r = \dfrac{14\sqrt{2}}{3},\ h = \dfrac{28}{3}$

8 a




Surface area: $4w^2 + 6wh$

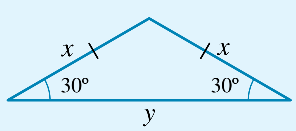
b $V = 100w - \dfrac{4}{3}w^3;\ \dfrac{1000}{3}$ $\text{cm}^3$

c 10 cm by 5 cm by $\dfrac{20}{3}$ cm

d $\left[\dfrac{568}{3}, \dfrac{1000}{3}\right]$

9 \$370

10 a




$A = \frac{1}{4}xy$

b $x = \sqrt{30},\ y = 2\sqrt{30}$

11 a $\theta = \dfrac{8 - 2r}{r}$

b $A = 4r - r^2$

c 2 radians

12 a 2 km due north

b $\text{TS} = \sqrt{(x-4)^2 + (\sqrt{x})^2}$; proof required — check with your teacher

c $x = 3.5$

d $\text{T}\left(\dfrac{7}{2}, \dfrac{\sqrt{14}}{2}\right)$

e 23 minutes

f Proof required — check with your teacher

13 a $r = \dfrac{5\theta}{\pi}$

b $h = \sqrt{100 - \dfrac{25\theta^2}{\pi^2}}$

c Proof required — check with your teacher

d 294°

14 4 $\text{units}^2$

## EXERCISE 13.7

1 a $r = \dfrac{1}{2}h$

b $V = \dfrac{1}{12}\pi h^3$

c $\dfrac{9\pi}{4}$ $\text{cm}^3/\text{cm}$

2 a $r = \sqrt{100 - h^2}$

b $V = \dfrac{1}{3}\pi(100h - h^3)$

c Volume is decreasing at the rate of $\dfrac{8\pi}{3}$ $\text{cm}^3/\text{cm}$.

3 a At the origin

b $v = 6t - 6$; initial velocity −6 m/s; move to the left

c After 1 second and 3 metres to the left of the origin

d 6 metres

e 3 m/s

f 0 m/s

4 a 1 metre to right of origin; 8 m/s

b $16\frac{2}{3}$ metres

c −6 $\text{m/s}^2$

5 a $0.4\pi$ $\text{m}^2/\text{m}$

b 96 $\text{mm}^2/\text{mm}$

c i Decreasing at 5 rabbits/month

ii −25 rabbits/month

6 a i $A = \dfrac{\sqrt{3}}{4}x^2$

ii $\sqrt{3}$ $\text{cm}^2/\text{cm}$

iii $8\sqrt{3}$ $\text{cm}^2/\text{cm}$

b 1 cm/cm

7 a i $s = \dfrac{x}{5}$

ii $\dfrac{ds}{dx} = \dfrac{1}{5}$, rate of change of shadow length with respect to distance of person from light pole is a constant value of $\dfrac{1}{5}$ m/m.

b $\dfrac{1}{3}$ $\text{m}^3/\text{m}$

c i $9\pi$ $\text{cm}^3/\text{cm}$

ii $4 \times 3^{\frac{4}{3}}\pi$ $\text{cm}^3/\text{cm}$ or $12 \times 3^{\frac{1}{3}}\pi$ $\text{cm}^3/\text{cm}$

8 a $\sqrt{2}x$ metres

b $V = \frac{1}{3}(200h - 2h^3)$

c $42\frac{2}{3}$ m$^3$/m

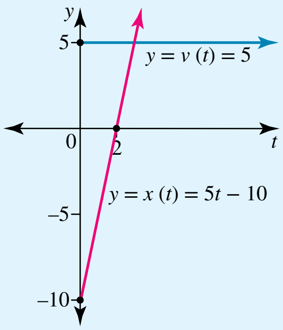
d $h = \frac{10\sqrt{3}}{3}$; gives the height for maximum volume

9 a 10 cm to left of origin, 5 cm to right of origin

b 15 cm

c $v = 5$ cm/s

d




Velocity graph is the gradient graph of the displacement.

10 a $v = 6 - 2t,\ a = -2$

b Displacement graph is quadratic with maximum turning point when $t = 3$; velocity graph is linear with $t$-intercept at $t = 3$; acceleration graph is horizontal with constant value of $-2$.

c $v = 0 \Rightarrow t = 3,\ t = 3 \Rightarrow x = 9$

d Displacement increases for $0 < t < 3$ and $v > 0$.

11 a 63 metres

b 12 m/s

c $-18$ m/s

d 9 seconds; 30 m/s

e 60 metres

f 10 m/s

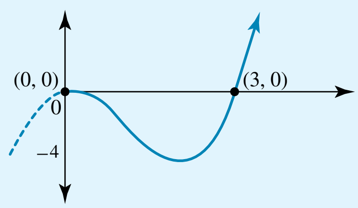
12 a $t = 0, x = 0, v = 0$

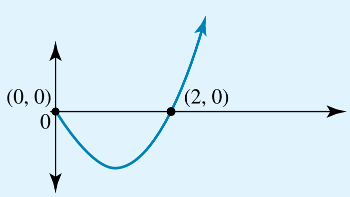
b 2 seconds; $1\frac{1}{3}$ cm left of origin

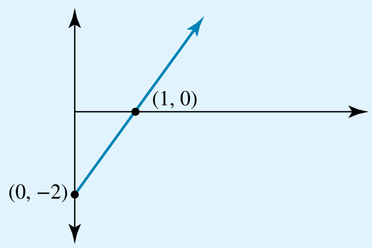
c 3 seconds

d 3 cm/s, 4 cm/s$^2$

e




At $t = 2$, displacement is most negative, velocity is zero, acceleration is 2 cm/s$^2$.

f Particle is moving to the left with greatest velocity in that direction; acceleration is momentarily zero as it changes from negative to positive.

13 a 20 m/s

b 10 m/s

c 5 seconds, travelling down towards the ground

d 4 seconds

e 80 metres

f 8 seconds, 40 m/s

14 a 3 seconds and 5 seconds

b $t \in (3, 5)$

c $t \in [0, 4)$

d $\sqrt{3}$ seconds

e 2 seconds and $(\sqrt{6} - 1)$ seconds

15 a 4 years, 5.6 butterflies per year

b 2.74 years

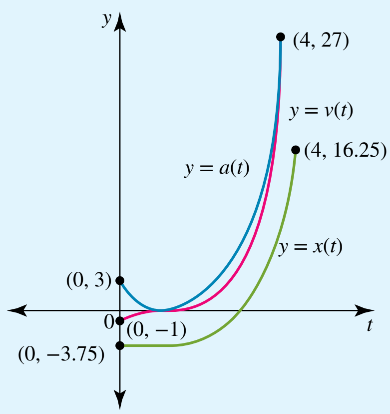
c As $t \to \infty, N \to 200$ and $\frac{dN}{dt} \to 0$.

16 a 20.5 metres

b 3 seconds

c $a = 3(t - 1)^2 \geq 0$

d




e Velocity and acceleration are positive and increasing for all $t$ values greater than 1. This causes the particle to forever move to the right for $t > 1$ passing through O at $t = 3$ and never returning.

# 14

# Anti-differentiation and introduction to integral calculus

# 14.1 Kick off with CAS

## Integral calculus

1 Using CAS technology, find the derivative of $y = x^2$ by using $\frac{d(x^2)}{dx}$.

2 Using CAS technology, find the template for the integral and calculate $\int 2x\,dx$.

3 Comparing the results in questions 1 and 2, what do you notice?

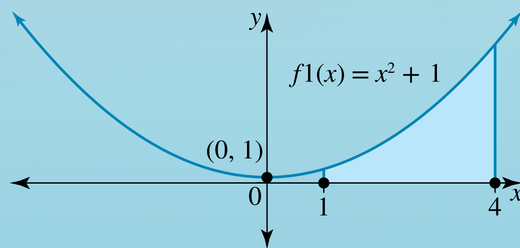
4 Using CAS technology, find the derivative of $y = 5x^3 + 2x$ by using $\frac{d(5x^3 + 2x)}{dx}$.

5 Using CAS technology, calculate $\int (15x^2 + 2)\,dx$.

6 Comparing the results in questions 4 and 5, what do you notice?

7 Using CAS technology, sketch the graph of $f(x) = x^2 + 1$.

8 Draw vertical lines from the $x$-axis to the graph at $x = 1$ and at $x = 4$. Estimate the area enclosed by the $x$-axis, the vertical lines and the graph.




9 Using CAS technology, calculate $\int_1^4 (x^2 + 1)\,dx$.

10 What do you notice about the answers to questions 8 and 9?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

# 14.2 Anti-derivatives

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Units 1 & 2
AOS 3
Topic 3
Concept 1

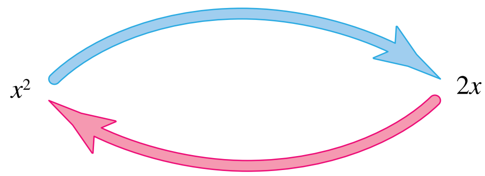
**Anti-derivatives**
Concept summary
Practice questions

Calculus is made up of two parts: differential calculus and integral calculus. Differential calculus arose from the need to measure an instantaneous rate of change, or gradient; integral calculus arose from the need to measure the area enclosed between curves. The fundamental theorem of calculus which established the connection between the two branches of calculus is one of the most important theorems in mathematics. Before we can gain some understanding of this theorem, we must first consider the reverse operation to differentiation.

## Anti-derivatives of polynomial functions

**Anti-differentiation** is the reverse process (or 'undoing' process) to differentiation.

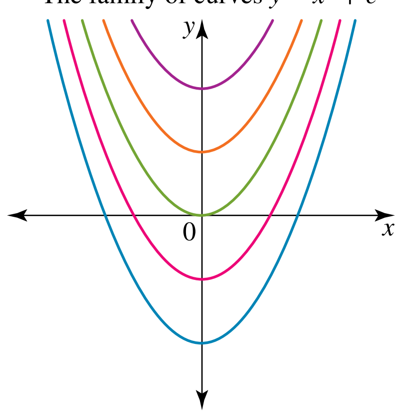
The derivative of $x^2$ is $2x$; hence, reversing the process, an anti-derivative of $2x$ is $x^2$.




The expression 'an' anti-derivative is of significance in the above example. For any constant $c$, $\frac{d}{dx}(x^2 + c) = 2x$, so 'the' anti-derivative of $2x$ is $x^2 + c$. The constant $c$ is referred to as an arbitrary constant.

If $\frac{dy}{dx} = 2x$, then $y = x^2 + c$. In this family of parabolas $y = x^2 + c$, each member has identical shape and differs only by the amount of vertical translation each has undergone. Each curve has the same gradient function of $\frac{dy}{dx} = 2x$. Some members of the family of anti-derivatives are shown in the diagram.

The family of curves $y = x^2 + c$




Given $\frac{dy}{dx}$ or $f'(x)$, the process of obtaining $y$ or $f(x)$ is called anti-differentiation. The process is also referred to as finding the **primitive function**, given the gradient function.

## The basic rule for anti-differentiation of polynomials

By studying the patterns in the following examples, the basic rule of anti-differentiation can be deduced. The values in the table can be verified by differentiating the polynomial $y$.

The rule the table illustrates is:

**If $\frac{dy}{dx} = x^n, n \in N$**

**then $y = \frac{1}{n+1}x^{n+1} + c$, where $c$ is an arbitrary constant.**

| $\frac{dy}{dx}$ | $y$ |
|---|---|
| 5 | $5x + c$ |
| $2x$ | $x^2 + c$ |
| $3x^2$ | $x^3 + c$ |
| $x^2$ | $\frac{1}{3}x^3 + c$ |
| $x^3$ | $\frac{1}{4}x^4 + c$ |

This rule can be verified by differentiating $y$ with respect to $x$.

The linearity properties allow anti-derivatives of sums and differences of polynomial terms to be calculated. For example, if $f'(x) = a_1 + a_2x + a_3x^2 + \ldots + a_nx^{n-1}, n \in N$, then

$$f(x) = a_1x + a_2\frac{x^2}{2} + a_3\frac{x^3}{3} + \ldots + a_n\frac{x^n}{n} + c$$

$$= a_1x + \frac{1}{2}a_2x^2 + \frac{1}{3}a_3x^3 + \ldots + \frac{1}{n}a_nx^n + c$$

Note that the degree of the primitive function is one higher than the degree of the gradient function.

The rule for anti-differentiation of a polynomial shows that differentiation and anti-differentiation are inverse operations. For $n \in N$, the derivative of $x^n$ is $n \times x^{n-1}$; the anti-derivative of $x^n$ is $x^{n+1} \div (n+1) = \dfrac{x^{n+1}}{n+1} + c$.

**WORKED EXAMPLE 1**

**a** If $\dfrac{dy}{dx} = 6x^8$, find $y$.

**b** Find an anti-derivative of $3x^2 - 5x + 2$.

**c** Given $f'(x) = x(x-1)(x+1)$, find the rule for the primitive function.

| THINK | WRITE |
|---|---|
| **a 1** State the reverse process required to find $y$. | **a** $\dfrac{dy}{dx} = 6x^8$<br>The derivative of $y$ is $6x^8$. Hence, the anti-derivative of $6x^8$ is $y$. |
| **2** Use the rule to anti-differentiate the polynomial term.<br>*Note:* It's a good idea to mentally differentiate the answer to check its derivative is $6x^8$. | $y = 6 \times \left(\frac{1}{8+1}x^{8+1}\right) + c$<br>$= 6 \times \frac{1}{9}x^9 + c$<br>$= \frac{2}{3}x^9 + c$ |
| **b 1** Apply the anti-differentiation rule to each term of the polynomial. | **b** $3x^2 - 5x + 2 = 3x^2 - 5x + 2x^0$<br>The anti-derivative is:<br>$3 \times \frac{1}{3}x^3 - 5 \times \frac{1}{2}x^2 + 2 \times \frac{1}{1}x^1 + c$<br>$= x^3 - \frac{5}{2}x^2 + 2x + c$ |
| **2** Choose any value for the constant $c$ to obtain an anti-derivative.<br>*Note:* The conventional choice is $c = 0$. | When $c = 0$, an anti-derivative is $x^3 - \frac{5}{2}x^2 + 2x$. |
| **c 1** Express the rule for the derivative function in expanded polynomial form. | **c** $f'(x) = x(x-1)(x+1)$<br>$= x(x^2 - 1)$<br>$= x^3 - x$ |
| **2** Calculate the anti-derivative and state the answer. | $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + c$<br>The rule for the primitive function is $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + c$. |

## The indefinite integral

Anti-differentiation is not only about undoing the effect of differentiation. It is an operation that can be applied to functions. There are different forms of notation for the anti-derivative just as there are for the derivative. The most common form of notation uses symbolism due to Leibniz.

It is customary to write 'the anti-derivative of $f(x)$' as $\int f(x)\,dx$. This is called the **indefinite integral**. Using this symbol, we could write 'the anti-derivative of $2x$ with respect to $x$ equals $x^2 + c$' as $\int 2x\,dx = x^2 + c$.

For any function:

$$\int f(x)\,dx = F(x) + c \text{ where } F'(x) = f(x)$$

$F(x)$ is an anti-derivative or primitive of $f(x)$. It can also be said that $F(x)$ is the indefinite integral, or just integral, of $f(x)$. The arbitrary constant $c$ is called the constant of integration. Here we take integration to be the process of using the integral to obtain an anti-derivative. A little more will be said about integration later.

The rule for anti-differentiation of a polynomial term could be written in this notation as:

$$\text{For } n \in N, \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

The linearity properties of anti-differentiation, or integration, could be expressed as:

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

### Anti-derivatives of power functions

The rule for obtaining the anti-derivative of a polynomial function also holds for power functions where the index may be rational. This is illustrated by applying the rule to $x^{\frac{1}{2}}$.

$$\begin{aligned}\int x^{\frac{1}{2}}\,dx &= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c\\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c\\ &= \frac{2}{3} \times x^{\frac{3}{2}} + c\\ &= \frac{2}{3}x^{\frac{3}{2}} + c\end{aligned}$$

To verify the result, differentiate:

$$\begin{aligned}\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}} + c\right) &= \frac{2}{3} \times \frac{3}{2}x^{\frac{1}{2}} + 0\\ &= x^{\frac{1}{2}}\end{aligned}$$

The rule for anti-differentiating $x^n$ is:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \in R\backslash\{-1\}$$

This rule must exclude $n = -1$ to prevent the denominator $n + 1$ becoming zero. This exception to the rule will be studied in Mathematical Methods Units 3 and 4.

**WORKED EXAMPLE 2**

**a** Use the linearity properties to calculate $\int (2x^3 + 4x - 3)\,dx$.

**b** Calculate $\int \frac{7x^5 - 8x^7}{5x}\,dx$.

**c** Given $f(x) = \frac{3}{2x^3}$, form $F(x)$, where $F'(x) = f(x)$.

**d** Find $\int \left(x - \frac{1}{x}\right)^2 dx$.

| THINK | WRITE |
|---|---|
| **a 1** Express the integral as the sum or difference of integrals of each term. | **a** $\int (2x^3 + 4x - 3)\,dx$<br>$= \int 2x^3\,dx + \int 4x\,dx - \int 3\,dx$<br>$= 2\int x^3\,dx + 4\int x\,dx - 3\int dx$ |
| **2** Anti-differentiate term by term.<br>*Note:* In practice, the calculation usually omits the steps illustrating the linearity properties. | $= 2 \times \frac{x^4}{4} + 4 \times \frac{x^2}{2} - 3x + c$<br>$= \frac{x^4}{2} + 2x^2 - 3x + c$ |
| **b 1** Express the term to be anti-differentiated in polynomial form. | **b** $\int \frac{7x^5 - 8x^7}{5x}\,dx$<br>$= \int \left(\frac{7x^5}{5x} - \frac{8x^7}{5x}\right) dx$<br>$= \int \left(\frac{7}{5}x^4 - \frac{8}{5}x^6\right) dx$ |
| **2** Calculate the anti-derivative. | $= \frac{7}{5} \times \frac{x^5}{5} - \frac{8}{5} \times \frac{x^7}{7} + c$<br>$= \frac{7}{25}x^5 - \frac{8}{35}x^7 + c$ |
| **c 1** Express the given function with a power of $x$ in its numerator. | **c** $f(x) = \frac{3}{2x^3}$<br>$= \frac{3x^{-3}}{2}$ |
| **2** Use the rule to obtain the anti-derivative of the power function. | $F(x)$ is the anti-derivative function.<br>$F(x) = \frac{3}{2} \times \frac{x^{-3+1}}{(-3+1)} + c$<br>$= \frac{3}{2} \times \frac{x^{-2}}{-2} + c$<br>$= \frac{3x^{-2}}{-4} + c$ |

3 Rewrite the anti-derivative function with positive indices.

$\therefore F(x) = -\frac{3}{4x^2} + c$

d 1 Prepare the expression to be anti-differentiated by expanding the perfect square and simplifying.

d $\int \left(x - \frac{1}{x}\right)^2 dx$

$= \int \left(x^2 - 2x \times \frac{1}{x} + \frac{1}{x^2}\right) dx$

$= \int (x^2 - 2 + x^{-2})\, dx$

2 Calculate the anti-derivative.

$= \frac{1}{3}x^3 - 2x + \frac{x^{-2+1}}{-1} + c$

$= \frac{1}{3}x^3 - 2x - x^{-1} + c$

$= \frac{1}{3}x^3 - 2x - \frac{1}{x} + c$

## EXERCISE 14.2 Anti-derivatives

**PRACTISE**

Work without CAS

1 WE1 a If $\frac{dy}{dx} = 12x^5$, find $y$.

b Find an anti-derivative of $4x^2 + 2x - 5$.

c Given $f'(x) = (x - 2)(3x + 8)$, obtain the rule for the primitive function.

2 Given the gradient function $\frac{dy}{dx} = \frac{2x^3 - 3x^2}{x}, x \neq 0$, obtain the primitive function.

3 WE2 a Use the linearity properties to calculate $\int (-7x^4 + 3x^2 - 6x)\, dx$.

b Calculate $\int \frac{5x^8 + 3x^3}{4x^2}\, dx$.

c Given $f(x) = \frac{2}{x^4}$, form $F(x)$, where $F'(x) = f(x)$.

d Obtain $\int \left(x + \frac{1}{x}\right)^2 dx$.

4 a Given $f(x) = \frac{1}{2\sqrt{x}}$, obtain $F(x)$, where $F'(x) = f(x)$.

b Calculate each expression to show that $\int 2(t^2 + 2)\, dt$ is the same as $2\int (t^2 + 2)\, dt$.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

5 Given $\frac{dy}{dx}$, obtain $y$ in terms of $x$ for each of the following.

a $\frac{dy}{dx} = 5x^9$

b $\frac{dy}{dx} = -3 + 4x^7$

c $\frac{dy}{dx} = 2(x^2 - 6x + 7)$

d $\frac{dy}{dx} = (8 - x)(2x + 5)$

6 For each $f'(x)$ expression, obtain an expression for $f(x)$.

a $f'(x) = \frac{1}{2}x^5 + \frac{7}{3}x^6$

b $f'(x) = \frac{4x^6}{3} + 5$

c $f'(x) = \frac{4x^4 - 6x^8}{x^2}, x \neq 0$

d $f'(x) = (3 - 2x^2)^2$

7 Calculate the following indefinite integrals.

a $\int \frac{3x^8}{5} dx$

b $\int 2\, dx$

c $4\int (20x - 5x^7)\, dx$

d $\int \frac{1}{100}(9 + 6x^2 - 5.5x^{10})\, dx$

8 a Calculate the primitive function of $2ax + b$, for $a, b \in R$.

b Calculate the anti-derivative of $0.05x^{99}$.

c Calculate an anti-derivative of $(2x + 1)^3$.

d Calculate the primitive of $7 - x(5x^3 - 4x - 8)$.

9 Find $F(x)$, where $F'(x) = f(x)$ for the following:

a $f(x) = \frac{3x^2 - 2}{4}$

b $f(x) = \frac{3x}{4} + \frac{2(1 - x)}{3}$

c $f(x) = 0.25(1 + 5x^{14})$

d $f(x) = \frac{12(x^5)^2 - (4x)^2}{3x^2}$

10 Given $\frac{dy}{dx}$, obtain $y$ in terms of $x$ for each of the following.

a $\frac{dy}{dx} = x^{\frac{3}{2}}$

b $\frac{dy}{dx} = x^{-\frac{3}{2}}$

11 For each $f'(x)$ expression, obtain an expression for $f(x)$.

a $f'(x) = \frac{5}{x^2}$

b $f'(x) = 6x^3 + 6x^{-3}$

12 Calculate the following indefinite integrals.

a $\int \frac{2x^5 + 7x^3 - 5}{x^2} dx$

b $\int (2\sqrt{x} + 1)^2 dx$

c $\int x^{\frac{1}{5}}(x^2 - 13x^{\frac{1}{10}} + 12)\, dx$

d $\int \frac{(x + 4)(x - 2)}{2x^4} dx$

13 For the following, find $F(x)$, where $F'(x) = f(x)$, given $a, b \in R$:

a $f(x) = (2ax)^2 + b^3$

b $f(x) = \sqrt{3x^4} + \sqrt{3}a^2$

14 a Calculate the primitive of $(x - 1)(x + 4)(x + 1)$.

b Calculate the anti-derivative of $\left(\frac{5}{x} - \frac{x}{5}\right)^2$.

c Calculate an anti-derivative of $\frac{x^p}{x^q}$, stating the restriction on the values of $p$ and $q$.

d Calculate an anti-derivative of $\frac{\sqrt[3]{x} - \sqrt[3]{x^5}}{x}$.

e Calculate $\frac{d}{dx}\left(\int (4x + 7)\, dx\right)$.

MASTER

15 The calculator enables the anti-derivative to be calculated through a number of menus. Familiarise yourself with these by calculating the anti-derivative of $x^3$ using each of these methods. Remember to insert the constant yourself if the calculator does not.

16 Write down the answers from the calculator to the following.

a $\int x^{\frac{3}{2}} dx$

b Given $\frac{dy}{dx} = (x^4 + 1)^2$, obtain $y$.

c If $f(t) = 100\left(t - \frac{5}{t^2}\right)$, obtain $F(t)$.

d $\int (y^5 - y^{-5})\, dy$

e $\int \sqrt{4u + 5}\, du$

f Choose your own function and use CAS technology to obtain its anti-derivative.

# 14.3 Anti-derivative functions and graphs

If some characteristic information is known about the anti-derivative function, such as a point on its curve, then this information allows the constant of integration to be calculated.

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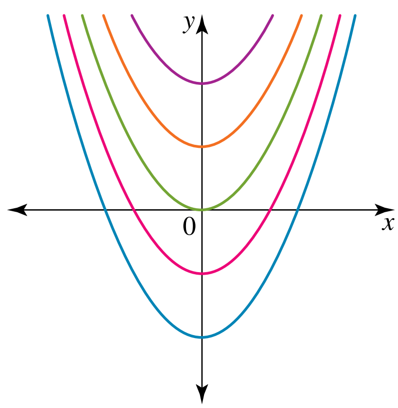
Units 1 & 2
AOS 3
Topic 3
Concept 2

**Anti-derivative functions and graphs**
Concept summary
Practice questions

## Determining the constant

Once the constant of integration is known, we can determine more than just a family of anti-derivatives. A specific anti-derivative function can be found since only one member of the family will possess the given characteristic information.

The family of curves $y = x^2 + c$




While there is a family of anti-derivative functions $y = x^2 + c$ for which $\frac{dy}{dx} = 2x$, if it is known the anti-derivative must contain the point $(0, 0)$, substituting $(0, 0)$ in $y = x^2 + c$ gives $0 = 0 + c$ and therefore $c = 0$. Only the specific anti-derivative function $y = x^2$ satisfies this condition.

eBookplus

**Interactivity**
Sketching the anti-derivative graph
**int-5965**

WORKED EXAMPLE 3

**a The gradient of a curve is given by $\frac{dy}{dx} = a + 3x$ where $a$ is a constant. Given the curve has a stationary point at $(2, 5)$, determine its equation.**

**b If $f'(x) = \frac{x^2 - 4}{2x^2}$ and $f(4) = 3$, obtain the value of $f(1)$.**

| THINK | WRITE |
| --- | --- |
| a 1 Use the given information to determine the value of $a$. | a $\frac{dy}{dx} = a + 3x$<br>At the stationary point $(2, 5)$, $\frac{dy}{dx} = 0$.<br>$\therefore 0 = a + 3 \times 2$<br>$\therefore a = -6$ |

**2** Calculate the anti-derivative.

$$\frac{dy}{dx} = -6 + 3x$$

$$y = -6x + 3 \times \frac{x^2}{2} + c$$

$$= -6x + \frac{3x^2}{2} + c$$

**3** Use the given information to calculate the constant of integration.

Substitute the point $(2, 5)$:

$$5 = -6(2) + \frac{3(2)^2}{2} + c$$

$$5 = -12 + 6 + c$$

$$c = 11$$

**4** State the answer.

$$y = -6x + \frac{3x^2}{2} + 11$$

The equation of the curve is $y = \frac{3}{2}x^2 - 6x + 11$.

**b 1** Express $f'(x)$ in the form in which it can be anti-differentiated.

**b** $$f'(x) = \frac{x^2 - 4}{2x^2}$$

$$= \frac{x^2}{2x^2} - \frac{4}{2x^2}$$

$$= \frac{1}{2} - 2x^{-2}$$

**2** Calculate the anti-derivative.

$$f(x) = \frac{1}{2}x - \frac{2x^{-1}}{-1} + c$$

$$= \frac{x}{2} + \frac{2}{x} + c$$

**3** Use the given information to calculate the constant of integration.

Substitute $f(4) = 3$ to calculate $c$.

$$3 = \frac{4}{2} + \frac{2}{4} + c$$

$$3 = 2 + \tfrac{1}{2} + c$$

$$c = \frac{1}{2}$$

**4** Calculate the required value.

$$f(x) = \frac{x}{2} + \frac{2}{x} + \frac{1}{2}$$

$$f(1) = \frac{1}{2} + \frac{2}{1} + \frac{1}{2}$$

$$= 3$$

## Sketching the anti-derivative graph

Given the graph of a function $y = f(x)$ whose rule is not known, the shape of the graph of the anti-derivative function $y = F(x)$ may be able to be deduced from the relationship $F'(x) = f(x)$. This means interpreting the graph of $y = f(x)$ as the gradient graph of $y = F(x)$.

- The $x$-intercepts of the graph of $y = f(x)$ identify the $x$-coordinates of the stationary points of $y = F(x)$.
- The nature of any stationary point on $y = F(x)$ is determined by the way the sign of the graph of $y = f(x)$ changes about its $x$-intercepts.
- If $f(x)$ is a polynomial of degree $n$ then $F(x)$ will be a polynomial of degree $n + 1$.

If the rule for $y = f(x)$ can be formed from its graph, then the equation of possible anti-derivative functions can also be formed. Additional information to determine the specific function would be needed, as in the previous Worked example.

**WORKED EXAMPLE 4**

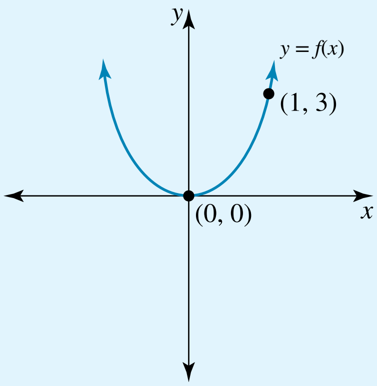
**Consider the graph of the quadratic function $y = f(x)$ shown.**

**a Describe the position and nature of any stationary point on the graph of $y = F(x)$ where $f(x) = F'(x)$.**

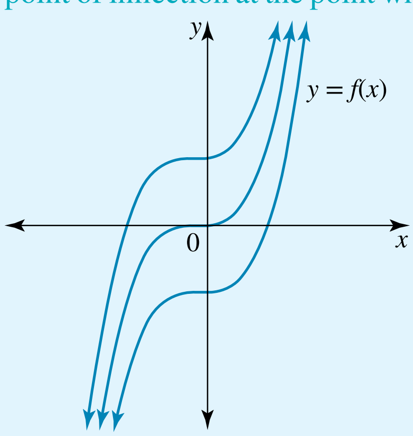
**b Draw three possible graphs for which $y = F(x)$.**

**c Obtain the rule for $f(x)$.**

**d Given $F(0) = -2$, determine the rule for $F(x)$, and sketch the graph of $y = F(x)$.**




| THINK | WRITE |
|---|---|
| **a 1** Identify the position of any stationary point on the graph of $y = F(x)$. | **a** The $x$-intercept on the given graph identifies a stationary point on $y = F(x)$. Therefore the graph of $y = F(x)$ has a stationary point at the point where $x = 0$. |
| **2** Determine the nature of the stationary point. | As $x$ increases, the sign of the graph of $y = f(x)$ changes from positive to zero to positive about its $x$-intercept. Therefore, there is a stationary point of inflection at the point where $x = 0$ on the graph of $y = F(x)$. |
| **b 1** State the degree of $F(x)$. | **b** Since $f(x)$ is quadratic, its anti-derivative $F(x)$ will be a polynomial of degree 3. |
| **2** Draw three possible graphs for the anti-derivative function. | Other than at $x = 0$, the graph of $y = f(x)$ is positive so the gradient of the graph of $y = F(x)$ is positive other than at $x = 0$. The $y$-coordinate of the stationary point on $y = F(x)$ is not known. Three graphs of an increasing cubic function with a stationary point of inflection at the point where $x = 0$ are shown for $y = F(x)$. |



c Determine the equation of the quadratic function using the information on its graph.

c The graph of $y = f(x)$ has a minimum turning point at $(0, 0)$ and contains the point $(1, 3)$.
Let $f(x) = a(x - h)^2 + k$.
The turning point is $(0, 0)$.
$\therefore f(x) = ax^2$
Substitute the point $(1, 3)$:
$3 = a(1)^2$
$\therefore \; a = 3$
The rule for the quadratic function is $f(x) = 3x^2$.

d 1 Use anti-differentiation to obtain the rule for $F(x)$.

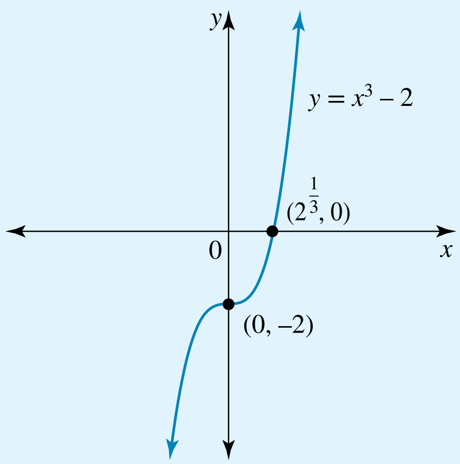
d $f(x) = 3x^2$
$\therefore F(x) = x^3 + c$

2 Determine the value of the constant of integration using the given information and state the rule for $F(x)$.

Since $F(0) = -2$,
$-2 = (0)^3 + c$
$c = -2$
$\therefore F(x) = x^3 - 2$

3 Sketch the graph of $y = F(x)$.

The point $(0, -2)$ is a stationary point of inflection and the $y$-intercept.
$x$-intercept: let $y = 0$
$x^3 - 2 = 0$
$x^3 = 2$
$x = \sqrt[3]{2}$
$(\sqrt[3]{2}, 0)$ is the $x$-intercept.




## EXERCISE 14.3 Anti-derivative functions and graphs

PRACTISE

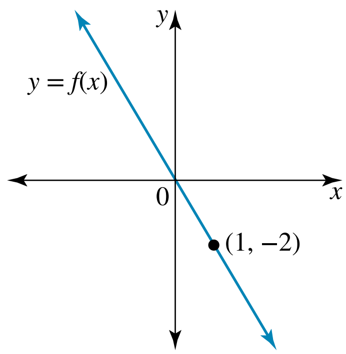
Work without CAS

1 WE3 a The gradient of a curve is given by $\dfrac{dy}{dx} = ax - 6$ where $a$ is a constant. Given the curve has a stationary point at $(-1, 10)$, determine its equation.

b If $f'(x) = \dfrac{2x^2 + 9}{2x^2}$ and $f(3) = 0$, find the value of $f(-1)$.

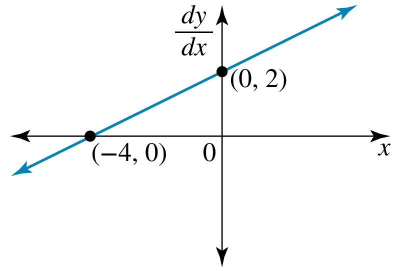
2 If $\dfrac{dy}{dx} = 2\sqrt{x}$ and $y = 10$ when $x = 4$, find $y$ when $x = 1$.

3 WE4 Consider the graph of the linear function $y = f(x)$ shown.




a Describe the position and nature of any stationary point on the graph of $y = F(x)$ where $f(x) = F'(x)$.
b Draw three possible graphs for which $y = F(x)$.
c Obtain the rule for $f(x)$.
d Given $F(0) = 1$, determine the rule for $F(x)$ and sketch the graph of $y = F(x)$.

4 The graph of the gradient, $\frac{dy}{dx}$, of a particular curve is shown.




Given that $(-2, 3)$ lies on the curve with this gradient, determine the equation of the curve and deduce the coordinates of, and the nature of, any stationary point on the curve.

**CONSOLIDATE**
Apply the most appropriate mathematical processes and tools

5 The gradient of a curve is given by $f'(x) = -3x^2 + 4$. Find the equation of the curve if it passes through the point $(-1, 2)$.

6 The gradient of a curve is given by $\frac{dy}{dx} = \frac{2x}{5} - 3$. It is also known that the curve passes through the point $(5, 0)$.
a Determine the equation of the curve.
b Find its $x$-intercepts.

7 The gradient of a curve is directly proportional to $x$ and at the point $(2, 5)$ on the curve the gradient is $-3$.
a Determine the constant of proportionality.
b Find the equation of the curve.

8 A function is defined by $f(x) = \frac{(4 - x)(5 - x)}{10x^4}$. Determine its primitive function if the point $(1, -1)$ lies on the primitive function.

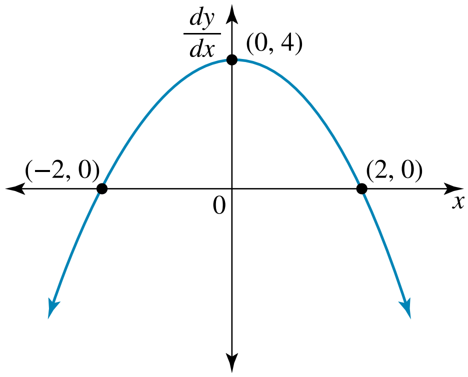
9 a Given $\frac{dy}{dx} = 2x(3 - x)$ and $y = 0$ when $x = 3$, obtain the value of $y$ when $x = 0$.
b Given $\frac{dz}{dx} = (10 - x)^2$ and $z = 200$ when $x = 10$, obtain the value of $z$ when $x = 4$.
c Given $\frac{dA}{dt} = \frac{4}{\sqrt{t}}$ and $A = 40$ when $t = 16$, obtain the value of $A$ when $t = 64$.
d Given $\frac{dx}{dy} = -\frac{3}{y^4}$ and $y = 1$ when $x = 12$, obtain the value of $y$ when $x = 75$.

10 At any point $(x, y)$ on a curve, the gradient of the tangent to the curve is given by $\frac{dy}{dx} = a - x^{\frac{2}{3}}$. The curve has a stationary point at $(8, 32)$.
a Find the value of $a$.
b Determine the equation of the curve.
c Calculate the equation of the tangent to the curve at the point where $x = 1$.

**11** A curve with a horizontal asymptote with equation $y = a$ passes through the point $(2, 3)$. If the gradient of the curve is given by $F'(x) = \frac{a}{x^2}$:

**a** Find the value of $a$.
**b** Determine the equation of the curve.
**c** Calculate, to the nearest degree, the angle at which the curve cuts the $x$-axis.

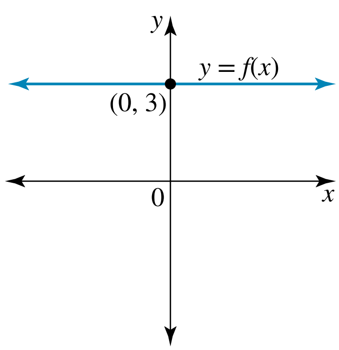
**12** The gradient function of a curve is illustrated in the diagram.



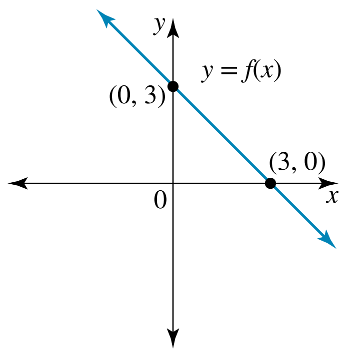

**a** Describe the position and nature of the stationary points of the curve with this gradient function.
**b** Sketch a possible curve which has this gradient function.
**c** Determine the rule for the gradient graph and hence obtain the equation which describes the family of curves with this gradient function.
**d** One of the curves belonging to the family of curves cuts the $x$-axis at $x = 3$. Obtain the equation of this particular curve and calculate the slope with which it cuts the $x$-axis at $x = 3$.

**13** For each of the following graphs of $y = f(x)$, draw a sketch of a possible curve for $y = F(x)$ given $F'(x) = f(x)$.

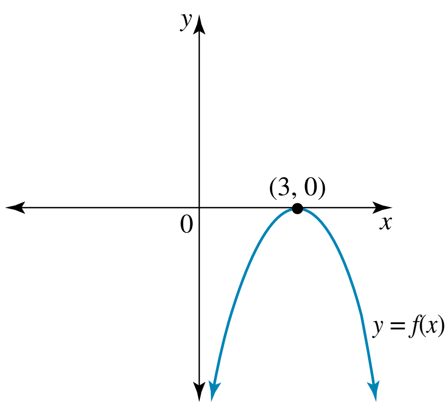
**a**



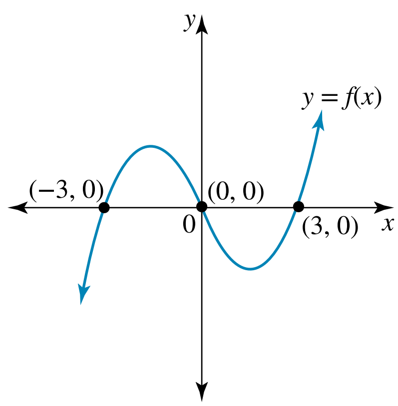

**b**



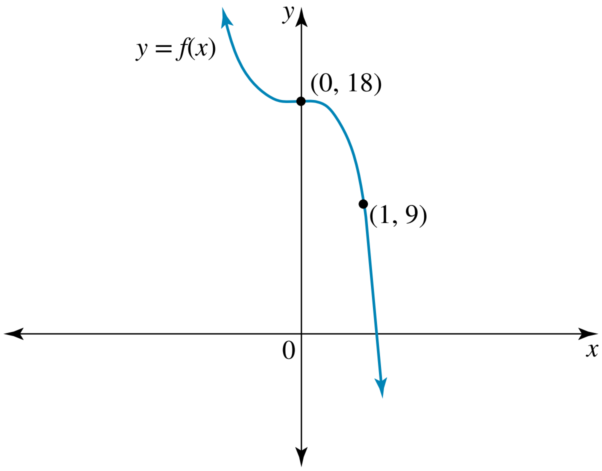

**c**




**d**



**14** The diagram shows the graph of a cubic function $y = f(x)$ with a stationary point of inflection at (0, 18) and passing through the point (1, 9).




**a** Find $\int f(x)\,dx$.

**b** A particular anti-derivative function of $y = f(x)$ passes through the origin. For this particular function:

**i** Determine its equation.

**ii** Find the coordinates of its other $x$-intercept and determine at which of its $x$-intercepts the graph of this function is steeper.

**iii** Deduce the exact $x$-coordinate of its turning point and its nature.

**iv** Hence show the $y$-coordinate of its turning point can be expressed as $2^p \times 3^q$, specifying the values of $p$ and $q$.

**v** Draw a sketch graph of this particular anti-derivative function.

MASTER

**15** Use the CAS calculator's keyboard template to sketch the shape of the graphs of the anti-derivative functions given by:

**a** $y_1 = \int (x - 2)(x + 1)\,dx$

**b** $y_2 = \int (x - 2)^2(x + 1)\,dx$

**c** $y_3 = \int (x - 2)^2(x + 1)^2 dx$.

**d** Comment on the effect of changing the multiplicity of each factor.

**16** Define $f(x) = x^2 - 6x$ and use the calculator to sketch on the same set of axes the graphs of $y = f(x)$, $y = f'(x)$ and $y = \int f(x)\,dx$. Comment on the connections between the three graphs.

# 14.4 Applications of anti-differentiation

studyon

Units 1 & 2
AOS 3
Topic 3
Concept 3

**Applications of anti-differentiation**
Concept summary
Practice questions

In this section, anti-differentiation is applied in calculations involving functions defined by their rates of change.

## The anti-derivative in kinematics

Recalling that velocity, $v$, is the rate of change of displacement, $x$, means that we can now interpret displacement as the anti-derivative of velocity.

$$v = \frac{dx}{dt} \Leftrightarrow x = \int v\,dt$$

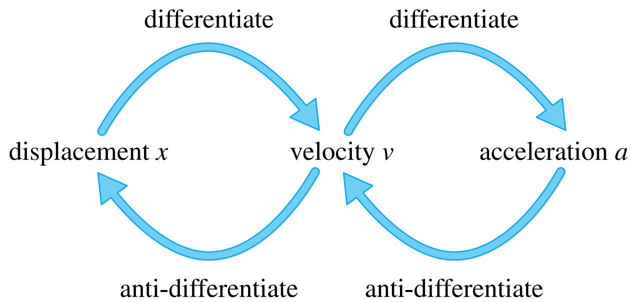
Further, since acceleration is the rate of change of velocity, it follows that velocity is the anti-derivative of acceleration.

$$a = \frac{dv}{dt} \Leftrightarrow v = \int a\,dt$$

The role of differentiation and anti-differentiation in kinematics is displayed in the diagram below.

Velocity is of particular interest for two reasons:

- The anti-derivative of velocity with respect to time gives displacement.
- The derivative of velocity with respect to time gives acceleration.




WORKED EXAMPLE 5

**A particle moves in a straight line so that its velocity at time $t$ seconds is given by $v = 3t^2 + 4t - 4,\ t \geq 0$. Initially the particle is 8 metres to the left of a fixed origin.**

**a** **After how many seconds does the particle reach the origin?**

**b** **Calculate the particle's acceleration when its velocity is zero.**

| THINK | WRITE |
|---|---|
| **a 1** Calculate the displacement function from the velocity function. | **a** $v = 3t^2 + 4t - 4$<br>Anti-differentiate the velocity to obtain displacement.<br>$x = \frac{3t^3}{3} + \frac{4t^2}{2} - 4t + c$<br>$= t^3 + 2t^2 - 4t + c$ |
| **2** Use the initial conditions to calculate the constant of integration. | When $t = 0,\ x = -8$.<br>$\therefore -8 = c$<br>$\therefore\ \ x = t^3 + 2t^2 - 4t - 8$ |
| **3** Set up and solve the equation which gives the time the particle is at the required position. | When the particle is at the origin, $x = 0$.<br>$t^3 + 2t^2 - 4t - 8 = 0$<br>$\therefore t^2(t + 2) - 4(t + 2) = 0$<br>$\therefore (t + 2)(t^2 - 4) = 0$<br>$\therefore (t + 2)^2(t - 2) = 0$<br>$\therefore t = -2, t = 2$<br>As $t \geq 0$, reject $t = -2$.<br>$\therefore t = 2$<br>The particle reaches the origin after 2 seconds. |

**b** **1** Calculate the acceleration function from the velocity function.

**b** $v = 3t^2 + 4t - 4$

Acceleration $a = \frac{dv}{dt}$

$\therefore a = 6t + 4.$

**2** Calculate the time when the velocity is zero.

When velocity is zero,

$3t^2 + 4t - 4 = 0$

$\therefore (3t - 2)(t + 2) = 0$

$\therefore t = \frac{2}{3}, t = -2$

$t \geq 0,\ \text{so } t = \frac{2}{3}$

**3** Calculate the acceleration at the time required.

When $t = \frac{2}{3}$,

$a = 6 \times \frac{2}{3} + 4$

$= 8$

The acceleration is 8 m/s$^2$ when the velocity is zero.

## Acceleration and displacement

To obtain the displacement from an expression for acceleration will require anti-differentiating twice in order to proceed from $a$ to $v$ to $x$. The first anti-differentiation operation will give velocity and the second will obtain displacement from velocity.

This will introduce two constants of integration, one for each step. Where possible, evaluate the first constant using given $v - t$ information before commencing the second anti-differentiation operation. To evaluate the second constant, $x - t$ information will be needed. The notation used for each constant should distinguish between them. For example, the first constant could be written as $c_1$ and the second as $c_2$.

WORKED EXAMPLE 6

**A particle moves in a straight line so that its acceleration at time $t$ seconds is given by $a = 4 + 6t,\ t \geq 0$. If $v = 30$ and $x = 2$ when $t = 0$, calculate the particle's velocity and position when $t = 1$.**

| THINK | WRITE |
|---|---|
| 1 Calculate the velocity function from the acceleration function. | $a = 4 + 6t$<br>Anti-differentiate the acceleration to obtain velocity.<br>$v = 4t + 3t^2 + c_1$<br>$= 3t^2 + 4t + c_1$ |
| 2 Evaluate the first constant of integration using the given $v - t$ information. | When $t = 0, v = 30$<br>$\Rightarrow 30 = c_1$<br>$\therefore v = 3t^2 + 4t + 30$ |
| 3 Calculate the displacement function from the velocity function. | Anti-differentiate the velocity to obtain displacement.<br>$x = t^3 + 2t^2 + 30t + c_2$ |
| 4 Evaluate the second constant of integration using the given $x - t$ information. | When $t = 0, x = 2$<br>$\Rightarrow 2 = c_2$<br>$\therefore x = t^3 + 2t^2 + 30t + 2$ |

| | |
|---|---|
| **5** Calculate the velocity at the given time. | $v = 3t^2 + 4t + 30$<br>When $t = 1$,<br>$v = 3 + 4 + 30$<br>$= 37$ |
| **6** Calculate the displacement at the given time. | $x = t^3 + 2t^2 + 30t + 2$<br>When $t = 1$,<br>$x = 1 + 2 + 30 + 2$<br>$= 35$ |
| **7** State the answer. | The particle has a velocity of 37 m/s and its position is 35 metres to the right of the origin when $t = 1$. |

## Other rates of change

The process of anti-differentiation can be used to solve problems involving other rates of change, not only those involved in kinematics. For example, if the rate at which the area of an oil spill is changing with respect to $t$ is given by $\frac{dA}{dt} = f(t)$, then the area of the oil spill at time $t$ is given by $A = \int f(t)\, dt$, the anti-derivative with respect to $t$.

**WORKED EXAMPLE 7**

**An ice block with initial volume $36\pi$ cm$^3$ starts to melt. The rate of change of its volume $V$ cm$^3$ after $t$ seconds is given by $\frac{dV}{dt} = -0.2$.**

**a** **Use anti-differentiation to express $V$ in terms of $t$.**

**b** **Calculate, to 1 decimal place, the number of minutes it takes for all of the ice block to melt.**

| THINK | WRITE |
|---|---|
| **a 1** Calculate the volume function from the given derivative function. | **a** $\frac{dV}{dt} = -0.2$<br>Anti-differentiate with respect to $t$:<br>$V = -0.2t + c$ |
| **2** Use the initial conditions to evaluate the constant of integration. | When $t = 0,\ V = 36\pi$<br>$\therefore 36\pi = c$<br>Therefore, $V = -0.2t + 36\pi$ |
| **b 1** Identify the value of $V$ and calculate the corresponding value of $t$. | **b** When the ice block has melted, $V = 0$.<br>$0 = -0.2t + 36\pi$<br>$\therefore t = \frac{36\pi}{0.2}$<br>$= 180\pi$ |
| **2** Express the time in the required units and state the answer to the required degree of accuracy.<br>*Note:* The rate of decrease of the volume is constant, so this problem could have been solved without calculus. | The ice block melts after $180\pi$ seconds.<br>Divide this by 60 to convert to minutes:<br>$180\pi$ seconds is equal to $3\pi$ minutes.<br>To 1 decimal place, the time for the ice block to melt is 9.4 minutes. |

## EXERCISE 14.4 Applications of anti-differentiation

**PRACTISE**

Work without CAS

**1** WE5 A particle moves in a straight line so that its velocity at time $t$ seconds is given by $v = 3t^2 - 10t - 8,\ t \geq 0$. Initially the particle is 40 metres to the right of a fixed origin.

a After how many seconds does the particle first reach the origin?

b Calculate the particle's acceleration when its velocity is zero.

**2** A particle moves in a straight line so that its velocity at time $t$ seconds is given by $v = \dfrac{1}{t^2} + 2,\ t > 0$. When $t = 1$, the particle is 1 metre to the left of a fixed origin. Obtain expressions for the particle's displacement and acceleration after $t$ seconds.

**3** WE6 A particle moves in a straight line so that its acceleration at time $t$ seconds is given by $a = 8 - 18t,\ t \geq 0$. If $v = 10$ and $x = -2$ when $t = 0$, calculate the particle's velocity and position when $t = 1$.

**4** A particle moves in a straight line so that its acceleration, in m/s$^2$, at time $t$ seconds is given by $a = 9.8,\ t \geq 0$. If $v = x = 0$ when $t = 0$, calculate the particle's displacement when $t = 5$.

**5** WE7 An ice block with initial volume $4.5\pi$ cm$^3$ starts to melt. The rate of change of its volume, $V$ cm$^3$, after $t$ seconds is given by $\dfrac{dV}{dt} = -0.25$.

a Use anti-differentiation to express $V$ in terms of $t$.

b Calculate, to 1 decimal place, the number of seconds it takes for all of the ice block to melt.

**6** When first purchased, the height of a small rubber plant was 50 cm. The rate of growth of its height over the first year after its planting is measured by $h'(t) = 0.2t,\ 0 \leq t \leq 12$, where $h$ is its height in cm $t$ months after being planted. Calculate its height at the end of the first year after its planting.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**7** The velocity, $v$ m/s, of a particle moving in a straight line at time $t$ seconds is given by $v = 8t^2 - 20t - 12,\ t \geq 0$. Initially the particle is 54 metres to the right of a fixed origin.

a Obtain an expression for the particle's displacement at time $t$ seconds.

b How far from its initial position is the particle after the first second?

c Determine the position of the particle when its velocity is zero.

**8** A particle starts from rest and moves in a straight line so that its velocity at time $t$ is given by $v = -3t^3,\ t \geq 0$. When its position is 1 metre to the right of a fixed origin, its velocity is $-24$ m/s. What was the initial position of the particle relative to the fixed origin?

**9** Starting from a point 9 metres to the right of a fixed origin, a particle moves in a straight line in such a way that its velocity after $t$ seconds is $v = 6 - 6t,\ t \geq 0$.

a Show that the particle moves with constant acceleration.

b Determine when and where its velocity is zero.

c How far does the particle travel before it reaches the origin?

d What is the average speed of this particle over the first three seconds?

e What is the average velocity of this particle over the first three seconds?

**10** After $t$ seconds the velocity $v$ m/s of a particle moving in a straight line is given by $v(t) = 3(t - 2)(t - 4),\ t \geq 0$.

**a** Calculate the particle's initial velocity and initial acceleration.
**b** Obtain an expression for $x(t)$, the particle's displacement from a fixed origin after time $t$ seconds, given that the particle was initially at this origin.
**c** Calculate $x(5)$.
**d** How far does the particle travel during the first 5 seconds?

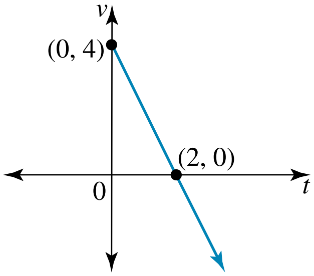
**11** A particle moving in a straight line has a velocity of 3 m/s and a displacement of 2 metres from a fixed origin after 1 second. If its acceleration after time $t$ seconds is $a = 8 + 6t$, obtain expressions in terms of $t$ for:

**a** its velocity
**b** its displacement.

**12** The acceleration of a moving object is given by $a = -10$. Initially, the object was at the origin and its initial velocity was 20 m/s.

**a** Determine when and where its velocity is zero.
**b** After how many seconds will the object return to its starting point?

**13** The diagram shows a velocity–time graph.




**a** Draw a possible displacement–time graph.
**b** Draw the acceleration–time graph.
**c** If it is known that the initial displacement is $-4$, determine the rule for the displacement–time graph.
**d** Sketch the displacement–time graph with the rule obtained in part **c**.

**14** Pouring boiling water on weeds is a means of keeping unwanted weeds under control. Along the cracks in an asphalt driveway, a weed had grown to cover an area of 90 $cm^2$, so it was given the boiling water treatment.

The rate of change of the area, $A$ $cm^2$, of the weed $t$ days after the boiling water is poured on it is given by $\frac{dA}{dt} = -18t$.

**a** Express the area as a function of $t$.
**b** According to this model, how many whole days will it take for the weed to be completely removed?
**c** For what exact values of $t$ is the model $\frac{dA}{dt} = -18t$ valid?

**15** Rainwater collects in a puddle on part of the surface of an uneven path. For the time period for which the rain falls, the rate at which the area, $A$ $m^2$, grows is modelled by $A'(t) = 4 - t$, where $t$ is the number of days of rain.

**a** After how many days does the area of the puddle stop increasing?
**b** Express the area function $A(t)$ in terms of $t$ and state its domain.
**c** What is the greatest area the puddle grows to?
**d** On the same set of axes, sketch the graphs of $A'(t)$ versus $t$ and $A(t)$ versus $t$ over an appropriate domain.

16 The rate of growth of a colony of microbes in a laboratory can be modelled by $\frac{dm}{dt} = k\sqrt{t}$ where $m$ is the number of microbes after $t$ days. Initially there were 20 microbes and after 4 days the population was growing at 300 microbes per day.

a Determine the value of $k$.

b Express $m$ as a function of $t$.

c Hence calculate the number of days it takes for the population size to reach 6420.

MASTER

17 The velocity of an object which moves in a straight line is $v = (2t + 1)^4$, $t \geq 0$. Use CAS technology to:

a state the acceleration

b calculate the displacement, given that initially the object's displacement was 4.2 metres from a fixed origin

c find the time and the velocity, to 2 decimal places, when the displacement is 8.4 metres.

18 With the aid of CAS technology, obtain an expression for $x$ in terms of $t$ if $a = \frac{1}{(t+1)^3}$ and $x = v = 0$ when $t = 0$.

# 14.5 The definite integral

studyon

Units 1 & 2

AOS 3

Topic 3

Concept 4

**The definite integral**

Concept summary

Practice questions

The indefinite integral has been used as a symbol for the anti-derivative. In this section we will look at the definite integral. We shall learn how to evaluate a definite integral, but only briefly explore its meaning, since this will form part of the Mathematical Methods Units 3 and 4 course.

## The definite integral

The definite integral is of the form $\int_a^b f(x)\,dx$; it is quite similar to the indefinite integral but has the numbers $a$ and $b$ placed on the integral symbol. These numbers are called terminals. They define the endpoints of the interval or the limits on the values of the variable $x$ over which the integration takes place. Due to their relative positions, $a$ is referred to as the **lower terminal** and $b$ as the **upper terminal**.

For example, $\int_1^3 2x\,dx$ is a definite integral with lower terminal 1 and upper terminal 3, whereas $\int 2x\,dx$ without terminals is an indefinite integral.

In the definite integral the term $f(x)$ is called the **integrand**. It is the expression being integrated with respect to $x$.

## Calculation of the definite integral

The definite integral results in a numerical value when it is evaluated.

It is evaluated by the calculation: $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is an anti-derivative of $f(x)$.

The calculation takes two steps:

- Anti-differentiate $f(x)$ to obtain $F(x)$.
- Substitute the terminals in $F(x)$ and carry out the subtraction calculation.

This calculation is commonly written as:

$$\int_a^b f(x)\,dx = [F(x)]_a^b$$
$$= F(b) - F(a)$$

For example, to evaluate $\int_1^3 2x\,dx$, we write:

$$\int_1^3 2x\,dx = [x^2]_1^3$$
$$= (3)^2 - (1)^2$$
$$= 8$$

Note that only 'an' anti-derivative is needed in the calculation of the definite integral. Had the constant of integration been included in the calculation, these terms would cancel out, as illustrated by the following:

$$\int_1^3 2x\,dx = [x^2 + c]_1^3$$
$$= ((3)^2 + c) - ((1)^2 + c)$$
$$= 9 + \cancel{c} - 1 - \cancel{c}$$
$$= 8$$

The value of a definite integral may be a positive, zero or negative real number. It is not a family of functions as obtained from an indefinite integral.

WORKED EXAMPLE 8

Evaluate $\int_{-1}^{2} (3x^2 + 1)\,dx$.

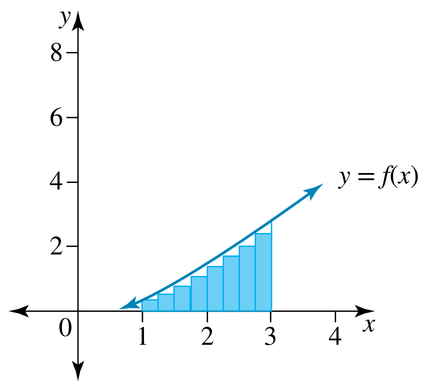
| THINK | WRITE |
|---|---|
| 1 Calculate an anti-derivative of the integrand. | $\int_{-1}^{2} (3x^2 + 1)\,dx$ <br> $= [x^3 + x]_{-1}^{2}$ |
| 2 Substitute the upper and then the lower terminal in place of $x$ and subtract the two expressions. | $= (2^3 + 2) - ((-1)^3 + (-1))$ |
| 3 Evaluate the expression. | $= (8 + 2) - (-1 - 1)$ <br> $= (10) - (-2)$ <br> $= 12$ |
| 4 State the answer. | $\int_{-1}^{2} (3x^2 + 1)\,dx = 12$ |

## Integration and area

Areas of geometric shapes such as rectangles, triangles, and trapeziums can be used to approximate areas enclosed by edges which are not straight. Leibniz invented the method for calculating the exact measure of an area enclosed by a curve, thereby establishing the branch of calculus known as integral calculus. His method bears some similarity to a method used in the geometry studies of Archimedes around 230 BC.

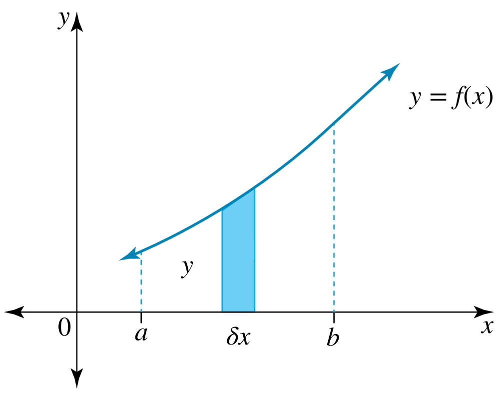
Lebniz's approach involved summing a large number of very small areas and then calculating the limiting sum, a process known as **integration**. Like differential calculus, integral calculus is based on the concept of a limit.

To calculate the area bounded by a curve $y = f(x)$ and the $x$-axis between $x = 1$ and $x = 3$, the area could be approximated by a set of rectangles as illustrated in the diagram below. Here, 8 rectangles, each of width 0.25 units, have been constructed.




The larger the number of rectangles, the better the accuracy of the approximation to the actual area; also, the larger the number, the smaller the widths of the rectangles.

To calculate the area between $x = a$ and $x = b$, Leibniz partitioned (divided) the interval $[a, b]$ into a large number of strips of very small width. A typical strip would have width $\delta x$ and area $\delta A$, with the total area approximated by the sum of the areas of such strips.




For $\delta x$, the approximate area of a typical strip is $\delta A \approx y\delta x$, so the total area $A$ is approximated from the sum of these areas. This is written as $A \approx \sum_{x=a}^{b} y\delta x$. The approximation improves as $\delta x \to 0$ and the number of strips increases.

The actual area is calculated as a limit; the limiting sum as $\delta x \to 0$, and therefore $A = \lim_{\delta x \to 0} \sum_{x=a}^{b} y\,\delta x$.

Leibniz chose to write the symbol for the limiting sum with the long capital 'S' used in his time. That symbol is the now familiar integral sign. This means that the area bounded by the curve, the $x$-axis and $x = a, x = b$ is $A = \int_a^b y\,dx$ or $A = \int_a^b f(x)\,dx$.

The process of evaluating this integral or limiting sum expression is called integration.

The fundamental theorem of calculus links together the process of anti-differentiation with the process of integration. The full proof of this theorem is left to Units 3 and 4.

## Signed area

As we have seen, the definite integral can result in a positive, negative or zero value. Area, however, can only be positive.

If $f(x) > 0$, the graph of $y = f(x)$ lies above the $x$-axis and $\int_a^b f(x)\,dx$ will be positive. Its value will be a measure of the area bounded by the curve $y = f(x)$ and the $x$-axis between $x = a$ and $x = b$.

However, if $f(x) < 0$, the graph of $y = f(x)$ is below the $x$-axis and $\int_a^b f(x)\,dx$ will be negative. Its value gives a **signed area**. By ignoring the negative sign, the value of the integral will still measure the actual area bounded by the curve $y = f(x)$ and the $x$-axis between $x = a$ and $x = b$.

If over the interval $[a, b]$ $f(x)$ is partly positive and partly negative, then $\int_a^b f(x)\,dx$ could be positive or negative, or even zero, depending on which of the positive or negative signed areas is the 'larger'.

$$\int_a^b f(x)\,dx \text{ is the sum of signed area measures}$$

This adds a complexity to using the signed area measure of the definite integral to calculate area. Perhaps you can already think of a way around this situation. However, for now, another treat awaiting you in Units 3 and 4 will be to explore signed areas.

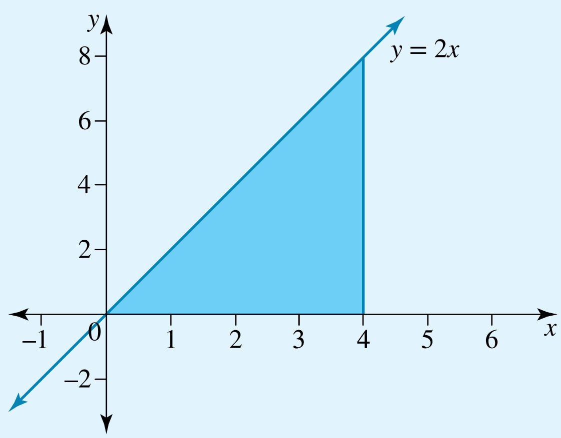
**WORKED EXAMPLE 9**

**The area bounded by the line $y = 2x$, the $x$-axis and $x = 0,\ x = 4$ is illustrated in the diagram.**

**a** **Calculate the area using the formula for the area of a triangle.**

**b** **Write down the definite integral which represents the measure of this area.**

**c** **Hence, calculate the area using calculus.**




**THINK**

**a** Calculate the area using the formula for the area of a triangle.

**WRITE**

**a** Base is 4 units; height is 8 units.

Area of triangle:

$$A = \tfrac{1}{2}bh$$
$$= \tfrac{1}{2} \times 4 \times 8$$
$$= 16$$

The area is 16 square units.

**THINK**

**b** State the definite integral which gives the area.

**WRITE**

**b** The definite integral $\int_a^b f(x)\,dx$ gives the area. For this area $f(x) = 2x,\ a = 0$ and $b = 4$.

$\int_0^4 2x\,dx$ gives the area measure.

**c 1** Evaluate the definite integral.

*Note:* There are no units, just a real number for the value of a definite integral.

**c** $\int_0^4 2x\,dx = [x^2]_0^4$
$= 4^2 - 0^2$
$= 16$

**2** State the area using appropriate units.

Therefore, the area is 16 square units.

## The definite integral in kinematics

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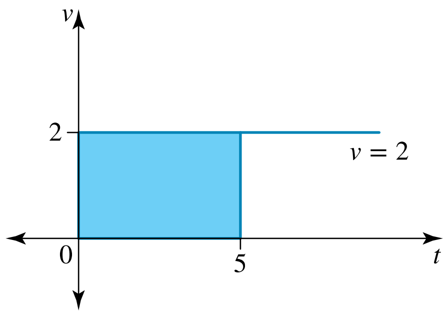
A particle which travels at a constant velocity of 2 m/s will travel a distance of 10 metres in 5 seconds. For this motion, the velocity–time graph is a horizontal line. Looking at the rectangular area under the velocity–time graph bounded by the horizontal axis between $t = 0$ and $t = 5$ shows that its area measure is also equal to 10.




- The area under the velocity–time graph gives the measure of the distance travelled by a particle.
- For a positive signed area, the definite integral $\int_{t_1}^{t_2} v\,dt$ gives the distance travelled over the time interval $t \in [t_1, t_2]$.

The velocity–time graph is the most important of the motion graphs as it gives the velocity at any time; the gradient of its tangent gives the instantaneous acceleration and the area under its graph gives the distance travelled.

WORKED EXAMPLE 10

**The velocity of a particle moving in a straight line is given by $v = 3t^2 + 1, t \geq 0$.**

**a** **Give the particle's velocity and acceleration when $t = 1$.**

**b i** **Sketch the velocity–time graph and shade the area which represents the distance the particle travels over the interval $t \in [1, 3]$.**

**ii** **Use a definite integral to calculate the distance.**

**Assume units for distance are in metres and time in seconds.**

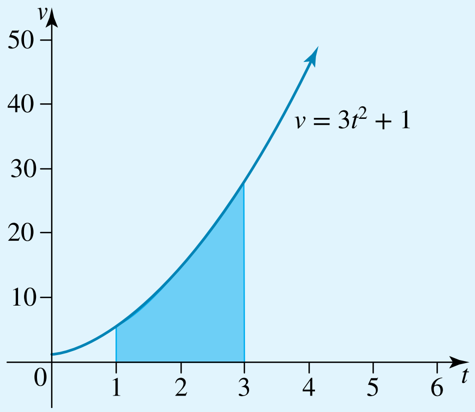
| THINK | WRITE |
|---|---|
| **a 1** Calculate the velocity at the given time. | **a** $v = 3t^2 + 1$<br>When $t = 1$,<br>$v = 3 \times 1^2 + 1$<br>$= 4$<br>The velocity is 4 m/s. |
| **2** Obtain the acceleration from the velocity function and evaluate it at the given time. | $a = \frac{dv}{dt}$<br>$= 6t$<br>When $t = 1, a = 6$<br>The acceleration is 6 $m/s^2$. |

**b** **i** Sketch the $v-t$ graph and shade the area required.

**b** **i** $v = 3t^2 + 1, t \geq 0$

The $v-t$ graph is part of the parabola which has turning point (0, 1) and passes through the point (1, 4).

The area under the graph bounded by the $t$-axis and $t = 1$, $t = 3$ represents the distance.




**ii** **1** State the definite integral which gives the measure of the area shaded.

*Note:* The integrand is a function of $t$ and is to be integrated with respect to $t$.

**ii** Area measure is given by $\int_1^3 (3t^2 + 1)\, dt$.

**2** Evaluate the definite integral.

$$\begin{aligned}\int_1^3 (3t^2 + 1)\, dt &= \left[t^3 + t\right]_1^3 \\ &= (3^3 + 3) - (1^3 + 1) \\ &= 28\end{aligned}$$

**3** State the distance travelled.

The distance travelled by the particle over the time interval [1, 3] is 28 metres.

## EXERCISE 14.5 The definite integral

**PRACTISE**

Work without CAS

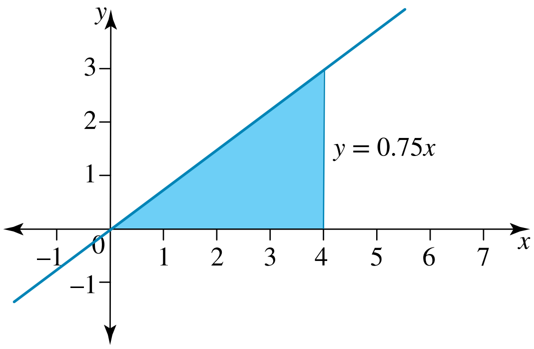
**1** WE8 Evaluate $\int_0^3 (3x^2 - 2x)\, dx$.

**2** Evaluate the following.

**a** $\int_{-2}^{2} (x - 2)(x + 2)\, dx$

**b** $\int_{-1}^{1} x^3\, dx$

**3** WE9 The area bounded by the line $y = 0.75x$, the $x$-axis and $x = 0$, $x = 4$ is illustrated in the diagram.

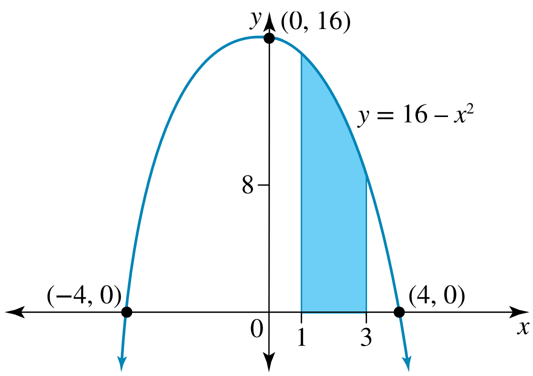



**a** Calculate the area using the formula for the area of a triangle.

**b** Write down the definite integral which represents the measure of this area.

**c** Hence, calculate the area using calculus.

**4** The area bounded by the curve $y = 16 - x^2$, the $x$-axis and $x = 1,\ x = 3$ is illustrated in the diagram.




**a** Write down the definite integral which represents the measure of this area.

**b** Hence, calculate the area.

**5** WE10 The velocity of a particle moving in a straight line is given by $v = t^3 + 2, t \geq 0$.

**a** Give the particle's velocity and acceleration when $t = 1$.

**b i** Sketch the velocity–time graph and shade the area which represents the distance the particle travels over the interval $t \in [2, 4]$.

**ii** Use a definite integral to calculate the distance.

Assume units for distance are in metres and time is in seconds.

**6** The velocity, $v$ m/s, of a particle moving in a straight line after $t$ seconds is given by $v = 3t^2 - 2t + 5, t \geq 0$.

**a** Show that velocity is always positive.

**b** Calculate the distance the particle travels in the first 2 seconds using:

**i** a definite integral

**ii** anti-differentiation.

**CONSOLIDATE**

Apply the most appropriate mathematical processes and tools

**7** Evaluate each of the following.

**a** $\int_{-2}^{2} 5x^4\,dx$

**b** $\int_{0}^{2} (7 - 2x^3)\,dx$

**c** $\int_{1}^{3} (6x^2 + 5x - 1)\,dx$

**d** $\int_{-3}^{0} 12x^2(x - 1)\,dx$

**e** $\int_{-4}^{-2} (x + 4)^2\,dx$

**f** $\int_{-\frac{1}{2}}^{\frac{1}{2}} (x + 1)(x^2 - x)\,dx$

**8** Show by calculation that the following statements are true.

**a** $\int_{1}^{2} (20x + 15)\,dx = 5\int_{1}^{2} (4x + 3)\,dx$

**b** $\int_{-1}^{2} (x^2 + 2)\,dx = \int_{-1}^{2} x^2\,dx + \int_{-1}^{2} 2\,dx$

**c** $\int_{1}^{3} 3x^2\,dx = \int_{1}^{3} 3t^2\,dt$

**d** $\int_{a}^{a} 3x^2\,dx = 0$

**e** $\int_{b}^{a} 3x^2\,dx = -\int_{a}^{b} 3x^2\,dx$

**f** $\int_{-a}^{a} dx = 2a$

**9** Evaluate each of the following.

**a** $\int_{1}^{2} \frac{1}{x^2}\,dx$

**b** $\int_{0}^{9} \sqrt{x}\,dx$

**c** $\int_1^4 5x^{\frac{3}{2}}\,dx$

**d** $\int_1^2 \frac{x^6 + 2x^3 + 4}{x^5}\,dx$

**e** $\int_3^5 2\sqrt{t}(\sqrt{t} + 3(\sqrt{t})^3)\,dt$

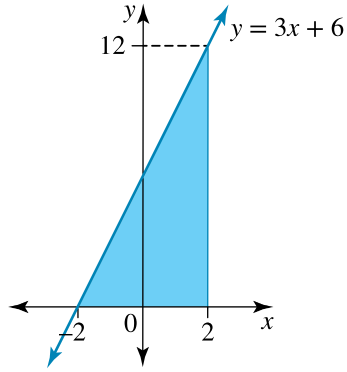
**f** $\int_{-2}^{-1} \left(2 - \frac{4}{u^3}\right) du$

**10 a** Determine the value of $n$ so that $\int_4^n 3\,dx = 9$.

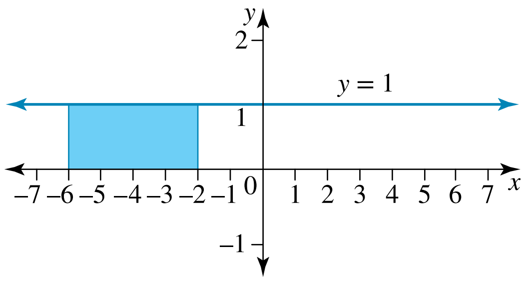
**b** Determine the value of $p$ so that $\int_0^p \sqrt{x}\,dx = 18$.

**11** Express the areas in each of the following diagrams in terms of a definite integral and calculate the area using both integration and a known formula for the area of the geometric shape.

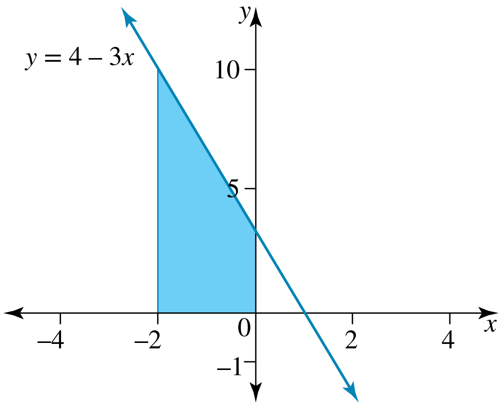
**a** The triangular area bounded by the line $y = 3x + 6$, the $x$-axis and $x = -2,\ x = 2$ as illustrated in the diagram




**b** The rectangular area bounded by the line $y = 1$, the $x$-axis and $x = -6,\ x = -2$ as illustrated in the diagram




**c** The trapezoidal area bounded by the line $y = 4 - 3x$, the coordinate axes and $x = -2$ as illustrated in the diagram

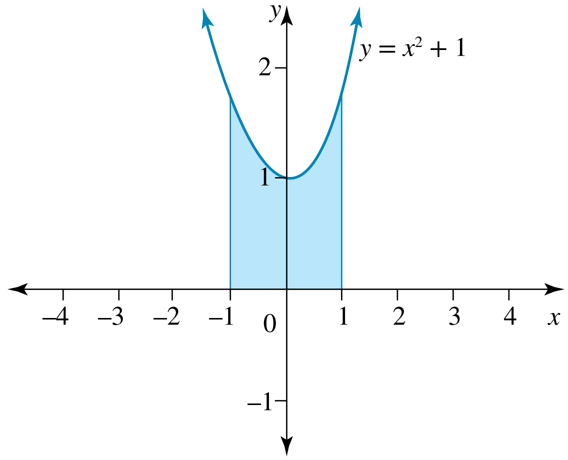


12 For each of the following areas:

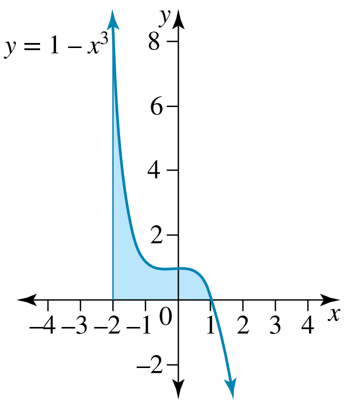
i write a definite integral, the value of which gives the area measure

ii calculate the area.

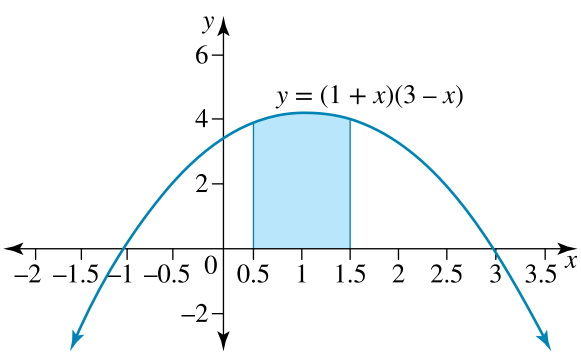
a The area bounded by the curve $y = x^2 + 1$, the $x$-axis and $x = -1$, $x = 1$ as illustrated in the diagram




b The area bounded by the curve $y = 1 - x^3$, the $x$-axis and $x = -2$ as illustrated in the diagram




c The area bounded by the curve $y = (1 + x)(3 - x)$, the $x$-axis and $x = 0.5$, $x = 1.5$ as illustrated in the diagram




13 Sketch a graph to show each of the areas represented by the given definite integrals and then calculate the area.

a $\int_0^3 4x^2\,dx$ b $\int_{-1}^{1} (1 - x^2)\,dx$ c $\int_{-2}^{4} dx$

14 Consider the function defined by $f(x) = 4 - \sqrt{x}$.

a Sketch the graph of $y = f(x)$, showing the intercepts with the coordinate axes.

b On the diagram, shade the area which is bounded by the curve and the coordinate axes.

c Express the shaded area in terms of a definite integral.

d Calculate the area.

**15** A particle starts from rest at the origin and moves in a straight line so that after $t$ seconds its velocity is given by $v = 10t - 5t^2,\ t \geq 0$.

**a** Calculate an expression for the displacement at time $t$.

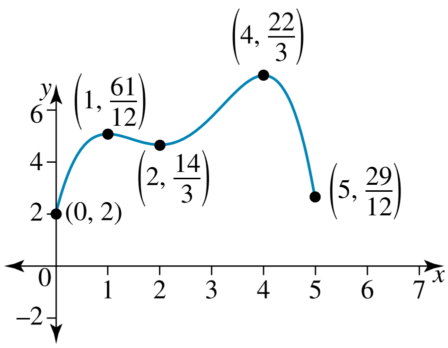
**b** When the particle is next at rest, how far will it have travelled from its starting point?

**c** Draw the velocity–time graph for $t \in [0, 2]$.

**d** Use a definite integral to calculate the area enclosed by the velocity–time graph and the horizontal axis for $t \in [0, 2]$.

**e** What does the area in part **d** measure?

**16** An athlete training to compete in a marathon race runs along a straight road with velocity $v$ km/h. The velocity–time graph over a 5-hour period for this athlete is shown below.




**a** What were the athlete's highest and lowest speeds over the 5-hour period?

**b** At what times did the athlete's acceleration become zero?

**c** Given the equation of the graph shown is $v = 2 + 8t - 7t^2 + \frac{7t^3}{3} - \frac{t^4}{4}$, use a definite integral to calculate the distance the athlete ran during this 5-hour training period.

**MASTER**

**17** Write down the values of:

**a** $\int_{-3}^{4} (2 - 3x + x^2)\,dx$

**b** $\int_{4}^{-3} (2 - 3x + x^2)\,dx$, and

**c** explain the relationship between parts **a** and **b**.

**18** **a** Use the calculator to obtain $\int_{-2}^{3} (x + 2)(x - 3)\,dx$.

**b** Draw the graph of $y = (x + 2)(x - 3)$.

**c** What is the area enclosed between the graph of $y = (x + 2)(x - 3)$ and the $x$-axis?

**d** Why do the answers to part **a** and part **c** differ? Write down a definite integral which does give the area.

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# 14.6 Review

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**Units 1 & 2** — **Anti-differentiation and introduction to integral calculus**

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# 14 Answers

## EXERCISE 14.2

1 a $y = 2x^6$

b $\frac{4}{3}x^3 + x^2 - 5x$

c $f(x) = x^3 + x^2 - 16x + c$

2 $y = \frac{2x^3}{3} - \frac{3x^2}{2} + c$

3 a $-\frac{7x^5}{5} + x^3 - 3x^2 + c$

b $\frac{5}{28}x^7 + \frac{3}{8}x^2 +$

c $F(x) = -\frac{2}{3x^3} + c$

d $\frac{x^3}{3} + 2x - \frac{1}{x} + c$

4 a $F(x) = \sqrt{x} + c$

b $\int (2t^2 + 4)\, dt = \frac{2t^3}{3} + 4t + c = 2\int (t^2 + 2)\, dt$

5 a $y = \frac{1}{2}x^{10} + c$

b $y = -3x + \frac{1}{2}x^8 + c$

c $y = \frac{2}{3}x^3 - 6x^2 + 14x + c$

d $y = -\frac{2}{3}x^3 + \frac{11}{2}x^2 + 40x +$

6 a $f(x) = \frac{1}{12}x^6 + \frac{1}{3}x^7 + c$

b $f(x) = \frac{4}{21}x^7 + 5x + c$

c $f(x) = \frac{4}{3}x^3 - \frac{6}{7}x^7 + c$

d $f(x) = 9x - 4x^3 + \frac{4x^5}{5} + c$

7 a $\frac{1}{15}x^9 + c$

b $2x + c$

c $40x^2 - \frac{5x^8}{2} + c$

d $\frac{1}{100}\left[9x + 2x^3 - 0.5x^{11}\right] + c$

8 a $ax^2 + bx + c$

b $0.0005x^{100} + c$

c $2x^4 + 4x^3 + 3x^2 + x$

d $7x - x^5 + \frac{4x^3}{3} + 4x^2 + c$

9 a $F(x) = \frac{1}{4}(x^3 - 2x) + c$

b $F(x) = \frac{1}{24}x^2 + \frac{2}{3}x + c$

c $F(x) = \frac{x}{4} + \frac{x^{15}}{12} + c$

d $F(x) = \frac{4x^9}{9} - \frac{16x}{3} + c$

10 a $y = \frac{2}{5}x^{\frac{5}{2}} + c$

b $y = -2x^{-\frac{1}{2}} + c$

11 a $f(x) = \frac{-5}{x} + c$

b $f(x) = \frac{3x^4}{2} - 3x^{-2} + c$

12 a $\frac{x^4}{2} + \frac{7x^2}{2} + \frac{5}{x} + c$

b $2x^2 + \frac{8}{3}x^{\frac{3}{2}} + x + c$

c $\frac{5}{16}x^{\frac{16}{5}} - 10x^{\frac{13}{10}} + 10x^{\frac{6}{5}} + c$

d $-\frac{1}{2x} - \frac{1}{2x^2} + \frac{4}{3x^3} + c$

13 a $F(x) = \frac{4a^2x^3}{3} + b^3x + c$

b $F(x) = \frac{\sqrt{3}x^3}{3} + \sqrt{3}a^2x + c$

14 a $\frac{1}{4}x^4 + \frac{4}{3}x^3 - \frac{1}{2}x^2 - 4x + c$

b $\frac{-25}{x} - 2x + \frac{x^3}{75} + c$

c $\frac{1}{p - q + 1}x^{p-q+1}, p \neq q - 1$

d $3x^{\frac{1}{3}} - \frac{3}{5}x^{\frac{5}{3}}$

e $4x + 7$

15 $\frac{x^4}{4} + c$

16 a $\frac{2x^{\frac{5}{2}}}{5}$

b $\frac{x^9}{9} + \frac{2x^5}{5} + x$

c $50t^2 + \frac{500}{t}$

d $\frac{2y^{10} + 3}{12y^4}$

e $\frac{(4u + 5)^{\frac{3}{2}}}{6}$

f Answers will vary.

## EXERCISE 14.3

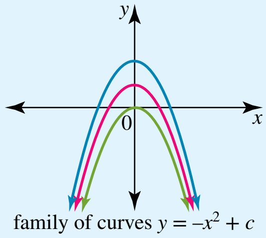
1 a $y = -3x^2 - 6x + 7$

b $f(-1) = 2$

2 $y = \frac{2}{3}$

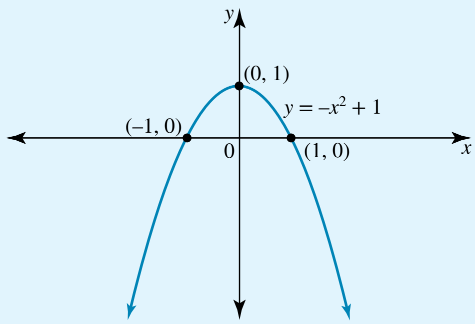
**3 a** Maximum turning point at $x = 0$

**b**




**c** $f(x) = -2x$

**d** $F(x) = -x^2 + 1$;




**4** $y = \frac{1}{4}x^2 + 2x + 6$; minimum turning point at $(-4, 2)$

**5** $f(x) = -x^3 + 4x + 5$

**6 a** $y = \frac{x^2}{5} - 3x + 10$

**b** $(5, 0), (10, 0)$

**7 a** $k = -\frac{3}{2}$

**b** $y = -\frac{3x^2}{4} + 8$

**8** $F(x) = \frac{-2}{3x^3} + \frac{9}{20x^2} - \frac{1}{10x} - \frac{41}{60}$

**9 a** $y = -9$

**b** $z = 128$

**c** $A = 72$

**d** $y = \frac{1}{4}$

**10 a** $a = 4$

**b** $y = 4x - \frac{3x^{\frac{5}{3}}}{5} + \frac{96}{5}$

**c** $5y - 15x = 98$

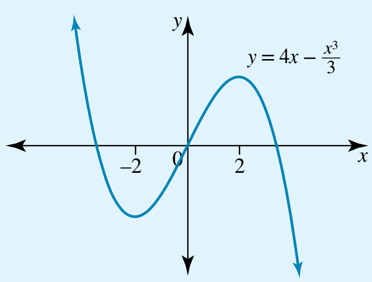
**11 a** $a = 6$

**b** $f(x) = -\frac{6}{x} + 6$

**c** 81°

**12 a** Minimum turning point at $x = -2$; maximum turning point at $x = 2$

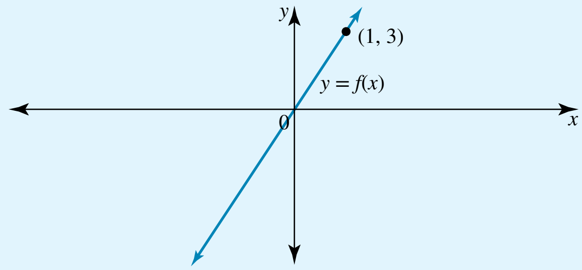
**b**



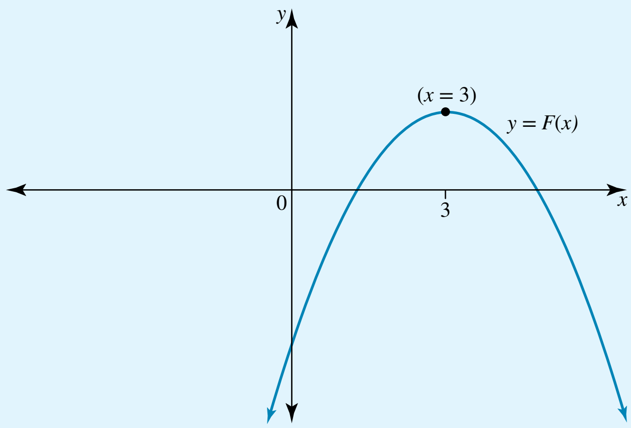

**c** $\frac{dy}{dx} = 4 - x^2$; $y = 4x - \frac{x^3}{3} + c$

**d** $y = -\frac{x^3}{3} + 4x - 3$; slope of $-5$

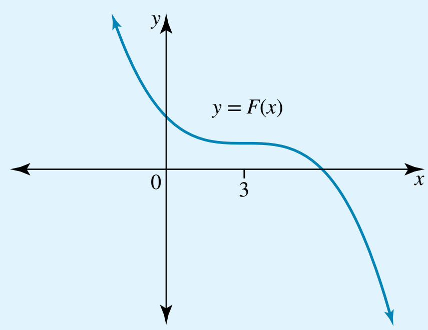
**13 a** Any line with gradient of 3



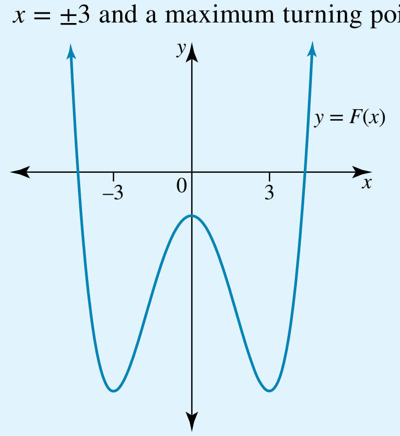

**b** Any parabola with a maximum turning point when $x = 3$




**c** Any inverted cubic with a stationary point of inflection when $x = 3$




**d** Any quartic with minimum turning points when $x = \pm 3$ and a maximum turning point when $x = 0$



**14 a** $-\dfrac{9x^4}{4} + 18x + c$

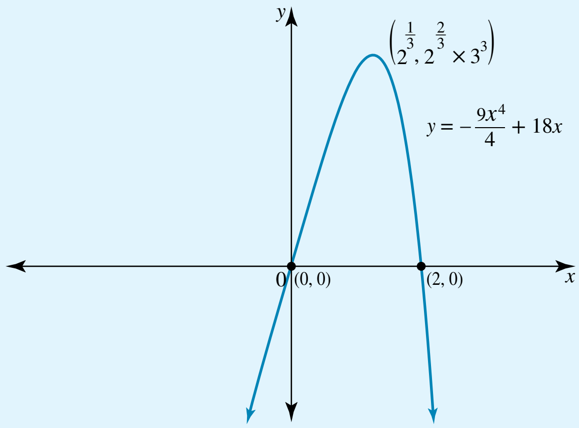
**b** **i** $y = -\dfrac{9x^4}{4} + 18x$

**ii** $(2, 0)$; steeper at $(2, 0)$

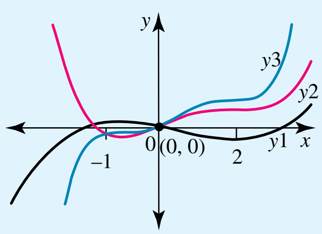
**iii** Maximum turning point when $x = \sqrt[3]{2}$

**iv** $p = -\frac{2}{3}, q = 3$

**v**



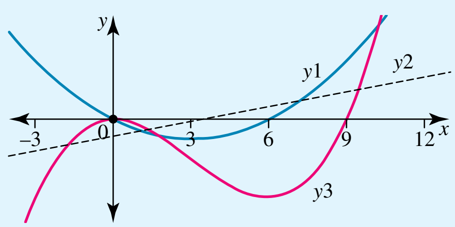

**15 a,b,c** Use your CAS technology to sketch the graphs.




**d** For factors of multiplicity 1, the anti-derivative graph has turning points at the zeros of each factor.

For factors of multiplicity 2, the anti-derivative graph has stationary points of inflection at the zeros of each factor.

**16** Use your CAS technology to sketch the graphs.




The $x$-intercept of $y = f'(x)$ is connected to the turning point of $y = f(x)$.

The turning points of $y = \int f(x)\, dx$ are connected to the $x$-intercepts of $y = f(x)$.

## EXERCISE 14.4

**1 a** $2\sqrt{2}$ seconds

**b** 14 m/s$^2$

**2** $a = -\dfrac{2}{t^3}, x = -\dfrac{1}{t} + 2t - 2$

**3** Velocity 9 m/s; position 9 metres to the right of the origin

**4** 122.5 metres

**5 a** $V = -0.25t + 4.5\pi$

**b** 56.5 seconds

**6** 64.4 cm

**7 a** $x = \dfrac{8t^3}{3} - 10t^2 - 12t + 54$

**b** $19\frac{1}{3}$ metres

**c** At the origin

**8** 13 metres to the right of the origin

**9 a** $a = -6$

**b** 1 second; 12 metres to the right of origin

**c** 15 metres

**d** 5 m/s

**e** $-3$ m/s

**10 a** 24 m/s; $-18$ m/s$^2$

**b** $x(t) = t^3 - 9t^2 + 24t$

**c** 20

**d** 28 metres

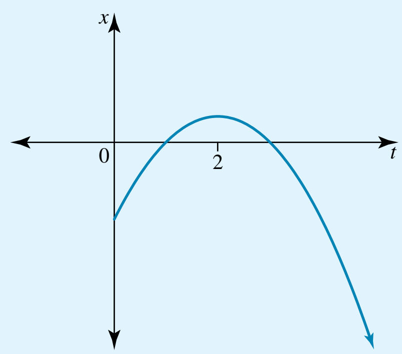
**11 a** $v = 3t^2 + 8t - 8$

**b** $x = t^3 + 4t^2 - 8t + 5$

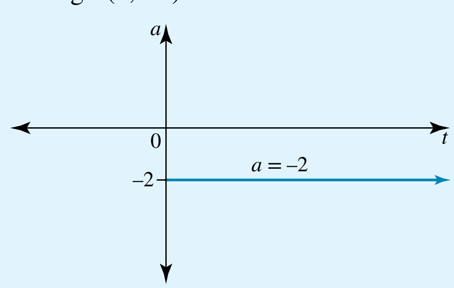
**12 a** 2 seconds; 20 metres to right of origin

**b** 4 seconds

**13 a** Part of a parabola with a maximum turning point at $t = 2$

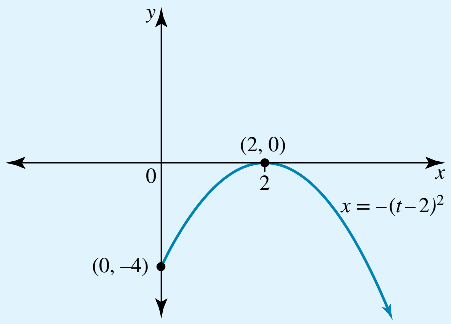



**b** Part of a horizontal line below the $t$-axis through $(0, -2)$



**c** $x = -(t - 2)^2, t \geq 0$

**d** Part of a parabola with maximum turning point at (2, 0)




**14 a** $A = -9t^2 + 90$

**b** 4 days

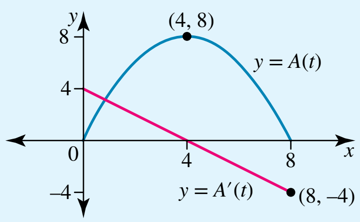
**c** $0 \leq t \leq \sqrt{10}$

**15 a** 4 days

**b** $A(t) = 4t - \frac{t^2}{2}$; domain [0, 8]

**c** 8 m$^2$

**d**




**16 a** $k = 150$

**b** $m = 100t^{\frac{3}{2}} + 20$

**c** 16 days

**17 a** $a = 8(2t + 1)^3$

**b** $x = \frac{(2t + 1)^5}{10} + 4.1$

**c** 0.56 seconds; 20.27 m/s

**18** $x = \frac{t}{2} + \frac{1}{2(t + 1)} - \frac{1}{2}$

## EXERCISE 14.5

**1** 18

**2 a** $-\frac{32}{3}$

**b** 0

**3 a** 6 square units

**b** $\int_0^4 0.75x\, dx$

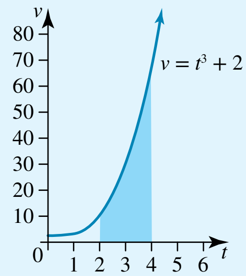
**c** 6 square units

**4 a** $\int_1^3 (16 - x^2)\, dx$

**b** $23\frac{1}{3}$ square units

**5 a** $v = 3$ m/s; $a = 3$ m/s$^2$

**b i**




**ii** $\int_2^4 (t^3 + 2)\, dt = 64$ so distance is 64 metres.

**6 a** Velocity is a positive definite quadratic function in $t$.

**b i** 14 metres

**ii** 14 metres

**7 a** 64

**b** 6

**c** 70

**d** $-351$

**e** $\frac{8}{3}$

**f** 0

**8** Proofs required — check with your teacher

**9 a** $\frac{1}{2}$

**b** 18

**c** 62

**d** $\frac{55}{16}$

**e** 212

**f** $\frac{7}{2}$

**10 a** $n = 7$

**b** $p = 9$

**11 a** $\int_{-2}^{2} (3x + 6)\, dx$; 24 square units

**b** $\int_{-6}^{-2} dx$; 4 square units

**c** $\int_{-2}^{0} (4 - 3x)\, dx$; 14 square units

**12 a i** $\int_{-1}^{1} (x^2 + 1)\, dx$

**ii** $2\frac{2}{3}$ square units

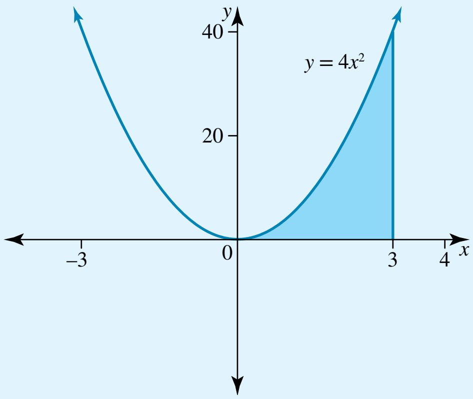
**b i** $\int_{-2}^{1} (1 - x^3)\, dx$

**ii** $6\frac{3}{4}$ square units

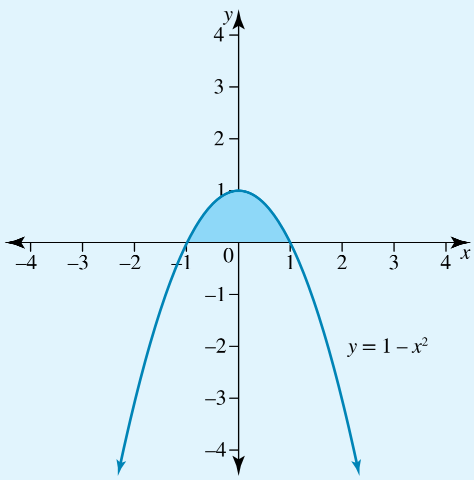
**c i** $\int_{0.5}^{1.5} (1 + x)(3 - x)\, dx$

**ii** $3\frac{11}{12}$ square units

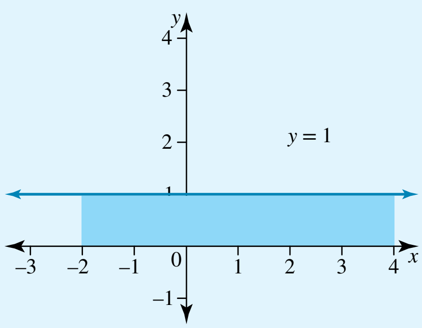
**13 a** Area enclosed by $y = 4x^2$, $x$-axis and $x = 0$, $x = 3$ is 36 square units.



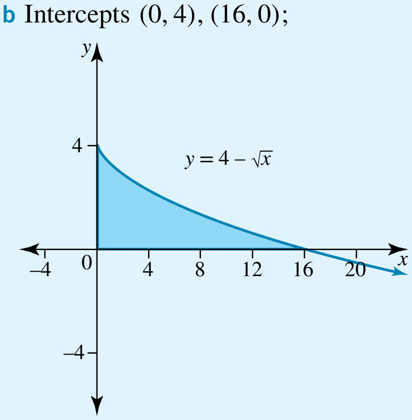

**b** Area enclosed by $y = 1 - x^2$ and $x$-axis is $\frac{4}{3}$ square units.




**c** Rectangular area enclosed by the line $y = 1$, $x$-axis and $x = -2$, $x = 4$ is 6 square units.




**14 a b** Intercepts $(0, 4)$, $(16, 0)$;



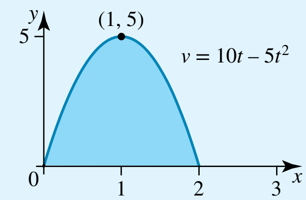

**c** Area measure equals $\int_0^{16} (4 - \sqrt{x})\,dx$.

**d** $\frac{64}{3}$ square units

**15 a** $x = 5t^2 - \frac{5t^3}{3}$

**b** $6\frac{2}{3}$ metres

**c**




**d** $6\frac{2}{3}$

**e** Distance travelled in first 2 seconds.

**16 a** $7\frac{1}{3}$ km/h; 2 km/h

**b** 1 hour, 2 hours and 4 hours

**c** $26\frac{2}{3}$ km

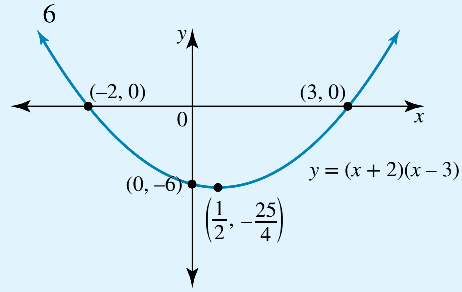
**17 a** $\frac{203}{6}$

**b** $-\frac{203}{6}$

**c** Interchanging the terminals changes the sign of the integral.

**18 a** $-\frac{125}{6}$

**b**




**c** $\frac{125}{6}$ square units

**d** Area lies below $x$-axis so signed area is negative.

Possible integrals are $-\int_{-2}^{3} (x + 2)(x - 3)\,dx$ or $\int_{3}^{-2} (x + 2)(x - 3)\,dx$.

# Index

## S



## T