MATHSQUEST 12

MATHEMATICAL METHODS

SOLUTIONS MANUAL

VCE UNITS 3 AND 4

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Topic 1 — Solving equations

Exercise 1.2 — Polynomials

1 **a** $15u^2 - u - 2 = (5u-2)(3u+1)$

b $6d^2 - 28d + 16 = 2\left(3d^2 - 14d + 8\right) = 2(3d-2)(d-4)$

c $3j^2 + 12j - 6$
$= 3\left(j^2 + 4j - 2\right)$
$= 3\left(j^2 + 4j + (2)^2 - (2)^2 - 2\right)$
$= 3\left((j+2)^2 - 6\right)$
$= 3\left((j+2)^2 - \left(\sqrt{6}\right)^2\right)$
$= 3\left(j+2-\sqrt{6}\right)\left(j+2+\sqrt{6}\right)$

2 **a** $f^2 - 12f - 28 = (f-14)(f+2)$

b $g^2 + 3g - 4 = (g+4)(g-1)$

c $b^2 - 1 = (b-1)(b+1)$

3 **a** $125a^3 - 27b^3 = (5a)^3 - (3b)^3$
$= (5a-3b)\left((5a)^2 + (5a)(3b) + (3b)^2\right)$
$= (5a-3b)\left(25a^2 + 15ab + 9b^2\right)$

b $2c^3 + 6c^2d + 6cd^2 + d^3$
$= 2\left(c^3 + 3c^2d + 3cd^2 + d^3\right)$
$= 2(c+d)^3$

c $40p^3 - 5 = 5\left(8p^3 - 1\right)$
$= 5\left((2p)^3 - 1^3\right)$
$= 5(2p-1)\left((2p)^2 + 2p + 1\right)$
$= 5(2p-1)\left(4p^2 + 2p + 1\right)$

4 **a** $27z^3 - 54z^2 + 36z - 8$
$= (3z)^3 - 3(3z)^2(2) + 3(3z)(2)^2 + 2^3$
$= (3z-2)^3$

b $m^3n^3 + 64$
$= (mn)^3 + 4^3$
$= (mn+4)\left((mn)^2 - 4mn + 4^2\right)$
$= (mn+4)\left(m^2n^2 - 4mn + 16\right)$

5 **a** $9x^2 - xy - 3x + y$
$= 9x^2 - 3x - xy + y$
$= 3x(x-1) - y(x-1)$
$= (x-1)(3x-y)$

b $3y^3 + 3y^2z^2 - 2zy - 2z^3$
$= 3y^2\left(y + z^2\right) - 2z(y + z^2)$
$= (y+z^2)\left(3y^2 - 2z\right)$

6 **a** $9a^2 - 16b^2 - 12a + 4$
$= 9a^2 - 12a + 4 - 16b^2$
$= (3a)^2 - 2(3a)(2) + 2^2 - (4b)^2$
$= (3a-2)^2 - (4b)^2$
$= (3a-2-4b)(3a-2+4b)$

b $n^2p^2 - 4m^2 - 4m - 1$
$= (np)^2 - \left(4m^2 + 4m + 1\right)$
$= (np)^2 - \left((2m)^2 + 2(2m) + 1\right)$
$= (np)^2 - (2m+1)^2$
$= \left(np - (2m+1)\right)\left(np + (2m+1)\right)$
$= (np - 2m - 1)(np + 2m + 1)$

7 Let $P(x) = x^3 - 2x^2 - 21x - 18$
$P(-1) = (-1)^3 - 2(-1)^2 - 21(-1) - 18$
$P(-1) = -1 - 2 + 21 - 18$
$P(-1) = 0$
Thus $(x+1)$ is a factor.
$x^3 - 2x^2 - 21x - 18 = (x+1)\left(x^2 - 3x - 18\right)$
$= (x+1)(x-6)(x+3)$

8 Let $P(x) = x^4 - 5x^3 - 32x^2 + 180x - 144$
$P(1) = (1)^4 - 5(1)^3 - 32(1)^2 + 180(1) - 144$
$P(1) = 1 - 5 - 32 + 180 - 144$
$P(1) = 181 - 181$
$P(1) = 0$
Thus $(x-1)$ is a factor.
$x^4 - 5x^3 - 32x^2 + 180x - 144 = (x-1)\left(x^3 - 4x^2 - 36x + 144\right)$
Let $Q(x) = x^3 - 4x^2 - 36x + 144$
$Q(2) = 2^3 - 4(2)^2 - 36(2) + 144 \neq 0$
$Q(4) = 4^3 - 4(4)^2 - 36(4) + 144$
$= 64 - 64 - 144 + 144$
$= 0$
Thus $(x-4)$ is a factor.
$x^3 - 4x^2 - 36x + 144 = (x-4)\left(x^2 - 36\right)$
$= (x-4)(x-6)(x+6)$
So $x^4 - 5x^3 - 32x^2 + 180x - 144 = (x-1)(x-4)(x-6)(x+6)$

9 **a** $x^4 - 8x^3 + 17x^2 + 2x - 24 = 0$
$(x-4)(x-3)(x-2)(x+1) = 0$
$x-4=0,\quad x-3=0\quad x-2=0\quad x+1=0$
$x=4,\quad x=3,\quad x=2,\quad x=-1$

b $a^4 + 2a^2 - 8 = 0$
Let $x = a^2$
$x^2 + 2x - 8 = 0$
$(x-2)(x+4) = 0$
$x = 2, -4$
Substitute $x = a^2$
$a^2 = 2,\qquad a^2 = -4$ (no solution)
$\therefore a = \pm\sqrt{2}$

10 **a** $2x^3 - x^2 - 10x + 5 = 0$
$x^2(2x-1) - 5(2x-1) = 0$
$(2x-1)(x^2-5) = 0$
$(2x-1)\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right) = 0$
$x = \frac{1}{2},\ \pm\sqrt{5}$

b $2a^2-5a=9$

$2a^2-5a-9=0$

$a=\dfrac{5\pm\sqrt{(-5)^2-4\times2\times-9}}{2\times2}$

$=\dfrac{5\pm\sqrt{97}}{4}$

11 $Ax^3+(B-1)x^2+(B+C)x+D\equiv 3x^3-x^2+2x-7$

$A=3,\quad B-1=-1,\quad B+C=2$ and $D=-7$

$B=0\qquad 0+C=2$

$C=2$

12 $x^3+9x^2-2x+1\equiv x^3+(dx+e)^2+4$

$x^3+9x^2-2x+1\equiv x^3+d^2x^2+2dex+e^2+4$

$d^2=9,\ 2(\pm3)e=-2$

$d=\pm3\qquad \pm6e=-2$

$e=\pm\dfrac{1}{3}$

13 a $7r^3-49r^2+r-7$

$=7r^2(r-7)+(r-7)$

$=(r-7)\left(7r^2+1\right)$

b $36v^3+6v^2+30v+5$

$=6v^2(6v+1)+5(6v+1)$

$=(6v+1)\left(6v^2+5\right)$

c $2m^3+3m^2-98m-147$

$=m^2(2m+3)-49(2m+3)$

$=(2m+3)\left(m^2-49\right)$

$=(2m+3)(m-7)(m+7)$

d $2z^3-z^2+2z-1$

$=z^2(2z-1)+(2z-1)$

$=(2z-1)\left(z^2+1\right)$

e $4x^2-28x+49-25y^2$

$=(2x-7)^2-(5y)^2$

$=(2x-7-5y)(2x-7+5y)$

f $16a^2-4b^2-12b-9$

$=(4a)^2-\left(4b^2+12b+9\right)$

$=(4a)^2-(2b+3)^2$

$=(4a-(2b+3))(4a+2b+3)$

$=(4a-2b-3)(4a+2b+3)$

g v^2-4-w^2+4w

$=v^2-\left(w^2-4w+4\right)$

$=v^2-(w-2)^2$

$=(v-(w-2))(v+(w-2))$

$=(v-w+2)(v+w-2)$

h $4p^2-1+4pq+q^2$

$=4p^2+4pq+q^2-1$

$=(2p+q)^2-1$

$=(2p+q-1)(2p+q+1)$

14 a $81y^2=1$

$81y^2-1=0$

$(9y)^2-1^2=0$

$(9y-1)[9y+1]=0$

$9y-1=0$ or $9y+1=0$

$y=\dfrac{1}{9}\qquad y=-\dfrac{1}{9}$

b $4z^2+28z+49=0$

$(2z)^2+2(2z)(7)+7^2=0$

$(2z+7)^2=0$

$2z+7=0$

$2z=-7$

$z=-\dfrac{7}{2}$

c $5m^2+3=10m$

$5m^2-10m+3=0$

$m=\dfrac{10\pm\sqrt{(-10)^2-4\times5\times3}}{2\times5}$

$=\dfrac{10\pm\sqrt{40}}{10}$

$=\dfrac{10\pm2\sqrt{10}}{10}$

$=\dfrac{5\pm\sqrt{10}}{5}$

d $x^2-4x=-3$

$x^2-4x+3=0$

$(x-3)(x-1)=0$

$x-3=0$ or $x-1=0$

$x=3\qquad x=1$

e $48p=24p^2+18$

$24p^2-48p+18=0$

$4p^2-8p+3=0$

$(2p-1)(2p-3)=0$

$2p-1=0$ or $2p-3=0$

$p=\dfrac{1}{2}\qquad p=\dfrac{3}{2}$

f $39k=4k^2+77$

$4k^2-39k+77=0$

$(4k-11)(k-7)=0$

$4k-11=0$ or $k-7=0$

$k=\dfrac{11}{4}\qquad k=7$

g $m^2+3m=4$

$m^2+3m-4=0$

$(m+4)(m-1)=0$

$m+4=0$ or $m-1=0$

$m=-4\qquad m=1$

h $4n^2=8-5n$

$4n^2+5n-8=0$

$n=\dfrac{-5\pm\sqrt{(5)^2-4\times4\times-8}}{2\times4}$

$=\dfrac{-5\pm\sqrt{153}}{8}$

$=\dfrac{-5\pm3\sqrt{17}}{8}$

15 a $2x^3+7x^2+2x-3=0$

Let $P(x)=2x^3+7x^2+2x-3$

$P(-1)=2(-1)^3+7(-1)^2+2(-1)-3$

$=-2+7-2-3$

$=0$

Thus $(x+1)$ is a factor.

$2x^3+7x^2+2x-3=0$
$(x+1)\left(2x^2+5x-3\right)=0$
$(x+1)(2x-1)(x+3)=0$
$x+1=0$ or $2x-1=0$ or $x+3=0$
$x=-1 \quad x=\frac{1}{2} \quad x=-3$

b $l^4-17l^2+16=0$
$\left(l^2-1\right)\left(l^2-16\right)=0$
$(l-1)(l+1)(l-4)(l+4)=0$
$l-1=0$ or $l+1=0$ or $l-4=0$ or $l+4=0$
$l=1 \quad l=-1 \quad l=4 \quad l=-4$

c $c^3+3c^2-4c-12=0$
Let $P(c)=c^4+c^3-10c^2-4c+24$
$P(2)=(2)^4+(2)^3-10(2)^2-4(2)+24$
$=16+8-40-8+24$
$=48-48$
$=0$
Thus $(c-2)$ is a factor.
$c^4+c^3-10c^2-4c+24=(c-2)\left(c^3+3c^2-4c-12\right)$
Let $Q(c)=c^3+3c^2-4c-12$
$Q(2)=2^3+3(2)^2-4(2)-12$
$=8+12-8-12$
$=0$
Thus $(c-2)$ is a factor.
$c^3+3c^2-4c-12=(c-2)\left(c^2+5c+6\right)$
$=(c-2)(c+2)(c+3)$
Therefore $c^4+c^3-10c^2-4c+24=(c-2)^2(c+2)(c+3)$
$(c-2)^2(c+2)(c+3)=0$
$c-2=0$ or $c+2=0$ or $c+3=0$
$c=2 \quad c=-2 \quad c=-3$

d $p^4-5p^3+5p^2+5p-6=0$
Let $P(p)=p^4-5p^3+5p^2+5p-6$
$P(1)=1^4-5(1)^3+5(1)^2+5(1)-6$
$=1-5+5$
$=0$
Thus $(p-1)$ is a factor.
$p^4-5p^3+5p^2+5p-6=(p-1)\left(p^3+4p^2+p+8\right)$
Let $Q(p)=p^3-4p^2+p+6$
$Q(2)=2^3-4(2)^2+2+6$
$=8-16+8$
$=0$
Thus $(p-2)$ is a factor.
$p^3-4p^2+p+6=(p-2)\left(p^2-2p-3\right)$
$=(p-2)(p+1)(p-3)$
Therefore
$p^4-5p^3+5p^2+5p-6=(p-1)(p-2)(p+1)(p-3)$
$(p-1)(p-2)(p+1)(p-3)=0$
$p-1=0$ or $p-2=0$ or $p+1=0$ or $p-3=0$
$p=1 \quad p=2 \quad p=-1 \quad p=3$

16 a Let $P(b)=b^3+5b^2+2b-8$
$P(1)=1^3+5(1)^2+2(1)-8=8-8=0$
Thus $b-1$ is a factor.
$b^3+5b^2+2b-8=(b-1)\left(b^2+6b+8\right)$
$=(b-1)(b+2)(b+4)$
If $b^3+5b^2+2b-8=0$
$(b-1)(b+2)(b+4)=0$
$b-1=0$ or $b+2=0$ or $b+4=0$
$b=1 \quad b=-2 \quad b=-4$

b $-2m^3+9m^2-m-12=0$
$2m^3-9m^2+m+12=0$
Let $P(m)=2m^3-9m^2+m+12$
$P(-1)=2(-1)^3-9(-1)^2-1+12$
$=-2-9-1+12$
$=-12+12$
$=0$
Thus $m+1$ is a factor.
$2m^3-9m^2+m+12=(m+1)\left(2m^2-11m+12\right)$
$=(m+1)(2m-3)(m-4)$
If $2m^3-9m^2+m+12=0$
$(m+1)(2m-3)(m-4)=0$
$m+1=0$ or $2m-3=0$ or $m-4=0$
$m=-1 \quad m=\frac{3}{2} \quad m=4$

c Let $P(x)=2x^3-x^2-6x+3$
$P\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3-\left(\frac{1}{2}\right)^2-6\left(\frac{1}{2}\right)+3$
$=\frac{1}{4}-\frac{1}{4}-3+3$
$=0$
Thus $(2x-1)$ is a factor.
$2x^3-x^2-6x+3=(2x-1)\left(x^2-3\right)$
If $2x^3-x^2-6x+3=0$
$(2x-1)\left(x^2-3\right)=0$
$2x-1=0$ or $x^2-3=0$
$x=\frac{1}{2} \quad x^2=3$
$x=\pm\sqrt{3}$

17 a Let $P(t)=3t^3+22t^2+37t+10$
$P(-5)=3(-5)^3+22(-5)^2+37(-5)+10$
$=-375+550-185+10$
$=-560+560$
$=0$
Thus $t+5$ is a factor.
$3t^3+22t^2+37t+10=(t+5)\left(3t^2+7t+2\right)$
$=(t+5)(t+2)(3t+1)$
If $3t^3+22t^2+37t+10=0$
$(t+5)(t+2)(3t+1)=0$
$t+5=0$ or $t+2=0$ or $3t+1=0$
$t=-5 \quad t=-2 \quad t=-\frac{1}{3}$

b Let $P(d)=3d^3-16d^2+12d+16$
$P(2)=3(2)^3-16(2)^2+12(2)+16$
$=24-64+24+16$
$=64-64$
$=0$
Thus $d-2$ is a factor.
$3d^3-16d^2+12d+16=(d-2)\left(3d^2-10d-8\right)$
$=(d-2)(d-4)(3d+2)$

If $3d^3-16d^2+12d+16=0$
$(d-2)(d-4)(3d+2)=0$
$d-2=0$ or $d-4=0$ or $3d+2=0$
$d=2 \quad d=4 \quad d=-\frac{2}{3}$

18 a $a^4-10a^2+9=0$
$\left(a^2-1\right)\left(a^2-9\right)=0$
$(a-1)(a+1)(a-3)(a+3)=0$
$a-1=0$ or $a+1=0$ or $a-3=0$ or $a+3=0$
$a=1 \quad a=-1 \quad a=3 \quad a=-3$

b $4k^4-101k^2+25=0$
$\left(4k^2-1\right)\left(k^2-25\right)=0$
$(2k-1)(2k+1)(k-5)(k+5)=0$
$2k-1=0$ or $2k-1=0$ or $k-5=0$ or $k+5=0$
$k=\frac{1}{2} \quad k=-\frac{1}{2} \quad k=5 \quad k=-5$

c $9z^4-145z^2+16=0$
$\left(9z^2-1\right)\left(z^2-16\right)=0$
$(3z-1)(3z+1)(z-4)(z+4)=0$
$3z-1=0$ or $3z+1=0$ or $z-4=0$ or $z+4=0$
$z=\frac{1}{3} \quad z=-\frac{1}{3} \quad z=4 \quad z=-4$

d $\left(x^2-2x\right)^2-47\left(x^2-2x\right)-48=0$
Let $A=\left(x^2-2x\right)$
$A^2-47A-48=0$
$(A-48)(A+1)=0$
$\left(x^2-2x-48\right)\left(x^2-2x+1\right)=0$
$(x-8)(x+6)(x-1)^2=0$
$x-8=0$ or $x+6=0$ or $x-1=0$
$x=8 \quad x=-6 \quad x=1$

19 a $5z^3-3z^2+4z-1\equiv az^3+bz^2+cz+d$
$a=5,\ b=-3,\ c=4$ and $d=-1$

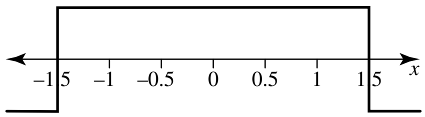
b $x^3-6x^2+9x-1\equiv x(x+a)^2-b$
$x^3-6x^2+9x-1\equiv x(x^2+2ax+a^2)-b$
$x^3-6x^2+9x-1\equiv x^3+2ax^2+a^2x-b$
$2a=-6 \quad b=1$
$a=-3$

20 $2x^3-5x^2+5x-5\equiv a(x-1)^3+b(x-1)^2+c(x-1)+d$
$\equiv a\left(x^3-3x^2+3x-1\right)+b\left(x^2-2x+1\right)+cx-c+d$
$\equiv ax^3-3ax^2+3ax-a+bx^2-2bx+b+cx-c+d$
$\equiv ax^3+(-3a+b)x^2+(3a-2b+c)x+(-a+b-c+d)$

Equating coefficients
$a=2 \quad -3a+b=-5 \quad 3a-2b+c=5 \quad -a+b-c+d=-5$
$-3(2)+b=-5 \quad 3(2)-2(1)+c=5 \quad -2+1-1+d=-5$
$-6+b=-5 \quad 6-2+c=5 \quad -2+d=-5$
$b=1 \quad 4+c=5 \quad d=-3$
$c=1$

Thus $2x^3-5x^2+5x-5\equiv 2(x-1)^3+(x-1)^2+(x-1)-3$

21 a $kx^2-3x+k=0$ has no solutions if the discriminant is less than zero.
$\Delta<0$
$(-3)^2-4(k)(k)<0$
$9-4k^2<0$
$(3-2k)(3+2k)<0$



Thus $\{k : k < -1.5\} \cup \{k : k > 1.5\}$

b $kx^2 + 4x - k + 2 = 0$

$\Delta = 16 - 4 \times k \times (-k + 2)$

$= 16 + 4k^2 - 8k$

$= 4(k^2 - 2k + 4)$

$= 4(k^2 - 2k + 1^2 - 1^2 + 4)$

$= 4\left[(k+1)^2 + 3\right]$

$= 4(k+1)^2 + 12$

As $(k+1)^2 > 0$,

$\therefore 4(k+1)^2 > 0$

and $4(k+1)^2 + 12 > 0$

Δ is always greater than zero, therefore the equation will always have a solution for all values of k.

22 $P(x) = ax^3 + bx^2 - 4x - 3$ where $(x+3)$ and $(x-1)$ are factors.

$a(-3)^3 + b(-3)^2 - 4(-3) - 3 = 0$

$-27a + 9b = -9$

$3a - b = 1$ ....(1)

$a(1)^3 + b(1)^2 - 4(1) - 3 = 0$

$a + b = 7$ ....(2)

(1) + (2)

$4a = 8$

$a = 2$

Substitute $a = 2$ into (2) $2 + b = 7$ so $b = 5$.

23 $P(x) = x^4 + ax^3 + bx^2 + cx + 24$ where $(x+2)$, $(x-3)$ and $(x+4)$ are factors.

$(-2)^4 + a(-2)^3 + b(-2)^2 + c(-2) + 24 = 0$

$16 - 8a + 4b - 2c + 24 = 0$

$-8a + 4b - 2c = -40$

$4a - 2b + c = 20$ ....(1)

$(3)^4 + a(3)^3 + b(3)^2 + c(3) + 24 = 0$

$81 + 27a + 9b + 3c + 24 = 0$

$27a + 9b + 3c = -105$

$9a + 3b + c = -35$ ....(2)

$(-4)^4 + a(-4)^3 + b(-4)^2 + c(-4) + 24 = 0$

$256 - 64a + 16b - 4c + 24 = 0$

$-64a + 16b - 4c = -280$

$16a - 4b + c = 70$ ....(3)

Solve using CAS: $a = 2,\ b = -13$ and $c = -14$

24 $(m-1)x^2 + \left(\dfrac{5-2m}{2}\right)x + 2m = 0$

This has two solutions if the discriminant is greater than zero.

$b^2 - 4ac > 0$

$\left(\dfrac{5-2m}{2}\right)^2 - 4(m-1)2m > 0$

Using CAS:

$-7m^2 + 3m + \dfrac{25}{4} > 0$

$m \in \left(\dfrac{3-2\sqrt{46}}{14}, \dfrac{3+2\sqrt{46}}{14}\right) \setminus \{1\}$

Note: $m \neq 1$ as the coefficient of x^2 would be zero if $m = 1$, therefore no parabola would exist.

Exercise 1.3 — Trigonometric symmetry properties

1 a $\sin(2\pi - \theta) = -\sin(\theta) = -0.4695$

b $\cos(\pi - \alpha) = -\cos(\alpha) = -0.5592$

c $\tan(-\beta) = -\tan(\beta) = -0.2680$

d $\sin(\pi + \theta) = -\sin(\theta) = -0.4695$

e $\cos(2\pi - \alpha) = \cos(\alpha) = 0.5592$

f $\tan(\pi + \beta) = \tan(\beta) = 0.2680$

2 a $\sin^2(\theta) + \cos^2(\theta) = 1$

$\cos^2(\theta) = 1 - \sin^2(\theta)$

$\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$

$\cos(\theta) = \pm\sqrt{1 - (0.4695)^2}$

$\cos(\theta) = \pm 0.8829$

As we are dealing with the fourth quadrant $\cos(\theta) = 0.8829$.

Thus $\cos(-\theta) = 0.8829$

b $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{0.4695}{0.8829} = 0.5318$

$\tan(180° - \theta) = -\tan(\theta) = -0.5318$

c $\sin^2(\alpha) + \cos^2(\alpha) = 1$

$\sin^2(\alpha) = 1 - \cos^2(\alpha)$

$\sin(\alpha) = \pm\sqrt{1 - \cos^2(\alpha)}$

$\sin(\alpha) = \pm\sqrt{1 - (0.5592)^2}$

$\sin(\alpha) = \pm 0.8290$

As we are dealing with the first quadrant $\sin(\alpha) = 0.8290$.

d $\tan(360° - \alpha) = -\tan(\alpha)$

$\tan(360° - \alpha) = -\dfrac{\sin(\alpha)}{\cos(\alpha)}$

$\tan(360° - \alpha) = -\dfrac{0.8290}{0.5592}$

$\tan(360° - \alpha) = -1.4825$

3 a $\sin\left(\dfrac{\pi}{2} + \theta\right) = \cos(\theta) = 0.8829$

b $\cos\left(\dfrac{3\pi}{2} - \theta\right) = -\sin(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\sin^2(\theta) = 1 - \cos^2(\theta)$

$\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$

$\sin(\theta) = \sqrt{1 - (0.8829)^2}$

$\sin(\theta) = 0.4696$

$\cos\left(\dfrac{3\pi}{2} - \theta\right) = -\sin(\theta) = -0.4696$

c $\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\sin\left(\dfrac{\pi}{2} - \theta\right)}{\cos\left(\dfrac{\pi}{2} - \theta\right)}$

$= \dfrac{\cos(\theta)}{\sin(\theta)}$

$= \dfrac{0.8829}{0.4695}$

$= 1.8803$

d $\sin\left(\frac{3\pi}{2}+\alpha\right)=-\cos(\alpha)$

$\sin^2(\alpha)+\cos^2(\alpha)=1$

$\cos^2(\alpha)=1-\sin^2(\alpha)$

$\cos^2(\alpha)=\sqrt{1-\sin^2(\alpha)}$

$\cos(\alpha)=\sqrt{1-(0.1736)^2}$

$\cos(\alpha)=0.9848$

$\sin\left(\frac{3\pi}{2}+\alpha\right)=-\cos(\alpha)=-0.9848$

e $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos(\alpha)=0.9848$

f $\tan\left(\frac{3\pi}{2}+\alpha\right)=-\cot(\alpha)$

$$\tan\left(\frac{3\pi}{2}+\theta\right)=\frac{-\sin\left(\frac{3\pi}{2}+\theta\right)}{\cos\left(\frac{3\pi}{2}+\theta\right)}=\frac{-\cos(\theta)}{\sin(\theta)}=\frac{-0.9848}{0.1736}=-5.6729$$

4 $\sin^2(\theta)+\cos^2(\theta)=1$

$\cos^2(\theta)=1-\sin^2(\theta)$

$\cos(\theta)=\sqrt{1-\sin^2(\theta)}$

$\cos(\theta)=\pm\sqrt{1-(0.8290)^2}$

$\cos(\theta)=\pm0.5592$

Since we are dealing with the first quadrant $\cos(\theta)=0.5592$.

$\sin^2(\beta)+\cos^2(\beta)=1$

$\sin^2(\beta)=1-\cos^2(\beta)$

$\sin(\beta)=\sqrt{1-\cos^2(\beta)}$

$\sin(\beta)=\pm\sqrt{1-(0.7547)^2}$

$\sin(\beta)=\pm0.6561$

Since we are dealing with the first quadrant $\sin(\beta)=0.6561$.

a $\sin(90°-\theta)=\cos(\theta)=0.5592$

b $\cos(270°+\theta)=\sin(\theta)=0.8290$

c $$\tan(90°+\theta)=\frac{\sin(90°+\theta)}{-\cos(90°+\theta)}=\frac{\cos(\theta)}{-\sin(\theta)}=\frac{-0.5592}{0.8290}=-0.6746$$

d $\sin(270°-\beta)=-\cos(\beta)=-0.7547$

e $$\tan(90°-\beta)=\frac{\sin(90°-\beta)}{\cos(90°-\beta)}=\frac{\cos(\beta)}{\sin(\beta)}=\frac{0.7547}{0.6561}=1.1503$$

f $\cos(270°-\beta)=-\sin(\beta)=-0.6561$

5 a $\tan\left(\frac{3\pi}{4}\right)=\tan\left(\pi-\frac{\pi}{4}\right)=-\tan\left(\frac{\pi}{4}\right)=-1$

b $\cos\left(\frac{5\pi}{6}\right)=\cos\left(\pi-\frac{\pi}{6}\right)=-\cos\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$

c $\sin\left(-\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

d $\cos\left(\frac{7\pi}{3}\right)=\cos\left(2\pi+\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

e $\tan\left(-\frac{\pi}{3}\right)=-\tan\left(\frac{\pi}{3}\right)=-\sqrt{3}$

f $\sin\left(\frac{11\pi}{6}\right)=\sin\left(2\pi-\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$

6 a $\tan\left(\frac{5\pi}{6}\right)=\tan\left(\pi-\frac{\pi}{6}\right)=-\tan\left(\frac{\pi}{6}\right)=-\frac{1}{\sqrt{3}}$

b $\cos\left(\frac{14\pi}{3}\right)=\cos\left(5\pi-\frac{\pi}{3}\right)=-\cos\left(\frac{\pi}{3}\right)=-\frac{1}{2}$

c $\tan\left(-\frac{5\pi}{4}\right)=-\tan\left(\frac{5\pi}{4}\right)=-\tan\left(\pi+\frac{\pi}{4}\right)=-\tan\left(\frac{\pi}{4}\right)=-1$

d $\cos\left(-\frac{3\pi}{4}\right)=\cos\left(\frac{3\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

e $\sin\left(-\frac{2\pi}{3}\right)=-\sin\left(\frac{2\pi}{3}\right)=-\sin\left(\pi-\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$

f $\sin\left(\frac{17\pi}{6}\right)=\sin\left(\frac{5\pi}{6}\right)=\sin\left(\pi-\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

7 a $\sin(\pi-\theta)=\sin(\theta)$

b $\cos(6\pi-\theta)=\cos(\theta)$

c $\tan(\pi+\theta)=\tan(\theta)$

d $\cos(-\theta)=\cos(\theta)$

e $\sin(180°+\theta)=-\sin(\theta)$

f $\tan(720°-\theta)=-\tan(\theta)$

8 a $\cos\left(\frac{\pi}{2}-\alpha\right)=\sin(\alpha)$

b $\tan(90°+\alpha)=-\frac{1}{\tan(\alpha)}$

c $\sin(270°-\alpha)=-\cos(\alpha)$

d $\tan\left(\frac{11\pi}{2}-\alpha\right)=\frac{1}{\tan(\alpha)}$

e $\cos\left(\frac{3\pi}{2}+\alpha\right)=\sin(\alpha)$

f $\sin(90°-\alpha)=\cos(\alpha)$

9 a $\sin(2\pi-\theta)=-\sin(\theta)=-0.9511$

b $\sin(\pi-\theta)=\sin(\theta)=0.9511$

c $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)=0.9511$

d $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$

$\sin^2(\theta)+\cos^2(\theta)=1$

$\cos^2(\theta)=1-\sin^2(\theta)$

$\cos(\theta)=\sqrt{1-\sin^2(\theta)}$

$\cos(\theta)=\sqrt{1-(0.9511)^2}$

$\cos(\theta)=0.3089$

$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}=\frac{0.9511}{0.3089}=3.0792$

e $\cos(3\pi+\theta)=-\cos(\theta)=-0.3089$

f $\tan(2\pi-\theta)=-\tan(\theta)=-3.0792$

10 **a** $\cos(180°-\alpha)=-\cos(\alpha)=-0.8572$

b $\cos(-\alpha)=\cos(\alpha)=0.8572$

c $\sin\left(\frac{3\pi}{2}+\alpha\right)=-\cos(\alpha)=-0.8572$

d $\tan(180°-\alpha)=-\tan(\alpha)$

$$\begin{aligned}\sin^2(\alpha)+\cos^2(\alpha)&=1\\ \sin^2(\alpha)&=1-\cos^2(\alpha)\\ \sin(\alpha)&=\sqrt{1-\cos^2(\alpha)}\\ \sin(\alpha)&=\sqrt{1-(0.8572)^2}\\ \sin(\alpha)&=0.5150\\ \tan(\alpha)&=\frac{\sin(\alpha)}{\cos(\alpha)}=\frac{0.515}{0.8572}=0.6008\\ \tan(180°-\alpha)&=-\tan(\alpha)\\ &=-0.6008\end{aligned}$$

e $\cos(360°-\alpha)=\cos(\alpha)=0.8572$

f $$\begin{aligned}\tan\left(\frac{\pi}{2}+\alpha\right)&=\frac{\sin\left(\frac{\pi}{2}+\alpha\right)}{\cos\left(\frac{\pi}{2}+\alpha\right)}\\ &=\frac{\cos(\alpha)}{-\sin(\alpha)}\\ &=\frac{0.8572}{-0.5150}\\ &=-1.6645\end{aligned}$$

11 For the right angle triangle of $(3,4,5)$ in the first quadrant

$\sin(\beta)=\frac{4}{5}$, $\cos(\beta)=\frac{3}{5}$ and $\tan(\beta)=\frac{4}{3}$

a $\cos(\beta)=-\frac{3}{5}$

b $\tan(\beta)=-\frac{4}{3}$

c $$\begin{aligned}\sin^2(\beta)+\cos^2(\beta)&=\left(\frac{4}{5}\right)^2+\left(-\frac{3}{5}\right)^2\\ \sin^2(\beta)+\cos^2(\beta)&=\frac{16}{25}+\frac{9}{25}\\ \sin^2(\beta)+\cos^2(\beta)&=1\end{aligned}$$

d $$\begin{aligned}\cos^2(\beta)-\sin^2(\beta)&=\left(-\frac{3}{5}\right)^2-\left(\frac{4}{5}\right)^2\\ \cos^2(\beta)-\sin^2(\beta)&=\frac{9}{25}-\frac{16}{25}=-\frac{7}{25}\end{aligned}$$

12 **a** $\sin(\theta)=\frac{5}{13}$

b $\tan(\theta)=\frac{5}{12}$

c $\cos(\theta)=\frac{12}{13}$

d $\sin(90°-\theta)=\cos(\theta)=\frac{12}{13}$

e $\cos(90°-\theta)=\sin(\theta)=\frac{5}{13}$

f $\tan(90°-\theta)=\frac{1}{\tan(\theta)}=\frac{12}{5}=2.4$

13 **a** $\sin\left(\frac{7\pi}{3}\right)=\sin\left(2\pi+\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$

b $\cos\left(\frac{7\pi}{3}\right)=\cos\left(2\pi+\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

c $\tan\left(\frac{5\pi}{6}\right)=\tan\left(\pi-\frac{\pi}{6}\right)=-\tan\left(\frac{\pi}{6}\right)=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$

d $\sin(150°)=\sin(180°-30°)=\sin(30°)=\frac{1}{2}$

e $\cos\left(\frac{7\pi}{6}\right)=\cos\left(\pi+\frac{\pi}{6}\right)=-\cos\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$

f $\tan\left(-\frac{7\pi}{6}\right)=\tan\left(-\frac{\pi}{6}\right)=-\tan\left(\frac{\pi}{6}\right)=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$

g $\cos\left(\frac{\pi}{2}\right)=0$

h $\tan(270°)=$ undefined

i $\sin(-4\pi)=0$

j $\tan(\pi)=0$

k $\cos(-6\pi)=1$

l $\sin\left(\frac{3\pi}{2}\right)=-1$

14 **a** $$\begin{aligned}\cos\left(\frac{7\pi}{6}\right)+\cos\left(\frac{2\pi}{3}\right)&=\cos\left(\pi+\frac{\pi}{6}\right)+\cos\left(\pi-\frac{\pi}{3}\right)\\ &=-\cos\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{3}\right)\\ &=-\frac{\sqrt{3}}{2}-\frac{1}{2}=-\frac{\left(\sqrt{3}+1\right)}{2}\end{aligned}$$

b $$\begin{aligned}2\sin\left(\frac{7\pi}{4}\right)+4\sin\left(\frac{5\pi}{6}\right)&=2\sin\left(2\pi-\frac{\pi}{4}\right)+4\sin\left(\pi-\frac{\pi}{6}\right)\\ &=-2\sin\left(\frac{\pi}{4}\right)+4\sin\left(\frac{\pi}{6}\right)\\ &=-2\times\frac{\sqrt{2}}{2}+4\times\frac{1}{2}=-\sqrt{2}+2\end{aligned}$$

c $$\begin{aligned}\sqrt{3}\tan\left(\frac{5\pi}{4}\right)-\tan\left(\frac{5\pi}{3}\right)&=\sqrt{3}\tan\left(\pi+\frac{\pi}{4}\right)-\tan\left(2\pi-\frac{\pi}{3}\right)\\ &=\sqrt{3}\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{3}\right)\\ &=\sqrt{3}+\sqrt{3}=2\sqrt{3}\end{aligned}$$

d $$\begin{aligned}\sin^2\left(\frac{8\pi}{3}\right)+\sin\left(\frac{9\pi}{4}\right)&=\sin^2\left(3\pi-\frac{\pi}{3}\right)+\sin\left(2\pi+\frac{\pi}{4}\right)\\ &=\sin^2\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{4}\right)\\ &=\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{\sqrt{2}}\\ &=\frac{3+2\sqrt{2}}{4}\end{aligned}$$

e $$\begin{aligned}2\cos^2\left(-\frac{5\pi}{4}\right)-1&=2\cos^2\left(\frac{5\pi}{4}\right)-1\\ &=2\left(-\frac{1}{\sqrt{2}}\right)^2-1\\ &=1-1\\ &=0\end{aligned}$$

f $$\frac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)} = \frac{\tan\left(4\pi+\frac{\pi}{4}\right)\cos(-\pi)}{-\sin\left(\frac{11\pi}{6}\right)}$$
$$= \frac{\tan\left(\frac{\pi}{4}\right)\times -1}{-\sin\left(\frac{\pi}{6}\right)}$$
$$= (1\times -1) \div -\frac{1}{2}$$
$$= 2$$

15 a $\sin^2(x)+\cos^2(x)=1$

$$\frac{\sin^2(x)}{\cos^2(x)}+\frac{\cos^2(x)}{\cos^2(x)}=\frac{1}{\cos^2(x)}$$
$$\tan^2(x)+1=\frac{1}{\cos^2(x)} \text{ as required}$$

b $\sin(x)=0.6157$

$$(0.6157)^2+\cos^2(x)=1$$
$$0.3791+\cos^2(x)=1$$
$$\cos^2(x)=1-0.3791$$
$$\cos(x)=\sqrt{1-0.3791}$$
$$\cos(x)=0.788$$
$$\tan^2(x)+1=\frac{1}{(0.788)^2}$$
$$\tan^2(x)=1.6105-1$$
$$\tan(x)=\sqrt{0.6165}=0.7814$$

16 Let height of tree be h metres and length of shadow be s metres.

$$\sin\left(\frac{\pi}{3}\right)=\frac{h}{30} \qquad \cos\left(\frac{\pi}{3}\right)=\frac{s}{30}$$
$$30\times\frac{\sqrt{3}}{2}=h \qquad 30\times\frac{1}{2}=s$$
$$15\sqrt{3}=h \qquad 15=s$$

Height of tree is $15\sqrt{3}$ metres and the length of the shadow is 15 metres.

17 a $v=12+3\sin\left(\frac{\pi t}{3}\right)$

Initially $t=0$

$v=12+3\sin(0)=12$ cm/s

b When $t=5$

$$v=12+3\sin\left(\frac{5\pi}{3}\right)$$
$$v=12+3\sin\left(2\pi-\frac{\pi}{3}\right)$$
$$v=12-3\sin\left(\frac{\pi}{3}\right)$$
$$v=12-\frac{3\sqrt{3}}{2} \text{ cm/s}$$

c When $t=12$

$$v=12+3\sin\left(\frac{12\pi}{3}\right)$$
$$v=12+3\sin(4\pi)=12 \text{ cm/s}$$

18 $h(t)=0.5\cos\left(\frac{\pi t}{12}\right)+1.0$

a At 6 am $t=0$

$h(0)=0.5\cos(0)+1.0=1.5$ m or $\frac{3}{2}$ m

b At 2 pm $t=8$

$$h(8)=0.5\cos\left(\frac{8\pi}{12}\right)+1.0$$
$$h(8)=0.5\cos\left(\frac{4\pi}{3}\right)+1.0$$
$$h(8)=0.5\cos\left(\pi-\frac{\pi}{3}\right)+1.0$$
$$h(8)=0.5\cos\left(\frac{\pi}{3}\right)+1.0$$
$$h(8)=\frac{1}{2}\times-\frac{1}{2}+1=0.75 \text{ m or } \frac{3}{4} \text{ m}$$

c At 10 pm $t=16$

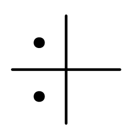
$$h(16)=0.5\cos\left(\frac{16\pi}{12}\right)+1.0$$
$$h(16)=0.5\cos\left(\frac{4\pi}{3}\right)+1.0$$
$$h(16)=0.5\cos\left(\pi+\frac{\pi}{3}\right)+1.0$$
$$h(16)=-0.5\cos\left(\frac{\pi}{3}\right)+1.0$$
$$h(16)=-\frac{1}{2}\times\frac{1}{2}+1=0.75 \text{ m or } \frac{3}{4} \text{ m}$$

Exercise 1.4 — Trigonometric equations and general solutions

1 a $2\cos(\theta)+\sqrt{3}=0 \qquad 0\le\theta\le 2\pi$

$$\cos(\theta)=-\frac{\sqrt{3}}{2}$$

$\frac{\sqrt{3}}{2}$ suggests 60°. Since cos is negative

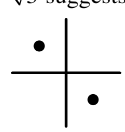


$$\theta=\pi-\frac{\pi}{6},\ \pi+\frac{\pi}{6}$$
$$\theta=\frac{5\pi}{6},\ \frac{7\pi}{6}$$

b $\tan(x)+\sqrt{3}=0 \qquad 0\le x\le 720°$

$\tan(x)=-\sqrt{3}$

$\sqrt{3}$ suggests 60°. Since tan is negative



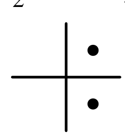
$x=180°-60°,\ 360°-60°,\ 540°-60°,\ 720°-60°$

$x=120°,\ 300°,\ 480°,\ 660°$

c $2\cos(\theta)=1 \qquad -\pi\le\theta\le\pi$

$$\cos(\theta)=\frac{1}{2}$$

$\frac{1}{2}$ suggests $\frac{\pi}{3}$. Since cos is positive

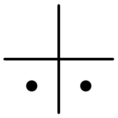


$$\theta=-\frac{\pi}{3},\ \frac{\pi}{3}$$

2 **a** $\sin(\theta)+0.5768=0$

$\sin(\theta)=-0.5768$

0.5768 suggests 36.2258°. Since sin is negative



$\theta=180°+36.2258°,\ 360°-36.2258°$

$\theta=215.23°,\ 324.77°$

Or solve on CAS

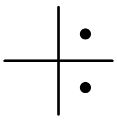
b $\sin(x)=1$

$x=-\frac{3\pi}{2},\ \frac{\pi}{2}$ for $-2\pi\le x\le 2\pi$.

3 **a** $2\cos(3\theta)-\sqrt{2}=0 \qquad 0\le\theta\le 2\pi$

$\cos(3\theta)=\frac{\sqrt{2}}{2} \qquad 0\le 3\theta\le 6\pi$

$\frac{\sqrt{2}}{2}$ suggests $\frac{\pi}{4}$. Since cos is positive



$3\theta=\frac{\pi}{4},\ 2\pi-\frac{\pi}{4},\ 2\pi+\frac{\pi}{4},\ 4\pi-\frac{\pi}{4},\ 4\pi+\frac{\pi}{4},\ 6\pi-\frac{\pi}{4}$

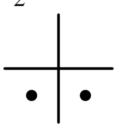
$3\theta=\frac{\pi}{4},\ \frac{7\pi}{4},\ \frac{9\pi}{4},\ \frac{15\pi}{4},\ \frac{17\pi}{4},\ \frac{23\pi}{4}$

$\theta=\frac{\pi}{12},\ \frac{7\pi}{12},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{17\pi}{12},\ \frac{23\pi}{12}$

b $2\sin(2x+\pi)+\sqrt{3}=0 \qquad -\pi\le x\le\pi$

$\sin(2x+\pi)=-\frac{\sqrt{3}}{2} \qquad -\pi\le 2x+\pi\le 3\pi$

$\frac{\sqrt{3}}{2}$ suggests $\frac{\pi}{3}$. Since sin is negative



$2x+\pi=-\pi+\frac{\pi}{3},\ -\frac{\pi}{3},\ \pi+\frac{\pi}{3},\ 2\pi-\frac{\pi}{3}$

$2x+\pi=-\frac{2\pi}{3},\ -\frac{\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}$

$2x=-\frac{2\pi}{3}-\pi,\ -\frac{\pi}{3}-\pi,\ \frac{4\pi}{3}-\pi,\ \frac{5\pi}{3}-\pi$

$2x=-\frac{5\pi}{3},\ -\frac{4\pi}{3},\ \frac{\pi}{3},\ \frac{2\pi}{3}$

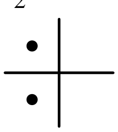
$x=-\frac{5\pi}{6},\ -\frac{2\pi}{3},\ \frac{\pi}{6},\ \frac{\pi}{3}$

4 $2\cos\left(3\theta-\frac{\pi}{2}\right)+\sqrt{3}=0 \qquad 0\le\theta\le 2\pi$

$\cos\left(3\theta-\frac{\pi}{2}\right)=-\frac{\sqrt{3}}{2} \qquad 0\le 3\theta\le 6\pi$

$\cos\left(3\theta-\frac{\pi}{2}\right)=-\frac{\sqrt{3}}{2} \qquad -\frac{\pi}{2}\le 3\theta-\frac{\pi}{2}\le 6\pi-\frac{\pi}{2}$

$\frac{\sqrt{3}}{2}$ suggests $\frac{\pi}{6}$, Since cos is negative



$$3\theta-\frac{\pi}{2}=\pi-\frac{\pi}{6},\ \pi+\frac{\pi}{6},\ 3\pi-\frac{\pi}{6},\ 3\pi+\frac{\pi}{6},\ 5\pi-\frac{\pi}{6},\ 5\pi+\frac{\pi}{6}$$
$$3\theta-\frac{\pi}{2}=\frac{5\pi}{6},\frac{7\pi}{6},\frac{17\pi}{6},\frac{19\pi}{6},\frac{29\pi}{6},\frac{31\pi}{6}$$
$$3\theta=\frac{5\pi}{6}+\frac{3\pi}{6},\ \frac{7\pi}{6}+\frac{3\pi}{6},\ \frac{17\pi}{6}+\frac{3\pi}{6},\ \frac{19\pi}{6}+\frac{3\pi}{6},\ \frac{29\pi}{6}+\frac{3\pi}{6},\ \frac{31\pi}{6}+\frac{3\pi}{6}$$
$$3\theta=\frac{4\pi}{3},\frac{5\pi}{3},\frac{10\pi}{3},\frac{11\pi}{3},\frac{16\pi}{3},\frac{17\pi}{3}$$
$$\theta=\frac{4\pi}{9},\frac{5\pi}{9},\frac{10\pi}{9},\frac{11\pi}{9},\frac{16\pi}{9},\frac{17\pi}{9}$$

5 $\cos^2(\theta)-\sin(\theta)\cos(\theta)=0 \qquad 0\le\theta\le 2\pi$

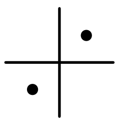
$\cos(\theta)\left(\cos(\theta)-\sin(\theta)\right)=0$

$\cos(\theta)=0$ or $\cos(\theta)-\sin(\theta)=0$

$\theta=\frac{\pi}{2},\frac{3\pi}{2}$ $\qquad \cos(\theta)=\sin(\theta)$

$$\frac{\cos(\theta)}{\cos(\theta)}=\frac{\sin(\theta)}{\cos(\theta)}$$
$$1=\tan(\theta)$$

1 suggests $\frac{\pi}{4}$. Since tan is positive



$\theta=\frac{\pi}{4},\ \pi+\frac{\pi}{4}$

$\theta=\frac{\pi}{4},\ \frac{5\pi}{4}$

Therefore $\theta=\frac{\pi}{4},\frac{\pi}{2},\frac{5\pi}{4},\frac{3\pi}{2}$

6 $2\cos^2(\theta)+3\cos(\theta)=-1 \qquad 0\le\theta\le 2\pi$

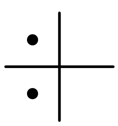
$2\cos^2(\theta)+3\cos(\theta)+1=0 \qquad 0\le\theta\le 2\pi$

$\left(2\cos(\theta)+1\right)\left(\cos(\theta)+1\right)=0$

$2\cos(\theta)+1=0$ or $\cos(\theta)+1=0$

$\cos(\theta)=-\frac{1}{2}$ $\qquad \cos(\theta)=-1$ so $\theta=\pi$

$\frac{1}{2}$ suggests $\frac{\pi}{3}$. Since cos is negative



$\theta=\pi-\frac{\pi}{3},\ \pi+\frac{\pi}{3}$

$\theta=\frac{2\pi}{3},\ \frac{4\pi}{3}$

Therefore $\theta=\frac{2\pi}{3},\ \pi,\ \frac{4\pi}{3}$

7 $2\sin(\theta)-\sqrt{3}=0$

$\sin(\theta)=\frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2}$ suggests $\frac{\pi}{3}$.

$\theta=2n\pi+\frac{\pi}{3}$ and $(2n+1)\pi-\frac{\pi}{3},\ n\in Z$

$=\frac{6n\pi+\pi}{3},\frac{3(2n+1)\pi-\pi}{3}$

$=\frac{6n\pi+\pi}{3},\frac{6n\pi+2\pi}{3},\ n\in Z$

8 $\sqrt{3}\tan(2\theta)+1=0$

$\tan(2\theta)=-\dfrac{1}{\sqrt{3}}$

Basic angle for $-\dfrac{1}{\sqrt{3}}$ is $-\dfrac{\pi}{6}$ (quadrant 4)

$2\theta=n\pi+\left(-\dfrac{\pi}{6}\right), n\in Z$

$=\dfrac{6n\pi-\pi}{6}$

$=\dfrac{6n\pi-\pi}{12}, n\in Z$

For $\theta\in[-\pi,\pi]$ solutions are:

$n=-2;\quad \theta=-\dfrac{13\pi}{12}$

$n=-1;\quad \theta=-\dfrac{7\pi}{12}$

$n=0;\quad \theta=-\dfrac{\pi}{12}$

$n=1;\quad \theta=\dfrac{5\pi}{12}$

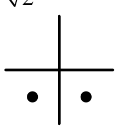
$n=2;\quad \theta=\dfrac{11\pi}{12}$

$\therefore \theta=-\dfrac{7\pi}{12},-\dfrac{\pi}{12},\dfrac{5\pi}{12},\dfrac{11\pi}{12}$

9 a $\sqrt{2}\sin(\theta)=-1 \qquad 0\le\theta\le2\pi$

$\sin(\theta)=-\dfrac{1}{\sqrt{2}}$

$\dfrac{1}{\sqrt{2}}$ suggests $\dfrac{\pi}{4}$. Since sin is negative



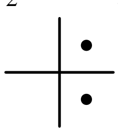
$\theta=\pi+\dfrac{\pi}{4}, 2\pi-\dfrac{\pi}{4}$

$\theta=\dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

b $2\cos(\theta)=1 \qquad 0\le\theta\le2\pi$

$\cos(\theta)=\dfrac{1}{2}$

$\dfrac{1}{2}$ suggests $\dfrac{\pi}{3}$. Since cos is positive



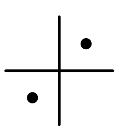
$\theta=\dfrac{\pi}{3}, 2\pi-\dfrac{\pi}{3}$

$\theta=\dfrac{\pi}{3}, \dfrac{5\pi}{3}$

c $\tan(3\theta)-\sqrt{3}=0 \qquad 0\le\theta\le2\pi$

$\tan(3\theta)=\sqrt{3} \qquad 0\le3\theta\le6\pi$

$\sqrt{3}$ suggests $\dfrac{\pi}{3}$. Since tan is positive



$3\theta=\dfrac{\pi}{3},\ \pi+\dfrac{\pi}{3},\ 2\pi+\dfrac{\pi}{3},\ 3\pi+\dfrac{\pi}{3},\ 4\pi+\dfrac{\pi}{3},\ 5\pi+\dfrac{\pi}{3}$

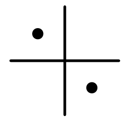
$3\theta=\dfrac{\pi}{3},\ \dfrac{4\pi}{3},\ \dfrac{7\pi}{3},\ \dfrac{10\pi}{3},\ \dfrac{13\pi}{3},\ \dfrac{16\pi}{3}$

$\theta=\dfrac{\pi}{9},\ \dfrac{4\pi}{9},\ \dfrac{7\pi}{9},\ \dfrac{10\pi}{9},\ \dfrac{13\pi}{9},\ \dfrac{16\pi}{9}$

d $\tan\left(\theta-\dfrac{\pi}{2}\right)+1=0 \qquad 0\le\theta\le2\pi$

$\tan\left(\theta-\dfrac{\pi}{2}\right)=-1 \qquad -\dfrac{\pi}{2}\le\theta-\dfrac{\pi}{2}\le2\pi-\dfrac{\pi}{2}$

1 suggests $\dfrac{\pi}{4}$. Since tan is negative



$\theta-\dfrac{\pi}{2}=-\dfrac{\pi}{4},\ \pi-\dfrac{\pi}{4}$

$\theta-\dfrac{\pi}{2}=-\dfrac{\pi}{4},\ \dfrac{3\pi}{4}$

$\theta=-\dfrac{\pi}{4}+\dfrac{\pi}{2},\ \dfrac{3\pi}{4}+\dfrac{\pi}{2}$

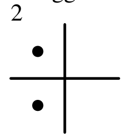
$\theta=\dfrac{\pi}{4},\ \dfrac{5\pi}{4}$

10 a $2\cos(x)+1=0 \qquad 0°\le x\le360°$

$2\cos(x)=-1$

$\cos(x)=-\dfrac{1}{2}$

$\dfrac{1}{2}$ suggests 60°. Since cos is negative



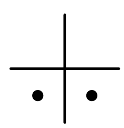
$x=180°-60°,\ 180°+60°$

$x=120°,\ 240°$

b $2\sin(2x)+\sqrt{2}=0 \qquad 0°\le x\le360°$

$\sin(2x)=-\dfrac{\sqrt{2}}{2} \qquad 0°\le2x\le720°$

$\dfrac{\sqrt{2}}{2}$ suggests 45°. Since sin is negative



$2x=180°+45°,\ 360°-45°,\ 540°+45°,\ 720°-45°$

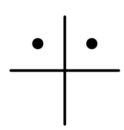
$2x=225°,\ 315°,\ 585°,\ 675°$

$x=112.5°,\ 157.5°,\ 292.5°,\ 337.5°$

11 a $3\sin(\theta)-2=0 \qquad 0\le\theta\le2\pi$

$\sin(\theta)=\dfrac{2}{3}$

$\dfrac{2}{3}$ suggests 0.7297^c. Since sin is positive

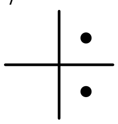


$\theta = 0.7297,\ \pi - 0.7297$
$\theta = 0.73,\ 2.41$
or solve on CAS

b $7\cos(x) - 2 = 0 \qquad 0° \le x \le 360°$

$\cos(x) = \frac{2}{7}$

$\frac{2}{7}$ suggests 73.3985°. Since cos is positive

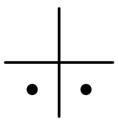


$x = 73.3985°,\ 360° - 73.3985°$
$x = 73.40°,\ 286.60°$
or solve on CAS

12 a $2\sin(2\theta) + \sqrt{3} = 0 \qquad -\pi \le \theta \le \pi$

$\sin(2\theta) = -\frac{\sqrt{3}}{2} \qquad -2\pi \le \theta \le 2\pi$

$\frac{\sqrt{3}}{2}$ suggests $\frac{\pi}{3}$. Since sin is negative



$2\theta = -\pi + \frac{\pi}{3},\ -\frac{\pi}{3},\ \pi + \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$

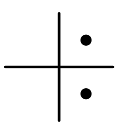
$2\theta = -\frac{2\pi}{3},\ -\frac{\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}$

$\theta = -\frac{\pi}{3},\ -\frac{\pi}{6},\ \frac{2\pi}{3},\ \frac{5\pi}{6}$

b $\sqrt{2}\cos(3\theta) = 1 \qquad -\pi \le \theta \le \pi$

$\cos(3\theta) = \frac{1}{\sqrt{2}} \qquad -3\pi \le \theta \le 3\pi$

$\frac{1}{\sqrt{2}}$ suggests $\frac{\pi}{4}$. Since cos is positive



$3\theta = -2\pi - \frac{\pi}{4},\ -2\pi + \frac{\pi}{4},\ -\frac{\pi}{4},\ \frac{\pi}{4},\ 2\pi - \frac{\pi}{4},\ 2\pi + \frac{\pi}{4}$

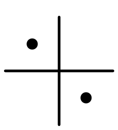
$3\theta = -\frac{9\pi}{4},\ -\frac{7\pi}{4},\ -\frac{\pi}{4},\ \frac{\pi}{4},\ \frac{7\pi}{4},\ \frac{9\pi}{4}$

$\theta = -\frac{3\pi}{4},\ -\frac{7\pi}{12},\ -\frac{\pi}{12},\ \frac{\pi}{12},\ \frac{7\pi}{12},\ \frac{3\pi}{4}$

c $\tan(2\theta) + 1 = 0 \qquad -\pi \le \theta \le \pi$

$\tan(2\theta) = -1 \qquad -2\pi \le \theta \le 2\pi$

1 suggests $\frac{\pi}{4}$. Since tan is negative



$2\theta = -\pi - \frac{\pi}{4},\ -\frac{\pi}{4},\ \pi - \frac{\pi}{4},\ 2\pi - \frac{\pi}{4}$

$2\theta = -\frac{5\pi}{4},\ -\frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{7\pi}{4}$

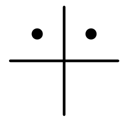
$\theta = -\frac{5\pi}{8},\ -\frac{\pi}{8},\ \frac{3\pi}{8},\ \frac{7\pi}{8}$

13 a $2\sin\left(2x + \frac{\pi}{4}\right) = \sqrt{2} \qquad -\pi \le x \le \pi$

$\sin\left(2x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad -2\pi \le 2x \le 2\pi$

$\sin\left(2x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad -2\pi + \frac{\pi}{4} \le 2x + \frac{\pi}{4} \le 2\pi + \frac{\pi}{4}$

$\frac{\sqrt{2}}{2}$ suggests $\frac{\pi}{4}$. Since sin is positive



$2x + \frac{\pi}{4} = -2\pi + \frac{\pi}{4},\ -\pi - \frac{\pi}{4},\ \frac{\pi}{4},\ \pi - \frac{\pi}{4},\ 2\pi + \frac{\pi}{4}$

$2x + \frac{\pi}{4} = -\frac{7\pi}{4},\ -\frac{5\pi}{4},\ \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{9\pi}{4}$

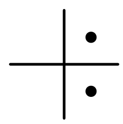
$2x = -2\pi,\ -\frac{3\pi}{2},\ 0,\ \frac{\pi}{2},\ 2\pi$

$x = -\pi,\ -\frac{3\pi}{4},\ 0,\ \frac{\pi}{4},\ \pi$

b $2\cos(x + \pi) = \sqrt{3} \qquad -\pi \le x \le \pi$

$\cos(x + \pi) = \frac{\sqrt{3}}{2} \qquad 0 \le x + \pi \le 2\pi$

$\frac{\sqrt{3}}{2}$ suggests $\frac{\pi}{6}$. Since cos is positive



$x + \pi = \frac{\pi}{6},\ 2\pi - \frac{\pi}{6}$

$x + \pi = \frac{\pi}{6},\ \frac{11\pi}{6}$

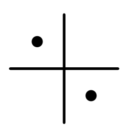
$x = \frac{\pi}{6} - \pi,\ \frac{11\pi}{6} - \pi$

$x = -\frac{5\pi}{6},\ \frac{5\pi}{6}$

c $\tan(x - \pi) = -1 \qquad -\pi \le x \le \pi$

$\tan(x - \pi) = -1 \qquad -2\pi \le x - \pi \le 0$

1 suggests $\frac{\pi}{4}$. Since tan is negative



$x - \pi = -\pi - \frac{\pi}{4},\ -\frac{\pi}{4}$

$x - \pi = -\frac{5\pi}{4},\ -\frac{\pi}{4}$

$x = -\frac{5\pi}{4} + \pi,\ -\frac{\pi}{4} + \pi$

$x = -\frac{\pi}{4},\ \frac{3\pi}{4}$

14 a $\tan^2(\theta) - 1 = 0 \qquad 0 \le \theta \le 2\pi$

$(\tan(\theta) - 1)(\tan(\theta) + 1) = 0$

$\tan(\theta) - 1 = 0$ or $\tan(\theta) + 1 = 0$

$\tan(\theta) = 1 \qquad \tan(\theta) = -1$

$\theta = \frac{\pi}{4},\ \frac{5\pi}{4} \qquad \theta = \frac{3\pi}{4},\ \frac{7\pi}{4}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b $4\sin^2(\theta) - \left(2 + 2\sqrt{3}\right)\sin(\theta) + \sqrt{3} = 0 \qquad 0 \le \theta \le 2\pi$

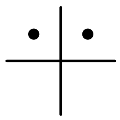
$\left(2\sin(\theta) - \sqrt{3}\right)\left(2\sin(\theta) - 1\right) = 0$

$2\sin(\theta) - \sqrt{3} = 0$ or $2\sin(\theta) - 1 = 0$

$\sin(\theta) = \frac{\sqrt{3}}{2} \qquad 2\sin(\theta) = \frac{1}{2}$

$\frac{\sqrt{3}}{2}$ suggests $\frac{\pi}{3}$ and $\frac{1}{2}$ suggests $\frac{\pi}{6}$

Since sin is positive



$\theta = \frac{\pi}{3}, \pi - \frac{\pi}{3} \qquad \theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3} \qquad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$

15 a $\sin(\alpha) - \cos^2(\alpha)\sin(\alpha) = 0 \qquad -\pi \le \alpha \le \pi$

$\sin(\alpha)\left(1 - \cos^2(\alpha)\right) = 0$

$\sin(\alpha)\left(1 - \cos(\alpha)\right)\left(1 + \cos(\alpha)\right) = 0$

$\sin(\alpha) = 0$ or $1 - \cos(\alpha) = 0$ or $1 + \cos(\alpha) = 0$

$\alpha = -\pi, 0, \pi \quad \cos(\alpha) = 1 \qquad \cos(\alpha) = -1$

$\alpha = 0 \qquad \alpha = -\pi, \pi$

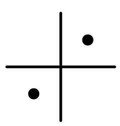
Thus $\alpha = -\pi, 0, \pi$.

b $\sin(2\alpha) = \sqrt{3}\cos(2\alpha) \qquad -\pi \le \alpha \le \pi$

$\frac{\sin(2\alpha)}{\cos(2\alpha)} = \sqrt{3}\frac{\cos(2\alpha)}{\cos(2\alpha)} \qquad -2\pi \le 2\alpha \le 2\pi$

$\tan(2\alpha) = \sqrt{3}$

$\sqrt{3}$ suggests $\frac{\pi}{3}$. Since tan is positive



$2\alpha = -2\pi + \frac{\pi}{3} - \pi + \frac{\pi}{3}, \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$2\alpha = -\frac{5\pi}{3}, -\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}$

$\alpha = -\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

c $\sin^2(\alpha) = \cos^2(\alpha) \qquad -\pi \le \alpha \le \pi$

$\sin^2(\alpha) - \cos^2(\alpha) = 0$

$\left(\sin(\alpha) - \cos(\alpha)\right)\left(\sin(\alpha) + \cos(\alpha)\right) = 0$

$\sin(\alpha) - \cos(\alpha) = 0$ or $\sin(\alpha) + \cos(\alpha) = 0$

$\sin(\alpha) = \cos(\alpha) \qquad \sin(\alpha) = -\cos(\alpha)$

$\tan(\alpha) = 1 \qquad \tan(\alpha) = -1$

$\alpha = -\frac{3\pi}{4}, \frac{\pi}{4} \qquad \alpha = -\frac{\pi}{4}, \frac{3\pi}{4} \qquad \alpha = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

d $4\cos^2(\alpha) - 1 = 0 \qquad -\pi \le \alpha \le \pi$

$\left(2\sin(\alpha)\right)^2 - 1^2 = 0$

$\left(2\sin(\alpha) - 1\right)\left(2\sin(\alpha) + 1\right) = 0$

$2\sin(\alpha) - 1 = 0$ or $2\ \sin(\alpha) + 1 = 0$

$\sin(\alpha) = \frac{1}{2} \qquad \sin(\alpha) = -\frac{1}{2}$

$\frac{1}{2}$ suggests $\frac{\pi}{3}$. Since sin is both positive and negative, all four quadrants.

$\alpha = -\pi + \frac{\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \pi - \frac{\pi}{3}$

$\alpha = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

16 a $2\cos(x) + 1 = 0$

$\cos(x) = -\frac{1}{2}$

Basic angle for $-\frac{1}{2}$ is $\frac{2\pi}{3}$ (quadrant 2)

General solution:

$x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$= \frac{6n\pi \pm 2\pi}{3}, n \in Z$

b $2\sin(x) - \sqrt{2} = 0$

$\sin(x) = \frac{\sqrt{2}}{2}$

$\frac{\sqrt{2}}{2}$ suggests $\frac{\pi}{4}$.

$\theta = 2n\pi + \frac{\pi}{4}$ and $(2n+1)\pi - \frac{\pi}{4}, \quad n \in Z$

$= \frac{8n\pi + \pi}{4}, \frac{4(2n+1)\pi - \pi}{4}$

$= \frac{8n\pi + \pi}{4}, \frac{8n\pi + 3\pi}{4}, \quad n \in Z$

17 $2\sin(2x) + 1 = 0$

$\sin(2x) = -\frac{1}{2}$

Basic angle for $-\frac{1}{2}$ is $-\frac{\pi}{6}$ (quadrant 4)

$2\theta = 2n\pi + \left(-\frac{\pi}{6}\right)$ and $(2n+1)\pi - \left(-\frac{\pi}{6}\right), \quad n \in Z$

$2\theta = \frac{12n\pi - \pi}{6}, \frac{6(2n+1)\pi + \pi}{6}$

$= \frac{12n\pi - \pi}{6}, \frac{12n\pi + 7\pi}{6}, \quad n \in Z$

$\theta = \frac{12n\pi - \pi}{12}, \frac{12n\pi + 7\pi}{12}, \quad n \in Z$

$n = 0: \quad \theta = -\frac{\pi}{12}, \frac{7\pi}{12}$

$n = 1: \quad \theta = \frac{11\pi}{12}, \frac{19\pi}{12}$

$n = 2: \quad \theta = \frac{23\pi}{12}, \frac{31\pi}{12}$

$\therefore \theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

18 $\sqrt{3}\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)$

$\sqrt{3}\tan\left(x + \frac{\pi}{2}\right) = 1$

$\tan\left(x + \frac{\pi}{2}\right) = \frac{1}{\sqrt{3}}$

$\frac{\pi}{6}$ is the base angle

General solution:

$x+\frac{\pi}{2}=n\pi+\frac{\pi}{6},\ n\in Z$

$x=n\pi-\frac{\pi}{3}$

$=\frac{3n\pi-\pi}{3},\ n\in Z$

$n=-1: \quad x=-\frac{4\pi}{3}$

$n=0: \quad x=-\frac{\pi}{3}$

$n=1: \quad x=\frac{2\pi}{3}$

$\therefore x=-\frac{\pi}{3},\frac{2\pi}{3}$

19 $\sin(3\theta)=\cos(2\theta)$

Using CAS:

$\theta=0.314,\ 1.571,\ 2.827,\ 4.084,\ 5.341$

20 $2\sin(2x)-1=-\frac{1}{2}x+1$

Using CAS:

$x=0.526,\ 1.179$

Exercise 1.5 — Literal and simultaneous equations

1 a $my-nx=4x+kx$

$my-kz=nx+4x$

$my-kz=x(n+4)$

$x=\frac{my-kz}{n+4}$

b $\frac{2p}{x}-\frac{m}{x-c}=\frac{3c}{x}$

$2p(x-c)-mx=3c(x-c)$

$2px-mx-3cx=2pc-3c^2$

$x(2p-m-3c)=2pc-3c^2$

$x=\frac{2pc-3c^2}{2p-m-3c}$

2 $\frac{x-my}{px+y}=2$

$x-my=2(px+y)$

$x-my=2px+2y$

$x-2px=my+2y$

$x(1-2p)=y(m+2)$

$y=\frac{x(1-2p)}{m+2}$

3 $x+y=2k$...................(1)

$mx+ny=d$................(2)

From (1)

$y=2k-x$..........(3)

Substitute (3) into (2)

$mx+n(2k-x)=d$

$mx+2nk-nx=d$

$mx-nx=d-2nk$

$x(m-n)=d-2nk$

$x=\frac{d-2nk}{m-n}$

Substitute $x=\frac{d-2nk}{m-n}$ into (3)

$y=2k-\frac{d-2nk}{m-n}$

$y=\frac{2k(m-n)-d+2nk}{m-n}$

$y=\frac{2km-d}{m-n}$

4 a $nx-my=k$...................(1)

$nx+my=2d$................(2)

(1) + (2)

$2nx=k+2d$

$x=\frac{k+2d}{2n}$

Substitute $x=\frac{k+2d}{2n}$ into (1)

$n\left(\frac{k+2d}{2n}\right)-my=k$

$\frac{1}{2}(k+2d)-my=k$

$\frac{1}{2}k+d-k=my$

$d-\frac{1}{2}k=my$

$y=\frac{2d-k}{2m}$

b $nx+my=m$..................(1)

$mx+ny=n$..................(2)

(1) × m and (2) × n

$mnx+m^2y=m^2$...........(3)

$mnx+n^2y=n$ ....(4)

(3) − (4)

$m^2y-n^2y=m^2-n^2$

$y\left(m^2-n^2\right)=m^2-n^2$

$y=\frac{m^2-n^2}{m^2-n^2}$

$y=1$

Substitute $y=1$ into (1)

$nx+m(1)=m$

$nx=0$

$x=0$

5 $2x+ky=4$...................(1)

$(k-3)x+2y=0$.......... (2)

There is a unique solution for all values of k except when the gradients are the same.

From (1) $\quad ky=-2x+4$

$y=-\frac{2}{k}x+\frac{4}{k}$ so $m=-\frac{2}{k}$

From (2) $\quad 2y=-(k-3)x$

$y=-\frac{(k-3)}{2}x$ so $m=-\frac{(k-3)}{2}$

Equating gradients, we have

$-\frac{2}{k}=-\frac{(k-3)}{2}$

$2(2)=k(k-3)$

$0=k^2-3k-4$

$0=(k-4)(k+1)$

$0=k-4$ or $0=k+1$

$k=4 \qquad k=-1$

If $k=-1$ or 4 the equations will have the same gradient so for all other values of k there will be a unique solutions. i.e. $k\in R\setminus\{-1,4\}$

6 $mx-2y=4$....................(1)

$x+(m-3)y=m$............(2)

There will be infinitely many solutions if the equations are identical.

From (1) $mx-4=2y$

$\frac{m}{2}x-2=y$

From (2) $x+(m-3)y=m$

$x-m=-(m-3)y$

$x-m=(3-m)y$

$y=\frac{1}{3-m}x-m$

For equations to be identical then $m=2$. To check we should look at the two gradients.

For (1) Gradient $=\frac{m}{2}=\frac{2}{2}=1$

For (2) Gradient $=\frac{1}{3-m}=\frac{1}{3-2}=1$

Gradients are also the same, therefore $m=2$ for an infinite number of solutions.

7 $2m-4n-p=1$..................(1)

$4m+n+p=5$..................(2)

$3m+3n-2p=22$............. (3)

$(1)\times 2$ and $(2)\times 2$

$4m-8n-2p=2$...............(4)

$8m+2n+2p=10$.............(5)

$(3)-(4)$

$-m+11n=20$...................(6)

$(3)+(5)$

$11m+5n=32$...................(7)

$(6)\times 11$

$-11m+121n=220$...........(8)

$(7)+(8)$

$126n=252$

$n=2$

Substitute $n=2$ into (6)

$-m+11(2)=20$

$22-20=m$

$m=2$

Substitute $m=2$ and $n=2$ into (2)

$4(2)+2+p=5$

$10+p=5$

$p=-5$

8 $2d-e-f=-2$...................(1)

$3d+2e-f=5$...................(2)

$d+3e+2f=11$..................(3)

$(1)\times 2$ and $(2)\times 2$

$4d-2e-2f=-4$...............(4)

$6d+4e-2f=10$................(5)

$(3)+(4)$

$5d+e=7$................(6)

$(3)+(5)$

$7d+7e=21.$

$d+e=3$...............(7)

$(6)-(7)$

$4d=4$

$d=1$

Substitute $d=1$ into (7)

$1+e=3$

$e=2$

Substitute $d=1$ and $e=2$ into (1)

$2(1)-2-f=-2$

$f=2$

9 $x+2y+2z=1$...................(1)

$2x-2y+z=2$..................(2)

a (1) + (2)

$3x+3z=3$

b Let $z=\lambda$

$3x+3\lambda=3$

$x+\lambda=1$

$x=1-\lambda$

Substitute $z=\lambda$ and $x=1-\lambda$ into (2)

$2(1-\lambda)-2y+\lambda=2$

$2-2\lambda-2y+\lambda=2$

$-\lambda=2y$

$y=-\frac{\lambda}{2}$

c This solution describes the line along which the two planes are intersecting.

10 $x+2y+4z=2$...................(1)

$x-y-3z=4$.....................(2)

$(2)-(1)$

$-3y-7z=2$

$3y+7z=-2$

Let $z=\lambda$

$3y+7\lambda=-2$

$3y=-7\lambda-2$

$y=\frac{-7\lambda-2}{3}$

$y=-\frac{7\lambda+2}{3}$

Substitute $z=\lambda$ and $y=-\frac{7\lambda+2}{3}$ into (2)

$x+\frac{7\lambda+2}{3}-3\lambda=4$

$x=3\lambda+4-\frac{7\lambda+2}{3}$

$x=\frac{9\lambda+12-7\lambda-2}{3}$

$x=\frac{2\lambda+10}{3}$

$x=\frac{2(\lambda+5)}{3}$

11 **a** $\frac{kx+dy}{x+3y}=-2k$

$kx+dy=-2k(x+3y)$

$kx+dy=-2kx-6ky$

$kx+2kx=-6ky-dy$

$3kx=-y(6k+d)$

$x=-\frac{y(6k+d)}{3k}$

b $\frac{mx+ny}{p}=x+q$

$mx-px=pq-ny$

$x(m-p)=pq-ny$

$x=\frac{pq-ny}{m-p}$

c $\frac{m}{x}-k=\frac{3k}{x}+m$

$m-kx=3k+mx$

$m-3k=mx+kx$

$m-3k=x(m+k)$

$x=\frac{m-3k}{m+k}$

d $\frac{k}{m+x}=\frac{2d}{m-x}$

$\frac{k}{m+x}=\frac{2d}{m-x}$

$k(m-x)=2d(m+x)$

$km-kx=2dm+2dx$

$km-2dm=2dx+kx$

$m(k-2d)=x(2d+k)$

$x=\frac{m(k-2d)}{k+2d}$

12 a $2mx+ny=3k$......................(1)

$mx+ny=-d$.....................(2)

$(1)-(2)$

$2mx-mx=3k+d$

$mx=3k+d$

$x=\frac{3k+d}{m}$

Substitute $x=\frac{3k+d}{m}$ into (2)

$m\left(\frac{3k+d}{m}\right)+ny=-d$

$3k+d+ny=-d$

$ny=-2d-3k$

$y=-\frac{2d+3k}{n}$

b $\frac{x}{2a}+\frac{y}{b}=2$.................(1)

$\frac{2x}{b}+\frac{4y}{a}=8$...............(2)

$(1)\times 2ab \Rightarrow xb+2ay=4ab$...........(3)

$(2)\times ab \Rightarrow 2ax+4by=8ab$............(4)

$(3)\times 2a \Rightarrow 2abx+4a^2y=8a^2b$.......(5)

$(4)\times b \Rightarrow 2abx+4b^2y=8ab^2$.........(6)

$(5)-(6)$

$4a^2y-4b^2y=8a^2b-8ab^2$

$a^2y-b^2y=2a^2b-2ab^2$

$y(a^2-b^2)=2ab(a-b)$

$y=\frac{2ab(a-b)}{(a-b)(a+b)}$

$=\frac{2ab}{a+b}$

Substitute into (3)

$xb+2a\left(\frac{2ab}{a+b}\right)=4ab$

$xb+\frac{4a^2b}{a+b}=4ab$

$xb(a+b)+4a^2b=4ab(a+b)$

$x(a+b)+4a^2=4a(a+b)$

$x(a+b)=4a^2+4ab-4a^2$

$x=\frac{4ab}{a+b}$

13 $x+my=3$...................(1)

$4mx+y=0$.................(2)

From (1) $my=-x+3$

$y=-\frac{1}{m}x+\frac{3}{m}$ where gradient$_1=-\frac{1}{m}$

From (2) $y=-4mx$ where gradient$_2=-4m$

There is no solution when the lines have the same gradient and are parallel.

Gradient$_1$ = Gradient$_2$

$-\frac{1}{m}=-4m$

$1=4m^2$

$\frac{1}{4}=m^2$

$m=\pm\frac{1}{2}$

No solution if $m=\pm\frac{1}{2}$.

14 $x+3ky=2$....................(1)

$(k-1)x-1=-6y$..........(2)

From (1) $3ky=-x+2$

$y=-\frac{1}{3k}x+\frac{2}{3k}$ where gradient $m_1=-\frac{1}{3k}$

From (2) $1-(k-1)x=6y$

$\frac{1}{6}-\frac{(k-1)}{6}x=y$ where gradient $m_2=-\frac{k-1}{6}$

The lines have a unique solution for all vales of k except for when the gradients are the same.

$m_1=m_2$

$-\frac{1}{3k}=-\frac{k-1}{6}$

$6=3k^2-3k$

$0=3k^2-3k-6$

$0=k^2-k-2$

$0=(k-2)(k+1)$

$k-2=0$ or $k+1=0$

$k=2$ $\quad k=-1$

Lines have a unique solution when $k\in R\setminus\{-1,2\}$.

15 a $-2x+my=1$...................(1)

$(m+3)x-2y=-2m$.......(2)

From (1) $my=2x+1$

$y=\frac{2}{m}x+\frac{1}{m}$ where gradient$_1=\frac{2}{m}$

From (2) $(m+3)x+2m=2y$

$\frac{(m+3)}{2}x+m=y$ where gradient$_2=\frac{m+3}{2}$

The lines have a unique solution for all vales of k except for when the gradients are the same.

Gradient$_1$ = Gradient$_2$

$\frac{2}{m}=\frac{m+3}{2}$

$4=m^2+3m$

$0=m^2+3m-4$

$0=(m+4)(m-1)$

$m+4=0$ or $m-1=0$

$m=-4$ $\quad m=1$

Lines have a unique solution when $m\in R\setminus\{-4,1\}$.

b The lines have a no solution when the gradients are the same because the lines are parallel.

If $m = -4$

$-2x - 4y = 1$.........(1)

$-x - 2y = 8$.........(2)

$(2) \times 2 \Rightarrow -2x - 4y = 16$

Therefore, if $m = -4$, the gradients are equal, but the y-intercepts are not, therefore the lines are parallel

c The lines have infinitely many solutions when both the equations are identical. This is when their gradients are the same and so too are the y-intercepts.

If $m = 1$

$-2x + y = 1$...........(1)

$4x - 2y = -2$.........(2)

$(1) \times -2 \Rightarrow 4x - 2y = -2$

The lines are identical, so there are infinitely many solutions when $m = 1$.

16 CAS can be used for all parts or the equations can be solved by hand.

a $2x + y - z = 12$.....................(1)

$-x - 3y + z = -13$................(2)

$-4x + 3y - z = -2$................(3)

(2) + (3)

$-5x = -15$

$x = 3$

$(1) \Rightarrow 6 + y - z = 12$

$y - z = 6$................(4)

$(2) \Rightarrow -3 - 3y + z = -13$

$-3y + z = -10$.......(5)

(4) + (5)

$-2y = -4$

$y = 2$

Substitute into (4)

$2 - z = 6$

$z = -4$

b $m + n - p = 6$........................(1)

$3m + 5n - 2p = 13$...............(2)

$5m + 4n - 7p = 34$...............(3)

$(1) \times 3$

$3m + 3n - 3p = 18$................(4)

$(2) - (4)$

$2n + p = -5$...............(5)

$(1) \times 5$

$5m + 5n - 5p = 30$...............(6)

$(6) - (3)$

$n + 2p = -4$...............(7)

$(7) \times 2$

$2n + 4p = -8$...............(8)

$(8) - (5)$

$3p = -3$

$p = -1$

Substitute $p = -1$ into (5)

$2n - 1 = -5$

$2n = -4$

$n = -2$

Substitute $p = -1$ and $n = -2$ into (1)

$m - 2 + 1 = 6$

$m = 7$

c $u + 2v - 4w = 23$.........................(1)

$3u + 4v - 2w = 37$......................(2)

$3u + v - 2w = 19$.........................(3)

$(1) \times 3$

$3u + 6v - 12w = 69$...................(4)

$(4) - (2)$

$2v - 10w = 32$

$v - 5w = 16$...................(5)

$(4) - (3)$

$5v - 10w = 50$

$v - 2w = 10$...................(6)

$(6) - (5)$

$3w = -6$

$w = -2$

Substitute $w = -2$ into (6)

$v - 2(-2) = 10$

$v + 4 = 10$

$v = 6$

Substitute $w = -2$ and $v = 6$ into (1)

$u + 2(6) - 4(-2) = 23$

$u + 12 + 8 = 23$

$u = 3$

d $a + b + c = 4$......................(1)

$2a - b + 2c = 17$...............(2)

$-a - 3b + c = 3$.................(3)

$(1) + (3)$

$-2a + 2c = 7$................(4)

$(3) \times 2$

$-2a = 6b + 2c = 6$.............(5)

$(4) \times -2$

$4b - 4c = -14$........(6)

$(6) + (7)$

$-3b = 9$

$b = -3$

Substitute $b = -3$ into (4)

$-2(-3) + 2c = 7$

$6 + 2c = 7$

$2c = 1$

$c = \frac{1}{2}$

Substitute $b = -3$ and $c = \frac{1}{2}$ into (1)

$a - 3 + \frac{1}{2} = 4$

$a - \frac{5}{2} = \frac{8}{2}$

$a = \frac{13}{2}$

17 Let a be the smallest angle, b be the largest angle and c be the third angle.

$b = a + 20$ .... (1)

$a + b = c + 60$ .... (2)

$a + b + c = 180$ ....(3)

Substitute (1) into (2) and (3)

$a + a + 20 = c + 60$

$2a - c = 40$.....................(4)

$a + a + 20 + c = 180$

$2a + c = 160$...................(5)

(5) + (4)

$4a = 200$

$a = 50°$

Substitute $a = 50°$ into (1)

$b = 50 + 20 = 70°$

Substitute $a = 50°$ and $b = 70°$ into (3)

$50 + 70 + c = 180$

$120 + c = 180$

$c = 60°$

The largest angle is 70°, the smallest angle is 50° and the third angle is 60°.

18 $x + y - 2z = 5$....................(1)

$x - 2y + 4z = 1$..................(2)

$(1) - (2)$

$3y - 6z = 4$

Let $z = \lambda$

$3y = 6\lambda + 4$

$y = \frac{2(3\lambda + 2)}{3}$

Substitute $y = \frac{2(3\lambda + 2)}{3}$ into (1)

$x + \frac{2(3\lambda + 2)}{3} - 2\lambda = 5$

$3x + 6\lambda + 4 - 6\lambda = 15$

$3x = 11$

$x = \frac{11}{3}$

19 $-2x + y + z = -2$....................(1)

$x - 3z = 0$..................... (2)

Let $z = \lambda$

From (2)

$x - 3\lambda = 0$

$x = 3\lambda$

Substitute $z = \lambda$ and $x = 3\lambda$ into (1)

$-2(3\lambda) + y + \lambda = -2$

$y - 5\lambda = -2$

$y = 5\lambda - 2$

20 $3x + 2y = -1$..................(1)

$mx + 4y = n$...................(2)

From (1) $2y = -3x - 1$

$y = -\frac{3}{2}x - \frac{1}{2}$ where gradient$_1 = -\frac{3}{2}$

From (2) $4y = -mx + n$

$y = -\frac{m}{4}x + \frac{n}{4}$ where gradient$_2 = -\frac{m}{4}$

If gradient$_1$ = gradient$_2$

$-\frac{3}{2} = -\frac{m}{4}$

$12 = 2m$

$m = 6$

a The lines have a unique solution for all values of k except for when the gradients are the same. Therefore $m \in R \setminus \{6\}$ and $n \in R$.

b The lines have infinitely many solutions when both the equations are identical. This is when the gradients are the same and so too are their "c" values. If the gradients are the same then $m = 6$ and if the "c" values are the same then

$-\frac{1}{2} = \frac{n}{4}$

$-4 = 2n$

$n = -2$

Therefore, for an infinite number of solutions, $m = 6, n = -2$

c The lines have no solution when the gradients are the same but the y-intercepts are different (lines are parallel). Therefore, $m = 6$ and $n \in R \setminus \{-2\}$.

21 $2x - y + az = 4$...............................(1)

$(a + 2)x + y - z = 2$....................(2)

$6x + (a + 1)y - 2z = 4$....................(3)

Solve using CAS:

$x = \frac{2(a + 2)}{a(a + 4)}$, $y = \frac{4(a + 2)}{a(a + 4)}$ and $z = \frac{4}{a}$

22 $w - 2x + 3y - z = 10$

$2w + x + y + z = 4$

$-w + x + 2y - z = -3$

$3w - 2x + y = 11$

Solve using CAS:

$w = 1$, $x = -3$, $y = 2$ and $z = 3$

Topic 2 — Functions and graphs

Exercise 2.2 — Polynomial functions

1 a A vertical line would cut the graph no more than once so the graph is of a function. A horizontal line could cut the graph up to two times. The function has a many-to-one correspondence.

b Domain $[-4, 2)$ and range $[0, 16]$.

c $f: [-4, 2) \to R, f(x) = x^2$

d Image of $-2\sqrt{3}$ is $f(-2\sqrt{3})$.

$$f(-2\sqrt{3}) = (-2\sqrt{3})^2 = 12$$

The image is 12.

2 $f(x) = x^2 - 4$

a The image of $\frac{2}{3}$ is $f\left(\frac{2}{3}\right)$

$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 - 4 = \frac{4}{9} - 4 = -\frac{32}{9}$$

b $f(2a)$

$$= (2a)^2 - 4$$
$$= 4a^2 - 4$$

c The implied domain is R.

3 $L = \{(x \cdot y): 3x - 4y = 12\}$.

a Let $y = 0$

$\therefore 3x = 12$

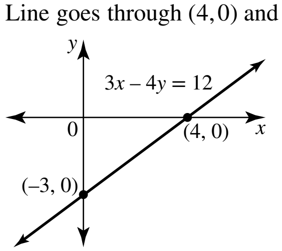
$\therefore x = 4$

Let $x = 0$

$\therefore -4y = 12$

$\therefore y = -3$

Line goes through $(4, 0)$ and $(0, -3)$.



b Rearranging the equation,

$3x - 4y = 12$

$\therefore 3x - 12 = 4y$

$\therefore y = \frac{3}{4}x - 3$

The gradient is $\frac{3}{4}$.

c Let the point on the line closest to the origin be P. Then the line OP is perpendicular to the given line.

The gradient of OP is $m = -\frac{4}{3}$ and OP contains the origin.

The equation of OP is $y = -\frac{4}{3}x$.

P is the point of intersection of $y = -\frac{4}{3}x$ and $3x - 4y = 12$.

At intersection,

$$3x - 4\left(-\frac{4}{3}x\right) = 12$$
$$\therefore 9x + 16x = 36$$
$$\therefore x = \frac{36}{25}$$
$$\therefore y = -\frac{4}{3} \times \frac{36}{25}$$
$$\therefore y = -\frac{48}{25}$$

The point closest to the origin is $\left(\frac{36}{25}, -\frac{48}{25}\right)$.

4 $A(-1, -3)$ and $B(5, -7)$

a Gradient of AB:

$$m = \frac{-7 + 3}{5 + 1} = -\frac{2}{3}$$

Equation of AB:

$$y + 3 = -\frac{2}{3}(x + 1)$$
$$\therefore 3y + 9 = -2x - 2$$
$$\therefore 2x + 3y = -11$$

b Midpoint of AB is $\left(\frac{-1 + 5}{2}, \frac{-3 - 7}{2}\right) = (2, -5)$.

Gradient of line perpendicular to AB is $\frac{3}{2}$.

Equation of the perpendicular bisector is

$$y + 5 = \frac{3}{2}(x - 2)$$
$$\therefore y = \frac{3x}{2} - 3 - 5$$
$$\therefore y = \frac{3x}{2} - 8$$

c $m = \tan(\theta)$

$$\therefore \tan(\theta) = \frac{3}{2}$$
$$\therefore \theta = \tan^{-1}(1.5)$$
$$\therefore \theta \approx 56.3°$$

5 a $y = 2(3x - 2)^2 - 8$

Turning point: When $3x - 2 = 0$, $x = \frac{2}{3}$.

Therefore the graph has a minimum turning point at $\left(\frac{2}{3}, -8\right)$.

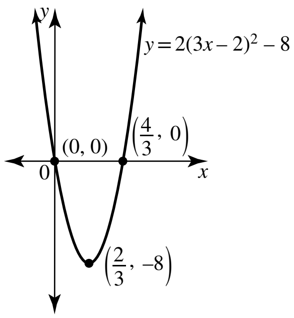
y intercept: Let $x = 0$

$\therefore y = 2(-2)^2 - 8$

$\therefore y = 0$

$(0, 0)$

Axis of symmetry $x = \frac{2}{3}$, so the other x intercept must be $\left(\frac{4}{3}, 0\right)$.



Domain R, range $[-8, \infty)$.

b x intercept at $x = -\frac{1}{2} \Rightarrow (2x+1)$ is a factor.

x intercept at $x = 4 \Rightarrow (x-4)$ is a factor.
Let the equation be $y = a(2x+1)(x-4)$
Substitute the point $(0, 2)$

$\therefore 2 = a(1)(-4)$

$\therefore a = -\frac{1}{2}$

The equation is $y = -\frac{1}{2}(2x+1)(x-4)$.

6 $f : R^+ \cup \{0\} \to R, f(x) = 4x^2 - 8x + 7$.

a The discriminant determines the number of x intercepts.

$\Delta = b^2 - 4ac,\ a = 4, b = -8, c = 7$

$\Delta = 64 - 4 \times 4 \times 7$

$= 64 - 112$

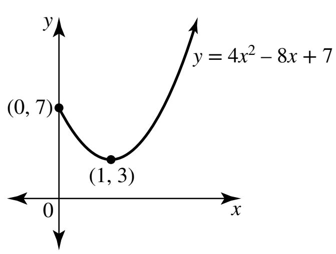
< 0

There are no x intercepts.

b $f(x) = 4x^2 - 8x + 7$
Completing the square,

$$f(x) = 4\left(x^2 - 2x + \frac{7}{4}\right)$$
$$= 4\left(\left(x^2 - 2x + 1\right) - 1 + \frac{7}{4}\right)$$
$$= 4\left((x-1)^2 + \frac{3}{4}\right)$$
$$= 4(x-1)^2 + 3$$

c Restricted domain is $R^+ \cup \{0\}$
Minimum turning point $(1, 3)$, y intercept and endpoint $(0, 7)$.



Range is $[3, \infty)$.

7 **a** $y = -4(x+2)^3 + 16$
Stationary point of inflection at $(-2, 16)$
y intercept: Let $x = 0$

$\therefore y = -4(2)^3 + 16$

$\therefore y = -16$

$(0, -16)$

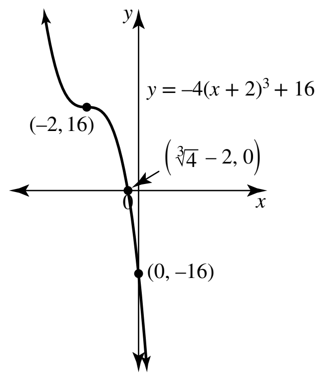
x intercept: Let $y = 0$

$\therefore 0 = -4(x+2)^3 + 16$

$\therefore (x+2)^3 = 4$

$\therefore x = \sqrt[3]{4} - 2$

$\left(\sqrt[3]{4} - 2, 0\right)$



b The x intercepts indicate the linear factor and its multiplicity. In the diagram each factor will have multiplicity 1.
Cut at $x = 0 \Rightarrow x$ is a factor.

Cut at $x = 0.8 = \frac{4}{5} \Rightarrow (5x-4)$ is a factor.

Cut at $x = 1.5 = \frac{3}{2} \Rightarrow (2x-3)$ is a factor.

Let the equation be $y = ax(5x-4)(2x-3)$
Substitute the point $(2, 24)$

$\therefore 24 = a(2)(6)(1)$

$\therefore 12a = 24$

$\therefore a = 2$

The equation is $y = 2x(5x-4)(2x-3)$.

8 $f : [-2, 4] \to R, f(x) = 4x^3 - 8x^2 - 16x + 32$.

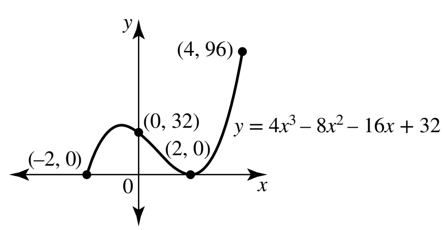
a $4x^3 - 8x^2 - 16x + 32$

$= 4x^2(x-2) - 16(x-2)$

$= 4(x-2)(x^2-4)$

$= 4(x-2)^2(x+2)$

b $f(x) = 4(x-2)^2(x+2),\ x \in [-2, 4]$
x intercepts: $x = 2$ (turning point), $x = -2$ which is also an endpoint.
y intercept: $f(0) = 32 \Rightarrow (0, 32)$
Right endpoint: $f(4) = 4(2)^2(6) = 96 \Rightarrow (4, 96)$



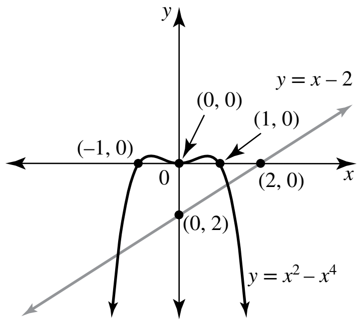
c Maximum value of the function f is 96 and its minimum value is 0.

9 **a** $y = x^2 - x^4$

$y = x^2(1 - x^2)$

$= x^2(1 - x)(1 + x)$

Turning point at $(0, 0)$, cuts x axis at $(\pm 1, 0)$.



Rearranging the equation $x^4 - x^2 + x - 2 = 0$ gives $x - 2 = x^2 - x^4$.

The number of intersections of the line $y = x - 2$ with the graph of $y = x^2 - x^4$ will be the number of solutions of the equation.

The line $y = x - 2$ passes through $(0, -2)$ and $(2, 0)$. It is drawn on the diagram, showing it makes two intersections with the quartic curve.

There are two solutions to the equation $x^4 - x^2 + x - 2 = 0$.

b $y = a(x + b)^4 + c$

The axis of symmetry $x = -b$ lies midway between the x intercepts.

$\therefore -b = \dfrac{-9 - 3}{2}$

$\therefore b = 6$

$y = a(x + 6)^4 + c$

As the range is $(-\infty, 7]$, the maximum turning point is $(-6, 7)$.

The equation becomes $y = a(x + 6)^4 + 7$.

Substitute the point $(-3, 0)$:

$\therefore 0 = a(3)^4 + 7$

$\therefore a = -\dfrac{7}{81}$

The equation is $y = -\dfrac{7}{81}(x + 6)^4 + 7$ with

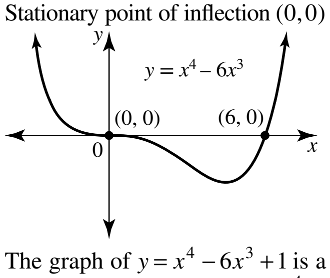
$a = -\dfrac{7}{81},\ b = 6,\ c = 7.$

Therefore, the turning point is (–6, 7).

10 $y = x^4 - 6x^3$

$\therefore y = x^3(x - 6)$

Stationary point of inflection $(0, 0)$ and graph cuts x axis at $(6, 0)$.



The graph of $y = x^4 - 6x^3 + 1$ is a vertical translation of 1 unit upwards of the graph of $y = x^4 - 6x^3$. Its point of inflection would lie above the axis but the graph would still intersect the x axis at a point between $x = 0$ and $x = 6$ as well as at a point where $x > 6$. There will two intersections.

Check: A point below the axis such as $(1, -5)$ for example, would still lie below the x axis if it was vertically translated up one unit so the graph must cross the axis to reach this point.

11 $y = (x + 1)^6 + 10$

Minimum turning point $(-1, 10)$.

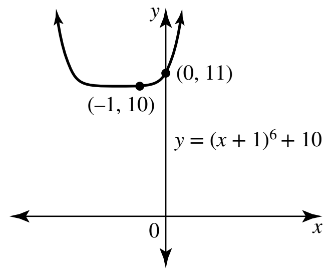
No x intercepts as turning point lies above x axis.

y intercept: Let $x = 0$

$\therefore y = (1)^6 + 10$

$\therefore y = 11$

$(0, 11)$



12 $y = (x + 4)(x + 2)^2(x - 2)^3(x - 5)$

Graph cuts x axis at $x = -4$, touches the axis at $x = -2$, saddle cuts the axis at $x = 2$ and cuts the axis at $x = 5$.

Its degree is 7 and the leading coefficient is positive. Its long term behaviour is as $x \to \pm\infty, y \to \pm\infty$.

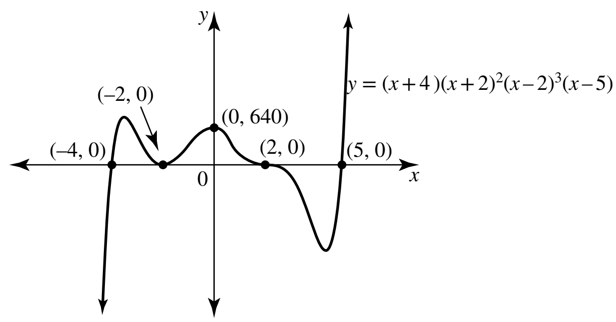
y intercept: Let $x = 0$

$\therefore y = (4)(2)^2(-2)^3(-5)$

$\therefore y = 640$

$(0, 640)$

y intercept is positive.



13

	Correspondence	Domain	Range	Function?
a	many-to-one	$[-3, 6)$	$[-9, 7]$	yes
b	one-to-many	$[0, \infty)$	R	no
c	many-to-many	$[-2, 2]$	$[-2, 2]$	no
d	one-to-one	R	R	yes
e	many-to-one	R	$\{2\}$	yes
f	one-to-one	R	R	yes

14 **a** $f: R \to R, f(x) = 9 - 4x$

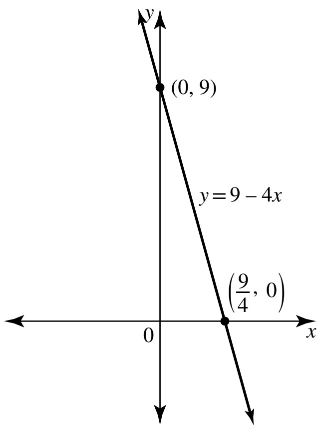
$f(0) = 9 \Rightarrow (0, 9)$

Let $f(x) = 0$

$\therefore 9 - 4x = 0$

$\therefore x = \dfrac{9}{4}$

$\Rightarrow \left(\dfrac{9}{4}, 0\right)$



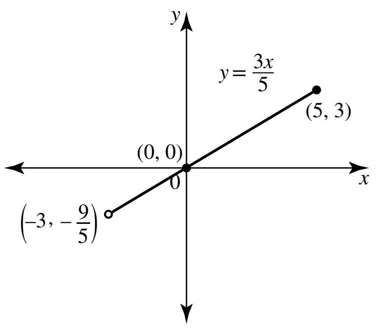
Range is R.

b $g:(-3,5]\to R, g(x)=\dfrac{3x}{5}$

$g(0)=0\Rightarrow(0,0)$

$g(-3)=-\dfrac{9}{5}\Rightarrow\left(-3,-\dfrac{9}{5}\right)$ is open endpoint

$g(5)=3\Rightarrow(5,3)$ is closed endpoint.



Range is $\left(-\dfrac{9}{5},3\right]$.

15 $A(5,-3)$, $B(7,8)$ and $C(-2,p)$

a The line $9x+7y=24$ has a gradient of $-\dfrac{9}{7}$.

This is the gradient of AC since it is parallel to this line.

$$m_{AC}=\frac{p+3}{-2-5}$$
$$\therefore -\frac{9}{7}=\frac{p+3}{-7}$$
$$\therefore 9=p+3$$
$$\therefore p=6$$

b A line perpendicular to AC would have a gradient of $\dfrac{7}{9}$.

The line through $B(7,8)$ perpendicular to AC has equation:

$$y-8=\frac{7}{9}(x-7)$$
$$\therefore 9y-72=7x-49$$
$$\therefore 9y-7x=23$$

c Let the point where $9y-7x=23$ meets AC be Q. The length of PQ is the shortest distance from B to AC.
To find Q, solve the pair of simultaneous equations
$9x+7y=24....(1)$
$9y-7x=23....(2)$

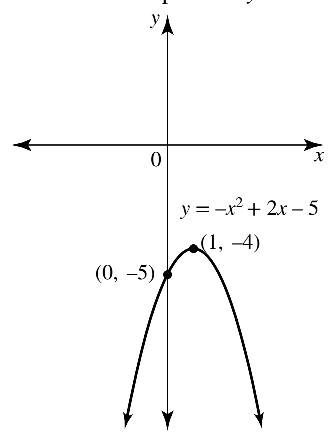
Q is the point $\left(\dfrac{11}{26},\dfrac{75}{26}\right)$.

The length of BQ is $\sqrt{\left(7-\dfrac{11}{26}\right)^2+\left(8-\dfrac{75}{26}\right)^2}\approx 8.3$ units.

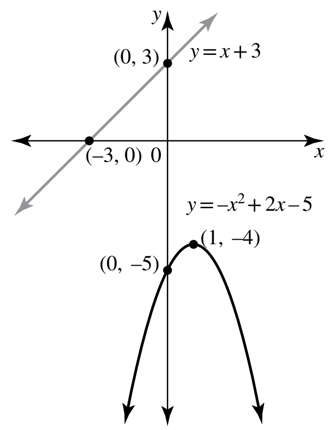
16 **a** $-x^2+2x-5$
$$=-\left(x^2-2x+5\right)$$
$$=-\left((x^2-2x+1)-1+5\right)$$
$$=-\left((x-1)^2+4\right)$$
$$=-(x-1)^2-4$$

b $y=-x^2+2x-5$
$\therefore y=-(x-1)^2-4$
Turning point is $(1,-4)$.

c Turning point is a maximum so the range is $(-\infty,-4]$. There are no x intercepts. The y intercept is $(0,-5)$.



d



The line $y=x+3$ passes through the points $(-3,0)$ and $(0,3)$. As the diagram shows, this line will never intersect the concave down parabola.

e The graphs of $y=x+k$ and $y=-x^2+2x-5$ will intersect when
$$x+k=-x^2+2x-5$$
$$\therefore x^2-x+k+5=0$$
For one intersection, $\Delta=0$.
$$\Delta=b^2-4ac,\ a=1, b=-1, c=k+5$$
$$=1-4(k+5)$$
$$=-4k-19$$
$$\therefore -4k-19=0$$
$$\therefore k=-\frac{19}{4}$$

For exactly one intersection, $k=-\dfrac{19}{4}$.

17 **a** Let the equation be $y=a(x-h)^2+k$
Turning point is $(-6,12)$
$\therefore y=a(x+6)^2+12$
Substitute the point $(4,-3)$

$\therefore -3 = a(10)^2 + 12$
$\therefore 100a = -15$
$\therefore a = -\frac{3}{20}$

The equation is $y = -\frac{3}{20}(x+6)^2 + 12$.

b The points $(-7, 0)$ and $\left(-2\frac{1}{2}, 0\right)$ are the two x intercepts.

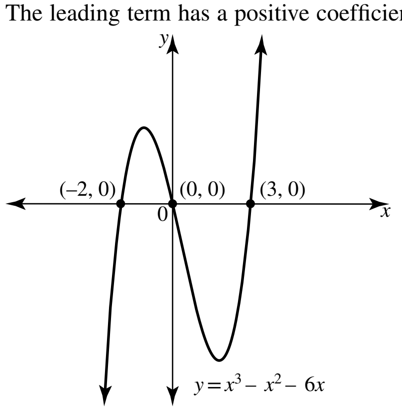
The equation has linear factors $(x+7)$ and $\left(x+2\frac{1}{2}\right)$ or $(2x+5)$.
Let the equation be $y = a(x+7)(2x+5)$
Substitute the point $(0, -20)$
$\therefore -20 = a(7)(5)$
$\therefore a = -\frac{4}{7}$

The equation is $y = -\frac{4}{7}(x+7)(2x+5)$.

c As the points $(-8, 11)$ and $(8, 11)$ have the same y co-ordinate, the turning point lies between them and the axis of symmetry lies midway between the,
Axis of symmetry: $x = \frac{-8+8}{2} \Rightarrow x = 0$.
The minimum value of a quadratic function is the y value of its minimum turning point.
Therefore, the turning point is $(0, -5)$.
Let the equation be $y = ax^2 - 5$
Substitute the point $(8, 11)$
$\therefore 11 = a(64) - 5$
$\therefore 64a = 16$
$\therefore a = \frac{1}{4}$

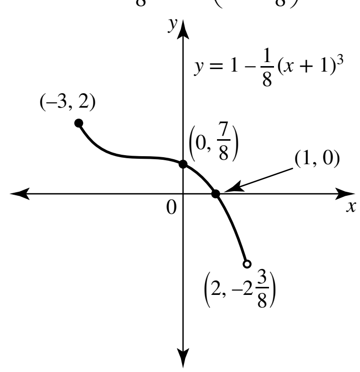
The equation is $y = \frac{1}{4}x^2 - 5$.

18 a $y = x^3 - x^2 - 6x$
$\therefore y = x(x^2 - x - 6)$
$\therefore y = x(x-3)(x+2)$
The factors show the graph cuts the x axis at $(0, 0)$, $(3, 0)$ and $(-2, 0)$.
The leading term has a positive coefficient.

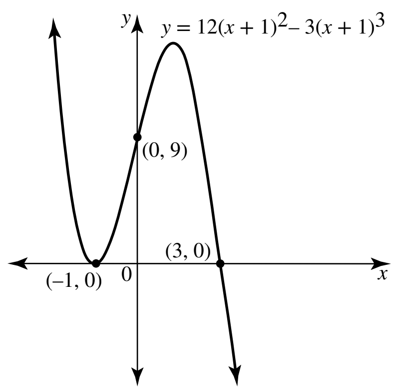


b $y = 1 - \frac{1}{8}(x+1)^3$
Stationary point of inflection $(-1, 1)$.
y intercept: Let $x = 0$
$\therefore y = 1 - \frac{1}{8}(1)^3$
$\therefore y = \frac{7}{8}$
$\left(0, \frac{7}{8}\right)$

x intercept: Let $y = 0$
$\therefore 0 = 1 - \frac{1}{8}(x+1)^3$
$\therefore (x+1)^3 = 8$
$\therefore x + 1 = 2$
$\therefore x = 1$
$(1, 0)$
End points:
$x = -3, y = 1 - \frac{1}{8}(-3+1)^3$
$= 1 - \frac{1}{8}(-2)^3$
$= 1 - \frac{1}{8} \times -8$
$= 2 \quad \therefore (-3, 2)$
$x = 2, y = 1 - \frac{1}{8}(2+1)^3$
$= 1 - \frac{1}{8}(3)^3$
$= 1 - \frac{27}{8}$
$= -2\frac{3}{8} \quad \therefore \left(2, -2\frac{3}{8}\right)$



c $y = 12(x+1)^2 - 3(x+1)^3$
$\therefore y = 3(x+1)^2(4 - (x+1))$
$\therefore y = 3(x+1)^2(3 - x)$
The factors show the graph touches the x axis at $(-1, 0)$ and cuts at $(3, 0)$.
y intercept: Let $x = 0$
$\therefore y = 3(1)^2(3) = 9$
$(0, 9)$



19 The x intercept at $x = -4 \Rightarrow (x+4)$ is a factor.
The x intercept at $x = \frac{5}{4} \Rightarrow (4x-5)$ is a repeated factor of multiplicity two.
Let the equation be $y = a(x+4)(4x-5)^2$
Substitute the point $(0, 10)$

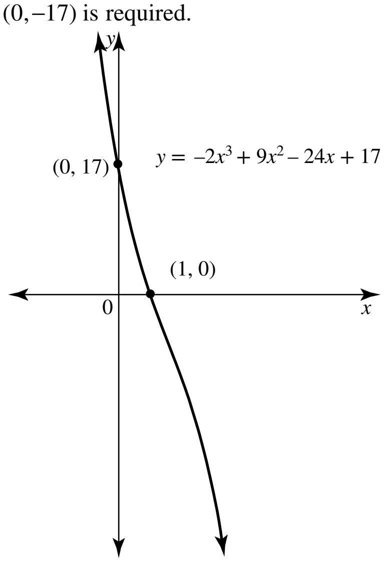
$\therefore 10 = a(4)(-5)^2$
$\therefore 10 = 100a$
$\therefore a = \dfrac{1}{10}$
The equation is $y = \dfrac{1}{10}(x+4)(4x-5)^2$.

20 a $f(x) = -2x^3 + 9x^2 - 24x + 17$
$f(1) = -2 + 9 - 24 + 17 = 0$
$\therefore (x-1)$ is a factor.
By inspection,
$-2x^3 + 9x^2 - 24x + 17 = (x-1)(-2x^2 + 7x - 17)$.
Consider the discriminant of the quadratic factor $-2x^2 + 7x - 17$.
$$\begin{aligned}\Delta &= 49 - 4(-2)(-17)\\ &= 49 - 136\\ &< 0\end{aligned}$$
Since the discriminant is negative, the quadratic cannot be factorised into real linear factors and therefore it has no real zeros.
For the cubic, this means there can only be one x intercept, the one which comes from the only linear factor $(x-1)$.

b For there to be a stationary point of inflection, the equation of the cubic function must be able to be written in the form $y = a(x+b)^3 + c$.
Let $-2x^3 + 9x^2 - 24x + 17 = a(x+b)^3 + c$
By inspection, the value of a must be -2.
$\therefore -2x^3 + 9x^2 - 24x + 17 = -2(x^3 + 3x^2b + 3xb^2 + b^3) + c$
Equate coefficients of like terms,
x^2: $9 = -6b \Rightarrow b = -\dfrac{3}{2}$
x: $-24 = -6b^2 \Rightarrow b^2 = 4$
It is not possible for b to have different values.
Therefore, it is not possible to express the equation of the function in the form $y = a(x+b)^3 + c$.
There is no stationary point of inflection on the graph of the function.

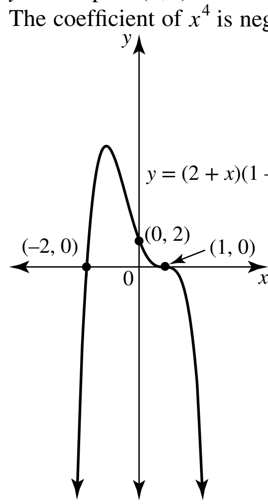
c The leading term has a negative coefficient. Therefore, as $x \to \pm\infty, y \to \mp\infty$.

d Given the function has a one-to-one correspondence, there cannot be any turning points on the graph. The graph of a decreasing function with no stationary point of inflection nor any turning points, and which passes through $(1, 0)$ and $(0, -17)$ is required.



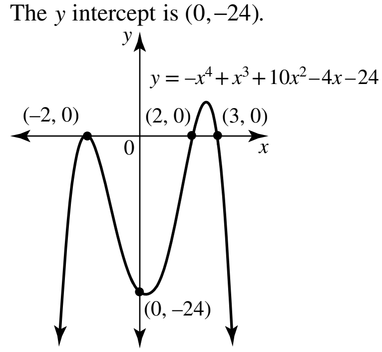
21 a Let the equation be $y = a(x-h)^4 + k$
Turning point is $(-5, 12)$
$\therefore y = a(x+5)^4 + 12$
Substitute the point $(-3, -36)$
$\therefore -36 = a(2)^4 + 12$
$\therefore 16a = -48$
$\therefore a = -3$
The equation is $y = -3(x+5)^4 + 12$.

b $y = (2+x)(1-x)^3$
x intercept at $(-2, 0)$ and at $(1, 0)$ there is a stationary point of inflection.
y intercept is $(0, 2)$.
The coefficient of x^4 is negative.

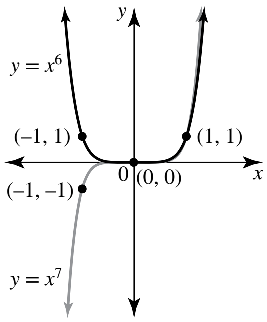


c i $-x^4 + x^3 + 10x^2 - 4x - 24$
Let $f(x) = -x^4 + x^3 + 10x^2 - 4x - 24$
By trial and error,
$f(2) = -16 + 8 + 40 - 8 - 24 = 0 \Rightarrow (x-2)$ is a factor
$f(-2) = -16 - 8 + 40 + 8 - 24 = 0 \Rightarrow (x+2)$ is a factor
Therefore, $(x-2)(x+2) = x^2 - 4$ is a factor.
By inspection,
$$\begin{aligned}-x^4 + x^3 + 10x^2 - 4x - 24 &= (x^2 - 4)(-x^2 + x + 6)\\ &= -(x^2 - 4)(x^2 - x - 6)\\ &= -(x-2)(x+2)(x-3)(x+2)\\ &= -(x+2)^2(x-2)(x-3)\end{aligned}$$

ii $y = -x^4 + x^3 + 10x^2 - 4x - 24$
$\therefore y = -(x+2)^2(x-2)(x-3)$
The factors indicate there is a turning point at $(-2, 0)$ and two other x intercepts at $(2, 0)$ and $(3, 0)$.
The y intercept is $(0, -24)$.



22 a i $y = x^6$ and $y = x^7$
$y = x^6$ is of even degree so its graph has similarities with $y = x^2$.
$y = x^7$ is of odd degree so its graph has similarities with $y = x^3$. As it has the higher degree it will steeper than $y = x^6$ for $x > 1$.



The points of intersection of the two graphs are $(0,0)$ and $(1,1)$.

ii $\{x : x^6 + x^7 \geq 0\}$

$x^6 - x^7 \geq 0$ when $x^6 \geq x^7$. The graph of $y = x^6$ lies above that of $y = x^7$ for $x < 0$ and $0 < x < 1$ and the two graphs intersect at $x = 0$ and $x = 1$.

Hence, $\{x : x^6 - x^7 \geq 0\} = \{x : x \leq 1\}$

b $y = 16 - (x+2)^4$

Even degree so maximum turning point at $(-2, 16)$.

y intercept: Let $x = 0$

$\therefore y = 0 \Rightarrow (0, 0)$.

Axis of symmetry $x = -2 \Rightarrow (-4, 0)$ is the other x intercept.

$y = 16 - (x+2)^5$

Odd degree so stationary point of inflection at $(-2, 16)$.

y intercept: Let $x = 0$

$\therefore y = 16 - 32 = -16 \Rightarrow (0, -16)$.

x intercept: Let $y = 0$

$\therefore 0 = 16 - (x+2)^5$

$\therefore (x+2)^5 = 16$

$\therefore x = -2 + \sqrt[5]{16}$

$\left(-2 + \sqrt[5]{16}, 0\right)$

Points of intersection of the two graphs occur when

$16 - (x+2)^4 = 16 - (x+2)^5$

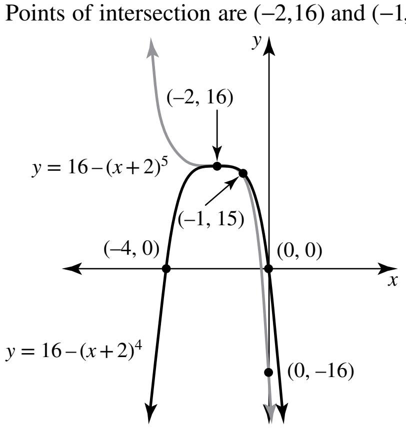
$\therefore (x+2)^4 = (x+2)^5$

$\therefore (x+2)^4[1 - (x+2)] = 0$

$\therefore (x+2)^4(-1-x) = 0$

$\therefore x = -2$ or $x = -1$

Points of intersection are $(-2, 16)$ and $(-1, 15)$.



c i The x intercepts determine the factors.

Graph touches x axis at $x = -3 \Rightarrow (x+3)^2$ is a factor.

Graph cuts x axis at $x = -1 \Rightarrow (x+1)$ is a factor.

Graph saddle cuts x axis at $x = 2 \Rightarrow (x-2)^3$ is a factor.

These factors imply the degree is $2 + 1 + 3 = 6$. The shape suggests the long term behaviour of an even degree polynomial function with positive leading term.

The equation is of the form $y = a(x+3)^2(x+1)(x-2)^3$.

Since it is a monic polynomial, $a = 1$.

Therefore the equation is $y = (x+3)^2(x+1)(x-2)^3$, degree 6.

ii An additional cut at $x = 10 \Rightarrow (x-10)$ is also a factor.

The graph would now have the behaviour that as $x \to \infty, y \to -\infty$, showing it to be an odd degree with a negative coefficient of its leading term.

The degree is 7 and a possible equation is $y = (x+3)^2(x+1)(x-2)^3(10-x)$.

23 Draw the graphs using CAS. At the intersection of the two graphs $x^4 - 2 = 2 - x^3$.

And therefore, $x^4 + x^3 - 4 = 0$. The roots of the equation are the x co-ordinates of the points of intersection of the two graphs. Use CAS to obtain these.

To two decimal places the roots are $x = -1.75$ and $x = 1.22$.

24 Sketch each graph and use the tools to obtain the points required.

a $y = (x^2 + x + 1)(x^2 - 4)$

Minimum turning points $(-1.31, -3.21)$ and $(1.20, -9.32)$, maximum turning point $(-0.636, -2.76)$.

b $y = 1 - 4x - x^2 - x^3$

No turning points or stationary point of inflection.

c $y = \frac{1}{4}\left((x-2)^5(x+3) + 80\right)$

Minimum turning point $(-2.17, -242)$, stationary point of inflection $(2, 20)$.

Exercise 2.3 — Other algebraic functions

1 a Let the equation be $y = \frac{a}{x-h} + k$.

Asymptotes are $x = -3$ and $y = 1$.

$\therefore\ y = \frac{a}{x+3} + 1$

Graph has an x intercept at $x = -9$.

Substitute the point $(-9, 0)$

$\therefore 0 = \frac{a}{-6} + 1$

$\therefore a = 6$

The equation is $y = \frac{6}{x+3} + 1$.

b i $y = \frac{5x-2}{x-1}$

If $x = 1$ the denominator would be zero and the function undefined. Its maximal domain is $R \setminus \{1\}$.

ii $\frac{5x-2}{x-1} = \frac{5(x-1)+3}{x-1}$

$= 5 + \frac{3}{x-1}$

$y = 5 + \frac{3}{x-1}$ has asymptotes $x = 1, y = 5$.

y intercept: Let $x = 0$, then $y = 2$. $(0, 2)$

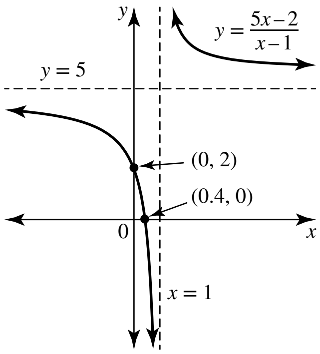
x intercept: Let $y = 0$

$\therefore \frac{5x-2}{x-1} = 0$

$\therefore 5x - 2 = 0$

$\therefore x = \frac{2}{5}$

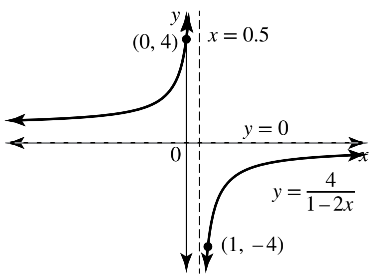
$\left(\frac{2}{5}, 0\right)$



Range is $R \setminus \{5\}$.

2 $y = \frac{4}{1-2x}$

Vertical asymptote: $1 - 2x = 0 \Rightarrow x = \frac{1}{2}$ is the vertical asymptote.
Horizontal asymptote is $y = 0$. There is no x intercept.
y intercept: Let $x = 0$
$\therefore y = 4 \quad (0, 4)$

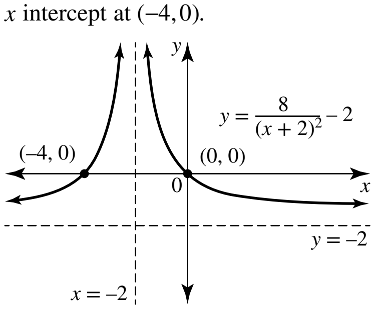


Domain $R \setminus \left\{\frac{1}{2}\right\}$ and range $R \setminus \{0\}$.

3 $y = \frac{8}{(x+2)^2} - 2$

Asymptotes: $x = -2,\ y = -2$
y intercept: Let $x = 0$
$\therefore y = \frac{8}{(2)^2} - 2$
$\therefore y = 0$

The origin is the intercept on both the axes. By symmetry about the vertical axis there is another x intercept at $(-4, 0)$.



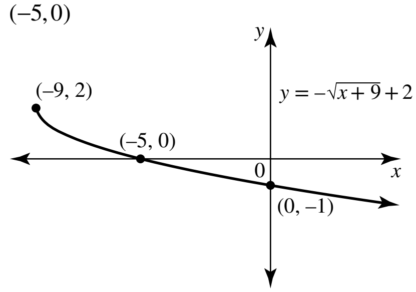
Domain $R \setminus \{-2\}$ and range $(-2, \infty)$.

4 Let the equation be $y = \frac{a}{(x-h)^2} + k$
Asymptotes are $x = 0$ and $y = -1$
$\therefore y = \frac{a}{x^2} - 1$
Substitute the point $\left(\frac{1}{2}, 0\right)$
$\therefore 0 = \frac{a}{\left(\frac{1}{2}\right)^2} - 1$
$\therefore 0 = 4a - 1$
$\therefore a = \frac{1}{4}$
The equation is $y = \frac{1}{4x^2} - 1$.

5 **a** $y = -\sqrt{x+9} + 2$

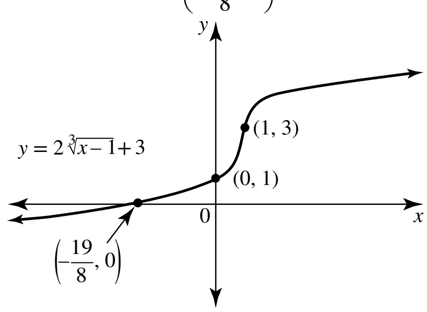
i For the function to have real values, $x + 9 \ge 0$. This means $x \ge -9$ so the maximal domain is $[-9, \infty)$.

ii endpoint: $(-9, 2)$
y intercept: Let $x = 0$
$\therefore y = -\sqrt{9} + 2$
$\therefore y = -1$
$(-1, 0)$
x intercept: Let $y = 0$
$\therefore 0 = -\sqrt{x+9} + 2$
$\therefore \sqrt{x+9} = 2$
$\therefore x + 9 = 4$
$\therefore x = -5$
$(-5, 0)$



Range is $(-\infty, 2]$.

b Let the equation be $y = a\sqrt[3]{(x-h)} + k$.
Point of inflection is $(1, 3)$
$\therefore y = a\sqrt[3]{x-1} + 3$
Substitute the point $(0, 1)$
$\therefore 1 = a\sqrt[3]{-1} + 3$
$\therefore 1 = -a + 3$
$\therefore a = 2$
The equation is $y = 2\sqrt[3]{x-1} + 3$.
x intercept: Let $y = 0$
$\therefore 2\sqrt[3]{x-1} + 3 = 0$
$\therefore \sqrt[3]{x-1} = -\frac{3}{2}$
$\therefore x - 1 = \left(-\frac{3}{2}\right)^3$
$\therefore x = 1 - \frac{27}{8}$
$\therefore x = -\frac{19}{8}$
The x intercept is $\left(-\frac{19}{8}, 0\right)$.



6 a $y = 3\sqrt{4x-9} - 6$

Domain requires $4x - 9 \ge 0 \Rightarrow x \ge \frac{9}{4}$.

Maximal domain is $\left[\frac{9}{4}, \infty\right)$.

Endpoint: When $4x - 9 = 0,\ x = \frac{9}{4}$

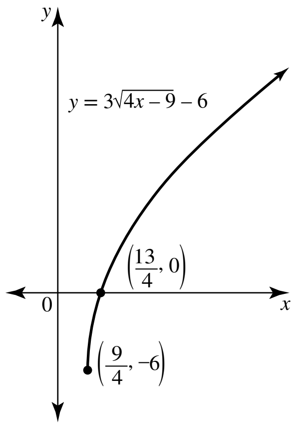
The endpoint is $\left(\frac{9}{4}, -6\right)$.

There is no y intercept.

x intercept: Let $y = 0$

$$\therefore 3\sqrt{4x-9} - 6 = 0$$
$$\therefore \sqrt{4x-9} = 2$$
$$\therefore 4x - 9 = 4$$
$$\therefore 4x = 13$$
$$\therefore x = \frac{13}{4}$$

$\left(\frac{13}{4}, 0\right)$



Range is $[-6, \infty)$.

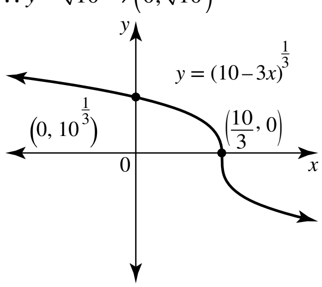
b $y = (10 - 3x)^{\frac{1}{3}}$. This is the cube root function $y = \sqrt[3]{10 - 3x}$.

When $10 - 3x = 0,\ x = \frac{10}{3}$, so the point of inflection is $\left(\frac{10}{3}, 0\right)$.

The point of inflection lies on the x axis.

y intercept: Let $x = 0$

$\therefore y = \sqrt[3]{10} \Rightarrow \left(0, \sqrt[3]{10}\right)$

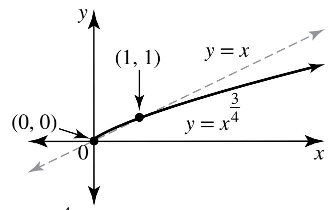


7 a $y = x^{\frac{3}{4}}$

$y = \sqrt[4]{x^3} \Rightarrow 4^{th}$ root of x^3

As the even root of the third quadrant section of the x^3 polynomial cannot be taken, the graph has one first quadrant branch with domain $R^+ \cup \{0\}$.

As $4 > 3$, the root shape dominates. The graph contains the points $(0,0)$ and $(1,1)$ and lies above $y = x$ for $0 < x < 1$ and below $y = x$ for $x > 1$.

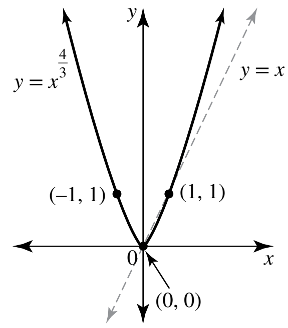


b $y = x^{\frac{4}{3}}$

Since $y = \sqrt[3]{x^4}$ the cube root of x^4 is required.

The x^4 polynomial lies in first and second quadrants. The cube root of both sections can be formed so the graph has two branches and domain R.

As $4 > 3$, the polynomial shape dominates. The graph contains the points $(0,0)$, $(1,1)$, $(-1,1)$ and lies below $y = x$ for $0 < x < 1$ and above $y = x$ for $x > 1$.



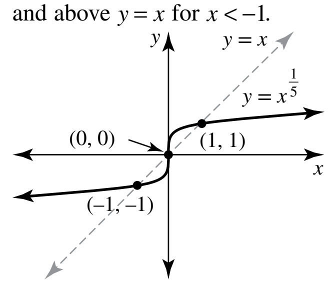
8 a $y = x^{\frac{1}{5}}$

$y = \sqrt[5]{x} \Rightarrow 5^{th}$ root of x

The line $y = x$ lies in first and third quadrants. The fifth root of both sections can be formed so the graph has two branches and domain R.

For the first quadrant, the graph contains the points $(0,0)$ and $(1,1)$ and lies above $y = x$ for $0 < x < 1$ and below $y = x$ for $x > 1$.

By symmetry for the third quadrant, the graph contains the points $(0,0)$ and $(-1,-1)$ and lies below $y = x$ for $-1 < x < 0$ and above $y = x$ for $x < -1$.

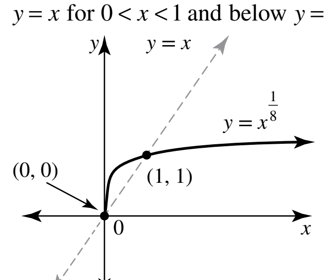


b $y = x^{\frac{1}{8}}$

$y = \sqrt[8]{x} \Rightarrow 8^{th}$ root of x.

As the even root of the third quadrant section of the $y = x$ line cannot be taken, the graph has one first quadrant branch with domain $R^+ \cup \{0\}$.

The graph contains the points $(0,0)$ and $(1,1)$ and lies above $y = x$ for $0 < x < 1$ and below $y = x$ for $x > 1$.



9 a $y = \dfrac{x-6}{x+9}$

If $x + 9 = 0$ then $x = -9$ and the function would be undefined.

Domain is $R \setminus \{-9\}$.

b $y = \sqrt{1-2x}$

The domain requires $1 - 2x \ge 0$

$\therefore x \le \dfrac{1}{2}$

Domain $\left(-\infty, \dfrac{1}{2}\right]$

c $\dfrac{-2}{(x+3)^2}$

Denominator would be zero if $x = -3$, so the domain is $R \setminus \{-3\}$.

d $\dfrac{1}{x^2+3}$

Since the denominator is the sum of two positive terms, it can never be zero, so the domain is R.

10 a $y = \dfrac{4}{x} + 5$

Asymptotes: $x = 0, y = 5$

No y intercept

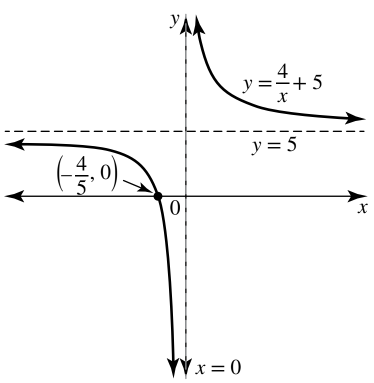
x intercept: Let $y = 0$

$\therefore \dfrac{4}{x} + 5 = 0$

$\therefore 4 = -5x$

$\therefore x = -\dfrac{4}{5}$

$\left(-\dfrac{4}{5}, 0\right)$



Domain $R \setminus \{0\}$ and range $R \setminus \{5\}$.

b $y = 2 - \dfrac{3}{x+1}$

Asymptotes: $x = -1, y = 2$

y intercept: Let $x = 0$

$\therefore y = 2 - \dfrac{3}{1} = -1$

$(0, -1)$

x intercept: Let $y = 0$

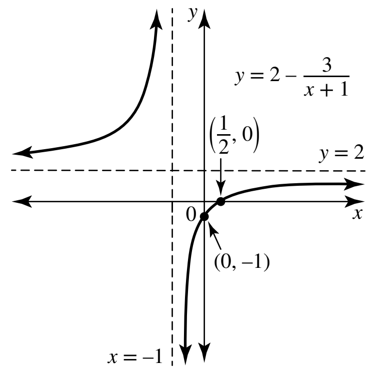
$\therefore 2 - \dfrac{3}{x+1} = 0$

$\therefore 2 = \dfrac{3}{x+1}$

$\therefore 2x + 2 = 3$

$\therefore x = \dfrac{1}{2}$

$\left(\dfrac{1}{2}, 0\right)$



Domain $R \setminus \{-1\}$ and range $R \setminus \{2\}$.

c $y = \dfrac{4x+3}{2x+1}$

$\therefore y = \dfrac{2(2x+1)+1}{2x+1}$

$\therefore y = 2 + \dfrac{1}{2x+1}$

Asymptotes: $2x + 1 = 0 \Rightarrow x = -\dfrac{1}{2},\ y = 2$

y intercept: Let $x = 0$

$\therefore y = \dfrac{3}{1} = 3$

$(0, 3)$

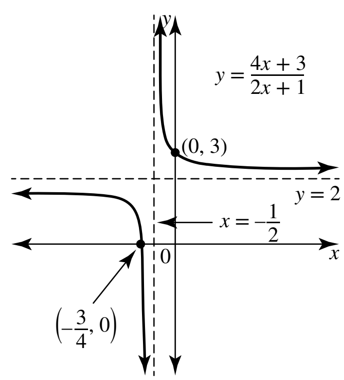
x intercept: Let $y = 0$

$\therefore \dfrac{4x+3}{2x+1} = 0$

$\therefore 4x + 3 = 0$

$\therefore x = -\dfrac{3}{4}$

$\left(-\dfrac{3}{4}, 0\right)$



Domain $R \setminus \left\{-\dfrac{1}{2}\right\}$ and range $R \setminus \{2\}$.

d $xy + 2y + 5 = 0$

$\therefore y(x+2) = -5$

$\therefore y = \dfrac{-5}{x+2}$

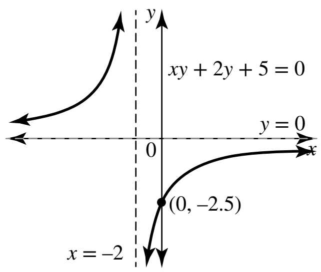
Asymptotes: $x = -2,\ y = 0$

No x intercept.

y intercept: Let $x = 0$

$\therefore y = \dfrac{-5}{2} = -2.5$

$(0, -2.5)$



Domain $R\setminus\{-2\}$ and range $R\setminus\{0\}$.

e $y=\dfrac{10}{5-x}-5$

Asymptotes: $5-x=0\Rightarrow x=5,\ y=-5$

y intercept: Let $x=0$

$\therefore y=\dfrac{10}{5}-5=-3$

$(0,-3)$

x intercept: Let $y=0$

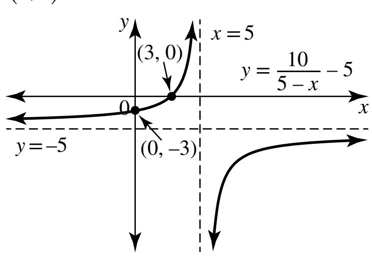
$\therefore \dfrac{10}{5-x}-5=0$

$\therefore 10=5(5-x)$

$\therefore 5-x=2$

$\therefore x=3$

$(3,0)$



Domain $R\setminus\{5\}$ and range $R\setminus\{-5\}$.

11 a Let the equation be $y=\dfrac{a}{x-h}+k$.

Asymptotes $x=-3$ and $y=6$

$\therefore y=\dfrac{a}{x+3}+6$

Substitute the point $(-4,8)$

$\therefore 8=\dfrac{a}{-1}+6$

$\therefore a=-2$

The equation is $y=\dfrac{-2}{x+3}+6$.

b Let the equation be $y=\dfrac{a}{x-h}+k$.

Asymptotes $x=-2$ and $y=-\dfrac{3}{2}$

$\therefore y=\dfrac{a}{x+2}-\dfrac{3}{2}$

Substitute the point $(-3,-2)$

$\therefore -2=\dfrac{a}{-1}-\dfrac{3}{2}$

$\therefore a=\dfrac{1}{2}$

The equation is $y=\dfrac{1}{2(x+2)}-\dfrac{3}{2}$.

12 a $y=\dfrac{2}{(3-x)^2}+1$ or $y=\dfrac{2}{(x-3)^2}+1$

Asymptotes: $x=3, y=1$

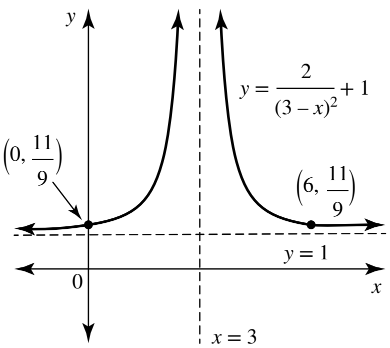
y intercept: Let $x=0$

$\therefore y=\dfrac{2}{9}+1=\dfrac{11}{9}$

$\left(0,\dfrac{11}{9}\right)$

No x intercepts as graph lies above its horizontal asymptote.

The point $\left(6,\dfrac{11}{9}\right)$ is symmetric about the vertical asymptote to $\left(0,\dfrac{11}{9}\right)$.



Domain $R\setminus\{3\}$ and range $(1,\infty)$

b $y=\dfrac{-3}{4(x-1)^2}-2$

Asymptotes: $x=1, y=-2$

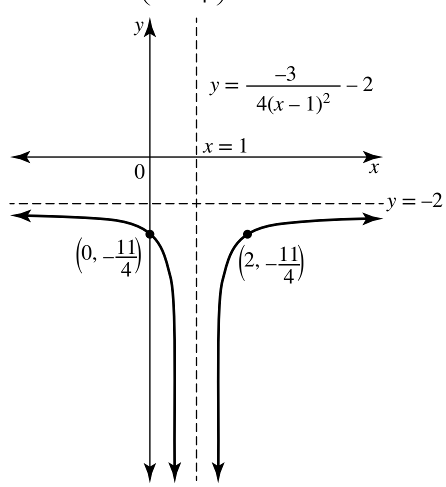
y intercept: Let $x=0$

$\therefore y=\dfrac{-3}{4}-2=-\dfrac{11}{4}$

$\left(0,-\dfrac{11}{4}\right)$

No x intercepts as graph lies below its horizontal asymptote.

The point $\left(2,-\dfrac{11}{4}\right)$ is symmetric about the vertical asymptote to $\left(0,-\dfrac{11}{4}\right)$.



Domain $R\setminus\{1\}$ and range $(-\infty,-2)$.

c $y=\dfrac{1}{(2x+3)^2}-1$

Asymptotes: $2x+3=0\Rightarrow x=-\dfrac{3}{2}, y=-1$

y intercept: Let $x=0$

$\therefore y=\dfrac{1}{9}-1=-\dfrac{8}{9}$

$\left(0,-\dfrac{8}{9}\right)$

x intercepts: Let $y=0$

$\therefore \frac{1}{(2x+3)^2}-1=0$

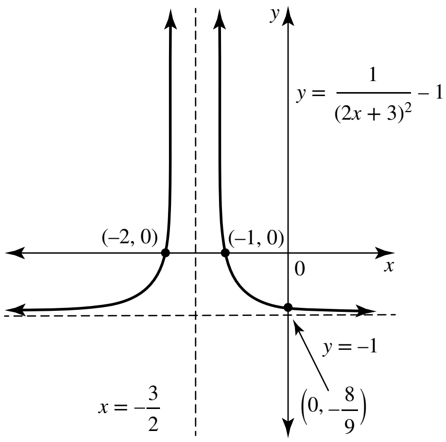
$\therefore 1=(2x+3)^2$

$\therefore 2x+3=\pm 1$

$\therefore 2x=-4 \text{ or } -2$

$\therefore x=-2 \text{ or } x=-1$

$(-2,0),(-1,0)$



Domain $R\setminus\left\{-\frac{3}{2}\right\}$ and range $(-1,\infty)$.

d $y=\frac{25x^2-1}{5x^2}$

$\therefore y=\frac{25x^2}{5x^2}-\frac{1}{5x^2}$

$\therefore y=5-\frac{1}{5x^2}$

Asymptotes: $x=0, y=5$

No y intercept

x intercepts: Let $y=0$

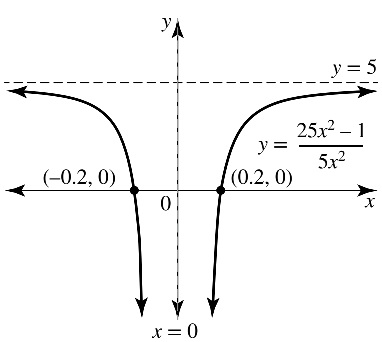
$\therefore \frac{25x^2-1}{5x^2}=0$

$\therefore 25x^2-1=0$

$\therefore x^2=\frac{1}{25}$

$\therefore x=\pm\frac{1}{5}$

$\left(\pm\frac{1}{5},0\right)$



Domain $R\setminus\{0\}$ and range $(-\infty,5)$.

13 a Let the equation be $y=\frac{a}{(x-h)^2}+k$

Asymptotes are $x=4$ and $y=2$

$\therefore y=\frac{a}{(x-4)^2}+2$

Substitute the point $(5,-1)$

$\therefore -1=\frac{a}{1}+2$

$\therefore a=-3$

The equation is $y=\frac{-3}{(x-4)^2}+2$.

b Let the equation of the graph be $y=\frac{a}{(x-h)^2}+k$

As the range is $(-4,\infty)$, the horizontal asymptote is $y=-4$.

$\therefore y=\frac{a}{(x-h)^2}-4$

Given $f(-1)=8$ and $f(2)=8$, the points $(-1,8)$ and $(2,8)$ lie on the graph. As these points have the same y coordinate, they must be symmetrically placed around the vertical asymptote.

Therefore, the vertical asymptote is $x=\frac{-1+2}{2}=\frac{1}{2}$

The equation becomes $y=\frac{a}{\left(x-\frac{1}{2}\right)^2}-4$

This can be written as $y=\frac{4a}{(2x-1)^2}-4$ or $y=\frac{b}{(2x-1)^2}-4$ where $b=4a$.

Substitute the point $(2,8)$

$\therefore 8=\frac{b}{9}-4$

$\therefore b=108$

The equation of the graph is $y=\frac{108}{(2x-1)^2}-4$.

The domain is $R\setminus\left\{\frac{1}{2}\right\}$ so the function is

$f: R\setminus\left\{\frac{1}{2}\right\}\to R, f(x)=\frac{108}{(2x-1)^2}-4$.

14 a i $(y-2)^2=4(x-3)$

$\therefore y-2=\pm\sqrt{4(x-3)}$

$\therefore y=2\pm 2\sqrt{(x-3)}$

The upper branch is the function with rule $y=2\sqrt{(x-3)}+2$. Its domain is $[3,\infty)$.
Its endpoint is $(3,2)$ so its range is $[2,\infty)$.
The other function is the lower branch $y=-2\sqrt{(x-3)}+2$ with the same domain $[3,\infty)$ but range $(-\infty,2]$.

ii $y^2+2y+2x=5$

Completing the square

$\left(y^2+2y+1\right)-1+2x=5$

$\therefore (y+1)^2=6-2x$

$\therefore y+1=\pm\sqrt{6-2x}$

$\therefore y=\pm\sqrt{6-2x}-1$

The upper branch is the function with rule $y=\sqrt{-2(x-3)}-1$ and endpoint $(3,-1)$.
Its domain requires $6-2x\ge 0\Rightarrow x\le 3$. The domain is $(-\infty,3]$ and range $[-1,\infty)$.
The lower branch is the function with rule $y=-\sqrt{-2(x-3)}-1$. Its domain is $(-\infty,3]$ and range $(-\infty,-1]$.

b i $y=1-\sqrt{3x}$

Domain: $3x\ge 0\Rightarrow x\ge 0$. Domain is $[0,\infty)$

Endpoint: $(0,1)$ which is also the y intercept.

x intercept: Let $y=0$

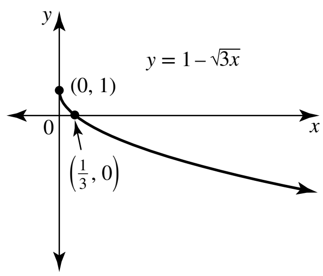
$\therefore 1-\sqrt{3x}=0$

$\therefore \sqrt{3x}=1$

$\therefore 3x=1$

$\therefore x=\frac{1}{3}$

$\left(\frac{1}{3},0\right)$



Range is $(-\infty,1]$.

ii $y=2\sqrt{-x}+4$

Domain: $-x\ge 0 \Rightarrow x\le 0$ Domain is $(-\infty,0]$.

Endpoint: $(0,4)$ which is also the y intercept.

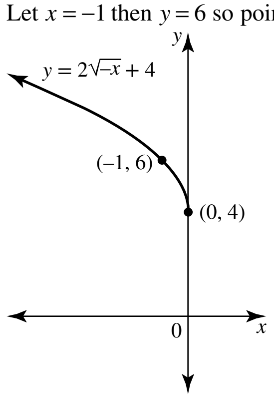
x intercept: Let $y=0$

$\therefore 2\sqrt{-x}+4=0$

$\therefore \sqrt{-x}=-2$

This is not possible so there is no x intercept (also possible to anticipate this as $a>0$).

Let $x=-1$ then $y=6$ so point on the graph is $(-1,6)$.



Range is $[4,\infty)$.

iii $y=2\sqrt{4+2x}+3$

Domain: $4+2x\ge 0 \Rightarrow x\ge -2$, $[-2,\infty)$

Endpoint: $(-2,3)$

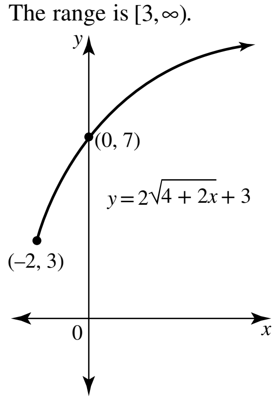
y intercept: Let $x=0$

$\therefore y=2\sqrt{4}+3=7$

$(0,7)$

There will not be an x intercept.

The range is $[3,\infty)$.



iv $y=-\sqrt{3}-\sqrt{12-3x}$

Domain: $12-3x\ge 0 \Rightarrow x\le 4$, $(-\infty,4]$.

Endpoint: $(4,-\sqrt{3})$

y intercept: Let $x=0$

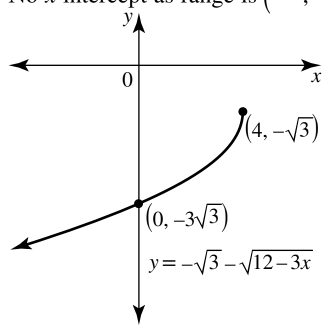
$\therefore y=-\sqrt{3}-\sqrt{12}$

$\therefore y=-\sqrt{3}-2\sqrt{3}$

$\therefore y=-3\sqrt{3}$

$\left(0,-3\sqrt{3}\right)$

No x intercept as range is $\left(-\infty,-\sqrt{3}\right]$.



15 a $f:[5,\infty)\to R, f(x)=a\sqrt{x+b}+c$

The endpoint of the graph is $(5,-2)$ so $f(x)=a\sqrt{x-5}-2$

The point $(6,0)$ is on the graph so $f(6)=0$.

$\therefore 0=a\sqrt{6-5}-2$

$\therefore 0=a-2$

$\therefore a=2$

Hence, $f(x)=2\sqrt{x-5}-2$ with $a=2,\ b=-5,\ c=-2$.

b $f:(-\infty,2]\to R, f(x)=\sqrt{ax+b}+c$

i Let $y=\sqrt{ax+b}+c$

Endpoint is $(2,-2)$ so $a(2)+b=0$ and $c=-2$

$\therefore b=-2a$ and $c=-2$

$y=\sqrt{ax-2a}-2$

Substitute $(0,0)$

$\therefore 0=\sqrt{-2a}-2$

$\therefore \sqrt{-2a}=2$

$\therefore -2a=4$

$\therefore a=-2$

Since $b=-2a$, $b=4$

$f(x)=\sqrt{-2x+4}-2$ with $a=-2, b=4, c=-2$.

ii Reflecting the graph in the x axis would make the endpoint $(2,2)$ and the range $(-\infty,2]$. The graph would still pass through the origin.

The equation of the reflected graph would be $y=-\sqrt{-2x+4}+2$.

16 a $\{(x,y): y=\sqrt[3]{x+2}-1\}$

$y=\sqrt[3]{x+2}-1$

Point of inflection $(-2,-1)$

y intercept: Let $x=0$

$\therefore y=\sqrt[3]{2}-1$

$\left(0,\sqrt[3]{2}-1\right)$

x intercept: Let $y=0$

$\therefore 0=\sqrt[3]{x+2}-1$

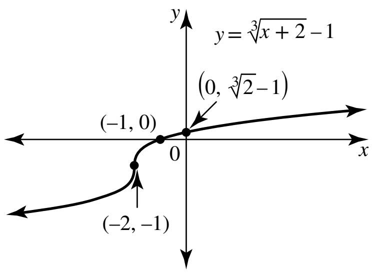
$\therefore \sqrt[3]{x+2}=1$

$\therefore x+2=1^3$

$\therefore x+2=1$

$\therefore x=-1$

$(-1,0)$



b $f(x)=\dfrac{1-\sqrt[3]{x+8}}{2}$

Let $y=f(x)$

$\therefore y=\dfrac{1}{2}\left(1-\sqrt[3]{x+8}\right)$

$=\dfrac{1}{2}-\dfrac{1}{2}\sqrt[3]{x+8}$

The implied domain is R and the range is R.

Point of inflection $\left(-8, \dfrac{1}{2}\right)$

y intercept: Let $x=0$

$\therefore y=\dfrac{1}{2}(1-\sqrt[3]{8})$

$=\dfrac{1}{2}(1-2)$

$=-\dfrac{1}{2}$

$\left(0, -\dfrac{1}{2}\right)$

x intercept: Let $y=0$

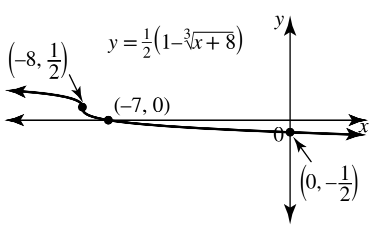
$\therefore 0=\dfrac{1}{2}\left(1-\sqrt[3]{x+8}\right)$

$\therefore \sqrt[3]{x+8}=1$

$\therefore x+8=1$

$\therefore x=-7$

$(-7, 0)$



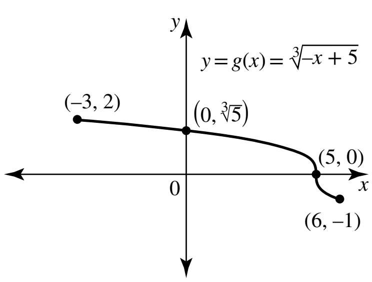
c $g:[-3,6]\to R, g(x)=\sqrt[3]{-x+5}$

Endpoints: $g(-3)=\sqrt[3]{8}=2$, so $(-3,2)$ is an endpoint.

$g(6)=\sqrt[3]{-1}=-1$, so $(6,-1)$ is an endpoint.

Point of inflection: $(5,0)$ which is also the x intercept.

y intercept: $g(0)=\sqrt[3]{5}\Rightarrow\left(0, \sqrt[3]{5}\right)$



Domain $[-3,6]$ and range $[-1,2]$.

d Let the equation be $y=a\sqrt[3]{x-h}+k$

Point of inflection is $(0,-2)$

$\therefore y=a\sqrt[3]{x}-2$

Substitute the point $(1,0)$

$\therefore 0=a\sqrt[3]{1}-2$

$\therefore 0=a-2$

$\therefore a=2$

The equation is $y=2\sqrt[3]{x}-2$.

e Let the equation be $y=a\sqrt[3]{x-h}+k$

The tangent is vertical at the point of inflection so $(-1,-2)$ is the point of inflection.

$\therefore y=a\sqrt[3]{x+1}-2$

Substitute the point $(-9,5)$

$\therefore 5=a\sqrt[3]{-8}-2$

$\therefore 5=-2a-2$

$\therefore 2a=-7$

$\therefore a=-\dfrac{7}{2}$

The equation is $y=-\dfrac{7\sqrt[3]{x+1}}{2}-2$

f $(y+2)^3=64x-128$

Take the cube root of each side

$\therefore y+2=\sqrt[3]{64x-128}$

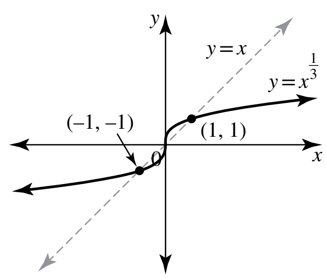
$\therefore y=\sqrt[3]{64(x-2)}-2$

$\therefore y=4\sqrt[3]{(x-2)}-2$

The point of inflection is $(2,-2)$.

17 a $y=x^{\frac{1}{3}}$ is $y=\sqrt[3]{x}$ and its graph could be formed by drawing the line $y=x$ and constructing its cube root. The root shape dominates since $3>1$.

b The two graphs both contain the points $(1,1),(0,0)$ and $(-1,-1)$. The line lies in quadrants 1 and 3. Since cube roots of negative numbers can be taken, the graph of $y=x^{\frac{1}{3}}$ will exist in both quadrants 1 and 3.



c $x^{\frac{1}{3}}-x>0$ when $x^{\frac{1}{3}}>x$.

From the diagram this occurs for $0<x<1$ and if $x<-1$

The solution set is $\{x: x<-1\}\cup\{x: 0<x<1\}$.

18 a $y=x^{\frac{5}{2}}$

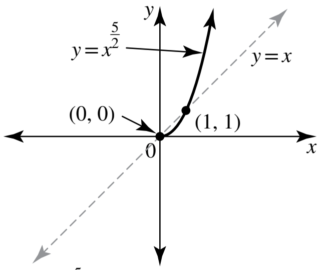
$\therefore y=\sqrt{x^5}$

The graph of $y=x^5$ lies in the first and third quadrants. However, where $x^5<0$, the square root of these values cannot be formed.

Therefore, the graph of $y=x^{\frac{5}{2}}$ lies only in the first quadrant and has domain $R^+\cup\{0\}$ and range $R^+\cup\{0\}$.

As $5>2$, the polynomial shape dominates the function $y=\sqrt{x^5}$.

The graph intersects the line $y=x$ at $(0,0)$ and $(1,1)$.



b $y = x^{\frac{5}{3}}$

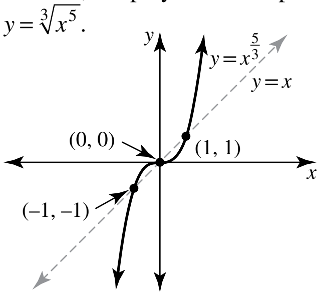
$\therefore y = \sqrt[3]{x^5}$

The cube root of both the negative and positive sections of $y = x^5$ can be formed.

Therefore, the graph of $y = x^{\frac{5}{3}}$ lies in both the first and third quadrants and has domain R and range R.

The graph intersects the line $y = x$ at $(0,0)$, $(1,1)$ and $(-1,-1)$.

As $5 > 3$, the polynomial shape dominates the function $y = \sqrt[3]{x^5}$.



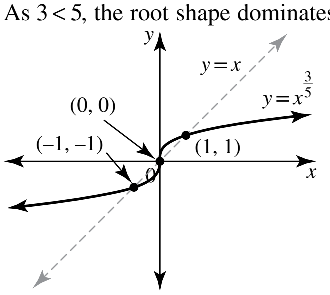
c $y = x^{\frac{3}{5}}$

$\therefore y = \sqrt[5]{x^3}$

The graph lies in both the first and third quadrants and has domain R and range R.

The graph intersects the line $y = x$ at $(0,0)$, $(1,1)$ and $(-1,-1)$.

As $3 < 5$, the root shape dominates the function $y = \sqrt[5]{x^3}$.



d $y = x^{0.25}$

$\therefore y = x^{\frac{1}{4}}$

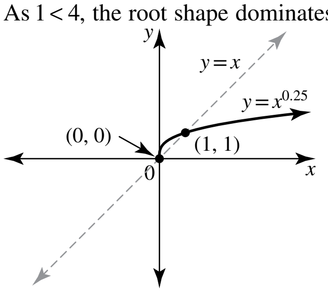
$\therefore y = \sqrt[4]{x}$

The line $y = x$ lies in the first and third quadrants. The fourth root of its negative sections cannot be formed.

Therefore the graph of $y = x^{0.25}$ lies only in the first quadrants and has domain $R^+ \cup \{0\}$ and range $R^+ \cup \{0\}$.

The graph intersects the line $y = x$ at $(0,0)$ and $(1,1)$.

As $1 < 4$, the root shape dominates the function $y = \sqrt[4]{x^1}$.



19 $y = \dfrac{1}{x^2 - 4}$, $y = \dfrac{1}{x^2 + 4}$ and $y = \dfrac{1}{(x-4)^2}$.

Draw each of the graphs on your device and determine where their asymptotes lie. Vertical asymptotes will exist for any value of x which makes the denominator zero, so these could be found by reasoning. Maximal domains may also be obtained by reasoning.

	Domain	**Range**	**Asymptotes**
$y = \dfrac{1}{x^2 - 4}$	$R \setminus \{\pm 2\}$	$R \setminus \left(-\frac{1}{4}, 0\right]$	$x = \pm 2, y = 0$
$y = \dfrac{1}{x^2 + 4}$	R	$\left(0, \frac{1}{4}\right]$	$y = 0$
$y = \dfrac{1}{(x-4)^2}$	$R \setminus \{4\}$	$(0, \infty)$	$x = 4, y = 0$

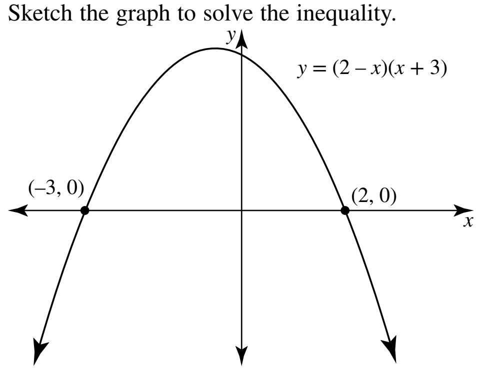
The graph of $y = \dfrac{1}{(x-4)^2}$ is a truncus.

20 $y = \sqrt{(2-x)(x+3)}$

For the graph to exist, $(2-x)(x+3) \geq 0$

Solve $(2-x)(x+3) = 0 \Rightarrow x = 2, -3$.

Sketch the graph to solve the inequality.



$x \in [-3, 2]$

Exercise 2.4 — Combinations of functions

1 a $f(x) = \begin{cases} -\sqrt[3]{x}, & x < -1 \\ x^3, & -1 \leq x \leq 1 \\ 2 - x, & x > 1 \end{cases}$

$f(-8)$: Use the rule $f(x) = -\sqrt[3]{x}$

$f(-8) = -\sqrt[3]{(-8)}$
$= 2$

$f(-1)$: Use the rule $f(x) = x^3$

$f(-1) = (-1)^3$
$= -1$

$f(2)$: Use the rule $f(x) = 2 - x$

$f(2) = 0$

b $f(x) = -\sqrt[3]{x}$

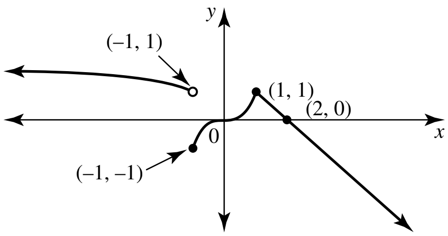
If $x = -1$, $f(-1) = 1$. Point $(-1,1)$ is open for the cube root function. The point $(-8,2)$ lies on this branch.

$f(x) = x^3$

Stationary point of inflection at the origin. The points $(-1,-1)$ and $(1,1)$ are closed points for this cubic.

$f(x) = 2 - x$

The point $(1,1)$ is an open point for the line and the point $(2,0)$ lies on the line.



c **i** The function is not continuous at $x = -1$.

ii Domain is R and range is R.

2 The line for which $x < 0$ has equation $y = x$, the horizontal line for $x \in (0, 4)$ is $y = 4$ and the line for $4 < x < 8$ is also $y = x$ closed at $x = 8$. The function is continuous so one way to express its rule is

$$y = \begin{cases} x+4, & x < 0 \\ 4, & 0 \le x < 4 \\ x, & 4 \le x \le 8 \end{cases}$$

3 $f(x) = -\sqrt{1+x}$ and $g(x) = -\sqrt{1-x}$

Domains: $x + 1 \ge 0 \Rightarrow x \ge -1$ and $1 - x \ge 0 \Rightarrow x \le -1$

$d_f = [-1, \infty)$ and $d_g = (-\infty, 1]$.

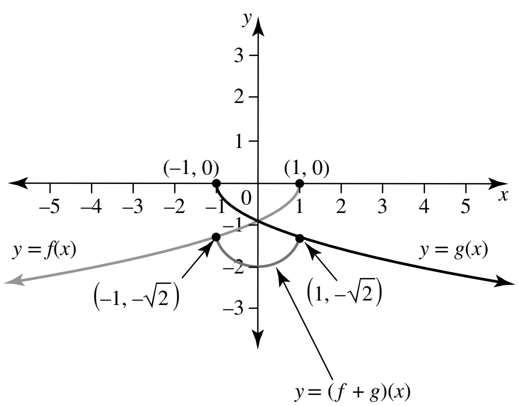
a Since $d_f = [-1, \infty)$, $d_g = (-\infty, 1]$ then $d_f \cap d_g = [-1, 1]$.

$$\begin{aligned} y &= (f+g)(x) \\ &= f(x) + g(x) \\ &= -\sqrt{1+x} - \sqrt{1-x} \end{aligned}$$

Domain is the same as $d_f \cap d_g = [-1, 1]$.

Graph is obtained from $f(x) = -\sqrt{1+x}$ and $g(x) = -\sqrt{1-x}$

x	-1	0	1
$f(x)$	0	-1	$-\sqrt{2}$
$g(x)$	$-\sqrt{2}$	-1	0
$f(x)+g(x)$	$-\sqrt{2}$	-2	$-\sqrt{2}$



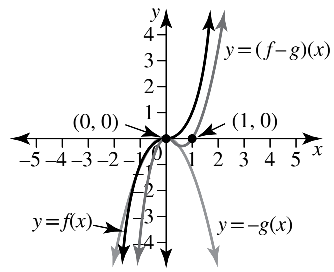
b Domain of fg is the same as $d_f \cap d_g = [-1, 1]$.

$$\begin{aligned} (fg)(x) &= f(x) \times g(x) \\ &= -\sqrt{1+x} \times -\sqrt{1-x} \\ &= \sqrt{(1+x)(1-x)} \\ &= \sqrt{1-x^2} \end{aligned}$$

This is the rule for a semi-circle, top half, centre $(0, 0)$, radius 1. Therefore the range of fg is $[0, 1]$.

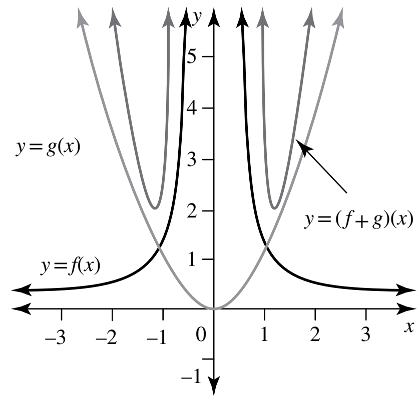
4 $f(x) = x^3$ and $g(x) = x^2$

$$\begin{aligned} (f-g)(x) &= f(x) - g(x) \\ &= x^3 - x^2 \end{aligned}$$



The graphs of f and g intersect when $x = 0, x = 1$ so these must be the x intercepts of $f - g$.

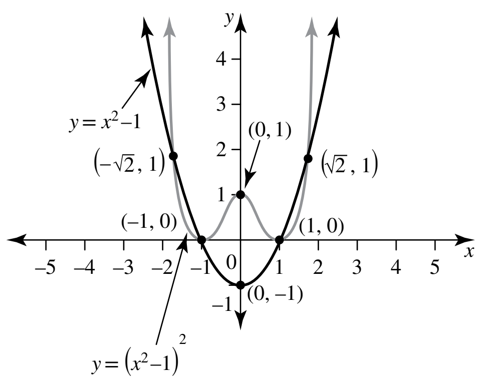
5



6 $y = x^2 - 1$

The parabola has turning point $(0, -1)$ and x intercepts $(\pm 1, 0)$.

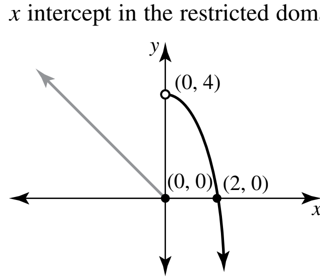
The graph of $y = (x^2 - 1)^2$ will have the same x intercepts but the point $(0, -1)$ will become the point $(0, 1)$ on this graph. This graph lies on or above the x axis. Its domain is R and its range is $[0, \infty)$.



7 **a** $y = \begin{cases} -2x, & x \le 0 \\ 4 - x^2, & x > 0 \end{cases}$

$y = -2x$ contains points $(0, 0)$ closed and $(-1, 2)$.

$y = 4 - x^2$ has open turning point at $(0, 4)$ and the x intercept in the restricted domain is $(2, 0)$.

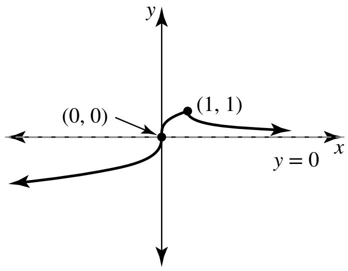


Domain R and range R. Discontinuous at $x = 0$.

b $y=\begin{cases}\sqrt[3]{x}, & x<1\\ \dfrac{1}{x}, & x\ge 1\end{cases}$

$y=\sqrt[3]{x}$ has open point $(1,1)$, inflection point $(0,0)$.

$y=\dfrac{1}{x}$ has horizontal asymptote $y=0$ (vertical asymptote is not in its domain) and closed point $(1,1)$.



Domain R and range $(-\infty,1]$. There is no point where the graph is discontinuous.

8 $f(x)=\begin{cases}\dfrac{1}{(x+1)^2}, & x<-1\\ x^2-x, & -1\le x\le 2\\ 8-2x, & x>2\end{cases}$

a **i** Use rule $f(x)=\dfrac{1}{(x+1)^2}$.

$f(-2)=\dfrac{1}{(-1)^2}=1$

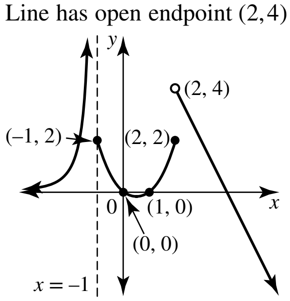
ii Use the rule $f(x)=x^2-x$

$f(2)=(2)^2-(2)=2$

b Truncus has a vertical asymptote $x=-1$ and horizontal asymptote $y=0$.

Parabola has closed endpoints $(-1,3)$ and $(2,2)$ and x intercepts at the origin and $(1,0)$.

Line has open endpoint $(2,4)$ and x intercept $(4,0)$.



c The domain over which the function is continuous is $R\setminus\{-1,2\}$.

9 $f:R\to R, f(x)=\begin{cases}\dfrac{1}{9}x^3+5, & x<-3\\ \sqrt{1-x}, & -3\le x\le 1\\ x-2, & x>1\end{cases}$

a The branch to the left of $x=1$ has the rule $f(x)=\sqrt{1-x}$, so $f(1)=0$.

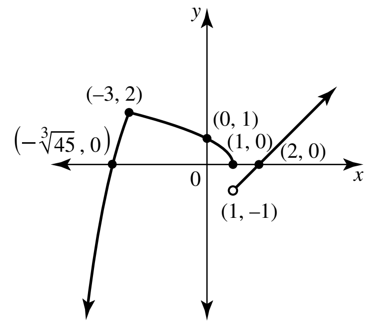
The branch to the right of $x=1$ has the rule $f(x)=x-2$, so $f(1)\to -1$ (open circle).

These branches do not join so the hybrid function is not continuous at $x=1$.

b The cubic function's point of inflection is not in its restricted domain. The point $(-3,2)$ is an open point.

The square root function has closed domain endpoints $(-3,2)$ and $(1,0)$.

The line has open endpoint $(1,-1)$ and contains point $(2,0)$.



Many-to-one correspondence

c $f(x)=4$

The graph shows only the linear branch has a point with $y=4$.

Let $x-2=4$

$\therefore x=6$

10 The left branch is a parabola on domain section where $x<0$.

Let its equation be $y=a(x+3)(x+1)$

Substitute point $(0,4)$

$\therefore 4=a(3)(1)$

$\therefore a=\dfrac{4}{3}$

Parabola branch has equation $y=\dfrac{4}{3}(x+3)(x+1)$.

The middle branch is $y=4$ for domain section $x\in[0,2]$.

The line through points $(3,2)$ and $(4,0)$ has gradient $m=-2$.

Its equation is $y=-2(x-4)$.

The rule for the hybrid function could be expressed as

$y=\begin{cases}\dfrac{4}{3}(x+3)(x+1), & x<0\\ 4, & 0\le x\le 2\\ -2x+8, & x\ge 3\end{cases}$

11 $f(x)=\begin{cases}x+a, & x\in(-\infty,-8]\\ \sqrt[3]{x}+2, & x\in(-8,8]\\ \dfrac{b}{x}, & x\in(8,\infty)\end{cases}$

a The branches must join at $x=-8$.

Left of $x=-8$, $f(x)=x+a$

$f(-8)=-8+a$

Right of $x=-8$, $f(x)=\sqrt[3]{x}+2$

$f(-8)=\sqrt[3]{(-8)}+2=0$

For continuity, $-8+a=0\Rightarrow a=8$

The branches must also join at $x=8$.

Left of $x=8$, $f(x)=\sqrt[3]{x}+2$

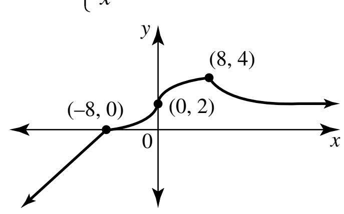
$f(8)=\sqrt[3]{(8)}+2=4$

Right of $x=8$, $f(x)=\dfrac{b}{x}$

$f(8)=\dfrac{b}{8}$

For continuity, $\dfrac{b}{8}=4\Rightarrow b=32$.

$f(x)=\begin{cases}x+8, & x\in(-\infty,-8]\\ \sqrt[3]{x}+2, & x\in(-8,8]\\ \dfrac{32}{x}, & x\in(8,\infty)\end{cases}$

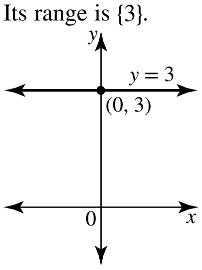


b $f(x)=k$ is a horizontal line.
Looking at the graph to determine in how many ways a horizontal line can intersect the graph gives the values for k.
i no solution if $k>4$
ii one solution if $k=4$ or $k\le 0$
iii two solutions if $0<k<4$

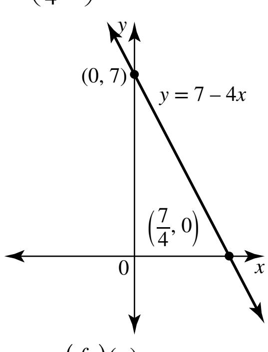
c $\{x : f(x)=1\}$
There will be two solutions, one on the cube root branch and one on the hyperbola branch.
Let $\sqrt[3]{x}+2=1$
$\therefore \sqrt[3]{x}=-1$
$\therefore x=-1$
Let $\dfrac{32}{x}=1$
$\therefore x=32$
The solution set is $\{-1,32\}$.

12 $f(x)=5-2x,\ d_f=R$ and $g(x)=2x-2,\ d_g=R$.

a $y=(f+g)(x)$
Rule:
$y=5-2x+2x-2$
$\therefore y=3$
Domain
$d_{f+g}=d_f\cap d_g$
$\therefore d_{f+g}=R$
Graph of $y=(f+g)(x)$ is the horizontal line through $(0,3)$. Its range is $\{3\}$.



b $y=(f-g)(x)$
Rule:
$y=5-2x-(2x-2)$
$\therefore y=7-4x$
Domain:
$d_{f-g}=d_f\cap d_g$
$\therefore d_{f-g}=R$
Graph of $y=(f-g)(x)$ is a straight line through $(0,7)$ and $\left(\dfrac{7}{4},0\right)$. Its range is R.



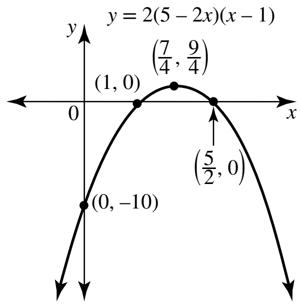
c $y=(fg)(x)$
Rule:
$y=(5-2x)(2x-2)$
$\therefore y=2(5-2x)(x-1)$
Domain: $d_{fg}=d_f\cap d_g=R$
Graph is a concave down parabola with x intercepts $\left(\dfrac{5}{2},0\right)$ and $(1,0)$; y intercept $(0,-10)$.

Turning point: $x=\dfrac{\frac{5}{2}+1}{2}=\dfrac{7}{4}$

$$\begin{aligned}y&=2\left(5-\frac{7}{2}\right)\left(\frac{7}{4}-1\right)\\&=2\times\frac{3}{2}\times\frac{3}{4}\\&=\frac{9}{4}\end{aligned}$$

$\left(\dfrac{7}{4},\dfrac{9}{4}\right)$ is the maximum turning point.

Range is $\left(-\infty,\dfrac{9}{4}\right]$



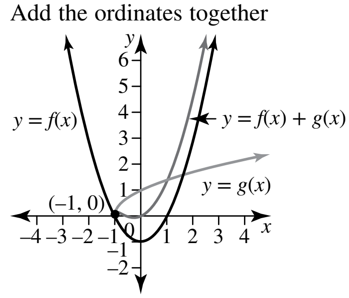
13 $f(x)=x^2-1$ and $g(x)=\sqrt{x+1}$

a **i** $(g-f)(3)$
$=g(3)-f(3)$
$=\sqrt{4}-8$
$=-6$

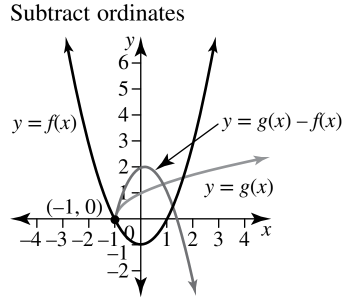
ii $(gf)(8)$
$=g(8)f(8)$
$=\sqrt{9}\times 63$
$=189$

b $d_f=R,\ d_g=[-1,\infty)$
$d_{f+g}=d_f\cap d_g$
$=[-1,\infty)$

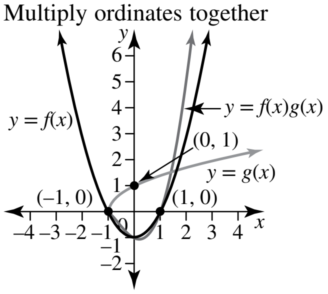
c **i** Graph of $f+g$
Add the ordinates together



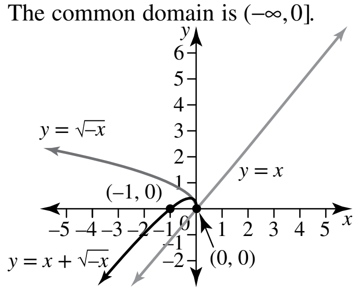
ii Graph of $g-f$
Subtract ordinates



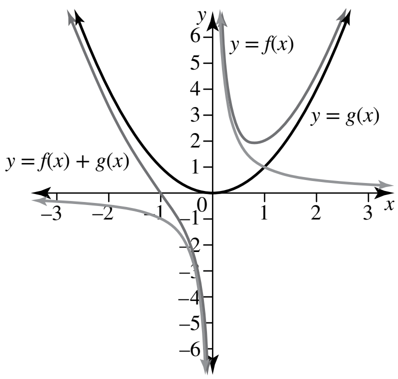
iii Graph of fg
Multiply ordinates together



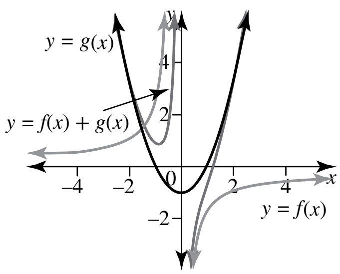
14 $y = x + \sqrt{-x}$
Draw the line $y_1 = x$ and the square root function $y_2 = \sqrt{-x}$.
The common domain is $(-\infty, 0]$.



15 a

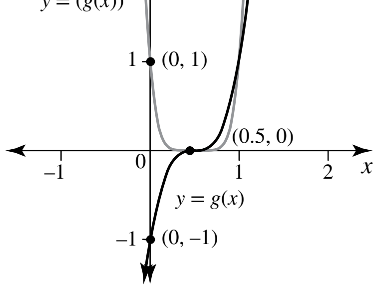


b



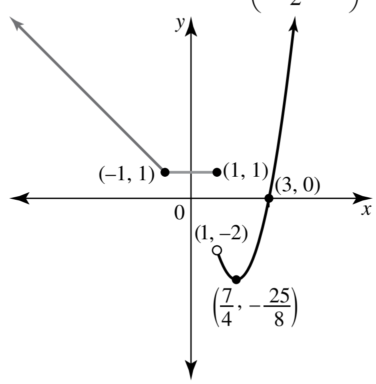
16 a $g(x) = (2x-1)^3$. There is a stationary point of inflection at $\left(\frac{1}{2}, 0\right)$.

$y = (g(x))^2$
Square ordinates



b The graphs of $y = f(x)$ and $y = (f(x))^2$ will intersect at places where $y = 0$ or $y = 1$.
For the function $f(x) = x^3 - 2x$, let $f(x) = 0$.
$\therefore x(x^2 - 2) = 0$
$\therefore x = 0, x = \pm\sqrt{2}$
Let $f(x) = 1$
$\therefore x^3 - 2x = 1$
$\therefore x^3 - 2x - 1 = 0$
$\therefore (x+1)(x^2 - x - 1) = 0$
$\therefore x = -1 \text{ or } x^2 - x - 1 = 0$
Consider $x^2 - x - 1 = 0$
$$x = \frac{1 \pm \sqrt{1+4}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$
The two graphs intersect at
$(0,0), \left(\pm\sqrt{2}, 0\right), (-1,1), \left(\frac{1 \pm \sqrt{5}}{2}, 1\right)$.

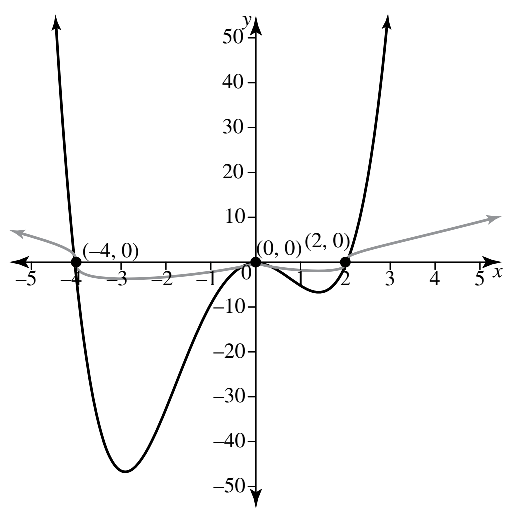
17



The minimum turning point of the parabolic branch needs to be obtained to form the range. The parabola $y = (2x-1)(x-3)$ has minimum turning point at $\left(\frac{7}{4}, -\frac{25}{8}\right)$.

The range of the hybrid function is $\left[-\frac{25}{8}, \infty\right)$.

18



Where the polynomial graph cuts the x axis, the cube root graph has vertical points of inflection; where the polynomial touches the x axis, the cube root graph also touches the x axis but at a sharp point.
Wherever the polynomial graph has the values $y = 0, y = -1$ and $y = 1$, the cube root graph must have the same value.
There are 9 points of intersection.

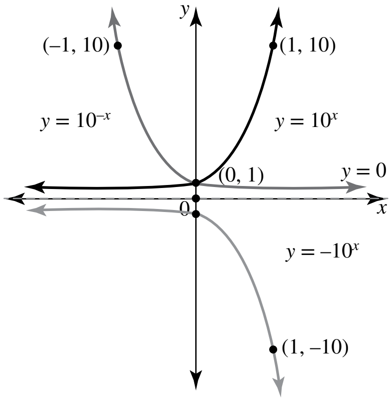
Exercise 2.5 — Non-algebraic functions

1 $f(x) = -10^x$

a $f(2) = -10^2$

$= -100$

b The graph of $y = 10^x$ contains $(0,1)$ and $(1,10)$.
The graph of $y = -10^x$ contains the points $(0,-1)$ and $(1,-10)$.
The graph of $y = 10^{-x}$ contains the points $(0,1)$ and $(-1,10)$.



c $y = 10^{-x}$ can be expressed as $y = \left(\frac{1}{10}\right)^x$ or $y = 0.1^x$.

2 Domain is R for both $f(x) = 2^x$ and $g(x) = 2^{-x} \Rightarrow d_f \cap d_g = R$
Domain of $f - g$ is R.

$y = (f - g)(x)$

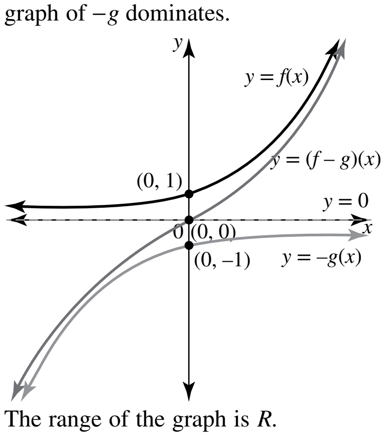
$= f(x) - g(x)$

$= 2^x - 2^{-x}$

To sketch the difference function, sketch it as $y = f(x) + (-g(x))$ and add y cordinates.

x	-1	0	1
$f(x)$	$\frac{1}{2}$	1	2
$-g(x)$	-2	-1	$-\frac{1}{2}$
$f(x) + (-g(x))$	$-1\frac{1}{2}$	0	$1\frac{1}{2}$

As $x \to \infty$, the graph of f dominates and as $x \to -\infty$, the graph of $-g$ dominates.



The range of the graph is R.

3 a $y = -2e^x - 3$

Asymptote: $y = -3$

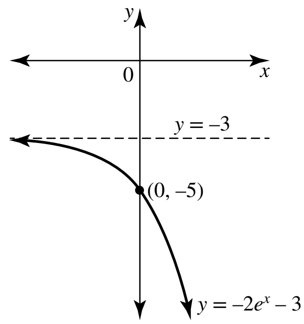
y intercept: Let $x = 0$

$\therefore y = -2e^0 - 3$

$\therefore y = -5$

$(0,-5)$

There will not be an x intercept.



Domain R, range $(-\infty, -3)$.

b $y = 4e^{-3x} - 4$

Asymptote: $y = -4$

y intercept: Let $x = 0$

$\therefore y = 4e^0 - 4$

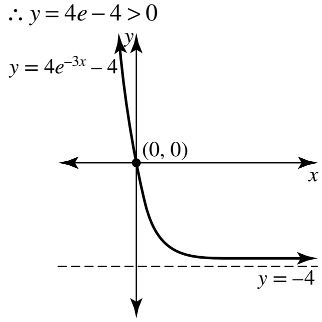
$\therefore y = 0$

$(0,0)$

The origin is also the x intercept.

Point: Let $x = -\frac{1}{3}$

$\therefore y = 4e - 4 > 0$



Domain R and range $(-4, \infty)$.

c $y = 5e^{x-2}$

Asymptote: $y = 0$

There is no x intercept.

y intercept: Let $x = 0$

$\therefore y = 5e^{-2}$

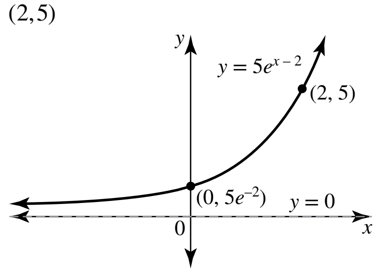
$(0, 5e^{-2})$

Point: Let $x = 2$

$\therefore y = 5e^0$

$\therefore y = 5$

$(2,5)$



Domain R and range R^+.

4 a $y = 2e^{1-3x} - 4$

Asymptote: $y = -4$

y intercept: Let $x = 0$

$\therefore y = 2e^1 - 4$

$(0, 2e - 4)$

This point lies above the asymptote so there will be an x intercept. Approximately, $2e - 4 = 1.4$.

x intercept: Let $y = 0$

$\therefore 2e^{1-3x} - 4 = 0$

$\therefore 2e^{1-3x} = 4$

$\therefore e^{1-3x} = 2$

Convert to logarithm form

$\therefore 1 - 3x = \log_e(2)$

$\therefore 3x = 1 - \log_e(2)$

$\therefore x = \frac{1}{3}(1 - \log_e(2))$

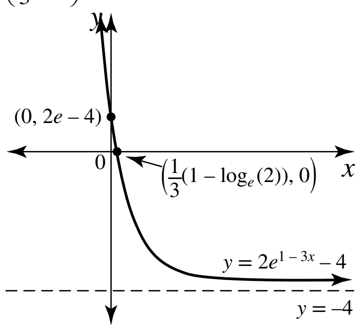
The x intercept is $\left(\frac{1}{3}(1 - \log_e(2)), 0\right)$ which is approximately $(-0.1, 0)$.

Point: Let $x = \frac{1}{3}$

$\therefore y = 2e^0 - 4$

$\therefore y = -2$

$\left(\frac{1}{3}, -2\right)$



b $y = 3 \times 2^x - 24$

Asymptote: $y = -24$

y intercept: Let $x = 0$

$\therefore y = 3 \times 2^0 - 24$

$= -21$

$(0, -21)$

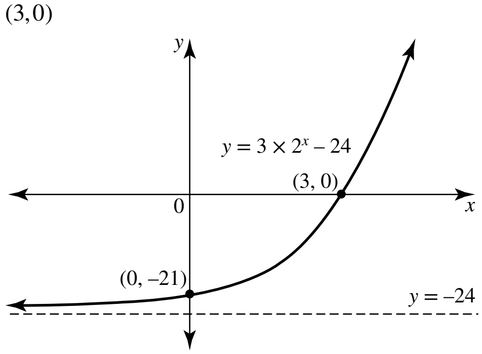
x intercept: Let $y = 0$

$\therefore 3 \times 2^x - 24 = 0$

$\therefore 2^x = 8$

$\therefore x = 3$

$(3, 0)$



Domain R and range $(-24, \infty)$.

5 a $y = ae^x + b$

From the graph, the asymptote is $y = 2$ so $b = 2$

The equation becomes $y = ae^x + 2$.

The graph passes through the origin.

Substitute $(0, 0)$

$\therefore 0 = ae^0 + 2$

$\therefore 0 = a + 2$

$\therefore a = -2$

The equation is $y = -2e^x + 2$ with $a = -2, b = 2$

b $y = a \times 10^{kx}$

Substitute point $(4, -20)$

$\therefore -20 = a \times 10^{4k}$ .....(1)

Substitute point $(8, -200)$

$\therefore -200 = a \times 10^{8k}$ .....(2)

Divide equation (2) by equation (1)

$\therefore \frac{-200}{-20} = \frac{a \times 10^{8k}}{a \times 10^{4k}}$

$\therefore 10 = 10^{4k}$

$\therefore 1 = 4k$

$\therefore k = \frac{1}{4}$

Substitute $k = \frac{1}{4}$ in equation (1)

$\therefore -20 = a \times 10^1$

$\therefore a = -2$

The equation is $y = -2 \times 10^{\frac{x}{4}}$.

6 $y = a \times e^{kx}$

Substitute point $(2, 36)$

$\therefore 36 = a \times e^{2k}$ .....(1)

Substitute point $(3, 108)$

$\therefore 108 = a \times e^{3k}$ .....(2)

Divide equation (2) by equation (1)

$\therefore \frac{108}{36} = \frac{a \times e^{3k}}{a \times e^{2k}}$

$\therefore 3 = e^k$

$\therefore k = \log_e(3)$

Substitute $e^k = 3$ in equation (1)

$36 = a \times e^{2k}$

$\therefore 36 = a \times (e^x)^2$

$\therefore 36 = a \times (3)^2$

$\therefore 36 = 9a$

$\therefore a = 4$

Answer is $a = 4, k = \log_e(3)$

7 a $y = 2\cos(4x) - 3,\ 0 \le x \le 2\pi$

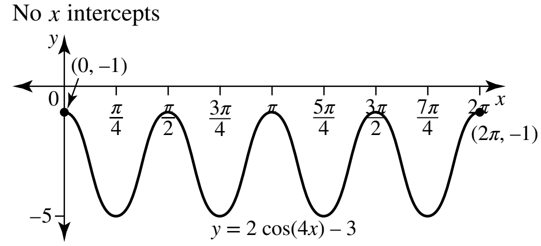
Period $\frac{2\pi}{4} = \frac{\pi}{2}$

Amplitude 2

Mean position $y = -3$

Range $[-3 - 2, -3 + 2] = [-5, -1]$

No x intercepts



b Let the equation be $y = a\sin(nx) + k$.

The period of the graph is 2.

$\therefore \frac{2\pi}{n} = 2$

$\therefore n = \pi$

The mean position is 5.

$y = 5 \Rightarrow k = 5$.

The range is $[-3, 13]$ which means the amplitude is 8.

As the graph has an inverted sine shape, $a = -8$

The equation is $y = -8\sin(\pi x) + 5$.

8 $f:[0,2\pi]\to R, f(x)=1-2\sin\left(\frac{3x}{2}\right)$

$y=f(x)=1-2\sin\left(\frac{3x}{2}\right)$

Period $2\pi\div\frac{3}{2}=\frac{4\pi}{3}$

Amplitude 2, inverted graph
Mean position $y=1$
Range $[-1,3]$
x intercepts: Let $y=0$

$0=1-2\sin\left(\frac{3x}{2}\right),\ 0\le x\le 2\pi$

$\therefore \sin\left(\frac{3x}{2}\right)=\frac{1}{2},\ 0\le\frac{3x}{2}\le 3\pi$

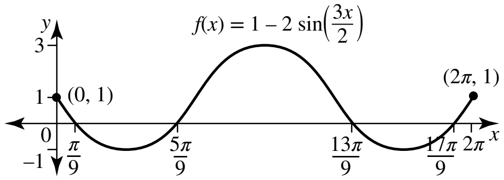
$\therefore \frac{3x}{2}=\frac{\pi}{6},\pi-\frac{\pi}{6},2\pi+\frac{\pi}{6},3\pi-\frac{\pi}{6}$

$\therefore \frac{3x}{2}=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}$

$\therefore 3x=\frac{\pi}{3},\frac{5\pi}{3},\frac{13\pi}{3},\frac{17\pi}{3}$

$\therefore x=\frac{\pi}{9},\frac{5\pi}{9},\frac{13\pi}{9},\frac{17\pi}{9}$

y intercept: Let $x=0$
$y=1-2\sin(0)=1$
$(0,1)$



9 a $f:\left[0,\frac{3\pi}{2}\right]\to R, f(x)=-6\sin\left(3x-\frac{3\pi}{4}\right)$

$y=f(x)=-6\sin\left(3x-\frac{3\pi}{4}\right)$

$\therefore y=-6\sin\left(3\left(x-\frac{\pi}{4}\right)\right)$

Horizontal translation $\frac{\pi}{4}$ units to the right.

Period $\frac{2\pi}{3}$

Amplitude 6, graph is inverted
Mean position $y=0$ so range is $[-6,6]$.
Endpoints:

$f(0)=-6\sin\left(-\frac{3\pi}{4}\right)$
$=-6\times\frac{-\sqrt{2}}{2}$
$=3\sqrt{2}$

$f\left(\frac{3\pi}{2}\right)=-6\sin\left(\frac{15\pi}{4}\right)$
$=-6\times\frac{-\sqrt{2}}{2}$
$=3\sqrt{2}$

Endpoints are $\left(0,3\sqrt{2}\right)$ and $\left(\frac{3\pi}{2},3\sqrt{2}\right)$.

x intercepts: Either translate those of $y=-6\sin(3x)$ $\frac{\pi}{4}$ units to the right or solve

$-6\sin\left(3x-\frac{3\pi}{4}\right)=0$

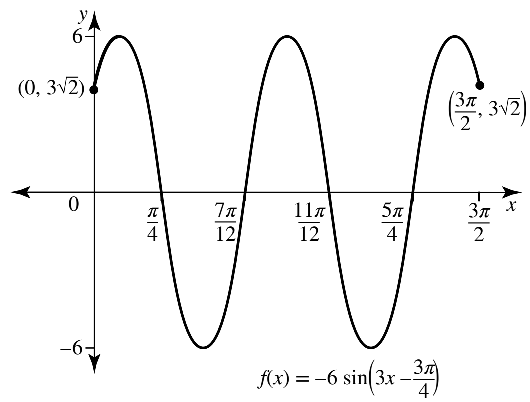
Solving the equation:

$\sin\left(3x-\frac{3\pi}{4}\right)=0$

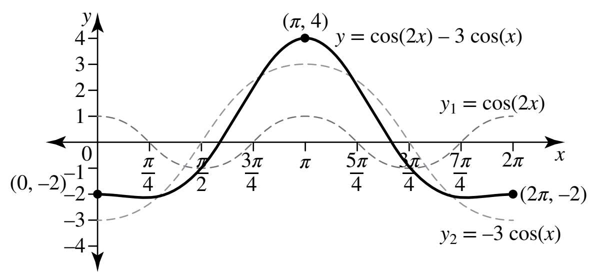
$\therefore 3x-\frac{3\pi}{4}=0,\pi,2\pi,3\pi$

$\therefore 3x=\frac{3\pi}{4},\frac{7\pi}{4},\frac{11\pi}{4},\frac{15\pi}{4}$

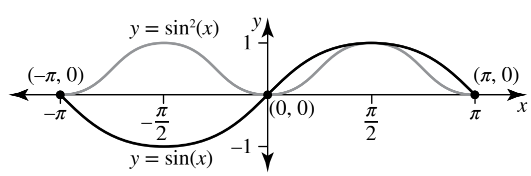
$\therefore x=\frac{\pi}{4},\frac{7\pi}{12},\frac{11\pi}{12},\frac{5\pi}{4}$



b $y=\cos(2x)-3\cos(x)$ for $x\in[0,2\pi]$.
$y=y_1+y_2$ where $y_1=\cos(2x)$ and $y_2=-3\cos(x)$
$y_1=\cos(2x)$ has period π, amplitude 1, range $[-1,1]$.
$y_2=-3\cos(x)$ has period 2π, amplitude 3, inverted graph, range $[-3,3]$.



10 To sketch the graph of $y=(\sin(x))^2=\sin^2(x)$ for $x\in[-\pi,\pi]$, remember that $(-1)^2=1, 0^2=0, 1^2=1$. The squared graph will not lie below the x axis.



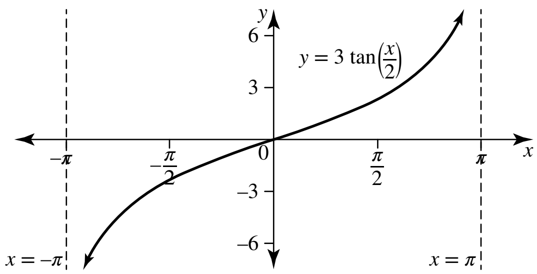
11 a $y=3\tan\left(\frac{x}{2}\right)$ for $x\in[-\pi,\pi]$

Period $\pi\div\frac{1}{2}=2\pi$

First positive asymptote: $\frac{x}{2}=\frac{\pi}{2}\Rightarrow x=\pi$

For $x\in[-\pi,\pi]$, there is only one other asymptote at $x=\pi-2\pi=-\pi$.
x intercept midway between the two asymptotes is $x=0$.
To illustrate the dilation effect: Let $x=\frac{\pi}{3}$ then

$y=3\tan\left(\frac{\pi}{6}\right)=\frac{3\sqrt{3}}{3}=\sqrt{3}$.



b $y = -\tan(2x - \pi)$ for $x \in [-\pi, \pi]$.

$y = -\tan\left(2\left(x - \frac{\pi}{2}\right)\right)$

Period $\frac{\pi}{2}$

Horizontal shift $\frac{\pi}{2}$ units to the right and the graph is inverted.

An asymptote occurs when $2x - \pi = \frac{\pi}{2}$

$\therefore 2x = \frac{3\pi}{2}$

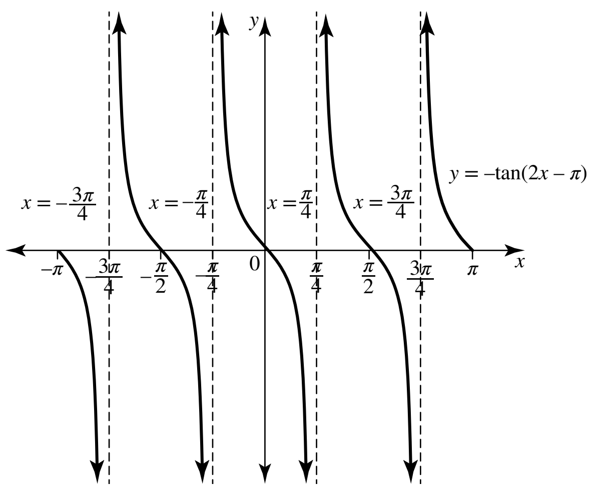
$\therefore x = \frac{3\pi}{4}$

Other asymptotes at $x = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$ and $x = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$ and $x = -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$.

The asymptotes are $x = -\frac{3\pi}{4}, x = -\frac{\pi}{4}, x = \frac{\pi}{4}, x = \frac{3\pi}{4}$.

x intercepts lie midway between the asymptotes as the mean position is $y = 0$.

They are at $x = -\frac{\pi}{2}, x = 0, x = \frac{\pi}{2}$ and at the endpoints $x = -\pi, x = \pi$.



12 The graph of $y = a\tan(nx)$

The distance between the asymptotes at $x = \pm\frac{\pi}{3}$ is $\frac{2\pi}{3}$ so this is the period.

$\therefore \frac{\pi}{n} = \frac{2\pi}{3}$

$\therefore 3 = 2n$

$\therefore n = \frac{3}{2}$

The equation becomes $y = a\tan\left(\frac{3x}{2}\right)$.

Substitute the point $\left(-\frac{\pi}{6}, -\frac{1}{2}\right)$

$\therefore -\frac{1}{2} = a\tan\left(\frac{3}{2} \times \frac{-\pi}{6}\right)$

$\therefore -\frac{1}{2} = a\tan\left(-\frac{\pi}{4}\right)$

$\therefore -\frac{1}{2} = a \times -1$

$\therefore a = \frac{1}{2}$

The equation is $y = \frac{1}{2}\tan\left(\frac{3x}{2}\right)$.

13 $y = 3\tan(2\pi x) - \sqrt{3}$ for $-\frac{7}{8} \le x \le \frac{7}{8}$.

Period $\frac{\pi}{2\pi} = \frac{1}{2}$

Mean position $x = -\sqrt{3}$

An asymptote occurs at $2\pi x = \frac{\pi}{2} \Rightarrow x = \frac{1}{4}$.

Other asymptotes at $x = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$, at $x = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$, at $x = -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4}$

The asymptotes are $x = -\frac{3}{4}, x = -\frac{1}{4}, x = \frac{1}{4}, x = \frac{3}{4}$

x intercepts: Let $y = 0$

$3\tan(2\pi x) - \sqrt{3} = 0$

$\therefore \tan(2\pi x) = \frac{\sqrt{3}}{3}, \quad -\frac{7\pi}{4} \le 2\pi x \le \frac{7\pi}{4}$

$\therefore 2\pi x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$ or $-\pi + \frac{\pi}{6}$

$\therefore 2\pi x = \frac{\pi}{6}, \frac{7\pi}{6}$ or $-\frac{5\pi}{6}$

$\therefore x = \frac{1}{12}, \frac{7}{12}, -\frac{5}{12}$

Endpoints: Let $x = -\frac{7}{8}$

$y = 3\tan\left(-\frac{7\pi}{4}\right)$

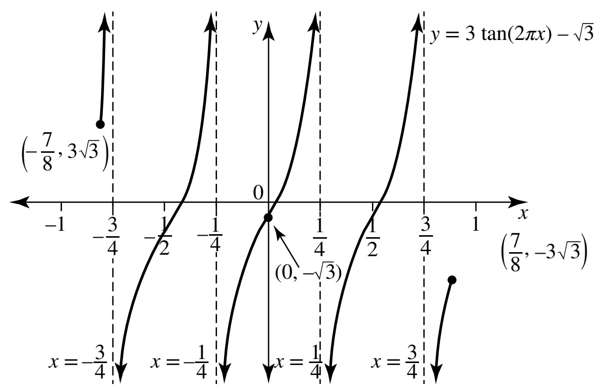
$= 3$

Let $x = \frac{7}{8}$,

$y = 3\tan\left(\frac{7\pi}{4}\right)$

$= -3$

Endpoints are $\left(-\frac{7}{8}, 3\right)$ and $\left(\frac{7}{8}, -3\right)$.



14 $y=1-\tan\left(x+\frac{\pi}{6}\right)$ for $0\le x\le 2\pi$.

Period π, horizontal translation $\frac{\pi}{6}$ units to the left, graph is inverted.

Mean position $y=1$.

An asymptote when $x+\frac{\pi}{6}=\frac{\pi}{2}\Rightarrow x=\frac{\pi}{3}$.

Other asymptote at $x=\frac{\pi}{3}+\pi=\frac{4\pi}{3}$ and $x=\frac{\pi}{3}-\pi=-\frac{2\pi}{3}$ but this one is not in the domain given.

The asymptotes are $x=\frac{\pi}{3}, x=\frac{4\pi}{3}$

x intercepts: Let $y=0$

$1-\tan\left(x+\frac{\pi}{6}\right)=0,\ 0\le x\le 2\pi$

$\therefore \tan\left(x+\frac{\pi}{6}\right)=1,\ \frac{\pi}{6}\le x\le 2\pi+\frac{\pi}{6}$

$\therefore x+\frac{\pi}{6}=\frac{\pi}{4},\pi+\frac{\pi}{4}$

$\therefore x+\frac{\pi}{6}=\frac{\pi}{4},\frac{5\pi}{4}$

$\therefore x=\frac{\pi}{12},\frac{13\pi}{12}$

Endpoints: Let $x=0$

$y=1-\tan\left(\frac{\pi}{6}\right)$

$=1-\frac{\sqrt{3}}{3}$

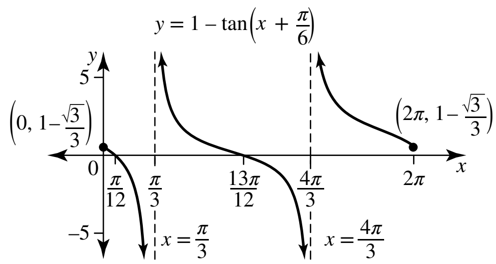
Let $x=2\pi$

$y=1-\tan\left(2\pi+\frac{\pi}{6}\right)$

$=1-\tan\left(\frac{\pi}{6}\right)$

$=1-\frac{\sqrt{3}}{3}$

Endpoints are $\left(0,1-\frac{\sqrt{3}}{3}\right)$ and $\left(2\pi,1-\frac{\sqrt{3}}{3}\right)$.



15 a $y=\frac{4}{5}\times 10^{x}$

Asymptote: $y=0$

y intercept: Let $x=0$

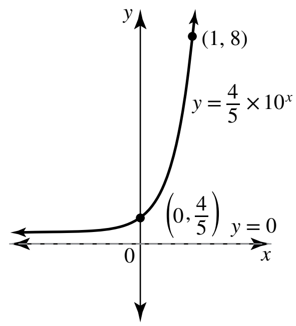
$y=\frac{4}{5}\times 10^{0}=\frac{4}{5}$

$\left(0,\frac{4}{5}\right)$

Point: Let $x=1$

$y=\frac{4}{5}\times 10^{1}=8$

$(1,8)$



As $x\to\infty,\ y\to\infty$.

b $y=3\times 4^{-x}$

Asymptote: $y=0$

y intercept: Let $x=0$

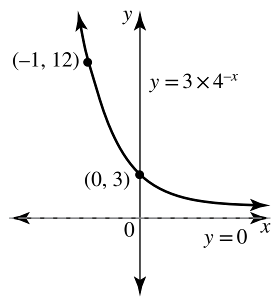
$y=3\times 4^{0}=3$

$(0,3)$

Point: Let $x=-1$

$y=3\times 4^{1}=12$

$(-1,12)$



As $x\to\infty,\ y\to 0^{+}$.

c $y=-5\times 3^{-\frac{x}{2}}$

Asymptote: $y=0$

y intercept: Let $x=0$

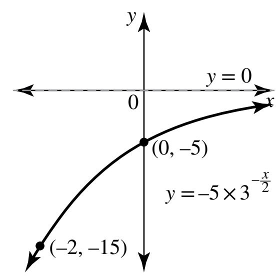
$y=-5\times 3^{0}=-5$

$(0,-5)$

Point: Let $x=-2$

$y=-5\times 3^{1}=-15$

$(-2,-15)$



As $x\to\infty,\ y\to 0^{-}$.

d $y=-\left(\frac{2}{3}\right)^{-x}$

Asymptote: $y=0$

y intercept: Let $x=0$

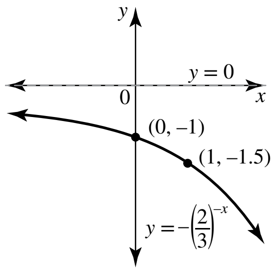
$y=-\left(\frac{2}{3}\right)^{0}=-1$

$(0,-1)$

Point: Let $x=1$

$y = -\left(\frac{2}{3}\right)^{-1} = -\frac{3}{2}$

$(1, -1.5)$



As $x \to \infty,\ y \to -\infty$

16 a $y = e^x - 3$

Asymptote: $y = -3$

y intercept: Let $x = 0$

$y = e^0 - 3 = -2$

$(0, -2)$

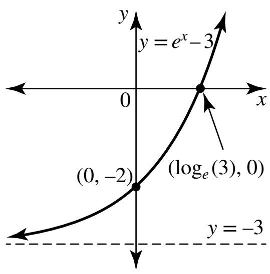
x intercept: Let $y = 0$

$e^x - 3 = 0$

$\therefore e^x = 3$

$\therefore x = \log_e(3)$

$\left(\log_e(3), 0\right)$



Range $(-3, \infty)$.

b $y = -2e^{2x} - 1$

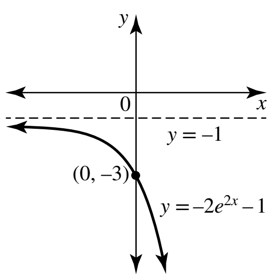
Asymptote: $y = -1$

y intercept: Let $x = 0$

$y = -2e^0 - 1 = -3$

$(0, -3)$

No x intercept



Range is $(-\infty, -1)$.

c $y = \frac{1}{2}e^{-4x} + 3$

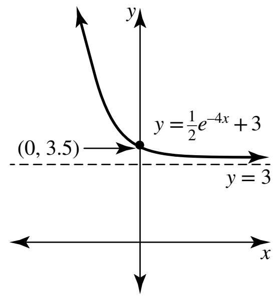
Asymptote: $y = 3$

y intercept: Let $x = 0$

$y = \frac{1}{2}e^0 + 3 = 3.5$

$(0, 3.5)$

No x intercept



Range is $(3, \infty)$.

d $y = 4 - e^{2x}$

Asymptote: $y = 4$

y intercept: Let $x = 0$

$y = 4 - e^0 = 3$

$(0, 3)$

x intercept: Let $y = 0$

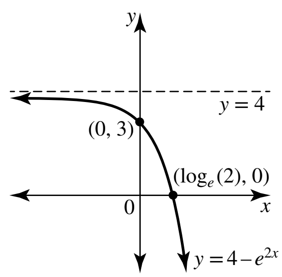
$4 - e^{2x} = 0$

$\therefore e^{2x} = 4$

$\therefore 2x = \log_e(4)$

$\therefore x = \frac{1}{2}\log_e(4)$

$\left(\frac{1}{2}\log_e(4), 0\right)$ or $\left(\log_e(2), 0\right)$



Range is $(-\infty, 4)$.

e $y = 4e^{2x-6} + 2$

$\therefore y = 4e^{2(x-3)} + 2$

Asymptote: $y = 2$

y intercept: Let $x = 0$

$y = 4e^{-6} + 2 \approx 2.01$

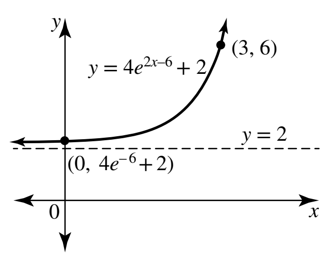
$(0, 4e^{-6} + 2)$

No x intercept

Point: Let $x = 3$

$\therefore y = 4e^0 + 2 = 6$

$(3, 6)$



Range is $(2, \infty)$.

f $y = 1 - e^{-\frac{x+1}{2}}$

Asymptote: $y = 1$

y intercept: Let $x = 0$

$y = 1 - e^{-0.5} \approx 0.39$

$\left(0, 1 - e^{-0.5}\right)$

x intercept: Let $y = 0$

$1 - e^{-\frac{x+1}{2}} = 0$

$\therefore e^{-\frac{x+1}{2}} = 1$

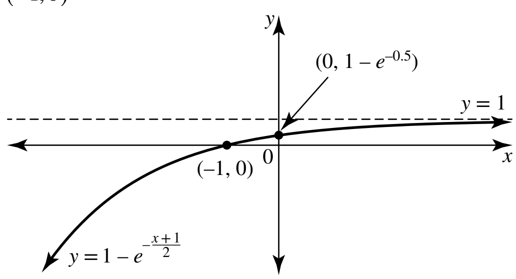
$\therefore e^{-\frac{x+1}{2}} = e^0$

$\therefore -\frac{x+1}{2} = 0$

$\therefore x + 1 = 0$

$\therefore x = -1$

$(-1, 0)$



Range is $(-\infty, 1)$.

17 a $f(x) = ae^x + b$

Asymptote is $y = 11$ so $b = 11$.

Equation becomes $f(x) = ae^x + 11$.

The graph passes through the origin so $f(0) = 0$.

$\therefore ae^0 + 11 = 0$

$\therefore a + 11 = 0$

$\therefore a = -11$

The rule for the function is $f(x) = -11e^x + 11$ with $a = -11,\ b = 11$.

The domain of the graph is R so as a mapping the function is written $f: R \to R, f(x) = -11e^x + 11$.

b $y = Ae^{nx} + k$

The asymptote is $y = 4$ so $k = 4$ and the equation becomes $y = ae^{nx} + 4$.

Substitute the point $(0, 5)$

$\therefore 5 = ae^0 + 4$

$\therefore 5 = a + 4$

$\therefore a = 1$

The equation becomes $y = e^{nx} + 4$.

Substitute the point $\left(-1, 4 + e^2\right)$

$\therefore 4 + e^2 = e^{-n} + 4$

$\therefore e^2 = e^{-n}$

$\therefore 2 = -n$

$\therefore n = -2$

The equation is $y = e^{-2x} + 4$

c i $y = 2^{x-b} + c$

Substitute the point $(0, -5)$

$-5 = 2^{-b} + c$ ....(1)

Substitute the point $(3, 9)$

$9 = 2^{3-b} + c$ ....(2)

Subtract equation (1) from equation (2)

$\therefore 14 = 2^{3-b} - 2^{-b}$

$\therefore 14 = 2^3 \times 2^{-b} - 2^{-b}$

$\therefore 14 = 8 \times 2^{-b} - 2^{-b}$

$\therefore 14 = 7 \times 2^{-b}$

$\therefore 2 = 2^{-b}$

$\therefore b = -1$

Substitute $b = -1$ in equation (1)

$\therefore -5 = 2 + c$

$\therefore c = -7$

Hence, $b = -1,\ c = -7$

ii The equation of the graph is $y = 2^{x+1} - 7$. Its asymptote is $y = -7$ and the given points lie above this asymptote. Therefore, the range of the graph is $(-7, \infty)$.

d i $y = Ae^{x-2} + B$

The long term behaviour $x \to -\infty,\ y \to -2$ means there is an asymptote at $y = -2$.

Therefore, $B = -2$ and the equation becomes $y = Ae^{x-2} - 2$.

Substitute the point $(2, 10)$

$\therefore 10 = Ae^0 - 2$

$\therefore A = 12$

Answer: $A = 12,\ B = -2$

ii The equation is $y = 12e^{x-2} - 2$

Substitute the point $\left(a, 2\left(\frac{6}{e} - 1\right)\right)$

$\therefore 2\left(\frac{6}{e} - 1\right) = 12e^{a-2} - 2$

$\therefore 12e^{-1} - 2 = 12e^{a-2} - 2$

$\therefore e^{-1} = e^{a-2}$

$\therefore a - 2 = -1$

$\therefore a = 1$

18 a $y = 6\sin(8x)$

Period $\frac{2\pi}{8} = \frac{\pi}{4}$, amplitude 6.

Mean position $y = 0$ so range is $[-6, 6]$.

b $y = 2 - 3\cos\left(\frac{x}{4}\right)$

Period $2\pi \div \frac{1}{4} = 8\pi$, amplitude 3

Mean position $y = 2$ so range is $[-1, 5]$.

c $y = -\sin(3x - 6)$

Period $\frac{2\pi}{3}$, amplitude 1.

Mean position $y = 0$ so range is $[-1, 1]$.

d $y = 3\left(5 + 2\cos(6\pi x)\right)$

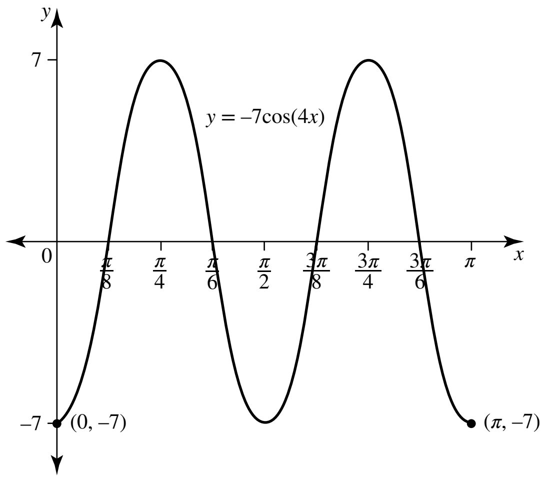
$y = 15 + 6\cos(6\pi x)$

Period $\frac{2\pi}{6\pi} = \frac{1}{3}$, amplitude 6.

Mean position $y = 15$ so range is $[9, 21]$.

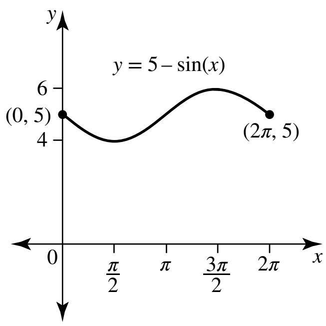
19 a $y = -7\cos(4x),\ 0 \le x \le \pi$

Period $\frac{2\pi}{4} = \frac{\pi}{2}$, amplitude 7, graph is inverted, mean position $y = 0$, range $[-7, 7]$.



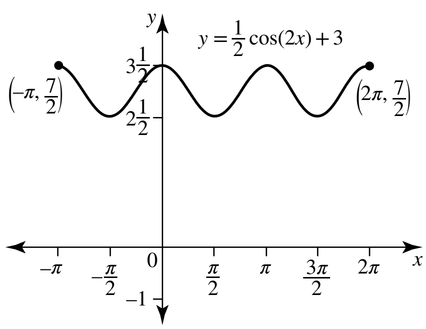
b $y = 5 - \sin(x),\ 0 \le x \le 2\pi$

Period 2π, amplitude 1, graph is inverted, mean position $y = 5$, range $[4, 6]$.



c $y = \frac{1}{2}\cos(2x) + 3,\ -\pi \le x \le 2\pi$

Period $\frac{2\pi}{2} = \pi$, amplitude $\frac{1}{2}$, mean position $y = 3$, range $[2.5, 3.5]$.



d $y = 2 - 4\sin(3x),\ 0 \le x \le 2\pi$

Period $\frac{2\pi}{3}$, amplitude 4, graph is inverted, mean position $y = 2$, range $[-2, 6]$.

x intercepts: Let $y = 0$

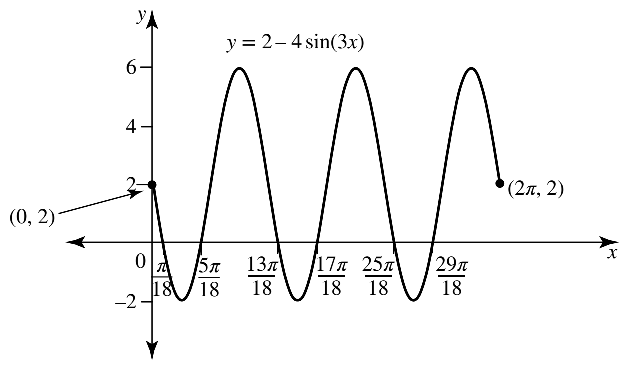
$\therefore 0 = 2 - 4\sin(3x),\ 0 \le x \le 2\pi$

$\therefore \sin(3x) = \frac{1}{2},\ 0 \le 3x \le 6\pi$

$\therefore 3x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, 4\pi + \frac{\pi}{6}, 5\pi - \frac{\pi}{6}$

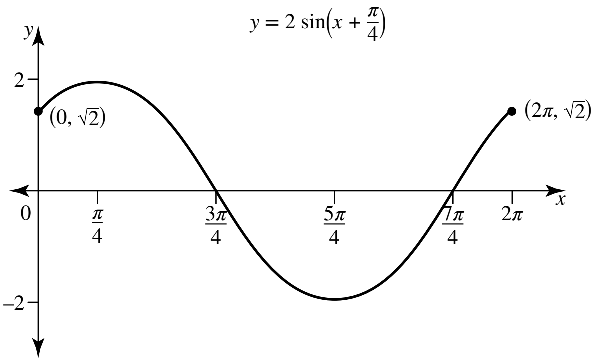
$\therefore 3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

$\therefore x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$



e $y = 2\sin\left(x + \frac{\pi}{4}\right),\ 0 \le x \le 2\pi$

Period 2π, amplitude 2, horizontal shift $\frac{\pi}{4}$ to left, mean position $y = 0$, range $[-2, 2]$.



Points on $y = 2\sin(x)$ are moved $\frac{\pi}{4}$ to the left.

f $y = -4\cos\left(3x - \frac{\pi}{2}\right) + 4,\ -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

$y = -4\cos\left(3\left(x - \frac{\pi}{6}\right)\right) + 4$

Period $\frac{2\pi}{3}$, amplitude 4, graph is inverted, horizontal translation $\frac{\pi}{6}$ to the right, mean position $y = 4$, range $[0, 8]$.

x intercepts: Let $y = 0$

$\therefore -4\cos\left(3x - \frac{\pi}{2}\right) + 4 = 0,\ -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

$\therefore \cos\left(3x - \frac{\pi}{2}\right) = 1,\ -2\pi \le 3x - \frac{\pi}{2} \le 2\pi$

$\therefore 3x - \frac{\pi}{2} = -2\pi, -\pi, 0, \pi, 2\pi$

$\therefore 3x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\therefore x = -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

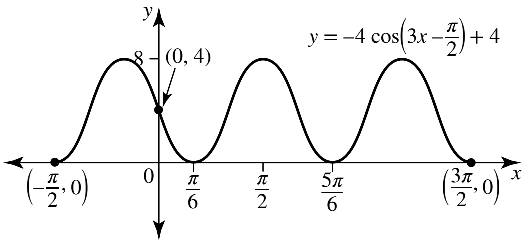
Endpoints: Let $x = -\frac{\pi}{2}$

$$\begin{aligned} y &= -4\cos\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) + 4 \\ &= -4\cos(-2\pi) + 4 \\ &= 0 \end{aligned}$$

Let $x = \frac{3\pi}{2}$

$$\begin{aligned} y &= -4\cos\left(\frac{9\pi}{2} - \frac{\pi}{2}\right) + 4 \\ &= -4\cos(4\pi) + 4 \\ &= 0 \end{aligned}$$

Endpoints are $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$.



20 a i $2\sin(2x)+\sqrt{3}=0$ for $x \in [0, 2\pi]$

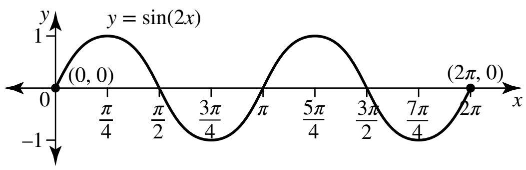
$\therefore \sin(2x) = -\frac{\sqrt{3}}{2},\ 2x \in [0, 4\pi]$

$\therefore 2x = \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 3\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}$

$\therefore 2x = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$

$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

ii Graph of $y = \sin(2x)$ for $x \in [0, 2\pi]$
Period π, amplitude 1, range $[-1, 1]$



iii $\left\{x : \sin(2x) < -\frac{\sqrt{3}}{2},\ 0 \le x \le 2\pi\right\}$

Draw the line $y = -\frac{\sqrt{3}}{2}$ on the graph of $y = \sin(x)$.

At their intersections, $\sin(x) = -\frac{\sqrt{3}}{2}$ and therefore $2\sin(x)+\sqrt{3}=0$, the solutions to which were found in part **ai** as $x = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$.

The sine curve lies below the line for $\frac{2\pi}{3} < x < \frac{5\pi}{6}$ and $\frac{5\pi}{3} < x < \frac{11\pi}{6}$.

The solution set is
$\left\{x : \frac{2\pi}{3} < x < \frac{5\pi}{6}\right\} \cup \left\{x : \frac{5\pi}{3} < x < \frac{11\pi}{6}\right\}$.

b The function $f(x) = 2 - 3\cos\left(x + \frac{\pi}{12}\right)$ has a range $[-1, 5]$ so its maximum value is 5.
This occurs when $\cos\left(x + \frac{\pi}{12}\right) = -1$. The first positive solution occurs when $x + \frac{\pi}{12} = \pi$, giving the value $x = \frac{11\pi}{12}$ for when the function is at its greatest value.

21 a $y = \tan(4x)$

Period $\frac{\pi}{4}$

Asymptote when $4x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{8}$

b $y = 9 + 8\tan\left(\frac{x}{7}\right)$

Period $\pi \div \frac{1}{7} = 7\pi$

Asymptote when $\frac{x}{7} = \frac{\pi}{2} \Rightarrow x = \frac{7\pi}{2}$

c $y = -\frac{3}{2}\tan\left(\frac{4x}{5}\right)$

Period $\pi \div \frac{4}{5} = \frac{5\pi}{4}$

Asymptote when $\frac{4x}{5} = \frac{\pi}{2} \Rightarrow x = \frac{5\pi}{8}$

d $y = 2\tan(6\pi x + 3\pi)$

Period $\frac{\pi}{6\pi} = \frac{1}{6}$

Asymptote when $6\pi x + 3\pi = \frac{\pi}{2}$

$\therefore 6\pi x + 3\pi = \frac{\pi}{2} - 3\pi$

$\therefore 6\pi x = -\frac{5\pi}{2}$

$\therefore x = -\frac{5}{12}$

For the first positive asymptote, add multiples of the period $\frac{1}{6}$.

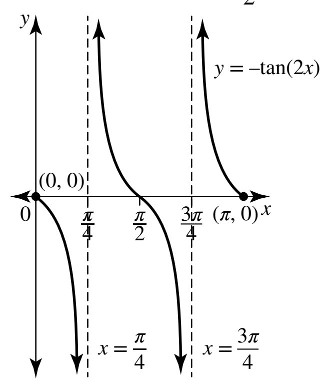
$-\frac{5}{12} + 3 \times \frac{1}{6} = \frac{1}{12}$

The first positive asymptote is $x = \frac{1}{12}$.

22 a i $y = -\tan(2x), x \in [0, \pi]$

Period $\frac{\pi}{2}$, graph is inverted.

Asymptotes when $2x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{4}$ and $x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$.



ii $y = 3\tan\left(x + \frac{\pi}{4}\right), x \in [0, 2\pi]$

Period π, horizontal shift $\frac{\pi}{4}$ to the left. The asymptote $y = \frac{\pi}{2}$ on $y = 3\tan(x)$ is moved by this translation to $x = \frac{\pi}{4}$. There must also be one other asymptote at $x = \frac{\pi}{4} + \pi \Rightarrow x = \frac{5\pi}{4}$.

x intercept midway between the pair of asymptotes is $x = \frac{3\pi}{4}$. One other x intercept is a period apart at $x = \frac{7\pi}{4}$. Others are outside the given domain.

Endpoints: Let $x = 0$ so $y = 3\tan\left(\frac{\pi}{4}\right) = 3$

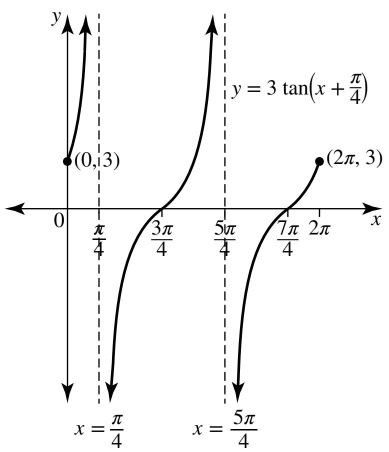
Let $x = 2\pi$

$y = 3\tan\left(2\pi + \frac{\pi}{4}\right)$

$= 3\tan\left(\frac{\pi}{4}\right)$

$= 3$

Endpoints are $(0, 3)$ and $(2\pi, 3)$.



iii $y = \tan\left(\frac{x}{3}\right) + \sqrt{3}, x \in [0, 6\pi]$

Period is 3π.

An asymptote occurs at $\frac{x}{3} = \frac{\pi}{2} \Rightarrow x = \frac{3\pi}{2}$. As the period is 3π there is another asymptote at $x = \frac{3\pi}{2} + 3\pi = \frac{9\pi}{2}$.

The asymptotes are $x = \frac{3\pi}{2}, x = \frac{9\pi}{2}$.

Mean position is $y = \sqrt{3}$

x intercepts: Let $y = 0$

$\therefore \tan\left(\frac{x}{3}\right) + \sqrt{3} = 0,\ x \in [0, 6\pi]$

$\therefore \tan\left(\frac{x}{3}\right) = -\sqrt{3},\ \frac{x}{3} \in [0, 2\pi]$

$\therefore \frac{x}{3} = \pi - \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$\therefore \frac{x}{3} = \frac{2\pi}{3}, \frac{5\pi}{3}$

$\therefore x = 2\pi, 5\pi$

Endpoints: Let $x = 0$

$y = \tan(0) + \sqrt{3}$

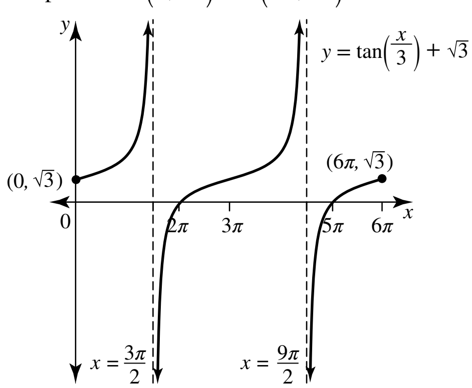
$= \sqrt{3}$

Let $x = 6\pi$

$y = \tan(2\pi) + \sqrt{3}$

$= \sqrt{3}$

Endpoints are $\left(0, \sqrt{3}\right)$ and $\left(6\pi, \sqrt{3}\right)$.



iv $y = 5\sqrt{3}\tan\left(\pi x - \frac{\pi}{2}\right) - 5, x \in (-2, 3)$

$\therefore y = 5\sqrt{3}\tan\left(\pi\left(x - \frac{1}{2}\right)\right) - 5$

Period $\frac{\pi}{\pi} = 1$, horizontal shift $\frac{1}{2}$ to the right, mean position $y = -5$.

An asymptote when

$\pi x - \frac{\pi}{2} = \frac{\pi}{2}$

$\therefore \pi x = \pi$

$\therefore x = 1$

Others are spaced 1 unit apart.

The asymptotes are

$x = -2, x = -1, x = 0, x = 1, x = 2, x = 3$

x intercepts: Let $y = 0$

$\therefore 5\sqrt{3}\tan\left(\pi x - \frac{\pi}{2}\right) - 5 = 0,\ x \in (-2, 3)$

$\therefore \tan\left(\pi x - \frac{\pi}{2}\right) = \frac{5}{5\sqrt{3}}$

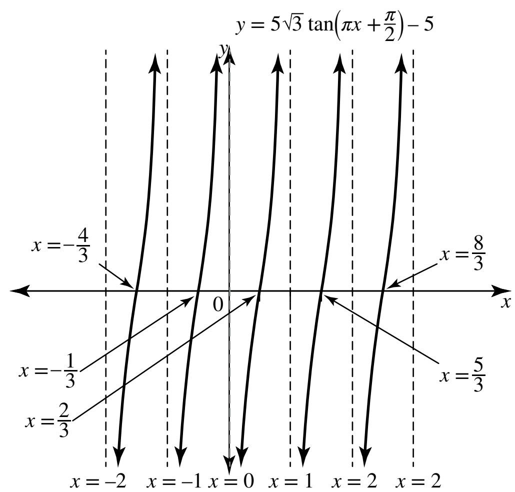
$\therefore \tan\left(\pi x - \frac{\pi}{2}\right) = \frac{1}{\sqrt{3}},\ \pi x - \frac{\pi}{2} \in \left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$

$\therefore \pi x - \frac{\pi}{2} = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$ or $-\frac{5\pi}{6}, -\frac{11\pi}{6}$

$\therefore \pi x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$ or $-\frac{\pi}{3}, -\frac{4\pi}{3}$

$\therefore x = \frac{2}{3}, \frac{5}{3}, \frac{8}{3}, -\frac{1}{3}, -\frac{4}{3}$

No endpoints as these are asymptotes



b **i** As the x intercepts lie midway between successive pairs of asymptotes, the mean position is $y = 0$. Hence there has been no horizontal translation applied.

ii The period is the distance between successive pairs of asymptotes. Using the asymptotes $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ shows the period is $\frac{\pi}{2}$.

iii Since only a 'possible' equation is required, let the equation be $y = a\tan(nx)$.

$\frac{\pi}{n} = \frac{\pi}{2} \Rightarrow n = 2$

The equation becomes $y = a\tan(2x)$.

Substitute the point $\left(\frac{\pi}{3}, \sqrt{3}\right)$

$\therefore \sqrt{3} = a\tan\left(\frac{2\pi}{3}\right)$

$\therefore \sqrt{3} = a \times -\sqrt{3}$

$\therefore a = -1$

A possible equation could be $y = -\tan(2x)$.

Another equation could be: Let $y = a\tan(2(x - h))$.

The graph passes through the origin.

$\therefore 0 = a\tan(-2h)$

$\therefore \tan(-2h) = 0$

$\therefore -\tan(2h) = 0$

$\therefore 2h = k\pi, k \in Z$

$\therefore h = k\left(\frac{\pi}{2}\right), k \in Z$

Choosing $k = -2$, for example, the equation becomes $y = a\tan(2x + 2\pi)$.

Substitute the point $\left(\frac{\pi}{3}, \sqrt{3}\right)$

$\sqrt{3} = a\tan\left(\frac{2\pi}{3} + 2\pi\right)$

$\therefore \sqrt{3} = a \times -\sqrt{3}$

$\therefore a = -1$

So the equation could be expressed as $y = -\tan(2x + 2\pi)$.

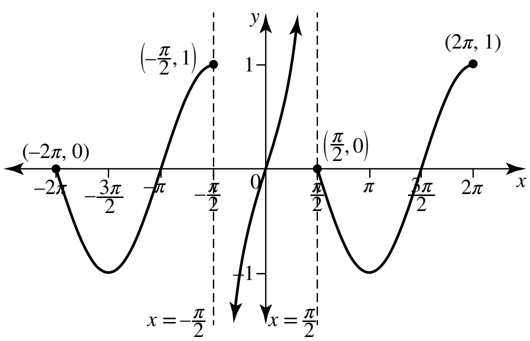
23 a $f(x) = \begin{cases} -\sin(x), & -2\pi \le x \le -\frac{\pi}{2} \\ \tan(x), & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos(x), & \frac{\pi}{2} \le x \le 2\pi \end{cases}$

i $f\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

ii $f(\pi) = \cos(\pi) = -1$

iii $f\left(-\frac{\pi}{2}\right) = -\sin\left(-\frac{\pi}{2}\right) = 1$

b

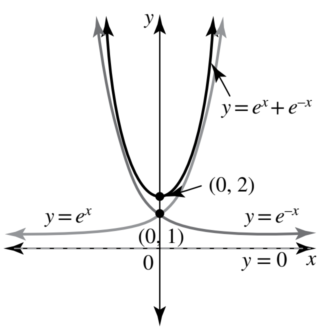


c The function is not continuous at $x = \pm\frac{\pi}{2}$.

d Domain is $[-2\pi, 2\pi]$ and range is R.

24 a $y = e^x + e^{-x}$

Draw the graphs of $y_1 = e^x$ and $y_2 = e^{-x}$ on the same set of axes and add their ordinates.

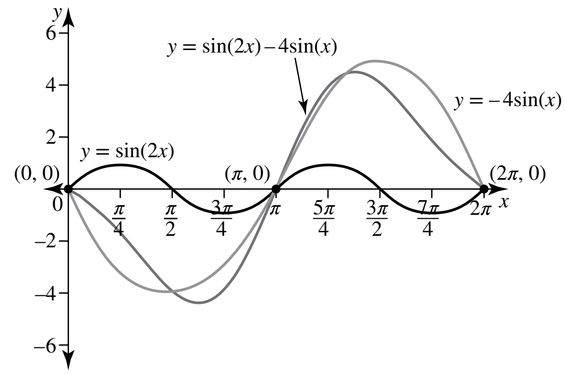


b $y = \sin(2x) - 4\sin(x),\ 0 \le x \le 2\pi$

$y_1 = \sin(2x)$ has period π and amplitude 1.

$y_2 = -4\sin(x)$ has period 2π, amplitude 4 and its graph is inverted.

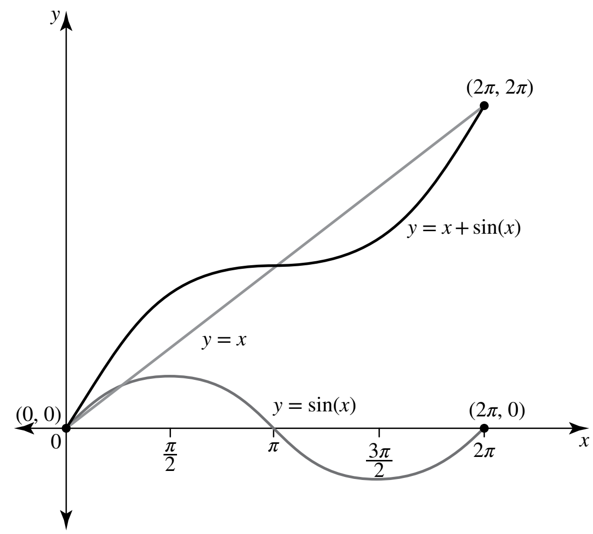
The required graph is $y = y_1 + y_2$.



c $y = x + \sin(x),\ 0 \le x \le 2\pi$

Mixing an algebraic function with a trigonometric functions creates a challenge with the scaling of the axes. This can be overcome by using the decimal approximations $\frac{\pi}{2} \approx 1.57,\ \pi \approx 3.14,\ \frac{3\pi}{2} \approx 4.71,\ 2\pi \approx 6.28$ for these key points.

	$x = 0$	$x = \frac{\pi}{2}$	$x = \pi$	$x = \frac{3\pi}{2}$	$x = 2\pi$
$y_1 = x$	0	1.57	3.14	4.71	6.28
$y_2 = \sin(x)$	0	1	0	−1	0
$y = y_1 + y_2$	0	2.57	3.14	3.71	6.28



25 a As there is no restriction given on the domain, both $y = e^x$ and $y = \cos(x)$ have domains of R. This means there are an infinite number of intersections of their graphs with negative x values. The graphs meet at $(0,1)$ but there are no intersections with positive x values.

b For $x \in [-2\pi, 2\pi]$ there are three solutions: $x \approx -4.721, -1.293$ and $x = 0$.

26 a The 13 points of intersections of the graphs of $y = \sin(2x)$ and $y = \tan(x)$ for $-2\pi \le x \le 2\pi$ are

$(\pm 2\pi, 0), (\pm\pi, 0), (0, 0),$

$\left(-\frac{5\pi}{4}, -1\right), \left(-\frac{\pi}{4}, -1\right), \left(\frac{3\pi}{4}, -1\right), \left(\frac{7\pi}{4}, -1\right),$

$\left(-\frac{7\pi}{4}, 1\right), \left(-\frac{3\pi}{4}, 1\right), \left(\frac{\pi}{4}, 1\right), \left(\frac{5\pi}{4}, 1\right)$

b The solutions to $\sin(2x) = \tan(x)$ are the x co-ordinates of the points of intersection of the graphs of $y = \sin(2x)$ and $y = \tan(x)$. These are integer multiples of π or odd integer multiples of $\frac{\pi}{4}$.

The general solution of the equation for $x \in R$ is

$x = n\pi, n \in Z$ or $x = (2n+1)\frac{\pi}{4}, n \in Z.$

Exercise 2.6 — Modelling and applications

1 a $P(t) = 83 - 65e^{-0.2t}, t \geq 0$

$P(0) = 83 - 65e^{0}$

$= 18$

There were 18 possums initially.

b $P(1) = 83 - 65e^{-0.2}$

≈ 30

The population has increased by 12.

c Let $P = 36$

$\therefore 36 = 83 - 65e^{-0.2t}$

$\therefore 65e^{-0.2t} = 47$

$\therefore e^{-0.2t} = \frac{47}{65}$

$\therefore -0.2t = \log_e\left(\frac{47}{65}\right)$

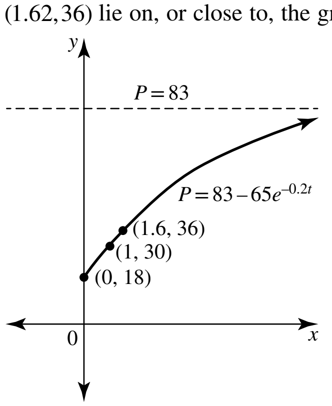
$\therefore t = -5\log_e\left(\frac{47}{65}\right)$

≈ 1.62

The population doubled in 1.62 months.

d $P(t) = 83 - 65e^{-0.2t}, t \geq 0$

Horizontal asymptote at $P = 83$. Points $(0, 18)$, $(1, 30)$ and $(1.62, 36)$ lie on, or close to, the graph.



e The presence of the asymptote at $P = 83$ shows that as $t \to \infty, P \to 83$. The population can never exceed 83 so the population cannot grow to 100.

2 a

x	0	1	3	4
y	4	2	10	8

The data points increase and decrease, so they cannot be modelled by a one-to-one function. Neither a linear model nor an exponential model is possible.

The data is not oscillating, however, so it is unlikely to be trigonometric. The jump between $x = 1$ and $x = 3$ is a concern, but the data could be modelled by a polynomial such as a cubic with a turning point between $x = 1$ and $x = 3$. However, $y = x^n$ requires the point $(0, 0)$ to be on it and that is not true for the data given.

b i $y = \frac{a}{x-2} + k$

Substitute the point $(0, 4)$

$\therefore 4 = \frac{a}{-2} + k$

$\therefore 8 = -a + 2k$ ...(1)

Substitute the point $(3, 10)$

$\therefore 10 = \frac{a}{1} + k$

$\therefore 10 = a + k$ ...(2)

Adding the two equations,

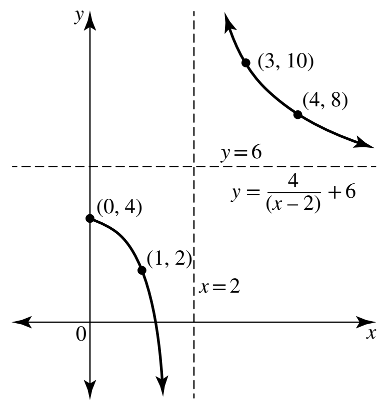
$3k = 18$

$\therefore k = 6$

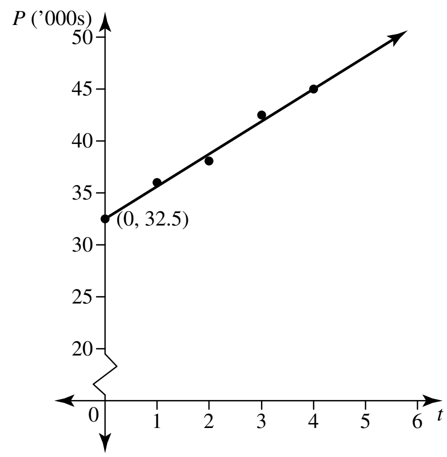
$\therefore a = 4$

The equation is $y = \frac{4}{x-2} + 6.$

ii The graph has a vertical asymptote at $x = 2$ and a horizontal asymptote at $y = 6$.



3 a



The data appears to be linear.

b $(2, 38.75)$ and $(4, 45)$

$m = \frac{45 - 38.75}{4 - 2}$

$= \frac{6.25}{2}$

$= 3.125$

$P - 45 = 3.125(t - 4)$

$\therefore P = 3.125t - 12.5 + 45$

$\therefore P = 3.125t + 32.5$

c Let $t=0$
$\therefore P=32.5$
There were 32 500 bees in January.

d The gradient gives the rate of increase
The bees are increasing at 3.125 thousand per month.

4 $F=a(r)^n$

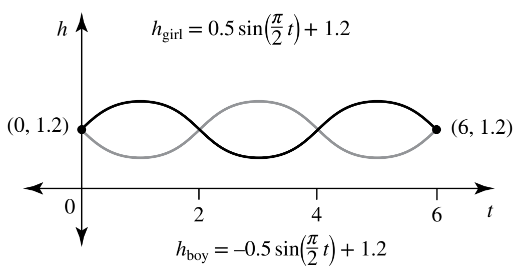
a Substitute $n=1, F=435$
$435=ar$......(1)
Substitute $n=2, F=655$
$655=ar^2$....(2)
Substitute equation (1) into equation (2)
$655=455r$
$\therefore r=\frac{655}{435}$
$\therefore r\approx 1.5$
$\therefore a=\frac{435}{1.5}$
$\therefore a=290$

b $F=290(1.5)^t$
Let $t=0$, then $F=290$.
Eric pays a fine of \$290.

c Let $t=6$
$F=290(1.5)^6$
≈ 3303
Eric would pay \$3303.

5 a $h(t)=a\sin(nt)+k$
The range of the function is between 0.7 and 1.7 metres travelled in 2 seconds.
Midway between these values is 1.2 so $k=1.2$ and amplitude is 0.5. As the girl rises from mean position, $a=0.5$.
It takes half a period for the function to range between its greatest and least values, so the period is 4.
$\frac{2\pi}{n}=4$
$\therefore n=\frac{\pi}{2}$
Hence, $a=0.5, n=\frac{\pi}{2}, k=1.2$.

b $h(t)=0.5\sin\left(\frac{\pi}{2}t\right)+1.2$
For $0\le t\le 6$, one and a half cycles will be covered.



c Let $h=1.45$. From the graph there will be four intersections of the line $h=1.45$ with the curve showing the girl's height.
$0.5\sin\left(\frac{\pi}{2}t\right)+1.2=1.45$
$\therefore \sin\left(\frac{\pi}{2}t\right)=0.5$
$\therefore \frac{\pi}{2}t=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}$
$\therefore t=\frac{1}{3},\frac{5}{3},\frac{13}{3},\frac{17}{3}$
The time interval between $\frac{1}{3}$ and $\frac{5}{3}$ is $\frac{4}{3}$ so over the first six seconds she is 1.45 metres or higher, for $\frac{8}{3}$ seconds.

d Graph is shown in part **b**, its equation is
$h_{boy}=-0.5\sin\left(\frac{\pi}{2}t\right)+1.2$.

6 a The point is $(0,2)$ so $a=2$.

b Reading from the diagram, the coordinates are $(2,2)$

c Turning point at $(5,0)$ so let the equation be of the form $y=a(x-5)^2$.
Substitute the point $(2,2)$
$\therefore 2=a(-3)^2$
$\therefore a=\frac{2}{9}$
The equation is $y=\frac{2}{9}(x-5)^2, 2\le x\le 9$

d For $0<x<1$, the line has gradient 2 and y intercept at $y=2$,
Its equation is $y=2x+2$.
The rule for the hybrid function can be expressed as

$$y=\begin{cases}2x+2, & -1<x<0\\ 2, & 0\le x\le 2\\ \frac{2}{9}(x-5)^2, & 2<x\le 9\end{cases}.$$

e There are three positions where the skateboarder would be at a height of 1.5 metres above the ground. One is when the person climbs the connecting ladder and the other two are on the parabolic ramp.
Consider
$2x+2=1.5$
$\therefore 2x=-0.5$
$\therefore x=-0.25$
Consider $\frac{2}{9}(x-5)^2=1.5$
$\therefore (x-5)^2=\frac{27}{4}$
$\therefore x=5\pm\frac{3\sqrt{3}}{2}$
Answer is $x=5\pm\frac{3\sqrt{3}}{2}$ or $x=-\frac{1}{4}$.

7 $T=20+75e^{-0.062t}$

a When $t=0$, $T=20+75=95$ so the initial temperature was 95°C.

b The exponential function has a horizontal asymptote at $T=20$ so as $t\to\infty, T=20$.
The temperature approaches 20°C.

c Let $T=65$
$\therefore 65=20+75e^{-0.062t}$
$\therefore e^{-0.062t}=\frac{45}{75}$
$\therefore e^{-0.062t}=\frac{3}{5}$
$\therefore -0.062t=\log_e\left(\frac{3}{5}\right)$
$\therefore t=-\frac{1}{0.062}\log_e\left(\frac{3}{5}\right)$
$\therefore t\approx 8.24$
It takes approximately 8.24 minutes to cool to 65°C.

d $T = A + Be^{-0.062t}$

As the temperature cannot exceed 85°C, $A = 85$

$T = 85 + Be^{-0.062t}$

Substitute (8.24, 65)

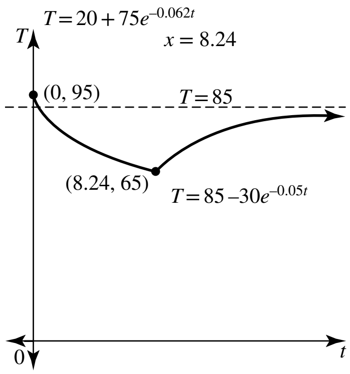
$\therefore 65 = 85 + Be^{-0.05\times 8.24}$

$\therefore Be^{-0.412} = -20$

$\therefore B = -20e^{0.412}$

$\therefore B \approx -30$

e



8 a $d = 1.5\sin\left(\frac{\pi t}{12}\right) + 12.5$

When $t = 0$, $d = 1.5\sin(0) + 12.5 = 12.5$

The boat is 12.5 metres above the seabed.

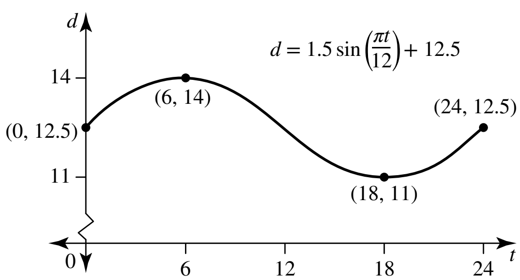
b Period is $2\pi \div \frac{\pi}{12} = 24$ hours

c Mean position is $d = 12.5$ and amplitude is 1.5.

Maximum height is $12.5 + 1.5 = 14$ metres

Minimum height is $12.5 - 1.5 = 11$ metres

d



e Maximum height occurs when $t = 6$, so between $t = 4$ and $t = 8$ the depth exceeds a value of $d = h$ for a continuous interval of 4 hours.

$\therefore h = 1.5\sin\left(\frac{\pi}{12}\times 4\right) + 12.5$

$\therefore h = 1.5\sin\left(\frac{\pi}{3}\right) + 12.5$

$\therefore h = 1.5 \times \frac{\sqrt{3}}{2} + 12.5$

$\therefore h = \frac{3\sqrt{3} + 50}{4} \approx 13.8$

f The first minimum after 12 hours half a cycle, have passed occurs when $t = 18$. It will be safe to return to shore 18 hours after 9 : 30 am. This makes the time 3 : 30 pm the following day.

9 a $N = 22 \times 2^t$

Let $N = 2816$

$\therefore 2816 = 22 \times 2^t$

$\therefore 2^t = 128$

$\therefore 2^t = 2^7$

$\therefore t = 7$

In 7 days the number of bacteria reaches 2816.

b As $t \to \infty, N \to \infty$, so the number of bacteria will increase without limit.

c i $N = \frac{66}{1 + 2e^{-0.2t}}$

Let $t = 0$,

$\therefore N = \frac{66}{1 + 2e^0}$

$\therefore N = \frac{66}{3}$

$\therefore N = 22$

Reconsider the first model $N = 22 \times 2^t$. If $t = 0$, $N = 22$.

Both models have an initial number of 22 bacteria.

ii As $t \to \infty, e^{-0.2t} \to 0$.

Therefore $N \to \frac{66}{1+0} = 66$.

The number of bacteria will never exceed 66.

10 a The garden area is the area of the entire square minus the area of the two right-angled triangles.

$A = 40 \times 40 - \frac{1}{2} \times x \times x - \frac{1}{2} \times (40 - x) \times 40$

$= 1600 - \frac{1}{2}x^2 - 20(40 - x)$

$= 1600 - \frac{1}{2}x^2 - 800 + 20x$

$= -\frac{1}{2}x^2 + 20x + 800$

b Both $x > 0$ and $40 - x > 0$ since these are lengths. The restriction that needs to be placed is that $0 < x < 40$.

c Completing the square

$A = -\frac{1}{2}\left(x^2 - 40x - 1600\right)$

$= -\frac{1}{2}\left(\left(x^2 - 40x + 400\right) - 400 - 1600\right)$

$= -\frac{1}{2}\left((x - 20)^2 - 1000\right)$

$= -\frac{1}{2}(x - 20)^2 + 1000$

The greatest area is 1000 m^2 when $x = 20$.

11 a The stationary point of inflection at $x = 0 \Rightarrow x^3$ is a factor of then graph's equation.

The cuts at $x = \pm\sqrt{5} \Rightarrow x + \sqrt{5}$ and $x - \sqrt{5}$are factors.

Let the equation be $y = ax^3\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$.

Substitute the point $\left(\sqrt{3}, -12\sqrt{3}\right)$

$\therefore -12\sqrt{3} = a\left(\sqrt{3}\right)^3\left(\sqrt{3} - \sqrt{5}\right)\left(\sqrt{3} + \sqrt{5}\right)$

$\therefore -12\sqrt{3} = a \times 3\sqrt{3} \times \left(\left(\sqrt{3}\right)^2 - \left(\sqrt{5}\right)^2\right)$

$\therefore -12\sqrt{3} = 3\sqrt{3}a \times (3 - 5)$

$\therefore -12\sqrt{3} = -6\sqrt{3}a$

$\therefore a = 2$

The equation is $y = 2x^3\left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right)$.

b $y = 2x^3\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)$

$y = 2x^3\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)$

$\therefore y = 2x^3(x^2-5)$

$\therefore y = 2x^5 - 10x^3$

c **i** The maximum turning point on the graph of $y = g(x)$ has coordinates of $\left(-\sqrt{3}, 12\sqrt{3}\right)$. A horizontal translation of $\sqrt{3}$ units to the right is required for this point to have a x coordinate of 0.
The minimum turning point on the graph of $y = g(x)$ has coordinates of $\left(\sqrt{3}, -12\sqrt{3}\right)$. A vertical translation of $12\sqrt{3}+1$ units upward is required for this point to have a y coordinate of 1.

ii The y co-ordinate of point A is $12\sqrt{3}+\left(12\sqrt{3}+1\right) = 24\sqrt{3}+1$. The height of A above the water is $\left(24\sqrt{3}+1\right) \approx 42.6$ metres.

iii B has an x value of $\sqrt{3}+\sqrt{3} = 2\sqrt{3}$ so the coordinates of B are $\left(2\sqrt{3}, 1\right)$.
C is the point $\left(\sqrt{3}, 12\sqrt{3}+1\right)$

12 $N: R^+ \cup \{0\} \to R, N(t) = \dfrac{at+b}{t+2}$

a $N(t) = \dfrac{at+b}{t+2}$

$N(0) = 10$

$\therefore 10 = \dfrac{b}{2}$

$\therefore b = 20$

$N(5) = 30$

$\therefore 30 = \dfrac{5a+20}{7}$

$\therefore 210 = 5a + 20$

$\therefore 5a = 190$

$\therefore a = 38$

Hence, $a = 38, b = 20$.

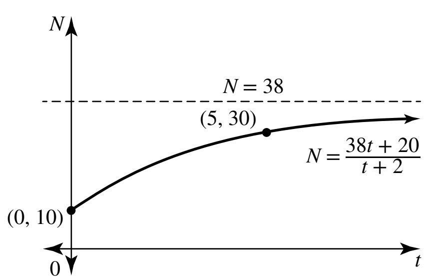
b $N(t) = \dfrac{38t+20}{t+2}$

$= \dfrac{38(t+2)-76+20}{t+2}$

$= \dfrac{38(t+2)}{t+2} - \dfrac{56}{t+2}$

$\therefore N(t) = 38 - \dfrac{56}{t+2}, t \ge 0.$

The graph is a hyperbola with horizontal asymptote $N = 38$. The vertical asymptote $t = -2$ lies outside the domain. Points (0,10) and (5,30) lie on the graph.



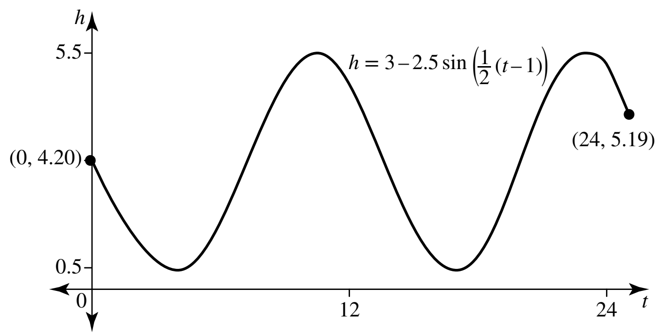
c The horizontal asymptote shows that as $t \to \infty, N \to 38$. The population of quolls will never exceed 38.

13 $h = 3 - 2.5\sin\left(\dfrac{1}{2}(t-1)\right)$

a At 7 : 30 am, $t = 0.5$, so $h = 3 - 2.5\sin(-0.25) \approx 3.619$.
The water level is approximately 3.619 metres below the jetty.

b Mean position is 3, amplitude is 2.5 so the distances below the jetty lie between $3 - 2.5 = 0.5$ and $3 + 2.5 = 5.5$.
The greatest distance below the jetty is 5.5 metres and least distance is 0.5 metres

c



The first maximum occurs for $t \approx 10.42$ and the first minimum for $t \approx 4.14$. (Solve using CAS)

d The difference between low and high tide is 5 metres so an additional 5 metres of rope is needed.

14 **a** In triangle OBC, OB is of length $h - 4$ cm, $h > 4$.
Using Pythagoras' theorem

$(h-4)^2 + r^2 = 4^2$

$\therefore r^2 = 16 - (h-4)^2$

$\therefore r = \sqrt{16-(h-4)^2}, r > 0$

$\therefore r = \sqrt{8h - h^2}$

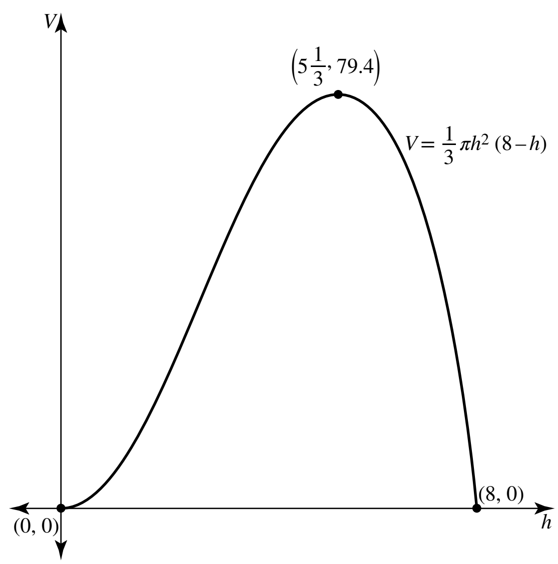
b $V = \dfrac{1}{3}\pi r^2 h$

$\therefore V = \dfrac{1}{3}\pi(8h - h^2)h$

$\therefore V = \dfrac{1}{3}\pi h^2(8-h)$

$h > 0$ and $8 - h > 0$ so the restriction on h is $0 < h < 8$. This would be seen on the graph to be the domain interval where $V > 0$.

c



d Using CAS the greatest volume is 79 cm^3.

Topic 3 — Composite functions, transformations and inverses

Exercise 3.2 — Composite functions and functional equations

1 $f(x)=(x-1)(x+3)$ and $g(x)=x^2$

	Dom	Ran
$f(x)$	R	$[-4,\infty)$
$g(x)$	R	$[0,\infty)$

For $f(g(x))$, the range of $g(x)$ is $[0,\infty)$, which is a subset of the domain of
$f(x)$, which is R so $f(g(x))$ exists.
$f(g(x))=(x^2-1)(x^2+3)=(x-1)(x+1)(x^2+3)$ where dom $=R$
For $g(f(x))$, the range of $f(x)$ is $[-4,\infty)$, which is a subset of the domain of
$g(x)$, which is R so $g(f(x))$ exists.
$g(f(x))=((x-1)(x+3))^2=(x-1)^2(x+3)^2$ where dom $=R$

2 $f(x)=2x-1$ and $g(x)=\dfrac{1}{x-2}$

	Dom	Ran
$f(x)$	R	R
$g(x)$	$R\backslash\{2\|$	$R\backslash\{0\}$

For $f(g(x))$, the range of $g(x)$ is $R\backslash\{0\}$, which is a subset of the domain of $f(x)$, which is R so $f(g(x))$ exists.
$f(g(x))=\dfrac{2}{x-2}-1$ where dom $=R\backslash\{2\}$
For $g(f(x))$, the range of $f(x)$ is R, which is not a subset of the domain of $g(x)$, which is $R\backslash\{2\}$ so $g(f(x))$ does not exist

3 $f(x)=\sqrt{x+3}$ and $g(x)=2x-5$

	Dom	Ran
$f(x)$	$[-3,\infty)$	$[0,\infty)$
$g(x)$	R	R

a $f(g(x))$ is not defined because the range of $g(x)$, which is R is not contained in the domain of $f(x)$, which is $[-3,\infty)$.
b We want ran $g=[-3,\infty)$
Solve $2x-5>-3$
If the domain of $g(x)$ is restricted to $[1,\infty)$ to produce the function $h(x)$, then $f(h(x))$ exists because the range of $h(x)$ will be $[-3,\infty)$, which equals the domain of $f(x)$.
$\therefore\ h(x)=2x-5,\ x\in[1,\infty)$
c $f(h(x))=\sqrt{2x-5+3}=\sqrt{2x-2}$ where $x\in[1,\infty)$

4 $f(x)=x^2$ and $g(x)=\dfrac{1}{x-4}$

	Dom	Ran
$f(x)$	R	$[0,\infty)$
$g(x)$	$R\backslash\{4\}$	$R\backslash\{0\}$

a $g(f(x))$ is not defined because the range of $f(x)$, which is $[0,\infty)$ is not contained in the domain of $g(x)$, which is $R\backslash\{4\}$.
b We want ran $f\neq 4$
Solve $x^2\neq 4$
If the domain of $f(x)$ is restricted to $R\backslash\{-2,2\}$ to produce the function $h(x)$, then $g(h(x))$ exists because the range of $h(x)$ will be $R\backslash\{4\}$, which equals the domain of $g(x)$.
$\therefore\ h(x)=x^2,\ x\in R\backslash\{-2,2\}$
c $g(h(x))=\dfrac{1}{x^2-4}$ where $x\in R\backslash\{-2,2\}$

5 $f(x)=-\dfrac{3}{x}$

LHS: $\dfrac{f(x)-f(y)}{f(xy)}$

$$=\frac{\frac{-3}{x}+\frac{3}{y}}{-\frac{3}{xy}}$$
$$=\frac{3x-3y}{xy}\times-\frac{xy}{3}$$
$$=-(x-y)$$
$$=y-x$$

RHS: $y-x$
$=$ LHS
Therefore $\dfrac{f(x)-f(y)}{f(xy)}=y-x$

6 $f(x)=e^{2x}$ and $f(y)=e^{2y}$
$f(x+y)=e^{2(x+y)}=e^{2x}\times e^{2y}=f(x)f(y)$
$f(x-y)=e^{2(x-y)}=e^{2x}\div e^{2y}=\dfrac{f(x)}{f(y)}$

7 $f(x)=x^2+1,\ g(x)=\sqrt{x}$ and $h(x)=\dfrac{1}{x}$

	Dom	Ran
$f(x)$	R	$[1,\infty)$
$g(x)$	$[0,\infty)$	$[0,\infty)$
$h(x)$	$R\backslash\{0\}$	$R\backslash\{0\}$

a $f(g(x))$ exists because the range of $g(x)$ which is $[0,\infty)$, is a subset of the domain of $f(x)$, which is R.
Dom $=[0,\infty)$.
b $g(f(x))$ exists because the range of $f(x)$, which is $[1,\infty)$, is a subset of the domain of $g(x)$,which is $[0,\infty)$
Dom $=R$.
c $h(g(x))$ does not exist because the range of $g(x)$, which is $[0,\infty)$, is not a subset of the domain of $h(x)$, which is $R\backslash\{0\}$.
d $h(f(x))$ does exist because the range of $f(x)$, which is $[1,\infty)$, is a subset of the domain of $h(x)$, which is $R\backslash\{0\}$.
Dom $=R$.

8 $f(x)=x^2,\ g(x)=\sqrt{x}$ and $h(x)=-\dfrac{1}{x}$

	Dom	Ran
$f(x)$	R	$[0,\infty)$
$g(x)$	$[0,\infty)$	$[0,\infty)$
$h(x)$	$R\backslash\{0\}$	$R\backslash\{0\}$

a $f(g(x))$ exists because the range of $g(x)$, which is $[0,\infty)$ is a subset of the domain of $f(x)$, which is R.
The rule is $f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$ where Dom $= [0,\infty)$.

b $g(f(x))$ exists because the range of $f(x)$, which is $[0,\infty)$ is equal to the domain of $g(x)$, which is $[0,\infty)$.
The rule is $g(f(x)) = f(x^2) = \sqrt{x^2} = |x|$ where Dom $= R$.

c $h(f(x))$ does not exist because the range of $f(x)$, which is $[0,\infty)$, is not a subset of the domain of $h(x)$, which is $R\setminus\{0\}$.

d $g(h(x))$ does not exist because the range of $h(x)$, which is, $R\setminus\{0\}$ is not a subset of the domain of $g(x)$, which is $[0,\infty)$.

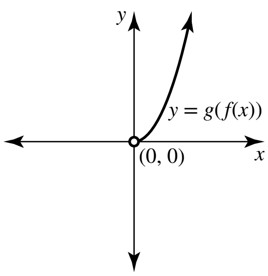
9 $f: R \to R,\ f(x) = x^2 + 1$ where Ran $= [1,\infty)$
$g: [-2,\infty) \to R,\ g(x) = \sqrt{x+2}$ where Ran $= [0,\infty)$
$f(g(x))$ exists because the range of $g(x)$, which is $[0,\infty)$, is a subset of the domain of $f(x)$, which is R.
$f(g(x)) = f(\sqrt{x+2}) = (\sqrt{x+2})^2 + 1 = x + 3$
where Dom $= [-2,\infty)$ and Ran $= [0,\infty)$.

10 $f: (0,\infty) \to R,\ f(x) = \dfrac{1}{x}$ where Ran $= (0,\infty)$
$g: R\setminus\{0\} \to R,\ f(x) = \dfrac{1}{x^2}$ where Ran $= (0,\infty)$

a $g(f(x))$ exists because the range of $f(x)$, which is $(0,\infty)$, is a subset of the domain of $g(x)$, which is $R\setminus\{0\}$.

b $g(f(x)) = g\left(\dfrac{1}{x}\right) = x^2$ where Dom $= (0,\infty)$ and Ran $= (0,\infty)$.

c



11 $g(x) = \dfrac{1}{(x-3)^2} - 2$ and $f(x) = \sqrt{x}$.

	Dom	Ran
$g(x)$	R\{3}	$(-2,\infty)$
$f(x)$	$[0,\infty)$	$[0,\infty)$

a $f(g(x))$ is not defined because the range of $g(x)$, which is $(-2,\infty)$ is not contained in the domain of $f(x)$, which is $[0,\infty)$.

b For $f(g(x))$ to have a domain for its existence

$$\frac{1}{(x-3)^2} - 2 = 0$$
$$\frac{1}{(x-3)^2} = 2$$
$$(x-3)^2 = \frac{1}{2}$$
$$x - 3 = \pm\frac{1}{\sqrt{2}}$$
$$x = \pm\frac{1}{\sqrt{2}} + 3$$

i.e. $x \in \left(-\infty, -\dfrac{1}{\sqrt{2}} + 3\right] \cup \left[\dfrac{1}{\sqrt{2}} + 3, \infty\right)$

12 $f: (-\infty, 2] \to R,\ f(x) = \sqrt{2-x}$ and $g: R \to R,\ g(x) = -\dfrac{1}{x-1} + 2$

	Dom	Ran
$f(x)$	$(-\infty,2]$	$[0,\infty)$
$g(x)$	$R\setminus\{1\}$	$R\setminus\{2\}$

a $g(f(x))$ is not defined because the range of $f(x)$, which is $[0,\infty)$ is not contained in the domain of $g(x)$, which is $R\setminus\{1\}$.

b For $g(f_1(x))$ to have a domain for its existence then the domain must be $(-\infty,2]\setminus\{1\}$.
$f_1(x) = \sqrt{2-x},\ x \in (-\infty,2]\setminus\{1\}$

c $g(f_1(x)) = -\dfrac{1}{\sqrt{2-x} - 1} + 2$ where $x \in (-\infty,2]\setminus\{1\}$

13 $f(x) = 5^x$

a To show that $f(x+y) = f(x) \times f(y)$.
LHS: $f(x+y)$
$= 5^{x+y}$
RHS: $f(x) \times f(y)$
$= 5^x \times 5^y$
$= 5^{x+y}$
$=$ LHS
Therefore $f(x+y) = f(x) \times f(y)$

b To show that $f(x-y) = \dfrac{f(x)}{f(y)}$.
LHS: $f(x-y)$
$= 5^{x-y}$
RHS: $\dfrac{f(x)}{f(y)}$
$= \dfrac{5^x}{5^y}$
$= 5^{x-y}$
$=$ LHS
Therefore $f(x-y) = \dfrac{f(x)}{f(y)}$

14 $f(x) = x^3,\ f(y) = y^3$ and $f(xy) = (xy)^3$
LHS: $f(xy) = (xy)^3$
RHS: $f(x) \times f(y)$
$= x^3 \times y^3$
$= (xy)^3$
$=$ LHS
Therefore $f(xy) = f(x) \times f(y)$

15 a $h(x) = 3x + 1$
$h(x+y) = 3(x+y) + 1$
$= 3x + 3y + 1$
$= (3x+1) + (3y+1) - 1$
$= h(x) + h(y) + c$
Therefore $c = -1$

b $h(x) = \dfrac{1}{x^3}$
LHS: $h(x) + (y)$
$= \dfrac{1}{x^3} + \dfrac{1}{y^3}$
$= \dfrac{y^3 + x^3}{x^3 y^3}$

RHS: $\left(x^3+y^3\right)h(xy)$

$=\left(x^3+y^3\right)\times\dfrac{1}{x^3y^3}$

$=\dfrac{x^3+y^3}{x^3y^3}$

$=$ LHS

Therefore $h(x)+h(y)=\left(x^3+y^3\right)h(xy)$

16 **a** $f(x)=x^2$

A:

LHS: $f(xy)=(xy)^2$

RHS: $f(x)f(y)=x^2\times y^2=(xy)^2$

LHS = RHS

B:

LHS: $f\left(\dfrac{x}{y}\right)=\left(\dfrac{x}{y}\right)^2$

RHS: $\dfrac{f(x)}{f(y)}=\dfrac{x^2}{y^2}=\left(\dfrac{x}{y}\right)^2$

LHS = RHS

C:

LHS: $f(-x)=(-x)^2=x^2$

RHS: $-f(x)=-x^2$

LHS ≠ RHS

D:

LHS: $f(x+y)=(x+y)^2$

RHS: $f(x)f(y)=x^2y^2$

LHS ≠ RHS

b $f(x)=\sqrt{x}$

A:

LHS: $f(xy)=\sqrt{xy}$

RHS: $f(x)f(y)=\sqrt{x}\times\sqrt{y}=\sqrt{xy}$

LHS = RHS

B:

LHS: $f\left(\dfrac{x}{y}\right)=\sqrt{\dfrac{x}{y}}$

RHS: $\dfrac{f(x)}{f(y)}=\dfrac{\sqrt{x}}{\sqrt{y}}=\sqrt{\dfrac{x}{y}}$

LHS = RHS

C:

LHS: $f(-x)=\sqrt{-x}$

RHS: $-f(x)=-\sqrt{x}$

LHS ≠ RHS

D:

LHS: $f(x+y)=\sqrt{x+y}$

RHS: $f(x)f(y)=\sqrt{x}\sqrt{y}=\sqrt{xy}$

LHS ≠ RHS

c $f(x)=\dfrac{1}{x}$

A:

LHS: $f(xy)=\dfrac{1}{xy}$

RHS: $f(x)f(y)=\dfrac{1}{x}\times\dfrac{1}{y}=\dfrac{1}{xy}$

LHS = RHS

B:

LHS: $f\left(\dfrac{x}{y}\right)=\dfrac{1}{\frac{x}{y}}=\dfrac{y}{x}$

RHS: $\dfrac{f(x)}{f(y)}=\dfrac{\frac{1}{x}}{\frac{1}{y}}=\dfrac{y}{x}$

LHS = RHS

C:

LHS: $f(-x)=\dfrac{1}{-x}$

RHS: $-f(x)=-\dfrac{1}{x}$

LHS = RHS

D:

LHS: $f(x+y)=\dfrac{1}{x+y}$

RHS: $f(x)f(y)=\dfrac{1}{x}\times\dfrac{1}{y}=\dfrac{1}{xy}$

LHS ≠ RHS

d $f(x)=e^x$

A:

LHS: $f(xy)=e^{xy}$

RHS: $f(x)f(y)=e^xe^y=e^{x+y}$

LHS ≠ RHS

B:

LHS: $f\left(\dfrac{x}{y}\right)=e^{\frac{x}{y}}$

RHS: $\dfrac{f(x)}{f(y)}=\dfrac{e^x}{e^y}=e^{x-y}$

LHS ≠ RHS

C:

LHS: $f(-x)=e^{-x}$

RHS: $-f(x)=-e^x$

LHS ≠ RHS

D:

LHS: $f(x+y)=e^{x+y}$

RHS: $f(x)f(y)=e^xe^y=e^{x+y}$

LHS = RHS

17 $f:[4,\infty)\to R,\ f(x)=\sqrt{x-4}$ and $g:R\to R,\ g(x)=x^2-2$

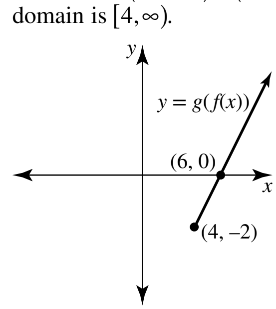
a

	Dom	**Ran**
$f(x)$	$[4,\infty)$	$[0,\infty)$
$g(x)$	R	$[-2,\infty)$

$g(f(x))$ exists because the range of $f(x)$, which is $[0,\infty)$ is the subset of the domain of $g(x)$ which is R.

b $g(f(x))=g\left(\sqrt{x-4}\right)=\left(\sqrt{x-4}\right)^2-2=x-6$ where the domain is $[4,\infty)$.

c



d $f(g(x))$ does not exist because the range of $g(x)$, which is $[-2,\infty)$ is not the subset of the domain of $f(x)$ which is $[4,\infty)$.

e If the domain of $g(x)$ is restricted to $\left(-\infty,-\sqrt{6}\right]\cup[6,\infty)$ then a new function is formed which is $g_1:\left(-\infty,-\sqrt{6}\right]\cup[6,\infty)\to R,\ g_1(x)=x^2-2$ so that $f\left(g(x)\right)$ exists.

f $f_1\left(g(x)\right)=\sqrt{x^2-2-4}=\sqrt{x^2-6}$ where $x\in\left(-\infty,-\sqrt{6}\right]\cup[6,\infty)$.

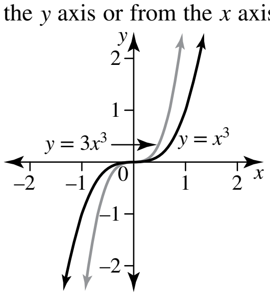
18 $f:[1,\infty)\to R,\ f(x)=-\sqrt{x}+k$ and $g:(-\infty,2]\to R,\ g(x)=x^2+k$

	Dom	**Ran**
$f(x)$	$[1,\infty)$	$(-\infty,-1+k]$
$g(x)$	$(-\infty,2]$	$[k,\infty)$

Ran $f\subseteq$ Dom g, so $[1,\infty)\subseteq[k,\infty)$, therefore $k\geq 1$.
Ran $g\subseteq$ Dom f, so $(-\infty,-1+k]\subseteq(-\infty,2]$, therefore $k\leq 3$.
Therefore, overall, $k\in[1,3]$

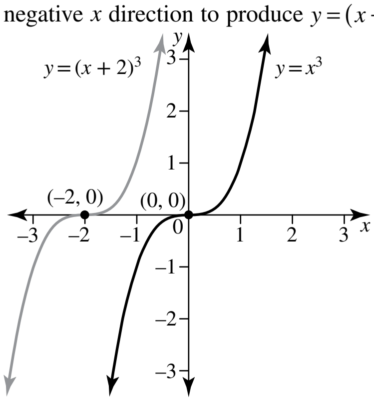
Exercise 3.3 — Transformations

1 a i $y=x^3$ has been dilated by a factor of three parallel to the y axis or from the x axis to produce $y=3x^3$.



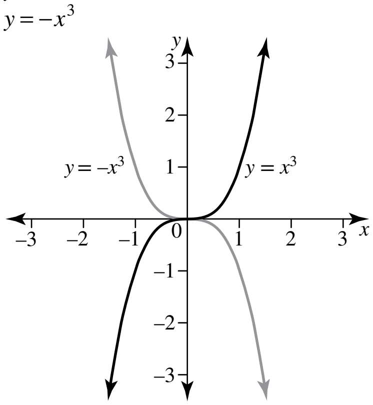
ii $(-2,-8)\to(-2,-24)$

b i $y=x^3$ has been translated two units to the left or in the negative x direction to produce $y=(x+2)^3$



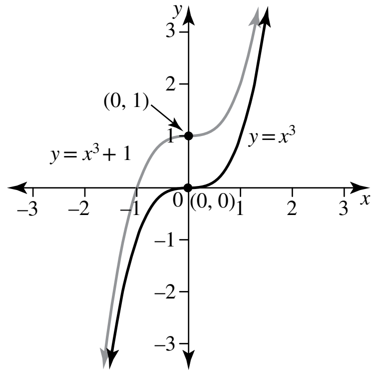
ii $(-2,-8)\to(-4,-8)$

c i $y=x^3$ has been reflected in the x axis two to produce $y=-x^3$



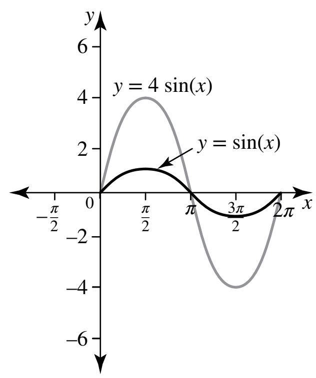
ii $(-2,-8)\to(-2,8)$

d i $y=x^3$ has been translated up one unit or in the positive in y axis direction to produce $y=x^3+1$

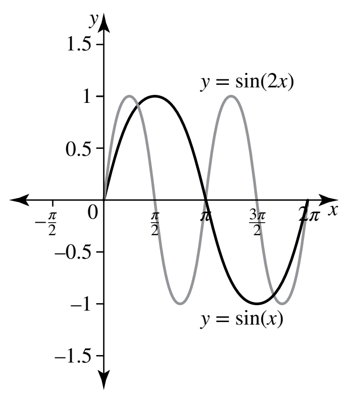


ii $(-2,-8)\to(-2,-7)$

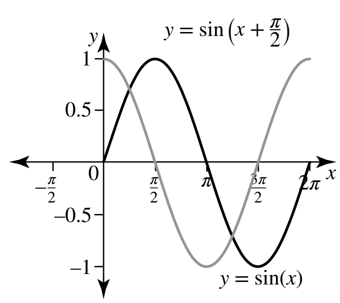
2 a $y=\sin(x)$ has been dilated by a factor of four parallel to the y axis or from the x axis to produce $y=4\sin(x)$



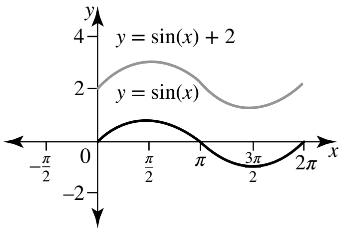
b $y=\sin(x)$ has been dilated by a factor of one half parallel to the x axis or from the y axis to produce $y=\sin(2x)$



c $y=\sin(x)$ has been translated $\dfrac{\pi}{2}$ units to the left or in the negative x direction to produce $y=\sin\left(x+\dfrac{\pi}{2}\right)$



d $y=\sin(x)$ has been translated up two units or in the positive y direction to produce $y=\sin(x)+2$



3 $y=\sin(x)\rightarrow y=-2\sin\left[2x-\frac{\pi}{2}\right]+1$

$y=\sin(x)\rightarrow y=-2\sin\left[2\left(x-\frac{\pi}{4}\right)\right]+1$

$y=\sin(x)$ has been
- Reflected in the x axis
- Dilated by a factor of 2 parallel to the y axis or from the x axis
- Dilated by a factor of $\frac{1}{2}$ parallel to the x axis or from the y axis
- Translated $\frac{\pi}{4}$ units to the right or in the positive x direction
- Translated 1 unit up or in the positive y direction

4 $y=e^x\rightarrow y=\frac{1}{3}e^{\left(\frac{x+1}{2}\right)}-2$

$y=e^x$ has been
- Dilated by a factor of $\frac{1}{3}$ parallel to the y axis or from the x axis
- Dilated by a factor of 2 parallel to the x axis or from the y axis
- Translated 1 units to the left or in the negative x direction
- Translated 2 units down or in the negative y direction

5 $g(x)=x^2$ is reflected in the y axis $\rightarrow(-x)^2=x^2$

is translated 4 units to the right $\rightarrow(x-4)^2$

is dilated by a factor of 2 from the y axis $\rightarrow\left(\frac{x}{2}-4\right)^2$

is translated 3 units down $\rightarrow\left(\frac{x}{2}-4\right)^2-3$

is dilated by a factor of $\frac{1}{3}$ from the x axis $\rightarrow\frac{1}{3}\left(\frac{x}{2}-4\right)^2-1$

$\therefore f(x)=\frac{1}{3}\left(\frac{x-8}{2}\right)^2-1$

6 $h(x)=\frac{1}{x}$ is dilated by a factor of 3 parallel to the x axis $\rightarrow\frac{3}{x}$

is translated up 2 units $\rightarrow\frac{3}{x}+2$

is reflected in the y axis $\rightarrow-\frac{3}{x}+2$

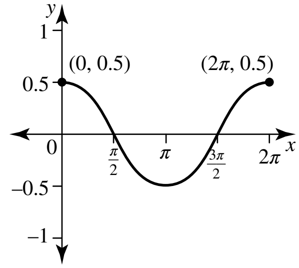
is translated 1 unit to the left $\rightarrow-\frac{3}{x+1}+2$

is reflected in the x axis $\rightarrow\frac{3}{x+1}-2$

$\therefore f(x)=\frac{3}{x+1}-2$

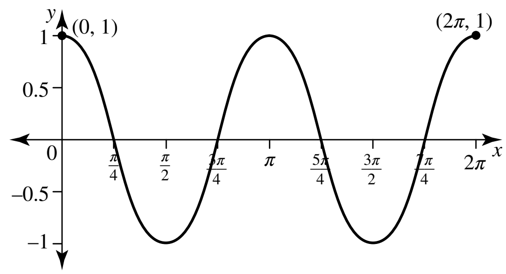
7 a $y=\cos(x)\rightarrow y=\frac{1}{2}\cos(x)$

$y=\cos(x)$ has been dilated by a factor of $\frac{1}{2}$ parallel to the y axis or from the x axis.



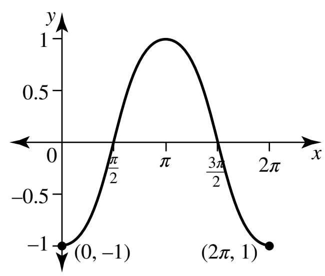
b $y=\cos(x)\rightarrow y=\cos(2x)$

$y=\cos(x)$ has been dilated by a factor of $\frac{1}{2}$ parallel to the x axis or from the y axis.



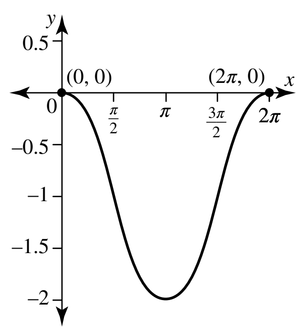
c $y=\cos(x)\rightarrow y=-\cos(x)$

$y=\cos(x)$ has been reflected in the x axis.



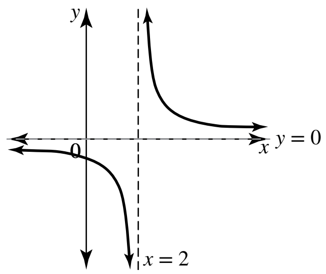
d $y=\cos(x)\rightarrow y=\cos(x)-1$

$y=\cos(x)$ has been translated down 1 unit or I unit in the negative y direction.



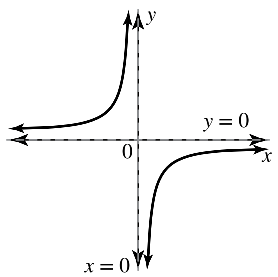
8 a $y=\frac{1}{x}\rightarrow y=f(x-2)$

$y=\frac{1}{x}$ has been translated by a factor of 2 parallel to the x axis or 2 units in the positive x direction.



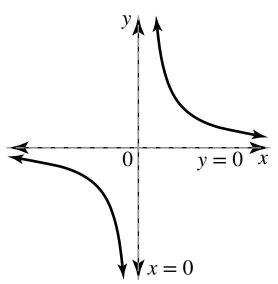
b $y=\frac{1}{x}\rightarrow y=-f(x)$

$y=\frac{1}{x}$ has been reflected in the x axis.



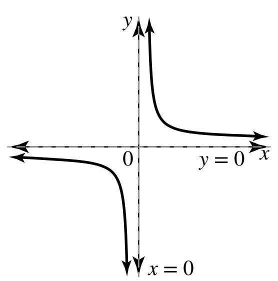
c $y=\frac{1}{x}\rightarrow y=3f(x)$

$y=\frac{1}{x}$ has been dilated by a factor of 3 parallel to the in the y axis or from the x axis.



d $y=\frac{1}{x}\rightarrow y=f(2x)$

$y=\frac{1}{x}$ has been dilated by a factor of $\frac{1}{2}$ parallel to the x axis or from the y axis



9 a $y=x^2\rightarrow y=\frac{1}{3}(x+3)^2-\frac{2}{3}$

$y=x^2$ has been dilated by a factor of $\frac{1}{3}$ parallel to the y axis or from the x axis, translated 3 units to the left or 3 units in the negative x direction and translated $\frac{2}{3}$ units down or $\frac{2}{3}$ units in the negative y direction.

b $y=x^3\rightarrow y=-2(1-x)^3+1$

$y=x^3$ has been reflected in the x and y axes, dilated by a factor of 2 parallel to the y axis or from the x axis direction, translated 1 unit to the right or 1 unit in the positive x direction and translated 1 unit up or 1 unit in the positive y direction.

c $y=\frac{1}{x}\rightarrow y=\frac{3}{(2x+6)}-1$ or $y=\frac{3}{2(x+3)}-1$

$y=\frac{1}{x}$ has been dilated by a factor of 3 parallel to the y axis or from the x axis direction, dilated by a factor of $\frac{1}{2}$ parallel to the x axis or from the y axis direction, translated 3 unit to the left or 3 unit in the negative x direction and translated 1 unit down 1 unit in the negative y direction.

10 a $(-2,4)\rightarrow\left(-2,\frac{4}{3}\right)\rightarrow\left(-5,\frac{4}{3}\right)\rightarrow\left(-5,\frac{2}{3}\right)$

b $(1,1)\rightarrow(-1,-1)\rightarrow(-1,-2)\rightarrow(0,-2)\rightarrow(0,-1)$

c $\left(2,\frac{1}{2}\right)\rightarrow(2,1)\rightarrow(1,1)\rightarrow(-2,1)\rightarrow(-2,0)$

11 a $y=\cos(x)\rightarrow y=2\cos\left[2\left(x-\frac{\pi}{2}\right)\right]+3$

$y=\cos(x)$ has been dilated by a factor of 2 parallel to the y axis or from the x axis, dilated by a factor of $\frac{1}{2}$ parallel to the x axis or from the y axis, translated $\frac{\pi}{2}$ units to the right or $\frac{\pi}{2}$ units in the positive x direction and translated 3 units up or 3 units in the positive y direction.

b $y=\tan(x)\rightarrow y=-\tan(-2x)+1$

$y=\tan(x)$ has been reflected in both axes, dilated by a factor of $\frac{1}{2}$ parallel to the x axis or from the y axis and translated 1 unit up or 1 unit in the positive y direction.

c $y=\sin(x)\rightarrow y=\sin(3x-\pi)-1$ or $y=\sin\left[3\left(x-\frac{\pi}{3}\right)\right]-1$

$y=\sin(x)$ has been dilated by a factor of $\frac{1}{3}$ parallel to the x axis or from the y axis, translated $\frac{\pi}{3}$ units to the right or $\frac{\pi}{3}$ units in the positive x direction and translated 1 down up or 1 unit in the negative y direction.

12 $h(x)=\sqrt[3]{x}\rightarrow\sqrt[3]{-x}\rightarrow\sqrt[3]{-(x-3)}\rightarrow\sqrt[3]{-\left(\frac{x}{2}-3\right)}=\sqrt[3]{-\left(\frac{x-6}{2}\right)}$

Therefore $f(x)=\sqrt[3]{-\left(\frac{x-6}{2}\right)}$

13 $h(x)=\frac{1}{x^2}\to\frac{1}{(x+2)^2}-3\to-\left(\frac{1}{(x+2)^2}-3\right)\to-3\left(\frac{1}{(x+2)^2}-3\right)=-\frac{3}{(x+2)^2}+9$

$\to-\frac{3}{(2-x)^2}+9$

Therefore $f(x)=-\frac{3}{(2-x)^2}+9$

14 $f(x)=2x^2-3\to-2x^2+3\to-2(3x)^2+3=-18x^2+3\to-18(x-1)^2+1$

Therefore $f(x)=-18(x-1)^2+1$

15 $h(x)=\frac{1}{x+2}\to\frac{1}{2x+2}\to\frac{1}{2(x+3)}-3\to\frac{1}{2(-x+3)}-3\to\frac{2}{2(3-x)}-6$

Therefore $f(x)=\frac{1}{(3-x)}-6$

16 $y=\frac{2x-5}{x-1}$

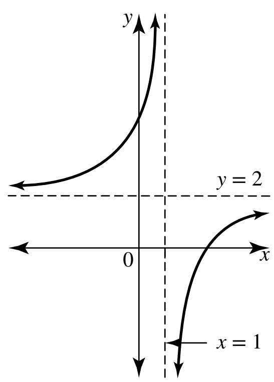
$=\frac{2(x-1)-3}{x-1}$

$=\frac{2(x-1)}{x-1}-\frac{3}{x-1}.$

$=2-\frac{3}{x-1}$

$y=\frac{1}{x}$ has been reflected in the y axis or the x axis, dilated by a factor of 3 parallel to the y axis or from the x axis, translated 1 unit to the right or in the positive x direction and translated 2 units up or in the positive y direction.

Dom $=R\setminus\{1\}$ and Ran $=R\setminus\{2\}$.



17 $y=3-\sqrt{\frac{5-x}{2}}\to y=\sqrt{x}$

$y=3-\sqrt{\frac{5-x}{2}}$ has been reflected in both axes, translated 5 units to the left or in the negative x direction, dilated by a factor of $\frac{1}{2}$ parallel to the x axis or from the y axis and translated down 3 units or in the negative y direction.

18 $y=-2(3x-1)^2+5\ \to y=(x-2)^2-1$

$y=-2(3x-1)^2+5$ has been reflected in the x axis, dilated by a factor of $\frac{1}{2}$ parallel to the y axis or from the x axis, dilated by a factor of 3 parallel to the x axis or from the y axis, translated 3 units to the left or in the negative x direction and translated 6 units down or in the negative y direction.

Exercise 3.4 — Transformations using matrices

1 $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \frac{x}{2} \\ y \end{bmatrix}$

$x'=\frac{1}{2}x$ or $2x'=x$ and $y'=y$

$y=\cos(x)\to y'=\cos(2x')$

The equation of the transformed graph is $y=\cos(2x)$.

2 $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ -y \end{bmatrix}$

$x' = x$ and $y' = -y$ or $-y' = y$

$y=\frac{1}{x^2}\rightarrow -y'=\frac{1}{(x')^2}$ or $y'=-\frac{1}{(x')^2}$

The equation of the transformed graph is $y=-\frac{1}{x^2}$.

3 $y=x^4$ is dilated by a factor of 2 from the x axis $\rightarrow y=2x^4$ is translated one unit in the negative x direction $\rightarrow y=2(x+1)^4$ is translated one unit in the negative y direction $\rightarrow y=2(x+1)^4-1$.

4 $y=\cos(x)$ is reflected in the x axis $\rightarrow y=-\cos(x)$ is dilated by a factor of $\frac{1}{2}$ parallel to the x axis $\rightarrow y=-\cos(2x)$ is translated $\frac{\pi}{2}$ units in the positive x direction $\rightarrow y=-\cos\left(2\left(x-\frac{\pi}{2}\right)\right)$ is translated 3 units in the negative y direction $\rightarrow y=-\cos\left(2\left(x-\frac{\pi}{2}\right)\right)-3$.

5 a $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right)+\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix}\right)+\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} -2 \\ -10 \end{bmatrix}+\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 0 \\ -10 \end{bmatrix}$

$(-2,5)\rightarrow(0,-10)$

b $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right)+\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 5 \end{bmatrix}\right)+\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 4 \\ 5 \end{bmatrix}+\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$(-2,5)\rightarrow(4,3)$

c $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} -2 \\ 5 \end{bmatrix}+\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} -5 \\ 6 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{5}{3} \\ 6 \end{bmatrix}$

$(-2,5)\rightarrow\left(\frac{5}{3},6\right)$

6 $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} -2 \\ 0 \end{bmatrix}\rightarrow\begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}+\begin{bmatrix} -2 \\ 0 \end{bmatrix}$

7 $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\rightarrow\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 3 \end{bmatrix}$

8 Dilated by a factor of $\frac{1}{2}$ parallel to the y axis or from the x axis, translated 3 units to the left or 3 units in the negative x direction and translated 2 units down or 2 units in the negative y direction.

$x'+3=x$ and $y'=\frac{y}{2}-2 \Rightarrow 2y'+4=y$

$$y=2(x-1)^2 \rightarrow 2y'+4=2(x'+3-1)^2$$
$$2y'+4=2(x'+2)^2$$
$$2y'=2(x'+2)^2-4$$
$$y'=(x'+2)^2-2$$

Transformed rule is $y=(x+2)^2-2$.

9 Reflection in the x axis, translation of $\frac{\pi}{4}$ to the right or $\frac{\pi}{4}$ in the positive x direction and translation of 2 units down or 2 units in the negative y direction.

$x'=x+\frac{\pi}{4}$ and $y'=-y-2$

$x=x'-\frac{\pi}{4}$ $\quad$ $y=-y'-2$

$$y=\cos(x) \rightarrow -y'-2=\cos\left(x'-\frac{\pi}{4}\right)$$
$$y'+2=-\cos\left(x'-\frac{\pi}{4}\right)$$
$$y'=-\cos\left(x'-\frac{\pi}{4}\right)-2$$

Transformed rule is $y=-\cos\left(x-\frac{\pi}{4}\right)-2$

10 Reflected in the y axis, dilated by a factor of 3 parallel to the y or from the x axis and translated 1 unit to the right or in the positive x direction.

$x'=-x+1$ and $y'=3y$

$x=1-x'$ $\quad$ $\frac{y'}{3}=y$

$$y=\sqrt{x+1}-2 \rightarrow \frac{y'}{3}=\sqrt{1-x'+1}-2$$
$$y'=3\sqrt{-x'}-6$$

Transformed rule is $y=3\sqrt{-x}-6$.

11 Dilated by a factor of 2 parallel to the x axis or from the y axis, reflected in the x axis and translated 4 units to the left or 4 units in the negative x direction.

$x'=2x-4$ and $y'=-y$

$x=\frac{1}{2}(x'+4)$ $\quad$ $-y'=y$

$$y=x^3 \rightarrow -y'=\left[\frac{1}{2}(x'+4)\right]^3$$
$$-y'=\frac{1}{8}(x'+4)^3$$
$$y'=-\frac{1}{8}(x'+4)^3$$

Transformed rule is $y=-\frac{1}{8}(x+4)^3$.

12 $x'=x-3$ and $y'=-2(y+1)$

$x=x'+3$ $\quad$ $-\frac{y'}{2}=y+1$

$$-\frac{y'}{2}-1=y$$
$$y=\frac{1}{x^2} \rightarrow -\frac{y'}{2}-1=\frac{1}{(x'+3)^2}$$
$$-\frac{y'}{2}=\frac{1}{(x'+3)^2}+1$$
$$y'=-\frac{2}{(x'+3)^2}-2$$

Therefore $y=-\frac{2}{(x+3)^2}-2$

13 $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

Check:

$x'=2x-4$ or $x=\frac{1}{2}(x'+4)$ and $y'=\frac{1}{2}y-1$ or $y=2(y'+1)$

$$y=x^2 \rightarrow 2(y'+1)=\left\{\frac{1}{2}(x'+4)\right\}^2$$
$$2(y'+1)=\left(\frac{(x'+4)}{2}\right)^2$$
$$y'+1=\frac{1}{2}\left(\frac{(x'+4)}{2}\right)^2$$
$$y'=\frac{1}{2}\left(\frac{(x'+4)}{2}\right)^2-1 \text{ as required}$$

14 $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} \frac{1}{3} \\ 4 \end{bmatrix}$$

Check:

$x'=\frac{1}{3}x+\frac{1}{3}$ so $3x'-1=x$ and $y'=-2y+4$ or $-\frac{1}{2}(y'-4)=y$

$$y=\frac{1}{x} \rightarrow -\frac{1}{2}(y'-4)=\frac{1}{3x'-1}$$
$$y'-4=-\frac{2}{3x'-1}$$
$$y'=-\frac{2}{3x'-1}+4 \text{ as required}$$

15 $y=\frac{2}{x+1}-3 \rightarrow y=\frac{1}{x}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} b \\ c \end{bmatrix}$$

$x'=x+b \rightarrow x=x'-b$

$y'=ay+c \rightarrow y=\frac{y'-c}{a}$

Substituting in the new points:

$$\frac{y'-c}{a}=\frac{2}{x'-b+1}-3=\frac{1}{x}$$
$$y'-c=\frac{2a}{x'-b+1}-3a=\frac{1}{x}$$
$$y'=\frac{2a}{x'-b+1}-3a+c=\frac{1}{x}$$

Now equate the components:

$2a = 1$

$a = \dfrac{1}{2}$

$-b + 1 = 0$

$b = 1$

$-3a + c = 0$

$c = 3a$

$= \dfrac{3}{2}$

16 $y = -(3x-1)^2 + 2 \rightarrow y = x^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right)$$

$x' = a(x+c)$ and $y' = b(y+d)$

$x = \dfrac{x'}{a} - c \qquad y = \dfrac{y'}{b} - d$

Substituting in the new points:

$$\frac{y'}{b} - d = -\left(3\left(\frac{x'}{a} - c\right) - 1\right)^2 + 2 = x^2$$

$$\frac{y'}{b} = -\left(\frac{3x'}{a} - 3c - 1\right)^2 + 2 + d = x^2$$

$$y' = -b\left(\frac{3x'}{a} - 3c - 1\right)^2 + b(2+d) = x^2$$

Now equate the components:

$\dfrac{3}{a} = 1$

$a = 3$

$-3c - 1 = 0$

$c = -\dfrac{1}{3}$

$-b = 1$

$b = -1$

$b(2+d) = 0$

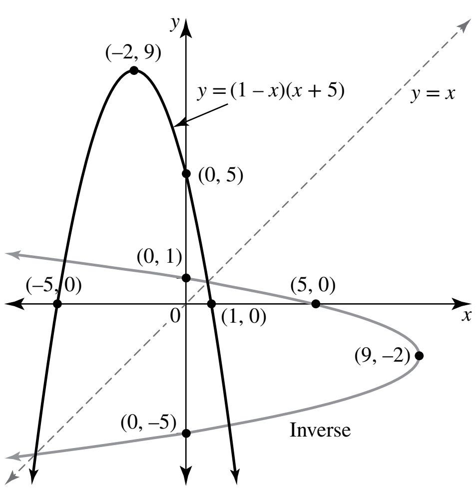
$-1(2+d) = 0$

$2 + d = 0$

$d = -2$

Exercise 3.5 — Inverse graphs and relations

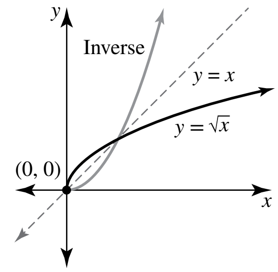
1 a $y = (1-x)(x+5)$ and inverse is $x = (1-y)(y+5)$



b $y = (1-x)(x+5)$ is a many-to-one mapping and is a function.
Inverse is a one-to-many mapping and is a relation.

c Function: Dom = R and Ran $= (-\infty, 9]$
Inverse: Dom $= (-\infty, 9]$ and Ran = R

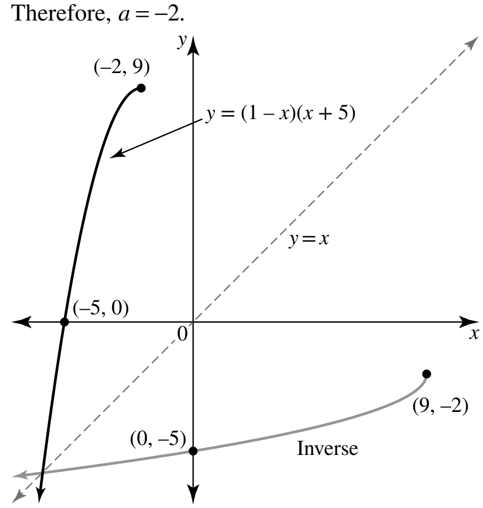
2 a & b



c $y = \sqrt{x}$ is a one-to-one mapping so is a function.
$y = x^2,\ x \geq 0$ is a one-to-one mapping so is a function.

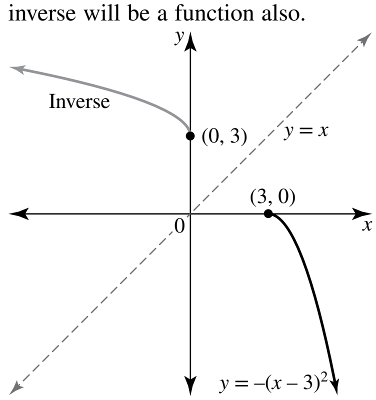
3 a x-intercepts occur at $x = 1$, $x = -5$. The TP is halfway between them, so the x-value of the TP is $x = -2$. To obtain the maximum domain, we restrict the parabola about the TP. Therefore, $a = -2$.

b



c For $y = (1-x)(x+5)$, Dom $= (-\infty, -2]$ and Ran $= (-\infty, 9]$
For $x = (1-y)(y+5)$, Dom $= (-\infty, 9]$ and Ran $= (-\infty, -2]$

4 If the domain for $y = -(x-3)^2$ is restricted to $[3, \infty)$ then the inverse will be a function also.



5 a $f(x) = \cos(x)$ is a many-to-one function.

b $g(x) = 1 - x^3$ is a one-to-one function.

c $h(x) = \sqrt{4 - x^2}$ is a many-to-one function.

d $k(x) = 2 + \dfrac{1}{x-3}$ is a one-to-one function.

6 **A** $y = x^2 - 1$ is a many-to-one function, so inverse in a one-to-many relation.

B $x^2 + y^2 = 1$ is a many-to-many relation, so inverse in a many-to-many relation.

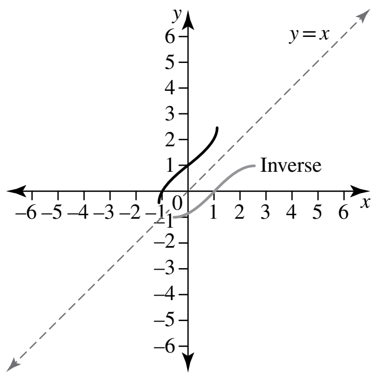
C $y = \dfrac{1}{x-1}$ is a one-to-one function, so inverse in a one-to-one function.

D $y = \sqrt{1 - x^2}$ is a many-to-one function, so inverse in a one-to-many relation.

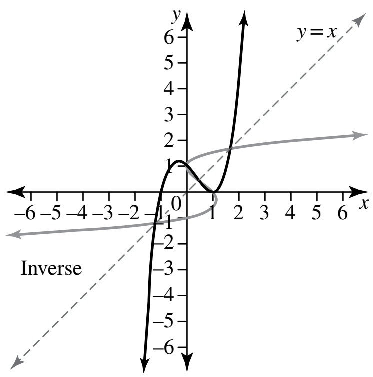
E $y = 10$ is a many-to-one function, so inverse in a one-to-many relation.

Therefore C is the answer.

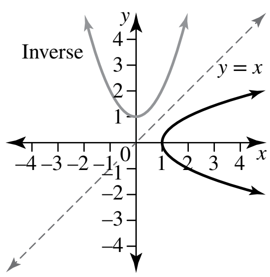
7 **a**



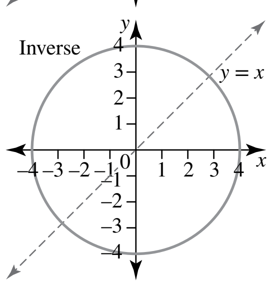
b



c



d



8 **A** $y = x^2$ a many-to-one function and $y = \pm\sqrt{x}$ a one-to-many relation.

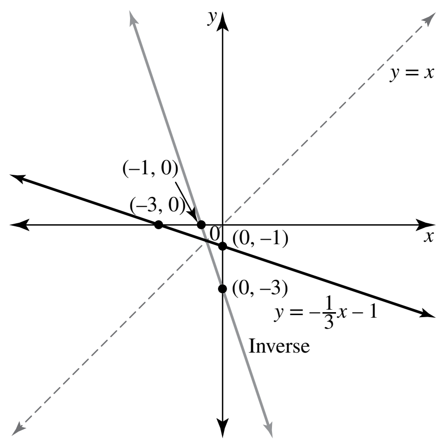
B $y = x^2, x \in (-\infty, 0]$ a one-to-one function and $y = \sqrt{x}$ a one-to-one function but not a pair.

C Two one-to-one functions.

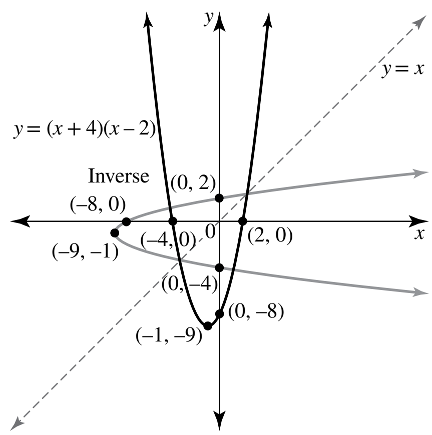
D $y = x^2 + 1,\ x \in [0, \infty)$ a one-to-one function and $y = -x^2 - 1,\ x \in [0, \infty)$ a one-to-one function but not a pair.

9 $x = (y-2)^2$ has a turning point at (0,2) and cuts the x axis at (0,4) so inverse will have to have a turning point at (2,0) and cut the y axis at (4,0), so Option A is the answer.

10 **a & b**



11 **a & b**



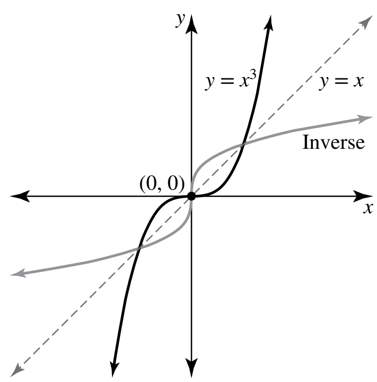
c Parabola is a many-to-one mapping and the inverse is a one-to-many relation.

d The inverse is not a function as functions must be one-to-one or many-to-one mappings.

e Function: Dom = R and Ran $= [-9, \infty)$.
Inverse: Dom $= [-9, \infty)$ and Ran = R.

f Largest domain for an inverse function is either $x \in (-\infty, -1]$ or $x \in [-1, \infty)$.

12 **a**

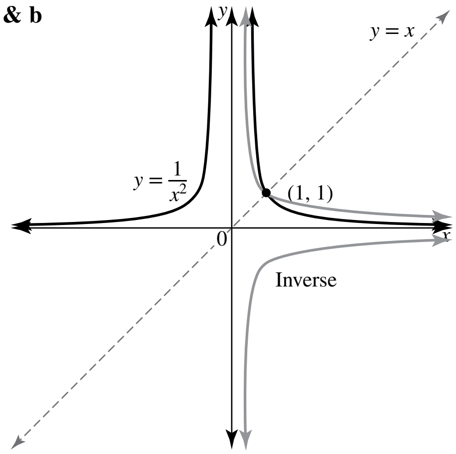


b Graph: One-to-one function.
Inverse: One-to-one function.

c The inverse is a function because it is a one-to-one mapping.

d Graph: Dom = R and Ran = R.
Inverse: Dom = R and Ran = R.

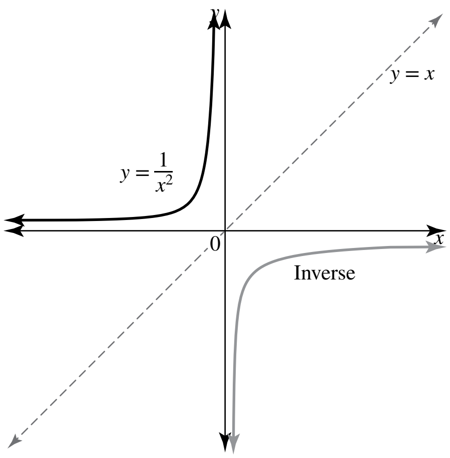
13 a & b



c Graph: many-to-one function.
Inverse: one-to-many relation.

d Restricted domain to produce an inverse function is $x \in (-\infty, 0)$.

e



Graph: Dom $= (-\infty, 0)$ and Ran $= (0, \infty)$.
Inverse: Dom $= (0, \infty)$ and Ran $= (-\infty, 0)$.

14 $y = 2x^2 - 12x + 13$ has a turning point where:

$$y = 2\left(x^2 - 6x + \frac{13}{2}\right)$$

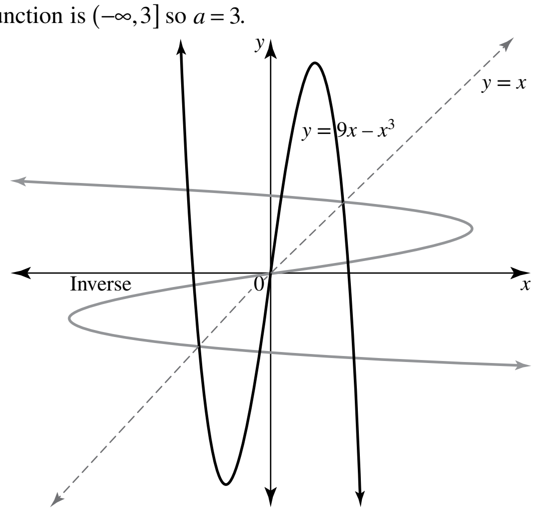
$$y = 2\left(x^2 - 6x + (3)^2 - (3)^2 + \frac{13}{2}\right)$$

$$y = 2(x-3)^2 - 18 + 13$$

$$y = 2(x-3)^2 - 5$$

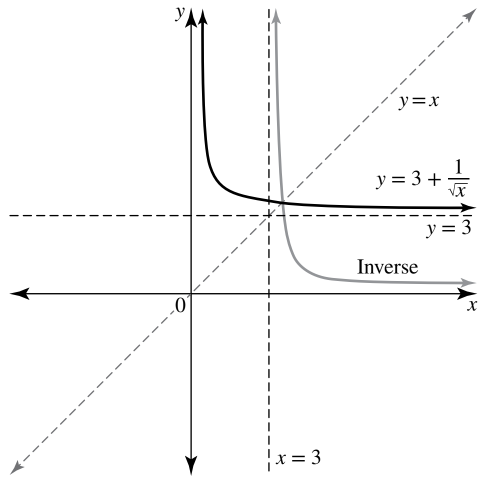
TP = $(3, 5)$ so largest possible domain for the inverse to be a function is $(-\infty, 3]$ so $a = 3$.

15 a



b (2.828, 2.828), (0, 0), (–2.828, –2.828)

16 a



b (3.532, 3.532)

Exercise 3.6 — Inverse functions

1 $y = x^3$ is a one-to-one function with Dom = R and Ran = R
Inverse: swap x and y

$$x = y^3$$

$$y = \sqrt[3]{x}$$

This is a one-to-one with Dom = R and Ran = R

2 $y = \frac{1}{x^2}$ is a many-to-one function with Dom = $R\backslash\{0\}$ and Ran $= (0, \infty)$
Inverse: swap x and y

$$x = \frac{1}{y^2}$$

$$\frac{1}{x} = y^2$$

$$\pm\frac{1}{\sqrt{x}} = y$$

This is a one-to-many relation and thus is not a function.
Dom $= (0, \infty)$ and Ran = $R\backslash\{0\}$.

3 $f: (-\infty, 2) \to R, f(x) = -\frac{1}{(x-2)^2}$ is a one-to-one function where Ran $= (-\infty, 0)$
Inverse: swap x and y

$$x = -\frac{1}{(y-2)^2}$$

$$-x = \frac{1}{(y-2)^2}$$

$$-\frac{1}{x} = (y-2)^2$$

$$\pm\sqrt{-\frac{1}{x}} = y - 2$$

$$-\sqrt{-\frac{1}{x}} = y - 2 \text{ since } x \in (-\infty, 0)$$

$$2 - \sqrt{-\frac{1}{x}} = y$$

This is a one-to-one function x where Ran $= (-\infty, 2)$.

$$f^{-1}: (-\infty, 0) \to R, f^{-1}(x) = 2 - \sqrt{-\frac{1}{x}}$$

4 $f: [3, \infty) \to R, f(x) = \sqrt{x-3}$ is a one-to-one function where Ran $= [0, \infty)$.

Inverse: swap x and y

$$x=\sqrt{y-3}$$
$$x^2=y-3$$
$$y=x^2+3 \text{ where } x\in[0,\infty) \text{ and } y\in[3,\infty)$$
$$f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=x^2+3$$

5 a $f(x)=(x+1)^2$ where Dom $=R$ and Ran $=[0,\infty)$. This is a many-to-one function.

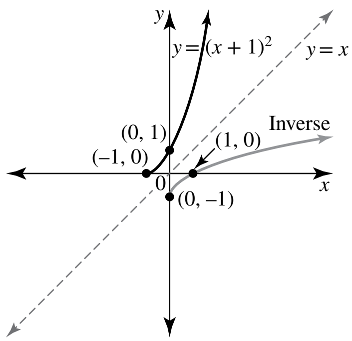
Inverse: swap x and y

$$x=(y+1)^2$$
$$\pm\sqrt{x}=y+1$$
$$\pm\sqrt{x}-1=y \text{ where Dom}=[0,\infty) \text{ and Ran}=R.$$

The inverse is not a function as $f(x)$ is not a one-to-one function.

b If the domain for the function is restricted to $[-1,\infty)$ then it is a one-to-one function so the inverse is also a function. Thus $b=-1$.

c



d $f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=\sqrt{x}-1$

e The graphs do not intersect.

6 $f(x)=2\sqrt{x+2}$ is a one-to-one mapping with Dom $=[-2,\infty)$ and Ran $=[0,\infty)$.

Inverse: swap x and y

$$x=2\sqrt{y+2}$$
$$\frac{x}{2}=\sqrt{y+2}$$
$$\frac{x^2}{4}=y+2$$
$$y=\frac{x^2}{4}-2$$

This is a one-to-one mapping with Dom $=[0,\infty)$ and Ran $=[-2,\infty)$.

The two functions intersect where

$$2\sqrt{x+2}=x$$
$$4(x+2)=x^2$$
$$4x+8=x^2$$
$$x^2-4x-8=0$$
$$x=\frac{4\pm\sqrt{(-4)^2-4\times-8\times1}}{2}$$
$$=\frac{4\pm\sqrt{48}}{2}$$
$$=\frac{4\pm4\sqrt{3}}{2}$$
$$=2+2\sqrt{3}$$

When $x=2+2\sqrt{3},\ y=2+2\sqrt{3}\ \therefore POI=\left(2+2\sqrt{3},\ 2+2\sqrt{3}\right)$

7 a $y=\frac{1}{3}(x-3)$ is one-to-one function where Dom $=R$ and Ran $=R$.

Inverse: swap x and y

$$x=\frac{1}{3}(y-3)$$
$$3x+3=y$$
$$y=3(x+1)$$

which is a one-to-one function where Dom $=R$ and Ran $=R$.

b $y=(x-5)^2$ is a many-to-one function where Dom $=R$ and Ran $=[0,\infty)$.

Inverse: swap x and y

$$x=(y-5)^2$$
$$\pm\sqrt{x}=y-5$$
$$y=5\pm\sqrt{x}$$

which is a one-to-many relation where Dom $=[0,\infty)$ and Ran $=R$.

c $f:[-4,0]\to R,\ f(x)=\sqrt{16-x^2}$ is a one-to-one function where Dom $=[-4,0]$ and Ran $=[0,4]$

Inverse: swap x and y

$$x=\sqrt{16-y^2}\quad x\in[0,4]$$
$$x^2=16-y^2$$
$$y^2=16-x^2$$
$$y=\pm\sqrt{16-x^2}$$
$$y=-\sqrt{16-x^2}\quad \text{as } y\in[-4,0]$$
$$\therefore f^{-1}(x)=-\sqrt{16-x^2}$$

which is a one-to-one function. Dom $=[0,4]$ and Ran $=[-4,0]$.

d $y=(x-1)^3$ is a one-to-one function where Dom $=R$ and Ran $=R$.

Inverse: swap x and y

$$x=(y-1)^3$$
$$\sqrt[3]{x}=y-1$$
$$y=\sqrt[3]{x}+1$$

which is a one-to-one function where Dom $=R$ and Ran $=R$.

e $y=\sqrt{x}$ is a one-to-one function. Dom $=[0,\infty)$ and Ran $=[0,\infty)$.

Inverse: swap x and y

$$x=\sqrt{y}$$
$$x^2=y \text{ where } x\in[0,\infty)$$

which is a one-to-one function. Dom $=[0,\infty)$ and Ran $=[0,\infty)$.

f $y=\frac{1}{(x-1)^2}+2$ is many-to-one function. Dom $=R\setminus\{1\}$ and Ran $=(2,\infty)$.

Inverse: swap x and y

$$x=\frac{1}{(y-1)^2}+2$$
$$x-2=\frac{1}{(y-1)^2}$$
$$\frac{1}{x-2}=(y-1)^2$$
$$\pm\frac{1}{\sqrt{x-2}}=y-1$$
$$y=1\pm\frac{1}{\sqrt{x-2}}$$

Which is a one-to-many relation with Dom $=(2,\infty)$ and Ran $=R\setminus\{1\}$.

8 $f(x)=\dfrac{1}{x+2},\ x\neq -2$

Inverse: swap x and y

$$x=\frac{1}{y+2}$$
$$\frac{1}{x}=y+2$$
$$y=\frac{1}{x}-2$$

$f^{-1}(x)=\dfrac{1}{x}-2,\ x\neq 0$

a $f\left(f^{1}(x)\right)=\dfrac{1}{\dfrac{1}{x}-2+2}=\dfrac{1}{\dfrac{1}{x}}=x$ as required.

b $f^{-1}\left(f(x)\right)=\dfrac{1}{\dfrac{1}{x+2}}-2=x+2-2=x$ as required.

9 $k(x)=x^3-1$

Inverse: Let $y=k(x)$, swap x and y

$$x=y^3-1$$
$$x+1=y^3$$
$$y=\sqrt[3]{x+1}$$
$$k^{-1}(x)=\sqrt[3]{x+1}$$

a $k\left(k^{1}(x)\right)=\left(\sqrt[3]{x+1}\right)^3-1$
$=x+1-1$
$=x$ as required.

b $k^{-1}\left(k(x)\right)=\sqrt[3]{x^3-1+1}$
$=\sqrt[3]{x^3}$
$=x$ as required.

10 **a** $f:R\to R,\ f(x)=x^4$ is a many-to-one function. The inverse will be a one-to-many relation.

b $f:R\to R,\ f(x)=2x^2-7x+3$ is a many-to-one function. The inverse will be a one-to-many relation.

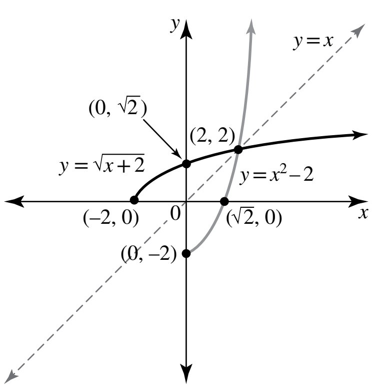
c $f:[-3,3]\to R,\ f(x)=\sqrt{9-x^2}$ is a many-to-one function. The inverse will be a one-to-many relation.

d $f:[-2,\infty)\to R,\ f(x)=\sqrt{x+2}$ is a one-to-one function with Dom $=[-2,\infty)$ and Ran $=[0,\infty)$.

Inverse: swap x and y

$$x=\sqrt{y+2},\ x\in[0,\infty)$$
$$x^2=y+2$$
$$y=x^2-2 \text{ where } y\in[-2,\infty)$$

Thus $f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=x^2-2$ where $y\in[-2,\infty)$.



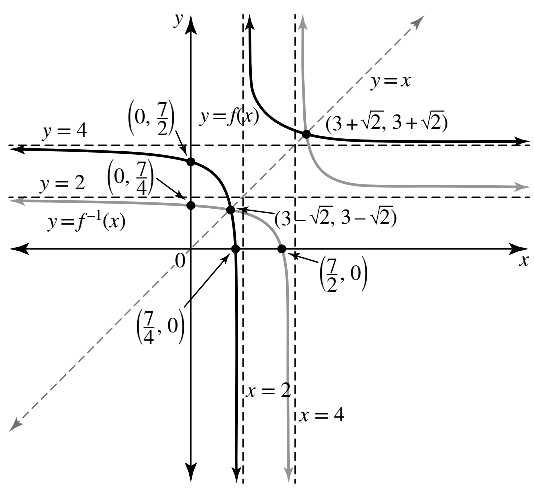
11 $f(x)=\dfrac{4x-7}{x-2}$
$=\dfrac{4(x-2)+1}{x-2}$
$=\dfrac{4(x-2)}{x-2}+\dfrac{1}{x-2}$
$=4+\dfrac{1}{x-2}$

$f(x)=\dfrac{1}{x-2}+4$ where Dom $=R\backslash\{2\}$ and Ran $=R\backslash\{4\}$

Inverse: swap x and y

$$x=\frac{1}{y-2}+4$$
$$x-4=\frac{1}{y-2}$$
$$\frac{1}{x-4}=y-2$$
$$y=\frac{1}{x-4}+2$$

$f^{-1}(x)=\dfrac{1}{x-4}+2$ where Dom $=R\backslash\{4\}$ and Ran $=R\backslash\{2\}$



12 **a** $f(x)=(x+2)^2$. If the maximal domain is $(-\infty,-2]$ then the inverse will be a function.

Inverse: swap x and y

$$x=(y+2)^2 \quad x\in[0,\infty)$$
$$-\sqrt{x}=y+2$$
$$y=-\sqrt{x}-2 \text{ where } y\in(-\infty,-2]$$

Thus $f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=-\sqrt{x}-2$ where Ran $=(-\infty,-2]$

b $f(x)=-\sqrt{25-x^2}$. If the maximal domain is $(0,5]$ then the inverse will be a function.

Inverse: swap x and y

$$x=-\sqrt{25-y^2} \quad x\in(-5,0]$$
$$x^2=25-y^2$$
$$y=\sqrt{25-x^2} \text{ where } y\in(0,5]$$

Thus $f^{-1}:(-5,0]\to R,\ f^{-1}(x)=\sqrt{25-x^2}$ where Ran $=(0,5]$

13 **a** $f(x)=x^2-10x+25=(x-5)^2$ which is a many-to-one function. For the inverse to be a function the domain must be restricted to $[5,\infty)$ so that $a=5$.

b Inverse: swap x and y

$x=(y-5)^2$ where $x\in[0,\infty)$

$\sqrt{x}=y-5$

$y=\sqrt{x}+5$ where $y\in[5,\infty)$

$f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=\sqrt{x}+5$ where Ran $=[5,\infty)$.

14 $f:[-2,4)\to R,\ f(x)=1-\dfrac{x}{3}$

a Dom $=[-2,4)$ and Ran $=\left(-\dfrac{1}{3},\dfrac{5}{3}\right]$.

b Inverse: swap x and y

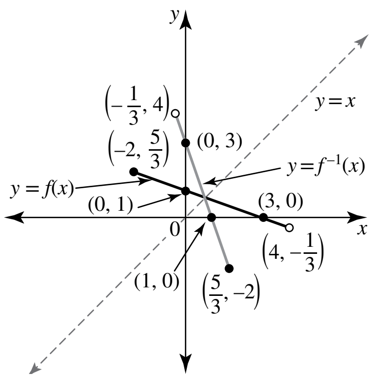
$x=1-\dfrac{y}{3}$

$\dfrac{y}{3}=1-x$

$y=3(1-x)$

$f^{-1}:\left(-\dfrac{1}{3},\dfrac{5}{3}\right]\to R,\ f^{-1}(x)=3(1-x)$ where Ran $=[-2,4)$

c



d $x=1-\dfrac{x}{3}$

$3x=3-x$

$4x=3$

$x=\dfrac{3}{4}$

$\therefore POI=\left(\dfrac{3}{4},\dfrac{3}{4}\right)$

15 **a** $f:D\to R,\ f(x)=\sqrt{1-3x}$

The maximal domain is $D=\left(-\infty,\dfrac{1}{3}\right]$.

b Inverse: swap x and y

$x=\sqrt{1-3y}$ where $x\in[0,\infty)$

$x^2=1-3y$

$3y=1-x^2$

$y=\dfrac{1}{3}(1-x^2)$ where Ran $=\left(-\infty,\dfrac{1}{3}\right]$

$f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=\dfrac{1}{3}(1-x^2)$ where Ran $=\left(-\infty,\dfrac{1}{3}\right]$.

c $x=\sqrt{1-3x}$

$x^2=1-3x$

$0=x^2+3x-1$

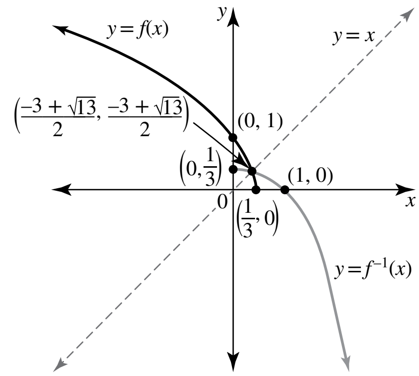
$x=\dfrac{-3\pm\sqrt{(3)^2-4\times1\times-1}}{2}$

$=\dfrac{-3\pm\sqrt{13}}{2}$

$=\dfrac{-3+\sqrt{13}}{2}$, dom $f=[0,\infty)$

$\therefore POI=\left(\dfrac{-3+\sqrt{13}}{2},\dfrac{-3+\sqrt{13}}{2}\right)$

d



16 **a** $f:(-\infty,a]\to R,\ f(x)=x^2-2x-1$

To find TP: $y=x^2-2x-1$

$y=x^2-2x+1-2$

$y=(x-1)^2-2$

TP is at $(1,-2)$ so largest possible value of a is 1.

Thus $f:(-\infty,1]\to R,\ f(x)=x^2-2x-1$

b Inverse: swap x and y

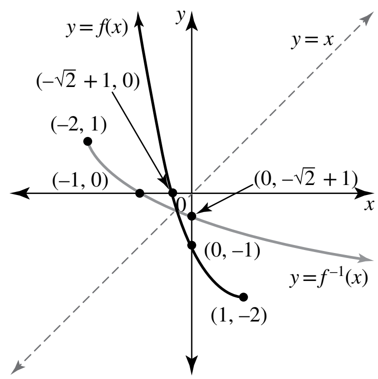
$x=(y-1)^2-2$

$x+2=(y-1)^2$

$\pm\sqrt{x+2}=y-1$

$y=-\sqrt{x+2}+1$ as dom $f=(-\infty,1]$

$f^{-1}:[-2,\infty)\to R,\ f^{-1}(x)=-\sqrt{x+2}+1$



c $x=x^2-2x-1$

$0=x^2-3x-1$

$x=\dfrac{3\pm\sqrt{(-3)^2-4\times1\times-1}}{2}$

$=\dfrac{3\pm\sqrt{13}}{2}$

$=\dfrac{3-\sqrt{13}}{2}$, dom $f=(-\infty,1]$

Therefore $POI=\left(\dfrac{3-\sqrt{13}}{2},\dfrac{3-\sqrt{13}}{2}\right)$

17 $f:[1,\infty)\to R,\ f(x)=\sqrt{x-1}$ where Ran $=[0,\infty)$

a Inverse: swap x and y

$x=\sqrt{y-1}\quad x\ge0,\ \ y\ge1$

$y=x^2+1$

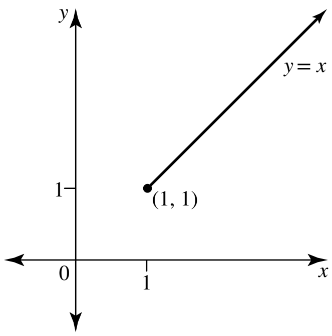
$f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=x^2+1$ where Ran $=[1,\infty)$

b $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x-1}\right)$

$= \left(\sqrt{x-1}\right)^2 + 1$

$= x - 1 + 1$

$= x$ where Dom $= [1, \infty)$



c $f^{-1}\left(-f\left(\frac{x+2}{3}\right)\right) = f^{-1}\left(-\sqrt{\frac{x+2}{3} - 1}\right)$

$= f^{-1}\left(-\sqrt{\frac{x+2-3}{3}}\right)$

$= f^{-1}\left(-\sqrt{\frac{x-1}{3}}\right)$

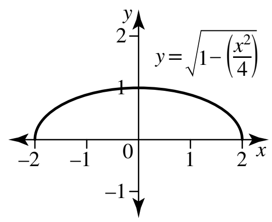
$= \left(-\sqrt{\frac{x-1}{3}}\right)^2 + 1$

$= \frac{x-1}{3} + 1$

$= \frac{x-1+3}{3}$

$= \frac{x+2}{3}$

18



Two inverse functions are:

$x = \sqrt{1 - \frac{y^2}{4}}$

$x^2 = 1 - \frac{y^2}{4}$

$x^2 + \frac{y^2}{4} = 1$

$\frac{y^2}{4} = 1 - x^2$

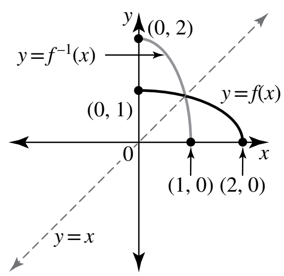
$\frac{y}{2} = \pm\sqrt{1 - x^2}$

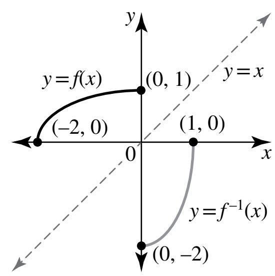
$y = \pm 2\sqrt{1 - x^2}$

Thus, we have

$f^{-1}: [0,1] \to R,\ f^{-1}(x) = 2\sqrt{1 - x^2}$ where Ran $= [0, 2]$ or

$f^{-1}: [0,1] \to R,\ f^{-1}(x) = -2\sqrt{1 - x^2}$ where Ran $= [-2, 0]$





Topic 4 — Logarithmic functions

Exercise 4.2 — Logarithm laws and equations

1 a $\log_7(49)+\log_2(32)-\log_5(125)$
$=\log_7(7)^2+\log_2(2)^5-\log_5(5)^3$
$=2\log_7(7)+5\log_2(2)-3\log_5(5)$
$=2+5-3$
$=4$

b $5\log_{11}(6)-5\log_{11}(66)$
$=5\left(\log_{11}(6)-\log_{11}(66)\right)$
$=5\left(\log_{11}\left(\frac{6}{66}\right)\right)$
$=5\left(\log_{11}\left(\frac{1}{11}\right)\right)$
$=5\log_{11}(11)^{-1}$
$=-5$

c $\frac{\log_4(25)}{\log_4(625)}$
$=\frac{\log_4(5)^2}{\log_4(5)^4}$
$=\frac{2\log_4(5)}{4\log_4(5)}$
$=\frac{1}{2}$

d $\log_2\left(\sqrt[7]{\frac{1}{128}}\right)$
$=\log_2\left(\left(2^{-7}\right)^{\frac{1}{7}}\right)$
$=\log_2(2)^{-1}$
$=-1$

2 a $7\log_4(x)-9\log_4(x)+2\log_4(x)=0$

b $\log_7(2x-1)+\log_7(2x-1)^2$
$=\log_7(2x-1)+2\log_7(2x-1)$
$=3\log_7(2x-1)$

c $\log_{10}(x-1)^3-2\log_{10}(x-1)$
$=3\log_{10}(x-1)-2\log_{10}(x-1)$
$=\log_{10}(x-1)$

3 a $\log_5(125)=\log_5(5)^3$
$=3\log_5(5)$
$=3$

b $\log_4(x-1)+2=\log_4(x+4)$
$\log_4(x-1)+2\log_4 4=\log_4(x+4)$
$\log_4(x-1)+\log_4 4^2=\log_4(x+4)$
$\log_4(16(x-1))=\log_4(x+4)$
$16(x-1)=x+4$
$16x-16=x+4$
$15x=20$
$x=\frac{4}{3}$

c $3\left(\log_2(x)\right)^2-2=5\log_2(x)$
$3\left(\log_2(x)\right)^2-5\log_2(x)-2=0$
$\left(3\log_2(x)+1\right)\left(\log_2(x)-2\right)=0$
$3\log_2(x)+1=0$ or $\log_2(x)-2=0$
$\log_2(x)=-\frac{1}{3}$ $\qquad$ $\log_2(x)=2$
$x=2^{-\frac{1}{3}}$ $\qquad$ $2^2=x$
$x=4$

d $\log_5(4x)+\log_5(x-3)=\log_5(7)$
$\log_5(4x(x-3))=\log_5(7)$
$4x(x-3)=7$
$4x^2-12x-7=0$
$(2x-7)(2x+1)=0$
$x=\frac{7}{2},-\frac{1}{2}$

$x=-\frac{1}{2}$ isn't a valid solution as $x>3$

Therefore $x=\frac{7}{2}$

4 a $\log_3(x)=5$
$3^5=x$
$x=243$

b $\log_3(x-2)-\log_3(5-x)=2$
$\log_3\left(\frac{x-2}{5-x}\right)=2$
$3^2=\frac{x-2}{5-x}$
$9=\frac{x-2}{5-x}$
$9(5-x)=x-2$
$45-9x=x-2$
$47=10x$
$x=\frac{47}{10}$

5 a i $\log_7(12)=\frac{\log_e(12)}{\log_e(7)}=1.2770$

ii $\log_3\left(\frac{1}{4}\right)=\frac{\log_e\left(\frac{1}{4}\right)}{\log_e(3)}=-1.2619$

b $z=\log_3(x)$
$3^z=x$

i $2x=2\times 3^z$

ii $\log_x(27)=\frac{\log_3(27)}{\log_3(x)}$
$=\frac{\log_3(3)^3}{\log_3(x)}$
$=\frac{3\log_3(3)}{\log_3(x)}$
$=\frac{3}{z}$

6 a $\log_5(9)=\dfrac{\log_{10}(9)}{\log_{10}(5)}$

b $\log_{\frac{1}{2}}(12)=\dfrac{\log_{10}(12)}{\log_{10}\left(\frac{1}{2}\right)}$

7 a $6^3=216$
$\log_6(216)=3$

b $2^8=256$
$\log_2(256)=8$

c $3^4=81$
$\log_3(81)=4$

d $10^{-4}=0.0001$
$\log_{10}(0.0001)=-4$

e $5^{-3}=0.008$
$\log_5(0.008)=-3$

f $7^1=7$
$\log_7(7)=1$

8 a $\log_3(81)=x$
$3^x=81$
$3^x=3^4$
$x=4$

b $\log_6\left(\frac{1}{216}\right)=x$
$6^x=\frac{1}{216}$
$6^x=6^{-3}$
$x=-3$

c $\log_x(121)=2$
$x^2=121$
$x^2=11^2$
$x=11$

d $\log_2(-x)=7$
$2^7=-x$
$128=-x$
$x=-128$

9 a $\log_2(256)+\log_2(64)-\log_2(128)$
$=\log_2\left(\frac{256\times 64}{128}\right)$
$=\log_2\left(2\times 2^6\right)$
$=\log_2(2)^7$
$=7\log_2(2)$
$=7$

b $5\log_7(49)-5\log_7(343)$
$=5\left(\log_7(49)-\log_7(343)\right)$
$=5\log_7\left(\frac{49}{343}\right)$
$=5\log_7\left(\frac{1}{7}\right)$
$=5\log_7(7)^{-1}$
$=-5\log_7(7)$
$=-5$

c $\log_4\left(\sqrt[6]{\frac{1}{64}}\right)$
$=\log_4\left((2)^{-6}\right)^{\frac{1}{6}}$
$=\log_4(2)^{-1}$
$=\log_4(4)^{-\frac{1}{2}}$
$=-\frac{1}{2}\log_4(4)$
$=-\frac{1}{2}$

d $\log_4\left(\frac{16}{256}\right)$
$=\log_4\left(\frac{1}{16}\right)$
$=\log_4(4)^{-2}$
$=-2\log_4(4)$
$=-2$

e $\dfrac{\log_5(32)}{3\log_5(16)}$
$=\dfrac{\log_2(2)^5}{3\log_2(2)^4}$
$=\dfrac{5\log_2(2)}{12\log_2(2)}$
$=\dfrac{5}{12}$

f $\dfrac{6\log_2\left(\sqrt[3]{x}\right)}{\log_2\left(x^5\right)}$
$=\dfrac{6\log_2(x)^{\frac{1}{3}}}{\log_2(x)^5}$
$=\dfrac{2\log_2(x)}{5\log_2(x)}$
$=\dfrac{2}{5}$

10 a $\log_3(x-4)+\log_3(x-4)^2$
$=\log_3(x-4)+2\log_3(x-4)$
$=3\log_3(x-4)$

b $\log_7(2x+3)^3-2\log_7(2x+3)$
$=3\log_7(2x+3)-2\log_7(2x+3)$
$=\log_7(2x+3)$

c $\log_5(x)^2+\log_5(x)^3-5\log_5(x)$
$=2\log_5(x)+3\log_5(x)-5\log_5(x)$
$=0$

d $\log_4(5x+1)+\log_4(5x+1)^3-\log_4(5x+1)^2$
$=\log_4(5x+1)+3\log_4(5x+1)-2\log_4(5x+1)$
$=2\log_4(5x+1)$

11 a $\log_3(7)=1.7712$

b $\log_2\left(\frac{1}{121}\right)=-6.9189$

12 If $n-\log_5(x)$ then $5^n=x$

a $5x=5\times 5^n=5^{n+1}$

b $\log_5\left(5x^2\right)=\log_5\left(5\times(5^n)^2\right)$
$=\log_5\left(5\times 5^{2n}\right)$
$=\log_5(5)^{2n+1}$
$=(2n+1)\log_5(5)$
$=2n+1$

c $\log_x(625)$
$= \frac{\log_5(625)}{\log_5(x)}$
$= \frac{\log_5(5^4)}{n}$
$= \frac{4}{n}$

13 a $\log_e(2x-1) = -3$
$e^{-3} = 2x-1$
$e^{-3}+1 = 2x$
$x = \frac{1}{2}(e^{-3}+1)$

b $\log_e\left(\frac{1}{x}\right) = 3$
$\log_e(x)^{-1} = 3$
$-\log_e(x) = 3$
$\log_e(x) = -3$
$x = e^{-3}$

c $\log_3(4x-1) = 3$
$3^3 = 4x-1$
$27+1 = 4x$
$28 = 4x$
$x = 7$

d $\log_{10}(x) - \log_{10}(3) = \log_{10}(5)$
$\log_{10}\left(\frac{x}{3}\right) = \log_{10}(5)$
$\frac{x}{3} = 5$
$x = 15$

e $3\log_{10}(x) + 2 = 5\log_{10}(x)$
$2 = 5\log_{10}(x) - 3\log_{10}(x)$
$2 = 2\log_{10}(x)$
$1 = \log_{10}(x)$
$x = 10$

f $\log_{10}(x^2) - \log_{10}(x+2) = \log_{10}(x+3)$
$\log_{10}\left(\frac{x^2}{x+2}\right) = \log_{10}(x+3)$
$\frac{x^2}{x+2} = x+3$
$x^2 = (x+3)(x+2)$
$x^2 = x^2+5x+6$
$0 = 5x+6$
$x = -\frac{6}{5}$

g $2\log_5(x) - \log_5(2x-3) = \log_5 x - 2$
$\log_5(x)^2 - \log_5(2x-3) = \log_5(x-2)$
$\log_5\left(\frac{x^2}{2x-3}\right) = \log_5(x-2)$
$\frac{x^2}{2x-3} = x-2$
$x^2 = (x-2)(2x-3)$
$x^2 = 2x^2 - 7x + 6$
$0 = x^2 - 7x + 6$
$0 = (x-1)(x-6)$
$x = 1 \quad x = 6$

h $\log_{10}(2x) - \log_{10}(x-1) = 1$
$\log_{10}\left(\frac{2x}{x-1}\right) = 1$
$10 = \frac{2x}{x-1}$
$10(x-1) = 2x$
$10x - 10 = 2x$
$10x - 2x = 10$
$8x = 10$
$x = \frac{5}{4}$

i $\log_3(x) + 2\log_3(4) - \log_3(2) = \log_3(10)$
$\log_3(x) + \log_3(4)^2 - \log_3(2) = \log_3(10)$
$\log_3(16x) - \log_3(2) = \log_3(10)$
$\log_3\left(\frac{16x}{2}\right) = \log_3(10)$
$8x = 10$
$x = \frac{5}{4}$

j $(\log_{10}(x))(\log_{10}(x)^2) - 5\log_{10}(x) + 3 = 0$
$(\log_{10}(x))(2\log_{10}(x)) - 5\log_{10}(x) + 3 = 0$
$2(\log_{10}(x))^2 - 5\log_{10}(x) + 3 = 0$
Let $a = \log_{10}(x)$
$2a^2 - 5a + 3 = 0$
$(2a-3)(a-1) = 0$
Substitute back for $a = \log_{10}(x)$
$(2\log_{10}(x) - 3)(\log_{10}(x) - 1) = 0$
$2\log_{10}(x) - 3 = 0$ or $\log_{10}(x) - 1 = 0$
$2\log_{10}(x) = 3 \qquad \log_{10}(x) = 1$
$\log_{10}(x) = \frac{3}{2} \qquad 10^1 = x$
$x = 10^{\frac{3}{2}} \qquad x = 10$

k $(\log_3(x))^2 = \log_3(x) + 2$
$(\log_3(x))^2 - \log_3(x) - 2 = 0$
$(\log_3(x) - 2)(\log_3(x) + 1) = 0$
$\log_3(x) - 2 = 0$ or $\log_3(x) + 1 = 0$
$\log_3(x) = 2 \qquad \log_3(x) = -1$
$3^2 = x \qquad 3^{-1} = x$
$x = 9 \qquad x = \frac{1}{3}$

l $\log_6(x-3) + \log_6(x+2) = 1$
$\log_6(x-3)(x+2) = 1$
$6 = (x-3)(x+2)$
$6 = x^2 - x - 6$
$0 = x^2 - x - 12$
$0 = (x-4)(x+3)$
$x - 4 = 0$ or $x + 3 = 0$
$x = 4 \qquad x = -3$
But $x > 3$, $\therefore x = 4$

14 a $\log_{19}(y) = 2\log_{10} 2 - 3\log_{10}(x)$
$\log_{19}(y) = \log_{10} 2^2 - \log_{10}(x)^3$
$\log_{19}(y) = \log_{10}\left(\frac{4}{x^3}\right)$
$y = \frac{4}{x^3}$

b $\log_4(y) = -2 + 2\log_4(x)$

$\log_4(y) = 2\log_4(x) - 2\log_4(4)$

$\log_4(y) = \log_4(x)^2 - \log_4(4^2)$

$\log_4(y) = \log_4\left(\frac{x^2}{16}\right)$

$y = \frac{x^2}{16}$

c $\log_9(3xy) = 1.5$

$\log_9(3xy) = \frac{3}{2}\log_9 9$

$\log_9(3xy) = \log_9\left(3^2\right)^{\frac{3}{2}}$

$\log_9(3xy) = \log_9 3^3$

$3xy = 27$

$xy = 9$

$y = \frac{9}{x}$

d $\log_8\left(\frac{2x}{y}\right) + 2 = \log_8(2)$

$\log_8\left(\frac{2x}{y}\right) + 2\log_8(8) = \log_8(2)$

$\log_8\left(\frac{2x}{y}\right) + \log_8(8)^2 = \log_8(2)$

$\log_8\left(\frac{128x}{y}\right) = \log_8(2)$

$\frac{128x}{y} = 2$

$y = 64x$

15 **a** $3\log_m(x) = 3\log_m(27)$

$3\log_m(x) = 3\log_m(m) + \log_m(3)^3$

$3\log_m(x) = 3\log_m(m) + 3\log_m(3)$

$\log_m(x) = \log_m(m) + \log_m(3)$

$\log_m(x) = \log_m(3m)$

$x = 3m$

b If $x = \log_{10}(m)$ and $y = \log_{10}(n)$ then

$10^x = m$ and $10^y = n$

$$\log_{10}\left(\frac{100n^2}{m^5\sqrt{n}}\right) = \log_{10}\left(\frac{100\left(10^y\right)^2}{\left(10^x\right)^5\left(10^y\right)^{\frac{1}{2}}}\right)$$

$$= \log_{10}\left(\frac{10^2 \times 10^{2y}}{10^{5x} \times 10^{\frac{y}{2}}}\right)$$

$$= \log_{10}\left(\frac{10^2 \times 10^{\frac{3y}{2}}}{10^{5x}}\right)$$

$$= \log_{10}\left(10^{2+\frac{3y}{2}-5x}\right)$$

$$= \left(2 + \frac{3y}{2} - 5x\right)\log_{10}(10)$$

$$= 2 + \frac{3y}{2} - 5x$$

16 $8\log_x(4) = \log_2(x)$

$\frac{8\log_2(4)}{\log_2(x)} = \frac{\log_2(x)}{\log_2(2)}$

$8\log_2(4) \times \log_2(2) = \left[\log_2(x)\right]^2$

$8\log_2(2^2) \times \log_2(2) = \left[\log_2(x)\right]^2$

$16\log_2(2) \times \log_2(2) = \left[\log_2(x)\right]^2$

$16 = \left[\log_2(x)\right]^2$

$\log_2(x) = \pm 4$

$x = 2^4, 2^{-4}$

$= 16, \frac{1}{16}$

17 **a** $e^{2x} - 3 = \log_e(2x+1)$

Solve using CAS

$x = -0.463,\ 0.675$

b $x^2 - 1 = \log_e(x)$

Solve using CAS

$x = 0.451,\ 1$

18 $\left(3\log_3(x)\right)\left(5\log_3(x)\right) = 11\log_3(x) - 2$

Solve using CAS

$x = 1.5518,\ 1.4422$

Exercise 4.3 — Logarithmic scales

1 $L = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$

When $L = 130$ dB,

$130 = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$

$13 = \log_{10}\left(I \times 10^{12}\right)$

$13 = \log_{10}(I) + \log_{10}(10)^{12}$

$13 = \log_{10}(I) + 12\log_{10}(10)$

$13 = \log_{10}(I) + 12$

$13 - 12 = \log_{10}(I)$

$1 = \log_{10}(I)$

$I = 10$

Intensity is 10 watt/m^2.

2 $M = 0.67\log_{10}\left(\frac{E}{K}\right)$

If $M = 5.5$ and $E = 10^{13}$ then

$5.5 = 0.67\log_{10}\left(\frac{10^{13}}{K}\right)$

$8.2090 = \log_{10}\left(\frac{10^{13}}{K}\right)$

$10^{8.2090} = \frac{10^{13}}{K}$

$K = \frac{10^{13}}{10^{8.2090}}$

$K = 10^{4.7910} = 61\,801.640$

Thus $K \simeq 61\,808$

3 $M = 0.67\log_{10}\left(\frac{E}{K}\right)$

When $M = 6.3$,

$$6.3 = 0.67\log_{10}\left(\frac{E_{6.3}}{K}\right)$$
$$\frac{6.3}{0.67} = \log_{10}\left(\frac{E_{6.3}}{K}\right)$$
$$9.403 = \log_{10}\left(\frac{E_{6.3}}{K}\right)$$
$$10^{9.403} = \frac{E_{6.3}}{K}$$
$$252\,911\,074K = E_{6.3}$$

When $M = 6.4$,

$$6.4 = 0.67\log_{10}\left(\frac{E_{6.4}}{K}\right)$$
$$\frac{6.4}{0.67} = \log_{10}\left(\frac{E_{6.4}}{K}\right)$$
$$9.5522 = \log_{10}\left(\frac{E_{6.4}}{K}\right)$$
$$10^{9.5522} = \frac{E_{6.4}}{K}$$
$$3\,566\,471\,895K = E_{6.4}$$

$$E_{6.4} : E_{6.3} = 3\,566\,471\,895K ; 252\,911\,074K$$
$$= 1.4101 : 1$$

6.4 earthquake is 1.41 times bigger than the 6.3 earthquake.

4 $M = 0.67\log_{10}\left(\frac{E}{K}\right)$

When $M = 9$ and $E = 10^{17}$

$$9 = 0.67\log_{10}\left(\frac{10^{17}}{K}\right)$$
$$13.4328 = \log_{10}(10)^{17} - \log_{10}(K)$$
$$\log_{10}(K) = 17\log_{10}(10) - 13.4328$$
$$\log_{10}(K) = 17 - 13.4328$$
$$\log_{10}(K) = 3.5672$$
$$10^{3.5672} = K$$
$$K = 3691.17$$

5 $L = 10\log_{10}\left(\frac{I}{I_0}\right) = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$

If $I = 20$;

$$L = 10\log_{10}\left(\frac{20}{10^{-12}}\right)$$
$$L = 10\log_{10}\left(20 \times 10^{12}\right)$$
$$L = 10\log_{10}\left(2 \times 10^{13}\right)$$
$$L = 10\log_{10}(2) + 10\log_{10}\left(10^{13}\right)$$
$$L = 10\log_{10}(2) + (13 \times 10)\log_{10}(10)$$
$$L = 10\log_{10}(2) + 130$$
$$L = 133.0103 \text{ dB}$$

If $I = 500$;

$$L = 10\log_{10}\left(\frac{500}{10^{-12}}\right)$$
$$L = 10\log_{10}\left(5 \times 10^2 \times 10^{12}\right)$$
$$L = 10\log_{10}\left(5 \times 10^{14}\right)$$
$$L = 10\log_{10}(5) + 10\log_{10}(10)^{14}$$
$$L = 10\log_{10}(5) + (10 \times 14)\log_{10}(10)$$
$$L = 10\log_{10}(5) + 140$$
$$L = 146.9897 \text{ dB}$$

A 500 watt amplifier is $146.9897 - 133.0103 = 13.98$ dB louder than the 20 watt amplifier.

6 $L = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$

When $I = 10^4$,

$$L = 10\log_{10}\left(\frac{10^4}{10^{-12}}\right)$$
$$L = 10\log_{10}\left(10^4 \times 10^{12}\right)$$
$$L = 10\log_{10}(10)^{16}$$
$$L = 160\log_{10}(10) = 160 \text{ dB}$$

Loudness is 160 dB

7 $pH = -\log_{10}\left[H^+\right]$

When $H^+ = 0.001$,

$$pH = -\log_{10}[0.001]$$
$$pH = -\log_{10}(10)^{-3}$$
$$pH = 3\log_{10}(10) = 3$$

Lemon juice has a pH of 3 which is acidic.

8 **a** $pH = -\log_{10}\left[H^+\right]$

When $pH = 0$,

$$0 = -\log_{10}\left[H^+\right]$$
$$0 = \log_{10}\left[H^+\right]$$
$$10^0 = \left[H^+\right]$$
$$1 \text{ moles/litre} = \left[H^+\right]$$

b When $pH = 4$,

$$4 = -\log_{10}\left[H^+\right]$$
$$-4 = \log_{10}\left[H^+\right]$$
$$10^{-4} = \left[H^+\right]$$
$$0.0001 \text{ moles/litre} = \left[H^+\right]$$

c $pH = -\log_{10}\left[H^+\right]$

When $pH = 8$,

$$8 = -\log_{10}\left[H^+\right]$$
$$-8 = \log_{10}\left[H^+\right]$$
$$10^{-8} = \left[H^+\right]$$
$$10^{-8} \text{ moles/litre} = \left[H^+\right]$$

d $pH = -\log_{10}\left[H^+\right]$

When $pH = 12$,

$$12 = -\log_{10}\left[H^+\right]$$
$$-12 = \log_{10}\left[H^+\right]$$
$$10^{-12} = \left[H^+\right]$$
$$10^{-12} \text{ moles/litre} = \left[H^+\right]$$

9 **a** $pH = -\log_{10}\left[H^+\right]$

$$\left[H^+\right] = 0.0000158 \text{ moles/litre}$$
$$pH = -\log_{10}(0.0000158)$$
$$pH = 4.8$$

My hair conditioner has a pH of 4.8 which is acidic.

b $pH = -\log_{10}\left[H^+\right]$

$$\left[H^+\right] = 0.00000275 \text{ moles/litre}$$

$pH = -\log_{10}(0.00000275)$
$pH = 5.56$
My shampoo has a pH of 5.56 which is acidic.

10 **a** $N(t) = 0.5N_0$

$$0.5N_0 = N_0 e^{-mt}$$
$$\frac{1}{2} = e^{-mt}$$
$$\log_e\left(\frac{1}{2}\right) = -mt$$
$$\log_e(2)^{-1} = -mt$$
$$-\log_e(2) = -mt$$
$$\log_e(2) = mt$$
$$t = \frac{\log_e(2)}{m} \text{ as required}$$

b $N(t) = 0.3N_0$
When $t = 5750$ years,

$$5750 = \frac{\log_e(2)}{m}$$
$$5750m = \log_e(2)$$
$$m = \frac{\log_e(2)}{5750} = 0.000121$$
$$0.3N_0 = N_0 e^{-0.000121t}$$
$$0.3 = e^{-0.000121t}$$
$$\log_e(0.3) = -0.000121t$$
$$\frac{\log_e(0.3)}{-0.000121} = t$$
$$t = 9987.55$$

The skeleton is 9988 years old.

11 $m_2 - m_1 = 2.5\log_{10}\left(\frac{b_1}{b_2}\right)$

Sirius: $m_1 = -1.5$ and $b_1 = -30.3$
Venus: $m_2 = -4.4$ and $b_2 = ?$

$$-4.4 - (-1.5) = 2.5\log_{10}\left(\frac{-30.3}{b_2}\right)$$
$$-2.9 = 2.5\log_{10}\left(\frac{-30.3}{b_2}\right)$$
$$\frac{-2.9}{2.5} = \log_{10}\left(\frac{-30.3}{b_2}\right)$$
$$-1.16 = \log_{10}\left(\frac{-30.3}{b_2}\right)$$
$$10^{-1.16} = \frac{-30.3}{b_2}$$
$$b_2 = \frac{-30.3}{10^{-1.16}}$$
$$b_2 = \frac{-30.3}{0.0692}$$
$$= -437.9683$$

Brightness of Venus is -437.97.

12 $n = 1200\log_{10}\left(\frac{f_2}{f_1}\right)$

$f_1 = 256, \quad f_2 = 512$

$$n = 1200\log_{10}\left(\frac{512}{256}\right)$$
$$n = 361 \text{ cents}$$

13 $L = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$

0.22 Rifle: $I = \left(2.5\times10^{13}\right)I_0 = 2.5\times10^{13}\times10^{-12} = 2.5\times10$

$$L = 10\log_{10}\left(\frac{2.5\times10}{10^{-12}}\right)$$
$$L = 10\left(\log_{10}(2.5\times10) - \log_{10}(10)^{-12}\right)$$
$$L = 10\left(\log_{10}(2.5) + \log_{10}(10) + 12\log_{10}(10)\right)$$
$$L = 10\left(\log_{10}(2.5) + 13\right)$$
$$L = 133.98$$

The loudness of the gunshot is about 133.98 dB so ear protection should be worn.

14 $M = 0.67\log_{10}\left(\frac{E}{K}\right)$

San Francisco: $M_{SF} = 8.3$

$$8.3 = 0.67\log_{10}\left(\frac{E_{SF}}{K}\right)$$
$$12.3881 = \log_{10}\left(\frac{E_{SF}}{K}\right)$$
$$10^{12.3881} = \frac{E_{SF}}{K}$$

South America: $M_{SA} = 4E_{SF}$

$$M_{SA} = 0.67\log_{10}\left(\frac{4E_{SF}}{K}\right)$$

Substitute $10^{12.3881} = \frac{E_{SF}}{K}$

$$M_{SA} = 0.67\log_{10}\left(4\times10^{12.3881}\right)$$
$$= 8.7$$

Magnitude of the South American earthquake was 8.7.

Exercise 4.4 — Indicial equations

1 **a**
$$3^{2x+1}\times27^{2-x} = 81$$
$$3^{2x+1}\times(3^3)^{2-x} = 3^4$$
$$3^{2x+1}\times3^{6-3x} = 3^4$$
$$3^{7-x} = 3^4$$

Equating indices

$$7 - x = 4$$
$$x = 3$$

b
$$10^{2x-1} - 5 = 0$$
$$10^{2x-1} = 5$$
$$\log_{10}(5) = 2x - 1$$
$$\log_{10}(5) + 1 = 2x$$
$$x = \frac{1}{2}\log_{10}(5) + \frac{1}{2}$$

c $\left(4^x - 16\right)\left(4^x + 3\right) = 0$

$4^x - 16 = 0$ or $4^x + 3 = 0$
$4^x = 16 \qquad 4^x = -3$
$4^x = 4^2 \qquad$ No solution
$x = 2$

d $2\left(10^{2x}\right)-7\left(10^{x}\right)+3=0$

$2\left(10^{x}\right)^{2}-7\left(10^{x}\right)+3=0$

$\left(2(10)^{x}-1\right)\left((10)^{x}-3\right)=0$

$2(10)^{x}-1=0$ or $(10)^{x}-3=0$

$10^{x}=\frac{1}{2}$ $\quad 10^{x}=3$

$x=\log_{10}\left(\frac{1}{2}\right)$ $\quad x=\log_{10}(3)$

2 a $2^{x+3}-\frac{1}{64}=0$

$2^{x+3}=\frac{1}{64}$

$2^{x+3}=2^{-6}$

Equating indices

$x+3=-6$

$x=-9$

b $2^{2x}-9=0$

$2^{2x}=9$

$\log_2(9)=2x$

$x=\frac{1}{2}\log_2(9)$

c $3e^{2x}-5e^{x}-2=0$

$3\left(e^{x}\right)^{2}-5e^{x}-2=0$

$\left(3e^{x}+1\right)\left(e^{x}-2\right)=0$

$3e^{x}+1=0$ or $e^{x}-2=0$

$3e^{x}=-1$ $\quad e^{x}=2$

No solution $\quad x=\log_e(2)$

d $e^{2x}-5e^{x}=0$

$e^{x}\left(e^{x}-5\right)=0$

$e^{x}=0$ or $e^{x}-5=0$

No solution $\quad e^{x}=5$

$x=\log_e(5)$

3 a $7^{2x-1}=5$

$\log_7(5)=2x-1$

$\log_7(5)+1=2x$

$x=\frac{1}{2}\log_7(5)+\frac{1}{2}$

b $\left(3^{x}-9\right)\left(3^{x}-1\right)=0$

$3^{x}-9=0$ or $3^{x}-1=0$

$3^{x}=9$ $\quad 3^{x}=1$

$3^{x}=3^{2}$ $\quad 3^{x}=3^{0}$

$x=2$ $\quad x=0$

c $25^{x}-5^{x}-6=0$

$\left(5^{2}\right)^{x}-5^{x}-6=0$

$\left(5^{x}\right)^{2}-5^{x}-6=0$

$\left(5^{x}-3\right)\left(5^{x}+2\right)=0$

$5^{x}-3=0$ or $5^{x}+2=0$

$5^{x}=3$ $\quad 5^{x}=-2$

$\log_5(3)=x$ $\quad$ No solution

d $6\left(9^{2x}\right)-19\left(9^{x}\right)+10=0$

$6\left(9^{x}\right)^{2}-19\left(9^{x}\right)+10=0$

$\left(3\left(9^{x}\right)-2\right)\left(2\left(9^{x}\right)-5\right)=0$

$3\left(9^{x}\right)-2=0$ or $2\left(9^{x}\right)-5=0$

$3\left(9^{x}\right)=2$ $\quad 2\left(9^{x}\right)=5$

$\left(9^{x}\right)=\frac{2}{3}$ $\quad \left(9^{x}\right)=\frac{5}{2}$

$x=\log_9\left(\frac{2}{3}\right)$ $\quad x=\log_9\left(\frac{5}{2}\right)$

4 a $16\times 2^{2x+3}=8^{-2x}$

$2^{4}\times 2^{2x+3}=2^{3(-2x)}$

$2^{2x+3+4}=2^{-6x}$

$2x+7=-6x$

$8x=-7$

$x=-\frac{7}{8}$

b $2\times 3^{x+1}=4$

$3^{x+1}=2$

$\log_3(2)=x+1$

$x=\log_3(2)-1$

c $2\left(5^{x}\right)-12=-\frac{10}{5^{x}}$

$2\left(5^{x}\right)^{2}-12\left(5^{x}\right)+10=0$

$\left(5^{x}\right)^{2}-6\left(5^{x}\right)+5=0$

$\left(5^{x}-1\right)\left(5^{x}-5\right)=0$

$5^{x}-1=0$ or $5^{x}-5=0$

$5^{x}=1$ $\quad 5^{x}=5$

$5^{x}=5^{0}$ $\quad 5^{x}=5^{1}$

$x=0$ $\quad x=1$

d $4^{x+1}=3^{1-x}$

$\log_e(4)^{x+1}=\log_e(3)^{1-x}$

$(x+1)\log_e(4)=(1-x)\log_e(3)$

$x\log_e(4)+\log_e(4)=\log_e(3)-x\log_e(3)$

$x\log_e(4)+x\log_e(3)=\log_e(3)-\log_e(4)$

$x\left(\log_e(4)+\log_e(3)\right)=\log_e\left(\frac{3}{4}\right)$

$$x=\frac{\log_e\left(\frac{3}{4}\right)}{\log_e(4)+\log_e(3)}$$

$$x=\frac{\log_e\left(\frac{3}{4}\right)}{\log_e(12)}$$

5 a $2\left(2^{x-1}-3\right)+4=0$

$2\left(2^{x-1}-3\right)=-4$

$2^{x-1}-3=-2$

$2^{x-1}=1$

$2^{x-1}=2^{0}$

$x-1=0$

$x=1$

b $2\left(5^{1-2x}\right)-3=7$

$2\left(5^{1-2x}\right)=10$

$5^{1-2x}=5$

$5^{1-2x}=5^1$

$1-2x=1$

$0=2x$

$x=0$

6 a $x^{-1}-\cfrac{1}{1-\cfrac{1}{1+x^{-1}}}$

$=x^{-1}-\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{x}}}$

$=x^{-1}-\cfrac{1}{1-\cfrac{1}{\cfrac{x+1}{x}}}$

$=x^{-1}-\cfrac{1}{1-\cfrac{x}{x+1}}$

$=x^{-1}-\cfrac{1}{\cfrac{x+1-x}{x+1}}$

$=x^{-1}-\cfrac{1}{\cfrac{1}{x+1}}$

$=\frac{1}{x}-(x+1)$

$=\frac{1}{x}-x-1$

b $2^{3-4x}\times3^{-4x+3}\times6^{x^2}=1$

$2^{3-4x}\times3^{-4x+3}\times(2\times3)^{x^2}=1$

$2^{3-4x}\times3^{-4x+3}\times2^{x^2}\times3^{x^2}=1$

$2^{x^2-4x+3}\times3^{x^2-4x+3}=1$

$6^{x^2-4x+3}=6^0$

$x^2-4x+3=0$

$(x-1)(x-3)=0$

$x-1=0$ or $x-3=0$

$x=1$ $\quad x=3$

7 a $e^{x-2}-2=7$

$e^{x-2}=9$

$\log_e(9)=x-2$

$\log_e(9)+2=x$

$\log_e(3)^2+2=x$

$x=2\log_e(3)+2$

b $e^{\frac{x}{4}}-1=3$

$e^{\frac{x}{4}}+1=3$

$e^{\frac{x}{4}}=2$

$\log_e(2)=\frac{x}{4}$

$x=4\log_e(2)$

c $e^{2x}=3e^x$

$e^{2x}-3e^x=0$

$e^x\left(e^x-3\right)=0$

$e^x=0$ or $e^x-3=0$

No solution $\quad e^x=3$

$x=\log_e(3)$

d $e^{x^2}+2=4$

$e^{x^2}=2$

$x^2=\log_e(2)$

$x=\pm\sqrt{\log_e(2)}$

8 a $e^{2x}=e^x+12$

$e^{2x}-e^x-12=0$

$\left(e^x\right)^2-\left(e^x\right)-12=0$

$\left(e^x-4\right)\left(e^x+3\right)=0$

$e^x-4=0$ or $e^x+3=0$

$e^x=4$ $\quad e^x=-3$

$\log_e(4)=x$ $\quad \log_e(-3)=x$

$2\log_e(2)=x$ $\quad$ No solution

b $e^x=12-32e^{-x}$

$e^x-12+32e^{-x}=0$

$\left(e^x\right)^2-12\left(e^x\right)+32=0$

$\left(e^x-4\right)\left(e^x-8\right)=0$

$e^x-4=0$ or $e^x-8=0$

$e^x=4$ $\quad e^x=8$

$\log_e(4)=x$ $\quad \log_e(8)=x$

$\log_e(2)^2=x$ $\quad \log_e(2^3)=x$

$2\log_e(2)=x$ $\quad 3\log_e(2)=x$

c $e^{2x}-4=2e^x$

$e^{2x}-2e^x-4=0$

$e^x=\frac{2\pm\sqrt{(-2)^2-4(1)(-4)}}{2}$

$e^x=\frac{2\pm\sqrt{20}}{2}$

$e^x=\frac{2\pm2\sqrt{5}}{2}$

$e^x=1\pm\sqrt{5}$

$x=\log_e\left(1\pm\sqrt{5}\right)$

Therefore $x=\log_e\left(1+\sqrt{5}\right)$ as $1-\sqrt{5}\not>0$

d $e^x-12=\frac{-5}{e^x}$

$e^{2x}-12e^x+5=0$

$e^x=\frac{12\pm\sqrt{144-4(1)(5)}}{2}$

$e^x=\frac{12\pm\sqrt{144-20}}{2}$

$e^x=\frac{12\pm\sqrt{124}}{2}$

$e^x=\frac{12\pm2\sqrt{31}}{2}$

$e^x=6\pm\sqrt{31}$

$x=\log_e\left(6\pm\sqrt{31}\right)$

9 $y=m(10)^{nx}$

When

$x=2,\ y=20;\ 20=m(10)^{2n}$ ....(1)

When

$x=4,\ y=200;\ 200=m(10)^{4n}$ ....(2)

$(2)\div(1)$

$$\frac{200}{20}=\frac{m(10)^{4n}}{m(10)^{2n}}$$
$$10=10^{2n}$$
$$\log_{10}(10)=2n$$
$$1=2n$$
$$n=\frac{1}{2}$$

Substitute $n=\frac{1}{2}$ into (1)

$$20=m(10)^{2\left(\frac{1}{2}\right)}$$
$$20=10m$$
$$m=2$$

10 a $2^x<0.3$

$$\log_2(0.3)<x$$
$$-1.737>x$$
$$x<-1.737$$

b $(0.4)^x<2$

$$\log_{0.4}(2)<x$$
$$-0.756<x$$
$$x>-0.756$$

11 $\left(\log_3(4m)\right)^2=25n^2$

$$\log_3(4m)=\pm5n$$
$$3^{-5n}=4m \quad\text{or}\quad 3^{5n}=4m$$
$$m=\frac{1}{4\times3^{5n}} \qquad m=\frac{3^{5n}}{4}$$

12 a $e^{m-kx}=2n$

$$m-kx=\log_e(2n)$$
$$-kx=\log_e(2n)-m$$
$$x=\frac{\log_e(2n)-m}{-k}$$
$$=\frac{m-\log_e(2n)}{k},\quad k\in R\setminus\{0\},n\in R^+$$

b $8^{mx}\times4^{2n}=16$

$$2^{3mx}\times2^{2(2n)}=2^4$$
$$2^{3mx+4n}=2^4$$
$$3mx+4n=4$$
$$3mx=4-4n$$
$$x=\frac{4-4n}{3m},\quad m\in R\setminus\{0\}$$

c $2e^{mx}=5+4e^{-mx}$

$$2\left(e^{mx}\right)^2-5e^{mx}-4=0$$
$$e^{mx}=\frac{5\pm\sqrt{(-5)^2-4(2)(-4)}}{2(2)}$$
$$e^{mx}=\frac{5\pm\sqrt{25+32}}{4}$$
$$e^{mx}=\frac{5\pm\sqrt{57}}{4}$$
$$e^{mx}=\frac{5+\sqrt{57}}{4},\ e^{mx}>0$$
$$mx=\log_e\left(\frac{5+\sqrt{57}}{4}\right)$$
$$x=\frac{1}{m}\log_e\left(\frac{5+\sqrt{57}}{4}\right),\quad m\in R\setminus\{0\}$$

13 $y=ae^{-kx}$

When

$x=2,\ y=3.033$ so $3.033=ae^{-2k}$ ....(1)

When

$x=6,\ y=1.1157$ so $1.1157=ae^{-6k}$ ....(2)

$(1)\div(2)$ $$\frac{3.033}{1.1157}=\frac{ae^{-2k}}{ae^{-6k}}$$
$$2.7185=e^{4k}$$
$$\log_e(2.7185)=4k$$
$$\frac{1}{4}\log_e(2.7185)=k$$
$$k=0.25$$

Substitute $k=0.25$ into (1)

$$3.033=ae^{-2(0.25)}$$
$$3.033=ae^{-0.5}$$
$$\frac{3.033}{e^{-0.5}}=a$$
$$a=5$$

14 $A=Pe^{rt}$

When

$t-5,\ A=\$12\,840.25 \qquad 12\,840.25=Pe^{5r}$.....................(1)

When

$t-7,\ A=\$14\,190.66 \qquad 14\,190.66=Pe^{7r}$ ....(2)

$(2)\div(1)$ $$\frac{14\,190.66}{12\,840.25}=\frac{Pe^{7r}}{Pe^{7r}}$$
$$1.1052=e^{2r}$$
$$\log_e(1.1052)=2r$$
$$\frac{1}{2}\log_e(1.1052)=r$$
$$0.05=r$$
$$r=5\%$$

Substitute $r-0.05$ into (1)

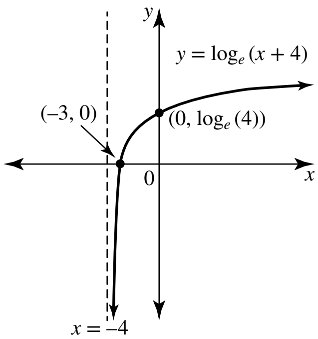
$$12\,840.25=Pe^{5(0.05)}$$
$$\frac{12\,840.25}{e^{0.25}}=P$$
$$P=\$10\,000$$

Exercise 4.5 — Logarithmic graphs

1 a Graph cuts y axis when $x = 0$,

$y = \log_e(4) = 1.386$

Domain $= (-4, \infty)$ and Range $= R$



b Graph cuts x axis when $y = 0$,

$\log_e(x) + 2 = 0$

$\log_e(x) = -2$

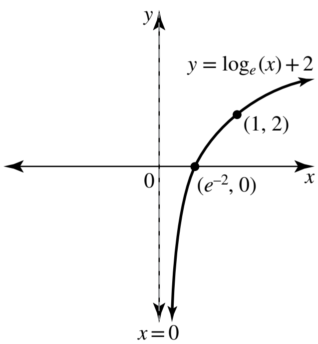
$e^{-2} = x$

$0.1353 = x$

When $x = 2$,

$y = \log_e(2) + 2 = 2.69$

Domain $= (0, \infty)$ and Range $= R$



c Graph cuts x axis when $y = 0$,

$4\log_e(x) = 0$

$\log_e(x) = 0$

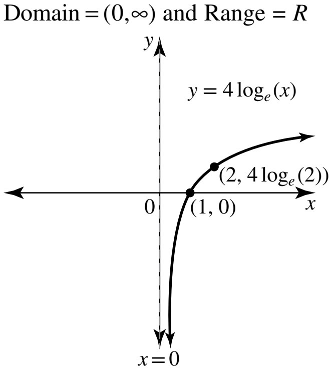
$e^0 = x$

$1 = x$

When $x = 2$,

$y = 4\log_e(2)$

Domain $= (0, \infty)$ and Range $= R$



d Graph cuts the x axis where $y = 0$,

$-\log_e(x-4) = 0$

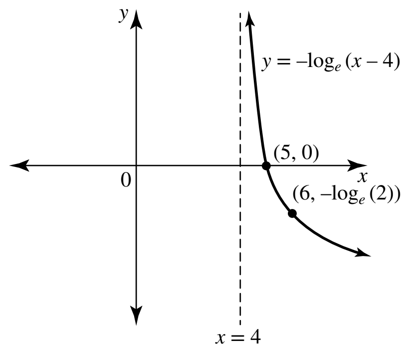
$\log_e(x-4) = 0$

$e^0 = x - 4$

$1 + 4 = x$

$5 = x$

Domain $= (4, \infty)$ and Range $= R$



2 a $y = \log_3(x+2) - 3$

Graph cuts the x axis where $y = 0$,

$\log_3(x+2) - 3 = 0$

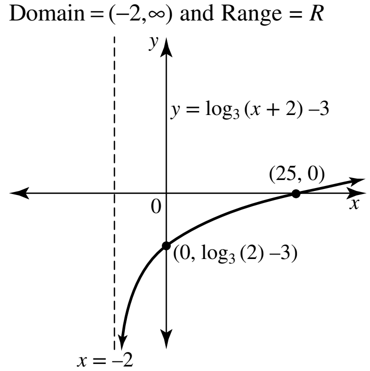
$\log_3(x+2) = 3$

$3^3 = x + 2$

$27 = x + 2$

$25 = x$

Domain $= (-2, \infty)$ and Range $= R$



b $y = 3\log_5(2-x)$

Graph cuts the x axis where $y = 0$,

$3\log_3(2-x) = 0$

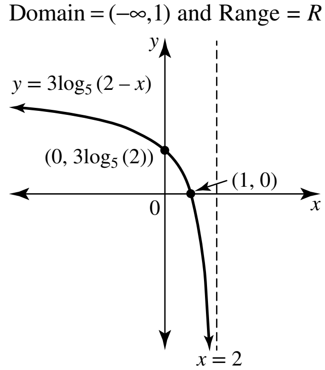
$\log_3(2-x) = 0$

$3^0 = 2 - x$

$x = 2 - 1$

$x = 1$

Domain $= (-\infty, 1)$ and Range $= R$



c $y = 2\log_{10}(x+1)$

Graph cuts the x axis where $y = 0$,

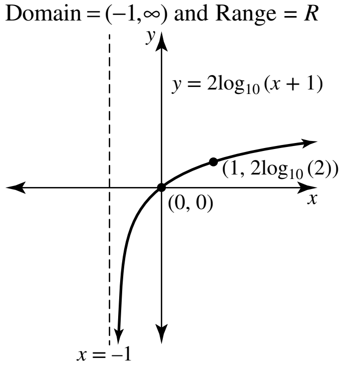
$2\log_{10}(x+1) = 0$

$\log_{10}(x+1) = 0$

$10^0 = x+1$

$0 = x$

Domain $= (-1, \infty)$ and Range $= R$



d $y = \log_2\left(\dfrac{-x}{2}\right)$

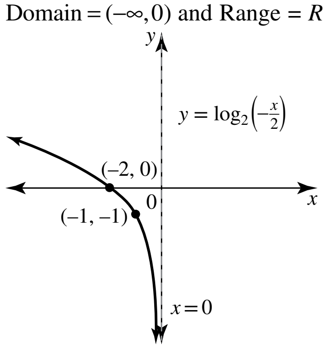
Graph cuts the x axis where $y = 0$,

$\log_2\left(\dfrac{-x}{2}\right) = 0$

$2^0 = \dfrac{-x}{2}$

$-2 = x$

Domain $= (-\infty, 0)$ and Range $= R$



3 $y = \log_e(x-m)+n$

Vertical asymptote is $x = 2$ so $m = 2$.

$y = \log_e(x-2)+n$

When $x = 4.71828,\ y = 3$

$3 = \log_e(4.71828-2)+n$

$3 = \log_e(2.71828)+n$

$n = 3-\log_e(2.71828)$

$n = 2$

$y = \log_e(x-2)+2$

4 $y = p\log_e(x-q)$

When $x = 0,\ y = 0$

$0 = p\log_e(-q)$........................(1)

When $x = 1,\ y = -0.35$

$-0.35 = p\log_e(1-q)$............(2)

From (1)

$0 = \log_e(-q)$

$e^0 = -q$

$q = -1$

Substitute $q = -1$ into (2)

$-0.35 = p\log_e(1-(-1))$

$-0.35 = p\log_e(2)$

$\dfrac{-0.35}{\log_e(2)} = p$

$p = \dfrac{-7}{20\log_e(2)}$

5 **a** Graph cuts x axis when $y = 0$.

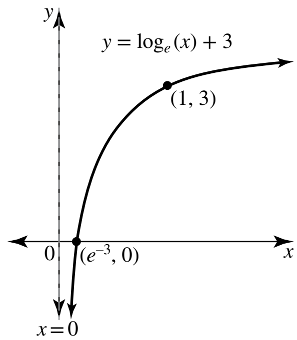
$\log_e(x)+3 = 0$

$\log_e(x) = -3$

$e^{-3} = x$

$0.05 \simeq x$

When $x = 1,\ y = \log_e 1 + 3 = 3$



b Graph cuts x axis when $y = 0$.

$\log_e(x)-5 = 0$

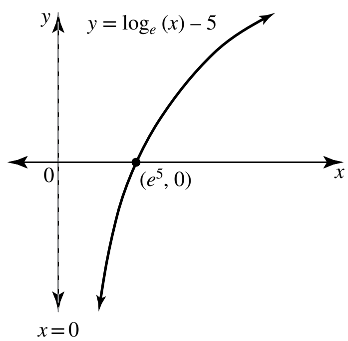
$\log_e(x) = 5$

$e^5 = x$

$148.4 \simeq x$

When $x = 200$,

$y = \log_e(200)-5 = 0.298$



c Graph cuts x axis when $y = 0$.

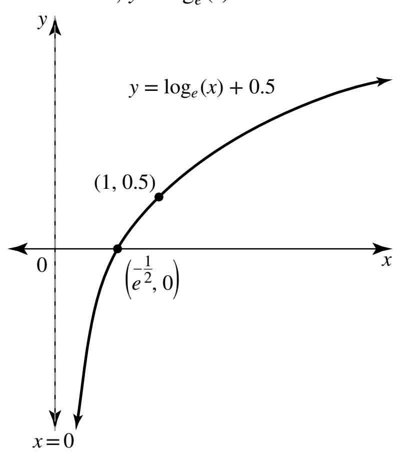
$\log_e(x)+0.5 = 0$

$\log_e(x) = -0.5$

$e^{-0.5} = x$

$0.6 \simeq x$

When $x = 1,\ y = \log_e(1)+0.5 = 0.5$



6 a Graph cuts x axis when $y = 0$.

$\log_e(x-4) = 0$

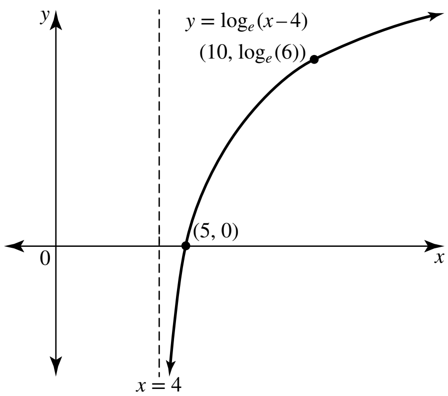
$e^0 = x - 4$

$1 \simeq x - 4$

$5 = x$

When $x = 10$,

$y = \log_e(10-4) = \log_e(6) \simeq 1.8$



b Graph cuts x axis when $y = 0$.

$\log_e(x+2) = 0$

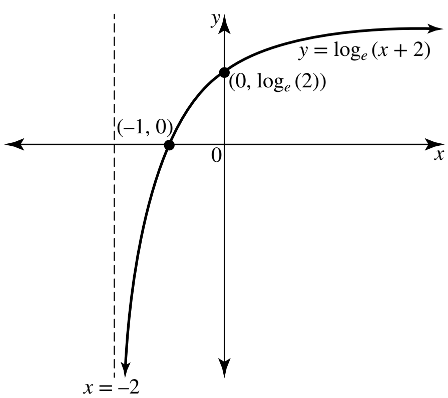
$e^0 = x + 2$

$1 \simeq x + 2$

$-1 = x$

When $x = 0$,

$y = \log_e(0+2) = \log_e(2) \simeq 0.7$



c Graph cuts x axis when $y = 0$.

$\log_e(x+0.5) = 0$

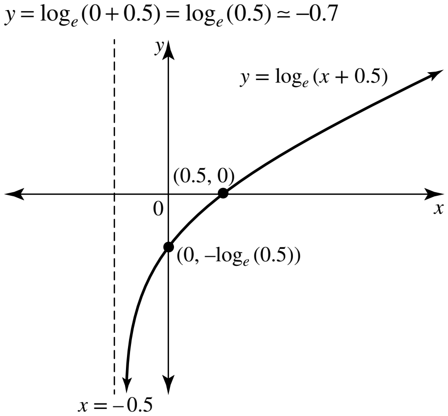
$e^0 = x + 0.5$

$1 \simeq x + 0.5$

$0.5 = x$

When $x = 0$,

$y = \log_e(0+0.5) = \log_e(0.5) \simeq -0.7$



7 a Graph cuts x axis when $y = 0$.

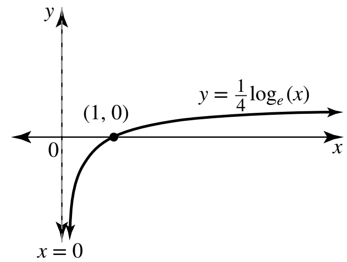
$\frac{1}{4}\log_e(x) = 0$

$\log_e(x) = 0$

$e^0 = x$

$1 = x$

Graph does not cut the y.



b Graph cuts x axis when $y = 0$.

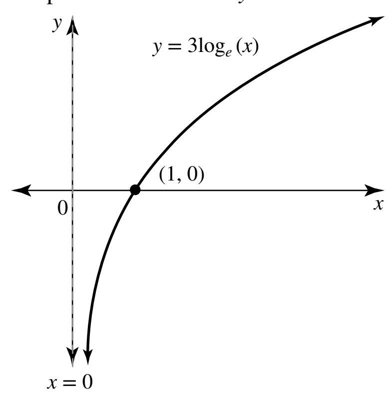
$3\log_e(x) = 0$

$\log_e(x) = 0$

$e^0 = x$

$1 = x$

Graph does not cut the y.



c Graph cuts x axis when $y = 0$.

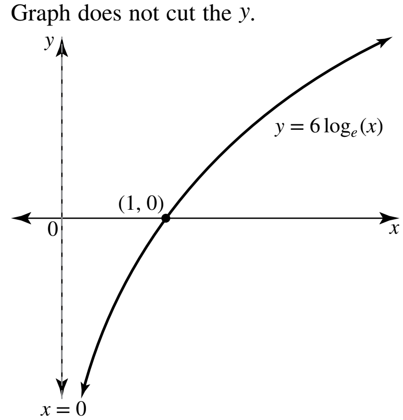
$6\log_e(x) = 0$

$\log_e(x) = 0$

$e^0 = x$

$1 = x$

Graph does not cut the y.



8 a Graph cuts x axis when $y = 0$.

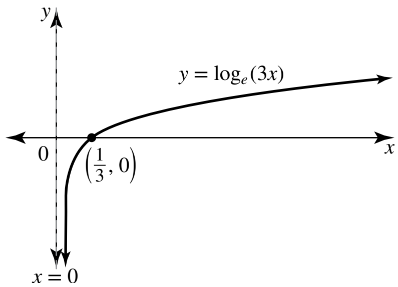
$\log_e(3x) = 0$

$e^0 = 3x$

$1 = 3x$

$\frac{1}{3} = x$

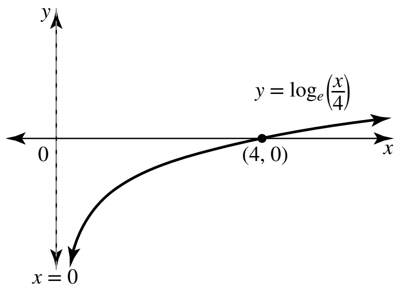
Graph does not cut the y.



b Graph cuts x axis when $y = 0$.

$$\log_e\left(\frac{x}{4}\right) = 0$$
$$e^0 = \frac{x}{4}$$
$$1 = \frac{x}{4}$$
$$4 = x$$

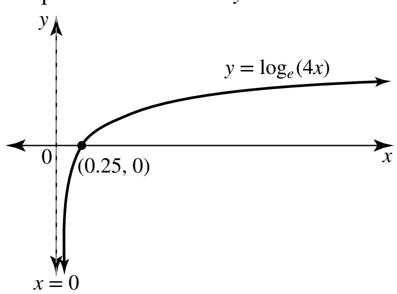
Graph does not cut the y.



c Graph cuts x axis when $y = 0$.

$$\log_e(4x) = 0$$
$$e^0 = 4x$$
$$1 = 4x$$
$$\frac{1}{4} = x$$

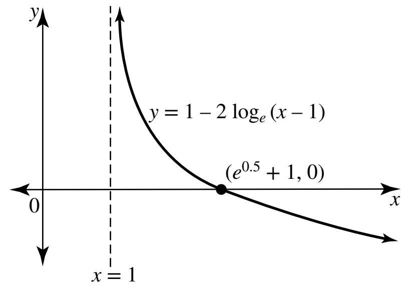
Graph does not cut the y.



9 a Graph cuts x axis when $y = 0$.

$$1 - 2\log_e(x-1) = 0$$
$$2\log_e(x-1) = 1$$
$$\log_e(x-1) = \frac{1}{2}$$
$$e^{\frac{1}{2}} = x - 1$$
$$e^{\frac{1}{2}} + 1 = x$$
$$2.6487 = x$$

Graph does not cut the y.

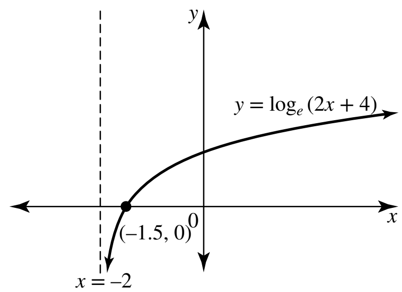


b Graph cuts x axis when $y = 0$.

$$\log_e(2x+4) = 0$$
$$e^0 = 2x + 4$$
$$1 - 4 = 2x$$
$$-\frac{3}{2} = x$$

Graph cuts the y axis where $x = 0$.

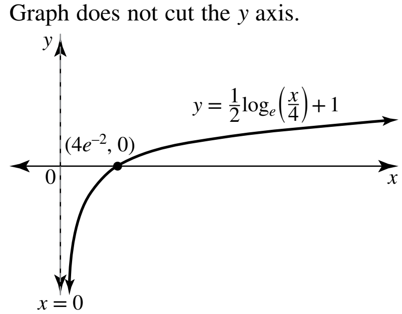
$$\log_e(2(0)+4) = y$$
$$\log_e(4) = y$$
$$1.3862 = y$$



c Graph cuts x axis when $y = 0$.

$$\frac{1}{2}\log_e\left(\frac{x}{4}\right) + 1 = 0$$
$$\frac{1}{2}\log_e\left(\frac{x}{4}\right) = -1$$
$$\log_e\left(\frac{x}{4}\right) = -2$$
$$e^{-2} = \frac{x}{4}$$
$$4e^{-2} = x$$
$$0.5413 = x$$

Graph does not cut the y axis.



10 a $f(x)=2\log_e(3x+3)$

Domain = $(-1,\infty)$ and Range = R

Inverse: swap x and y

$$x=2\log_e(3y+3)$$
$$\frac{x}{2}=\log_e(3y+3)$$
$$e^{\frac{x}{2}}=3y+3$$
$$e^{\frac{x}{2}}-3=3y$$
$$y=\frac{1}{3}e^{\frac{x}{2}}-1$$
$$f^{-1}(x)=\frac{1}{3}e^{\frac{x}{2}}-1$$

Domain = R and Range = $(-1,\infty)$

b $f(x)=\log_e\left(2(x-1)\right)+2$

Domain = $(1,\infty)$ and Range = R

Inverse: swap x and y

$$x=\log_e\left(2(y-1)\right)+2$$
$$x-2=\log_e\left(2(y-1)\right)$$
$$e^{x-2}=2(y-1)$$
$$\frac{1}{2}e^{x-2}=x-1$$
$$x=\frac{1}{2}e^{x-2}+1$$
$$f^{-1}(x)=\frac{1}{2}e^{x-2}+1$$

Domain = R and Range = $(1,\infty)$

c $f(x)=2\log_e(1-x)-2$

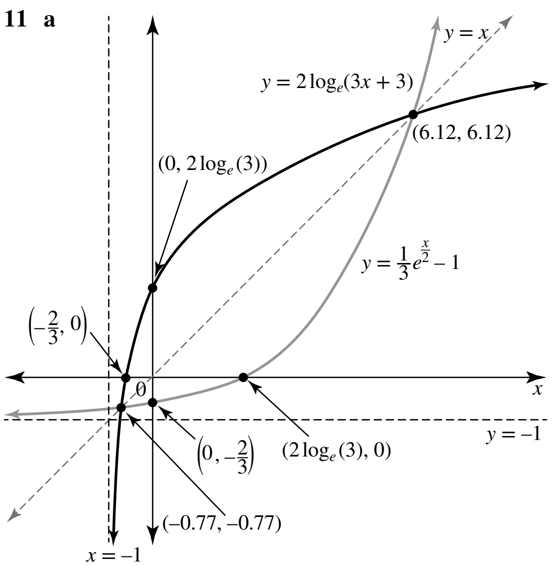
Domain = $(-\infty,1)$ and Range = R

Inverse: swap x and y

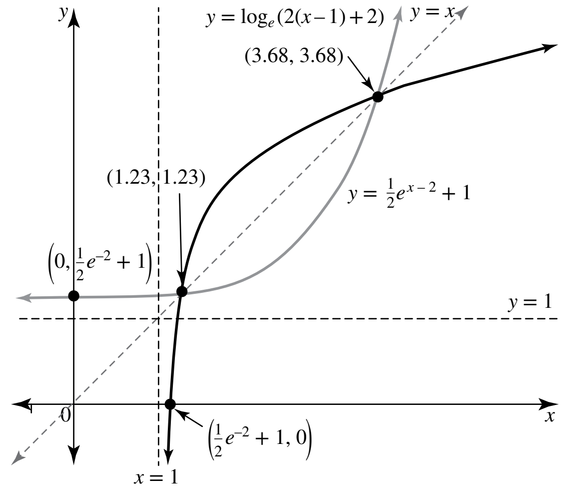
$$x=2\log_e(1-y)-2$$
$$x=2\log_e(1-y)-2$$
$$x+2=2\log_e(1-y)$$
$$\frac{1}{2}(x+2)=\log_e(1-y)$$
$$e^{\frac{1}{2}(x+2)}=1-y$$
$$y=1-e^{\frac{1}{2}(x+2)}$$
$$f^{-1}(x)=1-e^{\frac{1}{2}(x+2)}$$

Domain = R and Range = $(-\infty,1)$

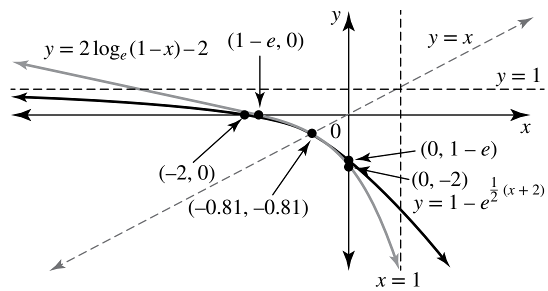
11 a



b



c



12 a $y=a\log_e(bx)$

When

$x=1,\ y=\log_e(2),\quad \log_e(2)=a\log_e(b)$..................(1)

When

$x=2,\ y=0,\quad 0=a\log_e(2b)$.........................(2)

(2) − (1)

$$0-\log_e(2)=a\log_e(2b)-a\log_e(b)$$
$$-\log_e(2)=a\left(\log_e(2b)-\log_e(b)\right)$$
$$-\log_e(2)=a\log_e\left(\frac{2b}{b}\right)$$
$$-\log_e(2)=a\log_e(2)$$
$$-\frac{\log_e(2)}{\log_e(2)}=a$$
$$-1=a$$

Substitute $a=-1$ into (1)

$$\log_e(2)=-\log_e(b)$$
$$\log_e(2)=\log_e(b)^{-1}$$
$$\log_e(2)=\log_e\left(\frac{1}{b}\right)$$
$$2=\frac{1}{b}$$
$$b=\frac{1}{2}$$

b When $x=3,\quad w=-\log_e\left(\frac{3}{2}\right)=-0.4055$

13 $y=a\log_e(x-h)+k$

Graph asymptotes to $x=-1$ so $h=-1$ and $y=a\log_e(x+1)+k$

Graph cuts the y axis at $y=-2$

$(0,-2)\Rightarrow -2=a\log_e(1)+k$

$$k=-2$$

$\therefore y=a\log_e(x+1)-2$

Graph cuts the x axis at $x=1$

$(1,0) \Rightarrow 0 = a\log_e(2) - 2$

$2 = a\log_e(2)$

$a = \dfrac{2}{\log_e(2)}$

Thus $y = \dfrac{2}{\log_e(2)}\log_e(x+1) - 2$

14 $y = m\log_2(nx)$

When

$x=-2,\ y=3$ so $3 = m\log_2(-2n)$.....................(1)

When

$x=-\dfrac{1}{2},\ y=$ so $\dfrac{1}{2} = m\log_2\left(-\dfrac{n}{2}\right)$...................(2)

(1) − (2)

$$3-\frac{1}{2} = m\log_2(-2n) - m\log_2\left(-\frac{n}{2}\right)$$
$$\frac{5}{2} = m\left(\log_2(-2n) - \log_2\left(-\frac{n}{2}\right)\right)$$
$$\frac{5}{2} = m\left(\log_2\left(-2n \div -\frac{n}{2}\right)\right)$$
$$\frac{5}{2} = m\log_2(4)$$
$$\frac{5}{2} = m\log_2(2)^2$$
$$\frac{5}{2} = 2m$$
$$m = \frac{5}{4}$$

Substitute $m = \dfrac{5}{4}$ into (1) $3 = \dfrac{5}{4}\log_2(-2n)$

$$\frac{12}{5} = \log_2(-2n)$$
$$2^{\frac{12}{5}} = -2n$$
$$-\frac{2^{\frac{12}{5}}}{2} = n$$
$$n = -2^{\frac{7}{5}}$$

Thus $m = 1.25$ and $n = -2^{\frac{7}{5}}$ as required.

15 **a** $x - 2 = \log_e(x)$

Solve on CAS

$x = 0.159$ or 3.146

b $1 - 2x = \log_e(x-1)$

Solve on CAS

$x = 1.2315$

16 **a** $x^2 - 2 < \log_e(x)$

Solve on CAS

$x \in (0.138, 1.564)$

b $x^2 - 2 \le \log_e(x)$

Solve on CAS

$x \in [0.136, 1.315]$

Exercise 4.6 — Applications

1 $A = Pe^{rt}$

Western bank: $P = \$4200,\ r = 5\% = 0.05$ so $A = 4200e^{0.05t}$

Common bank: $P = \$5500,\ r = 4.5\% = 0.045$ so $A = 5500e^{0.045t}$

Investments equal in value when

$$\frac{e^{0.05t}}{e^{0.045t}} = \frac{5500}{4200}$$
$$e^{0.005t} = \frac{55}{42}$$
$$\log_e\left(\frac{55}{42}\right) = 0.005t$$
$$\log_e\left(\frac{55}{42}\right) \div 0.005 = t$$
$$t = 53.9327$$

It takes 54 years for the amounts to be equal.

2 **a** $A = Pe^{rt}$

$A = 3P,\ t = 15$

$3P = Pe^{15r}$

$3 = e^{15r}$

$\log_e(3) = 15r$

$\dfrac{\log_e(3)}{15} = r$

$r = 0.0732$

Interest rate of investment is 7.32%.

b $A = Pe^{rt}$

$P = \$2000,\ r = 4.5\% = 0.045,\ A = \$9000$

$$9000 = 2000e^{0.045t}$$
$$\frac{9000}{2000} = e^{0.045t}$$
$$\log_e\left(\frac{9}{2}\right) = 0.045t$$
$$\log_e\left(\frac{9}{2}\right) \div 0.045 = t$$
$$t = 33.42$$

It takes 33 years and 5 months for the investment to grow to \$9000.

3 $t = -10\log_e\left(\dfrac{T-R}{37-R}\right)$

$T = 25^\circ\text{C},\ R = 20^\circ\text{C}$

$t = -10\log_e\left(\dfrac{25-20}{37-20}\right)$

$t = -10\log_e\left(\dfrac{5}{17}\right) = 12.2378$

Time of death is 9 am – 12.2378 hours = 8.7622 or 8.46 pm the day before. The person died $1\dfrac{3}{4}$ hours after the telephone call.

4 $n(t) = \log_e(t + e^2), \qquad t \ge 0$

a Initially $t = 0,\ n(0) = \log_e\left(e^2\right) = 2\log_e(e) = 2$

Initially there were 2 parts per million.

b When $t = 12,\ n(12) = \log_e\left(12 + e^2\right) = 2.9647$

After 12 hours there are 2.96 parts per million.

c When $n(t) = 4$,

$$4 = \log_e\left(t + e^2\right)$$
$$e^4 = t + e^2$$
$$e^4 - e^2 = t$$
$$t = 47.2$$

It takes 47.2 hours before the four parts in a million of fungal bloom exists.

5 $A = Pe^{rt}$

When $t = 10,\ P = \$1000$ and $r = \dfrac{5}{100} = 0.05$,

$A = 1000e^{0.05(10)}$

$A = \$1648.72$

6 $P(t) = 200^{kt} + 1000$

Initially $t = 0$ so $P(0) = 200^0 + 1000 = 1001$

When $t = 8$ and $P = 3 \times 1001 = 3003$,

$$3003 = 200^{8k} + 1000$$
$$2003 = 200^{8k}$$
$$\log_e(2003) = \log_e(200)^{8k}$$
$$\log_e(2003) = 8k\log_e(200)$$
$$\frac{\log_e(2003)}{\log_e(200)} = 8k$$
$$\frac{\log_e(2003)}{8\log_e(200)} = k$$
$$k = 0.1793$$

7 $P(t) = \dfrac{3}{4}\left(1 - e^{-kt}\right)$ and when $t = 3$ and $P = \dfrac{1}{1500}$,

$$\frac{1}{1500} = \frac{3}{4}\left(1 - e^{-3k}\right)$$
$$\frac{4}{4500} = 1 - e^{-3k}$$
$$e^{-3k} = 1 - \frac{4}{4500}$$
$$e^{-3k} = 0.999$$
$$\log_e(0.999) = -3k$$
$$-\frac{1}{3}\log_e(0.999) = k$$
$$k = 0.0003$$

8 $Q = Q_0e^{-0.000124t}$

a When $Q_0 = 100$ and $t = 1000$,

$Q = 100e^{-0.000124(1000)}$

$Q = e^{-0.124}$

$Q = 88.3$ milligrams

b When $Q = \dfrac{1}{2}Q_0 = 50$,

$$50 = 100e^{-0.000124t}$$
$$0.5 = e^{-0.000124t}$$
$$\log_e(0.5) = -0.000124t$$
$$\frac{\log_e(0.5)}{-0.000124} = t$$
$$t = 5589.897$$

It takes 5590 years for the amount of carbon-14 in the fossil to be halved.

9 $W = W_0(0.805)^t$

a When $t = 10$,

$W = W_0(0.805)^{10} = 0.11428W_0$

$0.114W_0$ are the words remaining after 10 millennia or 88.57% of the words have been lost.

b $W = \dfrac{2}{3}W_0$ since one third of the basic words have been lost

$$\frac{2}{3}W_0 = W_0(0.805)^t$$
$$\frac{2}{3} = (0.805)^t$$
$$\log_e\left(\frac{2}{3}\right) = \log_e(0.805)^t$$
$$\log_e\left(\frac{2}{3}\right) = t\log_e(0.805)$$
$$\log_e\left(\frac{2}{3}\right) \div \log_e(0.805) = t$$
$$t = 1.87$$

It takes 1.87 millennia to lose a third of the basic words.

10 **a** $M = a - \log_e(t + b)$

When $t = 0,\ M = 7.8948$,

$7.8948 = a - \log_e(b)$..............(1)

When $t = 80,\ M = 7.3070$,

$7.3070 = a - \log_e(80 + b)$........(2)

(1) − (2)

$$7.8948 - 7.3070 = a - \log_e(b) - \left(a - \log_e(80 + b)\right)$$
$$0.5878 = a - \log_e(b) - a + \log_e(80 + b)$$
$$0.5878 = \log_e(80 + b) - \log_e(b)$$
$$0.5878 = \log_e\left(\frac{(80 + b)}{b}\right)$$
$$e^{0.5878} = \frac{(80 + b)}{b}$$
$$1.8b = 80 + b$$
$$0.8b = 80$$
$$b = 100$$

Substitute $b = 100$ into (1):

$7.8948 = a - \log_e(100)$

$7.8948 + \log_e(100) = a$

$12.5 = a$

$M = 12.5 - \log_e(t + 100)$

Thus $a = 12.5$ and $b = 100$.

b When $t = 90$,

$M = 12.5 - \log_e(90 + 100)$

$M = 12.5 - \log_e(190) = 7.253$ g

11 **a** $P = a\log_e(t) + c$

When $t = 1,\ P = 10\ 000$,

$10\ 000 = a\log_e(1)$

$10\ 000 = c$

$P = a\log_e(t) + 10\ 000$

When $t = 4,\ P = 6000$,

$$6000 = a\log_e(4) + 10\ 000$$
$$-4000 = a\log_e(4)$$
$$\frac{-4000}{\log_e(4)} = a$$
$$a = -2885.4$$

b $P = -2885.4\log_e(t) + 10\,000$

$P = 10\,000 - 2885.4\log_e(t)$

When $t = 8$,

$P = 10\,000 - 2885.4\log_e(8) = 4000$

There are 4000 after 8 weeks.

c When $P = 1000$,

$1000 = 10\,000 - 2885.4\log_e(t)$

$2885.4\log_e(t) = 9000$

$\log_e(t) = \dfrac{9000}{2885.4}$

$\log_e(t) = 3.1192$

$e^{3.1192} = t$

$t = 22.6$

After 22.6 weeks there will be less than 1000 trout.

12 a $C = A\log_e(kt)$

When $t = 2,\ C = 0.1$,

$0.1 = A\log_e(2k)$...............(1)

When $t = 30,\ C = 4$,

$4 = A\log_e(30k)$...............(2)

$(2) \div (1)$

$\dfrac{A\log_e(30k)}{A\log_e(2k)} = \dfrac{4}{0.1}$

$\log_e(30k) = 40\log_e(2k)$

$\log_e(30) + \log_e(k) = 40\left(\log_e(2) + \log_e(k)\right)$

$\log_e(30) + \log_e(k) = 40\log_e(2) + 40\log_e(k)$

$\log_e(30) - 40\log_e(2) = 40\log_e(k) - \log_e(k)$

$\log_e(30) - 40\log_e(2) = 39\log_e(k)$

$-24.3247 = 39\log_e(k)$

$\dfrac{-24.3247}{39} = \log_e(k)$

$-0.6237 = \log_e(k)$

$e^{-0.6237} = k$

$k = 0.536$

Substitute $k = 0.536$ into (1):

$0.1 = A\log_e(2 \times 0.536)$

$0.1 = 0.0695A$

$A = 1.440$

$C = 1.438\log_e(0.536t)$

b When $t = 15$,

$C = 1.438\log_e(0.536 \times 15) = 3.002$ M

Concentration after 15 minutes is 3.002 M.

c When $C = 10$ M,

$10 = 1.438\log_e(0.536t)$

$6.9541 = \log_e(0.536t)$

$e^{6.9541} = 0.536t$

$1047.4385 = 0.536t$

$t = 1934$

After 1934 seconds or 32 minutes and 14 seconds the concentration is 10 M.

13 $F(t) = 10 + 2\log_e(t+2)$

a When $t = 0,\ F(0) = 10 + 2\log_e(2) = 11.3863$

b When $t = 4$,

$F(0) = 10 + 2\log_e(4+2)$

$= 10 + 2\log_e(6)$

$= 13.5835$

c When $F = 15$,

$15 = 10 + 2\log_e(t+2)$

$5 = 2\log_e(t+2)$

$\dfrac{5}{2} = \log_e(t+2)$

$e^{\frac{5}{2}} = t + 2$

$e^{\frac{5}{2}} - 2 = t$

$t = 10.18$

After 10.18 weeks Andrew's level of fitness is 10.

14 $Q = Q_0e^{-0.000124t}$

When $Q = 20\%$ of $Q_0 = 0.2Q_0$

$0.2Q_0 = Q_0e^{-0.000124t}$

$0.2 = e^{-0.000124t}$

$\log_e(0.2) = -0.000124t$

$\dfrac{\log_e(0.2)}{-0.000124} = t$

$t = 12\,979$

Age of painting is 12 979 years.

15 $R(x) = 800\log_e\left(2 + \dfrac{x}{100}\right)$ and $C(x) = 300 + 2x$

a $P(x) = R(x) - C(x)$

$P(x) = 800\log_e\left(2 + \dfrac{x}{100}\right) - 300 - 2x$

b When $P(x) = 0$,

$800\log_e\left(2 + \dfrac{x}{100}\right) - 300 - 2x = 0$

$800\log_e\left(2 + \dfrac{x}{100}\right) = 300 + 2x$

$x = 329.9728$

$x \simeq 330$

330 units are needed to break even.

16 a $V = ke^{mt}$

When $t = 0,\ V = 10\,000$;

$10\,000 = ke^0$

$10\,000 = k$

$V = 10\,000e^{mt}$

When $t = 12,\ V = 13\,500$;

$13\,500 = 10\,000e^{12m}$

$1.35 = e^{12m}$

$\log_e(1.35) = 12m$

$\dfrac{1}{12}\log_e(1.35) = m$

$0.025 = m$

$V = 10\,000e^{0.025}$

b When $t = 18$, $V = 10\,000e^{0.025(18)} = \$15\,685.58$

c Profit $= P$

$P = 1.375 \times 10\,000e^{0.025t} - 10\,000$

$P = 13\,750e^{0.025t} - 10\,000$

d When $t = 24$,

$P = 13\,750e^{0.025(24)} - 10\,000 = \$15\,059.38$

Topic 5 — Differentiation

Exercise 5.2 — Review of differentiation

1 a $f(x)=(2-x)^3+1$

$f(x)=2^3-3(2)^2x+3(2)x^2-x^3+1$

$f(x)=8-12x+6x^2-x^3+1$

$f(x+h)=(2-(x+h))^3+1$

$f(x+h)=2^3-3(2)^2(x+h)+3(2)(x+h)^2-(x+h)^3+1$

$f(x+h)=8-12(x+h)+6\left(x^2+2xh+h^2\right)-\left(x^3+3x^2h+3xh^2+h^3\right)+1$

$f(x+h)=8-12x-12h+6x^2+12xh+6h^2-x^3-3x^2h-3xh^2-h^3+1$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to0}\frac{8-12x-12h+6x^2+12xh+6h^2-x^3-3x^2h-3xh^2-h^3+1-8+12x-6x^2+x^3-1}{h}$$

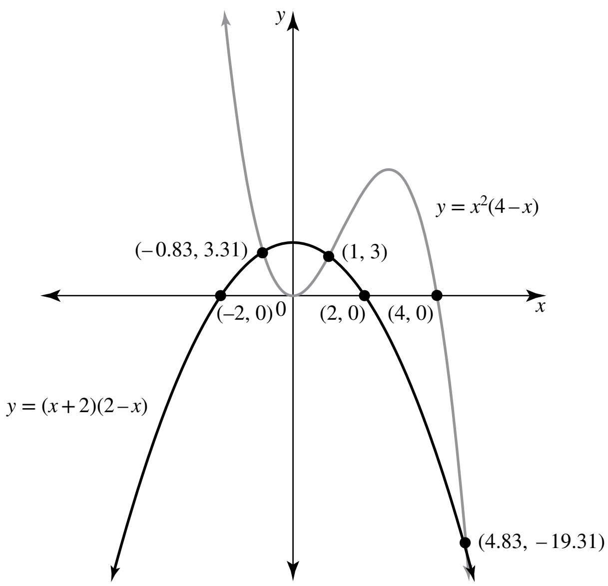
$$f'(x)=\lim_{h\to0}\frac{-12h+12xh+6h^2-3x^2h-3xh^2-h^3}{h}$$

$$f'(x)=\lim_{h\to0}\left(-12+12x+6h-3x^2-3xh-h^2\right),\ h\neq0$$

$$f'(x)=-12+12x-3x^2$$

b When $x=1$, $f'(1)=-12+12(1)-3(1)^2=-3$

2 a, b



(−0.83, 3.31), (1, 3) and (4.83, −19.31)

c $f(x)=(x+2)(2-x)=4-x^2$

$f(x+h)=((x+h)+2)(2-(x+h))$

$f(x+h)=4-(x+h)^2=4-x^2-2xh-h^2$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to0}\frac{4-x^2-2xh-h^2-4+x^2}{h}$$

$$f'(x)=\lim_{h\to0}\frac{-2xh-h^2}{h}$$

$$f'(x)=\lim_{h\to0}(-2x-h),\ h\neq0$$

$$f'(x)=-2x$$

When $x=1$, $f'(1)=-2(1)=-2$

$$f(x) = x^2(4 - x) = 4x^2 - x^3$$
$$f(x+h) = 4(x+h)^2 - (x+h)^3$$
$$f(x+h) = 4\left(x^2 + 2xh + h^2\right) - \left(x^3 + 3x^2h + 3xh^2 + h^3\right)$$
$$f(x+h) = 4x^2 + 8xh + 4h^2 - x^3 - 3x^2h - 3xh^2 - h^3$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - x^3 - 3x^2h - 3xh^2 - h^3 - 4x^2 + x^3}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{8xh + 4h^2 - 3x^2h - 3xh^2 - h^3}{h}$$
$$f'(x) = \lim_{h \to 0}\left(8x + 4h - 3x^2 - 3xh - h^2\right),\ h \ne 0$$
$$f'(x) = 8x - 3x^2$$

When $x = 1$, $f'(1) = 8(1) - 3(1)^2 = 5$

3 a $f(x) = 4x^3 + \dfrac{1}{3x^2} + \dfrac{1}{2} = 4x^3 + \dfrac{1}{3}x^{-2} + \dfrac{1}{2}$

$f'(x) = 12x^2 - \dfrac{2}{3}x^{-3} = 12x^2 - \dfrac{2}{3x^3}$

b $f(x) = \dfrac{2\sqrt{x} - x^4}{5x^3} = \dfrac{2}{5}x^{-\frac{5}{2}} - \dfrac{1}{5}x$

$f'(x) = -x^{-\frac{7}{2}} - \dfrac{1}{5} = -\dfrac{1}{x^{\frac{7}{2}}} - \dfrac{1}{5}$

4 a $f(x) = (x+3)(x^2+1) = x^3 + 3x^2 + x + 3$

$f'(x) = 3x^2 + 6x + 1$

b $f(x) = \dfrac{4 - \sqrt{x}}{\sqrt{x^3}} = 4x^{-\frac{3}{2}} - x^{-1}$

$f'(x) = -6x^{-\frac{5}{2}} + x^{-2} = -\dfrac{6}{x^{\frac{5}{2}}} + \dfrac{1}{x^2}$

5 a $f(x) = -\dfrac{1}{x^2} + 2x = -x^{-2} + 2x$

$f'(x) = 2x^{-3} + 2 = \dfrac{2}{x^3} + 2$

$$f'\left(-\frac{1}{2}\right) = \frac{2}{\left(-\frac{1}{2}\right)^3} + 2$$
$$= -16 + 2$$
$$= -14$$

b $f(x) = \dfrac{2x - 4}{x} = 2 - 4x^{-1}$

$f'(x) = 4x^{-2} = \dfrac{4}{x^2}$

If $f'(x) = 1$ then $\dfrac{4}{x^2} = 1$

$$4 = x^2$$
$$\pm 2 = x$$

When $x = -2$, $f(x) = \dfrac{2(-2) - 4}{-2} = 4$

When $x = 2$, $f(x) = \dfrac{2(2) - 4}{-2} = 0$

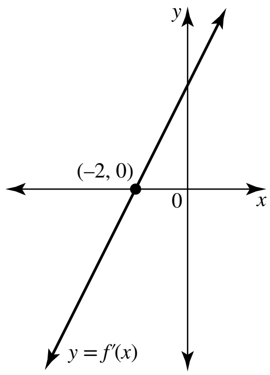
Therefore gradient = 1 at (2, 0) and (−2, 4)

6 $y = (x - a)(x^2 - 1) = x^3 - ax^2 - x + a$

$\dfrac{dy}{dx} = 3x^2 - 2ax - 1$

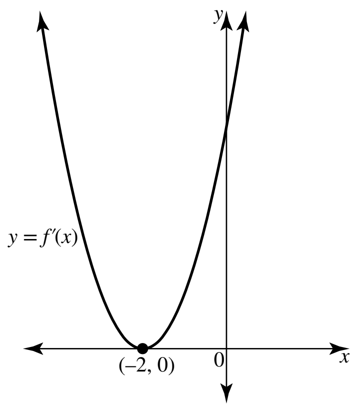
$x = -2$, $\dfrac{dy}{dx} = 3(-2)^2 - 2(-2)a - 1 = 12 + 4a - 1 = 11 + 4a$

7 a Turning point at $(-2,-9)$ so $f'(x)$ cuts the x axis at $x=-2$. For $x<-2$, $f'(x)$ is negative and for $x>-2$, $f'(x)$ is positive.



b Domain of the gradient function shown is $x\in(-\infty,2)\setminus\{-2\}$.

8 Stationary point of inflection at $x=-2$ so turning point at $x=-2$. Parabola lies above the x axis for all other values of x.



9 a $y=x(x-2)^2(x-4)=\left(x^2-4x\right)\left(x^2-4x+4\right)=x^4-8x^3+20x^2-16x$

$\frac{dy}{dx}=4x^3-24x^2+40x-16$

$x=3,\ \frac{dy}{dx}=4(3)^3-24(3)^2+40(3)-16=-4$

Equation of tangent which passes through $\left(x_1,y_1\right)\equiv(3,-3)$ which has $m_T=-4$

$y-y_1=m_T(x-x_1)$

$y+3=-4(x-3)$

$y+3=-4x+12$

$y=-4x+9$

b Equation of the line perpendicular to the tangent where $m_P=\frac{1}{4}$.

$y-y_1=m_P(x-x_1)$

$y+3=\frac{1}{4}(x-3)$

$y+3=\frac{1}{4}x-\frac{3}{4}$

$y=\frac{1}{4}x-\frac{15}{4}$

10 Tangent and perpendicular intersect where:

$y=-2x+5$....................(1)

$y=\frac{1}{2}x+\frac{5}{2}$....................(2)

(1) = (2)

$-2x+5=\frac{1}{2}x+\frac{5}{2}$

$-4x+10=x+5$

$5=5x$

$x=1$

Substitute $x=1$ into (1): $y=-2(1)+5=3$

General rule for the parabola is $y = ax^2 + bx + c$ and general rule for the derivative is $\frac{dy}{dx} = 2ax + b$.

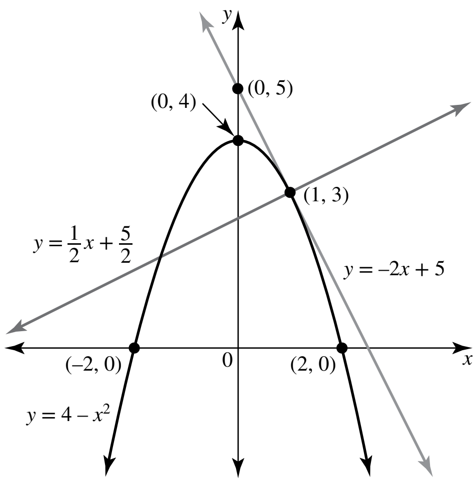
When $x = 0$, $y = 4$ so $c = 4$ or $y = ax^2 + bx + 4$.

When $x = 1$, $y = 3$ so $3 = a + b + 4$ or $-1 = a + b$.

When $x = 0$, $\frac{dy}{dx} = 0$ so $0 = b$.

If $b = 0$ then for $-1 = a + b$, $a = -1$.

Hence $y = 4 - x^2$ is the equation of the parabola.



11 a $y = \frac{3}{4x^5} - \frac{1}{2x} + 4 = \frac{3}{4}x^{-5} - \frac{1}{2}x^{-1} + 4$

$\frac{dy}{dx} = -\frac{15}{4}x^{-6} + \frac{1}{2}x^{-2} = -\frac{15}{4x^6} + \frac{1}{2x^2}$

b $f(x) = \frac{10x - 2x^3 + 1}{x^4} = 10x^{-3} - 2x^{-1} + x^{-4}$

$f'(x) = -30x^{-4} + 2x^{-2} - 4x^{-5} = -\frac{30}{x^4} + \frac{2}{x^2} - \frac{4}{x^5}$

c $y = \sqrt{x} - \frac{1}{2\sqrt{x}} = x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

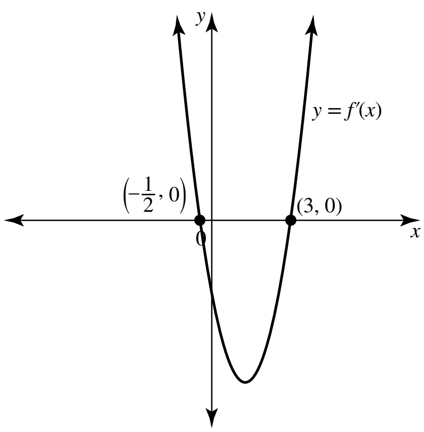
$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} + \frac{1}{4x^{\frac{3}{2}}}$

d $f(x) = \frac{(3-x)^3}{2x} = \frac{27 - 27x + 9x^2 - x^3}{2x} = \frac{27}{2}x^{-1} - \frac{27}{2} + \frac{9}{2}x - \frac{1}{2}x^2$

$f'(x) = -\frac{27}{2}x^{-2} + \frac{9}{2} - x = -\frac{27}{2x^2} - x + \frac{9}{2}$

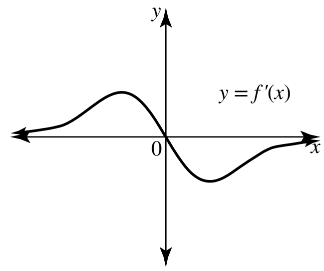
12 a i Domain = R

ii



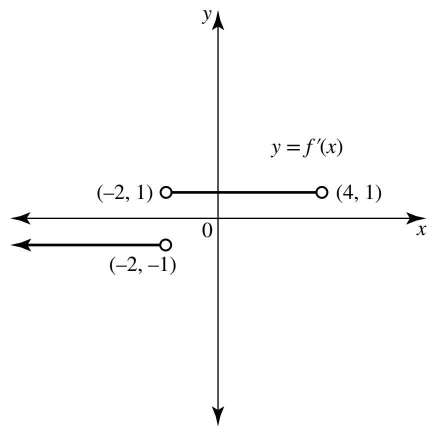
b i Domain = R

ii



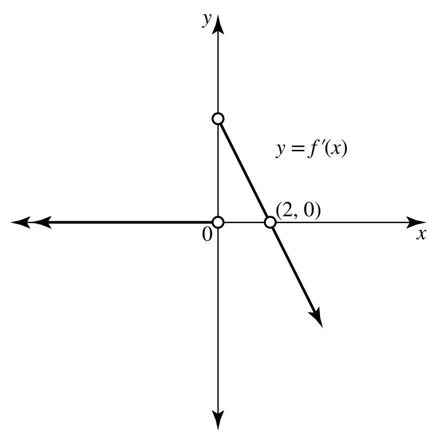
c i Domain = $(-\infty, 4) \setminus \{-2\}$

ii



d i Domain = $R \setminus \{0\}$

ii



13 a Gradient of secant = $\frac{f(x+h) - f(x)}{h}$

$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 5(x+h) + 6 - x^2 - 5x - 6}{h}$

$\frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2xh + h^2 + 5x + 5h + 6 - x^2 - 5x - 6}{h}$

$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 + 5h}{h}$

$\frac{f(x+h) - f(x)}{h} = 2x + h + 5$ prov $h \neq 0$

$\frac{f(x+h) - f(x)}{h} = 2(-1) + 1 + 5 = 4$ where $x = -1$ and $h = 1$

b Gradient of secant $= \dfrac{f(x+h)-f(x)}{h}$

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^3(x+h-3)-x^3(x-3)}{h}$$
$$\frac{f(x+h)-f(x)}{h}=\frac{\left(x^3+3x^2h+3xh^2+h^3\right)(x+h-3)-x^4+3x^3}{h}$$
$$\frac{f(x+h)-f(x)}{h}=\frac{x^4+x^3h-3x^3+3x^3h+3x^2h^2-9x^2h+3x^2h^2+3xh^3-9xh^2+xh^3+h^4-3h^3-x^4+3x^3}{h}$$
$$\frac{f(x+h)-f(x)}{h}=\frac{4x^3h+6x^2h^2-9x^2h+4xh^3-9xh^2-3h^3+h^4}{h}$$
$$\frac{f(x+h)-f(x)}{h}=4x^3+\ 6x^2h-9x^2+4xh^2-9xh-3h^2+h^3,\ h\neq 0$$
$$\frac{f(x+h)-f(x)}{h}=4(2)^3+6(2)^2(2)-9(2)^2+4(2)(2)^2-9(2)(2)-3(2)^2+(2)^3 \text{ where } x=2 \text{ and } h=2$$
$$\frac{f(x+h)-f(x)}{h}=36$$

c Gradient of secant $= \dfrac{f(x+h)-f(x)}{h}$

$$\frac{f(x+h)-f(x)}{h}=\frac{-(x+h)^3+x^3}{h}$$
$$\frac{f(x+h)-f(x)}{h}=\frac{-\left(x^3+3x^2h+3xh^2+h^3\right)+x^3}{h}$$
$$\frac{f(x+h)-f(x)}{h}=\frac{-x^3-3x^2h-3xh^2-h^3+x^3}{h}$$
$$\frac{f(x+h)-f(x)}{h}=-3x^2-3xh-h^2,\ h\neq 0$$
$$\frac{f(x+h)-f(x)}{h}=-3(3)^2-3(-3)(3)-(3)^2$$
$$=-27+27-9$$
$$=-9 \quad \text{where } x=-3 \text{ and } h=3$$

14 a $f(x)=12-x$

$f(x+h)=12-(x+h)=12-x-h$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{12-x-h-12+x}{h}$$
$$f'(x)=\lim_{h\to 0}(-1),\ h\neq 0$$
$$f'(x)=-1$$

b $f(x)=3x^2-2x-21$

$f(x+h)=3(x+h)^2-2(x+h)-21=3x^2+3xh+3h^2-2x-2h-21$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{3x^2+6xh+3h^2-2x-2h-21-3x^2+2x+21}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{6xh+3h^2-2h}{h}$$
$$f'(x)=\lim_{h\to 0}(6x+3h-2),\ h\neq 0$$
$$f'(x)=6x-2$$

15 a $f(x)=x^2-3$

$f'(x)=2x$

$f'(2)=2(2)=4$

b $f(x)=(3-x)(x-4)=-x^2+7x-12$

$f'(x)=-2x+7$

$f'(1)=-2(1)+7=5$

c $f(x)=(x-2)^3=x^3-3(x)^2(2)+3(x)(2)^2-(2)^3=x^3-6x^2+12x-8$

$f'(x)=3x^2-12x+12$

$f'(4)=3(4)^2-12(4)+12=12$

d $f(x)=\sqrt{x}-\frac{3}{x}+2x=x^{\frac{1}{2}}-3x^{-1}+2x$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}+3x^{-2}+2=\frac{1}{2\sqrt{x}}+\frac{3}{x^2}+2$

$f'(4)=\frac{1}{2\sqrt{4}}+\frac{3}{4^2}+2=\frac{1}{4}+\frac{3}{16}+2=\frac{4}{16}+\frac{3}{16}+\frac{32}{16}=\frac{39}{16}$

16 a $f(x)=(x+1)(x+3)=x^2+4x+3$

$f'(x)=2x+4$

$f'(-5)=2(-5)+4=-6$

When $x=-5,\ y=(-5+1)(-5+3)=8$

Equation of tangent which passes through the point $(x_1,\ y_1)\equiv(-5,8)$ where

$m_T=-6$ is

$y-y_1=m_T(x-x_1)$

$y-8=-6(x+5)$

$y-8=-6x-30$

$y=-6x-22$

b $f(x)=8-x^3$

$f'(x)=-3x^2$

$f'(a)=-3a^2$

When $x=a,\ y=8-a^3$

Equation of tangent which passes through the point $(x_1,\ y_1)\equiv(a,8-a^3)$ where

$m_T=3a^2$ is

$y-y_1=m_T(x-x_1)$

$y-(8-a^3)=-3a^2(x-a)$

$y-8+a^3=-3a^2x+3a^3$

$y=-3a^2x+2a^3+8$

c $f(x)=2\sqrt{x}-5=2x^{\frac{1}{2}}-5$

$f'(x)=x^{-\frac{1}{2}}=\frac{1}{\sqrt{x}}$

$f'(3)=\frac{1}{\sqrt{3}}$

When $x=3,\ y=2\sqrt{3}-5$

Equation of tangent which passes through the point $(x_1,y_1)\equiv(3,2\sqrt{3}-5)$ where $m_T=\frac{1}{\sqrt{3}}$ is

$y-y_1=m_T(x-x_1)$

$y-(2\sqrt{3}-5)=\frac{1}{\sqrt{3}}(x-3)$

$y-2\sqrt{3}+5=\frac{1}{\sqrt{3}}x-\frac{3}{\sqrt{3}}$

$y=\frac{1}{\sqrt{3}}x-\frac{3}{\sqrt{3}}+2\sqrt{3}-5$

$y=\frac{\sqrt{3}}{3}x+\sqrt{3}-5$

d $f(x)=-\frac{2}{x}-4x=-2x^{-1}-4x$

$f'(x)=2x^{-2}-4=\frac{2}{x^2}-4$

$f'(-2)=\frac{2}{(-2)^2}-4=-\frac{7}{2}$

When $x=-2,\ y=-\frac{2}{(-2)}-4(-2)=9$

Equation of tangent which passes through the point $(x_1,\ y_1)\equiv(-2,9)$ where $m_T=-\frac{7}{2}$ is

$y-y_1=m_T(x-x_1)$

$y-9=-\frac{7}{2}(x+2)$

$y-9=-\frac{7}{2}x-7$

$y=-\frac{7}{2}x+2$

17 a Equation of perpendicular line which passes through the point $(x_1,\ y_1)\equiv(-5,8)$

where $m_P=\frac{1}{6}$ is

$y-y_1=m_P(x-x_1)$

$y-8=\frac{1}{6}(x+5)$

$y-8=\frac{1}{6}x+\frac{5}{6}$

$y=\frac{1}{6}x+\frac{5}{6}+\frac{48}{6}$

$y=\frac{1}{6}x+\frac{53}{6}$

b Equation of perpendicular line which passes through the point $(x_{1,}y_1)\equiv(a,8-a^3)$

where $m_P=-\frac{1}{3a^2}$ is

$y-y_1=m_P(x-x_1)$

$y-(8-a^3)=-\frac{1}{3a^2}(x-a)$

$y-8+a^3=-\frac{1}{3a^2}x+\frac{1}{3a}$

$y=-\frac{1}{3a^2}x+\frac{1}{3a}+8-a^3$

c Equation of perpendicular line which passes through the point $(x_{1,}y_1)\equiv(3,2\sqrt{3}-5)$ where $m_P=-\sqrt{3}$ is

$y-y_1=m_P(x-x_1)$

$y-(2\sqrt{3}-5)=-\sqrt{3}(x-3)$

$y-2\sqrt{3}+5=-\sqrt{3}x+3\sqrt{3}$

$y=-\sqrt{3}x+5\sqrt{3}-5$

d Equation of perpendicular line which passes through the point $(x_{1,}y_1)\equiv(-2,9)$

where $m_P=-\frac{2}{7}$ is

$y-y_1=m_P(x-x_1)$

$y-9=\frac{2}{7}(x+2)$

$y-9=\frac{2}{7}x+\frac{4}{7}$

$y=\frac{2}{7}x+\frac{4}{7}+\frac{63}{7}$

$y=\frac{2}{7}x+\frac{67}{7}$

18 **a** Any line parallel to $y=3x+4$ has gradient of 3.

$f(x)=-(x-2)^2+3=-x^2+4x-4+3=-x^2+4x-1$

$f'(x)=-2x+4$

$3=-2x+4$

$\frac{1}{2}=x$

When $x=\frac{1}{2},\ y=-\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)-1=-\frac{1}{4}+1=\frac{3}{4}$

Equation of tangent which passes through the point $\left(x_1,y_1\right)\equiv\left(\frac{1}{2},\frac{3}{4}\right)$ where

$m_T=3$ is

$y-y_1=m_T\left(x-x_1\right)$

$y-\frac{3}{4}=3\left(x-\frac{1}{2}\right)$

$y-\frac{3}{4}=3x-\frac{3}{2}$

$y=3x-\frac{6}{4}+\frac{3}{4}$

$y=3x-\frac{3}{4}$

b A line perpendicular to $2y-2=-4x$ or $y=-2x+1$ will have a gradient of $\frac{1}{2}$.

$f(x)=-\frac{2}{x^2}+1=-2x^{-2}+1$

$f'(x)=4x^{-3}=\frac{4}{x^3}$

$\frac{1}{2}=\frac{4}{x^3}$

$2=\frac{x^3}{4}$

$8=x^3$

$2=x$

When $x=2,\ y=-\frac{2}{2^2}+1=-\frac{1}{2}+1=\frac{1}{2}$

Equation of perpendicular line which passes through the point $\left(x_1,y_1\right)\equiv\left(2,\frac{1}{2}\right)$

with $m_P=\frac{1}{2}$ is

$y-y_1=m_P\left(x-x_1\right)$

$y-\frac{1}{2}=\frac{1}{2}(x-2)$

$y-\frac{1}{2}=\frac{1}{2}x-1$

$y=\frac{1}{2}x-\frac{1}{2}$

19 Tangent to parabola at $x=4$ is $y=-x+6$. When $x=4,\ y=-4+6=2$ So parabola
$y=ax^2+bx+c$ passes through the points (4, 2), (0, −10) and (2, 0).
At (0, −10), $-10=a(0)^2+b(0)+c$ so $c=-10$.
At (2, 0), $0=a(2)^2+2b-10$ or $4a+2b=10\ \Rightarrow 2a+b=5$..........(1)
At (4, 2), $2=a(4)^2+4b-10$ or $16a+4b=12\ \Rightarrow 4a+b=3$............(2)
(2) − (1) $\quad 2a=-2\ \Rightarrow a=-1$
Substitute $a=-1$ into (1) $\quad 2(-1)+b=5$ so $b=7$
Equation of parabola is $y=-x^2+7x-10$.

20 Tangent to a cubic function at $x=2$ has the rule $y=11x-16$. If $x=2$ then $y=11(2)-16=6$. The cubic passes through the points (0, 0), (−1, 0) and (2, 6).
General rule for the cubic is $y=ax^3+bx^2+cx+d$

$d=0$

$\therefore y=ax^3+bx^2+cx$

$(-1, 0) \Rightarrow 0 = -a + b - c$...........(1)

$(2, 6) \Rightarrow 6 = 8a + 4b + 2c$...........(2)

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

$m = 11, x = 2$

$11 = 12a + 4b + c$..............(3)

Using CAS to solve:

$a = 1, b = 0, c = -1$

$\therefore y = x^3 - x$

21 Line perpendicular to $y = 2\sqrt{x} = 2x^{\frac{1}{2}}$ has the equation $y = -2x + m$. Thus the gradient of the curve is $\frac{1}{2}$.

If $y = 2x^{\frac{1}{2}}$ then $\frac{dy}{dx} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$.

If $\frac{1}{\sqrt{x}} = \frac{1}{2}$ then $\sqrt{x} = 2 \Rightarrow x = 4$

When $x = 4$, $y = 2\sqrt{4} = 4$.

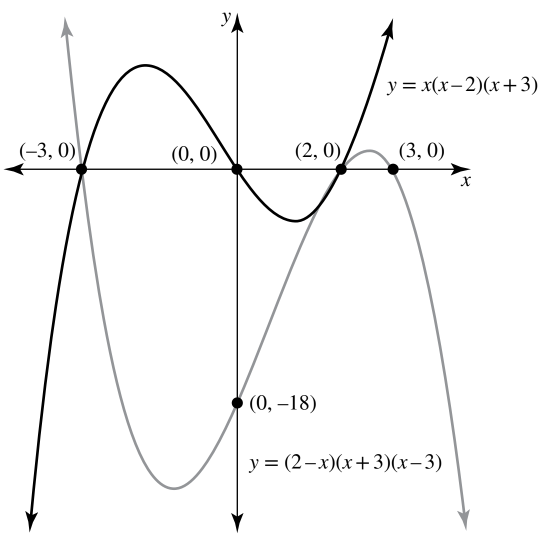
Perpendicular line passes through the point (4, 4). Thus

$4 = -2(4) + m$

$4 = -8 + m$

$m = 12$

22 a



b Point of intersection is $\left(\frac{3}{2}, -\frac{27}{8}\right)$.

c $y = x(x-2)(x+3) = x^3 + x^2 - 6x$

$\frac{dy}{dx} = 3x^2 + 2x - 6$

When

$$x = \frac{3}{2},\ \frac{dy}{dx} = 3\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right) - 6 = \frac{27}{4} - 3 = \frac{27}{4} - \frac{12}{4} = \frac{15}{4}$$

Equation of tangent which passes through the point $(x_1, y_1) \equiv \left(\frac{3}{2}, -\frac{27}{8}\right)$ which has a gradient of $m_T = \frac{15}{4}$ is

$$y - y_1 = m_T(x - x_1)$$

$$y + \frac{27}{8} = \frac{15}{4}\left(x - \frac{3}{2}\right)$$

$$y + \frac{27}{8} = \frac{15}{4}x - \frac{45}{8}$$

$$y = \frac{15}{4}x - \frac{45}{8} - \frac{27}{8}$$

$$y = \frac{15}{4}x - 9$$

Equation of perpendicular line which passes through the point $(x_1, y_1) \equiv \left(\frac{3}{2}, -\frac{27}{8}\right)$ which has a gradient of $m_P = -\frac{4}{15}$ is

$$y - y_1 = m_P(x - x_1)$$

$$y + \frac{27}{8} = -\frac{4}{15}\left(x + \frac{3}{2}\right)$$

$$y + \frac{27}{8} = -\frac{4}{15}x + \frac{2}{5}$$

$$y = -\frac{4}{15}x + \frac{2}{5} - \frac{27}{8}$$

$$y = -\frac{4}{15}x - \frac{119}{40}$$

Exercise 5.3 — Differentiation of exponential functions

1 a $y = e^{-\frac{1}{3}x}$

$$\frac{dy}{dx} = -\frac{1}{3}e^{-\frac{1}{3}x}$$

b $y = 3x^4 - e^{-2x^2}$

$$\frac{dy}{dx} = 12x^3 + e^{-2x^2}(-4x) = 12x^3 + 4xe^{-2x^2}$$

c $y = \frac{4e^x - e^{-x} + 2}{3e^{3x}}$

$$y = \frac{4}{3}e^{-2x} - \frac{1}{3}e^{-4x} + \frac{2}{3}e^{-3x}$$

$$\frac{dy}{dx} = -\frac{8}{3}e^{-2x} + \frac{4}{3}e^{-4x} - 2e^{-3x}$$

d $y = \left(e^{2x} - 3\right)^2$

$$y = e^{4x} - 6e^{2x} + 9$$

$$\frac{dy}{dx} = 4e^{4x} - 12e^{2x}$$

2 $f(x) = \frac{1}{2}e^{3x} + e^{-x}$

$$f'(x) = \frac{3}{2}e^{3x} - e^{-x}$$

$$f'(0) = \frac{3}{2}e^0 - e^0 = \frac{1}{2}$$

3 $y = e^{2x}$

$$\frac{dy}{dx} = 2e^{2x}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 2e^{2(0)} = 2$$

When $x = 0$, $y = e^0 = 1$

Equation of tangent which passes through $\left(x_1, y_1\right) \equiv (0,1)$ with $m_T = 2$ is

$y - y_1 = m_T\left(x - x_1\right)$

$y - 1 = 2x$

$y = 2x + 1$

4 $y = e^{-3x} + 4$

$\frac{dy}{dx} = -3e^{-3x}$

$\frac{dy}{dx}_{x=0} = -3e^{-3(0)} = -3$

When $x = 0,\ y = e^0 + 4 = 5$

Equation of tangent which passes through $\left(x_1, y_1\right) \equiv (0,5)$ with $m_T = -3$ is

$y - y_1 = m_T\left(x - x_1\right)$

$y - 5 = -3x$

$y = -3x + 5$

Equation of perpendicular line which passes through $\left(x_1, y_1\right) \equiv (0,5)$ with $m_P = \frac{1}{3}$ is

$y - y_1 = m_P\left(x - x_1\right)$

$y - 5 = \frac{1}{3}x$

$y = \frac{1}{3}x + 5$

5 **a** $\frac{d}{dx}\left(5e^{-4x}\right) = -20e^{-4x}$

b $\frac{d}{dx}\left(e^{-\frac{1}{2}x} + \frac{1}{3}x^3\right) = -\frac{1}{2}e^{-\frac{1}{2}x} + x^2$

c $\frac{d}{dx}\left(4e^{3x} - \frac{1}{2}e^{6\sqrt{x}} - 3e^{-3x+2}\right) = 12e^{3x} - \frac{3e^{6\sqrt{x}}}{2\sqrt{x}} + 9e^{-3x+2}$

d $\frac{d}{dx}\left(\frac{e^{5x} - e^{-x} + 2}{e^{2x}}\right) = \frac{d}{dx}\left(e^{3x} - e^{-3x} + 2e^{-2x}\right) = 3e^{3x} + 3e^{-3x} - 4e^{-2x}$

e $\frac{d}{dx}\left(\frac{e^x\left(2 - e^{-3x}\right)}{e^{-x}}\right) = \frac{d}{dx}\left(2e^{2x} - e^{-x}\right) = 4e^{2x} + e^{-x}$

f $\frac{d}{dx}\left(\left(e^{2x} + 3\right)\left(e^{-x} - 1\right)\right) = \frac{d}{dx}\left(e^x - e^{2x} + 3e^{-x} = 3\right) = e^x - 2e^{2x} - 3e^{-x}$

6 **a** $y = 2e^{-x}$

$\frac{dy}{dx} = -2e^{-x}$

$\frac{dy}{dx}_{x=0} = -2e^0 = -2$

b $y = \frac{4}{e^{2x}} = 4e^{-2x}$

$\frac{dy}{dx} = -8e^{-2x}$

$\frac{dy}{dx}_{x=\frac{1}{2}} = -8e^{-2\left(\frac{1}{2}\right)}$

$= -8e^{-1}$

$= -\frac{8}{e}$

c $y = \frac{1}{2}e^{3x}$

$\frac{dy}{dx} = \frac{3}{2}e^{3x}$

$\frac{dy}{dx}_{x=\frac{1}{3}} = \frac{3}{2}e^{3\left(\frac{1}{3}\right)} = \frac{3}{2}e$

d $y = 2x - e^x$

$\frac{dy}{dx} = 2 - e^x$

$\frac{dy}{dx}_{x=0} = 2 - e^0 = 2 - 1 = 1$

7 $y = e^{-2x}$

$m_T = \frac{dy}{dx} = -2e^{-2x}$

When $x = -\frac{1}{2}$, $m_T = -2e^{-2\left(-\frac{1}{2}\right)} = -2e$

When $x = -\frac{1}{2}$, $y = e^{-2\left(-\frac{1}{2}\right)} = e$

Equation of tangent with $m_T = -2e$ which passes through the point

$(x_1, y_1) = \left(-\frac{1}{2}, e\right)$ is given by

$y - y_1 = m_T(x - x_1)$

$y - e = -2e\left(x + \frac{1}{2}\right)$

$y = -2ex - e + e$

$y = -2ex$

8 $y = e^{-3x} - 2$

$m_T = \frac{dy}{dx} = -3e^{-3x}$

When $x = 0$, $m_T = -3e^{-3(0)} = -3$ and $m_P = \frac{1}{3}$

When $x = 0$, $y = e^{-3(0)} - 2 = 1 - 2 = -1$

Equation of tangent with $m_T = -3$ which passes through the point $(x_1, y_1) = (0, -1)$ is given by

$y - y_1 = m_T(x - x_1)$

$y + 1 = -3(x - 0)$

$y = -3x - 1$

Equation of perpendicular line with $m_P = \frac{1}{3}$ which passes through the point

$(x_1, y_1) = (0, -1)$ is given by

$y - y_1 = m_P(x - x_1)$

$y + 1 = \frac{1}{3}(x - 0)$

$y = \frac{1}{3}x - 1$

9 $y = e^{\sqrt{x}} + 1$

$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

$x = 3$, $\frac{dy}{dx} = \frac{e^{\sqrt{3}}}{2\sqrt{3}}$

When $x = 3$, $y = e^{\sqrt{3}} + 1$

Equation of tangent which passes through the point

$(x_1, y_1) \equiv \left(3, e^{\sqrt{3}} + 1\right)$ with

$m_T = \frac{e^{\sqrt{3}}}{2\sqrt{3}}$ is

$y - y_1 = m_T(x - x_1)$

$y - e^{\sqrt{3}} - 1 = \frac{e^{\sqrt{3}}}{2\sqrt{3}}(x - 3)$

$y = \frac{e^{\sqrt{3}}}{2\sqrt{3}}x - \frac{3e^{\sqrt{3}}}{2\sqrt{3}} + e^{\sqrt{3}} + 1$

Equation of perpendicular line which passes through $(3, e^{\sqrt{3}} + 1)$

with $m_P = -\frac{2\sqrt{3}}{e^{\sqrt{3}}}$ is

$y - y_1 = m_P(x - x_1)$

$y - e^{\sqrt{3}} - 1 = -\frac{2\sqrt{3}}{e^{\sqrt{3}}}(x - 3)$

$y = \frac{-2\sqrt{3}}{e^{\sqrt{3}}}x + \frac{6\sqrt{3}}{e^{\sqrt{3}}} + e^{\sqrt{3}} + 1$

10 **a** $f(x) = e^{-2x+3} - 2e$

$f'(x) = -2e^{-2x+3}$

$f'(-2) = -2e^{-2(-2)+3} = -2e^7$

b $-2e^{-2x+3} = -2$

$e^{-2x+3} = 1$

$e^{-2x+3} = e^0$

Equating indices

$-2x + 3 = 0$

$-2x = -3$

$x = \frac{3}{2}$

11 **a** $f(x) = \frac{e^{3x} + 2}{e^x} = e^{2x} + 2e^{-x}$

$f'(x) = 2e^{2x} - 2e^{-x}$

$f'(1) = 2e^2 - 2e^{-1} = 2e^2 - \frac{2}{e}$

$2e^{2x} - 2e^{-x} = 0$

$e^{2x} - e^{-x} = 0$

$e^{-x}\left(e^{3x} - 1\right) = 0$

b $e^{-x} = 0$ or $e^{3x} - 1 = 0$

Not possible $e^{3x} = 1$

$e^{3x} = e^0$

$x = 0$

12 $y = e^{x^2+3x-4}$

Let $u = x^2 + 3x - 4$ so $\frac{du}{dx} = 2x + 3$

$y = e^u$ so $\frac{dy}{du} = e^u$

By the Chain rule:

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (2x + 3)e^u$

$\frac{dy}{dx} = (2x + 3)e^{x^2+3x-4}$

$\frac{dy}{dx}_{x=1} = (2(1) + 3)e^{1^2+3(1)-4}$

$\frac{dy}{dx}_{x=1} = 5e^0 = 5$

When $x = 1$, $y = e^{1^2+3(1)-4} = e^0 = 1$

Equation of tangent which passes through $(x_1, y_1) \equiv (1, 1)$ where $m_T = 5$ is

$y - y_1 = m_T(x - x_1)$

$y - 1 = 5(x - 1)$

$y = 5x - 4$

13 $f(x) = Ae^{x} + Be^{-3x}$

$f'(x) = Ae^{x} - 3Be^{-3x}$

When $f'(x) = 0$.

$Ae^{x} - 3Be^{-3x} = 0$

$e^{-3x}(Ae^{4x} - 3B) = 0$

$e^{-3x} \neq 0$, no real solution

$0 = Ae^{4x} - 3B$

So, $3B = Ae^{4x}$

$e^{4x} = \frac{3B}{A}$

14 a $A = A_0 e^{-0.69t}$

When $t = 0,\ A = 2$

$2 = A_0 e^{-0.69(0)}$

$2 = A_0$

Thus $A = 2e^{-0.69t}$

b $\frac{dA}{dt} = -0.69 \times 2e^{-0.69t}$

$\frac{dA}{dt} = -1.38e^{-0.69t}$

When $t = 0,\ \frac{dA}{dt} = -1.38e^{-0.69(0)} = -1.38$

15 $f(x) = 3^{2x-4}$

$f'(x) = 2\log_e(3) \times 3^{2x-4}$

$f'(2) = 2\log_e(3) \times 3^{2(2)-4}$

$f'(2) = 2\log_e(3)$

16 a $y = 2e^{-2x} + 1$............(1)

$y = x^3 - 3x$...............(2)

Point of intersection occurs where (1) = (2)

$2e^{-2x} + 1 = x^3 - 3x$

$x \simeq 1.89$

$y = (1.8854)^3 - 3(1.8854)$

$y \simeq 1.05$

Therefore POI = (1.89, 1.05)

b $m_T = \frac{dy}{dx} = 3x^2 - 3$

When $x = 1.8854,\ m_T = 3(1.8854)^2 - 3 \simeq 7.66$.

Exercise 5.4 — Applications of exponential functions

1 a $M = Ae^{-0.00152t}$

When $t = 0,\ M = 20$

$20 = Ae^{-0.00152(0)}$

$20 = A$

Thus $M = 20e^{-0.00152t}$

b $\frac{dM}{dt} = -0.00152(20)e^{-0.00152t}$

$\frac{dM}{dt} = -0.0304e^{-0.00152t}$

c When $t = 30$ then

$\frac{dM}{dt} = -0.0304e^{-0.00152(30)}$

$\frac{dM}{dt} = -0.0291$

Rate of decay is 0.0291 g/year.

2 a $I = I_0 e^{-0.0022d}$

When $d = 315$ then

$I = I_0 e^{-0.0022(315)}$

$I = 0.5I_0$

b $\frac{dI}{dd} = -0.0022I_0 e^{-0.0022d}$

When $d = 315$ then

$\frac{dI}{dd} = -0.0022I_0 e^{-0.0022(315)}$

$\frac{dI}{dd} = -0.0022I_0 \times 0.05$

$\frac{dI}{dd} = -0.0011I_0$

Intensity is decreasing at a rate of $0.0011I_0$.

3 a $f(x) = e^{2x} + qe^{x} + 3$

When $x = 0,\ f(0) = 0$, so

$e^{2(0)} + qe^{0} + 3 = 0$

$1 + q + 3 = 0$

$q + 4 = 0$

$q = -4$

b $f(x) = e^{2x} - 4e^{x} + 3$

Graph cuts the x axis where $y = 0$.

$e^{2x} - 4e^{x} + 3 = 0$

$\left(e^{x}\right)^2 - 4\left(e^{x}\right) + 3 = 0$

$(e^{x} - 3)(e^{x} - 1) = 0$

$e^{x} - 3 = 0$ or $e^{x} - 1 = 0$

$e^{x} = 3$ $\quad$ $e^{x} = 1$

$x = \log_e 3$ $\quad$ $e^{x} = e^{0}$

$x = 0$

Thus $(m, 0) = (\log_e 3, 0)$.

c $f'(x) = 2e^{2x} - 4e^{x}$

d $f'(0) = 2e^{2(0)} - 4e^{0} = 2 - 4 = -2$

4 a $f(x) = e^{-2x} + ze^{-x} + 2$

$f(0) = 0$

$e^{-2(0)} + ze^{0} + 2 = 0$

$1 + z + 2 = 0$

$z + 3 = 0$

$z = -3$

b $y = f(x) = e^{-2x} - 3e^{-x} + 2$

Graph cuts the x axis where $y = 0$

$e^{-2x} - 3e^{-x} + 2 = 0$

$\left(e^{-x}\right)^2 - 3\left(e^{-x}\right) + 2 = 0$

$(e^{-x} - 1)(e^{-x} - 2) = 0$

$e^{-x} - 1 = 0$ or $e^{-x} - 2 = 0$

$e^{-x} = 1$ $\quad$ $e^{-x} = 2$

$e^{-x} = e^{0}$ $\quad$ $-x = \log_e 2$

$x = 0$ $\quad$ $x = -\log_e 2$

Thus $(n, 0) = (-\log_e(2), 0)$.

c $f(x) = e^{-2x} - 3e^{-x} + 2$

$f'(x) = -2e^{-2x} + 3e^{-x}$

d $f'(0) = -2e^{-2(0)} + 3e^{0} = -2 + 3 = 1$

5 a $A = A_0 e^{-kt}$

When $t = 0,\ A = 120$

$120 = A_0 e^{-k(0)}$

$120 = A_0$

Thus $A = 120e^{-kt}$

b $\frac{dA}{dt} = -120ke^{-kt}$

$\frac{dA}{dt} = -k \times 120e^{-kt}$

$\frac{dA}{dt} = -k \times A$

$\frac{dA}{dt} \alpha A$

c $A = 120e^{-kt}$

$90 = 120e^{-2k}$

$\frac{3}{4} = e^{-2k}$

$-2k = \log_e\left(\frac{3}{4}\right)$

$k = -\frac{1}{2}\log_e\left(\frac{3}{4}\right)$

$k = \frac{1}{2}\log_e\left(\frac{4}{3}\right)$

d $A = 120e^{kt}$, $k = \frac{1}{2}\log_e\left(\frac{4}{3}\right)$

$\frac{dA}{dt} = -120(0.139)e^{kt}$

$\frac{dA}{dt} = -16.68e^{kt}$

When $t = 5$

$\frac{dA}{dt} = -16.68e^{k(5)} \simeq -8.408$ units/min

Therefore the gas is decomposing at a rate of 8.408 units/min.

e As $t \to \infty, A \to 0$. Technically the graph approaches the line $A = 0$ (asymptotic behavior, so never reaches $A = 0$ exactly) however, the value of A would be so small, that in effect, after a long period of time, there is no gas left.

6 a $L = L_0e^{0.599t}$

When $t = 0,\ L = 11$

$11 = L_0e^{0.599(0)}$

$11 = L_0$

So $L = 11e^{0.599t}$

b $\frac{dL}{dt} = 0.599 \times 11e^{0.599t}$

$\frac{dL}{dt} = 6.589e^{0.599t}$

c When $t = 3$ then

$\frac{dL}{dt} = 6.589e^{0.599(3)} = 39.742$ mm/month

Therefore the temperature is decreasing at a rate of 1.531°C/min.

7 a When $t = 0,\ T = 95 - 20 = 75$

$75 = T_0e^{-z(0)}$

$75 = T_0$

So $T = 75e^{-zt}$

b $T = 75e^{-0.034t}$

$\frac{dT}{dt} = -0.034 \times 75e^{-0.034t}$

$\frac{dT}{dt} = -2.55e^{-0.034t}$

When $t = 15$ then

$\frac{dT}{dt} = -2.55e^{-0.034(15)} = -1.531$°C/min

Therefore the temperature is decreasing at a rate of 1.531°C/min.

Temperature is falling at rate of 1.531°C/min

8 a $P = P_0e^{0.016t}$

When $t = 0,\ P = 8.2$ million

$8.2 = P_0e^{0.016(0)}$

$8.2 = P_0$

Thus $P = 8.2e^{0.016t}$

When 2015, $t = 2015 - 1950 = 65$

$P = 8.2e^{0.016(65)} = 23.2$ million

b $20 = 8.2e^{0.016(t)}$

Solve for t using CAS:

$t = 55.72$

Therefore September, 2005

c $\frac{dP}{dt} = 0.016 \times 8.2e^{0.016t}$

$\frac{dP}{dt} = 0.1312e^{0.016t}$

When 2000, $t = 2000 - 1950 = 50$

$\frac{dP}{dt} = 0.1312e^{0.016(50)} = 0.29\,199$

Change in population is 0.29 million/year.

d $\frac{dP}{dt} = 0.016 \times 8.2e^{0.016t}$

$\frac{dP}{dt} = 0.1312e^{0.016t}$

$0.1312e^{0.016t} > 0.4$

Solve using CAS:

$0.1312e^{0.016t} > 0.4$

$t > 69.67$

Therefore in the year 2019.

9 a $P = P_0e^{-kh}$

When $h = 0.5,\ P = 66.7 \to 66.7 = P_0e^{-0.5k}$

When $h = 1.5,\ P = 52.3 \to 52.3 = P_0e^{-1.5k}$

Solve using CAS: $P_0 = 75.32$ cm of mercury, $k = 0.24$

So $P = 75.32e^{-0.24h}$

b $P = 75.32e^{-0.24h}$

$\frac{dP}{dh} = -0.24 \times 75.32e^{-0.24h}$

$\frac{dP}{dh} = -18.08e^{-0.24h}$

When $h = 5$

$\frac{dP}{dh} = -18.08e^{-0.24(5)} = -5.45$ cm of mercury/km

The rate is falling at 5.45 cm of mercury/km

10 a $h = 0.295(e^x + e^{-x})$

Height of post at $x = 0.6$, so

$h = 0.295(e^{0.6} + e^{-0.6})$

$h = 0.6994$ m

Height of chain at lowest point $x = 0$

$h = 0.295(e^0 + e^0) = 0.295(2) = 0.59$ m

Amount of sag is 0.6994–0.59 = 0.1094 metres or 10.94 centimetres.

b $h = 0.295(e^x + e^{-x})$

$\frac{dh}{dx} = 0.295(e^x - e^{-x})$

When $x = 0.6$

$\frac{dh}{dx} = 0.295(e^{0.6} - e^{-0.6})$

$\frac{dh}{dx} = 0.3756 \simeq 0.4$ metres

If θ is the angle the chain makes with the post then

$\tan(\theta) = 0.3756$

$\theta = \tan^{-1}(0.3756)$

$\theta = 20.6°$

11 a Graph cuts the y axis where $x = 0$.

$f(0) = e^0 - 0.5e^{2(0)} = 1 - 0.5 = 0.5$

Therefore (0, 0.5)

b $f'(x) = -e^{-x} + e^{-2x}$

c Max TP occurs where $f'(x) = 0$.

$-e^{-x} + e^{-2x} = 0$

$e^{-x}(e^{-x} - 1) = 0$

$e^{-x} - 1 = 0$ as $e^{-x} > 0$ for all x

$e^{-x} = 1$

$e^{-x} = e^0$

$x = 0$

Maximum TP at (0, 0.5).

d $f'(x) = -e^{-x} + e^{-2x}$

When $x = -\log_e(2)$

$f'(-\log_e(2)) = -e^{\log_e(2)} + e^{-2(-\log_e(2))}$

$= -2 + e^{\log_e(2^2)}$

$= -2 + 4$

$= 2$

Let θ be angle graph makes with the positive direction of the x axis.

$\tan(\theta) = 2$

$\theta = \tan^{-1}(2)$

$\theta = 63.4°$

e $m_T = f'(x) = -e^{-x} + e^{-2x}$

$m_T = f'(1) = -e^{-1} + e^{-2(1)} = -0.2325$

$f(1) = e^{-1} - 0.5e^{-2(1)} = 0.3002$

Equation of tangent with $m_T = -0.2325$ which passes through the point

$(x_1, y_1) = (1, 0.3002)$ is given by

$y - y_1 = m_T(x - x_1)$

$y - 0.3002 = -0.2325(x - 1)$

$y - 0.3002 = -0.2325x + 0.2325$

$y = -0.2325x + 0.5327$

f $m_N = \frac{1}{0.2325} = 4.3011$

Equation of perpendicular line with $m_P = 4.3011$ which passes through the point

$(x_1, y_1) = (1, 0.3002)$ is given by

$y - y_1 = m_P(x - x_1)$

$y - 0.3002 = 4.3011(x - 1)$

$y - 0.3002 = 4.3011x - 4.3011$

$y = 4.3011x - 4.0009$

12 a $T = T_0 e^{kt}$

When $t = 0,\ T = 30$

$30 = T_0 e^{k(0)}$

$30 = T_0$

Thus $T = 30e^{kt}$

b When $t = 7$ days and $k = 0.387$

$T = 30e^{0.378(7)}$

$T = 30e^{2.709}$

$T = 450\ 000$

c $\frac{dT}{dt} = 0.387 \times 30e^{0.387t}$

$\frac{dT}{dt} = 11.61e^{0.387t}$

When $t = 3$ then

$\frac{dT}{dt} = 11.61e^{0.387(3)} = 37\ 072.2$/day

d $C = C_0 e^{mt}$

When $t = 0,\ C = 450$

$450 = C_0 e^{m(0)}$

$450 = C_0$

So $C = 450e^{mt}$

e When $t = 7$ then

$C = 450 - \frac{90}{100} \times 450$

$C = 450 - 405$

$C = 45$

45 000 cane toads remain after 7 days.

f $45 = 450e^{7m}$

$0.1 = e^{7m}$

$\log_e(0.1) = 7m$

$\frac{1}{7}\log_e(0.1) = m$

$m = -\frac{1}{7}\log_e(10)$

$(m = -0.3289)$

$C = 450e^{-0.3289t}$

$\frac{dC}{dt} = -0.3289 \times 450e^{-0.3289t}$

$\frac{dC}{dt} = -148.0233e^{-0.3289t}$

When $t = 4$ then

$\frac{dC}{dt} = -148.0233e^{-0.3289(4)} = -39.710\,156$ thousand/day

Decrease of 39 711 cane toads per day.

13 a $y = Ae^{-x^2}$

When $x = 0,\ y = 5$

$5 = Ae^0$

$A = 5$

Thus $y = 5e^{-x^2}$.

b $\frac{dy}{dx} = -2x \times 5e^{-x^2}$

$\frac{dy}{dx} = -10xe^{-x^2}$

c i When $x = -0.5$, $\frac{dy}{dx} = -10(-0.5)e^{-(-0.5)^2} = 3.89$

ii When $x = 1$, $\frac{dy}{dx} = -10(1)e^{-(1)^2} = -3.68$

14 a $y = \frac{x^2 - 5}{2e^{x^2}}$

$\frac{dy}{dx} = \frac{x^3 - 6x}{e^{x^2}}$

b When $\frac{dy}{dx}=0$ then

$$\frac{x^3-6x}{e^{x^2}}=0$$
$$x^3-6x=0$$
$$x\left(x^2-6\right)=0$$
$$x=0 \quad \text{or} \quad x^2-6=0$$
$$x^2-\left(\sqrt{6}\right)^2=0$$
$$(x-\sqrt{6})(x+\sqrt{6})=0$$
$$x-\sqrt{6}=0 \quad \text{or} \quad x+\sqrt{6}=0$$
$$x=\sqrt{6} \qquad x=-\sqrt{6}$$

When $x=\pm\sqrt{6}$; $y=\frac{\left(\pm\sqrt{6}\right)^2-5}{2e^{\left(\pm\sqrt{6}\right)^2}}=\frac{6-5}{2e^6}=\frac{1}{2e^6}$

Co-ordinates are $\left(\pm\sqrt{6},\frac{1}{2e^6}\right)$

c When $x=1.5$, $\frac{dy}{dx}=\frac{(1.5)^3-6(1.5)}{e^{(1.5)^2}}=-0.593$

Exercise 5.5 — Differentiation of trigonometric functions

1 **a** $\frac{d}{dx}(5x+3\cos(x)+5\sin(x))=5-3\sin(x)+5\cos(x)$

b $\frac{d}{dx}\left(\sin(3x+2)-\cos(3x^2)\right)=3\cos(3x+2)+6x\sin(3x^2)$

c $\frac{d}{dx}\left(\frac{1}{3}\sin(9x)\right)=3\cos(9x)$

d $\frac{d}{dx}\left(5\tan(2x)-2x^5\right)=10\sec^2(2x)-10x^4$

e $\frac{d}{dx}\left(8\tan\left(\frac{x}{4}\right)\right)=2\sec^2\left(\frac{x}{4}\right)$

f $\frac{d}{dx}(\tan(9x°))=\frac{d}{dx}\left(\tan\left(\frac{\pi x}{20}\right)\right)=\frac{\pi}{20}\sec^2\left(\frac{\pi x}{20}\right)$

2 $\frac{\sin(x)\cos^2(2x)-\sin(x)}{\sin(x)\sin(2x)}$

$=\frac{\cos^2(2x)-1}{\sin(2x)},\ \sin(x)\neq 0$

$=\frac{-\sin^2(2x)}{\sin(2x)}$

$=-\sin(2x),\ \sin(2x)\neq 0$

Thus

$\frac{d}{dx}\left(\frac{\sin(x)\cos^2(2x)-\sin(x)}{\sin(x)\sin(2x)}\right)=\frac{d}{dx}(-\sin(2x))=-2\cos(2x)$

3 $y=-\cos(x)$

$m_T=\frac{dy}{dx}=\sin(x)$

When $x=\frac{\pi}{2}$; $m_T=\sin\left(\frac{\pi}{2}\right)=1$

When $x=\frac{\pi}{2}$; $y=-\cos\left(\frac{\pi}{2}\right)=0$

Equation of tangent with $m_T=1$ which passes through the point

$\left(x_1,y_1\right)=\left(\frac{\pi}{2},0\right)$ is given by

$$y-y_1=m_T(x-x_1)$$
$$y-0=1\left(x-\frac{\pi}{2}\right)$$
$$y=x-\frac{\pi}{2}$$

4 $y=\tan(2x)$

$m_T=\frac{dy}{dx}=2\sec^2(2x)$

When $x=-\frac{\pi}{8}$; $m_T=\frac{2}{\cos^2\left(-\frac{\pi}{4}\right)}=\frac{2}{\left(\frac{1}{\sqrt{2}}\right)^2}=2\div\frac{1}{2}=4$

When $x=-\frac{\pi}{8}$; $y=\tan\left(-\frac{\pi}{4}\right)=-1$

Equation of tangent with $m_T=4$ which passes through the point

$\left(x_1,y_1\right)=\left(-\frac{\pi}{8},-1\right)$ is given by

$$y-y_1=m_T(x-x_1)$$
$$y+1=4\left(x+\frac{\pi}{8}\right)$$
$$y=4x+\frac{\pi}{2}-1$$

5 **a** $y=2\cos(3x)$

$\frac{dy}{dx}=-6\sin(3x)$

b $y=\cos(x°)$

$y=\cos\left(\frac{\pi x}{180}\right)$

$\frac{dy}{dx}=-\frac{\pi}{180}\sin\left(\frac{\pi x}{180}\right)$

c $y=3\cos\left(\frac{\pi}{2}-x\right)$

$\frac{dy}{dx}=3\sin\left(\frac{\pi}{2}-x\right)$

d $y=-4\sin\left(\frac{x}{3}\right)$

$\frac{dy}{dx}=-\frac{4}{3}\cos\left(\frac{x}{3}\right)$

e $y=\sin(12x°)$

$y=\sin\left(\frac{\pi x}{15}\right)$

$\frac{dy}{dx}=\frac{\pi}{15}\cos\left(\frac{\pi x}{15}\right)$

f $y=2\sin\left(\frac{\pi}{2}+3x\right)$

$\frac{dy}{dx}=6\cos\left(\frac{\pi}{2}+3x\right)$

g $y=-\frac{1}{2}\tan\left(5x^2\right)$

$\frac{dy}{dx}=-\frac{1}{2}\times 10x\times\sec^2\left(5x^2\right)$

$=-5x\sec^2\left(5x^2\right)$

h $y=\tan(20x)$

$\frac{dy}{dx}=20\sec^2(20x)$

6 $y=-2\sin\left(\frac{x}{2}\right)$ for $x\in[0,2\pi]$

$\frac{dy}{dx}=-\cos\left(\frac{x}{2}\right)$

$\frac{1}{2}=-\cos\left(\frac{x}{2}\right)$

$-\frac{1}{2}=\cos\left(\frac{x}{2}\right)$ for $\frac{x}{2}\in[0,\pi]$

$\frac{1}{2}$ suggests $\frac{\pi}{3}$. Since cos is negative, the second quadrant.

$\frac{x}{2}=\pi-\frac{\pi}{3}$

$\frac{x}{2}=\frac{2\pi}{3}$

$x=\frac{4\pi}{3}$

When $x=\frac{4\pi}{3}$, $y=-2\sin\left(\frac{4\pi}{3}\times\frac{1}{2}\right)=-2\sin\left(\frac{2\pi}{3}\right)=-2\sin\left(\pi-\frac{\pi}{3}\right)=-2\sin\left(\frac{\pi}{3}\right)=-\sqrt{3}$

Point is $\left(\frac{4\pi}{3},-\sqrt{3}\right)$.

7 When $x=\frac{\pi}{6}$, $y=3\cos\left(\frac{\pi}{6}\right)=3\times\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$

$m_T=\frac{dy}{dx}=-3\sin(x)$

When $x=\frac{\pi}{6}$, $m_T=-3\sin\left(\frac{\pi}{6}\right)=-3\times\frac{1}{2}=-\frac{3}{2}$

Equation of tangent with $m_T=-\frac{3}{2}$ which passes through the point

$(x_1,y_1)=\left(\frac{\pi}{6},\frac{3\sqrt{3}}{2}\right)$ is given by

$y-y_1=m_T(x-x_1)$

$y-\frac{3\sqrt{3}}{2}=-\frac{3}{2}\left(x-\frac{\pi}{6}\right)$

$y-\frac{3\sqrt{3}}{2}=-\frac{3}{2}x+\frac{\pi}{4}$

$y=-\frac{3}{2}x+\frac{\pi}{4}+\frac{3\sqrt{3}}{2}$

8 When $x=\frac{\pi}{4}$, $y=2\tan\left(\frac{\pi}{4}\right)=2$

$m_T=\frac{dy}{dx}=2\sec^2(x)$

When $x=\frac{\pi}{4}$, $m_T=2\sec^2\left(\frac{\pi}{4}\right)=\frac{2}{\cos^2\left(\frac{\pi}{4}\right)}=2\div\frac{1}{2}=4$

Equation of tangent with $m_T=4$ which passes through the point

$(x_1,y_1)=\left(\frac{\pi}{4},2\right)$ is given by

$y-y_1=m_T(x-x_1)$

$y-2=4\left(x-\frac{\pi}{4}\right)$

$y-2=4x-\pi$

$y=4x+2-\pi$

9 $y=\sin(2x)$

$\frac{dy}{dx}=2\cos(2x)$

Let θ be the angle the curve makes with in the positive direction of the x axis.

$\tan\theta=2\cos\left(2\times\frac{\pi}{2}\right)$

$\tan\theta=2\cos(\pi)$

$\tan\theta=-2$

$\theta=\tan^{-1}(-2)$

$\theta=116.6°$

10 a $y=\sin(3x)$

When $x=\frac{2\pi}{3},\ y=\sin\left(3\times\frac{2\pi}{3}\right)=\sin(2\pi)=0$

$m_T=\frac{dy}{dx}=3\cos(3x)$

When

$x=\frac{2\pi}{3},\ m_T=3\cos\left(3\times\frac{2\pi}{3}\right)=3\cos(2\pi)=3$ and $m_P=-\frac{1}{3}$

Equation of tangent with $m_T=3$ which passes through the point

$(x_1,y_1)=\left(\frac{2\pi}{3},0\right)$ is given by

$$y-y_1=m_T(x-x_1)$$
$$y-0=3\left(x-\frac{2\pi}{3}\right)$$
$$y=3x-2\pi$$

Equation of perpendicular line with $m_P=-\frac{1}{3}$ which passes through the point

$(x_1,y_1)=\left(\frac{2\pi}{3},0\right)$ is given by

$$y-y_1=m_P(x-x_1)$$
$$y-0=-\frac{1}{3}\left(x-\frac{2\pi}{3}\right)$$
$$y=-\frac{1}{3}x+\frac{2\pi}{9}$$

b $y=\cos\left(\frac{x}{2}\right)$

When $x=\pi,\ y=\cos\left(\frac{\pi}{2}\right)=0$

$m_T=\frac{dy}{dx}=-\frac{1}{2}\sin\left(\frac{x}{2}\right)$

When $x=\pi,\ m_T=-\frac{1}{2}\sin\left(\frac{\pi}{2}\right)=-\frac{1}{2}$ and $m_N=2$

Equation of tangent with $m_T=-\frac{1}{2}$ which passes through the point

$(x_1,y_1)=(\pi,0)$ is given by

$$y-y_1=m_T(x-x_1)$$
$$y-0=-\frac{1}{2}(x-\pi)$$
$$y=-\frac{1}{2}x+\frac{\pi}{2}$$

Equation of perpendicular line with $m_P=2$ which passes through the point

$(x_1,y_1)=(\pi,0)$ is given by

$$y-y_1=m_P(x-x_1)$$
$$y-0=2(x-\pi)$$
$$y=2x-2\pi$$

11 a $f(x)=\sin(x)-\cos(x)$

$f(0)=\sin(0)-\cos(0)=-1$

b
$$f(x)=0$$
$$\sin(x)-\cos(x)=0$$
$$\sin(x)=\cos(x)$$
$$\tan(x)=1$$

1 suggests $\frac{\pi}{4}$. Since tan is positive 1st and 3rd quadrants.

$$x=\frac{\pi}{4},\pi+\frac{\pi}{4}$$
$$x=\frac{\pi}{4},\frac{5\pi}{4}$$

c $f'(x)=\cos(x)+\sin(x)$

d
$$f'(x)=0$$
$$\cos(x)+\sin(x)=0$$
$$\sin(x)=-\cos(x)$$
$$\tan(x)=-1$$

1 suggests $\frac{\pi}{4}$. Since tan is negative 2nd and 4th quadrants.

$$x=\pi-\frac{\pi}{4},2\pi-\frac{\pi}{4}$$
$$x=\frac{3\pi}{4},\frac{7\pi}{4}$$

12 a $f(x)=\sqrt{3}\cos(x)+\sin(x)$

$f(0)=\sqrt{3}\cos(0)+\sin(0)=\sqrt{3}$

b
$$f(x)=0$$
$$\sqrt{3}\cos(x)+\sin(x)=0$$
$$\sin(x)=-\sqrt{3}\cos(x)$$
$$\tan(x)=-\sqrt{3}$$

$\sqrt{3}$ suggests $\frac{\pi}{3}$. Since tan is negative 2nd and 4th quadrants.

$$x=-\frac{\pi}{3},\ \pi-\frac{\pi}{3}$$
$$x=-\frac{\pi}{3},\ \frac{2\pi}{3}$$

c $f'(x)=-\sqrt{3}\sin(x)+\cos(x)$

d
$$f'(x)=0$$
$$-\sqrt{3}\sin(x)+\cos(x)=0$$
$$\cos(x)=\sqrt{3}\sin(x)$$
$$1=\sqrt{3}\tan(x)$$
$$\frac{1}{\sqrt{3}}=\tan(x)$$

$\frac{1}{\sqrt{3}}$ suggests $\frac{\pi}{6}$. Since tan is positive 1st and 3rd quadrants.

$$x=-\pi+\frac{\pi}{6},\ \frac{\pi}{6}$$
$$x=-\frac{5\pi}{6},\ \frac{\pi}{6}$$

13 a
$$\frac{\sin(x)\cos(x)+\sin^2(x)}{\sin(x)\cos(x)+\cos^2(x)}=\frac{\sin(x)(\cos(x)+\sin(x))}{\cos(x)(\sin(x)+\cos(x))}$$
$$=\frac{\sin(x)}{\cos(x)}\ \text{prov}\ \sin(x)\neq-\cos(x)$$
$$=\tan(x)$$

b $\frac{d}{dx}\left(\frac{\sin(x)\cos(x)+\sin^2(x)}{\sin(x)\cos(x)+\cos^2(x)}\right)=\frac{d}{dx}(\tan(x))=\sec^2(x)$

14 $f(x)=\sin(2x)$ so $f'(x)=2\cos(2x)$

$f(x)=\cos(2x)$ so $f'(x)=-2\sin(2x)$

When the gradients are equal

$$2\cos(2x)=-2\sin(2x)\ \text{where}\ x\in[-\pi,\pi]$$
$$\cos(2x)=-\sin(2x)\ \text{where}\ 2x\in[-2\pi,2\pi]$$
$$\frac{\cos(2x)}{\cos(2x)}=\frac{-\sin(2x)}{\cos(2x)}$$
$$1=-\tan(2x)$$
$$-1=\tan(2x)$$

1 suggests $\frac{\pi}{4}$. Since tan is negative 2nd and 4th quadrants.

$$2x=-\pi-\frac{\pi}{4},-\frac{\pi}{4},\pi-\frac{\pi}{4}\ \text{and}\ 2\pi-\frac{\pi}{4}$$
$$2x=-\frac{5\pi}{4},-\frac{\pi}{4},\frac{3\pi}{4}\ \text{and}\ \frac{7\pi}{4}$$
$$x=-\frac{5\pi}{8},-\frac{\pi}{8},\frac{3\pi}{8}\ \text{and}\ \frac{7\pi}{8}$$

15 On CAS, solve $f'(x)=0,\ -\frac{\pi}{2}\le x\le\frac{\pi}{2}$.

$x=-0.524, 0.524$

Points where the gradient is zero are (–0.524, 0.342) and (0.524, –0.342)

16 On CAS, solve $f'(x)=0,\ 0\le x\le\frac{\pi}{2}$.

$x=0.243, 0.804$

Points where the gradient is zero are (0.243, 1.232) and (0.804, 0.863)

Exercise 5.6 — Applications of trigonometric functions

1 a $A=\frac{1}{2}ab\sin(c)$

$A=\frac{1}{2}(6)(7)\sin(\theta)$

$A=21\sin(\theta)$ as required

b $\frac{dA}{d\theta}=21\cos(\theta)$

c When $\theta=\frac{\pi}{3}$; $A=21\cos\left(\frac{\pi}{3}\right)=21\times\frac{1}{2}=10.5$

Rate of change is 10.5 cm^2/radian

2 a $BD=a\sin(\theta)$ and $CD=a\cos(\theta)$

b Length of sleepers required is

$=2a\sin(\theta)+2+a+2+a\cos(\theta)$

$=2a\sin(\theta)+a\cos(\theta)+a+4$

c $\frac{dL}{d\theta}=2a\cos(\theta)-a\sin(\theta)$

d $a=2\Rightarrow L=4\sin(\theta)+2\cos(\theta)+6$

Using CAS:

$\frac{dL}{d\theta}=0, \theta=1.1^c$

3 a $L(t)=2\sin(\pi t)+10$

When $t=0$; $L(0)=2\sin(0)+10=10$

b $\frac{dL}{dt}=2\pi\cos(\pi t)$

c When $t=1$ second then $\frac{dL}{dt}=2\pi\cos(\pi)=-2\pi$ cm/s.

4 a Period of function is $2\pi\div\frac{\pi}{6}=12$ hours

b Low tide occurs when $\sin\left(\frac{\pi t}{6}\right)=-1$ so

$H_{LOWTIDE}=1.5+0.5(-1)=1$ m.

$1.5+0.5\sin\left(\frac{\pi t}{6}\right)=1$

$0.5\sin\left(\frac{\pi t}{6}\right)=-0.5$

$\sin\left(\frac{\pi t}{6}\right)=-1$

1 suggests $\frac{\pi}{2}$. Since sin is negative 3rd quadrant.

$\frac{\pi t}{6}=\pi+\frac{\pi}{2}$

$\frac{\pi t}{6}=\frac{3\pi}{2}$

$t=\frac{3\pi}{2}\times\frac{6}{\pi}=9$ or 3 pm

Low tide = 1 metre at 3 pm

c $\frac{dH}{dt}=\frac{\pi}{6}\times\frac{1}{2}\cos\left(\frac{\pi t}{6}\right)=\frac{\pi}{12}\cos\left(\frac{\pi t}{6}\right)$

d When $t=7.30$ am then 1.5 hours.

$\frac{dH}{dt}=\frac{\pi}{12}\cos\left(\frac{\pi}{6}\times\frac{3}{2}\right)=\frac{\pi}{12}\cos\left(\frac{\pi}{4}\right)=\frac{\pi}{12}\times\frac{\sqrt{2}}{2}=\frac{\sqrt{2}\pi}{24}$

e $\frac{\pi}{12}\cos\left(\frac{\pi t}{6}\right)=\frac{\sqrt{2}\pi}{24}$

$\cos\left(\frac{\pi t}{6}\right)=\frac{\sqrt{2}\pi}{24}\times\frac{12}{\pi}=\frac{\sqrt{2}}{2}$

$\frac{\sqrt{2}}{2}$ suggests $\frac{\pi}{4}$. Since cos is positive then 1st and 4th quadrants.

$\frac{\pi t}{6}=\frac{\pi}{4}, 2\pi-\frac{\pi}{4}$

$\frac{\pi t}{6}=\frac{\pi}{4},\frac{7\pi}{4}$

$t=\frac{\pi}{4}\times\frac{6}{\pi},\frac{7\pi}{4}\times\frac{6}{\pi}$

$t=\frac{3}{2},\frac{21}{2}$

The second time when $t=10.5$ hours or at 4.30 pm.

5 a $2\sin(4x)+1=\frac{1}{2}$

$2\sin(4x)=-\frac{1}{2}$

$\sin(4x)=-\frac{1}{4}$

$x\simeq 0.849, 1.508$

b Max/min values occur when $f'(x)=0$

$f'(x)=8\cos(4x)$

$0=8\cos(4x)$

$0=\cos(4x)$

0 suggests $\frac{\pi}{2},\frac{3\pi}{2}$

$4x=\frac{\pi}{2},\frac{3\pi}{2}$

$x=\frac{\pi}{8},\frac{3\pi}{8}$

Maximum when $x=\frac{\pi}{8}$, $f\left(\frac{\pi}{8}\right)=2\sin\left(\frac{\pi}{2}\right)+1=2+1=3$

Minimum when

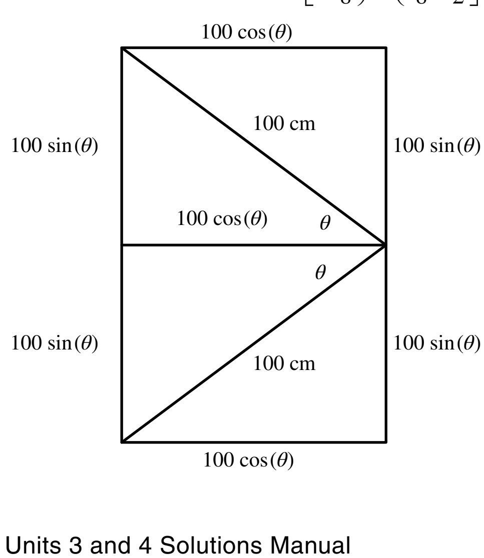
$x=\frac{3\pi}{8}$, $f\left(\frac{3\pi}{8}\right)=2\sin\left(\frac{3\pi}{2}\right)+1=-2+1=-1$

Coordinates are: $\left(\frac{\pi}{8},3\right),\left(\frac{3\pi}{8},-1\right)$

c $f'\left(\frac{\pi}{4}\right)=8\cos\left(4\times\frac{\pi}{4}\right)=-8$

d Rate of change is positive for $\left[0,\frac{\pi}{8}\right)\cup\left(\frac{3\pi}{8},\frac{\pi}{2}\right]$.

6 a



$L = 3\times100\cos(\theta)+4\times100\sin(\theta)+2\times100$
$L = 300\cos(\theta)+400\sin(\theta)+200$ as required

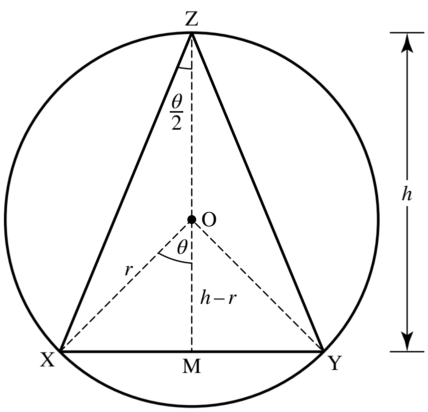
b $\dfrac{dL}{d\theta} = -300\sin(\theta)+400\cos(\theta)$

c Maximum length occurs when $\dfrac{dL}{d\theta}=0$.

$$-300\sin(\theta)+400\cos(\theta)=0$$
$$400\cos(\theta)=300\sin(\theta)$$
$$400=300\tan(\theta)$$
$$\frac{4}{3}=\tan(\theta)$$
$$\tan^{-1}\left(\frac{4}{3}\right)=\theta$$
$$\theta=0.93^c$$
$$L_{\max}=300\cos(0.9273)+400\sin(0.9273)+200$$
$$L_{\max}=700 \text{ cm}$$

Therefore the maximum length is 700 cm when $\theta = 0.93^c$.

7 a



$\angle XOY = 2\theta$ because the angle at the centre of the circle is twice the angle at the circumference.

$\angle XOM = \angle YOM = \dfrac{1}{2}\times 2\theta$

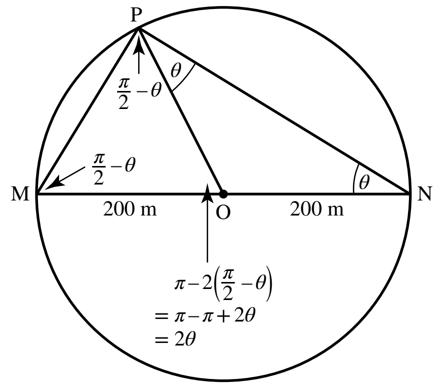
$\angle XOM = \theta$ as required

b
$$XM = r\sin(\theta)$$
$$\frac{XM}{h-r}=\tan(\theta)$$
$$\frac{r\sin(\theta)}{h-r}=\frac{\sin(\theta)}{\cos(\theta)}$$
$$\frac{r}{h-r}=\frac{1}{\cos(\theta)}$$
$$\frac{h-r}{r}=\cos(\theta)$$
$$\frac{h}{r}-1=\cos(\theta)$$
$$\frac{h}{r}=\cos(\theta)+1$$

c $r = 3$ cm, $\dfrac{h}{3}=\cos(\theta)+1$
$$h=3\cos(\theta)+3$$
$$\frac{dh}{d\theta}=-3\sin(\theta)$$

d When $\theta=\dfrac{\pi}{6}, \dfrac{dh}{d\theta}=-3\sin\left(\dfrac{\pi}{6}\right)=\dfrac{-3\sqrt{3}}{2}$.

8 a



Distance ÷ Time = Velocity
Distance = Velocity × Time
Distance ÷ Velocity = Time
So $d(PM)=400\cos(\theta)$.

$$T_{\text{obstacles}}=\frac{400\cos(\theta)}{2}$$
$$T_{\text{obstacles}}=200\cos(\theta)$$
$$d(PM)=200\times2\theta$$
$$d(PM)=400\theta$$
$$T_{\text{hurdles}}=\frac{400\theta}{5}=80\theta$$
$$T_{\text{total}}=T_{\text{obstacles}}+T_{\text{hurdles}}$$
$$T_{\text{total}}=200\cos(\theta)+80\theta$$
$$T_{\text{total}}=40(5\cos(\theta)+2\theta) \text{ as required}$$

b
$$T=40(5\cos(\theta)+2\theta)$$
$$\frac{dT}{d\theta}=40(-5\sin(\theta)+2)$$
$$0=40(-5\sin(\theta)+2)$$
$$0=-5\sin(\theta)+2$$
$$5\sin(\theta)=2$$
$$\sin(\theta)=\frac{2}{5}$$
$$\theta=\sin^{-1}\left(\frac{2}{5}\right)\quad 0<\theta<\frac{\pi}{2}$$
$$\theta=0.4115$$

c
$$T_{\max}=40(5\cos(0.4115)+2(0.4115))$$
$$T_{\max}=216.2244 \text{ seconds}$$
$$T_{\max}=3 \text{ mins } 36 \text{ seconds}$$

9 a $h=a\cos(nt)+c$

Amplitude = 50 so $a=50$

Period = 1 second so $1=\dfrac{2\pi}{n}$ and $n=2\pi$

Vertical translation is 50 so $c=50$

Thus $h=50\cos(2\pi t)+50$

b $\dfrac{dh}{dt}=-100\pi\sin(2\pi t)$

c When $t=0.25$ seconds

$\dfrac{dh}{dt}=-100\pi\sin(2\pi\times0.25)=-100\pi$ mm/sec

10 a $h=5-3.5\cos\left(\dfrac{\pi t}{30}\right)$

When $t=0$; $h=5-3.5\cos(0)=5-3.5=1.5$ m

b $h_{max} = 5 - 3.5(-1) = 8.5$ m

c period $= \dfrac{2\pi}{\frac{\pi}{30}} = 60$ s

Therefore 1 rotation takes 60 seconds

d Solve $5 - 3.5\cos\left(\frac{\pi t}{30}\right) > 7$ for $0 \le t \le 60$.

$t = 20.808, 39.192$

Time spent above 7 m = 39.192 − 20.808 = 18.4 seconds

e $\dfrac{dh}{dt} = \dfrac{3.5\pi}{30}\sin\left(\dfrac{\pi t}{30}\right)$

$\dfrac{dh}{dt} = \dfrac{7\pi}{60}\sin\left(\dfrac{\pi t}{30}\right)$

f $\dfrac{dh}{dt} = -0.2$ m/s

$-0.2 = \dfrac{7\pi}{60}\sin\left(\dfrac{\pi t}{30}\right)$

$\dfrac{-0.2 \times 60}{7\pi} = \sin\left(\dfrac{\pi t}{30}\right)$

$-0.5456 = \sin\left(\dfrac{\pi t}{30}\right)$

0.5456 suggests 0.5772. Since sin is negative 3rd and 4th quadrants.

$\dfrac{\pi t}{30} = \pi + 0.5772,\ 2\pi - 0.5772$

$\dfrac{\pi t}{30} = 3.7188,\ 5.7060$

$t = 3.7188 \times \dfrac{30}{\pi},\ 5.7060 \times \dfrac{30}{\pi}$

$t = 35.51$ s, 54.49 seconds

11 a $y = \dfrac{7}{2}\cos\left(\dfrac{\pi x}{20}\right) + \dfrac{5}{2} \quad 0 \le x \le 20$

$y_{max} = \dfrac{7}{2} \times 1 + \dfrac{5}{2} = 6$ m

b $\dfrac{dy}{dx} = -\dfrac{7\pi}{40}\sin\left(\dfrac{\pi x}{20}\right)$

c i When $x = 5$; $\dfrac{dy}{dx} = -\dfrac{7\pi}{40}\sin\left(\dfrac{\pi}{4}\right) = -0.3888$

ii When $x = 10$; $\dfrac{dy}{dx} = -\dfrac{7\pi}{40}\sin\left(\dfrac{\pi}{2}\right) = -0.5498$

d i When $y = 0$ then

$\dfrac{7}{2}\cos\left(\dfrac{\pi x}{20}\right) + \dfrac{5}{2} = 0$

$7\cos\left(\dfrac{\pi x}{20}\right) + 5 = 0$

$7\cos\left(\dfrac{\pi x}{20}\right) = -5$

$\cos\left(\dfrac{\pi x}{20}\right) = -\dfrac{5}{7}$

$\dfrac{\pi x}{20} = \cos^{-1}\left(-\dfrac{5}{7}\right)$

$\dfrac{\pi x}{20} = 2.3664$

$x = \dfrac{2.3664 \times 20}{\pi}$

$x = 15$ m

ii When $x = 15.0649$ then

$\dfrac{dy}{dx} = -\dfrac{7\pi}{40}\sin\left(\dfrac{\pi \times 15.0649}{20}\right)$

$\dfrac{dy}{dx} = -0.3786$

If θ is the required angle, then

$\tan(\theta) = -0.3786$

$\theta = \tan^{-1}(-0.3786)$

$\theta = 180° - 20.7343°$

$\theta = 159.27°$

12 a $h(x) = 10\cos\left(\dfrac{7x}{2}\right) - 5x + 90 \quad 0 \le x \le 4.5$

When $x = 0$; $h(0) = 10\cos(0) - 5(0) + 90 = 100$ cm

When

$x = 4.5$; $h(4.5) = 10\cos\left(\dfrac{7 \times 4.5}{2}\right) - 5(4.5) + 90 = 57.5$ cm

Therefore coordinates are (0, 100) and (4.5, 57.5)

b $h'(x) = -35\sin\left(\dfrac{7x}{2}\right) - 5$

Min value occurs when $h'(x) = 0$.

$-35\sin\left(\dfrac{7x}{2}\right) - 5 = 0$

$\sin\left(\dfrac{7x}{2}\right) = -\dfrac{1}{7}$

$\dfrac{7x}{2} = \sin^{-1}\left(-\dfrac{1}{7}\right)$

$\dfrac{1}{7}$ suggests 0.1433. Since sin is negative 3rd quadrant.

$\dfrac{7x}{2} = \pi + 0.1433$

$\dfrac{7x}{2} = 3.2849$

$x \simeq 0.94$

$x = 0.9386$, $h(0.9386) = 10\cos\left(\dfrac{7 \times 0.9386}{2}\right) - 5(0.9386) + 90$

$h(0.9386) = 75.41$ cm

Co-ordinates are (0.94, 75.41)

c When $x = 0.4$ then

$h'(0.4) = -35\sin\left(\dfrac{7 \times 0.4}{2}\right) - 5$

$h'(0.4) = -39.5$

13 a $P = -2\cos(mt) + n$

When $t = 0$, $P = 4$

$4 = -2\cos(0) + n$

$4 + 2 = n$

$n = 6$

Period:

$\dfrac{3}{2} = \dfrac{2\pi}{m}$

$3m = 4\pi$

$m = \dfrac{4\pi}{3}$

b $P = -2\cos\left(\dfrac{4\pi t}{3}\right) + 6$

$\dfrac{dP}{dt} = \dfrac{8\pi}{3}\sin\left(\dfrac{4\pi t}{3}\right)$

c When $t = 0.375$ then

$$\frac{dP}{dt} = \frac{8\pi}{3}\sin\left(\frac{4\pi \times 0.375}{3}\right) = \frac{8\pi}{3}\sin\left(\frac{\pi}{2}\right) = \frac{8\pi}{3} \text{ m/min}$$

14 a $h(x) = 2.5 - 2.5\cos\left(\frac{x}{4}\right) \quad -5 \le x \le 5$

$h(5) = 2.5 - 2.5\cos\left(\frac{5}{4}\right)$

$h(5) = 1.7117$

Maximum depth is 1.7 metres.

b $\frac{dh}{dx} = \frac{2.5}{4}\sin\left(\frac{x}{4}\right)$

$\frac{dh}{dx} = 0.625\sin\left(\frac{x}{4}\right)$

c When $x = 3$ then

$\frac{dh}{dx} = 0.625\sin\left(\frac{3}{4}\right)$

$\frac{dh}{dx} = 0.426$

d When $\frac{dh}{dx} = 0.58$ then

$0.58 = 0.625\sin\left(\frac{x}{4}\right)$

$0.928 = \sin\left(\frac{x}{4}\right) \quad \frac{-5}{4} \le \frac{x}{4} \le \frac{5}{4}$

0.928 suggests 1,1890. Since sin is positive 1st quadrant because of the domain.

$\frac{x}{4} = 1.1890$

$x = 4.756$ metres

15 a $x(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 1.5 \quad 0 \le t \le 12$

$y(t) = 2.0 - 2.0\cos\left(\frac{\pi t}{3}\right) \quad 0 \le t \le 12$

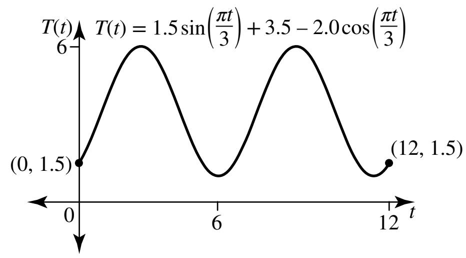
Solve $x(t) = y(t)$ on CAS.

First time emissions are equal is at $t = 1.9222$ or 1 hour and 55 minutes after 6 am. So at 7.55 am the emissions are both 2.86 units.

b i $T(t) = x(t) + y(t)$

$T(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 1.5 + 2 - 2\cos\left(\frac{\pi t}{3}\right)$

$T(t) = 3.5 + 1.5\sin\left(\frac{\pi t}{3}\right) - 2\cos\left(\frac{\pi t}{3}\right)$



ii Maximum emission of 6 units at $t = 2.3855$ or 2 hours and 23 minutes after 6 am which is 8.23 am, and at $t = 8.3855$ or 8 hours and 23 minutes after 6 am which is 2.23 pm.

Minimum emission of 1 unit at $t = 5.3855$ or 5 hours and 23 minutes after 6 am which is 11.23 am, and again at $t = 11.3855$ or 11 hours and 23 minutes after 6 am which is 5.23 pm.

c As emissions range is 1 – 6 units they lie within the required range.

16 a $N(t) = 45\sin\left(\frac{\pi t}{5}\right) - 35\cos\left(\frac{\pi t}{3}\right) + 68 \quad 0 \le t \le 15$

$N(0) = 45\sin\left(\frac{\pi(0)}{5}\right) - 35\cos\left(\frac{\pi(0)}{3}\right) + 68$

$N(0) = 33$

b Sketch the graph on CAS and find the minimum.
Coordinate is (6.4643, 1.2529)
Quietest time when $t = 6.4643$ or 6 hours and 28 minutes after 8 am which is 2.28 pm.

c When $t = 4$ (midday)

$N(0) = 45\sin\left(\frac{\pi(4)}{5}\right) - 35\cos\left(\frac{\pi(4)}{3}\right) + 68$

$N(0) = 112$

At midday there were 112 customers in line.

d Maximum number of customers in the queue between 3 pm ($t = 7$) and 7 pm ($t = 11$) is 86.

Topic 6 — Further differentiation and applications

Exercise 6.2 — The chain rule

1 a $y=\sqrt{x^2-7x+1}=\left(x^2-7x+1\right)^{\frac{1}{2}}$

$\frac{dy}{dx}=\frac{1}{2}(2x-7)\left(x^2-7x+1\right)^{-\frac{1}{2}}=\frac{2x-7}{2\sqrt{x^2-7x+1}}$

b $y=(3x^2+2x-1)^3$

$\frac{dy}{dx}=3(6x+2)(3x^2+2x-1)^2=6(3x+1)(3x^2+2x-1)^2$

2 a $y=\sin^2(x)=(\sin(x))^2$

$\frac{dy}{dx}=2\cos(x)\sin(x)$

b $y=e^{\cos(3x)}$

$\frac{dy}{dx}=-3\sin(3x)e^{\cos(3x)}$

3 $y=\sin^3(x)=(\sin(x))^3$

$\frac{dy}{dx}=3\cos(x)\sin^2(x)$

When $x=\frac{\pi}{3}$, $\frac{dy}{dx}=3\cos\left(\frac{\pi}{3}\right)\sin^2\left(\frac{\pi}{3}\right)=3\times\frac{1}{2}\times\left(\frac{\sqrt{3}}{2}\right)^2=\frac{9}{8}$

4 $y=e^{\sin^2(x)}$

$\frac{dy}{dx}=2\cos(x)\sin(x)e^{\sin^2(x)}$

When $x=\frac{\pi}{2}$, $\frac{dy}{dx}=2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)e^{\sin^2\left(\frac{\pi}{4}\right)}=2\times\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}e^{\left(\frac{\sqrt{2}}{2}\right)^2}=e^{\frac{1}{2}}=\sqrt{e}$

5 $y=\frac{1}{(2x-1)^2}=(2x-1)^{-2}$

$\frac{dy}{dx}=-2(2)(2x-1)^{-3}=-\frac{4}{(2x-1)^3}$

When $x=1$, $\frac{dy}{dx}=-\frac{4}{(2-1)^3}=-4$

When $x=1$, $y=\frac{1}{(2-1)^3}=1$

Equation of tangent with $m_T=-4$, which passes through the point $\left(x_1,\ y_1\right)\equiv(1,1)$, is given by

$y-y_1=m_T\left(x-x_1\right)$
$y-1=-4(x-1)$
$y-1=-4x+4$
$y=-4x+5$

6 $f(x)=(x-1)^3$ and $g(x)=e^x$

a $f(g(x))=f(e^x)=(e^x-1)^3$

b $h(x)=(e^x-1)^3$ and $h'(x)=3e^x(e^x-1)^2$

c At (0,0), $h'(0)=3e^0(e^0-1)^2=0$

Equation of tangent is $y=0$.

7 a $y=g(x)=3\left(x^2+1\right)^{-1}$

Let $u=x^2+1$ so $\frac{du}{dx}=2x$

Let $y=3u^{-1}$ so $\frac{dy}{du}=-3u^{-2}=-\frac{3}{u^2}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=-\frac{3}{u^2}\times 2x=-\frac{6x}{\left(x^2+1\right)^2}$

b $y=g(x)=e^{\cos(x)}$

Let $u=\cos(x)$ so $\frac{du}{dx}=-\sin(x)$

Let $y=e^u$ so $\frac{dy}{du}=e^u$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=e^u\times-\sin(x)=-\sin(x)e^{\cos(x)}$

c $y=g(x)=\sqrt{(x+1)^2+2}=\left(x^2+2x+3\right)^{\frac{1}{2}}$

Let $u=x^2+2x+3$ so $\frac{du}{dx}=2x+2$

Let $y=u^{\frac{1}{2}}$ so $\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=\frac{1}{2\sqrt{u}}\times 2(x+1)=\frac{x+1}{\sqrt{x^2+2x+3}}$

d $y=g(x)=\frac{1}{\sin^2(x)}=(\sin(x))^{-2}$

Let $u=\sin(x)$ so $\frac{du}{dx}=\cos(x)$

Let $y=u^{-2}$ so $\frac{dy}{du}=-2u^{-3}=-\frac{2}{u^3}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=-\frac{2}{u^3}\times\cos(x)=-\frac{2\cos(x)}{\sin^3(x)}$

e $y=f(x)=\sqrt{x^2-4x+5}=\left(x^2-4x+5\right)^{\frac{1}{2}}$

Let $u=x^2-4x+5$ so $\frac{du}{dx}=2x-4$

Let $y=u^{\frac{1}{2}}$ so $\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=\frac{1}{2\sqrt{u}}\times 2(x-2)=\frac{x-2}{\sqrt{x^2-4x+5}}$

f $y=f(x)=3\cos\left(x^2-1\right)$

Let $u=x^2-1$ so $\frac{du}{dx}=2x$

Let $y=3\cos(u)$ so $\frac{dy}{du}=-3\sin(u)$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=-3\sin(u)\times 2x=-6x\sin\left(x^2-1\right)$

g $y=f(x)=5e^{3x^2-1}$

Let $u=3x^2-1$ so $\frac{du}{dx}=6x$

Let $y=5e^u$ so $\frac{dy}{du}=5e^u$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=5e^u\times 6x=30xe^{3x^2-1}$

h $y=f(x)=\left(x^3-\frac{2}{x^2}\right)^{-2}=\left(x^3-2x^{-2}\right)^{-2}$

Let $u=x^3-2x^{-2}$ so $\frac{du}{dx}=3x^2+4x^{-3}=\left(3x^2+\frac{4}{x^3}\right)$

Let $y=u^{-2}$ so $\frac{dy}{du}=-2u^{-3}=-\frac{2}{u^3}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=-\frac{2}{u^3}\times\left(3x^2+\frac{4}{x^3}\right)$

$=-\frac{2}{\left(x^3-\frac{2}{x^2}\right)^3}\times\left(\frac{3x^5+4}{x^3}\right)$

$=-\frac{6x^5+8}{x^3\left(x^3-\frac{2}{x^2}\right)^3}$

i $y=f(x)=\frac{\sqrt{2-x}}{2-x}=\frac{1}{\sqrt{2-x}}=(2-x)^{-\frac{1}{2}}$

Let $u=2-x$ so $\frac{du}{dx}=-1$

Let $y=u^{-\frac{1}{2}}$ so $\frac{dy}{du}=-\frac{1}{2}u^{-\frac{3}{2}}=-\frac{1}{2u^{\frac{3}{2}}}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=-\frac{1}{2u^{\frac{3}{2}}}\times-1=\frac{1}{2(2-x)^{\frac{3}{2}}}$

j $y=f(x)=\cos^3(2x+1)=(\cos(2x+1))^3$

Let $u=\cos(2x+1)$ so $\frac{du}{dx}=-2\sin(2x+1)$

Let $y=u^3$ so $\frac{dy}{du}=3u^2$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=3u^2\times-2\sin(2x+1)=-6\sin(2x+1)\cos^2(2x+1)$

8 a $f(x)=\tan(4x+\pi)$

$f'(x)=\frac{4}{\cos^2(4x+\pi)}$

$f'\left(\frac{\pi}{4}\right)=\frac{4}{\cos^2\left(4\left(\frac{\pi}{4}\right)+\pi\right)}=\frac{4}{\cos^2(2\pi)}=4$

b $f(x)=(2-x)^{-2}$

$f'(x)=-2(-1)(2-x)^{-3}=\frac{2}{(2-x)^3}$

$f'\left(\frac{1}{2}\right)=\frac{2}{\left(2-\frac{1}{2}\right)^3}=2\div\frac{27}{8}=\frac{16}{27}$

c $f(x)=e^{2x^2}$

$f'(x)=4xe^{2x^2}$

$f'(-1)=4(-1)e^{2(-1)^2}=-4e^2$

d $f(x)=\sqrt[3]{\left(3x^2-2\right)^4}=\left(3x^2-2\right)^{\frac{4}{3}}$

$f'(x)=\frac{4}{3}\left(3x^2-2\right)^{\frac{1}{3}}\times 6x=8x\sqrt[3]{3x^2-2}$

$f'(1)=8(1)\sqrt[3]{3(1)^2-2}=8$

e $f(x)=(\cos(3x)-1)^5$

$f'(x)=5\times -3\sin(3x)(\cos(3x)-1)^4=-15\sin(3x)(\cos(3x)-1)^4$

$f'\left(\frac{\pi}{2}\right)=-15\sin\left(\frac{3\pi}{2}\right)\left(\cos\left(\frac{3\pi}{2}\right)-1\right)^4=-15(-1)\left(-1^4\right)=15$

9 $f(x)=\frac{1}{x^2}$ so $g(x)=f(f(x))=\frac{1}{\left(\frac{1}{x^2}\right)^2}=x^4$

$g'(x)=4x^3$

10 $f(x)=\sin^2(2x)=(\sin(2x))^2$

$f'(x)=2\cos(2x)\sin(2x),\qquad 0\le x\le\pi$

$0=2\cos(2x)\sin(2x),\qquad 0\le 2x\le 2\pi$

$\cos(2x)=0$ or $\sin(2x)=0$

$2x=\frac{\pi}{2},\ \frac{3\pi}{2}\qquad 2x=0,\ \pi,\ 2\pi$

$x=\frac{\pi}{4},\ \frac{3\pi}{4}\qquad x=0,\ \frac{\pi}{2},\ \pi$

$f(0)=\sin^2(2(0))=0$

$f\left(\frac{\pi}{4}\right)=\sin^2\left(2\left(\frac{\pi}{4}\right)\right)=1$

$f\left(\frac{\pi}{2}\right)=\sin^2\left(2\left(\frac{\pi}{2}\right)\right)=0$

$f\left(\frac{3\pi}{4}\right)=\sin^2\left(2\left(\frac{3\pi}{4}\right)\right)=1$

$f(\pi)=\sin^2(2(\pi))=0$

Therefore coordinates are: $(0,0),\left(\frac{\pi}{4},1\right),\left(\frac{\pi}{2},0\right),\left(\frac{3\pi}{4},1\right),(\pi,0)$

11 $z=4y^2-5$ and $y=\sin(3x)$

$z=4(\sin(3x))^2-5$

$\frac{dz}{dx}=4(2)(3\cos(3x))(\sin(3x))=24\cos(3x)\sin(3x)$

12 a $f(x)=g[\cos(x)]$

$f'(x)=-\sin(x)g'[\cos(x)]$

b $f(x)=g\left(2x^3\right)$

$f'(x)=6x^2g'\left(2x^3\right)$

c $f(x)=g\left(3e^{2x+1}\right)$

$f'(x)=6e^{2x+1}g'\left(3e^{2x+1}\right)$

d $f(x)=g\left(\sqrt{2x^2-x}\right)=g\left(\left(2x^2-x\right)^{\frac{1}{2}}\right)$

$f'(x)=\frac{1}{2}(4x-1)\left(2x^2-x\right)^{-\frac{1}{2}}g'\left(\left(2x^2-x\right)^{\frac{1}{2}}\right)=\frac{4x-1}{2\sqrt{2x^2-x}}g'\left(\sqrt{2x^2-x}\right)$

13 a $f(x)=[h(x)]^{-2}$ so $f'(x)=-2h'(x)[h(x)]^{-3}$

b $f(x)=\sin^2[h(x)]$ so $f'(x)=2h'(x)\sin[h(x)]$

c $f(x)=\sqrt[3]{2h(x)+3}$ so $f'(x)=\frac{1}{3}(2h'(x))(2h(x)+3)^{-\frac{2}{3}}=\frac{2h'(x)}{3(2h(x)+3)^{\frac{2}{3}}}$

d $f(x)=-2e^{h(x)+4}$ so $f'(x)=-2h'(x)e^{h(x)+4}$

14 a $g(x)=f(h(x))=f(2x-1)=\sqrt[3]{(2x-1)^2}$

b $g(x)=(2x-1)^{\frac{2}{3}}$ so $g'(x)=\frac{2}{3}(2)(2x-1)^{-\frac{1}{3}}=\frac{4}{3\sqrt[3]{2x-1}}$

c m_T at $x=1$: $g'(1)=\frac{4}{3\sqrt[3]{2(1)-1}}=\frac{4}{3}$

Equation of tangent with $m_T=\frac{4}{3}$ which passes through $(x_1,y_1)=(1,1)$ is given by

$$y-y_1=m_T(x-x_1)$$
$$y-1=\frac{4}{3}(x-1)$$
$$y=\frac{4}{3}x-\frac{4}{3}+1$$
$$y=\frac{4}{3}x-\frac{1}{3}$$

m_T at $x=0$: $g'(0)=\frac{4}{3\sqrt[3]{2(0)-1}}=-\frac{4}{3}$

Equation of tangent with $m_T=-\frac{4}{3}$ which passes through $(x_1,y_1)=(0,1)$ is given by

$$y-y_1=m_T(x-x_1)$$
$$y-1=-\frac{4}{3}(x-0)$$
$$y=-\frac{4}{3}x+1$$

d Tangents intersect where

$y=\frac{4}{3}x-\frac{1}{3}$...............(1)

$y=-\frac{4}{3}x+1$..............(2)

(1) = (2)

$$\frac{4}{3}x-\frac{1}{3}=-\frac{4}{3}x+1$$
$$\frac{8}{3}x=1+\frac{1}{3}$$
$$\frac{8}{3}x=\frac{4}{3}$$
$$x=\frac{4}{3}\times\frac{3}{8}=\frac{1}{2}$$

When $x=\frac{1}{2}$, $y=\frac{4}{3}\left(\frac{1}{2}\right)-\frac{1}{3}=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$

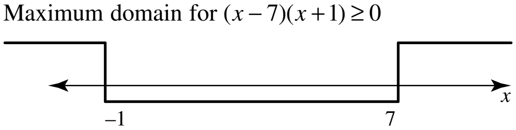
Tangents intersect at $\left(\frac{1}{2},\frac{1}{3}\right)$

15 a $h(x)=\sqrt{x^2-16}$ and $g(x)=x-3$

$$h(g(x))=\sqrt{(x-3)^2-16}$$
$$h(g(x))=\sqrt{x^2-6x+9-16}$$
$$h(g(x))=\sqrt{x^2-6x-7}$$
$$h(g(x))=\sqrt{(x-7)(x+1)}$$

If $h(g(x))=\sqrt{(x+m)(x+n)}$ then $m=-7$ and $n=1$

b Maximum domain for $(x-7)(x+1)\geq 0$



$\{x: x\leq -1\}\cup\{x: x\geq 7\}$

c $\frac{d}{dx}(h(g(x))=\frac{d}{dx}\left(\sqrt{x^2-6x-7}\right)$

$$\frac{d}{dx}(h(g(x))=\frac{d}{dx}\left(x^2-6x-7\right)^{\frac{1}{2}}$$
$$\frac{d}{dx}(h(g(x))=\frac{1}{2}(2x-6)\left(x^2-6x-7\right)^{-\frac{1}{2}}$$
$$\frac{d}{dx}(h(g(x))=\frac{x-3}{\sqrt{x^2-6x-7}}$$

d When $x=-2$, gradient

$$=\frac{-2-3}{\sqrt{(-2)^2-6(-2)-7}}=\frac{-5}{\sqrt{4+13-7}}=-\frac{5}{3}$$

16 $y=g\left(f(x)\right)=g\left(\frac{1}{x}\right)=\frac{1}{x}-\frac{1}{\left(\frac{1}{x}\right)^2}=\frac{1}{x}-x^2$

$$\frac{dy}{dx}=-x^{-2}-2x=-\frac{1}{x^2}-2x$$

Perpendicular equation is given by

$y=-x+a$ so $m_P=-1$ and $m_T=1$.

$$\frac{dy}{dx}=1$$
$$-\frac{1}{x^2}-2x=1$$
$$-1-2x^3=x^2$$
$$0=2x^3+x^2+1$$

Let $P(x)=2x^3+x^2+1$

$P(-1)=2(-1)^3+(-1)^2+1=0$

$(x+1)$ is a factor

$2x^3+x^2+1=(x+1)\left(2x^2-x+1\right)$

Quadratic can't be factorised,

$$x+1=0$$
$$x=-1$$

If $x=-1$, $y=\frac{1}{-1}-(-1)^2=-2$ and $-2=1+a\ \Rightarrow a=-3$

17 $f(x)=2\sin(x)$ and $h(x)=e^x$

a i $m(x)=f(h(x))=f\left(e^x\right)=2\sin\left(e^x\right)$

ii $n(x)=h(f(x))=h\left(2\sin(x)\right)=e^{2\sin(x)}$

b $m'(x)=2e^x\cos\left(e^x\right)$ and $n'(x)=2\cos(x)e^{2\sin(x)}$

Solve using CAS for $0\leq x\leq 3$

$$m'(x)=n'(x)$$
$$2e^x\cos\left(e^x\right)=2\cos(x)e^{2\sin(x)}$$
$$e^x\cos\left(e^x\right)=\cos(x)e^{2\sin(x)}$$
$$x=1.555,\ 2.105,\ 2.372$$

18 a $m(n(x))=m(x^2+4x-5)=3^{x^2+4x-5}$

b $\frac{d}{dx}\left(3^{x^2+4x-5}\right)=1.0986(2x+4)3^{x^2+4x-5}$

$$=2.1972(x+2)3^{x^2+4x-5}$$

When $x=1$,

$$\frac{d}{dx}\left(3^{x^2+4x-5}\right)6.5916=2.1972(1+2)3^{(1)^2+4(1)-5}$$
$$=2.1972\times 3\times 3^0=6.5916$$

Exercise 6.3 — The product rule

1 **a** $f(x)=\sin(3x)\cos(3x)$

$f'(x)=-3\sin(3x)\sin(3x)+3\cos(3x)\cos(3x)$

$f'(x)=3\cos^2(3x)-3\sin^2(3x)$

b $f(x)=x^2e^{3x}$

$f'(x)=3x^2e^{3x}+2xe^{3x}$

c $f(x)=(x^2+3x-5)e^{5x}$

$f'(x)=5(x^2+3x-5)e^{5x}+(2x+3)e^{5x}$

$f'(x)=(5x^2+17x-22)e^{5x}$

2 $f(x)=2x^4\cos(2x)$

$f'(x)=-4x^4\sin(2x)+8x^3\cos(2x)$

$$f'\left(\frac{\pi}{2}\right)=8\left(\frac{\pi}{2}\right)^3\cos\left(2\times\frac{\pi}{2}\right)-4\left(\frac{\pi}{2}\right)^4\sin\left(2\times\frac{\pi}{2}\right)$$

$$=\frac{8\pi^3}{8}(-1)$$

$$=-\pi^3$$

3 $f(x)=(x+1)\sin(x)$

$f'(x)=(x+1)\cos(x)+\sin(x)\times 1$

$f'(0)=\sin(0)+\cos(0)$

$=0+1$

$=1$

4 $y=(x^2+1)e^{3x}$

$$m_T=\frac{dy}{dx}=3(x^2+1)e^{3x}+2xe^{3x}$$

When $x=0,\ m_T=3(0+1)e^{3(0)}+2(0)e^{3(0)}=3$

When $x=0,\ y=(0+1)e^{3(0)}=1$

Equation of tangent with $m_T=3$ which passes through $(x_1,y_1)=(0,1)$ is given by

$y-y_1=m_T(x-x_1)$

$y-1=3(x-0)$

$y=3x+1$

5 Let $y=f(x)=2x^2(1-x)^3$

$f'(x)=2x^2\times -3(1-x)^2+(1-x)^3\times 4x$

$=-6x^2(1-x)^2+4x(1-x)^3$

$=-2x(1-x)^2(3x-2(1-x))$

$=-2x(1-x)^2(5x-2)$

If $f'(x)=0$

$-2x(1-x)^2(5x-2)=0$

$x=0$ or $1-x=0$ or $5x-2=0$

$$x=0,1,\frac{2}{5}$$

$f'(0)=2(0)^2(1-0)^3=0$

$f'(1)=2(1)^2(1-1)^3=0$

$$f'\left(\frac{2}{5}\right)=2\left(\frac{2}{5}\right)^2\left(1-\frac{2}{5}\right)^3$$

$$=2\times\frac{4}{25}\times\frac{27}{125}$$

$$=\frac{216}{3125}$$

Therefore the coordinates are: $(0,\ 0),(1,\ 0),\left(\frac{2}{5},\frac{216}{3125}\right)$

6 a $f(x)=e^{-\frac{x}{2}}\sin(x)$

$f(x)=0$ for $x\in[0,3\pi]$

$e^{-\frac{x}{2}}\sin(x)=0$

$\sin(x)=0$ since $e^{-\frac{x}{2}}>0$ for all x

$x=0,\ \pi,\ 2\pi,\ 3\pi$

b Max/min values occur when $f'(x)=0$.

$$f'(x)=e^{-\frac{x}{2}}\cos(x)-\frac{1}{2}e^{-\frac{x}{2}}\sin(x)$$

$$0=e^{-\frac{x}{2}}\left(\cos(x)-\frac{1}{2}\sin(x)\right)$$

$-\frac{1}{2}\sin(x)+\cos(x)=0$ since $e^{-\frac{x}{2}}>0$ for all x

$$\cos(x)=\frac{1}{2}\sin(x)$$
$$1=\frac{1}{2}\tan(x)$$
$$2=\tan(x)$$
$$x=1.11,\ 4.25,\ 7.39$$

7 a $y=x^2e^{5x}$

Let $u=x^2$ and $v=e^{5x}$ so $\frac{du}{dx}=2x$ and $\frac{dv}{dx}=5e^{5x}$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=5x^2e^{5x}+2xe^{5x}$

b $y=e^{2x+1}\tan(2x)$

Let $u=e^{2x+1}$ and $v=\tan(2x)$ so

$\frac{du}{dx}=2e^{2x+1}$ and $\frac{dv}{dx}=\frac{2}{\cos^2(2x)}$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=\frac{2e^{2x+1}}{\cos^2(2x)}+2e^{2x+1}\tan(2x)$

$\frac{dy}{dx}=\frac{2e^{2x+1}}{\cos^2(2x)}+\frac{2e^{2x+1}\sin(2x)}{\cos(2x)}$

$\frac{dy}{dx}=\frac{2e^{2x+1}+2e^{2x+1}\sin(2x)\cos(2x)}{\cos^2(2x)}$

$\frac{dy}{dx}=\frac{2e^{2x+1}(1+\sin(2x)\cos(2x))}{\cos^2(2x)}$

c $y=x^{-2}(2x+1)^3$

Let $u=x^{-2}$ and $v=(2x+1)^3$

so $\frac{du}{dx}=-2x^{-3}$ and $\frac{dv}{dx}=3(2)(2x+1)^2=6(2x+1)^2$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=6x^{-2}(2x+1)^2-2x^{-3}(2x+1)^3$

$\frac{dy}{dx}=\frac{6(2x+1)^2}{x^2}-\frac{2(2x+1)^3}{x^3}$

$\frac{dy}{dx}=\frac{6x(2x+1)^2-2(2x+1)^3}{x^3}$

$\frac{dy}{dx}=\frac{2(2x+1)^2(3x-(2x-1))}{x^3}$

$\frac{dy}{dx}=\frac{2(2x+1)^2(x-1)}{x^3}$

d $y=x\cos(x)$

Let $u=x$ and $v=\cos(x)$ so $\frac{du}{dx}=1$ and $\frac{dv}{dx}=-\sin(x)$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=-x\sin(x)+\cos(x)$

e $y=2\sqrt{x}(4-x)=2x^{\frac{1}{2}}(4-x)$

Let $u=2x^{\frac{1}{2}}$ and $v=4-x$ so $\frac{du}{dx}=x$ and $\frac{dv}{dx}=-1$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=2\sqrt{x}(-1)+\frac{4-x}{\sqrt{x}}$

$\frac{dy}{dx}=\frac{-2x+4-x}{\sqrt{x}}$

$\frac{dy}{dx}=\frac{4-3x}{\sqrt{x}}$

f $y=\sin(2x-\pi)e^{-3x}$

Let $u=\sin(2x-\pi)$ and $v=e^{-3x}$ so

$\frac{du}{dx}=2\cos(2x-\pi)$ and $\frac{dv}{dx}=-3e^{-3x}$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=-3e^{-3x}\sin(2x-\pi)+2e^{-3x}\cos(2x-\pi)$

g $y=3x^{-2}e^{x^2}$

Let $u=3x^{-2}$ and $v=e^{x^2}$ so $\frac{du}{dx}=-6x^{-3}$ and $\frac{dv}{dx}=2xe^{x^2}$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=3x^{-2}\times 2xe^{x^2}+e^{x^2}\times -6x^{-3}$

$\frac{dy}{dx}=\frac{6e^{x^2}}{x}-\frac{6e^{x^2}}{x^3}$

$\frac{dy}{dx}=\frac{6e^{x^2}\left(x^2-1\right)}{x^3}$

h $y=e^{2x}\sqrt{4x^2-1}=e^{2x}\left(4x^2-1\right)^{\frac{1}{2}}$

Let $u=e^{2x}$ and $v=\left(4x^2-1\right)^{\frac{1}{2}}$ so

$\frac{du}{dx}=2e^{2x}$ and $\frac{dv}{dx}=4x\left(4x^2-1\right)^{-\frac{1}{2}}=\frac{4x}{\sqrt{4x^2-1}}$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=\frac{4xe^{2x}}{\sqrt{4x^2-1}}+2e^{2x}\sqrt{4x^2-1}$

$\frac{dy}{dx}=\frac{4xe^{2x}+2e^{2x}\left(4x^2-1\right)}{\sqrt{4x^2-1}}$

$\frac{dy}{dx}=\frac{2e^{2x}\left(4x^2+2x-1\right)}{\sqrt{4x^2-1}}$

i $y=x^2\sin^3(2x)=x^2(\sin(2x))^3$

Let $u=x^2$ and $v=(\sin(2x))^3$ so

$\frac{du}{dx}=2x$ and $\frac{dv}{dx}=6\cos(2x)\sin^2(2x)$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=6x^2\cos(2x)\sin^2(2x)+2x\sin^3(2x)$

$\frac{dy}{dx}=2x\sin^2(2x)(3x\cos(2x)+\sin(2x))$

j $y=(x-1)^4(3-x)^{-2}$

Let $u=(x-1)^4$ and $v=(3-x)^{-2}$ so

$\frac{du}{dx}=4(x-1)^3$ and $\frac{dv}{dx}=-2(3-x)^{-3}\times-1=\frac{2}{(3-x)^3}$

$$\begin{aligned}\frac{dy}{dx}&=u\frac{dv}{dx}+v\frac{du}{dx}\\&=\frac{2(x-1)^4}{(3-x)^3}+\frac{4(x-1)^3}{(3-x)^2}\\&=\frac{2(x-1)^4+(3-x)4(x-1)^3}{(3-x)^3}\\&=\frac{2(x-1)^3(x-1+2(3-x))}{(3-x)^3}\\&=\frac{2(x-1)^3(5-x)}{(3-x)^3}\\&=\frac{2(x-1)^3(x-5)}{(x-3)^3}\end{aligned}$$

k $y=(3x-2)^2g(x)$

Let $u=(3x-2)^2$ and $v=g(x)$ so

$\frac{du}{dx}=6(3x-2)$ and $\frac{dv}{dx}=g'(x)$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=(3x-2)^2g'(x)+6(3x-2)g(x)$

$\frac{dy}{dx}=(3x-2)((3x-2)g'(x)+6g(x))$

l $y=-e^{5x}g\left(\sqrt{x}\right)$

Let $u=-e^{5x}$ and $v=g\left(\sqrt{x}\right)$ so $\frac{du}{dx}=-5e^{5x}$ and $\frac{dv}{dx}=\frac{g'(x)}{2\sqrt{x}}$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=-\frac{e^{5x}g'(x)}{2\sqrt{x}}-5e^{5x}g\left(\sqrt{x}\right)$

$\frac{dy}{dx}=-\frac{e^{5x}\left(g'(x)+10\sqrt{x}g\left(\sqrt{x}\right)\right)}{2\sqrt{x}}$

8 a $f(x)=xe^x$

$f'(x)=xe^x+e^x$

$f'(-1)=-e^{-1}+e^{-1}$
$=0$

b $f(x)=x\left(x^2+x\right)^4$

$$\begin{aligned}f'(x)&=+4x(2x+1)\left(x^2+x\right)^3+\left(x^2+x\right)^4\\&=\left(x^2+x\right)^3\left(x^2+x+8x^2+4x\right)\\&=\left(x^2+x\right)^3\left(9x^2+5x\right)\end{aligned}$$

$f'(1)=\left(1^2+1\right)^3\left(9(1)^2+5(1)\right)$
$=112$

c $f(x)=(1-x)\tan^2(x)$

$f'(x)=\frac{2(1-x)\tan(x)}{\cos^2(x)}-\tan^2(x)$

$$\begin{aligned}f'\left(\frac{\pi}{3}\right)&=\frac{2\times\left(1-\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)}{\left(\frac{1}{2}\right)^2}-\left[\tan\left(\frac{\pi}{3}\right)\right]^2\\&=\frac{2\times\left(1-\frac{\pi}{3}\right)\times\sqrt{3}}{\frac{1}{4}}-\left(\sqrt{3}\right)^2\\&=8\sqrt{3}\left(1-\frac{\pi}{3}\right)-3\end{aligned}$$

d $f(x)=\sqrt{x}\sin^2(2x^2)=x^{\frac{1}{2}}\left(\sin(2x^2)\right)^2$

$$\begin{aligned}f'(x)&=4x\sqrt{x}\cos(2x^2)\sin(2x^2)+\frac{\sin(2x^2)}{2\sqrt{x}}\\&=\frac{8x^2\cos(2x^2)\sin(2x^2)+\sin(2x^2)}{2\sqrt{x}}\end{aligned}$$

$$\begin{aligned}f'\left(\sqrt{\pi}\right)&=\frac{8\pi\cos(2\pi)\sin(2\pi)+\sin(2\pi)}{2\sqrt{\sqrt{\pi}}}\\&=\frac{8\pi(1)(0)+(0)}{2\sqrt{\sqrt{\pi}}}\\&=0\end{aligned}$$

9 $y=f(x)=x^4e^{-3x}$

Let $u=x^4$ and $v=e^{-3x}$ so $\frac{du}{dx}=4x^3$ and $\frac{dv}{dx}=-3e^{-3x}$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=-3x^4e^{-3x}+4x^3e^{-3x}$

$\frac{dy}{dx}=e^{-3x}\left(-3x^4+4x^3\right)=e^{-3x}(4x^3-3x^4)$

If $\frac{dy}{dx}=e^{-3x}(ax^3+bx^4)$ then $a=4$ and $b=-3$.

10 a $y=f(x)=(x-a)^2g(x)$

Let $u=(x-a)^2$ and $v=g(x)$ so

$\frac{du}{dx}=2(x-a)$ and $\frac{dv}{dx}=g'(x)$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=(x-a)^2g'(x)+2(x-a)g(x)$

b $f(x)=g(x)\sin(2x)$ where $g(x)=ax^2$

Let $u=ax^2$ and $v=\sin(2x)$ so $\frac{du}{dx}=2ax$ and $\frac{dv}{dx}=2\cos(2x)$

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx}=2ax^2\cos(2x)+2ax\sin(2x)$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}}=2a\left(\frac{\pi}{2}\right)^2\cos(\pi)+2a\left(\frac{\pi}{2}\right)\sin(\pi)=-3\pi$

$$\begin{aligned}-\frac{\pi^2}{2}a+0&=-3\pi\\\pi^2a&=6\pi\\a&=\frac{6}{\pi}\end{aligned}$$

11 $y = 2x\tan(2x)$

Let $u = 2x$ and $v = \tan(2x)$ so $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = \frac{2}{\cos^2(2x)}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = \frac{4x}{\cos^2(2x)} + 2\tan(2x)$

When $x = \frac{\pi}{12}$,

$\frac{dy}{dx} = \frac{4\left(\frac{\pi}{12}\right)}{\cos^2\left(\frac{\pi}{6}\right)} + 2\tan\left(\frac{\pi}{6}\right)$

$\frac{dy}{dx} = \frac{\pi}{3} \times \frac{1}{\cos^2\left(\frac{\pi}{6}\right)} + 2 \times \frac{1}{\sqrt{3}}$

$\frac{dy}{dx} = \frac{\pi}{3} \times \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{2}{\sqrt{3}}$

$\frac{dy}{dx} = \frac{\pi}{3} \times \frac{4}{3} + \frac{2\sqrt{3}}{3}$

$\frac{dy}{dx} = \frac{4\pi}{9} + \frac{2\sqrt{3}}{3} = \frac{4\pi + 6\sqrt{3}}{9}$

12 $y = xe^x$

Let $u = x$ and $v = e^x$ so $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = xe^x + e^x = e^x(x+1)$

When $x = 1$, $m_T = \frac{dy}{dx} = e^1(1+1) = 2e$ and $m_N = -\frac{1}{2e}$

When $x = 1$, $y = (1)e^1 = e$

Equation of tangent with $m_T = 2e$ which passes through the point $(x_1, y_1) = (1, e)$ is given by

$y - y_1 = m_T(x - x_1)$

$y - e = 2e(x-1)$

$y - e = 2ex - 2e$

$y = 2ex - e$

Equation of perpendicular with $m_P = -\frac{1}{2e}$ which passes through the point $(x_1, y_1) = (1, e)$ is given by

$y - y_1 = m_P(x - x_1)$

$y - e = -\frac{1}{2e}(x-1)$

$y - e = -\frac{1}{2e}x + \frac{1}{2e}$

$y = -\frac{1}{2e}x + \frac{1}{2e} + e$

$y = -\frac{1}{2e}x + \left(\frac{1+2e^2}{2e}\right)$

13 **a** $y = -\cos(x)\tan(x)$

$y = -\cos(x) \times \frac{\sin(x)}{\cos(x)}$

$y = -\sin(x)$, $\cos(x) \neq 0$

$\frac{dy}{dx} = -\cos(x)$

b $y = -\cos(x)\tan(x)$

Let $u = -\cos(x)$ and $v = \tan(x)$ so

$\frac{du}{dx} = \sin(x)$ and $\frac{dv}{dx} = \sec^2(x) = \frac{1}{\cos^2(x)}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -\cos(x) \times \frac{1}{\cos^2(x)} + \tan(x)\sin(x)$

$\frac{dy}{dx} = -\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)} \times \sin(x)$ prov $\cos(x) \neq 0$

$\frac{dy}{dx} = \frac{\sin^2(x) - 1}{\cos(x)}$

$\frac{dy}{dx} = \frac{1 - \cos^2(x) - 1}{\cos(x)}$

$\frac{dy}{dx} = -\frac{\cos(x)\cos(x)}{\cos(x)}$

$\frac{dy}{dx} = -\cos(x)$, $\quad \cos(x) \neq 0$

14 $y = e^{-x^2}(1-x)$

a Graph cuts the y axis where $x = 0$, $y = e^0(1-0) = 1$.

Graph cuts the x axis where $y = 0$

$e^{-x^2}(1-x) = 0$

$1 - x = 0$ as $e^{-x^2} > 0$ for all x

$x = 1$

Therefore, coordinates are: (0, 1) and (1, 0)

b $\frac{dy}{dx} = -e^{x^2} - 2xe^{x^2}(1-x)$

$= -e^{x^2}(1 + 2x(1-x))$

$= -e^{x^2}(1 + 2x - 2x^2)$

$\frac{dy}{dx} = e^{x^2}\left(2x^2 - 2x - 1\right)$

$0 = e^{x^2}\left(2x^2 - 2x - 1\right)$

$0 = 2x^2 - 2x - 1$ as $e^{x^2} > 0$ for all x

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$

$= \frac{2 \pm \sqrt{12}}{4}$

$x = -0.366,\ 1.366$

When $x = -0.366$, $y = e^{-\left(-0.366^2\right)}(1 + 0.366) = 1.1947$

When $x = 1.366$, $y = e^{-\left(1.366^2\right)}(1 - 1.366) = -0.057$

Therefore coordinates are: (–0.366, 1.195) and (1.366, –0.057)

c When $x = 1$, $m_T = e^{-(1)^2}\left(2(1)^2 - 2(1) - 1\right) = -\frac{1}{e}$

Equation of tangent with $m_T = -\frac{1}{e}$ which passes through $\left(x_1, y_1\right) \equiv (1, 0)$ is given by

$y - y_1 = m_T\left(x - x_1\right)$

$y - 0 = -\frac{1}{e}(x-1)$

$y = -\frac{1}{e}x + \frac{1}{e}$

d When $x = 0$, $m_T = e^{-(0)^2}\left(2(0)^2 - 2(0) - 1\right) = -1$ so $m_P = -1$

Equation of perpendicular with $m_P = 1$ which passes through $\left(x_1, y_1\right) \equiv (0, 1)$ is given by

$y - y_1 = m_P(x - x_1)$
$y - 1 = x$
$y = x + 1$

e Tangent and perpendicular intersect where

$x + 1 = -\frac{1}{e}x + \frac{1}{e}$
$x = -0.462$
$\therefore y = -0.462 + 1$
$= 0.538$
$POI = (-0.46, 0.54)$

15 a i $\frac{CD}{3} = \sin(\theta)$
$CD = 3\sin(\theta)$

ii $\frac{AD}{3} = \frac{BD}{3} = \cos(\theta)$
$AD = BD = 3\cos(\theta)$

b $S = 4 \times \frac{1}{2} \times 6\cos(\theta) \times 3\sin(\theta) + (6\cos(\theta))^2$

$S = 36\cos(\theta)\sin(\theta) + 36\cos^2(\theta)$

$S = 36\left(\cos^2(\theta) + \cos(\theta)\sin(\theta)\right)$ as required

c Let $S_1 = 36\cos^2(\theta) = 36(\cos(\theta))^2$

So $\frac{dS_1}{d\theta} = 2 \times 36 \times -\sin(\theta)\cos(\theta) = -72\sin(\theta)\cos(\theta)$

Let $S_2 = 36\cos(\theta)\sin(\theta)$

Let $u = 36\cos(\theta)$ and $v = \sin(\theta)$ so

$\frac{du}{d\theta} = -36\sin(\theta)$ and $\frac{dv}{d\theta} = \cos(\theta)$

$\frac{dS_2}{d\theta} = u\frac{dv}{d\theta} + v\frac{du}{d\theta}$

$\frac{dS_2}{d\theta} = 36\cos(\theta)\cos(\theta) - 36\sin(\theta)\sin(\theta)$

$\frac{dS_2}{d\theta} = 36(\cos^2(\theta) - \sin^2(\theta))$

$S = S_1 + S_2$

$\frac{dS}{d\theta} = \frac{dS_1}{d\theta} + \frac{dS_2}{d\theta}$

$\frac{dS}{d\theta} = -72\sin(\theta)\cos(\theta) + 36(\cos^2(\theta) - \sin^2(\theta))$

$\frac{dS}{d\theta} = -72\sin(\theta)\cos(\theta) + 36\cos^2(\theta) - 36\sin^2(\theta)$

$\frac{dS}{d\theta} = -72\sin(\theta)\cos(\theta) + 36\cos^2(\theta) - 36(1 - \cos^2(\theta))$

$\frac{dS}{d\theta} = -72\sin(\theta)\cos(\theta) + 36\cos^2(\theta) - 36 + 36\cos^2(\theta)$

$\frac{dS}{d\theta} = 72\cos^2(\theta) - 72\sin(\theta)\cos(\theta) - 36$

16 a $y = f(x) = 3x^3e^{-2x}$

Let $u = 3x^3$ and $v = e^{-2x}$ so $\frac{du}{dx} = 9x^2$ and $\frac{dv}{dx} = -2e^{-2x}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -6x^3e^{-2x} + 9x^2e^{-2x}$

$\frac{dy}{dx} = 3e^{-2x}(3x^2 - 2x^3)$

If $\frac{dy}{dx} = ae^{-2x}(bx^2 + cx^3)$ then $a = 3,\ b = 3$ and $c = -2$

b Stationary points occur where $\frac{dy}{dx} = 0$

$3e^{-2x}(3x^2 - 2x^3) = 0$
$x^2(3 - 2x) = 0$ as $e^{-2x} > 0$ for all x
$x = 0$ or $3 - 2x = 0$
$3 = 2x$
$\frac{3}{2} = x$

If $x = 0,\ y = 0$

If $x = \frac{3}{2},\ y = 3\left(\frac{3}{2}\right)^3 e^{-2\left(\frac{3}{2}\right)} = \frac{81}{8}e^{-3} = \frac{81}{8e^3}$

Stationary point (0,0) is a point of inflection and stationary point $\left(\frac{3}{2}, \frac{81}{8e^3}\right)$ is a maximum turning point.

c When $x = 1,\ y = 3(1)^3e^{-2(1)} = 3e^{-2} = \frac{3}{e^2}$

When $x = 1,\ m_T = \frac{dy}{dx} = 3e^{-2(1)}(3(1)^2 - 2(1)^3) = 3e^{-2} = \frac{3}{e^2}$

Equation of tangent with $m_T = \frac{3}{e^2}$ which passes through the point $(x_1, y_1) = \left(1, \frac{3}{e^2}\right)$ is given by

$y - y_1 = m_T(x - x_1)$

$y - \frac{3}{e^2} = \frac{3}{e^2}(x - 1)$

$y - \frac{3}{e^2} = \frac{3}{e^2}x - \frac{3}{e^2}$

$y = \frac{3}{e^2}x$

17 a When $x = -2,\ y = (4(-2)^2 - 5(-2))e^{-2} = 26e^{-2} \simeq 3.5187$ so they have made the correct decision.

b Graph cuts the x axis where $y = 0$.

$(4x^2 - 5x)e^x = 0$
$x(4x - 5) = 0$ as $e^x > 0$ for all x
$x = 0$ or $4x - 5 = 0$
$4x = 5$
$x = \frac{5}{4}$

T is the point $\left(\frac{5}{4}, 0\right)$

c $y = (4x^2 - 5x)e^x$

Let $u = 4x^{\to 2} - 5x$ and $v = e^x$ so $\frac{du}{dx} = 8x - 5$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = (4x^2 - 5x)e^x + (8x - 5)e^x$

$\frac{dy}{dx} = (4x^2 - 5x + 8x - 5)e^x$

$\frac{dy}{dx} = (4x^2 + 3x - 5)e^x$

Stationary points occur when $\frac{dy}{dx} = 0$.

$(4x^2+3x-5)e^x=0$

$4x^2+3x-5=0$ as $e^x>0$ for all x

$x=\dfrac{-3\pm\sqrt{(3)^2-4(4)(-5)}}{2(4)}$

$x=\dfrac{-3\pm\sqrt{9+80}}{8}$

$x=\dfrac{-3\pm\sqrt{89}}{8}$

Point B: When $x=\dfrac{-3+\sqrt{89}}{8}\simeq 0.804$,

$y=(4(0.804)^2-5(0.804))e^{0/804}\simeq -3.205$

B has the coordinates $(0.804,-3.205)$.

18 $y=2^x\sin(x)$

$\dfrac{dy}{dx}=2^x\cos(x)+\log_e(2)\times 2^x\sin(x)$

When $x=\dfrac{\pi}{2}$, $\dfrac{dy}{dx}=2^{\frac{\pi}{2}}\cos\left(\dfrac{\pi}{2}\right)+\log_e(2)\times 2^{\frac{\pi}{2}}\sin\left(\dfrac{\pi}{2}\right)\simeq 2.06$

Exercise 6.4 — The quotient rule

1 a $y=\dfrac{e^{2x}}{e^x+1}$

Let $u=e^{2x}$ and $v=e^x+1$

So $\dfrac{du}{dx}=2e^{2x}$ and $\dfrac{dv}{dx}=e^x$

$\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{2e^{2x}(e^x+1)-e^{2x}\times e^x}{(e^x+1)^2}$

$\dfrac{dy}{dx}=\dfrac{2e^{3x}+2e^{2x}-e^{3x}}{(e^x+1)^2}$

$\dfrac{dy}{dx}=\dfrac{e^{3x}+2e^{2x}}{(e^x+1)^2}$

b $y=\dfrac{\cos(3t)}{t^3}$

Let $u=\cos(3t)$ and $v=t^3$

So $\dfrac{du}{dt}=-3\sin(3t)$ and $\dfrac{dv}{dt}=3t^2$

$\dfrac{dy}{dt}=\dfrac{v\dfrac{du}{dt}-u\dfrac{dv}{dt}}{v^2}$

$\dfrac{dy}{dt}=\dfrac{-3t^3\sin(3t)-3t^2\cos(3t)}{t^6}$

$\dfrac{dy}{dt}=\dfrac{-3t^2(t\sin(3t)+\cos(3t))}{t^6}$

$\dfrac{dy}{dt}=\dfrac{-3(t\sin(3t)+\cos(3t))}{t^4}$, $t\neq 0$

2 $y=\dfrac{x+1}{x^2-1}$

Let $u=x+1$ and $v=x^2-1$

So $\dfrac{du}{dx}=1$ and $\dfrac{dv}{dx}=2x$

$\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$=\dfrac{(x^2-1)-2x(x+1)}{(x^2-1)^2}$

$=\dfrac{x^2-1-2x^2-2x}{(x^2-1)^2}$

$=\dfrac{-(x^2+2x+1)}{(x^2-1)^2}$

$=\dfrac{-(x+1)^2}{(x^2-1)^2}$

$=\dfrac{-(x+1)^2}{(x+1)^2(x-1)^2}$

$=\dfrac{-1}{(x-1)^2}$

3 $y=\dfrac{\sin(x)}{e^{2x}}$

Let $u=\sin(x)$ and $v=e^{2x}$

So $\dfrac{du}{dx}=\cos(x)$ and $\dfrac{dv}{dx}=2e^{2x}$

$\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{e^{2x}\cos(x)-2e^{2x}\sin(x)}{e^{4x}}$

$\dfrac{dy}{dx}=\dfrac{e^{2x}(\cos(x)-2\sin(x))}{e^{4x}}$

$\dfrac{dy}{dx}=\dfrac{\cos(x)-2\sin(x)}{e^{2x}}$

When $x=0$, $\dfrac{dy}{dx}=\dfrac{\cos(0)-2\sin(0)}{e^{2(0)}}=1$

4 $y=\dfrac{5x}{x^2+4}$

Let $u=5x$ and $v=x^2+4$

So $\dfrac{du}{dx}=5$ and $\dfrac{dv}{dx}=2x$

$\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{5(x^2+4)-5x\times 2x}{(x^2+4)^2}$

$\dfrac{dy}{dx}=\dfrac{5x^2+20-10x^2}{(x^2+4)^2}$

$\dfrac{dy}{dx}=\dfrac{20-5x^2}{(x^2+4)^2}$

$\dfrac{dy}{dx}=\dfrac{5(4-x^2)}{(x^2+4)^2}$

When $x=1$, $\dfrac{dy}{dx}=\dfrac{5(3)}{\left(1^2+4\right)^2}=\dfrac{15}{25}=\dfrac{3}{5}$

5 a $y=\dfrac{\sin(x)}{\sqrt{x}}$

Let $u=\sin(x)$ and $v=\sqrt{x}=x^{\frac{1}{2}}$ so

$\dfrac{du}{dx}=\cos(x)$ and $\dfrac{dv}{dx}=\dfrac{1}{2}x^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{x}}$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\left(\sqrt{x}\cos(x)-\frac{sin(x)}{2\sqrt{x}}\right)\div\left(\sqrt{x}\right)^2$$
$$\frac{dy}{dx}=\frac{2x\cos(x)-\sin(x)}{2\sqrt{x}}\times\frac{1}{x}$$
$$\frac{dy}{dx}=\frac{2x\cos(x)-\sin(x)}{2x\sqrt{x}}$$

b $y=\frac{\tan(2x)}{e^x}$

Let $u=\tan(2x)$ and $v=e^x$ so $\frac{du}{dx}=\frac{2}{\cos^2(2x)}$ and $\frac{dv}{dx}=e^x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\left(\frac{2e^x}{\cos^2(2x)}-e^x\tan(2x)\right)\div e^{2x}$$
$$\frac{dy}{dx}=\left(\frac{2e^x}{\cos^2(2x)}-\frac{e^x\sin(2x)}{\cos(2x)}\right)\times\frac{1}{e^{2x}}$$
$$\frac{dy}{dx}=\frac{e^x(2-\sin(2x)\cos(2x))}{\cos^2(2x)}\times\frac{1}{e^{2x}}$$
$$\frac{dy}{dx}=\frac{2-\sin(2x)\cos(2x)}{e^x\cos^2(2x)}$$

c $y=f(x)=\frac{(5-x)^2}{\sqrt{5-x}}=\frac{(5-x)^2}{(5-x)^{\frac{1}{2}}}$

Let $u=(5-x)^2$ and $v=(5-x)^{\frac{1}{2}}$ so $\frac{du}{dx}=-2(5-x)=2x-10$

and $\frac{dv}{dx}=-\frac{1}{2}(5-x)^{-\frac{1}{2}}=-\frac{1}{2\sqrt{5-x}}$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\left(\frac{-2(5-x)\sqrt{5-x}}{1}+\frac{(5-x)^2}{2\sqrt{5-x}}\right)\div(5-x)$$
$$\frac{dy}{dx}=\left(\frac{-4(5-x)^2+(5-x)^2}{2\sqrt{5-x}}\right)\div(5-x)$$
$$\frac{dy}{dx}=\frac{(5-x)^2-4(5-x)^2}{2\sqrt{5-x}(5-x)}$$
$$\frac{dy}{dx}=\frac{5-x-20+4x}{2\sqrt{5-x}}$$
$$\frac{dy}{dx}=\frac{3x-15}{2\sqrt{5-x}}$$
$$\frac{dy}{dx}=-\frac{3(5-x)}{2\sqrt{5-x}}$$
$$\frac{dy}{dx}=-\frac{3\sqrt{5-x}}{2}$$

d $y=\frac{\sin^2\left(x^2\right)}{x}$

Let $u=\left(\sin(x^2)\right)^2$ and $v=x$ so $\frac{du}{dx}=4x\cos(x)\sin(x)$ and

$\frac{dv}{dx}=1$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{4x^2\cos(x^2)\sin(x^2)-\sin^2\left(x^2\right)}{x^2}$$

e $y=\frac{3x-1}{2x^2-3}$

Let $u=3x-1$ and $v=2x^2-3$ so $\frac{du}{dx}=3$ and $\frac{dv}{dx}=4x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{3\left(2x^2-3\right)-4x(3x-1)}{\left(2x^2-3\right)^2}$$
$$\frac{dy}{dx}=\frac{6x^2-9-12x^2+4x}{\left(2x^2-3\right)^2}$$
$$\frac{dy}{dx}=\frac{-6x^2+4x-9}{\left(2x^2-3\right)^2}$$

f $y=f(x)=\frac{x-4x^2}{2\sqrt{x}}$

Let $u=x-4x^2$ and $v=2x^{\frac{1}{2}}$ so $\frac{du}{dx}=1-8x$ and

$\frac{dv}{dx}=x^{-\frac{1}{2}}=\frac{1}{\sqrt{x}}$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{2\sqrt{x}(1-8x)-\frac{x-4x^2}{\sqrt{x}}}{\left(2\sqrt{x}\right)^2}$$
$$\frac{dy}{dx}=\frac{2x(1-8x)-\left(x-4x^2\right)}{4x\sqrt{x}}$$
$$\frac{dy}{dx}=\frac{2x-16x^2-x+4x^2}{4x\sqrt{x}}$$
$$\frac{dy}{dx}=\frac{x-12x^2}{4x\sqrt{x}}$$
$$\frac{dy}{dx}=\frac{1}{4\sqrt{x}}-3\sqrt{x}$$

g $y=\frac{e^x}{\cos(2x+1)}$

Let $u=e^x$ and $v=\cos(2x+1)$ so

$\frac{du}{dx}=e^x$ and $\frac{dv}{dx}=-2\sin(2x+1)$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{e^x\cos(2x+1)+2e^x\sin(2x+1)}{\cos^2(2x+1)}$$

h $y=\frac{e^{-x}}{x-1}$

Let $u=e^{-x}$ and $v=x-1$ so $\frac{du}{dx}=-e^{-x}$ and $\frac{dv}{dx}=1$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{-e^{-x}(x-1)-e^{-x}}{(x-1)^2}$$
$$\frac{dy}{dx}=\frac{-e^{-x}x+e^{-x}-e^{-x}}{(x-1)^2}$$
$$\frac{dy}{dx}=-\frac{xe^{-x}}{(x-1)^2}$$

i $y=\frac{3\sqrt{x}}{x+2}$

Let $u=3x^{\frac{1}{2}}$ and $v=x+2$ so $\frac{du}{dx}=\frac{3}{2\sqrt{x}}$ and $\frac{dv}{dx}=1$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\left(\frac{3(x+2)}{2\sqrt{x}}-3\sqrt{x}\right)\div(x+2)^2$$
$$\frac{dy}{dx}=\frac{3(x+2)-6x}{2\sqrt{x}\,(x+2)^2}$$
$$\frac{dy}{dx}=\frac{3x+6-6x}{2\sqrt{x}\,(x+2)^2}$$
$$\frac{dy}{dx}=\frac{6-3x}{2\sqrt{x}\,(x+2)^2}$$

j $y=\frac{\cos(3x)}{\sin(3x)}$

Let $u=\cos(3x)$ and $v=\sin(3x)$ so $\frac{du}{dx}=-3\sin(3x)$ and $\frac{dv}{dx}=3\cos(3x)$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{-3\sin^2(3x)-3\cos^2(3x)}{\sin^2(3x)}$$
$$\frac{dy}{dx}=\frac{-3\left(\sin^2(3x)+\cos^2(3x)\right)}{\sin^2(3x)}$$
$$\frac{dy}{dx}=-\frac{3}{\sin^2(3x)}$$

k $y=\frac{x-2}{2x^2-x-6}$

Let $u=x-2$ and $v=2x^2-x-6$ so $\frac{du}{dx}=1$ and $\frac{dv}{dx}=4x-1$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{2x^2-x-6-(x-2)(4x-1)}{\left(2x^2-x-6\right)^2}$$
$$\frac{dy}{dx}=\frac{2x^2-x-6-4x^2+9x-2}{\left(2x^2-x-6\right)^2}$$
$$\frac{dy}{dx}=\frac{-2x^2+8x-8}{\left(2x^2-x-6\right)^2}$$
$$\frac{dy}{dx}=-\frac{2\left(x^2-4x+4\right)}{(2x+3)^2\,(x-2)^2}$$
$$\frac{dy}{dx}=-\frac{2(x-2)^2}{(2x+3)^2\,(x-2)^2}$$
$$\frac{dy}{dx}=-\frac{2}{(2x+3)^2},\quad x\neq 2$$

l $y=\frac{1-e^{2x}}{1+e^{2x}}$

Let $u=1-e^{2x}$ and $v=1+e^{2x}$ so $\frac{du}{dx}=-2e^{2x}$ and $\frac{dv}{dx}=2e^{2x}$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{-2e^{2x}\left(1+e^{2x}\right)-2e^{2x}\left(1-e^{2x}\right)}{\left(1+e^{2x}\right)^2}$$
$$\frac{dy}{dx}=\frac{-2e^{2x}-2e^{4x}-2e^{2x}+2e^{4x}}{\left(1+e^{2x}\right)^2}$$
$$\frac{dy}{dx}=\frac{-4e^{2x}}{\left(1+e^{2x}\right)^2}$$

6 a $y=f(x)=\frac{x+2}{\sin\left(g(x)\right)}$

Let $u=x+2$ and $v=\sin\left(g(x)\right)$ so $\frac{du}{dt}=1$ and

$$\frac{dv}{dt}=g'(x)\cos\left(g(x)\right)$$
$$\frac{dy}{dt}=\frac{v\frac{du}{dt}-u\frac{dv}{dt}}{v^2}$$
$$\frac{dy}{dt}=\frac{\sin\left(g(x)\right)-(x+2)g'(x)\cos\left(g(x)\right)}{\sin^2\left(g(x)\right)}$$

b $y=f(x)=\frac{g\left(e^{-2x}\right)}{e^x}$

Let $u=g\left(e^{-2x}\right)$ and $v=e^x$ so

$\frac{du}{dx}=-2e^{-2x}g'\left(e^{-2x}\right)$ and $\frac{dv}{dx}=e^x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$=\frac{e^x\times-2e^{-2x}g'\left(e^{-2x}\right)-e^xg\left(e^{-2x}\right)}{\left(e^x\right)^2}$$
$$=\frac{-2e^{-2x}g'\left(e^{-2x}\right)-g\left(e^{-2x}\right)}{e^x}$$

7 a $y=\frac{2x}{x^2+1}$

Let $u=2x$ and $v=x^2+1$ so $\frac{du}{dx}=2$ and $\frac{dv}{dx}=2x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$$
$$\frac{dy}{dx}=\frac{2x^2+2-4x^2}{(x^2+1)^2}$$
$$\frac{dy}{dx}=\frac{2-2x^2}{(x^2+1)^2}=\frac{2(1-x^2)}{(x^2+1)^2}$$

When $x=1$, $\frac{dy}{dx}=\frac{2(1-1^2)}{(1^2+1)^2}=0$

b $y=\dfrac{\sin(2x+\pi)}{\cos(2x+\pi)}$

Let $u=\sin(2x+\pi)$ and $v=\cos(2x+\pi)$ so

$\dfrac{du}{dx}=2\cos(2x+\pi)$ and $\dfrac{dv}{dx}=-2\sin(2x+\pi)$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{2\cos^2(2x+\pi)+2\sin^2(2x+\pi)}{\cos^2(2x+\pi)}$$
$$\frac{dy}{dx}=\frac{2\left(\cos^2(2x+\pi)+\sin^2(2x+\pi)\right)}{\cos^2(2x+\pi)}$$
$$\frac{dy}{dx}=\frac{2}{\cos^2(2x+\pi)}$$

When $x=\dfrac{\pi}{2},\ \dfrac{dy}{dx}=\dfrac{2}{\cos^2(2\pi)}=\dfrac{2}{1^2}=2$

c $y=\dfrac{x+1}{\sqrt{3x+1}}$

Let $u=x+1$ and $v=(3x+1)^{\frac{1}{2}}$ so $\dfrac{du}{dx}=1$ and $\dfrac{dv}{dx}=\dfrac{3}{2\sqrt{3x+1}}$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{\sqrt{3x+1}-\frac{3(x+1)}{2\sqrt{3x+1}}}{\left(\sqrt{3x+1}\right)^2}$$
$$\frac{dy}{dx}=\frac{2(3x+1)-3(x+1)}{2\sqrt{3x+1}(3x+1)}$$
$$\frac{dy}{dx}=\frac{6x+2-3x-3}{2\sqrt{3x+1}(3x+1)}$$
$$\frac{dy}{dx}=\frac{3x-1}{2\sqrt{3x+1}(3x+1)}$$

When $x=5,\ \dfrac{dy}{dx}=\dfrac{3(5)-1}{2\sqrt{3(5)+1}(3(5)+1)}=\dfrac{14}{2(4)(16)}=\dfrac{7}{64}$

d $y=\dfrac{5-x^2}{e^x}$

Let $u=5-x^2$ and $v=e^x$ so $\dfrac{du}{dx}=-2x$ and $\dfrac{dv}{dx}=e^x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{-2xe^x-e^x\left(5-x^2\right)}{\left(e^x\right)^2}$$
$$\frac{dy}{dx}=\frac{-2xe^x-5e^x+5e^xx^2}{e^{2x}}$$
$$\frac{dy}{dx}=\frac{5x^2-2x-5}{e^x}$$

When $x=0,\ \dfrac{dy}{dx}=-\dfrac{5}{e^0}=-5$

8 $y=\dfrac{2x}{(3x+1)^{\frac{3}{2}}}$

Let $u=2x$ and $v=(3x+1)^{\frac{3}{2}}$ so $\dfrac{du}{dx}=2$ and $\dfrac{dv}{dx}=\dfrac{9}{2}\sqrt{3x+1}$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{2(3x+1)^{\frac{3}{2}}-9x(3x+1)^{\frac{1}{2}}}{\left((3x+1)^{\frac{3}{2}}\right)^2}$$
$$\frac{dy}{dx}=\frac{2(3x+1)^{\frac{3}{2}}-9x(3x+1)^{\frac{1}{2}}}{(3x+1)^3}$$

When $x=1,\ \dfrac{dy}{dx}=\dfrac{2(4)^{\frac{3}{2}}-9(1)(4)^{\frac{1}{2}}}{(4)^3}=-\dfrac{1}{32}$

9 $y=\dfrac{e^x}{x^2+2}$

Let $u=e^x$ and $v=x^2+2$ so $\dfrac{du}{dx}=e^x$ and $\dfrac{dv}{dx}=2x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{e^x(x^2+2)-2xe^x}{(x^2+2)^2}$$
$$\frac{dy}{dx}=\frac{e^xx^2+2e^x-2xe^x}{(x^2+2)^2}$$
$$\frac{dy}{dx}=\frac{e^x(x^2-2x+2)}{(x^2+2)^2}$$

When $x=0,\ m_T=\dfrac{dy}{dx}=\dfrac{e^0(0^2-2(0)+2)}{(0^2+2)^2}=\dfrac{2}{4}=\dfrac{1}{2}$

When $x=0,\ y=\dfrac{e^0}{0^2+2}=\dfrac{1}{2}$

Equation of tangent with $m_T=\dfrac{1}{2}$ that passes through $(x_1y_1)=\left(0,\dfrac{1}{2}\right)$ is given by

$$y-y_1=m_T(x-x_1)$$
$$y-\frac{1}{2}=\frac{1}{2}(x-0)$$
$$y=\frac{1}{2}x+\frac{1}{2}$$

10 $\dfrac{d}{dx}\left(\dfrac{1+\cos(x)}{1-\cos(x)}\right)$

If $y=\dfrac{1+\cos(x)}{1-\cos(x)}$, let $u=1+\cos(x)$ and $v=1-\cos(x)$

$\dfrac{du}{dx}=-\sin(x)$ and $\dfrac{dv}{dx}=\sin(x)$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{(1-\cos(x))\times-\sin(x)-(1+\cos(x))\times\sin(x)}{(1-\cos(x))^2}$$
$$=\frac{-\sin(x)(1-\cos(x)+1+\cos(x)}{(1-\cos(x))^2}$$
$$=\frac{-2\sin(x)}{(-(\cos(x)-1))^2}$$
$$=\frac{-2\sin(x)}{(\cos(x)-1)^2}$$

11 $y=f(x)=\frac{\sqrt{2x-1}}{\sqrt{2x+1}}$

Let $u=\sqrt{2x-1}$ and $v=\sqrt{2x+1}$ so $\frac{du}{dx}=\frac{1}{\sqrt{2x-1}}$ and $\frac{dv}{dx}=\frac{1}{\sqrt{2x+1}}$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\left(\frac{\sqrt{2x+1}}{\sqrt{2x-1}}-\frac{\sqrt{2x-1}}{\sqrt{2x+1}}\right)\div\left(\sqrt{2x+1}\right)^2$$
$$\frac{dy}{dx}=\frac{(2x+1)-(2x-1)}{\sqrt{2x-1}\sqrt{2x+1}\,(2x+1)}$$
$$\frac{dy}{dx}=\frac{2x+1-2x-+1}{\sqrt{4x^2-1}\,(2x+1)}$$
$$\frac{dy}{dx}=\frac{2}{\sqrt{4x^2-1}\,(2x+1)}$$

If $f'(m)=\frac{2}{5\sqrt{15}}$ then

$$\left.\frac{dy}{dx}\right|_{x=m}=\frac{2}{\sqrt{4m^2-1}\,(2m+1)}=\frac{2}{5\sqrt{15}}$$

Then $2m+1=5$ or $4m^2-1=15$

$2m=4 \quad 4m^2=16$
$m=2 \quad m^2=4$
$m=2$

12 $y=\frac{e^{-3x}}{e^{2x}+1}$

Let $u=e^{-3x}$ and $v=e^{2x}+1$ so $\frac{du}{dx}=-3e^{-3x}$ and $\frac{dv}{dx}=2e^{2x}$

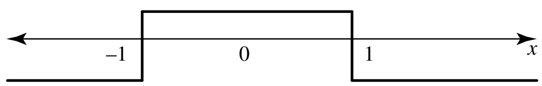
$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{-3e^{-3x}(e^{2x}+1)-2e^{2x}(e^{-3x})}{(e^{2x}+1)^2}$$
$$\frac{dy}{dx}=\frac{-3e^{-x}-3e^{-3x}-2e^{-x}}{(e^{2x}+1)^2}$$
$$\frac{dy}{dx}=\frac{-5e^{-x}-3e^{-3x}}{(e^{2x}+1)^2}$$
$$\frac{dy}{dx}=\frac{e^{-x}(-5-3e^{-2x})}{(e^{2x}+1)^2}$$

If $\frac{dy}{dx}=\frac{e^{-x}(a+be^{-2x})}{(e^{2x}+1)^2}$ then $a=-5$ and $b=-3$

13 $y=f(x)=\frac{10x}{x^2+1}$

Let $u=10x$ and $v=x^2+1$ so $\frac{du}{dx}=10$ and $\frac{dy}{dx}=2x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{10\left(x^2+1\right)-20x^2}{\left(x^2+1\right)^2}$$
$$\frac{dy}{dx}=\frac{10-10x^2}{\left(x^2+1\right)^2}$$
$$\frac{dy}{dx}<0$$
$$\frac{10-10x^2}{\left(x^2+1\right)^2}<0$$
$$10-10x^2<0$$
$$1-x^2<0$$



Thus $\{x : x<-1\}\cup\{x : x>1\}$ gives a negative gradient.

14 a $y=\frac{x-5}{x^2-5x-14}$

Function is undefined when

$x^2+5x-14=0$
$(x+7)(x-2)=0$
$x+7=0$ or $x-2=0$
$x=-7 \quad x=2$

b $y=\frac{x-5}{x^2+5x-14}$

Let $u=x-5$ and $v=x^2+5x-14$ so $\frac{du}{dx}=1$ and $\frac{dv}{dx}=2x+5$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{x^2+5x-14-(x-5)(2x+5)}{(x^2+5x-14)^2}$$
$$\frac{dy}{dx}=\frac{x^2+5x-14-(2x^2-5x-25)}{(x^2+5x-14)^2}$$
$$\frac{dy}{dx}=\frac{x^2+5x-14-2x^2+5x+25}{(x^2+5x-14)^2}$$
$$\frac{dy}{dx}=\frac{-x^2+10x+11}{(x^2+5x-14)^2}$$

When $\frac{dy}{dx}=0$

$$\frac{-x^2+10x+11}{(x^2+5x-14)^2}=0$$
$$-x^2+10x+11=0$$
$$11+10x-x^2=0$$
$$(11-x)(1+x)=0$$
$11-x=0$ or $1+x=0$
$x=11 \quad x=-1$

When $x=-1$, $y=\frac{-1-5}{(-1)^2+5(-1)-14}=\frac{-6}{-18}=\frac{1}{3}$

When

$x=11,\ y=\dfrac{11-5}{(11)^2+5(11)-14}=\dfrac{6}{121+55-14}=\dfrac{6}{162}=\dfrac{1}{27}$

Therefore coordinates are: $\left(-1,\dfrac{1}{3}\right),\left(11,\dfrac{1}{27}\right)$

c When $x=1,\ y=\dfrac{1-5}{(1)^2+5(1)-14}=\dfrac{-4}{-8}=\dfrac{1}{2}$ so $(x_1,y_1)\equiv\left(1,\dfrac{1}{2}\right)$

When $x=1,\ m_T=\dfrac{dy}{dx}=\dfrac{-(1)^2+10(1)+11}{((1)^2+5(1)-14)^2}=\dfrac{20}{64}=\dfrac{5}{16}$

Equation of tangent is

$$y-y_1=m_T(x-x_1)$$
$$y-\frac{1}{2}=\frac{5}{16}(x-1)$$
$$y-\frac{1}{2}=\frac{5}{16}x-\frac{5}{16}$$
$$y=\frac{5}{16}x-\frac{5}{16}+\frac{8}{16}$$
$$y=\frac{5}{16}x+\frac{3}{16}$$

15 a $y=f(x)=\dfrac{\sin(2x-3)}{e^x}$

Stationary points occur where $\dfrac{dy}{dx}=0.$

$$\frac{dy}{dx}=\frac{2\cos(2x-3)-\sin(2x-3)}{e^x}$$
$$0=\frac{2\cos(2x-3)-\sin(2x-3)}{e^x}$$
$$0=2\cos(2x-3)-\sin(2x-3)$$
$$x=0.25(6.2832n+8.2143)\text{ where } n\in Z$$

For the given domain let $n=-2,\ -1$

$x=-1.088,\ 0.483$

When $x=-1.088,\ y=\dfrac{\sin(2(-1.088)-3)}{e^{-1.088}}=2.655$

When $x=0.483,\ y=\dfrac{\sin(2(0.483)-3)}{e^{0.483}}=-0.552$

Thus $a=-1.088,\ b=2.655,\ c=0.483$ and $d=-0.552$

b $\dfrac{dy}{dx}_{x=1}=\dfrac{2\cos(2-3)-\sin(2-3)}{e^1}=0.707$

16 a $y=\dfrac{2x-1}{3x^2+1}$

$$\frac{dy}{dx}=\frac{-6x^2+6x+2}{(3x^2+1)^2}$$

b $\dfrac{-6x^2+6x+2}{(3x^2+1)^2}=0.875$

$x=-0.1466$ or 0.5746

Exercise 6.5 — Curve sketching

1 a $f(x)=\dfrac{2x^3}{3}+\dfrac{3x^2}{2}-2x+4$

Stationary points occur where $f'(x)=0$

$$f'(x)=2x^2+3x-2$$
$$0=2x^2+3x-2$$
$$0=(2x-1)(x+2)$$

Either $2x-1=0$ or $x+2=0$

$x=\dfrac{1}{2}\qquad x=-2$

$$f\left(\frac{1}{2}\right)=\frac{2}{3}\left(\frac{1}{2}\right)^3+\frac{3}{2}\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)+4$$
$$=\frac{1}{12}+\frac{3}{8}+3$$
$$=\frac{2}{24}+\frac{9}{24}+\frac{72}{24}$$
$$=\frac{83}{24}$$
$$f(-2)=\frac{2}{3}(-2)^3+\frac{3}{2}(-2)^2-2(-2)+4$$
$$=-\frac{16}{3}+14$$
$$=-\frac{16}{3}+\frac{42}{3}$$
$$=\frac{26}{3}$$

When

$x=-3,\ f'(-3)=2(-3)^2+3(-3)-2=18-9-2=7(+\text{ve})$

$x=-1,\ f'(-1)=2(-1)^2+3(-1)-2=2-3-2=-3(-\text{ve})$

$x=0,\ f'(0)=2(0)^2+3(0)-2=0+0-2=-2(-\text{ve})$

$x=1,\ f'(1)=2(1)^2+3(1)-2=2+3-2=3(+\text{ve})$

$x<-2$	$x=-2$	$-1<x<\frac{1}{2}$	$x=\frac{1}{2}$	$x>\frac{1}{2}$
/	—	\	—	/

Maximum TP at $\left(-2,\dfrac{26}{3}\right)$ Minimum TP at $\left(\dfrac{1}{2},\dfrac{83}{24}\right)$

b $y=ax^2+bx+c$

When $x=0,\ y=-8$ so $c=-8$

$y=ax^2+bx-8$

$\dfrac{dy}{dx}=2ax+b$

When $x=-1,\ y=-5;\ -5=a(-1)^2+b(-1)-8$

$3=a-b$....................(1)

When $x=-1,\ \dfrac{dy}{dx}=2a(-1)+b=0$

$-2a+b=0$...................(2)

$(1)+(2)$

$-a=3$

$a=-3$

Substitute $a=-3$ into (1) so $3=-3-b\ \Rightarrow b=-6$

Therefore $a=-3, b=-6, c=-8$

2 a $y=x^3+ax^2+bx-11$

Stationary point when $x=1$ and $x=\dfrac{5}{3}.$

$$\frac{dy}{dx}=3x^2+2ax+b$$

$$\frac{dy}{dx}_{x=1} = 3(1)^2 + 2a(1) + b = 0$$
$$3 + 2a + b = 0$$
$$2a + b = -3 \text{..................(1)}$$
$$\frac{dy}{dx}_{x=\frac{5}{3}} = 3\left(\frac{5}{3}\right)^2 + 2a\left(\frac{5}{3}\right) + b = 0$$
$$\frac{25}{3} + \frac{10}{3}a + b = 0$$
$$10a + 3b = -25 \text{............(2)}$$
$(1)\times 3$ $\qquad 6a + 3b = -9$ ....(3)

$(2)-(3)$ $\qquad 4a = -16$

$$a = -4$$

Substitute $a = -4$ into (1)

$$2(-4) + b = -3$$
$$-8 + b = -3$$
$$b = 5$$

b When $x = 1,\ y = (1)^3 - 4(1)^2 + 5(1) - 11 = -9$

When $x = \frac{5}{3},\ y = \left(\frac{5}{3}\right)^3 - 4\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) - 11 = \frac{125}{27} - \frac{100}{9} + \frac{15}{3} - 11$

$$= \frac{125}{27} - \frac{300}{27} + \frac{135}{27} - \frac{2700}{27} = -\frac{247}{27}$$

When $x = 0,\ \frac{dy}{dx} = 3(0)^2 - 8(0) + 5 = +\text{ve}$

When $x = 1.5,\ \frac{dy}{dx} = 3(1.5)^2 - 8(1.5) + 5 = -\text{ve}$

When $x = 2,\ \frac{dy}{dx} = 3(2)^2 - 8(2) + 5 = +\text{ve}$

$x < 1$	$x = 1$	$1 < x < \frac{5}{3}$	$x = \frac{5}{3}$	$x > \frac{5}{3}$
/	—	\	—	/

Maximum TP at $(1, -9)$ $\qquad$ Minimum TP at $\left(\frac{5}{3}, -\frac{247}{27}\right)$

3 a $y = f(x) = 2x^3 - x^2 = x^2(2x - 1)$

Graph cuts the y axis where $x = 0,\ y = 0.$

Graph cuts the x axis where $y = 0$

$x^2(2x - 1) = 0$

$x = 0$ or $2x - 1 = 0$

$$x = \frac{1}{2}$$

When $x = \frac{1}{2},\ y = 2\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 = 0$

Stationary points occur where $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 6x^2 - 2x = 0$$
$$3x^2 - x = 0$$
$$x(3x - 1) = 0$$

$x = 0$ or $3x - 1 = 0$

$$x = \frac{1}{3}$$

When $x = \frac{1}{3},\ y = 2\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 = \frac{2}{27} - \frac{3}{27} = -\frac{1}{27}$

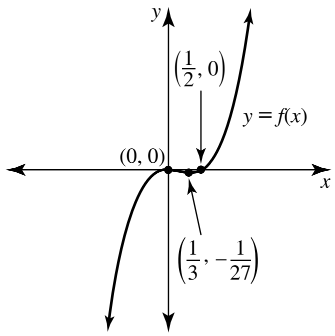
When $x = -1,\ \frac{dy}{dx} = 6(-1)^2 - 2(-1) = +\text{ve}$

When $x = \frac{1}{4},\ \frac{dy}{dx} = 6\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right) = \frac{3}{8} - \frac{4}{8} = -\text{ve}$

When $x = 1,\ \dfrac{dy}{dx} = 6(1)^2 - 2(1) = +\text{ve}$

$x < 0$	$x = 0$	$0 < x < \frac{1}{3}$	$x = \frac{1}{3}$	$x > \frac{1}{3}$
/	—	\	—	/

Maximum TP   Minimum TP



b Dom $x \in \left[0, \frac{1}{3}\right]$

4 a $f(x) = -x^4 + 2x^3 + 11x^2 - 12x = x\left(-x^3 + 2x^2 + 11x - 12\right)$

Let $P(x) = -x^3 + 2x^2 + 11x - 12$

$P(1) = -(1)^3 + 2(1)^2 + 11(1) - 12 = 0$

$(x-1)$ is a factor

$-x^3 + 2x^2 + 11x - 12 = (x-1)\left(-x^2 + x + 12\right) = (x-1)(4-x)(x+3)$

Thus $f(x) = -x^4 + 2x^3 + 11x^2 - 12x = x(x-1)(4-x)(x+3)$

Graph cuts the y axis where $x = 0, y = 0.$

Graph cuts the x axis where $y = 0$

$x(x-1)(4-x)(x+3) = 0$

$x = 0$ or $x - 1 = 0$ or $4 - x = 0$ or $x + 3 = 0$

$x = 1 \qquad x = 4 \qquad x = -3$

Stationary points occur where $f'(x) = 0.$

$f'(x) = -4x^3 + 6x^2 + 22x - 12 = -2\left(2x^3 - 3x^2 - 11x + 6\right)$

Let $P(x) = 2x^3 - 3x^2 - 11x + 6$

$P(-2) = 2(-2)^3 - 3(-2)^2 - 11(-2) + 6 = -16 - 12 + 22 + 6 = 0$

Thus $(x+2)$ is a factor

$-2\left(2x^3 - 3x^2 - 11x + 6\right) = -2(x+2)\left(2x^2 - 7x + 3\right) = -2(x+2)(2x-1)(x-3)$

Stationary points at $-2(x+2)(2x-1)(x-3) = 0$ where

$x + 2 = 0$ or $2x - 1 = 0$ or $x - 3 = 0$

$x = -2 \qquad x = \frac{1}{2} \qquad x = 3$

When $x = -2,\ f(-2) = -2(-2-1)(4+2)(-2+3) = 36$

When $x = \frac{1}{2},\ f\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(4 - \frac{1}{2}\right)\left(\frac{1}{2} + 3\right) = -\frac{49}{16}$

When $x = 3,\ f(3) = 3(3-1)(4-3)(3+3) = 36$

If $x = -3,\ f'(-3) = -4(-3)^3 + 6(-3)^2 + 22(-3) - 12 = 108 + 54 - 66 - 12 = +\text{ve}$

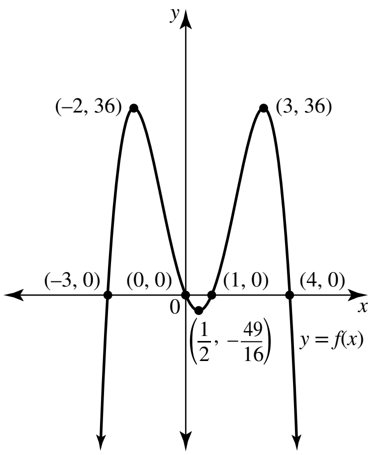
If $x = 0, f'(0) = -4(0)^3 + 6(0)^2 + 22(0) - 12 = -\text{ve}$

If $x = 1, f'(1) = -4(1)^3 + 6(1)^2 + 22(1) - 12 = -4 + 6 + 22 - 12 = +\text{ve}$

If $x = 4, f'(4) = -4(4)^3 + 6(4)^2 + 22(4) - 12 = -256 + 64 + 88 - 12 = -\text{ve}$

$x < -2$	$x = -2$	$-2 < x < \frac{1}{2}$	$x = \frac{1}{2}$	$\frac{1}{2} < x < 3$	$x = 3$	$x > 3$
/	—	\	—	/	—	\

Maximum TP   Minimum TP   Maximum TP



b $x \in (-\infty, -2] \cup \left[\frac{1}{2}, 3\right]$

5 a $f(x) = \frac{1}{4x} + x$ where $x \in \left[-2, -\frac{1}{4}\right]$

Endpoints are

$f(-2) = \frac{1}{4(-2)} - 2 = -\frac{17}{8}$ and $f\left(-\frac{1}{4}\right) = \frac{1}{4\left(-\frac{1}{4}\right)} - \frac{1}{4} = -\frac{5}{4}$

i.e. $\left(-2, -\frac{17}{8}\right)$ and $\left(-\frac{1}{4}, -\frac{5}{4}\right)$

b Stationary points occur where $f'(x) = 0$

$f(x) = \frac{1}{4}x^{-1} + x$

$f'(x) = -\frac{1}{4x^2} + 1$

$0 = -\frac{1}{4x^2} + 1$

$0 = 4x^2 - 1$

$0 = (2x-1)(2x+1)$

$x = \pm\frac{1}{2}$

But $x \in \left[-2, -\frac{1}{4}\right]$ so $x = -\frac{1}{2}$

If $x = -1$, $f'(-1) = -\frac{1}{4(-1)^2} + 1 = +\text{ve}$

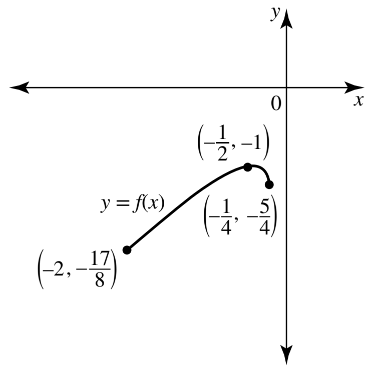
If $x = -\frac{3}{8}$, $f'\left(-\frac{3}{8}\right) = -\frac{1}{4\left(-\frac{3}{8}\right)^2} + 1 = -\frac{16}{9} + 1 = -\text{ve}$

$x < -\frac{1}{2}$	$x = -\frac{1}{2}$	$x > -\frac{1}{2}$
/	—	\

Maximum TP

Maximum TP at $\left(-\frac{1}{2}, -1\right)$

c



d Absolute minimum $= -\frac{17}{8}$. Absolute maximum is -1.

6 $f(x) = 2x^3 - 8x = 2x\left(x^2 - 4\right) = 2x(x-2)(x+2)$

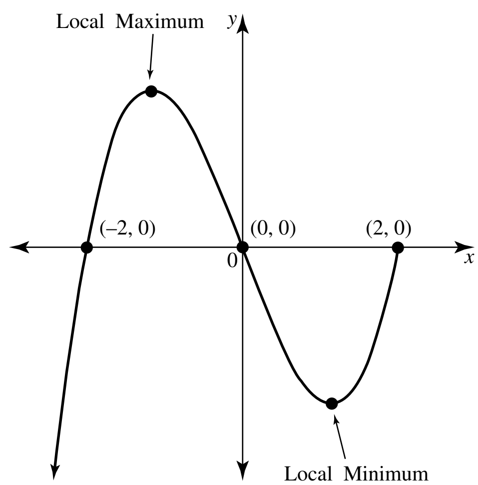
Graph cuts the y axis where $x = 0$, $y = 0$

Graph cuts the x axis where $y = 0$

$2x(x-2)(x+2) = 0$

$x = 0$ or $x - 2 = 0$ or $x + 2 = 0$

$x = 2 \qquad x = -2$



No absolute minimum. Absolute maximum occurs when $f'(x) = 0$ for $x \le 2$.

$f'(x) = 6x^2 - 8$

$0 = 6x^2 - 8$

$\frac{4}{3} = x^2$

$-\frac{2}{\sqrt{3}} = x$

$f\left(-\frac{2}{\sqrt{3}}\right) = 2\left(-\frac{2}{\sqrt{3}}\right)^3 - 8\left(-\frac{2}{\sqrt{3}}\right)$

$= -\frac{16}{3\sqrt{3}} + \frac{16}{\sqrt{3}}$

$= -\frac{16}{3\sqrt{3}} + \frac{48}{3\sqrt{3}}$

$= \frac{32}{3\sqrt{3}}$

Therefore no absolute minimum and the absolute maximum is $\frac{32}{3\sqrt{3}}$.

7 $y = f(x) = 16x^2 - x^4$

a Stationary points occur where $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 32x - 4x^3$

$0 = 4x\left(8 - x^2\right)$

$0 = 4x\left(2\sqrt{2} - x\right)\left(2\sqrt{2} + x\right)$

$x = 0$ or $2\sqrt{2} - x = 0$ or $2\sqrt{2} + x = 0$

$x = 2\sqrt{2}$ $\quad x = -2\sqrt{2}$

When $x = \pm 2\sqrt{2},\ y = 16\left(\pm 2\sqrt{2}\right)^2 - \left(\pm 2\sqrt{2}\right)^4 = 128 - 64 = 64$

Stationary points at $\left(\pm 2\sqrt{2}, 64\right)$ so $\left(2\sqrt{2}, 64\right)$ is a stationary point.

b When $x = -3,\ \frac{dy}{dx} = 32(-3) - 4(-3)^3 = +\text{ve}$

When $x = -1,\ \frac{dy}{dx} = 32(-1) - 4(-1)^3 = -\text{ve}$

When $x = 1,\ \frac{dy}{dx} = 32(1) - 4(1)^3 = +\text{ve}$

When $x = 3,\ \frac{dy}{dx} = 32(3) - 4(3)^3 = -\text{ve}$

x	$x < -2\sqrt{2}$	$x = -2\sqrt{2}$	$-2\sqrt{2} < x < 0$	$x = 0$	$0 < x < 2\sqrt{2}$	$x = 2\sqrt{2}$	$x > 2\sqrt{2}$
$\frac{dy}{dx}$	/	—	\	—	/	—	\
		Maximum TP		Minimum TP		Maximum TP	

Therefore $\left(2\sqrt{2}, 64\right)$ is a maximum TP

c The other stationary points are $\left(-2\sqrt{2}, 64\right)$ which is a maximum and (0,0) which is a minimum.

8 a $y = x(x+2)^2$

Stationary points occur where $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = (x+2)^2 + 2x(x+2)$

$0 = (x+2)(x+2+2x)$

$0 = (x+2)(3x+2)$

$x + 2 = 0$ or $3x + 2 = 0$

$x = -2$ $\quad x = -\frac{2}{3}$

When $x = -2,\ y = (-2)(-2+2) = 0$

When $x = -\frac{2}{3},\ y = \left(-\frac{2}{3}\right)\left(-\frac{2}{3} + 2\right)^2 = - -\frac{2}{3} \times \frac{16}{9} = -\frac{32}{27}$

When $x = -3,\ \frac{dy}{dx} = (-3+2)(3(-3)+2) = +\text{ve}$

When $x = -1,\ \frac{dy}{dx} = (-1+2)(3(-1)+2) = -\text{ve}$

When $x = 0,\ \frac{dy}{dx} = (0+2)(3(0)+2) = +\text{ve}$

x	$x < -2$	$x = -2$	$-2 < x < -\frac{2}{3}$	$x = -\frac{2}{3}$	$x > -\frac{2}{3}$
$\frac{dy}{dx}$	/	—	\	—	/

Maximum TP at (–2, 0) $\quad$ Minimum TP at $\left(-\frac{2}{3}, -\frac{32}{27}\right)$

b $y = x^3 + 3x^2 - 24x + 5$

Stationary points occur where $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 3x^2 + 6x - 24$

$0 = 3x^2 + 6x - 24$

$0 = x^2 + 2x - 8$

$0 = (x+4)(x-2)$

$x + 4 = 0$ or $x - 2 = 0$

$x = -4 \qquad x = 2$

When $x = -4$, $y = (-4)^3 + 3(-4)^2 - 24(-4) + 5 = 85$

When $x = 2$, $y = (2)^3 + 3(2)^2 - 24(2) + 5 = -23$

When $x = -5$, $\frac{dy}{dx} = 3(-5)^2 + 6(-5) - 24 = 75 - 30 - 24 = +\text{ve}$

When $x = -1$, $\frac{dy}{dx} = 3(-1)^2 + 6(-1) - 24 = -\text{ve}$

When $x = 3$, $\frac{dy}{dx} = 3(3)^2 + 6(3) - 24 = 27 + 18 - 24 = +\text{ve}$

x	$x < -4$	$x = -4$	$-4 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	/	—	\	—	/

Maximum TP at (–4, 85) Minimum TP at (2, –23)

c $y = \frac{x^2}{x+1}$

Stationary points occur where $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2x(x+1) - x^2}{(x+1)^2}$

$\frac{dy}{dx} = \frac{2x^2 + 2x - x^2}{(x+1)^2}$

$\frac{dy}{dx} = \frac{x^2 + 2x}{(x+1)^2}$

$0 = \frac{x^2 + 2x}{(x+1)^2}$

$0 = x(x+2)$

$x = 0$ or $x + 2 = 0$

$x = -2$

When $x = -2$, $y = \frac{(-2)^2}{-2+1} = -4$

When $x = 0$, $y = \frac{(0)^2}{0+1} = 0$

When $x = -3$, $\frac{dy}{dx} = \frac{(-3)^2 + 2(-3)}{(-3+1)^2} = +\text{ve}$

When $x = -0.5$, $\frac{dy}{dx} = \frac{(-0.5)^2 + 2(-0.5)}{(-0.5+1)^2} = -\text{ve}$

When $x = 1$, $\frac{dy}{dx} = \frac{(1)^2 + 2(1)}{(1+1)^2} = +\text{ve}$

x	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
$\frac{dy}{dx}$	/	—	\	—	/

Maximum TP at (–2, –4) Minimum TP at (0, 0)

d $y = (x-1)e^{-x}$

Stationary points occur where $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = e^{-x} - (x-1)e^{-x}$

$\frac{dy}{dx} = e^{-x} - xe^{-x} + e^{-x}$

$\frac{dy}{dx} = 2e^{-x} - xe^{-x}$

$0 = e^{-x}(2 - x)$

$2 - x = 0$ as $e^{-x} > 0$ for all x

$x = 2$

$x = 2, y = (2-1)e^{-2} = e^{-2}$

When $x = 1$, $\frac{dy}{dx} = e^{-1}(2-1) = +\text{ve}$

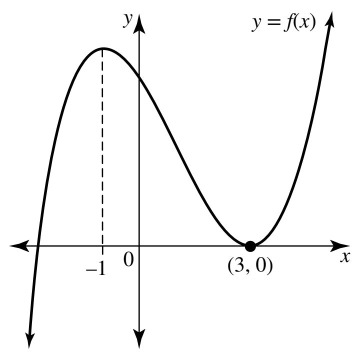
When $x = 3$, $\frac{dy}{dx} = e^{-3}(2-3) = -\text{ve}$

x	$x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	/	—	\

Maximum TP at $\left(2, e^{-2}\right)$

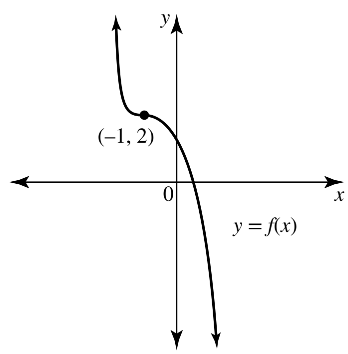
9 a

x	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
$\frac{dy}{dx}$	/	—	\	—	/



b

x	$x < -1$	$x = -1$	$-1 < x < 3$
$\frac{dy}{dx}$	\	—	\



10 a $y = x^3 + bx + c$

When $x = 2,\ y = -8$

$-8 = (2)^3 + 2b + c$

$-16 = 2b + c$..................(1)

$\dfrac{dy}{dx} = 3x^2 + b$

Stationary point at $(2, -8)$.

$0 = 3(2)^2 + b$

$0 = 12 + b$

$-12 = b$

Substitute $b = -12$ into (1)

$-16 = 2(-12) + c$

$-16 = -24 + c$

$8 = c$

$y = x^3 - 12x + 8$

b $y = ax^2 + bx + 15$

When $x = 1.5,\ y = 6$

$6 = a\left(\dfrac{3}{2}\right)^2 + \left(\dfrac{3}{2}\right)b + 15$

$-9 = \dfrac{9}{4}a + \dfrac{3}{2}b$

$-36 = 9a + 6b$

$-12 = 3a + 2b$..................(1)

$\dfrac{dy}{dx} = 2ax + b$

Stationary point at $(1.5, 6)$.

$0 = 2a\left(\dfrac{3}{2}\right) + b$

$0 = 3a + b$........................(2)

(1) − (2)

$-12 = b$

Substitute $b = -12$ into (2)

$0 = 3a - 12$

$12 = 3a$

$4 = a$

$y = 4x^2 - 12x + 15$

c $y = x^3 + bx^2 + cx + d$

When $x = -3,\ y = -10$

$-10 = (-3)^3 + b(-3)^2 - 3c + d$

$-10 = -27 + 9b - 3c + d$

$17 = 9b - 3c + d$....................(1)

When $x = 1,\ y = 6$

$6 = (1)^3 + b(1)^2 + c + d$

$6 = 1 + b + c + d$

$5 = b + c + d$.......................(2)

Stationary point at $(-3, -10)$

$\dfrac{dy}{dx} = 3x^2 + 2bx + c$

$0 = 3(-3)^2 + 2(-3)b + c$

$-27 = -6b + c$......................(3)

(1) − (2)

$12 = 8b - 4c$

$3 = 2b - c$..........................(4)

(3) + (4)

$-24 = -4b$

$6 = b$

Substitute $b = 6$ into (3)

$-27 = -6(6) + c$

$-27 = -36 + c$

$9 = c$

Substitute $b = 6$ and $c = 9$ into (2)

$5 = 6 + 9 + d$

$5 = 15 + d$

$-10 = d$

$y = x^3 + 6x^2 + 9x - 10$

11 a $y = f(x) = -\dfrac{1}{4}(x - 4)^3 + 2$

Graph cuts the y axis where $x = 0,\ y = 14 + 2 = 16.$

Graph cuts the x axis where $y = 0$

$(x - 4)^3 = 8$

$x - 4 = 2$

$x = 6$

Stationary points occur where $\dfrac{dy}{dx} = 0.$

$\dfrac{dy}{dx} = -\dfrac{3}{4}(x - 4)^2$

$0 = -\dfrac{3}{4}(x - 4)^2$

$x - 4 = 0$

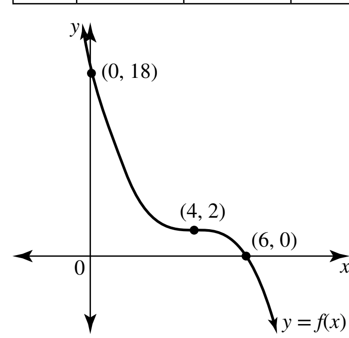
$x = 4$

When $x = 4,\ y = -\dfrac{1}{4}(4 - 4)^3 + 2 = 2$

When $x = 0,\ \dfrac{dy}{dx} = -\dfrac{3}{4}(0 - 4)^2 = -\text{ve}$

When $x = 5,\ \dfrac{dy}{dx} = -\dfrac{3}{4}(5 - 4)^2 = -\text{ve}$

x	$x < 4$	$x = 4$	$x > 4$
$\dfrac{dy}{dx}$	\	—	\



b $y = g(x) = 2x^3 - x^2, \quad x \in [-1, 1]$

Graph cuts the y axis where $x = 0,\ y = 0$.

Graph cuts the x axis where $y = 0$

$x^2(2x - 1) = 0$

$x = 0$ or $2x - 1 = 0$

$x = 0 \qquad x = \frac{1}{2}$

Stationary points occur where $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 6x^2 - 2x = 2x(3x - 1)$

$0 = 2x(3x - 1)$

$x = 0$ or $3x - 1 = 0$

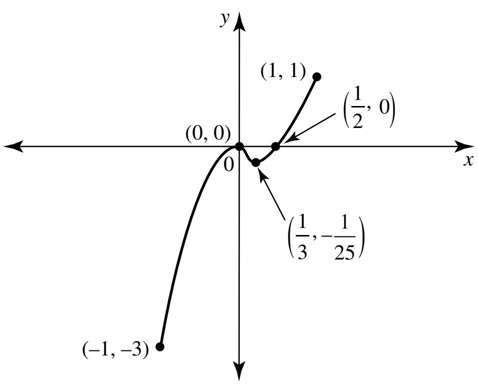
$x = \frac{1}{3}$

When $x = -1$, $\frac{dy}{dx} = 2(-1)(3(-1) - 1) = +\text{ve}$

When $x = 0.1$, $\frac{dy}{dx} = 2(0.1)(3(0.1) - 1) = -\text{ve}$

When $x = 1$, $\frac{dy}{dx} = 2(1)(3(1) - 1) = +\text{ve}$

x	$x < 0$	$x = 0$	$0 < x < \frac{1}{3}$	$x = \frac{1}{3}$	$x > \frac{1}{3}$
$\frac{dy}{dx}$	/	—	\	—	/



c $y = h(x) = x^3 - x^2 - x + 10$

Graph cuts the y axis where $x = 0,\ y = 10$.

Graph cuts the x axis where $y = 0$

$x^3 - x^2 - x + 10 = 0$

Let $P(x) = x^3 - x^2 - x + 10$

$P(2) = 2^3 - 2^2 - 2 + 10 \neq 0$

$P(-2) = (-2)^3 - (-2)^2 + 2 + 10 = -8 - 4 + 12 = 0$

$(x + 2)$ is a factor

$x^3 - x^2 - x + 10 = (x + 2)(x^2 - 3x + 5)$

$x^2 - 3x + 5$ cannot be further factorised as $\Delta < 0$

$(x + 2)(x^2 - 3x + 5) = 0$

$x + 2 = 0$

$x = -2$

Stationary points occur where $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 3x^2 - 2x - 1$

$0 = (3x + 1)(x - 1)$

$3x + 1 = 0$ or $x - 1 = 0$

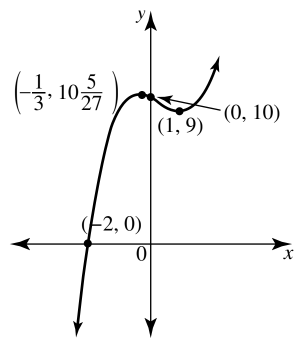
$x = -\frac{1}{3} \qquad x = 1$

When $x = -1$, $\frac{dy}{dx} = 3(-1)^2 - 2(-1) - 1 = +\text{ve}$

When $x = 0$, $\frac{dy}{dx} = 3(0)^2 - 2(0) - 1 = -\text{ve}$

When $x = 2$, $\frac{dy}{dx} = 2(2)^2 - 2(2) - 1 = +\text{ve}$

x	$x < -\frac{1}{3}$	$x = -\frac{1}{3}$	$-\frac{1}{3} < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	/	—	\	—	/



d $y = f(x) = x^4 - 6x^2 + 8$

Graph cuts the y axis where $x = 0,\ y = 8$.

Graph cuts the x axis where $y = 0$

$x^4 - 6x^2 + 8 = 0$

$(x^2 - 2)(x^2 - 4) = 0$

$(x - \sqrt{2})(x + \sqrt{2})(x - 2)(x + 2) = 0$

$x - \sqrt{2} = 0$ or $x + \sqrt{2} = 0$ or $x - 2 = 0$ or $x + 2 = 0$

$x = \sqrt{2} \qquad x = -\sqrt{2} \qquad x = 2 \qquad x = -2$

Stationary points occur where $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 4x^3 - 12x$

$0 = 4x(x^2 - 3)$

$0 = 4x(x - \sqrt{3})(x + \sqrt{3})$

$x = 0$ or $x - \sqrt{3} = 0$ or $x + \sqrt{3} = 0$

$x = \sqrt{3} \qquad x = -\sqrt{3}$

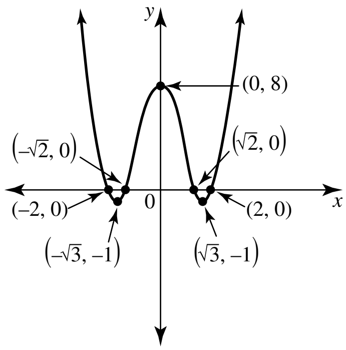
When $x = -2$, $\frac{dy}{dx} = 4(-2)^3 - 12(-2) = -\text{ve}$

When $x = -1$, $\frac{dy}{dx} = 4(-1)^3 - 12(-1) = +\text{ve}$

When $x = 1$, $\frac{dy}{dx} = 4(1)^3 - 12(1) = -\text{ve}$

When $x = 2$, $\frac{dy}{dx} = 4(2)^3 - 12(2) = +\text{ve}$

x	$x<-\sqrt{3}$	$x=-\sqrt{3}$	$-\sqrt{3}<x<0$	$x=0$	$0<x<\sqrt{3}$	$x=\sqrt{3}$	$x>-\frac{1}{3}$
$\frac{dy}{dx}$	\	—	/	—	\	—	/



e $y=f(x)=(x+3)^3(4-x)$

Graph cuts the y axis where $x=0,\ y=(3)^3(4)=108.$

Graph cuts the x axis where $y=0$

$(x+3)^3(4-x)=0$

$x+3=0$ or $4-x=0$

$x=-3 \qquad x=4$

Stationary points occur where $\frac{dy}{dx}=0.$

$\frac{dy}{dx}=3(x+3)^2(4-x)-(x+3)^3$

$0=(x+3)^2\{3(4-x)-(x+3)\}$

$0=(x+3)^2(12-3x-x-3)$

$0=(x+3)^2(9-4x)$

$x+3=0$ or $9-4x=0$

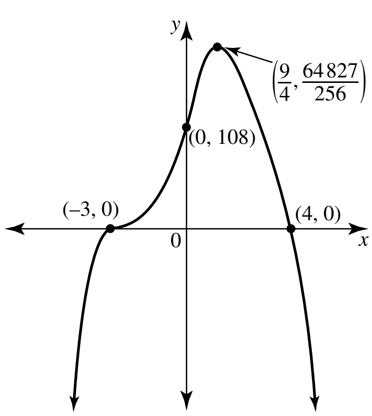
$x=-3 \qquad x=\frac{9}{4}$

When $x=-4,\ \frac{dy}{dx}=(-4+3)^2(9-4(-4))=+\text{ve}$

When $x=0,\ \frac{dy}{dx}=(0+3)^2(9-4(0))=+\text{ve}$

When $x=3,\ \frac{dy}{dx}=(3+3)^2(9-4(3))=-\text{ve}$

x	$x<-3$	$x=-3$	$-3<x<\frac{9}{4}$	$x=\frac{9}{4}$	$x>\frac{9}{4}$
$\frac{dy}{dx}$	/	—	/	—	\



f $y = f(x) = x^3 - 4x^2 - 3x + 12$

Graph cuts the y axis where $x = 0,\ y = 12$.

Graph cuts the x axis where $y = 0$

$$x^3 - 4x^2 - 3x + 12 = 0$$
$$x^2(x-4) - 3(x-4) = 0$$
$$(x-4)(x^2-3) = 0$$
$$(x-4)(x-\sqrt{3})(x+\sqrt{3}) = 0$$
$$x - 4 = 0 \text{ or } x - \sqrt{3} = 0 \text{ or } x + \sqrt{3} = 0$$
$$x = 4 \qquad x = \sqrt{3} \qquad x = -\sqrt{3}$$

Stationary points occur where $\frac{dy}{dx} = 0$.

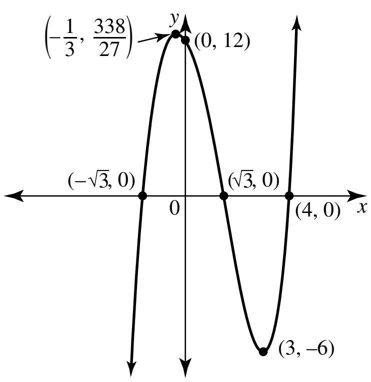
$$\frac{dy}{dx} = 3x^2 - 8x - 3$$
$$0 = (3x+1)(x-3)$$
$$3x + 1 = 0 \text{ or } x - 3 = 0$$
$$x = -\frac{1}{3} \qquad x = 3$$

When $x = -1$, $\frac{dy}{dx} = 3(-1)^2 - 8(-1) - 3 = +\text{ve}$

When $x = 0$, $\frac{dy}{dx} = 3(0)^2 - 8(0) - 3 = -\text{ve}$

When $x = 4$, $\frac{dy}{dx} = 3(4)^2 - 8(4) - 3 = +\text{ve}$

x	$x < -\frac{1}{3}$	$x = -\frac{1}{3}$	$-\frac{1}{3} < x < 3$	$x = 3$	$x > 3$
$\frac{dy}{dx}$	/	—	\	—	/



12 a $f(x) = \frac{1}{2}(2x-3)^4(x+1)^5$

Graph cuts the y axis where $f(0) = \frac{1}{2}(-3)^4(1)^5 = \frac{81}{2}$.

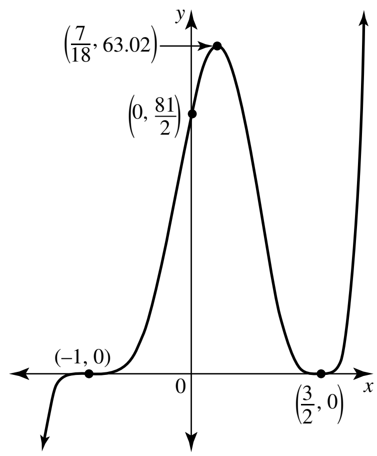
Graph cuts the y axis where $y = 0$

$$\frac{1}{2}(2x-3)^4(x+1)^5 = 0$$
$$2x - 3 = 0 \text{ or } x + 1 = 0$$
$$x = \frac{3}{2} \qquad x = -1$$

Stationary points $f'(x) = 0$

$$f'(x) = \frac{1}{2}\left[(2x-3)^4 \times 5(x+1)^4 + (x+1)^5 \times 4(2x-3)^3 \times 2\right]$$
$$= \frac{1}{2}(x+1)^4(2x-3)^3(5(2x-3) + 8(x+1))$$
$$= \frac{1}{2}(x+1)^4(2x-3)^3(18x-7)$$
$$0 = \frac{1}{2}(x+1)^4(2x-3)^3(18x-7)$$

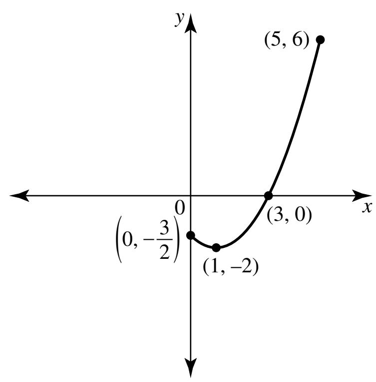
$$x = \frac{3}{2},\ -1,\ \frac{7}{18}$$
$$\Rightarrow \left(\frac{3}{2}, 0\right), (-1, 0), \left(\frac{7}{18}, 63.02\right)$$



b Strictly decreasing for $x \in \left[\frac{7}{18}, \frac{3}{2}\right]$

13 a $y = \frac{1}{2}(x-1)^2 - 2, \quad 0 \le x \le 5$

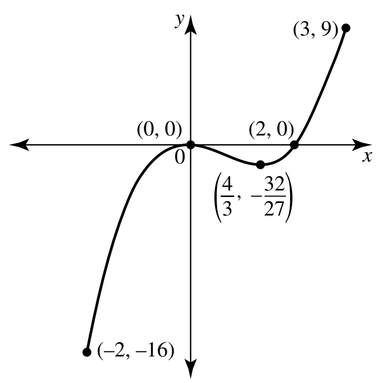
Turning point at $(1, -2)$ and cuts the y axis at $\left(0, -\frac{3}{2}\right)$



Absolute minimum is –2 and absolute maximum is 6.

b $y = x^3 - 2x^2 = x^2(x-2), \quad -2 \le x \le 3$

Graph cuts the y axis at (0,0) and the x axis at (0,0) and (2,0). There is a turning point at $(0,0)$ and $\left(\frac{1}{3}, -\frac{32}{27}\right)$

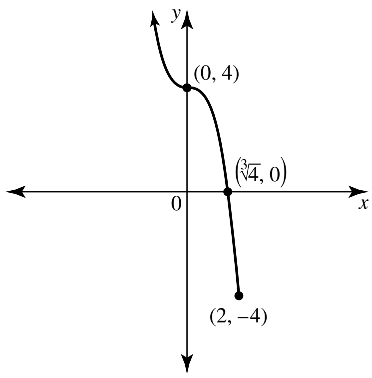


Absolute minimum is –16 and absolute maximum is 9.

c $y = 4 - x^3, \quad x \le 2$

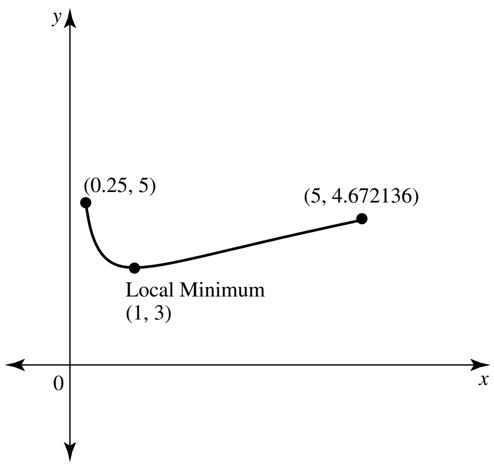
Graph cuts the y axis at (0,4) and the x axis at $\left(\sqrt[3]{4}, 0\right)$.

There is a point of inflection at $(0, 4)$.



No absolute maximum and absolute minimum is −4

14 $y = f(x) = 2\sqrt{x} + \dfrac{1}{x}, \quad 0.25 \le x \le 5.$



a When $x = 0.25,\ f(0.25) = 2\sqrt{0.25} + \dfrac{1}{0.25} = 5$ which is point A.

When $x = 5,\ f(5) = 2\sqrt{5} + 0.2 (= 4.672)$ which is point C.

Stationary point B occurs where $f'(x) = 0$

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2}$$
$$0 = \frac{1}{\sqrt{x}} - \frac{1}{x^2}$$
$$0 = x^2 - \sqrt{x}$$
$$0 = \sqrt{x}\left(x^{\frac{3}{2}} - 1\right)$$
$$x^{\frac{3}{2}} - 1 = 0$$
$$x^{\frac{3}{2}} = 1$$
$$x = 1$$

When $x = 1, f(1) = 2\sqrt{1} + \dfrac{1}{1} = 3$

Therefore $A = (0.25, 5), B = (1, 3), C = \left(5, 2\sqrt{5} + 0.2\right)$

b Absolute maximum occurs at A.

c Absolute minimum is 3 and absolute maximum is 5.

15 $y = f(x) = xe^x$

a Stationary points occur where $\dfrac{dy}{dx} = 0.$

$$\frac{dy}{dx} = e^x + xe^x = e^x(1 + x)$$
$$0 = e^x(1 + x)$$
$$x + 1 = 0 \text{ as } e^x > 0 \text{ for all } x$$
$$x = -1$$

When $x = -1,\ y = -e^{-1} = -\dfrac{1}{e}$

When $x = -2,\ \dfrac{dy}{dx} = e^{-2}(1 - 2) = -\text{ve}$

When $x = 0,\ \dfrac{dy}{dx} = e^{0}(1 + 1) = +\text{ve}$

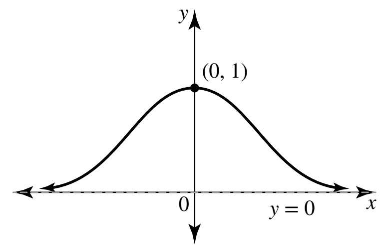
x	$x < -1$	$x = -1$	$x > -1$
$\dfrac{dy}{dx}$	\	—	/

Minimum TP at $\left(-1, -\dfrac{1}{e}\right)$

b $f'(x) > 0$ for $(-1, \infty)$.

c Absolute minimum is $-\dfrac{1}{e}$ and there is no absolute maximum.

16 a



b $y = 20e^{-2x^2 - 4x + 1}$

$$\frac{dy}{dx} = 20(-4x - 4)e^{-2x^2 - 4x + 1} = -80(x + 1)e^{-2x^2 - 4x + 1}$$

Stationary points occur where $\dfrac{dy}{dx} = 0$

$$-80(x + 1)e^{-2x^2 - 4x + 1} = 0$$
$$x + 1 = 0 \text{ as } e^{-2x^2 - 4x + 1} > 0 \text{ for all } x$$
$$x = -1$$

When $x = -2,\ \dfrac{dy}{dx} = -80(-2 + 1)e^{-2(-2)^2 - 4(-2) + 1} = +\text{ve}$

When $x = 0,\ \dfrac{dy}{dx} = -80(0 + 1)e^{1} = -\text{ve}$

x	$x < -1$	$x = -1$	$x > -1$
$\dfrac{dy}{dx}$	/	—	\

The function is strictly increasing when $x \in (-\infty, -1]$.

17 a $f(x) = (a - x)^2(x - 2)$ where $a > 2$

This is a positive cubic with a turning point at $(a, 0)$.

Stationary points occur where $f'(x) = 0$

$$f'(x) = -2(a - x)(x - 2) + (a - x)^2$$
$$f'(x) = -(a - x)(2(x - 2) - (a - x))$$
$$f'(x) = -(a - x)(3x - 4 - a)$$
$$0 = (a - x)(3x - 4 - a)$$
$$a - x = 0 \text{ or } 3x - 4 - a = 0$$
$$x = a \qquad x = \frac{a + 4}{3}$$

When $x = a,\ y = (a - a)^2(a - 2) = 0$

When $x = \dfrac{a + 4}{3}$,

$$y=\left(a-\frac{a+4}{3}\right)^2\left(\frac{a+4}{3}-2\right)$$
$$=\left(\frac{3a-a-4}{3}\right)^2\left(\frac{a+4-6}{3}\right)$$
$$=\left(\frac{2(a-2)}{3}\right)^2\left(\frac{a-2}{3}\right)$$
$$=\frac{4(a-2)^3}{27}$$

Therefore, stationary points are $(a,0)$ and $\left(\frac{a+4}{3},\frac{4(a-2)^3}{27}\right)$.

b

x	$x=a-1$	$x=a$	$x=\frac{2a+2}{3}$	$x=\frac{a+4}{3}$	$x=\frac{a+4}{3}+1=\frac{a+7}{3}$
$\frac{dy}{dx}$	╲	—	╱	—	╲

Minimum TP at $(a,0)$ and a maximum TP at $\left(\frac{a+4}{3},\frac{4(a-2)^3}{27}\right)$.

c $(3,4)=\left(\frac{a+4}{3},\frac{4(a-2)^3}{27}\right)$

$$\frac{a+4}{3}=3$$
$$a+4=9$$
$$a=5$$

18 a $f(x)=(x-a)(x-b)^3$ where $a<b$

This is a quartic graph with a stationary point of inflection at $x=b$ since $(x-b)$ raised to the power of three.

Graph cuts the x axis where $f(x)=0$.

$(x-a)(x-b)^3=0$

$x-a=0$ or $x-b=0$

$x=a \qquad x=b$

Stationary points are $(a, 0)$ and $(b, 0)$.

b Stationary points occur where $f'(x)=0$.

$f'(x)=(x-b)^3+3(x-a)(x-b)^2$

$f'(x)=(x-b)^2(x-b+3x-3a)$

$f'(x)=(x-b)^2(4x-3a-b)$

$0=(x-b)^2(4x-3a-b)$

$x-b=0$ or $4x-3a-b=0$

$x=b \qquad x=\frac{3a+b}{4}$

When $x=b$, $f(b)=(b-a)(b-b)^3=0$ This is a point of inflection.

When $x=\frac{3a+b}{4}$,

$$f\left(\frac{3a+b}{4}\right)=\left(\frac{3a+b}{4}-a\right)\left(\frac{3a+b}{4}-b\right)^3$$
$$=\left(\frac{3a+b-4a}{4}\right)\left(\frac{3a+b-4b}{4}\right)$$
$$=-\left(\frac{a-b}{4}\right)\left(\frac{3a-3b}{4}\right)$$
$$=-\frac{27(a-b)^4}{256}$$

Stationary points are $(b,0),\left(\frac{3a+b}{4},\frac{-27(a-b)^4}{256}\right)$

c

x	$x=\frac{3a+b-4}{4}$	$x=\frac{3a+b}{4}$	$x=\frac{3a+5b}{8}$	$x=b$	$x=b+1$
$\frac{dy}{dx}$	╲	—	╱	—	╱

There is a minimum TP at $\left(\frac{3a+b}{4},\frac{-27(a-b)^4}{256}\right)$ and a stationary point of inflection at $(b,0)$

d $(3,-27) \equiv \left(\frac{3a+b}{4}, -\frac{27(a-b)^4}{256}\right)$

$\frac{3a+b}{4} = 3$

$3a+b = 12$..................(1)

$-\frac{27(a-b)^4}{256} = -27$

$\frac{(a-b)^4}{256} = 1$

$(a-b)^4 = 256$

$a-b = \pm 4$ but $a <$

$a-b = -4$.....................(2)

$(1)+(2)$

$4a = 8$

$a = 2$

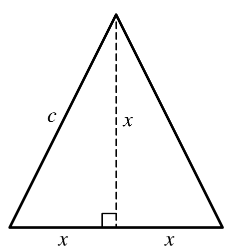
Substitute $a = 2$ into (2) so $2-b=-4 \Rightarrow b = 6$

Exercise 6.6 — Maximum and minimum problems

1 a Area = rectangular area plus triangular area

$A = 2xy + \frac{1}{2} \times 2x \times x$

$A = 2xy + x^2$



By Pythagoras $x^2 + x^2 = c^2$

$2x^2 = c^2$

$\sqrt{2}x = c, \quad c > 0$

Perimeter $= 150 = 2x + 2y + 2\sqrt{2}x$

$75 = y + \left(1+\sqrt{2}\right)x$

$75 - \left(1+\sqrt{2}\right)x = y$

Thus $A = 2x\left(75 - \left(1+\sqrt{2}\right)x\right) + x^2$

$A = 150x - \left(2\sqrt{2}+2\right)x^2 + x^2$

$A = 150x - \left(2\sqrt{2}+1\right)x^2$ as required

b Greatest area occurs when $\frac{dA}{dx} = 0$.

$\frac{dA}{dx} = 150 - 2\left(2\sqrt{2}+1\right)x$

$0 = 150 - 2\left(2\sqrt{2}+1\right)x$

$150 = 2\left(2\sqrt{2}+1\right)x$

$\frac{75}{2\sqrt{2}+1} = x$

$x = 19.59$

Width $= 2x = 39.2$ cm

Height $= 75 - \left(1+\sqrt{2}\right)(19.59) + 19.59 = 47.3$ cm

c Width $= 30$ cm

Height $= 75 - \left(1+\sqrt{2}\right)(15) + 15 = 53.8$ cm

2 a $V = x(16-2x)(10-2x)$

$V = x\left(160 - 42x + 4x^2\right)$

$V = 4x^3 - 42x^2 + 160x$

b Greatest volume occurs when $\frac{dV}{dx} = 0$.

$\frac{dV}{dx} = 12x^2 - 104x + 160 = 0$

$3x^2 - 26x + 40 = 0$

$(3x-20)(x-2) = 0$

$x = 2, \frac{20}{3}$

$x = 2, \ (0 < x < 5)$

Therefore, height = 2cm, width = 6 cm and length = 12 cm

$V_{\max} = 2(16-2(2))(10-2(2))$

$= 2 \times 12 \times 6$

$= 144 \text{ m}^3$

3 Point is $(x_1, y_1) \equiv (x, y)$. Let $(x_2, y_2) \equiv (0,0)$.

Minimum distance $D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$= \sqrt{(x-0)^2 + (y-0)^2}$

$= \sqrt{x^2 + (2x-5)^2}$

$= \sqrt{5x^2 - 20x + 25}$

$\frac{dD}{dx} = 0$ gives minimum distance

$\frac{dD}{dx} = \frac{1}{2}\left(5x^2 - 20x + 25\right)^{-\frac{1}{2}} \times (10x - 20)$

$0 = \frac{10x-20}{2\sqrt{5x^2-20x+25}}$

$0 = 10x - 20$

$10x = 20$

$x = 2$

$y = 2 \times 2 - 5$

$= 1$

Distance $= \sqrt{5x^2 - 20x + 25}$

$= \sqrt{5(2)^2 - 20(2) + 25}$

$= \sqrt{5}$ units

4 $P(t) = 200te^{-\frac{t}{4}} + 400, \quad 0 \le t \le 12$

a Initially $t = 0$

$P(0) = 200(0)e^0 + 400 = 400$ birds

b Largest number of birds when $P'(t) = 0$.

$P'(t) = 200e^{-\frac{t}{4}} - 50te^{-\frac{t}{4}} = 0$

$e^{-\frac{t}{4}}(4-t) = 0$

$4 - t = 0$ as $e^{-\frac{t}{4}} > 0$ for all t

$t = 4$

At the end of December the population was at its largest.

c $P(4) = 200(4)e^{-1} + 400 = 694$ birds

5 Let one number be m and the other number be n. P is the product of the two numbers.

$m+n=32$

$m=32-n$..................(1)

$P=mn$.............................(2)

Substitute (1) into (2)

$P=n(32-n)$

$P=32n-n^2$

Max/min values occur where $\frac{dP}{dn}=0$.

$\frac{dP}{dn}=32-2n$

$0=32-2n$

$2n=32$

$n=16$

Substitute $n=16$ into (1)

$m=32-16=16$

Therefore both numbers are 16.

6 **a** $A=lw$..............................(1)

$2l+5w=550$

$2l=550-5w$

$l=\frac{1}{2}(550-5w)$..............(2)

Substitute (2) into (1)

$A=\frac{1}{2}w(550-5w)$

$A=\frac{550w}{2}-\frac{5w^2}{2}$

Max/min values occur where $\frac{dA}{dw}=0$.

$\frac{dA}{dw}=\frac{550}{2}-5w$

$0=\frac{550}{2}-5w$

$5w=\frac{550}{2}$

$w=\frac{550}{10}$

$w=55$ m

Substitute $w=55$ into (2)

$l=\frac{1}{2}(550-5(55))$

$l=\frac{275}{2}$

$l=137.5$ m

b $A_{max}=137.5\times 55=7562.5\text{ m}^2$

7 $SA_{cylinder}=220\pi=2\pi rh+2\pi r^2$

$V_{cylinder}=\pi r^2h$............(1)

$110=rh+r^2$

$110-r^2=rh$

$\frac{110}{r}-r=h$.................(2)

Substitute (2)into (1)

$V=\pi r^2\left(\frac{110}{r}-r\right)$

$V=110\pi r-\pi r^3$

Max/min values occur when $\frac{dV}{dr}=0$.

$\frac{dV}{dr}=110\pi-3\pi r^2$

$0=110\pi-3\pi r^2$

$3r^2=110$

$r^2=\frac{110}{3}$

$r=\sqrt{\frac{110}{3}}\ r>0$

$r=6.06$ cm

Substitute $r=6.06$ into (1)

$h=\frac{110}{6.055}-6.06=12.11$ cm

$V_{max}=\pi(6.06)^2(12.11)=1395.04\text{ cm}^3$

8 If $y=2\sqrt{x}$ and the point $(x_1,y_1)=(5,0)$ the shortest distance is given by

$D=\sqrt{(x-x_1)^2+(y-y_1)^2}$

$D=\sqrt{(x-5)^2+(2\sqrt{x}-0)^2}$

$D=\sqrt{x^2-10x+25+4x}$

$D=\sqrt{x^2-6x+25}$

Min distance occurs when $\frac{dD}{dx}=0$.

$\frac{dD}{dx}=\frac{1}{2}\times\frac{2x-6}{\sqrt{x^2-6x=25}}$

$\frac{dD}{dx}=\frac{x-3}{\sqrt{x^2-6x=25}}$

$0=\frac{x-3}{\sqrt{x^2-6x=25}}$

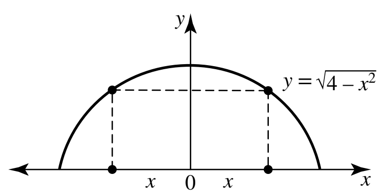
$0=x-3$

$x=3$

When $x=3,\ y=2\sqrt{3}$

$D_{min}=\sqrt{3^2-6(3)=25}=4$ units

9



$A=2x\sqrt{4-x^2}$

Max area occurs when $\frac{dA}{dx}=0$.

$\frac{dA}{dx}=-\frac{2x(-2x)}{2\sqrt{4-x^2}}+2\sqrt{4-x^2}$

$\frac{dA}{dx}=\frac{2(4-x^2)-2x^2}{\sqrt{4-x^2}}$

$0=8-4x^2$

$4x^2=8$

$x^2=2$

$x=\sqrt{2}\ x>0$

When

$x=\sqrt{2},\ A_{max}=2(\sqrt{2})\left(\sqrt{4-(\sqrt{2})^2}\right)=2(\sqrt{2})(\sqrt{2})=4\text{ units}^2.$

10 a $P = 96 = 2(2.5b) + 2(2a)$

$96 = 5b + 4a$

$48 = 2.5b + 2a$

$48 - 2.5b = 2a$

$24 - 1.25b = a$.................(1)

$A = ab + 2.5ab = 3.5ab$..........(2)

Substitute (1) into (2)

$A = 3.5b(24 - 1.25b)$

$A = 84b - 4.375b^2$

Max area occurs where $\frac{dA}{dx} = 0$.

$\frac{dA}{dx} = 84 - 8.75b$

$0 = 84 - 8.75b$

$8.75b = 84$

$b = 9.6$

Substitute $b = 9.6$ into (1)

$24 - 1.25(9.6) = a$

$12 = a$

b $A_{max} = 3.5(9.6)(12) = 403.2 \text{ m}^2$

11 a $A(t) = 1000 - 12te^{\frac{4-t^3}{8}}, \quad t \in [0,6]$

$A(0) = 1000 - 12(0)e^{\frac{4-0^3}{8}} = \$1000$

b Least amount of money occurs when $A'(t) = 0$.

$A'(t) = 12t \times \frac{3}{8}t^2 e^{\frac{4-t^3}{8}} - 12e^{\frac{4-t^3}{8}}$

$A'(t) = \frac{9}{2}t^3 e^{\frac{4-t^3}{8}} - 12e^{\frac{4-t^3}{8}}$

$A'(t) = e^{\frac{4-t^3}{8}}\left(\frac{9}{2}t^3 - 12\right)$

$0 = e^{\frac{4-t^3}{8}}\left(\frac{9}{2}t^3 - 12\right)$

$\frac{9}{2}t^3 - 12 = 0$ as $e^{\frac{4-t^3}{8}} > 0$ for all t

$\frac{9}{2}t^3 = 12$

$t^3 = \frac{24}{9}$

$t^3 = \frac{8}{3}$

$t = \sqrt[3]{\frac{8}{3}}$

$t = 1.387$

$A(1.387) = 1000 - 12(1 - 0.387)e^{\frac{4-1.3887^3}{8}} = \$980.34$

c The least amount of money occurred 1.387 years after January 1, 2009 which is May 2010.

d $A(6) = 1000 - 12(6)e^{\frac{4-6^3}{8}} = \$1000$

12 Area of pool is given by $A = 2lR + \frac{\pi}{2}R^2$ where A is a constant.

Perimeter of pool is given by $P = 2l + 2R + \pi R = 2l + (2+\pi)R$

From area equation $A - \frac{\pi}{2}R^2 = 2lR$

$A - \frac{\pi}{2}R^2 = 2lR$

$\frac{2A - \pi R^2}{2} = 2lR$

$\frac{2A - \pi R^2}{4R} = l$

Substitute $l = \frac{2A - \pi R^2}{4R}$ into perimeter equation.

$P = 2\left(\frac{2A - \pi R^2}{4R}\right) + (2+\pi)R$

$P = \frac{2A - \pi R^2 + 2(2+\pi)R^2}{2R}$

$P = \frac{2A - \pi R^2 + 2\pi R^2 + 4R^2}{2R}$

$P = \frac{2A + \pi R^2 + 4R^2}{2R}$

$P = \frac{A}{R} + \frac{(\pi+4)}{2}R$

Min value occurs when $\frac{dP}{dR} = 0$.

$\frac{dP}{dR} = -\frac{A}{R^2} + \frac{\pi+4}{2}$

$0 = \frac{-2A + (\pi+4)R^2}{2R^2}$

$0 = -2A + (\pi+4)R^2$

$2A = (\pi+4)R^2$

$\frac{2A}{\pi+4} = R^2$

$\sqrt{\frac{2A}{\pi+4}} = R, \quad R > 0$

Substitute $R = \sqrt{\frac{2A}{\pi+4}}$ into $A - \frac{\pi}{2}R^3 = 2lR$.

$A - \frac{\pi}{2}\left(\frac{2A}{\pi+4}\right) = 2l\left(\sqrt{\frac{2A}{\pi+4}}\right)$

$A - \frac{\pi A}{\pi+4} = 2l\left(\sqrt{\frac{2A}{\pi+4}}\right)$

$\frac{A(\pi+4) - \pi A}{\pi+4} = 2l\left(\sqrt{\frac{2A}{\pi+4}}\right)$

$\frac{2A}{\pi+4} = l\left(\sqrt{\frac{2A}{\pi+4}}\right)$

$\frac{2A}{\pi+4} \div \left(\sqrt{\frac{2A}{\pi+4}}\right) = l$

$\sqrt{\frac{2A}{\pi+4}} = l$

If both l and $R = \sqrt{\frac{2A}{\pi+4}}$, the perimeter is a minimum.

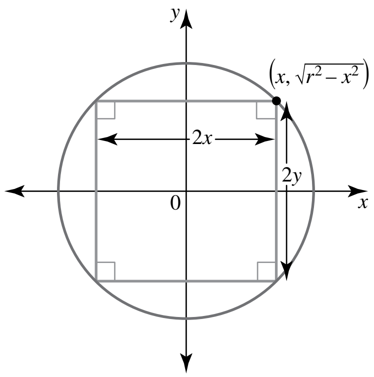
13 By Pythagoras

$$r^2+(h-12)^2=12^2$$
$$r^2=144-\left(h^2-24h+144\right)$$
$$r^2=24h-h^2$$
$$V_{\text{cone}}=\frac{1}{3}\pi r^2 h$$
$$V_{\text{cone}}=\frac{1}{3}\pi h\left(24h-h^2\right)$$
$$V_{\text{cone}}=8\pi h^2-\frac{1}{3}\pi h^3$$
$$\frac{dV}{dh}=16\pi h-\pi h^2$$

Max volume occurs when $\frac{dV}{dh}=0$.

$$16\pi h-\pi h^2=0$$
$$h(16-h)=0$$
$$h=0 \text{ or } 16-h=0$$
$$h=16,\quad (h>0)$$
$$V_{\max}=\frac{16\pi}{3}\left(24(16)-(16)^2\right)=2145\text{ cm}^3$$

14



By Pythagoras:

$$r^2=x^2+y^2$$
$$r^2-x^2=y^2$$
$$\sqrt{r^2-x^2}=y,\quad y>0$$

Area of rectangle is given by:

$$A=(2x)(2y)=4xy$$
$$A=4x\sqrt{r^2-x^2}$$
$$\frac{dA}{dx}=-\frac{8x^2}{2\sqrt{r^2-x^2}}+4\sqrt{r^2-x^2}$$
$$\frac{dA}{dx}=\frac{8\left(r^2-x^2\right)-8x^2}{2\sqrt{r^2-x^2}}$$
$$\frac{dA}{dx}=\frac{8r^2-8x^2-8x^2}{2\sqrt{r^2-x^2}}$$
$$\frac{dA}{dx}=\frac{4r^2-8x^2}{\sqrt{r^2-x^2}}$$

Max area occurs when $\frac{dA}{dx}=0$.

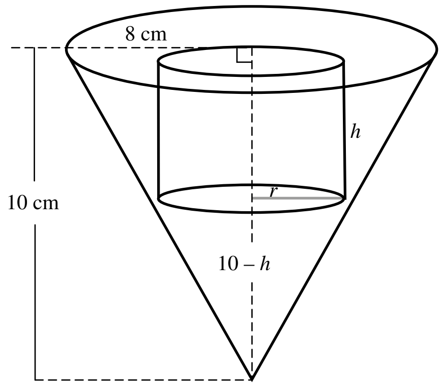
$$\frac{4r^2-8x^2}{\sqrt{r^2-x^2}}=0$$
$$4r^2-8x^2=0$$
$$4r^2=8x^2$$
$$\frac{1}{2}r^2=x^2$$
$$\frac{1}{\sqrt{2}}r=x,\quad x>0$$

Substitute $x=\frac{1}{\sqrt{2}}r$ into Pythagoras relationship.

$$r^2-x^2=y^2$$
$$r^2-\frac{1}{2}r^2=y^2$$
$$\frac{1}{2}r^2=y^2$$
$$y=\sqrt{\frac{1}{2}}r,\quad y>0$$

The x and y values are the same, thus, the largest rectangle is a square.

15



By similar triangles: $r:8$ as $10-h:h$

$$\frac{r}{8}=\frac{10-h}{10}$$
$$10r=80-8h$$
$$8h=80-10r$$
$$h=10-\frac{5}{4}r$$
$$V_{\text{cylinder}}=\pi r^2 h$$
$$V_{\text{cylinder}}=\pi r^2\left(10-\frac{5}{4}r\right)$$
$$V_{\text{cylinder}}=10\pi r^2-\frac{5}{4}\pi r^3$$
$$\frac{dV}{dr}=20\pi r-\frac{15}{4}r^2$$

Max volume occurs when $\frac{dV}{dr}=0$.

$$20\pi r-\frac{15}{4}\pi r^2=0$$
$$20r-\frac{15}{4}r^2=0$$
$$r\left(20-\frac{15}{4}r\right)=0$$
$$r=0 \text{ or } 20-\frac{15}{4}r=0$$
$$r=\frac{80}{15}=\frac{16}{3}\text{ cm},\quad r>0$$
$$h=10-\frac{5}{4}\left(\frac{16}{3}\right)=10-\frac{20}{3}=\frac{10}{3}\text{ cm}$$
$$V_{\max}=\pi\left(\frac{16}{3}\right)^2\left(\frac{10}{3}\right)=298\text{ cm}^3$$

16 Speed $= \dfrac{\text{distance}}{\text{time}}$

Rowing: $5 = \dfrac{AB}{t_r} = \dfrac{\sqrt{x^2+16}}{t_r}$ Walking: $8 = \dfrac{8-x}{t_w}$

$t_r = \dfrac{\sqrt{x^2+16}}{5}$ $\quad t_w = \dfrac{(8-x)}{8}$

Time for total journey is $T = t_r + t_w = \dfrac{\sqrt{x^2+16}}{5} + \dfrac{8-x}{8}$

$$\frac{dT}{dx} = \frac{2x}{10\sqrt{x^2+16}} - \frac{1}{8}$$
$$\frac{dT}{dx} = \frac{x}{5\sqrt{x^2+16}} - \frac{1}{8}$$

Min time occurs when $\dfrac{dT}{dx} = 0.$

$$\frac{x}{5\sqrt{x^2+16}} - \frac{1}{8} = 0$$
$$\frac{x}{5\sqrt{x^2+16}} = \frac{1}{8}$$
$$8x = 5\sqrt{x^2+16}$$
$$64x^2 = 25\left(x^2+16\right)$$
$$64x^2 = 25x^2 + 400$$
$$64x^2 - 25x^2 = 400$$
$$39x^2 = 400$$
$$x = \sqrt{\frac{400}{39}} = 3.2 \text{ km}$$

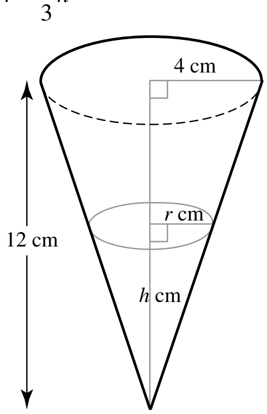
Therefore the rower will row to a point that is 3.2 km to the right of point O.

Exercise 6.7 — Rates of change

1 a By similar triangles

$r:4$ as $h:12$

$$\frac{r}{4} = \frac{h}{12}$$
$$r = \frac{1}{3}h$$



b $V = \dfrac{1}{3}\pi r^2 h$

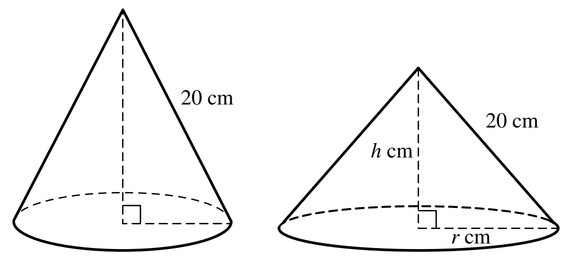
$$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$
$$V = \frac{\pi}{27}h^3$$

c $\dfrac{dV}{dh} = \dfrac{\pi}{9}h^2$

When $h = 5$ cm

$$\frac{dV}{dh} = \frac{\pi}{9}(5)^2 = \frac{25\pi}{9} \text{ cm}^3\text{ / cm}$$

2 a



By Pythagoras:

$$r^2 + h^2 = 20^2$$
$$r^2 = 400 - h^2$$
$$r = \sqrt{400-h^2}, \quad r > 0$$

b $V = \dfrac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi\left(400-h^2\right)h$$
$$V = \frac{400\pi h}{3} - \frac{\pi h^3}{3}$$

c $\dfrac{dV}{dh} = \dfrac{400\pi}{3} - \pi h^2$

When $h = 8$ cm

$$\frac{dV}{dh} = \frac{400\pi}{3} - \pi(8)^2$$
$$\frac{dV}{dh} = \frac{400\pi}{3} - \frac{192\pi}{3} = \frac{208\pi}{3} \text{ cm}^3\text{ / cm}$$

3 a Initially $t = 0$

$x = 2(0)^2 - 8(0) = 0$

It is at the origin initially.

b $v = \dfrac{dx}{dt} = 4t - 8$

When $t = 0$

$v = \dfrac{dx}{dt} = 4(0) - 8 = -8$

Initially it is moving with a velocity of 8 m/s to the left.

c

When $v = 0$	When $t = 2$
$0 = 4t - 8$ $8 = 4t$ $2 = t$	$x = 2(2)^2 - 8(2)$ $x = -8$ m

It is at rest after 2 seconds and is 8 metres to the left of the origin.

d When at the origin, $x = 0$.

$$2t^2 - 8t = 0$$
$$2t(t-4) = 0$$
$$t = 0 \text{ or } t - 4 = 0$$
$$t = 4$$

As expressed in **(a)** it is initially at the origin then it is there again after 4 seconds.

Initially it is at the origin, it then travels 8 metres to the left and at $t = 4$ it is back at the origin again so a total of 16 metres has been travelled.

e Average speed $= \dfrac{16-0}{4-0} = 4$ m/s

f Average velocity $= \dfrac{v_4 - v_0}{4} = \dfrac{8-8}{4} = 0$ m/s

4 **a** $x(t)=-\frac{1}{3}t^3+t^2+8t+1$ and $v(t)=-t^2+2t+8$

Initially $t=0$

$x(0)=-\frac{1}{3}(0)^3+(0)^2+8(0)+1$ and $v(0)=-(0)^2+2(0)+8$

$x(0)=1$ metre $\quad v(0)=8$ m/s

Initially it is 1 metre to the right of the origin travelling at 8 metres per second.

b It changes its direction of motion when $v=0$.

$-t^2+2t+8=0$

$(4-t)(2+t)=0$

$t=4,-2$

$t=4,\ t\geq 0$

$x(4)=-\frac{1}{3}(4)^3+(4)^2+8(4)+1=-\frac{64}{3}+49=-\frac{64}{3}+\frac{147}{3}=27\frac{2}{3}$ m

c $a(t)=\frac{dv}{dt}=-2t+2$

$a(4)=-2(4)+2=-6$ m/s^2

5 **a** $V_{\text{sphere}}=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$

When $r=10$ cm.

$\frac{dV}{dr}=4\pi(10)^2=400\pi$ cm^3/cm

b $SA_{\text{cube}}=6x^2$ where x is the side length of the cube.

$\frac{d(SA)}{dx}=-12x$ as the cube is melting

When $x=6$ mm.

$\frac{d(SA)}{dx}=-12(6)=-72$ mm^2/mm

6 $N=\frac{110}{t},\quad t>0$

a $\frac{dN}{dt}=-\frac{110}{t^2}$

When $t=5$ months.

$\frac{dN}{dt}=-\frac{110}{(5)^2}=-4.4$ rabbits per month

Population is decreasing at 4.4 rabbits per month.

b When $t=1,\ N=110$ and when $t=5,\ N=22$

Average rate of change is $\frac{22-110}{5-1}=-\frac{88}{4}=-22$ rabbits/month

c As $t\to\infty,\ N\to 0$ so the rabbits will eventually disappear.

7 $V=0.4(8-t)^3,\quad 0\leq t\leq 8$

a $\frac{dV}{dt}=-1.2(8-t)^2$

When $t=3$ minutes

$\frac{dV}{dt}=-1.2(8-3)^2=-30$ litres/min

Water is leaving the bath at a rate of 30 L/min

b When $t=0,\ V=0.4(8)^3=204.8$ and when $t=3,\ V=0.4(5)^3=50$

Average rate of change is $\frac{204.8-50}{3-0}=-51.6$ litres/minute

c $\frac{dV}{dt}=R(x)$

$R'(x)=0$

$R'(x)=-2.4(8-t)\times -1$

$0=2.4(8-t)$

$t=8$

$t=8$ corresponds to a minimum, therefore the maximum rate is when $t=0$

The rate of water leaving is greatest at the beginning which is when $t=0$.

8 $V = \frac{2}{3}t^2(15 - t), \quad 0 \le t \le 10$

a When $t = 10$, $V = \frac{2}{3}(10)^2(15 - 10) = 333\frac{1}{3}$ mL.

b $\frac{dV}{dt} = -\frac{2}{3}t^2 + \frac{4}{3}t(15 - t)$

$\frac{dV}{dt} = 20t - \frac{4}{3}t^2 - \frac{2}{3}t^2 = 20t - 2t^2$

c When $t = 3$ seconds.

$\frac{dV}{dt} = 20(3) - 2(3)^2 = 60 - 18 = 42$ mL/s

d The flow greatest when $\frac{d}{dx}\left(\frac{dV}{dt}\right) = 0$.

$\frac{d}{dx}\left(\frac{dV}{dt}\right) = 20 - 4t$

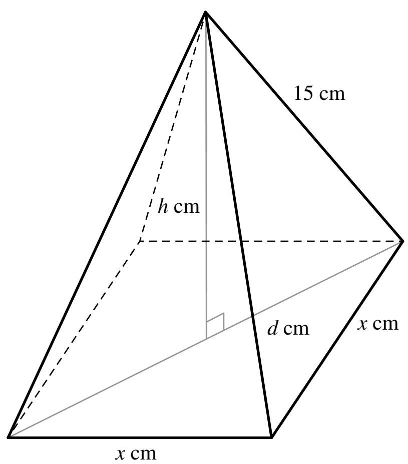
$0 = 20 - 4t$

$4t = 20$

$t = 5$

When $t = 5$, $\frac{dV}{dt} = 20(5) - 2(5)^2 = 50$ mL/s

9



a $V = \frac{1}{3}x^2h$

By Pythagoras

$d^2 = x^2 + x^2$

$d^2 = 2x^2$

$d = \sqrt{2}x$ m, $x > 0$

b By Pythagoras

$12^2 = h^2 + \left(\frac{\sqrt{2}}{2}x\right)^2$

$144 = h^2 + \frac{1}{2}x^2$

$144 - h^2 = \frac{1}{2}x^2$

$288 - 2h^2 = x^2$ as required

$V = \frac{1}{3}x^2h$

$V = \frac{1}{3}\left(288 - 2h^2\right)h$

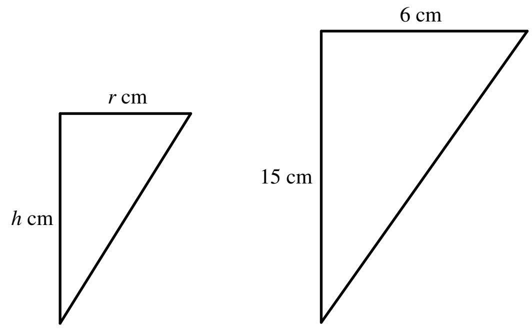
$V = \frac{1}{3}\left(288h - 2h^3\right)$

c $\frac{dV}{dh} = \frac{1}{3}(288 - 6h^2)$

When $h = 3\sqrt{3}$ metres.

$$\frac{dV}{dh} = \frac{1}{3}\left(288 - 6\left(3\sqrt{3}\right)^2\right)$$
$$= \frac{1}{3}(288 - 6 \times 27)$$
$$= \frac{126}{3}$$
$$= 42\text{m}^3 / \text{m}$$

10 $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$



By similar triangles: $r : 6$ as $h : 15$

$\frac{r}{6} = \frac{h}{15}$

$r = \frac{2h}{5}$

$V_{\text{cone}} = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$

$V = \frac{4\pi}{25}h^3$

$\frac{dV}{dh} = \frac{4\pi}{25}h^2$

a When $h = \frac{15}{2}$

$\frac{dV}{dh} = \frac{4\pi}{25}\left(\frac{15}{2}\right)^2 = 9\pi \text{ cm}^3 / \text{cm}$

b Container is one third full. When full, volume is 180π cm^3 so one third will be 60π cm^3.

$$60\pi = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$$
$$180 = \frac{4}{25}h^3$$
$$180 \times \frac{25}{4} = h^3$$
$$h^3 = 1125$$
$$h = 5 \times 3^{\frac{2}{3}}$$

$\frac{dV}{dh} = \frac{4\pi}{25}\left(5 \times 3^{\frac{2}{3}}\right)^2 = 12 \times 3^{\frac{1}{3}}\pi \text{ cm}^3 / \text{cm}$

11 $x(t) = 2t^2 - 16t - 18, \quad t \le 0$

a When $t = 2$ seconds

$x(2) = 2(2)^2 - 16(2) - 18 = -42$

The particle is 42 metres to the left of the origin.

b $\frac{dv}{dt} = 4t - 16$

When $t = 2$ seconds

$\frac{dv}{dt}_{t=2} = 4(2) - 16 = -8$

The speed is 8 m/s.

c When $t=0,\ \frac{dv}{dt}=-16$ and when $t=2,\ \frac{dv}{dt}=-8$

Average velocity is $\frac{-16+-8}{2-0}=-12$ m/s

d When $x(t)=0$

$2t^2-16t-18=0$

$t^2-8t-9=0$

$(t-9)(t+1)=0$

$t-9=0$

$t=9$ as $t\geq 0$

$\left.\frac{dv}{dt}\right|_{t=9}=4(9)-16=20$ m/s

Therefore it reaches O after 9 seconds at 20 m/s

12 $x=\frac{2}{3}t^3-4t^2,\quad t\geq 0$

a $v=\frac{dx}{dt}=2t^2-8t$

When $t=0$

$x_{t=0}=\frac{2}{3}(0)^3-4(0)^2=0$

$v_{t=0}=2(0)^2-8(0)=0$

The particle starts from rest at the origin.

b When $v=0$

$2t^2-8t=0$

$t^2-4t=0$

$t(t-4)=0$

$t=0$ or $t-4=0$

Initially $\quad t=4$

$x_{t=4}=\frac{2}{3}(4)^3-4(4)^2=\frac{128}{3}-\frac{192}{3}=-\frac{64}{3}=-21\frac{1}{3}$

Velocity is zero after 4 seconds when the particle is $21\frac{1}{3}$ metres to the left of the origin.

c When $x=0$

$\frac{2}{3}t^3-4t^2=0$

$t^2\left(\frac{2}{3}t-4\right)=0$

$t=0$ or $\frac{2}{3}t-4=0$

Initially $\quad \frac{2}{3}t=4$

$t=6$

The particle is at the origin again after 6 seconds.

d When t = 6 seconds

$v_{t=6}=2(6)^2-8(6)=72-48=24$ m/s

$a=\frac{dv}{dt}=4t-8$

$a_{t=6}=4(6)-8=24-8=16$ m/s^2

At the origin the particle's speed is 24 m/s and the acceleration is 16 m/s^2.

13 $h=50t-4t^2$

a $\frac{dh}{dt}=50-8t$

When $t=3$ seconds

$\left.\frac{dh}{dt}\right|_{t=3}=50-8(3)=50-24=26$ m/s

b When $t = 5$ seconds

$v_{t=5} = \frac{dh}{dt} = 50 - 8(5) = 10$ m/s

c When $v = -12$ m/s

$-12 = 50 - 8t$

$8t = 62$

$t = 7.75$

After 7.75 seconds the velocity of the ball is 12 m/s and it is travelling downwards.

d When $v = 0$

$50 - 8t = 0$

$8t = 50$

$t = 6.25$ seconds

The velocity is zero after 6.25 seconds.

e Greatest height is obtained when the velocity is zero.

$h_{t=6.25} = 50(6.25) - 4(6.25)^2 = 156.25$ metres

f When the ball strikes the ground, $h = 0$.

$0 = 50t - 4t^2$

$0 = 25t - 2t^2$

$0 = t(25 - 2t)$

$t = 0$ or $25 - 2t = 0$

Initially $\quad 2t = 25$

$t = 12.5$

The ball strikes the ground after 12.5 seconds.

$v_{t=12.5} = 50 - 8(12.5) = -50$ m/s

The ball hits the ground with a speed of 50 m/s.

14 $N(t) = \frac{2t}{(t+0.5)^2} + 0.5$

a Initially $t = 0$

$N(0) = \frac{2(0)}{(0+0.5)^2} + 0.5 = 0.5$ hundred thousand or 50 thousand

b $N(t) = \frac{2t}{(t+0.5)^2} + 0.5$

Let $u = 2t$ and $v = (t+0.5)^2$

$\frac{du}{dt} = 2 \qquad \frac{dv}{dt} = 2(t+0.5) = 2t + 1$

$$N'(t) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

$$= \frac{2(t+0.5)^2 - 2t(2t+1)}{(t+0.5)^4}$$

$$= \frac{2t^2 + 2t + 0.5 - 4t^2 - 2t}{(t+0.5)^4}$$

$$= \frac{-2t^2 + 0.5}{(t+0.5)^4}$$

c Maximum number of viruses occurs when $\frac{dN}{dt} = 0$.

$\frac{-2t^2 + 0.5}{(t+0.5)^4} = 0$

$-2t^2 + 0.5 = 0$

$2t^2 = 0.5$

$t^2 = 0.25$

$t = 0.5, \quad t \geq 0$

$N(1) = \frac{2(0.5)}{(0.5+0.5)^2} + 0.5 = 1.5$ hundred thousand after half an hour

d When $t = 10$

$$\left.\frac{dN}{dt}\right|_{t=10} = \frac{-2(10)^2 + 0.5}{(10+0.5)^4} = -\frac{199.5}{10.5^4} = -0.01641$$

After 10 hours the viruses were changing at a rate of −1641 viruses per hour.

15 $N = 220 - \dfrac{150}{t+1}$

a When $N = 190$

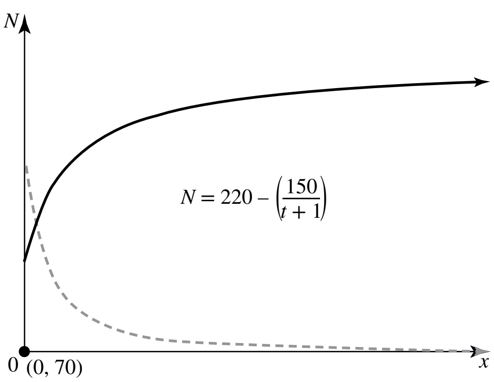
$$190 = 220 - \frac{150}{t+1}$$
$$220 - 190 = \frac{150}{t+1}$$
$$30(t+1) = 150$$
$$t+1 = 5$$
$$t = 4$$
$$\frac{dN}{dt} = \frac{150}{(t+1)^2}$$
$$t = 4, \frac{dN}{dt} = \frac{150}{(4+1)^2} = \frac{150}{25} = 6$$

Therefore after 4 years, butterflies are growing at a rate of 6 butterflies per year.

b Growth rate is 12 butterflies per year.

$$\frac{dN}{dt} = \frac{150}{(t+1)^2}$$
$$12 = \frac{150}{(t+1)^2}$$
$$12(t+1)^2 = 150$$
$$(t+1)^2 = 12.5$$
$$t+1 = 3.54, \quad t \geq 0$$
$$t = 2.54 \text{ years}$$

c



As $t \to \infty$, $N \to 220$ and $\dfrac{dN}{dt} \to 0$.

16 a $y = \dfrac{3t}{(4+t^2)}$

Let $u = 3t$ and $v = 4 + t^2$

$$\frac{du}{dt} = 3 \qquad \frac{dv}{dt} = 2t$$

$$\frac{dy}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$
$$\frac{dy}{dt} = \frac{3(4+t^2) - 3t(2t)}{(4+t^2)^2}$$
$$\frac{dy}{dt} = \frac{12 + 3t^2 - 6t^2}{(4+t^2)^2}$$
$$\frac{dy}{dt} = \frac{12 - 3t^2}{(4+t^2)^2}$$
$$\frac{dy}{dt} = \frac{3(4-t^2)}{(4+t^2)^2}$$

b Maximum concentration of painkiller in the blood occur when $\dfrac{dy}{dt} = 0$.

$$0 = \frac{3(4-t^2)}{(4+t^2)^2}$$
$$0 = 3(4-t^2)$$
$$0 = 4 - t^2$$
$$t = 2, -2$$
$$t = 2, \quad t > 0$$
$$t = 2, \quad y = \frac{3(2)}{(4+2^2)} = 0.75 \text{ mg/L}$$

Therefore max concentration is 0.75 mg/L after 2 hours

c $0.5 = \dfrac{3t}{(4+t^2)}$

$$2 + \frac{1}{2}t^2 = 3t$$
$$t^2 - 6t + 2 = 0$$
$$t = \frac{6 \pm \sqrt{(6)^2 - (4)(1)(2)}}{2(1)}$$
$$t = \frac{6 \pm \sqrt{36-16}}{2}$$
$$t = \frac{6 + 2\sqrt{5}}{2} \simeq 5.24 \text{ hours}, \quad (t > 2)$$

d

$$\left.\frac{dy}{dt}\right|_{t=1} = \frac{3(4-1^2)}{(4+1^2)^2}$$
$$\left.\frac{dy}{dt}\right|_{t=1} = \frac{9}{25}$$
$$\left.\frac{dy}{dt}\right|_{t=1} = 0.36 \text{ mg/L/h}$$

e

$$\frac{dy}{dt} = \frac{3(4-t^2)}{(4+t^2)^2}$$
$$-0.06 = \frac{3(4-t^2)}{(4+t^2)^2}$$
$$t = 2.45 \text{ and } 6 \text{ hours}$$

(solved on CAS)

Topic 7 — Antidifferentiation

Exercise 7.2 — Antidifferentiation

1 a $f'(x)=\frac{3}{2}x-4x^2+2x^3$

$f(x)=\frac{3}{4}x^2-\frac{4}{3}x^3+\frac{1}{2}x^4+c$

b $\int\left(\frac{3}{\sqrt{x}}-4x^3+\frac{2}{5x^3}\right)dx$

$=\int\left(3x^{-\frac{1}{2}}-4x^3+\frac{2}{5}x^{-3}\right)dx$

$=6x^{\frac{1}{2}}-x^4-\frac{1}{5}x^{-2}$

$=6\sqrt{x}-x^4-\frac{1}{5x^2}$

c $\int x(x-3)(2x+5)\,dx$

$=\int\left(2x^3-x^2-15x\right)dx$

$=\frac{1}{2}x^4-\frac{1}{3}x^3-\frac{15}{2}x^2+c$

d $\int\frac{3x^3-x}{2\sqrt{x}}dx$

$=\int\frac{3}{2}x^{\frac{5}{2}}-\frac{1}{2}x^{\frac{1}{2}}dx$

$=\frac{3}{7}x^{\frac{7}{2}}-\frac{1}{3}x^{\frac{3}{2}}+c$

2 a $\int\left(\frac{2}{\sqrt{x}}+\frac{3}{x^2}-\frac{1}{2x^3}\right)dx$

$=\int\left(2x^{-\frac{1}{2}}+3x^{-2}-\frac{1}{2}x^{-3}\right)dx$

$=4x^{\frac{1}{2}}-3x^{-1}+\frac{1}{4}x^{-2}+c$

$=4\sqrt{x}-\frac{3}{x}+\frac{1}{4x^2}+c$

b $\int(x+1)(2x^2-3x+4)\,dx$

$=\int\left(2x^3-x^2+x+4\right)dx$

$=\frac{1}{2}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2+4x+c$

3 a $\int(3x-5)^5\,dx=\frac{(3x-5)^6}{3\times 6}=\frac{1}{18}(3x-5)^6+c$

b $\int\frac{1}{(2x-3)^{\frac{5}{2}}}\,dx=\int(2x-3)^{-\frac{5}{2}}\,dx$

$=\frac{(2x-3)^{-\frac{3}{2}}}{2\times-\frac{3}{2}}$

$=-\frac{1}{3(2x-3)^{\frac{3}{2}}}+c$

4 a $\int(2x+3)^4dx=\frac{(2x+3)^5}{2\times 5}$

$=\frac{1}{10}(2x+3)^5+c$

b $\int(1-2x)^{-5}dx=\frac{(1-2x)^{-4}}{-2\times-4}$

$=\frac{1}{8}(1-2x)^{-4}$

$=\frac{1}{8(1-2x)^4}+c$

5 a $\int\left(2x^2+\frac{1}{x}\right)^3dx$

$=\int\left(\left(2x^2\right)^3+3\left(2x^2\right)^2\left(\frac{1}{x}\right)+3\left(2x^2\right)\left(\frac{1}{x}\right)^2+\left(\frac{1}{x}\right)^3\right)dx$

$=\int\left(8x^6+12x^3+6+x^{-3}\right)dx$

$=\frac{8}{7}x^7+3x^4-\frac{1}{2}x^{-2}+c$

$=\frac{8}{7}x^7+3x^4+6x-\frac{1}{2x^2}+c$

6 a $\int\left(\sqrt{x}-x\right)^2dx$

$=\int\left(\left(\sqrt{x}\right)^2-2\left(\sqrt{x}\right)(x)+(x)^2\right)dx$

$=\int\left(x-2x^{\frac{3}{2}}+x^2\right)dx$

$=\frac{1}{2}x^2-\frac{4}{5}x^{\frac{5}{2}}+\frac{1}{3}x^3+c$

b $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3dx$

$=\int\left(\left(x^{\frac{1}{2}}\right)^3+3\left(x^{\frac{1}{2}}\right)^2\left(x^{-\frac{1}{2}}\right)+3\left(x^{\frac{1}{2}}\right)\left(x^{-\frac{1}{2}}\right)^2+\left(x^{-\frac{1}{2}}\right)^3\right)dx$

$=\int\left(x^{\frac{3}{2}}+3x^{\frac{1}{2}}+3x^{-\frac{1}{2}}+x^{-\frac{3}{2}}\right)dx$

$=\frac{2}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+6x^{\frac{1}{2}}-2x^{-\frac{1}{2}}+c$

7 $y=(3x^2+2x-4)^3$

$\frac{dy}{dx}=3(6x+2)(3x^2+2x-4)^2$

$\frac{dy}{dx}=6(3x+1)(3x^2+2x-4)^2$

Therefore

$\int(3x+1)(3x^2+2x-4)^2\,dx=\frac{1}{6}\int 6(3x+1)(3x^2+2x-4)^2$

$=\frac{1}{6}(3x^2+2x-4)^3$

8 $y=\left(7x+\sqrt{x}-\frac{1}{\sqrt{x}}\right)^4$

$y=\left(7x+x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)^4$

$\frac{dy}{dx}=4\left(7+\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}\right)\left(7x+\sqrt{x}-\frac{1}{\sqrt{x}}\right)^3$

$\frac{dy}{dx}=4\left(7+\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x^3}}\right)\left(7x+\sqrt{x}-\frac{1}{\sqrt{x}}\right)^3$

Therefore

$\int\left(7+\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x^3}}\right)\left(7x+\sqrt{x}-\frac{1}{\sqrt{x}}\right)^3 dx$

$=\frac{1}{4}\int 4\left(7+\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x^3}}\right)\left(7x+\sqrt{x}-\frac{1}{\sqrt{x}}\right)^3 dx$

$=\frac{1}{4}\left(7x+\sqrt{x}-\frac{1}{\sqrt{x}}\right)^4$

9 $f'(x)=x^2-\frac{1}{x^2}$

$f'(x)=x^2-x^{-2}$

$f(x)=\frac{1}{3}x^3+x^{-1}+c$

$f(x)=\frac{1}{3}x^3+\frac{1}{x}+c$

10 a $\int x^3\,dx=\frac{1}{4}x^4+c$

b $\int\left(7x^2-\frac{2}{5x^3}\right)dx=\int\left(7x^2-\frac{2}{5}x^{-3}\right)dx=\frac{7}{3}x^3+\frac{1}{5}x^{-2}+c$

c $\int(4x^3-7x^2+2x-1)\,dx=x^4-\frac{7}{3}x^3+x^2-x+c$

d $\int\left(2\sqrt{x}\right)^3 dx$

$=\int 8x^{\frac{3}{2}}\,dx$

$=8\times\frac{2}{5}x^{\frac{5}{2}}+c$

$=\frac{16}{5}\sqrt{x^5}+c$

$=\frac{16}{5}x^2\sqrt{x}+c$

11 a $\int(3x-1)^3\,dx=\int\left((3x)^3-3(3x)^2(1)+3(3x)(1)^2+1^3\right)dx$

$=\int\left(27x^3-27x^2+9x-1\right)dx$

$=\frac{27}{4}x^4-9x^3+\frac{9}{2}x^2-x+c$

b $\int\frac{1}{4x^3}dx$

$=\int\frac{1}{4}x^{-3}\,dx$

$=-\frac{1}{8}x^{-2}+c$

$=-\frac{1}{8x^2}+c$

c $\int\left(x^{\frac{5}{2}}-3x^{\frac{2}{5}}\right)dx=\frac{2}{7}x^{\frac{7}{2}}-\frac{15}{7}x^{\frac{7}{5}}+c$

d $\int\frac{x^4-2x}{x^3}dx$

$=\int\left(x-2x^{-2}\right)dx$

$=\frac{1}{2}x^2+2x^{-1}+c$

$=\frac{1}{2}x^2+\frac{2}{x}+c$

e $\int\sqrt{x}\left(2x-\sqrt{x}\right)dx$

$=\int\left(2x^{\frac{3}{2}}-x\right)dx$

$=\frac{4}{5}x^{\frac{5}{2}}-\frac{1}{2}x^2+c$

f $\int\sqrt{4-x}dx$

$=\int(4-x)^{\frac{1}{2}}dx$

$=-\frac{2}{3}(4-x)^{\frac{3}{2}}+c$

12 a $\int(2x+3)(3x-2)\,dx$

$=\int\left(6x^2+5x-6\right)dx$

$=2x^3+\frac{5}{2}x^2-6x$

b $\int\frac{x^3+x^2+1}{x^2}\,dx$

$=\int\left(x+1+x^{-2}\right)dx$

$=\frac{1}{2}x^2+x-x^{-1}$

$=\frac{1}{2}x^2+x-\frac{1}{x}$

c $\int\left(2\sqrt{x}-\frac{4}{\sqrt{x}}\right)dx$

$=\int\left(2x^{\frac{1}{2}}-4x^{-\frac{1}{2}}\right)dx$

$=2\times\frac{2}{3}x^{\frac{3}{2}}-4\times 2x^{\frac{1}{2}}$

$=\frac{4}{3}x^{\frac{3}{2}}-8x^{\frac{1}{2}}$

$=\frac{4}{3}x\sqrt{x}-8\sqrt{x}$

d $\int\left(x^3-\frac{2}{x^3}\right)^2 dx$

$=\int\left((x^3)^2-2(x^3)\left(\frac{2}{x^3}\right)+\left(\frac{2}{x^3}\right)^2\right)dx$

$=\int\left(x^6-4+4x^{-6}\right)dx$

$=\frac{1}{7}x^7-4x-\frac{4}{5}x^{-5}$

$=\frac{1}{7}x^7-4x-\frac{4}{5x^5}$

e $\int 2(1-4x)^{-3}dx$

$= 2\int (1-4x)^{-3}dx$

$= 2\left(\frac{(1-4x)^{-2}}{-2\times -4}\right)$

$= \frac{1}{4(1-4x)^2}$

f $\int \frac{2}{(2x-3)^{\frac{5}{2}}}dx$

$= \int 2(2x-3)^{-\frac{5}{2}}dx$

$= 2\int (2x-3)^{-\frac{5}{2}}dx$

$= 2\left(-\frac{1}{3}(2x-3)^{-\frac{3}{2}}\right)$

$= -\frac{2}{3(2x-3)^{\frac{3}{2}}}$

13 $\frac{dy}{dx} = x^3 - 3\sqrt{x} = x^3 - 3x^{\frac{1}{2}}$

$y = \frac{1}{4}x^4 - 3\times\frac{2}{3}x^{\frac{3}{2}} + c$

$y = \frac{1}{4}x^4 - 2x\sqrt{x} + c$

14 $\frac{dy}{dx} = \frac{x^3+3x^2-3}{x^2} = x + 3 - 3x^{-2}$

$y = \frac{1}{2}x^2 + 3x + 3x^{-1} + c$

$y = \frac{1}{2}x^2 + 3x + \frac{3}{x} + c$

15 $\frac{dy}{dx} = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$y = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$

$y = \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + c$

16 $y = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(2x)(x^2+1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$

$\int \frac{5x}{\sqrt{x^2+1}}dx = 5\int \frac{x}{\sqrt{x^2+1}}dx$

$= 5\sqrt{x^2+1} + c$

17 $y = \left(5x^2+2x-1\right)^4$

$\frac{dy}{dx} = 4(10x+2)\left(5x^2+2x-1\right)^3$

$\frac{dy}{dx} = 8(5x+1)\left(5x^2+2x-1\right)^3$

$\int 16(5x+1)\left(5x^2+2x-1\right)^3 dx = 2\int 8(5x+1)\left(5x^2+2x-1\right)^3 dx$

$= 2\left(5x^2+2x-1\right)^4$

18 $y = \sqrt{5x^3+4x^2} = \left(5x^3+4x^2\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(15x^2+8x)\left(5x^3+4x^2\right)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{15x^2+8x}{2\sqrt{5x^3+4x^2}}$

$\int \frac{15x^2+8x}{\sqrt{5x^3+4x^2}}dx$

$= 2\int \frac{15x^2+8x}{2\sqrt{5x^3+4x^2}}dx$

$= 2\sqrt{5x^3+4x^2}$

19 $\int \frac{x^2}{\sqrt{x^3+1}}dx$

$= \frac{2}{3}\left(x^3+1\right)^{\frac{1}{2}} + c$

$= \frac{2}{3}\sqrt{x^3+1} + c$

20 $\int 2(3x+5)^{\frac{1}{2}}\left(7x^2+4x-1\right)dx$

$= \frac{4}{405}(3x+5)^{\frac{3}{2}}(135x^2-72x+35) + c$

Exercise 7.3 — Antiderivatives of exponential and trigonometric functions

1 a $\int\left(\frac{1}{2}\cos(3x+4) - 4\sin\left(\frac{x}{2}\right)\right)dx = \frac{1}{6}\sin(3x+4) + 8\cos\left(\frac{x}{2}\right) + c$

b $\int\left(\cos\left(\frac{2x}{3}\right) - \frac{1}{4}\sin(5-2x)\right)dx = \frac{3}{2}\sin\left(\frac{2x}{3}\right) - \frac{1}{8}\cos(5-2x) + c$

2 a $\int\left(\sin\left(\frac{x}{2}\right) - 3\cos\left(\frac{x}{2}\right)\right)dx = -2\cos\left(\frac{x}{2}\right) - 6\sin\left(\frac{x}{2}\right) + c$

b $f'(x) = 7\cos(2x) - \sin(3x)$

$f(x) = \frac{7}{2}\sin(2x) + \frac{1}{3}\cos(3x) + c$

3 a $\int\left(x^4 - e^{-4x}\right)dx = \frac{1}{5}x^5 + \frac{1}{4}e^{-4x} + c$

b $\int\left(\frac{1}{2}e^{2x} - \frac{2}{3}e^{-\frac{x}{2}}\right)dx = \frac{1}{4}e^{2x} + \frac{4}{3}e^{-\frac{x}{2}} + c$

4 a $\int\left(e^{\frac{x}{2}} + \sin\left(\frac{x}{3}\right) + \frac{x}{3}\right)dx = 3e^{\frac{x}{2}} - 3\cos\left(\frac{x}{3}\right) + \frac{1}{6}x^2 + c$

b $\int\left(\cos(4x) + 3e^{-3x}\right)dx = \frac{1}{4}\sin(4x) - e^{-3x} + c$

5 $\int\left(e^{2x} - e^{-3x}\right)^3 dx$

$= \int\left(\left(e^{2x}\right)^3 - 3\left(e^{2x}\right)^2\left(e^{-3x}\right) + 3\left(e^{2x}\right)\left(e^{-3x}\right)^2 - \left(e^{-3x}\right)^3\right)dx$

$= \int\left(e^{6x} - 3e^{x} + 3e^{-4x} - e^{-9x}\right)dx$

$= \frac{1}{6}e^{6x} - 3e^{x} - \frac{3}{4}e^{-4x} + \frac{1}{9}e^{-9x} + c$

6 $\int\left(e^{\frac{x}{2}}-\frac{1}{e^x}\right)^2 dx$

$=\int\left(\left(e^{\frac{x}{2}}\right)^2-2\left(e^{\frac{x}{2}}\right)\left(\frac{1}{e^x}\right)+\left(\frac{1}{e^x}\right)^2\right)dx$

$=\int\left(e^x-2e^{-\frac{x}{2}}+e^{-2x}\right)dx$

$=e^x+4e^{-\frac{x}{2}}-\frac{1}{2}e^{-2x}+c$

$=e^x+\frac{4}{e^{\frac{x}{2}}}-\frac{1}{2e^{2x}}+c$

7 $y=e^{\cos^2(x)}=e^{(\cos(x))^2}$

$\frac{dy}{dx}=-2\sin(x)\cos(x)e^{\cos^2(x)}$

Therefore

$\int\sin(x)\cos(x)e^{\cos^2(x)}\,dx=-\frac{1}{2}\int-2\sin(x)\cos(x)e^{\cos^2(x)}\,dx$

$=-\frac{1}{2}e^{\cos^2(x)}$

8 $y=e^{(x+1)^3}$

$\frac{dy}{dx}=3(1)(x+1)^2e^{(x+1)^3}$

$\frac{dy}{dx}=3(x+1)^2e^{(x+1)^3}$

Therefore

$\int 9(x+1)^2e^{(x+1)^3}dx=3\int 3(x+1)^2e^{(x+1)^3}\,dx$

$=3e^{(x+1)^3}+c$

9 **a** $\int\left(2e^{3x}-\sin(2x)\right)dx=\frac{2}{3}e^{3x}+\frac{1}{2}\cos(2x)+c$

b $\int\frac{e^{2x}+3e^{-5x}}{2e^x}dx=\int\left(\frac{1}{2}e^x+\frac{3}{2}e^{-6x}\right)dx$

$=\frac{1}{2}e^x-\frac{1}{4}e^{-6x}+c$

c $\int\left(0.5\cos(2x+5)-e^{-x}\right)dx=\frac{1}{4}\sin(2x+5)+e^{-x}+c$

d $\int\left(e^x-e^{2x}\right)^2dx$

$=\int\left((e^x)^2-2(e^x)(e^{2x})+(e^{2x})^2\right)dx$

$=\int\left(e^{2x}-2e^{3x}+e^{4x}\right)dx$

$=\frac{1}{2}e^{2x}-\frac{2}{3}e^{3x}+\frac{1}{4}e^{4x}+c$

10 $\int ae^{bx}\,dx=-3e^{3x}+c$

$\frac{a}{b}e^{bx}+c=-3e^{3x}+c$

$b=3$ and $\frac{a}{3}=-3$

$a=-9$

11 $\int\left(\frac{1}{4x^2}+\sin\left(\frac{3\pi x}{2}\right)\right)dx$

$=\int\left(\frac{1}{4}x^{-2}+\sin\left(\frac{3\pi x}{2}\right)\right)dx$

$=-\frac{1}{4}x^{-1}-\frac{2}{3\pi}\cos\left(\frac{3\pi x}{2}\right)$

$=-\frac{1}{4x}-\frac{2}{3\pi}\cos\left(\frac{3\pi x}{2}\right)$

12 $\frac{dy}{dx}=\cos(2x)-e^{-3x}$

$y=\frac{1}{2}\sin(2x)+\frac{1}{3}e^{-3x}+c$

13 **a** $\frac{dH}{dt}=1+\frac{\pi^2}{9}\sin\left(\frac{\pi t}{45}\right)$

$H(t)=t-\left(\frac{45}{\pi}\times\frac{\pi^2}{9}\right)\cos\left(\frac{\pi t}{45}\right)$

$H(t)=t-5\pi\cos\left(\frac{\pi t}{45}\right)$

b When $t=15$,

$H=15-5\pi\cos\left(\frac{\pi}{3}\right)=7.146$ kilojoules

14 $y=2xe^{3x}$

$\frac{dy}{dx}=2e^{3x}+6xe^{3x}$

$\int\left(2e^{3x}+6xe^{3x}\right)dx=2xe^{3x}$

$\int 2e^{3x}\,dx+6\int xe^{3x}\,dx=2xe^{3x}$

$6\int xe^{3x}\,dx=2xe^{3x}-\int 2e^{3x}\,dx$

$6\int xe^{3x}\,dx=2xe^{3x}-\frac{2}{3}e^{3x}$

$3\int xe^{3x}dx=xe^{3x}-\frac{1}{3}e^{3x}$

$\int xe^{3x}\,dx=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}$

15 $y=e^{2x^2+3x-1}$

$\frac{dy}{dx}=(4x+3)e^{2x^2+3x-1}$

$\int 2(4x+3)e^{2x^2+3x-1}\,dx=2\int(4x+3)e^{2x^2+3x-1}\,dx=2e^{2x^2+3x-1}$

16 $y=x\cos(x)$

$\frac{dy}{dx}=\cos(x)-x\sin(x)$

$\int(\cos(x)-x\sin(x))dx=x\cos(x)$

$\int\cos(x)\,dx-\int x\sin(x)\,dx=x\cos(x)$

$\int\cos(x)\,dx-x\cos(x)=\int x\sin(x)\,dx$

$\sin(x)-x\cos(x)=\int x\sin(x)\,dx$

17 $x(t)=20+\cos\left(\frac{\pi t}{4}\right)$

$\frac{dy}{dt}=\frac{\pi}{20}x(t)-\pi$

$\frac{dy}{dt}=\frac{\pi}{20}\left(20+\cos\left(\frac{\pi t}{4}\right)\right)-\pi$

$\frac{dy}{dt}=\pi+\frac{\pi}{20}\cos\left(\frac{\pi t}{4}\right)-\pi$

$\frac{dy}{dt}=\frac{\pi}{20}\cos\left(\frac{\pi t}{4}\right)$

$y=\frac{4}{\pi}\times\frac{\pi}{20}\sin\left(\frac{\pi t}{4}\right)+c$

$y=\frac{1}{5}\sin\left(\frac{\pi t}{4}\right)+c$

18 $f'(x)=a\sin(mx)-be^{nx}$

$f(x)=\cos(2x)-2e^{-2x}+3$

$f'(x)=-2\sin(2x)+4e^{-2x}$

Therefore $a=-2,\ b=-4,\ m=2$, and $n=-2$

19 $\int e^x \sin(x)\,dx = \int e^x \cos(x)\,dx$

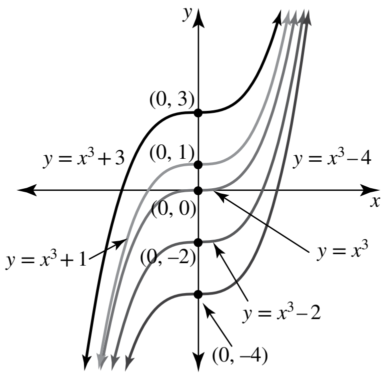
Solve using CAS:

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

20 $\int \frac{e^{2x} + e^x - 1}{e^x + 1}\,dx = e^x - x + \log_e(e^x + 1) + c$

Exercise 7.4 — Families of curves and applications

1 a $f'(x) = 3x^2$ so $f(x) = x^3 + c$



b $f(x) = x^3 + c$

$f(2) = 16$

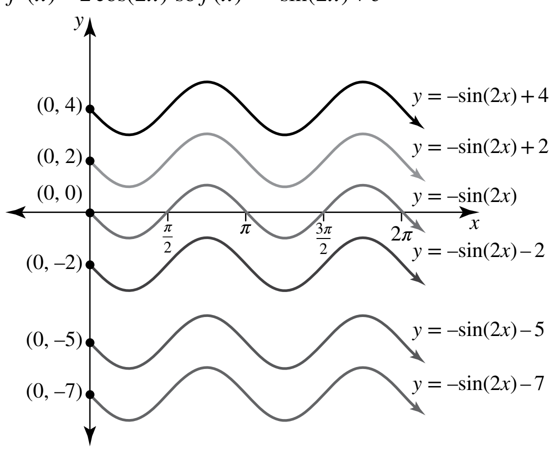
$2^3 + c = 16$

$8 + c = 16$

$c = 8$

$f(x) = x^3 + 8$

2 a $f'(x) = 2\cos(2x)$ so $f(x) = -\sin(2x) + c$



b $f(x) = -\sin(2x) + c$

$f\left(\frac{\pi}{2}\right) = 4$

$4 = -\sin(\pi) + c$

$4 = 0 + c$

$c = 4$

$f(x) = 4 - \sin(2x)$

3 $\frac{dy}{dx} = 2e^{2x} + e^{-x}$

$y = e^{2x} - e^{-x} + c$

When $x = 0,\ y = 3$

$3 = e^0 - e^0 + c$

$3 = 1 - 1 + c$

$c = 3$

$y = e^{2x} - e^{-x} + 3$

4 $f'(x) = \cos(2x) - \sin(2x)$

$f(x) = \frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x) + c$

$f(\pi) = 2$

$2 = \frac{1}{2}\sin(2\pi) + \frac{1}{2}\cos(2\pi) + c$

$2 = \frac{1}{2}(0) + \frac{1}{2}(1) + c$

$2 = \frac{1}{2} + c$

$c = \frac{3}{2}$

$f(x) = \frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x) + \frac{3}{2}$

5 $\frac{dV}{dt} = 20t^2 - t^3$

$V = \frac{20}{3}t^3 - \frac{1}{4}t^4 + c$

When $t = 0,\ V = 0$ so $c = 0$

$V = \frac{20}{3}t^3 - \frac{1}{4}t^4$

When $t = 20$,

$V = \frac{20}{3}(20)^3 - \frac{1}{4}(20)^4$

$V = 53\,333\frac{1}{3} - 40\,000$

$V = 13\,333\frac{1}{3}\ \text{cm}^3$

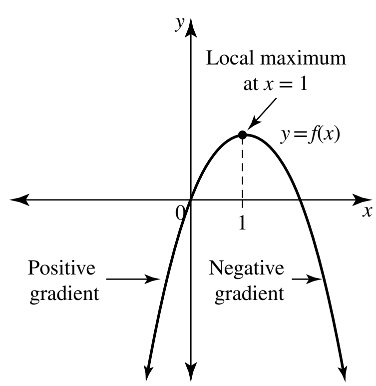
6 a $\frac{dV}{dr} = \pi r^2$

$V = \frac{1}{3}\pi r^3 + c$

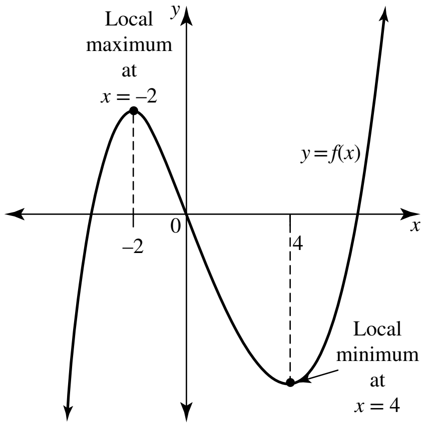
When $r = 0,\ V = 0$ so $c = 0$ and $V = \frac{1}{3}\pi r^3$.

b When $r = r4,\ V = \frac{1}{3}\pi(4)^3 = \frac{64}{3}\pi\ \text{cm}^3$

7 There is a stationary point at $x = 1$. When $x < 1$ the parabola has a positive gradient and when $x > 1$ the parabola has a negative gradient.



8 Stationary points occur at $x=-2$ and $x=4$. When $x<-2$ and $x>4$ the gradient is positive but when $-2<x<4$ the gradient is negative.



9 a $v=\dfrac{dx}{dt}=(3t+1)^{\frac{1}{2}}$

$x=\int(3t+1)^{\frac{1}{2}}dt$

$=\dfrac{1}{3\left(\frac{3}{2}\right)}(3t+1)^{\frac{3}{2}}+c$

$=\dfrac{2}{9}(3t+1)^{\frac{3}{2}}+c$

When $t=0,\ x=0$

$0=\dfrac{2}{9}(3(0)+1)^{\frac{3}{2}}+c$

$0=\dfrac{2}{9}+c$

$c=-\dfrac{2}{9}$

$x=\dfrac{2}{9}(3t+1)^{\frac{3}{2}}-\dfrac{2}{9}$

$x=\dfrac{2}{9}\sqrt{(3t+1)^3}-\dfrac{2}{9}$

b $v=\dfrac{dx}{dt}=\dfrac{1}{(t+2)^2}=(t+2)^{-2}$

$x=\int(t+2)^{-2}dt$

$=-\dfrac{1}{1}(t+2)^{-1}+c$

$=-\dfrac{1}{(t+2)}+c$

When $t=0,\ x=0$

$0=-\dfrac{1}{(0+2)}+c$

$0=-\dfrac{1}{2}+c$

$c=\dfrac{1}{2}$

$x=\dfrac{1}{2}-\dfrac{1}{(t+2)}$

c $v=\dfrac{dx}{dt}=(2t+1)^3$

$x=\int(2t+1)^3dt$

$=\dfrac{1}{2(4)}(2t+1)^4+c$

$=\dfrac{1}{8}(2t+1)^4+c$

When $x=0,\ t=0;$

$0=\dfrac{1}{8}(2(0)+1)^4+c$

$0=\dfrac{1}{8}+c$

$c=-\dfrac{1}{8}$

$x=\dfrac{1}{8}(2t+1)^4-\dfrac{1}{8}$

10 a $v=\dfrac{dx}{dt}=e^{(3t-1)}$

$x=\int e^{(3t-1)}dt$

$=\dfrac{1}{3}e^{(3t-1)}+c$

When $t=0,\ x=0$

$0=\dfrac{1}{3}e^{(3(0)-1)}+c$

$0=\dfrac{1}{3}e^{-1}+c$

$c=-\dfrac{1}{3e}$

$x=\dfrac{1}{3}e^{(3t-1)}-\dfrac{1}{3e}$

b $v=\dfrac{dx}{dt}=-\sin(2t+3)$

$x=\int-\sin(2t+3)dt$

$=\dfrac{1}{2}\cos(2t+3)+c$

When $t=0,\ x=0$

$0=\dfrac{1}{2}\cos(2(0)+3)+c$

$0=\dfrac{1}{2}\cos(3)+c$

$c=-\dfrac{1}{2}\cos(3)$

$x=\dfrac{1}{2}\cos(2t+3)-\dfrac{1}{2}\cos(3)$

11 $v=\dfrac{dx}{dt}=\sin(2t)+\cos(2t)$

a When $t=0$,

$v=\sin(0)+\cos(0)$

$v=1$ cm/s

b $x=\int(\sin(2t)+\cos(2t))dt$

$=-\dfrac{1}{2}\cos(2t)+\dfrac{1}{2}\sin(2t)+c$

When $t=0,\ x=0$

$0=-\dfrac{1}{2}\cos(0)+\dfrac{1}{2}\sin(0)+c$

$0=-\dfrac{1}{2}+c$

$c=\dfrac{1}{2}$

$x=\dfrac{1}{2}-\dfrac{1}{2}\cos(2t)+\dfrac{1}{2}\sin(2t)$

12 $v = 0.25t(50 - t) = 12.5t - 0.25t^2$

a Greatest velocity occurs when $\frac{dv}{dt} = 0$

$\frac{dv}{dt} = 12.5 - 0.5t$

$0 = 12.5 - 0.5t$

$0.5t = 12.5$

$t = 25$

When $t = 25,\ v = 0.25(25)(50 - 25) = 156.25$ m/s

b $x = \int 0.25t(50 - t)\,dt$

$x = \int 12.5t - 0.25t^2$

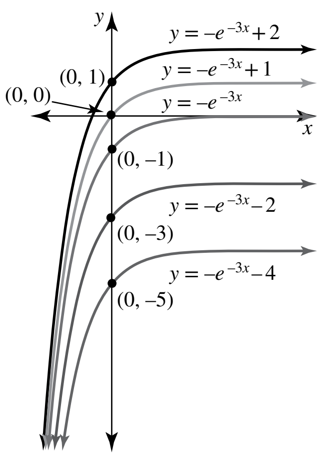
$x = 6.25t^2 - \frac{1}{12}t^3 + c$

When $t = 0,\ x = 0$ so $c = 0$

$x = 6.25t^2 - \frac{1}{12}t^3$

13 a $f'(x) = 3e^{-3x}$

$f(x) = -e^{-3x} + c$



b $f(x) = -e^{-3x} + c$

When $x = 0,\ y = 1$

$1 = -e^0 + c$

$1 = -1 + c$

$c = 2$

$f(x) = 2 - e^{-3x}$

14 $\frac{dy}{dx} = \cos(2x) + 3e^{-3x}$

$y = \int \left(\cos(2x) + 3e^{-3x}\right)dx$

$= \frac{1}{2}\sin(2x) - e^{-3x} + c$

When $x = 0,\ y = 4$

$4 = \frac{1}{2}\sin(0) - e^0 + c$

$4 = 0 - 1 + c$

$c = 5$

$y = \frac{1}{2}\sin(2x) - e^{-3x} + 5$

15 $\frac{dy}{dx} = e^{\frac{x}{2}}$

$y = 2e^{\frac{x}{2}} + c$

When $x = 0,\ y = 5$

$5 = 2e^0 + c$

$5 = 2 + c$

$c = 3$

$y = 2e^{\frac{x}{2}} + 3$

16 a $f'(x) = 5 - 2x$

$f(x) = 5x - x^2 + c$

When $f(1) = 4$

$4 = 5(1) - (1)^2 + c$

$4 = 4 + c$

$c = 0$

$f(x) = 5x - x^2$

b $f'(x) = \sin\left(\frac{x}{2}\right)$

$f(x) = -2\cos\left(\frac{x}{2}\right) + c$

When $f(\pi) = 3$

$3 = -2\cos\left(\frac{\pi}{2}\right) + c$

$3 = 0 + c$

$c = 3$

$f(x) = 3 - 2\cos\left(\frac{x}{2}\right)$

c $f'(x) = \frac{1}{(1 - x)^2} = (1 - x)^{-2}$

$f(x) = \frac{1}{(-1)(-1)}(1 - x)^{-1} + c$

$f(x) = \frac{1}{(1 - x)} + c$

When $f(0) = 4$

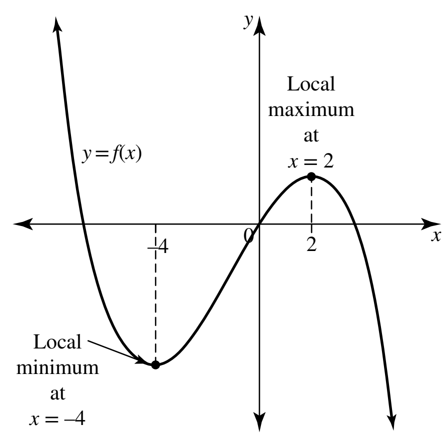
$4 = \frac{1}{(1 - 0)} + c$

$4 = 1 + c$

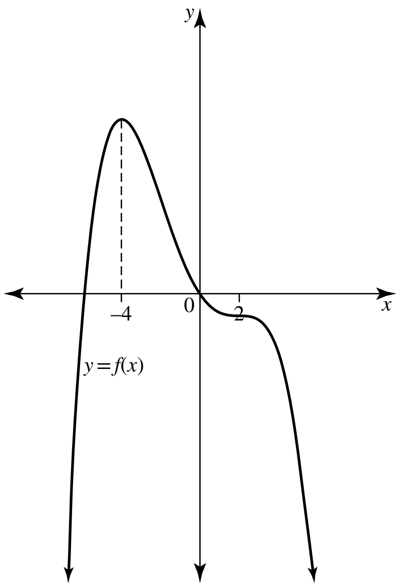
$c = 3$

$f(x) = \frac{1}{(1 - x)} + 3$

17 a



b



18 $v = \frac{dx}{dt} = 3t^2 + 7t$

$x = t^3 + \frac{7}{2}t^2 + c$

When $t = 0,\ x = 0;$

$0 = (0)^3 + \frac{7}{2}(0)^2 + c$

$c = 0$

$x = t^3 + \frac{7}{2}t^2$

19 a $v = \frac{dx}{dt} = 3\pi \sin\left(\frac{\pi t}{8}\right)$

$x = -\frac{8}{\pi} \times 3\cos\left(\frac{\pi t}{8}\right) + c$

$x = -24\cos\left(\frac{\pi t}{8}\right) + c$

When $t = 0,\ x = 0;$

$0 = -24\cos(0) + c$

$c = 24$

$x = 24 - 24\cos\left(\frac{\pi t}{8}\right)$

b $x_{MAX} = 24 - 24(-1) = 24 + 24 = 48$

$x_{MIN} = 24 - 24(1) = 24 - 24 = 0$

Maximum displacement is 48 metres.

c When $t = 4,\ x = 24 - 24\cos\left(\frac{\pi}{2}\right) = 24$

After 4 seconds the particle is 24 metres above the stationary position.

20 a $v = \frac{dx}{dt} = \frac{12}{(t-1)^2} + 6$

$v = \frac{dx}{dt} = 12(t-1)^{-2} + 6$

$x = -12(t-1)^{-1} + 6t + c$

$x = -\frac{12}{(t-1)} + 6t + c$

When $t = 0,\ x = 0;$

$x = -\frac{12}{(0-1)} + 6(0) + c$

$c = -12$

$x = 6t - \frac{12}{(t-1)} - 12$

b When $t = 3$,

$x = 6(3) - \frac{12}{(3-1)} - 12$

$= 18 - 6 - 12$

$= 0$

After 3 seconds the particle is at the origin again.

21 a $\frac{dP}{dt} = 30e^{0.3t}$

$P = \frac{30}{0.3}e^{0.3t} + c$

$P = 100e^{0.3t} + c$

When $t = 0,\ P = 50$

$50 = 100e^0 + c$

$50 = 100 + c$

$c = -50$

$P = 100e^{0.3t} - 50$

b When $t = 10,\ P = 100e^3 - 50 = 1959$

There are 1959 seals after 10 years.

22 $\frac{dh}{dt} = \frac{\pi}{2}\cos\left(\frac{\pi t}{4}\right)$

a $h = \frac{4}{\pi} \times \frac{\pi}{2}\sin\left(\frac{\pi t}{4}\right) + c = 2\sin\left(\frac{\pi t}{4}\right) + c$

When $t = 0,\ h = 3$

$3 = 2\sin(0) + c$

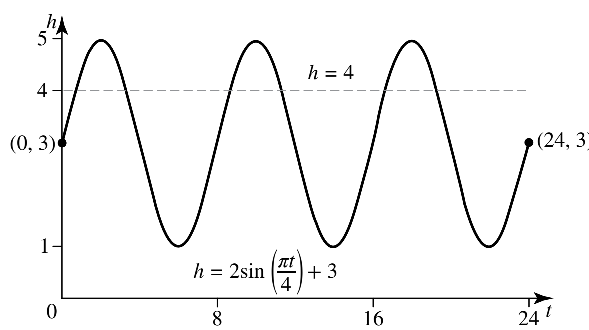
$c = 3$

$h = 2\sin\left(\frac{\pi t}{4}\right) + 3$

b Maximum depth = $2(1) + 3 = 5\,\text{m}$

Minimum depth = $2(-1) + 3 = 1\,\text{m}$

c



$4 = 2\sin\left(\frac{\pi t}{4}\right) + 3$

$1 = 2\sin\left(\frac{\pi t}{4}\right)$

$\frac{1}{2} = \sin\left(\frac{\pi t}{4}\right)$

$\frac{1}{2}$ indicates $\frac{\pi}{6}$. Since sin is positive then 1st and 2nd quadrants.

$\frac{\pi t}{4} = \frac{\pi}{6},\ \pi - \frac{\pi}{6},\ 2\pi + \frac{\pi}{6},\ 3\pi - \frac{\pi}{6},\ 4\pi + \frac{\pi}{6},\ 5\pi - \frac{\pi}{6}$

$\frac{\pi t}{4} = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{13\pi}{6},\ \frac{17\pi}{6},\ \frac{25\pi}{6},\ \frac{29\pi}{6}$

$t = \frac{\pi}{6} \times \frac{4}{\pi},\ \frac{5\pi}{6} \times \frac{4}{\pi},\ \frac{13\pi}{6} \times \frac{4}{\pi},\ \frac{17\pi}{6} \times \frac{4}{\pi},\ \frac{25\pi}{6} \times \frac{4}{\pi},\ \frac{29\pi}{6} \times \frac{4}{\pi}$

$t = \frac{2}{3},\ \frac{10}{3},\ \frac{26}{3},\ \frac{34}{3},\ \frac{50}{3},\ \frac{58}{3}$

$h \ge 4$ when $\left\{h : \frac{2}{3} \le t \le \frac{10}{3}\right\} \cup \left\{h : \frac{26}{3} \le t \le \frac{34}{3}\right\} \cup \left\{h : \frac{50}{3} \le t \le \frac{58}{3}\right\}$

This is $\frac{8}{3} + \frac{8}{3} + \frac{8}{3} = \frac{24}{3} = 8$ hours/day.

23 a $\frac{dN}{dt} = 400 + 1000\sqrt{t}$

$\frac{dN}{dt} = 400 + 1000t^{\frac{1}{2}}$

$N = 400t + \frac{2000}{3}t^{\frac{3}{2}} + c$

$N = 400t + \frac{2000}{3}\sqrt{t^3} + c$

When $t = 0,\ N = 40$

$40 = 400(0) + \frac{2000}{3}\sqrt{0^3} + c$

$c = 40$

$N = 400t + \frac{2000}{3}\sqrt{t^3} + 40$

b When $t = 5$,

$N = 400(5) + \frac{2000}{3}\sqrt{(5)^3} + 40$

$N = 2000 + \frac{2000}{3}\sqrt{125} + 40$

$N = 9494$ families

24 $v = \frac{dx}{dt} = 2t\cos(t)$

$x = 2t\sin(t) + 2\cos(t) + c$

When $t = 0,\ x = 0;\ \ 0 = 2(0)\sin(0) + 2\cos(0) + c$ so $c = -2$

$x = 2t\sin(t) + 2\cos(t) - 2$

Topic 8 — Integration

Exercise 8.2 — The fundamental theorem of integral calculus

1 a Left end-point rectangle rule:

$f(0.5)=2,\ f(1)=1,\ f(1.5)=\frac{2}{3},\ f(2)=0.5$

Approximate area

$=0.5\times 2+0.5\times 1+0.5\times\frac{2}{3}+0.5\times 0.5$

$=0.5\left(2+1+\frac{2}{3}+0.5\right)$

$=\frac{25}{12}$ units^2

b Right end-point rectangle rule:

$f(2.5)=0.4$

Approximate area

$=0.5\times 1+0.5\times\frac{2}{3}+0.5\times 0.5+0.5\times 0.4$

$=0.5\left(1+\frac{2}{3}+0.5+0.4\right)$

$=\frac{77}{60}$ units^2

2 $f(1)=-0.01(1)^3(1-5)(1+5)=0.24$

$f(2)=-0.01(2)^3(2-5)(2+5)=1.68$

$f(3)=-0.01(3)^3(3-5)(3+5)=4.32$

$f(4)=-0.01(4)^3(4-5)(4+5)=5.76$

Approximate area

$=1(0.24+1.68+4.32+5.76)$

$=12$ units^2

3 a $\int_0^1(4x^3+3x^2+2x+1)\,dx$

$=\left[x^4+x^3+x^2+x\right]_0^1$

$=\left(1^4+1^3+1^2+1\right)-0$

$=4$

b $\int_{-\pi}^{\pi}(\cos(x)+\sin(x))\,dx$

$=\left[\sin(x)-\cos(x)\right]_{-\pi}^{\pi}$

$=(\sin(\pi)-\cos(\pi))-(\sin(-\pi)-\cos(-\pi))$

$=(0-(-1))-(0-(-1))$

$=1-1$

$=0$

4 a $(x+1)^3 = x^3 + 3x^2 + 3x + 1$

$$\int_{-3}^{2} (x+1)^3\,dx = \int_{-3}^{2} (x^3 + x^2 + x + 1)\,dx$$
$$= \left[\frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x\right]_{-3}^{2}$$
$$= \left(\frac{1}{4}(2)^4 + (2)^3 + \frac{3}{2}(2)^2 + (2)\right) - \left(\frac{1}{4}(-3)^4 + (-3)^3 + \frac{3}{2}(-3)^2 + (-3)\right)$$
$$= (4+8+6+2) - \left(\frac{81}{4} - 27 + \frac{27}{2} - 3\right)$$
$$= 20 - \left(\frac{135}{4} - 30\right)$$
$$= \frac{80}{4} - \left(\frac{135}{4} - \frac{120}{4}\right)$$
$$= \frac{80}{4} - \frac{15}{4}$$
$$= \frac{65}{4}$$

b $\int_{0}^{1} \left(e^x + e^{-x}\right)^2\,dx$

$$= \int_{0}^{1} \left(e^{2x} + 2 + e^{-2x}\right)\,dx$$
$$= \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_{0}^{1}$$
$$= \left(\frac{1}{2}e^{2(1)} + 2(1) - \frac{1}{2}e^{-2(1)}\right) - \left(\frac{1}{2}e^{0} + 2(0) - \frac{1}{2}e^{0}\right)$$
$$= \frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2} - \frac{1}{2} + \frac{1}{2}$$
$$= 2 + 0.5e^2 - 0.5e^{-2}$$

5 $\int_{2}^{5} m(x)\,dx = 7$ and $\int_{2}^{5} n(x)\,dx = 3$

a $\int_{2}^{5} 3m(x)\,dx = 3\int_{2}^{5} m(x)\,dx = 3(7) = 21$

b $\int_{2}^{5} (2m(x) - 1)\,dx = 2\int_{2}^{5} m(x) - \int_{2}^{5} 1\,dx$

$$= 2(7) - [x]_2^5$$
$$= 14 - (5-2)$$
$$= 14 - 3$$
$$= 11$$

c $\int_{5}^{2} (m(x)+3)\,dx = -\int_{2}^{5} (m(x)+3)\,dx$

$$= -\int_{2}^{5} m(x)\,dx - \int_{2}^{5} 3\,dx$$
$$= -7 - [3x]_2^5$$
$$= -7 - (3(5) - 3(2))$$
$$= -7 - 15 + 6$$
$$= -16$$

d $\int_{2}^{5} (2m(x) + n(x) - 3)\,dx = 2\int_{2}^{5} m(x)\,dx + \int_{2}^{5} n(x)\,dx - \int_{2}^{5} 3\,dx$

$$= 2(7) + 3 - [3x]_2^5$$
$$= 14 + 3 - (3(5) - 3(2))$$
$$= 17 - 9$$
$$= 8$$

6 $$\int_k^1 \left(4x^3 - 3x^2 + 1\right) dx = 0$$

$$\left[x^4 - x^3 + x\right]_k^1 = 0$$

$$\left(1^4 - 1^3 + 1\right) - \left(k^4 - k^3 + k\right) = 0$$

$$1 - k + k^3 - k^4 = 0$$

$$(1-k) + k^3(1-k) = 0$$

$$(1-k)(1+k^3) = 0$$

$$(1-k)(1+k)(1-k+k^2) = 0$$

$k = \pm 1$ as $1 - k + k^2$ cannot be further factorised

Verification:

$$\int_{-1}^1 \left(4x^3 - 3x^2 + 1\right) dx = \left[x^4 - x^3 + x\right]_{-1}^1$$

$$= \left(1^4 - 1^3 + 1\right) - \left((-1)^4 - (-1)^3 - 1\right)$$

$$= 1 - 1$$

$$= 0$$

$$\int_1^1 \left(4x^3 - 3x^2 + 1\right) dx = 0$$

7 **a** Approximate area

$= 8 \times 1 + 9 \times 1 + 8 \times 1 + 5 \times 1$

$= 30$ units^2

b Approximate area

$= 9 \times 1 + 8 \times 1 + 5 \times 1$

$= 22$ units^2

8 Approximate area

$= 1(1.75) + 1(3) + 1(3.75) + 1(4) + 1(3.75) + 1(3) + 1(1.75)$

$= 2(1.75) + 2(3) + 2(3.75) + 4$

$= 21$ units^2

9 **a** $f(x) = \sqrt{x}(4 - x)$

Graph intersects the x axis where $y = 0$

$\sqrt{x}(4 - x) = 0$

$x = 0$ or $4 - x = 0$

$4 = x$

Thus $a = 4$.

b $f(0) = 0$

$f(1) = \sqrt{1}(4 - 1) = 3;$

$f(2) = \sqrt{2}(4 - 2) = 2.8284;$

$f(3) = \sqrt{3}(4 - 3) = 1.7321;$

$f(4) = 0$

Left End-point Rule:

Approximate area

$= 0.5\left(f(0) + f(1) + f(2) + f(3)\right)$

$= 0.5(0 + 3 + 2.8284 + 1.7321)$

$= 7.56$ units^2

Right End-point Rule:

Approximate area

$= 0.5\left(f(1) + f(2) + f(3) + f(4)\right)$

$= 0.5(3 + 2.8284 + 1.7321 + 0)$

$= 7.56$ units^2

10 a $\int_0^3 \left(3x^2-2x+3\right)\,dx = \left[x^3-x^2+3x\right]_0^3$

$= \left(3^3-3^2+3(3)\right)-0$

$= 27-9+9$

$= 27$

b $\int_1^2 \left(\frac{2x^3+3x^2}{x}\right)dx = \int_1^2 \left(2x^2+3x\right)dx,\ x \neq 0$

$= \left[\frac{2}{3}x^3+\frac{3}{2}x^2\right]_1^2$

$= \left(\frac{2}{3}(2)^3+\frac{3}{2}(2)^2\right)-\left(\frac{2}{3}(1)^3+\frac{3}{2}(1)^2\right)$

$= \frac{16}{3}+6-\frac{2}{3}-\frac{3}{2}$

$= \frac{28}{6}+\frac{36}{6}-\frac{9}{6}$

$= \frac{55}{6}$

c $\int_{-1}^{1} \left(e^{2x}-e^{-2x}\right)dx = \left[\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}\right]_{-1}^{1}$

$= \left(\frac{1}{2}e^{2(1)}+\frac{1}{2}e^{-2(1)}\right)-\left(\frac{1}{2}e^{2(-1)}+\frac{1}{2}e^{-2(-1)}\right)$

$= \frac{1}{2}e^2+\frac{1}{2}e^{-2}-\frac{1}{2}e^{-2}-\frac{1}{2}e^2$

$= 0$

d $\int_{2\pi}^{4\pi} \sin\left(\frac{x}{3}\right)dx = \left[-3\cos\left(\frac{x}{3}\right)\right]_{2\pi}^{4\pi}$

$= -3\cos\left(\frac{4\pi}{3}\right)+3\cos\left(\frac{2\pi}{3}\right)$

$= 1.5-1.5$

$= 0$

e $\int_{-3}^{-1} \frac{2}{\sqrt{1-3x}}\,dx = 2\int_{-3}^{-1} (1-3x)^{-\frac{1}{2}}\,dx$

$= 2\left[2\left(-\frac{1}{3}\right)(1-3x)^{\frac{1}{2}}\right]_{-3}^{-1}$

$= 2\left(-\frac{2}{3}(1+3)^{\frac{1}{2}}+\frac{2}{3}(1+9)^{\frac{1}{2}}\right)$

$= 2\left(-\frac{4}{3}+\frac{2\sqrt{10}}{3}\right)$

$= \frac{4}{3}\left(\sqrt{10}-2\right)$

f $\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\cos(2x)-\sin\left(\frac{x}{2}\right)\right)dx = \left[\frac{1}{2}\sin(2x)+2\cos\left(\frac{x}{2}\right)\right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \left(\frac{1}{2}\sin(\pi)+2\cos\left(\frac{\pi}{4}\right)\right)-\left(\frac{1}{2}\sin\left(-\frac{2\pi}{3}\right)+2\cos\left(-\frac{\pi}{6}\right)\right)$

$= \left(\frac{1}{2}(0)+\sqrt{2}\right)-\left(\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)+2\left(\frac{\sqrt{3}}{2}\right)\right)$

$= \sqrt{2}-\frac{3\sqrt{3}}{4}$

11 Given that $\int_0^5 f(x)\,dx = 7.5$ and $\int_0^5 g(x)\,dx = 12.5$

a $\int_0^5 -2f(x)\,dx = -2\int_0^5 f(x)\,dx = -2\times 7.5 = -15$

b $\int_5^0 g(x)\,dx = -\int_0^5 g(x)\,dx = -12.5$

c $\int_0^5 (3f(x)+2)\,dx = 3\int_0^5 f(x)\,dx + \int_0^5 2\,dx$

$$\begin{aligned} &= 3\times 7.5 + [2x]_0^5 \\ &= 22.5 + (2(5) - 0) \\ &= 22.5 + 10 \\ &= 32.5 \end{aligned}$$

d $\int_0^5 (g(x)+f(x))\,dx = \int_0^5 g(x)\,dx + \int_0^5 f(x)\,dx = 7.5 + 12.5 = 20$

e $\int_0^5 (8g(x) - 10f(x))\,dx = 8\int_0^5 g(x)\,dx - 10\int_0^5 f(x)\,dx = 8(12.5) - 10(7.5) = 25$

f $\int_0^3 g(x)\,dx + \int_3^5 g(x)\,dx = \int_0^5 g(x)\,dx = 12.5$

12 $\int_1^h \frac{3}{x^2}\,dx = -\frac{12}{5}$

$$\begin{aligned} \int_1^h 3x^{-2}\,dx &= -\frac{12}{5} \\ \left[3x^{-1}\right]_1^h &= -\frac{12}{5} \\ \left[\frac{3}{x}\right]_1^h &= -\frac{12}{5} \\ \left(\frac{3}{h}\right) - 3 &= -\frac{12}{5} \\ \frac{3}{h} &= -\frac{12}{5} + \frac{15}{5} \\ \frac{3}{h} &= \frac{27}{5} \\ \frac{h}{3} &= \frac{5}{27} \\ h &= \frac{5}{27}\times 3 = \frac{5}{9} \end{aligned}$$

13 $\int_0^a e^{-2x}\,dx = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$

$$\begin{aligned} \left[-\frac{1}{2}e^{-2x}\right]_0^a &= \frac{1}{2}\left(1 - \frac{1}{e^8}\right) \\ \left(-\frac{1}{2}e^{-2a}\right) - \left(-\frac{1}{2}e^0\right) &= \frac{1}{2}\left(1 - \frac{1}{e^8}\right) \\ \frac{1}{2} - \frac{1}{2e^{2a}} &= \frac{1}{2}\left(1 - \frac{1}{e^8}\right) \\ \frac{1}{2}\left(1 - \frac{1}{e^{2a}}\right) &= \frac{1}{2}\left(1 - \frac{1}{e^8}\right) \\ e^{2a} &= e^8 2a = 8 \\ a &= 4 \end{aligned}$$

14 a Graph cuts the x axis when $f(x) = 0$.

$f(x) = x^3 - 8x^2 + 21x - 14$

$f(1) = 1^3 - 8(1)^2 + 21(1) - 14 = 0$

Thus $(x-1)$ is a factor.

$x^3 - 8x^2 + 21x - 14 = (x-1)(x^2 - 7x + 14)$

As $x^2 - 7x + 14$ has a discriminant such that $\Delta = (-7)^2 - (4(1)14) = 49 - 56 = -7$ there are no factors.

$(x-1)(x^2 - 7x + 14) = 0$

$x - 1 = 0$

$x = 1$ so $a = 1$

b $\int_1^5 \left(x^3 - 8x^2 + 21x - 14\right) dx$

$= \left[\frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{21}{2}x^2 - 14x\right]_1^5$

$= \left(\frac{1}{4}(5)^4 - \frac{8}{3}(5)^3 + \frac{21}{2}(5)^2 - 14(5)\right) - \left(\frac{1}{4}(1)^4 - \frac{8}{3}(1)^3 + \frac{21}{2}(1)^2 - 14(1)\right)$

$= \frac{625}{4} - \frac{1000}{3} + \frac{525}{2} - 70 - \frac{1}{4} + \frac{8}{3} - \frac{21}{2} + 14$

$= \frac{624}{4} - \frac{992}{3} + \frac{504}{2} - 56$

$= 156 - 330\frac{2}{3} + 252 - 56$

$= 408 - 386\frac{2}{3}$

$= 21\frac{1}{3} \text{ units}^2$

15 **a** $y = x\sin(x)$

$\frac{dy}{dx} = x\cos(x) + \sin(x)$

b $\int_{-\pi}^{\frac{\pi}{2}} 2x\cos(x)\,dx = 2\int_{-\pi}^{\frac{\pi}{2}} x\cos(x)\,dx$

$= 2\left(x\sin(x) - \int_{-\pi}^{\frac{\pi}{2}} \sin(x)\right)$

$= 2\left[x\sin(x) + \cos(x)\right]_{-\pi}^{\frac{\pi}{2}}$

$= 2\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right) - 2\left(-\pi\sin(\pi) + \cos(\pi)\right)$

$= 2\left(\frac{\pi}{2}(1) + 0\right) - 2\left(-\pi(0) - 1\right)$

$= \pi + 2$

16 **a** $y = e^{x^3 - 3x^2} + 2$

$\frac{dy}{dx} = \left(3x^2 - 6x\right)e^{x^3 - 3x^2}$

$\frac{dy}{dx} = 3\left(x^2 - 2x\right)e^{x^3 - 3x^2}$

b $\int_0^1 3\left(x^2 - 2x\right)e^{x^3 - 3x^2}\,dx = \left[e^{x^3 - 3x^2}\right]_0^1$

$3\int_0^1 \left(x^2 - 2x\right)e^{x^3 - 3x^2}\,dx = \left(e^{1^3 - 3(1)^2} - e^0\right)$

$\int_0^1 \left(x^2 - 2x\right)e^{x^3 - 3x^2}\,dx = \frac{1}{3}\left(e^{-2} - 1\right)$

17 $\int_1^k (2x - 3)\,dx = 7 - 3\sqrt{5}$

$\left[x^2 - 3x\right]_1^k = 7 - 3\sqrt{5}$

$k^2 - 3k - \left(1^2 - 3(1)\right) = 7 - 3\sqrt{5}$

$k^2 - 3k + 2 = 7 - 3\sqrt{5}$

Solve using CAS:

$k = 3 - \sqrt{5}, \sqrt{5}$

$= \sqrt{5}, \quad k > 1$

18 $\int_{-2}^{0} \frac{1 + e^{2x} - 2xe^{2x}}{(e^{2x} + 1)^2}\,dx = 1.964$

(Solved using CAS)

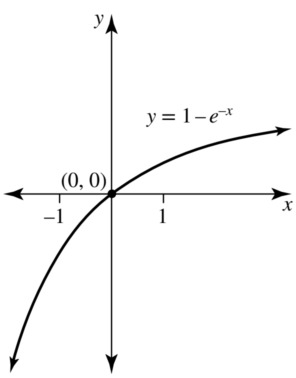
Exercise 8.3 — Areas under curves

1 Area is

$$\int_0^{25} 2\sqrt{x}\,dx = 2\int_0^{25} x^{\frac{1}{2}}\,dx$$
$$= 2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{25}$$
$$= 2\left(\frac{2}{3}(25)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}\right)$$
$$= \frac{4}{3}\left(5^2\right)^{\frac{3}{2}}$$
$$= \frac{4}{3}\times 125$$
$$= 166\frac{2}{3} \text{ units}^2$$

2 $$\int_0^{\pi}(2\sin(2x)+3)\,dx = \left[-\cos(2x)+3x\right]_0^{\pi}$$
$$= (-\cos(2\pi)+3\pi) - (-\cos(0)+3(0))$$
$$= 3\pi \text{ units}^2$$

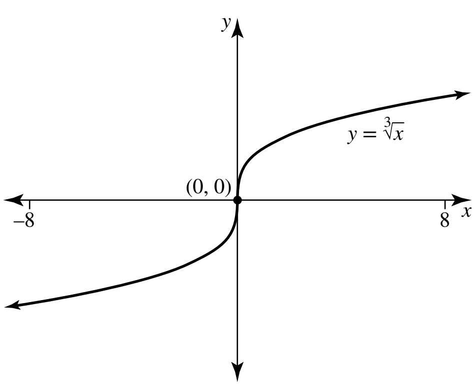
3



Area is

$$= -\int_{-1}^{0}\left(1-e^{-x}\right)dx + \int_0^1\left(1-e^{-x}\right)dx$$
$$= -\left[x+e^{-x}\right]_{-1}^{0} + \left[x+e^{-x}\right]_0^1$$
$$= -\left((0+e^0)-(-1+e^1)\right) + \left((1+e^{-1})-(0+e^0)\right)$$
$$= -(1+1-e) + \left(1+e^{-1}-1\right)$$
$$= -2+e+e^{-1}$$
$$= e+e^{-1}-2 \text{ units}^2$$

4



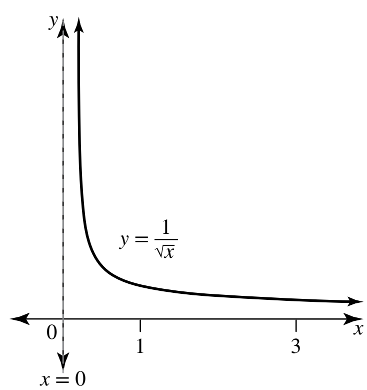
Area is

$$= -\int_{-8}^{0}\sqrt[3]{x}\,dx + \int_0^8 \sqrt[3]{x}\,dx$$
$$= 2\int_0^8 \sqrt[3]{x}\,dx \text{ by symmetry}$$
$$= 2\int_0^8 x^{\frac{1}{3}}\,dx$$
$$= 2\left[\frac{3}{4}x^{\frac{4}{3}}\right]_0^8$$
$$= 2\left(\frac{3}{4}\left(2^3\right)^{\frac{4}{3}}\right) - 0$$
$$= 2\times\frac{3}{4}\times 16$$
$$= 24 \text{ units}^2$$

5 Area $= \displaystyle\int_{-2.5}^{-0.5}\frac{1}{x^2}\,dx$

$$= \int_{-2.5}^{-0.5} x^{-2}\,dx$$
$$= \left[-x^{-1}\right]_{-2.5}^{-0.5}$$
$$= \left[-\frac{1}{x}\right]_{-2.5}^{-0.5}$$
$$= \left(\frac{1}{0.5}\right) - \left(\frac{1}{2.5}\right)$$
$$= 2 - 0.4$$
$$= 1.6 \text{ units}^2$$

6 a



b Area =

$$\int_1^3 \frac{1}{\sqrt{x}}\,dx = \int_1^3\left(x^{-\frac{1}{2}}\right)dx = \left[2x^{\frac{1}{2}}\right]_1^3 = 2\sqrt{3}-2\sqrt{1} = 2\sqrt{3}-2 \text{ units}^2$$

7 a Graph intersects x axis where $y = 0$.

$$2\sin(x)+\cos(x) = 0$$
$$2\sin(x) = -\cos(x)$$
$$2\tan(x) = -1$$
$$\tan(x) = -\frac{1}{2}$$

$\frac{1}{2}$ suggest 0.4636. Since tan is negative 2nd quadrant as $0 \le x \le \pi$.

$$x = \pi - 0.4636$$
$$x = 2.6779$$

Thus $(m, 0) = (2.6779, 0)$.

b Area

$= \int_0^{2.6779} (2\sin(x)+\cos(x))\, dx$

$= [-2\cos(x)+\sin(x)]_0^{2.6779}$

$= (-2\cos(2.6779)+\sin(2.6779)) - (-2\cos(0)+\sin(0))$

$= 1.7888 + 0.4473 + 2$

$= 4.2361 \text{ units}^2$

8 a $y = e^{x^2}$

$\frac{dy}{dx} = 2xe^{x^2}$

b $-\int_{-1}^{0} \left(2xe^{x^2}\right) dx + \int_0^1 \left(2xe^{x^2}\right) dx = 2\int_0^1 \left(2xe^{x^2}\right) dx$ by symmetry

$= 2\left[e^{x^2}\right]_0^1$

$= 2\left(e^{(1)^2} - e^0\right)$

$= 2(e-1) \text{ units}^2$

9 a $y = f(x) = -(x^2-1)(x^2-9)$

Graph cuts the y axis where $x = 0,\ y = -(-1)(-9) = -9$.

Graph cuts the x axis where $y = 0$

$-(x^2-1)(x^2-9) = 0$

$-(x-1)(x+1)(x-3)(x+3) = 0$

$x-1=0 \quad x+1=0 \quad x-3=0 \quad x+3=0$

$x = 1, \quad x = -1 \quad x = 3 \quad x = -3$

TP's occur where $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = -2x(x^2-9) - 2x(x^2-1)$

$\frac{dy}{dx} = -2x^3 + 18x - 2x^3 + 2x$

$\frac{dy}{dx} = -4x^3 + 20x$

$0 = -4x(x^2-5)$

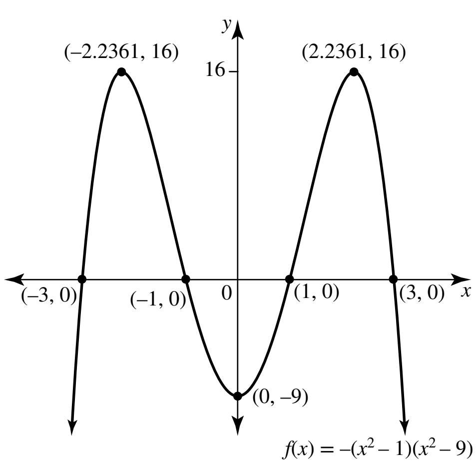
$0 = -4x(x-\sqrt{5})(x+\sqrt{5})$

$x = 0 \quad x-\sqrt{5} = 0 \quad x+\sqrt{5} = 0$

$x = \sqrt{5} \quad x = -\sqrt{5}$

When $x = 0,\ y = -9$.

When $x = \pm\sqrt{5},\ y = -\left(\left(\pm\sqrt{5}\right)^2 - 1\right)\left(\left(\pm\sqrt{5}\right)^2 - 9\right) = -4 \times -4 = 16$



b By symmetry area is

$$2\left(-\int_0^1\left(-(x^2-1)(x^2-9)\right)dx+\int_1^3\left(-(x^2-1)(x^2-9)\right)dx\right)$$
$$=2\left(-\int_0^1\left(-x^4+10x^2-9\right)dx+\int_1^3\left(-x^4+10x^2-9\right)dx\right)$$
$$=2\left(-\left[-\frac{1}{5}x^5+\frac{10}{3}x^3-9x\right]_0^1+\left[-\frac{1}{5}x^5+\frac{10}{3}x^3-9x\right]_1^3\right)$$
$$=2\left(-\left(-\frac{1}{5}(1)^5+\frac{10}{3}(1)^3-9(1)-0\right)+\left(-\frac{1}{5}(3)^5+\frac{10}{3}(3)^3-9(3)\right)-\left(-\frac{1}{5}(1)^5+\frac{10}{3}(1)^3-9(1)\right)\right)$$
$$=52.27\text{ units}^2$$

10 Required area

$$=4\left(\int_0^{\pi}(2\sin(x)+3\cos(x))dx\right)$$
$$=4\left(\left[-2\cos(x)+3\sin(x)\right]_0^{\pi}\right)$$
$$=4\left((-2\cos(\pi)+3\sin(\pi))-(-2\cos(0)+3\sin(0))\right)$$
$$=4(2+0+2-0)$$
$$=16\text{ units}^2$$

11 **a** $y=-0.5(x+2)(x+1)(x-2)(x-3)$

Graph cuts the x axis where $y=0$.

$-0.5(x+2)(x+1)(x-2)(x-3)=0$

$x+2=0,\ x+1=0,\ x-2=0,\ x-3=0$

$x=-2,\quad x=-1,\quad x=2,\quad x=3$

Thus $a=-2,\ b=-1,\ c=2$ and $d=3$.

b Area is

$$=-\int_{-1}^{2}(-0.5(x+2)(x+1)(x-2)(x-3))\,dx+2\int_2^3(-0.5(x+2)(x+1)(x-2)(x-3))\,dx$$
$$=-\int_{-1}^{2}\left(-0.5x^4+x^3+\frac{7}{2}x^2-4x-6\right)dx+2\int_2^3\left(-0.5x^4+x^3+\frac{7}{2}x^2-4x-6\right)dx$$
$$=13.05+2\times1.3167$$
$$=15.68\text{ units}^2$$

12 **a** $y=ax(x-2)$

When $x=1,\ y=3$

$3=a(1)(1-2)$

$3=-a$

$a=-3$

Thus $y=-3x(x-2)=-3x^2+6x$

b Area of glass is

$$\int_0^2(-3x^2+6x)\,dx$$
$$=\left[-x^3+3x^2\right]_0^2$$
$$=\left(-(2)^3+3(2)^2\right)-((0)-(0))$$
$$=-8+12-0$$
$$=4\text{ m}^2$$

c Two windows = 8 metres^2.

Cost = \$55 × 8 = \$440

13 a $\int_0^1 3x^3\,dx = \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}(1)^4 - 0 = \frac{3}{4}$ units^2

b Orange region $= 3\times 1 - \frac{3}{4} = 2\frac{1}{4}$ units^2

14 a $\int_0^{\frac{\pi}{2}} 2\sin(x)\,dx = [-2\cos(x)]_0^{\frac{\pi}{2}}$

$= -2\cos\left(\frac{\pi}{2}\right) - (-2\cos(0))$
$= 0 + 2$
$= 2$

b Shaded region is $2\left(2\times\frac{\pi}{2} - 2\right) = 2(\pi - 2)$ units^2

15 a $\int_0^4 \left(\sqrt{x}(x-3)^2\right)dx$

$= \int_0^4 \left(x^{\frac{1}{2}}(x^2 - 6x + 9)\right)dx$

$= \int_0^4 \left(x^{\frac{5}{2}} - 6x^{\frac{3}{2}} + 9x^{\frac{1}{2}}\right)dx$

$= \left[\frac{2}{7}x^{\frac{7}{2}} - \frac{12}{5}x^{\frac{5}{2}} + 6x^{\frac{3}{2}}\right]_0^4$

$= \left(\frac{2}{7}(4)^{\frac{7}{2}} - \frac{12}{5}(4)^{\frac{5}{2}} + 6(4)^{\frac{3}{2}}\right) - 0$

$= 36.5714 - 76.8 + 48$

$= 7.7714$ cm^2

b Area of whole motif is $7.7714 \times 2 = 15.5428 \simeq 15.5$ cm^2

c 0.34875 m$^2 = 3487.5$ cm^2
Scale factor is $3487.5 \div 15.5428 \simeq 224$

16 a $\int_{-2}^{2} e^{-x^2}\,dx = 1.7642$ units^2

Solved using CAS.

b $-\int_{-2}^{1} \frac{x^2+3x-4}{x^2+1}\,dx + \int_{1}^{3} \frac{x^2+3x-4}{x^2+1}\,dx$
$= 7.8371 + 2.0959$
$= 9.933$ units^2

Exercise 8.4 — Applications

1 $y = 4$.................(1)
$y = \sqrt{x}$.............(2)
$(1) = (2)$
$\sqrt{x} = 4$
$x = 16$
When $x = 16$, $y = 4$, therefore POI = (16, 4)
Shaded region is

$= \int_0^{16} \left(4 - \sqrt{x}\right)dx$

$= \int_0^{16} \left(4 - x^{\frac{1}{2}}\right)dx$

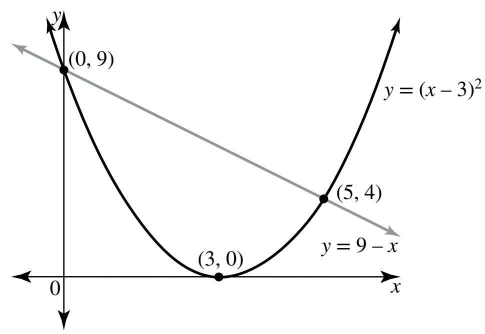
$= \left[4x - \frac{2}{3}x^{\frac{3}{2}}\right]_0^{16}$

$= \left(4(16) - \frac{2}{3}(4^2)^{\frac{3}{2}}\right) - 0$

$= 64 - \frac{2}{3}\times 4^3$

$= 21\frac{1}{3}$ units^2

2 $y = (x-3)^2$......................(1)
$y = 9 - x$..........................(2)
$(1) = (2)$
$(x-3)^2 = 9 - x$
$x^2 - 6x + 9 = 9 - x$
$x^2 - 5x = 0$
$x(x-5) = 0$
$x = 0$ or $x - 5 = 0$
$x = 5$
When $x = 0$, $y = 9 - 0 = 9$
When $x = 5$, $y = 9 - 5 = 4$
Graphs intersect at (0,9) and (5,4).



Area is

$= \int_0^5 \left(9 - x - (x-3)^2\right)dx$

$= \int_0^5 \left(9 - x - x^2 + 6x - 9\right)dx$

$= \int_0^5 \left(-x^2 + 5x\right)dx$

$= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2\right]_0^5$

$= \left(-\frac{1}{3}(5)^3 + \frac{5}{2}(5)^2\right) - 0$

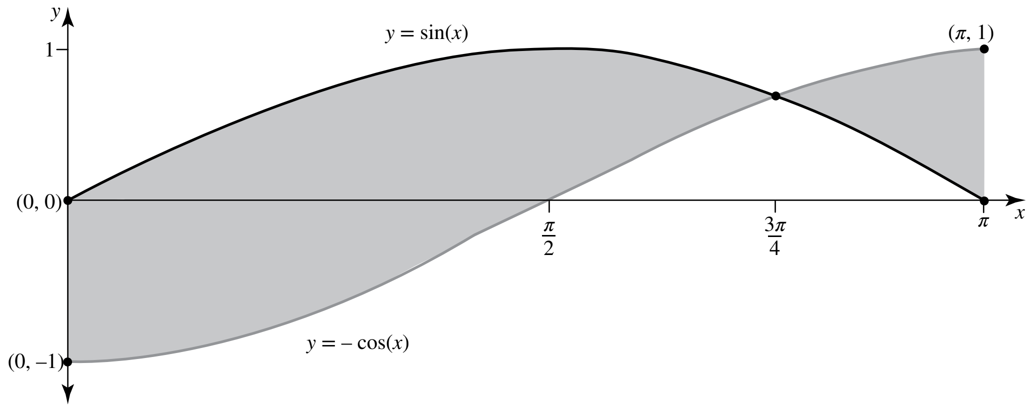
$= -\frac{125}{3} + \frac{125}{2}$

$= -\frac{250}{6} + \frac{375}{6}$

$= \frac{125}{6}$

$= 20\frac{5}{6}$ units^2

3 Point of intersection between the graphs:
$\sin(x) = -\cos(x) \quad 0 \le x \le \pi$
$\tan(x) = -1$
Basic angle $= \frac{\pi}{4}$, tan is negative in the 2nd quadrant
$x = \frac{3\pi}{4}$



Area is

$$A=\int_0^{\frac{3\pi}{4}}\left(\sin(x)+\cos(x)\right)dx+\int_{\frac{3\pi}{4}}^{\pi}\left(-\cos(x)-\sin(x)\right)dx$$

$$=\left[-\cos(x)+\sin(x)\right]_0^{\frac{3\pi}{4}}+\left[-\sin(x)+\cos(x)\right]_{\frac{3\pi}{4}}^{\pi}$$

$$=\left(-\cos\left(\frac{3\pi}{4}\right)+\sin\left(\frac{3\pi}{4}\right)-\left(-\cos(0)+\sin(0)\right)\right)+\left(-\sin(\pi)+\cos(\pi)-\left(-\sin\left(\frac{3\pi}{4}\right)+\cos\left(\frac{3\pi}{4}\right)\right)\right)$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+1-0+0-1+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$$

$$=2\sqrt{2}\text{ units}^2$$

4 Area is

$$=\int_0^{3.92}\left(4-x-4e^{-x}\right)dx+\int_{3.92}^{5}\left(4e^{-x}-4+x\right)dx$$

$$=\left[4x-\frac{1}{2}x^2+4e^{-4x}\right]_0^{3.92}+\left[-4e^{-x}-4x+\frac{1}{2}x^2\right]_{3.92}^{5}$$

$$=4.6254\text{ units}^2$$

5 Average value $=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$=\frac{1}{\left(\frac{1}{3}-0\right)}\int_0^{\frac{1}{3}}e^{3x}dx$$

$$=3\left[\frac{1}{3}e^{3x}\right]_0^{\frac{1}{3}}$$

$$=3\left(\frac{1}{3}e^1-\frac{1}{3}e^0\right)$$

$$=e-1$$

6 Average value $=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$=\frac{1}{1-0.5}\int_{0.5}^{1}\left(x^2-2x\right)dx$$

$$=2\left[\frac{1}{3}x^3-x^2\right]_{0.5}^{1}$$

$$=2\left(\left(\frac{1}{3}(1)^3-(1)^2\right)-\left(\frac{1}{3}\left(\frac{1}{2}\right)^3-\left(\frac{1}{2}\right)^2\right)\right)$$

$$=2\left(-\frac{2}{3}+\frac{5}{24}\right)$$

$$=2\left(-\frac{16}{24}+\frac{5}{24}\right)$$

$$=-\frac{11}{12}$$

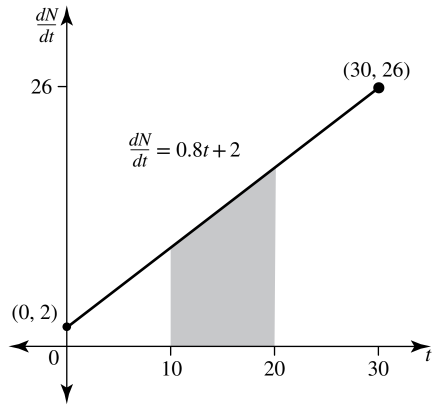
7 $\frac{dL}{dt} = \frac{4}{\sqrt{t}} = 4t^{-\frac{1}{2}}$

Average total increase in length is

$$\int_6^{36} 4t^{-\frac{1}{2}}dt = \left[8t^{\frac{1}{2}}\right]_6^{36}$$
$$= \left(8\sqrt{36}\right) - \left(8\sqrt{6}\right)$$
$$= 48 - 19.6$$
$$= 28.4 \text{ cm}$$

Therefore, the average total increase in length is 28.4 cm

8 **a** & **b** $\frac{dN}{dt} = 0.8t + 2$



c Number of bricks

$$= \int_{10}^{20} (0.8t + 2)\ dt$$
$$= \left[0.4t^2 + 2t\right]_{10}^{20}$$
$$= \left(0.4(20)^2 + 2(20)\right) - \left(0.4(10)^2 + 2(10)\right)$$
$$= 160 + 40 - 40 - 20$$
$$= 140 \text{ bricks}$$

9 $v = 1 + 3\sqrt{t+1} = 1 + 3(t+1)^{\frac{1}{2}}$

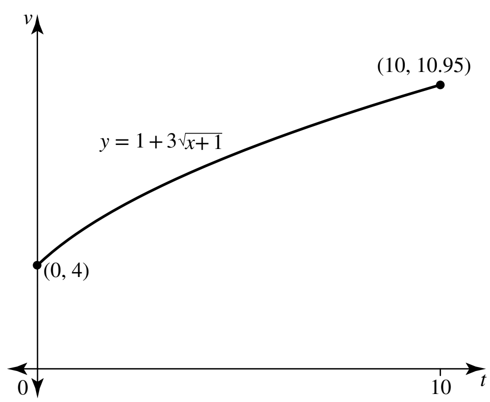
a Initially $t = 0$, $v = 1 + 3\sqrt{1} = 4$ m/s

b $a = \frac{dv}{dt} = \frac{3}{2}(1)(t+1)^{-\frac{1}{2}} = \frac{3}{2\sqrt{t+1}}$

i When $t = 0$, $a = \frac{3}{2\sqrt{1}} = \frac{3}{2} = 1.5$ m/s^2

ii When $t = 8$, $a = \frac{3}{2\sqrt{8+1}} = \frac{3}{6} = 0.5$ m/s^2

c



d Distance is

$$= \int_0^8 \left(1 + 3(t+1)^{\frac{1}{2}}\right) dt$$
$$= \left[t + 2(t+1)^{\frac{3}{2}}\right]_0^8$$
$$= \left(8 + 2(8+1)^{\frac{3}{2}}\right) - \left(0 + 2(0+1)^{\frac{3}{2}}\right)$$
$$= 8 + 2\left(3^2\right)^{\frac{3}{2}} - 2$$
$$= 8 + 54 - 2$$
$$= 60 \text{ metres}$$

10 **a** $v = \frac{dx}{dt} = 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$

$$x = \int 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)dt$$
$$x = 6\sin\left(\frac{t}{2} - \frac{\pi}{4}\right) + c$$

When $t = 0$, $x = -3\sqrt{2}$

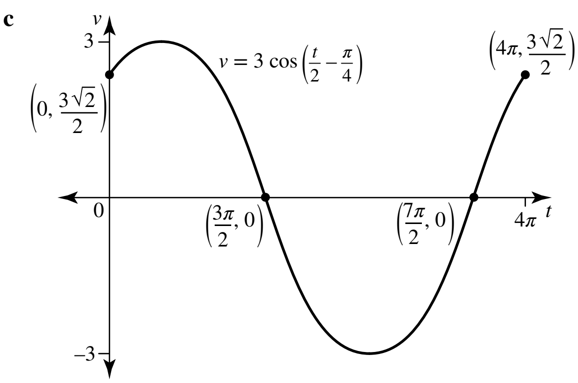
$$-3\sqrt{2} = 6\sin\left(-\frac{\pi}{4}\right) + c$$
$$-3\sqrt{2} = 6\left(-\frac{\sqrt{2}}{2}\right) + c$$
$$-3\sqrt{2} = -3\sqrt{2} + c$$
$$c = 0$$

Thus $x = 6\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)$

b When $t = 3\pi$,

$$x = 6\sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right)$$
$$x = 6\sin\left(\frac{6\pi}{4} - \frac{\pi}{4}\right)$$
$$x = 6\sin\left(\frac{5\pi}{4}\right)$$
$$x = 6 \times -\frac{\sqrt{2}}{2} = -3\sqrt{2} \text{ m}$$

c



d Distance is

$$= \int_0^{\frac{3\pi}{2}} 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right) dt - \int_{\frac{3\pi}{2}}^{3\pi} 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right) dt$$
$$= 2\int_0^{\frac{3\pi}{2}} 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right) dt$$

$= 2\left[6\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)\right]_0^{\frac{3\pi}{2}}$

$= 2\left(6\sin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) - 6\sin\left(-\frac{\pi}{4}\right)\right)$

$= 2\left(6\sin\left(\frac{\pi}{2}\right) - 6\sin\left(-\frac{\pi}{4}\right)\right)$

$= 2\left(6 + 3\sqrt{2}\right)$

$= 20.49$ m

e $v = 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$

$a = \frac{dv}{dt} = -\frac{3}{2}\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)$

f When $t = 3\pi$,

$a = -\frac{3}{2}\sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right)$

$a = -\frac{3}{2}\sin\left(\frac{6\pi}{4} - \frac{\pi}{4}\right)$

$a = -\frac{3}{2}\sin\left(\frac{5\pi}{4}\right)$

$a = -\frac{3}{2} \times -\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4}$ m/s^2

11 **a** $y = 3x^3 - x^4$....................(1)

$y = 3 - x$..........................(2)

(1) = (2)

$3x^3 - x^4 = 3 - x$

$0 = x^4 - 3x^3 - x + 3$

$0 = x^3(x-3) - (x-3)$

$0 = (x-3)(x^3 - 1)$

$0 = (x-3)(x-1)(x^2 + x + 1)$

$x - 3 = 0$ or $x - 1 = 0$ as $x^2 + x + 1$ cannot be further factorised

$x = 3$ $\quad$ $x = 1$

When $x - 1$, $y = 3 - 1 = 2$

When $x - 3$, $y = 3 - 3 = 0$

Thus $(a, b) = (1, 2)$ and $(c, 0) = (3, 0)$

b Area is

$= \int_1^3 \left(3x^3 - x^4 - (3 - x)\right) dx$

$= \int_1^3 \left(3x^3 - x^4 - 3 + x\right) dx$

$= \left[\frac{3}{4}x^4 - \frac{1}{5}x^5 - 3x + \frac{1}{2}x^2\right]_1^3$

$= \left[-\frac{1}{5}x^5 + \frac{3}{4}x^4 + \frac{1}{2}x^2 - 3x\right]_1^3$

$= \left(-\frac{1}{5}(3)^5 + \frac{3}{4}(3)^4 + \frac{1}{2}(3)^2 - 3(3)\right) - \left(-\frac{1}{5}(1)^5 + \frac{3}{4}(1)^4 + \frac{1}{2}(1)^2 - 3(1)\right)$

$= \left(-\frac{243}{5} + \frac{243}{4} + \frac{9}{2} - 9\right) - \left(-\frac{1}{5} + \frac{3}{4} + \frac{1}{2} - 3\right)$

$= -\frac{242}{5} + \frac{240}{4} + 4 - 6$

$= -\frac{968}{20} + \frac{1200}{20} - \frac{40}{20}$

$= \frac{192}{20}$

$= 9.6$ units^2

c Average value $= \dfrac{1}{b-a}\int_a^b f(x)\,dx$

$$= \frac{1}{2.5-1}\int_1^{2.5}\left(3x^3 - x^4\right)dx$$
$$= \frac{2}{3}\left[\frac{3}{4}x^4 - \frac{1}{5}x^5\right]_1^{2.5}$$
$$= \frac{2}{3}\left(\left(\frac{3}{4}\left(\frac{5}{2}\right)^4 - \frac{1}{5}\left(\frac{5}{2}\right)^5\right) - \left(\frac{3}{4}(1)^4 - \frac{1}{5}(1)^5\right)\right)$$
$$= \frac{2}{3}\left(\frac{1875}{64} - \frac{1250}{64} - \frac{11}{20}\right)$$
$$= 6.144$$

12 Area is

$$= \int_{-1.5}^{0.5}\left(\cos(x) - 0.5e^x\right)dx$$
$$= \left[\sin(x) - 0.5e^x\right]_{-1.5}^{0.5}$$
$$= \left(\sin(0.5) - 0.5e^{0.5}\right) - \left(\sin(-1.5) - 0.5e^{-1.5}\right)$$
$$= (0.4794 - 0.8244) - (-0.9975 - 0.1116)$$
$$= 0.7641 \text{ units}^2$$

13 a $y^2 = 4 - x$....................(1)

$y = x - 2$......................(2)

Substitute (2) into (1)

$$(x-2)^2 = 4 - x$$
$$x^2 - 4x + 4 = 4 - x$$
$$x^2 - 3x = 0$$
$$x(x-3) = 0$$
$$x = 0 \text{ or } x - 3 = 0$$
$$x = 3$$

When $x = 0,\ y = 0 - 2 = -2$

When $x = 3,\ y = 3 - 2 = 1$

Points of intersection are $(0,-2)$ and $(3,1)$.

b Green region is

$$= \int_2^3 (x-2)\,dx + \int_3^4 \sqrt{4-x}\,dx$$
$$= \left[\frac{1}{2}x^2 - 2x\right]_2^3 + \left[-\frac{2}{3}(4-x)^{\frac{3}{2}}\right]_3^4$$
$$= \left(\frac{1}{2}(3)^2 - 2(3)\right) - \left(\frac{1}{2}(2)^2 - 2(2)\right) + \left(-\frac{2}{3}(4-4)^{\frac{3}{2}}\right) - \left(-\frac{2}{3}(4-3)^{\frac{3}{2}}\right)$$
$$= \left(\frac{9}{2} - 6\right) - (2-4) + 0 + \frac{2}{3}$$
$$= -\frac{3}{2} + 2 + \frac{2}{3}$$
$$= \frac{7}{6} \text{ units}^2$$

c Orange region is

$$= \int_0^2\left(x - 2 - \left(-\sqrt{4-x}\right)\right)dx + \int_4^2\left(-\sqrt{4-x}\right)dx$$
$$= \int_0^2\left(x - 2 + (4-x)^{\frac{1}{2}}\right)dx - \int_4^2 (4-x)^{\frac{1}{2}}\,dx$$
$$= \left[\frac{1}{2}x^2 - 2x - \frac{2}{3}(4-x)^{\frac{3}{2}}\right]_0^2 - \left[-\frac{2}{3}(4-x)^{\frac{3}{2}}\right]_4^2$$

$=\left(\frac{1}{2}(2)^2-2(2)-\frac{2}{3}(4-2)^{\frac{3}{2}}\right)-\left(\frac{1}{2}(0)^2-2(0)-\frac{2}{3}(4-0)^{\frac{3}{2}}\right)-\left(-\frac{2}{3}(4-2)^{\frac{3}{2}}+\frac{2}{3}(4-4)^{\frac{3}{2}}\right)$

$=2-4-\frac{2}{3}(2)^{\frac{3}{2}}+\frac{2}{3}(4)^{\frac{3}{2}}+\frac{2}{3}(2)^{\frac{3}{2}}$

$=-2+5\frac{1}{3}$

$=3\frac{1}{3}$ units^2

d Area between graphs is = green region + orange region

$=1.17+3.3$

$\simeq 4.5$ units^2

14 Area is

$=\int_{-3}^{3}\left(-0.5x^2(x-3)(x+3)-0.25x^2(x-3)(x+3)\right)dx$

$=\int_{-3}^{3}\left(-0.75x^2(x-3)(x+3)\right)dx$

$=\int_{-3}^{3}\left(-0.75x^2(x^2-9)\right)dx$

$=\int_{-3}^{3}\left(-0.75x^4+6.75x^2\right)dx$

$=\left[\frac{-0.75}{5}x^5+\frac{6.75}{3}x^3\right]_{-3}^{3}$

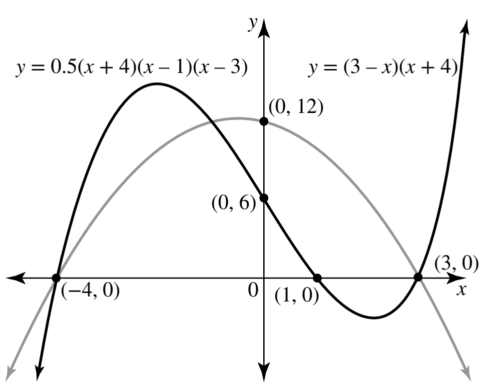
$=\left[-0.15x^5+2.25x^3\right]_{-3}^{3}$

$=\left(-0.15(3)^5+2.25(3)^3\right)-\left(-0.15(-3)^5+2.25(-3)^3\right)$

$=-36.45+60.75-36.45+60.75$

$=48.6$ units^2

15 a



b $y=0.5(x+4)(x-1)(x-3)$...........(1)

$y=(3-x)(x+4)$.........................(2)

$(1)=(2)$

$0.5(x+4)(x-1)(x-3)=(3-x)(x+4)$

$0.5(x+4)(x-1)(x-3)-(3-x)(x+4)=0$

$0.5(x+4)(x-1)(x-3)+(x-3)(x+4)=0$

$(x-3)(x+4)\left(0.5(x-1)+1\right)=0$

$(x-3)(x+4)\left(0.5x-0.5+1\right)=0$

$(x-3)(x+4)\left(0.5x+0.5\right)=0$

$x=3 \quad x=-4 \quad x=-1$

When $x=-4,\ y=(3+4)(-4+4)=0$

When $x=-1,\ y=(3+1)(-1+4)=12$

When $x=3,\ y=(3-1)(3+4)=0$

Therefore, the co-ordinates are $(-4,0),\ (-1,12)$ and $(3,0)$.

c Area between curves is

$$= \int_{-4}^{-1} \big(0.5(x+4)(x-1)(x-3) - (3-x)(x+4)\big)\,dx + \int_{-1}^{3} \big((3-x)(x+4) - 0.5(x+4)(x-1)(x-3)\big)\,dx$$

$$= \int_{-4}^{-1} \big((x-3)(x+4)(0.5(x-1)+1)\big)\,dx + \int_{-1}^{3} \big((3-x)(x+4)(1+0.5(x-1))\big)\,dx$$

$$= \int_{-4}^{-1} \big((x^2+x-12)(0.5x+0.5)\big)\,dx + \int_{-1}^{3} \big((-x^2-x+12)(0.5x+0.5)\big)\,dx$$

$$= \int_{-4}^{-1} \big(0.5x^3+0.5x^2-6x+0.5x^2+0.5x-6\big)\,dx + \int_{-1}^{3} \big(-0.5x^3-0.5x^2-0.5x^2-0.5x+6x+6\big)\,dx$$

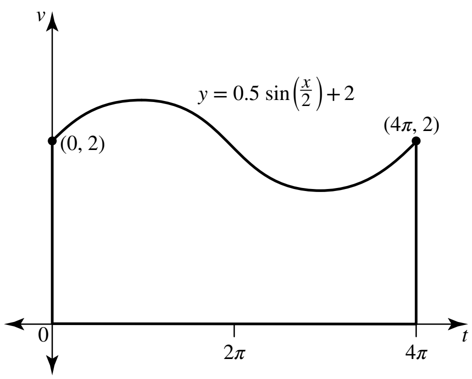
$$= \int_{-4}^{-1} \left(\frac{1}{2}x^3+x^2-\frac{11}{2}x-6\right)dx + \int_{-1}^{3} \left(-\frac{1}{2}x^3-x^2+\frac{11}{2}x+6\right)dx$$

$$= \left[\frac{1}{8}x^4+\frac{1}{3}x^3-\frac{11}{4}x^2-6x\right]_{-4}^{1} + \left[-\frac{1}{8}x^4-\frac{1}{3}x^3+\frac{11}{4}x^2+6x\right]_{-1}^{3}$$

$$= 12.375 + 26.7$$

$$= 39.04 \text{ units}^2$$

16 a



b Area is

$$= \int_0^{4\pi} \left(0.5\sin\left(\frac{x}{2}\right)+2\right)dx$$

$$= \left[-\cos\left(\frac{x}{2}\right)+2x\right]_0^{4\pi}$$

$$= \big(-\cos(2\pi)+2(4\pi)\big) - \big(-\cos(0)+2(0)\big)$$

$$= -1 + 8\pi + 1$$

$$= 8\pi$$

$$\simeq 25 \text{ m}^2$$

c Soil required is $0.5 \times 25 = 12.5$ m^3.

17 $v = e^{-0.5t} - 0.5$

a $a = \dfrac{dv}{dt} = -0.5e^{-0.5t}$

b $x = \int \left(e^{-0.5t} - 0.5\right) dt$

$\quad = -2e^{-0.5t} - 0.5t + c$

When $x = 0,\ t = 0$

$0 = -2e^{-0.5t(0)} - 0.5(0) + c$

$c = 2$

$x = -2e^{-0.5t} - 0.5t + 2$

c When $t = 4$,

$x = -2e^{-0.5(4)} - 0.5(4) + 2 = -0.2707$ metres

d Fourth second occurs between $t = 3$ and $t = 4$.

Distance

$= \int_4^3 \int \left(e^{-0.5t} - 0.5\right) dt$

$= \left[-2e^{-0.5t} - 0.5t\right]_4^3$

$= \left(-2e^{-0.5(3)} - 0.5(3)\right) - \left(-2e^{-0.5(4)} - 0.5(4)\right)$

$= -2e^{-1.5} - 1.5 + 2e^{-2} + 2$

$= 0.3244$ metres

18 a $y = a(x-5)(x+5)$

When $x = 0,\ y = 5$

$5 = a(-5)(5)$

$5 = -25a$

$-\dfrac{1}{5} = a$

$a = -0.2$

Thus equation of arch is $y = 5 - 0.2x^2$.

b $2\int_0^5 \left(5 - 0.2x^2\right) dx = 2\left[5x - \dfrac{0.2}{3}x^3\right]_0^5$

$= 2\left\{\left(5(5) - \dfrac{0.2}{3}(5)^3\right) - 0\right\}$

$= 2\left(25 - 8\dfrac{1}{3}\right)$

$= 33\dfrac{1}{3}\ \text{m}^2$

c Stone area $= (12 \times 7) - 33\dfrac{1}{3} = 50\dfrac{2}{3}\ \text{m}^2$.

d Volume of stones is $50\dfrac{2}{3} \times 3 = 152\ \text{m}^3$.

19 $N = \int_0^{17} 0.853e^{0.1333t} dt$

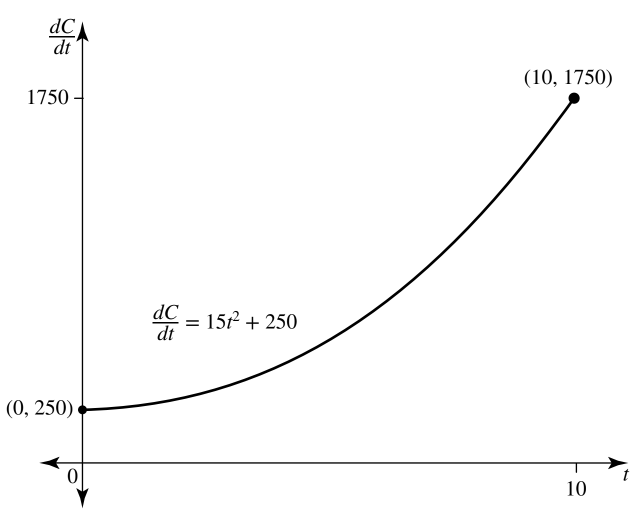
$= \left[6.3991e^{0.1333t}\right]_0^{17}$

$= 6.3991e^{0.1333(17)} - 6.3991e^{0.1333(0)}$

$= 6.3991(9.6417 - 1)$

$= 55.3$ million

20 a



b Total cost is

$= \int_5^{10} \left(15t^2 + 250\right) dt$

$= \left[5t^3 + 250t\right]_5^{10}$

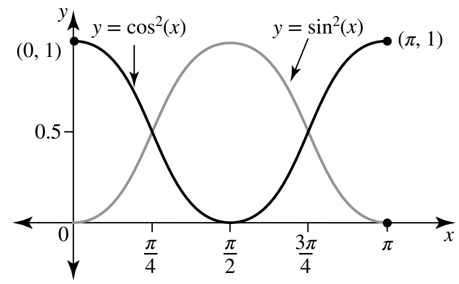
$= \left(5(10)^3 + 250(10)\right) - \left(5(5)^3 + 250(5)\right)$

$= (5000 + 2500) - (625 = 1250)$

$= 7500 - 1875$

$= \$5625$

21 a



b Area between curves is

$= \int_0^{\frac{\pi}{4}} \left(\cos^2(x) - \sin^2(x)\right) dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\sin^2(x) - \cos^2(x)\right) dx$

$+ \int_{\frac{3\pi}{4}}^{\pi} \left(\cos^2(x) - \sin^2(x)\right) dx$

$= 0.5 + 1 + 0.5$

$= 2\ \text{units}^2$

22 Parabola, of the form $y = a(x-2)(x+2)$

$(0, -3) \Rightarrow -3 = a(-2)(2)$

$-3 = -4a$

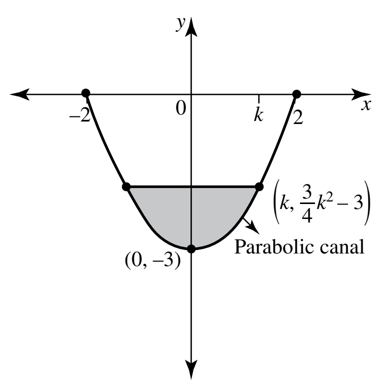
$a = \dfrac{3}{4}$

Equation of parabola is

$y = \dfrac{3}{4}(x-2)(x+2)$

$= \dfrac{3}{4}(x^2 - 4)$

$= \dfrac{3}{4}x^2 - 3$



If the canal were full with water, the cross sectional area would be

$$-\int_2^{-2}\left(\frac{3}{4}x^2-3\right)dx$$
$$=-2\int_0^2\left(\frac{3}{4}x^2-3\right)dx$$
$$=-2\left[\frac{1}{4}x^3-3x\right]_0^2$$
$$=-2\left(\left(\frac{1}{4}(2)^3-3(2)\right)-0\right)$$
$$=-2(2-6)$$
$$=8\text{ m}^2$$

When the canal is one-third full, the cross sectional area $=\frac{8}{3}\text{ m}^2$.

Cross-sectional area for a canal which is one third full is given by

$$A=2\int_0^k\frac{3}{4}k^2-3-\left(\frac{3}{4}x^2-3\right)dx$$
$$\frac{8}{3}=2\int_0^k\left(\frac{3}{4}k^2-\frac{3}{4}x^2\right)dx$$
$$=2\left[\frac{3}{4}k^2x-\frac{1}{4}x^3\right]_0^k$$
$$=2\left(\frac{3}{4}k^3-\frac{1}{4}k^3-0\right)$$
$$=2\left(\frac{1}{2}k^3\right)$$
$$=k^3$$
$$k=\sqrt[3]{\frac{8}{3}}$$
$$=1.39$$
$$y=\frac{3}{4}\times(1.39)^2-3$$
$$=-1.58$$
Therefore the depth of water = −1.58 − (−3) = 1.44 m.

Topic 9 — Logarithmic functions using calculus

Exercise 9.2 — The derivative of $f(x) = \log_e(x)$

1 a $\frac{d}{dx}\left(7\log_e\left(\frac{x}{3}\right)\right) = \frac{7}{x}$

b $\frac{d}{dx}\left(2\log_e\left(x^3+2x^2-1\right)\right) = \frac{2(3x^2+4x)}{x^3+2x^2-1}$

c $\frac{d}{dx}\left(\sin(x)\log_e(x-2)\right) = \sin(x)\times\frac{1}{(x-2)} + \cos(x)\log_e(x-2)$

$= \frac{\sin(x)}{x-2} + \cos(x)\log_e(x-2)$

d $y = \frac{\log_e(x^2)}{(2x-1)}$

Let $u = \log_e(x^2)$ so $\frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x}$

Let $v = 2x-1$ so $\frac{dv}{dx} = 2$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{(2x-1)\times\frac{2}{x} - 2\log_e(x^2)}{(2x-1)^2}$

$\frac{dy}{dx} = \left(\frac{2(2x-1)}{x} - 2\log_e(x^2)\right)\times\frac{1}{(2x-1)^2}$

$\frac{dy}{dx} = \frac{2(2x-1) - 2x\log_e(x^2)}{x(2x-1)^2}$

e $\frac{d}{dx}\left(3\log_e\left(e^{2x}-e^{-x}\right)\right) = \frac{3\left(2e^{2x}+e^{-x}\right)}{\left(e^{2x}-e^{-x}\right)} = \frac{3e^{-x}(2e^{3x}+1)}{e^{-x}(e^{3x}-1)} = \frac{3(2e^{3x}+1)}{(e^{3x}-1)}$

f $\frac{d}{dx}\left(\sqrt{\log_e(3-2x)}\right) = \frac{d}{dx}\left(\log_e(3-2x)\right)^{\frac{1}{2}}$

$= \frac{-2}{(3-2x)}\times\frac{1}{2}\left(\log_e(3-2x)\right)^{-\frac{1}{2}}$

$= \frac{-1}{(3-2x)\sqrt{\log_e(3-2x)}}$

$= \frac{1}{(2x-3)\sqrt{\log_e(3-2x)}}$

2 a $y = -5\log_e(2x)$

$\frac{dy}{dx} = -5\times\frac{1}{x}$

$\frac{dy}{dx} = -\frac{5}{x},\ x\in(0,\infty)$

b $y = \log_e\left(\frac{1}{x-2}\right)$

$= \log_e(x-2)^{-1}$

$= -\log_e(x-2)$

$\frac{dy}{dx} = -\frac{1}{x-2},\ x\in(2,\infty)$

c $y = \log_e\left(\frac{x+3}{x+1}\right) = \log_e(x+3) - \log_e(x+1)$

$\frac{dy}{dx} = \frac{1}{x+3} - \frac{1}{x+1}$

$= \frac{x+1-(x+3)}{(x+3)(x+1)}$

$= \frac{-2}{(x+3)(x+1)}$

for $x\in(-\infty,-3)\cup(-1,\infty)$

d $y=\log_e\left(x^2-x-6\right)$

$\frac{dy}{dx}=\frac{2x-1}{x^2-x-6},\ x\in(-\infty,-2)\cup(3,\infty)$

3 **a** Dom $=(2,\infty)$ and Ran $=R$

b Graph cuts the x axis where $y=0$

$2\log_e(x-2)=0$

$\log_e(x-2)=0$

$e^0=x-2$

$1=x-2$

$x=3$

Thus $(a,0)=(3,0)$ so $a=3$

c $y=2\log_e(x-2)$

$\frac{dy}{dx}=\frac{2}{x-2}$

When $x=3,\ m_T=\frac{dy}{dx}=\frac{2}{(3-2)}=2$

Equation of tangent with $m_T=2$ which passes through $\left(x_1,y_1\right)=(3,0)$ is given by

$y-y_1=m_T\left(x-x_1\right)$

$y-0=2(x-3)$

$y=2x-6$

d Equation of perpendicular line with $m_P=-\frac{1}{2}$ which passes through $\left(x_1,y_1\right)=(3,0)$ is given by

$y-y_1=m_P\left(x-x_1\right)$

$y-0=-\frac{1}{2}(x-3)$

$y=-\frac{1}{2}x+\frac{3}{2}$

4 $y=4\log_e(3x-1)$

$\frac{dy}{dx}=\frac{12}{3x-1}$

If the tangent is parallel to $6x-y+2=0$ or $y=6x+2$ then the gradient is 6.

$m_T=\frac{12}{3x-1}=6$

$12=6(3x-1)$

$12=18x-6$

$18=18x$

$1=x$

When $x=1,\ y=4\log_e(3(1)-1)=4\log_e(2)$

Equation of tangent with $m_T=6$ which passes through $(x_1,y_1)=\left(1,4\log_e(2)\right)$ is given by

$y-y_1=m_T(x-x_1)$

$y-4\log_e(2)=6(x-1)$

$y-4\log_e(2)=6x-6$

$y=6x+4\log_e(2)-6$

5 $y=\frac{1}{10x}+\log_e(x)$

Turning point occurs where $\frac{dy}{dx}=0.$

$\frac{1}{x}-\frac{1}{10x^2}=0$

$\frac{10x-1}{10x^2}=0$

$10x-1=0$

$10x=1$

$x=0.1$

When $x=0.1,\ y=\frac{1}{10(0.1)}+\log_e\left(\frac{1}{10}\right)=1+\log_{e(1)}-\log_e(10)=1-\log_e(10)$

Minimum TP at $(0.1,\ 1-\log_e(10))$.

6 a $f(x) = 2x\log_e(x)$

Local max/min values occur where $f'(x) = 0$

$$f'(x) = \frac{2x}{x} + 2\log_e(x) = 2 + 2\log_e(x)$$

$$0 = 2 + 2\log_e(x)$$
$$-2 = 2\log_e(x)$$
$$-1 = \log_e(x)$$
$$x = e^1$$

When $x = 0.1$, $f'(x) = 2\log_e(0.1) + 2 = -2.6052$

When $x = 1$, $f'(x) = 2\log_e(1) + 2 = 2$

x	$x < e^{-1}$	$x = e^{-1}$	$x > e^{-1}$
$f'(x)$	\	—	/

When $x = e^{-1}$,

$y = 2e^{-1}\log_e(e^{-1})$

$y = -2e^{-1}$

$y = -\frac{2}{e}$

$\left(\frac{1}{e}, -\frac{2}{e}\right)$ is a local minimum TP.

b $y = f(x) = \frac{\log_e(2x)}{x}$

Let $u = \log_e(2x)$ so $\frac{du}{dx} = \frac{1}{x}$

Let $v = x$ so $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \log_e(2x) \times 1}{x^2}$$

$$\frac{dy}{dx} = \frac{1 - \log_e(2x)}{x^2}$$

Max/min values occur where $\frac{dy}{dx} = 0$

$$\frac{1 - \log_e(2x)}{x^2} = 0$$
$$1 - \log_e(2x) = 0$$
$$1 = \log_e(2x)$$
$$e^1 = 2x$$
$$x = \frac{1}{2}e$$

When $x = 0.6$, $\frac{dy}{dx} = \frac{1 - \log_e(2 \times 0.6)}{(0.6)^2} = 1.7649$

When $x = 2$, $\frac{dy}{dx} = \frac{1 - \log_e(2 \times 2)}{(2)^2} = -0.0966$

x	$x < \frac{1}{2}e$	$x = \frac{1}{2}e$	$x > \frac{1}{2}e$
$\frac{dy}{dx}$	/	—	\

When $x = \frac{1}{2}e$, $y = \log_e(e) \div \frac{1}{2}e = \frac{2}{e}$

There is a local maximum at $\left(\frac{e}{2}, \frac{2}{e}\right)$.

c $f(x) = x\log_e\left(\frac{3}{x}\right)$

$f'(x) = \log_e\left(\frac{3}{x}\right) + x \times \left(-\frac{1}{x}\right)$

$f'(x) = \log_e\left(\frac{3}{x}\right) - 1$

Max/min values occur where $f'(x) = 0$

$$\log_e\left(\frac{3}{x}\right) - 1 = 0$$
$$\log_e\left(\frac{3}{x}\right) = 1$$
$$\frac{3}{x} = e$$
$$x = \frac{3}{e}$$

When $x = 1$, $f'(1) = \log_e(3) - 1 = 0.0986$

When $x = 2$, $f'(1) = \log_e(1.5) - 1 = -0.5945$

x	$x < \frac{3}{e}$	$x = \frac{3}{e}$	$x > \frac{3}{e}$
$f'(x)$	/	—	\

When $x = \frac{3}{e}$, $f\left(\frac{3}{e}\right) = \frac{3}{e}\log_e\left(3 \div \frac{3}{e}\right) = \frac{3}{e}\log_e(e) = \frac{3}{e}$

There is a local maximum at $\left(\frac{3}{e}, \frac{3}{e}\right)$.

7 a $\frac{d}{dx}\left(4\log_e\left(\frac{x}{2}\right)\right) = \frac{4}{x}$

b $\frac{d}{dx}\left(\frac{1}{2}\log_e\left(\sqrt{x-2}\right)\right) = \frac{d}{dx}\left(\frac{1}{2}\log_e(x-2)^{\frac{1}{2}}\right)$

$$= \frac{d}{dx}\left(\frac{1}{2} \times \frac{1}{2}\log_e(x-2)\right)$$
$$= \frac{1}{4} \times \frac{1}{x-2}$$
$$= \frac{1}{4(x-2)}$$

c $\frac{d}{dx}\left(\log_e\left(x^3 - 3x^2 + 7x - 1\right)\right) = \frac{3x^2 - 6x + 7}{x^3 - 3x^2 + 7x - 1}$

d $\frac{d}{dx}\left(-6\log_e(\cos(x)\right) = 6\frac{\sin(x)}{\cos(x)} = 6\tan(x)$

e $\frac{d}{dx}\left(\sqrt{\log_e(3x+1)}\right) = \frac{d}{dx}\left(\log_e(3x+1)\right)^{\frac{1}{2}}$

$$= \frac{1}{2}\left(\log_e(3x+1)\right)^{-\frac{1}{2}} \times \frac{3}{3x+1}$$
$$= \frac{3}{2(3x+1)\sqrt{\log_e(3x+1)}}$$

f $y = \frac{2\log_e(2x)}{e^{2x} + 1}$

Let $u = 2\log_e(2x)$ so $\frac{du}{dx} = \frac{2}{x}$

Let $v = e^{2x} + 1$ so $\frac{dv}{dx} = 2e^{2x}$

$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx}=\frac{\frac{2}{x}(e^{2x}+1)-4e^{2x}\log_e(2x)}{(e^{2x}+1)^2}$

$\frac{dy}{dx}=\frac{2(e^{2x}+1)-4xe^{2x}\log_e(2x)}{x(e^{2x}+1)^2}$

$\frac{dy}{dx}=\frac{2e^{2x}+2-4xe^{2x}\log_e(2x)}{x(e^{2x}+1)^2}$

8 a $y=(x^2-3x+7)\log_e(2x-1)$

$\frac{dy}{dx}=(x^2-3x+7)\times\frac{2}{(2x-1)}+(2x-3)\log_e(2x-1)$

$\frac{dy}{dx}=(2x-3)\log_e(2x-1)+\frac{2(x^2-3x+7)}{(2x-1)},\ x\in\left(\frac{1}{2},\infty\right)$

b $y=\sin(x)\log_e\left(x^2\right)$

$\frac{dy}{dx}=\sin(x)\times\frac{2x}{x^2}+\cos(x)\log_e\left(x^2\right)$

$\frac{dy}{dx}=\frac{2\sin(x)}{x}+\cos(x)\log_e\left(x^2\right)$

$\frac{dy}{dx}=\frac{x\cos(x)\log_e\left(x^2\right)+2\sin(x)}{x},\ x\in(0,\infty)$

c $y=\frac{\log_e(3x)}{(x^3-x)}$

Let $u=\log_e(3x)$ so $\frac{du}{dx}=\frac{1}{x}$

Let $v=x^3-x$ so $\frac{dv}{dx}=3x^2-1$

$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx}=\frac{\frac{1}{x}(x^3-x)-(3x^2-1)\log_e(3x)}{(x^3-x)^2}$

$\frac{dy}{dx}=\frac{(x^2-1)-(3x^2-1)\log_e(3x)}{(x^3-x)^2}$

$\frac{dy}{dx}=\frac{x^2-3x^2\log_e(3x)-1+\log_e(3x)}{(x^3-x)^2},\ x\in(0,\infty)/\{1\}$

d $y=\log_e\left(\frac{4-x}{x+2}\right)=\log_e(4-x)-\log_e(x+2)$

$\frac{dy}{dx}=\frac{-1}{4-x}-\frac{1}{x+2}$

$=\frac{-(x+2)-(4-x)}{(4-x)(x+2)}$

$=\frac{-6}{(4-x)(x+2)}$

where $x\in(-2,4)$

9 Use CAS for this question

a $\frac{d}{dx}\left(2\log_5(x)\right)$ given $x=5$ is $\frac{2}{5}\log_5(e)$

b $\frac{d}{dx}\left(\frac{1}{3}\log_3(x+1)\right)$ given $x=2$ is $\frac{1}{9}\log_3(e)$

c $\frac{d}{dx}\left(\log_6(x^2-3)\right)$ given $x=3$ is $\log_6(e)$

10 a $y=\log_e(2x-2)$

Gradient of tangent is $m_T=\frac{dy}{dx}=\frac{2}{2x-2}=\frac{1}{x-1}$

When $x=1.5$, $m_T=\frac{1}{1.5-1}=2$

Equation of tangent with $m_T=2$ which passes through $\left(x_1,y_1\right)=(1.5,0)$ is given by

$y-y_1=m_T\left(x-x_1\right)$

$y-0=2(x-1.5)$

$y=2x-3$

b $y=3\log_e(x)$

Gradient of tangent is $m_T=\frac{dy}{dx}=\frac{3}{x}$

When $x=e$, $m_T=\frac{3}{e}$

Equation of tangent with $m_T=\frac{3}{e}$ which passes through $\left(x_1,y_1\right)=(e,3)$ is given by

$y-y_1=m_T\left(x-x_1\right)$

$y-3=\frac{3}{e}(x-e)$

$y-3=\frac{3}{e}x-3$

$y=\frac{3}{e}x$

c $y=\frac{1}{2}\log_e\left(x^2\right)=\log_e(x)$

Gradient of tangent $m_T=\frac{dy}{dx}=\frac{1}{x}$

When $x=e$, $m_T=\frac{1}{e}$

Equation of tangent with $m_T=\frac{1}{e}$ which passes through $\left(x_1,y_1\right)=(e,1)$ is given by

$y-y_1=m_T\left(x-x_1\right)$

$y-1=\frac{1}{e}(x-e)$

$y-1=\frac{1}{e}x-1$

$y=\frac{1}{e}x$

11 a $y=2\log_e(2x)$

$\frac{dy}{dx}=\frac{2}{x}$

b Gradient of tangent at $\left(\frac{e}{2},2\right)$ is $m_T=2\div\frac{e}{2}=\frac{4}{e}$.

Equation of tangent with $m_T=\frac{4}{e}$ which passes through $(x_1,y_1)=\left(\frac{e}{2},e\right)$ is given by

$y-y_1=m_T(x-x_1)$

$y-e=\frac{4}{e}\left(x-\frac{e}{2}\right)$

$y-e=\frac{4}{e}x-2$

$y=\frac{4}{e}x-2+e$

12 $y=x$ is a tangent to $y=\log_e(x-1)+b$

Gradient of tangent is $m_T=1$

Also gradient of tangent is $m_T=\frac{dy}{dx}=\frac{1}{x-1}$

Thus

$\frac{1}{x-1}=1$

$1=x-1$

$x=2$

When $x=2,\ y=2$

$2=\log_e(2-1)+b$

$2=\log_e(1)+b$

$b=2$

Thus $y=\log_e(x-1)+2$

13 $y=-2x+k$ is perpendicular to $y=\log_e(2(x-1))$

Gradient of perpendicular line is $m_P=-2$

Gradient of tangent is $m_T=\frac{1}{2}$

Also gradient of tangent is $m_T=\frac{dy}{dx}=\frac{1}{x-1}$

Thus

$\frac{1}{x-1}=\frac{1}{2}$

$x-1=2$

$x=3$

When $x=3,\ y=\log_e(2(3-1))=\log_e(4)\simeq 1.3863$

$1.3863=-2(3)+k$

$7.3863=k$

$k\simeq 7.4$

Thus $y=-2x+7.4$

14 a The graph cuts the x axis where $y=0$.

$\frac{1}{x^2}-2\log_e(x+3)=0$

$\frac{1}{x^2}=2\log_e(x+3)$

By calculator $x=-1.841$ and -0.795

Therefore coordinates are (−1.841, 0) and (−0.795, 0)

b $y=\frac{1}{x^2}-2\log_e(x+3)$

$y=x^{-2}-2\log_e(x+3)$

$\frac{dy}{dx}=-2x^{-3}-\frac{2}{x+3}$

$\frac{dy}{dx}=-\frac{2}{x^3}-\frac{2}{x+3}$

$\frac{dy}{dx}=\frac{-2(x+3)-2x^3}{x^3(x+3)}$

$\frac{dy}{dx}=-\frac{(2x^3+2x+6)}{x^3(x+3)}$

At $(-1.841,0),\ m_T=-\frac{(2(-1.841)^3+2(-1.841)+6)}{(-1.841)^3(-1.841+3)}=-1.1989.$

Equation of tangent with $m_T=-1.1989$ which passes through $(x_1,y_1)=(-1.841,0)$ is given by

$y-y_1=m_T(x-x_1)$

$y-0=-1.1989(x+1.841)$

$y=-1.1989x-2.2072$

At $(-0.795,0),\ m_T=-\frac{(2(-0.795)^3+2(-0.795)+6)}{(-0.795)^3(-0.795+3)}=3.0725.$

Equation of tangent with $m_T=3.0735$ which passes through $(x_1,y_1)=(-0.795,0)$ is given by

$y-y_1=m_T(x-x_1)$

$y-0=3.0735(x+0.795)$

$y=3.0735x+2.4434$

c Minimum TP occurs when $\frac{dy}{dx}=0$

$$-\frac{(2x^3+2x+6)}{x^3(x+3)}=0$$
$$2x^3+2x+6=0$$
$$x^3+x+6=0$$
$$x=-1.2134 \text{ since } x<0$$

When

$x=-1.2134,\ y=\frac{1}{(-1.2134)^2}-2\log_e(-1.2134+3)=-0.4814.$

Minimum TP is $(-1.2134,-0.4814)$.

15 a $$\left(2\log_e(x)\right)^2=2\log_e(x)$$
$$\left(2\log_e(x)\right)^2-2\log_e(x)=0$$
$$2\log_e(x)\left(2\log_e(x)-1\right)=0$$
$$\log_e(x)=0 \text{ or } 2\log_e(x)-1=0$$
$$e^0=x \qquad 2\log_e(x)=1$$
$$x=1 \qquad \log_e(x)=0.5$$
$$x=e^{0.5}$$

When $x=1,\ y=2\log_e(1)=0$

When $x=e^{0.5},\ y=2\log_e(e)^{0.5}=1$

The points of intersection are (1,0) and $\left(e^{0.5},1\right)$.

b If $y=\left(2\log_e(x)\right)^2$

$$\frac{dy}{dx}=2\left(\frac{2}{x}\right)\left(2\log_e(x)\right)=\frac{8\log_e(x)}{x}$$

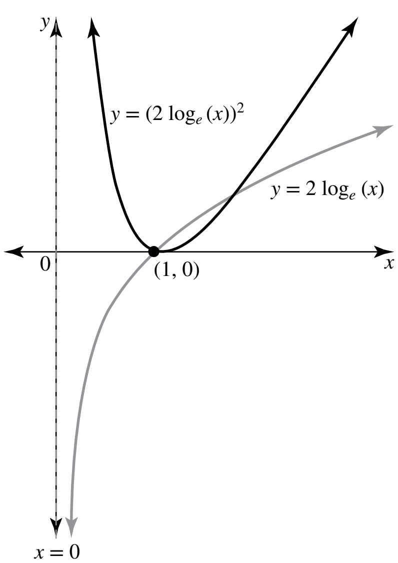
At (1,0) then $\frac{dy}{dx}=\frac{8\log_e(1)}{1}=0$

If $y=2\log_e(x)$

$$\frac{dy}{dx}=\frac{2}{x}$$

At (1,0) then $\frac{dy}{dx}=\frac{2}{1}=2$

c



d $2\log_e(x)>\left(2\log_e(x)\right)^2$ when $\left\{x: 1<x<e^{0.5}\right\}$.

16 a Graph cuts the x axis where $y=0$.

$$-\frac{1}{x^2}-8\log_e(x)=0$$
$$-8\log_e(x)=\frac{1}{x^2}$$
$$x=0.3407 \text{ or } 0.8364$$

Therefore coordinates are (0.3407, 0) and (0.8364, 0)

b $$y=-\frac{1}{x^2}-8\log_e(x)$$
$$y=-x^{-2}-8\log_e(x)$$
$$\frac{dy}{dx}=2x^{-3}-\frac{8}{x}$$
$$\frac{dy}{dx}=\frac{2}{x^3}-\frac{8}{x}$$
$$\frac{dy}{dx}=\frac{2-8x^2}{x^3}$$

When $x=0.3407,\ \frac{dy}{dx}=\frac{2-8(0.3407)^2}{(0.3407)^3}=27.09$

When $x=0.8364,\ \frac{dy}{dx}=\frac{2-8(0.8364)^2}{(0.8364)^3}=-6.15$

c At $(1,-1)$, $m_T=\frac{dy}{dx}=\frac{2-8(1)^2}{(1)^3}=-6$

Equation of tangent with $m_T=-6$ which passes through $\left(x_1,y_1\right)=(1,-1)$ is given by

$$y-y_1=m_T\left(x-x_1\right)$$
$$y+1=-6(x-1)$$
$$y+1=-6x+6$$
$$y=-6x+5$$

Equation of perpendicular line with $m_P=\frac{1}{6}$ which passes through $\left(x_1,y_1\right)=(1,-1)$ is given by

$$y-y_1=m_P\left(x-x_1\right)$$
$$y+1=\frac{1}{6}(x-1)$$
$$y+1=\frac{1}{6}x-\frac{1}{6}$$
$$y=\frac{1}{6}x-\frac{7}{6} \text{ or } x-6y=7$$

d TP occurs where $\frac{dy}{dx}=0.$

$$\frac{2-8x^2}{x^3}=0$$
$$2-8x^2=0$$
$$1-4x^2=0$$
$$(1-2x)(1+2x)=0$$
$$1-2x=0 \text{ or } 1+2x=0$$
$$1=2x \qquad 2x=-1$$
$$x=\frac{1}{2} \qquad x=-\frac{1}{2} \text{ but } x>0$$

When $x=0.5,\ y=-\frac{1}{(0.5)^2}-8\log_e(0.5)=-4+8\log_e(2)$

Maximum TP at $\left(\frac{1}{2},-4+8\log_e(2)\right)$.

17 a $f:\ y=-2\log_e(2-x)-1$ where dom = $(-\infty,2)$ and ran = R

f^{-1} : Interchange x and y.

$$x=-2\log_e(2-y)-1$$
$$x+1=-2\log_e(2-y)$$
$$-\frac{1}{2}(x+1)=\log_e(2-y)$$
$$e^{-\frac{1}{2}(x+1)}=2-y$$
$$y=2-e^{-\frac{1}{2}(x+1)}$$

$f^{-1}: R\to R,\ f^{-1}(x)=2-e^{-\frac{1}{2}(x+1)}$ where Dom = R and Ran = $(-\infty,2)$

b If $f(x)=x$ then

$-2\log_e(2-x)-1=x$

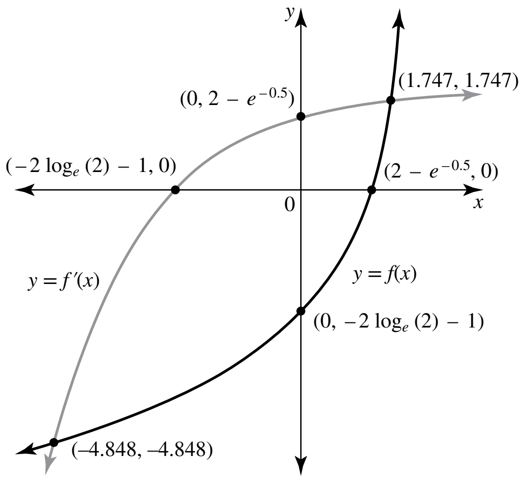
$x=-4.8479$ or 1.7467

When $x=-4.8479,\ y=-2\log_e(2+4.8479)-1=-4.8479$

When $x=1.7467,\ y=-2\log_e(2-1.7467)-1=1.7467$

Points of intersection are (–4.8479, –4.8479) and (1.7467, 1.7467).

c



18 Equation of tangent is $y=m_Tx+c$

When $y=0,\ x=0.3521$

$0=0.3521m_T+c$

$-0.3521m_T=c$

Thus

$y=m_Tx-0.3521m_T$

$y=m_T(x-0.3521)$

But $m_T=\dfrac{dy}{dx}=\dfrac{2}{2x-1}$

Thus $y=\dfrac{2}{2x-1}(x-0.3521)$

At $x=n,\ y=\dfrac{2}{2n-1}(n-0.3521)$.................(1)

Also at $x=n,\ y=\log_e(2n-1)$.....................(2)

(1) = (2)

$\dfrac{2}{2n-1}(n-0.3521)=\log_e(2n-1)$

$n=2$ as n is an integer

Exercise 9.3 — The antiderivative of $f(x)=\frac{1}{x}$

1 a $\int\frac{2}{5x}dx=\frac{2}{5}\int\frac{1}{x}dx=\frac{2}{5}\log_e(x)+c,\ x>0$

b $\int_1^3\frac{3}{4x-1}dx=\frac{3}{4}\int_1^3\frac{4}{4x-1}dx$

$=\left[\frac{3}{4}\log_e(4x-1)\right]_1^3$

$=\left(\frac{3}{4}\log_e(4(3)-1)\right)-\left(\frac{3}{4}\log_e(4(1)-1)\right)$

$=\frac{3}{4}\log_e(11)-\frac{3}{4}\log_e(3)$

$=\frac{3}{4}\log_e\left(\frac{11}{3}\right)$

2 a $\int\frac{x^2+2x-3}{x^2}dx=\int\left(1+2x^{-1}-3x^{-2}\right)dx$

$=x+2\log_e(x)+3x^{-1}+c$

$=x+2\log_e(x)+\frac{3}{x}+c,\ x>0$

b $f(x)=\int\left(x^3-\frac{1}{x}\right)dx$

$=\frac{x^4}{4}-\log_e(x)+c,\quad x>0$

$f(1)=\frac{1}{4}$

$\frac{1}{4}=\frac{1}{4}-\log_e(1)+c$

$c=0$

$f(x)=\frac{x^4}{4}-\log_e(x),\quad x>0$

3 $y=2\log_e(\cos(2x))$

$\frac{dy}{dx}=\frac{-2\sin(2x)}{\cos(2x)}=-\tan(2x)$

$\int(-2\tan(2x))\,dx=\log_e(\cos(2x))$

$-2\int(\tan(2x))\,dx=\log_e(\cos(2x))$

$\int(\tan(2x))\,dx=-\frac{1}{2}\log_e(\cos(2x))$

4 $y=\left(\log_e(x)\right)^2$

Let $u=\log_e(x)$ so $\frac{du}{dx}=\frac{1}{x}$

Thus $y=u^2$ so $\frac{dy}{du}=2u$

$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$

$\frac{dy}{dx}=\frac{1}{x}\times 2u$

$\frac{dy}{dx}=\frac{1}{x}\times 2\log_e(x)$

$\frac{dy}{dx}=\frac{2}{x}\log_e(x)$

$\int_1^e\left(\frac{4}{x}\log_e(x)\right)dx=2\int_1^e\left(\frac{2}{x}\log_e(x)\right)dx$

$=2\left[\left(\log_e(x)\right)^2\right]_1^e$

$=2\left(\left(\log_e(e)\right)^2-\left(\log_e(1)\right)^2\right)$

$=2(1-0)$

$=2$

5 a $f(x)=\frac{1}{x+2}-1$ cuts the x axis where $f(x)=0$.

Thus $\frac{1}{x+2}-1=0$

$\frac{1}{x+2}=1$

$1=x+2$

$x=-1$

So $(a,0)=(-1,0),\ a=-1$

b Area is

$\int_{-1}^2\left(\frac{1}{x+2}-1\right)dx=\left[\log_e(x+2)-x\right]_{-1}^2$

$=\left(\log_e(4)-2\right)-\left(\log_e(1)+1\right)$

$=\log_e(4)-3$

c $\frac{1}{x+2}-1=-\frac{1}{2}x+\frac{1}{4}$

$\frac{1}{x+2}=-\frac{1}{2}x+\frac{5}{4}$

$\frac{4}{x+2}=-2x+5$

$4=(x+2)(-2x+5)$

$4=-2x^2+x+10$

$0=2x^2-x-6$

$=(2x+3)(x-2)$

$x=-\frac{3}{2},2$

$x=-\frac{3}{2}, y=\frac{1}{-\frac{3}{2}+2}-1=1 \quad \therefore\left(-\frac{3}{2},1\right)$

$x=2, y=\frac{1}{2+2}-1=-\frac{3}{4} \quad \therefore\left(2,-\frac{3}{4}\right)$

d $A=\int_{-\frac{3}{2}}^{2}\left(-\frac{1}{2}x+\frac{1}{4}-\left(\frac{1}{x+2}-1\right)\right)dx$

$=\int_{-\frac{3}{2}}^{2}\left(-\frac{1}{2}x+\frac{5}{4}-\frac{1}{x+2}\right)dx$

$=\left[-\frac{1}{4}x^2+\frac{5}{4}x-\log_e(x+2)\right]_{-\frac{3}{2}}^{2}$

$=-\frac{1}{4}(2)^2+\frac{5}{4}(2)-\log_e(4)-\left(-\frac{1}{4}\left(\frac{-3}{2}\right)^2+\frac{5}{4}\left(\frac{-3}{2}\right)-\log_e\left(\frac{1}{2}\right)\right)$

$=-1+\frac{5}{2}-\log_e(4)+\frac{9}{16}+\frac{15}{8}+\log_e\left(\frac{1}{2}\right)$

$=\frac{63}{16}-3\log_e(2)$

6 a Graphs intersect where

$\frac{1}{x+1}=-\frac{1}{(x+1)^2}+3$

$x+1=-1+3(x+1)^2$ prov $x\neq-1$

$x+1=-1+3x^2+6x+3$

$0=3x^2+5x+1$

$x=\frac{-5\pm\sqrt{(5)^2-4(3)(1)}}{2(3)}$

$x=\frac{-5\pm\sqrt{13}}{6}$

When $x=\frac{-5+\sqrt{13}}{6},\ y=\frac{1}{\frac{-5+\sqrt{13}}{6}+1}=\frac{1}{\frac{-5+\sqrt{13}+6}{6}}=\frac{6}{\sqrt{13}+1}$

$\therefore \text{POI}=\left(\frac{-5+\sqrt{13}}{6},\frac{6}{\sqrt{13}+1}\right)$

b Required area is

$\int_{-0.2324}^{2}\left(-\frac{1}{(x+1)^2}+3-\frac{1}{x+1}\right)dx$

$=\int_{-0.2324}^{2}\left(-(x+1)^{-2}+3-\frac{1}{x+1}\right)dx$

$=\left[(x+1)^{-1}+3x-\log_e(x+1)\right]_{-0.2324}^{2}$

$=\left[\frac{1}{(x+1)}+3x-\log_e(x+1)\right]_{-0.2324}^{2}$

$=4.3647\text{ units}^2$

7 $A = -\int_{-4}^{-2} \frac{1}{x}\,dx$

$= \int_{2}^{4} \frac{1}{x}\,dx$

$= \left[\log_e(x)\right]_2^4$

$= \log_e(4) - \log_e(2)$

$= \log_e(2)$ units^2

8 $A = \int_{-2}^{-1} \left(\frac{1}{x-1} + 2\right) dx$

$= \int_{1}^{2} \left(\frac{1}{-x-1} + 2\right) dx$

$= \int_{1}^{2} \left(-\frac{1}{x+1} + 2\right) dx$

$= \left[-\log_e(x+1) + 2x\right]_1^2$

$= -\log_e(3) + 4 - \left(-\log_e(2) + 2\right)$

$= -\log_e(3) + 4 + \log_3(2) - 2$

$= \log_e\left(\frac{2}{3}\right) + 2$ units^2

9 **a** $\int -\frac{4}{x}\,dx = -4\log_e(x) + c,\ x > 0$

b $\int \frac{3}{4x+7}\,dx = \frac{3}{4}\int \frac{4}{4x+7}\,dx = \frac{3}{4}\log_e(4x+7) + c,\ x > -\frac{7}{4}$

c $\int \frac{x^3 + 2x^2 + 3x - 1}{x^2}\,dx = \int \left(x + 2 + \frac{3}{x} - x^{-2}\right) dx$

$= \frac{1}{2}x^2 + 2x + 3\log_e(x) + \frac{1}{x} + c,\ x > 0$

d $\int \left(\frac{3}{2-x} + \cos(4x)\right) dx$

$= -3\int \frac{-1}{(2-x)}\,dx + \int \cos(4x)\,dx = -3\log_e(2-x) + \frac{1}{4}\sin(4x) + c,\ x < 2$

10 **a** $\int_{2}^{4} \frac{3}{1-2x}\,dx = 3\int_{2}^{4} \frac{1}{1-2x}$

$= 3\left[-\frac{1}{2}\log_e(1-2x)\right]_2^4$

$= 3\left(-\frac{1}{2}\log_e(1-2(4)) + \frac{1}{2}\log_e(1-2(2))\right)$

$= -\frac{3}{2}\left(\log_e(7) - \log_e(3)\right)$

$= -\frac{3}{2}\log_e\left(\frac{7}{3}\right)$

b $\int_{-3}^{-1} \left(\frac{2}{x+4}\right) dx = 2\int_{-3}^{-1} \left(\frac{1}{x+4}\right) dx$

$= 2\left[\log_e(x+4)\right]_{-3}^{-1}$

$= 2\left(\log_e(-1+4) - \log_e(-3+4)\right)$

$= 2\left(\log_e(3) - \log_e(1)\right)$

$= 2\log_e(3)$

c $\int_{1}^{4} \left(e^{2x} + \frac{2}{x}\right) dx = \left[\frac{1}{2}e^{2x} + 2\log_e(x)\right]_1^4$

$= \left(\frac{1}{2}e^8 + 2\log_e(4)\right) - \left(\frac{1}{2}e^2 + 2\log_e(1)\right)$

$= 1489.56$

11 a $\frac{dy}{dx}=\frac{5}{2x+4}$ is equivalent to

$$\int\frac{5}{2x+4}dx=5\int\frac{1}{2x+4}dx$$
$$=\frac{5}{2}\log_e(2x+4)+c,\ \ x>-2$$

Thus $y=\frac{5}{2}\log_e(2(x+2))+c$ and when $x=-\frac{3}{2},\ y=3.$

$$3=\frac{5}{2}\log_e\left(2\left(-\frac{3}{2}\right)+4\right)+c$$
$$3=\frac{5}{2}\log_e(1)+c$$
$$c=3$$
$$y=\frac{5}{2}\log_e(2x+4)+3$$
$$y=\frac{5}{2}\log_e(2(x+2))+3$$

b $\frac{dy}{dx}=\frac{3}{2-5x}$ is equivalent to

$$\int\frac{3}{2-5x}dx=3\int\frac{1}{2-5x}dx$$
$$=-\frac{3}{5}\log_e(2-5x)+c,\quad x<\frac{2}{5}$$

Thus $y=-\frac{3}{5}\log_e(2-5x)+c$ and when $x=\frac{1}{5},\ y=1.$

$$1=-\frac{3}{5}\log_e\left(2-5\left(\frac{1}{5}\right)\right)+c$$
$$1=-\frac{3}{5}\log_e(1)+c$$
$$c=1$$
$$y=-\frac{3}{5}\log_e(2-5x)+1$$

12 Let $y=f(x)=2x\log_e(mx)$

$$\frac{dy}{dx}=2\log_e(mx)+2x\times\frac{1}{x}$$
$$\frac{dy}{dx}=2\log_e(mx)+2$$
$$\int(2\log_e(mx)+2)\,dx=2x\log_e(mx)$$
$$2\int\log_e(mx)\,dx+\int 2\,dx=2x\log_e(mx)$$
$$2\int\log_e(mx)\,dx=2x\log_e(mx)-\int 2dx$$
$$2\int\log_e(mx)\,dx=2x\log_e(mx)-2x$$
$$\int\log_e(mx)\,dx=x\log_e(mx)-x+c$$

13 If $y=3x\log_e(x)$ then $\frac{dy}{dx}=3\log_e(x)+\frac{3x}{x}=3\log_e(x)+3$

$$\int(3\log_e(x)+3)\,dx=3x\log_e(x)$$
$$3\int\log_e(x)\,dx+\int 3\,dx=3x\log_e(x)$$
$$3\int\log_e(x)\,dx=3x\log_e(x)-\int 3\,dx$$
$$3\int\log_e(x)\,dx=3x\log_e(x)-3x$$
$$\int\log_e(x)\,dx=x\log_e(x)-x$$
$$2\int\log_e(x)\,dx=2x\log_e(x)-2x$$
$$\int 2\log_e(x)\,dx=2x\log_e(x)-2x$$

14 If $y=\log_e\left(3x^3-4\right)$

$$\frac{dy}{dx}=\frac{9x^2}{3x^3-4}$$
$$\int_2^3\frac{9x^2}{3x^3-4}dx=\left[\log_e\left(3x^3-4\right)\right]_2^3$$
$$9\int_2^3\frac{x^2}{3x^3-4}dx=\left[\log_e\left(3x^3-4\right)\right]_2^3$$
$$\int_2^3\frac{x^2}{3x^3-4}dx=\frac{1}{9}\left[\log_e\left(3x^3-4\right)\right]_2^3$$
$$\int_2^3\frac{x^2}{3x^3-4}dx=\frac{1}{9}\left(\log_e\left(3(3)^3-4\right)-\log_e\left(3(2)^3-4\right)\right)$$
$$\int_2^3\frac{x^2}{3x^3-4}dx=\frac{1}{9}\left(\log_e(77)-\log_e(20)\right)$$
$$\int_2^3\frac{x^2}{3x^3-4}dx=\frac{1}{9}\log_e\left(\frac{77}{20}\right)$$

15 If $y=\log_e\left(e^x+1\right)^2$

Let $u=\left(e^x+1\right)^2$ so $\frac{du}{dx}=2\left(e^x\right)\left(e^x+1\right)=2e^{2x}+2e^x$

Let $y=\log_e(u)$ so $\frac{dy}{du}=\frac{1}{u}$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$
$$\frac{dy}{dx}=\left(2e^{2x}+2e^x\right)\times\frac{1}{u}$$
$$\frac{dy}{dx}=\left(2e^{2x}+2e^x\right)\times\frac{1}{\left(e^x+1\right)^2}$$
$$\frac{dy}{dx}=\frac{2e^x\left(e^x+1\right)}{\left(e^x+1\right)^2}$$
$$\frac{dy}{dx}=\frac{2e^x}{\left(e^x+1\right)},\ e^x\neq-1$$

Thus

$$\int_1^5\frac{2e^x}{\left(e^x+1\right)}dx=\left[\log_e\left(\left(e^x+1\right)^2\right)\right]_1^5$$
$$2\int_1^5\frac{e^x}{\left(e^x+1\right)}dx=\left[\log_e\left(e^x+1\right)^2\right]_1^5$$
$$\int_1^5\frac{e^x}{\left(e^x+1\right)}dx=\frac{1}{2}\left[\log_e\left(e^x+1\right)^2\right]_1^5$$
$$\int_1^5\frac{e^x}{\left(e^x+1\right)}dx=\frac{1}{2}\left(\log_e\left(e^5+1\right)^2-\log_e\left(e^2+1\right)^2\right)$$
$$\int_1^5\frac{e^x}{\left(e^x+1\right)}dx=\frac{1}{2}(10.0134-2.6265)$$
$$\int_1^5\frac{e^x}{\left(e^x+1\right)}dx=3.6935$$

16 a $y=\frac{10x}{5+x^2}$

Let $u=10x$ so $\frac{du}{dx}=10$

Let $v=5+x^2$ so $\frac{dv}{dx}=2x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{10\left(5+x^2\right)-10x(2x)}{(5+x^2)^2}$$
$$\frac{dy}{dx}=\frac{50+10x^2-20x^2}{(5+x^2)^2}$$
$$\frac{dy}{dx}=\frac{50-10x^2}{(5+x^2)^2}$$

Maximum TP occurs when $\frac{dy}{dx}=0$

$$\frac{50-10x^2}{(5+x^2)^2}=0$$
$$50-10x^2=0$$
$$50=10x^2$$
$$5=x^2$$
$$\pm\sqrt{5}=x$$

When $x=-\sqrt{5},\ y=\frac{10\left(-\sqrt{5}\right)}{5+\left(-\sqrt{5}\right)^2}=-\frac{10\sqrt{5}}{5+5}=-\frac{10\sqrt{5}}{10}=-\sqrt{5}$

So $\left(-\sqrt{5},-\sqrt{5}\right)$ is a minimum TP

When $x=\sqrt{5},\ y=\frac{10\left(\sqrt{5}\right)}{5+\left(\sqrt{5}\right)^2}=\frac{10\sqrt{5}}{5+5}=\frac{10\sqrt{5}}{10}=\sqrt{5}$

So $\left(\sqrt{5},\sqrt{5}\right)$ is a maximum TP

b $\frac{d}{dx}\left(\log_e(5+x^2)\right)=\frac{2x}{5+x^2}$

$$\int\frac{10x}{5+x^2}dx=5\int\frac{2x}{5+x^2}dx=5\log_e(5+x^2)$$

c $y=\log_e\left(5+x^2\right)$ so $\frac{dy}{dx}=\frac{2x}{5+x^2}$

$$\int_{\sqrt{5}}^{6}\frac{10x}{5+x^2}dx=5\int_{\sqrt{5}}^{6}\frac{2x}{5+x^2}dx$$
$$\int_{\sqrt{5}}^{6}\frac{10x}{5+x^2}dx=5\left[\log_e\left(5+x^2\right)\right]_{\sqrt{5}}^{6}$$
$$\int_{\sqrt{5}}^{6}\frac{10x}{5+x^2}dx=5\left\{\log_e\left(5+6^2\right)-\log_e\left(5+\left(\sqrt{5}\right)^2\right)\right\}$$
$$\int_{\sqrt{5}}^{6}\frac{10x}{5+x^2}dx=5\log_e\left(\frac{41}{10}\right)\text{ units}^2$$

17 a Graph of $y=\frac{5x}{x^2+1}$

Tangent at $x=-\frac{1}{2},\ y=\frac{5\left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)^2+1}$

$$y=-\frac{5}{2}\div\frac{5}{4}$$
$$y=-\frac{5}{2}\times\frac{4}{5}$$
$$y=-2$$

Gradient of tangent = $m_T=\frac{dy}{dx}$

Let $u=5x$ so $\frac{du}{dx}=5$

Let $v=x^2+1$ so $\frac{dv}{dx}=2x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{5(x^2+1)-5x(2x)}{(x^2+1)^2}$$
$$\frac{dy}{dx}=\frac{5x^2+5-10x^2}{(x^2+1)^2}$$
$$\frac{dy}{dx}=\frac{5-5x^2}{(x^2+1)^2}$$

At $x=-\frac{1}{2},\ \frac{dy}{dx}=\frac{5-5\left(-\frac{1}{2}\right)^2}{\left(\left(-\frac{1}{2}\right)^2+1\right)^2}$

$$\frac{dy}{dx}=\left(5-\frac{5}{4}\right)\div\left(\left(\frac{1}{4}+1\right)^2\right)$$
$$\frac{dy}{dx}=\left(\frac{20-5}{4}\right)\div\frac{25}{16}$$
$$\frac{dy}{dx}=\frac{15}{4}\times\frac{16}{25}$$
$$\frac{dy}{dx}=\frac{12}{5}$$

Equation of tangent with $m_T=\frac{12}{5}$ which passes through $\left(x_1,y_1\right)\equiv\left(-\frac{1}{2},-2\right)$ is given by

$$y-y_1=m_T(x-x_1)$$
$$y+2=\frac{12}{5}\left(x+\frac{1}{2}\right)$$
$$y=\frac{12}{5}x+\frac{6}{5}-\frac{10}{5}$$
$$y=\frac{12}{5}x-\frac{4}{5}\text{ or }12x-5y=4$$

b If $y=\log_e\left(x^2+1\right)$ then $\frac{dy}{dx}=\frac{2x}{x^2+1}$

$$\int\frac{5x}{x^2+1}dx=\frac{5}{2}\int\frac{2x}{x^2+1}dx$$
$$=\frac{5}{2}\log_e\left(1+x^2\right)$$

c $y=\frac{5x}{x^2+1}$..................(1)

$y=\frac{12}{5}x-\frac{4}{5}$...............(2)

(1) = (2)

$$\frac{5x}{x^2+1}=\frac{12}{5}x-\frac{4}{5}$$
$$25x=(12x-4)(x^2+1)$$
$$25x=12x^3-4x^2+12x-4$$
$$0=12x^3-4x^2+12x-4-25x$$
$$0=12x^3-4x^2-13x-4$$

Let $P(x)=12x^3-4x^2-13x-4$

$$P\left(-\frac{1}{2}\right)=12\left(-\frac{1}{2}\right)^3-4\left(-\frac{1}{2}\right)^2-13\left(-\frac{1}{2}\right)-4$$
$$P\left(-\frac{1}{2}\right)=-\frac{12}{8}-\frac{4}{4}+\frac{13}{2}-\frac{8}{2}$$
$$P\left(-\frac{1}{2}\right)=-\frac{3}{2}-\frac{2}{2}+\frac{13}{2}-\frac{8}{2}=0$$

$\left(x+\frac{1}{2}\right)$ or $(2x+1)$ is a factor

$12x^3-4x^2-13x-4=(2x+1)\left(6x^2-5x-4\right)$

$12x^3-4x^2-13x-4=(2x+1)(2x+1)(3x-4)$

Thus if

$(2x+1)^2(3x-4)=0$

$x=-\frac{1}{2}$ or $\frac{4}{3}$

Shaded area is $\int_{-\frac{1}{2}}^{\frac{4}{3}}\left(\frac{5x}{x^2+1}-\frac{12}{5}x+\frac{4}{5}\right)dx$

$=\int_{-\frac{1}{2}}^{\frac{4}{3}}\left(\frac{5x}{x^2+1}\right)dx-\frac{12}{5}\int_{-\frac{1}{2}}^{\frac{4}{3}}x\,dx+\int_{-\frac{1}{2}}^{\frac{4}{3}}\frac{4}{5}dx$

$=\frac{5}{2}\int_{-\frac{1}{2}}^{\frac{4}{3}}\left(\frac{2x}{x^2+1}\right)dx-\frac{12}{5}\int_{-\frac{1}{2}}^{\frac{4}{3}}x\,dx+\int_{-\frac{1}{2}}^{\frac{4}{3}}\frac{4}{5}dx$

$=\left[\frac{5}{2}\log_e(x^2+1)-\frac{12}{10}x^2+\frac{4}{5}x\right]_{-\frac{1}{2}}^{\frac{4}{3}}$

$=2.5541-2.13+1.07-0.5579+0.3+0.4$

$=1.6295$ units^2

18 a The curve $y=\frac{1}{x}+x^3-4$

Tangent at $x=1,\ y=\frac{1}{1}+1^3-4=-2$

Gradient of tangent

$=m_T=\frac{dy}{dx}=-x^{-2}+3x^2=\frac{1}{x^2}+3x^2=\frac{3x^4-1}{x^2}$

When $x=1,\ m_T=\frac{3(1)^4-1}{(1)^2}=2$

Equation of tangent with $m_T=2$ which passes through $(x_1,y_1)\equiv(1,-2)$ is given by

$y-y_1=m_T(x-x_1)$

$y+2=2(x-1)$

$y+2=2x-2$

$y=2x-4$

b Shaded region is $\int_1^2\left(\frac{1}{x}+x^3-4-(2x-4)\right)dx$

$=\int_1^2\left(\frac{1}{x}+x^3-2x\right)dx$

$=\left[\log_e(x)+\frac{1}{4}x^4-x^2\right]_1^2$

$=\left(\log_e(2)+\frac{1}{4}(2)^4-(2)^2\right)-\left(\log_e(1)+\frac{1}{4}(1)^4-(1)^2\right)$

$=\log_e(2)+4-4-0-\frac{1}{4}+1$

$=\frac{3}{4}+\log_e(2)$ units^2

19 a Dom $=(1,\infty)$ and ran $=R$

b Graph cuts the x axis where $y=0$

$2\log_e(x-1)=0$

$\log_e(x-1)=0$

$e^0=x-1$

$1=x-1$

$x=2$

Thus $(a,0)\equiv(2,0),\ a=2$

c $\int_2^5 2\log_e(x-1)\,dx=5.0904$ units^2

Solved using CAS

20 a Shaded region is $\int_4^6 5\log_e(x-3)\,dx=6.4792$ units^2

b $y=\log_e(x-3)$

For inverse, swap x and y

$x=5\log_e(y-3)$

$\frac{x}{5}=\log_e(y-3)$

$e^{\frac{x}{5}}=y-3$

$y=e^{\frac{x}{5}}+3$

c Area = Area of rectangle – area under the curve

$=6\times5\log_e(3)-\int_0^{5\log_e(3)}\left(e^{\frac{x}{5}}+3\right)dx$

$=30\log_e(3)-\left[5e^{\frac{x}{5}}+3x\right]_0^{5\log_e(3)}$

$=30\log_e(3)-\left(5e^{\log_e(3)}+3\times5\log_e(3)-(5+0)\right)$

$=30\log_e(3)-15-15\log_e(3)+5$

$=6.4792$ units^2

Exercise 9.4 — Applications

1 a $y=f(x)=2\log_e(4x),\ x\in\left[\frac{1}{4},\infty\right)$

Inverse is $x=2\log_e(4y),\ x\in[0,\infty)$

$\frac{1}{2}x=\log_e(4y)$

$e^{\frac{1}{2}x}=4y$

$\frac{1}{4}e^{\frac{1}{2}x}=y$

Thus $f^{-1}(x)=\frac{1}{4}e^{\frac{x}{2}}$

b $\int_0^{2\log_e 8}\left(\frac{1}{4}e^{\frac{1}{2}x}\right)dx=\frac{1}{4}\int_0^{2\log_e 8}\left(e^{\frac{1}{2}x}\right)dx$

$=\frac{1}{4}\left[2e^{\frac{1}{2}x}\right]_0^{2\log_e 8}$

$=\frac{1}{4}\left\{2e^{\frac{1}{2}(2\log_e 8)}-2e^0\right\}$

$=\frac{1}{2}(8-1)$

$=\frac{7}{2}$ units^2

c Required area is $4\log_e(8)-\frac{7}{2}$ units^2.

2 a Graph cuts the y axis where $x=0,\ y=2\log_e(5)$

Graph cuts the x axis where $y=0$

$$2\log_e(x+5)+1=0$$
$$2\log_e(x+5)=-1$$
$$\log_e(x+5)=-\frac{1}{2}$$
$$e^{-\frac{1}{2}}=x+5$$
$$x=e^{-\frac{1}{2}}-5$$

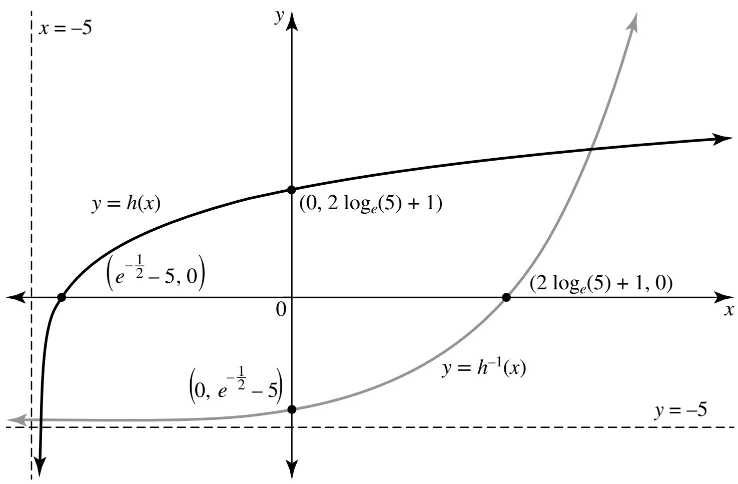
At the point $\left(e^{-0.5}-5,0\right)$

b h : dom = $(-5,\infty)$ and ran = R
h^{-1} : dom = R and ran = $(-5,\infty)$
If $y=2\log_e(x+5)+1$
Inverse is $x=2\log_e(y+5)+1$

$$x-1=2\log_e(y+5)$$
$$\frac{1}{2}(x-1)=\log_e(y+5)$$
$$e^{\frac{1}{2}(x-1)}=y+5$$
$$y=e^{\frac{1}{2}(x-1)}-5$$

Thus $h^{-1}(x)=e^{\frac{1}{2}(x-1)}-5$

c



d
$$h(x)=h^{-1}(x)$$
$$2\log_e(x+5)+1=e^{\frac{1}{2}(x-1)}-5$$
$$x=-4.9489,\ 5.7498$$

e $$\int_{-4.9489}^{5.7498}\left(2\log_e(x+5)+1-e^{\frac{1}{2}(x-1)}+5\right)dx=72.7601\text{ units}^2$$

3 a Graph cuts the x axis where $y=0$

$$2x\log_e(2x)=0$$
$$x=0 \text{ or } \log_e(2x)=0$$
$$e^0=2x$$
$$1=2x$$
$$x=\frac{1}{2}$$

Cuts x axis at $\left(\frac{1}{2},0\right)$.

b $y=x^2\log_e(2x)$

$$\frac{dy}{dx}=2x\log_e(2x)+\frac{x^2}{x}$$
$$\frac{dy}{dx}=2x\log_e(2x)+x$$

c Shaded region is $\int_{\frac{1}{2}}^{2} 2x\log_e(2x)\,dx$.

$$\int_{\frac{1}{2}}^{2}\left(2x\log_e(2x)+x\right)dx=\left[x^2\log_e(2x)\right]_{\frac{1}{2}}^{2}$$

$$\int_{\frac{1}{2}}^{2}\left(2x\log_e(2x)\right)dx+\int_{\frac{1}{2}}^{2}x\,dx=\left((2)^2\log_e(2(2))\right)-\left(\left(\frac{1}{2}\right)^2\log_e\left(2\left(\frac{1}{2}\right)\right)\right)$$

$$\int_{\frac{1}{2}}^{2}\left(2x\log_e(2x)\right)dx=4\log_e(4)-\frac{1}{4}\log_e(1)-\int_{\frac{1}{2}}^{2}x\,dx$$

$$\int_{\frac{1}{2}}^{2}\left(2x\log_e(2x)\right)dx=5.5452-\left[\frac{1}{2}x^2\right]_{\frac{1}{2}}^{2}$$

$$\int_{\frac{1}{2}}^{2}\left(2x\log_e(2x)\right)dx=5.5452-\left(\frac{1}{2}(2)^2-\frac{1}{2}\left(\frac{1}{2}\right)^2\right)$$

$$\int_{\frac{1}{2}}^{2}\left(2x\log_e(2x)\right)dx=5.5452-2+\frac{1}{8}$$

$$\int_{\frac{1}{2}}^{2}\left(2x\log_e(2x)\right)dx=3.670\text{ units}^2$$

4 a Graph intersects the x axis where $y=0$

$$-\log_e(5x)=0$$
$$\log_e(5x)=0$$
$$e^0=5x$$
$$1=5x$$
$$x=\frac{1}{5}$$

Cuts x axis at $\left(\frac{1}{5},0\right)$.

b $y=-x\log_e(5x)+x$

$$\frac{dy}{dx}=-x\times\frac{1}{x}+\log_e(5x)\times-1+1$$

$$\frac{dy}{dx}=-\log_e(5x)$$

c Shaded region is $-\int_{\frac{1}{5}}^{2}\left(-\log_e(5x)\right)dx=\int_{\frac{1}{5}}^{2}\left(\log_e(5x)\right)dx$

$$\int_{\frac{1}{5}}^{2}\left(-\log_e(5x)-1\right)dx=\int_{\frac{1}{5}}^{2}\left(-\log_e(5x)\right)dx-\int_{\frac{1}{5}}^{2}(1)\,dx=\left[-x\log_e(5x)\right]_{\frac{1}{5}}^{2}$$

$$\int_{\frac{1}{5}}^{2}\left(\log_e(5x)\right)dx+[x]_{\frac{1}{5}}^{2}=-\left[-x\log_e(5x)\right]_{\frac{1}{5}}^{2}$$

$$\int_{\frac{1}{5}}^{2}\left(\log_e(5x)\right)dx=-\left[-x\log_e(5x)\right]_{\frac{1}{5}}^{2}-[x]_{\frac{1}{5}}^{2}$$

$$\int_{\frac{1}{5}}^{2}\left(\log_e(5x)\right)dx=-\left(-2\log_e(10)+\frac{1}{5}\log_e(1)\right)-\left(2-\frac{1}{5}\right)$$

$$\int_{\frac{1}{5}}^{2}\left(\log_e(5x)\right)dx=2\log_e(10)-\frac{1}{5}\log_e(1)-2+\frac{1}{5}$$

$$\int_{\frac{1}{5}}^{2}\left(\log_e(5x)\right)dx=2\log_e(10)-\frac{9}{5}\text{ units}^2$$

5 **a** Graph cuts the y axis where $x=0,\ y=\log_e(2)$, i.e. at $\left(0,\log_e(2)\right)$.
Graph cuts the x axis where $y=0$

$$\log_e(2-4x)=0$$
$$e^0=2-4x$$
$$1=2-4x$$
$$4x=1$$
$$x=\frac{1}{4}$$

At the point $\left(\frac{1}{4},0\right)$.

b Largest possible domain occurs when

$$2-4x>0$$
$$-4x>-2$$
$$x<\frac{1}{2}$$

Dom $=\left(-\infty,\frac{1}{2}\right)$

c $y=\log_e(2-4x)$

$$\frac{dy}{dx}=\frac{-4}{2-4x}$$
$$\frac{dy}{dx}=\frac{-2}{1-2x}$$

Since $x<\frac{1}{2}$ then $2x<1$ and $2-4x<0$ for all x. Therefore $\frac{dy}{dx}<0$.

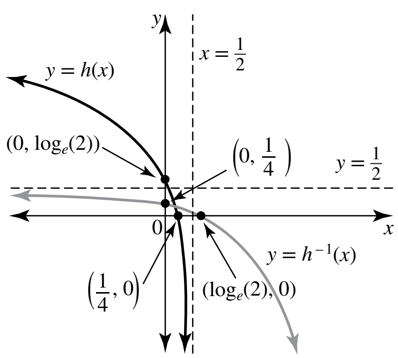
d **i** $y=\log_e(2-4x)$
Inverse: swap x and y

$$x=\log_e(2-4y)$$
$$e^x=2-4y$$
$$4y=2-e^x$$
$$y=\frac{1}{4}\left(2-e^x\right)$$

So $h^{-1}(x)=\frac{1}{4}\left(2-e^x\right)$

ii Dom = R and ran $=\left(-\infty,\frac{1}{2}\right)$

e



6 **a** $y=x\log_e(x)$

$$\frac{dy}{dx}=\log_e(x)+x\times\frac{1}{x}$$
$$\frac{dy}{dx}=\log_e(x)+1$$

Consequentially

$$\int_1^{e^2} \left(\log_e(x)+1\right)dx = \left[x\log_e(x)\right]_1^{e^2}$$

$$\int_1^{e^2} \log_e(x)\,dx + \int_1^{e^2} 1\,dx = \left[x\log_e(x)\right]_1^{e^2}$$

$$\int_1^{e^2} \log_e(x)\,dx = \left(e^2\log_e(e^2)-(1)\log_e(1)\right) - \int_1^{e^2} 1\,dx$$

$$\int_1^{e^2} \log_e(x)\,dx = 2e^2\log_e(e) - [x]_1^{e^2}$$

$$\int_1^{e^2} \log_e(x)\,dx = 2e^2 - e^2 + 1$$

$$\int_1^{e^2} \log_e(x)\,dx = e^2 + 1$$

b $y = x\left(\log_e(x)\right)^m$

$$\frac{dy}{dx} = \left(\log_e(x)\right)^m + mx\left(\log_e(x)\right)^{m-1} \times \frac{1}{x}$$

$$\frac{dy}{dx} = \left(\log_e(x)\right)^m + m\left(\log_e(x)\right)^{m-1}$$

c Consequentially

$$\int_1^{e^2} \left(\left(\log_e(x)\right)^m + m\left(\log_e(x)\right)^{m-1}\right)dx = \left[x\left(\log_e(x)\right)^m\right]_1^{e^2}$$

$$\int_1^{e^2} \left(\log_e(x)\right)^m dx + m\int_1^{e^2} \left(\log_e(x)\right)^{m-1} dx = \left[x\left(\log_e(x)\right)^m\right]_1^{e^2}$$

$$\int_1^{e^2} \left(\log_e(x)\right)^m dx + m\int_1^{e^2} \left(\log_e(x)\right)^{m-1} dx = e^2\left(\log_e(e^2)\right)^m - (1)\left(\log_e(1)\right)^m$$

$$\int_1^{e^2} \left(\log_e(x)\right)^m dx + m\int_1^{e^2} \left(\log_e(x)\right)^{m-1} dx = e^2\left(2\log_e(e)\right)^m - 0$$

$$\int_1^{e^2} \left(\log_e(x)\right)^m dx + m\int_1^{e^2} \left(\log_e(x)\right)^{m-1} dx = 2^m e^2$$

If $I_m = \int_1^{e^2} \left(\log_e(x)\right)^m dx$ then

$I_m + mI_{m-1} = 2^m e^2$ as required

d $I_3 = \int_1^{e^2} \left(\log_e(x)\right)^3 dx$ if $I_m = \int_1^{e^2} \left(\log_e(x)\right)^m dx$

Remembering that $I_1 = e^2 + 1$

$$I_2 + 2I_1 = 2^2 e^2$$
$$I_2 = 2^2 e^2 - 2I_1$$
$$I_2 = 4e^2 - 2(e^2+1)$$
$$I_2 = 4e^2 - 2e^2 - 2$$
$$I_2 = 2e^2 - 2$$
$$I_3 + 3I_2 = 2^3 e^2$$
$$I_3 = 2^3 e^2 - 3I_2$$
$$I_3 = 8e^2 - 3(2e^2 - 2)$$
$$I_3 = 8e^2 - 6e^2 + 6$$
$$I_3 = 2e^2 + 6$$

So $\int_1^{e^2} \left(\log_e(x)\right)^3 dx = 2e^2 + 6$

7 a $y = 4x\log_e(x+1)$

When

$x = -0.5,\ y = 4(-0.5)\log_e(1-0.5) = -2\log_e(0.5) = -2\log_e(2^{-1}) = 2\log_e(2)$

Gradient of tangent $m_T = \dfrac{dy}{dx} = \dfrac{4x}{x+1} + 4\log_e(x+1)$

When $x = -0.5,\ m_T = 4\log_e(-0.5+1) + \dfrac{4(-0.5)}{-0.5+1} = 4\log_e(0.5) - 4$

Equation of line with $m_T = 4\log_e(0.5) - 4$ which passes through $(x_1, y_1) \equiv (-0.5, 2\log_e(2))$ is given by

$$y - y_1 = m_T(x - x_1)$$
$$y - 2\log_e(2) = (4\log_e(0.5) - 4)(x + 0.5)$$
$$y = (4\log_e(0.5) - 4)x + 0.5(4\log_e(0.5) - 4) + 2\log_e(0.5)$$
$$y = (4\log_e(0.5) - 4)x - 2\log_e(0.5) - 2 + 2\log_e(0.5)$$
$$y = (-4\log_e(2) - 4)x - 2$$

b $y = 2(x^2-1)\log_e(x+1)$

$$\frac{dy}{dx} = 4x\log_e(x+1) + \frac{2(x^2-1)}{(x+1)}$$
$$\frac{dy}{dx} = 4x\log_e(x+1) + \frac{2(x-1)(x+1)}{(x+1)}$$
$$\frac{dy}{dx} = 4x\log_e(x+1) + 2x - 2,\ x > -1$$

c $\int 4x\log_e(x+1)\,dx = 2(x^2-1)\log_e(x+1) + \int(2x-2)\,dx$

$$= 2(x^2-1)\log_e(x+1) + x^2 - 2x$$

$$\int_{-0.5}^{0.5} 4x\log_e(x+1)\,dx = \left[2(x^2-1)\log_e(x+1) - x^2 + 2x\right]_{-0.5}^{0.5}$$
$$= \left(2\times(0.5^2-1)\log_e(1.5) - 0.5^2 + 1\right) - \left(2\times((-0.5)^2-1)\log_e(0.5) - (-0.5)^2 - 1\right)$$
$$= -\frac{3}{2}\log_e\left(\frac{3}{2}\right) + \frac{3}{4} + \frac{3}{2}\log_e\left(\frac{1}{2}\right) + \frac{5}{4}$$
$$= \frac{3}{2}\log_e\left(\frac{1}{3}\right) + 2$$
$$= -\frac{3}{2}\log_e(3) + 2 \text{ units}^2$$

8 a $m(x) = \log_e\left(x^2+1\right)$ so $m'(x) = \dfrac{2x}{x^2+1}$

$m'(-2) = \dfrac{2(-2)}{(-2)^2+1} = -\dfrac{4}{5} = -0.8$

b Area $= \int_0^3 \log_e\left(x^2+1\right)dx = 3.4058 \text{ units}^2$

c Area $= \int_{-3}^3 \log_e\left(x^2+1\right)dx = 3.4058\times 2 = 6.8117 \text{ units}^2$

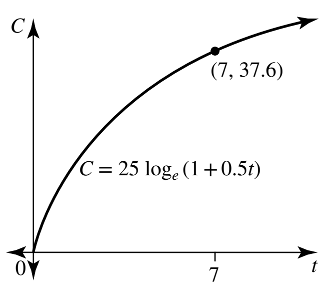
9 a When $t = 7,\ C_7 = 25\log_e(1+0.5(7))$

$C_7 = 25\log_e(1+3.5)$

$C_7 = 25\log_e(4.5)$

$C_7 = 37.6$ mg

b



c $\dfrac{dC}{dt} = \dfrac{25\times 0.5}{1+0.5t} = \dfrac{12.5}{1+0.5t}$

When $t = 3,\ \dfrac{dC}{dt} = \dfrac{12.5}{1+0.5(3)} = \dfrac{12.5}{2.5} = 5$ mg/day

d $\int_0^7 25\log_e(1+0.5t)\,dt$

$= 25\int_0^7 \log_e(1+0.5t)\,dt$

$= 25\times 6.5367$

$= 163.4$ mg

10 POI between $y=-4$ and $y=\log_e\left(\frac{x}{2}\right)$ is $x=-0.03663$

$A = 2\int_0^1 2e^x\,dx - \int_{0.03663}^{1} \log_e\left(\frac{x}{2}\right)dx + 0.03663\times 4$

$= 5.093\,\text{km}^2$

$= 5.1\ \text{km}^2$

11 $A = 4.6\log_e(t-4)$

a Graph intersects the t axis where $A=0$

$4.6\log_e(t-4)=0$

$\log_e(t-4)=0$

$e^0 = t-4$

$1 = t-4$

$t=5$

Thus $(a,0)\equiv(5,0)$, $a=5$

b When $A=15$,

$15 = 4.6\log_e(t-4)$

$\frac{15}{4.6} = \log_e(t-4)$

$e^{3.2609} = t-4$

$e^{3.2609}+4 = t$

$t=30$

It takes the patient 30 minutes to reach level 15.

c $\frac{dA}{dt} = \frac{4.6}{t-4}$

When $t=10$, $\frac{dA}{dt} = \frac{4.6}{10-4} = \frac{4.6}{6} = \frac{46}{60} = \frac{23}{30}$ units/min

d Total change is given by

$\int_5^{30} 4.6\log_e(t-4)\,dt$

$= 4.6\int_5^{30} \log_e(t-4)\,dt$

$= 4.6\times 59.7105$

$= 274.6683$

Change in alertness is 274.6683 units.

12 a Graphs intersect where $g(x)=h(x)$.

$x^2\log_e(x) = \log_e(x+1)$

$x = 1.5017$

Consequently $y=\log_e(1.5017+1)=0.9170$.

Therefore POI = (1.5017, 0.9170)

b Area between curves is given by

$\int_0^{1.5017}\left(\log_e(x+1) - x^2\log_e(x)\right)dx = 0.7096\ \text{units}^2$

13 a $y = f(x) = e^{\frac{1}{2}(x-1)}+3$

Inverse: $x = e^{\frac{1}{2}(y-1)}+3$

$x-3 = e^{\frac{1}{2}(y-1)}$

$\log_e(x-3) = \frac{1}{2}(y-1)$

$2\log_e(x-3) = y-1$

$y = 2\log_e(x-3)+1$

Thus $f^{-1}:(3,\infty)\to R,\ f^{-1}(x) = 2\log_e(x-3)+1$

b Graph cuts the y axis when $x=0$.

$y = e^{-\frac{1}{2}}+3$

Point is $\left(0, e^{-\frac{1}{2}}+3\right)$

Graph cuts x axis where $y=0$.

$2\log_e(x-3)+1=0$

$2\log_e(x-3)=-1$

$\log_e(x-3) = -\frac{1}{2}$

$e^{-\frac{1}{2}} = x-3$

$x = e^{-\frac{1}{2}}+3$

Point is $\left(e^{-\frac{1}{2}}+3, 0\right)$

c $\int_0^5\left(e^{\frac{1}{2}(x-1)}+3\right)dx - \int_{e^{-0.5}+3}^{5}\left(2\log_e(x-3)+1\right)dx$

$= 28.5651 - 1.9857$

$= 26.58\ \text{m}^2$

d Solve $\left(-5 = 2\log_e(x-3)+1\right)$ for x

$x = 3.049787$

$5\left(e^{-0.5}+3\right) - \int_{3.049787}^{e^{-0.5}+3}\left(2\log_e(x-3)+1\right)dx$

$= 18.90\ \text{m}^2$

e Total area of Australian native garden is

$26.5794 + 18.90 = 45.5\ \text{m}^2$

14 a i When $x=2.3942$, $y=e^{-2.3942}+3=3.0915$ i.e the point (2.4, 3.1)

ii When $x=-2.8654$, $y=-\log_e(-2.8654+3)=2.0054$ i.e the point $(-2.9, 2.0)$

iii When $x=-1$, $y=e^1+3=5.7128$ i.e the point $(-1.0, 5.7)$

iv When $x=-1$, $y=-\log_e(-1+3)=-0.691$ i.e the point $(-1.0, -0.7)$

v When $x=2.3942$, $y=-\log_e(2.3942+3)=-1.6853$ i.e the point $(2.4, -1.7)$

vi When $x=-1$, $y=2e^{(-1-3)}+3=2.0366$ i.e the point $(-1.0, 2.0)$

b i Region I: Area is

$\int_{-1}^{2.3942}\left(2e^{(x-3)}+2+\log_e(x+3)\right)dx$

$= 12.1535\ \text{units}^2$

ii Region II: Area is

$\int_{-1}^{2.3942}\left(e^{-x}+3-2e^{(x-3)}-2\right)dx$

$= \int_{-1}^{2.3942}\left(e^{-x}-2e^{(x-3)}+1\right)dx$

$= 4.9666\ \text{units}^2$

iii Region III: Area is

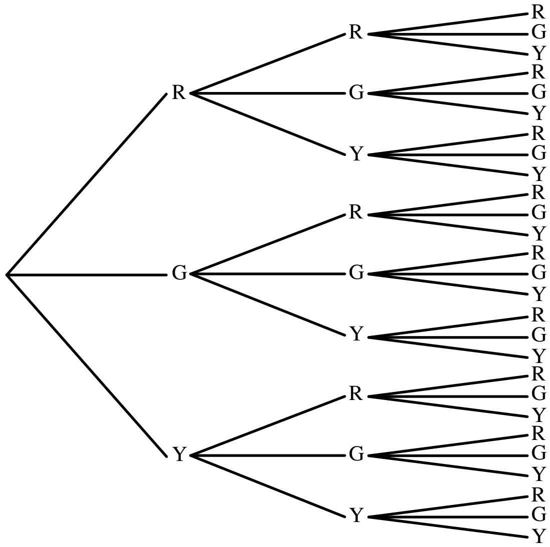
$\int_{-2.8654}^{-1}\left(2e^{(x-3)}+2+\log_e(x+3)\right)dx$

$= 3.5526\ \text{units}^2$

Topic 10 — Discrete random variables

Exercise 10.2 — Discrete random variables

1 a



ξ = {RRR, RRG, RRY, RGR, RGG, RGY, RYR, RYG, RYY, GRR, GRG, GRY, GGR, GGG, GGY, GYR, GYG, GYY, YRR, YRG, YRY, YGR, YGG, YGY, YYR, YYG, YYY}

b Y is the number of green balls obtained.

$Y = \{0,\ 1,\ 2,\ 3\}$

$$\Pr(Y=3) = \Pr(\text{GGG}) = \frac{3}{10}\times\frac{3}{10}\times\frac{3}{10} = \frac{27}{1000}$$

$$\Pr(Y=2) = \Pr(\text{RGG} + \Pr(\text{GRG}) + \Pr(\text{GGR}) + \Pr(\text{GGY}) + \Pr(\text{GYG}) + \Pr(\text{YGG})$$

$$\Pr(Y=2) = \frac{3}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{3}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{3}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{3}{10}\times\frac{3}{10}\times\frac{4}{10} + \frac{3}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{4}{10}\times\frac{3}{10}\times\frac{3}{10}$$

$$\Pr(Y=2) = \frac{27}{1000}\times 3 + \frac{36}{1000}\times 3 = \frac{189}{1000}$$

$$\Pr(Y=1) = \Pr(\text{RRG}) + \Pr(\text{RGR}) + \Pr(\text{RGY}) + \Pr(\text{RYG}) + \Pr(\text{GRR}) + \Pr(\text{GRY}) + \Pr(\text{GYR}) + \Pr(\text{GYY}) + \Pr(\text{YRG}) + \Pr(\text{YGR}) + \Pr(\text{YGY}) + \Pr(\text{YYG})$$

$$\Pr(Y=1) = \frac{3}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{3}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{3}{10}\times\frac{3}{10}\times\frac{4}{10} + \frac{3}{10}\times\frac{4}{10}\times\frac{3}{10} + \frac{3}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{3}{10}\times\frac{3}{10}\times\frac{4}{10} + \frac{3}{10}\times\frac{4}{10}\times\frac{3}{10} + \frac{3}{10}\times\frac{4}{10}\times\frac{4}{10} + \frac{4}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{4}{10}\times\frac{3}{10}\times\frac{3}{10} + \frac{4}{10}\times\frac{3}{10}\times\frac{4}{10} + \frac{4}{10}\times\frac{4}{10}\times\frac{3}{10}$$

$$\Pr(Y=1) = \frac{27}{1000}\times 3 + \frac{36}{1000}\times 6 + \frac{48}{1000}\times 3$$

$$\Pr(Y=1) = \frac{441}{1000}$$

$$\Pr(Y=0) = 1 - (P(Y=1) + P(Y=2) + P(Y=3))$$

$$\Pr(Y=0) = 1 - \left(\frac{441}{1000} + \frac{189}{1000} + \frac{27}{1000}\right)$$

$$\Pr(Y=0) = \frac{1000}{1000} - \frac{657}{10\,000} = \frac{343}{1000}$$

c

y	0	1	2	3
$\Pr(Y = y)$	$\frac{343}{1000}$	$\frac{441}{1000}$	$\frac{189}{1000}$	$\frac{27}{1000}$

d $\sum_{\text{all } y} \Pr(Y = y) = 1$ and all probabilities are between 0 and 1, therefore this is a discrete probability function.

2 Let X be the number of sixes obtained.

$\xi = \{11, 12, 13, 14, 15, 16$
$21, 22, 23, 24, 25, 26$
$31, 32, 33, 34, 35, 36$
$41, 42, 43, 44, 45, 46$
$51, 52, 53, 54, 55, 56$
$61, 62, 63, 64, 65, 66\}$

$X = 1, 2, 3$

$\Pr(X=2) = \Pr(66) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$\Pr(X=1) = \Pr(61, 62, 63, 64, 65, 16, 26, 36, 46, 56)$

$\Pr(X=1) = 10 \times \frac{1}{6} \times \frac{1}{6} = \frac{10}{36}$

$\Pr(X=0) = 1 - (\Pr(X=1) + \Pr(X=2))$

$\Pr(X=0) = 1 - \left(\frac{1}{36} + \frac{10}{36}\right) = \frac{36}{36} - \frac{11}{36} = \frac{25}{36}$

x	0	1	2
$\Pr(X=x)$	$\frac{25}{36}$	$\frac{10}{36} = \frac{5}{18}$	$\frac{1}{36}$

3 a i $0 \le \Pr(Y=y) \le 1$ for all y and the sum of the probabilities is 1.
This is a discrete probability density function.

ii $0 \le \Pr(Y=y) \le 1$ for all y and the sum of the probabilities is 1.
This is a discrete probability density function.

b $\sum_{\text{all } x} \Pr(X=x) = 1$

$$5k + 3k - 0.1 + 2k + k + 0.6 - 3k = 1$$
$$8k + 0.5 = 1$$
$$8k = 0.5$$
$$k = \frac{0.5}{8} = \frac{1}{16}$$

4 a i $0 \le \Pr(Y=y) \le 1$ for all y and the sum of the probabilities is 0.9.
This is not a discrete probability density function.

ii Negative probabilities so this is not a discrete probability density function.

b $\sum_{\text{all } x} \Pr(X=x) = 1$

$$0.5k^2 + 0.5k + 0.25(k+1) + 0.5 + 0.5k^2 = 1$$
$$k^2 + 0.75k + 0.75 = 1$$
$$k^2 + 0.75k + 0.25 = 0$$
$$100k^2 + 75k + 25 = 0$$
$$4k^2 + 3k - 1 = 0$$
$$(4k-1)(k+1) = 0$$
$$k = \frac{1}{4}, -1$$
$$k = \frac{1}{4}, \text{ as } k \ge 0$$

5 a $\xi = \{11, 12, 13, 14, 15, 16$
$21, 22, 23, 24, 25, 26$
$31, 32, 33, 34, 35, 36$
$41, 42, 43, 44, 45, 46$
$51, 52, 53, 54, 55, 56$
$61, 62, 63, 64, 65, 66\}$

b Z = number of even numbers so $Z = \{0, 1, 2\}$

$\Pr(Z=0) = \Pr(11) + \Pr(13) + \Pr(15) + \Pr(31) + \Pr(33) + \Pr(35) + \Pr(51) + \Pr(53) + \Pr(55)$

$\Pr(Z=0) = (0.1 \times 0.1) \times 9$

$\Pr(Z=0) = 0.09$

$\Pr(Z=2)=\Pr(22)+\Pr(24)+\Pr(26)+\Pr(42)+\Pr(44)+\Pr(46)+\Pr(62)+\Pr(64)+\Pr(66)$

$\Pr(Z=2)=(0.2\times 0.2)+(0.2\times 0.25)+(0.2\times 0.25)+(0.25\times 0.2)+(0.25\times 0.25)$
$+(0.25\times 0.25)+(0.25\times 0.2)+(0.25\times 0.25)+(0.25\times 0.25)$

$\Pr(Z=2)=0.49$

$\Pr(Z=1)=\Pr(12)+\Pr(14)+\Pr(16)+\Pr(21)+\Pr(23)+\Pr(25)+\Pr(32)+\Pr(34)+\Pr(36)+\Pr(41)+\Pr(43)+\Pr(45)$
$+\Pr(52)+\Pr(54)+\Pr(56)+\Pr(61)+\Pr(63)+\Pr(65)$

$\Pr(Z=1)=1-(\Pr(Z=2)+\Pr(Z=0))=1-(0.49+0.09)=0.42$

z	0	1	2
$\Pr(Z=z)$	0.09	0.42	0.49

c $\Pr(Z=1)=0.42$

6 a $\xi=\{\text{SSS, SSA, SAS, SAA, ASS, ASA, AAS, AAA}\}$ where S means Simon won and A means Samara won.

b X = number of sets Simon wins

$\Pr(X=0)=\Pr(AAA)=0.6^3=0.216$

$\Pr(X=1)=\Pr(AAS)+\Pr(ASA)+\Pr(SAA)=(0.6)^2\times 0.4\times 3=0.432$

$\Pr(X=2)=\Pr(SSA)+\Pr(SAS)+\Pr(ASS)=(0.6)^2\times 0.4\times 3=0.288$

$\Pr(x=3)=\Pr(SSS)=0.4^3=0.064$

x	0	1	2	3
$\Pr(X=x)$	0.216	0.432	0.288	0.064

c $\Pr(X\le 2)=1-\Pr(X=3)=1-0.064=0.936$

7 a $0\le \Pr(X=x)\le 1$ for all x and $\sum_{\text{all } x}\Pr(X=x)=0.1+0.5+0.5+0.1=1.2\ne 1$

This is not a discrete probability density function.

b Negative probabilities so this is not a discrete probability density function.

c $0\le \Pr(Z=z)\le 1$ for all z and $\sum_{\text{all } z}\Pr(Z=z)=0.25+0.15+0.45+0.35=1.1$ so $\sum_{\text{all } z}\Pr(Z=z)\ne 1$. This is not a discrete probability density function.

d $0\le \Pr(X=x)\le 1$ for all x and $\sum_{\text{all } x}\Pr(X=x)=0.1+0.25+0.3+0.25+0.1=1$

This is a discrete probability density function.

8 a $\sum_{\text{all } x}\Pr(X=x)=1$

$$3d+0.5-3d+2d+0.4-2d+d-0.05=1$$
$$d+0.85=1$$
$$d=0.15$$

b $\sum_{\text{all } y}\Pr(Y=y)=1$

$$0.5k+1.5k+2k+1.5k+0.5k=1$$
$$6k=1$$
$$k=\frac{1}{6}$$

c $\sum_{\text{all } z}\Pr(Z=z)=1$

$$\frac{1}{3}-a^2+\frac{1}{3}-a^2+\frac{1}{3}-a^2+a=1$$
$$1+a-3a^2=1$$
$$a-3a^2=0$$
$$a(1-3a)=0$$
$$1-3a=0 \text{ as } a>0$$
$$1=3a \text{ or } a=\frac{1}{3}$$

9 a $p(x)=\frac{1}{7}(5-x),\ p(1)=\frac{1}{7}(5-1)=\frac{4}{7},$

$p(3)=\frac{1}{7}(5-3)=\frac{2}{7},\ p(4)=\frac{1}{7}(5-4)=\frac{1}{7}$

Each probability lies between 0 and 1. Sum of probabilities is 1 so this is a discrete probability function.

b $p(x)=\frac{x^2-x}{40}$

$p(-1)=\frac{(-1)^2+1}{40}=\frac{2}{40},\ p(1)=\frac{1^2-1}{40}=0$

$p(2)=\frac{2^2-2}{40}=\frac{2}{40},\ p(3)=\frac{3^2-3}{40}=\frac{6}{40}$

$p(4)=\frac{4^2-4}{40}=\frac{12}{40},\ p(5)=\frac{5^2-5}{40}=\frac{20}{40}$

Each probability lies between 0 and 1. Sum of probabilities is greater than 1 so this is not a discrete probability function.

c $p(x)=\frac{1}{15}\sqrt{x}$

$p(1)=\frac{1}{15}\sqrt{1}=\frac{1}{15},\ p(4)=\frac{1}{15}\sqrt{4}=\frac{2}{15},\ p(9)=\frac{1}{15}\sqrt{9}=\frac{3}{15}$

$p(16)=\frac{1}{15}\sqrt{16}=\frac{4}{15},\ p(25)=\frac{1}{15}\sqrt{25}=\frac{5}{15}$

Each probability lies between 0 and 1. Sum of probabilities is 1 so this is a discrete probability function.

10 $p(x)=\frac{1}{a}(15-3x)$

$p(1)=\frac{1}{a}(15-31)=\frac{12}{a},\ p(2)=\frac{1}{a}(15-3(2))=\frac{9}{a},\ p(3)=\frac{1}{a}(15-3(3))=\frac{6}{a}$

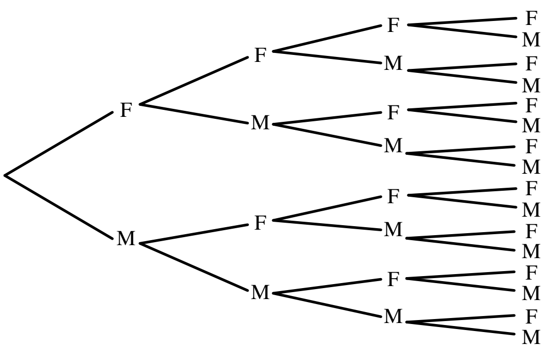
$p(4)=\frac{1}{a}(15-3(4))=\frac{3}{a},\ p(5)=\frac{1}{a}(15-3(5))=0$

$$\frac{12}{a}+\frac{9}{a}+\frac{6}{a}+\frac{3}{a}+0=1$$

$$\frac{30}{a}=1$$

$$a=30$$

11 a F = female and M = male



$$\xi=\left\{\begin{matrix}FFFF, FFFM, FFMF, FFMM, FMFF, FMFM,\\ FMMF, FMMM, MFFF, MFFM, MFMF, MFMM,\\ MMFF, MMFM, MMMF, MMMM,\end{matrix}\right\}$$

b X is the number of females in the litter

$X=\{0,1,2,3,4\}$

$\Pr(X=0)=\left(\frac{1}{2}\right)^4=\frac{1}{16},\ \Pr(X=1)=4\left(\frac{1}{2}\right)^4=\frac{4}{16},\ \Pr(X=2)=6\left(\frac{1}{2}\right)^4=\frac{6}{16}$

$\Pr(X=3)=4\left(\frac{1}{2}\right)^4=\frac{4}{16},\ \Pr(X=4)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$

x	0	1	2	3	4
$\Pr(X=x)$	$\frac{1}{16}$	$\frac{4}{16}=\frac{1}{4}$	$\frac{6}{16}=\frac{3}{8}$	$\frac{4}{16}=\frac{1}{4}$	$\frac{1}{16}$

c $\Pr(X=4)=\frac{1}{16}$

d $\Pr(X\geq 1)=1-\Pr(X=0)=1-\frac{1}{16}=\frac{15}{16}$

e $\Pr(X\leq 2)=\Pr(X=0)+\Pr(X=1)+\Pr(X=2)=\frac{1}{16}+\frac{4}{16}+\frac{6}{16}=\frac{11}{16}$

12 a $\xi=\{$11, 12, 13, 14, 15, 16, 17, 18, 19, 110, 111, 112
21, 22, 23, 24, 25, 26, 27, 28, 29, 210, 211, 212
31, 32, 33, 34, 35, 36, 37, 38, 39, 310, 311, 312
41, 42, 43, 44, 45, 46, 47, 48, 49, 410, 411, 412
51, 52, 53, 54, 55, 56, 57, 58, 59, 510, 511, 512
61, 62, 63, 64, 65, 66, 67, 68, 69, 610, 611, 612
71, 72, 73, 74, 75, 76, 77, 78, 79, 710, 711, 712
81, 82, 83, 84, 85, 86, 87, 88, 89, 810, 811, 812$\}$

b X is the number of primes obtained as a result of a toss

$\Pr(X=0)=\Pr($11, 14, 16, 18, 19, 1 10, 1 12, 41, 44, 46, 48, 49, 4 10, 4 12,
61, 64, 66, 68, 69, 6 10, 6 12, 81, 84, 86, 88, 89, 8 10, 8 12$)$

$$=28\times\left(\frac{1}{8}\times\frac{1}{12}\right)$$
$$=\frac{28}{96}$$

$\Pr(X=1)=\Pr($12, 13, 15, 17, 1 11, 21, 24, 26, 28, 29, 2 10, 2 12,
31, 34, 36, 38, 39, 3 10, 3 12, 42, 43, 45, 47, 4 11,
51, 54, 56, 58, 59, 5 10, 5 12, 62, 63, 65, 67, 6 11,
71, 74, 76, 78, 79, 7 10, 7 12, 82, 83, 85, 87, 8 11$)$

$$=48\times\left(\frac{1}{8}\times\frac{1}{12}\right)$$
$$=\frac{48}{96}$$

$\Pr(X=2)=\Pr($22, 23, 25, 27, 2 11, 32, 33, 35, 37, 3 11,
52, 53, 55, 57, 5 11, 72, 73, 75, 77, 7 11$)$

$$=20\times\left(\frac{1}{8}\times\frac{1}{12}\right)$$
$$=\frac{20}{96}$$

c $\Pr(\text{Win})=\Pr(X=2)\times\Pr(X=2)\times\Pr(X=2)$

$$=\left(\frac{5}{24}\right)^3=0.009$$

13 a Possible scores are:
PP = 20 points, PJ or JP = 15 points, PS or SP = 12 points, JJ = 10 points
JS or SJ = 7 points, SS = 4 points

b $\Pr(20)=\frac{3}{13}\times\frac{3}{13}=\frac{9}{169}$, $\Pr(15)=2\left(\frac{3}{13}\times\frac{1}{13}\right)=\frac{6}{169}$,
$\Pr(12)=2\left(\frac{3}{13}\times\frac{9}{13}\right)=\frac{54}{169}$, $\Pr(10)=\frac{1}{13}\times\frac{1}{13}=\frac{1}{169}$,
$\Pr(7)=2\left(\frac{1}{13}\times\frac{9}{13}\right)=\frac{18}{169}$, $\Pr(4)=\frac{9}{13}\times\frac{9}{13}=\frac{81}{169}$

x	4	7	10	12	15	20
$\Pr(X=x)$	$\frac{81}{169}$	$\frac{18}{169}$	$\frac{1}{169}$	$\frac{54}{169}$	$\frac{6}{169}$	$\frac{9}{169}$

c i $\Pr(X=15)=\frac{6}{169}$

ii $\Pr(X\geq 12)=\frac{54}{169}+\frac{6}{169}+\frac{9}{169}=\frac{69}{169}$

iii $\Pr(X=15\mid X\geq 12)=\frac{\Pr(X=15\cap X\geq 12)}{\Pr(X\geq 12)}=\frac{6}{169}\times\frac{169}{69}=\frac{6}{69}=\frac{2}{23}$

14 a $\sum_{\text{all } x} \Pr(X = x) = 1$

$3m + 2n = 1$.....................(1)

$3\Pr(X = 0) = \Pr(X = 5)$

$3m = n$.....................(2)

Substitute (2) into (1)

$n + 2n = 1$

$3n = 1$

$n = \frac{1}{3}$

Substitute $n = \frac{1}{3}$ into (2)

$3m = \frac{1}{3}$

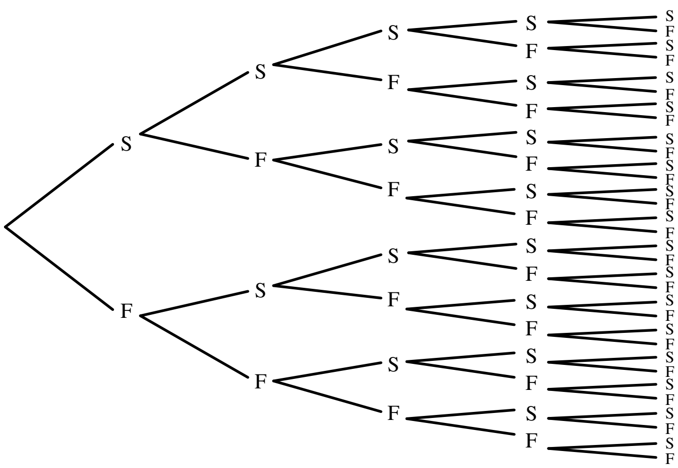
$m = \frac{1}{9}$

b i $\Pr(X \geq 0) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 5)$

$\Pr(X \geq 0) = \frac{1}{9} + \frac{1}{9} + \frac{1}{3} = \frac{5}{9}$

ii $\Pr(X = 1 \mid X \geq 0) = \frac{\Pr(X = 1) \cap \Pr(X \geq 0)}{\Pr(X \geq 0)} = \frac{1}{9} \div \frac{5}{9} = \frac{1}{5}$

15 a S = Success and F – Failure



E = {SSSSS, SSSSF, SSSFS, SSSFF, SSFSS, SSFSF, SSFFS, SSFFF, SFSSS, SFSSF, SFSFS, SFSFF, SFFSS, SFFSF, SFFFS, SFFFF, FSSSS, FSSSF, FSSFS, FSSFF, FSFSS, FSFSF, FSFFS, FSFFF, FFSSS, FFSSF, FFSFS, FFSFF, FFFSS, FFFSF, FFFFS, FFFFF}

$X = \{0, 1, 2, 3, 4, 5\}$

$\Pr(X = 0) = \Pr(5 \text{ failures}) = 0.4^5 = 0.01\,024$

$\Pr(X = 1) = \Pr(4 \text{ failures}) = 5 \times 0.4^4 \times 0.6 = 0.0768$

$\Pr(X = 2) = \Pr(3 \text{ failures}) = 10 \times 0.4^3 \times 0.6^2 = 0.2304$

$\Pr(X = 3) = \Pr(2 \text{ failures}) = 10 \times 0.4^2 \times 0.6^3 = 0.3456$

$\Pr(X = 4) = \Pr(1 \text{ failure}) = 5 \times 0.4 \times 0.6^4 = 0.2592$

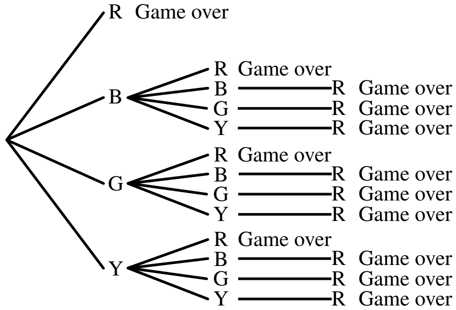
$\Pr(X = 5) = P(0 \text{ failures}) = 0.6^5 = 0.0778$

x	0	1	2	3	4	5
$\Pr(X = x)$	0.0102	0.0768	0.2304	0.3456	0.2592	0.0778

b $\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5) = 0.3456 + 0.2592 + 0.0778 = 0.6826$

It is a success helping three or more patients.

16 a



b Wins \$10 with BBB, GGG or YYY

c $X = \{0,\ 1,\ 10\}$

$$\begin{aligned}\Pr(X=0) &= \frac{2}{5} + 3\left(\frac{1}{5}\times\frac{2}{5}\right) + 9\left(\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}\right)\\ &= \frac{2}{5} + \frac{6}{25} + \frac{9}{125}\\ &= \frac{50}{125} + \frac{30}{125} + \frac{9}{125} = \frac{98}{125}\end{aligned}$$

$$\Pr(X=10) = 3\left(\frac{1}{5}\right)^3 = \frac{3}{125}$$

$$\begin{aligned}\Pr(X=1) &= 1 - (\Pr(X=0) + \Pr(x=10))\\ &= \frac{125}{125} - \left(\frac{98}{125} + \frac{3}{125}\right) = \frac{24}{125}\end{aligned}$$

x	\$0	\$1	\$10
$\Pr(X = x)$	$\frac{98}{125}$	$\frac{24}{125}$	$\frac{3}{125}$

17 a $\Pr(H) = \frac{2}{3}$ and $\Pr(T) = \frac{1}{3}$ and X = the number of heads obtained.

$$\Pr(X=0) = \left(\frac{1}{3}\right)^6 = 0.0014$$

$$\Pr(X=1) = {}^6C_1\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) = 0.0165$$

$$\Pr(X=2) = {}^6C_2\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 = 0.0823$$

$$\Pr(X=3) = {}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 = 0.2195$$

$$\Pr(X=4) = {}^6C_4\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 = 0.3292$$

$$\Pr(X=5) = {}^6C_5\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^5 = 0.2634$$

$$\Pr(X=0=6) = \left(\frac{2}{3}\right)^6 = 0.0878$$

x	0	1	2	3	4	5	6
$\Pr(X = x)$	0.0014	0.0165	0.0823	0.2195	0.3292	0.2634	0.0878

b **i** $\Pr(X > 2) = \Pr(X=3) + \Pr(X=4) + \Pr(X=5) + \Pr(X=6)$

$$\begin{aligned}&= 0.2195 + 0.3292 + 0.2634 + 0.0878\\ &= 0.8999\end{aligned}$$

ii $\Pr(X > 2 \mid X < 5)$

$$\begin{aligned}&= \frac{\Pr(X=3) + \Pr(X=4)}{1 - (\Pr(X=5) + \Pr(X=6))}\\ &= \frac{0.2195 + 0.3292}{1 - (0.2634 + 0.0878)}\\ &= 0.8457\end{aligned}$$

18 $\sum_{\text{all } y} \Pr(Y=y)=1$

$$0.5k^2+0.3-0.2k+0.1+0.5k^2+0.3=1$$
$$k^2-0.2k+0.7=1$$
$$k^2-0.2k-0.3=0$$
$$k=-0.4568 \text{ or } k=0.6568$$

k can be positive or negative due to the two places of k: $0.5k^2$ and $0.3-0.2k$

For both values of k, $0<0.5k^2<1$ and $0<0.3-0.2k<1$

Exercise 10.3 — Measures of centre and spread

1 **a** $p(x)=\frac{1}{16}(2x-1),\ x\in\{1, 2, 3, 4\}$

$p(1)=\frac{1}{16}(2(1)-1)=\frac{1}{16}$

$p(2)=\frac{1}{16}(2(2)-1)=\frac{3}{16}$

$p(3)=\frac{1}{16}(2(3)-1)=\frac{5}{16}$

$p(4)=\frac{1}{16}(2(4)-1)=\frac{7}{16}$

y	1	2	3	4
$\Pr(X=x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

b $E(X)=1\left(\frac{1}{16}\right)+2\left(\frac{3}{16}\right)+3\left(\frac{5}{16}\right)+4\left(\frac{7}{16}\right)$

$E(X)=\frac{1}{16}+\frac{6}{16}+\frac{15}{16}+\frac{28}{16}=\frac{50}{16}=3.125 \text{ or } 3\frac{1}{8}$

2 **a** $\Pr(Y)=\Pr(-\$2)=\frac{1}{4},\ \Pr(G)=\Pr(\$3)=\frac{1}{4},\ \Pr(R)=\Pr(\$6)=\frac{1}{4},\ \Pr(P)=\Pr(\$8)=\frac{1}{4}$

x	−\$2	\$3	\$6	\$8
$\Pr(X=x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

b $E(X)=-2\left(\frac{1}{4}\right)+3\left(\frac{1}{4}\right)+6\left(\frac{1}{4}\right)+8\left(\frac{1}{4}\right)$

$E(X)=-\frac{2}{4}+\frac{3}{4}+\frac{6}{4}+\frac{8}{4}=\frac{15}{4}=\$3.75$

3 **a** $E(Y)=-5\left(\frac{1}{10}\right)+0\left(\frac{3}{10}\right)+5\left(\frac{2}{10}\right)+d\left(\frac{3}{10}\right)+25\left(\frac{1}{10}\right)=\frac{15}{2}$

$$-\frac{5}{10}+0+\frac{10}{10}+\frac{3d}{10}+\frac{25}{10}=\frac{15}{2}$$
$$\frac{3d+30}{10}=\frac{15}{2}$$
$$3d+30=75$$
$$3d=45$$
$$d=15$$

b **i** $E(2Y+3)=2E(Y)+3=2(7.5)+3=18$

ii $E(5-Y)=5-E(Y)=5-7.5=-2.5$

iii $E(-2Y)=-2E(Y)=-2(7.5)=-15$

4 $p(z)=\frac{1}{38}(z^2-4),\quad 2\le z\le 5$

$p(2)=\frac{1}{38}(2^2-4)=0,\ p(3)=\frac{1}{38}(3^2-4)=\frac{5}{38},$

$p(4)=\frac{1}{38}(4^2-4)=\frac{12}{38},\ \ p(5)=\frac{1}{38}(5^2-4)=\frac{21}{38}$

$\text{E}(Z)=2(0)+3\left(\frac{5}{38}\right)+4\left(\frac{12}{38}\right)+5\left(\frac{21}{38}\right)$

$\text{E}(Z)=0+\frac{15}{38}+\frac{48}{38}+\frac{105}{38}$

$\text{E}(Z)=\frac{168}{38}\simeq 4.42$

5 a $\text{E}(X)=1(0.3)+2(0.15)+3(0.4)+4(0.1)+5(0.05)$

$\text{E}(X)\simeq \$2.45$

b $\text{E}(X^2)=1^2(0.3)+2^2(0.15)+3^2(0.4)+4^2(0.1)+5^2(0.05)$

$\text{E}(X^2)=7.35$

$\text{Var}(X)=\text{E}(X^2)-[\text{E}(X)]^2$

$\text{Var}(X)=7.35-2.45^2=\$1.35$

$\text{SD}(X)=\sqrt{1.35}=\$1.16$

6 a $k+k+2k+3k+3k=1$

$10k=1$

$k=\frac{1}{10}$

b $\text{E}(X)=-2\left(\frac{1}{10}\right)+0\left(\frac{1}{10}\right)+2\left(\frac{2}{10}\right)+4\left(\frac{3}{10}\right)+6\left(\frac{3}{10}\right)$

$\text{E}(X)=-\frac{2}{10}+0+\frac{4}{10}+\frac{12}{10}+\frac{18}{10}=\frac{32}{10}=3.2$

c $\text{E}(X^2)=(-2)^2\left(\frac{1}{10}\right)+0^2\left(\frac{1}{10}\right)+2^2\left(\frac{2}{10}\right)+4^2\left(\frac{3}{10}\right)+6^2\left(\frac{3}{10}\right)$

$\text{E}(X^2)=\frac{4}{10}+0+\frac{8}{10}+\frac{48}{10}+\frac{108}{10}=\frac{168}{10}=16.8$

$\text{Var}(X)=\text{E}(X^2)-[\text{E}(X)]^2=16.8-3.2^2=6.56$

$\text{SD}(X)=\sqrt{6.56}=2.56$

7 a $p(x)=\frac{x^2}{30}\quad x=1,\ 2,\ 3,\ 4.$

$p(1)=\frac{1^2}{30}=\frac{1}{30},\ \ p(2)=\frac{2^2}{30}=\frac{4}{30},\ p(3)=\frac{3^2}{30}=\frac{9}{30},\ p(4)=\frac{4^2}{30}=\frac{16}{30}$

x	1	2	3	4
$\Pr(X=x)$	$\frac{1}{30}$	$\frac{4}{30}=\frac{2}{15}$	$\frac{9}{30}=\frac{3}{10}$	$\frac{16}{30}=\frac{8}{15}$

$\sum_{\text{all } x}\Pr(X=x)=\frac{1}{30}+\frac{4}{30}+\frac{9}{30}+\frac{16}{30}=1$

b i $\text{E}(X)=1\left(\frac{1}{30}\right)+2\left(\frac{4}{30}\right)+3\left(\frac{9}{30}\right)+4\left(\frac{16}{30}\right)$

$\text{E}(X)=\frac{1}{30}+\frac{8}{30}+\frac{27}{30}+\frac{64}{30}=\frac{100}{30}=\frac{10}{3}$

ii $\text{E}(X^2)=1^2\left(\frac{1}{30}\right)+2^2\left(\frac{4}{30}\right)+3^2\left(\frac{9}{30}\right)+4^2\left(\frac{16}{30}\right)$

$\text{E}(X^2)=\frac{1}{30}+\frac{16}{30}+\frac{81}{30}+\frac{256}{30}$

$\text{E}(X^2)=\frac{354}{30}=\frac{118}{10}$

$\text{Var}(X) = \text{E}\left(X^2\right) - [\text{E}(X)]^2$

$\text{Var}(X) = \frac{118}{10} - \left(\frac{10}{3}\right)^2$

$\text{Var}(X) = \frac{1062}{90} - \frac{1000}{90} = \frac{62}{90} = \frac{31}{45} = 0.69$

c **i** $\text{Var}(4X+3) = 4^2\,\text{Var}(X) = 16(0.69) = 11.02$

ii $\text{Var}(2-3X) = (-3)^2\,\text{Var}(X) = 9(0.689) = 6.2$

8 **a** $\text{E}(Z) = -7(0.21) + m(0.34) + 23(0.33) + 31(0.12) = 14.94$

$-1.47 + 0.34m + 7.59 + 3.72 = 14.94$

$0.34m + 9.84 = 14.94$

$0.34m = 5.1$

$m = \frac{5.1}{0.34}$

$m = 15$

b $\text{E}(Z^2) = (-7)^2(0.21) + 15^2(0.34) + 23^2(0.33) + 31^2(0.12)$

$\text{E}(Z^2) = 10.29 + 76.5 + 174.57 + 115.32$

$\text{E}(Z^2) = 376.68$

$\text{Var}(Z) = \text{E}(Z^2) - [\text{E}(Z)]^2$

$\text{Var}(Z) = 376.68 - 14.94^2$

$\text{Var}(Z) = 153.48$

$\text{Var}(2(Z-1)) = \text{VAR}(2Z-2)$

$\text{Var}(2(Z-1)) = 2^2\,\text{VAR}(Z)$

$\text{Var}(2(Z-1)) = 4 \times 153.48$

$\text{Var}(2(Z-1)) = 613.91$

$\text{Var}(3-Z) = (-1)^2\,\text{Var}(Z)$

$\text{Var}(3-Z) = 153.48$

9 **a** $\sum_{\text{all } x} \Pr(X = x) = 1$

i $\text{E}(X) = -3\left(\frac{1}{9}\right) + (-2)\left(\frac{1}{9}\right) + (-1)\left(\frac{1}{9}\right) + 0\left(\frac{2}{9}\right) + 1\left(\frac{2}{9}\right) + 2\left(\frac{1}{9}\right) + 3\left(\frac{1}{9}\right)$

$\text{E}(X) = -\frac{3}{9} - \frac{2}{9} - \frac{1}{9} + 0 + \frac{2}{9} + \frac{2}{9} + \frac{3}{9} = \frac{1}{9}$

ii $\text{E}(X^2) = (-3)^2\left(\frac{1}{9}\right) + (-2)^2\left(\frac{1}{9}\right) + (-1)^2\left(\frac{1}{9}\right) + 0^2\left(\frac{2}{9}\right) + 1^2\left(\frac{2}{9}\right) + 2^2\left(\frac{1}{9}\right) + 3^2\left(\frac{1}{9}\right)$

$\text{E}(X^2) = \frac{9}{9} + \frac{4}{9} + \frac{1}{9} + 0 + \frac{2}{9} + \frac{4}{9} + \frac{9}{9}$

$\text{E}(X^2) = \frac{29}{9}$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$\text{Var}(X) = \frac{29}{9} - \left(\frac{1}{9}\right)^2$

$\text{Var}(X) = \frac{261}{81} - \frac{1}{81} = \frac{260}{81} = 3.2099$

$\text{SD}(X) = \sqrt{3.2099} = 1.7916$

b $\sum_{\text{all } y} \Pr(Y = y) = 1$

i $\text{E}(Y) = 1(0.15) + 4(0.2) + 7(0.3) + 10(0.2) + 13(0.15)$

$\text{E}(Y) = 0.15 + 0.8 + 2.1 + 2 + 1.95 = 7$

ii $\text{E}(Y^2) = 1^2(0.15) + 4^2(0.2) + 7^2(0.3) + 10^2(0.2) + 13^2(0.15)$

$\text{E}(Y^2) = 0.15 + 3.2 + 14.7 + 20 + 25.35$

$\text{E}(Y^2) = 63.4$

$\text{Var}(Y) = \text{E}(Y^2) - [\text{E}(Y)]^2$

$\text{Var}(Y) = 63.4 - 7^2$

$\text{Var}(Y) = 63.4 - 49 = 14.4$

$\text{SD}(Y) = \sqrt{14.4} = 3.7947$

c $\sum_{\text{all } z} \Pr(Z = z) = 1$

i $E(Z) = 1\left(\frac{1}{12}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{12}\right) + 6\left(\frac{1}{12}\right)$

$E(Z) = \frac{1}{12} + \frac{6}{12} + \frac{12}{12} + \frac{8}{12} + \frac{5}{12} + \frac{6}{12}$

$E(Z) = \left(\frac{38}{12}\right) = \frac{19}{6}$

ii $E(Z^2) = 1^2\left(\frac{1}{12}\right) + 2^2\left(\frac{1}{4}\right) + 3^2\left(\frac{1}{3}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{12}\right) + 6^2\left(\frac{1}{12}\right)$

$E(Z^2) = \frac{1}{12} + \frac{12}{12} + \frac{36}{12} + \frac{32}{12} + \frac{25}{12} + \frac{36}{12}$

$E(Z^2) = \frac{142}{12}$

$\text{Var}(Z) = E(Z^2) - [E(Z)]^2$

$\text{Var}(Z) = \frac{142}{12} - \left(\frac{19}{6}\right)^2$

$\text{Var}(Z) = \frac{1704}{144} - \frac{1444}{144}$

$\text{Var}(Z) = \frac{65}{36} = 1.8056$

$\text{SD}(Z) = \sqrt{\frac{65}{36}} = 1.3437$

10 a $\sum_{\text{all } y} \Pr(Y = y) = 1$

$1 - 2c + 3c^2 + 1 - 2c = 1$

$3c^2 - 4c + 1 = 0$

$(3c - 1)(c - 1) = 0$

$3c - 1 = 0$ or $c - 1 = 0$

$3c = 1 \quad c = 1$

$c = \frac{1}{3}$

$\therefore c = \frac{1}{3}$ as $0 < c < 1$

b

y	-1	1	3	5	7
$\Pr(Y = y)$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$

$E(Y) = -1\left(\frac{1}{3}\right) + 1\left(\frac{1}{9}\right) + 3\left(\frac{1}{9}\right) + 5\left(\frac{1}{9}\right) + 7\left(\frac{1}{3}\right)$

$E(Y) = -\frac{1}{3} + \frac{1}{9} + \frac{3}{9} + \frac{5}{9} + \frac{7}{3}$

$E(Y) = -\frac{3}{9} + \frac{1}{9} + \frac{3}{9} + \frac{5}{9} + \frac{21}{9}$

$E(Y) = \frac{27}{9} = 3$

c $E(Y^2) = (-1)^2\left(\frac{1}{3}\right) + 1^2\left(\frac{1}{9}\right) + 3^2\left(\frac{1}{9}\right) + 5^2\left(\frac{1}{9}\right) + 7^2\left(\frac{1}{3}\right)$

$E(Y^2) = \frac{3}{9} + \frac{1}{9} + \frac{9}{9} + \frac{25}{9} + \frac{147}{9}$

$E(Y^2) = \frac{185}{9}$

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$

$\text{Var}(Y) = \frac{185}{9} - 3^2$

$\text{Var}(Y) = \frac{185}{9} - \frac{81}{9}$

$\text{Var}(Y) = \frac{104}{9}$

$\text{Var}(Y) = 11.56$

$\text{SD}(Y) = \sqrt{11.56} = 3.40$

11 **a** $E(2X-1)=2E(X)-1$

$E(2X-1)=2(4.5)-1=8$

b $E(5-X)=5-E(X)$

$E(5-X)=5-4.5=0.5$

c $E(3X+1)=3E(X)+1$

$E(3X+1)=3(4.5)+1=14.5$

12 $SD(X)=2.5$ so $Var(X)=2.5^2=6.25$

a $Var(6X)=6^2\,Var(X)=36\times 6.25=225$

b $Var(2X+3)=2^2\,Var(X)=4\times 6.25=25$

c $Var(-X)=(-1)^2\,VAR(X)=6.25$

13 **a** $p(x)=h(3-x)(x+1)$

$p(0)=h(3)(1)=3h$

$p(1)=h(3-1)(1+1)=4h$

$p(2)=h(3-2)(2+1)=3h$

$3h+4h+3h=1$

$10h=1$

$h=\frac{1}{10}$

b

x	0	1	2
$\Pr(X=x)$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

$E(X)=0\left(\frac{3}{10}\right)+1\left(\frac{4}{10}\right)+2\left(\frac{3}{10}\right)$

$E(X)=0+\frac{4}{10}+\frac{6}{10}=\frac{10}{10}=1$

$E(X^2)=0^2\left(\frac{3}{10}\right)+1^2\left(\frac{4}{10}\right)+2^2\left(\frac{3}{10}\right)$

$E(X^2)=0+\frac{4}{10}+\frac{12}{10}=\frac{16}{10}=1.6$

$Var(X)=E(X^2)-[E(X)]^2$

$Var(X)=\frac{16}{10}-(1)^2$

$Var(X)=\frac{16}{10}-\frac{10}{10}=\frac{6}{10}=0.6$

$SD(X)=\sqrt{\frac{6}{10}}=0.7746$

14 **a** $\sum_{\text{all } x}\Pr(X=x)=1$

$a+0.2+0.3+b+0.1=1$

$a+b+0.6=1$

$a+b=0.4$.............................(1)

$E(X)=2.5$

$1(a)+2(0.2)+3(0.3)+4(b)+5(0.1)=2.5$

$a+0.4+0.9+4b+0.5=2.5$

$a+4b+1.8=2.5$

$a+4b=0.7$......(2)

(2) = (1)

$3b=0.3$

$b=0.1$

Substitute $b=0.1$ into (1)

$a+0.1=0.4$

$a=0.3$

b $E(X^2)=1^2(0.3)+2^2(0.2)+3^2(0.3)+4^2(0.1)+5^2(0.1)$

$E(X^2)=0.3+0.8+2.7+1.6+2.5$

$E(X^2)=7.9$

$\text{Var}(X)=E(X^2)-[E(X)]^2$

$\text{Var}(X)=7.9-2.5^2$

$\text{Var}(X)=7.9-6.25$

$\text{Var}(X)=1.65$

$\text{SD}(X)=\sqrt{1.65}=1.2845$

15 a $\text{Var}(X)=2a-2$ and $E(X)=a$

$\text{Var}(X)=E(X^2)-[E(X)]^2$

$2a-2=E(X^2)--a^2$

$a^2+2a-2=E(X^2)$

b $E(X^2)=6$

$a^2+2a-2=6$

$a^2+2a-8=0$

$(a+4)(a-2)=0$

$a+4=0$ or $a-2=0$

$a=-4 \qquad a=2$

$\therefore a=2,\ a>0$

Thus $E(X)=a=2$ and $\text{Var}(X)=2a-2=2(2)-2=2$

16 a $p(n)=\begin{cases} ny & y=1,\ 2,\ 3,\ 4 \\ n(7-y) & y=5,\ 6 \end{cases}$

$p(1)=n,\ p(2)=2n,\ p(3)=3n,\ p(4)=4n,\ p(5)=2n,\ \text{p}(6)=n$

$\sum_{\text{all } x} \Pr(X=x)=1$

$n+2n+3n+4n+2n+n=1$

$13n-1=0$

$n=\frac{1}{13}$

b

y	1	2	3	4	5	6
$\Pr(Y=y)$	$\frac{1}{13}$	$\frac{2}{13}$	$\frac{3}{13}$	$\frac{4}{13}$	$\frac{2}{13}$	$\frac{1}{13}$

$E(Y)=1\left(\frac{1}{13}\right)+2\left(\frac{2}{13}\right)+3\left(\frac{3}{13}\right)+4\left(\frac{4}{13}\right)+5\left(\frac{2}{13}\right)+6\left(\frac{1}{13}\right)$

$E(Y)=\frac{1}{13}+\frac{4}{13}+\frac{9}{13}+\frac{16}{13}+\frac{10}{13}+\frac{6}{13}=\frac{46}{13}=3.5385$

$E(Y^2)=1^2\left(\frac{1}{13}\right)+2^2\left(\frac{2}{13}\right)+3^2\left(\frac{3}{13}\right)+4^2\left(\frac{4}{13}\right)+5^2\left(\frac{2}{13}\right)+6^2\left(\frac{1}{13}\right)$

$E(Y^2)=\frac{1}{13}+\frac{8}{13}+\frac{27}{13}+\frac{64}{13}+\frac{50}{13}+\frac{36}{13}=\frac{186}{13}$

$\text{Var}(Y)=E(Y^2)-[E(Y)]^2$

$\text{Var}(Y)=\frac{186}{13}-\left(\frac{46}{13}\right)^2$

$\text{Var}(Y)=\frac{2418}{169}-\frac{2116}{169}=\frac{302}{169}=1.7870$

$\text{SD}(Y)=\sqrt{1.7870}=1.3368$

17 a E = {11, 12, 13, 14, 15, 16, 17, 18
21, 22, 23, 24, 25, 26, 27, 28
31, 32, 33, 34, 35, 36, 37, 38
41, 42, 43, 44, 45, 46, 47, 48
51, 52, 53, 54, 55, 56, 57, 58
61, 62, 63, 64, 65, 66, 67, 68
71, 72, 73, 74, 75, 76, 77, 78
81, 82, 83, 84, 85, 86, 87, 88}

b $\Pr(Z=1)=\Pr(11)=\left(\frac{1}{8}\right)^2=\frac{1}{64}$

$\Pr(Z=2)=\Pr(12,\ 21,\ 22)=3\left(\frac{1}{8}\right)^2=\frac{3}{64}$

$\Pr(Z=3)=\Pr(13,\ 23,\ 31,\ 32,\ 33)=5\left(\frac{1}{8}\right)^2=\frac{5}{64}$

$\Pr(Z=4)=\Pr(14,\ 24,\ 34,\ 41,\ 42,\ 43,\ 44)=7\left(\frac{1}{8}\right)^2=\frac{7}{64}$

$\Pr(Z=5)=\Pr(15,\ 25,\ 35,\ 45,\ 51,\ 52,\ 53,\ 54,\ 55)=9\left(\frac{1}{8}\right)^2=\frac{9}{64}$

$\Pr(Z=6)=\Pr(16,\ 26,\ 36,\ 46,\ 56,\ 61,\ 62,\ 63,\ 64,\ 65,\ 66)$

$\Pr(Z=6)=11\left(\frac{1}{8}\right)^2=\frac{11}{64}$

$\Pr(Z=7)=\Pr(17,\ 27,\ 37,\ 47,\ 57,\ 67,\ 71,\ 72,\ 73,\ 74,\ 75,\ 76,\ 77)$

$\Pr(Z=7)=13\left(\frac{1}{8}\right)^2=\frac{13}{64}$

$\Pr(Z=8)=\Pr(18,\ 28,\ 38,\ 48,\ 58,\ 68,\ 78,\ 81,\ 82,\ 83,\ 84,\ 85,\ 86,\ 87,\ 88)$

$\Pr(Z=8)=15\left(\frac{1}{8}\right)^2=\frac{15}{64}$

z	1	2	3	4	5	6	7	8
$\Pr(Z-z)$	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{5}{64}$	$\frac{7}{64}$	$\frac{9}{64}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{15}{64}$

c $E(Z)=1\left(\frac{1}{64}\right)+2\left(\frac{3}{64}\right)+3\left(\frac{5}{64}\right)+4\left(\frac{7}{64}\right)+5\left(\frac{9}{64}\right)+6\left(\frac{11}{64}\right)+7\left(\frac{13}{64}\right)+8\left(\frac{15}{64}\right)$

$E(Z)=\frac{1}{64}+\frac{6}{64}+\frac{15}{64}+\frac{28}{64}+\frac{45}{64}+\frac{66}{64}+\frac{91}{64}+\frac{120}{64}$

$E(Z)=\frac{372}{64}=5.8125$

$E(Z^2)=1^2\left(\frac{1}{64}\right)+2^2\left(\frac{3}{64}\right)+3^2\left(\frac{5}{64}\right)+4^2\left(\frac{7}{64}\right)+5^2\left(\frac{9}{64}\right)+6^2\left(\frac{11}{64}\right)+7^2\left(\frac{13}{64}\right)+8^2\left(\frac{15}{64}\right)$

$E(Z^2)=\frac{1}{64}+\frac{12}{64}+\frac{45}{64}+\frac{112}{64}+\frac{225}{64}+\frac{396}{64}+\frac{637}{64}+\frac{960}{64}=\frac{2388}{64}=37.3125$

$\text{Var}(Z)=E(Z^2)-[E(Z)]^2$

$\text{Var}(Z)=\frac{2388}{64}-\left(\frac{372}{64}\right)^2$

$\text{Var}(Z)=\frac{152\,832}{4096}-\frac{138\,384}{4096}=\frac{14\,448}{4096}=3.5273$

$\text{SD}(Z)=\sqrt{\frac{14\,448}{4096}}=1.8781$

18 **a** Area of whole board is $\pi(4\times 5)^2=400\pi$

B and A $=\pi(4)^2=16\pi$ and $\Pr(A)=\frac{16\pi}{400\pi}=\frac{1}{25}$

B and B $=\pi(8)^2-16\pi=64\pi-16\pi=48\pi$ and $\Pr(B)=\frac{48\pi}{400\pi}=\frac{3}{25}$

B and C $=\pi(12)^2-64\pi=144\pi-64\pi=80\pi$ and $\Pr(C)=\frac{80\pi}{400\pi}=\frac{5}{25}$

B and D $=\pi(16)^2-144\pi=256\pi-144\pi=112\pi$ and $\Pr(D)=\frac{112\pi}{400\pi}=\frac{7}{25}$

B and E $=\pi(20)^2-256\pi=400\pi-256\pi=144\pi$ and $\Pr(E)=\frac{144\pi}{400\pi}=\frac{9}{25}$

b X is the gain in dollars

$\Pr(E)=-\$1,\ \Pr(D)=\$0,\ \Pr(C)=\$1,\ \Pr(B)=\$4,\ \Pr(A)=\$9$

x	–\$1	\$0	\$1	\$4	\$9
$\Pr(X=x)$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{5}{25}=\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{25}$

c **i** $E(X)=-1\left(\frac{9}{25}\right)+0\left(\frac{7}{25}\right)+1\left(\frac{5}{25}\right)+4\left(\frac{3}{25}\right)+9\left(\frac{1}{25}\right)$

$E(X)=-\frac{9}{25}+0+\frac{5}{25}+\frac{12}{25}+\frac{9}{25}$

$E(X)=\frac{17}{25}=0.68$ cents

ii $E(X^2)=(-1)^2\left(\frac{9}{25}\right)+0^2\left(\frac{7}{25}\right)+1^2\left(\frac{5}{25}\right)+4^2\left(\frac{3}{25}\right)+9^2\left(\frac{1}{25}\right)$

$E(X^2)=\frac{9}{25}+0+\frac{5}{25}+\frac{48}{25}+\frac{81}{25}$

$E(X^2)=\frac{143}{25}$

$\text{Var}(X)=E(X^2)-[E(X)]^2$

$\text{Var}(X)=\frac{143}{25}-\left(\frac{17}{25}\right)^2$

$\text{Var}(X)-\frac{3575}{625}-\frac{289}{625}=\frac{3286}{625}=\$5.26$

$\text{SD}(X)=\sqrt{\frac{3286}{625}}=\$2.29$

19 **a** Possible Scores:

$$E=\begin{Bmatrix}\overbrace{1-1}^{2}, & \overbrace{1-3}^{4}, & \overbrace{1-5}^{6}, & \overbrace{1-7}^{8}, & \overbrace{1-10}^{11} \\ \overbrace{3-1}^{4}, & \overbrace{3-3}^{6}, & \overbrace{3-5}^{8}, & \overbrace{3-7}^{10}, & \overbrace{3-10}^{13} \\ \overbrace{5-1}^{6}, & \overbrace{5-3}^{8}, & \overbrace{5-5}^{10}, & \overbrace{5-7}^{12}, & \overbrace{5-10}^{15} \\ \overbrace{7-1}^{8}, & \overbrace{7-3}^{10}, & \overbrace{7-5}^{12}, & \overbrace{7-7}^{14}, & \overbrace{7-10}^{17} \\ \overbrace{10-1}^{11}, & \overbrace{10-3}^{13}, & \overbrace{10-5}^{15}, & \overbrace{10-7}^{17}, & \overbrace{10-10}^{20}\end{Bmatrix}$$

$\therefore$ Possible Scores are 2, 4, 6, 8, 10, 11, 12, 13, 14, 15, 17, 20

b $\Pr(2)=\Pr(11)=0.2\times0.2=0.04$

$\Pr(4)=\Pr(13,\ 31)=(0.2\times0.2)^2=0.08$

$\Pr(6)=\Pr(15,\ 33,\ 51)=(0.2\times0.3)+(0.2\times0.2)+(0.3\times0.2)=0.16$

$\Pr(8)=\Pr(17,\ 35,\ 53,\ 71)=(0.2\times0.2)+(0.2\times0.3)+(0.3\times0.2)+(0.2\times0.2)=0.20$

$\Pr(10)=\Pr(37,\ 55,\ 73)=(0.2\times0.2)+(0.3\times0.3)+(0.2\times0.2)=0.17$

$\Pr(11)=\Pr(1\ 10,\ 10\ 1)=(0.2\times0.1)+(0.1\times0.2)=0.04$

$\Pr(12)=\Pr(57,\ 75)=(0.3\times0.2)+(0.2\times0.3)=0.12$

$\Pr(13)=\Pr(1\ 10,\ 10\ 1)=(0.2\times0.1)+(0.1\times0.2)=0.04$

$\Pr(14)=\Pr(77)=0.2\times0.2=0.04$

$\Pr(15)=\Pr(5\ 10,10\ 5)=(0.3\times0.1)+(0.1\times0.3)=0.06$

$\Pr(17)=\Pr(7\ 10,\ 10\ 7)=(0.2\times0.1)+(0.1\times0.2)=0.04$

$\Pr(20)=\Pr(10\ 10)=0.1\times0.1=0.01$

x	2	4	6	8	10	11	12	13	14	15	17	20
$\Pr(X=x)$	0.04	0.08	0.16	0.2	0.17	0.04	0.12	0.04	0.04	0.06	0.04	0.01

c $E(X)=9.4$ and $\text{SD}(X)=3.7974$

20 **a** $\sum_{\text{all } x}\Pr(X=x)=1$

$0.5k^2+0.5k^2+k+k^2+4k+2k+2k+k^2+7k^2=1$

$10k^2+9k-1=0$

$k=-1 \quad \text{or} \quad k=0.1$

$\therefore\ k=0.1, k>0$

b $E(X)=1.695$

c $\text{SD}(X)=1.167$

Exercise 10.4 — Applications

1 **a** $E(X)=5(0.05)+10(0.25)+15(0.4)+20(0.25)+25(0.05)$
$E(X)=0.25+2.5+6+5+1.25$
$E(X)=15$

b $E(X^2)=5^2(0.05)+10^2(0.25)+15^2(0.4)+20^2(0.25)+25^2(0.05)$
$E(X^2)=1.25+25+90+100+31.25$
$E(X^2)=247.5$
$Var(X)=E(X^2)-[E(X)]^2$
$Var(X)=247.5-15^2$
$Var(X)=22.5$
$SD(X)=\sqrt{22.5}=4.7434$

c $\mu-2\sigma=15-2(4.7434)=5.5132$
$\mu+2\sigma=15+2(4.7434)=24.4947$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=\Pr(5.5132\le X\le 24.4947)$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=\Pr(X=10)+\Pr(x=15)+\Pr(X=20)$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=1-(\Pr(X=5)+\Pr(X=25))$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=1-0.1=0.9$

2 $E(X)=0(0.012)+1(0.093)+2(0.243)+3(0.315)+4(0.214)+5(0.1)+6(0.023)$
$E(X)=0+0.093+0.486+0.945+0.856+0.5+0.138$
$E(X)=3.018$
$E(X^2)=0^2(0.012)+1^2(0.093)+2^2(0.243)+3^2(0.315)+4^2(0.214)+5^2(0.1)+6^2(0.023)$
$E(X^2)=0+0.093+0.972+2.835+3.424+2.5+0.828$
$E(X^2)=10.652$
$Var(X)=E(X^2)-[E(X)]^2$
$Var(X)=10.652-3.018^2$
$Var(X)=1.5437$
$SD(X)=\sqrt{1.5437}=1.2424$
$\mu-2\sigma=3.018-2(1.2424)=0.5332$
$\mu+2\sigma=3.018+2(1.2424)=5.5028$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=\Pr(0.5332\le X\le 5.5028)$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=\Pr(X=1)+\Pr(X=2)+\Pr(x=3)+\Pr(X=4)+\Pr(X=5)$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=1-(\Pr(X=0)+\Pr(X=1))$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=1-(0.012+0.023)$
$\Pr(\mu-2\sigma\le X\le\mu+2\sigma)=0.965$

3

x	\$100	\$250	\$500	\$750	\$1000
$\Pr(X=x)$	0.1	0.2	0.3	0.3	0.1

where X is hundreds of thousands of dollars.

a $\Pr(X\le \$500)=\Pr(X=\$100)+\Pr(X=\$250)+\Pr(X=\$500)$
$\Pr(X\le \$500)=0.1+0.2+0.3=0.6$

b $\Pr(X\ge \$250 \mid X\le \$750)=\dfrac{\Pr(X\ge \$250)\cap\Pr(X\le \$750)}{\Pr(X\le \$750)}$
$\Pr(X\ge \$250 \mid X\le \$750)=\dfrac{\Pr(X=\$250)+\Pr(X=\$500)+\Pr(X=\$750)}{1-\Pr(X=\$1000)}$
$\Pr(X\ge \$250 \mid X\le \$750)=\dfrac{0.2+0.3+0.3}{1-0.1}$
$\Pr(X\ge \$250 \mid X\le \$750)=\dfrac{0.8}{0.9}=\dfrac{8}{9}$

c $E(X)=100(0.1)+250(0.2)+500(0.3)+750(0.3)+1000(0.1)$
$E(X)=10+50+150+225+100=\$535$
$\therefore$ the expected profit is \$535 000

d $E(X^2)=100^2(0.1)+250^2(0.2)+500^2(0.3)+750^2(0.3)+1000^2(0.1)$
$E(X^2)=1000+12500+75000+168750+100000$
$E(X^2)=357250$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$
$\text{Var}(X) = 357\,250 - (535)^2 = 71\,025$
$\text{SD}(X) = \sqrt{71\,025} = 266.51$

$\mu - 2\sigma = 535 - 2(266.51) = 1.98$
$\mu + 2\sigma = 535 + 2(266.51) = 1068.02$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(1.98 \le X \le 1068.02)$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1$

4 a $\sum_{\text{all } z} \Pr(Z = z) = 1$

$3m + 3n = 1$..................................(1)
$\Pr(Z < 2) = 3\Pr(Z > 4)$
$\Pr(Z = 0) + \Pr(Z = 1) = 3\Pr(Z = 5)$
$2m = 3n$...............(2)

Substitute (2) into (1)
$3m + 2m = 1$
$5m = 1$
$m = \frac{1}{5}$

Substitute $m = \frac{1}{5}$ into (2)

$2\left(\frac{1}{5}\right) = 3n$
$n = \frac{2}{15}$

b $\text{E}(Z) = 0\left(\frac{3}{15}\right) + 1\left(\frac{3}{15}\right) + 2\left(\frac{3}{15}\right) + 3\left(\frac{2}{15}\right) + 4\left(\frac{2}{15}\right) + 5\left(\frac{2}{15}\right)$

$\text{E}(Z) = 0 + \frac{3}{15} + \frac{6}{15} + \frac{6}{15} + \frac{8}{15} + \frac{10}{15}$

$\text{E}(Z) = \frac{33}{15} = \frac{11}{5}$ as required

$\text{E}(Z^2) = 0^2\left(\frac{3}{15}\right) + 1^2\left(\frac{3}{15}\right) + 2^2\left(\frac{3}{15}\right) + 3^2\left(\frac{2}{15}\right) + 4^2\left(\frac{2}{15}\right) + 5^2\left(\frac{2}{15}\right)$

$\text{E}(Z^2) = 0 + \frac{3}{15} + \frac{12}{15} + \frac{18}{15} + \frac{32}{15} + \frac{50}{15}$

$\text{E}(Z^2) = \frac{115}{15} = \frac{23}{3}$

$\text{Var}(Z) = \text{E}(Z^2) - [\text{E}(Z)]^2$

$\text{Var}(Z) = \frac{23}{3} - \left(\frac{11}{5}\right)^2$

$\text{Var}(Z) = \frac{23}{3} - \frac{121}{25}$

$\text{Var}(Z) = \frac{575}{75} - \frac{363}{75}$

$\text{Var}(Z) = \frac{212}{75} = 2.8267$

$\text{SD}(Z) = \sqrt{\frac{212}{75}} \simeq 1.6813$

c $\mu - 2\sigma = \frac{11}{5} - 2(1.6813) = -1.1626$

$\mu + 2\sigma = \frac{11}{5} + 2(1.6813) = 5.5626$

$\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) = \Pr(-1.1626 \le Z \le 5.5626) = 1$

5 a $E(Y)=3.5$

$1(0.3)+2(0.2)+d(0.4)+8(0.1)=3.5$

$0.3+0.4+0.4d+0.8=3.5$

$0.4d+1.5=3.5$

$0.4d=2$

$d=\frac{2}{0.4}$

$d=5$

b $\Pr(Y\geq 2\,|\,Y\leq 5)=\frac{\Pr(Y\geq 2)\cap\Pr(Y\leq 5)}{\Pr(Y\leq 5)}$

$\Pr(Y\geq 2\,|\,Y\leq 5)=\frac{\Pr(Y=2)+\Pr(Y=5)}{1-\Pr(Y=8)}$

$\Pr(Y\geq 2\,|\,Y\leq 5)=\frac{0.2+0.4}{1-0.1}$

$\Pr(Y\geq 2\,|\,Y\leq 5)=\frac{0.6}{0.9}=\frac{2}{3}$

c $E(Y^2)=1^2(0.3)+2^2(0.2)+5^2(0.4)+8^2(0.1)$

$E(Y^2)=0.3+0.8+10+6.4=17.5$

$\text{Var}(Y)=E(Y^2)-[E(Y)]^2$

$\text{Var}(Y)=17.5-3.5^2$

$\text{Var}(Y)=5.25$

d $SD(Y)=\sqrt{5.25}=2.2913$

6 a $\sum_{\text{all } z}\Pr(Z=z)=1$

$0.2+0.15+a+b+0.05=1$

$a+b+0.4=1$

$a+b=0.6$..................................(1)

$E(Z)=4.6$

$1(0.2)+3(0.15)+5a+7b+9(0.05)=4.6$

$0.2+0.45+5a+7b+0.45=4.6$

$5a+7b+1.1=4.6$

$5a+7b=3.5$................(2)

From (1) $a=0.6-b$..........(3)

Substitute (3) into (2)

$5(0.6-b)+7b=3.5$

$3-5b+7b=3.5$

$2b=0.5$

$b=0.25$

Substitute $b=0.25$ into (3)

$a=0.6-0.25=0.35$

b $E(Z^2)=1^2(0.2)+3^2(0.15)+5^2(0.35)+7^2(0.25)+9^2(0.05)$

$E(Z^2)=0.2+1.35+8.75+12,25+4.05$

$E(Z^2)=26.6$

$\text{Var}(Z)=E(Z^2)-[E(Z)]^2$

$\text{Var}(Z)=26.6-4.6^2=5.44$

$SD(Z)=\sqrt{5.44}=2.3324$

c i $E(3Z+2)=3E(Z)+3$

$=3(4.6)+3$

$=15.8$

ii $\text{Var}(3Z+2)=3^2\,\text{Var}(Z)=9\times 5.44=48.96$

7 a

z	0	1	2	3	4	5
$\Pr(Z=z)$	m	m	m	m	n	n

$\sum_{\text{all } z}\Pr(Z=z)=1$

$4m+2n=1$...................(1)

$\Pr(Z\leq 3)=\Pr(Z\geq 4)$

$4m=2n$

$2m=n$....................(2)

Substitute (2) into (1)

$4m+2(2m)=1$

$4m+4m=1$

$8m=1$

$m=\frac{1}{8}$

Substitute $m=\frac{1}{8}$ into (2)

$2\left(\frac{1}{8}\right)=n$

$n=\frac{1}{4}$

b i $E(Z)=0\left(\frac{1}{8}\right)+1\left(\frac{1}{8}\right)+2\left(\frac{1}{8}\right)+3\left(\frac{1}{8}\right)+4\left(\frac{1}{4}\right)+5\left(\frac{1}{4}\right)$

$E(Z)=0+\frac{1}{8}+\frac{2}{8}+\frac{3}{8}+\frac{8}{8}+\frac{10}{8}$

$E(Z)=\frac{24}{8}=3$

ii $E(Z^2)=0^2\left(\frac{1}{8}\right)+1^2\left(\frac{1}{8}\right)+2^2\left(\frac{1}{8}\right)+3^2\left(\frac{1}{8}\right)+4^2\left(\frac{1}{4}\right)+5^2\left(\frac{1}{4}\right)$

$E(Z^2)=0+\frac{1}{8}+\frac{4}{8}+\frac{9}{8}+\frac{32}{8}+\frac{50}{8}$

$E(Z^2)=\frac{96}{8}=12$

$\text{Var}(Z)=E(Z^2)-[E(Z)]^2$

$\text{Var}(Z)=12-3^2=3$

c $SD(Z)=\sqrt{3}=1.732$

$\mu-2\sigma=3-2(1.732)=-0.464$

$\mu+2\sigma=3+2(1.732)=6.464$

$\Pr(\mu-2\sigma\leq Z\leq\mu+2\sigma)=\Pr(-0.464\leq Z\leq 6.464)=1$

8

	M	M’	
N	0.216	0.264	0.480
N’	0.234	0.286	0.520
	0.450	0.550	1.000

a As M and N are independent

$\Pr(M\cap N)=\Pr(M)\Pr(N)=0.45\times 0.48=0.216$

b $\Pr(M'\cap N')=0.286$

c Y is the number of times M and N occur. $Y=\{0,\ 1,\ 2\}$

$\Pr(Y=0)=0.286$

$\Pr(Y=1)=0.264+0.234=0.498$

$\Pr(Y=2)=0.216$

y	0	1	2
$\Pr(Y=y)$	0.286	0.498	0.216

d **i** $E(Y) = 0(0.286) + 1(0.498) + 2(0.216) = 0 + 0.498 + 0.432 = 0.93$

ii $E(Y^2) = 0^2(0.286) + 1(0.498) + 2^2(0.216) = 0 + 0.498 + 0.864 = 1.362$

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$

$\text{Var}(Y) = 1.362 - 0.93^2$

$\text{Var}(Y) = 0.4971$

iii $SD(Y) = \sqrt{0.4971} = 0.7050$

9 **a** If $p(x) = \frac{1}{9}(4 - x)$, where $x = \{0, 1, 2\}$

$p(0) = \frac{4}{9}, \; p(1) = \frac{3}{9} = \frac{1}{3}, \; p(2) = \frac{2}{9}.$

b $\sum_{x=1}^{\infty} p(x) = 1$ so this is a probability density function.

i $E(X) = \mu = \sum_{x=1}^{\infty} xp(x) = 0\left(\frac{4}{9}\right) + 1\left(\frac{3}{9}\right) + 2\left(\frac{2}{9}\right) = \frac{7}{9}$

ii $\text{Var}(X) = \sigma^2 = \sum_{x=1}^{\infty} (x-2)^2 p(x) = \frac{11}{9} - \left(\frac{7}{9}\right)^2 = \frac{99}{81} - \frac{49}{81} = \frac{50}{81}$

iii $SD(X) = \sqrt{\frac{50}{81}} = 0.7857$

c $\mu - 2\sigma = 2 - 2(1.03) = -0.06$

$\mu + 2\sigma = 2 + 2(1.03) = 4.06$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(-0.06 \le X \le 4.06)$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(0) + \Pr(1) + \Pr(2)$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1$

10 **a** $\sum_{\text{all } x} \Pr(X = x) = 1$

$\frac{k^2}{4} + \frac{5k-1}{12} + \frac{3k-1}{12} + \frac{4k-1}{12} = 1$

$3k^2 + 5k - 1 + 3k - 1 + 4k - 1 = 12$

$3k^2 + 12k - 3 = 12$

$3k^2 + 12k - 15 = 0$

$k^2 + 4k - 5 = 0$

$(k + 5)(k - 1) = 0$

$k = -5, \; k = 1$

$k = -5$ is not applicable $\therefore \; k = 1$

x	0	1	2	3
$\Pr(X = x)$	$\frac{3}{12} = \frac{1}{4}$	$\frac{4}{12} = \frac{1}{3}$	$\frac{2}{12} = \frac{1}{6}$	$\frac{3}{12}$

b $E(X) = 0\left(\frac{3}{12}\right) + 1\left(\frac{4}{12}\right) + 2\left(\frac{2}{12}\right) + 3\left(\frac{3}{12}\right) = 0 + \frac{4}{12} + \frac{4}{12} + \frac{9}{12} = \frac{17}{12} = 1.4$

c $\Pr(X < 1.4) = \Pr(X = 0) + \Pr(X = 1) = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$

11 **a**

Money	\$1000	\$15 000	\$50 000	\$100 000	\$200 000
Probability	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

E (Bank offer) is

$= 1000\left(\frac{1}{5}\right) + 15\,000\left(\frac{1}{5}\right) + 50\,000\left(\frac{1}{5}\right) + 100\,000\left(\frac{1}{5}\right) + 200\,000\left(\frac{1}{5}\right)$

= \$73 200

b

Money	\$1000	\$15 000	\$50 000	\$100 000
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

E (Bank offer) is

$=1000\left(\frac{1}{4}\right)+15\,000\left(\frac{1}{4}\right)+50\,000\left(\frac{1}{4}\right)+200\,000\left(\frac{1}{4}\right)$

$=\$66\,500$

12 a

Autobiography		Probability	Cook Book		Probability
New	\$65	0.40	New	\$54	0.40
Good used	\$30	0.30	Good used	\$25	0.25
Worn used	\$12	0.30	Worn used	\$15	0.35

New Autobiography + New Cook Book \$65 + \$54 = \$119 $0.4 \times 0.40 = 0.16$
New Autobiography + Good Cook Book \$65 + \$25 = \$90 $0.4 \times 0.25 = 0.10$
New Autobiography + Worn Cook Book \$65 + \$15 = \$80 $0.4 \times 0.35 = 0.14$
Good Autobiography + New Cook Book \$30 + \$54 = \$84 $0.3 \times 0.40 = 0.12$
Good Autobiography + Good Cook Book \$30 + \$25 = \$55 $0.3 \times 0.25 = 0.075$
Good Autobiography + Worn Cook Book \$30 + \$15 = \$45 $0.3 \times 0.35 = 0.105$
Worn Autobiography + New Cook Book \$12 + \$54 = \$66 $0.3 \times 0.40 = 0.12$
Worn Autobiography + Good Cook Book \$12 + \$25 = \$37 $0.3 \times 0.25 = 0.075$
Worn Autobiography + Worn Cook Book \$12 + \$15 = \$27 $0.3 \times 0.35 = 0.105$
X = cost of two books.

x	\$119	\$90	\$84	\$80	\$66	\$55	\$45	\$37	\$27
$\Pr(X = x)$	0.16	0.10	0.12	0.14	0.12	0.075	0.105	0.075	0.105

b $\text{E}(X)=119(0.16)+90(0.1)+84(0.12)+80(0.14)+66(0.12)+55(0.075)+45(0.105)+37(0.075)+27(0.105)=\$71.70$

13 a Let Y be the net profit per day.

y	−\$120	\$230	\$580	\$930
$\Pr(Y = y)$	0.3	0.4	0.2	0.1

b $\text{E}(Y)=-120(0.3)+230(0.4)+580(0.2)+930(0.1)$

$\text{E}(Y)=\$265$

c $\text{E}(Y^2)=(-120)^2(0.3)+230^2(0.4)+580^2(0.2)+930^2(0.1)$

$\text{E}(Y^2)=179\,250$

$\text{Var}(Y)=\text{E}(Y^2)-[\text{E}(Y)]^2$

$\text{Var}(Y)=179\,250-265^2$

$\text{Var}(Y)=109\,025$

$\text{SD}(Y)=\sqrt{109\,025}=\$330$

$\mu-2\sigma=265-2(330)=-\$395$

$\mu+2\sigma=265+2(330)=\$925$

$\Pr(\mu-2\sigma\le Y\le\mu+2\sigma)=\Pr(-\$395\le Y\le\$925)$

$\Pr(\mu-2\sigma\le Y\le\mu+2\sigma)=1-\Pr(Y=\$930)$

$\Pr(\mu-2\sigma\le Y\le\mu+2\sigma)=1-0.1=0.9$

14 a Coin: $\Pr(H)=\frac{3}{4}$ and $\Pr(T)=\frac{1}{4}$

Die: $\Pr(1)=\frac{1}{12},\ \Pr(2)=\frac{1}{12},\ \Pr(3)=\frac{1}{4},\ \Pr(4)=\frac{1}{4},\ \Pr(5)=\frac{1}{12},\ \Pr(6)=\frac{1}{4}$

$\text{E}=\{\overbrace{1H}^{5},\ \overbrace{2H}^{5},\ \overbrace{3H}^{1},\ \overbrace{4H}^{1},\ \overbrace{5H}^{5},\ \overbrace{6H}^{1},\ \overbrace{1T}^{10},\ \overbrace{2T}^{10},\ \overbrace{3T}^{1},\ \overbrace{4T}^{1},\ \overbrace{5T}^{10},\ \overbrace{6T}^{1}\}$

$\Pr(10) = \Pr(1T,\ 2T,\ 5T) = \left(\frac{1}{12}\times\frac{1}{4}\right)+\left(\frac{1}{12}\times\frac{1}{4}\right)+\left(\frac{1}{12}\times\frac{1}{4}\right)=\frac{3}{48}=\frac{1}{16}$

$\Pr(5) = \Pr(1H,\ 2H,\ 5H) = \left(\frac{1}{12}\times\frac{3}{4}\right)+\left(\frac{1}{12}\times\frac{3}{4}\right)+\left(\frac{1}{12}\times\frac{3}{4}\right)=\frac{9}{48}=\frac{3}{16}$

$\Pr(1) = \Pr(3H,\ 3T,\ 4H,\ 4T,\ 6H,\ 6T)$

$= \left(\frac{1}{4}\times\frac{3}{4}\right)+\left(\frac{1}{4}\times\frac{1}{4}\right)+\left(\frac{1}{4}\times\frac{3}{4}\right)+\left(\frac{1}{4}\times\frac{1}{4}\right)+\left(\frac{1}{4}\times\frac{3}{4}\right)+\left(\frac{1}{4}\times\frac{1}{4}\right)$

$= 3\left(\frac{3}{16}\right)+3\left(\frac{1}{16}\right)$

$= \frac{12}{16}$

x	1	5	10
$\Pr(X=x)$	$\frac{12}{16}=\frac{3}{4}$	$\frac{3}{16}$	$\frac{1}{16}$

b $E(X)=1\left(\frac{12}{16}\right)+5\left(\frac{3}{16}\right)+10\left(\frac{1}{16}\right)=\frac{12}{16}+\frac{15}{16}+\frac{10}{16}=\frac{37}{16}=2.3$

c $E(25 \text{ tosses}) = 25\times 2.3125 = 57.8$

d Let n be the number of tosses

$2.3125n = 100$

$n=\frac{100}{2.3125}=43.243$

Minimum number of tosses required is 44.

15 **a** $\Pr(V\cup W)=0.7725$ and $\Pr(V\cap W)=0.2275$.

$\Pr(V\cup W)=\Pr(W)+\Pr(V)-\Pr(W\cap V)$

$0.7725=\Pr(W)+\Pr(V)-0.2275$

$1.0000=\Pr(W)+\Pr(V)$...............................(1)

V and W are independent events.

$\Pr(W\cap V)=\Pr(W)\Pr(V)$

$0.2275=\Pr(W)\Pr(V)$

$\frac{0.2275}{\Pr(W)}=\Pr(V)$.................................(2)

Substitute (2) into (1)

$1=\frac{0.2275}{P(W)}+\Pr(W)$

$\Pr(W)=0.2275+[\Pr(W)]^2$

$0=[\Pr(W)]^2-\Pr(W)+0.2275$

$\Pr(W)=0.65$ or 0.35

But $\Pr(V)<\Pr(W)$ so $\Pr(W)=0.65$ and $\Pr(V)=0.35$

b

	W	W’	
V	0.2275	0.1225	0.35
V’	0.4225	0.2275	0.65
	0.35	0.65	1.000

Note: $\Pr(V'\cap W')=0.2275$

c

x	0	1	2
$\Pr(X=x)$	0.2275	0.545	0.2275

d **i** $E(X)=0(0.2275)+1(0.545)+2(0.2275)=1$

ii $E(X^2)=0^2(0.2275)+1^2(0.545)+2^2(0.2275)$

$=0+0.545+0.91=1.455$

$\text{Var}(X)=E(X^2)-[E(X)]^2$

$\text{Var}(X)=1.455-1^2$

$\text{Var}(X)=0.455$

iii $SD(X)=\sqrt{0.455}=0.6745$

16 a $\sum_{\text{all } z} \Pr(Z = z) = 1$

$\frac{k^2}{7} + \frac{5-2k}{7} + \frac{8-3k}{7} = 1$

$k = 2 \text{ or } 3$

But if $k = 3$, $\frac{8-3(3)}{7} = -\frac{1}{7}$ so this is not applicable.

$\therefore k = 2$

b

z	1	3	5
$\Pr(Z = z)$	$\frac{2^2}{7} = \frac{4}{7}$	$\frac{5-2(2)}{7} = \frac{1}{7}$	$\frac{8-3(2)}{7} = \frac{2}{7}$

i $E(Z) = 2.4286$

ii $\text{Var}(Z) = E(Z^2) - [E(Z)]^2 = 9 - 2.4286^2 = 3.1019$

iii $\text{SD}(Z) = \sqrt{3.1019} = 1.7613$

c $\mu - 2\sigma = 2.4286 - 2(1.7613) = -1.094$

$\mu + 2\sigma = 2.4286 + 2(1.7613) = 5.9512$

$\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) = \Pr(-1.094 \le Z \le 5.9512) = 1$

Topic 11 — The binomial distribution

Exercise 11.2 — Bernoulli trials

1 a This is not a Bernoulli distribution as a successful outcome is not specified.

b This is a Bernoulli distribution as a success is getting a hole in one and a failure is not getting a hole in one.

c This is a Bernoulli distribution as a success is withdrawing an ace and a failure is withdrawing any other card.

2 a

x	0	1
$\Pr(X = x)$	0.58	0.42

b $E(X) = 0.42$

c i $\text{Var}(X) = 0.58 \times 0.42 = 0.2436$

ii $SD(X) = \sqrt{0.2436} = 0.4936$

3 a This is a Bernoulli distribution as the arthritis drug is either successful or not.

b This is a Bernoulli distribution as the child is either a girl or not.

c This is not a Bernoulli distribution as the probability of success is unknown.

d This is a Bernoulli distribution as the next person either subscribes or not.

4 a The friend does not replace the ball before I choose a ball, so this cannot be a Bernoulli distribution.

b There are 6 outcomes not 2, so this is not a Bernoulli distribution.

c The probability of success is unknown so this is not a Bernoulli distribution.

5 a $E(Z) = p = 0.63$

b $\text{Var}(Z) = pq = 0.63 \times 0.37 = 0.2331$

c $SD(Z) = \sqrt{0.2331} = 0.4828$

6 a

y	0	1
$\Pr(Y = y)$	0.32	0.68

b i $E(Y) = p = 0.68$

ii $\text{Var}(Y) = pq = 0.68 \times 0.32 = 0.2176$

iii $SD(Y) = \sqrt{0.2176} = 0.4665$

c $\mu - 2\sigma = 0.68 - 2(0.4665) = -0.253$

$\mu + 2\sigma = 0.68 + 2(0.4665) = 1.613$

$$\begin{aligned}\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma) &= \Pr(-0.253 \le Y \le 1.613)\\ &= \Pr(Y = 0) + \Pr(Y = 1)\\ &= 1\end{aligned}$$

7 a

x	0	1
$\Pr(X = x)$	0.11	0.89

b i $E(X) = p = 0.89$

ii $\text{Var}(X) = pq = 0.89 \times 0.11 = 0.0979$

iii $SD(X) = \sqrt{0.0979} = 0.3129$

c $\mu - 2\sigma = 0.89 - 2(0.3129) = 0.2642$

$\mu + 2\sigma = 0.89 + 2(0.3129) = 1.5158$

$$\begin{aligned}\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) &= \Pr(0.2642 \le X \le 1.5158)\\ &= \Pr(X = 1)\\ &= 0.89\end{aligned}$$

8 a $\text{Var}(X) = p(1-p) = 0.21$

$$p - p^2 = 0.21$$

$$0 = p^2 - p + 0.21$$

Therefore $p = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(0.21)}}{2(1)}$

$p = \dfrac{1 \pm \sqrt{1 - 0.84}}{2}$

$p = \dfrac{1 \pm 0.4}{2}$

$p = 0.3$ or 0.7

But $p > 1 - p$ so $p = 0.7$

b $E(X) = p = 0.7$

9 a $SD(Y) = 0.4936$

$\text{Var}(Y) = 0.4936^2 = 0.2436$

b $\text{Var}(Y) = p(1-p) = 0.2436$

$$p - p^2 = 0.2436$$

$$0 = p^2 - p + 0.2436$$

Therefore $p = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(0.21)436}}{2(1)}$

$p = \dfrac{1 \pm \sqrt{1 - 0.9744}}{2}$

$p = \dfrac{1 \pm 0.16}{2}$

$p = \dfrac{0.84}{2}$ or $\dfrac{1.16}{2}$

$p = 0.42$ or 0.58

But $p > 1 - p$ so $p = 0.58$

c $E(Y) = p = 0.58$

10 a $\Pr(\text{breast cancer}) = 0.0072$

b

z	0	1
$\Pr(Z = z)$	0.9928	0.0072

c $\mu = E(Z) = 0.0072$

$\text{Var}(Z) = pq = 0.0072 \times 0.9928 = 0.0071$

$\sigma = SD(Z) = \sqrt{0.0071} = 0.0845$

$\mu - 2\sigma = 0.0072 - 2(0.0845) = -0.1618$

$\mu + 2\sigma = 0.0072 + 2(0.0845) = 0.1762$

$$\begin{aligned}\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) &= \Pr(-0.1618 \le Z \le 0.1762)\\ &= \Pr(Z = 0)\\ &= 0.9928\end{aligned}$$

11 a

y	0	1
$\Pr(Y = y)$	0.67	0.33

b $\mu = E(Y) = p = 0.33$

c $\text{Var}(Y) = pq = 0.33 \times 0.67 = 0.2211$

$\sigma = SD(Y) = \sqrt{0.2211} = 0.4702$

$\mu - 2\sigma = 0.33 - 2(0.4702) = -0.6104$

$\mu + 2\sigma = 0.33 + 2(0.4702) = 1.2704$

$$\begin{aligned}\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma) &= \Pr(-0.6104 \le Y \le 1.2704)\\ &= \Pr(Y = 0) + \Pr(Y = 1)\\ &= 1\end{aligned}$$

12 a

x	0	1
$\Pr(X = x)$	$\frac{4}{5}$	$\frac{1}{5}$

b $E(X) = p = \frac{1}{5}$

c $\Pr(X = 5) = \left(\frac{1}{5}\right)^5 = 0.00032$

13 a $\text{Var}(Z) = p(1-p) = 0.1075$

$p - p^2 = 0.1075$

$0 = p^2 - p + 0.1075$

$p = 0.1225$ or 0.8775

Since $p > 1 - p$. $p = 0.8775$.

b

z	0	1
$\Pr(Z = z)$	0.1225	0.8775

c $E(Z) = p = 0.8775$

14 a $SD(X) = 0.3316$

$\text{Var}(X) = 0.3316^2 = 0.11$

b $\text{Var}(Z) = p(1-p) = 0.11$

$p - p^2 = 0.11$

$0 = p^2 - p + 0.11$

$p = 0.1258$ or 0.8742

Since $p > 1 - p$, $p = 0.8742$.

Exercise 11.3 — The binomial distribution

1 a $Y \sim \text{Bi}\left(5, \frac{3}{7}\right)$

$\Pr(Y = 0) = \left(\frac{4}{7}\right)^5 = \frac{1024}{16\,807} = 0.0609$

$\Pr(Y = 1) = 5\left(\frac{4}{7}\right)^4\left(\frac{3}{7}\right) = \frac{3840}{16\,807} = 0.2285$

$\Pr(Y = 2) = 10\left(\frac{4}{7}\right)^3\left(\frac{3}{7}\right)^2 = \frac{5760}{16\,807} = 0.3427$

$\Pr(Y = 3) = 10\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)^3 = \frac{4320}{16\,807} = 0.2570$

$\Pr(Y = 4) = 5\left(\frac{4}{7}\right)\left(\frac{3}{7}\right)^4 = \frac{1620}{16\,807} = 0.0964$

$\Pr(Y = 5) = \left(\frac{3}{7}\right)^5 = \frac{243}{16\,807} = 0.0145$

y	0	1	2	3	4	5
$\Pr(Y = y)$	0.0609	0.2285	0.3427	0.2570	0.0964	0.0145

b $\Pr(Y \le 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3)$

$= 0.0609 + 0.2285 + 0.3427 + 0.2570$

$= 0.8891$

c $\Pr(Y \ge 1 \mid Y \le 3) = \frac{\Pr(Y \ge 1) \cap \Pr(Y \le 3)}{\Pr(Y \le 3)}$

$= \frac{\Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3)}{0.8891}$

$= \frac{0.2285 + 0.3427 + 0.2570}{0.8891}$

$= \frac{0.8282}{0.8891}$

$= 0.9315$

d $\Pr(\text{Miss, Bulls-eye, Miss, Miss}) = \frac{4}{7} \times \frac{3}{7} \times \frac{4}{7} \times \frac{4}{7} = 0.0800$

2 a $Y \sim \text{Bi}(10,\ 0.42)$

$\Pr(Z=0) = (0.58)^{10} = 0.0043$

$\Pr(Z=1) = 10(0.58)^{9}(0.42) = 0.0312,$

$\Pr(Z=2) = 45(0.58)^{8}(0.42)^{2} = 0.1017$

$\Pr(Z=3) = 120(0.58)^{7}(0.42)^{3} = 0.1963$

$\Pr(Z=4) = 210(0.58)^{6}(0.42)^{4} = 0.2488$

$\Pr(Z=5) = 252(0.58)^{5}(0.42)^{5} = 0.2162$

$\Pr(Z=6) = 210(0.58)^{4}(0.42)^{6} = 0.1304$

$\Pr(Z=7) = 4120(0.58)^{3}(0.42)^{7} = 0.0540$

$\Pr(Z=8) = 45(0.58)^{2}(0.42)^{8} = 0.0147$

$\Pr(Z=9) = 10(0.58)(0.42)^{9} = 0.0024$

$\Pr(Z=10) = (0.42)^{10} = 0.0002$

b

z	0	1	2	3	4	5	6	7	8	9	10
$\Pr(Z=z)$	0.0043	0.0312	0.1017	0.1963	0.2488	0.2162	0.1304	0.0540	0.0147	0.0024	0.0002

c
$$\begin{aligned}\Pr(Z \ge 5 \mid Z \le 8) &= \frac{\Pr(Z \ge 5) \cap \Pr(Z \le 8)}{\Pr(Z \le 8)} \\ &= \frac{\Pr(Z=5)+\Pr(Z=6)+\Pr(Z=7)+\Pr(Z=8)}{1-\left(\Pr(Z=9)+\Pr(Z=10)\right)} \\ &= \frac{0.2162+0.1304+0.0540+0.0147}{1-(0.0024+0.0002)} \\ &= \frac{0.4153}{1-0.0026} \\ &= 0.4164\end{aligned}$$

3 a $X \sim \text{Bi}\left(25,\ \frac{1}{6}\right)$

$\text{E}(X) = np = 25 \times \frac{1}{6} = 4\frac{1}{6} \simeq 4.1667$

b $\text{Var}(X) = npq = 25 \times \frac{1}{6} \times \frac{5}{6} = 3\frac{17}{36} \simeq 3.472$

$\text{SD}(X) = \sqrt{3.472} = 1.8634$

4 a $\text{E}(Z) = np = 32.535$

$\text{Var}(Z) = npq = np(1-p) = 9.02195$

Re-iterating, we have

$np = 32.535$....................(1)

$np(1-p) = 9.02195$........(2)

$(2) \div (1)$

$$\begin{aligned}\frac{np(1-p)}{np} &= \frac{9.02195}{32.535} \\ 1-p &= 0.2773 \\ 1-0.2773 &= p \\ 0.7227 &= p\end{aligned}$$

b Substitute $p = 0.7227$ into (1):

$$\begin{aligned}0.7227n &= 32.535 \\ n &= \frac{32.535}{0.7227} = 45\end{aligned}$$

5 $X \sim \text{Bi}(n,\ 0.2)$

$$\begin{aligned}\Pr(X \ge 1) &\ge 0.85 \\ 1-\Pr(X=0) &\ge 0.85 \\ 1-0.8^{n} &\ge 0.85 \\ 1-0.85 &\ge 0.8^{n} \\ n &\ge 8.50\end{aligned}$$

Thus 9 tickets would be required.

6 $X \sim \text{Bi}(n,\ 0.33)$

$$\begin{aligned} \Pr(X \geq 1) &> 0.9 \\ 1-\Pr(X=0) &> 0.9 \\ 1-0.67^n &> 0.9 \\ 1-0.9 &> 0.67^n \\ n &> 5.75 \end{aligned}$$

They need to play 6 games.

7 $X \sim \text{Bi}(15,\ 0.62)$

a $\Pr(X=10) = {}^{10}C_{10}(0.62)^{10}(0.38)^0 = 0.1997$

b $\Pr(X \geq 10) = 0.4665$

c $\Pr(X < 4 \mid X \leq 8) = \dfrac{\Pr(X<4)}{\Pr(X \leq 8)}$

$= \dfrac{0.0011}{0.3295}$

$= 0.0034$

8 $X \sim \text{Bi}(15,\ 0.45)$

a $\text{E}(X) = np = 15 \times 0.45 = 6.75$

b $\Pr(X=4) = 0.0780$

c $\Pr(X \leq 8) = 0.8182$

d If T = buys a ticket and N = does not buy a ticket

$\Pr(T,T,N,N) = (0.45)^2 \times (0.55)^2 = 0.0613$

9 $X \sim \text{Bi}(8,\ 0.63)$

a

x	0	1	2	3	4	5	6	7	8
$\Pr(X=x)$	0.0004	0.0048	0.0285	0.0971	0.2067	0.2815	0.2397	0.1166	0.0248

b $\Pr(X \leq 7) = 1 - \Pr(X=8) = 1 - 0.0248 = 0.9752$

c $\Pr(X \geq 3 \mid X \leq 7) = \dfrac{\Pr(X \geq 3) \cap \Pr(X \leq 7)}{\Pr(X \leq 7)}$

$= \dfrac{\Pr(3 \leq X \leq 7)}{0.9752}$

$= \dfrac{0.9416}{0.9752}$

$= 0.9655$

d $\Pr(B', B, B, B, B, B) = 0.37 \times 0.63^5 = 0.0367$

10 a $X \sim \text{Bi}(45,\ 0.72)$

i $\text{E}(X) = np = 45 \times 0.72 = 32.4$

ii $\text{Var}(Z) = np(1-p) = 45 \times 0.72 \times 0.28 = 9.072$

b $Y \sim \text{Bi}\left(100,\ \dfrac{1}{5}\right)$

i $\text{E}(Y) = np = 100 \times \dfrac{1}{5} = 20$

ii $\text{Var}(Y) = np(1-p) = 100 \times \dfrac{1}{5} \times \dfrac{4}{5} = 16$

c $Z \sim \text{Bi}\left(72,\ \dfrac{2}{9}\right)$

i $\text{E}(Z) = np = 72 \times \dfrac{2}{9} = 16$

ii $\text{Var}(Z) = np(1-p) = 72 \times \dfrac{2}{9} \times \dfrac{7}{9} = 12\dfrac{4}{9} \simeq 12.\dot{4}$

11 $Z \sim \text{Bi}(7,\ 0.32)$

a

z	0	1	2	3	4	5	6	7
$\Pr(Z=z)$	0.0672	0.2215	0.3127	0.2452	0.1154	0.0326	0.0051	0.0003

b $\text{E}(Z) = np = 7 \times 0.32 = 2.24$

$\text{Var}(Z) = np(1-p) = 7 \times 0.32 \times 0.68 = 1.5232$

c $\text{SD}(Z)=\sqrt{1.5232}=1.2342$

$\mu-2\sigma=2.24-2(1.2342)=-0.2284$

$\mu+2\sigma=2.24+2(1.2342)=4.7084$

$$\begin{aligned}\Pr(\mu-2\sigma\le Z\le\mu+2\sigma)&=\Pr(-0.2284\le Y\le 4.7084)\\&=1-\left(\Pr(X=5)+\Pr(X=6)+\Pr(X=7)\right)\\&=1-(0.0326+0.0051+0.0003)\\&=0.9623\end{aligned}$$

12 a Let X be the number of females on the executive.

$X\sim \text{Bi}\left(12,\ \frac{1}{2}\right)$

$\Pr(X\ge 8)=0.1938$

b Let Y be the number of females on the executive.

$Y\sim \text{Bi}(12,\ 0.58)$

$\Pr(Y\ge 8)=0.3825$

13 a Let $X\sim \text{Bi}(n,\ p)$

$\text{E}(X)=np=9.12$........................(1)

$\text{Var}(X)=np(1-p)=5.6544$.....(2)

(2) ÷ (1)

$$\frac{np(1-p)}{np}=\frac{5.6544}{9.12}$$

$1-p=0.62$

$1-0.62=p$

$0.38=p$

b Substitute $p=0.38$ into (1):

$n\times 0.38=9.12$

$$n=\frac{9.12}{0.38}=24$$

14 a Let $X\sim \text{Bi}(n,\ p)$

$\text{E}(X)=np=3.8325$........................(1)

$\text{Var}(X)=np(1-p)=3.4128$.........(2)

(2) ÷ (1)

$$\frac{np(1-p)}{np}=\frac{3.4128}{3.8325}$$

$1-p=0.8905$

$1-0.8905=p$

$0.1095=p$

b Substitute $p=0.1095$ into (1):

$n\times 0.1095=3.8325$

$$n=\frac{3.8325}{0.1095}=35$$

15 a Let $X\sim \text{Bi}(16,\ p)$

$\text{E}(X)=np=10.16$

$\text{E}(X)=16p=10.16$

$$p=\frac{10.16}{16}=0.635$$

b $\text{Var}(X)=np(1-p)$

$\text{Var}(X)=16(0.635)(0.365)$

$\text{Var}(X)=3.7084$

$\text{SD}(X)=\sqrt{3.7084}=1.9257$

16 $X\sim \text{Bi}\left(10,\ \frac{1}{7}\right)$

a $\text{E}(X)=np=10\times\frac{1}{7}=1.4286$

$\text{Var}(X)=np(1-p)=10\times\frac{1}{7}\times\frac{6}{7}=1.2245$

b $\text{SD}(X)=\sqrt{1.2245}=1.1066$

$\mu-2\sigma=1.4286-2(1.1066)=-0.7846$

$\mu+2\sigma=1.4286+2(1.1066)=3.6410$

$$\begin{aligned}\Pr(\mu-2\sigma\le X\le\mu+2\sigma)&=\Pr(-0.7846\le Y\le 3.6410)\\&=\Pr(0\le Y\le 3)\\&=0.9574\end{aligned}$$

17 $X\sim\text{Bi}(n,\ 0.75)$

$$\begin{aligned}\Pr(X\ge 1)&\ge 0.95\\1-\Pr(X=0)&\ge 0.95\\1-0.25^n&\ge 0.95\\1-0.95&\ge 0.25^n\\n&\ge 2.16\end{aligned}$$

Thus 3 shots would be required.

18 **a** $X\sim\text{Bi}(12,0.2)$

$\Pr(x=3)=0.2362$

b $Y\sim\text{Bi}(14,0.2362)$

$\Pr(Y\ge 6)=0.0890$

Exercise 11.4 — Applications

1 $Y\sim\text{Bi}(10,\ 0.3)$

a $\Pr(Y\ge 7)=0.0106$

b $\text{E}(Y)=np=10\times 3=3$

$\text{Var}(Y)=np(1-p)=10\times 0.3\times 0.7=2.1$

$\text{SD}(Y)=\sqrt{2.1}=1.4491$

2 $Z\sim\text{Bi}(5,\ 0.01)$

a $\Pr(Z\le 3)=0.951+0.0480+0.0010+0\times 3=1$

b **i** $\text{E}(Z)=np=5\times 0.01=0.05$

ii $\text{Var}(Z)=np(1-p)=5\times 0.01\times 0.99=0.0495$

$\text{SD}(Z)=\sqrt{0.0495}=0.2225$

c $\mu-2\sigma=0.05-2(0.0495)=-0.049$

$\mu+2\sigma=0.05+2(0.0495)=0.149$

$$\begin{aligned}\Pr(\mu-2\sigma\le Z\le\mu+2\sigma)&=\Pr(-0.049\le Z\le 0.149)\\&=\Pr(Z=0)\\&=0.9510\end{aligned}$$

3 $X\sim\text{Bi}(15,\ 0.3)$

a $\Pr(X\le 5)=0.7216$

b **i** $\text{E}(X)=np=15\times 0.3=4.5$

ii $\text{Var}(X)=np(1-p)=15\times 0.3\times 0.7=3.15$

$\text{SD}(X)=\sqrt{3.15}=1.7748$

4 $Y\sim\text{Bi}(6,\ 0.08)$

a **i** $\text{E}(Y)=np=6\times 0.08=0.48$

ii $\text{Var}(Y)=np(1-p)=6\times 0.08\times 0.92=0.4416$

$\text{SD}(Y)=\sqrt{0.4416}=0.6645$

b $\mu-2\sigma=0.48-2(0.6645)=-0.849$

$\mu+2\sigma=0.48+2(0.6645)=1.809$

$$\begin{aligned}\Pr(\mu-2\sigma\le Y\le\mu+2\sigma)&=\Pr(-0.849\le Y\le 1.809)\\&=\Pr(Y=0)+P(Y=1)\\&=0.6064+0.3164\\&=0.9228\end{aligned}$$

c $Z\sim\text{Bi}(6,\ 0.01)$

i $\text{E}(Z)=np=6\times 0.01=0.06$

ii $\text{Var}(Z)=np(1-p)=6\times 0.01\times 0.99=0.0594$

$\text{SD}(Z)=\sqrt{0.0594}=0.2437$

d $\mu-2\sigma=0.06-2(0.2437)=-0.4274$

$\mu+2\sigma=0.06+2(0.2437)=0.5474$

$$\begin{aligned}\Pr(\mu-2\sigma\le Z\le\mu+2\sigma)&=\Pr(-0.4274\le Z\le 0.5474)\\&=\Pr(Z=0)\\&=0.9415\end{aligned}$$

e There is a probability of 0.9228 that a maximum of 1 male will be colour blind, whereas there is a probability of 0.9415 that no females will be colourblind.

5 $Z\sim\text{Bi}(12,\ 0.85)$

a $\Pr(Z\le 8)=0.0922$

b
$$\begin{aligned}\Pr(Z\ge 5\mid Z\le 8)&=\frac{\Pr(Z\ge 5)\cap\Pr(Z\le 8)}{\Pr(Z\le 8)}\\&=\frac{\Pr(5\le Z\le 8)}{0.0922}\\&=\frac{0.092213}{0.0922}\\&=0.9992\end{aligned}$$

c **i** $\text{E}(Z)=np=12\times 0.85=10.2$

ii $\text{Var}(Z)=np(1-p)+12\times 0.85\times 0.15=1.53$

$\text{SD}(Z)=\sqrt{1.53}=1.2369$

6 Let Z be the number of offspring with genotype XY.

$Z\sim\text{Bi}\left(7,\ \frac{1}{2}\right)$

$\Pr(Z=6)={}^7C_6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^6=\frac{7}{64}=0.0547$

7 Let Z be the number of chips that fail the test.

$Z\sim\text{Bi}(250,\ 0.02)$

$\Pr(Z=7)={}^{250}C_7(0.98)^{243}(0.02)^7=0.1051$

8 **a** $X\sim\text{Bi}(3,\ p)$

$\Pr(X=0)=(1-p)^3,\quad \Pr(X=1)=3(1-p)^2p,$

$\Pr(X=2)=3(1-p)p^2,\quad \Pr(X=3)=p^3$

x	0	1	2	3
$\Pr(X=x)$	$(1-p)^3$	$3(1-p)^2p$	$3(1-p)p^2$	p^3

b $\Pr(X=0)=\Pr(X=1)$

$$\begin{aligned}(1-p)^3&=3(1-p)^2p\\(1-p)^3-3(1-p)^2p&=0\\(1-p)^2(1-p-3p)&=0\\(1-p)^2(1-4p)&=0\\(1-p)(1+p)(1-4p)&=0\end{aligned}$$

$1-p=0,\ 1+p=0$ or $1-4p=0$

$p=1 \qquad p=-1 \qquad 1=4p$

$p=\frac{1}{4}$

$\therefore p=\frac{1}{4}$ because $0<p<1$

c **i** $\mu=\text{E}(X)=np=3\times\frac{1}{4}=\frac{3}{4}$

ii $\text{Var}(X)=np(1-p)=3\times\frac{1}{4}\times\frac{3}{4}=\frac{9}{16}$

$\sigma=\text{SD}(X)=\sqrt{\frac{9}{16}}=\frac{3}{4}$

9 **a** Let X be the number of people who suffer from anaemia.

$X\sim\text{Bi}(100,\ 0.013)$

$\Pr(X\ge 5)=0.0101$

b $\Pr(X=4 \mid X<10)=\dfrac{\Pr(X=4)}{\Pr(X<10)}$

$\Pr(X=4)=0.0319$

$\Pr(X<10)=0.9999$

$\Pr(X=4 \mid X<10)=\dfrac{\Pr(X=4)}{\Pr(X<10)}=\dfrac{0.0319}{0.9999}=0.0319$

c $\mu=np=100\times 0.013=1.3$

$\text{Var}(X)=np(1-p)=100\times 0.013\times 0.987=1.2831$

$\sigma=\text{SD}(X)=\sqrt{1.2831}=1.1327$

$\mu-2\sigma=1.3-2(1.1327)=-0.9654$

$\mu+2\sigma=1.3+2(1.1327)=3.5654$

$$\begin{aligned}\Pr(\mu-2\sigma\le X\le\mu+2\sigma)&=\Pr(-0.9654\le X\le 3.5654)\\&=\Pr(0\le X\le 3)\\&=0.9580\end{aligned}$$

This means there is a 96% chance that a maximum of 3 people per 100 will suffer from anaemia.

10 $X\sim\text{Bi}(20, 0.2)$

a $\Pr(X\ge 10)=0.0026$

b $\Pr(X\ge 10)=1\times1\times1\times1\times\Pr(X\ge 6)$
$=0.0489$

11 $X\sim\text{Bi}(6,0.7)$ $X=$ kicking 50 m

a **i** $\Pr(YYYNNN)=(0.7)^3(0.3)^3$
$=0.0093$

ii $\Pr(X=3)={}^6C_3(0.7)^3(0.3)^3$
$=0.1852$

iii $\Pr(X\ge 3 \mid \text{1st kick} > 50\text{ m})=\dfrac{0.7\times\Pr(X\ge 2)}{0.7}$
$=\dfrac{0.678454}{0.7}$
$=0.9692$

b $X\sim\text{Bi}(n,0.95)$

$$\begin{aligned}\Pr(X\ge 1)&\ge 0.95\\1-\Pr(X=0)&\ge 0.95\\1-0.3^n&\ge 0.95\\1-0.95&\ge 0.3^n\\n&\ge 2.48\end{aligned}$$

Therefore, 3 footballers are needed.

12 $X\sim\text{Bi}(12,0.85)$

a $\Pr(X\ge 9)=0.9078$

b $\Pr(3M,9G)=(0.15)^3(0.85)^9=0.0008$

c $\Pr(X=10 \mid \text{last 9 are goals})=\dfrac{\Pr(X=1)\times\text{last 9 are goals}}{\Pr(\text{last 9 are goals})}$
$=\dfrac{0.057375\times(0.85)^9}{(0.85)^9}$
$=0.0574$

13 $X\sim\text{Bi}(n,0.08)$

$$\begin{aligned}\Pr(X\ge 2)&>0.8\\1-(\Pr(X=0)+\Pr(X=1))&>0.8\\1-0.8&>\Pr(X=0)+\Pr(X=1)\\0.2&>(0.92)^n+n(0.92)^{n-1}(0.08)\end{aligned}$$

$n=36.4179$ so at least 37 tickets must be bought.

14 $X\sim\text{Bi}(10,\ p)$

$$\begin{aligned}\Pr(X\le 8)&=1-\Pr(X\ge 9)\\&=1-(\Pr(X=9)+\Pr(X=10))\\&=1-\left(10(1-p)p^9+p^{10}\right)\end{aligned}$$

If $\Pr(X\le 8)=0.9$ thus solve $0.9=1-\left(10(1-p)p^9+p^{10}\right)$

$0.9=1-\left(10(1-p)p^9+p^{10}\right)$

$p^{10}+10(1-p)p^9-0.1=0$

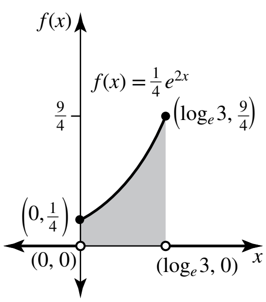
$p=0.6632$

Topic 12 — Continuous probability distributions

Exercise 12.2 — Continuous random variables and probability functions

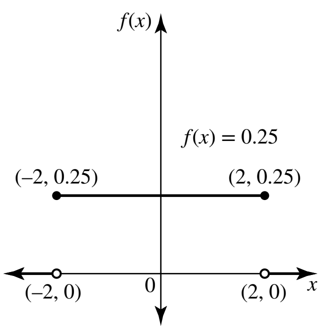
1 a $f(x)=\begin{cases}\frac{1}{4}e^{2x}, & 0 \le x \le \log_e 3 \\ 0, & \text{elsewhere}\end{cases}$

$$\begin{aligned}A &= \int_0^{\log_e 3} \frac{1}{4}e^{2x}\,dx \\ &= \left[\frac{1}{4}\times\frac{1}{2}e^{2x}\right]_0^{\log_e 3} \\ &= \left[\frac{1}{8}e^{2x}\right]_0^{\log_e 3} \\ &= \frac{1}{8}e^{2\log_e 3}-\frac{1}{8}e^{0} \\ &= \frac{1}{8}e^{\log_e 9}-\left(\frac{1}{8}\times 1\right) \\ &= \frac{1}{8}\left(e^{\log_e 9}-1\right) \\ &= \frac{1}{8}(9-1) \\ &= 1\end{aligned}$$



This is a probability function as the area is 1 units^2.

b



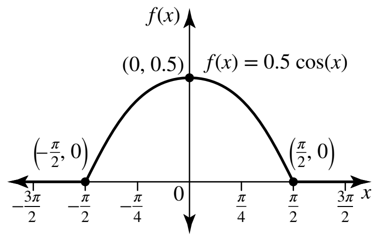
$$\int_{-2}^{2} 0.25\,dx = [0.25x]_{-2}^{2}$$

$$\int_{-2}^{2} 0.25\,dx = 0.25(2)-0.25(-2)$$

$$\int_{-2}^{2} 0.25\,dx = 0.5+0.5=1$$

This is a probability density function.

2 a



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos(x)\,dx = \left[\frac{1}{2}\sin(x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

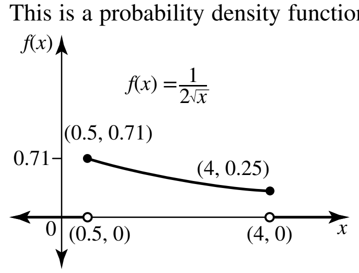
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos(x)\,dx = \frac{1}{2}\sin\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos(x)\,dx = \frac{1}{2}+\frac{1}{2}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos(x)\,dx = 1$$

This is a probability density function.

b



$$\int_{0.25}^{4} 0.5x^{-0.5}\,dx = \left[x^{0.5}\right]_{0.5}^{4}$$

$$\int_{0.25}^{4} 0.5x^{-0.5}\,dx = \sqrt{4}-\sqrt{0.5}$$

$$\int_{0.25}^{4} 0.5x^{-0.5}\,dx = 2-0.7071$$

$$\int_{0.25}^{4} 0.5x^{-0.5}\,dx = 1.2929$$

This is not a probability density function.

3

$$\begin{aligned}\int_1^3 n(x^3-1)\,dx &= 1 \\ n\left[\frac{1}{4}x^4-x\right]_1^3 &= 1 \\ n\left(\left(\frac{1}{4}(3)^4-3\right)-\left(\frac{1}{4}(1)^4-1\right)\right) &= 1 \\ n\left(\frac{81}{4}-3-\frac{1}{4}+1\right) &= 1 \\ 18n &= 1 \\ n &= \frac{1}{18}\end{aligned}$$

4
$$\int_{-2}^{0}(-ax)\,dx+\int_{0}^{3}(2ax)\,dx=1$$
$$\left[-\frac{1}{2}ax^2\right]_{-2}^{0}+\left[ax^2\right]_0^3=1$$
$$\left(0-\left(-\frac{1}{2}a(-2)^2\right)\right)+\left(a(3)^2-0\right)=1$$
$$2a+9a=1$$
$$11a=1$$
$$a=\frac{1}{11}$$

5 a i $\Pr(X\le 2)=\frac{10+26}{100}=\frac{36}{100}=\frac{9}{25}$

ii $\Pr(X>4)=\frac{16}{100}=\frac{4}{25}$

b i $\Pr(1<X\le 4)=\frac{26+28+20}{100}=\frac{74}{100}=\frac{37}{50}$

ii $\Pr(X>1\mid X\le 4)=\frac{P(X>1\cap X\le 4)}{P(X\le 4)}=\frac{P(1<X\le 4)}{P(X\le 4)}$

$=\frac{37}{50}\div\frac{84}{100}=\frac{37}{50}\times\frac{100}{84}=\frac{37}{42}$

6 a Number of batteries is 100.

b $\Pr(X>45)=\frac{29}{100}$

c $\Pr(15<X\le 60)=\frac{82}{100}=\frac{41}{50}$

d $\Pr(X>60)=\frac{3}{100}$

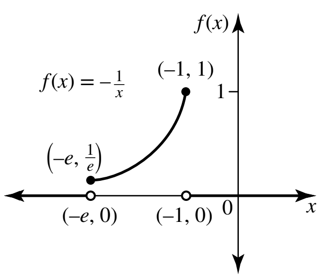
7 a 200 shot-put throws were measured.

b i $\Pr(X>0.5)=\frac{200-75}{100}=\frac{125}{200}=\frac{5}{8}$

ii $\Pr(1<X\le 2)=\frac{62}{200}=\frac{31}{100}$

c $\Pr(X<0.5\mid X<1)=\frac{\Pr(0.5<X<1)}{\Pr(X<1)}=\frac{63}{200}\div\frac{138}{200}=\frac{63}{200}\times\frac{200}{138}=\frac{63}{138}=\frac{21}{46}$

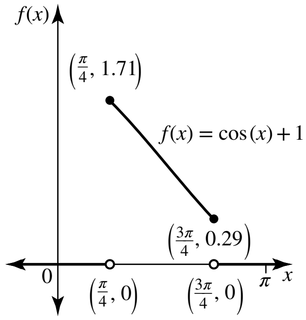
8 a



$$\int_{-e}^{-1}-\frac{1}{x}\,dx$$
$$=\left[-\log_e(x)^{-1}\right]_{-e}^{-1}$$
$$=\left[\log_e\left(\frac{1}{x}\right)\right]_{-e}^{-1}$$
$$=1\text{ units}^2$$

This is a probability density function.

b



$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos(x)+1)\,dx$$

$$= \left[\sin(x)+x\right]_{\frac{\pi}{3}}^{\frac{3\pi}{4}}$$

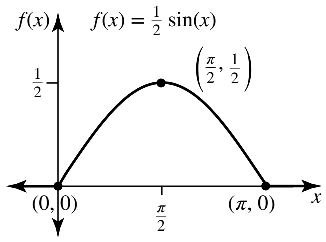
$$= \left(\sin\left(\frac{3\pi}{4}\right)+\frac{3\pi}{4}\right)-\left(\sin\left(\frac{\pi}{4}\right)+\frac{\pi}{4}\right)$$

$$= 0.7071+\frac{3\pi}{4}-0.7071-\frac{\pi}{4}$$

$$= \frac{\pi}{2}\text{ units}^2$$

This is not a probability density function.

c



$$\int_0^{\pi} \frac{1}{2}\sin(x)\,dx$$

$$= \left[-\frac{1}{2}\cos(x)\right]_0^{\pi}$$

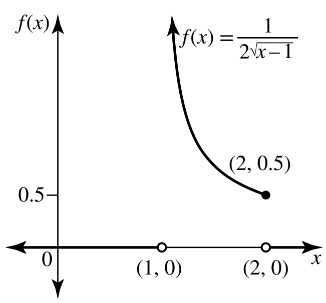
$$= \left(-\frac{1}{2}\cos(\pi)\right)-\left(-\frac{1}{2}\cos(0)\right)$$

$$= \frac{1}{2}+\frac{1}{2}$$

$$= 1\text{ units}^2$$

This is a probability density function.

d



$$\int_1^2 \frac{1}{2\sqrt{x-1}}\,dx = 1\text{ units}^2$$

This is a probability density function.

9

$$\int_{0.25}^{1.65} c\,dx = 1$$

$$[cx]_{0.25}^{1.65} = 1$$

$$1.65c - 0.25c = 1$$

$$1.4c = 1$$

$$c = \frac{1}{1.4}$$

$$c = \frac{5}{7}$$

10

$$\int_{-1}^{5} f(z)\,dz = 1$$

$$A_{triangle} = 1$$

$$\frac{1}{2}\times 6\times z = 1$$

$$3z = 1$$

$$z = \frac{1}{3}$$

11 a

$$\int_0^2 m(6-2x)\,dx = 1$$

$$m\int_0^2 (6-2x)\,dx = 1$$

$$m\left[6x-x^2\right]_0^2 = 1$$

$$m\left(6(2)-(2)^2-6(0)+0^2\right) = 1$$

$$8m = 1$$

$$m = \frac{1}{8}$$

b

$$\int_0^{\infty} me^{-2x} = 1$$

$$m\int_0^{\infty} e^{-2x} = 1$$

$$m\left[-\frac{1}{2e^{2x}}\right]_0^{\infty} = 1$$

$$m\left(0+\frac{1}{2}\right) = 1$$

$$\frac{1}{2}m = 1$$

$$m = 2$$

c

$$\int_0^{\log_e(3)} me^{2x}\,dx = 1$$

$$m\int_0^{\log_e(3)} e^{2x}\,dx = 1$$

$$m\left[\frac{1}{2}e^{2x}\right]_0^{\log_e(3)} = 1$$

$$m\left(\frac{1}{2}e^{2\log_e(3)}-\frac{1}{2}e^0\right) = 1$$

$$m\left(\frac{1}{2}e^{\log_e(9)}-\frac{1}{2}\right) = 1$$

$$m\left(\frac{9}{2}-\frac{1}{2}\right) = 1$$

$$4m = 1$$

$$m = \frac{1}{4}$$

12

$$\int_0^3 \left(x^2 + 2kx + 1\right) dx = 1$$

$$\left[\frac{1}{3}x^3 + kx^2 + x\right]_0^3 = 1$$

$$\left(\frac{1}{3}(3)^3 + k(3)^2 + 3\right) - 0 = 1$$

$$9 + 9k + 3 = 1$$

$$9k = -11$$

$$k = -\frac{11}{9}$$

13

$$\int_2^a f(x)\,dx = 1$$

$$\int_2^a \frac{1}{2}\log_e\left(\frac{x}{2}\right) dx = 1$$

$$\left[\frac{1}{2}(x\log_e(x) - 1.6934x)\right]_2^a = 1$$

$$\frac{1}{2}\left(a\log_e(a) - 1.6934a\right) - \frac{1}{2}\left(2\log_e(2) - 1.6934(2)\right) = 1$$

$$a\log_e(a) - 1.6934a - 2\log_e(2) + 1.6934(2) = 2$$

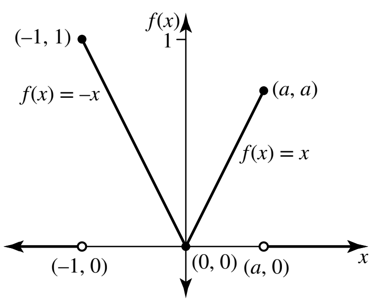
$$a\log_e(a) - 1.6934a - 1.3863 + 3.3868 - 2 = 0$$

$$a\log_e(a) - 1.6934a + 0.0005 = 0$$

$$a = 4.2521,\ 5.4374$$

Only $a = 5.4374 = 2e$ gives the answer of 1.

14 a



$$\int_{-1}^a f(x)\,dx$$

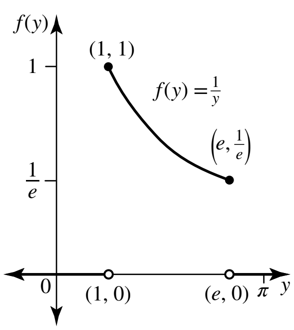
$$= \int_{-1}^0 -x\,dx + \int_0^a x\,dx$$

$$= \left[-\frac{1}{2}x^2\right]_{-1}^0 + \left[\frac{1}{2}x^2\right]_0^a$$

$$= -\frac{1}{2}(0)^2 + \frac{1}{2}(-1)^2 + \frac{1}{2}a^2 - \frac{1}{2}(0)^2$$

$$= \frac{a^2 + 1}{2}$$

b



$$\int_1^e f(y)\,dy$$
$$=\int_1^e \frac{1}{y}\,dy$$
$$=\left[\log_e(y)\right]_1^e$$
$$=\log_e(e)-\log_e(1)$$
$$=1$$

c $\int_{-1}^{a} f(x)\,dx = \int_1^e f(y)\,dy$

$$\frac{a^2+1}{2}=1$$
$$a^2+1=2$$
$$a^2=1$$
$$a=\pm 1$$
$$a=1 \text{ since } a>0$$

15 $\int_0^{\frac{\pi}{12}} n\sin(3x)\cos(3x)\,dx=1$

$$n\int_0^{\frac{\pi}{12}} \sin(3x)\cos(3x)\,dx=1$$
$$0.08\dot{3}n=1$$
$$n=12$$

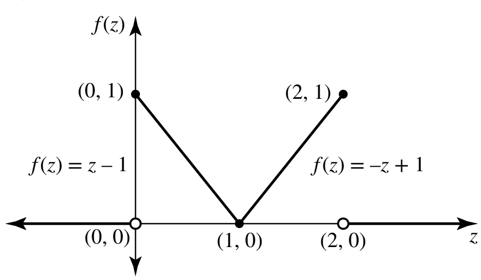
16 a $\int_1^a \log_e(x)\,dx=1$

$$\left[x\log_e(x)-x\right]_1^a=1$$
$$\left(a\log_e(a)-a\right)-\left(\log_e(1)-1\right)=1$$
$$a\log_e(a)-a+1=1$$
$$a\log_e(a)-a=0$$
$$a\log_e(a)=a$$
$$\log_e(a)=1$$
$$e^1=a$$
$$a=e$$

b As $f(x)\ge 0$ and $\int_1^e f(x)\,dx=1$, this is a probability density function.

Exercise 12.3 — The continuous probability density function

1 a



b $\Pr(Z<0.75)=\int_0^{0.75}(-z+1)\,dx$

$$\Pr(Z<0.75)=\left[-\frac{1}{2}z^2+z\right]_0^{0.75}$$
$$\Pr(Z<0.75)=\left(-\frac{1}{2}\left(\frac{3}{4}\right)^2+\frac{3}{4}\right)-0$$
$$\Pr(Z<0.75)=\frac{15}{32}$$

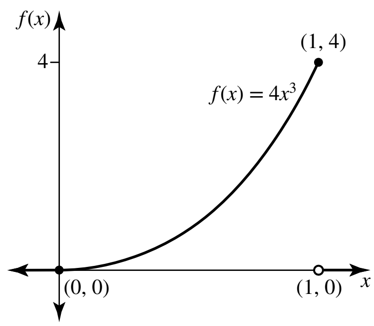
c $\Pr(Z>0.5)=\int_{0.5}^{2} f(z)\,dz$

$$\Pr(Z>0.5)=\int_{0.5}^{1}(1-z)\,dz+\int_1^2(z-1)\,dz$$
$$\Pr(Z>0.5)=\left[z-\frac{1}{2}z^2\right]_{0.5}^{1}+\frac{1}{2}$$
$$\Pr(Z>0.5)=\left(1-\frac{1}{2}(1)^2\right)-\left(\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)^2\right)+\frac{1}{2}$$
$$\Pr(Z>0.5)=\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{8}\right)+\frac{1}{2}$$
$$\Pr(Z>0.5)=1-\frac{3}{8}$$
$$\Pr(Z>0.5)=\frac{5}{8}$$

2 a $\int_0^a 4x^3\,dx=1$

$$\left[x^4\right]_0^a=1$$
$$a^4-0=1$$
$$a^4=1$$
$$a=\pm 1$$
$$a=1 \text{ since } a>0$$

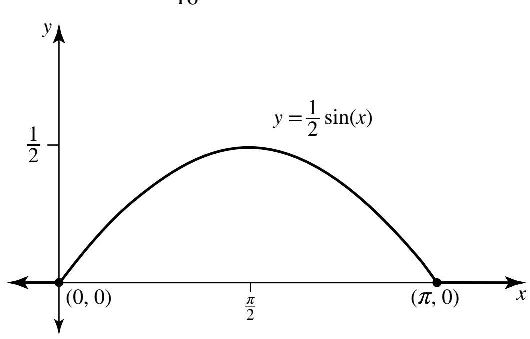
b



c $\Pr(0.5\le X\le 1)=\int_{0.5}^{1}4x^3\,dx$

$$\Pr(0.5\le X\le 1)=\left[x^4\right]_{0.5}^{1}$$
$$\Pr(0.5\le X\le 1)=1^4-\frac{1}{2}^4$$
$$\Pr(0.5\le X\le 1)=1-\frac{1}{16}$$
$$\Pr(0.5\le X\le 1)=\frac{15}{16}$$

3 a



b $\Pr\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right) = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(x)\,dx$

$= \frac{1}{2}[-\cos(x)]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$= \frac{1}{2}\left(-\cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right)$

$= \frac{1}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{2}$

c $\Pr\left(X > \frac{\pi}{4} \mid X < \frac{3\pi}{4}\right) = \dfrac{\Pr\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right)}{\Pr\left(X < \frac{3\pi}{4}\right)}$

$\Pr\left(X < \frac{3\pi}{4}\right) = \frac{1}{2}\int_{0}^{\frac{3\pi}{4}} \sin(x)\,dx$

$= \frac{1}{2}[-\cos(x)]_{0}^{\frac{3\pi}{4}}$

$= \frac{1}{2}\left(-\cos\left(\frac{3\pi}{4}\right) + \cos(0)\right)$

$= \frac{1}{2}\left(\frac{\sqrt{2}}{2} + 1\right)$

$= \frac{\sqrt{2}}{4} + \frac{1}{2}$

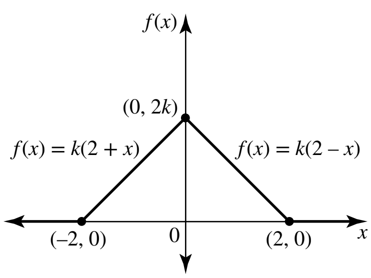
$\Pr\left(X > \frac{\pi}{4} \mid X < \frac{3\pi}{4}\right) = \dfrac{\Pr\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right)}{\Pr\left(X < \frac{3\pi}{4}\right)}$

$= \dfrac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}+2}{4}}$

$= \dfrac{2\sqrt{2}}{\sqrt{2}+2}$

$= 2\sqrt{2} - 2$

4 a



b $A = \frac{1}{2}bh$

$1 = \frac{1}{2} \times 4 \times 2 \times k$

$1 = 4k$

$k = \frac{1}{4}$

c $\Pr(-1 \le X \le 1) = \int_{-1}^{0} \frac{1}{4}(2 + x)\,dx + \int_{0}^{1} \frac{1}{4}(2 - x)\,dx$

$\Pr(-1 \le X \le 1) = \int_{-1}^{0} \left(\frac{1}{2} + \frac{1}{4}x\right)dx + \int_{0}^{1} \left(\frac{1}{2} - \frac{1}{4}x\right)dx$

$\Pr(-1 \le X \le 1) = \left[\frac{1}{2}x + \frac{1}{8}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x - \frac{1}{8}x^2\right]_{0}^{1}$

$\Pr(-1 \le X \le 1) = 0 - \left(\frac{1}{2}(-1) + \frac{1}{8}(-1)^2\right) + \left(\frac{1}{2}(1) - \frac{1}{8}(1)^2\right) - 0$

$\Pr(-1 \le X \le 1) = \left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{1}{2} - \frac{1}{8}\right)$

$\Pr(-1 \le X \le 1) = \frac{3}{8} + \frac{3}{8}$

$\Pr(-1 \le X \le 1) = \frac{3}{4}$

d $\Pr(X \ge -1 \mid X \le 1) = \dfrac{\Pr(-1 \le X \le 1)}{\Pr(X \le 1)}$

$\Pr(X \le 1) = \int_{-2}^{0} \frac{1}{4}(2 + x)\,dx + \int_{0}^{1} \frac{1}{4}(2 - x)dx$

$\Pr(X \le 1) = \left[\frac{1}{2}x + \frac{1}{8}x^2\right]_{-2}^{0} + \frac{3}{8}$

$\Pr(X \le 1) = 0 - \left(\frac{1}{2}(-2) + \frac{1}{8}(-2)^2\right) + \frac{3}{8}$

$\Pr(X \le 1) = 1 - \frac{1}{2} + \frac{3}{8}$

$\Pr(X \le 1) = \frac{1}{2} + \frac{3}{8}$

$\Pr(X \le 1) = \frac{7}{8}$

$\Pr(X \ge -1 \mid X \le 1) = \frac{3}{4} \div \frac{7}{8} = \frac{3}{4} \times \frac{8}{7} = \frac{6}{7}$

5 Let X be the amount of petrol sold in thousands of litres.

a $\int_{18}^{30} k\,dx = 1$

$[kx]_{18}^{30} = 1$

$(30k) - (18k) = 1$

$12k = 1$

$k = \frac{1}{12}$

b $\Pr(20 < X < 25) = \int_{20}^{25} \frac{1}{12}dx$

$\Pr(20 < X < 25) = \left[\frac{1}{12}x\right]_{20}^{25}$

$\Pr(20 < X < 25) = \left(\frac{1}{12}(25)\right) - \left(\frac{1}{12}(20)\right)$

$\Pr(20 < X < 25) = \frac{5}{12}$

c $\Pr(X \ge 26 | X \ge 22) = \dfrac{\Pr(X \ge 26 \cap X \ge 22)}{P(X \ge 22)}$

$\Pr(X \ge 26 | X \ge 22) = \dfrac{\Pr(X \ge 26)}{\Pr(X \ge 22)}$

$\Pr(X \ge 22) = \int_{22}^{30} \frac{1}{12}\,dx$

$\Pr(X \ge 22) = \left[\frac{1}{12}x\right]_{22}^{30}$

$\Pr(X \ge 22) = \frac{1}{12}(30) - \frac{1}{12}(22)$

$\Pr(X \ge 22) = \frac{8}{12}$

$\Pr(X \ge 22) = \frac{2}{3}$

$\Pr(X \ge 26) = \int_{26}^{30} \frac{1}{12}\,dx$

$\Pr(X \ge 26) = \left[\frac{1}{12}x\right]_{26}^{30}$

$\Pr(X \ge 26) = \frac{1}{12}(30) - \frac{1}{12}(26)$

$\Pr(X \ge 26) = \frac{4}{12}$

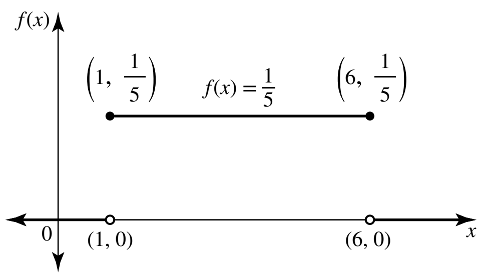
$\Pr(X \ge 26) = \frac{1}{3}$

$\dfrac{\Pr(X \ge 26)}{\Pr(X \ge 22)} = \frac{1}{3} \div \frac{2}{3}$

$\dfrac{\Pr(X \ge 26)}{\Pr(X \ge 22)} = \frac{1}{3} \times \frac{3}{2}$

$\dfrac{\Pr(X \ge 26)}{\Pr(X \ge 22)} = \frac{1}{2}$

6 a



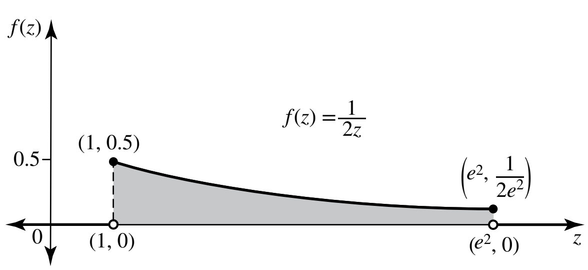
b $\Pr(2 \le X \le 5) = \int_2^5 \frac{1}{5}\,dx$

$\Pr(2 \le X \le 5) = \left[\frac{1}{5}x\right]_2^5$

$\Pr(2 \le X \le 5) = \frac{1}{5}(5) - \frac{1}{5}(2)$

$\Pr(2 \le X \le 5) = \frac{3}{5}$

7 a



b $\int_1^{e^2} f(z)\,dz$

$= \int_1^{e^2} \frac{1}{2z}\,dz$

$= \frac{1}{2}\int_1^{e^2} \frac{1}{z}\,dz$

$= \frac{1}{2}\left[\log_e(z)\right]_1^{e^2}$

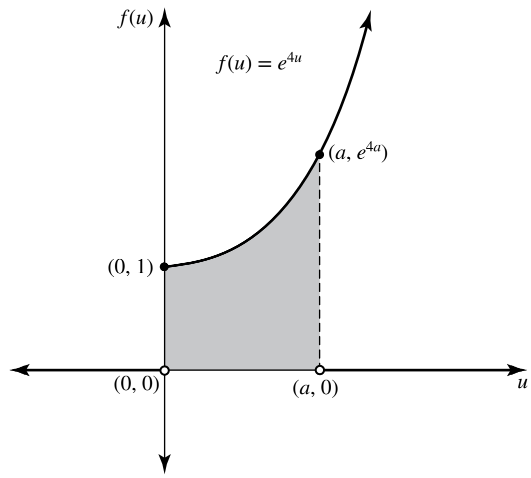
$= \frac{1}{2}\left(\log_e\left(e^2\right) - \log_e\left(1^2\right)\right)$

$= \frac{1}{2} \times 2\log_e(e)$

$= 1$

As $f(z) \ge 0$ and $\int_1^{e^2} f(z)\,dz = 1$, this is a probability density function.

c



d $\int_0^a f(u)\,du$

$= \int_0^a e^{4u}\,du$

$= \left[\frac{1}{4}e^{4u}\right]_0^a$

$= \frac{1}{4}e^{4a} - \frac{1}{4}e^0$

$= \frac{1}{4}e^{4a} - \frac{1}{4}$

$\int_1^{e^2} f(z)\,dz = \int_0^a f(u)\,du$

$1 = \frac{1}{4}e^{4a} - \frac{1}{4}$

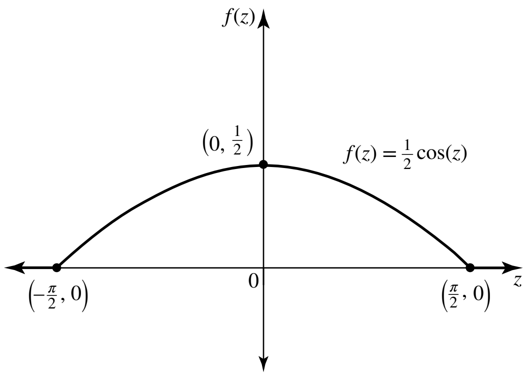
$\frac{5}{4} = \frac{1}{4}e^{4a}$

$5 = e^{4a}$

$\log_e(5) = 4a$

$\frac{1}{4}\log_e(5) = a$

8 a



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos(z)\,dz$$

$$=\left[\frac{1}{2}\sin(z)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\sin\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$$

$$=\frac{1}{2}+\frac{1}{2}$$

$$=1$$

This is a probability density function.

b $\Pr\left(-\frac{\pi}{6}\le Z\le\frac{\pi}{4}\right)=\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}}\cos(z)\,dz$

$$=\frac{1}{2}\left[\sin(z)\right]_{-\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\left[\sin\left(\frac{\pi}{4}\right)-\sin\left(-\frac{\pi}{6}\right)\right]$$

$$=\frac{1}{2}\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right)$$

$$=\frac{\sqrt{2}+1}{4}$$

9 a $\int_0^a f(u)\,du=1$

$$\int_0^a\left(1-\frac{1}{4}\left(2u-3u^2\right)\right)du=1$$

$$\int_0^a\left(1-\frac{1}{2}u+\frac{3}{4}u^2\right)du=1$$

$$\left[u-\frac{1}{4}u^2+\frac{1}{4}u^3\right]_0^a=1$$

$$\left(a-\frac{1}{4}a^2+\frac{1}{4}a^3\right)-0=1$$

$$\frac{1}{4}a^3-\frac{1}{4}a^2+a-1=0$$

$$\frac{1}{4}a^2(a-1)+(a-1)=0$$

$$(a-1)\left(\frac{1}{4}a^2+1\right)=0$$

$$a=1$$

b $\Pr(U<0.75)=\int_0^{0.75}\left(1-\frac{1}{4}\left(2u-3u^2\right)\right)du$

$\Pr(U<0.75)=\int_0^{0.75}\left(1-\frac{1}{2}u-\frac{3}{4}u^2\right)du$

$\Pr(U<0.75)=\left[u-\frac{1}{4}u^2+\frac{1}{4}u^3\right]_0^{0.75}$

$\Pr(U<0.75)=\left(0.75-\frac{1}{4}(0.75)^2+\frac{1}{4}(0.75)^3\right)-0$

$\Pr(U<0.75)=\frac{183}{256}$

c $\Pr(0.1<U<0.5)=\int_{0.1}^{0.5}\left(1-\frac{1}{4}\left(2u-3u^2\right)\right)du$

$\Pr(0.1<U<0.5)=\int_{0.1}^{0.5}\left(1-\frac{1}{2}u-\frac{3}{4}u^2\right)du$

$\Pr(0.1<U<0.5)=\left[u-\frac{1}{4}u^2+\frac{1}{4}u^3\right]_{0.1}^{0.5}$

$\Pr(0.1<U<0.5)=\left(0.5-\frac{1}{4}(0.5)^2+\frac{1}{4}(0.5)^3\right)-\left(0.1-\frac{1}{4}(0.1)^2+\frac{1}{4}(0.1)^3\right)$

$\Pr(0.1<U<0.5)=0.371$

d $\Pr(U=0.8)=0$

10 a $\Pr(X>1.2)=\int_{1.2}^{2}\left(\frac{3}{8}x^2\right)dx$

$=\frac{3}{8}\left[\frac{1}{3}x^3\right]_{1.2}^{2}$

$=\frac{3}{8}\left(\frac{1}{3}(2)^3-\frac{1}{3}\left(\frac{6}{5}\right)^3\right)$

$=\frac{3}{8}\left(\frac{8}{3}-\frac{72}{125}\right)$

$=\frac{98}{125}$

b $\Pr(X>1|X>0.5)=\frac{\Pr(X>1)}{\Pr(X>0.5)}$

$=\frac{\int_1^2\left(\frac{3}{8}x^2\right)dx}{\int_{0.5}^2\left(\frac{3}{8}x^2\right)dx}$

$=\frac{\frac{7}{8}}{\frac{63}{64}}$

$=\frac{8}{9}$

c $\Pr(X\le n)=\int_0^n\left(\frac{3}{8}x^2\right)dx$

$\frac{3}{4}=\frac{3}{8}\left[\frac{1}{3}x^3\right]_0^n$

$\frac{3}{4}=\frac{3}{8}\left(\frac{1}{3}n^3\right)$

$\frac{3}{4}=\frac{1}{8}n^3$

$n^3=6$

$n=6^{\frac{1}{3}}$

11 a

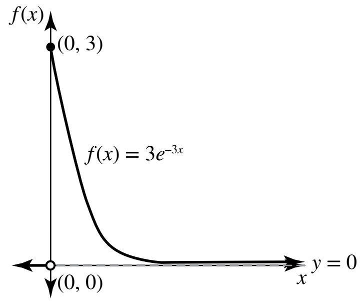
$$\int_0^a f(z)\,dz = 1$$
$$\int_0^a e^{-\frac{z}{3}}\,dz = 1$$
$$\left[-3e^{-\frac{z}{3}}\right]_0^a = 1$$
$$-3e^{-\frac{a}{3}} + 3e^0 = 1$$
$$-3e^{-\frac{a}{3}} + 3 = 1$$
$$-3e^{-\frac{a}{3}} = -2$$
$$e^{-\frac{a}{3}} = \frac{2}{3}$$
$$\log_e\left(\frac{2}{3}\right) = -\frac{a}{3}$$
$$-3\log_e\left(\frac{2}{3}\right) = a$$
$$-\log_e\left(\frac{3}{2}\right)^{-1} = a$$
$$a = 3\log_e\left(\frac{3}{2}\right)$$

b
$$\Pr(0 < Z < 0.7) = \int_0^{0.7} e^{-\frac{z}{3}}\,dz$$
$$\Pr(0 < Z < 0.7) = \left[-3e^{-\frac{z}{3}}\right]_0^{0.7}$$
$$\Pr(0 < Z < 0.7) = -3e^{-\frac{0.7}{3}} + 3e^0$$
$$\Pr(0 < Z < 0.7) = -3e^{-\frac{0.7}{3}} + 3$$
$$\Pr(0 < Z < 0.7) = 0.6243$$

c
$$\Pr(Z < 0.7 \mid Z > 0.2) = \frac{\Pr(0.2 < Z < 0.7)}{\Pr(Z > 0.2)}$$
$$\Pr(0.2 < Z < 0.7) = \int_{0.2}^{0.7} e^{-\frac{z}{3}}\,dz$$
$$\Pr(0.2 < Z < 0.7) = \left[-3e^{-\frac{z}{3}}\right]_{0.2}^{0.7}$$
$$\Pr(0.2 < Z < 0.7) = -3e^{-\frac{0.7}{3}} + 3e^{-\frac{0.2}{3}}$$
$$\Pr(0.2 < Z < 0.7) = -2.3757 + 2.8065$$
$$\Pr(0.2 < Z < 0.7) = 0.4308$$
$$\Pr(Z > 2) = 1 - \Pr(Z \le 2)$$
$$\Pr(0 \le Z \le 0.2) = \int_0^{0.2} e^{-\frac{z}{3}}\,dz$$
$$\Pr(0 \le Z \le 0.2) = \left[-3e^{-\frac{z}{3}}\right]_0^{0.2}$$
$$\Pr(0 \le Z \le 0.2) = -3e^{-\frac{0.2}{3}} + 3e^0$$
$$\Pr(0 \le Z \le 0.2) = 0.1935$$
$$\frac{\Pr(0.2 < Z < 0.7)}{\Pr(Z > 2)} = \frac{0.43085}{1 - 0.1935} = 0.5342$$

d
$$\Pr(Z \le \alpha) = 0.54$$
$$\int_0^{\alpha} e^{-\frac{z}{3}}\,dz = 0.54$$
$$\left[-3e^{-\frac{z}{3}}\right]_0^{\alpha} = 0.54$$
$$-3e^{-\frac{0\alpha}{3}} + 3e^0 = 0.54$$
$$-3e^{-\frac{\alpha}{3}} + 3 = 0.54$$
$$-3e^{-\frac{\alpha}{3}} = -2.46$$
$$e^{-\frac{\alpha}{3}} = 0.82$$
$$\log_e(0.82) = -\frac{\alpha}{3}$$
$$-3\log_e(0.82) = \alpha$$
$$\alpha = 0.60$$

12 a



b
$$\Pr(0 \le X \le 1) = \int_0^1 f(x)\,dx$$
$$\Pr(0 \le X \le 1) = \int_0^1 3e^{-3x}\,dx$$
$$\Pr(0 \le X \le 1) = \left[-e^{-3x}\right]_0^1$$
$$\Pr(0 \le X \le 1) = -e^3 + e^0$$
$$\Pr(0 \le X \le 1) = 0.9502$$

c
$$\Pr(X > 2) = \int_2^{\infty} 3e^{-3x}\,dx$$
$$\Pr(X > 2) = 0.0025$$

13 a
$$\int_1^a \log_e\left(x^2\right)dx = 1$$
$$\left[x\log_e\left(x^2\right) - 2x\right]_1^a = 1$$
$$\left(a\log_e\left(a^2\right) - 2a\right) - \left(\log_e\left(1^2\right) - 2(1)\right) = 1$$
$$a\log_e\left(a^2\right) - 2a + 2 - 1 = 0$$
$$2a\log_e(a) - 2a + 1 = 0$$
$$a = 2.1555$$

b
$$\Pr(1.25 \le X \le 2) = \int_{1.25}^{2} \log_e(x^2)\,dx$$
$$\Pr(1.25 \le X \le 2) = 0.7147$$

14 a
$$\Pr(-0.25 < Z < 0.25) = \int_{-0.25}^{0.25} \frac{1}{\pi\left(z^2+1\right)}dz$$
$$\Pr(-0.25 < Z < 0.25) = 0.1560$$

b $$\int_0^a \frac{1}{x^2+1}\,dx = 0.5$$
$$\left[\tan^{-1}(x)\right]_0^a = 0.5$$
$$\tan^{-1}(a) - \tan^{-1}(0) = 0.5$$
$$\tan^{-1}(a) = 0.5$$
$$a = \tan\left(\frac{1}{2}\right)$$
$$a = 0.5463$$

Exercise 12.4 — Measures of centre and spread

1 a $$\int_1^a \frac{1}{\sqrt{z}}\,dz = 1$$
$$\int_1^a \left(z^{-\frac{1}{2}}\right)dz = 1$$
$$\left[2z^{\frac{1}{2}}\right]_1^a = 1$$
$$2\sqrt{a} - 2\sqrt{1} = 1$$
$$2\sqrt{a} = 3$$
$$\sqrt{a} = \frac{3}{2}$$
$$a = \frac{9}{4}$$

b i $$E(Z) = \int_1^{\frac{9}{4}} z f(z)\,dz$$
$$E(Z) = \int_1^{\frac{9}{4}} \sqrt{z}\,dz$$
$$E(Z) = \left[\frac{2}{3}\sqrt{z^3}\right]_1^{\frac{8}{4}}$$
$$E(Z) = \frac{2}{3}\sqrt{\left(\frac{9}{4}\right)^3} - \frac{2}{3}\sqrt{1^3}$$
$$E(Z) = \frac{2}{3}\left(\left(\frac{3}{2}\right)^2\right)^{\frac{3}{2}} - \frac{2}{3}$$
$$E(Z) = \frac{2}{3}\times\frac{27}{8} - \frac{2}{3}$$
$$E(Z) = \frac{9}{4} - \frac{2}{3}$$
$$E(Z) = \frac{27}{12} - \frac{8}{12}$$
$$E(Z) = \frac{19}{12}$$

ii $$\int_1^m \frac{1}{\sqrt{z}}\,dz = 0.5$$
$$\int_1^m z^{-\frac{1}{2}}\,dz = 0.5$$
$$\left[2\sqrt{z}\right]_1^m = 0.5$$
$$2\sqrt{m} - 2\sqrt{1} = 0.5$$
$$2\sqrt{m} = 2.5$$
$$\sqrt{m} = 1.25$$
$$m = 1.5625 \text{ or } \frac{25}{16}$$

2 a $$\int_0^a y^{\frac{1}{2}}\,dy = 1$$
$$\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^a = 1$$
$$\frac{2}{3}a^{\frac{3}{2}} - \frac{2}{3}0^{\frac{3}{2}} = 1$$
$$a^{\frac{3}{2}} = \frac{3}{2}$$
$$a = 1.3104$$

b $$E(Y) = \int_0^{1.3104} y\sqrt{y}\,dy$$
$$E(Y) = \int_0^{1.3104} y^{\frac{3}{2}}\,dy$$
$$E(Y) = \left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^{1.3104}$$
$$E(Y) = \frac{2}{5}(1.3104)^{\frac{5}{2}} - \frac{2}{5}(0)^{\frac{5}{2}}$$
$$E(Y) = 0.7863$$

c $$\int_0^m \sqrt{y}\,dy = 0.5$$
$$\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^m = 0.5$$
$$\frac{2}{3}m^{\frac{3}{2}} - \frac{2}{3}0^{\frac{3}{2}} = 0.5$$
$$m = 0.8255$$

3 $$E(Z) = \int_1^{\frac{e}{2}} 2z\log_e(2z)\,dz$$
$$E(Z) = 0.7305$$
$$\text{Var}(Z) = E(Z^2) - [E(Z)]^2$$
$$E(Z^2) = \int_1^{\frac{e}{2}} 2z^2\log_e(2z)\,dz$$
$$E(Z^2) = 0.8760$$
$$\text{Var}(Z) = 0.876 - (0.7305)^2$$
$$\text{Var}(Z) = 0.3424$$
$$\text{SD}(Z) = \sqrt{0.3424}$$
$$\text{SD}(Z) = 0.5851$$

Median:
$$0.5 = \int_1^m 2\log_e(2z)dz$$
$$0.5 = \left[2z\log_e(z) - 0.6137z\right]_1^m$$
$$0.5 = 2m\log_e(m) - 0.6137m - \left(2(1)\log_e(1) - 0.6137(1)\right)$$
$$m = 1.3010,\ 0.0120$$
$$\therefore m = 1.3010\ (m > 1)$$

4 $$E(X) = \int_0^\infty 3xe^{-3xdx}$$
$$E(X) = \frac{1}{3}$$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$\text{E}(X^2) = \int_0^{\infty} 3x^2 e^{-3x}\, dx$

$\text{E}(X^2) = \frac{2}{9}$

$\text{Var}(X) = \frac{2}{9} - \left(\frac{1}{3}\right)^2$

$\text{Var}(X) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$

$\text{SD}(X) = \sqrt{\frac{1}{9}}$

$\text{SD}(X) = \frac{1}{3}$

Median:

$\int_0^m 3e^{-3x}\, dx = 0.5$

$\left[-e^{-3x}\right]_o^m = 0.5$

$-e^{-3m} + e^0 = 0.5$

$1 - e^{-3m} = 0.5$

$0.5 = e^{-3m}$

$\log_e(0.5) = -3m$

$-\frac{1}{3}\log_e(0.5) = m$

$m = 0.2310$

5 a $\int_0^1 \frac{1}{2\sqrt{x}}\, dx = \int_0^1 \frac{1}{2} x^{-\frac{1}{2}}\, dx$

$\int_0^1 \frac{1}{2\sqrt{x}}\, dx = \frac{1}{2}\int_0^1 x^{-\frac{1}{2}}\, dx$

$\int_0^1 \frac{1}{2\sqrt{x}}\, dx = \frac{1}{2}\left[2x^{\frac{1}{2}}\right]_0^1$

$\int_0^1 \frac{1}{2\sqrt{x}}\, dx = \frac{1}{2}\left(2\sqrt{1} - 2\sqrt{0}\right)$

$\int_0^1 \frac{1}{2\sqrt{x}}\, dx = \frac{1}{2} \times 2$

$\int_0^1 \frac{1}{2\sqrt{x}}\, dx = 1$

As $f(x) \geq 0$ for all x-values, and the area under the curve = 1, $f(x)$ is a probability density function.

b $\text{E}(X) = \int_0^1 x f(x)\, dx$

$\text{E}(X) = \int_0^1 \frac{x}{2\sqrt{x}}\, dx$

$\text{E}(X) = \frac{1}{2}\int_0^1 \sqrt{x}\, dx$

$\text{E}(X) = \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$

$\text{E}(X) = \frac{1}{2}\left(\frac{2}{3}\sqrt{1^3} - \frac{2}{3}\sqrt{0^3}\right)$

$\text{E}(X) = \frac{1}{2} \times \frac{2}{3}$

$\text{E}(X) = \frac{1}{3}$

c $\int_0^m \frac{1}{2\sqrt{x}}\, dx = 0.5$

$\frac{1}{2}\int_0^m x^{-\frac{1}{2}}\, dx = 0.5$

$\frac{1}{2}\left[2x^{\frac{1}{2}}\right]_0^m = 0.5$

$2\sqrt{m} - 2\sqrt{0} = 1$

$\sqrt{m} = 0.5$

$m = 0.25$

6 a $\text{E}(T) = \int_0^{\infty} t f(t)\, dt$

$\text{E}(T) = \int_0^{\infty} 2te^{-2t}\, dt$

$\text{E}(T) = 0.5$ min

b $\text{E}(T^2) = \int_0^{\infty} t^2 f(t)\, dt$

$\text{E}(T^2) = \int_0^{\infty} 2t^2 e^{-2t}\, dt$

$\text{E}(T^2) = 0.5$

$\text{Var}(T) = \text{E}(T^2) - [\text{E}(T)]^2$

$\text{Var}(T) = 0.5 - 0.5^2$

$\text{Var}(T) = 0.5 - 0.25$

$\text{Var}(T) = 0.25$

$\text{SD}(T) = \sqrt{0.25} = 0.5$ min

c $\int_0^m f(t)\, dt = 0.5$

$\int_0^m 2e^{-2t}\, dt = 0.5$

$\left[-e^{-2t}\right]_0^m = 0.5$

$-e^{-2m} + e^0 = 0.5$

$1 - 0.5 = e^{-2m}$

$0.5 = e^{-2m}$

$\log_e(0.5) = -2m$

$-\frac{1}{2}\log_e(0.5) = m$

$m = 0.35$ min

7 a $\text{E}(Y) = \int_0^{\sqrt[3]{9}} y f(y)\, dy$

$\text{E}(Y) = \int_0^{\sqrt[3]{9}} \frac{y^3}{3}\, dy$

$\text{E}(Y) = \left[\frac{y^4}{12}\right]_0^{\sqrt[3]{9}}$

$\text{E}(Y) = \frac{\left(\sqrt[3]{9}\right)^4}{12} - \frac{\left(\sqrt[3]{0}\right)^4}{12}$

$\text{E}(Y) = \frac{\left(3^2\right)^{\frac{4}{3}}}{12}$

$\text{E}(Y) = 1.5601$

b $\int_0^m f(y)\,dy = 0.5$

$\int_0^m \frac{y^2}{3}\,dy = 0.5$

$\left[\frac{y^3}{9}\right]_0^m = 0.5$

$\frac{m^3}{9} - \frac{0^3}{9} = 0.5$

$m^3 = 4.5$

$m = \sqrt[3]{4.5}$

$m = 1.6510$

c $\int_0^{Q_1} f(y)\,dy = 0.25$

$\int_0^{Q_1} \frac{y^2}{3}\,dy = 0.25$

$\left[\frac{y^3}{9}\right]_0^{Q_1} = 0.25$

$\frac{Q_1^3}{9} - \frac{0^3}{9} = 0.25$

$Q_1^3 = 2.25$

$Q_1 = \sqrt[3]{2.25}$

$Q_1 = 1.3104$

$\int_0^{Q_3} f(y)\,dy = 0.75$

$\int_0^{Q_3} \frac{y^2}{3}\,dy = 0.75$

$\left[\frac{y^3}{9}\right]_0^{Q_3} = 0.75$

$\frac{Q_3^3}{9} - \frac{0^3}{9} = 0.75$

$Q_3^3 = 6.75$

$Q_3 = \sqrt[3]{6.75}$

$Q_3 = 1.8899$

d Inter-quartile range is $Q_3 - Q_1 = 1.8899 - 1.3104 = 0.5795$

8 a $\int_1^8 \frac{a}{z}\,dz = 1$

$a\int_1^8 \frac{1}{z}\,dz = 1$

$a\left[\log_e(z)\right]_1^8 = 1$

$a\left(\log_e(8) - \log_e(1)\right) = 1$

$a\log_e(8) = 1$

$a = \frac{1}{\log_e(8)}$

$a = 0.4809$

b $E(Z) = \int_1^8 \left(z \times \frac{0.4809}{z}\right)dz$

$E(Z) = \int_1^8 0.4809\,dz$

$E(Z) = [0.4809z]_1^8$

$E(Z) = 0.4809(8) - 0.4809(1)$

$E(Z) = 3.3663$

c $E(Z^2) = \int_1^8 \left(z^2 \times \frac{0.4809}{z}\right)dz$

$E(Z^2) = \int_1^8 0.4809z\,dz$

$E(Z^2) = \left[0.2405z^2\right]_1^8$

$E(Z^2 = 0.2405(8)^2 - 0.2405(1)^2$

$E(Z^2) = 15.1515$

$\text{Var}(Z) = E(Z^2) - [E(Z)]^2$

$\text{Var}(Z) = 15.1515 - 3.3663^2$

$\text{Var}(Z) = 3.8195$

$\text{SD}(Z) = \sqrt{3.8195} = 1.9571$

d $\int_1^{Q_1} \frac{0.4809}{z}\,dz = 0.25$

$0.4809\left[\log_e(z)\right]_1^{Q_1} = 0.25$

$\log_e(Q_1) - \log_e(1) = 0.5199$

$\log_e(Q_1) = 0.5199$

$Q_1 = e^{0.5199}$

$Q_1 = 1.6817$

$\int_1^{Q_3} \frac{0.4809}{z}\,dz = 0.75$

$0.4809\left[\log_e(z)\right]_1^{Q_3} = 0.75$

$\log_e(Q_3) - \log_e(1) = 1.5596$

$\log_e(Q_3) = 1.5596$

$Q_3 = e^{1.5596}$

$Q_3 = 4.7568$

Inter-quartile range is $Q_3 - Q_1 = 4.7568 - 1.6817 = 3.0751$

e Range = 8 – 1 = 7

9 a $\int_0^\pi \frac{1}{\pi}\left(\sin(2x) + 1\right)dx$

$= \frac{1}{\pi}\int_0^\pi \left(\sin(2x) + 1\right)dx$

$= \frac{1}{\pi}\left[-\frac{1}{2}\cos(2x) + x\right]_0^\pi$

$= \frac{1}{\pi}\left(\left(-\frac{1}{2}\cos(2\pi) + \pi\right) - \left(-\frac{1}{2}\cos(0) + 0\right)\right)$

$= \frac{1}{\pi}\left(-\frac{1}{2} + \pi + \frac{1}{2}\right)$

$= 1$

As $f(x) \geq 0$ for all values of x and the area under the curve is 1, $f(x)$ is a probability density function.

b $E(X) = \int_0^\pi \frac{x}{\pi}\left(\sin(2x) + 1\right)dx$

$E(X) = 1.0708$

c **i** $E(X^2) = \int_0^\pi \frac{x^2}{\pi}\left(\sin(2x) + 1\right)dx$

$E(X^2) = 1.7191$

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$\text{Var}(X) = 1.7191 - 1.0708^2$

$\text{Var}(X) = 0.5725$

ii $\text{SD}(X) = \sqrt{0.5725} = 0.7566$

d

$$\int_0^m \left(\frac{1}{\pi}(\sin(2x)+1)\right)dx = 0.5$$
$$\frac{1}{\pi}\left[-\frac{1}{2}\cos(2x)+x\right]_0^m = 0.5$$
$$\left(-\frac{1}{2}\cos(2m)+m\right)-\left(-\frac{1}{2}\cos(0)+0\right)=1.5708$$
$$-\frac{1}{2}\cos(2m)+m+\frac{1}{2}=1.5708$$
$$m-\frac{1}{2}\cos(2m)=1.0708$$
$$m=0.9291$$

10

$$\text{E}(X)=\int_0^2 xf(x)\,dx=1$$
$$\int_0^2 x\left(ax-bx^2\right)dx=1$$
$$\int_0^2 \left(ax^2-bx^3\right)dx=1$$
$$\left[\frac{a}{3}x^3-\frac{b}{4}x^4\right]_0^2=1$$
$$\left(\frac{a}{3}(2)^3-\frac{b}{4}(2)^4\right)-0=1$$
$$\left[\frac{a}{3}x^3-\frac{b}{4}x^4\right]_0^2=1$$
$$\left(\frac{a}{3}(2)^3-\frac{b}{4}(2)^4\right)-0=1$$
$$\frac{8a}{3}-4b=1$$
$$8a-12b=3 \text{....................(1)}$$
$$\int_0^2 f(x)\,dx=1$$
$$\int_0^2 \left(ax-bx^2\right)dx=1$$
$$\left[\frac{a}{2}x^2-\frac{b}{3}x^3\right]_0^2=1$$
$$\left(\frac{a}{2}(2)^2-\frac{b}{3}(2)^3\right)-0=1$$
$$2a-\frac{8}{3}b=1$$
$$6a-8b=3 \text{....................(2)}$$

$8a-12b=3$..................(1)
$6a-8b=3$....................(2)
$(1)\times 3$
$24a-36b=9$................(3)
$(2)\times 4$
$24a-32b=12$...............(4)
$(4)-(3)$

$$4b=3$$
$$b=\frac{3}{4}$$

Substitute $b=\frac{3}{4}$ into (1)

$$8a-12\left(\frac{3}{4}\right)=3$$
$$8a-9=3$$
$$8a=12$$
$$a=\frac{3}{2}$$

11 a

$$\int_1^a \frac{3}{z^2}\,dz=1$$
$$\int_1^a 3z^{-2}\,dz=1$$
$$\left[-3z^{-1}\right]_1^a=1$$
$$\left[-\frac{3}{z}\right]_1^a=1$$
$$-\frac{3}{a}+\frac{3}{1}=1$$
$$3a-3=a$$
$$2a=3$$
$$a=\frac{3}{2}$$

b

$$\text{E}(Z)=\int_1^{\frac{3}{2}} z\,f(z)\,dz$$
$$\text{E}(Z)=\int_1^{\frac{3}{2}}\left(z\times\frac{3}{z^2}\right)dz$$
$$\text{E}(Z)=\int_1^{\frac{13}{2}}\frac{3}{z}\,dz$$
$$\text{E}(Z)=\left[3\log_e(z)\right]_1^{\frac{3}{2}}$$
$$\text{E}(Z)=3\log_e\left(\frac{3}{2}\right)-3\log_e(1)$$
$$\text{E}(Z)=1.2164$$

$$\text{E}(Z^2)=\int_1^{\frac{3}{2}} z^2 f(z)\,dz$$
$$\text{E}(Z^2)=\int_1^{\frac{3}{2}}\left(z^2\times\frac{3}{z^2}\right)dz$$
$$\text{E}(Z^2)=\int_1^{\frac{3}{2}} 3\,dz$$
$$\text{E}(Z^2)=[3z]_1^{\frac{3}{2}}$$
$$\text{E}(Z^2)=3\left(\frac{3}{2}\right)-3(1)$$
$$\text{E}(Z^2)=\frac{9}{2}-\frac{6}{2}$$
$$\text{E}(Z^2)=\frac{3}{2}$$
$$\text{Var}(Z)=\text{E}(Z^2)-\left[\text{E}(Z)\right]^2$$
$$\text{Var}(Z)=\frac{3}{2}-1.2164^2$$
$$\text{Var}(Z)=0.0204$$

c $\int_1^m f(z)\,dz = 0.5$

$\int_1^m \frac{3}{z^2}\,dz = 0.5$

$\left[-\frac{3}{z}\right]_1^m = \frac{1}{2}$

$-\frac{3}{m} + \frac{3}{1} = \frac{1}{2}$

$-6 + 6m = m$

$5m = 6$

$m = \frac{6}{5}$

$\int_1^{Q_1} f(z)\,dz = 0.25$

$\int_1^{Q_1} \frac{3}{z^2}\,dz = 0.25$

$\left[-\frac{3}{z}\right]_1^{Q_1} = \frac{1}{4}$

$-\frac{3}{Q_1} + \frac{3}{1} = \frac{1}{4}$

$-12 + 12Q_1 = Q_1$

$11Q_1 = 12$

$Q_1 = \frac{12}{11}$

$\int_1^{Q_3} f(z)\,dz = 0.75$

$\int_1^{Q_3} \frac{3}{z^2}\,dz = 0.75$

$\left[-\frac{3}{z}\right]_1^{Q_3} = \frac{3}{4}$

$-\frac{3}{Q_3} + \frac{3}{1} = \frac{3}{4}$

$-12 + 12Q_3 = 3Q_3$

$9Q_3 = 12$

$Q_3 = \frac{12}{9}$

$Q_3 = \frac{4}{3}$

Inter-quartile range is $\frac{4}{3} - \frac{12}{11} = \frac{44}{33} - \frac{36}{33} = \frac{8}{33}$

12 a $y = \sqrt{4 - x^2}$

$y = \left(4 - x^2\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(-2x)\left(4 - x^2\right)^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{x}{\sqrt{4 - x^2}}$

b $E(X)=\int_0^{\sqrt{3}} xf(x)\,dx$

$E(X)=\int_0^{\sqrt{3}} \frac{3x}{\pi\sqrt{4-x^2}}\,dx$

$E(X)=-\frac{3}{\pi}\int_0^{\sqrt{3}}\left(-\frac{x}{\sqrt{4-x^2}}\right)dx$

$E(X)=-\frac{3}{\pi}\left[\sqrt{4-x^2}\right]_0^{\sqrt{3}}$

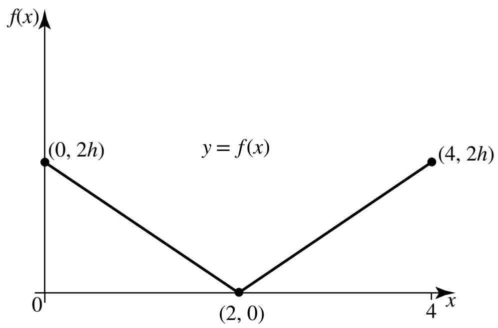
$E(X)=-\frac{3}{\pi}\left(\sqrt{4-\left(\sqrt{3}\right)^2}-\sqrt{4-0^2}\right)$

$E(X)=-\frac{3}{\pi}\left(\sqrt{1}-\sqrt{4}\right)$

$E(X)=-\frac{3}{\pi}\times -1$

$E(X)=\frac{3}{\pi}$

13 a



$$\int_0^4 f(x)\,dx=1$$

$$\left(\frac{1}{2}\times 2\times 2h\right)+\left(\frac{1}{2}\times 2\times 2h\right)=1$$

$$2h+2h=1$$

$$4h=1$$

$$h=\frac{1}{4}$$

b $E(X)=\int_0^4 xf(x)\,dx$

$E(X)=\int_0^2\left(-\frac{1}{4}x^2+\frac{1}{2}x\right)dx+\int_2^4\left(\frac{1}{4}x^2-\frac{1}{2}x\right)dx$

$E(X)=\left[-\frac{1}{12}x^3+\frac{1}{4}x^2\right]_0^2+\left[\frac{1}{12}x^3-\frac{1}{4}x^2\right]_2^4$

$E(X)=\left(-\frac{1}{12}(2)^3+\frac{1}{4}(2)^2\right)-0+\left(\frac{1}{12}(4)^3-\frac{1}{4}(4)^2\right)-\left(\frac{1}{12}(2)^3-\frac{1}{4}(2)^2\right)$

$E(X)=-\frac{2}{3}+1+\frac{16}{3}-4-\frac{2}{3}+1$

$E(X)=4-4+2$

$E(X)=2$

c $E(X^2)=\int_0^4 x^2 f(x)\,dx$

$E(X^2)=\int_0^2\left(-\frac{1}{4}x^3+\frac{1}{2}x^2\right)dx+\int_2^4\left(\frac{1}{4}x^3-\frac{1}{2}x^2\right)dx$

$E(X^2)=\left[-\frac{1}{16}x^4+\frac{1}{6}x^3\right]_0^2+\left[\frac{1}{16}x^4-\frac{1}{6}x^3\right]_2^4$

$E(X^2)=\left(-\frac{1}{16}(2)^4+\frac{1}{6}(2)^3\right)-0+\left(\frac{1}{16}(4)^4-\frac{1}{6}(4)^3\right)-\left(\frac{1}{16}(2)^4-\frac{1}{6}(2)^3\right)$

$E(X^2)=-1+\frac{4}{3}+16-\frac{32}{3}-1+\frac{4}{3}$

$E(X^2)=14-8$

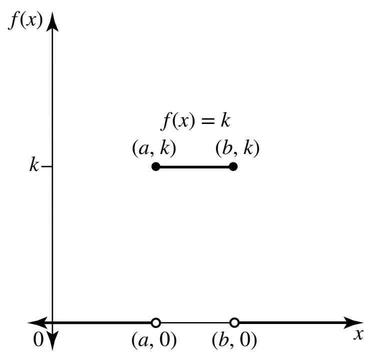
$E(X^2)=6$

$\text{Var}(X)=E(X^2)-[E(X)]^2$

$\text{Var}(X)=6-2^2$

$\text{Var}(X)=2$

14 a



b $\int_a^b k\,dx=1$

$[kx]_a^b=1$

$kb-ka=1$

$k(b-a)=1$

$k=\frac{1}{b-a}$

c $E(X)=\int_a^b xf(x)\,dx$

$E(X)=\int_a^b kx\,dx$

$E(X)=\left[\frac{k}{2}x^2\right]_a^b$

$E(X)=\frac{k}{2}b^2-\frac{k}{2}a^2$

$E(X)=\frac{k}{2}\left(b^2-a^2\right)$

$E(X)=\frac{1}{2(b-a)}\times\frac{(b-a)(b+a)}{1}$

$E(X)=\frac{b+a}{2}$

d $\mathrm{E}(X^2)=\int_a^b x^2 f(x)\,dx$

$\mathrm{E}(X^2)=\int_a^b kx^2\,dx$

$\mathrm{E}(X^2)=\left[\frac{k}{3}x^3\right]_a^b$

$\mathrm{E}(X^2)=\frac{k}{3}b^3-\frac{k}{3}a^3$

$\mathrm{E}(X^2)=\frac{k}{3}\left(b^3-a^3\right)$

$\mathrm{E}(X^2)=\frac{1}{3(b-a)}\times\frac{(b-a)(b^2+ba+a^2)}{1}$

$\mathrm{E}(X^2)=\frac{b^2+ba+a^2}{3}$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$

$\mathrm{Var}(X)=\frac{b^2+ba+a^2}{3}-\left(\frac{b+a}{2}\right)^2$

$\mathrm{Var}(X)=\frac{b^2+ba+a^2}{3}-\frac{b^2+2ba+a^2}{4}$

$\mathrm{Var}(X)=\frac{4b^2+4ba+4a^2-3b^2-6ba-3a^2}{12}$

$\mathrm{Var}(X)=\frac{b^2-2ba+a^2}{12}$

$\mathrm{Var}(X)=\frac{(b-a)^2}{12}=\frac{(a-b)^2}{12}$

15 a $\int_2^{7.9344} f(y)\,dy=\int_2^{7.9344} 0.2\log_e\left(\frac{y}{2}\right)dy=1$

b $\mathrm{E}(Y)=\int_2^{7.9344} 0.2y\log_e\left(\frac{y}{2}\right)dy=5.7278$

c $\mathrm{E}(Y^2)=\int_2^{7.9344} 0.2y^2\log_e\left(\frac{y}{2}\right)dy=34.9677$

$\mathrm{Var}(Y)=\mathrm{E}(Y^2)-[\mathrm{E}(Y)]^2$

$\mathrm{Var}(Y)=34.9677-5.7278^2$

$\mathrm{Var}(Y)=2.1600$

$\mathrm{SD}(Y)=\sqrt{2.1600}=1.4697$

d $\int_2^m 0.2\log_e\left(\frac{y}{2}\right)dy=0.5$

$0.5=\left[0.05\left(2x^2\log_e(x)-2.3863x^2\right)\right]_2^m$

$0.5=0.05\left(2m^2\log_e(m)-2.3863m^2\right)-0.05\left(0.05\left(2(2)^2\log_e(2)-2.3863(2)^2\right)\right)$

$\frac{0.5}{0.05}=2m^2\log_e(m)-2.3863m^2-(5.5452-9.5472)$

$10=2m^2\log_e(m)-2.3863m^2+4.0218$

$0=2m^2\log_e(m)-2.3863m^2-5.9782$

$m=3.9816$

e Range is $7.9344-2=5.9344$

16 a $\int_1^a \sqrt{z-1}\,dz=1$

$\int_1^a (z-1)^{\frac{1}{2}}dz=1$

$\left[\frac{2}{3}(z-1)^{\frac{3}{2}}\right]_1^a=1$

$$\left[\frac{2}{3}\sqrt{(z-1)^3}\right]_1^a = 1$$
$$\frac{2}{3}\sqrt{(a-1)^3} - \frac{2}{3}\sqrt{(1-1)^3} = 1$$
$$\sqrt{(a-1)^3} = \frac{3}{2}$$
$$(a-1)^3 = \frac{9}{4}$$
$$a = 2.3104$$

b **i** $E(Z) = \int_1^{2.3104} z\sqrt{z-1}\,dz = 1.7863$

ii $E(Z^2) = \int_1^{2.3104} z^2\sqrt{z-1}\,dz = 3.3085$

iii $\text{Var}(Z) = E(Z^2) - [E(Z)]^2$
$\text{Var}(Z) = 3.3085 - 1.7863^2$
$\text{Var}(Z) = 0.1176$

iv $\text{SD}(Z) = \sqrt{0.1176} = 0.3430$

Exercise 12.5 — Linear transformations

1 $E(Y) = 4$ and $\text{Var}(Y) = 3$

a $E(2Y-3) = 2E(Y) - 3 = 2(4) - 3 = 8 - 3 = 5$

b $\text{Var}(2Y-3) = 2^2\,\text{Var}(Y) = 4 \times 3 = 12$

c $\text{Var}(Y) = E(Y^2) - [E(Y)]^2$
$3 = E(Y^2) - (4)^2$
$3 = E(Y^2) - 16$
$19 = E(Y^2)$

d $E(Y(Y-1)) = E(Y^2 - Y) = E(Y^2) - E(Y) = 19 - 4 = 15$

2 $E(X) = 9$ and $\text{Var}(X) = 2$

a $Y = aX + 5$
$E(Y) = E(aX + 5)$
$E(Y) = aE(X) + 5$
$E(Y) = 9a + 5$
$\text{Var}(Y) = \text{Var}(aX + 5) = a^2\,\text{Var}(X) = 2a^2$
$E(Y) = \text{Var}(Y)$
$9a + 5 = 2a^2$
$0 = 2a^2 - 9a - 5$
$0 = (2a+1)(a-5)$
$a = 5$ since a is a positive integer

b $E(Y) = 9a + 5 = 9(5) + 5 = 50$
$\text{Var}(Y) = 2a^2 = 2(5)^2 = 50$

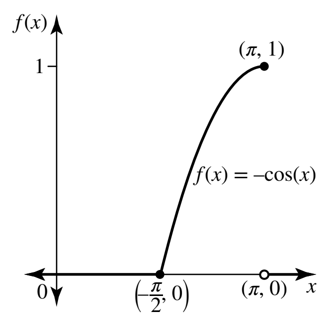
3 **a**
$$\int_{-2}^{2} f(x)\,dx = 1$$
$$\int_{-2}^{0}(-kx)\,dx + \int_{0}^{2}(kx)\,dx = 1$$
$$\left[-\frac{k}{2}x^2\right]_{-2}^{0} + \left[\frac{k}{2}x^2\right]_{0}^{2} = 1$$
$$\left(-\frac{k}{2}(0)^2 + \frac{k}{2}(-2)^2\right) + \left(\frac{k}{2}(2)^2 - \frac{k}{2}(0)^2\right) = 1$$
$$2k + 2k = 1$$
$$4k = 1$$
$$k = \frac{1}{4}$$

b $E(X) = \int_{-2}^{2} xf(x)\,dx$
$$E(X) = \int_{-2}^{0}\left(x \times -\frac{x}{4}\right)dx + \int_{0}^{2}\left(x \times \frac{x}{4}\right)dx$$
$$E(X) = \int_{-2}^{0} -\frac{x^2}{4}\,dx + \int_{0}^{2}\frac{x^2}{4}\,dx$$
$$E(X) = \left[-\frac{x^3}{12}\right]_{-2}^{0} + \left[\frac{x^3}{12}\right]_{0}^{2}$$
$$E(X) = \left(-\frac{0^3}{12} + \frac{(-2)^3}{12}\right) + \left(\frac{2^3}{12} - \frac{0^3}{12}\right)$$
$$E(X) = -\frac{3}{4} + \frac{3}{4}$$
$$E(X) = 0$$
$$\text{Var}(X) = E(X^2) - [E(X)]^2$$
$$E(X^2) = \int_{-2}^{2} x^2 f(x)\,dx$$
$$E(X^2) = \int_{-2}^{0}\left(x^2 \times -\frac{x}{4}\right)dx + \int_{0}^{2}\left(x^2 \times \frac{x}{4}\right)dx$$
$$E(X^2) = \int_{-2}^{0}\left(-\frac{x^3}{4}\right)dx + \int_{0}^{2}\left(\frac{x^3}{4}\right)dx$$
$$E(X^2) = \left[-\frac{x^4}{16}\right]_{-2}^{0} + \left[\frac{x^4}{16}\right]_{0}^{2}$$
$$E(X^2) = \left(-\frac{0^4}{16} + \frac{(-2)^4}{16}\right) + \left(\frac{2^4}{16} - \frac{0^4}{16}\right)$$
$$E(X^2) = 1 + 1$$
$$E(X^2) = 2$$
$$\text{Var}(X) = 2 - 0^2 = 2$$

c $E(5X+3) = 5E(X) + 3 = 5(0) + 3 = 3$
$\text{Var}(5X+3) = 5^2\,\text{Var}(X) = 25 \times 2 = 50$

d
$$\begin{aligned} E\left((3X-2)^2\right) &= E\left(9X^2 - 12X + 4\right) \\ &= 9E(X^2) - 12E(X) + 4 \\ &= 9(2) - 12(0) + 4 \\ &= 22 \end{aligned}$$

4 **a**



$$\int_{\frac{\pi}{2}}^{\pi}(-\cos(x))\,dx$$
$$= [-\sin(x)]_{\frac{\pi}{2}}^{\pi}$$
$$= -\sin(\pi) + \sin\left(\frac{\pi}{2}\right)$$
$$= 0 + 1$$
$$= 1$$

This is a probability density function.

b $\mathrm{E}(X)=\int_{\frac{\pi}{2}}^{\pi} xf(x)\,dx$

$\mathrm{E}(X)=\int_{\frac{\pi}{2}}^{\pi} -x\cos(x)\,dx$

$\mathrm{E}(X)=2.5708$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$

$\mathrm{E}(X^2)=\int_{\frac{\pi}{2}}^{\pi} x^2 f(x)\,dx$

$\mathrm{E}(X^2)=\int_{\frac{\pi}{2}}^{\pi} -x^2\cos(x)\,dx$

$\mathrm{E}(X^2)=6.7506$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$

$\mathrm{Var}(X)=6.7506-(2.5708)^2$

$\mathrm{Var}(X)=0.1416$

c $\mathrm{E}(3X+1)=3\mathrm{E}(X)+1$

$\mathrm{E}(3X+1)=3(2.5708)+1$

$\mathrm{E}(3X+1)=8.7124$

$\mathrm{Var}(3X+1)=3^2\,\mathrm{Var}(X)$

$\mathrm{Var}(3X+1)=9(0.1416)$

$\mathrm{Var}(3X+1)=1.2743$

d $\mathrm{E}((2X-1)(3X-2))=\mathrm{E}\left(6X^2-7X+2\right)$

$\mathrm{E}((2X-1)(3X-2))=6\mathrm{E}(X^2)-7\mathrm{E}(X)+2$

$\mathrm{E}((2X-1)(3X-2))=6(6.7606)-7(2.5708)+2$

$\mathrm{E}((2X-1)(3X-2))=24.5079$

5 $\mathrm{E}(Z)=5$ and $\mathrm{Var}(Z)=2$

a $\mathrm{E}(3Z-2)=3\mathrm{E}(Z)-2=3(5)-2=13$

b $\mathrm{Var}(3Z-2)=3^2\,\mathrm{Var}(Z)=9(2)=18$

c $\mathrm{Var}(Z)=\mathrm{E}\left(Z^2\right)-[\mathrm{E}(Z)]^2$

$2=\mathrm{E}\left(Z^2\right)-5^2$

$2=\mathrm{E}\left(Z^2\right)-25$

$27=\mathrm{E}\left(Z^2\right)$

d $\mathrm{E}\left(\frac{1}{3}Z^2-1\right)=\frac{1}{3}\mathrm{E}\left(Z^2\right)-1=\frac{27}{3}-1=8$

6 $\mathrm{E}(Y)=3.5$ and $\mathrm{SD}(Y)=1.2$

a $\mathrm{E}(2-Y)=2-\mathrm{E}(Y)=2-3.5=-1.5$

b $\mathrm{E}\left(\frac{Y}{2}\right)=\frac{1}{2}\mathrm{E}(Y)=\frac{1}{2}\times 3.5=1.75$

c $\mathrm{Var}(Y)=[\mathrm{SD}(Y)]^2=(1.2)^2=1.44$

d $\mathrm{Var}(2-Y)=(-1)^2\,\mathrm{Var}(Y)=1.44$

e $\mathrm{Var}\left(\frac{Y}{2}\right)=\left(\frac{1}{2}\right)^2\mathrm{Var}(Y)=\frac{1}{4}\times 1.44=0.36$

7 Let T be the time for the kettle to boil.

$\mathrm{E}(T)=1.5$ and $\mathrm{SD}(T)=1.1$ so $\mathrm{Var}(T)=1.21$

$\mathrm{E}(5T)=5\mathrm{E}(T)=5\times 1.5=7.5$ minutes

$\mathrm{Var}(5T)=5^2\,\mathrm{Var}(T)=25\times 1.21=30.25$

$\mathrm{SD}(5T)=\sqrt{30.25}=5.5$ minutes

8 **a** $\int_0^2 mx(2-x)\,dx=1$

$\int_0^2 (2mx-mx^2)\,dx=1$

$\left[mx^2-\frac{m}{3}x^3\right]_0^2=1$

$\left(m(2)^2-\frac{m}{3}(2)^3\right)-0=1$

$4m-\frac{8}{3}m=1$

$\frac{12m-8m}{3}=1$

$4m=3$

$m=\frac{3}{4}$

b $\mathrm{E}(X)=\int_0^2 xf(x)\,dx$

$\mathrm{E}(X)=\int_0^2 \frac{3}{4}x^2(2-x)\,dx$

$\mathrm{E}(X)=\int_0^2 \left(\frac{3}{2}x^2-\frac{3}{4}x^3\right)dx$

$\mathrm{E}(X)=\left[\frac{1}{2}x^3-\frac{3}{16}x^4\right]_0^2$

$\mathrm{E}(X)=\left(\frac{1}{2}(2)^3-\frac{3}{16}(2)^4\right)-0$

$\mathrm{E}(X)=4-3$

$\mathrm{E}(X)=1$

$\mathrm{E}(X^2)=\int_0^2 x^2 f(x)\,dx$

$\mathrm{E}(X^2)=\int_0^2 \frac{3}{4}x^3(2-x)\,dx$

$\mathrm{E}(X^2)=\int_0^2 \left(\frac{3}{2}x^3-\frac{3}{4}x^4\right)dx$

$\mathrm{E}(X^2)=\left[\frac{3}{8}x^4-\frac{3}{20}x^5\right]_0^2$

$\mathrm{E}(X^2)=\left(\frac{3}{8}(2)^4-\frac{3}{20}(2)^5\right)-0$

$\mathrm{E}(X^2)=6-4.8$

$\mathrm{E}(X^2)=1.2$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$

$\mathrm{Var}(X)=1.2-1^2$

$\mathrm{Var}(X)=0.2$

c $\mathrm{E}(5-2X)=5-2\mathrm{E}(X)=5-2(1)=3$

$\mathrm{Var}(5-2X)=(-2)^2\,\mathrm{Var}(X)=4\times 0.2=0.8$

9 **a** $\int_0^a \frac{2}{z+1}\,dz=1$

$\left[2\log_e(z+1)\right]_0^a=1$

$2\log_e(a+1)-2\log_e(0+1)=1$

$\log_e(a+1)=0.5$

$e^{0.5}=a+1$

$e^{0.5}-1=a$

$0.6487=a$

b $\mathrm{E}(Z)=\int_{0}^{0.6487}\frac{2z}{z+1}dz=0.2974$

$\mathrm{E}(Z^2)=\int_{0}^{0.6487}\frac{2z^2}{z+1}dz=0.1234$

$\mathrm{Var}(Z)=\mathrm{E}(Z^2)-[\mathrm{E}(Z)]^2$

$\mathrm{Var}(Z)=0.1234-0.2974^2$

$\mathrm{Var}(Z)=0.0349$

c **i** $\mathrm{E}(3Z+1)=3\mathrm{E}(Z)+1=3(0.2974)+1=1.8922$

ii $\mathrm{Var}(3Z+1)=3^2\,\mathrm{Var}(Z)=9(0.0349)=0.3141$

iii $\mathrm{E}(Z^2+2)=\mathrm{E}(Z^2)+2=0.1234+2=2.1234$

10 $Y=aX+3$ and $\mathrm{E}(X)=5$ as well as $\mathrm{Var}(X)=2$

$\mathrm{E}(Y)=\mathrm{Var}(Y)$

$\mathrm{E}(aX+3)=\mathrm{Var}(aX+3)$

$a\mathrm{E}(X)+3=a^2\mathrm{Var}(X)$

$5a+3=2a^2$

$0=2a^2-5a-3$

$0=(2a+1)(a-3)$

$a=-\frac{1}{2},3$

$\therefore a=3$ since a is a positive integer

Thus $\mathrm{E}(Y)=\mathrm{E}(3X+3)=3\mathrm{E}(X)+3=3(5)+3=18$

and $\mathrm{Var}(Y)=\mathrm{Var}(3X+3)=3^2\,\mathrm{Var}(X)=9(2)=18$

11 $Z=aY-3$ and $\mathrm{E}(Y)=4$ as well as $\mathrm{Var}(Y)=1$

$\mathrm{E}(Z)=\mathrm{Var}(Z)$

$\mathrm{E}(aY-3)=\mathrm{Var}(aY-3)$

$a\mathrm{E}(Y)-3=a^2\mathrm{Var}(Y)$

$4a-3=a^2$

$0=a^2-4a+3$

$0=(q-3)(a-1)$

$a=1$ or 3

Thus $\mathrm{E}(Z)=\mathrm{E}(Y-3)=\mathrm{E}(Y)-3=4-3=1$ and $\mathrm{E}(Z)=\mathrm{E}(3Y-3)=3\mathrm{E}(Y)-3=3(4)-3=9$

Also $\mathrm{Var}(Z)=\mathrm{Var}(Y-3)=1^2\,\mathrm{Var}(Y)=1$ and $\mathrm{Var}(Z)=\mathrm{Var}(3Y-3)=3^2\,\mathrm{Var}(Y)=9(1)=9$

12 **a** $\int_{1}^{a}f(z)dz=1$

$\int_{1}^{a}3z^{-\frac{1}{2}}dz=1$

$\left[6z^{\frac{1}{2}}\right]_1^a=1$

$\left[6\sqrt{z}\right]_1^a=1$

$6\sqrt{a}-6\sqrt{1}=1$

$6\sqrt{a}-6=1$

$6\sqrt{a}=7$

$\sqrt{a}=\frac{7}{6}$

$a=\frac{49}{36}$

b $\mathrm{E}(Z)=\int_{1}^{\frac{49}{36}} zf(z)\,dz$

$\mathrm{E}(Z)=\int_{1}^{\frac{49}{36}} 6z^{\frac{1}{2}}\,dz$

$\mathrm{E}(Z)=\left[2z^{\frac{3}{2}}\right]_{1}^{\frac{49}{36}}$

$\mathrm{E}(Z)=2\left(\frac{49}{36}\right)^{\frac{3}{2}}-2(1)^{\frac{3}{2}}$

$\mathrm{E}(Z)=2\left(\left(\frac{7}{6}\right)^{2}\right)^{\frac{3}{2}}-2$

$\mathrm{E}(Z)=2\left(\frac{7}{6}\right)^{3}-2$

$\mathrm{E}(Z)=1.1759$

$\mathrm{E}(Z^2)=\int_{1}^{\frac{49}{36}} z^2 f(z)\,dz$

$\mathrm{E}(Z^2)=\int_{1}^{\frac{49}{36}} 3z^{\frac{3}{2}}\,dz$

$\mathrm{E}(Z^2)=\left[\frac{6}{5}z^{\frac{5}{2}}\right]_{1}^{\frac{49}{36}}$

$\mathrm{E}(Z^2)=\frac{6}{5}\left(\frac{49}{36}\right)^{\frac{5}{2}}-\frac{6}{5}(1)^{\frac{5}{2}}$

$\mathrm{E}(Z^2)=\frac{6}{5}\left(\left(\frac{7}{6}\right)^{2}\right)^{\frac{5}{2}}-\frac{6}{5}$

$\mathrm{E}(Z^2)=\frac{6}{5}\left(\frac{7}{6}\right)^{5}-\frac{6}{5}$

$\mathrm{E}(Z^2)=1.3937$

$\mathrm{Var}(Z)=\mathrm{E}(Z^2)-[\mathrm{E}(Z)]^2$
$\mathrm{Var}(Z)=1.3937-1.1759^2$
$\mathrm{Var}(Z)=0.0109$

c **i** $\mathrm{E}(4-3Z)=4-3\mathrm{E}(Z)=4-3(1.1759)=0.4722$

ii $\mathrm{Var}(4-3Z)=(-3)^2\,\mathrm{Var}(Z)=9\times 0.011=0.0978$

13 **a**

$$\int_{0}^{3\pi}\frac{x}{k\pi}\sin\left(\frac{x}{3}\right)dx=1$$

$$\frac{1}{k\pi}\int_{0}^{3\pi}x\sin\left(\frac{x}{3}\right)dx=1$$

$$\frac{1}{k\pi}\left[-3x\cos\left(\frac{x}{3}\right)+9\sin\left(\frac{x}{3}\right)\right]_{0}^{3\pi}=1$$

$$\left(-3(3\pi)\cos(\pi)+9\sin(\pi)\right)-\left(-3(0)\cos(0)+9\sin(0)\right)=k\pi$$

$$9\pi=k\pi$$

$$k=9$$

b $E(X)=\int_{0}^{3\pi} xf(x)\,dx$

$E(X)=\int_{0}^{3\pi} \frac{x^2}{9\pi}\sin\left(\frac{x}{3}\right)dx$

$E(X)=5.61 \text{ mm}$

c $W=2X-1$

$E(W)=E(2X-1)$

$E(W)=2E(X)-1$

$E(W)=2(5.6051)-1$

$E(W)=10.21 \text{ mm}$

14 **a**

$$\int_{0.9}^{1.25} k(2y+1)\,dy=1$$

$$k\int_{0.9}^{1.25} (2y+1)\,dy=1$$

$$k\left[y^2+y\right]_{0.9}^{1.25}=1$$

$$k\left(\left(1.25^2+1.25\right)-\left(0.9^2+0.9\right)\right)=1$$

$$k(2.8125-1.71)=1$$

$$1.1025k=1$$

$$k=\frac{400}{441}$$

b $E(Y)=\int_{0.9}^{1.25} yf(y)\,dy$

$E(Y)=\int_{0.9}^{1.25} \left(0.907y(2y+1)\right)dy$

$E(Y)=\int_{0.9}^{1.25} \left(1.814y^2+0.907y\right)dy$

$E(Y)=\left[0.604\dot{6}y^3+0.4535y^2\right]_{0.9}^{1.25}$

$E(Y)=\left(0.604\dot{6}(1.25)^3+0.4535(1.25)^2\right)-\left(0.604\dot{6}(0.9)^3+0.4535(0.9)^2\right)$

$E(Y)=1.8895-0.8081$

$E(Y)=1.081 \text{ kg}$

c $Z=0.75Y+0.45$

$E(Z)=E(0.75Y+0.45)$

$E(Z)=0.75E(Y)+0.45$

$E(Z)=0.75(1.0814)+0.45$

$E(Z)=1.261 \text{ kg}$

15 **a**

$$\int_{1}^{a}\frac{5\log_e(z)}{\sqrt{z}}dz=1$$

$$\left[10\sqrt{z}\log_e(z)-20\sqrt{z}\right]_1^a=1$$

$$\left(10\sqrt{a}\log_e(a)-20\sqrt{a}\right)-\left(10\sqrt{1}\log_e(1)-20\sqrt{1}\right)=1$$

$$10\sqrt{a}\log_e(a)-20\sqrt{a}+20=1$$

$$10\sqrt{a}\log_e(a)-20\sqrt{a}+19=0$$

$$a=1.7755$$

b $\mathrm{E}(Z)=\int_{1}^{1.7755} zf(z)\,dz$

$\mathrm{E}(Z)=\int_{1}^{1.7755} 5\sqrt{z}\log_e(z)\,dz$

$\mathrm{E}(Z)=1.4921$

$\mathrm{E}(Z^2)=\int_{1}^{1.7755} z^2 f(z)\,dz$

$\mathrm{E}(Z^2)=\int_{1}^{1.7755} 5z^{\frac{3}{2}}\log_e(z)\,dz$

$\mathrm{E}(Z^2)=2.2625$

$\mathrm{Var}(Z)=\mathrm{E}(Z^2)-[\mathrm{E}(Z)]^2$

$\mathrm{Var}(Z)=2.2625-1.4921^2$

$\mathrm{Var}(Z)=0.0361$

c $\mathrm{E}(3-2Z)=3-2\mathrm{E}(Z)=3-2(1.4921)=0.0158$

$\mathrm{Var}(3-2Z)=(-2)^2\,\mathrm{Var}(Z)=4\times 0.0361=0.1444$

16 $Y=aX+1$ and $\mathrm{E}(X)=2$ as well as $\mathrm{Var}(X)=7$

$\mathrm{E}(Y)=\mathrm{Var}(Y)$

$\mathrm{E}(aX+1)=\mathrm{Var}(aX+1)$

$a\mathrm{E}(X)+1=a^2\mathrm{Var}(X)$

$2a+1=a^2$

$0=a^2-2a-1$

$a=0.5469$

$\mathrm{E}(Y)=\mathrm{E}(0.5469X+1)=0.5469\mathrm{E}(X)+1=0.5469(2)+1=2.0938$

$\mathrm{Var}(Y)=\mathrm{Var}(0.5469X+1)=0.5469^2\,\mathrm{Var}(X)=0.5469^2(7)=2.0938$

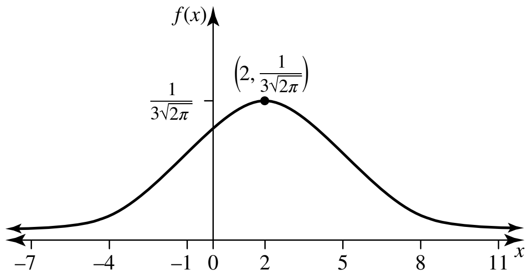
Topic 13 — The normal distribution

Exercise 13.2 — The normal distribution

1 $f(x)=\dfrac{1}{3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

a $\mu=2$ and $\sigma=3$

b

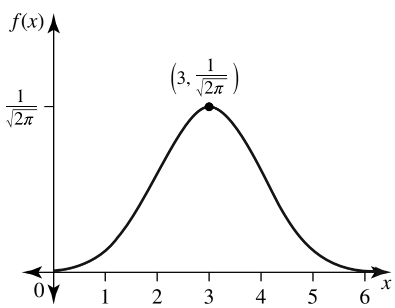


2 **a** Input $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-3)^2}dx\simeq 1$ on CAS.

b $f(x)=\int_{-\infty}^{\infty}\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-3)^2}=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)}$

$\mu=3$ and $\sigma=1$

c



3 **a** Let X = the results on the Mathematical Methods test

$X\sim N\left(72,8^2\right)$

$\mu+\sigma=72+8=80$
$\mu-\sigma=72-8=64$
$\mu+2\sigma=72+2(8)=88$
$\mu-2\sigma=72-2(8)=56$

So $\Pr(56<X<88)=0.95$
and $\Pr(X<56)\cup\Pr(X>88)=0.05$
Thus $\Pr(X<56)=\Pr(X>88)=0.05\div 2=0.025$

So $\Pr(X>88)=0.025$

b $\mu+3\sigma=88+8=96$
$\mu-3\sigma=58-8=50$

$\Pr(48<X<96)=0.997$
$\Pr(X<48)\cup\Pr(X>96)=0.003$
$\Pr(X<48)=\Pr(X>96)=0.003\div 2=0.0015$

c $\Pr(64<X<80)=0.68$
$\Pr(X<64)\cup\Pr(X>80)=0.32$
$\Pr(X<64)=\Pr(X>80)=0.32\div 2=0.16$
$\Pr(X<80)=1-\Pr(X>80)=1-0.16=0.84$

4 Let X = the length of pregnancy for a human $X\sim N\left(275,14^2\right)$

$\mu+\sigma=275+14=289$ $\quad\Pr(261<X<289)=0.68$
$\mu-\sigma=275-14=261$

$\mu+2\sigma=275+2(14)=303$ $\quad\Pr(247<X<303)=0.95$
$\mu-2\sigma=275-2(14)=247$

$\mu+3\sigma=275+3(14)=317$ $\quad\Pr(233<X<317)=0.997$
$\mu-3\sigma=275-3(14)=233$

$\Pr(X<233)\cup\Pr(X>317)=0.003$
$\Pr(X<233)=\Pr(X>317)=0.003\div 2=0.0015$
So $\Pr(X<233)=0.0015$

5 **a** Input $\int_{-\infty}^{\infty}\frac{1}{4\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x+2}{4}\right)^2}dx=0.9999\simeq 1$ using CAS

b $f(x)=\dfrac{1}{4\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x+2}{4}\right)^2}=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

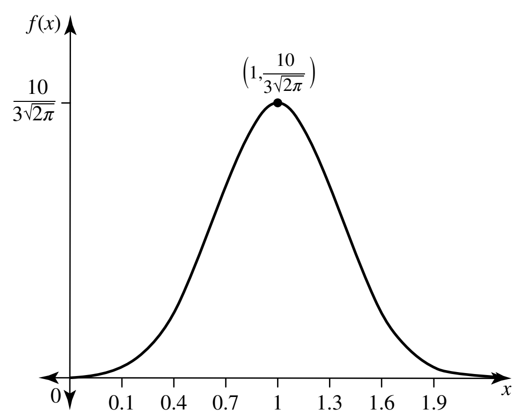
$\mu=-2$

6 **a** $f(x)=\dfrac{10}{3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{10(x-1)}{3}\right)^2}=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\frac{1}{\sigma}=\frac{10}{3}$ so $\sigma=\frac{3}{10}=0.3$ and $\mu=1$

b Dilation by a factor of $\frac{10}{3}$ parallel to the y-axis or from the x-axis. Dilation by a factor of $\frac{3}{10}$ parallel to the x-axis or from the y-axis and a translation 1 unit in the positive x-direction.

c



7 **a** $f(x)=\dfrac{1}{10\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2}=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\mu=-4,\ \sigma=10$

b Dilation factor $\frac{1}{10}$ from the x-axis, dilation factor 10 from the y-axis, translation 4 units in the negative x-direction.

c **i** $\sigma=SD(X)=10$

$Var(X)=\sigma^2=10^2=100$

ii $\text{Var}(X) = \text{E}\left(X^2\right) - [\text{E}(X)]^2$

$100 = \text{E}\left(X^2\right) - (-4)^2$

$100 = \text{E}\left(X^2\right) - 16$

$116 = \text{E}\left(X^2\right)$

d $\int_{-\infty}^{\infty} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2} dx = 0.9999 \simeq 1$ $f(x) \geq 0$ for all values of x and the area under the curve is 1 so this is a probability density function.

8 a $f(x) = \frac{5}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{5(x-2)}{2}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\frac{1}{\sigma} = \frac{5}{2}$ so $\sigma = \frac{2}{5}$ and $\mu = 2$

b $\text{Var}(X) = \sigma^2 = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$

$\text{Var}(X) = \text{E}\left(X^2\right) - [\text{E}(X)]^2$

$\frac{4}{25} = \text{E}\left(X^2\right) - 2^2$

$\frac{4}{25} = \text{E}\left(X^2\right) - 4$

$\frac{4}{25} + \frac{100}{25} = \text{E}\left(X^2\right)$

$\frac{104}{25} = \text{E}\left(X^2\right)$

$4.16 = \text{E}\left(X^2\right)$

c i $\text{E}(5X) = 5\text{E}(X) = 5(2) = 10$

ii $\text{E}\left(5X^2\right) = 5\text{E}(X^2) = 5 \times \frac{104}{25} = \frac{104}{5} = 2.8$

9 a Let X = the scores on an IQ test $X \sim \text{N}\left(120, 20^2\right)$

i $\mu - \sigma = 120 - 20 = 100$

$\mu + \sigma = 120 + 20 = 140$

ii $\mu - 2\sigma = 120 - 2(20) = 80$

$\mu + 2\sigma = 120 + 2(20) = 160$

iii $\mu - 3\sigma = 120 - 3(20) = 60$

$\mu + 3\sigma = 120 + 2(20) = 180$

b i $\Pr(X < 80) = 0.5 - 0.475 = 0.025$

ii $\Pr(X > 180) = 0.003 \div 2 = 0.0015$

10 Let X = the results on a biology exam $X \sim \text{N}\left(70, 6^2\right)$

$\mu + 3\sigma = 70 + 3(6) = 88$

$\Pr(X > 88) = \frac{1 - 0.997}{2} = 0.0015 = 0.15\%$ get a mark which is greater than 88.

11 $X \sim \text{N}\left(15, 5^2\right)$

a 68% of values lie between $15 - 5 = 10$ and $15 + 5 = 20$.

b 95% of values lie between $15 - 2(5) = 5$ and $15 + 2(5) = 25$.

c 99.7% of values lie between $15 - 3(5) = 0$ and $15 + 3(5) = 30$.

12 $X \sim \text{N}\left(24, 7^2\right)$

a $\Pr(X < 31) = 0.16 + 0.68 = 0.84$

b $\Pr(10 < X < 31) = 1 - (0.025 + 0.16) = 1 - 0.105 = 0.815$

c $\Pr(X > 10 \mid X < 31) = \frac{\Pr(10 < X < 31)}{\Pr(X < 31)} = \frac{0.815}{0.84} = 0.9702$

13 Let X = the number of pears per tree $X \sim \text{N}\left(230, 25^2\right)$

a $\Pr(X < 280) = 1 - \Pr(X > 280) = 1 - 0.025 = 0.975$

b $\Pr(180 < X < 280) = \Pr(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95$

c $\Pr(X > 180 \mid X < 280) = \frac{\Pr(180 < X < 280)}{\Pr(X < 280)} = \frac{0.95}{0.975} = 0.9744$

14 Let X = the rainfall in millimetres $X \sim \text{N}\left(305, 50^2\right)$

a $\Pr(205 < X < 355) = 1 - (0.025 + 0.16) = 1 - 0.185 = 0.815$

b 0.025 signifies 2σ

$\Pr(X < k) = 0.025$

$\Pr(X < 205) = 0.025$

So $k = 205$

c $\mu - 3\sigma = 155$

$\Pr(X < 155) = \frac{1 - 0.997}{2} = 0.0015$

0.0015 signifies 3σ

$\Pr(X < h) = 0.0015$

$\Pr(X < 155) = 0.05$

So $h = 155$

15 a $f(x) = \frac{5}{\sqrt{2\pi}} e^{-\frac{1}{2}(5(x-1))^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\text{SD}(X) = \sigma = \frac{1}{5}$

$\text{Var}(X) = \sigma^2 = \frac{1}{25} = 0.04$

b $\text{Var}(X) = \text{E}\left(X^2\right) - [\text{E}(X)]^2$

$\text{E}(X) = \mu = 1$

$0.04 = \text{E}\left(X^2\right) - 1^2$

$1.04 = \text{E}\left(X^2\right)$

c i $\text{E}(2X + 3) = 2\text{E}(X) + 3 = 2(1) + 3 = 5$

ii $\text{E}((X+1)(2X-3)) = \text{E}\left(2X^2 - X - 3\right)$

$\text{E}((X+1)(2X-3)) = 2\text{E}\left(X^2\right) - \text{E}(X) - 3$

$\text{E}((X+1)(2X-3)) = 2(1.04) - 1 - 3$

$\text{E}((X+1)(2X-3)) = -1.92$

16 $X \sim \text{N}\left(72.5, 8.4^2\right)$

a $\Pr(64.1 < X < 89.3) = 1 - (0.16 + 0.025) = 1 - 0.185 = 0.815$

b $\Pr(X < 55.7) = 0.025$

c $\Pr(X < 55.7) = \frac{1 - 0.95}{2} = \frac{0.05}{2} = 0.025$

$\Pr(X < 47.3) = \frac{1 - 0.997}{2} = \frac{0.003}{2} = 0.0015$

$\Pr(47.3 < X < 55.7) = 0.025 - 0.0015 = 0.0235$

$\Pr(X > 47.3 \mid X < 55.7) = \frac{\Pr(47.3 < X < 55.7)}{\Pr(X < 55.7)}$

$\Pr(X > 47.3 \mid X < 55.7) = \frac{0.025 - 0.0015}{0.025}$

$\Pr(X > 47.3 \mid X < 55.7) = 0.94$

d $\Pr(X > m) = 0.16$

$\Pr(X > \mu + \sigma) = \frac{1 - 0.68}{2} = \frac{0.32}{2} = 0.16$

So $m = 80.9$

Exercise 13.3 — Calculating probabilities and the standard normal distribution

1 a i $\Pr(Z < 1.2) = 0.8849$

ii $\Pr(-2.1 < Z < 0.8) = 0.7703$

b $X \sim N(45, 6^2)$

i $\Pr(X > 37) = 0.9088$

ii $z = \frac{37-45}{6}$

$= -\frac{4}{3}$

2 $X \sim N\left(50, 15^2\right)$

$\Pr(50 < X < 70) = 0.4088$

3 a $\Pr(Z \le 2) = 0.9772$

b $\Pr(Z \le -2) = 0.0228$

c $\Pr(-2 \le \Pr \le 2) = 0.9545$

d $\Pr(Z < -1.95) \cup \Pr(Z > 1.95)$

By symmetry

$= 2\Pr(Z < -1.95)$
$= 2 \times 0.0256$
$= 0.0512$

4 a $\Pr(X < 61) = \Pr\left(Z < \frac{61-65}{3}\right)$

$\Pr(X < 61) = \Pr\left(Z < -\frac{4}{3}\right)$

$\Pr(X < 61) = 0.0912$

b $\Pr(X \ge 110) = \Pr\left(Z \ge \frac{110-98}{15}\right)$

$\Pr(X \ge 110) = \Pr\left(Z \ge \frac{12}{15}\right)$

$\Pr(X \ge 110) = \Pr\left(Z \ge \frac{4}{5}\right)$

$\Pr(X \ge 110) = 0.2119$

c $\Pr(-2 < X \le 5) = \Pr\left(\frac{-2-2}{3} < Z \le \frac{5-2}{3}\right)$

$\Pr(-2 < X \le 5) = \Pr\left(-\frac{4}{3} < Z \le 1\right)$

$\Pr(-2 < X \le 5) = 0.7501$

5 Let X = the speed of cars $X \sim N\left(98, 6^2\right)$

a $\Pr(X > 110) = 0.0228$

b $\Pr(X < 90) = 0.0912$

c $\Pr(90 < X < 110) = 0.8860$

6 Let X = the score on a physics test $X \sim N\left(72, 12^2\right)$

a $\Pr(X > 95) = 0.0276 = 2.76\%$

b $\Pr(X \ge 55) = 0.9217 = 92.17\%$

7 a $\Pr(X < a) = 0.35$ and $\Pr(X < b) = 0.62$

i $\Pr(X > a) = 1 - 0.35 = 0.65$

ii $\Pr(a < X < b) = \Pr(X < b) - \Pr(X < a) = 0.62 - 0.35 = 0.27$

b $\Pr(X < c) = 0.27$ and $\Pr(X < d) = 0.56$

i $\Pr(c < X < d) = \Pr(X < d) - \Pr(X < c) = 0.56 - 0.27 = 0.29$

ii $\Pr(X > c \mid X < d) = \frac{\Pr(c < X < d)}{\Pr(X < d)} = \frac{0.29}{0.56} = 0.5179$

c i $\Pr(X > 32) = \Pr(Z > k)$

$k = \frac{x-\mu}{\sigma} = \frac{32-20}{5} = 2.4$

ii $\Pr(X > 32) = \Pr(Z > k) = \Pr(Z < -k)$

$-k = \frac{x-\mu}{\sigma} = \frac{12-20}{5} = -1.6$ so $k = 1.6$

8 Let X = the score for Jing Jing and Y = the score for Rani

$X \sim N\left(72, 9^2\right)$	$Y \sim N\left(15, 4^2\right)$
$z = \frac{x-\mu}{\sigma}$	$z = \frac{y-\mu}{\sigma}$
$z = \frac{85-72}{9}$	$z = \frac{18-15}{4}$
$z = \frac{13}{9}$	$z = \frac{3}{4}$
$z = 1.4$	$z = 0.75$

Jing Jing did better.

9 Let X = the length of Tasmanian salmon

$X \sim N\left(38, 2.4^2\right)$

$\Pr(X > 39.5) = 0.2659$

This is not in the top 15% so is not a gourmet fish.

10 Chemistry $X \sim N\left(68, 5^2\right)$

$z = \frac{x-\mu}{\sigma} = \frac{72-68}{5} = \frac{4}{5} = 0.8$

Mathematical Methods $X \sim N\left(69, 7^2\right)$

$z = \frac{x-\mu}{\sigma} = \frac{77-69}{7} = \frac{6}{7} = 0.86$

Physics $X \sim N\left(61, 8^2\right)$

$z = \frac{x-\mu}{\sigma} = \frac{68-61}{8} = \frac{7}{8} = 0.875$

Justine did best in physics compared to her peers.

11 Let X = the pulse rate in beats per minute $X \sim N\left(80, 5^2\right)$

a $\Pr(X > 85) = 0.1587$

b $\Pr(X \le 75) = 0.1587$

c $\Pr(78 \le X < 82 \mid X > 75) = \frac{03\,108}{1-0.1587} = 0.3695$

12 Let X = the weight of a bag of sugar $X \sim N\left(1.025, 0.01^2\right)$

a $\Pr(X > 1.04) = 0.0668 = 6.68\%$

b $\Pr(X < 0.996) = 0.0019 = 0.19\%$

13 a $\Pr(Z \le -2.125) \cup \Pr(Z \ge 2.125)$

By symmetry

$= 2\Pr(Z \le -2.125)$
$= 2 \times 0.0168$
$= 0.0336$

b $\Pr(X < 252.76) = 0.5684$

c $\Pr(-3.175 \le Z \le 1.995) = 0.9762$

d $\Pr(X < 5.725) = 0.3081$

14 Solve using CAS

$k = 25.2412$

Exercise 13.4 — The inverse normal distribution

1 **a** $\Pr(X \le a) = 0.16,\ \mu = 41$ and $\sigma = 6.7$

$a = 34.34$

b $\Pr(X \le a) = 0.21,\ \mu = 12.5$ and $\sigma = 7.7$

$a = 14.68$

c $\Pr(15 - a < X < 15 + a) = 0.32,\ \mu = 15$ and $\sigma = 4$

By symmetry

$$\Pr(X < 15 - a) = \frac{0.68}{2} = 0.34$$
$$15 - a = 13.35$$
$$a = 1.65$$

2 $\Pr(m \le X \le n) = 0.92,\ \mu = 27.3$ and $\sigma = 8.2$

$$\Pr(X < m) = \frac{0.08}{2}$$
$$\Pr(X < m) = 0.04$$
$$m = 12.9444$$
$$\Pr(X < n) = 0.96$$
$$n = 41.6556$$

3 $X \sim \mathrm{N}\left(112, \sigma^2\right)$

$$\Pr(X < 108.87) = 0.42$$
$$\Pr\left(Z < \frac{108.87 - 112}{\sigma}\right) = 0.42$$
$$\frac{108.87 - 112}{\sigma} = -0.2019$$
$$3.13 = -0.2019\sigma$$
$$\sigma = 15.5$$

4 $X \sim \mathrm{N}\left(\mu, 4.45^2\right)$

$$\Pr(X < 32.142) = 0.11$$
$$\Pr\left(Z < \frac{32.142 - \mu}{4.45}\right) = 0.11$$
$$\frac{32.142 - \mu}{4.45} = -1.2265$$
$$32.142 - \mu = -1.2265 \times 4.45$$
$$32.142 - \mu = -5.4579$$
$$32.142 + 5.4579 = \mu$$
$$\mu = 37.6$$

5 **a** $\Pr(Z < z) = 0.39$

So $z = -0.2793$

b $\Pr(Z \ge z) = 0.15$ or $\Pr(Z < z) = 0.85$

So $z = 1.0364$

c $\Pr(-z < Z < z) = 0.28$

$$\Pr(Z < -z) = 0.36$$

So $-z = -0.3585$

Therefore $z = 0.3585$

6 $X \sim \mathrm{N}\left(37.5, 8.62^2\right)$

a $\Pr(X < a) = 0.72$

So $a = 42.52$

b $\Pr(X < a) = 0.68$

So $a = 41.53$

c $\Pr(37.5 - a \le X < 37.5 + a) = 0.88$

$$\Pr(X < 37.5 - a) = \frac{0.12}{2} = 0.06$$
$$37.5 - a = 24.10$$
$$a = 13.40$$

7 $Z \sim \mathrm{N}\left(0, 1^2\right)$

a $\Pr(Z < z) = 0.57$

$z = 0.1764$

b $\Pr(Z < z) = 0.63$

$z = 0.3319$

8 $X \sim \mathrm{N}\left(43.5, 9.7^2\right)$

a $\Pr(X < a) = 0.73$

$a = 49.4443$

b $\Pr(X < a) = 0.24$

$a = 36.6489$

9 $X \sim \mathrm{N}\left(\mu, 5.67^2\right)$

$$\Pr(X > 20.952) = 0.09$$
$$\Pr\left(Z > \frac{20.952 - \mu}{5.67}\right) = 0.09$$
$$\frac{20.952 - \mu}{5.67} = 1.3408$$
$$20.952 - \mu = 1.3408 \times 5.67$$
$$20.952 - \mu = 7.6023$$
$$20.952 - 7.6023 = \mu$$
$$13.3497 = \mu$$
$$\mu = 13.35$$

10 $X \sim \mathrm{N}\left(\mu, 3.5^2\right)$

$$\Pr(X < 23.96) = 0.28$$
$$\Pr\left(Z < \frac{23.96 - \mu}{3.5}\right) = 0.28$$
$$\frac{23.96 - \mu}{3.5} = -0.5828$$
$$23.96 - \mu = -0.5828 \times 3.5$$
$$23.96 - \mu = -2.038$$
$$23.96 + 2.038 = \mu$$
$$\mu = 26$$

11 $X \sim \mathrm{N}\left(115, \sigma^2\right)$

$$\Pr(X < 122.42) = 0.76$$
$$\Pr\left(Z < \frac{122.42 - 115}{\sigma}\right) = 0.76$$
$$\frac{122.42 - 115}{\sigma} = 0.7063$$
$$7.42 = 0.7063\sigma$$
$$\frac{7.42}{0.7063} = \sigma$$
$$\sigma = 10.5$$

12 $X \sim \mathrm{N}\left(41, \sigma^2\right)$

$$\Pr(X > 55.9636) = 0.11$$
$$\Pr\left(Z > \frac{55.9636 - 41}{\sigma}\right) = 0.11$$
$$\frac{55.9636 - 41}{\sigma} = 1.2265$$
$$14.9636 = 1.2265\sigma$$
$$\frac{14.9636}{1.2265} = \sigma$$
$$\sigma = 12.2$$

13 $X \sim N(\mu, \sigma^2)$

$$\Pr(X < 33.711) = 0.36$$
$$\Pr\left(Z < \frac{33.711 - \mu}{\sigma}\right) = 0.36$$
$$\frac{33.711 - \mu}{\sigma} = -0.3585$$
$$33.711 - \mu = -0.3585\sigma$$
$$33.711 = \mu - 0.3585\sigma \ldots\ldots\ldots(1)$$
$$\Pr(X < 34.10) = 0.42$$
$$\Pr\left(Z < \frac{34.10 - \mu}{\sigma}\right) = 0.42$$
$$\frac{34.10 - \mu}{\sigma} = -0.2019$$
$$34.10 - \mu = -0.2019\sigma$$
$$34.10 = \mu - 0.2019\sigma \ldots\ldots\ldots(2)$$
$$33.711 = \mu - 0.3585\sigma \ldots\ldots\ldots(1)$$

(1) − (2)

$$33.711 - 34.10 = -0.3585\sigma + 0.2019\sigma$$
$$-0.389 = -0.1566\sigma$$
$$\frac{-0.389}{-0.1566} = \sigma$$
$$\sigma = 2.5$$

Substitute $\sigma = 2.5$ into (1)

$$33.711 = \mu - 0.3585(2.5)$$
$$33.711 = \mu - 0.8963$$
$$\mu = 34.6$$

14 $X \sim N(\mu, \sigma^2)$

$$\Pr(X > 18.35) = 0.31$$
$$\Pr\left(Z > \frac{18.35 - \mu}{\sigma}\right) = 0.31$$
$$\frac{18.35 - \mu}{\sigma} = 0.4959$$
$$18.35 - \mu = 0.4959\sigma$$
$$18.35 = \mu + 0.4959\sigma \ldots\ldots\ldots(1)$$
$$\Pr(X > 15.09) = 0.45$$
$$\Pr\left(Z > \frac{15.09 - \mu}{\sigma}\right) = 0.45$$
$$\frac{15.09 - \mu}{\sigma} = -0.1257$$
$$15.09 - \mu = -0.1257\sigma$$
$$15.09 = \mu - 0.1257\sigma \ldots\ldots\ldots(2)$$
$$18.35 = \mu + 0.4959\sigma \ldots\ldots\ldots(1)$$

(1) − (2)

$$18.35 - 15.09 = 0.4959\sigma + 0.1257\sigma$$
$$3.26 = 0.6216\sigma$$
$$\frac{3.26}{0.6216} = \sigma$$
$$\sigma = 5.2$$

Substitute $\sigma = 5.2$ into (1)

$$18.35 = \mu + 0.4959(5.2)$$
$$18.35 = \mu + 2.5787$$
$$18.35 - 2.5787 = \mu$$
$$\mu = 15.8$$

15 $\Pr(a < X < b) = 0.52$ and $X \sim N(42.5, 10.3^2)$

$\Pr(X < a) = 0.24$ and $\Pr(X > b) = 0.24$

$a = 35.2251$ $\quad b = 49.7749$

So $\Pr(35.2251 < X < 49.7749) = 0.52$

$$\Pr(X > a \mid X < b) = \frac{\Pr(a < X < b)}{\Pr(X < b)}$$
$$\Pr(X < b) = 0.24 + 0.52 = 0.76$$
$$\Pr(X > a \mid X < b) = \frac{0.52}{0.76}$$
$$\Pr(X > a \mid X < b) = 0.6842$$

16 $X \sim N(\mu, \sigma^2)$

$$\Pr(X < 39.9161) = 0.5789$$
$$\Pr\left(Z < \frac{39.9161 - \mu}{\sigma}\right) = 0.5789$$
$$\frac{39.9161 - \mu}{\sigma} = 0.1991$$
$$39.9161 - \mu = 0.1991\sigma$$
$$39.9161 = \mu + 0.1991\sigma \ldots\ldots\ldots(1)$$
$$\Pr(X > 38.2491) = 0.4799$$
$$\Pr\left(Z > \frac{38.2491 - \mu}{\sigma}\right) = 0.4799$$
$$\frac{38.2491 - \mu}{\sigma} = 0.0504$$
$$38.2491 - \mu = 0.0504\sigma$$
$$38.2491 = \mu + 0.0504\sigma \ldots\ldots\ldots(2)$$
$$39.9161 = \mu + 0.1991\sigma \ldots\ldots\ldots(1)$$

(1) − (2)

$$39.9161 - 38.2491 = 0.1991\sigma - 0.0504\sigma$$
$$1.667 = 0.1487\sigma$$
$$\frac{1.667}{0.1487} = \sigma$$
$$\sigma = 11.21$$

Substitute $\sigma = 11.21$ into (1)

$$39.9161 = \mu + 0.1991(11.21)$$
$$39.9161 = \mu + 2.2319$$
$$39.9161 - 2.2319 = \mu$$
$$\mu = 37.68$$

Exercise 13.5 — Mixed probability application problems

1 a i $W \sim N(508, 3^2)$

$$\Pr(W < 500) = 0.0038$$

ii $\Pr(W < w) = 0.01$

$$w = 501.0210$$

b

$$\Pr(W < 500) \le 0.01$$
$$\Pr\left(Z < \frac{500 - 508}{\sigma}\right) \le 0.01$$
$$\frac{500 - 508}{\sigma} \le -2.3263$$
$$-8 \le -2.3263\sigma$$
$$\frac{-8}{-2.3263} \le \sigma$$
$$3.4389 \le \sigma$$

Or $\sigma \ge 3.4389$ grams

2 a

$$\sum \Pr(X = x) = 1$$
$$3k^2 + 2k + 6k^2 + 2k + k^2 + 2k + 3k = 1$$
$$10k^2 + 9k - 1 = 0 \text{ as required}$$

b $10k^2 + 9k - 1 = 0$
$(10k - 1)(k + 1) = 0$
$k = \frac{1}{10}$ as $k \neq -1$

c Let X = the chocolate surprises containing a ring
$X \sim \text{Bi}(8,\ 0.25)$
$\text{E}(X) = 8 \times 0.25 = 2$

d $\Pr(X = 2) = 0.3115$

e $\Pr(X = 0) \leq 0.09$
${}^nC_0(0.75)^n().25)^0 \leq 0.09$
$(0.75)^n \leq 0.09$
$n\log_e(0.75) \leq \log_e(0.09)$
$n \geq \frac{\log_e(0.09)}{\log_e(0.75)}$
$n \geq 8.3701$
Sample size has to be 9.

f Let W = the weight of the defective chocolate surprises
$W \sim \text{N}(125, \sigma^2)$
$\Pr(W < 100) = 0.082$
$\Pr\left(Z < \frac{100 - 125}{\sigma}\right) = 0.082$
$\frac{100 - 125}{\sigma} = -1.3917$
$-25 = -1.3917\sigma$
$\frac{-25}{-1.3917} = \sigma$
$17.9636 = \sigma$
$\sigma = 18$

3 Let X = the error in a speedometer
$X \sim \text{N}(0, 0.76^2)$
$\Pr(\text{Unacceptable}) = \Pr(X < -1.5) \cup \Pr(X > 1.5)$
$\Pr(\text{Unacceptable}) = 2\Pr(X < -1.5)$ by symmetry
$\Pr(\text{Unacceptable}) = 2 \times 0.0242$
$\Pr(\text{Unacceptable}) = 0.0484$

4 Let X = the height of Perth adult males
$X \sim \text{N}(174, 8^2)$

a $\Pr(X \geq 180) = 0.2266$ or 22.66%

b $\Pr(X \geq x) = 0.25$
$x = 179.396 = 179$ cm

5 **a** Let X = average weight of David's avocados
$X \sim \text{N}(410, 20^2)$

i $\Pr(X < 360) = 0.0062$

ii $\Pr(X > 340 \mid X < 360) = \frac{\Pr(340 < X < 360)}{\Pr(X < 360)}$
$\Pr(X > 340 \mid X < 360) = \frac{0.005977}{0.0062}$
$\Pr(X > 340 \mid X < 360) = 0.9625$

b Let Y = average weight of Jane's avocados
$Y \sim \text{N}(\mu, \sigma^2)$
$\Pr(Y < 400) = 0.4207$
$\Pr\left(Z < \frac{400 - \mu}{\sigma}\right) = 0.4207$
$\frac{400 - \mu}{\sigma} = -0.2001$
$400 - \mu = -0.2001\sigma$
$400 = \mu - 0.2001\sigma$ ....(1)
$\Pr(Y > 415) = 0.3446$
$\Pr\left(Z > \frac{415 - \mu}{\sigma}\right) = 0.3446$
$\frac{415 - \mu}{\sigma} = 0.3999$
$415 = \mu + 0.3999\sigma$ ....(2)
$400 = \mu - 0.2001\sigma$ ....(1)
$(2) - (1)$
$415 - 400 = 0.3999\sigma + 0.2001\sigma$
$15 = 0.6\sigma$
$\frac{15}{0.6} = \sigma$
$25 = \sigma$
Substitute $\sigma = 25$ into (1)
$400 = \mu - 0.2001(25)$
$400 = \mu - 3.0015$
$\mu = 405$

6 Let X = the length of metal rods
$X \sim \text{N}(145, 1.4^2)$

a $\Pr(X > 146.5) = 0.1420$

b $\Pr(X < \mu - d) = \frac{0.15}{2}$
$\Pr(X < 145 - d) = 0.075$
$145 - d = 142.9847$
$145 - 142.9847 = d$
$2.0153 = d$
$d = 2.0$

c Let Y = the number of rods with a size fault
$Y \sim \text{Bi}(12,\ 0.15)$
$\Pr(Y = 2) = 0.2924$

d **i** $a + 0.15 + 0.17 = 1$
$a + 0.32 = 1$
$a = 1 - 0.32$
$a = 0.68$

ii $\text{E}(Y) = 0.68(x - 5) + 0.15(0) + 0.17(x - 8)$
$\text{E}(Y) = 0.68x - 3.4 + 0.17x - 1.36$
$\text{E}(Y) = 0.85x - 4.76$

iii If $\text{E}(Y) = 0$ then
$0.85x - 4.76 = 0$
$0.85x = 4.76$
$x = 5.6$
Selling price of good rods will be $5.6 - 5 = 6$ cents

iv Production of good rods $= \frac{0.68}{0.68 + 0.17}$
$= 0.8$
i.e. 80%

7 $X \sim \text{N}\left(2500, 700^2\right)$ and $Y \sim \text{N}\left(3000, 550^2\right)$

a $\Pr(X < 1250) = 0.0371$

b $\Pr(Y < 1500) = 0.0032$

c $\Pr(\text{Both "special"}) = \Pr(X \cap Y)$
$\Pr(\text{Both "special"}) = \Pr(X) \times \Pr(Y)$ as they are independent events
$\Pr(\text{Both "special"}) = 0.0371 \times 0.0032$
$\Pr(\text{Both "special"}) = 0.0001$

d **i** $\Pr(\text{One "special"}) = 0.4 \times 0.0371 + 0.6 \times 0.0032$
$\Pr(\text{One "special"}) = 0.0167$

ii $\Pr(X\text{"special"} \mid \text{One "special"}) = \dfrac{\Pr(X \cap \text{One "special"})}{\Pr(\text{One "special"})}$

$\Pr(X\text{"special"} \mid \text{One "special"}) = \dfrac{0.4 \times 0.0371}{0.0167}$

$\Pr(X\text{"special"} \mid \text{One "special"}) = \dfrac{0.00744}{0.0167}$

$\Pr(X\text{"special"} \mid \text{One "special"}) = 0.8856$

8 **a** Let X = the height of plants
$X \sim \text{N}\left(18, 5^2\right)$
$\Pr(X < 10) = 0.0548$
$\Pr(10 < X < 25) = 0.8644$
$\Pr(X > 25) = 0.0808$

Plant	**Size**	**Probability**
Small	$X < 10$	0.0548
Medium	$10 < X < 25$	0.8644
Large	$X > 25$	0.0808

b $\text{E(Cost of one plant)} = 2(0.0548) + 3.5(0.8644) + 5(0.0808) = \$3.54$
$\text{E(Cost of 150 plants)} = 150 \times \$3.54 = \$531$

9 Let W = the weight of perch
$W \sim \text{N}\left(185, 20^2\right)$

a $\Pr(W > 205) = 0.1587 = 15.87\%$ (Cannery, 60 cents)

b $\Pr(165 < W < 205) = 0.6827 = 68.27\%$ (Market, 45 cents)

c $\Pr(W < 165) = 0.1587 = 15.87\%$ (Jam, 30 cents)
$\text{E(Profit)} = 60(0.1587) + 45(0.6827) + 30(0.1587) = 45.0045 = 45$ cents

10 Let X = the diameter of the tennis ball
$X \sim \text{N}\left(70, 1.5^2\right)$

a $\Pr(X < 71.5) = 0.8413$

b $\Pr(68.6 < X < 71.4) = 0.6494$

c Let Y = the tennis balls in range
$Y \sim \text{Bi}(5,\ 0.3506)$
$\Pr(Y \geq 1) = 1 - \Pr(Y = 0)$
$\Pr(Y \geq 1) = 1 - (0.64935)^5$
$\Pr(Y \geq 1) = 0.8845$

d

$$(68.6 < X < 71.4) = 0.995$$

$$\Pr\left(\frac{68.6 - 70}{\sigma} < Z < \frac{71.4 - 70}{\sigma}\right) = 0.995$$

$$\Pr\left(\frac{-1.4}{\sigma} < Z < \frac{1.4}{\sigma}\right) = 0.995$$

$$\Pr\left(Z > \frac{1.4}{\sigma}\right) = 0.0025$$

$$\frac{1.4}{\sigma} = 2.807$$

$$\frac{1.4}{2.807} = \sigma$$

$$0.4987 = \sigma$$

11 Let X = the diameter of a Fugee apple

$X \sim \mathrm{N}\left(71, 12^2\right)$

a $\mu + 2\sigma$ will be the largest possible diameter
(71+24) mm will be the largest possible diameter
95 mm will be the largest possible diameter

b $\Pr(X < 85) = 0.8783$

c $\Pr(X < 60) = 0.1797 = 18\%$

d $\Pr(X \le x) = 0.85$

$x = 83.4372$ mm

83 mm is the minimum diameter

e $\Pr(x > 100) = 0.0078$

f $\mathrm{E}(\text{Cost of one apple}) = 0.1797(0.12) + 0.6703(0.15) + 0.15(0.25) = 0.1596$ or 16 cents

$\mathrm{E}(\text{Cost of 2500 apples}) = 2500 \times 0.1596 = \$399$

g Let Y = Jumbo apples in a bag

$Y \sim \mathrm{Bi}(6,\ 0.15)$

$$\Pr(Y \ge 2) = 1 - (\Pr(Y = 0) + \Pr(Y = 1))$$
$$\Pr(Y \ge 2) = 1 - (0.3771 + 0.3993)$$
$$\Pr(Y \ge 2) = 1 - 0.7764$$
$$\Pr(Y \ge 2) = 0.2236$$

12 Let X_S = the amount of disinfectant in a standard bottle

$X_S \sim \mathrm{N}\left(0.765, 0.007^2\right)$

Let X_L = the amount of disinfectant in a large bottle

$X_L \sim \mathrm{N}\left(1.015, 0.009^2\right)$

a $\Pr(X_S < 0.75) = 0.0161$

b $\Pr(X_L < 1.00) = 0.0478$

Let Y = the large bottles with less than 0.95 litres in them

$Y \sim \mathrm{Bi}(12,\ 0.0478)$

$$\Pr(Y \ge 4) = 1 - \Pr(Y < 4)$$
$$\Pr(Y \ge 4) = 1 - (\Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) + \Pr(Y = 3))$$
$$\Pr(Y \ge 4) = 1 - (0.5556 + 0.3347 + 0.0924 + 0.0155)$$
$$\Pr(Y \ge 4) = 1 - 0.9982$$
$$\Pr(Y \ge 4) = 0.0019$$

13 Let L = the length of antenna of a lemon emigrant butterfly

$L \sim \mathrm{N}\left(22, 1.5^2\right)$

a $\Pr(L < 18) = 0.0038$

Let Y = the length of antenna of a yellow emigrant butterfly

b
$$\Pr(Y < 15.5) = 0.08$$
$$\Pr\left(Z < \frac{15.5 - \mu}{\sigma}\right) = 0.08$$
$$\frac{15.5 - \mu}{\sigma} = -1.4051$$
$$15.5 - \mu = -1.4051\sigma$$
$$15.5 = \mu - 1.4051\sigma \text{....................(1)}$$
$$\Pr(Y > 22.5) = 0.08$$
$$\Pr\left(Z > \frac{22.5 - \mu}{\sigma}\right) = 0.08$$
$$\frac{22.5 - \mu}{\sigma} = 1.4051$$
$$22.5 - \mu = 1.4051\sigma$$
$$22.5 = \mu + 1.4051\sigma \text{...................(2)}$$
$$15.5 = \mu - 1.4051\sigma \text{....................(1)}$$

(2) – (1)

$$22.5-15.5=1.4051\sigma+1.4051\sigma$$
$$2=2.8102\sigma$$
$$\frac{2}{2.8102}=\sigma$$
$$\sigma=2.5\text{ mm}$$

Substitute $\sigma=2.5$ into (1)

$$15.5=\mu-1.4051(2.5)$$
$$15.5=\mu-3.5128$$
$$15.5+3.5128=\mu$$
$$\mu=19.0\text{ mm}$$

c $\Pr(\text{Yellow})=0.45$ and $\Pr(\text{Lemon})=0.55$

Let B = the number of yellow emigrants

$B\sim\text{Bi}(12,\ 0.45)$

$\Pr(B=5)=0.2225$

14 a Let X = the error in seconds of a clock

$X\sim\text{N}(\mu,\sigma^2)$

The clock can be up to 3 seconds fast or 3 seconds slow.

$$\Pr(X>3)=0.025$$
$$\mu=0$$
$$\Pr\left(Z>\frac{3-0}{\sigma}\right)=0.025$$
$$\frac{3}{\sigma}=1.95996$$
$$3=1.95996\sigma$$
$$\frac{3}{1.95996}=\sigma$$
$$\sigma=1.5306$$

b Let Y = the number of rejected clocks

$Y\sim\text{Bi}(12,\ 0.05)$

$\Pr(Y<2)=\Pr(Y\le 1)$

$\Pr(Y<2)=0.8816$

Topic 14 — Statistical inference

Exercise 14.2 — Population parameters and sample statistics

1 Mr Parker teaches on average 120 students per day. This is the population size. $N = 120$
He asks one class of 30 about the amount of homework they have that night. This is the sample size. $n = 30$

2 Lois is able to hem 100 shirts per day. This is the population size. $N = 100$
Each day she checks 5 to make sure that they are suitable. This is the sample size. $n = 5$

3 Mr Banner tests his joke on this year's class (15 students). This is the sample size. $n = 15$
We don't know how many students Mr Banner will teach. The population size is unknown.

4 Lee-Yin asks 9 friends what they think. This is the sample size. $n = 9$
We don't know how many people will eventually eat Lee-Yin's cake pops. The population size is unknown.

5 **a** Population parameter
b Sample statistic

6 **a** Population parameter
b Sample statistic

7 Number of boys: $\frac{523}{523+621} \times 75 = 34.29$. Therefore 34 boys.
Number of girls: $\frac{621}{523+621} \times 75 = 40.71$. Therefore 41 girls.

8 Number of boarders: 23% of 90 = 20.7. Therefore 21 boarders.
The rest of the sample will be day students. $90 - 21 = 69$ day students.

9 **a** We don't know how many people will eventually eat the pudding. The population size is unknown.
b 40 volunteers to taste test your recipe. This is the sample size. $n = 40$

10 **a** We don't know how many people will eventually receive the vaccine. The population size is unknown.
b 247 suitable people test the vaccine. This is the sample size. $n = 247$

11 Sample statistic

12 Population parameter

13 Sample statistics

14 Population parameter

15 **a** A systematic sample with $k = 10$
b Yes – assuming that the order of patients is random

16 The sample is not random, therefore the results are not likely to be random

17 It is probably not random. Tony is likely to ask people that he knows, or people that approach him.

18 Number of male staff: 60% of 1500 = 900
Number of full time male staff: 95% of 900 = 855
Number to sample: $\frac{855}{1500} \times 80 = 45.6$
Number of part-time male staff: 900 – 855 = 45
Number to sample: $\frac{45}{1500} \times 80 = 2.4$
Number of female staff: 1500 – 900 = 600
Number of full time female staff: 78% of 600 = 460
Number to sample: $\frac{460}{1500} \times 80 = 24.96$
Number of part time female staff: 600 – 460 = 140
Number to sample: $\frac{140}{1500} \times 80 = 7.47$
The sample consists of:
Full time male staff: 46
Part time male staff: 2
Full time female staff: 25
Part time female staff: 7

19 Use the random number generator on your calculator to produce numbers from 1 to 100. Keep generating numbers until you have 10 different numbers. Answers will vary.

20 First assign every person in your class a number e.g. 1–25. Decide how many students will be in your sample, e.g. 5. Then use the random number generator on your calculator to produce numbers from 1 to 25. Keep generating numbers until you have 5 different numbers. The students that were assigned these numbers are the 5 students in your random sample. Answers will vary.

Exercise 14.3 — The distribution of $\hat{p}$

1 $\hat{p} = \frac{6}{15}$
$= \frac{2}{5}$

2 $\hat{p} = \frac{6}{20}$
$= \frac{3}{10}$

3 $N = 1000$
$n = 50$
$p = 0.85$
Is $10n \le N$? $10n = 500$, therefore $10n \le N$.
Is $np \ge 10$? $np = 50 \times 0.85$, therefore $np \ge 10$.
$= 42.5$
Is $nq \ge 10$? $nq = 50 \times 0.15$, therefore $nq \not\ge 10$
$= 7.5$
The sample is not large.
For the sample to be large, $nq = 10$
$0.15n = 10$
$n = 66.7$
$n = 67$ is the smallest sample size that can be considered large.

4 $np = 10$ As $p < q$, if $np \le 10$, then $nq \le 10$.
$0.05n = 10$
$n = 200$
Is $10n \le N$? $10n = 2000$, therefore $n = 200$ is a large sample.

5 **a** $\mu_{\hat{p}} = p$
$= 0.5$
b $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$
$= \sqrt{\frac{0.5 \times 0.5}{50}}$
$= 0.07$

6 If $N = 1000$, $n = 100$ and $p = 0.8$.

a $\mu_{\hat{p}} = p$
$= 0.8$

b $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$
$= \sqrt{\frac{0.8 \times 0.2}{100}}$
$= 0.04$

7 $\hat{p} = \frac{27}{30}$
$= \frac{9}{10}$

8 $\hat{p} = \frac{147}{537}$

9 **a** $p = \frac{12}{21} = \frac{4}{7}$

b 0 females chosen out of 4, 1 chosen out of 4, 2 chosen out of 4, 3 chosen out of 4 or 4 chosen out of 4.

Therefore the possible values for $\hat{p}$ are $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

c

X	$\hat{p}$	Number of samples	Relative frequency
0	0	${}^{12}C_0 \, {}^9C_4 = 126$	0.021
1	$\frac{1}{4}$	${}^{12}C_1 \, {}^9C_3 = 1008$	0.168
2	$\frac{1}{2}$	${}^{12}C_2 \, {}^9C_2 = 2376$	0.397
3	$\frac{3}{4}$	${}^{12}C_3 \, {}^9C_1 = 1980$	0.331
4	1	${}^{12}C_4 \, {}^9C_0 = 495$	0.083
	TOTAL samples	5985	

Probability distribution table:

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	0.021	0.168	0.397	0.331	0.083

d $\Pr(\hat{P} > 0.6) = \Pr\left(\hat{P} = \frac{3}{4}\right) + \Pr(\hat{P} = 1)$
$= 0.331 + 0.083$
$= 0.414$

e $\Pr(\hat{P} > 0.5 \mid \hat{P} > 0.3) = \frac{\Pr\left(\hat{P} > 0.5\right)}{\Pr\left(\hat{P} > 0.3\right)}$
$= \frac{0.414}{0.414 + 0.397}$
$= 0.510$

10 **a** $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

b

X	$\hat{p}$	$\Pr(\hat{P} = \hat{p})$
0	0	${}^5C_0\,(0.62)^0\,(0.38)^5 = 0.008$
1	$\frac{1}{5}$	${}^5C_1\,(0.62)^1\,(0.38)^4 = 0.064$
2	$\frac{2}{5}$	${}^5C_2\,(0.62)^2\,(0.38)^3 = 0.211$
3	$\frac{3}{5}$	${}^5C_3\,(0.62)^3\,(0.38)^2 = 0.344$
4	$\frac{4}{5}$	${}^5C_4\,(0.62)^4\,(0.38)^1 = 0.281$
5	1	${}^5C_5\,(0.62)^5\,(0.38)^0 = 0.092$
	TOTAL samples	5985

Probability distribution table:

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	0.008	0.064	0.211	0.344	0.281	0.092

c $\Pr(\hat{P} > 0.5) = \Pr\left(\hat{P} = \frac{3}{5}\right) + \Pr\left(\hat{P} = \frac{4}{5}\right) + \Pr(\hat{P} = 1)$
$= 0.344 + 0.281 + 0.092$
$= 0.717$

11 $np = 10$ As $p < q$, if $np \le 10$, then $nq \le 10$.
$0.01n = 10$
$n = 1000$

Is $10n \le N$? $10n = 10\,000$, therefore $n = 1000$ is a large sample.

12 $\mu_{\hat{p}} = p$
$= 0.15$

$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$
$= \sqrt{\frac{0.15 \times 0.85}{150}}$
$= 0.029$

13 $\mu_{\hat{p}} = p$
$= 0.75$

$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$
$= \sqrt{\frac{0.75 \times 0.25}{100}}$
$= 0.043$

14 $\mu_{\hat{p}} = p$
$p = 0.12$

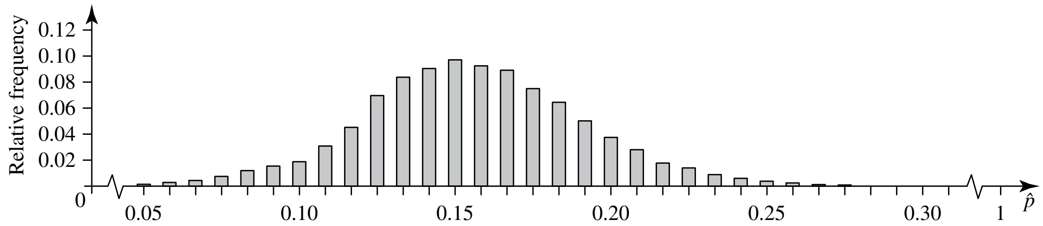
$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$
$0.0285 = \sqrt{\frac{0.12 \times 0.88}{n}}$
$8.1225 \times 10^{-4} = \frac{0.1056}{n}$
$n = \frac{0.1056}{8.1225 \times 10^{-4}}$
$= 130$

15 $\mu_{\hat{p}} = p$
$p = 0.81$

$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$
$0.0253 = \sqrt{\frac{0.81 \times 0.19}{n}}$
$6.4009 \times 10^{-4} = \frac{0.1539}{n}$
$n = \frac{0.1539}{6.4009 \times 10^{-4}}$
$= 240.4$

Therefore $n = 240$

16



17

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$
$$0.015 = \sqrt{\frac{p(1-p)}{510}}$$
$$2.25 \times 10^{-4} = \frac{p(1-p)}{510}$$
$$0.11\,475 = p(1-p)$$
$$= p - p^2$$

The quadratic $p^2 - p + 0.11\,475 = 0$ can be solved using the quadratic formula.

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{1 \pm \sqrt{1 - 4 \times 1 \times 0.11\,475}}{2}$$
$$= \frac{1 \pm \sqrt{0.541}}{2}$$

$p = 0.87$ or $p = 0.13$

As $\hat{p} > 0.5$, the population proportion is $p = 0.87$.

18

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$
$$0.0255 = \sqrt{\frac{p(1-p)}{350}}$$
$$6.5025 \times 10^{-4} = \frac{p(1-p)}{350}$$
$$0.2\,275\,875 = p(1-p)$$
$$= p - p^2$$

The quadratic $p^2 - p + 0.2\,275\,875 = 0$ can be solved using the quadratic formula.

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{1 \pm \sqrt{1 - 4 \times 1 \times 0.2\,275\,875}}{2}$$
$$= \frac{1 \pm \sqrt{0.08\,965}}{2}$$

$p = 0.65$ or $p = 0.35$

Exercise 14.4 — Confidence intervals

1 $n = 30$

$\hat{p} = 0.78$

$z = 1.96$

The 95% confidence interval is $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$$0.78 \pm 1.96\sqrt{\frac{0.78 \times 0.22}{30}}$$

63%–93%

2 $n = 53$

$\hat{p} = 0.82$

$z = 1.96$

The 95% confidence interval is $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$0.82 \pm 1.96\sqrt{\dfrac{0.82 \times 0.18}{53}}$

72%–92%

3 $n = 116$

$\hat{p} = 0.86$

$z = 2.58\ (\Pr(Z < z) = 0.005)$

The 99% confidence interval is $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

$0.86 \pm 2.58\sqrt{\dfrac{0.86 \times 0.14}{116}}$

78%–94%

4 $n = 95$

$\hat{p} = 0.3$

$z = 1.64\ (\Pr(Z < z) = 0.05)$

The 90% confidence interval is $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

$0.3 \pm 1.64\sqrt{\dfrac{0.3 \times 0.7}{95}}$

22%–38%

5 95% confidence interval $z = 1.96$

$\hat{p}$ will be at the centre of the interval, $\hat{p} = 0.4$

The confidence interval is $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$. This means that $z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = 0.05$.

$$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.05$$
$$1.96\sqrt{\frac{0.4 \times 0.6}{n}} = 0.05$$
$$\sqrt{\frac{0.24}{n}} = 0.0255$$
$$\frac{0.24}{n} = 6.5077 \times 10^{-4}$$
$$n = \frac{0.24}{6.5077 \times 10^{-4}} = 368.8$$

A sample of size 369 is needed.

6 90% confidence interval $z = 1.64$

$\hat{p}$ will be at the centre of the interval, $\hat{p} = 0.8$

The confidence interval is $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$. This means that $z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = 0.05$.

$$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.05$$
$$1.64\sqrt{\frac{0.8 \times 0.2}{n}} = 0.05$$
$$\sqrt{\frac{0.16}{n}} = 0.0305$$
$$\frac{0.16}{n} = 9.285 \times 10^{-4}$$
$$n = \frac{0.16}{9.285 \times 10^{-4}} = 172.1$$

A sample of size 173 is needed.

7 $n = 250$

$\hat{p} = \dfrac{20}{250} = 0.08$

$z = 1.96$

The 95% confidence interval is $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

$0.08 \pm 1.96\sqrt{\dfrac{0.08 \times 0.92}{250}}$

4.6%–11.4% of complaints take more than 1 day to resolve.

8 $n = 250$

$\hat{p} = \dfrac{230}{250} = 0.92$

$z = 2.58$

$0.92 \pm 2.58\sqrt{\dfrac{0.08 \times 0.92}{250}}$

87.6%–96.4% of complaints are resolved within 1 day.

9 $n = 250$

$\hat{p} = \dfrac{92}{250} = 0.368$

$z = 1.64\,485$

The 90% confidence interval $0.368 \pm 1.64\sqrt{\dfrac{0.368 \times 0.632}{250}}$

31.8%–41.8% of Australians have Type A blood.

10 $n = 250, p = 0.65$

Since n is large, we can approximate the distribution of $\hat{P}$ to that of a normal curve. Therefore $\mu = p = 0.65$ and

$\sigma = \sqrt{\dfrac{0.65 \times 0.35}{250}} = 0.030$

$\Pr(\hat{P} < 0.6) = 0.0487$

11 $n = 200, p = 0.8$

Since n is large, we can approximate the distribution of $\hat{P}$ to that of a normal curve. Therefore $\mu = p = 0.8$ and

$\sigma = \sqrt{\dfrac{0.8 \times 0.2}{200}} = 0.0283$

$$\Pr\left(0.8 < \hat{P} < 0.9 \mid \hat{P} > 0.65\right) = \frac{\Pr\left(0.8 < \hat{P} < 0.9\right)}{\Pr\left(\hat{P} > 0.65\right)} = \frac{0.4998}{0.9999} = 0.4998$$

12 $z = 1.96$

$\hat{p}$ will be at the centre of the interval, $\hat{p} = 0.3$

The confidence interval is $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$. This means that $z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = 0.05$.

$$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.05$$
$$1.96\sqrt{\frac{0.3 \times 0.7}{n}} = 0.05$$
$$\sqrt{\frac{0.21}{n}} = 0.0255$$
$$\frac{0.21}{n} = 6.5077 \times 10^{-4}$$
$$n = \frac{0.21}{6.5077 \times 10^{-4}} = 322.7$$

A sample of size 323 is needed.

13 $z = 2.58$

$\hat{p}$ will be at the centre of the interval, $\hat{p} = 0.25$

The confidence interval is $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. This means that

$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.05.$

$$\begin{aligned} z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.05 \\ 2.58\sqrt{\frac{0.25 \times 0.75}{n}} &= 0.05 \\ \sqrt{\frac{0.1875}{n}} &= 0.0194 \\ \frac{0.1875}{n} &= 3.756 \times 10^{-4} \\ n &= \frac{0.1875}{3.756 \times 10^{-4}} \\ &= 497.62 \end{aligned}$$

A sample of size 498 is needed.

14 95% confidence interval means that $z = 1.96$

The interval is 0%–5%

$\hat{p}$ will be at the centre of the interval, $\hat{p} = 0.025$ (2.5%)

The confidence interval is $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. This means that

$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.025.$

$$\begin{aligned} z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.025 \\ 1.96\sqrt{\frac{0.025 \times 0.975}{n}} &= 0.025 \\ \sqrt{\frac{0.024\,375}{n}} &= 0.012\,755 \\ \frac{0.024\,375}{n} &= 1.6269 \times 10^{-4} \\ n &= \frac{0.024\,375}{1.6269 \times 10^{-4}} \\ &= 149.8 \end{aligned}$$

A sample of size 150 is needed.

15 99% confidence interval means that $z = 2.58$

$\hat{p}$ will be at the centre of the interval, $\hat{p} = 0.94$ (94%)

The confidence interval is $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. This means that

$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.04.$

$$\begin{aligned} z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.04 \\ 2.58\sqrt{\frac{0.94 \times 0.06}{n}} &= 0.04 \\ \sqrt{\frac{0.0564}{n}} &= 0.0155 \\ \frac{0.0564}{n} &= 2.404 \times 10^{-4} \\ n &= \frac{0.0564}{2.404 \times 10^{-4}} \\ &= 234.6 \end{aligned}$$

A sample of size 235 is needed.

16 $M = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$$\begin{aligned} 0.03 &= 1.96\sqrt{\frac{0.15 \times 0.85}{n}} \\ n &= 544 \end{aligned}$$

The sample size needed is 544 people.

17 $\hat{p}$ will be at the centre of the interval, $\hat{p} = 0.90$

The confidence interval is $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. This means that

$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.05.$

$$\begin{aligned} z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.05 \\ z\sqrt{\frac{0.9 \times 0.1}{100}} &= 0.05 \\ 0.03z &= 0.05 \\ z &= 1.67 \end{aligned}$$

$\Pr(-1.67 < z < 1.67) = 0.9$

Benton's is 90% sure of their claim.

18 $\hat{p}$ will be at the centre of the interval, $\hat{p} = 0.775$

The confidence interval is $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. This means that

$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.025.$

$$\begin{aligned} z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.025 \\ z\sqrt{\frac{0.775 \times 0.225}{250}} &= 0.025 \\ 0.026z &= 0.025 \\ z &= 0.947 \end{aligned}$$

$\Pr(-0.947 < z < 0.947) = 0.66$

Benton's is 66% sure of their claim.