MATHSQUEST 12

MATHEMATICAL METHODS

VCE UNITS 3 AND 4

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VCE UNITS 3 AND 4

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Introduction

At Jacaranda, we are deeply committed to the ideal that learning brings life-changing benefits to all students. By continuing to provide resources for Mathematics of exceptional and proven quality, we ensure that all VCE students have the best opportunity to excel and to realise their full potential.

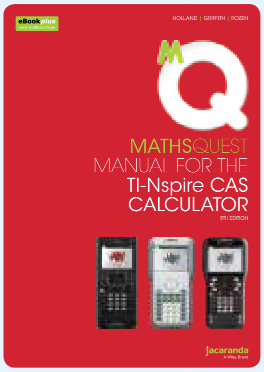
Maths Quest 12 Mathematical Methods VCE Units 3 and 4 comprehensively covers the requirements of the revised Study Design 2016–2018.

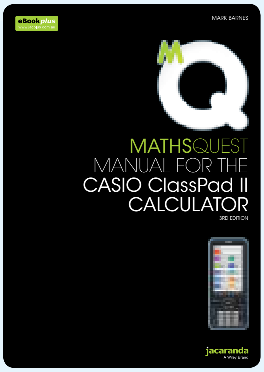
Features of the new *Maths Quest* series

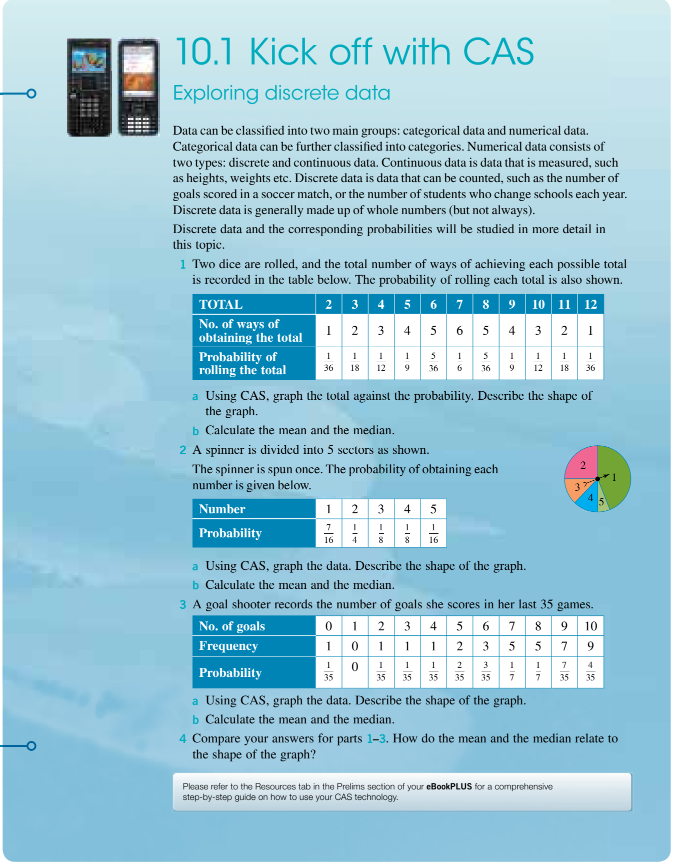
CAS technology

Each topic opens with an engaging **Kick off with CAS** activity designed to stimulate students' interest and curiosity and to highlight the important applications of CAS technology in developing deep understanding of the mathematical concepts presented.

For up-to-date, step-by-step instructions on how to use CAS technology, we have provided the *Manual for the TI-Nspire CAS calculator* and the *Manual for the Casio ClassPad II* in the Prelims section of the eBook.







10.1 Kick off with CAS

Exploring discrete data

Data can be classified into two main groups: categorical data and numerical data. Categorical data can be further classified into categories. Numerical data consists of two types: discrete and continuous data. Continuous data is data that is measured, such as heights, weights etc. Discrete data is data that can be counted, such as the number of goals scored in a soccer match, or the number of students who change schools each year. Discrete data is generally made up of whole numbers (but not always).

Discrete data and the corresponding probabilities will be studied in more detail in this topic.

1 Two dice are rolled, and the total number of ways of achieving each possible total is recorded in the table below. The probability of rolling each total is also shown.

TOTAL	2	3	4	5	6	7	8	9	10	11	12
No. of ways of obtaining the total	1	2	3	4	5	6	5	4	3	2	1
Probability of rolling the total	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

a Using CAS, graph the total against the probability. Describe the shape of the graph.

b Calculate the mean and the median.

2 A spinner is divided into 5 sectors as shown.

The spinner is spun once. The probability of obtaining each number is given below.

Number	1	2	3	4	5
Probability	$\frac{7}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$

a Using CAS, graph the data. Describe the shape of the graph.

b Calculate the mean and the median.

3 A goal shooter records the number of goals she scores in her last 35 games.

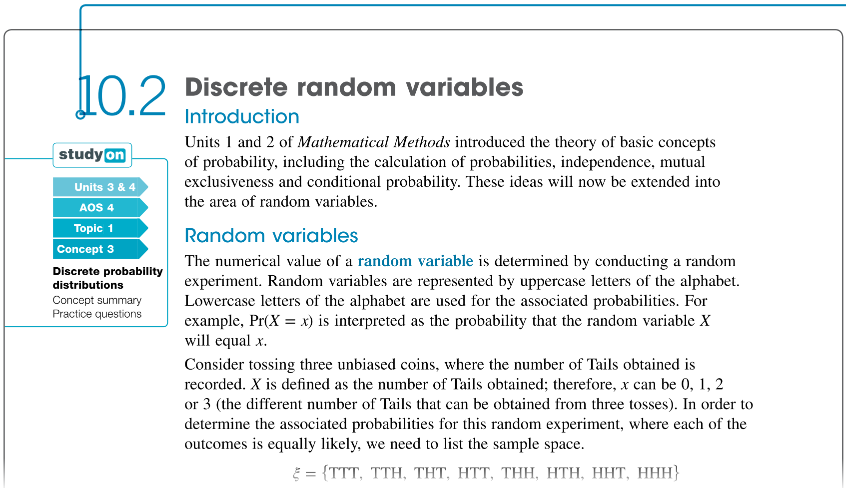
No. of goals	0	1	2	3	4	5	6	7	8	9	10
Frequency	1	0	1	1	1	2	3	5	5	7	9
Probability	$\frac{1}{35}$	0	$\frac{1}{35}$	$\frac{1}{35}$	$\frac{1}{35}$	$\frac{2}{35}$	$\frac{3}{35}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{7}{35}$	$\frac{4}{35}$

a Using CAS, graph the data. Describe the shape of the graph.

b Calculate the mean and the median.

4 Compare your answers for parts 1–3. How do the mean and the median relate to the shape of the graph?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.



10.2 **Discrete random variables**

Introduction

Units 1 and 2 of *Mathematical Methods* introduced the theory of basic concepts of probability, including the calculation of probabilities, independence, mutual exclusiveness and conditional probability. These ideas will now be extended into the area of random variables.

Random variables

The numerical value of a **random variable** is determined by conducting a random experiment. Random variables are represented by uppercase letters of the alphabet. Lowercase letters of the alphabet are used for the associated probabilities. For example, $\Pr(X = x)$ is interpreted as the probability that the random variable X will equal x.

Consider tossing three unbiased coins, where the number of Tails obtained is recorded. X is defined as the number of Tails obtained; therefore, x can be 0, 1, 2 or 3 (the different number of Tails that can be obtained from three tosses). In order to determine the associated probabilities for this random experiment, where each of the outcomes is equally likely, we need to list the sample space.

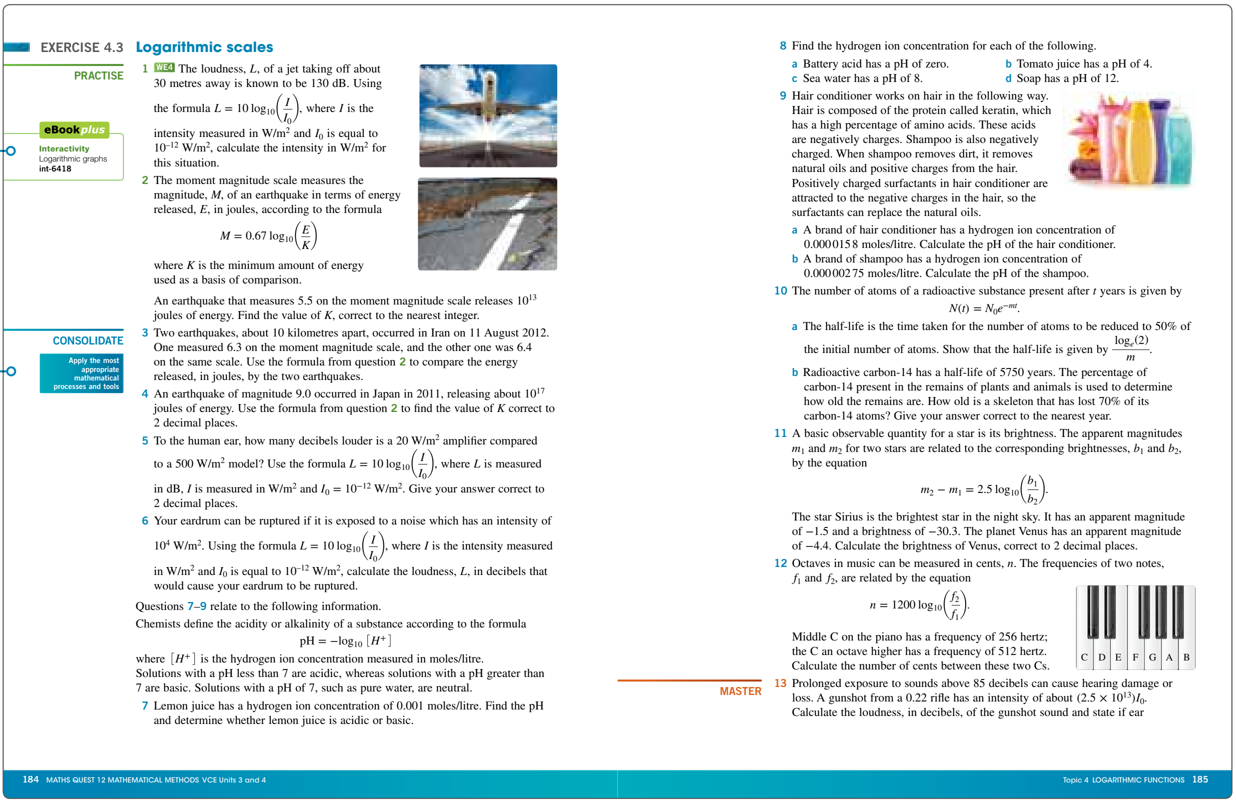
$$\xi = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$$

studyON links

Link to **studyON**, an interactive and highly visual study, revision and exam practice tool for instant feedback and on-demand progress reports.

Interactivities

Many **NEW** interactivities in the resources tab of the eBookPLUS bring difficult concepts to life to engage and excite and to consolidate understanding.



EXERCISE 4.3 **Logarithmic scales**

PRACTISE

eBook*plus*
Interactivity
Logarithmic graphs
int-6418

1 **WE4** The loudness, L, of a jet taking off about 30 metres away is known to be 130 dB. Using the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity measured in W/m^2 and I_0 is equal to 10^{-12} W/m^2, calculate the intensity in W/m^2 for this situation.

2 The moment magnitude scale measures the magnitude, M, of an earthquake in terms of energy released, E, in joules, according to the formula

$$M = 0.67\log_{10}\left(\frac{E}{K}\right)$$

where K is the minimum amount of energy used as a basis of comparison.

An earthquake that measures 5.5 on the moment magnitude scale releases 10^{13} joules of energy. Find the value of K, correct to the nearest integer.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 Two earthquakes, about 10 kilometres apart, occurred in Iran on 11 August 2012. One measured 6.3 on the moment magnitude scale, and the other one was 6.4 on the same scale. Use the formula from question 2 to compare the energy released, in joules, by the two earthquakes.

4 An earthquake of magnitude 9.0 occurred in Japan in 2011, releasing about 10^{17} joules of energy. Use the formula from question 2 to find the value of K correct to 2 decimal places.

5 To the human ear, how many decibels louder is a 20 W/m^2 amplifier compared to a 500 W/m^2 model? Use the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where L is measured in dB, I is measured in W/m^2 and $I_0 = 10^{-12}$ W/m^2. Give your answer correct to 2 decimal places.

6 Your eardrum can be ruptured if it is exposed to a noise which has an intensity of 10^4 W/m^2. Using the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity measured in W/m^2 and I_0 is equal to 10^{-12} W/m^2, calculate the loudness, L, in decibels that would cause your eardrum to be ruptured.

Questions 7–9 relate to the following information.

Chemists define the acidity or alkalinity of a substance according to the formula

$$\text{pH} = -\log_{10}\left[H^+\right]$$

where $\left[H^+\right]$ is the hydrogen ion concentration measured in moles/litre.
Solutions with a pH less than 7 are acidic, whereas solutions with a pH greater than 7 are basic. Solutions with a pH of 7, such as pure water, are neutral.

7 Lemon juice has a hydrogen ion concentration of 0.001 moles/litre. Find the pH and determine whether lemon juice is acidic or basic.

8 Find the hydrogen ion concentration for each of the following.
  a Battery acid has a pH of zero.
  b Tomato juice has a pH of 4.
  c Sea water has a pH of 8.
  d Soap has a pH of 12.

9 Hair conditioner works on hair in the following way. Hair is composed of the protein called keratin, which has a high percentage of amino acids. These acids are negatively charges. Shampoo is also negatively charged. When shampoo removes dirt, it removes natural oils and positive charges from the hair. Positively charged surfactants in hair conditioner are attracted to the negative charges in the hair, so the surfactants can replace the natural oils.
  a A brand of hair conditioner has a hydrogen ion concentration of 0.000 015 8 moles/litre. Calculate the pH of the hair conditioner.
  b A brand of shampoo has a hydrogen ion concentration of 0.000 002 75 moles/litre. Calculate the pH of the shampoo.

10 The number of atoms of a radioactive substance present after t years is given by

$$N(t) = N_0e^{-mt}.$$

  a The half-life is the time taken for the number of atoms to be reduced to 50% of the initial number of atoms. Show that the half-life is given by $\frac{\log_e(2)}{m}$.
  b Radioactive carbon-14 has a half-life of 5750 years. The percentage of carbon-14 present in the remains of plants and animals is used to determine how old the remains are. How old is a skeleton that has lost 70% of its carbon-14 atoms? Give your answer correct to the nearest year.

11 A basic observable quantity for a star is its brightness. The apparent magnitudes m_1 and m_2 for two stars are related to the corresponding brightnesses, b_1 and b_2, by the equation

$$m_2 - m_1 = 2.5\log_{10}\left(\frac{b_1}{b_2}\right).$$

The star Sirius is the brightest star in the night sky. It has an apparent magnitude of −1.5 and a brightness of −30.3. The planet Venus has an apparent magnitude of −4.4. Calculate the brightness of Venus, correct to 2 decimal places.

12 Octaves in music can be measured in cents, n. The frequencies of two notes, f_1 and f_2, are related by the equation

$$n = 1200\log_{10}\left(\frac{f_2}{f_1}\right).$$

Middle C on the piano has a frequency of 256 hertz; the C an octave higher has a frequency of 512 hertz. Calculate the number of cents between these two Cs.

MASTER

13 Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss. A gunshot from a 0.22 rifle has an intensity of about $(2.5 \times 10^{13})I_0$. Calculate the loudness, in decibels, of the gunshot sound and state if ear

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Topic 4 LOGARITHMIC FUNCTIONS 185

Graded questions

A wide variety of questions at **Practise**, **Consolidate** and **Master** levels allow students to build, apply and extend their knowledge independently and progressively.

Review

Each topic concludes with a customisable **Review**, available in the resources tab of the **eBookPLUS**, giving students the opportunity to revise key concepts covered throughout the topic. A variety of typical question types is available including short-answer, multiple-choice and extended response.

Summary

A comprehensive and fully customisable topic summary is available in the resources tab of the e**BookPLUS**, enabling students to add study notes and key information relevant to their personal study needs.

eBook*plus* **ONLINE ONLY** 1.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

eBook*plus* **ONLINE ONLY** Activities

To access eBookPLUS activities, log on to

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Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

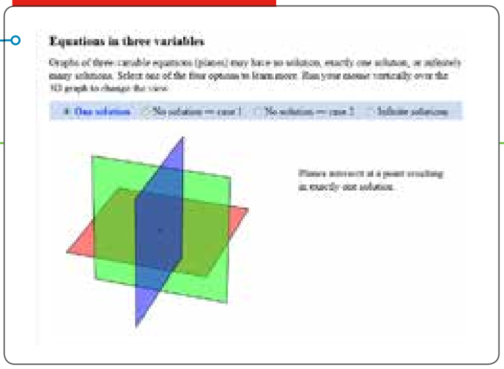
eBook*plus*

The **eBookPLUS** is available for students and teachers and contains:

- the full text online in HTML format, including PDFs of all topics
- the *Manual for the TI-Nspire CAS calculator* for step-by-step instructions
- the *Manual for the Casio ClassPad II calculator* for step-by-step instructions
- **interactivities** to bring concepts to life
- topic reviews in a customisable format
- topic summaries in a customisable format
- links to **studyON**.

Interactivities bring concepts to life

Interactivity



Equations in three variables

Graphs of three-variable equations (planes) may have no solution, exactly one solution, or infinitely many solutions. Select one of the four options to learn more. Run your mouse vertically over the 3D graph to change the view.

● One solution ○ No solution — case 1 ○ No solution — case 2 ○ Infinite solutions

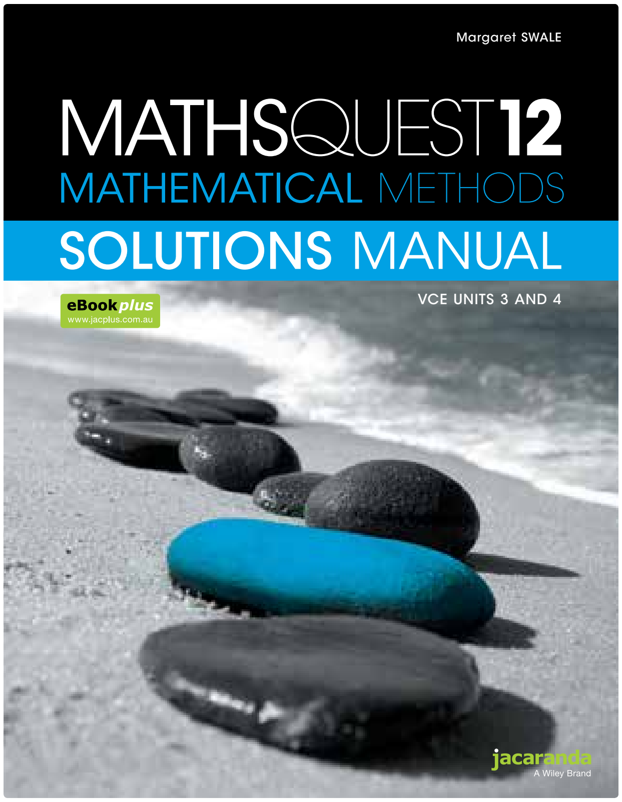
Planes intersect at a point resulting in exactly one solution.

eGuide*plus*

The **eGuidePLUS** is available for teachers and contains:

- the full **eBookPLUS**
- a Work Program to assist with planning and preparation
- School-assessed Coursework — Application task and Modelling and Problem-solving tasks, including fully worked solutions
- two tests per topic with fully worked solutions.

Maths Quest 12 Mathematical Methods Solutions Manual VCE Units 3 and 4



Available to students and teachers to purchase separately, the Solutions Manual provides fully worked solutions to every question in the corresponding student text. The Solutions Manual is designed to encourage student independence and to model best practice. Teachers will benefit by saving preparation and correction time.

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eGuidePLUS features assessment and curriculum material to support teachers.

studyON is an interactive and highly visual online study, revision and exam practice tool designed to help students and teachers maximise exam results.

studyON features:

- **Concept summary screens** provide concise explanations of key concepts, with relevant examples.
- **Access 1000+ past VCAA questions** or custom-authored practice questions at a concept, topic or entire course level, and receive immediate feedback.
- **Sit past VCAA exams** (Units 3 & 4) or **topic tests** (Units 1 & 2) in exam-like situations.
- **Video animations and interactivities** demonstrate concepts to provide a deep understanding (Units 3 & 4 only).
- **All results and performance in practice and sit questions** are tracked to a concept level to pinpoint strengths and weaknesses.

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- **Contact** John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
  Phone: 1800 JAC PLUS (1800 522 7587)

Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

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1

Solving equations

1.1 Kick off with CAS

Literal and simultaneous equations

1 Using the calculator application in CAS, solve the following literal equations for m.

a $2m - 3ab = md - 1$

b $\dfrac{3c(2 - m)}{k} = mp$

c $\dfrac{3}{m} - \dfrac{a}{m + a} = \dfrac{p}{5m}$

2 **a** Using the calculator application, solve the following pairs of simultaneous equations.

i $3x - 2y = 10$
$x + 5y = 2$

ii $2x - y = 8$
$8x - 4y = 1$

iii $x + y = 3$
$4x + 4y = 12$

b Interpret and explain the CAS results for each pair of equations.

3 **a** Using the calculator application, solve the following sets of simultaneous equations.

i $2x - 3y + z = 12$
$x + y - 3z = -13$
$-x + 2y - z = -9$

ii $2x - 3y + z = 12$
$4x - 6y + 2z = 3$

b An equation involving three variables defines a plane. Using the graph application in CAS, sketch each set of equations from part **a** on a different set of axes. Interpret the solutions found in part **a** after sketching the graphs.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

1.2 Polynomials

Factorisation

Review of quadratic expressions

eBook*plus*

Interactivities
Perfect square form of a quadratic
int-2558
Completing the square
int-2559

The following techniques are used to **factorise** quadratic expressions.

- Perfect squares: $a^2 \pm 2ab + b^2 = (a \pm b)^2$
- Difference of perfect squares: $a^2 - b^2 = (a - b)(a + b)$
- Trial and error (trinomials): To factorise $x^2 - x - 6$, we look for the factors of x^2 and –6 that, when combined, form the middle term of $-x$.

$x - 3$

$x + 2$

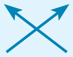
$$x^2 - x - 6 = (x - 3)(x + 2)$$

- Completing the square: The method of completing the square will work for any **quadratic** that can be factorised.

WORKED EXAMPLE 1

Use an appropriate technique to factorise each of the following quadratic expressions.

a $9a^2 - 24ab + 16b^2$ **b** $6x^2 - 17x + 7$ **c** $2t^2 + 8t - 14$

THINK	WRITE
a The first and last terms are perfect squares, so check if the expression fits the perfect square formula.	**a** $9a^2 - 24ab + 16b^2 = (3a)^2 - 2(3a)(4b) + (4b)^2$ $= (3a - 4b)^2$
b **1** Always try to factorise by the trial and error method before applying the method of completing the square.	**b** $6x^2 - 17x + 7$ Possible factors are: $3x - 7$  $2x - 1$
2 Write the answer.	$6x^2 - 17x + 7 = (3x - 7)(2x - 1)$
c **1** Take out the common factor of 2.	**c** $2t^2 + 8t - 14 = 2(t^2 + 4t - 7)$
2 Trial and error is not an appropriate method here, as the only factors of 7 are 1 and 7, and these cannot be combined to give a middle coefficient of 4. Thus, the completion of the square method is required.	$= 2(t^2 + 4t + (2)^2 - (2)^2 - 7)$ $= 2((t + 2)^2 - 4 - 7)$ $= 2((t + 2)^2 - 11)$ $= 2\left(t + 2 - \sqrt{11}\right)\left(t + 2 + \sqrt{11}\right)$

Polynomial expressions of degree 3 or higher

For **polynomials** of degree 3, it is necessary to remember the perfect cube patterns as well as the sum and difference of two cubes.

Perfect cube:

$$\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3}$$
$$\boldsymbol{a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3}$$

Sum and difference of two cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

WORKED EXAMPLE 2

Use an appropriate technique to factorise each of the following cubic expressions.

a $27y^3 - 27y^2 + 9y - 1$

b $x^3 + 8$

c $3y^3 - 81$

d $8m^3 + 60m^2 + 150m + 125$

THINK	WRITE
a This is a perfect cube pattern. Check to see that it has the pattern of $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$.	a $27y^3 - 27y^2 + 9y - 1$ $= (3y)^3 - 3(3y)^2(1) + 3(3y)(1)^2 - (1)^3$ $= (3y - 1)^3$
b This is a sum of two cubes pattern.	b $x^3 + 8 = x^3 + 2^3$ $= (x + 2)(x^2 - 2x + 2^2)$ $= (x + 2)(x^2 - 2x + 4)$
c 1 Remember to take out a common factor first.	c $3y^3 - 81 = 3(y^3 - 27)$
2 Now factorise using the difference of two cubes pattern.	$= 3(y^3 - 3^3)$ $= 3(y - 3)(y^2 + 3y + 3^2)$ $= 3(y - 3)(y^2 + 3y + 9)$
d This is a perfect cube pattern. Check to see that it has the pattern of $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$.	d $8m^3 + 60m^2 + 150m + 125$ $= (2m)^3 + 3(2m)^2(5) + 3(2m)(5)^2 + (5)^3$ $= (2m + 5)^3$

Factorising cubics using the technique of grouping

To factorise $x^3 - 3x^2 + 4x - 12$, start by grouping two and two in the following manner.

$$\underbrace{x^3 - 3x^2}_{\text{TWO}} + \underbrace{4x - 12}_{\text{TWO}}$$
$$= x^2(x - 3) + 4(x - 3)$$
$$= (x - 3)(x^2 + 4)$$

The other possible method is to group three and one. For example, to factorise $x^2 - z^2 + 4x + 4$, rearrange the expression to become

$$x^2 + 4x + 4 - z^2$$
$$= \underbrace{x^2 + 4x + 4}_{\text{THREE}} - \underbrace{z^2}_{\text{ONE}}$$
$$= (x + 2)^2 - z^2$$
$$= (x + 2 - z)(x + 2 + z)$$

WORKED EXAMPLE 3

Factorise the following polynomials.

a $m^2 - n^2 - 36 - 12n$ **b** $p^3 + 2p^2 - 4p - 8$

THINK	WRITE
a 1 Group the polynomial one and three and rearrange, taking out –1 as a common factor.	**a** $m^2 - n^2 - 36 - 12n$ $= m^2 - n^2 - 12n - 36$ $= m^2 - (n^2 + 12n + 36)$
2 Factorise the group of three terms as a perfect square.	$= m^2 - (n + 6)^2$
3 Apply the difference of perfect squares method.	$= (m - (n + 6))(n + (n + 6))$ $= (m - n - 6)(n + m + 6)$
b 1 Group the polynomial two and two.	**b** $\underbrace{p^3 + 2p^2}_{\text{TWO}} \underbrace{-4p - 8}_{\text{TWO}}$
2 Factorise each pair.	$p^3 + 2p^2 - 4p - 8$ $= p^2(p + 2) - 4(p + 2)$ $= (p + 2)(p^2 - 4)$
3 Finish the factorisation by applying the difference of perfect squares method.	$= (p + 2)(p - 2)(p + 2)$ $= (p + 2)^2(p - 2)$

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The remainder and factor theorems
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Practice questions

Factor theorem

When the previous methods are not appropriate for a third degree polynomial or a higher degree polynomial, then knowledge of the **factor theorem** is essential. The factor theorem is an algebraic theorem that links the zeros of a polynomial. It states the following:

> **A polynomial, $P(x)$, has a factor $(x - a)$ if and only if $P(a) = 0$; that is, if a is a root of the polynomial.**

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Interactivities
Long division of polynomials
int-2564
The remainder and factor theorems
int-2565

Consider the factorisation of $x^3 + 3x^2 - 13x - 15$.

Let $P(x) = x^3 + 3x^2 - 13x - 15$.

By substituting integer values of x that are factors of the constant term, we aim to achieve a zero remainder, that is, to achieve $P(x) = 0$. If this is so, we have found one linear factor of the **cubic** polynomial.

$$\begin{aligned} P(1) &= 1^3 + 3(1)^2 - 13(1) - 15 \\ &= 1 + 3 - 13 - 15 \\ &\neq 0 \\ P(-1) &= (-1)^3 + 3(-1)^2 - 13(-1) - 15 \\ &= -1 + 3 + 13 - 15 \\ &= 0 \end{aligned}$$

Thus, $(x + 1)$ is a factor. The quadratic factor can then be found by long division or by inspection.

$$\begin{array}{r r}
 & x^2 + 2x - 15 \\ \cline{2-2}
x + 1 \big) & x^3 + 3x^2 - 13x - 15 \\
 & -\ (x^3 + x^2) \\ \cline{2-2}
 & 2x^2 - 13x - 15 \\
 & -\ (2x^2 + 2x) \\ \cline{2-2}
 & -15x - 15 \\
 & -\ (-15x - 15) \\ \cline{2-2}
 & 0 \\ \cline{2-2}
\end{array}$$

$$\text{or } x^3 + 3x^2 - 13x - 15 = \underset{2x^2 + x^2}{\overset{-15x + 2x}{(x + 1)(x^2 + 2x - 15)}}$$

Completing the factorisation gives:

$$\begin{aligned} x^3 + 3x^2 - 13x - 15 &= (x + 1)(x^2 + 2x - 15) \\ &= (x + 1)(x + 5)(x - 3) \end{aligned}$$

WORKED EXAMPLE 4 **Fully factorise $x^4 - 4x^3 - x^2 + 16x - 12$.**

THINK

WRITE

1 Let $P(x)$ equal the quartic polynomial.

$$P(x) = x^4 - 4x^3 - x^2 + 16x - 12$$

2 Try $P(1)$, $P(-1)$, $P(2)$, $P(-2)$ etc. to get a zero remainder.

$$\begin{aligned} P(1) &= 1^4 - 4(1)^3 - (1)^2 + 16(1) - 12 \\ &= 17 - 17 \\ &= 0 \end{aligned}$$

Thus $(x - 1)$ is a factor.

3 Use long division to obtain the cubic factor.

$$\begin{array}{r r}
 & x^3 - 3x^2 - 4x + 12 \\ \cline{2-2}
x - 1 \big) & x^4 - 4x^3 - x^2 + 16x - 12 \\
 & -\ (x^4 - x^3) \\ \cline{2-2}
 & -3x^3 - x^2 + 16x - 12 \\
 & -\ (-3x^3 + 3x^2) \\ \cline{2-2}
 & -4x^2 + 16x - 12 \\
 & -\ (-4x^2 + 4x) \\ \cline{2-2}
 & 12x - 12 \\
 & -\ (12x - 12) \\ \cline{2-2}
 & 0 \\ \cline{2-2}
\end{array}$$

4 Let $H(x)$ equal the cubic polynomial. Apply the factor theorem again to find a linear factor of the cubic.

$$\begin{aligned} H(x) &= x^3 - 3x^2 - 4x + 12 \\ H(1) &= 1^3 - 3(1)^2 - 4(1) + 12 \\ &= 13 - 7 \\ &\neq 0 \\ H(2) &= 2^3 - 3(2)^2 - 4(2) + 12 \\ &= 20 - 20 \\ &= 0 \end{aligned}$$

Thus $(x - 2)$ is a factor.

5 Use long division to obtain the quadratic factor.

$$\begin{array}{r} x^2 - x - 6 \\ x - 2\overline{)x^3 - 3x^2 - 4x + 12} \\ -\ \underline{(x^3 - 2x^2)} \\ -x^2 - 4x + 12 \\ -\ \underline{(-x^2 + 2x)} \\ -6x + 12 \\ -\ \underline{(-6x + 12)} \\ \underline{0} \end{array}$$

6 Complete the quartic factorisation by factorising the quadratic factor into its two linear factors.

$$\begin{aligned} P(x) &= x^4 - 4x^3 - x^2 + 16x - 12 \\ &= (x - 1)(x - 2)(x^2 - x - 6) \\ &= (x - 1)(x - 2)(x - 3)(x + 2) \end{aligned}$$

Solving polynomial equations

Polynomial equations, whether they be quadratics, cubic polynomials, **quartic** polynomials or polynomials of a higher degree, can be solved using the **Null Factor Law**.

Consider again the cubic polynomial $x^3 + 3x^2 - 13x - 15$.

$$\begin{aligned} x^3 + 3x^2 - 13x - 15 &= (x + 1)(x^2 + 2x - 15) \\ &= (x + 1)(x + 5)(x - 3) \end{aligned}$$

We will equate it to zero so that we have a cubic equation to solve.

$$\begin{aligned} x^3 + 3x^2 - 13x - 15 &= 0 \\ (x + 1)(x^2 + 2x - 15) &= 0 \\ (x + 1)(x + 5)(x - 3) &= 0 \end{aligned}$$

Applying the Null Factor Law, $x = -1$, $x = -5$ or $x = 3$.

The quadratic formula

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Interactivities

The discriminant
int-2560
The quadratic formula
int-2561

Quadratic equations of the form $0 = ax^2 + bx + c$ can also be solved by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned} \text{The discriminant} &= \Delta \\ &= b^2 - 4ac \end{aligned}$$

If $\Delta > 0$, there are two real solutions to the equation.
If $\Delta = 0$, there is one real solution to the equation.
If $\Delta < 0$, there are no real solutions to the equation.

WORKED EXAMPLE 5

a Solve $x^4 - 4x^3 - x^2 + 16x - 12 = 0.$

b Solve $2a^4 - 5a^2 - 3 = 0.$

THINK	WRITE
a 1 The quartic expression was factorised in Worked example 4.	**a** $x^4 - 4x^3 - x^2 + 16x - 12 = 0$ $(x - 1)(x - 2)(x - 3)(x + 2) = 0$
2 Use the Null Factor Law to solve the quartic polynomial for x.	$(x - 1)(x - 2)(x - 3)(x + 2) = 0$ $x = 1, 2, 3, -2$
b 1 The left-hand side is in quadratic form. Let $m = a^2$ to help with the factorisation.	**b** $2a^4 - 5a^2 - 3 = 0$ Let $m = a^2$. $2m^2 - 5m - 3 = 0$
2 Factorise the quadratic.	$(2m + 1)(m - 3) = 0$
3 Substitute $m = a^2$ and factorise further where possible. *Note:* There is no factorisation technique for the addition of perfect squares.	$(2a^2 + 1)(a^2 - 3) = 0$ $(2a^2 + 1)(a - \sqrt{3})(a + \sqrt{3}) = 0$
4 Solve the equation.	$2a^2 + 1 = 0$ has no real solution. $\therefore a = \pm\sqrt{3}$

study on

- Units 3 & 4
- AOS 2
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- Concept 2

Equating coefficients
Concept summary
Practice questions

Equality of polynomials

Two polynomials $P(x)$ and $Q(x)$ are such that

$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2x^2 + a_1x + a_0$ and

$Q(x) = b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + ... + b_2x^2 + b_1x + b_0.$

$P(x)$ is identically equal to $Q(x)$ for all values of x (that is, $P(x) \equiv Q(x)$) if and only if

$a_n = b_n,\ a_{n-1} = b_{n-1},\ a_{n-2} = b_{n-2}...a_2 = b_2,\ a_1 = b_1$ and $a_0 = b_0$.

For instance, $3x^3 + (m - 2)x^2 + (m + n)x \equiv kx^3 + x^2$. You are required to find the values of m, n and k. As the polynomials are equal to each other, we can equate coefficients to give:

$$k = 3 \qquad \begin{aligned} m - 2 &= 1 \\ m &= 3 \end{aligned} \qquad \begin{aligned} m + n &= 0 \\ 3 + n &= 0 \\ n &= -3 \end{aligned}$$

WORKED EXAMPLE 6

If $mx^4 + (n - 3)x^3 + (m + p)x^2 + (p + q)x \equiv -2x^4 - 2x^3 + x^2$, find the values of m, n, p and q.

THINK	WRITE
1 Equate the coefficients of the x^4 terms.	$m = -2$
2 Equate the coefficients of the x^3 terms and solve for n.	$n - 3 = -2$ $n = 1$

3 Equate the coefficients of the x^2 terms and solve for p.	$m + p = 1$ $-2 + p = 1$ $p = 3$
4 Equate the coefficients of the x terms and solve for q. Note, as there is no x term on the right-hand side, the coefficient is zero.	$p + q = 0$ $3 + q = 0$ $q = -3$
5 Write the answer.	$m = -2,\ n = 1,\ p = 3,\ q = -3$

EXERCISE 1.2 Polynomials

PRACTISE

Work without CAS

1 **WE1** Use an appropriate method to factorise each of the following quadratic expressions.

a $15u^2 - u - 2$

b $6d^2 - 28d + 16$

c $3j^2 + 12j - 6$

2 Use an appropriate method to factorise each of the following quadratic expressions.

a $f^2 - 12f - 28$

b $g^2 + 3g - 4$

c $b^2 - 1$

3 **WE2** Use an appropriate method to factorise each of the following cubic expressions.

a $125a^3 - 27b^3$

b $2c^3 + 6c^2d + 6cd^2 + 2d^3$

c $40p^3 - 5$

4 Use an appropriate method to factorise each of the following cubic expressions.

a $27z^3 - 54z^2 + 36z - 8$

b $m^3n^3 + 64$

5 **WE3** Fully factorise the following polynomials.

a $9x^2 - xy - 3x + y$

b $3y^3 + 3y^2z^2 - 2zy - 2z^3$

6 Fully factorise the following polynomials.

a $9a^2 - 16b^2 - 12a + 4$

b $n^2p^2 - 4m^2 - 4m - 1$

7 **WE4** Fully factorise $x^3 - 2x^2 - 21x - 18$.

8 Fully factorise $x^4 - 5x^3 - 32x^2 + 180x - 144$.

9 **WE5** Solve the following for x.

a $x^4 - 8x^3 + 17x^2 + 2x - 24 = 0$

b $a^4 + 2a^2 - 8 = 0$

10 Solve the following for x.

a $2x^3 - x^2 - 10x + 5 = 0$

b $2a^2 - 5a = 9$

11 **WE6** If $Ax^3 + (B - 1)x^2 + (B + C)x + D \equiv 3x^3 - x^2 + 2x - 7$, find the values of A, B, C and D.

12 If $x^3 + 9x^2 - 2x + 1 \equiv x^3 + (dx + e)^2 + 4$, find the values of d and e.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

13 Factorise the following expressions.

a $7r^3 - 49r^2 + r - 7$

b $36v^3 + 6v^2 + 30v + 5$

c $2m^3 + 3m^2 - 98m - 147$

d $2z^3 - z^2 + 2z - 1$

e $4x^2 - 28x + 49 - 25y^2$

f $16a^2 - 4b^2 - 12b - 9$

g $v^2 - 4 - w^2 + 4w$

h $4p^2 - 1 + 4pq + q^2$

14 Fully factorise and solve the following quadratic equations over R.

a $81y^2 = 1$

b $4z^2 + 28z + 49 = 0$

c $5m^2 + 3 = 10m$

d $x^2 - 4x = -3$

e $48p = 24p^2 + 18$

f $39k = 4k^2 + 77$

g $m^2 + 3m = 4$

h $4n^2 = 8 - 5n$

15 Solve each of the following equations over R.

a $2x^3 + 7x^2 + 2x - 3 = 0$

b $l^4 - 17l^2 + 16 = 0$

c $c^4 + c^3 - 10c^2 - 4c + 24 = 0$

d $p^4 - 5p^3 + 5p^2 + 5p - 6 = 0$

16 Solve each of the following equations over R.

a $b^3 + 5b^2 + 2b - 8 = 0$

b $-2m^3 + 9m^2 - m - 12 = 0$

c $2x^3 - x^2 - 6x + 3 = 0$

17 a Show that $3t^3 + 22t^2 + 37t + 10$ is divisible by $(t + 5)$ and hence solve the equation $3t^3 + 22t^2 + 37t + 10 = 0$.

b Show that $3d^3 - 16d^2 + 12d + 16$ is divisible by $(d - 2)$ and hence solve the equation $3d^3 + 22d^2 + 37d + 10 = 0$.

18 Solve each of the following equations over R.

a $a^4 - 10a^2 + 9 = 0$

b $4k^4 - 101k^2 + 25 = 0$

c $9z^4 - 145z^2 + 16 = 0$

d $(x^2 - 2x)^2 - 47(x^2 - 2x) - 48 = 0$

19 a Given that $P(z) = 5z^3 - 3z^2 + 4z - 1$ and $Q(z) = az^3 + bz^2 + cz + d$, find the values of a, b, c and d if $P(z) \equiv Q(z)$.

b Given that $P(x) = x^3 - 6x^2 + 9x - 1$ and $Q(x) = x(x + a)^2 - b$, find the values of a and b if $P(x) \equiv Q(x)$.

20 If $2x^3 - 5x^2 + 5x - 5 \equiv a(x - 1)^3 + b(x - 1)^2 + c(x - 1) + d$, find the values of a, b, c and d and hence express $2x^3 - 5x^2 + 5x - 5$ in the form $a(x - 1)^3 + b(x - 1)^2 + c(x - 1) + d$.

21 a For what values of k does the equation $kx^2 - 3x + k = 0$ have no solutions?

b If $kx^2 + 4x - k + 2 = 0$, show that the equation has a solution for all values of k.

22 Given $(x + 3)$ and $(x - 1)$ are factors of $ax^3 + bx^2 - 4x - 3$, find the values of a and b.

MASTER

23 If $(x + 2)$, $(x - 3)$ and $(x + 4)$ are factors of $x^4 + ax^3 + bx^2 + cx + 24$, find the values of a, b and c.

24 A quadratic equation has the rule $(m - 1)x^2 + \left(\frac{5 - 2m}{2}\right)x + 2m = 0$. Find the value(s) of m for which the quadratic equation has two solutions.

1.3 Trigonometric symmetry properties

The unit circle

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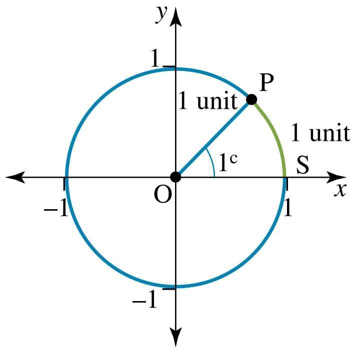
Interactivities

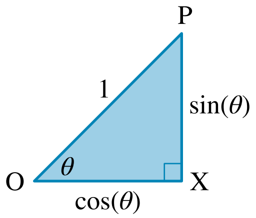
The unit circle
int-2582

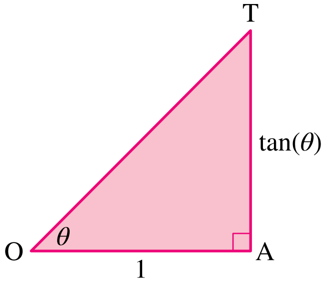
Symmetry points and quadrants
int-2584

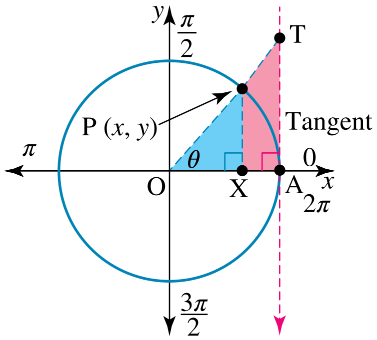


Angles are measured in degrees or **radians**. To define a radian, a circle with a radius of one unit is used. This circle is called the unit circle. When the point P is moved around the circle such that the arc length from S to P is 1 unit, the angle SOP is defined. The measure of this angle is 1 radian, 1^c.

For the blue right-angled triangle, where θ is the angle at the origin, we know that, by definition, the distance along the x-axis is defined as $\cos(\theta)$ and the distance along the y-axis is defined as $\sin(\theta)$. In addition, if we consider the similar triangles POX (blue) and TOA (pink), the following important facts can be observed.







For triangle TOA, by definition, $\tan(\theta) = \frac{\text{TA}}{\text{OA}} = \frac{\text{TA}}{1}$; hence, $\text{TA} = \tan(\theta)$

Using similar triangles, we can say that

$$\frac{\tan(\theta)}{\sin(\theta)} = \frac{1}{\cos(\theta)}$$

So

$$\boxed{\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}}$$

This result is known as one of the trigonometric identities. You should learn and remember it, as it will be used frequently in later sections.

Also, if we consider the triangle POX, then by **Pythagoras' theorem**,

$$(\sin(\theta))^2 + (\cos(\theta))^2 = 1$$

or

$$\boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$$

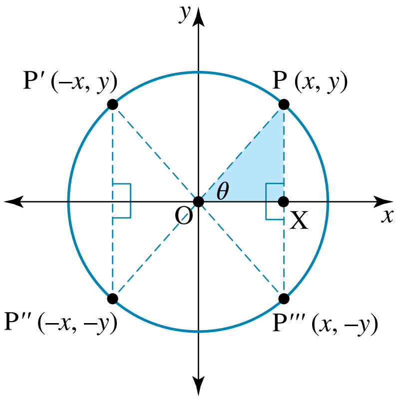
This is known as the **Pythagorean identity** and should also be learned and remembered.

Special values for sine, cosine and tangent

Using the unit circle and rotating anticlockwise, we can determine the values of sine and cosine for the angles 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π by reading off the x- or y-axis. The value for tangent is determined by the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

Angle (θ)	**sin(θ)**	**cos(θ)**	**tan(θ)**
0	0	1	0
$\frac{\pi}{2}$	1	0	Undefined
π	0	–1	0
$\frac{3\pi}{2}$	–1	0	Undefined
2π	0	1	0

The first quadrant: $0° < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$



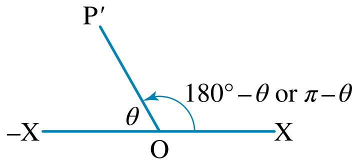
For $0° < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$,

$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$$

The second quadrant: $90° < \theta < 180°$ or $\frac{\pi}{2} < \theta < \pi$

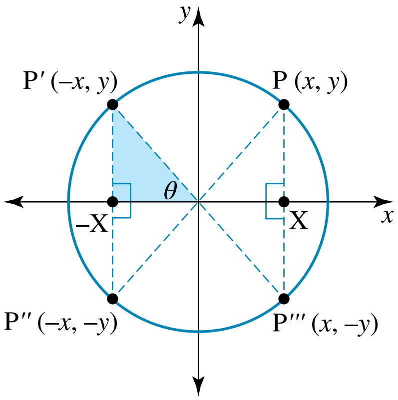


Consider the point P′ in the second quadrant. When θ is the angle in the blue triangle at the origin, angles in the second quadrant are usually expressed as $180° - \theta$ or $\pi - \theta$. The angle refers to the angle made with respect to the positive direction of the x-axis and in an anticlockwise direction.

All angles in the second, third and fourth quadrants can be related back to the first quadrant.

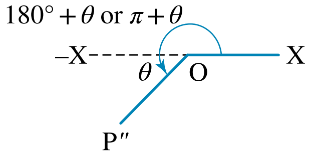
Remember that in the first quadrant, $x = \cos(\theta)$, $y = \sin(\theta)$ and $\tan(\theta) = \frac{y}{x}$.

So in the second quadrant, using symmetry, the angles are:



$$\begin{aligned}
\cos(\pi - \theta) &= -x = -\cos(\theta) \\
\sin(\pi - \theta) &= y = \sin(\theta) \\
\tan(\pi - \theta) &= \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} \\
&= \frac{y}{-x} \\
&= -\tan(\theta)
\end{aligned}$$

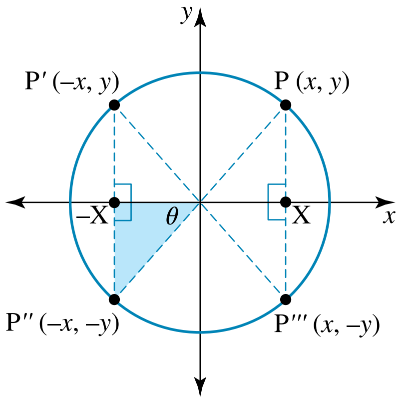
The third quadrant: $180° < \theta < 270°$ or $\pi < \theta < \frac{3\pi}{2}$



When the point P″ is in the third quadrant and θ is the angle in the blue triangle at the origin, angles in the third quadrant are usually expressed as $180° + \theta$ or $\pi + \theta$. The angle refers to the angle made with respect to the positive direction of the x-axis and in an anticlockwise direction.

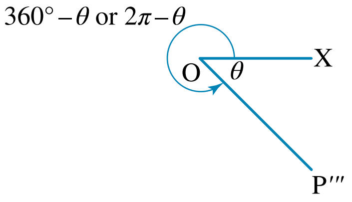
Remember that in the first quadrant, $x = \cos(\theta)$, $y = \sin(\theta)$ and $\tan(\theta) = \frac{y}{x}$.

So in the third quadrant, using symmetry, the angles are:



$$\begin{aligned}
\cos(\pi + \theta) &= -x = -\cos(\theta)\\
\sin(\pi + \theta) &= -y = -\sin(\theta)\\
\tan(\pi + \theta) &= \frac{\sin(\pi + \theta)}{\cos(\pi + \theta)}\\
&= \frac{-y}{-x}\\
&= \tan(\theta)
\end{aligned}$$

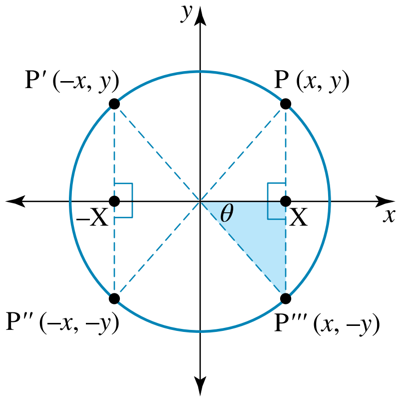
The fourth quadrant: $270° < \theta < 360°$ or $\frac{3\pi}{2} < \theta < 2\pi$



When P‴ is a point in the fourth quadrant and θ is the angle in the blue triangle at the origin, angles in the fourth quadrant are usually expressed as $360° - \theta$ or $2\pi - \theta$. The angle refers to the angle made with respect to the positive direction of the x-axis and in an anticlockwise direction.

Remember that in the first quadrant, $x = \cos(\theta)$, $y = \sin(\theta)$ and $\tan(\theta) = \frac{y}{x}$.

So in the fourth quadrant, using symmetry, the angles are:



$$\begin{aligned}
\cos(2\pi - \theta) &= x = \cos(\theta)\\
\sin(2\pi - \theta) &= -y = -\sin(\theta)\\
\tan(2\pi - \theta) &= \frac{\sin(2\pi - \theta)}{\cos(2\pi - \theta)}\\
&= \frac{-y}{x}\\
&= -\tan(\theta)
\end{aligned}$$

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All sin cos tan
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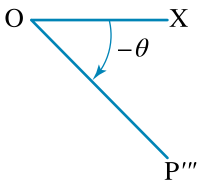
The summary of the results from all four quadrants is as follows.

2nd quadrant $\sin(\pi - \theta) = \sin(\theta)$ $\cos(\pi - \theta) = -\cos(\theta)$ $\tan(\pi - \theta) = -\tan(\theta)$ **S** **Sin positive**	**1st quadrant** $\sin(\theta)$ $\cos(\theta)$ $\tan(\theta)$ **A** **All positive**
T **Tan positive** $\sin(\pi + \theta) = -\sin(\theta)$ $\cos(\pi + \theta) = -\cos(\theta)$ $\tan(\pi + \theta) = \tan(\theta)$ **3rd quadrant**	**C** **Cos positive** $\sin(2\pi - \theta) = -\sin(\theta)$ $\cos(2\pi - \theta) = \cos(\theta)$ $\tan(2\pi - \theta) = -\tan(\theta)$ **4th quadrant**

Negative angles

Angles measured in a clockwise direction rather than in an anticlockwise direction are called negative angles.

$$\begin{aligned}
\cos(-\theta) &= x = \cos(\theta) \\
\sin(-\theta) &= -y = -\sin(\theta) \\
\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} \\
&= \frac{-y}{x} \\
&= -\tan(\theta)
\end{aligned}$$



So

$$\begin{aligned}
\cos(-\theta) &= \cos(\theta) \\
\sin(-\theta) &= -\sin(\theta) \\
\tan(-\theta) &= -\tan(\theta)
\end{aligned}$$

Note: These relationships are true no matter what quadrant the negative angle is in.

WORKED EXAMPLE 7

If $\sin(\theta) = \frac{\sqrt{3}}{2}$ and $\cos(\alpha) = \frac{4}{5}$, and θ and α are in the first quadrant, find the values of the following.

- **a** $\sin(\pi + \theta)$
- **b** $\cos(-\alpha)$
- **c** $\tan(\theta)$
- **d** $\cos(\pi - \theta)$
- **e** $\sin(\pi + \alpha)$
- **f** $\tan(2\pi - \alpha)$

THINK	WRITE
a 1 $(\pi + \theta)$ means the 3rd quadrant, where sine is negative.	**a** $\sin(\pi + \theta) = -\sin(\theta)$
2 Substitute the appropriate value.	$= -\frac{\sqrt{3}}{2}$
b 1 $(-\alpha)$ means the 4th quadrant, where cosine is positive.	**b** $\cos(-\alpha) = \cos(\alpha)$
2 Substitute the appropriate value.	$= \frac{4}{5}$

c **1** Use the Pythagorean identity to find the value of $\cos(\theta)$.

c
$$\begin{aligned}\sin^2(\theta) + \cos^2(\theta) &= 1\\ \cos^2(\theta) &= 1 - \sin^2(\theta)\\ \cos^2(\theta) &= 1 - \left(\frac{\sqrt{3}}{2}\right)^2\\ \cos^2(\theta) &= 1 - \frac{3}{4}\\ \cos^2(\theta) &= \frac{1}{4}\\ \cos(\theta) &= \pm\frac{1}{2}\end{aligned}$$

θ is in the first quadrant, so cosine is positive. Hence, $\cos(\theta) = \frac{1}{2}$.

2 Use the identity $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ to find $\tan(\theta)$.

$$\begin{aligned}\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)}\\ &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\\ &= \frac{\sqrt{3}}{2} \times \frac{2}{1}\\ &= \sqrt{3}\end{aligned}$$

d **1** $(\pi - \theta)$ means the 2nd quadrant, where cosine is negative.

d $\cos(\pi - \theta) = -\cos(\theta)$

2 In part **c** we determined $\cos(\theta) = \frac{1}{2}$, so we can substitute this value.

$$= -\frac{1}{2}$$

e **1** In order to find the value of $\sin(\alpha)$, apply the Pythagorean identity.

e
$$\begin{aligned}\sin^2(\alpha) + \cos^2(\alpha) &= 1\\ \sin^2(\alpha) &= 1 - \cos^2(\alpha)\\ \sin^2(\alpha) &= 1 - \left(\frac{4}{5}\right)\\ \sin^2(\alpha) &= 1 - \frac{16}{25}\\ \sin^2(\alpha) &= \frac{9}{25}\\ \sin(\alpha) &= \pm\frac{3}{5}\end{aligned}$$

α is in the first quadrant, so sine is positive. Hence, $\sin(\alpha) = \frac{3}{5}$.

2 $(\pi + \alpha)$ means the 3rd quadrant, where sine is negative.

$\sin(\pi + \alpha) = -\sin(\alpha)$

3 Substitute the appropriate value.

$$= -\frac{3}{5}$$

f Use the identity $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ and simplify.

$$\textbf{f}\quad \tan(2\pi - \alpha) = \frac{\sin(2\pi - \alpha)}{\cos(2\pi - \alpha)}$$
$$= \frac{-\sin(\alpha)}{\cos(\alpha)}$$
$$= \frac{-\frac{3}{5}}{\frac{4}{5}}$$
$$= -\frac{3}{4}$$

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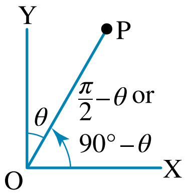
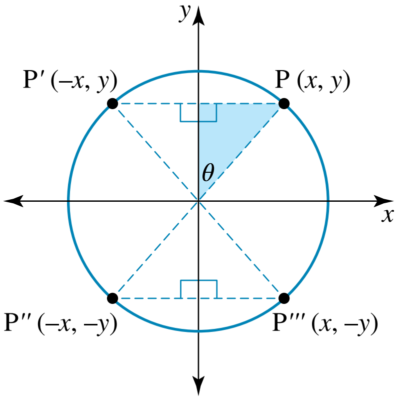
Complementary angles

Sometimes angles are named relative to the y-axis rather than the x-axis, for example $\frac{\pi}{2} \pm \theta$ or $\frac{3\pi}{2} \pm \theta$. These are special cases, and great care should be taken with these types of examples.

The first quadrant: reference angle 90° – θ or $\frac{\pi}{2} - \theta$

Remember that in the first quadrant the distance along the x-axis is defined as $\cos(\theta)$ while the distance along the y-axis is defined as $\sin(\theta)$.





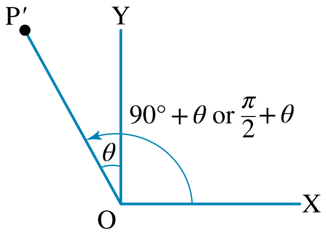
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

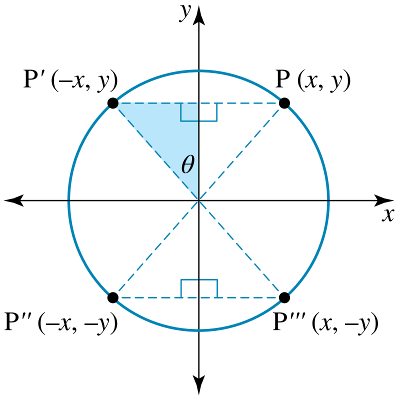
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sin\left(\frac{\pi}{2} + \theta\right)}{\cos\left(\frac{\pi}{2} + \theta\right)} = \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)}$$

Note: $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$ is a **complementary** relationship because the sum of their angles adds to $\frac{\pi}{2}$.

The second quadrant: reference angle 90° + θ or $\frac{\pi}{2} + \theta$



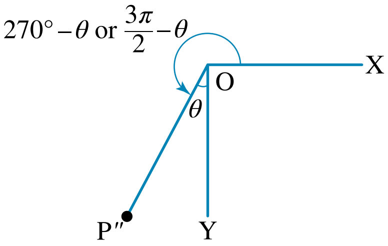


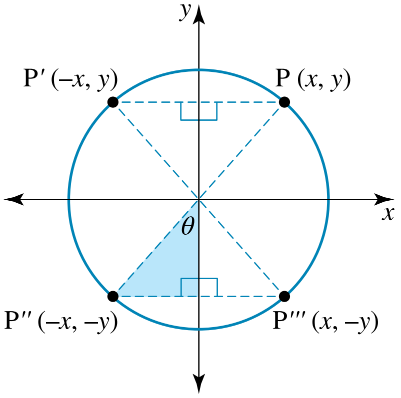
$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta)$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = \frac{\sin\left(\frac{\pi}{2} + \theta\right)}{\cos\left(\frac{\pi}{2} + \theta\right)} = \frac{\cos(\theta)}{-\sin(\theta)} = -\frac{1}{\tan\theta}$$

The third quadrant: reference angle 270° ± θ or $\frac{3\pi}{2} - \theta$



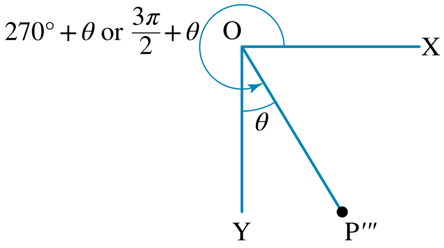


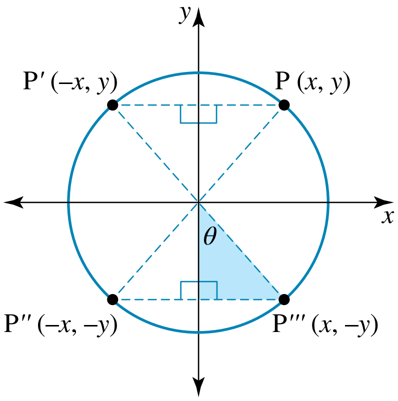
$$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin(\theta)$$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos(\theta)$$

$$\tan\left(\frac{3\pi}{2} - \theta\right) = \frac{\sin\left(\frac{3\pi}{2} - \theta\right)}{\cos\left(\frac{3\pi}{2} - \theta\right)} = \frac{-\cos(\theta)}{-\sin(\theta)} = \frac{1}{\tan\theta}$$

The fourth quadrant: reference angle 270° + θ or $\frac{3\pi}{2} + \theta$





$$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin(\theta)$$

$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos(\theta)$$

$$\tan\left(\frac{3\pi}{2} + \theta\right) = \frac{\sin\left(\frac{3\pi}{2} + \theta\right)}{\cos\left(\frac{3\pi}{2} + \theta\right)} = \frac{-\cos(\theta)}{\sin(\theta)} = -\frac{1}{\tan\theta}$$

WORKED EXAMPLE 8

If $\cos(\theta) = 0.5300$ and θ is in the first quadrant, find the values of the following, correct to 4 decimal places.

a $\sin\left(\frac{\pi}{2} - \theta\right)$ **b** $\cos\left(\frac{3\pi}{2} - \theta\right)$ **c** $\tan\left(\frac{3\pi}{2} + \theta\right)$

THINK	WRITE
a 1 $\left(\frac{\pi}{2} - \theta\right)$ is in the 1st quadrant, so all trigonometric ratios are positive.	**a** $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$
2 Substitute the appropriate value.	$= 0.5300$
b 1 $\left(\frac{3\pi}{2} - \theta\right)$ is in the 3rd quadrant, so sine is negative.	**b** $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin(\theta)$
2 Use the Pythagorean identity to find $\sin(\theta)$.	$\sin^2(\theta) + \cos^2(\theta) = 1$ $\sin^2(\theta) = 1 - \cos^2(\theta)$ $\sin^2(\theta) = 1 - (0.5300)^2$ $\sin(\theta) = \pm\sqrt{1 - (0.5300)^2}$ $\sin(\theta) = 0.8480$ as θ is in the first quadrant.
3 Substitute the appropriate values to determine $\cos\left(\frac{3\pi}{2} - \theta\right)$.	$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin(\theta)$ $= -0.8480$

c **1** $\left(\frac{3\pi}{2}+\theta\right)$ is in the 4th quadrant, so tangent is negative.

c $\tan\left(\frac{3\pi}{2}+\theta\right)=-\frac{1}{\tan\theta}$

2 Use the identity $\tan\theta=\frac{\sin(\theta)}{\cos(\theta)}$ to find the reciprocal.

$$\frac{1}{\tan(\theta)}=\frac{\cos(\theta)}{\sin(\theta)}$$

$$=\frac{0.5300}{0.8480}$$

$$=0.6250$$

$$\therefore\tan\left(\frac{3\pi}{2}+\theta\right)=-0.6250$$

study on

Units 3 & 4

AOS 2

Topic 3

Concept 1

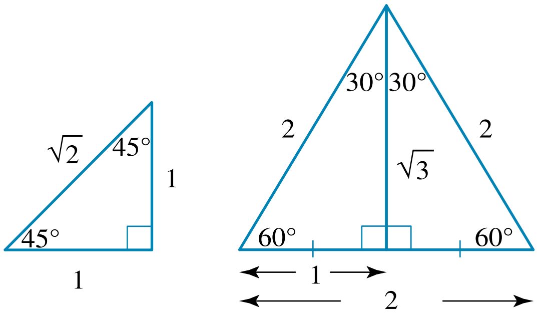
Exact values

Concept summary

Practice questions

Exact values

In Mathematical Methods Unit 2, you studied the exact trigonometric ratios for the angles $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ (30°, 45°, 60°). These values come from an isosceles triangle and an equilateral triangle.



The table below provides a summary of these angles and their ratios.

Angle (θ)	**sin(θ)**	**cos(θ)**	**tan(θ)**
30° or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45° or $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60° or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

WORKED EXAMPLE 9

Give exact values for each of the following trigonometric expressions.

a $\cos\left(\frac{2\pi}{3}\right)$

b $\tan\left(\frac{7\pi}{4}\right)$

c $\cos\left(-\frac{\pi}{6}\right)$

d $\sin\left(\frac{11\pi}{3}\right)$

THINK

a **1** Rewrite the angle in terms of π and find the corresponding angle in the first quadrant.

WRITE

a $\cos\left(\frac{2\pi}{3}\right)=\cos\left(\pi-\frac{\pi}{3}\right)$

2 The angle is in the 2nd quadrant, so cosine is negative.	$= -\cos\left(\frac{\pi}{3}\right)$
3 Write the answer.	$= -\frac{1}{2}$
b **1** Rewrite the angle in terms of 2π and find the corresponding angle in the first quadrant.	**b** $\tan\left(\frac{7\pi}{4}\right) = \tan\left(2\pi - \frac{\pi}{4}\right)$
2 The angle is in the 4th quadrant, so tangent is negative.	$= -\tan\left(\frac{\pi}{4}\right)$
3 Write the answer.	$= -1$
c **1** Rewrite the negative angle as $\cos(-\theta) = \cos(\theta)$.	**c** $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$
2 Write the answer.	$= \frac{\sqrt{3}}{2}$
d **1** Rewrite the angle in terms of a multiple of 2π.	**d** $\sin\left(\frac{11\pi}{3}\right) = \sin\left(4\pi - \frac{\pi}{3}\right)$
2 Subtract the extra multiple of 2π so the angle is within one revolution of the unit circle.	$= \sin\left(2\pi - \frac{\pi}{3}\right)$
3 The angle is in the 4th quadrant, so sine is negative.	$= -\sin\left(\frac{\pi}{3}\right)$
4 Write the answer.	$= -\frac{\sqrt{3}}{2}$

EXERCISE 1.3 Trigonometric symmetry properties

PRACTISE

Work without CAS
Questions 1, 5 and 6

1 **WE7** Evaluate the following expressions correct to 4 decimal places, given that $\sin(\theta) = 0.4695$, $\cos(\alpha) = 0.5592$ and $\tan(\beta) = 0.2680$, where θ, α and β are in the first quadrant.

a $\sin(2\pi - \theta)$
b $\cos(\pi - \alpha)$
c $\tan(-\beta)$
d $\sin(\pi + \theta)$
e $\cos(2\pi - \alpha)$
f $\tan(\pi + \beta)$

2 Evaluate the following expressions correct to 4 decimal places, given that $\sin(\theta) = 0.4695$ and $\cos(\alpha) = 0.5592$, where θ and α are in the first quadrant.

a $\cos(-\theta)$
b $\tan(180° - \theta)$
c $\sin(360° + \alpha)$
d $\tan(360° - \alpha)$

3 **WE8** Evaluate the following expressions correct to 4 decimal places, given that $\cos(\theta) = 0.8829$ and $\sin(\alpha) = 0.1736$, where θ and α are in the first quadrant.

a $\sin\left(\frac{\pi}{2} + \theta\right)$
b $\cos\left(\frac{3\pi}{2} - \theta\right)$
c $\tan\left(\frac{\pi}{2} - \theta\right)$
d $\sin\left(\frac{3\pi}{2} + \alpha\right)$
e $\sin\left(\frac{\pi}{2} - \alpha\right)$
f $\tan\left(\frac{3\pi}{2} + \alpha\right)$

4 Evaluate the following expressions correct to 4 decimal places, given that $\sin(\theta) = 0.8290$ and $\cos(\beta) = 0.7547$, where θ and β are in the first quadrant.

a $\sin(90° - \theta)$
b $\cos(270° + \theta)$
c $\tan(90° + \theta)$
d $\sin(270° - \beta)$
e $\tan(90° - \beta)$
f $\cos(270° - \beta)$

5 **WE9** Find the exact values of each of the following.

a $\tan\left(\frac{3\pi}{4}\right)$
b $\cos\left(\frac{5\pi}{6}\right)$
c $\sin\left(-\frac{\pi}{4}\right)$
d $\cos\left(\frac{7\pi}{3}\right)$
e $\tan\left(-\frac{\pi}{3}\right)$
f $\sin\left(\frac{11\pi}{6}\right)$

6 Find the exact values of each of the following.

a $\tan\left(\frac{5\pi}{6}\right)$
b $\cos\left(\frac{14\pi}{3}\right)$
c $\tan\left(-\frac{5\pi}{4}\right)$
d $\cos\left(-\frac{3\pi}{4}\right)$
e $\sin\left(-\frac{2\pi}{3}\right)$
f $\sin\left(\frac{17\pi}{6}\right)$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Simplify the following.

a $\sin(\pi - \theta)$
b $\cos(6\pi - \theta)$
c $\tan(\pi + \theta)$
d $\cos(-\theta)$
e $\sin(180° + \theta)$
f $\tan(720° - \theta)$

8 Simplify the following.

a $\cos\left(\frac{\pi}{2} - \alpha\right)$
b $\tan(90° + \alpha)$
c $\sin(270° - \alpha)$
d $\tan\left(\frac{11\pi}{2} - \alpha\right)$
e $\cos\left(\frac{3\pi}{2} + \alpha\right)$
f $\sin(90° - \alpha)$

9 Given that $\sin(\theta) = 0.9511$ and θ is in the first quadrant, evaluate the following correct to 4 decimal places.

a $\sin(2\pi - \theta)$
b $\sin(\pi - \theta)$
c $\cos\left(\frac{\pi}{2} - \theta\right)$
d $\tan(\theta)$
e $\cos(3\pi + \theta)$
f $\tan(2\pi - \theta)$

10 Given that $\cos(\alpha) = 0.8572$ and α is in the first quadrant, evaluate the following correct to 4 decimal places.

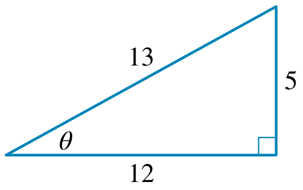
a $\cos(180° + \alpha)$
b $\cos(-\alpha)$
c $\sin\left(\frac{3\pi}{2} + \alpha\right)$
d $\tan(180° - \alpha)$
e $\cos(360° - \alpha)$
f $\tan\left(\frac{\pi}{2} + \alpha\right)$

11 If $\sin(\beta) = \frac{4}{5}$ and $\frac{\pi}{2} < \beta < \pi$, find the exact values of:

a $\cos(\beta)$
b $\tan(\beta)$
c $\cos^2(\beta) + \sin^2(\beta)$
d $\cos^2(\beta) - \sin^2(\beta)$

12 For the given triangle, find the values of:

a $\sin(\theta)$
b $\tan(\theta)$
c $\cos(\theta)$
d $\sin(90° - \theta)$
e $\cos(90° - \theta)$
f $\tan(90° - \theta)$



13 Find exact value answers for each of the following.

a $\sin\left(\frac{7\pi}{3}\right)$
b $\cos\left(\frac{7\pi}{3}\right)$
c $\tan\left(\frac{5\pi}{6}\right)$
d $\sin(150°)$
e $\cos\left(\frac{7\pi}{6}\right)$
f $\tan\left(-\frac{7\pi}{6}\right)$
g $\cos\left(\frac{\pi}{2}\right)$
h $\tan(270°)$
i $\sin(-4\pi)$
j $\tan(\pi)$
k $\cos(-6\pi)$
l $\sin\left(\frac{3\pi}{2}\right)$

14 Calculate the exact values of the following.

a $\cos\left(\frac{7\pi}{6}\right) + \cos\left(\frac{2\pi}{3}\right)$

b $2\sin\left(\frac{7\pi}{4}\right) + 4\sin\left(\frac{5\pi}{6}\right)$

c $\sqrt{3}\tan\left(\frac{5\pi}{4}\right) - \tan\left(\frac{5\pi}{3}\right)$

d $\sin^2\left(\frac{8\pi}{3}\right) + \sin\left(\frac{9\pi}{4}\right)$

e $2\cos^2\left(-\frac{5\pi}{4}\right) - 1$

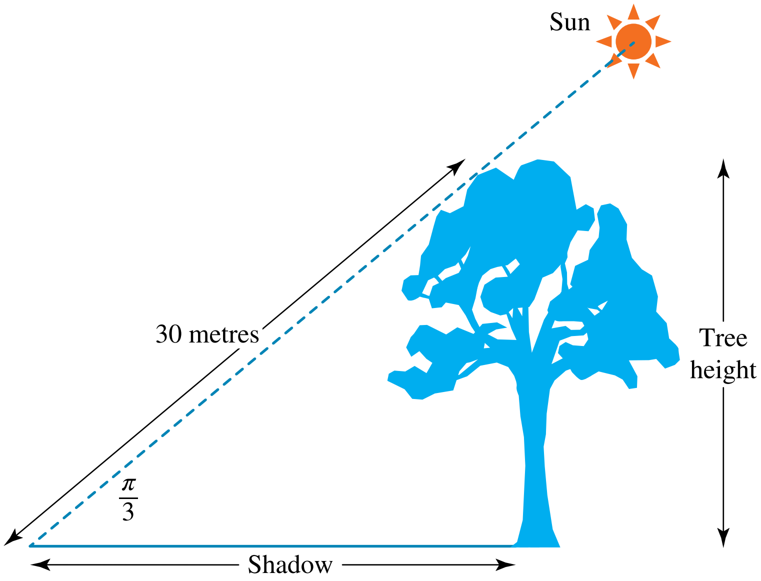
f $\dfrac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

15 **a** Use the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$ to show that

$$\tan^2(x) + 1 = \frac{1}{\cos^2(x)}$$

b Hence, find the value of $\tan(x)$ correct to 4 decimal places, given that $\sin(x) = 0.6157$ and $0 \le x \le \frac{\pi}{2}$.

16 The diagram shows a tree casting a shadow. Find the exact value of the height of the tree and the length of the shadow cast by the tree.



MASTER

17 The weight on a spring moves in such a way that its speed, v cm/s, is given by the rule

$$v = 12 + 3\sin\left(\frac{\pi t}{3}\right).$$

a Find the initial speed of the weight.

b Find the exact value of the speed of the weight after 5 seconds.

c Find the exact value of the speed of the weight after 12 seconds.

18 The height, $h(t)$ metres, that the sea water reaches up the side of the bank of the Yarra river is determined by the rule

$$h(t) = 0.5\cos\left(\frac{\pi t}{12}\right) + 1.0$$

where t is the number of hours after 6 am. Find the height of the water up the side of the bank at:

a 6 am **b** 2 pm **c** 10 pm.

Give your answers in exact form.

1.4 Trigonometric equations and general solutions

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Units 3 & 4

AOS 2

Topic 3

Concept 5

Solving trigonometric equations

Concept summary

Practice questions

Trigonometric equations

Trigonometric equations frequently involve working with the special angles that have exact values previously discussed but may also require the use of CAS.

To solve the basic type of equation $\sin(x) = a,\ 0 \le x \le 2\pi$, remember the following:

- Identify the quadrants in which solutions lie from the sign of a:
  - if $a > 0$, x must lie in quadrants 1 and 2 where sine is positive
  - if $a < 0$, x must be in quadrants 3 and 4 where sine is negative.
- Obtain the base value or first quadrant value by solving $\sin(x) = a$ if $a > 0$ or ignoring the negative sign if $a < 0$ (to ensure the first quadrant value is obtained).
- Use the base value to generate the values for the quadrants required from their symmetric forms.

The basic equations $\cos(x) = a$ or $\tan(x) = a,\ 0 \le x \le 2\pi$ are solved in a similar manner, with the sign of a determining the quadrants in which solutions lie.

For $\cos(x) = a$: if $a > 0$, x must lie in quadrants 1 and 4 where cosine is positive; if $a < 0$, x must be in quadrants 2 and 3 where cosine is negative.

For $\tan(x) = a$: if $a > 0$, x must lie in quadrants 1 and 3 where tangent is positive; if $a < 0$, x must be in quadrants 2 and 4 where tangent is negative.

In technology active questions, by defining the **domain** of the equation, the CAS technology will solve the problem without having to determine a base value or first quadrant value.

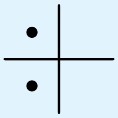
WORKED EXAMPLE 10

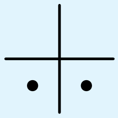
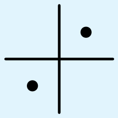
Solve the following equations.

a $\sqrt{2}\cos(x) + 1 = 0,\ 0 \le x \le 2\pi$

b $2\sin(x) = -1.5,\ 0° \le x \le 720°$, **correct to 2 decimal places**

c $\tan(\theta) - 1 = 0,\ -\pi \le \theta \le \pi$

THINK	WRITE
a 1 Express the equation with the trigonometric function as the subject.	**a** $\sqrt{2}\cos(x) + 1 = 0$ $\sqrt{2}\cos(x) = -1$ $\cos(x) = -\dfrac{1}{\sqrt{2}}$
2 Identify the quadrants in which the solutions lie.	Cosine is negative in quadrants 2 and 3. 
3 Use knowledge of exact values to state the first quadrant base.	The base is $\dfrac{\pi}{4}$, since $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$.
4 Generate the solutions using the appropriate quadrant forms.	$x = \pi - \dfrac{\pi}{4},\ \pi + \dfrac{\pi}{4}$
5 Calculate the solutions from their quadrant forms.	$x = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

b 1 Express the equation with the trigonometric function as the subject.	**b** $2\sin(x) = -1.5$ $\sin(x) = -0.75$
2 Identify the quadrants in which the solutions lie.	Sine is negative in quadrants 3 and 4. 
3 Calculate the base using CAS, as an exact value is not possible.	The base is $\sin^{-1}(0.75) = 48.59°$.
4 Generate the solutions using the appropriate quadrant forms. As $x \in [0°, 720°]$, there will be four positive solutions from two anticlockwise rotations.	$x = 180° + 48.59°, 360° - 48.59°, 540° + 48.59°, 720° - 48.59°$
5 Calculate the solutions from their quadrant forms. Alternatively, the solve function on CAS can be used to find the solutions (but remember to define the domain).	$x = 228.59°, 311.41°, 588.59°, 671.41°$
c 1 Express the equation with the trigonometric function as the subject.	**c** $\tan(\theta) - 1 = 0$ $\tan(\theta) = 1$
2 Identify the quadrants in which the solutions lie.	Tangent is positive in quadrants 1 and 3. 
3 Use knowledge of exact values to state the first quadrant base.	The base is $\frac{\pi}{4}$, since $\tan\left(\frac{\pi}{4}\right) = 1$.
4 Generate the solutions using the appropriate quadrant forms. As the domain is $x \in [-\pi, \pi]$, there will be one positive solution and one negative solution.	$x = \frac{\pi}{4}, -\pi + \frac{\pi}{4}$
5 Calculate the solutions from their quadrant forms.	$x = \frac{\pi}{4}, \frac{-3\pi}{4}$

Changing the domain

Equations such as $\sin(2x) = 1, 0 \le x \le 2\pi$ can be expressed in the basic form by the substitution $\theta = 2x$. However, the accompanying domain must be changed to be the domain for θ. This requires the endpoints of the domain for x to be multiplied by 2. Hence, $0 \le x \le 2\pi \Rightarrow 2 \times 0 \le 2x \le 2 \times 2\pi$ gives the domain requirement for θ as $0 \le \theta \le 4\pi$.

This allows the equation to be written as $\sin(\theta) = 1, 0 \le \theta \le 4\pi$.

WORKED EXAMPLE 11

Solve the following for x.

a $2\sin(2x) - 1 = 0, 0 \le x \le 2\pi$

b $2\cos(2x - \pi) - 1 = 0, -\pi \le x \le \pi$.

THINK	WRITE
a 1 Change the domain to be that for the given multiple of the variable.	**a** $2\sin(2x) - 1 = 0,\ 0 \le x \le 2\pi$ Multiply each value by 2: $2\sin(2x) - 1 = 0,\ 0 \le 2x \le 4\pi$

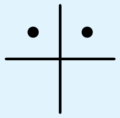
2 Express the equation with the trigonometric function as the subject.

$$\begin{aligned} 2\sin(2x) - 1 &= 0 \\ 2\sin(2x) &= 1 \\ \sin(2x) &= \frac{1}{2} \end{aligned}$$

3 Solve the equation for $2x$.

As $2x \in [0, 4\pi]$, each of the 2 revolutions will generate 2 solutions, giving a total of 4 values for $2x$.

Sine is positive in quadrants 1 and 2.



The base is $\frac{\pi}{6}$.

$$2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

4 Calculate the solutions for x.

Note: By dividing by 2 at the very end, the solutions lie back within the domain originally specified, namely $0 \le x \le 2\pi$.

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

b 1 Change the domain to that for the given multiple of the variable.

b $2\cos(2x - \pi) - 1 = 0, -\pi \le x \le \pi$

Multiply each value by 2:

$2\cos(2x - \pi) - 1 = 0, -2\pi \le 2x \le 2\pi$

Subtract π from each value:

$2\cos(2x - \pi) - 1 = 0, -3\pi \le 2x - \pi \le \pi$

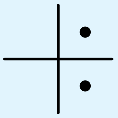
2 Express the equation with the trigonometric function as the subject.

$$\begin{aligned} 2\cos(2x - \pi) - 1 &= 0 \\ 2\cos(2x - \pi) &= 1 \\ \cos(2x - \pi) &= \frac{1}{2} \end{aligned}$$

3 Solve the equation for $(2x - \pi)$.

The domain of $[-3\pi, \pi]$ involves 2 complete rotations of the unit circle, so there will be 4 solutions, 3 of which will be negative and 1 of which will be positive.

Cosine is positive in quadrants 1 and 4.



The base is $\frac{\pi}{3}$.

$$2x - \pi = \frac{\pi}{3}, -\frac{\pi}{3}, -2\pi + \frac{\pi}{3}, -2\pi - \frac{\pi}{3}$$

$$2x - \pi = \frac{\pi}{3}, -\frac{\pi}{3}, -\frac{5\pi}{3}, -\frac{7\pi}{3}$$

4 Calculate the solutions for x.

$$\begin{aligned} 2x &= \frac{\pi}{3} + \pi, -\frac{\pi}{3} + \pi, -\frac{5\pi}{3} + \pi, -\frac{7\pi}{3} + \pi \\ &= \frac{4\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}, -\frac{4\pi}{3} \\ x &= \frac{2\pi}{3}, \frac{\pi}{3}, -\frac{\pi}{3}, -\frac{2\pi}{3} \end{aligned}$$

study on

Units 3 & 4
AOS 2
Topic 3
Concept 6

Solving $\sin(nx) = k\cos(nx)$
Concept summary
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Further types of trigonometric equations

Trigonometric equations may require algebraic techniques or the use of relationships between the functions before they can be reduced to the basic form $f(x) = a$, where f is either sin, cos or tan.

- Equations of the form $\sin(x) = a\cos(x)$ can be converted to $\tan(x) = a$ by dividing both sides of the equation by $\cos(x)$.
- Equations of the form $\sin^2(x) = a$ can be converted to $\sin(x) = \pm\sqrt{a}$ by taking the square roots of both sides of the equation.
- Equations of the form $\sin^2(x) + b\sin(x) + c = 0$ can be converted to standard quadratic equations by using the substitution $A = \sin(x)$.

Because $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$, neither $\sin(x)$ nor $\cos(x)$ can have values greater than 1 or less than –1. This may have implications requiring the rejection of some steps when working with sine or cosine trigonometric equations. As $\tan(x) \in R$, there is no restriction on the values the tangent function can take.

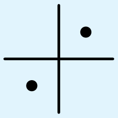
WORKED EXAMPLE 12

Solve the following equations.

a $\sin(2x) = \cos(2x),\ 0 \leq x \leq 2\pi.$

b $2\sin^2(\theta) + 3\sin(\theta) - 2 = 0,\ 0 \leq x \leq 2\pi.$

c $\cos^2(2\alpha) - 1 = 0,\ -\pi \leq \alpha \leq \pi$

THINK	WRITE
a 1 Change the domain to that for the given multiple of the variable.	**a** $0 \leq x \leq 2\pi$ Multiply through by 2: $0 \leq 2x \leq 4\pi$
2 Reduce the equation to one trigonometric function by dividing through by $\cos(2x)$.	$\sin(2x) = \cos(2x)$ $\frac{\sin(2x)}{\cos(2x)} = \frac{\cos(2x)}{\cos(2x)}$ providing $\cos(2x) \neq 0$ $\tan(2x) = 1$
3 Solve the equation for $2x$.	Tangent is positive in quadrants 1 and 3.  The base is $\frac{\pi}{4}$. $2x = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}$ $= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$
4 Calculate the solutions for x. Note that the answers are within the prescribed domain of $0 \leq x \leq 2\pi$.	$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$
b 1 Use substitution to form a quadratic equation.	**b** $2\sin^2(\theta) + 3\sin(\theta) - 2 = 0$ Let $A = \sin(\theta)$. $2A^2 + 3A - 2 = 0$

2 Solve the quadratic equation.

$$(2A - 1)(A + 2) = 0$$

$A = \frac{1}{2}$ or $A = -2$

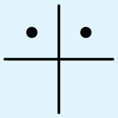
But $A = \sin(\theta)$.

$\sin(\theta) = \frac{1}{2}$ or $\sin(\theta) = -2$

3 Solve each trigonometric equation separately.

$\sin(\theta) = \frac{1}{2}$

Sine is positive in quadrants 1 and 2.



The base is $\frac{\pi}{6}$.

$$\theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$\sin(\theta) = -2$

There is no solution as $-1 \le \sin(\theta) \le 1$.

4 Write the answer.

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

c **1** Change the domain to that for the given multiple of the variable.

c $-\pi \le \alpha \le \pi$

Multiply through by 2:

$-2\pi \le 2\alpha \le 2\pi$

2 Use substitution to form a quadratic equation and factorise by applying the difference of perfect squares method.

$$\cos^2(2\alpha) - 1 = 0$$

Let $A = \cos(2\alpha)$.

$$A^2 - 1 = 0$$

3 Solve the quadratic equation.

$(A - 1)(A + 1) = 0$

$A = 1, -1$

But $A = \cos(2\alpha)$.

$\therefore \cos(2\alpha) = 1$ or $\cos(2\alpha) = -1$

4 Solve each trigonometric equation separately.

$\cos(2\alpha) = 1$

$$2\alpha = -2\pi, 0, 2\pi$$

$$\alpha = -\pi, 0, \pi$$

$\cos(2\alpha) = -1$

$$2\alpha = -\pi, \pi$$

$$\alpha = -\frac{\pi}{2}, \frac{\pi}{2}$$

5 Write the answers in numerical order.

$$\therefore \alpha = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$$

General solutions to trigonometric equations

studyon

Units 3 & 4

AOS 2

Topic 3

Concept 7

General solutions of circular functions
Concept summary
Practice questions

eBookplus

Interactivity
Trigonometric equations and general solutions
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All of the trigonometric equations solved so far have been solved over a specific domain and therefore have defined numbers of solutions. However, if no domain is given, then there will be an infinite number of solutions to the general equation. This is because multiples of 2π can be added and subtracted to any solutions within a specific domain. In cases such as this, a general solution is given in terms of the **parameter** n, where n is an integer.

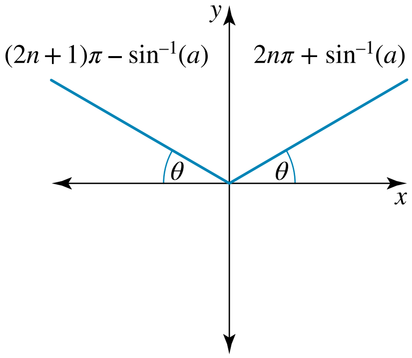
The general solution for the sine function

Consider $\sin(\theta) = a$, where a is a positive value. The solutions are found in quadrant 1 and 2, and the basic angle is in quadrant 1 and determined by $\theta = \sin^{-1}(a)$. The angle in quadrant 2 is found by $(\pi - \theta)$. If we keep cycling around the unit circle in either direction, the two solutions can be summarised as even numbers of π adding on θ and odd numbers of π subtracting θ.

The general solution is

$$\boldsymbol{\theta = 2n\pi + \sin^{-1}(a) \text{ or } \theta = (2n+1)\pi - \sin^{-1}(a) \text{ where } n \in Z \text{ and } a \in [-1, 1].}$$

The solutions if a is positive are represented in the diagram.



Note: If a is negative, choose the basic angle to be in quadrant 4 (therefore a negative angle).

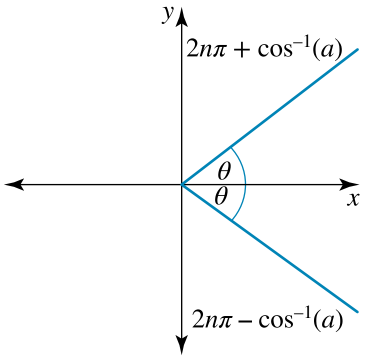
The general solution for the cosine function

Consider $\cos(\theta) = a$, where a is a positive value. The solutions are found in quadrant 1 and 4, and the basic angle is in quadrant 1 and determined by $\theta = \cos^{-1}(a)$. The angle in quadrant 4 is found by $(2\pi - \theta)$. If we keep cycling around the unit circle in either direction, the two solutions can be summarised as even numbers of π adding on θ or subtracting θ.

The general solution is

$$\boldsymbol{\theta = 2n\pi \pm \cos^{-1}(a) \text{ where } n \in Z \text{ and } a \in [-1, 1].}$$

The solutions if a is positive are represented in the diagram.



Note: If a is negative, choose the basic angle to be in quadrant 2 (therefore a positive angle).

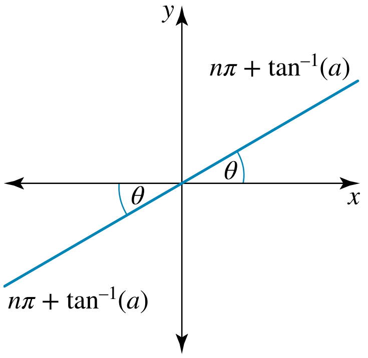
The general solution for the tangent function

Consider $\tan(\theta) = a$, where a is a positive value. The solutions are found in quadrant 1 and 3, and the basic angle is in quadrant 1 and determined by $\theta = \tan^{-1}(a)$. The angle in quadrant 3 is found by $(\pi + \theta)$. If we keep cycling around the unit circle in either direction, the two solutions can be summarised as multiples of π adding on θ.

The general solution is

$$\boldsymbol{\theta = n\pi + \tan^{-1}(a) \text{ where } n \in Z \text{ and } a \in R.}$$

The solutions if a is positive are represented in the diagram.



Note: If a is negative, choose the basic angle to be in quadrant 4 (therefore a negative angle).

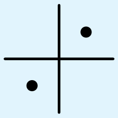
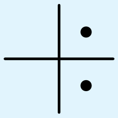
We can summarise the general solutions for sine, cosine and tangent as follows:

- **If $\sin(\theta) = a$, then $\theta = 2n\pi + \sin^{-1}(a)$ or $\theta = (2n + 1)\pi - \sin^{-1}(a)$ where $a \in [-1, 1]$ and $n \in Z$.**
- **If $\cos(\theta) = a$, then $\theta = 2n\pi \pm \cos^{-1}(a)$ where $a \in [-1, 1]$ and $n \in Z$.**
- **If $\tan(\theta) = a$, then $\theta = n\pi + \tan^{-1}(a)$ where $a \in R$ and $n \in Z$.**

WORKED EXAMPLE 13

a **Find the general solution of the equation $\tan(x) - \sqrt{3} = 0$.**

b **Find the general solution of the equation $2\cos(2\theta) - \sqrt{2} = 0$ and hence find all the solutions for $\theta \in [-\pi, \pi]$.**

THINK	**WRITE**
a 1 Express the equation with the trigonometric function as the subject.	**a** $\tan(x) - \sqrt{3} = 0$ $\tan(x) = \sqrt{3}$
2 Recognise the exact value and determine the quadrants in which the tangent function is positive.	Tangent is positive in quadrants 1 and 3.  The base is $\frac{\pi}{3}$.
3 Write the general solution for $\tan(x) = a$.	$x = n\pi + \tan^{-1}(a)$ $= n\pi + \tan^{-1}(\sqrt{3})$
4 Substitute the basic angle for $\tan^{-1}(\sqrt{3})$ and simplify. Note that $n \in Z$ must always be included as part of the solution.	$= n\pi + \frac{\pi}{3}, n \in Z$ $= \frac{3n\pi + \pi}{3}$ $= \frac{(3n + 1)\pi}{3}, n \in Z$
b 1 Express the equation with the trigonometric function as the subject.	**b** $2\cos(2\theta) - \sqrt{2} = 0$ $2\cos(2\theta) = \sqrt{2}$ $\cos(2\theta) = \frac{\sqrt{2}}{2}$
2 Recognise the exact value and determine the quadrants in which the cosine function is positive.	Cosine is positive in quadrants 1 and 4.  The base is $\frac{\pi}{4}$.
3 Write the general solution for $\cos(2\theta) = a$.	$2\theta = 2n\pi \pm \cos^{-1}(a)$ $= 2n\pi \pm \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
Substitute the basic angle for $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ and simplify.	$= 2n\pi \pm \frac{\pi}{4}, n \in Z$ $= \frac{8n\pi}{4} \pm \frac{\pi}{4}$ $= \frac{8n\pi + \pi}{4}, \frac{8n\pi - \pi}{4}$ $= \frac{(8n + 1)\pi}{4}, \frac{(8n - 1)\pi}{4}, n \in Z$

4 Divide through by 2 to find the solution for θ. This is always best done once the solutions are written with common denominators.

$\theta = \frac{(8n+1)\pi}{8}, \frac{(8n-1)\pi}{8}, n \in Z$

5 Substitute appropriate values of n to achieve solutions for $\theta \in [-\pi, \pi]$.

If $n = -1$, $\theta = -\frac{9\pi}{8}$ or $\theta = -\frac{7\pi}{8}$.

$\theta = -\frac{9\pi}{8}$ is outside the domain.

If $n = 0$, $\theta = \frac{\pi}{8}$ or $\theta = -\frac{\pi}{8}$.

Both are within the domain.

If $n = 1$, $\theta = \frac{7\pi}{8}$ or $\theta = \frac{9\pi}{8}$.

$\theta = \frac{9\pi}{8}$ lies outside the domain.

$\therefore$ the solutions for $\theta \in [-\pi, \pi]$ are

$\theta = -\frac{7\pi}{8}, -\frac{\pi}{8}, \frac{\pi}{8}, \frac{7\pi}{8}$.

EXERCISE 1.4 Trigonometric equations and general solutions

PRACTISE

Work without CAS
Questions 1, 3–8

1 **WE10** Solve the following equations.
- a $2\cos(\theta) + \sqrt{3} = 0$ for $0 \le \theta \le 2\pi$
- b $\tan(x) + \sqrt{3} = 0$ for $0° \le x \le 720°$
- c $2\cos(\theta) = 1$ for $-\pi \le \theta \le \pi$

2 a Solve the equation $\sin(\theta) + 0.5768 = 0$, $0° \le \theta \le 360°$, correct to 2 decimal places.
- b Solve $\sin(x) = 1$, $-2\pi \le x \le 2\pi$.

3 **WE11** Solve the following equations.
- a $2\cos(3\theta) - \sqrt{2} = 0$ for $0 \le \theta \le 2\pi$.
- b $2\sin(2x + \pi) + \sqrt{3} = 0$ for $-\pi \le x \le \pi$.

4 Solve $2\cos\left(3\theta - \frac{\pi}{2}\right) + \sqrt{3} = 0$, $0 \le \theta \le 2\pi$.

5 **WE12** Solve the equation $\cos^2(\theta) - \sin(\theta)\cos(\theta) = 0$ for $0 \le \theta \le 2\pi$.

6 Solve $\{\theta : 2\cos^2(\theta) + 3\cos(\theta) = -1, 0 \le \theta \le 2\pi\}$.

7 **WE13** Find the general solution of the equation $2\sin(\theta) - \sqrt{3} = 0$.

8 Find the general solution of the equation $\sqrt{3}\tan(2\theta) + 1 = 0$ and then find all the solutions for $\theta \in [-\pi, \pi]$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 Solve the following trigonometric equations for $0 \le \theta \le 2\pi$.
- a $\sqrt{2}\sin(\theta) = -1$
- b $2\cos(\theta) = 1$
- c $\tan(3\theta) - \sqrt{3} = 0$
- d $\tan\left(\theta - \frac{\pi}{2}\right) + 1 = 0$

10 Solve the following trigonometric equations for $0° \le x \le 360°$.
- a $2\cos(x) + 1 = 0$
- b $2\sin(2x) + \sqrt{2} = 0$

11 Solve the following, correct to 2 decimal places.

a $3\sin(\theta) - 2 = 0$ given that $0 \leq \theta \leq 2\pi$.

b $7\cos(x) - 2 = 0$ given that $0° \leq x \leq 360°$.

12 Solve the following for θ given that $-\pi \leq \theta \leq \pi$.

a $2\sin(2\theta) + \sqrt{3} = 0$ **b** $\sqrt{2}\cos(3\theta) = 1$ **c** $\tan(2\theta) + 1 = 0$

13 Solve the following for x given that $-\pi \leq x \leq \pi$.

a $2\sin\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$ **b** $2\cos(x + \pi) = \sqrt{3}$ **c** $\tan(x - \pi) = -1$

14 Solve the following for θ given that $0 \leq \theta \leq 2\pi$.

a $\tan^2(\theta) - 1 = 0$

b $4\sin^2(\theta) - (2 + 2\sqrt{3})\sin(\theta) + \sqrt{3} = 0$

15 Solve the following for α where $-\pi \leq \alpha \leq \pi$.

a $\sin(\alpha) - \cos^2(\alpha)\sin(\alpha) = 0$

b $\sin(2\alpha) = \sqrt{3}\cos(2\alpha)$

c $\sin^2(\alpha) = \cos^2(\alpha)$

d $4\cos^2(\alpha) - 1 = 0$

16 Find the general solutions for the following.

a $2\cos(x) + 1 = 0$

b $2\sin(x) - \sqrt{2} = 0$

17 Find the general solution of $2\sin(2x) + 1 = 0$ and hence find all solutions for $0 \leq x \leq 2\pi$.

18 Find the general solution of $\sqrt{3}\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)$ and hence find all solutions for $-\pi \leq x \leq \pi$.

MASTER

19 Solve $\sin(3\theta) = \cos(2\theta)$ for $0 \leq \theta \leq 2\pi$, correct to 3 decimal places.

20 Solve $2\sin(2x) - 1 = -\frac{1}{2}x + 1$ for $0 \leq x \leq 2$, correct to 3 decimal places.

1.5 Literal and simultaneous equations

Literal equations

Equations with several pronumerals are called **literal equations**. Rather than the solution having a numerical answer, the solution will be expressed in terms of pronumerals, also called parameters.

WORKED EXAMPLE 14

Solve the following equations for x.

a $mx + ny = kx - z$ **b** $\frac{p}{x} - \frac{2m}{m + x} = \frac{3y}{x}$

THINK	WRITE
a 1 Collect the x terms on the left-hand side.	**a** $mx + ny = kx - z$ $mx - kx = -z - ny$
2 Take out the common factor of x to leave only one instance of x on the left-hand side.	$x(m - k) = -z - ny$

3 Divide both sides by $m - k$.

$$x = \frac{-z - ny}{m - k}$$
$$= -\frac{z + ny}{m - k}$$

b **1** Multiply both sides by the common denominator of $x(m + x)$.

b $$\frac{p}{x} - \frac{2m}{m + x} = \frac{3y}{x}$$
$$p(m + x) - 2mx = 3y(m + x)$$

2 Expand the brackets.

$$pm + px - 2mx = 3my + 3xy$$

3 Collect the x terms on the left-hand side.

$$px - 2mx - 3xy = 3my - pm$$

4 Take out the common factor of x to leave only one instance of x on the left-hand side.

$$x(p - 2m - 3y) = 3my - pm$$

5 Divide both sides by $p - 2m - 3y$.

$$x = \frac{3my - pm}{p - 2m - 3y}$$

Simultaneous literal equations

These equations are solved by applying the methods of elimination and substitution. Once again, the solutions will be in terms of parameters. As a rule, if you are solving for n pronumerals, you will need n equations to solve for all the unknowns.

WORKED EXAMPLE 15

Solve the pair of simultaneous equations for x and y.

$$mx - y = k$$
$$x + ny = 2d$$

THINK	WRITE
1 Label the equations.	$mx - y = k \quad (1)$ $x + ny = 2d \quad (2)$
2 Use the elimination method to solve these equations. Multiply equation (2) by m so that the coefficients of x are the same in both equations, and label this equation (3).	$(2) \times m$: $\Rightarrow mx + mny = 2dm \quad (3)$
3 Subtract (3) from (1) to eliminate the x terms.	$mx - y = k \quad (1)$ $mx + mny = 2dm \quad (3)$ $(1) - (3)$: $-y - mny = k - 2dm$
4 Take out the common factor of y to leave only one instance of y on the left-hand side.	$y(-1 - mn) = k - 2dm$

5 Divide both sides by $-1 - mn$ and simplify.

$$\begin{aligned} y &= \frac{k - 2dm}{-1 - mn} \\ &= -\frac{k - 2dm}{1 + mn} \end{aligned}$$

6 Substitute $y = -\frac{k - 2dm}{1 + mn}$ into (1).

Note: Equation (2) could also have been chosen.

$$\begin{aligned} mx - \left(-\frac{k - 2dm}{1 + mn}\right) &= k \\ mx + \frac{k - 2dm}{1 + mn} &= k \\ mx &= k - \frac{k - 2dm}{1 + mn} \end{aligned}$$

7 Simplify the right-hand side.

$$\begin{aligned} mx &= k - \frac{k - 2dm}{1 + mn} \\ &= \frac{k(1 + mn)}{1 + mn} - \frac{k - 2dm}{1 + mn} \\ &= \frac{k + kmn - (k - 2dm)}{1 + mn} \\ &= \frac{kmn + 2dm}{1 + mn} \\ &= \frac{m(kn + 2d)}{1 + mn} \end{aligned}$$

8 Divide both sides by m.

$$\begin{aligned} x &= \frac{m(kn + 2d)}{m(1 + mn)} \\ &= \frac{kn + 2d}{1 + mn} \end{aligned}$$

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Units 3 & 4

AOS 2

Topic 4

Concept 1

Simultaneous equations with two variables
Concept summary
Practice questions

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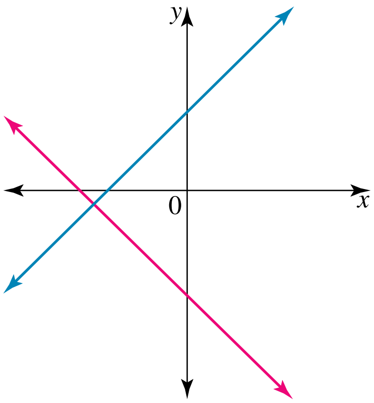
Interactivity
Intersecting, parallel and identical lines
int-2552

Solving systems of equations

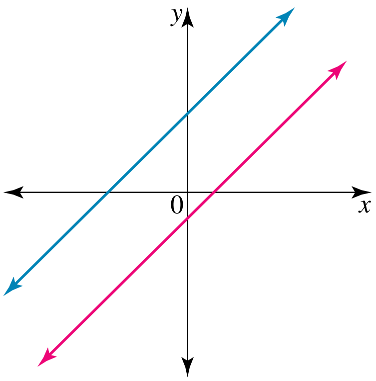
Solving simultaneous equations with two variables

Three possible scenarios exist when we are dealing with two linear **simultaneous equations**. There may be one solution only, there may be no solutions, or there may be infinitely many solutions.

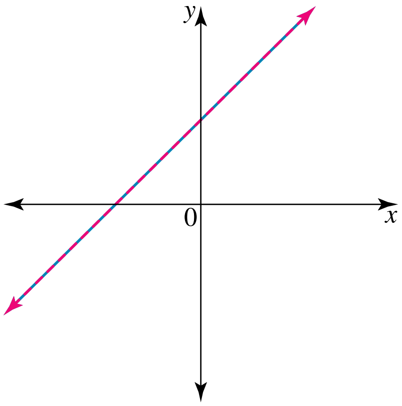
If the two straight lines intersect each other at only one place, we have one solution. This indicates that the **gradients** of the two equations are different.



If the two straight lines have the same gradient, they are parallel lines, so they never meet. Therefore, there are no solutions to the simultaneous equations. Although the gradients of the lines are the same, the y-intercepts are different.



If the two straight lines have the same equation, one line lies on top of the other and there are infinitely many solutions. Both the gradients and the y-intercepts are identical.



WORKED EXAMPLE 16

Find the value of k for which the simultaneous equations

$$kx + 3y = 1$$
$$4x + 3ky = 0$$

have a unique solution.

THINK

1 Label the equations.

2 There will be a unique solution for all values of k, except when the gradients of the two lines are the same. To find the gradient, write the equations in the general form, $y = mx + c$.

3 Equate the gradients and solve for k.

WRITE

$$kx + 3y = 1 \qquad (1)$$
$$4x + 3ky = 0 \qquad (2)$$

$$(1) \Rightarrow kx + 3y = 1$$
$$3y = 1 - kx$$
$$y = \frac{1 - kx}{3} \qquad \therefore m = -\frac{k}{3}$$
$$(2) \Rightarrow 4x + 3ky = 0$$
$$3ky = -4x$$
$$y = \frac{-4x}{3k} \qquad \therefore m = -\frac{4}{3k}$$

$$-\frac{k}{3} = -\frac{4}{3k}$$
$$3k^2 = 12$$
$$k^2 = 4$$
$$k = \pm 2$$

4 Write the solution.

This solution tells us that if $k = \pm 2$, the equations will have the same gradient, so for any other value of k, there will be a unique solution.

$k \in R\backslash\{-2, 2\}$

5 *Note:* In this example, we weren't required to further analyse the values of k and work out if the equations were identical or were parallel lines. To do this, we would substitute the values of k into the equations and interpret the results, as shown here.

$$k = 2$$

$(1) \quad \Rightarrow 2x + 3y = 1$

$(2) \quad \Rightarrow 4x + 6y = 0$

$(2) \div 2 \Rightarrow 2x + 3y = 0$

The gradients are the same but the y-intercepts are not, so these lines are parallel.

$$k = -2$$

$(1) \quad \Rightarrow -2x + 3y = 1$

$(2) \quad \Rightarrow 4x - 6y = 0$

$(2) \div -2 \Rightarrow 2x + 3y = 0$

The gradients are the same but the y-intercepts are not, so these lines are also parallel.

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Units 3 & 4
AOS 2
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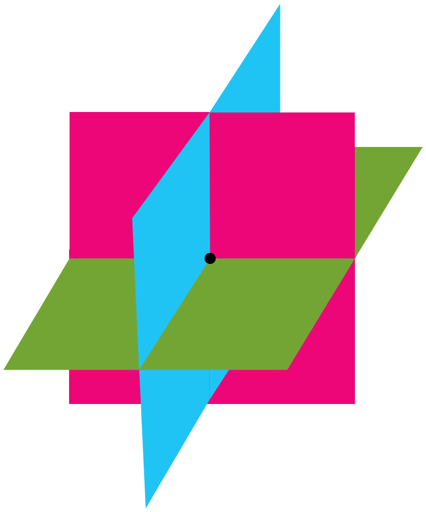
Concept 2

Simultaneous equations with three variables
Concept summary
Practice questions

Simultaneous equations with three variables

An equation with two variables defines a line. An equation with three variables defines a **plane**. If an equation has three variables, there needs to be three different equations for us to be able to solve for the point at which the three planes intersect (if in fact they do intersect at a single point).

There are a number of different possible outcomes when planes intersect.

No solution	One unique solution	Infinitely many solutions
 Planes are parallel to each other.	 There is a single point at which all three planes intersect.	 The planes are identical.
 There is no common point of intersection.		 The planes intersect along a line.

eBook*plus*

Interactivity
Equations in
3 variables
int-2550

When solving three simultaneous equations without technology, the strategy is to eliminate one of the variables and reduce the three equations with three unknowns to two equations with two unknowns.

Solving simultaneous equations with technology becomes a straightforward problem in CAS by using the inbuilt functions.

WORKED EXAMPLE 17

Solve the following system of simultaneous equations.

$$2x - 3y + 2z = -5$$
$$x - 5y + z = 1$$
$$2x + 3y + z = -2$$

THINK	WRITE
1 Label the equations and determine which of the three pronumerals you are going to eliminate. Either x or z would be appropriate choices as the coefficients in all three equations are either the same or a multiple of the other. Let us eliminate z.	$2x - 3y + 2z = -5$ (1) $x - 5y + z = 1$ (2) $2x + 3y + z = -2$ (3)
2 Subtract equation (2) from (3) to eliminate z and label this equation (4).	(3) − (2) ⇒ $x + 8y = -3$ (4)
3 We need another equation without z. In order to subtract equation (1) from (3), multiply equation (3) by 2. Label this equation (5).	(3) × 2 ⇒ $4x + 6y + 2z = -4$ (5)
4 Subtract equation (1) from the newly formed (5) to eliminate z and label this equation (6).	(5) − (1) ⇒ $2x + 9y = 1$ (6)
5 We now have two equations with only x and y.	$x + 8y = -3$ (4) $2x + 9y = 1$ (6)
6 The standard elimination method will be used to solve this pair of simultaneous equations. Multiply equation (4) by 2 so that the coefficients of x are the same. Label this equation (7).	(4) × 2 ⇒ $2x + 16y = -6$ (7)
7 Subtract equation (6) from (7) and solve for y.	(7) − (6) ⇒ $7y = -7$ $y = -1$
8 Substitute $y = -1$ back into the system of equations in order to find x. Choose one of the equations containing only x and y.	Substitute $y = -1$ into (4): $x - 8 = -3$ $x = 5$

9 Substitute the values for x and y into one of the original equations and solve for z.	Substitute $y = -1$ and $x = 5$ into (2): $5 + 5 + z = 1$ $10 + z = 1$ $z = -9$
10 Write the final solution. Alternatively, CAS can be used to solve the three simultaneous equations if the question is technology active.	$x = 5,\ y = -1,\ z = -9$

Simultaneous equations involving parameters

When there are infinitely many solutions to a system of equations, such as when planes intersect along a line, we can describe the set of solutions through the use of a parameter. Conventionally, the parameter chosen is λ.

WORKED EXAMPLE 18

The simultaneous equations shown have infinitely many solutions.

$$\begin{aligned} 2x + y - 4z &= 2 \\ x + y + 3z &= -1 \end{aligned}$$

a **Eliminate y by subtracting the second equation from the first equation.**

b **Let $z = \lambda$ and solve the equations in terms of λ.**

c **Explain what this solution represents.**

THINK	WRITE
a **1** Label the equations.	**a** $2x + y - 4z = 2 \quad (1)$ $x + y + 3z = -1 \quad (2)$
2 Subtract equation (2) from equation (1).	$(2) - (1) \Rightarrow x - 7z = 3$
b **1** Substitute $z = \lambda$ and solve for x.	**b** $z = \lambda$ $x - 7\lambda = 3$ $x = 3 + 7\lambda$
2 Substitute $z = \lambda$ and $x = 3 + 7\lambda$ into equation (2) and solve for y. *Note:* Equation (1) could also have been chosen.	Substitute $z = \lambda$ and $x = 3 + 7\lambda$ into (2): $3 + 7\lambda + y + 3\lambda = -1$ $y + 10\lambda + 3 = -1$ $y = -4 - 10\lambda$
3 Write the solution.	$x = 3 + 7\lambda,\ y = -4 - 10\lambda,\ z = \lambda$
c Interpret the solution.	**c** This solution describes the line along which the two planes intersect.

EXERCISE 1.5 Literal and simultaneous equations

PRACTISE

Work without CAS
Questions 1–6

1 **WE14** Solve the following equations for x.

a $my - nx = 4x + kz$

b $\dfrac{2p}{x} - \dfrac{m}{x - c} = \dfrac{3c}{x}$

2 Given that $\dfrac{x - my}{px + y} = 2$, solve the equation for y.

3 **WE15** Solve the pair of simultaneous equations for x and y.

$$x + y = 2k$$
$$mx + ny = d$$

4 Solve the following pairs of simultaneous equations for x and y.

a $nx - my = k$
$nx + my = 2d$

b $nx + my = m$
$mx + ny = n$

5 **WE16** Find the value of k for which the simultaneous equations

$$2x + ky = 4$$
$$(k - 3)x + 2y = 0$$

have a unique solution.

6 Find the value of m for which the simultaneous equations

$$mx - 2y = 4$$
$$x + (m - 3)y = m$$

have infinitely many solutions.

7 **WE17** Solve the following system of simultaneous equations.

$$2m - 4n - p = 1$$
$$4m + n + p = 5$$
$$3m + 3n - 2p = 22$$

8 Solve the following system of simultaneous equations.

$$2d - e - f = -2$$
$$3d + 2e - f = 5$$
$$d + 3e + 2f = 11$$

9 **WE18** The simultaneous equations shown have infinitely many solutions.

$$x + 2y + 2z = 1$$
$$2x - 2y + z = 2$$

a Eliminate y by adding the second equation to the first equation.
b Let $z = \lambda$ and solve the equations in terms of λ.
c Explain what this solution represents.

10 Solve the pair of simultaneous equations through the use of the parameter λ.

$$x + 2y + 4z = 2$$
$$x - y - 3z = 4$$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

11 Solve the following equations for x.

a $\dfrac{kx + dy}{x + 3y} = -2k$

b $\dfrac{mx + ny}{p} = x + q$

c $\dfrac{m}{x} - k = \dfrac{3k}{x} + m$

d $\dfrac{k}{m + x} = \dfrac{2d}{m - x}$

12 Solve the following pairs of simultaneous equations for x and y.

a $2mx + ny = 3k$
$mx + ny = -d$

b $\dfrac{x}{2a} + \dfrac{y}{b} = 2$
$\dfrac{2x}{b} + \dfrac{4y}{a} = 8$

13 Find the value of m for which the simultaneous equations

$$x + my = 3$$
$$4mx + y = 0$$

have no solution.

14 Find the value of k for which the simultaneous equations

$$\begin{aligned} x + 3ky &= 2 \\ (k - 1)x - 1 &= -6y \end{aligned}$$

have a unique solution.

15 Find the value of m for which the simultaneous equations

$$\begin{aligned} -2x + my &= 1 \\ (m + 3)x - 2y &= -2m \end{aligned}$$

have:

a a unique solution
b no solution
c an infinite number of solutions.

16 Solve the following systems of simultaneous equations.

a $$\begin{aligned} 2x + y - z &= 12 \\ -x - 3y + z &= -13 \\ -4x + 3y - z &= -2 \end{aligned}$$

b $$\begin{aligned} m + n - p &= 6 \\ 3m + 5n - 2p &= 13 \\ 5m + 4n - 7p &= 34 \end{aligned}$$

c $$\begin{aligned} u + 2v - 4w &= 23 \\ 3u + 4v - 2w &= 37 \\ 3u + v - 2w &= 19 \end{aligned}$$

d $$\begin{aligned} a + b + c &= 4 \\ 2a - b + 2c &= 17 \\ -a - 3b + c &= 3 \end{aligned}$$

17 The measure of the largest angle of a triangle is 20° more than the smallest angle, and the sum of the largest and smallest angles is 60° more than the third angle. Find the angle sizes of the triangle using simultaneous equations.

18 Solve the pair of simultaneous equations through the use of the parameter λ.

$$\begin{aligned} x + y - 2z &= 5 \\ x - 2y + 4z &= 1 \end{aligned}$$

19 Solve the pair of simultaneous equations through the use of the parameter λ.

$$\begin{aligned} -2x + y + z &= -2 \\ x - 3z &= 0 \end{aligned}$$

20 Find the values of m and n for which the equations

$$\begin{aligned} 3x + 2y &= -1 \\ mx + 4y &= n \end{aligned}$$

have:

a a unique solution
b an infinite number of solutions
c no solution.

MASTER

21 Solve the following system of simultaneous equations in terms of a.

$$\begin{aligned} 2x - y + az &= 4 \\ (a + 2)x + y - z &= 2 \\ 6x + (a + 1)y - 2z &= 4 \end{aligned}$$

22 Solve the following system of simultaneous equations.

$$\begin{aligned} w - 2x + 3y - z &= 10 \\ 2w + x + y + z &= 4 \\ -w + x + 2y - z &= -3 \\ 3w - 2x + y &= 11 \end{aligned}$$

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1.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

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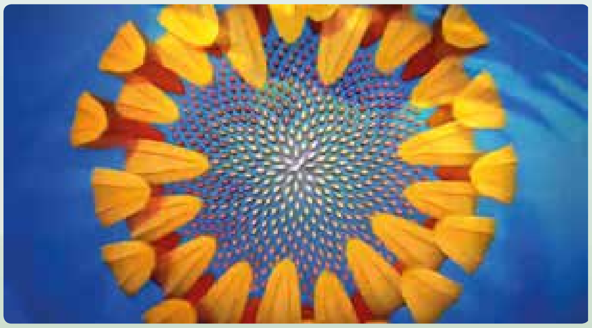
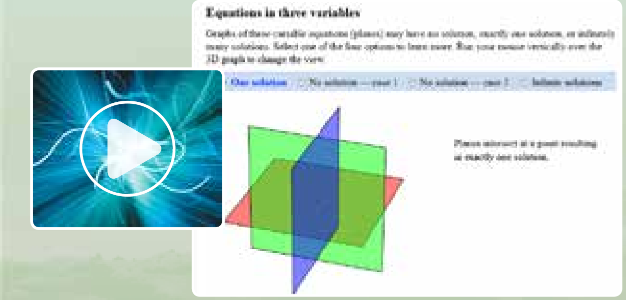
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





Equations in three variables

Graphs of three-variable equations (planes) may have no solution, exactly one solution, or infinitely many solutions. Select one of the four options to learn more. Run your mouse vertically over the 3D graph to change the view.

Planes intersect at a point resulting in exactly one solution.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

1 Answers

EXERCISE 1.2

1 a $(5u - 2)(3u + 1)$ **b** $2(3d - 2)(d - 4)$

c $3(j + 2 - \sqrt{6})(j + 2 + \sqrt{6})$

2 a $(f - 14)(f + 2)$ **b** $(g + 4)(g - 1)$

c $(b - 1)(b + 1)$

3 a $(5a - 3b)(25a^2 + 15ab + 9b^2)$

b $2(c + d)^3$

c $5(2p - 1)(4p^2 + 2p + 1)$

4 a $(3z - 2)^3$

b $(mn + 4)(m^2n^2 - 4mn + 16)$

5 a $(x - 1)(3x - y)$ **b** $(y + z^2)(3y^2 - 2z)$

6 a $(3a - 2 - 4b)(3a - 2 + 4b)$

b $(np - 2m - 1)(np + 2m + 1)$

7 $(x + 1)(x - 6)(x + 3)$

8 $(x - 1)(x - 4)(x - 6)(x + 6)$

9 a $x = -1, 2, 3, 4$ **b** $a = \pm\sqrt{2}$

10 a $x = \pm\sqrt{5}, \frac{1}{2}$

b $a = \dfrac{5 \pm \sqrt{97}}{4}$

11 $A = 3, B = 0, C = 2$ and $D = -7$

12 $d = \pm 3, e = \pm\frac{1}{3}$

13 a $(r - 7)(7r^2 + 1)$

b $(6v + 1)(6v^2 + 5)$

c $(2m + 3)(m - 7)(m + 7)$

d $(2z - 1)(z^2 + 1)$

e $(2x - 7 - 5y)(2x - 7 + 5y)$

f $(4a - 2b - 3)(4a + 2b + 3)$

g $(v - w + 2)(v + w - 2)$

h $(2p + q - 1)(2p + q + 1)$

14 a $y = \pm\frac{1}{9}$ **b** $z = -\frac{7}{2}$

c $m = \dfrac{5 \pm \sqrt{10}}{5}$ **d** $x = 1$ and 3

e $p = \frac{1}{2}$ and $\frac{3}{2}$ **f** $k = \frac{11}{4}$ and 7

g $m = -4$ and 1 **h** $n = \dfrac{-5 \pm 3\sqrt{17}}{8}$

15 a $x = -3, -1$ and $\frac{1}{2}$ **b** $l = \pm 4$ and ± 1

c $c = -3$ and ± 2 **d** $p = \pm 1,\ 2$ and 3

16 a $b = -4,\ -2, 1$

b $m = -1,\ \frac{3}{2},\ 4$

c $x = \pm\sqrt{3}, \frac{1}{2}$

17 a Let $P(t) = 3t^3 + 22t^2 + 37t + 10$.

$$\begin{aligned} P(-5) &= 3(-5)^3 + 22(-5)^2 + 37(-5) + 10 \\ &= 3 \times 125 + 22 \times 25 - 185 + 10 \\ &= -375 + 550 - 175 \\ &= 0 \end{aligned}$$

$\therefore t + 5$ is a factor

$t = -5,\ -2$ and $-\frac{1}{3}$

b Let $P(d) = 3d^3 - 16d^2 + 12d + 16$.

$$\begin{aligned} P(2) &= 3(2)^3 - 16(2)^2 + 12(2) + 16 \\ &= 24 - 64 + 24 + 16 \\ &= 0 \end{aligned}$$

$\therefore d - 2$ is a factor

$d = -\frac{2}{3},\ 2$ and 4

18 a $a = \pm 1$ and ± 3 **b** $k = \pm\frac{1}{2}$ and ± 5

c $z = \pm\frac{1}{3}$ and ± 4 **d** $x = 8, -6, 1$

19 a $a = 5, b = -3, c = 4$ and $d = -1$

b $a = -3$ and $b = 1$

20 $a = 2, b = 1, c = 1$ and $d = -3$;

$2(x - 1)^3 + (x - 1)^2 + (x - 1) - 3$

21 a $k \in \left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$

b $kx^2 + 4x - k + 2 = 0$

$$\begin{aligned} \Delta &= 16 - 4 \times k \times (-k + 2) \\ &= 16 + 4k^2 - 8k \\ &= 4(k^2 - 2k + 4) \\ &= 4(k^2 - 2k + 1^2 - 1^2 + 4) \\ &= 4[(k + 1)^2 + 3] \\ &= 4(k + 1)^2 + 12 \end{aligned}$$

As $(k + 1)^2 > 0$,

$\therefore 4(k + 1)^2 > 0$

and $4(k + 1)^2 + 12 > 0$

Δ is always greater than zero, therefore the equation will always have a solution for all values of k.

22 $a = 2, b = 5$

23 $a = 2, b = -13, c = -14$

24 $m \in \left(\dfrac{3 - 2\sqrt{46}}{14}, \dfrac{3 + 2\sqrt{46}}{14}\right) \backslash \{1\}$

EXERCISE 1.3

1 a -0.4695 **b** -0.5592

c -0.2680 **d** -0.4695

e 0.5592 **f** 0.2680

2 a 0.8829 **b** -0.5318

c 0.8290 **d** -1.4825

3 a 0.8829 **b** −0.4696

c 1.8803 **d** −0.9848

e 0.9848 **f** −5.6729

4 a 0.5592 **b** 0.8290

c −0.6746 **d** −0.7547

e 1.1503 **f** −0.6561

5 a −1 **b** $-\dfrac{\sqrt{3}}{2}$

c $-\dfrac{1}{\sqrt{2}}$ **d** $\dfrac{1}{2}$

e $-\sqrt{3}$ **f** $-\dfrac{1}{2}$

6 a $-\dfrac{1}{\sqrt{3}}$ **b** $-\dfrac{1}{2}$

c −1 **d** $-\dfrac{1}{\sqrt{2}}$

e $-\dfrac{\sqrt{3}}{2}$ **f** $\dfrac{1}{2}$

7 a $\sin(\theta)$ **b** $\cos(\theta)$

c $\tan(\theta)$ **d** $\cos(\theta)$

e $-\sin(\theta)$ **f** $-\tan(\theta)$

8 a $\sin(\alpha)$ **b** $-\dfrac{1}{\tan(\alpha)}$

c $-\cos(\alpha)$ **d** $\dfrac{1}{\tan(\alpha)}$

e $\sin(\alpha)$ **f** $\cos(\alpha)$

9 a −0.9511 **b** 0.9511

c 0.9511 **d** 3.0792

e −0.3089 **f** −3.0792

10 a −0.8572 **b** 0.8572

c −0.8572 **d** −0.6008

e 0.8572 **f** −1.6645

11 a $-\dfrac{3}{5}$ **b** $-\dfrac{4}{3}$

c 1 **d** $-\dfrac{7}{25}$

12 a $\dfrac{5}{13}$ **b** $\dfrac{5}{12}$

c $\dfrac{12}{13}$ **d** $\dfrac{12}{13}$

e $\dfrac{5}{13}$ **f** $\dfrac{12}{5}$

13 a $\dfrac{\sqrt{3}}{2}$ **b** $\dfrac{1}{2}$

c $-\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$ **d** $\dfrac{1}{2}$

e $-\dfrac{\sqrt{3}}{2}$ **f** $-\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$

g 0 **h** Undefined

i 0 **j** 0

k 1 **l** −1

14 a $\dfrac{-\sqrt{3}+1}{2}$ **b** $-\sqrt{2}+2$

c $2\sqrt{3}$ **d** $\dfrac{3+2\sqrt{2}}{4}$

e 0 **f** 2

15 a
$$\begin{aligned}\tan^2(x)+1 &= \frac{\sin^2(x)}{\cos^2(x)}+\frac{\cos^2(x)}{\cos^2(x)}\\ &= \frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\end{aligned}$$

b 0.7814

16 Height $h = 15\sqrt{3}$ m and shadow $s = 15$ m

17 a 12 cm/s

b $12 - \dfrac{3\sqrt{3}}{2}$ cm/s

c 12 cm/s

18 a 1.5 m

b 0.75 m

c 0.75 m

EXERCISE 1.4

1 a $\theta = \dfrac{5\pi}{6}$ and $\dfrac{7\pi}{6}$

b $x = 120°, 300°, 480°, 660°$

c $\theta = -\dfrac{\pi}{3}, \dfrac{\pi}{3}$

2 a $\theta = 215.23°, 324.77°$ **b** $x = -\dfrac{3\pi}{2}, \dfrac{\pi}{2}$

3 a $\theta = \dfrac{\pi}{12}, \dfrac{7\pi}{12}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{17\pi}{12}, \dfrac{23\pi}{12}$

b $x = -\dfrac{5\pi}{6}, -\dfrac{2\pi}{3}, \dfrac{\pi}{6}, \dfrac{\pi}{3}$

4 $\theta = \dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{10\pi}{9}, \dfrac{11\pi}{9}, \dfrac{16\pi}{9}, \dfrac{17\pi}{9}$

5 $\theta = \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{5\pi}{4},$ and $\dfrac{3\pi}{2}$

6 $\theta = \dfrac{2\pi}{3}, \pi,$ and $\dfrac{4\pi}{3}$

7 $\theta = \dfrac{(6n+1)\pi}{3}$ and $\dfrac{(3n+1)2\pi}{3}, n \in Z$

8 $\theta = \dfrac{(6n-1)\pi}{12}, n \in Z$. Solutions within $[-\pi, \pi]$ are $\theta = -\dfrac{7\pi}{12}, -\dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{11\pi}{12}$.

9 a $\theta = \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ **b** $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

c $\theta = \dfrac{\pi}{9}, \dfrac{4\pi}{9}, \dfrac{7\pi}{9}, \dfrac{10\pi}{9}, \dfrac{13\pi}{9}, \dfrac{16\pi}{9}$

d $\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

10 a $x = 120^\circ$ and 240°

b $x = 112.5^\circ, 157.5^\circ, 292.5^\circ, 337.5^\circ$

11 a $\theta = 0.73$ and 2.41 **b** $x = 73.40^\circ$ and 286.60°

12 a $\theta = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$

b $\theta = -\frac{3\pi}{4}, -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$

c $\theta = -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$

13 a $x = -\pi, -\frac{3\pi}{4}, 0, \frac{\pi}{4}, \pi$

b $x = -\frac{5\pi}{6}, \frac{5\pi}{6}$

c $x = -\frac{\pi}{4}, \frac{3\pi}{4}$

14 a $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b $\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$

15 a $\alpha = -\pi, 0, \pi$

b $\alpha = -\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

c $\alpha = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

d $\alpha = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

16 a $x = \frac{6n\pi \pm 2\pi}{3}, n \in Z$

b $x = \frac{8n\pi + \pi}{4}, \frac{8n\pi + 3\pi}{4}, n \in Z$

17 $x = \frac{12n\pi - \pi}{12}, x = \frac{12n\pi + 7\pi}{12}, n \in Z; x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

18 $x = \frac{3n\pi - \pi}{3}, n \in Z; x = -\frac{\pi}{3}, \frac{2\pi}{3}$

19 $\theta = 0.314, 1.571, 2.827, 4.084, 5.341$

20 $x = 0.526, 1.179$

EXERCISE 1.5

1 a $x = \frac{my - kz}{n + 4}$

b $x = \frac{2pc - 3c^2}{2p - m - 3c}$

2 $y = \frac{x(1 - 2p)}{m + 2}$

3 $x = \frac{d - 2kn}{m - n}, y = \frac{2km - d}{m - n}$

4 a $x = \frac{k + 2d}{2n}, y = \frac{2d - k}{2m}$

b $x = 0, y = 1$

5 $k \in R \backslash \{-1, 4\}$

6 $m = 2$

7 $m = 2,\ n = 2$ and $p = -5$

8 $d = 1,\ e = 2$ and $f = 2$

9 a $3x + 3z = 3$

b $x = 1 - \lambda,\ y = -\frac{\lambda}{2},\ z = \lambda$

c This solution describes the line along which the two planes are intersecting.

10 $x = \frac{2(\lambda + 5)}{3},\ y = -\frac{7\lambda + 2}{3},\ z = \lambda$

11 a $x = -\frac{y(6k + d)}{3k}$ **b** $x = \frac{pq - ny}{m - p}$

c $x = \frac{m - 3k}{m + k}$ **d** $x = \frac{km - 2dm}{2d + k}$

12 a $x = \frac{3k + d}{m},\ y = -\frac{2d + 3k}{n}$

b $x = \frac{4ab}{a + b},\ y = \frac{2ab}{a + b}$

13 $m = \pm\frac{1}{2}$

14 $k \in R \backslash \{-1, 2\}$

15 a $m \in R \backslash \{-4, 1\}$

b $m = -4$

c $m = 1$

16 a $x = 3,\ y = 2$ and $z = -4$

b $m = 7,\ n = -2$ and $p = -1$

c $u = 3,\ v = 6$ and $w = -2$

d $a = \frac{13}{2},\ b = -3$ and $c = \frac{1}{2}$

17 The largest angle is 70°, the smallest angle is 50° and the third angle is 60°.

18 $x = \frac{11}{3},\ y = \frac{2(3\lambda + 2)}{3},\ z = \lambda$

19 $x = 3\lambda,\ y = 5\lambda - 2,\ z = \lambda$

20 a $m \in R \backslash \{6\}, n \in R$

b $m = 6, n = -2$

c $m = 6, n \in R \backslash \{-2\}$

21 $x = \frac{2(a + 2)}{a(a + 4)},\ y = \frac{4(a + 2)}{a(a + 4)},\ z = \frac{4}{a}$

22 $w = 1, x = -3,\ y = 2$ and $z = 3$

2

Functions and graphs

2.1 Kick off with CAS

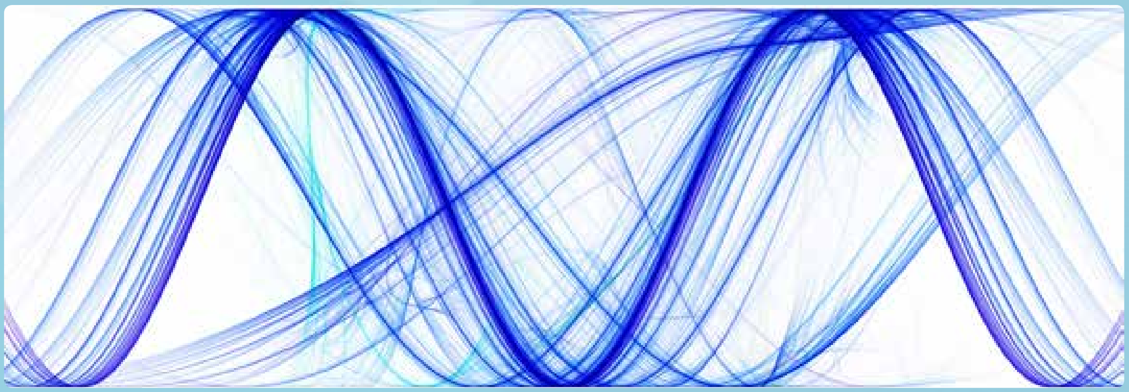
Graph sketching

1 a Using the graph application on CAS, sketch $y = x^4 - x^3 - 23x^2 + 3x + 60$, $x \in [-5, 5]$.
  b Determine the x- and y-intercepts.
  c Find the coordinates of the turning points.
  d Find the coordinates of the end points.
  e Determine the solution(s) to the equation $-40 = x^4 - x^3 - 23x^2 + 3x + 60$, $x \in [-5, 5]$.
  f Hence, solve $-40 < x^4 - x^3 - 23x^2 + 3x + 60$, $x \in [-5, 5]$.
  g For what values of k does the equation $k = x^4 - x^3 - 23x^2 + 3x + 60$, $x \in [-5, 5]$ have:
    i 1 solution
    ii 2 solutions
    iii 3 solutions
    iv 4 solutions?

2 a Using CAS, sketch

$$f(x) = \begin{cases} x^2 - 2, & x \leq 1 \\ 2, & 2 < x < 3. \\ -2x + 8, & x \geq 3 \end{cases}$$

  b Determine $f(-1), f(1), f(2)$ and $f(5)$.
  c Solve $f(x) = 1$.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

2.2 Polynomial functions

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Interactivity
Vertical and horizontal line tests
int-2570

Familiar graphs include those of the straight line and the parabola. This section reviews how the key aspects of these graphs and the graphs of other polynomial **functions** can be deduced from their equations.

Functions

A function is a set of ordered pairs in which each x-value is paired to a unique y-value. A vertical line will intersect the graph of a function at most once. This is known as the **vertical line test** for a function.

A horizontal line may intersect the graph of a function once, in which case the function has a one-to-one correspondence, or the horizontal line may intersect the graph more than once, in which case the function has a many-to-one correspondence.

The **domain** of a function is the set of x-values in the ordered pairs, and the **range** is the set of the y-values of the ordered pairs.

As a mapping, a function is written $f: D \to R,\ f(x) = \ldots$, where the ordered pairs of the function f are formed using each of the x-values in the domain D and pairing them with a unique y-value drawn from the **co-domain** set R according to the function rule $f(x) = \ldots$ Not all of the available y-values may be required for a particular mapping; this is dependent on the function rule.

For any polynomial function, the **implied** or **maximal** domain is R. For example, the mapping or function notation for the straight line $y = 2x$ is $f: R \to R,\ f(x) = 2x$. Under this mapping, the **image** of 3, or the value of f at 3, is $f(3) = 2 \times 3 = 6$, and the ordered pair $(3, 6)$ lies on the line of the function.

If only that part of the line $y = 2x$ where the x-values are positive was required, then this straight line function would be defined on a **restricted domain**, a subset of the maximal domain, and this would be written as $g: R^+ \to R,\ g(x) = 2x$.

WORKED EXAMPLE 1

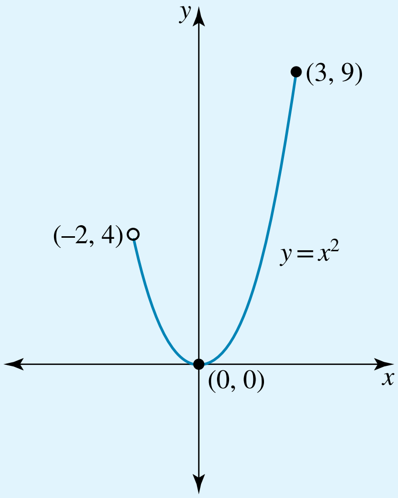
Part of the graph of the parabola $y = x^2$ is shown in the diagram.

a Explain why the graph is a function and state the type of correspondence.

b State the domain and range.

c Express the given parabola using function notation.

d Calculate the value of y when $x = -\sqrt{2}$.



THINK	WRITE
a 1 Use the vertical line test to explain why the graph is of a function.	**a** This is a function, because any vertical line that intersects the graph does so in exactly one place.
2 State the type of correspondence.	A horizontal line could cut the graph in up to two places. The correspondence is many-to-one.

b **1** State the domain.	**b** Reading from left to right horizontally in the x-axis direction, the domain is (–2, 3].
2 State the range.	Reading from bottom to top vertically in the y-axis direction, the range is [0, 9].
c Use the domain and the function rule to form the mapping.	**c** Let the function be f. As a mapping, it is $f:(-2, 3] \rightarrow R,\ f(x) = x^2$.
d Calculate the required value.	**d** $f(x) = x^2$ Let $x = -\sqrt{2}$. $f(-\sqrt{2}) = (-\sqrt{2})^2$ $= 2$

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Interactivities
Equations from point–gradient and gradient–y-intercept
int-2551
Midpoint of a line segment and the perpendicular bisector
int-2553
Roots, zeros and factors
int-2557

The linear polynomial function

Two points are needed in order to determine the equation of a line. When sketching an oblique line by hand, usually the two points used are the x- and y-intercepts. If the line passes through the origin, then one other point needs to be determined from its equation.

Gradient

The gradient, or slope, of a line may be calculated from $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. This remains constant between any pair of points (x_1, y_1) and (x_2, y_2) on the line. The linear function either increases or decreases steadily.

Parallel lines have the same gradient, and the product of the gradients of perpendicular lines is equal to -1. That is,

$m_1 = m_2$ for parallel lines
and $m_1 m_2 = -1$ for perpendicular lines.

The angle of inclination of an oblique line with the positive direction of the x-axis can be calculated from the gradient by the relationship $m = \tan(\theta)$. The angle θ is acute if the gradient is positive and obtuse if the gradient is negative.

Equation of a line

The equation of a straight line can be expressed in the form $y = mx + c$, where m is the gradient of the line and c is the y-value of the intercept the line makes with the y-axis.

If a point (x_1, y_1) and the gradient m are known, the equation of a line can be calculated from the point–gradient form $y - y_1 = m(x - x_1)$.

Oblique lines are one-to-one functions.

Horizontal lines run parallel to the x-axis and have the equation $y = c$. These are many-to-one functions.

Vertical lines rise parallel to the y-axis and have the equation $x = k$. These lines are not functions.

The perpendicular bisector of a line segment is the line that passes through the midpoint of the line segment at right angles to the line segment. The midpoint of the line segment with end points (x_1, y_1) and (x_2, y_2) has coordinates $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

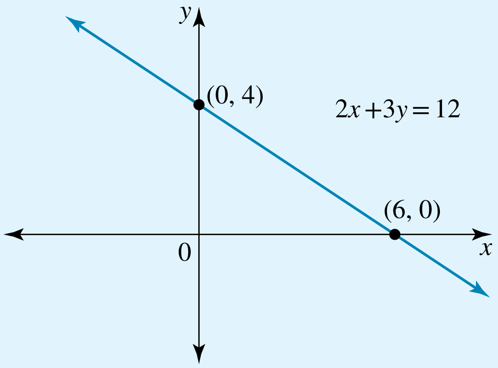
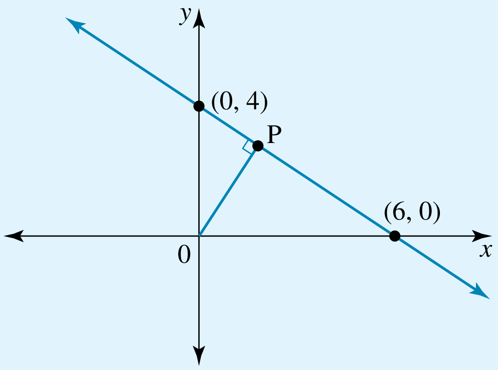
WORKED EXAMPLE 2

Consider the line L where $L = \{(x, y) : 2x + 3y = 12\}$.

a Sketch the line.

b Calculate the gradient of the line.

c Determine the coordinates of the point on the line that is closest to the origin.

THINK	WRITE/DRAW
a 1 Calculate the x- and y-intercepts.	**a** $2x + 3y = 12$ y-intercept: Let $x = 0$. $3y = 12$ $y = 4$ The y-intercept is (0, 4). x-intercept: Let $y = 0$. $2x = 12$ $x = 6$ The x-intercept is (6, 0).
2 Sketch the graph.	
b Rearrange the equation in the form $y = mx + c$ and state the gradient. *Note:* The gradient could also be calculated using $m = \dfrac{\text{rise}}{\text{run}}$ from the diagram.	**b** $2x + 3y = 12$ $3y = -2x + 12$ $y = \dfrac{-2x}{3} + 4$ The gradient is $m = -\dfrac{2}{3}$.
c 1 Draw a diagram describing the position of the required point.	**c** Let P be the point on the line which is closest to the origin. This means that OP is perpendicular to the line L. 

2 Form the equation of a second line on which the required point must lie.	The gradient of the line L is $-\frac{2}{3}$. As OP is perpendicular to L, its gradient is $\frac{3}{2}$. OP contains the origin point (0, 0). Hence, the equation of OP is $y = \frac{3}{2}x$.
3 Set up a pair of simultaneous equations that can be used to find the required point.	The point P is the intersection of the lines: $y = \frac{3}{2}x \quad [1]$ $2x + 3y = 12 \quad [2]$
4 Solve the equations to obtain the coordinates of the required point.	Substitute [1] into [2]: $2x + 3 \times \frac{3x}{2} = 12$ $4x + 9x = 24$ $x = \frac{24}{13}$ Substitute $x = \frac{24}{13}$ into [1]: $y = \frac{3}{2} \times \frac{24}{13}$ $y = \frac{36}{13}$
5 State the coordinates of the required point.	The point $\left(\frac{24}{13}, \frac{36}{13}\right)$ is the point on the line that is closest to the origin.

study on
Units 3 & 4
AOS 1
Topic 1
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The quadratic polynomial function

The function $f: R \to R, f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$, is the quadratic polynomial function. If $a > 0$, its graph is a concave-up parabola with a minimum turning point; if $a < 0$, its graph is a concave-down parabola with a maximum turning point.

General form $y = ax^2 + bx + c$

As the x-intercepts of the graph of $y = ax^2 + bx + c$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, there may be zero, one or two x-intercepts as determined by the **discriminant** $\Delta = b^2 - 4ac$.

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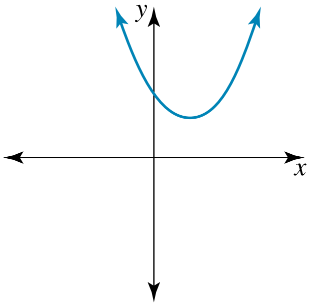
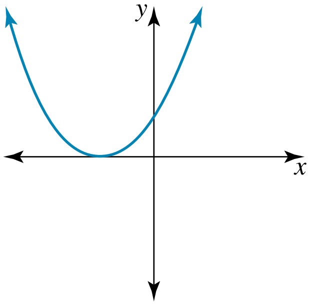
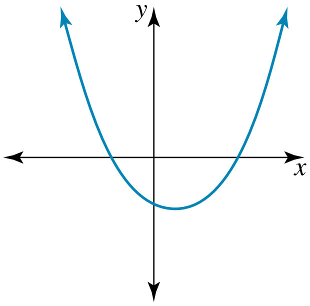
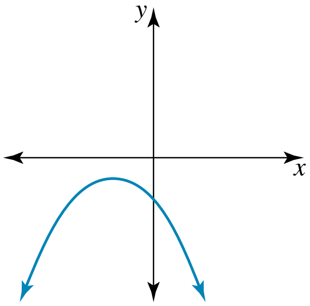
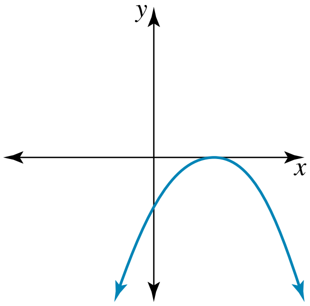
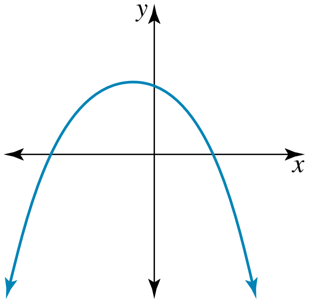
Interactivity
Quadratic functions
int-2562

	$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
$a > 0$			
$a < 0$			

If $\Delta < 0$, there are no x-intercepts; the quadratic function is either positive or negative, depending whether $a > 0$ or $a < 0$ respectively.

If $\Delta = 0$, there is one x-intercept, a turning point where the graph touches the x axis.

If $\Delta > 0$, there are two distinct x-intercepts and the graph crosses the x-axis at these places.

As the roots of the quadratic equation are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the axis of symmetry of the parabola has the equation $x = \dfrac{-b}{2a}$. This is also the x-coordinate of the turning point, so by substituting this value into the parabola's equation, the y-coordinate of the turning point can be calculated.

Turning point form, $y = a(x - h)^2 + k$

The simplest parabola has the equation $y = x^2$. Its turning point is the origin, (0, 0), which is unaltered by a **dilation** from the x-axis in the y-direction. However, if the graph of this parabola undergoes a horizontal translation of h units and a vertical translation of k units, the turning point moves to the point (h, k).

Thus, $y = a(x - h)^2 + k$ is the equation of a parabola with turning point (h, k) and axis of symmetry $x = h$.

If $y = a(x - h)^2 + k$ is expanded, then the general form $y = ax^2 + bx + c$ is obtained. Conversely, when the technique of completing the square is applied to the equation $y = ax^2 + bx + c$, the turning point form is obtained.

x-intercept form, $y = a(x - x_1)(x - x_2)$

When the equation of a quadratic function is expressed as the product of its two linear factors, the x-intercepts at $x = x_1$ and $x = x_2$ can be obtained by inspection. The axis of symmetry lies midway between the intercepts, so the equation for this axis must be $x = \frac{x_1 + x_2}{2}$, and this gives the x-coordinate of the turning point. The y-coordinate of the turning point can be calculated from the equation once the x-coordinate is known.

Expanding the equation $y = a(x - x_1)(x - x_2)$ will return it to general form, and factorising the general equation $y = ax^2 + bx + c$ will convert it to x-intercept form.

Key features of the graph of a quadratic function

When sketching the graph of a parabola by hand, identify:

- **the *y*-intercept**
- **any *x*-intercepts**
- **the turning point**
- **the axis of symmetry, if it is helpful to the sketch**
- **any end point coordinates if the function is given on a restricted domain.**

The methods used to identify these features will depend on the form in which the equation of the graph is expressed.

Similarly, when determining the equation of a parabola given a key feature, you should select the form of the equation that emphasises that key feature.

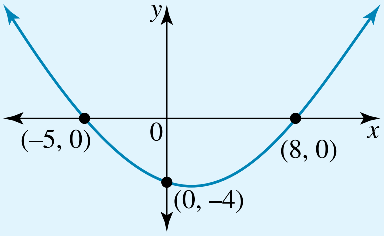
- If the turning point is given, use the $y = a(x - h)^2 + k$ form.
- If the x-intercepts are given, use the $y = a(x - x_1)(x - x_2)$ form.
- Otherwise, use the $y = ax^2 + bx + c$ form.

Three pieces of information are always required to determine the equation, as each form involves 3 constants or parameters.

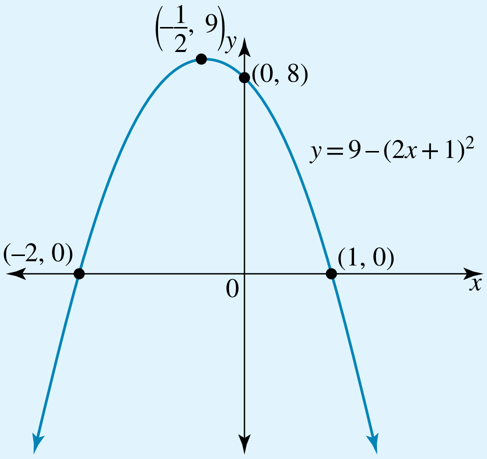
WORKED EXAMPLE 3

a **Sketch the graph of $y = 9 - (2x + 1)^2$ and state its domain and range.**

b **Determine the equation of the given graph and hence obtain the coordinates of the turning point.**



THINK	WRITE/DRAW
a **1** Rewrite the equation so it is in a standard form.	**a** $y = 9 - (2x + 1)^2$ $y = -(2x + 1)^2 + 9$ or $y = -\left(2\left(x + \frac{1}{2}\right)\right)^2 + 9$ $y = -4\left(x + \frac{1}{2}\right)^2 + 9$

2 State the coordinates and type of turning point.	The graph has a maximum turning point at $\left(-\frac{1}{2}, 9\right)$.
3 Calculate the y-intercept.	y-intercept: Let $x = 0$. $y = 9 - (1)^2$ $y = 8$ The y-intercept is $(0, 8)$.
4 Calculate any x-intercepts.	As the graph has a maximum turning point with a positive y-value, there will be x-intercepts. Let $y = 0$. $\begin{aligned} 9 - (2x + 1)^2 &= 0 \\ (2x + 1)^2 &= 9 \\ 2x + 1 &= \pm 3 \\ 2x &= -4 \text{ or } 2 \\ x &= -2 \text{ or } 1 \end{aligned}$ The x-intercepts are $(-2, 0)$ and $(1, 0)$.
5 Sketch the graph.	
6 State the domain and range.	The domain is R and the range is $(-\infty, 9]$.
b 1 Select a form of the equation.	**b** As the two x-intercepts are known, the x-intercept form of the equation will be used.
2 Use the key features to partially determine the equation.	There is an x-intercept at $x = -5$. $\Rightarrow (x + 5)$ is a factor. There is an x-intercept at $x = 8$. $\Rightarrow (x - 8)$ is a factor. The equation is $y = a(x + 5)(x - 8)$.

3 Use the third piece of information to fully determine the equation.

The point (0, –4) lies on the graph. Substitute this point in $y = a(x + 5)(x - 8)$.

$$\begin{aligned}-4 &= a(5)(-8)\\ -4 &= -40a\\ a &= \frac{1}{10}\end{aligned}$$

The equation is $y = \frac{1}{10}(x + 5)(x - 8)$.

4 Determine the equation of the axis of symmetry.

The axis of symmetry lies midway between the x-intercepts.

$$\begin{aligned}\therefore\quad x &= \frac{-5 + 8}{2}\\ &= \frac{3}{2}\end{aligned}$$

5 Calculate the coordinates of the turning point.

The turning point has $x = \frac{3}{2}$.

Substitute $x = \frac{3}{2}$ in the equation of the graph.

$$y = \frac{1}{10}\left(\frac{3}{2} + 5\right)\left(\frac{3}{2} - 8\right)$$

$$y = \frac{1}{10} \times \frac{13}{2} \times \frac{-13}{2}$$

$$y = -\frac{169}{40}$$

The turning point is $\left(\frac{3}{2}, -\frac{169}{40}\right)$.

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Units 3 & 4
AOS 1
Topic 1
Concept 4

Cubic functions
Concept summary
Practice questions

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Interactivities
Cubic polynomials
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x-intercepts of cubic graphs
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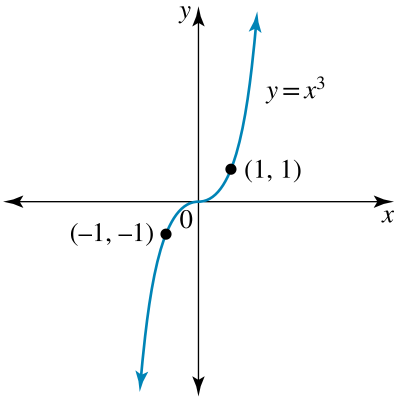
Cubic functions

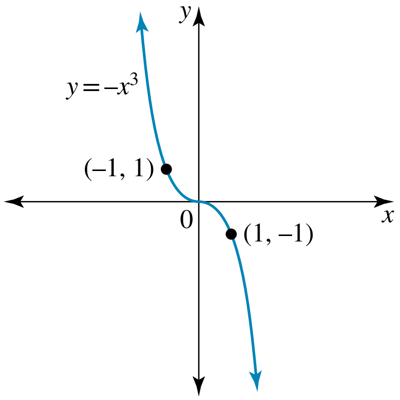
The function $f : R \to R,\ f(x) = ax^3 + bx^2 + cx + d,\ a, b, c, d \in R,\ a \neq 0$ is the cubic polynomial function. Although the shape of its graph may take several forms, for its maximal domain the function has a range of R. Its long-term behaviour is dependent on the sign of the coefficient of the x^3 term.

If $a > 0$, then as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

If $a < 0$, then as $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$.

This behaviour is illustrated in the graph of $y = x^3$, the simplest cubic function, and that of $y = -x^3$.





Cubic functions of the form $y = a(x - h)^3 + k$

A significant feature of both of the graphs of $y = x^3$ and $y = -x^3$ is the stationary point of inflection at the origin. This point is constant under a dilation but becomes the point (h, k) following a horizontal and vertical translation of h and k units respectively.

Cubic functions with equations of the form $y = a(x - h)^3 + k$ have:

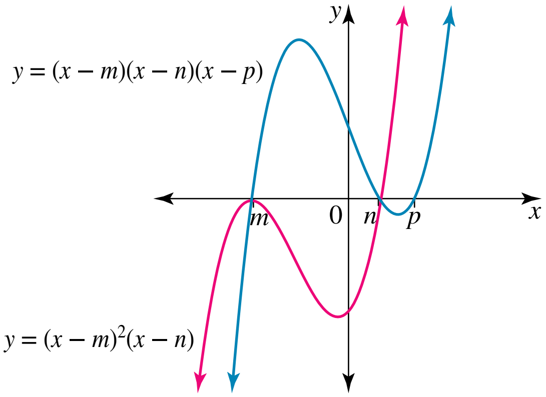
- **a stationary point of inflection at (h, k)**
- **one x-intercept**
- **long-term behaviour dependent on the sign of a.**

The coordinates of the stationary point of inflection are read from the equation in exactly the same way the turning points of a parabola are read from its equation in turning point form.

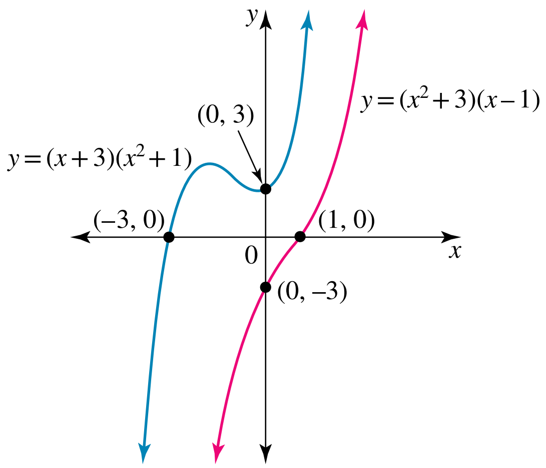
Cubic functions expressed in factorised form

A cubic function may have one, two or three x-intercepts, and hence its equation may have up to three linear factors. Where the equation can be expressed as the product of linear factors, we can readily deduce the behaviour of the function and sketch its graph without finding the positions of any turning points. Unlike the quadratic function, the turning points are not symmetrically placed between pairs of x-intercepts.

- **If there are three linear factors, that is $y = (x - m)(x - n)(x - p)$, the graph cuts the x-axis at $x = m$, $x = n$ and $x = p$.**
- **If there is one factor of multiplicity 2 and one other linear factor, that is $y = (x - m)^2(x - n)$, the graph touches the x-axis at a turning point at $x = m$ and cuts the x-axis at $x = n$.**



If the equation of the cubic function has one linear factor and one irreducible quadratic factor, it is difficult to deduce its behaviour without either technology or calculus. For example, the diagram shows the graphs of $y = (x + 3)(x^2 + 1)$ and $y = (x^2 + 3)(x - 1)$.



The intercepts made with the coordinate axes can be located and the long-term behaviour is known. However, at this stage we could not predict that $y = (x^2 + 3)(x - 1)$ has no turning points or stationary point of inflection (it has a non-stationary point of inflection). Nor could we predict, without numerical calculations, that there is a maximum and a minimum turning point on the graph of $y = (x + 3)(x^2 + 1)$.

Determining the equation of a cubic function from its graph

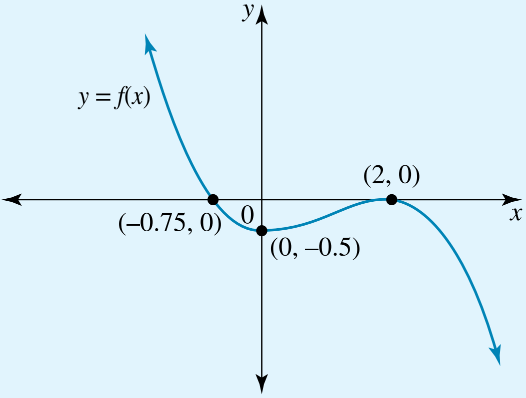
Depending on the information given, one form of the cubic equation may be preferable over another.

- If there is a stationary point of inflection given, use the $y = a(x - h)^3 + k$ form.
- If the x-intercepts are given, use the $y = a(x - m)(x - n)(x - p)$ form, or the repeated factor form $y = a(x - m)^2(x - n)$ if there is a turning point at one of the x-intercepts.
- If an x-intercept occurs at $x = \frac{b}{c}$, then $\left(x - \frac{b}{c}\right)$ is a factor. Alternatively, the rational root theorem allows this factor to be expressed as $(cx - b)$.
- Use the general form $y = ax^3 + bx^2 + cx + d$ if, for example, neither x-intercepts nor a stationary point of inflection are given.

WORKED EXAMPLE 4

a **Sketch the graph of $y = 2(x - 1)^3 + 8$, labelling the intercepts with the coordinate axes with their exact coordinates.**

b **Determine the function f whose graph is shown in the diagram, expressing its rule as the product of linear factors with integer coefficients.**



THINK	WRITE/DRAW
a 1 State the key feature that can be deduced from the equation.	**a** $y = 2(x - 1)^3 + 8$ This equation shows there is a stationary point of inflection at (1, 8).

2 Calculate the y-intercept.

y-intercept: Let $x = 0$.

$y = 2(-1)^3 + 8$

$y = 6$

The y-intercept is $(0, 6)$.

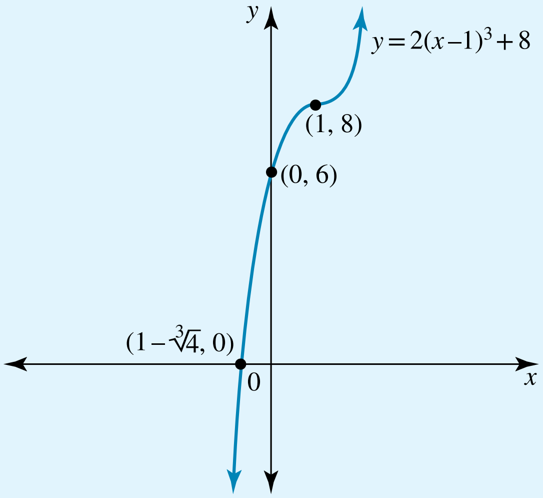
3 Calculate the x-intercept in exact form.

x-intercept: Let $y = 0$.

$$\begin{aligned}2(x-1)^3 + 8 &= 0\\ (x-1)^3 &= -4\\ x - 1 &= \sqrt[3]{-4}\\ x &= 1 + \sqrt[3]{-4}\\ x &= 1 - \sqrt[3]{4}\end{aligned}$$

The x-intercept is $(1 - \sqrt[3]{4}, 0)$.

4 Sketch the graph and label the intercepts with the coordinate axes.



b **1** Obtain a linear factor of the equation of the graph that has integer coefficients.

b The graph has an x-intercept at $x = -0.75$.

In fraction form, this is $x = -\frac{3}{4}$.

$\therefore (4x + 3)$ is a factor.

2 State a second factor.

The graph has a turning point on the x-axis at $x = 2$. This means $(x - 2)^2$ is a factor.

3 State the form of the equation.

The equation is of the form $y = a(4x + 3)(x - 2)^2$.

4 Determine the equation fully.

The point $(0, -0.5)$ or $\left(0, -\frac{1}{2}\right)$ lies on the graph.

Substitute this point into $y = a(4x + 3)(x - 2)^2$.

$$-\frac{1}{2} = a(3)(-2)^2$$

$$-\frac{1}{2} = 12a$$

$$a = -\frac{1}{24}$$

The graph has the equation $y = -\frac{1}{24}(4x + 3)(x - 2)^2$.

5 State the required function.

The domain of the graph is R. Hence, the function f is given by $f: R \to R,\ f(x) = -\frac{1}{24}(4x + 3)(x - 2)^2$.

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Units 3 & 4

AOS 1

Topic 1

Concept 7

Quartic functions

Concept summary

Practice questions

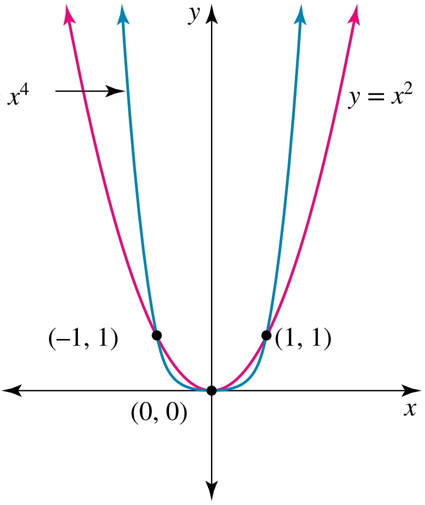
Quartic and higher degree polynomial functions

The function $f: R \to R, f(x) = ax^4 + bx^3 + cx^2 + dx + e$ where $a, b, c, d, e \in R, a \neq 0$ is the general form of a quartic polynomial function. Its graph can take various shapes, but all of them exhibit the same long-term behaviour. If the x^4 term has a positive coefficient, $y \to \infty$ as $x \to \pm\infty$; if the x^4 term has a negative coefficient, $y \to -\infty$ as $x \to \pm\infty$. Particular forms of the quartic equation enable some shapes of the graphs to be predicted.

Quartic functions of the form $y = a(x - h)^4 + k$

The simplest quartic function is $y = x^4$. It has a graph that has much the same shape as $y = x^2$, as shown in the diagram.

This leads to the conclusion that the graph of $y = a(x - h)^4 + k$ will be much the same shape as that of $y = a(x - h)^2 + k$ and will have the following characteristics.



For $y = a(x - h)^4 + k$:

- **If $a > 0$, the graph will be concave up with a minimum turning point (h, k).**
- **If $a < 0$, the graph will be concave down with a maximum turning point (h, k).**
- **The axis of symmetry has the equation $x = h$.**
- **There may be zero, one or two x intercepts.**

Quartic functions with linear factors.

Not all quartic functions can be factorised. However, if it is possible to express the equation as the product of linear factors, then the multiplicity of each factor will determine the behaviour of its graph.

A quartic polynomial may have up to 4 linear factors as it is of fourth degree. The possible combinations of these linear factors are:

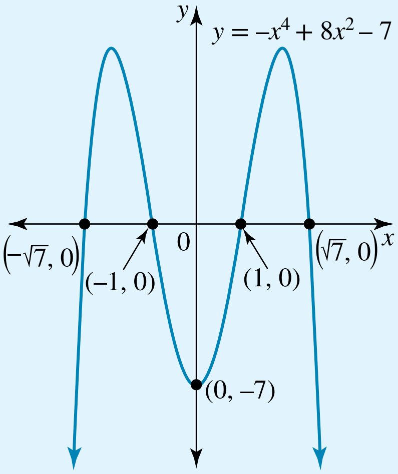
- **four distinct linear factors: $y = (x - a)(x - b)(x - c)(x - d)$**
- **one repeated linear factor: $y = (x - a)^2(x - b)(x - c)$, where the graph has a turning point that touches the x-axis at $x = a$**
- **two repeated linear factors: $y = (x - a)^2(x - b)^2$, where the graph has turning points that touch the x-axis at $x = a$ and $x = b$.**
- **one factor of multiplicity three: $y = (x - a)^3(x - b)$, where the graph has a stationary point of inflection that cuts the x-axis at $x = a$.**

The factorised forms may be derived from the general equation using standard algebraic techniques. Technology or calculus is required to accurately identify the position of turning points that do not lie on the x-axis.

WORKED EXAMPLE 5

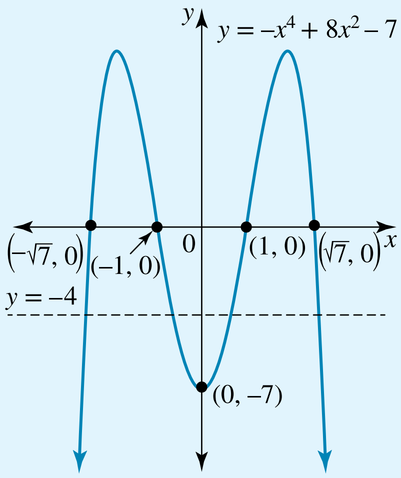
a Sketch the graph of $y = -x^4 + 8x^2 - 7$ and hence determine graphically the number of solutions to the equation $x^4 - 8x^2 + 3 = 0$.

b A quartic function has the equation $y = a(x + b)^4 + c$. The points (0, 5), (–2, 9) and (4, 9) lie on the graph of the function. Calculate the values of a, b and c and state the coordinates of the turning point.

THINK	**WRITE/DRAW**
a **1** Express the equation in factorised form.	**a** $y = -x^4 + 8x^2 - 7$ This is a quadratic in x^2. $y = -(x^4 - 8x^2 + 7)$ Let $a = x^2$. $y = -(a^2 - 8a + 7)$ $= -(a - 7)(a - 1)$ Substitute back for a: $y = -(x^2 - 7)(x^2 - 1)$ $= -(x + \sqrt{7})(x - \sqrt{7})(x + 1)(x - 1)$
2 State the x- and y-values of the intercepts with the axes.	x-intercepts: Let $y = 0$. $-(x + \sqrt{7})(x - \sqrt{7})(x + 1)(x - 1) = 0$ $\therefore x = \pm\sqrt{7},\ x = \pm 1$ y-intercept: $y = -x^4 + 8x^2 - 7$ Let $x = 0$. $\therefore y = -7$.
3 What will be the long-term behaviour?	As the coefficient of x^4 is negative, $y \to -\infty$ as $x \to \pm\infty$.
4 Sketch the graph.	
5 Rearrange the given equation so that the graph's equation appears on one of its sides.	The given equation is $x^4 - 8x^2 + 3 = 0$. This rearranges to $3 = -x^4 + 8x^2$ $3 - 7 = -x^4 + 8x^2 - 7$ $-x^4 + 8x^2 - 7 = -4$
6 Explain how the number of solutions to the equation could be solved graphically.	The number of intersections of the graph of $y = -x^4 + 8x^2 - 7$ with the horizontal line $y = -4$ will determine the number of solutions to the equation $x^4 - 8x^2 + 3 = 0$.

7 Specify the number of solutions.

The line $y = -4$ lies parallel to the x-axis between the origin and the y-intercept of the graph $y = -x^4 + 8x^2 - 7$.



There are four points of intersection, so there are four solutions to the equation $x^4 - 8x^2 + 3 = 0$.

b **1** Deduce the equation of the axis of symmetry.

b $y = a(x + b)^4 + c$

As the points (–2, 9) and (4, 9) have the same y-value, the axis of symmetry must pass midway between them.

The axis of symmetry is the line

$$x = \frac{-2 + 4}{2}$$
$$x = 1$$
$$\therefore b = -1$$

2 Use the given points given to form a pair of simultaneous equations.

The equation is $y = a(x - 1)^4 + c$.

Substitute the point (4, 9):

$$a(3)^4 + c = 9$$
$$81a + c = 9 \qquad [1]$$

Substitute the point (0, 5):

$$a(-1)^4 + c = 5$$
$$a + c = 5 \qquad [2]$$

3 Solve the equations.

Subtract equation [2] from equation [1]:

$$80a = 4$$
$$a = \frac{1}{20}$$
$$\therefore c = 5 - \frac{1}{20}$$
$$c = \frac{99}{20}$$

4 State the values required.

$a = \frac{1}{20}$, $b = -1$ and $c = \frac{99}{20}$.

5 Give the coordinates of the turning point.

The equation is $y = \frac{1}{20}(x - 1)^4 + \frac{99}{20}$.

The minimum turning point is $\left(1, \frac{99}{20}\right)$.

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Units 3 & 4

AOS 1

Topic 2

Concept 2

Graphs of $f(x) = x^n$, where n = even positive integer

Concept summary

Practice questions

eBook plus

Interactivity

Patterns of functions

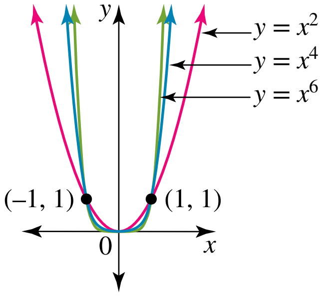
int-6415

The family of polynomial functions $y = x^n$ where $n \in N$

One classification of the polynomial functions is to group them according to whether their degree is even or odd.

The graph of $y = x^n$ where n is an even positive integer

The similarities shown between the graphs of $y = x^2$ and $y = x^4$ continue to hold for all polynomial functions of even degree. A comparison of the graphs of $y = x^2$, $y = x^4$ and $y = x^6$ is shown in the diagram.



The graphs each have a minimum turning point at (0, 0) and each contains the points (–1, 1) and (1, 1). They exhibit the same long-term behaviour that as $x \to \pm\infty$, $y \to \infty$.

The graph of the function with the highest degree, $y = x^6$, rises more steeply than the other two graphs for $x < -1$ and $x > 1$. However, for $-1 < x < 0$ and $0 < x < 1$, the function with the highest degree lies below the other graphs.

For $y = a(x - h)^n + k$, where n is an even positive integer:

- **If $a > 0$, the graph will be concave up with a minimum turning point (h, k).**
- **If $a < 0$, the graph will be concave down with a maximum turning point (h, k).**
- **The axis of symmetry has the equation $x = h$.**
- **There may be zero, one or two x-intercepts.**
- **The shape of the graph will be similar to that of $y = a(x - h)^2 + k$.**
- **If $a > 0$, the range is $[k, \infty)$.**
- **If $a < 0$, the range is $(-\infty, k]$.**

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Units 3 & 4

AOS 1

Topic 2

Concept 3

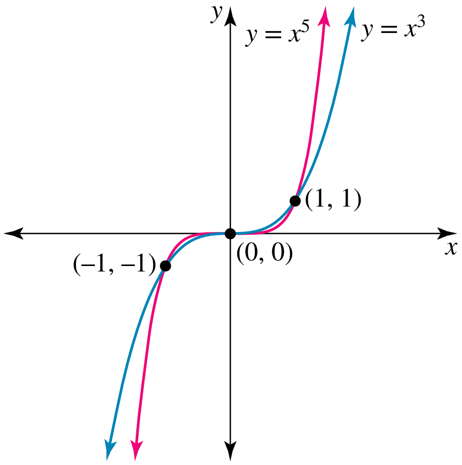
Graphs of $f(x) = x^n$, where n = odd positive integer

Concept summary

Practice questions

The graph of $y = x^n$ where n is an odd positive integer, $n > 1$

Polynomials of odd degree also share similarities, as the graphs of $y = x^3$ and $y = x^5$ illustrate.



Both $y = x^3$ and $y = x^5$ have a stationary point of inflection at (0, 0), and both pass through the points (–1, –1) and (1, 1), as does the linear function $y = x$. The three graphs display the same long-term behaviour that as $x \to \pm\infty$, $y \to \pm\infty$.

As observed for even degree polynomials, the graph of the function with the highest degree, $y = x^5$, rises more steeply than the other two graphs for $x < -1$ and $x > 1$. However, for $-1 < x < 0$ and $0 < x < 1$, the function with the highest degree lies below the other graphs.

The graphs of $y = a(x - h)^n + k$ where n is an odd positive integer, $n \neq 1$, have the following characteristics:

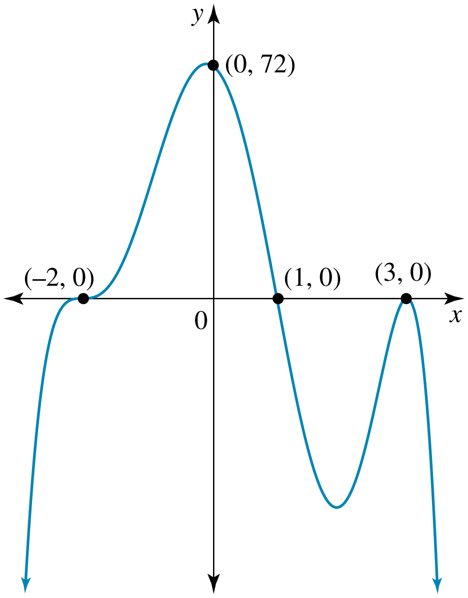
For $y = a(x - h)^n + k$, where n is an odd positive integer and $n \neq 1$:

- **There is a stationary point of inflection at (h, k).**
- **If $a > 0$, the long-term behaviour is as $x \to \pm\infty$, $y \to \pm\infty$.**
- **If $a < 0$, the long-term behaviour is as $x \to \pm\infty$, $y \to \mp\infty$.**
- **There will be one x-intercept.**
- **The shape of the graph is similar to that of the cubic function $y = a(x - h)^3 + k$.**

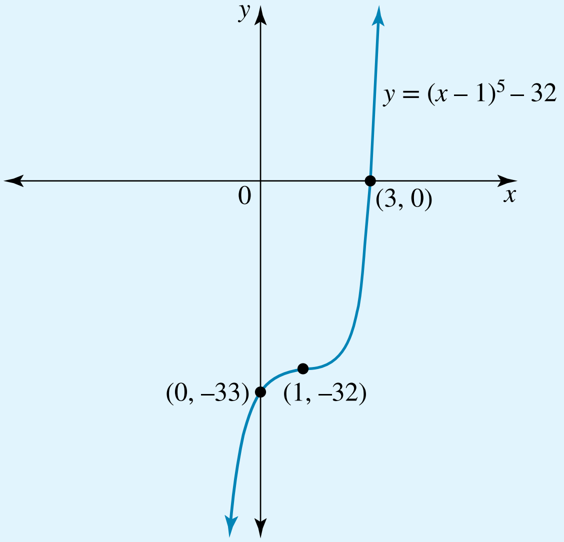
Polynomial functions that can be expressed as the product of linear factors

A degree n polynomial function may have up to n linear factors and therefore up to n intercepts with the x-axis. Where the polynomial can be specified completely as the product of linear factors, its graph can be drawn by interpreting the multiplicity of each linear factor together with the long-term behaviour determined by the sign of the coefficient of x^n.

For example, consider $y = (x + 2)^3(1 - x)(x - 3)^2$. The equation indicates there are x-intercepts at -2, 1 and 3. The x-intercept $(-2, 0)$ has a multiplicity of 3, meaning that there is a stationary point of inflection at this point. The x-intercept $(3, 0)$ has a multiplicity of 2, so this point is a turning point. The point $(1, 0)$ is a standard x-intercept. The polynomial is of degree 6 and the coefficient of x^6 is negative; therefore, as $x \to \pm\infty$, $y \to -\infty$.



WORKED EXAMPLE 6 Sketch the graph of $y = (x - 1)^5 - 32$.

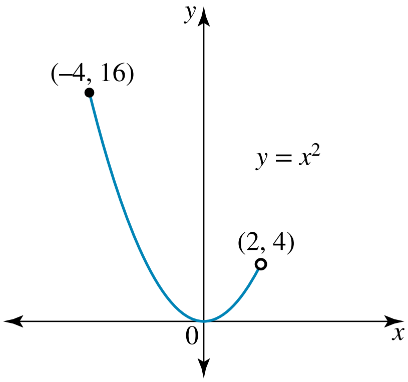
THINK	WRITE/DRAW
1 State whether the graph has a turning point or a point of inflection, and give the coordinates of the key point.	$y = (x - 1)^5 - 32$ As the degree is odd, the graph will have a stationary point of inflection at (1, −32).
2 Calculate the intercepts with the coordinate axes.	y-intercept: Let $x = 0$. $y = (-1)^5 - 32$ $= -33$ The y-intercept is (0, −33). x-intercepts: Let $y = 0$. $0 = (x - 1)^5 - 32$ $(x - 1)^5 = 32$ $x - 1 = 2$ $x = 3$ The x-intercept is (3, 0).
3 Sketch the graph.	

EXERCISE 2.2 Polynomial functions

PRACTISE

Work without CAS

1 **WE1** Part of the graph of the parabola $y = x^2$ is shown in the diagram.



a Explain why the graph shows a function and state the type of correspondence.
b State the domain and range.
c Express the given parabola using function notation.
d Calculate the value of y when $x = -2\sqrt{3}$.

2 Given $f(x) = x^2 - 4$:

a calculate the value of y when $x = \frac{2}{3}$
b calculate $f(2a)$
c state the implied domain of the function.

3 **WE2** Consider the line L where $L = \{(x, y) : 3x - 4y = 12\}$.

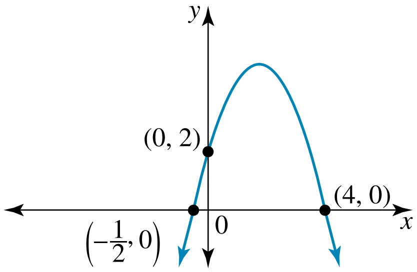
- **a** Sketch the line.
- **b** Calculate the gradient of the line.
- **c** Determine the coordinates of the point on the line that is closest to the origin.

4 Consider the points A (–1, –3) and B (5, –7).

- **a** Form the equation of the line that passes through points A and B.
- **b** Calculate the equation of the perpendicular bisector of the line segment joining the points A and B.
- **c** Calculate the angle at which the perpendicular bisector of AB cuts the x-axis.

5 **a** **WE3** Sketch the graph of $y = 2(3x - 2)^2 - 8$ and state its domain and range.

- **b** Determine the equation of the given graph and hence obtain the coordinates of the turning point.

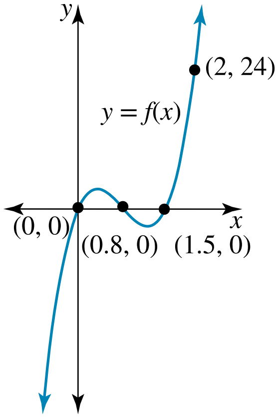


6 Consider the quadratic function $f : R^+ \cup \{0\} \rightarrow R,\ f(x) = 4x^2 - 8x + 7$.

- **a** Determine the number of intercepts the graph of $y = f(x)$ makes with the x-axis.
- **b** Express the equation of the function in the form $f(x) = a(x + b)^2 + c$.
- **c** Sketch the graph of $y = f(x)$ and state its domain and range.

7 **a** **WE4** Sketch the graph of $y = -4(x + 2)^3 + 16$, labelling the intercepts with the coordinate axes with their exact coordinates.

- **b** Determine the function f whose graph is shown in the diagram, expressing its rule as the product of linear factors with integer coefficients.



8 Consider the function $f : [-2, 4] \rightarrow R$, $f(x) = 4x^3 - 8x^2 - 16x + 32$.

- **a** Factorise $4x^3 - 8x^2 - 16x + 32$.
- **b** Sketch the graph of $y = f(x)$.
- **c** State the maximum and minimum values of the function f.

9 **a** **WE5** Sketch the graph of $y = x^2 - x^4$ and hence determine graphically the number of solutions to the equation $x^4 - x^2 + x - 2 = 0$.

- **b** A quartic function has the equation $y = a(x + b)^4 + c$. The graph of the function cuts the x-axis at $x = -9$ and $x = -3$. The range of the graph is $(-\infty, 7]$. Calculate the values of a, b and c and state the coordinates of the turning point.

10 Sketch the graph of $y = x^4 - 6x^3$ and hence state the number of intersections the graph of $y = x^4 - 6x^3 - 1$ would make with the x-axis.

11 **WE6** Sketch the graph of $y = (x + 1)^6 + 10$.

12 Sketch the graph of $y = (x + 4)(x + 2)^2(x - 2)^3(x - 5)$.

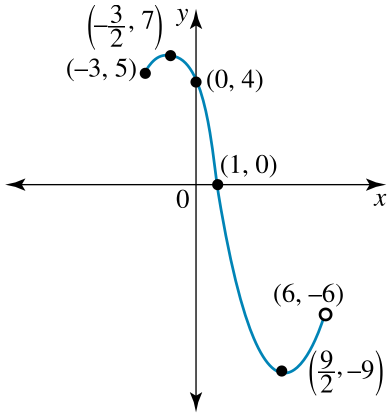
CONSOLIDATE

Apply the most appropriate mathematical processes and tools

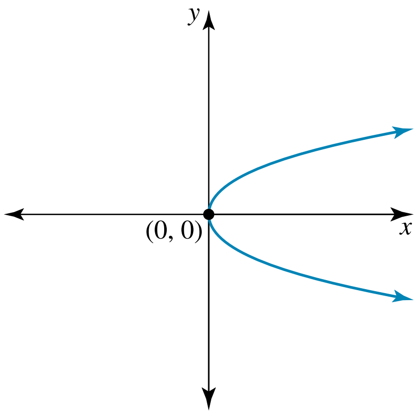
13 For each of the following, state:

i the type of correspondence
ii the domain and the range
iii whether or not the relation is a function.

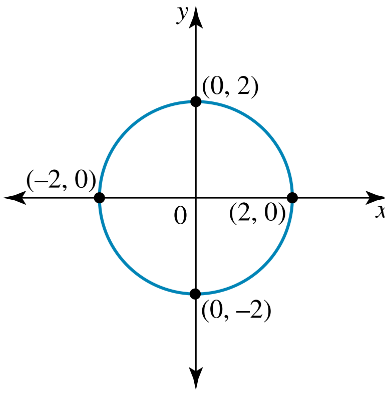
a



b



c



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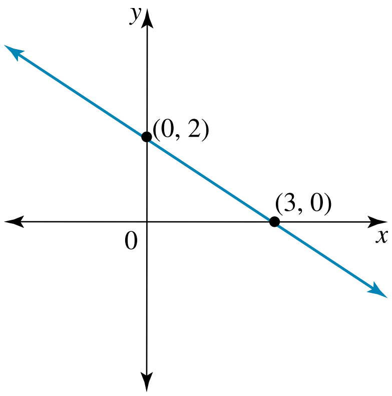

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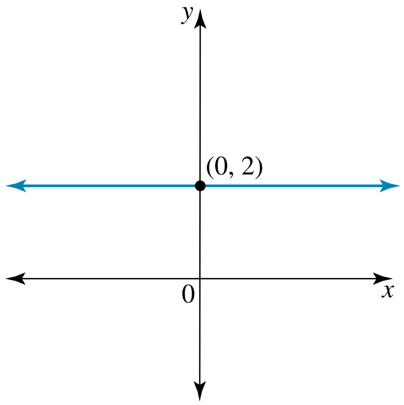
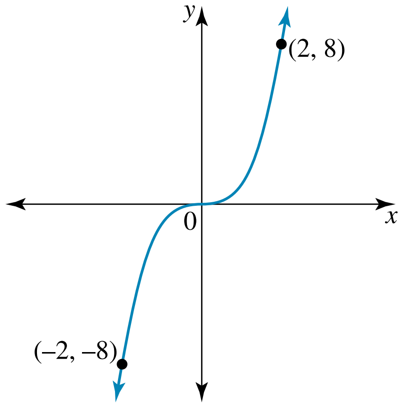


f



14 Sketch the following linear functions and state the range of each.

a $f : R \rightarrow R,\ f(x) = 9 - 4x$

b $g : (-3, 5] \rightarrow R,\ g(x) = \dfrac{3x}{5}$

15 Consider the three points A (5, –3), B (7, 8) and C (–2, p). The line through A and C is parallel to $9x + 7y = 24$.

a Calculate the value of p.
b Determine the equation of the line through B that is perpendicular to AC.
c Calculate the shortest distance from B to AC, expressing the value to 1 decimal place.

16 **a** Express $-x^2 + 2x - 5$ in the form $a(x + b)^2 + c$.

b Hence, state the coordinates of the turning point of the graph of $y = -x^2 + 2x - 5$.

c Sketch the graph of $y = -x^2 + 2x - 5$ and state its range.

d Use a graphical method to show that the graphs of $y = x + 3$ and $y = -x^2 + 2x - 5$ never intersect.

e Determine the value of k so that the graphs of $y = x + k$ and $y = -x^2 + 2x - 5$ will intersect exactly once.

17 Determine the equations of the following quadratic functions.

a The turning point has coordinates (–6, 12) and the graph of the function passes through the point (4, –3).

b The points (–7, 0), (0, –20) and $(-2\frac{1}{2}, 0)$ lie on the graph.

c The minimum value of the function is –5 and it contains the points (–8, 11) and (8, 11).

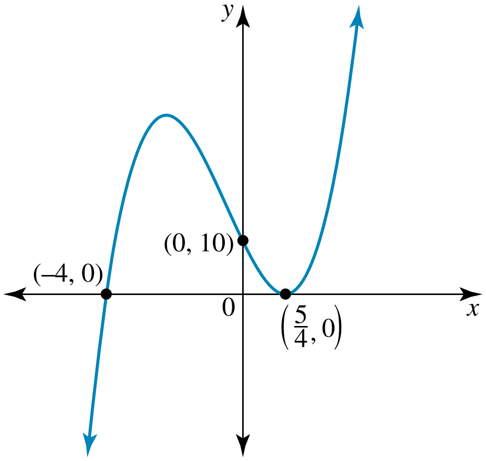
18 Sketch the graphs of the following cubic functions without attempting to locate any turning points that do not lie on the coordinate axes.

a $y = x^3 - x^2 - 6x$

b $y = 1 - \frac{1}{8}(x + 1)^3, x \in [-3, 2)$

c $y = 12(x + 1)^2 - 3(x + 1)^3$

19 Form a possible equation for the cubic graph shown.



20 **a** Show that the graph of $y = f(x)$ where $f(x) = -2x^3 + 9x^2 - 24x + 17$ has exactly one x-intercept.

b Show that there is no stationary point of inflection on the graph.

c State the long-term behaviour of the function.

d Given the function has a one-to-one correspondence, draw a sketch of the graph.

21 **a** A quartic function has exactly one turning point at (–5, 12) and also contains the point (–3, –36). Form its equation.

b Sketch the graph of $y = (2 + x)(1 - x)^3$.

c **i** Factorise $-x^4 + x^3 + 10x^2 - 4x - 24$.

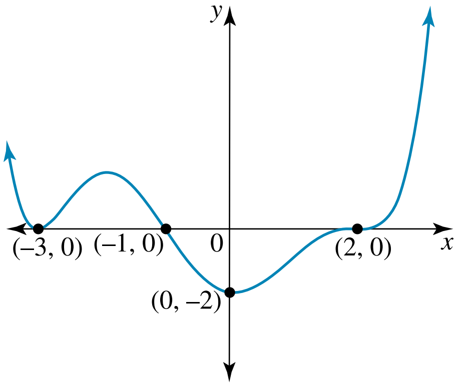
ii Hence sketch $y = -x^4 + x^3 + 10x^2 - 4x - 24$.

22 **a** **i** Sketch the graphs of $y = x^6$ and $y = x^7$ on the same set of axes, labelling any points of intersection with their coordinates.

ii Hence state the solutions to $\{x : x^6 - x^7 \geq 0\}$.

b Sketch the graphs of $y = 16 - (x + 2)^4$ and $y = 16 - (x + 2)^5$ on the same set of axes, identifying the key features of each graph and any points of intersection.

c Consider the graph of the polynomial function shown.



i Assuming the graph is a monic polynomial that maintains the long-term behaviour suggested in the diagram, give a possible equation for the graph and state its degree.

ii In fact the graph cuts straight through the x-axis once more at $x = 10$. This is not shown on the diagram. Given this additional information, state the degree and a possible equation for the function.

MASTER

23 Use CAS technology to sketch the graphs of $y = x^4 - 2$ and $y = 2 - x^3$, and hence state to 2 decimal places the values of the roots of the equation $x^4 + x^3 - 4 = 0$.

24 Use CAS technology to obtain the coordinates of any turning points or stationary points of inflection on the graphs of:

a $y = (x^2 + x + 1)(x^2 - 4)$

b $y = 1 - 4x - x^2 - x^3$

c $y = \frac{1}{4}((x - 2)^5(x + 3) + 80)$.

Express answers to 3 decimal places, where appropriate.

2.3 Other algebraic functions

The powers of the variable in a polynomial function must be natural numbers. In this section we consider functions where the power of the variable may be rational.

Maximal domain

The maximal domain of any function must exclude:

- any value of x for which the denominator would become zero
- any value of x which would create a negative term under a square root sign.

The maximal or implied domain of rational functions of the form $y = \dfrac{g(x)}{f(x)}$, where both $f(x)$ and $g(x)$ are polynomials, must exclude any values of x for which $f(x) = 0$. The domain would be $R\backslash\{x : f(x) = 0\}$.

Likewise, the maximal domain of square root functions of the form $y = \sqrt{f(x)}$ would be $\{x : f(x) \geq 0\}$.

For a function of the form $y = \dfrac{g(x)}{\sqrt{f(x)}}$, the maximal domain would be $\{x : f(x) > 0\}$.

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Units 3 & 4
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Graphs of $f(x) = \frac{1}{x^n}$, where n = positive odd integer
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Interactivity
Hyperbola
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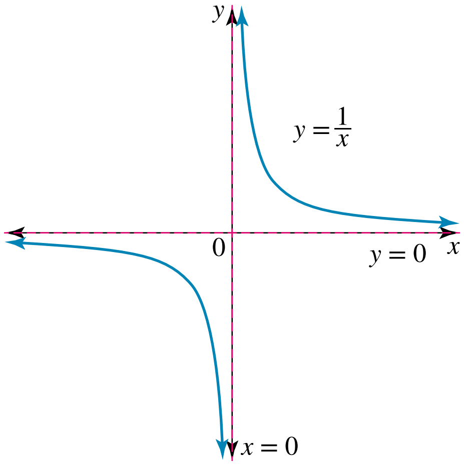
The rectangular hyperbola

The equation of the simplest hyperbola is $y = \frac{1}{x}$. In power form this is written as $y = x^{-1}$. Its maximal domain is $R \backslash \{0\}$, as the function is undefined if $x = 0$. The graph of this function has the following characteristics.

- There is a vertical **asymptote** with equation $x = 0$.
- There is a horizontal asymptote with equation $y = 0$.
- As $x \to \infty$, $y \to 0$ from above the horizontal asymptote, and as $x \to -\infty$, $y \to 0$ from below the horizontal asymptote.
- As $x \to 0^+$, $y \to \infty$, and as $x \to 0^-$, $y \to -\infty$.
- The function has one-to-one correspondence.
- The domain is $R \backslash \{0\}$ and the range is $R \backslash \{0\}$.



As the asymptotes are perpendicular to each other, the graph is called a rectangular hyperbola. The graph lies in the first and third quadrants formed by its asymptotes. The graph of $y = -\frac{1}{x}$ would lie in the second and fourth quadrants.

Hyperbolas of the form $y = \frac{a}{x - h} + k$

The asymptotes are the key feature of the graph of a hyperbola. Their positions are unaffected by a dilation, but if the graph of $y = \frac{1}{x}$ is horizontally or vertically translated, then the vertical and horizontal asymptotes are moved accordingly.

The graph of $y = \frac{a}{x - h} + k$ has:

- **a vertical asymptote $x = h$**
- **a horizontal asymptote $y = k$**
- **a domain of $R \backslash \{h\}$**
- **a range of $R \backslash \{k\}$.**

If $a > 0$, the graph lies in quadrants 1 and 3 as formed by its asymptotes.
If $a < 0$, then the graph lies in quadrants 2 and 4 as formed by its asymptotes.

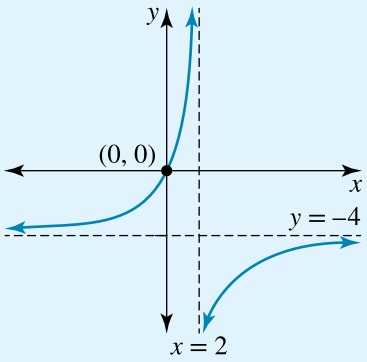
Identifying the asymptotes

The presence of a vertical asymptote at $x = h$ on the graph of $y = \frac{a}{x - h} + k$ could also be recognised by solving $x - h = 0$. The hyperbola $y = \frac{a}{bx + c}$ has a vertical asymptote when $bx + c = 0$, and its maximal domain is $R \backslash \left\{-\frac{c}{b}\right\}$.

The horizontal asymptote is identified from the equation of a hyperbola expressed in proper rational form, that is, when the numerator is of lower degree than the denominator. The equation $y = \frac{1 + 2x}{x}$ should be rewritten as $y = \frac{1}{x} + 2$ in order to identify the horizontal asymptote $y = 2$.

WORKED EXAMPLE 7

a **Determine an appropriate equation for the hyperbola shown.**



b **i** **Obtain the maximal domain of $y = \frac{2x + 5}{x + 1}$.**

ii **Sketch the graph of $y = \frac{2x + 5}{x + 1}$ and state its range.**

THINK	WRITE/DRAW
a **1** Write the general equation of a hyperbola.	**a** Let the equation be $y = \frac{a}{x - h} + k$.
2 Identify the asymptotes and enter them into the equation.	The graph shows there is a vertical asymptote at $x = 2$. $\therefore y = \frac{a}{x - 2} + k$ There is a horizontal asymptote at $y = -4$. $\therefore y = \frac{a}{x - 2} - 4$
3 Identify the known point through which the graph passes and use this to fully determine the equation.	The graph passes through the origin. Substitute (0, 0): $0 = \frac{a}{-2} - 4$ $4 = -\frac{a}{2}$ $a = -8$ The equation is $y = \frac{-8}{x - 2} - 4$.
b **i** **1** Identify what must be excluded from the domain.	**b** **i** $y = \frac{2x + 5}{x + 1}$ The function is undefined if its denominator is zero. When $x + 1 = 0$, $x = -1$. This value must be excluded from the domain.
2 State the maximal domain.	The maximal domain is $R\backslash\{-1\}$.

ii **1** Express the equation in proper rational form.

ii $$\frac{2x+5}{x+1} = \frac{2(x+1)+3}{x+1}$$
$$= \frac{2(x+1)}{x+1} + \frac{3}{x+1}$$
$$= 2 + \frac{3}{x+1}$$

The equation is $y = \dfrac{3}{x+1} + 2$.

2 State the equations of the asymptotes.

The graph has a vertical asymptote $x = -1$ and a horizontal asymptote $y = 2$.

3 Calculate any intercepts with the coordinate axes.

x-intercept: Let $y = 0$ in $y = \dfrac{2x+5}{x+1}$.

$$0 = \frac{2x+5}{x+1}$$
$$0 = 2x + 5$$
$$x = -\frac{5}{2}$$

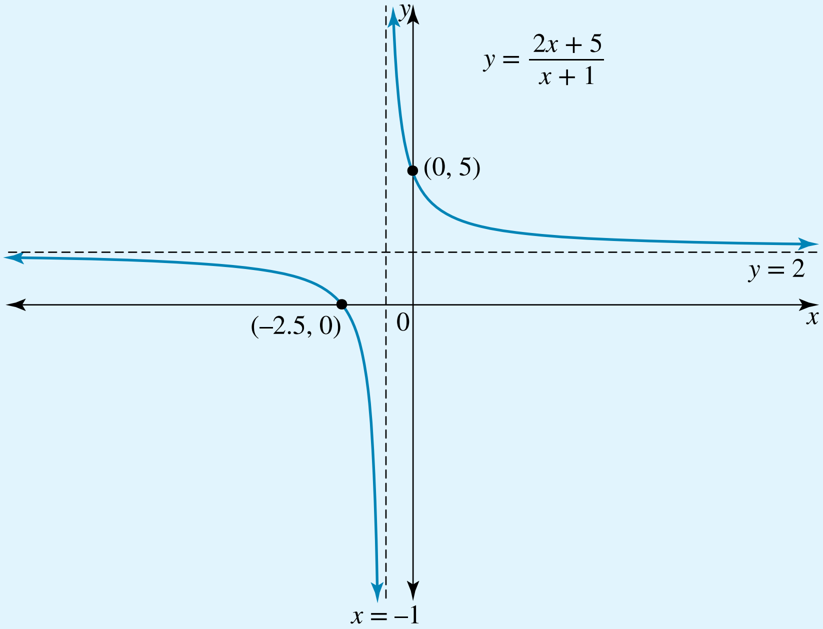
The x-intercept is $\left(-\dfrac{5}{2}, 0\right)$

y-intercept: Let $x = 0$.

$$y = \frac{5}{1}$$
$$= 5$$

The y-intercept is $(0, 5)$.

4 Sketch the graph.



5 State the range.

The range is $R\backslash\{2\}$.

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Units 3 & 4

AOS 1

Topic 2

Concept 5

Graphs of $f(x) = \frac{1}{x^n}$, where n = positive even integer

Concept summary

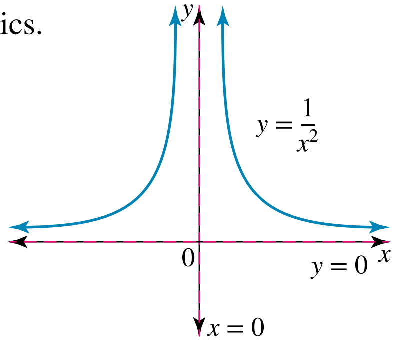
Practice questions

The truncus

The graph of the function $y = \frac{1}{x^2}$ is called a truncus. Its rule can be written as a power function, $y = x^{-2}$.

The graph of this function has the following characteristics.

- There is a vertical asymptote with equation $x = 0$.
- There is a horizontal asymptote with equation $y = 0$.
- The domain is $R\backslash\{0\}$.
- The range is R^+.
- The function has many-to-one correspondence.
- The graph is symmetric about its vertical asymptote.



The graph of $y = \frac{1}{x^2}$ lies in the first and second quadrants that are created by its asymptotes. The graph of $y = -\frac{1}{x^2}$ would lie in the third and fourth quadrants.

The truncus is steeper than the hyperbola for $x \in (-1, 0)$ and $x \in (0, 1)$. However, a similar approach is taken to sketching both functions.

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Interactivity

Patterns of functions

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The general form of the truncus $y = \frac{a}{(x-h)^2} + k$

The graph of the truncus with the equation $y = \frac{a}{(x-h)^2} + k$ has the following characteristics.

- **There is a vertical asymptote at $x = h$.**
- **There is a horizontal asymptote at $y = k$.**
- **The domain is $R\backslash\{h\}$.**
- **If $a > 0$, then the range is (k, ∞).**
- **If $a < 0$, then the range is $(-\infty, k)$.**

WORKED EXAMPLE 8

Sketch the graph of $y = 8 - \frac{2}{(x-3)^2}$ and state its domain and range.

THINK	WRITE/DRAW
1 State the equations of the asymptotes.	$y = 8 - \frac{2}{(x-3)^2}$ The vertical asymptote is $x = 3$. The horizontal asymptote is $y = 8$.
2 Calculate the y-intercept.	y-intercept: Let $x = 0$. $y = 8 - \frac{2}{(-3)^2}$ $y = 7\frac{7}{9}$ The y-intercept is $\left(0, 7\frac{7}{9}\right)$.

3 Calculate any x-intercepts.

x-intercepts: Let $y = 0$.

$$0 = 8 - \frac{2}{(x-3)^2}$$

$$\frac{2}{(x-3)^2} = 8$$

$$2 = 8(x-3)^2$$

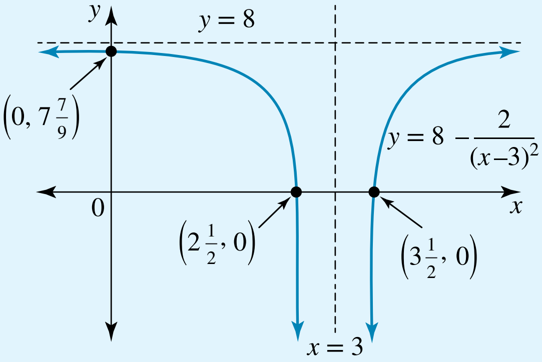
$$(x-3)^2 = \frac{1}{4}$$

$$x - 3 = \pm\frac{1}{2}$$

$$x = 2\tfrac{1}{2} \text{ or } x = 3\tfrac{1}{2}$$

The x-intercepts are $\left(2\tfrac{1}{2}, 0\right)$, $\left(3\tfrac{1}{2}, 0\right)$.

4 Sketch the graph.



5 State the domain and range.

The domain is $R\backslash\{3\}$ and the range is $(-\infty, 8)$.

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Interactivity
The relation $y^2 = x$
int-2574

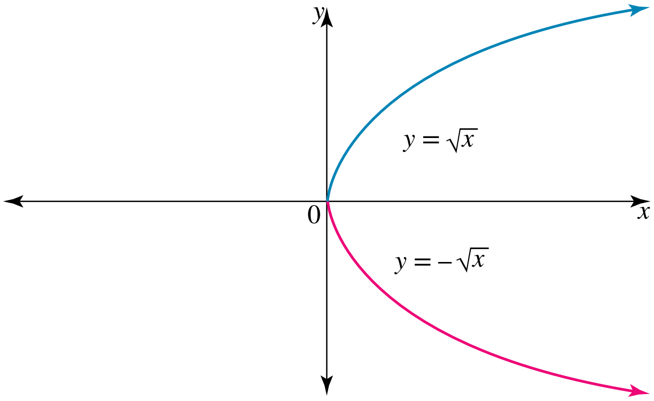
The square root and cube root functions

The square root function has the rule $y = \sqrt{x}$, and the rule for the cube root function is $y = \sqrt[3]{x}$. As power functions these rules can be expressed as $y = x^{\frac{1}{2}}$ and $y = x^{\frac{1}{3}}$ respectively.

The maximal domain of $y = \sqrt{x}$ is $[0, \infty)$, because negative values under a square root must be excluded. However, cube roots of negative numbers are real, so the maximal domain of the cube root function $y = \sqrt[3]{x}$ is R.

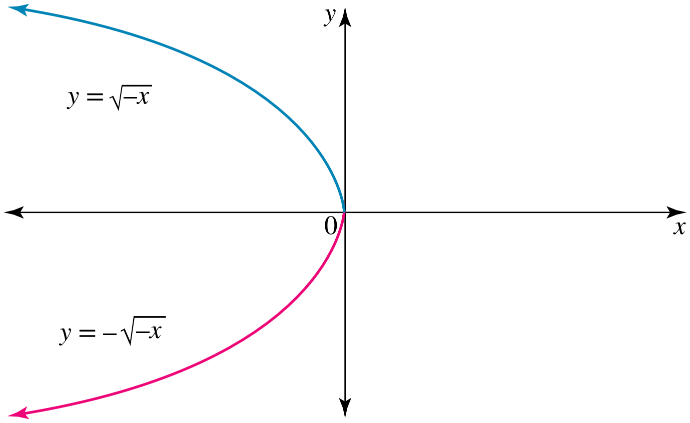
The graph of the square root function

The function $y = \sqrt{x}$ is the top half of the 'sideways' parabola $y^2 = x$. The bottom half of this parabola is the function $y = -\sqrt{x}$.



The parabola $y^2 = x$ is not a function, but its two halves are. The equation $y^2 = x$ could also be written as $y = \pm\sqrt{x}$. The turning point or vertex of the parabola is the end point for the square root functions $y = \sqrt{x}$ and $y = -\sqrt{x}$. These functions both have domain $[0, \infty)$, but their ranges are $[0, \infty)$ and $(-\infty, 0]$ respectively.

The parabola $y^2 = -x$ would open to the left of its vertex. Its two branches would be the square root functions $y = \sqrt{-x}$ and $y = -\sqrt{-x}$, with domain $(-\infty, 0]$ and ranges $[0, \infty)$ and $(-\infty, 0]$ respectively.



The four square root functions show the different orientations that can be taken. Calculation of the maximal domain and the range will identify which form a particular function takes.

Square root functions of the form $y = a\sqrt{x - h} + k$ have the following characteristics.

- **The end point is (h, k).**
- **The domain is $[h, \infty)$.**
- **If $a > 0$, the range is $[k, \infty)$; if $a < 0$, the range is $(-\infty, k]$.**

Square root functions of the form $y = a\sqrt{-(x - h)} + k$ have the following characteristics.

- **The end point is (h, k).**
- **The domain is $(-\infty, h]$.**
- **If $a > 0$, the range is $[k, \infty)$; if $a < 0$, the range is $(-\infty, k]$.**

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Interactivity
Fractional power functions
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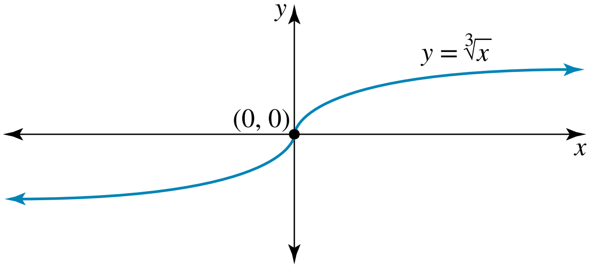
The graph of the cube root function

The graph of the cubic function $y = x^3$ has a stationary point of inflection at the origin. The graph of $y^3 = x$ has a 'sideways' orientation but still has a point of inflection at the origin.

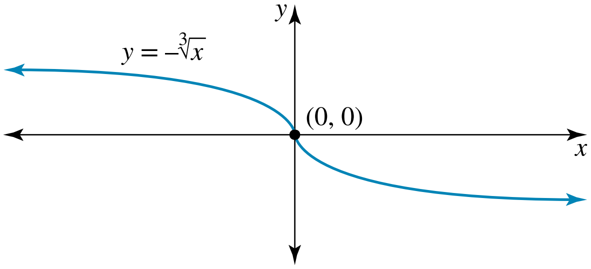
The rule $y^3 = x$ can also be expressed as $y = \sqrt[3]{x}$. The graph of $y = \sqrt[3]{x}$ is shown in the diagram.



The graph $y = \sqrt[3]{x}$ has the following characteristics.

- There is a point of inflexion at (0, 0) where the tangent drawn to the curve would be vertical.
- The domain is R and the range is R.
- The function has one-to-one correspondence.

The graph of $y = -\sqrt[3]{x}$ would be the reflection of $y = \sqrt[3]{x}$ in the x axis.



This would also be the graph of $y = \sqrt[3]{-x}$, as $\sqrt[3]{-x} = -\sqrt[3]{x}$.

The general equation $y = a\sqrt[3]{x-h} + k$ shows the graph has the following characteristics.

- **There is a point of inflection at (h, k).**
- **The domain is R and the range is R.**
- **One x-intercept can be located by solving $a\sqrt[3]{x-h} + k = 0$.**
- **If $a > 0$, the long-term behaviour is $x \to \pm\infty$, $y \to \pm\infty$.**
- **If $a < 0$, the long-term behaviour is $x \to \pm\infty$, $y \to \mp\infty$.**

The long-term behaviour of the cube root function resembles that of the cubic function.

WORKED EXAMPLE 9

a i State the maximal domain of $y = \sqrt{4-x} - 1$.

ii Sketch the graph of $y = \sqrt{4-x} - 1$ and state its range.

b The graph of a cube root function has its point of inflection at $(1, 5)$ and the graph cuts the y-axis at $(0, 2)$. Determine the rule and sketch the graph.

THINK	WRITE/DRAW
a i Form the maximal domain.	**a i** $y = \sqrt{4-x} - 1$ The term under the square root cannot be negative. $4 - x \geq 0$ $x \leq 4$ The maximal domain is $(-\infty, 4]$.
ii 1 State the coordinates of the end point.	**ii** The end point is (4, –1).
2 Calculate the yintercept, if there is one.	With the domain $(-\infty, 4]$, the graph opens to the left, so it will cut the y-axis. y-intercept: Let $x = 0$. $y = \sqrt{4} - 1$ $y = 1$ The y-intercept is (0, 1).

3 Calculate the x intercept, if there is one.

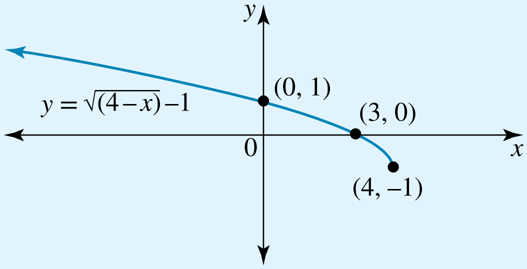
The end point lies below the x-axis and the y-intercept lies above the x-axis. There will be an x-intercept.

x-intercept: Let $y = 0$.

$$0 = \sqrt{4 - x} - 1$$
$$\sqrt{4 - x} = 1$$
$$4 - x = 1$$
$$x = 3$$

The x-intercept is (3, 0).

4 Sketch the graph.



5 State the range.

The range is $[-1, \infty)$.

b **1** Write the general equation of a cube root function.

b Let the equation be $y = a\sqrt[3]{x - h} + k$.

2 Insert the information about the point of inflection.

The point of inflection is (1, 5).

$$\therefore y = a\sqrt[3]{x - 1} + 5$$

3 Fully determine the equation using the other piece of information given.

Substitute the point (0, 2):

$$2 = a\sqrt[3]{-1} + 5$$
$$2 = -a + 5$$
$$a = 3$$

The equation is $y = 3\sqrt[3]{x - 1} + 5$.

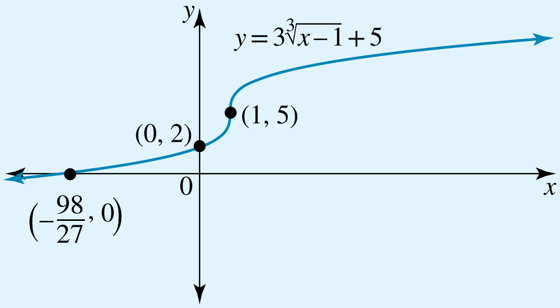
4 Calculate the x-intercept.

x-intercept: Let $y = 0$.

$$0 = 3\sqrt[3]{x - 1} + 5$$
$$\sqrt[3]{x - 1} = \frac{-5}{3}$$
$$x - 1 = \left(-\frac{5}{3}\right)^3$$
$$x = 1 - \frac{125}{27}$$
$$x = -\frac{98}{27}$$

The x-intercept is $\left(-\frac{98}{27}, 0\right)$.

5 Sketch the graph.



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Units 3 & 4

AOS 1

Topic 2

Concept 6

Graphs of functions with fractional powers

Concept summary

Practice questions

Power functions of the form $y = x^{\frac{p}{q}}$, $p, q \in N$

The square root and cube root functions are examples of power functions of the form $y = x^{\frac{p}{q}}$, $p, q \in N$. For the square root function, $y = \sqrt{x} = x^{\frac{1}{2}}$ so $p = 1$ and $q = 2$; for the cube root function, $y = \sqrt[3]{x} = x^{\frac{1}{3}}$, so $p = 1$ and $q = 3$.

In this section we consider some other functions that have powers which are positive rational numbers and deduce the shape of their graphs through an analysis based on index laws.

Index laws enable $x^{\frac{p}{q}}$ to be expressed as $\sqrt[q]{x^p}$.

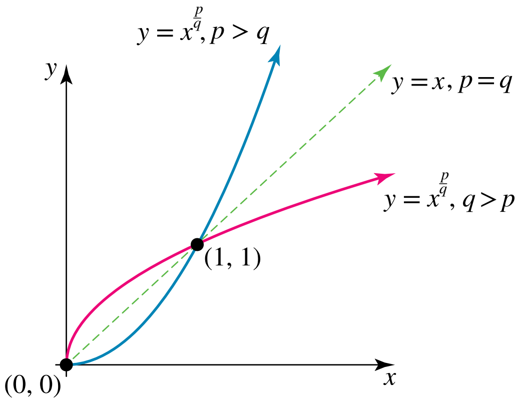
With $p, q \in N$, the function is formed as the qth root of the polynomial x^p. As polynomial shapes are known, this interpretation allows the shape of the graph of the function to be deduced. Whichever is the larger of p and q will determine whether the polynomial or the root shape will be the **dominant function**.

For the graph of $y = x^{\frac{p}{q}}$, $p, q \in N$:

- **If $p > q$, the polynomial shape dominates, because the index $\frac{p}{q} > 1$.**
- **If $q > p$, the root shape dominates, because the index must be in the interval $0 < \frac{p}{q} < 1$.**
- **If $p = q$, the index is 1 and the graph is that of $y = x$.**
- **Even roots of the polynomial x^p cannot be formed in any section where the polynomial graph is negative.**
- **The points (0, 0) and (1, 1) will always lie on the graph.**

The basic polynomial or root shape for the first quadrant is illustrated for $p > q \Rightarrow$ index > 1, $p = q \Rightarrow$ index $= 1$ and $q > p \Rightarrow$ index < 1.

Note that the polynomial shape lies below $y = x$ for $0 < x < 1$ and above $y = x$ for $x > 1$, whereas the root shape lies above $y = x$ for $0 < x < 1$ and below $y = x$ for $x > 1$. It is always helpful to include the line $y = x$ when sketching a graph of the form $y = x^{\frac{p}{q}}$.



WORKED EXAMPLE 10

Give the domain and deduce the shape of the graph of:

a $y = x^{\frac{2}{3}}$

b $y = x^{\frac{3}{2}}$.

THINK	WRITE/DRAW
a 1 Express the function rule in surd form and deduce how the function can be formed.	**a** $y = x^{\frac{2}{3}}$ $= \sqrt[3]{x^2}$ The function is formed as the cube root of the quadratic polynomial $y = x^2$.

2 Use the nature of the operation forming the function to determine the domain of the function.

Cube roots of both positive and negative numbers can be calculated. However, the graph of $y = x^2$ lies in quadrants 1 and 2 and is never negative. Therefore, there will be two non-negative branches to the power function, giving it a domain of R.

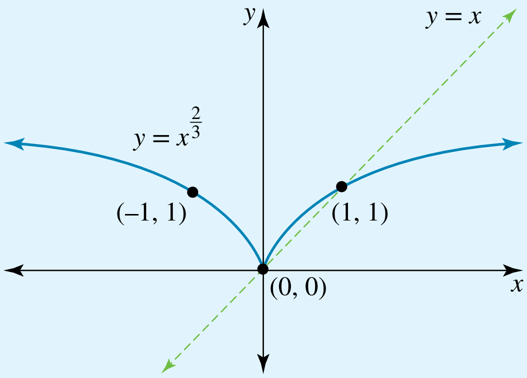
3 Reason which shape, the root or the polynomial, will dominate.

As $3 > 2$ (or as the index is less than 1), the root shape dominates the graph. This means the graph lies above $y = x$ for $0 < x < 1$ and below it for $x > 1$.

4 Draw the required graph, showing its position relative to the line $y = x$.

Note: There is a sharp point at the origin.

The points (0, 0) and (1, 1) lie on the graph, and by symmetry the graph will also pass through the point (–1, 1).



b **1** Express the function rule in surd form and deduce how the function can be formed.

b $y = x^{\frac{3}{2}}$

$= \sqrt{x^3}$

The function is formed as the square root of the cubic polynomial $y = x^3$.

2 Use the nature of the operation forming the function to determine the domain of the function.

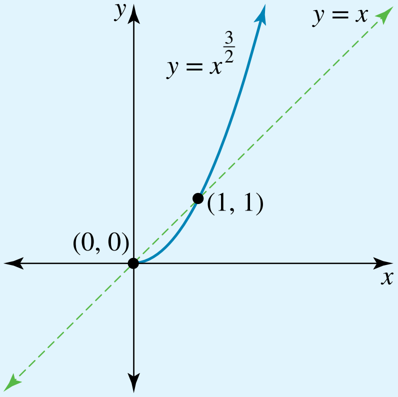
The graph of $y = x^3$ is positive in quadrant 1 and negative in quadrant 3, so the square root can only be taken of the section in quadrant 1. There will be one branch and its domain will be $R^+ \cup \{0\}$.

3 Reason which shape, the root or the polynomial, will dominate.

As $3 > 2$ (or as the index is greater than 1), the polynomial shape dominates. The graph will lie below $y = x$ for $0 < x < 1$ and above it for $x > 1$.

4 Draw the required graph, showing its position relative to the line $y = x$.

The points (0, 0) and (1, 1) lie on the graph.

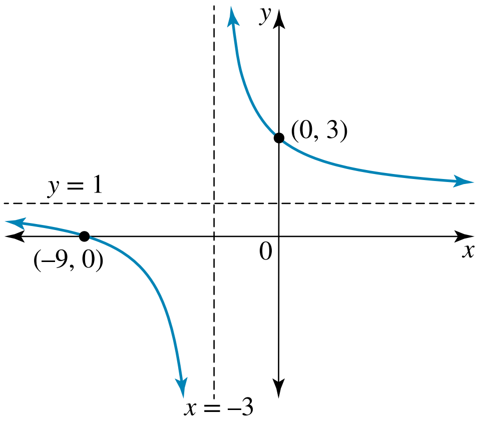


EXERCISE 2.3 Other algebraic functions

PRACTISE

Work without CAS

1 a WE7 Determine an appropriate equation for the hyperbola shown.



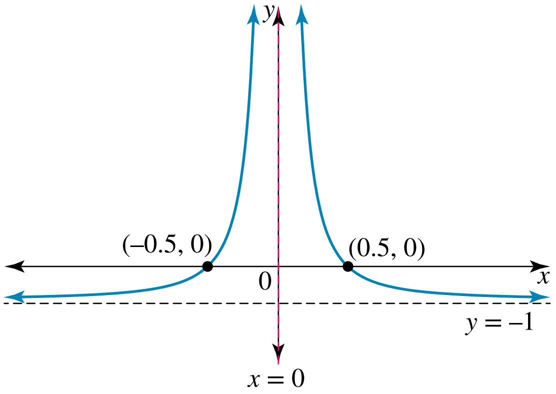
b i Find the maximal domain of $y = \dfrac{5x - 2}{x - 1}$.

ii Sketch the graph of $y = \dfrac{5x - 2}{x - 1}$ and state the range.

2 Sketch the graph of $y = \dfrac{4}{1 - 2x}$, stating its domain and range.

3 WE8 Sketch the graph of $y = \dfrac{8}{(x + 2)^2} - 2$ and state its domain and range.

4 Determine an appropriate equation for the truncus shown.



5 a i WE9 State the maximal domain of $y = -\sqrt{x + 9} + 2$

ii Sketch the graph of $y = -\sqrt{x + 9} + 2$ and state its range.

b The graph of a cube root function has its point of inflection at (1, 3) and the graph cuts the y-axis at (0, 1). Determine its rule and sketch its graph, locating its x-intercept.

6 a Determine the maximal domain and the range of $y = 3\sqrt{4x - 9} - 6$, and sketch its graph.

b State the coordinates of the point of inflection of the graph of $y = (10 - 3x)^{\frac{1}{3}}$ and sketch the graph.

7 WE10 Give the domain and deduce the shape of the graph of:

a $y = x^{\frac{3}{4}}$ **b** $y = x^{\frac{4}{3}}$

8 Give the domain and deduce the shape of the graph of:

a $y = x^{\frac{1}{5}}$ **b** $y = x^{\frac{1}{8}}$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 Determine the maximal domains of each of the following functions.

a $y = \dfrac{x - 6}{x + 9}$ **b** $y = \sqrt{1 - 2x}$

c $\dfrac{-2}{(x + 3)^2}$ **d** $\dfrac{1}{x^2 + 3}$

10 Sketch the following hyperbolas and state the domain and range of each.

a $y = \dfrac{4}{x} + 5$

b $y = 2 - \dfrac{3}{x + 1}$

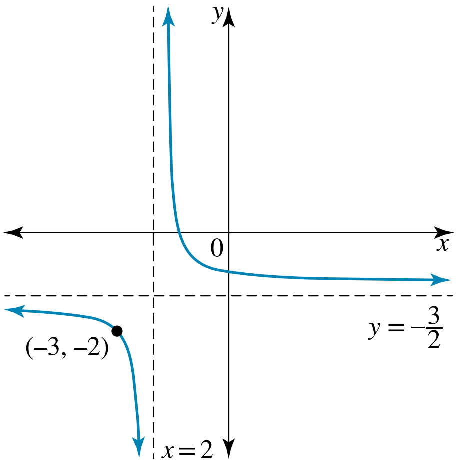
c $y = \dfrac{4x + 3}{2x + 1}$

d $xy + 2y + 5 = 0$

e $y = \dfrac{10}{5 - x} - 5$

11 a The graph of a hyperbola has a vertical asymptote at $x = -3$ and a horizontal asymptote at $y = 6$. The point $(-4, 8)$ lies on the graph. Form the equation of this graph.

b Form a possible equation for the graph shown.



12 Sketch each of the following and state the domain and range of each.

a $y = \dfrac{2}{(3 - x)^2} + 1$

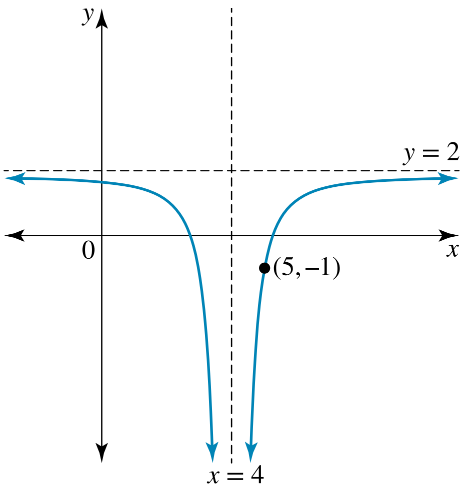
b $y = \dfrac{-3}{4(x - 1)^2} - 2$

c $y = \dfrac{1}{(2x + 3)^2} - 1$

d $y = \dfrac{25x^2 - 1}{5x^2}$

13 a The diagram shows the graph of a truncus. Form its equation.

b A function f defined on its maximal domain has a graph $y = f(x)$ in the shape of a truncus with range $(-4, \infty)$. Given $f(-1) = 8$ and $f(2) = 8$, determine the equation of the graph and state the function f using function notation.



14 a Give the equations of the two square root functions that form the branches of each of the following 'sideways' parabolas, and state the domain and range of each function.

i $(y - 2)^2 = 4(x - 3)$

ii $y^2 + 2y + 2x = 5$

b Sketch the following square root functions and state the domain and range of each.

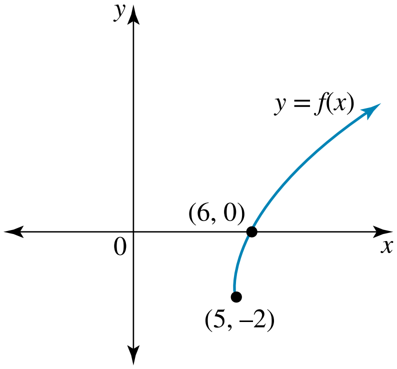
i $y = 1 - \sqrt{3x}$

ii $y = 2\sqrt{-x} + 4$

iii $y = 2\sqrt{4 + 2x} + 3$

iv $y = -\sqrt{3} - \sqrt{12 - 3x}$

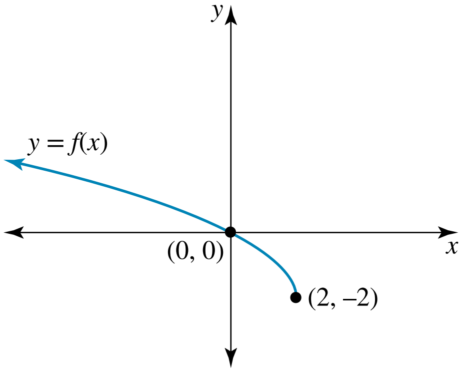
15 a The graph of the function $f:[5, \infty) \rightarrow R$, $f(x) = a\sqrt{x+b} + c$ is shown in the diagram. Determine the values of a, b and c.



b The graph of the function $f:(-\infty, 2] \rightarrow R$, $f(x) = \sqrt{ax+b} + c$ is shown in the diagram.

i Determine the values of a, b and c.

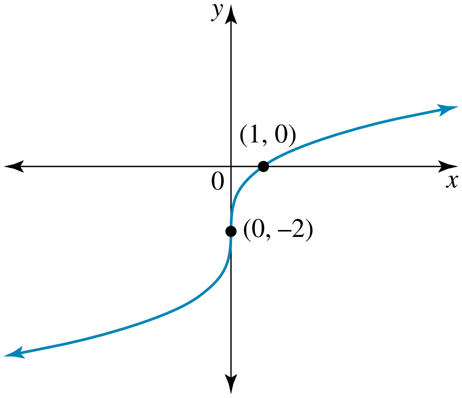
ii If the graph of $y = f(x)$ is reflected in the x-axis, what would the equation of the reflection be?



16 a Sketch the graph of $\{(x, y): y = \sqrt[3]{x+2} - 1\}$, labelling the intercepts with the coordinate axes with their exact coordinates.

b Sketch the graph of $y = f(x)$ where $f(x) = \dfrac{1 - \sqrt[3]{x+8}}{2}$, stating its implied domain and range.

c Sketch the graph of $g: [-3, 6] \rightarrow R$, $g(x) = \sqrt[3]{-x+5}$ and state its domain and range.



d Form a possible equation for the cube root function whose graph is shown.

e The graph of a cube root function passes through the points (–9, 5) and (–1, –2). At the point (–1, –2), the tangent drawn to the curve is vertical. Determine the equation of the graph.

f Express y as the subject of the equation $(y + 2)^3 = 64x - 128$ and hence state the coordinates of the point of inflection of its graph.

17 a Explain how the graph of $y = x^{\frac{1}{3}}$ could be drawn using the graph of $y = x$.

b On the same set of axes, sketch the graphs of $y = x$ and $y = x^{\frac{1}{3}}$.

c Hence, obtain $\{x : x^{\frac{1}{3}} - x > 0\}$.

18 For each of the following, identify the domain and the quadrants in which the graph lies, and sketch the graph, showing its position relative to the line $y = x$.

a $y = x^{\frac{5}{2}}$

b $y = x^{\frac{5}{3}}$

c $y = x^{\frac{3}{5}}$

d $y = x^{0.25}$

MASTER

19 Use CAS technology to draw the graphs of $y = \dfrac{1}{x^2 - 4}$, $y = \dfrac{1}{x^2 + 4}$ and $y = \dfrac{1}{(x - 4)^2}$. Hence or otherwise, determine the domain and range of each graph, and the equations of the asymptotes. Which graph is a truncus?

20 What is the maximal domain of the function $y = \sqrt{(2 - x)(x + 3)}$? Use CAS technology to investigate the shape of the graph.

2.4 Combinations of functions

studyon

Units 3 & 4
AOS 1
Topic 4
Concept 6

Hybrid functions
Concept summary
Practice questions

By combining together pieces of different functions defined over restricted domains, a ‘piecewise’ function can be created. By combining together different functions using arithmetic operations, other functions can be created. In this section we consider some of these combinations.

Hybrid functions

A **hybrid function**, or piecewise function, is a function whose rule takes a different form over different subsets of its domain. An example of a hybrid function is the one defined by the rule

$$f(x) = \begin{cases} \sqrt[3]{x}, & x \le 0 \\ 2, & 0 < x < 2. \\ x, & x \ge 2 \end{cases}$$

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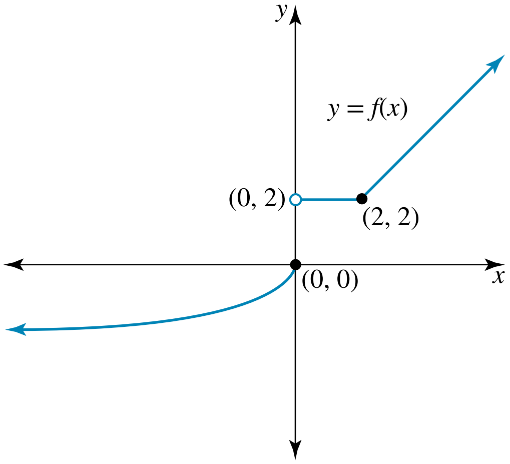
Interactivity
Hybrid functions
int-6414

To sketch its graph, the three functions that combine to form its branches, $y = \sqrt[3]{x}$, $y = 2$ and $y = x$, are drawn on their respective restricted domains on the same set of axes. If the branches do not join, then it is important to indicate which end points are open and which are closed, as each of the x-values of any function must have a unique y-value. The graph of this hybrid function $y = f(x)$ is shown in the diagram.



The function is not continuous when $x = 0$ as the branches do not join for that value of x. It is said to be discontinuous at that point of its domain. As the rule shows, $x = 0$ lies in the domain of the cube root section, the point (0, 0) is closed and the point (0, 2) is open.

The function is continuous at $x = 2$ as there is no break or gap in the curve. There is no need for a closed point to be shown at $x = 2$, because its two neighbouring branches run ‘naturally’ into each other at this point.

To calculate the value of the function for a given value of x, choose the function rule of that branch defined for the section of the domain to which the x-value belongs.

WORKED EXAMPLE 11

Consider the function for which $f(x) = \begin{cases} \sqrt{-x}, & x \leq -1 \\ 2 - x^2, & -1 < x < 1 \\ \sqrt{x} + 1, & x \geq 1 \end{cases}$.

a Evaluate $f(-1)$, $f(0)$ and $f(4)$.

b Sketch the graph of $y = f(x)$.

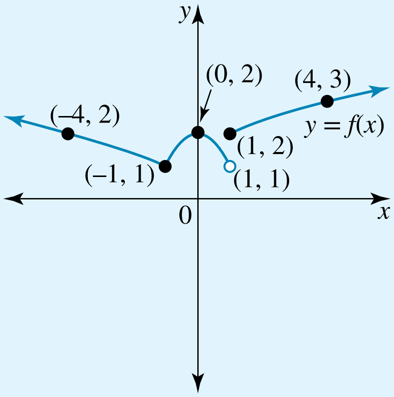
c State:

  i any value of x for which the function is not continuous

  ii the domain and range.

THINK	**WRITE/DRAW**
a For each x-value, decide which section of the domain it is in and calculate its image using the branch of the hybrid function's rule applicable to that section of the domain.	**a** $f(x) = \begin{cases} \sqrt{-x}, & x \leq -1 \\ 2 - x^2, & -1 < x < 1 \\ \sqrt{x} + 1, & x \geq 1 \end{cases}$ $f(-1)$: Since $x = -1$ lies in the domain section $x \leq -1$, use the rule $f(x) = \sqrt{-x}$. $f(-1) = \sqrt{-(-1)}$ $= \sqrt{1}$ $= 1$ $f(0)$: Since $x = 0$ lies in the domain section $-1 < x < 1$, use the rule $f(x) = 2 - x^2$. $f(0) = 2 - 0^2$ $= 2 - 0$ $= 2$ $f(4)$: Since $x = 4$ lies in the domain section $x \geq 1$, use the rule $f(x) = \sqrt{x} + 1$. $f(4) = \sqrt{4} + 1$ $= 2 + 1$ $= 3$
b 1 Obtain the information needed to sketch each of the functions forming the branches of the hybrid function.	**b** $y = \sqrt{-x}$, $x \leq -1$ is a square root function. The points (–1, 1) and (–4, 2) lie on its graph. $y = 2 - x^2$, $-1 < x < 1$ is a parabola with maximum turning point (0, 2). At $x = -1$ or $x = 1$, $y = 1$. The points (–1, 1) and (1, 1) are open for the parabola. $y = \sqrt{x} + 1$, $x \geq 1$ is a square root function. The points (1, 2) and (4, 3) lie on its graph.

2 Sketch each branch on the same set of axes to form the graph of the hybrid function.



c **i** State any value of x where the branches of the graph do not join

c The function is not continuous at $x = 1$.

ii State the domain and range.

The domain is R.
The range is $[1, \infty)$.

studyon
Units 3 & 4
AOS 1
Topic 4
Concept 1

Sum of two functions
Concept summary
Practice questions

Sums, differences and products of functions

New functions are formed when two given functions are combined together under the operations of addition, subtraction and multiplication. The given functions can only be combined where they both exist, so the domain of the new function formed must be the domain common to both the given functions. For functions f and g with domains d_f and d_g respectively, the common domain is $d_f \cap d_g$.

- **The sum and difference functions $f \pm g$ are defined by $(f \pm g)(x) = f(x) \pm g(x)$ with domain $d_f \cap d_g$.**
- **The product function fg is defined by $(fg)(x) = f(x)g(x)$with domain $d_f \cap d_g$.**

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Interactivity
Sums, differences and products of functions
int-6416

Graphs of the functions $f \pm g$ and fg may be able to be recognised from their rules. If not, the graphs may be deduced by sketching the graphs of f and g and combining by addition, subtraction or multiplication, as appropriate, the values of $f(x)$and $g(x)$ for selected x-values in their common domain. The difference function $f - g$ can be considered to be the sum function $f + (-g)$.

WORKED EXAMPLE 12

Consider the functions f and g defined by $f(x) = \sqrt{4 + x}$ and $g(x) = \sqrt{4 - x}$ respectively.

a **Form the rule for the sum function $f + g$, stating its domain, and sketch the graph of $y = (f + g)(x)$.**

b **Form the rule for the product function fg and state its domain and range.**

THINK | **WRITE/DRAW**

a **1** Write the domains of the functions f and g.

a $f(x) = \sqrt{4 + x}$

Domain: $4 + x \geq 0$

$x \geq -4$

$d_f = [-4, \infty)$

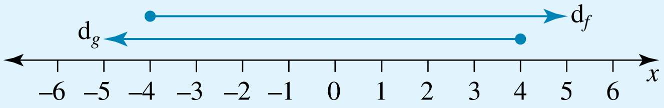
$g(x) = \sqrt{4 - x}$

Domain: $4 + x \geq 0$

$x \leq 4$

$d_g = (-\infty, 4]$

2 State the common domain.

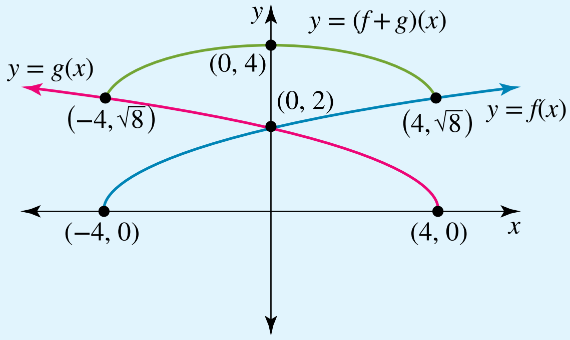


$d_f \cap d_g = [-4, 4]$

3 Form the sum function and state its domain.

$$(f + g)(x) = f(x) + g(x)$$
$$= \sqrt{4 + x} + \sqrt{4 - x}$$
$$d_{f+g} = [-4, 4]$$

4 Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same set of axes. Add the y-coordinates of key points together to form the graph of $y = (f + g)(x)$.



x	–4	0	4
$f(x)$	0	2	$\sqrt{8}$
$g(x)$	$\sqrt{8}$	2	0
$f(x) + g(x)$	$\sqrt{8}$	4	$\sqrt{8}$

b **1** Form the product function and state its domain.

b $(fg)(x) = f(x)g(x)$

$$= (\sqrt{4 + x}) \times (\sqrt{4 - x})$$
$$= \sqrt{(4 + x)(4 - x)}$$
$$= \sqrt{16 - x^2}$$
$$d_{fg} = [-4, 4]$$

2 State the range of the function.

The rule $(fg)(x) = \sqrt{16 - x^2}$ is that of the top half of a semicircle with centre (0, 0) and radius 4. Therefore, the range is [0, 4].

Graphical techniques

Given the graphs of functions whose rules are not necessarily known, it may be possible to deduce the shape of the graph of the function that is the sum or other combination of the functions whose graphs are given.

Addition of ordinates

Given the graphs of $y_1 = f(x)$ and $y_2 = g(x)$, the graphing technique known as addition of ordinates adds together the y-values, or ordinates, of the two given graphs over the common domain to form the graph of the sum function $y = y_1 + y_2 = f(x) + g(x)$.

Note the following points when applying this technique over the common domain $d_f \cap d_g$:

- **If the graphs of f and g intersect at (a, b), then the point $(a, 2b)$ lies on the graph of $f + g$.**
- **Where $f(x) = -g(x)$, the graph of $f + g$ cuts the x-axis.**
- **If one of $f(x)$ or $g(x)$ is positive and the other is negative, the graph of $f + g$ lies between the graphs of f and g.**
- **If one of $f(x)$ or $g(x)$ is zero, then the graph of $f + g$ cuts the other graph.**
- **If $f(x) \to 0^+$, then the graph of $f + g$ approaches the graph of g from above.**
- **If $f(x) \to 0^-$, then the graph of $f + g$ approaches the graph of g from below.**
- **Any vertical asymptote of f or g will be a vertical asymptote on the graph of $f + g$.**

The subtraction of ordinates is usually simpler to achieve as the addition of the ordinates of $y_1 = f(x)$ and $y_2 = -g(x)$.

Squaring ordinates

Given the graph of $y = f(x)$, the graph of $y = (f(x))^2$ can be deduced by squaring the y-values, or ordinates, noting in particular that $0^2 = 0$, $1^2 = 1$ and $(-1)^2 = 1$.

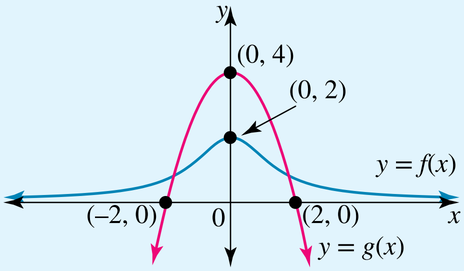
- The graph of f and its square will intersect at any point on f where $y = 0$ or $y = 1$.
- If the point $(a, -1)$ is on the graph of f, then $(a, 1)$ lies on the graph of the squared function.
- The squared function's graph can never lie below the x-axis.
- Where $0 < f(x) < 1$, $(f(x))^2 < f(x)$, and where $f(x) > 1$ or $f(x) < -1$, $(f(x))^2 > f(x)$.

Similar reasoning about the ordinates and their square roots and the domain will allow the graph of $y = \sqrt{f(x)}$ to be deduced.

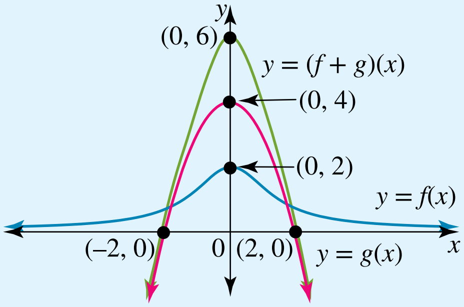
These graphing techniques can be applied to combinations of known functions where the first step would be to draw their graphs.

WORKED EXAMPLE 13

The graphs of the functions f and g are shown.



Draw the graph of $y = (f + g)(x)$.

THINK	WRITE/DRAW
1 State the domain common to both functions.	Both of the functions have a domain of R, so $d_f \cap d_g = R$.
2 Determine the coordinates of a key point on the required graph.	At $x = 0$, $f(x) = 2$ and $g(x) = 4$. Hence the point $(0, 6)$ lies on the graph of $f + g$.
3 Deduce the behaviour of the required graph where one of the given graphs cuts the x-axis.	At $x = \pm 2$, $g(x) = 0$. Hence, the graph of $f + g$ will cut the graph of f when $x = \pm 2$.
4 Use the long-term behaviour of one of the given graphs to deduce the long-term behaviour of the required graph.	As $x \to \pm\infty$, $f(x) \to 0^+$. Hence $(f + g)(x) \to g(x)$ from above as $x \to \pm\infty$.
5 Draw a sketch of the required graph.	

EXERCISE 2.4 Combinations of functions

PRACTISE

Work without CAS

1 **WE11** Consider the function for which

$$f(x) = \begin{cases} -\sqrt[3]{x}, & x < -1 \\ x^3, & -1 \leq x \leq 1 \\ 2 - x, & x > 1 \end{cases}$$

a Evaluate $f(-8)$, $f(-1)$ and $f(2)$.

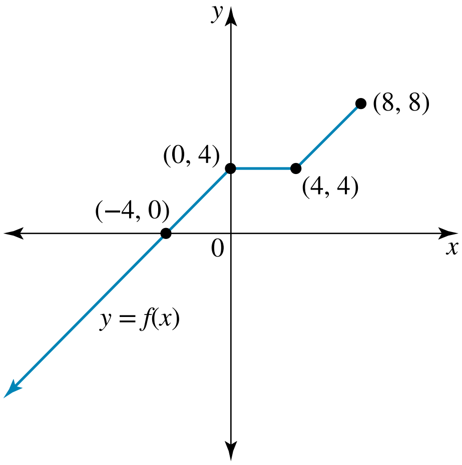
b Sketch the graph of $y = f(x)$.

c State:

  i any value of x for which the function is not continuous

  ii the domain and range.

2 Form the rule for the hybrid function shown in the diagram.



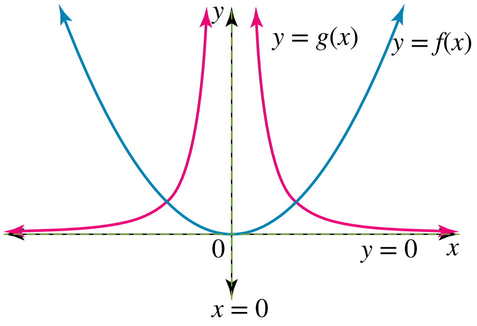
3 **WE12** Consider the functions f and g defined by $f(x) = -\sqrt{1+x}$ and $g(x) = -\sqrt{1-x}$ respectively.

a Form the rule for the sum function $f + g$, stating its domain, and sketch the graph of $y = (f + g)(x)$.

b Form the rule for the product function fg, stating its domain and range.

4 Given $f(x) = x^3$ and $g(x) = x^2$, form the rule $(f - g)(x)$ for the difference function and sketch the graphs of $y = f(x)$, $y = -g(x)$ and $y = (f - g)(x)$ on the same set of axes. Comment on the relationship of the graphs at the places where $y = (f - g)(x)$ cuts the axes.

5 **WE13** The graphs of the functions f and g are shown. Draw the graph of $y = (f + g)(x)$.



6 Sketch the graph of $y = x^2 - 1$ and hence draw the graph of $y = (x^2 - 1)^2$, stating the domain and range.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Sketch the graphs of each of the following hybrid functions and state their domains, ranges and any points of discontinuity.

a $y = \begin{cases} -2x, & x \le 0 \\ 4 - x^2, & x > 0 \end{cases}$

b $y = \begin{cases} \sqrt[3]{x}, & x < 1 \\ \dfrac{1}{x}, & x \ge 1 \end{cases}$

8 A hybrid function is defined by

$$f(x) = \begin{cases} \dfrac{1}{(x+1)^2}, & x < -1 \\ x^2 - x, & -1 \le x \le 2. \\ 8 - 2x, & x > 2 \end{cases}$$

a Evaluate:

i $f(-2)$

ii $f(2)$.

b Sketch the graph of $y = f(x)$.

c State the domain over which the hybrid function is continuous.

9 Consider the function

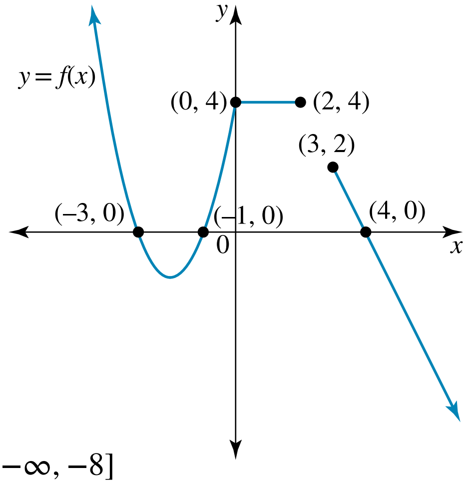
$$f: R \to R,\ f(x) = \begin{cases} \dfrac{1}{9}x^3 + 5, & x < -3 \\ \sqrt{1-x}, & -3 \le x \le 1. \\ x - 2, & x > 1 \end{cases}$$

a Show the function is not continuous at $x = 1$.

b Sketch the graph of $y = f(x)$ and state the type of correspondence it displays.

c Determine the value(s) of x for which $f(x) = 4$.

10 Form the rule for the function whose graph is shown in the diagram.



11 A hybrid function is defined by

$$f(x) = \begin{cases} x + a, & x \in (-\infty, -8] \\ \sqrt[3]{x} + 2, & x \in (-8, 8] \\ \dfrac{b}{x}, & x \in (8, \infty) \end{cases}.$$

a Determine the values of a and b so that the function is continuous for $x \in R$, and for these values, sketch the graph of $y = f(x)$.

Use the values of a and b from part **a** for the remainder of this question.

b Determine the values of k for which the equation $f(x) = k$ has:

i no solutions **ii** one solution **iii** two solutions.

c Find $\{x : f(x) = 1\}$.

12 Consider the functions f and g defined by $f(x) = 5 - 2x$ and $g(x) = 2x - 2$ respectively. For each of the following, give the rule, state the domain and the range, and sketch the graph.

a $y = (f + g)(x)$ **b** $y = (f - g)(x)$ **c** $y = (fg)(x)$

13 Consider the functions $f(x) = x^2 - 1$ and $g(x) = \sqrt{x + 1}$.

a Evaluate:

i $(g - f)(3)$ **ii** $(gf)(8)$

b State the domain of the function $f + g$.

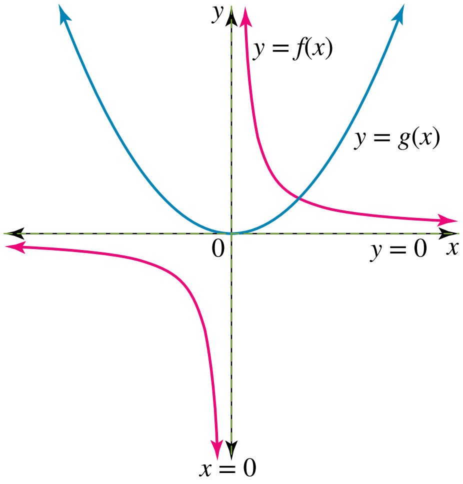
c Draw a possible graph for each of the following functions.

i $f + g$ **ii** $g - f$ **iii** fg

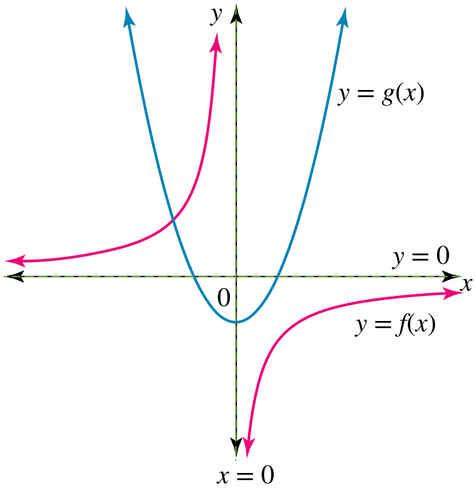
14 Use addition of ordinates to sketch $y = x + \sqrt{-x}$.

15 The graphs of two functions $y = f(x)$ and $y = g(x)$ are drawn in the following diagrams. Use the addition of ordinates technique to sketch $y = f(x) + g(x)$ for each diagram.

a



b



16 a Consider the function defined by $g(x) = (2x - 1)^3$. Sketch the graph of $y = g(x)$ and hence sketch $y = (g(x))^2$.

b Calculate the coordinates of the points of intersection of the graphs of $y = f(x)$ and $y = (f(x))^2$ if $f(x) = x^3 - 2x$.

MASTER

17 Use CAS technology to draw on screen the hybrid function defined by the rule

$$f(x) = \begin{cases} -x, & x < -1 \\ 1, & -1 \le x \le 1. \\ (2x-1)(x-3), & x > 1 \end{cases}$$

State the range of the function.

18 Draw the graph of $y = \sqrt[3]{f(x)}$ for the function $f(x) = x^2(x + 4)(x - 2)$ using CAS technology and describe the shape of the graph. How many points of intersection are there for the two graphs $y = \sqrt[3]{f(x)}$ and $f(x) = x^2(x + 4)(x - 2)$?

2.5 Non-algebraic functions

studyon

Units 3 & 4
AOS 1
Topic 3
Concept 1

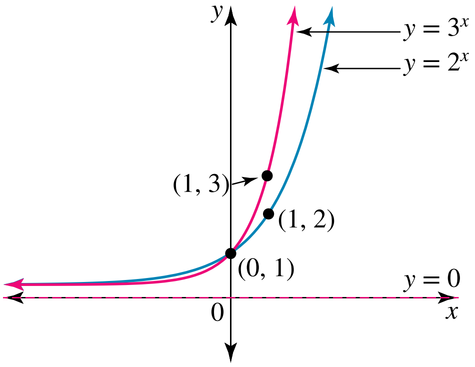
Exponential functions
Concept summary
Practice questions

Although polynomial functions of high degrees can be used to approximate exponential and trigonometric functions, these functions have no exact algebraic form. Such functions are called transcendental functions.

Exponential functions

Exponential functions are those of the form $f: R \to R,\ f(x) = a^x$ where the base $a \in R^+ \backslash \{1\}$.

The index law $a^0 = 1$ explains why the graph of $y = a^x$ must contain the point (0, 1). The graph of $y = 2^x$ would also contain the point (1, 2), while the graph of $y = 3^x$ would contain the point (1, 3).



As the base becomes larger, exponential functions increase more quickly. This can be seen in the diagram comparing the graphs of $y = 2^x$ and $y = 3^x$.

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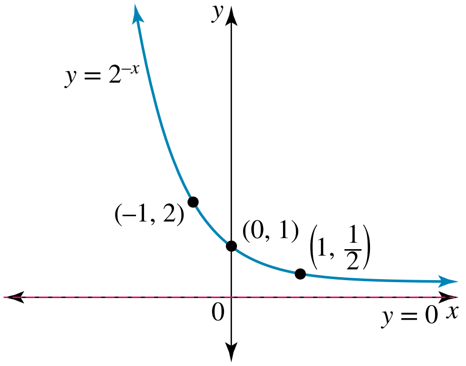
Interactivity
Exponential functions
int-5959

The index law $a^{-x} = \dfrac{1}{a^x}$ explains why for negative values of x the graphs of $y = 2^x$ and $y = 3^x$ approach the x-axis but always lie above the x-axis. The x-axis is a horizontal asymptote for both of their graphs and for any graph of the form $y = a^x$.

Exponential functions of the form $f: R \to R,\ f(x) = a^{-x}$ where $a \in R^+ \backslash \{1\}$ have base $\dfrac{1}{a}$. This is again explained by index laws, as $a^{-x} = \dfrac{1}{a^x} = \left(\dfrac{1}{a}\right)^x$. However, it is often preferable to write $y = 2^{-x}$ rather than $y = \left(\dfrac{1}{2}\right)^x$. The graph of $y = 2^{-x}$ or $\left(\dfrac{1}{2}\right)^x$ must contain the point (0, 1), and other points on this graph include (–1, 2) and $\left(1, \dfrac{1}{2}\right)$.



The graph of $y = 2^x$ illustrates a 'growth' form, whereas the graph of $y = 2^{-x}$ takes a 'decay' form. The two graphs are reflections of each other in the y-axis.

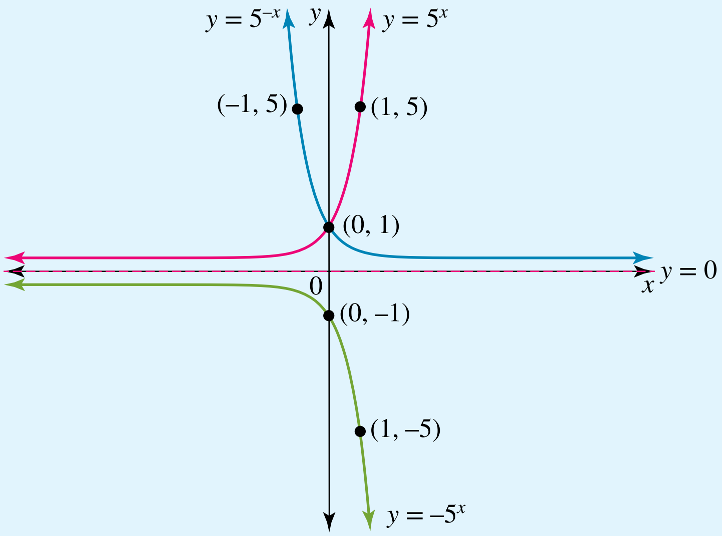
WORKED EXAMPLE 14

Consider the function $f(x) = -5^x$.

a **Evaluate $f(2)$.**

b **On the same set of axes sketch the graphs of $y = 5^x$, $y = -5^x$ and $y = 5^{-x}$.**

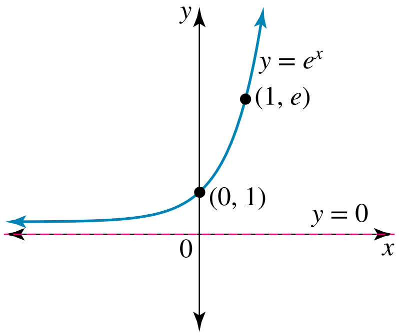
c **Express $y = 5^{-x}$ in an equivalent form.**

THINK	**WRITE/DRAW**
a Calculate the required value. *Note:* $-5^2 \neq (-5)^2$	**a** $f(x) = -5^x$ $f(2) = -5^2$ $= -25$
b **1** Identify points on each curve.	**b** $y = 5^x$ contains the points (0, 1) and (1, 5). $y = -5^x$ contains the points (0, –1) and (1, –5). $y = 5^{-x}$ contains the points (0, 1) and (–1, 5).
2 Sketch the graphs on the same axes.	
c Write an equivalent form for the given rule.	**c** Since $5^{-x} = \left(\frac{1}{5}\right)^x$, an alternative form for the rule is $y = \left(\frac{1}{5}\right)^x$ or $y = 0.2^x$.

The exponential function $y = e^x$

The number e is known as Euler's number after the eminent Swiss mathematician Leonhard Euler, who first used the symbol. Its value is 2.718 281 828 45… Like π, e is an irrational number of great importance in mathematics. It appears in later topic on calculus. Most calculators have keys for both π and e.

As $2 < e < 3$, the graph of $y = e^x$ lies between those of $y = 2^x$ and $y = 3^x$, and has much the same shape.



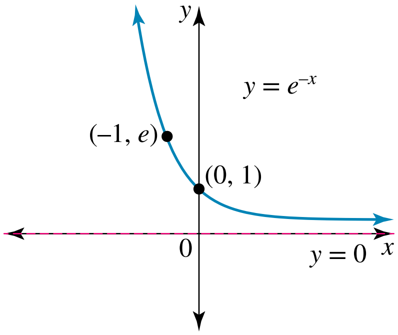
The graph of $y = e^x$ has the following key features.

- **The points (0, 1) and (1, e) lie on the graph.**
- **There is a horizontal asymptote at $y = 0$.**
- **The domain is R.**
- **The range is R^+.**
- **The function has one-to-one correspondence.**
- **As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to 0^+$.**

The graph shows an 'exponential growth' shape. Mathematical models of such phenomena, for example population growth, usually involve the exponential function $y = e^x$.

Exponential decay models usually involve the function $y = e^{-x}$.

The graph of $y = e^{-x}$ is shown.



The graph of $y = e^{-x}$ has the following key features.

- **The points (0, 1) and (–1, e) lie on the graph.**
- **There is a horizontal asymptote at $y = 0$.**
- **The domain is R.**
- **The range is R^+.**
- **The function has one-to-one correspondence.**
- **As $x \to -\infty$, $y \to \infty$, and as $x \to \infty$, $y \to 0^+$.**
- **The graph is a reflection of $y = e^x$ in the y-axis.**

Sketching the graph of $y = ae^{nx} + k$

A vertical translation affects the position of the horizontal asymptote of an exponential graph in the same way it does for a hyperbola or truncus. The graph of $y = e^x + k$ has a horizontal asymptote with equation $y = k$. If $k < 0$, then $y = e^x + k$ will cut through the x-axis and its x-intercept will need to be calculated.

To sketch the graph of an exponential function:

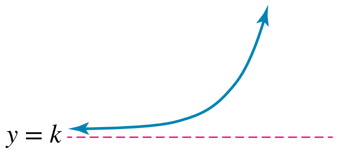
- identify the equation of its asymptote
- calculate its y-intercept
- calculate its x-intercept if there is one.

If the function has no x-intercept, it may be necessary to obtain the coordinates of another point on its graph.

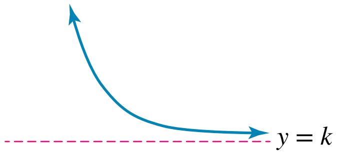
The graph of $y = ae^{nx} + k$ has the following key features.

- **There is a horizontal asymptote at $y = k$.**
- **There is one y-intercept, obtained by letting $x = 0$.**
- **There is either one or no x-intercept. The relative position of the asymptote and y-intercept will determine whether there is an x-intercept.**
- **If $a > 0$, the range is (k, ∞).**
- **If $a < 0$, the range is $(-\infty, k)$.**

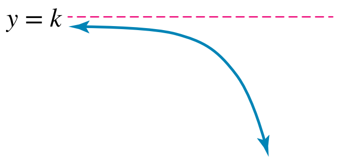
The values of a and n in the equation $y = ae^{nx} + k$ are related to dilation factors, and their signs will affect the orientation of the graph. The possibilities are shown in the following diagrams and table.



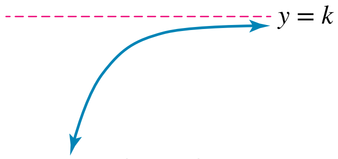
$a > 0, n > 0$



$a > 0, n < 0$



$a < 0, n > 0$



$a < 0, n < 0$

Dilation factors	Graph behaviour
$a > 0, n > 0$	As $x \to -\infty, y \to k^+$
$a > 0, n < 0$	As $x \to \infty, y \to k^+$
$a < 0, n > 0$	As $x \to -\infty, y \to k^-$
$a < 0, n < 0$	As $x \to \infty, y \to k^-$

Calculating the *x*-intercept

To calculate the x-intercept of $y = e^x - 2$, the exponential equation $e^x - 2 = 0$ needs to be solved. This can be solved using technology. Alternatively, we can switch $e^x = 2$ to its logarithm form to obtain the exact value of the x-intercept as $x = \log_e(2)$. In decimal form, $\log_e(2) \approx 0.69$, with the value obtained by using the ln key on a calculator. The symbol 'ln' stands for the natural logarithm, $\log_e$. Base e logarithms as functions are studied in Topic 4.

Sketching the graph of $y = ae^{n(x-h)} + k$

Under a horizontal translation of h units, the point (0, 1) on $y = e^x$ is translated to the point (h, 1) on the graph of $y = e^{x-h}$. The y-intercept is no longer (0, 1), so it will need to be calculated.

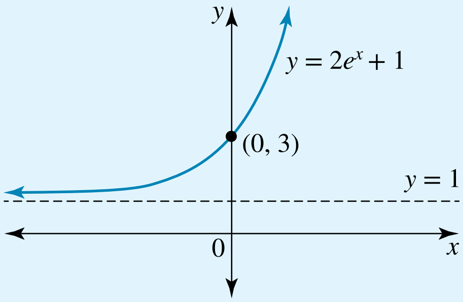
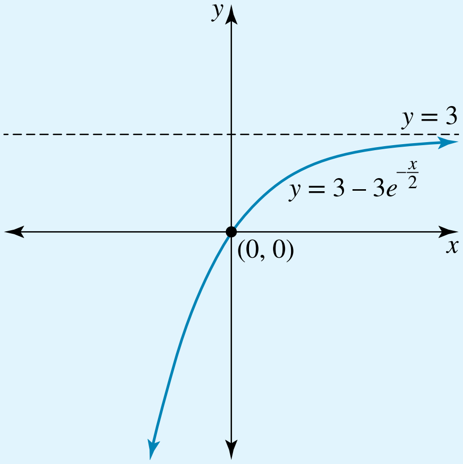
By letting $x = h$ for the horizontal translation, the index for the exponential will be zero. This simplifies the calculation to obtain another point on the graph. For the graph of $y = ae^{n(x-h)} + k$ when $x = h$, $y = ae^0 + k \Rightarrow y = a + k$.

WORKED EXAMPLE 15

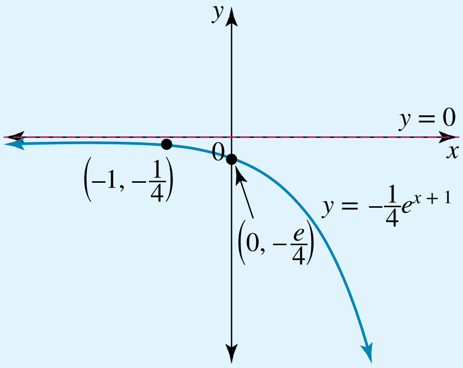
Sketch the following graphs and state the domain and range of each graph.

a $y = 2e^x + 1$ **b** $y = 3–3e^{-\frac{x}{2}}$ **c** $y = -\frac{1}{4}e^{x+1}$

THINK	WRITE/DRAW
a 1 State the equation of the asymptote.	$y = 2e^x + 1$ The asymptote is $y = 1$.
2 Calculate the y-intercept.	y-intercept: Let $x = 0$. $y = 2e^0 + 1$ $y = 2 + 1$ $y = 3$ The y-intercept is (0, 3).
3 Calculate any x-intercepts.	As the y-intercept is above the positive asymptote, there is no x-intercept.

	4 Locate another point if necessary and sketch the graph.	Growth shape 
	5 State the domain and range.	The domain is R and the range is $(1, \infty)$.
b	**1** State the equation of the asymptote.	$y = 3 - 3e^{-\frac{x}{2}}$ The asymptote is $y = 3$.
	2 Calculate the y-intercept.	y-intercept: Let $x = 0$. $y = 3 - 3e^0$ $y = 0$ The y-intercept is $(0, 0)$.
	3 Calculate any x-intercepts.	$(0, 0)$ is also the x-intercept.
	4 Locate another point if necessary and sketch the graph.	If $x = -2$, then $y = 3 - 3e < 0$. 
	5 State the domain and range.	The domain is R and the range is $(-\infty, 3)$.
c	**1** State the equation of the asymptote.	$y = -\frac{1}{4}e^{x+1}$ The asymptote is $y = 0$.
	2 Calculate the y-intercept.	y-intercept: Let $x = 0$. $y = -\frac{1}{4}e^{0+1}$ $= -\frac{1}{4}e$ The y-intercept is $\left(0, -\frac{e}{4}\right)$.
	3 Calculate any x-intercepts.	There are no x-intercepts as the x-axis is an asymptote.

4 Locate another point if necessary and sketch the graph.	Let $x = -1$. $y = -\frac{1}{4}e^0$ $= -\frac{1}{4}e$ Another point on the graph is $\left(-1, -\frac{1}{4}\right)$.



5 State the domain and range.	The domain is R and the range is R^-.

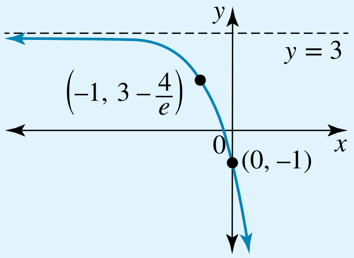
Determining the equation of an exponential function

The form of the equation is usually specified along with the given information needed to determine the equation. This is necessary because it could be difficult to decide whether the base is e or some other value. The number of pieces of information given will also need to match the number of parameters or unknown constants in the equation.

The asymptote is a key piece of information to obtain. If a graph is given, the equation of the asymptote will be apparent. Insert this value into the equation and then substitute coordinates of known points on the graph. Simultaneous equations may be required to calculate all the parameters in the equation.

WORKED EXAMPLE 16

a The diagram shows the graph of $y = ae^x + b$. Determine the values of a and b.



b The graph of $y = a \times 10^{kx}$ contains the points (2, 30) and (4, 300). Form its equation.

THINK	WRITE
a 1 Insert the equation of the asymptote into the equation of the graph.	**a** $y = ae^x + b$ The asymptote is $y = 3$. This means $b = 3$. The equation becomes $y = ae^x + 3$.

2 Use a known point on the graph to fully determine the equation.

The graph passes through the point $\left(-1, 3 - \frac{4}{e}\right)$.

Substitute this point into the equation.

$$3 - \frac{4}{e} = ae^{-1} + 3$$

$$-\frac{4}{e} = \frac{a}{e}$$

$$a = -4$$

The equation is $y = -4e^x + 3$.

3 State the values required.

$a = -4, b = 3$

b **1** Substitute the given points in the equation.

b $y = a \times 10^{kx}$

$$(2, 30) \Rightarrow 30 = a \times 10^{2k}$$

$$(4, 300) \Rightarrow 300 = a \times 10^{4k}$$

2 Solve the simultaneous equations.

$$a \times 10^{2k} = 30 \quad [1]$$

$$a \times 10^{4k} = 300 \quad [2]$$

Divide equation [2] by equation [1]:

$$\frac{a \times 10^{4k}}{a \times 10^{2k}} = \frac{300}{30}$$

$$10^{2k} = 10$$

$$2k = 1$$

$$k = \frac{1}{2}$$

Substitute $k = \frac{1}{2}$ in equation [1]:

$$a \times 10^1 = 30$$

$$a = 3$$

3 State the equation.

The equation is $y = 3 \times 10^{\frac{x}{2}}$.

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AOS 1

Topic 3

Concept 3

The sine function
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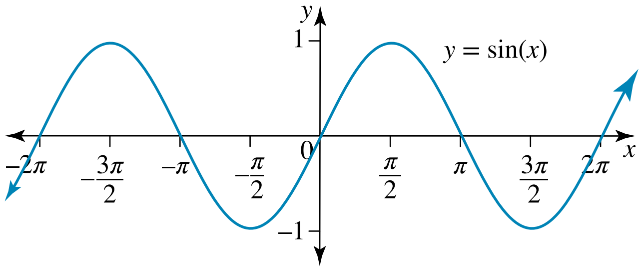
Circular functions

Circular functions, or trigonometric functions, are periodic functions such as $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$.

The graph of the sine function has a wave shape that repeats itself every 2π units. Its **period** is 2π as shown in its graph.



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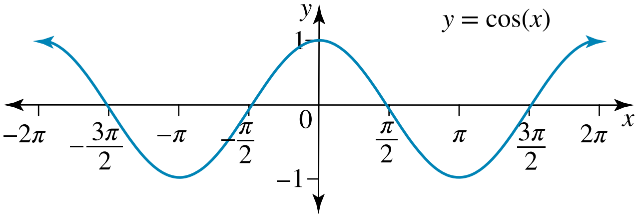
The graph oscillates about the line $y = 0$ (the x-axis), rising and falling by up to 1 unit. This gives the graph its range of $[-1, 1]$ with a mean, or equilibrium, position $y = 0$ and an **amplitude** of 1.

The graph of the cosine function has the same wave shape with period 2π.



The characteristics of both the sine function and the cosine function are:

- **period 2π**
- **amplitude 1**
- **mean position $y = 0$**
- **domain R**
- **range $[-1, 1]$**
- **many-to-one correspondence.**

Although the domain of both the sine and cosine functions is R, they are usually sketched on a given restricted domain.

The two graphs of $y = \sin(x)$ and $y = \cos(x)$ are 'out of phase' by $\frac{\pi}{2}$; that is, $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$. In other words, a horizontal shift of the cosine graph by $\frac{\pi}{2}$ units to the right gives the sine graph. Likewise, a horizontal shift of the sine graph by $\frac{\pi}{2}$ units to the left gives the cosine graph; $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$.

The periodicity of the functions is expressed by $f(x) = f(x + n2\pi)$, $n \in Z$ where f is sin or cos.

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Graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$

The value of a affects the amplitude of the sine and cosine functions.

Because $-1 \le \sin(nx) \le 1$, $-a \le a\sin(nx) \le a$. This means the graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$ have amplitude $|a|$.

As the amplitude measures a distance, the rise or fall from the mean position, it is always positive. If $a < 0$, the graphs will be inverted, or reflected in the x-axis.

The value of n affects the period of the sine and cosine functions.

Since one cycle of $y = \sin(x)$ is completed for $0 \le x \le 2\pi$, one cycle of $y = \sin(nx)$ is completed for $0 \le nx \le 2\pi$. This means one cycle is covered over the interval $0 \le x \le \frac{2\pi}{n}$, assuming $n > 0$.

The graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$ have:
- **period $\frac{2\pi}{n}$**
- **amplitude $|a|$**
- **range $[-a, a]$.**

The graphs of $y = a\sin(nx) + k$ and $y = a\cos(nx) + k$

Any vertical translation affects the equilibrium or mean position about which the sine and cosine graphs oscillate.

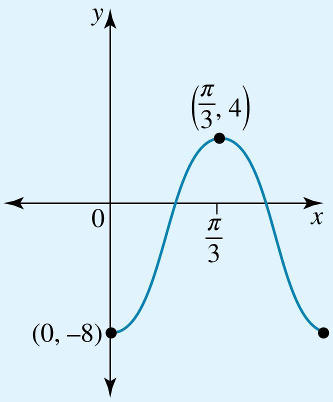
The graphs of $y = a\sin(nx) + k$ and $y = a\cos(nx) + k$ have:
- **mean position $y = k$**
- **range $[k - a, k + a]$.**

Where the graph of $y = f(x)$ crosses the x-axis, the intercepts are found by solving the trigonometric equation $f(x) = 0$.

WORKED EXAMPLE 17

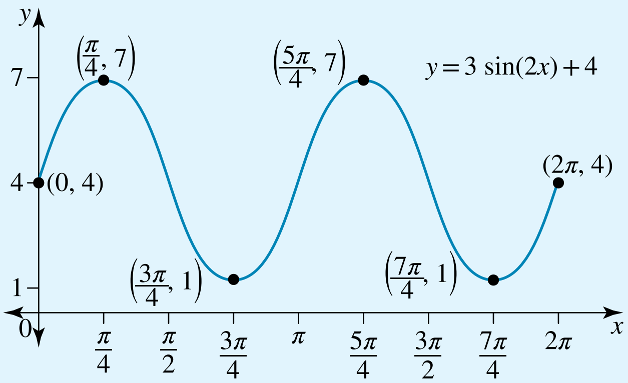
a **Sketch the graph of $y = 3\sin(2x) + 4$, $0 \le x \le 2\pi$**

b **The diagram shows the graph of a cosine function. State its mean position, amplitude and period, and give a possible equation for the function.**



THINK	WRITE/DRAW
a **1** State the period and amplitude of the graph.	**a** $y = 3\sin(2x) + 4$, $0 \le x \le 2\pi$ The period is $\frac{2\pi}{2} = \pi$. The amplitude is 3.
2 State the mean position and the range.	The mean position is $y = 4$. The range of the graph is $[4 - 3, 4 + 3] = [1, 7]$.

3 Construct appropriate scales on the axes and sketch the graph.



b 1 Deduce the mean position.

b The minimum value is –8 and the maximum value is 4, so the mean position is

$$y = \frac{-8 + 4}{2} = -2.$$

2 State the amplitude.

The amplitude is the distance from the mean position to either its maximum or minimum. The amplitude is 6.

3 State the period.

At $x = \frac{\pi}{3}$, the graph is halfway through its cycle, so its period is $\frac{2\pi}{3}$.

4 Determine a possible equation for the given graph.

Let the equation be $y = a\cos(nx) + k$.

The graph is an inverted cosine shape, so $a = -6$.

The period is $\frac{2\pi}{n}$.

$$\frac{2\pi}{n} = \frac{2\pi}{3}$$

$$n = 3$$

The mean position is $y = -2$, so $k = -2$.

The equation is $y = -6\cos(3x) - 2$.

Horizontal translations of the sine and cosine graphs

Horizontal translations do not affect the period, amplitude or mean position of the graphs of sine or cosine functions. The presence of a horizontal translation of h units is recognised from the equation in the form given as $y = a\sin(n(x - h)) + k$ in exactly the same way it is for any other type of function. The graph will have the same shape as $y = a\sin(nx) + k$, but it will be translated to the right or to the left, depending on whether h is positive or negative respectively.

Translations affect the position of the maximum and minimum points and any x- and y-intercepts. However, successive maximum points would remain one period apart, as would successive minimum points.

The equation in the form $y = a\sin(bx - c) + k$ must be rearranged into the form $y = a\sin\left(b\left(x - \frac{c}{b}\right)\right) + k$ to identify the horizontal translation $h = \frac{c}{b}$.

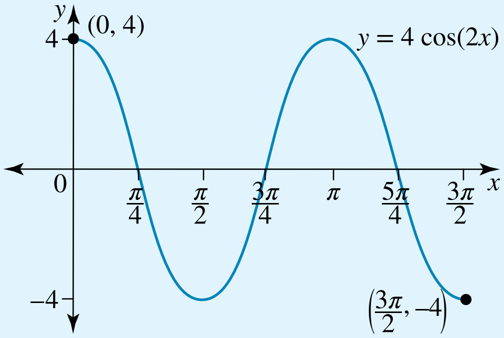
Combinations of the sine and cosine functions

Trigonometric functions such as $y = \sin(x) + \cos(x)$ can be sketched using addition of ordinates. In this example, both of the functions being combined under addition have the same period. If the functions have different periods, then to observe the periodic nature of the sum function the graphs should be sketched over a domain that allows both parts to complete at least one full cycle. For example, the function $y = \sin(2x) + \cos(x)$ would be drawn over $[0, 2\pi]$, with two cycles of the sine function and one cycle of the cosine function being added together.

WORKED EXAMPLE 18

a Sketch the graph of the function $f: \left[0, \frac{3\pi}{2}\right] \to R, f(x) = 4\cos\left(2x + \frac{\pi}{3}\right)$.

b Sketch the graph of $y = \cos(x) + \frac{1}{2}\sin(2x)$ for $x \in [0, 2\pi]$.

THINK	WRITE/DRAW
a 1 State the period, amplitude, mean position and horizontal translation.	**a** $f: \left[0, \frac{3\pi}{2}\right] \to R, f(x) = 4\cos\left(2x + \frac{\pi}{3}\right)$ $f(x) = 4\cos\left(2\left(x + \frac{\pi}{6}\right)\right)$ The period is $\frac{2\pi}{2} = \pi$. The amplitude is 4. The mean position is $y = 0$. The horizontal translation is $\frac{\pi}{6}$ to the left.
2 Sketch the graph without the horizontal translation, $y = 4\cos(2x)$.	
3 Calculate the coordinates of the end points of the domain of the given function.	$f(0) = 4\cos\left(\frac{\pi}{3}\right)$ $= 4 \times \frac{1}{2}$ $= 2$ $f\left(\frac{3\pi}{2}\right) = 4\cos\left(3\pi + \frac{\pi}{3}\right)$ $= 4 \times \frac{-1}{2}$ $= -2$ The end points of the graph are (0, 2) and $\left(\frac{3\pi}{2}, -2\right)$.

4 Calculate or deduce the positions of the x-intercepts.

Each x-intercept on $y = 4\cos(2x)$ is translated $\frac{\pi}{6}$ units to the left.

Alternatively, let $y = 0$.

$$4\cos\left(2x + \frac{\pi}{3}\right) = 0$$

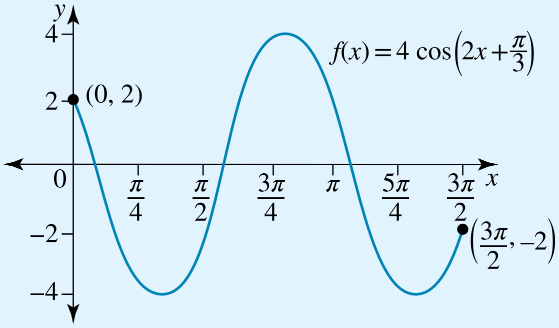
$$\cos\left(2x + \frac{\pi}{3}\right) = 0,\ \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le 3\pi + \frac{\pi}{3}$$

$$2x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

$$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$$

5 Apply the horizontal translation to key points on the graph already sketched and hence sketch the function over its given domain.



b 1 Identify the two functions forming the sum function.

b $y = \cos(x) + \frac{1}{2}\sin(2x)$

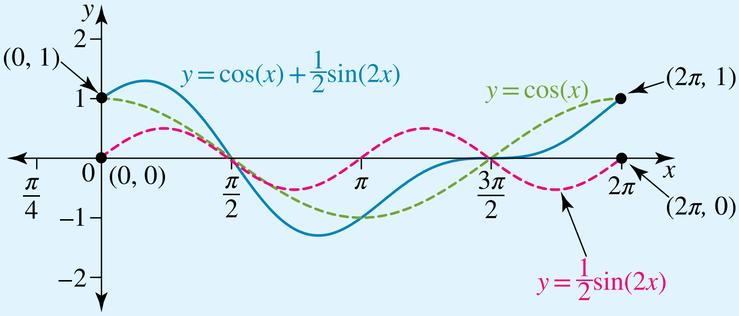
$y = y_1 + y_2$ where $y_1 = \cos(x)$ and $y_2 = \frac{1}{2}\sin(2x)$.

2 State the key features of the two functions.

$y_1 = \cos(x)$ has period 2π and amplitude 1.

$y_2 = \frac{1}{2}\sin(2x)$ has period π and amplitude $\frac{1}{2}$.

3 Sketch the two functions on the same set of axes and add together y-values of known points.



The tangent function

The domain of the tangent function, $y = \tan(x)$, can be deduced from the relationship $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Whenever $\cos(x) = 0$, the tangent function will be undefined and its graph will have vertical asymptotes. Because $\cos(x) = 0$ when x is an odd multiple of $\frac{\pi}{2}$, the domain is $R\backslash\{x : x = (2n + 1)\frac{\pi}{2}, n \in Z\}$.

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Topic 3

Concept 5

The tangent function
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The tangent function
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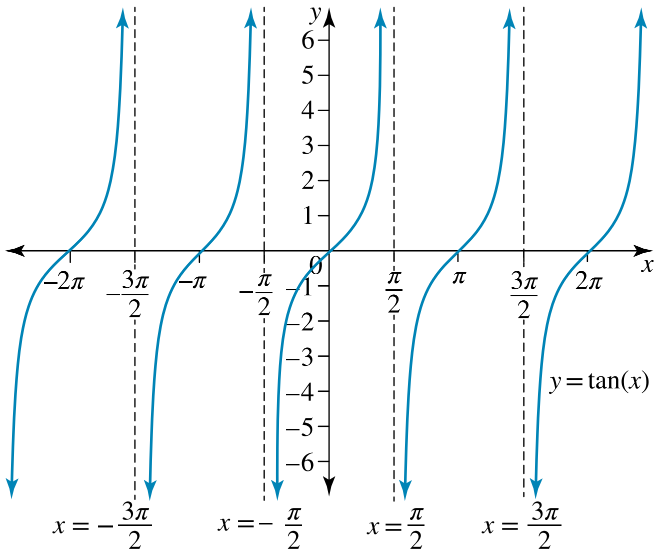
If $\sin(x) = 0$, then $\tan(x) = 0$; therefore, its graph will have x-intercepts when $x = n\pi$, $n \in Z$.

The graph of $y = \tan(x)$ is shown.



The key features of the graph of $y = \tan(x)$ are:

- **period π**
- **range R, which implies it is not meaningful to refer to an amplitude**
- **vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ i.e. $x = (2k+1)\frac{\pi}{2}, k \in Z$**
- **asymptotes spaced one period apart**
- **x-intercepts at $x = 0, \pm\pi, \pm 2\pi, \ldots$ i.e. $x = k\pi, k \in Z$**
- **mean position $y = 0$**
- **domain $R\backslash\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\}$.**
- **many-to-one correspondence.**

The graphs of $y = \tan(nx)$ and $y = a\tan(x)$

The period of $y = \tan(x)$ is π, so the period of $y = \tan(nx)$ will be $\frac{\pi}{n}$.

Altering the period alters the position of the vertical asymptotes, as these will now be $\frac{\pi}{n}$ units apart. An asymptote occurs when $nx = \frac{\pi}{2}$. Once one asymptote is found, others can be generated by adding or subtracting multiples of the period.

The mean position remains at $y = 0$, so the x-intercepts will remain midway between successive pairs of asymptotes.

The dilation factor a affects the steepness of the tangent graph $y = a\tan(x)$. Its effect is illustrated by comparing the values of the functions $f(x) = \tan(x)$ and $g(x) = 2\tan(x)$ at the point where $x = \frac{\pi}{4}$.

Because $f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$ and $g\left(\frac{\pi}{4}\right) = 2\tan\left(\frac{\pi}{4}\right) = 2$, the point $\left(\frac{\pi}{4}, 1\right)$ lies on the graph of $y = f(x)$ but the point $\left(\frac{\pi}{4}, 2\right)$ lies on the dilated graph $y = g(x)$.

The graph of $y = a\tan(nx)$ has:

- period $\dfrac{\pi}{n}$
- vertical asymptotes $\dfrac{\pi}{n}$ units apart
- mean position $y = 0$ with x-intercepts on this line midway between pairs of successive asymptotes.

Also note that the graph has an inverted shape if $a < 0$.

The x-intercepts can be located using their symmetry with the asymptotes. Alternatively, they can be calculated by solving the equation $a\tan(nx) = 0$.

The graph of $y = \tan(x - h)$

A horizontal translation of h units will move the vertical asymptotes h units in the same direction, but the translation will not affect the period of the graph. An asymptote for the graph of $y = \tan(x - h)$ occurs when $x - h = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{2} + h$.

Other asymptotes can be generated by adding or subtracting multiples of the period π.

The x-intercepts will remain midway between successive pairs of asymptotes, as the mean position is unaffected at a horizontal translation. They may be found by this means or alternatively found by solving the equation $\tan(x - h) = 0$.

The graph of $y = a\tan(nx - b))$ has:

- **period $\dfrac{\pi}{n}$**
- **horizontal translation of $h = \dfrac{b}{n}$, as the equation is $y = a\tan\left(n\left(x - \dfrac{b}{n}\right)\right)$**
- **mean position $y = 0$.**

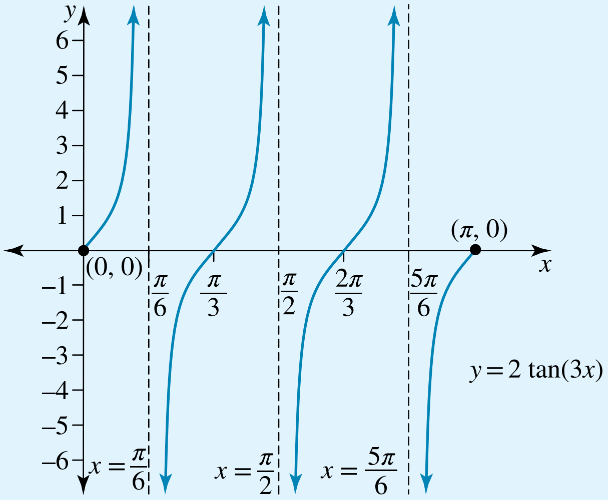
WORKED EXAMPLE 19

Sketch the graphs of:

a $y = 2\tan(3x)$ for $x \in [0, \pi]$ **b** $y = -\tan\left(2x + \dfrac{\pi}{2}\right)$ for $x \in (0, 2\pi)$.

THINK	WRITE/DRAW
a 1 State the period.	**a** $y = 2\tan(3x)$ The period is $\dfrac{\pi}{3}$.
2 Calculate the positions of the asymptotes.	An asymptote occurs when $3x = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{6}$. Others are formed by adding multiples of the period. Asymptotes occur at $x = \dfrac{\pi}{6} + \dfrac{\pi}{3} = \dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2} + \dfrac{\pi}{3} = \dfrac{5\pi}{6}$ within the domain constraint $x \in [0, \pi]$.
3 Calculate the positions of the x-intercepts. *Note:* An alternative method is to let $y = 0$ and solve the trigonometric equation for x.	The mean position is $y = 0$, and the x-intercepts occur midway between the asymptotes. One occurs at $x = \dfrac{1}{2}\left(\dfrac{\pi}{6} + \dfrac{\pi}{2}\right) = \dfrac{\pi}{3}$. The next is a period apart at $x = \dfrac{\pi}{3} + \dfrac{\pi}{3} = \dfrac{2\pi}{3}$, and the one after that is at $x = \dfrac{2\pi}{3} + \dfrac{\pi}{3} = \pi$.

4 Sketch the graph.



b **1** State the period.

b $y = -\tan\left(2x + \frac{\pi}{2}\right)$

$\therefore y = -\tan\left(2\left(x + \frac{\pi}{4}\right)\right)$

The period is $\frac{\pi}{2}$.

2 Calculate the positions of the asymptotes.

An asymptote occurs when $2x + \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow x = 0$.
Adding multiples of the period, another occurs at $x = 0 + \frac{\pi}{2} = \frac{\pi}{2}$, another at $x = \frac{\pi}{2} + \frac{\pi}{2} = \pi$, another at $x = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$, and another at $x = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi$.

The asymptotes are $x = 0, x = \frac{\pi}{2}, x = \pi, x = \frac{3\pi}{2}, x = 2\pi$.

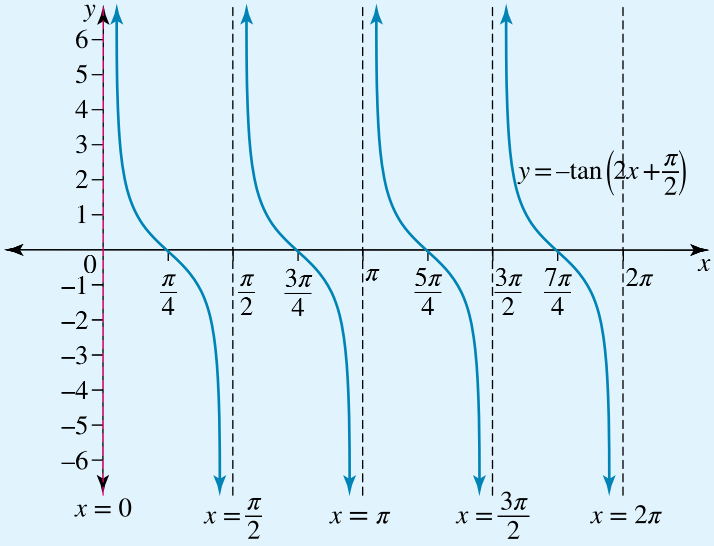
3 Calculate the positions of the *x*-intercepts.

The mean position is $y = 0$, and the *x*-intercepts are midway between the asymptotes.

x-intercepts occur at $x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$.

4 Sketch the graph, noting its orientation.

The graph is inverted due to the presence of the negative coefficient in its equation, $y = -\tan\left(2x + \frac{\pi}{2}\right)$.



The graph of $y = a\tan(n(x-h)) + k$

Under a vertical translation of k units, the mean position becomes $y = k$. The points that are midway between the asymptotes will now lie on this line $y = k$, not on the x-axis, $y = 0$. The x-intercepts must be calculated by letting $y = 0$ and solving the ensuing trigonometric equation this creates.

The vertical translation does not affect either the asymptotes or the period.

The graph of $y = a\tan(n(x-h)) + k$ has

- **period $\frac{\pi}{n}$**
- **vertical asymptotes when $n(x-h) = (2k+1)\frac{\pi}{2}$, $k \in Z$**
- **mean position $y = k$**
- **x-intercepts where $a\tan(n(x-h)) + k = 0$.**

WORKED EXAMPLE 20 Sketch the graph of $y = 3\tan(2\pi x) + \sqrt{3}$ over the interval $-\frac{7}{8} \le x \le \frac{7}{8}$.

THINK	WRITE/DRAW
1 State the period and mean position.	$y = 3\tan(2\pi x) + \sqrt{3}$ The period is $\frac{\pi}{2\pi} = \frac{1}{2}$. The mean position is $y = \sqrt{3}$.
2 Calculate the positions of the asymptotes.	An asymptote occurs when $2\pi x = \frac{\pi}{2} \Rightarrow x = \frac{1}{4}$. Others are formed by adding and subtracting a period. For the interval $-\frac{7}{8} \le x \le \frac{7}{8}$, the asymptotes occur at $x = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$, $x = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ and $x = -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4}$. The asymptotes are $x = -\frac{3}{4}, x = -\frac{1}{4}, x = \frac{1}{4}, x = \frac{3}{4}$.
3 Calculate the positions of the x-intercepts.	x-intercepts: Let $y = 0$. $3\tan(2\pi x) + \sqrt{3} = 0,\ -\frac{7}{8} \le x \le \frac{7}{8}$ $\tan(2\pi x) = -\frac{\sqrt{3}}{3},\ -\frac{7\pi}{4} \le 2\pi x \le \frac{7\pi}{4}$ $2\pi x = -\frac{\pi}{6}, -\pi - \frac{\pi}{6}, \pi - \frac{\pi}{6}, \cancel{2\pi - \frac{\pi}{6}}$ $2\pi x = -\frac{\pi}{6}, -\frac{7\pi}{6}, \frac{5\pi}{6}$ $x = -\frac{1}{12}, -\frac{7}{12}, \frac{5}{12}$

4 Obtain the y-intercept.

When $x = 0$, $y = 3\tan(0) + \sqrt{3} = \sqrt{3}$.
The point $(0, \sqrt{3})$ is on the mean position.

5 Calculate the coordinates of the end points.

End points: Let $x = -\frac{7}{8}$.

$$y = 3\tan\_2\pi \times -\frac{7}{8} \rightarrow + \sqrt{3}$$

$$= 3\tan\left(-\frac{7\pi}{4}\right) + \sqrt{3}$$
$$= 3 \times 1 + \sqrt{3}$$
$$= 3 + \sqrt{3}$$

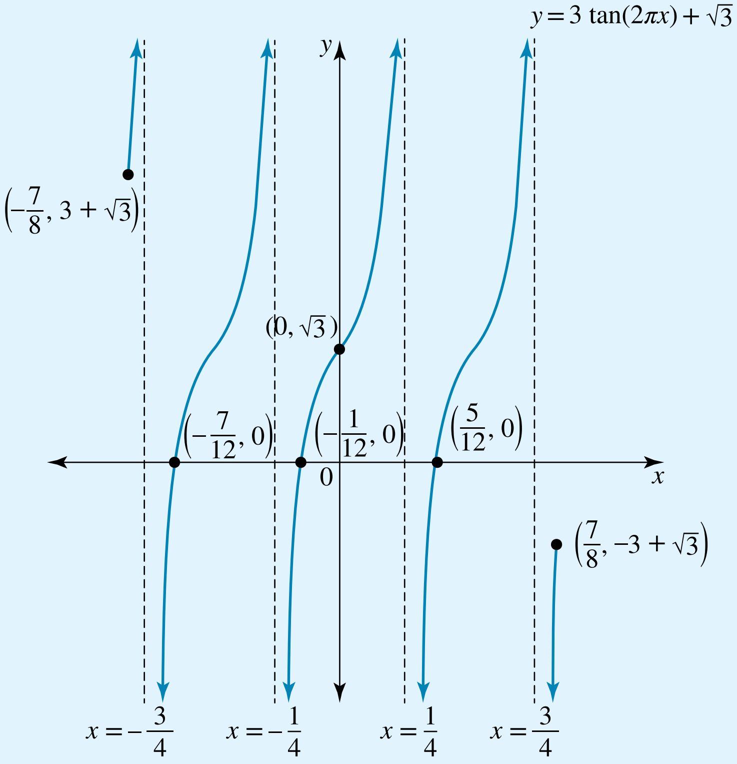
One end point is $\left(-\frac{7}{8}, 3 + \sqrt{3}\right)$.

Let $x = \frac{7}{8}$.

$$y = 3\tan\left(\frac{7\pi}{4}\right) + \sqrt{3}$$
$$= 3 \times -1 + \sqrt{3}$$
$$= -3 + \sqrt{3}$$

The other end point is $\left(\frac{7}{8}, -3 + \sqrt{3}\right)$.

6 Sketch the graph.



EXERCISE 2.5 Non-algebraic functions

PRACTISE

Work without CAS

1 **WE14** Consider the function $f(x) = -10^x$.

a Evaluate $f(2)$.

b On the same set of axes, sketch the graphs of $y = 10^x$, $y = -10^x$ and $y = 10^{-x}$.

c Express $y = 10^{-x}$ in an equivalent form.

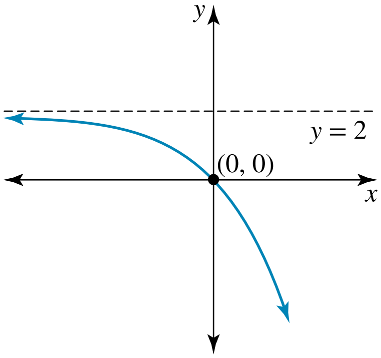
2 For the functions defined by $f(x) = 2^x$ and $g(x) = 2^{-x}$, sketch the graph of the difference function $y = (f - g)(x)$ and state its domain, range and rule.

3 **WE15** Sketch the following graphs and state the domain and range of each.

a $y = -2e^x - 3$ b $y = 4e^{-3x} - 4$ c $y = 5e^{x-2}$

4 a Sketch the graph of $y = 2e^{1-3x} - 4$, labelling any intercepts with the coordinate axes with their exact coordinates.

b Sketch the graph of $y = 3 \times 2^x - 24$ and state its domain and range.



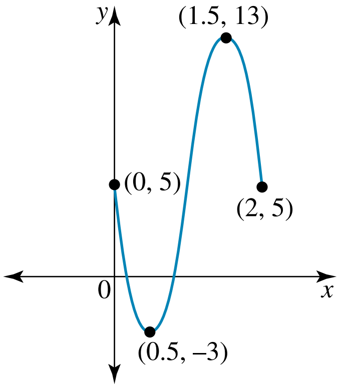
5 a **WE16** The diagram shows the graph of $y = ae^x + b$. Determine the values of a and b.

b The graph of $y = a \times 10^{kx}$ contains the points (4, –20) and (8, –200). Form its equation.

6 The graph of $y = a \times e^{kx}$ contains the points (2, 36) and (3, 108). Calculate the exact values of a and k.

7 a **WE17** Sketch the graph of $y = 2\cos(4x) - 3$, $0 \le x \le 2\pi$.

b The diagram shows the graph of a sine function. State its mean position, amplitude, and period, and give a possible equation for the function.



8 Sketch the graph of $f: [0, 2\pi] \to R$, $f(x) = 1 - 2\sin\left(\frac{3x}{2}\right)$, locating any intercepts with the coordinate axes.

9 a **WE18** Sketch the graph of the function

$$f: \left[0, \frac{3\pi}{2}\right] \to R,\ f(x) = -6\sin\left(3x - \frac{3\pi}{4}\right).$$

b Sketch the graph of $y = \cos(2x) - 3\cos(x)$ for $x \in [0, 2\pi]$.

10 Sketch the graph of $y = (\sin(x))^2 = \sin^2(x)$ for $x \in [-\pi, \pi]$.

11 **WE19** Sketch the graphs of:

a $y = 3\tan\left(\frac{x}{2}\right)$ for $x \in [-\pi, \pi]$

b $y = -\tan(2x - \pi)$ for $x \in [-\pi, \pi]$.

12 The graph of $y = a\tan(nx)$ has the domain $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ with vertical asymptotes at $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$ only. The graph passes through the origin and the point $\left(-\frac{\pi}{6}, -\frac{1}{2}\right)$. Determine its equation.

13 **WE20** Sketch the graph of $y = 3\tan(2\pi x) - \sqrt{3}$ over the interval $-\frac{7}{8} \le x \le \frac{7}{8}$.

14 Sketch the graph of $y = 1 - \tan\left(x + \frac{\pi}{6}\right)$ over the interval $0 \le x \le 2\pi$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

15 Sketch the graph of each of the following exponential functions and state their long-term behaviour as $x \to \infty$.

a $y = \frac{4}{5} \times 10^x$

b $y = 3 \times 4^{-x}$

c $y = -5 \times 3^{-\frac{x}{2}}$

d $y = -\left(\frac{2}{3}\right)^{-x}$

16 For each of the following functions, sketch the graph, state the range and identify the exact position of any intercepts the graph makes with the coordinate axes.

a $y = e^x - 3$

b $y = -2e^{2x} - 1$

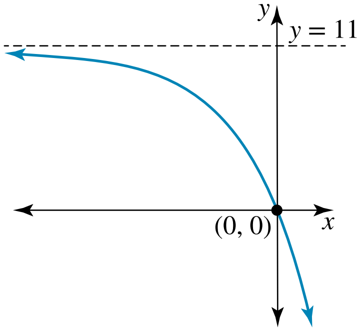
c $y = \frac{1}{2}e^{-4x} + 3$

d $y = 4 - e^{2x}$

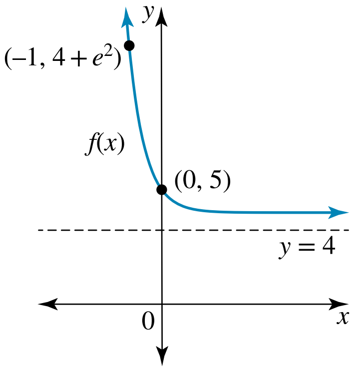
e $y = 4e^{2x-6} + 2$

f $y = 1 - e^{-\frac{x+1}{2}}$

17 a The graph shown is of the function $f(x) = ae^x + b$. Determine the values of a and b and write the function as a mapping.



b The graph shown has an equation of the form $y = Ae^{nx} + k$. Determine its equation.



c The graph of $y = 2^{x-b} + c$ contains the points (0, –5)and (3, 9).

i Calculate the values of b and c.

ii State the range of the graph.

d The graph of $y = Ae^{x-2} + B$ contains the point (2, 10). As $x \to -\infty,\ y \to -2$.

i Calculate the values of A and B.

ii The graph passes through the point $\left(a, 2\left(\frac{6}{e} - 1\right)\right)$. Find the value of a.

18 State the period, amplitude and range of each of the following.

a $y = 6\sin(8x)$

b $y = 2 - 3\cos\left(\frac{x}{4}\right)$

c $y = -\sin(3x - 6)$

d $y = 3(5 + 2\cos(6\pi x))$

19 Sketch the following over the intervals specified.

a $y = -7\cos(4x),\ 0 \le x \le \pi$

b $y = 5 - \sin(x),\ 0 \le x \le 2\pi$

c $y = \frac{1}{2}\cos(2x) + 3,\ -\pi \le x \le 2\pi$

d $y = 2 - 4\sin(3x),\ 0 \le x \le 2\pi$

e $y = 2\sin\left(x + \frac{\pi}{4}\right),\ 0 \le x \le 2\pi$

f $y = -4\cos\left(3x - \frac{\pi}{2}\right) + 4,\ -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

20 **a** **i** Solve the equation $2\sin(2x) + \sqrt{3} = 0$ for $x \in [0, 2\pi]$.

ii Sketch the graph of $y = \sin(2x)$ for $x \in [0, 2\pi]$.

iii Hence find $\left\{x: \sin(2x) < -\frac{\sqrt{3}}{2},\ 0 \le x \le 2\pi\right\}$.

b State the maximum value of the function $f(x) = 2 - 3\cos\left(x + \frac{\pi}{12}\right)$ and give the first positive value of x for when this maximum occurs.

21 State the period and calculate the equation of the first positive asymptote for each of the following.

a $y = \tan(4x)$

b $y = 9 + 8\tan\left(\frac{x}{7}\right)$

c $y = -\frac{3}{2}\tan\left(\frac{4x}{5}\right)$

d $y = 2\tan(6\pi x + 3\pi)$

22 a Sketch the following graphs over the intervals specified.

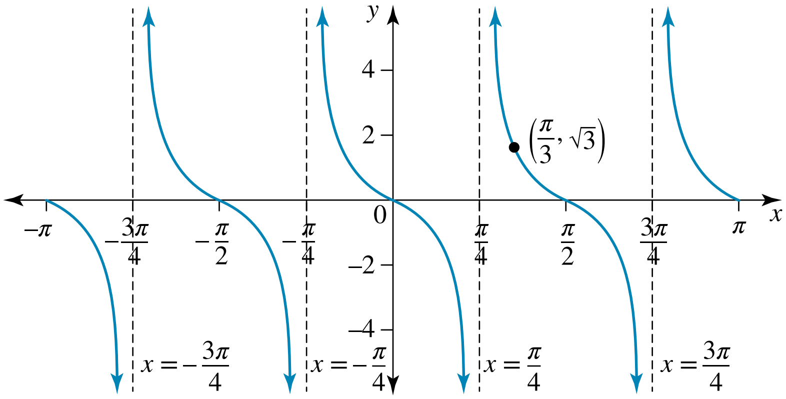
i $y = -\tan(2x),\ x \in [0,\ \pi]$

ii $y = 3\tan\left(x + \dfrac{\pi}{4}\right),\ x \in [0,\ 2\pi]$

iii $y = \tan\left(\dfrac{x}{3}\right) + \sqrt{3},\ x \in [0,\ 6\pi]$

iv $y = 5\sqrt{3}\tan\left(\pi x - \dfrac{\pi}{2}\right) - 5,\ x \in (-2,\ 3)$

b The graph of $y = \tan(x)$ undergoes a set of transformations to form that of the graph shown.



i Explain why there was no horizontal translation among the set of transformations applied to $y = \tan(x)$ to obtain this graph.

ii State the period of the graph shown.

iii Form a possible equation for the graph.

23 A hybrid function is defined by the rule

$$f(x) = \begin{cases} -\sin(x), & -2\pi \le x \le -\dfrac{\pi}{2} \\ \tan(x), & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \cos(x), & \dfrac{\pi}{2} \le x \le 2\pi \end{cases}.$$

a Evaluate:

i $f\left(\dfrac{\pi}{3}\right)$ **ii** $f(\pi)$ **iii** $f\left(-\dfrac{\pi}{2}\right)$.

b Sketch the graph of $y = f(x)$.

c Identify any points of the domain where the function is not continuous.

d State the domain and range of the function.

24 Use addition of ordinates to sketch the graphs of:

a $y = e^{-x} + e^{x}$ **b** $y = \sin(2x) - 4\sin(x),\ 0 \le x \le 2\pi$

c $y = x + \sin(x),\ 0 \le x \le 2\pi$

MASTER

25 **a** Use a graphical method to determine the number of the roots of the equation $e^x = \cos(x)$.

b Calculate the values of the roots that lie in the interval $[-2\pi, 2\pi]$, giving their values correct to 3 decimal places.

26 **a** Use CAS technology to find the coordinates of the points of intersection of the graphs of $y = \sin(2x)$ and $y = \tan(x)$ for $-2\pi \le x \le 2\pi$.

b Hence, or otherwise, give the general solution to the equation $\sin(2x) = \tan(x)$, $x \in R$.

2.6 Modelling and applications

People in research occupations, such as scientists, engineers and economists, analyse data though the use of mathematical models in order to increase our understanding of natural phenomena and to draw inferences about future behaviour. In this section we consider some applications of the functions that are discussed in the previous sections of this topic.

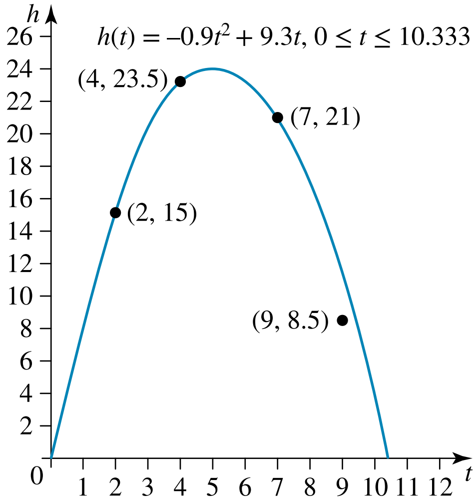
Modelling with data

Consider the set of data shown in the table.

t	2	4	7	9
h	15	23.5	21	8.5

In deciding what type of model this data might best fit, a linear model would be ruled out as the data is not steadily increasing or decreasing. The values increase and then decrease; there are no obvious signs that the data is oscillating or showing asymptotic behaviour. Observations such as these would rule out an exponential model, a trigonometric model and a hyperbola or truncus model.

The data is likely to be a polynomial model with a many-to-one correspondence. Plotting the points can help us recognise a possible model. If the variables t and h are time and height respectively, then we may suspect the polynomial would be a quadratic one. Three of the data points could be used to form the model in the form $h = at^2 + bt + c$, or the entire set of data could be used to obtain the model through a quadratic regression function on CAS. The quadratic model $h(t) = -0.9t^2 + 9.3t$ shows a good fit with the data.



Applications of mathematical models

The variables in a mathematical model are usually treated as continuous, even though they may represent a quantity that is discrete in reality, such as the number of foxes in a region. Values obtained using the model need to be considered in context and rounded to whole numbers where appropriate.

Domain restrictions must also be considered. A variable representing a physical quantity such as length must be positive. Similarly, a variable representing time usually cannot be negative. However, it is important to read carefully how the variables are defined. For example, if t is the time in hours after 10 am, then $t = -2$ would be possible as it refers to the time 8am.

For some exponential models that are functions of time, the behaviour or limiting value as $t \to \infty$ may be of interest.

WORKED EXAMPLE 21

The population of foxes on the outskirts of a city is starting to increase. Data collected suggests that a model for the number of foxes is given by $N(t) = 480 - 320e^{-0.3t}$, $t \geq 0$, where N is the number of foxes t years after the observations began.

a **How many foxes were present initially at the start of the observations?**

b **By how many had the population of foxes grown at the end of the first year of observations?**

c **After how many months does the model predict the number of foxes would double its initial population?**

d **Sketch the graph of N versus t.**

e **Explain why this model does not predict the population of foxes will grow to 600.**

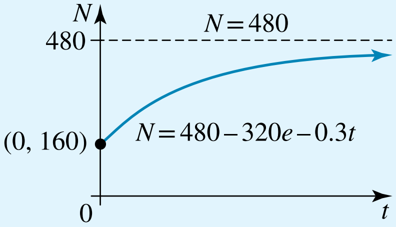
THINK	WRITE/DRAW
a Calculate the initial number.	**a** $N(t) = 480 - 320e^{-0.3t}$ When $t = 0$, $N(0) = 480 - 320e^{0}$ $= 480 - 320$ $= 160$ There were 160 foxes present initially.
b **1** Calculate the number after 1 year.	**b** When $t = 1$, $N(1) = 480 - 320e^{-0.3}$ ≈ 242.94 After the first year 243 foxes were present.
2 Express the change over the first year in context.	Over the first year the population grew from 160 to 243, an increase of 83 foxes.
c **1** Calculate the required value of t. *Note:* An algebraic method requiring logarithms has been used here. CAS technology could also be used to solve the equation.	**c** Let $N = 2 \times 160 = 320$. $320 = 480 - 320e^{-0.3t}$ $320e^{-0.3t} = 160$ $e^{-0.3t} = \frac{1}{2}$ $-0.3t = \log_e\left(\frac{1}{2}\right)$ $t = \frac{1}{0.3}\log_e\left(\frac{1}{2}\right)$ $t \approx 2.31$

2 Answer the question.

$0.31 \times 12 \approx 4$
The population doubles after 2 years and 4 months.

d Sketch the graph.

d $N(t) = 480 - 320e^{-0.3t}$
The horizontal asymptote is $N = 480$.
The y-intercept is $(0, 160)$.



e Give an explanation for the claim.

e The presence of an asymptote on the graph shows that as $t \to \infty$, $N \to 480$. Hence N can never reach 600. The population will never exceed 480 according to this model.

EXERCISE 2.6 Modelling and applications

PRACTISE

1 **WE21** The population of possums in an inner city suburb is starting to increase. Observations of the numbers present suggest a model for the number of possums in the suburb given by $P(t) = 83 - 65e^{-0.2t}$, $t \geq 0$, where P is the number of possums observed and t is the time in months since observations began.

a How many possums were present at the start of the observations?

b By how many had the population of possums grown at the end of the first month of observations?

c When does the model predict the number of possums would double its initial population?

d Sketch the graph of P versus t.

e Explain why this model does not predict the population of possums will grow to 100.

2 Consider the data points shown.

x	0	1	3	4
y	4	2	10	8

a Discuss why neither a linear, trigonometric, exponential nor a power function of the form $y = x^n$ is a likely fit for the data.

b Assuming the data set fits a hyperbola of the form $y = \dfrac{a}{x - 2} + k$, $x \in [0, \infty)\backslash\{2\}$:

i use the data to determine the equation of the hyperbola

ii sketch the model, showing the data points.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 The population, in thousands, of bees in a particular colony increases as shown in the table.

Month (t)	1	2	3	4
Population in thousands (P)	36	38.75	42.5	45

a Plot the data points P against t and suggest a likely model for the data.

b Use the values when $t = 2$ and $t = 4$ to form a rule for the model expressing P in terms of t.

c If the variable t measures the number of months since January, how many bees were in the colony in January, according to the model?

d What is the rate of increase in the population of bees according to the model?

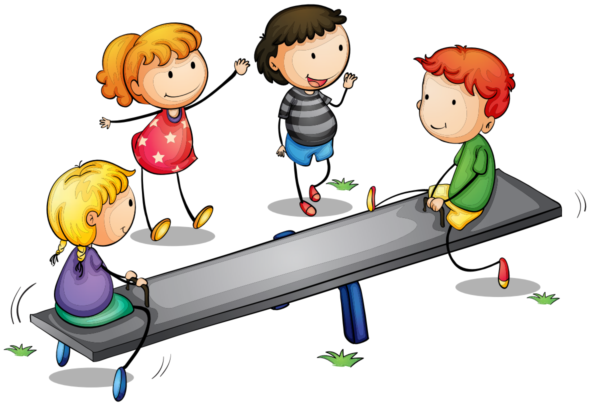
4 Eric received a speeding ticket on his way home from work. If he pays the fine now, there will be no added penalty. If he delays the payment by one month his fine will be \$435, and a delay of two months will result in a fine of \$655. The relationship between the fine, F, in dollars, and the number of months, n, that payment is delayed is given by $F = a(r)^n$ where a and r are constants.

a Calculate the value of a to the nearest integer and the value of r to 1 decimal place.

b If Eric pays the fine immediately, how much will it cost him?

c If Eric delays the payment for six months, how much can he expect to pay? Express the value to the nearest dollar.

5 A young girl and boy are lifted onto a seesaw in a playground. At this time the seesaw is horizontal with respect to the ground.



Initially the girl's end of the seesaw rises. Her height above the ground, h metres, t seconds after the seesaw starts to move is modelled by $h(t) = a\sin(nt) + k$.

The greatest height above the ground that the girl reaches is 1.7 metres, and the least distance above the ground that she reaches is 0.7 metres. It takes 2 seconds for her to seesaw between these heights.

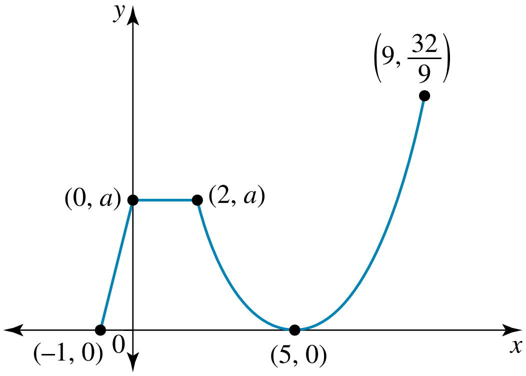
a Find the values of a, n and k.

b Draw the graph showing the height of the girl above the ground for $0 \le t \le 6$.

c For what length of time during the first 6 seconds of the motion of the seesaw is the girl's height above the ground 1.45 metres or higher?

d Sketch the graph showing the height of the boy above the ground during the first 6 seconds and state its equation.

6 A parabolic skate ramp has been built at a local park. It is accessed by climbing a ladder to a platform as shown. The platform is 2 metres long. The horizontal distance from the origin is x, and the vertical distance from the origin is y. The lowest point on the skate ramp is at (5, 0) and the highest point is at $\left(9, \frac{32}{9}\right)$.



- **a** Find the value of a where $(0, a)$ is the point where the ladder connects with the platform.
- **b** What are the coordinates of the point where the platform and the skate ramp meet?
- **c** Find the equation of the parabolic section of the skate ramp.
- **d** Write a hybrid function rule to define the complete skate ramp system for $\{x : -1 \leq x \leq 9\}$.
- **e** Determine the exact values of x when the skateboarder is 1.5 metres above the ground.

7 Manoj pours himself a mug of coffee but gets distracted by a phone call before he can drink the coffee. The temperature of the cooling mug of coffee is given by $T = 20 + 75e^{-0.062t}$, where T is the temperature of the coffee t minutes after it was initially poured into the mug.

- **a** What was its initial temperature when it was first poured?
- **b** To what temperature will the coffee cool if left unattended?
- **c** How long does it take for the coffee to reach a temperature of 65°C? Give your answer correct to 2 decimal places.
- **d** Manoj returns to the coffee when it has reached 65°C and decides to reheat the coffee in a microwave. The temperature of the coffee in this warming stage is $T = A + Be^{-0.05t}$. Given that the temperature of the reheated coffee cannot exceed 85°C, calculate the values of A and B.
- **e** Sketch a graph showing the temperature of the coffee during its cooling and warming stage.

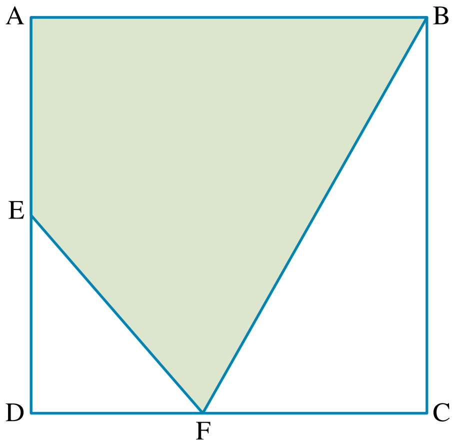
8 James is in a boat out at sea fishing. The weather makes a change for the worse and the water becomes very choppy. The depth of water above the sea bed can be modelled by the function with equation $d = 1.5 \sin\left(\frac{\pi t}{12}\right) + 12.5$, where d is the depth of water in metres and t is the time in hours since the change of weather began.

- **a** How far from the sea bed was the boat when the change of weather began?
- **b** What is the period of the function?
- **c** What are the maximum and minimum heights of the boat above the sea bed?
- **d** Sketch one cycle of the graph of the function.
- **e** If the boat is h metres above the seabed for a continuous interval of 4 hours, calculate h correct to 1 decimal place.
- **f** James has heard on the radio that the cycle of weather should have passed within 12 hours, and when the height of water above the sea bed is at a minimum after that, it will be safe to return to shore. If the weather change occurred at 9.30 am, when will he be able to return to shore?

9 A biologist conducts an experiment to determine conditions that affect the growth of bacteria. Her initial experiment finds the growth of the population of bacteria is modelled by the rule $N = 22 \times 2^t$, where N is the number of bacteria present after t days.

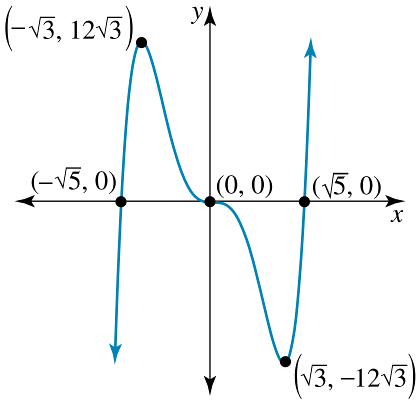
- **a** How long will it take for the number of bacteria to reach 2816?
- **b** What will happen to the number of bacteria in the long term according to this model?
- **c** The biologist changes the conditions of her experiment and starts with a new batch of bacteria. She finds that under the changed conditions the growth of the population of bacteria is modelled by the rule $N = \dfrac{66}{1 + 2e^{-0.2t}}$.
  - **i** Show that in both of her experiments the biologist used the same initial number of bacteria.
  - **ii** What will happen to the number of bacteria in the long term according to her second model?

10 ABCD is a square field of side length 40 metres. The points E and F are located on AD and DC respectively so that ED = DF = x m. A gardener wishes to plant an Australian native garden in the region that is shaded green in the diagram.



- **a** Show that the area, A m^2, to be used for the Australian native garden is given by $A = 800 + 20x - \frac{1}{2}x^2$.
- **b** What restrictions must be placed on x?
- **c** **i** Calculate the value of x for which the area of the Australian native garden is greatest.
  - **ii** Calculate the greatest possible area of the native garden.

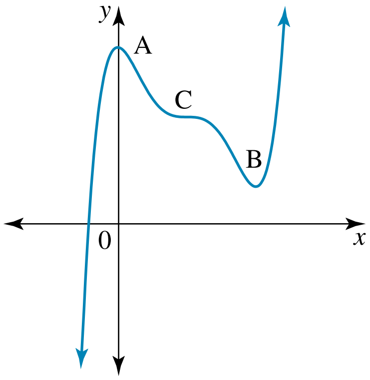
11 The graph of $y = g(x)$ is shown. The graph has a stationary point of inflection at the origin and also crosses the x-axis at the points where $x = -\sqrt{5}$ and $x = \sqrt{5}$. The coordinates of the maximum turning point and the minimum turning point are $(-\sqrt{3}, 12\sqrt{3})$ and $(\sqrt{3}, -12\sqrt{3})$respectively.



- **a** Use the above information to form the equation of the graph.
- **b** Hence, show that $g(x) = 2x^5 - 10x^3$.

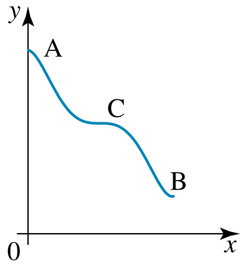
c A water slide is planned for a new theme park and its cross section shape is to be designed using a horizontal and vertical translation of the curve $g(x) = 2x^5 - 10x^3$.

The point A, the maximum turning point of the original curve, now lies on the y-axis. The point B, the minimum turning point of the original curve, now lies 1 unit above the x-axis. The point C is the image of the origin (0, 0) after the original curve is translated.



The water slide is modelled by the section of the curve from A to B with the x-axis as the water level.

- **i** State the values of the horizontal and vertical translations required to achieve this model.
- **ii** Give the height of A above the water level to 1 decimal place.
- **iii** State the coordinates of the points C and B.



12 In an effort to understand more about the breeding habits of a species of quoll, 10 quolls were captured and relocated to a small reserve where their behaviour could be monitored. After 5 years the population size grew to 30 quolls.

A model for the size of the quoll population, N, after t years on the reserve is thought to be defined by the function $N: R^+ \cup \{0\} \to R, N(t) = \dfrac{at + b}{t + 2}$.

a Calculate the values of a and b.

b Sketch the graph of N against t.

c Hence or otherwise, determine how large can the quoll population can grow to.

MASTER

13 The water level in a harbor, h metres below a level jetty, at time t hours after 7 am is given by $h = 3 - 2.5 \sin\left(\dfrac{1}{2}(t - 1)\right)$.

a How far below the jetty is the water level in the harbor at 7:30 am? Give your answer correct to 3 decimal places.

b What are the greatest and least distances below the jetty?

c Sketch the graph of h versus t and hence determine the values of t at which the low and high tides first occur. Give your answers correct to 2 decimal places.

d A boat ties up to the jetty at high tide. How much extra rope will have to be left so that the boat is still afloat at low tide?

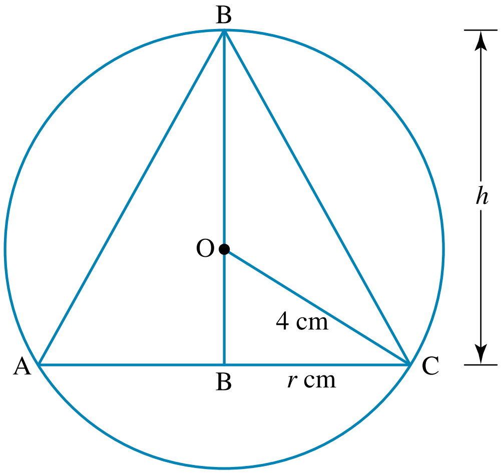
14 A right circular cone is inscribed in a sphere of radius 4 cm, as shown in the cross section below.

a Express the radius, r cm, of the cone in terms of h.

b Write an equation expressing the volume of the cone, V cm^3, in terms of h and state any restrictions on h.

c Sketch the graph of V versus h.

d Use the graph to find the maximum volume for the cone to the nearest cm^3.



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- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
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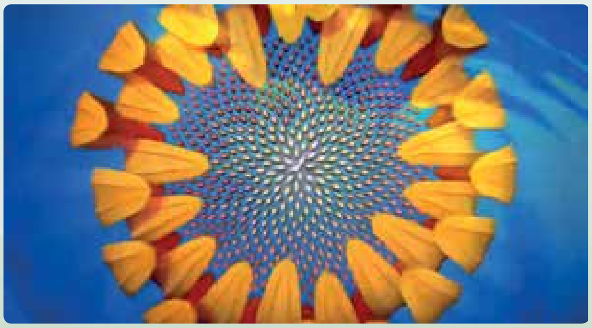

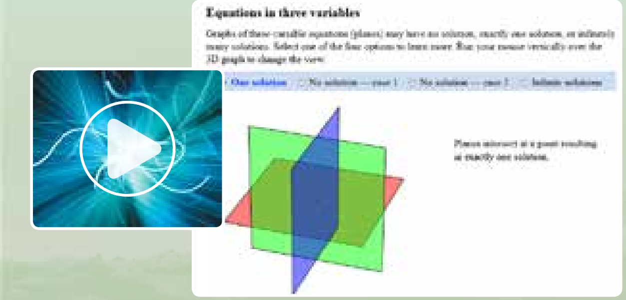

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2 Answers

EXERCISE 2.2

1 a Many-to-one correspondence

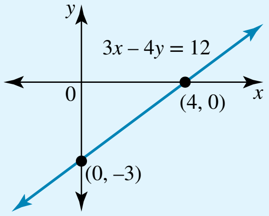
b Domain $[-4, 2)$, range $[0, 16]$

c $f: [-4, 2) \rightarrow R, f(x) = x^2$

d 12

2 a $-\frac{32}{9}$ **b** $4a^2 - 4$ **c** R

3 a

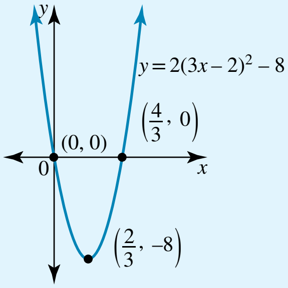


b $\frac{3}{4}$

c $\left(\frac{36}{25}, -\frac{48}{25}\right)$

4 a $2x + 3y = -11$ **b** $y = \frac{3x}{2} - 8$ **c** 56.3°

5 a

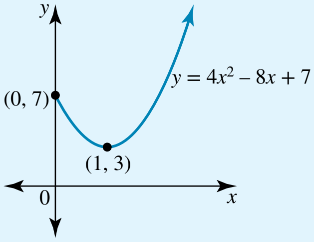


Domain R, range $[-8, \infty)$.

b $y = -\frac{1}{2}(2x + 1)(x - 4)$

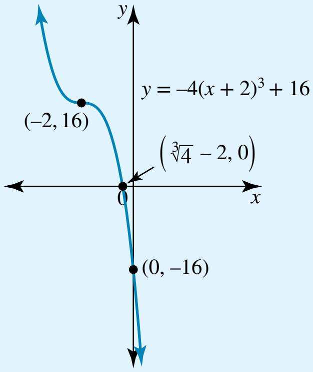
6 a None **b** $f(x) = 4(x - 1)^2 + 3$

c



Domain $R^+ \cup \{0\}$, range $[3, \infty)$.

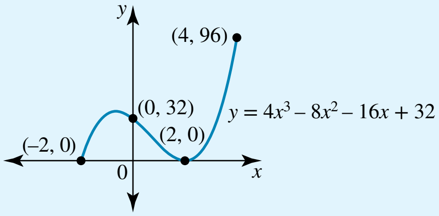
7 a



b $y = 2x(5x - 4)(2x - 3)$

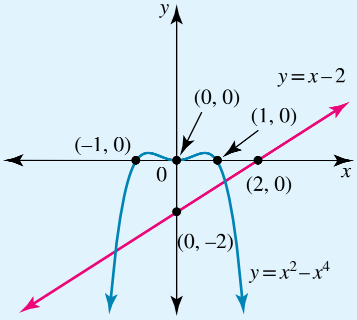
8 a $4(x - 2)^2(x + 2)$

b



c Maximum value 96, minimum value 0.

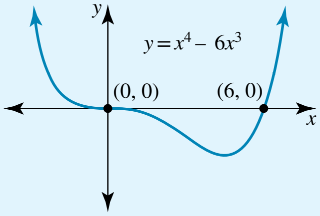
9 a



$\therefore$ 2 solutions.

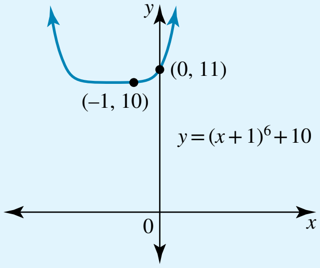
b $a = -\frac{7}{81}, b = 6, c = 7, (-6, 7)$

10

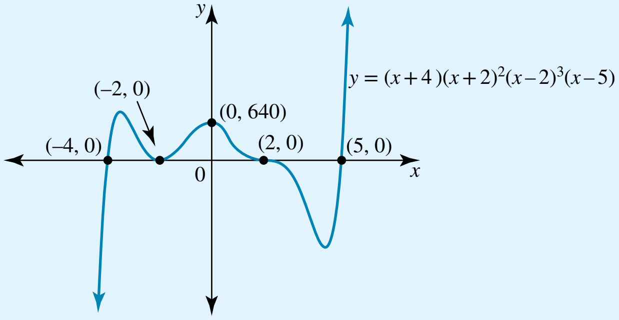


$y = x^4 - 6x^3 - 1$ will make 2 intersections with the x-axis.

11



12



13 a **i** Many-to-one **ii** $[-3, 6), [-9, 7]$ **iii** Yes

b **i** One-to-many **ii** $[0, \infty), R$ **iii** No

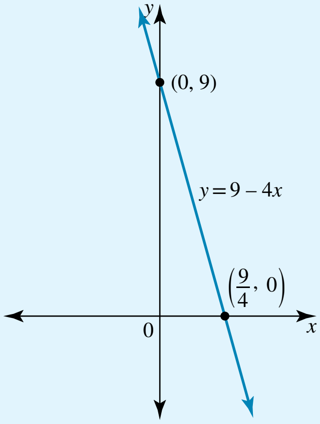
c **i** Many-to-many **ii** $[-2, 2], [-2, 2]$ **iii** No

d **i** One-to-one **ii** R, R **iii** Yes

e **i** Many-to-one **ii** $R, \{2\}$ **iii** Yes

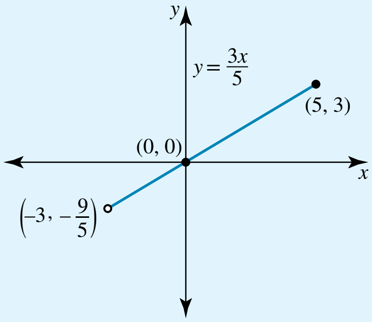
f **i** One-to-one **ii** R, R **iii** Yes

14 a



Range R

b



Range $\left(-\frac{9}{5}, 3\right]$

15 a $p = 6$

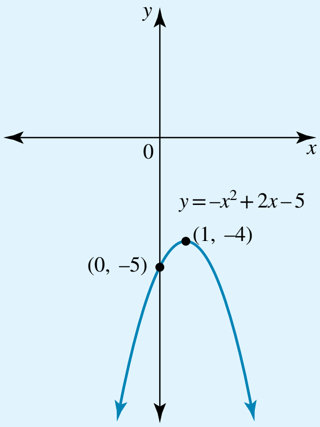
b $9y - 7x = 23$

c 8.3 units

16 a $-(x - 1)^2 - 4$

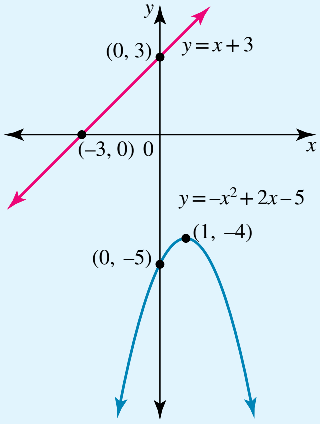
b $(1, -4)$

c



Range $(-\infty, -4]$

d



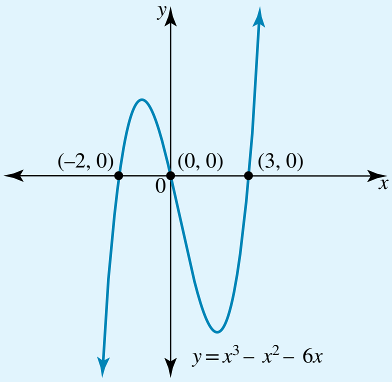
e $k = -\frac{19}{4}$

17 a $y = -\frac{3}{20}(x + 6)^2 + 12$

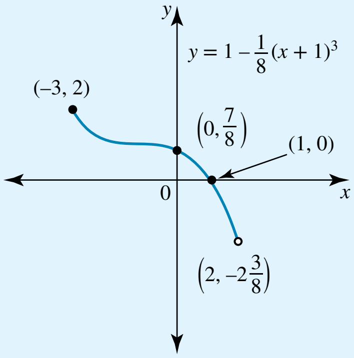
b $y = -\frac{4}{7}(x + 7)(2x + 5)$

c $y = \frac{1}{4}x^2 - 5$

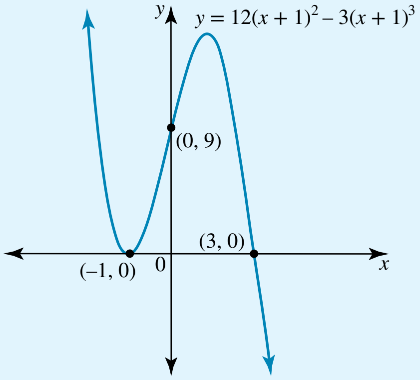
18 a



b



c



19 $y = \frac{1}{10}(x + 4)(4x - 5)^2$

20 a $f(x) = -2x^3 + 9x^2 - 24x + 17$

$f(1) = -2 + 9 - 24 + 17 = 0$

$\therefore (x - 1)$ is a factor.

By inspection,
$-2x^3 + 9x^2 - 24x + 17 = (x - 1)(-2x^2 + 7x - 17)$.

Consider the discriminant of the quadratic factor $-2x^2 + 7x - 17$.

$$\begin{aligned}\Delta &= 49 - 4(-2)(-17)\\ &= 49 - 136\\ &< 0\end{aligned}$$

As the discriminant is negative, the quadratic cannot be factorised into real linear factors; therefore, it has no real zeros.

For the cubic, this means there can only be one x-intercept, the one which comes from the only linear factor $(x - 1)$.

b For there to be a stationary point of inflection, the equation of the cubic function must be able to be written in the form $y = a(x + b)^3 + c$.

Let $-2x^3 + 9x^2 - 24x + 17 = a(x + b)^3 + c$

By inspection, the value of a must be -2.

$$\begin{aligned}\therefore\ &-2x^3 + 9x^2 - 24x + 17\\ &= -2(x^3 + 3x^2b + 3xb^2 + b^3) + c\end{aligned}$$

Equate coefficients of like terms:

x^2: $9 = -6b \Rightarrow b = -\frac{3}{2}$

x: $-24 = -6b^2 \Rightarrow b^2 = 4$

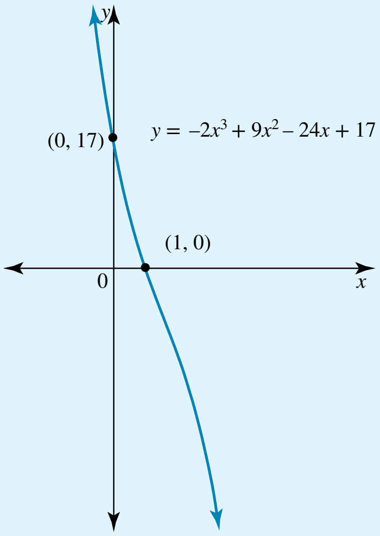
It is not possible for b to have different values.

Therefore, it is not possible to express the equation of the function in the form $y = a(x + b)^3 + c$.

There is no stationary point of inflection on the graph of the function.

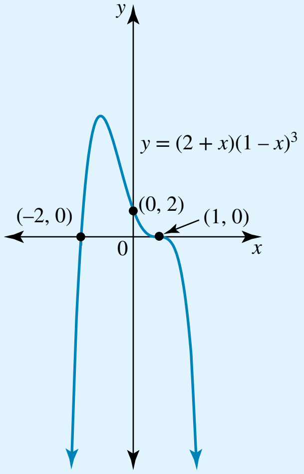
c $x \to \pm\infty, y \to \mp\infty$

d



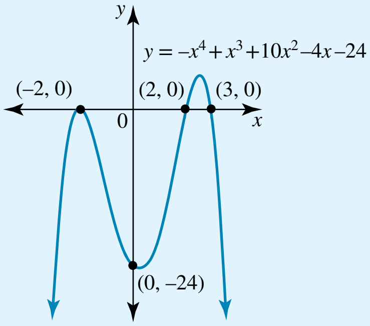
21 a $y = -3(x + 5)^4 + 12$

b

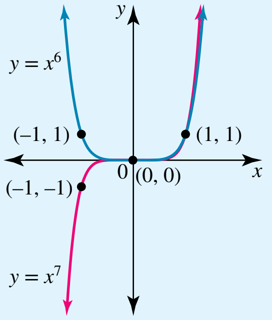


c i $-(x + 2)^2(x - 2)(x - 3)$

ii

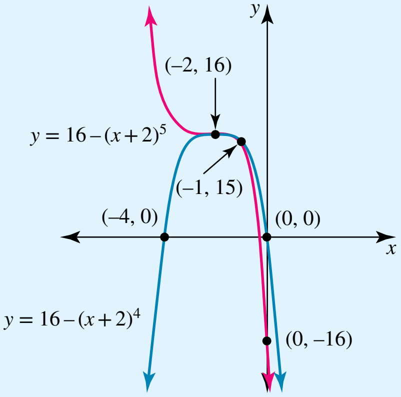


22 a i



ii $\{x : x \le 1\}$

b



c i $y = (x + 3)^2(x + 1)(x - 2)^3$, degree 6

ii $y = (x + 3)^2(x + 1)(x - 2)^3(10 - x)$, degree 7

23 $x = -1.75, x = 1.22$

24 a Minimum turning points (–1.31, –3.21) and (1.20, –9.32), maximum turning point (–0.636, –2.76)

b None

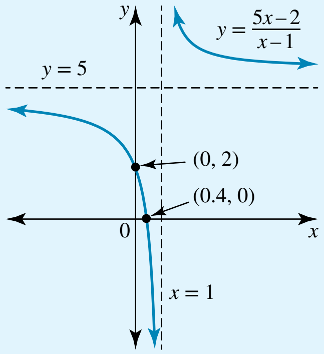
c Minimum turning point (–2.17, –242), stationary point of inflection (2, 20).

EXERCISE 2.3

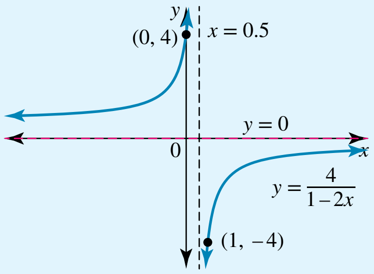
1 a $y = \dfrac{6}{x+3} + 1$

b i Maximal domain $R\backslash\{1\}$

ii Range $R\backslash\{5\}$

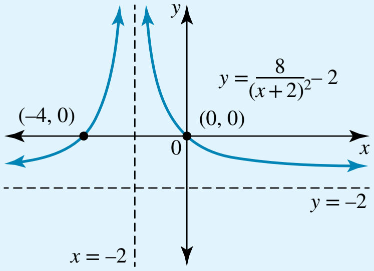


2



Domain $R\backslash\left\{\frac{1}{2}\right\}$, range $R\backslash\{0\}$.

3

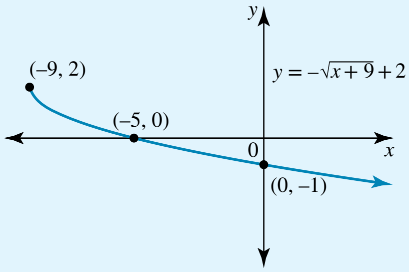


Domain $R\backslash\{-2\}$, range $(-2, \infty)$.

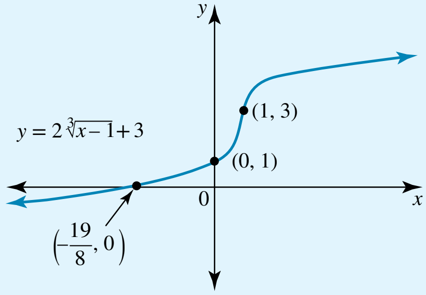
4 $y = \dfrac{1}{4x^2} - 1$

5 a i Maximal domain $[-9, \infty)$

ii Range $(-\infty, 2]$

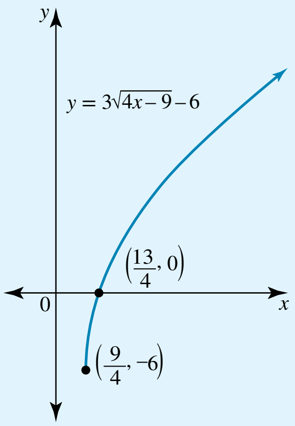


b $y = 2\sqrt[3]{x-1} + 3$

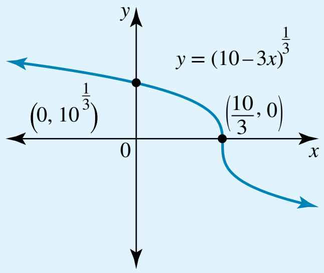


x-intercept $\left(-\dfrac{19}{8}, 0\right)$.

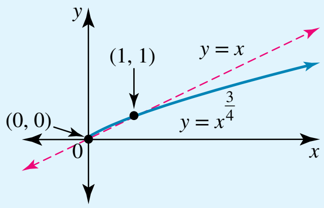
6 a Maximal domain $\left[\dfrac{9}{4}, \infty\right)$, range $[-6, \infty)$



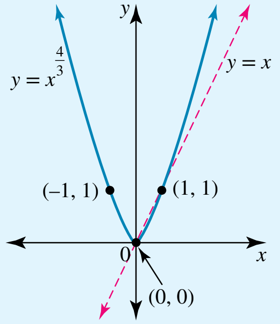
b $\left(\dfrac{10}{3}, 0\right)$



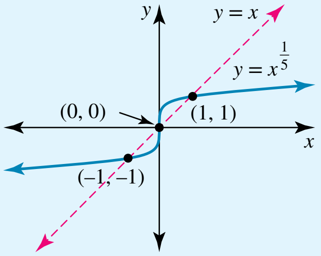
7 a Domain $R^+ \cup \{0\}$



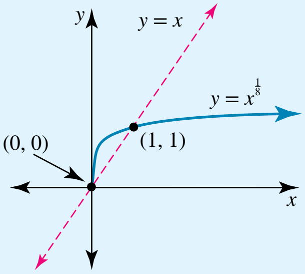
b Domain R



8 a Domain R



b Domain $R^+ \cup \{0\}$



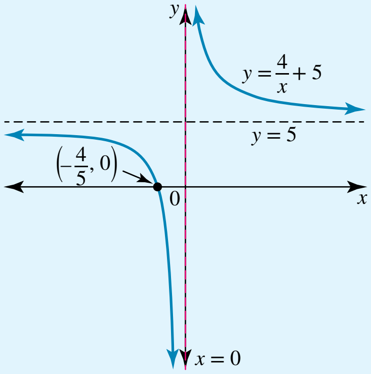
9 a $R \backslash \{-9\}$

b $\left(-\infty, \frac{1}{2}\right]$

c $R \backslash \{-3\}$

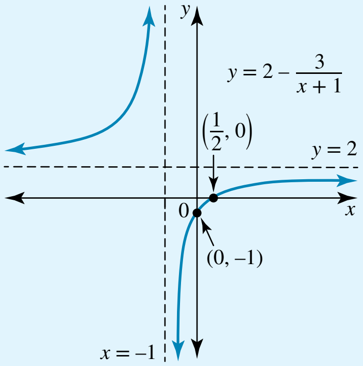
d R

10 a



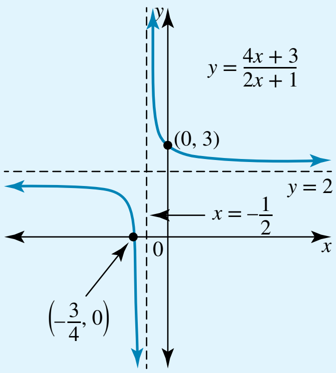
Domain $R \backslash \{0\}$, range $R \backslash \{5\}$

b



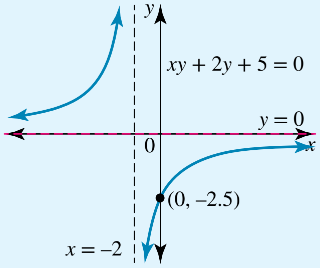
Domain $R \backslash \{-1\}$, range $R \backslash \{2\}$

c



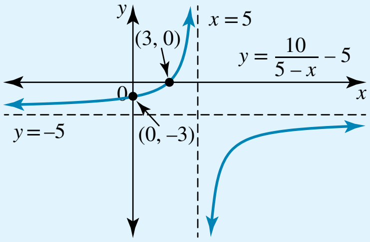
Domain $R \backslash \left\{-\frac{1}{2}\right\}$, range $R \backslash \{2\}$

d



Domain $R \backslash \{-2\}$, range $R \backslash \{0\}$

e

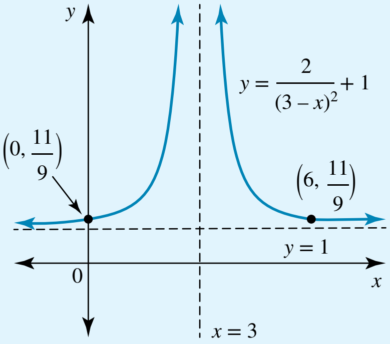


Domain $R \backslash \{5\}$, range $R \backslash \{-5\}$

11 a $y = \dfrac{-2}{x+3} + 6$

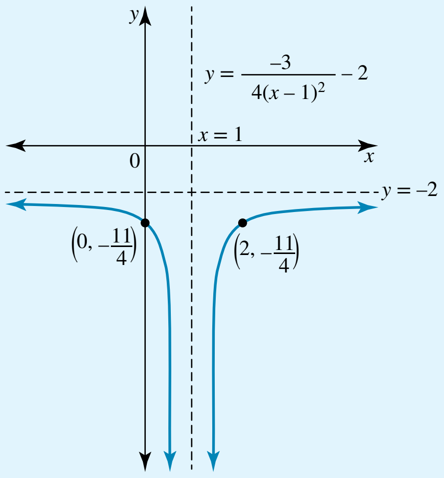
b $y = \dfrac{1}{2(x+2)} - \dfrac{3}{2}$

12 a



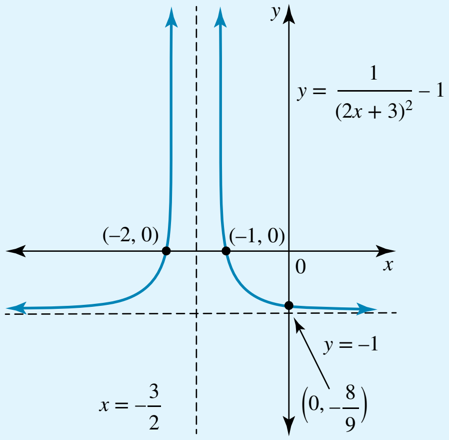
Domain $R \backslash \{3\}$, range $(1, \infty)$

b



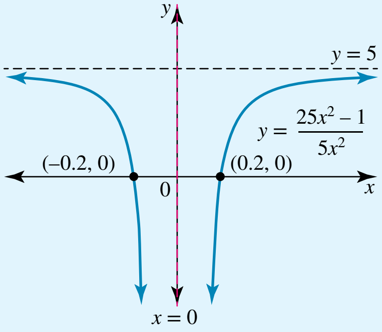
Domain $R\backslash\{1\}$, range $(-\infty, -2)$

c



Domain $R\backslash\left\{-\frac{3}{2}\right\}$, range $(-1, \infty)$

d



Domain $R\backslash\{0\}$, range$(-\infty, 5)$

13 a $y = \frac{-3}{(x-4)^2} + 2$

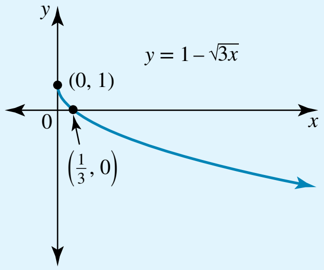
b $y = \frac{108}{(2x-1)^2} - 4,$

$f: R\backslash\left\{\frac{1}{2}\right\} \to R, f(x) = \frac{108}{(2x-1)^2} - 4.$

14 a **i** $y = 2\sqrt{(x-3)} + 2$, domain $[3, \infty)$, range $[2, \infty)$; $y = -2\sqrt{(x-3)} + 2$, domain $[3, \infty)$, range $(-\infty, 2]$.

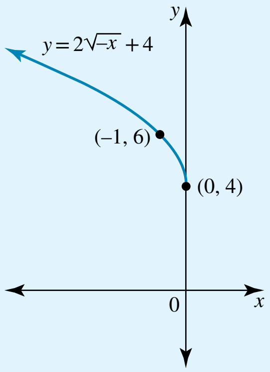
ii $y = \sqrt{-2(x-3)} - 1$, domain $(-\infty, 3]$, range $[-1, \infty)$; $y = -\sqrt{-2(x-3)} - 1$, domain $(-\infty, 3]$, range $(-\infty, -1]$.

b **i**



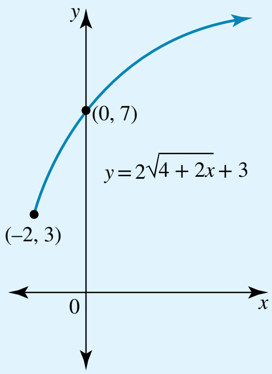
Domain $[0, \infty)$, range $(-\infty, 1]$

ii



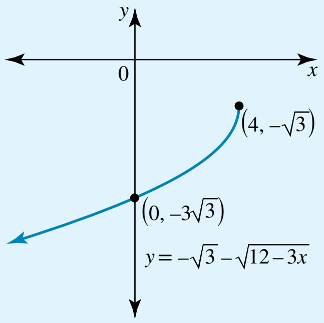
Domain$(-\infty, 0]$, range $[4, \infty)$

iii



Domain $[-2, \infty)$, range $[3, \infty)$

iv

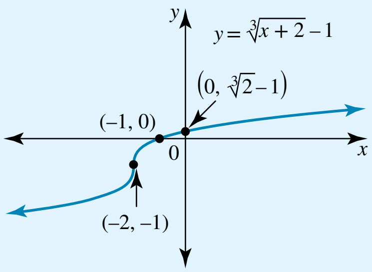


Domain $(-\infty, 4]$, range $(-\infty, -\sqrt{3}]$

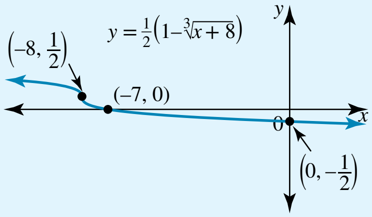
15 a $a = 2,\ b = -5,\ c = -2$

b **i** $a = -2, b = 4, c = -2$ **ii** $y = -\sqrt{-2x+4} + 2$

16 a

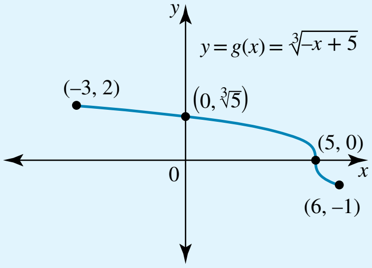


b



Domain R, range R

c



Domain $[-3, 6]$, range $[-1, 2]$

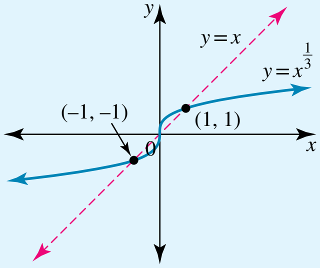
d $y = 2\sqrt[3]{x} - 2$

e $y = -\dfrac{7\sqrt[3]{x+1}}{2} - 2$

f $y = 4\sqrt[3]{(x-2)} - 2,\ (2, -2)$

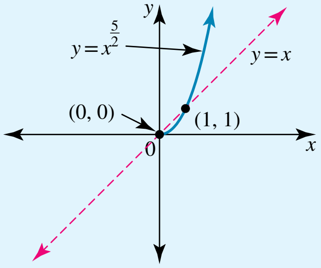
17 a Draw $y = x$ and construct its cube root.

b

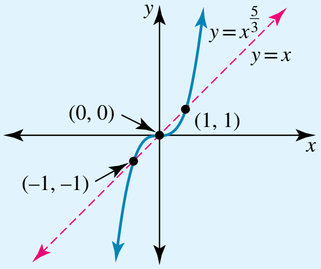


c $\{x : x < -1\} \cup \{x : 0 < x < 1\}$

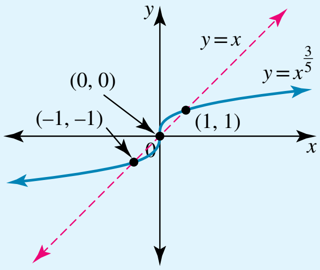
18 a Domain $[0, \infty)$, quadrant 1



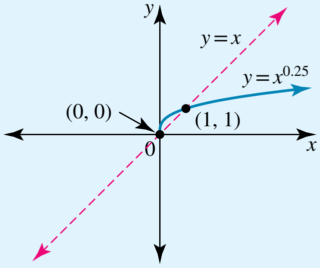
b Domain R, quadrants 1 and 3



c Domain R, quadrants 1 and 3



d Domain $[0, \infty)$, quadrant 1



19

Function	Domain	Range	Asymptotes
$y = \dfrac{1}{x^2 - 4}$	$R\backslash\{\pm 2\}$	$R\backslash\left(-\dfrac{1}{4}, 0\right]$	$x = \pm 2, y = 0$
$y = \dfrac{1}{x^2 + 4}$	R	$\left(0, \dfrac{1}{4}\right]$	$y = 0$
$y = \dfrac{1}{(x-4)^2}$	$R\backslash\{4\}$	$(0, \infty)$	$x = 4, y = 0$

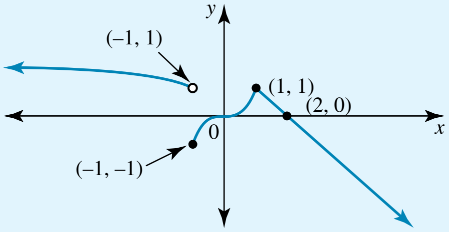
$y = \dfrac{1}{(x-4)^2}$ is a truncus.

20 The maximal domain is $x \in [-3, 2]$.

EXERCISE 2.4

1 a $f(-8) = 2,\ f(-1) = -1,\ f(2) = 0$

b

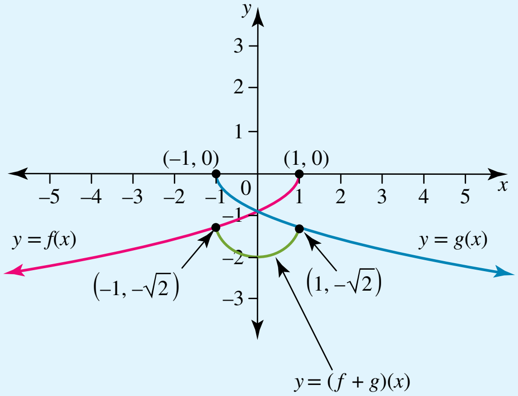


c i $x = -1$

ii Domain R, range R.

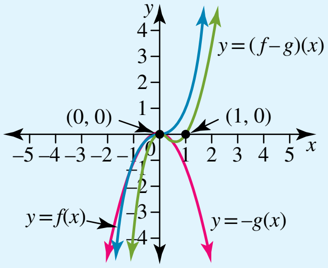
2 $y = \begin{cases} x + 4, & x < 0 \\ 4, & 0 \le x < 4 \\ x, & 4 \le x \le 8 \end{cases}$

3 a $y = -\sqrt{1+x} - \sqrt{1-x}$, domain $[-1, 1]$



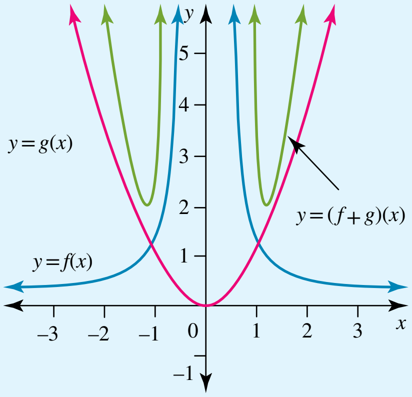
b $y = \sqrt{1 - x^2}$, domain $[-1, 1]$, range $[0, 1]$.

4 $(f - g)(x) = x^3 - x^2$

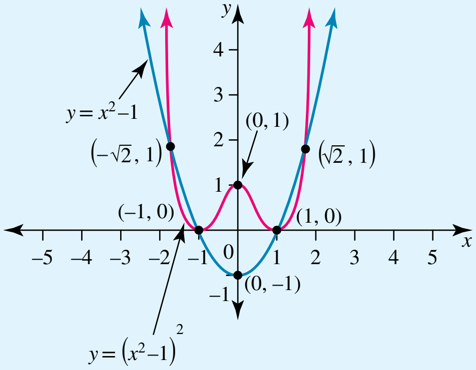


The graphs of f and g intersect when $x = 0, x = 1$, which gives the places where the difference function has x-intercepts.

5

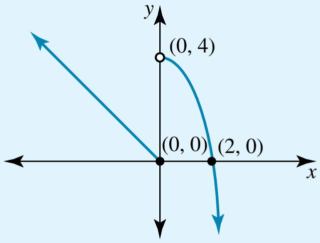


6



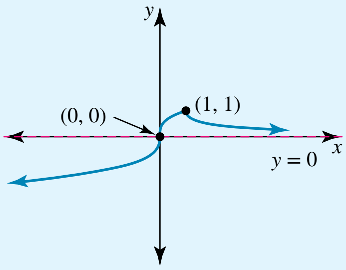
Domain R, range $[0, \infty)$

7 a



Domain R, range R, $x = 0$

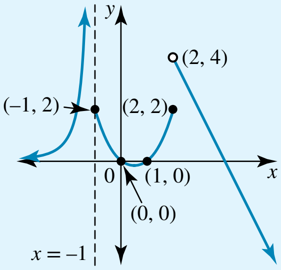
b



Domain R, range $(-\infty, 1]$, no point of discontinuity

8 a i 1 **ii** 2

b



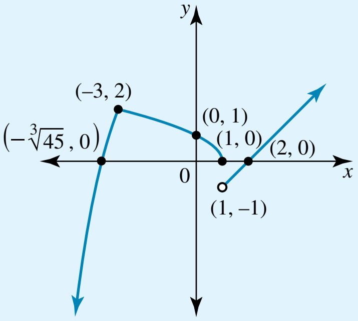
c $R \backslash \{-1, 2\}$

9 a The branch to the left of $x = 1$ has the rule $f(x) = \sqrt{1-x}$, so $f(1) = 0$.

The branch to the right of $x = 1$ has the rule $f(x) = x - 2$, so $f(1) \to -1$ (open circle).

These branches do not join, so the hybrid function is not continuous at $x = 1$.

b

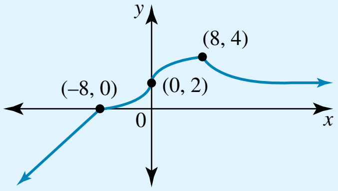


Many-to-one correspondence

c $x = 6$

10 $$y = \begin{cases} \frac{4}{3}(x+3)(x+1), & x < 0 \\ 4, & 0 \le x \le 2 \\ -2x + 8, & x \ge 3 \end{cases}$$

11 a $a = 8, b = 32$

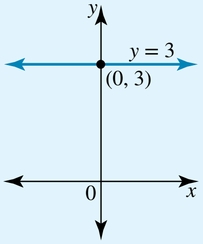


b **i** $k > 4$ **ii** $k = 4$ or $k \leq 0$

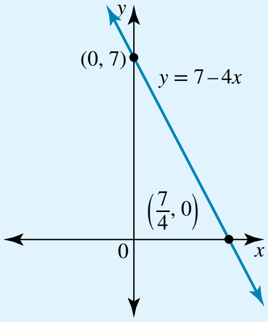
iii $0 < k < 4$

c $\{-1, 32\}$

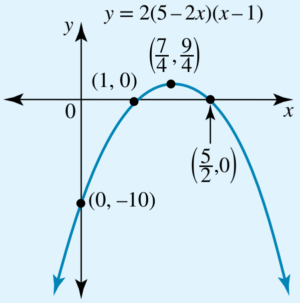
12 a $y = 3$, domain R, range $\{3\}$, horizontal line through $(0, 3)$



b $y = 7 - 4x$, domain R, range R, straight line through $(0, 7)$ and $\left(\frac{7}{4}, 0\right)$



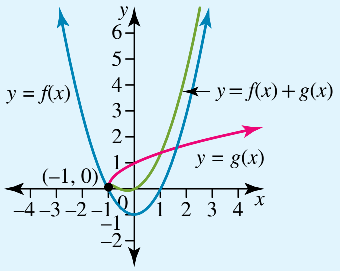
c $y = 2(5 - 2x)(x - 1)$, domain R, range $\left(-\infty, \frac{9}{4}\right]$, concave down parabola with turning point $\left(\frac{7}{4}, \frac{9}{4}\right)$ and passing through $(0, -10)$, $(1, 0)$ and $(2.5, 0)$



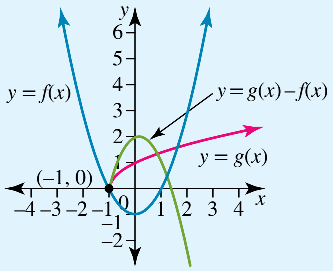
13 a **i** -6 **ii** 189

b $[-1, \infty)$

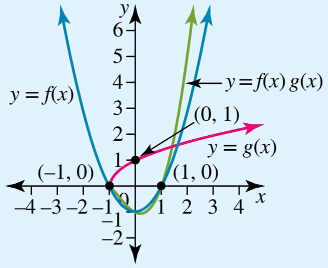
c **i**



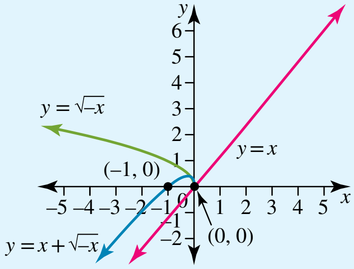
ii



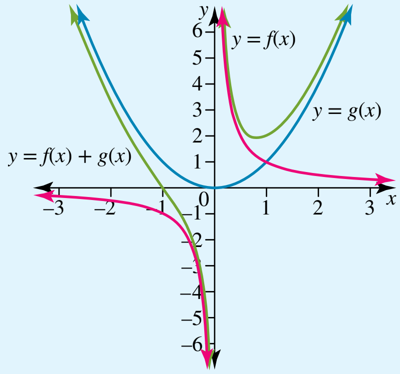
iii



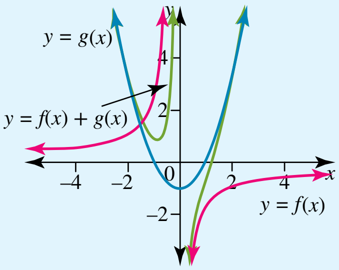
14



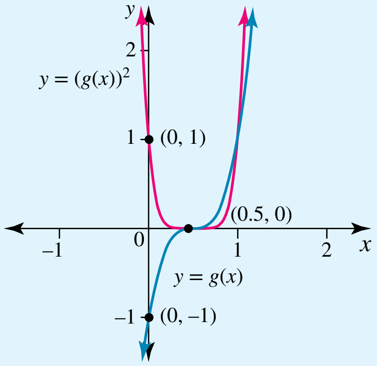
15 a



b

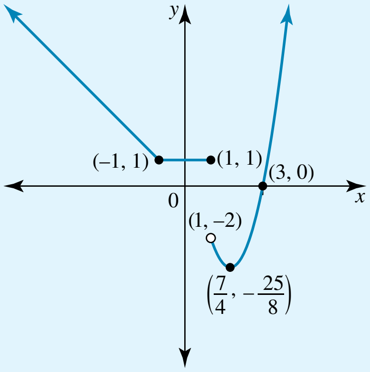


16 a



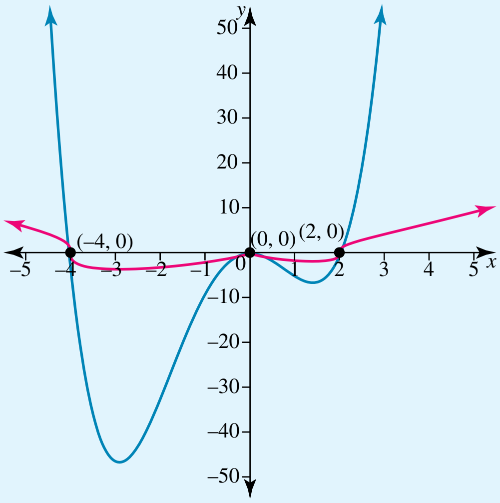
b $(0, 0), (\pm\sqrt{2}, 0), (-1, 1), \left(\frac{1\pm\sqrt{5}}{2}, 1\right)$

17



Range $= \left[-\frac{25}{8}, \infty\right)$

18



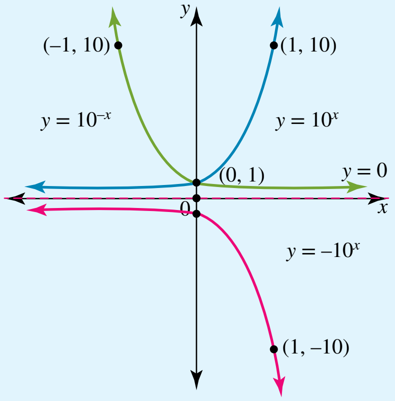
Where the polynomial graph cuts the x-axis, the cube root graph has vertical points of inflection; where the polynomial touches the x-axis, the cube root graph also touches the x-axis but at a sharp point.

Wherever the polynomial graph has the values $y = 0, y = -1$ and $y = 1$, the cube root graph must have the same value. There are 9 points of intersection.

EXERCISE 2.5

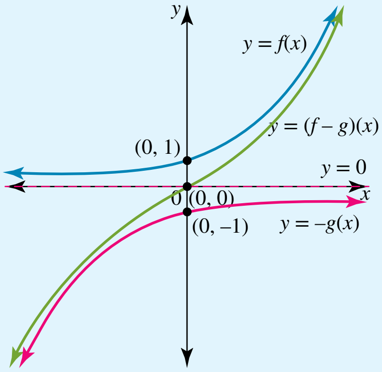
1 a –100

b

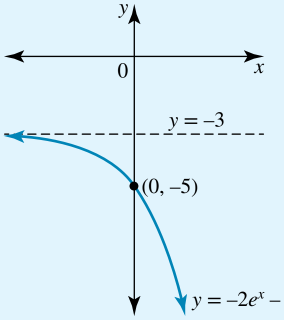


c $y = \left(\frac{1}{10}\right)^x$ or $y = 0.1^x$.

2 $y = 2^x - 2^{-x}$, domain R, range R

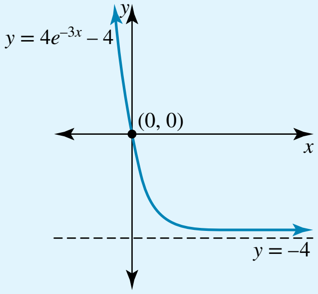


3 a



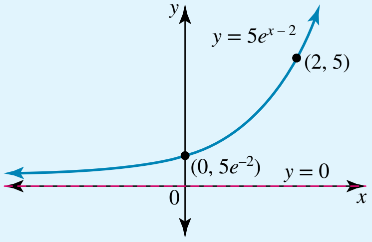
Domain R, range $(-\infty, -3)$

b



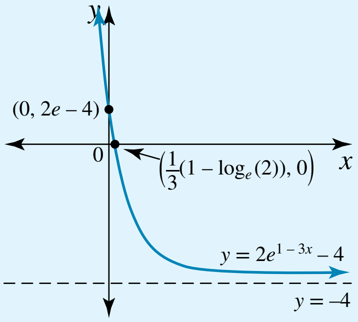
Domain R, range $(-4, \infty)$

c

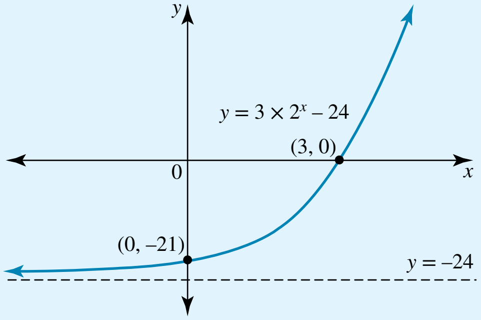


Domain R, range R^+

4 a



b



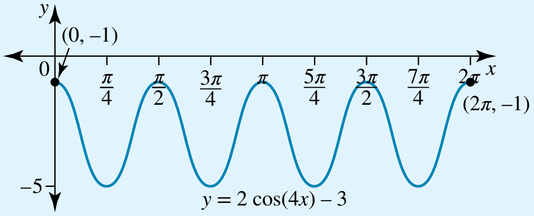
Domain R, range $(-24, \infty)$.

5 a $a = -2, b = 2$

b $y = -2 \times 10^{\frac{x}{4}}$

6 $a = 4$, $k = \log_e(3)$.

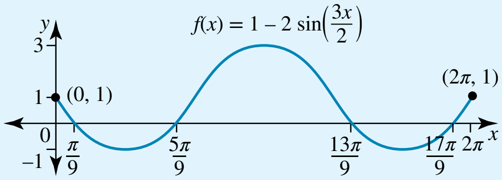
7 a



b Mean position = 5, amplitude = 8, period = 2

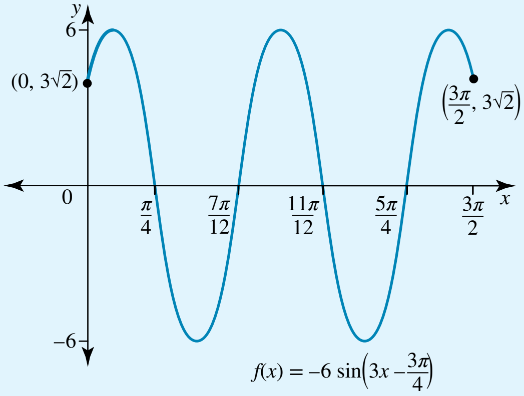
$y = -8\sin(\pi x) + 5$

8



x-intercepts at $x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$; y-intercept = (0, 1)

9 a



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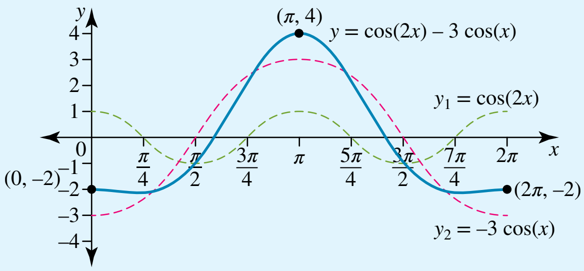

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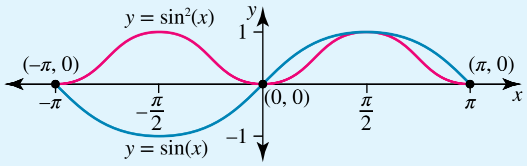
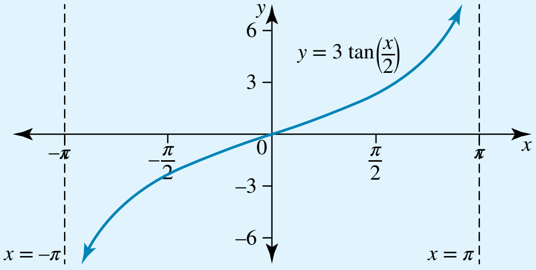
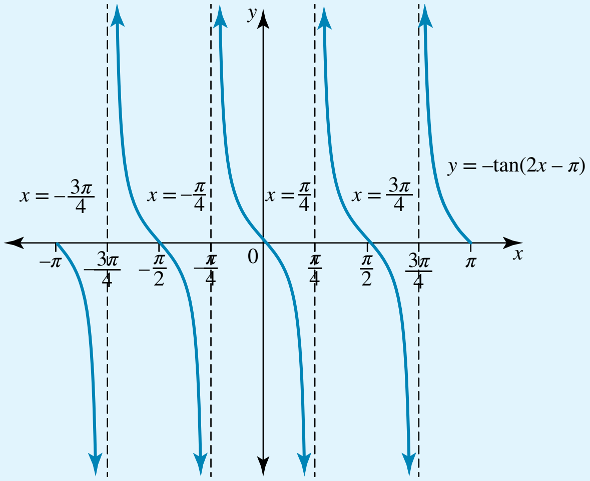


11 a

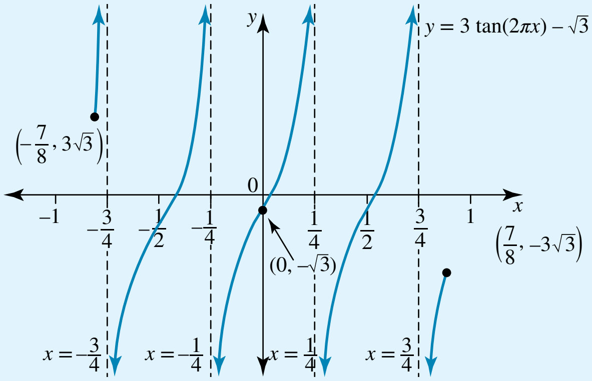


b



12 $y = \frac{1}{2}\tan\left(\frac{3x}{2}\right)$

13



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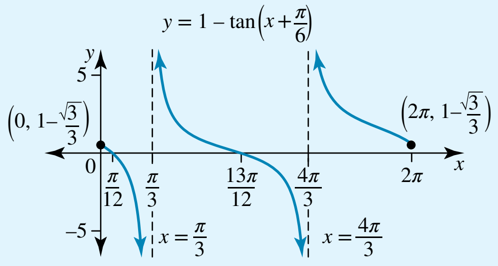
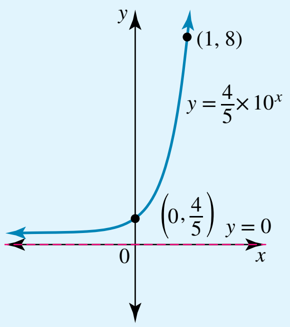
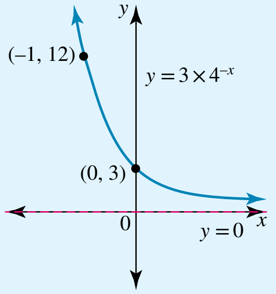


15 a



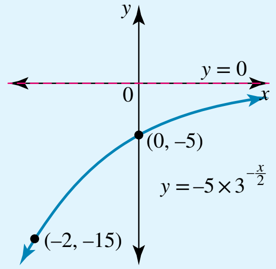
$x \to \infty, y \to \infty$

b



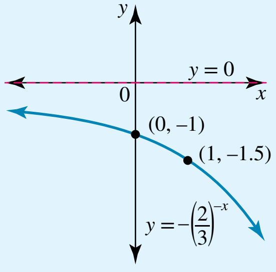
$x \to \infty, y \to 0^+$

c



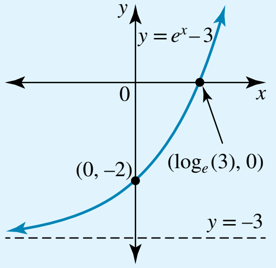
$x \to \infty, y \to 0^-$

d



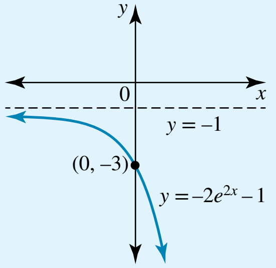
$x \to \infty, y \to -\infty$

16 a



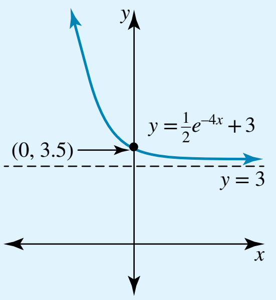
Range $(-3, \infty)$

b



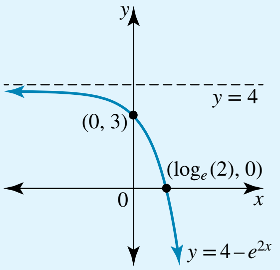
Range $(-\infty, -1)$

c



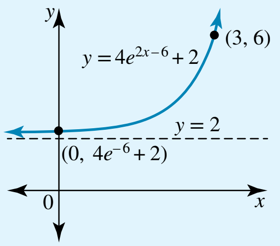
Range $(3, \infty)$

d



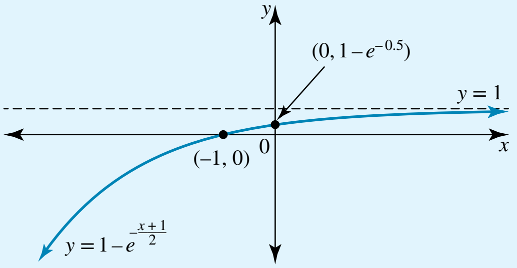
Range $(-\infty, 4)$

e



Range $(2, \infty)$

f



Range $(-\infty, 1)$

17 a $a = -11, b = 11, f: R \to R, f(x) = -11e^x + 11$

b $y = e^{-2x} + 4$

c **i** $b = -1, c = -7$ **ii** $(-7, \infty)$

d **i** $A = 12, B = -2$ **ii** $a = 1$

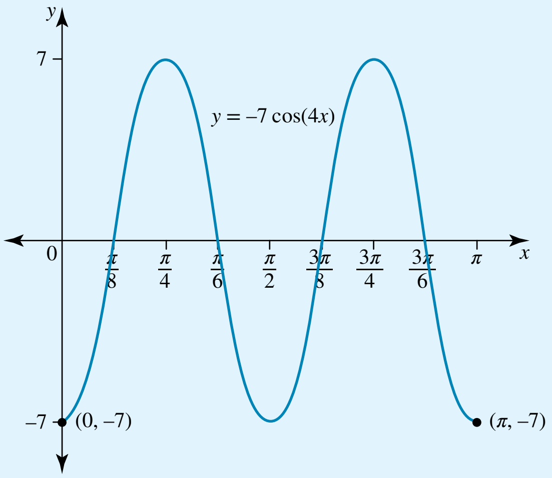
18 a Period $\frac{\pi}{4}$, amplitude 6, range $[-6, 6]$

b Period 8π, amplitude 3, range $[-1, 5]$

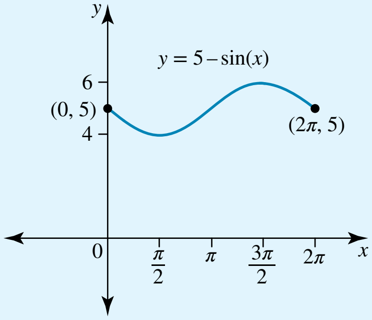
c Period $\frac{2\pi}{3}$, amplitude 1, range $[-1, 1]$

d Period $\frac{1}{3}$, amplitude 6, range $[9, 21]$

19 a



b



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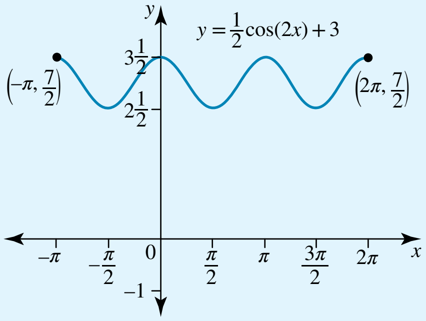

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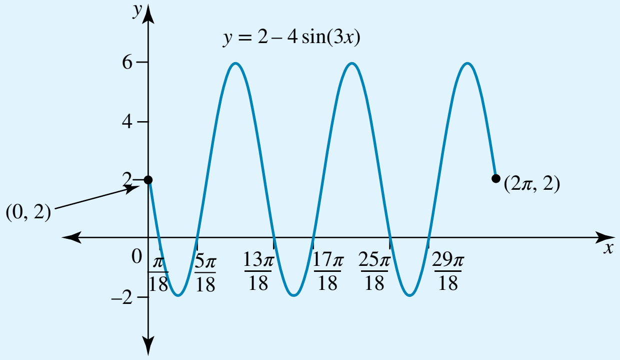
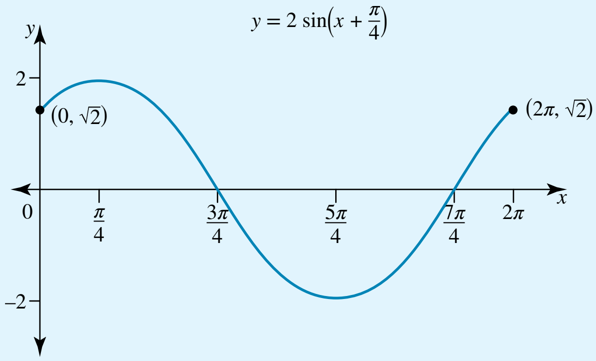
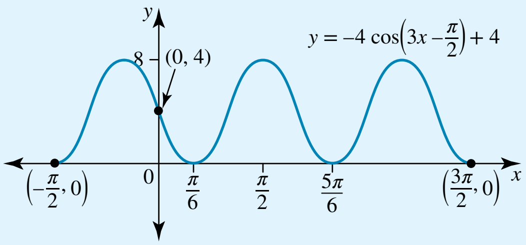


e

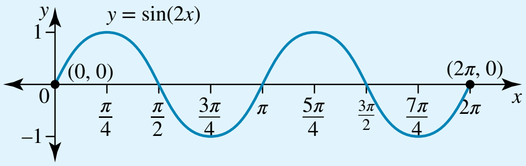


f



20 a **i** $x = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

ii



iii $\left\{x : \frac{2\pi}{3} < x < \frac{5\pi}{6}\right\} \cup \left\{x : \frac{5\pi}{3} < x < \frac{11\pi}{6}\right\}$

b $5, x = \frac{11\pi}{12}$

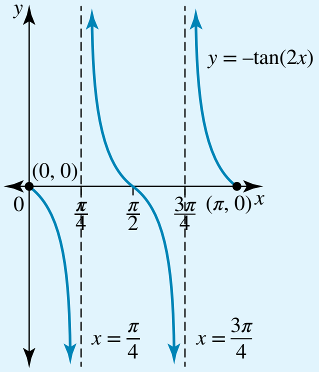
21 a Period $\frac{\pi}{4}$, asymptote $x = \frac{\pi}{8}$

b Period 7π, asymptote $x = \frac{7\pi}{2}$

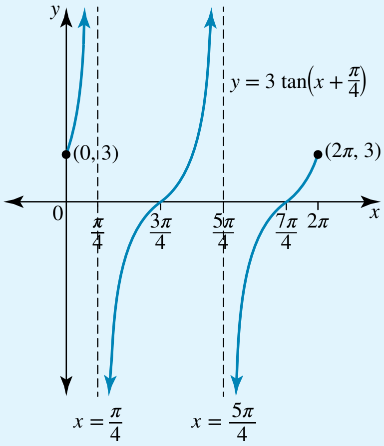
c Period $\frac{5\pi}{4}$, asymptote $x = \frac{5\pi}{8}$

d Period $\frac{1}{6}$, asymptote $x = \frac{1}{12}$

22 a **i**



ii



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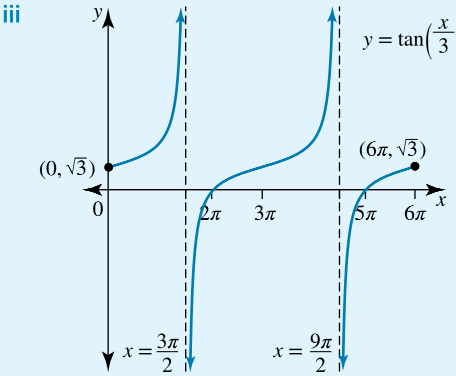

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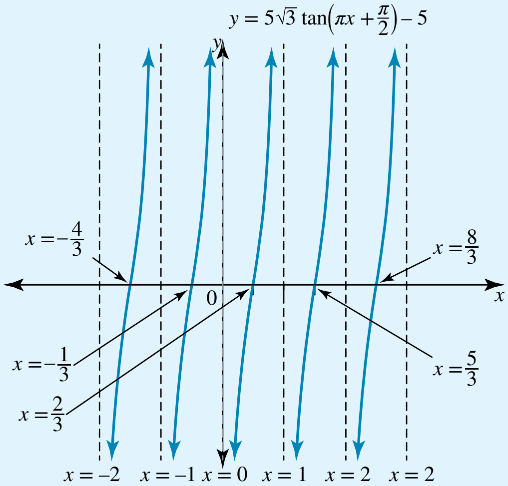


b **i** Mean position unaltered

ii $\frac{\pi}{2}$

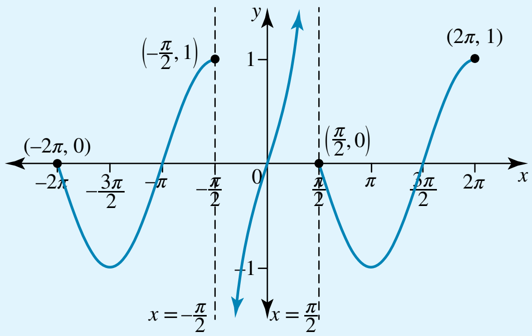
iii $y = -\tan(2x)$ (other answers possible)

23 a **i** $\sqrt{3}$

ii −1

iii 1

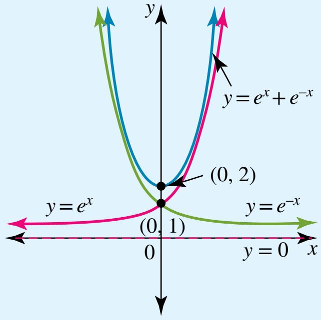
b



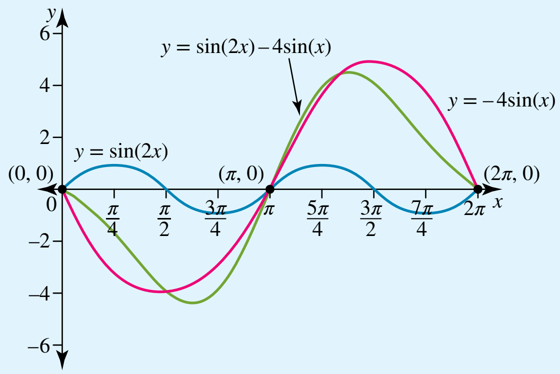
c Not continuous at $x = \pm\frac{\pi}{2}$

d Domain $[-2\pi, 2\pi]$, range R

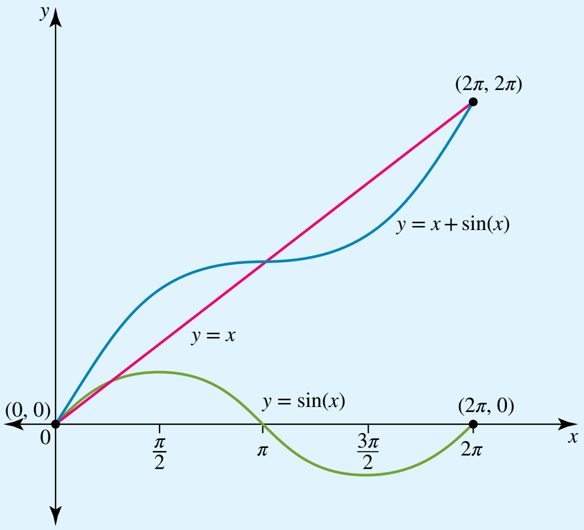
24 a



b



c



25 a Infinite solutions; one is $x = 0$ and the rest have negative values.

b $x \approx -4.721, -1.293$ and $x = 0$

26 a $(\pm 2\pi, 0), (\pm\pi, 0), (0, 0), \left(-\frac{5\pi}{4}, -1\right), \left(-\frac{\pi}{4}, -1\right), \left(\frac{3\pi}{4}, -1\right), \left(\frac{7\pi}{4}, -1\right), \left(-\frac{7\pi}{4}, 1\right), \left(-\frac{3\pi}{4}, 1\right), \left(\frac{\pi}{4}, 1\right), \left(\frac{5\pi}{4}, 1\right).$

b $x = n\pi, n \in Z$ or $x = (2n + 1)\frac{\pi}{4}, n \in Z$

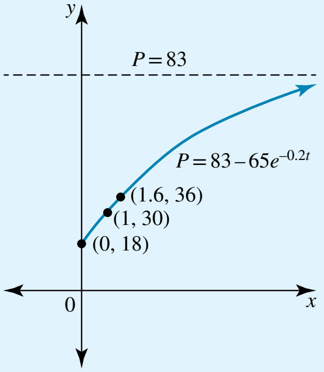
EXERCISE 2.6

1 a 18

b 12

c 1.62 months

d



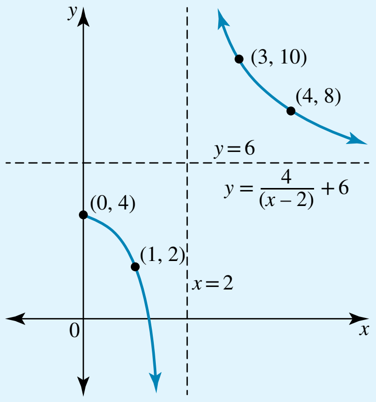
e The population cannot exceed 83.

2 a The data points increase and decrease, so they cannot be modelled by a one-to-one function. Neither a linear model nor an exponential model is possible.

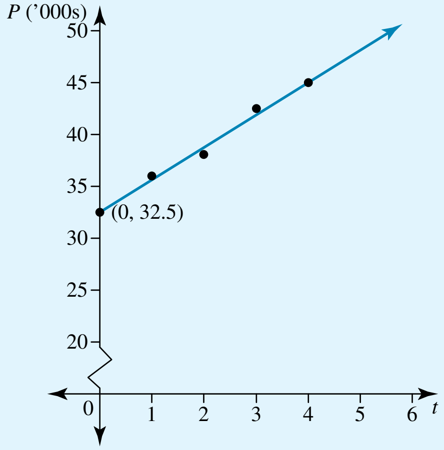
The data is not oscillating, however, so it is unlikely to be trigonometric. The jump between $x = 1$ and $x = 3$ is a concern, but the data could be modelled by a polynomial such as a cubic with a turning point between $x = 1$ and $x = 3$. However, $y = x^n$ requires the point (0, 0) to be on it and that is not true for the data given.

b i $y = \dfrac{4}{x - 2} + 6$

ii



3 a



The data appears to be linear.

b $P = 3.125t + 32.5$

c 32 500 bees

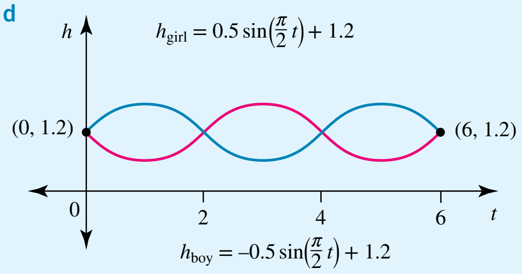
d 3.125 thousand per month

4 a $a = 290, r = 1.5$ **b** \$290

c \$3303

5 a $a = 0.5, n = \dfrac{\pi}{2}, k = 1.2$

b, d



c $\dfrac{8}{3}$ seconds

d $h_{\text{boy}} = -0.5 \sin\left(\dfrac{\pi}{2} t\right) + 1.2$

6 a $a = 2$

b (2, 2)

c $y = \frac{2}{9}(x - 5)^2, 2 \le x \le 9$

d $y = \begin{cases} 2x + 2, & -1 < x < 0 \\ 2, & 0 \le x \le 2 \\ \frac{2}{9}(x - 5)^2, & 2 < x \le 9 \end{cases}$

e $x = 5 \pm \dfrac{3\sqrt{3}}{2}$ or $x = -\dfrac{1}{4}$

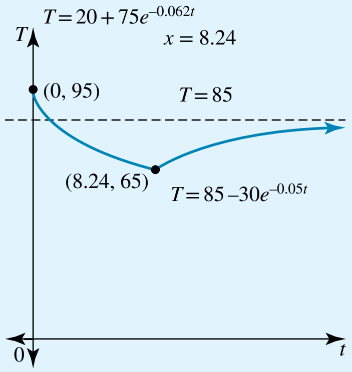
7 a 95°C

b The temperature approaches 20°C.

c 8.24 minutes

d d $A = 85, B \approx -30$

e

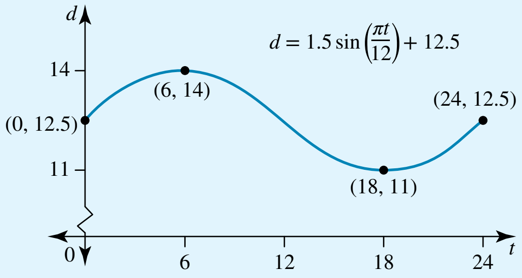


8 a 12.5 metres

b 24 hours

c Maximum is 14 metres, minimum is 11 metres

d



e $h = \dfrac{3\sqrt{3} + 50}{4} \approx 13.8$

f 3:30 pm the following day

9 a 7 days

b As $t \to \infty, N \to \infty$.

c i $N = 22 \times 2^t, t = 0$

$$\begin{aligned} N &= 22 \times 2^0 \\ &= 22 \times 1 \\ &= 22 \end{aligned}$$

$$\begin{aligned} N &= \frac{66}{1 + 2e^{-0.2t}}, t = 0 \\ &= \frac{66}{1 + 2e^{-0.2 \times 0}} \\ &= \frac{66}{3} \\ &= 22 \end{aligned}$$

Initially there are 22 bacteria in each model.

ii The population will never exceed 66.

10 a The garden area is the area of the entire square minus the area of the two right-angled triangles.

$$A = 40 \times 40 - \frac{1}{2} \times x \times x - \frac{1}{2} \times (40 - x) \times 40$$
$$= 1600 - \frac{1}{2}x^2 - 20(40 - x)$$
$$= 1600 - \frac{1}{2}x^2 - 800 + 20x$$
$$= -\frac{1}{2}x^2 + 20x + 800$$

b $0 < x < 40$

c **i** 20 **ii** 1000 m^2

11 a $y = 2x^3(x - \sqrt{5})(x + \sqrt{5})$

b $y = 2x^3(x - \sqrt{5})(x + \sqrt{5})$

$$= 2x^3(x^2 - 5)$$
$$= 2x^5 - 10x^3$$

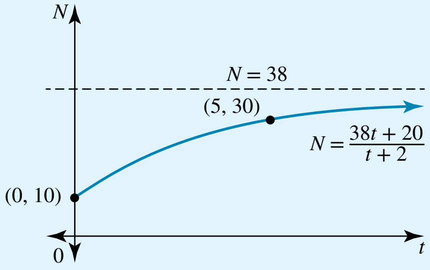
c **i** Horizontal translation of $\sqrt{3}$ units to the right and vertical translation of $12\sqrt{3} + 1$ units upward

ii $(24\sqrt{3} + 1) \approx 42.6$ metres

iii B $(2\sqrt{3}, 1)$, C $(\sqrt{3}, 12\sqrt{3} + 1)$

12 a $a = 38, b = 20$

b

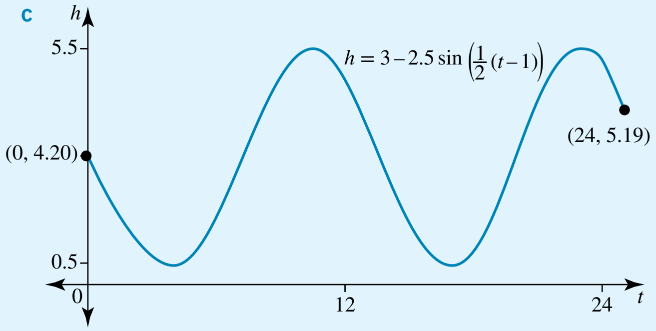


c The population will never exceed 38.

13 a 3.619 m below the jetty

b 5.5 m and 0.5 m

c



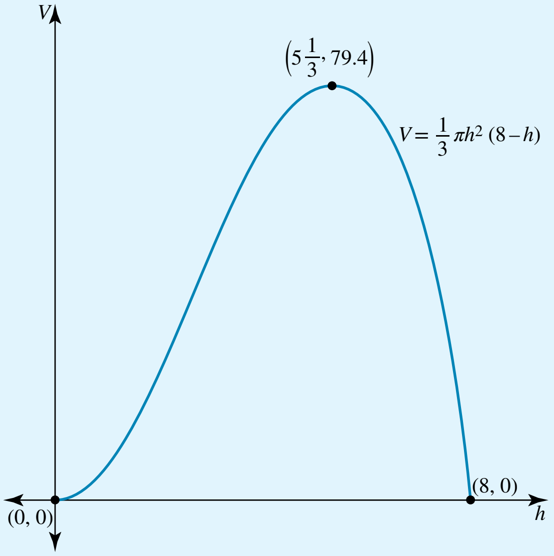
First maximum at $t \approx 10.42$, first minimum at $t \approx 4.14$

d 5 m of extra rope

14 a $r = \sqrt{8h - h^2}$

b $V = \frac{1}{3}\pi h^2(8 - h), 0 < h < 8$

c



d 79 cm^3

3

Composite functions, transformations and inverses

3.1 Kick off with CAS

Inverses

Part 1

1 On a calculation screen in CAS, define the function $f(x) = 2x - 5$.

2 To find the inverse, solve $f(y) = x$ for y.

3 Define the inverse as $g(x)$.

4 Determine $f(g(x))$ and $g(f(x))$. What do you notice?

5 Repeat steps 1–4 for the following functions.

  a $f(x) = \sqrt{x - 2}$

  b $f(x) = -(x - 1)^3$

6 Is the result in step 4 the same? What can you conclude about the relationship between a function and its inverse?

Part 2

7 Open a graph screen and sketch $f(x) = 5 - 3x$.

8 Using the features of CAS, sketch the inverse on the same set of axes.

9 Repeat steps 7 and 8 for the following functions.

  a $f(x) = \dfrac{1}{x - 1}$

  b $f(x) = (x + 1)^3$

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

3.2 Composite functions and functional equations

Composite functions

studyon

Units 3 & 4

AOS 1

Topic 4

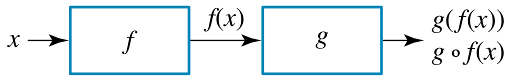
Concept 4

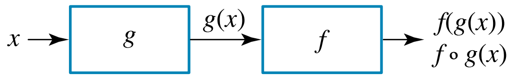
Composite functions

Concept summary

Practice questions

Composition of two functions occurs when the output of one function becomes the input for a second function.





Suppose $f(x) = x^2$ and $g(x) = 3x - 1$.

The composite function $f \circ g(x) = f(g(x))$ (pronounced 'f of g of x') involves expressing $g(x)$ in terms of $f(x)$.

$$\begin{aligned} f \circ g(x) = f(g(x)) &= f(3x - 1) \\ &= (3x - 1)^2 \\ &= 9x^2 - 6x + 1 \end{aligned}$$

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Interactivity

Composite functions

int-6417

The domain of $f(g(x)) = \text{dom } g(x) = R$.

On the other hand, the composite function $g \circ f(x) = g(f(x))$ involves expressing $f(x)$ in terms of $g(x)$.

$$\begin{aligned} g \circ f(x) = g(f(x)) &= g(x^2) \\ &= 3(x^2) - 1 \\ &= 3x^2 - 1 \end{aligned}$$

The domain of $g(f(x)) = \text{dom } f(x) = R$.

Note that the order of the composition will affect the result of the composition. This means that $f(g(x))$ will generally not equal $g(f(x))$.

Existence of composite functions

When two functions are composed, the output of the first function (the inner function) becomes the input for the second function (the outer function). This means that if the composition is possible, the range of the inner function must be a subset of or equal to the domain of the outer function. It may be necessary to restrict the domain of the inner function to ensure that its range lies completely within the domain of the outer function. The domain of the inner function is always the domain of the composite function.

For $f(g(x))$ to exist, ran $g \subseteq$ dom f. The domain of $g(x) =$ dom $f(g(x))$.

For $g(f(x))$ to exist, ran $f \subseteq$ dom g. The domain of $f(x) =$ dom $g(f(x))$.

Looking again at the functions $f(x) = x^2$ and $g(x) = 3x - 1$, we can investigate why they exist by listing the domains and ranges.

Function	Domain	Range
$f(x)$	R	$[0, \infty)$
$g(x)$	R	R

For $f(g(x))$, the range of g is R, which is equal to the domain of f, R.

$$\begin{gathered} R \subseteq R \\ \text{ran } g \subseteq \text{dom } f \\ \text{Therefore, } f(g(x)) \text{ exists.} \end{gathered}$$

For $g(f(x))$, the range of f is $[0, \infty)$, which is a subset of the domain of g, R.

$$\begin{gathered} [0, \infty) \subseteq R \\ \text{ran } f \subseteq \text{dom } g \\ \text{Therefore, } g(f(x)) \text{ exists.} \end{gathered}$$

WORKED EXAMPLE 1

If $f(x) = e^x$ and $g(x) = \sqrt{x-2}$, investigate whether the composite functions $f(g(x))$ and $g(f(x))$ exist. If they do, form the rule for the composite function and state the domain.

THINK

1 Construct a table to investigate the domains and ranges of the two functions.

WRITE

$f(x) = e^x$ and $g(x) = \sqrt{x-2}$

Function	Domain	Range
$f(x)$	R	$(0, \infty)$
$g(x)$	$[2, \infty)$	$[0, \infty)$

2 Investigate whether $f(g(x))$ exists by comparing the range of g to the domain of f.

$[0, \infty) \subseteq R$

ran $g \subseteq$ dom f

Therefore $f(g(x))$ exists.

3 Form the rule for $f(g(x))$ and state the domain.

$$\begin{aligned} f(g(x)) &= f(\sqrt{x-2}) \\ &= e^{\sqrt{x-2}} \end{aligned}$$

Domain $= [2, \infty)$

4 Investigate whether $g(f(x))$ exists.

$(0, \infty) \not\subseteq [2, \infty)$

ran $f \not\subseteq$ dom g

Therefore $g(f(x))$ does not exist.

WORKED EXAMPLE 2

For the functions $f(x) = \sqrt{4-x}$ and $g(x) = x - 1$:

a **state why $f(g(x))$ is not defined**

b **restrict the domain of $g(x)$ to form a new function, $h(x)$, such that $f(h(x))$ is defined**

c **find $f(h(x))$.**

THINK

a 1 Construct a table to investigate the domains and ranges of the two functions.

WRITE

a $f(x) = \sqrt{4-x}$ and $g(x) = x - 1$

Function	Domain	Range
$f(x)$	$(-\infty, 4]$	$[0, \infty)$
$g(x)$	R	R

2 For $f(g(x))$ to be defined, the range of g must be a subset of the domain of f.

$R \not\subseteq (-\infty, 4]$

ran $g \not\subseteq$ dom f

$\therefore f(g(x))$ is not defined.

b 1 For $f(g(x))$ to be defined, the range g must be a subset of or equal to the domain of f. The maximal range of g will be when ran $g =$ dom f.

b We want ran $g =$ dom $f = (-\infty, 4]$.

2 Use the restriction of the range of g to solve for the new domain of g.

$$x - 1 \leq 4$$
$$x \leq 5$$

3 Define $h(x)$.

$$h(x) = x - 1, \quad x \in (-\infty, 5]$$

c Find $f(h(x))$ by substituting $h(x)$ into $f(x)$. Make sure the domain is stated.

c $$f(h(x)) = \sqrt{4 - (x - 1)}$$
$$= \sqrt{5 - x}, \quad x \in (-\infty, 5]$$

Functional equations

Sometimes you may be asked to investigate whether certain functions satisfy rules such as $f(x + y) = f(x) + f(y)$ or $f(x - y) = \dfrac{f(x)}{f(y)}$. In these cases the function will be defined for you. Such equations are called functional equations.

Consider $f(x) = e^x$. To investigate whether $f(x + y) = f(x) \times f(y)$ is true we set up a LHS–RHS investigation.

LHS: $f(x + y)$
$$= e^{(x+y)}$$
RHS: $f(x) \times f(y)$
$$= e^x \times e^y$$
$$= e^{x+y}$$

LHS = RHS, so $f(x) = e^x$ satisfies $f(x + y) = f(x) \times f(y)$.

WORKED EXAMPLE 3 Investigate whether $f(x) = \dfrac{1}{x}$ satisfies the equation $\dfrac{f(x) + f(y)}{f(xy)} = x + y$.

THINK	WRITE
1 Simplify the left-hand side of the equation.	LHS: $\dfrac{f(x) + f(y)}{f(xy)}$ where $f(x) = \dfrac{1}{x}$ $= \left(\dfrac{1}{x} + \dfrac{1}{y}\right) \div \dfrac{1}{xy}$ $= \dfrac{y + x}{xy} \times \dfrac{xy}{1}$ $= x + y$
2 Simplify the right-hand side of the equation.	RHS: $x + y$
3 Answer the question.	LHS = RHS, so $f(x) = \dfrac{1}{x}$ satisfies the equation $\dfrac{f(x) + f(y)}{f(xy)} = x + y$.

EXERCISE 3.2 Composite functions and functional equations

PRACTISE

Work without CAS

1 **WE1** If $f(x) = (x - 1)(x + 3)$ and $g(x) = x^2$, investigate whether the composite functions $f(g(x))$ and $g(f(x))$ exist. If they do, form the rule for the composite function and state the domain.

2 If $f(x) = 2x - 1$ and $g(x) = \frac{1}{x - 2}$, investigate whether the composite functions $f(g(x))$ and $g(f(x))$ exist, and if they do, form the rule for the composite function.

3 **WE2** For the functions, $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 5$:

a state why $f(g(x))$ is not defined

b restrict the domain of g to form a new function, $h(x)$, such that $f(h(x))$ is defined

c find $f(h(x))$.

4 For the functions, $f(x) = x^2$ and $g(x) = \frac{1}{x - 4}$:

a state why $g(f(x))$ is not defined

b restrict the domain of f to form a new function, $h(x)$, such that $g(h(x))$ is defined

c find $g(h(x))$.

5 **WE3** If $f(x) = -\frac{3}{x}$, show that this equation satisfies $\frac{f(x) - f(y)}{f(xy)} = y - x$.

6 If $f(x) = e^{2x}$, state the functional equations for $f(x + y)$ and $f(x - y)$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 For the functions given, determine if the following compositions are defined or undefined. If the composite function exists, identify its domain.

$f(x) = x^2 + 1,\ g(x) = \sqrt{x},\ h(x) = \frac{1}{x}$

a $f \circ g(x)$ **b** $g(f(x))$ **c** $h(g(x))$ **d** $h \circ f(x)$

8 For the functions given, determine if the following compositions are defined or undefined. If the composite function exists, state the rule and its domain.

$f(x) = x^2,\ g(x) = \sqrt{x},\ h(x) = -\frac{1}{x}$

a $f \circ g(x)$ **b** $g(f(x))$ **c** $h(f(x))$ **d** $g(h(x))$

9 The functions f and g are defined by $f: R \rightarrow R,\ f(x) = x^2 + 1$ and $g: [-2, \infty) \rightarrow R,\ g(x) = \sqrt{x + 2}$. Show that $f(g(x))$ exists and find the rule for $f(g(x))$, stating its domain and range.

10 If $f: (0, \infty) \rightarrow R,\ f(x) = \frac{1}{x}$ and $g: R\backslash\{0\} \rightarrow R,\ g(x) = \frac{1}{x^2}$:

a prove that $g(f(x))$ exists

b find $g(f(x))$ and state its domain and range

c sketch the graph of $y = g(f(x))$.

11 If $g(x) = \frac{1}{(x - 3)^2} - 2$ and $f(x) = \sqrt{x}$:

a prove that $f(g(x))$ is not defined

b restrict the domain of g to obtain a function $g_1(x)$ such that $f(g_1(x))$ exists.

12 For the equations $f: (-\infty, 2] \rightarrow R,\ f(x) = \sqrt{2 - x}$ and

$g: R \rightarrow R,\ g(x) = -\frac{1}{x - 1} + 2$:

a prove that $g(f(x))$ is not defined

b restrict the domain of f to obtain a function $f_1(x)$ such that $g(f_1(x))$ exists

c find $g(f_1(x))$.

13 If $f(x) = 5^x$, show that:

a $f(x + y) = f(x) \times f(y)$ **b** $f(x - y) = \frac{f(x)}{f(y)}$.

14 Investigate whether $f(x) = x^3$ satisfies the equation $f(xy) = f(x) \times f(y)$.

15 **a** For $h(x) = 3x + 1$, show that $h(x + y)$ can be written in the form $h(x) + h(y) + c$, and find the value of c.

b Show that if $h(x) = \frac{1}{x^3}$, then $h(x) + h(y) = (x^3 + y^3)h(xy)$.

16 Determine if the functions **a**–**d** satisfy the equations **A**–**D**.

a $f(x) = x^2$ **b** $f(x) = \sqrt{x}$ **c** $f(x) = \frac{1}{x}$ **d** $f(x) = e^x$

A $f(xy) = f(x)f(y)$ **B** $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ **C** $f(-x) = f(-x)$ **D** $f(x + y) = f(x)f(y)$

MASTER

17 For the equations $f: [4, \infty) \to R,\ f(x) = \sqrt{x - 4}$ and $g: R \to R,\ g(x) = x^2 - 2$:

a prove that $g(f(x))$ is defined

b find the rule for $g(f(x))$ and state the domain

c sketch the graph of $y = g(f(x))$

d prove that $f(g(x))$ is not defined

e restrict the domain of g to obtain a function $g_1(x)$ such that $f(g_1(x))$ exists

f find $f(g_1(x))$.

18 If $f: [1, \infty) \to R,\ f(x) = -\sqrt{x} + k$ and $g: (-\infty, 2] \to R,\ g(x) = x^2 + k$, where k is a positive constant, find the value(s) for k such that both $f(g(x))$ and $g(f(x))$ are defined.

3.3 Transformations

Dilations, reflections and translations

There are three commonly used transformations.

- **Dilations**:
  The point (1, 2) when dilated by factor 2 parallel to the y-axis, or from the x-axis, becomes the point (1, 4).
  The point (2, 7) when dilated by factor $\frac{1}{2}$ parallel to the x-axis, or from the y-axis, becomes the point (1, 7).
- **Reflections**:
  When the point (1, 2) is reflected in the x-axis it becomes the point (1, –2).
  When the point (1, 2) is reflected in the y-axis it becomes the point (–1, 2).
- **Translations**:
  The point (2, 7) when translated 2 units in the positive x-direction becomes (4, 7).
  The point (2, 7) when translated 4 units in the negative y-direction becomes (2, 3).

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Units 3 & 4
AOS 1
Topic 5
Concept 1

Transformations from $y = f(x)$ to $y = af(n(x + b)) + c$
Concept summary
Practice questions

The general rule for transformations

When a function of the form $y = f(x)$ has a number of different transformations applied to it, the general equation becomes

$$y = af[n(x - h)] + k.$$

The following transformations have been applied to $y = f(x)$:

- It has been dilated by a factor of $|a|$ parallel to the y-axis or from the x-axis. Each y-value has been multiplied by $|a|$, so each point is now $(x, |a|y)$.
- It has been dilated by a factor of $\frac{1}{|n|}$ parallel to the x-axis or from the y-axis. Each x-value has been multiplied by $\frac{1}{|n|}$, so each point is now $\left(\frac{x}{|n|}, y\right)$.
- If a is negative, the graph has been reflected in the x-axis. Each y-value has changed sign, so each point is now $(x, -y)$.

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Transformations of functions
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- If n is negative, the graph has been reflected in the y-axis.
  Each x-value has changed sign, so each point is now $(-x, y)$.
- It has been translated h units parallel to the x-axis
  Each x-value has increased by h, so each point is now $(x + h, y)$.
- It has been translated k units parallel to the y-axis
  Each y-value has increased by k, so each point is now $(x, y + k)$

As a general rule, transformations should be read from left to right, as the order is important. Sometimes there may be more than one way to describe the order of transformations, but reading from left to right is a consistent and safe approach.

WORKED EXAMPLE 4

i Describe the transformation that has been applied to the graph of $y = x^2$ in each of the following examples. Sketch both graphs on the one set of axes.

ii Find the image of the point (2, 4) after it has undergone each of the transformations.

a $y = 2x^2$ **b** $y = (2x)^2$ **c** $y = -x^2$

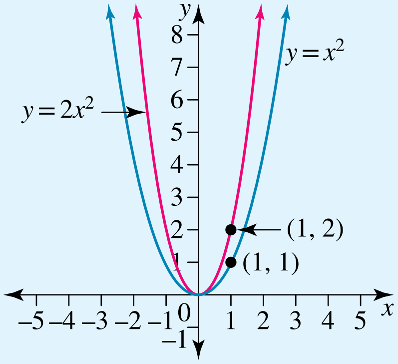
d $y = (x + 1)^2$ **e** $y = (x - 2)^2$ **f** $y = x^2 - 2$

THINK

a i Specify the transformation that has been applied to $y = x^2$, then sketch both graphs on the one set of axes.

WRITE/DRAW

a $y = x^2$ has been dilated by factor 2 parallel to the y-axis or from the x-axis.

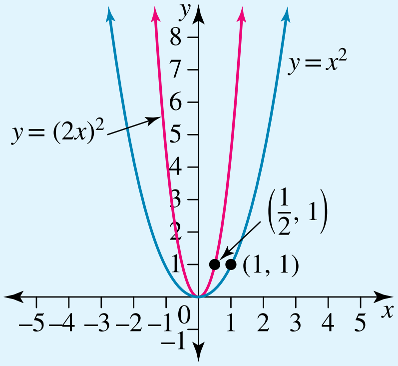


ii Each y-value is doubled for its corresponding x-value.

$(2, 4) \rightarrow (2, 8)$

b i Specify the transformation that has been applied to $y = x^2$, then sketch both graphs on the one set of axes.

b $y = x^2$ has been dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis.

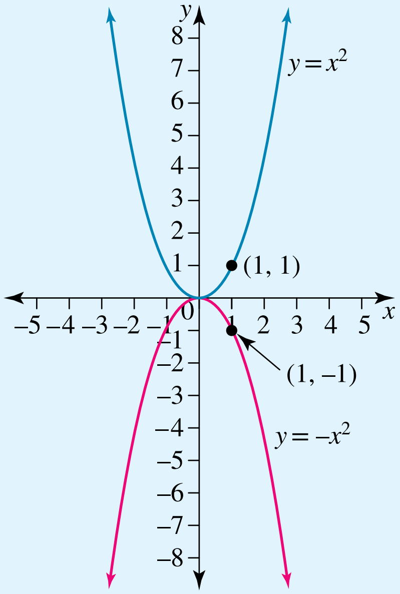


ii Each x-value is halved for its corresponding y-value.

$(2, 4) \rightarrow (1, 4)$

c **i** Specify the transformation that has been applied to $y = x^2$, then sketch both graphs on the one set of axes.

c $y = x^2$ has been reflected in the x-axis.

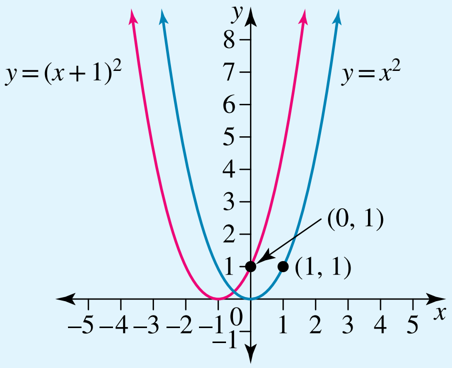


ii All y-values change sign.

$(2, 4) \rightarrow (2, -4)$

d **i** Specify the transformation that has been applied to $y = x^2$, then sketch both graphs on the one set of axes.

d $y = x^2$ has been translated 1 unit to the left.

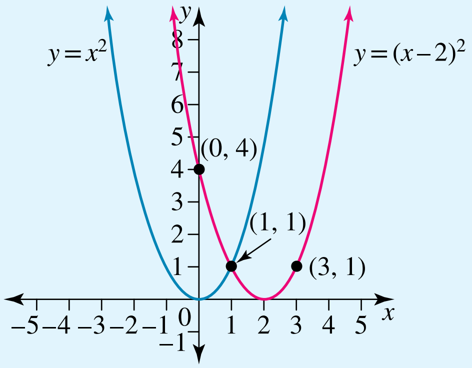


ii All x-values subtract 1 unit.

$(2, 4) \rightarrow (1, 4)$

e **i** Specify the transformation that has been applied to $y = x^2$, then sketch both graphs on the one set of axes.

e $y = x^2$ has been translated 2 units to the right.

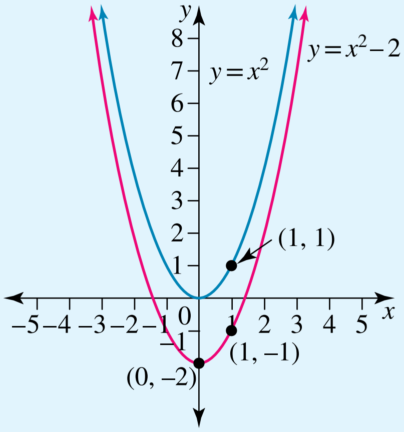


ii All x-values gain 2 units.

$(2, 4) \rightarrow (4, 4)$

f **i** Specify the transformation that has been applied to $y = x^2$, then sketch both graphs on the one set of axes.

f $y = x^2$ has been translated down 2 units.



ii All y-values subtract 2 units.

$(2, 4) \rightarrow (2, 2)$

WORKED EXAMPLE 5

The graph $y = x^2$ is transformed so that its equation becomes $y = \frac{1}{2}(2x + 3)^2 - 4$. Define the transformations which have been applied to $y = x^2$.

THINK	**WRITE**
1 Rewrite the equation with x by itself.	$y = \frac{1}{2}\left(2\left(x + \frac{3}{2}\right)\right)^2 - 4$
2 Define the transformations, reading from left to right. *Note:* There are other ways of writing the order of the transformations. However, the best method is to read the transformations from right to left.	$y = x^2$ has been: 1. dilated by factor $\frac{1}{2}$ parallel to the y-axis or from the x-axis 2. dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis 3. translated $\frac{3}{2}$ units to the left or $\frac{3}{2}$ units in the negative x direction 4. translated 4 units down or 4 units in the negative y direction.

Sometimes you may be asked to build up an equation from a series of transformations. In these cases, you must apply the transformation in the order that it is mentioned.

WORKED EXAMPLE 6

The graph of $g(x) = \sqrt{x}$ undergoes the following transformations:

- **translation 2 units right**
- **dilation of factor 3 from the x-axis**
- **dilation of factor $\frac{1}{2}$ from the y-axis**
- **translation 1 unit down**
- **reflection in the x-axis.**

Find the rule of the image of the graph.

THINK	WRITE
1 Translation 2 units right means to replace x with $x - 2$ in the equation.	$\sqrt{x} \rightarrow \sqrt{x-2}$
2 Dilation of factor 3 from the x-axis means multiply the equation by 3.	$\sqrt{x-2} \rightarrow 3\sqrt{x-2}$
3 Dilation of factor $\frac{1}{2}$ from the y-axis means replace x with $2x$ in the equation.	$3\sqrt{x-2} \rightarrow 3\sqrt{2x-2}$
4 Translation 1 unit down means to subtract 1 unit from the equation.	$3\sqrt{2x-2} \rightarrow 3\sqrt{2x-2} - 1$
5 Reflection in the x-axis means to multiply the equation through by –1.	$3\sqrt{2x-2} - 1 \rightarrow -3\sqrt{2x-2} + 1$
6 Write the final answer. *Note:* $g(x)$ has not been used to denote the transformed equation because $g(x)$ has already been defined as $g(x) = \sqrt{x}$.	$g(x) = \sqrt{x} \rightarrow h(x) = -3\sqrt{2x-2} + 1$

EXERCISE 3.3 Transformations

PRACTISE

Work without CAS

1 **WE4** i Describe the transformation that has been applied to the graph of $y = x^3$ in each of the following examples. Sketch both graphs on the one set of axes.

ii Find the image of the point (–2, –8) after it has undergone each of the transformations.

a $y = 3x^3$ b $y = (x + 2)^3$ c $y = -x^3$ d $y = x^3 + 1$

2 Describe the transformation that has been applied to the graph of $y = \sin(x)$ for $x \in [0, 2\pi]$ in each of the following cases. In each case, sketch both graphs on the one set of axes.

a $y = 4\sin(x)$ b $y = \sin(2x)$

c $y = \sin\left(x + \frac{\pi}{2}\right)$ d $y = \sin(x) + 2$

3 **WE5** The graph of $y = \sin(x)$ is transformed so that its equation becomes $y = -2\sin\left[2x - \frac{\pi}{2}\right] + 1$. Define the transformations which have been applied to $y = \sin(x)$.

4 The graph of $y = e^x$ is transformed so that its equation becomes $y = \frac{1}{3}e^{\left(\frac{x+1}{2}\right)} -2$. Define the transformations which have been applied to $y = e^x$.

5 **WE6** The graph of $g(x) = x^2$ undergoes the following transformations:

- reflection in the y-axis
- translation 4 units right
- dilation of factor 2 from the y-axis
- translation 3 units down
- dilation of factor $\frac{1}{3}$ from the x-axis.

Find the rule of the image of the graph.

6 The graph of $h(x) = \frac{1}{x}$ undergoes the following transformations:

- dilation of factor 3 parallel to the x-axis
- translation 2 units up

- reflection in the y-axis
- translation 1 unit left
- reflection in the x-axis.

Find the rule of the image of the graph.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Describe and sketch the transformation that has been applied to the graph of $y = \cos(x)$ for $x \in [0, 2\pi]$ in each of the following cases.

a $y = \frac{1}{2}\cos(x)$ **b** $y = \cos(2x)$ **c** $y = -\cos(x)$ **d** $y = \cos(x) - 1$

8 Describe and sketch the transformation that has been applied to the graph of $f(x) = \dfrac{1}{x}$ in each of the following cases. Give the equations of any asymptotes in each case.

a $y = f(x - 2)$ **b** $y = -f(x)$ **c** $y = 3f(x)$ **d** $y = f(2x)$

9 State the sequence of transformations that has been applied to the first function in order to achieve the transformed function.

a $y = x^2 \rightarrow y = \frac{1}{3}(x + 3)^2 - \frac{2}{3}$

b $y = x^3 \rightarrow y = -2(1 - x)^3 + 1$

c $y = \dfrac{1}{x} \rightarrow y = \dfrac{3}{(2x + 6)} - 1$

10 For the corresponding sequence of transformations in question **9**, find the image of the point:

a $(-2, 4)$ **b** $(1, 1)$ **c** $\left(2, \frac{1}{2}\right)$.

11 State the sequence of transformations that has been applied to the first function in order to obtain the second function.

a $y = \cos(x) \rightarrow y = 2\cos\left[2\left(x - \dfrac{\pi}{2}\right)\right] + 3$

b $y = \tan(x) \rightarrow y = -\tan(-2x) + 1$

c $y = \sin(x) \rightarrow y = \sin(3x - \pi) - 1$

12 The graph of $h(x) = \sqrt[3]{x}$ undergoes the following transformations:

Reflection in the y-axis, then a translation of 3 units in the positive x direction, followed by a dilation of factor 2 parallel to the x-axis.

Find the rule of the image of the graph.

13 The graph of $h(x) = \dfrac{1}{x^2}$ undergoes the following transformations:

Translation of 2 units left and 3 units down, then a reflection in the x-axis, followed by a dilation of factor 3 from the x-axis, and a reflection in the y-axis.

Find the rule of the image of the graph.

14 The graph of $h(x) = 2x^2 - 3$ undergoes the following transformations:

Reflection in the x-axis, then a dilation of factor $\frac{1}{3}$ from the y-axis, followed by a translation of 1 unit in the positive x direction and 2 units in the negative y direction.

Find the rule of the image of the graph.

15 The graph of $h(x) = \dfrac{1}{x + 2}$ undergoes the following transformations:

Dilation of factor $\frac{1}{2}$ parallel to the x-axis, then a translation of 3 units down and

3 units left, then a reflection in the y-axis, followed by a dilation of factor 2 from the x-axis.

Find the rule of the image of the graph.

16 Show that $\frac{2x-5}{x-1} = 2 - \frac{3}{x-1}$ and hence describe the transformations which have been applied to $y = \frac{1}{x}$. Sketch the graph of $y = \frac{2x-5}{x-1}$. State the domain and range and give the equations of any asymptotes.

MASTER

17 State the transformations that have been applied to the first function in order to obtain the second function.

$$y = 3 - \sqrt{\frac{5-x}{2}} \rightarrow y = \sqrt{x}$$

18 State the transformations that have been applied to the first function in order to obtain the second function.

$$y = -2(3x-1)^2 + 5 \rightarrow y = (x+2)^2 - 1$$

3.4 Transformations using matrices

All of the transformations that we have studied to date can also be achieved by using matrices. Let (x', y') be the image of (x, y). The following is a summary of the various transformations of dilations, reflections and translations.

Mapping	Equation
Dilation of factor a parallel to the x-axis or from the y-axis	$x' = ax \quad = ax + 0y$ $y' = y \quad = 0x + y$
Dilation of factor a parallel to the y-axis or from the x-axis	$x' = x \quad = x + 0y$ $y' = ay \quad = 0x + ay$
Reflection in the x-axis	$x' = x \quad = x + 0y$ $y' = -y \quad = 0x - y$
Reflection in the y-axis	$x' = -x \quad = -x + 0y$ $y' = y \quad = 0x + y$
Reflection in the line $y = x$	$x' = y \quad = 0x + y$ $y' = x \quad = x + 0y$
Translation defined by shifting b units horizontally and c units vertically	$x' = x + b$ $y' = y + c$

It is possible to summarise the application of matrices to map the transformations, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$, of points on a curve.

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Interactivity
Dilation and enlargement matrices
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Dilation matrices

Dilation matrices can be expressed as follows.

- **Dilation of factor a parallel to the x-axis or from the y-axis:**

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ y \end{bmatrix}$$

- **Dilation of factor a parallel to the y-axis or from the x-axis:**

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ay \end{bmatrix}$$

Consider the function $y = x^2$ dilated by factor 2 parallel to the y-axis or from the x-axis.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$$

Thus, $x' = x$ and $y' = 2y$ or $\dfrac{y'}{2} = y$.

So, $y = x^2 \to \dfrac{y'}{2} = (x')^2$ or $y' = 2(x')^2$.

Thus the transformed equation is $y = 2x^2$.

Reflection matrices

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Interactivity
Reflection matrix
int-6295

Reflection matrices can be expressed as follows.

- **Reflection in the y-axis:**

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

- **Reflection in the x-axis:**

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

- **Reflection in the line $y = x$:**

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

Consider the function $y = \sin(x)$ reflected in the x-axis.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

So, $x' = x$ and $y' = -y$ or $-y' = y$.

So, $y = \sin(x) \to -y' = \sin(x')$ or $y' = -\sin(x')$.

Thus the transformed equation is $y = -\sin(x)$.

Translation matrix

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Interactivity
Translation matrix
int-6294

When the matrix $\begin{bmatrix} b \\ c \end{bmatrix}$ is added to $\begin{bmatrix} x \\ y \end{bmatrix}$, this is a translation of b units in the x direction and c units in the y direction.

Consider the function $y = \sqrt{x}$ translated by 1 unit in the positive x direction and –2 units in the y direction.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} x+1 \\ y-2 \end{bmatrix}$$

So, $x' = x + 1$ or $x' - 1 = x$ and $y' = y - 2$ or $y' + 2 = y$

$y = \sqrt{x} \to y' + 2 = \sqrt{x' - 1}$ or $y' = \sqrt{x' - 1} - 2$.

Thus the transformed equation is $y = \sqrt{x - 1} - 2$.

WORKED EXAMPLE 7

Use matrices to find the equations of the following transformed functions.

a The graph of $y = \frac{1}{x}$ is dilated by factor 2 parallel to the y-axis.

b The graph of $y = x^3$ is reflected in the y-axis.

c The graph of $y = \tan(x)$ is translated $\frac{\pi}{4}$ units in the negative x direction.

THINK	WRITE
a 1 Set up the matrix transformation for a dilation of factor 2 parallel to the y-axis.	**a** $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ $= \begin{bmatrix}x\\2y\end{bmatrix}$
2 State the transformed values for x and y.	$x' = x$ and $y' = 2y$ or $y = \frac{y'}{2}$
3 Determine the rule for the transformed function by substituting for x and y.	$\frac{y'}{2} = \frac{1}{x'}$ $y' = \frac{2}{x'}$ The transformed rule is $y = \frac{2}{x}$.
b 1 Set up the matrix transformation for reflection in the y-axis.	**b** $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\y\end{bmatrix}$
2 State the transformed values for x and y.	$x' = -x$ or $x = -x'$ and $y' = y$
3 Determine the rule for the transformed function.	$y' = (-x')^3$ $y' = -(x')^3$ The transformed rule is $y = -x^3$.
c 1 Set up the matrix transformation for a translation of $\frac{\pi}{4}$ in the negative x direction.	**c** $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}-\frac{\pi}{4}\\0\end{bmatrix} = \begin{bmatrix}x-\frac{\pi}{4}\\y\end{bmatrix}$
2 State the transformed values for x and y.	$x' = x - \frac{\pi}{4}$ or $x = x' + \frac{\pi}{4}$ and $y' = y$
3 Determine the rule for the transformed function.	The transformed rule is $y = \tan\left(x + \frac{\pi}{4}\right)$.

Combining transformations

It is possible to combine transformations without having to take each transformation step by step. For instance, suppose the following transformations are applied to $y = x^3$.

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}$$

This suggests that the image is obtained by dilating by factor 2 from the x-axis, then reflecting in the x-axis and finally translating 1 unit in the negative x direction and 1 unit in the positive y direction.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ -2y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$[x'y'] = \begin{bmatrix} x - 1 \\ -2y + 1 \end{bmatrix}$$

Thus $x' = x - 1$ or $x = x' + 1$ and $y' = -2y + 1$ or $y = -\frac{1}{2}(y' - 1)$.

So,

$$y = x^3 \rightarrow -\tfrac{1}{2}(y' - 1) = (x' + 1)^3$$
$$y' = -2(x' + 1)^3 + 1$$

The transformed equation is $y = -2(x + 1)^3 + 1$.

WORKED EXAMPLE 8

Use matrices to dilate the function $y = \sqrt[3]{x}$ by factor $\frac{1}{2}$ parallel to the x-axis, reflect it in the y-axis and then translate it 2 units right. Write the equation of the transformed function.

THINK	WRITE
1 Specify the transformation matrix for the dilation.	The dilation matrix is $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$
2 Specify the transformation matrix for the reflection.	The reflection matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
3 Specify the transformation matrix for the translation.	The translation matrix is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$
4 Form the matrix equation by combining the transformations in the correct order.	$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
5 Apply the transformations to form the equation of the image.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} -\frac{1}{2}x + 2 \\ y \end{bmatrix}$

6 State the transformed values for x and y.	$x' = -\frac{1}{2}x + 2$ or $x = -2(x' - 2)$ and $y' = y$
7 Determine the rule for the transformed function.	Thus $y = \sqrt[3]{x}$ becomes $y' = \sqrt[3]{-2(x' - 2)}$. The equation of the transformed function is $y = \sqrt[3]{-2(x - 2)}$.

EXERCISE 3.4 Transformations using matrices

PRACTISE

Work without CAS

1 **WE7** The graph of $y = \cos(x)$ is dilated by factor $\frac{1}{2}$ parallel to the x-axis. Use matrices to find the equation of the transformed function.

2 The graph of $y = \dfrac{1}{x^2}$ is reflected in the x-axis. Use matrices to find the equation of the transformed function.

3 **WE8** Consider the graph of $y = x^4$. This graph is dilated by factor 2 from the x-axis. It is then translated 1 unit in the negative x direction and 1 unit in the negative y direction. Find the equation of the transformed function.

4 Consider the graph of $y = \cos(x)$. This graph is reflected in the x-axis, dilated by factor $\frac{1}{2}$ parallel to the x-axis and finally translated $\dfrac{\pi}{2}$ units in the positive x direction as well as 3 units in the negative y direction. Find the equation of the transformed function.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Using matrix methods, find the image of the point (–2, 5) under the following sets of transformations.

- **a** Reflection in the x-axis, then a dilation of factor 2 from the x-axis, followed by a translation 2 units right
- **b** Dilation of factor 2 from the y-axis, reflection in the y-axis, then a shift down 2 units
- **c** Translations 3 units left and 1 unit up, a reflection in the y-axis, then a dilation of factor $\frac{1}{3}$ parallel to the x-axis

6 The graph of $y = \sqrt{x}$ is reflected in the y-axis, dilated by factor $\frac{1}{2}$ from the x-axis, and then is shifted left 2 units. Write the transformation matrix that represents these transformations.

7 The graph of $y = e^x$ is dilated by factor $\frac{1}{4}$ from the y-axis, then dilated by factor 3 from the x-axis. Write the transformation matrix that represents these transformations.

8 The following matrix equation was applied to the function defined by $y = 2(x - 1)^2$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

Find the rule for the transformed function and describe the transformations that were applied to $y = x^2$.

9 The following matrix equation was applied to the function defined by $y = \cos(x)$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{4} \\ -2 \end{bmatrix}$$

Find the rule for the transformed function and describe the transformations that were applied to $y = \cos(x)$.

10 The transformation $T: R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

Find the image of the curve $y = \sqrt{x + 1} - 2$ under the transformation T. Describe the transformations that were applied to the curve.

11 The following matrix equation was applied to the function defined by $y = x^3$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

Find the rule for the transformed function and describe the transformations that were applied to $y = x^3$.

12 The transformation $T: R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right).$$

Find the image of the curve $y = \frac{1}{x^2}$ under the transformation T.

13 If $f(x) = x^2$ and $g(x) = \frac{1}{2}\left(\frac{x + 4}{2}\right)^2 - 1$, create a matrix equation which would map $f(x)$ to $g(x)$.

14 If $f(x) = \frac{1}{x}$ and $g(x) = \frac{-2}{3x - 1} + 4$, create a matrix equation which would map $f(x)$ to $g(x)$.

MASTER

15 The transformation $T: R^2 \to R^2$ is defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b \\ c \end{bmatrix}$, where a, b and c are non-zero real numbers. If the image of the curve $g(x) = \frac{2}{x + 1} - 3$ is $h(x) = \frac{1}{x}$, find the values of a, b and c.

16 The transformation $T: R^2 \to R^2$ is defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right)$, where a, b, c and d are non-zero real numbers. If the image of the curve $g(x) = -(3x - 1)^2 + 2$ is $h(x) = x^2$, find the values of a, b, c and d.

3.5 Inverse graphs and relations

Inverses

The **relation** $A = \{(-1, 4), (0, 3), (1, 5)\}$ is formed by the mapping

$$-1 \to 4$$
$$0 \to 3$$
$$1 \to 5$$

The **inverse** relation is formed by the 'undoing' mapping:

$$4 \rightarrow -1$$
$$3 \rightarrow 0$$
$$5 \rightarrow 1$$

The inverse of A is the relation $\{(4, -1), (3, 0), (5, 1)\}$.

The x- and y-coordinates of the points in relation A have been interchanged in its inverse. This causes the domains and ranges to be interchanged also.

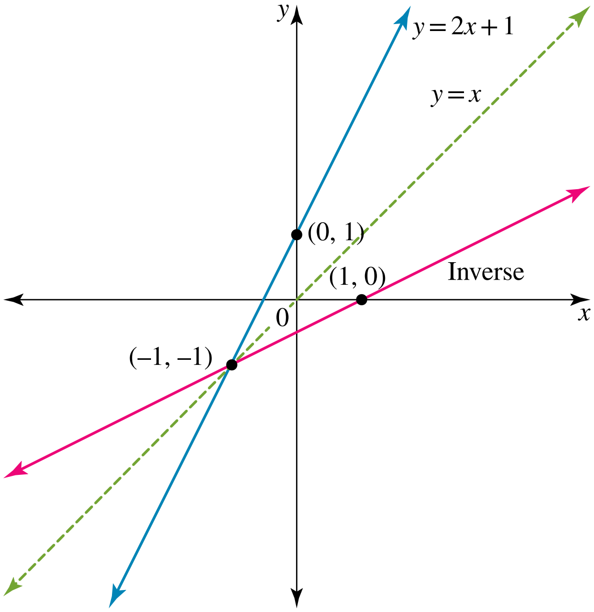
The domain of $A = \{-1, 0, 1\}$ = range of its inverse, and

The range of $A = \{3, 4, 5\}$ = domain of its inverse.

- **For any relation, the inverse is obtained by interchanging the *x*- and *y*-coordinates of the ordered pairs.**
- **Domains and ranges are interchanged between a pair of inverse relations.**

Graphs of inverses

When finding the inverse of a relation graphically, we reflect the relation in the line $y = x$. Consider the equation $y = 2x + 1$.



The line $y = x$ acts as a mirror. The inverse is the same distance from the line as is the original graph. The coordinates of known points, such as the axial intercepts, are interchanged by this reflection through the mirror. We can see that the line $y = 2x + 1$ cuts the y-axis at (0, 1) and the x-axis at (–0.5, 0), whereas the inverse graph cuts the y-axis at (–0.5, 0) and the x-axis at (1, 0). When sketching a graph and its inverse, the line $y = x$ should always be sketched.

WORKED EXAMPLE 9

a Sketch the graph of $y = x^2 - 2x - 3$.

b On the same set of axes, sketch the graph of the inverse.

c State the type of mapping for the parabola and its inverse, and whether the relations are functions.

d Give the domain and range for each of the graphs.

THINK

a 1 Determine where the given function cuts the x- and y-axes.

WRITE/DRAW

a y-intercept, $x = 0$:

$y = -3$

x-intercept, $y = 0$:

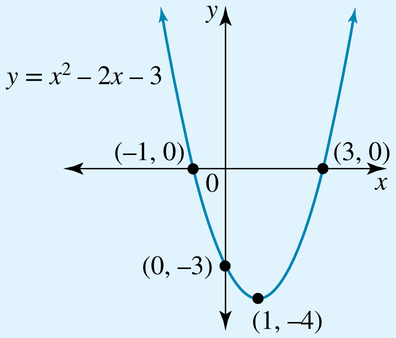
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x = 3 \text{ or } x = -1$$

A turning point occurs when $x = 1$:

$$y = (1)^2 - 2(1) - 3$$
$$= -4$$
$$\therefore \text{TP} = (1, -4)$$

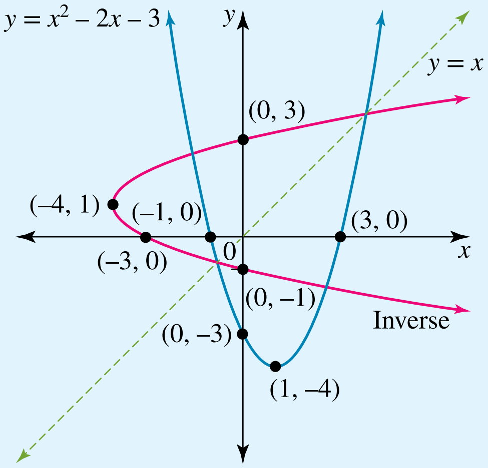
2 Sketch the graph of the parabola.

Note: When sketching graphs and their inverses, the scale on both axes needs to be relatively accurate so that distortions do not occur.



b On the same set of axes, sketch the inverse by interchanging the coordinates of all important points such as axial intercepts and the turning point.

b



c Comment on the types of mapping for the two graphs.

c The parabola $y = x^2 - 2x - 3$ is a many-to-one mapping, so it is a function. However, the inverse is a one-to-many mapping and as such is not a function.

d State the domains and ranges for both graphs. Remember that the domain of the original graph becomes the range of the inverse and vice versa.

d For $y = x^2 - 2x - 3$:
Domain $= R$
Range $= [-4, \infty)$
For the inverse:
Domain $= [-4, \infty)$
Range $= R$

Inverse functions

As we have seen in the previous examples, the inverses produced are not always functions. Any function that is many-to-one will have an inverse that is one-to-many, and hence this inverse will not be a function.

Only one-to-one functions will have an inverse that is also a function.

If we require the inverse of a many-to-one function to also be a function, the domain of the original graph must be restricted in order to ensure its correspondence is one-to-one. Achieving the maximum possible domain is always preferred, so many-to-one graphs are often restricted about the turning point or an asymptote.

When sketching a graph and its inverse function, if the graphs intersect, they will do so on the line $y = x$, since interchanging the coordinates of any point on $y = x$ would not cause any alteration to the coordinates.

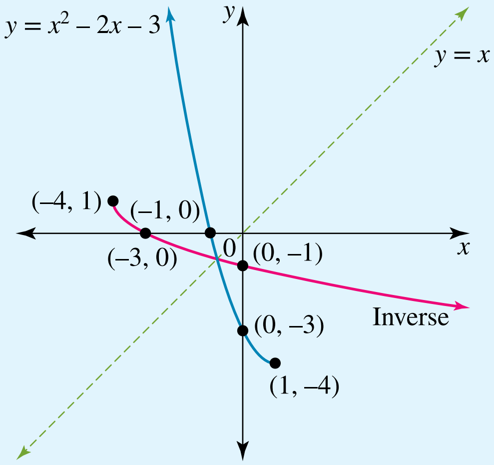
WORKED EXAMPLE 10

a **Consider the graph of $y = x^2 - 2x - 3$ from Worked example 9. The domain is restricted to $x \in (-\infty, a]$, where a is the largest possible value such that the inverse function exists. Determine the value of a.**

b **Sketch the restricted graph of y and its inverse on the same set of axes.**

c **Give the domain and range for both graphs.**

THINK	WRITE/DRAW
a The turning point is (1, –4), so to maximise the domain, we restrict y about this point.	**a** The x-value of the turning point is 1, so $a = 1$.
b Sketch the graph of $y = x^2 - 2x - 3$ for $x \in (-\infty, 1]$. Due to the restriction, there is only one x-intercept. Interchange the coordinates of the x-intercept and turning point, and sketch the graph of the inverse by reflecting the graph in the line $y = x$.	**b** For $y = x^2 - 2x - 3$, $x \in (-\infty, 1]$: x-intercept = (–1, 0), y-intercept = (0, –3) and TP = (1, –4). For the inverse, x-intercept = (–3, 0), y-intercept = (0, –1) and sideways TP = (–4, 1).



c State the domain and range for this function and its inverse.

c For $y = x^2 - 2x - 3$:
Domain: $x \in (-\infty, 1]$
Range: $y \in (-4, \infty]$
Inverse:
Domain: $x \in (-4, \infty]$
Range: $y \in (-\infty, 1]$

EXERCISE 3.5 Inverse graphs and relations

PRACTISE

Work without CAS

1 **WE9** **a** Sketch the graph of $y = (1 - x)(x + 5)$ and its inverse on the one set of axes. Show all axis intercepts and turning point coordinates.

b State the mapping for each graph and whether it is a function or a relation.

c Give the domain and range for the function and its inverse.

2 **a** Sketch the graph of $y = \sqrt{x}$.

b By reflecting this function in the line $y = x$, sketch the graph of the inverse relation.

c State the type of mapping for y and its inverse and state whether the inverse is a relation or a function.

3 **WE10** **a** Consider the graph of $y = (1 - x)(x + 5)$. The domain is restricted to $x \in (-\infty, a]$, where a is the largest possible value such that the inverse function exists. Determine the value of a.

b Sketch the restricted graph of y and its inverse on the same set of axes.

c Give the domain and range for both graphs.

4 Consider the graph of $y = -(x - 3)^2$. State the largest positive domain for the given function so that its inverse is a function. Sketch the restricted function with its inverse on the one set of axes.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Identify which of the following functions are one-to-one functions.

a $f(x) = \cos x$

b $g(x) = 1 - x^3$

c $h(x) = \sqrt{4 - x^2}$

d $k(x) = 2 + \dfrac{1}{x - 3}$

6 Which of the following has an inverse that is a function?

A $y = x^2 - 1$

B $x^2 + y^2 = 1$

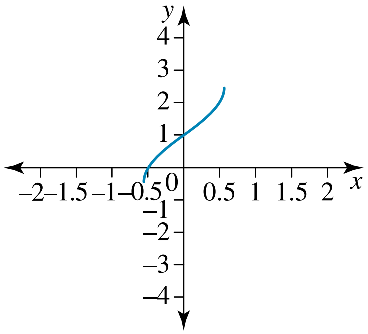
C $y = \dfrac{1}{x - 1}$

D $y = \sqrt{1 - x^2}$

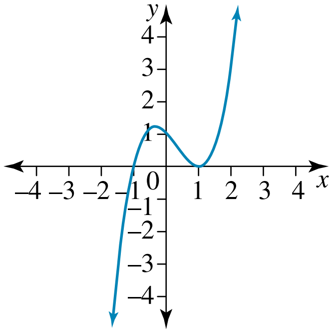
E $y = 10$

7 For each of the following relations, sketch the graph and its inverse on the same set of axes. Include the line $y = x$.

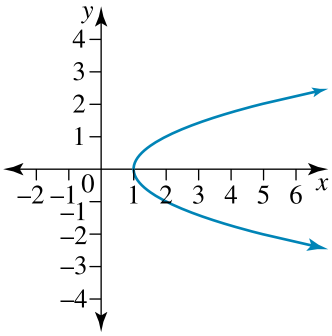
a



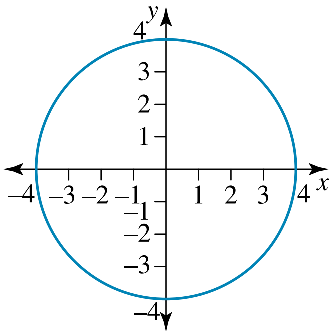
b



c

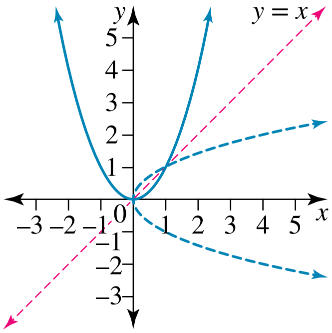


d

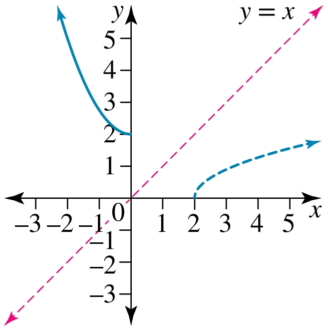


8 Identify the function and inverse function pair.

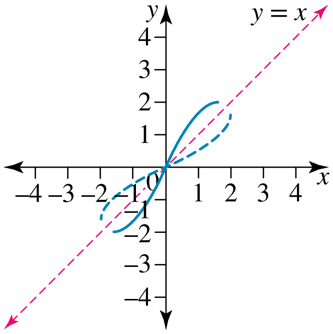
A



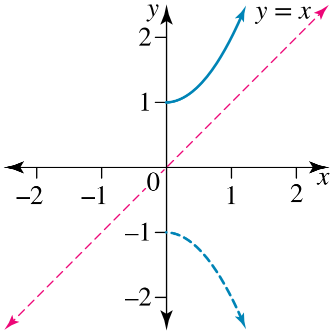
B



C

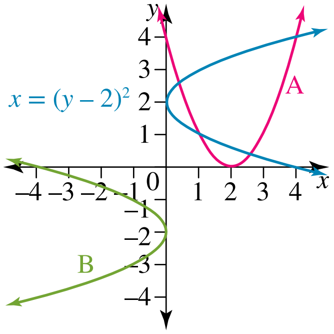


D



9 The graph of $x = (y - 2)^2$ is shown in blue.

The inverse relation is one of the other two graphs shown. Choose whether Option A or Option B is the inverse, giving clear reasons for your decision.



10 a Sketch the graph of $y = -\frac{1}{3}x - 1$, showing all important features.

b On the same set of axes, sketch the inverse function, again showing axis intercepts.

11 a Sketch the graph of $y = (x + 4)(x - 2)$.

b On the same set of axes, sketch the graph of the inverse relation.

c State the type of mapping for the parabola and its inverse.

d Is the inverse a function? Give a reason for your answer.

e Give the domain and range for each of the graphs.

f What is the largest domain to which y could be restricted so that its inverse is a function?

12 a Sketch the graph of $y = x^3$ and its inverse on the same set of axes.

b State the type of mapping for the graph and its inverse.

c Is the inverse a function? Give a reason for your answer.

d Give the domain and range for each of the graphs.

13 a Sketch the graph of $y = \frac{1}{x^2}$.

b On the same set of axes, sketch the inverse relation.

c State the mapping for each graph and indicate whether the rule describes a function or a relation.

d Restrict the domain of $y = \frac{1}{x^2}$, where x consists of negative values only, so that its inverse is a function. State this domain.

e Using this restricted domain for y, sketch the graph of y and its inverse on a new set of axes. State the domain and range of each function.

14 Given $y = 2x^2 - 12x + 13$ with a domain of $(-\infty, a]$, find the largest value of a so that the inverse of y is a function.

MASTER

15 a Use CAS to sketch the graph of $y = 9x - x^3$ and its inverse on the one set of axes.

b Determine the points of intersection of y and its inverse that occur along the line $y = x$, correct to 3 decimal places.

16 a Use CAS to sketch the graph of $y = 3 + \frac{1}{\sqrt{x}}$ and its inverse on the one set of axes.

b Determine the points of intersection of y and its inverse, correct to 3 decimal places.

3.6 Inverse functions

Notation for inverse functions

If the inverse of a function f is itself a function, then the inverse function is denoted by f^{-1}.

For example, the equation of the inverse of the square root function, $f(x) = \sqrt{x}$, can be written as $f^{-1}(x) = x^{2\cdot}$, $x \geq 0$.

In mapping notation, if $f:[0, \infty) \rightarrow R$, $f(x) = \sqrt{x}$, then the inverse function is $f^{-1}:[0, \infty) \rightarrow R$, $f^{-1}(x) = x^2$.

The domain of f^{-1} equals the range of f, and the range of f^{-1} equals the domain of f; that is, $d_{f^{-1}} = r_f$ and $r_{f^{-1}} = d_f$.

Note that f^{-1} is a function notation and thus cannot be used for relations which are not functions.

Note also that the inverse function f^{-1} and the reciprocal function $\frac{1}{f}$ represent different functions: $f^{-1} \neq \frac{1}{f}$.

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Finding the equation of an inverse

In the previous section, we saw that an inverse is graphed by reflecting the given function in the line $y = x$. We follow the same procedure to determine the rule of an inverse. That is, the x and y variables are interchanged.

Consider the linear function $f(x) = 2x + 1$. As f is a one-to-one function, its inverse will also be a function. To obtain the rule for the inverse function, f^{-1}, the x and y variables are interchanged.

Inverse: Let $y = f(x)$, swap x and y

$$x = 2y + 1$$

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Rearrange the rule to make y the subject of the equation.

$$2y = x - 1$$
$$y = \tfrac{1}{2}(x - 1)$$
$$\therefore f^{-1}(x) = \tfrac{1}{2}(x - 1)$$

The function $f(x) = 2x + 1$ has a domain of R and range of R.

The inverse function $f^{-1}(x) = \tfrac{1}{2}(x - 1)$ has a domain of R and range of R.

WORKED EXAMPLE 11

Consider the function $y = (x + 2)^2$. Find the rule for the inverse and indicate whether this inverse is a function or a relation. Give the domain and range for both.

THINK	WRITE
1 To obtain the inverse, interchange the x and y variables.	$y = (x + 2)^2$ Inverse: swap x and y. $x = (y + 2)^2$
2 Rearrange to make y the subject of the equation.	$(y + 2)^2 = x$ $y + 2 = \pm\sqrt{x}$ $y = \pm\sqrt{x} - 2$
3 Comment on whether the inverse is a function or a relation.	As $y = (x + 2)^2$ is a many-to-one function, the inverse will be a one-to-many relation. Therefore, it is not a function.
4 State the domain and range for both rules.	$y = (x + 2)^2$ has a domain of R and a range of $[0, \infty)$. $y = \pm\sqrt{x} - 2$ has a domain of $[0, \infty)$ and a range of R.

Restricting domains

In some cases, the domain will need to be included when we state the equation of the inverse. For example, to find the equation of the inverse of the function $y = \sqrt{x}$, interchanging coordinates gives $x = \sqrt{y}$. Expressing $x = \sqrt{y}$ with y as the subject gives $y = x^2$. This rule is not unexpected since 'square root' and 'squaring' are inverse operations. However, as the range of the function $y = \sqrt{x}$ is $[0, \infty)$, this must be the domain of its inverse. Hence, the equation of the inverse of $y = \sqrt{x}$ is $y = x^2$ with the restriction that $x \geq 0$.

Other examples involve restricting the domain of f so that the inverse is a function.

WORKED EXAMPLE 12

Consider the function $f: [0, \infty) \to R$, $f(x) = x^2 + 2$. Fully define the inverse, f^{-1}.

THINK	WRITE
1 Let $y = f(x)$, then interchange the x and y variables.	Let $y = f(x)$. Swap x and y. Inverse: $x = y^2 + 2$

2 Rearrange to make y the subject of the equation.	$y^2 = x - 2$ $y = \pm\sqrt{x - 2}$ dom f = ran f^{-1} $\therefore y = \sqrt{x - 2}$
3 Determine the domain of f^{-1}.	dom f^{-1} = ran f = $[2, \infty]$
4 Use the full function notation to define the inverse.	$f^{-1}: [2, \infty) \to R, f^{-1}(x) = \sqrt{x - 2}$

The point of intersection of $f(x)$ and $f^{-1}(x)$

The point where $f(x)$ intersects with its inverse can be found by solving $f(x) = f^{-1}(x)$. However, this can often be a difficult equation to solve. As $y = f(x)$ intersects with $y = f^{-1}(x)$ along the line $y = x$, there is actually a three-way point of intersection: $f(x) = f^{-1}(x) = x$. Therefore, it is preferable to solve either $f(x) = x$ or $f^{-1}(x) = x$ to find the point of intersection.

WORKED EXAMPLE 13

Consider the quadratic function defined by $f(x) = 2 - x^2$.

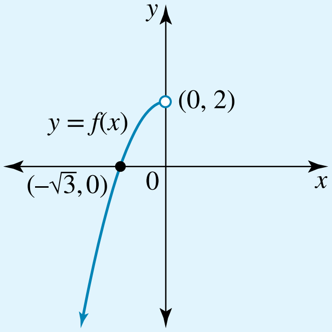
a Form the rule for its inverse and explain why the inverse is not a function.

b If the domain of f is restricted to $(-\infty, a)$, find the maximum value of a so that the inverse exists.

c Sketch the graph of $f(x) = 2 - x^2$ over this restricted domain, and use this to sketch its inverse on the same diagram.

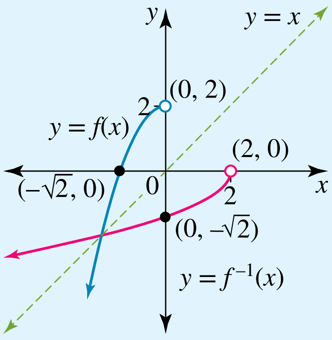
d Form the equation of the inverse, $y = f^{-1}(x)$.

e At what point do the two graphs intersect?

THINK	WRITE/DRAW
a 1 Interchange x and y coordinates to form the rule for the inverse.	a Let $y = f(x)$. Inverse: swap x and y. $x = 2 - y^2$ $y^2 = 2 - x$ $y = \pm\sqrt{2 - x}$
2 Explain why the inverse is not a function.	The quadratic function is many-to-one, so its inverse has a one-to-many correspondence. Therefore, the inverse is not a function.
b To maximise the domain, restrict the graph about the turning point.	b TP = (0, 2) Therefore, $a = 0$.
c 1 Sketch the graph of the function for the restricted domain.	c $f(x) = 2 - x^2$ y-intercept: (0, 2) x-intercept: Let $y = 0$. $2 - x^2 = 0$ $x^2 = 2$ $x = \pm\sqrt{2}$ $\Rightarrow x = -\sqrt{2}$ since $x \in (-\infty, 0)$. x-intercept: $(-\sqrt{2}, 0)$ Turning point: (0, 2) 

2 Deduce the key features of the inverse. Sketch its graph and the line $y = x$ on the same diagram as the graph of the function.

For the inverse, $(2, 0)$ is an open point on the x-axis and $(0, -\sqrt{2})$ is the y-intercept. Its graph is the reflection of the graph of $f(x) = 2 - x^2$, $x \in (-\infty, 0)$ in the line $y = x$.



d Use the range of the inverse to help deduce its equation.

Note: When you write the answer, the domain must also be included.

d From part **a**, the inverse of $f(x) = 2 - x^2$ is

$$y^2 = 2 - x$$
$$\therefore y = \pm\sqrt{2 - x}$$

The range of the inverse must be $(-\infty, 0)$ (the domain of the original graph), so the branch with the negative square root is required.
Therefore the equation of the inverse is $y = -\sqrt{2 - x}$.
$f^{-1}(x) = -\sqrt{2 - x}$, domain $= (-\infty, 2]$

e Choose two of the three equations that contain the required point and solve this system of simultaneous equations.

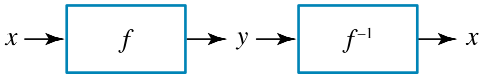
Note: As the graph and its inverse intersect along the line $y = x$, then the y-value of the coordinate will be the same as the x-value.

e The point of intersection lies on $y = x$.
Solving $x = f(x)$:

$$\begin{aligned} x &= 2 - x^2, x \in (-\infty, 0) \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x &= -2, 1 \end{aligned}$$

Reject $x = 1$ since $x \in (-\infty, 0)$, $\therefore x = -2$.
Therefore the point of intersection is $(-2, -2)$.

Composite functions with inverse functions



Because each output of a one-to-one function is different for each input, it is possible to reverse the process and turn the outputs back into the original inputs. The inverse is the function that results from reversing a one-to-one function. Essentially, the inverse function is an 'undoing' function.

So, if we take the inverse function of the original function or evaluate the function of the inverse function, in effect the two operations cancel each other out, leaving only x.

Therefore,

$$\boxed{f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.}$$

Consider $g(x) = 3x + 1$ and $g^{-1}(x) = \dfrac{x - 1}{3}$.

$$\begin{aligned} g(g^{-1}(x)) &= g\left(\frac{x-1}{3}\right) \\ &= 3\left(\frac{x-1}{3}\right) + 1 \\ &= x - 1 + 1 \\ &= x \end{aligned} \quad \text{and} \quad \begin{aligned} g^{-1}(g(x)) &= g^{-1}(3x + 1) \\ &= \frac{3x + 1 - 1}{3} \\ &= \frac{3x}{3} \\ &= x \end{aligned}$$

EXERCISE 3.6 Inverse functions

PRACTISE

Work without CAS

1 WE11 Consider the function $y = x^3$. Find the rule for the inverse and indicate whether this inverse is a function or a relation. Give the domain and range for both.

2 Consider the function $y = \frac{1}{x^2}$. Find the rule for the inverse and indicate whether this inverse is a function or a relation. Give the domain and range for both.

3 WE12 Consider the function $f:(-\infty, 2) \to R,\ f(x) = -\frac{1}{(x-2)^2}$. Fully define the inverse, f^{-1}.

4 Consider the function $f:[3, \infty) \to R,\ f(x) = \sqrt{x-3}$. Fully define the inverse, f^{-1}.

5 WE13 Consider the quadratic function $f(x) = (x+1)^2$ defined on its maximal domain.

- **a** Form the rule for its inverse and explain why the inverse is not a function.
- **b** If the domain of f is restricted to $[b, \infty)$, find the minimum value of b so that the inverse exists.
- **c** Sketch the graph of $f(x) = (x+1)^2$ over this restricted domain, and use this to sketch its inverse on the same diagram.
- **d** Form the equation of the inverse, $y = f^{-1}(x)$.
- **e** At what point do the two graphs intersect?

6 Find the point of intersection between $f(x) = 2\sqrt{x+2}$ and its inverse.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 For each of the following functions, find the rule for the inverse and indicate whether this inverse is a function or a relation. Give the domain and range for the inverse.

- **a** $y = \frac{1}{3}(x-3)$
- **b** $y = (x-5)^2$
- **c** $f:[-4, 0] \to R,\ f(x) = \sqrt{16 - x^2}$
- **d** $y = (x-1)^3$
- **e** $y = \sqrt{x}$
- **f** $y = \frac{1}{(x-1)^2} + 2$

8 If $f(x) = \frac{1}{x+2},\ x \neq -2$, verify that:

- **a** $f(f^{-1}(x)) = x$
- **b** $f^{-1}(f(x)) = x$.

9 If $k(x) = x^3 - 1$, verify that:

- **a** $k(k^{-1}(x)) = x$
- **b** $k^{-1}(k(x)) = x$.

10 Indicate whether each of the following functions has an inverse function. In each case, give a reason for your decision. If the inverse is a function, write the rule for the inverse in function notation and sketch $y = f(x)$ and $y = f^{-1}(x)$ on the one set of axes, including the point of intersection if it exists.

- **a** $f:R \to R,\ f(x) = x^4$
- **b** $f:R \to R,\ f(x) = 2x^2 - 7x + 3$
- **c** $f:[-3, 3] \to R,\ f(x) = \sqrt{9 - x^2}$
- **d** $f:[-2, \infty) \to R,\ f(x) = \sqrt{x+2}$

11 Given $f(x) = \frac{4x-7}{x-2}$, find the rule for f^{-1}, then sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. Include the point(s) of intersection on your graph.

12 **a** Given $f(x) = (x+2)^2$, restrict the maximal domain of f to only negative x-values so that its inverse is also a function. Write the inverse in function notation.

b Given $f(x) = -\sqrt{25 - x^2}$, restrict the maximal domain of f to only positive x-values so that its inverse is also a function. Write the inverse in function notation.

13 Given $f(x) = x^2 - 10x + 25$ with a domain of $[a, \infty)$, find:

- **a** the smallest value of a so that f^{-1} exists
- **b** $f^{-1}(x)$.

14 Consider $f:[-2, 4) \to R,\ f(x) = 1 - \frac{x}{3}$.

- **a** State the domain and determine the range of f.
- **b** Obtain the rule for f^{-1} and state its domain and range.
- **c** Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram.
- **d** Calculate the coordinates of any point of intersection of the two graphs.

15 Consider $f:D \to R, f(x) = \sqrt{1 - 3x}$:

- **a** Find D, the maximal domain of f.
- **b** Obtain the rule for $f^{-1}(x)$ and state its domain and range.
- **c** Evaluate the point(s) of intersection between $y = f(x)$ and $y = f^{-1}(x)$.
- **d** Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

16 If $f:(-\infty, a] \to R,\ f(x) = x^2 - 2x - 1$:

- **a** find the largest possible value of a so that f^{-1} exists
- **b** find $f^{-1}(x)$ and sketch both graphs on the same set of axes
- **c** find the point(s) of intersection between $y = f(x)$ and $y = f^{-1}(x)$.

MASTER

17 If $f:[1, \infty) \to R, f(x) = \sqrt{x - 1}$:

- **a** find $f^{-1}(x)$
- **b** sketch the graph of $y = f^{-1}(f(x))$ over its maximal domain
- **c** find $f^{-1}\left(-f\left(\frac{x+2}{3}\right)\right)$.

18 Given that $f(x) = \sqrt{1 - \frac{x^2}{4}}$, use CAS to view the graph and hence define two inverse functions, f^{-1}, using function notation with maximal domains. Sketch each pair of functions on separate axes.

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3.7 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

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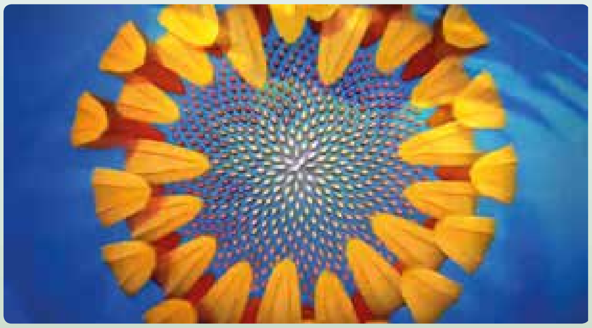
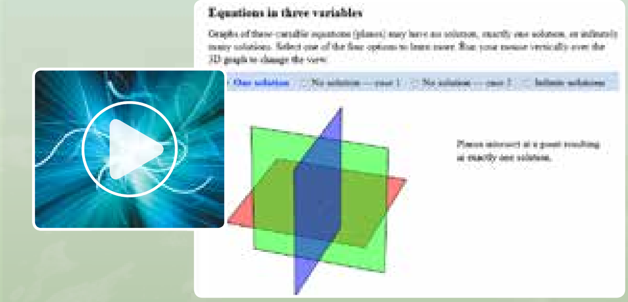
A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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3 Answers

EXERCISE 3.2

1 $f(g(x)) = (x-1)(x+1)(x^2+3)$, domain $= R$

$g(f(x)) = (x-1)^2(x+3)^2$, domain $= R$

2 $f(g(x)) = \frac{2}{x-2} - 1$, domain $= R\backslash\{2\}$

$g(f(x))$ does not exist.

3 a ran $g \not\subseteq$ dom f

$R \not\subseteq [-3, \infty)$

b $h(x) = 2x - 5,\ x \in [1, \infty)$

c $f(h(x)) = \sqrt{2x-2},\ x \in [1, \infty)$

4 a ran $f \not\subseteq$ dom g

$[0, \infty) \not\subseteq R\backslash\{4\}$

b $h(x) = x^2,\ \ x \in R\backslash\{-2, 2\}$

c $g(h(x)) = \frac{1}{x^2-4},\ \ x \in R\backslash\{-2, 2\}$

5 LHS: $$\frac{f(x)-f(y)}{f(xy)} = \frac{-\frac{3}{x} - \frac{-3}{y}}{-\frac{3}{xy}}$$
$$= \frac{-3y+3x}{xy} \div -\frac{3}{xy}$$
$$= \frac{-3(y-x)}{xy} \times \frac{xy}{-3}$$
$$= y - x$$

LHS = RHS; therefore, $\frac{f(x)-f(y)}{f(xy)} = y - x.$

6 $f(x+y) = f(x)f(y)$

$f(x-y) = \frac{f(x)}{f(y)}.$

7 a $f \circ g(x)$ is defined, domain $= [0, \infty)$.

b $g(f(x))$ is defined, domain $= R$.

c $h(g(x))$ is not defined.

d $h \circ f(x)$ is defined, domain $= R$.

8 a $f \circ g(x)$ is defined, $f \circ g(x) = x$, domain $= [0, \infty)$.

b $g(f(x))$ is defined, $g(f(x)) = |x|$, domain $= R$.

c $h(f(x))$ is not defined.

d $g(h(x))$ is not defined.

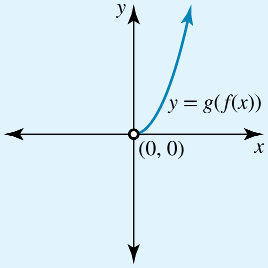
9 ran $g \subseteq$ dom f
$[0, \infty) \subseteq R$
Therefore $f(g(x))$ is defined.

$f(g(x)) = x + 3$ where domain $= [-2, \infty)$ and range $= [0, \infty)$.

10 a ran $f \subseteq$ dom g
$(0, \infty) \subseteq R\backslash\{0\}$
Therefore $g(f(x))$ is defined

b $g(f(x)) = x^2$, domain $x \in (0, \infty)$, range $= (0, \infty)$

c



11 a ran $g \not\subseteq$ dom f
$(-2, \infty) \not\subseteq [0, \infty)$
Therefore $g(f(x))$ is not defined.

b $g_1(x) = \frac{1}{(x-3)^2} - 2,$

$$x \in \left(-\infty,\ -\frac{1}{\sqrt{2}} + 3\right] \cup \left[\frac{1}{\sqrt{2}} + 3, \infty\right)$$

12 a ran $f \not\subseteq$ dom g
$[0, \infty) \not\subseteq R\backslash\{1\}$

Therefore $g(f(x))$ is not defined.

b $f_1(x) = \sqrt{2-x},\ x \in (-\infty, 2]\backslash\{1\}$

c $g(f_1(x)) = -\frac{1}{\sqrt{2-x}-1} + 2,\ x \in (-\infty, 2]\backslash\{1\}$

13 a LHS: $f(x+y) = 5^{x+y}$

RHS: $f(x) \times f(y) = 5^x \times 5^y$
$\qquad = 5^{x+y}$

LHS = RHS; therefore, $f(x+y) = f(x) \times f(y)$.

b LHS: $f(x-y) = 5^{x-y}$

RHS: $\frac{f(x)}{f(y)} = \frac{5^x}{5^y} = 5^{x-y}$

LHS = RHS; therefore, $f(x-y) = \frac{f(x)}{f(y)}$

14 $f(x) = x^3$ satisfies $f(xy) = f(x)f(y)$.

15 a $$\begin{aligned} h(x+y) &= 3(x+y) + 1 \\ &= 3x + 3y + 1 \\ &= (3x+1) + (3y+1) - 1 \\ &= h(x) + h(y) - 1 \end{aligned}$$

Therefore, $c = -1$.

b LHS: $$h(x) + h(y) = \frac{1}{x^3} + \frac{1}{y^3} = \frac{y^3 + x^3}{x^3y^3}$$

RHS: $$(x^3+y^3)h(xy) = (x^3+y^3) \times \frac{1}{(xy)^3} = \frac{x^3+y^3}{x^3y^3}$$

LHS = RHS; therefore, $h(x) + h(y) = (x^3+y^3)h(xy)$.

16 a Satisfies A and B

b Satisfies A and B

c Satisfies A, B and C

d Satisfies D

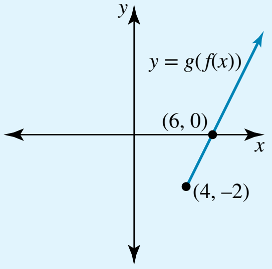
17 a $\text{ran} f \subseteq \text{dom}\, g$

$[0, \infty) \subseteq R$

Therefore $g(f(x))$ is defined.

b $g(f(x)) = x - 6$, domain $= [4, \infty)$

c



d $\text{ran}\, g \not\subseteq \text{dom} f$

$[-2, \infty) \not\subseteq [4, \infty)$

Therefore $f(g(x))$ is not defined.

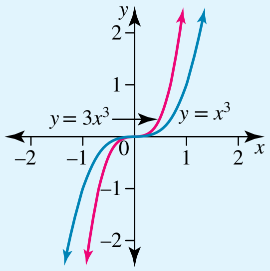
e $g_1(x) = x^2 - 2,\ x \in \left(-\infty, -\sqrt{6}\right\} \cup \left[\sqrt{6}, \infty\right)$

f $f(g_1(x)) = \sqrt{x^2 - 6}$,
domain $= \left(-\infty, -\sqrt{6}\right\} \cup \left[\sqrt{6}, \infty\right)$

18 $k \in [1, 3]$

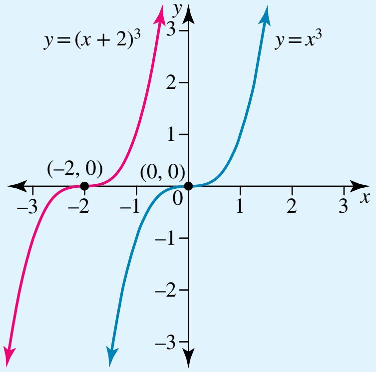
EXERCISE 3.3

1 a i Dilated by factor 3 parallel to the y-axis or from the x-axis.



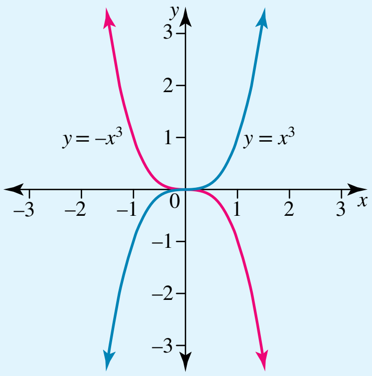
ii $(-2, -8) \rightarrow (-2, -24)$

b i Translated 2 units to the left or in the negative x direction.



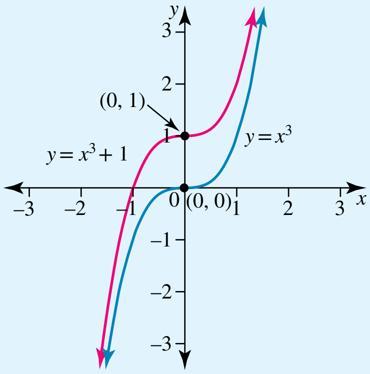
ii $(-2, -8) \rightarrow (-4, -8)$

c i Reflected in the x-axis



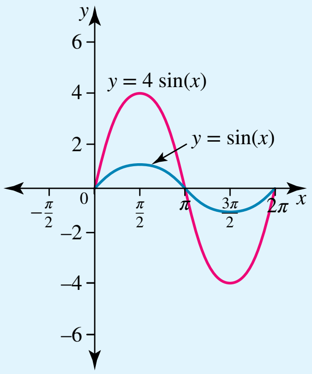
ii $(-2, -8) \rightarrow (-2, 8)$

d i Translated up 1 unit or in the positive y direction.

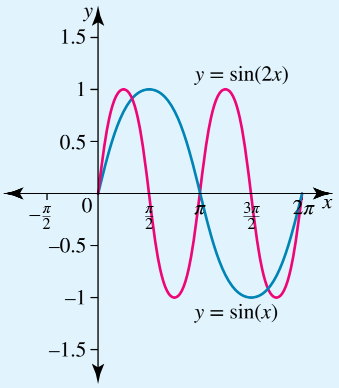


ii $(-2, -8) \rightarrow (-2, -7)$

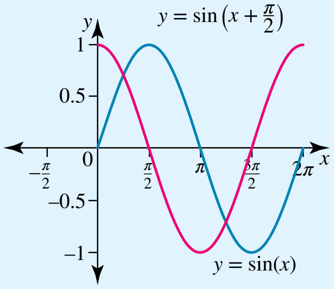
2 a Dilated by factor 4 parallel to the y-axis or from the x-axis



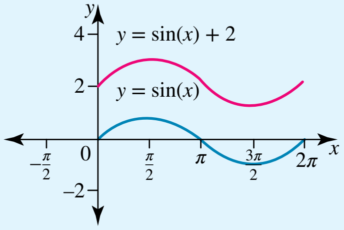
b Dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis



c Translated $\frac{\pi}{2}$ units to the left or in the negative x direction.



d Translated up 2 units or in the positive xy direction.



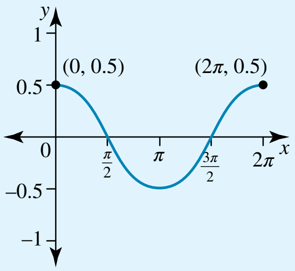
3 Reflected in the x-axis, dilated by factor 2 parallel to the y-axis or from the x-axis, dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis, translated $\frac{\pi}{4}$ units to the right or in the positive x direction and translated up 1 unit or in the positive y direction

4 Dilated by factor $\frac{1}{3}$ parallel to the y-axis or from the x-axis, dilated by factor 2 parallel to the x-axis or from the y-axis, translated 1 unit to the left or in the negative x direction and translated down 2 units or in the negative y-direction

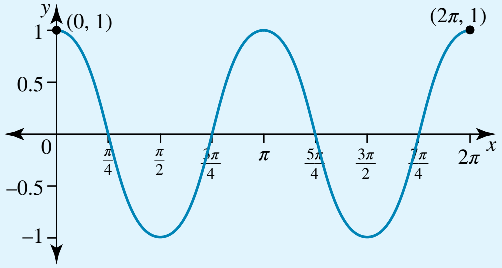
5 $f(x) = \frac{1}{3}\left(\frac{x-8}{2}\right)^2 - 1$

6 $f(x) = \frac{3}{x+1} - 2$

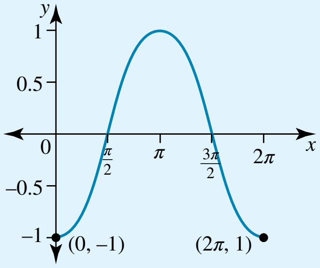
7 a Dilated by factor $\frac{1}{2}$ parallel to the y-axis or from the x-axis.



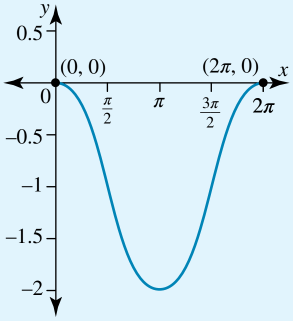
b Dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis



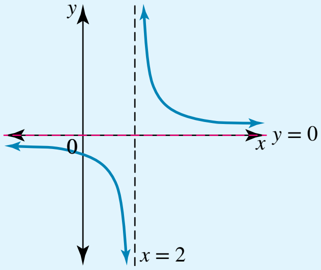
c Reflected in the x-axis



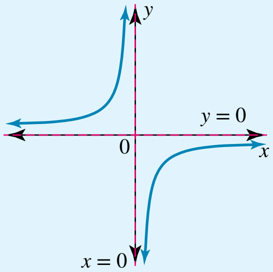
d Translated down 1 unit or in the negative y direction



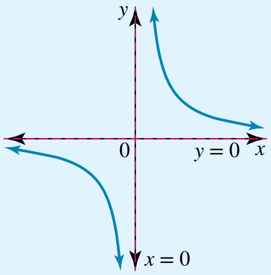
8 a Translated 2 units to the right or in the positive x direction, $y = \frac{1}{x-2}$; asymptotes $x = 2$, $y = 0$



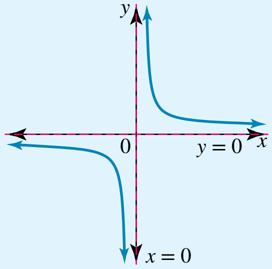
b Reflected in the x axis, $y = -\frac{1}{x}$; asymptotes $x = 0$, $y = 0$



c Dilated by factor 3 parallel to the y-axis or from the x-axis, $y = \frac{3}{x}$; asymptotes $x = 0$, $y = 0$.



d Dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis, $y = \frac{1}{2x}$; asymptotes $x = 0$, $y = 0$.



9 a $y = x^2$ has been dilated by factor $\frac{1}{3}$ parallel to the y-axis or from the x-axis, translated 3 units to the left or in the negative x direction, and translated down $\frac{2}{3}$ units or in the negative y direction.

b $y = x^3$ has been reflected in the x-axis, dilated by factor 2 parallel to the y-axis or from the x-axis, reflected in the y-axis, translated 1 unit to the right or in the positive x direction, and translated 1 unit up or in the positive y direction.

c $y = \frac{1}{x}$ has been dilated by a factor of 3 parallel to the y-axis or from the x-axis, dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis, translated 3 units to the left or in the negative x direction, and translated down 1 unit or in the negative y direction.

10 a $(-2, 4) \to \left(-5, \frac{2}{3}\right)$

b $(1, 1) \to (0, -1)$

c $\left(2, \frac{1}{2}\right) \to (-2, 0)$

11 a $y = \cos(x)$ has been dilated by factor 2 parallel to the y-axis or from the x-axis, dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis, translated $\frac{\pi}{2}$ units to the right or in the positive x direction, and translated up 3 units up or in the positive y direction.

b $y = \tan(x)$ has been reflected in both axes, dilated by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis, and translated up 1 unit or in the positive y direction.

c $y = \sin(x)$ has been dilated by factor $\frac{1}{3}$ parallel to the x-axis or from the y-axis, translated $\frac{\pi}{3}$ units to the right or in the positive x direction, and translated down 1 unit or in the negative y direction.

12 $f(x) = \sqrt[3]{-\frac{x-6}{2}}$

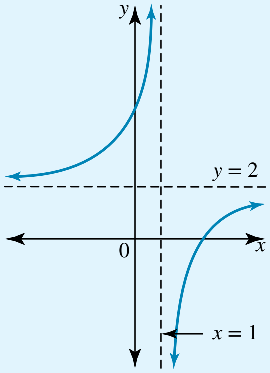
13 $f(x) = -\frac{3}{(2-x)^2} + 9$

14 $f(x) = -18(x-1)^2 + 1$

15 $f(x) = \frac{1}{3-x} - 6$

16
$$\begin{aligned} y &= \frac{2x-5}{x-1} \\ &= \frac{2(x-1)-3}{x-1} \\ &= \frac{2(x-1)}{x-1} - \frac{3}{x-1} \\ &= 2 - \frac{3}{x-1} \end{aligned}$$

Relative to $y = \frac{1}{x}$, $y = \frac{2x-5}{x-1}$ has been reflected in the y-axis or the x-axis, dilated by factor 3 parallel to the y-axis or from the x-axis, translated 1 unit to the right or in the positive x direction, and translated 2 units up or in the positive y direction. Domain = $R\backslash\{1\}$ and range = $R\backslash\{2\}$; asymptotes $x = 1$ and $y = 2$



17 Reflection in the x-axis, reflection in the y-axis, translation 5 units left, dilation by factor $\frac{1}{2}$ parallel to the x-axis or from the y-axis, translation 3 units down

18 Reflection in the x-axis, dilation by factor $\frac{1}{2}$ parallel to the y-axis or from the x-axis, dilation by factor 3 parallel to the x-axis or from the y-axis, translation 3 units left, translation 6 units down

EXERCISE 3.4

1 $y = \cos(2x)$

2 $y = -\frac{1}{x^2}$

3 $y = 2(x+1)^4 - 1$

4 $y = -\cos\left[2\left(x - \frac{\pi}{2}\right)\right] - 3$

5 a $(-2, 5) \to (0, -10)$

b $(-2, 5) \to (4, 3)$

c $(-2, 5) \to \left(\frac{5}{3}, 6\right)$

6 $\begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

7 $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 3 \end{bmatrix}$

8 $y = (x+2)^2 - 2$

The original function, $y = x^2$, has been dilated by factor $\frac{1}{2}$ parallel to the y-axis or from the x-axis, translated 3 units to the left or in the negative x direction, and translated down 2 units or in the negative y direction.

9 $y = -\cos\left(x - \frac{\pi}{4}\right) - 2$

The original function, $y = \cos(x)$, has been reflected in the x-axis, translated $\frac{\pi}{4}$ units to the right or in the positive x direction, and translated 2 units down or in the negative y direction.

10 $y = 3\sqrt{-x} - 6$

The original function, $y = \sqrt{x+1} - 2$, has been translated 1 unit to the right or in the positive x direction, reflected in the y-axis, and dilated by factor 3 parallel to the y-axis or from the x-axis.

11 $y = -\left(\frac{1}{2}(x+4)\right)^3$

The original function, $y = x^3$, has been dilated by factor 2 parallel to the x-axis or from the y-axis, reflected in the x-axis, and translated 4 units to the left or in the negative x direction.

or $y = -\frac{1}{8}(x+4)^3$

12 $y = -\frac{2}{(x+3)^2} - 2$

13 $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

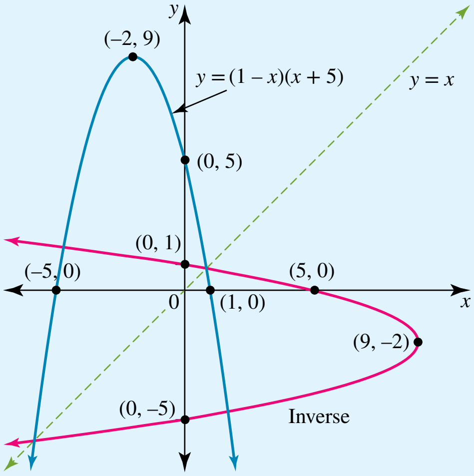
14 $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 4 \end{bmatrix}$

15 $a = \frac{1}{2}, b = 1, c = \frac{3}{2}$

16 $a = 3, b = -1, c = -\frac{1}{3}, d = -2$

EXERCISE 3.5

1 a



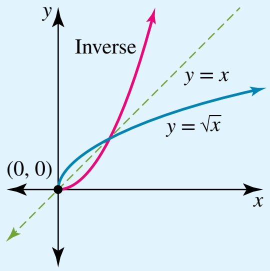
b $y = (1-x)(x+5)$ is a many-to-one function

The inverse is a one-to-many relation.

c $y = (1-x)(x+5)$: domain = R, range = $(-\infty, 9]$

Inverse: domain = $(-\infty, 9]$, range = R

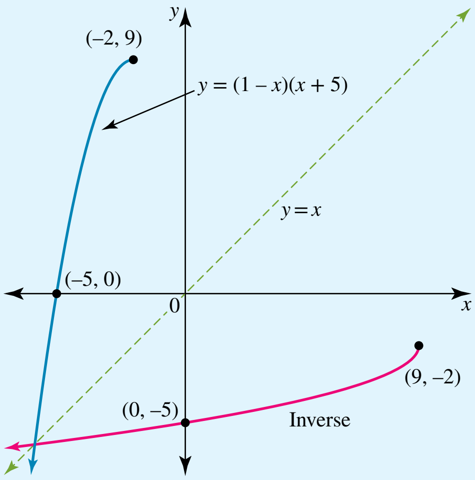
2 a and **b**



c $y = \sqrt{x}$ is a one-to-one function. The inverse is a one-to-one function.

3 a $a = -2$

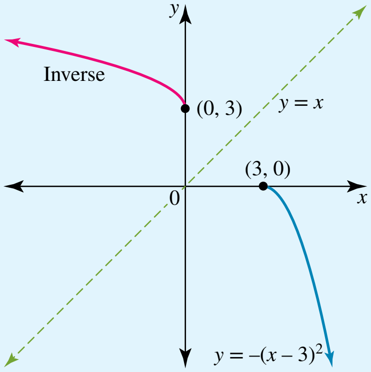
b



c y: domain $(-\infty, -2]$, range $(-\infty, 9]$

Inverse: domain $(-\infty, 9]$, range $(-\infty, -2]$

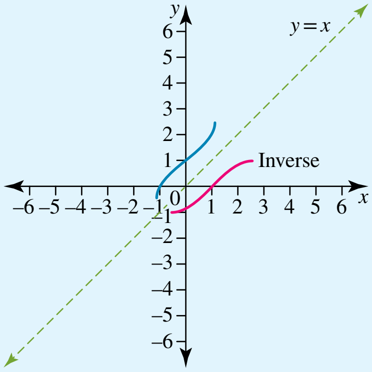
4 Domain = $[3, \infty)$



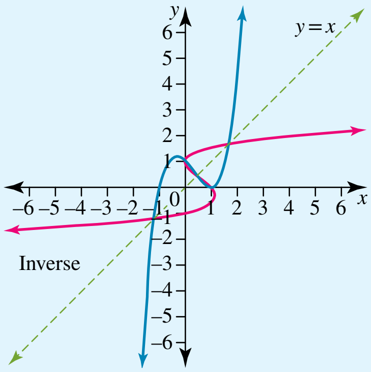
5 b, d

6 C

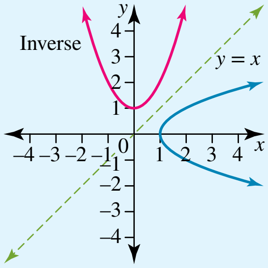
7 a



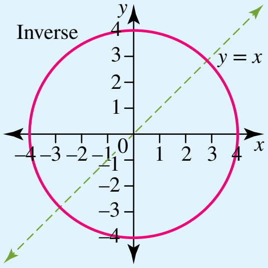
b



c



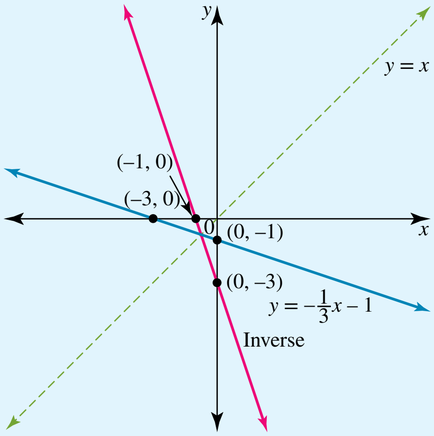
d



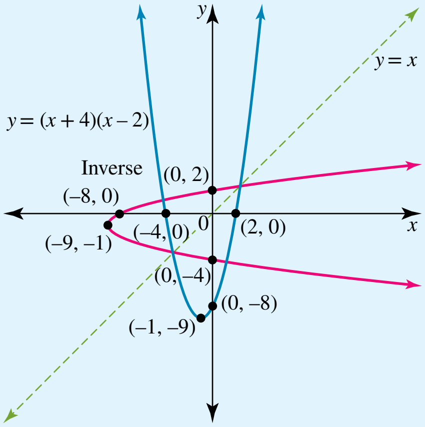
8 C

9 A is the correct option, as the given function has a turning point at (0, 2) and option A has a turning point at (2, 0).

10 a and **b**



11 a and **b**



c $y = (x + 4)(x - 2)$: many-to-one
Inverse: one-to-many

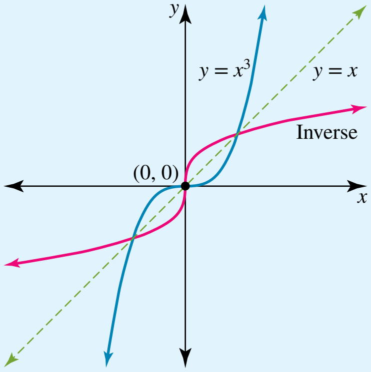
d The inverse is not a function, as a one-to-many correspondence indicates a relation. Also, the inverse can only be a function if the original graph is a one-to-one function, and this graph is a many-to-one function.

e y: domain $= R$, range $= [-9, \infty)$

Inverse: domain $= [-9, \infty)$, range $= R$

f $(-\infty, -1]$ or $[-1, \infty)$

12 a



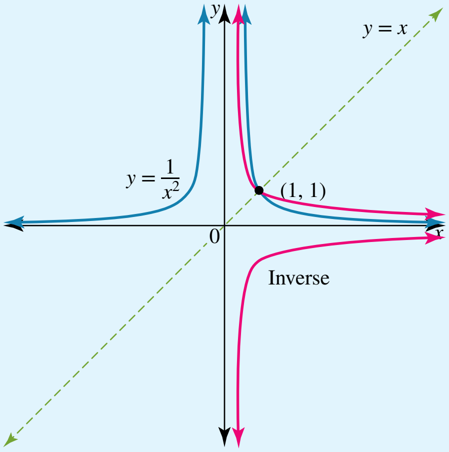
b $y = x^3$: one-to-one; inverse: one-to-one

c The inverse of y is a function because y is a one-to-one function.

d $y = x^3$: domain $= R$, range $= R$

Inverse: domain $= R$, range $= R$

13 a and **b**



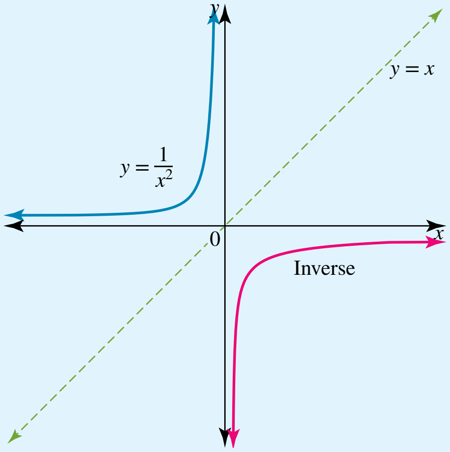
c $y = \dfrac{1}{x^2}$: many-to-one function

Inverse: one-to-many relation

d $(-\infty, 0)$

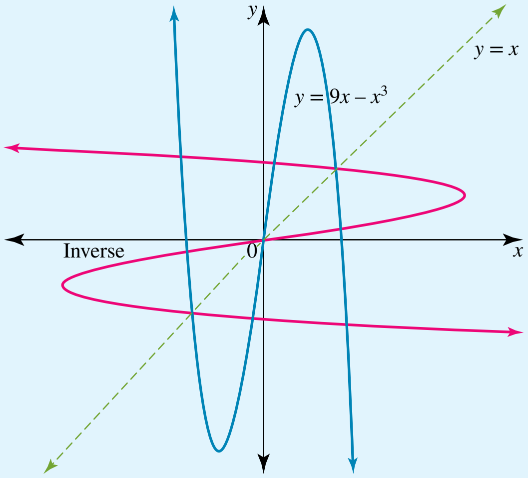
e $y = \dfrac{1}{x^2}$: domain $= (-\infty, 0)$, range $= (0, \infty)$

Inverse: domain $= (0, \infty)$, range $= (-\infty, 0)$



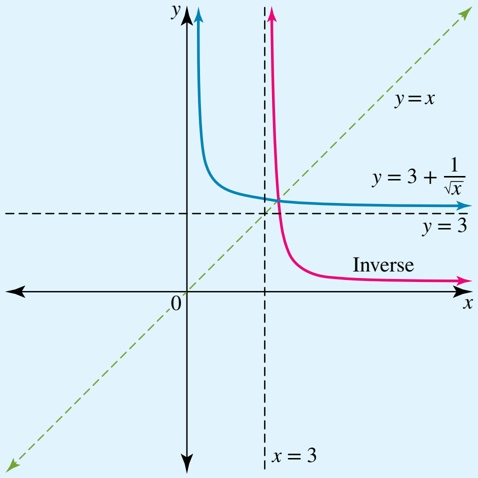
14 $a = 3$

15 a



b (2.828, 2.828), (0, 0), (–2.828, –2.828)

16 a



b Point of intersection = (3.532, 3.532)

EXERCISE 3.6

1 Inverse: $y = \sqrt[3]{x}$, one-to-one function.

Domain for both $= R$, range for both $= R$

2 Inverse: $y = \pm\frac{1}{\sqrt{x}}$, one-to-many relation (therefore not a function)

y: domain $= R\backslash\{0\}$, range $= (0, \infty)$

Inverse: domain $= (0, \infty)$, range $= R\backslash\{0\}$

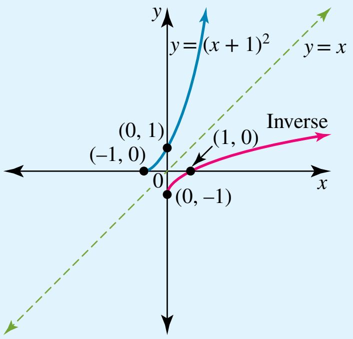
3 $f^{-1}: (-\infty\ , 0) \to R,\ f^{-1}(x) = -\sqrt{-\frac{1}{x}} + 2$

4 $f^{-1}: [0, \infty) \to R,\ f^{-1}(x) = x^2 + 3$

5 a $y = \pm\sqrt{x} - 1$; the inverse is not a function as $f(x)$ is not a one-to-one function.

b $b = -1$

c



d $f^{-1}(x) = \sqrt{x} - 1$, domain $= [0, \infty)$

e No intersection

6 $(2 + 2\sqrt{3}, 2 + 2\sqrt{3})$

7 a $y = 3(x + 1)$; one-to-one function with domain $= R$ and range $= R$

b $y = 5 \pm \sqrt{x}$; one-to-many relation with domain $= [0, \infty)$ and range $= R$

c $f^{-1}(x) = -\sqrt{16 - x^2}$; one-to-one function with domain $= [0, 4]$ and range $= [-4, 0]$

d $y = 1 + \sqrt[3]{x}$; one-to-one function with domain $= R$ and range $= R$

e $y = x^2$; one-to-one function with domain $= [0, \infty)$ and range $= [0, \infty)$

f $y = \frac{1}{\pm\sqrt{x - 2}} + 1$; one-to-many relation with domain $= (2, \infty)$ and range $= R\backslash\{1\}$

8 $f^{-1}(x) = \frac{1}{x} - 2$

a
$$\begin{aligned} f(f^{-1}(x)) &= \frac{1}{\frac{1}{x} - 2 + 2} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

b
$$\begin{aligned} f^{-1}(f(x)) &= \frac{1}{\frac{1}{x + 2}} - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

9 $k^{-1}(x) = \sqrt[3]{x + 1}$

a
$$\begin{aligned} k(k^{-1}(x)) &= (\sqrt[3]{x + 1})^3 - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

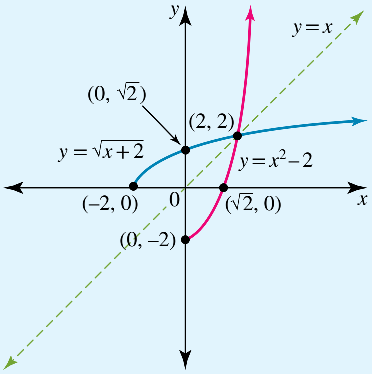
b
$$\begin{aligned} k^{-1}(k(x)) &= \sqrt[3]{x^3 - 1 + 1} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

10 a $f(x) = x^4$: many-to-one function; inverse: one-to-many relation

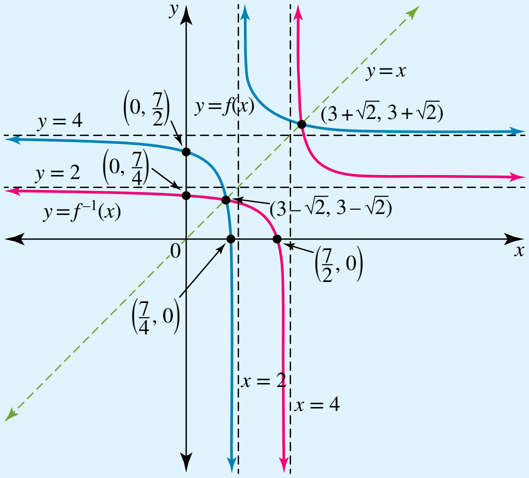
b $f(x) = 2x^2 - 7x + 3$: many-to-one function; inverse: one-to-many relation

c $f(x) = \sqrt{9 - x^2},\ x \in [-3, 3]$: many-to-one function; inverse: one-to-many relation

d $f(x) = \sqrt{x + 2},\ x \in [-2, \infty)$: one-to-one function; f^{-1}:$[0, \infty) \to R,\ f^{-1}(x) = x^2 - 2$



11 $f^{-1}(x) = 2 + \dfrac{1}{x-4}$



12 a Restrict the domain to $(-\infty, -2]$.

$f^{-1}: [0, \infty) \to R,\ f^{-1}(x) = -\sqrt{x} - 2$

b Restrict the domain to $(0, 5]$.

$f^{-1}: (-5, 0] \to R,\ f^{-1}(x) = \sqrt{25 - x^2}$

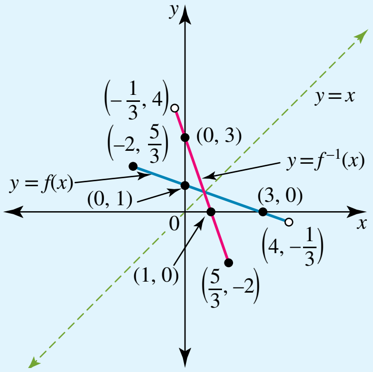
13 a $a = 5$

b $f^{-1}(x) = 5 + \sqrt{x},\ x \in [0, \infty\ )$

14 a Domain $= [-2, 4)$, range $= \left(-\frac{1}{3}, \frac{5}{3}\right]$

b $f^{-1}(x) = -3(x - 1)$; domain $= \left(-\frac{1}{3}, \frac{5}{3}\right]$, range $= [-2, 4)$

c



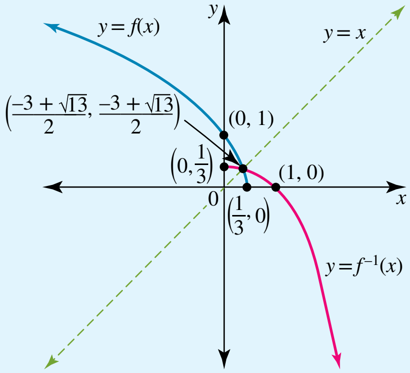
d The point of intersection is $\left(\frac{3}{4}, \frac{3}{4}\right)$.

15 a $D = \left(-\infty, \frac{1}{3}\right]$

b $f^{-1}(x) = \dfrac{1}{3} - \dfrac{x^2}{3}$; domain $= [0, \infty)$, range $= \left(-\infty, \frac{1}{3}\right]$

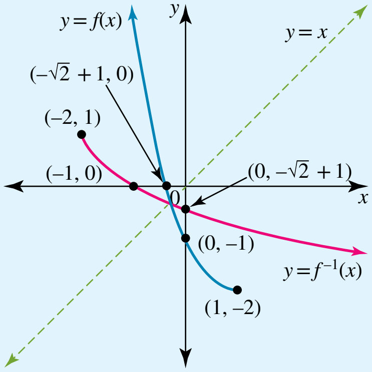
c The point of intersection is $\left(\dfrac{-3 + \sqrt{13}}{2}, \dfrac{-3 + \sqrt{13}}{2}\right)$.

d



16 a $a = 1$

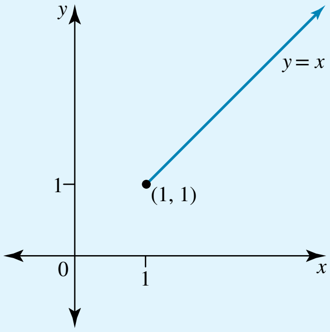
b $f^{-1}(x) = -\sqrt{x + 2} + 1,\ x \in [-2, \infty)$



c The point of intersection is $\left(\dfrac{3 - \sqrt{13}}{2}, \dfrac{3 - \sqrt{13}}{2}\right)$.

17 a $f^{-1}(x) = x^2 + 1,\ x \in [0, \infty)$

b $f^{-1}(f(x)) = x$; domain $= [1, \infty)$

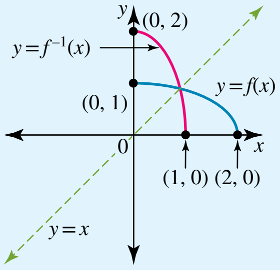


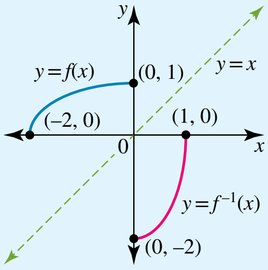
c $\dfrac{x + 2}{3}$

18 Two possible domains are [–2, 0] and [0, 2]. Both have a range of [0, 1].

$f^{-1}(x): [0, 1] \to R,\ f^{-1}(x) = -2\sqrt{1 - x^2}$ or

$f^{-1}(x): [0, 1] \to R,\ f^{-1}(x) = 2\sqrt{1 - x^2}$.





4

Logarithmic functions

4.1 Kick off with CAS

Exponentials and logarithms

Part 1

1 On a calculation screen in CAS, define the function $f(x) = e^{2x-1}$.

2 To find the inverse, solve $f(y) = x$ for y.

3 Define the inverse as $h(x)$.

4 Determine $f(h(x))$ and $h(f(x))$.

5 Repeat steps 1–4 for the function $f(x) = \log_e(x + 5)$.

6 What can you conclude about the relationship between exponentials and logarithms?

Part 2

7 Use CAS to solve the following equations for x.

a $3e^{kx} - 2 = 3e^{-kx}$

b $k\log_e(3mx + 2) = d$

c $2^x > 3$

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

4.2 Logarithm laws and equations

Introduction

Logarithm is another name for the **exponent** or **index**. Consider the following indicial equations:

Exponent or index ↓   Exponent or index ↓

Base number → $10^2 = 100$ Base number → $e^x = y$

Written as logarithms, they become:

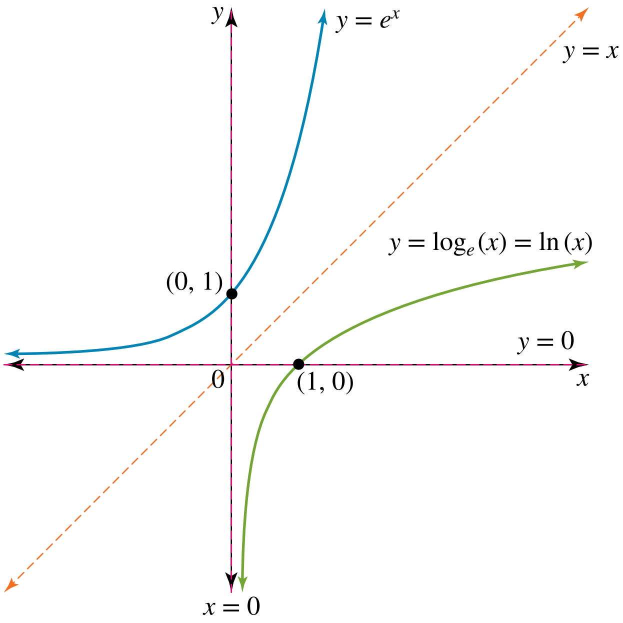
$\log_{10}100 = 2$ ← Exponent or index $\log_e(y) = x$ ← Exponent or index

↑ Base number   ↑ Base number

The logarithmic function can also be thought of as the inverse of the exponential function.

Consider the exponential function $y = e^x$. To achieve the inverse, the x and y variables are interchanged. Therefore, $y = e^x$ becomes $x = e^y$. If we make y the subject of the equation, we have $y = \log_e(x)$.

This can also be shown graphically.



Rule	$y = e^x$	$y = \log_e(x) = \ln(x)$
Type of mapping	One-to-one	One-to-one
Domain	R	$(0, \infty)$
Range	$(0, \infty)$	R

The expression $\log_e(x)$ or $\ln(x)$ is called the natural or Napierian logarithm, and can be found on your calculator as 'ln'. The expression $\log_{10}(x)$ is the standard logarithm, which traditionally is written as $\log(x)$ and can be found on your calculator as log.

The logarithms have laws that have been developed from the indicial laws.

Laws of logarithms

studyon

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Logarithmic laws
Concept summary
Practice questions

1. $a^m \times a^n = a^{m+n} \Leftrightarrow \log_a(m) + \log_a(n) = \log_a(mn)$
   To prove this law:
   Let $x = \log_a(m)$ and $y = \log_a(n)$.
   So $a^x = m$ and $a^y = n$.
   Now $a^m \times a^n = a^{m+n}$
   or $mn = a^{x+y}$.
   By applying the definition of a logarithm to this statement, we get
   $\log_a(mn) = x + y$
   or $\log_a(mn) = \log_a(m) + \log_a(n)$.
2. $a^m \div a^n = a^{m-n} \Leftrightarrow \log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
   To prove this law:
   Let $x = \log_a(m)$ and $y = \log_a(n)$.
   So $a^x = m$ and $a^y = n$.
   Now $\frac{a^x}{a^y} = a^{x-y}$
   or $\frac{m}{n} = a^{x-y}$.
   By converting the equation into logarithm form, we get
   $\log_a\left(\frac{m}{n}\right) = x - y$
   or $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$.

Note: Before the first or second law can be applied, each logarithmic term must have a coefficient of 1.

3. $(a^m)^n = a^{mn} \Leftrightarrow \log_a(m^n) = n\log_a(m)$
   To prove this law:
   Let $x = \log_a(m)$.
   So $a^x = m$.
   Now $(a^x)^n = m^n$
   or $a^{nx} = m^n$.
   By converting the equation into logarithm form, we have
   $\log_a(m^n) = nx$
   or $\log_a(m^n) = n\log_a(m)$

Applying these laws, we can also see that:

4. As $a^0 = 1$, then by the definition of a logarithm, $\log_a(1) = 0$.
5. As $a^1 = a$, then by the definition of a logarithm, $\log_a(a) = 1$.
6. $a^x > 0$, therefore $\log_a(0)$ is undefined, and $\log_a(x)$ is only defined for $x > 0$ and $a \in R^+ \backslash \{1\}$.

Another important fact related to the definition of a logarithm is

$$a^{\log_a(m)} = m.$$

This can be proved as follows:

Let $y = a^{\log_a(m)}$.

Converting index form to logarithm form, we have

$$\log_a(y) = \log_a(m).$$

Therefore $y = m$.

Consequently $a^{\log_a(m)} = m$.

In summary, the logarithm laws are:

1. $\log_a(m) + \log_a(n) = \log_a(mn)$
2. $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
3. $\log_a(m^n) = n\log_a(m)$
4. $\log_a(1) = 0$
5. $\log_a(a) = 1$
6. $\log_a(0) =$ undefined
7. $\log_a(x)$ is defined for $x > 0$ and $a \in R^+\backslash\{1\}$
8. $a^{\log_a(m)} = m$.

WORKED EXAMPLE 1

Simplify:

a $\log_4(64) + \log_4(16) - \log_4(256)$

b $2\log_3(7) - 2\log_3(21)$

c $\dfrac{\log_3(27)}{\log_3(81)}$

d $\log_5\left(\sqrt[4]{\dfrac{1}{625}}\right)$.

THINK	WRITE
a 1 Express all the numbers in base 4 and, where possible, apply the log law $\log_a(m^n) = n\log_a(m)$.	**a** $\log_4(64) + \log_4(16) - \log_4(256)$ $= \log_4(4^3) + \log_4(4^2) - \log_4(4^4)$ $= 3\log_4(4) + 2\log_4(4) - 4\log_4(4)$
2 Apply $\log_a(a) = 1$ and simplify.	$= 3 \times 1 + 2 \times 1 - 4 \times 1$ $= 1$
b 1 Apply the law $n\log_a(m) = \log_a(m^n)$.	**b** $2\log_3(7) - 2\log_3(21)$ $= \log_3(7^2) - \log_3(21^2)$
2 Apply the law $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$ and simplify.	$= \log_3\left(\frac{7^2}{21^2}\right)$ $= \log_3\left(\frac{7}{21}\right)^2$ $= 2\log_3\left(\frac{1}{3}\right)$
3 Convert $\frac{1}{3}$ to 3^{-1} and apply $\log_a a = 1$.	$= 2\log_3(3^{-1})$ $= -2\log_3(3)$ $= -2$
c 1 Apply the law $n\log_a(m) = \log_a(m^n)$. *Note:* The 16 and 64 cannot be cancelled, as when they are with the log function, they represent single numbers. Therefore, the 16 and 64 cannot be separated from their logarithm components.	**c** $\dfrac{\log_3(16)}{\log_3(64)}$ $= \dfrac{\log_3(2^4)}{\log_3(2^6)}$ $= \dfrac{4\log_3(2)}{6\log_3(2)}$

2 Cancel the logs as they are identical.

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

d **1** Convert the surd into a fractional power and simplify.

$$\text{d } \log_5\left(\sqrt[4]{\frac{1}{625}}\right)$$

$$= \log_5\left(\left(\frac{1}{5^4}\right)^{\frac{1}{4}}\right)$$

$$= \log_5\left(\left(5^{-4}\right)^{\frac{1}{4}}\right)$$

$$= \log_5(5^{-1})$$

2 Apply the laws $n \log_a m = \log_a m^n$ and $\log_a a = 1$.

$$= -\log_5(5)$$

$$= -1$$

Solving logarithmic equations involves the use of the logarithm laws as well as converting to index form. As $\log_a(x)$ is only defined for $x > 0$ and $a \in R^+\backslash\{1\}$, always check the validity of your solution.

WORKED EXAMPLE 2

Solve the following equations for x.

a $\log_4(64) = x$

b $\log_2(3x) + 3 = \log_2(x - 2)$

c $(\log_2(x))^2 = 3 - 2\log_2(x)$

d $\log_2(2x) + \log_2(x + 2) = \log_2(6)$

THINK	WRITE
a **1** Convert the equation into index form.	**a** $\log_4(64) = x$ $4^x = 64$
2 Convert 64 to base 4 and evaluate.	$4^x = 4^3$ $\therefore x = 3$
b **1** Rewrite 3 in log form, given $\log_2 2 = 1$.	**b** $\log_2(3x) + 3 = \log_2(x - 2)$ $\log_2(3x) + 3\log_2(2) = \log_2(x - 2)$
2 Apply the law $\log_a(m^n) = n\log_a(m)$.	$\log_2(3x) + \log_2(2^3) = \log_2(x - 2)$
3 Simplify the left-hand side by applying $\log_a(mn) = \log_a(m) + \log_a(n)$.	$\log_2(3x \times 8) = \log_2(x - 2)$
4 Equate the logs and simplify.	$24x = x - 2$ $23x = -2$ $x = -\frac{2}{23}$
c **1** Identify the quadratic form of the log equation. Let $a = \log_2(x)$ and rewrite the equation in terms of a.	**c** $(\log_2(x))^2 = 3 - 2\log_2(x)$ Let $a = \log_2(x)$. $a^2 = 3 - 2a$

2 Solve the quadratic.	$a^2 + 2a - 3 = 0$ $(a - 1)(a + 3) = 0$ $a = 1, -3$
3 Substitute in $a = \log_2(x)$ and solve for x.	$\log_2(x) = 1 \qquad \log_2(x) = -3$ $x = 2^1 \qquad x = 2^{-3}$ $\therefore x = 2, \frac{1}{8}$
d 1 Simplify the left-hand side by applying $\log_a(mn) = \log_a(m) + \log_a(n)$.	**d** $\log_2(2x) + \log_2(x + 2) = \log_2(6)$ $\log_2(2x(x + 2)) = \log_2(6)$
2 Equate the logs and solve for x.	$2x(x + 2) = 6$ $2x^2 + 4x - 6 = 0$ $x^2 + 2x - 3 = 0$ $(x - 1)(x + 3) = 0$ $x = 1, -3$
3 Check the validity of both solutions.	$x = -3$ is not valid, as $x > 0$.
4 Write the answer.	$x = 1$

Change of base rule

The definition of a logarithm, together with the logarithmic law $n \log_a(m) = \log_a(m^n)$, is important when looking at the change of base rule.

Suppose $y = \log_a(m)$.

By definition, $a^y = m$.

Take the logarithm to the same base of both sides.

$$\log_b(a^y) = \log_b(m)$$

$$y \log_b(a) = \log_b(m)$$

$$y = \frac{\log_b(m)}{\log_b(a)}$$

Therefore,

$$\boxed{\log_a(m) = \frac{\log_b(m)}{\log_b(a)}.}$$

WORKED EXAMPLE 3

a Evaluate the following, correct to 4 decimal places.

i $\log_7(5)$ **ii** $\log_{\frac{1}{3}}(11)$

b If $p = \log_5(x)$, find the following in terms of p.

i x **ii** $\log_x(81)$

THINK	WRITE
a i Input the logarithm into your calculator.	**a i** $\log_7(5) = 0.8271$
ii Input the logarithm into your calculator.	**ii** $\log_{\frac{1}{3}}(11) = -2.1827$
b i Rewrite the logarithm in index form to find an expression for x.	**b i** $p = \log_5(x)$ $x = 5^p$

ii 1 Rewrite $\log_x(81)$ using $\log_a(m^n) = n\log_a(m)$.	**ii** $\log_x(81)$ $= \log_x(9^2)$ $= 2\log_x(9)$
2 Apply the change-of-base rule so that x is no longer a base. *Note:* Although 9 has been chosen as the base in this working, a different value could be applied, giving a different final answer.	$= 2\dfrac{\log_9(9)}{\log_9(x)}$ $= 2\dfrac{1}{\log_9(x)}$
3 Replace x with 5^p and apply the law $\log_a(m^n) = n\log_a(m)$.	$= 2\dfrac{1}{\log_9(5^p)}$ $= \dfrac{2}{p\log_9(5)}$

EXERCISE 4.2 Logarithm laws and equations

PRACTISE

Work without CAS
Questions 1–4, 6

1 **WE1** Simplify the following.

a $\log_7(49) + \log_2(32) - \log_5(125)$

b $5\log_{11}(6) - 5\log_{11}(66)$

c $\dfrac{\log_4 25}{\log_4 625}$

d $\log_2\left(\sqrt[7]{\frac{1}{128}}\right)$

2 Simplify the following.

a $7\log_4(x) - 9\log_4(x) + 2\log_4(x)$

b $\log_7(2x-1) + \log_7(2x-1)^2$

c $\log_{10}(x-1)^3 - 2\log_{10}(x-1)$

3 **WE2** Solve the following for x.

a $\log_5(125) = x$

b $\log_4(x-1) + 2 = \log_4(x+4)$

c $3(\log_2(x))^2 - 2 = 5\log_2(x)$

d $\log_5(4x) + \log_5(x-3) = \log_5(7)$

4 Solve the following for x.

a $\log_3(x) = 5$

b $\log_3(x-2) - \log_3(5-x) = 2$

5 **WE3** **a** Evaluate the following, correct to 4 decimal places.

i $\log_7(12)$

ii $\log_3\left(\frac{1}{4}\right)$

b If $z = \log_3(x)$, find the following in terms of z.

i $2x$

ii $\log_x(27)$

6 Rewrite the following in terms of base 10.

a $\log_5(9)$

b $\log_{\frac{1}{2}}(12)$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Express each of the following in logarithmic form.

a $6^3 = 216$

b $2^8 = 256$

c $3^4 = 81$

d $10^{-4} = 0.0001$

e $5^{-3} = 0.008$

f $7^1 = 7$

8 Find the value of x.

a $\log_3(81) = x$

b $\log_6\left(\frac{1}{216}\right) = x$

c $\log_x(121) = 2$

d $\log_2(-x) = 7$

9 Simplify the following.

a $\log_2(256) + \log_2(64) - \log_2(128)$
b $5\log_7(49) - 5\log_7(343)$
c $\log_4\left(\sqrt[6]{\frac{1}{64}}\right)$
d $\log_4\left(\frac{16}{256}\right)$
e $\dfrac{\log_5(32)}{3\log_5(16)}$
f $\dfrac{6\log_2\left(\sqrt[3]{x}\right)}{\log_2(x^5)}$

10 Simplify the following.

a $\log_3(x-4) + \log_3(x-4)^2$
b $\log_7(2x+3)^3 - 2\log_7(2x+3)$
c $\log_5(x^2) + \log_5(x^3) - 5\log_5(x)$
d $\log_4(5x+1) + \log_4(5x+1)^3 - \log_4(5x+1)^2$

11 Evaluate the following, correct to 4 decimal places.

a $\log_3(7)$
b $\log_2\left(\frac{1}{121}\right)$

12 If $n = \log_5(x)$, find the following in terms of n.

a $5x$
b $\log_5(5x^2)$
c $\log_x(625)$

13 Solve the following for x.

a $\log_e(2x-1) = -3$
b $\log_e\left(\frac{1}{x}\right) = 3$
c $\log_3(4x-1) = 3$
d $\log_{10}(x) - \log_{10}(3) = \log_{10}(5)$
e $3\log_{10}(x) + 2 = 5\log_{10}(x)$
f $\log_{10}(x^2) - \log_{10}(x+2) = \log_{10}(x+3)$
g $2\log_5(x) - \log_5(2x-3) = \log_5(x-2)$
h $\log_{10}(2x) - \log_{10}(x-1) = 1$
i $\log_3(x) + 2\log_3(4) - \log_3(2) = \log_3(10)$
j $(\log_{10}(x))(\log_{10}(x^2)) - 5\log_{10}(x) + 3 = 0$
k $(\log_3 x)^2 = \log_3(x) + 2$
l $\log_6(x-3) + \log_6(x+2) = 1$

14 Express y in terms of x for the following equations.

a $\log_{10}(y) = 2\log_{10}(2) - 3\log_{10}(x)$
b $\log_4(y) = -2 + 2\log_4(x)$
c $\log_9(3xy) = 1.5$
d $\log_8\left(\frac{2x}{y}\right) + 2 = \log_8(2)$

15 **a** Find the value of x in terms of m for which $3\log_m(x) = 3 + \log_m 27$, where $m > 0$ and $x > 0$.

b If $\log_{10}(m) = x$ and $\log_{10}(n) = y$, show that $\log_{10}\left(\dfrac{100n^2}{m^5\sqrt{n}}\right) = 2 + \dfrac{3y}{2} - 5x$.

16 Solve the equation $8\log_x(4) = \log_2(x)$ for x.

MASTER

17 Solve the following for x, correct to 3 decimal places.

a $e^{2x} - 3 = \log_e(2x+1)$
b $x^2 - 1 = \log_e(x)$

18 Find x, correct to 4 decimal places, if $(3\log_3(x))(5\log_3(x)) = 11\log_3(x) - 2$.

4.3 Logarithmic scales

Logarithmic scales are used in the calculation of many scientific and mathematical quantities, such as the loudness of sound, the strength (magnitude) of an earthquake, octaves in music, pH in chemistry and the intensity of the brightness of stars.

WORKED EXAMPLE 4

Loudness, in decibels (dB), is related to the intensity, I, of the sound by the equation

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where I_0 is equal to 10^{-12} watts per square metre (W/m^2). (This value is the lowest intensity of sound that can be heard by human ears).

a **An ordinary conversation has a loudness of 60 dB. Calculate the intensity in W/m^2.**

b **If the intensity is doubled, what is the change in the loudness, correct to 2 decimal places?**

THINK	WRITE
a **1** Substitute $L = 60$ and simplify.	**a** $L = 10\log_{10}\left(\frac{I}{I_0}\right)$ $60 = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$ $60 = 10\log_{10}(10^{12}I)$ $6 = \log_{10}(10^{12}I)$
2 Convert the logarithm to index form and solve for I.	$10^6 = 10^{12}I$ $I = 10^{-6}$ W/m^2
b **1** Determine an equation for L_1.	**b** $L_1 = 10\log_{10}\left(\frac{I_1}{10^{-12}}\right)$ $= 10\log_{10}(10^{12}I_1)$ $= 10\log_{10}(10^{12}) + 10\log_{10}(I_1)$ $= 120\log_{10}(10) + 10\log_{10}(I_1)$ $= 120 + 10\log_{10}(I_1)$
2 The intensity has doubled, therefore $I_2 = 2I_1$. Determine an equation for L_2.	$L_2 = 10\log_{10}\left(\frac{2I_1}{10^{-12}}\right)$ $= 10\log_{10}(2 \times 10^{12}I_1)$ $= 10\log_{10}(2) + 10\log_{10}(10^{12}) + 10\log_{10}(I_1)$ $= 3.010 + 120\log_{10}(10) + 10\log_{10}(I_1)$ $= 3.01 + 120 + 10\log_{10}(I_1)$
3 Replace $120 + 10\log_{10}(I_1)$ with L_1.	$= 3.01 + L_1$
4 Answer the question.	Doubling the intensity increases the loudness by 3.01 dB.

EXERCISE 4.3 Logarithmic scales

PRACTISE

1 **WE4** The loudness, L, of a jet taking off about 30 metres away is known to be 130 dB. Using the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity measured in W/m^2 and I_0 is equal to 10^{-12} W/m^2, calculate the intensity in W/m^2 for this situation.

2 The moment magnitude scale measures the magnitude, M, of an earthquake in terms of energy released, E, in joules, according to the formula

$$M = 0.67\log_{10}\left(\frac{E}{K}\right)$$

where K is the minimum amount of energy used as a basis of comparison.

An earthquake that measures 5.5 on the moment magnitude scale releases 10^{13} joules of energy. Find the value of K, correct to the nearest integer.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 Two earthquakes, about 10 kilometres apart, occurred in Iran on 11 August 2012. One measured 6.3 on the moment magnitude scale, and the other one was 6.4 on the same scale. Use the formula from question **2** to compare the energy released, in joules, by the two earthquakes.

4 An earthquake of magnitude 9.0 occurred in Japan in 2011, releasing about 10^{17} joules of energy. Use the formula from question **2** to find the value of K correct to 2 decimal places.

5 To the human ear, how many decibels louder is a 20 W/m^2 amplifier compared to a 500 W/m^2 model? Use the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where L is measured in dB, I is measured in W/m^2 and $I_0 = 10^{-12}$ W/m^2. Give your answer correct to 2 decimal places.

6 Your eardrum can be ruptured if it is exposed to a noise which has an intensity of 10^4 W/m^2. Using the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity measured in W/m^2 and I_0 is equal to 10^{-12} W/m^2, calculate the loudness, L, in decibels that would cause your eardrum to be ruptured.

Questions **7–9** relate to the following information.

Chemists define the acidity or alkalinity of a substance according to the formula

$$\text{pH} = -\log_{10}\left[H^+\right]$$

where $\left[H^+\right]$ is the hydrogen ion concentration measured in moles/litre. Solutions with a pH less than 7 are acidic, whereas solutions with a pH greater than 7 are basic. Solutions with a pH of 7, such as pure water, are neutral.

7 Lemon juice has a hydrogen ion concentration of 0.001 moles/litre. Find the pH and determine whether lemon juice is acidic or basic.

8 Find the hydrogen ion concentration for each of the following.

a Battery acid has a pH of zero.
b Tomato juice has a pH of 4.
c Sea water has a pH of 8.
d Soap has a pH of 12.

9 Hair conditioner works on hair in the following way. Hair is composed of the protein called keratin, which has a high percentage of amino acids. These acids are negatively charges. Shampoo is also negatively charged. When shampoo removes dirt, it removes natural oils and positive charges from the hair. Positively charged surfactants in hair conditioner are attracted to the negative charges in the hair, so the surfactants can replace the natural oils.

a A brand of hair conditioner has a hydrogen ion concentration of 0.000 015 8 moles/litre. Calculate the pH of the hair conditioner.
b A brand of shampoo has a hydrogen ion concentration of 0.000 002 75 moles/litre. Calculate the pH of the shampoo.

10 The number of atoms of a radioactive substance present after t years is given by

$$N(t) = N_0 e^{-mt}.$$

a The half-life is the time taken for the number of atoms to be reduced to 50% of the initial number of atoms. Show that the half-life is given by $\frac{\log_e(2)}{m}$.
b Radioactive carbon-14 has a half-life of 5750 years. The percentage of carbon-14 present in the remains of plants and animals is used to determine how old the remains are. How old is a skeleton that has lost 70% of its carbon-14 atoms? Give your answer correct to the nearest year.

11 A basic observable quantity for a star is its brightness. The apparent magnitudes m_1 and m_2 for two stars are related to the corresponding brightnesses, b_1 and b_2, by the equation

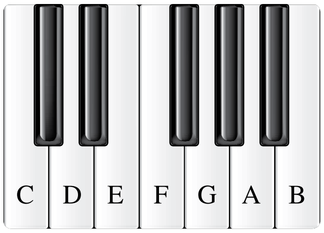
$$m_2 - m_1 = 2.5 \log_{10}\left(\frac{b_1}{b_2}\right).$$

The star Sirius is the brightest star in the night sky. It has an apparent magnitude of −1.5 and a brightness of −30.3. The planet Venus has an apparent magnitude of −4.4. Calculate the brightness of Venus, correct to 2 decimal places.

12 Octaves in music can be measured in cents, n. The frequencies of two notes, f_1 and f_2, are related by the equation

$$n = 1200 \log_{10}\left(\frac{f_2}{f_1}\right).$$

Middle C on the piano has a frequency of 256 hertz; the C an octave higher has a frequency of 512 hertz. Calculate the number of cents between these two Cs.



MASTER

13 Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss. A gunshot from a 0.22 rifle has an intensity of about $(2.5 \times 10^{13})I_0$. Calculate the loudness, in decibels, of the gunshot sound and state if ear protection should be worn when a person goes to a rifle range for practice shooting. Use the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where I_0 is equal to 10^{-12} W/m^2, and give your answer correct to 2 decimal places.

14 Early in the 20th century, San Francisco had an earthquake that measured 8.3 on the magnitude scale. In the same year, another earthquake was recorded in South America that was four times stronger than the one in San Francisco. Using the equation $M = 0.67\log_{10}\left(\frac{E}{K}\right)$, calculate the magnitude of the earthquake in South America, correct to 1 decimal place.

4.4 Indicial equations

study on

Units 3 & 4

AOS 2

Topic 1

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Practice questions

When we solve an equation such as $3^x = 81$, the technique is to convert both sides of the equation to the same base. For example, $3^x = 3^4$; therefore, $x = 4$.

When we solve an equation such as $x^3 = 27$, we write both sides of the equation with the same index. In this case, $x^3 = 3^3$; therefore, $x = 3$.

If an equation such as $5^{2x} = 2$ is to be solved, then we must use logarithms, as the sides of the equation cannot be converted to the same base or index. To remove x from the power, we take the logarithm of both sides.

$$\begin{aligned}\log_5(5^{2x}) &= \log_5(2)\\ 2x &= \log_5(2)\\ x &= \frac{1}{2}\log_5(2)\end{aligned}$$

Note: If $a^x = b$, a solution for x exists only if $b > 0$.

Remember the index laws:

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^m = a^m b^m$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad b \neq 0$
- $a^0 = 1, \quad a \neq 0$
- $a^{-m} = \frac{1}{a^m}, \quad a \neq 0$
- $a^{\frac{1}{m}} = \sqrt[m]{a}$
- $a^{\frac{n}{m}} = \sqrt[m]{a^n}$.

Also remember that $a^x > 0$ for all x.

WORKED EXAMPLE 5

Solve the following equations for x, giving your answers in exact form.

a $4^{3x} \times 16^{3-x} = 256$

b $7^{x-3} - 3 = 0$

c $(5^x - 25)(5^x + 1) = 0$

d $3^{2x} - 9(3^x) + 14 = 0$

THINK	WRITE
a 1 Convert the numbers to the same base.	a $4^{3x} \times 16^{3-x} = 256$ $4^{3x} \times (4^2)^{3-x} = 4^4$
2 Simplify and add the indices on the left-hand side of the equation.	$4^{3x} \times 4^{6-2x} = 4^4$ $4^{x+6} = 4^4$
3 As the bases are the same, equate the indices and solve the equation.	$x + 6 = 4$ $x = -2$
b 1 Rearrange the equation.	b $7^{x-3} - 3 = 0$ $7^{x-3} = 3$
2 Take the logarithm of both sides to base 7 and simplify.	$\log_7(7^{x-3}) = \log_7(3)$ $x - 3 = \log_7(3)$
3 Solve the equation.	$x = \log_7(3) + 3$
c 1 Apply the Null Factor Law to solve each bracket.	c $(5^x - 25)(5^x + 1) = 0$ $5^x - 25 = 0$ or $5^x + 1 = 0$ $5^x = 25$ $\quad$ $5^x = -1$
2 Convert 25 to base 5. $5^x > 0$, so there is no real solution for $5^x = -1$.	$5^x = 5^2$ $x = 2$
d 1 Let $a = 3^x$ and substitute into the equation to create a quadratic to solve.	d $3^{2x} - 9(3^x) + 14 = 0$ Let $a = 3^x$: $a^2 - 9a + 14 = 0$
2 Factorise the left-hand side.	$(a - 7)(a - 2) = 0$
3 Apply the Null Factor Law to solve each bracket for a.	$a - 7 = 0$ or $a - 2 = 0$ $a = 7$ $\quad$ $a = 2$
4 Substitute back in for a.	$3^x = 7$ $\quad$ $3^x = 2$
5 Take the logarithm of both sides to base 3 and simplify.	$\log_3(3^x) = \log_3(7)$ $\quad$ $\log_3(3^x) = \log_3(2)$ $x = \log_3(7)$ $\quad$ $x = \log_3(2)$

EXERCISE 4.4 Indicial equations

PRACTISE

Work without CAS

1 **WE5** Solve the following equations for x.

a $3^{2x+1} \times 27^{2-x} = 81$

b $10^{2x-1} - 5 = 0$

c $(4^x - 16)(4^x + 3) = 0$

d $2(10^{2x}) - 7(10^x) + 3 = 0$

2 Solve the following equations for x.

a $2^{x+3} - \frac{1}{64} = 0$

b $2^{2x} - 9 = 0$

c $3e^{2x} - 5e^x - 2 = 0$

d $e^{2x} - 5e^x = 0$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 Solve the following equations for x.

a $7^{2x-1} = 5$ **b** $(3^x - 9)(3^x - 1) = 0$

c $25^x - 5^x - 6 = 0$ **d** $6(9^{2x}) - 19(9^x) + 10 = 0$

4 Solve the following equations for x.

a $16 \times 2^{2x+3} = 8^{-2x}$ **b** $2 \times 3^{x+1} = 4$

c $2(5^x) - 12 = -\dfrac{10}{5^x}$ **d** $4^{x+1} = 3^{1-x}$

5 Solve the following equations for x.

a $2(2^{x-1} - 3) + 4 = 0$ **b** $2(5^{1-2x}) - 3 = 7$

6 **a** Simplify $x^{-1} - \dfrac{1}{1 - \dfrac{1}{1 + x^{-1}}}$. **b** Solve $2^{3-4x} \times 3^{-4x+3} \times 6^{x^2} = 1$ for x.

7 Solve the following equations for x.

a $e^{x-2} - 2 = 7$ **b** $e^{\frac{x}{4}} + 1 = 3$

c $e^{2x} = 3e^x$ **d** $e^{x^2} + 2 = 4$

8 Solve the following equations for x.

a $e^{2x} = e^x + 12$ **b** $e^x = 12 - 32e^{-x}$

c $e^{2x} - 4 = 2e^x$ **d** $e^x - 12 = -\dfrac{5}{e^x}$

9 If $y = m(10)^{nx}$, $y = 20$ when $x = 2$ and $y = 200$ when $x = 4$, find the values of the constants m and n.

10 Solve the following for x, correct to 3 decimal places.

a $2^x < 0.3$ **b** $(0.4)^x < 2$

11 Solve $(\log_3(4m))^2 = 25n^2$ for m.

12 Solve the following for x.

a $e^{m-kx} = 2n$, where $k \in R\backslash\{0\}$ and $n \in R^+$

b $8^{mx} \times 4^{2n} = 16$, where $m \in R\backslash\{0\}$

c $2e^{mx} = 5 + 4e^{-mn}$, where $m \in R\backslash\{0\}$

MASTER

13 If $y = ae^{-kx}$, $y = 3.033$ when $x = 2$ and $y = 1.1157$ when $x = 6$, find the values of the constants a and k. Give answers correct to 2 decimal places.

14 The compound interest formula $A = Pe^{rt}$ is an indicial equation. If a principal amount of money, P, is invested for 5 years, the interest earned is \$12 840.25, but if this same amount is invested for 7 years, the interest earned is \$14 190.66. Find the integer rate of interest and the principal amount of money invested, to the nearest dollar.

4.5 Logarithmic graphs

The graph of $y = \log_a(x)$

study on

Units 3 & 4
AOS 1
Topic 3
Concept 2

Logarithmic functions
Concept summary
Practice questions

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Interactivity
Logarithmic graphs
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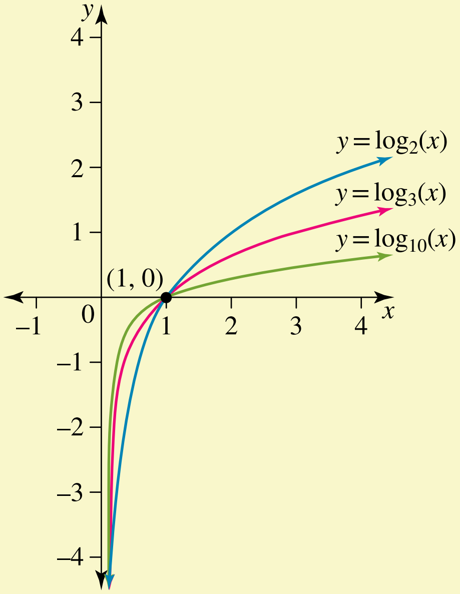
The graph of the logarithmic function $f: R^+ \to R$, $f(x) = \log_a(x)$, $a > 1$ has the following characteristics.

For $f(x) = \log_a(x)$, $a > 1$:

- **the domain is $(0, \infty)$**
- **the range is R**
- **the graph is an increasing function**
- **the graph cuts the x-axis at $(1, 0)$**
- **as $x \to 0$, $y \to -\infty$, so the line $x = 0$ is an asymptote**
- **as a increases, the graph rises more steeply for $x \in (0, 1)$ and is flatter for $x \in (1, \infty)$.**



Dilations

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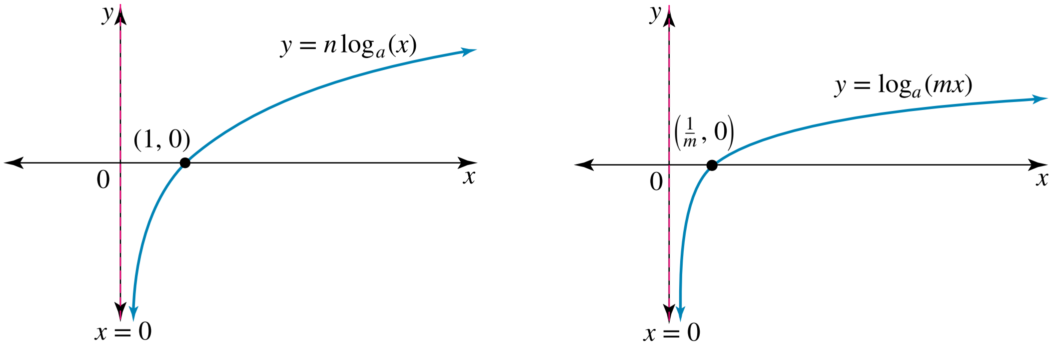
Units 3 & 4
AOS 1
Topic 5
Concept 5

Transformations of logarithmic functions
Concept summary
Practice questions

Graphs of the form $y = n\log_a(x)$ and $y = \log_a(mx)$

The graph of $y = n\log_a(x)$ is the basic graph of $y = \log_a x$ dilated by factor n parallel to the y-axis or from the x-axis. The graph of $y = \log_a(mx)$ is the basic graph of $y = \log_a x$ dilated by factor $\frac{1}{m}$ parallel to the x-axis or from the y-axis. The line $x = 0$ or the y-axis remains the vertical asymptote and the domain remains $(0, \infty)$.

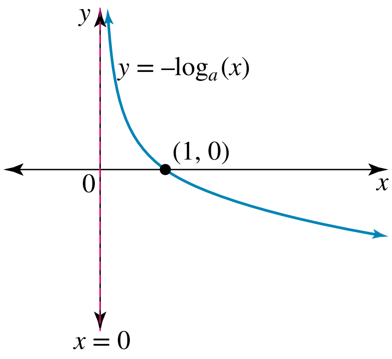


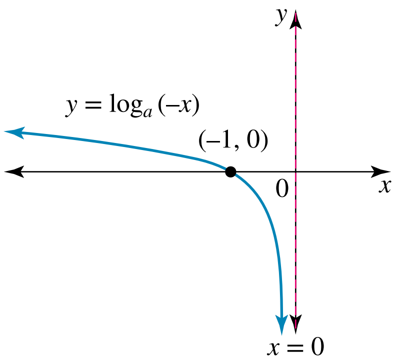
Reflections

Graphs of the form $y = -\log_a(x)$ and $y = \log_a(-x)$

The graph of $y = -\log_a(x)$ is the basic graph of $y = \log_a x$ reflected in the x-axis. The line $x = 0$ or the y-axis remains the vertical asymptote and the domain remains $(0, \infty)$.

The graph of $y = \log_a(-x)$ is the basic graph of $y = \log_a x$ reflected in the y-axis. The line $x = 0$ or the y-axis remains the vertical asymptote but the domain changes to $(-\infty, 0)$.



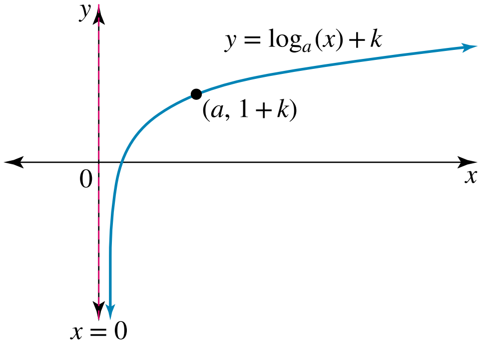


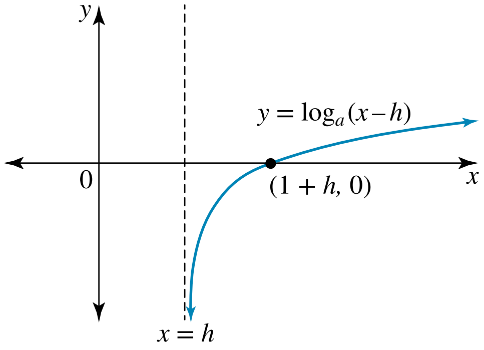
Translations

Graphs of the form $y = \log_a(x) - k$ and $y = \log_a(x - h)$

The graph of $y = \log_a(x) + k$ is the basic graph of $y = \log_a(x)$ translated k units parallel to the y-axis. Thus the line $x = 0$ or the y-axis remains the vertical asymptote and the domain remains $(0, \infty)$.

The graph of $y = \log_a(x - h)$ is the basic graph of $y = \log_a x$ translated h units parallel to the x-axis. Thus the line $x = 0$ or the y-axis is no longer the vertical asymptote. The vertical asymptote is $x = h$ and the domain is (h, ∞).





WORKED EXAMPLE 6

Sketch the graphs of the following, showing all important characteristics. State the domain and range in each case.

a $y = \log_e(x - 2)$

b $y = \log_e(x + 1) + 2$

c $y = \frac{1}{4}\log_e(2x)$

d $y = -\log_e(-x)$

THINK	WRITE/DRAW
a 1 The basic graph of $y = \log_e x$ has been translated 2 units to the right, so $x = 2$ is the vertical asymptote.	**a** $y = \log_e(x - 2)$ The domain is $(2, \infty)$. The range is R.

2 Find the x-intercept.

x-intercept, $y = 0$:

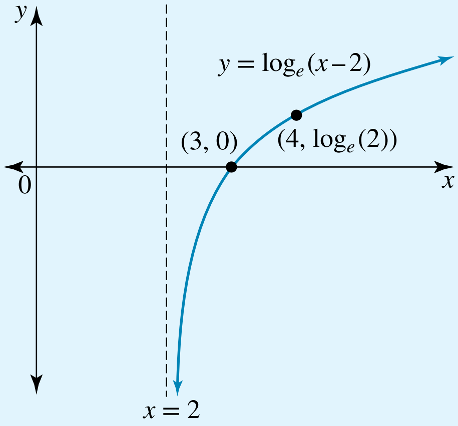
$$\begin{aligned} \log_e(x - 2) &= 0 \\ e^0 &= x - 2 \\ 1 &= x - 2 \\ x &= 3 \end{aligned}$$

3 Determine another point through which the graph passes.

When $x = 4,\ y = \log_e 2$.

The point is $(4, \log_e(2))$.

4 Sketch the graph.



b 1 The basic graph of $y = \log_e x$ has been translated up 2 units and 1 unit to the left, so $x = -1$ is the vertical asymptote.

b $y = \log_e(x + 1) + 2$

The domain is $(-1, \infty)$.

The range is R.

2 Find the x-intercept.

The graph cuts the x-axis where $y = 0$.

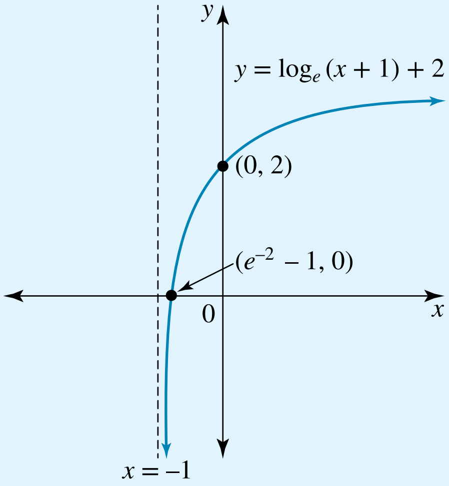
$$\begin{aligned} \log_e(x + 1) + 2 &= 0 \\ \log_e(x + 1) &= -2 \\ e^{-2} &= x + 1 \\ x &= e^{-2} - 1 \end{aligned}$$

3 Find the y-intercept.

The graph cuts the y-axis where $x = 0$.

$$\begin{aligned} y &= \log_e(1) + 2 \\ &= 2 \end{aligned}$$

4 Sketch the graph.



c **1** The basic graph of $y = \log_e x$ has been dilated by factor $\frac{1}{4}$ from the x-axis and by factor $\frac{1}{2}$ from the y-axis. The vertical asymptote remains $x = 0$.

c $y = \frac{1}{4}\log_e(2x)$

The domain is $(0, \infty)$.

The range is R.

2 Find the x-intercept.

x-intercept, $y = 0$:

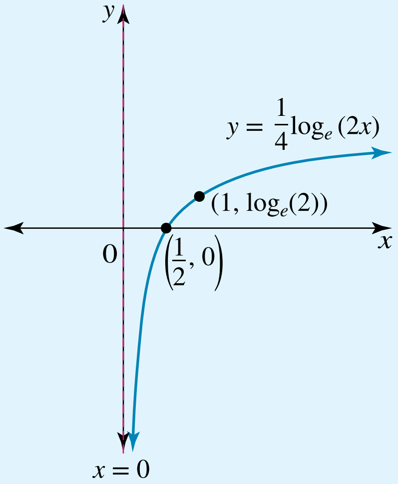
$$\begin{aligned}\tfrac{1}{4}\log_e(2x) &= 0\\ \log_e(2x) &= 0\\ e^0 &= 2x\\ 1 &= 2x\\ x &= \tfrac{1}{2}\end{aligned}$$

3 Determine another point through which the graph passes.

When $x = 1$, $y = \log_e(2)$.

The point is $(1, \log_e(2))$.

4 Sketch the graph.



d **1** The basic graph of $y = \log_e x$ has been reflected in both axes. The vertical asymptote remains $x = 0$.

d $y = -\log_e(-x)$

The domain is $(-\infty, 0)$.

The range is R.

2 Find the x-intercept.

x-intercept, $y = 0$:

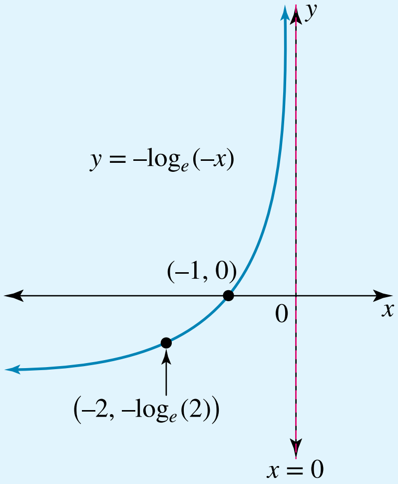
$$\begin{aligned}-\log_e(-x) &= 0\\ \log_e(-x) &= 0\\ e^0 &= -x\\ x &= -1\end{aligned}$$

3 Determine another point through which the graph passes.

When $x = -2$, $y = -\log_e(2)$.

The point is $(-2, -\log_e(2))$.

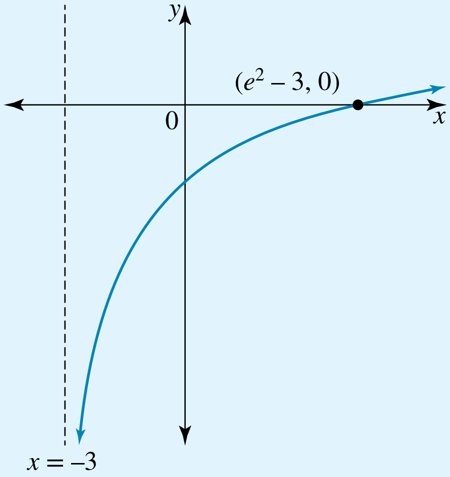
4 Sketch the graph.



The situation may arise where you are given the graph of a translated logarithmic function and you are required to find the rule. Information that could be provided to you is the equation of the asymptote, the intercepts and/or other points on the graph. As a rule, the number of pieces of information is equivalent to the number of unknowns in the equation.

WORKED EXAMPLE 7

The rule for the function shown is of the form $y = \log_e(x - a) + b$. Find the values of the constants a and b.



THINK	WRITE
1 The vertical asymptote corresponds to the value of a.	The vertical asymptote is $x = -3$, therefore a must be -3. So $y = \log_e(x + 3) + b$.
2 Substitute in the x-intercept to find b.	The graph cuts the x-axis at $(e^2 - 3, 0)$. $0 = \log_e(e^2 - 3 + 3) + b$ $-b = \log_e(e^2)$ $-b = 2$ $b = -2$ So $y = \log_e(x + 3) - 2$
3 Write the answer.	$a = -3, b = -2$

EXERCISE 4.5 Logarithmic graphs

PRACTISE

Work without CAS

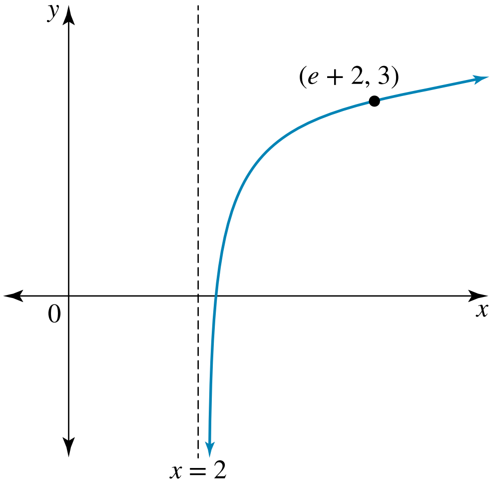
1 **WE6** Sketch the graphs of the following functions, showing all important characteristics. State the domain and range for each graph.

a $y = \log_e(x + 4)$
b $y = \log_e(x) + 2$
c $y = 4\log_e(x)$
d $y = -\log_e(x - 4)$

2 Sketch the graphs of the following functions, showing all important characteristics.

a $y = \log_3(x + 2) - 3$
b $y = 3\log_5(2 - x)$
c $y = 2\log_{10}(x + 1)$
d $y = \log_2\left(-\frac{x}{2}\right)$

3 WE7 The rule for the function shown is $y = \log_e(x - m) + n$. Find the values of the constants m and n.



4 The logarithmic function with the rule of the form $y = p\log_e(x - q)$ passes through the points (0, 0) and (1, −0.35). Find the values of the constants p and q.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

a $y = \log_e(x) + 3$ **b** $y = \log_e(x) - 5$ **c** $y = \log_e(x) + 0.5$

6 Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

a $y = \log_e(x - 4)$ **b** $y = \log_e(x + 2)$ **c** $y = \log_e(x + 0.5)$

7 Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

a $y = \frac{1}{4}\log_e(x)$ **b** $y = 3\log_e(x)$ **c** $y = 6\log_e(x)$

8 Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

a $y = \log_e(3x)$ **b** $y = \log_e\left(\frac{x}{4}\right)$ **c** $y = \log_e(4x)$

9 Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

a $y = 1 - 2\log_e(x - 1)$ **b** $y = \log_e(2x + 4)$ **c** $y = \frac{1}{2}\log_e\left(\frac{x}{4}\right) + 1$

10 For each of the following functions, state the domain and range. Define the inverse function, f^{-1}, and state the domain and range in each case.

a $f(x) = 2\log_e(3x + 3)$
b $f(x) = \log_e(2(x - 1)) + 2$
c $f(x) = 2\log_e(1 - x) - 2$

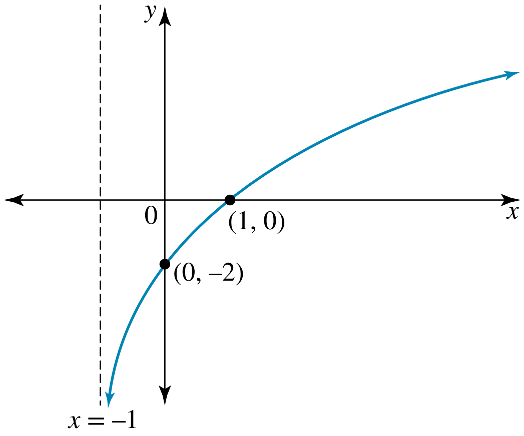
11 For each of the functions in question **10**, sketch the graphs of f and f^{-1} on the same set of axes. Give the coordinates of any points of intersection, correct to 2 decimal places.

12 The equation $y = a\log_e(bx)$ relates x to y. The table below shows values for x and y.

x	1	2	3
y	$\log_e(2)$	0	w

a Find the integer values of the constants a and b.
b Find the value of w correct to 4 decimal places.

13 The graph of a logarithmic function of the form $y = a\log_e(x - h) + k$ is shown below. Find the values of a, h and k.



14 The graph of $y = m\log_2(nx)$ passes through the points $(-2, 3)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Show that the values of m and n are 1.25 and $-2^{\frac{7}{5}}$ respectively.

MASTER

15 Solve the following equations for x. Give your answers correct to 3 decimal places.

a $x - 2 = \log_e(x)$ **b** $1 - 2x = \log_e(x - 1)$

16 Solve the following equations for x. Give your answers correct to 3 decimal places.

a $x^2 - 2 < \log_e(x)$ **b** $x^3 - 2 \leq \log_e(x)$

4.6 Applications

Logarithmic functions can be used to model many real-life situations directly. They can be used to solve exponential functions that are used to model other real-life scenarios.

WORKED EXAMPLE 8

If P dollars is invested into an account that earns interest at a rate of r for t years and the interest is compounded continuously, then $A = Pe^{rt}$, where A is the accumulated dollars.

A deposit of \$6000 is invested at the Western Bank, and \$9000 is invested at the Common Bank at the same time. Western offers compound interest continuously at a nominal rate of 6%, whereas the Common Bank offers compound interest continuously at a nominal rate of 5%. In how many years will the two investments be the same?

THINK	WRITE
1 Write the compound interest equation for each of the two investments.	$A = Pe^{rt}$ Western Bank: $A = 6000e^{0.06t}$ Common Bank: $A = 9000e^{0.05t}$

2 Equate the two equations and solve for t.

CAS could also be used to determine the answer.

$$6000e^{0.06t} = 9000e^{0.05t}$$

$$\frac{e^{0.06t}}{e^{0.05t}} = \frac{9000}{6000}$$

$$e^{0.01t} = \tfrac{3}{2}$$

$$0.01t = \log_e\left(\tfrac{3}{2}\right)$$

$$0.01t = 0.4055$$

$$t = \frac{0.4055}{0.01}$$

$$t = 40.5 \text{ years}$$

WORKED EXAMPLE 9

A coroner uses a formula derived from Newton's Law of Cooling to calculate the elapsed time since a person died. The formula is

$$t = -10\log_e\left(\frac{T-R}{37-R}\right)$$

where t is the time in hours since the death, T is the body's temperature measured in °C and R is the constant room temperature in °C. An accountant stayed late at work one evening and was found dead in his office the next morning. At 10 am the coroner measured the body temperature to be 29.7°C. A second reading at noon found the body temperature to be 28°C. Assuming that the office temperature was constant at 21°C, determine the accountant's estimated time of death.

THINK / **WRITE**

1 Determine the time of death for the 10 am information.

$R = 21°\text{C}$

$T = 29.7°\text{C}$

Substitute the values into the equation and evaluate.

$$t = -10\log_e\left(\frac{T-R}{37-R}\right)$$

$$t = -10\log_e\left(\frac{29.7-21}{37-21}\right)$$

$$= -10\log_e\left(\tfrac{8.7}{16}\right)$$

$$= -10\log_e(0.54375)$$

$$= 6.09 \text{ h}$$

2 Determine the time of death for the 12 pm information.

$R = 21°\text{C}$

$T = 28°\text{C}$

Substitute the values into the equation and evaluate.

$$t = -10\log_e\left(\frac{T-R}{37-R}\right)$$

$$t = -10\log_e\left(\frac{28-21}{37-21}\right)$$

$$= -10\log_e\left(\tfrac{7}{16}\right)$$

$$= -10\log_e(0.4376)$$

$$= 8.27 \text{ h}$$

3 Determine the estimated time of death for each reading.

$10 - 6.09 = 3.91$ or 3.55 am

$12 - 8.27 = 3.73$ or 3.44 am

4 Write the answer.

The estimated time of death is between 3.44 and 3.55 am.

EXERCISE 4.6 Applications

PRACTISE

1 **WE8** A deposit of \$4200 is invested at the Western Bank, and \$5500 is invested at the Common Bank at the same time. Western offers compound interest continuously at a nominal rate of 5%, whereas the Common bank offers compound interest continuously at a nominal rate of 4.5%. In how many years will the two investments be the same? Give your answer to the nearest year.

2 **a** An investment triples in 15 years. What is the interest rate that this investment earns if it is compounded continuously? Give your answer correct to 2 decimal places.

b An investment of \$2000 earns 4.5% interest compounded continuously. How long will it take for the investment to have grown to \$9000? Give your answer to the nearest month.

3 **WE9** An elderly person was found deceased by a family member. The two had spoken on the telephone the previous evening around 7 pm. The coroner attended and found the body temperature to be 25°C at 9 am. If the house temperature had been constant at 20°C, calculate how long after the telephone call the elderly person died. Use Newton's Law of Cooling, $t = -10 \log_e\left(\dfrac{T - R}{37 - R}\right)$, where R is the room temperature in °C and T is the body temperature in °C.

4 The number of parts per million, n, of a fungal bloom in a stream t hours after it was detected can be modelled by $n(t) = \log_e (t + e^2),\ t \geq 0$.

a How many parts per million were detected initially?

b How many parts of fungal bloom are in the stream after 12 hours? Give your answer to 2 decimal places.

c How long will it take before there are 4 parts per million of the fungal bloom? Give your answer correct to 1 decimal place.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 If \$1000 is invested for 10 years at 5% interest compounded continuously, how much money will have accumulated after the 10 years?

6 Let $P(t) = 200^{kt} + 1000$ represent the number of bacteria present in a petri dish after t hours. Suppose the number of bacteria trebles every 8 hours. Find the value of the constant k correct to 4 decimal places.

7 An epidemiologist studying the progression of a flu epidemic decides that the function

$$P(t) = \tfrac{3}{4}(1 - e^{-kt}),\ k > 0$$

will be a good model for the proportion of the earth's population that will contract the flu after t months. If after 3 months $\dfrac{1}{1500}$ of the earth's population has the flu, find the value of the constant k, correct to 4 decimal places.

8 Carbon-14 dating works by measuring the amount of carbon-14, a radioactive element, that is present in a fossil. All living things have a constant level of carbon-14 in them. Once an organism dies, the carbon-14 in its body starts to decay according to the rule

$$Q = Q_0 e^{-0.000\,124t}$$

where t is the time in years since death, Q_0 is the amount of carbon-14 in milligrams present at death and Q is the quantity of carbon-14 in milligrams present after t years.

- **a** If it is known that a particular fossil initially had 100 milligrams of carbon-14, how much carbon-14, in milligrams, will be present after 1000 years? Give your answer correct to 1 decimal place.
- **b** How long will it take before the amount of carbon-14 in the fossil is halved? Give your answer correct to the nearest year.

9 Glottochronology is a method of dating a language at a particular stage, based on the theory that over a long period of time linguistic changes take place at a fairly constant rate. Suppose a particular language originally has W_0 basic words and that at time t, measured in millennia, the number, $W(t)$, of basic words in use is given by $W(t) = W_0(0.805)^t$.

- **a** Calculate the percentage of basic words lost after ten millennia.
- **b** Calculate the length of time it would take for the number of basic words lost to be one-third of the original number of basic words. Give your answer correct to 2 decimal places.

10 The mass, M grams, of a radioactive element, is modelled by the rule

$$M = a - \log_e(t + b)$$

where t is the time in years. The initial mass is 7.8948 grams, and after 80 years the mass is 7.3070 grams.

- **a** Find the equation of the mass remaining after t years. Give a correct to 1 decimal place and b as an integer.
- **b** Find the mass remaining after 90 years.

11 The population, P, of trout at a trout farm is declining due to deaths of a large number of fish from fungal infections.

The population is modelled by the function

$$P = a\log_e(t) + c$$

where t represents the time in weeks since the infection started. The population of trout was 10 000 after 1 week and 6000 after 4 weeks.

- **a** Find the values of the constants a and c. Give your answers correct to 1 decimal place where appropriate.
- **b** Find the number of trout, correct to the nearest whole trout, after 8 weeks.
- **c** If the infection remains untreated, how long will it take for the population of trout to be less than 1000? Give your answer correct to 1 decimal place.

12 In her chemistry class, Hei is preparing a special solution for an experiment that she has to complete. The concentration of the solution can be modelled by the rule

$$C = A\log_e(kt)$$

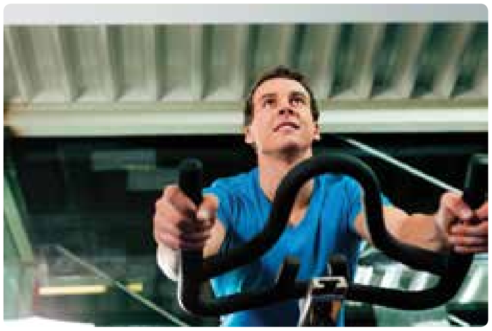
where C is the concentration in moles per litre (M) and t represents the time of mixing in seconds. The concentration of the solution after 30 seconds of mixing is 4 M, and the concentration of the solution after 2 seconds of mixing was 0.1 M.

a Find the values of the constants A and k, giving your answers correct to 3 decimal places.
b Find the concentration of the solution after 15 seconds of mixing.
c How long does it take, in minutes and seconds, for the concentration of the solution to reach 10 M?

13 Andrew believes that his fitness level can be modelled by the function

$$F(t) = 10 + 2\log_e(t + 2)$$

where $F(t)$ is his fitness level and t is the time in weeks since he started training.



a What was Andrew's level of fitness before he started training?
b After 4 weeks of training, what was Andrew's level of fitness?
c How long will it take for Andrew's level of fitness to reach 15?

14 In 1947 a cave with beautiful prehistoric paintings was discovered in Lascaux, France. Some charcoal found in the cave contained 20% of the carbon-14 that would be expected in living trees. Determine the age of the paintings to the nearest whole number if

$$Q = Q_0 e^{-0.000\,124t}$$

where Q_0 is the amount of carbon-14 originally and t is the time in years since the death of the prehistoric material. Give your answer correct to the nearest year.

MASTER

15 The sales revenue, R dollars, that a manufacturer receives for selling x units of a certain product can be modelled by the function

$$R(x) = 800\log_e\left(2 + \frac{x}{250}\right).$$

Furthermore, each unit costs the manufacturer 2 dollars to produce, and the initial cost of adjusting the machinery for production is \$300, so the total cost in dollars, C, of production is

$$C(x) = 300 + 2x.$$

a Write the profit, $P(x)$ dollars, obtained by the production and sale of x units.
b Find the number of units that need to be produced and sold to break even, that is, $P(x) = 0$. Give your answer correct to the nearest integer.

16 The value of a certain number of shares, \$$V$, can be modelled by the equation

$$V = ke^{mt}$$

where t is the time in months. The original value of the shares was \$10 000, and after one year the value of the shares was \$13 500.

a Find the values of the constants k and m, giving answers correct to 3 decimal places where appropriate.
b Find the value of the shares to the nearest dollar after 18 months.
c After t months, the shares are sold for 1.375 times their value at the time. Find an equation relating the profit made, P, over the time the shares were owned.
d If the shares were kept for 2 years, calculate the profit made on selling the shares at that time.

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4.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

To access eBookPLUS activities, log on to

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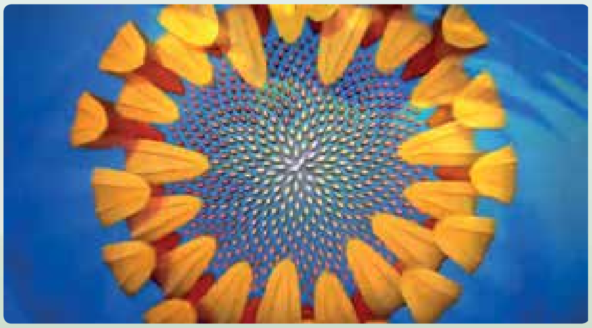
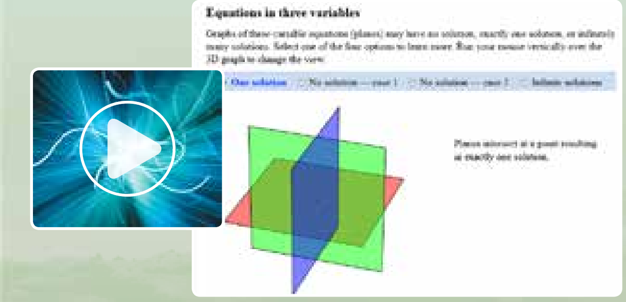
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





Equations in three variables

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

4 Answers

EXERCISE 4.2

1 a 4 **b** –5

c $\frac{1}{2}$ **d** –1

2 a 0

b $3\log_7(2x - 1)$

c $\log_{10}(x - 1)$

3 a 3 **b** $\frac{4}{3}$

c $2^{-\frac{1}{3}}, 4$ **d** $\frac{7}{2}$

4 a 243 **b** $\frac{47}{10}$

5 a i 1.2770

ii –1.2619

b i 2×3^z

ii $\frac{3}{z}$

6 a $\dfrac{\log_{10}(9)}{\log_{10}(5)}$ **b** $\dfrac{\log_{10}(12)}{\log_{10}\left(\frac{1}{2}\right)}$

7 a $\log_6 216 = 3$ **b** $\log_2 256 = 8$

c $\log_3 81 = 4$ **d** $\log_{10} 0.0001 = -4$

e $\log_5 0.008 = -3$ **f** $\log_7 7 = 1$

8 a 4 **b** –3

c 11 **d** –128

9 a 7 **b** –5

c $-\frac{1}{2}$ **d** –2

e $\frac{5}{12}$ **f** $\frac{2}{5}$

10 a $3\log_3(x - 4)$ **b** $\log_7(2x + 3)$

c 0 **d** $2\log_4(5x + 1)$

11 a 1.7712 **b** –6.9189

12 a 5^{n+1} **b** $1 + 2n$

c $\dfrac{4}{n}$

13 a $\frac{1}{2}(e^{-3} + 1)$ **b** e^{-3}

c 7 **d** 15

e 10 **f** $-\frac{6}{5}$

g 6, 1 **h** $\frac{5}{4}$

i $\frac{5}{4}$ **j** $x = 10, 10^{\frac{3}{2}}$

k $x = 9, \frac{1}{3}$ **l** 4

14 a $y = \dfrac{4}{x^3}$ **b** $y = \dfrac{x^2}{16}$

c $y = \dfrac{9}{x}$ **d** $y = 64x$

15 a $x = 3m$

b $\log_{10} m = x$, so $10^x = m$, and $\log_{10} n = y$, so $10^y = n$.

$$\begin{aligned}
\log_{10}\left(\frac{100n^2}{m^5\sqrt{n}}\right) &= \log_{10}\left(\frac{100(10^y)^2}{(10^x)^5(10^y)^{\frac{1}{2}}}\right)\\
&= \log_{10}\left(\frac{10^2 \times 10^{2y}}{10^{5x} \times 10^{\frac{y}{2}}}\right)\\
&= \log_{10}\left(\frac{10^2 \times 10^{\frac{3y}{2}}}{10^{5x}}\right)\\
&= \log_{10}\left(10^{2 + \frac{3y}{2} - 5x}\right)\\
&= \left(2 + \frac{3y}{2} - 5x\right)\log_{10} 10\\
&= 2 + \frac{3y}{2} - 5x
\end{aligned}$$

16 16, $\frac{1}{16}$

17 a $x = -0.463, 0.675$ **b** $x = 0.451, 1$

18 1.5518, 1.4422

EXERCISE 4.3

1 10 W/m^2

2 61 808

3 The magnitude 6.4 earthquake is 1.41 times stronger than the magnitude 6.3 earthquake.

4 3691.17

5 The 500 W/m^2 amplifier is 13.98 dB louder.

6 160 dB

7 Lemon is acidic with a pH of 3.

8 a 1 M/litre **b** 0.0001 M/litre

c 10^{-8} M/litre **d** 10^{-12} M/litre

9 a 4.8, acidic **b** 5.56, acidic

10 a

$$\begin{aligned}
0.5N_0 &= N_0e^{-mt}\\
\tfrac{1}{2} &= e^{-mt}\\
\log_e\left(\tfrac{1}{2}\right) &= -mt\\
\log_e(2^{-1}) &= -mt\\
-\log_e(2) &= -mt\\
\log_e(2) &= mt\\
t &= \frac{\log_e(2)}{m}
\end{aligned}$$

b 9988 years

11 –437.97

12 361 cents

13 133.98 dB, so protection should be worn.

14 The magnitude of the South American earthquake was 8.7.

EXERCISE 4.4

1 a 3 **b** $\frac{1}{2}\log_{10}(5) + \frac{1}{2}$

c 2 **d** $\log_{10}(3)$, $\log_{10}\left(\frac{1}{2}\right)$

2 a –9 **b** $\frac{1}{2}\log_2(9)$

c $\log_e(2)$ **d** $\log_e(5)$

3 a $\frac{1}{2}\log_7(5) + \frac{1}{2}$ **b** 0, 2

c $\log_5(3)$ **d** $\log_9\left(\frac{2}{3}\right)$ or $\log_9\left(\frac{5}{2}\right)$

4 a $-\frac{7}{8}$ **b** $\log_3(2) - 1$

c 0, 1 **d** $\dfrac{\log_e\left(\frac{3}{4}\right)}{\log_e(12)}$

5 a 1 **b** 0

6 a $\frac{1}{x} - x - 1$ **b** 1 or 3

7 a $2\log_e(3) + 2$

b $4\log_e(2)$

c $\log_e(3)$

d $-\sqrt{\log_e(2)}, \sqrt{\log_e(2)}$

8 a $2\log_e(2)$

b $2\log_e(2), 3\log_e(2)$

c $\log_e(\sqrt{5} + 1)$

d $\log_e(6 - \sqrt{31})$, $\log_e(\sqrt{31} + 6)$

9 $m = 2$ and $n = \frac{1}{2}$

10 a $x < -1.737$ **b** $x > -0.756$

11 $\frac{3^{5n}}{4}, \frac{1}{4 \times 3^{5n}}$

12 a $\frac{m}{k} - \frac{1}{k}\log_e(2n), k \in R\backslash\{0\}$ and $n \in R^+$

b $\frac{4 - 4n}{3m}, m \in R\backslash\{0\}$

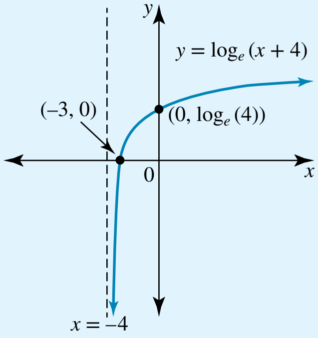
c $\frac{1}{m}\log_e\left(\frac{\sqrt{57} + 5}{4}\right), m \in R\backslash\{0\}$

13 $a = 5, k = 0.25$

14 $P = \$10\,000, r = 5\%$

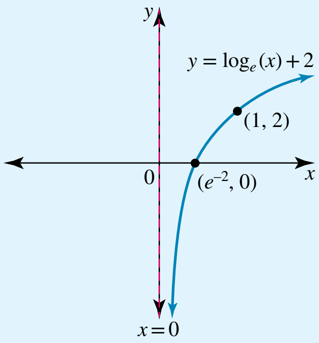
EXERCISE 4.5

1 a



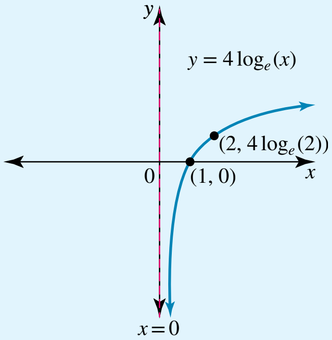
Domain = (–4, ∞), range = R

b



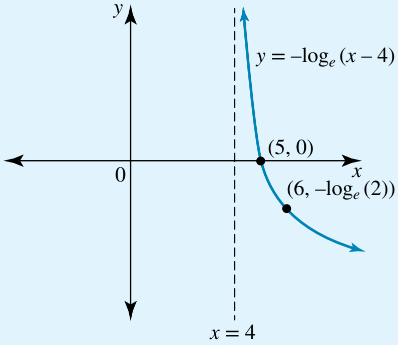
Domain = (0, ∞), range = R

c



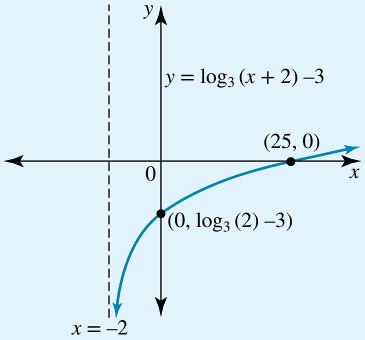
Domain = (0, ∞), range = R

d



Domain = (4, ∞), range = R

2 a



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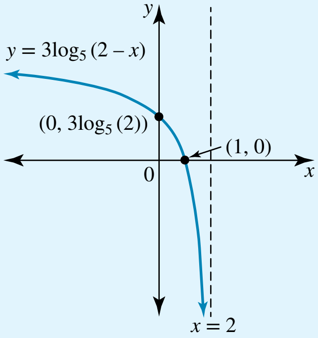

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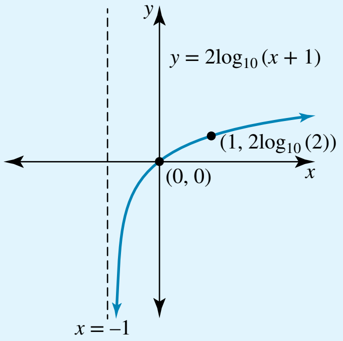
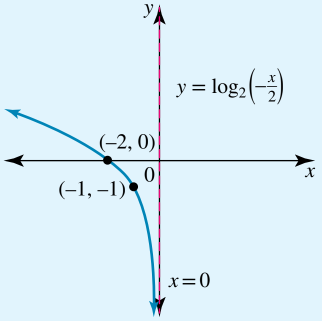


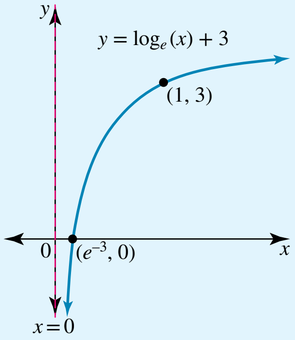
d



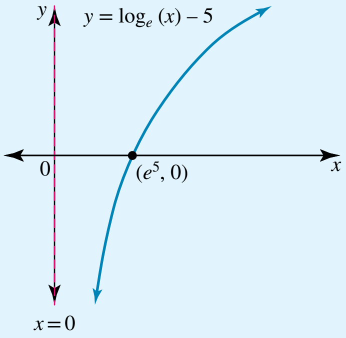
3 $m = 2, n = 2$

4 $p = -\dfrac{7}{20\log_e(2)}, q = -1$

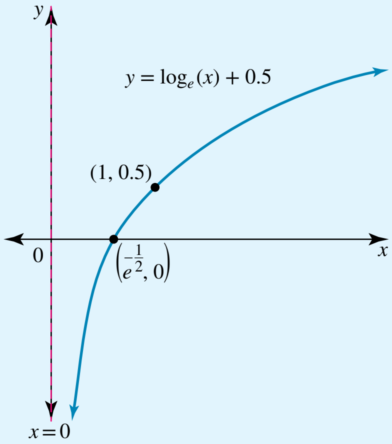
5 a



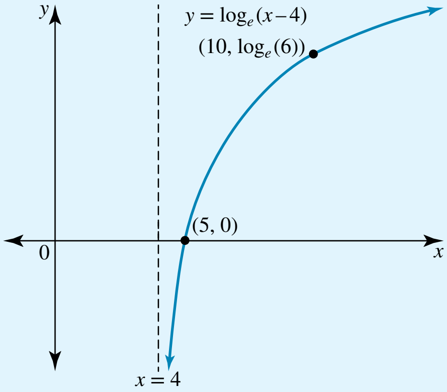
b



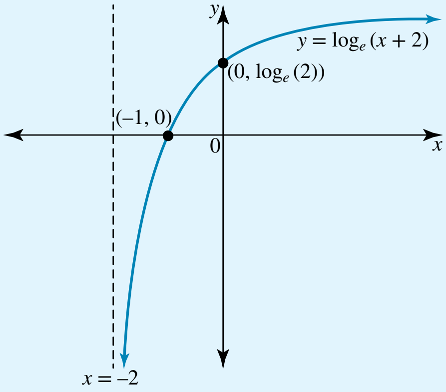
c



6 a



b



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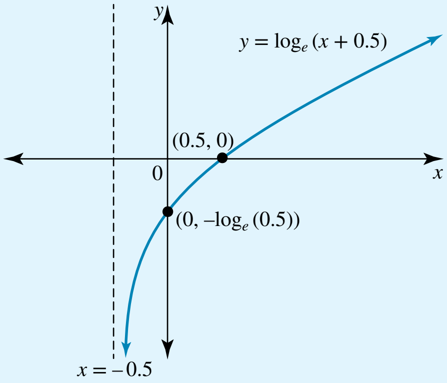
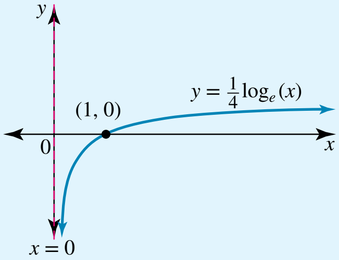
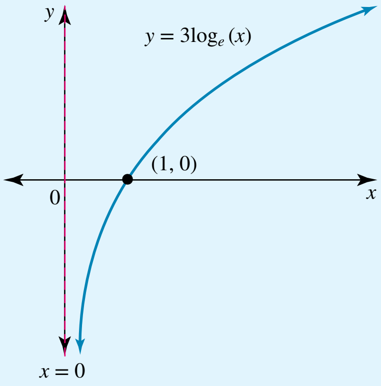


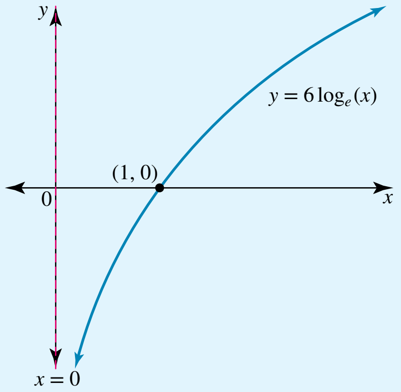
7 a $y = \frac{1}{4}\log_e(x)$



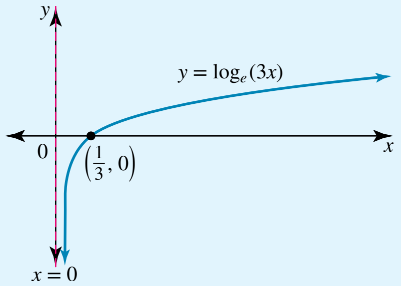
b $y = 3\log_e(x)$



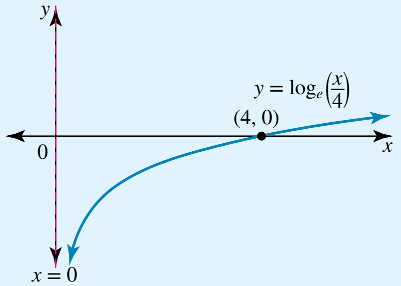
c $y = 6\log_e(x)$



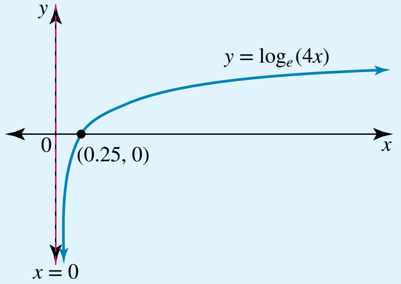
8 a $y = \log_e(3x)$



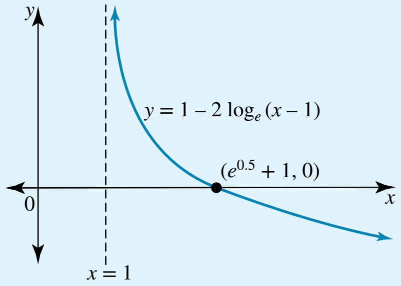
b $y = \log_e\left(\frac{x}{4}\right)$



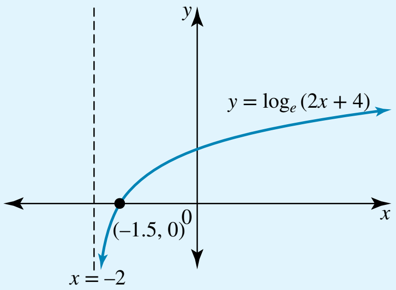
c $y = \log_e(4x)$



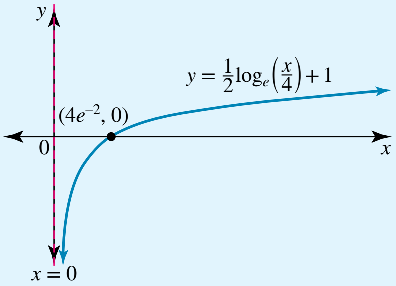
9 a $y = 1 - 2\log_e(x - 1)$



b $y = \log_e(2x + 4)$



c $y = \frac{1}{2}\log_e\left(\frac{x}{4}\right) + 1$



10 a $f(x) = 2\log_e(3(x + 1))$, domain = $(-1, \infty)$ and range = R

$f^{-1}(x) = \frac{1}{3}e^{\frac{x}{2}} - 1$, domain = R and range = $(-1, \infty)$

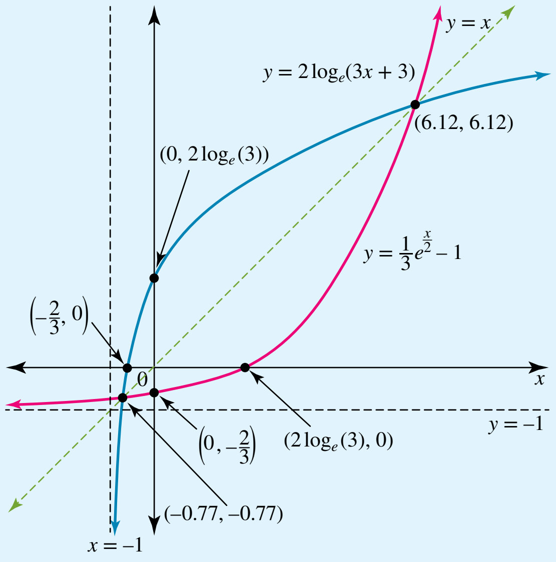
b $f(x) = \log_e(2(x - 1)) + 2$, domain = $(1, \infty)$ and range = R

$f^{-1}(x) = \frac{1}{2}e^{x-2} + 1$, domain = R and range = $(1, \infty)$

c $f(x) = 2\log_e(1 - x) - 2$, domain = $(-\infty, 1)$ and range = R

$f^{-1}(x) = 1 - e^{\frac{1}{2}(x+2)}$, domain = R and range = $(-\infty, 1)$

11 a



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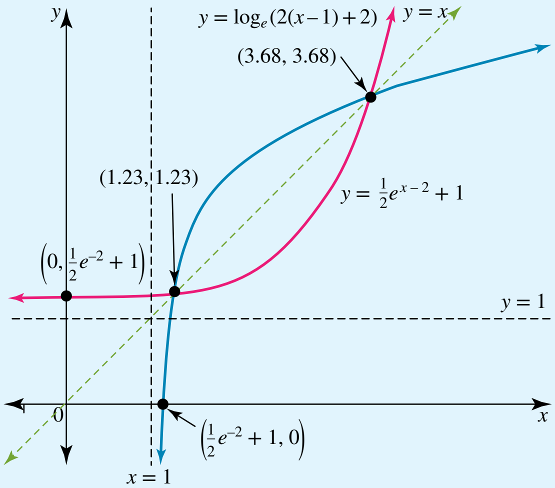
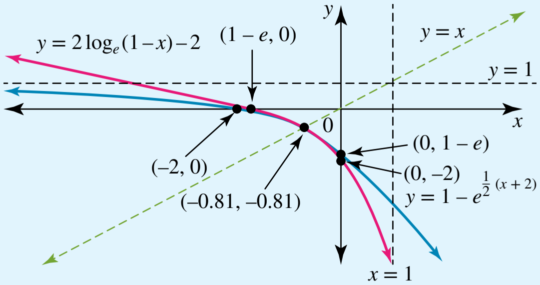


c



12 a $a = -1, b = \frac{1}{2}$

b –0.4055

13 $a = \dfrac{2}{\log_e(2)}, \; h = -1, \; k = -2$

14 $(-2, 3) \Rightarrow 3 = m\log_2(-2n)$ [1]

$\left(-\frac{1}{2}, \frac{1}{2}\right) \Rightarrow \frac{1}{2} = m\log_2\left(-\dfrac{n}{2}\right)$ [2]

[1] – [2]:

$$\begin{aligned}
3 - \tfrac{1}{2} &= m\log_2(-2n) - m\log_2\left(-\frac{n}{2}\right) \\
\tfrac{5}{2} &= m\left(\log_2(-2n) - \log_2\left(-\frac{n}{2}\right)\right) \\
&= m\left(\log_2\left(\frac{-2n}{-\frac{n}{2}}\right)\right) \\
&= m\log_2(4) \\
&= m\log_2 2^2 \\
&= 2m \\
m &= \tfrac{5}{4}
\end{aligned}$$

Substitute $m = \frac{5}{4}$ into [1]:

$$\begin{aligned}
3 &= \tfrac{5}{4}\log_2(-2n) \\
\tfrac{12}{5} &= \log_2(-2n) \\
2^{\frac{12}{5}} &= -2n \\
n &= 2^{\frac{12}{5}} \div -2 \\
&= -2^{\frac{7}{5}}
\end{aligned}$$

15 a 0.159 or 3.146

b 1.2315

16 a $x \in (0.138, 1.564)$

b $x \in [0.136, 1.315]$

EXERCISE 4.6

1 54 years

2 a 7.32%

b 33 years 5 months

3 8.46 pm, so the person died one and three quarter hours after the phone call.

4 a 2 parts per million

b 2.96 parts per million

c 47.2 hours

5 \$1648.72

6 0.1793

7 0.0003

8 a 88.3 milligrams

b 5590 years

9 a 88.57% lost

b 1.87 millennia

10 a $a = 12.5, b = 100$

b 7.253 g

11 a $a = -2885.4, c = 10000$

b 4000

c 22.6 weeks

12 a $A = 1.440, k = 0.536$

b 3.002 M

c 32 minutes 14 seconds

13 a 11.3863

b 13.5835

c 10.18 weeks

14 12 979 years

15 a $P(x) = 800\log_e\left(2 + \dfrac{x}{250}\right) - 300 - 2x$

b 330

16 a $k = 10\,000, m = 0.025$

b \$15 685.58

c $P = 13\,750e^{0.025t} - 10000$

d \$15 059.38

5

Differentiation

5.1 Kick off with CAS

Gradients and tangents of a curve

1 a Using the graph application on CAS, sketch the graph of $f(x) = \frac{1}{2}x^3 - 2x^2$.

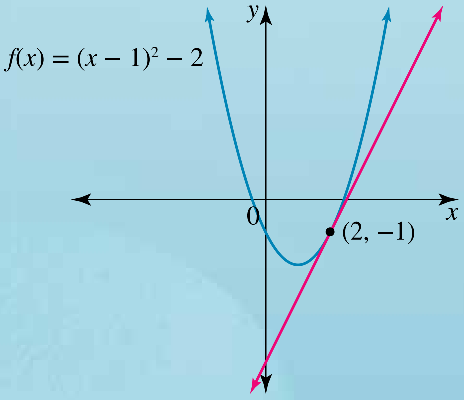
b Using the derivative template, sketch the graph of the gradient function, $f'(x)$, on the same set of axes as $f(x)$.

c Repeat this process for $f(x) = -(x - 1)^2 + 3$ and $f(x) = x^4 + x^3 - 7x^2 - x + 6$. Use a different set of axes for each $f(x)$ and $f'(x)$ pair.

d What do you notice about the positions of the turning points and x-intercepts? What do you notice about the shapes of the graphs?

2 a On a new set of axes, sketch the graph of $f(x) = (x - 1)^2 - 2$.

b Draw a tangent to the curve at $x = 2$.



c What is the equation of this tangent, and hence, what is the gradient of the parabolic curve at $x = 2$?

d Draw a tangent to the curve at $x = -1$. What is the gradient of the curve at this point?

3 a Open the calculation application on CAS and define $f(x) = (x - 1)^2 - 2$.

b Determine the derivative function, $f'(x)$, and calculate $f'(2)$ and $f'(-1)$.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

5.2 Review of differentiation

The derivative of a function

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Units 3 & 4
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Concept 1

Differentiation
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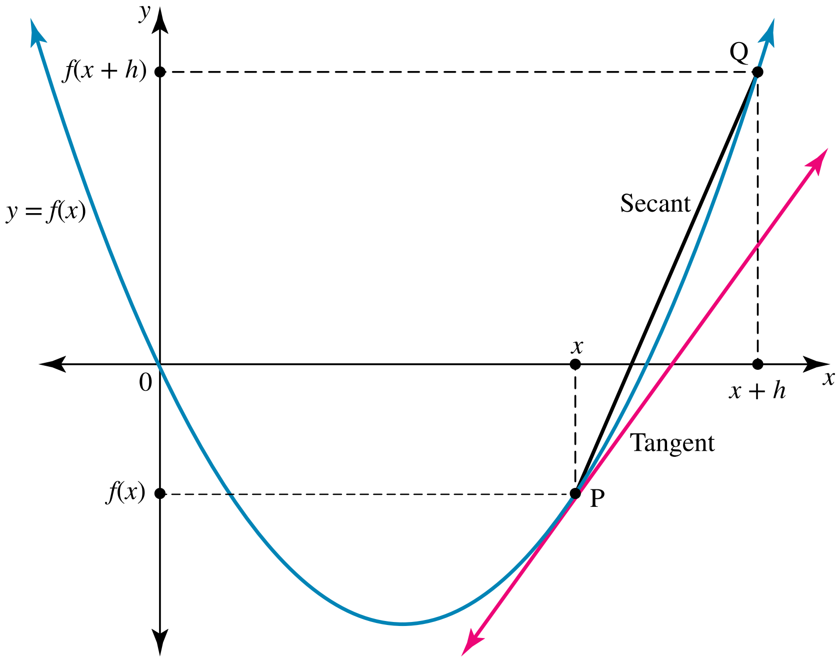
Practice questions

The **gradient** of a curve is the rule for the instantaneous **rate of change** of the function at any point. The gradient at any point (x, y) can be found using **differentiation by first principles**.

Consider the **secant** PQ on the curve $y = f(x)$. The coordinates of P are $(x, f(x))$, and the coordinates of Q are $(x + h, f(x + h))$.

The gradient of the secant, otherwise known as the average rate of change of the function, is found in the following way.



$$\textbf{Average rate of change} = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$

As Q gets closer and closer to P, h gets smaller and smaller and in fact is approaching zero. When Q is effectively the same point as P, the secant becomes the **tangent** to the curve at P. This is called the limiting situation.

$$\textbf{Gradient of the tangent at P} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{or } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In this notation, $f'(x)$ is the **derivative** of the function, or the gradient of the tangent to the curve at the point $(x, f(x))$. $f'(x)$ is also the gradient function of $f(x)$, and $\frac{dy}{dx}$ is the gradient equation for y with respect to x.

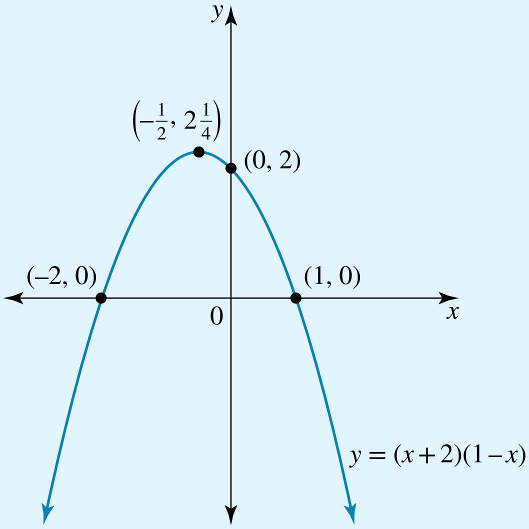
WORKED EXAMPLE 1

Consider the function $f(x) = (x + 2)(1 - x)$.

a **Sketch the graph of the parabolic function, showing axis intercepts and the coordinates of any turning points.**

b **If P is the point $(x, f(x))$ and Q is the point $(x + h, f(x + h))$, find the gradient of the secant PQ using first principles.**

c **If P is the point $(-2, 0)$, determine the gradient of the tangent to the curve at $x = -2$.**

THINK	WRITE/DRAW
a **1** Find the axis intercepts and the coordinates of the turning points.	**a** y-intercept: $x = 0$ $\therefore y = 2$ x-intercept: $y = 0$ $(x + 2)(1 - x) = 0$ $x + 2 = 0$ or $1 - x = 0$ $x = -2, 1$ Turning point: $x = \frac{-2 + 1}{2} = -\frac{1}{2}$ $y = \left(-\frac{1}{2} + 2\right)\left(1 + \frac{1}{2}\right)$ $= \frac{3}{2} \times \frac{3}{2}$ $= \frac{9}{4}$ The turning point is $\left(-\frac{1}{2}, \frac{9}{4}\right)$.
2 Sketch the graph of the function.	
b **1** Expand the quadratic.	**b** $f(x) = (x + 2)(1 - x)$ $= -x^2 - x + 2$

2 Find the gradient of the secant PQ by applying the rule gradient of secant $= \dfrac{f(x+h)-f(x)}{h}$ and simplifying.	Gradient of secant: $\text{Gradient} = \dfrac{f(x+h)-f(x)}{h}$ $= \dfrac{-(x+h)^2-(x+h)+2-(-x^2-x+2)}{h}$ $= \dfrac{-(x^2+2xh+h^2)-x-h+2+x^2+x-2}{h}$ $= \dfrac{-x^2-2xh-h^2-h+x^2}{h}$ $= \dfrac{-2xh-h^2-h}{h}$ $= \dfrac{h(-2x-h-1)}{h}$ $= -2x-h-1$
c 1 Find the rule for the gradient of the tangent at P by applying $\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}$.	**c** The gradient of the tangent at P is given by $\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0}(-2x-h-1)$ $= -2x-1$
2 Substitute $x=-2$ into the formula for the gradient of the tangent.	$\text{Gradient at P} = -2x-1.$ $x=-2 \Rightarrow \text{gradient} = -2\times-2-1$ $= 3$

The derivative of x^n

Differentiating from first principles is quite a tedious method, but there are rules to shortcut the process, depending on the function. Units 1 and 2 of Mathematical Methods cover differentiation when $f(x) = x^n$.

If $f(x) = ax^n, f'(x) = nax^{n-1}$ where $n \in R$ and $a \in R$

and

if $f(x) = g(x) \pm h(x), f'(x) = g'(x) \pm h'(x)$

WORKED EXAMPLE 2

Differentiate:

a $f(x) = x^3 - \dfrac{1}{2x} + 4$

b $y = \dfrac{\sqrt{x} - 3x^3}{4x^2}$.

THINK	WRITE/DRAW
a 1 Rewrite the equation with negative indices.	**a** $f(x) = x^3 - \dfrac{1}{2x} + 4$ $= x^3 - \frac{1}{2}x^{-1} + 4$
2 Differentiate each term separately.	$f'(x) = 3x^2 + \frac{1}{2}x^{-2}$
3 Write the answer with positive indices.	$f'(x) = 3x^2 + \dfrac{1}{2x^2}$

b **1** Split the fraction into two terms and rewrite $\sqrt{x}$ using a fractional index.	**b** $y = \frac{\sqrt{x} - 3x^3}{4x^2}$ $= \frac{x^{\frac{1}{2}}}{4x^2} - \frac{3x^3}{4x^2}$
2 Simplify each term by applying the index laws.	$= \frac{x^{-\frac{3}{2}}}{4} - \frac{3}{4}x$
3 Differentiate each term separately.	$\frac{dy}{dx} = -\frac{3}{2} \times \frac{x^{-\frac{5}{2}}}{4} - \frac{3}{4}$
4 Simplify and write the answer with positive indices.	$= -\frac{3}{8x^{\frac{5}{2}}} - \frac{3}{4}$

WORKED EXAMPLE 3

a **If $f(x) = \frac{3}{x} - x^2$, what is the gradient of the curve when $x = -2$?**

b **If $f(x) = 2\sqrt{x} - 4$, determine the coordinates of the point where the gradient is 2.**

THINK	WRITE
a **1** Rewrite the equation with negative indices and differentiate each term.	**a** $f(x) = \frac{3}{x} - x^2$ $= 3x^{-1} - x^2$ $f'(x) = -3x^{-2} - 2x$ $= -\frac{3}{x^2} - 2x$
2 The gradient of the curve when $x = -2$ is $f'(-2)$.	$f'(-2) = -\frac{3}{(-2)^2} - 2 \times -2$ $= -\frac{3}{4} + 4$ $= \frac{13}{4}$
b **1** Rewrite $\sqrt{x}$ with a fractional index and differentiate each term.	**b** $f(x) = 2\sqrt{x} - 4$ $= 2x^{\frac{1}{2}} - 4$ $f'(x) = x^{-\frac{1}{2}}$ $= \frac{1}{\sqrt{x}}$
2 Finding where the gradient is 2 means solving $f'(x) = 2$.	$\frac{1}{\sqrt{x}} = 2$ $1 = 2\sqrt{x}$ $\frac{1}{2} = \sqrt{x}$ $x = \frac{1}{4}$

3 Find $f\left(\frac{1}{4}\right)$ to determine the y-value where the gradient is 2.

$$f\left(\frac{1}{4}\right) = 2\sqrt{\frac{1}{4}} - 4$$
$$= 2 \times \frac{1}{2} - 4$$
$$= -3$$

4 Write the answer.

The gradient is 2 at the point $\left(\frac{1}{4}, -3\right)$.

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Units 3 & 4
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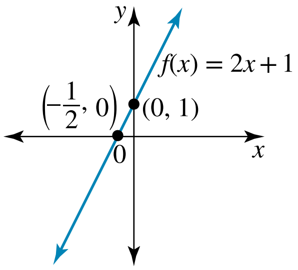
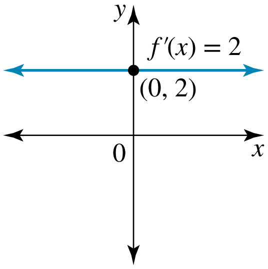
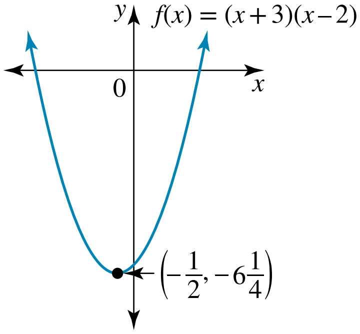
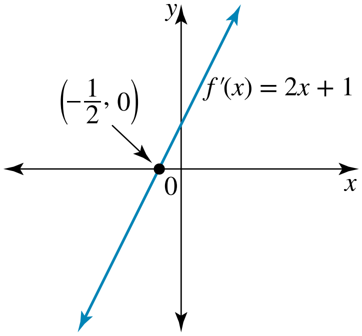
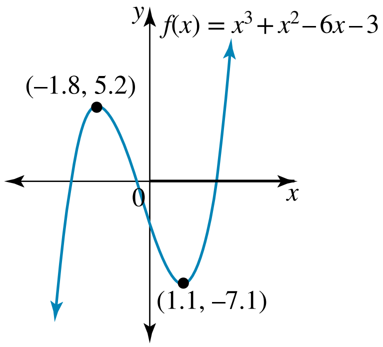
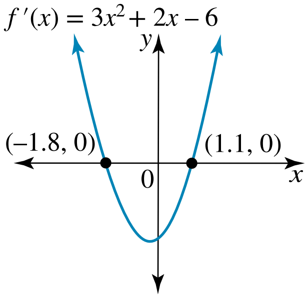
Concept 4

Graphs of the gradient function
Concept summary
Practice questions

Graphs of the gradient function

The previous section shows that the derivative of the function $f(x) = x^n$ is one degree lower: $f'(x) = nx^{n-1}$. This also applies to the gradient graphs of these functions. For example, if $f(x)$ is a quadratic graph, $f'(x)$ will be a linear graph; if $f(x)$ is a cubic graph, $f'(x)$ will be a quadratic graph, and so on.

Given function $f(x)$	**Gradient function $f'(x)$**
A line of the form $y = mx + c$ is degree one, and the gradient is m. Example:  	The gradient is a constant value, so the gradient graph is a line parallel to the x-axis, $y = m$, degree zero. 
A quadratic of the form $y = ax^2 + bx + c$ is degree two. Example: The function shown has a local minimum at $x = -\frac{1}{2}$.  	A line of the form $y = mx + c$ is degree one. The line shown has an x-intercept at $x = -\frac{1}{2}$. 

Given function $f(x)$	**Gradient function $f'(x)$**
A cubic of the form $y = ax^3 + bx^2 + cx + d$ is degree three. Example: The function shown has turning points at $x \simeq -1.8$ and $x \simeq 1.1$.  	A quadratic of the form $y = ax^2 + bx + c$ is degree two. The curve shown has x-intercepts at $x \simeq -1.8$ and $x \simeq 1.1$. 

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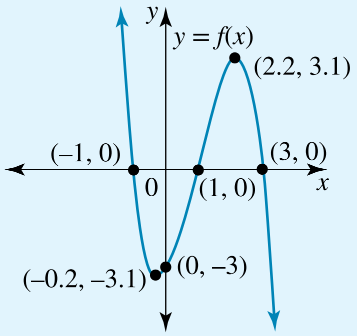
Interactivity
Graph of a derivative function
int-5961

Sometimes the graph of $f(x)$ may not be a known function, so the features of $f(x)$ need to be studied carefully in order to sketch the gradient graph.

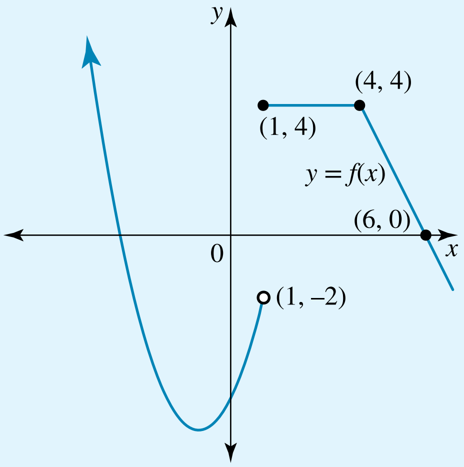
- **Turning points on the graph of $f(x)$ represent x-intercepts on the graph of $f'(x)$.**
- **Where the graph of $f(x)$ has a positive gradient, the graph of $f'(x)$ is above the x-axis.**
- **Where the graph of $f(x)$ has a negative gradient, the graph of $f'(x)$ is below the x-axis.**

WORKED EXAMPLE 4

a The graph of the cubic function $f(x)$ is shown. Sketch the derivative function $f'(x)$ on the same set of axes as $f(x)$.



b State the domain of the gradient function, $f'(x)$, for the function shown.



THINK

a The turning points are points of zero gradient, so these points will correspond to x-intercepts on $f'(x)$.

$f'(x)$ lies above the x-axis where the gradient is positive.

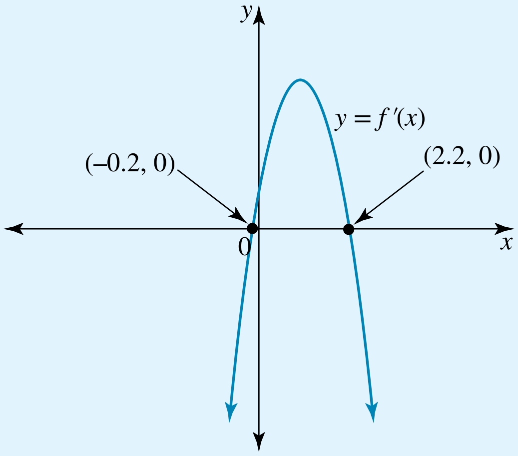
Where the gradient is negative is where $f'(x)$ lies below the x-axis.

Use this information to sketch $f'(x)$.

b For the gradient to exist, the graph must be smooth and continuous. The gradient doesn't exist at $x = 1$ or at $x = 4$.

WRITE/DRAW

a



b The domain $= R\backslash\{1, 4\}$

The gradient of a function only exists where the graph is smooth and continuous. That is, a single tangent must be able to be drawn at $x = a$ for $f'(a)$ to exist.

Equations of tangents and perpendicular lines

study on

Units 3 & 4
AOS 3
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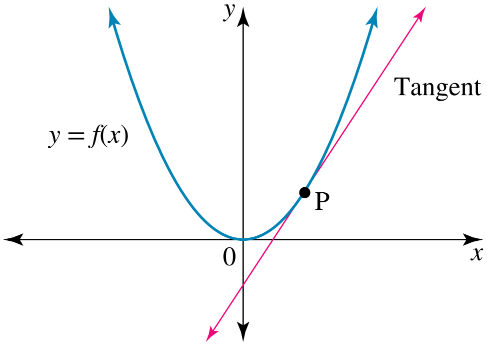
Equations of tangents
Concept summary
Practice questions

A tangent is a straight line. Therefore, to form the equation of the tangent, its gradient and a point on the line are needed. The equation can then be formed using $y - y_1 = m(x - x_1)$.

For the tangent to a curve $y = f(x)$ at a point P, the gradient m is found by evaluating the curve's derivative, $f'(x)$, at P, the point of contact or point of tangency. The coordinates of P provide the point (x_1, y_1) on the line.



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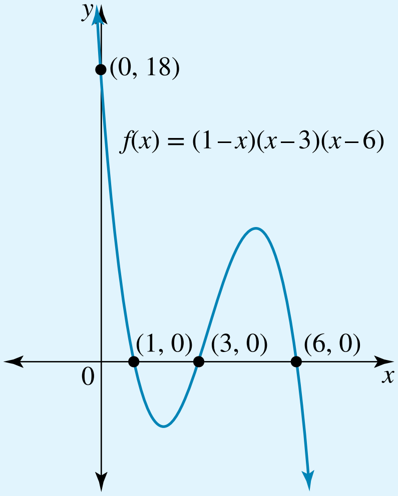
Interactivity
Equations of tangents
int-5962

Other important features of the tangent include:

- **The angle of inclination of the tangent to the horizontal can be calculated using $m = \tan(\theta)$.**
- **Tangents that are parallel to each other have the same gradient.**
- **The gradient of a line perpendicular to the tangent is found using $m_T m_P = -1$. That is, if the gradient of a tangent is m_T, then the gradient of a perpendicular line is $-\dfrac{1}{m_P}$.**
- **The gradient of a horizontal tangent is zero.**
- **The gradient of a vertical tangent is undefined.**

WORKED EXAMPLE 5

Consider the function $f(x) = (1 - x)(x - 3)(x - 6)$. The graph of this function is shown.



a **Find the equation of the tangent to the curve at the point (4, 6).**
b **Find the equation of the line perpendicular to the tangent at the point (4, 6).**

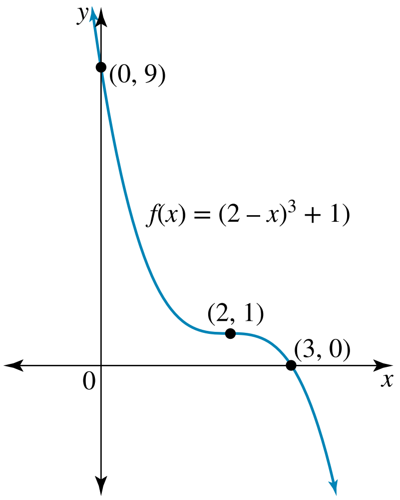
THINK	WRITE
a 1 Expand $f(x)$.	**a** $f(x) = (1 - x)(x - 3)(x - 6)$ $= (1 - x)(x^2 - 9x + 18)$ $= -x^3 + 10x^2 - 27x + 18$
2 Find the derivative of $f(x)$.	$f'(x) = -3x^2 + 20x - 27$
3 Find the gradient at $x = 4$.	$f'(4) = -3(4)^2 + 20(4) - 27$ $= -48 + 80 - 27$ $= 5$
4 Substitute the appropriate values into the formula $y - y_1 = m(x - x_1)$.	$m = 5$ and $(x_1, y_1) = (4, 6)$ $y - y_1 = m(x - x_1)$ $y - 6 = 5(x - 4)$ $y - 6 = 5x - 20$ $y = 5x - 14$
b 1 Find the gradient of the line perpendicular to the tangent.	**b** $m_P = -\frac{1}{m_T}$ $= -\frac{1}{5}$
2 Find the equation of the perpendicular line.	$m = -\frac{1}{5}$ and $(x_1, y_1) = (4, 6)$ $y - y_1 = m(x - x_1)$ $y - 6 = -\frac{1}{5}(x - 4)$ $y - 6 = -\frac{1}{5}x + \frac{4}{5}$ $y = -\frac{1}{5}x + \frac{4}{5} + \frac{30}{5}$ $y = -\frac{1}{5}x + \frac{34}{5}$ or $5y = -x + 34$ $x + 5y = 34$

EXERCISE 5.2 Review of differentiation

PRACTISE

Work without CAS

1 **WE1** The graph of $y = (2 - x)^3 + 1$ is shown.



a Using first principles, find the equation for the gradient of the tangent to the curve at any point along the curve.

b Hence, find the gradient of the tangent to the curve at the point (1, 2).

2 a Sketch the graphs of $y = (x + 2)(2 - x)$ and $y = x^2(4 - x)$ on the one set of axes.

b Find the point(s) of intersection of the two curves, giving coordinates correct to 2 decimal places where appropriate.

c If P is the point of intersection where $x \in Z$, use first principles to find the gradient of the tangents to each of the curves at this point.

3 **WE2** Differentiate:

a $f(x) = 4x^3 + \dfrac{1}{3x^2} + \dfrac{1}{2}$

b $y = \dfrac{2\sqrt{x} - x^4}{5x^3}$.

4 Differentiate:

a $f(x) = (x + 3)(x^2 + 1)$

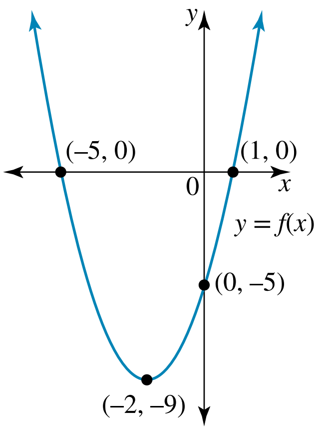
b $y = \dfrac{4 - \sqrt{x}}{\sqrt{x^3}}$.

5 **WE3** a If $f(x) = -\dfrac{1}{x^2} + 2x$, what is the gradient of the curve when $x = -\frac{1}{2}$?

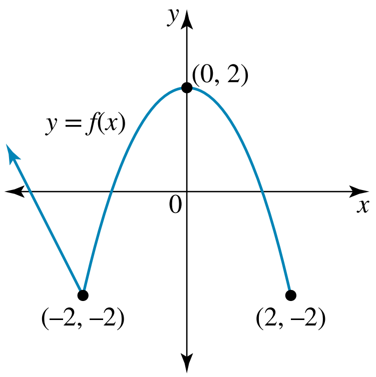
b If $f(x) = \dfrac{2x - 4}{x}$, determine the coordinates of the point where the gradient is 1.

6 If $y = (x - a)(x^2 - 1)$, find the gradient of the curve when $x = -2$ in terms of a.

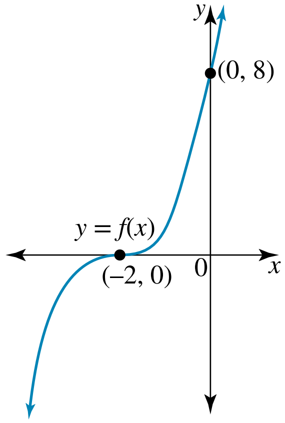
7 **WE4** a The graph of $f(x)$ is shown. Analyse this function and sketch the graph of $f'(x)$.



b State the domain of the gradient function, $f'(x)$, for the function shown.



8 The graph of $f(x)$ is shown. Analyse this function and sketch the graph of $f'(x)$.



9 **WE5** **a** Find the equation of the tangent to the curve with equation $y = x(x - 2)^2(x - 4)$ at the point $(3, -3)$.

b Find the equation of the line perpendicular to the tangent at the point (3, –3).

10 The equation of a tangent to a given parabola is $y = -2x + 5$. The equation of the line perpendicular to this tangent is $y = \frac{1}{2}x + \frac{5}{2}$. The parabola also has a stationary point at (0, 4). Determine the equation of the parabola and hence sketch the parabola, the tangent and the line perpendicular to the tangent, on the one set of axes.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

11 Differentiate:

a $y = \dfrac{3}{4x^5} - \dfrac{1}{2x} + 4$

b $f(x) = \dfrac{10x - 2x^3 + 1}{x^4}$

c $y = \sqrt{x} - \dfrac{1}{2\sqrt{x}}$

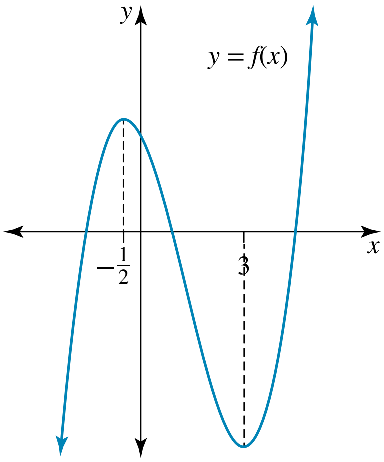
d $f(x) = \dfrac{(3 - x)^3}{2x}$

12 For the following graphs:

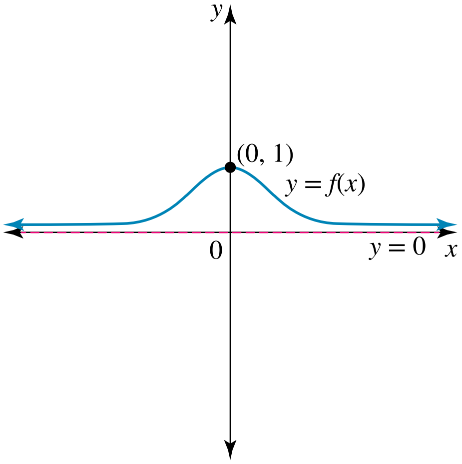
i state the domain of the gradient function, $f'(x)$

ii sketch the graph of $f'(x)$.

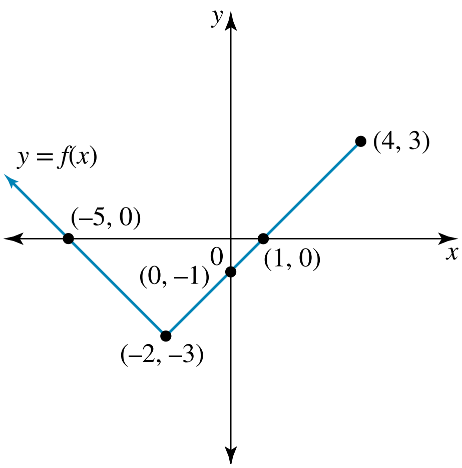
a



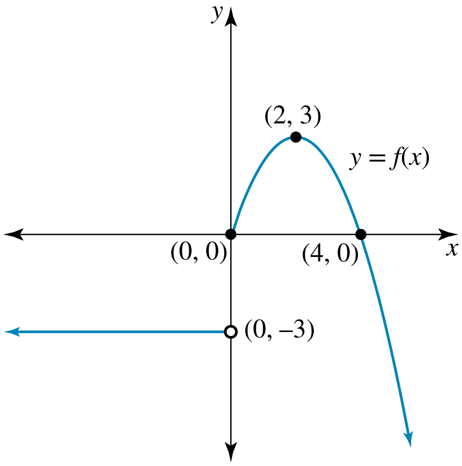
b



c



d



13 Find the gradient of the secant in each of the following cases.

- **a** $f: R \rightarrow R, f(x) = x^2 + 5x + 6$ between $x = -1$ and $x = 0$
- **b** $f: R \rightarrow R, f(x) = x^3(x - 3)$ between $x = 2$ and $x = 4$
- **c** $f: R \rightarrow R, f(x) = -x^3$ between $x = -3$ and $x = 0$

14 Use first principles to find the rule for the derived function where the function is defined by:

- **a** $f(x) = 12 - x$
- **b** $f(x) = 3x^2 - 2x - 21$

15 Find the gradient of the tangent to each of the following curves at the specified point.

- **a** $f(x) = x^2 - 3$ at $x = 2$
- **b** $f(x) = (3 - x)(x - 4)$ at $x = 1$
- **c** $f(x) = (x - 2)^3$ at $x = 4$
- **d** $f(x) = \sqrt{x} - \dfrac{3}{x} + 2x$ at $x = 4$

16 Find the equations of the tangents to the following curves at the specified points.

- **a** $f(x) = (x + 1)(x + 3)$ at $x = -5$
- **b** $f(x) = 8 - x^3$ at $x = a$
- **c** $f(x) = 2\sqrt{x} - 5$ at $x = 3$
- **d** $f(x) = -\dfrac{2}{x} - 4x$ at $x = -2$

17 Find the equation of the line perpendicular to the tangent for each of the functions defined in question **16**.

18 a Find the equation of the tangent to the curve $f(x) = -(x - 2)^2 + 3$ that is parallel to the line $y = 3x + 4$.

b Find the equation of the tangent to the curve $f(x) = -\dfrac{2}{x^2} + 1$ that is perpendicular to the line $2y - 2 = -4x$.

19 The tangent to a parabolic curve at $x = 4$ has the equation $y = -x + 6$. The curve also passes through the points $(0, -10)$ and $(2, 0)$. Find the equation of the curve.

20 The tangent to a cubic function at the point $x = 2$ has a rule defined by $y = 11x - 16$. The cubic passes through the origin as well as the point $(-1, 0)$. Find the equation of the cubic function.

MASTER

21 A line perpendicular to the graph of $y = 2\sqrt{x}$ has the equation $y = -2x + m$, where m is a real constant. Determine the value of m.

22 **a** Use CAS technology to sketch $y = x(x - 2)(x + 3)$ and $y = (2 - x)(x + 3)(x - 3)$ on the same set of axes.

b Find the coordinates of the point of intersection between the graphs where $1 < x < 2$.

c Find the equation of the tangent and the line perpendicular to the tangent at the point defined in part **b** for the cubic function defined by $y = x(x - 2)(x + 3)$.

5.3 Differentiation of exponential functions

The derivative of the exponential function from first principles

We can find the derivative of the exponential function as follows.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
f(x) &= e^x \\
f(x + h) &= e^{x+h} \\
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
f'(x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
f'(x) &= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\
f'(x) &= e^x \lim_{h \to 0} \frac{e^h - 1}{h}
\end{aligned}
$$

We don't know the value of $\lim_{h \to 0} \frac{e^h - 1}{h}$, but we can investigate by substituting different values for h and looking at what happens to the **limit** as the value of h approaches zero.

$$
\begin{aligned}
\text{If } h = 1, &\quad \frac{e^h - 1}{h} = 1.7183 \\
h = 0.1, &\quad \frac{e^h - 1}{h} = 1.0517 \\
h = 0.01, &\quad \frac{e^h - 1}{h} = 1.0050 \\
h = 0.001, &\quad \frac{e^h - 1}{h} = 1.0005 \\
h = 0.0001, &\quad \frac{e^h - 1}{h} = 1.000\,05
\end{aligned}
$$

From these results, we can see that as the value of h gets smaller and approaches zero, the value of $\frac{e^h - 1}{h}$ approaches 1:

$$f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

As $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, therefore

$$f'(x) = e^x$$

So the derivative of the exponential function, $f(x) = e^x$, is itself.

Note: This rule only applies to exponential functions of base e.

It can be shown using the **chain rule**, which will be introduced in the next topic, that:

If $f(x) = e^{kx}$, then $f'(x) = ke^{kx}$
and
if $f(x) = e^{g(x)}$, then $f'(x) = g'(x)e^{g(x)}$.

WORKED EXAMPLE 6

Find the derivative of each of the following with respect to x.

a $y = e^{-\frac{1}{2}x}$

b $y = \frac{1}{4}e^{2x} + e^{x^2}$

c $y = \dfrac{e^{2x} + 3e^x - 1}{e^{2x}}$

d $y = (e^x - 2)^2$

THINK	WRITE
a 1 Write the equation to be differentiated.	**a** $y = e^{-\frac{1}{2}x}$
2 Apply the rule for $\frac{d}{dx}(e^{kx})$ with $k = -\frac{1}{2}$.	$\frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x}$
b 1 Write the equation to be differentiated.	**b** $y = \frac{1}{4}e^{2x} + e^{x^2}$
2 Apply the rule $\frac{d}{dx}\left(e^{g(x)}\right) = g'(x)e^{g(x)}$, and differentiate each term separately.	$\frac{dy}{dx} = \frac{1}{4} \times 2e^{2x} + 2xe^{x^2}$ $= \frac{1}{2}e^{2x} + 2xe^{x^2}$
c 1 Write the equation to be differentiated.	**c** $y = \dfrac{e^{2x} + 3e^x - 1}{e^{2x}}$
2 Split the right-hand side into three separate terms and divide through by e^{2x}.	$y = \dfrac{e^{2x}}{e^{2x}} + \dfrac{3e^x}{e^{2x}} - \dfrac{1}{e^{2x}}$ $= 1 + 3e^{-x} - e^{-2x}$
3 Apply the rule $\frac{d}{dx}(e^{kx}) = ke^{kx}$ and differentiate each term separately.	$\frac{dy}{dx} = -3e^{-x} + 2e^{-2x}$ $= -\dfrac{3}{e^x} + \dfrac{2}{e^{2x}}$
d 1 Write the equation to be differentiated.	**d** $y = (e^x - 2)^2$
2 Expand the right side.	$y = e^{2x} - 4e^x + 4$
3 Differentiate each term separately.	$\frac{dy}{dx} = 2e^{2x} - 4e^x$

WORKED EXAMPLE 7

a Determine the gradient of the tangent to the curve with equation $y = e^{-x}$ at the point where $x = 1$.

b Determine the equation of the tangent to the curve $y = e^{-x}$ at the point where $x = 1$. What is the equation of the line perpendicular to this tangent?

THINK	WRITE
a **1** The gradient of the tangent is given by $\frac{dy}{dx}$.	**a** $y = e^{-x}$ $\frac{dy}{dx} = -e^{-x}$
2 Substitute $x = 1$.	$\frac{dy}{dx} = -e^{-1}$ $= -\frac{1}{e}$ The gradient of the curve is $-\frac{1}{e}$.
b **1** We have the gradient but we need a point. Determine the corresponding y-value when $x = 1$.	**b** $x = 1$: $y = e^{-1}$ $= \frac{1}{e}$
2 Use $y - y_1 = m(x - x_1)$ to find the equation of the tangent.	If $(x_1, y_1) = \left(1, \frac{1}{e}\right)$ and $m = -\frac{1}{e}$: $y - \frac{1}{e} = -\frac{1}{e}(x - 1)$ $y - \frac{1}{e} = -\frac{1}{e}x + \frac{1}{e}$ $y = -\frac{1}{e}x + \frac{2}{e}$ $= -\frac{1}{e}(x - 2)$ The equation of the tangent is $y = -\frac{1}{e}(x - 2)$.
3 A line perpendicular to a tangent has a gradient that is the negative reciprocal of the gradient of the tangent. $m_P = -\frac{1}{m_T}$.	$m_P = -(-e)$ $= e$
4 The perpendicular line passes through the same point, $\left(1, \frac{1}{e}\right)$. Use $y - y_1 = m(x - x_1)$ to find the equation of the perpendicular line.	If $(x_1, y_1) = \left(1, \frac{1}{e}\right)$ and $m = e$: $y - \frac{1}{e} = e(x - 1)$ $y - \frac{1}{e} = ex - e$ $y = ex - e + \frac{1}{e}$ The equation of the perpendicular line is $y = ex - e + \frac{1}{e}$.

EXERCISE 5.3 Differentiation of exponential functions

PRACTISE

Work without CAS

1 **WE6** Find the derivative of each of the following functions.

a $e^{-\frac{1}{3}x}$
b $3x^4 - e^{-2x^2}$
c $y = \dfrac{4e^x - e^{-x} + 2}{3e^{3x}}$
d $y = (e^{2x} - 3)^2$

2 Consider the function defined by the rule

$$f(x) = \tfrac{1}{2}e^{3x} + e^{-x}.$$

Find the gradient of the curve when $x = 0$.

3 **WE7** Find the equation of the tangent to the curve with equation $y = e^{2x}$ at the point where $x = 0$.

4 Find the equations of the tangent and the line perpendicular to the curve with equation $y = e^{-3x} + 4$ at the point where $x = 0$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Differentiate the following with respect to x.

a $5e^{-4x} + 2e$
b $e^{-\frac{1}{2}x} + \frac{1}{3}x^3$
c $4e^{3x} - \frac{1}{2}e^{6\sqrt{x}} - 3e^{-3x+2}$
d $\dfrac{e^{5x} - e^{-x} + 2}{e^{2x}}$
e $\dfrac{e^x(2 - e^{-3x})}{e^{-x}}$
f $(e^{2x} + 3)(e^{-x} - 1)$

6 Find the exact gradients of the tangents to the given functions at the specified points.

a $y = 2e^{-x}$ at $x = 0$
b $y = \dfrac{4}{e^{2x}}$ at $x = \frac{1}{2}$
c $y = \frac{1}{2}e^{3x}$ at $x = \frac{1}{3}$
d $y = 2x - e^x$ at $x = 0$

7 Determine the equation of the tangent to the curve with equation $y = e^{-2x}$ at the point where $x = -\frac{1}{2}$.

8 Determine the equations of the tangent and the line perpendicular to the curve $y = e^{-3x} - 2$ at the point where $x = 0$.

9 Determine the equations of the tangent and the line perpendicular to the curve $y = e^{\sqrt{x}} + 1$ at the point where $x = 3$.

10 Determine the derivative of the function $f(x) = e^{-2x+3} - 4e$ and hence find:

a $f'(-2)$ in exact form
b $\{x : f'(x) = -2\}$.

11 Determine the derivative of the function $f(x) = \dfrac{e^{3x} + 2}{e^x}$ and hence find:

a $f'(1)$ in exact form
b $\{x : f'(x) = 0\}$.

12 Find the equation of the tangent to the curve $y = e^{x^2+3x-4}$ at the point where $x = 1$.

13 For the function with the rule $f(x) = Ae^x + Be^{-3x}$, where A and B are non-zero real constants, find $f'(x)$ and show that $f'(x) = 0$ when $e^{4x} = \dfrac{3B}{A}$.

14 The curve with the rule $A = A_0e^{-0.69t}$ passes through the point (0, 2).

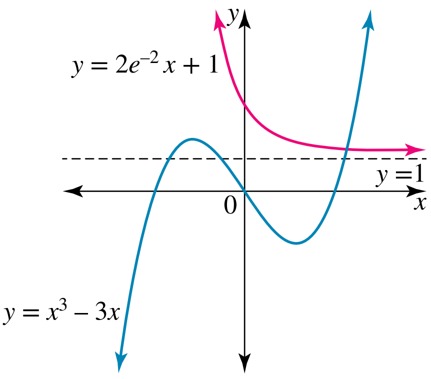
a Find the value of A_0.
b Find $\dfrac{dA}{dt}$ when $t = 0$.

MASTER

15 Determine the exact value for $f'(2)$ if $f(x) = 3^{2x-4}$.

16 a The graphs of the equations $y = 2e^{-2x} + 1$ and $y = x^3 - 3x$ are shown. Find the coordinates of the point of intersection, giving your answer correct to 2 decimal places.

b Find the gradient of the tangent to the cubic at this point.



5.4 Applications of exponential functions

Exponential functions are commonly used to model a number of real world applications, including Newton's Law of Cooling, population **growth** and **decay**, cell growth and decay, and radioactive decay.

A general equation to represent exponential growth and decay is given by

$$A = A_0e^{kt}$$

where A_0 is the initial amount and k is a constant.

If the equation represents growth, then k is a positive value. If the equation represents decay, then k is a negative value.

WORKED EXAMPLE 8

The number of bacteria on a culture plate, N, can be defined by the rule

$$N(t) = 2000e^{0.3t},\ t \geq 0$$

where t is the time, in seconds, the culture has been multiplying.

a How many bacteria are initially present?

b How many bacteria, to the nearest whole number, are present after 10 seconds?

c At what rate is the bacteria population multiplying after 10 seconds? Give your answer correct to the nearest whole number.

THINK	WRITE
a 1 Initially $t = 0$, so substitute this value into the rule.	a $N(0) = 2000e^{0.3(0)}$ $= 2000$
2 Write the answer.	Initially there are 2000 bacteria present.
b 1 Substitute $t = 10$.	b $N(10) = 2000e^{0.3(10)}$ $= 2000e^3$ $= 40\,171$
2 Write the answer.	After 10 seconds there are 40 171 bacteria present.
c 1 $\dfrac{dN}{dt}$ represents the required rate.	c $\dfrac{dN}{dt} = 600e^{0.3t}$
2 Substitute $t = 10$.	$\dfrac{dN}{dt} = 600e^{0.3(10)}$ $= 600e^3$ $= 12\,051$
3 Write the answer with the correct units.	After 10 seconds the bacteria are growing at a rate of 12 051 per second.

EXERCISE 5.4 Applications of exponential functions

PRACTISE

1 **WE8** The mass, M grams, of a radioactive substance is initially 20 grams; 30 years later its mass is 19.4 grams. If the mass in any year is given by

$$M = M_0e^{-0.00152t}$$

where t is the time in years and M_0 is a constant, find:

a the value of M_0

b the annual rate of decay

c the rate of decay after 30 years.

2 The intensity of light decreases as it passes through water. The phenomenon can be modelled by the equation

$$I = I_0 e^{-0.0022d}$$

where I_0 is the intensity of light at the surface of the water and I is the intensity of light at a depth of d metres below the surface of the water.

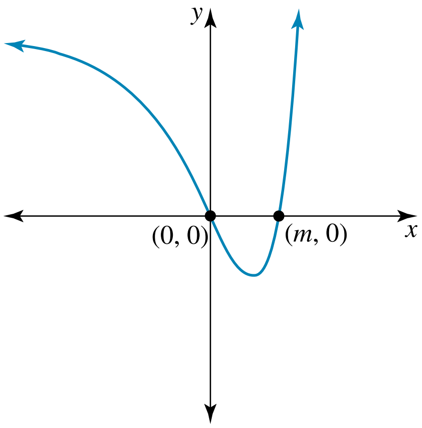
a What is the intensity of light at a depth of 315 metres?

b What is the rate at which the intensity of light is decreasing at 315 metres?

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 The graph shown is that of the function $f: R \to R, f(x) = e^{2x} + qe^{x} + 3$, where q is a constant.



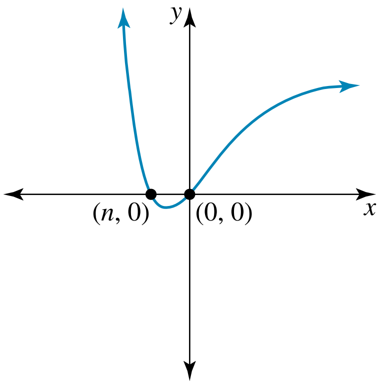
a Find the value of q.

b Find the exact value of m, where m is a constant and $(m, 0)$ are the coordinates of the point where the function intersects the x-axis.

c Find the derivative function, $f'(x)$.

d Find the gradient of the curve where it intersects the y-axis.

4 The graph shown is that of the function $f: R \to R, f(x) = e^{-2x} + ze^{-x} + 2$, where z is a constant.



a Find the value of z.

b Find the exact value of n, where n is a constant and $(n, 0)$ are the coordinates of the point where the graph intersects the x-axis.

c Find the derivative function, $f'(x)$.

d Find the gradient of the curve where it passes through the origin.

5 An unstable gas decomposes in such a way that the amount present, A units, at time t minutes is given by the equation

$$A = A_0 e^{-kt}$$

where k and A_0 are constants. It was known that initially there were 120 units of unstable gas.

a Find the value of A_0.

b Show that $\frac{dA}{dt}$ is proportional to A.

c After 2 minutes there were 90 units of the gas left. Find the value of k.

d At what rate is the gas decomposing when $t = 5$? Give your answer correct to 3 decimal places.

e Will there ever be no gas left? Explain your answer.

6 The bilby is an endangered species that can be found in the Kimberley in Western Australia as well as some parts of South Australia, the Northern Territory and Queensland. The gestation time for a bilby is 2–3 weeks and when they are born, they are only about 11 mm in length. The growth of a typical bilby can be modelled by the rule

$$L = L_0 e^{0.599t}$$

where L_0 is its length in millimetres at birth and L is the length of the bilby in millimetres t months after its birth.

a Determine the value of L_0.
b What is the rate of change of length of the bilby at time t months?
c At what rate is the bilby growing when it is 3 months old? Give your answer correct to 3 decimal places.

7 A body that is at a higher temperature than its surroundings cools according to Newton's Law of Cooling, which states that

$$T = T_0 e^{-zt}$$

where T_0 is the original *excess* of temperature, T is the excess of temperature in degrees centigrade after t minutes, and z is a constant.

a The original temperature of the body was 95°C and the temperature of the surroundings was 20°C. Find the value of T_0.
b At what rate is the temperature decreasing after a quarter of an hour if it is known that $z = 0.034$? Give your answer correct to 3 decimal places.

8 The population of Australia since 1950 can be modelled by the rule

$$P = P_0 e^{0.016t}$$

where P_0 is the population in millions at the beginning of 1950 and P is the population in millions t years after 1950. It is known that there were 8.2 million people in Australia at the beginning of 1950.

a Calculate the population in millions at the beginning of 2015, correct to 1 decimal place.
b During which year and month does the population reach 20 million?
c Determine the rate of change of population at the turn of the century, namely the year 2000, correct to 2 decimal places.
d In which year is the rate of increase of the population predicted to exceed 400 000 people per year?

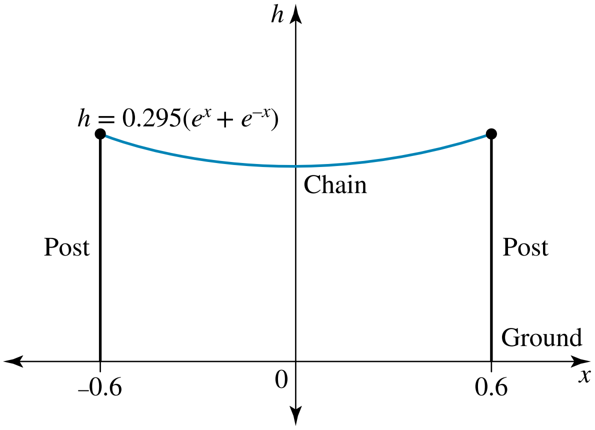
9 The pressure of the atmosphere, P cm of mercury, decreases with the height, h km above sea level, according to the law

$$P = P_0 e^{-kh}$$

where P_0 is the pressure of the atmosphere at sea level and k is a constant. At 500 m above sea level, the pressure is 66.7 cm of mercury, and at 1500 m above sea level, the pressure is 52.3 cm of mercury.

a What are the values of P_0 and k, correct to 2 decimal places?
b Find the rate at which the pressure is falling when the height above sea level is 5 km. Give your answer correct to 2 decimal places.

10 An entrance to a local suburban park has a series of posts connected with heavy chains as shown.



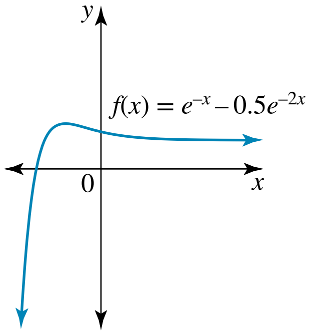
The chain between any two posts can be modelled by the curve defined by

$$h = 0.295(e^x + e^{-x}), -0.6 \le x \le 0.6$$

where h metres is the height of the chain above the ground and x is the horizontal distance between the posts in metres. The x-axis represents the ground. The posts are positioned at $x = -0.6$ and $x = 0.6$.

- **a** Calculate the amount of sag in the chain (i.e. the difference in height between the highest points of the chain and the lowest point of the chain). Give your answer in centimetres.
- **b** Calculate the angle the chain makes with the post positioned on the right-hand side of the structure (i.e. at $x = 0.6$). Give your answer correct to 1 decimal place.

11 The graph of the function $f: R \to R, f(x) = e^{-x} - 0.5e^{-2x}$ is shown.



- **a** Find the coordinates of the point at which the graph crosses the y-axis.
- **b** Determine $f'(x)$.
- **c** Find the coordinates of the point at which the gradient is equal to zero.
- **d** What is the angle, correct to 1 decimal place, which the graph makes with the positive direction of the x-axis if it is known that the graph cuts the x-axis at $(-\log_e(2), 0)$?
- **e** What is the equation of the tangent to the curve when $x = 1$?
- **f** What is the equation of the line perpendicular to the curve when $x = 1$?

12 The cane toad, originally from South America, is an invasive species in Australia. Cane toads were introduced to Australia from Hawaii in June 1935 in an attempt to control cane beetles, though this proved to be ineffective. The long-term effects of the toad's introduction in Australia include the depletion of native species, because the cane toads produce a poison that kills most animals that try to eat them.

One study has found that cane toads are especially vulnerable to a common native predator, the meat ant. At waterholes in tropical Australia, interactions between cane toads and meat ants were observed. It was found that the ants are very effective in capturing and killing small young toads as they emerge from the water because the ants are not overpowered by the toad's poison.

In a controlled experiment at a particular waterhole, it was observed that at the beginning of the experiment there were an estimated 30 000 tadpoles (future cane toads) in the water. The number of tadpoles increased by about 60 000 a day over the period of a week. This growth pattern can be defined by the equation

$$T = T_0 e^{kt}$$

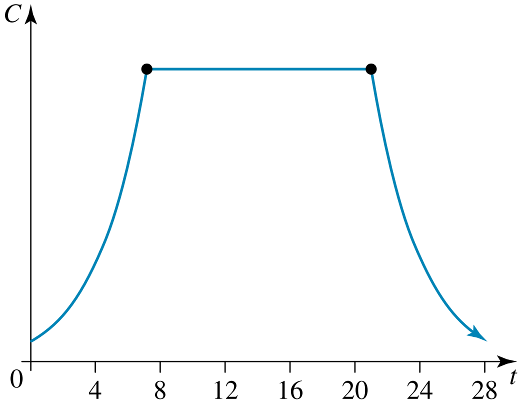
where T_0 is the initial number of cane toad tadpoles (in thousands) at the waterhole during the time of the experiment, T is the number of cane toad tadpoles (in thousands) at the waterhole t days into the experiment, and k is a constant.

a Calculate the value of T_0.

b How many cane toad tadpoles are in the waterhole after a week if it is known that $k = 0.387$? Give your answer to the nearest thousand.

c Find the rate at which the cane toad tadpole numbers are increasing after 3 days.

After a week, no more tadpoles could be supported by the habitat. In favourable conditions, tadpoles take about two weeks to develop into small cane toads, at which point they leave the water. Once the small cane toads emerged, meat ants were introduced into their environment. This caused 90% of the cane toads to be killed off over a period of a week. The growth and decline of the tadpoles/cane toads is shown.



The decline in the number of young cane toads can be defined by the equation

$$C = C_0 e^{mt}$$

where C_0 is the number of young cane toads (in thousands) just before the meat ants were introduced, C is the number of young cane toads (in thousands) t days after the meat ants were introduced and m is a constant.

d Determine the value of C_0.

e How many young cane toads still survived a week after the meat ants were introduced?

f Find m and the rate of decline in the number of cane toads after 4 days.

MASTER

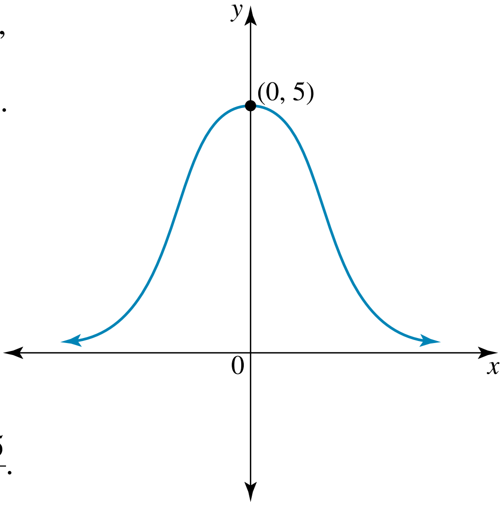
13 The graph of $y = Ae^{-x^2}$, where A is a constant, is shown. Answer the following questions correct to 2 decimal places where appropriate.

a If the gradient of the graph is zero at the point (0, 5), determine the value of A.

b Find $\frac{dy}{dx}$.

c Determine the gradient of the tangent to the curve at the point where:

i $x = -0.5$ **ii** $x = 1$



14 Consider the curve with equation $y = \frac{x^2 - 5}{2e^{x^2}}$. Using CAS:

a find $\frac{dy}{dx}$

b find the exact coordinates of the points on the curve where the gradient is equal to zero

c find the gradient of the tangent to the curve at $x = \frac{3}{2}$, giving your answer correct to 3 decimal places.

5.5 Differentiation of trigonometric functions

The derivatives of sin(x) and cos(x)

The derivative of sin(x) can be investigated using differentiation from first principles. Consider $f : R \to R, f(x) = \sin(x)$ where x is an angle measurement in radians.

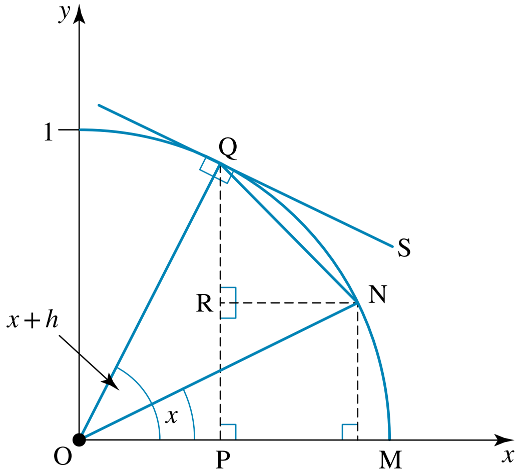
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ f(x) &= \sin(x) \\ f(x+h) &= \sin(x+h) \\ \frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin(x)}{h} \\ f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \end{aligned}$$

To evaluate this limit, we must look at the unit circle.

$$\angle \text{NOM} = x, \quad \angle \text{QOM} = x + h$$

$$\angle \text{PQO} = \frac{\pi}{2} - (x + h)$$

$$\begin{aligned} \angle \text{RQS} &= \frac{\pi}{2} - \left(\frac{\pi}{2} - (x + h)\right) \\ &= x + h \end{aligned}$$



By definition

$$\begin{aligned} \sin(x) &= MN \\ \sin(x+h) &= PQ \\ \sin(x+h)-\sin(x) &= PQ-MN = QR \\ \frac{\sin(x+h)-\sin(x)}{h} &= \frac{QR}{h} \end{aligned}$$

From the diagram, it can be seen that ∠RQS = $x + h$ and the arc QN has length h. As $h \to 0$, ∠RQS approaches ∠RQN, which approaches x. Furthermore, the arc QN approaches the chord QN.

Consequently $\frac{QR}{h} \to \frac{QR}{QN}$, but by definition, $\frac{QR}{QN} = \cos(x)$.

Hence

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{QR}{h} \\ &= \cos(x) \end{aligned}$$

If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

It can also be shown using the chain rule, which will be introduced in the next topic, that:

If $f(x) = \sin(kx)$, then $f'(x) = k\cos(kx)$, where k is a constant

and

if $f(x) = \sin(g(x))$, then $f'(x) = g'(x)\cos(g(x))$.

The derivative of $\cos(x)$ can also be investigated geometrically, using the same method as shown for $\sin(x)$ and yielding the following result.

If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

It can also be shown using the chain rule that:

If $f(x) = \cos(kx)$, then $f'(x) = -k\sin(kx)$, where k is a constant

and

if $f(x) = \cos(g(x))$, then $f'(x) = -g'(x)\sin(g(x))$.

The derivative of tan(x)

Consider the function $f: R \to R, f(x) = \tan(x)$.

If $f(x) = \tan(x)$, then $f'(x) = \frac{1}{\cos^2(x)} = \sec^2(x)$.

In order to prove this differentiation, we would use the trigonometric identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$ in conjunction with the **quotient rule**, which will also be introduced in the next topic.

It can also be shown using the chain rule that:

$$\textbf{If } f(x) = \tan(kx)\textbf{, then } f'(x) = \frac{k}{\cos^2(kx)} = k\sec^2(kx)\textbf{, where } k \textbf{ is a constant}$$

and

$$\textbf{if } f(x) = \tan(g(x)) \textbf{ then } f'(x) = \frac{g'(x)}{\cos^2(g(x))} = g'(x)\sec^2(g(x)).$$

Remember that these rules can only be applied if the angle x is measured in radians.

WORKED EXAMPLE 9

Determine the derivative of each of the following functions.

a $\sin(8x) + x^4$

b $\tan(5x) + 2\cos(x^2)$

c $\dfrac{1 - \sin^2(x)}{\cos(x)}$

d $\sin(6x°)$

THINK	WRITE
a Apply the rule $\frac{d}{dx}(\sin(kx)) = k\cos(kx)$ and differentiate each term separately.	**a** $y = \sin(8x) + x^4$ $\frac{dy}{dx} = 8\cos(8x) + 4x^3$
b Apply the rules $\frac{d}{dx}(\cos(g(x))) = -g'(x)\sin(g(x))$ and $\frac{d}{dx}(\tan(kx)) = \frac{k}{\cos^2(x)}$.	**b** $y = \tan(5x) + 2\cos(x^2)$ $\frac{dy}{dx} = \frac{5}{\cos^2(5x)} - 2 \times 2x\sin(x^2)$ $= \frac{5}{\cos^2(5x)} - 4x\sin(x^2)$
c 1 Remember the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$. Use this to simplify the equation.	**c** $y = \frac{1 - \sin^2(x)}{\cos(x)}$ $= \frac{\cos^2(x)}{\cos(x)}$ $= \cos(x), \cos(x) \neq 0$
2 Differentiate the simplified function.	$\frac{dy}{dx} = -\sin(x)$
d 1 The function $\sin(6x°)$ cannot be differentiated as the angle is not measured in radians. Convert the angle to radian measures by multiplying by $\frac{\pi}{180}$, as $1° = \frac{\pi^c}{180}$.	**d** $\sin(6x°) = \sin\left(6 \times \frac{\pi}{180}x\right)$ $= \sin\left(\frac{\pi x}{30}\right)$
2 Differentiate the resultant function by applying the rule $\frac{d}{dx}(\sin(kx)) = k\cos(kx)$.	$y = \sin\left(\frac{\pi x}{30}\right)$ $\frac{dy}{dx} = \frac{\pi}{30}\cos\left(\frac{\pi x}{30}\right)$

WORKED EXAMPLE 10 Find the equation of the tangent to the curve $y = \sin(3x) + 1$ at the point where $x = \frac{\pi}{3}$.

THINK	WRITE
1 First find the coordinates of the point; that is, determine the y-value when $x = \frac{\pi}{3}$.	When $x = \frac{\pi}{3}$, $y = \sin\left(3 \times \frac{\pi}{3}\right) + 1$ $= \sin(\pi) + 1$ $= 0 + 1$ $= 1$ The point is $\left(\frac{\pi}{3}, 1\right)$.
2 Find the derivative of the function.	$\frac{dy}{dx} = 3\cos(3x)$
3 Determine the gradient at the point where $x = \frac{\pi}{3}$.	$x = \frac{\pi}{3}, \frac{dy}{dx} = 3\cos\left(3 \times \frac{\pi}{3}\right)$ $= 3\cos(\pi)$ $= 3(-1)$ $= -3$
4 Substitute the appropriate values into the rule $y - y_1 = m(x - x_1)$ to find the equation of the tangent.	$m = -3, (x_1, y_1) = \left(\frac{\pi}{3}, 1\right)$ $y - y_1 = m(x - x_1)$ $y - 1 = -3\left(x - \frac{\pi}{3}\right)$ $y - 1 = -3x + \pi$ $y = -3x + \pi + 1$ The equation of the tangent is $y = 1 + \pi - 3x$.

EXERCISE 5.5 Differentiation of trigonometric functions

PRACTISE

Work without CAS

1 WE9 Differentiate each of the following functions with respect to x.

a $5x + 3\cos(x) + 5\sin(x)$

b $\sin(3x + 2) - \cos(3x^2)$

c $\frac{1}{3}\sin(9x)$

d $5\tan(2x) - 2x^5$

e $8\tan\left(\frac{x}{4}\right)$

f $\tan(9x°)$

2 Simplify and then differentiate $\frac{\sin(x)\cos^2(2x) - \sin(x)}{\sin(x)\sin(2x)}$ with respect to x.

3 WE10 Find the equation of the tangent to the curve $y = -\cos(x)$ at the point where $x = \frac{\pi}{2}$.

4 Find the equation to the tangent to the curve $y = \tan(2x)$ at the point where $x = -\frac{\pi}{8}$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 For each of the following functions, find $\frac{dy}{dx}$.

a $y = 2\cos(3x)$

b $y = \cos(x°)$

c $y = 3\cos\left(\frac{\pi}{2} - x\right)$

d $y = -4\sin\left(\frac{x}{3}\right)$

e $y = \sin(12x°)$

f $y = 2\sin\left(\frac{\pi}{2} + 3x\right)$

g $y = -\frac{1}{2}\tan(5x^2)$

h $y = \tan(20x)$

6 Determine the point on the curve with equation $y = -2\sin\left(\frac{x}{2}\right), x \in [0, 2\pi]$ where the gradient is equal to $\frac{1}{2}$.

7 Find the equation of the tangent to the curve with equation $y = 3\cos(x)$ at the point where $x = \frac{\pi}{6}$.

8 Find the equation of the tangent to the curve with equation $y = 2\tan(x)$ at the point where $x = \frac{\pi}{4}$.

9 Find the angle that the curve with equation $y = \sin(2x)$ makes with the positive direction of the x-axis the first time it intersects the x-axis when $x > 0$. Give your answer correct to 1 decimal place.

10 Find the equations of the tangent and the line perpendicular to each of the following graphs at the points indicated.

a $y = \sin(3x)$ at $\left(\frac{2\pi}{3}, 0\right)$

b $y = \cos\left(\frac{x}{2}\right)$ at $(\pi, 0)$

11 Consider the function $f: [0, 2\pi] \rightarrow R, f(x) = \sin(x) - \cos(x)$. Find:

a $f(0)$

b $\{x : f(x) = 0\}$

c $f'(x)$

d $\{x : f'(x) = 0\}$.

12 Consider the function $f: [-\pi, \pi] \rightarrow R, f(x) = \sqrt{3}\cos(x) + \sin(x)$. Find:

a $f(0)$

b $\{x : f(x) = 0\}$

c $f'(x)$

d $\{x : f'(x) = 0\}$.

13 a Use both or either of the trigonometric identities $\sin^2(\theta) + \cos^2(\theta) = 1$ and $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ to simplify $\frac{\sin(x)\cos(x) + \sin^2(x)}{\sin(x)\cos(x) + \cos^2(x)}$.

b Hence, find $\frac{d}{dx}\left(\frac{\sin(x)\cos(x) + \sin^2(x)}{\sin(x)\cos(x) + \cos^2(x)}\right)$.

14 Determine the x-values over the domain $x \in [-\pi, \pi]$ for which the gradients of the functions $f(x) = \sin(2x)$ and $f(x) = \cos(2x)$ are equal.

MASTER

15 For the function $f(x) = x - \sin(2x), -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, find the point(s) where the gradient is zero. Give your answer correct to 3 decimal places.

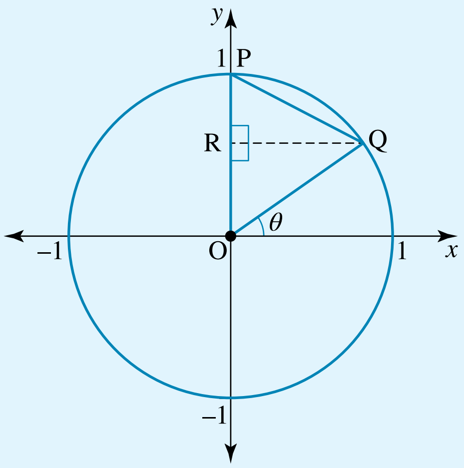
16 For the function, $f(x) = 2x + \cos(3x), 0 \le x \le \frac{\pi}{2}$, find the point(s) where the gradient is zero. Give your answer correct to 3 decimal places.

5.6 Applications of trigonometric functions

Trigonometric functions in real-life situations are usually used to model geometric scenarios or situations where cyclic phenomena are being investigated.

WORKED EXAMPLE 11

The circle shown has a radius of 1 unit.



a **Show that the area of the triangle OQP is equal to $A = \frac{1}{2}\cos(\theta)$, where $\angle QOX = \theta$ and θ is in radian measure.**

b **Find $\frac{dA}{d\theta}$ when $\theta = \frac{\pi}{6}$.**

THINK	**WRITE**
a **1** Find the magnitude of ∠RQO.	**a** As RQ is parallel to the x-axis, $\angle RQO = \theta$ because it is alternate to ∠QOX.
2 Find $d(\overline{RQ})$ and apply the formula for the area of a triangle, $A = \frac{1}{2} \times \text{base} \times \text{height}$.	$\cos(\theta) = \frac{RQ}{OQ}$ $= \frac{RQ}{1}$ $\cos(\theta) = RQ$ and $OP = 1$ $\text{Area} = \frac{1}{2} \times OP \times RQ$ $\text{Area} = \frac{1}{2} \times 1 \times \cos(\theta)$ $= \frac{1}{2}\cos(\theta)$ (as required)
b **1** Find $\frac{dA}{d\theta}$.	**b** $\frac{dA}{d\theta} = -\frac{1}{2}\sin(\theta)$
2 Substitute $\theta = \frac{\pi}{6}$.	$\theta = \frac{\pi}{6}, \frac{dA}{d\theta} = -\frac{1}{2}\sin\left(\frac{\pi}{6}\right)$ $= -\frac{1}{2} \times \frac{1}{2}$ $= -\frac{1}{4}$

The previous example involved a geometric application question, but everyday application questions can also be solved using trigonometric functions.

WORKED EXAMPLE 12

The temperature on a particular day can be modelled by the function

$$T(t) = -3\cos\left(\frac{\pi t}{9}\right) + 18, 0 \le t \le 18$$

where t is the time in hours after 5.00 am and T is the temperature in degrees Celsius.

For the remaining 6 hours of the 24-hour period, the temperature remains constant.

a Calculate the temperature at 8.00 am.

b At what time(s) of the day is the temperature 20°C? Give your answer correct to the nearest minute.

c Find $\frac{dT}{dt}$.

d What is the rate of change of temperature at the time(s) found in part **b**, correct to 2 decimal places?

THINK	WRITE
a At 8.00 am $t = 3$. Substitute this value into the equation.	**a** $T(3) = -3\cos\left(\frac{3\pi}{9}\right) + 18$ $= -3\cos\left(\frac{\pi}{3}\right) + 18$ $= -3 \times \frac{1}{2} + 18$ $= -1.5 + 18$ $= 16.5^\circ\text{C}$
b **1** Substitute $T = 20$ into the equation.	**b** $20 = -3\cos\left(\frac{\pi t}{9}\right) + 18$
2 Solve the equation for $0 \le t \le 18$ using CAS.	$20 = -3\cos\left(\frac{\pi t}{9}\right) + 18$ $t = 6.6, 11.4$
3 Interpret your answers and convert the t values to times of the day.	$t = 6.6 \Rightarrow$ 11.36 am $t = 11.4 \Rightarrow$ 4.24 pm
4 Write the answer.	The temperature is 20°C at 11.36 am and 4.24 pm.
c Determine $\frac{dT}{dt}$.	**c** $\frac{dT}{dt} = 3 \times \frac{\pi}{9}\sin\left(\frac{\pi t}{9}\right)$ $= \frac{\pi}{3}\sin\left(\frac{\pi t}{9}\right)$
d **1** Substitute $t = 6.6$ (11.36 am) and $t = 11.4$ (4.24 pm) into $\frac{dT}{dt}$.	**d** When $t = 6.6$ (11.36 am), $\frac{dT}{dt} = \frac{\pi}{3}\sin\left(\frac{6.6 \times \pi}{9}\right)$ $= 0.78$

When $t = 11.4$ (4.24 pm)

$$\frac{dT}{dt} = \frac{\pi}{3}\sin\left(\frac{11.4 \times \pi}{9}\right)$$
$$= -0.78$$

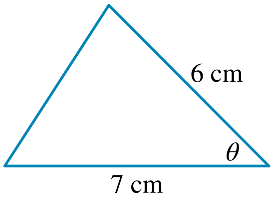
2 Write the answer.

At 11.36 am the temperature is increasing at a rate of 0.78°C per hour.
At 4.24 pm the temperature is decreasing at a rate of 0.78°C per hour.

EXERCISE 5.6 Applications of trigonometric functions

PRACTISE

1 **WE11** Consider the following triangle.

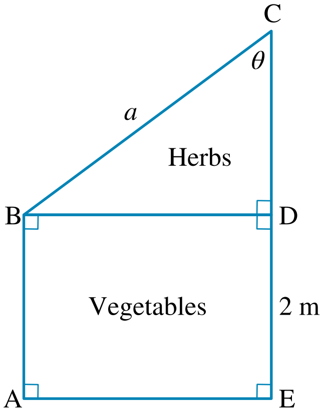


a Show that the area, A cm^2, is given by $A = 21\sin(\theta)$.

b Find $\frac{dA}{d\theta}$.

c What is the rate of change of area with respect to θ when $\theta = \frac{\pi}{3}$?

2 The diagram shows a garden bed bordered by wooden sleepers. BDC is a triangular herb garden and ABDE is a rectangular garden for vegetables.



a Find BD and CD in terms of a and θ, where a is a constant, θ is ∠BCD as shown and $0 < \theta < \frac{\pi}{2}$.

b Find the total length, L metres, of sleepers required to surround the garden bed. (This should include BD as well as the sleepers defining the perimeter.)

c Find $\frac{dL}{d\theta}$ in terms of θ and a.

d Let $a = 2$ and use CAS to sketch $\frac{dL}{d\theta}$ for $0 < \theta < \frac{\pi}{2}$. Hence, find when $\frac{dL}{d\theta} = 0$, correct to 1 decimal place.

3 **WE12** A mass oscillates up and down at the end of a metal spring. The length of the spring, L cm, after time t seconds, is modelled by the function $L(t) = 2\sin(\pi t) + 10$ for $t \geq 0$.

a What is the length of the spring when the mass is not oscillating, that is, when it is at the mean position, P?

b Find $\frac{dL}{dt}$.

c Find the exact value of $\frac{dL}{dt}$ after 1 second.

4 Between 6 am and 6 pm on a given day the height, H metres, of the tide in a harbour is given by

$$H(t) = 1.5 + 0.5\sin\left(\frac{\pi t}{6}\right), 0 \leq t \leq 12.$$

a What is the period of the function?

b What is the value of H at low tide and when does low tide occur?

c Find $\frac{dH}{dt}$.

d Find the exact value of $\frac{dH}{dt}$ at 7.30 am.

e Find the second time during the given time interval that $\frac{dH}{dt}$ equals the value found in part **d**.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Given that $f:\left[0, \frac{\pi}{2}\right] \to R, f(x) = 2\sin(4x) + 1$, find:

a the values of x for which $f(x) = 0.5$, giving your answer correct to 3 decimal places

b the coordinates where the gradient of the function is zero

c the value of $f'(x)$ when $x = \frac{\pi}{4}$

d the interval over which the gradient is positive.

6 A wire frame is shaped in the following way.

The diagonals shown are 100 cm long, and each diagonal makes an angle of θ with the horizontal.

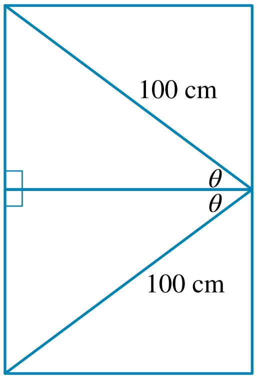
a Show that the length of wire required to form the shape is given by

$$L = 300\cos(\theta) + 400\sin(\theta) + 200, 0 \leq \theta \leq \frac{\pi}{2}$$

where L is the total length of wire in centimetres and θ is the angle shown in radians.

b Find $\frac{dL}{d\theta}$.

c Use CAS to sketch the graph of L. Find the maximum length of the wire required and the value of θ, correct to 2 decimal places, for which this occurs.



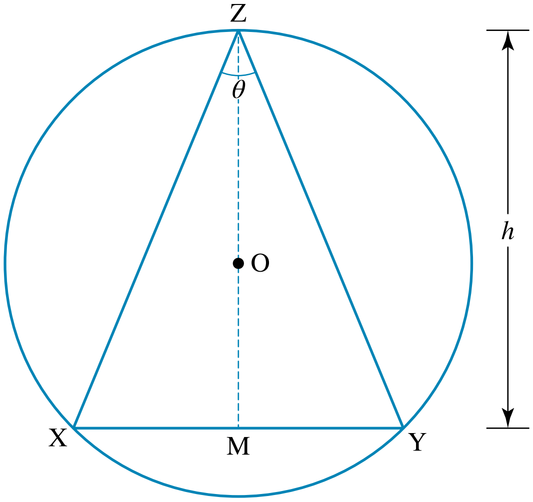
7 The triangle XYZ is inscribed by a circle with radius, r cm. The actual placement of the triangle is dependent on the size of the angle XZY, θ radians, and the length of ZM, where M is the midpoint of XY.

a Show that $\angle XOM = \theta$.

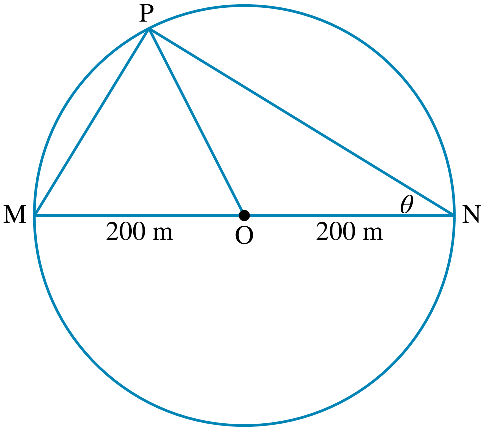
b Show that the relationship between θ, r and h, where $h = d(\overline{ZM})$, is given by $\frac{h}{r} = \cos(\theta) + 1$.

c If the radius of the circle is 3 cm, find $\frac{dh}{d\theta}$.

d Find the exact value of $\frac{dh}{d\theta}$ when $\theta = \frac{\pi}{6}$.



8 The figure shows a circular running track with centre O. The track has a radius of 200 metres.



An athlete at a morning training session completes an obstacle course from N to P at a rate of 2 m/s and then a series of hurdles from P to M along the running track at a rate of 5 m/s.

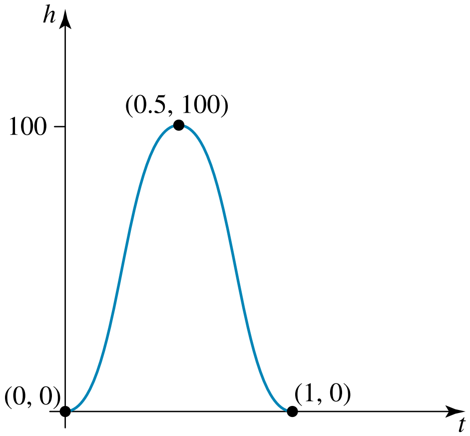
a If $\angle MNP = \theta$ radians and the total time taken to complete the total course is T seconds, show that

$$T = 40(5\cos(\theta) + 2\theta), 0 < \theta \leq \frac{\pi}{2}.$$

b Find the value of θ when $\frac{dT}{d\theta} = 0$.

c What is the maximum time taken to complete to whole course? Give your answer in minutes and seconds.

9 A very young girl is learning to skip. The graph showing this skipping for one cycle is given.



The general equation for this graph is given by $h = a\cos(nt) + c$, where h is the height in millimetres of the girl's feet above the ground and t is the time in seconds the girl has been skipping.

a Find the values of the constants a, n and c, and hence restate the equation for one cycle of the skipping.

b Find $\frac{dh}{dt}$.

c What is the value of $\frac{dh}{dt}$ when $t = 0.25$ seconds?

10 The height, h metres, above ground level of a chair on a rotating Ferris wheel is modelled by the function

$$h = 5 - 3.5\cos\left(\frac{\pi t}{30}\right)$$

where t is measured in seconds.

a People can enter a chair when it is at its lowest position, at the bottom of the rotation. They enter the chair from a platform. How high is the platform above ground level?

b What is the highest point reached by the chair?

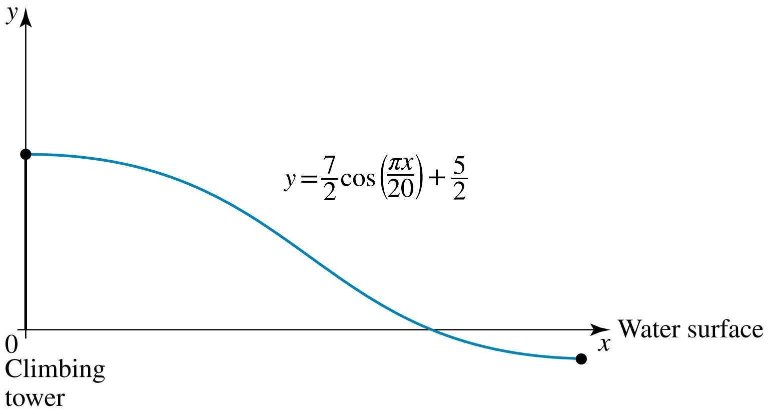
c How long does 1 rotation of the wheel take?

d During a rotation, for how long is a chair higher than 7 m off the ground? Give your answer to 1 decimal place.

e Find $\frac{dh}{dt}$.

f Find the first two times, correct to 2 decimal places, when a chair is descending at a rate of 0.2 m/sec.

11 A section of a water slide at a local aquatic complex is shown.



The water slide can be defined by the rule

$$y = \frac{7}{2}\cos\left(\frac{\pi x}{20}\right) + \frac{5}{2}, 0 \le x \le 20$$

where y is the height in metres of the water slide above the water surface and x is the horizontal distance in metres between the start of the slide and the end of the slide. (*Note*: The x-axis represents the water surface.)

a How high must a person climb in order to reach the top of the water slide?

b Find $\frac{dy}{dx}$.

c What is the exact gradient of the water slide:

i when $x = 5$ **ii** when $x = 10$?

d **i** How far, to the nearest whole metre, from the climbing tower does the slide come into contact with the water surface.

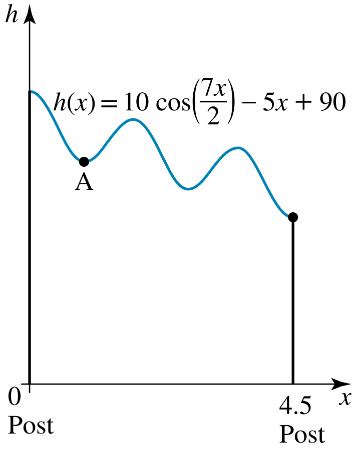
ii What angle does the slide make with the water surface at this point? Give your answer correct to 2 decimal places.

12 The following represents the cross-section of a waterfall feature in an Australian native garden.

It consists of an undulating surface of corrugated plastic with vertical posts at each end. The relationship that defines this surface can be expressed by

$$h(x) = 10\cos\left(\frac{7x}{2}\right) - 5x + 90, 0 \le x \le 4.5$$

where h centimetres represents the vertical height of the water feature and x metres is the horizontal distance between the upright posts supporting the undulating surface. The posts supporting the undulating surface over which the water falls are situated at the points $x = 0$ and $x = 4.5$, as shown.

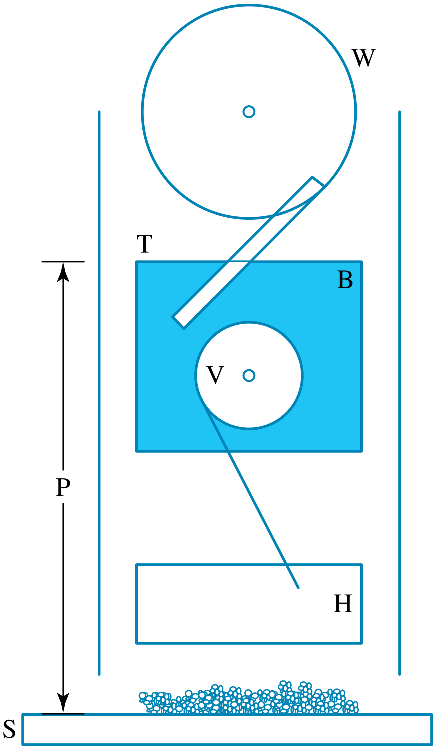


a Find the coordinates of the end points of the undulating surface. Give your answers correct to the nearest centimetre.

b Find the coordinates of point A, the first point in the interval [0, 4.5] where the gradient of the undulating surface is zero. Give your answer correct to 2 decimal places.

c What is the slope of the undulating surface at $x = 0.4$? Give your answer correct to 1 decimal place.

13 A mechanism for crushing rock is shown. Rocks are placed on a steel platform, S, and a device raises and lowers a heavy mallet, H. The wheel, W, rotates, causing the upper block, B, to move up and down. The other wheel, V, is attached to the block, B, and rotates independently, causing the mallet to move up and down. T is the top of the block B.



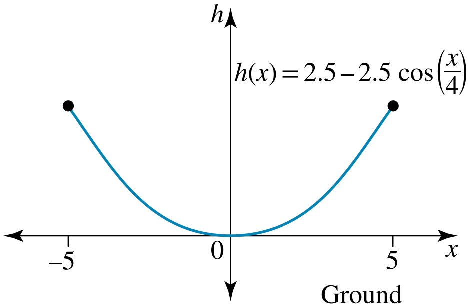
The distance, P metres, between T and the steel platform, S, is modelled by the equation $P = -2\cos(mt) + n$, where t is the time in minutes and m and n are constants. When $t = 0$, T is at its lowest point, 4 metres above the steel platform. The wheel, W, rotates at a rate of 1 revolution per 1.5 minutes.

a Show that $n = 6$ and $m = \dfrac{4\pi}{3}$.

b Find $\dfrac{dP}{dt}$.

c What is the exact rate of change of distance when $t = 0.375$ minutes?

14 At a skateboard park, a new skateboard ramp has been constructed. A cross-section of the ramp is shown.



The equation that approximately defines this curve is given by

$$h(x) = 2.5 - 2.5\cos\left(\frac{x}{4}\right), -5 \le x \le 5$$

where h is the height in metres above the ground level and x is the horizontal distance in metres from the lowest point of the ramp to each end of the ramp.

a Determine the maximum depth of the skateboard ramp, giving your answer correct to 1 decimal place.

b Find the gradient of the ramp, $\dfrac{dh}{dx}$.

c Find $\dfrac{dh}{dx}$ when $x = 3$, giving your answer correct to 3 decimal places.

d Find where $\dfrac{dh}{dx} = 0.58$, giving your answer correct to 3 decimal places.

MASTER

15 An industrial process is known to cause the production of two separate toxic gases that are released into the atmosphere. At a factory where this industrial process occurs, the technicians work a 12-hour day from 6.00 am until 6.00 pm.

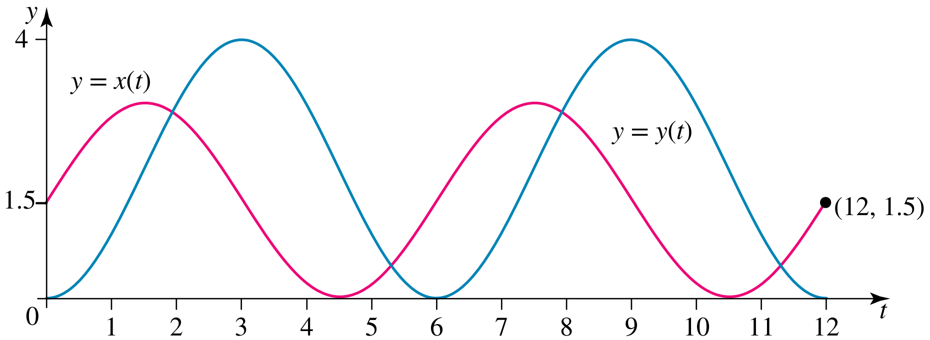
The emission of the toxic gas X can be modelled by the rule

$$x(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 1.5, 0 \le t \le 12$$

and the emission of the toxic gas Y can be modelled by the rule

$$y(t) = 2.0 - 2.0\cos\left(\frac{\pi t}{3}\right), 0 \le t \le 12.$$

The graphs of these two functions are shown.



a At what time of the day are the emissions the same for the first time, and how many units of each gas are emitted at that time? Give your answer correct to 2 decimal places, and remember to note whether the time is am or pm.

b The Environment Protection Authority (EPA) has strict rules about the emissions of toxic gases. The total emission of toxic gases for this particular industrial process is given by

$$T(t) = x(t) + y(t)$$

  i Sketch the graph of the function $T(t)$.

  ii Find the maximum and minimum emissions in a 12-hour working day and the times at which these occur.

c If the EPA rules state that all toxic emissions from any one company must lie within the range 0 to 7 units at any one time, indicate whether this company works within the guidelines.

16 At a suburban shopping centre, one of the stores sells electronic goods such as digital cameras, laptop computers and printers. The store had a one-day sale towards the end of the financial year. The doors opened at 7.55 am and the cash registers opened at 8.00 am. The store closed its doors at 11.00 pm. The total number of people queuing at the six cash registers at any time during the day once the cash registers opened could be modelled by the equation

$$N(t) = 45\sin\left(\frac{\pi t}{5}\right) - 35\cos\left(\frac{\pi t}{3}\right) + 68, 0 \le t \le 15$$

where $N(t)$ is the total number of people queuing t hours after the cash registers opened at 8.00 am.

a Many people ran into the store and quickly grabbed bargain items. How many people were queuing when the cash registers opened?

b When was the quietest time of the day and how many people were in the queue at this time?

c How many people were in the queue at midday?

d What was the maximum number of people in the queue between 3.00 pm and 7.00 pm?

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5.7 Review

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- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
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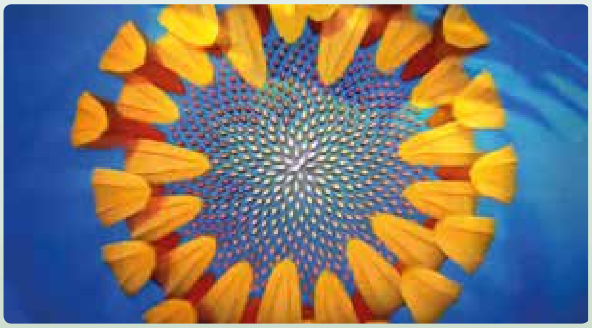

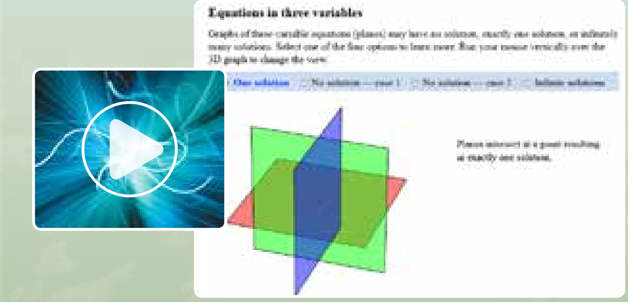

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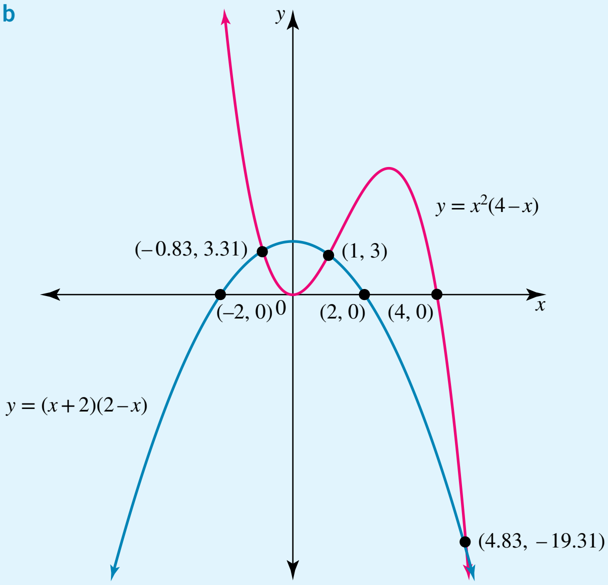
5 Answers

EXERCISE 5.2

1 a $\frac{dy}{dx} = -12 + 12x - 3x^2$ **b** -3

2 a, b



(−0.83, 3.31), (1, 3) and (4.83, −19.31)

c The gradient of $y = (x + 2)(2 - x)$ when $x = 1$ is -2.

The gradient of $y = x^2(4 - x)$ when $x = 1$ is 5.

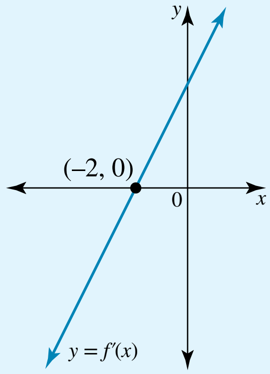
3 a $12x^2 - \frac{2}{3x^3}$ **b** $-\frac{1}{x^{\frac{7}{2}}} - \frac{1}{5}$

4 a $3x^2 + 6x + 1$ **b** $-\frac{6}{x^{\frac{5}{2}}} + \frac{1}{x^2}$

5 a -14 **b** (2, 0) and (–2, 4)

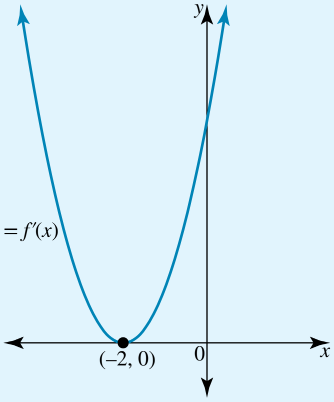
6 $11 + 4a$

7 a



b $x \in (-\infty, 2)\backslash\{-2\}$

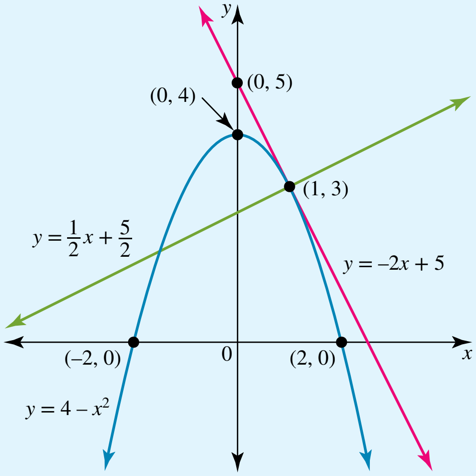
8



9 a $y = -4x + 9$

b $y = \frac{1}{4}x - \frac{15}{4}$

10 $y = 4 - x^2$

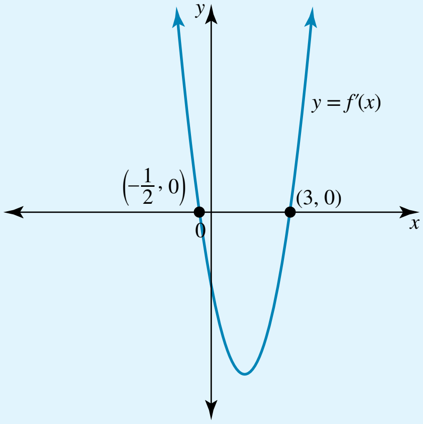


11 a $\frac{dy}{dx} = -\frac{15}{4x^6} + \frac{1}{2x^2}$ **b** $f'(x) = -\frac{30}{x^4} + \frac{2}{x^2} - \frac{4}{x^5}$

c $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{4x^{\frac{3}{2}}}$ **d** $f'(x) = -\frac{27}{2x^2} - x + \frac{9}{2}$

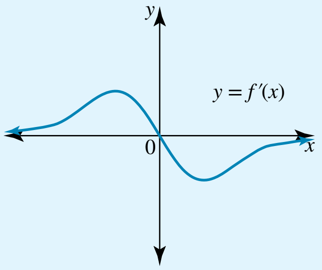
12 a i Domain = R

ii



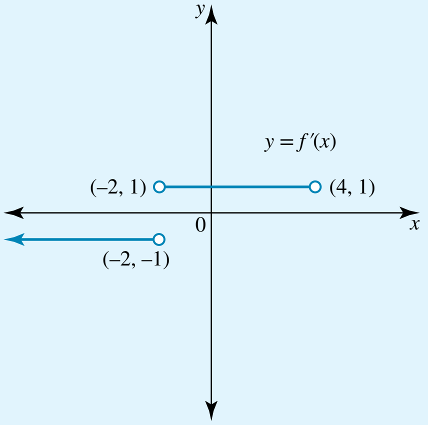
b i Domain = R

ii



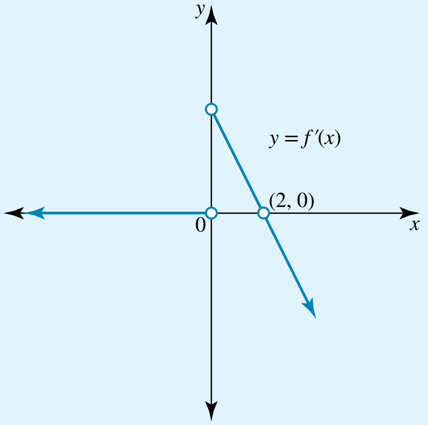
c i Domain = $(-\infty, 4)\backslash\{-2\}$

ii



d i Domain = $R\backslash\{0\}$

ii



13 a 4 **b** 36 **c** -9

14 a $f'(x) = -1$ **b** $f'(x) = 6x - 2$

15 a 4 **b** 5

c 12 **d** $\frac{39}{16}$

16 a $y = -6x - 22$ **b** $y = -3a^2x + 2a^3 + 8$

c $y = \frac{\sqrt{3}}{3}x + \sqrt{3} - 5$ **d** $y = -\frac{7}{2}x + 2$

17 a $y = \frac{1}{6}x + \frac{53}{6}$

b $y = -\frac{1}{3a^2}x + 8 - a^3 + \frac{1}{3a}$

c $y = -\sqrt{3}x + 5\sqrt{3} - 5$

d $y = \frac{2}{7}x + \frac{67}{7}$

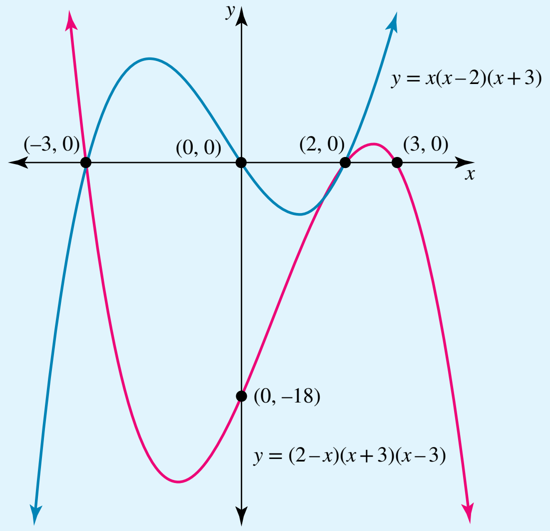
18 a $y = 3x - \frac{3}{4}$ **b** $y = \frac{1}{2}x - \frac{1}{2}$

19 $y = -x^2 + 7x - 10$

20 $y = x^3 - x$

21 $m = 12$

22 a



b $\left(1\frac{1}{2}, -3\frac{3}{8}\right)$

c Tangent: $y = \frac{15}{4}x - 9$

Line perpendicular to tangent: $y = -\frac{4}{15}x - \frac{119}{40}$.

EXERCISE 5.3

1 a $-\frac{1}{3}e^{-\frac{1}{3}x}$ **b** $12x^3 + 4xe^{-2x^2}$

c $-\frac{8}{3}e^{-2x} + \frac{4}{3}e^{-4x} - 2e^{-3x}$ **d** $4e^{4x} - 12e^{2x}$

2 $\frac{1}{2}$

3 $y = 2x + 1$

4 $y_T = -3x + 5,\ y_P = \frac{1}{3}x + 5$

5 a $-20e^{-4x}$ **b** $-\frac{1}{2}e^{-\frac{1}{2}x} + x^2$

c $12e^{3x} - \frac{3e^{6\sqrt{x}}}{2\sqrt{x}} + 9e^{-3x+2}$ **d** $3e^{3x} + 3e^{-3x} - 4e^{-2x}$

e $4e^{2x} + e^{-x}$ **f** $e^x - 2e^{2x} - 3e^{-x}$

6 a -2 **b** $-\frac{8}{e}$

c $\frac{3e}{2}$ **d** 1

7 $y = -2ex$

8 $y_T = -3x - 1,\ y_P = \frac{1}{3}x - 1$

9 $y_T = \frac{e^{\sqrt{3}}}{2\sqrt{3}}x + e^{\sqrt{3}} + 1 - \frac{3e^{\sqrt{3}}}{2\sqrt{3}}$

$y_P = -\frac{2\sqrt{3}}{e^{\sqrt{3}}}x + e^{\sqrt{3}} + 1 + \frac{6\sqrt{3}}{e^{\sqrt{3}}}$

10 a $-2e^7$ **b** $\frac{3}{2}$

11 a $2e^2 - \frac{2}{e}$ **b** 0

12 $y = 5x - 4$

13
$$\begin{aligned} f'(x) &= Ae^x - 3Be^{-3x} \\ 0 &= Ae^x - 3Be^{-3x} \\ &= e^{-3x}(Ae^{4x} - 3B) \\ e^{-3x} &\neq 0, \text{ no real solution} \\ \text{So, } 0 &= Ae^{4x} - 3B \\ 3B &= Ae^{4x} \\ e^{4x} &= \frac{3B}{A} \end{aligned}$$

14 a 2 **b** −1.38

15 $2\log_e(3)$

16 a (1.89, 1.05) **b** 7.66

EXERCISE 5.4

1 a 20 **b** $-0.0304e^{-0.00152t}$ **c** 0.0291 g/year

2 a $0.5I_0$ **b** $0.0011I_0$

3 a $q = -4$ **b** $(m, 0) = (\log_e 3, 0)$

c $f'(x) = 2e^{2x} - 4e^x$ **d** $f'(0) = -2$

4 a $z = -3$ **b** $(n, 0) = (-\log_e(2), 0)$

c $f'(x) = -2e^{-2x} + 3e^{-x}$ **d** 1

5 a 120

b
$$\begin{aligned} \frac{dA}{dt} &= -120ke^{-kt} \\ \frac{dA}{dt} &= -k \times 120e^{-kt} \\ \frac{dA}{dt} &= -k \times A \\ \frac{dA}{dt} &\propto A \end{aligned}$$

c $k = \frac{1}{2}\log_e\left(\frac{4}{3}\right)$

d 8.408 units/min

e As $t \to \infty$, $A \to 0$. Technically, the graph approaches the line $A = 0$ (with asymptotic behaviour, so it never reaches $A = 0$ exactly). However, the value of A would be so small that in effect, after a long period of time, there will be no gas left.

6 a 11

b $\frac{dL}{dt} = 6.589e^{0.599t}$

c 39.742 mm/month

7 a $T_0 = 75$

b 1.531°C/min

8 a 23.2 million

b 2005, September

c 0.29 million/year

d 2019

9 a $P_0 = 75.32$ cm of mercury, $k = 0.24$

b 5.45 cm of mercury/km

10 a 10.94 cm

b 20.6°

11 a (0, 0.5)

b $f'(x) = -e^{-x} + e^{-2x}$

c (0, 0.5)

d 63.4°

e $y = -0.2325x + 0.5327$

f $y = 4.3011x - 4.0009$

12 a 30 **b** 450 000

c 37 072.2/day **d** 450

e 45 000

f $m = -\frac{1}{7}\log_e(10)$, 39 711/day

13 a 5

b $-10xe^{-x^2}$

c i 3.89

ii −3.68

14 a $\frac{x^3 - 6x}{e^{x^2}}$ **b** $\left(\pm\sqrt{6}, \frac{1}{2e^6}\right)$ **c** −0.593

EXERCISE 5.5

1 a $5 - 3\sin(x) + 5\cos(x)$

b $3\cos(3x + 2) + 6x\sin(3x^2)$

c $3\cos(9x)$

d $10\sec^2(2x) - 10x^4$

e $2\sec^2\left(\frac{x}{4}\right)$

f $\frac{\pi}{20}\sec^2\left(\frac{\pi x}{20}\right)$

2 $-2\cos(2x)$

3 $y = x - \frac{\pi}{2}$

4 $y = 4x + \frac{\pi}{2} - 1$

5 a $-6\sin(3x)$ **b** $-\frac{\pi}{180}\sin\left(\frac{\pi x}{180}\right)$

c $3\sin\left(\frac{\pi}{2} - x\right)$ **d** $-\frac{4}{3}\cos\left(\frac{x}{3}\right)$

e $\frac{\pi}{15}\cos\left(\frac{\pi x}{15}\right)$ **f** $6\cos\left(\frac{\pi}{2} + 3x\right)$

g $-5x\sec^2(5x^2)$ **h** $20\sec^2(20x)$

6 $\left(\frac{4\pi}{3}, -\sqrt{3}\right)$

7 $y = -\frac{3}{2}x + \frac{\pi}{4} + \frac{3\sqrt{3}}{2}$

8 $y = 4x + 2 - \pi$

9 116.6°

10 a $y_T = 3x - 2\pi, y_P = -\frac{1}{3}x + \frac{2\pi}{9}$

b $y_T = -\frac{1}{2}x + \frac{\pi}{2}, y_P = 2x - 2\pi$

11 a -1 **b** $\frac{\pi}{4}, \frac{5\pi}{4}$

c $\cos(x) + \sin(x)$ **d** $\frac{3\pi}{4}, \frac{7\pi}{4}$

12 a $\sqrt{3}$ **b** $-\frac{\pi}{3}, \frac{2\pi}{3}$

c $-\sqrt{3}\sin(x) + \cos(x)$ **d** $-\frac{5\pi}{6}, \frac{\pi}{6}$

13 a $\tan(x)$ **b** $\sec^2(x)$

14 $x = -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$

15 $(-0.524, 0.342), (0.524, -0.342)$

16 $(0.243, 1.232), (0.804, 0.863)$

EXERCISE 5.6

1 a $A = \frac{1}{2}ab\sin(c)$

$= \frac{1}{2} \times 6 \times 7\cos(\theta)$

$= 21\sin(\theta)$

b $21\cos(\theta)$

c 10.5 cm^2/radian

2 a $BD = a\sin(\theta)$, $CD = a\cos(\theta)$

b $L = a + 2a\sin(\theta) + a\cos(\theta) + 4$

c $2a\cos(\theta) - a\sin(\theta)$

d $\theta = 1.1^c$

3 a 10 **b** $2\pi\cos(\pi t)$ **c** -2π cm/s

4 a 12 hours

b Low tide = 1 metre at 3.00 pm

c $\frac{\pi}{12}\cos\left(\frac{\pi t}{6}\right)$

d $\frac{\sqrt{2}\pi}{24}$

e 4.30 pm

5 a 0.849, 1.508 **b** $\left(\frac{\pi}{8}, 3\right), \left(\frac{3\pi}{8}, -1\right)$

c -8 **d** $\left[0, \frac{\pi}{8}\right) \cup \left(\frac{3\pi}{8}, \frac{\pi}{2}\right]$

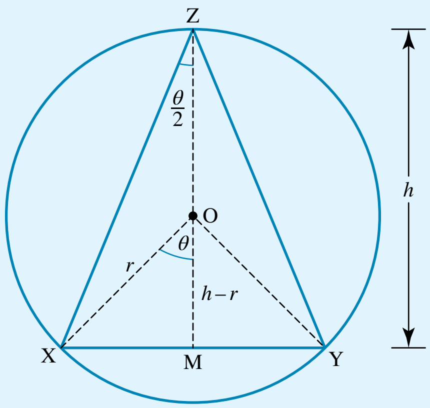
6 a $L = 3 \times 100\cos(\theta) + 4 \times 100\sin(\theta) + 2 \times 100$

$L = 300\cos(\theta) + 400\sin(\theta) + 200$ as required

b $-300\sin(\theta) + 400\cos(\theta)$

c 700 cm, $\theta = 0.93^c$

7 a



$\angle XOY = 2\theta$ because the angle at the centre of the circle is twice the angle at the circumference.

$\angle XOM = \angle YOM = \frac{1}{2} \times 2\theta$

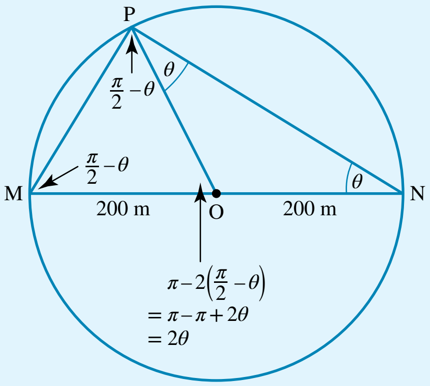
$\angle XOM = \theta$ as required

b
$$XM = r\sin(\theta)$$
$$\frac{XM}{h-r} = \tan(\theta)$$
$$\frac{r\sin(\theta)}{h-r} = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\frac{r}{h-r} = \frac{1}{\cos(\theta)}$$
$$\frac{h-r}{r} = \cos(\theta)$$
$$\frac{h}{r} - 1 = \cos(\theta)$$
$$\frac{h}{r} = \cos(\theta) + 1$$

c $-3\sin(\theta)$

d $-\frac{3\sqrt{3}}{2}$

8 a



Distance ÷ time = velocity

Distance = velocity × time

Distance ÷ velocity = time

So $d(PM) = 400\cos(\theta)$.

$$\begin{aligned} T_{\text{obstacles}} &= \frac{400\cos(\theta)}{2} \\ T_{\text{obstacles}} &= 200\cos(\theta) \\ d(PM) &= 200 \times 2\theta \\ d(PM) &= 400\theta \\ T_{\text{hurdles}} &= \frac{400\theta}{5} \\ &= 80\theta \\ T_{\text{total}} &= T_{\text{obstacles}} + T_{\text{hurdles}} \\ T_{\text{total}} &= 200\cos(\theta) + 80\theta \\ T_{\text{total}} &= 40(5\cos(\theta) + 2\theta) \end{aligned}$$

b 0.4115

c $T_{\max} = 3$ min 36 s

9 a $h = 50\cos(2\pi t) + 50$

b $-100\pi\sin(2\pi t)$

c -100π mm/s

10 a 1.5 m

b 8.5 m

c 60 s

d 18.4 s

e $\frac{7\pi}{60}\sin\left(\frac{\pi t}{30}\right)$

f 35.51 s, 54.49 s

11 a 6 m

b $-\frac{7\pi}{40}\sin\left(\frac{\pi x}{20}\right)$

c **i** $-\frac{7\sqrt{2}\pi}{80} = -0.3888$

ii $-\frac{7\pi}{40} = -0.5498$

d **i** 15 m

ii 159.27°

12 a (0, 100), (4.5, 57.5)

b (0.94, 75.41)

c −39.5

13 a $P = -2\cos(mt) + n$

When $t = 0,\ P = 4$:

$$\begin{aligned} 4 &= -2\cos(0) + n \\ 4 + 2 &= n \\ 6 &= n \end{aligned}$$

Period: $\frac{3}{2} = \frac{2\pi}{m}$

$$\begin{aligned} 3m &= 4\pi \\ m &= \frac{4\pi}{3} \end{aligned}$$

b $\frac{8\pi}{3}\sin\left(\frac{4\pi t}{3}\right)$

c $\frac{8\pi}{3}$ m/min

14 a 1.7 m

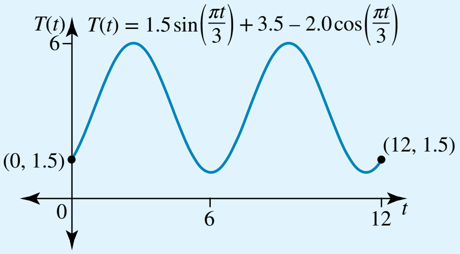
b $0.625\sin\left(\frac{x}{4}\right)$

c 0.426

d 4.756 m

15 a 2.86 units at 7.55 am ($t = 1.92$)

b **i**



ii Minimum 1 unit at 11.23 am ($t = 5.39$) and 5.23 pm ($t = 11.39$)

Maximum 6 units at 8.23 am ($t = 2.39$) and 2.23 pm ($t = 8.39$)

c Emissions of 1 unit and 6 units lie within the guidelines.

16 a 33

b 1 person at 2.28 pm ($t = 6, 46$)

c 112

d 86

6

Further differentiation and applications

6.1 Kick off with CAS

Sketching curves

The following steps demonstrate how to use CAS to sketch $f(x) = (2x-a)(x-b)^2$, $0 < a < b$.

1 On a calculation screen, define $f(x) = (2x-a)(x-b)^2$.

2 To find the x-intercepts, solve $f(x) = 0$ for x.

3 To find the y-intercepts, evaluate $f(0)$.

4 Determine $f'(x)$.

5 To find where the stationary points occur, solve $f'(x) = 0$ for x.

6 Substitute each x-value where the stationary points occur into $f(x)$ to find the corresponding y-values.

7 To find the nature of each stationary point, set up a standard table, placing the x-values of the stationary points in the x_1 and x_2 positions. As the x-values of the stationary points contain parameters, add and subtract 1 for the upper and lower limits respectively. To ensure a value between x_1 and x_2 is found, determine the midpoint between the two values.

x	x_1-1	x_1	$\frac{x_1+x_2}{2}$	x_2	x_2+1
$f'(x)$					

8 Determine $f'(x)$ for $x = x_1$, $\frac{x_1+x_2}{2}$ and x_2, and hence determine the nature of each stationary point.

9 Use all the information found to sketch the graph.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

6.2 The chain rule

Composite functions

studyon
Units 3 & 4
AOS 3
Topic 2
Concept 3
The chain rule
Concept summary
Practice questions

A **composite function**, also known as a function of a function, consists of two or more functions nested within each other. Consider the functions $g(x) = x^4$ and $h(x) = 2x + 1$.

If $f(x) = g(h(x))$, we are actually determining the rule for $g(2x + 1)$, so $f(x) = g(h(x)) = g(2x + 1) = (2x + 1)^4$.

It is worth noting, however, that $g(h(x))$ is not necessarily equal to $h(g(x))$. In this instance, $h(g(x)) = h(x^4) = 2x^4 + 1$.

The chain rule for differentiation is another name for the derivative of a composite function.

The proof of the chain rule

Consider again $f(x) = (2x + 1)^4$. If this is expanded, it is possible to find the derivative.

$$\begin{aligned} f(x) &= (2x + 1)^4 \\ &= (2x)^4 + 4(2x)^3(1) + 6(2x)^2(1)^2 + 4(2x)(1)^3 + 1^4 \\ &= 16x^4 + 32x^3 + 24x^2 + 8x + 1 \end{aligned}$$

Therefore, the derivative can be given by

$$\begin{aligned} f'(x) &= 64x^3 + 96x^2 + 48x + 8 \\ &= 8(8x^3 + 12x^2 + 6x + 1) \\ &= 8(2x + 1)^3 \end{aligned}$$

The chain rule allows us to reach this same outcome without having to expand the function first.

The proof of the chain rule is as follows.

If $f(x) = m(n(x))$,

then $f(x + h) = m(n(x + h))$.

Therefore, $\dfrac{f(x + h) - f(x)}{h} = \dfrac{m(n(x + h)) - m(n(x))}{h}$.

Multiply the numerator and the denominator by $n(x + h) - n(x)$, as it is expected that at some stage $n'(x)$ will appear somewhere in the rule.

$$\frac{f(x + h) - f(x)}{h} = \frac{n(x + h) - n(x)}{h} \times \frac{m(n(x + h)) - m(n(x))}{n(x + h) - n(x)}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0}\left[\frac{m(n(x + h)) - m(n(x))}{n(x + h) - n(x)} \times \frac{n(x + h) - n(x)}{h}\right] \\ &= \lim_{h \to 0}\left[\frac{m(n(x + h)) - m(n(x))}{n(x + h) - n(x)}\right] \times \lim_{h \to 0}\left[\frac{n(x + h) - n(x)}{h}\right] \end{aligned}$$

By definition, $n'(x) = \lim\limits_{h \to 0} \dfrac{n(x+h) - n(x)}{h}$. Also, if we let $n(x) = A$ and

$n(x+h) = A + B$, then $n(x+h) - n(x) = A + B - A$, so that

$$\frac{m(n(x+h)) - m(n(x))}{n(x+h) - n(x)} = \frac{m(A+B) - n(A)}{B}.$$

Also, as $h \to 0$, $B \to 0$.

Consequently, $\lim\limits_{B \to 0} \dfrac{m(A+B) - m(A)}{B} = m'(A)$.

Therefore, $\lim\limits_{h \to 0} \left[\dfrac{m(n(x+h)) - m(n(x))}{n(x+h) - n(x)} \right] = m'(n(x))$.

Bringing this all together, we can see that

If $f(x) = m(n(x))$,

$f'(x) = m'(n(x)) \times n'(x)$.

Using Leibniz notation, this becomes

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$, where $y = f(u)$ and u is a function of x.

Consider again $y = f(x) = (2x+1)^4$. The chain rule can be used to find the derivative of this function.

Let $u = 2x + 1$, $\therefore \dfrac{du}{dx} = 2$

and $y = u^4$, $\therefore \dfrac{dy}{du} = 4u^3$.

By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$$= 4u^3 \times 2$$

$$= 8u^3$$

Since $u = 2x + 1$, $\dfrac{dy}{dx} = 8(2x+1)^3$.

In Topic 5, the following derivatives were given. We are now in a position to derive these using the chain rule.

$$\frac{d}{dx}(\sin(kx)) = k\cos(kx)$$

$$\frac{d}{dx}(\cos(kx)) = -k\sin(kx)$$

$$\frac{d}{dx}(\tan(kx)) = k\sec^2(kx) = \frac{k}{\cos^2(kx)}$$

We will prove the last of these facts.

If $y = \tan(kx)$,

let $u = kx, \;\therefore \dfrac{du}{dx} = k$

and $y = \tan(u), \therefore \dfrac{dy}{du} = \sec^2(u)$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \sec^2(u) \times k$$
$$\frac{dy}{dx} = k\sec^2(u)$$

But $u = kx \therefore \dfrac{dy}{dx} = k\sec^2(kx) = \dfrac{k}{\cos^2(kx)}$

The other derivatives can be shown in a similar way.

The chain rule can also be applied without showing so much detail. Suppose you are asked to differentiate $y = \sin(e^{2x})$. Firstly, differentiate the inner function.

$$\frac{d}{dx}(e^{2x}) = 2e^{2x}$$

Then differentiate the outer function.

$$\frac{d}{dx}(\sin(e^{2x})) = \cos(e^{2x})$$

The required derivative is the product of the two derivatives.

$$\frac{dy}{dx} = 2e^{2x} \times \cos(e^{2x}) = 2e^{2x}\cos(e^{2x})$$

WORKED EXAMPLE 1

Use the chain rule to find the derivative of $y = (x^3 + 2x^2 - x^{-2})^{-7}$.

THINK	WRITE
1 Write the function to be derived.	$y = (x^3 + 2x^2 - x^{-2})^{-7}$
2 Let u equal the inner function.	Let $u = x^3 + 2x^2 - x^{-2}$.
3 Differentiate to find $\dfrac{du}{dx}$.	$\dfrac{du}{dx} = 3x^2 + 4x + 2x^{-3}$
4 State the equation relating y and u.	$y = u^{-7}$
5 Differentiate to find $\dfrac{dy}{du}$.	$\dfrac{dy}{du} = -7x^{-8}$
6 Apply the chain rule.	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
	$\dfrac{dy}{dx} = -7u^{-8} \times (3x^2 + 4x + 2x^{-3})$
7 Substitute back in for u.	$\dfrac{dy}{dx} = -7(3x^2 + 4x + 2x^{-3})(x^3 + 2x^2 - x^{-2})^{-8}$

The chain rule must often be applied first before application problems involving the derivative can be solved.

WORKED EXAMPLE 2 **Use the chain rule to find the derivative of $y = \cos^2(e^x)$ and evaluate the derivative when $x = 0$, giving your answer correct to 4 decimal places.**

THINK	WRITE
1 Write the function to be derived.	$y = \cos^2(e^x) = [\cos(e^x)]^2$
2 Consider first the inner function.	$\frac{d}{dx}(\cos(e^x))$
3 Use the chain rule to differentiate this inner function.	$\frac{d}{dx}(\cos\,(e^x)) = -e^x\sin(e^x)$
4 Consider the outer function.	$\frac{d}{dx}(\cos(e^x))^2$
5 Apply the chain rule to differentiate this function.	$\frac{d}{dx}(\cos(e^x))^2 = 2\cos(e^x)$
6 The required derivative is the product of the two previously found derivatives.	$\frac{dy}{dx} = 2\cos(e^x) \times -e^x\sin(e^x)$ $\frac{dy}{dx} = -2e^x\cos(e^x)\sin(e^x)$
7 Evaluate $\frac{dy}{dx}$ when $x = 0$.	Let $x = 0$. $\frac{dy}{dx} = -2e^0\cos(e^0)\sin(e^0)$ $= -2(1)\cos(1)\sin(1)$ $= -0.9093$

WORKED EXAMPLE 3 **For the function with the rule $y = (x - 1)^{\frac{2}{3}}$, find:**

a $\frac{dy}{dx}$

b **the equations of the tangents at (2, 1) and (0, 1)**

c **the coordinates of the point where these two tangents intersect.**

THINK	WRITE
a 1 Write the function to be derived.	a $y = (x - 1)^{\frac{2}{3}}$
2 Apply the chain rule to find the derivative. Multiply the derivative of the outer function with the derivative of the inner function. Write the answer in surd form.	$\frac{dy}{dx} = \frac{2}{3}(x - 1)^{-\frac{1}{3}} \times (1)$ $\frac{dy}{dx} = \frac{2}{3\sqrt[3]{(x-1)}}$

b **1** Find the gradient at $x = 2$.	**b** When $x = 2$: $\frac{dy}{dx} = \frac{2}{3\sqrt[3]{2-1}}$ $= \frac{2}{3}$
2 Find the equation of the tangent at $x = 2, y = 1$.	If $m_T = \frac{2}{3}, (x_1, y_1) = (2, 1)$. $y - y_1 = m(x - x_1)$ $y - 1 = \frac{2}{3}(x - 2)$ $y - 1 = \frac{2}{3}x - \frac{4}{3}$ $y = \frac{2}{3}x - \frac{1}{3}$ or $2x - 3y = 1$
3 Find the gradient at $x = 0$.	When $x = 0$: $\frac{dy}{dx} = \frac{2}{3\sqrt[3]{0-1}}$ $= -\frac{2}{3}$
4 Find the equation of the tangent at $x = 0, y = 1$.	If $m_T = -\frac{2}{3}, (x_1, y_1) = (0, 1)$. $y - y_1 = m(x - x_1)$ $y - 1 = -\frac{2}{3}(x - 0)$ $y - 1 = -\frac{2}{3}x$ $y = -\frac{2}{3}x + 1$ or $2x + 3y = 3$
c **1** Write the simultaneous equations to be solved and label them.	**c** $y = \frac{2}{3}x - \frac{1}{3}$ (1) $y = -\frac{2}{3}x + 1$ (2)
2 Equate the two equations and solve for x.	$(1) = (2)$ $\frac{2}{3}x - \frac{1}{3} = -\frac{2}{3}x + 1$ $\frac{4}{3}x = \frac{4}{3}$ $x = 1$
3 Solve for y.	Substitute $x = 1$ into (2): $y = -\frac{2}{3}(1) + 1$ $= \frac{1}{3}$
4 Write the answer.	The tangents intersect at the point $\left(1, \frac{1}{3}\right)$.

EXERCISE 6.2 The chain rule

PRACTISE

Work without CAS

1 **WE1** Use the chain rule to find the derivatives of the following.

a $y = \sqrt{x^2 - 7x + 1}$ **b** $y = (3x^2 + 2x - 1)^3$

2 Use the chain rule to find the derivatives of the following.

a $y = \sin^2(x)$ **b** $y = e^{\cos(3x)}$

3 **WE2** If $y = \sin^3(x)$, find the exact value of $\frac{dy}{dx}$ when $x = \frac{\pi}{3}$.

4 If $f(x) = e^{\sin^2(x)}$, find $f'\left(\frac{\pi}{4}\right)$.

5 **WE3** Use the chain rule to find the derivative of $y = \frac{1}{(2x-1)^2}$ and hence find the equation of the tangent to the curve at the point where $x = 1$.

6 Let $f(x) = (x-1)^3$ and $g(x) = e^x$.

a Write the rule for $f(g(x))$.

b Find $h'(x)$ where $h(x) = f(g(x))$.

c Find the equation of the tangent of $y = h(x)$ at the origin.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Find the derivatives of the following functions.

a $g(x) = 3(x^2+1)^{-1}$

b $g(x) = e^{\cos(x)}$

c $g(x) = \sqrt{(x+1)^2+2}$

d $g(x) = \frac{1}{\sin^2(x)}$

e $f(x) = \sqrt{x^2-4x+5}$

f $f(x) = 3\cos(x^2-1)$

g $f(x) = 5e^{3x^2-1}$

h $f(x) = \left(x^3 - \frac{2}{x^2}\right)^{-2}$

i $f(x) = \frac{\sqrt{2-x}}{2-x}$

j $\cos^3(2x+1)$

8 Find the derivatives of the following functions, and hence find the gradients at the given x-values.

a $f(x) = \tan(4x+\pi)$; find $f'\left(\frac{\pi}{4}\right)$.

b $f(x) = (2-x)^{-2}$; find $f'\left(\frac{1}{2}\right)$.

c $f(x) = e^{2x^2}$; find $f'(-1)$.

d $f(x) = \sqrt[3]{(3x^2-2)^4}$; find $f'(1)$.

e $f(x) = (\cos(3x)-1)^5$; find $f'\left(\frac{\pi}{2}\right)$.

9 Let $f: R^+ \to R$, $f(x) = \frac{1}{x^2}$. Find $g(x) = f(f(x))$ and hence find $g'(x)$.

10 If $f(x) = \sin^2(2x)$, find the points where $f'(x) = 0$ for $x \in [0, \pi]$.

11 If $z = 4y^2 - 5$ and $y = \sin(3x)$, find $\frac{dz}{dx}$.

12 Find $f'(x)$ for each of the following.

a $f(x) = g[\cos(x)]$

b $f(x) = g(2x^3)$

c $f(x) = g(3e^{2x+1})$

d $f(x) = g\left(\sqrt{2x^2-x}\right)$

13 Find $f'(x)$ for each of the following.

a $f(x) = [h(x)]^{-2}$

b $f(x) = \sin^2[h(x)]$

c $f(x) = \sqrt[3]{2h(x)+3}$

d $f(x) = -2e^{h(x)+4}$

14 For the functions with the rules $f(x) = \sqrt[3]{x^2}$ and $h(x) = 2x - 1$:

a define the rule for $g(x) = f(h(x))$

b find $g'(x)$

c find the equations of the tangents at the points (1, 1) and (0, 1)

d find the coordinates of the point of intersection of these two tangents.

15 The function h has a rule $h(x) = \sqrt{x^2 - 16}$ and the function g has the rule $g(x) = x - 3$.

a Find the integers m and n such that $h(g(x)) = \sqrt{(x + m)(x + n)}$.

b State the maximal domain of $h(g(x))$.

c Find the derivative of $h(g(x))$.

d Find the gradient of the function $h(g(x))$ at the point when $x = -2$.

16 The line perpendicular to the graph $y = g(f(x))$ where $f(x) = \frac{1}{x}$ and $g(x) = x - \frac{1}{x^2}$ is given by $y = -x + a$, where a is a real constant. Find the possible value(s) of a.

MASTER

17 For the functions $f(x) = 2\sin(x)$ and $h(x) = e^x$:

a state the rule for:

i $m(x) = f(h(x))$

ii $n(x) = h(f(x))$

b determine when $m'(x) = n'(x)$ over the interval $x \in [0, 3]$, correct to 3 decimal places.

18 For the functions $m(x) = 3^x$ and $n(x) = x^2 + 4x - 5$:

a state the rule of $m(n(x))$

b find the gradient of the function at the point where $x = 1$.

6.3 The product rule

The proof of the product rule

studyon

Units 3 & 4

AOS 3

Topic 2

Concept 4

The product rule

Concept summary

Practice questions

There are many functions that have rules which are the product of two simpler functions, such as $x\sin(x)$ or $e^x(2x + 1)$. In order to differentiate such functions, we need to apply the **product rule**.

If $f(x) = g(x) \times h(x)$, then
$f'(x) = g(x) \times h'(x) + h(x) \times g'(x)$.

Or if $y = uv$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

This rule can be proven as follows.

Let $f(x) = u(x)v(x)$

so $f(x + h) = u(x + h)v(x + h)$

$$\frac{f(x + h) - f(x)}{h} = \frac{u(x + h)v(x + h) - u(x)v(x)}{h}$$

Add and subtract $u(x)v(x + h)$, as it is expected that at some stage $v'(x)$ will appear somewhere in the rule.

$$\frac{f(x + h) - f(x)}{h} = \frac{u(x + h)v(x + h) - u(x)v(x + h) + u(x)v(x + h) - u(x)v(x)}{h}$$

$$= \frac{[u(x + h) - u(x)]v(x + h) + u(x)[v(x + h) - v(x)]}{h}$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{[u(x+h)-u(x)]v(x+h)+u(x)[v(x+h)-v(x)]}{h}\\
&= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\times v(x+h)\right]+\lim_{h\to 0}\left[\frac{v(x+h)-v(x)}{h}\times u(x)\right]\\
&= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\times\lim_{h\to 0}v(x+h)+\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}\times\lim_{h\to 0}u(x)\\
&= u'(x)v(x)+v'(x)u(x)\\
&= u(x)v'(x)+v(x)u'(x)
\end{aligned}
$$

The Leibnitz notation states that if $y = uv$,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Apply this to differentiate $y = x\sin(x)$:

Let $u = x$. Therefore, $\frac{du}{dx} = 1$.

Let $v = \sin(x)$, so $\frac{dv}{dx} = \cos(x)$

$$
\begin{aligned}
\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\
\frac{dy}{dx} &= x\times\cos(x)+\sin(x)\times 1\\
&= x\cos(x)+\sin(x)
\end{aligned}
$$

To differentiate $y = e^x(2x+1)$:

Let $u = e^x$, therefore $\frac{du}{dx} = e^x$

Let $v = 2x+1$, so $\frac{dv}{dx} = 2$

$$
\begin{aligned}
\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\
&= e^x\times 2+(2x+1)\times e^x\\
&= 2e^x+2xe^x+e^x\\
&= 3e^x+2xe^x
\end{aligned}
$$

WORKED EXAMPLE 4

If $g(x) = x^3 \sin(3x)$, find $g'(x)$.

THINK	WRITE
1 Define u and v as functions of x.	$g(x) = x^3 \sin(3x)$ Let $u(x) = x^3$ and $v(x) = \sin(3x)$.
2 Differentiate u and v with respect to x.	$u'(x) = 3x^2$ $v'(x) = 3\cos(3x)$
3 Apply the product rule to determine $g'(x)$.	$g'(x) = u(x)v'(x) + v(x)u'(x)$ $= x^3 \times 3\cos(3x) + \sin(3x) \times 3x^2$ $= 3x^3 \cos(3x) + 3x^2 \sin(3x)$

The product rule may have to be used first before an application problem can be solved.

WORKED EXAMPLE 5

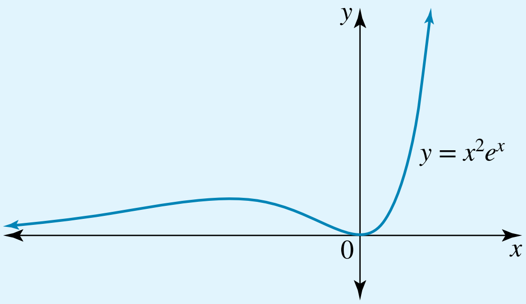
Given that $y = e^{2x}(x+1)^2$, find $\frac{dy}{dx}$ and hence find the equation of the tangent to the curve at the point (0, 1).

THINK	WRITE
1 Define u and v as functions of x.	$y = e^{2x}(x+1)^2$ Let $u = e^{2x}$ and $v = (x+1)^2$.
2 Differentiate u and v with respect to x.	$\frac{du}{dx} = 2e^{2x}$ $\frac{dv}{dx} = 2(x+1)$
3 Apply the product rule to determine $\frac{dy}{dx}$ and simplify.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $\frac{dy}{dx} = e^{2x} \times 2(x+1) + (x+1)^2 \times 2e^{2x}$ $= 2e^{2x}(x+1) + 2e^{2x}(x+1)^2$ $= 2e^{2x}(x+1)(1+x+1)$ $= 2e^{2x}(x+1)(x+2)$
4 Evaluate $\frac{dy}{dx}$ when $x = 0$.	When $x = 0$, then $\frac{dy}{dx} = 2e^0(0+1)(0+2)$ $= 4$
5 Find the equation of the tangent.	If $m = 4$ and $(x_1, y_1) = (0, 1)$, $y - y_1 = m(x - x_1)$ $y - 1 = 4(x - 0)$ $y - 1 = 4x$ $y = 4x + 1$

Frequently, problems may involve graphs of a function being given so that aspects of the function can be investigated.

WORKED EXAMPLE 6

The graph of $f: R \rightarrow R$, $f(x) = x^2e^x$ is shown. Using calculus, find the coordinates where $f'(x) = 0$.



THINK	WRITE
1 Define u and v as functions of x.	$f(x) = x^2e^x$ Let $u(x) = x^2$ and $v(x) = e^x$.
2 Differentiate u and v with respect to x.	$u'(x) = 2x$ $v'(x) = e^x$
3 Apply the product rule to determine $f'(x)$.	$f'(x) = u(x)v'(x) + v(x)u'(x)$ $= x^2 \times e^x + e^x \times 2x$ $= x^2e^x + 2xe^x$
4 Solve $f'(x) = 0$.	$x^2e^x + 2xe^x = 0$ $e^xx(x + 2) = 0$ $e^x > 0$ for all values of x. Either $x = 0$ or $x + 2 = 0$. $\therefore x = 0, -2$
5 Substitute the x-values to find the corresponding y-values.	When $x = -2$, $y = (-2)^2e^{-2}$ $= 4e^{-2}$ When $x = 0$, $y = (0)^2e^0$ $= 0$
6 Write the answer.	The coordinates where the gradient is zero are (0, 0) and $(-2, 4e^{-2})$.

EXERCISE 6.3 The product rule

PRACTISE

Work without CAS
Questions 1–5

1 WE4 For each of the following functions, find the derivative function.

a $f(x) = \sin(3x)\cos(3x)$

b $f(x) = x^2e^{3x}$

c $f(x) = (x^2 + 3x - 5)e^{5x}$

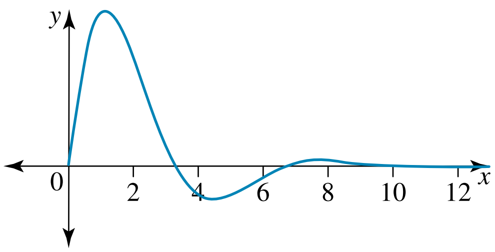
2 If $f(x) = 2x^4\cos(2x)$, find $f'\left(\dfrac{\pi}{2}\right)$.

3 WE5 Given the function $f(x) = (x + 1)\sin(x)$, find $f'(x)$ and hence find the gradient of the function when $x = 0$.

4 Given that $y = (x^2 + 1)e^{3x}$, find the equation of the tangent to the curve at $x = 0$.

5 **WE6** Given $f(x) = 2x^2(1 - x)^3$, use calculus to determine the coordinates where $f'(x) = 0$.

6 The graph of $f: R^+ \to R,\ f(x) = e^{-\frac{x}{2}}\sin(x)$ is shown.



- **a** Find the values of x when $f(x) = 0$ for $x \in [0, 3\pi]$.
- **b** Use calculus to find the values of x when $f'(x) = 0$ for $x \in [0, 3\pi]$. Give your answers correct to 2 decimal places.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Differentiate the following.

- **a** x^2e^{5x}
- **b** $e^{2x+1}\tan(2x)$
- **c** $x^{-2}(2x + 1)^3$
- **d** $x\cos(x)$
- **e** $2\sqrt{x}(4 - x)$
- **f** $\sin(2x - \pi)e^{-3x}$
- **g** $3x^{-2}e^{x^2}$
- **h** $e^{2x}\sqrt{4x^2 - 1}$
- **i** $x^2\sin^3(2x)$
- **j** $(x - 1)^4(3 - x)^{-2}$
- **k** $(3x - 2)^2g(x)$
- **l** $-e^{5x}g(\sqrt{x})$

8 Find the derivative of the following functions, and hence find the gradient at the given point.

- **a** $f(x) = xe^x$; find $f'(-1)$.
- **b** $f(x) = x(x^2 + x)^4$; find $f'(1)$.
- **c** $f(x) = (1 - x)\tan^2(x)$; find $f'\left(\frac{\pi}{3}\right)$.
- **d** $f(x) = \sqrt{x}\sin(2x^2)$; find $f'(\sqrt{\pi})$.

9 Given that $f: R \to R,\ f(x) = x^4e^{-3x}$ and that $f'(x)$ may be written in the form $f'(x) = e^{-3x}(ax^3 + bx^4)$, find the constants a and b.

10 **a** If $f(x) = (x - a)^2g(x)$, find the derivative of f.

b If $f(x) = g(x)\sin(2x)$ and $f'\left(\frac{\pi}{2}\right) = -3\pi$, find the constant a if $g(x) = ax^2$.

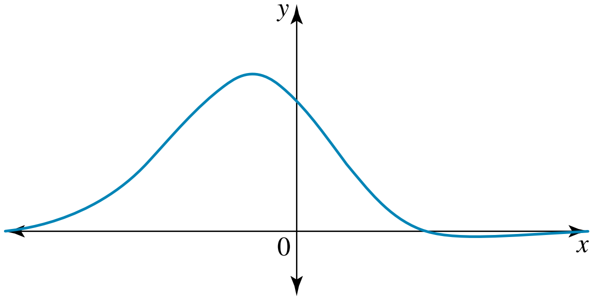
11 Let $y = 2x\tan(2x)$. Evaluate $\frac{dy}{dx}$ when $x = \frac{\pi}{12}$, giving your answer in exact form.

12 For the function with the rule $y = xe^x$, find the equations of the tangent and the line perpendicular to the curve at the point where $x = 1$.

13 Find the derivative of $y = -\cos(x)\tan(x)$ by:

- **a** simplifying the expression first
- **b** applying the product rule and then simplifying.

14 The graph of $y = e^{-x^2}(1 - x)$ is shown.

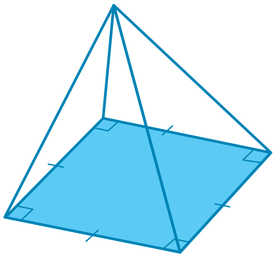


- **a** Find the coordinates of the points where the graph cuts the x- and y-axes.
- **b** Find the coordinates of the points where the gradient is zero, giving your answers correct to 3 decimal places.
- **c** Find the equation of the tangent to the curve at the point where the curve intersects the x-axis.

d Find the equation of the line perpendicular to the curve where the curve crosses the y-axis.

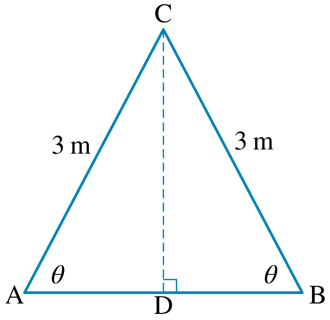
e Where do the tangent and the perpendicular line from parts **c** and **d** intersect? Give your answer correct to 2 decimal places.

15 An artist has been commissioned to produce a sculpture for an art gallery. The artist intends to construct a Perspex square-based pyramid as shown.



She also plans to have an animal-themed sculpture inside the pyramid. Each face of the pyramid is an isosceles triangle as shown.

D is the midpoint of AB. Angles CAB and CBA are each θ radians.

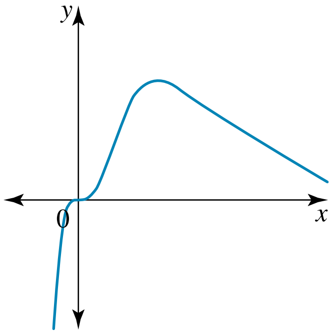


a **i** Find CD in terms of θ.
ii Find BD in terms of θ.

b Show that the total surface area, S m^2, of the pyramid, including the base, is given by $S = 36\left(\cos^2(\theta) + \cos(\theta)\sin(\theta)\right)$.

c Find $\dfrac{dS}{d\theta}$.

16 The graph of the function $f: R \to R, f(x) = 3x^3e^{-2x}$ is shown. The derivative may be written as $f'(x) = ae^{-2x}(bx^2 + cx^3)$ where a, b and c are constants.



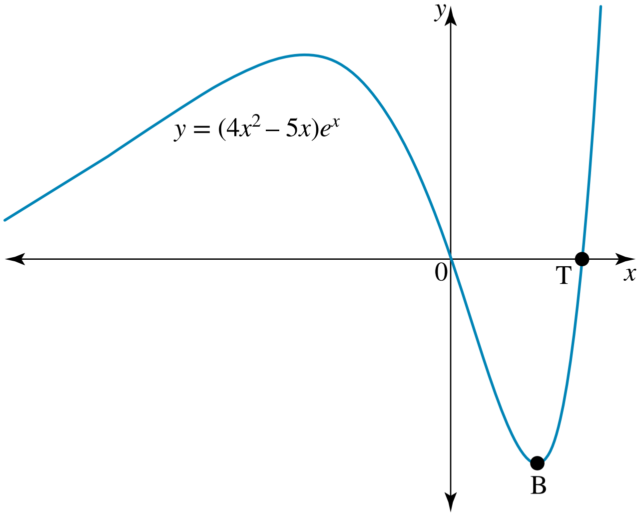
a Find the exact values of a, b and c.

b Find the exact coordinates where $f'(x) = 0$.

c Find the equation of the tangent to the curve at $x = 1$.

MASTER

17 A country town has decided to construct a new road. The x-axis is also the position of the railway line that connects Sydney with Brisbane. The road can be approximated by the equation $y = (4x^2 - 5x)e^x$.



a The post office for the town is positioned at (–2, 3.5). They want the new road to be adjacent to the post office. Have they made a sensible decision regarding the placement of the road?

b Find the coordinates of the point T where the road crosses the railway line.

c Use calculus to find the coordinates of the point B. Give your answer correct to 3 decimal places.

18 Differentiate $y = 2^x \sin(x)$ and find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$. Give your answer correct to 2 decimal places.

6.4 The quotient rule

study on

Units 3 & 4

AOS 3

Topic 2

Concept 5

The quotient rule

Concept summary

Practice questions

The proof of the quotient rule

When one function is divided by a second function, for example $f(x) = \frac{x}{x^2 - 1}$ or $f(x) = \frac{e^x}{\cos(x)}$, we have the quotient of the two functions. For such functions, there is a rule for finding the derivative. It is called the quotient rule.

If $f(x) = \frac{u(x)}{v(x)}$ where $v(x) \neq 0$, then

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}.$$

Or if $y = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

This rule can be proven as follows by using the product rule.

If $f(x) = \frac{u(x)}{v(x)}$, then $f(x) = u(x) \times [v(x)]^{-1}$

$$\begin{aligned} f'(x) &= u(x) \times -1 \times [v(x)]^{-2} \times v'(x) + [v(x)]^{-1} \times u'(x) \\ &= -\frac{u(x)v'(x)}{[v(x)]^2} + \frac{u'(x)}{[v(x)]} \\ &= \frac{u'(x)v(x)}{[v(x)]^2} - \frac{u(x)v'(x)}{[v(x)]^2} \\ &= \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} \end{aligned}$$

The Leibnitz notation for the quotient rule states that if $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

So to differentiate $y = \frac{x}{x^2 - 1}$:

Let $u = x$ so that $\frac{du}{dx} = 1$ and let $v = x^2 - 1$ so that $\frac{dv}{dx} = 2x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{(x^2-1)(1) - x(2x)}{(x^2-1)^2}\\ &= \frac{x^2 - 1 - 2x^2}{(x^2-1)^2}\\ &= \frac{-x^2-1}{(x^2-1)^2}\\ &= -\frac{x^2+1}{(x^2-1)^2}\end{aligned}$$

Note that although the numerator has been factorised and simplified, it is more common not to expand the denominator.

Always check that the quotient rule is the best method to use to differentiate the function. For example, $y = \frac{x-2}{\sqrt{x-2}}$ can be broken down to $y = \sqrt{x-2}$; therefore, the chain rule should be used. Also, $y = \frac{5x^2 - 2x}{\sqrt{x}}$ can be split into separate fractions, and each term can be differentiated using the basic differentiation rule.

Before applying the quotient rule, always check if the function can be simplified first.

WORKED EXAMPLE 7 **Find the derivative of $y = \frac{\sin(2t)}{t^2}$ with respect to t.**

THINK	WRITE
1 Define u and v as functions of t.	$y = \frac{\sin(2t)}{t^2}$ Let $u = \sin(2t)$ and $v = t^2$.
2 Differentiate u and v with respect to t.	$\frac{du}{dt} = 2\cos(2t)$ $\frac{dv}{dt} = 2t$
3 Apply the product rule to determine $\frac{dy}{dt}$ and simplify.	$\frac{dy}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$ $= \frac{t^2\,2\cos(2t) - \sin(2t)\times 2t}{(t^2)^2}$ $= \frac{2t(t\cos(2t) - \sin(2t))}{t^4}$ $= \frac{2(t\cos(2t) - \sin(2t))}{t^3}$

WORKED EXAMPLE 8

Find the derivative of $f(x) = \dfrac{\cos(3x)}{2e^x - x}$ and hence find the gradient at the point where $x = 0$.

THINK	WRITE
1 Define u and v as functions of x.	$f(x) = \dfrac{\cos(3x)}{2e^x - x}$ Let $u(x) = \cos(3x)$ and $v(x) = 2e^x - x$.
2 Differentiate u and v with respect to x.	$u'(x) = -3\sin(3x)$ $v'(x) = 2e^x - 1$
3 Apply the product rule to determine $\dfrac{dy}{dx}$ and simplify.	$f'(x) = \dfrac{v(x)u'(x) - u(x)v'(x)}{v^2}$ $= \dfrac{(2e^x - x) \times -3\sin(3x) - \cos(3x) \times (2e^x - 1)}{(2e^x - x)^2}$ $= \dfrac{-3(2e^x - x)\sin(3x) - (2e^x - 1)\cos(3x)}{(2e^x - x)^2}$
4 Evaluate $f'(0)$.	$f'(0) = \dfrac{-3(2e^0 - 0)\sin(0) - (2e^0 - 1)\cos(0)}{(2e^0 - 0)^2}$ $= -\dfrac{0 - 1}{4}$ $= \dfrac{1}{4}$

EXERCISE 6.4 The quotient rule

PRACTISE

Work without CAS

1 WE7 Use the quotient rule to find the derivatives of:

a $\dfrac{e^{2x}}{e^x + 1}$

b $\dfrac{\cos(3t)}{t^3}$.

2 Find the derivative of $\dfrac{x + 1}{x^2 - 1}$.

3 WE8 If $y = \dfrac{\sin(x)}{e^{2x}}$, find the gradient of the function at the point where $x = 0$.

4 If $y = \dfrac{5x}{x^2 + 4}$, find the gradient of the function at the point where $x = 1$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Differentiate the following.

a $\dfrac{\sin(x)}{\sqrt{x}}$

b $\dfrac{\tan(2x)}{e^x}$

c $f(x) = \dfrac{(5 - x)^2}{\sqrt{5 - x}}$

d $y = \dfrac{\sin^2(x^2)}{x}$

e $y = \dfrac{3x - 1}{2x^2 - 3}$

f $f(x) = \dfrac{x - 4x^2}{2\sqrt{x}}$

g $\dfrac{e^x}{\cos(2x + 1)}$

h $\dfrac{e^{-x}}{x - 1}$

i $y = \dfrac{3\sqrt{x}}{x + 2}$

j $y = \dfrac{\cos(3x)}{\sin(3x)}$

k $\dfrac{x - 2}{2x^2 - x - 6}$

l $\dfrac{1 - e^{2x}}{1 + e^{2x}}$

6 Differentiate the following functions.

a $f(x) = \dfrac{x+2}{\sin(g(x))}$

b $f(x) = \dfrac{g(e^{-2x})}{e^x}$

7 Find the gradient at the stated point for each of the following functions.

a $y = \dfrac{2x}{x^2+1}$, $x = 1$

b $y = \dfrac{\sin(2x+\pi)}{\cos(2x+\pi)}$, $x = \dfrac{\pi}{2}$

c $y = \dfrac{x+1}{\sqrt{3x+1}}$, $x = 5$

d $y = \dfrac{5-x^2}{e^x}$, $x = 0$

8 Find the gradient of the tangent to the curve with equation $y = \dfrac{2x}{(3x+1)^{\frac{3}{2}}}$ at the point where $x = 1$.

9 Find the equation of the tangent to the curve $y = \dfrac{e^x}{x^2+2}$ when $x = 0$.

10 Show that $\dfrac{d}{dx}\left(\dfrac{1+\cos(x)}{1-\cos(x)}\right) = -\dfrac{-2\sin(x)}{(\cos(x)-1)^2}$.

11 Given that $f(x) = \dfrac{\sqrt{2x-1}}{\sqrt{2x+1}}$, find m such that $f'(m) = \dfrac{2}{5\sqrt{15}}$.

12 If $\dfrac{d}{dx}\left(\dfrac{e^{-3x}}{e^{2x}+1}\right) = \dfrac{e^{-x}(a+be^{-2x})}{(e^{2x}+1)^2}$, find the exact values of a and b.

13 Find the derivative of the function $f: R \rightarrow R$, $f(x) = \dfrac{10x}{x^2+1}$ and determine when the gradient is negative.

14 For the curve with the rule $y = \dfrac{x-5}{x^2+5x-14}$:

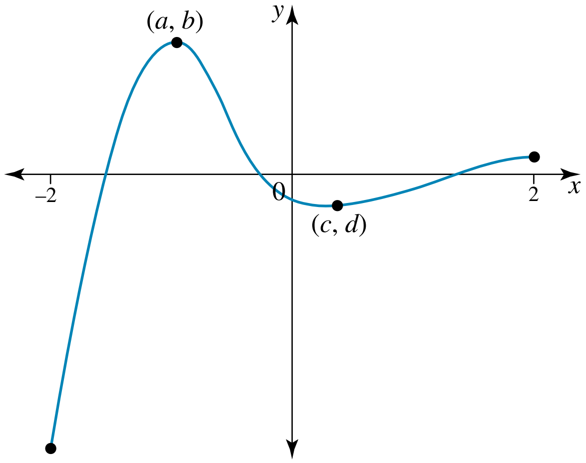
a state when the function is undefined

b find the coordinates when the gradient is zero

c find the equation of the tangent to the curve at the point where $x = 1$.

MASTER

15 Let $f: [-2, 2] \rightarrow R$, $f(x) = \dfrac{\sin(2x-3)}{e^x}$. The graph of this function is shown.



a The stationary points occur at (a, b) and (c, d). Find the values of a, b, c and d, giving your answers correct to 3 decimal places.

b Find the gradient of the tangent to the curve at the point where $x = 1$, correct to 3 decimal places.

16 Consider the curve defined by the rule $y = \dfrac{2x - 1}{3x^2 + 1}$.

a Find the rule for the gradient.

b For what value(s) of x is the gradient equal to 0.875? Give your answers correct to 4 decimal places.

6.5 Curve sketching

At the points where a differentiable function is neither increasing nor decreasing, the function is stationary and its gradient is zero. Identifying such **stationary points** provides information that assists curve sketching.

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AOS 3
Topic 3
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Stationary points
Concept summary
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Interactivity
Stationary points
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Stationary points

There are three types of stationary points:

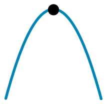
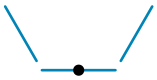
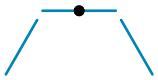
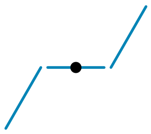
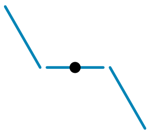
- (local) minimum turning point
- (local) maximum turning point
- stationary point of inflection.

At a stationary point of a curve $y = f(x)$, $f'(x) = 0$.

The word '**local**' means that the point is a minimum or a maximum in a particular locality or neighbourhood. Beyond this section of the graph, there could be other points on the graph which are lower than the local minimum or higher than the local maximum. Our purpose for the time being is simply to identify the turning points and their nature, so we shall continue to refer to them just as minimum or maximum turning points.

Nature of a stationary point

At each of the three types of stationary points, $f'(x) = 0$. This means that the tangents to the curve at these points are horizontal. By examining the slope of the tangent to the curve immediately before and immediately after the stationary point, the nature or type of stationary point can be determined.

	Minimum turning point	**Maximum turning point**	**Stationary point of inflection**
Stationary point			 or 
Slope of tangent			 or 

For a minimum turning point, the behaviour of the function changes from decreasing just before the point, to stationary at the point, to increasing just after the point. The slope of the tangent changes from negative to zero to positive.

For a maximum turning point, the behaviour of the function changes from increasing just before the point, to stationary at the point, to decreasing just after the point. The slope of the tangent changes from positive to zero to negative.

For a stationary point of inflection, the behaviour of the function remains either increasing or decreasing before and after the point and stationary at the point. The slope of the tangent is zero at the point but does not change sign either side of the point.

To identify stationary points and their nature:

- establish where $f'(x) = 0$
- determine the nature by testing the slope of the tangent at selected points either side of, and in the neighbourhood of, the stationary point.

WORKED EXAMPLE 9

a **Determine the stationary points of $f(x) = 2 + 4x - 2x^2 - x^3$ and justify their nature.**

b **The curve $y = ax^2 + bx - 24$ has a stationary point at $(-1, -25)$. Calculate the values of a and b.**

THINK

WRITE

a 1 Derive the function.

a $f(x) = 2 + 4x - 2x^2 - x^3$
$f'(x) = 4 - 4x - 3x^2$

2 Calculate the x-coordinates of the stationary points by solving $f'(x) = 0$.

Note: Always include the reason why $f'(x) = 0$.

At stationary points, $f'(x) = 0$.

$$4 - 4x - 3x^2 = 0$$
$$(2 - 3x)(2 + x) = 0$$
$$x = \frac{2}{3} \text{ or } x = -2$$

3 Calculate the corresponding y-coordinates.

When $x = \frac{2}{3}$,

$$f\left(\frac{2}{3}\right) = 2 + 4\left(\frac{2}{3}\right) - 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$$
$$= \frac{94}{27}$$

When $x = -2$,

$$f(-2) = 2 + 4(-2) - 2(-2)^2 - (-2)^3$$
$$= -6$$

4 Write the answer.

The stationary points are $\left(\frac{2}{3}, \frac{94}{27}\right)$, $(-2, -6)$.

5 To justify the nature of the stationary points, draw a table to show the gradient of the curve either side of the stationary points.

Note: The shape of the cubic graph would suggest the nature of the stationary points.

x	–3	–2	0	$\frac{2}{3}$	1
$f'(x)$	–11	0	4	0	–3
Slope	╲	—	╱	—	╲

6 Identify the nature of each stationary point by examining the sign of the gradient before and after each point.

At $x = -2$, the gradient changes from negative to positive, so (–2, –6) is a minimum turning point.
At $x = \frac{2}{3}$, the gradient changes from positive to negative, so $\left(\frac{2}{3}, \frac{94}{27}\right)$ is a maximum turning point.

b **1** Use the coordinates of the given point to form an equation. *Note:* As there are two unknowns to determine, two pieces of information are needed to form two equations in the two unknowns.	**b** $y = ax^2 + bx - 24$ The point (–1, –25) lies on the curve. $-25 = a(-1)^2 + b(-1) - 24$ $a - b = -1 \quad (1)$
2 Use the other information given about the point to form a second equation.	The point (–1, –25) is a stationary point, so $\frac{dy}{dx} = 0$ at this point. $y = ax^2 + bx - 24$ $\frac{dy}{dx} = 2ax + b$ At $(-1, -25)$, $\frac{dy}{dx} = 2a(-1) + b$ $= -2a + b$ $-2a + b = 0 \quad (2)$
3 Solve the simultaneous equations and state the answer.	$a - b = -1 \quad (1)$ Add the equations: $-a = -1$ $\therefore a = 1$ Substitute $a = 1$ in equation (2): $-2 + b = 0$ $\therefore b = 2$ The values are $a = 1$ and $b = 2$.

Curve sketching

To sketch the graph of any function $y = f(x)$:

- Obtain the y-intercept by evaluating $f(0)$.
- Obtain any x-intercepts by solving, if possible, $f(x) = 0$. This may require the use of factorisation techniques including the factor theorem.
- Calculate the x-coordinates of the stationary points by solving $f'(x) = 0$. Use the equation of the curve to obtain the corresponding y-coordinates.
- Identify the nature of the stationary points.
- Calculate the coordinates of the end points of the domain, where appropriate.
- Identify any other key features of the graph, where appropriate.

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Units 3 & 4
AOS 3
Topic 3
Concept 5

Strictly increasing or decreasing functions
Concept summary
Practice questions

Strictly increasing and decreasing

A function is strictly increasing on an interval $x \in [a, b]$ if, for every value in that interval, $f(b) > f(a)$.

Similarly, a function is strictly decreasing on an interval $x \in [a, b]$ if, for every value in that interval, $f(b) < f(a)$.

WORKED EXAMPLE 10

a Sketch the function $y = \frac{1}{2}x^3 - 3x^2 + 6x - 8$. Locate any intercepts with the coordinate axes and any stationary points, and justify their nature.

b State the domain over which the function is strictly increasing.

THINK

WRITE

a 1 State the y-intercept.

a $y = \frac{1}{2}x^3 - 3x^2 + 6x - 8$

y-intercept: $(0, -8)$

2 Calculate any x-intercepts.

x intercepts:

When $y = 0$,

$$\frac{1}{2}x^3 - 3x^2 + 6x - 8 = 0$$

$$x^3 - 6x^2 + 12x - 16 = 0$$

Let $P(x) = x^3 - 6x^2 + 12x - 16$.

$$\begin{aligned} P(4) &= 64 - 96 + 48 - 16 \\ &= 0 \end{aligned}$$

$\therefore (x - 4)$ is a factor.

$$0 = x^3 - 6x^2 + 12x - 16$$

$$0 = (x - 4)(x^2 - 2x + 4)$$

$\therefore x = 4$ or $x^2 - 2x + 4 = 0$

The discriminant of $x^2 - 2x + 4$ is $\Delta = 4 - 16 < 0$.

Therefore, there is only one x-intercept, $(4, 0)$.

3 Obtain the derivative in order to locate any stationary points.

Stationary points:

$$y = \frac{1}{2}x^3 - 3x^2 + 6x - 8$$

$$\frac{dy}{dx} = \frac{3}{2}x^2 - 6x + 6$$

At stationary points, $\frac{dy}{dx} = 0$:

$$\begin{aligned} \frac{3}{2}x^2 - 6x + 6 &= 0 \\ \frac{3}{2}(x^2 - 4x + 4) &= 0 \\ \frac{3}{2}(x - 2)^2 &= 0 \\ x &= 2 \end{aligned}$$

Substitute $x = 2$ into the function's equation:

$$\begin{aligned} y &= \frac{1}{2}(2)^3 - 3(2)^2 + 6(2) - 8 \\ &= -4 \end{aligned}$$

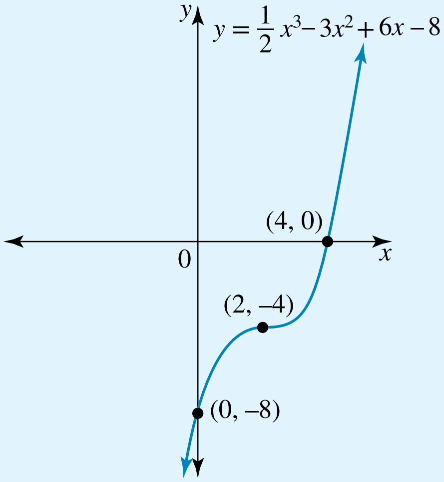
The stationary point is $(2, -4)$.

4 Identify the type of stationary point by evaluating the slope either side of the stationary point.

x	0	2	4
$\frac{dy}{dx}$	6	0	6
Slope	/	—	/

The point $(2, -4)$ is a stationary point of inflection.

5 Sketch the curve, showing the intercepts with the axes and the stationary point.



b Identify the domain over which $f(b) > f(a)$, where $b > a$.

b $x \in R$

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Units 3 & 4

AOS 3

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Concept 6

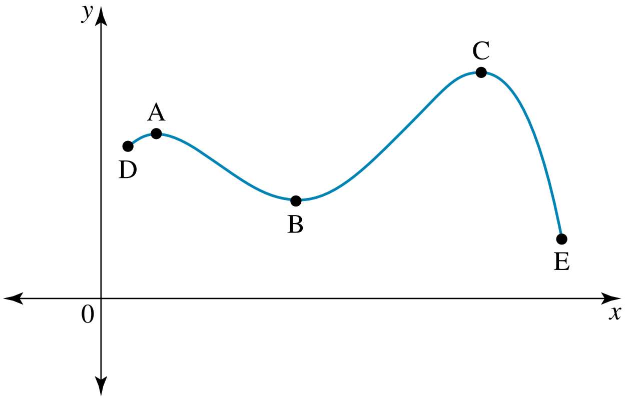
Interval end point maximum and minimum values

Concept summary

Practice questions

Local and absolute maxima and minima

The diagram shows the graph of a function sketched over a domain with end points D and E.



There are three turning points: A and C are maximum turning points, and B is a minimum turning point.

The y-coordinate of point A is greater than those of its neighbours, so A is a local maximum point. However, the y-coordinate of point C is not only greater than those of its neighbours; it is greater than that of any other point on the graph. For this reason, C is called the **absolute maximum** point.

The **absolute minimum** point is the point whose y-coordinate is less than any others on the graph. For this function, point E, an end point of the domain, is the absolute minimum point. Point B is a local minimum point; it is not the absolute minimum point.

Absolute maximums and minimums may not exist for all functions. For example, a cubic function on its maximal domain may have one local maximum turning point and one local minimum turning point, but there is neither an absolute maximum nor an absolute minimum point, because as $x \to \pm\infty, y \to \pm\infty$ (assuming a positive coefficient of x^3).

If a differentiable function has an absolute maximum or an absolute minimum value, then this will occur at either a turning point or an end point of the domain. The y-coordinate of such a point gives the value of the absolute maximum or the absolute minimum.

WORKED EXAMPLE 11

A function defined on a restricted domain has the rule $y = \frac{x}{2} + \frac{2}{x}$, $x \in \left[\frac{1}{4}, 4\right]$.

a Specify the coordinates of the end points of the domain.

b Obtain the coordinates of any stationary point and determine its nature.

c Sketch the graph of the function.

d State the absolute maximum and the absolute minimum values of the function, if they exist.

THINK	WRITE/DRAW
a Use the given domain to calculate the coordinates of the end points.	**a** $y = \frac{x}{2} + \frac{2}{x}$ for the domain $\frac{1}{4} \le x \le 4$ Substitute each of the end values of the domain in the function's rule. Left end point: when $x = \frac{1}{4}$, $y = \frac{x}{2} + \frac{2}{x}$ $= \frac{1}{8} + 8$ $= 8\frac{1}{8}$ Right end point: when $x = 4$, $y = 2 + \frac{1}{2}$ $= 2\frac{1}{2}$ The end points are $\left(\frac{1}{4}, \frac{65}{8}\right)$ and $\left(4, \frac{5}{2}\right)$.
b 1 Calculate the derivative of the function.	**b** $y = \frac{x}{2} + \frac{2}{x}$ $y = \frac{x}{2} + 2x^{-1}$ $\frac{dy}{dx} = \frac{1}{2} - 2x^{-2}$ $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{x^2}$
2 Calculate the coordinates of any stationary point.	At a stationary point, $\frac{dy}{dx} = 0$. $\frac{1}{2} - \frac{2}{x^2} = 0$ $\frac{1}{2} = \frac{2}{x^2}$ $x^2 = 4$ $x = \pm 2$ $x = 2, x \in \left[\frac{1}{4}, 4\right]$ When $x = 2, y = \frac{2}{2} + \frac{2}{2}$ $= 2$ (2, 2) is a stationary point.

3 Test the gradient at two selected points either side of the stationary point.

x	1	2	3
$\dfrac{dy}{dx}$	$\frac{1}{2}-\frac{2}{1}=-\frac{3}{2}$	0	$\frac{1}{2}-\frac{2}{9}=\frac{5}{18}$
Slope	╲	—	╱

4 State the nature of the stationary point. The gradient changes from negative to zero to positive about the stationary point.

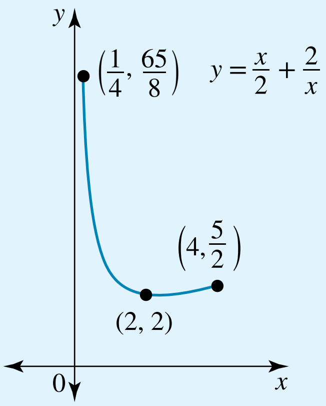
The point (2, 2) is a minimum turning point.

c **1** Calculate any intercepts with the coordinate axes.

c There is no y-intercept since $x = 0$ is not in the given domain, nor is $y = \frac{x}{2} + \frac{2}{x}$ defined at $x = 0$.

There is no x-intercept since the end points and the minimum turning point all have positive y-coordinates, and there are no other turning points.

2 Sketch the graph using the three known points.



d Examine the graph and the y-coordinates to identify the absolute extremes.

d The function has an absolute maximum of $\frac{65}{8}$ at the left end point and an absolute minimum, and local minimum, of 2 at its turning point.

EXERCISE 6.5 Curve sketching

PRACTISE

Work without CAS

1 WE9 **a** Determine the stationary points of $f(x) = \dfrac{2x^3}{3} + \dfrac{3x^2}{2} - 2x + 4$ and justify their nature.

b The curve $y = ax^2 + bx + c$ passes through the point (0, –8) and has a stationary point at (−1, −5). Calculate the values of a, b and c.

2 The curve $y = x^3 + ax^2 + bx - 11$ has stationary points when $x = 1$ and $x = \frac{5}{3}$.

a Calculate a and b.

b Determine the coordinates of the stationary points and their nature.

3 **WE10** **a** Sketch the function $f(x) = 2x^3 - x^2$. Locate any intercepts with the coordinate axes and any stationary points, and justify their nature.

b State the domain over which the function is strictly decreasing.

4 **a** Sketch the function $f(x) = -x^4 + 2x^3 + 11x^2 - 12x$. Locate any intercepts with the coordinate axes and any stationary points, and justify their nature.

b State the domain over which the function is strictly increasing.

5 **WE11** A function defined on a restricted domain has the rule $f(x) = \dfrac{1}{4x} + x$, $x \in \left[-2, -\frac{1}{4}\right]$.

a Specify the coordinates of the end points of the domain.

b Obtain the coordinates of any stationary point and determine its nature.

c Sketch the graph of the function.

d State the absolute maximum and absolute minimum values of the function, if they exist.

6 Find, if possible, the absolute maximum and minimum values of the function $f(x) = 2x^3 - 8x$ over the domain $\{x : x \leq 2\}$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Consider the function defined by $f(x) = 16x^2 - x^4$.

a Show that $(2\sqrt{2}, 64)$ is a stationary point of the function.

b Determine the nature of this stationary point.

c State the coordinates of any other stationary points and state their nature.

8 Obtain any stationary points of the following curves and justify their nature.

a $y = x(x + 2)^2$

b $y = x^3 + 3x^2 - 24x + 5$

c $y = \dfrac{x^2}{x + 1}$

d $y = (x - 1)e^{-x}$

9 Sketch a possible graph of the function $y = f(x)$ for which:

a $f'(-1) = 0$, $f'(3) = 0$, $f(3) = 0$, $f'(x) > 0$ for $x < -1$, $f'(x) < 0$ for $-1 < x < 3$ and $f'(x) > 0$ for $x > 3$

b $f'(-1) = 0$, $f'(x) < 0$ for $x \in R\backslash\{-1\}$ and $f(-1) = 2$.

10 **a** The point (2, –8) is a stationary point of the curve $y = x^3 + bx + c$. Find the values of b and c.

b The point (1.5, 6) is a stationary point of the curve $y = ax^2 + bx + 15$. Determine the values of a and b.

c A curve has equation $y = x^3 + bx^2 + cx + d$. The curve has a stationary point at (–3, –10) and passes through the point (1, 6). Determine the values of b, c and d.

11 Sketch the graphs of each of the following functions. Label any intercepts and any stationary points with their coordinates, and justify the nature of the stationary points.

a $f : R \to R,\ f(x) = -\frac{1}{4}(x - 4)^3 + 2$

b $g(x) = 2x^3 - x^2, x \in [-1, 1]$

c $h : R \to R,\ h(x) = x^3 - x^2 - x + 10$

d $f(x) = x^4 - 6x^2 + 8$

e $f(x) = (x + 3)^3(4 - x)$

f $f(x) = x^3 - 4x^2 - 3x + 12$

12 **a** Sketch the graph of $f(x) = \frac{1}{2}(2x - 3)^4(x + 1)^5$, showing all intercepts and stationary points.

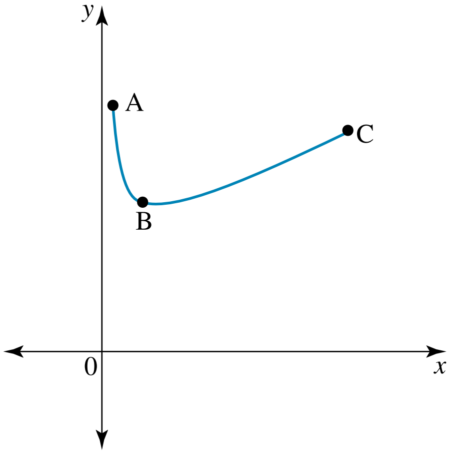
b State the domain over which the function is strictly decreasing.

13 Sketch the graphs and state the absolute maximum and minimum values over the given domain for each of the following functions.

a $y = \frac{1}{2}(x-1)^2 - 2,\ 0 \le x \le 5$ **b** $y = x^3 - 2x^2,\ \ -2 \le x \le 3$

c $y = 4 - x^3,\ x \le 2$

14 The graph of $f(x) = 2\sqrt{x} + \frac{1}{x},\ 0.25 \le x \le 5$ is shown.



a Determine the coordinates of the end points A and C and the stationary point B.

b At which point does the absolute maximum occur?

c State the absolute maximum and absolute minimum values.

15 Let $f(x) = xe^x$.

a Determine any stationary points and state their nature.

b Find $\{x : f'(x) > 0\}$.

c State, if possible, the absolute maximum and minimum values.

16 a Sketch the graph of $y = e^{-x^2}$.

b If $y = 20e^{-2x^2-4x+1}$, use calculus to find the values of x for which the function is strictly increasing.

MASTER

17 Consider the function $f(x) = (a-x)^2(x-2)$ where $a > 2$.

a Find the coordinates of the stationary points.

b State the nature of the stationary points.

c Find the value of a if the graph of $y = f(x)$ has a turning point at (3, 4).

18 Consider the function $f(x) = (x-a)(x-b)^3$, where $a > 0$, $b > 0$ and $a < b$.

a Determine the x-intercepts

b Find the coordinates of the stationary points

c State the nature of the stationary points

d If one of the stationary points has coordinates (3, −27), find a and b.

6.6 Maximum and minimum problems

How to solve maximum and minimum problems

In many practical situations, it is necessary to find the maximum or minimum value of the function that describes it. For example, if you were running your own business, you would always want to minimise the production costs while maximising the profits.

To solve maximum or minimum problems, apply the following steps.

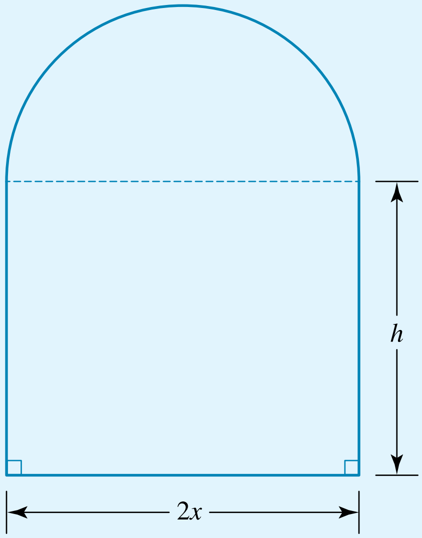
- Draw a diagram if possible and label it with as few variables as possible.
- If there is more than one variable, find a connection between the variables from the information given. For instance, if you are finding the area of a rectangle, where the dimensions are length, l, and width, w, the width needs to be expressed in terms of the length or vice versa, so that the area can be expressed in terms of one variable only. Pythagoras' theorem, trigonometry, similar triangles, standard formulas, or given information could be required to express one variable in terms of another.
- Find an expression for the quantity to be maximised or minimised in terms of the one nominated variable.

- Differentiate the expression, equate to zero and solve for the unknown variable.
- Reject any unrealistic solutions.
- Check the nature of the stationary point.
- Check whether the answer is the absolute maximum or minimum by evaluating the end points of the domain.
- Always sketch the shape of the graph.
- Answer the actual question.

WORKED EXAMPLE 12

The new owner of an apartment wants to install a window in the shape of a rectangle surmounted by a semicircle in order to allow more light into the apartment.

The owner has 336 cm of wood for a frame to surround the window. They want to determine the dimensions of the window that will allow as much light into the apartment as possible.



a **Show that the area, A in cm^2, of the window is $A = 336x - \frac{1}{2}(4 + \pi)x^2$.**

b **Hence determine, to the nearest cm, the width and the height of the window for which the area is greatest.**

c **Structural limitations mean that the width of the window should not exceed 84 cm. What should the dimensions of the window of maximum area now be?**

THINK	WRITE/DRAW
a **1** Form an expression for the total area. The total area is the sum of the areas of the rectangle and semicircle. *Note:* This expression involves more than one variable.	**a** Rectangle: length $2x$ cm, width h cm. $\therefore A_{\text{rectangle}} = 2xh$ Semicircle: diameter $2x$ cm, radius x cm $\therefore A_{\text{semicircle}} = \frac{1}{2}\pi x^2$ The total area of the window is $A = 2xh + \frac{1}{2}\pi x^2$.
2 Use the perimeter information to form an expression connecting the two variables.	$P_{\text{window}} = 336$ cm $C_{\text{semicircle}} = \frac{1}{2}(2\pi x)$ $\therefore P_{\text{shape}} = h + 2x + h + \frac{1}{2}(2\pi x)$ Hence, $2h + 2x + \pi x = 336$.
3 Express one appropriately chosen variable in terms of the other. The required expression for the area is in terms of x, so express h in terms of x.	$2h = 336 - 2x - \pi x$ $h = \frac{1}{2}(336 - 2x - \pi x)$
4 Write the area as a function of x by substituting for h.	$A = x(2h) + \frac{1}{2}\pi x^2$ $= x(336 - 2x - \pi x) + \frac{1}{2}\pi x^2$ $= 336x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$ $= 336x - \left(2 + \frac{1}{2}\pi\right)x^2$ $\therefore A = 336x - \frac{1}{2}(4 + \pi)x^2$ as required.

b **1** Determine where the stationary point occurs and justify its nature.

b At the stationary point, $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = 336 - (4 + \pi)x$$

$$0 = 336 - (4 + \pi)x$$

$$x = \frac{336}{4 + \pi}$$

$$x = 47.05$$

x	40	$\frac{336}{4+\pi}$	50
$\frac{dA}{dx}$	$176 - 40\pi$ ≈ 50.3	0	$136 - 50\pi$ $\approx\ -21.1$
Slope	/	—	\

Maximum area is obtained when $x = \frac{336}{4 + \pi}$ cm.

2 State the values of both variables.

When $x = \frac{336}{4 + \pi}$,

$$2h = 336 - 2 \times \left(\frac{336}{4 + \pi}\right) - \pi \times \left(\frac{336}{4 + \pi}\right)$$

$$\therefore h = 47.05 \text{ cm}$$

3 Calculate the required dimensions and state the answer.

The width of the window is $2x \simeq 94$ cm.
The total height of the window is $h + x \simeq 94$ cm.
Therefore the area of the window will be greatest if its width is 94 cm and its height is 94 cm.

c **1** Give the restricted domain of the area function.

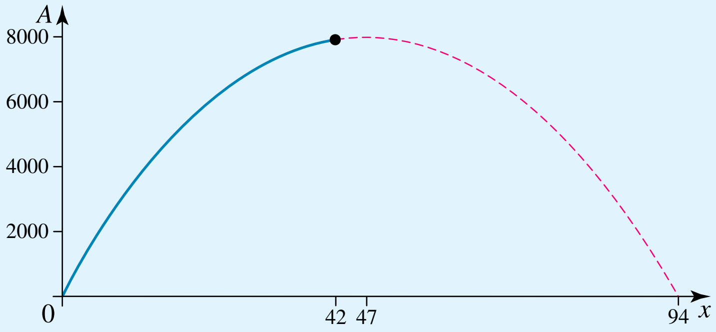
c If the width is not to exceed 84 cm, then

$$2x \le 84$$

$$\therefore x \le 42.$$

With the restriction, the domain of the area function is [0, 42].

2 Determine where the function is greatest. As the stationary point occurs when $x \approx 47$, for the domain [0, 42] there is no stationary point, so the greatest area must occur at an end point of the domain.

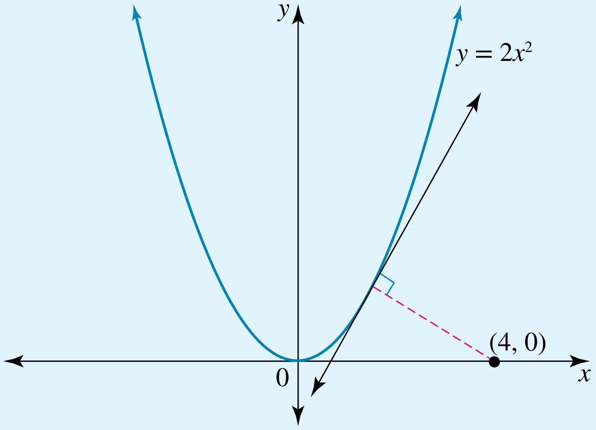


The maximum occurs when $x = 42$.

3 Calculate the required dimensions and state the answer.	When $x = 42$, $h = \frac{1}{2}(336 - 84 - 42\pi)$ $\therefore h = 60.03$ cm ≈ 60 cm The width of the window is $2x = 84$ cm. The height of the window is $h + x = 102$ cm. With the restriction, the area of the window will be greatest if its width is 84 cm and its height is 102 cm.

WORKED EXAMPLE 13

Find the minimum distance from the curve $y = 2x^2$ to the point (4, 0), correct to 2 decimal places. You do not need to justify your answer.



THINK	WRITE
1 Let P be the point on the curve such that the distance from P to the point (4, 0) is a minimum.	$P = (x, y)$
2 Write the formula for the distance between the two points.	$d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(x - 4)^2 + (y - 0)^2}$ $= \sqrt{(x - 4)^2 + y^2}$
3 Express the distance between the two points as a function of x only.	$y = 2x^2$ $\therefore d(x) = \sqrt{(x - 4)^2 + (2x^2)^2}$ $= (x^2 - 8x + 16 + 4x^4)^{\frac{1}{2}}$
4 Differentiate $d(x)$.	$d'(x) = \frac{1}{2} \times (4x^4 + x^2 - 8x + 16)^{-\frac{1}{2}} \times (16x^3 + 2x - 8)$ $= \frac{16x^3 + 2x - 8}{2\sqrt{4x^4 + x^2 - 8x + 16}}$ $= \frac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$

5 Solve $d'(x) = 0$ using CAS.

$$0 = \frac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$$
$$0 = 8x^3 + x - 4$$
$$x = 0.741$$

6 Evaluate $d(0.846)$.

$$d(0.741) = \sqrt{(0.741)^2 - 8(0.741) + 16 + 4(0.741)^4}$$
$$= 3.439$$

7 Write the answer.

The minimum distance is 3.44 units.

Note: When finding the minimum distance between two points, one of which is on a curve, the line joining the points is always perpendicular to the curve. This fact can also be used to determine the minimum distance between two points.

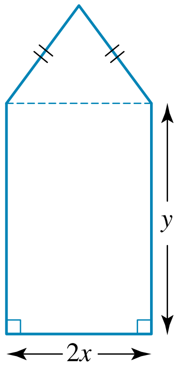
EXERCISE 6.6 Maximum and minimum problems

PRACTISE

Work without CAS Questions 2, 3

1 **WE12** The owner of an apartment wants to create a stained glass feature in the shape of a rectangle surmounted by an isosceles triangle of height equal to half its base. This will be adjacent to a door opening on to a balcony.

The owner has 150 cm of plastic edging to place around the perimeter of the figure, and wants to determine the dimensions of the figure with the greatest area.



a Show that the area, A in cm^2, of the stained glass figure is $A = 150x - (2\sqrt{2} + 1)x^2$.

b Hence determine, correct to 1 decimal place, the width and the height of the figure for which the area is greatest.

c Due to structural limitations, the width of the figure should not exceed 30 cm. What should the dimensions of the stained glass figure of maximum area now be?

2 A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 16 cm by 10 cm. The box will be made by cutting equal squares of side length x cm out of the four corners and folding the flaps up.

a Express the volume as a function of x.

b Determine the dimensions of the box with greatest volume and give this maximum volume.

3 **WE13** Find the minimum distance from the line $y = 2x - 5$ to the origin.

4 A colony of blue wrens, also known as superb fairy wrens, survives in a national park in Ringwood Victoria because the wooded areas have rich undergrowth and a plentiful supply of insects, the wrens' main food source. Breeding begins in spring and continues until late summer.

The population of the colony any time t months after 1 September can be modelled by the function

$$P(t) = 200te^{-\frac{t}{4}} + 400,\ 0 \le t \le 12$$

where P is the number of birds in the colony. Find:

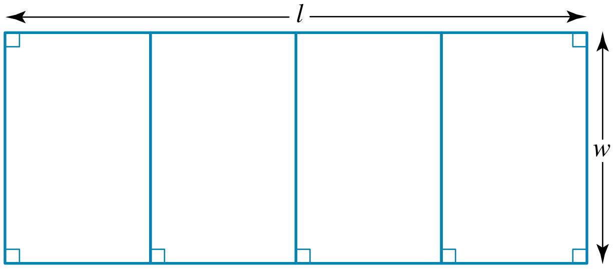
a the initial population of the birds
b when the largest number of birds is reached
c the maximum number of birds, to the nearest bird.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 The sum of two positive numbers is 32. Find the numbers if their product is a maximum.

6 A pen for holding farm animals has dimensions $l \times w$ metres. This pen is to be partitioned so that there are four spaces of equal area as shown.

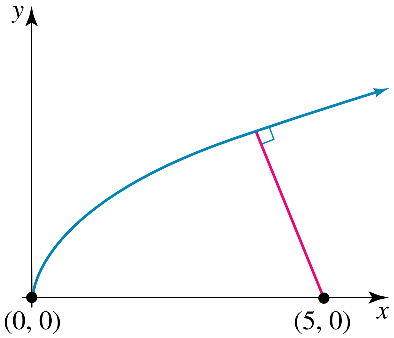


The farmer has 550 metres of fencing material to construct this pen.

a Find the required length and width in order to maximise the area of the pen.
b Find the maximum area.

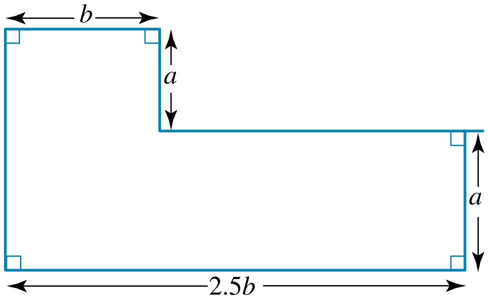
7 A cylinder has a surface area of 220π cm^2. Find the height and radius of each end of the cylinder so that the volume of the cylinder is maximised, and find the maximum volume for the cylinder. Give answers correct to 2 decimal places.

8 Find the minimum distance from the line $y = 2\sqrt{x}$ to the point (5, 0).



9 Calculate the area of the largest rectangle with its base on the x-axis that can be inscribed in the semicircle $y = \sqrt{4 - x^2}$.

10 A playground is being constructed by the local council. The shape of the playground is shown below. All measurements are in metres.



The perimeter of the playground is known to be 96 metres.

a Determine the values of a and b that give a maximum area for the playground.

b Find the maximum area.

11 The amount of money in a savings account t years after the account was opened on 1 January 2009 is given by the equation

$$A(t) = 1000 - 12te^{\frac{4-t^3}{8}} \text{ for } t \in [0, 6].$$

a How much money was in the account when the account was first opened?

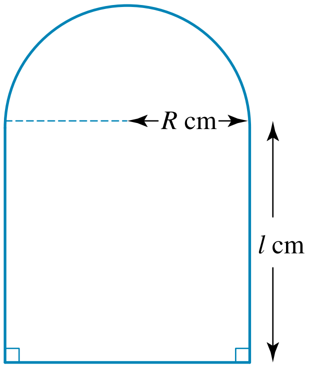
b What was the least amount of money in the account?

c When did the account contain its lowest amount? Give the year and month.

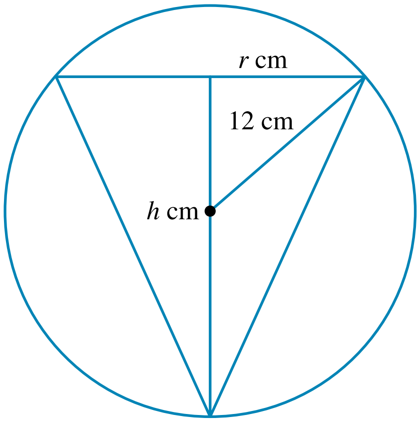
d How much money was in the account at the end of the six years?

12 An ornamental fish pond has the following shape.

A plastic cover is being made for the pond for the winter months. If the surface area of the pond, A, is a constant, show that the perimeter of the pond is a minimum when both R and l are equal to $\sqrt{\dfrac{2A}{\pi + 4}}$.



13 Find the volume of the largest cone, correct to the nearest cubic centimetre, that can be inscribed in a sphere of radius 12 centimetres. Let the base radius of the cone be r cm and the vertical height h cm.

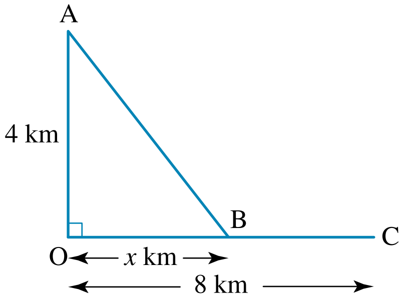


14 Prove that the rectangle of largest area that can be inscribed in a circle of a fixed radius is a square.

MASTER

15 A cone is 10 cm high and has a base radius of 8 cm. Find the radius and height of a cylinder which is inscribed in the cone such that the volume of the cylinder is a maximum. Find the maximum volume of the cylinder, correct to the nearest cubic centimetre.

16 A rower is in a boat 4 kilometres from the nearest point, O, on a straight beach. His destination is 8 kilometres along the beach from O. If he is able to row at 5 km/h and walk at 8 km/h, what point on the beach should he row to in order to reach his destination in the least possible time? Give your answer correct to 1 decimal place.



6.7 Rates of change

Calculus enables the behaviour of a quantity that changes to be analysed. Many topics of interest in the biological, physical and social sciences involve the study of rates of change. In this section we consider the application of calculus to rates of change in general contexts and then as applied to the motion of a moving object.

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Units 3 & 4
AOS 3
Topic 1
Concept 5

Types of rates of change
Concept summary
Practice questions

Rates of change

The **average rate of change** of a function, f, over the interval x_1 to x_2 is calculated by finding the gradient of the line connecting the two points.

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

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Interactivity
Rates of change
int-5960

The instantaneous rate of change, or rate of change, of the function $y = f(x)$ is given by the derivative, $\frac{dy}{dx}$ or $f'(x)$.

For example, the derivative $\frac{dV}{dt}$ could be the rate of change of volume with respect to time with possible units being litres per minute; the rate of change of volume with respect to height would be $\frac{dV}{dh}$, with possible units being litres per cm. To calculate these rates, V would need to be expressed as a function of one independent variable, either time or height. Similar methods to those encountered in optimisation problems are often required to connect variables together in order to obtain this function of one variable.

To solve rates of change problems, apply the following steps.

- Draw a diagram of the situation where appropriate.
- Identify the rate of change required and define the variables involved.
- Express the quantity that is changing as a function of one independent variable, the variable the rate is measured with respect to.
- Calculate the derivative that measures the rate of change.
- To obtain the rate at a given value or instant, substitute the given value into the derivative expression.
- Remember that a negative value for the rate of change means the quantity is decreasing (negative gradient), whereas a positive value for the rate of change means the quantity is increasing (positive gradient).

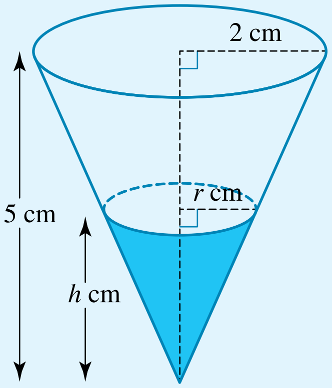
WORKED EXAMPLE 14

A container in the shape of an inverted right cone of radius 2 cm and depth 5 cm is being filled with water. When the depth of water is h cm, the radius of the water level is r cm.

a Use similar triangles to express r in terms of h.

b Express the volume of the water as a function of h.

c At what rate with respect to the depth of water is the volume of water changing when its depth is 1 cm?

THINK	WRITE/DRAW
a **1** Draw a diagram of the situation.	**a** 
2 Obtain the required relationship between the variables using similar triangles.	$\frac{r}{h}=\frac{2}{5}$ $\therefore r=\frac{2h}{5}$
b Express the function in the required form.	**b** $V_{\text{cone}}=\frac{1}{3}\pi r^2h$ Therefore, the volume of water is $V=\frac{1}{3}\pi r^2h$. Substitute $r=\frac{2h}{5}$: $V=\frac{1}{3}\pi\left(\frac{2h}{5}\right)^2h$ $=\frac{4\pi h^3}{75}$ $\therefore V=\frac{4\pi}{75}h^3$
c **1** Calculate the derivative of the function. The derivative gives the rate of change at any depth.	**c** $\frac{dV}{dh}=\frac{4\pi}{75}\times 3h^2$ $=\frac{4\pi}{25}h^2$
2 Evaluate the derivative at the given value.	When $h=1$, $\frac{dV}{dh}=\frac{4\pi}{25}$ $=0.16\pi$
3 Write the answer in context, with the appropriate units.	At the instant the depth is 1 cm, the volume of water is increasing at the rate of 0.16π cm^3/cm.

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Interactivity
Kinematics
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Kinematics

Many quantities change over time, so many rates measure that change with respect to time. Motion is one such quantity. The study of the motion of a particle without considering the causes of the motion is called **kinematics**. Analysing motion requires interpretation of the **displacement**, **velocity** and **acceleration**, and this analysis depends on calculus. For the purpose of our course, only motion in a straight line, also called **rectilinear motion**, will be considered.

Displacement

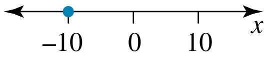
The displacement, x, gives the position of a particle by specifying both its **distance** and its direction from a fixed origin.

Common units for displacement and distance are cm, m and km.

The commonly used conventions for motion along a horizontal straight line are:

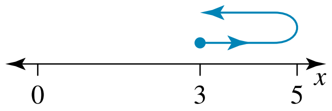
- if $x > 0$, the particle is to the right of the origin
- if $x < 0$, the particle is to the left of the origin
- if $x = 0$, the particle is at the origin.

For example, if $x = -10$, this means the particle is 10 units to the left of origin O.



Note that its distance from the origin of 10 units is the same as the distance from the origin of a particle with displacement $x = 10$.

Distance is not concerned with the direction of motion. This can have implications if there is a change of direction in a particle's motion. For example, suppose a particle that is initially 3 cm to the right of the origin travels 2 cm further to the right and then 2 cm to the left, thus returning to where it started.



Its change in displacement is zero, but the distance it has travelled is 4 cm.

Velocity

Velocity, v, measures the rate of change of displacement, which means that $v = \frac{dx}{dt}$.

For a particle moving in a horizontal straight line, the sign of the velocity indicates that:

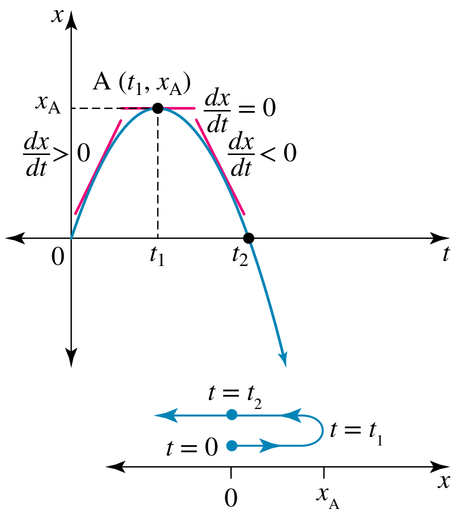
- if $v > 0$, the particle is moving to the right
- if $v < 0$, the particle is moving to the left
- if $v = 0$, the particle is stationary (instantaneously at rest).

Common units for velocity and **speed** include m/s and km/h.

Just as for distance, speed is not concerned with the direction the particle travels and is never negative. A velocity of -10 m/s means the particle is travelling at 10 m/s to the left. Its speed, however, is 10 m/s, regardless of whether the particle is moving to the left or to the right; that is, the speed is 10 m/s for $v = \pm 10$ m/s.

The position or displacement, x, of a particle can be plotted against time t to create a position–time graph, $x = f(t)$. Because $v = \frac{dx}{dt}$, the gradient of the tangent to the curve $f(t)$ at any point represents the velocity of the particle at that point: $v = \frac{dx}{dt} = f'(t)$.

This position–time graph shows the displacement or position of a particle that starts at the origin and initially moves to the right, as the gradient of the graph, that is the velocity, is positive.



At the point A the tangent is horizontal and the velocity is zero, indicating the particle changes its direction of motion at that point.

The particle then starts to move to the left as indicated by the gradient of the graph, that is the velocity, having a negative sign. The particle returns to the origin and continues to move to the left, so its displacement becomes negative.

The same motion is also shown along the horizontal displacement line.

Average velocity

Average velocity is the average rate of change of the displacement. It is measured by the gradient of the chord joining two points on the position–time graph. It must be evaluated using coordinate geometry, not calculus.

- **Average velocity** $= \dfrac{\text{change in displacement}}{\text{change in time}} = \dfrac{x_2 - x_1}{t_2 - t_1}$
- **Average speed** $= \dfrac{\text{distance travelled}}{\text{time taken}}$

Acceleration

Acceleration, a, measures the rate of change of velocity; thus, $a = \dfrac{dv}{dt}$.

Common units for acceleration include m/s^2.

Displacement, velocity and acceleration are linked by calculus. Differentiation enables us to obtain the velocity function from the displacement function, and to obtain the acceleration function from the velocity function.

$$x \rightarrow v \rightarrow \text{a}$$

$$x \rightarrow \frac{dx}{dt} \rightarrow \frac{dv}{dt}$$

Average acceleration

Acceleration acts tangentially to the velocity-time graph, whereas average acceleration measures the gradient of the chord joining two points on the velocity-time graph.

$$\textbf{Average acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

WORKED EXAMPLE 15

A particle moves in a straight line such that its displacement, x metres, from a fixed origin at time t seconds is modelled by $x = t^2 - 4t - 12,\ \ t \geq 0$.

a **Identify its initial position.**

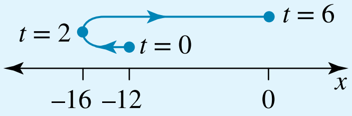
b **Obtain its velocity and hence state its initial velocity and describe its initial motion.**

c **At what time and position is the particle momentarily at rest?**

d Show the particle is at the origin when $t = 6$, and calculate the distance it has travelled to reach the origin.

e Calculate the average speed over the first 6 seconds.

f Calculate the average velocity over the first 6 seconds.

THINK	WRITE/DRAW
a Calculate the value of x when $t = 0$.	**a** $x = t^2 - 4t - 12,\quad t \geq 0$ When $t = 0$, $x = -12$. Initially the particle is 12 metres to the left of the origin.
b **1** Calculate the rate of change required.	**b** $v = \frac{dx}{dt}$ $v = 2t - 4$
2 Calculate the value of v at the given instant.	When $t = 0$, $v = -4$. The initial velocity is -4 m/s.
3 Describe the initial motion.	Since the initial velocity is negative, the particle starts to move to the left with an initial speed of 4 m/s.
c **1** Calculate when the particle is momentarily at rest. *Note:* This usually represents a change of direction of motion.	**c** The particle is momentarily at rest when its velocity is zero. When $v = 0$, $2t - 4 = 0$ $t = 2$ The particle is at rest after 2 seconds.
2 Calculate where the particle is momentarily at rest.	The position of the particle when $t = 2$ is $x = (2)^2 - 4(2) - 12$ $= -16$ Therefore, the particle is momentarily at rest after 2 seconds at the position 16 metres to the left of the origin.
d **1** Calculate the position to show the particle is at the origin at the given time.	**d** When $t = 6$, $x = 36 - 24 - 12$ $= 0$ The particle is at the origin when $t = 6$.
2 Track the motion on a horizontal displacement line and calculate the required distance.	The motion of the particle for the first 6 seconds is shown.  Distances travelled are 4 metres to the left, then 16 metres to the right. The total distance travelled is the sum of the distances in each direction. The particle has travelled a total distance of 20 metres.

e Calculate the value required.

e $$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$
$$= \frac{20}{6}$$
$$= 3\tfrac{1}{3}$$
The average speed is $3\frac{1}{3}$ m/s.

f Calculate the average rate of change required.

Note: As there is a change of direction, the average velocity will not be the same as the average speed.

f Average velocity is the average rate of change of displacement.

For the first 6 seconds,

$(t_1, x_1) = (0, -12), \quad (t_2, x_2) = (6, 0)$

$$\text{average velocity} = \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{0 - (-12)}{6 - 0}$$
$$= 2$$
The average velocity is 2 m/s.

EXERCISE 6.7 Rates of change

PRACTISE

1 **WE14** A container in the shape of an inverted right cone of radius 4 cm and depth 12 cm is being filled with water. When the depth of water is h cm, the radius of the water level is r cm.

- **a** Use similar triangles to express r in terms of h.
- **b** Express the volume of the water as a function of h.
- **c** At what rate with respect to the depth of water is the volume of water changing when its depth is 5 cm?

2 A cone has a slant height of 20 cm. The diameter of its circular base is increased in such a way that the cone maintains its slant height of 10 cm while its perpendicular height decreases. When the base radius is r cm, the perpendicular height of the cone is h cm.

- **a** Use Pythagoras's theorem to express r in terms of h.
- **b** Express the volume of the cone as a function of h.
- **c** What is the rate of change of the volume with respect to the perpendicular height when the height is 8 cm?

3 **WE15** A particle moves in a straight line such that its displacement, x metres, from a fixed origin at time t seconds is given by $x = 2t^2 - 8t, \quad t \geq 0$.

- **a** Identify its initial position.
- **b** Obtain its velocity and hence state its initial velocity and describe its initial motion.
- **c** At what time and position is the particle momentarily at rest?
- **d** Show that the particle is at the origin when $t = 4$ and calculate the distance it has travelled to reach the origin.
- **e** Calculate the average speed over the first 4 seconds.
- **f** Calculate the average velocity over the first 4 seconds.

4 The position, in metres, of a particle after t seconds is given by $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1,\ t \geq 0$.

a Find its initial position and initial velocity.

b Calculate the distance travelled before it changes its direction of motion.

c What is its acceleration at the instant it changes direction?

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 a A spherical balloon of radius r is expanding. Find the rate of change of the volume with respect to the radius when the radius is 10 cm.

b An ice cube melts in such a way as to maintain its shape as a cube. Calculate the rate at which its surface area is changing with respect to its side length at the instant the side length is 6 mm.

6 The number of rabbits on a farm is modelled by $N = \dfrac{110}{t},\ t > 0$, where N is the number of rabbits present after t months.

a At what rate is the population of rabbits changing after 5 months?

b Calculate the average rate of change of the population over the interval $t \in [1, 5]$.

c What will happen to the population of rabbits in the long term?

7 The volume of water, V litres, in a bath t minutes after the plug is removed is given by $V = 0.4(8 - t)^3,\ 0 \leq t \leq 8$.

a At what rate is the water leaving the bath after 3 minutes?

b What is the average rate of change of the volume for the first 3 minutes?

c When is the rate of water leaving the bath the greatest?

8 Water is being poured into a vase. The volume, V mL, of water in the vase after t seconds is given by $V = \frac{2}{3}t^2(15 - t),\ 0 \leq t \leq 10$.

a What is the volume after 10 seconds?

b At what rate is the water flowing into the vase at t seconds?

c What is the rate of flow after 3 seconds?

d When is the rate of flow the greatest, and what is the rate of flow at this time?

9 A tent in the shape of a square-based right pyramid has perpendicular height h metres, base side length x metres and volume $\frac{1}{3}Ah$, where A is the area of its base.

a Express the length of the diagonal of the square base in terms of x.

b If the slant height of the pyramid is 12 metres, show that $x^2 = 288 - 2h^2$ and hence express the volume of air inside the tent in terms of h.

c Calculate the rate of change of the volume with respect to height when the height is $3\sqrt{3}$ metres.

10 A container in the shape of an inverted right circular cone is being filled with water. The cone has a height of 15 cm and a radius of 6 cm. At what rate is the volume of water changing with respect to the depth of water when:

a the depth of water reaches half the height of the cone?

b the container is one-third full?

11 A particle moves in a straight line so that at time t seconds its displacement, x metres, from a fixed origin O is given by $x(t) = 2t^2 - 16t - 18, \quad t \geq 0$.

a Calculate the distance the particle is from O after 2 seconds.
b With what speed is it travelling after 2 seconds?
c What was the average velocity of the particle over the first 2 seconds of motion?
d At what time and with what velocity does it reach O?

12 The position, x m, relative to a fixed origin of a particle moving in a straight line at time t seconds is $x = \frac{2}{3}t^3 - 4t^2, t \geq 0$.

a Show the particle starts at the origin from rest.
b At what time and at what position is the particle next at rest?
c When does the particle return to the origin?
d What are the particle's speed and acceleration when it returns to the origin?

13 A ball is thrown vertically upwards into the air so that after t seconds its height h metres above the ground is $h = 50t - 4t^2$.

a At what rate is its height changing after 3 seconds?
b Calculate its velocity when $t = 5$.
c At what time is its velocity -12 m/s and in what direction is the ball then travelling?
d When is its velocity zero?
e What is the greatest height the ball reaches?
f At what time and with what speed does the ball strike the ground?

14 A colony of viruses can be modelled by the rule

$$N(t) = \frac{2t}{(t + 0.5)^2} + 0.5$$

where N hundred thousand is the number of viruses on a nutrient plate t hours after they started multiplying.

a How many viruses were present initially?
b Find $N'(t)$.
c What is the maximum number of viruses, and when will this maximum occur?
d At what rate were the virus numbers changing after 10 hours?

MASTER

15 A population of butterflies in an enclosure at a zoo is modelled by

$$N = 220 - \frac{150}{t + 1}, \quad t \geq 0$$

where N is the number of butterflies t years after observations of the butterflies commenced.

a How long did it take for the butterfly population to reach 190 butterflies, and at what rate was the population growing at that time?
b At what time was the growth rate 12 butterflies per year? Give your answer correct to 2 decimal places.
c Sketch the graphs of population versus time and rate of growth versus time, and explain what happens to each as $t \to \infty$.

16 A veterinarian has administered a painkiller by injection to a sick horse. The concentration of painkiller in the blood, y mg/L, can be defined by the rule

$$y = \frac{3t}{(4 + t^2)}$$

where t is the number of hours since the medication was administered.

a Find $\frac{dy}{dt}$.

b What is the maximum concentration of painkiller in the blood, and at what time is this achieved?

c The effect of the painkiller is considerably reduced once the concentration falls below 0.5 mg/L, when a second dose needs to be given to the horse. When does this occur?

d Find the rate of change of concentration of painkiller in the blood after one hour. Give your answer correct to 2 decimal places.

e When is the rate of change of painkiller in the blood equal to −0.06 mg/L/hour? Give your answer correct to 2 decimal places.

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6.8 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

To access eBookPLUS activities, log on to

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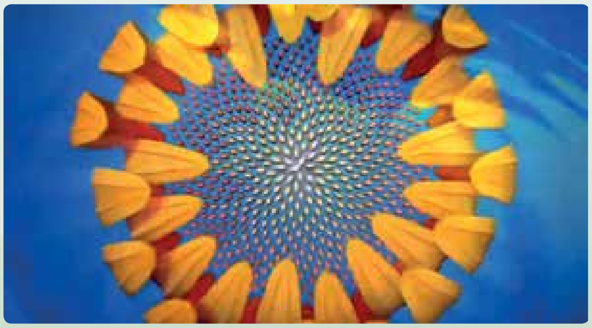
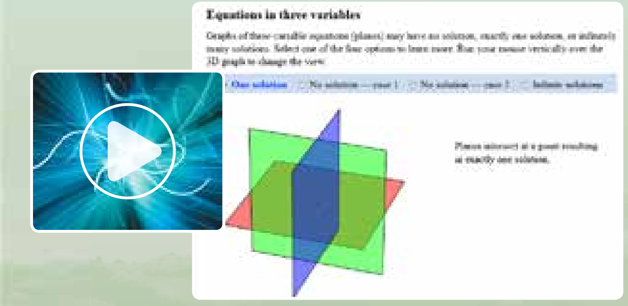
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





Equations in three variables

Graphs of three-variable equations (planes) may have no solution, exactly one solution, or infinitely many solutions. Select one of the four options to learn more. Run your mouse vertically over the 3D graph to change the view.

Planes intersect at a point resulting in exactly one solution.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

6 Answers

EXERCISE 6.2

1 a $\dfrac{2x-7}{2\sqrt{x^2-7x+1}}$

b $6(3x+1)(3x^2+2x-1)^2$

2 a $2\cos(x)\sin(x)$

b $-3\sin(3x)e^{\cos(3x)}$

3 $\dfrac{dy}{dx} = 3\cos(x)\sin^2(x)$; $\dfrac{dy}{dx} = \frac{9}{8}$

4 $\dfrac{dy}{dx} = 2\cos(x)\sin(x)e^{\sin^2(x)}$; $\dfrac{dy}{dx} = e^{\frac{1}{2}} = \sqrt{e}$

5 $y = -4x + 5$

6 a $f(g(x)) = (e^x - 1)^3$

b $h'(x) = 3e^x(e^x - 1)^2$

c $y = 0$

7 a $-\dfrac{6x}{(x^2+1)^2}$

b $-\sin(x)e^{\cos(x)}$

c $\dfrac{x+1}{\sqrt{x^2+2x+3}}$

d $\dfrac{-2\cos(x)}{\sin^3(x)}$

e $\dfrac{x-2}{\sqrt{x^2-4x+5}}$

f $-6x\sin(x^2-1)$

g $30xe^{3x^2-1}$

h $-\dfrac{6x^5+8}{x^3\left(x^3-\dfrac{2}{x^2}\right)^3}$

i $\dfrac{1}{2(2-x)^{\frac{3}{2}}}$

j $-6\sin(2x+1)\cos^2(2x+1)$

8 a $f'(x) = \dfrac{4}{\cos^2(4x+\pi)}$, $f'\left(\dfrac{\pi}{4}\right) = 4$

b $f'(x) = \dfrac{2}{(2-x)^3}$, $f'\left(\frac{1}{2}\right) = \frac{16}{27}$

c $f'(x) = 4xe^{2x^2}, f'(-1) = -4e^2$

d $f'(x) = 8x\sqrt[3]{3x^2-2}$, $f'(1) = 8$

e $f'(x) = -15\sin(3x)(\cos(3x)-1))^4$; $f'\left(\dfrac{\pi}{2}\right) = 15$

9 $f(f(x)) = x^4$, $g'(x) = 4x^3$

10 $(0, 0), \left(\dfrac{\pi}{4}, 1\right), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{4}, 1\right), (\pi, 0)$

11 $24\cos(3x)\sin(3x)$

12 a $f'(x) = -\sin(x)g'[\cos(x)]$

b $f'(x) = 6x^2g'(2x^3)$

c $f'(x) = 6e^{2x+1}g'(3e^{2x+1})$

d $f'(x) = \dfrac{(4x-1)g'(\sqrt{2x^2-x})}{2\sqrt{2x^2-x}}$

13 a $f'(x) = -2h'(x)[h(x)]^{-3}$

b $f'(x) = 2h'(x)\sin[h(x)]$

c $f'(x) = \dfrac{2h'(x)}{3(2h(x)+3)^{\frac{2}{3}}}$

d $f'(x) = -2h'(x)e^{h(x)+4}$

14 a $g(x) = \sqrt[3]{(2x-1)^2}$

b $g'(x) = \dfrac{4}{3\sqrt[3]{2x-1}}$

c At (1, 1): $y = \frac{4}{3}x - \frac{1}{3}$

At (0, 1): $y = -\frac{4}{3}x + 1$

d $\left(\frac{1}{2}, \frac{1}{3}\right)$

15 a $m = -7$, $n = 1$

b $\{x : x \le -1\} \cup \{x : x \ge 7\}$

c $\dfrac{x-3}{\sqrt{x^2-6x-7}}$

d $-\frac{5}{3}$

16 $a = -3$

17 a **i** $f(h(x)) = 2\sin(e^x)$

ii $h(f(x)) = e^{2\sin(x)}$

b 1.555, 2.105, 2.372

18 a 3^{x^2+4x-5}

b 6.5916

EXERCISE 6.3

1 a $f'(x) = 3\cos^2(3x) - 3\sin^2(3x)$

b $f'(x) = 3x^2e^{3x} + 2xe^{3x}$

c $f'(x) = (5x^2 + 17x - 22)e^{5x}$

2 $f'(x) = 8x^3\cos(2x) - 4x^4\sin(2x)$; $f'\left(\dfrac{\pi}{2}\right) = -\pi^3$

3 $f'(x) = \sin(x) + (x+1)\cos(x)$; $f'(0) = 1$

4 $\dfrac{dy}{dx} = 2xe^{3x} + 3(x^2+1)e^{3x}$; $\dfrac{dy}{dx} = 3$. The equation of the tangent is $y = 3x + 1$.

5 $(0, 0), (1, 0), \left(\frac{2}{5}, \frac{216}{3125}\right)$

6 a $x = 0, \pi, 2\pi, 3\pi$

b $x = 1.11, 4.25, 7.39$

7 a $5x^2e^{5x} + 2xe^{5x}$

b $2e^{2x+1}\sec^2(2x) + 2e^{2x+1}\tan(2x)$

c $\dfrac{2(x-1)(2x+1)^2}{x^3}$

d $-x\sin(x) + \cos(x)$

e $\dfrac{4-3x}{\sqrt{x}}$

f $-3e^{-3x}\sin(2x-\pi) + 2e^{-3x}\cos(2x-\pi)$

g $\dfrac{6e^{x^2}(x^2-1)}{x^3}$

h $\dfrac{2e^{2x}(4x^2+2x-1)}{\sqrt{4x^2-1}}$

i $2x\sin^2(2x)[3x\cos(2x) + \sin(2x)]$

j $\dfrac{2(x-5)(x-1)^3}{(x-3)^3}$

k $(3x-2)((3x-2)g'(x) + 6g(x))$

l $\dfrac{-e^{5x}\left(g'(\sqrt{x}) + 10\sqrt{x}g(\sqrt{x})\right)}{2\sqrt{x}}$

8 a $f'(x) = e^x(x+1),\ f'(-1) = 0$

b $f'(x) = (x^2+x)^3(9x^2+5x),\ f'(1) = 112$

c $f'(x) = \dfrac{2(1-x)\tan(x)}{\cos^2(x)} - \tan^2(x),$

$f'\left(\dfrac{\pi}{3}\right) = 8\sqrt{3}\left(1 - \dfrac{\pi}{3}\right) - 3$

d $f'(x) = \dfrac{8x^2\cos(2x^2)\sin(2x^2) + \sin(2x^2)}{2\sqrt{x}},\ f'(\sqrt{\pi}) = 0$

9 $a = 4,\ b = -3$

10 a $f'(x) = (x-a)^2g'(x) + 2(x-a)g(x)$

b $\dfrac{6}{\pi}$

11 $\dfrac{4\pi + 6\sqrt{3}}{9}$

12 Tangent $y = 2ex - e$;

perpendicular line $y = -\dfrac{x}{2e} + \left(\dfrac{1}{2e} + e\right)$

13 a Simplified $y = -\sin(x)$; $\dfrac{dy}{dx} = -\cos(x)$

b $-\cos(x)$

14 a (0, 1) and (1, 0)

b (1.366, −0.057) and (−0.366, 1.195)

c $y = -\dfrac{1}{e}x + \dfrac{1}{e}$

d $y = x + 1$

e (−0.46, 0.54)

15 a i $CD = 3\sin(\theta)$

ii $BD = 3\cos(\theta)$

b S = area of 4 triangles + area of square base

$S = 4 \times \frac{1}{2} \times 6\cos(\theta) \times 3\sin(\theta) + (6\cos(\theta))^2$

$= 36\cos(\theta)\sin(\theta) + 36\cos^2(\theta)$

$= 36(\cos^2(\theta) + \cos(\theta)\sin(\theta))$ as required

c $\dfrac{dS}{d\theta} = 72\cos^2(\theta) - 72\sin(\theta)\cos(\theta) - 36$

16 a $a = 3,\ b = 3,\ c = -2$

b (0,0) and $\left(\frac{3}{2}, \dfrac{81}{8e^3}\right)$

c $y = \dfrac{3}{e^2}x$

17 a The decision is appropriate because the road goes through the point (−2, 3.5187).

b $\left(\frac{5}{4}, 0\right)$

c (0.804, −3.205)

18 2.06

EXERCISE 6.4

1 a $\dfrac{e^{3x} + 2e^{2x}}{(e^x+1)^2}$

b $\dfrac{-3(t\sin(3t) + \cos(3t))}{t^4}$

2 $-\dfrac{1}{(x-1)^2}$

3 $\dfrac{dy}{dx} = \dfrac{\cos(x) - 2\sin(x)}{e^{2x}};\ x = 0, \dfrac{dy}{dx} = 1$

4 $\frac{3}{5}$

5 a $\dfrac{2x\cos(x) - \sin(x)}{2x\sqrt{x}}$

b $\dfrac{2 - \sin(2x)\cos(2x)}{e^x\cos^2(2x)}$

c $-\dfrac{3\sqrt{5-x}}{2}$

d $\dfrac{4x^2\sin(x^2)\cos(x^2) - \sin^2(x^2)}{x^2}$

e $\dfrac{-6x^2 + 4x - 9}{(2x^2-3)^2}$

f $\dfrac{1}{4\sqrt{x}} - 3\sqrt{x}$

g $\dfrac{e^x\cos(2x+1) + 2e^x\sin(2x+1)}{\cos^2(2x+1)}$

h $-\dfrac{xe^{-x}}{(x-1)^2}$

i $\dfrac{6-3x}{2\sqrt{x}(x+2)^2}$

j $-\dfrac{3}{\sin^2(3x)}$

k $-\dfrac{2}{(2x+3)^2}$

l $\dfrac{-4e^{2x}}{(1+e^{2x})^2}$

6 a $f'(x) = \dfrac{\sin[g(x)] - (x+2)g'(x)\cos[g(x)]}{\sin^2[g(x)]}$

b $f'(x) = \dfrac{-2e^{-2x}g'(e^{-2x}) - g(e^{-2x})}{e^x}$

7 a 0 **b** 2

c $\frac{7}{64}$ **d** −5

8 $-\frac{1}{32}$

9 $y = \frac{1}{2}x + \frac{1}{2}$

10 $y = \dfrac{1+\cos(x)}{1-\cos(x)}$, $u = 1 + \cos(x)$, $v = 1 - \cos(x)$

$$\frac{du}{dx} = -\sin(x),\ \frac{dv}{dx} = \sin(x)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}\\ &= \frac{(1-\cos(x)) \times -\sin(x) - (1+\cos(x)) \times \sin(x)}{(1-\cos(x))^2}\\ &= \frac{-\sin(x)(1-\cos(x)+1+\cos(x))}{(1-\cos(x))^2}\\ &= \frac{-2\sin(x)}{(1-\cos(x))^2}\\ &= \frac{-2\sin(x)}{(-(\cos(x)-1))^2}\\ &= \frac{-2\sin(x)}{(\cos(x)-1)^2}\end{aligned}$$

11 $m = 2$

12 $a = -5,\ b = -3$

13 $\dfrac{dy}{dx} = \dfrac{10(1-x^2)}{(x^2+1)^2}$; negative gradient when $(-\infty, -1) \cup (1, \infty)$

14 a Undefined function when $x = 2, -7$

b $\left(-1, \frac{1}{3}\right)$ and $\left(11, \frac{1}{27}\right)$ **c** $y = \frac{5}{16}x + \frac{3}{16}$

15 a $a = -1.088,\ b = 2.655,\ c = 0.483,\ d = -0.552$

b 0.707

16 a $\dfrac{-6x^2 + 6x + 2}{(3x^2+1)^2}$

b $x = -0.1466,\ 0.5746$

EXERCISE 6.5

1 a $\left(\frac{1}{2}, \frac{83}{24}\right)$, minimum turning point; $\left(-2, \frac{26}{3}\right)$, maximum turning point

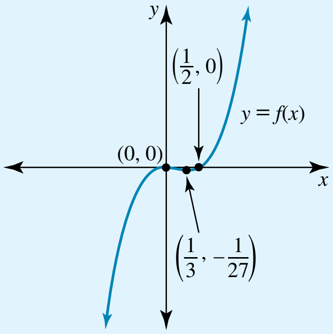
b $a = -3, b = -6, c = -8$

2 a $a = -4, b = 5$

b $(1, -9)$, maximum turning point

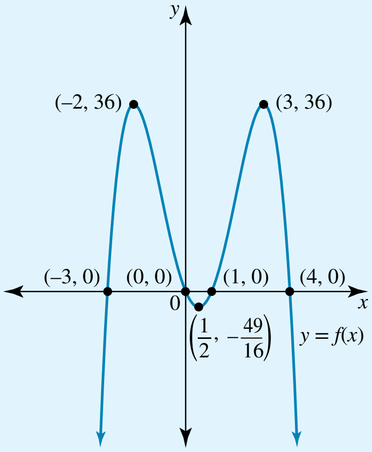
$\left(\frac{5}{3}, \frac{-247}{27}\right)$, minimum turning point

3 a



b $x \in \left[0, \frac{1}{3}\right]$

4 a

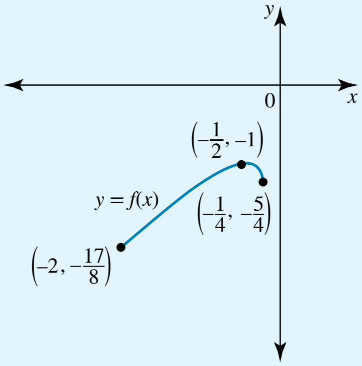


b $x \in (-\infty, -2] \cup \left[\frac{1}{2}, 3\right]$

5 a $\left(-2, -\frac{17}{8}\right), \left(-\frac{1}{4}, -\frac{5}{4}\right)$

b $\left(-\frac{1}{2}, -1\right)$, maximum turning point

c



d Absolute minimum $-\frac{17}{8}$; absolute maximum is −1.

6 No absolute minimum; absolute maximum is $\frac{32}{3\sqrt{3}}$.

7 a

$$\begin{aligned}f'(x) &= 32x - 4x^3\\ 0 &= 32x - 4x^3\\ &= 4x(8 - x^2)\\ &= 4x(2\sqrt{2} - x)(2\sqrt{2} + x)\\ x &= 0, \pm 2\sqrt{2}\\ f(2\sqrt{2}) &= 16(2\sqrt{2})^2 - (2\sqrt{2})^4\\ &= 16 \times 8 - 16 \times 4\\ &= 64\end{aligned}$$

$\therefore (2\sqrt{2}, 64)$ is a stationary point.

b Maximum

c $(-2\sqrt{2}, 64)$ maximum turning point, $(0, 0)$ minimum turning point

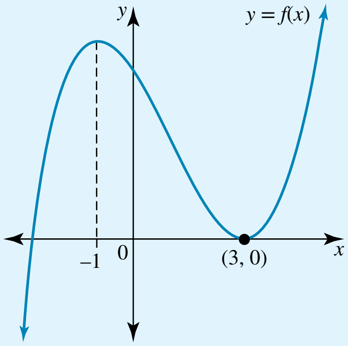
8 a $(-2, 0)$ maximum turning point, $\left(-\frac{2}{3}, -\frac{32}{27}\right)$ minimum turning point

b $(-4, 85)$ maximum turning point, $(2, -23)$ minimum turning point

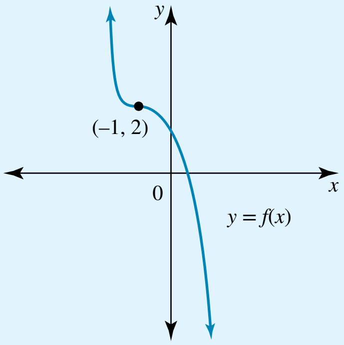
c $(-2, -4)$ maximum turning point, $(0, 0)$ minimum turning point

d $(2, e^{-2})$ maximum turning point

9 a



b

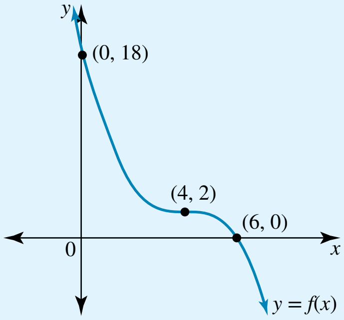


10 a $b = -12, c = 8$

b $a = 4, b = -12$

c $b = 6,\ c = 9,\ d = -10$

11 a



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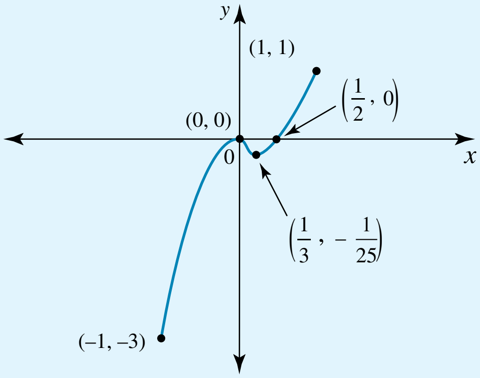
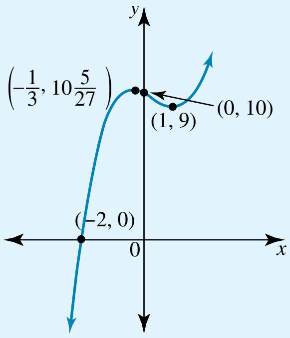


c



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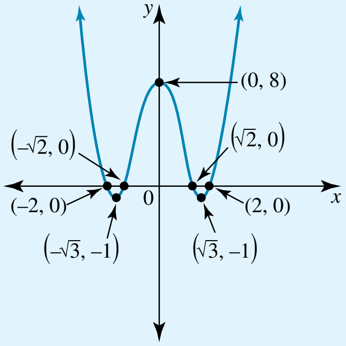
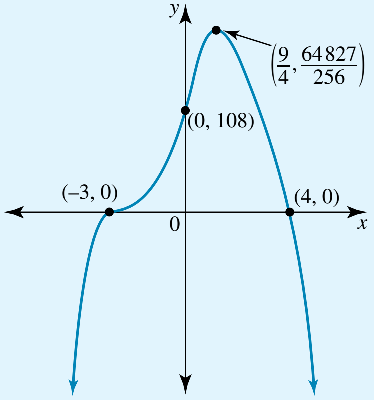
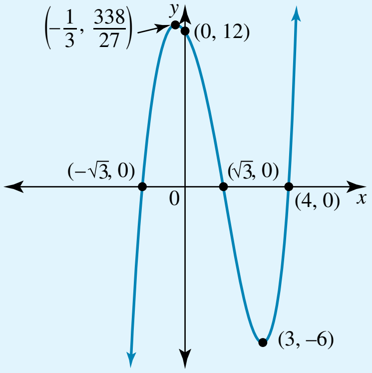


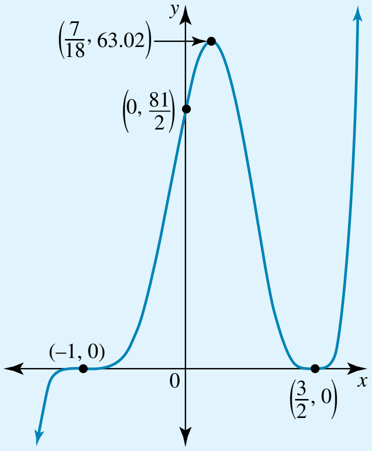
e



f

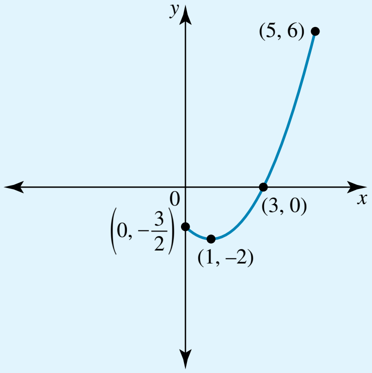


12 a



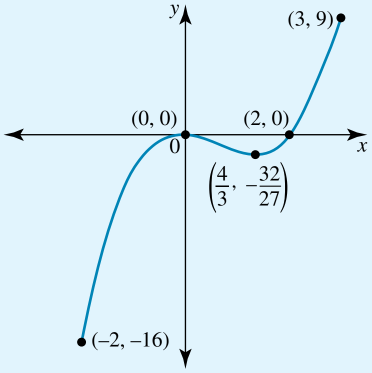
b $x \in \left[\frac{7}{18}, \frac{3}{2}\right]$

13 a



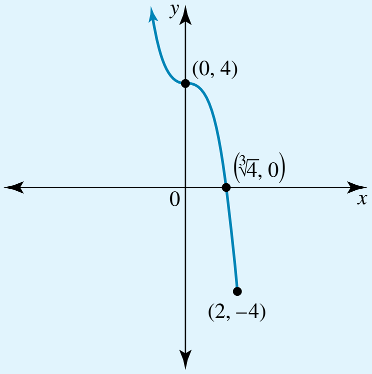
Absolute maximum = 6, absolute minimum = −2

b



Absolute maximum = 9, absolute minimum = −16

c



No absolute maximum; absolute minimum = −4

14 a A (0,25, 5), B (1, 3), C $(5, 2\sqrt{5} + 0.2)$

b A

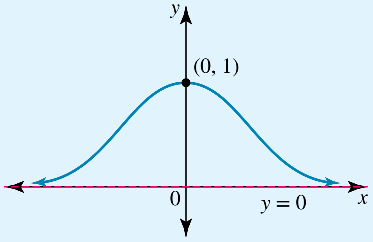
c Absolute minimum = 3, absolute maximum = 5

15 a $\left(-1, -\frac{1}{e}\right)$, minimum turning point

b $x \in (-1, \infty)$

c Absolute minimum $= -\frac{1}{e}$, no absolute maximum

16 a



b $x \in (-\infty, -1]$

17 a Stationary points $(a, 0)$, $\left(\frac{a+4}{3}, \frac{4(a-2)^3}{27}\right)$

b Minimum turning point $(a, 0)$, maximum turning point $\left(\frac{a+4}{3}, \frac{4(a-2)^3}{27}\right)$

c $a = 5$

18 a $(a, 0)$, $(b, 0)$

b $(b, 0)$, $\left(\frac{3a+b}{4}, \frac{-27(a-b)^4}{256}\right)$

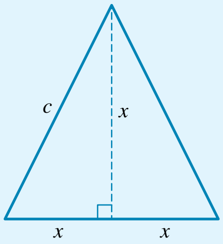
c $(b, 0)$ is a stationary point of inflection;

$\left(\frac{3a+b}{4}, \frac{-27(a-b)^4}{256}\right)$ is a minimum turning point.

d $a = 2, b = 6$

EXERCISE 6.6

1 a



Using Pythagoras' theorem, $c^2 = x^2 + x^2$, so the sloping sides have lengths $\sqrt{2}x$ cm.

Since the perimeter of the figure is 150 cm,

$$2x + 2y + 2\sqrt{2}x = 150$$

$$\therefore y = 75 - x - \sqrt{2}x$$

The area of the figure is the sum of the areas of the rectangle and the triangle, base $2x$, height x.

$$A = 2xy + \tfrac{1}{2}(2x)x$$

$$\therefore A = 2x(75 - x - \sqrt{2}x) + x^2$$

$$= 150 - 2x^2 - 2\sqrt{2}x^2 + x^2$$

$$\therefore A = 150x - (1 + 2\sqrt{2})x^2$$

b Width = 39.2 cm and height = 47.3 cm

c Width = 30 cm and height = 53.8 cm.

2 a $V = x(16 - 2x)(10 - 2x)$

b $x = 2$; therefore, height is 2 cm, length is 12 cm and width is 6 cm. Volume is 144 cm^3.

3 The point on the curve is $(2, -1)$. The minimum distance is $\sqrt{5}$ units.

4 a 400

b $t = 4$, so after 4 months (i.e. end of December)

c 694 birds

5 Both numbers are 16.

6 a $l = 137.5$ m, $w = 55$ m

b $A_{\max} = 7562.5 \text{ m}^2$

7 $r = 6.06$ cm, $h = 12.11$ cm and $V_{\max} = 1395.04 \text{ cm}^3$

8 The point on the line is $(3, 2\sqrt{3})$. Minimum distance is 4 units.

9 4 square units

10 a $a = 12, \quad b = 9.6$

b $A_{\max} = 403.2 \;\text{ m}^2$

11 a \$1000

b $A_{\text{least}} = \$980.34$

c May 2010

d \$1000

12

$$P = 2l + 2R + \pi R$$

$$A = 2Rl + \frac{\pi R^2}{2}$$

$$A - \frac{\pi R^2}{2} = 2Rl$$

$$l = \frac{A}{2R} - \frac{\pi R}{4}$$

Substitute for l into the perimeter formula:

$$\begin{aligned} P &= 2\left(\frac{A}{2R} - \frac{\pi R}{4}\right) + 2R + \pi R \\ &= \frac{A}{R} - \frac{\pi R}{2} + 2R + \pi R \\ &= \frac{A}{R} + \frac{\pi R}{2} + 2R \end{aligned}$$

For minimum perimeter, solve $\dfrac{dP}{dR} = 0$.

$$\frac{dP}{dR} = -\frac{A}{R^2} + \frac{\pi}{2} + 2$$

$$0 = -\frac{A}{R^2} + \frac{\pi + 4}{2}$$

$$\frac{A}{R^2} = \frac{\pi + 4}{2}$$

$$R^2 = \frac{2A}{\pi + 4}$$

$$\begin{aligned} R &= \pm\sqrt{\frac{2A}{\pi + 4}} \\ &= \sqrt{\frac{2A}{\pi + 4}}, \quad R > 0 \end{aligned}$$

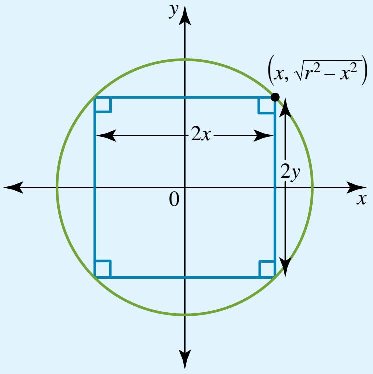
Substitute R into the length equation to determine l:

$$\begin{aligned} l &= \frac{A}{2R} - \frac{\pi R}{4} \\ &= \frac{2A - \pi R^2}{4R} \\ &= \frac{2A - \dfrac{2A\pi}{\pi + 4}}{4\sqrt{\dfrac{2A}{\pi + 4}}} \\ &= \frac{\dfrac{2A(\pi + 4) - 2A\pi}{\pi + 4}}{4\sqrt{\dfrac{2A}{\pi + 4}}} \\ &= \frac{\dfrac{8A\pi}{\pi + 4}}{4\sqrt{\dfrac{2A}{\pi + 4}}} \\ &= \frac{4 \times \dfrac{2A\pi}{\pi + 4}}{4\sqrt{\dfrac{2A}{\pi + 4}}} \\ &= \sqrt{\frac{2A}{\pi + 4}} \end{aligned}$$

If l and R are both $\sqrt{\dfrac{2A}{\pi + 4}}$ m, the perimeter is a minimum (as required).

13 $h = 16$ cm, $V_{\max} = 2145 \text{ cm}^3$

14



$$\begin{aligned} A &= 2x \times 2y \\ &= 2x \times 2\sqrt{r^2 - x^2} \\ &= 4x\sqrt{r^2 - x^2} \end{aligned}$$

For maximum area, solve $\dfrac{dA}{dx} = 0$.

$$\frac{dA}{dx} = 4x \times \frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}} \times -2x + (r^2 - x^2)^{\frac{1}{2}} \times 4$$

$$\begin{aligned}
0 &= \frac{-4x^2}{\sqrt{r^2 - x^2}} + 4\sqrt{r^2 - x^2} \\
&= \frac{-4x^2 + 4(r^2 - x^2)}{\sqrt{r^2 - x^2}} \\
&= -4x^2 + 4r^2 - 4x^2 \\
&= -8x^2 + 4r \\
8x^2 &= 4r^2 \\
x^2 &= \frac{r^2}{2} \\
x &= \pm\frac{r}{\sqrt{2}} \\
&= \frac{r}{\sqrt{2}},\ x > 0
\end{aligned}$$

Substitute for x into the equation for y:

$$\begin{aligned}
y &= \sqrt{r^2 - x^2} \\
&= \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} \\
&= \sqrt{r^2 - \frac{r^2}{2}} \\
&= \sqrt{\frac{r^2}{2}} \\
&= \frac{r}{\sqrt{2}}
\end{aligned}$$

The x and y values are the same; therefore, the rectangle of largest area is a square.

15 $h = \frac{10}{3}$ cm, $r = \frac{16}{3}$ cm and $V_{\text{max}} = 298\text{ cm}^3$

16 Row to a point that is 3.2 km to the right of O.

EXERCISE 6.7

1 **a** $r = \dfrac{h}{3}$ **b** $V = \dfrac{\pi h^3}{27}$ **c** $\dfrac{dV}{dh} = \dfrac{25\pi}{9}$ cm³/cm

2 **a** $r = \sqrt{400 - h^2}$

b $V = \dfrac{400\pi h}{3} - \dfrac{\pi h^3}{3}$

c $\dfrac{dV}{dh} = \dfrac{208\pi}{3}$ cm³/cm

3 **a** 0 m

b $v = 4t - 8$, initial velocity is -8 m/s, so the object is moving left at a speed of 8 m/s.

c After 2 seconds, 8 m to the left of the origin

d
$$\begin{aligned}
2t^2 - 8t &= 0 \\
2t(t - 4) &= 0 \\
t &= 0, 4
\end{aligned}$$

The distance travelled is 16 m.

e 4 m/s

f 0 m/s

4 **a** 1 m right, 8 m/s

b $27\frac{2}{3}$ m

c -6 m/s^2

5 **a** $400\pi\text{ cm}^3/\text{cm}$

b $-72\text{ mm}^2/\text{mm}$

6 **a** -4.4 rabbits/month

b -22 rabbits/month

c $t \to \infty, N \to 0$. Effectively the population of rabbits will be zero in the long term.

7 **a** $\dfrac{dV}{dt} = 30$ L/min

b -51.6 L/min

c $t = 0$

8 **a** $V = 333\frac{1}{3}$ mL

b $\dfrac{dV}{dt} = 20t - 2t^2$

c $\dfrac{dV}{dt} = 42$ mL/s

d $t = 5s,\ \dfrac{dV}{dt} = 50$ mL/s

9 **a** $\sqrt{2}x$ metres

b Right-angled triangle with lengths $h, \dfrac{\sqrt{2}x}{2}$ and hypotenuse 12

$$\begin{aligned}
h^2 + \left(\frac{\sqrt{2}x}{2}\right)^2 &= 144 \\
\frac{2x^2}{4} &= 144 - h^2 \\
x^2 &= 288 - 2h^2
\end{aligned}$$

(as required)

$V = \frac{1}{3}(288h - 2h^3)$

c $42\text{ m}^3/\text{m}$

10 **a** $9\pi\text{ cm}^3/\text{cm}$

b $12 \times 3^{\frac{1}{3}}\pi\text{ cm}^3/\text{cm}$

11 **a** 42 metres to the left

b 8 m/s

c -12 m/s

d 9 seconds, 20 m/s

12 **a**
$$\begin{aligned}
t = 0, x &= \frac{2}{3}(0)^3 - 4(0)^2 \\
&= 0\text{ m} \\
v = \frac{dx}{dt} &= 2t^2 - 8t \\
t = 0, v &= 2(0)^2 - 8(0) \\
&= 0\text{ m/s}
\end{aligned}$$

b 4 seconds, $21\frac{1}{3}$ metres left of origin

c 6 seconds

d 24 m/s, 16 m/s^2

13 **a** 26 m/s

b 10 m/s

c 7.75 seconds, travelling down towards the ground

d 6.25 seconds

e 156.25 metres

f 12.5 seconds, 50 m/s

14 **a** 0.5 hundred thousand or 50 000

b $N'(t) = \dfrac{-2t^2 + 0.5}{(t + 0.5)^4}$

c $N_{\text{max}} = 1.5$ hundred thousand or 150 000 after half an hour.

d −1641 viruses/hour

15 **a** 4 years, 6 butterflies per year

b 2.54 years

c As $t \to \infty, N \to 220$ and $\dfrac{dN}{dt} \to 0$.

16 **a** $\dfrac{3(4 - t^2)}{(4 + t^2)^2}$

b $y_{\text{max}} = 0.75$ mg/L after 2 hrs

c Next dose after 5.24 hours

d 0.36 mg/L/h

e $t = 2.45$ h and $t = 6$ h

7

Antidifferentiation

7.1 Kick off with CAS

The antiderivative function

Part 1

1 a Using the graph application on CAS, sketch the graph of $f(x) = x^2 - 2x$.

b Using the integral template, sketch a possible graph of the antiderivative function $y = \int f(x)dx$ on the same set of axes as $f(x)$.

c Repeat this process for $f(x) = 3 - x$ and $f(x) = (x - 2)^3 + 1$. Use a different set of axes for each $f(x)$ and $y = \int f(x)dx$ pair.

d What do you notice about the positions of the stationary points and x-intercepts? What do you notice about the shapes of the graph and where the graphs have positive and negative gradients?

Part 2

2 Using the calculation application on CAS, use the integral template and find the following.

a $\int (3x + \sqrt{x} - 2)dx$

b $\int \left(\frac{2x^2 - 4x + 2}{5x^4}\right)dx$

c $\int \left(\frac{1}{\sqrt[3]{x}} - \frac{7}{2}x + 4x^{-2}\right)dx$

d $\int (2 - 3x)^4 dx$

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

7.2 Antidifferentiation

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Units 3 & 4

AOS 3

Topic 4

Concept 1

Antidifferentiation

Concept summary

Practice questions

Antidifferentiation, also known as **integration**, is the reverse process of differentiation. Antidifferentiation allows us to find $f(x)$ when we are given $f'(x)$.

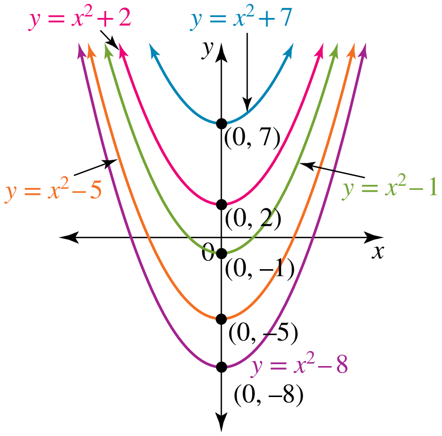
In Topic 5, you learned that $\frac{d}{dx}(x^2) = 2x$. Alternatively, this can be expressed in function notation: if $f(x) = x^2$, then $f'(x) = 2x$. So if you were given $f'(x) = 2x$ and asked to find $f(x)$, you might expect that $f(x) = x^2$.

However, this is not quite as simple as it first appears. Consider each of the following derivatives.

$$\frac{d}{dx}(x^2 + 7) = 2x$$
$$\frac{d}{dx}(x^2 + 2) = 2x$$
$$\frac{d}{dx}(x^2 - 1) = 2x$$
$$\frac{d}{dx}(x^2 - 5) = 2x$$

If we are asked to find $f(x)$ given that $f'(x) = 2x$, how do we know which of the equations above is the correct answer? To give a totally correct answer, additional information about the function must be given.

If $f'(x) = 2x$, then $f(x) = x^2 + c$, where c is an arbitrary constant. This means we have a family of curves that fit the criteria for the function f.



To know which specific curve matches f, we must know additional information such as a point through which the curve passes.

Notation

The notation that is commonly used for antidifferentiation was introduced by the German mathematician Gottfried Leibniz (1646–1716). An example of this notation is:

$$\int 2x dx = x^2 + c$$

This equation indicates that the **antiderivative** of $2x$ with respect to x is equal to x^2 plus an unknown constant, c. $\int f(x)dx$ is known as the indefinite integral. It is read as

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Interactivity
Integration of ax^n
int-6419

'the integral of $f(x)$ with respect to x'. The symbol $\int$ on the left and the dx on the right can be thought of as the bread of a sandwich, with the $f(x)$ being the sandwich filling. The $\int$ tells us to antidifferentiate and the dx tells us that x is the variable.

Remembering that the reverse process of differentiation is integration, consider the following:

$$\frac{d}{dx}(x^3) = 3x^2 \quad \therefore \int 3x^2 dx = x^3 + c$$

$$\frac{d}{dx}(x^4) = 4x^3 \quad \therefore \int 4x^3 dx = x^4 + c$$

$$\frac{d}{dx}(2x^5) = 10x^4 \quad \therefore \int 10x^4 dx = 2x^5 + c$$

$$\frac{d}{dx}(3x^6) = 18x^5 \quad \therefore \int 18x^5 dx = 3x^6 + c$$

From this series of derivatives and integrals, two important observations can be made.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \; n \neq -1$$

$$\int ax^n dx = a\int x^n dx = \frac{ax^{n+1}}{n+1} + c, \; n \neq -1$$

This is the general rule for the antiderivative of ax^n.

Properties of integrals

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As differentiation is a linear operation, so too is antidifferentiation.

$$\int (f(x) \pm g(x)) = \int f(x)dx \pm \int g(x)dx$$

$$\int af(x)dx = a\int f(x)dx, \text{ where } a \text{ is a constant.}$$

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Units 3 & 4
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Properties of indefinite integrals
Concept summary
Practice questions

That is, we can antidifferentiate the separate components of an expression. For example,

$$\int (4x^3 + 6x^2 - 9x + 7)dx = \frac{4x^{3+1}}{4} + \frac{6x^{2+1}}{3} - \frac{9x^{1+1}}{2} + 7x + c$$
$$= x^4 + 2x^3 - \tfrac{9}{2}x^2 + 7x + c$$

To check your antiderivative is correct, it is always good to differentiate your answer to see if the derivative matches the original expression.

Note: If you are asked to find 'the' antiderivative of an expression, then the '+ c' component must be part of the answer. However, if you are asked to find 'an' antiderivative, then you can choose what the value of c is. The convention when finding 'an' antiderivative is to let c equal 0. So the example above would have an antiderivative of $x^4 + 2x^3 - \frac{9}{2}x^2 + 7x$.

WORKED EXAMPLE 1

a Find $\int(3x^4 - x^3 + 2)dx$.

b If $f'(x) = 2\sqrt{x} + \frac{1}{x^2} - 7$, find the rule for f.

c Find an antiderivative of $(2x - 3)(4 - x)$.

d Determine $\int\frac{x^4 - 2x^3 + 5}{x^3}dx$.

THINK	WRITE
a Antidifferentiate each term separately by applying the rule.	**a** $\int(3x^4 - x^3 + 2)dx = \frac{3x^5}{5} - \frac{x^4}{4} + 2x + c$
b 1 First rewrite any surds as fractional powers, and rewrite any powers in the denominator as negative powers. That is, write each term in the form ax^n.	**b** $f(x) = \int\left(2\sqrt{x} + \frac{1}{x^2} - 7\right)dx$ $= \int(2x^{\frac{1}{2}} + x^{-2} - 7)dx$
2 Antidifferentiate each term separately by applying the rule.	$= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{-1}}{-1} - 7x + c$
3 Simplify.	$= \frac{4x^{\frac{3}{2}}}{3} - \frac{1}{x} - 7x + c$
c 1 First expand the expression.	**c** $(2x - 3)(4 - x) = 8x - 2x^2 - 12 + 3x$ $= -2x^2 + 11x - 12$
2 Antidifferentiate each term separately by applying the rule. *Note:* The '+ c' is not needed as the question asked for 'an' antiderivative.	$\int(-2x^2 + 11x - 12)dx = -\frac{2x^3}{3} + \frac{11x^2}{2} - 12x$
d 1 Rewrite the expression as separate lfractions.	**d** $\int\frac{x^4 - 2x^3 + 5}{x^3}dx = \int\left(\frac{x^4}{x^3} - \frac{2x^3}{x^3} + \frac{5}{x^3}\right)dx$ $= \int(x - 2 + 5x^{-3})dx$
2 Antidifferentiate each term separately by applying the rule.	$= \frac{x^2}{2} - 2x + \frac{5x^{-2}}{-2} + c$
3 Simplify. *Note:* Simplest form usually assumes positive indices only.	$= \frac{x^2}{2} - 2x - \frac{5}{2x^2} + c$

Note: It is extremely useful to differentiate the answer of an antiderivative in order to check its validity.

Integrals of the form $\int f(ax+b)\,dx,\ n \neq 1$

Consider the function $f: R \to R,\ f(x) = (ax+b)^n$.

Using the chain rule, $f'(x) = an(ax+b)^{n-1}$.

Hence,

$$\int an(ax+b)^{n-1}dx = (ax+b)^n$$

$$an\int (ax+b)^{n-1}dx = (ax+b)^n$$

$$\int (ax+b)^{n-1}dx = \frac{1}{an}(ax+b)^n + c$$

Thus we obtain the general rule:

$$\boldsymbol{\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.}$$

WORKED EXAMPLE 2

Antidifferentiate:

a $(2x+3)^5$

b $2(3x-1)^{-2}$.

THINK	WRITE
a Apply the rule $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$.	**a** $\int (2x+3)^5 dx$ $= \dfrac{(2x+3)^6}{2(6)} + c$ $= \dfrac{(2x+3)^6}{12} + c$
b **1** Take 2 out as a factor.	**b** $\int 2(3x-1)^{-2}dx$ $= 2\int (3x-1)^{-2}dx$
2 Apply the rule $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$.	$= \dfrac{2(3x-1)^{-1}}{3(-1)} + c$ $= \dfrac{2(3x-1)^{-1}}{-3} + c$ $= -\dfrac{2}{3(3x-1)} + c$

Note: The rules described above only apply if the expression inside the brackets is linear. If the expression is of any other kind, it must be expanded, if possible, before integrating, or you must use CAS to integrate the expression.

WORKED EXAMPLE 3

Find $\int\left(x+\frac{1}{x}\right)^2 dx$.

THINK	WRITE
1 As the inner function is not linear, there is no antidifferentiation rule we can apply, so the expression must first be expanded.	$\int\left(x+\frac{1}{x}\right)^2 dx$ $= \int\left(x^2+2x\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)^2\right)dx$ $= \int\left(x^2+2+\frac{1}{x^2}\right)dx$
2 Write all terms in the form ax^n.	$= \int(x^2+2+x^{-2})dx$
3 Apply the rules for antidifferentiation.	$= \frac{x^3}{3}+2x-\frac{x^{-1}}{1}+c$ $= \frac{1}{3}x^3+2x-\frac{1}{x}+c$

Integration by recognition

Sometimes you may be required to find an antiderivative of a very complex function. In order to complete this task, you will first be given a function to differentiate. The technique is then to recognise the patterns between the derivative you have found and the function you have been given to antidifferentiate.

In general, if $f(x) = g'(x)$, then $\int g'(x)dx = f(x) + c$.

WORKED EXAMPLE 4

If $y = (3x^2 + 4x - 7)^5$, find $\frac{dy}{dx}$. Hence find an antiderivative of $20(3x + 2)(3x^2 + 4x - 7)^4$.

THINK	WRITE
1 Use the chain rule to differentiate the given function.	$y = (3x^2+4x-7)^5$ $\frac{dy}{dx} = 5(6x+4)(3x^2+4x-7)^4$
2 Remove 2 as a factor from the linear bracket.	$= 10(3x+2)(3x^2+4x-7)^4$
3 Rewrite the result as an integral.	$\int 10(3x+2)(3x^2+4x-7)^4dx = (3x^2+4x-7)^5$
4 Adjust the left-hand side so that it matches the expression to be integrated.	$2 \times \int 10(3x+2)(3x^2+4x-7)^4dx = 2 \times (3x^2+4x-7)^5$
5 Write the answer. *Note:* c is not required because 'an' antiderivative was required.	$\int 20(3x+2)(3x^2+4x-7)^4dx = 2(3x^2+4x-7)^5$

EXERCISE 7.2 Antidifferentiation

PRACTISE

Work without CAS

1 **WE1** Find:

a $f(x)$ if $f'(x) = \frac{3}{2}x - 4x^2 + 2x^3$

b an antiderivative of $\dfrac{3}{\sqrt{x}} - 4x^3 + \dfrac{2}{5x^3}$

c $\int x(x-3)(2x+5)dx$

d $\int \dfrac{3x^3 - x}{2\sqrt{x}}dx.$

2 Find:

a $\int \left(\dfrac{2}{\sqrt{x}} + \dfrac{3}{x^2} - \dfrac{1}{2x^3}\right)dx$

b $\int (x+1)(2x^2 - 3x + 4)dx.$

3 **WE2** Antidifferentiate:

a $(3x-5)^5$

b $\dfrac{1}{(2x-3)^{\frac{5}{2}}}.$

4 Find:

a $\int (2x+3)^4 dx$

b $\int (1-2x)^{-5}dx.$

5 **WE3** Find $\int \left(2x^2 + \dfrac{1}{x}\right)^3 dx.$

6 Find:

a $\int (\sqrt{x} - x)^2 dx$

b $\int \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^3 dx.$

7 **WE4** If $y = (3x^2 + 2x - 4)^3$, find $\dfrac{dy}{dx}$. Hence, find an antiderivative of $(3x+1)(3x^2+2x-4)^2$.

8 If $y = \left(7x + \sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^4$, find $\dfrac{dy}{dx}$. Hence, find an antiderivative of

$$y = \left(7 + \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}\right)\left(7x + \sqrt{x} - \frac{1}{\sqrt{x}}\right)^3.$$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 Given that $f'(x) = x^2 - \dfrac{1}{x^2}$, find the rule for f.

10 Find:

a $\int x^3 dx$

b $\int 7x^2 - \dfrac{2}{5x^3}\,dx$

c $\int (4x^3 - 7x^2 + 2x - 1)dx$

d $\int (2\sqrt{x})^3 dx.$

11 Find the indefinite integral of each of the following functions.

a $(3x-1)^3$

b $\dfrac{1}{4x^3}$

c $x^{\frac{5}{2}} - 3x^{\frac{2}{5}}$

d $\dfrac{x^4 - 2x}{x^3}$

e $\sqrt{x}(2x - \sqrt{x})$

f $\sqrt{4-x}$

12 Find an antiderivative for each of the following functions.

a $(2x + 3)(3x - 2)$

b $\dfrac{x^3 + x^2 + 1}{x^2}$

c $2\sqrt{x} - \dfrac{4}{\sqrt{x}}$

d $\left(x^3 - \dfrac{2}{x^3}\right)^2$

e $2(1 - 4x)^{-3}$

f $\dfrac{2}{(2x - 3)^{\frac{5}{2}}}$

13 The gradient function for a particular curve is given by $\dfrac{dy}{dx} = x^3 - 3\sqrt{x}$. Find the general rule for the function, y.

14 Find the general equation of the curve whose gradient at any point is given by $\dfrac{x^3 + 3x^2 - 3}{x^2}$.

15 Find the general equation of the curve whose gradient at any point on the curve is given by $\sqrt{x} + \dfrac{1}{\sqrt{x}}$.

16 If $y = \sqrt{x^2 + 1}$, find $\dfrac{dy}{dx}$ and hence find the antiderivative of $\dfrac{5x}{\sqrt{x^2 + 1}}$.

17 If $y = (5x^2 + 2x - 1)^4$, find $\dfrac{dy}{dx}$ and hence find an antiderivative of $16(5x + 1)(5x^2 + 2x - 1)^3$.

18 If $y = \sqrt{5x^3 + 4x^2}$, find $\dfrac{dy}{dx}$ and hence find an antiderivative of $\dfrac{15x^2 + 8x}{\sqrt{5x^3 + 4x^2}}$.

MASTER

19 Find $\displaystyle\int \frac{x^2}{\sqrt{x^3 + 1}}\,dx$.

20 Find $\displaystyle\int 2(3x + 5)^{\frac{1}{2}}(7x^2 + 4x - 1)\,dx$.

7.3 Antiderivatives of exponential and trigonometric functions

Antiderivatives of cos(*x*) and sin(*x*).

We know that:

- if $y = \sin(x)$, then $\dfrac{dy}{dx} = \cos(x)$
- if $y = \sin(ax + b)$, then $\dfrac{dy}{dx} = a\cos(ax + b)$
- if $y = \cos(x)$, then $\dfrac{dy}{dx} = -\sin(x)$
- if $y = \cos(ax + b)$, then $\dfrac{dy}{dx} = -a\sin(ax + b)$.

Hence,

$$\int \cos(x)dx = \sin(x) + c \text{ and } \int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$$

$$\int \sin(x)dx = -\cos(x) + c \text{ and } \int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$$

WORKED EXAMPLE 5

Find an antiderivative of $f(x) = 2\sin(5x) + 3\cos\left(\frac{x}{3}\right)$.

THINK	WRITE
1 Separate the two terms.	$\int\left(2\sin(5x) + 3\cos\left(\frac{x}{3}\right)\right)dx$ $= \int 2\sin(5x)dx + \int 3\cos\left(\frac{x}{3}\right)dx$
2 Take out 2 and 3 as factors.	$= 2\int \sin(5x)dx + 3\int \cos\left(\frac{x}{3}\right)dx$
3 Apply the antidifferentiation rules for sin and cos.	$= -\frac{2}{5}\cos(5x) + 9\sin\left(\frac{x}{3}\right)$

The antiderivative of e^x

As we have seen in Topic 5, $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(e^{kx}) = ke^{kx}$. Therefore, it follows that

$$\int e^x dx = e^x + c \text{ and } \int e^{kx}dx = \frac{1}{k}e^{kx} + c$$

WORKED EXAMPLE 6

Find:

a $\int (x^7 - e^{3x})dx$

b $\int 8e^{2x}dx$.

THINK	WRITE
a Integrate each term separately.	**a** $\int (x^7 - e^{3x})dx$ $= \frac{1}{8}x^8 - \frac{1}{3}e^{3x} + c$
b 1 Apply the rule $\int e^{kx}dx = \frac{1}{k}e^{kx} + c$.	**b** $\int 8e^{2x}dx$ $= 8\int e^{2x}dx$ $= 8 \times \frac{1}{2}e^{2x} + c$
2 Simplify.	$= 4e^{2x} + c$

Algebraic expansion may also be necessary for questions involving e^{kx}.

WORKED EXAMPLE 7 **Find y if it is known that $\frac{dy}{dx} = (e^x + e^{-x})^3$.**

THINK

1 Expand the brackets.

2 Antidifferentiate each term separately.

WRITE

$$\begin{aligned}\frac{dy}{dx} &= (e^x + e^{-x})^3\\ &= (e^x)^3 + 3(e^x)^2(e^{-x}) + 3(e^x)(e^{-x})^2 + (e^{-x})^3\\ &= e^{3x} + 3e^{2x}e^{-x} + 3e^xe^{-2x} + e^{-3x}\\ &= e^{3x} + 3e^x + 3e^{-x} + e^{-3x}\end{aligned}$$

$$\begin{aligned}y &= \int(e^{3x} + 3e^x + 3e^{-x} + e^{-3x})dx\\ &= \tfrac{1}{3}e^{3x} + 3e^x - 3e^{-x} - \tfrac{1}{3}e^{-3x} + c\end{aligned}$$

For particularly difficult antidifferentiation problems, you may first be asked to differentiate a function so that you can use this result to carry out the antidifferentiation.

Recall that if $f(x) = g'(x)$, then $\int g'(x)dx = f(x) + c$

WORKED EXAMPLE 8 **Given that $y = e^{x^2}$, find $\frac{dy}{dx}$ and hence find an antiderivative of xe^{x^2}.**

THINK

1 Use the chain rule to differentiate the given function.

2 Rewrite the result as an integral.

3 Adjust the left-hand side so that it matches the expression to be integrated.

4 Write the answer.

WRITE

$$y = e^{x^2}$$
$$\frac{dy}{dx} = 2xe^{x^2}$$

$$\int 2xe^{x^2}dx = e^{x^2}$$

$$2\int xe^{x^2}dx = e^{x^2}$$
$$\tfrac{1}{2} \times 2\int xe^{x^2}dx = \tfrac{1}{2} \times e^{x^2}$$

$$\int xe^{x^2}dx = \tfrac{1}{2}e^{x^2}$$

EXERCISE 7.3 Antiderivatives of exponential and trigonometric functions

PRACTISE

Work without CAS

1 **WE5** **a** Find the indefinite integral of $\frac{1}{2}\cos(3x + 4) - 4\sin\left(\frac{x}{2}\right)$.

b Find an antiderivative of $\cos\left(\frac{2x}{3}\right) - \frac{1}{4}\sin(5 - 2x)$.

2 **a** Find $\int\left(\sin\left(\frac{x}{2}\right) - 3\cos\left(\frac{x}{2}\right)\right)dx$.

b If $f'(x) = 7\cos(2x) - \sin(3x)$, find a general rule for f.

3 **WE6** Find:

a $\int(x^4 - e^{-4x})dx$

b $\int\left(\frac{1}{2}e^{2x} - \frac{2}{3}e^{-\frac{1}{2}x}\right)dx$.

4 Find the indefinite integral of:

a $e^{\frac{x}{3}} + \sin\left(\frac{x}{3}\right) + \frac{x}{3}$

b $\cos(4x) + 3e^{-3x}$.

5 **WE7** Find $\int(e^{2x} - e^{-3x})^3dx$.

6 Find the indefinite integral of $\left(e^{\frac{x}{2}} - \frac{1}{e^x}\right)^2$.

7 **WE8** Find the derivative of $e^{\cos^2(x)}$ and hence find an antiderivative of $\sin(x)\cos(x)e^{\cos^2(x)}$.

8 Find the derivative of $e^{(x+1)^3}$ and hence find $\int 9(x+1)^2e^{(x+1)^3}dx$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 Find:

a $\int(2e^{3x} - \sin(2x))dx$

b $\int\frac{e^{2x} + 3e^{-5x}}{2e^x}dx$

c $\int(0.5\cos(2x + 5) - e^{-x})dx$

d $\int(e^x - e^{2x})^2dx$.

10 If it is known that $\int ae^{bx}dx = -3e^{3x} + c$, find the exact values of the constants a and b.

11 Find an antiderivative of $\frac{1}{4x^2} + \sin\left(\frac{3\pi x}{2}\right)$.

12 The gradient of a tangent to a curve is given by $\frac{dy}{dx} = \cos(2x) - e^{-3x}$. Find a possible general rule for the curve y.

13 Heat escapes from a storage tank at a rate of kilojoules per day. This rate can be modelled by

$$\frac{dH}{dt} = 1 + \frac{\pi^2}{9}\sin\left(\frac{\pi t}{45}\right),\ 0 \leq t \leq 100$$

where $H(t)$ is the total accumulated heat loss in kilojoules, t days after June 1.

a Determine $H(t)$.

b What is the total accumulated heat loss after 15 days? Give your answer correct to 3 decimal places.

14 Differentiate $y = 2xe^{3x}$ and hence find an antiderivative of xe^{3x}.

15 Given that $y = e^{2x^2+3x-1}$, find $\frac{dy}{dx}$ and hence find an antiderivative of $2(4x + 3)e^{2x^2+3x-1}$.

16 Find $\frac{d}{dx}(x\cos(x))$ and hence find an antiderivative of $x\sin(x)$.

17 If $x(t) = 20 + \cos\left(\frac{\pi t}{4}\right)$ and $\frac{dy}{dt} = \frac{\pi}{20}x(t) - \pi$, find a possible rule for y in terms of t.

18 If $f'(x) = a\sin(mx) - be^{nx}$ and $f(x) = \cos(2x) - 2e^{-2x} + 3$, find the exact constants a, b, c, m and n.

MASTER

19 Find $\{x : \int e^x \sin(x)dx = \int e^x \cos(x)dx, \quad 0 \le x \le 2\pi\}$.

20 Find $\int \frac{e^{2x} + e^x - 1}{e^x + 1}dx$.

7.4 Families of curves and applications

Initial conditions

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Families of curves
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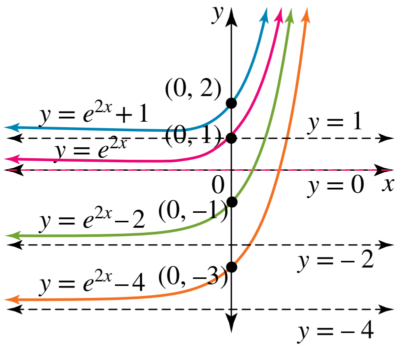
Suppose we are asked to investigate $\frac{dy}{dx} = 2e^{2x}$. Because $y = e^{2x} + c$, this is a series of an infinite number of exponential functions. We call this a family of curves. The functions with c values of 1, 0, –2 and –4, shown below, are four of the possible functions for $y = e^{2x} + c$.



A specific function can only be found if we are given some additional information to allow us to evaluate the constant, c. For example, we might be told that the curve passes through the origin. This lets us know that when $x = 0$, $y = 0$.

Hence, $$y = e^{2x} + c$$
$$(0, 0) \Rightarrow 0 = e^{2(0)} + c$$
$$0 = 1 + c$$
$$c = -1$$

Therefore, $$y = e^{2x} - 1$$

This additional information is referred to as an initial condition. The question could have been given as follows: 'If $f'(x) = 2e^{2x}$, find f given that $f(0) = 0$.'

WORKED EXAMPLE 9

a Sketch a family of curves that have the derivative function $f'(x) = 2\cos(2x)$ for $0 \le x \le 2\pi$.

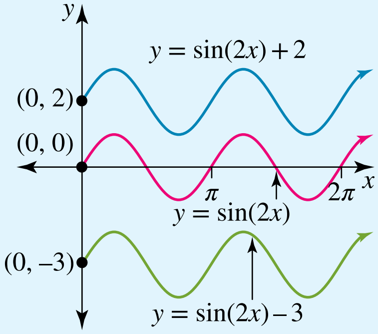
b Find the specific rule for this function if $f(\pi) = 2$.

THINK	WRITE/DRAW
a 1 Apply the rule $\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$ to antidifferentiate the function.	**a** $f(x) = \int 2\cos(2x)dx$ $= \sin(2x) + c$

2 Graph the function, first with $c = 0$. That is, sketch $f(x) = \sin(2x)$. Then translate this graph up or down to sketch the graphs with different c values. Any c values may be used.

$f(x) = \sin(2x) + c$

Amplitude $= 1$ and period $= \frac{2\pi}{2} = \pi$



b **1** Substitute the known point into the equation.	**b** $f(x) = \sin(2x) + c$ $f(\pi) = 2 \Rightarrow 2 = \sin(2\pi) + c$
2 Simplify and determine the value for c.	$2 = 0 + c$ $c = 2$
3 State the rule for $f(x)$.	$f(x) = \sin(2x) + 2$

WORKED EXAMPLE 10

Find the equation of the curve that passes through the point (1, 0) if the gradient is given by $\frac{dy}{dx} = 3x^2 - 2x + 2$.

THINK	WRITE
1 Write the gradient rule and antidifferentiate to find y.	$\frac{dy}{dx} = 3x^2 - 2x + 2$ $y = \int (3x^2 - 2x + 2)dx$ $= x^3 - x^2 + 2x + c$
2 Substitute the known point into the equation.	When $x = 1$, $y = 0$: $0 = 1 - 1 + 2 + c$ $c = -2$
3 State the rule for y.	$y = x^3 - x^2 + 2x - 2$

Application questions such as those involving rates of change may also be given in terms of the derivative function. Integrating the equation for the rate of change allows us to determine the original function.

WORKED EXAMPLE 11

A young boy bought an ant farm. It is known that the ant population is changing at a rate defined by $\frac{dN}{dt} = 20e^{0.2t}$, $0 \le t \le 20$, where N is the number of ants in the colony and t is the time in days since the ant farm has been set up.

a **Find a rule relating N to t if initially there were 50 ants.**

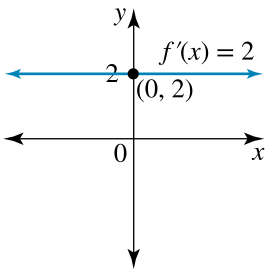
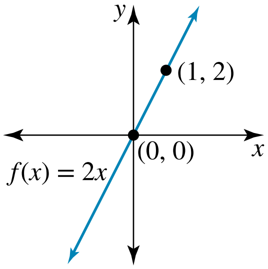
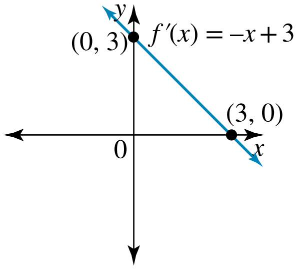
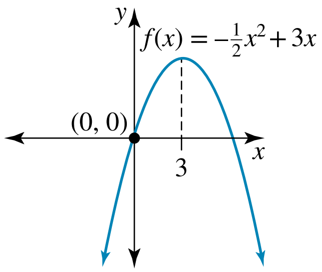
b **How many ants make up the colony after 8 days?**

THINK	WRITE
a **1** Write the rate rule and antidifferentiate to find the function for N.	**a** $\frac{dN}{dt} = 20e^{0.2t}$ $N = \int (20e^{0.2t})dt$ $= \frac{20}{0.2}e^{0.2t} + c$ $= 100e^{0.2t} + c$
2 Use the initial condition to determine the value of c.	When $t = 0$, $N = 50$: $50 = 100e^{0.2(0)} + c$ $50 = 100 + c$ $c = -50$
3 State the equation for N.	$N = 100e^{0.2t} - 50$
b **1** Substitute $t = 8$ into the population equation.	**b** When $t = 8$: $N = 100e^{0.2(8)} - 50$ $= 100e^{1.6} - 50$ $= 445.3$
2 Answer the question.	There are 445 ants after 8 days.

Sketching the graph of a function given the graph of $f'(x)$

When $f'(x)$ represents the equation of a polynomial function, the graph of $f(x)$ can be drawn by raising the degree of $f'(x)$ by one.

Note: When we sketch $f(x)$ from the gradient function, the '+ c' component is unknown, so there is not just one single answer. Often we will choose c to be zero, but any vertical translation of this general graph is correct.

Gradient function, $f'(x)$	Original function, $f(x)$
A line parallel to the x-axis ($y = m$) is degree 0. Example:  	A line of the form $y = mx + c$ is degree 1 and its gradient is m. 
A line of the form $y = mx + c$ is degree 1. Example: The line shown has an x-intercept at $x = 3$.  	A quadratic of the form $y = ax^2 + bx + c$ is degree 2. The function shown has a turning point at $x = 3$. 

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Topic 4
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Graphs of antiderivative functions
Concept summary
Practice questions

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Sketching the antiderivative graph
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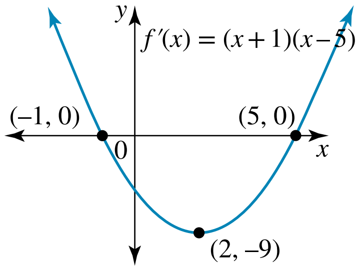
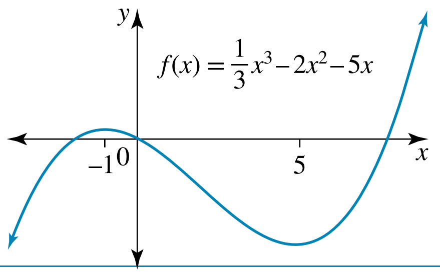
Gradient function, $f'(x)$	**Original function, $f(x)$**
A quadratic of the form $y = ax^2 + bx + c$ is degree 2. Example: The graph shown has x-intercepts at $x = -1$ and $x = 5$.  	A cubic of the form $y = ax^3 + bx^2 + cx + d$ is degree 3. The graph has turning points at $x = -1$ and $x = 5$. 

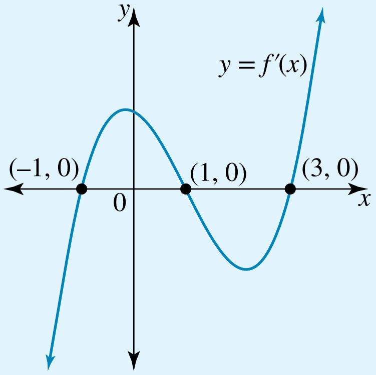
The derivative functions will not always be polynomial functions, so it is important to analyse the graph of the derivative carefully, as it will give you key information about the antiderivative graph.

Features of derivative graphs include the following:

- **x-intercepts on the graph of $y = f'(x)$ give the x-coordinates of stationary points on the graph of $y = f(x)$, that is, points where the derivative is zero.**
- **When the graph of $y = f'(x)$ is above the x-axis, this indicates the graph of $y = f(x)$ has a positive gradient for these x-values.**
- **When the graph of $y = f'(x)$ is below the x-axis, this indicates the graph of $y = f(x)$ has a negative gradient for these x-values.**

WORKED EXAMPLE 12

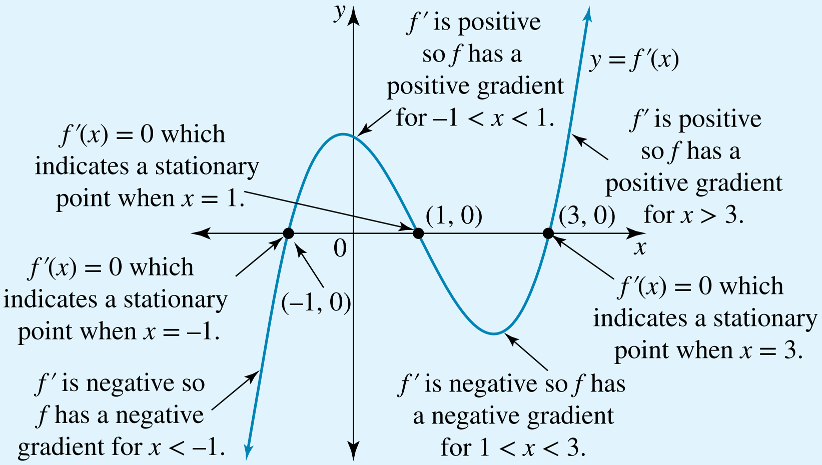
The graph of the gradient function $y = f'(x)$ is shown. Analyse this derivative function and sketch a possible graph for $y = f(x)$.



THINK

1 Use the derivative graph to determine the key features of f — where f' lies above or below the x axis (this indicates where f has positive or negative gradient) and where the x-intercepts are located (this indicates the position of stationary points on f). As $y = f'(x)$ is a positive cubic, $y = f(x)$ will be a positive quartic.

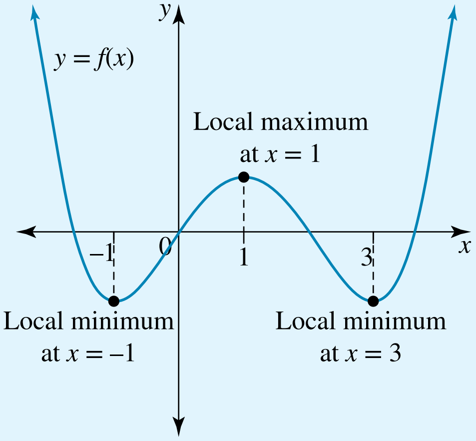
WRITE/DRAW



$y = f(x)$ has stationary points when $x = -1, 1, 3$.
The gradient of $y = f(x)$ is positive when $x \in (-1, 1) \cup (3, \infty)$.
The gradient of $y = f(x)$ is negative when $x \in (-\infty, -1) \cup (1, 3)$.

2 Use this knowledge to sketch the function $y = f(x)$.

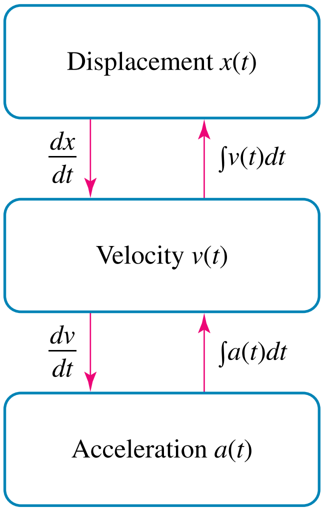
Note: In this graph of $y = f(x)$, we have chosen $c = 0$. Your graph may have a different c value. However, it should be of the same basic shape, just translated vertically.



Linear motion

From Topic 6 we know that the study of the motion of a particle in a straight line is called kinematics. When this motion is only in a straight line, it is referred to as rectilinear motion.

Displacement–velocity–acceleration relationship



Because velocity is the derivative of position (displacement) with respect to time, it follows that position is the antiderivative of velocity.

Consider a particle whose position, x metres, from the origin at time t seconds is defined by

$$x(t) = t^2 - 5t - 6,\ t \geq 0.$$

Initially, at $t = 0$, the particle is 6 metres to the left of the origin. The velocity of the particle can be defined as $v = x'(t) = 2t - 5$ metres/second.

The initial velocity of the particle is –5 metres/second.

This same situation could have been approached in the following way. A particle has an instantaneous velocity defined by

$$v = x'(t) = 2t - 5 \text{ metres/second.}$$

If it is known that the particle is initially 6 metres to the left of the origin, then the displacement can be given by

$$\begin{aligned} x &= \int (2t - 5)dt \\ &= t^2 - 5t + c \end{aligned}$$

When $t = 0,\ x = -6$: $\quad -6 = 0 + c$

$$c = -6$$

So: $\quad x = t^2 - 5t - 6$

WORKED EXAMPLE 13

In each of the following cases, find the displacement as a function of t if initially the particle is at the origin.

a $v = t^3 - t$ **b** $v = (2t - 3)^3$ **c** $v = \dfrac{1}{(t-1)^2}$

THINK	WRITE
a **1** Write the velocity equation and antidifferentiate to find the displacement function, x.	**a** $v = \frac{dx}{dt} = t^3 - t$ $x = \int (t^3 - t)dt$ $x = \frac{1}{4}t^4 - \frac{1}{2}t^2 + c$
2 Substitute the initial condition into the formula for x and determine c.	When $t = 0$, $x = 0$: $0 = 0 + c$ $c = 0$
3 State the rule.	$x = \frac{1}{4}t^4 - \frac{1}{2}t^2$
b **1** Write the velocity equation and antidifferentiate to find the displacement function, x.	**b** $v = \frac{dx}{dt}$ $= (2t - 3)^3$ $x = \int (2t - 3)^3 dt$ $= \frac{(2t-3)^4}{2(4)} + c$ $= \frac{1}{8}(2t - 3)^4 + c$
2 Substitute the initial condition into the formula for x and determine c.	When $t = 0$, $x = 0$: $0 = \frac{1}{8}(-3)^4 + c$ $0 = \frac{81}{8} + c$ $c = -\frac{81}{8}$
3 State the rule.	$x = \frac{1}{8}(2t - 3)^4 - \frac{81}{8}$
c **1** Write the velocity equation and antidifferentiate to find the displacement function, x.	**c** $v = \frac{dx}{dt}$ $= \frac{1}{(t-1)^2}$ $= (t - 1)^{-2}$ $x = \int (t - 1)^{-2} dt$ $= \frac{(t-1)^{-1}}{-1} + c$ $= -(t - 1)^{-1} + c$ $= -\frac{1}{(t-1)} + c$

2 Substitute the initial condition into the formula for x and determine c.	When $t = 0$, $x = 0$: $0 = -\frac{1}{(-1)} + c$ $0 = 1 + c$ $c = -1$
3 State the rule.	$x = -\frac{1}{(t-1)} - 1$

WORKED EXAMPLE 14

The velocity of a particle moving in a straight line along the x-axis is given by

$$v = \frac{dx}{dt} = 9 - 9e^{-3t}$$

where t is the time in seconds and x is the displacement in metres.

a Show that the particle is initially at rest.

b Find the equation relating x to t if it is known that initially the particle was 3 metres to the left of the origin.

THINK	WRITE
a 1 Substitute $t = 0$ and evaluate.	**a** $v = 9 - 9e^{-3t}$ $t = 0 \Rightarrow v = 9 - 9e^{0}$ $= 9 - 9 \times 1$ $= 0$ m/s
2 Answer the question.	Initially the particle is at rest as its velocity is 0 m/s.
b 1 Write the velocity equation and antidifferentiate to find the position equation, x.	**b** $v = \frac{dx}{dt}$ $= 9 - 9e^{-3t}$ $x = \int (9 - 9e^{-3t})dt$ $= 9t + 3e^{-3t} + c$
2 Substitute the initial condition to determine c. Remember, left of the origin means the position is negative.	When $t = 0$, $x = -3$: $-3 = 9 \times 0 + 3e^{0} + c$ $-3 = 3 + c$ $c = -6$
3 State the equation.	$x = 9t + 3e^{-3t} - 6$

EXERCISE 7.4 Families of curves and applications

PRACTISE

Work without CAS
Questions 1–4, 6–12

1 WE9 **a** Sketch a family of curves which have the derivative function $f'(x) = 3x^2$.

b Find the rule for the function that belongs to this family of curves and passes through the point (2, 16).

2 a Sketch a family of curves related to the derivative function $f'(x) = -2\cos(2x)$.

b Find the rule for the function that belongs to this family of curves and passes through the point $\left(\frac{\pi}{2}, 4\right)$.

3 **WE10** Find the equation of the curve that passes through the point (0, 3) if the gradient is given by $\frac{dy}{dx} = 2e^{2x} + e^{-x}$.

4 The gradient function of a particular curve is given by $f'(x) = \cos(2x) - \sin(2x)$ Find the rule for this function if it is known that the curve passes through the point $(\pi, 2)$.

5 **WE11** A chemical factory has permission from the Environment Protection Authority to release particular toxic gases into the atmosphere for a period of 20 seconds no more than once every 3 hours. This maintains safe levels of the gases in the atmosphere. This rate of emission is given by

$$\frac{dV}{dt} = 20t^2 - t^3 \text{ cm}^3\text{/s}$$

where $0 \leq t \leq 20$ and V cm^3 is the total volume of toxic gases released over t seconds. Find the total volume of toxic gases released during a 20-second release period.

6 The rate of change of volume of a balloon as it is being blown up can be modelled by

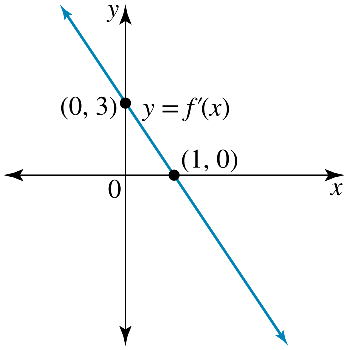
$$\frac{dV}{dr} = \pi r^2$$

where V cm^3 is the volume of the balloon and r cm is the radius of the balloon.

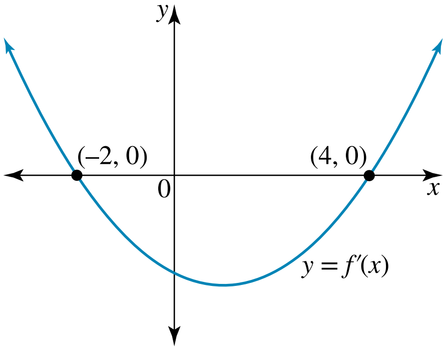
a Find the rule for the volume of the balloon.

b What is the volume of the balloon when its radius is 4 cm?

7 **WE12** The graph of the gradient function $y = f'(x)$ is shown. Analyse this derivative function and sketch the given function $y = f(x)$.



8 The graph of the gradient function $y = f'(x)$ is shown. Analyse this derivative function and sketch the given function $y = f(x)$.



9 **WE13** In each of the following cases, find the displacement as a function of t if initially the particle is at the origin.

a $v = (3t + 1)^{\frac{1}{2}}$

b $v = \frac{1}{(t + 2)^2}$

c $v = (2t + 1)^3$

10 Find the displacement of a particle that starts from the origin and has a velocity defined by:

a $v = e^{(3t-1)}$ **b** $v = -\sin(2t + 3)$.

11 **WE14** A particle is oscillating so that its velocity, v cm/s, can be defined by

$$v = \frac{dx}{dt} = \sin(2t) + \cos(2t)$$

where t is the time in seconds and x centimetres is its displacement.

a Show that initially the particle is moving at 1 cm/s.
b Find the equation relating x to t if it is known that initially the particle was at the origin.

12 When a bus travels along a straight road in heavy traffic from one stop to another stop, the velocity at time t seconds is given by $v = 0.25t(50 - t)$, where v is the velocity in m/s.

a Find the greatest velocity reached by the bus.
b Find the rule for the position of the bus, x metres, in terms of t.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

13 **a** Sketch a family of curves with the derivative function $f'(x) = 3e^{-3x}$.
b Find the rule for the function that belongs to this family of curves and passes through the point (0, 1).

14 Find the antiderivative of $\cos(2x) + 3e^{-3x}$ if $y = 4$ when $x = 0$.

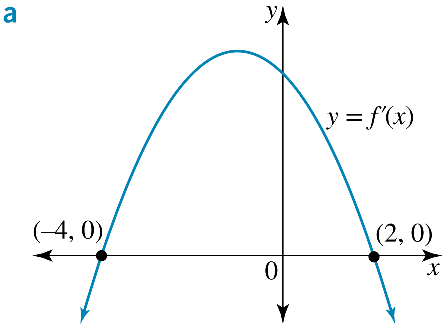
15 Find the equation of the curve defined by $\frac{dy}{dx} = e^{\frac{1}{2}x}$, given that it passes through the point (0, 5).

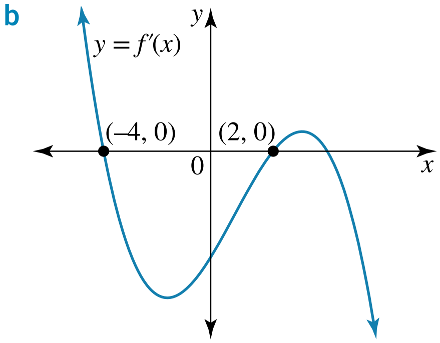
16 Find $f(x)$ for each of the following.

a $f'(x) = 5 - 2x$ and $f(1) = 4$ **b** $f'(x) = \sin\left(\frac{x}{2}\right)$ and $f(\pi) = 3$

c $f'(x) = \frac{1}{(1-x)^2}$ and $f(0) = 4$

17 The graphs of two gradient functions are shown. Sketch the corresponding antiderivative graphs.





18 A particle moves in a straight line so that its velocity, in metres per second, can be defined by the rule $v = 3t^2 + 7t$, $t \geq 0$. Find the rule relating the position of the particle, x metres, to t, if it is known that the particle started from the origin.

19 A particle attached to a spring moves up and down in a straight line so that at time t seconds its velocity, v metres per second, is given by

$$v = 3\pi \sin\left(\frac{\pi t}{8}\right), t \geq 0.$$

Initially the particle is stationary.

a Find the rule relating the position of the particle, x centimetres, to t.

b What is the maximum displacement of the particle?

c Where is the particle, relative to the stationary position, after 4 seconds?

20 A particle starting from rest at the origin moves in a straight line with a velocity of $\frac{12}{(t-1)^2} + 6$ metres per second after t seconds.

a Find the rule relating the position of the particle, x metres, to t.

b Find the position of the particle after 3 seconds.

21 A population of sea lions on a distant island is growing according to the model

$$\frac{dP}{dt} = 30e^{0.3t},\ 0 \le t \le 10$$

where P is the number of sea lions present after t years.

a If initially there were 50 sea lions on the island, find the rule for the number of sea lions present, P, after t years.

b Find the number of sea lions on the island after 10 years. Give your answer correct to the nearest whole sea lion.

22 The rate of change of the depth of water in a canal is modelled by the rule

$$\frac{dh}{dt} = \frac{\pi}{2}\cos\left(\frac{\pi t}{4}\right)$$

where h is the height of the water in metres and t is the number of hours since 6 am.

a Find an expression for h in terms of t if the water is 3 metres deep at 6 am.

b What are the maximum and minimum depths of the water?

c For how many hours a day is the water level 4 metres or more?

MASTER

23 A newly established suburban area of Perth is growing at a rate modelled by the rule

$$\frac{dN}{dt} = 400 + 1000\sqrt{t},\ 0 \le t \le 10$$

where N is the number of families living in the suburb t years after the suburb was established in 2015.

a Find a rule relating N and t if initially there were 40 families living in this suburb.

b How many families will be living in the suburb 5 years after its establishment? Give your answer correct to the nearest number of families.

24 If $v = 2t\cos(t)$ metres per second, find a rule relating the position x metres to t if it is known the particle starts from rest at the origin.

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7.5 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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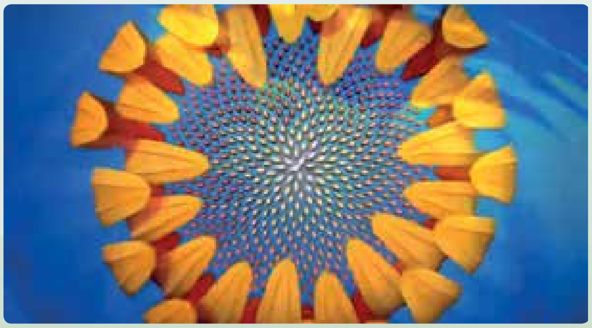
Activities

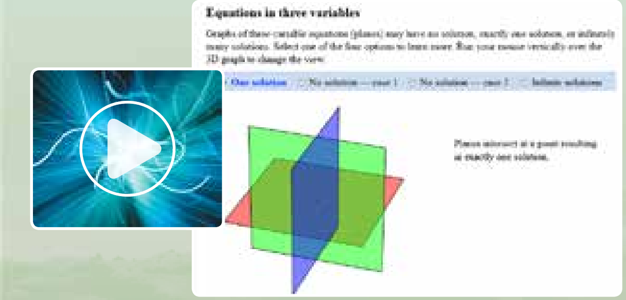
To access eBookPLUS activities, log on to

www.jacplus.com.au

Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

7 Answers

EXERCISE 7.2

1 a $\frac{3x^2}{4} - \frac{4x^3}{3} + \frac{x^4}{2} + c$

b $6\sqrt{x} - x^4 - \frac{1}{5x^2}$

c $\frac{x^4}{2} - \frac{x^3}{3} - \frac{15x^2}{2} + c$

d $\frac{3x^{\frac{7}{2}}}{7} - \frac{x^{\frac{3}{2}}}{3} + c$

2 a $4\sqrt{x} - \frac{3}{x} + \frac{1}{4x^2} + c$

b $\frac{1}{2}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x + c$

3 a $\frac{(3x-5)^6}{18} + c$

b $-\frac{1}{3(2x-3)^{\frac{3}{2}}} + c$

4 a $\frac{(2x+3)^5}{10} + c$

b $\frac{1}{8(1-2x)^4} + c$

5 $\frac{8x^7}{7} + 3x^4 + 6x - \frac{1}{2x^2} + c$

6 a $\frac{x^3}{3} - \frac{4x^{\frac{5}{2}}}{5} + \frac{x^2}{2} + c$

b $\frac{2}{5}x^2\sqrt{x} + 2x\sqrt{x} + 6\sqrt{x} - \frac{2}{\sqrt{x}}$

7 $\frac{dy}{dx} = 6(3x+1)(3x^2+2x-4)^2$

$$\int(3x+1)(3x^2+2x-4)^2 = \frac{1}{6}(3x^2+2x-4)^3$$

8 $\frac{dy}{dx} = 4\left(7 + \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}\right)\left(7x + \sqrt{x} - \frac{1}{\sqrt{x}}\right)^3$

$$\int\left(7 + \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}\right)\left(7x + \sqrt{x} - \frac{1}{\sqrt{x}}\right)^3 dx$$

$$= \frac{1}{4}\left(7x + \sqrt{x} - \frac{1}{\sqrt{x}}\right)^4$$

9 $f(x) = \frac{1}{3}x^3 + \frac{1}{x} + c$

10 a $\frac{1}{4}x^4 + c$

b $\frac{7}{3}x^3 + \frac{1}{5x^2} + c$

c $x^4 - \frac{7}{3}x^3 + x^2 - x + c$

d $\frac{16}{5}x^2\sqrt{x} + c$

11 a $\frac{27}{4}x^4 - 9x^3 + \frac{9}{2}x^2 - x + c$

b $-\frac{1}{8x^2} + c$

c $\frac{2}{7}x^{\frac{7}{2}} - \frac{15}{7}x^{\frac{7}{5}} + c$

d $\frac{1}{2}x^2 + \frac{2}{x} + c$

e $\frac{4x^{\frac{5}{2}}}{5} - \frac{1}{2}x^2 + c$

f $-\frac{2(4-x)^{\frac{3}{2}}}{3} + c$

12 a $2x^3 + \frac{5}{2}x^2 - 6x$

b $\frac{1}{2}x^2 + x - \frac{1}{x}$

c $\frac{4}{3}x^{\frac{3}{2}} - 8x^{\frac{1}{2}}$

d $\frac{1}{7}x^7 - 4x - \frac{4}{5x^5}$

e $\frac{1}{4(1-4x)^2}$

f $-\frac{2}{3(2x-3)^{\frac{3}{2}}}$

13 $y = \frac{1}{4}x^4 - 2x\sqrt{x} + c$

14 $y = \frac{1}{2}x^2 + 3x + \frac{3}{x} + c$

15 $y = \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + c$

16 $\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$

$$\int\frac{5x}{\sqrt{x^2+1}} = 5\sqrt{x^2+1} + c$$

17 $\frac{dy}{dx} = 8(5x+1)(5x^2+2x-1)^3$

$$\int 16(5x+1)(5x^2+2x-1)^3 dx = 2(5x^2+2x-1)^4$$

18 $\frac{dy}{dx} = \frac{15x^2+8x}{2\sqrt{5x^3+4x^2}}$

$$\int\frac{15x^2+8x}{\sqrt{5x^3+4x^2}}dx = 2\sqrt{5x^3+4x^2}$$

19 $\frac{2}{3}\sqrt{x^3+1} + c$

20 $\frac{4}{405}(3x+5)^{\frac{3}{2}}(135x^2-72x+35) + c$

EXERCISE 7.3

1 a $\frac{1}{6}\sin(3x+4) + 8\cos\left(\frac{x}{2}\right) + c$

b $\frac{3}{2}\sin\left(\frac{2x}{3}\right) - \frac{1}{8}\cos(5-2x)$

2 a $-2\cos\left(\frac{x}{2}\right) - 6\sin\left(\frac{x}{2}\right) + c$

b $f(x) = \frac{7}{2}\sin(2x) + \frac{1}{3}\cos(3x) + c$

3 a $\frac{x^5}{5} + \frac{1}{4}e^{-4x} + c$

b $\frac{1}{4}e^{2x} + \frac{4}{3}e^{-\frac{1}{2}x} + c$

4 a $3e^{\frac{x}{2}} - 3\cos\left(\frac{x}{3}\right) + \frac{x^2}{6} + c$

b $\frac{1}{4}\sin(4x) - e^{-3x} + c$

5 $\frac{1}{6}e^{6x} - 3e^x - \frac{3}{4}e^{-4x} + \frac{1}{9}e^{-9x} + c$

6 $e^x + 4e^{-\frac{x}{2}} - \frac{1}{2}e^{-2x} + c$

7 $\dfrac{dy}{dx} = -2\sin(x)\cos(x)e^{\cos^2(x)}$

$\int \sin(x)\cos(x)e^{\cos^2(x)} = -\frac{1}{2}e^{\cos^2(x)}$

8 $\dfrac{dy}{dx} = 3(x+1)^2e^{(x+1)^3}$

$\int 9(x+1)^2e^{(x+1)^3}dx = 3e^{(x+1)^3} + c$

9 a $\frac{2}{3}e^{3x} + \frac{1}{2}\cos(2x) + c$ **b** $\frac{1}{2}e^{x} - \frac{1}{4}e^{-6x} + c$

c $\frac{1}{4}\sin(2x+5) + e^{-x} + c$ **d** $\frac{1}{2}e^{2x} - \frac{2}{3}e^{3x} + \frac{1}{4}e^{4x} + c$

10 $a = -9,\ b = 3$

11 $-\dfrac{1}{4x} - \dfrac{2}{3\pi}\cos\left(\dfrac{3\pi x}{2}\right)$

12 $y = \frac{1}{2}\sin(2x) + \dfrac{e^{-3x}}{3} + c$

13 a $H(t) = t - 5\pi\cos\left(\dfrac{\pi t}{45}\right)$

b $H = 7.146$ kilojoules

14 $\dfrac{dy}{dx} = 6xe^{3x} + 2e^{3x}$

$\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$

15 $\dfrac{dy}{dx} = (4x+3)e^{2x^2+3x-1}$

$\int 2(4x+3)e^{2x^2+3x-1}dx = 2e^{2x^2+3x-1}$

16 $\dfrac{dy}{dx} = -x\sin(x) + \cos(x)$

$\int x\sin(x)dx = \sin(x) - x\cos(x)$

17 $\dfrac{dy}{dt} = \dfrac{\pi}{20}\cos\left(\dfrac{\pi t}{4}\right)$ $y = \dfrac{1}{5}\sin\left(\dfrac{\pi t}{4}\right) + c$

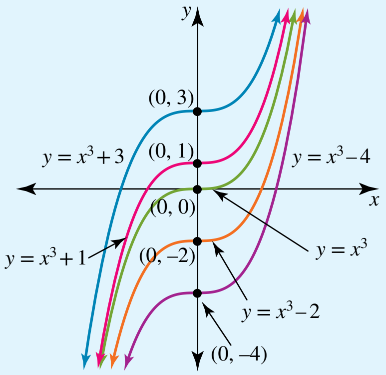
18 $a = -2,\ b = -4,\ m = 2,\ n = -2$

19 $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

20 $e^x - x + \log_e(e^x + 1) + c$

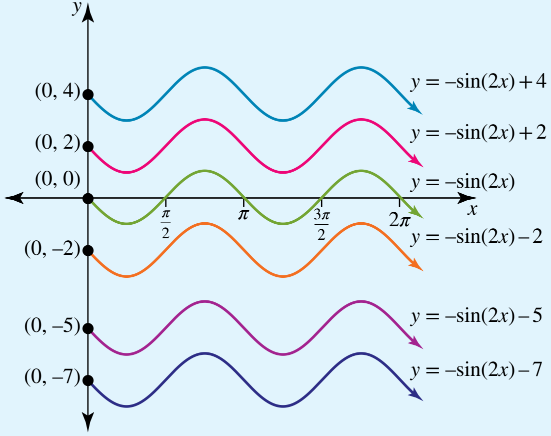
EXERCISE 7.4

1 a $f(x) = x^3 + c$



b $f(x) = x^3 + 8$

2 a $f(x) = -\sin(2x) + c$



b $f(x) = 4 - \sin(2x)$

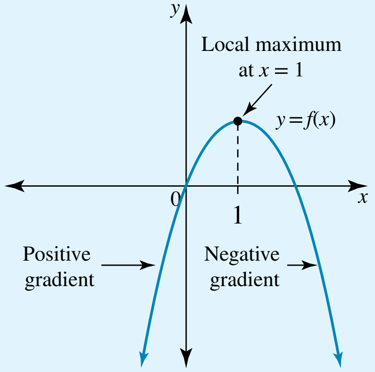
3 $e^{2x} - e^{-x} + 3$

4 $\frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x) + \frac{3}{2}$

5 $V = 13\,333\frac{1}{3}\text{ cm}^3$

6 a $V = \dfrac{\pi}{3}r^3$ **b** $\dfrac{64\pi}{3}\text{ cm}^3$

7



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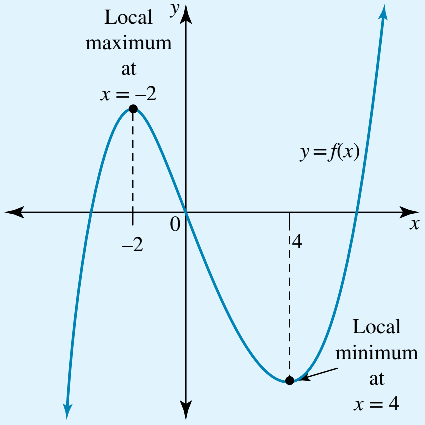


9 a $x = \frac{2}{9}\sqrt{(3t+1)^3} - \frac{2}{9}$

b $x = \frac{1}{2} - \dfrac{1}{t+2}$

c $x = \frac{1}{8}(2t+1)^4 - \frac{1}{8}$

10 a $x = \frac{1}{3}e^{3t-1} - \dfrac{1}{3e}$

b $x = \frac{1}{2}\cos(2t+3) - \frac{1}{2}\cos(3)$

11 a $v = \sin(2t) + \cos(2t)$

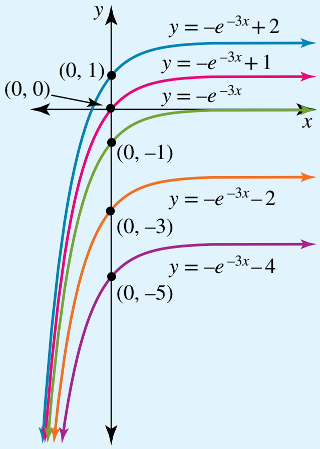
$$t = 0 \Rightarrow v = \sin(0) + \cos(0)$$
$$= 0 + 1$$
$$= 1 \text{ cm/s}$$

b $x = -\frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t) + \frac{1}{2}$

12 a 156.25 m/s

b $x = 6.25t^2 - \frac{1}{12}t^3$

13 a $f(x) = -e^{-3x} + c$



b $f(x) = 2 - e^{-3x}$

14 $y = \frac{1}{2}\sin(2x) - e^{-3x} + 5$

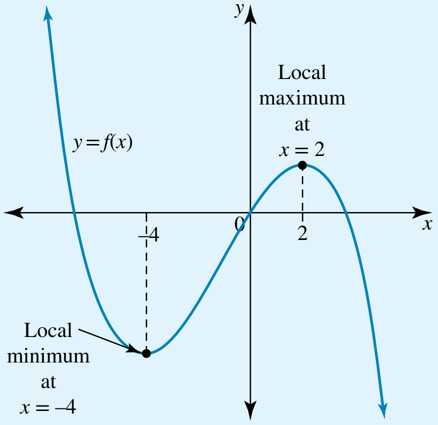
15 $y = 2e^{\frac{x}{2}} + 3$

16 a $f(x) = 5x - x^2$

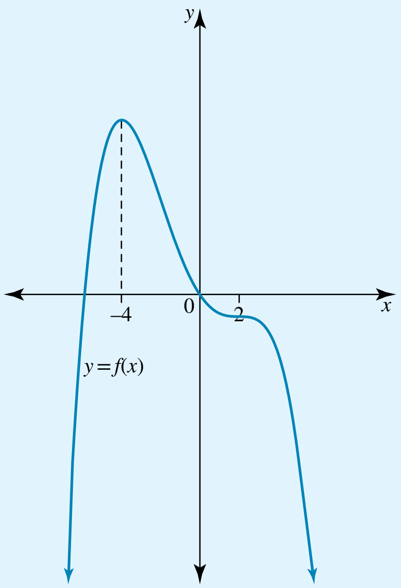
b $f(x) = 3 - 2\cos\left(\frac{x}{2}\right)$

c $f(x) = \frac{1}{1 - x} + 3$

17 a



b



18 $x = t^3 + \frac{7}{2}t^2$

19 a $x = 24 - 24\cos\left(\frac{\pi t}{8}\right)$

b Maximum displacement = 48 metres

c After 4 seconds the particle is 24 metres above the stationary position.

20 a $x = 6t - \frac{12}{(t - 1)} - 12$

b At the origin

21 a $P = 100e^{0.3t} - 50$

b 1959 seals

22 a $h = 2\sin\left(\frac{\pi t}{4}\right) + 3$

b Minimum depth is 1 metre and maximum depth is 5 metres.

c 8 hours a day

23 a $N = 400t + \frac{2000}{3}\sqrt{t^3} + 40$

b 9494 families

24 $x = 2t\sin(t) + 2\cos(t) - 2$

8

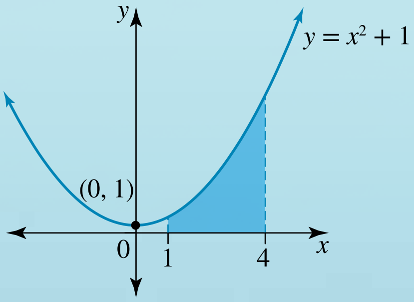
Integration

8.1 Kick off with CAS

Area under curves

1 a Using the graph application on CAS, sketch the graph of $f(x) = x^2 + 1$.

b Estimate the area enclosed by the x-axis, the vertical lines $x = 1$ and $x = 4$, and the curve.



c Using the calculation application, find the integral template and calculate $\int_1^4 (x^2 + 1)dx$.

d What do you notice about the answers to parts **b** and **c**?

2 a Sketch the graph of $f(x) = x^3$.

b Using one calculation, find the area enclosed by the x-axis, the vertical lines $x = -3$ and $x = 3$, and the curve.

c How do you explain your answer? Is there a method you can use to obtain the correct area under the curve?

d Using the calculation application, how could you apply the integral template to calculate the area?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

8.2 The fundamental theorem of integral calculus

Estimation of the area under a curve

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Concept 4

Approximating areas under curves

Concept summary

Practice questions

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Interactivity

Estimation of area under a curve

int-6422

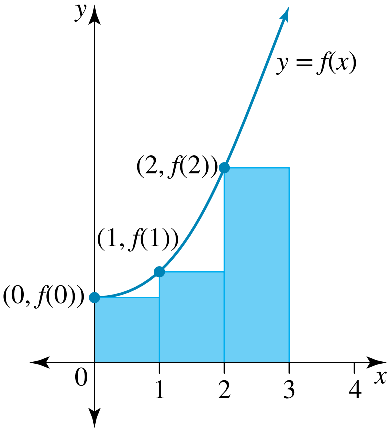
There are several different ways to approximate or estimate the area between a curve and the x-axis. This section will cover the left end-point rectangle rule and the right end-point rectangle rule.

The left end-point rectangle rule

Consider the curve defined by the rule $f: R \to R, f(x) = x^2 + 2$.

Suppose we wish to know the area between this curve and the x-axis from $x = 0$ to $x = 3$. This can be achieved by constructing rectangles of width 1 unit and height such that the top left corner touches the curve.

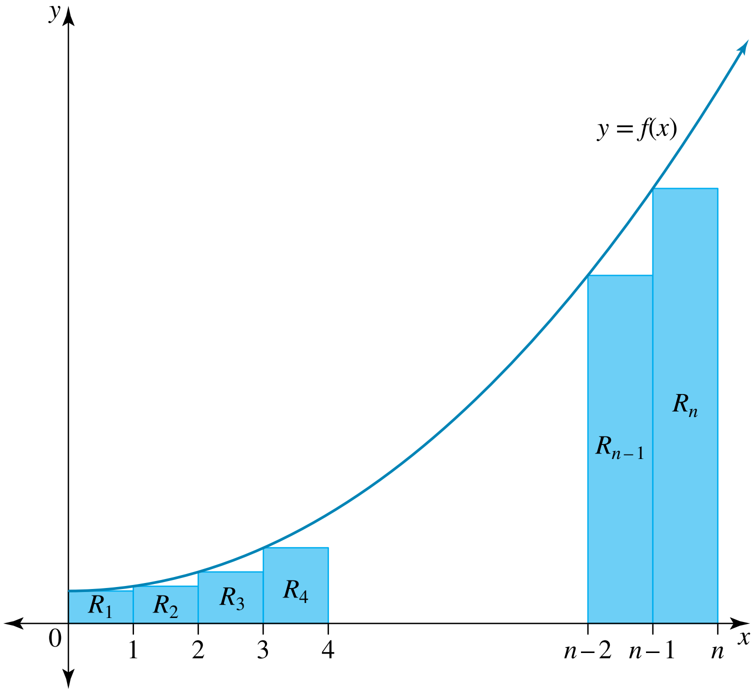
The height of the first rectangle is obtained by evaluating $f(0)$, the height of the second rectangle is obtained by evaluating $f(1)$, and the height of the third rectangle is obtained by evaluating $f(2)$. Each rectangle has a width of 1 unit.



Approximate area:

$$
\begin{aligned}
A &= 1 \times f(0) + 1 \times f(1) + 1 \times f(2) \\
&= f(0) + f(1) + f(2) \\
&= 2 + 3 + 6 \\
&= 11
\end{aligned}
$$

The theoretical explanation of this method can be explained for the general function $y = f(x)$ as follows.



The height of R_1 is $f(0)$.

The height of R_2 is $f(1)$.

The height of R_3 is $f(2)$.

The height of R_4 is $f(3)$.

The height of R_{n-1} is $f(n - 2)$.

The height of R_n is $f(n - 1)$.

$$
\begin{aligned}
\text{Approximate area} &= 1 \times f(0) + 1 \times f(1) + 1 \times f(2) + 1 \times f(3) + \ldots \\
&\quad + 1 \times f(n - 2) + 1 \times f(n - 1) \\
&= f(0) + f(1) + f(2) + f(3) + \ldots + f(n - 2) + f(n - 1)
\end{aligned}
$$

If the same area was approximated using rectangles of width 0.5 units, there would be twice as many rectangles, so

$$A \simeq 0.5 \times f(0) + 0.5 \times f(0.5) + 0.5 \times f(1) + \ ... \ + 0.5 \times f(n - 1.5)$$
$$+ 0.5 \times f(n - 1) + 0.5 \times f(n - 0.5).$$

The narrower the rectangles are, the closer the approximation is to the actual area under the curve.

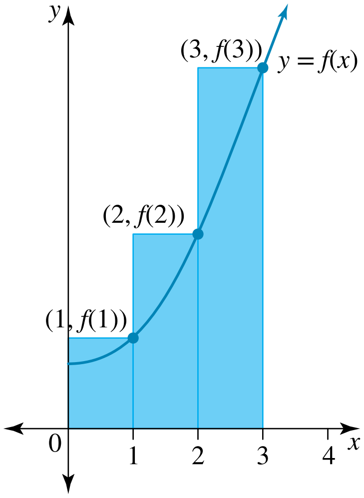
In the example shown, the approximate area is less than the actual area. However, if the function was a decreasing function, for example $y = -x^2$, then the area of the left end-point rectangles would be greater than the actual area under the curve.

The right end-point rectangle rule

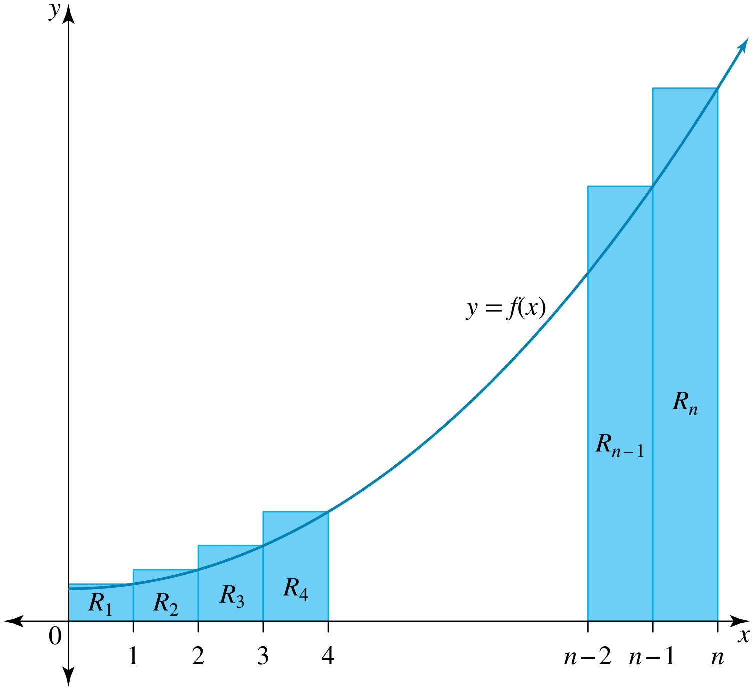
We will now approximate the area between the curve $f(x) = x^2 + 2$, the x-axis, $x = 0$ and $x = 3$ by constructing rectangles of width 1 unit and height such that the top right corner touches the curve.

The height of the first rectangle is obtained by evaluating $f(1)$, the height of the second rectangle is obtained by evaluating $f(2)$, and the height of the third rectangle is obtained by evaluating $f(3)$. Each rectangle has a width of 1 unit.

$$\begin{aligned} \text{Approximate area, } A &= 1 \times f(1) + 1 \times f(2) + 1 \times f(3) \\ &= f(1) + f(2) + f(3) \\ &= 3 + 6 + 11 \\ &= 20 \end{aligned}$$



The theoretical explanation of this method can be explained for the general function $y = f(x)$ as follows.



The height of R_1 is $f(1)$.

The height of R_2 is $f(2)$.

The height of R_3 is $f(3)$.

The height of R_4 is $f(4)$.

The height of R_{n-1} is $f(n - 1)$.

The height of R_n is $f(n)$.

$$\begin{aligned} \text{Approximate area, } A &= 1 \times f(1) + 1 \times f(2) + 1 \times f(3) + 1 \times f(4) + ... \\ &\quad + 1 \times f(n - 1) + 1 \times f(n) \\ &= f(1) + f(2) + f(3) + f(4) + \ ... + f(n - 1) + f(n) \end{aligned}$$

The narrower the rectangles are, the closer the approximation is to the actual area under the curve.

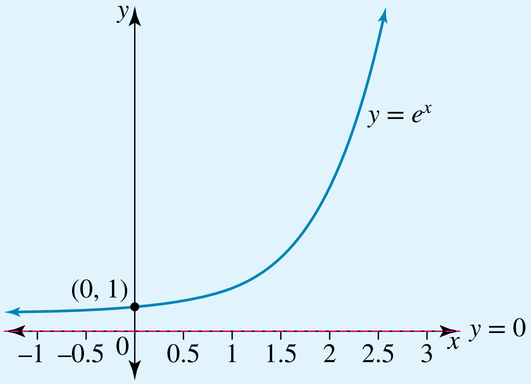
In the example shown, the approximate area is greater than the actual area. However, if the function was a decreasing function, say $y = -x^2$, then the area of the right end-point rectangles would be less than the actual area under the curve.

WORKED EXAMPLE 1

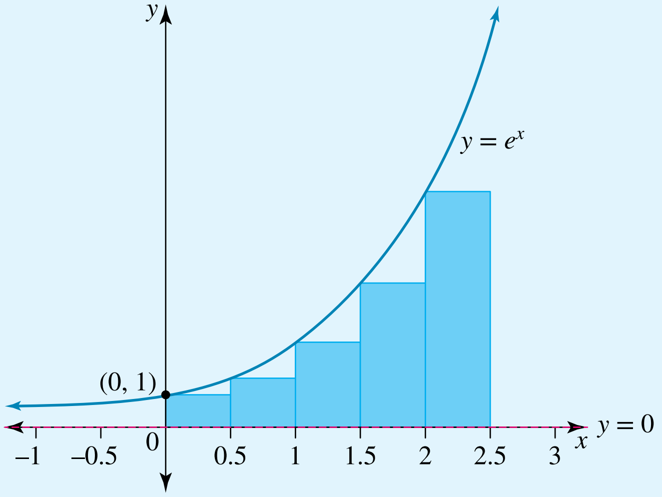
The graph of the function defined by the rule $f(x) = e^x$ is shown. Give your answers to the following correct to 2 decimal places.

a **Use the left end-point rectangle method with rectangles of width 0.5 units to estimate the area between the curve and the x-axis from $x = 0$ to $x = 2.5$.**

b **Use the right end-point rectangle method with rectangles of width 0.5 units to estimate the area between the curve and the x-axis from $x = 0$ to $x = 2.5$.**



THINK	WRITE/DRAW
a **1** Draw the left end-point rectangles on the graph. State the widths and heights of the rectangles.	**a**



Rectangle widths = 0.5 units
Rectangle heights are $f(0), f(0.5), f(1), f(1.5)$ and $f(2)$.

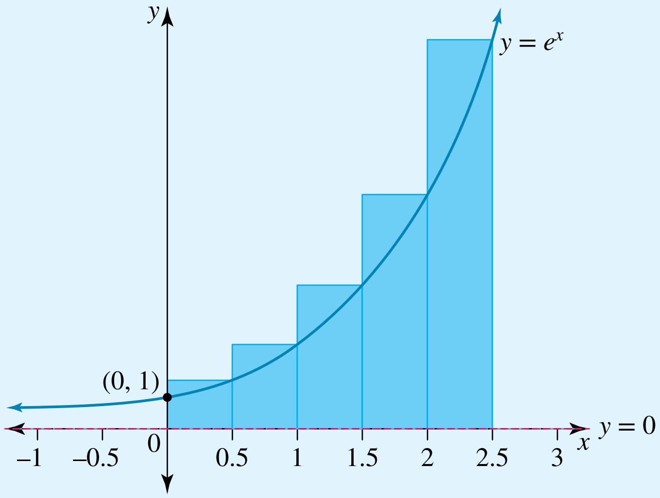
2 Find the approximate area by adding the areas of all the rectangles.

$$\begin{aligned} A &= 0.5f(0) + 0.5f(0.5) + 0.5f(1) \\ &\quad + 0.5f(1.5) + 0.5f(2) \\ &= 0.5\left[f(0) + f(0.5) + f(1) + f(1.5) + f(2)\right] \\ &= 0.5\left[e^0 + e^{0.5} + e^1 + e^{1.5} + e^2\right] \\ &= 0.5 \times 17.2377 \\ &\approx 8.62 \end{aligned}$$

The area is approximately 8.62 units^2.

b 1 Draw the right end-point rectangles on the graph. State the widths and heights of the rectangles.

b



Rectangle widths = 0.5 units

Rectangle heights are $f(0.5), f(1), f(1.5), f(2)$ and $f(2.5)$.

2 Find the approximate area by adding the areas of all the rectangles.

$$\begin{aligned}A &= 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) \\ &\quad + 0.5f(2) + 0.5f(2.5) \\ &= 0.5\left[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)\right] \\ &= 0.5\left[e^{0.5} + e^1 + e^{1.5} + e^2 + e^{2.5}\right] \\ &= 0.5 \times 28.4202 \\ &\approx 14.21\end{aligned}$$

The area is approximately 14.21 units^2.

The definite integral

The **definite integral**, $\int_a^b f(x)dx$, is similar to the indefinite integral, $\int f(x)dx$, except that it has end points, or terminals, a and b. The **indefinite integral** involves finding only an antiderivative of f, but the presence of the end points means that the definite integral requires further calculation involving these values. In fact, the end points a and b indicate the range of the values of x over which the integral is taken.

Consider $\int_{-1}^{1} (1 - x^2)dx$

$$\begin{aligned}&= \left[x - \tfrac{1}{3}x^3\right]_{-1}^{1} \\ &= \left(1 - \tfrac{1}{3}(1)^3\right) - \left(-1 - \tfrac{1}{3}(-1)^3\right) \\ &= 1 - \tfrac{1}{3} + 1 - \tfrac{1}{3} \\ &= \tfrac{4}{3}\end{aligned}$$

Note: For the definite integral, no arbitrary constant is required for the antidifferentiation, as this would only be eliminated once the end points were used in the calculation.

$$= \left[x - \tfrac{1}{3}x^3 + c\right]_{-1}^{1}$$

$$= \left(1 - \tfrac{1}{3}(1)^3 + c\right) - \left(-1 - \tfrac{1}{3}(-1)^3 + c\right)$$

$$= 1 - \frac{1}{3} + c + 1 - \frac{1}{3} - c$$

$$= 2 - \frac{2}{3}$$

$$= \frac{4}{3}$$

Properties of the definite integral

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Units 3 & 4
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Properties of definite integrals
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Practice questions

If f and g are continuous functions on an interval where $a < x < b$ and k is a constant, then the following rules apply.

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \text{ providing } a < c < b.$$

WORKED EXAMPLE 2

Evaluate:

a $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)dx$

b $\displaystyle\int_0^{2} (e^{-x} + 2)dx.$

THINK	WRITE
a 1 Antidifferentiate the given function and specify the end points for the calculation using square brackets.	**a** $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)dx$ $= \left[\sin(x)\right]_0^{\frac{\pi}{2}}$

2 Substitute the upper and lower end points into the antiderivative and calculate the difference between the two values.	$= \sin\left(\frac{\pi}{2}\right) - \sin(0)$ $= 1 - 0$ $= 1$
b 1 Antidifferentiate the given function and specify the end points for the calculation using square brackets.	**b** $\int_0^2 (e^{-x} + 2)dx$ $= \left[-e^{-x} + 2x\right]_0^2$
2 Substitute the upper and lower end points into the antiderivative and calculate the difference between the two values.	$= (-e^{-2} + 2(2)) - (-e^{0} + 2(0))$ $= -\frac{1}{e^2} + 4 + 1$ $= -\frac{1}{e^2} + 5$

Sometimes, definite integral questions take more of a theoretical approach to problem solving. Even if the function is unknown, we can use the properties of definite integrals to find the values of related integrals.

WORKED EXAMPLE 3

a **Given that $\int_1^3 f(x)dx = 8$, find:**

i $\int_1^3 2f(x)dx$

ii $\int_1^3 (f(x) + 1)dx$

iii $\int_3^1 f(x)dx$

iv $\int_1^3 (f(x) - x)dx$.

b **Find k if $\int_1^k (x + 2)dx = 0$.**

THINK	WRITE
a i Apply the definite integral property $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.	**a i** $\int_1^3 2f(x)dx = 2\int_1^3 f(x)dx$ $= 2 \times 8$ $= 16$
ii 1 Apply the definite integral property: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$	**ii** $\int_1^3 (f(x) + 1)dx = \int_1^3 f(x)dx + \int_1^3 1dx$

2 Integrate the second function and evaluate.

$$= 8 + \Big[x\Big]_1^3$$
$$= 8 + (3 - 1)$$
$$= 10$$

iii Apply the definite integral property:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

iii $$\int_3^1 f(x)dx = -\int_1^3 f(x)dx$$
$$= -8$$

iv **1** Apply the definite integral property:

$$\int_a^b \Big(f(x) \pm g(x)\Big)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

iv $$\int_1^3 (f(x) - x)dx = \int_1^3 f(x)dx - \int_1^3 xdx$$

2 Integrate the second function and evaluate.

$$= 8 - \left[\tfrac{1}{2}x^2\right]_1^3$$
$$= 8 - \left(\tfrac{1}{2}(3)^2 - \tfrac{1}{2}(1)^2\right)$$
$$= 8 - \left(\tfrac{9}{2} - \tfrac{1}{2}\right)$$
$$= 8 - 4$$
$$= 4$$

b **1** Antidifferentiate and substitute the values of 1 and k.

b $$0 = \int_1^k (x + 2)dx$$
$$0 = \left[\tfrac{1}{2}x^2 + 2x\right]_1^k$$
$$0 = \left(\tfrac{1}{2}k^2 + 2k\right) - \left(\tfrac{1}{2}(1)^2 + 2(1)\right)$$

2 Simplify and solve for k.

$$0 = \tfrac{1}{2}k^2 + 2k - \tfrac{5}{2}$$
$$0 = k^2 + 4k - 5$$
$$0 = (k + 5)(k - 1)$$
$$k = -5 \quad \text{or} \quad k = 1$$

3 Write the answer.

$$k = 1, -5$$

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Interactivity
The fundamental theorem of integral calculus
int-6423

The fundamental theorem of integral calculus

In this section, the variable t is used and the function f is defined as a continuous function on the interval $[a, b]$ where $x \in [a, b]$. $A(x)$ is defined as

$$A(x) = \int_a^x f(t)dt$$

where $A(x)$ is the area between the curve $y = f(x)$ and the t-axis from $t = a$ to $t = x$. $A(x + \delta x)$ represents the area between the curve $y = f(x)$ and the t-axis from $t = a$ to $t = x + \delta x$.

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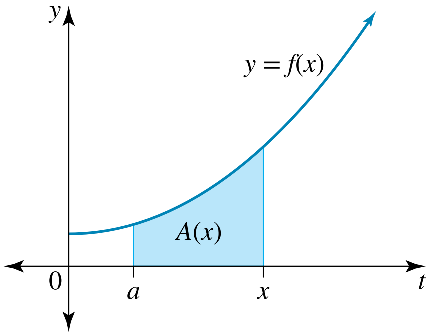
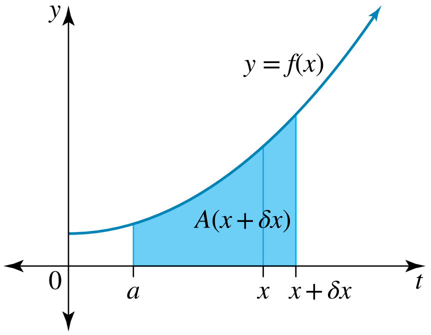
The fundamental theorem of calculus
Concept summary
Practice questions

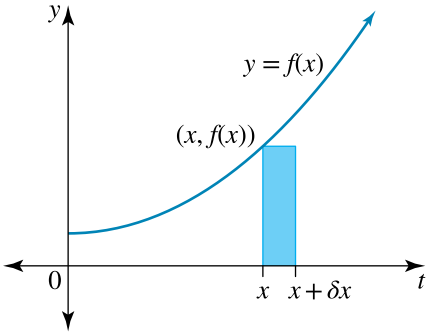


Area between the curve, the *t*-axis and the lines *t* = *a* and *t* = *x*

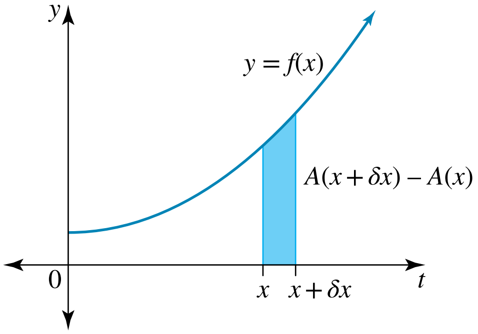


Area between the curve, the *t*-axis and the lines *t* = *a* and *t* = *x* + δ*x*

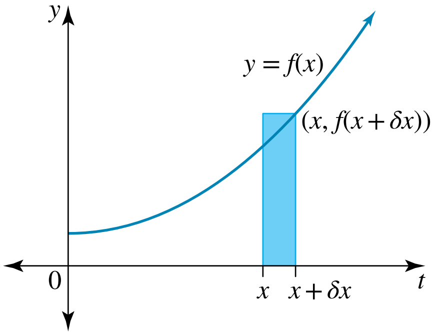
The difference between the areas is therefore $A(x + \delta x) - A(x)$. As we have already seen, this area lies between the areas of the left end-point rectangle and the right end-point rectangle.



Left end-point rectangle method



Actual area under the curve



Right end-point rectangle method

Therefore, $f(x)\delta x \le A(x + \delta x) - A(x) \le f(x + \delta x)\delta x$.

If we divide by δx, we have

$$f(x) \le \frac{A(x + \delta x) - A(x)}{\delta x} \le f(x + \delta x).$$

Of course, the width of each rectangular strip becomes smaller and smaller as $\delta x \to 0$, which results in an increasingly accurate area calculation between the curve and the *t*-axis. This concept is a limiting situation.

By definition,

$$\lim_{\delta x \to 0} \frac{A(x + \delta x) - A(x)}{\delta x} = \frac{d}{dx}(A(x))$$

and as $\delta x \to 0$, then $f(x + \delta x) \to f(x)$.

Consequently, we can say that

$$\frac{d}{dx}(A(x)) = f(x).$$

If we then integrate both sides with respect to x, we have

$$\int \frac{d}{dx}(A(x))dx = \int f(x)dx \qquad \text{or} \qquad A(x) = \int f(x)dx.$$

To further investigate this theorem, we will let F be any antiderivative of f, and A be the special antiderivative defined as $\int_a^x f(t)dt$.

From our knowledge of antidifferentiation

$$A(x) - F(x) = c \text{ where } c \text{ is a number.}$$

Therefore, $\displaystyle\int_a^x f(t)dt - F(x) = c.$

If we let $x = a$, then $\displaystyle\int_a^a f(t)dt = 0$, so

$$0 - F(a) = c \quad \text{or} \quad -F(a) = c.$$

Therefore, $\displaystyle\int_a^x f(t)dt - F(x) = -F(a)$

If we now let $x = b$, then

$$\int_a^b f(t)dt - F(b) = -F(a) \quad \text{or} \quad \int_a^b f(t)dt = F(b) - F(a).$$

It is customary that $F(b) - F(a)$ is represented by $\Big[F(x)\Big]_a^b$.
Therefore,

$$\begin{aligned}\int_a^b f(t)dt &= \Big[F(x)\Big]_a^b \\ &= F(b) - F(a)\end{aligned}$$

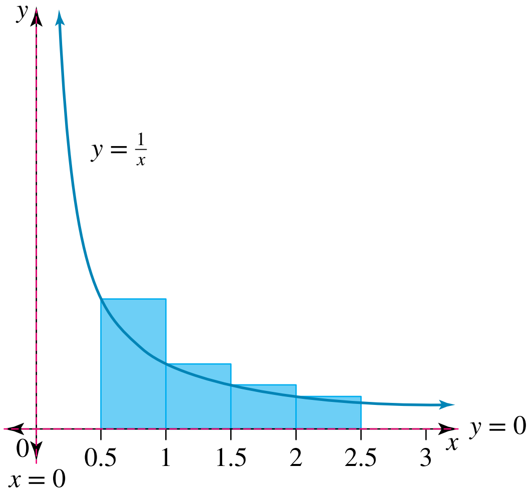
Finding the area between the curve and the x-axis from a to b is the same as finding the definite integral with the end-point terminals a and b.

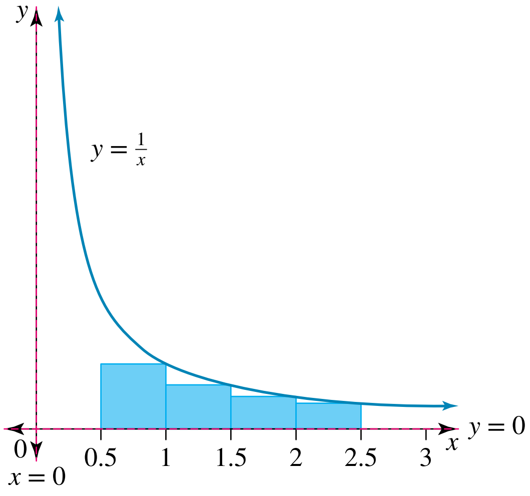
EXERCISE 8.2 The fundamental theorem of integral calculus

PRACTISE

Work without CAS

1 **WE1** The left end-point rectangle method and the right end-point rectangle method are shown for the calculation of the approximate area between the curve $f(x) = \dfrac{1}{x}$, $x > 0$, and the x-axis from $x = 0.5$ to $x = 2.5$.



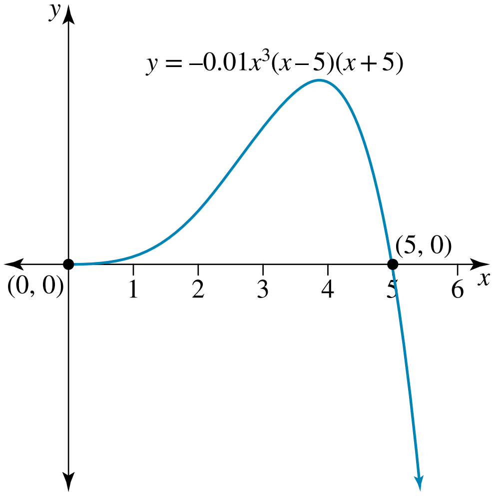


Calculate the approximate area under the curve:

a using the left end-point rectangle rule

b using the right end-point rectangle rule.

2 Consider the function defined by the rule $f:R \to R$, $f(x) = -0.01x^3(x - 5)(x + 5)$, $x \geq 0$. The graph of the function is shown. Use the left end-point rule with rectangles 1 unit wide to approximate the area bound by the curve and the x-axis.



3 **WE2** Evaluate:

a $\int_0^1 (4x^3 + 3x^2 + 2x + 1)dx$

b $\int_{-\pi}^{\pi} (\cos(x) + \sin(x))dx.$

4 Evaluate:

a $\int_{-3}^{2} (x + 1)^3 dx$

b $\int_0^1 (e^x + e^{-x})^2 dx.$

5 **WE3** Given that $\int_2^5 m(x)dx = 7$ and $\int_2^5 n(x)dx = 3$, find:

a $\int_2^5 3m(x)dx$

b $\int_2^5 (2m(x) - 1)dx$

c $\int_5^2 (m(x) + 3)dx$

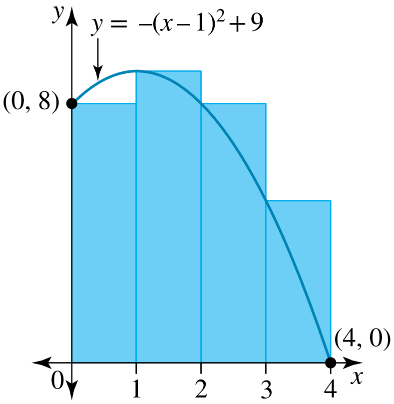
d $\int_2^5 (2m(x) + n(x) - 3)dx.$

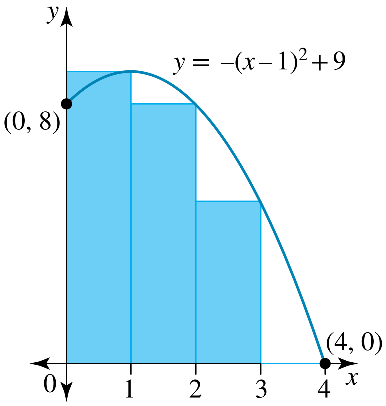
6 Find k if $\int_k^1 (4x^3 - 3x^2 + 1)dx = 0.$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 The graph of $f:[0, 4] \to R$, $f(x) = -(x - 1)^2 + 9$ is shown.

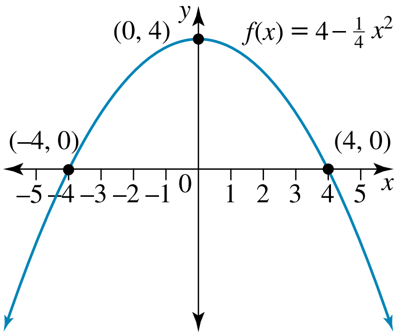




a Use the left end point rule with rectangles 1 unit wide to estimate the area between the curve and the x-axis from $x = 0$ to $x = 4$.

b Use the right end point rule with rectangles 1 unit wide to estimate the area between the curve and the x-axis from $x = 0$ to $x = 4$.

8 The graph of the function $f(x) = 4 - \frac{1}{4}x^2$ is shown.

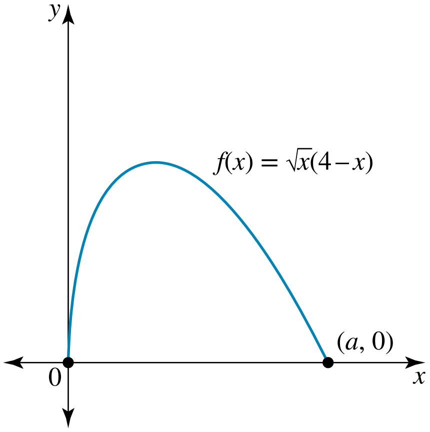


Estimate the area bound by the curve and the x-axis using the right end point method and using rectangles of width 1.

9 The graph of $f(x) = \sqrt{x}(4 - x)$ for $x \in [0, a]$ is shown.

a The graph intersects the x-axis at the point $(a, 0)$ as shown. Find the value of the constant a.

b Use both the left end point and the right end point rules to determine the approximate area between the curve and the x-axis from $x = 0$ to $x = a$. Use a rectangle width of 1 and give your answers correct to 2 decimal places.



10 Evaluate the following.

a $\displaystyle\int_0^3 (3x^2 - 2x + 3)dx$

b $\displaystyle\int_1^2 \frac{2x^3 + 3x^2}{x}dx$

c $\displaystyle\int_{-1}^1 (e^{2x} - e^{-2x})dx$

d $\displaystyle\int_{2\pi}^{4\pi} \sin\left(\frac{x}{3}\right)dx$

e $\displaystyle\int_{-3}^{-1} \frac{2}{\sqrt{1 - 3x}}dx$

f $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\cos(2x) - \sin\left(\frac{x}{2}\right)\right]dx.$

11 Given that $\displaystyle\int_0^5 f(x)dx = 7.5$ and $\displaystyle\int_0^5 g(x)dx = 12.5$, find:

a $\displaystyle\int_0^5 -2f(x)dx$

b $\displaystyle\int_5^0 g(x)dx$

c $\displaystyle\int_0^5 (3f(x) + 2)dx$

d $\displaystyle\int_0^5 (g(x) + f(x))dx$

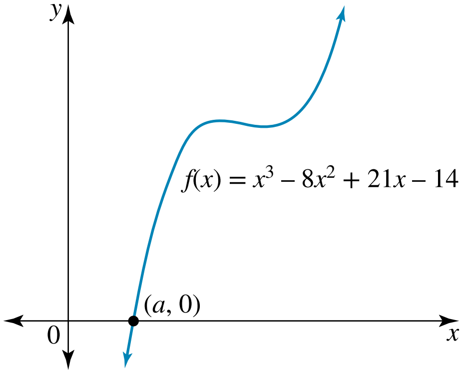
e $\displaystyle\int_0^5 (8g(x) - 10f(x))dx$

f $\displaystyle\int_0^3 g(x)dx + \int_3^5 g(x)dx.$

12 Determine h if $\int_1^h \frac{3}{x^2}dx = -\frac{12}{5}$.

13 Determine a if $\int_0^a e^{-2x}dx = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$.

14 The graph of the function $f: R \to R$, $f(x) = x^3 - 8x^2 + 21x - 14$ is shown



a The graph cuts the x-axis at the point $(a, 0)$. Find the value of the constant a.

b Evaluate $\int_a^5 (x^3 - 8x^2 + 21x - 14)dx$

15 a If $y = x\sin(x)$, find $\frac{dy}{dx}$.

b Hence, find the value of $\int_{-\pi}^{\frac{\pi}{2}} 2x\cos(x)dx$.

16 a If $y = e^{x^3-3x^2} + 2$, find $\frac{dy}{dx}$.

b Hence, find the value of $\int_0^1 (x^2 - 2x)e^{x^3-3x^2}dx$.

MASTER

17 If $\int_1^k (2x - 3)dx = 7 - 3\sqrt{5}$, find k, given $k > 1$.

18 Find $\int_{-2}^0 \frac{1 + e^{2x} - 2xe^{2x}}{(e^{2x} + 1)^2}dx$, correct to 3 decimal places.

8.3 Areas under curves

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Units 3 & 4

AOS 3

Topic 4

Concept 9

Area under a curve
Concept summary
Practice questions

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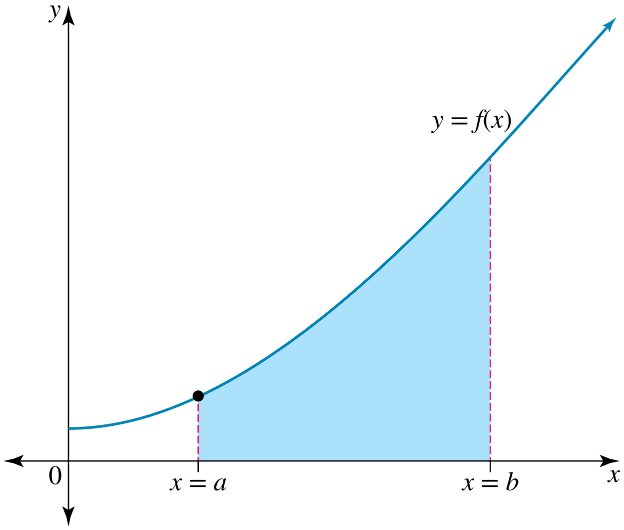
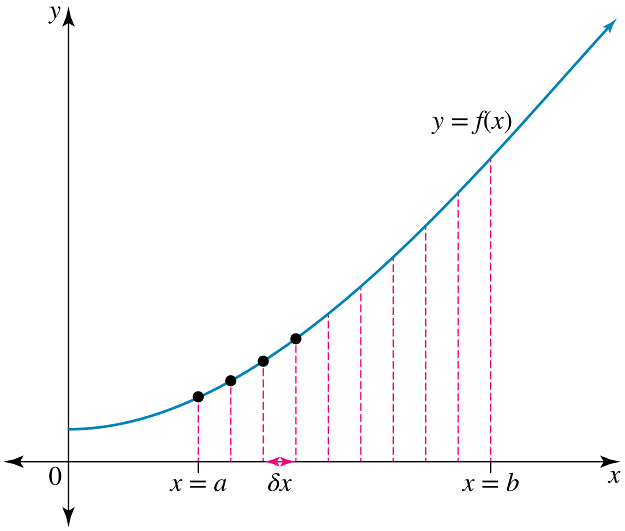
Interactivity
Area under a curve
int-5966

If we are interested in the area between a curve that is a continuous function, $y = f(x)$, and the x-axis between $x = a$ and $x = b$, then the following graph shows us exactly what we require.



As we have already seen, this area can be approximated by dividing it into a series of thin vertical strips or rectangles. The approximate value of the area is the sum of the areas of all the rectangles, whether they are left end-point or right end-point rectangles.

Suppose A represents the sum of the areas of all the rectangular strips between $x = a$ and $x = b$, where each strip has a width of δx.



Providing there is a very large number of rectangular strips so that δx is extremely small,

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x \qquad (1)$$

where $\sum\limits_{x=a}^{x=b}$ means 'the sum from $x = a$ to $x = b$'.

Also, since each strip can have its area defined as $\delta A \simeq \delta x \times y$,

$$\frac{\delta A}{\delta x} \simeq y.$$

Therefore, if the area under the curve is divided into a very large number of strips, then

$$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} = y$$

$$\frac{dA}{dx} = y$$

This leads to the statement that

$$A = \int y dx.$$

But since $x = a$ and $x = b$ are the boundary points or end points, then

$$A = \int_a^b y dx \qquad (2)$$

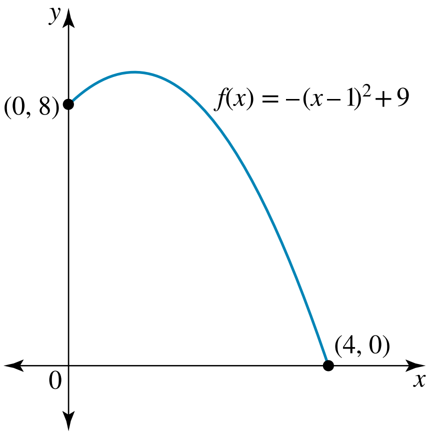
Equating (1) and (2), we have

$$\boxed{A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x = \int_a^b y\,dx.}$$

This statement allows us to calculate the area between a curve and the x-axis from $x = a$ to $x = b$.

Consider the function defined by the rule $f: [0, 4] \to R$, $f(x) = -(x-1)^2 + 9$.

$$\begin{aligned}
A &= \int_0^4 (-(x-1)^2 + 9)dx \\
&= \left[-\frac{(x-1)^3}{3} + 9x\right]_0^4 \\
&= \left(-\frac{(3)^3}{3} + 9(4)\right) - \left(-\frac{(-1)^3}{3} + 9(0)\right) \\
&= -9 + 36 - \tfrac{1}{3} + 0 \\
&= 27 - \tfrac{1}{3} \\
&= 26\tfrac{2}{3} \text{ units}^2
\end{aligned}$$

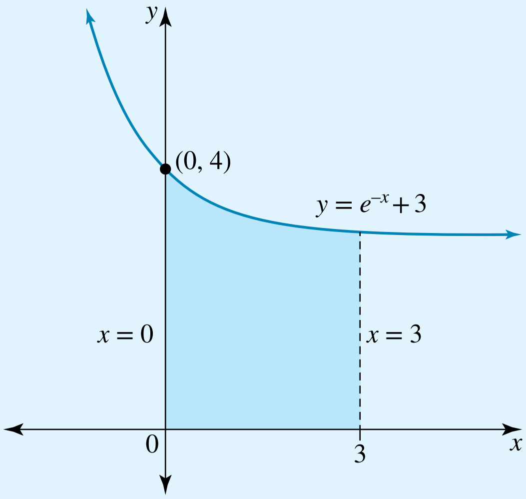


WORKED EXAMPLE 4 **Determine the area bound by the curve defined by the rule $y = e^{-x} + 3$ and the x-axis from $x = 0$ to $x = 3$.**

THINK

WRITE/DRAW

1 Sketch the graph of the given function and shade the required area.



2 Write the integral needed to find the area.

$$A = \int_0^3 (e^{-x} + 3)dx$$

3 Antidifferentiate the function and evaluate.

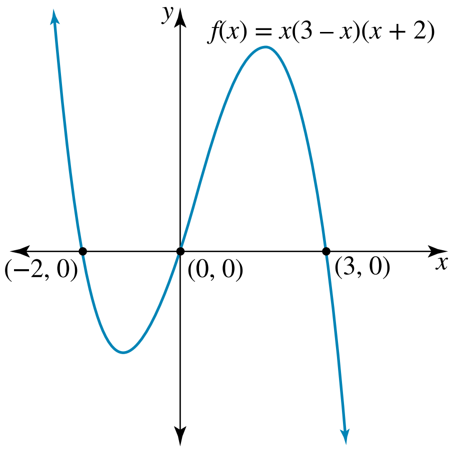
$$\begin{aligned} A &= \left[-e^{-x} + 3x\right]_0^3 \\ &= (-e^{-3} + 3(3)) - (-e^0 + 3(0)) \\ &= -e^{-3} + 9 + 1 \\ &= -e^{-3} + 10 \end{aligned}$$

4 Write the answer.

The area is $-e^{-3} + 10$ square units.

Signed areas

When we calculate the area between a graph $y = f(x)$ and the x-axis from $x = a$ to $x = b$ using the definite integral $\int_a^b f(x)dx$, the area can either be positive or negative.



Consider the function defined by the rule $f: R \to R$, $f(x) = x(3 - x)(x - 2)$, which is shown.

If we evaluate $\int_{-2}^{0} x(3 - x)(x + 2)dx$

$$\begin{aligned} &= \int_{-2}^{0} (6x + x^2 - x^3)dx \\ &= \left[3x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_{-2}^{0} \\ &= 0 - \left(3(-2)^2 + \tfrac{1}{3}(-2)^3 - \tfrac{1}{4}(-2)^4\right) \\ &= 0 - \left(12 - \tfrac{8}{3} - 4\right) \\ &= -\left(8 - \tfrac{8}{3}\right) \\ &= -\tfrac{16}{3} \end{aligned}$$

This area is negative because the region lies below the x-axis.

Whereas $\int_{0}^{3} x(3 - x)(x + 2)dx$

$$\begin{aligned} &= \int_{0}^{3} (6x + x^2 - x^3)dx \\ &= \left[3x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_{0}^{3} \end{aligned}$$

$$= \left(3(3)^2 + \tfrac{1}{3}(3)^3 - \tfrac{1}{4}(3)^4\right) - 0$$
$$= \left(27 + 9 - \tfrac{81}{4}\right) - 0$$
$$= \left(36 - \tfrac{81}{4}\right)$$
$$= \tfrac{63}{4}$$

This area is positive as the region lies above the x-axis.

If we want an accurate answer for the area bound by the curve from $x = -2$ to $x = 3$, we counteract the negative region by subtracting it from the positive region. By subtracting the negative area, we are actually adding the area.

$$A = \int_0^3 x(3 - x)(x + 2)dx - \int_{-2}^0 x(3 - x)(x + 2)dx$$
$$= \tfrac{63}{4} - \left(-\tfrac{16}{3}\right)$$
$$= \tfrac{253}{12}$$
$$= 21\tfrac{1}{12} \text{ units}^2$$

The total area bound by the curve, the x-axis and the lines $x = -2$ and $x = 3$ is $21\frac{1}{12}$ square units.

This confirms the theory that if $f(x) > 0$, then the region above the x-axis has a positive area, but if $f(x) < 0$, then the region below the x-axis has a negative area.

Had we not broken up the interval and calculated $\int_{-2}^{3}(6x + x^2 - x^3)dx$, the result would have been

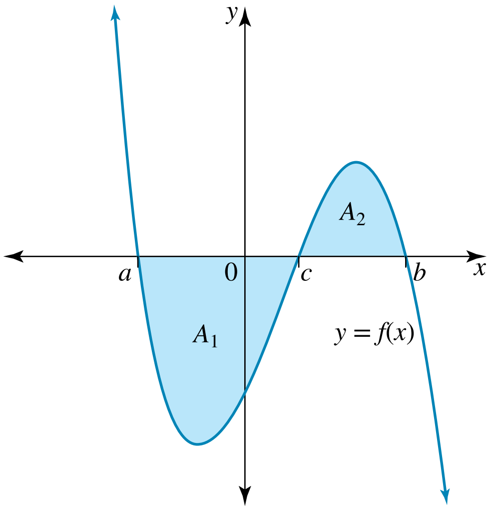
$$\int_{-2}^{3}(6x + x^2 - x^3)dx$$
$$= \left[3x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_{-2}^{3}$$
$$= \left(3(3)^2 + \tfrac{1}{3}(3)^3 - \tfrac{1}{4}(3)^4\right) - \left(3(-2)^2 + \tfrac{1}{3}(-2)^3 - \tfrac{1}{4}(-2)^4\right)$$
$$= \tfrac{63}{4} - \tfrac{16}{3}$$
$$= 10\tfrac{5}{12}$$

This result would not have given us the required area. The value of $10\frac{5}{12}$ is the value of the definite integral, but not the area under the curve.

This shows that it is imperative to have a ‘picture’ of the function to determine when $f(x) > 0$ and when $f(x) < 0$; otherwise, we are just evaluating the definite integral rather than finding the necessary area.

The total area between the function $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is given by

$$\begin{aligned} A_{\text{total}} &= \int_c^b f(x)dx - \int_a^c f(x)dx \\ &= A_2 - A_1 \end{aligned}$$



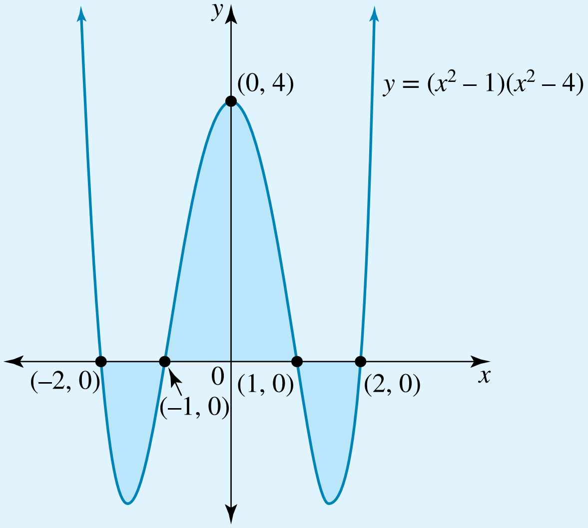
The other method to account for the negative area is to switch the terminals within the integral for the negative region.

So,

$$\begin{aligned} A_{\text{total}} &= \int_c^b f(x)dx + \int_c^a f(x)dx \\ &= A_2 + A_1 \end{aligned}$$

WORKED EXAMPLE 5

Find the area bound by the curve $y = (x^2 - 1)(x^2 - 4)$ and the x-axis from $x = -2$ to $x = 2$.

THINK	WRITE/DRAW
1 Make a careful sketch of the given function. Shade the required region.	The graph cuts the y-axis where $x = 0$. ∴ the y-intercept is $(0, 4)$. The graph cuts the x-axis where $y = 0$: $(x^2 - 1)(x - 4)^2 = 0$ $(x - 1)(x + 1)(x - 2)(x + 2) = 0$ $x = \pm 1, x = \pm 2$ 

2 Express the area using definite integrals. Account for the negative regions by subtracting these from the positive areas.

Note that the region from $x = -2$ to $x = -1$ is the same as the region from $x = 1$ to $x = 2$ due to the symmetry of the graph.

$$A = \int_{-1}^{1} (x^2 - 1)(x^2 - 4)dx - 2\int_{1}^{2} (x^2 - 1)(x^2 - 4)dx$$

$$= \int_{-1}^{1} (x^4 - 5x^2 + 5)dx - 2\int_{1}^{2} (x^4 - 5x^2 + 5)dx$$

3 Antidifferentiate and evaluate.

$$= \left[\tfrac{1}{5}x^5 - \tfrac{5}{3}x^3 + 5x\right]_{-1}^{1} - 2\left[\tfrac{1}{5}x^5 - \tfrac{5}{3}x^3 + 5x\right]_{1}^{2}$$

$$= \left(\tfrac{1}{5}(1)^5 - \tfrac{5}{3}(1)^3 + 5(1)\right) - \left(\tfrac{1}{5}(-1)^5 - \tfrac{5}{3}(-1)^3 + 5(-1)\right)$$

$$- 2\left[\left(\tfrac{1}{5}(2)^5 - \tfrac{5}{3}(2)^3 + 5(2)\right) - \left(\tfrac{1}{5}(1)^5 - \tfrac{5}{3}(1)^3 + 5(1)\right)\right]$$

$$= \left(\tfrac{1}{5} - \tfrac{5}{3} + 5\right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 5\right)$$

$$- 2\left[\left(\tfrac{32}{5} - \tfrac{40}{3} + 10\right) - \left(\tfrac{1}{5} - \tfrac{5}{3} + 5\right)\right]$$

$$= \tfrac{1}{5} - \tfrac{5}{3} + 5 + \tfrac{1}{5} - \tfrac{5}{3} + 5 - 2\left(\tfrac{32}{5} - \tfrac{40}{3} + 10 - \tfrac{1}{5} + \tfrac{5}{3} - 5\right)$$

$$= \tfrac{2}{5} - \tfrac{10}{3} + 10 - \tfrac{64}{5} + \tfrac{80}{3} - 20 + \tfrac{2}{5} - \tfrac{10}{3} + 10$$

$$= -12 + 20$$

$$= 8$$

4 Write the answer.

The area is 8 units^2.

EXERCISE 8.3 Areas under curves

PRACTISE

Work without CAS

1 WE4 Determine the area bound by the curve defined by the rule $y = 2\sqrt{x}$, $x \geq 0$ and the x-axis from $x = 0$ to $x = 25$.

2 Find the area bounded by the curve $y = 2\sin(2x) + 3$, the x-axis and the lines $x = 0$ and $x = \pi$.

3 WE5 Sketch the graph of $y = 1 - e^{-x}$ and hence find the exact area between the curve and the x-axis from $x = -1$ to $x = 1$.

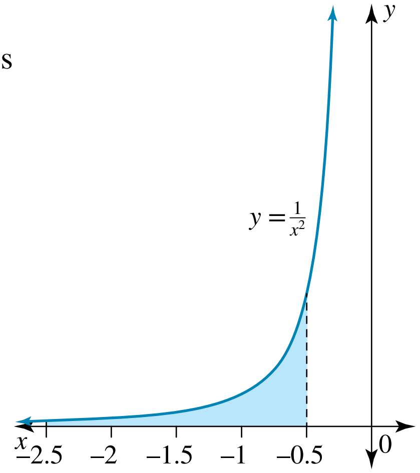
4 Sketch the graph of $y = \sqrt[3]{x}$ and hence find the area between the curve and the x-axis from $x = -8$ to $x = 8$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 The graph of $y = \dfrac{1}{x^2}$, $x < 0$ is shown.

Find the area of the shaded region (i.e. for $-2.5 \leq x \leq -0.5$).



6 Consider the function defined by the rule $f: R \backslash \{0\} \rightarrow R,\ f(x) = \frac{1}{\sqrt{x}}$.

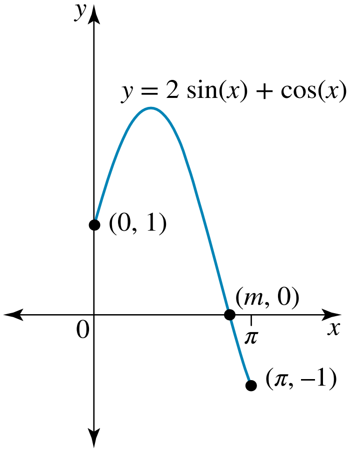
a Sketch the graph of f for $x > 0$.

b Using calculus, find the area enclosed by the function, the lines $x = 1$ and $x = 3$, and the x-axis.

7 The graph of $y = 2\sin(x) + \cos(x)$ for $0 \leq x \leq \pi$ is shown.

a The graph intersects the x axis at $(m, 0)$. Find the value of the constant m, correct to 4 decimal places.

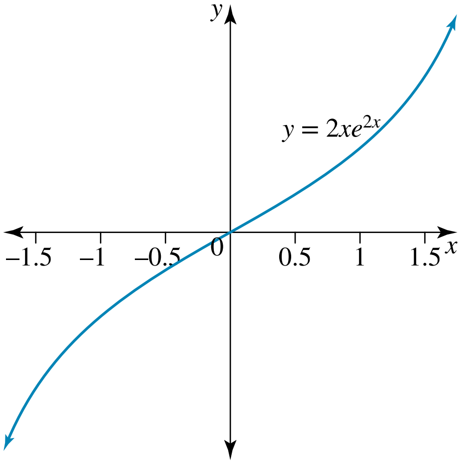
b Find $\int_0^m (2\sin(x) + \cos(x))dx$, correct to 4 decimal places.



8 The graph of $y = 2xe^{2x}$ is shown.

a Find $\frac{d}{dx}\left(e^{x^2}\right)$.

b Hence, find the exact area between the curve $y = 2xe^{x^2}$ and the x-axis from $x = -1$ to $x = 1$.



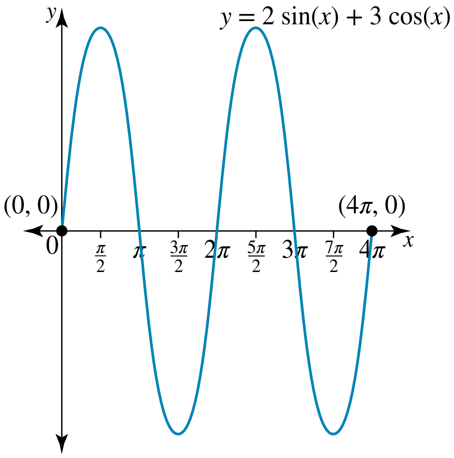
9 Consider the function $f: R \rightarrow R,\ f(x) = -(x^2 - 1)(x^2 - 9)$.

a Sketch the graph of f, showing the axis intercepts and turning points.

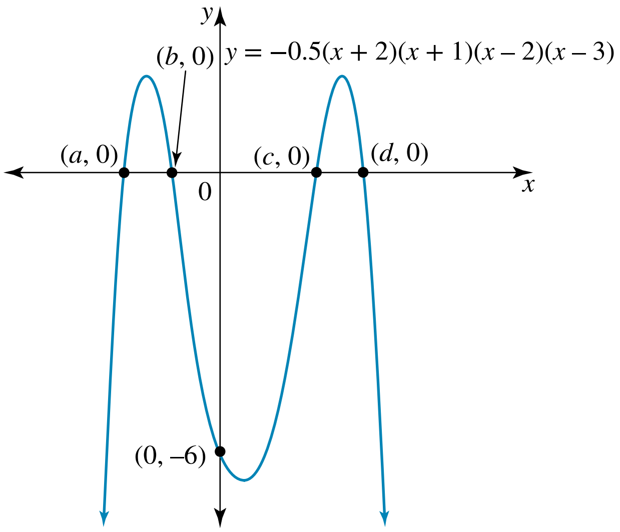
b Using calculus, find the area enclosed by the function, the lines $x = -3$ and $x = 3$, and the x-axis, correct to 2 decimal places.

10 The graph of the function $y = 2\sin(x) + 3\cos(x)$ is shown.

Using calculus, find the area between the curve and the x-axis from $x = 0$ to $x = 4\pi$.



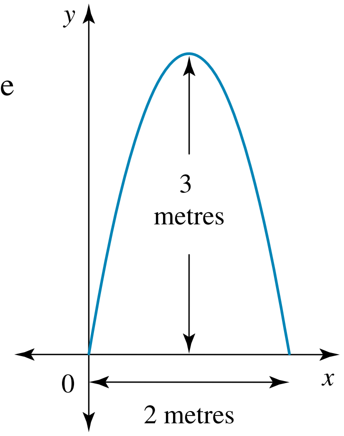
11 The graph of $y = -0.5(x + 2)(x + 1)(x - 2)(x - 3)$ is shown.



a The graph intersects the x-axis at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$. Find the values of the constants a, b, c and d.

b Find the area between the curve and the x-axis from $x = a$ to $x = d$, correct to 2 decimal places.

12 The ‘Octagon Digital’ store on the corner of two main roads in the north-eastern suburbs of a large Australian city has two very distinctive parabolic windows, each one facing one of the main roads. In the early hours of a Sunday morning, a motorist smashed through one of the windows. The owner decided it would be beneficial to replace both windows with strongly reinforced and quite heavily tinted glass. Each window has the dimensions shown in the diagram.

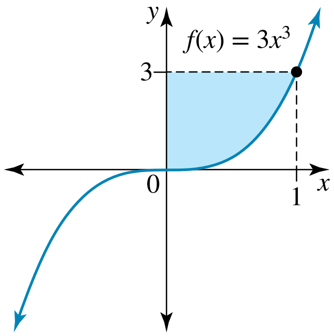


a Find the equation of the parabola that defines the shape of each window.

b Find the area of glass required to replace each window.

c If the cost per square metre of the replacement reinforced and tinted glass is \$55, find the cost of replacing the two windows.

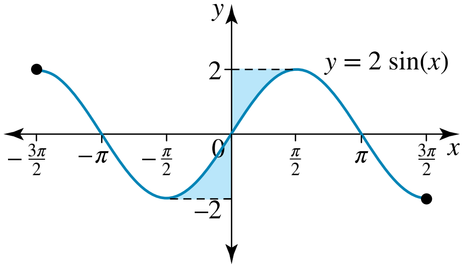
13 The graph of $f: R \to R$, $f(x) = 3x^3$ is shown.



a Find the area bounded by the curve and the x-axis from $x = 0$ to $x = 1$.

b Hence, or otherwise, find the area of the shaded region.

14 The graph of $y = 2\sin(x)$, $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$ is shown.



a Calculate $\int_0^{\frac{\pi}{2}} 2\sin(x)dx$.

b Hence, or otherwise, find the area of the shaded region.

MASTER

15 The Red Fish Restaurant is a new restaurant about to open. The owners commissioned a graphic artist to design a logo that will be seen on the menus and on advertisements for the restaurant, and will also be etched into the front window of the restaurant. The logo is shown in Figure 1.

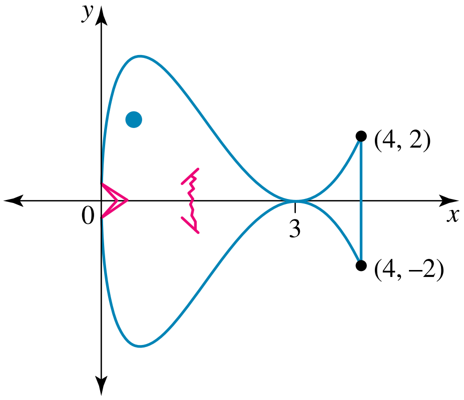


Figure 1

As the logo is to appear in a number of different scenarios, the owners need to know the area of the original to allow for enlargement or diminishing processes. The graphic artist formed the shape by using the rule

$$y = \sqrt{x}(x - 3)^2, 0 \leq x \leq 4$$

for the upper part of the fish and

$$y = -\sqrt{x}(x - 3)^2, 0 \leq x \leq 4$$

for the lower part of the fish.

The original outline is shown in Figure 2.

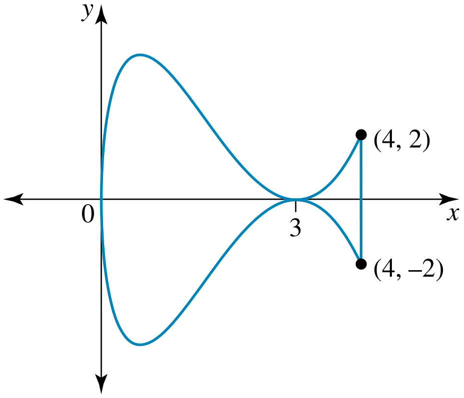


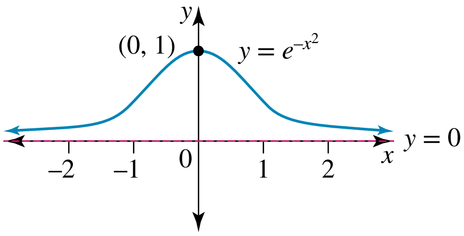
Figure 2

a Calculate the area between the upper curve and the x-axis from $x = 0$ to $x = 4$.

b Calculate the area of the entire fish logo. (All measurements are in centimetres.) Give your answer correct to 1 decimal place.

c The etched fish on the front window of the restaurant has an area of 0.348 75 m^2. What was the scale factor, to the nearest integer, used to enlarge the fish motif?

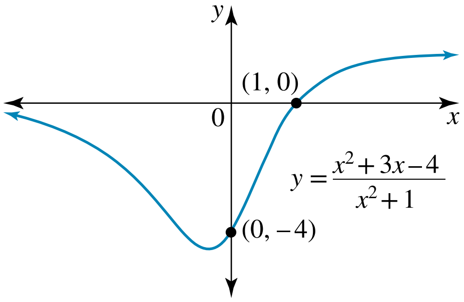
16 **a** The graph of $y = e^{-x^2}$ is shown. Find the area between the curve and the x-axis from $x = -2$ to $x = 2$, giving your answer correct to 4 decimal places.



b The graph of the function

$$y = \frac{x^2 + 3x - 4}{x^2 + 1}$$ is shown.

Calculate the area, correct to 3 decimal places, between the curve and the x-axis from $x = -2$ to $x = 3$.



8.4 Applications

The definite integral is extremely useful in solving a large variety of applied problems including the average value or mean value of a function, the area between curves, total change as the integral of instantaneous change, and kinematic problems.

study on

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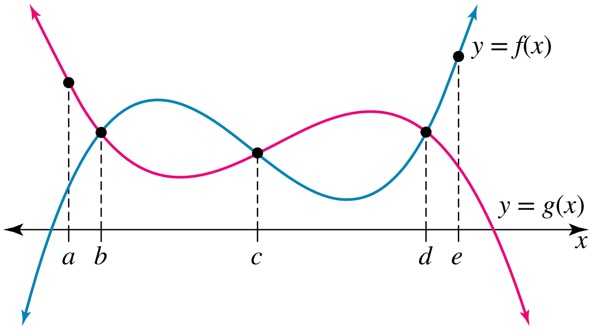
Areas between curves



Consider the functions f and g, which are both continuous on the interval $[a, e]$.

Sometimes $f > g$ and on other occasions $f < g$. It is absolutely critical to know when $f > g$ or $f < g$, so a graphic representation of the situation is essential, particularly to show the points of intersection of the graphs. We can find the area between the curves, providing we take each section one at a time. Within each section, the area is found by subtracting the lower function from the higher function. As we are finding the area between two curves, we don't need to worry about whether the region is above or below the x-axis.

Area between the curves:

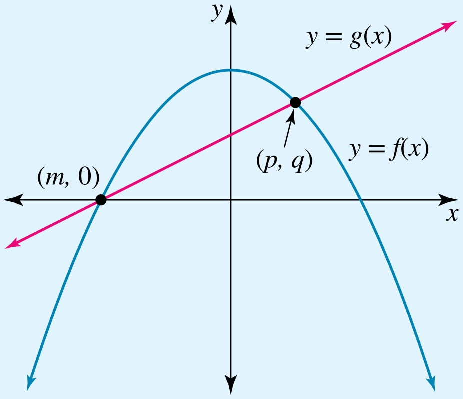
$$A = \underbrace{\int_a^b (g(x) - f(x))\,dx}_{f<g} + \underbrace{\int_b^c (f(x) - g(x))\,dx}_{f>g} + \underbrace{\int_c^d (g(x) - f(x))\,dx}_{f<g} + \underbrace{\int_d^e (f(x) - g(x))\,dx}_{f>g}$$

WORKED EXAMPLE 6

The functions $f: R \to R$, $f(x) = (x + 2)(2 - x)$ and $g: R \to R$, $g(x) = x + 2$ are shown.

a The graphs intersect at $(m, 0)$ and (p, q). Find the values of the constants m, p and q.

b Find the area bound by the two curves.



THINK	WRITE
a 1 Points of intersection are found by solving the equations simultaneously, so equate the equations and solve for x.	**a** $x + 2 = (x + 2)(2 - x)$ $x + 2 = 4 - x^2$ $x^2 + x - 2 = 0$ $(x + 2)(x - 1) = 0$ $x = -2, x = 1$

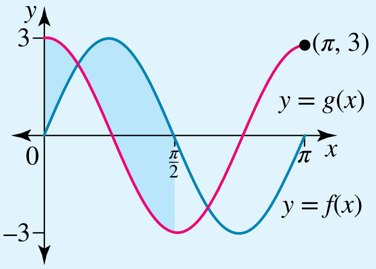
2 Find the corresponding y-values.	When $x = -2$, $y = -2 + 2 = 0$. When $x = 1$, $y = 1 + 2 = 3$.
3 State the solution.	$m = -2$, $p = 1$, $q = 3$
b 1 Determine whether $f > g$ or $f < g$.	**b** As $f(x) = 4 - x^2$ lies above $g(x) = x + 2$, $f > g$.
2 Express the area in definite integral notation and simplify the expression within the integral.	$A = \int_{-2}^{1} (f(x) - g(x))dx$ $= \int_{-2}^{1} (4 - x^2 - (x + 2))dx$ $= \int_{-2}^{1} (-x^2 - x + 2)dx$
3 Antidifferentiate and evaluate.	$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1}$ $= \left(-\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 + 2(1)\right) - \left(-\frac{1}{3}(-2)^3 - \frac{1}{2}(-2)^2 + 2(-2)\right)$ $= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4$ $= -3 - \frac{1}{2} + 8$ $= 4\frac{1}{2}$
4 Write the answer.	The area is 4.5 units^2.

WORKED EXAMPLE 7

The graphs of $f(x) = 3\sin(2x)$ and $g(x) = 3\cos(2x)$ are shown for $x \in [0, \pi]$.

a **Find the coordinates of the point(s) of intersection of f and g for the interval $\left[0, \frac{\pi}{2}\right]$.**

b **Using calculus, determine the area enclosed between the curves on the interval $\left[0, \frac{\pi}{2}\right]$.**



THINK	WRITE
a 1 Use simultaneous equations to find where the graphs intersect, and equate the two equations.	**a** $3\sin(2x) = 3\cos(2x)$ $\frac{3\sin(2x)}{3\cos(2x)} = 1, 0 \le x \le \frac{\pi}{2}$ $\tan(2x) = 1, 0 \le x \le \frac{\pi}{2}$
2 Solve for $2x$ for $x \in \left[0, \frac{\pi}{2}\right]$.	$2x = \frac{\pi}{4}$ $\therefore x = \frac{\pi}{8}$

3 Find the corresponding y-value.

$$\begin{aligned} f\left(\frac{\pi}{8}\right) &= 3\sin\left(\frac{\pi}{4}\right) \\ &= 3 \end{aligned}$$

4 Write the solution.

The coordinates are $\left(\frac{\pi}{8}, 3\right)$.

b **1** Determine when $f > g$ and $f < g$.

b When $0 < x < \frac{\pi}{8}$, $g > f$.

When $\frac{\pi}{8} < x < \frac{\pi}{2}$, $f > g$.

2 Express each area individually in definite integral notation.

The area is equal to:

$$A = \int_0^{\frac{\pi}{8}} (3\cos(2x) - 3\sin(2x))dx + \int_{\frac{\pi}{8}}^{\frac{\pi}{2}} (3\sin(2x) - 3\cos(2x))dx$$

3 Use calculus to antidifferentiate and evaluate.

$$\begin{aligned}
&= \left[\frac{3}{2}\sin(2x) + \frac{3}{2}\cos(2x)\right]_0^{\frac{\pi}{8}} + \left[-\frac{3}{2}\cos(2x) - \frac{3}{2}\sin(2x)\right]_{\frac{\pi}{8}}^{\frac{\pi}{2}} \\
&= \frac{3}{2}\sin\left(\frac{\pi}{4}\right) + \frac{3}{2}\cos\left(\frac{\pi}{4}\right) - \left(\frac{3}{2}\sin(0) + \frac{3}{2}\cos(0)\right) \\
&\quad + -\frac{3}{2}\cos(\pi) - \frac{3}{2}\sin(\pi) - \left(-\frac{3}{2}\cos\left(\frac{\pi}{4}\right) - \frac{3}{2}\sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{3}{2} \times \frac{\sqrt{2}}{2} + \frac{3}{2} \times \frac{\sqrt{2}}{2} - 0 - \frac{3}{2} + \frac{3}{2} - 0 + \frac{3}{2} \times \frac{\sqrt{2}}{2} + \frac{3}{2} \times \frac{\sqrt{2}}{2} \\
&= \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} \\
&= 3\sqrt{2}
\end{aligned}$$

4 Write the answer.

The area is $3\sqrt{2}$ square units.

studyon

Units 3 & 4

AOS 3

Topic 4

Concept 8

Average value of a function
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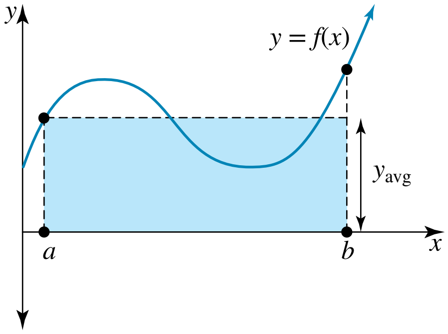
Practice questions

The average or mean value of a function

The average or mean value of a function $f(x)$ over the interval $[a, b]$ is given by

$$\textbf{Average} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Geometrically, the average value of a function is the height of a rectangle, y_{avg}, with a width of $(b - a)$, that has the same area as the area under the curve $y = f(x)$ for the interval $[a, b]$.



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Interactivity
Average value of a function
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WORKED EXAMPLE 8 Find the average value for the function defined by $f(x) = \sin(2x)$ for the interval

$$x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right].$$

THINK	WRITE
1 Write the rule for the average or mean value of a function.	$\text{Average} = \frac{1}{b-a}\int_a^b f(x)dx$
2 Substitute the appropriate values into the rule.	$= \frac{1}{\left(\frac{3\pi}{8}-\frac{\pi}{8}\right)}\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin(2x)dx$ $= \frac{4}{\pi}\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin(2x)dx$
3 Antidifferentiate and evaluate.	$= \frac{4}{\pi}\left[-\frac{1}{2}\cos(2x)\right]_{\frac{8}{\pi}}^{\frac{3\pi}{8}}$ $= \frac{4}{\pi}\left(-\frac{1}{2}\cos\left(2\times\frac{3\pi}{8}\right)+\frac{1}{2}\cos\left(2\times\frac{\pi}{8}\right)\right)$ $= \frac{4}{\pi}\left(-\frac{1}{2}\times-\frac{\sqrt{2}}{2}+\frac{1}{2}\times\frac{\sqrt{2}}{2}\right)$ $= \frac{4}{\pi}\left(\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}\right)$ $= \frac{4}{\pi}\times\frac{\sqrt{2}}{2}$ $= \frac{2\sqrt{2}}{\pi}$

Total change as the integral of instantaneous change

If we are given the equation for the rate of change and we want to find the amount that has changed over a particular time period, we would integrate the rate of change equation using the starting and finishing times as the terminals. For example, if we know the rate of water flowing, $\frac{dV}{dt}$ in L/min, and we want to find the amount of liquid that has flowed in the first 30 minutes, we would evaluate $\int_0^{30}\frac{dV}{dt}dt$.

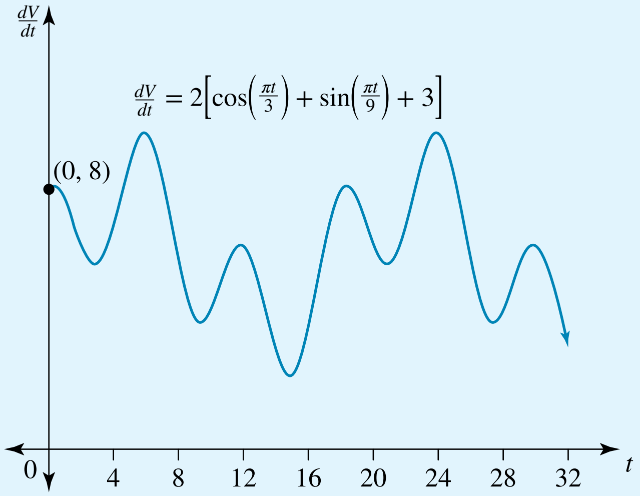
WORKED EXAMPLE 9

It is a common practice to include heating in concrete slabs when new residential homes or units are being constructed, because it is more economical than installing heating later. A typical reinforced concrete slab, 10–15 centimetres thick, has tubing installed on top of the reinforcement, then concrete is poured on top. When the system is complete, hot water runs through the tubing. The concrete slab absorbs the heat from the water and releases it into the area above.

The number of litres/minute of water flowing through the tubing over t minutes can be modelled by the rule

$$\frac{dV}{dt} = 2\left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right].$$

The graph of this function is shown.



a What is the rate of flow of water, correct to 2 decimal places, at:

- **i** 4 minutes
- **ii** 8 minutes?

b State the period of the given function.

c Find the volume of water that flows through the tubing during the time period for one whole cycle.

THINK	WRITE
a i Substitute $t = 4$ into the given equation and evaluate.	**a i** $\frac{dV}{dt} = 2\left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right]$ When $t = 4$, $\frac{dV}{dt} = 2\left[\cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{9}\right) + 3\right]$ $= 6.97$ The rate at 4 minutes is 6.97 litres/minute.
ii Substitute $t = 8$ into the given equation and evaluate.	**ii** When $t = 8$, $\frac{dV}{dt} = 2\left[\cos\left(\frac{8\pi}{3}\right) + \sin\left(\frac{8\pi}{9}\right) + 3\right]$ $= 5.68$ The rate at 8 minutes is 5.68 litres/minute.

b Determine the cycle for the function by analysing the shape of the graph.

b A complete cycle for the function occurs between $t = 6$ and $t = 24$, so the period is $24 - 6 = 18$ minutes.

c 1 The area under the curve of the equation of the rate of flow gives the total volume that has flowed through the tubing.

c $$A = \int_6^{24} 2\left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right]dt$$

$$= 2\int_6^{24} \left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right]dt$$

$$= 2\left[\frac{3}{\pi}\sin\left(\frac{\pi t}{3}\right) - \frac{9}{\pi}\cos\left(\frac{\pi t}{9}\right) + 3t\right]_6^{24}$$

2 Antidifferentiate and evaluate.

$$= 2\left(\frac{3}{\pi}\sin(8\pi) - \frac{9}{\pi}\cos\left(\frac{8\pi}{3}\right) + 72\right)$$

$$-\left(\frac{3}{\pi}\sin(2\pi) - \frac{9}{\pi}\cos\left(\frac{2\pi}{3}\right) + 18\right)$$

$$= 2\left(-\frac{9}{\pi}\cos\left(\frac{2\pi}{3}\right) + 72 + \frac{9}{\pi}\cos\left(\frac{2\pi}{3}\right) - 18\right)$$

$$= 2 \times 54$$

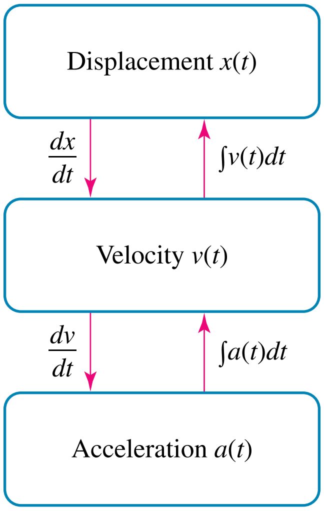
$$= 108$$

3 Write the answer.

The volume of water that passes through the tubing during one cycle is 108 litres.

Kinematics

You are already aware of the relationships between displacement, velocity and acceleration.
However, our knowledge about the definite integral and the area under curves now gives us additional skills for the calculation of facts related to kinematics.



WORKED EXAMPLE 10

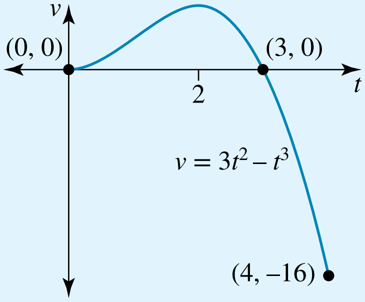
A particle starting from rest accelerates according to the rule $a = 3t(2 - t)$.

a **Find a relationship between the velocity of the particle, v metres/second, and the time, t seconds.**

b **Find the displacement of the particle after 4 seconds.**

c **Sketch the graph of velocity versus time for the first 4 seconds of the motion.**

d **Calculate the distance travelled by the particle in the first 4 seconds.**

THINK	WRITE/DRAW
a **1** Antidifferentiate the acceleration equation to find the velocity equation.	**a** $v = \int a(t)dt$ $= \int (3t(2-t))dt$ $= \int (6t - 3t^2)dt$ $= 3t^2 - t^3 + c$
2 Apply the initial conditions to find v in terms of t.	When $t = 0$, $v = 0$, so $c = 0$. $\therefore v = 3t^2 - t^3$
b **1** Integrate v between $t = 0$ and $t = 4$. As we are finding displacement, there is no need to sketch the graph.	**b** $x = \int_0^4 (3t^2 - t^3)dt$ $= \left[t^3 - \frac{1}{4}t^4\right]_0^4$ $= \left(4^3 - \frac{1}{4}(4)^4\right) - \left(0^3 - \frac{1}{4}(0)^4\right)$ $= 0$
2 Write the answer.	After 4 seconds the displacement is zero.
c Sketch a graph of v versus t.	**c** y-intercept: (0, 0) t-intercepts: $0 = 3t^2 - t^3$ $= t^2(3 - t)$ $t = 0, 3$ When $t = 4$, $v = 3 \times 4^2 - 4^3 = -16$. 
d **1** The area under the curve of a velocity-time graph gives the distance covered. Set up the integrals and subtract the negative region.	**d** $D = \int_0^3 (3t^2 - t^3)dt - \int_3^4 (3t^2 - t^3)dt$
2 Antidifferentiate and evaluate.	$= \left[t^3 - \frac{1}{4}t^4\right]_0^3 - \left[t^3 - \frac{1}{4}t^4\right]_3^4$ $= \left(\left(3^3 - \frac{3^4}{4}\right) - \left(0^3 - \frac{0^4}{4}\right)\right) - \left(\left(4^3 - \frac{4^4}{4}\right) - \left(3^3 - \frac{3^4}{4}\right)\right)$ $= 27 - \frac{81}{4} - 0 - 64 + 64 + 27 - \frac{81}{4}$ $= 54 - \frac{162}{4}$ $= 13.5$
3 Write the answer.	The distance travelled by the particle in 4 seconds is 13.5 metres.

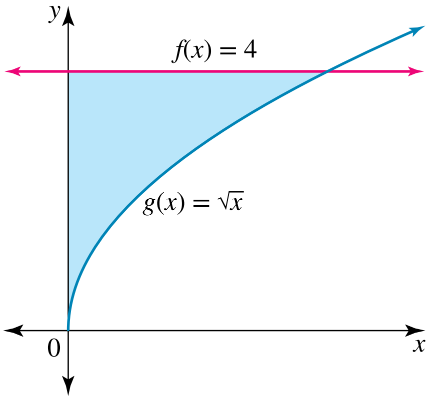
EXERCISE 8.4 Applications

PRACTISE

Work without CAS Questions 1–3, 5, 6, 8, 9

1 **WE6** The graphs of $g(x) = \sqrt{x}$ and the line $f(x) = 4$ are shown.

Find the coordinates of the point of intersection between f and g, and hence find the area of the shaded region.

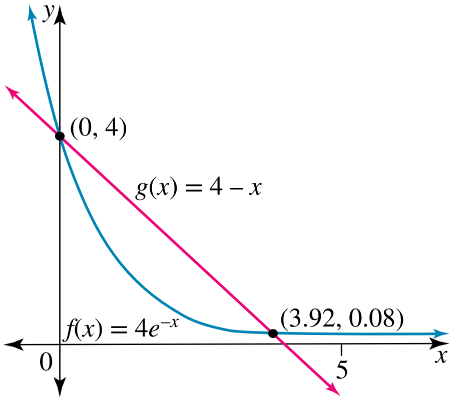


2 Find the area enclosed between the curve $f(x) = (x - 3)^2$ and the line $g(x) = 9 - x$.

3 **WE7** Using calculus, find the area enclosed between the curves $f(x) = \sin(x)$ and $g(x) = -\cos(x)$ from $x = 0$ to $x = \pi$.

4 The graphs of $f(x) = 4e^{-x}$ and $g(x) = 4 - x$ are shown.

Using calculus, find the area enclosed between $y = f(x)$, $y = g(x)$ and the lines $x = 0$ and $x = 5$. Give your answer correct to 4 decimal places.



5 **WE8** Find the average value or mean value of the function defined by the rule $f(x) = e^{3x}$ for $x \in \left[0, \frac{1}{3}\right]$.

6 Find the average value or mean value of the function defined by the rule $f(x) = x^2 - 2x$ for $x \in [0.5, 1]$.

7 **WE9** The average rate of increase, in cm/month, in the length of a baby boy from birth until age 36 months is given by the rule

$$\frac{dL}{dt} = \frac{4}{\sqrt{t}}$$

where t is the time in months since birth and L is the length in centimetres. Find the average total increase in length of a baby boy from 6 months of age until 36 months of age. Give your answer correct to 1 decimal place.

8 A number of apprentice bricklayers are competing in a competition in which they are required to build a fence. The competitors must produce a fence that is straight, neatly constructed and level. The winner will also be judged on how many bricks they have laid during a 30-minute period.

The winner laid bricks at a rate defined by the rule

$$\frac{dN}{dt} = 0.8t + 2$$

where N is the number of bricks laid after t minutes.

a Sketch the graph of the given function for $0 \leq t \leq 30$.

b Shade the region defined by $10 \leq t \leq 20$.

c How many bricks in total did the winner lay in the 10-minute period defined by $10 \leq t \leq 20$?

9 **WE10** A particle moves in a line so that its velocity v metres/second from a fixed point, O, is defined by $v = 1 + 3\sqrt{t + 1}$, where t is the time in seconds.

a Find the initial velocity of the particle.

b What is the acceleration of the particle when:

i $t = 0$ **ii** $t = 8$?

c Sketch the graph of v versus t for the first 10 seconds.

d Find the distance covered by the particle in the first 8 seconds.

10 An object travels in a line so that its velocity, v metres/second, at time t seconds is given by

$$v = 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right), t \geq 0.$$

Initially the object is $-3\sqrt{2}$ metres from the origin.

a Find the relationship between the displacement of the object, x metres, and time, t seconds.

b What is the displacement of the object when time is equal to 3π seconds?

c Sketch the graph of v versus t for $0 \leq t \leq 4\pi$.

d Find the distance travelled by the object after 3π seconds. Give your answer in metres, correct to 2 decimal places.

e Find a relationship between the acceleration of the object, a metres/second^2, and time, t seconds.

f What is the acceleration of the object when $t = 3\pi$ seconds?

CONSOLIDATE

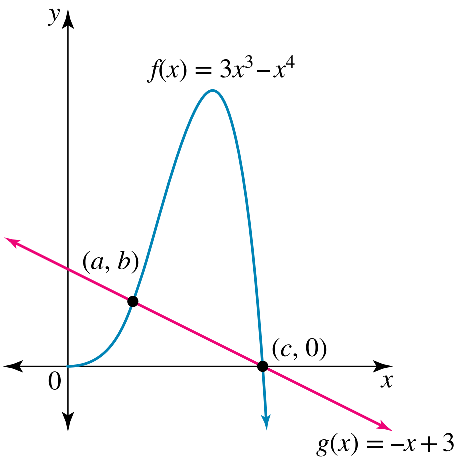
Apply the most appropriate mathematical processes and tools

11 The graphs of $f(x) = 3x^3 - x^4$ and $g(x) = -x + 3$ are shown.

a The graphs intersect at the points (a, b) and $(c, 0)$. Find the constants a, b and c.

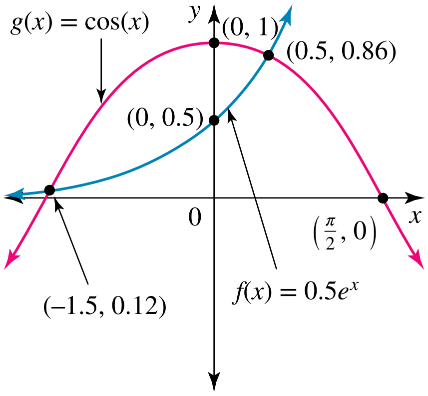
b Find the area enclosed between the curves from $x = a$ to $x = c$.

c Find the average value or mean value of the function $f(x) = 3x^3 - x^4$ for $x \in [1, 2.5]$. Give your answer correct to 3 decimal places.



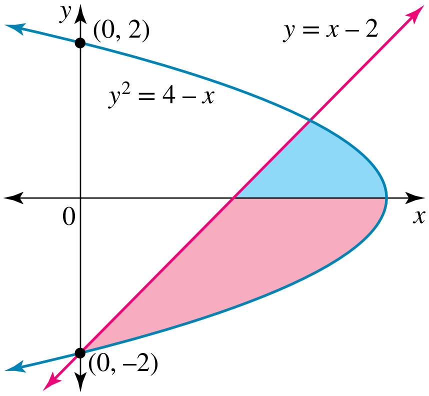
12 The graphs of $f(x) = 0.5e^x$ and $g(x) = \cos(x)$ are shown. The graphs intersect at (–1.5, 0.12) and (0.5, 0.86).

Using calculus, find the area enclosed between the curves from $x = -1.5$ to $x = 0.5$. Give your answer correct to 4 decimal places.

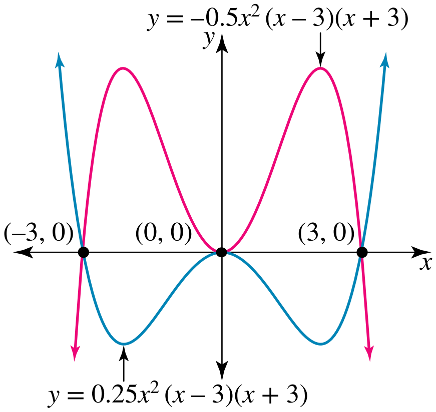


13 The graphs of $y^2 = 4 - x$ and $y = x - 2$ are shown.

- **a** Find the points of intersection of the two graphs.
- **b** Find the blue shaded area.
- **c** Find the pink shaded area.
- **d** Hence, find the area enclosed between the two graphs that is represented by the blue and pink shaded regions.



14 The graphs of $y = 0.25x^2(x - 3)(x + 3)$ and $y = -0.5x^2(x - 3)(x + 3)$ are shown.



Using calculus, find the area of the region enclosed between the curves and the lines $x = -3$ and $x = 3$.

15 **a** Sketch the graphs of $y = 0.5(x + 4)(x - 1)(x - 3)$ and $y = (3 - x)(x + 4)$ on the one set of axes.

- **b** Show that the three coordinate pairs of the points of intersection of the two graphs are (–4, 0), (–1, 12) and (3, 0).
- **c** Find the area, correct to 2 decimal places, enclosed between the curves from $x = -4$ to $x = 3$.

16 The edge of a garden bed can be modelled by the rule

$$y = 0.5\sin\left(\frac{x}{2}\right) + 2.$$

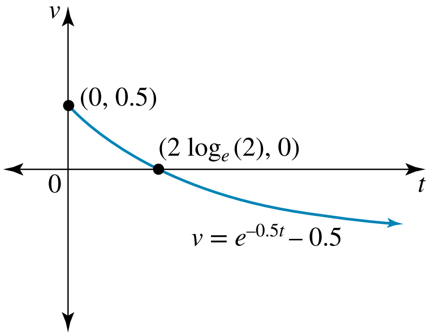
The bed has edges defined by $y = 0$, $x = 0$ and $x = 4\pi$. All measurements are in metres.

a Sketch the graph of $y = 0.5\sin\left(\dfrac{x}{2}\right) + 2$ along with $y = 0$, $x = 0$ and $x = 4\pi$ as edges to show the shape of the garden bed.

b Calculate the area of the garden bed, correct to the nearest square metre.

c Topsoil is going to be used on the garden bed in preparation for new planting for spring. The topsoil is to be spread so that it is uniformly 50 cm thick. Find the amount of soil, in cubic metres, that will be needed for the garden bed.

17 A particle moves in a straight line. At time t seconds its velocity, v metres per second, is defined by the rule $v = e^{-0.5t} - 0.5$, $t \geq 0$. The graph of the motion is shown.



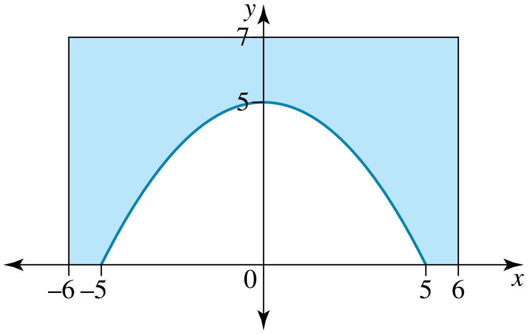
a Find the acceleration of the particle, a m/s^2, in terms of t.

b Find the displacement of the particle, x m, if $x = 0$ when $t = 0$.

c Find the displacement of the particle after 4 seconds.

d Find the distance covered by the particle in the fourth second. Give your answer correct to 4 decimal places.

18 A stone footbridge over a creek is shown along with the mathematical profile of the bridge.



The arch of the footbridge can be modelled by a quadratic function for $x \in [-5, 5]$, with all measurements in metres.

a Find the equation for the arch of the bridge

b Find the area between the curve and the x-axis from $x = -5$ to $x = 5$.

c Find the area of the side of the bridge represented by the shaded area.

d The width of the footbridge is 3 metres. Find the volume of stones used in the construction of the footbridge.

19 The rate of growth of mobile phone subscribers with a particular company in the UK can be modelled by the rule

$$\frac{dN}{dt} = 0.853e^{0.1333t}$$

where N million is the number of subscribers with the company since 1998 and t is the number of years since 1998, the year the company was established.

Find how many millions of mobile phone subscribers have joined the company between 1998 and 2015, correct to 1 decimal place.

20 The maintenance costs for a car increase as the car gets older. It has been suggested that the increase in maintenance costs of dollars per year could be modelled by

$$\frac{dC}{dt} = 15t^2 + 250$$

where t is the age of the car in years and C is the total accumulated cost of maintenance for t years.

a Sketch the graph of the given function for $0 \leq t \leq 10$.

b Find the total accumulated cost of maintenance for $t = 5$ to $t = 10$ years.

MASTER

21 Consider the functions $f(x) = \sin^2(x)$ and $g(x) = \cos^2(x)$.

a Sketch the graphs on the same set of axes for $0 \leq x \leq \pi$.

b Find the area between the curves for $0 \leq x \leq \pi$.

22 The cross-section of a waterway is parabolic. Its depth is 3 metres, and the width across the top of the waterway is 4 metres. When the waterway is one-third full, what is the depth of the water in metres, to 2 decimal places?

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8.5 Review

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- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

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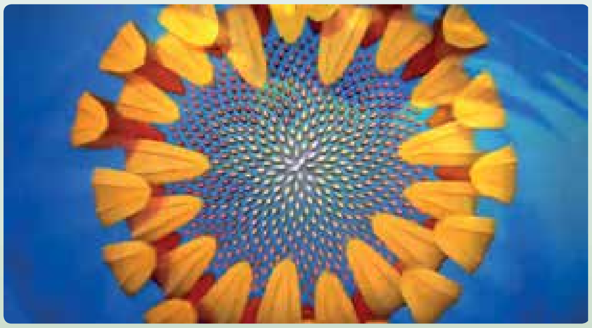

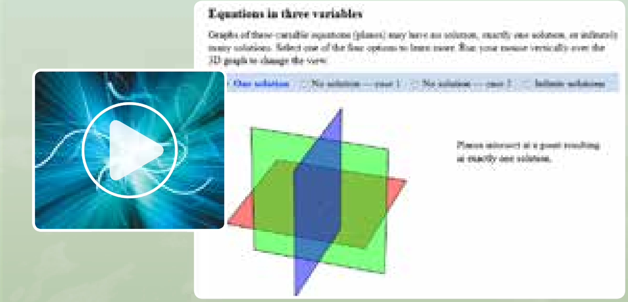

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8 Answers

EXERCISE 8.2

1 a $\frac{25}{12}$ units^2 **b** $\frac{77}{60}$ units^2

2 12 units^2

3 a 4 **b** 0

4 a $\frac{65}{4}$ **b** $2 + 0.5e^2 - 0.5e^{-2}$

5 a 21 **b** 11 **c** -16 **d** 8

6 $k = \pm 1$

7 a 30 units^2 **b** 22 units^2

8 21 units^2

9 a $a = 4$

b Left end point: 7.56 units^2
Right end point: 7.56 units^2

10 a 27 **b** $\frac{55}{6}$ **c** 0

d 0 **e** $\frac{4}{3}(\sqrt{10} - 2)$ **f** $\sqrt{2} - \frac{3\sqrt{3}}{4}$

11 a -15 **b** -12.5 **c** 32.5

d 20 **e** 25 **f** 12.5

12 $h = \frac{5}{9}$

13 $a = 4$

14 a $a = 1$ **b** $21\frac{1}{3}$ units^2

15 a $\frac{dy}{dx} = x\cos(x) + \sin(x)$ **b** $\pi + 2$

16 a $\frac{dy}{dx} = 3(x^2 - 2x)e^{x^3 - 3x^2}$ **b** $\frac{1}{3}(e^{-2} - 1)$

17 $k = \sqrt{5}$

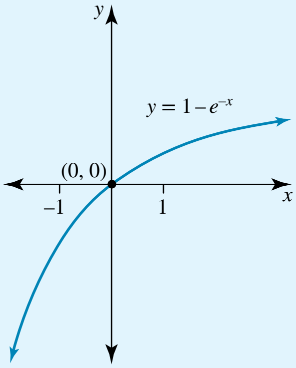
18 1.964

EXERCISE 8.3

1 $166\frac{2}{3}$ units^2

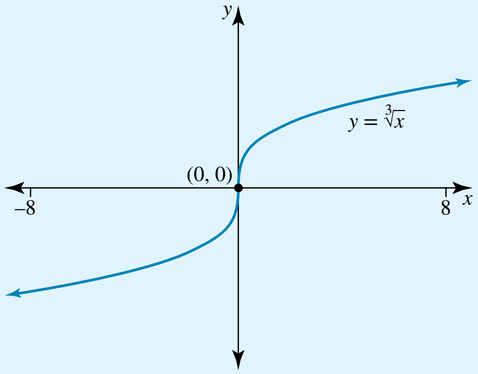
2 3π units^2

3



$e + e^{-1} - 2$ units^2

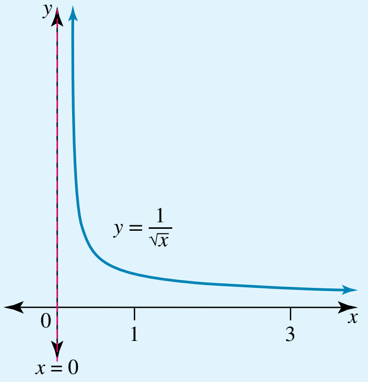
4



24 units^2

5 1.6 units^2

6 a

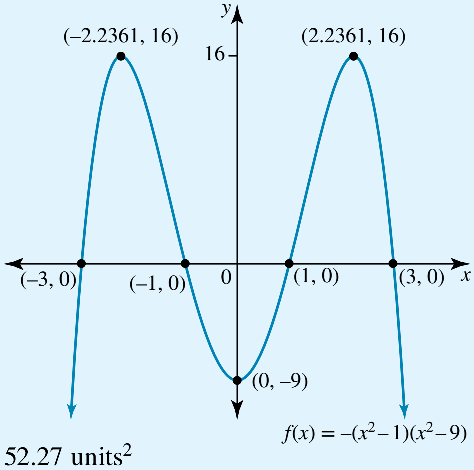


b $2\sqrt{3} - 2$ units^2

7 a $m = 2.6779$ **b** 4.2361 units^2

8 a $\frac{dy}{dx} = 2xe^{x^2}$ **b** $2(e - 1)$ units^2

9 a



b 52.27 units^2

10 16 units^2

11 a $a = -2,\ b = -1,\ c = 2,\ d = 3$

b 15.68 units^2

12 a $y = -3x^2 + 6x$ **b** 4 metres^2

c $440

13 a $\frac{3}{4}$ units^2 **b** $2\frac{1}{4}$ units^2

14 a 2 **b** $2(\pi - 2)$ units^2

15 a 7.7714 cm^2 **b** 15.5 cm^2

c Scale factor = 224

16 a 1.7642 units^2 **b** 9.933 units^2

EXERCISE 8.4

1 Point of intersection = (16, 4), area = $21\frac{1}{3}$ units^2

2 Area = $20\frac{5}{6}$ units^2

3 $2\sqrt{2}$ units^2

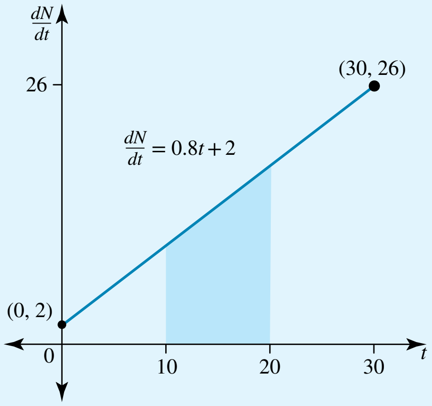
4 4.6254 units^2

5 $e - 1$

6 $-\frac{11}{12}$

7 28.4 cm

8 a and **b**



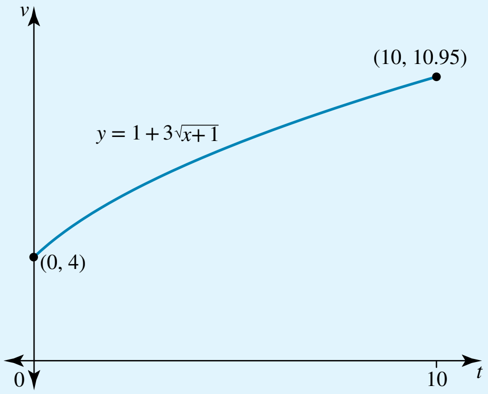
c 140 bricks

9 a 4 m/s

b $a = \dfrac{3}{2\sqrt{t+1}}$

i 1.5 m/s^2 **ii** 0.5 m/s^2

c

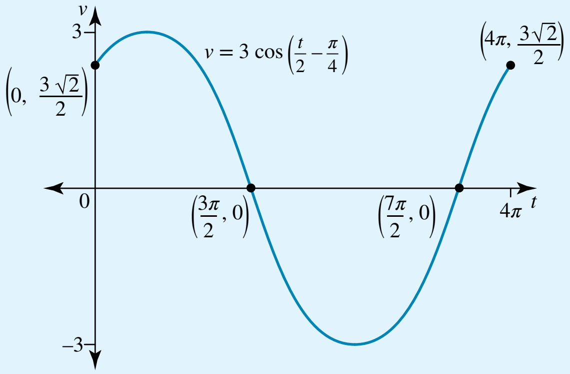


d 60 m

10 a $x = 6\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)$

b $-3\sqrt{2}$ m

c



d 20.49 m

e $a = -\frac{3}{2}\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)$

f $\frac{3\sqrt{2}}{4}$ m/s^2

11 a $a = 1,\ b = 2,\ c = 3$

b 9.6 units^2

c 6.144

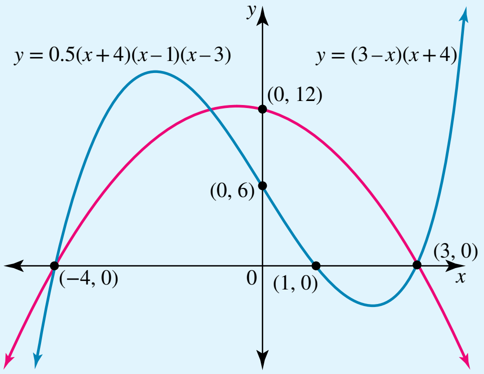
12 0.7641 units^2

13 a (0, –2), (3, 1) **b** $\frac{7}{6}$ units^2

c $3\frac{1}{3}$ units^2 **d** 4.5 units^2

14 48.6 units^2

15 a



b $y = 0.5(x + 4)(x - 1)(x - 3)$ (1)

$y = (3 - x)(x + 4)$ (2)

(1) = (2)

$$
\begin{aligned}
0.5(x + 4)(x - 1)(x - 3) &= (3 - x)(x + 4) \\
0.5(x + 4)(x - 1)(x - 3) - (3 - x)(x + 4) &= 0 \\
0.5(x + 4)(x - 1)(x - 3) + (x - 3)(x + 4) &= 0 \\
(x - 3)(x + 4)(0.5(x - 1) + 1) &= 0 \\
(x - 3)(x + 4)(0.5x - 0.5 + 1) &= 0 \\
(x - 3)(x + 4)(0.5x + 0.5) &= 0
\end{aligned}
$$

$x - 3 = 0,\ x + 4 = 0$ or $0.5x + 0.5 = 0$

$x = 3 \qquad x = -4 \qquad x = -1$

When $x = -4$, $y = (3 + 4)(-4 + 4) = 0$.

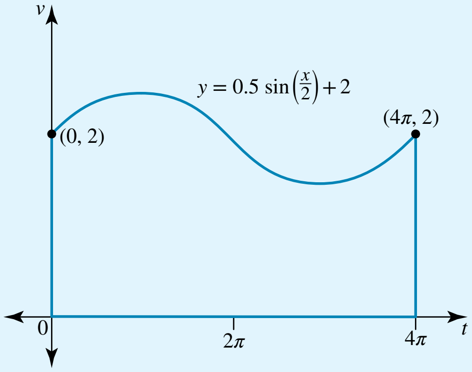
When $x = -1$, $y = (3 + 1)(-1 + 4) = 12$.

When $x = 3$, $y = (3 - 1)(3 + 4) = 0$.

Therefore, the coordinates are (–4, 0), (–1, 12) and (3, 0).

c 39.04 units^2

16 a



b 25 m^2

c 12.5 m^3

17 a $a = -0.5e^{-0.5t}$

b $x = -2e^{-0.5t} - 0.5t + 2$

c −0.2707 m

d 0.3244 m

18 a $y = 5 - 0.2x^2$

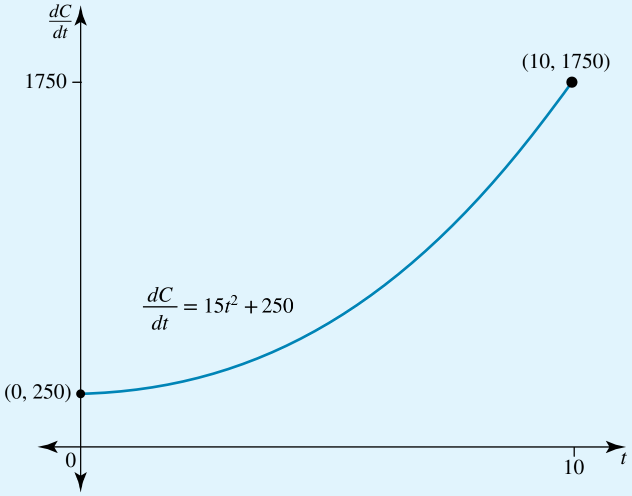
b $33\frac{1}{3}$ m^2

c $50\frac{2}{3}$ m^2

d 152 m^3

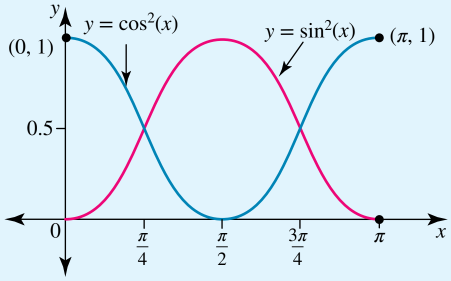
19 55.3 million

20 a



b $5625

21 a



b 2 units^2

22 1.44 m

9

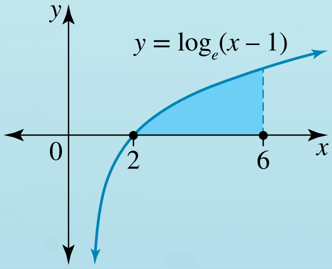
Logarithmic functions using calculus

9.1 Kick off with CAS

Logarithmic functions using calculus

1 a Using the graph application on CAS, sketch the graph of $f(x) = \log_e(x - 1)$.

b Estimate the area enclosed by the x-axis, the vertical lines $x = 2$ and $x = 6$, and the curve.



c Using the calculation application, use the integral template to calculate

$$\int_2^6 \log_e(x - 1)dx.$$

2 a Sketch the graph of $f(x) = \frac{1}{x}$.

b Estimate the area enclosed by the x-axis, the vertical lines $x = 1$ and $x = 3$, and the curve.

c Estimate the area enclosed by the x-axis, the vertical lines $x = –3$ and $x = –1$, and the curve.

d What do you notice about your answers to parts **b** and **c**? Why is this?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

9.2 The derivative of $f(x) = \log_e(x)$

The derivative of the logarithmic function

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Units 3 & 4
AOS 3
Topic 2
Concept 2

Rules for common derivatives
Concept summary
Practice questions

The proof for the derivative of $y = \log_e(x)$ relies heavily on its link to its inverse function $y = e^x$.

$$\text{If} \quad y = \log_e(x)$$

then, by the definition of a logarithm,

$$e^y = x.$$

$$\text{If} \quad x = e^y$$

then applying the exponential derivative rule gives

$$\frac{dx}{dy} = e^y.$$

However, it is known that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\text{so} \quad \frac{dy}{dx} = \frac{1}{e^y}.$$

$$\text{But} \quad e^y = x$$

$$\text{so therefore,} \quad \frac{dy}{dx} = \frac{1}{x}.$$

In summary,

$$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}.$$

It is worth noting that

$$\frac{d}{dx}(\log_e(kx)) = \frac{1}{x} \text{ also.}$$

This can be shown by applying the chain rule.

If $\quad y = \log_e(kx)$

let $\quad u = kx$ so that $\dfrac{du}{dx} = k.$

Thus, $\quad y = \log_e(u)$ and $\dfrac{dy}{du} = \dfrac{1}{u}.$

By the chain rule,

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = k \times \frac{1}{kx} \text{ when } u = kx.$$

$$\text{So} \quad \frac{dy}{dx} = \frac{1}{x}.$$

By using the chain rule it can also be shown that

$$\frac{d}{dx}(\log_e(g(x))) = \frac{g'(x)}{g(x)}.$$

In summary,

$$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_e(kx)) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_e(g(x))) = \frac{g'(x)}{g(x)}$$

Note: The above rules are only applicable for logarithmic functions of base *e*.

WORKED EXAMPLE 1

Find the derivative of each of the following.

a $3\log_e(2x)$

b $3\log_e(\sqrt{x})$

c $\log_e(\sin(x))$

d $\log_e(x^2 - 7x + 6)$

e $\dfrac{\log_e(x^3)}{x^2 - 1}$

f $(x^3 + 1)\log_e(x - 1)$

THINK	WRITE
a 1 Use the rule $\frac{d}{dx}(\log_e(kx)) = \frac{1}{x}$ to differentiate the function.	**a** $\frac{d}{dx}(3\log_e(2x)) = 3 \times \frac{1}{x}$
2 Simplify the answer.	$= \frac{3}{x}$
b 1 Rewrite the function using $\sqrt{x} = x^{\frac{1}{2}}$.	**b** $3\log_e(\sqrt{x}) = 3\log_e\left(x^{\frac{1}{2}}\right)$
2 Simplify the function by applying log laws.	$= 3 \times \frac{1}{2}\log_e(x)$ $= \frac{3}{2}\log_e(x)$
3 Differentiate the function and simplify.	$\frac{d}{dx}\left(\frac{3}{2}\log_e(x)\right) = \frac{3}{2} \times \frac{1}{x}$ $= \frac{3}{2x}$
c 1 Use the rule $\frac{d}{dx}(\log_e(g(x))) = \frac{g'(x)}{g(x)}$ to differentiate the function. State $g(x)$ and $g'(x)$.	**c** If $g(x) = \sin(x)$, $g'(x) = \cos(x)$
2 Substitute $g(x)$ and $g'(x)$ into the derivative rule.	$\frac{d}{dx}(\log_e(\sin(x))) = \frac{\cos(x)}{\sin(x)}$ or $\frac{1}{\tan(x)}$

d **1** Use the rule $\frac{d}{dx}(\log_e(g(x))) = \frac{g'(x)}{g(x)}$ to differentiate the function. State $g(x)$ and $g'(x)$.

d If $g(x) = x^2 - 7x + 6$,
$g'(x) = 2x - 7$

2 Substitute $g(x)$ and $g'(x)$ into the derivative rule.

$$\frac{d}{dx}(x^2 - 7x + 6) = \frac{2x - 7}{x^2 - 7x + 6}$$

e **1** Use the quotient rule to differentiate the function. Identify u and v.

Note: Simplify the logarithmic function by applying log laws.

e If $y = \frac{\log_e(x^3)}{x^2 - 1}$,
let $u = \log_e(x^3) = 3\log_e(x)$
and let $v = x^2 - 1$.

2 Differentiate u and v.

$$\frac{du}{dx} = \frac{3}{x}$$

$$\frac{dv}{dx} = 2x$$

3 Substitute the appropriate functions into the quotient rule.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2 - 1) \times \frac{3}{x} - (\log_e(x^3)) \times 2x}{(x^2 - 1)^2}$$

4 Simplify.

$$= \frac{(x^2 - 1) \times \frac{3}{x} - (\log_e(x^3)) \times 2x}{(x^2 - 1)^2} \times \frac{x}{x}$$

$$= \frac{3x^2 - 3 - 2x^2 \log_e(x^3)}{x(x^2 - 1)^2}$$

f **1** Use the product rule to differentiate the function. Identify u and v.

f If $y = (x^3 + 1)\log_e(x - 1)$,
let $u = x^3 + 1$
and let $v = \log_e(x - 1)$.

2 Differentiate u and v.

$$\frac{du}{dx} = 3x^2$$

$$\frac{dv}{dx} = \frac{1}{x - 1}$$

3 Substitute the appropriate functions into the product rule.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (x^3 + 1) \times \frac{1}{x - 1} + \log_e(x - 1) \times 3x^2$$

4 Simplify where possible.

$$= \frac{(x^3 + 1)}{(x - 1)} + 3x^2\log_e(x - 1)$$

Questions may also involve the differentiation of logarithmic functions to find the gradient of a curve at a given point or to find the equations of the tangent at a given point.

WORKED EXAMPLE 2

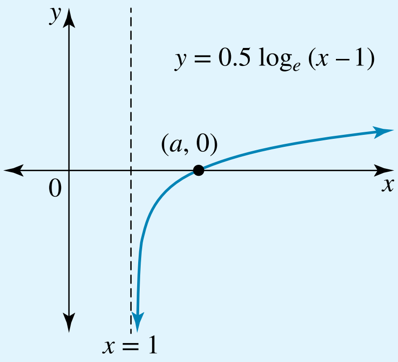
The graph of the function $f(x) = 0.5 \log_e(x - 1)$ is shown.

a **State the domain and range of f.**

b **Find the value of the constant a given that $(a, 0)$ is the x-axis intercept.**

c **Find the gradient of the curve at $(a, 0)$.**

d **Find the equation of the tangent at $(a, 0)$.**

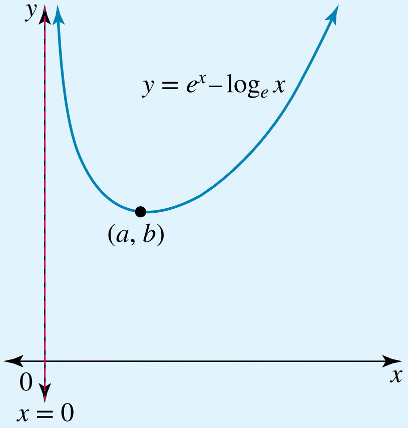


THINK	WRITE
a State the domain and range of the function.	**a** Domain $= (1, \infty)$. Range $= R$.
b **1** To find the x-intercept, $f(x) = 0$.	**b** $0.5 \log_e(x - 1) = 0$
2 Solve $0.5 \log_e(x - 1) = 0$ for x.	$\log_e(x - 1) = 0$ $e^0 = x - 1$ $1 = x - 1$ $x = 2$
3 Answer the question.	$(a, 0) \equiv (2, 0)$ $\therefore a = 2$
c **1** Determine the derivative of the function.	**c** $f(x) = 0.5 \log_e(x - 1)$ $f'(x) = \frac{1}{2} \times \frac{1}{x - 1}$ $= \frac{1}{2(x - 1)}$
2 Substitute $x = 2$ into the derivative to find the gradient at this point.	$f'(2) = \frac{1}{2(2 - 1)}$ $= \frac{1}{2}$ The gradient at $x = 2$ is $\frac{1}{2}$.
d **1** State the general equation for a tangent.	**d** The equation of the tangent is $y - y_1 = m_T(x - x_1)$.
2 State the known information.	The gradient of the tangent at $(x_1, y_1) = (2, 0)$ is $m_T = \frac{1}{2}$.
3 Substitute the values into the general equation.	$y - 0 = \frac{1}{2}(x - 2)$
4 Simplify.	$y = \frac{1}{2}x - 1$

Questions involving the derivative of the logarithmic function may involve maximum/minimum applications.

WORKED EXAMPLE 3

The graph of the function $y = e^x - \log_e(x)$ is shown.



Use calculus to find the values of the constants a and b, where (a, b) is the local minimum turning point. Give your answer correct to 3 decimal places.

THINK	WRITE
1 Determine the derivative of the function.	$y = e^x - \log_e(x)$ $\frac{dy}{dx} = e^x - \frac{1}{x}$
2 The minimum turning point occurs where $\frac{dy}{dx} = 0$.	$e^x - \frac{1}{x} = 0$ $e^x = \frac{1}{x}$
3 Use CAS to solve for x.	$x = 0.567$
4 Find the corresponding y-value.	When $x = 0.567$, $y = e^{0.567} - \log_e(0.567)$ $= 2.330$
5 Write the answer.	$a = 0.567, b = 2.330$

EXERCISE 9.2 The derivative of $f(x) = \log_e(x)$

PRACTISE

Work without CAS Questions 2–5

1 **WE1** Differentiate the following functions with respect to x.

a $7\log_e\left(\frac{x}{3}\right)$

b $2\log_e(x^3 + 2x^2 - 1)$

c $\sin(x)\log_e(x - 2)$

d $\frac{\log_e(x^2)}{2x - 1}$

e $3\log_e(e^{2x} - e^{-x})$

f $\sqrt{\log_e(3 - 2x)}$

2 Differentiate the following functions with respect to x and state any restrictions on x.

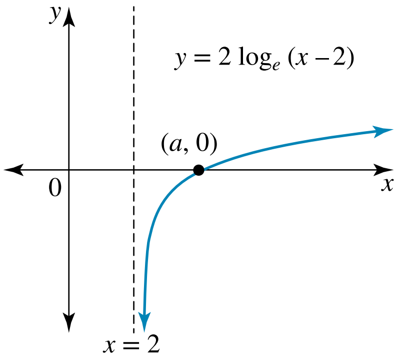
a $y = -5\log_e(2x)$

b $y = \log_e\left(\frac{1}{x - 2}\right)$

c $y = \log_e\left(\frac{x + 3}{x + 1}\right)$

d $y = \log_e(x^2 - x - 6)$

3 **WE2** The graph of the function $f: (2, \infty) \to R,\ f(x) = 2\log_e(x-2)$ is shown.



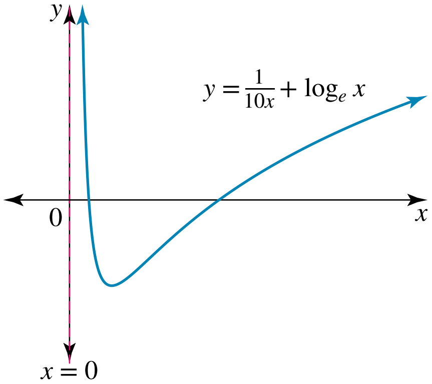
- **a** State the domain and range of f.
- **b** Find the value of the constant a, given that $(a, 0)$ is the x-axis intercept.
- **c** Find the equations of the tangent at $(a, 0)$.
- **d** Find the equation of the line perpendicular to the curve at $(a, 0)$.

4 Find the equation of the tangent to the curve $y = 4\log_e(3x-1)$ at the point where the tangent is parallel to the line $6x - y + 2 = 0$.

5 **WE3** The graph of the function

$f: R^+ \to R,\ f(x) = \dfrac{1}{10x} + \log_e(x)$ is shown.

Use calculus to determine the coordinates of the minimum turning point.



6 Use calculus to determine the local maximum or minimum value of the function defined by:

- **a** $f(x) = 2x\log_e(x),\ x > 0$
- **b** $f(x) = \dfrac{\log_e(2x)}{x},\ x > 0$
- **c** $f(x) = x\log_e\left(\dfrac{3}{x}\right),\ x > 0.$

In each case, investigate the nature of the turning point to determine whether it is a maximum or a minimum.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 Find the derivative of each of the following functions.

- **a** $4\log_e\left(\dfrac{x}{2}\right)$
- **b** $\frac{1}{2}\log_e(\sqrt{x-2})$
- **c** $\log_e(x^3 - 3x^2 + 7x - 1)$
- **d** $-6\log_e(\cos(x))$
- **e** $\sqrt{\log_e(3x+1)}$
- **f** $\dfrac{2\log_e(2x)}{e^{2x}+1}$

8 Find the derivative of each of the following functions. State any restrictions on x.

- **a** $(x^2 - 3x + 7)\log_e(2x-1)$
- **b** $\sin(x)\log_e(x^2)$
- **c** $\dfrac{\log_e(3x)}{(x^3 - x)}$
- **d** $\log_e\left(\dfrac{4-x}{x+2}\right)$

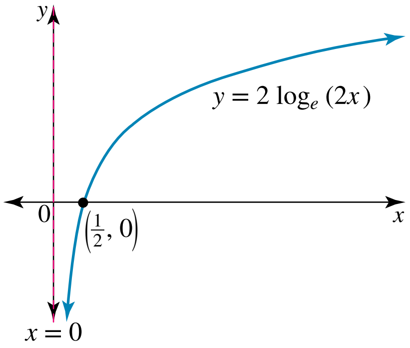
9 Find the gradient of each of the following functions at the specified point.

- **a** $y = 2\log_5(x);\ x = 5$
- **b** $y = \frac{1}{3}\log_3(x+1);\ x = 2$
- **c** $y = \log_6(x^2 - 3);\ x = 3$

10 Find the equation of the tangent to each of the given curves at the specified point.

- **a** $y = \log_e(2x-2)$ at $\left(\frac{3}{2}, 0\right)$
- **b** $y = 3\log_e(x)$ at $(e, 3)$
- **c** $y = \frac{1}{2}\log_e(x^2)$ at $(e, 1)$

11 The graph of the function defined by the rule $y = 2\log_e(2x)$ is shown.



a Find the derivative of y with respect to x.

b Find the equation of the tangent at $\left(\dfrac{e}{2}, e\right)$.

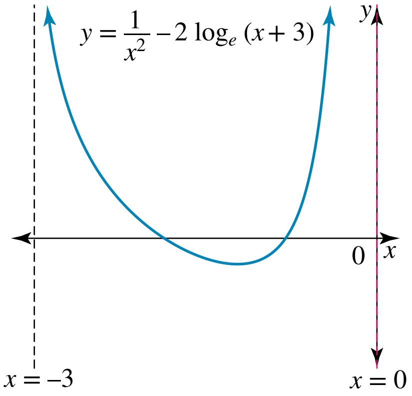
12 The line $y = x$ is a tangent to the curve $y = \log_e(x-1) + b$, where b is a constant. Find the possible value of b.

13 The equation of a line perpendicular to the curve $y = \log_e(2(x-1))$ has the equation $y = -2x + k$, where k is a constant. Find the value of k, correct to 1 decimal place.

14 The graph of the function

$$f : R^- \rightarrow R, f(x) = \frac{1}{x^2} - 2\log_e(x+3)$$

is shown.



a Determine the coordinates of the x-intercepts, correct to 3 decimal places.

b Find the equations of the tangents at the x-axis intercepts.

c Find the coordinates of the minimum turning point. Give your answer correct to 4 decimal places.

15 a The function f is defined by $f : [0, \infty) \rightarrow R, f(x) = (2\log_e(x))^2$, and the function g is defined by $g : [0, \infty) \rightarrow R, f(x) = 2\log_e(x)$. Find the coordinates of the points of intersection between f and g.

b Find the gradient of each graph at the point (1, 0).

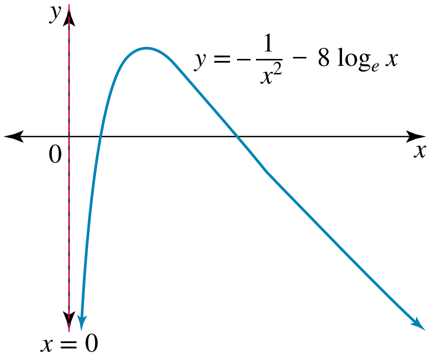
c Sketch both graphs on the same set of axes.

d For what x-values is $2\log_e x > (2\log_e(x))^2$?

16 The graph of the function

$$f : R^+ \rightarrow R, f(x) = -\frac{1}{x^2} - 8\log_e(x)$$

is shown.



a Find the coordinates of the x-intercepts, giving your answers correct to 4 decimal places.

b Find the gradient of the curve at the points found in part **a**, giving your answers correct to 2 decimal places.

c Find the equation of the tangent at $(1, -1)$ and the equation of the line perpendicular to the curve at $(1, -1)$.

d Show that the coordinates of the maximum turning point are $\left(\frac{1}{2}, -4 + 8\log_e(2)\right)$.

MASTER

17 **a** The function defined by the rule $f: (-\infty, 2) \to R$, $f(x) = -2\log_e(2 - x) - 1$ has an inverse function, f^{-1}. State the rule, domain and range for f^{-1}.

b Find the coordinates of the point(s) of intersection of f and f^{-1}. Give your answers correct to 4 decimal places.

c On the one set of axes, sketch the graphs of f and f^{-1}, showing all relevant features.

18 The tangent to the curve $y = \log_e(2x - 1)$ at $x = n$ intersects the x-axis at $x = 0.3521$. Find the value of the integer constant n.

9.3 The antiderivative of $f(x) = \frac{1}{x}$

studyon

Units 3 & 4
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Rules for common antiderivatives
Concept summary
Practice questions

We have found previously that if $y = \log_e(x)$, then $\frac{dy}{dx} = \frac{1}{x}$.

Therefore,

$$\boxed{\int \frac{1}{x}dx = \log_e(x) + c, x > 0}$$

Also,

$$\boxed{\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e(ax+b) + c, x > -\frac{b}{a}}$$

WORKED EXAMPLE 4

a Find $\int \frac{3}{2x}dx$.

b Evaluate $\int_0^3 \frac{4}{2x+1}dx$.

THINK	WRITE
a 1 Remove $\frac{3}{2}$ as a factor.	$\int \frac{3}{2x}dx = \frac{3}{2}\int \frac{1}{x}dx$
2 Apply the integration rule.	$= \frac{3}{2}\log_e(x) + c, x > 0$
b 1 Remove 4 as a factor.	$\int_0^3 \frac{4}{2x+1}dx = 4\int_0^3 \frac{1}{2x+1}dx$
2 Apply the integration rule.	$= \left[4 \times \frac{1}{2}\log_e(2x+1)\right]_0^3, x > -\frac{1}{2}$ $= [2\log_e(2x+1)]_0^3, x > -\frac{1}{2}$
3 Substitute the end points and evaluate.	$= (2\log_e(2(3)+1)) - (2\log_e(2(0)+1))$ $= 2\log_e(7) - 2\log_e(1)$ $= 2\log_e(7)$

Integration by recognition

Integration by recognition is used when we want to antidifferentiate more complex functions that we don't have an antiderivative rule for. This method involves finding the derivative of a related function and using this derivative to find the antiderivative.

WORKED EXAMPLE 5 Differentiate $y = \frac{x^2}{4}\log_e(x)$ and hence find $\int x\log_e(x)dx$.

THINK	WRITE
1 Use the product rule to differentiate the given function.	$y = \frac{x^2}{4}\log_e(x)$ Let $u = \frac{x^2}{4}$, so $\frac{du}{dx} = \frac{x}{2}$. Let $v = \log_e(x)$, so $\frac{dv}{dx} = \frac{1}{x}$. $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $= \frac{x^2}{4} \times \frac{1}{x} + \log_e(x) \times \frac{x}{2}$ $= \frac{x}{4} + \frac{x}{2}\log_e(x)$
2 Express the answer in integral form. *Note:* There is no need to include $+c$ as the question asked for 'an' antiderivative.	$\int\left(\frac{x}{4} + \frac{x}{2}\log_e(x)\right)dx = \frac{x^2}{4}\log_e(x),\ x > 0$
3 Separate the two parts of the integral.	$\int\left(\frac{x}{4}\right)dx + \int\left(\frac{x}{2}\log_e(x)\right)dx = \frac{x^2}{4}\log_e(x)$
4 Subtract $\int\left(\frac{x}{4}\right)dx$ from both sides to make $\int\frac{x}{2}\log_e(x)dx$ the subject. (Remember we are determining $\int x\log_e(x)dx$).	$\int\frac{x}{2}\log_e(x)dx = \frac{x^2}{4}\log_e(x) - \int\left(\frac{x}{4}\right)dx$
5 Antidifferentiate the function on the right side of the equation, $\frac{x}{4}$.	$\int\frac{x}{2}\log_e(x)dx = \frac{x^2}{4}\log_e(x) - \frac{x^2}{8}$
6 Remove $\frac{1}{2}$ as a factor so that the function to be integrated matches the one in the question.	$\frac{1}{2}\int x\log_e(x)dx = \frac{x^2}{4}\log_e(x) - \frac{x^2}{8}$
7 Multiply the equation through by 2.	$2 \times \frac{1}{2}\int x\log_e(x)dx = 2\left(\frac{x^2}{4}\log_e(x) - \frac{x^2}{8}\right)$
8 State the answer.	$\int x\log_e(x)dx = \frac{x^2}{2}\log_e(x) - \frac{x^2}{4},\ x > 0$

Area under and between curves

Evaluating the area under a curve or the area between curves can also involve logarithmic functions as part of the method.

WORKED EXAMPLE 6

The graph of $y = \dfrac{1}{x-2} - 3, \quad x > 2$ is shown.

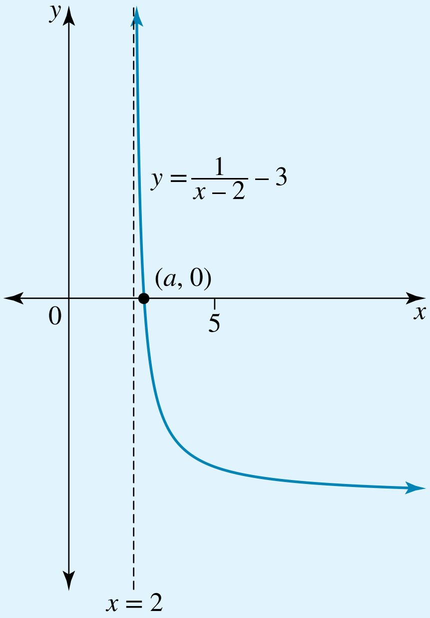
a Find the value of the constant a given that $(a, 0)$ is the x-axis intercept.

b Find the area between the curve and the x-axis from $x = a$ to $x = 5$.

c Find the equation of the straight line that joins the points $(0, 1)$ and $(a, 0)$.

d Find the other point of intersection between the line and the curve.

e Use calculus to find the area between the curve and the line to 2 decimal places.



THINK

a The x-intercept is found by equating the function to zero.

WRITE

a
$$\frac{1}{x-2} - 3 = 0$$
$$\frac{1}{x-2} = 3$$
$$1 = 3(x-2)$$
$$1 = 3x - 6$$
$$7 = 3x$$
$$x = \tfrac{7}{3}$$

Therefore, $a = \frac{7}{3}$.

b **1** State the integral needed to find the area under the curve from $\frac{7}{3}$ to 5. Remember to account for the region being underneath the x-axis.

b $$A = -\int_{\frac{7}{3}}^{5} \left(\frac{1}{x-2} - 3\right) dx$$

2 Antidifferentiate and evaluate.

$$= -\left[\log_e(x-2) - 3x\right]_{\frac{7}{3}}^{5}$$
$$= -\left[\log_e(5-2) - 3(5) - \left(\log_e\left(\tfrac{7}{3} - 2\right) - 3\left(\tfrac{7}{3}\right)\right)\right]$$
$$= -\left[\log_e(3) - 15 - \log_e\left(\tfrac{1}{3}\right) + 7\right]$$

3 Simplify.

$$= -\left[\log_e(3) - 8 - \log_e(3^{-1})\right]$$
$$= -\left[\log_e(3) - 8 + \log_e(3)\right]$$
$$= -\left[2\log_e(3) - 8\right]$$
$$= -2\log_e(3) + 8 \text{ units}^2$$

c Find the equation of the line joining $(0, 1)$ and $\left(\frac{7}{3}, 0\right)$,

c $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = \left(\frac{7}{3}, 0\right)$

$$m = \frac{0 - 1}{\frac{7}{3} - 0} = -\frac{3}{7}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -\tfrac{3}{7}(x - 0) \\ y &= -\tfrac{3}{7}x + 1 \end{aligned}$$

d 1 Solve the two equations simultaneously by equating the two equations.

d

$$\begin{aligned} -\tfrac{3}{7}x + 1 &= \frac{1}{x - 2} - 3 \\ -\tfrac{3}{7}x &= \frac{1}{x - 2} - 4 \\ -3x &= \frac{7}{x - 2} - 28 \\ -3x(x - 2) &= 7 - 28(x - 2) \\ -3x^2 + 6x &= 7 - 28x + 56 \\ 0 &= 3x^2 - 34x + 63 \\ 0 &= (3x - 7)(x - 9) \\ x &= \tfrac{7}{3}, 9 \end{aligned}$$

2 Identify the required x-value and find the corresponding y-value.

$$\begin{aligned} x &= 9 \\ y &= -\tfrac{3}{7} \times 9 + 1 \\ &= -\tfrac{20}{7} \end{aligned}$$

3 State the answer.

The point of intersection is $\left(9, -\frac{20}{7}\right)$.

e 1 Write the rule to find the area between the curves from $x = \frac{7}{3}$ to $x = 9$.

e

$$\begin{aligned} A &= \int_{\frac{7}{3}}^{9} \left(-\tfrac{3}{7}x + 1 - \left(\frac{1}{x - 2} - 3\right)\right) dx \\ &= \int_{\frac{7}{3}}^{9} \left(-\tfrac{3}{7}x - \frac{1}{x - 2} + 4\right) dx \end{aligned}$$

2 Antidifferentiate.

$$= \left[-\tfrac{3}{14}x^2 - \log_e(x - 2) + 4x\right]_{\frac{7}{3}}^{9}$$

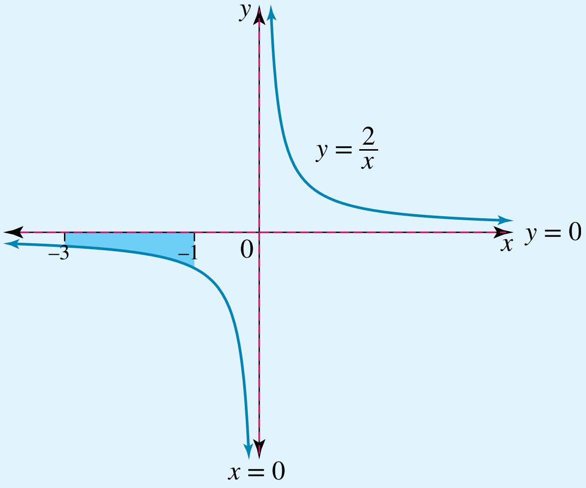
3 Use CAS technology to find the area.

Note: If you are asked to use calculus, it is important that you write the rule and show the antidifferentiation, even when you are using CAS technology to find the result.

The area between the curves is 7.43 units^2.

WORKED EXAMPLE 7

Using calculus, find the area enclosed between the curve $y = \frac{2}{x}$, the x-axis and the lines $x = -3$ and $x = -1$.

WRITE	THINK
1 Sketch a graph of the required area.	
2 State the integral needed to find the area under the curve from $x = -3$ to $x = -1$.	$A = -\int_{-3}^{-1} \left(\frac{2}{x}\right) dx$
3 The integral of $y = \frac{1}{x}$ is $y = \log_e(x)$. Negative values cannot be substituted, so symmetry must be used to find the area.	$A = \int_{1}^{3} \left(\frac{2}{x}\right) dx$
4 Antidifferentiate and evaluate.	$= \left[2\log_e(x)\right]_1^3$ $= 2\log_e(3) - 2\log_e(1)$ $= 2\log_e(3)$ units^2

EXERCISE 9.3 The antiderivative of $f(x) = \frac{1}{x}$

PRACTISE

Work without CAS
Questions 1–5, 7, 8

1 **WE4** **a** Find $\int \frac{2}{5x} dx$. **b** Evaluate $\int_1^3 \frac{3}{4x - 1} dx$.

2 **a** Find $\int \frac{x^2 + 2x - 3}{x^2} dx$.

b Find $f(x)$ if $f'(x) = x^3 - \frac{1}{x}$ and $f(1) = \frac{1}{4}$.

3 **WE5** Differentiate $y = 2\log_e(\cos(2x))$ and hence find an antiderivative of $\int \tan(2x) dx$.

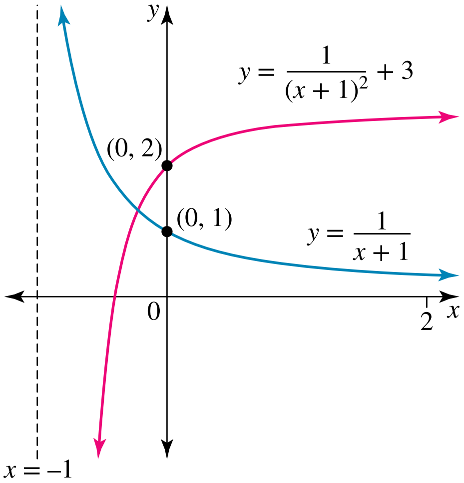
4 Differentiate $y = (\log_e(x))^2$ and hence evaluate $\int_1^e \frac{4\log_e x}{x} dx$.

5 **WE6** Consider $f(x) = \dfrac{1}{x+2} - 1, x > -2$.

a Find the value of the constant a, where $(a, 0)$ is the x-axis intercept.
b Find the area between the curve and the x-axis from $x = a$ to $x = 2$.
c A straight line given by $y = -\frac{1}{2}x + \frac{1}{4}$ intersects $y = f(x)$ in two places. What are the coordinates of the points of intersection?
d Use calculus to find the area between the curve and the line.

6 The graphs of $y = \dfrac{1}{x+1}, x > -1$ and $y = -\dfrac{1}{(x+1)^2} + 3, x > -1$ are shown.

a Find the coordinates of the point of intersection of the two graphs.
b Using calculus, find the area enclosed between the curves and the line $x = 2$. Give your answer correct to 4 decimal places.



7 **WE7** Using calculus, find the area enclosed between the curve $y = \dfrac{1}{x}$, the x-axis and the lines $x = -4$ and $x = -2$.

8 Using calculus, find the area enclosed between the curve $y = \dfrac{1}{x-1} + 2$, the x-axis and the lines $x = -2$ and $x = -1$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 Antidifferentiate the following.

a $-\dfrac{4}{x}$

b $\dfrac{3}{4x+7}$

c $\dfrac{x^3 + 2x^2 + 3x - 1}{x^2}$

d $\dfrac{3}{2-x} + \cos(4x)$

10 Evaluate:

a $\displaystyle\int_2^4 \frac{3}{1-2x}dx$

b $\displaystyle\int_{-3}^{-1} \frac{2}{x+4}dx$

c $\displaystyle\int_1^4 \left(e^{2x} + \frac{2}{x}\right)dx$, correct to 2 decimal places.

11 **a** Given that $\dfrac{dy}{dx} = \dfrac{5}{2x+4}$ and $y = 3$ when $x = -\frac{3}{2}$, find an expression for y in terms of x.

b Given that $\dfrac{dy}{dx} = \dfrac{3}{2-5x}$ and $y = 1$ when $x = \frac{1}{5}$, find an expression for y in terms of x.

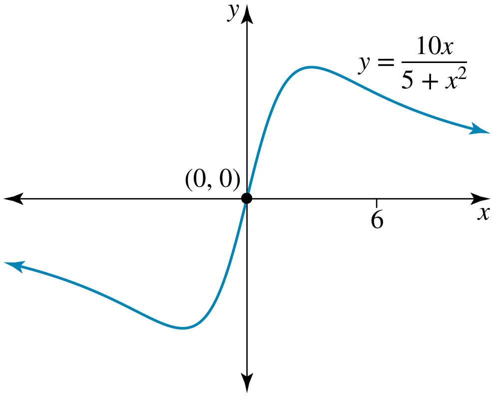
12 If $f(x) = 2x\log_e(mx)$, find $f'(x)$ and hence find $\int \log_e(mx)dx$ where m is a constant.

13 Differentiate $3x\log_e(x)$ and hence find an antiderivative for $2\log_e(x)$.

14 Differentiate $\log_e(3x^3 - 4)$ and hence evaluate $\int_2^3 \frac{x^2}{3x^3 - 4}dx$.

15 Differentiate $\log_e(e^x + 1)^2$ and hence find $\int_1^5 \frac{e^x}{e^x + 1}dx$, correct to 4 decimal places.

16 The graph of $y = \frac{10x}{5 + x^2}$ is shown.

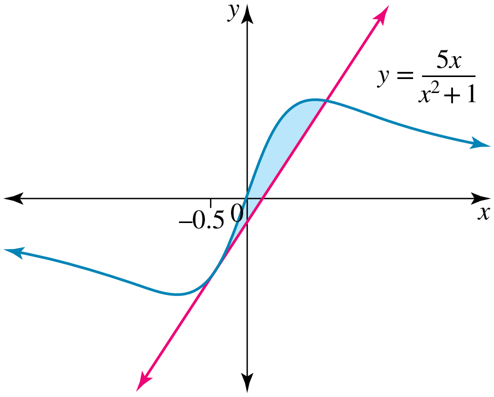


a Find the exact coordinates of the minimum and maximum turning points.

b Find the derivative of $\log_e(5 + x^2)$ and hence find an antiderivative for $\frac{10x}{5 + x^2}$.

c Find the area enclosed between the curve, the x-axis, the line where x equals the x-coordinate of the maximum turning point, and the line $x = 6$.

17 The graph of $y = \frac{5x}{x^2 + 1}$ is shown. The tangent to the curve at $x = -0.5$ is also shown.

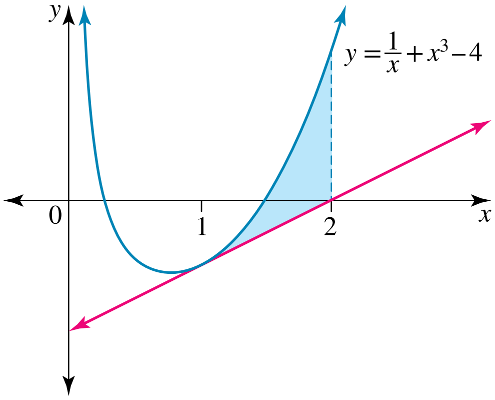


a Find the equation of the tangent to the curve at $x = -0.5$.

b Find the derivative of $\log_e(x^2 + 1)$ and hence find an antiderivative for $\frac{5x}{x^2 + 1}$.

c Using calculus, find the area of the shaded region. Give your answer correct to 4 decimal places.

18 The graph of the function $y = \frac{1}{x} + x^3 - 4$ is shown. The tangent to the curve at $x = 1$ is also shown.



a Find the equation of the tangent to the curve at $x = 1$.

b Find the area of the shaded region.

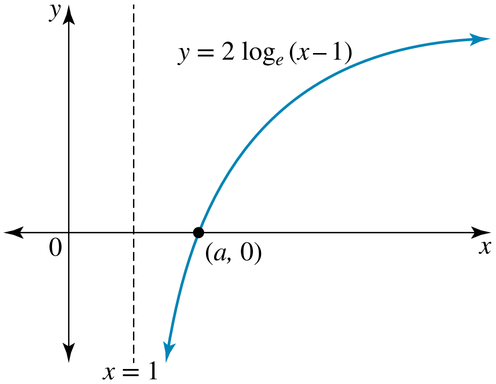
MASTER

19 The graph of the function $f: (1, \infty) \to R,\ f(x) = 2\log_e(x - 1)$ is shown.

a State the domain and range of f.

b Find the value of the constant a, given that $(a, 0)$ is the x-axis intercept.

c Find the area between the curve and the x-axis from $x = a$ to $x = 5$, correct to 4 decimal places.

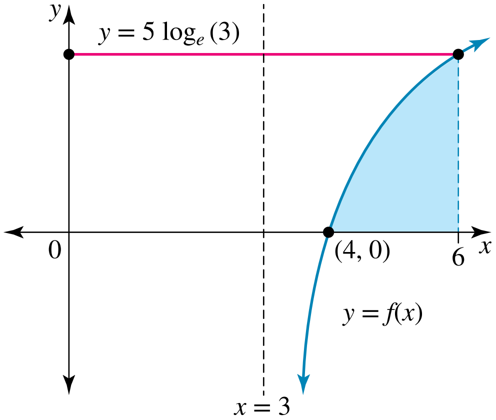


20 The graph of the function $f: (3, \infty) \to R,\ f(x) = 5\log_e(x - 3)$ is shown.

a Find the area of the shaded region, correct to 3 decimal places.

b Find the rule for the inverse function, $y = f^{-1}(x)$.

c Verify your answer to part **a** by finding the area enclosed between the curve $y = f^{-1}(x)$, the y-axis and the line $y = 6$.



9.4 Applications

eBook*plus*

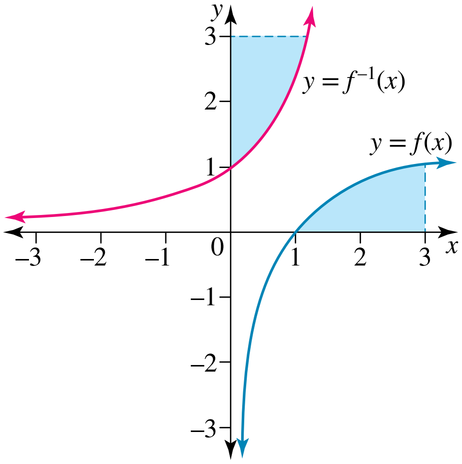
Interactivity
Area under a curve
int-6426

Application problems can involve real-life applications of logarithms. We cannot antidifferentiate a logarithmic function without technology, so if we want to find the area under a logarithmic curve, we require another method. One option is to use integration by recognition. Another is to link the areas bound by the curve of the inverse of the required function and the axes.

To find the inverse of a function, all components relating to x of the original function will relate to y of the inverse. Similarly, all components relating to y of the original function will relate to x of the inverse. This is also true for areas bound by the curve and the axes.

If $f(x) = \log_e(x),\ f^{-1}(x) = e^x$.

The area bound by the curve of $f(x)$ and the x-axis from $x = 1$ to $x = 3$ is shown. This area is equivalent to the area bound by the curve of $f^{-1}(x)$ and the y-axis from $y = 1$ to $y = 3$.



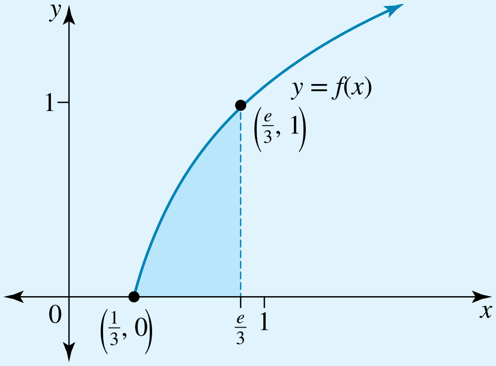
WORKED EXAMPLE 8

The graph of the function $f:\left[\frac{1}{3}, \infty\right) \to R,\ f(x) = \log_e(3x)$ is shown.

a Find $f^{-1}(x)$.

b Calculate $\int_0^1 f^{-1}(x)dx$.

c Hence, find the exact area of the shaded region.



THINK

WRITE/DRAW

a To find the inverse, swap x and y, and solve for y.

a Let $y = f(x)$.

Swap x and y:

$$\begin{aligned} \Rightarrow x &= \log_e(3y) \\ e^x &= 3y \\ y &= \tfrac{1}{3}e^x \\ \therefore f^{-1}(x) &= \tfrac{1}{3}e^x \end{aligned}$$

b 1 Set up the appropriate integral and antidifferentiate.

b
$$\int_0^1 f^{-1}(x)dx = \int_0^1 \tfrac{1}{3}e^x dx$$
$$= \left[\tfrac{1}{3}e^x\right]_0^1$$

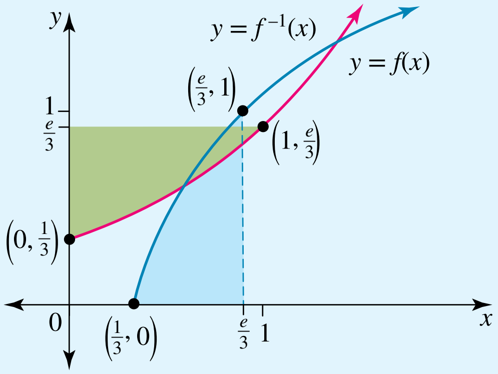
2 Evaluate.

$$= \tfrac{1}{3}e^1 - \tfrac{1}{3}e^0$$
$$= \frac{e}{3} - \frac{1}{3}$$

c 1 The required shaded area is $\int_{\frac{1}{3}}^{\frac{e}{3}} f(x)dx$, the blue area.

This is equivalent to the area bound by the curve of $f^{-1}(x)$ and the y-axis from $y = \dfrac{1}{3}$ to $y = \dfrac{e}{3}$.

c



2 To find the area bound by the y-axis, the green shaded area, we need to find the area of the rectangle with coordinates $(0, 0)$, $\left(0, \dfrac{e}{3}\right)$, $\left(1, \dfrac{e}{3}\right)$, and $(1, 0)$.

$$A_{\text{rectangle}} = 1 \times \frac{e}{3}$$
$$= \frac{e}{3}$$

3 Subtract the area underneath $f^{-1}(x)$, from $x = 0$ to $x = 1$ (worked out in part **b**). This answer is the required green shaded area.

$$A = A_{\text{rectangle}} - \int_0^1 f^{-1}(x)dx$$
$$= \frac{e}{3} - \left(\frac{e}{3} - \frac{1}{3}\right)$$
$$= \frac{1}{3}$$

4 State the answer.

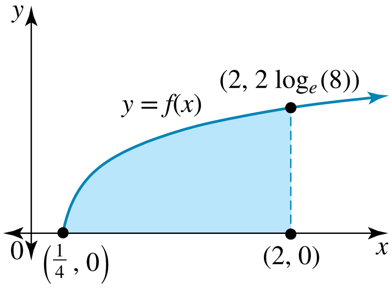
$$\int_{\frac{1}{3}}^{\frac{e}{3}} f(x)dx = \frac{1}{3}\text{ units}^2$$

EXERCISE 9.4 Applications

PRACTISE

Work without CAS
Question 1

1 **WE8** The graph of the function $f : \left[\frac{1}{4}, \infty\right) \to R,\ f(x) = 2\log_e(4x)$ is shown.

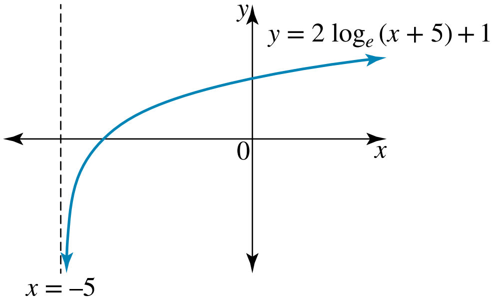


a Find $f^{-1}(x)$.

b Calculate $\int_0^{2\log_e 8} f^{-1}(x)dx$.

c Hence, find the exact area of the shaded region.

2 Part of the graph of the function $h : (-5, \infty) \to R,\ h(x) = 2\log_e(x+5) + 1$ is shown.



a Find the coordinates of the axial intercepts.

b Find the rule and domain for h^{-1}, the inverse of h.

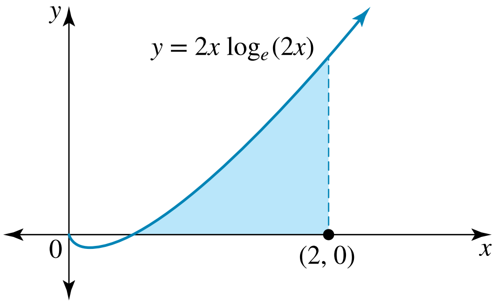
c On the one set of axes, sketch the graphs of $y = h(x)$ and $y = h^{-1}(x)$. Clearly label the axial intercepts with exact values and any asymptotes.

d Find the values of x, correct to 4 decimal places, for which $h(x) = h^{-1}(x)$.

e Find the area of the region enclosed by the graphs of h and h^{-1}. Give your answer correct to 4 decimal places.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

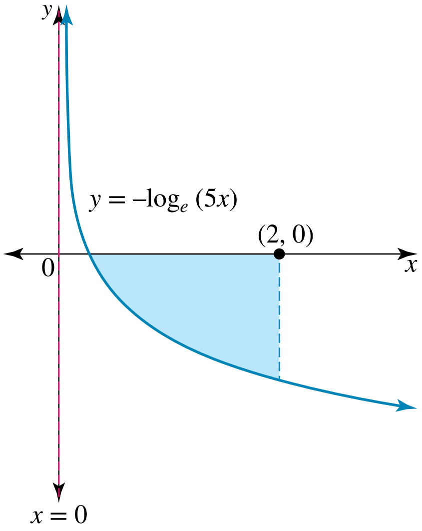


3 Part of the graph of the function $f : R^+ \to R$, $f(x) = 2x\log_e(2x)$ is shown.

a Find the coordinates of the point where the graph intersects the x-axis when $x > 0$.

b Find the derivative of $x^2\log_e(3x)$.

c Use your answer to part **b** to find the area of the shaded region, correct to 3 decimal places.



4 Part of the graph of the function $g : \left[\frac{1}{5}, \infty\right) \to R$, $g(x) = -\log_e(5x)$ is shown.

a Find the coordinates of the point where the graph intersects the x-axis.

b If $y = -x\log_e(5x) + x$, find $\frac{dy}{dx}$.

c Use your result from part **b** to find the area of the shaded region.

5 Let h be the graph of the function $h : D \to R$, $h(x) = \log_e(2 - 4x)$, where D is the largest possible domain over which h is defined.

a Find the exact coordinates of the axial intercepts of the graph $y = h(x)$.

b Find D as the largest possible domain over which h is defined.

c Use calculus to show that the rate of change of h with respect to x is always negative.

d **i** Find the rule for h^{-1}.

ii State the domain and range of h^{-1}.

e On the one set of axes, sketch the graphs of $y = h(x)$ and $y = h^{-1}(x)$, clearly labelling intercepts with the x- and y-axes with exact values. Label any asymptotes with their equations.

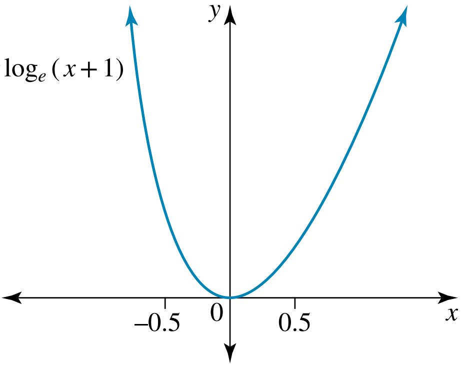
6 **a** If $y = x\log_e(x)$, find $\frac{dy}{dx}$. Hence find the exact value of $\int_1^{e^2} \log_e(x)dx$.

b If $y = x(\log_e(x))^m$ where m is a positive integer, find $\frac{dy}{dx}$.

c Let $I_m = \int_1^{e^2} (\log_e(x))^m dx$ for $m > 1$. Show that $I_m + mI_{m-1} = 2^m e^2$.

d Hence, find the value of $\int_1^{e^2} (\log_e(x))^3 dx$.

7 The graph of $y = 4x \log_e (x + 1)$ is shown.

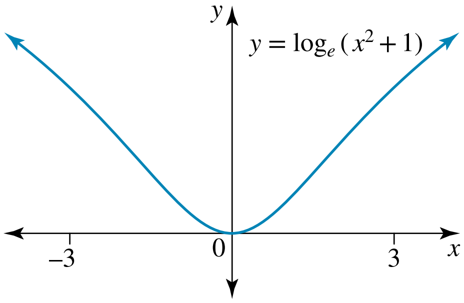


a Find the equation of the tangent to the curve at $x = -0.5$.

b Differentiate $y = 2(x^2 - 1) \log_e (x + 1)$ with respect to x.

c Hence, find the area enclosed between the curve, the x-axis and the lines $x = -0.5$ and $x = 0.5$.

8 The graph of the function $m : R \rightarrow R$, $m(x) = \log_e (x^2 + 1)$ is shown.



a Find the gradient of the curve at $x = -2$.

b Determine the area enclosed between the curve, the x-axis and the lines $x = 0$ and $x = 3$, correct to 4 decimal places.

c Use your result to part **b** to find the area enclosed between the curve, the x-axis and the lines $x = -3$ and $x = 3$, correct to 4 decimal places.

9 Bupramorphine patches are used to assist people with their management of pain. The patches are applied to the skin and are left on for the period of a week. When a patient applies a patch for the first time, the amount of morphine in their blood system can be modelled by the equation $C = 25 \log_e (1 + 0.5t)$, where C mg is the amount of morphine in the subject's blood system and t is the time in days since the patch was applied.

a Find the concentration of morphine in the patient's blood system, C_7, seven days after the patch was applied to the skin, correct to 1 decimal place.

b Sketch the graph of C versus t.

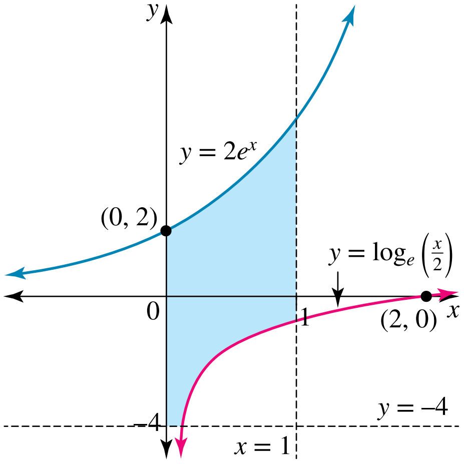
c Find the rate at which the morphine is released into the blood system after three days.

d Use the inverse function of C to determine the total amount of morphine released into the patient's blood system over the seven days. That is, find

$$\int_0^7 25 \log_e (1 + 0.5t)dt,$$ correct to 1 decimal place.

10 The shaded area in the diagram is the plan of a mine site. All distances are in kilometres.

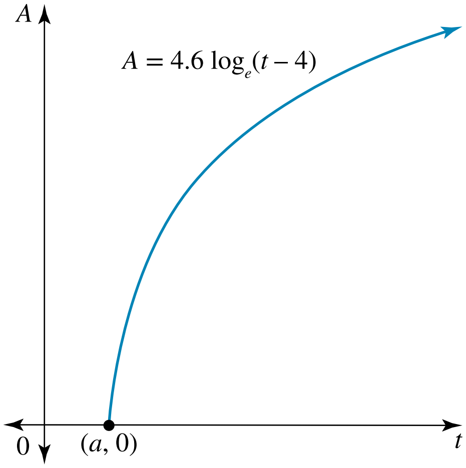
Two of the boundaries of the mine site are in the shape of graphs defined by the functions with equations $f : R \rightarrow R$, $f(x) = 2e^x$ and $g : R^+ \rightarrow R$, $g(x) = \log_e\left(\dfrac{x}{2}\right)$, where $g(x)$ is the inverse function of $f(x)$.



Calculate the area of the region bounded by the graphs of f and g, the y-axis and the lines $x = 1$ and $y = -4$. Give your answer correct to 1 decimal place.

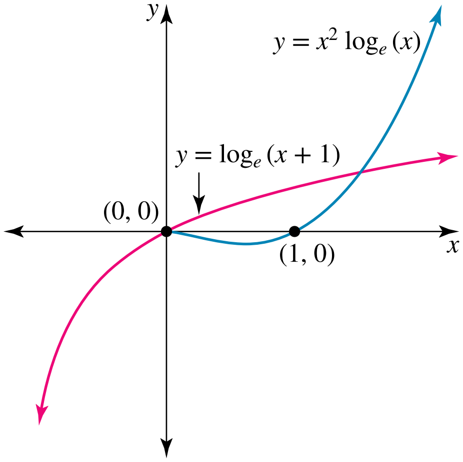
11 A patient has just had a medical procedure that required a general anaesthetic. Five minutes after the end of the procedure was completed, the patient starts to show signs of awakening. The alertness, A, of the patient t minutes after the completion of the procedure can be modelled by the rule $A = 4.6\log_e(t - 4)$.

The graph of the function is shown.



- **a** Find the value of the constant a, given that $(a, 0)$ is the x-axis intercept.
- **b** When the patient has an alertness of 15, they are allowed to have water to sip, and 15 minutes later they can be given a warm drink and something to eat. How long does it take for the patient to reach an alertness of 15? Give your answer correct to the nearest minute.
- **c** Find the rate at which the alertness of the patient is changing 10 minutes after the completion of the medical procedure.
- **d** Use the inverse function of A to determine the total change of alertness for 30 minutes after the completion of the medical procedure. That is, calculate the area between the curve and the t axis from $t = 5$ to $t = 30$.

12 The graphs of the functions $g : R^+ \to R, g(x) = x^2\log_e(x)$ and $h : (-1, \infty) \to R, h(x) = \log_e(x + 1)$ are shown.



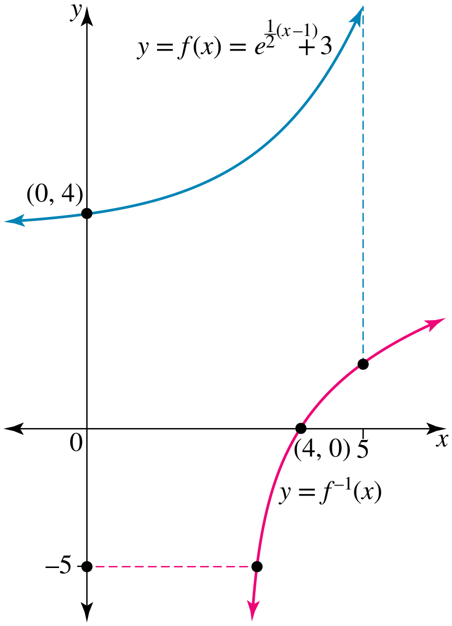
- **a** Find the coordinates of the point of intersection of the two graphs to the right of the origin. Give your answer correct to 4 decimal places.
- **b** Find the area enclosed between the curves between the points of intersection found in part **a**, correct to 4 decimal places.

MASTER

13 At the Royal Botanical Gardens, a new area of garden is being prepared for native Australian plants.

The area of garden has two curved walking paths as borders. One of the paths can be modelled by the rule $f(x) = e^{\frac{1}{2}(x-1)} + 3$.

a The other curved walking path is defined by the rule for the inverse of f, f^{-1}. State the rule for f^{-1}.



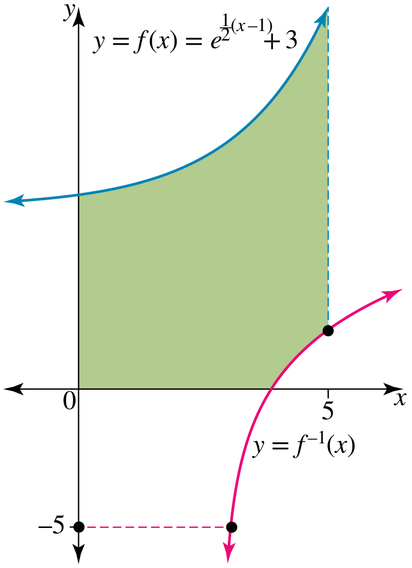
The other borders are given by $x = 5$ and $y = -5$ as shown. The remaining border is formed by the y-axis, as shown. All measurements are in metres.

b Determine the respective axis intercepts of the graphs of f and f^{-1}.

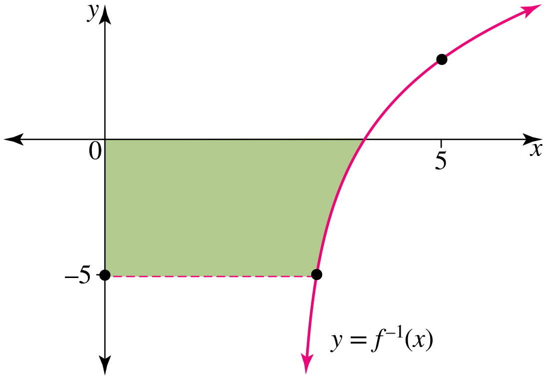
c Find the area of the garden above the x-axis, as shown in the diagram below, by calculating

$$\int_0^5 \left(e^{\frac{1}{2}(x-1)} + 3\right)dx - \int_{e^{-0.5}+3}^{5} (2\log_e(x-3)+1)dx.$$

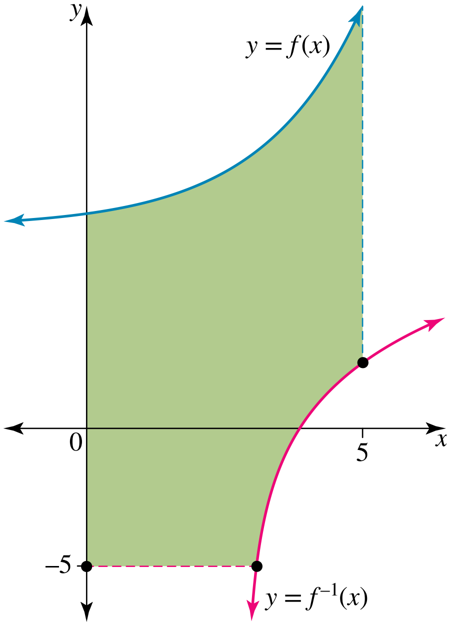
Give your answer correct to 2 decimal places.



d Find the area of the garden below the *x*-axis, as shown in the diagram below, correct to 2 decimal places.

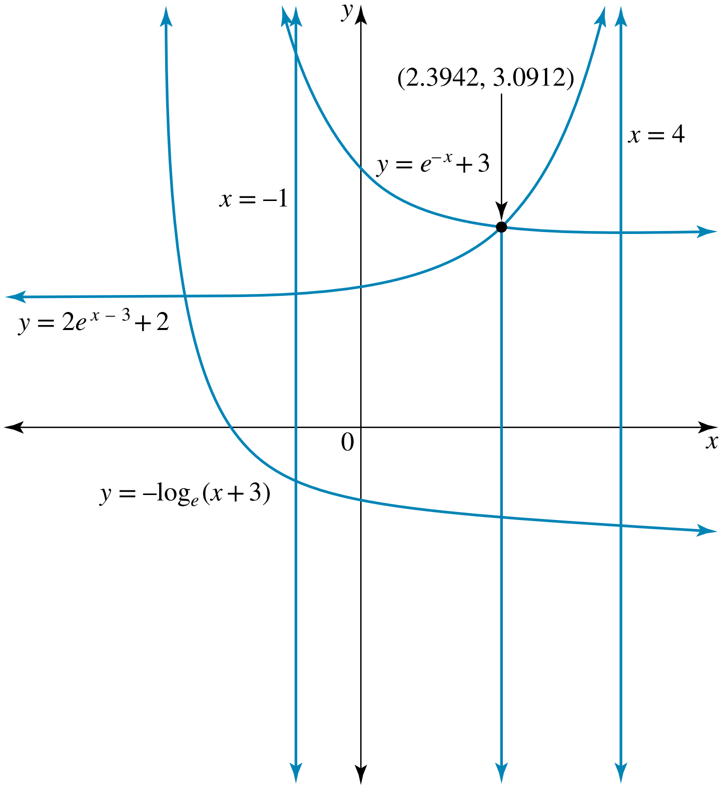


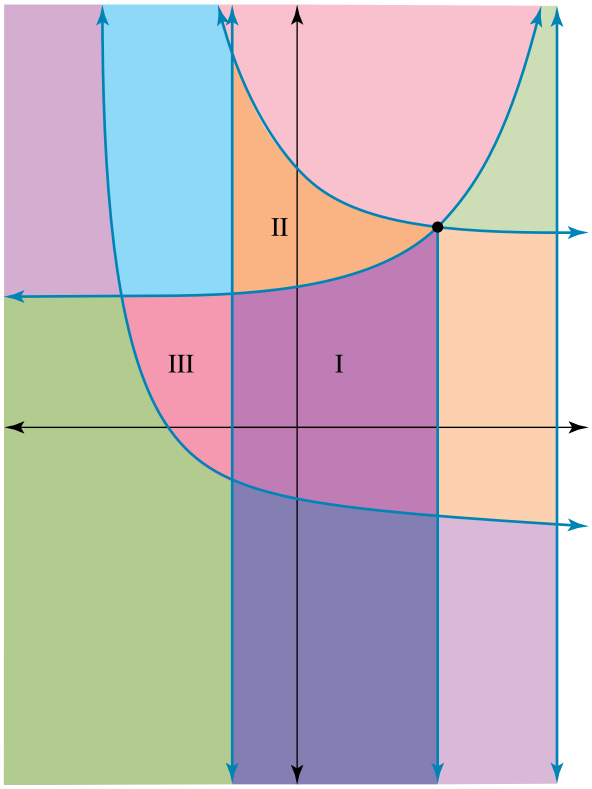
e Hence, determine the total area of the garden correct to 1 decimal place.



14 A young couple are participating on a reality television show in which they are renovating an apartment.

They have commissioned an up-and-coming artist to create a modern art piece to be featured in their living/dining room. The artist has decided to use exponential and logarithmic curves as well as some straight lines to create the art piece. The curves and lines are shown along with the colour sketch for the finished piece.





a Find the coordinates of the points of intersection between each of the following pairs of graphs. Give your answers correct to 1 decimal place.

i $y = e^{-x} + 3$ and $y = 2e^{x-3} + 2$

ii $y = 2e^{x-3} + 2$ and $y = -\log_e(x + 3)$

iii $x = -1$ and $y = e^{-x} + 3$

iv $x = -1$ and $y = -\log_e(x + 3)$

v $x = 2.3942$ and $y = -\log_e(x + 3)$

vi $x = -1$ and $y = 2e^{x-3} + 2$

b Calculate, correct to 4 decimal places, the area of:

i region I

ii region II

iii region III.

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9.5 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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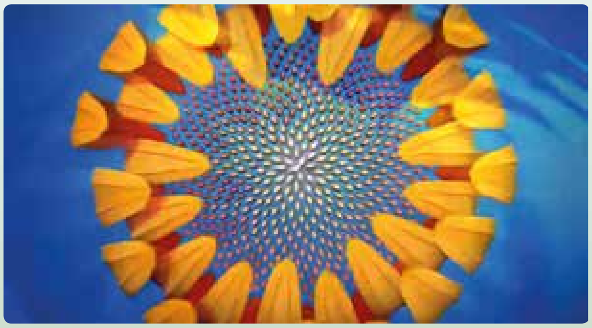
Activities

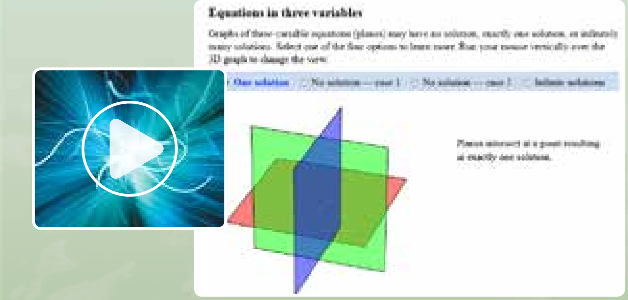
To access eBookPLUS activities, log on to

www.jacplus.com.au

Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





Equations in three variables

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

9 Answers

EXERCISE 9.2

1 a $\frac{7}{x}$

b $\frac{2(3x^2 + 4x)}{x^3 + 2x^2 - 1}$

c $\frac{\sin(x)}{x - 2} + \cos(x)\log_e(x - 2)$

d $\frac{2(2x - 1) - 2x\log_e(x^2)}{x(2x - 1)^2}$

e $\frac{3(2e^{3x} + 1)}{e^{3x} - 1}$

f $\frac{1}{(2x - 3)\sqrt{\log_e(3 - 2x)}}$

2 a $\frac{dy}{dx} = -\frac{5}{x}, x \in (0, \infty)$

b $\frac{dy}{dx} = -\frac{1}{x - 2}, x \in (2, \infty)$

c $\frac{dy}{dx} = \frac{-2}{(x + 3)(x + 1)}, x \in (-\infty, -3) \cup (-1, \infty)$

d $\frac{dy}{dx} = \frac{2x - 1}{x^2 - x - 6}, x \in (-\infty, -2) \cup (3, \infty)$

3 a Domain = $(2, \infty)$, range = R

b $a = 3$

c $y = 2x - 6$

d $y = -\frac{1}{2}x + \frac{3}{2}$

4 Tangent: $y = 6x + 4\log_e(2) - 6$

5 Minimum turning point at $(0.1, 1 – \log_e(10))$

6 a Local minimum at $\left(\frac{1}{e}, -\frac{2}{e}\right)$

b Local maximum at $\left(\frac{e}{2}, \frac{2}{e}\right)$

c Local maximum at $\left(\frac{3}{e}, \frac{3}{e}\right)$

7 a $\frac{4}{x}$

b $\frac{1}{4(x - 2)}$

c $\frac{3x^2 - 6x + 7}{x^3 - 3x^2 + 7x - 1}$

d $6\tan(x)$

e $\frac{3}{2(3x + 1)\sqrt{\log_e(3x + 1)}}$

f $\frac{2e^{2x} + 2 - 4e^{2x}x\log_e(2x)}{x(e^{2x} + 1)^2}$

8 a $(2x – 3)\log_e(2x – 1) + \frac{2(x^2 – 3x + 7)}{2x – 1}, x \in \left(\frac{1}{2}, \infty\right)$

b $\frac{x\cos(x)\log_e(x^2) + 2\sin(x)}{x}, x \in (0, \infty)$

c $\frac{x^2 – 3x^2\log_e(3x) – 1 + \log_e(3x)}{(x^3 – x)^2}, x \in (0, \infty)/\{1\}$

d $\frac{-6}{(4 – x)(x + 2)}, x \in (-2, 4)$

9 a $\frac{2}{5}\log_5(e)$

b $\frac{1}{9}\log_3(e)$

c $\log_6(e)$

10 a $y = 2x - 3$

b $y = \frac{3}{e}x$

c $y = \frac{1}{e}x$

11 a $\frac{dy}{dx} = \frac{2}{x}$

b $y = \frac{4}{e}x + e - 2$

12 $b = 2$

13 $k = 7.4$

14 a $x = -1.841, \quad -0.795$

b At $(-1.841, 0)$; $y = -1.1989x - 2.2072$

At $(-0.795, 0)$; $y = 3.0735x + 2.4434$

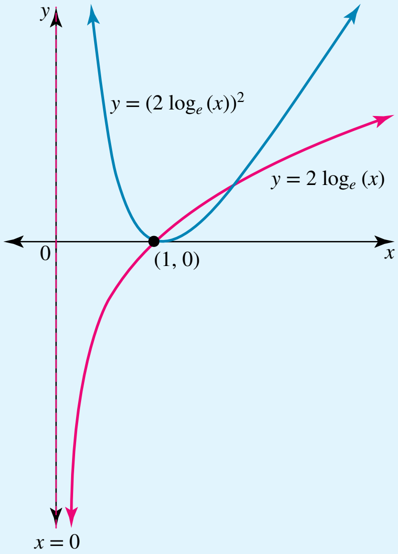
c Minimum turning point = $(-1.2134, -0.4814)$

15 a $(1, 0)$ and $(e^{0.5}, 1)$

b For f : at $(1, 0)$, $\frac{dy}{dx} = 0$.

For g : at $(1, 0)$, $\frac{dy}{dx} = 2$.

c



d $\{x : 1 < x < e^{0.5}\}$

16 a $x = 0.3407, \quad 0.8364$

b At $(0,3407, 0)$, $\frac{dy}{dx} = 27.09$; at $(0,8364, 0)$, $\frac{dy}{dx} = -6.15$.

c Tangent: $y = -6x + 5$

Perpendicular line: $y = \frac{1}{6}x - \frac{7}{6}$ or $x - 6y = 7$

d The turning point occurs where $\frac{dy}{dx} = 0$.

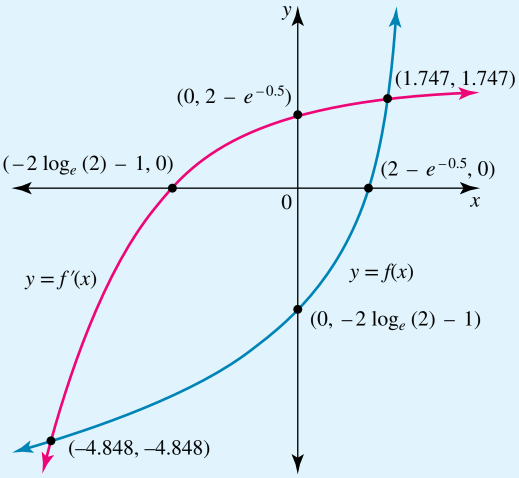
$$\begin{aligned}\frac{2-8x^2}{x^3}&=0\\ 2-8x^2&=0\\ 1-4x^2&=0\\ (1-2x)(1+2x)&=0\\ x=\tfrac{1}{2},-\tfrac{1}{2}\text{ but }x&>0\\ x=\tfrac{1}{2},\ y=-\frac{1}{\left(\frac{1}{2}\right)^2}&-8\log_e\left(\tfrac{1}{2}\right)\\ &=-\frac{1}{\frac{1}{2}}-8\log_e(2^{-1})\\ &=-4+8\log_e(2)\end{aligned}$$

The maximum turning point is at $\left(\frac{1}{2}, -4+8\log_e(2)\right)$.

17 a $f^{-1}: R \to R,\ f^{-1}(x) = 2 - e^{-\frac{1}{2}(x+1)}$; domain $= R$, range $= (-\infty\ , 2)$

b $(-4.8479, -4.8479)$, and $(1.7467, 1.7467)$

c



18 $n = 2$

EXERCISE 9.3

1 a $\frac{2}{5}\log_e(x) + c, x > 0$ **b** $\frac{3}{4}\log_e\left(\frac{11}{3}\right)$

2 a $x + 2\log_e(x) + \frac{3}{x} + c, x > 0$

b $f(x) = \frac{1}{4}x^4 - \log_e(x), x > 0$

3 $\int \tan(2x)dx = -\frac{1}{2}\log_e(\cos(2x))$

4 $\frac{dy}{dx} = \frac{2}{x}\log_e(x)$ and $\int_1^e \frac{4}{x}\log_e(x)dx = 2$

5 a $a = -1$ **b** $\log_e(4) - 3$

c $\left(-\frac{3}{2}, 1\right), \left(2, -\frac{3}{4}\right)$ **d** $\frac{63}{16} - 3\log_e(2)$

6 a $\left(\frac{-5+\sqrt{13}}{6}, \frac{6}{\sqrt{13}+1}\right)$ or $\left(\frac{-5+\sqrt{13}}{6}, \frac{\sqrt{13}-1}{2}\right)$

b 4.3647 units^2

7 $\log_e(2)$ units^2

8 $\log_e\left(\frac{2}{3}\right) + 2$

9 a $-4\log_e(x) + c, x > 0$

b $\frac{3}{4}\log_e(4x+7) + c, x > -\frac{7}{4}$

c $\frac{1}{2}x^2 + 2x + 3\log_e(x) + \frac{1}{x} + c, x > 0$

d $-3\log_e(2-x) + \frac{1}{4}\sin(4x) + c, x < 2$

10 a $-\frac{3}{2}\log_e\left(\frac{7}{3}\right)$ **b** $2\log_e(3)$ **c** 1489.56

11 a $y = \frac{5}{2}\log_e(2(x+2)) + 3$

b $y = -\frac{3}{5}\log_e(2-5x) + 1$

12 $f'(x) = 2\log_e(mx) + 2$ and

$$\int \log_e(mx)dx = x\log_e(mx) - x + c$$

13 $f'(x) = 3 + 3\log_e(x)$ and

$$\int 2\log_e(x)dx = 2x\log_e(x) - 2x$$

14 $\frac{dy}{dx} = \frac{9x}{3x^3-4}$ and $\int_2^3 \frac{x^2}{3x^3-4}dx = \frac{1}{9}\log_e\left(\frac{77}{20}\right)$

15 $\frac{dy}{dx} = \frac{2e^x}{(e^x+1)}$ and $\int_1^5 \frac{e^x}{e^x+1}dx = 3.6935$

16 a Maximum turning point $= (\sqrt{5}, \sqrt{5})$, minimum turning point $= (-\sqrt{5}, -\sqrt{5})$

b $5\log_e(5+x^2)$

c $5\log_e\left(\frac{41}{10}\right)$ units^2

17 a $y = \frac{12}{5}x - \frac{4}{5}$ or $12x - 5y = 4$

b $\frac{5}{2}\log_e(1+x^2)$

c 1.6295 units^2

18 a $y = 2x - 4$ **b** $\frac{3}{4} + \log_e(2)$ units^2

19 a Domain $= (1, \infty)$, range $= R$

b $a = 2$

c 5.0904 units^2

20 a 6.4792 units^2 **b** $f^{-1}(x) = e^{\frac{x}{5}} + 3, x \in R$

c 6.4792 units^2

EXERCISE 9.4

1 a $f^{-1}(x) = \frac{1}{4}e^{\frac{x}{2}}$ **b** $\frac{7}{2}$ units^2

c $4\log_e(8) - \frac{7}{2}$ units^2

2 a $(0, 2\log_e 5)$ and $(e^{-0.5} - 5, 0)$

b $h^{-1}: R \to R,\ h^{-1}(x) = e^{\frac{1}{2}(x-1)} - 5$

c See the figure at the foot of this page.*

d $x = -4.9489, 5.7498$

e 72.7601 units^2

3 a $\left(\frac{1}{2}, 0\right)$ **b** $\frac{dy}{dx} = 2x\log_e(2x) + x$

c 3.670 units^2

4 a $\left(\frac{1}{5}, 0\right)$ **b** $\frac{dy}{dx} = -\log_e(5x)$

c $2\log_e(10) - \frac{9}{5}$ units^2

5 a $(0, \log_e(2))$ and $\left(\frac{1}{4}, 0\right)$

b $D = \left(-\infty, \frac{1}{2}\right)$

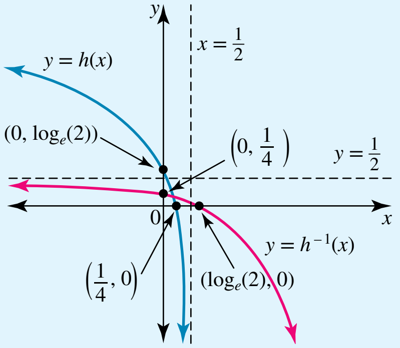
c $\frac{dy}{dx} = \frac{-2}{1-2x}$ where $1 - 2x > 0$ always, since $x < \frac{1}{2}$,

so $\frac{dy}{dx} < 0$ always.

d **i** $h^{-1}(x) = \frac{1}{4}(2 - e^x)$

ii Dom $= R$, range $= \left(-\infty, \frac{1}{2}\right)$

e



6 a $\frac{dy}{dx} = \log_e(x) + 1$ and $\int_1^{e^2} \log_e(x)dx = e^2 + 1$

b $\frac{dy}{dx} = (\log_e(x))^m + m(\log_e(x))^{m-1}$

c Check with your teacher.

d $\int_1^{e^2} (\log_e(x))^3 dx = 2e^2 + 6$

7 a $y = (-4\log_e(2) - 4)x - 2$

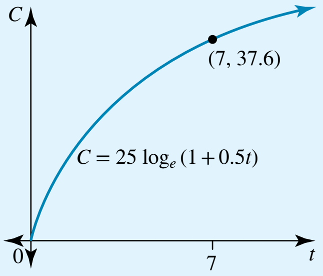
b $\frac{dy}{dx} = 4x\log_e(x+1) + 2(x-1) = 4x\log_e(x+1) + 2x - 2$

c $2 - 1.5\log_e(3)$ units^2

8 a -0.8 **b** 3.4058 units^2 **c** 6.8117 units^2

9 a $C_7 = 37.6$ mg

b



c $\frac{dC}{dt} = 5$ mg/day

d 163.4 mg

10 5.1 km^2

11 a $a = 5$ **b** 30 minutes

c $\frac{23}{30}$ units/minute **d** 274.6683 units

12 a (1.5017, 0.9170) **b** 0.7096 units^2

13 a $f^{-1}(x) = 2\log_e(x-3) + 1$

b $(0, e^{-\frac{1}{2}} + 3)$ and $(e^{-\frac{1}{2}} + 3, 0)$

c 26.58 m^2

d 18.90 m^2

e 45.5 m^2

14 a **i** (2.4, 3.1) **ii** (−2.9, 2.0) **iii** (−1.0, 5.7)

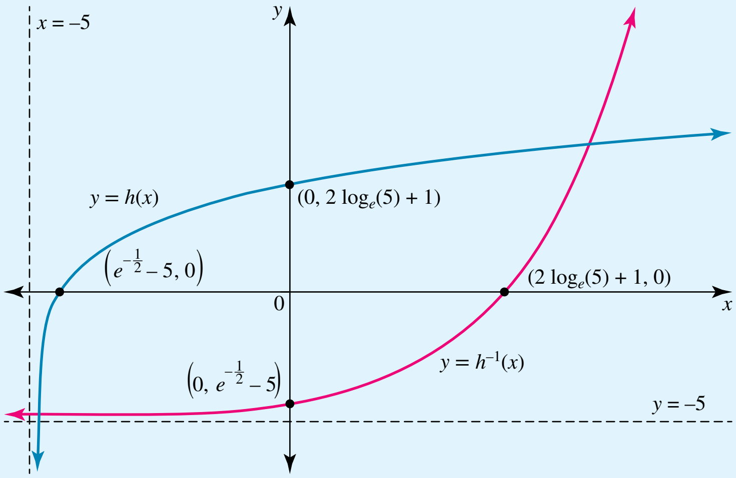
iv (−1.0, −0.7) **v** (2.4, −1.7) **vi** (−1.0, 2.0)

b **i** $A_{\text{I}} = 12.1535$ units^2

ii $A_{\text{II}} = 4.9666$ units^2

iii $A_{\text{III}} = 3.5526$ units^2

***2 c**



10

Discrete random variables

10.1 Kick off with CAS

Exploring discrete data

Data can be classified into two main groups: categorical data and numerical data. Categorical data can be further classified into categories. Numerical data consists of two types: discrete and continuous data. Continuous data is data that is measured, such as heights, weights etc. Discrete data is data that can be counted, such as the number of goals scored in a soccer match, or the number of students who change schools each year. Discrete data is generally made up of whole numbers (but not always).

Discrete data and the corresponding probabilities will be studied in more detail in this topic.

1 Two dice are rolled, and the total number of ways of achieving each possible total is recorded in the table below. The probability of rolling each total is also shown.

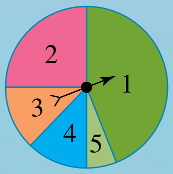
TOTAL	2	3	4	5	6	7	8	9	10	11	12
No. of ways of obtaining the total	1	2	3	4	5	6	5	4	3	2	1
Probability of rolling the total	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

a Using CAS, graph the total against the probability. Describe the shape of the graph.

b Calculate the mean and the median.

2 A spinner is divided into 5 sectors as shown.

The spinner is spun once. The probability of obtaining each number is given below.



Number	1	2	3	4	5
Probability	$\frac{7}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$

a Using CAS, graph the data. Describe the shape of the graph.

b Calculate the mean and the median.

3 A goal shooter records the number of goals she scores in her last 35 games.

No. of goals	0	1	2	3	4	5	6	7	8	9	10
Frequency	1	0	1	1	1	2	3	5	5	7	9
Probability	$\frac{1}{35}$	0	$\frac{1}{35}$	$\frac{1}{35}$	$\frac{1}{35}$	$\frac{2}{35}$	$\frac{3}{35}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{7}{35}$	$\frac{4}{35}$

a Using CAS, graph the data. Describe the shape of the graph.

b Calculate the mean and the median.

4 Compare your answers for parts 1–3. How do the mean and the median relate to the shape of the graph?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

10.2 Discrete random variables

Introduction

study on

Units 3 & 4
AOS 4
Topic 1
Concept 3

Discrete probability distributions
Concept summary
Practice questions

Units 1 and 2 of *Mathematical Methods* introduced the theory of basic concepts of probability, including the calculation of probabilities, independence, mutual exclusiveness and conditional probability. These ideas will now be extended into the area of random variables.

Random variables

The numerical value of a **random variable** is determined by conducting a random experiment. Random variables are represented by uppercase letters of the alphabet. Lowercase letters of the alphabet are used for the associated probabilities. For example, $\Pr(X = x)$ is interpreted as the probability that the random variable X will equal x.

Consider tossing three unbiased coins, where the number of Tails obtained is recorded. X is defined as the number of Tails obtained; therefore, x can be 0, 1, 2 or 3 (the different number of Tails that can be obtained from three tosses). In order to determine the associated probabilities for this random experiment, where each of the outcomes is equally likely, we need to list the sample space.

$$\xi = \{\text{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}\}$$

$$\begin{aligned}
\Pr(X = 0) &= \Pr(\text{HHH}) \\
&= \left(\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\right) \\
&= \tfrac{1}{8} \\
\Pr(X = 1) &= \Pr(\text{HHT}) + \Pr(\text{HTH}) + \Pr(\text{THH}) \\
&= \left(\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\right) \\
&= \tfrac{3}{8} \\
\Pr(X = 2) &= \Pr(\text{TTH}) + \Pr(\text{THT}) + \Pr(\text{HTT}) \\
&= \left(\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\right) \\
&= \tfrac{3}{8} \\
\Pr(X = 3) &= \Pr(\text{TTT}) \\
&= \left(\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\right) \\
&= \tfrac{1}{8}
\end{aligned}$$

This is an example of a **discrete random variable**. A discrete random variable can have only countable numbers or integer values. For the tossing of the three coins, $X = \{0, 1, 2, 3\}$.

Other examples of discrete random variables include the number of pups in a litter, the number of soft-centred chocolates in a box of mixed chocolates, the number of rainy days in the month of March, the number of blue smarties in a standard 15-gram packet and the number of traffic accidents at a main intersection over the period of three months.

The **probability distribution** of a discrete random variable defines the probabilities associated with each value the random variable can assume. For the experiment of the tossing of three coins, the distribution can be displayed in a table.

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Discrete random variables always display both of the following properties.

1. **The probability of each outcome is restricted to a value from 0 to 1; that is, $0 \le \Pr(X = x) \le 1$.**
2. **The sum of the probabilities of each outcome add up to 1; that is, $\sum_{\text{all } x} \Pr(X = x) = 1$.**

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Units 3 & 4

AOS 4

Topic 1

Concept 4

Representations of a probability distribution

Concept summary

Practice questions

eBook*plus*

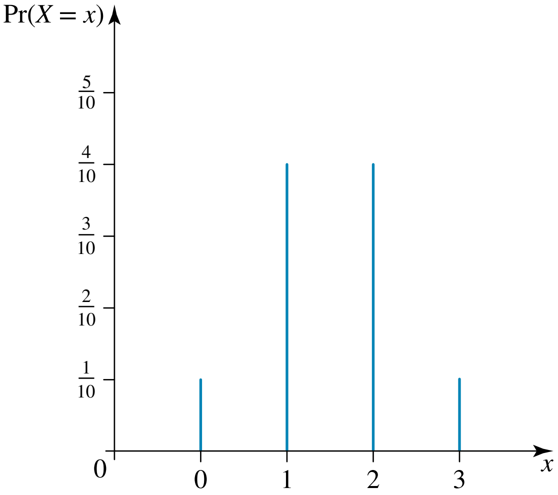
Interactivity

Using a tree diagram to depict a sample space

int-6427

The probability distribution can also be represented graphically, with probability on the vertical axis and the possible x-values on the horizontal axis. For the probability distribution shown in the table, the graph would appear as shown.

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{4}{10}$	$\frac{1}{10}$



WORKED EXAMPLE 1

A motorist travels along a main road in Brisbane. In doing so they must travel through three intersections with traffic lights over a stretch of two kilometres. The probability that the motorist will have to stop because of a red light at any of the intersections is $\frac{2}{5}$.

Let X be the number of red lights encountered by the motorist.

a Use a tree diagram to produce a sample space for this situation.

b Determine the probability of each outcome.

c Find the probability distribution for this random variable.

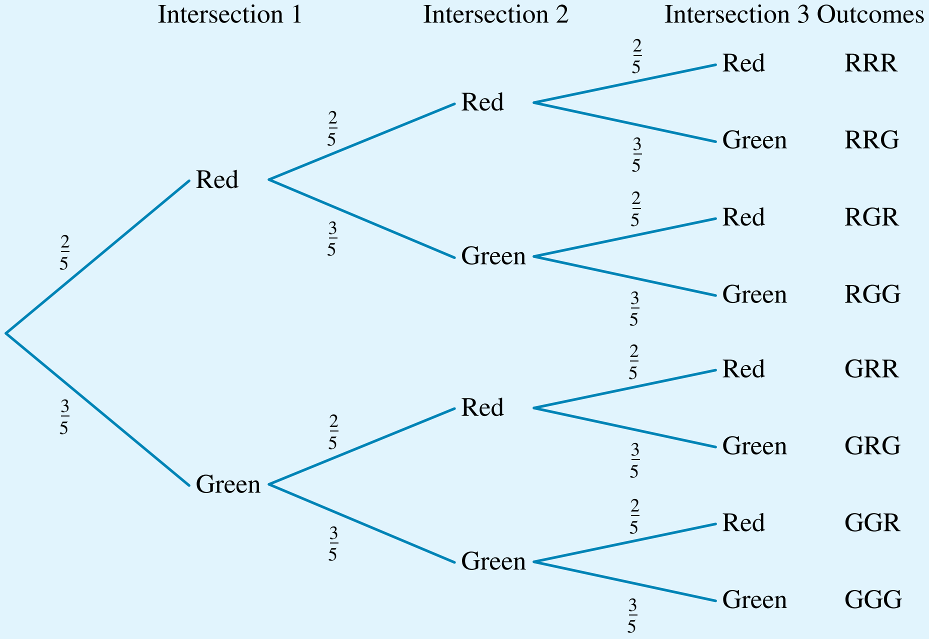
d Test whether this probability distribution obeys the necessary properties for a discrete random variable distribution.

THINK **WRITE**

a 1 Set up a tree diagram to show the sample space.

Note: $\Pr(\text{R}) = \frac{2}{5}$,
$\Pr(\text{G}) = \frac{3}{5}$.

a Let R = a red light and G = a green light.



2 List the event or sample space.

$\xi = \{\text{RRR, RRG, RGR, RGG, GRR, GRG, GGR, GGG}\}$

b Calculate the probability of each outcome.

b
$$\Pr(\text{RRR}) = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{125}$$
$$\Pr(\text{RRG}) = \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{12}{125}$$
$$\Pr(\text{RGR}) = \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{12}{125}$$
$$\Pr(\text{RGG}) = \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{18}{125}$$
$$\Pr(\text{GRR}) = \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{12}{125}$$
$$\Pr(\text{GRG}) = \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{18}{125}$$
$$\Pr(\text{GGR}) = \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{18}{125}$$
$$\Pr(\text{GGG}) = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$$

c **1** Set up the probability distribution by combining the outcomes related to each possible value of x.

c $\Pr(X = 0) = \Pr(GGG) = \frac{27}{125}$

$\Pr(X = 1) = \Pr(RGG) + \Pr(GRG) + \Pr(GGR)$

$= \frac{18}{125} + \frac{18}{125} + \frac{18}{125}$

$= \frac{54}{125}$

$\Pr(X = 2) = \Pr(RRG) + \Pr(RGR) + \Pr(GRR)$

$= \frac{12}{125} + \frac{12}{125} + \frac{12}{125}$

$= \frac{36}{125}$

$\Pr(X = 3) = \Pr(RRR) = \frac{8}{125}$

2 Enter the combined results into a table.

X = number of red lights

x	0	1	2	3
Pr($X = x$)	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

d Test whether the two properties of a discrete random variable are obeyed.

d Each $\Pr(X = x)$ is such that $0 \le \Pr(X = x) \le 1$ and

$\sum \Pr(X = x) = \frac{27}{125} + \frac{54}{125} + \frac{36}{125} + \frac{8}{125}$

$= \frac{125}{125}$

$\sum_{\text{all } x} \Pr(X = x) = 1$

Therefore, both properties of a discrete random distribution are obeyed.

WORKED EXAMPLE 2

a **State, giving reasons, whether each of the following represents a discrete probability distribution.**

i

x	0	2	4	6
Pr($X = x$)	−0.1	0.3	0.4	0.2

ii

x	−3	−1	4	6
Pr($X = x$)	0.01	0.32	0.52	0.15

iii

x	−1	0	1	2
Pr($X = x$)	0.2	0.1	0.2	0.3

b **A random variable, X, has the following probability distribution.**

x	1	2	3	4	5
Pr($X = x$)	b	$2b$	$0.5b$	$0.5b$	b

Find the value of the constant b.

THINK	WRITE
a i 1 Check that each probability is a value from 0 to 1.	**a i** Each probability does not meet the requirement $0 \le \Pr(X = x) \le 1$, as $\Pr(X = 0) = -0.1$.
2 If this condition is satisfied, add the probabilities together to see if they add to 1.	As one of the probabilities is a negative value, there is no point checking the sum of the probabilities.
3 Answer the question.	This is not a discrete probability distribution.
ii 1 Check that each probability is a value from 0 to 1.	**ii** Each probability does meet the requirement $0 \le \Pr(X = x) \le 1$.
2 If this condition is satisfied, add the probabilities together to see if they add to 1.	$\begin{aligned}\sum \Pr(X = x) &= 0.01 + 0.32 + 0.52 + 0.15 \\ &= 1\end{aligned}$
3 Answer the question.	Yes, this is a discrete probability function, as both of the conditions have been satisfied.
iii 1 Check that each probability is a value from 0 to 1.	**iii** Each probability does meet the requirement $0 \le \Pr(X = x) \le 1$.
2 If this condition is satisfied, add the probabilities together to see if they add to 1.	$\begin{aligned}\sum \Pr(X = x) &= 0.2 + 0.1 + 0.2 + 0.3 \\ &= 0.8\end{aligned}$
3 Answer the question.	As the sum of the probabilities is not equal to 1, this is not a discrete probability distribution.
b 1 As we know this is a probability distribution, we can equate the probabilities to 1.	**b** $\sum \Pr(X = x) = 1$ $b + 2b + 0.5b + 0.5b + b = 1$
2 Simplify.	$5b = 1$
3 Solve for *b*.	$b = \frac{1}{5}$

The tossing of an unbiased die 3 times to see how many sixes are obtained is an example of a **uniform distribution**, because all of the outcomes are equally likely. Another example is seeing how many Heads are obtained when a single coin is tossed *n* times. However, a non-uniform distribution exists when a biased coin is used, because all of the outcomes are not equally likely.

WORKED EXAMPLE 3

A coin is biased so that there are twice as many chances of it landing with Heads up. The coin is tossed 3 times.

a List the sample space and calculate the associated probabilities for each of the possible outcomes.

b Find the probability distribution for this non-uniform distribution.

c Find $\Pr(X > 1)$.

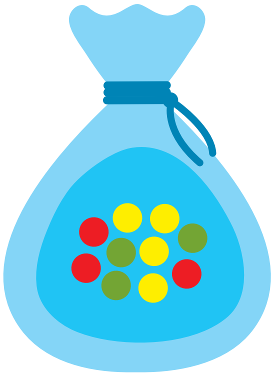
THINK	WRITE
a **1** Determine the probability for each event.	**a** If a Head is twice as likely to happen, then $\Pr(\text{H}) = \frac{2}{3}$ and $\Pr(\text{T}) = \frac{1}{3}$.
2 List the sample space.	$\xi = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}$
3 Calculate the individual probabilities.	$\Pr(\text{HHH}) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$ $\Pr(\text{HHT}) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$ $\Pr(\text{HTH}) = \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{27}$ $\Pr(\text{THH}) = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27}$ $\Pr(\text{HTT}) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{27}$ $\Pr(\text{THT}) = \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{27}$ $\Pr(\text{TTH}) = \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$ $\Pr(\text{TTT}) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$
b **1** Group the outcomes that contain the same number of Heads.	**b** Let X be the number of Heads. $\Pr(X = 0) = \Pr(\text{TTT}) = \frac{8}{27}$ $\Pr(X = 1) = \Pr(\text{HHT}) + \Pr(\text{HTH}) + \Pr(\text{THH})$ $= 3 \times \frac{4}{27}$ $= \frac{4}{9}$ $\Pr(X = 2) = \Pr(\text{HTT}) + \Pr(\text{TTH}) + \Pr(\text{THT})$ $= 3 \times \frac{2}{27}$ $= \frac{2}{9}$ $\Pr(X = 3) = \Pr(\text{HHH}) = \frac{1}{27}$
2 Check that the probabilities add to 1.	Check: $\frac{1}{27} + \frac{6}{27} + \frac{12}{27} + \frac{8}{27} = \frac{27}{27}$ $= 1$
3 Write the answer.	$\begin{array}{c\|c\|c\|c\|c} \boldsymbol{x} & 0 & 1 & 2 & 3 \\ \hline \mathbf{Pr}(\boldsymbol{X} = \boldsymbol{x}) & \frac{1}{27} & \frac{2}{9} & \frac{4}{9} & \frac{8}{27} \end{array}$
c **1** Define what probabilities are included in this inequality.	**c** $\Pr(X > 1) = \Pr(X = 2) + \Pr(X = 3)$
2 Add the probabilities.	$\Pr(X > 1) = \frac{4}{9} + \frac{8}{27}$ $= \frac{12}{27} + \frac{8}{27}$
3 Write the answer.	$\Pr(X > 1) = \frac{20}{27}$

EXERCISE 10.2 Discrete random variables

PRACTISE

Work without CAS
Questions 1–4

1 **WE1** A bag contains 3 red, 3 green and 4 yellow balls. A ball is withdrawn from the bag, its colour is noted, and then the ball is returned to the bag. This process is repeated on two more occasions. Let Y be the number of green balls obtained.



a Use a tree diagram to produce the sample space for the experiment.

b Determine the probability of each outcome.

c Find the probability distribution for this random variable.

d Test whether this probability distribution obeys the necessary properties for a discrete random variable distribution.

2 An unbiased die is tossed twice. Let the random variable X be the number of sixes obtained. Find the probability distribution for this discrete random variable.

3 **WE2** a State, giving reasons, whether each of the following represent a discrete probability distribution.

i

y	3	6	9	12
$\Pr(Y = y)$	0.2	0.3	0.3	0.2

ii

y	-2	-1	0	1	2
$\Pr(Y = y)$	0.15	0.2	0.3	0.2	0.15

b Find the value(s) of k if the table represents a discrete probability distribution.

x	2	3	4	5	6
$\Pr(X = x)$	$5k$	$3k - 0.1$	$2k$	k	$0.6 - 3k$

4 a State, giving reasons, whether each of the following represents a discrete probability distribution.

i

y	5	10	15	20
$\Pr(Y = y)$	0.15	0.35	0.35	0.05

ii

y	0	1	2	3	4
$\Pr(Y = y)$	-0.2	-0.1	0.1	0.2	0.3

b Find the value(s) of k if the table represents a discrete probability distribution.

x	1	2	3	4	5
$\Pr(X = x)$	$0.5k^2$	$0.5k$	$0.25(k + 1)$	0.5	$0.5k^2$

5 **WE3** Two dice are weighted so that $\Pr(2) = 0.2$, $\Pr(1) = \Pr(3) = \Pr(5) = 0.1$ and $\Pr(4) = \Pr(6) = 0.25$. They are both rolled at the same time. Let Z be the number of even numbers obtained.

a List the sample space.

b List the possible values of Z and construct a probability distribution table.

c Find $\Pr(Z = 1)$.

6 Samara and Simon are going to play tennis together. Samara has been playing tennis for longer than Simon, and the probability that she wins a set is 0.6. They intend to play 3 sets of tennis. Let X be the number of sets that Simon wins.

a List the sample space.
b List the possible values of X and construct a probability distribution table.
c Find $\Pr(X \leq 2)$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 State, with reasons, which of the following could be a probability distribution.

a

x	−3	−1	1	3
$\Pr(X = x)$	0.1	0.5	0.5	0.1

b

y	1	2	3	4	5
$\Pr(Y = y)$	−0.2	−0.1	0.4	0.1	0.2

c

z	−2	0	2	4
$\Pr(Z = z)$	0.25	0.15	0.45	0.25

d

x	1	2	3	4	5
$\Pr(X = x)$	0.1	0.25	0.3	0.25	0.1

8 Each of the following tables shows a discrete probability distribution. Find the unknown value in each case. (Assume the unknown value is not zero.)

a

x	2	4	6	8	10
$\Pr(X = x)$	$3d$	$0.5 - 3d$	$2d$	$0.4 - 2d$	$d - 0.05$

b

y	−6	−3	0	3	6
$\Pr(Y = y)$	$0.5k$	$1.5k$	$2k$	$1.5k$	$0.5k$

c

z	1	3	5	7
$\Pr(Z = z)$	$\frac{1}{3} - a^2$	$\frac{1}{3} - a^2$	$\frac{1}{3} - a^2$	a

9 State, with reasons, whether the following are discrete probability distributions.

a $p(x) = \frac{1}{7}(5 - x), x \in \{1, 3, 4\}$

b $p(x) = \dfrac{x^2 - x}{40}, x \in \{-1, 1, 2, 3, 4, 5\}$

c $p(x) = \frac{1}{15}\sqrt{x}, x \in \{1, 4, 9, 16, 25\}$

10 Find the value of a if the following is a discrete probability function.

$$p(x) = \frac{1}{a}(15 - 3x), x \in \{1, 2, 3, 4, 5\}$$

11 A mature British Blue female cat has just given birth to 4 kittens. Assume that there is an equally likely chance of a kitten being of either sex.

a Use a tree diagram to list the sample space for the possible number of males and females in the litter.

b Let X be the number of females in the litter. Construct a probability distribution table for the gender of the kittens.

c Find the probability that 4 females will be born.

d Find the probability that at least 1 female will be born.

e Find the probability that at most 2 females will be born.

12 Matthew likes to collect differently shaped dice. Currently he has two tetrahedrons (4 sides), an icosahedron (20 sides), two dodecahedrons (12 sides) and an octahedron (8 sides) as well as two standard six-sided cubes.

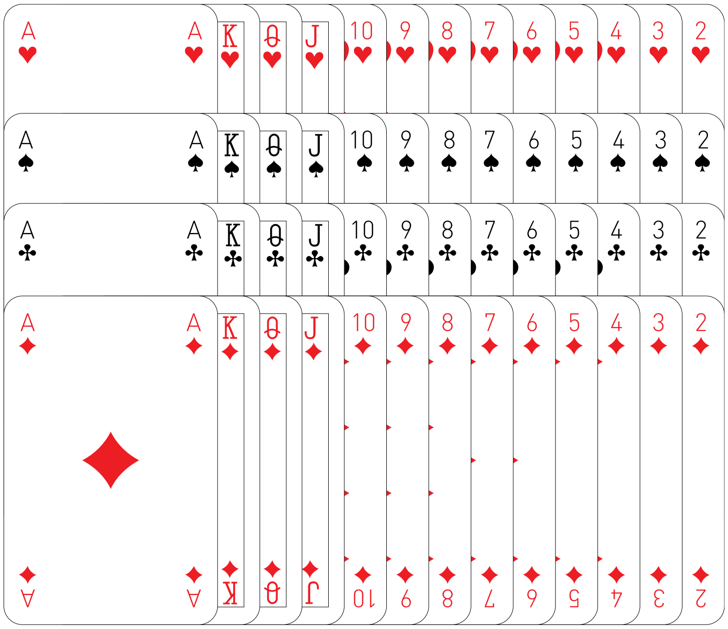
He has decided to play a game of chance using the octahedral die (with sides numbered 1 to 8) and one dodecahedral die (with sides numbered 1 to 12). He tosses the dice simultaneously and notes the number showing uppermost on both dice.

a List the sample space for the simultaneous tossing of the two dice.

b Let X be the number of primes obtained as a result of a toss. Find $\Pr(X = 0)$, $\Pr(X = 1)$ and $\Pr(X = 2)$.

c This particular game of chance involves tossing the two dice simultaneously on three occasions. The winner of the game must obtain two primes with each of the three tosses. Find the probability of being a winner. Give your answer correct to 3 decimal places.

13 A card game has the following rules. A card is chosen at random from a standard deck of 52 cards. Each card is awarded a numerical score. The premium cards — aces, kings and queens (P) — are each awarded 10 points. Each jack (J) is awarded 5 points, and each standard card (S) is awarded 2 points. The game is played twice.



a List the possible total points scored when two games are played.

b If X is the total points scored when two games are played, find the probability distribution.

c Find:

i $\Pr(X = 15)$ **ii** $\Pr(X \geq 12)$ **iii** $\Pr(X = 15 | X \geq 12)$.

14 A discrete random variable, X, can take the values -5, -1, 0, 1 and 5. The probability distribution is defined in the following manner.

$\Pr(X = -1) = \Pr(X = 0) = \Pr(X = 1) = m$

$\Pr(X = -5) = \Pr(X = 5) = n$

$3\Pr(X = 0) = \Pr(X = 5)$

a Determine the values of m and n.

b Find:

i $\Pr(X \geq 0)$ **ii** $\Pr(X = 1 | X \geq 0)$.

15 Diabetes is the name of a group of diseases that affect how the body uses blood glucose. If you have diabetes, it means that you have too much glucose in your blood. This can lead to serious health problems. Treatment for type 2 diabetes primarily involves monitoring your blood sugar level along with medications, insulin or both.

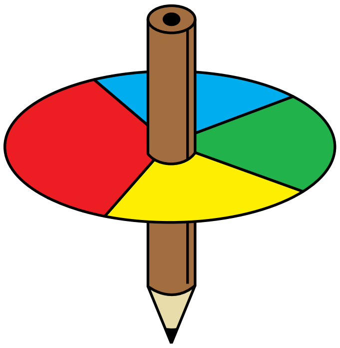
A new diabetes medication is to be trialled by 5 patients. From experiments that have been performed with mice, the success rate of the new medication is about 60%.

a Let X denote the number of patients who improve their health with the new medication. Find the probability distribution.

b The new medication will be considered a success if 68% or more of the patients improve their health. Find $\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$ and comment on the success of the new medication.

16 A game is played using a spinner that has been loaded so that it is more likely to land on the red side. In fact, $\Pr(\text{red}) = \frac{2}{5}$, and $\Pr(\text{blue}) = \Pr(\text{green}) = \Pr(\text{yellow}) = \frac{1}{5}$.

Each player pays \$2 to play. The player spins the spinner a total of 3 times; however, once the spinner lands on the red side the game is over. If a player has a combination of any 3 colours, they win \$1, but if the player has a combination of 3 colours that are all the same, they win \$10. There are a total of 40 different outcomes for the game.



a List the possible ways in which the game could end.

b List the possible ways in which the player could win \$10.

c Suppose X equals the amount of money won by playing the game, excluding the amount the person pays to play, so $X = \{0, 1, 10\}$. Find the probability distribution. Give your answers correct to 4 decimal places.

MASTER

17 A biased coin is tossed 6 times.

a If $\Pr(H) = \frac{2}{3}$ and X defines the number of Tails obtained, construct a probability distribution for this discrete probability experiment. Give your answers correct to 4 decimal places.

b Calculate, correct to 4 decimal places:

i the probability of more than 2 Heads

ii the probability of more than 2 Heads, given that fewer than 5 Heads come up.

18 A discrete random variable has the following probability distribution.

y	1	2	3	4	5
$\Pr(Y = y)$	$0.5k^2$	$0.3 - 0.2k$	0.1	$0.5k^2$	0.3

Find the value(s) of k, correct to 4 decimal places, that meet the criteria for this to be a valid probability distribution function.

10.3 Measures of centre and spread

The mean, median, variance and standard deviation are common statistical measurements that give us insight about sets of data, including discrete random variable distributions.

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The expected value or mean

The **expected value** or **mean** of a discrete probability function represents the 'average' outcome for the random experiment. When we find the mean, we are not stating the actual outcome; we are stating the outcome that we expect to happen.

Consider again a weighted spinner where $\Pr(\text{red}) = \frac{2}{5}$ and $\Pr(\text{blue}) = \Pr(\text{green}) = \Pr(\text{yellow}) = \frac{1}{5}$. The player spins the spinner 3 times, although the game is over if the spinner lands on its red side. If a player obtains a combination of 3 colours they win \$1, and if a player obtains a combination of 3 colours that are the same, they win \$10. The game costs \$2 to play. The calculated probabilities are $\Pr(\text{Win } \$10) = \frac{3}{125}$, $\Pr(\text{Win } \$1) = \frac{24}{125}$ and $\Pr(\text{Game over}) = \frac{98}{125}$.

If we consider the profit made by the person conducting the game, then they can lose \$8 (the player pays \$2 to play but wins \$10), gain \$1 (the player pays \$2 but wins \$1) or gain \$2 (the player pays \$2 and wins nothing). If Y represents the profit made by the person conducting the game, the following table would represent this situation.

y	−\$8	\$1	\$2
$\Pr(Y = y)$	$\frac{3}{125}$	$\frac{24}{125}$	$\frac{98}{125}$

The expected profit

$$= -8 \times \frac{3}{125} + 1 \times \frac{24}{125} + 2 \times \frac{98}{125}$$

$$= -\frac{24}{125} + \frac{24}{125} + \frac{196}{125}$$

$$= \$1.57$$

That is, on average, the person conducting the game makes a profit of \$1.57 per game.

The expected value of a random variable, X, is denoted by $\text{E}(X)$ or μ (mu). If a random variable assumes the values $x_1,\ x_2,\ x_3 \ldots x_{n-1},\ x_n$ with associated probabilities $\Pr(X = x_1),\ \Pr(X = x_2),\ \Pr(X = x_3) \ldots \Pr(X = x_{n-1}),\ \Pr(X = x_n)$ then

$$\mathbf{E}(X) = x_1\Pr(X = x_1) + x_2\Pr(X = x_2) + \ldots + x_{n-1}\Pr(X = x_{n-1}) + x_n\Pr(X = x_n)$$

$$= \sum_{x=1}^{x=n} x_n\Pr(X = x_n)$$

WORKED EXAMPLE 4

Find the expected value of the random variable with the following probability distribution.

x	10	20	30	40	50
$\Pr(X = x)$	0.42	0.34	0.16	0.07	0.01

THINK	WRITE
1 Write the rule to find the expected value.	$E(X) = \sum_{x=1}^{x=n} x_n \Pr(X = x_n)$
2 Substitute the appropriate values into the rule.	$E(X) = 10(0.42) + 20(0.34) + 30(0.16) + 40(0.07) + 50(0.01)$
3 Simplify.	$E(X) = 4.2 + 6.8 + 4.8 + 2.8 + 0.5$ $= 19.1$

Linear properties of the expected value

Sometimes we may be required to find the expected value of a linear function $aX + b$.

$$\begin{aligned} E(aX + b) &= E(aX) + E(b) \\ &= aE(X) + b \end{aligned}$$

Also, $E(X + Y) = E(X) + E(Y)$

where X and Y are discrete random variables and a and b are constants.

Note that the above properties are linear in nature, so

$$E(X^2) \neq [E(X)]^2$$

WORKED EXAMPLE 5

A discrete random variable, X, has the following probability distribution.

x	$5 - d$	$3 - d$	$-d$	$3 + d$
$\Pr(X = x)$	$\frac{7}{20}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{10}$

If $E(X) = 1$, find:

a **the value of the constant d**

b i **$E(7X)$** **ii** **$E(5X + 3)$** **iii** **$E(3X - 2)$.**

THINK	WRITE
a 1 Write the rule to find the expected value.	a $E(X) = \sum_{x=1}^{x=n} x_n \Pr(X = x_n)$
2 Substitute the appropriate values into the rule.	$E(X) = \frac{7}{20}(5 - d) + \frac{9}{20}(3 - d) - \frac{2d}{20} + \frac{2}{20}(3 + d)$

3 Simplify.

$$E(X) = \frac{35}{20} - \frac{7d}{20} + \frac{27}{20} - \frac{9d}{20} - \frac{2d}{20} + \frac{6}{20} + \frac{2d}{20}$$

$$E(X) = \frac{68 - 16d}{20}$$

4 Substitute $E(X) = 1$ and solve for d.

$$\begin{aligned} 1 &= \frac{68 - 16d}{20} \\ 20 &= 68 - 16d \\ 16d &= 48 \\ d &= 3 \end{aligned}$$

b i 1 Apply the linear property of $E(X)$: $E(aX + b) = aE(X) + b$.

b i $E(7X) = 7E(X)$

2 Substitute in the value of $E(X)$ and evaluate.

$$\begin{aligned} E(7X) &= 7 \times 1 \\ &= 7 \end{aligned}$$

ii 1 Apply the linear property of $E(X)$: $E(aX + b) = aE(X) + b$.

ii $E(5X + 3) = 5E(X) + 3$

2 Substitute in the value of $E(X)$ and evaluate.

$$\begin{aligned} E(5X + 3) &= 5 \times 1 + 3 \\ &= 8 \end{aligned}$$

iii 1 Apply the linear property of $E(X)$: $E(aX + b) = aE(X) + b$.

iii $E(3X - 2) = 3E(X) - 2$

2 Substitute in the value of $E(X)$ and evaluate.

$$\begin{aligned} E(3X - 2) &= 3 \times 1 - 2 \\ &= 1 \end{aligned}$$

The variance and standard deviation

The measure of spread of a random variable distribution tells us how the data is dispersed. The measure of spread is called the **variance**, and the square root of the variance gives the **standard deviation**. The variance is denoted by $\text{Var}(X)$ or σ^2 (sigma squared) and is defined as

$$\mathbf{Var}(X) = \sigma^2 = \mathbf{E}(X^2) - \mathbf{E}[(X)]^2$$

This may also be written as

$$\text{Var}(X) = \sigma^2 = E(X^2) - [\mu]^2, \text{ where } \mu = E(X).$$

The derivation of this rule is as follows:

$$\begin{aligned} \text{Var}(X) &= E(X - \mu)^2 \\ &= E(X^2 - 2X\mu + \mu^2) \\ &= E(X^2) - E(2X\mu) + E(\mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 \end{aligned}$$

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Since $\mathrm{E}(X) = \mu$,

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$$
$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

The standard deviation of X is the square root of the variance of X and is denoted by $\mathrm{SD}(X)$ or σ.

$$\boxed{\mathrm{SD}(X) = \sigma = \sqrt{\mathrm{Var}(X)}}$$

If the standard deviation is large, the spread of the data is large. If the standard deviation is small, the data is clumped together, close to the mean.

WORKED EXAMPLE 6

A discrete random variable, X, has the following probability distribution.

x	1	2	3	4	5
$\Pr(X = x)$	0.15	0.25	0.3	0.2	0.1

Find:

a $\mathrm{E}(X)$ **b** $\mathrm{Var}(X)$ **c** $\mathrm{SD}(X)$.

THINK	WRITE
a 1 Write the rule to find the expected value.	**a** $\mathrm{E}(X) = \sum_{\text{all } x} x\Pr(X = x)$
2 Substitute the appropriate values into the rule.	$\mathrm{E}(X) = 1(0.15) + 2(0.25) + 3(0.3) + 4(0.2) + 5(0.1)$
3 Simplify.	$\mathrm{E}(X) = 0.15 + 0.5 + 0.9 + 0.8 + 0.5$ $= 2.85$
b 1 Evaluate $\mathrm{E}(X^2)$.	**b** $\mathrm{E}(X^2) = \sum_{\text{all } x} x^2\Pr(X = x)$ $\mathrm{E}(X^2) = 1^2(0.15) + 2^2(0.25) + 3^2(0.3) + 4^2(0.2) + 5^2(0.1)$ $= 0.15 + 1 + 2.7 + 3.2 + 2.5$ $= 9.55$
2 Write the rule for the variance.	$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$
3 Substitute in the appropriate values and evaluate.	$\mathrm{Var}(X) = 9.55 - (2.85)^2$ $= 9.55 - 8.1225$ $= 1.4225$
c 1 Write the rule for the standard deviation.	**c** $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$
2 Substitute in the variance and evaluate.	$\mathrm{SD}(X) = \sqrt{1.4275}$ $= 1.1948$

Properties of the variance

The variance of a linear function has rules similar to those for the expectation of a linear function.

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

This can be proved in the following manner.

$$\begin{aligned}\text{Var}(aX + b) &= \text{E}(aX + b)^2 - [\text{E}(aX + b]^2\\ &= \text{E}(a^2X^2 + 2abX + b^2) - [a\text{E}(X) + b]^2\\ &= \text{E}(a^2X^2) + \text{E}(2abX) + \text{E}(b^2) - (a^2[\text{E}(X)]^2 + 2ab\text{E}(X) + b^2)\\ &= a^2\text{E}(X^2) + 2ab\text{E}(X) + b^2 - a^2[\text{E}(X)]^2 - 2ab\text{E}(X) - b^2\\ &= a^2(\text{E}(X^2) - [\text{E}(X)]^2)\\ \text{But } \text{Var}(X) &= \text{E}(X^2) - [\text{E}(X)]^2, \text{ so}\end{aligned}$$

$$\boxed{\mathbf{Var}(aX + b) = a^2\mathbf{Var}(X)}$$

WORKED EXAMPLE 7

A discrete probability function is defined by the rule $p(y) = \frac{1}{12}(10 - 3y)$, $y \in \{1, 2, 3\}$.

a **Show that the sum of the probabilities is equal to one.**

b **Find: i $\text{E}(Y)$ ii $\text{Var}(Y)$**

c **Find: i $\text{Var}(3Y - 1)$ ii $\text{Var}(4 - 5Y)$**

THINK	WRITE
a **1** Evaluate the probabilities for the given values of *y*.	**a** $p(y) = \frac{1}{12}(10 - 3y),\ y \in \{1,\ 2,\ 3\}$ $p(1) = \frac{1}{12}(10 - 3(1)) = \frac{7}{12}$ $p(2) = \frac{1}{12}(10 - 3(2)) = \frac{4}{12} = \frac{1}{3}$ $p(3) = \frac{1}{12}(10 - 3(3)) = \frac{1}{12}$
2 Add the probabilities.	$\Pr(Y = 1) + \Pr(Y = 2) + \Pr(Y = 3)$ $= \frac{7}{12} + \frac{4}{12} + \frac{1}{12}$ $= \frac{12}{12}$ $= 1$
b **i** **1** Write the rule to find the expected value.	**b i** $\text{E}(Y) = \sum_{\text{all } y} y\Pr(Y = y)$
2 Substitute the appropriate values into the rule.	$\text{E}(Y) = 1\left(\frac{7}{12}\right) + 2\left(\frac{4}{12}\right) + 3\left(\frac{1}{12}\right)$
3 Simplify.	$= \frac{7}{12} + \frac{8}{12} + \frac{3}{12}$ $= \frac{18}{12}$ $= \frac{3}{2}$
ii **1** Evaluate $\text{E}(Y^2)$.	**ii** $\text{E}(Y) = 1^2\left(\frac{7}{12}\right) + 2^2\left(\frac{4}{12}\right) + 3^2\left(\frac{1}{12}\right)$ $= \frac{7}{12} + \frac{16}{12} + \frac{9}{12}$ $= \frac{32}{12}$ $= \frac{8}{3}$

2 Write the rule for the variance.

$\text{Var}(Y) = \text{E}(Y^2) - [\text{E}(Y)]^2$

3 Substitute in the appropriate values and evaluate.

$$\begin{aligned}\text{Var}(Y) &= \frac{32}{12} - \left(\frac{3}{2}\right)^2 \\ &= \frac{32}{12} - \frac{9}{4} \\ &= \frac{32 - 27}{12} \\ &= \frac{5}{12}\end{aligned}$$

c i 1 Apply the property of the variance: $\text{Var}(aY + b) = a^2\text{Var}(Y)$.

c i $\text{Var}(3Y - 1) = 3^2\text{Var}(Y)$

2 Substitute in the value of Var(Y) and evaluate.

$$\begin{aligned}\text{Var}(3Y - 1) &= 9 \times \frac{5}{12} \\ &= \frac{15}{4}\end{aligned}$$

ii 1 Apply the property of the variance: $\text{Var}(aY + b) = a^2\text{Var}(Y)$.

ii $\text{Var}(4 - 5Y) = (-5)^2\text{Var}(Y)$

2 Substitute in the value of Var(Y) and evaluate.

$$\begin{aligned}\text{Var}(4 - 5Y) &= 25 \times \frac{5}{12} \\ &= \frac{125}{12}\end{aligned}$$

EXERCISE 10.3 Measures of centre and spread

PRACTISE

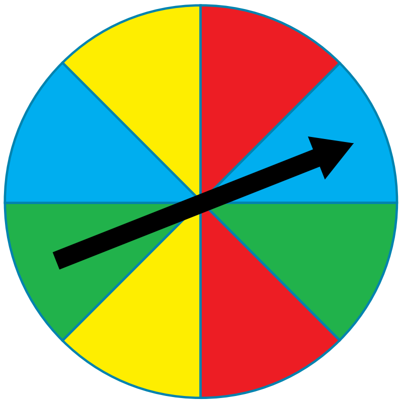
Work without CAS
Questions 1–3

1 **WE4** A discrete random variable is defined by the function $p(x) = \frac{1}{16}(2x - 1)$, $x \in \{1, 2, 3, 4\}$.

a Construct a probability distribution table for this function.

b Find the expected value of the function.

2 A game of chance is played with a spinner.



Each sector represents $\frac{1}{8}$ of the circular spinner. If the pointer lands on yellow, the player receives nothing. If the pointer lands on green, the player receives \$5. If the pointer lands on red, the player receives \$8, and if the pointer lands on blue, the player receives \$10. The game costs \$2 to play. Let X represent the net profit made by the player.

a Construct a probability distribution table for the net profit.

b Find the expected net profit in dollars for any player.

3 **WE5** The discrete random variable, Y, has the following probability distribution.

y	-5	0	5	d	25
$\Pr(Y = y)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{10}$

If E(Y) = 7.5, find:

a the value of the constant d

b i $\text{E}(2Y + 3)$ **ii** $\text{E}(5 - Y)$ **iii** $\text{E}(-2Y)$.

4 Find the mean of the discrete random variable, Z, for a probability function defined by $p(z) = \frac{1}{38}(z^2 - 4)$, $2 \le z \le 5$.

Give your answer correct to 2 decimal places.

5 **WE6** Recently the large supermarket chains have been waging a price war on bread.

On a particular Tuesday, a standard loaf of bread was purchased from a number of outlets of different chains. The following table shows the probability distribution for the price of the bread, X.

x	\$1	\$2	\$3	\$4	\$5
$\Pr(X = x)$	0.3	0.15	0.4	0.1	0.05

a Calculate the expected cost of a loaf of bread on that given Tuesday.
b Calculate the variance and the standard deviation of that loaf of bread, correct to 2 decimal places.

6 A discrete random variable, X, has the following probability distribution.

x	-2	0	2	4	6
$\Pr(X = x)$	k	k	$2k$	$3k$	$3k$

a Determine the value of the constant k.
b Find the expected value of X.
c Find the variance and the standard deviation of X, correct to 2 decimal places.

7 **WE7** A discrete probability function is defined by $p(x) = \dfrac{x^2}{30}$, $x = 1, 2, 3, 4$.

Where appropriate, give your answers to the following to 2 decimal places.

a Construct a probability distribution table and show that $\sum_{\text{all } x} \Pr(X = x) = 1$.
b Find: **i** $\text{E}(X)$ **ii** $\text{Var}(X)$
c Find: **i** $\text{Var}(4X + 3)$ **ii** $\text{Var}(2 - 3X)$.

8 **a** Find the value of the constant m if the discrete random variable Z has the probability distribution shown and $\text{E}(Z) = 14.94$.

z	-7	m	23	31
$\Pr(Z = z)$	0.21	0.34	0.33	0.12

b Find $\text{Var}(Z)$ and hence find $\text{Var}(2(Z - 1))$ and $\text{Var}(3 - Z)$, correct to 2 decimal places.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 For each of the following probability distributions, find:

i the expected value
ii the standard deviation, correct to 4 decimal places.

a

x	-3	-2	-1	0	1	2	3
$\Pr(X = x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

b

y	1	4	7	10	13
$\Pr(Y = y)$	0.15	0.2	0.3	0.2	0.15

c

z	1	2	3	4	5	6
$\Pr(Z = z)$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

10 A random variable, Y, has the following probability distribution.

y	-1	1	3	5	7
$\Pr(Y = y)$	$1 - 2c$	c^2	c^2	c^2	$1 - 2c$

a Find the value of the constant c.

b Find E(Y), the mean of Y.

c Find Var(Y), and hence find the standard deviation of Y, correct to 2 decimal places.

11 Given that $E(X) = 4.5$, find:

a $E(2X - 1)$ **b** $E(5 - X)$ **c** $E(3X + 1)$.

12 Given that $SD(X) = \sigma = 2.5$, find:

a $\text{Var}(6X)$ **b** $\text{Var}(2X + 3)$ **c** $\text{Var}(-X)$.

13 A discrete probability function is defined by the rule $p(x) = h(3 - x)(x + 1)$, $x = 0, 1, 2$.

a Show that the value of h is $\frac{1}{10}$.

b Hence, find the mean, variance and standard deviation of X. Where appropriate, give your answers to 4 decimal places.

14 A discrete probability function has the following distribution.

x	1	2	3	4	5
$\Pr(X = x)$	a	0.2	0.3	b	0.1

The expected value of the function is 2.5.

a Find the values of the constants a and b.

b Hence, find the variance and standard deviation of X. Where appropriate, give your answers to 4 decimal places.

15 For a given discrete random variable, X, it is known that $E(X) = a$ and $\text{Var}(X) = 2a - 2$, where a is a constant that is greater than zero.

a Find $E(X^2)$ in terms of a.

b If $E(X^2)$ is known to be 6, find $E(X)$ and $\text{Var}(X)$.

16 For a discrete random variable, Y, the probability function is defined by

$$p(y) = \begin{cases} ny & y \in \{1, 2, 3, 4\} \\ n(7 - y) & y \in \{5, 6\} \end{cases}$$

a Find the value of the constant n.

b Find the expected value, the variance and the standard deviation of Y, correct to 4 decimal places.

17 Two octahedral dice (with faces numbered 1 to 8) are rolled simultaneously and the two numbers are recorded.

a List the probability or event space and find $n(\xi)$.

Let Z be the larger of the two numbers on the two dice.

b State the probability distribution for Z.

c Find the expected value and standard deviation of Z, correct to 4 decimal places.

18 A dart competition at a local sports centre allows each player to throw one dart at the board, which has a radius of 20 centimetres. The board consists of five concentric circles, each with the same width.

The inner circle has a radius of 4 cm. The probability of landing on each band is determined by the area of that band available on the board.

a Calculate the probability of landing on each of the bands.

The outer red band is called band E, the next white band is called band D and so on until you get to the inner red circle, which is band A.

The competition costs \$1 to enter and the prizes are as follows:
If a dart hits band E, the player receives nothing.
If a dart hits band D, the player receives \$1.
If a dart hits band C, the player receives \$2.
If a dart hits band B, the player receives \$5.
If a dart hits band A, the player receives \$10.

b If X is a discrete random variable that represents the profit in dollars for the player, construct a probability distribution table for this game.

c Calculate:

i the expected profit a player could make in dollars

ii the standard deviation.

MASTER

19 At a beginner's archery competition, each archer has two arrows to shoot at the target. A target is marked with ten evenly spaced concentric rings.

The following is a summary of the scoring for the beginner's competition.

Yellow – 10 points
Red – 7 points
Blue – 5 points
Black – 3 points
White – 1 point

Let X be the total score after a beginner shoots two arrows.

a List the possible score totals.

The probability of a beginner hitting each of the rings has been calculated as follows:

Pr(yellow) = 0.1, Pr(red) = 0.2, Pr(blue) = 0.3, Pr(black) = 0.2 and Pr(white) = 0.2.

b Construct a probability distribution table for the total score achieved by a beginner archer.

c Calculate the expected score and the standard deviation for a beginner. Where appropriate, give your answers correct to 4 decimal places.

20 A discrete random variable, X, has the following probability distribution.

x	-2	-1	0	1	2	3	4
$\Pr(X = x)$	$0.5k^2$	$0.5k^2$	$k + k^2$	$4k$	$2k$	$2k + k^2$	$7k^2$

a Find the value of the constant k.

b Find the expected value of X.

c Find the standard deviation of X, correct to 4 decimal places.

10.4 Applications

One important application of the expected value and standard deviation of a random variable is that approximately 95% of the distribution lies within two standard deviations of the mean.

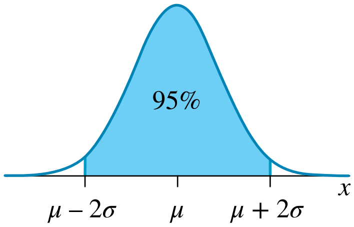
studyon

Units 3 & 4
AOS 4
Topic 1
Concept 8

95% confidence intervals
Concept summary
Practice questions

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

This can be illustrated by the normal distribution curve. This type of distribution is covered in Topic 13.



WORKED EXAMPLE 8

Let Y be a discrete random variable with the following probability distribution.

y	0	1	2	3	4
$\Pr(Y = y)$	0.08	0.34	0.38	0.17	0.03

a Find the expected value of Y.

b Find the standard deviation of Y.

c Find $\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.

THINK	WRITE
a 1 Write the rule to find the expected value.	**a** $E(Y) = \sum_{\text{all } y} y\Pr(Y = y)$
2 Substitute the appropriate values into the rule.	$E(Y) = 0(0.08) + 1(0.34) + 2(0.38) + 3(0.17) + 4(0.03)$
3 Simplify.	$E(Y) = 0 + 0.34 + 0.76 + 0.51 + 0.12$ $= 1.73$
b 1 Find $E(Y^2)$.	**b** $E(Y^2) = 0^2(0.08) + 1^2(0.34) + 2^2(0.38) + 3^2(0.17) + 4^2(0.03)$ $= 0 + 0.34 + 1.52 + 1.53 + 0.48$ $= 3.87$
2 Write the rule for the variance.	$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$
3 Substitute in the appropriate values and evaluate.	$\text{Var}(Y) = 3.87 - 1.73^2$ $= 0.8771$
4 Write the rule for the standard deviation.	$\text{SD}(Y) = \sqrt{\text{Var}(Y)}$
5 Substitute in the variance and evaluate.	$\text{SD}(Y) = \sqrt{0.8771}$ $= 0.9365$
c 1 Find $\mu - 2\sigma$.	**c** $\mu - 2\sigma = 1.73 - 2(0.9365)$ $= -0.143$
2 Find $\mu + 2\sigma$.	$\mu + 2\sigma = 1.73 + 2(0.9365)$ $= 3.603$
3 Substitute the values into $\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.	$\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma)$ $= \Pr(-0.143 \le Y \le 3.603)$

4 Interpret this interval in the context of a discrete distribution. The smallest y-value in the distribution table is 0, so -0.173 is rounded up to 0. The largest y-value in the distribution table that is smaller than 3.573 is 3.	$\Pr(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$ $= \Pr(0 \leq Y \leq 3)$ $= 0.08 + 0.34 + 0.38 + 0.17$ $= 0.97$ *Note:* This is very close to the estimated value of 0.95.

WORKED EXAMPLE 9

A biased die has a probability distribution for the outcome of the die being rolled as follows.

x	1	2	3	4	5	6
$\Pr(X = x)$	0.1	0.1	0.2	0.25	0.25	0.1

a **Find Pr(even number).**

b **Find $\Pr(X \geq 3 | X \leq 5)$.**

c **Find $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.**

THINK	WRITE
a 1 State the probabilities to be added.	**a** $\Pr(\text{even number}) = \Pr(X = 2) + \Pr(X = 4) + \Pr(X = 6)$
2 Substitute the values and simplify.	$= 0.1 + 0.25 + 0.1$ $= 0.45$
b 1 Define the rule.	**b** $\Pr(X \geq 3 \vert X \leq 5) = \dfrac{\Pr(X \geq 3 \cap X \leq 5)}{\Pr(X \leq 5)}$
2 Find $\Pr(X \geq 3 \cap X \leq 5)$.	$\Pr(X \geq 3 \cap X \leq 5) = \Pr(3 \leq X \leq 5)$ $= \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$ $= 0.2 + 0.25 + 0.25$ $= 0.7$
3 Calculate $\Pr(X \leq 5)$.	$\Pr(X \leq 5) = 1 - \Pr(X = 6)$ $= 1 - 0.1$ $= 0.9$
4 Substitute the appropriate values into the formula.	$\Pr(X \geq 3 \vert X \leq 5) = \dfrac{\Pr(X \geq 3 \cap X \leq 5)}{\Pr(X \leq 5)}$ $= \dfrac{\Pr(3 \leq X \leq 5)}{\Pr(X \leq 5)}$ $= \dfrac{0.7}{0.9}$
5 Evaluate and simplify.	$= \dfrac{7}{9}$
c 1 Calculate the expected value.	**c** $E(X) = 1(0.1) + 2(0.1) + 3(0.2) + 4(0.25)$ $+ 5(0.25) + 6(0.1)$ $= 0.1 + 0.2 + 0.6 + 1 + 1.25 + 0.6$ $= 3.75$

2 Calculate $E(X^2)$.	$E(X^2) = 1^2(0.1) + 2^2(0.1) + 3^2(0.2) + 4^2(0.25)$ $+ 5^2(0.25) + 6^2(0.1)$ $= 0.1 + 0.4 + 1.8 + 4 + 6.25 + 3.6$ $= 16.15$
3 Calculate the variance.	$\text{Var}(X) = E(X^2) - [E(X)]^2$ $= 16.15 - 3.75^2$ $= 2.0875$
4 Calculate the standard deviation	$\text{SD}(X) = \sqrt{\text{Var}(X)}$ $= \sqrt{2.0875}$ $= 1.4448$
5 Calculate $\mu - 2\sigma$.	$\mu - 2\sigma = 3.75 - 2(1.4448)$ $= 0.8604$
6 Calculate $\mu + 2\sigma$.	$\mu + 2\sigma = 3.75 + 2(1.4448)$ $= 6.6396$
7 Substitute the appropriate values into $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$.	$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$ $= \Pr(0.8604 \le X \le 6.6396)$
8 Interpret this interval in the context of a discrete distribution.The smallest x-value in the distribution table is 1, so 0.8604 is rounded up to 1. The largest x-value in the distribution table that is smaller than 6.6396 is 6.	$= \Pr(1 \le X \le 6)$ $= 1$ *Note:* This is very close to the estimated value of 0.95.

EXERCISE 10.4 Applications

PRACTISE

1 WE8 A discrete random variable, X, has the following probability distribution.

x	5	10	15	20	25
$\Pr(X = x)$	0.05	0.25	0.4	0.25	0.05

a Find the expected value of X.
b Find the standard deviation of X, correct to 4 decimal places.
c Find $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$.

2 The number of Tails, X, when a fair coin is tossed six times has the following probability distribution.

x	0	1	2	3	4	5	6
$\Pr(X = x)$	0.012	0.093	0.243	0.315	0.214	0.1	0.023

Find $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$.

3 WE9 A financial adviser for a large company has put forward a number of options to improve the company's profitability, X (measured in hundreds of thousands of dollars). The decision to implement the options will be based on the cost of the options as well as their profitability. The company stands to make an extra profit of 1 million dollars with a probability of 0.1, an extra profit of \$750 000 with a probability of 0.3, an extra profit of \$500 000 with a probability of 0.3, an extra profit of \$250 000 with a probability of 0.2 and an extra profit of \$100 000 with a probability of 0.1.

Find:

a $\Pr(X \leq \$500\,000)$

b $\Pr(X \geq \$250\,000 \mid X \leq \$750\,000)$

c the expected profit

d $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

4 A discrete random variable, Z, can take the values 0, 1, 2, 3, 4 and 5. The probability distribution of Z is as follows:

$\Pr(Z = 0) = \Pr(Z = 1) = \Pr(Z = 2) = m$

$\Pr(Z = 3) = \Pr(Z = 4) = \Pr(Z = 5) = n$

and $\Pr(Z < 2) = 3\Pr(Z > 4)$ where m and n are constants.

a Determine the values of m and n.

b Show that the expected value of Z is $\frac{11}{5}$, and determine the variance and standard deviation for Z, correct to 4 decimal places.

c Find $\Pr(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 A discrete random variable, Y, has the following probability distribution.

y	1	2	d	8
$\Pr(Y = y)$	0.3	0.2	0.4	0.1

a Find the value of the constant d if it is known that $E(Y) = 3.5$.

b Determine $\Pr(Y \geq 2 \mid Y \leq d)$.

c Find $\text{Var}(Y)$.

d Find $\text{SD}(Y)$ correct to 4 decimal places.

6 A discrete random variable, Z, has the following probability distribution.

z	1	3	5	7	9
$\Pr(Z = z)$	0.2	0.15	a	b	0.05

The expected value of Z is known to be equal to 4.6.

a Find the values of the constants a and b.

b Determine the variance and standard deviation of Z, correct to 4 decimal places where appropriate.

c Find:

i $E(3Z + 2)$

ii $\text{Var}(3Z + 2)$.

7 A probability distribution is such that

$\Pr(Z = 0) = \Pr(Z = 1) = \Pr(Z = 2) = \Pr(Z = 3) = m$

$\Pr(Z = 4) = \Pr(Z = 5) = n$

and $\Pr(Z \leq 3) = \Pr(Z \geq 4)$.

a Find the values of the constants m and n.

b Find:

i $E(Z)$

ii $\text{Var}(Z)$.

c Find $\Pr(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

8 In a random experiment the events M and N are independent events where $\Pr(M) = 0.45$ and $\Pr(N) = 0.48$.

a Find the probability that both M and N occur.
b Find the probability that neither M nor N occur.

Let Y be the discrete random variable that defines the number of times M and N occur.

$Y = 0$ if neither M nor N occurs.

$Y = 1$ if only one of M and N occurs.

$Y = 2$ if both M and N occur.

c Specify the probability distribution for Y.
d Determine, correct to 4 decimal places where appropriate:
i $\text{E}(Y)$ **ii** $\text{Var}(Y)$ **iii** $\text{SD}(Y)$.

9 A probability function is defined as $p(x) = \frac{1}{9}(4 - x)$, $x \in \{0, 1, 2\}$.

a Construct a probability distribution table.
b Find, correct to 4 decimal places where appropriate:
i $\text{E}(X)$ **ii** $\text{Var}(X)$ **iii** $\text{SD}(X)$.
c Find $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

10 The number of customers, X, waiting in line at a bank just before closing time has a probability distribution as follows.

x	0	1	2	3
$\Pr(X = x)$	$\frac{k^2}{4}$	$\frac{5k-1}{12}$	$\frac{3k-1}{12}$	$\frac{4k-1}{12}$

a Find the value of the constant k.
b Determine the expected number of customers waiting in line just before closing time.
c Calculate the probability that the number of customers waiting in line just before closing time is no greater than $\text{E}(X)$.

11 The television show *Steal or No Steal* features 26 cases with various amounts of money ranging from 50 cents to \$200 000. The contestant chooses one case and then proceeds to open the other cases. At the end of each round, the banker makes an offer to end the game. The game ends when the contestant accepts the offer or when all the other 25 cases have been opened; in the latter event, the contestant receives the amount of money in the case they first chose.

Suppose a contestant has five cases left and the amounts of \$200 000, \$100 000, \$50 000, \$15 000 and \$1000 are still to be found.

a Find the expected amount that the banker should offer the contestant to end the game.
b The contestant turned down the offer and opened a case containing \$100 000. What would you expect the banker to offer the contestant at this stage?

12 A bookstore sells both new and secondhand books. A particular new autobiography costs \$65, a good-quality used autobiography costs \$30 and a worn autobiography costs \$12. A new cookbook costs \$54, a good-quality used cookbook costs \$25 and a worn cookbook costs \$15. Let X denote the total cost of buying two books (an autobiography and a cookbook). Assume that the purchases are independent of one another.

a Construct a probability distribution table for the cost of the two textbooks if the following probabilities apply.

- The probability of buying a new autobiography is 0.4.
- The probability of buying a good-quality used autobiography is 0.3.
- The probability of buying a worn used autobiography is 0.3.
- The probability of buying a new cookbook is 0.4.
- The probability of buying a good-quality used cookbook is 0.25.
- The probability of buying a worn used cookbook is 0.35.

b Find the expected cost of the two books.

13 Let X be the number of dining suites sold by the dining suite department of a large furniture outlet on any given day. The probability function for this discrete random variable is as follows.

x	0	1	2	3
$\Pr(X = x)$	0.3	0.4	0.2	0.1

The dining suite department receives a profit of \$350 for every dining setting sold. The daily running costs for the sales operation of the department are \$120. The net profit per day is a function of the random variable such that $y(x) = 350x - 120$ dollars.

a Set up a probability distribution table for the net profit, $\$Y$, per day.

b Find the expected daily profit for the dining suite department.

c Determine $\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.

14 A loaded six-sided die and a biased coin are tossed simultaneously. The coin is biased such that the probability of obtaining a Head is three times the probability of obtaining a Tail. The loaded die has the following probabilities for each of the numbers 1 to 6.

$$\Pr(1) = \Pr(2) = \Pr(5) = \frac{1}{12}$$

$$\Pr(3) = \Pr(4) = \Pr(6) = \frac{1}{4}$$

When a player tosses the coin and die simultaneously, they receive the following outcomes.

$$\overbrace{\text{1T 2T 5T}}^{\text{10 points}} \qquad \overbrace{\text{1H 2H 5H}}^{\text{5 points}} \qquad \overbrace{\text{All other results}}^{\text{1 point}}$$

Let X be the number of points scored from a simultaneous toss.

a Construct a probability distribution table for the number of points scored.
b Find the expected points received from a single toss, correct to 1 decimal place.
c If 25 simultaneous tosses occurred, what would the expected score be, correct to 1 decimal place?
d What is the minimum number of simultaneous tosses that would have to occur for the expected total to be a score of 100?

MASTER

15 In a certain random experiment the events V and W are independent events.

a If $\Pr(V \cup W) = 0.7725$ and $\Pr(V \cap W) = 0.2275$, find $\Pr(V)$ and $\Pr(W)$, given $\Pr(V) < \Pr(W)$.
b Find the probability that neither V nor W occur.

Let X be the discrete random variable that defines the number of times events V and W occur.

$X = 0$ if neither V nor W occurs.
$X = 1$ if only one of V and W occurs.
$X = 2$ if both V and W occur.

c Specify the probability distribution for X.
d Determine, correct to 4 decimal places where appropriate:
  i $\mathrm{E}(X)$
  ii $\mathrm{Var}(X)$
  iii $\mathrm{SD}(X)$.

16 The probability distribution table for the discrete random variable, Z, is as follows.

z	1	3	5
$\Pr(Z = z)$	$\frac{k^2}{7}$	$\frac{5-2k}{7}$	$\frac{8-3k}{7}$

a Find the value(s) of the constant k.
b Find, correct to 4 decimal places:
  i $\mathrm{E}(Z)$
  ii $\mathrm{Var}(Z)$
  iii $\mathrm{SD}(Z)$.
c Find $\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma)$.

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10.5 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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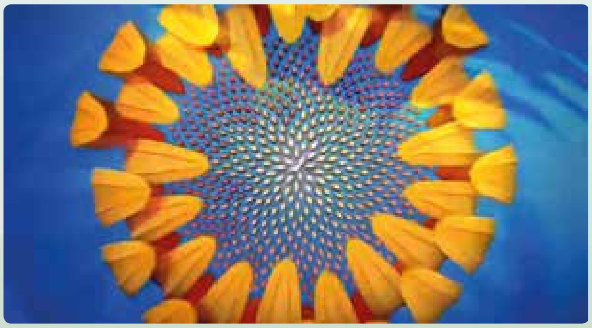
Activities

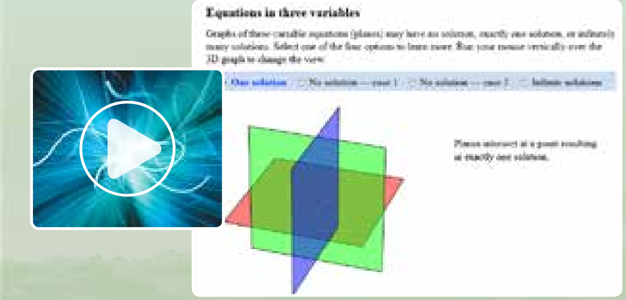
To access eBookPLUS activities, log on to

www.jacplus.com.au

Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





Equations in three variables

Graphs of three-variable equations (planes) may have no solution, exactly one solution, or infinitely many solutions. Select one of the four options to learn more. Run your mouse vertically over the 3D graph to change the view.

Planes intersect at a point resulting in exactly one solution.

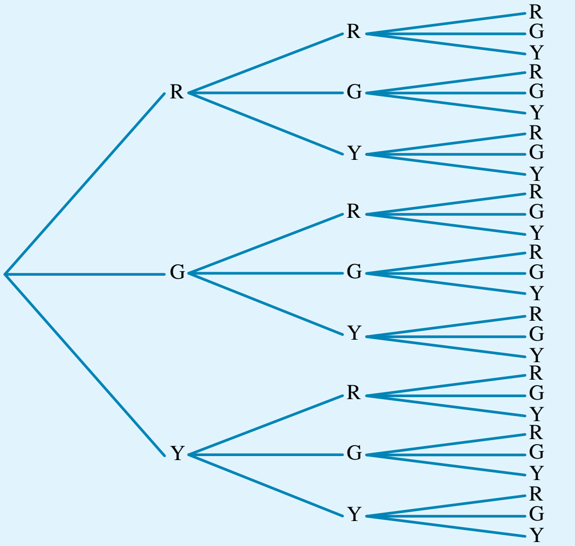
studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

10 Answers

EXERCISE 10.2

1 a



ξ = {RRR, RRG, RRY, RGR, RGG, RGY, RYR, RYG, RYY, GRR, GRG, GRY, GGR, GGG, GGY, GYR, GYG, GYY, YRR, YRG, YRY, YGR, YGG, YGY, YYR, YYG, YYY}

b $\Pr(Y = 3) = \frac{27}{1000}$, $\Pr(Y = 2) = \frac{189}{1000}$, $\Pr(Y = 1) = \frac{441}{1000}$, $\Pr(Y = 0) = \frac{343}{1000}$

c

y	0	1	2	3
$\Pr(Y = y)$	$\frac{343}{1000}$	$\frac{441}{1000}$	$\frac{189}{1000}$	$\frac{27}{1000}$

d All probabilities are $0 \le \Pr(X = x) \le 1$. $\sum \Pr(X = x) = 1$; therefore, this is a discrete probability function.

2

x	0	1	2
$\Pr(X = x)$	$\frac{25}{36}$	$\frac{10}{36} = \frac{5}{18}$	$\frac{1}{36}$

3 a i $0 \le \Pr(Y = y) \le 1$ for all y and $\sum_{\text{all } y} \Pr(Y - y) = 1$

Yes, this is a discrete probability function.

ii $0 \le \Pr(Y = y) \le 1$ for all y and $\sum_{\text{all } y} \Pr(Y - y) = 1$

Yes, this is a discrete probability function.

b $k = \frac{1}{16}$

4 a i $0 \le \Pr(Y = y) \le 1$ for all y and $\sum_{\text{all } y} \Pr(Y - y) = 0.9$

No, this is not a discrete probability function.

ii Probabilities cannot have negative values. No, this is not a discrete probability function.

b $k = \frac{1}{4}$

5 a ξ = {11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66}

b

z	0	1	2
$\Pr(Z = z)$	0.09	0.42	0.49

c 0.42

6 a ξ = {SSS, SSA, SAS, SAA, ASS, ASA, AAS, AAA}

b

x	0	1	2	3
$\Pr(X = x)$	0.216	0.432	0.288	0.064

c 0.936

7 a $0 \le \Pr(X = x) \le 1$ for all x but $\sum_{\text{all } x} \Pr(X = x) \neq 1$.

This is not a probability distribution.

b Probabilities cannot have negative values. This is not a probability distribution.

c $0 \le \Pr(Z = z) \le 1$ for all z but $\sum_{\text{all } z} \Pr(Z = z) = 1.1$.

This is not a probability distribution.

d $0 \le \Pr(Z = z) \le 1$ for all x and $\sum_{\text{all } x} \Pr(X = x) = 1$.

This is a probability distribution.

8 a $d = 0.15$

b $k = \frac{1}{6}$

c $a = \frac{1}{3}$

9 a $0 \le \Pr(X = x) \le 1$ for all x and $\sum_{\text{all } x} \Pr(X = x) = 1$.

This is a discrete probability distribution.

b $0 \le \Pr(Z = z) \le 1$ for all x but $\sum_{\text{all } x} \Pr(X = x) \neq 1$.

This is not a discrete probability distribution.

c $0 \le \Pr(Z = z) \le 1$ for all x and $\sum_{\text{all } x} \Pr(X = x) = 1$.

This is a discrete probability distribution.

10 $a = 30$

11 a ξ = {FFFF, FFFM, FFMF, FFMM, FMFF, FMFM, FMMF, FMMM, MFFF, MFFM, MFMF, MFMM, MMFF, MMFM, MMMF, MMMM}

b

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{16}$	$\frac{4}{16} = \frac{1}{4}$	$\frac{6}{16} = \frac{3}{8}$	$\frac{4}{16} = \frac{1}{4}$	$\frac{1}{16}$

c $\frac{1}{16}$

d $\frac{15}{16}$

e $\frac{11}{16}$

12 a ξ = {11, 12, 13, 14, 15, 16, 17, 18, 19, 110, 111, 112, 21, 22, 23, 24, 25, 26, 27, 28, 29, 210, 211, 212, 31, 32, 33, 34, 35, 36, 37, 38, 39, 310, 311, 312, 41, 42, 43, 44, 45, 46, 47, 48, 49, 410, 411, 412, 51, 52, 53, 54, 55, 56, 57, 58, 59, 510, 511, 512, 61, 62, 63, 64, 65, 66, 67, 68, 69, 610, 611, 612, 71, 72, 73, 74, 75, 76, 77, 78, 79, 710, 711, 712, 81, 82, 83, 84, 85, 86, 87, 88, 89, 810, 811, 812}

b $\Pr(x = 0) = \frac{28}{96}$, $\Pr(x = 1) = \frac{48}{96}$, $\Pr(X = 2) = \frac{20}{96}$

c Pr(win) = 0.009

13 a Possible scores = 4, 7, 10, 12, 15 and 20 points

b

x	4	7	10	12	15	20
$\Pr(X = x)$	$\frac{81}{169}$	$\frac{18}{169}$	$\frac{1}{169}$	$\frac{54}{169}$	$\frac{6}{169}$	$\frac{9}{169}$

c **i** $\frac{6}{169}$ **ii** $\frac{69}{169}$ **iii** $\frac{2}{23}$

14 a $m = \frac{1}{9}$, $n = \frac{1}{3}$

b **i** $\frac{5}{9}$ **ii** $\frac{1}{5}$

15 a See the table at the foot of the page.*

b 0.6826. It is a success, helping 3 or more patients.

16 a The game can end as follows:

Throw 1 red — game over.

Throw 1 blue, green or yellow, throw 2 red — game over.

Throws 1 and 2 combinations of blue, green and yellow, throw 3 red – game over.

b Wins $10 with BBB, GGG or YYY

c

x	$0	$1	$10
$\Pr(X = x)$	$\frac{98}{125}$	$\frac{24}{125}$	$\frac{3}{125}$

17 a See the table at the foot of the page.*

b **i** 0.8999 **ii** 0.8457

18 $k = 0.6568$ or $k = -0.4568$

EXERCISE 10.3

1 a

y	1	2	3	4
$\Pr(X = x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

b 3.125

2 a

x	−$2	$3	$6	$8
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

b $3.75

3 a $d = 15$

b **i** 18 **ii** −2.5 **iii** −15

4 $E(Z) = 4.42$

5 a $2.45

b $\text{Var}(X) = \$1.35$, $\text{SD}(X) = \$1.16$

6 a $k = \frac{1}{10}$

b 3.2

c Var(X) = 6.56, SD(X) = 2.56

7 a

x	1	2	3	4
$\Pr(X = x)$	$\frac{1}{30}$	$\frac{4}{30} = \frac{2}{15}$	$\frac{9}{30} = \frac{3}{10}$	$\frac{16}{30} = \frac{8}{15}$

b **i** $\frac{10}{3}$ **ii** $\frac{31}{45} = 0.69$

c **i** 11.02 **ii** 6.2

8 a $m = 15$

b $\text{Var}(Z) = 153.48$, $\text{Var}(2(Z - 1)) = 613.91$, $\text{Var}(3 - Z) = 153.48$

9 a **i** $E(X) = \frac{1}{9}$ **ii** $\text{SD}(X) = 1.7916$

b **i** $E(Y) = 7$ **ii** $\text{SD}(Y) = 3.7947$

c **i** $E(Z) = \frac{19}{6}$ **ii** $\text{SD}(Z) = 1.3437$

10 a $c = \frac{1}{3}$

b $E(Y) = 3$

c $\text{Var}(Y) = 11.56$, $\text{SD}(Y) = 3.40$

11 a 8 **b** 0.5 **c** 14.5

12 a 225 **b** 25 **c** 6.25

13 a $p(x) = h(3 - x)(x + 1)$

$$\begin{aligned}p(0) &= h(3)(1) = 3h\\ p(1) &= h(3 - 1)(1 + 1) = 4h\\ p(2) &= h(3 - 2)(2 + 1) = 3h\\ 3h + 4h + 3h &= 1\\ 10h &= 1\\ h &= \frac{1}{10}\end{aligned}$$

b $E(X) = 1$, $\text{Var}(X) = 0.6$, $\text{SD}(X) = 0.7746$

14 a $a = 0.3$, $b = 0.1$

b $\text{Var}(X) = 1.65$, $\text{SD}(X) = 1.2845$

15 a $E(X^2) = a^2 + 2a - 2$

b $E(X) = a = 2$, $\text{Var}(X) = 2a - 2 = 2$

16 a $n = \frac{1}{13}$

b $E(Y) = \frac{46}{13} = 3.5385$, $\text{Var}(Y) = 1.7870$, $\text{SD}(Y) = 1.3368$

17 a ξ = {11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 36, 37, 38, 41, 42, 43, 44, 45, 46, 47, 48, 51, 52, 53, 54, 55, 56, 57, 58, 61, 62, 63, 64, 65, 66, 67, 68, 71, 72, 73, 74, 75, 76, 77, 78, 81, 82, 83, 84, 85, 86, 87, 88}

---

*15a

x	0	1	2	3	4	5
$\Pr(X = x)$	0.0102	0.0768	0.2304	0.3456	0.2592	0.0778

*17a

x	0	1	2	3	4	5	6
$\Pr(X = x)$	0.0014	0.0165	0.0823	0.2195	0.3292	0.2634	0.0878

b

z	1	2	3	4	5	6	7	8
$\Pr(Z = z)$	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{5}{64}$	$\frac{7}{64}$	$\frac{9}{64}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{15}{64}$

c $\text{E}(Z) = 5.8125$, $\text{SD}(Z) = 1.8781$

18 a $\Pr(\text{Band A}) = \frac{1}{25}$, $\Pr(\text{Band B}) = \frac{3}{25}$, $\Pr(\text{Band C}) = \frac{5}{25}$,

$\Pr(\text{Band D}) = \frac{7}{25}$, $\Pr(\text{Band E}) = \frac{9}{25}$

b

x	−\$1	\$0	\$1	\$4	\$9
$\Pr(X = x)$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{5}{25} = \frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{25}$

c **i** $\text{E}(X) = 0.68$ cents **ii** $\text{SD}(X) = \$2.29$

19 a Score totals = $\{1 + 1 = 2, 1 + 3 = 4, 1 + 5 = 6, 1 + 7 = 8, 1 + 10 = 11, 3 + 1 = 4, 3 + 3 = 6, 3 + 5 = 8, 3 + 7 = 10, 3 + 10 = 13, 5 + 1 = 6, 5 + 3 = 8, 5 + 5 = 10, 5 + 7 = 12, 5 + 10 = 15, 7 + 1 = 8, 7 + 3 = 10, 7 + 5 = 12, 7 + 7 = 14, 7 + 10 = 17, 10 + 1 = 11, 10 + 3 = 13, 10 + 5 = 15, 10 + 7 = 17, 10 + 10 = 20\}$

b See the table at the foot of the page.*

c $\text{E}(X) = 9.4$, $\text{SD}(X) = 3.7974$

20 a $k = 0.1$

b $\text{E}(X) = 1.695$

c $\text{SD}(X) = 1.1670$

EXERCISE 10.4

1 a 15 **b** 4.7434 **c** 0.9

2 0.965

3 a 0.6

b $\frac{8}{9}$

c \$535 000

d 1

4 a $m = \frac{1}{5}$, $n = \frac{2}{15}$

b $\text{Var}(Z) = 2.8267$, $\text{SD}(Z) = 1.6813$

c 1

5 a $d = 5$

b $\frac{2}{3}$

c $\text{Var}(Y) = 5.25$

d $\text{SD}(Y) = 2.2913$

6 a $a = 0.35$, $b = 0.25$

b $\text{Var}(Z) = 5.44$, $\text{SD}(Z) = 2.3324$

c **i** 15.8 **ii** 48.96

7 a $m = \frac{1}{8}$, $n = \frac{1}{4}$

b **i** $\text{E}(Z) = 3$ **ii** $\text{Var}(Z) = 3$

c $\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) = 1$

8 a 0.216

b 0.286

c

y	0	1	2
$\Pr(Y = y)$	0.286	0.498	0.216

d **i** $\text{E}(Y) = 0.93$

ii $\text{Var}(Y) = 0.4971$

iii $\text{SD}(Y) = 0.7050$

9 a

x	0	1	2
$\Pr(X = x)$	$\frac{4}{9}$	$\frac{3}{9} = \frac{1}{3}$	$\frac{2}{9}$

b **i** $\frac{7}{9}$ **ii** $\frac{50}{81} = 0.6173$ **iii** 0.7857

c 1

10 a $k = 1$ **b** 1.4 **c** $\frac{7}{12}$

11 a \$73 200

b \$66 500

12 a See the table at the foot of the page.*

b \$71.70

13 a

y	−\$120	\$230	\$580	\$930
$\Pr(Y = y)$	0.3	0.4	0.2	0.1

b $\text{E}(Y) = \$265$

c $\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma) = 0.9$

14 a

x	1	5	10
$\Pr(X = x)$	$\frac{12}{16} = \frac{3}{4}$	$\frac{3}{16}$	$\frac{1}{16}$

b 2.3

c 57.8

d 44 tosses

15 a $\Pr(\text{V}) = 0.35$, $\Pr(\text{W}) = 0.65$

b 0.2275

c

x	0	1	2
$\Pr(X = x)$	0.2275	0.5450	0.2275

d **i** $\text{E}(X) = 1$

ii $\text{Var}(X) = 0.455$

iii $\text{SD}(X) = 0.6745$

16 a $k = 2$

b **i** $\text{E}(Z) = 2.4286$

ii $\text{Var}(Z) = 3.1019$

iii $\text{SD}(Z) = 1.7613$

c $\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) = 1$

---

*19b

x	2	4	6	8	10	11	12	13	14	15	17	20
$\Pr(X = x)$	0.04	0.08	0.16	0.2	0.17	0.04	0.12	0.04	0.04	0.06	0.04	0.01

*12a

x	119	90	84	80	66	55	45	37	27
$\Pr(X = x)$	0.16	0.1	0.12	0.14	0.12	0.075	0.105	0.075	0.105

11

The binomial distribution

11.1 Kick off with CAS

Exploring binomial probability distributions

The binomial distribution is a type of discrete probability distribution. There are only two possible outcomes—success and failure. For example, if success is rolling a 2 on a standard die, then failure is rolling a 1, 3, 4, 5 or 6. The pronumeral n represents the number of independent trials, and p represents the probability of success.

To find the probability of each possible scenario, the binomial probability function on CAS can be used.

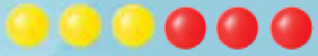
1 Six balls are in a bag — 1 red ball and 5 yellow balls. A ball is selected from the bag, its colour recorded, and then it is returned to the bag. This is repeated until 5 balls have been selected. $n = 5,\ p = \frac{1}{6}$

a Using CAS, complete the table below by finding the probabilities for each possible number of selected red balls.

Number of red balls selected	0	1	2	3	4	5
Probability						

b Graph the number of red balls against the probability. Describe the shape of the graph.

2 Six balls are in a bag — 3 red balls and 3 yellow balls. A ball is selected from the bag, its colour recorded, and then it is returned to the bag. This is repeated until 5 balls have been selected. $n = 5,\ p = \frac{1}{2}$



a Using CAS, complete the table below.

Number of red balls selected	0	1	2	3	4	5
Probability						

b Graph the number of red balls against the probability. Describe the shape of the graph.

3 Six balls are in a bag — 5 red balls and 1 yellow ball. A ball is selected from the bag, its colour recorded, and then it is returned to the bag. This is repeated until 5 balls have been selected. $n = 5,\ p = \frac{5}{6}$

a Using CAS, complete the table below.

Number of red balls selected	0	1	2	3	4	5
Probability						

b Graph the number of red balls against the probability. Describe the shape of the graph.

4 Compare your answers for parts 1–3. How does the value of p relate to the shape of the graph?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

11.2 Bernoulli trials

Introduction

eBook*plus*

Interactivity
The Bernoulli distribution
int-6430

In probability theory, the **Bernoulli distribution** is a discrete probability distribution of the simplest kind. It is named after the Swiss mathematician Jakob Bernoulli (1654–1705). The term '**Bernoulli trial**' refers to a single event that has only 2 possible outcomes, a success or a failure, with each outcome having a fixed probability. The following are examples of Bernoulli trials.

- Will a coin land Heads up?
- Will a newborn child be a male or a female?
- Are a random person's eyes blue or not?
- Will a person vote for a particular candidate at the next local council elections or not?
- Will you pass or fail an examination?

The Bernoulli distribution has only one controlling parameter: the probability of success, p. The alternative to success is failure, which is denoted by $1 - p$ (and can also be denoted by q).

For a discrete probability distribution that has a Bernoulli random variable, X:

$$\mathbf{Pr}(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

The mean and variance for a Bernoulli distribution

If X is a Bernoulli random variable with the following distribution,

x	0	1
$\Pr(X = x)$	$1 - p$	p

then

$$\begin{aligned} \mathrm{E}(X) &= \sum_{\text{all } x} x\Pr(X = x) \\ &= 0(1 - p) + 1 \times p \\ &= p \\ \mathrm{E}(X^2) &= 0^2(1 - p) + 1^2 \times p \\ &= p \\ \mathrm{Var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= p - p^2 \\ &= p(1 - p) \end{aligned}$$

For a Bernoulli distribution:

$$E(X) = \mu = p$$

$$\mathrm{Var}(X) = \sigma^2 = p(1 - p)$$

WORKED EXAMPLE 1

a Determine which of the following can be defined as a Bernoulli trial.

i Interviewing a random person to see if they have had a flu injection this year

ii Rolling a die in an attempt to obtain an even number

iii Choosing a ball from a bag which contains 3 red balls, 5 blue balls and 4 yellow balls

b A new cream has been developed for the treatment of dermatitis. In laboratory trials the cream was found to be effective in 72% of the cases. Hang's doctor has prescribed the cream for her. Let X be the effectiveness of the cream.

i Construct a probability distribution table for X.

ii Find $E(X)$.

iii Find the variance and the standard deviation of X, correct to 4 decimal places.

THINK	WRITE
a **i** Check for the characteristics of a Bernoulli trial.	**a** Yes, this is a Bernoulli trial, as there are 2 possible outcomes. A person either has or has not had a flu injection this year.
ii Check for the characteristics of a Bernoulli trial.	Yes, this is a Bernoulli trial, as there are 2 possible outcomes. The die will show either an odd number or an even number.
iii Check for the characteristics of a Bernoulli trial.	No, this is not a Bernoulli trial, as success has not been defined.
b **i** Construct a probability distribution table and clearly state the value of p.	**b** $p =$ success with cream $= 0.72$

x	0	1
$\Pr(X = x)$	0.28	0.72

THINK	WRITE
ii **1** State the rule for the expected value.	$E(X) = \sum_{\text{all } x} x\Pr(X = x)$
2 Substitute the appropriate values and evaluate.	$E(X) = 0 \times 0.28 + 1 \times 0.72$ $= 0.72$
iii **1** Find $E(X^2)$.	$E(X^2) = 0^2 \times 0.28 + 1^2 \times 0.72$ $= 0.72$
2 Calculate the variance.	$\text{Var}(X) = E(X^2) - [E(X)]^2$ $\text{Var}(X) = 0.72 - (0.72)^2$ $= 0.2016$
3 Calculate the standard deviation.	$SD(X) = \sqrt{0.2016}$ $= 0.4490$

EXERCISE 11.2 Bernoulli trials

PRACTISE

Work without CAS
Question 1

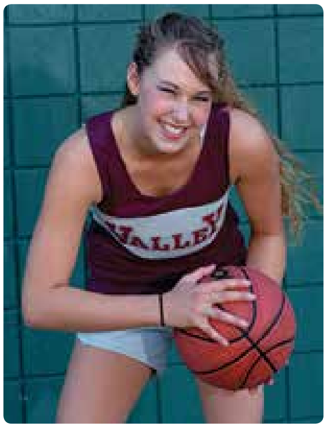
1 WE1 Determine which of the following can be defined as a Bernoulli trial.

a Spinning a spinner with 3 coloured sections

b A golfer is at the tee of the first hole of a golf course. As she is an experienced golfer, the chance of her getting a hole in one is 0.15. Will she get a hole in one at this first hole?

c A card is drawn from a standard pack of 52 cards. What is the chance of drawing an ace?

2 Caitlin is playing basketball for her local club. The chance that Caitlin scores a goal is 0.42. The ball has just been passed to her and she shoots for a goal. Let X be the random variable that defines Caitlin getting a goal. (Assume X obeys the Bernoulli distribution).



a Set up a probability distribution for this discrete random variable.

b Find $\mathrm{E}(X)$.

c Find:

  i $\mathrm{Var}(X)$

  ii $\mathrm{SD}(X)$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 Determine which of the following can be defined as a Bernoulli trial.

a A new drug for arthritis is said to have a success rate of 63%. Jing Jing has just been prescribed the drug to treat her arthritis, and her doctor is interested in whether her symptoms improve or not.

b Juanita has just given birth to a baby, and we are interested in the gender of the baby, in particular whether the baby is a girl.

c You are asked what your favourite colour is.

d A telemarketer rings random telephone numbers in an attempt to sell a magazine subscription and has a success rate of 58%. Will the next person he rings subscribe to the magazine?

4 State clearly why the following are not Bernoulli trials.

a A bag contains 12 balls, 5 of which are black, 3 of which are white and 4 of which are red. Paul has just drawn a ball from the bag without returning it. Now it is Alice's turn to draw a ball from the bag. Does she get a red one?

b A die is tossed and the outcome is recorded.

c A fairy penguin colony at Phillip Island in Victoria is being studied by an ecologist. Will the habitat be able to sustain the colony in the future?

5 A discrete random variable, Z, has a Bernoulli distribution as follows.

z	0	1
$\Pr(Z = z)$	0.37	0.63

a Find $\mathrm{E}(Z)$. b Find $\mathrm{Var}(Z)$. c Find $\mathrm{SD}(Z)$.

6 Eli and Jacinta are about to play a game of chess. As Eli is a much more experienced chess player, the chance that he wins is 0.68. Let Y be the discrete random variable that defines the fact the Eli wins.

a Construct a probability distribution table for Y.

b Evaluate:

i $\mathrm{E}(Y)$

ii $\mathrm{Var}(Y)$

iii $\mathrm{SD}(Y)$

c Find $\Pr(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$.

7 During the wet season, the probability that it rains on any given day in Cairns in northern Queensland is 0.89. I am going to Cairns tomorrow and it is the wet season. Let X be the chance that it rains on any given day during the wet season.

a Construct a probability distribution table for X.

b Evaluate:

i $\mathrm{E}(X)$

ii $\mathrm{Var}(X)$

iii $\mathrm{SD}(X)$

c Find $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

8 X is a discrete random variable that has a Bernoulli distribution. It is known that the variance for this distribution is 0.21.

a Find the probability of success, p, where $p > 1 - p$.

b Find $\mathrm{E}(X)$.

9 Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation for this distribution is 0.4936.

a Find the variance of Y correct to 4 decimal places.

b Find the probability of success, p, if $p > 1 - p$.

c Find $\mathrm{E}(Y)$.

10 It has been found that when breast ultrasound is combined with a common mammogram, the rate in which breast cancer is detected in a group of women is 7.2 per 1000. Louise is due for her two-yearly mammography testing, which will involve an ultrasound combined with a mammogram. Let Z be the discrete random variable that breast cancer is detected.

a What is the probability that Louise has breast cancer detected at this next test?

b Construct a probability distribution table for Z.

c Find $\Pr(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

11 A manufacturer of sweets reassures their customers that when they buy a box of their ‘All Sorts’ chocolates there is a 33% chance that the box will contain one or more toffees. Kasper bought a box of ‘All Sorts’ and selected one. Let Y be the discrete random variable that Kasper chose a toffee.

a Construct a probability distribution table for Y.
b Find $\mathrm{E}(Y)$.
c Find $\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.

12 Cassandra is sitting for a Mathematics examination. She has just started question 1, which is a multiple choice question with 5 possible answer choices. Cassandra plans to randomly guess the answer to the question. Let X be the discrete random variable that Cassandra answers the question correctly.

a Construct a probability distribution table for X.
b Find $\mathrm{E}(X)$.
c In total the test has 5 multiple choice questions to be answered. What is the probability that Cassandra answers all five questions correctly?

MASTER

13 Z is a discrete random variable that has a Bernoulli distribution. It is known that the variance of Z is 0.1075.

a Find the probability of success, correct to 4 decimal places, if $\Pr(\text{success}) > \Pr(\text{failure})$.
b Construct a probability distribution table for Z.
c Evaluate the expected value of Z.

14 Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation of Y is 0.3316.

a Find the variance correct to 2 decimal places.
b Find the probability of success correct to 4 decimal places if $\Pr(\text{success}) > \Pr(\text{failure})$.

11.3 The binomial distribution

When a Bernoulli trial is repeated a number of times, we have a **binomial distribution**. A binomial distribution is characterised by the following rules:

- It is made up of n Bernoulli trials or n identical trials.
- Each trial is an independent trial.
- There are two possible outcomes for each trial, a success, p, and a failure, $1 - p$.

Consider again Question 12 from Exercise 11.2. Cassandra has 5 multiple choice questions to answer on her mathematics examination. Each question has 5 different choices for the correct answer, and she plans to randomly guess every question. Cassandra can get all 5 questions incorrect; 1 correct and 4 incorrect; 2 correct and 3 incorrect; 3 correct and 2 incorrect; 4 correct and 1 incorrect; or all 5 correct. This situation represents a binomial distribution and can be analysed as follows.

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If X represents the number of questions answered correctly, then $p = \frac{1}{5}$ and $1 - p = \frac{4}{5}$.

Let I = an incorrect answer and C = a correct answer

0 correct answers: outcome = IIIII

$$\begin{aligned}\Pr(X = 0) &= \left(\tfrac{4}{5}\right)^5 \\ &= \tfrac{1024}{3125} \\ &= 0.3277\end{aligned}$$

1 correct answer: outcomes = IIIIC, IIICI, IICII, ICIII, CIIII

$$\begin{aligned}\Pr(X = 1) &= 5 \times \left(\tfrac{4}{5}\right)^4\left(\tfrac{1}{5}\right) \\ &= \tfrac{1280}{3125} \\ &= 0.4096\end{aligned}$$

2 correct answers: outcomes = IIICC, IICIC, ICIIC, CIIIC, IICCI, ICICI, CIICI, ICCII, CICII, CCIII

$$\begin{aligned}\Pr(X = 2) &= 10 \times \left(\tfrac{4}{5}\right)^3\left(\tfrac{1}{5}\right)^2 \\ &= \tfrac{640}{3125} \\ &= 0.2048\end{aligned}$$

3 correct answers: outcomes = IICCC, ICICC, CIICC, ICCIC, CICIC, CCIIC, ICCCI, CICCI, CCICI, CCCII

$$\begin{aligned}\Pr(X = 3) &= 10 \times \left(\tfrac{4}{5}\right)^2\left(\tfrac{1}{5}\right)^3 \\ &= \tfrac{160}{3125} \\ &= 0.0512\end{aligned}$$

4 correct answers: outcomes = ICCCC, CICCC, CCICC, CCCIC, CCCCI

$$\begin{aligned}\Pr(X = 4) &= 5 \times \left(\tfrac{4}{5}\right)\left(\tfrac{1}{5}\right)^4 \\ &= \tfrac{20}{3125} \\ &= 0.0064\end{aligned}$$

5 correct answers: outcome = CCCCC

$$\begin{aligned}\Pr(X = 5) &= \left(\tfrac{1}{5}\right)^5 \\ &= \tfrac{1}{3125} \\ &= 0.0003\end{aligned}$$

This can then be represented in a table:

x	0	1	2	3	4	5
$\Pr(X = x)$	0.3277	0.4096	0.2048	0.0512	0.0064	0.0003

where $\sum_{\text{all } x} \Pr(X - x) = 1$.

It can be quite tedious to work out all the possible outcomes, especially when the number of trials is large. However, we usually just want to know how many different ways there are of obtaining each number of correct answers, not the actual specific order of the incorrect and correct answers.

From your probability studies in Units 1 and 2, you will recall that the number of ways of obtaining x successes from n independent trials is given by ${}^nC_x$, which can also be written as $\binom{n}{x}$.

$$ {}^nC_x = \frac{n!}{(n-x)!x!} $$

If a discrete random variable, X, has a binomial distribution, we say that

$$X \sim \text{Bi}(n, p)$$

where n is the number of independent trials and p is the probability of success.

If $X \sim \text{Bi}(n, p)$, then $Pr(X = x) = {}^nC_x(1-p)^{n-x}p^x$, where $x = 0,\ 1,\ 2,\ 3 \ldots n$.

Note: If the order is specified for a particular scenario, then the binomial probability distribution rule cannot be used. The probabilities need to be multiplied in the given order.

Graphing the binomial distribution

The probability distribution for the previous example, where Cassandra answered 5 questions on a mathematics exam, can be graphed as follows:

x	0	1	2	3	4	5
$Pr(X = x)$	0.3277	0.4096	0.2048	0.0512	0.0064	0.0003

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The shape of this graph (Figure 1) is positively skewed. It indicates that the probability of success is low (in this case $p = 0.2$), as the larger x-values (number of successful outcomes) have corresponding low probabilities.

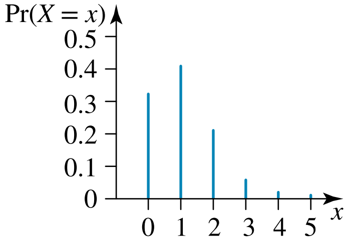

Figure 1

If the value of p was higher, for example if Cassandra was 80% sure of getting a question right ($p = 0.8$), then the graph would look like Figure 2.

The shape of this graph is negatively skewed. It indicates that the probability of success is high (in this case $p = 0.8$), as the larger x-values (number of successful outcomes) have corresponding high probabilities.

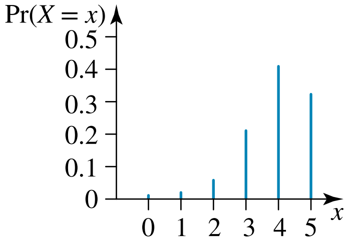

Figure 2

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If the value of p was 0.5, for example if Cassandra was 50% sure of getting a question right, then the graph would look like Figure 3.

The shape of this graph is symmetrical. It indicates that the probability of success is equal to the probability of failure. If the number of trials increased, the graph would approach the shape of a bell-shaped curve.

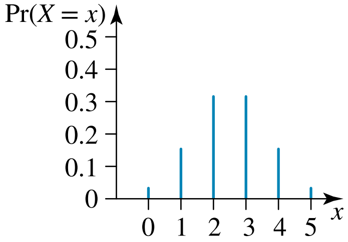

Figure 3

WORKED EXAMPLE 2

It is known that 52% of the population participates in sport on a regular basis. Five random individuals are interviewed and asked whether they participate in sport on a regular basis. Let X be the number of people who regularly participate in sport.

a Construct a probability distribution table for X.

b Find the probability that 3 people or less play sport.

c Find the probability that at least one person plays sport, given that no more than 3 people play sport.

d Find the probability that the first person interviewed plays sport but the next 2 do not.

THINK	WRITE
a 1 Write the rule for the probabilities of the binomial distribution.	a $X \sim \text{Bi}(5, 0.52)$ $\Pr(X = x) = {}^n\text{C}_x(1 - p)^{n-x}p^x$
2 Substitute $x = 0$ into the rule and simplify.	$\Pr(X = 0) = {}^5\text{C}_0(0.48)^5$ $= 0.025\,48$
3 Substitute $x = 1$ into the rule and simplify.	$\Pr(X = 1) = {}^5\text{C}_1(0.48)^4(0.52)$ $= 0.138\,02$
4 Substitute $x = 2$ into the rule and simplify.	$\Pr(X = 2) = {}^5\text{C}_2(0.48)^3(0.52)^2$ $= 0.299\,04$
5 Substitute $x = 3$ into the rule and simplify.	$\Pr(X = 3) = {}^5\text{C}_3(0.48)^2(0.52)^3$ $= 0.323\,96$
6 Substitute $x = 4$ into the rule and simplify.	$\Pr(X = 4) = {}^5\text{C}_4(0.48)(0.52)^4$ $= 0.175\,48$
7 Substitute $x = 5$ into the rule and simplify.	$\Pr(X = 5) = {}^5\text{C}_5(0.52)^5$ $= 0.038\,02$
8 Construct a probability distribution table and check that $\sum_{\text{all } x} \Pr(X = x) = 1$.	see table below

x	$\Pr(X = x)$
0	0.025 48
1	0.138 02
2	0.299 04
3	0.323 96
4	0.175 48
5	0.038 02

$$\sum_{\text{all } x} \Pr(X = x) = 1$$

THINK	WRITE
b 1 Interpret the question and write the probability to be found.	b $\Pr(X \le 3)$
2 State the probabilities included in $\Pr(X \le 3)$.	$\Pr(X \le 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3)$ $\Pr(X \le 3) = 1 - (\Pr(X = 4) + \Pr(X = 5))$

	3 Substitute the appropriate probabilities and evaluate.	$\Pr(X \le 3) = 1 - (0.17548 + 0.03802)$ $\Pr(X \le 3) = 0.7865$ *Note:* CAS technology can be used to add up multiple probabilities.
c	**1** State the rule for conditional probability.	**c** $\Pr(X \ge 1 \mid X \le 3) = \dfrac{\Pr(X \ge 1 \cap X \le 3)}{\Pr(X \le 3)}$ $= \Pr(1 \le X \le 3)$
	2 Evaluate $\Pr(X \ge 1 \cap X \le 3)$.	$\Pr(X \ge 1 \cap X \le 3) = \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3)$ $= 0.13802 + 0.29904 + 0.32396$ $= 0.76102$
	3 Substitute the appropriate values into the rule.	$\Pr(X \ge 1 \mid X \le 3) = \dfrac{\Pr(X \ge 1 \cap X \le 3)}{\Pr(X \le 3)}$ $= \dfrac{0.76102}{0.7865}$
	4 Simplify.	$\Pr(X \ge 1 \mid X \le 3) = 0.9676$
d	**1** Order has been specified for this question. Therefore, the binomial probability distribution rule cannot be used. The probabilities must be multiplied together in order.	**d** S = plays sport, N = doesn't play sport $\Pr(SNN) = \Pr(S) \times \Pr(N) \times \Pr(N)$
	2 Substitute the appropriate values and evaluate.	$\Pr(SNN) = 0.52 \times 0.48 \times 0.48$ $= 0.1198$

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The mean and variance of the binomial distribution

If $X \sim \text{Bi}(n, p)$, then $\Pr(X = x) = {}^nC_x(1-p)^{n-x}p^x$, where $x = 0, 1, 2, 3 \ldots n$.

Suppose $n = 3$:

$$\begin{aligned}
\Pr(X = 0) &= {}^3C_0(1-p)^3p^0 \\
&= (1-p)^3 \\
\Pr(X = 1) &= {}^3C_1(1-p)^2p \\
&= 3p(1-p)^2 \\
\Pr(X = 2) &= {}^3C_2(1-p)p^2 \\
&= 3p^2(1-p) \\
\Pr(X = 3) &= {}^3C_3(1-p)^0p^3 \\
&= p^3
\end{aligned}$$

$$\begin{aligned}
\text{E}(X) = \mu &= \sum_{\text{all } x} x\Pr(X = x) \\
&= 0(1-p)^3 + 1 \times 3p(1-p)^2 + 2 \times 3p^2(1-p) + 3p^3 \\
&= 0 + 3p(1-p)^2 + 6p^2(1-p) + 3p^3
\end{aligned}$$

$$= 3p[(1-p)^2 + 2p(1-p) + p^2]$$
$$= 3p[(1-p) + p]^2$$
$$= 3p(1-p+p)^2$$
$$= 3p$$

Suppose now $n = 4$:

$$\begin{aligned}
\Pr(X = 0) &= {}^4C_0(1-p)^4p^0 \\
&= (1-p)^4 \\
\Pr(X = 1) &= {}^4C_1(1-p)^3p \\
&= 4p(1-p)^3 \\
\Pr(X = 2) &= {}^4C_2(1-p)^2p^2 \\
&= 6p^2(1-p)^2 \\
\Pr(X = 3) &= {}^4C_3(1-p)p^3 \\
&= 4p^3(1-p) \\
\Pr(X = 4) &= {}^4C_4(1-p)^0p^4 \\
&= p^4
\end{aligned}$$

$$\begin{aligned}
\text{E}(X) = \mu &= \sum_{\text{all } x} x\Pr(X = x) \\
&= 0(1-p)^4 + 1 \times 4p(1-p)^3 + 2 \times 6p^2(1-p)^2 + 3 \times 4p^3(1-p) + 4p^4 \\
&= 0 + 4p(1-p)^3 + 12p^2(1-p)^2 + 12p^3(1-p) + 4p^4 \\
&= 4p[(1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) + p^3] \\
&= 4p[(1-p) + p]^3 \\
&= 4p(1-p+p)^3 \\
&= 4p
\end{aligned}$$

In both cases the expected value of X is the number chosen for n times p. The same result can be achieved for any value of n.

If $X \sim \text{Bi}(n, p)$, then $E(X) = \mu = np$.

Consider again $n = 3$:

$$\begin{aligned}
\text{E}(X^2) &= 0^2(1-p)^3 + 1^2 \times 3p(1-p)^2 + 2^2 \times 3p^2(1-p) + 3^2p^3 \\
&= 0 + 3p(1-p)^2 + 12p^2(1-p) + 9p^3 \\
&= 3p(1-p)^2 + 12p^2(1-p) + 9p^3 \\
\text{Var}(X) &= \text{E}(X^2) - [\text{E}(X)]^2 \\
&= 3p(1-p)^2 + 12p^2(1-p) + 9p^3 - (3p)^2 \\
&= 3p(1-p)^2 + 12p^2(1-p) + 9p^3 - 9p^2 \\
&= 3p(1-p)^2 + 12p^2(1-p) - 9p^2(1-p) \\
&= 3p(1-p)^2 + 3p^2(1-p) \\
&= 3p(1-p)(1-p+p) \\
&= 3p(1-p)
\end{aligned}$$

Consider again $n = 4$:

$$\begin{aligned}\mathrm{E}(X^2) &= 0^2(1-p)^4 + 1^2 \times 4p(1-p)^3 + 2^2 \times 6p^2(1-p)^2 + 3^2 \times 4p^3(1-p) + 4^2p^4 \\ &= 0 + 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) + 16p^4 \\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) + 16p^4\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) + 16p^4 - (4p)^2 \\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) + 16p^4 - 16p^2 \\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) + 16p^2(p^2-1) \\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) - 16p^2(1-p^2) \\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) - 16p^2(1-p)(1+p) \\ &= 4p(1-p)[(1-p)^2 + 6p(1-p) + 9p^2 - 4p(1+p)] \\ &= 4p(1-p)(1 - 2p + p^2 + 6p - 6p^2 + 9p^2 - 4p - 4p^2) \\ &= 4p(1-p)\end{aligned}$$

Again, in both cases, the variance of X is the number chosen for n times $p(1-p)$. This same result can be shown for any value of n.

If $X \sim \mathrm{Bi}(n, p)$, then $\mathrm{Var}(X) = \sigma^2 = np(1-p)$ and $\mathrm{SD}(X) = \sigma = \sqrt{np(1-p)}$.

WORKED EXAMPLE 3

a A test consists of 20 multiple choice questions, each with 5 alternatives for the answer. A student has not studied for the test so she chooses the answers at random. Let X be the discrete random variable that describes the number of correct answers.

i Find the expected number of correct questions answered.

ii Find the standard deviation of the correct number of questions answered, correct to 4 decimal places.

b A binomial random variable, Z, has a mean of 8.4 and a variance of 3.696.

i Find the probability of success, p.

ii Find the number of trials, n.

THINK	WRITE
a i 1 Write the rule for the expected value.	**a i** $\mu = np$
2 Substitute the appropriate values and simplify.	$n = 20,\ p = \frac{1}{5}$ $\mu = np$ $= 20 \times \frac{1}{5}$ $= 4$
3 Write the answer.	The expected number of questions correct is 4.

ii 1 Write the rule for the variance.	**ii** $\sigma^2 = np(1-p)$
2 Substitute the appropriate values and evaluate.	$\text{Var}(X) = 20 \times \frac{1}{5} \times \frac{4}{5}$ $= \frac{16}{5}$ $= 3.2$
3 Write the rule for the standard deviation.	$\sigma = \sqrt{\text{Var}(X)}$
4 Substitute the variance and evaluate.	$\sigma = \sqrt{\frac{16}{5}}$ $= 1.7889$
b i 1 Write the rules for the variance and expected value.	**b i** $\mu = np$ $\text{Var}(Z) = np(1-p)$
2 Substitute the known information and label the two equations.	$8.4 = np \qquad (1)$ $\text{Var}(Z) = np(1-p)$ $3.696 = np(1-p) \qquad (2)$
3 To cancel out the *n*, divide equation (2) by equation (1).	$(2) \div (1): \frac{np(1-p)}{np} = \frac{3.696}{8.4}$
4 Simplify.	$1 - p = 0.44$ $p = 0.56$
5 Write the answer.	The probability of success is 0.56.
ii 1 Substitute $p = 0.56$ into $E(Z) = np$ and solve for *n*.	**ii** $\mu = np$ $8.4 = n \times 0.56$ $n = 15$
2 Write the answer.	There are 15 trials.

WORKED EXAMPLE 4

The probability of an Olympic archer hitting the centre of the target is 0.7. What is the smallest number of arrows he must shoot to ensure that the probability he hits the centre at least once is more than 0.9?

THINK	WRITE
1 Write the rule for the probabilities of the binomial distribution.	$X \sim \text{Bi}(n, 0.7)$ $\Pr(X \geq 1) > 0.9$
2 The upper limit of successes is unknown, because *n* is unknown. Therefore, $\Pr(X \geq 1)$cannot be found by adding up the probabilities. However, the required probability can be found by subtracting from 1 the only probability not included in $\Pr(X \geq 1)$.	$\Pr(X \geq 1) = 1 - \Pr(X = 0)$

3 Substitute in the appropriate values and simplify.

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$
$$1 - \Pr(X = 0) > 0.9$$
$$1 - {}^{n}\text{C}_{x}(1 - p)^{n-x}p^{x} > 0.9$$
$$1 - {}^{n}\text{C}_{0}(0.3)^{n}(0.7)^{0} > 0.9$$
$$1 - 1 \times (0.3)^{n} \times 1 > 0.9$$
$$1 - (0.3)^{n} > 0.9$$

4 Solve for n using CAS.

$$n > 1.91249$$

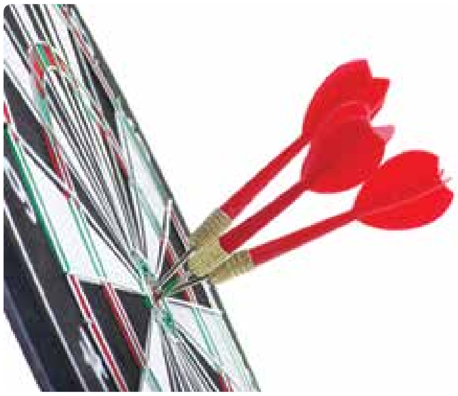
5 Interpret the result and answer the question.

$n = 2$ (as n must be an integer)
The smallest number of arrows the archer needs to shoot in order to guarantee a probability of 0.9 of hitting the centre is 2.

EXERCISE 11.3 The binomial distribution

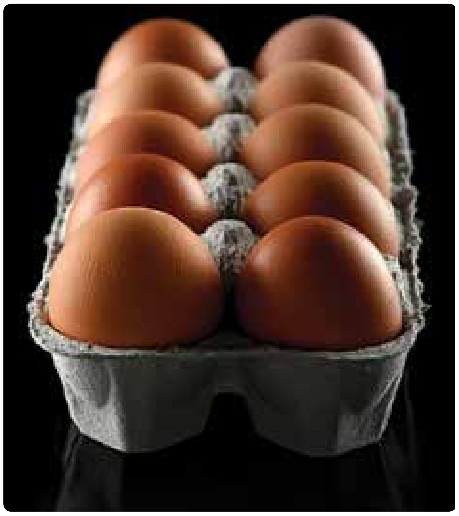
PRACTISE

1 **WE2** Jack is an enthusiastic darts player and on average is capable of achieving a bullseye 3 out of 7 times. Jack will compete in a five-round tournament. Let Y be the discrete random variable that defines the number of bullseyes Jack achieves.



- a Construct a probability distribution table for Y, giving your answers correct to 4 decimal places.
- b Find the probability that Jack will score at most 3 bullseyes.
- c Find the probability that Jack will score more than 1 bullseye, given that he scored at most 3 bullseyes.
- d Find the probability that his first shot missed, his second shot was a bullseye and then his next 2 shots missed.

2 At a poultry farm, eggs are collected daily and classified as large or medium. Then they are packed into cartons containing 12 eggs of the same classification. Experience has enabled the director of the poultry farm to know that 42% of all eggs produced at the farm are considered to be large. Ten eggs are randomly chosen from a conveyor belt on which the eggs are to be classified. Let Z be the discrete random variable that gives the number of large eggs.



- a Find $\Pr(Z = 0), \Pr(Z = 1) \ldots \Pr(Z = 9)$, $\Pr(Z = 10)$ for this binomial distribution.
- b Construct a probability distribution table for Z.
- c Find $\Pr(Z \geq 5 | Z \leq 8)$.

3 **WE3** A fair die is tossed 25 times. Let X be the discrete random variable that represents the number of ones achieved. Find, correct to 4 decimal places:

a the expected number of ones achieved
b the standard deviation of the number of ones achieved.

4 A binomial random variable, Z, has a mean of 32.535 and a variance of 9.021 95.

a Find the probability of success, p.
b Find the number of trials, n.

5 **WE4** The probability of winning a prize in a particular competition is 0.2. How many tickets would someone need to buy in order to guarantee them a probability of at least 0.85 of winning a prize?

6 Lizzie and Matt enjoy playing card games. The probability that Lizzie will beat Matt is 0.67. How many games do they need to play so that the probability of Matt winning a game is more than 0.9?

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 If X has a binomial distribution so that $n = 15$ and $p = 0.62$, calculate:

a $\Pr(X = 10)$
b $\Pr(X \geq 10)$
c $\Pr(X < 4 | X \leq 8)$

8 Jenna is selling raffle tickets for cancer research outside her local supermarket. As people pass her table, there is a probability of 0.45 that they will stop and buy a ticket. During the course of 15 minutes, 15 people walked past her table. Let X be the binomial random variable for the number of people who stopped and bought a ticket. Find:

a the expected value for the number of people who will stop and buy a ticket
b the probability that 4 people will stop and buy a ticket
c the probability that no more than 8 people will buy a ticket
d the probability that the first 2 people will buy a ticket but the next 2 won't.

9 A particular medication used by asthma sufferers has been found to be beneficial if used 3 times a day. In a trial of the medication it was found to be successful in 63% of the cases. Eight random asthma sufferers have had the medication prescribed for them.

a Construct a probability distribution table for the number of sufferers who have benefits from the medication, X.
b Find the probability that no more than 7 people will benefit from the medication.
c Find the probability that at least 3 people will benefit from the medication, given that no more than 7 will.
d Find the probability that the first person won't benefit from the medication, but the next 5 will.

10 For each of the following binomial random variables, calculate:

i the expected value
ii the variance.

a $X \sim \text{Bi}(45, 0.72)$ **b** $Y \sim \text{Bi}\left(100, \frac{1}{5}\right)$ **c** $Z \sim \text{Bi}\left(72, \frac{2}{9}\right)$

11 At midday at the local supermarket, three checkouts are in operation. The probability that a customer can walk up to a register without queuing is 0.32. Larissa visits the supermarket at noon on 7 different occasions.

a Construct a probability distribution table for the number of times Larissa doesn't have to queue to pay for her purchases, Z.

b Find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.

c Find $\Pr(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

12 The executive committee for an independent school consists of 12 members. Find the probability that there are 8 or more women on the executive committee if:

a it is equally likely that a man or a woman is chosen for the executive position

b women have a 58% chance of being chosen for an executive position.

13 A binomial random variable has an expected value of 9.12 and a variance of 5.6544.

a Find the probability of success, p. **b** Find the number of trials, n.

14 A binomial random variable has an expected value of 3.8325 and a variance of 3.4128.

a Find the probability of success, p. **b** Find the number of trials, n.

15 A binomial experiment is completed 16 times and has an expected value of 10.16.

a Find the probability of success, p.

b Find the variance and the standard deviation.

16 A large distributor of white goods has found that 1 in 7 people who buy goods from them do so by using their layby purchasing system. On one busy Saturday morning, 10 customers bought white goods. Let X be the number of people who use the lay-by purchasing system to buy their goods.

a Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

b Find $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

MASTER

17 Lilly knows that the chance of her scoring a goal during a basketball game is 0.75. What is the least number of shots that Lilly must attempt to ensure that the probability of her scoring at least 1 goal in a match is more than 0.95?

18 The tram that stops outside Maia's house is late 20% of the time. If there are 12 times during the day that the tram stops outside Maia's house, find, correct to 4 decimal places:

a the probability that the tram is late 3 times

b the probability that the tram is late 3 times for at least 6 out of the next 14 days.

11.4 Applications

The binomial distribution has important applications in medical research, quality control, simulation and genetics. In this section we will explore some of these areas.

WORKED EXAMPLE 5

It has been found that 9% of the population have diabetes. A sample of 15 people were tested for diabetes. Let X be the random variable that gives the number of people who have diabetes.

a **Find $\Pr(X \leq 5)$.**

b **Find $E(X)$ and $SD(X)$.**

THINK	WRITE
a 1 Summarise the information using binomial notation.	**a** $X \sim \text{Bi}(15, 0.09)$
2 Use CAS technology to add up the required probabilities.	$\Pr(X \leq 5) = 0.9987$
b 1 State the rule for the expected value.	**b** $E(X) = np$
2 Substitute the appropriate values and simplify.	$E(X) = 15 \times 0.09$ $= 1.35$
3 Find the variance.	$\text{Var}(X) = np(1 - p)$ $= 15 \times 0.09 \times 0.91$ $= 1.2285$
4 Find the standard deviation.	$SD(X) = \sqrt{\text{Var}(X)}$ $= \sqrt{1.2285}$ $= 1.1084$

EXERCISE 11.4 Applications

PRACTISE

1 **WE5** It is thought that about 30% of teenagers receive their spending money from part-time jobs. Ten random teenagers were interviewed about their spending money and how they obtained it. Let Y be the random variable that defines the number of teenagers who obtain their spending money by having a part-time job.

- **a** Find $\Pr(Y \geq 7)$.
- **b** Find $E(Y)$ and $SD(Y)$.

2 A mobile phone manufacturer has a relatively simple but important quality test at the end of the manufacturing, which is that the phone should be dropped onto a hard surface. If the phone cracks or breaks in any way it is rejected and destroyed. The probability that a phone is rejected and destroyed is 0.01. Let Z be the number of mobile phones that are dropped and broken when 5 mobile phones are tested.

- **a** Find $\Pr(Z \leq 3)$.
- **b** Find:
  - **i** $E(Z)$
  - **ii** $SD(Z)$.
- **c** Find $\Pr(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 In Australia, it is estimated that 30% of the population over the age of 25 have hypertension. A statistician wishes to investigate this, so he arranges for 15 random adults over the age of 25 to be tested to see if they have high blood pressure. Let X be the random variable which defines the number of adults over the age of 25 with hypertension.

a Find $\Pr(X \leq 5)$.

b Find:

i $E(X)$ **ii** $SD(X)$.

4 It is estimated that about 8% of men and 1% of women have colour blindness. Six men and 6 women are checked for any signs of colour blindness. Let Y be the discrete random variable that defines the number of men who have colour blindness, and let Z be the discrete random variable that defines the number of women who have colour blindness.

a Find:

i $E(Y)$ **ii** $SD(Y)$.

b Find $\Pr(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$.

c Find:

i $E(Z)$ **ii** $SD(Z)$.

d Find $\Pr(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

e Compare the two distributions.

5 Suppose that 85% of adults with allergies report systematic relief with a new medication that has just been released. The medication has just been given to 12 patients who suffer from allergies. Let Z be the discrete random variable that defines the number of patients who get systematic relief from allergies with the new medication.

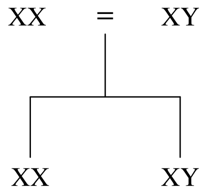
a Find the probability that no more than 8 people get relief from allergies.

b Given that no more than 8 people get relief from allergies after taking the medication, find the probability that at least 5 people do.

c Find:

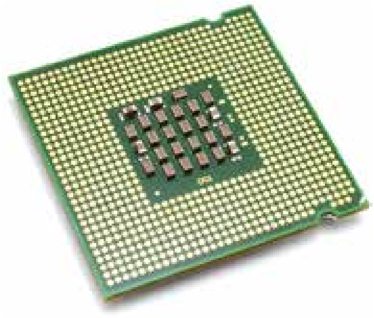
i $E(Z)$ **ii** $SD(Z)$.

6 Consider a woman with the genotype XX and a man with the genotype XY. Their offspring have an equal chance of inheriting one of these genotypes.



What is the probability that 6 of their 7 offspring have the genotype XY?

7 Silicon chips are tested at the completion of the fabrication process. Chips either pass or fail the inspection, and if they fail they are destroyed. The probability that a chip fails an inspection is 0.02. What is the probability that in a manufacturing run of 250 chips, only 7 will fail the inspection?



8 A manufacturer of electric kettles has a process of randomly testing the kettles as they leave the assembly line to see if they are defective. For every 50 kettles produced, 3 are selected and tested for any defects. Let X be the binomial random variable that is the number of kettles that are defective so that $X \sim \text{Bi}(3, p)$.

a Construct a probability distribution table for X, giving your probabilities in terms of p.

b Assuming $\Pr(X = 0) = \Pr(X = 1)$, find the value of p where $0 < p < 1$.

c Find:

i μ **ii** σ.

9 The probability of a person in Australia suffering anaemia is 1.3%. A group of 100 different Australians of differing ages were tested for anaemia.

a Find the probability that more than 5 of the 100 Australians suffer from anaemia. Give your answer correct to 4 decimal places.

b Find the probability that 4 of the 100 Australians suffer from anaemia, given that less than 10 do. Give your answer correct to 4 decimal places.

c Find the value of $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$ and interpret this value.

10 Edie is completing a multiple choice test of 20 questions. Each question has 5 possible answers.

a If Edie randomly guesses every question, what is the probability, correct to 4 decimal places, that she correctly answers 10 or more questions?

b If Edie knows the answers to the first 4 questions but must randomly guess the answers to the other questions, find the probability that she correctly answers 10 or more questions. Give your answer correct to 4 decimal places.

11 Six footballers are chosen at random and asked to kick a football. The probability of a footballer being able to kick at least 50 m is 0.7.

a Determine the probability, correct to 4 decimal places, that:

i only the first three footballers chosen kick the ball at least 50 m

ii exactly three of the footballers chosen kick the ball at least 50 m

iii at least three of the footballers chosen kick the ball at least 50 m, given that the first footballer chosen kicks it at least 50 m.

b What is the minimum number of footballers required to ensure that the probability that at least one of them can kick the ball 50 m is at least 0.95?

12 Lori is a goal shooter for her netball team. The probability of her scoring a goal is 0.85. In one particular game, Lori had 12 shots at goal. Determine the probability, correct to 4 decimal places, that:

a she scored more than 9 goals

b only her last 9 shots were goals

c she scored 10 goals, given that her last 9 shots were goals.

MASTER

13 The chance of winning a prize in the local raffle is 0.08. What is the least number of tickets Siena needs to purchase so that the chance of both her and her sister each winning at least one prize is more than 0.8?

14 A regional community is trying to ensure that their local water supply has fluoride added to it, as a medical officer found that a large number of children aged between eight and twelve have at least one filling in their teeth. In order to push their cause, the community representatives have asked a local dentist to check the teeth of ten 8–12-year-old children from the community. Let X be the binomial random variable that defines the number of 8–12-year-old children who have at least one filling in their teeth; $X \sim \text{Bi}(10, p)$. Find the value of p, correct to 4 decimal places, if $\Pr(X \leq 8) = 0.9$.

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11.5 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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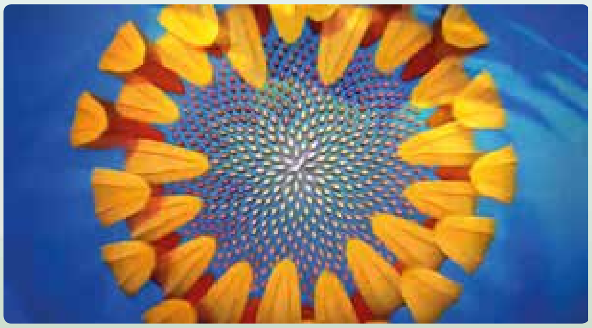
Activities

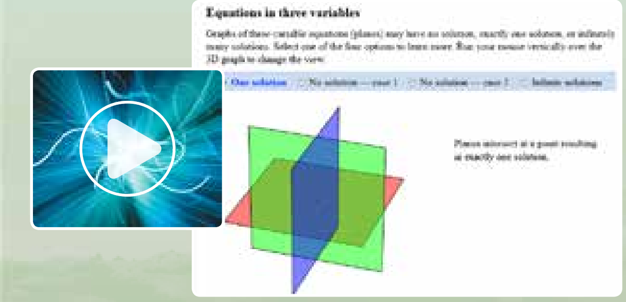
To access eBookPLUS activities, log on to

www.jacplus.com.au

Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





Equations in three variables

Graphs of three-variable equations (planes) may have no solution, exactly one solution, or infinitely many solutions. Select one of the four options to learn more. Run your mouse vertically over the 3D graph to change the view.

Planes intersect at a point resulting in exactly one solution.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

11 Answers

EXERCISE 11.2

1 a No

b Yes

c Yes

2 a

x	0	1
$\Pr(X = x)$	0.58	0.42

b 0.42

c i 0.2436

ii 0.4936

3 a Yes

b Yes

c No

d Yes

4 a No replacement of ball

b There are 6 outcomes, not 2.

c Success unknown

5 a 0.63

b 0.2331

c 0.4828

6 a

y	0	1
$\Pr(Y = y)$	0.32	0.68

b i 0.68

ii 0.2176

iii 0.4665

c 1

7 a

x	0	1
$\Pr(X = x)$	0.11	0.89

b i 0.89

ii 0.0979

iii 0.3129

c 0.89

8 a 0.7

b 0.7

9 a 0.2436

b 0.58

c 0.58

10 a 0.0072

b

x	0	1
$\Pr(X = x)$	0.9928	0.0072

c 0.9928

11 a

y	0	1
$\Pr(Y = y)$	0.67	0.33

b 0.33

c 1

12 a

x	0	1
$\Pr(X = x)$	$\frac{4}{5}$	$\frac{1}{5}$

b $\frac{1}{5}$

c 0.000 32

13 a 0.8775

b

x	0	1
$\Pr(X = x)$	0.1225	0.8775

c 0.8775

14 a 0.11

b 0.8742

EXERCISE 11.3

1 a See the table at the foot of the page.*

b 0.8891

c 0.9315

d 0.0800

2 a $\Pr(Z = 0) = 0.0043$,
$\Pr(Z = 1) = 0.0312$,
$\Pr(Z = 2) = 0.1017$,
$\Pr(Z = 3) = 0.1963$,
$\Pr(Z = 4) = 0.2488$,
$\Pr(Z = 5) = 0.2162$,
$\Pr(Z = 6) = 0.1304$,
$\Pr(Z = 7) = 0.0540$,
$\Pr(Z = 8) = 0.0147$,
$\Pr(Z = 9) = 0.0024$,
$\Pr(Z = 10) = 0.0002$

---

*1 a

x	0	1	2	3	4	5
$\Pr(X = x)$	0.0609	0.2285	0.3427	0.2570	0.0964	0.0145

b See the table at the foot of the page.*

c 0.4164

3 a 4.1667

b 1.8634

4 a 0.7227

b 45

5 9 tickets

6 6 games

7 a 0.1997

b 0.4665

c 0.0034

8 a $E(X) = 6.75$

b 0.0780

c 0.8182

d 0.0613

9 a See the table at the foot of the page.*

b 0.9752

c 0.9655

d 0.0367

10 a i 32.4

ii 9.072

b i 20

ii 16

c i 16

ii $12.\dot{4}$

11 a See the table at the foot of the page.*

b $E(Z) = 2.24$, $\text{Var}(Z) = 1.5232$

c 0.9623

12 a 0.1938

b 0.3825

13 a 0.38

b 24

14 a 0.1095

b 35

15 a 0.6350

b $\text{Var}(X) = 3.7084$, $\text{SD}(X) = 1.9257$

16 a $E(X) = 1.4286$, $\text{Var}(X) = 1.2245$

b 0.9574

17 3 shots

18 a 0.2362

b 0.0890

EXERCISE 11.4

1 a 0.0106

b $E(Y) = 3$, $\text{SD}(Y) = 1.4491$

2 a 1

b i 0.05

ii $\text{Var}(Z) = 0.0495$, $\text{SD}(Z) = 0.2225$

c 0.9510

3 a 0.7216

b i 4.5

ii 1.7748

4 a i 0.48

ii 0.6645

b 0.9228

c i 0.06

ii 0.2437

d 0.9415

e There is a probability of 0.9228 that a maximum of 1 male will have colour blindness, whereas there is a probability of 0.9415 that no females will have colour blindness.

5 a 0.0922

b 0.9992

c i 10.2

ii 1.2369

6 0.0547

---

*2 b

z	0	1	2	3	4	5
$\Pr(Z = z)$	0.0043	0.0312	0.1017	0.1963	0.2488	0.2162

z	6	7	8	9	10
$\Pr(Z = z)$	0.1304	0.0540	0.0147	0.0024	0.0002

*9 a

x	0	1	2	3	4	5	6	7	8
$\Pr(X = x)$	0.0004	0.0048	0.0285	0.0971	0.2067	0.2815	0.2397	0.1166	0.0248

*11 a

z	0	1	2	3	4	5	6	7
$\Pr(Z = z)$	0.0672	0.2215	0.3127	0.2452	0.1154	0.0326	0.0051	0.0003

7 0.1051

8 a

x	0	1	2	3
$\Pr(X = x)$	$(1 - p)^3$	$3(1 - p)^2p$	$3(1 - p)p^2$	p^3

b $p = \frac{1}{4}$

c **i** $\frac{3}{4}$

ii $\frac{3}{4}$

9 a 0.0101

b 0.0319

c 0.9580. This means there is a probability of 0.9580 that a maximum of three people will suffer from anaemia per 100.

10 a 0.0026

b 0.0489

11 a **i** 0.0093

ii 0.1852

iii 0.9692

b 3

12 a 0.9078

b 0.0008

c 0.0574

13 37

14 0.6632

12

Continuous probability distributions

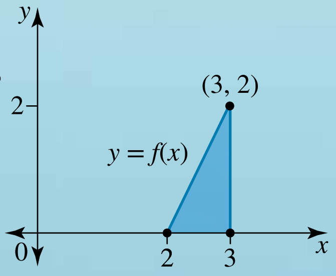
12.1 Kick off with CAS

Exploring probability density functions

Continuous data is data that is measured, such as heights and weights, and may assume any value within a given range. Often the data is recorded in intervals, such as heights between 150 and 160 cm, and the probability of a value falling within a particular interval can be calculated. These probabilities can be sketched via a histogram, and the midpoints of each column on the histogram can be joined to form a graph. Topic 8 covered areas under curves; recall that as the intervals get smaller and smaller, the graph joining the midpoints approaches a smooth curve. Some curves can be modelled by specific functions.

If the area under a function is exactly 1, and if $f(x) \geq 0$ for all x-values, then this function can be called a probability density function. These two conditions must be met.

1 Sketch the graph of $f(x) = 2(x - 2)$ over the domain $2 \leq x \leq 3$.
  - **a** What is the area under the curve for the given domain?
  - **b** Is $f(x) \geq 0$?
  - **c** Is this graph a probability distribution function?



2 Sketch the graph of $f(x) = 5e^{-5x}$ over the domain $x \geq 0$.
  - **a** What is the area under the curve for the given domain?
  - **b** Is $f(x) \geq 0$?
  - **c** Is this graph a probability distribution function?

3 Sketch the graph of $f(x) = \sin(x)$ over the domain $0 \leq x \leq \pi$.
  - **a** What is the area under the curve for the given domain?
  - **b** Is $f(x) \geq 0$?
  - **c** Is this graph a probability distribution function?

4 If time permits, design your own probability density function, ensuring that the two key conditions are met.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

12.2 Continuous random variables and probability functions

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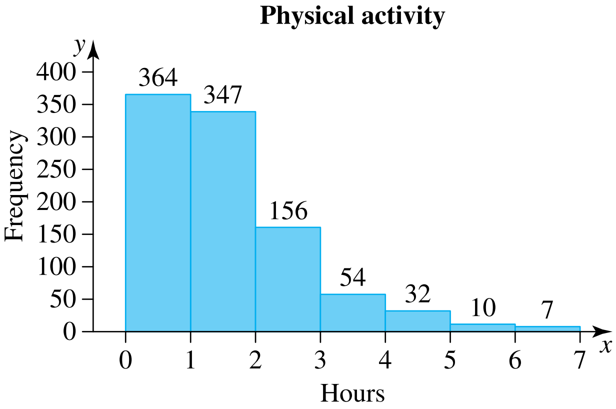
Interactivity
Continuous random variables
int-6433

Continuous random variables

Discrete data is data that is finite or countable, such as the number of soft-centred chocolates in a box of soft- and hard-centred chocolates.

A **continuous random variable** assumes an uncountable or infinite number of possible outcomes between two values. That is, the variable can assume any value within a given range. For example, the birth weights of babies and the number of millimetres of rain that falls in a night are continuous random variables. In these examples, the measurements come from an interval of possible outcomes. If a newborn boy is weighed at 4.46 kilograms, that is just what the weight scale's output said. In reality, he may have weighed 4.463 279 . . . kilograms. Therefore, a possible range of outcomes is valid, within an interval that depends on the precision of the scale.

Consider an Australian health study that was conducted. The study targeted young people aged 5 to 17 years old. They were asked to estimate the average number of hours of physical activity they participated in each week. The results of this study are shown in the following histogram.



Remember, continuous data has no limit to the accuracy with which it is measured. In this case, for example, $0 \le x < 1$ means from 0 seconds to 59 minutes and 59 seconds, and so on, because x is not restricted to integer values. In the physical activity study, x taking on a particular value is equivalent to x taking on a value in an appropriate interval. For instance,

$$\Pr(X = 0.5) = \Pr(0 \le X < 1)$$

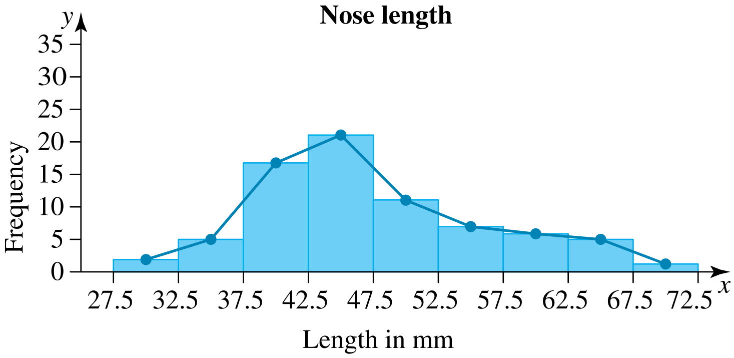
$$\Pr(X = 1.5) = \Pr(1 \le X < 2)$$

and so on. From the histogram,

$$\begin{aligned}\Pr(X = 2.5) &= \Pr(2 \le X < 3)\\ &= \frac{156}{(364 + 347 + 156 + 54 + 32 + 10 + 7)}\\ &= \frac{156}{970}\end{aligned}$$

In another study, the nose lengths, X millimetres, of 75 adults were measured. This data is continuous because the results are measurements. The result of the study is shown in the table and accompanying histogram.

Nose length	Frequency
$27.5 < X \le 32.5$	2
$32.5 < X \le 37.5$	5
$37.5 < X \le 42.5$	17
$42.5 < X \le 47.5$	21
$47.5 < X \le 52.5$	11
$52.5 < X \le 57.5$	7
$57.5 < X \le 62.5$	6
$62.5 < X \le 67.5$	5
$67.5 < X \le 72.5$	1



It is possible to use the histogram to find the number of people who have a nose length of less than 47.5 mm.

$$\begin{aligned}\Pr(\text{nose length is} < 47.5) &= \frac{2 + 5 + 17 + 21}{75}\\ &= \frac{45}{75}\\ &= \frac{3}{5}\end{aligned}$$

It is worth noting that we cannot find the probability that a person has a nose length which is less than 45 mm, as this is not the end point of any interval. However, if we had a mathematical formula to approximate the shape of the graph, then the formula could give us the answer to this important question.

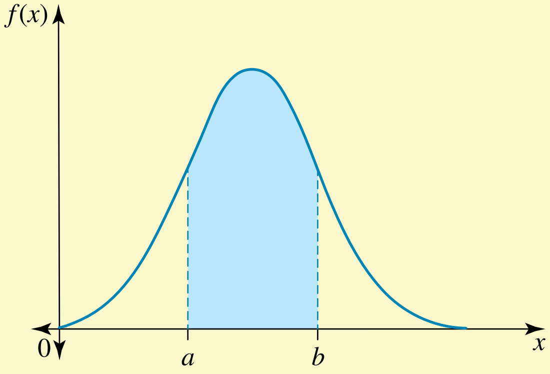
In the histogram, the midpoints at the top of each bar have been connected by line segments. If the class intervals were much smaller, say 1 mm or even less, these line segments would take on the appearance of a smooth curve. This smooth curve is of considerable importance for continuous random variables, because it represents the **probability density function** for the continuous data.

This problem for a continuous random variable can be addressed by using calculus.

For any continuous random variable, X, the probability density function is such that

$$\Pr(a < X < b) = \int_a^b f(x)dx$$

which is the area under the curve from $x = a$ to $x = b$.



A probability density function must satisfy the following conditions:

- **$f(x) \geq 0$ for all $x \in [a, b]$**
- **$\int_a^b f(x)dx = 1$; this is absolutely critical.**

Other properties are:

- **$\Pr(X = x) = 0$, where $x \in [a, b]$**
- **$\Pr(a < X < b) = P(a \leq X < b) = \Pr(a < X \leq b) = \Pr(a \leq X \leq b) = \int_a^b f(x)dx$**
- **$\Pr(X < c) = \Pr(X \leq c) = \int_a^c f(x)dx$ when $x \in a, b$ and $a < c < b$.**

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Probability density functions
Concept summary
Practice questions

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Probability density functions

In theory, the domain of a continuous probability density function is R, so that

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

However, if we must address the condition that

$$\int_a^b f(x)dx = 1,$$

then the function must be zero everywhere else.

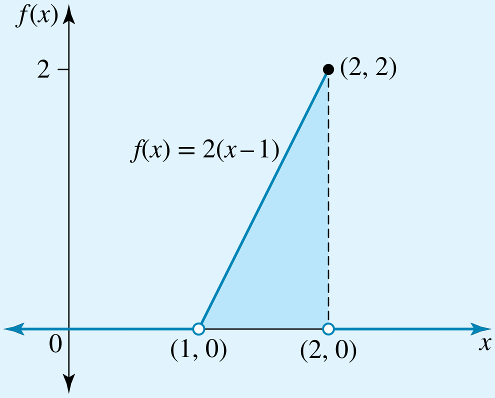
WORKED EXAMPLE 1

Sketch the graph of each of the following functions and state whether each function is a probability density function.

a $f(x) = \begin{cases} 2(x-1), & 1 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

b $f(x) = \begin{cases} 0.5, & 2 \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$

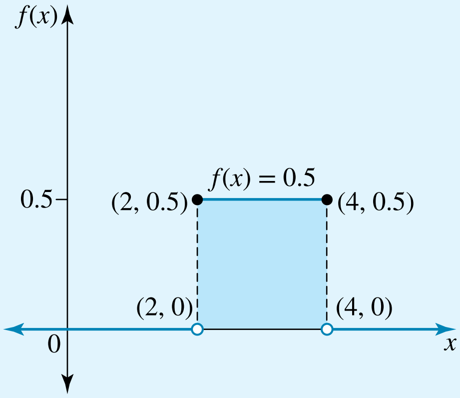
c $f(x) = \begin{cases} 2e^{-x}, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

THINK	WRITE/DRAW
a 1 Sketch the graph of $f(x) = 2(x - 1)$ over the domain $1 \le x \le 2$, giving an x-intercept of 1 and an end point of (2, 2). Make sure to include the horizontal lines for $y = 0$ either side of this graph. *Note:* This function is known as a triangular probability function because of its shape.	**a** 
2 Inspect the graph to determine if the function is always positive or zero, that is, $f(x) \ge 0$ for all $x \in [a, b]$.	Yes, $f(x) \ge 0$ for all x-values.
3 Calculate the area of the shaded region to determine if $\int_1^2 2(x-1)dx = 1$.	Method 1: Using the area of triangles $\begin{aligned} \text{Area of shaded region} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times 1 \times 2 \\ &= 1 \end{aligned}$ Method 2: Using calculus $\begin{aligned} \text{Area of shaded region} &= \int_1^2 2(x-1)dx \\ &= \int_1^2 (2x-2)dx \\ &= \left[x^2 - 2x\right]_1^2 \\ &= (2^2 - 2(2)) - (1^2 - 2(1)) \\ &= 0 - 1 + 2 \\ &= 1 \end{aligned}$
4 Interpret the results.	$f(x) \ge 0$ for all values, and the area under the curve = 1. Therefore, this is a probability density function.

b **1** Sketch the graph of $f(x) = 0.5$ for $2 \leq x \leq 4$. This gives a horizontal line, with end points of (2, 0.5) and (4, 0.5). Make sure to include the horizontal lines for $y = 0$ on either side of this graph.

Note: This function is known as a uniform or rectangular probability density function because of its rectangular shape.

b



2 Inspect the graph to determine if the function is always positive or zero, that is, $f(x) \geq 0$ for all $x \in [a, b]$.

Yes, $f(x) \geq 0$ for all x-values.

3 Calculate the area of the shaded region to determine if $\int_2^4 0.5dx = 1$.

Again, it is not necessary to use calculus to find the area.

Method 1:

$$\begin{aligned}\text{Area of shaded region} &= \text{length} \times \text{width}\\ &= 2 \times 0.5\\ &= 1\end{aligned}$$

Method 2:

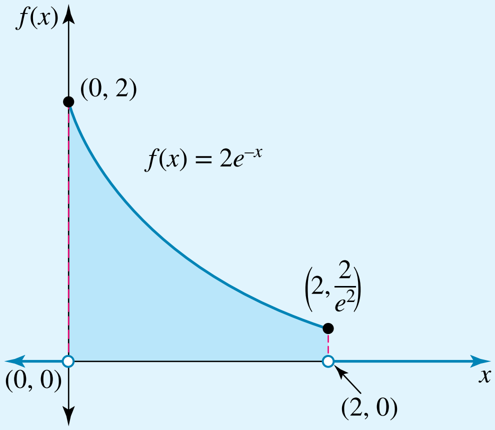
$$\begin{aligned}\text{Area of shaded region} &= \int_2^4 0.5dx\\ &= [0.5x]_2^4\\ &= 0.5(4) - 0.5(2)\\ &= 2 - 1\\ &= 1\end{aligned}$$

4 Interpret the results.

$f(x) \geq 0$ for all values, and the area under the curve = 1. Therefore, this is a probability density function.

c **1** Sketch the graph of $f(x) = 2e^{-x}$ for $0 \leq x \leq 2$. End points will be (0, 2) and $(2, e^{-2})$. Make sure to include the horizontal lines for $y = 0$ on either side of this graph.

c



2 Inspect the graph to determine if the function is always positive or zero, that is, $f(x) \geq 0$ for all $x \in [a, b]$.

Yes, $f(x) \geq 0$ for all x-values.

3 Calculate the area of the shaded region to determine if $\int_0^2 2e^{-x}dx = 1$.

$$\begin{aligned}\int_0^2 2e^{-x}dx &= 2\int_0^2 e^{-x}dx \\ &= 2\left[-e^{-x}\right]_0^2 \\ &= 2(-e^{-2} + e^0) \\ &= 2(-e^{-2} + 1) \\ &= 1.7293\end{aligned}$$

4 Interpret the results.

$f(x) \geq 0$ for all values. However, the area under the curve $\neq 1$. Therefore this is not a probability density function.

WORKED EXAMPLE 2

Given that the functions below are probability density functions, find the value of a in each function.

a $f(x) = \begin{cases} a(x-1)^2, & 0 \leq x \leq 4 \\ 0, & \text{elsewhere} \end{cases}$

b $f(x) = \begin{cases} ae^{-4x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$

THINK

WRITE

a 1 As the function has already been defined as a probability density function, this means that the area under the graph is definitely 1.

a

$$\int_0^4 f(x)dx = 1$$

$$\int_0^4 a(x-1)^2 dx = 1$$

2 Remove a from the integral, as it is a constant.

$$a\int_0^4 (x-1)^2 dx = 1$$

3 Antidifferentiate and substitute in the terminals.

$$a\int_0^4 (x-1)^2 dx = 1$$

$$a\left[\frac{(x-1)^3}{3}\right]_0^4 = 1$$

$$a\left[\frac{3^3}{3} - \frac{(-1)^3}{3}\right] = 1$$

4 Solve for a.

$$a\left(9 + \frac{1}{3}\right) = 1$$

$$a \times \frac{28}{3} = 1$$

$$a = \frac{3}{28}$$

b **1** As the function has already been defined as a probability density function, this means that the area under the graph is definitely 1.

b
$$\int_0^\infty f(x)dx = 1$$
$$\int_0^\infty ae^{-4x}dx = 1$$

2 Remove a from the integral, as it is a constant.

$$a\int_0^\infty e^{-4x}dx = 1$$

3 To evaluate an integral containing infinity as one of the terminals, we find the appropriate limit.

$$a \times \lim_{k\to\infty}\int_0^k e^{-4x}dx = 1$$

4 Antidifferentiate and substitute in the terminals.

$$a \times \lim_{k\to\infty}\int_0^k e^{-4x}dx = 1$$
$$a \times \lim_{k\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_0^k = 1$$
$$a \times \lim_{k\to\infty}\left(-\frac{e^{-4k}}{4} + \frac{1}{4}\right) = 1$$
$$a \times \lim_{k\to\infty}\left(-\frac{e^{-4k}}{4} + \frac{1}{4}\right) = 1$$
$$a \times \lim_{k\to\infty}\left(-\frac{1}{4e^{4k}} + \frac{1}{4}\right) = 1$$

5 Solve for a. Remember that a number divided by an extremely large number is effectively zero, so $\lim\limits_{k\to\infty}\left(\frac{1}{e^{4k}}\right) = 0$.

$$a\left(0 + \frac{1}{4}\right) = 1$$
$$\frac{a}{4} = 1$$
$$a = 4$$

EXERCISE 12.2 Continuous random variables and probability functions

PRACTISE

Work without CAS

1 **WE1** Sketch each of the following functions and determine whether each one is a probability density function.

a $f(x) = \begin{cases} \frac{1}{4}e^{2x}, & 0 \le x \le \log_e 3 \\ 0, & \text{elsewhere} \end{cases}$

b $f(x) = \begin{cases} 0.25, & -2 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

2 Sketch each of the following functions and determine whether each one is a probability density function.

a $f(x) = \begin{cases} \frac{1}{2}\cos(x), & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases}$

b $f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & \frac{1}{2} \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$

3 **WE2** Given that the function is a probability density function, find the value of n.

$$f(x) = \begin{cases} n(x^3 - 1), & 1 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

4 Given that the function is a probability density function, find the value of a.

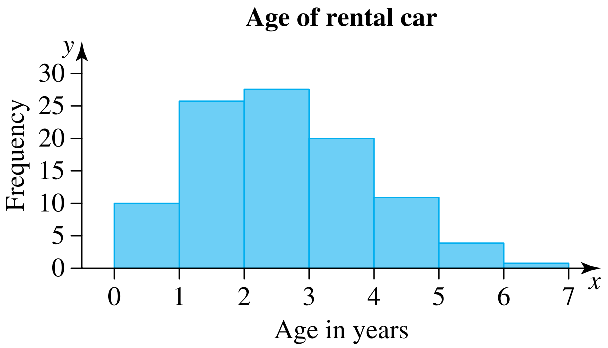
$$f(x) = \begin{cases} -ax, & -2 \le x < 0 \\ 2ax, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 A small car-hire firm keeps note of the age and kilometres covered by each of the cars in their fleet. Generally, cars are no longer used once they have either covered 350 000 kilometres or are more than five years old. The following information describes the ages of the cars in their current fleet.

Age	Frequency
$0 < x \le 1$	10
$1 < x \le 2$	26
$2 < x \le 3$	28
$3 < x \le 4$	20
$4 < x \le 5$	11
$5 < x \le 6$	4
$6 < x \le 7$	1



a Determine:

i $\Pr(X \le 2)$

ii $\Pr(X > 4)$.

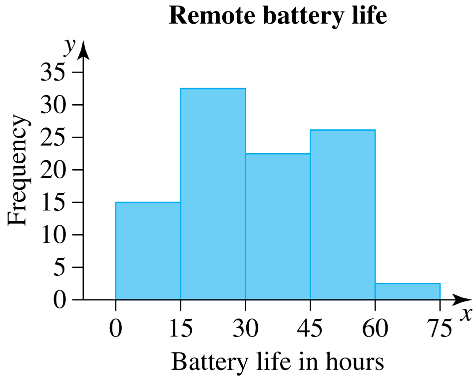
b Determine:

i $\Pr(1 < X \le 4)$

ii $\Pr(X > 1 \mid X \le 4)$.

6 The battery life for batteries in television remote controls was investigated in a study.

Hours of life	Frequency
$0 < x \le 15$	15
$15 < x \le 30$	33
$30 < x \le 45$	23
$45 < x \le 60$	26
$60 < x \le 75$	3



a How many remote control batteries were included in the study?

b What is the probability that a battery will last more than 45 hours?

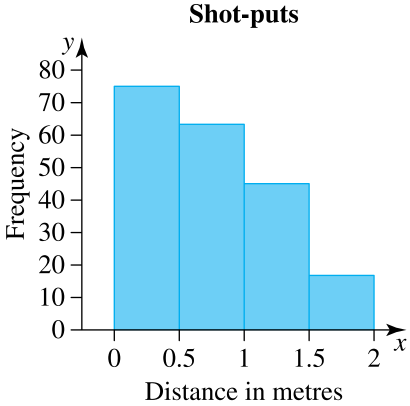
c What is the probability that a battery will last between 15 and 60 hours?

d A new battery producer is advocating that their batteries have a long life of 60+ hours. If it is known that this is just advertising hype because these batteries are no different from the batteries in the study, what is the probability that these new batteries will have a life of 60+ hours?

7 A number of experienced shot-putters were asked to aim for a line 10 metres away.

After each of them put their shot, its distance from the 10-metre line was measured. All of the shots were on or between the 8- and 10-metre lines. The results of the measurements are shown, where X is the distance in metres from the 10-metre line.

Metres	Frequency
$0 < x \le 0.5$	75
$0.5 < x \le 1$	63
$1 < x \le 1.5$	45
$1.5 < x \le 2$	17



a How many shot-put throws were measured?

b Calculate:

i $\Pr(X > 0.5)$ **ii** $\Pr(1 < X \le 2)$

c A guest shot-putter is visiting the athletics club where the measurements are being conducted. His shot-putting ability is equivalent to the abilities of the club members. Find the probability that he puts the shot within 50 cm of the 10-metre line if it is known that he put the shot within 1 metre of the 10-metre line.

8 Sketch each of the following functions and determine whether each function is a probability density function. *Note:* Use CAS where appropriate.

a $f(x) = \begin{cases} -\frac{1}{x}, & -e \le x \le -1 \\ 0, & \text{elsewhere} \end{cases}$

b $f(x) = \begin{cases} \cos(x) + 1, & \frac{\pi}{4} \le x \le \frac{3\pi}{4} \\ 0, & \text{elsewhere} \end{cases}$

c $f(x) = \begin{cases} \frac{1}{2}\sin(x), & 0 \le x \le \pi \\ 0, & \text{elsewhere} \end{cases}$

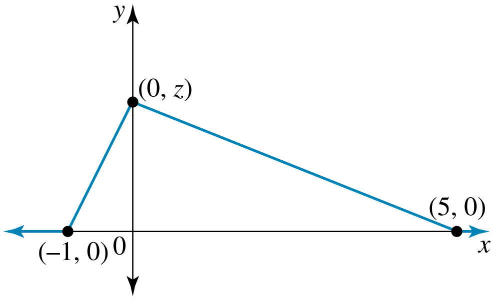
d $f(x) = \begin{cases} \frac{1}{2\sqrt{x-1}}, & 1 < x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

9 The rectangular function, f, is defined by the rule

$$f(x) = \begin{cases} c, & 0.25 < x < 1.65 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the value of the constant c, given that f is a probability density function.

10 The graph of a function, f, is shown.



If f is known to be a probability density function, show that the value of z is $\frac{1}{3}$.

11 Find the value of the constant m in each of the following if each function is a probability density function.

a $f(x) = \begin{cases} m(6 - 2x), & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

b $f(x) = \begin{cases} me^{-2x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

c $f(x) = \begin{cases} me^{2x}, & 0 \leq x \leq \log_e 3 \\ 0, & \text{elsewhere} \end{cases}$

12 Let X be a continuous random variable with the probability density function

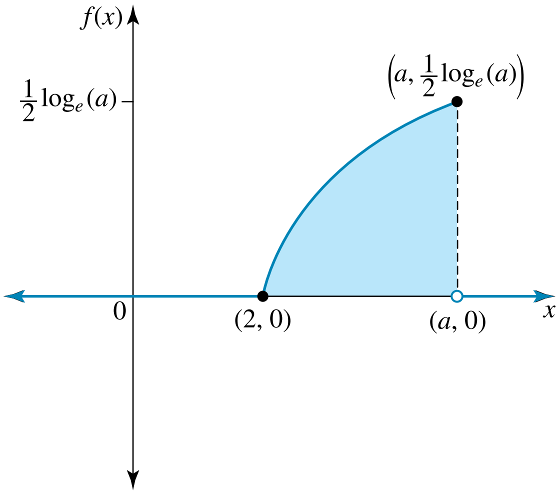
$$f(x) = \begin{cases} x^2 + 2kx + 1, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Show that the value of k is $-\frac{11}{9}$.

13 X is a continuous random variable such that

$$f(x) = \begin{cases} \frac{1}{2}\log_e\left(\frac{x}{2}\right), & 2 \leq x \leq a \\ 0, & \text{elsewhere} \end{cases}$$

and $\int_2^a f(x)dx = 1$. The graph of this function is shown.



Find the value of the constant a.

14 X is a continuous random variable such that

$$f(x) = \begin{cases} -x, & -1 \leq x < 0 \\ x, & 0 \leq x \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

Y is another continuous random variable such that

$$f(y) = \begin{cases} \frac{1}{y}, & 1 \leq y \leq e \\ 0, & \text{elsewhere} \end{cases}.$$

a Sketch the graph of the function for X and find $\int_{-1}^{a} f(x)dx$.

b Sketch the graph of the function for Y and find $\int_{1}^{e} f(y)dy$.

c Find the value of the constant a if $\int_{-1}^{a} f(x)dx = \int_{1}^{e} f(y)dy$.

MASTER

15 X is a continuous random variable such that

$$f(x) = \begin{cases} n\sin(3x)\cos(3x), & 0 < x < \dfrac{\pi}{12} \\ 0, & \text{elsewhere} \end{cases}.$$

If f is known to be a probability density function, find the value of the constant, n.

16 A function f is defined by the rule

$$f(x) = \begin{cases} \log_e(x), & x > 0 \\ 0, & \text{elsewhere} \end{cases}.$$

a If $\int_{1}^{a} f(x)dx = 1$, find the value of the real constant a.

b Does this function define a probability density function?

12.3 The continuous probability density function

study on

Units 3 & 4

AOS 4

Topic 3

Concept 2

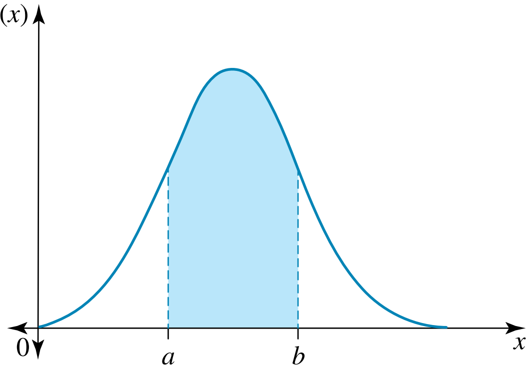
Calculating probabilities

Concept summary

Practice questions

As stated in section 12.2, if X is a continuous random variable, then

$$\Pr(a \le X \le b) = \int_{a}^{b} f(x)dx.$$



In other words, by finding the area between the curve of the continuous probability function, the x-axis, the line $x = a$ and the line $x = b$, providing $f(x) \ge 0$, then we are finding $\Pr(a \le X \le b)$. It is worth noting that because we are dealing with a continuous random variable, $\Pr(X = a) = 0$, and consequently:

$$\Pr(a \le X \le b) = \Pr(a < X \le b) = \Pr(a \le X < b) = \Pr(a < X < b)$$

$$\Pr(a \le X \le b) = \Pr(a \le X \le c) + \Pr(c < X \le b), \text{ where } a < c < b$$

This property is particularly helpful when the probability density function is a hybrid function and the required probability encompasses two functions.

WORKED EXAMPLE 3

A continuous random variable, Y, has a probability density function, f, defined by

$$f(y) = \begin{cases} -ay, & -3 \le y \le 0 \\ ay, & 0 < y \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a Sketch the graph of f.

b Find the value of the constant, a.

c Determine $\Pr(1 \le Y \le 3)$.

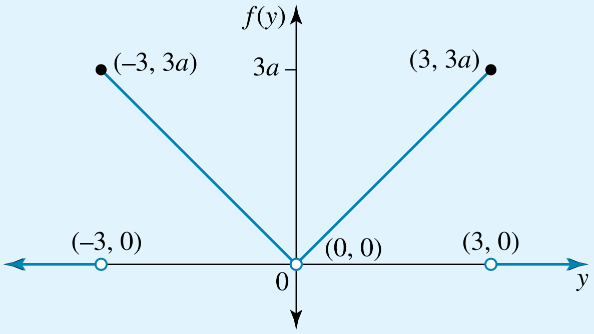
d Determine $\Pr(Y < 2 \mid Y > -1)$

THINK

a The hybrid function contains three sections. The first graph, $f(y) = -ay$, is a straight line with end points of (0, 0) and (–3, 3a). The second graph is also a straight line and has end points of (0, 0) and (3, 3a). Don't forget to include the $f(y) = 0$ lines for $x > 3$ and $x < -3$.

WRITE/DRAW

a $f(-3) = 3a$ and $f(3) = 3a$



THINK

b Use the fact that $\int_{-3}^{3} f(y)dy = 1$ to solve for a.

WRITE/DRAW

b $\int_{-3}^{3} f(y)dy = 1$

Using the area of a triangle, we find:

$$\frac{1}{2} \times 3 \times 3a + \frac{1}{2} \times 3 \times 3a = 1$$

$$\frac{9a}{2} + \frac{9a}{2} = 1$$

$$9a = 1$$

$$a = \frac{1}{9}$$

THINK

c Identify the part of the function that the required y-values sit within: the values $1 \le Y \le 3$ are within the region where $f(y) = \frac{1}{9}y$.

WRITE/DRAW

c

$$\begin{aligned}\Pr(1 \le Y \le 3) &= \int_{1}^{3} f(y)dy \\ &= \int_{1}^{3} \left(\tfrac{1}{9}y\right)dy \\ &= \left[\tfrac{1}{18}y^2\right]_1^3 \\ &= \tfrac{1}{18}(3)^2 - \tfrac{1}{18}(1)^2 \\ &= \tfrac{8}{18} \\ &= \tfrac{4}{9}\end{aligned}$$

Note: The method of finding the area of a trapezium could also be used.

d **1** State the rule for the conditional probability.

d $$\Pr(Y < 2 \mid Y > -1) = \frac{\Pr(Y < 2 \cap Y > -1)}{\Pr(Y > -1)}$$
$$= \frac{\Pr(-1 < Y < 2)}{\Pr(Y > -1)}$$

2 Find $\Pr(-1 < Y < 2)$. As the interval is across two functions, the interval needs to be split.

$$\Pr(-1 < Y < 2) = \Pr(-1 < Y < 0) + \Pr(0 \le Y < 2)$$

3 To find the probabilities we need to find the areas under the curve.

$$= \int_{-1}^{0} -\frac{1}{9}y\,dy + \int_{0}^{2} \frac{1}{9}y\,dy$$
$$= -\int_{-1}^{0} \frac{1}{9}y\,dy + \int_{0}^{2} \frac{1}{9}y\,dy$$

4 Antidifferentiate and evaluate after substituting the terminals.

$$= -\left[\frac{1}{18}y^2\right]_{-1}^{0} + \left[\frac{1}{18}y^2\right]_{0}^{2}$$
$$= -\left(\frac{1}{18}(0)^2 - \frac{1}{18}(-1)^2\right) + \frac{1}{18}(2)^2 - \frac{1}{18}(0)^2$$
$$= \frac{1}{18} + \frac{4}{18}$$
$$= \frac{5}{18}$$

5 Find $\Pr(Y > -1)$. As the interval is across two functions, the interval needs to be split.

$$\Pr(Y > -1) = \Pr(-1 < Y < 0) + \Pr(0 \le Y \le 3)$$

6 To find the probabilities we need to find the areas under the curve. As $\Pr(0 \le Y \le 3)$ covers exactly half the area under the curve, $\Pr(0 \le Y \le 3) = \frac{1}{2}$. (The entire area under the curve is always 1 for a probability density function.)

$$= \int_{-1}^{0} -\frac{1}{9}y\,dy + \int_{0}^{3} \frac{1}{9}y\,dy$$
$$= -\int_{-1}^{0} \frac{1}{9}y\,dy + \frac{1}{2}$$

7 Antidifferentiate and evaluate after substituting the terminals.

$$= -\left[\frac{1}{18}y^2\right]_{-1}^{0} + \frac{1}{2}$$
$$= -\left(\frac{1}{18}(0)^2 - \frac{1}{18}(-1)^2\right) + \frac{1}{2}$$
$$= \frac{1}{18} + \frac{9}{18}$$
$$= \frac{10}{18}$$
$$= \frac{5}{9}$$

8 Now substitute into the formula to find $\Pr(Y < 2 \mid Y > -1) = \dfrac{\Pr(-1 < Y < 2)}{\Pr(Y > -1)}$.

$$\Pr(Y < 2 \mid Y > -1) = \frac{\Pr(-1 < Y < 2)}{\Pr(Y > -1)}$$
$$= \frac{5}{18} \div \frac{5}{9}$$
$$= \frac{5}{18} \times \frac{9}{5}$$
$$= \frac{1}{2}$$

EXERCISE 12.3 The continuous probability density function

PRACTISE

Work without CAS

1 **WE3** The continuous random variable Z has a probability density function given by

$$f(z) = \begin{cases} -z + 1, & 0 \leq z < 1 \\ z - 1, & 1 \leq z \leq 2 \\ 0, & \text{elsewhere} \end{cases}.$$

a Sketch the graph of f.

b Find $\Pr(Z < 0.75)$.

c Find $\Pr(Z > 0.5)$.

2 The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 4x^3, & 0 \leq x \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a Find the value of the constant a.

b Sketch the graph of f.

c Find $\Pr(0.5 \leq X \leq 1)$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 Let X be a continuous random variable with a probability density function defined by

$$f(x) = \begin{cases} \frac{1}{2}\sin(x), & 0 \leq x \leq \pi \\ 0, & \text{elsewhere} \end{cases}.$$

a Sketch the graph of f.

b Find $\Pr\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right)$.

c Find $\Pr\left(X > \frac{\pi}{4} \,\middle|\, X < \frac{3\pi}{4}\right)$.

4 A probability density function is defined by the rule

$$f(x) = \begin{cases} k(2 + x), & -2 \leq x < 0 \\ k(2 - x), & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

where X is a continuous random variable and k is a constant.

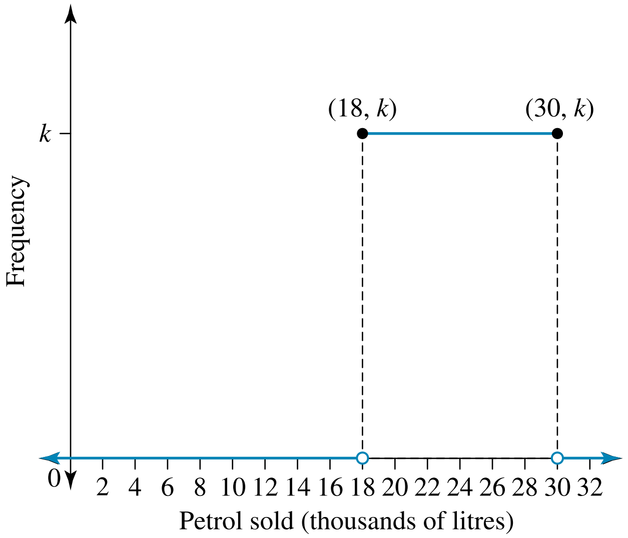
a Sketch the graph of f.

b Show that the value of k is $\frac{1}{4}$.

c Find $\Pr(-1 \leq X \leq 1)$.

d Find $\Pr(X \geq -1 \mid X \leq 1)$.

5 The amount of petrol sold daily by a busy service station is a uniformly distributed probability density function. A minimum of 18 000 litres and a maximum of 30 000 litres are sold on any given day. The graph of the function is shown.



a Find the value of the constant k.

b Find the probability that between 20 000 and 25 000 litres of petrol are sold on a given day.

c Find the probability that as much as 26 000 litres of petrol were sold on a particular day, given that it was known that at least 22 000 litres were sold.

6 The continuous random variable X has a uniform rectangular probability density function defined by

$$f(x) = \begin{cases} \dfrac{1}{5}, & 1 \le x \le 6 \\ 0, & \text{elsewhere} \end{cases}.$$

a Sketch the graph of f.

b Determine $\Pr(2 \le X \le 5)$.

7 The continuous random variable Z has a probability density function defined by

$$f(z) = \begin{cases} \dfrac{1}{2z}, & 1 \le z \le e^2 \\ 0, & \text{elsewhere} \end{cases}.$$

a Sketch the graph of f and shade the area that represents $\displaystyle\int_1^{e^2} f(z)dz$.

b Find $\displaystyle\int_1^{e^2} f(z)dz$. Explain your result.

The continuous random variable U has a probability function defined by

$$f(u) = \begin{cases} e^{4u}, & u \ge 0 \\ 0, & \text{elsewhere} \end{cases}.$$

c Sketch the graph of f and shade the area that represents $\displaystyle\int_0^{a} f(u)du$, where a is a constant.

d Find the exact value of the constant a if $\displaystyle\int_1^{e^2} f(z)dz$ is equal to $\displaystyle\int_0^{a} f(u)du$.

8 The continuous random variable Z has a probability density function defined by

$$f(z) = \begin{cases} \dfrac{1}{2}\cos(z), & -\dfrac{\pi}{2} \le z \le \dfrac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases}.$$

a Sketch the graph of f and verify that $y = f(z)$ is a probability density function.

b Find $\Pr\left(-\frac{\pi}{6} \le Z \le \frac{\pi}{4}\right)$.

9 The continuous random variable U has a probability density function defined by

$$f(u) = \begin{cases} 1 - \frac{1}{4}(2u - 3u^2), & 0 \le u \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant. Find:

a the value of the constant a

b $\Pr(U < 0.75)$

c $\Pr(0.1 < U < 0.5)$

d $\Pr(U = 0.8)$.

10 The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} \frac{3}{8}x^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

Find:

a $\Pr(X > 1.2)$

b $\Pr(X > 1 \mid X > 0.5)$, correct to 4 decimal places

c the value of n such that $\Pr(X \le n) = 0.75$.

11 The continuous random variable Z has a probability density function defined by

$$f(z) = \begin{bmatrix} e^{-\frac{z}{3}}, & 0 \le z \le a \\ 0, & \text{elsewhere} \end{bmatrix}$$

where a is a constant. Find:

a the value of the constant a such that $\int_0^a f(z)dz = 1$

b $\Pr(0 < Z < 0.7)$

c $\Pr(Z < 0.7 \mid Z > 0.2)$, correct to 4 decimal places

d the value of α, correct to 2 decimal places, such that $\Pr(Z \le \alpha) = 0.54$.

12 The continuous random variable X has a probability density function given as

$$f(x) = \begin{cases} 3e^{-3x}, & x \ge 0 \\ 0, & \text{elsewhere} \end{cases}.$$

a Sketch the graph of f.

b Find $\Pr(0 \le X \le 1)$, correct to 4 decimal places.

c Find $\Pr(X > 2)$, correct to 4 decimal places.

MASTER

13 The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} \log_e(x^2), & x \ge 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Find, correct to 4 decimal places:

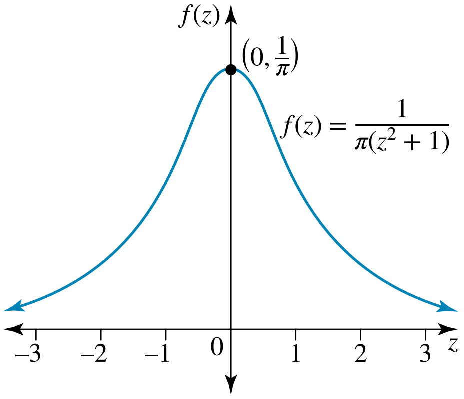
a the value of the constant a if $\int_1^a f(x)dx = 1$

b $\Pr(1.25 \le X \le 2)$.

14 The graph of the probability function

$$f(z) = \frac{1}{\pi(z^2 + 1)}$$

is shown.



a Find, correct to 4 decimal places, $\Pr(-0.25 < Z < 0.25)$.

b Suppose another probability density function is defined as

$$f(x) = \begin{cases} \frac{1}{x^2 + 1}, & -a \leq x \leq a \\ 0, & \text{elsewhere} \end{cases}.$$

Find the value of the constant a.

12.4 Measures of centre and spread

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Units 3 & 4
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Topic 3
Concept 3

Mean and median
Concept summary
Practice questions

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Mean
int-6435

The commonly used measures of central tendency and spread in statistics are the mean, median, variance, standard deviation and range. These same measurements are appropriate for continuous probability functions.

Measures of central tendency

The mean

Remember that for a discrete random variable,

$$\text{E}(X) = \mu = \sum_{x=1}^{x=n} x_n \Pr(X = x_n).$$

This definition can also be applied to a continuous random variable.

We define $\text{E}(X) = \mu = \int_{-\infty}^{\infty} x f(x)dx.$

If $f(x) = 0$ everywhere except for $x \in [a, b]$, where the function is defined, then

$$\mathbf{E}(X) = \mu = \int_a^b xf(x)dx.$$

Consider the continuous random variable, X, which has a probability density function defined by

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

For this function,

$$\begin{aligned} \mathrm{E}(X) = \mu &= \int_0^1 x f(x)\,dx \\ &= \int_0^1 x(x^2)\,dx \\ &= \int_0^1 x^3\,dx \\ &= \left[\frac{x^4}{4}\right]_0^1 \\ &= \frac{1^4}{4} - 0 \\ &= \frac{1}{4} \end{aligned}$$

Similarly, if the continuous random variable X has a probability density function of

$$f(x) = \begin{cases} 7e^{-7x}, & x \ge 0 \\ 0, & \text{elsewhere,} \end{cases}$$

then

$$\begin{aligned} \mathrm{E}(X) = \mu &= \int_0^\infty x f(x)\,dx \\ &= \lim_{k \to \infty} \int_0^k 7xe^{-7x}\,dx \\ &= 0.1429 \end{aligned}$$

where CAS technology is required to determine the integral.

The mean of a function of X is similarly found.

The function of X, $g(x)$, has a mean defined by:

$$\mathbf{E}(g(x)) = \mu = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

So if we again consider

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

then

$$\begin{aligned}\mathrm{E}(X^2) &= \int_0^1 x^2 f(x)dx \\ &= \int_0^1 x^4 dx \\ &= \left[\frac{x^5}{5}\right]_0^1 \\ &= \frac{1^5}{5} - 0 \\ &= \frac{1}{5}\end{aligned}$$

This definition is important when we investigate the variance of a continuous random variable.

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Interactivity
Median and percentiles
int-6436

Median and percentiles

The median is also known as the 50th **percentile**, Q_2, the halfway mark or the middle value of the distribution.

For a continuous random variable, X, defined by the probability function f, the median can be found by solving $\int_{-\infty}^{m} f(x)dx = 0.5$.

Other percentiles that are frequently calculated are the 25th percentile or lower quartile, Q_1, and the 75th percentile or upper quartile, Q_3.

The interquartile range is calculated as:
$IQR = Q_3 - Q_1$

Consider a continuous random variable, X, that has a probability density function of

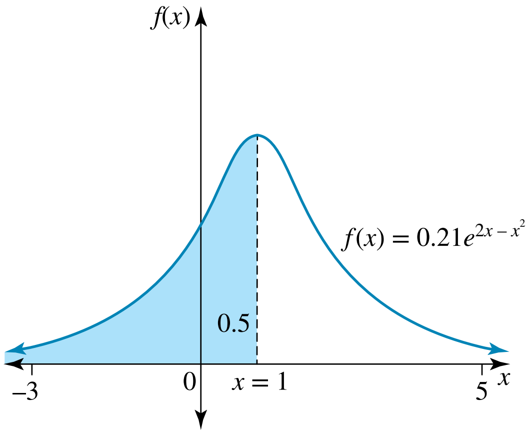
$$f(x) = \begin{cases} 0.21e^{2x-x^2}, & -3 \le x \le 5 \\ 0, & \text{elsewhere} \end{cases}.$$

To find the median, m, we solve for m as follows:

$$\int_{-3}^{m} 0.21e^{2x-x^2}dx = 0.5$$

The area under the curve is equated to 0.5, giving half of the total area and hence the 50th percentile. Solving via CAS, the result is that $m = 0.9897 \simeq 1$.

This can be seen on a graph as shown.



Consider the continuous random variable X, which has a probability density function of

$$f(x) = \begin{cases} \dfrac{x^3}{4}, & 0 \leq x \leq 2 \\ 0, & \text{elsewhere.} \end{cases}$$

The median is given by $\Pr(0 \leq x \leq m) = 0.5$:

$$\begin{aligned} \int_0^m \frac{x^3}{4} dx &= 0.5 \\ \left[\frac{x^4}{16}\right]_0^m &= \frac{1}{2} \\ \frac{m^4}{16} - 0 &= \frac{1}{2} \\ m^4 &= 8 \\ m &= \pm\sqrt[4]{8} \\ m &= 1.6818 \ (0 \leq m \leq 2) \end{aligned}$$

To find the lower quartile, we make the area under the curve equal to 0.25. Thus the lower quartile is given by $\Pr(0 \leq x \leq a) = 0.25$:

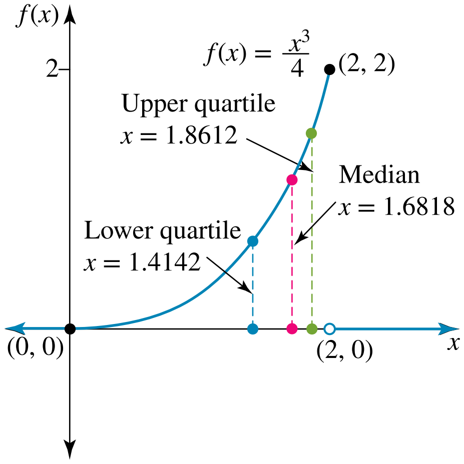
$$\begin{aligned} \int_0^a \frac{x^3}{4} dx &= 0.25 \\ \left[\frac{x^4}{16}\right]_0^a &= \frac{1}{4} \\ \frac{a^4}{16} - 0 &= \frac{1}{4} \\ a^4 &= 4 \\ a &= \pm\sqrt[4]{4} \\ a &= Q_1 = 1.4142 \ (0 \ \leq a \leq m) \end{aligned}$$

Similarly, to find the upper quartile, we make the area under the curve equal to 0.75. Thus the upper quartile is given by $\Pr(0 \leq x \leq n) = 0.75$:

$$\begin{aligned} \int_0^n \frac{x^3}{4} dx &= 0.75 \\ \left[\frac{x^4}{16}\right]_0^n &= \frac{3}{4} \\ \frac{n^4}{16} - 0 &= \frac{3}{4} \\ n^4 &= 12 \\ n &= \pm\sqrt[4]{12} \\ n &= Q_3 = 1.8612 \ (m \leq x \leq 2) \end{aligned}$$

So the **interquartile range** is given by $Q_3 - Q_1 = 1.8612 - 1.4142$
$= 0.4470.$

These values are shown on the following graph.



WORKED EXAMPLE 4

A continuous random variable, Y, has a probability density function, f, defined by

$$f(y) = \begin{cases} ky, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant.

a **Sketch the graph of f.**

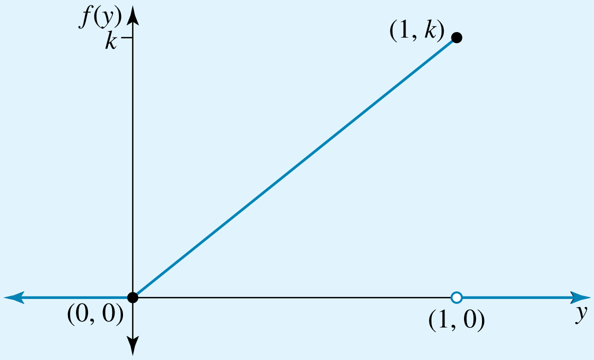
b **Find the value of the constant k.**

c **Find:**

  i **the mean of Y**

  ii **the median of Y.**

d **Find the interquartile range of Y.**

THINK	WRITE/DRAW
a The graph $f(y) = ky$ is a straight line with end points at (0, 0) and (1, k). Remember to include the lines $f(y) = 0$ for $y > 1$ and $y < 0$.	**a** 

b Solve $\int_0^1 ky\,dy = 1$ to find the value of k.

b

$$\int_0^1 ky\,dy = 1$$

$$k\int_0^1 y\,dy = 1$$

$$k\left[\frac{y^2}{2}\right]_0^1 = 1$$

$$\frac{k(1)^2}{2} - 0 = 1$$

$$\frac{k}{2} = 1$$

$$k = 2$$

Using the area of a triangle also enables you to find the value of k.

$$\frac{1}{2} \times 1 \times k = 1$$

$$\frac{k}{2} = 1$$

$$k = 2$$

c i 1 State the rule for the mean.

c i

$$\mu = \int_0^1 y(2y)dy$$

$$= \int_0^1 2y^2dy$$

2 Antidifferentiate and simplify.

$$= \left[\frac{2}{3}y^3\right]_0^1$$

$$= \frac{2(1)^3}{3} - 0$$

$$= \frac{2}{3}$$

ii 1 State the rule for the median.

ii

$$\int_0^m f(y)dy = 0.5$$

$$\int_0^m 2ydy = 0.5$$

2 Antidifferentiate and solve for m. Note that m must be a value within the domain of the function, so within $0 \leq y \leq 1$.

$$[y^2]_0^m = 0.5$$

$$m^2 - 0 = 0.5$$

$$m = \pm\sqrt{\frac{1}{2}}$$

$$m = \frac{1}{\sqrt{2}}\ (0 < m < 1)$$

3 Write the answer.	$\text{Median} = \frac{1}{\sqrt{2}}$
d **1** State the rule for the lower quartile, Q_1.	**d** $\int_0^a f(y)dy = 0.25$ $\int_0^a 2ydy = 0.25$
2 Antidifferentiate and solve for Q_1.	$\left[y^2\right]_0^a = 0.25$ $a^2 - 0 = 0.25$ $a = \pm\sqrt{0.25}$ $a = Q_1 = 0.5 \; \left(0 < Q_1 < \frac{1}{\sqrt{2}}\right)$
3 State the rule for the upper quartile, Q_3.	$\int_0^n f(y)dy = 0.75$ $\int_0^n 2ydy = 0.75$
4 Antidifferentiate and solve for Q_3.	$\left[y^2\right]_0^n = 0.75$ $n^2 - 0 = 0.75$ $n = \pm\sqrt{0.75}$ $n = Q_3 = 0.8660 \; \left(\frac{1}{\sqrt{2}} < Q_3 < 1\right)$
5 State the rule for the interquartile range.	$IQR = Q_3 - Q_1$
6 Substitute the appropriate values and simplify.	$= 0.8660 - 0.5$ $= 0.3660$

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Units 3 & 4

AOS 4

Topic 3

Concept 4

Variance and standard deviation
Concept summary
Practice questions

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Interactivity
Variance, standard deviation and range
int-6437

Measures of spread

Variance, standard deviation and range

The variance and standard deviation are important measures of spread in statistics. From previous calculations for discrete probability functions, we know that

$$\boxed{\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2 \text{ and } \text{SD}(X) = \sqrt{\text{Var}(X)}}$$

For continuous probability functions,

$$\begin{aligned}\text{Var}(X) &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx \\ &= \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x)dx\end{aligned}$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \int_{-\infty}^{\infty} 2xf(x)\mu dx + \int_{-\infty}^{\infty} \mu^2 f(x)dx$$

$$= \mathrm{E}(X^2) - 2\mu \int_{-\infty}^{\infty} xf(x)dx + \mu^2 \int_{-\infty}^{\infty} 1f(x)dx$$

$$= \mathrm{E}(X^2) - 2\mu \times \mathrm{E}(X) + \mu^2$$

$$= \mathrm{E}(X^2) - 2\mu^2 + \mu^2$$

$$= \mathrm{E}(X^2) - \mu^2$$

$$= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

Two important facts were used in this proof: $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\int_{-\infty}^{\infty} xf(x)dx = \mu = \mathrm{E}(X)$.

Substituting this result into $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$ gives us

$$\mathrm{SD}(X) = \sqrt{\mathrm{E}(X^2) - [\mathrm{E}(X)]^2}.$$

The range is calculated as the highest value minus the lowest value, so for the probability density function given by $f(x) = \begin{cases} \frac{1}{5}, & 1 \le x \le 6 \\ 0, & \text{elsewhere} \end{cases}$, the highest possible x-value is 6 and the lowest is 1. Therefore, the range for this function $= 6 - 1$
$= 5$.

WORKED EXAMPLE 5

For a continuous random variable, X, with a probability density function, f, defined by

$$f(x) = \begin{cases} \frac{1}{2}x + 2, & -4 \le x \le -2 \\ 0, & \text{elsewhere} \end{cases}$$

find:

a **the mean**

b **the median**

c **the variance**

d **the standard deviation, correct to 4 decimal places.**

THINK

a **1** State the rule for the mean and simplify.

WRITE

a $\mu = \int_{-4}^{-2} xf(x)dx$

$$= \int_{-4}^{-2} x\left(\frac{1}{2}x + 2\right)dx$$

$$= \int_{-4}^{-2} \left(\frac{1}{2}x^2 + 2x\right)dx$$

2 Antidifferentiate and evaluate.

$$= \left[\frac{1}{6}x^3 + x^2\right]_{-4}^{-2}$$

$$= \left(\frac{1}{6}(-2)^3 + (-2)^2\right) - \left(\frac{1}{6}(-4)^3 + (-4)^2\right)$$

$$= \frac{4}{3} + 4 + \frac{32}{3} - 16$$

$$= -2\frac{2}{3}$$

b **1** State the rule for the median.

b $$\int_{-4}^{m} f(x)dx = 0.5$$

$$\int_{-4}^{m} \left(\frac{1}{2}x + 2\right)dx = 0.5$$

2 Antidifferentiate and solve for *m*. The quadratic formula is needed as the quadratic equation formed cannot be factorised. Alternatively, use CAS to solve for *m*.

$$\left[\frac{1}{4}x^2 + 2x\right]_{-4}^{m} = 0.5$$

$$\left(\frac{1}{4}m^2 + 2m\right) - \left(\frac{(-4)^2}{4} + 2(-4)\right) = 0.5$$

$$\frac{1}{4}m^2 + 2m + 4 = 0.5$$

$$m^2 + 8m + 16 = 2$$

$$m^2 + 8m + 14 = 0$$

$$\text{So } m = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(14)}}{2(1)}$$

$$m = \frac{-8 \pm \sqrt{8}}{2}$$

$$= -4 \pm \sqrt{2}$$

$$\therefore m = -4 + \sqrt{2} \text{ as } m \in [-4, 2].$$

3 Write the answer.

The median is $-4 + \sqrt{2}$.

c **1** Write the rule for variance.

c $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

2 Find $\text{E}(X^2)$ first.

$$\text{E}(X^2) = \int_{a}^{b} x^2 f(x)dx$$

$$= \int_{-4}^{-2} x^2\left(\frac{1}{2}x + 2\right)dx$$

$$= \int_{-4}^{-2} \left(\frac{1}{2}x^3 + 2x^2\right)dx$$

$$= \left[\frac{1}{8}x^4 + \frac{2}{3}x^3\right]_{-4}^{-2}$$

$$= \left(\frac{1}{8}(-2)^4 + \frac{2}{3}(-2)^3\right) - \left(\frac{1}{8}(-4)^4 + \frac{2}{3}(-4)^3\right)$$

$$= 2 - \frac{16}{3} - 32 + \frac{128}{3}$$

$$= -30 + \frac{112}{3}$$

$$= \frac{22}{3}$$

3 Substitute E(X) and E(X^2) into the rule for variance.

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - [\text{E}(X)]^2\\ &= \frac{22}{3} - \left(-\frac{8}{3}\right)^2\\ &= \frac{22}{3} - \frac{64}{9}\\ &= \frac{66}{9} - \frac{64}{9}\\ &= \frac{2}{9}\end{aligned}$$

d **1** Write the rule for standard deviation.

d $\text{SD}(X) = \sqrt{\text{Var}(X)}$

2 Substitute the variance into the rule and evaluate.

$$\begin{aligned}&= \sqrt{\frac{2}{9}}\\ &= 0.4714\end{aligned}$$

EXERCISE 12.4 Measures of centre and spread

PRACTISE

Work without CAS
Question 1

1 **WE4** The continuous random variable Z has a probability density function of

$$f(z) = \begin{cases} \dfrac{1}{\sqrt{z}}, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a Find the value of the constant a.

b Find:

i the mean of Z **ii** the median of Z.

2 The continuous random variable, Y, has a probability density function of

$$f(y) = \begin{cases} \sqrt{y}, & 0 \le y \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

Find, correct to 4 decimal places:

a the value of the constant a **b** E(Y)

c the median value of Y.

3 **WE5** For the continuous random variable Z, the probability density function is

$$f(z) = \begin{cases} 2\log_e(2z), & 1 \le z \le \dfrac{e}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean, median, variance and standard deviation, correct to 4 decimal places.

4 The function

$$f(x) = \begin{cases} 3e^{-3x}, & x \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$

defines the probability density function for the continuous random variable, X. Find the mean, median, variance and standard deviation of X.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Let X be a continuous random variable with a probability density function of

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

- **a** Prove that f is a probability density function.
- **b** Find $\mathrm{E}(X)$.
- **c** Find the median value of f.

6 The time in minutes that an individual must wait in line to be served at the local bank branch is defined by

$$f(t) = 2e^{-2t}, t \ge 0$$

where T is a continuous random variable.

- **a** What is the mean waiting time for a customer in the queue, correct to 1 decimal place?
- **b** Calculate the standard deviation for the waiting time in the queue, correct to 1 decimal place.
- **c** Determine the median waiting time in the queue, correct to 2 decimal places.

7 The continuous random variable Y has a probability density function defined by

$$f(y) = \begin{cases} \dfrac{y^2}{3}, & 0 \le y \le \sqrt[3]{9} \\ 0, & \text{elsewhere} \end{cases}.$$

Find, correct to 4 decimal places:

- **a** the expected value of Y
- **b** the median value of Y
- **c** the lower and upper quartiles of Y
- **d** the inter-quartile range of Y.

8 The continuous random variable Z has a probability density function defined by

$$f(y) = \begin{cases} \dfrac{a}{z}, & 1 \le z \le 8 \\ 0, & \text{elsewhere} \end{cases}$$

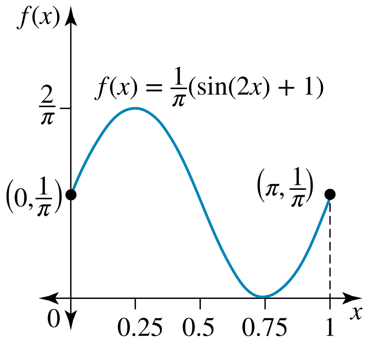
where a is a constant.

- **a** Find the value, correct to four decimal places, of the constant a.
- **b** Find $\mathrm{E}(Z)$ correct to 4 decimal places.
- **c** Find $\mathrm{Var}(Z)$ and $\mathrm{SD}(Z)$.
- **d** Determine the interquartile range for Z.
- **e** Determine the range for Z.

9 X is a continuous random variable. The graph of the probability density function

$$f(x) = \frac{1}{\pi}(\sin(2x) + 1) \text{ for } 0 \le x \le \pi$$

is shown.



- **a** Show that $f(x)$ is a probability density function.
- **b** Calculate $\mathrm{E}(X)$ correct to 4 decimal places.
- **c** Calculate, correct to 4 decimal places:
  - **i** $\mathrm{Var}(X)$
  - **ii** $\mathrm{SD}(X)$.
- **d** Find the median value of f correct to 4 decimal places.

10 The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} ax - bx^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the values of the constants a and b if $\mathrm{E}(X) = 1$.

11 The continuous random variable, Z, has a probability density function of

$$f(z) = \begin{cases} \dfrac{3}{z^2}, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a Show that the value of a is $\frac{3}{2}$.

b Find the mean value and variance of f correct to 4 decimal places.

c Find the median and interquartile range of f.

12 a Find the derivative of $\sqrt{4 - x^2}$.

b Hence, find the mean value of the probability density function defined by

$$f(x) = \begin{cases} \dfrac{3}{\pi\sqrt{4 - x^2}}, & 0 \le x \le \sqrt{3} \\ 0, & \text{elsewhere} \end{cases}.$$

13 Consider the continuous random variable X with a probability density function of

$$f(x) = \begin{cases} h(2 - x), & 0 \le x \le 2 \\ h(x - 2), & 2 < x \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

where h is a constant.

a Find the value of the constant h.

b Find $\mathrm{E}(X)$.

c Find $\mathrm{Var}(X)$.

14 Consider the continuous random variable X with a probability density function of

$$f(x) = \begin{cases} k, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

where a, b and k are positive constants.

a Sketch the graph of the function f.

b Show that $k = \dfrac{1}{b - a}$.

c Find $\mathrm{E}(X)$ in terms of a and b.

d Find $\mathrm{Var}(X)$ in terms of a and b.

MASTER

15 The continuous random variable Y has a probability density function

$$f(y) = \begin{cases} 0.2 \log_e\left(\dfrac{y}{2}\right), & 2 \le y \le 7.9344 \\ 0, & \text{elsewhere} \end{cases}.$$

a Verify that f is a probability density function.

b Find $\mathrm{E}(Y)$ correct to 4 decimal places.

c Find $\mathrm{Var}(Y)$ and $\mathrm{SD}(Y)$ correct to 4 decimal places.

d Find the median value of Y correct to 4 decimal places.

e State the range.

16 The continuous random variable Z has a probability density function

$$f(z) = \begin{cases} \sqrt{z-1}, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a Find the value of the constant a correct to 4 decimal places.

b Determine, correct to 4 decimal places:

i $\mathrm{E}(Z)$

ii $\mathrm{E}(Z^2)$

iii $\mathrm{Var}(Z)$

iv $\mathrm{SD}(Z)$.

12.5 Linear transformations

Sometimes it is necessary to apply transformations to a continuous random variable. A transformation is a change that is applied to the random variable. The change may consist of one or more operations that may involve adding or subtracting a constant or multiplying or dividing the variable by a constant.

Suppose a linear transformation is applied to the continuous random variable X to create a new continuous random variable, Y. For instance

$$Y = aX + b$$

It can be shown that $\mathrm{E}(Y) = \mathrm{E}(aX + b) = a\mathrm{E}(X) + b$

and $\mathrm{Var}(Y) = \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$.

First let us show that $\mathrm{E}(Y) = \mathrm{E}(aX + b) = a\mathrm{E}(X) + b$.

Since $\mathrm{E}(X) = \displaystyle\int_{-\infty}^{\infty} xf(x)\,dx$,

then $\mathrm{E}(aX + b) = \displaystyle\int_{-\infty}^{\infty} (ax + b)f(x)\,dx$.

Using the distributive law, it can be shown that this is equal to

$$\begin{aligned} \mathrm{E}(aX + b) &= \int_{-\infty}^{\infty} axf(x)\,dx + \int_{-\infty}^{\infty} bf(x)\,dx \\ &= a\int_{-\infty}^{\infty} xf(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx \end{aligned}$$

But $\mathrm{E}(X) = \displaystyle\int_{-\infty}^{\infty} xf(x)\,dx$, so

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b\int_{-\infty}^{\infty} f(x)\,dx.$$

Also, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$, so

$$\boxed{\mathbf{E}(aX + b) = a\mathbf{E}(X) + b.}$$

Also note that $\mathrm{E}(aX) = a\mathrm{E}(X)$ and $\mathrm{E}(b) = b$.

Now let us show that $\mathrm{Var}(Y) = \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$.

Since $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$,

then

$$\begin{aligned}\mathrm{Var}(aX + b) &= \mathrm{E}((aX + b)^2) - [\mathrm{E}(aX + b)]^2 \\ &= \int_{-\infty}^{\infty} (ax + b)^2 f(x)\,dx - (a\mathrm{E}(X) + b)^2 \\ &= \int_{-\infty}^{\infty} (a^2x^2 + 2abx + b^2)f(x)\,dx - [a^2[\mathrm{E}(X)]^2 + 2ab\mathrm{E}(X) + b^2]\end{aligned}$$

Using the distributive law to separate the first integral, we have

$$\begin{aligned}\mathrm{Var}(aX + b) &= \int_{-\infty}^{\infty} a^2x^2 f(x)\,dx + \int_{-\infty}^{\infty} 2abxf(x)\,dx + \int_{-\infty}^{\infty} b^2 f(x)\,dx - a^2[\mathrm{E}(X)]^2 \\ &\quad - 2ab\mathrm{E}(X) - b^2 \\ &= a^2\int_{-\infty}^{\infty} x^2 f(x)\,dx + 2ab\int_{-\infty}^{\infty} xf(x)\,dx + b^2\int_{-\infty}^{\infty} f(x)\,dx - a^2[\mathrm{E}(X)]^2 \\ &\quad - 2ab\mathrm{E}(X) - b^2\end{aligned}$$

But $\mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)\,dx$, $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for a probability density function. Thus,

$$\begin{aligned}\mathrm{Var}(aX + b) &= a^2\mathrm{E}(X^2) + 2ab\mathrm{E}(X) + b^2 - a^2[\mathrm{E}(X)]^2 - 2ab\mathrm{E}(X) - b^2 \\ &= a^2\mathrm{E}(X^2) - a^2[\mathrm{E}(X)]^2 \\ &= a^2(\mathrm{E}(X^2) - [\mathrm{E}(X)]^2) \\ &= a^2\mathrm{Var}(X)\end{aligned}$$

Thus,

$$\mathbf{E}(aX + b) = a\mathbf{E}(X) + b$$

and

$$\mathbf{Var}(aX + b) = a^2\mathbf{Var}(X).$$

WORKED EXAMPLE 6

A continuous random variable, X, has a mean of 3 and a variance of 2. Find:

a $\mathrm{E}(2X + 1)$
b $\mathrm{Var}(2X + 1)$
c $\mathrm{E}(X^2)$
d $\mathrm{E}(3X^2)$
e $\mathrm{E}(X^2 - 5)$.

THINK	WRITE
a Use $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ to find $\mathrm{E}(2X + 1)$.	**a** $\mathrm{E}(2X + 1) = 2\mathrm{E}(X) + 1$ $= 2(3) + 1$ $= 7$
b Use $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$ to find $\mathrm{Var}(2X + 1)$.	**b** $\mathrm{Var}(2X + 1) = 2^2\mathrm{Var}(X)$ $= 4 \times 2$ $= 8$

c Use $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$ to find $\text{E}(X^2)$.

c
$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - [\text{E}(X)]^2\\ 2 &= \text{E}(X^2) - 3^2\\ 2 &= \text{E}(X^2) - 9\\ \text{E}(X^2) &= 11\end{aligned}$$

d Use $\text{E}(aX^2) = a\text{E}(X^2)$ to find $\text{E}(3X^2)$.

d
$$\begin{aligned}\text{E}(3X^2) &= 3\text{E}(X^2)\\ &= 3 \times 11\\ &= 33\end{aligned}$$

e Use $\text{E}(aX^2 + b) = a\text{E}(X^2) + b$ to find $\text{E}(X^2 - 5)$.

e
$$\begin{aligned}\text{E}(X^2 - 5) &= \text{E}(X^5) - 5\\ &= 11 - 5\\ &= 6\end{aligned}$$

It may also be necessary to find the expected value and variance before using the facts that $\text{E}(aX + b) = a\text{E}(X) + b$ and $\text{Var}(aX + b) = a^2\text{Var}(X)$.

WORKED EXAMPLE 7

The graph of the probability density function for the continuous random variable X is shown. The rule for the probability density function is given by

$$f(x) = \begin{cases} 3kx, & 0 \le x \le 1\\ 0, & \text{elsewhere}\end{cases}$$

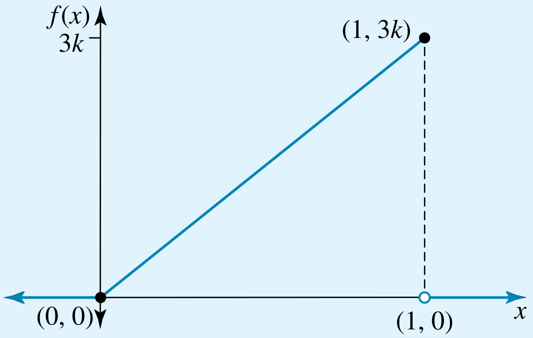
where k is a constant.

a **Find the value of the constant k.**

b **Calculate $\text{E}(X)$ and $\text{Var}(X)$.**

c **Find $\text{E}(3X - 1)$ and $\text{Var}(3X - 1)$.**

d **Find $\text{E}(2X^2 + 3)$.**



THINK

a Solve $\int_0^1 kx\,dx = 1$ to find k, or alternatively use the formula for the area of a triangle to find k.

WRITE

a Method 1:

$$\int_0^1 3kxdx = 1$$

$$\left[\frac{3kx^2}{2}\right]_0^1 = 1$$

$$\frac{3k(1)^2}{2} - 0 = 1$$

$$k = \tfrac{2}{3}$$

Method 2:

$$\begin{aligned}\tfrac{1}{2} \times 1 \times 3k &= 1\\ \frac{3k}{2} &= 1\\ 3k &= 2\\ k &= \frac{2}{3}\end{aligned}$$

b **1** Write the rule for the mean.

b
$$\begin{aligned}\mathrm{E}(X) &= \int_0^1 xf(x)dx\\ &= \int_0^1 (x \times 2x)dx\\ &= \int_0^1 (2x^2)dx\end{aligned}$$

2 Antidifferentiate and evaluate.

$$\begin{aligned}&= \left[\tfrac{2}{3}x^3\right]_0^1\\ &= \tfrac{2}{3}(1)^3 - 0\\ &= \tfrac{2}{3}\end{aligned}$$

3 Write the rule for the variance.

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

4 Find $\mathrm{E}(X^2)$.

$$\begin{aligned}\mathrm{E}(X^2) &= \int_0^1 x^2f(x)dx\\ &= \int_0^1 2x^3dx\\ &= \left[\tfrac{1}{2}x^4\right]_0^1\\ &= \tfrac{1}{2}(1)^4 - 0\\ &= \tfrac{1}{2}\end{aligned}$$

5 Substitute the appropriate values into the variance formula.

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2\\ &= \tfrac{1}{2} - \left(\tfrac{2}{3}\right)^2\\ &= \tfrac{1}{2} - \tfrac{4}{9}\\ &= \tfrac{9}{18} - \tfrac{8}{18}\\ &= \tfrac{1}{18}\end{aligned}$$

c **1** Use the property $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ to work out $\mathrm{E}(3X - 1)$.

c
$$\begin{aligned}\mathrm{E}(3X - 1) &= 3\mathrm{E}(X) - 1\\ &= 3\left(\tfrac{2}{3}\right) - 1\\ &= 2 - 1\\ &= 1\end{aligned}$$

2 Use the property $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$ to calculate $\mathrm{Var}(3X - 1)$.

$$\begin{aligned}\mathrm{Var}(3X - 1) &= 3^2\mathrm{Var}(X)\\ &= 9\left(\tfrac{1}{18}\right)\\ &= \tfrac{1}{2}\end{aligned}$$

d Use the property $\mathrm{E}(aX^2 + b) = a\mathrm{E}(X^2) + b$ to calculate $\mathrm{E}(2X^2 + 3)$.

d
$$\begin{aligned}\mathrm{E}(2X^2 + 3) &= 2\mathrm{E}(X^2) + 3\\ &= 2\left(\tfrac{1}{2}\right) + 3\\ &= 4\end{aligned}$$

EXERCISE 12.5 Linear transformations

PRACTISE

Work without CAS
Questions 1–3

1 **WE6** If the continuous random variable Y has a mean of 4 and a variance of 3, find:

a $E(2Y - 3)$ **b** $\text{Var}(2Y - 3)$ **c** $E(Y^2)$ **d** $E(Y(Y - 1))$.

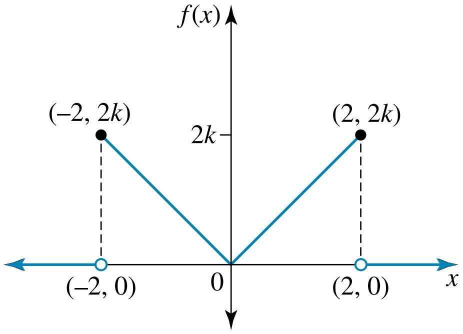
2 Two continuous random variables, X and Y, are related such that $Y = aX + 5$ where a is a positive integer and $E(aX + 5) = \text{Var}(aX + 5)$. The mean of X is 9 and the variance of X is 2.

a Find the value of the constant a. **b** Find $E(Y)$ and $\text{Var}(Y)$.

3 **WE7** The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} -kx, & -2 \le x \le 0 \\ kx, & 0 < x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant. The graph of the function is shown.



a Find the value of the constant k.
b Determine $E(X)$ and $\text{Var}(X)$.
c Find $E(5X + 3)$ and $\text{Var}(5X + 3)$.
d Find $E((3X - 2)^2)$.

4 The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} -\cos(x), & \frac{\pi}{2} \le x \le \pi \\ 0, & \text{elsewhere} \end{cases}.$$

a Sketch the graph of f and verify that it is a probability density function.
b Calculate $E(X)$ and $\text{Var}(X)$.
c Calculate $E(3X + 1)$ and $\text{Var}(3X + 1)$.
d Calculate $E((2X - 1)(3X - 2))$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 For a continuous random variable Z, where $E(Z) = 5$ and $\text{Var}(Z) = 2$, find:

a $E(3Z - 2)$ **b** $\text{Var}(3Z - 2)$

c $E(Z^2)$ **d** $E\left(\frac{1}{3}Z^2 - 1\right)$.

6 The mean of the continuous random variable Y is known to be 3.5, and its standard deviation is 1.2. Find:

a $E(2 - Y)$ **b** $E\left(\frac{Y}{2}\right)$ **c** $\text{Var}(Y)$ **d** $\text{Var}(2 - Y)$ **e** $\text{Var}\left(\frac{Y}{2}\right)$.

7 The length of time it takes for an electric kettle to come to the boil is a continuous random variable with a mean of 1.5 minutes and a standard deviation of 1.1 minutes.

If each time the kettle is brought to the boil is an independent event and the kettle is boiled five times a day, find the mean and standard deviation of the total time taken for the kettle to boil during a day.

8 The probability density function for the continuous random variable X is

$$f(x) = \begin{cases} mx(2 - x), & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

where m is a constant. Find:

a the value of the constant m

b $E(X)$ and $\text{Var}(X)$

c $E(5 - 2X)$ and $\text{Var}(5 - 2X)$.

9 The continuous random variable Z has a probability density function given by

$$f(z) = \begin{cases} \dfrac{2}{z + 1}, & 0 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant. Calculate, correct to 4 decimal places:

a the value of the constant a

b the mean and variance of Z

c i $E(3Z + 1)$ ii $\text{Var}(3Z + 1)$ iii $E(Z^2 + 2)$.

10 The continuous random variable X is transformed so that $Y = aX + 3$ where a is a positive integer. If $E(X) = 5$ and $\text{Var}(X) = 2$, find the value of the constant a, given that $E(Y) = \text{Var}(Y)$. Then calculate both $E(Y)$ and $\text{Var}(Y)$ to verify this statement.

11 The continuous random variable Y is transformed so that $Z = aY - 3$ where a is a positive integer. If $E(Y) = 4$ and $\text{Var}(Y) = 1$, find the value(s) of the constant a, given that $E(Z) = \text{Var}(Z)$. Then calculate both $E(Z)$ and $\text{Var}(Z)$ to verify this statement.

12 The continuous random variable Z has a probability density function given by

$$f(z) = \begin{cases} \dfrac{3}{\sqrt{z}}, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a Find the value of the constant a.

b Calculate the mean and variance of Z correct to 4 decimal places.

c Find, correct to 4 decimal places:
i $E(4 - 3Z)$ ii $\text{Var}(4 - 3Z)$.

13 The daily rainfall, X mm, in a particular Australian town has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{x}{k\pi} \sin\left(\dfrac{x}{3}\right), & 0 \le x \le 3\pi \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant.

a Find the value of the constant k.

b What is the expected daily rainfall, correct to 2 decimal places?

c During the winter the daily rainfall is better approximated by $W = 2X - 1$. What is the expected daily rainfall during winter, correct to 2 decimal places?

14 The mass, Y kilograms, of flour sold in bags labelled as 1 kilogram is known to have a probability density function given by

$$f(y) = \begin{cases} k(2y+1), & 0.9 \le y \le 1.25 \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant.

a Find the value of the constant k.

b Find the expected mass of a bag of flour, correct to 3 decimal places.

c On a particular day, the machinery packaging the bags of flour needed to be recalibrated and produced a batch which had a mass of Z kilograms, where the probability density function for Z was given by $Z = 0.75Y + 0.45$. What was the expected mass of a bag of flour for this particular batch, correct to 3 decimal places?

MASTER

15 The continuous random variable Z has a probability density function defined by

$$f(z) = \begin{cases} \dfrac{5\log_e(z)}{\sqrt{z}}, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant. Determine, correct to 4 decimal places:

a the value of the constant a

b $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$

c $\mathrm{E}(3 - 2Z)$ and $\mathrm{Var}(3 - 2Z)$.

16 A continuous random variable, X, is transformed so that $Y = aX + 1$, where a is a positive constant. If $\mathrm{E}(X) = 2$ and $\mathrm{Var}(X) = 7$, find the value of the constant a, given $\mathrm{E}(Y) = \mathrm{Var}(Y)$. Then calculate both $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ to verify this statement. Give your answers correct to 4 decimal places.

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12.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

To access eBookPLUS activities, log on to

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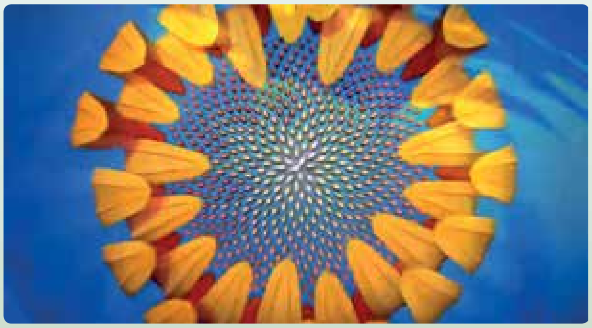
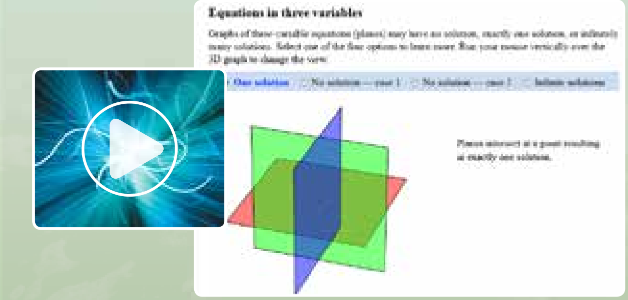
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





Equations in three variables

Graphs of three-variable equations (planes) may have no solution, exactly one solution, or infinitely many solutions. Select one of the four options to learn more. Run your mouse vertically over the 3D graph to change the view.

Planes intersect at a point resulting in exactly one solution.

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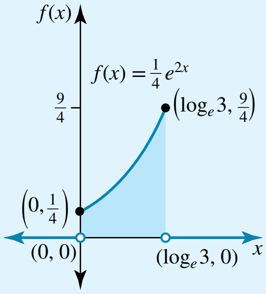
studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

12 Answers

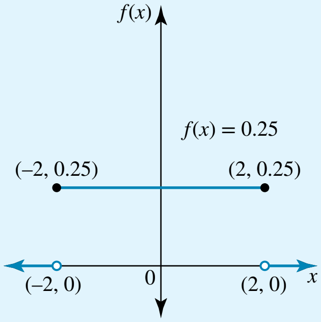
EXERCISE 12.2

1 a



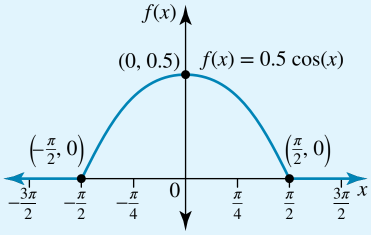
This is a probability density function as the area is 1 unit^2.

b



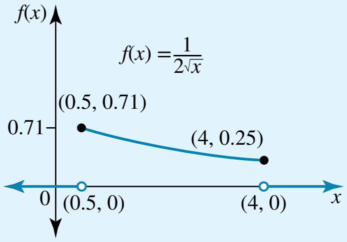
This is a probability density function as the area is 1 unit^2.

2 a



This is a probability density function as the area is 1 unit^2.

b



This is not a probability density function as the area is 1.2929 units^2.

3 $n = \frac{1}{18}$

4 $a = \frac{1}{11}$

5 a i $\frac{9}{25}$

ii $\frac{4}{25}$

b i $\frac{37}{50}$

ii $\frac{37}{42}$

6 a 100 **b** $\frac{29}{100}$ **c** $\frac{41}{50}$ **d** $\frac{3}{100}$

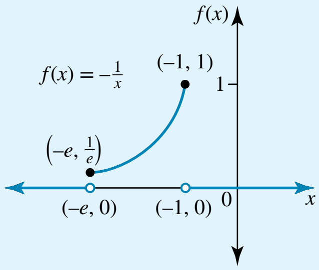
7 a 200

b i $\frac{5}{8}$

ii $\frac{31}{100}$

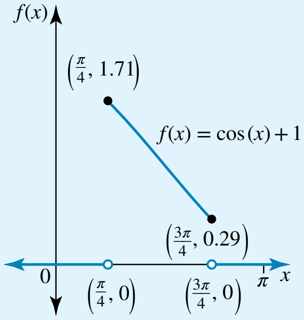
c $\frac{21}{46}$

8 a



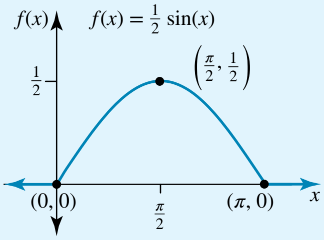
This is a probability density function as the area is 1 unit^2.

b



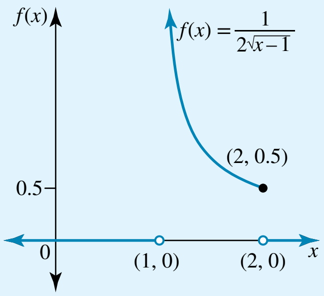
This is not a probability density function as the area is $\frac{\pi}{2}$ units^2.

c



This is a probability density function as the area is 1 unit^2.

d



This is a probability density function as the area is 1 unit^2.

9 $c = \frac{5}{7}$

10 $$\int_{-1}^{5} f(z)dz = 1$$

$$A_{\text{triangle}} = 1$$
$$\tfrac{1}{2}bh = 1$$
$$\tfrac{1}{2} \times 6 \times z = 1$$
$$3z = 1$$
$$z = \tfrac{1}{3}$$

11 a $m = \frac{1}{8}$

b $m = 2$

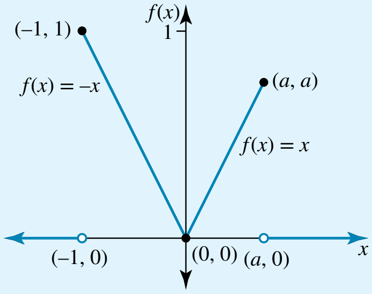
c $m = \frac{1}{4}$

12 $$\int_{0}^{3} (x^2 + 2kx + 1)dx = 1$$

$$\left[\tfrac{1}{3}x^3 + kx^2 + x\right]_0^3 = 1$$
$$\left(\tfrac{1}{3}(3)^3 + k(3)^2 + 3\right) - 0 = 1$$
$$9 + 9k + 3 = 1$$
$$9k + 12 = 1$$
$$9k = -11$$
$$k = -\tfrac{11}{9}$$

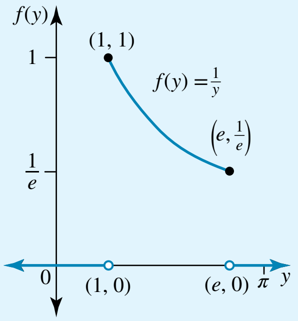
13 $a = 2e$

14 a



$$\int_{-1}^{0} -xdx + \int_{0}^{a} xdx = \frac{a^2 + 1}{2}$$

b



$$\int_{1}^{e} \frac{1}{y}dy = 1$$

c $a = 1$

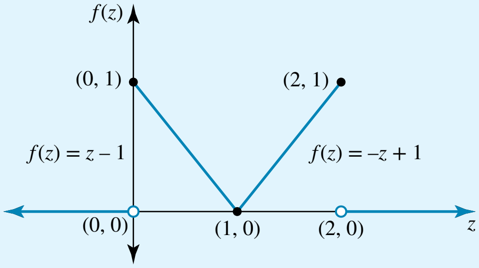
15 $n = 12$

16 a $a = e.$

b As $f(x) \geq 0$ and $\int_{1}^{e} f(x)dx = 1$, this is a probability density function.

EXERCISE 12.3

1 a

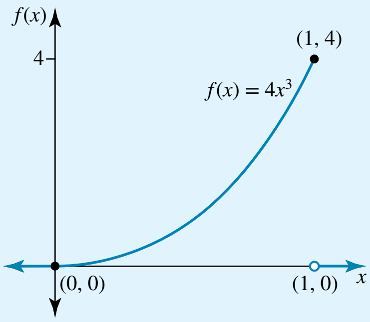


b $\frac{15}{32}$

c $\frac{5}{8}$

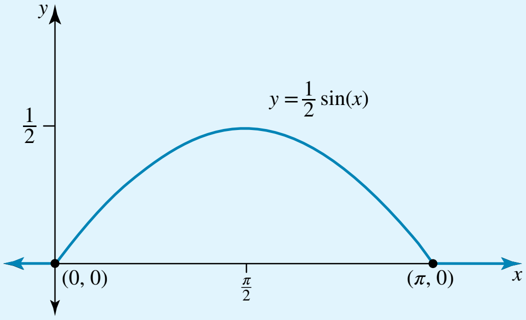
2 a $a = 1$

b



c $\frac{15}{16}$

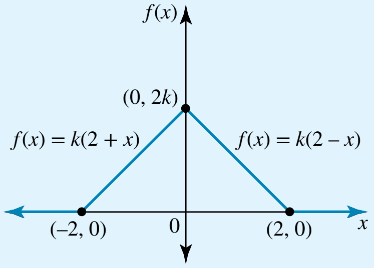
3 a



b $\frac{\sqrt{2}}{2}$

c $2\sqrt{2}-2$

4 a



b
$$\begin{aligned} A &= \tfrac{1}{2}bh \\ 1 &= \tfrac{1}{2} \times 4 \times 2k \\ 1 &= 4k \\ k &= \tfrac{1}{4} \end{aligned}$$

c $\frac{3}{4}$

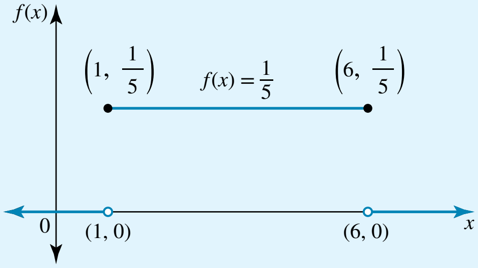
d $\frac{6}{7}$

5 a $\frac{1}{12}$

b $\frac{5}{12}$

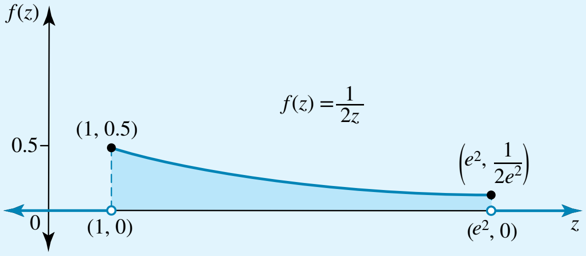
c $\frac{1}{2}$

6 a



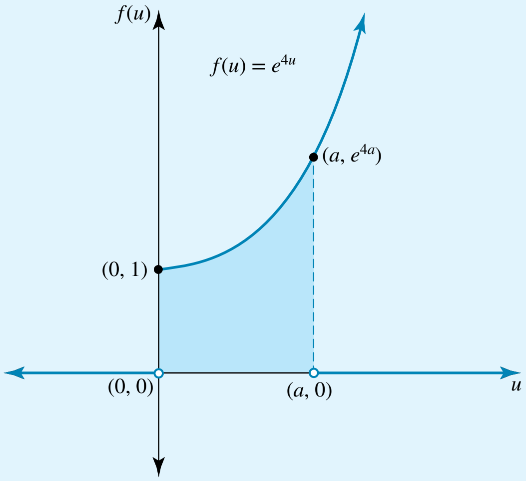
b $\frac{3}{5}$

7 a



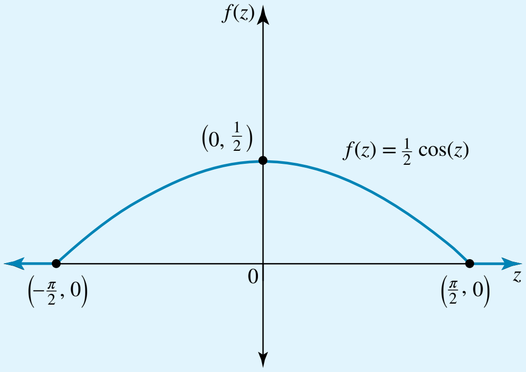
b $\int_{1}^{e^2} \frac{1}{2z}dz = 1$. As $f(z) \geq 0$ and $\int_{1}^{e^2} f(z)dz = 1$, this is a probability density function.

c



d $\int_{0}^{a} e^{4u}du = \frac{1}{4}e^{4a} - \frac{1}{4}$ and $a = \frac{1}{4}\log_e 5$

8 a



$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos(z)dz &= \left[\frac{1}{2}\sin(z)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1 \end{aligned}$$

This is a probability density function as the area under the curve is 1 and $f(z) \geq 0$ for all values of z.

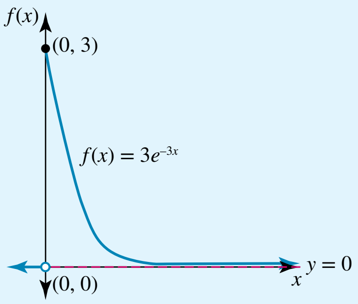
b $\frac{\sqrt{2}+1}{4}$

9 a $a = 1$
b $\frac{183}{256}$
c 0.371
d 0

10 a $\frac{98}{125}$
b $\frac{8}{9}$
c $6^{\frac{1}{3}}$

11 a $a = 3\log_e\left(\frac{3}{2}\right)$
b 0.6243
c 0.5342
d 0.60

12 a



b 0.9502

c 0.0025

13 a $a = 2.1555$ **b** 0.7147

14 a 0.1560 **b** $a = \tan\left(\frac{1}{2}\right) \approx 0.5463$

EXERCISE 12.4

1 a $a = \frac{9}{4}$

b i $\frac{19}{12}$ **ii** 1.5625 or $\frac{25}{16}$

2 a 1.3104

b 0.7863

c Median = 0.8255

3 $\text{E}(Z) = 0.7305$, $m = 1.3010$, $\text{Var}(z) = 0.3424$, $\text{SD}(Z) = 0.5851$

4 $\text{E}(X) = \frac{1}{3}$, $m = 0.2310$, $\text{Var}(X) = \frac{1}{9}$, $\text{SD}(X) = \frac{1}{3}$

5 a
$$\begin{aligned}\int_0^1 \frac{1}{2\sqrt{x}}dx &= \int_0^1 \frac{1}{2}x^{\frac{1}{2}}dx\\ &= \frac{1}{2}\int_0^1 x^{-\frac{1}{2}}dx\\ &= \frac{1}{2}\left[2x^{\frac{1}{2}}\right]_0^1\\ &= \frac{1}{2}(2\sqrt{1} - 2\sqrt{0})\\ &= \frac{1}{2} \times 2\\ &= 1\end{aligned}$$

As $f(x) \geq 0$ for all x-values, and the area under the curve is 1, $f(x)$ is a probability density function.

b $\frac{1}{3}$

c $m = 0.25$

6 a 0.5 min

b 0.5 min

c $m = 0.35$ min

7 a 1.5601

b $m = 1.6510$

c $Q_1 = 1.3104$, $Q_3 = 1.8899$

d 0.5795

8 a $a = 0.4809$

b 3.3663

c $\text{Var}(Z) = 3.8195$, $\text{SD}(Z) = 1.9571$

d 3.0751

e 7

9 a
$$\begin{aligned}\int_0^\pi \frac{1}{\pi}(\sin(2x) + 1)dx &= \frac{1}{\pi}\int_0^\pi (\sin(2x) + 1)dx\\ &= \frac{1}{\pi}\left[-\tfrac{1}{2}\cos(2x) + x\right]_0^\pi\\ &= \frac{1}{\pi}\left(\left(-\tfrac{1}{2}\cos(2\pi) + \pi\right) - \left(-\tfrac{1}{2}\cos(0) + 0\right)\right)\\ &= \frac{1}{\pi}\left(-\tfrac{1}{2} + \pi + \tfrac{1}{2}\right)\\ &= 1\end{aligned}$$

As $f(x) \geq 0$ for all x-values, and the area under the curve is 1, $f(x)$ is a probability density function.

b 1.0708

c i 0.5725

ii 0.7566

d $m = 0.9291$

10 $a = \frac{3}{2}$, $b = \frac{3}{4}$

11 a
$$\begin{aligned}\int_1^a \frac{3}{z^2}dz &= 1\\ \int_1^a 3z^{-2}dz &= 1\\ \left[-3z^{-1}\right]_1^a &= 1\\ \left[-\frac{3}{z}\right]_1^a &= 1\\ -\frac{3}{a} + \frac{3}{1} &= 1\\ -\frac{3}{a} + 3 &= 1\\ -\frac{3}{a} &= -2\\ 3 &= 2a\\ a &= \frac{3}{2}\end{aligned}$$

b $\text{E}(Z) = 1.2164$, $\text{Var}(Z) = 0.0204$

c $m = \frac{6}{5}$, interquartile range $= \frac{8}{33}$

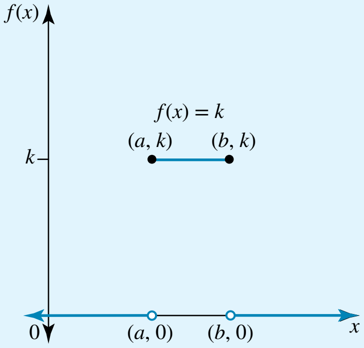
12 a $-\dfrac{x}{\sqrt{4 - x^2}}$ **b** $\dfrac{3}{\pi}$

13 a $h = \frac{1}{4}$

b 2

c 2

14 a



b $$\int_a^b k\,dx = 1$$

$$[kx]_a^b = 1$$

$$kb - ka = 1$$

$$k(b-a) = 1$$

$$k = \frac{1}{b-a}$$

c $\dfrac{b+a}{2}$

d $\dfrac{(a-b)^2}{12}$

15 a $$\int_2^{7.9344} f(y)dy = \int_2^{7.9344} 0.2\log_e\left(\frac{y}{2}\right)dy = 1$$

b 5.7278

c $\text{Var}(Y) = 2.1600$, $\text{SD}(Y) = 1.4697$

d $m = 3.9816$

e 5.9344

16 a 2.3104

b i 1.7863 **ii** 3.3085

iii 0.1176 **iv** 0.3430

EXERCISE 12.5

1 a 5 **b** 12

c 19 **d** 15

2 a $a = 5$

b $E(Y) = 50$, $\text{Var}(Y) = 50$

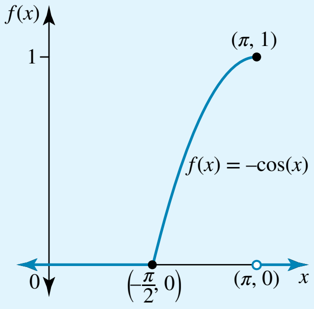
3 a $k = \frac{1}{4}$

b $E(X) = 0$, $\text{Var}(X) = 2$

c $E(5X+3) = 3$, $\text{Var}(5X+3) = 50$

d $E((3X-2)^2) = 22$

4 a



$$\int_{\frac{\pi}{2}}^{\pi}(-\cos(x))dx = \Big[-\sin(x)\Big]_{\frac{\pi}{2}}^{\pi}$$
$$= -\sin(\pi) + \sin\left(\frac{\pi}{2}\right)$$
$$= 0 + 1$$
$$= 1$$

As $f(x) \geq 0$ for all x-values and the area under the curve is 1, $f(x)$ is a probability density function.

b $E(X) = 2.5708$, $\text{Var}(X) = 0.1416$

c $E(3X+1) = 8.7124$, $\text{Var}(3X+1) = 1.2743$

d $E((2X-1)(3X-2)) = 24.5079$

5 a 13 **b** 18

c 27 **d** 8

6 a –1.5 **b** 1.75

c 1.44 **d** 1.44

e 0.36

7 $E(5T) = 7.5$ minutes, $\text{SD}(5T) = 5.5$ minutes

8 a $m = \frac{3}{4}$

b $E(X) = 1$, $\text{Var}(X) = 0.2$

c $E(5-2X) = 3$, $\text{Var}(5-2X) = 0.8$

9 a $a = 0.6487$

b $E(Z) = 0.2974$, $\text{Var}(Z) = 0.0349$

c i 1.8922 **ii** 0.3141 **iii** 2.1234

10 $a = 3$, $E(Y) = 18$, $\text{Var}(Y) = 18$

11 $a = 1$ or 3, $E(Z) = 1$ or 9, $\text{Var}(Z) = 1$ or 9

12 a $a = \frac{49}{36}$

b $E(Z) = 1.1759$, $\text{Var}(Z) = 0.0109$

c i 0.4722 **ii** 0.0978

13 a $k = 9$ **b** 5.61 mm **c** 10.21 mm

14 a $k = \frac{400}{441}$ **b** 1.081 kg **c** 1.261 kg

15 a $a = 1.7755$

b $E(Z) = 1.4921$, $\text{Var}(Z) = 0.0361$

c $E(3-2Z) = 0.0158$, $\text{Var}(3-2Z) = 0.1444$

16 $a = 0.5469$, $E(Y) = 2.0938$, $\text{Var}(Y) = 2.0938$

13

The normal distribution

13.1 Kick off with CAS

The normal distribution

The normal distribution is a type of continuous probability distribution. It has a distinct bell-shaped curve and is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2}$$

where μ is the mean of the data and σ is the standard deviation.

A normal distribution curve can be sketched in CAS via the inbuilt probability programs or by using the equation of the function.

Exploring the effect of μ on the normal distribution

1 On the same set of axes, sketch the following normal distributions.

  a $\mu = 50, \sigma = 10$

  b $\mu = 20, \sigma = 10$

  c $\mu = 75, \sigma = 10$

2 What effect does changing the value of the mean have on the shape of the normal distribution curve?

Exploring the effect of σ on the normal distribution

3 On the same set of axes, sketch the following normal distributions.

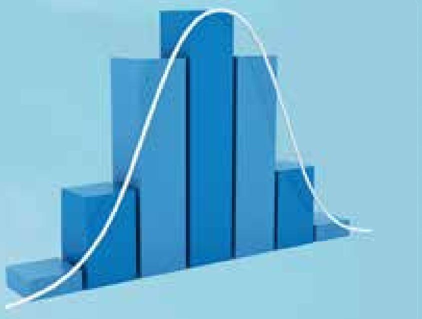
  a $\mu = 50, \sigma = 5$

  b $\mu = 50, \sigma = 10$

  c $\mu = 50, \sigma = 15$

4 What effect does changing the value of the standard deviation have on the shape of the normal curve?

5 Given that the normal distribution curve is a probability density function, can you explain the effect of σ on the graph?



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

13.2 The normal distribution

The **normal distribution** is arguably the most important distribution in statistics. It is characterised by the well-known bell-shaped curve, which is symmetrical about the mean (as well as the median and mode). Continuous random variables such as height, weight, time and other naturally occurring phenomena are frequently analysed with normal distribution calculations.

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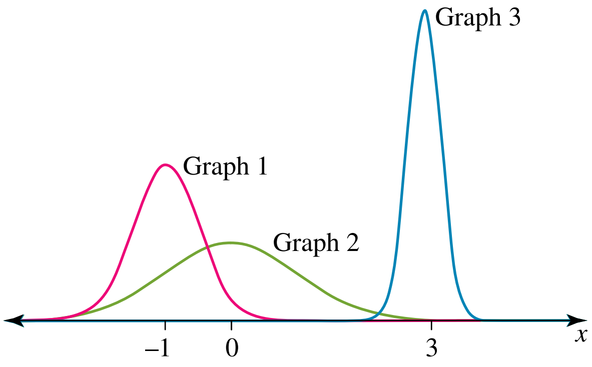
Normal distributions may vary depending on their means and standard deviations. The following diagram shows three different normal distributions.



Graph 1 has mean of −1 and a standard deviation of 0.5

Graph 2 has a mean of 0 and a standard deviation of 1.

Graph 3 has a mean of 3 and a standard deviation of 0.25.

The probability density function for the normal distribution is given by

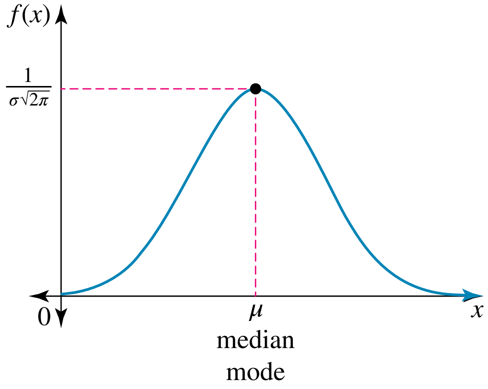
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where the parameters μ and σ are the mean and standard deviation of the distribution respectively.

We say that

$$X \sim \text{N}(\mu, \sigma^2)$$

meaning X is distributed normally with the mean and variance specified.



There are five important characteristics of the normal distribution.

1. A normal distribution is symmetrical about the mean.
2. The mean, median and mode are equal.
3. The area under the curve is equal to 1. That is, $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
4. The majority of the values feature around the centre of the curve with fewer values at the tails of the curve.
5. Normal distributions are defined by two parameters – the mean, μ, and the standard deviation, σ.

As the mean and standard deviation can vary, and the area under the graph must be constant and equal to 1, in effect, changing the mean and the standard deviation transforms the normal curve.

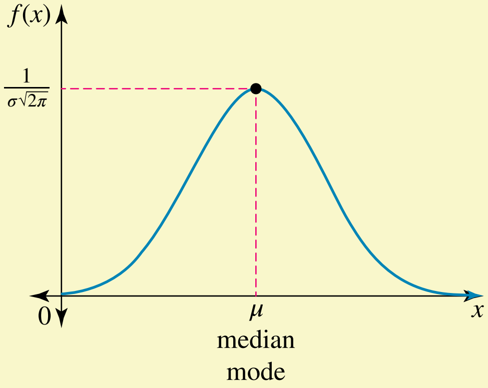
Changing the standard deviation affects the normal curve twofold. The transformed curve will display:

- **dilation by a factor $\frac{1}{\sigma}$ parallel to the y-axis**
- **dilation by a factor σ parallel to the x-axis.**

Changing the mean has the effect of a translation parallel to the x-axis.

The importance of the normal distribution stems from the fact that the distributions of many naturally occurring phenomena can be approximated by this distribution. The pattern was first noticed by astronomers in the seventeenth century. Galileo realised that errors in astronomical observation formed a symmetrical curve and that small errors occurred more frequently than large errors. However, it was not until the nineteenth century that the formula to describe this distribution was developed, by the German mathematician Carl Friedrich Gauss.

In summary, the normal probability density function has the following characteristics:



- $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $x \in R$
- **The distribution is symmetrical about the mean.**
- **μ = median = mode**
- **The maximum value is $\frac{1}{\sigma\sqrt{2\pi}}$ when $x = \mu$.**
- **The curve continues infinitely in both directions.**
- $\int_{-\infty}^{\infty} f(x)dx = 1$

Important intervals and their probabilities

Often we are required to find the proportion of a population for a given interval. Using the property that the symmetry of the normal distribution is about the mean, we are able to predict with certainty the following facts.

- **Approximately 68% of the population will fall within 1 standard deviation of the mean:**

$$\Pr(\mu - \sigma < X < \mu + \sigma) \approx 0.68.$$

- **Approximately 95% of the population will fall within 2 standard deviations of the mean:**

$$\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95.$$

We say that a randomly chosen member of the population will most probably be or is highly likely to be within 2 standard deviations of the mean.

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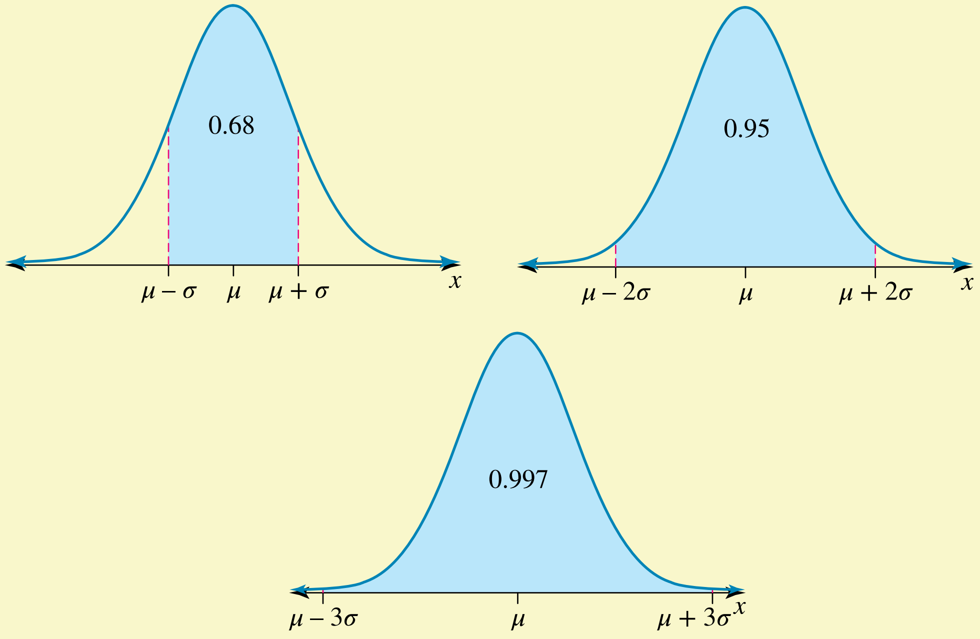
- **Approximately 99.7% of the population will fall within 3 standard deviations of the mean:**

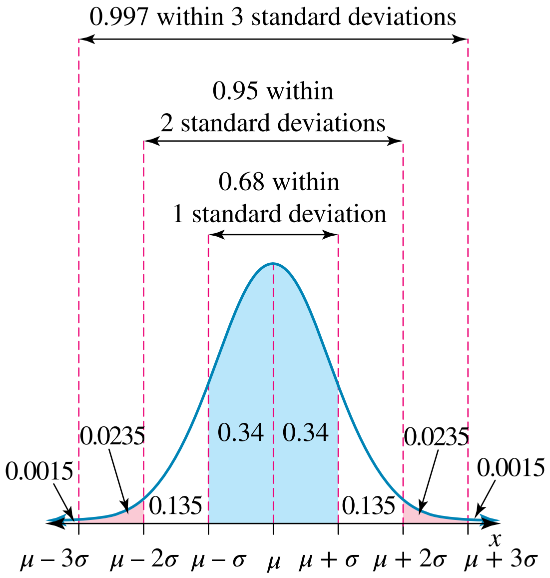
$$\mathbf{Pr(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997.}$$

We say that a randomly chosen member of the population will almost certainly be within 3 standard deviations of the mean.

This is shown on the following graphs.



A more comprehensive breakdown of the proportion of the population for each standard deviation is shown on the graph below.



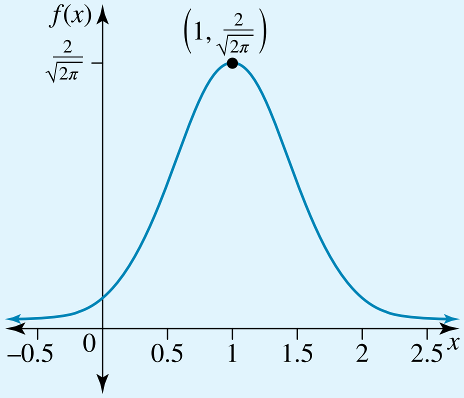
WORKED EXAMPLE 1

The probability density function for a normal distribution is given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(2(x-1))^2}, \ x \in R.$$

a **State the mean and standard deviation of the distribution.**

b **Sketch the graph of the function.**

THINK	WRITE/DRAW
a Use $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ to determine μ and σ.	**a** $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ $= \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}(2(x-1))^2}$ $\frac{1}{\sigma} = 2$, so $\sigma = \frac{1}{2}$ and $\mu = 1$.
b Sketch the graph with a mean of 1 and a standard deviation of 0.5. The x-axis needs to be scaled with markings at μ, $\mu \pm \sigma$, $\mu \pm 2\sigma$ and $\mu \pm 3\sigma$. The peak of the graph must also be labelled with its coordinates.	**b** 

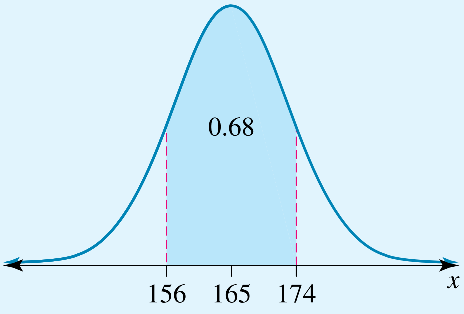
WORKED EXAMPLE 2

The heights of the women in a particular town are normally distributed with a mean of 165 centimetres and a standard deviation of 9 centimetres.

a **What is the approximate probability that a woman chosen at random has a height which is between 156 cm and 174 cm?**

b **What is the approximate probability that a woman chosen at random is taller than 174 cm?**

c **What approximate percentage of the women in this particular town are shorter than 147 cm?**

THINK	WRITE/DRAW
a Determine how many standard deviations from the mean the 156–174 cm range is.	**a** Let X be the height of women in this particular town. $\mu + \sigma = 165 + 9$ $= 174$ $\mu - \sigma = 165 - 9$ $= 156$ Since the range is one standard deviation from the mean, $\Pr(156 \le X \le 174) \approx 0.68$.
b Use the fact that $P(156 \le X \le 174) \approx 0.68$ to calculate the required probability. Sketch a graph to help.	**b** 

Since $\Pr(156 \leq X \leq 174) \approx 0.68$,

$$\begin{aligned}\Pr(X < 156) \cup \Pr(X > 174) &\approx 1 - 0.68\\ &= 0.32\end{aligned}$$

Because of symmetry,

$$\begin{aligned}\Pr(X < 156) &= \Pr(X > 174)\\ &= \frac{0.32}{2}\\ &= 0.16\end{aligned}$$

Thus, $\Pr(X > 174) \approx 0.16$.

c **1** Determine how many standard deviations 147 cm is from the mean.

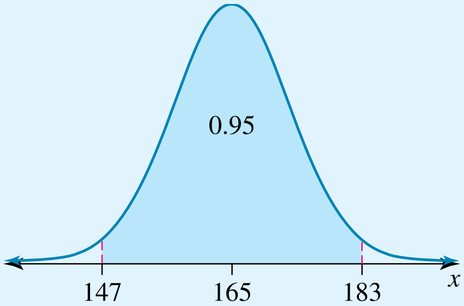
c

$$\begin{aligned}\mu - \sigma &= 165 - 9\\ &= 156\\ \mu - 2\sigma &= 165 - 2 \times 9\\ &= 147\end{aligned}$$

147 cm is 2 standard deviations from the mean. The corresponding upper value is 183 ($165 + 2 \times 9$).

$\Pr(147 \leq X \leq 183) \approx 0.95$

2 Using symmetry, calculate $\Pr(X < 147)$.



Thus,

$$\begin{aligned}\Pr(X < 147) \cup \Pr(X > 183) &\approx 1 - 0.95\\ &= 0.05\end{aligned}$$

and by symmetry,

$$\begin{aligned}\Pr(X < 147) = \Pr(X > 183) &\approx \frac{0.05}{2}\\ &= 0.025\end{aligned}$$

Thus, approximately 2.5% of the population of women in this particular town are shorter than 147 cm.

EXERCISE 13.2 The normal distribution

PRACTISE

Work without CAS

1 **WE1** The probability density function of a normal distribution is given by

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - 2}{3}\right)^2}.$$

a State the mean and the standard deviation of the distribution.

b Sketch the graph of the probability function.

2 A normal distribution has a probability density function of

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x - 3)^2}.$$

a Using CAS technology, verify that $\int_{-\infty}^{\infty} f(x)dx = 1$.

b State μ and σ.

c Sketch the graph of the probability function.

3 **WE2** The results of a Mathematical Methods test are normally distributed with a mean of 72 and a standard deviation of 8.

a What is the approximate probability that a student who sat the test has a score which is greater than 88?

b What approximate proportion of the students who sat the test had a score which was less than 48?

c What approximate percentage of the students who sat the test scored less than 80?

4 The length of pregnancy for a human is normally distributed with a mean of 275 days and a standard deviation of 14 days. A mother gave birth after less than 233 days. What is the approximate probability of this happening for the general population?

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 Consider the normal probability density function

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+2}{4}\right)^2}, x \in R.$$

a Using CAS technology, verify that $\int_{-\infty}^{\infty} f(x)dx = 1$.

b State μ.

6 A normal probability density function is defined by

$$f(x) = \frac{10}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{10(x-1)}{3}\right)^2}, x \in R.$$

a Find the values of μ and σ.

b State what effect the mean and standard deviation have on the graph of the normal distribution.

c Sketch the graph of the function, f.

7 A normal probability density function is given by

$$f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2}, x \in R.$$

a Find the values of μ and σ.

b State what effect the mean and standard deviation have on the graph of the normal distribution.

c Determine:

i Var(X)

ii E(X^2).

d Verify that this is a probability density function.

8 $f(x) = \frac{5}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{5(x-2)}{2}\right)^2}$, $x \in R$ defines a normal probability density function.

a Find the values of μ and σ.

b Calculate E(X^2).

c Determine:

i E($5X$)

ii E($5X^2$).

9 Scores on a commonly used IQ test are known to be normally distributed with a mean of 120 and a standard deviation of 20.

a Determine:

i $\mu \pm \sigma$ **ii** $\mu \pm 2\sigma$ **iii** $\mu \pm 3\sigma$.

b Find:

i $\Pr(X < 80)$ **ii** $\Pr(X > 180)$.

10 The results of a Year 12 Biology examination are known to be normally distributed with a mean of 70 and a standard deviation of 6. What approximate percentage of students sitting for this examination can be expected to achieve a score that is greater than 88?

11 A continuous random variable, X, is known to be normally distributed with a mean of 15 and a standard deviation of 5. Find the range between which approximately:

a 68% of the values lie

b 95% of the values lie

c 99.7% of the values lie.

12 A normal probability density function, X, has a mean of 24 and a standard deviation of 7. Find the approximate values for:

a $\Pr(X < 31)$

b $\Pr(10 < X < 31)$

c $\Pr(X > 10 | X < 31)$.

13 The number of pears harvested from each tree in a large orchard is normally distributed with a mean of 230 and a standard deviation of 25. Find the approximate probability that the number of pears harvested from a randomly selected tree is:

a less than 280

b between 180 and 280

c is greater than 180, given that less than 280 pears were harvested.

14 The annual rainfall in a particular area of Australia, X mm, is known to be normally distributed with a mean of 305 mm and a standard deviation of 50 mm.

a Calculate the approximate value of $\Pr(205 < X < 355)$.

b Find k such that $\Pr(X < k) \approx 0.025$.

c Find h such that $\Pr(X < h) \approx 0.0015$.

MASTER

15 A normally distributed probability density function is given by

$$f(x) = \frac{5}{\sqrt{2\pi}} e^{-\frac{1}{2}(5(x-1))^2}, x \in R.$$

a Calculate $\text{Var}(X)$, giving your answer correct to 2 decimal places.

b Calculate $\text{E}(X^2)$, giving your answer correct to 2 decimal places.

c Find:

i $\text{E}(2X + 3)$

ii $\text{E}((X + 1)(2X - 3))$.

16 A continuous random variable, X, is normally distributed with a mean of 72.5 and a standard deviation of 8.4. Find the approximate values for:

a $\Pr(64.1 < X < 89.3)$

b $\Pr(X < 55.7)$

c $\Pr(X > 47.3 | X < 55.7)$

d m such that $\Pr(X > m) \approx 0.16$.

13.3 Calculating probabilities and the standard normal distribution

The standard normal distribution

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Suppose we are comparing the results of two students on two similar IQ tests. Michelle obtained 92 on one IQ test, for which the results were known to be normally distributed with a mean of 80 and a standard deviation of 6. Samara obtained 88 on a similar IQ test, for which the results were known to be normally distributed with a mean of 78 and a standard deviation of 10. Which student was the most successful?

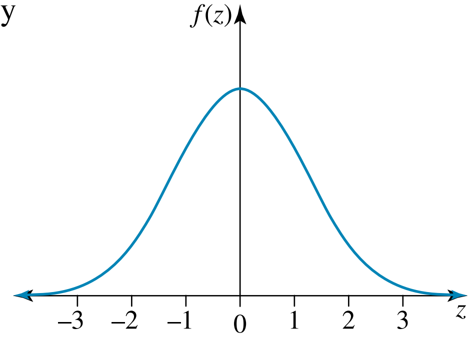
This question is very difficult to answer unless we have some common ground for a comparison. This can be achieved by using a transformed or standardised form of the normal distribution called the **standard normal distribution**. The variable in a standard normal distribution is always denoted by Z, so that it is immediately understood that we are dealing with the standard normal distribution. The standard normal distribution always has a mean of 0 and a standard deviation of 1, so that Z indicates how many standard deviations the corresponding X-value is from the mean. To find the value of Z, we find the difference between the x-value and the mean, $x - \mu$. To find how many standard deviations this equals, we divide by the standard deviation, σ.

$$\boxed{z = \frac{x - \mu}{\sigma}}$$

Therefore, if $z = \frac{x - \mu}{\sigma}$, $\mu = 0$ and $\sigma = 1$, the probability density function is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, z \in R.$$

Remember that $\mu \pm 3\sigma$ encompasses approximately 99.7% of the data, so for the standard normal curve, these figures are $0 \pm 3 \times 1 = 0 \pm 3$. Therefore, approximately 99.7% of the data lies between -3 and 3.



For the standard normal distribution, we say $Z \sim \text{N}(0,\ 1)$.

Let us return to the comparison between Michelle and Samara.

For Michelle: $X \sim \text{N}(80,\ 6^2)$, $z = \frac{x - \mu}{\sigma}$

$$= \frac{92 - 80}{6}$$
$$= \frac{12}{6}$$
$$= 2$$

For Samara: $X \sim \text{N}(78,\ 10^2)$, $z = \frac{x - \mu}{\sigma}$

$$= \frac{88 - 78}{10}$$
$$= \frac{10}{10}$$
$$= 1$$

Michelle's mark lies within 2 standard deviations of the mean, so it lies in the top 2.5%, whereas Samara's mark is 1 standard deviation from the mean, so it is in the top 16%. Hence, Michelle performed better than Samara.

Obviously, not all data values will lie exactly 1, 2 or 3 standard deviations from the mean. In these cases technology such as a CAS calculator is needed to calculate the required probability. CAS can be used to calculate probabilities associated with the normal distribution for any value of μ and σ.

WORKED EXAMPLE 3

a **Calculate the values of the following probabilities, correct to 4 decimal places.**

i **$\Pr(Z < 2.5)$** **ii** **$\Pr(-1.25 \leq Z \leq 1.25)$**

b **X is a normally distributed random variable such that $X \sim N(25, 3^2)$.**

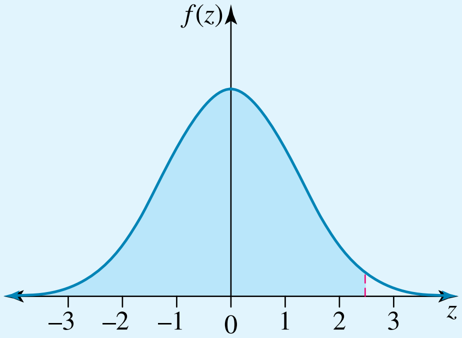
i **Calculate $\Pr(X > 27)$ correct to 4 decimal places.**

ii **Convert X to a standard normal variable, Z.**

THINK / **WRITE/DRAW**

a i 1 Sketch a graph to help understand the problem.

a i



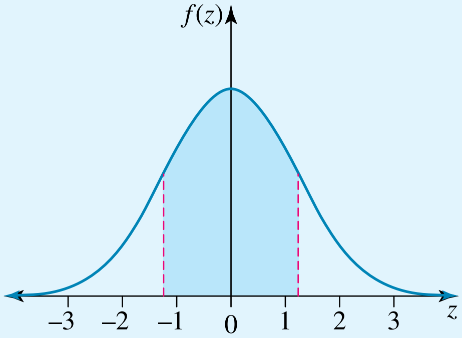
2 Use CAS to find the probability.

The upper limit is 2.5 and the lower limit is –∞.
The mean is 0 and the standard deviation is 1.

$\Pr(z < 2.5) = 0.9938$

ii 1 Sketch a graph to help understand the problem.

ii



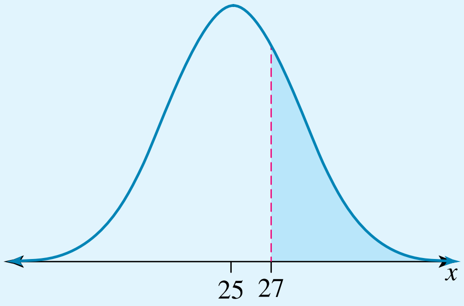
2 Use CAS to find the probability.

The upper limit is 1.25 and the lower limit is –1.25.

$\Pr(-1.25 < Z < 1.25) = 0.7887$

b i 1 Sketch a graph to help understand the problem.

b i



2 Use CAS to find the probability. The upper limit is ∞ and the lower limit is 27. The mean is 25 and the standard deviation is 3.

$\Pr(X > 27) = 0.2525$

ii 1 Write the rule to standardise X.

ii $z = \dfrac{x - \mu}{\sigma}$

2 Substitute the mean and standard deviation.

$$z = \frac{27 - 25}{3}$$
$$= \frac{2}{3}$$

EXERCISE 13.3 Calculating probabilities and the standard normal distribution

PRACTISE

1 **WE3** a Calculate the values of the following probabilities correct to 4 decimal places.

   i $\Pr(Z < 1.2)$
   ii $\Pr(-2.1 < Z < 0.8)$

   b X is a normally distributed random variable such that $X \sim N(45, 6^2)$.

   i Calculate $\Pr(X > 37)$ correct to 4 decimal places.
   ii Convert X to a standard normal variable, Z.

2 For a particular type of laptop computer, the length of time, X hours, between charges of the battery is normally distributed such that

$$X \sim N(50, 15^2).$$

Find $\Pr(50 < X < 70)$.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 If $Z \sim N(0, 1)$, find:

   a $\Pr(Z \leq 2)$
   b $\Pr(Z \leq -2)$
   c $\Pr(-2 < Z \leq 2)$
   d $\Pr(Z > 1.95) \cup \Pr(Z < -1.95)$.

4 Convert the variable in the following expressions to a standard normal variable, Z, and use it to write an equivalent expression. Use your calculator to evaluate each probability.

   a $\Pr(X < 61)$, $X \sim N(65, 9)$
   b $\Pr(X \geq 110)$, $X \sim N(98, 225)$
   c $\Pr(-2 < X \leq 5)$, $X \sim N(2, 9)$.

5 A radar gun is used to measure the speeds of cars on a freeway.

The speeds are normally distributed with a mean of 98 km/h and a standard deviation of 6 km/h. What is the probability that a car picked at random is travelling at:

   a more than 110 km/h
   b less than 90 km/h
   c a speed between 90 km/h and 110 km/h?

6 A large number of students took a test in Physics. Their final grades have a mean of 72 and a standard deviation of 12. If the distribution of these grades can be approximated by a normal distribution, what percentage of students, correct to 2 decimal places:

a gained a score of more than 95

b should pass the test if grades greater than or equal to 55 are considered passes?

7 X is a continuous random variable and is known to be normally distributed.

a If $\Pr(X < a) = 0.35$ and $\Pr(X < b) = 0.62$, find:

i $\Pr(X > a)$ **ii** $\Pr(a < X < b)$.

b If $\Pr(X < c) = 0.27$ and $\Pr(X < d) = 0.56$, find:

i $\Pr(c < X < d)$ **ii** $\Pr(X > c \mid X < d)$.

c A random variable, X, is normally distributed with a mean of 20 and a standard deviation of 5.

i Find k if $\Pr(X > 32) = \Pr(Z > k)$.

ii Find k if $\Pr(X < 12) = \Pr(Z > k)$.

8 Jing Jing scored 85 on the mathematics section of a scholarship examination, the results of which were known to be normally distributed with a mean of 72 and a standard deviation of 9. Rani scored 18 on the mathematics section of a similar examination, the results of which were normally distributed with a mean of 15 and a standard deviation of 4. Assuming that both tests measure the same kind of ability, which student has the higher score?

9 A salmon farm in Tasmania has a very large number of salmon in its ponds.

It is known that the lengths of the salmon from this farm are normally distributed with a mean of 38 cm and a standard deviation of 2.4 cm. A randomly chosen fish from this farm was measured as 39.5 cm. If salmon with lengths in the top 15% are considered to be gourmet salmon, determine whether this particular fish can be classified as gourmet.

10 The results by Justine in Chemistry, Mathematical Methods and Physics are shown in the table below. The marks, X, the mean, μ, and standard deviation, σ, for each examination are given.

Subject	Mark, X	Mean, μ	Standard deviation, σ	Standardised mark, Z
Chemistry	72	68	5	
Maths Methods	75	69	7	
Physics	68	61	8	

Complete the table by finding Justine's standardised mark for each subject and use this to determine in which subject she did best when compared to her peers.

11 Teresa has taken her pulse each day for a month after going for a brisk walk. Her pulse rate in beats per minute is known to be normally distributed with a mean of 80 beats per minute and a standard deviation of 5 beats per minute. After her most recent walk she took her pulse rate. What is the probability that her pulse rate was:

- **a** in excess of 85 beats per minute
- **b** equal to or less than 75 beats per minute
- **c** between 78 and 82 beats per minute, given that it was higher than 75 beats per minute?

12 The labels on packets of sugar say the bags have a weight of 1 kg. The actual mean weight of the bags is 1.025 kg in order to minimise the number of bags which may be underweight. If the weight of the bags is normally distributed with a standard deviation of 10 g, find the percentage of bags that would be expected to weigh:

- **a** more than 1.04 kg
- **b** less than 996 g, the legal meaning of underweight?

MASTER

13 If $Z \sim \text{N}(0,\ 1)$, find:

- **a** $\Pr(Z \geq 2.125) \cup \Pr(Z < -2.125)$
- **b** $\Pr(X < 252.76)$ if $\mu = 248.55$, $\sigma = 24.45$ and X is normally distributed
- **c** $\Pr(-3.175 \leq Z \leq 1.995)$
- **d** $\Pr(X < 5.725)$ if $\mu = 7.286$, $\sigma = 3.115$ and X is normally distributed.

14 A continuous random variable, Z, has a probability density function defined by $f(z) = 0.025e^{-0.025z}$, $z \geq 0$.

A second continuous random variable, Y, is distributed normally with a mean of 25 and a standard deviation of 3. In order to find k such that

$$\Pr(Z > k) = \Pr(Y < k),$$

$\int_{k}^{\infty} f(z)\,dz = \int_{-\infty}^{k} g(y)dy$ must be solved. Find the value of k.

13.4 The inverse normal distribution

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AOS 4

Topic 4

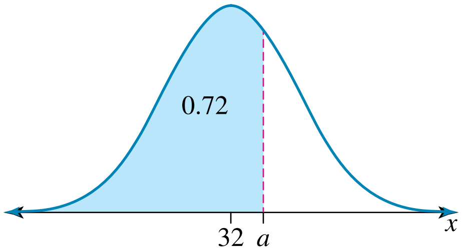
Concept 6

Inverse cumulative normal distribution

Concept summary

Practice questions

CAS technology provides an easy way to find a Z or X value, given a probability for a normal distribution. Suppose X is normally distributed with a mean of 32 and a standard deviation of 5. We wish to find $\Pr(X \leq a) = 0.72$.



The key information to enter into your calculator is the known probability, that is, the area under the curve. It is essential to input the correct area so that your calculator knows if you are inputting the ‘less than’ area or the ‘greater than’ area.

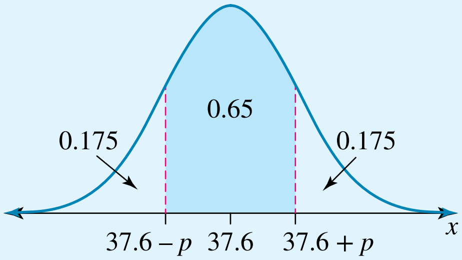
WORKED EXAMPLE 4

If X is a normally distributed random variable, find:

a **m given that $\Pr(X \le m) = 0.85$, $X \sim N(15.2, 1.5^2)$**

b **n given that $\Pr(X > n) = 0.37$, $X \sim N(22, 2.75^2)$**

c **p given that $\Pr(37.6 - p \le X \le 37.6 + p) = 0.65$, $X \sim N(37.6, 12^2)$.**

THINK	WRITE/DRAW
a Use the probability menus on the CAS calculator to find the required X value.	**a** $\Pr(X \le m) = 0.85$, $\mu = 15.2$, $\sigma = 1.5$ $m = 16.7547$
b Use the probability menus on the CAS calculator to find the required X value. *Note*: It may be a requirement to input the 'less than' area, so $\Pr(X < n) = 1 - 0.37$ $= 0.63$	**b** $\Pr(X > n) = 0.37$, $\mu = 22$, $\sigma = 2.75$ $n = 22.9126$
c **1** Sketch a graph to visualise the problem. Due to symmetry, the probabilities either side of the upper and lower limits can be calculated.	**c**  $1 - 0.65 = 0.35$ $\Pr(X < 37.6-p) = \Pr(X > 37.6+p)$ $= \frac{0.35}{2}$ $= 0.175$
2 Determine p by finding X given that $\Pr(X < 37.6 - p) = 0.175$. *Note: p* could also be found by using the upper limit.	$\Pr(X < 37.6 - p) = 0.175$ $37.6 - p = 26.38$ $p = 37.6 - 26.38$ $= 11.22$

Quantiles and percentiles

Quantiles and percentiles are terms that enable us to convey information about a distribution. Quantiles refer to the value below which there is a specified probability that a randomly selected value will fall. For example, to find the 0.7 quantile of a standard normal distribution, we find a such that $\Pr(Z < a) = 0.7$.

Percentiles are very similar to quantiles. For the example of $\Pr(Z < a) = 0.7$, we could also be asked to find the 70th percentile for the standard normal distribution.

Finding the mean or standard deviation

If the mean or standard deviation is unknown, then the known probability needs to be linked to the standard normal distribution and the corresponding *z*-value calculated via CAS. Once the *z*-value has been found, the missing mean or standard deviation can be calculated via the rule $z = \frac{x - \mu}{\sigma}$.

WORKED EXAMPLE 5

a For the normally distributed variable X, the 0.15 quantile is 1.9227 and the mean is 2.7. Find the standard deviation of the distribution.

b X is normally distributed so that the 63rd percentile is 15.896 and the standard deviation is 2.7. Find the mean of X.

THINK	WRITE
a 1 Write the probability statement.	**a** The 0.15 quantile is 1.9227. $\Pr(X < 1.9227) = 0.15$
2 Find the corresponding standardised value, Z, by using CAS.	$\Pr(Z < z) = 0.15$ $z = -1.0364$
3 Write the standardised formula connecting z and x.	$z = \frac{x - \mu}{\sigma}$
4 Substitute the appropriate values and solve for σ.	$-1.0364 = \frac{1.9227 - 2.7}{\sigma}$ $-1.0364\sigma = -0.7773$ $\sigma = 0.75$
b 1 Write the probability statement.	**b** The 63rd percentile is 15.896. $\Pr(X < 15.896) = 0.63$
2 Find the corresponding standardised value, Z, by using CAS.	$\Pr(Z < z) = 0.63$ $z = 0.3319$
3 Write the standardised formula connecting z and x.	$z = \frac{x - \mu}{\sigma}$
4 Substitute in the appropriate values and solve for μ.	$0.3319 = \frac{15.896 - \mu}{2.7}$ $0.8960 = 15.896 - \mu$ $\mu = 15$

EXERCISE 13.4 The inverse normal distribution

PRACTISE

1 **WE4** Find the value of a, correct to 2 decimal places, if X is normally distributed and:

a $\Pr(X \leq a) = 0.16,\ X \sim \text{N}(41, 6.7^2)$

b $\Pr(X > a) = 0.21,\ X \sim \text{N}(12.5, 2.7^2)$

c $\Pr(15 - a \leq X \leq 15 + a) = 0.32,\ X \sim \text{N}(15, 4^2)$.

2 Find the values of m and n if X is normally distributed and $\Pr(m \leq X \leq n) = 0.92$ when $\mu = 27.3$ and $\sigma = 8.2$. The specified interval is symmetrical about the mean.

3 **WE5** X is distributed normally with a mean of 112, and the 42nd percentile is 108.87. Find the standard deviation of the distribution, correct to 1 decimal place.

4 X is a normally distributed random variable such that $X \sim \mathrm{N}(\mu, 4.45^2)$. If the 0.11 quantile is 32.142, find the value of μ, correct to 1 decimal place.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

5 If $Z \sim \mathrm{N}(0, 1)$ find the value of z given that:

a $\Pr(Z < z) = 0.39$

b $\Pr(Z \geq z) = 0.15$

c $\Pr(-z < Z < z) = 0.28$.

6 If $X \sim \mathrm{N}(37.5, 8.62^2)$, find a correct to 2 decimal places such that:

a $\Pr(X < a) = 0.72$

b $\Pr(X > a) = 0.32$

c $\Pr(37.5 - a < X < 37.5 + a) = 0.88$.

7 For a standard normal distribution, find:

a the 0.57 quantile

b the 63rd percentile.

8 If X is distributed normally with $\mu = 43.5$ and $\sigma = 9.7$, find:

a the 0.73 quantile

b the 24th percentile.

9 X is distributed normally with a standard deviation of 5.67, and $\Pr(X > 20.952) = 0.09$. Find the mean of X, giving your answer correct to 2 decimal places.

10 X is distributed normally with a standard deviation of 3.5, and $\Pr(X < 23.96) = 0.28$. Find the mean for X, rounded to the nearest whole number.

11 $X \sim \mathrm{N}(115, \sigma^2)$ and the 76th percentile is 122.42. Find the value of σ, giving your answer correct to 1 decimal place.

12 X is distributed normally with $\mu = 41$ and $\Pr(X > 55.9636) = 0.11$. Find σ, giving your answer correct to 1 decimal place.

13 X is distributed normally and $\Pr(X < 33.711) = 0.36$ while $\Pr(X < 34.10) = 0.42$. Find the mean and the standard deviation of X, giving your answer correct to 1 decimal place.

14 X is distributed normally such that $\Pr(X > 18.35) = 0.31$ and the 45th percentile is 15.09. Find μ and σ for X, giving your answer correct to 1 decimal place.

MASTER

15 $\Pr(a < X < b) = 0.52$ and the specified interval is symmetrical about the mean. If X is normally distributed with a mean of 42.5 and a standard deviation of 10.3, find $\Pr(X > a \mid X < b)$.

16 X is distributed normally such that $\Pr(X < 39.9161) = 0.5789$ and $\Pr(X > 38.2491) = 0.4799$. Find the mean and the standard deviation of X, giving your answers correct to 2 decimal places.

13.5 Mixed probability application problems

Application problems involving the normal distribution cover a wide range of topics. Such questions will not only incorporate theory associated with the normal distribution but may also include other areas of probability you have previously studied.

WORKED EXAMPLE 6

The amount of instant porridge oats in packets packed by a particular machine is normally distributed with a mean of μ grams and a standard deviation of 6 grams. The advertised weight of a packet is 500 grams.

a **Find the proportion of packets that will be underweight (less than 500 grams) when $\mu = 505$ grams.**

b **Find the value of μ required to ensure that only 1% of packets are under-weight.**

c **As a check on the setting of the machine, a random sample of 5 boxes is chosen and the setting is changed if more than one of them is under-weight. Find the probability that the setting on the machine is changed when $\mu = 505$ grams.**

THINK	WRITE
a 1 Rewrite the information in the question using appropriate notation.	**a** X is the amount of instant porridge oats in a packet and $X \sim \text{N}(505, 6^2)$.
2 Use CAS to find $\Pr(X < 500)$.	$\Pr(X < 500) = 0.2023$
b 1 State the known probability.	**b** $\Pr(X < 500) = 0.01$
2 Find the corresponding standardised value, Z, by using CAS.	$\Pr(Z < z) = 0.01$ $z = -2.3263$
3 Write the standardised formula connecting z and x.	$z = \dfrac{x - \mu}{\sigma}$
4 Substitute the appropriate values and solve for μ.	$-2.3263 = \dfrac{500 - \mu}{6}$ $-13.9581 = 500 - \mu$ $\mu = 513.96$ g
c 1 The wording of the question (sample of 5 boxes) indicates that this is now a binomial distribution. Rewrite the information in the question using appropriate notation.	**c** Let Y = the number of underweight packets. $Y \sim \text{Bi}(5, 0.2023)$
2 Using CAS, calculate the probability.	$\Pr(Y > 1) = 1 - \Pr(Y \leq 1)$ $= 1 - 0.7325$ $= 0.2674$

EXERCISE 13.5 Mixed probability application problems

PRACTISE

1 **WE6** Packages of butter with a stated weight of 500 grams have an actual weight of W grams, which is normally distributed with a mean of 508 grams.

a If the standard deviation of W is 3.0 grams, find:

i the proportion of packages that weigh less than 500 grams

ii the weight that is exceeded by 99% of the packages.

- **b** If the probability that a package weighs less than 500 grams is not to exceed 0.01, find the maximum allowable standard deviation of W.

2 Chocolate Surprise is a toy that is packed inside an egg-shaped chocolate. A certain manufacturer provides four different types of Chocolate Surprise toy — a car, an aeroplane, a ring and a doll — in the proportions given in the table.

Toy	Proportion
Car	$3k^2 + 2k$
Aeroplane	$6k^2 + 2k$
Ring	$k^2 + 2k$
Doll	$3k$

- **a** Show that k must be a solution to the equation $10k^2 + 9k - 1 = 0$.
- **b** Find the value of k.

In response to customer demand, the settings on the machine that produce Chocolate Surprise have been changed so that 25% of all Chocolate Surprises produced contain rings. A sample of 8 Chocolate Surprises is randomly selected from a very large number produced by the machine.

- **c** What is the expected number of Chocolate Surprises that contain rings? Give your answer correct to the nearest whole number.
- **d** What is the probability, correct to 4 decimal places, that this sample has exactly 2 Chocolate Surprises that contain rings?
- **e** What is the smallest sample size that should be taken so that the probability of selecting no Chocolate Surprise that contain a ring is at most 0.09?

A Chocolate Surprise is considered defective if it weighs less than 100 grams. The weight of a Chocolate Surprise is known to be normally distributed with a mean of 125 grams.

- **f** If 8.2% of the Chocolate Surprises produced are defective, find, to the nearest gram, the standard deviation for the weight of the Chocolate Surprises.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

3 A particular brand of car speedometer was tested for accuracy. The error measured is known to be normally distributed with a mean of 0 km/h and a standard deviation of 0.76 km/h.

Speedometers are considered unacceptable if the error is more than 1.5 km/h. Find the proportion of speedometers which are unacceptable.

4 The heights of adult males in Perth can be taken as normally distributed with a mean of 174 cm and a standard deviation of 8 cm. Suppose the West Australian Police Force accepts recruits only if they are at least 180 cm tall.

- **a** What percentage of Perth adult males satisfy the height requirement for the West Australian Police Force?
- **b** What minimum height, to the nearest centimetre, would the West Australian Police Force have to accept if it wanted a quarter of the Perth adult male population to satisfy the height requirement?

5 **a** Farmer David grows avocados on a farm on Mount Tamborine, Queensland.

The average weight of his avocados is known to be normally distributed with a mean weight of 410 grams and a standard deviation of 20 grams.

i Find the probability that an avocado chosen at random weighs less than 360 grams.

ii Find the probability that an avocado that weighs less than 360 grams weighs more than 340 grams.

b Farmer Jane grows avocados on a farm next to farmer David's. If Y represents the average weight of Jane's avocados, the weights of which are also normally distributed where $\Pr(Y < 400) = 0.4207$ and $\Pr(Y > 415) = 0.3446$, find the mean and standard deviation of the weights of Jane's avocados. Give answers correct to the nearest integer.

6 A manufacturer produces metal rods whose lengths are normally distributed with a mean of 145.0 cm and a standard deviation 1.4 cm.

a Find the probability, correct to 4 decimal places, that a randomly selected metal rod is longer than 146.5 cm.

b A metal rod has a size fault if its length is not within d cm either side of the mean. The probability of a metal rod having a size fault is 0.15. Find the value of d, giving your answer correct to 1 decimal place.

c A random sample of 12 metal rods is taken from a crate containing a very large number of metal rods. Find the probability that there are exactly 2 metal rods with a size fault, giving your answer correct to 4 decimal places.

d The sales manager is considering what price, x dollars, to sell each of the metal rods for, whether they are good or have some kind of fault. The materials cost is \$5 per rod. The metal rods are sorted into three bins. The staff know that 15% of the manufactured rods have a size fault and another 17% have some other fault. The profit, Y dollars, is a random variable whose probability distribution is shown in the following table.

Bin	Description	Profit (\$$y$)	$\Pr(Y = y)$
A	Good metal rods that are sold for x dollars each	$x - 5$	a
B	Metal rods with a size fault — these are not sold but recycled.	0	0.15
C	Metal rods with other faults — these are sold at a discount of \$3 each.	$x - 8$	0.17

i Find the value of a, correct to 2 decimal places.

ii Find the mean of Y in terms of x.

iii Hence or otherwise, find, correct to the nearest cent, the selling price of good rods so that the mean profit is zero.

iv The metal rods are stored in the bins until a large number is ready to be sold. What proportion of the rods ready to be sold are good rods?

7 A company sells two different products, X and Y, for \$5.00 and \$6.50 respectively. Regular markets exist for both products, with sales being normally distributed and averaging 2500 units (standard deviation 700) and 3000 units (standard deviation 550) respectively each week. It is company policy that if in any one week the market for a particular product falls below half the average, that product is advertised as a 'special' the following week.

- **a** Find the probability, correct to 4 decimal places, that product X will be advertised as a 'special' next week.
- **b** Find the probability, correct to 4 decimal places, that product Y will be advertised as a 'special' next week.
- **c** Find the probability, correct to 4 decimal places, that both products will be advertised as a 'special' next week.
- **d** If 40% of the company's product is product X and 60% is product Y, find the probability that:
  - **i** one product is a 'special'
  - **ii** if one product is advertised as 'special', then it is product X.

8 The height of plants sold at a garden nursery supplier are normally distributed with a mean of 18 cm and a standard deviation of 5 cm.

- **a** Complete the following table by finding the proportions for each of the three plant sizes, correct to 4 decimal places.

Description of plant	Plant size (cm)	Cost in \$	Proportion of plants
Small	Less than 10 cm	2.00	
Medium	10–25 cm	3.50	
Large	Greater than 25 cm	5.00	

- **b** Find the expected cost, to the nearest dollar, for 150 plants chosen at random from the garden nursery.

9 A fruit grower produces peaches whose weights are normally distributed with a mean of 185 grams and a standard deviation of 20 grams.

Peaches whose weights exceed 205 grams are sold to the cannery, yielding a profit of 60 cents per peach. Peaches whose weights are between 165 grams and 205 grams are sold to wholesale markets at a profit of 45 cents per peach. Peaches whose weights are less than 165 grams are sold for jam at a profit of 30 cents per peach.

- **a** Find the percentage of peaches sold to the canneries.
- **b** Find the percentage of peaches sold to the wholesale markets.
- **c** Find the mean profit per peach.

10 The Lewin Tennis Ball Company makes tennis balls whose diameters are distributed normally with a mean of 70 mm and a standard deviation of 1.5 mm. The tennis balls are packed and sold in cylindrical tins that each hold five tennis balls. A tennis ball fits in the tin if the diameter is less than 71.5 mm.

a What is the probability, correct to 4 decimal places, that a randomly chosen tennis ball produced by the Lewin company fits into the tin?

The Lewin management would like each ball produced to have a diameter between 68.6 mm and 71.4 mm.

b What is the probability, correct to 4 decimal places, that a randomly chosen tennis ball produced by the Lewin company is in this range?

c A tin of five balls is selected at random. What is the probability, correct to 4 decimal places, that at least one ball has a diameter outside the range of 68.6 mm to 71.4 mm?

Lewin management wants engineers to change the manufacturing process so that 99.5% of all balls produced have a diameter between 68.6 mm and 71.4 mm. The mean is to stay at 70 mm but the standard deviation is to be changed.

d What should the new standard deviation be, correct to 4 decimal places?

11 The Apache Orchard grows a very juicy apple called the Fugee apple. Fugee apples are picked and then sorted by diameter in three categories:

- small — diameter less than 60 mm
- jumbo — the largest 15% of the apples
- standard — all other apples.

Diameters of Fugee apples are found to be normally distributed with a mean of 71 mm and a standard deviation of 12 mm.

a A particular apple is the largest possible whose diameter lies within two standard deviations of the mean. What is the diameter? Give your answer correct to the nearest millimetre.

b Find, correct to 4 decimal places, the probability that a Fugee apple, selected at random, has a diameter less than 85 mm.

c What percentage of apples (to the nearest 1 per cent) is sorted into the small category?

d Find, correct to the nearest millimetre, the minimum diameter of a jumbo Fugee.

e An apple is selected at random from a bin of jumbo apples. What is the probability, correct to 4 decimal places, that it has a diameter greater than 100 mm?

f The Apache Orchard receives the following prices for Fugee apples:

- small — 12 cents each
- standard — 15 cents each
- jumbo — 25 cents each.

What is the expected income, correct to the nearest dollar, for a container of 2500 unsorted apples?

g Some apples are selected before sorting and are packed into bags of six to be sold at the front gate of the orchard. Find the probability, correct to 4 decimal places, that one of these bags contains at least two jumbo apples.

12 A brand of disinfectant is sold in two sizes: standard and large. For each size, the contents, in litres, of a randomly chosen bottle is normally distributed with a mean and standard deviation as shown in the following table.

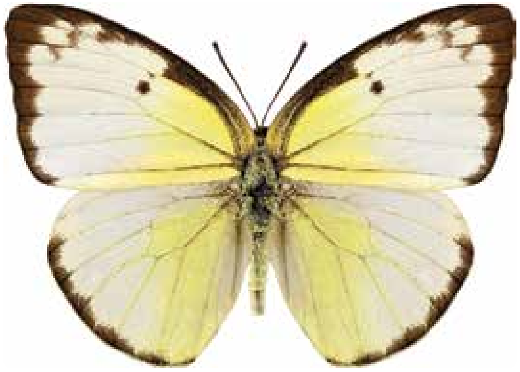
Bottle size	Mean	Standard deviation
Standard	0.765 L	0.007 L
Large	1.015 L	0.009 L

a Find the probability, correct to 4 decimal places, that a randomly chosen standard bottle contains less than 0.75 litres.

b Find the probability that a box of 12 randomly chosen large bottles contains at least 4 bottles whose contents are each less than 1 litre.

MASTER

13 Amalie is gathering data on two particular species of yellow butterflies: the lemon emigrant and the yellow emigrant, which can be very difficult to tell apart as the intensity of the yellow can be confusing. Both species are equally likely to be caught in a particular area of Australia. One technique for telling them apart is by measuring the lengths of their antennae. For the lemon emigrant, the antennae are distributed normally with a mean of 22 mm and a standard deviation of 1.5 mm.



a Find the probability, correct to 4 decimal places, that a randomly chosen lemon emigrant butterfly will have antennae which are shorter than 18 mm.

b Amalie knows that 8% of the yellow emigrants have antennae which are shorter than 15.5 mm, and 8% of yellow emigrant butterflies have antennae which are longer than 22.5 mm. Assuming that the antenna lengths are normally distributed, find the mean and standard deviation of the antenna length of yellow emigrant butterflies, giving your answers correct to the nearest 0.1 mm.

In the region where Amalie is hunting for yellow butterflies, 45% of the yellow butterflies are lemon emigrants and 55% are lemon emigrants.

c Find the probability, correct to 4 decimal places, that a random sample of 12 butterflies from the region will contain 5 yellow emigrant butterflies.

14 The daily error (in seconds) of a particular brand of clock is known to be normally distributed. Only those clocks with an error of less than 3 seconds are acceptable.

a Find the mean and standard deviation of the distribution of error if 2.5% of the clocks are rejected for losing time and 2.5% of the clocks are rejected for gaining time.

b Determine the probability that fewer than 2 clocks are rejected in a batch of 12 such clocks.

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13.6 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

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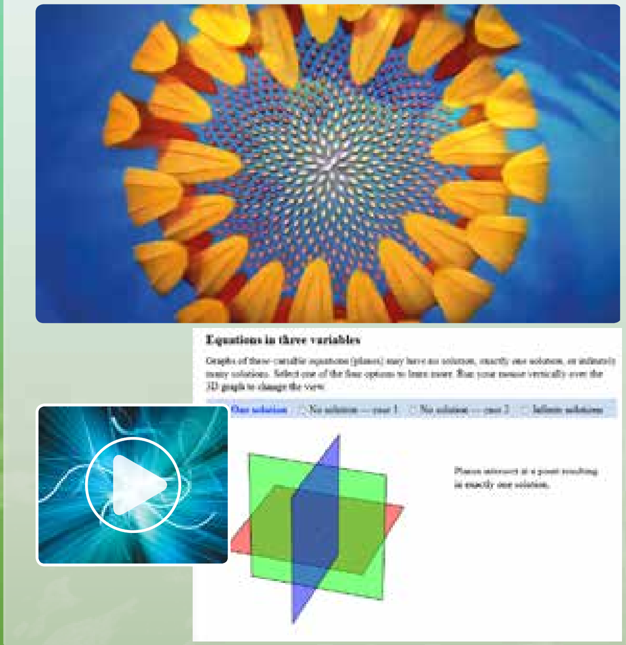

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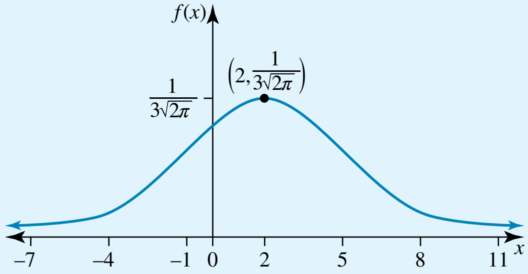
studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

13 Answers

EXERCISE 13.2

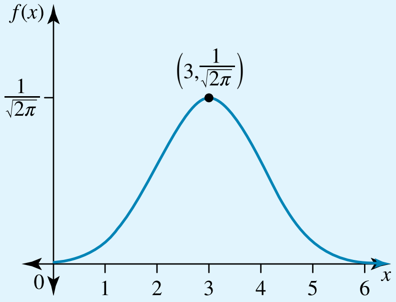
1 a $\mu = 2, \sigma = 3$

b



2 b $\mu = 3, \sigma = 1$

c



3 a 0.025 **b** 0.0015 **c** 0.84

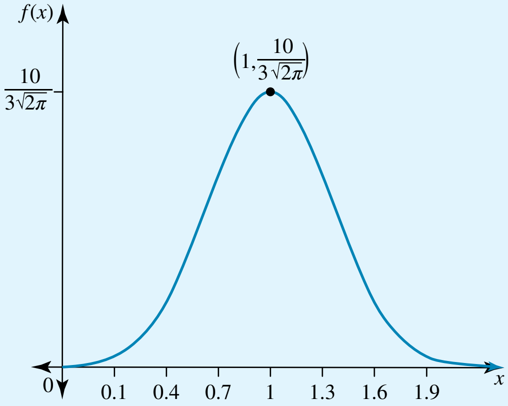
4 0.0015

5 b $\mu = -2$

6 a $\mu = 1, \sigma = 0.3$ or $\frac{3}{10}$

b Dilation factor $\frac{10}{3}$ parallel to the y-axis, dilation factor $\frac{3}{10}$ parallel to the x-axis, translation 1 unit in the positive x-direction

c



7 a $\mu = -4, \sigma = 10$

b Dilation factor $\frac{1}{10}$ from the x-axis, dilation factor 10 from the y-axis, translation 4 units in the negative x-direction

c i 100 **ii** 116

d $\displaystyle\int_{-\infty}^{\infty} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2} dx = 0.9999 \simeq 1$

$f(x) \geq 0$ for all values of x, and the area under the curve is 1. Therefore, this function is a probability density function.

8 a $\mu = 2, \sigma = \frac{2}{5}$ **b** $\frac{104}{25} = 4.16$

c i 10 **ii** 2.8

9 a i 100 and 140 **ii** 80 and 160

iii 60 and 180

b i 0.025 **ii** 0.0015

10 0.15 %

11 a 10 and 20 **b** 5 and 25 **c** 0 and 30

12 a 0.84 **b** 0.815 **c** 0.9702

13 a 0.975 **b** 0.95 **c** 0.9744

14 a 0.815 **b** $k = 205$ **c** $h = 155$

15 a 0.04 **b** 1.04

c i 5 **ii** –1.92

16 a 0.815 **b** 0.025 **c** 0.94

d $m = 80.9$

EXERCISE 13.3

1 a i 0.8849 **ii** 0.7703

b i 0.9088 **ii** $-\frac{4}{3}$

2 0.4088

3 a 0.9772 **b** 0.0228 **c** 0.9545

d 0.0512

4 a 0.0912 **b** 0.2119 **c** 0.7501

5 a 0.0228 **b** 0.0912 **c** 0.8860

6 a 2.76% **b** 92.17%

7 a i 0.65 **ii** 0.27

b i 0.29 **ii** 0.5179

c i $k = 2.4$ **ii** $k = 1.6$

8 Jing Jing

9 The salmon is in the top 26.6%, so it is not gourmet.

10 Chemistry 0.8, Maths Methods 0.86, Physics 0.875 so Physics best

11 a 0.1587 **b** 0.1587 **c** 0.3695

12 a 6.68% **b** 0.19%

13 a 0.0336 **b** 0.5684 **c** 0.9762

d 0.3081

14 $k = 25.2412$

EXERCISE 13.4

1 a $a = 34.34$ **b** $a = 14.68$ **c** $a = 1.65$

2 $m = 12.9444, n = 41.6556$

3 $\sigma = 15.5$

4 $\mu = 37.6$

5 a −0.2793 **b** 1.0364 **c** 0.3585

6 a $a = 42.52$ **b** $a = 41.53$ **c** $a = 13.40$

7 a 0.1764 **b** 0.3319

8 a 49.4443 **b** 36.6489

9 $\mu = 13.35$

10 $\mu = 26$

11 $\sigma = 10.5$

12 $\mathrm{SD}(X) = 12.2$

13 $\mu = 34.6, \sigma = 2.5$

14 $\mu = 15.8, \sigma = 5.2$

15 0.6842

16 $\mu = 37.68, \sigma = 11.21$

EXERCISE 13.5

1 a i 0.0038 **ii** 501.0210

b 3.4389 grams

2 a $3k^2 + 2k + 6k^2 + 2k + k^2 + 2k + 3k = 1$

$$10k^2 + 9k - 1 = 0$$

b $k = \frac{1}{10}$ **c** 2 **d** 0.3115

e 9 **f** $\sigma = 18$

3 0.0484

4 a 22.66% **b** 179 cm

5 a i 0.0062 **ii** 0.9625

b $\mu = 405, \sigma = 25$

6 a 0.1420 **b** $d = 2.0$ **c** 0.2924

d i $a = 0.68$ **ii** $0.85x - 4.76$

iii 6 cents **iv** 80%

7 a 0.0371 **b** 0.0032 **c** 0.0001

d i 0.0167 **ii** 0.8856

8 a Small: 0.0548, medium: 0.8644, large: 0.0808

b $531

9 a 15.87% **b** 68.27% **c** 45 cents

10 a 0.8413 **b** 0.6494 **c** 0.8845

d $\sigma = 0.4987$

11 a 95 mm **b** 0.8783 **c** 18%

d 83 mm **e** 0.0078 **f** $399

g 0.2236

12 a 0.0161 **b** 0.0019

13 a 0.0038

b $\mu = 19.0$ mm, $\sigma = 2.5$ mm

c 0.2225

14 a $\mu = 0, \sigma = 1.5306$

b 0.8816

14

Statistical inference

14.1 Kick off with CAS

Sample distributions

Data gathered from the last census showed that 40% of all households owned a dog. It would be nearly impossible to replicate this data, as to do this would mean surveying millions of people. However, samples of the population can be surveyed. The percentage of dog owners will vary from sample to sample, but the percentages from each set of samples can be graphed to reveal any patterns.

Whether or not a household owns a dog is a binomial random variable, so the percentages generated will be based on the binomial distribution. The proportion of successes in the population is 40%.

1 Generate data for 100 random samples of 5 people who were asked whether their household had a dog, and graph the data using a dot plot or histogram.

2 Generate data for 100 random samples of 50 people and graph the data using a dot plot or histogram.

3 Generate the data for 100 random samples of 200 people and graph the data using a dot plot or histogram.

4 a Do the data from each set of samples give a clear result as to the percentage of households that own a dog?

b Which set of samples gives more reliable information? Why?

c As the sample size increases, what happens to the shape of the graph?

d Is the graph symmetrical, and if so, about what value is it symmetrical?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

14.2 Population parameters and sample statistics

study on

Units 3 & 4

AOS 4

Topic 5

Concept 1

Populations and samples

Concept summary

Practice questions

Suppose you were interested in the percentage of Year 12 graduates who plan to study Mathematics once they complete school. It is probably not practical to question every student. There must be a way that we can ask a smaller group and then use this information to make generalisations about the whole group.

Samples and populations

A **population** is a group that you want to know something about, and a **sample** is the group within the population that you collect the information from. Normally, a sample is smaller than the population; the exception is a census, where the whole population is the sample.

The number of members in a sample is called the **sample size** (symbol n), and the number of members of a population is called the **population size** (symbol N). Sometimes the population size is unknown.

WORKED EXAMPLE 1

Cameron has uploaded a popular YouTube video. He thinks that the 133 people in his year group at school have seen it, and he wants to know what they think. He decides to question 10 people. Identify the population and sample size.

THINK	WRITE
1 Cameron wants to know what the people in his year at school think. This is the population.	$N = 133$
2 He asks 10 people. This is the sample.	$n = 10$

WORKED EXAMPLE 2

A total of 137 people volunteer to take part in a medical trial. Of these, 57 are identified as suitable candidates and are given the medication. Identify the population and sample size.

THINK	WRITE
1 57 people are given the medication. This is the sample size.	$n = 57$
2 We are interested in the group of people who might receive the drug in the future. This is the population.	The population is unknown, as we don't know how many people may be given this drug in the future.

Statistics and parameters

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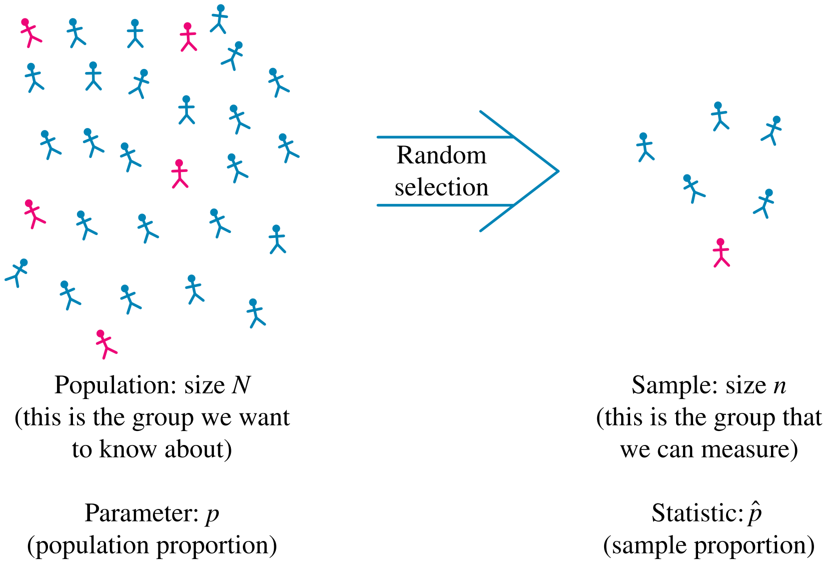
Interactivity
Population parameters and sample statistics
int-6442

A **parameter** is a characteristic of a population, whereas a **statistic** is a characteristic of a sample. This means that a statistic is always known exactly (because it is measured from the sample that has been selected). A parameter is usually estimated from a sample statistic. (The exception is if the sample is a census, in which case the parameter is known exactly.)

In this unit, we will study binomial data (that means that each data point is either yes/no or success/failure) with special regard to the proportion of successes.

The relationship between populations and samples



WORKED EXAMPLE 3

Identify the following as either sample statistics or population parameters.

a **Forty-three per cent of voters polled say that they are in favour of banning fast food.**

b **According to Australian Bureau of Statistics census data, the average family has 1.7 children.**

c **Between 18% and 23% of Australians skip breakfast regularly.**

d **Nine out of 10 kids prefer cereal for breakfast.**

THINK	WRITE
a 43% is an exact value that summarises the sample asked.	**a** Sample statistic
b The information comes from census data. The census questions the entire population.	**b** Population parameter
c 18%–23% is an estimate about the population.	**c** Population parameter
d Nine out of 10 is an exact value. It is unlikely that all kids could have been asked; therefore, it is from a sample.	**d** Sample statistic

Random samples

studyon

Units 3 & 4

AOS 4

Topic 5

Concept 2

Random sampling methods

Concept summary

Practice questions

A good sample should be representative of the population. If we consider our initial interest in the proportion of Year 12 graduates who intend to study Mathematics once they finished school, we could use a Mathematical Methods class as a sample. This would not be a good sample because it does not represent the population — it is a very specific group.

In a **random sample**, every member of the population has the same probability of being selected. The Mathematical Methods class is not a random sample because students who don't study Mathematical Methods have no chance of being selected; furthermore, students who don't attend that particular school have no chance of being selected.

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Interactivity

Random samples

int-6443

A **systematic sample** is almost as good as a random sample. In a systematic sample, every kth member of the population is sampled. For example, if $k = 20$, a customs official might choose to sample every 20th person who passes through the arrivals gate. The reason that this is almost as good as random sample is that there is an assumption that the group passing the checkpoint during the time the sample is taken is representative of the population. This assumption may not always be true; for example, people flying for business may be more likely to arrive on an early morning flight. Depending on the information you are collecting, this may influence the quality of the data.

In a **stratified random sample**, care is taken so that subgroups within a population are represented in a similar proportion in the sample. For example, if you were collecting information about students in Years 9–12 in your school, the proportions of students in each year group should be the same in the sample and the population. Within each subgroup, each member has the same chance of being selected.

A **self-selected sample**, that is one where the participants choose to participate in the survey, is almost never representative of the population. This means, for example, that television phone polls, where people phone in to answer yes or no to a question, do not accurately reflect the opinion of the population.

WORKED EXAMPLE 4

A survey is to be conducted in a middle school that has the distribution detailed in the table below. It is believed that students in different year levels may respond differently, so the sample chosen should reflect the subgroups in the population (that is, it should be a stratified random sample). If a sample of 100 students is required, determine how many from each year group should be selected.

Year level	Number of students
7	174
8	123
9	147

THINK	WRITE
1 Find the total population size.	Total population = 174 + 123 + 147 = 444
2 Find the number of Year 7s to be surveyed.	Number of Year 7s $= \frac{174}{444} \times 100$ $= 39.1$ Survey 39 Year 7s.
3 Find the number of Year 8s to be surveyed.	Number of Year 8s $= \frac{123}{444} \times 100$ $= 27.7$ Survey 28 Year 8s.

4 Find the number of Year 9s to be surveyed.	$\text{Number of Year 9s} = \frac{147}{444} \times 100$ $= 33.1$ Survey 33 Year 9s.
5 There has been some rounding, so check that the overall sample size is still 100.	Sample size = 39 + 28 + 33 = 100 The sample should consist of 39 Year 7s, 28 Year 8s and 33 Year 9s.

Using technology to select a sample

If you know the population size, it should also be possible to produce a list of population members. Assign each population member a number (from 1 to *N*). Use the random number generator on your calculator to generate a random number between 1 and *N*. The population member who was allocated that number becomes the first member of the sample. Continue generating random numbers until the required number of members has been picked for the sample. If the same random number is generated more than once, ignore it and continue selecting members until the required number has been chosen.

EXERCISE 14.2 Population parameters and sample statistics

PRACTISE

Work without CAS

1 **WE1** On average, Mr Parker teaches 120 students per day. He asks one class of 30 about the amount of homework they have that night. Identify the population and sample size.

2 Bruce is able to hem 100 shirts per day. Each day he checks 5 to make sure that they are suitable. Identify the population and sample size.

3 **WE2** Ms Lane plans to begin her Statistics class each year by telling her students a joke. She tests her joke on this year's class (15 students). She plans to retire in 23 years time. Identify the population and sample size.

4 Lee-Yin is trying to perfect a recipe for cake pops. She tries 5 different versions before she settles on her favourite. She takes some samples to school and asks 9 friends what they think. Identify the population and sample size.

5 **WE3** Identify the following as either sample statistics or population parameters.

a Studies have shown that between 85% and 95% of lung cancers are related to smoking.

b About 50% of children aged between 9 and 15 years eat the recommended daily amount of fruit.

6 Identify the following as either sample statistics or population parameters.

a According to the 2013 census, the ratio of male births per 100 female births is 106.3.

b About 55% of boys and 40% of girls reported drinking at least 2 quantities of 500 ml of soft drink every day.

7 **WE4** A school has 523 boys and 621 girls. You are interested in finding out about their attitudes to sport and believe that boys and girls may respond differently. If a sample of 75 students is required, determine how many boys and how many girls should be selected.

8 In a school, 23% of the students are boarders. For this survey, it is believed that boarders and day students may respond differently. To select a sample of 90 students, how many boarders and day students should be selected?

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

9 You are trying out a new chocolate pudding recipe. You found 40 volunteers to taste test your new recipe compared to your normal pudding. Half of the volunteers were given a serving the new pudding first, then a serving of the old pudding. The other half were given the old pudding first and then the new pudding. The taste testers did not know the order of the puddings they were trying. The results show that 31 people prefer the new pudding recipe.

a What is the population size?
b What is the sample size?

10 You want to test a new flu vaccine on people with a history of chronic asthma. You begin with 500 volunteers and end up with 247 suitable people to test the vaccine.

a What is the population size?
b What is the sample size?

11 In a recent survey, 1 in 5 students indicated that they ate potato crisps or other salty snacks at least four times per week. Is this a sample statistic or a population parameter?

12 Around 25 to 30% of children aged 0–15 years eat confectionary at least four times a week. Is this a sample statistic or a population parameter?

13 According to the Australian Bureau of Statistics, almost a quarter (24%) of internet users did not make an online purchase or order in 2012–13. The three most commonly reported main reasons for not making an online purchase or order were: 'Has no need' (33%); 'Prefers to shop in person/see the product' (24%); and 'Security concerns/concerned about providing credit card details online' (12%). Are these sample statistics or population parameters?

14 According to the 2011 census, there is an average of 2.6 people per household. Is this a sample statistic or a population parameter?

15 A doctor is undertaking a study about sleeping habits. She decides to ask every 10th patient about their sleeping habits.

a What type of sample is this?
b Is this a valid sampling method?

16 A morning television show conducts a viewer phone-in poll and announces that 95% of listeners believe that Australia should become a republic. Comment on the validity of this type of sample.

17 Tony took a survey by walking around the playground at lunch and asking fellow students questions. Why is this not the best sampling method?

18 A company has 1500 staff members, of whom 60% are male; 95% of the male staff work full time, and 78% of the female staff work full time. If a sample of

80 staff is to be selected, identify the numbers of full-time male staff, part-time male staff, full-time female staff and part-time female staff that should be included in the sample.

MASTER

19 Use CAS technology to produce a list of 10 random numbers between 1 and 100.

20 Use CAS technology to select a random sample from students in your Mathematical Methods class.

14.3 The distribution of $\hat{p}$

studyon

Units 3 & 4
AOS 4
Topic 5
Concept 4

Sample proportion
Concept summary
Practice questions

eBookplus

Interactivity
Distribution of $\hat{p}$
int-6444

Let us say that we are interested in the following collection of balls. As you can see in Figure 1, there are 20 balls, and $\frac{1}{4}$ of them are red. This means that the population parameter, p, is $\frac{1}{4}$ and the population size, N, is 20.

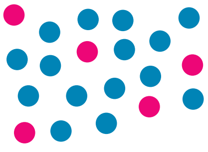

Figure 1

Normally we wouldn't know the population parameter, so we would choose a sample from the population and find the sample statistic. In this case, we are going to use a sample size of 5, that is, $n = 5$.

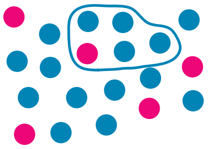

Figure 2

If our sample is the group shown in Figure 2, then as there is 1 red ball, the **sample proportion** would be $\hat{p} = \frac{1}{5}$.

A different sample could have a different sample proportion. In the case shown in Figure 3, $\hat{p} = \frac{2}{5}$.

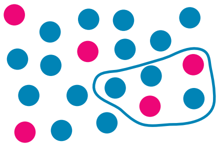

Figure 3

In the case shown in Figure 4, $\hat{p} = 0$.

It would also be possible to have samples for which $\hat{p} = \frac{3}{5}$, $\hat{p} = \frac{4}{5}$ or $\hat{p} = 1$, although these samples are less likely to occur.

In summary,

$$\hat{p} = \frac{\text{number of successful outcomes in the sample}}{\text{sample size}}$$

Figure 4

It might seem that using a sample does not give a good estimate about the population. However, the larger the sample size, the more likely that the sample proportions will be close to the population proportion.

WORKED EXAMPLE 5

You are trying out a new chocolate tart recipe. You found 40 volunteers to taste test your new recipe compared to your normal one. Half the volunteers were given a serving of the new tart first, then a serving of the old tart. The other half were given the old tart first and then the new one. The taste testers did not know the order of the tarts they were trying. The results show that 31 people prefer the new tart recipe.

What is the sample proportion, $\hat{p}$?

THINK	WRITE
1 There are 40 volunteers. This is the sample size.	$n = 40$
2 31 people prefer the new recipe.	Number of successes $= 31$
3 Calculate the sample proportion.	$\hat{p} = \frac{31}{40}$

Revision of binomial distributions

In a set of binomial data, each member of the population can have one of two possible values. We define one value as a success and the other value as a failure. (A success isn't necessarily a good thing, it is simply the name for the condition we are counting. For example, a success may be having a particular disease and a failure may be not having the disease).

The proportion of successes in a population is called p and is a constant value.

$$p = \frac{\text{number in the population with the favourable attribute}}{\text{population size}}$$

The proportion of failures in a population is called q, where $q = 1 - p$.

The sample size is called n.

The number of successes in the sample is called X.

The proportion of successes in the sample, $\hat{p}$, will vary from one sample to another.

$$\begin{aligned}\hat{p} &= \frac{\text{number in the sample with the favourable attribute}}{\text{sample size}}\\ &= \frac{X}{n}\end{aligned}$$

Sampling distribution of $\hat{p}$

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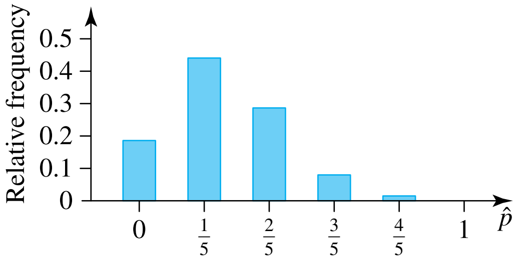
Interactivity
Sampling distribution of $\hat{p}$
int-6445

Normally, you would take one sample from a population and make some inferences about the population from that sample. In this section, we are going to explore what would happen if you took lots of samples of the same size. (Assume you return each sample back to the population before selecting again.)

Consider our population of 20 balls (5 red and 15 blue). There are ${}^{20}C_5 = 15\,504$ possible samples that could be chosen. That is, there are 15 504 possible ways of choosing 5 balls from a population of 20 balls. A breakdown of the different samples is shown in the table, where X is the number of red balls in the sample.

X	$\hat{p}$	Number of samples	Relative frequency
0	0	${}^5C_0\ {}^{15}C_5 = 3003$	0.194
1	$\frac{1}{5}$	${}^5C_1\ {}^{15}C_4 = 6825$	0.440
2	$\frac{2}{5}$	${}^5C_2\ {}^{15}C_3 = 4550$	0.293
3	$\frac{3}{5}$	${}^5C_3\ {}^{15}C_2 = 1050$	0.068
4	$\frac{4}{5}$	${}^5C_4\ {}^{15}C_1 = 75$	0.005
5	1	${}^5C_5\ {}^{15}C_0 = 1$	6.450×10^{-5}
	Total number of samples	**15 504**	

Graphing the distribution of $\hat{p}$ against the relative frequency of $\hat{p}$ results in the following.



As the value of $\hat{p}$, the sample proportion, varies depending on the sample, these values can be considered as the values of the random variable, $\hat{P}$.

The graph of the distribution of $\hat{p}$ can also be represented in a probability distribution table. This distribution is called a **sampling distribution**.

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	0.194	0.440	0.293	0.068	0.005	6.450×10^{-5}

We can find the average value of $\hat{p}$ as shown.

$\hat{p}$	Frequency, f	$f \cdot \hat{p}$
0	3 003	0
$\frac{1}{5}$	6 825	1365
$\frac{2}{5}$	4 550	1820
$\frac{3}{5}$	1 050	630
$\frac{4}{5}$	75	60
1	1	1
Totals	**15 504**	**3876**

$$\text{The average value of } \hat{p} = \frac{3876}{15\,504}$$
$$= 0.25$$

For this distribution, the average value for $\hat{p}$ is equal to the population proportion, p.

Sampling where the population is large

studyon

Units 3 & 4
AOS 4
Topic 5
Concept 6

Normal approximation to sampling distribution of proportion

Concept summary
Practice questions

It was mentioned earlier that larger samples give better estimates of the population.

Expected value

The proportion of $\hat{p}$ in a large sample conforms to $\hat{P} = \frac{X}{n}$. As the sample is from a large population, X can be assumed to be a binomial variable.

$$\therefore \mathrm{E}(\hat{P}) = \mathrm{E}\left(\frac{X}{n}\right)$$
$$= \frac{1}{n}\mathrm{E}(X) \qquad \left(\text{because } \frac{1}{n} \text{ is a constant}\right)$$
$$= \frac{1}{n} \times np$$
$$= p$$

Variance and standard deviation

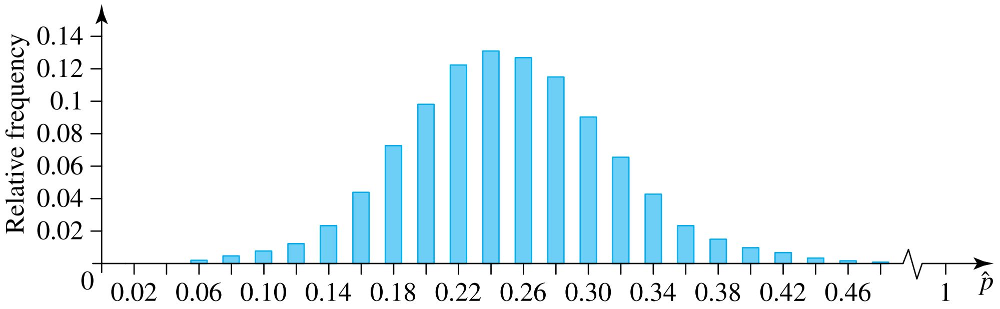
The variance and standard deviation can be found as follows.

$$\begin{aligned}\text{Var}(\hat{P}) &= \text{Var}\left(\frac{X}{n}\right)\\ &= \left(\frac{1}{n}\right)^2 \text{Var}(X)\\ &= \frac{1}{n^2} \times npq\\ &= \frac{pq}{n}\\ &= \frac{p(1-p)}{n}\\ \therefore \text{SD}(\hat{P}) &= \sqrt{\frac{p(1-p)}{n}}\end{aligned}$$

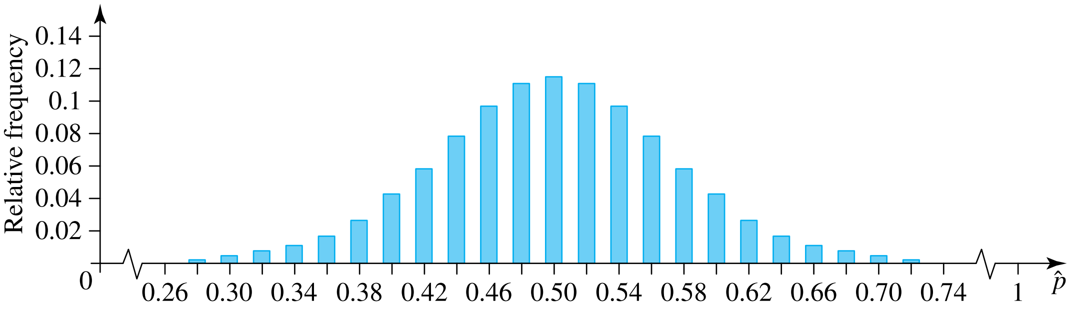
For large samples, the distribution of $\hat{p}$ is approximately normal with a mean or expected value of $\mu_{\hat{p}} = p$ and a standard deviation of $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$.

There are a number of different ways to decide if a sample is large. One generally accepted method that we will adopt for this section is that if $np \geq 10$, $nq \geq 10$ and $10n \leq N$, then the sample can be called large.

Consider the distribution of $\hat{p}$ when $N = 1000$, $n = 50$ and $p = 0.25$.



And consider this distribution of $\hat{p}$ when $N = 1000$, $n = 50$ and $p = 0.5$.



As these graphs show, the value of p doesn't matter. The distribution of $\hat{p}$ is symmetrical about p.

WORKED EXAMPLE 6 Consider a population size of 1000 and a sample size of 50. If $p = 0.1$, would this still be a large sample? If not, how big would the sample need to be?

THINK	WRITE
1 Is $10n \leq N$?	$n = 50$ and $N = 1000$ $10n = 500$ Therefore, $10n \leq N$.
2 Is $np \geq 10$?	$p = 0.1$ $np = 0.1 \times 50$ $= 5$ $5 \ngeq 10$ The sample is not large.
3 Find a value for n to make a large sample by solving $np = 10$.	$np = 10$ $0.1n = 10$ $n = 100$
4 Check the other conditions.	$10n = 10 \times 100$ $= 1000$ $= N$ $nq = 100 \times 0.9$ $= 90$ $nq \geq 10$ A sample size of 100 would be needed for a large sample.

WORKED EXAMPLE 7 If $N = 600$, $n = 60$ and $p = 0.3$:

a find the mean of the distribution

b find the standard deviation of the distribution, correct to 2 decimal places.

THINK	WRITE
a The mean is p.	a $\mu_{\hat{p}} = p$ $= 0.3$
b 1 Write the rule for the standard deviation.	b $\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$
2 Substitute the appropriate values and simplify.	$= \sqrt{\dfrac{0.3 \times (1-0.3)}{60}}$ $= 0.06$

EXERCISE 14.3 The distribution of $\hat{p}$

PRACTISE

1 **WE5** In a 99-g bag of lollies, there were 6 green lollies out of the 15 that were counted. What is the sample proportion, $\hat{p}$?

2 Hang is interested in seedlings that can grow to more than 5 cm tall in the month of her study period. She begins with 20 seedlings and finds that 6 of them are more than 5 cm tall after the month. What is the sample proportion, $\hat{p}$?

3 **WE6** Consider a population size of 1000 and a sample size of 50. If $p = 0.85$, is this a large sample? If not, how big does the sample need to be?

4 If the population size was 10 000 and $p = 0.05$, what would be a large sample size?

5 **WE7** If $N = 500$, $n = 50$ and $p = 0.5$:

a find the mean of the distribution

b find the standard deviation of the distribution, correct to 2 decimal places.

6 If $N = 1000$, $n = 100$ and $p = 0.8$:

a find the mean of the distribution

b find the standard deviation of the distribution, correct to 2 decimal places.

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

7 A car manufacturer has developed a new type of bumper that is supposed to absorb impact and result in less damage than previous bumpers. The cars are tested at 25 km/h. If 30 cars are tested and only 3 are damaged, what is the proportion of undamaged cars in the sample?

8 A standard warranty lasts for 1 year. It is possible to buy an extended warranty for an additional 2 years. The insurer decides to use the sales figures from Tuesday to estimate the proportion of extended warranties sold. If 537 units were sold and 147 of them included extended warranties, estimate the proportion of sales that will include extended warranties.

9 A Year 12 Mathematical Methods class consists of 12 girls and 9 boys. A group of 4 students is to be selected at random to represent the school at an inter-school Mathematics competition.

a What is the value of p, the proportion of girls in the class?

b What could be the possible values of the sample proportion, $\hat{p}$, of girls?

c Use this information to construct a probability distribution table to represent the sampling distribution of the sample proportion of girls in the small group.

d Find $\Pr(\hat{P} > 0.6)$. That is, find the probability that the proportion of girls in the small group is greater than 0.6.

e Find $\Pr(\hat{P} > 0.5 \mid \hat{P} > 0.3)$.

10 In a particular country town, the proportion of employment in the farming industry is 0.62. Five people aged 15 years and older are selected at random from the town.

a What are the possible values of the sample proportion, $\hat{p}$, of workers in the farming industry?

b Use this information to construct a probability distribution table to represent the sampling distribution of the sample proportion of workers in the farming industry.

c Find the probability that the proportion of workers in the farming industry in the sample is greater than 0.5.

11 In a population of 1.2 million, it is believed that $p = 0.01$. What would be the smallest sample size that could be considered large?

12 If $N = 1500$, $n = 150$ and $p = 0.15$, find the mean and standard deviation for the distribution of $\hat{p}$. Give your answers correct to 3 decimal places where appropriate.

13 If $N = 1200$, $n = 100$ and $p = 0.75$, find the mean and standard deviation for the distribution of $\hat{p}$. Give your answers correct to 3 decimal places where appropriate.

14 A distribution for $\hat{p}$ has a mean of 0.12 and a standard deviation of 0.0285. Find the population proportion and the sample size.

15 A distribution for $\hat{p}$ has a mean of 0.81 and a standard deviation of 0.0253. Find the population proportion and the sample size.

16 If $N = 1500$, $n = 150$ and $p = 0.15$, use CAS technology to graph the distribution for $\hat{p}$.

MASTER

17 A distribution for $\hat{p}$ has a standard deviation of 0.015. If the sample size was 510 and $\hat{p} > 0.5$, what was the population proportion, correct to 2 decimal places?

18 A distribution for $\hat{p}$ has a standard deviation of 0.0255. If the sample size was 350, what was the population proportion, correct to 2 decimal places?

14.4 Confidence intervals

We have just learned that different samples can have different values for $\hat{p}$. So what can one sample tell us about a population?

Let us say that you are interested in the proportion of the school that buys their lunch. You decide that your class is a reasonable sample and find out that 25% of the class will buy their lunch today. What can you say about the proportion of the whole school that will buy their lunch today? Assuming that your class is in fact a representative sample, you may say that around 25% of the school will buy their lunch. Is it possible to be more specific? By using **confidence intervals**, it is possible to say how confident you are that a population parameter will lie in a particular interval.

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Topic 5

Concept 7

Confidence limits for the population proportion

Concept summary

Practice questions

Normal approximation to the distribution of $\hat{p}$

We have learned that when we consider the distributions of $\hat{p}$, they are normally distributed with a $\mu_{\hat{p}} = p$ and $\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$. As we don't know the exact value for p, the best estimate is $\hat{p}$. This means that the best estimate of the standard deviation is $\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.

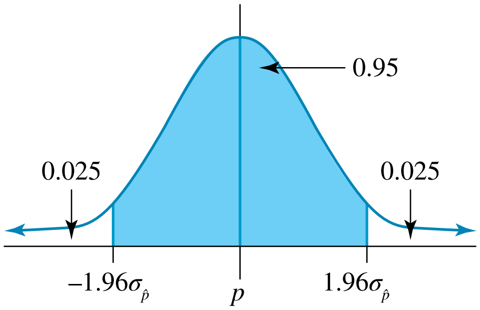
We know that for normal distributions, $z = \dfrac{x-\mu}{\sigma}$. This means that, to find the upper and lower values of z, we can use $z = \dfrac{\hat{p} \pm p}{\sigma_{\hat{p}}}$. Rearranging gives us $p = \hat{p} \pm z\sigma_{\hat{p}}$.

An approximate confidence interval for a population proportion is given by $(\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$, where $\sigma_{\hat{p}} = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.

A 95% confidence interval means that 95% of the distribution is in the middle area of the distribution. This means that the tails combined contain 5% of the distribution (2.5% on each end). The z-score for this distribution is 1.96.

The confidence interval for this distribution can be expressed as

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right).$$

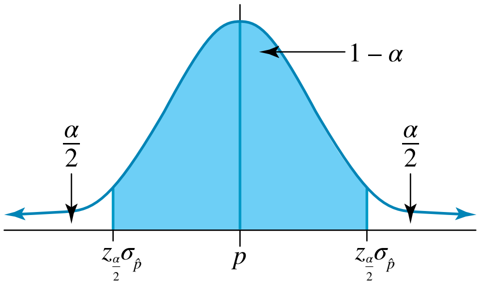


WORKED EXAMPLE 8

There are 20 people in your class and 25% are planning on buying their lunch. Estimate the proportion of the school population that will purchase their lunch today. Find a 95% confidence interval for your estimate, given $z = 1.96$.

THINK	WRITE
1 There are 20 people in the class. This is the sample size. 25% are buying their lunch. This is the sample proportion.	$n = 20$ $\hat{p} = 0.25$
2 For a 95% confidence interval, $z = 1.96$.	$z = 1.96$
3 The confidence interval is $\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$. Find $z\sigma_{\hat{p}}$.	$z\sigma_{\hat{p}} = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $= 1.96\sqrt{\frac{0.25 \times 0.75}{20}}$ $= 0.1898$
4 Identify the 95% confidence interval by finding the upper and lower values.	$\hat{p} - z\sigma_{\hat{p}} = 0.25 - 0.1898$ $= 0.0602$ $\hat{p} + z\sigma_{\hat{p}} = 0.25 + 0.1898$ $= 0.4398$ We can be 95% confident that between 6% and 44% of the population will buy their lunch today.

To find other confidence intervals, we can talk in general about a $1 - \alpha$ confidence interval. In this case, the tails combined will have an area of α (or $\frac{\alpha}{2}$ in each tail). In this case, the z-score that has a tail area of $\frac{\alpha}{2}$ is used.



WORKED EXAMPLE 9

Paul samples 102 people and finds that 18 of them like drinking coconut milk. Estimate the proportion of the population that likes drinking coconut milk. Find a 99% confidence interval for your estimate, correct to 1 decimal place.

THINK	WRITE
1 There are 102 people in the sample. This is the sample size. 18 like drinking coconut milk.	$n = 102$ $\hat{p} = \frac{18}{102}$ $= 0.18$
2 For a 99% confidence interval, find the z score using the inverse standard normal distribution.	For the 99% confidence interval, 1% will be in the tails, so 0.5% in each tail. Therefore, the area under the normal distribution curve to the left of z is 0.995. $z = 2.58$
3 The confidence interval is $(\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$. Find $z\sigma_{\hat{p}}$.	$z\sigma_{\hat{p}} = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $= 2.58\sqrt{\frac{0.18 \times 0.82}{102}}$ $= 0.098$
4 Identify the 99% confidence interval by finding the upper and lower values, correct to 1 decimal place.	$\hat{p} - z\sigma_{\hat{p}} = 0.18 - 0.098$ $= 0.082$ $\hat{p} + z\sigma_{\hat{p}} = 0.18 + 0.098$ $= 0.278$ We can be 99% confident that between 8.2% and 27.8% of the population like drinking coconut milk.

WORKED EXAMPLE 10

Grow Well are 95% sure that 30% to 40% of shoppers prefer their mulch. What sample size was needed for this level of confidence?

THINK	WRITE
1 The confidence interval is symmetric about $\hat{p}$: $(\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$, so the value of $\hat{p}$ must be halfway between the upper and lower values of the confidence interval.	$\hat{p} = \frac{30 + 40}{2}$ $= 35\%$ $= 0.35$
2 State the z-score related to the 95% confidence interval.	$z = 1.96$
3 The lower value of the confidence interval, 30%, is equivalent to $\hat{p} - z\sigma_{\hat{p}}$. Substitute the appropriate values. *Note:* The equation $0.4 = \hat{p} + z\sigma_{\hat{p}}$ could also have been used.	$0.3 = \hat{p} - z\sigma_{\hat{p}}$ $= \hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $= 0.35 - 1.96\sqrt{\frac{0.35(1-0.35)}{n}}$
4 Solve for n.	$n = 349.586$
5 Write the answer.	The sample size needed was 350 people.

Margin of error

The distance between the endpoints of the confidence interval and the sample estimate is called the **margin of error**, M.

Worked example 10 considered a 95% confidence interval, $(\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$. In this case the margin of error would be $M = z\sigma_{\hat{p}}$.

For a 95% level of confidence, $M = 1.96\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.

Note that the larger the sample size, the smaller the value of M will be. This means that one way to reduce the size of a confidence interval without changing the level of confidence is to increase the sample size.

EXERCISE 14.4 Confidence intervals

PRACTISE

1 **WE8** Of 30 people surveyed, 78% said that they like breakfast in bed. Estimate the proportion of the population that like breakfast in bed. Find a 95% confidence interval for the estimate.

2 Of the 53 people at swimming training today, 82% said that their favourite stroke is freestyle. Estimate the proportion of the population whose favourite stroke is freestyle. Find a 95% confidence interval for the estimate.

3 **WE9** Jenny samples 116 people and finds that 86% of them plan to go swimming over the summer holidays. Estimate the proportion of the population that plan to go swimming over the summer holidays. Find a 99% confidence interval for your estimate.

4 Yuki samples 95 people and finds that 30% of them eat chocolate daily. Estimate the proportion of the population that eats chocolate daily. Find a 90% confidence interval for your estimate.

5 **WE10** In a country town, the owners of Edie's Eatery are 95% sure that 35% to 45% of their customers love their homemade apple pie. What sample size was needed for this level of confidence?

6 If Parkers want to be 90% confident that between 75% and 85% of their customers will shop in their store for more than 2 hours, what sample size will be needed?

CONSOLIDATE

Apply the most appropriate mathematical processes and tools

The following information relates to question 7 and 8.

Teleco is being criticised for its slow response time when handling complaints. The company claims that it will respond within 1 day. Of the 3760 complaints in a given week, a random sample of 250 was selected. Of these, it was found that 20 of them had not been responded to within 1 day.

7 Find the 95% confidence interval for the proportion of claims that take more than 1 day to resolve.

8 What is the 99% confidence interval for the proportion of claims that take less than 1 day to resolve?

9 A sample of 250 blood donors have their blood types recorded. Of this sample, 92 have Type A blood. What is the 90% confidence interval for the proportion of Australians who have Type A blood?

10 It is believed that 65% of people have brown hair. A random selection of 250 people were asked the colour of their hair. Applying the normal approximation, find the probability that less than 60% of the people in the sample have brown hair.

11 Nidya is a top goal shooter. The probability of her getting a goal is 0.8. To keep her skills up, each night she has 200 shots on goal. Applying the normal approximation, find the probability that on Monday the proportion of times she scores a goal is between 0.8 and 0.9, given that it is more than 0.65.

12 Smooth Writing are 95% sure that 25% to 35% of shoppers prefer their pen. What sample size was needed for this level of confidence?

13 An online tutoring company is 99% sure that 20% to 30% of students prefer to use their company. What sample size was needed for this level of confidence?

14 Teleco want to be 95% sure that less than 5% of their complaints take more than 1 day to resolve. What sample proportion do they need and how large does the sample need to be to support this claim?

15 Barton's Dentistry want to be able to claim that 90% to 98% of people floss daily. They would like 99% confidence about their claim. How many people do they need to survey?

16 Tatiana is conducting a survey to estimate the proportion of Year 12 students who will take a gap year after they complete their VCE. Previous surveys have shown the proportion to be approximately 15%. Determine the required size of the sample so that the margin of error for the survey is 3% in a confidence interval of approximately 95% for this proportion.

MASTER

17 Bentons claim that between 85% and 95% of their customers stay for more than 2 hours when they shop. If they surveyed 100 people, how confident are they about that claim?

18 The Hawthorn Football Club claim that between 75% and 80% of their members remain members for at least 10 years. If they surveyed 250 people, how confident are they about that claim? Give your answer to the nearest whole number.

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14.5 Review

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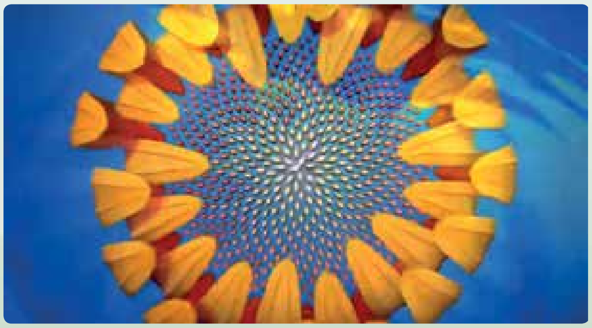

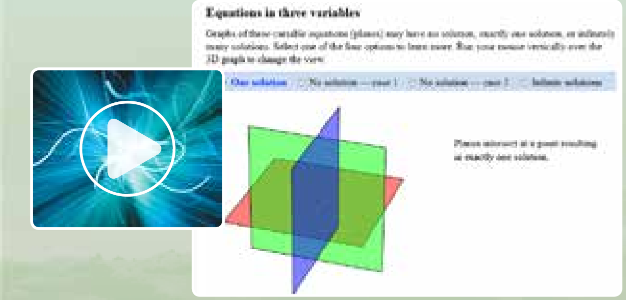

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14 Answers

EXERCISE 14.2

1 $N = 120$, $n = 30$

2 $N = 100$, $n = 5$

3 $n = 15$, population size is unknown

4 $n = 9$, population size is unknown

5 a Population parameter **b** Sample statistic

6 a Population parameter **b** Sample statistic

7 34 boys and 41 girls

8 21 boarders, 69 day students

9 a The population size is unknown.

b $n = 40$

10 a The population is people who will receive the vaccine in the future. The size is unknown.

b $n = 247$

11 Sample statistic

12 Population parameter

13 Sample statistics

14 Population parameter

15 a A systematic sample with $k = 10$

b Yes, assuming that the order of patients is random

16 The sample is not random; therefore, the results are not likely to be random.

17 It is probably not random. Tony is likely to ask people who he knows or people who approach him.

18 Full-time male staff: 46

Part-time male staff: 2

Full-time female staff: 25

Part-time female staff: 7

19 Use the random number generator on your calculator to produce numbers from 1 to 100. Keep generating numbers until you have 10 different numbers.

20 First, assign every person in your class a number, e.g. 1 to 25 if there are 25 students in your class. Decide how many students will be in your sample, e.g. 5. Then use the random number generator on your calculator to produce numbers from 1 to 25. Keep generating numbers until you have 5 different numbers. The students that were assigned these numbers are the 5 students in your random sample.

EXERCISE 14.3

1 $\hat{p} = \frac{2}{5}$ **2** $\hat{p} = \frac{3}{10}$

3 This is not a large sample; $n = 67$ would be a large sample.

4 $n = 200$

5 a $\mu_{\hat{p}} = 0.5$ **b** $\sigma_{\hat{p}} = 0.07$

6 a $\mu_{\hat{p}} = 0.8$ **b** $\sigma_{\hat{p}} = 0.04$

7 $\hat{p} = \frac{9}{10}$ **8** $\hat{p} = \frac{147}{537}$

9 a $p = \frac{4}{7}$

b $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

c

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	0.021	0.168	0.397	0.331	0.083

d 0.414

e 0.510

10 a $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

b

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	0.008	0.064	0.211	0.344	0.281	0.092

c 0.717

11 $n = 1000$ **12** $\mu_{\hat{p}} = 0.15$, $\sigma_{\hat{p}} = 0.029$

13 $\mu_{\hat{p}} = 0.75$, $\sigma_{\hat{p}} = 0.043$ **14** $p = 0.12$, $n = 130$

15 $p = 0.81$, $n = 240$

16 See the figure at the foot of the page.*

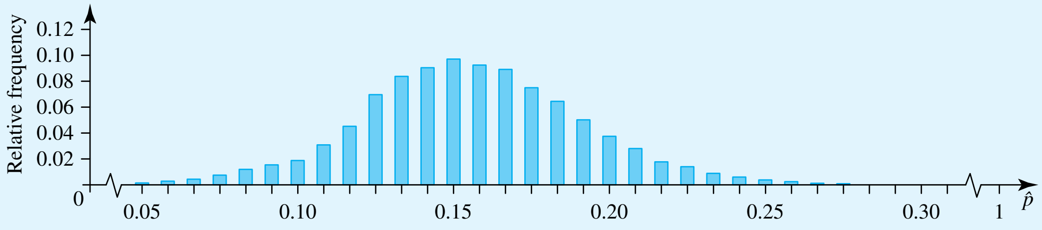
17 $p = 0.87$ **18** $p = 0.35$ or $p = 0.65$

EXERCISE 14.4

1 63%–93% **2** 72%–92%

3 78%–94% **4** 22%–38%

***16**



5 $n = 369$ **6** $n = 173$

7 4.6%–11.4% of complaints take more than 1 day to resolve.

8 87.6%–96.4% of complaints are resolved within 1 day.

9 31.8%–41.8% of Australians have Type A blood.

10 0.0487

11 0.4998

12 $n = 323$

13 $n = 498$

14 $\hat{p} = 2.5\%$, $n = 150$

15 $n = 235$

16 544 people

17 90% sure

18 66%

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