Topic 1 — Lines and linear relationships

1.2 Linear equations and inequations

1.2 Exercise

1 a

$$\begin{aligned} 3(5x-1) &= 4x-14 \\ \therefore 15x-3 &= 4x-14 \\ \therefore 11x &= -14+3 \\ \therefore 11x &= -11 \\ \therefore x &= -1 \end{aligned}$$

b

$$\begin{aligned} \frac{4-x}{3}+\frac{3x-2}{4} &= 5 \\ \therefore \frac{4(4-x)+3(3x-2)}{12} &= 5 \\ \therefore \frac{16-4x+9x-6}{12} &= 5 \\ \therefore \frac{10+5x}{12} &= 5 \\ \therefore 10+5x &= 60 \\ \therefore 5x &= 50 \\ \therefore x &= 10 \end{aligned}$$

2 a

$$\begin{aligned} 2x-5 &= 11 \\ 2x &= 16 \\ x &= 8 \end{aligned}$$

b

$$\begin{aligned} \frac{x}{2}+4 &= -1 \\ \frac{x}{2} &= -5 \\ x &= -10 \end{aligned}$$

c

$$\begin{aligned} 2x-4 &= 5x+3 \\ -4 &= 3x+3 \\ -7 &= 3x \\ x &= -\frac{7}{3} \end{aligned}$$

d

$$\begin{aligned} 5x+3 &= -4(1-x) \\ 5x+3 &= -4+4x \\ x+3 &= -4 \\ x &= -7 \end{aligned}$$

e

$$\begin{aligned} \frac{3-2x}{6} &= \frac{x+4}{4} \quad (\text{LCD} = 12) \\ \frac{3-2x}{6}\times\frac{12}{1} &= \frac{x+4}{4}\times\frac{12}{1} \\ 2(3-2x) &= 3(x+4) \\ 6-4x &= 3x+12 \\ 6 &= 7x+12 \\ -6 &= 7x \\ x &= -\frac{6}{7} \end{aligned}$$

f

$$\begin{aligned} \frac{4x-5}{6}-\frac{x+4}{12} &= 1 \quad (\text{LCD} = 12) \\ \frac{4x-5}{6}\times\frac{12}{1}-\frac{x+4}{12}\times\frac{12}{1} &= 1\times 12 \\ 2(4x-5)-(x+4) &= 12 \\ 8x-10-x-4 &= 12 \\ 7x-14 &= 12 \\ 7x &= 26 \\ x &= \frac{26}{7} \end{aligned}$$

3 a

$$\begin{aligned} 7(2x-3) &= 5(3+2x) \\ \therefore 14x-21 &= 15+10x \\ \therefore 14x-10x &= 15+21 \\ \therefore 4x &= 36 \\ \therefore x &= 9 \end{aligned}$$

b

$$\begin{aligned} \frac{4x}{5}-9 &= 7 \\ \therefore \frac{4x}{5} &= 16 \\ \therefore 4x &= 80 \\ \therefore x &= 20 \end{aligned}$$

c

$$\begin{aligned} 4-2(x-6) &= \frac{2x}{3} \\ \therefore 4-2x+12 &= \frac{2x}{3} \\ \therefore 16-2x &= \frac{2x}{3} \\ \therefore 48-6x &= 2x \\ \therefore 48 &= 8x \\ \therefore x &= 6 \end{aligned}$$

d

$$\begin{aligned} \frac{3x+5}{9} &= \frac{4-2x}{5} \\ \therefore 5(3x+5) &= 9(4-2x) \\ \therefore 15x+25 &= 36-18x \\ \therefore 33x &= 11 \\ \therefore x &= \frac{11}{33} \\ \therefore x &= \frac{1}{3} \end{aligned}$$

e

$$\begin{aligned} \frac{x+2}{3}+\frac{x}{2}-\frac{x+1}{4} &= 1 \\ \therefore \frac{4(x+2)+6x-3(x+1)}{12} &= 1 \\ \therefore 4x+8+6x-3x-3 &= 12 \\ \therefore 7x &= 7 \\ \therefore x &= 1 \end{aligned}$$

f

$$\begin{aligned} \frac{7x}{5}-\frac{3x}{10} &= 2\left(x+\frac{9}{2}\right) \\ \therefore \frac{14x-3x}{10} &= 2x+9 \\ \therefore 11x &= 10(2x+9) \\ \therefore 11x &= 20x+90 \\ \therefore -9x &= 90 \\ \therefore x &= -10 \end{aligned}$$

4 $\frac{d-x}{a}=\frac{a-x}{d}$

$\therefore d(d-x)=a(a-x)$

$\therefore d^2-dx=a^2-ax$

$\therefore ax-dx=a^2-d^2$

$\therefore x(a-d)=a^2-d^2$

$\therefore x=\frac{a^2-d^2}{a-d}$

$\therefore x=\frac{(a-d)(a+d)}{a-d}$

$\therefore x=a+d$

5 **a** $ax+b=c$

$\therefore ax=c-b$

$\therefore x=\frac{c-b}{a}$

b $a(x-b)=bx$

$\therefore ax-ab=bx$

$\therefore ax-bx=ab$

$\therefore x(a-b)=ab$

$\therefore x=\frac{ab}{a-b}$

c $a^2x+a^2=ab+abx$

$\therefore a^2x-abx=ab-a^2$

$\therefore x\left(a^2-ab\right)=-\left(a^2-ab\right)$

$\therefore x=-1$

d $\frac{x}{a}+\frac{x}{b}=a+b$

$\therefore \frac{bx+ax}{ab}=a+b$

$\therefore bx+ax=ab(a+b)$

$\therefore x(b+a)=ab(a+b)$

$\therefore x=ab$

e $\frac{bx-a}{c}=\frac{cx+a}{b}$

$\therefore b(bx-a)=c(cx+a)$

$\therefore b^2x-ba=c^2x+ca$

$\therefore b^2x-c^2x=ca+ba$

$\therefore x\left(b^2-c^2\right)=a(c+b)$

$\therefore x=\frac{a(c+b)}{b^2-c^2}$

$\therefore x=\frac{a(c+b)}{(b+c)(b-c)}$

$\therefore x=\frac{a}{b-c}$

f $\frac{x+a}{b}-2=\frac{x-b}{a}$

$\therefore \frac{x+a}{b}-\frac{x-b}{a}=2$

$\therefore \frac{a(x+a)-b(x-b)}{ba}=2$

$\therefore ax+a^2-bx+b^2=2ba$

$\therefore ax-bx=2ba-a^2-b^2$

$\therefore x(a-b)=-\left(a^2-2ab+b^2\right)$

$\therefore x=\frac{-(a-b)^2}{a-b}$

$\therefore x=-(a-b)$

$\therefore x=b-a$

6 $7-\frac{3x}{8}\le -2$

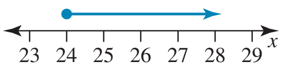
$\therefore -\frac{3x}{8}\le -2-7$

$\therefore -\frac{3x}{8}\le -9$

$\therefore -3x\le -72$

$\therefore x\ge \frac{-72}{-3}$

$\therefore x\ge 24$



7 **a** $4-2x\ge 5$

$-2x\ge 1$

$x\le -\frac{1}{2}$

b $\frac{x-6}{3}+4<1$

$\frac{x-6}{3}<-3$

$x-6<-9$

$x<-3$

c $2x-3<4x+1$

$-3<2x+1$

$-4<2x$

$-2<x$

$x>-2$

d $-2(x-5)-x>3(x+4)$

$-2x+10-x>3x+12$

$-3x+10>3x+12$

$-6x+10>12$

$-6x>2$

$x<-\frac{2}{6}$

$x<-\frac{1}{3}$

e $1-\frac{4-x}{2}>-1$

$-\frac{4-x}{2}>-2$

$4-x<4$

$-x<0$

$x>0$

f $\frac{5-x}{2}<-\frac{3x+2}{8}$ (LCD = 8)

$8\times\left(\frac{5-x}{2}\right)<8\times\left(-\frac{3x+2}{8}\right)$

$4(5-x)<-(3x+2)$

$20-4x<-3x-2$

$-x<-22$

$x>22$

8 **a** $3x-5\le -8$

$\therefore 3x\le -8+5$

$\therefore 3x\le -3$

$\therefore x\le -1$

b $4(x-6)+3(2-2x)<0$

$\therefore 4x-24+6-6x<0$

$\therefore -2x-18<0$

$\therefore -2x<18$

$\therefore x>-9$

c $1-\dfrac{2x}{3}\geq -11$

$\therefore -\dfrac{2x}{3}\geq -12$

$\therefore -2x\geq -36$

$\therefore x\leq 18$

d $\dfrac{5x}{6}-\dfrac{4-x}{2}>2$

$\therefore \dfrac{5x-3(4-x)}{6}>2$

$\therefore 5x-12+3x>12$

$\therefore 8x>24$

$\therefore x>3$

e $8x+7(1-4x)\leq 7x-3(x+3)$

$\therefore 8x+7-28x\leq 7x-3x-9$

$\therefore -20x+7\leq 4x-9$

$\therefore -24x\leq -16$

$\therefore x\geq \dfrac{16}{24}$

$\therefore x\geq \dfrac{2}{3}$

f $\dfrac{2}{3}(x-6)-\dfrac{3}{2}(x+4)>1+x$

$\therefore \dfrac{2(x-6)}{3}-\dfrac{3(x+4)}{2}>1+x$

$\therefore \dfrac{4(x-6)-9(x+4)}{6}>1+x$

$\therefore 4x-24-9x-36>6(1+x)$

$\therefore -5x-60>6+6x$

$\therefore -11x>66$

$\therefore x<-6$

9 $\dfrac{7(x-3)}{8}+\dfrac{3(2x+5)}{4}=\dfrac{3x}{2}+1$

$\therefore \dfrac{7(x-3)}{8}+\dfrac{3(2x+5)}{4}-\dfrac{3x}{2}=1$

$\therefore \dfrac{7(x-3)+6(2x+5)-12x}{8}=1$

$\therefore \dfrac{7x-21+12x+30-12x}{8}=1$

$\therefore \dfrac{7x+9}{8}=1$

$\therefore 7x+9=8$

$\therefore 7x=-1$

$\therefore x=-\dfrac{1}{7}$

10 $b(x+c)=a(x-c)+2bc$

$\therefore bx+bc=ax-ac+2bc$

$\therefore bx-ax=-ac+bc$

$\therefore x(b-a)=c(b-a)$

$\therefore x=\dfrac{c(b-a)}{b-a}$

$\therefore x=c$

11 $4(2+3x)>8-3(2x+1)$

$\therefore 8+12x>8-6x-3$

$\therefore 12x+6x>8-3-8$

$\therefore 18x>-3$

$\therefore x>\dfrac{-3}{18}$

$\therefore x>-\dfrac{1}{6}$

12 Use CAS technology to obtain the solutions for parts **a** and **b**.

a $\{x=0\}$ (This gives $x=0$ as the solution.)

b The answer is given as $\{x>4\}$.
Therefore, $x>4$.

13 a Let the numbers be x and $x+2$.
Then $x+(x+2)=9\,((x+2)-x)$.
Solving,

$\therefore 2x+2=9\,(2)$

$\therefore 2x=16$

$\therefore x=8$

The numbers are 8 and 10.

b Let the number be x.
Then $4(x-3)=72$.
Solving,

$x-3=18$

$\therefore x=21$

The number is 21.

c Let the three consecutive numbers be x, $x+1$ and $x+2$.
Then $x+(x+1)+(x+2)=36+\dfrac{1}{4}x$.
Solving,

$3x+3=36+\dfrac{x}{4}$

$\therefore 3x-\dfrac{x}{4}=33$

$\therefore \dfrac{12x-x}{4}=33$

$\therefore 11x=132$

$\therefore x=12$

The numbers are 12, 13 and 14.

d Let the width of the rectangle be x cm. Therefore, its length is $(2x+12)$ cm.
The perimeter is 48 cm.

$\therefore 2x+2(2x+12)=48$

$\therefore 2x+4x+24=48$

$\therefore 6x=24$

$\therefore x=4$

The width is 4 cm and the length is 20 cm.

e Let the width be x cm.
As length : width : height = 2 : 1 : 3, the length is $2x$ cm and the height is $3x$ cm.
The sum of the lengths of the 12 edges of the rectangular prism is 360 cm.
Therefore, $4(2x)+4(x)+4(3x)=360$.

$\therefore 24x=360$

$\therefore x=15$

Since the height is $3x$ cm, the height is 45 cm.

14 Let n be the number of books sold and $\$P$ be the profit made.

a Revenue = $2.5n$ dollars and costs = $100 + 0.2n$ dollars.

$\therefore P = 2.5n - (100 + 0.2n)$

$\therefore P = 2.3n - 100$

b To make a profit,

$$\begin{aligned} \therefore 2.3n - 100 &> 0 \\ \therefore 2.3n &> 100 \\ \therefore n &> \frac{100}{2.3} \\ \therefore n &> 43.478 \end{aligned}$$

At least 44 books must be sold to ensure a profit is made.

15 Let the number of mobile phone covers be n.

Profit on one mobile phone cover = $\$12 - \$1.80 = \$10.20$

Profit on n mobile phone covers = $\$10.2n$

To cover fixed costs:

$$\begin{aligned} 10.2n &\geq 250 \\ n &\geq 24.5098 \end{aligned}$$

25 mobile phone covers are needed to cover the costs.

16 Let the number of drinks sold be n.

Profit on one drink = $\$4 - \$1.20 = \$2.80$

Profit on n drinks = $\$2.8n$

Total costs = $2.8n - 65$

To make at least $\$450$:

$$\begin{aligned} 2.8n - 65 &\geq 450 \\ 2.8n &\geq 515 \\ n &\geq 183.9 \end{aligned}$$

184 drinks need to be sold to raise $\$450$ for the new equipment.

1.2 Exam questions

1

$$\frac{x-a}{b} - 2 = \frac{b-x}{a}$$

$$\frac{a(x-a)}{ab} - \frac{2ab}{ab} = \frac{b(b-x)}{ab} \qquad \text{[1 mark]}$$

$$\begin{aligned} ax - a^2 - 2ab &= b^2 - bx \\ ax + bx &= b^2 + 2ab + a^2 && \text{[1 mark]} \\ x(a+b) &= (a+b)^2 \\ x &= a + b && \text{[1 mark]} \end{aligned}$$

2

$$\begin{aligned} 3(x-4) &\leq 6 \\ 3x - 12 &\leq 6 \\ 3x &\leq 18 \\ x &\leq 6 \end{aligned}$$

The correct answer is **E**.

3

$$\frac{x}{5} - \frac{2x-1}{3} \geq -2$$

$$\begin{aligned} \frac{3x - 5(2x-1)}{15} &\geq -2 && \text{[1 mark]} \\ 3x - 10x + 5 &\geq -30 \\ -7x &\geq -35 && \text{[1 mark]} \\ x &\leq 5 && \text{[1 mark]} \end{aligned}$$

1.3 Systems of simultaneous linear equations

1.3 Exercise

1 a

$$\begin{aligned} x &= 2y + 5 \\ 4x - 3y &= 25 \end{aligned}$$

Substitute the first equation into the second equation:

$$\begin{aligned} 4(2y+5) - 3y &= 25 \\ \therefore 8y + 20 - 3y &= 25 \\ \therefore 5y + 20 &= 25 \\ \therefore 5y &= 5 \\ \therefore y &= 1 \end{aligned}$$

In the first equation, replace y with 1 to obtain $x = 7$.

Therefore, $x = 7$, $y = 1$.

b

$$\begin{aligned} 5x + 9y &= -38 && [1] \\ -3x + 2y &= 8 && [2] \end{aligned}$$

Choosing to eliminate x:

Multiply equation [1] by 3 and equation [2] by 5:

$$\begin{aligned} 15x + 27y &= -114 && [3] \\ -15x + 10y &= 40 && [4] \end{aligned}$$

Add equations [3] and [4]:

$$\begin{aligned} 37y &= -74 \\ \therefore y &= -2 \end{aligned}$$

Substitute $y = -2$ in equation [2]:

$$\begin{aligned} \therefore -3x - 4 &= 8 \\ \therefore -3x &= 12 \\ \therefore x &= -4 \end{aligned}$$

Therefore, $x = -4$, $y = -2$.

2 a

$$\begin{aligned} 4x + 3y &= 5 && [1] \\ y &= x - 3 && [2] \end{aligned}$$

Substitute [2] into [1]:

$$\begin{aligned} 4x + 3(x-3) &= 5 \\ 4x + 3x - 9 &= 5 \\ 7x - 9 &= 5 \\ 7x &= 14 \\ x &= 2 \end{aligned}$$

Substitute $x = 2$ into [2]:

$$\begin{aligned} y &= 2 - 3 \\ &= -1 \end{aligned}$$

b

$$\begin{aligned} x &= 2y - 4 && [1] \\ x &= 1 - 8y && [2] \end{aligned}$$

Substitute [1] into [2]:

$$\begin{aligned} 2y - 4 &= 1 - 8y \\ 10y - 4 &= 1 \\ 10y &= 5 \\ y &= \frac{1}{2} \end{aligned}$$

Substitute $y = \frac{1}{2}$ into [1]:

$$\begin{aligned} x &= 2 \times \frac{1}{2} - 4 \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

c $7x - 3y = 11$ [1]

$2x + 3y = 7$ [2]

Add equation [1] and [2]:

$$\begin{aligned}9x &= 18\\ x &= 2\end{aligned}$$

Substitute $x = 2$ into equation [1]:

$$\begin{aligned}7 \times 2 - 3y &= 11\\ -3y + 14 &= 11\\ -3y &= -3\\ y &= 1\end{aligned}$$

d

$$\begin{aligned}2x + 3y &= 10 && [1]\\ x - y &= -5 && [2]\\ [2] \times 2 \Rightarrow 2x - 2y &= -10 && [3]\end{aligned}$$

Subtract equation [2] from equation [1]:

$$\begin{aligned}5y &= 20\\ y &= 4\end{aligned}$$

Substitute $y = 4$ into equation [2]:

$$\begin{aligned}x - 4 &= -5\\ x &= -1\end{aligned}$$

e

$$\begin{aligned}2x + 3y &= 11 && [1]\\ 3x + 5y &= 18 && [2]\\ [1] \times 3 \Rightarrow 6x + 9y &= 33 && [3]\\ [2] \times 2 \Rightarrow 6x + 10y &= 36 && [4]\end{aligned}$$

Subtract equation [4] from equation [3]:

$$\begin{aligned}-y &= -3\\ y &= 3\end{aligned}$$

Substitute $y = 3$ into equation [1]:

$$\begin{aligned}2x + 3 \times 3 &= 11\\ 2x + 9 &= 11\\ 2x &= 2\\ x &= 1\end{aligned}$$

f

$$\begin{aligned}4x - 3y &= -38 && [1]\\ 5x + 2y &= -13 && [2]\\ [1] \times 2 \Rightarrow 8x - 6y &= -76 && [3]\\ [2] \times 3 \Rightarrow 15x + 6y &= -39 && [4]\end{aligned}$$

Add equations [3] and [4] :

$$\begin{aligned}23x &= -115\\ x &= -5\end{aligned}$$

Substitute $x = -5$ into equation [1]:

$$\begin{aligned}4 \times -5 - 3y &= -38\\ -3y - 20 &= -38\\ -3y &= -18\\ y &= 6\end{aligned}$$

3 a $y = 5x - 1$ [1]

$x + 2y = 9$ [2]

Substitute equation [1] in equation [2]:

$$\begin{aligned}\therefore x + 2(5x - 1) &= 9\\ \therefore x + 10x - 2 &= 9\\ \therefore 11x &= 11\\ \therefore x &= 1\end{aligned}$$

Substitute $x = 1$ in equation [1]:

$\therefore y = 4$

Answer: $x = 1, y = 4$

b $x = 5 + \dfrac{y}{2}$ [1]

$-4x - 3y = 35$ [2]

Substitute equation [1] in equation [2]:

$$\begin{aligned}\therefore -4\left(5 + \frac{y}{2}\right) - 3y &= 35\\ \therefore -20 - 2y - 3y &= 35\\ \therefore -5y &= 55\\ \therefore y &= -11\end{aligned}$$

Substitute $y = -11$ in equation [1]:

$$\begin{aligned}\therefore x &= 5 - \frac{11}{2}\\ \therefore x &= \frac{10}{2} - \frac{11}{2}\\ \therefore x &= -\frac{1}{2}\end{aligned}$$

Answer: $x = -\dfrac{1}{2}, y = -11$

4 a $8x + 3y = 8$ [1]

$-2x + 11y = \dfrac{35}{6}$ [2]

Multiply equation [2] by 4:

$\therefore -8x + 44y = \dfrac{70}{3}$ [3]

Add equation [3] to equation [1]:

$$\begin{aligned}\therefore 47y &= \frac{24}{3} + \frac{70}{3}\\ \therefore 47y &= \frac{94}{3}\\ \therefore y &= \frac{2}{3}\end{aligned}$$

Substitute $y = \dfrac{2}{3}$ in equation [1]:

$$\begin{aligned}\therefore 8x + 2 &= 8\\ \therefore 8x &= 6\\ \therefore x &= \frac{3}{4}\end{aligned}$$

Answer: $x = \dfrac{3}{4}, y = \dfrac{2}{3}$

b $\dfrac{x}{2} + \dfrac{y}{3} = 8$ [1]

$\dfrac{x}{3} + \dfrac{y}{2} = 7$ [2]

Rearrange equation [1]:

$$\frac{3x + 2y}{6} = 8$$

$\therefore 3x + 2y = 48$ [3]

Rearrange equation [2]:

$$\frac{2x + 3y}{6} = 7$$

$\therefore 2x + 3y = 42$ [4]

Multiply equation [3] by 3 and equation [4] by 2:

$9x + 6y = 144$ [5]

$4x + 6y = 84$ [6]

Subtract equation [5] from equation [6]:

$$\begin{aligned}\therefore 5x &= 60\\ \therefore x &= 12\end{aligned}$$

Substitute $x = 12$ in equation [1]:

$\therefore 6 + \frac{y}{3} = 8$

$\therefore \frac{y}{3} = 2$

$\therefore y = 6$

Answer: $x = 12, y = 6$

5 a $2x - y = 7$ [1]

$7x - 5y = 42$ [2]

Either of the substitution or elimination methods could be used.

Using substitution:

From equation [1], $2x - 7 = y$. Substitute in equation [2]:

$\therefore 7x - 5(2x - 7) = 42$

$\therefore 7x - 10x + 35 = 42$

$\therefore -3x = 7$

$\therefore x = -\frac{7}{3}$

Substitute $x = -\frac{7}{3}$ in equation [1]:

$\therefore y = -\frac{14}{3} - 7$

$\therefore y = -\frac{14}{3} - \frac{21}{3}$

$\therefore y = -\frac{35}{3}$

Answer: $x = -\frac{7}{3}, y = -\frac{35}{3}$

b $ax - by = a$ [1]

$bx + ay = b$ [2]

Multiply equation [1] by a and equation [2] by b:

$a^2x - aby = a^2$ [3]

$b^2x + aby = b^2$ [4]

Add equations [3] and [4]:

$\left(a^2 + b^2\right)x = a^2 + b^2$

$\therefore x = 1$

Substitute $x = 1$ in equation [2]:

$b + ay = b$

$\therefore ay = 0$

$\therefore y = 0$

Answer: $x = 1, y = 0$

6 a $4x - 3y = 23$ [1]

$7x + 4y = 31$ [2]

Multiply equation [1] by 4 and equation [2] by 3:

$16x - 12y = 92$ [3]

$21x + 12y = 93$ [4]

Add equations [3] and [4]:

$37x = 185$

$x = 5$

Substitute $x = 5$ into equation [1]:

$4(5) - 3y = 23$

$20 - 3y = 23$

$-3y = 3$

$y = -1$

Answer: $x = 5, y = -1$

b $3(x + 2) = 2y$ [1]

$7x - 6y = 146$ [2]

Rearrange equation [1]:

$3x - 2y = -6$ [3]

Multiply equation [3] by 3:

$9x - 6y = -18$ [4]

Subtract equation [2] from equation [4]:

$2x = -164$

$x = -82$

Substitute $x = -82$ in equation [1]:

$3(-82 + 2) = 2y$

$-240 = 2y$

$y = -120$

Answer: $x = -82, y = -120$

7 a Let \$$a$ be the cost of an adult ticket and \$$c$ be the cost of a child's ticket.

$4a + 5c = 160$ [1]

$3a + 7c = 159$ [2]

$3 \times$ equation [1]: $12a + 15c = 480$ [3]

$4 \times$ equation [2]: $12a + 28c = 636$ [4]

Subtract equation [3] from equation [4]:

$13c = 156$

$\therefore c = 12$

Substitute $c = 12$ in equation [1]:

$\therefore 4a + 5(12) = 160$

$\therefore 4a + 60 = 160$

$\therefore 4a = 100$

$\therefore a = 25$

An adult ticket costs \$25 and a child's ticket costs \$12.

b Let the cost, \$$C$, of the bill for the use of n units of electricity be $C = a + kn$, where \$$a$ is the fixed amount and \$$k$ the charge per unit.

$n = 1428, C = 235.90 \Rightarrow 235.90 = a + 1428k$ [1]

$n = 2240, C = 353.64 \Rightarrow 353.64 = a + 2240k$ [2]

Equations [2] − [1] $\Rightarrow 117.74 = 812k$

$\therefore k = \frac{117.74}{812}$

$\therefore k = 0.145$

Substitute in equation [1]:

$\therefore 235.90 = a + 1428(0.145)$

$\therefore a = 235.90 - 1428 \times 0.145$

$\therefore a = 28.84$

Therefore, $C = 28.84 + 0.145n$.

When $n = 3050$,

$C = 28.84 + 0.145 \times 3050$

$= 471.09$

The bill will be \$471.09.

8 Use CAS technology to obtain the following solution.

$x = 5,\ y = 8,\ z = -6$

9 Use CAS technology to obtain the following solution.

$x = 1,\ y = -2,\ z = 3$

10 Use CAS technology to obtain the following solutions.

a $x = 2, y = 1, z = 4$

b $x = 4, y = 2, z = -1$

c $x = -6, y = 8, z = 1$

d $x = -2,\ y = 5,\ z = 10$

e $x = 4, y = 3, z = 2$

f $x = -2, y = -11, z = 40$

11 Let each adult, concession and child's ticket cost \$$a$, \$$b$ and \$$c$ respectively.

$3a + 2b + 3c = 96$ [1]

$2a + b + 6c = 100$ [2]

$a + 4b + c = 72$ [3]

An adult ticket costs \$14, a concession ticket costs \$12 and a children's ticket costs \$10.

12 Let the hourly rates of pay for Agnes, Bjork and Chi be \$$a$, \$$b$ and \$$c$ respectively.

$2a + 3b + 4c = 194$ [1]

$4a + 2b + 3c = 191$ [2]

$2a + 5b + 2c = 180$ [3]

Agnes earns \$20/hour, Bjork earns \$18/hour and Chi earns \$25/hour.

13 Let the student use x fifty-cent coins, y twenty-cent coins and z ten-cent coins.

The student spends \$4.20: $0.50x + 0.20y + 0.10z = 4.20$

$\therefore 5x + 2y + z = 42$ [1]

The number of twenty-cent coins: $y = \frac{1}{2}z + 4x$

$\therefore 2y = z + 8x$

$\therefore 8x - 2y + z = 0$ [2]

Total number of coins: $x + y + z = 22$ [3]

The student uses 2 fifty-cent coins, 12 twenty-cent coins and 8 ten-cent coins.

14 Use CAS technology to obtain the following solutions.

a $x = 3,\ y = 1.5,\ z = -2.6$

b $x = 10,\ y = -6,\ z = -0.5,\ w = 5$

c $x_1 = -2,\ x_2 = -4,\ x_3 = \frac{2}{3},\ x_4 = \frac{1}{6}$

15 Let the amount of each of the supplements X, Y and Z used be x kg, y kg and z kg respectively.

Unsaturated fat: $0.06x + 0.10y + 0.08z = 6.8$ [1]

Saturated fat: $0.03x + 0.04y + 0.04z = 3.1$ [2]

Trans fat: $0.01x + 0.02y + 0.03z = 1.4$ [3]

The food compound needs to use 50 kg of supplement X, 30 kg of supplement Y and 10 kg of supplement Z.

1.3 Exam questions

1 $3y = -4x - 4$ [1]

$2x - \frac{y}{4} = 5$ [2]

In [1], move the term to the left-hand side. Multiply [2] by 2:

$4x + 3y = -4$

$4x - \frac{y}{2} = 10$

Subtract [2] from [1]:

$$3y + \frac{y}{2} = -14$$

$$\frac{7y}{2} = -14$$

$$7y = -28$$

$$y = -4$$

Substitute $y = -4$ into [1]:

$$-12 = -4x - 4$$

$$x = 2$$

$\therefore (x,\ y) = (2,\ -4)$

The correct answer is **B**.

2 $x - 4 = 4y + 8$ [1]

$3x - 6 = 2y + 20$ [2]

Multiply [1] by 3:

$3x - 12 = 12y + 24$

$3x - 6 = 2y + 20$

Subtract the equations:

$$-6 = 10y + 4$$

$$10y = -10$$

$$y = -1$$

The correct answer is **C**.

3 $y = x + 2$ [1]

$\frac{1}{4}x + y = 0$ [2]

Substitute [1] into [2]:

$$\frac{1}{4}x + (x + 2) = 0$$

$$\frac{5}{4}x = -2$$

$$5x = -8$$

$$x = \frac{-8}{5}$$

$$x = -1.6$$

Substitute $x = -1.6$ into [1]

$$y = -1.6 + 2$$

$$y = 0.4$$

Therefore, the solution is $(-1.6, 0.4)$.

The correct answer is **E**.

1.4 Linear graphs and their equations

1.4 Exercise

1 The two points are $(-3, 0)$ and $(0, -4)$.

$$m = \frac{-4 - 0}{0 - (-3)}$$

$$\therefore m = -\frac{4}{3}$$

2 a The two points are $(-2, 0)$ and $(0, 4)$.

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{0 - -2} = \frac{4}{2} = 2$$

b The two points are $(3, 0)$ and $(0, -2)$.

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 0}{0 - 3} = \frac{-2}{-3} = \frac{2}{3}$$

c The two points are $(0, 0)$ and $(-3, 1)$

$$\begin{aligned}\text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{1-0}{-3-0}\\ &= -\frac{1}{3}\end{aligned}$$

d The two points are $(7, -2)$ and $(2, 5)$.

$$\begin{aligned}\text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{5--2}{2-7}\\ &= -\frac{7}{5}\end{aligned}$$

3 a $(-3, 8), (-7, 18)$

$$\begin{aligned}m &= \frac{y_2 - y_1}{x_2 - x_1}\\ \therefore m &= \frac{18-8}{-7-(-3)}\\ &= \frac{10}{-4}\\ &= -2.5\end{aligned}$$

b $(0, -4), (12, 56)$

$$\begin{aligned}\therefore m &= \frac{56-(-4)}{12-0}\\ &= \frac{60}{12}\\ &= 5\end{aligned}$$

c $(-2, -5), (10, -5)$. Note that the line joining these points is horizontal.

$$\begin{aligned}\therefore m &= \frac{-5-(-5)}{10-(-2)}\\ &= \frac{0}{12}\\ &= 0\end{aligned}$$

d $(3, -3), (3, 15)$. Note that the line joining these points is vertical.

$$\begin{aligned}\therefore m &= \frac{15-(-3)}{3-3}\\ &= \frac{18}{0}\end{aligned}$$

Therefore, the gradient is undefined.

4 Gradient of the line through points (a, b) and $(-b, -a)$:

$$m = \frac{-a-b}{-b-a}$$

$\therefore m = 1$

Gradient of the line through points $(-c, d)$ and $(-d, c)$:

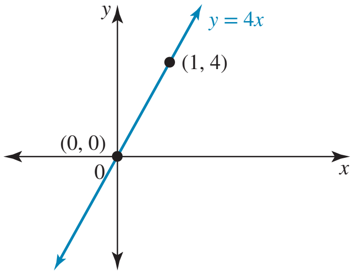
$$m = \frac{c-d}{-d-(-c)}$$

$$\therefore m = \frac{c-d}{-d+c}$$

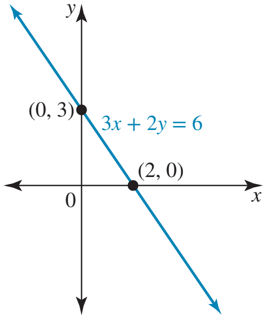
$\therefore m = 1$

As the gradients are the same, the lines are parallel.

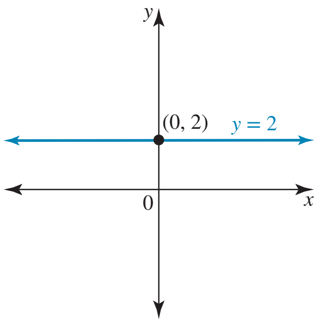
5 a



b



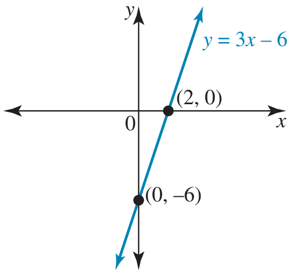
c



6 a $y = 3x - 6$

y-intercept: when $x = 0, y = -6$

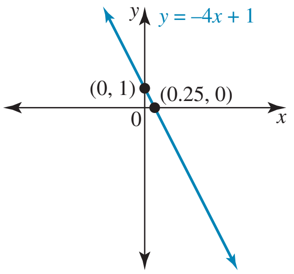
x-intercept: when $y = 0, x = 2$



b $y = -4x + 1$

y-intercept: when $x = 0, y = 1$

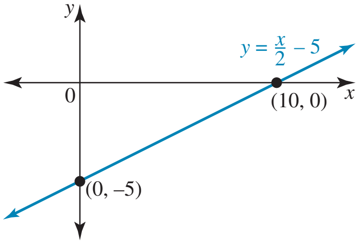
x-intercept: when $y = 0, x = \dfrac{1}{4}$



c $y = \frac{x}{2} - 5$

y-intercept: when $x = 0,\ y = -5$

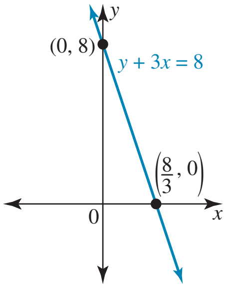
x-intercept: when $y = 0,\ x = 10$



d $y + 3x = 8$

y-intercept: when $x = 0,\ y = 8$

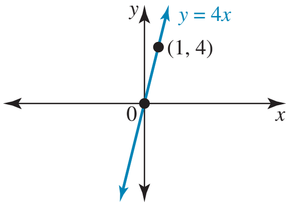
x-intercept: when $y = 0,\ x = \frac{8}{3}$



e $y = 4x$

Line passing through the origin: $(0, 0)$

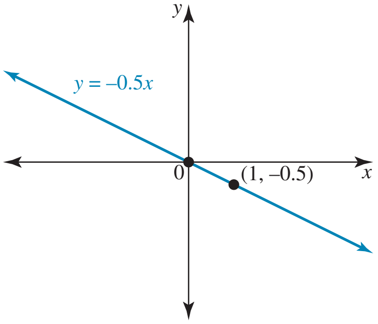
Additional point: when $x = 1,\ y = 4$, so the graph also passes through $(1, 4)$.



f $y = -0.5x$

Line passing through the origin: $(0, 0)$

Additional point: when $x = 1,\ y = -0.5$, so the graph also passes through $(1, -0.5)$.



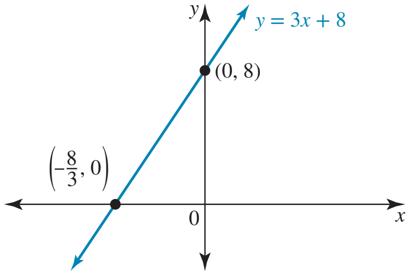
7 a $y = 3x + 8$

y-intercept $(0, 8)$

x-intercept: when $y = 0, 3x + 8 = 0$

$\therefore x = -\frac{8}{3}$

x-intercept $\left(-\frac{8}{3}, 0\right)$



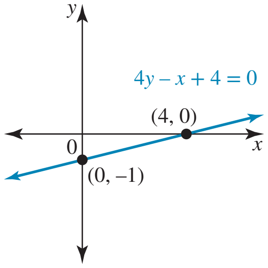
b $4y - x + 4 = 0$

y-intercept: when $x = 0, 4y + 4 = 0 \Rightarrow y = -1$

y-intercept $(0, -1)$

x-intercept: when $y = 0, -x + 4 = 0 \Rightarrow x = 4$

x-intercept $(4, 0)$



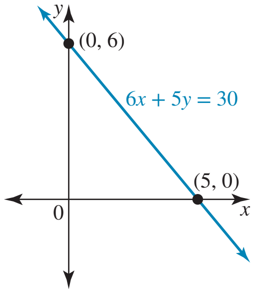
c $6x + 5y = 30$

y-intercept: when $x = 0, 5y = 30 \Rightarrow y = 6$

y-intercept $(0, 6)$

x-intercept: when $y = 0, 6x = 30 \Rightarrow x = 5$

x-intercept $(5, 0)$

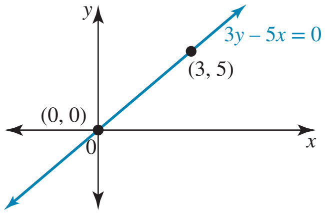


d $3y - 5x = 0$

When $x = 0, y = 0$, so the line contains the origin $(0, 0)$.

Second point: let $x = 3$, so $3y - 15 = 0 \Rightarrow y = 5$

A second point is $(3, 5)$.



e $\frac{x}{2} - \frac{3y}{4} = 6$

y-intercept: when $x = 0, -\frac{3y}{4} = 6$

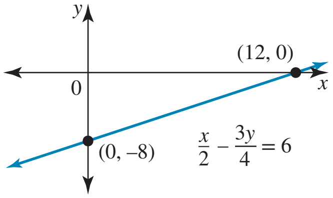
$\therefore -3y = 24$

$\therefore y = -8$

y-intercept $(0, -8)$

x-intercept: when $y = 0,\ \frac{x}{2} = 6 \Rightarrow x = 12$

x-intercept $(12, 0)$

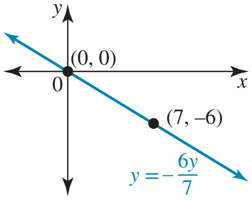


f $y = -\dfrac{6x}{7}$

When $x = 0$, $y = 0$, so the line contains the origin $(0, 0)$.

Second point: let $x = 7 \Rightarrow y = -6$

A second point is $(7, -6)$.



8 a Gradient -2, point $(-8, 3)$

$y - 3 = -2(x + 8)$

$\therefore y = -2x - 13$

b Points $(4, -1)$ and $(-3, 1)$

$m = \dfrac{1 + 1}{-3 - 4}$

$\therefore m = -\dfrac{2}{7}$

Equation of the line:

$y + 1 = -\dfrac{2}{7}(x - 4)$

$\therefore 7y + 7 = -2x + 8$

$\therefore 7y + 2x = 1$

c The gradient of the line is $-\dfrac{6}{4} = -\dfrac{3}{2}$ and its y-intercept gives $c = 6$. Therefore, the equation is $y = -\dfrac{3}{2}x + 6$.

d $6y - 5x - 18 = 0$

$\therefore 6y = 5x + 18$

$\therefore y = \dfrac{5x}{6} + 3$

$m = \dfrac{5}{6}, \; c = 3$

The gradient is $\dfrac{5}{6}$ and the coordinates of the y-intercept are $(0, 3)$.

9 Horizontal lines are parallel to the x-axis.
The horizontal line through the point $(2, 10)$ has the equation $y = 10$.

10 a $y = mx + c$

$m = 5, \; c = 2$

$\therefore y = 5x + 2$

b The two points are $\left(-\dfrac{1}{2}, 0\right)$ and $(0, -1)$.

$$\begin{aligned}\text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{-1 - 0}{0 - -\frac{1}{2}}\\ &= -2\end{aligned}$$

So, $m = -2$.

y-intercept $= (0, -1)$, so $c = -1$.

The equation of the line is found using:

$y = mx + c$

$y = -2x - 1$

c The equation of the line is found using:

$y - y_1 = m(x - x_1)$

$m = 3$ and $(x_1, y_1) = (1, 2)$

$y - 2 = 3(x - 1)$

$y - 2 = 3x - 3$

$y = 3x - 1$

d The equation of the line is found using $m = -5$ and $c = 0$.

$y = mx + c$

$y = -5x$

e The two points are $(-1, 0)$ and $(3, -2)$.

$$\begin{aligned}\text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{-2 - 0}{3 - -1}\\ &= -\frac{2}{4}\\ &= -\frac{1}{2}\end{aligned}$$

The equation of the line is found using:

$y - y_1 = m(x - x_1)$

$m = -\dfrac{1}{2}$ and $(x_1, y_1) = (-1, 0)$

$y - 0 = -\dfrac{1}{2}(x - -1)$

$2y = -(x + 1)$

$2y = -x - 1$

$x + 2y + 1 = 0$

f The equation of the line is found using $y - y_1 = m(x - x_1)$:

$y - 5 = -\dfrac{3}{4}(x - -3)$

$y - 5 = -\dfrac{3}{4}(x + 3)$

$4y - 20 = -3(x + 3)$

$4y - 20 = -3x - 9$

$3x + 4y - 11 = 0$

11 a i $m = \dfrac{\text{rise}}{\text{run}}$

$\therefore m = \dfrac{5}{4}$

The line contains the origin, so $c = 0$.

The equation is $y = \dfrac{5x}{4}$.

ii $m = \dfrac{\text{rise}}{\text{run}}$

$\therefore m = \dfrac{-9}{3}$

$\therefore m = -3$

The line cuts the y-axis at $(0, 9)$, so $c = 9$.

The equation is $y = -3x + 9$.

iii $m = \dfrac{\text{rise}}{\text{run}}$

$\therefore m = \dfrac{2}{3}$

The line cuts the y-axis at $(0, -2)$, so $c = -2$.

The equation is $y = \frac{2x}{3} - 2$.

iv $m = \frac{\text{rise}}{\text{run}}$

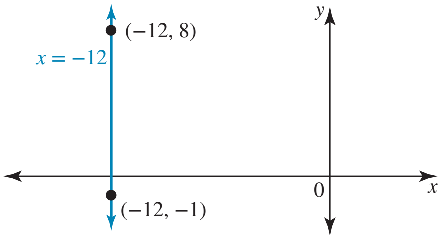
$\therefore m = \frac{-1}{2}$

The line cuts the y-axis at $(0, -1)$, so $c = -1$.

The equation is $y = -\frac{x}{2} - 1$.

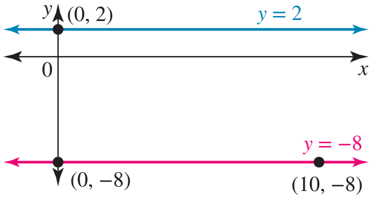
b Points $(-12, 8), (-12, -1)$

The vertical line with equation $x = -12$ contains these two points.



c The line $y = 2$ is horizontal, so any parallel line to it must also be horizontal.

The equation of the horizontal line containing $(10, -8)$ is $y = -8$.



12 a $y - y_1 = m(x - x_1), m = -5$ and point $(7, 2)$

$\therefore y - 2 = -5(x - 7)$

$\therefore y = -5x + 37$

b $y - y_1 = m(x - x_1), m = \frac{2}{3}$ and point $(-4, -6)$

$\therefore y + 6 = \frac{2}{3}(x + 4)$

$\therefore 3(y + 6) = 2(x + 4)$

$\therefore 3y + 18 = 2x + 8$

$\therefore 3y - 2x + 10 = 0$

c $y = mx + c, m = -\frac{7}{4}, c = -9$

$\therefore y = -\frac{7}{4}x - 9$

$\therefore 4y = -7x - 36$

$\therefore 4y + 7x + 36 = 0$

d $y - y_1 = m(x - x_1), m = -0.8$ and point $(0.5, -0.2)$

$\therefore y + 0.2 = -0.8(x - 0.5)$

$\therefore y = -0.8x + 0.4 - 0.2$

$\therefore y = -0.8x + 0.2$

e Points $(-1, 8), (-4, -2)$

Gradient: $m = \frac{-2 - 8}{-4 + 1} = \frac{10}{3}$

Equation: $y - y_1 = m(x - x_1), m = \frac{10}{3}$ and point $(-1, 8)$

$\therefore y - 8 = \frac{10}{3}(x + 1)$

$\therefore 3(y - 8) = 10(x + 1)$

$\therefore 3y - 24 = 10x + 10$

$\therefore 3y - 10x = 34$

f Points $(0, 10), (10, -10)$

Gradient: $m = \frac{-10 - 10}{10 - 0} = -2$

Equation: $y = mx + c, m = -2, c = 10$

$\therefore y = -2x + 10$

13 a $y = 2x - 8$ is in the form $y = mx + c$ with $m = 2$, $c = -8$.

The gradient is 2 and the y-intercept has coordinates $(0, -8)$.

b Rearrange the equation to make y the subject.

$5x - 3y - 6 = 0$

$5x - 6 = 3y$

$y = \frac{5}{3}x - 2$

The gradient is $\frac{5}{3}$.

c Rearrange the equation to make y the subject.

$4y - 3x = 4$

$4y = 3x + 4$

$y = \frac{3}{4}x + 1$

$y = \frac{3}{4}x + 1$ is in the form $y = mx + c$ with $m = \frac{3}{4}$, $c = 1$.

The gradient is $\frac{3}{4}$ and the y-intercept has coordinates $(0, 1)$.

d Rearrange each equation to make y the subject.

i $3x - 4y - 4 = 0$

$3x - 4 = 4y$

$y = \frac{3}{4}x - 1$

Gradient $= \frac{3}{4}$

ii $4y - 3x - 6 = 0$

$4y = 3x + 6$

$y = \frac{3}{4}x + \frac{3}{2}$

Gradient $= \frac{3}{4}$

iii $6x - 8y - 6 = 0$

$6x - 6 = 8y$

$y = \frac{6}{8}x - \frac{6}{8}$

$= \frac{3}{4}x - \frac{3}{4}$

Gradient $= \frac{3}{4}$

iv $2y - 6x - 12 = 0$

$2y = 6x + 12$

$y = 3x + 6$

Gradient $= 3$

Lines **i**, **ii** and **iii** are parallel as they all have the same gradient.

14 a $4x + 5y = 20$

Rearrange to the form $y = mx + c$.

$\therefore 5y = -4x + 20$

$\therefore y = \dfrac{-4x}{5} + \dfrac{20}{5}$

$\therefore y = -\dfrac{4}{5}x + 4$

$m = -\dfrac{4}{5}, c = 4$, so the gradient is $-\dfrac{4}{5}$ and the y-intercept is $(0, 4)$.

b $\dfrac{2x}{3} - \dfrac{y}{4} = -5$

Rearrange to the form $y = mx + c$.

$\therefore \dfrac{2x}{3} + 5 = \dfrac{y}{4}$

$\therefore y = 4\left(\dfrac{2x}{3} + 5\right)$

$\therefore y = \dfrac{8}{3}x + 20$

$m = \dfrac{8}{3}, c = 20$, so the gradient is $\dfrac{8}{3}$ and the y-intercept is $(0, 20)$.

c $x - 6y + 9 = 0$

Rearrange to the form $y = mx + c$.

$\therefore x + 9 = 6y$

$\therefore y = \dfrac{x}{6} + \dfrac{9}{6}$

$\therefore y = \dfrac{1}{6}x + \dfrac{3}{2}$

$m = \dfrac{1}{6}, c = \dfrac{3}{2}$, so the gradient is $\dfrac{1}{6}$ and the y-intercept is $\left(0, \dfrac{3}{2}\right)$.

d $2y - 3 = 0$

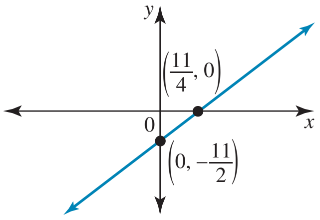
$\therefore y = \dfrac{3}{2}$, which is a horizontal line.

The gradient is 0 and the y-intercept is $\left(0, \dfrac{3}{2}\right)$.

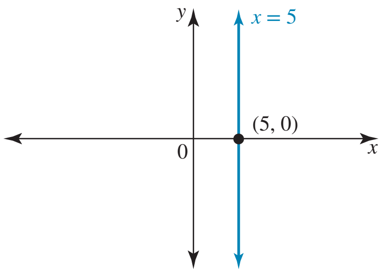
15 a To sketch the line $2y - 4x = -11$, one method is to type the equation in the Main Home screen, tap the graphing window so this screen is split in two areas, and then highlight and drag the equation to the graphing area. Alternatively, open the Graph and Tab menu.

The equation $2y - 4x = -11$ needs to be expressed with y as the subject. This can be done using solve $2y - 4x = -11$ on the Main screen for y (or do it yourself).

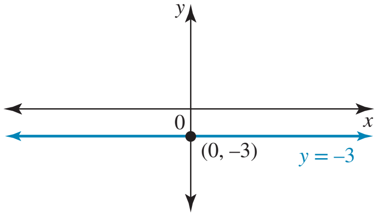
Enter the equation as $y_1 = 2x - \dfrac{11}{2}$ and tap the box to the left of y_1 to signal this is the graph to be sketched. Tap the graph symbol to activate the sketch, adjusting the window view if necessary.



b To sketch a vertical line, tap Type and select $x =$ Type. The equation $x = 5$ can then be entered and the graph obtained.



c Use $y =$ Type, enter the equation $y = 3$ and proceed to obtain the graph.



16 a $5y = -3x + 4$

The point $(2a, 2 - a)$ lies on the line.

Substitute $x = 2a, y = 2 - a$ in $5y = -3x + 4$.

$\therefore 5(2 - a) = -3(2a) + 4$

$\therefore 10 - 5a = -6a + 4$

$\therefore a = -6$

b Points $(p, q), (-p, -q)$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\therefore m = \dfrac{-q - q}{-p - p}$

$= \dfrac{-2q}{-2p}$

$= \dfrac{q}{p}$

Equation $y - y_1 = m(x - x_1)$, $m = \dfrac{q}{p}$ and point (p, q)

$\therefore y - q = \dfrac{q}{p}(x - p)$

$\therefore p(y - q) = q(x - p)$

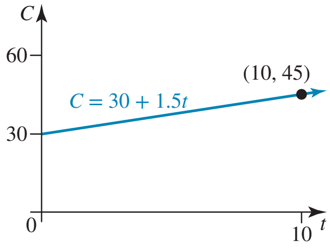
$\therefore py - pq = qx - qp$

$\therefore py = qx$

$\therefore y = \dfrac{qx}{p}$

17 a $C = 30 + 1.5t$

b



The gradient is 1.5.

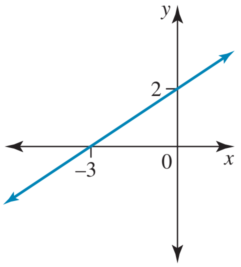
1.4 Exam questions

1 Let $(x_1, y_1) = (-3, 5)$.
Let $(x_2, y_2) = (1, -4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-4 - 5}{1 - (-3)}$$
$$= -\frac{9}{4}$$

The correct answer is **B**.

2



$-3y + 2x = -6$
x-intercept when $y = 0$:
$-3(0) + 2x = -6$
$x = -3$
Therefore, the x-intercept is $(-3,\ 0)$.
y-intercept when $x = 0$:
$-3y + 2(0) = -6$
$y = 2$
Therefore, the x-intercept is $(0,\ 2)$.
The correct answer is **C**.

3 $y - y_1 = m(x - x_1)$

$$m = \frac{3 - 1}{-1 - 3}$$
$$= \frac{2}{-4} \qquad \text{[1 mark]}$$
$$y - 1 = -\frac{1}{2}(x - 3)$$
$$y - 1 = -\frac{1}{2}x + \frac{3}{2}$$
$$y = -\frac{1}{2}x + \frac{5}{2} \qquad \text{[1 mark]}$$

1.5 Intersections of lines and their applications

1.5 Exercise

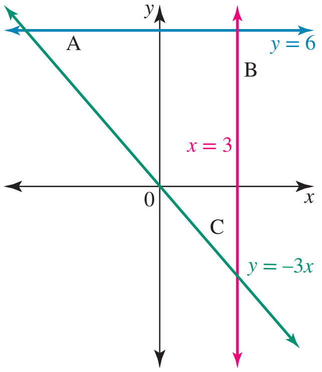
1 $3x - 2y = 15 \quad [1]$
$x + 4y = 54 \quad [2]$
$2 \times [1] + [2]$:
$6x - 4y = 30$
$x + 4y = 54$
$\therefore 7x = 84$
$\therefore x = 12$
Substitute into equation [2]:
$\therefore 12 + 4y = 54$
$\therefore 4y = 42$
$\therefore y = 10.5$
The point of intersection is $(12, 10.5)$.

2 **a** Use simultaneous equations to obtain the point of intersection.
$4x - 3y = 13 \quad [1]$
$2y - 6x = -7 \quad [2]$
Eliminate y:
$2\times$ equation [1]: $8x - 6y = 26 \quad [3]$
$3\times$ equation [2]: $-18x + 6y = -21 \quad [4]$
Add equations [3] and [4]:
$\therefore -10x = 5$
$$\therefore x = -\frac{5}{10}$$
$$\therefore x = -\frac{1}{2}$$
Substitute $x = -\frac{1}{2}$ in equation [2]:
$\therefore 2y + 3 = -7$
$\therefore 2y = -10$
$\therefore y = -5$
The point of intersection is $\left(-\frac{1}{2}, -5\right)$.

b $$y = \frac{3x}{4} - 9 \quad [1]$$
$x + 5y + 7 = 0 \quad [2]$
Substitute equation [1] in equation [2]:
$$\therefore x + 5\left(\frac{3x}{4} - 9\right) + 7 = 0$$
$$\therefore x + \frac{15x}{4} - 45 + 7 = 0$$
$$\therefore \frac{19x}{4} = 38$$
$$\therefore x = 38 \times \frac{4}{19}$$
$$\therefore x = 8$$
Substitute $x = 8$ in equation [1]:
$$\therefore y = \frac{3}{4} \times 8 - 9$$
$\therefore y = -3$
The point of intersection is $(8, -3)$.

c The horizontal line $y = -5$ and the vertical line $x = 7$ would intersect at the point $(7, -5)$.

3 **a**



Let the triangle enclosed by the horizontal, vertical and oblique line through the origin have vertices A, B and C. Point B has coordinates $(3, 6)$.

Point A is the intersection of $y = -3x$ and $y = 6$.
$\therefore 6 = -3x$
$\therefore x = -2$
A has coordinates $(-2, 6)$.
Point C is the intersection of $y = -3x$ and $x = 3$.
$\therefore y = -3 \times 3$
$\therefore y = -9$
C has coordinates $(3, -9)$.
The vertices of the triangle are $(-2, 6)$, $(3, 6)$ and $(3, -9)$.

b The triangle is right-angled at B. Side AB has a length of 5 units and side BC has a length of 15 units.

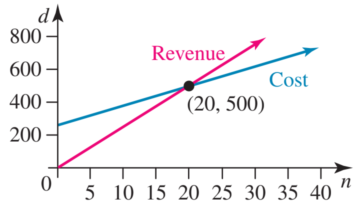
Area of triangle ABC: $A = \frac{1}{2}bh, b = 5, h = 15$

$$\therefore A = \frac{1}{2} \times 5 \times 15$$
$$= 37.5$$

The area of the triangle is 37.5 square units.

4 Revenue $d_R = 25n$ and cost $d_C = 260 + 12n$

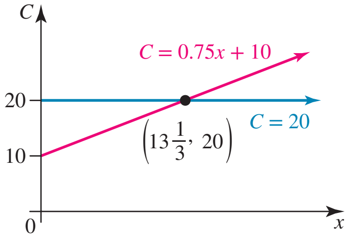
a At the point of intersection (break-even point), $d_R = d_C$.
$\therefore 25n = 260 + 12n$
$\therefore 13n = 260$
$\therefore n = 20$
If $n = 20, d = 500$, so the point of intersection is $(20, 500)$.
Revenue: $(0, 0), (20, 500)$
Cost: $(0, 260), (20, 500)$



b As the gradient of the revenue graph is larger than the gradient of the cost graph, if $n > 20, d_R > d_C$.
Therefore, at least 21 items must be sold for a profit to be made.
An alternative method is to solve the inequation $d_R > d_C$.
$\therefore 25n > 260 + 12n$
$\therefore 13n > 260$
$\therefore n > 20$

5 **a** Let the cost be \$$C$ for a distance of x km.
Pedal On: $C = 10 + 0.75x$
Bikes R Gr8: $C = 20$

b Points on the line $C = 10 + 0.75x$ could be $(0, 10)$ and $(4, 13)$.
$C = 20$ is a horizontal line.



c When costs are equal, $10 + 0.75x = 20$.
$\therefore 0.75x = 10$
$\therefore \frac{3}{4}x = 10$
$\therefore x = 10 \times \frac{4}{3}$
$\therefore x = \frac{40}{3}$
After $13\frac{1}{3}$ km, the costs are equal.

d Shay needs to estimate whether the number of kilometres to be ridden will be greater than or less than $13\frac{1}{3}$ km.
If the distance is less than $13\frac{1}{3}$ km, Pedal On is cheaper, but if the distance is greater than $13\frac{1}{3}$ km, then Bikes R Gr8 is cheaper.

6 $2mx + 3y = 2m$
$4x + y = 5$
For no solutions, the two equations must represent parallel lines and therefore must have the same gradient.
The system becomes:

$$2mx + 3y = 2m \Rightarrow y = -\frac{2mx}{3} + \frac{2m}{3}$$
$$4x + y = 5 \Rightarrow y = -4x + 5$$

Hence, equating gradients, $-\frac{2m}{3} = -4$
$\therefore m = 6$

Checking y-intercepts are not equal, $\frac{2m}{3} = 4 \neq 5$
So the lines are parallel, not the same line.
Therefore, $m = 6$ for the system to not have any solutions.

7 $ax + y = b$
$3x - 2y = 4$
For infinitely many solutions, the two equations must be identical.
The system becomes:
$y = -ax + b$
$y = \frac{3x}{2} - 2$
Equating gradients and y-intercepts,
$-a = \frac{3}{2}$
$\therefore a = -1.5$
Therefore, $a = -1.5$ and $b = -2$ for infinite solutions.

8 The lines will not intersect if they are parallel, in which case they will have the same gradients but different y-intercepts.
Rearranging $2x + 3y = 23$,
$3y = -2x + 23$
$\therefore y = \frac{-2}{3}x + \frac{23}{3}$
$\therefore m_1 = -\frac{2}{3},\ c_1 = \frac{23}{3}$
Rearranging $7x + py = 8$,
$py = -7x + 8$
$\therefore y = \frac{-7}{p}x + \frac{8}{p}$
$\therefore m_2 = -\frac{7}{p},\ c_2 = \frac{8}{p}$
The lines have the same gradients.

$\therefore -\frac{2}{3} = -\frac{7}{p}$

$\therefore 2p = 21$

$\therefore p = \frac{21}{2}$

With $p = \frac{21}{2}, c_2 = \frac{8}{\frac{21}{2}}$ and $c_1 = \frac{23}{3}$.

$\therefore c_2 = \frac{16}{21}$

$\therefore c_2 \neq c_1$

If $p = \frac{21}{2}$, the lines will not intersect.

9 **a** $px + 5y = q$

$\therefore 5y = -px + q$

$\therefore y = -\frac{px}{5} + \frac{q}{5}$

$3x - qy = 5q$

$\therefore 3x - 5q = qy$

$\therefore y = \frac{3x}{q} - 5$

b For infinitely many solutions, the two equations must be identical. This means they will have the same gradients and the same y-intercepts.

Equating gradients: $-\frac{p}{5} = \frac{3}{q}$ [1]

Equating y-intercepts: $\frac{q}{5} = -5$ [2]

Equation [2] gives $q = -25$. Substitute this in equation [1]:

$\therefore -\frac{p}{5} = \frac{3}{-25}$

$\therefore p = \frac{3}{-25} \times -5$

$\therefore p = +\frac{3}{5}$

For infinitely many solutions, $p = \frac{3}{5}$ and $q = -25$.

c Two lines will intersect if they do not have the same gradients.

Therefore, $-\frac{p}{5} \neq \frac{3}{q}$ or $pq \neq -15$.

10 Graph the two lines $y = \frac{17 + 9x}{5}$ and $y = 8 - \frac{3x}{2}$.

Then tap Analysis → G-Solve → intersect to obtain the coordinates of the point of intersection to 2 decimal places as $(1.39, 5.91)$.

11 The equations of each line need to be formed before using simultaneous equations to obtain their point of intersection.

Line 1: $y - y_1 = m(x - x_1)$, $m = -2$ and a point is $(4, -8)$.

$\therefore y + 8 = -2(x - 4)$

$\therefore y = -2x$ [1]

Line 2: $y = mx + c, m = 3, c = 5$

$\therefore y = 3x + 5$ [2]

At intersection at point Q,

$-2x = 3x + 5$

$\therefore -5x = 5$

$\therefore x = -1$

When $x = -1$, equation [1] gives $y = 2$.

The point Q has coordinates $(-1, 2)$.

12 $2x + 3y = 0$

$x - 8y = 19$

Multiply the second equation by 2, then subtract from the first equation:

$2x + 3y = 0$

$2x - 16y = 38$

$\therefore 19y = -38$

$\therefore y = -2$

$\therefore x = 3$

Two of the lines intersect at $(3, -2)$. Test if this point lies on the third line, $9x + 5y = 17$.

LHS $= 9(3) + 5(-2)$

$= 27 - 10$

$= 17$

$=$ RHS

Therefore, the three lines are concurrent since they meet at the point $(3, -2)$.

13 Point of intersection of $x + 4y = 13$ and $5x - 4y = 17$:

$x + 4y = 13$

$5x - 4y = 17$

Add the equations:

$6x = 30$

$\therefore x = 5$

$\therefore 5 + 4y = 13$

$\therefore 4y = 8$

$\therefore y = 2$

This point $(5, 2)$ must lie on the third line, $-3x + ay = 5$, since the lines are concurrent.

$-3(5) + a(2) = 5$

$\therefore 2a = 20$

$\therefore a = 10$

14 If the three lines are concurrent, they must intersect at the same point.

$3x - y + 3 = 0$ [1]

$5x + 2y + 16 = 0$ [2]

$9x - 5y + 3 = 0$ [3]

Find the point of intersection of the lines given by equations [1] and [2].

$2 \times$ equation [1] + equation [2]:

$11x + 22 = 0$

$\therefore x = -2$

Substitute $x = -2$ in equation [1]:

$\therefore -6 - y + 3 = 0$

$\therefore y = -3$

Therefore, the lines [1] and [2] intersect at $[-2, -3]$.

Test if this point satisfies equation [3]:

LHS $= 9x - 5y + 3$

$= 9 \times -2 - 5 \times -3 + 3$

$= -18 + 15 + 3$

$= 0$

$=$ RHS

Since all three lines contain the point $(-2, -3)$, the lines are concurrent and their point of concurrency is $(-2, -3)$.

15 Obtain the point of intersection of two of the lines.

$x + 4y = 9$ [1]

$3x - 2y = -1$ [2]

Equation [1] + $2 \times$ equation [2]:

$7x = 7$

$\therefore x = 1$

Substitute $x = 1$ in equation [1]:

$\therefore 1 + 4y = 9$

$\therefore 4y = 8$

$\therefore y = 2$

The two lines intersect at $(1, 2)$.

For the three lines to not be concurrent, the point $(1, 2)$ cannot be on the line $4x + 3y = d$.

$\therefore d \neq 4 \times 1 + 3 \times 2$

$\therefore d \neq 10$

The lines will not be concurrent if d is any real number except 10.

16 **a** Model A has the higher y-intercept.

Therefore, the cost model for A is $C = 300 + 0.05x$ and the cost model for B is $C = 250 + 0.25x$.

b The gradients give the cost per kilometre.

c
$$\begin{aligned} y &= C_A - C_B \\ &= (300 + 0.05x) - (250 + 0.25x) \\ &= 50 - 0.2x \end{aligned}$$

$\therefore y = 50 - 0.2x$

d y-intercept: $(0, 50)$

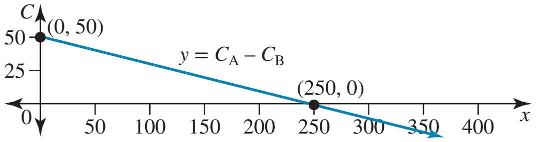
x-intercept: let $y = 0$.

$\therefore 0 = 50 - 0.2x$

$\therefore 0.2x = 50$

$\therefore x = 250 \Rightarrow (250, 0)$

The graph of $y = C_A - C_B$ is shown in the diagram.



e **i** If $C_A = C_B$, then $y = 0$. Since $y = 0$, when $x = 250$, the costs are equal when the number of kilometres travelled is 250 km.

ii If $C_A < C_B$, then $C_A - C_B < 0$, so $y < 0$. This occurs when $x > 250$. Therefore, company A is cheaper if the distance travelled is more than 250 km.

17 **a** At 6 am the boat's position is at $(0, 2)$, and at 7 am its position is at $(6, 3)$. It travels on a straight line through these points. For this line, $m = \dfrac{3-2}{6-0} = \dfrac{1}{6}$, $c = 2$.

The equation of the boat's path is $y = \dfrac{x}{6} + 2$, $x \geq 0$.

b When $x = 6t$,

$$\begin{aligned} y &= \frac{6t}{6} + 2 \\ &= t + 2 \end{aligned}$$

The boat's position north of the lookout is $(t + 2)$ km.

c As t measures the time after 6 am, at 9:30 am, $t = 3.5$.

When $t = 3.5$,

$x = 6(3.5)$ and $y = 3.5 + 2$

$= 21 \qquad = 5.5$

Therefore, the boat is at the position $(21, 5.5)$.

d At t hours after 6 am, the trawler's position is given by $x = \dfrac{4t-1}{3}, y = t$.

At 6 am, $t = 0$, so its position is $\left(\dfrac{1}{3}, 0\right)$.

At 7 am, $t = 1$, so its position is $(1, 1)$.

The line joining these points has gradient

$$\begin{aligned} m &= \frac{1-0}{1+\frac{1}{3}} \\ &= \frac{1}{\frac{4}{3}} \\ &= \frac{3}{4} \end{aligned}$$

Its equation is $y - y_1 = m(x - x_1)$ with $m = \dfrac{3}{4}$ and a point $\left(-\dfrac{1}{3}, 0\right)$.

$\therefore y = \dfrac{3}{4}\left(x + \dfrac{1}{3}\right)$

$\therefore y = \dfrac{3x}{4} + \dfrac{1}{4}$

The equation of the trawler's path is $y = \dfrac{3x}{4} + \dfrac{1}{4}$, $x \geq -\dfrac{1}{3}$.

e The common point of the two paths is their point of intersection.

$y = \dfrac{x}{6} + 2 \qquad [1]$

$y = \dfrac{3x}{4} + \dfrac{1}{4} \qquad [2]$

At the intersection,

$\dfrac{x}{6} + 2 = \dfrac{3x}{4} + \dfrac{1}{4}$

Multiply both sides by 12:

$\therefore 2x + 24 = 9x + 3$

$\therefore 21 = 7x$

$\therefore x = 3$

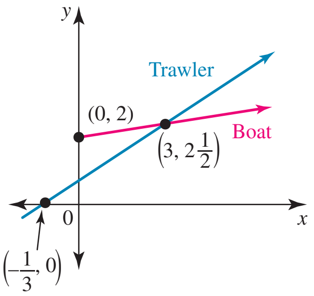
Substitute $x = 3$ in equation [1]:

$\therefore y = \dfrac{3}{6} + 2$

$\therefore y = 2\dfrac{1}{2}$

The common point of the paths is $\left(3, 2\dfrac{1}{2}\right)$.

f



Although there is a common point to the paths, the boat and the trawler will only collide if they are at this common point at the same time.

Boat: $x = 6t$, $y = t + 2$

At the common point where $x = 3, y = 2.5, t = 0.5$, so the boat is at this point at 6:30 am.

Trawler: $x = \dfrac{4t-1}{3}, y = t$

At the common point where $x = 3, y = 2.5, t = 2.5$, so the trawler is not at this point until 8:30 am.

Therefore, the two boats do not collide.

18 **a** In the graphing screen select Param Type and then enter $xt1 = t, yt1 = 3 + 2t$ and $xt2 = t + 1, yt2 = 4t - 1$. Graph both lines. To graph the lines simultaneously, alter the t step to 1, for example.

Tapping Analysis → G-Solve → trace gives an estimate of the common point as occurring when $t = 4$.

Tap the table of values symbol to compare the positions of each graph around $t = 4$. The table of values shows that when $t = 4$, particle P_1 is at $(4, 11)$, but the second particle P_2 is at $(4, 11)$ when $t = 3$. This identifies the common point on the paths to be $(4, 11)$, but the two particles are in this position at different times and therefore do not collide.

b From the table of values, the common point is $(4, 11)$.

1.5 Exam questions

1 $y = mx + c$

$y = 2x - 1$

$m = 2,\ c = -1$

$y = 2x + 1$

$m = 2,\ c = 1$

Therefore, the two lines have the same gradient but different y-intercepts, so they are parallel and will never intersect.

The correct answer is **C**.

2 Parallel lines have the same gradient and do not intersect.

For the gradients to be equal:

$4(2a + 1) = 2(a - 1)$

$8a + 4 = 2a - 2$

$a = -1$

The correct answer is **B**.

3 Linear equations that have equal gradients and equal y-intercepts have infinitely many solutions.

$ax + y = b$

$x + 3y = 2$

Rewrite the equations in $y = mx + c$ form:

$y = -ax + b$ [1]

$3y = -x + 2$ [2]

$y = -\frac{1}{3}x + \frac{2}{3}$

$m_1 = -a,\ m_2 = -\frac{1}{3}$ [1 mark]

$-a = -\frac{1}{3}$

$\therefore a = \frac{1}{3}$ [1 mark]

$c_1 = b,\ c_2 = \frac{2}{3}$ [1 mark]

$\therefore b = \frac{2}{3}$ [1 mark]

Therefore, the equations will have infinitely many solutions when $a = \frac{1}{3}$ and $b = \frac{2}{3}$. [1 mark]

1.6 Straight lines and gradients

1.6 Exercise

1 A $(-3, -12)$, B $(0, 3)$

$$m = \frac{3 - (-12)}{0 - (-3)}$$

$\therefore m_{\text{AB}} = 5$

B $(0, 3)$, C $(4, 23)$

$$m = \frac{23 - 3}{4 - 0}$$

$\therefore m_{\text{BC}} = 5$

Since $m_{\text{AB}} = m_{\text{BC}}$ and point B is common, the points A, B and C are collinear.

2 A $(-4, 13)$, B $(7, -9)$ and C $(12, -19)$

$$m_{\text{AB}} = \frac{-9 - 13}{7 + 4} \text{ and } m_{\text{BC}} = \frac{-19 + 9}{12 - 7}$$

$$= -\frac{22}{11} \qquad = -\frac{10}{5}$$

$$= -2 \qquad = -2$$

Since $m_{\text{AB}} = m_{\text{BC}}$ and point B is common, the three points are collinear.

3 Let the points be A $(3, b)$, B $(4, 2b)$ and C $(8, 5 - b)$.

The points are collinear if $m_{\text{AB}} = m_{\text{BC}}$.

$$\therefore \frac{2b - b}{4 - 3} = \frac{5 - b - 2b}{8 - 4}$$

$$\therefore b = \frac{5 - 3b}{4}$$

$\therefore 4b = 5 - 3b$

$\therefore 7b = 5$

$\therefore b = \frac{5}{7}$

4 P $(-6, -8)$, Q $(6, 4)$

$$m_{\text{PQ}} = \frac{-8 - 4}{-6 - 6}$$

$\therefore m_{\text{PQ}} = 1$

Equation of PQ:

$y - 4 = 1(x - 6)$

$\therefore y = x - 2$

Test point R $(-32, 34)$.

If $x = -32$,

$y = -32 - 2$

$= -34$

$\neq 34$

Therefore, P, Q and R are not collinear.

5 A $(-15, -95)$, B $(12, 40)$ and C $(20, 75)$

If the points can be joined to form a triangle, then they cannot be collinear.

$$m_{\text{AB}} = \frac{40 + 95}{12 + 15} \text{ and } m_{\text{BC}} = \frac{75 - 40}{20 - 12}$$

$$= \frac{135}{27} \qquad = \frac{35}{8}$$

$$= 5 \qquad \neq 5$$

Since $m_{\text{AB}} \neq m_{\text{BC}}$, the points A, B and C are not collinear, so they may be joined to form a triangle ABC.

6 **a** $3y - 6x = 1$

$\therefore 3y = 6x + 1$

$\therefore y = 2x + \frac{1}{3}$

$m = 2$

i The parallel line has gradient 2.

ii The perpendicular line has gradient $-\frac{1}{2}$.

b The gradient of $y = x$ is $m_1 = 1$ and the gradient of $y = -x$ is $m_2 = -1$.

Since $m_1 m_2 = -1$, the lines are perpendicular.

c The gradient of $y = 5x + 10$ is 5, so the gradient of the perpendicular line is $-\frac{1}{5}$. The point is $(1, 1)$.

Equation of the line:

$$y - 1 = -\frac{1}{5}(x - 1)$$

$\therefore 5y - 5 = -x + 1$

$\therefore 5y + x = 6$

7 **a** Rearrange to make y the subject.

$$x - 2y = 6$$
$$2y = x - 6$$
$$y = \frac{x}{2} - 3$$

The gradient of the line is $\frac{1}{2}$.

The gradient of the parallel line is $\frac{1}{2}$.

b Rearrange to make y the subject.

$$3y - 4x + 2 = 0$$
$$3y = 4x - 2$$
$$y = \frac{4}{3}x - \frac{2}{3}$$

The gradient of the line is $\frac{4}{3}$, so the gradient of the parallel line is $\frac{4}{3}$.

c The gradient of the line $y = 3x - 4$ is 3.

The relationship between perpendicular lines is $m_1m_2 = -1$.

Therefore, the gradient of the perpendicular line is $-\frac{1}{3}$.

d Rearrange to make y the subject.

$$4y - 2x = 8$$
$$4y = 2x + 8$$
$$y = \frac{x}{2} + 2$$

The gradient of the line is $\frac{1}{2}$.

The relationship between perpendicular lines is $m_1m_2 = -1$.

Therefore, the gradient of the perpendicular line is -2

8 **a** Rearrange to make y the subject.

$$5x + 4y - 1 = 0$$
$$4y = -5x + 1$$
$$y = -\frac{5}{4} + \frac{1}{4}$$

The gradient of the line is $-\frac{5}{4}$.

The relationship between perpendicular lines is $m_1m_2 = -1$.

Therefore, the gradient of the perpendicular line is $\frac{4}{5}$.

b The two points are $(-3, 5)$ and $(2, -7)$.

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-7 - 5}{2\ -\ -3}$$
$$= -\frac{12}{5}$$

The relationship between perpendicular lines is $m_1m_2 = -1$.

Therefore, the gradient of the perpendicular line is $\frac{5}{12}$.

c The two points are $(2, 4)$ and $(7, -1)$.

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1 - 4}{7 - 2}$$
$$= -\frac{5}{5}$$
$$= -1$$

The relationship between perpendicular lines is $m_1m_2 = -1$.

Therefore, the gradient of the perpendicular line is 1.

d The gradient of $y = 0.2x$ is 0.2 and the gradient of $y = -5x$ is -5.

If the lines are perpendicular, $m_1m_2 = -1$.

$0.2 \times -5 = -1$

The lines are perpendicular.

9 **a** The line $7y - 5x = 0$ rearranges to $y = \frac{5x}{7}$. Its gradient is $\frac{5}{7}$.

The parallel line has the same gradient of $\frac{5}{7}$ and a y-intercept at $(0,\ 6)$. Its equation is

$y = mx + c,\ m = \frac{5}{7}, c = 6$

$$\therefore y = \frac{5x}{7} + 6$$

Rearranging to the required form,

$$\therefore 7y = 5x + 42$$
$$\therefore 5x - 7y = -42$$

b $3y + 4x = 2$

$$3y + 4x = 2$$
$$\therefore 3y = -4x + 2$$
$$\therefore y = -\frac{4x}{3} + \frac{2}{3}$$

The parallel line has $m = -\frac{4}{3}$.

Equation of the parallel line: $y - y_1 = m(x - x_1)$, $m = -\frac{4}{3}$, point $\left(-2, \frac{4}{5}\right)$

$$\therefore y - \frac{4}{5} = -\frac{4}{3}(x + 2)$$

Multiply both sides by 15:

$$\therefore 15y - 12 = -20(x + 2)$$
$$\therefore 15y - 12 = -20x - 40$$
$$\therefore 20x + 15y = -28$$

c $2x - 3y + 7 = 0$

$$\therefore 2x + 7 = 3y$$
$$\therefore y = \frac{2x}{3} + \frac{7}{3}$$

$m_1 = \frac{2}{3}$, so the gradient of the perpendicular line is $m_2 = -\frac{3}{2}$, since $m_1m_2 = -1$.

Equation of the perpendicular line: $y - y_1 = m(x - x_1)$, $m = -\frac{3}{2}$, point $\left(-\frac{3}{4}, 1\right)$

$$\therefore y-1=-\frac{3}{2}\left(x+\frac{3}{4}\right)$$
$$\therefore 2y-2=-3\left(x+\frac{3}{4}\right)$$
$$\therefore 2y-2=-3x-\frac{9}{4}$$
$$\therefore 8y-8=-12x-9$$
$$\therefore 12x+8y=-1$$

d $3x-y=2$

$$\therefore y=3x-2$$
$$m_1=3\Rightarrow m_2=-\frac{1}{3}$$

Equation of perpendicular line through the origin:

$$y=mx+c,\ m=-\frac{1}{3}, c=0$$
$$\therefore y=-\frac{1}{3}x$$
$$\therefore 3y=-x$$
$$\therefore x+3y=0$$

10 $2y=4x+7$

$$\therefore y=2x+\frac{7}{2}$$
$$m=2$$

The parallel line has gradient 2 and point $(8,-2)$.

The equation of the line is:

$$y+2=2(x-8)$$
$$\therefore y=2x-18$$

For the x-intercept, let $y=0$.

$$\therefore 0=2x-18$$
$$\therefore x=9$$

The coordinates of the x-intercept are $(9,0)$.

11 **a** Points $(1,-8)$ and $(5,-2)$

$$m=\frac{-2+8}{5-1}$$
$$\therefore m=\frac{3}{2}$$
$$\therefore \tan\theta=\frac{3}{2}$$
$$\therefore \theta=\tan^{-1}(1.5)$$
$$\therefore \theta=56.31^\circ$$

b Gradient of -2

$$\tan\theta=-2$$
$$\therefore \theta=180^\circ-\tan^{-1}(2)$$
$$\therefore \theta=116.57^\circ$$

c $m=\tan(135)^\circ$, point $(2,7)$

$$\therefore m=-1$$

The equation of the line is:

$$y-7=-(x-2)$$
$$\therefore y=-x+9$$

12 **a** $m=\tan\ \theta$

$$=\tan\ (40^\circ)$$
$$=0.839$$

b $m=\tan\theta$

$$=\tan\ (145^\circ)$$
$$=-0.7$$

13 **a** $m=\tan\theta$

$$0.5=\tan\theta$$
$$\theta=\tan^{-1}(0.5)$$
$$\theta=26.6^\circ$$

b $m=\tan\theta$

$$-0.5=\tan\theta$$
$$\theta=180^\circ-\tan^{-1}(0.5)$$
$$\theta=180^\circ-26.6^\circ$$
$$\theta=153.4^\circ$$

14 **a** The gradient of the line is obtained using the two given points $(-2,0)$ and $(3,9)$.

$$m=\frac{9-0}{3+2}$$
$$=\frac{9}{5}$$
$$=1.8$$
$$\therefore \tan\theta=1.8$$
$$\therefore \theta=\tan^{-1}(1.8)$$
$$\therefore \theta\simeq 60.95^\circ$$

The angle is 60.95° to 2 decimal places.

b Points $(4,0)$ and $(0,3)$

$$m=\frac{3-0}{0-4}$$
$$=-\frac{3}{4}$$
$$\therefore \tan\theta=-\frac{3}{4}$$
$$\therefore \theta=180^\circ-\tan^{-1}(0.75)$$
$$\therefore \theta\simeq 143.13^\circ$$

The angle is 143.13° to 2 decimal places.

c A line parallel to the y-axis must be a vertical line. Therefore, the angle of inclination to the x axis is 90°.

d Given $m=-7$, $\tan\theta=-7$

$$\therefore \theta=180^\circ-\tan^{-1}(7)$$
$$\therefore \theta\simeq 98.13^\circ$$

15 Angle of inclination to horizontal of line with gradient of 5:

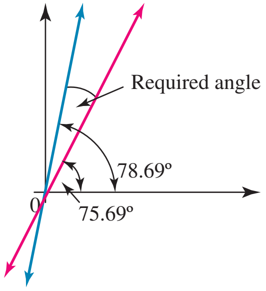
$$\tan\theta=5$$
$$\therefore \theta=\tan^{-1}(5)$$
$$\therefore \theta=78.69^\circ$$

Angle of inclination to horizontal of line with gradient of 4:

$$\tan\theta=4$$
$$\therefore \theta=\tan^{-1}(4)$$
$$\therefore \theta=75.96^\circ$$

Therefore, the angle between the two lines is the difference between the larger and the smaller angle:

$78.69^\circ-75.96^\circ=2.73^\circ$



16 Point $(-6,12)$

$m=\tan\theta$ where $\theta=\tan^{-1}(1.5)$

$$\therefore m=1.5$$

Equation: $y-y_1=m(x-x_1)$

$\therefore y - 12 = 1.5(x + 6)$

$\therefore y = 1.5x + 9 + 12$

$\therefore y = 1.5x + 21$

$\therefore 2y = 3x + 42$

$\therefore 3x - 2y = -42$

17 **a** Point $(4, 0)$

Gradient: $m = \tan(123.69°)$

$\therefore m \simeq -1.5$

Equation: $y - 0 = -1.5(x - 4)$

Therefore, the equation of line L is $y = -1.5x + 6$.

b K and L are perpendicular lines.

$$m_L = -1.5$$
$$= -\frac{3}{2}$$
$$\therefore m_K = \frac{2}{3}$$

Equation of K: $m = \frac{2}{3}$, point $(4, 0)$

$$\therefore y = \frac{2}{3}(x - 4)$$
$$\therefore y = \frac{2x}{3} - \frac{8}{3}$$

or $2x - 3y = 8$

c The y-intercepts of lines K and L are $\left(0, -\frac{8}{3}\right)$ and $(0, 6)$ respectively, so the distance between them is $6 + \frac{8}{3} = \frac{26}{3}$ units.

18 **a** **i** $ax - 7y = 8$ is parallel to $3y + 6x = 7$.

Rearranging each equation,

$$ax - 7y = 8$$
$$\therefore ax - 8 = 7y$$
$$\therefore y = \frac{ax}{7} - \frac{8}{7}$$
$$\therefore m_1 = \frac{a}{7}$$
$$3y + 6x = 7$$
$$\therefore 3y = -6x + 7$$
$$\therefore y = -2x + \frac{7}{3}$$
$$\therefore m_2 = -2$$

For parallel lines, $m_1 = m_2$

$$\therefore \frac{a}{7} = -2$$
$$\therefore a = -14$$

ii For perpendicular lines, $m_1 m_2 = -1$

$$\therefore \frac{a}{7} \times -2 = -1$$
$$\therefore -2a = -7$$
$$\therefore a = \frac{7}{2}$$

b The gradient of the line through $(2c, -c)$ and $(c, -c - 2)$ is:

$$m = \frac{-c - 2 + c}{c - 2c}$$
$$= \frac{-2}{-c}$$
$$= \frac{2}{c}$$

Since $m = \tan\left(45^{\text{o}}\right)$,

$$\frac{2}{c} = 1$$
$$\therefore c = 2$$

c Points $(d + 1, d - 1)$ and $(4, 8)$

$$m = \frac{d - 1 - 8}{d + 1 - 4}$$
$$= \frac{d - 9}{d - 3}$$

i Parallel to the line through $(7, 0)$ and $(0, -2)$ with gradient $= \frac{\text{rise}}{\text{run}} = \frac{2}{7}$

$$\therefore \frac{d - 9}{d - 3} = \frac{2}{7}$$
$$\therefore 7d - 63 = 2d - 6$$
$$\therefore 5d = 57$$
$$\therefore d = \frac{57}{5}$$
$$\therefore d = 11.4$$

ii Parallel to the x-axis $\Rightarrow m = 0$

$$\therefore \frac{d - 9}{d - 3} = 0$$
$$\therefore d - 9 = 0$$
$$\therefore d = 9$$

Alternatively, a horizontal line through $(4, 8)$ has equation $y = 8$. As this line passes through $(d + 1, d - 1)$, then:

$$d - 1 = 8$$
$$\therefore d = 9$$

iii Perpendicular to the x-axis $\Rightarrow m$ is undefined.

$\frac{d - 9}{d - 3}$ is undefined when its denominator is zero.

$\therefore d = 3$

Alternatively, a vertical line through $(4, 8)$ has equation $x = 4$. As this line passes through $(d + 1, d - 1)$, then:

$$d + 1 = 4$$
$$\therefore d = 3$$

d Let the lines with gradients $m_1 = -1.25$ and $m_2 = 0.8$ be inclined at θ_1 and θ_2 with the horizontal.

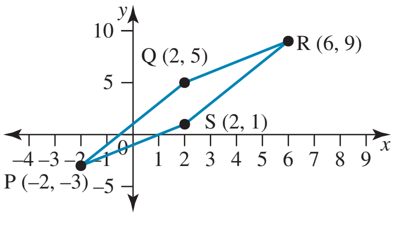
$$\tan\theta_1 = -1.25$$
$$\therefore \theta_1 = 180° - \tan^{-1}(1.25)$$
$$\therefore \theta_1 \simeq 128.66°$$
$$\tan\theta_2 = 0.8$$
$$\therefore \theta_2 = \tan^{-1}(0.8)$$
$$\therefore \theta_2 \simeq 38.66°$$

The angle between the two lines is:

$$\alpha = \theta_1 - \theta_2$$
$$= 128.66° - 38.66°$$
$$= 90°$$

The value of α, the angle between the two lines, is 90°.

19 P $(-2, -3)$, Q $(2, 5)$, R $(6, 9)$ and S $(2, 1)$



PQRS will be a parallelogram if its opposite sides are parallel, that is if PQ and SR are parallel and PS and QR are parallel.

$$m_{PQ} = \frac{5+3}{2+2} \text{ and } m_{SR} = \frac{9-1}{6-2}$$
$$= 2 \qquad = 2$$

Since $m_{PQ} = m_{SR}$, PQ is parallel to SR, which is sufficient for PQRS to be a parallelogram. However, to show the other pair of opposite sides are parallel:

$$m_{PS} = \frac{1+3}{2+2} \text{ and } m_{QR} = \frac{9-5}{6-2}$$
$$= 1 \qquad = 1$$

Since $m_{PS} = m_{QR}$, PS is parallel to QR and therefore PQRS is a parallelogram.

To be a rectangle, the adjacent sides need to be perpendicular. Testing PQ and QR gives

$$\begin{aligned} m_{PQ} \times m_{QR} &= 2 \times 1 \\ &= 2 \\ &\neq -1 \end{aligned}$$

The sides PQ and QR are not perpendicular, so PQRS is not a rectangle.

20 Obtain the point of intersection by solving the simultaneous equations:

$2x - 3y = 18$ [1]

$5x + y = 11$ [2]

Equation [1] + 3 × equation [2]:

$17x = 51$

$\therefore x = 3$

Substitute $x = 3$ in equation [2]:

$\therefore 15 + y = 11$

$\therefore y = -4$

The point of intersection is $(3, -4)$.

If the line is perpendicular to the horizontal line $y = 8$, then the vertical line through the point $(3, -4)$ is required.

Therefore, the equation is $x = 3$.

1.6 Exam questions

1 Gradient, $m = \frac{0-(-3)}{6-0}$

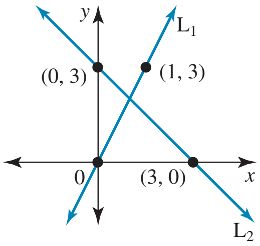
$$= \frac{1}{2}$$

y-intercept, $c = -3$

$$y = mx + c$$
$$y = \frac{1}{2}x - 3$$

The correct answer is **B**.

2 a



Line 1:

$y = 3x$

$x = 0,\ y = 0$

$x = 1,\ y = 3$

$\therefore$ points at $(0, 0)$ and $(1, 3)$

Line 2:

$y = 3 - x$

$x = 0,\ y = 3$

$y = 0,\ x = 3$

$\therefore$ points at $(0, 3)$ and $(3, 0)$

Award 1 mark for correctly drawing line 1.

Award 1 mark for correctly drawing line 2.

b L$_1$: $m = 3$

L$_2$: x-intercept is point $(3, 0)$ [1 mark]

Equation of a line:

$y - y_1 = m(x - x_1)$

$y - 0 = 3x - 9$

$y = 3x - 9$ [1 mark]

3 $6x + 10y - 3 = 0$

$$\begin{aligned} 10y &= -6x + 3 \\ y &= -\frac{6}{10}x + \frac{3}{10} \\ m &= -\frac{6}{10} \\ &= -0.6 \\ &= \tan\theta \end{aligned}$$

Since $\tan\theta$ is negative,

$\theta = 180° - \tan^{-1}(0.6)$

$\approx 149°$

The correct answer is **D**.

1.7 Bisection and lengths of line segments

1.7 Exercise

1 Points $(12, 5)$ and $(-9, -1)$

Midpoint:

$$x = \frac{12+(-9)}{2} \qquad y = \frac{5+(-1)}{2}$$
$$= \frac{3}{2} \qquad = \frac{4}{2}$$
$$= 2$$

The midpoint is $(1.5, 2)$.

2 a (x_1, y_1) (x_2, y_2)

$(-2, 8)$ $(12, -2)$

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-2+12}{2}, \frac{8+-2}{2}\right) \\ &= (5, 3) \end{aligned}$$

b (x_1, y_1) (x_2, y_2)

$(1, 0)$ $(-5, 4)$

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{1+-5}{2}, \frac{0+4}{2}\right) \\ &= (-2, 2) \end{aligned}$$

3 **a** $(x_1, y_1) \quad (x_2, y_2)$

$(7, 3) \quad (-4, 2)$

$$\begin{aligned}\text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{7 + -4}{2}, \frac{3 + 2}{2}\right)\\ &= \left(\frac{3}{2}, \frac{5}{2}\right)\end{aligned}$$

b $(x_1, y_1) \quad (x_2, y_2)$

$(24, 12) \quad (16, 12)$

$$\begin{aligned}\text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{24 + 16}{2}, \frac{12 + 12}{2}\right)\\ &= (20,\ 12)\end{aligned}$$

4 Midpoint M $(5, 6)$, A $(3, 7)$, N (x, y)

$$5 = \frac{3 + x}{2} \qquad 6 = \frac{7 + y}{2}$$
$$10 = 3 + x \qquad 12 = 7 + y$$
$$x = 7 \qquad y = 5$$

N has coordinates $(7, 5)$.

The answer is **C**.

5 The midpoint of PQ is $(3, 0)$ and Q is $(-10, 10)$.

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$
$$\therefore 3 = \frac{x_1 + (-10)}{2} \qquad \therefore 0 = \frac{y_1 + 10}{2}$$
$$\therefore x_1 - 10 = 6 \qquad \therefore y_1 + 10 = 0$$
$$\therefore x_1 = 16 \qquad \therefore y_1 = -10$$

Therefore, point P has coordinates $(16, -10)$.

6 $(x_1, y_1) \quad (x_2, y_2)$

$(5, -4) \quad (1, 0)$

$$\begin{aligned}\text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{5 + 1}{2}, \frac{-4 + 0}{2}\right)\\ &= (3, -2)\end{aligned}$$

$m = -3,\ (x_1, y_1) = (3, -2)$

Equation:

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y + 2 &= -3(x - 3)\\ y + 2 &= -3x + 9\\ y &= -3x + 7\end{aligned}$$

7 A $(3, 0)$, B $(9, 4)$, C $(5, 6)$ and D $(-1, 2)$

If the two line segments AC and BD bisect each other, they must have the same midpoint.

Midpoint of AC:

$$x = \frac{3 + 5}{2}, y = \frac{0 + 6}{2}$$
$$= 4 \qquad = 3$$

Therefore, the midpoint of AC is $(4,\ 3)$.

Midpoint of BD:

$$x = \frac{9 - 1}{2}, y = \frac{4 + 2}{2}$$
$$= 4 \qquad = 3$$

Therefore, the midpoint of BD is $(4,\ 3)$.

Hence, AC and BD bisect each other.

8 A $(-4, 4)$, B $(-3, 10)$

The midpoint of AB is $\left(\frac{-4 + (-3)}{2}, \frac{4 + 10}{2}\right) = \left(-\frac{7}{2}, 7\right)$.

The gradient of AB is $\frac{10 - 4}{-3 - (-4)} = 6$.

Therefore, the gradient of the perpendicular is $-\frac{1}{6}$.

Equation: point $\left(-\frac{7}{2}, 7\right)$, gradient $-\frac{1}{6}$

$$\begin{aligned}y - 7 &= -\frac{1}{6}\left(x + \frac{7}{2}\right)\\ \therefore 6y - 42 &= -\left(x + \frac{7}{2}\right)\\ \therefore 12y - 84 &= -2x - 7\\ \therefore 12y + 2x &= 77\end{aligned}$$

9 A $(-6, 0)$ and B $(2, 4)$

$$\begin{aligned}\text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{4 - 0}{2 - -6}\\ &= \frac{4}{8}\\ &= \frac{1}{2}\end{aligned}$$

Since $m_{AB} = \frac{1}{2}$, the gradient of a line perpendicular to AB is $m_{\perp} = -2$.

$$\begin{aligned}\text{Midpoint of AB} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{-6 + 2}{2}, \frac{0 + 4}{2}\right)\\ &= (-2,\ 2)\end{aligned}$$

Point $(-2, 2)$, gradient -2

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - 2 &= -2(x + 2)\\ y - 2 &= -2x - 4\\ y &= -2x - 2\end{aligned}$$

The equation of the perpendicular bisector is $y = -2x - 2$.

10 A $(1, 2)$ and B $(-3, 5)$

$$\begin{aligned}\text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{5 - 2}{-3 - 1}\\ &= -\frac{3}{4}\end{aligned}$$

Since $m_{AB} = -\frac{3}{4}$, the gradient of a line perpendicular to AB is $m_{\perp} = \frac{4}{3}$.

$$\begin{aligned}\text{Midpoint of AB} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{1 + -3}{2}, \frac{2 + 5}{2}\right)\\ &= \left(-1, \frac{7}{2}\right)\end{aligned}$$

Point $\left(-1, \frac{7}{2}\right)$, gradient $\frac{4}{3}$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \frac{7}{2} &= \frac{4}{3}(x + 1) \\ y - \frac{7}{2} &= \frac{4x}{3} + \frac{4}{3} \qquad \text{LCD} = 6 \\ 6y - 21 &= 8x + 8 \end{aligned}$$

$6y - 8x - 29 = 0$

The equation of the perpendicular bisector is

$6y - 8x - 29 = 0$.

11 $ax + by = c$ is the perpendicular bisector of CD where C is $(-2, -5)$ and D is $(2, 5)$.

Perpendicular bisector of CD:

The midpoint of CD is $\left(\frac{-2+2}{2}, \frac{-5+5}{2}\right) = (0, 0)$.

The gradient of CD is $\frac{5-(-5)}{2-(-2)} = \frac{5}{2}$.

Therefore, the gradient of the perpendicular is $-\frac{2}{5}$.

Equation: point $(0, 0)$, gradient $-\frac{2}{5}$

Therefore, the perpendicular bisector has the equation

$y = -\frac{2}{5}x$.

Rearranging,

$$5y = -2x$$

$\therefore 2x + 5y = 0$

Comparing with $ax + by = c$, $a = 2, b = 5, c = 0$.

12 Points $(6, -8)$ and $(-4, -5)$

$$\begin{aligned} d &= \sqrt{(6 - (-4))^2 + (-8 - (-5))^2} \\ &= \sqrt{(10)^2 + (-3)^2} \\ &= \sqrt{109} \\ &\simeq 10.44 \end{aligned}$$

13 (x_1, y_1) (x_2, y_2)

$(10, -3)$ $(-2, 6)$

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-2 - 10)^2 + (6 + 3)^2} \\ &= \sqrt{144 + 81} \\ &= \sqrt{225} \\ &= 15 \end{aligned}$$

14 **a** (x_1, y_1) (x_2, y_2)

$(-3, 2)$ $(3, -4)$

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - -3)^2 + (-4 - 2)^2} \\ &= \sqrt{36 + 36} \\ &= \sqrt{72} \\ &= 6\sqrt{2} \end{aligned}$$

b (x_1, y_1) (x_2, y_2)

$(-1, -5)$ $(5, -1)$

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5 - -1)^2 + (-1 - -5)^2} \\ &= \sqrt{36 + 16} \\ &= \sqrt{52} \\ &= 2\sqrt{13} \end{aligned}$$

15 A $(-2, 0)$, B $(2, 3)$ and C $(3, 0)$

$$\begin{aligned} d_{\text{AB}} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-2 - 2)^2 + (0 - 3)^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\begin{aligned} d_{\text{AC}} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-2 - 3)^2 + (0 - 0)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\begin{aligned} d_{\text{BC}} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - 3)^2 + (3 - 0)^2} \\ &= \sqrt{1 + 9} \\ &= \sqrt{10} \end{aligned}$$

As $\sqrt{10} < \sqrt{25}$, the shortest side is BC.

16 A $(-1, 1)$ and B $(6, -1)$

The midpoint of AB is $\left(\frac{-1+6}{2}, \frac{1+(-1)}{2}\right) = \left(\frac{5}{2}, 0\right)$.

Distance between points $(3, 10)$ and $\left(\frac{5}{2}, 0\right)$

$$\begin{aligned} &= \sqrt{\left(3 - \frac{5}{2}\right)^2 + (10 - 0)^2} \\ &= \sqrt{(0.5)^2 + (10)^2} \\ &= \sqrt{0.25 + 100} \\ &= \sqrt{100.25} \\ &= 10.01 \text{ (2 d.p.)} \end{aligned}$$

17 The distance between the points $(p, 8)$ and $(0, -4)$ is 13 units.

$$\begin{aligned} \therefore\ 13 &= \sqrt{(p - 0)^2 + (8 + 4)^2} \\ \therefore\ 13 &= \sqrt{p^2 + 144} \\ \therefore\ 169 &= p^2 + 144 \\ \therefore\ p^2 &= 25 \\ \therefore\ p &= \pm 5 \end{aligned}$$

18 A $(-7, 2)$ and B $(-13, 10)$

a
$$\begin{aligned} d_{\text{AB}} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-13 + 7)^2 + (10 - 2)^2} \\ &= \sqrt{36 + 64} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

The distance is 10 units.

b The coordinates of the midpoint are given by

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}$$

$$\therefore x = \frac{-7 - 13}{2}, \qquad y = \frac{2 + 10}{2}$$

$$\therefore x = -10, \qquad y = 6$$

The midpoint is $(-10, 6)$.

c The perpendicular bisector of AB contains the midpoint $(-10, 6)$.

$$\begin{aligned} m_{\text{AB}} &= \frac{10-2}{-13+7} \\ &= \frac{8}{-6} \\ &= -\frac{4}{3} \end{aligned}$$

Therefore, the gradient of the perpendicular bisector is $\frac{3}{4}$.

Equation of perpendicular bisector:

$$y - 6 = \frac{3}{4}(x+10)$$

$\therefore 4y - 24 = 3x + 30$

$\therefore 4y - 3x = 54$

d $4y - 3x = 54 \quad [1]$

$4y + 3x = 24 \quad [2]$

Add equations [1] and [2]:

$\therefore 8y = 78$

$\therefore y = \frac{39}{4}$

Subtract equation [1] from equation [2]:

$\therefore 6x = -30$

$\therefore x = -5$

The point of intersection is $(-5, 9.75)$.

19 C $(-8, 5)$, D $(2, 4)$ and E $(0.4, 0.8)$

a The perimeter is the sum of the side lengths.

$$\begin{aligned} CD &= \sqrt{(2+8)^2 + (4-5)^2} \\ &= \sqrt{100+1} \\ &= \sqrt{101} \end{aligned}$$

$$\begin{aligned} DE &= \sqrt{(2-0.4)^2 + (4-0.8)^2} \\ &= \sqrt{1.6^2 + 3.2^2} \\ &= \sqrt{12.8} \end{aligned}$$

$$\begin{aligned} EC &= \sqrt{(-8-0.4)^2 + (5-0.8)^2} \\ &= \sqrt{8.4^2 + 4.2^2} \\ &= \sqrt{88.2} \end{aligned}$$

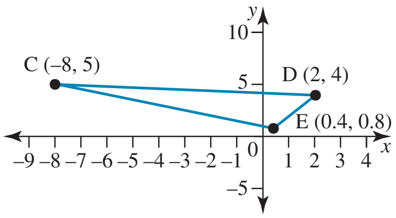
Perimeter

$= \sqrt{101} + \sqrt{12.8} + \sqrt{88.2}$

$\simeq 23$

The perimeter is 23 units.

b If angle CED is a right angle, CE is perpendicular to ED and CD is the hypotenuse.



Test if Pythagoras's theorem $CD^2 = CE^2 + ED^2$ holds.

$$\begin{aligned} \text{LHS} &= CD^2 \\ &= 101 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= CE^2 + ED^2 \\ &= 88.2 + 12.8 \\ &= 101 \\ &= \text{LHS} \end{aligned}$$

Therefore, angle CED is a right angle.

An alternative method is to show CE and ED are perpendicular.

$$m_{\text{CE}} = \frac{5-0.8}{-8-0.4} \text{ and } m_{\text{ED}} = \frac{4-0.8}{2-0.4}$$

$$= -\frac{4.2}{8.4} \qquad = \frac{3.2}{1.6}$$

$$= -\frac{1}{2} \qquad = 2$$

Since $m_{\text{CE}} \times m_{\text{ED}} = -1$, CE is perpendicular to ED and therefore angle CED is 90°.

c The midpoint, M, of the hypotenuse CD is

$$\left(\frac{-8+2}{2}, \frac{5+4}{2}\right) = (-3, 4.5).$$

d M is equidistant from C and D, so $MC = MD = \frac{1}{2}CD$.

$$\therefore MC = MD = \frac{1}{2}\sqrt{101}$$

$$\begin{aligned} ME &= \sqrt{(-3-0.4)^2 + (4.5-0.8)^2} \\ &= \sqrt{25.25} \\ &= \sqrt{\frac{101}{4}} \\ &= \frac{1}{2}\sqrt{101} \end{aligned}$$

$\therefore ME = MC = MD$, which shows M is equidistant from each of the vertices of the triangle.

20 **a** The centre of the circle is the midpoint of the diameter.

$\therefore 4 = \frac{-2+x_2}{2}$, and $8 = \frac{-2+y_2}{2}$, where (x_2, y_2) are the coordinates of the other end point of the diameter.

$\therefore 8 = -2 + x_2$ and $\therefore 16 = -2 + y_2$

$\therefore x_2 = 10 \qquad \therefore y_2 = 18$

The other end of the diameter is the point $(10, 18)$.

b Area formula: $A = \pi r^2$

The radius is the distance between the centre $(4, 8)$ and $(-2, -2)$, an end point of the diameter.

$$\begin{aligned} r &= \sqrt{(4+2)^2 + (8+2)^2} \\ &= \sqrt{36+100} \\ &= \sqrt{136} \end{aligned}$$

$$\begin{aligned} \therefore A &= \pi\left(\sqrt{136}\right)^2 \\ &= \pi \times 136 \\ &= 136\pi \end{aligned}$$

The area is 136π square units.

21 Anna: The distance of $(-2.3, 1.5)$ from $(0, 0)$ is:

$$\begin{aligned} d_{\text{A}} &= \sqrt{2.3^2 + 1.5^2} \\ &= \sqrt{7.54} \end{aligned}$$

Liam: The distance of $(1.7, 2.1)$ from $(0, 0)$ is:

$$\begin{aligned} d_{\text{L}} &= \sqrt{1.7^2 + 2.1^2} \\ &= \sqrt{7.3} \end{aligned}$$

Since $d_{\text{A}} > d_{\text{L}}$, Anna carries the rucksack.

22 P $(3, 7)$ and R $(5, 3)$

a Distance PR:

$$\begin{aligned} d_{\text{PR}} &= \sqrt{(5-3)^2 + (3-7)^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \\ &\simeq 4.5 \end{aligned}$$

The petrol station and the restaurant are approximately 4.5 km apart.

b $m_{PR} = \dfrac{3-7}{5-3}$, point $(5, 3)$

$= -2$

Equation of PR:

$y - 3 = -2(x - 5)$

$\therefore y = -2x + 13$

c H $(2, 3.5)$

Since PR has gradient -2, the perpendicular line through H has gradient $\dfrac{1}{2}$.

Its equation is:

$y - 3.5 = 0.5(x - 2)$

$\therefore y = 0.5x + 2.5$

d B is the intersection of lines PR and HB.

$y = -2x + 13 \quad [1]$

$y = 0.5x + 2.5 \quad [2]$

At the intersection,

$-2x + 13 = 0.5x + 2.5$

$\therefore 10.5 = 2.5x$

$\therefore x = \dfrac{10.5}{2.5}$

$\therefore x = 4.2$

Substitute $x = 4.2$ in equation [1]:

$\therefore y = -8.4 + 13$

$\therefore y = 4.6$

The coordinates of B are $(4.2, 4.6)$.

e The total distance Ada cycles is HB + BR.

$d_{HB} = \sqrt{(4.2-2)^2 + (4.6-3.5)^2}$

$\simeq 2.460$

$d_{BR} = \sqrt{(4.2-5)^2 + (4.6-3)^2}$

$\simeq 1.789$

Ada cycles 4.249 km at an average speed of 10 km/h.

Ada's time taken is $\dfrac{4.249}{10} = 0.4249$ hours.

In minutes, the time is $0.4249 \times 60 = 25.494$ minutes.

This figure, when rounded to the nearest minute, is 25 minutes. However, Ada would not be at R at this time. Ada takes 26 minutes to reach the restaurant, to the nearest minute.

23 a Select Geometry from the menu.

Tap Draw → Point three times to obtain points A, B and C.

Tap Draw → Line Segment and join the points to form triangle ABC.

Tap a line segment, then Draw → Construct → Perp. Bisector. Do this for each side to create the three perpendicular bisectors.

Move a vertex around to create different triangles.

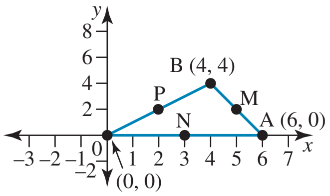
For every case, the perpendicular bisectors are concurrent.

b In the Geometry menu, select the grid template.

Plot the three points A $(0, 0)$, B $(4, 4)$ and C $(6, 0)$, checking they are accurately placed using the coordinate tab. Then construct the triangle ABC and the perpendicular bisectors of each side. Select Intersection to mark the point D where these perpendicular bisectors meet.

Tap the coordinate button for point D. This gives D as the point $(3, 1)$.

Checking algebraically:



BA: The midpoint is M $\left(\dfrac{4+6}{2}, \dfrac{4+0}{2}\right) = (5, 2)$.

$m_{BA} = \dfrac{4-0}{4-6}$

$= -2$

Therefore, the gradient of the perpendicular bisector of BA is $\dfrac{1}{2}$.

Equation of perpendicular bisector:

$y - 2 = \dfrac{1}{2}(x - 5)$

$\therefore y = \dfrac{x}{2} - \dfrac{1}{2} \quad [1]$

OA: The midpoint is N $(3, 0)$, so the equation of the perpendicular bisector of OA is $x = 3$ [2].

OB: The midpoint is P $(2, 2)$ and the gradient is 1. The perpendicular bisector of OB has gradient -1.

Equation of perpendicular bisector:

$y - 2 = -(x - 2)$

$\therefore y = -x + 4 \quad [3]$

Substitute $x = 3$ from equation [2] into each of equations [1] and [3].

Equation [1]: $y = \dfrac{3}{2} - \dfrac{1}{2} = 1$

Equation [3]: $y = -3 + 4 = 1$

All three lines contain the point $(3, 1)$, so this is the point of intersection of the perpendicular bisectors.

24 a Create three points and join each pair by a line segment to create the triangle ABC.

Mark the midpoints of each side using Draw → Construct → Midpoint.

Join each vertex to the midpoint of the opposite side using Draw → Line Segment.

Move a vertex around to create different triangles.

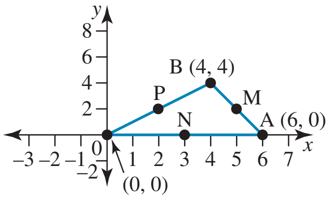
For every case, the medians are concurrent.

b In the Geometry menu, select the grid template.

Plot the three points A $(0, 0)$, B $(4, 4)$ and C $(6, 0)$, checking they are accurately placed using the coordinate tab. Then construct the triangle ABC and the midpoints of each side. Draw the medians from each vertex to the midpoint of the opposite side and select Intersection to mark the point P where these medians meet.

Tap the coordinate button for point P. This gives the coordinates as repeated decimals $(3.33333, 1.33333)$ or $\left(3.\overline{3}, 1.\overline{3}\right)$, giving the point of intersection of the medians as $\left(\dfrac{10}{3}, \dfrac{4}{3}\right)$.

Check algebraically:



M $(5, 2)$, N $(3, 0)$ and P $(2, 2)$

Median OM: $m = \frac{2}{5}, c = 0$, so the equation of OM is $y = \frac{2x}{5}$ [1].

Median NB: $m = 4$, point $(3, 0)$

Equation of NB:

$$y = 4(x - 3)$$

$$\therefore y = 4x - 12 \quad [2]$$

Median of AP: point $(6, 0)$

$$m = \frac{2}{-4}$$

$$\therefore m = -\frac{1}{2}$$

Equation of AP:

$$y = -\frac{1}{2}(x - 6)$$

$$\therefore y = -\frac{x}{2} + 3 \quad [3]$$

Substitute equation [1] into equation [2]:

$$\therefore \frac{2x}{5} = 4x - 12$$

$$\therefore 2x = 20x - 60$$

$$\therefore 60 = 18x$$

$$\therefore x = \frac{60}{18}$$

$$\therefore x = \frac{10}{3}$$

Substitute $x = \frac{10}{3}$ into equation [1]:

$$\therefore y = \frac{2}{5} \times \frac{10}{3}$$

$$\therefore y = \frac{4}{3}$$

Test if $x = \frac{10}{3}, y = \frac{4}{3}$ satisfies equation [3]:

$$y = -\frac{x}{2} + 3$$

$$\text{LHS} = \frac{4}{3} \qquad \text{RHS} = -\frac{1}{2} \times \frac{10}{3} + 3$$

$$= -\frac{5}{3} + \frac{9}{3}$$

$$= \frac{4}{3}$$

$$= \text{LHS}$$

Therefore, the three medians meet at $\left(\frac{10}{3}, \frac{4}{3}\right)$.

1.7 Exam questions

1 A: $(-7, 0) = (x_1, y_1)$

B: $(-3, -3) = (x_2, y_2)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-3 - (-7))^2 + (-3 - 0)^2}$$

$$= \sqrt{16 + 9}$$

$$= 5$$

The correct answer is **D**.

2 Let P $(-5, -3) = (x_1, y_1)$.

Let Q $(2, -1) = (x_2, y_2)$.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{-5 + 2}{2}, \frac{-3 + (-1)}{2}\right)$$

$$= \left(-\frac{3}{2}, -2\right)$$

The correct answer is **E**.

3 **a** Let A $(1, -5) = (x_1, y_1)$

Let B $(4, -2) = (x_2, y_2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-2 - (-5)}{4 - 1}$$

$$= \frac{3}{3}$$

$$= 1$$ [1 mark]

b For perpendicular lines, $m_1 m_2 = -1$.

The gradient of the perpendicular line to AB $= -1$. [1 mark]

The equation of a line through a point: $y - y_1 = m(x - x_1)$

$A = (x_1, y_1) = (1, -5), \ m = -1$ [1 mark]

$$y - (-5) = -1(x - 1)$$

$$y + 5 = -x + 1$$

$$y = -x - 4$$

The equation of the line perpendicular to AB and passing through A is $y = -x - 4$. [1 mark]

1.8 Review

1.8 Exercise

Technology free: short answer

1 **a** $3(5x - 2) + 5(3x - 2) = 8(x - 2)$

$$\therefore 15x - 6 + 15x - 10 = 8x - 16$$

$$\therefore 30x - 16 = 8x - 16$$

$$\therefore 22x = 0$$

$$\therefore x = 0$$

b $\frac{2x - 1}{5} + \frac{3 - 2x}{4} = \frac{3}{20}$

$$\therefore \frac{4(2x - 1) + 5(3 - 2x)}{20} = \frac{3}{20}$$

$$\therefore 8x - 4 + 15 - 10x = 3$$

$$\therefore -2x = -8$$

$$\therefore x = 4$$

c $ax + 3c = 3a + cx$

$\therefore ax - cx = 3a - 3c$

$\therefore x(a - c) = 3(a - c)$

$\therefore x = \dfrac{3(a - c)}{a - c}$

$\therefore x = 3$ (for $a \neq c$)

d $\dfrac{5x - b}{2b} - \dfrac{2x}{b} = 2$

$\therefore \dfrac{5x - b - 2(2x)}{2b} = 2$

$\therefore \dfrac{x - b}{2b} = 2$

$\therefore x - b = 4b$

$\therefore x = 5b$

2 $2x + y = 6$ [1]

$5x - 2y = 24$ [2]

Multiply equation [1] by 2:

$4x + 2y = 12$ [3]

$5x - 2y = 24$

Add equations [2] and [3]:

$9x = 36$

$x = 4$

Substitute $x = 4$ into equation [1]:

$8 + y = 6$

$y = -2$

Check $x = 4$ and $y = -2$ satisfy equation [2]:

$\text{LHS} = 5 \times 4 - 2 \times (-2)$

$= 24$

$= \text{RHS}$

Answer: $x = 4,\ y = -2$

3 **a** $3 - 2x \leq 1$

$\therefore -2x \leq -2$

$\therefore x \geq 1$

b $\dfrac{2x}{3} - 5 > 2 + 3x$

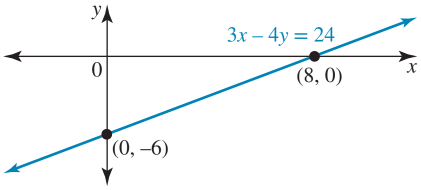
$\therefore \dfrac{2x}{3} - 3x > 7$

$\therefore \dfrac{2x - 9x}{3} > 7$

$\therefore -7x > 21$

$\therefore x < -3$

4 $3x - 4y = 24$



5 **a** Rearranging the line $2x - 7y = 2$:

$2x - 2 = 7y$

$\therefore y = \dfrac{2x}{7} - \dfrac{2}{7}$

The gradient is $\dfrac{2}{7}$.

The gradient of the parallel line is also $\dfrac{2}{7}$ and it passes through the point $(-5, 8)$.

Its equation is:

$y - 8 = \dfrac{2}{7}(x + 5)$

$\therefore 7(y - 8) = 2(x + 5)$

$\therefore 7y - 56 = 2x + 10$

$\therefore 7y - 2x = 66$

b The line perpendicular to $2x - 7y = 2$ has gradient $-\dfrac{7}{2}$. It contains the point $(4, 0)$.

Its equation is:

$y - 0 = -\dfrac{7}{2}(x - 4)$

$\therefore 2y = -7(x - 4)$

$\therefore 2y = -7x + 28$

$\therefore 2y + 7x = 28$

c Points $(9, 2)$ and $(6, -7)$

$m = \dfrac{2 + 7}{9 - 6}$

$= 3$

The equation of the line is:

$y - 2 = 3(x - 9)$

$\therefore y = 3x - 25$

6 Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \dfrac{6 - (-2)}{-4 - 2}$

$m = \dfrac{8}{-6}$

$m = \dfrac{-4}{3}$

$y = mx + c$

$y = \dfrac{-4}{3}x + c$

Substitute in one of the points: $(2, -2)$

$-2 = \dfrac{-4}{3}(2) + c$

$-2 = \dfrac{-8}{3} + c$

$c = \dfrac{2}{3}$

$\therefore y = -\dfrac{4}{3}x + \dfrac{2}{3}$

Technology active: multiple choice

7 $4(1 - 3x) + 2(3 + x) > 5$

$\therefore 4 - 12x + 6 + 2x > 5$

$\therefore -10x > -5$

$\therefore x < \dfrac{-5}{-10}$

$\therefore x < \dfrac{1}{2}$

The correct answer is **D**.

8 $7x - 2y = 11$ [1]

$3x + y = 1$ [2]

Equation [1] + 2 × equation [2]:

$13x = 13$

$\therefore x = 1$

Substitute $x = 1$ in equation [2]:

$\therefore 3 + y = 1$

$\therefore y = -2$

$x = 1, y = -2$

The correct answer is **D**.

9 The value of the calculator is \$200 less the depreciation amount.

$\therefore V = 200 - 25t$

The correct answer is **C**.

10 Gradient: $\dfrac{\text{rise}}{\text{run}} = \dfrac{3}{6} = \dfrac{1}{2}$

Equation: $y = mx + c,\ m = \dfrac{1}{2},\ c = -3$

$$\therefore y = \frac{1}{2}x - 3$$

$$\therefore 2y = x - 6$$

$\therefore x - 2y = 6$

The correct answer is **A**.

11 Let the points be A $(-1, -2)$, B $(4, 3)$ and C $(9,\ b)$. They will be collinear if $m_{\text{AB}} = m_{\text{BC}}$.

$$\therefore \frac{3+2}{4+1} = \frac{b-3}{9-4}$$

$$\therefore \frac{5}{5} = \frac{b-3}{5}$$

$$\therefore 5 = b - 3$$

$$\therefore b = 8$$

The correct answer is **D**.

12 The gradient of the line $\dfrac{x}{3} - \dfrac{y}{2} = 1$ is required.

Rearranging,

$$\frac{x}{3} - 1 = \frac{y}{2}$$

$$\therefore y = 2\left(\frac{x}{3} - 1\right)$$

$$\therefore y = \frac{2}{3}x - 2$$

$$\therefore m = \frac{2}{3}$$

For the required angle θ,

$$m = \tan\theta$$

$$\therefore\ \tan\theta = \frac{2}{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \theta = 33.69°$$

$$\therefore \theta = 33.7°$$

The correct answer is **B**.

13 The line through $(9, 5)$ parallel to the horizontal line $y = -2$ is the horizontal line $y = 5$.

The correct answer is **E**.

14 A is the point $(13, 11)$. Let B be the point (x_2, y_2).

Midpoint: $x = \dfrac{x_1 + x_2}{2}, y = \dfrac{y_1 + y_2}{2}$

Since the midpoint is the point $(3, -5)$,

$$3 = \frac{13 + x_2}{2}, \qquad -5 = \frac{11 + y_2}{2}$$

$$\therefore 6 = 13 + x_2 \qquad -10 = 11 + y_2$$

$$\therefore x_2 = -7, \qquad y_2 = -21$$

The coordinates of B are $(-7, -21)$

The correct answer is **D**.

15 Rearranging the equations:

$ay + 3x = 4$

$$\therefore ay = -3x + 4$$

$$\therefore y = \frac{-3x}{a} + \frac{4}{a} \Rightarrow m_1 = -\frac{3}{a}, c_1 = \frac{4}{a}$$

and

$2y + 4x = 3$

$$\therefore 2y = -4x + 3$$

$$\therefore y = -2x + \frac{3}{2} \Rightarrow m_2 = -2,\ c_2 = \frac{3}{2}$$

The lines do not intersect if they are parallel lines with $m_1 = m_2$ and $c_1 \neq c_2$.

$$\therefore\ -\frac{3}{a} = -2$$

$$\therefore\ -3 = -2a$$

$$\therefore a = \frac{3}{2}$$

If $a = \dfrac{3}{2}$,

$$c_1 = \frac{4}{\frac{3}{2}}$$

$$= \frac{8}{3}$$

$$\neq c_2$$

There is no intersection if $a = 1.5$.

The correct answer is **B**.

16 Points $(-3, 5)$ and $(-6, 12)$

Distance between the points

$$= \sqrt{(-6+3)^2 + (12-5)^2}$$

$$= \sqrt{9 + 49}$$

$$= \sqrt{58}$$

The correct answer is **C**.

Technology active: extended response

17 **a** Let the distance between the school and the golf range be $3x$ km.

Tenzin cycles x km at 20 km/h in $\dfrac{x}{20}$ hours and cycles $2x$ km at 10 km/h in $\dfrac{2x}{10}$ hours.

The total time taken is 45 minutes, which is $\dfrac{3}{4}$ hour.

Therefore, $\dfrac{x}{20} + \dfrac{2x}{10} = \dfrac{3}{4}$.

Solving this equation,

$$\frac{x + 4x}{20} = \frac{3}{4}$$

$$\therefore \frac{5x}{20} = \frac{3}{4}$$

$$\therefore \frac{x}{4} = \frac{3}{4}$$

$$\therefore x = 3$$

The distance between the school and the golf range is 9 km.

b The distance between T $(1.5, 4)$ and O $(0, 0)$ is

$$\sqrt{1.5^2 + 4^2}$$

$$= \sqrt{18.25}$$

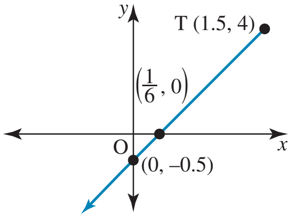
$$= 4.272$$

The golf ball is approximately 4.27 metres from the hole.

c The gradient of TO is $\dfrac{4}{1.5} = \dfrac{8}{3}$. TO contains the origin, so its equation would be $y = \dfrac{8x}{3}$.

d **i** $6x - 2y = 1$

x-intercept $\left(\frac{1}{6}, 0\right)$ and y-intercept $\left(0, -\frac{1}{2}\right)$



ii The gradient of the golf ball's path is:

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{\frac{1}{2}}{\frac{1}{6}}$$
$$= \frac{1}{2} \times \frac{6}{1}$$
$$= 3$$

The line perpendicular to the path has gradient $-\frac{1}{3}$ and contains the origin.

Its equation is $y = -\frac{x}{3}$.

The point on the path closest to the hole at the origin is the intersection of $y = -\frac{x}{3}$ with $6x - 2y = 1$.

Substitute $y = -\frac{x}{3}$ into $6x - 2y = 1$:

$$\therefore 6x + \frac{2x}{3} = 1$$
$$\therefore \frac{18x + 2x}{3} = 1$$
$$\therefore 20x = 3$$
$$\therefore x = \frac{3}{20}$$

Substitute $x = \frac{3}{20}$ into $y = -\frac{x}{3}$

$$\therefore y = -\frac{\frac{3}{20}}{3}$$
$$\therefore y = -\frac{3}{20} \times \frac{1}{3}$$
$$\therefore y = -\frac{1}{20}$$

The closest point to the hole is

$$\left(\frac{3}{20}, \frac{-1}{20}\right) = (0.15, -0.05)$$

Distance from the hole

$$= \sqrt{0.15^2 + (-0.05)^2}$$
$$= \sqrt{0.025}$$
$$= 0.16$$

The closest distance is 16 cm.

18 **a** **i** $m = \tan(45°) \Rightarrow m = 1$, point $(0, 1) \Rightarrow c = 1$

The equation of the path is $y = x + 1$.

At E, $y = 10$, so $x = 9$ for E to lie on the path.

The coordinates of E are (9, 10).

ii The line perpendicular to the path has $m = -1$ and point E (9, 10).

Its equation is

$$y - 10 = -(x - 9)$$
$$\therefore y = -x + 19$$

When $y = 0$, $x = 19$.

The debris reaches the ground 19 metres horizontally from the origin and therefore 10 metres horizontally from E.

b S (0, 1) and E (k, 10)

$$m_{\text{SE}} = \frac{10 - 1}{k - 0}$$
$$= \frac{9}{k}$$

and $c = 1$.

The equation of SE is $y = \frac{9x}{k} + 1$.

$$\therefore y - 1 = \frac{9x}{k}$$
$$\therefore ky - k = 9x$$
$$\therefore 9x - ky + k = 0$$

c **i** The line perpendicular to SE has gradient $-\frac{k}{9}$ and point E (k, 10).

Its equation is

$$y - 10 = -\frac{k}{9}(x - k)$$
$$\therefore 9(y - 10) = -k(x - k)$$
$$\therefore 9y - 90 = -kx + k^2$$
$$\therefore 9y + kx = k^2 + 90$$

When $y = 0$,

$$kx = k^2 + 90$$
$$\therefore x = k + \frac{90}{k}$$

This point is a horizontal distance of $\left(k + \frac{90}{k}\right) - k$ from E.

Therefore, $r = \frac{90}{k}$.

ii $4 \le r \le 6$

$$\therefore 4 \le \frac{90}{k} \le 6$$

Since $k > 0$,

$$4k \le 90 \text{ and } 90 \le 6k$$
$$\therefore k \le 22.5 \text{ and } k \ge 15$$
$$\therefore 15 \le k \le 22.5$$

19 **a** Since the volume of water in the tank increased by 250 litres in 10 days, the rate of increase is 25 litres/day.

b $V = 1000 + 25t$

c On March 30, two days before April 1, $t = -2$.

$$\therefore V = 1000 - 25 \times 2$$
$$\therefore V = 950$$

The model predicts there would have been 950 litres in the tank.

d **i** Let $\$C$ be the cost and t be the number of hours of construction.

Latasi company: $C_L = 500 + 26t$

Natano company: $C_N = 600 + 18t$

ii When $C_L = C_N$,

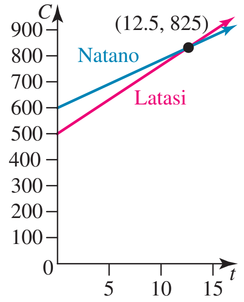
$$500 + 26t = 600 + 18t$$
$$\therefore 8t = 100$$
$$\therefore t = 12.5$$

If $t = 12.5$,

$$C = 500 + 26 \times 12.5$$
$$\therefore C = 825$$

Both companies charge the same cost of \$825 for 12.5 hours of construction.

iii Both graphs contain the point $(12.5, 825)$. Latasi has initial point $(0, 500)$ and Natano has initial point $(0, 600)$.



iv When $t = 8$, the diagram shows the Latasi company would be cheaper.

20 Let:
$x = 1$ bag of chips
$y = 1$ bag of lollies
$z = 1$ box of biscuits

$$\begin{aligned} 3x + y + 2z &= 24.10 \quad [1] \\ 2x + 2y + 4z &= 33.40 \quad [2] \\ x + 3y + 3z &= 29.50 \quad [3] \end{aligned}$$

Using CAS technology:
$x = 3.70,\ y = 4.20,\ z = 4.40$

a. 1 bag of chips costs \$3.70.
b. 1 bag of lollies costs \$4.20.
c. 1 box of biscuits costs \$4.40.

1.8 Exam questions

1 Let Sam's Mathematical Methods mark be x, Chemistry mark be y, and Physics mark be z.

$$\begin{aligned} x + y + z &= 256 \quad [1] \\ x - y &= 8 \quad [2] \\ y &= z + 1 \quad [3] \end{aligned} \qquad [1 \text{ mark}]$$

Using CAS technology, Sam's Mathematical Methods mark is 91, his Chemistry mark is 83 and his Physics mark is 82. [1 mark]

2
$$\begin{aligned} y &= mx + c \\ 3x + y - 5 &= 0 \\ y &= -3x + 5 \\ \therefore m &= -3,\ c = 5 \end{aligned}$$

The correct answer is **D**.

3 A system of linear equations that has no solution has equations with equal gradients and different y-intercepts (parallel lines).
Rewrite the equations in $y = mx + c$ form:

$$\begin{aligned} 2ax + 3y &= 2a \\ 3y &= -2ax + 2a \\ y &= -\frac{2}{3}ax + \frac{2}{3}a \end{aligned}$$

$$m_1 = -\frac{2}{3}a, \quad c_1 = \frac{2}{3}a \qquad [1 \text{ mark}]$$

$$\begin{aligned} 8x - 10 &= 2y \\ y &= 4x - 5 \\ m_2 = 4, &\quad c_2 = -5 \end{aligned} \qquad [1 \text{ mark}]$$

Equate the gradients:

$$\begin{aligned} -\frac{2}{3}a &= 4 \\ 2a &= -12 \\ a &= -6 \end{aligned} \qquad [1 \text{ mark}]$$

$$m_1 = -\frac{2}{3}(-6) = 4$$
$$c_1 = \frac{2}{3}(-6) = -4$$
$$\Rightarrow m_1 = m_2 \text{ and } c_1 \neq c_2$$

Therefore, the lines are parallel and there is no solution. [1 mark]

4 Gradient, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Let $(x_1,\ y_1) = (-1,\ -1)$ and $(x_2,\ y_2) = (1,\ 5)$.

$$\begin{aligned} m &= \frac{5 - (-1)}{1 - (-1)} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

$$\begin{aligned} m &= \tan\theta \\ \therefore \theta &= \tan^{-1}(m) \\ \theta &= \tan^{-1}(3) \\ \theta &\approx 72^\circ \end{aligned}$$

The correct answer is **C**.

5
$$\begin{aligned} 2x - 3y + 1 &= 0 \\ 3y &= 2x + 1 \\ y &= \frac{2}{3}x + \frac{1}{3} \end{aligned}$$

Gradient: $m = \dfrac{2}{3}$

Gradient of the perpendicular line: $m = -\dfrac{3}{2}$

Equation of the line through $(3,\ -2)$ and with $m = -\dfrac{3}{2}$:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= -\frac{3}{2}(x - 3) \\ y + 2 &= -\frac{3}{2}x + \frac{9}{2} \\ y &= -\frac{3}{2}x + \frac{5}{2} \\ 2y &= -3x + 5 \\ 2y + 3x - 5 &= 0 \end{aligned}$$

The correct answer is **B**.

Topic 2 — Algebraic foundations

2.2 Algebraic skills

2.2 Exercise

1 a $4m(m-2)+3m$
$=4m^2-8m+3m$
$=4m^2-5m$

b $5\left(m^2-3m+2\right)-(m+2)$
$=5m^2-15m+10-m-2$
$=5m^2-16m+8$

c $(x-3)(x+5)$
$=x^2+5x-3x-15$
$=x^2+2x-15$

d $(3m-2)(5m-4)$
$=15m^2-12m-10m+8$
$=15m^2-22m+8$

e $(4-3x)(4+3x)$
$=4^2-(3x)^2$
$=16-9x^2$

f $(2x-5)^2$
$=(2x)^2-2\times 2x\times 5+5^2$
$=4x^2-20x+25$

2 a $2(x-5)(x+5)-3(2x-3)$
$=2\left(x^2-5^2\right)-6x+9$
$=2\left(x^2-25\right)-6x+9$
$=2x^2-50-6x+9$
$=2x^2-6x-41$

b $3-2(x+5)(3x-2)$
$=3-2\left(3x^2-2x+15x-10\right)$
$=3-2\left(3x^2+13x-10\right)$
$=3-6x^2-26x+20$
$=23-6x^2-26x$

c $(3-2x)(x-5)-(x+3)(x+4)$
$=\left(3x-15-2x^2+10x\right)-\left(x^2+4x+3x+12\right)$
$=13x-15-2x^2-x^2-7x-12$
$=6x-3x^2-27$

d $3(2x-1)(2x+1)+(x-5)^2$
$=3\left((2x)^2-1^2\right)+x^2-2\times x\times 5+5^2$
$=3\left(4x^2-1\right)+x^2-10x+25$
$=12x^2-3+x^2-10x+25$
$=13x^2-10x+22$

3 $3(2x+1)^2+(7x+11)(7x-11)-(3x+4)(2x-1)$
$=3\left(4x^2+4x+1\right)+\left(49x^2-121\right)-\left(6x^2-3x+8x-4\right)$
$=12x^2+12x+3+49x^2-121-6x^2-5x+4$
$=55x^2+7x-114$
The coefficient of x is 7.

4 a $(2x+3)^2=(2x)^2+2(2x)(3)+(3)^2$
$\therefore (2x+3)^2=4x^2+12x+9$

b $4a(b-3a)(b+3a)$
$=4a\left(b^2-9a^2\right)$
$=4ab^2-36a^3$

c $10-(c+2)(4c-5)$
$=10-\left(4c^2-5c+8c-10\right)$
$=10-\left(4c^2+3c-10\right)$
$=10-4c^2-3c+10$
$=20-3c-4c^2$

d $(5-7y)^2=25-70y+49y^2$

e $\left(3m^3+4n\right)\left(3m^3-4n\right)$
$=\left(3m^3\right)^2-(4n)^2$
$=9m^6-16n^2$

f $(x+1)^3$
$=(x+1)(x+1)^2$
$=(x+1)\left(x^2+2x+1\right)$
$=x^3+2x^2+x+x^2+2x+1$
$=x^3+3x^2+3x+1$

5 a $2(2x-3)(x-2)+(x+5)(2x-1)$
$=2\left(2x^2-7x+6\right)+\left(2x^2+9x-5\right)$
$=4x^2-14x+12+2x^2+9x-5$
$=6x^2-5x+7$
The coefficient of x is -5.

b $(2+3x)\left(4-6x-5x^2\right)-(x-6)(x+6)$
$=8-12x-10x^2+12x-18x^2-15x^3-\left(x^2-36\right)$
$=8-28x^2-15x^3-x^2+36$
$=44-29x^2-15x^3$
The coefficient of x is 0.

c $(4x+7)(4x-7)(1-x)$
$=\left(16x^2-49\right)(1-x)$
$=16x^2-16x^3-49+49x$
$=-16x^3+16x^2+49x-49$
The coefficient of x is 49.

d $(x+1-2y)(x+1+2y)+(x-1)^2$
$=((x+1)-2y)((x+1)+2y)+(x-1)^2$
$=(x+1)^2-4y^2+(x-1)^2$
$=x^2+2x+1-4y^2+x^2-2x+1$
$=2x^2-4y^2+2$
The coefficient of x is 0.

e $(3-2x)(2x+9)-3(5x-1)(4-x)$
$=6x+27-4x^2-18x-3\left(20x-5x^2-4+x\right)$
$=-12x+27-4x^2-60x+15x^2+12-3x$
$=39-75x+11x^2$
The coefficient of x is -75.

f $x^2+x-4\left(x^2+x-4\right)$
$=x^2+x-4x^2-4x+16$
$=-3x^2-3x+16$
The coefficient of x is -3.

6 a x^2-36
$=x^2-6^2$
$=(x-6)(x+6)$

b $4 - 25a^2$
$= 2^2 - (5a)^2$
$= (2 - 5a)(2 + 5a)$

c $9m^2 - 1$
$= (3m + 1)(3m - 1)$

d $4a^2 - 64$
$= 4\left(a^2 - 16\right)$
$= 4(a - 4)(a + 4)$

e $2m^2 - 98x^2$
$= 2\left(m^2 - 49x^2\right)$
$= 2(m - 7x)(m + 7x)$

f $1 - 9(1 - m)^2$
$= 1 - (3(1 - m))^2$
$= (1 - 3(1 - m))(1 + 3(1 - m))$
$= (1 - 3 + 3m)(1 + 3 - 3m)$
$= (3m - 2)(4 - 3m)$

7 **a** $x^2 - 9x + 18$
$= (x - 3)(x - 6)$

b $x^2 - 6x + 9$
$= (x - 3)(x - 3)$
$= (x - 3)^2$

c $x^2 + 7x - 60 = (x + 12)(x - 5)$

d $4x^2 + 4x - 15$
$= (2x - 3)(2x + 5)$

e $4x^2 - 20x + 25$
$= (2x - 5)(2x - 5)$
$= (2x - 5)^2$

f $8x^2 - 48xy + 72y^2$
$= 8\left(x^2 - 6xy + 9y^2\right)$
$= 8(x - 3y)^2$

8 **a** $5x^2 - 45y^2$
$= 5(x^2 - 9y^2)$
$= 5(x^2 - (3y)^2)$
$= 5(x - 3y)(x + 3y)$

b $x^2 - 9x - 10$
$= x^2 - 10x + 1x - 10$
$= x(x - 10) + 1(x - 10)$
$= (x - 10)(x + 1)$

c $8x^2 - 14x - 15$
$= 8x^2 - 20x + 6x - 15$
$= 4x(2x - 5) + 3(2x - 5)$
$= (2x - 5)(4x + 3)$

d $4x^3 - 8x^2y - 12xy^2$
$= 4x\left(x^2 - 2xy - 3y^2\right)$
$= 4x(x - 3y)(x + y)$

e $9y^2 - x^2 - 8x - 16$
$= 9y^2 - \left(x^2 + 8x + 16\right)$
$= 9y^2 - (x + 4)^2$
$= [3y - (x + 4)][3y + (x + 4)]$
$= (3y - x - 4)(3y + x + 4)$

f $4(x - 3)^2 - 3(x - 3) - 22$
$= 4a^2 - 3a - 22$ where $a = (x - 3)$
$= (4a - 11)(a + 2)$
$= [4(x - 3) - 11][(x - 3) + 2]$
$= (4x - 23)(x - 1)$

9 **a** $49 - 168x + 144x^2$
$= (7 - 12x)(7 - 12x)$
$= (7 - 12x)^2$

b $2(x - 1)^2 + 13(x - 1) + 20$
Let $a = x - 1$.
$2a^2 + 13a + 20$
$= (2a + 5)(a + 4)$
Substitute $a = x - 1$ back in.
$(2(x - 1) + 5)(x - 1 + 4)$
$= (2x + 3)(x + 3)$

c $40(x + 2)^2 - 18(x + 2) - 7$
Let $a = x + 2$.
$40a^2 - 18a - 7$
$= (4a + 1)(10a - 7)$
Substitute $a = x + 2$ back in.
$(4(x + 2) + 1)(10(x + 2) - 7)$
$= (4x + 9)(10x + 13)$

d $144x^2 - 36y^2$
$= 36\left(4x^2 - y^2\right)$
$= 36(2x + y)(2x - y)$

e Factorise by grouping ‘2 and 2’:
$3a^2x + 9ax - a - 3 = 3ax(a + 3) - 1(a + 3)$
$= (3ax - 1)(a + 3)$

f Factorise by grouping ‘3 and 1’:
$16x^2 + 8x + 1 - y^2$
$= \left(16x^2 + 8x + 1\right) - y^2$
$= (4x + 1)^2 - y^2$
$= ((4x + 1) - y)((4x + 1) + y)$
$= (4x + 1 - y)(4x + 1 + y)$

10 **a** $x^3 + 2x^2 - 25x - 50$
$= x^2(x + 2) - 25(x + 2)$
$= (x + 2)\left(x^2 - 25\right)$
$= (x + 2)(x - 5)(x + 5)$

b $100p^3 - 81pq^2$
$= p\left(100p^2 - 81q^2\right)$
$= p(10p - 9q)(10p + 9q)$

c $4n^2 + 4n + 1 - 4p^2$
$= \left(4n^2 + 4n + 1\right) - 4p^2$
$= (2n + 1)^2 - (2p)^2$
$= (2n + 1 - 2p)(2n + 1 + 2p)$

d $49(m + 2n)^2 - 81(2m - n)^2$
$= [7(m + 2n) - 9(2m - n)][7(m + 2n) + 9(2m - n)]$
$= (7m + 14n - 18m + 9n)(7m + 14n + 18m - 9n)$
$= (23n - 11m)(5n + 25m)$
$= (23n - 11m) \times 5(n + 5m)$
$= 5(n + 5m)(23n - 11m)$

e $13(a-1)+52(1-a)^3$
$= 13(a-1)-52(a-1)^3$
$= 13(a-1)\left(1-4(a-1)^2\right)$
$= 13(a-1)[1-2(a-1)][1+2(a-1)]$
$= 13(a-1)(1-2a+2)(1+2a-2)$
$= 13(a-1)(3-2a)(2a-1)$

f $a^2-b^2-a+b+(a+b-1)^2$
$= (a-b)(a+b)-(a-b)+(a+b-1)^2$
$= (a-b)((a+b)-1)+(a+b-1)^2$
$= (a+b-1)[(a-b)+(a+b-1)]$
$= (a+b-1)(a-b+a+b-1)$
$= (a+b-1)(2a-1)$

11 **a** $(x+5)^2+(x+5)-56$
$= a^2+a-56$ where $a=(x+5)$
$= (a+8)(a-7)$
$= ((x+5)+8)((x+5)-7)$
$= (x+13)(x-2)$

b $2(x+3)^2-7(x+3)-9$
$= 2a^2-7a-9$ where $a=(x+3)$
$= (2a-9)(a+1)$
$= (2(x+3)-9)((x+3)+1)$
$= (2x-3)(x+4)$

c $70(x+y)^2-y(x+y)-6y^2$
$= 70a^2-ya-6y^2$ where $a=(x+y)$
$= (7a+2y)(10a-3y)$
$= (7(x+y)+2y)(10(x+y)-3y)$
$= (7x+9y)(10x+7y)$

d x^4-8x^2-9
$= \left(x^2\right)^2-8\left(x^2\right)-9$
$= a^2-8a-9$ where $a=x^2$
$= (a-9)(a+1)$
$= \left(x^2-9\right)\left(x^2+1\right)$
$= (x-3)(x+3)\left(x^2+1\right)$

e $9(p-q)^2+12\left(p^2-q^2\right)+4(p+q)^2$
$= 9(p-q)^2+12(p-q)(p+q)+4(p+q)^2$
$= 9a^2+12ab+4b^2$ where $a=(p-q), b=(p+q)$
$= (3a+2b)^2$
$= (3(p-q)+2(p+q))^2$
$= (3p-3q+2p+2q)^2$
$= (5p-q)^2$

f $a^2\left(a+\dfrac{1}{a}\right)^2-4a^2\left(a+\dfrac{1}{a}\right)+4a^2$
$= a^2x^2-4a^2x+4a^2$ where $x=\left(a+\dfrac{1}{a}\right)$
$= a^2\left(x^2-4x+4\right)$
$= a^2(x-2)^2$
$= (a(x-2))^2$
$= \left(a\left(a+\dfrac{1}{a}-2\right)\right)^2$
$= \left(a^2+1-2a\right)^2$
$= \left((a-1)^2\right)^2$
$= (a-1)^4$

12 **a** $x^3-125=x^3-5^3$
$= (x-5)\left(x^2+5x+25\right)$

b $3+3x^3=3\left(1^3+x^3\right)$
$= 3(1+x)\left(1-x+x^2\right)$

13 **a** x^3-8
$= x^3-2^3$
$= (x-2)\left(x^2+2x+4\right)$

b x^3+1000
$= x^3+10^3$
$= (x+10)\left(x^2-10x+100\right)$

c $1-x^3$
$= 1^3-x^3$
$= (1-x)\left(1+x+x^2\right)$

d $27x^3+64y^3$
$= (3x)^3+(4y)^3$
$= (3x+4y)\left(9x^2-12xy+16y^2\right)$

e x^4-125x
$= x\left(x^3-125\right)$
$= x\left(x^3-5^3\right)$
$= x(x-5)\left(x^2+5x+25\right)$

f $(x-1)^3+216$
$= (x-1)^3+6^3$
$= ((x-1)+6)\left((x-1)^2-6(x-1)+36\right)$
$= (x+5)\left(x^2-2x+1-6x+6+36\right)$
$= (x+5)\left(x^2-8x+43\right)$

14 **a** $xy^3-27x=x\left(y^3-27\right)$
$= x\left(y^3-3^3\right)$
$= x(y-3)\left(y^2+3y+9\right)$

b $-x^3-216=-\left(x^3+216\right)$
$= -\left(x^3+6^3\right)$
$= -(x+6)\left(x^2-6x+36\right)$

c $3-81x^3=3\left(1-27x^3\right)$
$= 3\left(1^3-(3x)^3\right)$
$= 3(1-3x)\left(1^2+1\times3x+(3x)^2\right)$
$= 3(1-3x)\left(1+3x+9x^2\right)$

d $32x^3+4m^3=4\left(8x^3+m^3\right)$
$= 4\left((2x)^3+m^3\right)$
$= 4(2x+m)\left((2x)^2-2mx+m^2\right)$
$= 4(2x+m)\left(4x^2-2mx+m^2\right)$

e $27m^3+64n^3=(3m)^3+(4n)^3$
$= (3m+4n)\left((3m)^2-3m\times4n+(4n)^2\right)$
$= (3m+4n)\left(9m^2-12mn+16n^2\right)$

f $250x^3-128m^3=2\left(125x^3-64m^3\right)$
$= 2\left((5x)^3-(4m)^3\right)$
$= 2(5x-4m)\left((5x)^2+5x\times4m+(4m)^2\right)$
$= 2(5x-4m)\left(25x^2+20mx+16m^2\right)$

15 **a** $24x^3-81y^3$
$= 3\left(8x^3-27y^3\right)$
$= 3\left((2x)^3-(3y)^3\right)$
$= 3(2x-3y)\left(4x^2+6xy+9y^2\right)$

b $8x^4y^4 + xy$

$$\begin{aligned}&= xy\left(8x^3y^3 + 1\right)\\&= xy\left((2xy)^3 + 1^3\right)\\&= xy\,(2xy + 1)\left(4x^2y^2 - 2xy + 1\right)\end{aligned}$$

c $125(x + 2)^3 + 64(x - 5)^3$

$$\begin{aligned}&= [5\,(x + 2) + 4\,(x - 5)]\left[25(x + 2)^2 - 20\,(x + 2)\,(x - 5)\right.\\&\quad\left. + 16(x - 5)^2\right]\\&= (5x + 10 + 4x - 20)\left(25\left(x^2 + 4x + 4\right)\right.\\&\quad\left. - 20\left(x^2 - 3x - 10\right) + 16\left(x^2 - 10x + 25\right)\right)\\&= (9x - 10)\left(25x^2 + 100x + 100 - 20x^2 + 60x + 200\right.\\&\quad\left. + 16x^2 - 160x + 400\right)\\&= (9x - 10)\left(21x^2 + 700\right)\\&= 7\,(9x - 10)\left(3x^2 + 100\right)\end{aligned}$$

d $2(x - y)^3 - 54(2x + y)^3$

$$\begin{aligned}&= 2\left((x - y)^3 - 27(2x + y)^3\right)\\&= 2\,((x - y) - 3\,(2x + y))\left((x - y)^2 + 3\,(x - y)\,(2x + y)\right.\\&\quad\left. + 9(2x + y)^2\right)\\&= 2\,(x - y - 6x - 3y)\left(x^2 - 2xy + y^2 + 3\left(2x^2 - xy - y^2\right)\right.\\&\quad\left. + 9\left(4x^2 + 4xy + y^2\right)\right)\\&= 2\,(-5x - 4y)\left(43x^2 + 31xy + 7y^2\right)\\&= -2\,(5x + 4y)\left(43x^2 + 31xy + 7y^2\right)\end{aligned}$$

e $a^5 - a^3b^2 + a^2b^3 - b^5$

$$\begin{aligned}&= a^3\left(a^2 - b^2\right) + b^3\left(a^2 - b^2\right)\\&= \left(a^2 - b^2\right)\left(a^3 + b^3\right)\\&= (a - b)\,(a + b)\,(a + b)\left(a^2 - ab + b^2\right)\\&= (a - b)\,(a + b)^2\left(a^2 - ab + b^2\right)\end{aligned}$$

f $x^6 - y^6$

$$\begin{aligned}&= \left(x^3\right)^2 - \left(y^3\right)^2\\&= \left(x^3 - y^3\right)\left(x^3 + y^3\right)\\&= (x - y)\left(x^2 + xy + y^2\right)(x + y)\left(x^2 - xy + y^2\right)\\&= (x - y)\,(x + y)\left(x^2 + xy + y^2\right)\left(x^2 - xy + y^2\right)\end{aligned}$$

16 a $\dfrac{x^2 + 4x}{x^2 + 2x - 8}$

$$\begin{aligned}&= \frac{x\,(x + 4)}{(x + 4)\,(x - 2)}\\&= \frac{x}{x - 2}\end{aligned}$$

b $\dfrac{x^4 - 64}{5 - x} \div \dfrac{x^2 + 8}{x - 5}$

$$\begin{aligned}&= \frac{x^4 - 64}{5 - x} \times \frac{x - 5}{x^2 + 8}\\&= \frac{\left(x^2 + 8\right)\left(x^2 - 8\right)}{-\,(x - 5)} \times \frac{x - 5}{x^2 + 8}\\&= -\left(x^2 - 8\right)\\&= 8 - x^2\end{aligned}$$

17 a $\dfrac{x}{(x + 1)\,(x - 2)} \times \dfrac{x - 2}{3x} = \dfrac{1}{3\,(x + 1)}$

b $\dfrac{5}{x\,(3x + 1)} \div \dfrac{10}{x\,(x + 3)}$

$$\begin{aligned}&= \frac{5}{x\,(3x + 1)} \times \frac{x\,(x + 3)}{10}\\&= \frac{x + 3}{2\,(3x + 1)}\end{aligned}$$

c $\dfrac{x^2 + 5x + 6}{4x + 8}$

$$\begin{aligned}&= \frac{(x + 3)\,(x + 2)}{4\,(x + 2)}\\&= \frac{x + 3}{4}\end{aligned}$$

d $\dfrac{16 - 9x^2}{8 + 6x} \times \dfrac{2x + 10}{3x - 4}$

$$\begin{aligned}&= \frac{(4 + 3x)\,(4 - 3x)}{2\,(4 + 3x)} \times \frac{2\,(x + 5)}{-\,(4 - 3x)}\\&= \frac{x + 5}{-1}\\&= -x - 5\end{aligned}$$

e $\dfrac{4x}{3x^2 + 5x + 2} \div \dfrac{18x^2 - 6x}{9x^2 - 1}$

$$\begin{aligned}&= \frac{4x}{(3x + 2)\,(x + 1)} \times \frac{9x^2 - 1}{18x^2 - 6x}\\&= \frac{4x}{(3x + 2)\,(x + 1)} \times \frac{(3x - 1)\,(3x + 1)}{6x\,(3x - 1)}\\&= \frac{2\,(3x + 1)}{3\,(3x + 2)\,(x + 1)}\end{aligned}$$

f $\dfrac{2x^2 - 3x - 5}{2x^2 - 11x + 15} \times \dfrac{3x^2 - 5x - 12}{3x^2 + 7x + 4}$

$$\begin{aligned}&= \frac{(2x - 5)\,(x + 1)}{(x - 3)\,(2x - 5)} \times \frac{(3x + 4)\,(x - 3)}{(x + 1)\,(3x + 4)}\\&= 1\end{aligned}$$

18 a $\dfrac{3x^2 - 7x - 20}{25 - 9x^2}$

$$\begin{aligned}&= \frac{\cancel{(3x + 5)}\,(x - 4)}{(5 - 3x)\,\cancel{(5 + 3x)}}\\&= \frac{x - 4}{5 - 3x}\end{aligned}$$

b $\dfrac{x^3 + 4x^2 - 9x - 36}{x^2 + x - 12}$

$$\begin{aligned}&= \frac{x^2\,(x + 4) - 9\,(x + 4)}{(x + 4)\,(x - 3)}\\&= \frac{(x + 4)\left(x^2 - 9\right)}{(x + 4)\,(x - 3)}\\&= \frac{(x + 4)\,(x - 3)\,(x + 3)}{(x + 4)\,(x - 3)}\\&= x + 3\end{aligned}$$

c $\dfrac{(x + h)^3 - x^3}{h}$

$$\begin{aligned}&= \frac{((x + h) - x)\left((x + h)^2 + x\,(x + h) + x^2\right)}{h}\\&= \frac{\cancel{(h)}\left(x^2 + 2xh + h^2 + x^2 + xh + x^2\right)}{\cancel{h}}\\&= 3x^2 + 3xh + h^2\end{aligned}$$

d $\dfrac{2x^2}{9x^3 + 3x^2} \times \dfrac{1 - 9x^2}{18x^2 - 12x + 2}$

$$\begin{aligned}&= \frac{2x^2}{3x^2\,(3x + 1)} \times \frac{(1 - 3x)\,(1 + 3x)}{2\left(9x^2 - 6x + 1\right)}\\&= \frac{\cancel{2}}{3\cancel{(3x + 1)}} \times \frac{(1 - 3x)\,\cancel{(1 + 3x)}}{\cancel{2}(3x - 1)^2}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \times \frac{(1-3x)}{(1-3x)^2}\\
&= \frac{1}{3} \times \frac{1}{1-3x}\\
&= \frac{1}{3\,(1-3x)}
\end{aligned}$$

e $\dfrac{m^3-2m^2n}{m^3+n^3} \div \dfrac{m^2-4n^2}{m^2+3mn+2n^2}$

$$\begin{aligned}
&= \frac{m^3-2m^2n}{m^3+n^3} \times \frac{m^2+3mn+2n^2}{m^2-4n^2}\\
&= \frac{m^2\,(m-2n)}{(m+n)\left(m^2-mn+n^2\right)} \times \frac{(m+2n)\,(m+n)}{(m-2n)\,(m+2n)}\\
&= \frac{m^2}{m^2-mn+n^2}
\end{aligned}$$

f $\dfrac{1-x^3}{1+x^3} \times \dfrac{1-x^2}{1+x^2} \div \dfrac{1+x+x^2}{1-x+x^2}$

$$\begin{aligned}
&= \frac{1-x^3}{1+x^3} \times \frac{1-x^2}{1+x^2} \times \frac{1-x+x^2}{1+x+x^2}\\
&= \frac{(1-x)\left(1+x+x^2\right)}{(1+x)\left(1-x+x^2\right)} \times \frac{(1-x)\,(1+x)}{1+x^2} \times \frac{1-x+x^2}{1+x+x^2}\\
&= \frac{(1-x)}{1} \times \frac{(1-x)}{1+x^2} \times \frac{1}{1}\\
&= \frac{(1-x)^2}{1+x^2}
\end{aligned}$$

19 $\dfrac{6}{5x-25} + \dfrac{1}{x-1} - \dfrac{2x}{x^2-6x+5}$

$$\begin{aligned}
&= \frac{6}{5\,(x-5)} + \frac{1}{(x-1)} - \frac{2x}{(x-5)\,(x-1)}\\
&= \frac{6\,(x-1)+5\,(x-5)-10x}{5\,(x-5)\,(x-1)}\\
&= \frac{x-31}{5\,(x-5)\,(x-1)}
\end{aligned}$$

20 a
$$\begin{aligned}
\frac{2x}{3} + \frac{5x}{4} &= \frac{2x}{3} \times \frac{4}{4} + \frac{5x}{4} \times \frac{3}{3}\\
&= \frac{8x}{12} + \frac{15x}{12}\\
&= \frac{23x}{12}
\end{aligned}$$

b
$$\begin{aligned}
\frac{x}{7} - \frac{3x}{2} &= \frac{x}{7} \times \frac{2}{2} - \frac{3x}{2} \times \frac{7}{7}\\
&= \frac{2x}{14} - \frac{21x}{14}\\
&= -\frac{19x}{14}
\end{aligned}$$

c
$$\begin{aligned}
\frac{4}{x-3} + \frac{5}{x+5} &= \frac{4}{(x-3)} \times \frac{(x+5)}{(x+5)} + \frac{5}{(x+5)} \times \frac{(x-3)}{(x-3)}\\
&= \frac{4(x+5)}{(x-3)(x+5)} + \frac{5(x-3)}{(x+5)(x-3)}\\
&= \frac{4x+20+5x-15}{(x-3)(x+5)}\\
&= \frac{9x+5}{(x-3)(x+5)}
\end{aligned}$$

d
$$\begin{aligned}
\frac{x}{3x-1} - \frac{5}{1-2x} &= \frac{x}{(3x-1)} \times \frac{(1-2x)}{(1-2x)}\\
&\quad - \frac{5}{(1-2x)} \times \frac{(3x-1)}{(3x-1)}\\
&= \frac{x(1-2x)}{(3x-1)(1-2x)} - \frac{5(3x-1)}{(1-2x)(3x-1)}\\
&= \frac{x-2x^2-15x+5}{(3x-1)(1-2x)}\\
&= \frac{-2x^2-14x+5}{(3x-1)(1-2x)}
\end{aligned}$$

e
$$\begin{aligned}
\frac{x-3}{2x+1} - \frac{x+2}{x-1} &= \frac{(x-3)}{(2x+1)} \times \frac{(x-1)}{(x-1)}\\
&\quad - \frac{(x+2)}{(x-1)} \times \frac{(2x+1)}{(2x+1)}\\
&= \frac{(x-3)(x-1)}{(2x+1)(x-1)} - \frac{(x+2)(2x+1)}{(x-1)(2x+1)}\\
&= \frac{x^2-4x+3-2x^2-5x-2}{(2x+1)(x-1)}\\
&= \frac{-x^2-9x+1}{(2x+1)(x-1)}
\end{aligned}$$

f $\dfrac{3}{x^2-9} - \dfrac{1}{x+3} + \dfrac{5}{x-3}$

$$\begin{aligned}
&= \frac{3}{(x^2-9)} - \frac{1}{(x+3)} \times \frac{(x-3)}{(x-3)} + \frac{5}{(x-3)} \times \frac{(x+3)}{(x+3)}\\
&= \frac{3}{(x^2-9)} - \frac{(x-3)}{(x^2-9)} + \frac{5(x+3)}{(x^2-9)}\\
&= \frac{3-x+3+5x+15}{(x^2-9)}\\
&= \frac{4x+21}{(x^2-9)}
\end{aligned}$$

21 a $\dfrac{4}{x^2+1} + \dfrac{4}{x-x^2}$

$$\begin{aligned}
&= \frac{4}{x^2+1} + \frac{4}{x\,(1-x)}\\
&= \frac{4x\,(1-x)+4\left(x^2+1\right)}{\left(x^2+1\right)x\,(1-x)}\\
&= \frac{4x-4x^2+4x^2+4}{x\,(1-x)\left(x^2+1\right)}\\
&= \frac{4\,(x+1)}{x\,(1-x)\left(x^2+1\right)}
\end{aligned}$$

b $\dfrac{4}{x^2-4} - \dfrac{3}{x+2} + \dfrac{5}{x-2}$

$$\begin{aligned}
&= \frac{4}{(x-2)\,(x+2)} - \frac{3}{x+2} + \frac{5}{x-2}\\
&= \frac{4-3\,(x-2)+5\,(x+2)}{(x-2)\,(x+2)}\\
&= \frac{4-3x+6+5x+10}{(x-2)\,(x+2)}\\
&= \frac{2x+20}{(x-2)\,(x+2)}\\
&= \frac{2\,(x+10)}{(x-2)\,(x+2)} = \frac{2x+20}{x^2-4}
\end{aligned}$$

c $\frac{5}{x+6}+\frac{4}{5-x}+\frac{3}{x^2+x-30}$

$=\frac{5}{x+6}-\frac{4}{x-5}+\frac{3}{(x+6)(x-5)}$

$=\frac{5(x-5)-4(x+6)+3}{(x+6)(x-5)}$

$=\frac{5x-25-4x-24+3}{(x+6)(x-5)}$

$=\frac{x-46}{(x+6)(x-5)}$

d $\frac{1}{4y^2-36y+81}+\frac{2}{4y^2-81}-\frac{1}{2y^2-9y}$

$=\frac{1}{(2y-9)^2}+\frac{2}{(2y-9)(2y+9)}-\frac{1}{y(2y-9)}$

$=\frac{y(2y+9)+2y(2y-9)-(2y-9)(2y+9)}{(2y-9)^2(2y+9)y}$

$=\frac{2y^2+9y+4y^2-18y-\left(4y^2-81\right)}{y(2y+9)(2y-9)^2}$

$=\frac{2y^2-9y+81}{y(2y-9)^2(2y+9)}$

22 a $(2+3x)(x+6)(3x-2)(6-x)$

$=(3x+2)(3x-2)(6+x)(6-x)$

$=\left(9x^2-4\right)\left(36-x^2\right)$

$=324x^2-9x^4-144+4x^2$

$=-9x^4+328x^2-144$

b $x^2-6x+9-xy+3y$

$=\left(x^2-6x+9\right)+(-xy+3y)$

$=(x-3)^2-y(x-3)$

$=(x-3)[(x-3)-y]$

$=(x-3)(x-3-y)$

c First take out the common factors.

$2y^4+2y(x-y)^3$

$=2y[y^3+(x-y)^3]$

The term in the square brackets is a sum of two cubes (y^3+b^3) where $b=(x-y)$, so the next step is to factorise this sum of two cubes.

$=2y(y+b)\left(y^2-yb+b^2\right)$

$=2y[y+(x-y)][y^2-y(x-y)+(x-y)^2]$

Finally, the inner brackets in each factor need to be expanded and like terms collected.

$=2y(x)\left(y^2-yx+y^2+x^2-2xy+y^2\right)$

$=2xy\left(x^2-3xy+3y^2\right)$

23 a $(g+12+h)^2$

$=((g+12)+h)^2$

$=(g+12)^2+2h(g+12)+h^2$

$=g^2+24g+144+2gh+24h+h^2$

$=g^2+h^2+2gh+24g+24h+144$

b $(2p+7q)^2(7q-2p)$

$=(7q+2p)^2(7q-2p)$

$=(7q+2p)((7q+2p)(7q-2p))$

$=(7q+2p)\left(49q^2-4p^2\right)$

$=343q^3+98q^2p-28qp^2-8p^3$

c $(x+10)(5+2x)(10-x)(2x-5)$

$=(10+x)(10-x)(2x+5)(2x-5)$

$=\left(100-x^2\right)\left(4x^2-25\right)$

$=400x^2-2500-4x^4+25x^2$

$=-4x^4+425x^2-2500$

24 a $\frac{x^3-125}{x^2-25}\times\frac{5}{x^3+5x^2+25x}$

$=\frac{(x-5)\left(x^2+5x+25\right)}{(x-5)(x+5)}\times\frac{5}{x\left(x^2+5x+25\right)}$

$=\frac{5}{x(x+5)}$

b $\left(\frac{4}{x+1}-\frac{3}{(x+1)^2}\right)\div\frac{16x^2-1}{x^2+2x+1}$

$=\left(\frac{4(x+1)-3}{(x+1)^2}\right)\times\frac{x^2+2x+1}{16x^2-1}$

$=\frac{4x+1}{(x+1)^2}\times\frac{(x+1)^2}{(4x+1)(4x-1)}$

$=\frac{1}{4x-1}$

c $\frac{1}{p-q}-\frac{p}{p^2-q^2}-\frac{q^3}{p^4-q^4}$

$=\frac{1}{p-q}-\frac{p}{(p-q)(p+q)}-\frac{q^3}{(p^2-q^2)(p^2+q^2)}$

$=\frac{1}{p-q}-\frac{p}{(p-q)(p+q)}-\frac{q^3}{(p-q)(p+q)(p^2+q^2)}$

$=\frac{(p+q)(p^2+q^2)-p(p^2+q^2)-q^3}{(p-q)(p+q)(p^2+q^2)}$

$=\frac{p^3+pq^2+qp^2+q^3-p^3-pq^2-q^3}{(p-q)(p+q)(p^2+q^2)}$

$=\frac{qp^2}{(p-q)(p+q)(p^2+q^2)}$

$=\frac{p^2q}{p^4-q^4}$

d $(a+6b)\div\left(\frac{7}{a^2-3ab+2b^2}-\frac{5}{a^2-ab-2b^2}\right)$

$=(a+6b)\div\left(\frac{7}{(a-b)(a-2b)}-\frac{5}{(a-2b)(a+b)}\right)$

$=(a+6b)\div\frac{7(a+b)-5(a-b)}{(a-b)(a-2b)(a+b)}$

$=(a+6b)\div\frac{2a+12b}{(a-b)(a-2b)(a+b)}$

$=(a+6b)\times\frac{(a-b)(a-2b)(a+b)}{2(a+6b)}$

$=\frac{(a-b)(a-2b)(a+b)}{2}$

$=\frac{1}{2}(a-2b)(a-b)(a+b)$

25 a On the Main window of a CAS calculator, tap menu → algebra → expand and enter the expression $(x+5)(2-x)(3x+7)$.

$(x+5)(2-x)(3x+7)=-3x^3-16x^2+9x+70$

b Tap menu → algebra → factor and enter the expression $27(x-2)^3+64(x+2)^3$.

$27(x-2)^3+64(x+2)^3=\left(13x^2+28x+148\right)(7x+2)$

c Tap menu → algebra → factor and enter the expression as $3 / (x-1) + 8 / (x+8)$.

$$\frac{3}{x-1}+\frac{8}{x+8}=\frac{11x+16}{(x+8)(x-1)}$$

2.2 Exam questions

1 $$\begin{aligned} x(x-2)(x+1)+(x-3)^2 &= x\left(x^2-x-2\right)+x^2-6x+9 \\ &= x^3-x^2-2x+x^2-6x+9 \\ \text{Coefficient of } x &= -2-6 \\ &= -8 \end{aligned}$$

The correct answer is **E**.

2 $$\begin{aligned} 9(2x-1)^2-9 &= 9\left[(2x-1)^2-1\right] \\ &= 9\left[(2x-1+1)(2x-1-1)\right] \\ &= 9(2x)(2x-2) \\ &= 18x(2x-2) \\ &= 36x(x-1) \end{aligned}$$

Therefore, the factors are $36x$ and $x-1$.
The correct answer is **D**.

3 $8(x+3)^2+24(x+3)+16$

Let $(x+3)=a$.

$$\begin{aligned} 8a^2+24a+16 &= 8\left(a^2+3a+2\right) \\ &= 8(a+2)(a+1) \end{aligned}$$

Substitute $(x+3)=a$.

$$\begin{aligned} &= 8(x+3+2)(x+3+1) \\ &= 8(x+5)(x+4) \end{aligned}$$ [1 mark]

2.3 Pascal's triangle and binomial expansions

2.3 Exercise

1 Use $a=3x$ and $b=2$ in the expansion of $(a-b)^3$.

$$\begin{aligned} &(3x-2)^3 \\ &= (3x)^3-3(3x)^2(2)+3(3x)(2)^2-(2)^3 \\ &= 27x^3-54x^2+36x-8 \end{aligned}$$

2 a $$\begin{aligned} &(x-3)^3 \\ &= x^3-3\times x^2\times 3+3\times x\times 3^2-3^3 \\ &= x^3-9x^2+27x-27 \end{aligned}$$

b $$\begin{aligned} &(2x-1)^3 \\ &= (2x)^3-3\times(2x)^2\times 1+3\times 2x\times 1^2-1^3 \\ &= 8x^3-3\times 4x^2+6x-1 \\ &= 8x^3-12x^2+6x-1 \end{aligned}$$

c $$\begin{aligned} &(x+4)^3 \\ &= x^3+3\times x^2\times 4+3\times x\times 4^2+4^3 \\ &= x^3+12x^2+3\times x\times 16+64 \\ &= x^3+12x^2+48x+64 \end{aligned}$$

3 a $$\begin{aligned} &(3x+1)^3 \\ &= (3x)^3+3(3x)^2(1)+3(3x)(1)^2+(1)^3 \\ &= 27x^3+27x^2+9x+1 \end{aligned}$$

b $$\begin{aligned} &(1-2x)^3 \\ &= (1)^3-3(1)^2(2x)+3(1)(2x)^2-(2x)^3 \\ &= 1-6x+12x^2-8x^3 \end{aligned}$$

c $$\begin{aligned} &(5x+2y)^3 \\ &= (5x)^3+3(5x)^2(2y)+3(5x)(2y)^2+(2y)^3 \\ &= 125x^3+150x^2y+60xy^2+8y^3 \end{aligned}$$

4 $$\begin{aligned} &(x+2)^3 \\ &= x^3+3x^2(2)+3x(2)^2+(2)^3 \\ &= x^3+6x^2+12x+8 \end{aligned}$$

Statement A is the only correct statement.

5 $$\begin{aligned} &\left(a+2b^2\right)^3 \\ &= a^3+3a^2\left(2b^2\right)+3a\left(2b^2\right)^2+\left(2b^2\right)^3 \\ &= a^3+3a^2\times 2b^2+3a\times 4b^4+8b^6 \\ &= a^3+6a^2b^2+12ab^4+8b^6 \end{aligned}$$

The coefficient of a^2b^2 is 6.

6

Binomial power	Expansion	Number of terms in the expansion	Sum of indices in each term
$(x+a)^2$	$x^2+2xa+a^2$	3	2
$(x+a)^3$	$x^3+3x^2a+3xa^2+a^3$	4	3
$(x+a)^4$	$x^4+4x^3a+6x^2a^2$ $+4xa^3+a^4$	5	4
$(x+a)^5$	$x^5+5x^4a+10x^3a^2$ $+10x^2a^3+5xa^4+a^5$	6	5

7 $(a-b)^6$

The binomial coefficients for row 6 are:
1, 6, 15, 20, 15, 6, 1

Thus,

$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$
$$(a-b)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$$

$$\begin{aligned} &(2x-1)^6 \\ &= (2x)^6-6(2x)^5(1)+15(2x)^4(1)^2-20(2x)^3(1)^3 \\ &\quad +15(2x)^2(1)^4-6(2x)(1)^5+(1)^6 \\ &= 64x^6-192x^5+240x^4-160x^3+60x^2-12x+1 \end{aligned}$$

8 $$\begin{aligned} &(3x+2y)^4 \\ &= (3x)^4+4(3x)^3(2y)+6(3x)^2(2y)^2+4(3x)(2y)^3+(2y)^4 \\ &= 81x^4+216x^3y+216x^2y^2+96xy^3+16y^4 \end{aligned}$$

9 a $$\begin{aligned} &(x+4)^5 \\ &= x^5+5x^4(4)+10x^3(4)^2+10x^2(4)^3+5x(4)^4+(4)^5 \\ &= x^5+20x^4+160x^3+640x^2+1280x+1024 \end{aligned}$$

b Alternating the signs in part **a** gives

$$(x-4)^5=x^5-20x^4+160x^3-640x^2+1280x-1024$$

c $(xy+2)^5$

$$\begin{aligned} (xy+2)^5 &= (xy)^5+5(xy)^4(2)+10(xy)^3(2)^2+10(xy)^2(2)^3 \\ &\quad +5(xy)(2)^4+(2)^5 \\ &= x^5y^5+10x^4y^4+40x^3y^3+80x^2y^2+80xy+32 \end{aligned}$$

10 a $$\begin{aligned} &(3x-5y)^4 \\ &= (3x)^4-4(3x)^3(5y)+6(3x)^2(5y)^2-4(3x)(5y)^3+(5y)^4 \\ &= 81x^4-540x^3y+1350x^2y^2-1500xy^3+625y^4 \end{aligned}$$

b $$\begin{aligned} &\left(3-x^2\right)^4 \\ &= (3)^4-4(3)^3\left(x^2\right)+6(3)^2\left(x^2\right)^2-4(3)\left(x^2\right)^3+\left(x^2\right)^4 \\ &= 81-108x^2+54x^4-12x^6+x^8 \end{aligned}$$

11 $$\begin{aligned} &(1+x)^6-(1-x)^6 \\ &= 1+6x+15x^2+20x^3+15x^4+6x^5+x^6 \\ &\quad -\left(1-6x+15x^2-20x^3+15x^4-6x^5+x^6\right) \\ &= 12x+40x^3+12x^5 \end{aligned}$$

12 a $(x+1)^3 - 3x(x+2)^2$

$= x^3 + 3x^2 + 3x + 1 - 3x\left(x^2 + 4x + 4\right)$

$= -2x^3 - 9x^2 - 9x + 1$

The coefficient of x^2 is -9.

b $3x^2(x+5)(x-5) + 4(5x-3)^3$

$= 3x^2\left(x^2 - 25\right) + 4\left(125x^3 - 3 \times 25x^2 \times 3 + 3 \times 5x \times 9 - 27\right)$

$= 3x^4 - 75x^2 + 500x^3 - 900x^2 + 540x - 108$

$= 3x^4 + 500x^3 - 975x^2 + 540x - 108$

The coefficient of x^2 is -975.

c $(x-1)(x+2)(x-3) - (x-1)^3$

$= (x-1)\left(x^2 - x - 6\right) - \left(x^3 - 3x^2 + 3x - 1\right)$

$= x^3 - x^2 - 6x - x^2 + x + 6 - x^3 + 3x^2 - 3x + 1$

$= x^2 - 8x + 7$

The coefficient of x^2 is 1.

d $\left(2x^2 - 3\right)^3 + 2\left(4 - x^2\right)^3$

$= \left(2x^2\right)^3 - 3\left(2x^2\right)^2(3) + 3\left(2x^2\right)(3)^2 - (3)^3$

$+ 2\left(4^3 - 3(4)^2\left(x^2\right) + 3(4)\left(x^2\right)^2 - \left(x^2\right)^3\right)$

$= 8x^6 - 36x^4 + 54x^2 - 27 + 2\left(64 - 48x^2 + 12x^4 - x^6\right)$

$= 8x^6 - 36x^4 + 54x^2 - 27 + 128 - 96x^2 + 24x^4 - 2x^6$

$= 6x^6 - 12x^4 - 42x^2 + 101$

The coefficient of x^2 is $-$ 42.

13 $[(x-1)+y]^4$

$= (x-1)^4 + 4(x-1)^3y + 6(x-1)^2y^2 + 4(x-1)y^3 + y^4$

$= (x^4 - 4x^3 + 6x^2 - 4x + 1) + 4y(x^3 - 3x^2 + 3x - 1)$

$+ 6y^2(x^2 - 2x + 1) + 4y^3(x-1) + y^4$

$= (x^4 - 4x^3 + 6x^2 - 4x + 1)$

$+ (4yx^3 - 12yx^2 + 12yx - 4y)$

$+ (6y^2x^2 - 12y^2x + 6y^2) + (4y^3x - 4y^3) + y^4$

$= x^4 + y^4 - 4x^3 - 4y^3 + 6x^2 + 6y^2 - 4x - 4y + 4x^3y$

$+ 4y^3x - 12x^2y - 12xy^2 + 6x^2y^2 + 12xy + 1$

14 $\left(\dfrac{x}{2} + \dfrac{2}{x}\right)^6$

$= \left(\dfrac{x}{2}\right)^6 + 6\left(\dfrac{x}{2}\right)^5\left(\dfrac{2}{x}\right) + 15\left(\dfrac{x}{2}\right)^4\left(\dfrac{2}{x}\right)^2$

$+ 20\left(\dfrac{x}{2}\right)^3\left(\dfrac{2}{x}\right)^3 + 15\left(\dfrac{x}{2}\right)^2\left(\dfrac{2}{x}\right)^4$

$+ 6\left(\dfrac{x}{2}\right)\left(\dfrac{2}{x}\right)^5 + \left(\dfrac{2}{x}\right)^6$

For the term independent of x, the x power in the numerator and denominator must be the same. This will occur in the fourth term, $20\left(\dfrac{x}{2}\right)^3\left(\dfrac{2}{x}\right)^3$.

$20\left(\dfrac{x}{2}\right)^3\left(\dfrac{2}{x}\right)^3$

$= 20 \times \dfrac{x^3}{8} \times \dfrac{8}{x^3}$

$= 20$

15 $(x+ay)^4 = x^4 + 4x^3ay + 6x^2a^2y^2 + 4xa^3y^3 + a^4y^4$

The coefficient of x^2y^2 is $6a^2$.

$\left(ax^2 - y\right)^4 = \left(ax^2\right)^4 - 4\left(ax^2\right)^3y + 6\left(ax^2\right)^2y^2 - 4\left(ax^2\right)y^3 + y^4$

$= a^4x^8 - 4a^3x^6y + 6a^2x^4y^2 - 4ax^2y^3 + y^4$

The coefficient of x^2y^3 is $-4a$.

$\therefore 6a^2 = 3 \times (-4a)$

$\therefore 6a^2 = -12a$

$\therefore 6a = -12$ since $a \neq 0$

$\therefore a = -2$

16 $(1+2x)(1-x)^5$

$= (1+2x)\left(1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5\right)$

$= 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5 + 2x(1 - 5x + 10x^2$

$- 10x^3 + 5x^4 - x^5)$

$= 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5 + 2x - 10x^2 + 20x^3$

$- 20x^4 + 10x^5 - 2x^6$

$= 1 - 3x + 10x^3 - 15x^4 + 9x^5 - 2x^6$

The coefficient of x is -3.

17 a $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

b $1.1^4 = (1+0.1)^4$

$= (1+x)^4$ for $x = 0.1$

$(1+0.1)^4 = 1 + 4(0.1) + 6(0.1)^2 + 4(0.1)^3 + (0.1)^4$

$\therefore 1.1^4 = 1 + 0.4 + 0.06 + 0.004 + 0.0001$

$\therefore 1.1^4 = 1.4641$

18 a $(x+1)^5 - (x+1)^4$

$= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$- \left(x^4 + 4x^3 + 6x^2 + 4x + 1\right)$

$= x^5 + 4x^4 + 6x^3 + 4x^2 + x$

Hence,

$(x+1)^5 - (x+1)^4$

$= x^5 + 4x^4 + 6x^3 + 4x^2 + x$

$= x\left(x^4 + 4x^3 + 6x^2 + 4x + 1\right)$

$= x(x+1)^4$

19 $(x+1)^{n+1} - (x+1)^n = (x+1)(x+1)^n - (x+1)^n$

$= (x+1)^n[(x+1) - 1]$

$= (x+1)^n[x]$

$\therefore (x+1)^{n+1} - (x+1)^n = x(x+1)^n$

20 Since each term is formed by adding the two terms to its left and right from the preceding row,

$b + 165 = 220$

$\therefore b = 55$

$165 + 330 = c$

$\therefore c = 495$

$45 + a = 165$

$\therefore a = 120$

2.3 Exam questions

1 For $(x+2)^5$, the power of the binomial is 5.

Therefore, the binomial coefficients are in row 5.

1 5 10 10 5 1

$(x+2)^5 = x^5 + 5x^4(2)^1 + 10x^3(2)^2 \ldots$

The required coefficient of x^3 is $10 \times 2^2 = 40$.

The correct answer is **B**.

2 For $\left(x - \dfrac{2}{x}\right)^4$, the power of the binomial is 4.

Therefore, the binomial coefficients are in row 4.

1 4 6 4 1

$\left(x - \dfrac{2}{x}\right)^4 = x^4 - 4x^3\left(\dfrac{2}{x}\right) + 6x^2\left(\dfrac{2}{x}\right)^2 - 4x\left(\dfrac{2}{x}\right)^3 \ldots$

The coefficient of term independent of x is the x^0 term, that is $6x^2\left(\frac{2}{x}\right)^2$.

The coefficient $= 6 \times (2)^2 = 24$.

The correct answer is **E**.

3 $(x-2)^3$

The power of the binomials is 4. Therefore, use row 4 coefficients.

$$(x-2)^4 = x^4 - 4x^3(2) + 6x^2(2)^2 - 4x(2)^3 + 2^4$$
$$(x-2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16$$

$(x-2)^3$

The power of the binomials is 3. Therefore, use row 3 coefficients.

$$(x-2)^3 = x^3 - 3x^2(2) + 3x(2)^2 - (2)^3$$
$$(x-2)^3 = x^3 - 6x^2 + 12x - 8$$
$$(x-2)^4 - (x-2)^3 = \left(x^4 - 8x^3 + 24x^2 - 32x + 16\right) - \left(x^3 - 6x^2 + 12x - 8\right)$$
$$= x^4 - 9x^3 + 30x^2 - 44x + 24$$

Therefore, the coefficient of x is -44. [1 mark]

2.4 The binomial theorem

2.4 Exercise

1 a $3! = 3 \times 2 \times 1 = 6$

b $4! = 4 \times 3 \times 2 \times 1 = 24$

c $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

d $2! = 2 \times 1 = 2$

e Since $0! = 1$ and $1! = 1$, $0! \times 1! = 1 \times 1 = 1$.

f Since $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, it can also be expressed as $7 \times 6!$.

Therefore, $\frac{7!}{6!} = \frac{7 \times 6!}{6!}$

Cancelling 6! from numerator and denominator, $\frac{7!}{6!} = 7$.

2 a $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ can be written as 7!.

b $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ can be written as 8!.

c $8 \times 7! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, so it can be written as 8!.

d $9 \times 8! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, so it can be written as 9!.

3 $6! + 4! - \frac{10!}{9!}$

$$= 6 \times 5! + 4 \times 3! - \frac{10 \times 9!}{9!}$$
$$= 6 \times 120 + 4 \times 6 - 10$$
$$= 734$$

4 $\frac{n!}{(n-2)!}$

$$= \frac{n(n-1)(n-2)!}{(n-2)!}$$
$$= n(n-1)$$

5 a $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

b $4! + 2! = 24 + 2 = 26$

c $7 \times 6 \times 5! = 7! = 5040$

d $\frac{6!}{3!}$

$$= \frac{6 \times 5 \times 4 \times 3!}{3!}$$
$$= 6 \times 5 \times 4$$
$$= 120$$

6 a $\frac{26!}{24!}$

$$= \frac{26 \times 25 \times 24!}{24!}$$
$$= 26 \times 25$$
$$= 650$$

b $\frac{42!}{43!}$

$$= \frac{42!}{43 \times 42!}$$
$$= \frac{1}{43}$$

c $\frac{49!}{50!} \div \frac{69!}{70!}$

$$\frac{49!}{50!} \times \frac{70!}{69!}$$
$$= \frac{49!}{50 \times 49!} \times \frac{70 \times 69!}{69!}$$
$$= \frac{1}{50} \times \frac{70}{1}$$
$$= \frac{7}{5}$$

d $\frac{11! + 10!}{11! - 10!}$

$$= \frac{10!(11+1)}{10!(11-1)}$$
$$= \frac{12}{10}$$
$$= \frac{6}{5}$$

7 $\binom{7}{4}$

$$= \frac{7!}{4!(7-4)!}$$
$$= \frac{7!}{4!\,3!}$$
$$= \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1}$$
$$= \frac{7 \times 6 \times 5}{3 \times 2}$$
$$= 35$$

8 $\binom{n}{2} = \frac{n!}{2!(n-2)!}$

$$= \frac{n(n-1)(n-2)!}{2 \times 1 \times (n-2)!}$$
$$= \frac{n(n-1)}{2}$$

$$\binom{21}{2} = \frac{21 \times 20}{2} = 210$$

9 a $\binom{5}{2}$

$$= \frac{5!}{2! \times (5-2)!}$$
$$= \frac{5!}{2! \times 3!}$$
$$= \frac{5 \times 4 \times 3!}{2 \times 3!}$$
$$= \frac{20}{2}$$
$$= 10$$

b $\binom{5}{3}$

$$= \frac{5!}{3! \times (5-3)!}$$
$$= \frac{5!}{3! \times 2!}$$
$$= 10$$

c $\binom{12}{12}$

$$= \frac{12!}{12! \times 0!}$$
$$= \frac{12!}{12!}$$
$$= 1$$

d ${}^{20}C_3$

$$= \binom{20}{3}$$
$$= \frac{20!}{3! \times 17!}$$
$$= \frac{20 \times 19 \times 18 \times 17!}{3! \times 17!}$$
$$= \frac{20 \times 19 \times 18}{3 \times 2 \times 1}$$
$$= 1140$$

e $\binom{7}{0}$

$$= \frac{7!}{0! \times 7!}$$
$$= 1$$

f $\binom{13}{10}$

$$= \frac{13!}{10! \times 3!}$$
$$= \frac{13 \times 12 \times 11 \times 10!}{10! \times 3!}$$
$$= \frac{13 \times 12 \times 11}{3 \times 2 \times 1}$$
$$= 286$$

10 a $\binom{n}{3}$

$$= \frac{n!}{3! \times (n-3)!}$$
$$= \frac{n \times (n-1) \times (n-2) \times \cancel{(n-3)!}}{3! \times \cancel{(n-3)!}}$$
$$= \frac{n(n-1)(n-2)}{6}$$

b $\binom{n}{n-3}$

$$= \frac{n!}{(n-3)! \times 3!}$$
$$= \frac{n(n-1)(n-2)}{6}$$

c $\binom{n+3}{3}$

$$= \frac{(n+3)!}{3! \times ((n+3)-3)!}$$
$$= \frac{(n+3)!}{3! \times n!}$$
$$= \frac{(n+3) \times (n+2) \times (n+1) \times n!}{3! \times n!}$$
$$= \frac{(n+3)(n+2)(n+1)}{6}$$

d $\binom{2n+1}{2n-1}$

$$= \frac{(2n+1)!}{(2n-1)! \times ((2n+1)-(2n-1))!}$$
$$= \frac{(2n+1)!}{(2n-1)! \times 2!}$$
$$= \frac{(2n+1) \times (2n) \times (2n-1)!}{(2n-1)! \times 2!}$$
$$= \frac{2n(2n+1)}{2}$$
$$= n(2n+1)$$

e $\binom{n}{2} + \binom{n}{3}$

$$= \frac{n!}{2! \times (n-2)!} + \frac{n!}{3! \times (n-3)!}$$
$$= \frac{n(n-1)(n-2)!}{2! \times (n-2)!} + \frac{n(n-1)(n-2)(n-3)!}{3! \times (n-3)!}$$
$$= \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}$$
$$= \frac{3n(n-1) + n(n-1)(n-2)}{6}$$
$$= \frac{n(n-1)[3 + (n-2)]}{6}$$
$$= \frac{n(n-1)(n+1)}{6}$$

f $\binom{n+1}{3}$

$$= \frac{(n+1)!}{3! \times ((n+1)-3)!}$$
$$= \frac{(n+1)!}{3! \times (n-2)!}$$

$$= \frac{(n+1)\,n\,(n-1)\,(n-2)\,!}{6\,(n-2)\,!}$$

$$= \frac{n\,(n-1)\,(n+1)}{6}$$

11 **a** $(n+1)\times n! = (n+1)\,!$

b $(n-1)\times(n-2)\times(n-3)\,! = (n-1)\,!$

c $\dfrac{n!}{(n-3)\,!}$

$$= \frac{n\times(n-1)\times(n-2)\times(n-3)\,!}{(n-3)\,!}$$

$$= n\,(n-1)\,(n-2)$$

d $\dfrac{(n-1)\,!}{(n+1)\,!}$

$$= \frac{(n-1)\,!}{(n+1)\times n\times(n-1)\,!}$$

$$= \frac{1}{n\,(n+1)}$$

e $\dfrac{(n-1)\,!}{n!} - \dfrac{(n+1)\,!}{(n+2)\,!}$

$$= \frac{(n-1)\,!}{n\times(n-1)\,!} - \frac{(n+1)\,!}{(n+2)\times(n+1)\,!}$$

$$= \frac{1}{n} - \frac{1}{n+2}$$

$$= \frac{(n+2)-n}{n\,(n+2)}$$

$$= \frac{2}{n\,(n+2)}$$

f $\dfrac{n^3-n^2-2n}{(n+1)\,!}\times\dfrac{(n-2)\,!}{n-2}$

$$= \frac{n\left(n^2-n-2\right)}{(n+1)\times n\times(n-1)\times(n-2)\,!}\times\frac{(n-2)\,!}{n-2}$$

$$= \frac{n\,(n-2)\,(n+1)}{(n+1)\,n\,(n-1)}\times\frac{1}{n-2}$$

$$= \frac{1}{n-1}$$

12 $(2x+3)^5$

$$= (2x)^5 + \binom{5}{1}(2x)^4\,(3) + \binom{5}{2}(2x)^3\,(3)^2$$

$$+ \binom{5}{3}(2x)^2\,(3)^3 + \binom{5}{4}(2x)\,(3)^4 + (3)^5$$

$$= 32x^5 + 5\times16x^4\times3 + 10\times8x^3\times9$$

$$+ 10\times4x^2\times27 + 5\times2x\times81 + 243$$

$$= 32x^5 + 240x^4 + 720x^3 + 1080x^2$$

$$+ 810x + 243$$

13 $(x-2)^7$

$$= (x)^7 - \binom{7}{1}x^6\,(2) + \binom{7}{2}(x)^5(2)^2 - \binom{7}{3}x^4(2)^3$$

$$+ \binom{7}{4}x^3(2)^4 - \binom{7}{5}x^2(2)^5 + \binom{7}{6}x(2)^6 - (2)^7$$

$$= x^7 - 7\times x^6\times2 + 21\times x^5\times4 - 35\times x^4\times8$$

$$+ 35\times x^3\times16 - 21\times x^2\times32 + 7\times x\times64 - 128$$

$$= x^7 - 14x^6 + 84x^5 - 280x^4$$

$$+ 560x^3 - 672x^2 + 448x - 128$$

14 **a** $(x+1)^5$

$$= x^5 + \binom{5}{1}x^4 + \binom{5}{2}x^3 + \binom{5}{3}x^2 + \binom{5}{4}x + 1$$

$$= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

b $(2-x)^5$

$$= (2)^5 - \binom{5}{1}(2)^4x + \binom{5}{2}(2)^3x^2 - \binom{5}{3}(2)^2x^3$$

$$+ \binom{5}{4}(2)\,x^4 - x^5$$

$$= 32 - 5\times16x + 10\times8x^2 - 10\times4x^3 + 5\times2x^4 - x^5$$

$$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$

c $(2x+3y)^6$

$$= (2x)^6 + \binom{6}{1}(2x)^5(3y) + \binom{6}{2}(2x)^4(3y)^2 + \binom{6}{3}(2x)^3(3y)^3$$

$$+ \binom{6}{4}(2x)^2(3y)^4 + \binom{6}{5}(2x)(3y)^5 + (3y)^6$$

$$= 64x^6 + 6\times32x^5\times3y + 15\times16x^4\times9y^2 + 20\times8x^3\times27y^3$$

$$+ 15\times4x^2\times81y^4 + 6\times2x\times243y^5 + 729y^6$$

$$= 64x^6 + 576x^5y + 2160x^4y^2 + 4320x^3y^3$$

$$+ 4860x^2y^4 + 2916xy^5 + 729y^6$$

d $\left(\dfrac{x}{2}+2\right)^7$

$$= \left(\frac{x}{2}\right)^7 + \binom{7}{1}\left(\frac{x}{2}\right)^6(2) + \binom{7}{2}\left(\frac{x}{2}\right)^5(2)^2 + \binom{7}{3}\left(\frac{x}{2}\right)^4(2)^3$$

$$+ \binom{7}{4}\left(\frac{x}{2}\right)^3(2)^4 + \binom{7}{5}\left(\frac{x}{2}\right)^2(2)^5 + \binom{7}{6}\left(\frac{x}{2}\right)(2)^6 + (2)^7$$

$$= \frac{x^7}{128} + 7\times\frac{x^6}{64}\times2 + 21\times\frac{x^5}{32}\times4 + 35\times\frac{x^4}{16}\times8 + 35\times\frac{x^3}{8}$$

$$\times16 + 21\times\frac{x^2}{4}\times32 + 7\times\frac{x}{2}\times64 + 128$$

$$= \frac{x^7}{128} + \frac{7x^6}{32} + \frac{21x^5}{8} + \frac{35x^4}{2} + 70x^3 + 168x^2 + 224x + 128$$

e $\left(x-\dfrac{1}{x}\right)^8$

$$= x^8 - \binom{8}{1}x^7\left(\frac{1}{x}\right) + \binom{8}{2}x^6\left(\frac{1}{x}\right)^2 - \binom{8}{3}x^5\left(\frac{1}{x}\right)^3$$

$$+ \binom{8}{4}x^4\left(\frac{1}{x}\right)^4 - \binom{8}{5}x^3\left(\frac{1}{x}\right)^5 + \binom{8}{6}x^2\left(\frac{1}{x}\right)^6$$

$$- \binom{8}{7}x\left(\frac{1}{x}\right)^7 + \left(\frac{1}{x}\right)^8$$

$$= x^8 - \frac{8x^7}{x} + \frac{28x^6}{x^2} - \frac{56x^5}{x^3} + \frac{70x^4}{x^4} - \frac{56x^3}{x^5} + \frac{28x^2}{x^6}$$

$$- \frac{8x}{x^7} + \frac{1}{x^8}$$

$$= x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - \frac{56}{x^2} + \frac{28}{x^4} - \frac{8}{x^6} + \frac{1}{x^8}$$

f $\left(x^2+1\right)^{10}$

$$= \left(x^2\right)^{10} + \binom{10}{1}\left(x^2\right)^9 + \binom{10}{2}\left(x^2\right)^8 + \binom{10}{3}\left(x^2\right)^7$$

$$+\binom{10}{4}\left(x^2\right)^6 + \binom{10}{5}\left(x^2\right)^5 + \binom{10}{6}\left(x^2\right)^4$$

$$+\binom{10}{7}\left(x^2\right)^3 + \binom{10}{8}\left(x^2\right)^2 + \binom{10}{9}\left(x^2\right) + 1$$

$$= x^{20} + 10x^{18} + 45x^{16} + 120x^{14} + 210x^{12} + 252x^{10} + 210x^8$$

$$+120x^6 + 45x^4 + 10x^2 + 1$$

15 In the general term, let $r = 3$ and $n = 7$.

$$t_4 = \binom{7}{3}\left(\frac{x}{3}\right)^4\left(-\frac{y}{2}\right)^3$$

$$= -35 \times \frac{x^4}{81} \times \frac{y^3}{8}$$

$$= -\frac{35}{648}x^4y^3$$

Checking, the fourth term is an even term so its coefficient is negative.

16 There are 11 terms, so the middle term is the sixth term.
For t_6, let $r = 5$ and $n = 10$.

$$t_6 = \binom{10}{5}\left(x^2\right)^5\left(\frac{y}{2}\right)^5$$

$$= 252x^{10} \times \frac{y^5}{32}$$

$$= \frac{63x^{10}y^5}{8}$$

17 $\left(4+3x^3\right)^8$

$$t_{r+1} = \binom{8}{r}(4)^{8-r}\left(3x^3\right)^r$$

$$= \binom{8}{r}(4)^{8-r}(3)^r x^{3r}$$

For x^{15} we require $3r = 15$, so $r = 5$.
Hence, $t_{r+1} = t_6$, so the sixth term contains x^{15}.

$$t_6 = \binom{8}{5}(4)^3(3)^5x^{15}$$

The coefficient of x^{15} is $\binom{8}{5}(4)^3(3)^5$.

$$\binom{8}{5}(4)^3(3)^5$$

$$= \frac{8\times7\times6}{3\times2\times1}\times\left(2^2\right)^3\times3^5$$

$$= 8\times7\times2^6\times3^5$$

$$= 2^3\times7\times2^6\times3^5$$

$$= 2^9\times3^5\times7$$

Therefore, the coefficient of x^{15} is $2^9 \times 3^5 \times 7$.

18 Find the term independent of x in $\left(x+\frac{2}{x}\right)^6$.

The question is equivalent to finding the coefficient of x^0.

$$t_{r+1} = \binom{6}{r}x^{6-r}\left(\frac{2}{x}\right)^r$$

$$= \binom{6}{r}2^r\frac{x^{6-r}}{x^r}$$

$$= \binom{6}{r}2^rx^{6-2r}$$

For x^0, $6 - 2r = 0$, which means $r = 3$.
So,

$$t_4 = \binom{6}{3}2^3$$

$$= 160$$

19 **a** $(5x+2)^6$ has the general term $t_{r+1} = \binom{n}{r}(5x)^{n-r}(2)^r$ with $n = 6$.
For t_4, $r = 3$.

$$\therefore t_4 = \binom{6}{3}(5x)^{6-3}(2)^3$$

$$= 20\times(5x)^3\times8$$

$$= 160\times125x^3$$

$$\therefore t_4 = 20\,000x^3$$

b $\left(3x^2-1\right)^6$

$$t_{r+1} = \binom{6}{r}\left(3x^2\right)^{6-r}(-1)^r$$

Put $r = 2$.

$$\therefore t_3 = \binom{6}{2}\left(3x^2\right)^4(-1)^2$$

$$= 15\times81x^8\times1$$

$$\therefore t_3 = 1215x^8$$

c $(x+2y)^7$ has 8 terms in its expansion, so there are two middle terms: t_4 and t_5.

$$t_{r+1} = \binom{7}{r}(x)^{7-r}(2y)^r$$

For t_4, put $r = 3$.

$$\therefore t_4 = \binom{7}{3}(x)^4(2y)^3$$

$$= 35\times x^4\times8y^3$$

$$\therefore t_4 = 280x^4y^3$$

For t^5, put $r = 4$.

$$\therefore t_5 = \binom{7}{4}(x)^3(2y)^4$$

$$= 35\times x^3\times16y^4$$

$$\therefore t_5 = 560x^3y^4$$

The middle terms are $280x^4y^3$ and $560x^3y^4$.

20 **a** $\left(1-2x^2\right)^9$

$$t_{r+1} = \binom{9}{r}(1)^{9-r}\left(-2x^2\right)^r$$

$$= \binom{9}{r}(-2)^rx^{2r}$$

For the term in x^6, $2r = 6 \Rightarrow r = 3$.

$$\therefore t_4 = \binom{9}{3}(-2)^3 x^6$$

$$= 84 \times (-8)\, x^6$$

$$\therefore t_4 = -672x^6$$

The coefficient of x^6 is -672.

b $(3 + 4x)^{11}$

$$t_{r+1} = \binom{11}{r}(3)^{11-r}(4x)^r$$

$$= \binom{11}{r}(3)^{11-r}(4)^r x^r$$

For the term in x^5, $r = 5$.

$$\therefore t_6 = \binom{11}{5}(3)^6(4)^5 x^5$$

The coefficient of x^5 is $\binom{11}{5}(3)^6(4)^5$, which needs to be expressed as the product of its prime factors.

$$\binom{11}{5}(3)^6(4)^5$$

$$= \frac{11 \times \cancel{10}^2 \times \cancel{9}^3 \times \cancel{8} \times 7}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1} \times 3^6 \times \left(2^2\right)^5$$

$$= 11 \times 2 \times 3 \times 7 \times 3^6 \times 2^{10}$$

$$= 11 \times 7 \times 3^7 \times 2^{11}$$

c $\left(x^2 + \frac{1}{x^3}\right)^{10}$

$$t_{r+1} = \binom{10}{r}\left(x^2\right)^{10-r}\left(\frac{1}{x^3}\right)^r$$

$$= \binom{10}{r} x^{20-2r} \times \frac{1}{x^{3r}}$$

$$= \binom{10}{r} x^{20-5r}$$

For the term independent of x, $20 - 5\,r = 0 \Rightarrow r = 4$.

$$\therefore t_5 = \binom{10}{4} x^0$$

$$= 210$$

The term independent of x is $t_5 = 210$.

21 a The factorial symbol is found in the main menu → Keyboard → mth → CALC.

$15! = 1\,307\,674\,368\,000$

b $\binom{15}{10}$ is evaluated using the nCr key obtained from the main menu by tapping

Keyboard → mth → CALC → nCr and entering 15, 10).

$$\binom{15}{10} = 3003$$

22 a $\binom{n}{2} = 1770$

First use Simplify to express nCr $(n,\ 2)$ as $\frac{n\,(n-1)}{2}$.

Drag this expression down to the next prompt position and complete the equation $\frac{n\,(n-1)}{2} = 1770$.

Highlight the equation and drop it into the equation solver, solving for n.

This returns $n = -59, n = 60$.

Since n must be a positive integer, $n = 60$

b $\binom{12}{r} = 220$

As the method used in part **a** does not enable a solution to be obtained, an alternative approach is through the table of values in the graphing menu.

In the Main window, define f (x) = nCr $(12, x)$ by tapping Interactive → Define and completing the boxes as follows:

Func name: f obtained from Keyboard → abc

Variable/s: x

Expression: nCr $(12, x)$

In the graphing window, enter f (x) at y_1:

Tap the table symbol to open Table Input and complete the boxes as follows:

Start: 1

End: 12

Step: 1

Tap the symbol to open the table of values. Scroll down the y_1 column to locate 220. This will occur at both the values $x = 3$ and $x = 9$.

Hence, the solutions to $\binom{12}{r} = 220$ are $r = 3,\ r = 9$.

There are two solutions, since $\binom{12}{r} = \binom{12}{12-r}$.

2.4 Exam questions

1 $\frac{4! + 3! - 2!}{4! - 3! + 2!}$

$$= \frac{24 + 6 - 2}{24 - 6 + 2}$$

$$= \frac{28}{20}$$

$$= \frac{7}{5}$$

The correct answer is **D**.

2 $3 \times \binom{5}{3} + 4 \times \binom{4}{2}$

$$= 3 \times \frac{5!}{3! \times 2!} + 4 \times \frac{4!}{2! \times 2!}$$

$$= 3 \times \frac{5 \times 4}{2} + 4 \times \frac{4 \times 3}{2}$$

$$= 3 \times 10 + 4 \times 6$$

$$= 30 + 24$$

$$= 54$$

The correct answer is **D**.

3 $t_{r+1} = \binom{n}{r} x^{n-r} y^r$

$t_{r+1} = \binom{7}{r} 3^{7-r}\left(-4x^3\right)^r$ [1 mark]

For x^{12}, we require $r = 4$.

$r + 1 = 5$, so the 5th term contains x^{12}.

$t_5 = t_{4+1} = \binom{7}{4} 3^{7-4}\left(-4x^3\right)^4$ [1 mark]

$$\begin{aligned}\text{The coefficient} &= \frac{7!}{4!\,3!} \times 3^3 \times (-4)^4 && \text{[1 mark]}\\ &= \frac{7\times6\times5}{3\times2\times1}\times 3^3\times\left[(-2)^2\right]^4\\ &= \frac{7\times3\times2\times5}{3\times2\times1}\times 3^3\times 2^8\\ &= 7\times5\times3^3\times2^8 && \text{[1 mark]}\end{aligned}$$

Therefore, the coefficient of x^{12} is $7\times5\times3^3\times2^8$.

2.5 Sets of real numbers

2.5 Exercise

1 a $\frac{6}{11} \in Q$

b $\sqrt{27} \in I$ as 27 is not a perfect square.

c $(6-2)\times3 \in N$ since $(6-2)\times3 = 12$. Other possible answers are $12 \in Z$ or $12 \in Q$.

d $\sqrt{0.25} \in Q$ since $\sqrt{0.25} = 0.5 = \frac{1}{2}$.

2 a $17 \in N$ is correct since 17 is a natural number.

b $Q \subset N$ is incorrect since $N \subset Q$.

c $Q \cup I = R$ is correct.

3 a $\frac{12}{\sqrt{9}} = \frac{12}{3} = 4$. This is a counting number, so in set N.

b 3π. Any multiple of π is an irrational number, so in set I.

c $\frac{\sqrt{25}}{2} = \frac{5}{2}$. This is a fraction, a rational number, so in set Q.

d $-\sqrt{4} = -2$. Since $2^2 = 4$, the answer is a negative integer, so in set Z.

4 $\frac{x-5}{(x+1)(x-3)}$ is undefined if its denominator is zero.
Since $(x+1)(x-3) = 0$ when $x = -1$ or $x = 3$, the expression is undefined for these values.
Note that if $x = 5$, which makes the numerator zero, then $\frac{0}{(1)(-3)} = \frac{0}{-3} = 0$, so the expression also equals zero (and is defined).

5 Option E does not represent a real number since $\frac{(8-4)\times2}{8-4\times2} = \frac{8}{0}$ and division by zero is not defined.

6 a The statement $\sqrt{16+25} \in Q$ is false because $\sqrt{16+25} = \sqrt{41}$ and this surd is not a rational number since 41 is not a perfect square.
A correct statement is $\sqrt{16+25} \in I$.

b The statement $\left(\frac{4}{9}-1\right) \in Z$ is false because $\frac{4}{9} - 1 = -\frac{5}{9}$ and this fraction is not an integer.
A correct statement is $\left(\frac{4}{9}-1\right) \in Q$.

c The statement $R^+ = \{x : x \geq 0\}$ is false because 0 is included in the set.
A correct statement is $R^+ = \{x : x > 0\}$.

d The statement $\sqrt{2.25} \in I$ is false because $\sqrt{2.25} = 1.5$, which is rational.
A correct statement is $\sqrt{2.25} \in Q$.

7 a $\frac{1}{x+5}$ is undefined when its denominator is zero.
Therefore, it is undefined if $x = -5$.

b $\frac{x+2}{x-2}$ is undefined when its denominator is zero.
Therefore, it is undefined if $x = 2$.

c $\frac{x+8}{(2x+3)(5-x)}$ is undefined if either of the factor terms in its denominator are zero.
If $2x+3 = 0$, then $x = -\frac{3}{2}$.
If $5 - x = 0$, then $x = 5$.
Therefore, $\frac{x+8}{(2x+3)(5-x)}$ is undefined if $x = -\frac{3}{2}$ or 5.

d As $\frac{4}{x^2-4x} = \frac{4}{x(x-4)}$, it is undefined if either of the factor terms in its denominator are zero. Therefore, it is undefined if $x = 0$ or 4.

8 a $R^- \subset R$ is a true statement since the negative reals are a subset of the real numbers.

b $N \subset R^+$ is a true statement since the set of natural numbers $\{1, 2, 3, ...\}$ is a subset of the positive real numbers.

c $Z \cup N = R$ is a false statement since the union of the integers and the natural numbers is just the set of integers: $Z \cup N = Z$, not R.

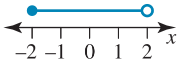
d $Q \cap Z = Z$ is a true statement since the intersection of the rational numbers with the integers is the set of integers.

e $I \cup Z = R \setminus Q$ is a false statement. Excluding the rationals from the real numbers leaves the set of irrationals so $R \setminus Q = I$. However, the union of the irrationals with the integers does not form only the set of irrationals, so $I \cup Z \neq I$.

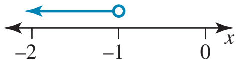
f $Z \setminus N = Z^-$ is a false statement since excluding the natural numbers from the integers gives $Z^- \cup \{0\}$, not Z^-.

9 $\sqrt{11}$ is not a rational number; $\frac{2}{11}$ is a rational number; 11^{11} is a large positive integer; 11π and 2^π are irrational as π is irrational, and $\sqrt{121} = 11$, which is a positive integer.
The irrational numbers are $\sqrt{11}$, 11π and 2^π.

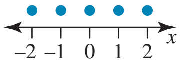
10 a $[-2, 2) = \{x : -2 \leq x < 2\}$



b $\{x : x < -1\} = (-\infty, -1)$



c $\{-2, -1, 0, 1, 2\} = Z \cap [-2, 2]$ or $\{x : -2 \leq x \leq 2, x \in Z\}$



11 a $[3, 5]$

b $R \backslash [3, 5]$ or $(-\infty, 3) \cup (5, \infty)$

12 a $[-2, 3)$

b $(1,\ 9)$

c $(-\infty,\ 5)$

d $(0, 4]$

13 a $[-5, -1]$ since the interval is closed at both the left and right ends.

b $(-2, \infty)$ since the interval is open at -2 and extends indefinitely towards positive infinity.

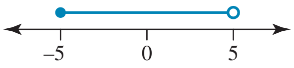
c $[-3, 2) \cup (2, 3]$
This is the union of two intervals, the first closed at -3 and open at -2, and the second open at 2 and closed at 3.

d The union of the two intervals $(-\infty, 2)$ and $(4, \infty)$ can be expressed as $(-\infty, 2) \cup (4, \infty)$ or as $R \setminus [2, 4]$.

14 $R \setminus \{x : 1 < x \le 4\}$
This is the set of real numbers excluding those numbers in the set $(1, 4]$.
This is equivalent to $(-\infty, 1] \cup (4, \infty)$.

15 **a** $\{x : 4 < x \le 8\} = (4, 8]$
b $\{x : x > -3\} = (-3, \infty)$
c $\{x : x \le 0\} = (-\infty, 0]$
d $\{x : -2 \le x \le 0\} = [-2, 0]$

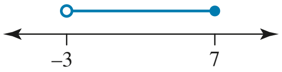
16 **a** $[-5, 5)$



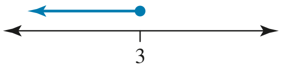
b $(4, \infty)$

c $[-3, 7]$

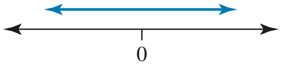
d $(-3, 7]$



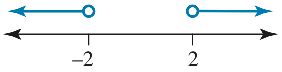
e $(-\infty, 3]$



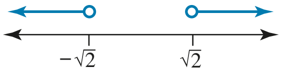
f $(-\infty, \infty) = R$



17 **a** $R\backslash[-2, 2]$



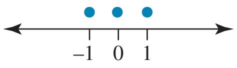
b $\left(-\infty, -\sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$



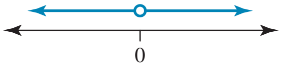
c $[-4, 2) \cap (0, 4) = (0, 2)$

d $[-4, 2) \cup (0, 4) = [-4, 4)$

e $\{-1, 0, 1\}$



f $R\backslash\{0\}$



18 **a** $\{x : 2 < x < 6, x \in Z\}$ is the set of integers between 2 and 6 not including 2 and 6.
Therefore, $\{x : 2 < x < 6, x \in Z\} = \{3, 4, 5\}$.
b $R\backslash(-1, 5]$ is the set of real numbers excluding those that lie between -1 and 5 but not excluding -1 since $-1 \notin (-1, 5]$.
Therefore, $R\,(-1, 5] = (-\infty, -1] \cup (5, \infty)$.
c R^- is the set of negative real numbers, so $R^- = (-\infty, 0)$.
d $(-\infty, -4) \cup [2, \infty)$ is the set of all real numbers excluding those that lie in $(-4, 2]$.
Therefore, $(-\infty, -4) \cup [2, \infty) = R\backslash[-4, 2]$.

19 **a** With the main menu on Standard mode, enter $\sqrt{7225}$ (the square root symbol is found in Keyboard mth).
$\sqrt{7225} = 85$ so it is rational.
b CAS on Standard mode gives $\sqrt{75\,600} = 60\sqrt{21}$, so $\sqrt{75\,600}$ is irrational.
c 0.234 234 234 … is a recurring decimal. Enter it as 0.234234234234234234234 so it occupies the entire row.
On Standard mode this gives the value as $\frac{26}{111}$, so it is rational.

20 The formula for the area of a circle is $A = \pi r^2$.
With $r = \frac{1}{2}d$, $A = \frac{d^2\pi}{4}$.
The actual area of the circle is $A = \frac{\pi}{4}d^2$.
The Egyptian formula gives $A = \frac{64}{81}d^2$.

$$\therefore\ \frac{\pi}{4} = \frac{64}{81}$$
$$\therefore\ \pi = \frac{64}{81} \times 4$$
$$\therefore\ \pi = \frac{256}{81}$$

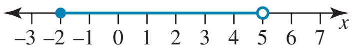
Evaluating on Decimal mode gives the estimate $\pi = 3.160\,493\,827$.
CAS on Decimal mode gives $\frac{22}{7} = 3.142\,857\,143$ to 9 decimal places and $\pi = 3.141\,592\,654$ to 9 decimal places.
This shows that $\frac{22}{7}$ is a better approximation.

2.5 Exam questions

1 $\sqrt{9} + \sqrt{16} = 3 + 4$
$= 7$, which is a rational number
The correct answer is **D**.

2 If $x = 0, 1, -3$, the expression is undefined since, for any of these values, the denominator is 0.
Division by 0 is not possible. Therefore, any value that makes the denominator of a fraction 0 causes the expression to be undefined. [1 mark]

3



The number line shows the numbers to be excluded, from -2 up to but not including 5.
$\{x : x < -2\} \cup \{x : x \ge 5\}$
The correct answer is **B**.

2.6 Surds

2.6 Exercise

1 **a** $3\sqrt{3} = \sqrt{9 \times 3} = \sqrt{27}$, $4\sqrt{5} = \sqrt{16 \times 5} = \sqrt{80}$,
$5\sqrt{2} = \sqrt{25 \times 2} = \sqrt{50}$ and $5 = \sqrt{25}$.
In increasing order the set is written as
$\left\{5,\ 3\sqrt{3},\ 5\sqrt{2},\ 4\sqrt{5}\right\}$.
b **i** $\sqrt{84} = \sqrt{4 \times 21} = 2\sqrt{21}$

ii $2\sqrt{108ab^2}$
$= 2\sqrt{36 \times 3ab^2}$
$= 2 \times 6 \times b \times \sqrt{3a}$
$= 12b\sqrt{3a}$

2 $\sqrt{900} = 30$, $\sqrt{\frac{4}{9}} = \frac{2}{3}$, $\sqrt{1.44} = 1.2$ and $\sqrt[3]{27} = 3$, so these are all rational.
Although π is irrational, it does not contain a radical sign, so it is not a surd.
The surds are $\sqrt{8}$, $\sqrt{10^3}$ and $\sqrt[3]{36}$.

3 a $4\sqrt{5}$
$= \sqrt{16 \times 5}$
$= \sqrt{80}$

b $2\sqrt[3]{6}$
$= \sqrt[3]{8} \times \sqrt[3]{6}$
$= \sqrt[3]{48}$

c $\frac{9\sqrt{7}}{4}$
$= \frac{\sqrt{81} \times \sqrt{7}}{\sqrt{16}}$
$= \frac{\sqrt{567}}{\sqrt{16}}$
$= \sqrt{\frac{567}{16}}$

d $\frac{3}{\sqrt{3}}$
$= \frac{\sqrt{9}}{\sqrt{3}}$
$= \sqrt{\frac{9}{3}}$
$= \sqrt{3}$

e $ab\sqrt{c} = \sqrt{a^2b^2c}$

f $m\sqrt[3]{n} = \sqrt[3]{m^3n}$

4 a $\sqrt{32} = \sqrt{16 \times 2}$
$= 4\sqrt{2}$

b $2\sqrt{44} = 2\sqrt{4} \times 11$
$= 2 \times 2\sqrt{11}$
$= 4\sqrt{11}$

c $\sqrt{52} = \sqrt{4 \times 13}$
$= 2\sqrt{13}$

d $3\sqrt{80} = 3\sqrt{16 \times 5}$
$= 3 \times 4\sqrt{5}$
$= 12\sqrt{5}$

e $\sqrt{45} = \sqrt{9 \times 5}$
$= 3\sqrt{5}$

f $\frac{\sqrt{99}}{18} = \frac{\sqrt{9 \times 11}}{18}$
$= \frac{3\sqrt{11}}{18}$
$= \frac{\sqrt{11}}{6}$

5 a $\sqrt{75}$
$= \sqrt{25 \times 3}$
$= 5\sqrt{3}$

b $5\sqrt{48}$
$= 5\sqrt{16 \times 3}$
$= 5 \times 4\sqrt{3}$
$= 20\sqrt{3}$

c $\sqrt{2000}$
$= \sqrt{20 \times 100}$
$= 10\sqrt{20}$
$= 10\sqrt{4 \times 5}$
$= 10 \times 2\sqrt{5}$
$= 20\sqrt{5}$

d $3\sqrt{288}$
$= 3\sqrt{2 \times 144}$
$= 3 \times 12\sqrt{2}$
$= 36\sqrt{2}$

e $2\sqrt{72}$
$= 2\sqrt{36 \times 2}$
$= 2 \times 6\sqrt{2}$
$= 12\sqrt{2}$

f $\sqrt[3]{54}$
$= \sqrt[3]{27 \times 2}$
$= 3\sqrt[3]{2}$

6 a $\sqrt{5} - 4\sqrt{7} - 7\sqrt{5} + 3\sqrt{7}$
$\sqrt{5} - 7\sqrt{5} + 3\sqrt{7} - 4\sqrt{7}$
$= -6\sqrt{5} - \sqrt{7}$

b $3\sqrt{48} - 4\sqrt{27} + 3\sqrt{32}$
$= 3\sqrt{16 \times 3} - 4\sqrt{9 \times 3} + 3\sqrt{16 \times 2}$
$= 3 \times 4\sqrt{3} - 4 \times 3\sqrt{3} + 3 \times 4\sqrt{2}$
$= 12\sqrt{3} - 12\sqrt{3} + 12\sqrt{2}$
$= 12\sqrt{2}$

c $3\sqrt{5} \times 7\sqrt{15}$
$= 21\sqrt{75}$
$= 21\sqrt{25 \times 3}$
$= 21 \times 5\sqrt{3}$
$= 105\sqrt{3}$

7 a $5\sqrt{11}-2\sqrt{3}-9\sqrt{3}+4\sqrt{11}$

$=5\sqrt{11}+4\sqrt{11}-2\sqrt{3}-9\sqrt{3}$

$=9\sqrt{11}-11\sqrt{3}$

b $5\sqrt{2}+\sqrt{3}+4\sqrt{27}-\frac{1}{2}\sqrt{72}$

$=5\sqrt{2}+\sqrt{3}+4\sqrt{9\times 3}-\frac{1}{2}\sqrt{36\times 2}$

$=5\sqrt{2}+\sqrt{3}+4\times 3\sqrt{3}-\frac{1}{2}\times 6\sqrt{2}$

$=5\sqrt{2}+\sqrt{3}+12\sqrt{3}-3\sqrt{2}$

$=2\sqrt{2}+13\sqrt{3}$

c $4\sqrt{3}+2\sqrt{8}-7\sqrt{2}+5\sqrt{48}$

$=4\sqrt{3}+2\sqrt{4\times 2}-7\sqrt{2}+5\sqrt{16\times 3}$

$=4\sqrt{3}+2\times 2\sqrt{2}-7\sqrt{2}+5\times 4\sqrt{3}$

$=4\sqrt{3}+4\sqrt{2}-7\sqrt{2}+20\sqrt{3}$

$=24\sqrt{3}-3\sqrt{2}$

d $2\sqrt{3}-3\sqrt{20}+4\sqrt{12}-\sqrt{5}$

$=2\sqrt{3}-3\sqrt{4\times 5}+4\sqrt{4\times 3}-\sqrt{5}$

$=2\sqrt{3}-3\times 2\sqrt{5}+4\times 2\sqrt{3}-\sqrt{5}$

$=2\sqrt{3}-6\sqrt{5}+8\sqrt{3}-\sqrt{5}$

$=10\sqrt{3}-7\sqrt{5}$

e $2\sqrt{12}-\sqrt{125}-\sqrt{50}+2\sqrt{180}$

$=2\sqrt{4\times 3}-\sqrt{25\times 5}-\sqrt{25\times 2}+2\sqrt{36\times 5}$

$=2\times 2\sqrt{3}-5\sqrt{5}-5\sqrt{2}+2\times 6\sqrt{5}$

$=4\sqrt{3}-5\sqrt{5}-5\sqrt{2}+12\sqrt{5}$

$=4\sqrt{3}+7\sqrt{5}-5\sqrt{2}$

f $\sqrt{36}-\sqrt{108}+2\sqrt{75}-4\sqrt{300}$

$=6-\sqrt{36\times 3}+2\sqrt{25\times 3}-4\sqrt{100\times 3}$

$=6-6\sqrt{3}+2\times 5\sqrt{3}-4\times 10\sqrt{3}$

$=6-6\sqrt{3}+10\sqrt{3}-40\sqrt{3}$

$=6-36\sqrt{3}$

8 a $3\sqrt{7}+8\sqrt{3}+12\sqrt{7}-9\sqrt{3}$

$=3\sqrt{7}+12\sqrt{7}+8\sqrt{3}-9\sqrt{3}$

$=15\sqrt{7}-\sqrt{3}$

b $10\sqrt{2}-12\sqrt{6}+4\sqrt{6}-8\sqrt{2}$

$=10\sqrt{2}-8\sqrt{2}-12\sqrt{6}+4\sqrt{6}$

$=2\sqrt{2}-8\sqrt{6}$

c $3\sqrt{50}-\sqrt{18}$

$=3\times 5\sqrt{2}-3\sqrt{2}$

$=15\sqrt{2}-3\sqrt{2}$

$=12\sqrt{2}$

d $8\sqrt{45}+2\sqrt{125}$

$=8\times 3\sqrt{5}+2\times 5\sqrt{5}$

$=24\sqrt{5}+10\sqrt{5}$

$=34\sqrt{5}$

e $\sqrt{6}+7\sqrt{5}+4\sqrt{24}-8\sqrt{20}$

$=\sqrt{6}+7\sqrt{5}+4\times 2\sqrt{6}-8\times 2\sqrt{5}$

$=\sqrt{6}+7\sqrt{5}+8\sqrt{6}-16\sqrt{5}$

$=9\sqrt{6}-9\sqrt{5}$

f $2\sqrt{12}-7\sqrt{243}+\frac{1}{2}\sqrt{8}-\frac{2}{3}\sqrt{162}$

$=2\times 2\sqrt{3}-7\times 9\sqrt{3}+\frac{1}{2}\times 2\sqrt{2}-\frac{2}{3}\times 9\sqrt{2}$

$=4\sqrt{3}-63\sqrt{3}+\sqrt{2}-6\sqrt{2}$

$=-59\sqrt{3}-5\sqrt{2}$

9 a $4\sqrt{6}\times\sqrt{21}=4\sqrt{126}$

$=4\sqrt{9\times 14}$

$=4\times 3\sqrt{14}$

$=12\sqrt{14}$

b $-4\sqrt{27}\times -\sqrt{28}=-4\sqrt{9\times 3}\times -\sqrt{4\times 7}$

$=-4\times 3\sqrt{3}\times -2\sqrt{7}$

$=-12\sqrt{3}\times -2\sqrt{7}$

$=24\sqrt{21}$

c $4\sqrt{5}\left(\sqrt{5}+\sqrt{10}\right)=4\sqrt{25}+4\sqrt{50}$

$=4\times 5+4\sqrt{25\times 2}$

$=20+4\times 5\sqrt{2}$

$=20+20\sqrt{2}$

d $3\sqrt{7}\left(2\sqrt{7}-3\sqrt{14}\right)=6\sqrt{49}-9\sqrt{98}$

$=6\times 7-9\sqrt{49\times 2}$

$=42-9\times 7\sqrt{2}$

$=42-63\sqrt{2}$

e $\left(\sqrt{5}-2\right)\left(4-\sqrt{5}\right)=4\sqrt{5}-\sqrt{25}-8+2\sqrt{5}$

$=6\sqrt{5}-5-8$

$=6\sqrt{5}-13$

f $\left(4\sqrt{3}-2\right)\left(2\sqrt{5}+3\sqrt{7}\right)$

$=8\sqrt{15}+12\sqrt{21}-4\sqrt{5}-6\sqrt{7}$

10 a $4\sqrt{5}\times 2\sqrt{7}=8\sqrt{35}$

b $-10\sqrt{6}\times\left(-8\sqrt{10}\right)$

$=80\sqrt{60}$

$=80\times 2\sqrt{15}$

$=160\sqrt{15}$

c $3\sqrt{8}\times 2\sqrt{5}$

$=6\sqrt{40}$

$=6\times 2\sqrt{10}$

$=12\sqrt{10}$

d $\sqrt{18}\times\sqrt{72}$

As the numbers are large, simplify each surd before multiplying them together.

$=3\sqrt{2}\times 6\sqrt{2}$

$=18\sqrt{4}$

$=18\times 2$

$=36$

e $\dfrac{4\sqrt{27}\times\sqrt{147}}{2\sqrt{3}}$

$= \dfrac{4\times 3\sqrt{3}\times\sqrt{49\times 3}}{2\sqrt{3}}$

$= \dfrac{\overset{6}{\cancel{12}}\sqrt{3}\times 7\cancel{\sqrt{3}}}{\cancel{2}\cancel{\sqrt{3}}}$

$= 6\sqrt{3}\times 7$

$= 42\sqrt{3}$

f $5\sqrt{2}\times\sqrt{3}\times 4\sqrt{5}\times\dfrac{\sqrt{6}}{6}+3\sqrt{2}\times 7\sqrt{10}$

$= \dfrac{20\sqrt{2\times 3\times 5\times 6}}{6}+21\sqrt{20}$

$= \dfrac{10\sqrt{180}}{3}+21\times 2\sqrt{5}$

$= \dfrac{10\times 6\sqrt{5}}{3}+42\sqrt{5}$

$= 20\sqrt{5}+42\sqrt{5}$

$= 62\sqrt{5}$

11 a Difference of two squares

$\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)$

$= \left(\sqrt{7}\right)^2-\left(\sqrt{3}\right)^2$

$= 7-3$

$= 4$

b $\left(\sqrt{2}-8\right)\left(\sqrt{2}+8\right)$

$= \left(\sqrt{2}\right)^2-(8)^2$

$= 2-64$

$= -62$

c Perfect square

$\left(\sqrt{7}-2\right)^2$

$= \left(\sqrt{7}\right)^2-2\times\sqrt{7}\times 2+2^2$

$= 7-4\sqrt{7}+4$

$= 11-4\sqrt{7}$

d $\left(\sqrt{11}+\sqrt{2}\right)^2$

$= \left(\sqrt{11}\right)^2+2\times\sqrt{11}\times\sqrt{2}+\left(\sqrt{2}\right)^2$

$= 11+2\sqrt{22}+2$

$= 13+2\sqrt{22}$

e Difference of two squares

$\left(4-2\sqrt{5}\right)\left(4+2\sqrt{5}\right)$

$= (4)^2-\left(2\sqrt{5}\right)^2$

$= 16-4\times 5$

$= 16-20$

$= -4$

f Perfect square

$\left(3\sqrt{5}+2\sqrt{3}\right)^2$

$= \left(3\sqrt{5}\right)^2+2\times 3\sqrt{5}\times 2\sqrt{3}+\left(2\sqrt{3}\right)^2$

$= 9\times 5+12\sqrt{15}+4\times 3$

$= 45+12\sqrt{15}+12$

$= 57+12\sqrt{15}$

12 a $2\sqrt{3}\left(4\sqrt{15}+5\sqrt{3}\right)$

$= 8\sqrt{45}+10\sqrt{9}$

$= 8\times 3\sqrt{5}+10\times 3$

$= 24\sqrt{5}+30$

b $\left(\sqrt{3}-8\sqrt{2}\right)\left(5\sqrt{5}-2\sqrt{21}\right)$

$= 5\sqrt{15}-2\sqrt{63}-40\sqrt{10}+16\sqrt{42}$

$= 5\sqrt{15}-2\times 3\sqrt{7}-40\sqrt{10}+16\sqrt{42}$

$= 5\sqrt{15}-6\sqrt{7}-40\sqrt{10}+16\sqrt{42}$

c $\left(4\sqrt{3}-5\sqrt{2}\right)^2$

$= 16\times 3-2\times 20\sqrt{6}+25\times 2$

$= 48-40\sqrt{6}+50$

$= 98-40\sqrt{6}$

d $\left(3\sqrt{5}-2\sqrt{11}\right)\left(3\sqrt{5}+2\sqrt{11}\right)$

$= 9\times 5-4\times 11$

$= 45-44$

$= 1$

13 a $\sqrt{2}\left(3\sqrt{5}-7\sqrt{6}\right)$

$= 3\sqrt{10}-7\sqrt{12}$

$= 3\sqrt{10}-14\sqrt{3}$

b $5\sqrt{3}\left(7-3\sqrt{3}+2\sqrt{6}\right)$

$= 35\sqrt{3}-15\sqrt{9}+10\sqrt{18}$

$= 35\sqrt{3}-15\times 3+30\sqrt{2}$

$= 35\sqrt{3}+30\sqrt{2}-45$

c $2\sqrt{10}-3\sqrt{6}\left(3\sqrt{15}+2\sqrt{6}\right)$

$= 2\sqrt{10}-9\sqrt{90}-6\times 6$

$= 2\sqrt{10}-27\sqrt{10}-36$

$= -25\sqrt{10}-36$

d $\left(2\sqrt{3}+\sqrt{5}\right)\left(3\sqrt{2}+4\sqrt{7}\right)$

$= 6\sqrt{6}+8\sqrt{21}+3\sqrt{10}+4\sqrt{35}$

14 a $\left(2\sqrt{2}+3\right)^2$

$= \left(2\sqrt{2}\right)^2+2\times\left(2\sqrt{2}\right)\times 3+3^2$

$= 4\times 2+12\sqrt{2}+9$

$= 17+12\sqrt{2}$

b $\left(3\sqrt{6}-2\sqrt{3}\right)^2$

$= 9\times 6-2\times 6\sqrt{18}+4\times 3$

$= 54-2\times 18\sqrt{2}+12$

$= 66-36\sqrt{2}$

c $\left(\sqrt{7}-\sqrt{5}\right)^2$

$= \left(\sqrt{7}\right)^2-2\left(\sqrt{7}\right)\left(\sqrt{5}\right)+\left(\sqrt{5}\right)^2$

$= 7-2\sqrt{35}+5$

$= 12-2\sqrt{35}$

d $\left(2\sqrt{5}+\sqrt{3}\right)\left(2\sqrt{5}-\sqrt{3}\right)$

$= \left(2\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2$

$= 20 - 3$

$= 17$

e $\left(10\sqrt{2}-3\sqrt{5}\right)\left(10\sqrt{2}+3\sqrt{5}\right)$

$= \left(10\sqrt{2}\right)^2 - \left(3\sqrt{5}\right)^2$

$= 200 - 45$

$= 155$

f $\left(\sqrt{3}+\sqrt{2}+1\right)\left(\sqrt{3}+\sqrt{2}-1\right)$

$= \left(\left(\sqrt{3}+\sqrt{2}\right)+1\right)\left(\left(\sqrt{3}+\sqrt{2}\right)-1\right)$

$= \left(\sqrt{3}+\sqrt{2}\right)^2 - (1)^2$

$= \left(3+2\sqrt{6}+2\right) - 1$

$= 4 + 2\sqrt{6}$

15 a $\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\sqrt{15}}{\sqrt{9}}$

$= \frac{\sqrt{15}}{3}$

b $\frac{2\sqrt{3}}{3\sqrt{2}} = \frac{2\sqrt{3}}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2\sqrt{6}}{3\sqrt{4}}$

$= \frac{2\sqrt{6}}{3 \times 2}$

$= \frac{\sqrt{6}}{3}$

c $\frac{\sqrt{2}+3\sqrt{5}}{\sqrt{6}} = \frac{\left(\sqrt{2}+3\sqrt{5}\right)}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$

$= \frac{\sqrt{12}+3\sqrt{30}}{\sqrt{36}}$

$= \frac{\sqrt{4\times 3}+3\sqrt{30}}{6}$

$= \frac{2\sqrt{3}+3\sqrt{30}}{6}$

d $\frac{3\sqrt{5}-5\sqrt{2}}{2\sqrt{10}} = \frac{\left(3\sqrt{5}-5\sqrt{2}\right)}{2\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$

$= \frac{3\sqrt{50}-5\sqrt{20}}{2\sqrt{100}}$

$= \frac{3\sqrt{25\times 2}-5\sqrt{4\times 5}}{2\times 10}$

$= \frac{3\times 5\sqrt{2}-5\times 2\sqrt{5}}{20}$

$= \frac{15\sqrt{2}-10\sqrt{5}}{20}$

$= \frac{5\left(3\sqrt{2}-2\sqrt{5}\right)}{20}$

$= \frac{3\sqrt{2}-2\sqrt{5}}{2}$

e Use the conjugate of the denominator.

$\frac{3}{\sqrt{5}-\sqrt{2}} = \frac{3}{\sqrt{5}-\sqrt{2}} \times \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

$= \frac{3\left(\sqrt{5}+\sqrt{2}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2}$

$= \frac{3\left(\sqrt{5}+\sqrt{2}\right)}{5-2}$

$= \frac{3\left(\sqrt{5}+\sqrt{2}\right)}{3}$

$= \sqrt{5}+\sqrt{2}$

f Use the conjugate of the denominator.

$\frac{\sqrt{5}}{2\sqrt{2}+3} = \frac{\sqrt{5}}{2\sqrt{2}+3} \times \frac{2\sqrt{2}-3}{2\sqrt{2}-3}$

$= \frac{\sqrt{5}\left(2\sqrt{2}-3\right)}{\left(2\sqrt{2}\right)^2-(3)^2}$

$= \frac{\sqrt{5}\left(2\sqrt{2}-3\right)}{4\times 2-9}$

$= \frac{2\sqrt{10}-3\sqrt{5}}{-1}$

$= 3\sqrt{5}-2\sqrt{10}$

16 a $\frac{3\sqrt{2}}{4\sqrt{3}}$

$= \frac{3\sqrt{2}}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{3\sqrt{6}}{4\times 3}$

$= \frac{\sqrt{6}}{4}$

b $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{5}+\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{10}+2}{2}$

c $\frac{\sqrt{12}-3\sqrt{2}}{2\sqrt{18}}$

$= \frac{2\sqrt{3}-3\sqrt{2}}{6\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2\sqrt{6}-3\times 2}{6\times 2}$

$$=\frac{2\left(\sqrt{6}-3\right)}{12}$$

$$=\frac{\sqrt{6}-3}{6}$$

d $\frac{1}{\sqrt{6}+\sqrt{2}}$

$$=\frac{1}{\sqrt{6}+\sqrt{2}}\times\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}$$

$$=\frac{\sqrt{6}-\sqrt{2}}{6-2}$$

$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

e $\frac{2\sqrt{10}+1}{5-\sqrt{10}}$

$$=\frac{2\sqrt{10}+1}{5-\sqrt{10}}\times\frac{5+\sqrt{10}}{5+\sqrt{10}}$$

$$=\frac{\left(2\sqrt{10}+1\right)\left(5+\sqrt{10}\right)}{25-10}$$

$$=\frac{10\sqrt{10}+2\times 10+5+\sqrt{10}}{15}$$

$$=\frac{25+11\sqrt{10}}{15}$$

f $\frac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

$$=\frac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

$$=\frac{\left(3\sqrt{3}+2\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}{3-2}$$

$$=\left(3\sqrt{3}+2\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$$

$$=3\times 3-3\sqrt{6}+2\sqrt{6}-2\times 2$$

$$=5-\sqrt{6}$$

17 a i $\frac{6}{7\sqrt{2}}$

$$=\frac{6}{7\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{6\sqrt{2}}{7\times 2}$$

$$=\frac{3\sqrt{2}}{7}$$

ii $\frac{3\sqrt{5}+7\sqrt{15}}{2\sqrt{3}}$

$$=\frac{3\sqrt{5}+7\sqrt{15}}{2\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\frac{3\sqrt{15}+7\sqrt{45}}{6}$$

$$=\frac{3\sqrt{15}+21\sqrt{5}}{6}$$

$$=\frac{3\left(\sqrt{15}+7\sqrt{5}\right)}{6}$$

$$=\frac{\sqrt{15}+7\sqrt{5}}{2}$$

b $14\sqrt{6}+\frac{12}{\sqrt{6}}-5\sqrt{24}$

$$=14\sqrt{6}+\frac{12}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}-5\times 2\sqrt{6}$$

$$=14\sqrt{6}+\frac{12\sqrt{6}}{6}-10\sqrt{6}$$

$$=14\sqrt{6}+2\sqrt{6}-10\sqrt{6}$$

$$=6\sqrt{6}$$

c $\frac{10}{4\sqrt{3}+3\sqrt{2}}$

$$=\frac{10}{4\sqrt{3}+3\sqrt{2}}\times\frac{4\sqrt{3}-3\sqrt{2}}{4\sqrt{3}-3\sqrt{2}}$$

$$=\frac{10\left(4\sqrt{3}-3\sqrt{2}\right)}{48-18}$$

$$=\frac{10\left(4\sqrt{3}-3\sqrt{2}\right)}{30}$$

$$=\frac{4\sqrt{3}-3\sqrt{2}}{3}$$

d $\frac{1}{p^2-1}, p=4\sqrt{3}+1$

$$=\frac{1}{\left(4\sqrt{3}+1\right)^2-1}$$

$$=\frac{1}{\left(48+8\sqrt{3}+1\right)-1}$$

$$=\frac{1}{48+8\sqrt{3}}$$

$$=\frac{1}{8\left(6+\sqrt{3}\right)}$$

$$=\frac{1}{8\left(6+\sqrt{3}\right)}\times\frac{6-\sqrt{3}}{6-\sqrt{3}}$$

$$=\frac{6-\sqrt{3}}{8\,(36-3)}$$

$$=\frac{6-\sqrt{3}}{264}$$

18 Use perfect cube factors.

$$\sqrt[3]{384}$$

$$=\sqrt[3]{4\times 4\times 4\times 6}$$

$$=\sqrt[3]{4^3\times 6}$$

$$=4\sqrt[3]{6}$$

19 $\frac{1}{2}\sqrt{12}-\frac{1}{5}\sqrt{80}+\frac{\sqrt{10}}{\sqrt{2}}+\sqrt{243}+5$

$$=\frac{1}{2}\sqrt{4\times 3}-\frac{1}{5}\sqrt{16\times 5}+\sqrt{\frac{10}{2}}+\sqrt{81\times 3}+5$$

$$=\frac{1}{2}\times 2\sqrt{3}-\frac{1}{5}\times 4\sqrt{5}+\sqrt{5}+9\sqrt{3}+5$$

$$=\sqrt{3}-\frac{4}{5}\sqrt{5}+\sqrt{5}+9\sqrt{3}+5$$

$$=10\sqrt{3}+\frac{1}{5}\sqrt{5}+5$$

20 **a** $\left(\sqrt{2}+1\right)^3$

$= \left(\sqrt{2}\right)^3 + 3\left(\sqrt{2}\right)^2(1) + 3\left(\sqrt{2}\right)(1)^2 + (1)^3$

$= 2\sqrt{2} + 6 + 3\sqrt{2} + 1$

$= 7 + 5\sqrt{2}$

b $\left(\sqrt{2}+\sqrt{6}\right)^2 - 2\sqrt{3}\left(\sqrt{2}+\sqrt{6}\right)\left(\sqrt{2}-\sqrt{6}\right) = a + b\sqrt{3}$

Consider:

$\left(\sqrt{2}+\sqrt{6}\right)^2 - 2\sqrt{3}\left(\sqrt{2}+\sqrt{6}\right)\left(\sqrt{2}-\sqrt{6}\right)$

$= 2 + 2\sqrt{12} + 6 - 2\sqrt{3}\,(2-6)$

$= 8 + 4\sqrt{3} - 2\sqrt{3} \times -4$

$= 8 + 4\sqrt{3} + 8\sqrt{3}$

$= 8 + 12\sqrt{3}$

$\therefore 8 + 12\sqrt{3} = a + b\sqrt{3}$

$\therefore a = 8,\ b = 12$

21 **a** $\dfrac{2\sqrt{3}-1}{\sqrt{3}+1} - \dfrac{\sqrt{3}}{\sqrt{3}+2}$

$= \dfrac{2\sqrt{3}-1}{\sqrt{3}+1} \times \dfrac{\sqrt{3}-1}{\sqrt{3}-1} - \dfrac{\sqrt{3}}{\sqrt{3}+2} \times \dfrac{\sqrt{3}-2}{\sqrt{3}-2}$

$= \dfrac{\left(2\sqrt{3}-1\right)\left(\sqrt{3}-1\right)}{3-1} - \dfrac{\sqrt{3}\left(\sqrt{3}-2\right)}{3-4}$

$= \dfrac{6-3\sqrt{3}+1}{2} - \dfrac{3-2\sqrt{3}}{-1}$

$= \dfrac{7-3\sqrt{3}}{2} + \dfrac{3-2\sqrt{3}}{1}$

$= \dfrac{7-3\sqrt{3}+6-4\sqrt{3}}{2}$

$= \dfrac{13-7\sqrt{3}}{2}$

b $\dfrac{\sqrt{3}-1}{\sqrt{3}+1} + \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

$= \dfrac{\left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right) + \left(\sqrt{3}+1\right)\left(\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}$

$= \dfrac{\left(3-2\sqrt{3}+1\right) + \left(3+2\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}$

$= \dfrac{8}{3-1} = 4$

This is a rational number.

22 **a** $4\sqrt{5} - 2\sqrt{6} + \dfrac{3}{\sqrt{6}} - \dfrac{10}{3\sqrt{5}}$

$= 4\sqrt{5} - 2\sqrt{6} + \dfrac{3}{\sqrt{6}} \times \dfrac{\sqrt{6}}{\sqrt{6}} - \dfrac{10}{3\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$

$= 4\sqrt{5} - 2\sqrt{6} + \dfrac{3\sqrt{6}}{6} - \dfrac{10\sqrt{5}}{15}$

$= 4\sqrt{5} - 2\sqrt{6} + \dfrac{\sqrt{6}}{2} - \dfrac{2\sqrt{5}}{3}$

$= \dfrac{24\sqrt{5} - 12\sqrt{6} + 3\sqrt{6} - 4\sqrt{5}}{6}$

$= \dfrac{20\sqrt{5} - 9\sqrt{6}}{6}$

b $\sqrt{2}\left(2\sqrt{10} + 9\sqrt{8}\right) - \dfrac{\sqrt{5}}{\sqrt{5}-2}$

$= 2\sqrt{20} + 9\sqrt{16} - \dfrac{\sqrt{5}}{\sqrt{5}-2} \times \dfrac{\sqrt{5}+2}{\sqrt{5}+2}$

$= 4\sqrt{5} + 36 - \dfrac{\sqrt{5}\left(\sqrt{5}+2\right)}{5-4}$

$= 4\sqrt{5} + 36 - \sqrt{5}\left(\sqrt{5}+2\right)$

$= 4\sqrt{5} + 36 - 5 - 2\sqrt{5}$

$= 2\sqrt{5} + 31$ or $\dfrac{2\sqrt{5}+31}{1}$

c $\dfrac{3}{2\sqrt{3}\left(\sqrt{2}+\sqrt{3}\right)^2} + \dfrac{2}{\sqrt{3}}$

$= \dfrac{3}{2\sqrt{3}\left(2+2\sqrt{6}+3\right)} + \dfrac{2}{\sqrt{3}}$

$= \dfrac{3}{2\sqrt{3}\left(5+2\sqrt{6}\right)} + \dfrac{2}{\sqrt{3}}$

$= \dfrac{3}{10\sqrt{3}+4\sqrt{18}} + \dfrac{2}{\sqrt{3}}$

$= \dfrac{3}{10\sqrt{3}+12\sqrt{2}} + \dfrac{2}{\sqrt{3}}$

$= \dfrac{3}{10\sqrt{3}+12\sqrt{2}} \times \dfrac{10\sqrt{3}-12\sqrt{2}}{10\sqrt{3}-12\sqrt{2}} + \dfrac{2}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{3\left(10\sqrt{3}-12\sqrt{2}\right)}{300-288} + \dfrac{2\sqrt{3}}{3}$

$= \dfrac{3\left(10\sqrt{3}-12\sqrt{2}\right)}{12} + \dfrac{2\sqrt{3}}{3}$

$= \dfrac{3\left(10\sqrt{3}-12\sqrt{2}\right) + 8\sqrt{3}}{12}$

$= \dfrac{30\sqrt{3} - 36\sqrt{2} + 8\sqrt{3}}{12}$

$= \dfrac{38\sqrt{3} - 36\sqrt{2}}{12}$

$= \dfrac{2\left(19\sqrt{3} - 18\sqrt{2}\right)}{12}$

$= \dfrac{19\sqrt{3} - 18\sqrt{2}}{6}$

d $\dfrac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}}+\dfrac{2\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$

As the denominators are a pair of conjugates, express each fraction on the common denominator rather than rationalising each denominator.

$$=\frac{\left(2\sqrt{3}-\sqrt{2}\right)^2+\left(2\sqrt{3}+\sqrt{2}\right)^2}{\left(2\sqrt{3}+\sqrt{2}\right)\left(2\sqrt{3}-\sqrt{2}\right)}$$

$$=\frac{\left(12-4\sqrt{6}+2\right)+\left(12+4\sqrt{6}+2\right)}{4\times 3-2}$$

$$=\frac{28}{10}$$

$$=\frac{14}{5}$$

e $\dfrac{\sqrt{2}}{16-4\sqrt{7}}-\dfrac{\sqrt{14}+2\sqrt{2}}{9\sqrt{7}}$

$$=\frac{\sqrt{2}}{4\left(4-\sqrt{7}\right)}\times\frac{4+\sqrt{7}}{4+\sqrt{7}}-\frac{\sqrt{14}+2\sqrt{2}}{9\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}$$

$$=\frac{\sqrt{2}\left(4+\sqrt{7}\right)}{4(16-7)}-\frac{\sqrt{7}\left(\sqrt{14}+2\sqrt{2}\right)}{63}$$

$$=\frac{4\sqrt{2}+\sqrt{14}}{36}-\frac{\sqrt{7}\times\sqrt{2}\left(\sqrt{7}+2\right)}{63}$$

$$=\frac{7\left(4\sqrt{2}+\sqrt{14}\right)-4\sqrt{7}\times\sqrt{2}\left(\sqrt{7}+2\right)}{4\times 9\times 7}$$

$$=\frac{28\sqrt{2}+7\sqrt{14}-4\times 7\times\sqrt{2}-8\sqrt{14}}{252}$$

$$=-\frac{\sqrt{14}}{252}$$

f $\dfrac{\left(2-\sqrt{3}\right)^2}{2+\sqrt{3}}+\dfrac{2\sqrt{3}}{4-3\sqrt{2}}$

$$=\frac{4-4\sqrt{3}+3}{2+\sqrt{3}}+\frac{2\sqrt{3}}{4-3\sqrt{2}}$$

$$=\frac{7-4\sqrt{3}}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}+\frac{2\sqrt{3}}{4-3\sqrt{2}}\times\frac{4+3\sqrt{2}}{4+3\sqrt{2}}$$

$$=\frac{\left(7-4\sqrt{3}\right)\left(2-\sqrt{3}\right)}{4-3}+\frac{2\sqrt{3}\left(4+3\sqrt{2}\right)}{16-9\times 2}$$

$$=\left(7-4\sqrt{3}\right)\left(2-\sqrt{3}\right)+\frac{2\sqrt{3}\left(4+3\sqrt{2}\right)}{-2}$$

$$=\left(7-4\sqrt{3}\right)\left(2-\sqrt{3}\right)-\sqrt{3}\left(4+3\sqrt{2}\right)$$

$$=14-15\sqrt{3}+12-4\sqrt{3}-3\sqrt{6}$$

$$=26-19\sqrt{3}-3\sqrt{6}\text{ or }\frac{26-19\sqrt{3}-3\sqrt{6}}{1}$$

23 a $x=2\sqrt{3}-\sqrt{10}$

i $x+\dfrac{1}{x}$

$$=2\sqrt{3}-\sqrt{10}+\frac{1}{2\sqrt{3}-\sqrt{10}}$$

$$=2\sqrt{3}-\sqrt{10}+\frac{1}{2\sqrt{3}-\sqrt{10}}\times\frac{2\sqrt{3}+\sqrt{10}}{2\sqrt{3}+\sqrt{10}}$$

$$=2\sqrt{3}-\sqrt{10}+\frac{2\sqrt{3}+\sqrt{10}}{12-10}$$

$$=2\sqrt{3}-\sqrt{10}+\frac{2\sqrt{3}+\sqrt{10}}{2}$$

$$=\frac{4\sqrt{3}-2\sqrt{10}+2\sqrt{3}+\sqrt{10}}{2}$$

$$=\frac{6\sqrt{3}-\sqrt{10}}{2}$$

ii $x^2-4\sqrt{3}x$

$$=\left(2\sqrt{3}-\sqrt{10}\right)^2-4\sqrt{3}\left(2\sqrt{3}-\sqrt{10}\right)$$

$$=12-4\sqrt{30}+10-8\times 3+4\sqrt{30}$$

$$=-2$$

b $y=\dfrac{\sqrt{7}+2}{\sqrt{7}-2}$

i $y-\dfrac{1}{y}$

$$=\frac{\sqrt{7}+2}{\sqrt{7}-2}-\frac{\sqrt{7}-2}{\sqrt{7}+2}$$

$$=\frac{\left(\sqrt{7}+2\right)^2-\left(\sqrt{7}-2\right)^2}{\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)}$$

$$=\frac{7+4\sqrt{7}+4-\left(7-4\sqrt{7}+4\right)}{7-4}$$

$$=\frac{8\sqrt{7}}{3}$$

ii $\dfrac{1}{y^2-1}$

$$=\frac{1}{\frac{\left(\sqrt{7}+2\right)^2}{\left(\sqrt{7}-2\right)^2}-1}$$

$$=1\div\left(\frac{\left(\sqrt{7}+2\right)^2-\left(\sqrt{7}-2\right)^2}{\left(\sqrt{7}-2\right)^2}\right)$$

$$=\frac{\left(\sqrt{7}-2\right)^2}{\left(\sqrt{7}+2\right)^2-\left(\sqrt{7}-2\right)^2}$$

$$=\frac{7-4\sqrt{7}+4}{11+4\sqrt{7}-\left(11-4\sqrt{7}\right)}$$

$$=\frac{11-4\sqrt{7}}{8\sqrt{7}}$$

$$= \frac{11-4\sqrt{7}}{8\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$
$$= \frac{11\sqrt{7}-28}{56}$$

c i $$\frac{1}{\sqrt{7}+\sqrt{3}} - \frac{1}{\sqrt{7}-\sqrt{3}} = m\sqrt{7}+n\sqrt{3}$$
$$\therefore \frac{\left(\sqrt{7}-\sqrt{3}\right)-\left(\sqrt{7}+\sqrt{3}\right)}{\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)} = m\sqrt{7}+n\sqrt{3}$$
$$\therefore \frac{-2\sqrt{3}}{7-3} = m\sqrt{7}+n\sqrt{3}$$
$$\therefore -\frac{1}{2}\sqrt{3} = m\sqrt{7}+n\sqrt{3}$$
$$\therefore m = 0, n = -\frac{1}{2}$$

ii $$\left(2+\sqrt{3}\right)^4 - \frac{7\sqrt{3}}{\left(2+\sqrt{3}\right)^2} = m+\sqrt{n}$$
$$\left(2+\sqrt{3}\right)^2 = 4+4\sqrt{3}+3$$
$$= 7+4\sqrt{3}$$
$$\therefore \left(2+\sqrt{3}\right)^4 = \left(7+4\sqrt{3}\right)^2$$
$$= 49+56\sqrt{3}+48$$
$$= 97+56\sqrt{3}$$

Hence,

$$97+56\sqrt{3} - \frac{7\sqrt{3}}{7+4\sqrt{3}} = m+\sqrt{n}$$
$$\therefore 97+56\sqrt{3} - \frac{7\sqrt{3}}{7+4\sqrt{3}} \times \frac{7-4\sqrt{3}}{7-4\sqrt{3}} = m+\sqrt{n}$$
$$\therefore 97+56\sqrt{3} - \frac{7\sqrt{3}\left(7-4\sqrt{3}\right)}{49-48} = m+\sqrt{n}$$
$$\therefore 97+56\sqrt{3} - 49\sqrt{3} + 28\times 3 = m+\sqrt{n}$$
$$\therefore 181+7\sqrt{3} = m+\sqrt{n}$$
$$\therefore 181+\sqrt{49\times 3} = m+\sqrt{n}$$
$$\therefore 181+\sqrt{147} = m+\sqrt{n}$$
$$\therefore m = 181,\ n = 147$$

d $$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

i $x_1 = -\dfrac{b}{2a} + \dfrac{\sqrt{b^2-4ac}}{2a}$ so the conjugate is

$$x_2 = -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-b-\sqrt{b^2-4ac}}{2a}$$

ii $x_1 + x_2$
$$= -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} + -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$
$$= -\frac{2b}{2a}$$
$$= -\frac{b}{a}$$

iii x_1x_2
$$= \left(-\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right)\left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right)$$
$$= \left(-\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2$$
$$= \frac{b^2}{4a^2} - \frac{b^2-4ac}{4a^2}$$
$$= \frac{b^2-b^2+4ac}{4a^2}$$
$$= \frac{4ac}{4a^2}$$
$$= \frac{c}{a}$$

24 $A\left(\sqrt{2}, -1\right), B\left(\sqrt{5}, \sqrt{10}\right), C\left(\sqrt{10}, \sqrt{5}\right)$

a $d(A,B) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$$\therefore AB = \sqrt{\left(\sqrt{5}-\sqrt{2}\right)^2+\left(\sqrt{10}+1\right)^2}$$

On the CAS calculator, use Simplify to obtain $AB = 3\sqrt{2}$.

$$BC = \sqrt{\left(\sqrt{10}-\sqrt{5}\right)^2+\left(\sqrt{5}-\sqrt{10}\right)^2}$$
$$= \sqrt{2\left(\sqrt{10}-\sqrt{5}\right)^2}$$
$$= \sqrt{20}-\sqrt{10}$$
$$= 2\sqrt{5}-\sqrt{10}$$

(Alternatively, use CAS to obtain $-\sqrt{10}+2\sqrt{5}$.)

$$AC = \sqrt{\left(\sqrt{10}-\sqrt{2}\right)^2+\left(\sqrt{5}+1\right)^2}$$
$$\therefore AC = \sqrt{18-2\sqrt{5}}$$
$$\sqrt{a^2 \pm b} \simeq a \pm \frac{b}{2a}$$

The surds present in the lengths of the three sides are $\sqrt{2}, \sqrt{5}, \sqrt{10}$.

$$\sqrt{2} = \sqrt{1^2+1},\ a = 1, b = 1$$
$$\simeq 1 + \frac{1}{2\times 1}$$
$$= 1.5$$
$$\sqrt{5} = \sqrt{2^2+1},\ a = 2, b = 1$$
$$\simeq 2 + \frac{1}{2\times 2}$$
$$= 2.25$$
$$\sqrt{10} = \sqrt{3^2+1},\ a = 3, b = 1$$
$$\simeq 3 + \frac{1}{2\times 3}$$
$$= 3 + \frac{1}{6}$$
$$\simeq 3.167$$

$$\begin{aligned}
AB &= 3\sqrt{2}\\
\therefore AB &\simeq 3 \times 1.5\\
\therefore AB &= 4.5\\
BC &= -\sqrt{10} + 2\sqrt{5}\\
\therefore BC &\simeq -3.167 + 2 \times 2.25\\
\therefore BC &= 4.5 - 3.167\\
\therefore BC &\simeq 1.333
\end{aligned}$$

$$\begin{aligned}
AC &= \sqrt{18 - 2\sqrt{5}}\\
&\simeq \sqrt{18 - 4.5}\\
&= \sqrt{13.5}
\end{aligned}$$

Using the approximation formula,

$$\begin{aligned}
\sqrt{13.5} &= \sqrt{4^2 - 2.5},\ a = 4, b = 2.5\\
&\simeq 4 - \frac{2.5}{2 \times 4}\\
&= 4 - \frac{2.5}{8}\\
&= 4 - 0.3125\\
&\simeq 3.6875
\end{aligned}$$

$\therefore AC \simeq 3.6875$

b From the approximation values, the side AB is the longest.
Switch the calculator to decimal mode and calculate $3\sqrt{2}$.
The longest side is $AB \simeq 4.2$ units.

25 a The formula for the area of a rectangle is $A = lw$.

$$A = \left(\sqrt{6} + \sqrt{3} + 1\right)\left(\sqrt{3} + 2\right)$$

Expand using CAS.

$$\therefore A = 2\sqrt{6} + 3\sqrt{3} + 3\sqrt{2} + 5$$

The area is $\left(2\sqrt{6} + 3\sqrt{3} + 3\sqrt{2} + 5\right)$ square metres.

b Cost for mowing the area is \$50 − \$23.35 = \$26.65.

Cost per square metre is $\dfrac{26.65}{2\sqrt{6} + 3\sqrt{3} + 3\sqrt{2} + 5}$ dollars.

This evaluates to \$1.38 per square metre.

2.6 Exam questions

1
$$\begin{aligned}
5\sqrt{6} \times 3\sqrt{2} - \sqrt{3}\left(4\sqrt{3} + 2\right) &= 15\sqrt{12} - 12 - 2\sqrt{3}\\
&= 15\sqrt{4 \times 3} - 12 - 2\sqrt{3}\\
&= 30\sqrt{3} - 12 - 2\sqrt{3}\\
&= 28\sqrt{3} - 12
\end{aligned}$$

The correct answer is **A**.

2
$$\begin{aligned}
&\left(3\sqrt{2} - 2\sqrt{3}\right)\left(3\sqrt{2} + 2\sqrt{3}\right)\\
&= \left(3\sqrt{2}\right)^2 - \left(2\sqrt{3}\right)^2\\
&= 9 \times 2 - 4 \times 3\\
&= 6 \qquad \text{[1 mark]}
\end{aligned}$$

3
$$\begin{aligned}
\frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} + \sqrt{3}} &= \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} + \sqrt{3}} \times \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} - \sqrt{3}}\\
&= \frac{2 - \sqrt{6} - \sqrt{6} + 3}{2 - 3}\\
&= \frac{5 - 2\sqrt{6}}{-1}\\
&= 2\sqrt{6} - 5
\end{aligned}$$

The correct answer is **E**.

2.7 Review

2.7 Exercise

Technology free: short answer

1 a
$$\begin{aligned}
&(5x + 2y)^2 - 3(1 + 2y)(1 - 2y)\\
&= 25x^2 + 20xy + 4y^2 - 3\left(1 - 4y^2\right)\\
&= 25x^2 + 20xy + 4y^2 - 3 + 12y^2\\
&= 25x^2 + 20xy + 16y^2 - 3
\end{aligned}$$

b
$$\begin{aligned}
&(2x - 3)\left(x^2 - 6x + 2\right) - (5 - x)(5 + x)\\
&= 2x^3 - 12x^2 + 4x - 3x^2 + 18x - 6 - \left(25 - x^2\right)\\
&= 2x^3 - 15x^2 + 22x - 6 - 25 + x^2\\
&= 2x^3 - 14x^2 + 22x - 31
\end{aligned}$$

c
$$\begin{aligned}
&\left(2x + \sqrt{3}\right)^3\\
&= (2x)^3 + 3(2x)^2\left(\sqrt{3}\right) + 3(2x)\left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^3\\
&= 8x^3 + 12\sqrt{3}x^2 + 18x + 3\sqrt{3}
\end{aligned}$$

d
$$\begin{aligned}
&(2x - 5)^4\\
&= (2x)^4 - 4(2x)^3(5) + 6(2x)^2(5)^2 - 4(2x)(5)^3 + (5)^4\\
&= 16x^4 - 160x^3 + 600x^2 - 1000x + 625
\end{aligned}$$

e
$$\begin{aligned}
&\left(1 - \frac{2x}{3}\right)^5\\
&= 1 - \binom{5}{1}\left(\frac{2x}{3}\right) + \binom{5}{2}\left(\frac{2x}{3}\right)^2 - \binom{5}{3}\left(\frac{2x}{3}\right)^3\\
&\quad + \binom{5}{4}\left(\frac{2x}{3}\right)^4 - \left(\frac{2x}{3}\right)^5\\
&= 1 - \frac{5 \times 2x}{3} + \frac{10 \times 4x^2}{9} - \frac{10 \times 8x^3}{27} + \frac{5 \times 16x^4}{81} - \frac{32x^5}{243}\\
&= 1 - \frac{10x}{3} + \frac{40x^2}{9} - \frac{80x^3}{27} + \frac{80x^4}{81} - \frac{32x^5}{243}
\end{aligned}$$

f
$$\begin{aligned}
&(x + 1 - y)^2\\
&= ((x + 1) - y)^2\\
&= (x + 1)^2 - 2y(x + 1) + y^2\\
&= x^2 + 2x + 1 - 2yx - 2y + y^2\\
&= x^2 + 2x + 1 - 2xy - 2y + y^2
\end{aligned}$$

2 a $72x^2 + 41xy - 45y^2 = (9x - 5y)(8x + 9y)$

b
$$\begin{aligned}
&x^2y^3 - 36x^2y\\
&= x^2y\left(y^2 - 36\right)\\
&= x^2y(y - 6)(y + 6)
\end{aligned}$$

c
$$\begin{aligned}
&4y^2 - x^2 + 18x - 81\\
&= 4y^2 - \left(x^2 - 18x + 81\right)\\
&= 4y^2 - (x - 9)^2\\
&= \left[2y - (x - 9)\right]\left[2y + (x - 9)\right]\\
&= (2y - x + 9)(2y + x - 9)
\end{aligned}$$

d
$$\begin{aligned}
&64x^3 + 1\\
&= (4x)^3 + 1^3\\
&= (4x + 1)\left((4x)^2 - (4x) + 1\right)\\
&= (4x + 1)\left(16x^2 - 4x + 1\right)
\end{aligned}$$

e
$$\begin{aligned}
&(x + 2)^3 - 8(x - 1)^3\\
&= (x + 2)^3 - (2(x - 1))^3\\
&= [(x + 2) - 2(x - 1)]\\
&\qquad \left[(x + 2)^2 + (x + 2)2(x - 1) + 4(x - 1)^2\right]
\end{aligned}$$

$= (x + 2 - 2x + 2)$
$\left[x^2 + 4x + 4 + 2\left(x^2 + x - 2\right) + 4\left(x^2 - 2x + 1\right)\right]$
$= (4 - x)\left(x^2 + 4x + 4 + 2x^2 + 2x - 4 + 4x^2 - 8x + 4\right)$
$= (4 - x)\left(7x^2 - 2x + 4\right)$

f $2x^3 + 6x^2y + 6xy^2 + 2y^3$
$= 2\left(x^3 + 3x^2y + 3xy^2 + y^3\right)$
$= 2\left[\left(x^3 + y^3\right) + 3xy(x + y)\right]$
$= 2\left[(x + y)\left(x^2 - xy + y^2\right) + 3xy(x + y)\right]$
$= 2(x + y)\left(\left(x^2 - xy + y^2\right) + 3xy\right)$
$= 2(x + y)\left(x^2 + 2xy + y^2\right)$
$= 2(x + y)(x + y)^2$
$= 2(x + y)^3$

Note that $2\left(x^3 + 3x^2y + 3xy^2 + y^3\right)$ in the second line could be recognised as the expansion of $(x + y)^3$.

3 a $\binom{10}{6}$
$= \dfrac{10!}{6! \times 4!}$
$= \dfrac{10 \times 9 \times 8 \times 7 \times 6!}{6! \times 4!}$
$= \dfrac{10 \times {}^{3}\cancel{9} \times \cancel{8} \times 7}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$
$= 210$

b i Row 5 of Pascal's triangle is formed from row 4.

		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Row 5 is 1 5 10 10 5 1.

ii $(1 + 2xy)^5$
$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
$a = 1,\ b = 2xy$
$(1 + 2xy)^5$
$= (1)^5 + 5(1)^4(2xy) + 10(1)^3(2xy)^2 + 10(1)^2(2xy)^3$
$+ 5(1)(2xy)^4 + (2xy)^5$
$= 1 + 10xy + 40x^2y^2 + 80x^3y^3 + 80x^4y^4 + 32x^5y^5$

c There are 7 terms in the expansion of $(3 - 2x)^6$, so the middle term is t_4.
Row 6 of Pascal's triangle is 1 6 15 20 15 6 1.
The 4th coefficient is 20.
The middle term is $t_4 = 20 \times (3)^3(-2x)^3$.
$t_4 = 20 \times (3)^3(-2x)^3$
$= -20 \times 27 \times 8x^3$
$= -20 \times 216x^3$
$= -4320x^3$

4 a $4\sqrt{3} + 3\sqrt{8} - 2\sqrt{72} - \sqrt{75}$
$= 4\sqrt{3} + 3 \times 2\sqrt{2} - 2 \times 6\sqrt{2} - 5\sqrt{3}$
$= 4\sqrt{3} + 6\sqrt{2} - 12\sqrt{2} - 5\sqrt{3}$
$= -\sqrt{3} - 6\sqrt{2}$

b $\left(\sqrt{7} - \sqrt{3}\right)\left(\sqrt{7} + \sqrt{3}\right)$
$= 7 - 3$
$= 4$

c $\dfrac{5}{\sqrt{50}} + \sqrt{108} \div \sqrt{3}$
$= \dfrac{5}{\sqrt{50}} + \dfrac{\sqrt{108}}{\sqrt{3}}$
$= \dfrac{5}{5\sqrt{2}} + \sqrt{\dfrac{108}{3}}$
$= \dfrac{1}{\sqrt{2}} + \sqrt{36}$
$= \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} + 6$
$= \dfrac{\sqrt{2}}{2} + 6$

d $\dfrac{11}{2\sqrt{3} - 3\sqrt{5}}$
$= \dfrac{11}{2\sqrt{3} - 3\sqrt{5}} \times \dfrac{2\sqrt{3} + 3\sqrt{5}}{2\sqrt{3} + 3\sqrt{5}}$
$= \dfrac{11\left(2\sqrt{3} + 3\sqrt{5}\right)}{4 \times 3 - 9 \times 5}$
$= \dfrac{11\left(2\sqrt{3} + 3\sqrt{5}\right)}{-33}$
$= -\dfrac{2\sqrt{3} + 3\sqrt{5}}{3}$

e $4\sqrt{5} - \dfrac{2}{3\sqrt{5}} - \dfrac{\sqrt{125}}{2}$
$= 4\sqrt{5} - \dfrac{2}{3\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}} - \dfrac{5\sqrt{5}}{2}$
$= 4\sqrt{5} - \dfrac{2\sqrt{5}}{15} - \dfrac{5\sqrt{5}}{2}$
$= \dfrac{120\sqrt{5} - 4\sqrt{5} - 75\sqrt{5}}{30}$
$= \dfrac{41\sqrt{5}}{30}$

f $\dfrac{3}{\sqrt{5} - \sqrt{2}} - \dfrac{3}{\sqrt{5} + \sqrt{2}}$
$= \dfrac{3\left(\sqrt{5} + \sqrt{2}\right) - 3\left(\sqrt{5} - \sqrt{2}\right)}{\left(\sqrt{5} - \sqrt{2}\right)\left(\sqrt{5} + \sqrt{2}\right)}$
$= \dfrac{3\sqrt{5} + 3\sqrt{2} - 3\sqrt{5} + 3\sqrt{2}}{5 - 2}$
$= \dfrac{6\sqrt{2}}{3}$
$= 2\sqrt{2}$

5 a $\dfrac{x^2+7x+12}{(x+1)^2-9}$

$= \dfrac{(x+3)(x+4)}{((x+1)-3)((x+1)+3)}$

$= \dfrac{(x+3)(x+4)}{(x-2)(x+4)}$

$= \dfrac{x+3}{x-2}$

b $\dfrac{4}{x^2-9} + \dfrac{3}{x^2-6x+9}$

$= \dfrac{4}{(x-3)(x+3)} + \dfrac{3}{(x-3)^2}$

$= \dfrac{4(x-3)+3(x+3)}{(x+3)(x-3)^2}$

$= \dfrac{4x-12+3x+9}{(x+3)(x-3)^2}$

$= \dfrac{7x-3}{(x+3)(x-3)^2}$

c $\dfrac{5x}{x^3+27} \div \dfrac{5x^2-15x}{9-x^2}$

$= \dfrac{5x}{x^3+27} \times \dfrac{9-x^2}{5x^2-15x}$

$= \dfrac{{}^{1}\cancel{5x}}{\cancel{(x+3)}\left(x^2-3x+9\right)} \times \dfrac{(3-x)\cancel{(3+x)}}{\cancel{5x}(x-3)}$

$= \dfrac{1}{x^2-3x+9} \times \dfrac{-(x-3)}{x-3}$

$= \dfrac{-1}{x^2-3x+9}$

d $\dfrac{2}{3\sqrt{5}} + \dfrac{4\sqrt{5}}{3\sqrt{5}-1}$

$= \dfrac{2}{3\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}} + \dfrac{4\sqrt{5}}{3\sqrt{5}-1} \times \dfrac{3\sqrt{5}+1}{3\sqrt{5}+1}$

$= \dfrac{2\sqrt{5}}{15} + \dfrac{4\sqrt{5}\left(3\sqrt{5}+1\right)}{9\times 5-1}$

$= \dfrac{2\sqrt{5}}{15} + \dfrac{4\sqrt{5}\left(3\sqrt{5}+1\right)}{44}$

$= \dfrac{2\sqrt{5}}{15} + \dfrac{\sqrt{5}\left(3\sqrt{5}+1\right)}{11}$

$= \dfrac{22\sqrt{5}+15\sqrt{5}\left(3\sqrt{5}+1\right)}{165}$

$= \dfrac{22\sqrt{5}+45\times 5+15\sqrt{5}}{165}$

$= \dfrac{37\sqrt{5}+225}{165}$

e $\dfrac{3}{\sqrt{5}-\sqrt{2}} \times \dfrac{\sqrt{10}}{\sqrt{5}-\sqrt{2}}$

$= \dfrac{3\sqrt{10}}{\left(\sqrt{5}-\sqrt{2}\right)^2}$

$= \dfrac{3\sqrt{10}}{5-2\sqrt{10}+2}$

$= \dfrac{3\sqrt{10}}{7-2\sqrt{10}} \times \dfrac{7+2\sqrt{10}}{7+2\sqrt{10}}$

$= \dfrac{3\sqrt{10}\left(7+2\sqrt{10}\right)}{49-40}$

$= \dfrac{3\sqrt{10}\left(7+2\sqrt{10}\right)}{9}$

$= \dfrac{\sqrt{10}\left(7+2\sqrt{10}\right)}{3}$

$= \dfrac{7\sqrt{10}+20}{3}$

f $\dfrac{3\sqrt{2}}{\sqrt{5}+2\sqrt{3}} - \dfrac{2\sqrt{5}+\sqrt{6}}{2\sqrt{2}-1}$

$= \dfrac{3\sqrt{2}}{\sqrt{5}+2\sqrt{3}} \times \dfrac{\sqrt{5}-2\sqrt{3}}{\sqrt{5}-2\sqrt{3}} - \dfrac{2\sqrt{5}+\sqrt{6}}{2\sqrt{2}-1} \times \dfrac{2\sqrt{2}+1}{2\sqrt{2}+1}$

$= \dfrac{3\sqrt{2}\left(\sqrt{5}-2\sqrt{3}\right)}{5-12} - \dfrac{\left(2\sqrt{5}+\sqrt{6}\right)\left(2\sqrt{2}+1\right)}{8-1}$

$= -\dfrac{3\sqrt{2}\left(\sqrt{5}-2\sqrt{3}\right)}{7} - \dfrac{\left(2\sqrt{5}+\sqrt{6}\right)\left(2\sqrt{2}+1\right)}{7}$

$= \dfrac{-3\sqrt{10}+6\sqrt{6}-4\sqrt{10}-2\sqrt{5}-2\sqrt{12}-\sqrt{6}}{7}$

$= \dfrac{5\sqrt{6}-4\sqrt{3}-2\sqrt{5}-7\sqrt{10}}{7}$

6 a $2\sqrt{6}=\sqrt{24},\ 4\sqrt{3}=\sqrt{48},\ 3\sqrt{5}=\sqrt{45},$
$5\sqrt{2}=\sqrt{50},\ 2\sqrt{10}=\sqrt{40}$

In decreasing order, $\sqrt{50}, \sqrt{48}, \sqrt{45}, \sqrt{40}, \sqrt{24}$

Therefore, with its elements in decreasing order the set becomes $\left\{5\sqrt{2}, 4\sqrt{3}, 3\sqrt{5}, 2\sqrt{10}, 2\sqrt{6}\right\}$.

b $7! = 5040$

$(4!)^2 = 24^2 = 576$ $\qquad \dfrac{16!}{14!} = 16\times 15 = 240$ $\qquad 4!\times 5! = 24\times 120 = 2880$

With its elements in decreasing order the set becomes $\left\{7!,\ 4!\times 5!,\ (4!)^2,\ \dfrac{16!}{14!}\right\}$.

c i $\sqrt{10} \simeq 3.162$

$\dfrac{1}{\sqrt{10}} = \dfrac{1}{\sqrt{10}} \times \dfrac{\sqrt{10}}{\sqrt{10}}$

$= \dfrac{\sqrt{10}}{10}$

$\therefore \dfrac{1}{\sqrt{10}} \simeq \dfrac{3.162}{10}$

$\therefore \dfrac{1}{\sqrt{10}} \simeq 0.3162$

ii $\dfrac{\left(\sqrt{5}-\sqrt{2}\right)^2}{3-\sqrt{10}}$

$= \dfrac{5-2\sqrt{10}+2}{3-\sqrt{10}} \times \dfrac{3+\sqrt{10}}{3+\sqrt{10}}$

$= \dfrac{\left(7-2\sqrt{10}\right)\left(3+\sqrt{10}\right)}{9-10}$

$= -\left(7-2\sqrt{10}\right)\left(3+\sqrt{10}\right)$

$= -\left(21+7\sqrt{10}-6\sqrt{10}-20\right)$

$= -\left(1+\sqrt{10}\right)$

$\simeq -4.162$

d $x = 1-2\sqrt{5}$

$\therefore x^2 = \left(1-2\sqrt{5}\right)^2$

$= 1-4\sqrt{5}+4\times 5$

$\therefore x^2 = 21-4\sqrt{5}$

Hence, $\left(1-2\sqrt{5}\right)^2 = 21-4\sqrt{5}$.

$\therefore 1-2\sqrt{5} = \pm\sqrt{21-4\sqrt{5}}$

However, $1-2\sqrt{5}<0$, so $1-2\sqrt{5} = -\sqrt{21-4\sqrt{5}}$.

$\therefore \sqrt{21-4\sqrt{5}} = -\left(1-2\sqrt{5}\right)$

$\therefore \sqrt{21-4\sqrt{5}} = 2\sqrt{5}-1$

Technology active: multiple choice

7 $2(x+1)^2 + 9(x+1)-5$

$= 2a^2+9a-5$ where $a = x+1$

$= (2a-1)(a+5)$

$= (2(x+1)-1)((x+1)+5)$

$= (2x+1)(x+6)$

The correct answer is **A**.

8 $(1-x)^3 = 1-3x+3x^2-x^3$

The correct answer is **D**.

9 $24x^3-81y^3$

$= 3\left(8x^3-27y^3\right)$

$= 3\left((2x)^3-(3y)^3\right)$

$= 3(2x-3y)\left((2x)^2+(2x)(3y)+(3y)^2\right)$

$= 3(2x-3y)\left(4x^2+6xy+9y^2\right)$

The correct answer is **C**.

10 $\dfrac{(n+1)!+n!}{(n+1)!-n!}$

$= \dfrac{(n+1)n!+n!}{(n+1)n!-n!}$

$= \dfrac{n!((n+1)+1)}{n!((n+1)-1)}$

$= \dfrac{n+2}{n}$

The correct answer is **A**.

11 $\begin{pmatrix} n \\ 1 \end{pmatrix} = 10$

$\therefore \dfrac{n!}{1!(n-1)!} = 10$

$\therefore \dfrac{n(n-1)!}{(n-1)!} = 10$

$\therefore n = 10$

The correct answer is **D**.

12 $(2x-3)^5$ has 6 terms in its expansion, so the answer is **E**.

13 $0! = 1$ and 1 is a natural number, so the statement $0! \notin N$ is incorrect.

The correct answer is **C**.

14 A: $\sqrt{4+9} = \sqrt{13}$, which is not rational.

B: $\sqrt{11-4\times 9} = \sqrt{11-36} = \sqrt{-25}$, which is not real.

D: $\sqrt[3]{\dfrac{4}{9}}$ is not rational as neither 4 nor 9 is a perfect cube.

E: $\dfrac{4}{\sqrt{9}-3} = \dfrac{4}{0}$, which is undefined.

C: $\dfrac{1}{\sqrt{4}+\sqrt{9}} = \dfrac{1}{2+3} = \dfrac{1}{5}$, which is rational.

The correct answer is **C**.

15 $\dfrac{6}{\sqrt{a}}+\dfrac{\sqrt{a}}{\sqrt{6}},\ a = 18$

$= \dfrac{6}{\sqrt{18}}+\dfrac{\sqrt{18}}{\sqrt{6}}$

$= \dfrac{6}{3\sqrt{2}}+\sqrt{\dfrac{18}{6}}$

$= \dfrac{2}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}+\sqrt{3}$

$= \sqrt{2}+\sqrt{3}$

The correct answer is **B**.

16 The interval shown is $(-\infty, 1) \cup [4, \infty]$. This is all of the real numbers excluding those that lie between 1 and 4 but not excluding 4.

This is $R\backslash[1,4)$.

The correct answer is **D**.

Technology active: extended response

17 a $5\sqrt{3}x-y = 12 \qquad [1]$

$2\sqrt{3}x+3y = 15 \qquad [2]$

$3\times$ equation [1]: $15\sqrt{3}x-3y = 36 \qquad [3]$

Add equations [3] and [2]:

$\therefore 17\sqrt{3}x = 51$

$\therefore x = \dfrac{51}{17\sqrt{3}}$

$\therefore x = \dfrac{3}{\sqrt{3}}$

$\therefore x = \dfrac{3}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}$

$\therefore x = \sqrt{3}$

Substitute $x = \sqrt{3}$ in equation [1]:

$\therefore 5\sqrt{3}\times\sqrt{3}-y = 12$

$\therefore 15-y = 12$

$\therefore y = 3$

Answer: $x = \sqrt{3}, y = 3$

b $x-\sqrt{2}x < \sqrt{2}$

$\therefore x\left(1-\sqrt{2}\right) < \sqrt{2}$

As $\sqrt{2} > 1$, $1-\sqrt{2} < 0$

$$\therefore x > \frac{\sqrt{2}}{1-\sqrt{2}}$$

$$\therefore x > \frac{\sqrt{2}}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}}$$

$$\therefore x > \frac{\sqrt{2}\left(1+\sqrt{2}\right)}{1-2}$$

$$\therefore x > -\sqrt{2}\left(1+\sqrt{2}\right)$$

$$\therefore x > -\sqrt{2}-2$$

The solution set is $\left(-2-\sqrt{2}, \infty\right)$.

c $\dfrac{a^3-1}{a^3+1} - \dfrac{a^3+1}{a^3-1} + \dfrac{3a^3}{a^6-1}$

$$= \frac{a^3-1}{a^3+1} - \frac{a^3+1}{a^3-1} + \frac{3a^3}{\left(a^3-1\right)\left(a^3+1\right)}$$

$$= \frac{\left(a^3-1\right)\left(a^3-1\right)-\left(a^3+1\right)\left(a^3+1\right)+3a^3}{\left(a^3-1\right)\left(a^3+1\right)}$$

$$= \frac{a^6-2a^3+1-\left(a^6+2a^3+1\right)+3a^3}{\left(a^3-1\right)\left(a^3+1\right)}$$

$$= \frac{-a^3}{\left(a^3-1\right)\left(a^3+1\right)}$$

$$= -\frac{a^3}{a^6-1}$$

$$= \frac{a^3}{1-a^6}$$

d **i** $(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)$

$$\begin{aligned} &= a^3+ab^2+ac^2-a^2b-abc-ca^2 \\ &\quad +ba^2+b^3+bc^2-ab^2-b^2c-bca \\ &\quad +ca^2+cb^2+c^3-cab-bc^2-c^2a \\ &= a^3+b^3+c^3-3abc \end{aligned}$$

ii If $a+b+c=0$,

$$\begin{aligned} (0)\left(a^2+b^2+c^2-ab-bc-ca\right) &= a^3+b^3+c^3-3abc \\ \therefore 0 &= a^3+b^3+c^3-3abc \\ \therefore a^3+b^3+c^3 &= 3abc \end{aligned}$$

iii $102^3-100^3-2^3$

Since $102+(-100)+(-2)=0$,

$102^3+(-100)^3+(-2)^3 = 3 \times 102 \times (-100) \times (-2)$

$\therefore 102^3-100^3-2^3 = 61200$

18 **a** $\dfrac{2}{3} = \dfrac{4}{6} = \dfrac{8}{12}$ and $\dfrac{5}{6} = \dfrac{10}{12}$

As $\dfrac{8}{12} < \dfrac{9}{12} < \dfrac{10}{12}$, a rational number that lies between $\dfrac{2}{3}$ and $\dfrac{5}{6}$ is $\dfrac{3}{4}$.

b **i** Suppose $a < b$.

$\therefore a+a < a+b$ and $a+b < b+b$

$\therefore 2a < a+b < 2b$

$\therefore a < \dfrac{a+b}{2} < b$

Therefore, the real number $\dfrac{a+b}{2}$ lies between a and b, showing it is always possible to find another real number between them.

ii If $a, b \in Z$, it is not always possible to find another integer between them, because if they are consecutive integers, for example if $a=3$ and $b=4$, there is no other integer between them.

c $\left(1+\sqrt{3}\right)^6+\left(1-\sqrt{3}\right)^6$

$$\begin{aligned} &= \left(1+\binom{6}{1}\sqrt{3}+\binom{6}{2}\left(\sqrt{3}\right)^2+\binom{6}{3}\left(\sqrt{3}\right)^3+\binom{6}{4}\left(\sqrt{3}\right)^3\right. \\ &\quad \left.+\binom{6}{5}\left(\sqrt{3}\right)^4+\binom{6}{5}\left(\sqrt{3}\right)^5+\left(\sqrt{3}\right)^6\right) \\ &\quad +\left(1-\binom{6}{1}\sqrt{3}+\binom{6}{2}\left(\sqrt{3}\right)^2-\binom{6}{3}\left(\sqrt{3}\right)^3\right. \\ &\quad \left.+\binom{6}{4}\left(\sqrt{3}\right)^4-\binom{6}{5}\left(\sqrt{3}\right)^5+\left(\sqrt{3}\right)^6\right) \\ &= 2\left(1+\binom{6}{2}\left(\sqrt{3}\right)^2+\binom{6}{2}\left(\sqrt{3}\right)^4+\left(\sqrt{3}\right)^6\right) \\ &= 2\,(1+15\times 3+15\times 9+27) \\ &= 416 \end{aligned}$$

Hence, $\left(1+\sqrt{3}\right)^6+\left(1-\sqrt{3}\right)^6$ is rational.

d $\dfrac{n}{\sqrt{m}-\sqrt{2}} + \dfrac{n}{\sqrt{m}+\sqrt{2}} = 8\sqrt{3}$

$$\therefore \frac{n\left(\sqrt{m}+\sqrt{2}\right)+n\left(\sqrt{m}-\sqrt{2}\right)}{\left(\sqrt{m}-\sqrt{2}\right)\left(\sqrt{m}+\sqrt{2}\right)} = 8\sqrt{3}$$

$$\therefore \frac{n\left(\sqrt{m}+\sqrt{2}+\sqrt{m}-\sqrt{2}\right)}{m-2} = 8\sqrt{3}$$

$$\therefore \frac{2n\sqrt{m}}{m-2} = 8\sqrt{3}$$

Since m is prime, $m=3$.

$$\therefore \frac{2n}{m-2} = 8$$

$$\therefore \frac{2n}{3-2} = 8$$

$$\therefore n = 4$$

Answer: $m=3,\ n=4$

19 **a** A square of edge $80\sqrt{2}$ metres has a perimeter of $4 \times 80\sqrt{2} = 320\sqrt{2}$ metres. The friends walk $0.320\sqrt{2}$ kilometres in 20 minutes or $\dfrac{1}{3}$ hour.

$$\frac{\text{distance}}{\text{time}}$$

$$= \frac{0.32\sqrt{2}}{\frac{1}{3}}$$

$$= 0.96\sqrt{2}$$

The average walking speed was $0.96\sqrt{2}$ km/h.

b The diameter of the circle is $80\sqrt{2}$ metres. Its circumference is given by

$C = \pi d$

$= 80\pi\sqrt{2}$

The friends walk two circuits of the circle, which is a distance of $160\pi\sqrt{2}$ metres or $0.16\pi\sqrt{2}$ kilometres at an average speed of $\sqrt{8}$ km/h.

The time taken in hours is $\dfrac{0.16\pi\sqrt{2}}{\sqrt{8}}$.

In minutes, the time taken is

$$\frac{0.16\pi\sqrt{2}}{\sqrt{8}} \times 60$$

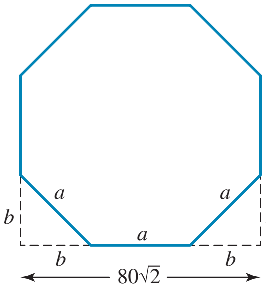
$$= \frac{0.16\pi\sqrt{2}}{2\sqrt{2}} \times 60$$

$$= 0.08\pi \times 60$$

$$= 4.8\pi$$

The time taken is 4.8π minutes.

c



Let the octagonal path have edge length a metres, so its perimeter is $8a$ metres.

Let b metres be the edge length of the corner of the square cut off by the octagon.

The edge of the square is $b + a + b$, so $a + 2b = 80\sqrt{2}$.

Using Pythagoras' theorem,

$$a^2 = b^2 + b^2$$

$$\therefore a^2 = 2b^2$$

$$\therefore a = \sqrt{2}b$$

$$\therefore b = \frac{a}{\sqrt{2}}$$

$$\therefore b = \frac{\sqrt{2}a}{2}$$

Substitute $b = \dfrac{\sqrt{2}a}{2}$ into $a + 2b = 80\sqrt{2}$:

$$\therefore a + 2 \times \frac{\sqrt{2}a}{2} = 80\sqrt{2}$$

$$\therefore a\left(1 + \sqrt{2}\right) = 80\sqrt{2}$$

$$\therefore a = \frac{80\sqrt{2}}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$$

$$\therefore a = 80\sqrt{2}\left(\sqrt{2}-1\right)$$

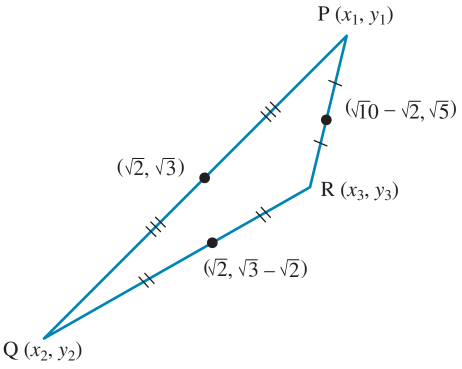
$$\therefore a = 160 - 80\sqrt{2}$$

As the perimeter is $8a$ metres, the perimeter is

$$8\left(160 - 80\sqrt{2}\right) = \left(1280 - 640\sqrt{2}\right)$$

$$= 640\left(2 - \sqrt{2}\right) \text{ metres.}$$

d i



Let the vertices of the triangle be $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$.

The midpoint of PQ has coordinates $\left(\sqrt{2}, \sqrt{3}\right)$.

$$\therefore \frac{x_1 + x_2}{2} = \sqrt{2} \text{ and } \frac{y_1 + y_2}{2} = \sqrt{3}$$

$$\therefore x_1 + x_2 = 2\sqrt{2} \quad [1] \text{ and } y_1 + y_2 = 2\sqrt{3} \quad [2]$$

The midpoint of QR has coordinates $\left(\sqrt{2}, \sqrt{3} - \sqrt{2}\right)$.

$$\therefore \frac{x_2 + x_3}{2} = \sqrt{2} \text{ and } \frac{y_2 + y_3}{2} = \sqrt{3} - \sqrt{2}$$

$$\therefore x_2 + x_3 = 2\sqrt{2} \quad [3] \text{ and } y_2 + y_3 = 2\sqrt{3} - 2\sqrt{2} \quad [4]$$

The midpoint of PR has coordinates $\left(\sqrt{10} - \sqrt{2}, \sqrt{5}\right)$.

$$\therefore \frac{x_1 + x_3}{2} = \sqrt{10} - \sqrt{2} \text{ and } \frac{y_1 + y_3}{2} = \sqrt{5}$$

$$\therefore x_1 + x_2 = 2\sqrt{10} - 2\sqrt{2} \quad [5] \text{ and } y_1 + y_3 = 2\sqrt{5} \quad [6]$$

Solving for the x-coordinates:

$$x_1 + x_2 = 2\sqrt{2} \quad [1]$$

$$x_2 + x_3 = 2\sqrt{2} \quad [3]$$

$$x_1 + x_3 = 2\sqrt{10} - 2\sqrt{2} \quad [5]$$

Equation [1] − equation [3]:

$$x_1 - x_3 = 0$$

$$\therefore x_1 = x_3$$

Substitute in equation [5]:

$$\therefore x_3 + x_3 = 2\sqrt{10} - 2\sqrt{2}$$

$$\therefore x_3 = \sqrt{10} - \sqrt{2}$$

$$\therefore x_1 = \sqrt{10} - \sqrt{2}$$

Equation [3] becomes:

$$x_2 + \sqrt{10} - \sqrt{2} = 2\sqrt{2}$$

$$\therefore x_2 = 3\sqrt{2} - \sqrt{10}$$

Solving for the y-coordinates:

$$y_1 + y_2 = 2\sqrt{3} \quad [2]$$

$$y_2 + y_3 = 2\sqrt{3} - 2\sqrt{2} \quad [4]$$

$$y_1 + y_3 = 2\sqrt{5} \quad [6]$$

Equation [2] − equation [6]:

$$y_2 - y_3 = 2\sqrt{3} - 2\sqrt{5} \quad [7]$$

Equation [4] + equation [7]:

$$2y_2 = 4\sqrt{3} - 2\sqrt{2} - 2\sqrt{5}$$

$$\therefore y_2 = 2\sqrt{3} - \sqrt{2} - \sqrt{5}$$

Substitute $y_2 = 2\sqrt{3} - \sqrt{2} - \sqrt{5}$ in equation [2]:

$$\therefore y_1 + 2\sqrt{3} - \sqrt{2} - \sqrt{5} = 2\sqrt{3}$$

$$\therefore y_1 = \sqrt{2} + \sqrt{5}$$

Substitute $y_1 = \sqrt{2} + \sqrt{5}$ in equation [6]:

$$\therefore \sqrt{2} + \sqrt{5} + y_3 = 2\sqrt{5}$$

$$\therefore y_3 = \sqrt{5} - \sqrt{2}$$

The vertices of the triangle are

$P\left(\sqrt{10} - \sqrt{2}, \sqrt{5} + \sqrt{2}\right)$,

$Q\left(3\sqrt{2} - \sqrt{10}, 2\sqrt{3} - \sqrt{5} - \sqrt{2}\right)$ and

$R\left(\sqrt{10} - \sqrt{2}, \sqrt{5} - \sqrt{2}\right)$.

ii $P\left(\sqrt{10} - \sqrt{2}, \sqrt{5} + \sqrt{2}\right)$ and $R\left(\sqrt{10} - \sqrt{2}, \sqrt{5} - \sqrt{2}\right)$ have the same x-cooordinates. The distance between these points is $\left(\sqrt{5} + \sqrt{2}\right) - \left(\sqrt{5} - \sqrt{2}\right) = 2\sqrt{2}$ km.

20 **a** Expand $\left(1-2x+3x^2\right)^5$ as far as the term in x^3.

$$\left(1-2x+3x^2\right)^5$$
$$=\left((1-2x)+3x^2\right)^5$$
$$=(1-2x)^5+\binom{5}{1}(1-2x)^4\left(3x^2\right)+\binom{5}{2}(1-2x)^3\left(3x^2\right)^2$$
$$+\ldots..+\left(3x^2\right)^5$$
$$=(1-2x)^5+15x^2(1-2x)^4+90x^4(1-2x)^3+\ldots..+\left(3x^2\right)^5$$

The term in x^3 is the sum of the term in x^3 from $(1-2x)^5$ and $15x^2 \times$ the term in x from $(1-2x)^4$.

Consider $(1-2x)^5$.

$$t_{r+1}=\binom{5}{r}1^{5-r}(-2x)^r$$
$$=\binom{5}{r}(-2)^r x^r$$

For x^3, put $r=3$.

$$\therefore t_4=\binom{5}{3}(-2)^3x^3$$
$$=-80x^3$$

Consider $(1-2x)^4$.

$$t_{r+1}=\binom{4}{r}1^{4-r}(-2x)^r$$
$$=\binom{4}{r}(-2)^r x^r$$

For x, put $r=1$.

$$\therefore t_2=\binom{4}{1}(-2)x$$
$$=-8x$$

Therefore, the required term in x^3 is $-80x^3+15x^2\times-8x=-200x^3$.

The coefficient of x^3 in the expansion of $\left(1-2x+3x^2\right)^5$ is -200.

b $(x+1)^{17}$

i $t_{r+1}=\binom{17}{r}x^{17-r}1^r$

$$=\binom{17}{r}x^{17-r}$$

The coefficient of the $(r+1)$th term is $\binom{17}{r}$.

$$t_{r+2}=\binom{17}{r+1}x^{16-r}1^{r+1}$$
$$=\binom{17}{r+1}x^{16-r}$$

The coefficient of the $(r+2)$th term is $\binom{17}{r+1}$.

The ratio of the coefficients is

$$\binom{17}{r}:\binom{17}{r+1}$$
$$=\frac{17!}{r!\,(17-r)!}\div\frac{17!}{(r+1)!\,(16-r)!}$$
$$=\frac{17!}{r!\,(17-r)!}\times\frac{(r+1)!\,(16-r)!}{17!}$$
$$=\frac{(r+1)r!\,(16-r)!}{r!\,(17-r)(16-r)!}$$
$$=\frac{r+1}{17-r}$$

ii $\dfrac{r+1}{17-r}=\dfrac{2}{1}$

$$\therefore r+1=34-2r$$
$$\therefore 3r=33$$
$$\therefore r=11$$

The $(r+1)$th term is the 12th term and the $(r+2)$th term is the 13th term.

$t_{12}=\binom{17}{11}x^6$ and $t_{13}=\binom{17}{12}x^5$ or

$t_{12}=12\,376x^6,\ t_{13}=6188x^5$.

iii For $(x+1)^n$, replacing 17 by n gives the ratio of the coefficients of the consecutive $(r+1)$th and $(r+2)$th terms as $\binom{n}{r}:\binom{n}{r+1}$.

$$\therefore \frac{r+1}{n-r}=\frac{2}{1}$$
$$\therefore r+1=2n-2r$$
$$\therefore 3r=2n-1$$

Since $r\in N$, we need the values of n so that integer multiples of 3 are odd and $n<17$.

Odd multiples of 3 are 3, 9, 15, 21...

If $r=1$,

$3=2n-1$

$\therefore n=2$

If $r=3$,

$9=2n-1$

$\therefore n=5$

If $r=5$,

$15=2n-1$

$\therefore n=8$

As r increases by 2, n increases by 3.

Therefore, the values of n could be 2, 5, 8, 11, 14.

2.7 Exam questions

1 $$\frac{(a-b)^3}{a^3-b^3}=\frac{\cancel{(a-b)}(a-b)^2}{\cancel{(a-b)}\left(a^2+ab+b^2\right)}$$
$$=\frac{(a-b)^2}{a^2+ab+b^2}$$

The correct answer is **C**.

2 $$\frac{p^2-9q^2}{(p-3q)^2}\div\frac{(p+3q)^2}{4p-12q}$$
$$=\frac{(p-3q)(p+3q)}{(p-3q)(p-3q)}\times\frac{4(p-3q)}{(p+3q)(p+3q)}$$
$$=\frac{4}{p+3q}$$

[1 mark]

3 R is the set of real numbers, and both π and $\sqrt{169}\,(=13)$ belong to the set of real numbers.

Z is the set of integers and $\sqrt{169}\,(=13)$ is an integer.

The correct answer is **C**.

4 

[1 mark]

$\{x : -3 < x \leq 1\}$ [1 mark]

5
$$\begin{aligned}\frac{5\sqrt{6}-2\sqrt{2}}{\sqrt{8}} &= \frac{5\sqrt{6}-2\sqrt{2}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}\left(5\sqrt{6}-2\sqrt{2}\right)}{2\times 2}\\ &= \frac{5\sqrt{12}-4}{4}\\ &= \frac{10\sqrt{3}-4}{4}\\ &= \frac{2\left(5\sqrt{3}-2\right)}{4}\\ &= \frac{5\sqrt{3}-2}{2}\end{aligned}$$

The correct answer is **B**.

Topic 3 — Quadratic relationships

3.2 Quadratic equations with rational roots

3.2 Exercise

1 a $3x(5-x)=0$

$\therefore 3x=0 \quad \text{or} \quad 5-x=0$

$\therefore x=0, x=5$

b $(3-x)(7x-1)=0$

$\therefore 3-x=0 \quad \text{or} \quad 7x-1=0$

$\therefore x=3, x=\frac{1}{7}$

c $(x+8)^2=0$

$\therefore x=-8$

d $2(x+4)(6+x)=0$

$\therefore x=-4,\ x=-6$

2 a $(3x-4)(2x+1)=0$

$3x-4=0 \quad \text{or} \quad 2x+1=0$

$3x=4 \quad \text{or} \quad 2x=-1$

$x=\frac{4}{3} \quad \text{or} \quad x=-\frac{1}{2}$

$x=\frac{4}{3},-\frac{1}{2}$

b $x^2-7x+12=0$

$(x-4)(x-3)=0$

$x=4,3$

c $8x^2+26x+21=0$

$(2x+3)(4x+7)=0$

$x=-\frac{3}{2},-\frac{7}{4}$

d $10x^2-2x=0$

$2x(5x-1)=0$

$x=0,\frac{1}{5}$

e $12x^2+40x-32=0$

$4(3x^2+10x-8)=0$

$4(3x-2)(x+4)=0$

$x=\frac{2}{3},-4$

f $\frac{1}{2}x^2-5x=0$

$\frac{1}{2}x(x-10)=0$

$x=0,10$

3 a $10x^2+23x=21$

$10x^2+23x-21=0$

$\therefore (10x-7)(x+3)=0$

$\therefore x=\frac{7}{10},\ x=-3$

b Zero of $x=-5 \;\Rightarrow\; (x-(-5))=(x+5)$ is a factor and zero of $x=0 \;\Rightarrow\; (x-0)=x$ is a factor.

The quadratic takes the form $(x+5)x=x^2+5x$.

4 a $6x^2+5x+1=0$

$\therefore (3x+1)(2x+1)=0$

$\therefore 3x+1=0 \quad \text{or} \quad 2x+1=0$

$\therefore x=-\frac{1}{3},x=-\frac{1}{2}$

b $12x^2-7x=10$

$\therefore 12x^2-7x-10=0$

$\therefore (4x-5)(3x+2)=0$

$\therefore 4x-5=0 \quad \text{or} \quad 3x+2=0$

$\therefore x=\frac{5}{4},x=-\frac{2}{3}$

c $49=14x-x^2$

$\therefore x^2-14x+49=0$

$\therefore (x-7)^2=0$

$\therefore x=7$

d $5x+25-30x^2=0$

$\therefore -5(6x^2-x-5)=0$

$\therefore 6x^2-x-5=0$

$\therefore (6x+5)(x-1)=0$

$\therefore 6x+5=0 \quad \text{or} \quad x-1=0$

$\therefore x=-\frac{5}{6},x=1$

5 $(5x-1)^2-16=0$

$\therefore\ (5x-1)^2=16$

$\therefore 5x-1=\pm\sqrt{16}$

$\therefore 5x-1=4 \quad \text{or} \quad 5x-1=-4$

$\therefore 5x=5 \qquad 5x=-3$

$\therefore x=1,-\frac{3}{5}$

6 a $(x+2)^2=9$

$x+2=\pm\sqrt{9}$

$x=\pm3-2$

$x=-5,1$

b $(x-1)^2-25=0$

$(x-1)^2=25$

$x-1=\pm\sqrt{25}$

$x=\pm5+1$

$x=-4,6$

c $(x-7)^2+4=0$

$(x-7)^2=-4$

$x-7=\pm\sqrt{-4}$

No real solutions

d $(2x+11)^2=81$

$2x+11=\pm\sqrt{81}$

$2x+11=\pm9$

$2x=\pm9-11$

$2x=-20,-2$

$x=-\frac{20}{2},-\frac{2}{2}$

$x=-10,-1$

e $(7-x)^2=0$

$7-x=\pm\sqrt{0}$

$7-x=0$

$x=7$

f $8 - \frac{1}{2}(x-4)^2 = 0$

$\frac{1}{2}(x-4)^2 = 8$

$(x-4)^2 = 16$

$x - 4 = \pm\sqrt{16}$

$x = \pm 4 + 4$

$x = 0, 8$

7 a $x^2 = 121$

$\therefore x = \pm 11$

b $9x^2 = 16$

$\therefore x^2 = \frac{16}{9}$

$\therefore x = \pm\frac{4}{3}$

c $(x-5)^2 = 1$

$\therefore x - 5 = \pm 1$

$\therefore x = 1 + 5 \quad \text{or} \quad x = -1 + 5$

$\therefore x = 6, x = 4$

d $(5-2x)^2 - 49 = 0$

$\therefore \ [(5-2x) - 7]\,[(5-2x) + 7] = 0$

$\therefore (-2-2x)(12-2x) = 0$

$\therefore -2 - 2x = 0 \quad \text{or} \quad 12 - 2x = 0$

$\therefore -2 = 2x \quad \text{or} \quad 12 = 2x$

$\therefore x = -1, x = 6$

e $2(3x-1)^2 - 8 = 0$

$\therefore 2\,(3x-1)^2 = 8$

$\therefore \ (3x-1)^2 = 4$

$\therefore 3x - 1 = \pm 2$

$\therefore 3x = 3 \quad \text{or} \quad 3x = -1$

$\therefore x = 1, \ x = -\frac{1}{3}$

f $\left(x^2+1\right)^2 = 100$

$\therefore \ \left(x^2+1\right) = \pm 10$

$\therefore x^2 + 1 = 10 \quad \text{or} \quad x^2 + 1 = -10$

$\therefore x^2 = 9 \quad \text{or} \quad x^2 = -11$

Reject $x^2 = -11$ since x^2 cannot be negative.

$\therefore x^2 = 9$

$\therefore x = \pm 3$

8 $9x^4 + 17x^2 - 2 = 0$

Let $u = x^2$.

$9u^2 + 17u - 2 = 0$

$\therefore (9u-1)(u+2) = 0$

$\therefore u = \frac{1}{9}, u = -2$

Substitute back for x^2.

$x^2 = \frac{1}{9}$

$\therefore x = \pm\sqrt{\frac{1}{9}}$ or $x^2 = -2$; no real solutions, therefore reject

$\therefore x = \pm\frac{1}{3}$

9 a $18(x-3)^2 + 9(x-3) - 2 = 0$

Let $x - 3 = u$.

$18u^2 + 9u - 2 = 0$

$(6u-1)(3u+2) = 0$

Substitute $x - 3 = u$.

$(6(x-3)-1)(3(x-3)+2) = 0$

$(6x-18-1)(3x-9+2) = 0$

$(6x-19)(3x-7) = 0$

$x = \frac{19}{6}, \frac{7}{3}$

b $5(x+2)^2 + 23(x+2) + 12 = 0$

Let $x + 2 = u$.

$5u^2 + 23u + 12 = 0$

$(u+4)(5u+3) = 0$

Substitute $x + 2 = u$.

$(x+2+4)(5(x+2)+3) = 0$

$(x+6)(5x+10+3) = 0$

$(x+6)(5x+13) = 0$

$x = -6, -\frac{13}{5}$

c $x + 6 + \frac{8}{x} = 0$

$x \times \left(x + 6 + \frac{8}{x}\right) = x \times 0$

$x^2 + 6x + 8 = 0$

$(x+2)(x+4) = 0$

$x = -2, -4$

d $2x + \frac{3}{x} = 7$

$x \times \left(2x + \frac{3}{x}\right) = x \times 7$

$2x^2 + 3 = 7x$

$2x^2 - 7x + 3 = 0$

$(2x-1)(x-3) = 0$

$x = \frac{1}{2}, 3$

10 a $(3x+4)^2 + 9(3x+4) - 10 = 0$

Let $u = 3x + 4$.

$\therefore u^2 + 9u - 10 = 0$

$\therefore (u+10)(u-1) = 0$

$\therefore u = -10 \quad \text{or} \quad u = 1$

$\therefore 3x + 4 = -10 \quad \text{or} \quad 3x + 4 = 1$

$\therefore 3x = -14 \quad \text{or} \quad 3x = -3$

$\therefore x = -\frac{14}{3}, x = -1$

b $2(1+2x)^2 + 9(1+2x) = 18$

Let $u = 1 + 2x$.

$\therefore 2u^2 + 9u = 18$

$\therefore 2u^2 + 9u - 18 = 0$

$\therefore (2u-3)(u+6) = 0$

$\therefore u = \frac{3}{2}$ or $u = -6$

$\therefore 1 + 2x = \frac{3}{2}$ or $1 + 2x = -6$

$\therefore 2x = \frac{1}{2} \quad \text{or} \quad 2x = -7$

$\therefore x = \frac{1}{4}, x = -\frac{7}{2}$

c $x^2 - 29x^2 + 100 = 0$

Let $u = x^2$.

$\therefore u^2 - 29u + 100 = 0$

$\therefore (u - 25)(u - 4) = 0$

$\therefore u = 25 \text{ or } u = 4$

$\therefore x^2 = 25 \text{ or } x^2 = 4$

$\therefore x = \pm 5, x = \pm 2$

d $2x^4 = 31x^2 + 16$

Let $u = x^2$.

$\therefore 2u^2 = 31u + 16$

$\therefore 2u^2 - 31u - 16 = 0$

$\therefore (2u + 1)(u - 16) = 0$

$\therefore u = -\frac{1}{2} \text{ or } u = 16$

$\therefore x^2 = -\frac{1}{2} \text{ or } x^2 = 16$

Reject $x^2 = -\frac{1}{2}$.

$\therefore x^2 = 16$

$\therefore x = \pm 4$

11 a $x(x - 7) = 8$

$\therefore x^2 - 7x = 8$

$\therefore x^2 - 7x - 8 = 0$

$\therefore (x - 8)(x + 1) = 0$

$\therefore x = 8, x = -1$

b $4x(3x - 16) = 3(4x - 33)$

$\therefore 12x^2 - 64x = 12x - 99$

$\therefore 12x^2 - 76x + 99 = 0$

$\therefore (6x - 11)(2x - 9) = 0$

$\therefore x = \frac{11}{6}, x = \frac{9}{2}$

c $(x + 4)^2 + 2x = 0$

$\therefore x^2 + 8x + 16 + 2x = 0$

$\therefore x^2 + 10x + 16 = 0$

$\therefore (x + 8)(x + 2) = 0$

$\therefore x = -8, x = -2$

d $(2x + 5)(2x - 5) + 25 = 2x$

$\therefore 4x^2 - 25 + 25 = 2x$

$\therefore 4x^2 - 2x = 0$

$\therefore 2x(2x - 1) = 0$

$\therefore x = 0, x = \frac{1}{2}$

12 a $2 - 3x = \frac{1}{3x}$

$\therefore 3x(2 - 3x) = 1$

$\therefore 6x - 9x^2 = 1$

$\therefore 9x^2 - 6x + 1 = 0$

$\therefore (3x - 1)^2 = 0$

$\therefore x = \frac{1}{3}$

b $\frac{4x + 5}{x + 125} = \frac{5}{x}$

$\therefore x(4x + 5) = 5(x + 125)$

$\therefore 4x^2 + 5x = 5x + 625$

$\therefore 4x^2 - 625 = 0$

$\therefore (2x - 25)(2x + 25) = 0$

$\therefore x = \frac{25}{2}, x = -\frac{25}{2}$

c $7x - \frac{2}{x} + \frac{11}{5} = 0$

$\therefore \frac{35x^2 - 10 + 11x}{5x} = 0$

$\therefore 35x^2 + 11x - 10 = 0$

$\therefore (7x + 5)(5x - 2) = 0$

$\therefore x = -\frac{5}{7}, x = \frac{2}{5}$

d $\frac{12}{x + 1} - \frac{14}{x - 2} = 19$

$\therefore \frac{12(x - 2) - 14(x + 1)}{(x + 1)(x - 2)} = 19$

$\therefore 12(x - 2) - 14(x + 1) = 19(x + 1)(x - 2)$

$\therefore 12x - 24 - 14x - 14 = 19(x^2 - x - 2)$

$\therefore -2x - 38 = 19x^2 - 19x - 38$

$\therefore 0 = 19x^2 - 17x$

$\therefore x(19x - 17) = 0$

$\therefore x = 0, x = \frac{17}{19}$

13 a $x^4 = 81$

$\therefore x^2 = \pm 9$

Reject $x^2 = -9$.

$\therefore x^2 = 9$

$\therefore x = \pm 3$

b $(9x^2 - 16)^2 = 20(9x^2 - 16)$

Let $u = 9x^2 - 16$.

$\therefore u^2 = 20u$

$\therefore u^2 - 20u = 0$

$\therefore u(a - 20) = 0$

$\therefore u = 0 \text{ or } u = 20$

$\therefore 9x^2 - 16 = 0 \text{ or } 9x^2 - 16 = 20$

$\therefore (3x - 4)(3x + 4) = 0 \text{ or } 9x^2 = 36$

$\therefore x = \frac{4}{3}, x = -\frac{4}{3} \text{ or } x^2 = 4$

$\therefore x = \pm\frac{4}{3}, x = \pm 2$

14 a $\left(x - \frac{2}{x}\right)^2 - 2\left(x - \frac{2}{x}\right) + 1 = 0$

Let $u = x - \frac{2}{x}$.

$\therefore u^2 - 2u + 1 = 0$

$\therefore (u - 1)^2 = 0$

$\therefore u = 1$

$\therefore x - \frac{2}{x} = 1$

$$\therefore x^2 - 2 = x$$
$$\therefore x^2 - x - 2 = 0$$
$$\therefore (x-2)(x+1) = 0$$
$$\therefore x = 2, x = -1$$

b $2\left(1+\frac{3}{x}\right)^2 + 5\left(1+\frac{3}{x}\right) + 3 = 0$

Let $u = 1 + \frac{3}{x}$.

$$\therefore 2u^2 + 5u + 3 = 0$$
$$\therefore (2u+3)(u+1) = 0$$
$$\therefore u = -\frac{3}{2} \quad \text{or} \quad u = -1$$
$$\therefore 1 + \frac{3}{x} = -\frac{3}{2} \quad \text{or} \quad 1 + \frac{3}{x} = -1$$
$$\therefore \frac{3}{x} = -\frac{5}{2} \quad \text{or} \quad \frac{3}{x} = -2$$
$$\therefore 6 = -5x \quad \text{or} \quad 3 = -2x$$
$$\therefore x = -\frac{6}{5}, x = -\frac{3}{2}$$

c $\left(x+\frac{1}{x}\right)^2 - 4\left(x+\frac{1}{x}\right) + 4 = 0$

Let $u = x + \frac{1}{x}$.

$$\therefore u^2 - 4u + 4 = 0$$
$$\therefore (u-2)^2 = 0$$
$$\therefore u = 2$$

Substitute back.

$$\therefore x + \frac{1}{x} = 2$$
$$\therefore x^2 + 1 = 2x$$
$$\therefore x^2 - 2x + 1 = 0$$
$$\therefore (x-1)^2 = 0$$
$$\therefore x = 1$$

15 $(px+q)^2 = r^2$

$$\therefore (px+q) = \pm\sqrt{r^2}$$
$$\therefore px + q = r \text{ or } px + q = -r$$
$$\therefore px = r - q \; px = -r - q$$
$$\therefore x = \frac{r-q}{p}, x = -\frac{r+q}{p}$$

16 **a** $(x-2b)(x+3a) = 0$

$$\therefore x = 2b, x = -3a$$

b $2x^2 - 13ax + 15a^2 = 0$

$$\therefore (2x-3a)(x-5a) = 0$$
$$\therefore x = \frac{3a}{2}, x = 5a$$

c $(x-a-b)^2 = 4b^2$

$$\therefore x - a - b = \pm 2b$$
$$\therefore x = 2b + a + b \text{ or } x = -2b + a + b$$
$$\therefore x = a + 3b, x = a - b$$

d $(x+a)^2 - 3b(x+a) + 2b^2 = 0$

Let $u = x + a$.

$$\therefore u^2 - 3bu + 2b^2 = 0$$
$$\therefore (u-2b)(u-b) = 0$$
$$\therefore u = 2b \text{ or } u = b$$
$$\therefore x + a = 2b \text{ or } x + a = b$$
$$\therefore x = 2b - a, x = b - a$$

17 $(x-\alpha)(x-\beta) = 0$

a If the roots are $x = 1, x = 7$, the equation must be $(x-1)(x-7) = 0$.

b If the roots are $x = -5, x = 4$, the equation must be $(x+5)(x-4) = 0$.

c If the roots are $x = 0, x = 10$, the equation must be $(x-0)(x-10) = 0$.
$\therefore x(x-10) = 0$.

d If the quadratic equation has one root only of $x = 2$, the equation must be $(x-2)(x-2) = 0$.
$\therefore (x-2)^2 = 0$.

18 As the zeros of $4x^2 + bx + c$ are $x = -4$ and $x = \frac{3}{4}$, then $(x+4)$ and $\left(x - \frac{3}{4}\right)$ are factors of $4x^2 + bx + c$.

However, the coefficient of x^2 must be 4, so
$4x^2 + bx + c = 4(x+4)\left(x - \frac{3}{4}\right)$.

$$\therefore 4x^2 + bx + c = 4\left(x - \frac{3}{4}\right)(x+4)$$
$$= (4x-3)(x+4)$$
$$\therefore 4x^2 + bx + c = 4x^2 + 13x - 12$$

Hence, $b = 13$ and $c = -12$.

19 **a** $60x^2 + 113x - 63 = 0$

Solve using Interactive → Equation/inequality on Standard mode.

$$\therefore x = -\frac{7}{3}, x = \frac{9}{20}$$

b $4x(x-7) + 8(x-3)^2 = x - 26$

$$\therefore x = \frac{7}{4}, x = \frac{14}{3}$$

20 **a** The roots are the solutions.

$$32x^2 - 96x + 72 = 0$$
$$\therefore 8(4x^2 - 12x + 9) = 0$$
$$\therefore 4x^2 - 12x + 9 = 0$$
$$\therefore (2x-3)^2 = 0$$
$$\therefore 2x - 3 = 0$$
$$\therefore x = \frac{3}{2}$$

b $44 + 44x^2 = 250x$

$$\therefore 44x^2 - 250x + 44 = 0$$
$$\therefore 2(22x^2 - 125x + 22) = 0$$
$$\therefore 22x^2 - 125x + 22 = 0$$
$$\therefore (11x-2)(2x-11) = 0$$
$$\therefore x = \frac{2}{11}, x = \frac{11}{2}$$

3.2 Exam questions

1 $(4a+5)(3a-4) = 12a^2 - 16a + 15a - 20$

$$= 12a^2 - a - 20$$
$$= (4a+5)(3a-4)$$

The correct answer is **B**.

2 $3x^2 - 243 = 0$

$$x^2 - 81 = 0$$
$$(x-9)(x+9) = 0$$
$$x = \pm 9$$

The correct answer is **D**.

3 Let $(a-1)=x$. [1 mark]

$2x^2-7x+6=0$

$(2x-3)(x-2)=0$

$2x-3=0, x-2=0$

$x=\frac{3}{2}, \quad x=2$ [1 mark]

Substitute back for $x=a-1$.

$a-1=\frac{3}{2}, \quad a-1=2$

$\therefore a=\frac{5}{2}, \quad a=3$ [1 mark]

3.3 Quadratics over *R*

3.3 Exercise

1 **a** $x^2+10x+25=(x+5)^2$

b $x^2-7x+\left(\frac{7}{2}\right)^2=\left(x-\frac{7}{2}\right)^2$

$\therefore x^2-7x+\frac{49}{4}=\left(x-\frac{7}{2}\right)^2$

c $x^2+x+\left(\frac{1}{2}\right)^2=\left(x+\frac{1}{2}\right)^2$

$\therefore x^2+x+\frac{1}{4}=\left(x+\frac{1}{2}\right)^2$

2 **a** $x^2-10x-7$

$=(x^2-10x+25)-25-7$

$=(x-5)^2-32$

$=\left(x-5-\sqrt{32}\right)\left(x-5+\sqrt{32}\right)$

$=\left(x-5-4\sqrt{2}\right)\left(x-5+4\sqrt{2}\right)$

b $3x^2+7x+3$

$=3\left(x^2+\frac{7}{3}x+1\right)$

$=3\left(x^2+\frac{7}{3}x+\left(\frac{7}{6}\right)^2-\left(\frac{7}{6}\right)^2+1\right)$

$=3\left[\left(x+\frac{7}{6}\right)^2-\frac{49}{36}+1\right]$

$=3\left[\left(x+\frac{7}{6}\right)^2-\frac{49}{36}+\frac{36}{36}\right]$

$=3\left[\left(x+\frac{7}{6}\right)^2-\frac{13}{36}\right]$

$=3\left(x+\frac{7}{6}-\sqrt{\frac{13}{36}}\right)\left(x+\frac{7}{6}+\sqrt{\frac{13}{36}}\right)$

$=3\left(x+\frac{7}{6}-\frac{\sqrt{3}}{6}\right)\left(x+\frac{7}{6}+\frac{\sqrt{3}}{6}\right)$

$=3\left(x+\frac{7-\sqrt{13}}{6}\right)\left(x+\frac{7+\sqrt{13}}{6}\right)$

c $5x^2-9$

$=\left(\sqrt{5}x\right)^2-3^2$

$=\left(\sqrt{5}x-3\right)\left(\sqrt{5}x+3\right)$

3 $3x^2-8x+5$

$=3\left(x^2-\frac{8x}{3}+\frac{5}{3}\right)$

$=3\left(\left(x^2-\frac{8x}{3}+\left(\frac{4}{3}\right)^2\right)-\left(\frac{4}{3}\right)^2+\frac{5}{3}\right)$

$=3\left(\left(x-\frac{4}{3}\right)^2-\frac{16}{9}+\frac{5}{3}\right)$

$=3\left(\left(x-\frac{4}{3}\right)^2-\frac{16}{9}+\frac{15}{9}\right)$

$=3\left(\left(x-\frac{4}{3}\right)^2-\frac{1}{9}\right)$

$=3\left[\left(x-\frac{4}{3}-\frac{1}{3}\right)\right]\left[\left(x-\frac{4}{3}+\frac{1}{3}\right)\right]$

$=3\left(x-\frac{5}{3}\right)\left(x-\frac{3}{3}\right)$

$=3\left(x-\frac{5}{3}\right)(x-1)$

$=(3x-5)(x-1)$

Factorisation by inspection gives

$=3x^2-8x+5$

$=(3x-5)(x-1)$

4 **a** $3(x-8)^2-6$

$=3\left[(x-8)^2-2\right]$

$=3\left(x-8-\sqrt{2}\right)\left(x-8+\sqrt{2}\right)$

b $(x-7)^2+9$ is the sum of two squares so does not factorise over *R*.

5 **a** $x^2-6x+7=x^2-6x+\left(\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2+7$

$=x^2-6x+(3)^2-(3)^2+7$

$=(x-3)^2-9+7$

$=(x-3)^2-2$

$=\left(x-3-\sqrt{2}\right)\left(x-3+\sqrt{2}\right)$

b $x^2+4x-3=x^2+4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2-3$

$=x^2+4x+(2)^2-(2)^2-3$

$=(x+2)^2-4-3$

$=(x+2)^2-7$

$=\left(x+2-\sqrt{7}\right)\left(x+2+\sqrt{7}\right)$

c $x^2-2x+6=x^2-2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+6$

$=x^2-2x+(1)^2-(1)^2+6$

$=(x-1)^2-1+6$

$=(x-1)^2+5$

The sum of two squares cannot be factorised over *R*.

d $$\begin{aligned}2x^2+5x-2&=2\left(x^2+\frac{5}{2}x-1\right)\\&=2\left(x^2+\frac{5}{2}x+\left(\frac{5}{4}\right)^2-\left(\frac{5}{4}\right)^2-1\right)\\&=2\left(\left(x+\frac{5}{4}\right)^2-\frac{25}{16}-\frac{16}{16}\right)\\&=2\left(\left(x+\frac{5}{4}\right)^2-\frac{41}{16}\right)\\&=2\left(x+\frac{5}{4}-\frac{\sqrt{41}}{4}\right)\left(x+\frac{5}{4}+\frac{\sqrt{41}}{4}\right)\end{aligned}$$

e $$\begin{aligned}-x^2+8x-8&=-(x^2-8x+8)\\&=-\left(x^2-8x+\left(\frac{8}{2}\right)^2-\left(\frac{8}{2}\right)^2+8\right)\\&=-\left(x^2-8x+(4)^2-(4)^2+8\right)\\&=-\left((x-4)^2-16+8\right)\\&=-\left((x-4)^2-8\right)\\&=-\left(x-4-\sqrt{8}\right)\left(x-4+\sqrt{8}\right)\\&=-\left(x-4-2\sqrt{2}\right)\left(x-4+2\sqrt{2}\right)\end{aligned}$$

f $$\begin{aligned}3x^2+4x-6&=3\left(x^2+\frac{4}{3}x-2\right)\\&=3\left(x^2+\frac{4}{3}x+\left(\frac{4}{6}\right)^2-\left(\frac{4}{6}\right)^2-2\right)\\&=3\left(\left(x+\frac{4}{6}\right)^2-\frac{16}{36}-\frac{72}{36}\right)\\&=3\left(\left(x+\frac{2}{3}\right)^2-\frac{22}{9}\right)\\&=3\left(x+\frac{2}{3}-\frac{\sqrt{22}}{3}\right)\left(x+\frac{2}{3}+\frac{\sqrt{22}}{3}\right)\end{aligned}$$

6 **a** $$\begin{aligned}x^2-12&=\left(x-\sqrt{12}\right)\left(x+\sqrt{12}\right)\\&=\left(x-2\sqrt{3}\right)\left(x+2\sqrt{3}\right)\end{aligned}$$

b $$\begin{aligned}x^2-12x+4&=\left(x^2-12x+36\right)-36+4\\&=(x-6)^2-32\\&=\left(x-6-\sqrt{32}\right)\left(x-6+\sqrt{32}\right)\\&=\left(x-6-4\sqrt{2}\right)\left(x-6+4\sqrt{2}\right)\end{aligned}$$

c $$\begin{aligned}x^2+9x-3&=\left(x^2+9x+\left(\frac{9}{2}\right)^2\right)-\left(\frac{9}{2}\right)^2-3\\&=\left(x+\frac{9}{2}\right)^2-\frac{81}{4}-\frac{12}{4}\\&=\left(x+\frac{9}{2}\right)^2-\frac{93}{4}\\&=\left(x+\frac{9}{2}-\sqrt{\frac{93}{4}}\right)\left(x+\frac{9}{2}+\sqrt{\frac{93}{4}}\right)\\&=\left(x+\frac{9-\sqrt{93}}{2}\right)\left(x+\frac{9+\sqrt{93}}{2}\right)\end{aligned}$$

d $$\begin{aligned}2x^2+5x+1&=2\left(x^2+\frac{5x}{2}+\frac{1}{2}\right)\\&=2\left(\left(x^2+\frac{5x}{2}+\left(\frac{5}{4}\right)^2\right)-\left(\frac{5}{4}\right)^2+\frac{1}{2}\right)\\&=2\left(\left(x+\frac{5}{4}\right)^2-\frac{25}{16}+\frac{8}{16}\right)\\&=2\left(\left(x+\frac{5}{4}\right)^2-\frac{17}{16}\right)\\&=2\left(x+\frac{5-\sqrt{17}}{4}\right)\left(x+\frac{5+\sqrt{17}}{4}\right)\end{aligned}$$

e $$\begin{aligned}3x^2+4x+3&=3\left(x^2+\frac{4x}{3}+1\right)\\&=3\left(\left(x^2+\frac{4x}{3}+\left(\frac{2}{3}\right)^2\right)-\left(\frac{2}{3}\right)^2+1\right)\\&=3\left(\left(x+\frac{2}{3}\right)^2-\frac{4}{9}+\frac{9}{9}\right)\\&=3\left(\left(x+\frac{2}{3}\right)^2+\frac{5}{9}\right)\end{aligned}$$

Since the sum of two squares does not factorise over R, there are no linear factors.

f $$\begin{aligned}1+40x-5x^2&=-5\left(x^2-8x-\frac{1}{5}\right)\\&=-5\left[\left(x^2-8x+16\right)-16-\frac{1}{5}\right]\\&=-5\left[(x-4)^2-\frac{80}{5}-\frac{1}{5}\right]\\&=-5\left[(x-4)^2-\frac{81}{5}\right]\\&=-5\left(x-4-\frac{9}{\sqrt{5}}\right)\left(x-4+\frac{9}{\sqrt{5}}\right)\\&=-5\left(x-4-\frac{9\sqrt{5}}{5}\right)\left(x-4+\frac{9\sqrt{5}}{5}\right)\end{aligned}$$

7 $$\begin{aligned}&(2x+1)(x+5)-1=0\\&\therefore 2x^2+11x+5-1=0\\&\quad\therefore 2x^2+11x+4=0\\&x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},\ a=2, b=11, c=4\\&\therefore x=\frac{-11\pm\sqrt{(11)^2-4\times(2)\times(4)}}{2\times(2)}\\&\quad=\frac{-11\pm\sqrt{121-32}}{4}\\&\quad=\frac{-11\pm\sqrt{89}}{4}\end{aligned}$$

8 a $3x^2 - 5x + 1 = 0$

$$a = 3,\ b = -5,\ c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times 1}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25 - 12}}{6}$$

$$= \frac{5 \pm \sqrt{13}}{6}$$

b $-5x^2 + x + 5 = 0$

$$a = -5,\ b = 1,\ c = 5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(1) \pm \sqrt{(1)^2 - 4 \times (-5) \times 5}}{2(-5)}$$

$$= \frac{-1 \pm \sqrt{1 - (-100)}}{-10}$$

$$= \frac{-1 \pm \sqrt{101}}{-10}$$

$$= \frac{1 \pm \sqrt{101}}{10}$$

c $2x^2 + 3x + 4 = 0$

$$a = 2,\ b = 3,\ c = 4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(3) \pm \sqrt{(3)^2 - 4 \times 2 \times 4}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{9 - 32}}{4}$$

$$= \frac{-3 \pm \sqrt{-23}}{4}$$

There are no real solutions since $\Delta < 0$.

d $x(x + 6) = 8$

First express the equation in the form $ax^2 + bx + c = 0$.

$$x^2 + 6x = 8$$

$$x^2 + 6x - 8 = 0$$

$$a = 1,\ b = 6,\ c = -8$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(6) \pm \sqrt{(6)^2 - 4 \times 1 \times (-8)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 + 32}}{2}$$

$$= \frac{-6 \pm \sqrt{68}}{2}$$

$$= \frac{-6 \pm \sqrt{4 \times 17}}{2}$$

$$= \frac{-6 \pm 2\sqrt{17}}{2}$$

$$= \frac{2\left(-3 \pm \sqrt{17}\right)}{2}$$

$$= -3 \pm \sqrt{17}$$

9 a $9x^2 - 3x - 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 9, b = -3, c = -4$$

$$\therefore x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 9 \times (-4)}}{2 \times 9}$$

$$= \frac{3 \pm \sqrt{9 + 144}}{18}$$

$$= \frac{3 \pm \sqrt{153}}{18}$$

$$= \frac{3 \pm 3\sqrt{17}}{18}$$

$$= \frac{3\left(1 \pm \sqrt{17}\right)}{18}$$

$$\therefore x = \frac{1 \pm \sqrt{17}}{6}$$

b $5x(4 - x) = 12$

$$\therefore 20x - 5x^2 = 12$$

$$\therefore 5x^2 - 20x + 12 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a = 5,\ b = -20,\ c = 12$$

$$\therefore x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \times 5 \times 12}}{2 \times 5}$$

$$= \frac{20 \pm \sqrt{400 - 240}}{10}$$

$$= \frac{20 \pm \sqrt{160}}{10}$$

$$= \frac{20 \pm 4\sqrt{10}}{10}$$

$$= \frac{4\left(5 \pm \sqrt{10}\right)}{10}$$

$$\therefore x = \frac{10 \pm 2\sqrt{10}}{5}$$

c $(x - 10)^2 = 20$

$$\therefore x - 10 = \pm\sqrt{20}$$

$$\therefore x = 10 \pm 2\sqrt{5}$$

d $x^2 + 6x - 3 = 0$

Completing the square,

$$\therefore \left(x^2 + 6x + 9\right) - 9 - 3 = 0$$

$$\therefore (x + 3)^2 = 12$$

$$\therefore x + 3 = \pm\sqrt{12}$$

$$\therefore x = -3 \pm 2\sqrt{3}$$

e $6x = x^2$

$$0 = x^2 - 6x$$

$$0 = x(x - 6)$$

$$x = 0, 6$$

f $8x^2 - 22x + 12 = 0$

$$2(4x^2 - 11x + 6) = 0$$

$$4x^2 - 11x + 6 = 0$$

$$(4x - 3)(x - 2) = 0$$

$$x = \frac{3}{4}, 2$$

10 a $4x^2 + 5x + 10$

$\Delta = b^2 - 4ac, a = 4, b = 5, c = 10$

$\therefore \Delta = (5)^2 - 4 \times 4 \times 10$

$= 25 - 160$

$= -135$

Since $\Delta < 0$, there are no real factors.

b $169x^2 - 78x + 9$

$\Delta = b^2 - 4ac, a = 169, b = -78, c = 9$

$\therefore \Delta = (-78)^2 - 4 \times 169 \times 9$

$= 6084 - 6084$

$= 0$

Since $\Delta = 0$, the quadratic is a perfect square and factorises over Q. There is one repeated rational factor. Completing the square is not essential to obtain the factor.

Check: $169x^2 - 78x + 9 = (13x - 3)^2$

c $-3x^2 + 11x - 10$

$\Delta = b^2 - 4ac, a = -3, b = 11, c = -10$

$\therefore \Delta = (11)^2 - 4 \times (-3)(-10)$

$= 121 - 120$

$= 1$

Since $\Delta > 0$ and a perfect square, the quadratic factorises over Q and has two rational factors. Completing the square is not essential to obtain the factors.

Check: $-3x^2 + 11x - 10 = (-3x + 5)(x - 2)$

d $\frac{1}{3}x^2 - \frac{8}{3}x + 2$

$\Delta = b^2 - 4ac, a = \frac{1}{3}, b = -\frac{8}{3}, c = 2$

$\therefore \Delta = \left(-\frac{8}{3}\right)^2 - 4 \times \frac{1}{3} \times 2$

$= \frac{64}{9} - \frac{8}{3}$

$= \frac{64 - 24}{9}$

$= \frac{40}{9}$

Since $\Delta > 0$ but is not a perfect square, the quadratic factorises over R and has two real linear factors. Completing the square is needed to obtain the factors.

11 a $5x^2 + 9x - 2$

$\Delta = b^2 - 4ac$

$a = 5, b = 9, c = -2$

$\therefore \Delta = (9)^2 - 4 \times 5 \times (-2)$

$= 81 + 40$

$\therefore \Delta = 121$

Since Δ is a perfect square, there are two rational factors.

b $12x^2 - 3x + 1$

$\Delta = b^2 - 4ac$

$a = 12, b = -3, c = 1$

$\therefore \Delta = (-3)^2 - 4 \times 12 \times 1$

$= 9 - 48$

$\therefore \Delta = -39$

Since $\Delta < 0$, there are no real linear factors.

c $121x^2 + 110x + 25$

$\Delta = b^2 - 4ac$

$a = 121, b = 110, c = 25$

$\therefore \Delta = (110)^2 - 4 \times 121 \times 25$

$= 12100 - 12100$

$\therefore \Delta = 0$

Since $\Delta = 0$, there is one repeated rational factor.

d $x^2 + 10x + 23$

$\Delta = b^2 - 4ac$

$a = 1, b = 10, c = 23$

$\therefore \Delta = (10)^2 - 4 \times 1 \times 23$

$= 100 - 92$

$\therefore \Delta = 8$

Since $\Delta > 0$ but not a perfect square, there are two irrational factors.

12 a $3x^2 - 4x + 1 = 0$

$a = 3,\ b = -4,\ c = 1$

$\Delta = b^2 - 4ac$

$= (-4)^2 - 4(3)(1)$

$= 16 - 12$

$= 4$

b Since the discriminant is positive and a perfect square, the equation has two solutions, both of which are rational.

c $-x^2 - 4x + 3 = 0$

$a = -1,\ b = -4,\ c = 3$

$\Delta = b^2 - 4ac$

$= (-4)^2 - 4(-1)(3)$

$= 16 + 12$

$= 28$

This equation has two irrational solutions as the discriminant is positive but not a perfect square.

d $2x^2 - 20x + 50 = 0$

$a = 2, b = -20, c = 50$

$\Delta = b^2 - 4ac$

$= (-20)^2 - 4(2)(50)$

$= 400 - 400$

$= 0$

This equation has one rational solution.

e $x^2 + 4x + 7 = 0$

$a = 1, b = 4, c = 7$

$\Delta = b^2 - 4ac$

$= 4^2 - 4(1)(7)$

$= 16 - 28$

$= -12$

This equation has no real solutions as the discriminant is negative.

f $1 = x^2 + 5x$

$1 = x^2 + 5x$

$0 = x^2 + 5x - 1$

$a = 1, b = 5, c = -1$

$\Delta = b^2 - 4ac$

$= 5^2 - 4(1)(-1)$

$= 25 + 4$

$= 29$

This equation has two irrational solutions as the discriminant is positive but not a perfect square.

13 a $-5x^2 - 8x + 9 = 0$

$\Delta = b^2 - 4ac, a = -5, b = -8, c = 9$

$\therefore \Delta = (-8)^2 - 4 \times (-5) \times 9$

$= 64 + 180$

$\therefore \Delta = 244$

Since $\Delta > 0$ but not a perfect square, there are two irrational roots.

b $4x^2 + 3x - 7 = 0$

$\Delta = b^2 - 4ac, a = 4, b = 3, c = -7$

$\therefore \Delta = (3)^2 - 4 \times 4 \times (-7)$

$= 9 + 112$

$\therefore \Delta = 121$

Since Δ is a perfect square, there are two rational roots.

c $4x^2 + x + 2 = 0$

$\Delta = b^2 - 4ac, a = 4, b = 1, c = 2$

$\therefore \Delta = (1)^2 - 4 \times 4 \times 2$

$= 1 - 32$

$\therefore \Delta = -31$

Since $\Delta < 0$, there are no real roots.

d $28x - 4 - 49x^2 = 0$

$\Delta = b^2 - 4ac, a = -49, b = 28, c = -4$

$\therefore \Delta = (28)^2 - 4 \times (-49) \times (-4)$

$= 784 - 784$

$\therefore \Delta = 0$

Since $\Delta = 0$, there is one rational root (or two equal roots).

e $4x^2 + 25 = 0$

As $4x^2 \geq 0$, the sum $4x^2 + 25$ cannot equal zero. Therefore, there are no real roots.

f $3\sqrt{2}x^2 + 5x + \sqrt{2} = 0$

$\Delta = b^2 - 4ac, a = 3\sqrt{2}, b = 5, c = \sqrt{2}$

$\therefore \Delta = (5)^2 - 4 \times 3\sqrt{2} \times \sqrt{2}$

$= 25 - 24$

$\therefore \Delta = 1$

Since $\Delta > 0$, there are two roots. However, despite Δ being a perfect square, the coefficient of x^2 in the quadratic equation is irrational, so the two roots are irrational.

14 $3(2x + 1)^4 - 16(2x + 1)^2 - 35 = 0$

Let $u = (2x + 1)^2$.

$3u^2 - 16u - 35 = 0$

$\therefore (3u + 5)(u - 7) = 0$

$\therefore u = -\frac{5}{3},\ u = 7$

$\therefore (2x + 1)^2 = -\frac{5}{3}$, which is not possible since a perfect square cannot be negative

or $(2x + 1)^2 = 7$

$\therefore 2x + 1 = \pm\sqrt{7}$

$\therefore 2x = -1 \pm \sqrt{7}$

$\therefore x = \frac{-1 \pm \sqrt{7}}{2}$

15 $\sqrt{2}x^2 + 4\sqrt{3}x - 8\sqrt{2} = 0$

$$x = \frac{-4\sqrt{3} \pm \sqrt{\left(4\sqrt{3}\right)^2 - 4 \times \sqrt{2} \times -8\sqrt{2}}}{2 \times \sqrt{2}}$$

$$= \frac{-4\sqrt{3} \pm \sqrt{48 + 64}}{2\sqrt{2}}$$

$$= \frac{-4\sqrt{3} \pm \sqrt{112}}{2\sqrt{2}}$$

$$= \frac{-4\sqrt{3} \pm 4\sqrt{7}}{2\sqrt{2}}$$

$$= \frac{-2\sqrt{3} \pm 2\sqrt{7}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{-2\sqrt{6} \pm 2\sqrt{14}}{2}$$

$$= -\sqrt{6} \pm \sqrt{14}$$

16 **a** $(x^2 - 3)^2 - 4(x^2 - 3) + 4 = 0$

Let $u = x^2 - 3$.

$\therefore u^2 - 4u + 4 = 0$

$\therefore (u - 2)^2 = 0$

$\therefore u = 2$

$\therefore x^2 - 3 = 2$

$\therefore x^2 = 5$

$\therefore x = \pm\sqrt{5}$

b $5x^4 - 39x^2 - 8 = 0$

Let $u = x^2$.

$\therefore 5u^2 - 39u - 8 = 0$

$\therefore (5u + 1)(u - 8) = 0$

$\therefore u = -\frac{1}{5}$ or $u = 8$

$\therefore x^2 = -\frac{1}{5}$ (reject) or $x^2 = 8$

$\therefore x^2 = 8$

$\therefore x = \pm 2\sqrt{2}$

c $x^2(x^2 - 12) + 11 = 0$

Let $u = x^2$.

$\therefore u(u - 12) + 11 = 0$

$\therefore u^2 - 12u + 11 = 0$

$\therefore (u - 1)(u - 11) = 0$

$\therefore u = 1$ or $u = 11$

$\therefore x^2 = 1$ or $x^2 = 11$

$\therefore x = \pm 1,\ x = \pm\sqrt{11}$

d $\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) - 3 = 0$

Let $u = x + \frac{1}{x}$.

$\therefore u^2 + 2u - 3 = 0$

$\therefore (u + 3)(u - 1) = 0$

$\therefore u = -3$ or $u = 1$

$\therefore x + \frac{1}{x} = -3$ or $x + \frac{1}{x} = 1$

$\therefore x^2 + 1 = -3x$ or $x^2 + 1 = x$

$\therefore x^2 + 3x + 1 = 0$ or $x^2 - x + 1 = 0$

$\therefore x = \frac{-3 \pm \sqrt{9 - 4(1)(1)}}{2}$ or $x = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2}$

$\therefore x = \frac{-3 \pm \sqrt{5}}{2}$ or $x = \frac{1 \pm \sqrt{-3}}{2}$

$\therefore x = \frac{-3 \pm \sqrt{5}}{2}$

(since $x = \frac{1 \pm \sqrt{-3}}{2}$ is not real).

e $(x^2 - 7x - 8)^2 = 3(x^2 - 7x - 8)$

Let $u = x^2 - 7x - 8$.

$$\begin{aligned}
\therefore u^2 &= 3u\\
\therefore u^2 - 3u &= 0\\
\therefore u(a-3) &= 0\\
\therefore u &= 0 \text{ or } u = 3\\
\therefore x^2 - 7x - 8 = 0 &\text{ or } x^2 - 7x - 8 = 3\\
\therefore (x+1)(x-8) = 0 &\text{ or } x^2 - 7x - 11 = 0\\
\therefore x = -1, x = 8 &\text{ or } x = \frac{7 \pm \sqrt{49 - 4(1)(-11)}}{2}\\
\therefore x = -1, x = 8 &\text{ or } x = \frac{7 \pm \sqrt{93}}{2}
\end{aligned}$$

f $3\left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) - 2 = 0$, given that $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$.

Let $u = x + \frac{1}{x}$ so $x^2 + \frac{1}{x^2} = u^2 - 2$.

$$\begin{aligned}
\therefore 3(u^2 - 2) + 2u - 2 &= 0\\
\therefore 3u^2 + 2u - 8 &= 0\\
\therefore (3u - 4)(u + 2) &= 0\\
\therefore u &= \frac{4}{3} \text{ or } u = -2\\
\therefore x + \frac{1}{x} = \frac{4}{3} &\text{ or } x + \frac{1}{x} = -2\\
\therefore 3x^2 + 3 = 4x &\text{ or } x^2 + 1 = -2x\\
\therefore 3x^2 - 4x + 3 = 0 &\text{ or } x^2 + 2x + 1 = 0
\end{aligned}$$

Consider $3x^2 - 4x + 3 = 0$.

$$\begin{aligned}
\Delta &= 16 - 4(3)(3)\\
&= -20
\end{aligned}$$

Since $\Delta < 0$, there are no real roots.

$$\begin{aligned}
\therefore x^2 + 2x + 1 &= 0\\
\therefore (x+1)^2 &= 0\\
\therefore x &= -1
\end{aligned}$$

17 **a** $0.2x^2 - 2.5x + 10 = 0$

$$\begin{aligned}
\Delta &= b^2 - 4ac,\ a = 0.2, b = -2.5, c = 10\\
\therefore \Delta &= (-2.5)^2 - 4 \times (0.2) \times (10)\\
&= 6.25 - 8\\
&= -1.75
\end{aligned}$$

Since $\Delta < 0$, there are no real roots to the equation.

b $kx^2 - (k+3)x + k = 0$

$$\begin{aligned}
\Delta &= b^2 - 4ac,\ a = k,\ b = -(k+3),\ c = k\\
\therefore \Delta &= (-(k+3))^2 - 4 \times (k) \times (k)\\
&= (k+3)^2 - 4k^2\\
&= (k + 3 - 2k)(k + 3 + 2k)\\
&= (3 - k)(3k + 3)
\end{aligned}$$

For two equal solutions, $\Delta = 0$.

$$\begin{aligned}
\therefore (3 - k)(3k + 3) &= 0\\
\therefore k &= 3,\ k = -1
\end{aligned}$$

18 $mx^2 + (m-4)x = 4$

$$\begin{aligned}
&\therefore mx^2 + (m-4)x - 4 = 0\\
&\Delta = b^2 - 4ac, a = m, b = (m-4), c = -4\\
&\Delta = (m-4)^2 - 4 \times (m) \times (-4)\\
&\quad = (m-4)^2 + 16m\\
&\quad = m^2 - 8m + 16 + 16m\\
&\quad = m^2 + 8m + 16\\
&\quad = (m+4)^2
\end{aligned}$$

Since $\Delta \geq 0$ for all m, the equation will always have real roots.

19 **a** $x^2 + (m+2)x - m + 5 = 0$

For one root, $\Delta = 0$.

$$\begin{aligned}
\Delta &= b^2 - 4ac,\ a = 1, b = m + 2, c = -m + 5\\
\therefore \Delta &= (m+2)^2 - 4 \times 1 \times (-m + 5)\\
&= m^2 + 4m + 4 + 4m - 20\\
\therefore \Delta &= m^2 + 8m - 16
\end{aligned}$$

Therefore, for one root, $m^2 + 8m - 16 = 0$.

$$\begin{aligned}
\therefore \left(m^2 + 8m + 16\right) - 16 - 16 &= 0\\
\therefore (m+4)^2 &= 32\\
\therefore m + 4 &= \pm\sqrt{32}\\
\therefore m &= -4 \pm 4\sqrt{2}
\end{aligned}$$

b $(m+2)x^2 - 2mx + 4 = 0$

For one root, $\Delta = 0$.

$$\begin{aligned}
\therefore (-2m)^2 - 4(m+2)(4) &= 0\\
\therefore 4m^2 - 16m - 32 &= 0\\
\therefore m^2 - 4m - 8 &= 0\\
\therefore \left(m^2 - 4m + 4\right) - 4 - 8 &= 0\\
\therefore (m-2)^2 - 12 &= 0\\
\therefore m - 2 &= \pm\sqrt{12}\\
\therefore m &= 2 \pm 2\sqrt{3}
\end{aligned}$$

c $3x^2 + 4x - 2(p-1) = 0$

For no roots, $\Delta < 0$.

$$\begin{aligned}
&\therefore 16 - 4\,(3)\,(-2\,(p-1)) < 0\\
&\therefore 16 + 24\,(p-1) < 0\\
&\therefore 24p - 8 < 0\\
&\therefore 24p < 8\\
&\therefore p < \frac{1}{3}
\end{aligned}$$

d $kx^2 - 4x - k = 0$

The discriminant determines the number of solutions.

$$\begin{aligned}
\Delta &= (-4)^2 - 4(k)(-k)\\
&= 16 + 4k^2
\end{aligned}$$

Since $k \in R \setminus \{0\}$, $k^2 > 0$.

$\therefore 16 + 4k^2 > 16$

Thus, Δ is always positive. Therefore, the equation always has two solutions.

e $px^2 + (p+q)x + q = 0$

$$\begin{aligned}
\Delta &= (p+q)^2 - 4pq\\
&= p^2 + 2pq + q^2 - 4pq\\
&= p^2 - 2pq + q^2\\
&= (p-q)^2
\end{aligned}$$

As Δ is a perfect square and $p, q \in Q$, the roots are always rational.

20 **a** $x^2 - 20\sqrt{5}x + 100 = 0$

$$\begin{aligned}
\therefore \left(x^2 - 20\sqrt{5x} + \left(10\sqrt{5}\right)^2\right) - \left(10\sqrt{5}\right) + 100 &= 0\\
\therefore \left(x - 10\sqrt{5}\right)^2 - 500 + 100 &= 0\\
\therefore \left(x - 10\sqrt{5}\right)^2 &= 400\\
\therefore x - 10\sqrt{5} &= \pm 20\\
\therefore x &= 10\sqrt{5} \pm 20
\end{aligned}$$

b Since one root is $x = 1 - \sqrt{2}$, the other root is $x = 1 + \sqrt{2}$, as the roots occur in conjugate surd pairs.

Since $x = 1 - \sqrt{2}$ is a root, $\left(x - (1 - \sqrt{2})\right)$ is a factor of the equation, and since $x = 1 + \sqrt{2}$ is a root, $\left(x - (1 + \sqrt{2})\right)$ is a factor of the equation.

Therefore, the equation is $\left(x - 1 + \sqrt{2}\right)\left(x - 1 - \sqrt{2}\right) = 0$.

Expanding,

$$\left(x - 1 + \sqrt{2}\right)\left(x - 1 - \sqrt{2}\right) = 0$$

$$\therefore \left((x-1) + \sqrt{2}\right)\left((x-1) - \sqrt{2}\right) = 0$$

$$\therefore (x-1)^2 - \left(\sqrt{2}\right)^2 = 0$$

$$\therefore x^2 - 2x + 1 - 2 = 0$$

$$\therefore x^2 - 2x - 1 = 0$$

$\therefore b = -2, c = -1$

21 a If $x = \sqrt{2}$ is a zero, then $\left(x - \sqrt{2}\right)$ is a factor, and if $x = -\sqrt{2}$ is a zero, then $\left(x + \sqrt{2}\right)$ is a factor.

The product of these factors is

$\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$

$= x^2 - \left(\sqrt{2}\right)^2$

$= x^2 - 2$

b If $x = -4 + \sqrt{2}$ is a zero, then $\left(x - \left(-4 + \sqrt{2}\right)\right) = \left(x + 4 - \sqrt{2}\right)$ is a factor, and if $x = -4 - \sqrt{2}$ is a zero, then $\left(x + 4 + \sqrt{2}\right)$ is a factor.

The product of these factors is

$\left(x + 4 - \sqrt{2}\right)\left(x + 4 + \sqrt{2}\right)$

$= \left((x+4) - \sqrt{2}\right)\left((x+4) + \sqrt{2}\right)$

$= (x+4)^2 - \left(\sqrt{2}\right)^2$

$= x^2 + 8x + 16 - 2$

$= x^2 + 8x + 14$

22 a i $x^2 + 6\sqrt{2}x + 18 = 0$

$$\therefore x^2 + 6\sqrt{2}x + \left(3\sqrt{2}\right)^2 - \left(3\sqrt{2}\right)^2 + 18 = 0$$

$$\therefore \left(x^2 + 6\sqrt{2}x + \left(3\sqrt{2}\right)^2\right) - 18 + 18 = 0$$

$$\therefore \left(x + 3\sqrt{2}\right)^2 = 0$$

$$\therefore x = -3\sqrt{2}$$

ii $2\sqrt{5}x^2 - 3\sqrt{10}x + \sqrt{5} = 0$

Divide both sides by $\sqrt{5}$.

$\therefore 2x^2 - 3\sqrt{2}x + 1 = 0$

$$\therefore x = \frac{3\sqrt{2} \pm \sqrt{\left(3\sqrt{2}\right)^2 - 4 \times 2 \times 1}}{4}$$

$$\therefore x = \frac{3\sqrt{2} \pm \sqrt{18 - 8}}{4}$$

$$\therefore x = \frac{3\sqrt{2} \pm \sqrt{10}}{4}$$

b i If $x = \dfrac{-1 + \sqrt{5}}{2}$ is a root of a quadratic equation $x^2 + bx + c = 0$ with rational coefficients, then its conjugate $x = \dfrac{-1 - \sqrt{5}}{2}$ is also a root.

ii In factorised form, the equation is

$$\left(x - \frac{-1 + \sqrt{5}}{2}\right)\left(x - \frac{-1 - \sqrt{5}}{2}\right) = 0$$

$$\therefore \left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right) = 0$$

$$\therefore \left(\left(x + \frac{1}{2}\right) - \frac{\sqrt{5}}{2}\right)\left(\left(x + \frac{1}{2}\right) + \frac{\sqrt{5}}{2}\right) = 0$$

$$\therefore \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 = 0$$

$$\therefore x^2 + x + \frac{1}{4} - \frac{5}{4} = 0$$

$$\therefore x^2 + x - 1 = 0$$

Therefore, $b = 1,\ c = -1$.

23 a $4x - 3\sqrt{x} = 1$

$\therefore 4x - 1 = 3\sqrt{x}$

$\therefore (4x - 1)^2 = 9x$

$\therefore 16x^2 - 8x + 1 = 9x$

$\therefore 16x^2 - 17x + 1 = 0$

$\therefore (16x - 1)(x - 1) = 0$

$\therefore x = \dfrac{1}{16}, x = 1$

Check: substitute $x = \dfrac{1}{16}$ and $x = 1$ in $4x - 3\sqrt{x} = 1$.

$\text{LHS} = \frac{1}{4} - 3 \times \frac{1}{4}$ $\quad$ $\text{LHS} = 4 - 3 \times 1$

$= -0.5$ $\quad$ $= 1$

$\neq \text{RHS}$ $\quad$ $= \text{RHS}$

Reject $x = \dfrac{1}{16}$.

The answer is $x = 1$.

b Let $u = \sqrt{x}$ in $4x - 3\sqrt{x} = 1$

$\therefore 4u^2 - 3u = 1$

$\therefore 4u^2 - 3u - 1 = 0$

$\therefore (4u + 1)(u - 1) = 0$

$\therefore u = -\dfrac{1}{4}, u = 1$

$\therefore \sqrt{x} = -1$ or $\sqrt{x} = 1$

Since $\sqrt{x}$ cannot be negative, reject $\sqrt{x} = -1$.

$\therefore \sqrt{x} = 1$

$\therefore x = 1$

24 $2\sqrt{x} = 8 - x$

a Squaring both sides,

$$\left(2\sqrt{x}\right)^2 = (8-x)^2$$
$$\therefore 4x = 64 - 16x + x^2$$
$$\therefore x^2 - 20x + 64 = 0$$
$$\therefore (x-4)(x-16) = 0$$
$$\therefore x = 4, x = 16$$

Check: substitute $x = 4$ in $2\sqrt{x} = 8 - x$.

$\text{LHS} = 2\sqrt{4}$ $\quad \text{RHS} = 8 - 4$

$= 4$ $\quad = 4$

$\therefore \text{LHS} = \text{RHS}$

Hence, $x = 4$ is a solution.

Substitute $x = 16$ in $2\sqrt{x} = 8 - x$.

$\text{LHS} = 2\sqrt{16}$ $\quad \text{RHS} = 8 - 16$

$= 8$ $\quad = -8$

$\text{LHS} \neq \text{RHS}$

Hence, reject $x = 16$.

The answer is $x = 4$.

b Let $u = \sqrt{x}$.

$$\therefore 2u = 8 - u^2$$
$$\therefore u^2 + 2u - 8 = 0$$
$$\therefore (u-2)(u+4) = 0$$
$$\therefore u = 2 \text{ or } u = -4$$
$$\therefore \sqrt{x} = 2 \text{ or } \sqrt{x} = -4$$

Reject $\sqrt{x} = -4$ since $\sqrt{x} \geq 0$.

$$\therefore \sqrt{x} = 2$$
$$\therefore x = 2^2$$
$$\therefore x = 4$$

25 $1 + \sqrt{x+1} = 2x$

$$\therefore \sqrt{x+1} = 2x - 1$$
$$\therefore x + 1 = (2x-1)^2$$
$$\therefore x + 1 = 4x^2 - 4x + 1$$
$$\therefore 4x^2 - 5x = 0$$
$$\therefore x(4x-5) = 0$$
$$\therefore x = 0, x = \frac{5}{4}$$

Check: Substitute $x = 0$ in $1 + \sqrt{x+1} = 2x$.

$\text{LHS} = 1 + \sqrt{1}$ $\quad \text{RHS} = 2 \times 0$

$= 2$

$= 0$

$\therefore \text{LHS} \neq \text{RHS}$

Reject $x = 0$.

Substitute $x = \frac{5}{4}$ in $1 + \sqrt{x+1} = 2x$.

$\text{LHS} = 1 + \sqrt{\frac{5}{4} + 1}$ $\quad \text{RHS} = 2 \times \frac{5}{4}$

$= 1 + \sqrt{\frac{9}{4}}$ $\quad = \frac{5}{2}$

$= \frac{5}{2}$

$\therefore \text{LHS} = \text{RHS}$

The solution is $x = \frac{5}{4}$.

3.3 Exam questions

1 $\Delta = b^2 - 4ac$

$a = 3,\ b = 5,\ c = -1$

$\Delta = (5)^2 - 4(3)(-1)$

$= 37$

The correct answer is **A**.

2 $2x^2 - 7x + 1 = 0$

$a = 2,\ b = -7,\ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(1)}}{2(2)}$$
$$= \frac{7 \pm \sqrt{49 - 8}}{4}$$
$$= \frac{7 \pm \sqrt{41}}{4}$$

The correct answer is **E**.

3 $mx^2 + (m+3)x = -3$

$mx^2 + (m+3)x + 3 = 0$ [1 mark]

For all roots, $\Delta \geq 0$.

$\Delta = b^2 - 4ac$

$a = m,\ b = (m+3), c = +3$

$\Delta = (m+3)^2 - 4m(+3)$ [1 mark]

$= m^2 + 6m + 9 - 12m$

$= m^2 - 6m + 9$

$= (m-3)^2$ [1 mark]

As Δ is a perfect square, $\Delta \geq 0$.

Therefore, the quadratic will have real roots for any real value of m. [1 mark]

3.4 Applications of quadratic equations

3.4 Exercise

1 The salmon costs $\frac{400}{x}$ dollars per kilogram at the market.

$(x-2)$ kilograms of salmon is sold at $\left(\frac{400}{x} + 10\right)$ dollars per kilogram for \$540.

$$\therefore (x-2) \times \left(\frac{400}{x} + 10\right) = 540$$
$$\therefore 400 + 10x - \frac{800}{x} - 20 = 540$$
$$\therefore 10x - \frac{800}{x} = 160$$
$$\therefore 10x^2 - 800 = 160x$$
$$\therefore x^2 - 16x - 80 = 0$$
$$\therefore (x-20)(x+4) = 0$$
$$\therefore x = 20,\ x = -4$$

Reject $x = -4$ since $x \in N$.

Therefore, 20 kilograms of salmon were bought at the market.

2 Let the area of a sphere of radius r cm be A cm^2.

$A = kr^2$

Substitute $r = 5$, $A = 100\pi$.

$\therefore 100\pi = k(25)$

$\therefore k = \frac{100\pi}{25}$

$\therefore k = 4\pi$

$\therefore A = 4\pi r^2$

When $A = 360\pi$,

$360\pi = 4\pi r^2$

$\therefore r^2 = 90$

$\therefore r = \pm 3\sqrt{10}$

$r > 0, \ \therefore r = 3\sqrt{10}$

The radius is $3\sqrt{10}$ cm.

3 Let C dollars be the cost of hire for t hours.

$C = 10 + kt^2$

$t = 3, C = 32.50$

$\Rightarrow 32.5 = 10 + k(9)$

Solving,

$\therefore 9k = 22.5$

$\therefore k = 2.5$

$\therefore C = 10 + 2.5t^2$

When $C = 60$,

$60 = 10 + 2.5t^2$

$\therefore 2.5t^2 = 50$

$\therefore t^2 = \frac{50}{2.5}$

$\therefore t^2 = 20$

Since $t > 0$, $t = \sqrt{20}$.

$\therefore t \simeq 4.472$

The chainsaw was hired for approximately $4\frac{1}{2}$ hours.

4 $A = kx^2$, where A is the area of an equilateral triangle of side length x and k is the constant of proportionality.

When $x = 2\sqrt{3}, A = 3\sqrt{3}$.

$\therefore 3\sqrt{3} = k\left(2\sqrt{3}\right)^2$

$\therefore 3\sqrt{3} = 12k$

$\therefore k = \frac{\sqrt{3}}{4}$

Hence, $A = \frac{\sqrt{3}}{4}x^2$.

If $A = 12\sqrt{3}$,

$12\sqrt{3} = \frac{\sqrt{3}}{4}x^2$

$\therefore x^2 = 48$

$\therefore x = 4\sqrt{3}$

(the negative square root is not appropriate for the length).

The side length is $4\sqrt{3}$ cm.

5 Cost in dollars, $C = 20 + 5x$

Revenue in dollars, $R = 1.5x^2$

Profit in dollars, $P = R - C$

$\therefore P = 1.5x^2 - 5x - 20$

If $P = 800$,

$800 = 1.5x^2 - 5x - 20$

$\therefore 1.5x^2 - 5x - 820 = 0$

$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1.5 \times (-820)}}{2 \times 1.5}$$

$$\therefore x = \frac{5 \pm \sqrt{25 + 4920}}{3}$$

$$\therefore x = \frac{5 \pm \sqrt{4945}}{3}$$

$\therefore x = 25.107, \ x = -21.773$

Reject the negative value, so $x \simeq 25.107$.

The number of litres is $100x = 2510.7$. To the nearest litre, 2511 litres must be sold.

6 Let the numbers be n and $n + 2$.

The product is 440.

$\therefore n(n + 2) = 440$

$\therefore n^2 + 2n - 440 = 0$

$\therefore (n - 20)(n + 22) = 0$

$\therefore n = 20, -22$

Since $n \in N, \ n = 20$.

Therefore, the numbers are 20 and 22.

7 Let the natural numbers be n and $n + 1$.

$n^2 + (n + 1)^2 + (n + (n + 1))^2 = 662$

$\therefore n^2 + (n + 1)^2 + (2n + 1)^2 = 662$

$\therefore n^2 + n^2 + 2n + 1 + 4n^2 + 4n + 1 = 662$

$\therefore 6n^2 + 6n - 660 = 0$

$\therefore n^2 + n - 110 = 0$

$\therefore (n + 11)(n - 10) = 0$

$\therefore n = -11 \text{ (reject)}, n = 10$

$\therefore n = 10$

The two consecutive natural numbers are 10 and 11.

8 $A = \frac{1}{2}bh$, where A is the area of a triangle with base b and height h.

Given $h : b = \sqrt{2} : 1$, then

$\frac{h}{b} = \frac{\sqrt{2}}{1}$

$\therefore h = \sqrt{2}b$

$\therefore A = \frac{1}{2}\sqrt{2}b^2$

When $A = \sqrt{32}$,

$\sqrt{32} = \frac{\sqrt{2}}{2}b^2$

$\therefore b^2 = 4\sqrt{2} \times \frac{2}{\sqrt{2}}$

$\therefore b^2 = 8$

$\therefore b = 2\sqrt{2}$

(reject the negative square root)

With $b = 2\sqrt{2}$,

$h = \sqrt{2} \times 2\sqrt{2}$

$\therefore h = 4$

The base is $2\sqrt{2}$ cm and the height is 4 cm.

9 Using Pythagoras's theorem,

$(3x + 3)^2 = (3x)^2 + (x - 3)^2$

$\therefore 9x^2 + 18x + 9 = 9x^2 + x^2 - 6x + 9$

$\therefore x^2 - 24x = 0$

$\therefore x(x - 24) = 0$

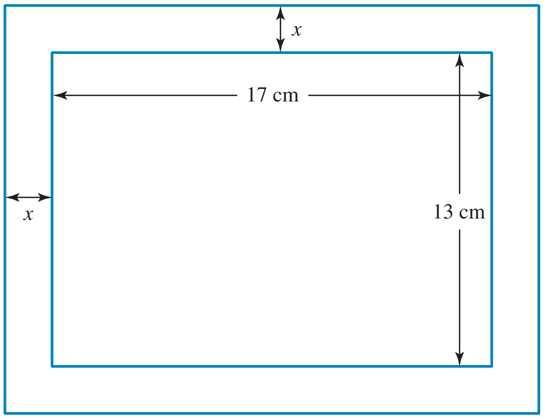
$\therefore x = 0, x = 24$

Reject $x = 0$, because $3x$ would be zero and $x - 3$ would be negative.

$\therefore x = 24$

The three side lengths are 72 cm, 21 cm and 75 cm, so the perimeter is 168 cm.

10



Let the width of the border be x cm.
The frame has length $(17 + 2x)$ cm and width $(13 + 2x)$.
The area of the border is the difference between the area of the frame and the area of the photo.

$$\therefore 260 = (17 + 2x)(13 + 2x) - 17 \times 13$$
$$\therefore 260 = 221 + 60x + 4x^2 - 221$$
$$\therefore 4x^2 + 60x - 260 = 0$$
$$\therefore x^2 + 15x - 65 = 0$$
$$\therefore x = \frac{-15 \pm \sqrt{15^2 - 4 \times 1 \times (-65)}}{2}$$
$$\therefore x = \frac{-15 \pm \sqrt{225 + 260}}{2}$$
$$\therefore x = \frac{-15 \pm \sqrt{485}}{2}$$
$$\therefore x = 3.511,\ x \simeq -18.51$$

Reject the negative value.
$\therefore x \simeq 3.511$
The length of the frame is $17 + 2 \times 3.511 = 24.0$ cm or 240 mm.
The width of the frame is $13 + 2 \times 3.511 = 20.0$ cm or 200 mm.

11 **a** Let the length be l metres.

$$x + l + x = 16$$
$$\therefore 2x + l = 16$$
$$\therefore l = 16 - 2x$$

b Area is the product of the length and width.

$$k = x(16 - 2x)$$
$$\therefore k = 16x - 2x^2$$
$$\therefore 2x^2 - 16x + k = 0$$

c Discriminant,

$$\Delta = (-16)^2 - 4 \times 2 \times k$$
$$= 256 - 8k$$

i No solutions if $\Delta < 0$

$$\therefore 256 - 8k < 0$$
$$\therefore 256 < 8k$$
$$\therefore k > 32$$

ii One solution if $\Delta = 0$

$$\therefore 256 - 8k = 0$$
$$\therefore k = 32$$

iii Two solutions if $\Delta > 0$

$$\therefore 256 - 8k > 0$$
$$\therefore k < 32$$

However, k is the area measure, so there are two solutions if $0 < k < 32$.

d There can only be solutions to $2x^2 - 16x + k = 0$ if $\Delta \geq 0$. This means $0 < k \leq 32$. The greatest value of k is therefore 32.

The largest area is 32 square metres.

To find the dimensions, the value of x needs to be found when $k = 32$.

$$2x^2 - 16x + 32 = 0$$
$$\therefore x^2 - 8x + 16 = 0$$
$$\therefore (x - 4)^2 = 0$$
$$\therefore x = 4$$

The length is $16 - 2 \times 4 = 8$ metres and the width is 4 metres.

e Put $k = 15$ in the equation $2x^2 - 16x + k = 0$.

$$\therefore 2x^2 - 16x + 15 = 0$$
$$\therefore x^2 - 8x + 7.5 = 0$$
$$\therefore (x^2 - 8x + 16) - 16 + 7.5 = 0$$
$$\therefore (x - 4)^2 - 8.5 = 0$$
$$\therefore x - 4 = +\sqrt{8.5}$$
$$\therefore x = 4 \pm \sqrt{8.5}$$
$$\therefore x \simeq 6.9 \text{ or } x \simeq 1.08$$

If $x = 6.91$, the width is 6.91 m and the length is $16 - 2 \times 6.91 = 2.18$ m.
If $x = 1.08$, the width is 1.08 m and the length is $16 - 2 \times 1.08 = 13.84$ m.
To use as much of the back fence as possible, $x = 1.08$.
The dimensions of the rectangle are width 1.1 metres and length 13.8 metres (to 1 d.p.).

12 Let the parcel of cards purchased for \$10 at the fete contain x cards.

Therefore, the cost per card is $\frac{10}{x}$ dollars.

The collector sells $(x - 2)$ cards to the friend at a cost per card of $\left(\frac{10}{x} + 1\right)$ dollars for a total of \$16, since the collector makes a profit of \$6.

$$\therefore (x - 2) \times \left(\frac{10}{x} + 1\right) = 16$$
$$\therefore 10 + x - \frac{20}{x} - 2 = 16$$
$$\therefore x - 8 = \frac{20}{x}$$
$$\therefore x^2 - 8x = 20$$
$$\therefore x^2 - 8x - 20 = 0$$
$$\therefore (x - 10)(x + 2) = 0$$
$$\therefore x = 10$$

(reject $x = -2$)
The friend receives 8 cards.

13 **a** One method is to define $A = \pi r^2 + \pi rl$ using Func name: A, Variable/s: r, l and Expression $\pi r^2 + \pi rl$.
Then use Equation/inequality to solve $20 = A(r, 5)$ for r.
This gives, to 3 decimal places, $\{r = -6.0525, r = 1.052\}$.
Rejecting the negative value, the radius of the base of the cone is 1.052 m.

b Using Equation/inequality to solve $20 = A(r, l)$ for r using Standard mode gives

$$\left\{ r = \frac{-\left(l\pi - \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi},\ r = \frac{-\left(l\pi + \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi} \right\}$$

Rejecting the negative value, $r = \dfrac{-l\pi + \sqrt{l^2\pi^2 + 80\pi}}{2\pi}$.

Highlight this expression and drag down to the next prompt. Add in the condition $l = 5$ from the keyboard Math Optn and evaluate on Decimal mode to obtain the same value for r as in Question 11.

$$\frac{-\left(l\pi - \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi}\,|\,l = 5$$

$= 1.052$

14 $d = kt^2$, where d is the distance fallen after time t and k is the constant of proportionality.

Replace t with $2t$.

$\therefore d \to k(2t)^2$

$\therefore d = 4\left(kt^2\right)$

The distance is quadrupled.

15 $H = kV^2$, where H is the number of calories of heat in a wire with voltage V and k is the constant of proportionality.

If the voltage is reduced by 20%, then 80% of it remains.

Replace V with $0.80V$.

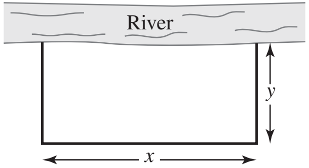
$\therefore H \quad \to \quad k(0.80V)^2$

$\therefore H \quad = \quad 0.64\left(kV^2\right)$

This means H is now 64% of what it was, so the effect of reducing the voltage by 20% is to reduce the number of calories of heat by 36%.

3.4 Exam questions

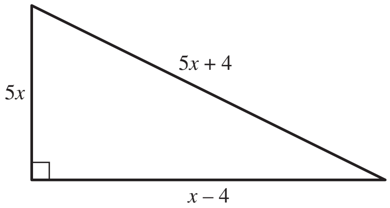
1 Let the length of each of the other sides be y.



$$\begin{aligned} 2y + x &= 100 \\ y &= 50 - \frac{x}{2} \\ \text{Area} &= xy \\ &= x\left(50 - \frac{x}{2}\right) \\ &= 50x - \frac{x^2}{2} \end{aligned}$$

The correct answer is **A**.

2



Use Pythagoras' theorem:

$$\begin{aligned} h^2 &= a^2 + b^2 \\ (5x+4)^2 &= (5x)^2 + (x-4)^2 \\ 25x^2 + 40x + 16 &= 25x^2 + x^2 - 8x + 16 \\ x^2 - 48x &= 0 \\ x(x-48) &= 0 \\ x &= 0, x = 48 \end{aligned}$$

$x = 0$ is not a solution in this context.

$\therefore x = 48$

The correct answer is **E**.

3 Let x be the first number. Then the second number is $x + 44$. [1 mark]

$$\begin{aligned} x(x+44) &= -483 \\ x^2 + 44x + 483 &= 0 \qquad \text{[1 mark]} \\ (x+23)(x+21) &= 0 \\ x &= -23, -21 \qquad \text{[1 mark]} \end{aligned}$$

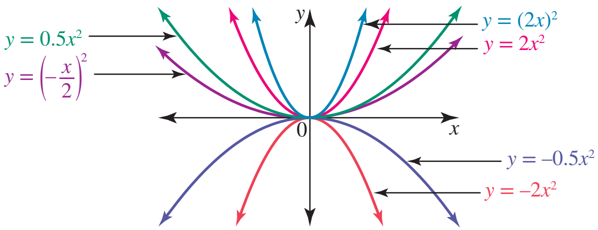
Two pairs of numbers: $-23, 21$ and $-21, 23$

The smallest number is -23. [1 mark]

3.5 Graphs of quadratic polynomials

3.5 Exercise

1



2 **A i** $y = x^2 - 2$

B ii $y = -2x^2$

C iii $y = -(x+2)^2$

3 a $y = x^2 + 8$

The equation represents a parabola formed by translating $y = x^2$ vertically upwards 8 units. Its turning point is $(0, 8)$.

b $y = x^2 - 8$

The equation represents a parabola formed by translating $y = x^2$ vertically downwards 8 units. Its turning point is $(0, -8)$.

c $y = 1 - 5x^2$

$y = -5x^2 + 1$

Parabolas with equations $y = ax^2 + k$ have a turning point at $(0, k)$.

Therefore, the turning point is $(0, 1)$.

d $y = (x - 8)^2$

The equation represents a parabola formed by translating $y = x^2$ horizontally 8 units to the right. Its turning point is $(8, 0)$.

e $y = (x + 8)^2$

The equation represents a parabola formed by translating $y = x^2$ horizontally 8 units to the left. Its turning point is $(-8, 0)$.

f $y = -\frac{1}{2}(x + 12)^2$

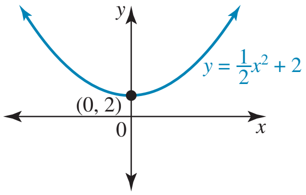
The equation can be written as $y = -\frac{1}{2}(x - (-12))^2$, which is in the form $y = a(x - h)^2$.

The turning point is $(-12, 0)$.

4 a $y = \frac{1}{2}x^2 + 2$

The turning point is $(0, 2)$, which is also the y-intercept.

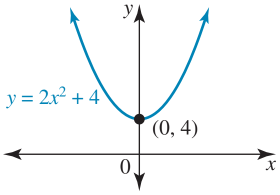
The coefficient of x^2 is positive, so the graph is concave up.



b $y = 2x^2 + 4$

The turning point is $(0, 4)$, which is also the y-intercept.

The coefficient of x^2 is positive, so the graph is concave up.



c $y = (x - 2)^2$

The turning point is $(2, 0)$, which is also the x-intercept.

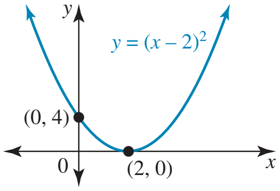
y-intercept: let $x = 0$.

$y = (0 - 2)^2$

$y = 4$

The y-intercept is $(0, 4)$.

The graph is concave up.



d $y = -\frac{1}{4}(x + 1)^2$

The turning point is $(-1, 0)$, which is also the x-intercept.

y-intercept: when $x = 0$,

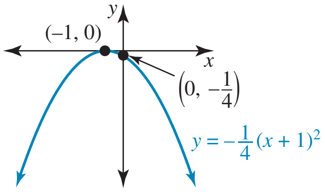
$y = -\frac{1}{4}(0 + 1)^2$

$y = -\frac{1}{4}(1)$

$y = -\frac{1}{4}$

The y-intercept is $\left(0, -\frac{1}{4}\right)$.

The parabola is concave down.



e $y = x^2 - 4$

The turning point is $(0, -4)$, which is also the y-intercept.

x-intercepts: when $y = 0$,

$0 = x^2 - 4$

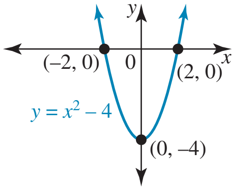
$4 = x^2$

$x = \pm\sqrt{4}$

$x = \pm 2$

The x-intercepts are $(-2, 0)$ and $(2, 0)$.

The graph is concave up.



f $y = -x^2 + 2$

The turning point is $(0, 2)$, which is also the y-intercept.

x-intercepts: let $y = 0$.

$0 = -x^2 + 2$

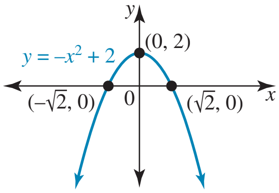
$-2 = -x^2$

$2 = x^2$

$x = \pm\sqrt{2}$

The x-intercepts are $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, 0\right)$.

The graph is concave down.



5 $y = \frac{1}{3}x^2 + x - 6$

Axis of symmetry equation:

$$x = -\frac{1}{\frac{2}{3}}$$

$$\therefore x = -\frac{3}{2}$$

Turning point: substitute $x = -\frac{3}{2}$.

$$\therefore y = \frac{1}{3} \times \left(-\frac{3}{2}\right)^2 + -\frac{3}{2} - 6$$

$$\therefore y = \frac{3}{4} - \frac{3}{2} - 6$$

$$\therefore y = -\frac{27}{4}$$

$\Rightarrow (-1.5, -6.75)$

y-intercept: $(0, -6)$

x-intercepts: put $y = 0$.

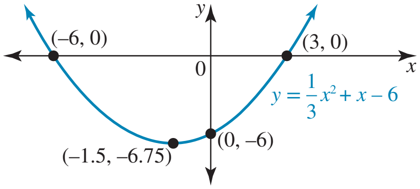
$$\therefore \frac{1}{3}x^2 + x - 6 = 0$$

$$\therefore x^2 + 3x - 18 = 0$$

$$\therefore (x + 6)(x - 3) = 0$$

$$\therefore x = -6, 3$$

$\Rightarrow (-6, 0), (3, 0)$



6 **a** $y = 9x^2 + 18x + 8$

y-intercept: let $x = 0$.

$y = 9(0)^2 + 18(0) + 8$

$y = 8$

The y-intercept is $(0, 8)$.

x-intercepts: let $y = 0$.

$0 = 9x^2 + 18x + 8$

$0 = (3x + 2)(3x + 4)$

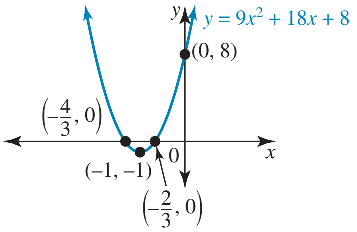
$x = -\frac{2}{3}, -\frac{4}{3}$

The x-intercepts are $\left(-\frac{4}{3}, 0\right)$ and $\left(-\frac{2}{3}, 0\right)$.

The axis of symmetry gives the x-coordinate of the turning point.

$$x_{\mathrm{TP}} = \frac{-b}{2a}, a = 1, b = 18$$
$$= \frac{-18}{18}$$
$$= -1$$
$$y_{\mathrm{TP}} = 9x^2 + 18x + 8$$
$$= 9(-1)^2 + 18(-1) + 8$$
$$= 9 - 18 + 8$$
$$= -1$$

The turning point is $(-1, -1)$.



b $y = -x^2 + 7x - 10$

y-intercept: when $x = 0$, $y = -10$.

The y-intercept is $(0, -10)$.

x-intercepts: let $y = 0$.

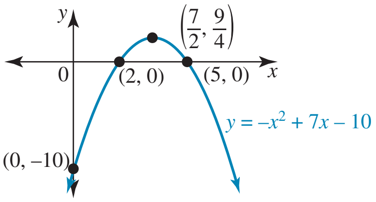
$$0 = -x^2 + 7x - 10$$
$$x^2 - 7x + 10 = 0$$
$$(x - 2)(x - 5) = 0$$
$$x = 2, 5$$

The x-intercepts are $(2, 0)$ and $(5, 0)$.

The axis of symmetry gives the x-coordinate of the turning point.

$$x_{\mathrm{TP}} = \frac{-b}{2a}, a = -1, b = 7$$
$$= \frac{-7}{-2}$$
$$= \frac{7}{2}$$
$$y_{\mathrm{TP}} = -x^2 + 7x - 10$$
$$= -\left(\frac{7}{2}\right)^2 + 7 \times \frac{7}{2} - 10$$
$$= -\frac{49}{4} + \frac{49}{2} - 10$$
$$= -\frac{49}{4} + \frac{98}{4} - \frac{40}{4}$$
$$= \frac{9}{4}$$

The turning point is $\left(\frac{7}{2}, \frac{9}{4}\right)$.



c $y = -x^2 - 2x - 3$

y-intercept: when $x = 0$, $y = -3$.

The y-intercept is $(0, -3)$.

x-intercepts: let $y = 0$.

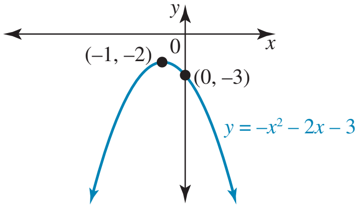
$$0 = -x^2 - 2x - 3$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \ a = -1, b = -2, c = -3$$
$$= \frac{2 \pm \sqrt{(-2)^2 - 4(-1)(-3)}}{2(-1)}$$
$$= \frac{2 \pm \sqrt{-8}}{-2}$$

There are no real solutions and hence no x-intercepts.

The axis of symmetry gives the x-coordinate of the turning point.

$$x_{\mathrm{TP}} = \frac{-b}{2a}$$
$$= \frac{2}{-2}$$
$$= -1$$
$$y_{\mathrm{TP}} = -x^2 - 2x - 3$$
$$= -(-1)^2 - 2(-1) - 3$$
$$= -1 + 2 - 3$$
$$= -2$$

The turning point is $(-1, -2)$.



d $y = x^2 - 4x + 2$

y-intercept: when $x = 0$, $y = 2$.

The y-intercept is $(0, 2)$.

x-intercepts: let $y = 0$.

$$0 = x^2 - 4x + 2$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a = 1, b = -4, c = 2$$
$$= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{8}}{2}$$
$$= \frac{4 \pm 2\sqrt{2}}{2}$$
$$= \frac{2\left(2 \pm \sqrt{2}\right)}{2}$$
$$= 2 \pm \sqrt{2}$$

The x-intercepts are $\left(2 - \sqrt{2}, 0\right)$ and $\left(2 + \sqrt{2}, 0\right)$.

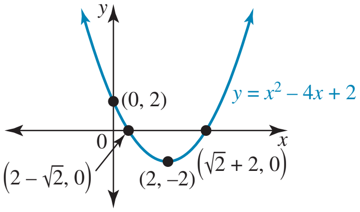
The axis of symmetry gives the x-coordinate of the turning point.

$$x_{\mathrm{TP}} = \frac{-b}{2a}$$
$$= \frac{4}{2}$$
$$= 2$$
$$y_{\mathrm{TP}} = x^2 - 4x + 2$$
$$= (2)^2 - 4(2) + 2$$

$= 4 - 8 + 2$

$= -2$

The turning point is $(2, -2)$.



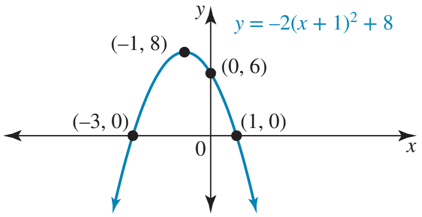
7 **a** $y = -2(x+1)^2 + 8$

Turning point: $(-1, 8)$ Type: maximum

y-intercept: put $x = 0 \therefore y = 6 \Rightarrow (0, 6)$

x-intercepts: put $y = 0$.

$$\begin{aligned}\therefore -2(x+1)^2 + 8 &= 0\\ \therefore (x+1)^2 &= 4\\ \therefore x + 1 &= \pm 2\\ \therefore x &= -3, x = 1\\ &\Rightarrow (-3, 0), (1, 0)\end{aligned}$$



b **i** $y = -x^2 + 10x - 30$

$= -(x^2 - 10x + 30)$

$= -[(x^2 - 10x + 25) - 25 + 30]$

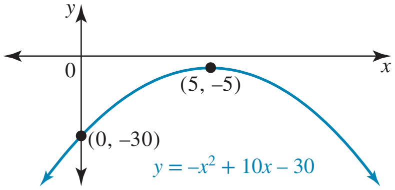
$= -\left[(x-5)^2 + 5\right]$

$= -(x-5)^2 - 5$

The vertex is $(5, -5)$, a maximum turning point.

ii y-intercept: $(0, -30)$

No x-intercepts, since the graph is concave down with a maximum y-value of -5.



8 **a** $y = 4 - 3x^2$

$\therefore y = -3x^2 + 4$

Maximum turning point at $(0, 4)$

b $y = (4 - 3x)^2$

$(4 - 3x) = 0 \Rightarrow x = \dfrac{4}{3}$

Therefore, there is a minimum turning point at $\left(\dfrac{4}{3}, 0\right)$

or

$y = (4 - 3x)^2$

$= (3x - 4)^2$

$= \left(3\left(x - \dfrac{4}{3}\right)\right)^2$

$= 9\left(x - \dfrac{4}{3}\right)^2$

Minimum turning point at $\left(\dfrac{4}{3}, 0\right)$

9 $y = a(x-h)^2 + k$ has turning point (h, k).

Testing each option:

A $y = -5x^2 + 2$ has turning point $(0, 2)$. Incorrect.

B $y = 2 - (x-5)^2$ rearranged is $y = -(x-5)^2 + 2$. The turning point is $(5, 2)$. Incorrect.

C $y = (x+2)^2 - 5$ has turning point $(-2, -5)$. Incorrect.

D $y = -(x+5)^2 + 2$ has turning point $(-5, 2)$. Correct.

E $y = (x+5)^2 - 2$ has turning point $(-5, -2)$. Incorrect.

The correct answer is **D**.

10 **a** $y = (x+4)^2 - 1$

The turning point is $(-4, -1)$.

y-intercept: let $x = 0$.

$y = (0+4)^2 - 1$

$y = 16 - 1$

$y = 15$

The y-intercept is $(0, 15)$.

x-intercepts: let $y = 0$.

$(x+4)^2 - 1 = 0$

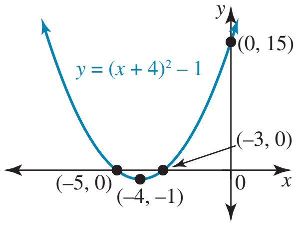
$(x+4)^2 = 1$

$x + 4 = \pm 1$

$x = 1 - 4 \quad \text{or} \quad x = -1 - 4$

$x = -3 \quad \text{or} \quad x = -5$

The x-intercepts are $(-5, 0)$ and $(-3, 0)$.



b $y = 3 - (x+4)^2$ rearranges to $y = -(x+4)^2 + 3$.

The turning point is $(-4, 3)$.

y-intercept: let $x = 0$.

$y = 3 - (0+4)^2$

$y = 3 - 16$

$y = -13$

The y-intercept is $(0, -13)$.

x-intercepts: let $y = 0$.

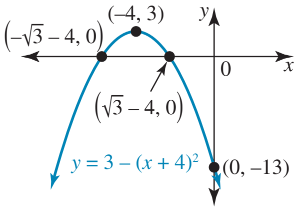
$0 = 3 - (x+4)^2$

$(x+4)^2 = 3$

$x + 4 = \pm\sqrt{3}$

$x = -4 \pm \sqrt{3}$

The x-intercepts are $\left(-4 - \sqrt{3}, 0\right)$ and $\left(-4 + \sqrt{3}, 0\right)$.



c **i** $x^2 + 6x + 12$

$= x^2 + 6x + \left(\dfrac{6}{2}\right)^2 - \left(\dfrac{6}{2}\right)^2 + 12$

$= (x^2 + 6x + 9) - 9 + 12$

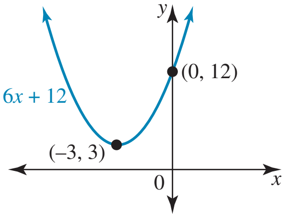
$= (x+3)^2 + 3$

$x^2 + 6x + 12 = (x+3)^2 + 3$

ii Since $y = x^2 + 6x + 12$ can be expressed as $y = (x + 3)^2 + 3$, its turning point is $(-3, 3)$.
Use the form $y = x^2 + 6x + 12$ to obtain the y-intercept.
When $x = 0$, $y = 12$, so $(0, 12)$ is the y-intercept.
x-intercepts: let $y = 0$ in $y = (x + 3)^2 + 3$.

$$(x + 3)^2 + 3 = 0$$
$$(x + 3)^2 = -3$$

There are no real solutions and hence no x-intercepts.



d $y = -(2x + 5)^2$

The equation can be written as $y = -\left(2\left(x + \dfrac{5}{2}\right)\right)^2$ or

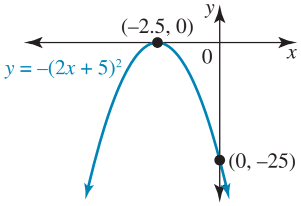
$$y = 4\left(x + \frac{5}{2}\right)^2.$$

The turning point is $\left(-\dfrac{5}{2}, 0\right)$.

The turning point is also the x-intercept.
y-intercept: let $x = 0$.

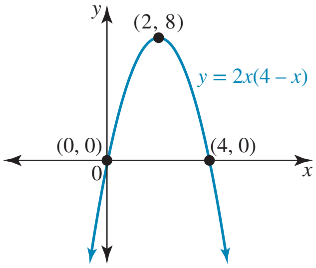
$$y = -(2(0) + 5)^2$$
$$y = -25$$

The y-intercept is $(0, -25)$.

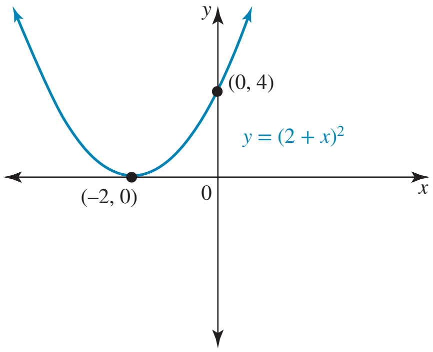


11 $y = 2x(4 - x)$
x-intercepts: $2x(4 - x) = 0$
$\therefore x = 0, x = 4$
$\Rightarrow (0, 0), (4, 0)$
Turning point:

$$x = \frac{0 + 4}{2}$$
$$\therefore x = 2$$
$$\therefore y = 4(2)$$
$$\therefore y = 8$$
$$\Rightarrow (2, 8)$$



12 $y = (2 + x)^2$
x-intercept and turning point: $(-2, 0)$
y-intercept: $(0, 4)$



13 **a** $y = (x + 1)(x - 3)$
y-intercept: let $x = 0$.

$$y = (0 + 1)(0 - 3)$$
$$y = -3$$

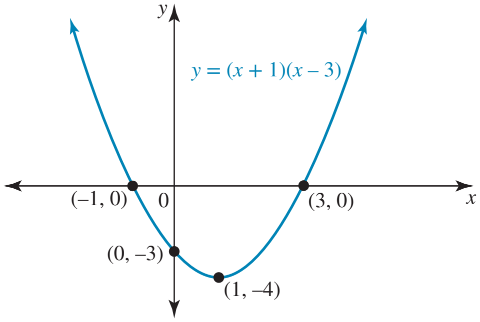
The y-intercept is $(0, -3)$.
x-intercepts: let $y = 0$.

$$(x + 1)(x - 3) = 0$$
$$x = -1, -3$$

The x-intercepts are $(-1, 0)$ and $(3, 0)$.
The axis of symmetry lies halfway between the x-intercepts. It gives the x-coordinate of the turning point.

$$x_{\text{TP}} = \frac{-1 + 3}{2}$$
$$= 1$$
$$y_{\text{TP}} = (x + 1)(x - 3)$$
$$= (1 + 1)(1 - 3)$$
$$= -4$$

The turning point is $(1, -4)$.



b $y = (x - 5)(2x + 1)$
y-intercept: let $x = 0$.

$$y = (0 - 5)(2 \times 0 + 1)$$
$$y = -5$$

The y-intercept is $(0, -5)$.
x-intercepts: let $y = 0$.

$$(2x + 1)(x - 5) = 0$$
$$x = -\frac{1}{2}, 5$$

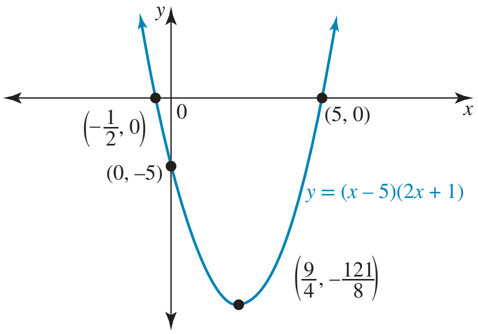
The x-intercepts are $\left(-\dfrac{1}{2}, 0\right)$ and $(5, 0)$.

The axis of symmetry lies halfway between the x-intercepts. It gives the x-coordinate of the turning point.

$$x_{\text{TP}} = \frac{-\frac{1}{2} + 5}{2}$$
$$= \frac{\frac{9}{2}}{2}$$
$$= \frac{9}{4}$$

$$
\begin{aligned}
y_{\text{TP}} &= (2x+1)(x-5)\\
&= \left(2\times\frac{9}{4}+1\right)\left(\frac{9}{4}-5\right)\\
&= \frac{11}{2}\times-\frac{11}{4}\\
&= -\frac{121}{8}
\end{aligned}
$$

The turning point is $\left(\frac{9}{4}, \frac{121}{8}\right)$.



14 a $y = x^2 - 9$

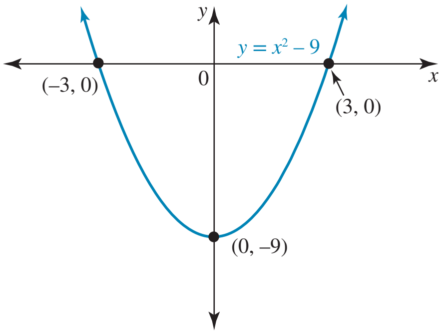
Min TP $(0, -9)$ and this is also the y-intercept.

x-intercepts: $0 = x^2 - 9$

$\therefore 0 = (x-3)(x+3)$

$\therefore x = 3, x = -3$

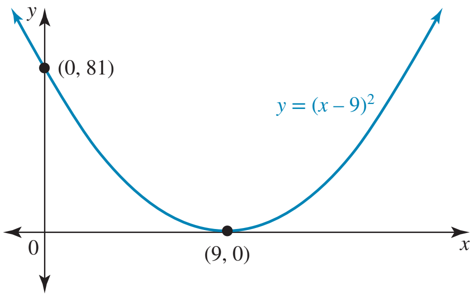
$(3, 0), (-3, 0)$



b $y = (x-9)^2$

Min TP $(9, 0)$ and this is also the x-intercept.

y-intercept: $y = (-9)^2 \Rightarrow (0, 81)$



c $y = 6 - 3x^2$

$\therefore y = -3x^2 + 6$

Max TP $(0, 6)$, which is also the y-intercept.

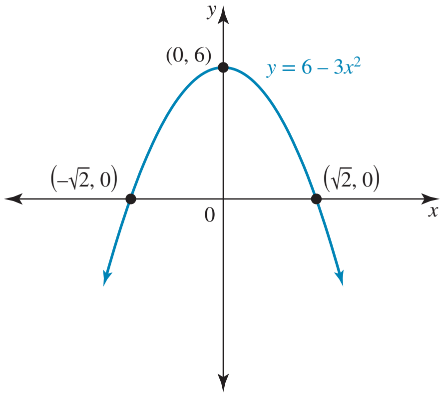
x-intercept: $0 = 6 - 3x^2$

$\therefore 3x^2 = 6$

$\therefore x^2 = 2$

$\therefore x = \pm\sqrt{2}$

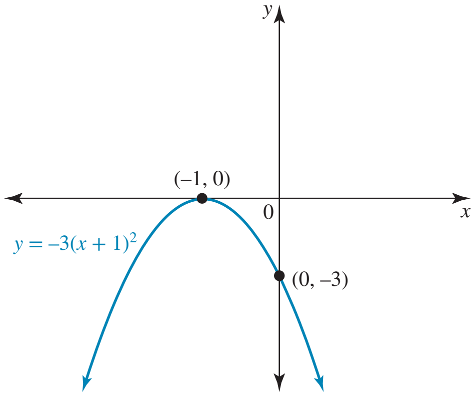
$\left(\sqrt{2}, 0\right), \left(-\sqrt{2}, 0\right)$



d $y = -3(x+1)^2$

Max TP $(-1, 0)$ and this is also the x-intercept.

y-intercept: $y = -3(1)^2 \Rightarrow (0, -3)$



15 a $y = x^2 + 6x - 7$

TP $x = \frac{-b}{2a}$

$\therefore x = \frac{-6}{2} = -3$

$\therefore y = 9 - 18 - 7 = -16$

Min TP $(-3, -16)$

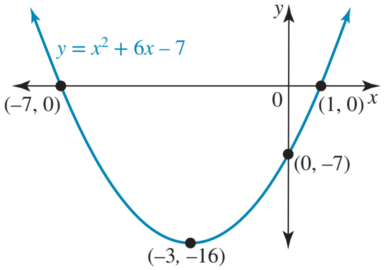
y-intercept $(0, -7)$

x-intercepts: $0 = x^2 + 6x - 7$

$\therefore (x+7)(x-1) = 0$

$\therefore x = -7, x = 1$

$(-7, 0), (1, 0)$



b $y = 3x^2 - 6x - 7$

TP $x = \frac{-b}{2a}$

$\therefore x = \frac{6}{6}$

$\therefore x = 1$

$y = 3 - 6 - 7$

$\therefore y = -10$

Min TP $(1, -10)$

y-intercept $(0, -7)$

x-intercept: $0 = 3x^2 - 6x - 7$

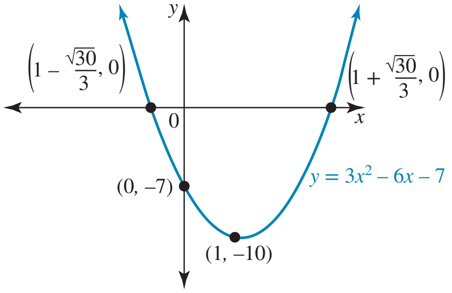
$$\therefore x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 3 \times (-7)}}{6}$$

$$= \frac{6 \pm \sqrt{120}}{6}$$

$$= \frac{6 \pm 2\sqrt{30}}{6}$$

$$\therefore x = \frac{3 \pm \sqrt{30}}{3}$$

$$\left(\frac{3+\sqrt{30}}{3}, 0\right), \left(\frac{3-\sqrt{30}}{3}, 0\right)$$



c $y = 5 + 4x - 3x^2$

$\therefore y = -3x^2 + 4x + 5$

TP $x = \dfrac{-b}{2a}$

$$\therefore x = \frac{-4}{-6}$$

$$\therefore x = \frac{2}{3}$$

$$y = -3 \times \frac{4}{9} + 4 \times \frac{2}{3} + 5$$

$$\therefore y = \frac{19}{3}$$

Max TP $\left(\dfrac{2}{3}, \dfrac{19}{3}\right)$

y-intercept $(0, 5)$

x-intercept: $0 = -3x^2 + 4x + 5$

$\therefore 3x^2 - 4x - 5 = 0$

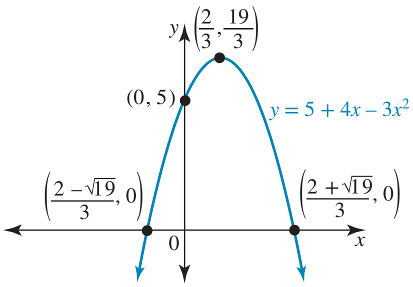
$$\therefore x = \frac{4 \pm \sqrt{4^2 - 4 \times 3 \times (-5)}}{6}$$

$$= \frac{4 \pm \sqrt{76}}{6}$$

$$= \frac{4 \pm 2\sqrt{19}}{6}$$

$$\therefore x = \frac{2 \pm \sqrt{19}}{3}$$

$$\left(\frac{2 \pm \sqrt{19}}{3}, 0\right)$$



d $y = 2x^3 - x - 4$

Min TP $x = \dfrac{-b}{2a}$

$$\therefore x = \frac{1}{4}$$

$$y = 2 \times \frac{1}{16} - \frac{1}{4} - 4$$

$$= -\frac{33}{8}$$

$$\left(\frac{1}{4}, -\frac{33}{8}\right)$$

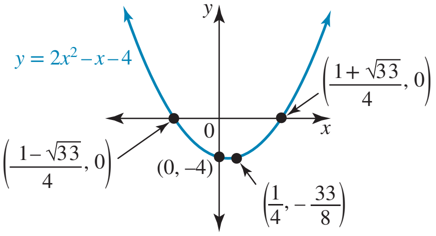
y-intercept $(0, -4)$

x-intercepts $0 = 2x^2 - x - 4$

$$\therefore x = \frac{1 \pm \sqrt{1 + 32}}{4}$$

$$\therefore x = \frac{1 \pm \sqrt{33}}{4}$$

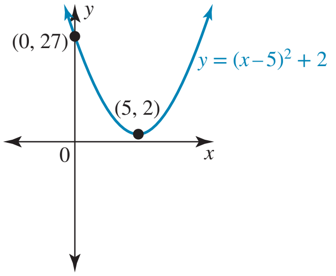
$$\left(\frac{1 \pm \sqrt{33}}{4}, 0\right)$$



16 **a** $y = (x - 5)^2 + 2$

Min TP $(5, 2)$ so there are no x-intercepts.

y-intercept $(0, 27)$



b $y = 2(x + 1)^2 - 2$

Min TP $(-1, -2)$

y-intercept $(0, 0)$

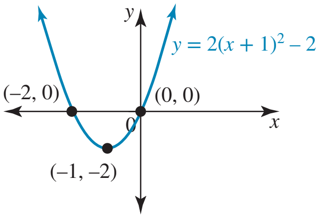
x-intercepts: $0 = 2(x + 1)^2 - 2$

$\therefore (x+1)^2 = 1$

$\therefore x + 1 = \pm 1$

$\therefore x = -2, x = 0$

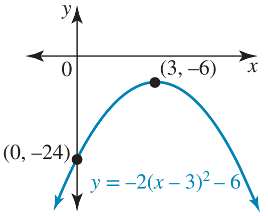
$(-2, 0), (0, 0)$



c $y = -2(x-3)^2 - 6$

Max TP $(3, -6)$ so no x-intercepts

y-intercept $(0, -24)$



d $y = -(x-4)^2 + 1$

Max TP $(4, 1)$

y-intercept $(0, -15)$

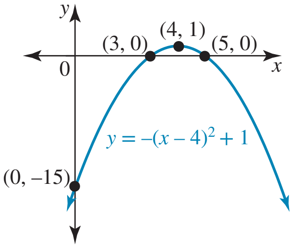
x-intercepts: $0 = -(x-4)^2 + 1$

$\therefore (x-4)^2 = 1$

$\therefore x - 4 = \pm 1$

$\therefore x = 3, x = 5$

$(3, 0), (5, 0)$



17 a $y = -2(1+x)(2-x)$

x-intercepts: $(-1, 0), (2, 0)$

Min TP $x = \dfrac{-1+2}{2}$

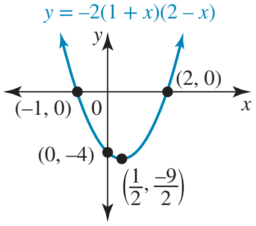
$\therefore x = \dfrac{1}{2}$

$y = -2 \times \dfrac{3}{2} \times \dfrac{3}{2}$

$\therefore y = -\dfrac{9}{2}$

$\left(\dfrac{1}{2}, -\dfrac{9}{2}\right)$

y-intercept $(0, -4)$



b $y = (2x+1)(2-3x)$

x-intercepts: $2x + 1 = 0, 2 - 3x = 0$

$\therefore x = -\dfrac{1}{2},\ x = \dfrac{2}{3}$

$\left(-\dfrac{1}{2}, 0\right), \left(\dfrac{2}{3}, 0\right)$

Max TP $x = \dfrac{-\frac{1}{2} + \frac{2}{3}}{2}$

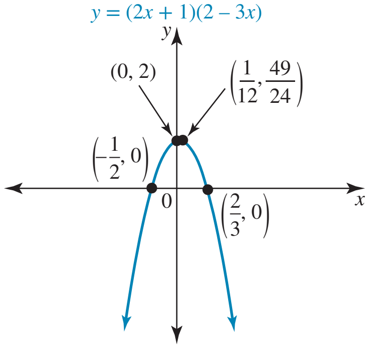
$\therefore x = \dfrac{1}{12}$

$y = \left(\dfrac{1}{6} + 1\right) \times \left(2 - \dfrac{1}{4}\right)$

$\therefore y = \dfrac{49}{24}$

$\left(\dfrac{1}{12}, \dfrac{49}{24}\right)$

y-intercept $(0, 2)$



c $y = 0.8x(10x - 27)$

x-intercepts: $(0, 0), (2.7, 0)$

Min TP $x = \dfrac{0 + 2.7}{2}$

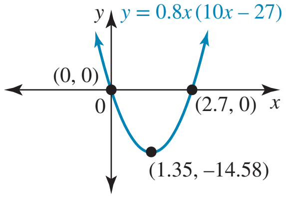
$\therefore x = 1.35$

$y = 0.8 \times 1.35 \times (13.5 - 27)$

$\therefore y = -14.58$

$(1.35, -14.58)$

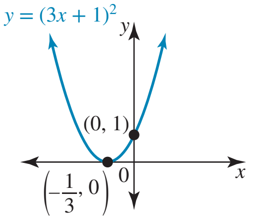
y-intercept $(0, 0)$



d $y = (3x+1)^2$

x-intercept and minimum turning point at $\left(-\dfrac{1}{3}, 0\right)$

y-intercept $(0, 1)$



18 **a** $y = \frac{1}{4}(1-2x)^2$

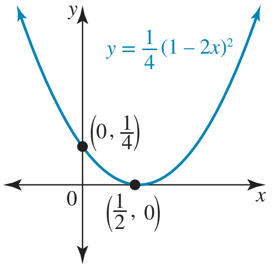
TP occurs when

$1 - 2x = 0$

$\therefore x = \frac{1}{2}$

Min TP $\left(\frac{1}{2}, 0\right)$ is also the x-intercept.

y-intercept: $y = \frac{1}{4}(1)^2 \Rightarrow \left(0, \frac{1}{4}\right)$



b $y = -0.25(1+2x)^2$

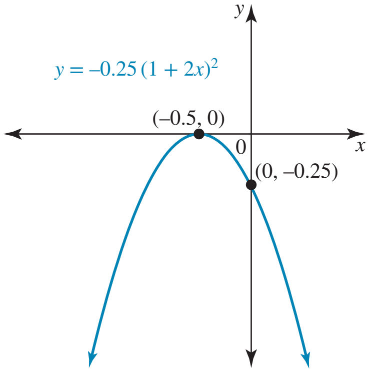
TP occurs when

$1 + 2x = 0$

$\therefore x = -0.5$

Max TP $(-0.5, 0)$ is also the x-intercept.

y-intercept: $y = -0.25(1)^2 \Rightarrow (0, -0.25)$



c $y = -2x^2 + 3x - 4$

Max TP: $x = \frac{-b}{2a}$

$\therefore x = \frac{3}{4}$

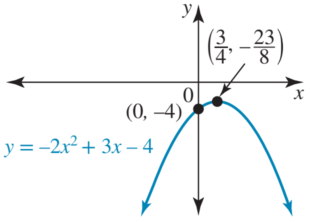
$y = -2 \times \frac{9}{16} + 3 \times \frac{3}{4} - 4$

$= -\frac{23}{8}$

$\left(\frac{3}{4}, -\frac{23}{8}\right)$

y-intercept $(0, -4)$

As the TP $\left(\frac{3}{4}, -\frac{23}{8}\right)$ is a maximum, there are no x-intercepts.



d $y = 10 - 2x^2 + 8x$

$\therefore y = -2x^2 + 8x + 10$

Max TP: $x = \frac{-b}{2a}$

$\therefore x = \frac{-8}{-4}$

$\therefore x = 2$

$y = -2(2)^2 + 8(2) + 10$

$\therefore y = 18$

$(2, 18)$

y-intercept $(0, 10)$

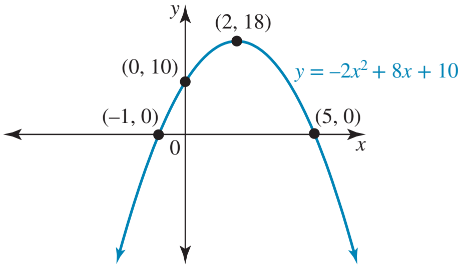
x-intercept: $0 = -2x^2 + 8x + 10$

$\therefore x^2 - 4x - 5 = 0$

$\therefore (x-5)(x+1) = 0$

$\therefore x = 5, x = -1$

$(5, 0), (-1, 0)$



19 **a** **i** $2x^2 - 12x + 9$

$$= 2\left[x^2 - 6x + \frac{9}{2}\right]$$

$$= 2\left[\left(x^2 - 6x + 9\right) - 9 + \frac{9}{2}\right]$$

$$= 2\left[(x-3)^2 - \frac{18}{2} + \frac{9}{2}\right]$$

$$= 2\left[(x-3)^2 - \frac{9}{2}\right]$$

$$= 2\,(x-3)^2 - 9$$

ii TP $(3, -9)$

iii The minimum value is given by the y-coordinate of the minimum turning point.

The minimum value is -9.

b **i** $-x^2 - 18x + 5 = -\left[x^2 + 18x - 5\right]$

$$= -\left[\left(x^2 + 18x + 81\right) - 81 - 5\right]$$

$$= -\left[(x+9)^2 - 86\right]$$

$$= -(x+9)^2 + 86$$

ii Maximum TP $(-9, 86)$

iii The maximum value is 86.

20 **a** $y = 42x - 18x^2$

$\Delta = b^2 - 4ac, a = -18, b = 42, c = 0$

$\therefore \Delta = 42^2 - 4 \times (-18) \times 0$

$= 42^2$

Since the discriminant is a perfect square, there are 2 rational x-intercepts (obvious from the factors of the equation).

b Factored form is $y = 6x(7 - 3x)$.

x-intercepts: $(0, 0), \left(\frac{7}{3}, 0\right)$

Turning point:

$x = \frac{0 + \frac{7}{3}}{2}$

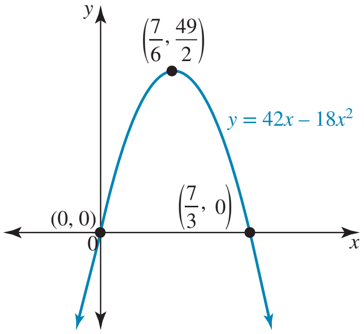
$$\therefore x = \frac{7}{6}$$

$$\therefore y = 6 \times \frac{7}{6}\left(7 - 3 \times \frac{7}{6}\right)$$

$$\therefore y = 7\left(\frac{7}{2}\right)$$

$$\therefore y = \frac{49}{2}$$

$$\Rightarrow \left(\frac{7}{6}, \frac{49}{2}\right)$$



21 $7x^2 - 4x + 9$

$$\Delta = b^2 - 4ac, \; a = 7, \; b = -4, \; c = 9$$
$$\therefore \Delta = (-4)^2 - 4 \times 7 \times 9$$
$$= -236$$

Since $\Delta < 0$ and $a > 0$, $0.7x^2 - 4x + 9$ is positive definite.

22 **a** $y = 9x^2 + 17x - 12$

$$\Delta = b^2 - 4ac \; a = 9, \; b = 17, \; c = -12$$
$$\therefore \Delta = 289 - 4 \times 9 \times (-12)$$
$$\therefore \Delta = 721$$

Since $\Delta > 0$ but not a perfect square, there are two irrational intercepts with the x-axis.

b $y = -5x^2 + 20x - 21$

$$\Delta = b^2 - 4ac \; a = -5, \; b = 20, \; c = -21$$
$$\therefore \Delta = 400 - 4 \times (-5) \times (-21)$$
$$\therefore \Delta = -20$$

Since $\Delta < 0$, there are no intercepts with the x-axis.

c $y = -3x^2 - 30x - 75$

$$\Delta = b^2 - 4ac \quad a = -3, \; b = -30, \; c = -75$$
$$\therefore \Delta = 900 - 4 \times (-3) \times (-75)$$
$$\therefore \Delta = 0$$

Since $\Delta = 0$, there is one rational intercept with the x-axis.

d $y = 0.02x^2 + 0.5x + 2$

$$\Delta = b^2 - 4ac \quad a = 0.02, \; b = 0.5, \; c = 2$$
$$\therefore \Delta = 0.25 - 4 \times 0.02 \times 2$$
$$\therefore \Delta = 0.09$$
$$\therefore \Delta = (0.3)^2$$

Since $\Delta > 0$ and it is a perfect square, there are two rational intercepts with the x-axis.

23 $y = 5x^2 + 10x - k$

a For one x-intercept, $\Delta = 0$.

$$\Delta = (10)^2 - 4 \times (5) \times (-k)$$
$$= 100 + 20k$$

Therefore, $100 + 20k = 0 \Rightarrow k = -5$

b For two x-intercepts, $\Delta > 0$.

Therefore, $100 + 20k > 0 \Rightarrow k > -5$

c For no x-intercepts, $\Delta < 0$.

Therefore, $100 + 20k < 0 \Rightarrow k < -5$.

24 **a** $mx^2 - 2x + 4$ is positive definite if $\Delta < 0$ and $m > 0$.

$$\Delta = (-2)^2 - 4 \times m \times 4$$
$$= 4 - 16m$$
$$\Delta < 0 \Rightarrow 4 - 16m < 0$$
$$\therefore m > \frac{1}{4}$$

The graph is positive definite for $m > \frac{1}{4}$.

b **i** $px^2 + 3x - 9$ is positive definite if $\Delta < 0$ and $p > 0$.

$$\Delta = (3)^2 - 4 \times p \times (-9)$$
$$= 9 + 36p$$
$$\Delta < 0 \Rightarrow 9 + 36p < 0 \therefore p < -\frac{1}{4}$$

There is no positive value of p for which $\Delta < 0$. Hence, there is no real value of p for which $px^2 + 3x - 9$ is positive definite.

ii If $p = 3, \; y = 3x^2 + 3x - 9$.

The equation of the axis of symmetry is $x = -\frac{b}{2a}$.

$$\therefore x = -\frac{3}{6}$$
$$\therefore x = -\frac{1}{2}$$

The axis of symmetry has equation $x = -\frac{1}{2}$.

c **i** $y = 2x^2 - 3tx + 12$

If the turning point lies on the x-axis, there is only one x-intercept, so $\Delta = 0$.

$$\therefore (-3t)^2 - 4 \times 2 \times 12 = 0$$
$$\therefore 9t^2 = 96$$
$$\therefore t^2 = \frac{32}{3}$$
$$\therefore t = \pm\sqrt{\frac{32}{3}}$$
$$\therefore t = \pm\frac{4\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$\therefore t = \pm\frac{4\sqrt{6}}{3}$$

ii The equation of the axis of symmetry is $x = -\frac{b}{2a}$.

$$\therefore x = -\frac{-3t}{4}$$
$$\therefore x = \frac{3t}{4}$$

For the equation of the axis of symmetry to be $x = 3t^2$,

$$3t^2 = \frac{3t}{4}$$
$$\therefore 12t^2 - 3t = 0$$
$$\therefore 3t(4t - 1) = 0$$
$$\therefore t = 0 \quad \text{or} \quad t = \frac{1}{4}$$

25 **a** **i** As the x-coordinate of the turning point is $x = -\frac{b}{2a}$, one method is to enter $y = 12x^2 - 18x + 5|x = -(-18)/24$ in the main screen on Standard mode and press EXE to calculate the y-coordinate.

The turning point has $y = \frac{7}{4}$, and simplifying the x-coordinate gives $x = \frac{3}{4}$.

The turning point is $\left(\frac{3}{4}, -\frac{7}{4}\right)$.

ii In the main screen, use Equation/Inequality to solve $12x^2 - 18x + 5 = 0$. This gives the x-intercepts as $x = \frac{-\sqrt{21}+9}{12}, x = \frac{\sqrt{21}+9}{12}$.

b i Entering $y = -8x^2 + 9x + 12|x = -9/-16$ gives $y = \frac{465}{32}$.

The turning point is $\left(\frac{9}{16}, \frac{465}{32}\right)$.

ii Solving $-8x^2 + 9x + 12 = 0$ gives the x-intercepts as $x = \frac{-\left(\sqrt{465}-9\right)}{16}, x = \frac{\sqrt{465}+9}{16}$.

3.5 Exam questions

1 $y = a(x-h)^2 + k$

h is a horizontal translation; k is a vertical translation.

$y = (x-2)^2 + 3$

$\therefore$ translated 2 unit to the right and 3 up

The correct answer is **B**.

2 Turning point: $x = \frac{-b}{2a}$

$a = 1, b = -6$

$x = \frac{-(-6)}{2(1)}$

$= 3$

Substitute $x = 3$.

$f(3) = 3^2 - 6(3) + 7$

$= -2$

$\therefore$ turning point $(3, -2)$

The correct answer is **C**.

3 $y = 2x^2 + x - 3$

y-intercept, $x = 0$:

$y = 2(0)^2 + (0) - 3$

$y = -3$

y-intercept $(0, -3)$ [1 mark]

$x-$ intercept, $y = 0$:

$(0) = 2x^2 + x - 3$

$(2x+3)(x-1) = 0$

$x = \frac{-3}{2}, 1$

x-intercepts $\left(\frac{-3}{2}, 0\right), (1, 0)$ [1 mark]

Equation of axis of symmetry:

$x = \frac{-b}{2a}$

$a = 2,\ b = 1,\ c = -3$

$x = -\frac{1}{2(2)}$

$x = -\frac{1}{4}$ [1 mark]

y-coordinate of axis of symmetry:

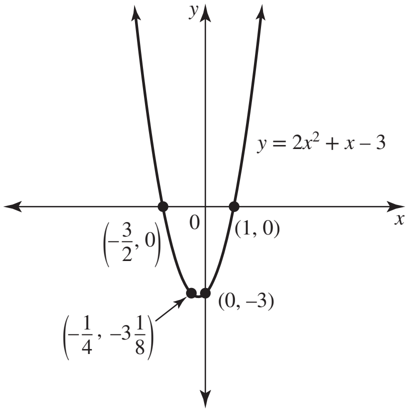
$y = 2\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right) - 3$

$y = \frac{1-2-24}{8}$

$y = -\frac{25}{8}$

$y = -3\frac{1}{8}$

$\therefore$ turning point $= \left(-\frac{1}{4}, -3\frac{1}{8}\right)$ [1 mark]



[1 mark]

3.6 Determining the rule from a graph of a quadratic polynomial

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3.6 Exercise

1 a $y = x^2 + c$

Substitute the given point $(1, 5)$ in the equation.

$5 = 1^2 + c$

$5 = 1 + c$

$c = 4$

The equation is $y = x^2 + 4$.

b $y = ax^2$.

Substitute the given point $(6, -2)$ in the equation.

$-2 = a(6)^2$

$-2 = 36a$

$a = -\frac{2}{36}$

$a = -\frac{1}{18}$

The equation is $y = -\frac{1}{18}x^2$.

c $y = a(x-2)^2$

Substitute the given point $(0, -12)$.

$-12 = a(0-2)^2$

$-12 = 4a$

$a = -3$

The equation is $y = -3(x-2)^2$.

2 $y = x^2 + bx$

a Substitute the point $(-3, 3)$ in the equation.

$3 = (-3)^2 + b(-3)$

$3 = 9 - 3b$

$3b = 6$

$b = 2$

The equation is $y = x^2 + 2x$.

b For x-intercepts, let $y = 0$.

$x^2 + 2x = 0$

$x(x + 2) = 0$

$x = 0, -2$

The x-intercepts are $(-2, 0), (0, 0)$.

3 a $y = (x - a)^2 + c$

Turning point $(4, -8)$

The equation is $y = (x - 4)^2 - 8$.

b $y = -2(x - h)^2 + k$

Turning point $(-1, 3)$

The equation is $y = -2(x + 1)^2 + 3$.

4 a $y = A(x - h)^2 + k$

Vertex $(5, 12)$

The equation becomes $y = A(x - 5)^2 + 12$.

b $y = A(x - 5)^2 + 12$

Substitute the point $(0, 7)$.

$7 = A(0 - 5)^2 + 12$

$7 = 25A + 12$

$-5 = 25A$

$\frac{-5}{25} = A$

$A = -\frac{1}{5}$

5 a For an x-intercept at $x = 3$, $x - 3$ would be a factor of the equation.

For an x-intercept at $x = 8$, $x - 8$ would be a factor of the equation.

A possible equation is $y = (x - 3)(x - 8)$, but any answer in the form $y = a(x - 3)(x - 8)$ would be correct.

b For an x-intercept at $x = -11$, $x + 11$ would be a factor of the equation.

For an x-intercept at $x = 2$, $x - 2$ would be a factor of the equation.

A possible equation is $y = (x + 11)(x - 2)$, but any answer in the form $y = a(x + 11)(x - 2)$ would be correct.

6 a Given information: minimum turning point $(-2, 1)$ and point $(0, 5)$

Let $y = a(x - h)^2 + k$.

$\therefore y = a(x + 2)^2 + 1$

Substitute $(0, 5)$.

$\therefore 5 = a(2)^2 + 1$

$\therefore a = 1$

Therefore, the equation of the parabola is $y = (x + 2)^2 + 1$.

b Given information: x-intercepts at $x = 0, x = 2$, and point $(-1, 6)$

Let $y = a(x - x_1)(x - x_2)$.

Since the x-intercepts are at $x = 0, x = 2$,

$\therefore y = ax(x - 2)$

Substitute $(-1, 6)$.

$\therefore 6 = a(-1)(-1 - 2)$

$\therefore 6 = 3a$

$\therefore a = 2$

Therefore, the equation of the parabola is $y = 2x(x - 2)$.

7 Given information: minimum turning point and x-intercept $(-2, 0)$, point $(2, 2)$

Let $y = a(x - h)^2 + k$ or $y = a(x - x_1)(x - x_2)$

$\therefore y = a(x + 2)^2$

Substitute $(2, 2)$.

$\therefore 2 = a(2 + 2)^2$

$\therefore 2 = a(16)$

$\therefore a = \frac{1}{8}$

Therefore, the equation is $y = \frac{1}{8}(x + 2)^2$.

To obtain polynomial form:

$y = \frac{1}{8}(x + 2)^2$

$\therefore y = \frac{1}{8}(x^2 + 4x + 4)$

$\therefore y = \frac{1}{8}x^2 + \frac{1}{2}x + \frac{1}{2}$

8 a From the diagram, the maximum turning point is $(0, 6)$.

$\therefore y = ax^2 + 6$

The point $(1, 4)$ lies on the graph.

$\therefore 4 = a(1)^2 + 6$

$\therefore a = -2$

The equation is $y = -2x^2 + 6$.

b From the diagram, the x-intercepts are $x = -6$ and $x = -1$.

$\therefore y = a(x + 6)(x + 1)$

The point $(-9, 4.8)$ lies on the graph.

$\therefore 4.8 = a(-9 + 6)(-9 + 1)$

$\therefore 4.8 = 24a$

$\therefore a = 0.2$

The equation is $y = 0.2(x + 6)(x + 1)$.

9 a TP $(2, -4) \Rightarrow y = a(x - 2)^2 - 4$

Point $(0, -2) \Rightarrow -2 = a(4) - 4$

$\therefore 4a = 2$

$\therefore a = \frac{1}{2}$

The equation is $y = \frac{1}{2}(x - 2)^2 - 4$.

b x-intercepts: $0 = \frac{1}{2}(x - 2)^2 - 4$

$\therefore (x - 2)^2 = 8$

$\therefore x - 2 = \pm\sqrt{8}$

$\therefore x = 2 \pm 2\sqrt{2}$

The distance between $x = 2 - 2\sqrt{2}$ and $x = 2 + 2\sqrt{2}$ is $2 \times 2\sqrt{2} = 4\sqrt{2}$.

The length of the intercept cut off on the x-axis is $4\sqrt{2}$ units.

10 a $y = 3x^2 \rightarrow y = 3(x + 1)^2 - 5$ or $y = 3x^2 + 6x - 2$

b $y = (x - 3)^2$ has vertex $(3, 0)$. Translating 8 units to the left, $(3, 0) \rightarrow (-5, 0)$. The equation of the image is $y = (x + 5)^2$.

11 a Let the equation be in the form $y = a(x - h)^2 + k$.

The given graph has a turning point at $\left(\frac{1}{2}, 4\right)$.

The equation becomes $y = a\left(x - \frac{1}{2}\right)^2 + 4$.

Substitute the point $(0, 5)$ to find a.

$5 = a\left(0 - \frac{1}{2}\right)^2 + 4$

$5 = \frac{1}{4}a + 4$

$1 = \frac{1}{4}a$

$a = 4$

The equation is $y = 4\left(x - \frac{1}{2}\right)^2 + 4$.

This can be written as

$y = 2^2\left(x - \frac{1}{2}\right)^2 + 4$

$y = \left(2\left(x - \frac{1}{2}\right)\right)^2 + 4$

$y = (2x - 1)^2 + 4$

Equation A is correct.

b Let the equation be in the form $y = a(x - h)^2 + k$.

The given graph has a turning point at $(5, 0)$.

The equation becomes $y = a\,(x - 5)^2$,

Substitute the point $\left(0, \frac{25}{2}\right)$ to find a.

$\frac{25}{2} = a(-5)^2$

$\frac{25}{2} = 25a$

$a = \frac{1}{2}$

The equation is $y = \frac{1}{2}(x - 5)^2$.

Since $(x - 5)^2$ is the same as $(5 - x)^2$, the equation can be expressed as $y = \frac{1}{2}(5 - x)^2$.

Equation A is correct.

c As the x-intercepts are given, let the equation be $y = a(x - x_1)(x - x_2)$.

Substitute the intercept values of $-\sqrt{2}$ and $\sqrt{2}$.

$y = a\left(x, -, -, \sqrt{2}\right)\left(x, -, \sqrt{2}\right)$

$y = a\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$

Expanding the difference of two squares,

$y = a\left(x^2 - \left(\sqrt{2}\right)^2\right)$

$y = a(x^2 - 2)$

Substitute the given point $(4, -7)$ to find a.

$-7 = a(4^2 - 2)$

$-7 = a(14)$

$a = -\frac{7}{14}$

$a = -\frac{1}{2}$

The equation is $y = -\frac{1}{2}(x^2 - 2)$.

Equation A is correct.

12 Let $y = ax^2 + bx + c$.

$(-1, -7) \Rightarrow -7 = a(-1)^2 + b(-1) + c$

$\therefore a - b + c = -7 \qquad [1]$

$(2, -10) \Rightarrow -10 = a(2)^2 + b(2) + c$

$\therefore 4a + 2b + c = -10 \qquad [2]$

$(4, -32) \Rightarrow -32 = a(4)^2 + b(4) + c$

$\therefore 16a + 4b + c = -32 \qquad [3]$

Use CAS technology to solve the equations.

Alternatively:

Eliminate c.

[2] − [1]

$\therefore 3a + 3b = -3$

$\therefore a + b = -1 \qquad [4]$

[3] − [1]

$\therefore 15a + 5b = -25$

$\therefore 3a + b = -5 \qquad [5]$

Solving equations [4] and [5]:

[5] − [4]

$\therefore 2a = -4$

$\therefore a = -2$

$\therefore b = 1$

In equation [1],

$-2 - 1 + c = -7$

$\therefore c = -4$

Therefore, the equation of the parabola is $y = -2x^2 + x - 4$.

13 Let $y = ax^2 + bx + c$.

$(0, -2) \Rightarrow -2c = a(0)^2 + b(0) + c$

$\therefore c = -2$

$\therefore y = ax^2 + bx - 2$

$(-1, 0) \Rightarrow 0 = a(-1)^2 + b(-1) - 2$

$\therefore a - b - 2 = 0$

$\therefore a - b = 2 \qquad [1]$

$(4, 0) \Rightarrow 0 = a(4)^2 + b(4) - 2$

$\therefore 16a + 4b - 2 = 0$

$\therefore 8a + 2b = 1 \qquad [2]$

Solving equations [1] and [2],

[2] + 2 × [1]

$10a = 5$

$\therefore a = \frac{1}{2}$

$\therefore \frac{1}{2} - b = 2$

$\therefore b = -\frac{3}{2}$

Therefore, the equation is $y = \frac{1}{2}x^2 - \frac{3}{2}x - 2$.

In Worked example 17b, the equation $y = \frac{1}{2}(x + 1)(x - 4)$ was obtained.

$y = \frac{1}{2}x^2 - \frac{3}{2}x - 2$

$\therefore y = \frac{1}{2}(x^2 - 3x - 4)$

$\therefore y = \frac{1}{2}(x + 1)(x - 4)$

So the equations represent the same parabola.

14 A $(-1, 10)$, B $(1, 0)$, C $(2, 4)$

a Let the equation be $y = ax^2 + bx + c$.

$(-1, 10) \Rightarrow 10 = a - b + c \qquad [1]$

$(1, 0) \Rightarrow 0 = a + b + c \qquad [2]$

$(2, 4) \Rightarrow 4 = 4a + 2b + c \qquad [3]$

Use CAS technology to solve the equations.

Alternatively:

Equation [2] − equation [1]:

$-10 = 2b$

$\therefore b = -5$

Substitute $b = -5$ in equations [2] and [3]:

Equation [2] $\Rightarrow 5 = a + c \quad [4]$

Equation [3] $\Rightarrow 4 = 4a - 10 + c$

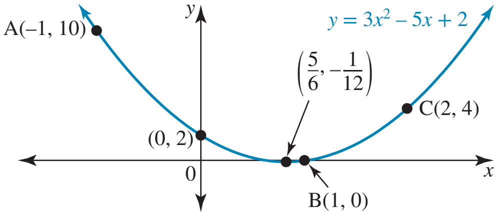
$\therefore 14 = 4a + c \quad [5]$

Equation [5] − equation [4]:
$9 = 3a$
$\therefore a = 3$
Substitute $a = 3$ in equation [4]:
$\therefore c = 2$
The equation is $y = 3x^2 - 5x + 2$.

b y-intercept $(0, 2)$
x-intercepts: $3x^2 - 5x + 2 = 0$
$\therefore (3x - 2)(x - 1) = 0$
$\therefore x = \frac{2}{3}, x = 1$
x-intercepts $\left(\frac{2}{3}, 0\right), (1, 0)$

c TP $x = \dfrac{\frac{2}{3} + 1}{2}$
$\therefore x = \frac{5}{6}$
$y = \left(3 \times \frac{5}{6} - 2\right)\left(\frac{5}{6} - 1\right)$
$\therefore y = \frac{1}{2} \times \frac{-1}{6}$
$\therefore y = -\frac{1}{12}$
The vertex is $\left(\frac{5}{6}, -\frac{1}{12}\right)$.

d



15 a Answers may vary but all are of the form
$y = a(x + 3)(x - 5)$.
Three possible answers are $y = (x + 3)(x - 5)$, $y = 2(x + 3)(x - 5)$ and $y = -3(x + 3)(x - 5)$.

b The point $(0, 45)$ also lies on the parabola.
$\therefore 45 = a(3)(-5)$
$\therefore -15a = 45$
$\therefore a = -3$
The equation is $y = -3(x + 3)(x - 5)$.
Vertex $x = \frac{-3 + 5}{2}$
$\therefore x = 1$
$y = -3(4)(-4)$
$\therefore y = 48$
The vertex is $(1, 48)$.

16 a The axis of symmetry at $x = 4$ means the equation is of the form $y = a(x - 4)^2 + k$.
Point $(0, 6) \Rightarrow 6 = 16a + k$ [1]
Point $(6, 0) \Rightarrow 0 = 4a + k$ [2]
Equation [1] − equation [2]:
$6 = 12a$
$\therefore a = \frac{1}{2}$
Substitute $a = \frac{1}{2}$ in equation [2]:
$\therefore k = -2$
The equation is $y = \frac{1}{2}(x - 4)^2 - 2$
$\therefore y = \frac{1}{2}(x^2 - 8x + 16) - 2$
$\therefore y = \frac{1}{2}x^2 - 4x + 6$

17 a $y = (ax + b)(x + c)$
Point $(5, 0) \Rightarrow 0 = (5a + b)(5 + c)$
$\therefore 5a + b = 0$ or $5 + c = 0$
$\therefore b = -5a$ or $c = -5$
Point $(0, -10) \Rightarrow -10 = (b)(c)$ [1]
Consider the case where $c = -5$.
Equation [1] becomes
$-10 = -5b$
$\therefore b = 2$
Substitute in $y = (ax + b)(x + c)$.
$\therefore y = (ax + 2)(x - 5)$
$\therefore y = ax^2 - 5ax + 2x - 10$
$\therefore y = ax^2 + x(2 - 5a) - 10$
Consider the case $b = -5a$.
Equation [1] becomes
$-10 = -5ac$
$\therefore c = \frac{2}{a}$
Substitute in $y = (ax + b)(x + c)$.
$\therefore y = (ax - 5a)\left(x + \frac{2}{a}\right)$
$\therefore y = ax^2 + 2x - 5ax - 10$
$\therefore y = ax^2 + (2 - 5a)x - 10$
Either way, $y = ax^2 + (2 - 5a)x - 10$.

b $\Delta = (2 - 5a)^2 - 4 \times a \times (-10)$
$\therefore \Delta = (2 - 5a)^2 + 40a$
$\therefore \Delta = 4 - 20a + 25a^2 + 40a$
$\therefore \Delta = 4 + 20a + 25a^2$
$\therefore \Delta = (2 + 5a)^2$

c Given $\Delta = 4$
$\therefore (2 + 5a)^2 = 4$
$\therefore 2 + 5a = \pm 2$
$\therefore 5a = -4$ or $5a = 0$
$\therefore a = -\frac{4}{5}, a = 0$
As a is the coefficient of x^2, for a parabola $a \neq 0$.
Therefore, reject $a = 0$.
$\therefore a = -\frac{4}{5}$
With $a = -\frac{4}{5}$, the equation of the parabola becomes
$y = -\frac{4}{5}x^2 + \left(2 - 5 \times \frac{-4}{5}\right)x - 10$
$\therefore y = -\frac{4}{5}x^2 + 6x - 10$
Factorising,
$y = -\frac{2}{5}(2x^2 - 15x + 25)$
$\therefore y = -\frac{2}{5}(2x - 5)(x - 5)$
The x-intercepts are at $x = \frac{5}{2}, x = 5$. Since $x = 5$ was given, the other x-intercept is $\left(\frac{5}{2}, 0\right)$.

18 **a** The point $(-4, 0)$ is the turning point.

$\therefore y = a(x+4)^2$

Point $(2, 9) \Rightarrow 9 = a(6)^2$

$\therefore a = \frac{9}{36}$

$\therefore a = \frac{1}{4}$

The equation is $y = \frac{1}{4}(x+4)^2$.

b The point $(p, 0)$ is the turning point.

$\therefore y = a(x-p)^2$

Point $(2, 9) \Rightarrow 9 = a(2-p)^2 \quad [1]$

Point $(0, 36) \Rightarrow 36 = a(-p)^2$

$\therefore 36 = ap^2 \quad [2]$

Equation [2] ÷ equation [1]:

$$\frac{36}{9} = \frac{ap^2}{a(2-p)^2}$$

$$\therefore 4 = \frac{p^2}{(2-p)^2}$$

$\therefore 4\left(4 - 4p + p^2\right) = p^2$

$\therefore 3p^2 - 16p + 16 = 0$

$\therefore (3p-4)(p-4) = 0$

$$\therefore p = \frac{4}{3}, p = 4$$

Therefore, there are two possible values for p.

If $p = \frac{4}{3}$, equation $[2] \Rightarrow 36 = a \times \frac{16}{9}$

$36 = a \times \frac{16}{9}$

$\therefore a = \frac{36 \times 9}{16}$

$\therefore a = \frac{81}{4}$

The equation of the parabola $y = a(x-p)^2$ becomes

$y = \frac{81}{4}\left(x - \frac{4}{3}\right)^2$.

$\therefore y = \frac{81}{4}\left(\frac{3x-4}{3}\right)^2$

$\therefore y = \frac{81}{4} \times \frac{(3x-4)^2}{9}$

$\therefore y = \frac{9}{4}(3x-4)^2$

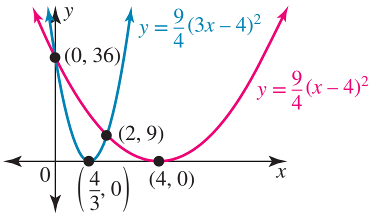
If $p = 4$, equation $[2] \Rightarrow 36 = a \times 16$

$\therefore a = \frac{9}{4}$

The equation of the parabola would be $y = \frac{9}{4}(x-4)^2$.

The graphs of both these parabolas contain the points $(2, 9)$ and $(0, 10)$.

The turning point of $y = \frac{9}{4}(3x-4)^2$ is $\left(\frac{4}{3}, 0\right)$ and the turning point of $y = \frac{9}{4}(x-4)^2$ is $(4, 0)$.



3.6 Exam questions

1 $y = a(x-h)^2 + k$

$(h, k) = (-4, -11)$

$y = a(x+4)^2 - 11$

Substitute $(-3, -10)$.

$-10 = a(-3+4)^2 - 11$

$\therefore a = 1$

$y = (x+4)^2 - 11$

$= x^2 + 8x + 16 - 11$

$= x^2 + 8x + 5$

The correct answer is **A**.

2 Using x-intercepts: $y = a(x-4)(x+3)$

Substitute the y – intercept, $(0, -12)$.

$-12 = a(-4)(3)$

$-12 = -12a$

$a = 1$ [1 mark]

Equation: $y = 1(x-4)(x+3)$

$y = x^2 - x - 12$ [1 mark]

3 Turning point $(2, 3)$

$\Rightarrow y = a(x-2)^2 + 3$

$(0, 1) \Rightarrow 1 = a(-2)^2 + 3$

$-2 = 4a$

$a = -\frac{1}{2}$

$y = -\frac{1}{2}(x-2)^2 + 3$

The correct answer is **D**.

3.7 Quadratic inequations

3.7 Exercise

1 **a** The zeros of $(x+5)(x-5)$ are $x = -5$ and $x = 5$.

The coefficient of the x^2 term is positive, so the shape of the sign diagram is that of a concave up graph.

The sign diagram is

b For $(x+5)(x-5) \geq 0$, look at the sections above the x-axis.

$(x+5)(x-5) \geq 0$ when $x \leq -5$ or $x \geq 5$

2 **a** The sign diagram is a 'squashed' version of the graph.

b Look at the section of the sign diagram, or the graph, that lies below the x-axis. $(x-3)(x-7) \leq 0$ when $3 \leq x \leq 7$.

3 **a** $\{x : x(3-x) < 0\}$

There are two sections of the sign diagram that lie below the x-axis. The union of these sets of x-values is required.

The required set of x-values is $\{x : x < 0\} \cup \{x : x > 3\}$.

b $\{x : x(3-x) > 0\}$

The section of the sign diagram that lies above the x-axis is required.

This is $\{x : 0 < x < 3\}$.

4 **a** $(x-6)^2$

The zero is $x = 6$ and the coefficient of x^2 is positive.

The sign diagram will touch the x-axis at $x = 6$ and take the shape of a concave up graph.

b $(x-6)^2 > 0$ for all values of x except $x = 6$.

This can be written as $R\backslash\{6\}$.

5 a $9 + 3x - 2x^2$ is a quadratic trinomial.

$9 + 3x - 2x^2 = (3 - x)(3 + 2x)$.

b Let $9 + 3x - 2x^2 = 0$.

$(3 - x)(3 + 2x) = 0$

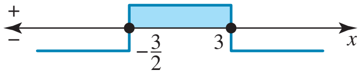
$3 - x = 0$ or $3 + 2x = 0$

$3 = x$ or $2x = -3$

$x = 3,\ x = -\frac{3}{2}$

The zeros are $x = -\frac{3}{2}, x = 3$.

The sign diagram of $9 + 3x - 2x^2$ will be concave down with zeros at $x = -\frac{3}{2}, x = 3$.

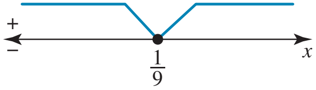


c $9 + 3x - 2x^2 \geq 0$ for the section of the sign diagram that lies above the x-axis. This is the interval for which $-\frac{3}{2} \leq x \leq 3$.

6 Zeros $x = -3, x = 4$, concave up

7 $81x^2 - 18x + 1 = (9x - 1)^2$

Zero $x = \frac{1}{9}$, multiplicity 2



8 a $(4 - x)(2x - 3) \leq 0$

The solution is $x \leq \frac{3}{2}$ or $x \geq 4$.

Check: let $x = -2$.

$(4 - (-2))(2(-2) - 3)$

$= 6 \times -7$

$= -42$

< 0

Let $x = 5$.

$(4 - 5)(10 - 3)$

$= -1 \times 7$

$= -7$

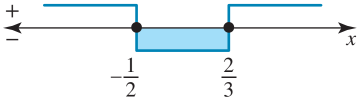
< 0

b $6x^2 < x + 2$

$\therefore 6x^2 - x - 2 < 0$

$\therefore (3x - 2)(2x + 1) < 0$

The zeros are $x = \frac{2}{3}, x = -\frac{1}{2}$.



The solution set is $\left\{x : -\frac{1}{2} < x < \frac{2}{3}\right\}$.

9 $6x < x^2 + 9$

$\therefore 0 < x^2 - 6x + 9$

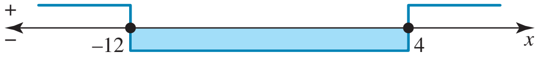
$\therefore (x - 3)^2 > 0$

This will always be true except if $x = 3$.

Therefore, the solution set is $R\backslash\{3\}$.

10 a $x^2 + 8x - 48 \leq 0$

$\therefore (x + 12)(x - 4) \leq 0$



$\therefore\ -12 \leq x \leq 4$

b $-x^2 + 3x + 4 \leq 0$

$\therefore (-x + 4)(x + 1) \leq 0$

$\therefore x \leq -1$ or $x \geq 4$

c $3(3 - x) < 2x^2$

$\therefore 9 - 3x < 2x^2$

$\therefore 2x^2 + 3x - 9 > 0$

$\therefore (2x - 3)(x + 3) > 0$

$\therefore x < -3$ or $x > \frac{3}{2}$

d $(x + 5)^2 < 9$

$\therefore (x + 5)^2 - 9 < 0$

$\therefore (x + 5 - 3)(x + 5 + 3) < 0$

$\therefore (x + 2)(x + 8) < 0$

$\therefore\ -8 < x < -2$

e $9x < x^2$

$\therefore 9x - x^2 < 0$

$\therefore x(9 - x) < 0$

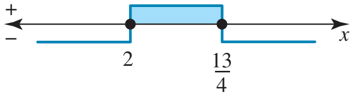
$\therefore x < 0$ or $x > 9$

f $5(x - 2) \geq 4(x - 2)^2$

$\therefore 5(x - 2) - 4(x - 2)^2 \geq 0$

$\therefore (x - 2)[5 - 4(x - 2)] \geq 0$

$\therefore (x - 2)(13 - 4x) \geq 0$

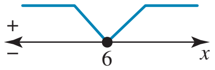


$\therefore 2 \leq x \leq \frac{13}{4}$

11 a $36 - 12x + x^2 > 0$

$36 - 12x + x^2 > 0$

$\therefore (6 - x)^2 > 0$



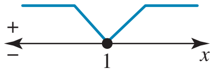
$\therefore x \in R\backslash\{6\}$

The solution set is $R\backslash\{6\}$.

b $6x^2 - 12x + 6 \leq 0$

$\therefore 6(x^2 - 2x + 1) \leq 0$

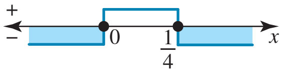
$\therefore 6(x - 1)^2 \leq 0$



The solution set is $\{1\}$.

c $-8x^2 + 2x < 0$

$\therefore\ -2x(4x - 1) < 0$



$\therefore x < 0$ or $x > \dfrac{1}{4}$

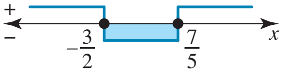
The solution set is $x : x < 0 < \cup \left\{x : x > \dfrac{1}{4}\right\}$.

d $x(1 + 10x) \le 21$

$\therefore x + 10x^2 \le 21$

$\therefore 10x^2 + x - 21 \le 0$

$\therefore (5x - 7)(2x + 3) \le 0$



$\therefore -\dfrac{3}{2} \le x \le \dfrac{7}{5}$

The solution set is $\left\{x : -\dfrac{3}{2} \le x \le \dfrac{7}{5}\right\}$.

12 a $y = x^2 + 3x - 10 \quad [1]$

$y + x = 2 \quad [2]$

From equation [2], $y = 2 - x$.

Substitute in [1]:

$2 - x = x^2 + 3x - 10$

$\therefore x^2 + 4x - 12 = 0$

$\therefore (x + 6)(x - 2) = 0$

$\therefore x = -6, x = 2$

If $x = -6, y = 8$, and if $x = 2, y = 0$.

The points of intersection are $(-6, 8)$ and $(2, 0)$.

b $y = 6x + 1$ and $y = -x^2 + 9x - 5$

For intersection,

$6x + 1 = -x^2 + 9x - 5$

$\therefore x^2 - 3x + 6 = 0$

$\Delta = (-3)^2 - 4(1)(6)$

$= 9 - 24$

$= -15$

Since $\Delta < 0$, there are no intersections.

13 a $y = 5x + 2 \quad [1]$

$y = x^2 - 4 \quad [2]$

Substitute [1] in [2]:

$\therefore 5x + 2 = x^2 - 4$

$\therefore x^2 - 5x - 6 = 0$

$\therefore (x - 6)(x + 1) = 0$

$\therefore x = 6, \ x = -1$

In [1], when $x = 6, y = 32$ and when $x = -1, y = -3$.

The answer is $x = 6, y = 32$ or $x = -1, y = -3$.

b $4x + y = 3 \quad [1]$

$y = x^2 + 3x - 5 \quad [2]$

From [1], $y = 3 - 4x$

Substitute in [2]:

$\therefore 3 - 4x = x^2 + 3x - 5$

$\therefore x^2 + 7x - 8 = 0$

$\therefore (x - 1)(x + 8) = 0$

$\therefore x = 1, \ x = -8$

In [1], when $x = 1, y = -1$ and when $x = -8, y = 35$.

The answer is $x = 1, y = -1$ or $x = -8, y = 35$.

c $2y + x - 4 = 0 \quad [1]$

$y = (x - 3)^2 + 4 \quad [2]$

From [1], $x = 4 - 2y$

Substitute in [2]:

$\therefore y = (4 - 2y - 3)^2 + 4$

$\therefore y = (1 - 2y)^2 + 4$

$\therefore y = 1 - 4y + 4y^2 + 4$

$\therefore 4y^2 - 5y + 5 = 0$

Test the discriminant:

$\Delta = (-5)^2 - 4 \times 4 \times 5$

$= -55$

Since $\Delta < 0$, there are no solutions.

d $\dfrac{x}{3} + \dfrac{y}{5} = 1 \quad [1]$

$x^2 - y + 5 = 0 \quad [2]$

From [2], $y = x^2 + 5$.

Substitute in [1]:

$\therefore \dfrac{x}{3} + \dfrac{x^2 + 5}{5} = 1$

$\therefore \dfrac{5x + 3(x^2 + 5)}{15} = 1$

$\therefore 3x^2 + 5x + 15 = 15$

$\therefore 3x^2 + 5x = 0$

$\therefore x(3x + 5) = 0$

$\therefore x = 0, \ x = -\dfrac{5}{3}$

In [2], when $x = 0, y = 5$, and when $x = -\dfrac{5}{3}, y = \dfrac{25}{9} + 5$.

The answer is $x = 0, y = 5$ or $x = -\dfrac{5}{3}, y = \dfrac{70}{9}$.

14 a $y = 2x + 5 \quad [1]$

$y = -5x^2 + 10x + 2 \quad [2]$

At intersection, $2x + 5 = -5x^2 + 10x + 2$.

$\therefore 5x^2 - 8x + 3 = 0$

$\therefore (5x - 3)(x - 1) = 0$

$\therefore x = \dfrac{3}{5}, x = 1$

In [1], when $x = \dfrac{3}{5}, y = \dfrac{6}{5} + 5$.

$\therefore x = \dfrac{3}{5}, y = \dfrac{31}{5}$

In [1], when $x = 1, y = 7$.

The points of intersection are $\left(\dfrac{3}{5}, \dfrac{31}{5}\right), (1, 7)$.

b $y = -5x - 13 \quad [1]$

$y = 2x^2 + 3x - 5 \quad [2]$

At intersection, $-5x - 13 = 2x^2 + 3x - 5$.

$\therefore 2x^2 + 8x + 8 = 0$

$\therefore x^2 + 4x + 4 = 0$

$\therefore (x + 2)^2 = 0$

$\therefore x = -2$

In [1], when $x = -2, y = -3$.

The point of intersection is $(-2, -3)$.

c $y = 10 \quad [1]$

$y = (5 - x)(6 + x) \quad [2]$

At intersection, $10 = (5 - x)(6 + x)$.

$\therefore 10 = 30 - x - x^2$

$\therefore x^2 + x - 20 = 0$

$\therefore (x - 4)(x + 5) = 0$

$\therefore x = 4, x = -5$

The points of intersection are $(4, 10), (-5, 10)$

d $19x - y = 46 \quad [1]$

$y = 3x^2 - 5x + 2 \quad [2]$

From [1], $y = 19x - 46$.

Substitute in [2].

$\therefore 19x - 46 = 3x^2 - 5x + 2$

$\therefore 3x^2 - 24x + 48 = 0$

$\therefore x^2 - 8x + 16 = 0$

$\therefore (x-4)^2 = 0$

$\therefore x = 4$

Substitute $x = 4$ in [1].

$\therefore y = 19 \times 4 - 46$

$\therefore y = 30$

The point of intersection is $(4, 30)$.

15 a $y = 4 - 2x \quad [1]$

$y = 3x^2 + 8 \quad [2]$

At intersection, $4 - 2x = 3x^2 + 8$.

$\therefore 3x^2 + 2x + 4 = 0$

$\Delta = 2^2 - 4 \times 3 \times 4$

$\therefore \Delta = -44$

Since $\Delta < 0$, there are no intersections.

b $y = 2x + 1 \quad [1]$

$y = -x^2 - x + 2 \quad [2]$

At intersection, $2x + 1 = -x^2 - x + 2$.

$\therefore x^2 + 3x - 1 = 0$

$\Delta = 3^2 - 4 \times 1 \times (-1)$

$\therefore \Delta = 13$

Since $\Delta > 0$, there are two intersections.

c $y = 0 \quad [1]$

$y = -2x^2 + 3x - 2 \quad [2]$

At intersection, $0 = -2x^2 + 3x - 2$.

$\Delta = 3^2 - 4 \times (-2) \times (-2)$

$\therefore \Delta = -7$

Since $\Delta < 0$, there are no intersections.

16 $y = 4x$ and $y = x^2 + 4$

At intersection,

$4x = x^2 + 4$

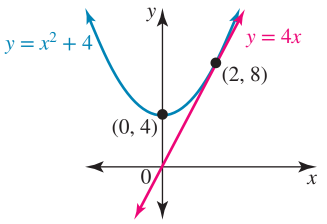
$\therefore 0 = x^2 - 4x + 4$

$\therefore (x-2)^2 = 0$

$\therefore x = 2$

Since there is only one value, the line is a tangent to the parabola.

When $x = 2, y = 8$, so the point of contact is $(2, 8)$.



17 $y = mx - 7$ and $y = 3x^2 + 6x + 5$

For intersection,

$3x^2 + 6x + 5 = mx - 7$

$\therefore 3x^2 + x(6-m) + 12 = 0$

$\Delta = (6-m)^2 - 4(3)(12)$

$= (6-m)^2 - 144$

$= (6 - m - 12)(6 - m + 12)$

$= (-6 - m)(18 - m)$

$= -(6 + m)(18 - m)$

For at least one intersection, $\Delta \geq 0$

Sign diagram of the discriminant:

Therefore, there will be at least one intersection if $m \leq -6$ or $m \geq 18$

18 $y = kx + 9$ and $y = x^2 + 14$

For intersection,

$x^2 + 14 = kx + 9$

$\therefore x^2 - kx + 5 = 0$

$\Delta = (-k)^2 - 4(1)(5)$

$= k^2 - 20$

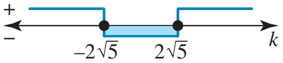
For no intersection, $\Delta < 0$.

$\therefore k^2 - 20 < 0$

$\therefore \left(k + \sqrt{20}\right)\left(k - \sqrt{20}\right) < 0$

The zeros are $k = \pm 2\sqrt{5}$.

Sign diagram of the discriminant:



Therefore, there are no intersections if $-2\sqrt{5} < k < 2\sqrt{5}$.

19 $y = (k-2)x + k \quad [1]$

$y = x^2 - 5x \quad [2]$

At intersection, $(k-2)x + k = x^2 - 5x$.

$\therefore x^2 - 5x - (k-2)x - k = 0$

$\therefore x^2 - x(5 + k - 2) - k = 0$

$\therefore x^2 - (3 + k)x - k = 0$

$\Delta = [-(3+k)]^2 - 4 \times (1) \times (-k)$

$\therefore \Delta = 9 + 6k + k^2 + 4k$

$\therefore \Delta = k^2 + 10k + 9$

$\therefore \Delta = (k+1)(k+9)$

Sign diagram of Δ:

a For no intersections, $\Delta < 0$.

$\therefore -9 < k < -1$

b For one point of intersection, $\Delta = 0$.

$\therefore k = -1, k = -9$

c For two points of intersection, $\Delta > 0$.

$\therefore k < -9$ or $k > -1$

20 a The equation $px^2 - 2px + 4 = 0$ will have real roots if $\Delta \geq 0$ and $p \neq 0$.

$\Delta = (-2p)^2 - 4 \times p \times 4$

$= 4p^2 - 16p$

$= 4p(p-4)$

Sign diagram of Δ:

$\Delta \geq 0 \Rightarrow p \leq 0$ or $p \geq 4$.

The equation will have real roots if $p < 0$ or $p \geq 4$ since $p \neq 0$.

b $y = tx + 1 \quad [1]$

$y = 2x^2 + 5x + 11 \quad [2]$

At intersection, $tx + 1 = 2x^2 + 5x + 11$.

$\therefore 2x^2 + 5x - tx + 10 = 0$

$\therefore 2x^2 + (5-t)x + 10 = 0$

For no intersection, there are no solutions to $2x^2 + (5-t)x + 10 = 0$ and its discriminant must be negative.

$$\begin{aligned}\Delta &= (5-t)^2 - 4\times 2\times 10\\ &= (5-t)^2 - 80\\ &= \left(5-t-\sqrt{80}\right)\left(5-t+\sqrt{80}\right)\end{aligned}$$

$$\therefore \Delta = \left(5-4\sqrt{5}-t\right)\left(5+4\sqrt{5}-t\right)$$

The zeros are $t = 5-4\sqrt{5}$ and $t = 5+4\sqrt{5}$.

Sign diagram of Δ:

$\Delta < 0$ when $5-4\sqrt{5} < t < 5+4\sqrt{5}$

The line does not intersect the parabola for $5-4\sqrt{5} < t < 5+4\sqrt{5}$.

c $y = x$ [1]

$y = 9x^2 + nx + 1$ [2]

At intersection, $x = 9x^2 + nx + 1$.

$$\therefore 9x^2 + nx - x + 1 = 0$$

$$\therefore 9x^2 + (n-1)x + 1 = 0$$

If the line is a tangent to the parabola, there is one solution, so $\Delta = 0$.

$$\begin{aligned}\Delta &= (n-1)^2 - 4\times 9\times 1\\ &= (n-1)^2 - 36\\ &= (n-1-6)(n-1+6)\end{aligned}$$

$$\therefore \Delta = (n-7)(n+5)$$

$\Delta = 0$ when $n = 7, n = -5$.

The line is a tangent to the parabola if $n = 7$ or $n = -5$.

21 a $2y - 3x = 6$ [1]

$y = x^2$ [2]

Substitute [2] in [1]

$$\therefore 2x^2 - 3x = 6$$

$$\therefore 2x^2 - 3x - 6 = 0$$

$$\therefore x = \frac{3 \pm \sqrt{(-3)^2 - 4\times 2\times(-6)}}{4}$$

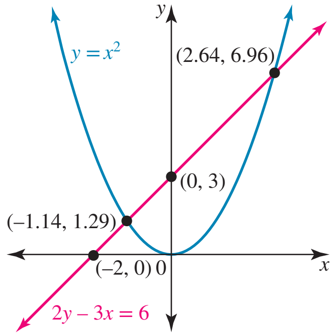
$$\therefore x = \frac{3\pm\sqrt{57}}{4}$$

Therefore, $x = -1.137...$ or $x = 2.637...$

Substitute each into equation [2] to obtain $y = 1.29$ or $y = 6.96$ respectively.

To 2 decimal places, the points of intersection are $(-1.14, 1.29), (2.64, 6.96)$.

b The line $2y - 3x = 6$ intersects the axes at $(-2, 0), (0, 3)$. The parabola $y = x^2$ has a minimum turning point at $(0, 0)$. Both graphs contain the points $(-1.14, 1.29), (2.64, 6.96)$.



c The region enclosed between the two graphs lies below the line and above the parabola. The closed region can be described by the inequalities $2y - 3x \le 6$ and $y \ge x^2$. That is, the region is $\{(x,y) : 2y - 3x \le 6\} \cap \left\{(x,y) : y \ge x^2\right\}$.

d $2y - 3x = 6 \Rightarrow y = \frac{3}{2}x + 3$ The gradient is $\frac{3}{2}$.

Let the parallel line that is to be a tangent to the parabola have the equation $y = \frac{3}{2}x + c$.

At intersection with the parabola, $x^2 = \frac{3}{2}x + c$.

$$\therefore 2x^2 - 3x - 2c = 0$$

For the line to be a tangent, $\Delta = 0$.

$$\begin{aligned}\therefore 9 - 4\times 2\times(-2c) &= 0\\ \therefore 9 + 16c &= 0\\ \therefore c &= -\frac{9}{16}\end{aligned}$$

The tangent line has the equation $y = \frac{3}{2}x - \frac{9}{16}$. Its y-intercept is $\left(0, -\frac{9}{16}\right)$.

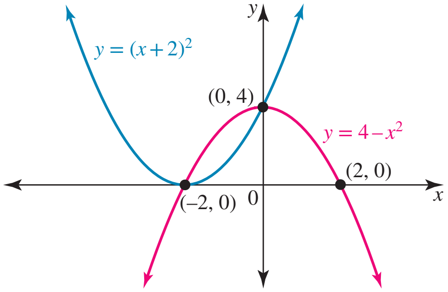
22 a On the main screen in Standard mode, tap Interactive → Equation/Inequality → solve and enter $19 - 3x - 5x^2 < 0$. The < symbol is found in Keyboard → Mth → OPTN.

The answer given is $x < \frac{-\left(\sqrt{389}+3\right)}{10}, x > \frac{\sqrt{389}-3}{10}$.

b Use the procedure in part **a**. The solution to $6x^2 + 15x \le 10$ is $\frac{-\left(\sqrt{465}+15\right)}{12} \le x \le \frac{\sqrt{465}-15}{12}$.

23 a $y = (x+2)^2$ has a minimum turning point and x-intercept at $(-2, 0)$; its y-intercept is at $(0, 4)$.

$y = 4 - x^2$ has a maximum turning point and y-intercept at $(0, 4)$; its x-intercepts are at $(-2, 0)$ and $(2, 0)$.



The points of intersection are $(-2, 0)$ and $(0, 4)$.

b **i** $y = (x+2)^2$ and $y = k - x^2$ intersect when $(x+2)^2 = k - x^2$.

$$\therefore x^2 + 4x + 4 = k - x^2$$

$$\therefore 2x^2 + 4x + 4 - k = 0$$

$$\Delta = 4^2 - 4\times 2\times(4-k)$$

$$\therefore \Delta = -16 + 8k$$

For one point of intersection, $\Delta = 0$.

$$\therefore -16 + 8k = 0$$

$$\therefore k = 2$$

ii $y = 2 - x^2$ has a maximum turning point and y-intercept at $(0, 2)$; its x-intercepts are at $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, 0\right)$.

Point of intersection: substitute $k = 2$ in $2x^2 + 4x + 4 - k = 0$.

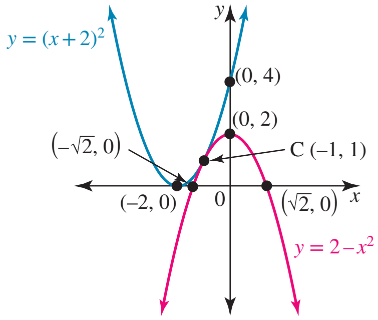
$$\therefore 2x^2 + 4x + 2 = 0$$

$$\therefore x^2 + 2x + 1 = 0$$

$\therefore (x+1)^2 = 0$

$\therefore x = -1$

When $x = -1, y = 1$. The common point C has coordinates $(-1, 1)$.



iii $(-1, 1)$ lies on $y = ax + b$.

$\therefore 1 = -a + b$

$\therefore b = a + 1$

The equation of the tangent line becomes $y = ax + a + 1$.

There is one solution to the simultaneous equations $y = ax + a + 1$ and $y = 2 - x^2$.

$$\therefore ax + a + 1 = 2 - x^2$$

$$\therefore x^2 + ax + a - 1 = 0$$

$$\begin{aligned}\Delta &= a^2 - 4 \times 1 \times (a-1)\\ &= a^2 - 4a + 4\\ &= (a-2)^2\end{aligned}$$

$\Delta = 0 \Rightarrow a = 2$ and $a = 2 \Rightarrow b = 3$

The tangent to $y = 2 - x^2$ is the line $y = 2x + 3$.

To show this line is also a tangent to $y = (x+2)^2$:

$$2x + 3 = (x+2)^2$$

$$\therefore 2x + 3 = x^2 + 4x + 4$$

$$\therefore x^2 + 2x + 1 = 0$$

$$\Delta = 4 - 4 \times 1 \times 1$$

$$\therefore \Delta = 0$$

The line $y = 2x + 3$ is a tangent to $y = (x+2)^2$.

The equation of the common tangent is $y = 2x + 3$.

24 a The equation $x^2 - 5x + 4 = 0$ can be expressed as $x^2 = 5x - 4$. The intersection of $y = x^2$ with the line $y = 5x - 4$ would form this equation.

b Rearranging the equation $3x^2 + 9x - 2 = 0$ gives

$$9x = 2 - 3x^2$$

$$\therefore 9x + 3 = 2 - 3x^2 + 3$$

$$\therefore 3(3x+1) = -3x^2 + 5$$

$$\therefore 3x + 1 = -x^2 + \frac{5}{3}$$

The equation gives the x-coordinates of the points of intersection of the line $y = 3x + 1$ with the parabola $y = -x^2 + \frac{5}{3}$.

25 a In the Graph & Tab screen, enter $y1 : 2x^2 - 10x$.

The family of lines could be graphed using the Dynamic Graph facility.

Enter $y2 : bx + a$ and tap the diamond shape on the ribbon → Dynamic Graph.

Enter for a:

Start: −6

End: 0

Step: 2

Although $b = -4$, enter for b:

Start: −4

End: −3

Step: 1

Then graph by ticking the boxes and tapping the graphing symbol.

To alter the values of a, the cursor is moved sideways (The values of b are controlled by the vertical cursor but this is not relevant for the line $y = -4x + a$.)

For each line, tap Analysis → G-Solve → Intersect to obtain the points of intersection.

$a = -6$: Intersections are not found.

$a = -4$: $(1, -8), (2, -12)$

$a = -2$: $(0.38, -3.53), (2.62, -12.47)$ to 2 decimal places

$a = 0$: $(0, 0), (3, -12)$

b Solve $2x^2 - 10x = -4x + a$ for x using Equation/Inequality, having cleared All Variables first.

The solution is $x = \dfrac{-\left(\sqrt{2a+9} - 3\right)}{2}, x = \dfrac{\sqrt{2a+9} + 3}{2}$

c For the line $y = -4x + a$ to be a tangent, there should be only one solution.

$$\therefore \sqrt{2a+9} = 0$$

$$\therefore 2a + 9 = 0$$

$$\therefore a = -\frac{9}{2}$$

For the point of contact, $x = \dfrac{0+3}{2}$.

When $x = \dfrac{3}{2}$ and $a = -\dfrac{9}{2}$,

$$y = -4 \times \frac{3}{2} - \frac{9}{2}$$

$$\therefore y = -\frac{21}{2}$$

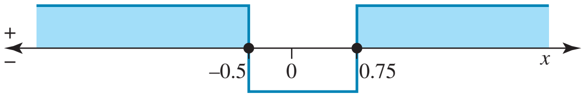
For the line to be a tangent, $a = -4.5$ and the point of contact is $(1.5, -10.5)$.

3.7 Exam questions

1 $(4x-3)(2x+1) > 0$

Solve $(4x-3)(2x+1) = 0$ to find the zeros.

$x = 0.75, \ x = -0.5$



The solution is $\{x : x < -0.5\} \cup \{x : x > 0.75\}$

The correct answer is **E**.

2 Solve

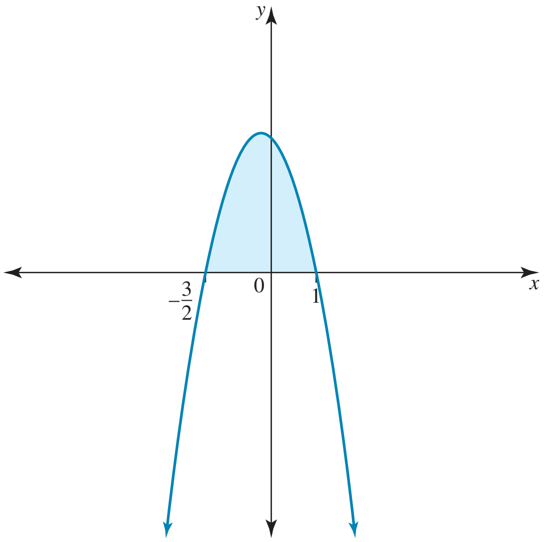
$-2x^2 - x + 3 = 0$

$-\left(2x^2 + x - 3\right) = 0$

$-(x-1)(2x+3) = 0$

$x = 1, \ -\dfrac{3}{2}$

Sketch the graph to find the given region.



For $-(x-1)(2x+3) \geq 0, x \in \left[-\frac{3}{2}, 1\right]$.

The correct answer is **E**.

3 $x^2 + 4x + 33 = mx + 24$

$x^2 + (4-m)x + 9 = 0$

$a = 1,\ b = (4-m),\ c = 9$ [1 mark]

$\Delta = b^2 - 4ac$

$= (4-m)^2 - 4(1)(9)$

$= (4-m)^2 - 36$

For the line to be a tangent, $\Delta = 0$. [1 mark]

$(4-m)^2 - 36 = 0$

$(4-m-6)(4-m+6) = 0$

$(-m-2)(10-m) = 0$

$m = -2,\ m = 10$ [1 mark]

The line $y = mx + 24$ will be a tangent to the parabola $x^2 + 4x + 33$ for $m = -2$ or $m = 10$.

3.8 Quadratic models and applications

3.8 Exercise

1 30 metres of edging using the back fence as one edge

a Width x metres, length $30 - 2x$ metres

$\therefore A = x(30 - 2x)$

$\therefore A = 30x - 2x^2$

b $A = 0 \Rightarrow x(30 - 2x) = 0$

$\therefore x = 0, x = 15$

Therefore, the turning point is

$x = 7.5, A = 7.5(30 - 15)$

$\Rightarrow (7.5, 112.5)$

The dimensions of the garden for maximum area are width 7.5 m and length 15 m.

c The greatest area occurs at the turning point. The greatest area is 112.5 square metres.

2 $h = 100 + 38t - \frac{19}{12}t^2$

a At the turning point, $t = -\frac{b}{2a}$.

$\therefore t = -\frac{38}{2\left(-\frac{19}{12}\right)}$

$\therefore t = 12$

$\therefore h = 100 + 38(12) - \frac{19}{12} \times 12^2$

$\therefore h = 100 + 19 \times 12$

$\therefore h = 328$

Therefore, the greatest height the missile reaches is 328 metres.

b It reaches its greatest height after 12 seconds.

c Time to return to the ground: $0 = 100 + 38t - \frac{19}{12}t^2$

$\therefore 19t^2 - 456t - 1200 = 0$

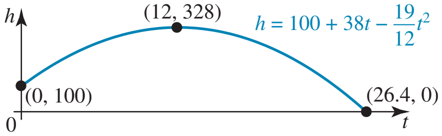
$$\therefore t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore t = \frac{456 \pm \sqrt{456^2 - 4 \times 19 \times (-1200)}}{38}$$

$\therefore t = -2.39, 26.39$

$\therefore t = 26.4$

(Reject the negative value.)



3 $10h = 16t + 4 - 9t^2$

a The ball reaches the ground when $h = 0$.

$\therefore 0 = 16t + 4 - 9t^2$

$\therefore 9t^2 - 16t - 4 = 0$

$\therefore (9t + 2)(t - 2) = 0$

$\therefore t = -\frac{2}{9},\ t = 2$

Reject $t = -\frac{2}{9}$ since $t > 0$.

$\therefore t = 2$

It takes the ball 2 seconds to reach the ground.

b Let $h = 1.6$

$\therefore 10 \times 1.6 = 16t + 4 - 9t^2$

$\therefore 9t^2 - 16t + 12 = 0$

$\Delta = (-16)^2 - 4 \times 9 \times 12$

$\therefore \Delta = -176$

Since $\Delta < 0$, there is no value of t for which $h = 1.6$.

The ball does not strike the overhanging foliage.

c $10h = 16t + 4 - 9t^2$

$10h = 16t + 4 - 9t^2$

$\therefore h = 1.6t + 0.4 - 0.9t^2$

$\therefore h = -0.9t^2 + 1.6t + 0.4$

At the turning point, $t = -\frac{b}{2a}$.

$\therefore t = -\frac{1.6}{2 \times (-0.9)}$

$\therefore t = \frac{1.6}{1.8}$

$\therefore t = \frac{8}{9}$

When $t = \frac{8}{9}$,

$$10h = -9 \times \frac{64}{81} + 16 \times \frac{8}{9} + 4$$

$$\therefore 10h = \frac{100}{9}$$

$$\therefore h = \frac{10}{9}$$

The greatest height the ball reaches is $\frac{10}{9}$ metres.

4 **a** $y = 1.2 + 2.2x - 0.2x^2$

$$\therefore y = -0.2(x^2 - 11x - 6)$$

$$\therefore y = -0.2\left[\left(x^2 - 11x + \left(\frac{11}{2}\right)^2\right) - \left(\frac{11}{2}\right)^2 - 6\right]$$

$$\therefore y = -0.2\left[\left(x - \frac{11}{2}\right)^2 - \frac{121}{4} - \frac{24}{4}\right]$$

$$\therefore y = -0.2\left[\left(x - \frac{11}{2}\right)^2 - \frac{145}{4}\right]$$

$$\therefore y = -0.2(x - 5.5)^2 + 0.2 \times \frac{145}{4}$$

$$\therefore y = -0.2(x - 5.5)^2 + 7.25$$

b Since the maximum turning point is (5.5, 7.25), the greatest height the volleyball reaches is 7.25 metres.

c The court is 18 metres in length, so the net is 9 metres horizontally from the back of the court.

When $x = 9$, the height of the volleyball is

$$y = 1.2 + 2.2 \times 9 - 0.2 \times 81$$

$$\therefore y = 4.8$$

The volleyball is 4.8 metres high and the net is 2.43 metres high. Therefore, the ball clears the net by 2.37 metres.

5 **a** The total length of hosing for the edges is 120 metres.

$$\therefore 2l + 4w = 120$$

$$\therefore l + 2w = 60$$

$$\therefore l = 60 - 2w$$

The total area of the garden is $A = l \times w$.

$$\therefore A = (60 - 2w) \times w$$

$$\therefore A = 60w - 2w^2$$

Completing the square,

$$A = -2(w^2 - 30w)$$

$$\therefore A = -2\left[\left(w^2 - 30w + 15^2\right) - 15^2\right]$$

$$\therefore A = -2\left[(w - 15)^2 - 225\right]$$

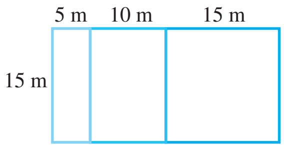
$$\therefore A = -2(w - 15)^2 + 450$$

Maximum area of 450 sq m when $w = 15$, and if $w = 15$, then $l = 30$.

Width 15 m, length 30 m

b The total area is divided into three sections in the ratio 1 : 2 : 3.

Dividing the length $l = 30$ into 6 parts makes each part 5 m. The lengths of each section are 5, 10, 15 metres respectively.



The smallest section has width 15 metres and length 5 metres with area 75 sq m. The amount of hosing required for its four sides is its perimeter of 40 m.

The middle section has width 15 metres and length 10 metres with area 150 sq m. The amount of hosing required is for three sides, since it shares one side with the smallest section. Therefore, the amount of hosing is $2 \times 10 + 15 = 35$ metres.

The largest section has width 15 metres and length 15 metres with area 225 sq m. The amount of hosing required is for three sides, since it shares one side with the middle section. Therefore, the amount of hosing is $2 \times 15 + 15 = 45$ metres.

Total length = 40 + 30 + 45 = 120 metres

6 **a** $N = 100 + 46t + 2t^2$

Initially, $t = 0 \Rightarrow N = 100$.

When $N = 200$,

$$200 = 100 + 46t + 2t^2$$

$$\therefore 2t^2 + 46t - 100 = 0$$

$$\therefore t^2 + 23t - 50 = 0$$

$$\therefore (t + 25)(t - 2) = 0$$

$$\therefore t = -25 \text{ (reject) or } t = 2$$

$$\therefore t = 2$$

It takes 2 hours for the initial number of bacteria to double.

b At 1 pm, $t = 5$.

$$\therefore N = 100 + 46 \times 5 + 2 \times 25$$

$$\therefore N = 380$$

At 1 pm there are 380 bacteria present.

c $N = 380 - 180t + 30t^2$ where t is the time since 1 pm.

The minimum number of bacteria occurs at the minimum turning point.

At the turning point,

$$t = -\frac{-180}{2 \times 30}$$

$$\therefore t = 3$$

$$N = 380 - 180 \times 3 + 30 \times 9$$

$$\therefore N = 110$$

The minimum number of bacterial cells is 110 reached at 4 pm.

7 $z = 5x^2 + 4xy + 6y^2$ [1]

$x + y = 2$ [2]

From equation [2], $y = 2 - x$

Substitute in equation [1].

$$\therefore z = 5x^2 + 4x(2 - x) + 6(2 - x)^2$$

$$\therefore z = 5x^2 + 8x - 4x^2 + 6(4 - 4x + x^2)$$

$$\therefore z = 7x^2 - 16x + 24$$

At the turning point,

$$x = -\frac{-16}{2 \times 7}$$

$$\therefore x = \frac{8}{7}$$

$$z = 7 \times \frac{64}{49} - 16 \times \frac{8}{7} + 24$$

$$= \frac{64}{7} - \frac{128}{7} + \frac{168}{7}$$

$$\therefore z = \frac{104}{7}$$

The minimum turning point is $\left(\frac{8}{7}, \frac{104}{7}\right)$.

The minimum value of z is $\frac{104}{7}$. This occurs when $x = \frac{8}{7}$ and

$$y = 2 - \frac{8}{7}$$
$$= \frac{6}{7}$$

8 **a** Each piece of wire forms the perimeter of each square. The perimeter of a square of side length 4 cm is 16 cm. As the total length of the original piece of wire is 20 cm, the perimeter of the second square is 20 − 16 = 4 cm. This makes the side length of the second square 1 cm.

The sum of the areas of the two squares gives the value of S.

$\therefore S = 4 \times 4 + 1 \times 1$

$\therefore S = 17$

b The square of side length x cm has area x^2 sq cm and perimeter $4x$ cm.

The second square has perimeter $(20 - 4x)$ cm. Each of its sides is $\frac{20 - 4x}{4} = 5 - x$ cm. Hence, the area of the second square is $(5 - x)^2$ sq cm.

$\therefore S = x^2 + (5 - x)^2$

$= x^2 + 25 - 10x + x^2$

$\therefore S = 2x^2 - 10x + 25$

The minimum turning point occurs when

$x = -\frac{-10}{2 \times 2} \Rightarrow x = \frac{5}{2}$.

If $x = \frac{5}{2}$, the perimeter of the first square is $4 \times \frac{5}{2} = 10$ cm.

For the sum of the areas of the two squares to be a minimum, the piece of wire needs to be cut into two equal pieces of 10 cm.

9 **a** Let $C = 20 + an + bn^2$, where a and b are the constants of proportionality.

$n = 5, C = 195 \Rightarrow 195 = 20 + 5a + 25b$

$5a + 25b = 175$

$a + 5b = 35 \quad [1]$

$n = 8,\ C = 420 \Rightarrow 420 = 20 + 8a + 64b$

$8a + 64b = 400$

$a + 8b = 50 \quad [2]$

Solving equations [1] and [2] simultaneously:

[2] – [1] gives

$3b = 15$

$b = 5$

Substitute into [1]:

$a + 25 = 35$

$a = 10$

The relationship is $C = 20 + 10n + 5n^2$.

b $C \le 1000$

$\therefore 20 + 10n + 5n^2 \le 1000$

$\therefore n^2 + 2n + 4 \le 200$

$\therefore n^2 + 2n - 196 \le 0$

Consider $n^2 + 2n - 196 = 0$.

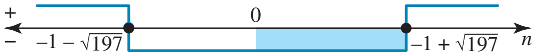
$\therefore \left(n^2 + 2n + 1\right) - 1 - 196 = 0$

$\therefore (n + 1)^2 = 197$

$\therefore n + 1 = \pm\sqrt{197}$

$\therefore n = -1 \pm \sqrt{197}$

Sign diagram of $n^2 + 2n - 196$

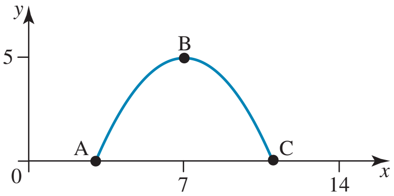


$n \in N$, and $-1 + \sqrt{197} \simeq 13.04$, $n^2 + 2n - 196 \le 0$ when $0 < n < -1 + \sqrt{197}$.

$\therefore 1 \le n \le 13$

Therefore, 13 is the maximum number of tables that can be manufactured if the costs are not to exceed $1000.

10 **a**



The width of the bridge is 14 metres, so the turning point B has coordinates (7, 5).

As AC = 8 metres, by symmetry, OA = 3 metres and OC = 11 metres.

The coordinates of A are (3, 0) and the coordinates of C are (11, 0).

b The equation has the form $y = a(x - 7)^2 + 5$.

$C(11, 0) \Rightarrow 0 = a(11 - 7)^2 + 5$

$\therefore 16a + 5 = 0$

$\therefore a = -\frac{5}{16}$

The equation of the parabola is $y = -\frac{5}{16}(x - 7)^2 + 5$.

c When $y = 1.5$,

$\frac{3}{2} = -\frac{5}{16}(x - 7)^2 + 5$

$\therefore 24 = -5(x - 7)^2 + 80$

$\therefore 5(x - 7)^2 = 56$

$\therefore (x - 7)^2 = \frac{56}{5}$

$\therefore x - 7 = \pm\sqrt{\frac{56}{5}}$

$\therefore x = 7 \pm 3.35$

$\therefore x = 3.65,\ x = 10.35$

The width of the water level is 10.35 − 3.65 = 6.7 metres, to 1 decimal place.

11 Cost: $C = 15 + 10x$

Revenue: $R = vx$

$\therefore R = (50 - x)x$

Profit = Revenue − Cost

$\therefore P = (50 - x)x - (15 + 10x)$, where P dollars is the profit from a sale of x kg of fertiliser.

$\therefore P = 50x - x^2 - 15 - 10x$

$\therefore P = -x^2 + 40x - 15$

Completing the square,

$P = -\left[\left(x^2 - 40x + 400\right) - 400 + 15\right]$

$\therefore P = -\left[(x - 20)^2 - 385\right]$

$\therefore P = -(x - 20)^2 + 385$

The maximum turning point is (20, 385). The profit is greatest when $x = 20$.

If $x = 20$, the cost per kilogram, v, equals 30.

For maximum profit, the cost is $30 per kilogram.

12 **a** Let the two numbers be x and y.

Given $x + y = 16$, then $y = 16 - x$.

i Let P be the product of the two numbers.

$P = xy$

$\therefore P = x(16 - x)$

The x-intercepts of the graph of P against x are $x = 0, x = 16$, so the axis of symmetry is $x = 8$.

When $x = 8$, $P = 64$, so $(8, 64)$ is the maximum turning point. Also, when $x = 8$, $y = 8$.

The product is greatest when the numbers are both 8.

ii Let S be the sum of the squares of the numbers.

$\therefore S = x^2 + (16 - x)^2$

$\therefore S = x^2 + 256 - 32x + x^2$

$\therefore S = 2x^2 - 32x + 256$

The graph of this function would have a minimum turning point.

Coordinates of the turning point: $x = -\dfrac{-32}{4} \Rightarrow x = 8$

$S = 8^2 + 8^2$

$\therefore S = 128$

TP $(8, 128)$, so S is least when $x = 8$ and therefore $y = 8$.

The sum of the squares of the two numbers is least when both are 8.

b $x + y = k$, so $y = k - x$

i Product, $P = x(k - x)$

Greatest when $x = \dfrac{0 + k}{2}$

When $x = \dfrac{k}{2}$,

$P = \dfrac{k}{2} \times \dfrac{k}{2}$

$\therefore P = \dfrac{k^2}{4}$

The greatest product of the two numbers is $\dfrac{k^2}{4}$.

ii Sum of squares, $S = x^2 + (k - x)^2$

If $S = P$, then $x^2 + (k - x)^2 = x(k - x)$. [1]

$\therefore x^2 + k^2 - 2kx + x^2 = kx - x^2$

$\therefore 3x^2 - 3kx + k^2 = 0$

Use the discriminant to test if there are solutions.

$\Delta = (-3k)^2 - 4 \times 3 \times k^2$

$= 9k^2 - 12k^2$

$= -3k^2$

$\Delta < 0$ unless $k = 0$.

Substitute $k = 0$ in equation [1].

$2x^2 = -x^2$

$\therefore 3x^2 = 0$

$\therefore x = 0$

But the numbers were non-zero, so $k \neq 0$.

There are no non-zero numbers for which the sum of their squares and their product are equal.

13 a The times of day need to be converted to the number of hours after midnight.

4:21 pm is 16 hours plus $\dfrac{21}{60}$ hours after midnight, giving the t value of 16.35.

Enter the values into the Statistics menu as

List1	**List2**
10.25	1.05
16.35	3.26
22.5	0.94

Tap Calc → Quadratic Reg to obtain, to 2 decimal places,

$a = -0.06$

$b = 1.97$

$c = -12.78$

The equation of the quadratic model is
$h = -0.06t^2 + 1.97t - 12.78$.

b Enter the equation as $y1 - 0.06x^2 + 1.97x - 12.78$ in the Graph&Tab screen and sketch the graph. Tap Analysis → G-Solve → Max to obtain the maximum turning point at $x = 16.42, y = 3.39$ to 2 decimal places.

The greatest height the tide reaches is 3.39 metres above sea level.

The time of day is 16.42 hours after midnight, which is 4 pm plus $0.42 \times 60 = 25.2$ minutes.

The greatest tide is predicted to occur at 4:25 pm.

14 a The two sections of the wire form the perimeters of the two shapes. For the square of side length x cm, its perimeter is $4x$ cm. Therefore, the perimeter of the circle is $20 - 4x$ cm.

The area of the square is x^2 sq cm.

The area of the circle is given by $A = \pi r^2$.

The circle's circumference, or perimeter, is $20 - 4x$ cm.

$\therefore 2\pi r = 20 - 4x$

$\therefore r = \dfrac{20 - 4x}{2\pi}$

$\therefore r = \dfrac{10 - 2x}{\pi}$

The area of the circle is $\pi\left(\dfrac{10 - 2x}{\pi}\right)^2 = \dfrac{(10 - 2x)^2}{\pi}$ sq cm.

The sum of the areas: $S = x^2 + \dfrac{(10 - 2x)^2}{\pi}$.

b Use the graphing screen to obtain the graph, then use the Analysis menu to obtain the coordinates of the minimum turning point as $(2.8, 14.0)$ to 1 decimal place.

For minimum S, $x = 2.8$. Therefore, the perimeter of the square is $4 \times 2.8 = 11.2$ cm.

Hence, the wire should be cut into two sections with one of length 11.2 cm and the other 8.8 cm in order to minimise the sum of the areas of the square and the circle these sections respectively enclose.

c Since the perimeters of the square and the circle are $4x$ and $20 - 4x$ respectively,

$4x \geq 0$ and $20 - 4x \geq 0$

$\therefore x \geq 0$ and $x \leq 5$

$\therefore 0 \leq x \leq 5$

Sketch the graph of $S = x^2 + \dfrac{(10 - 2x)^2}{\pi}$ for $0 \leq x \leq 5$.

The screen suggests the greatest value for S occurs when $x = 0$. Check this by tapping Analysis → G-Solve → y-intercept to obtain $S = 31.83$.

Then tap Analysis → G-Solve → y-Cal, and enter the value 5 for x. This gives $S = 25$.

The value of x for which S is a minimum is $x = 0$.
(This means all the wire is used to form a circle.)

3.8 Exam questions

1 Let the length of side opposite the back of her house be y and let the other two sides be x.

80 metres of wire to enclose three sides:

$2x + y = 80$

$y = 80 - 2x$

Area $= xy$

$= x(80 - 2x)$

$= 80x - 2x^2$

Turning point:

$$x = \frac{-b}{2a}$$

$a = -2,\ b = 80$

$$x = \frac{-80}{-4}$$

$$= 20$$

$$\therefore y = 80(20) - 2(20)^2$$

$$= 1600 - 800$$

$$= 800$$

$\therefore$ maximum area is 800 m^2.

The correct answer is **B**.

2 Square 1:

The length of the sides is $4x$

Square 2:

The sum of the four sides is $40 - x$

$\therefore$ one side is $\frac{40 - x}{4} = 10 - x$

$\text{Area} = x^2 + (10 - x)^2$

$\text{Area} = x^2 + 100 - 20x + x^2$

$\text{Area} = 2x^2 - 20x + 100$

The correct answer is **A**.

3 $h = 1.5 + 4t - 0.2t^2$

$a = -0.2,\ b = 4,\ c = 1.5$

Turning point:

Axis of symmetry has equation $t = -\frac{b}{2a}$

$$\therefore t = -\frac{4}{2(-0.2)}$$

$$= \frac{4}{0.4}$$

$= 10$ [1 mark]

$$h = 1.5 + 4t - 0.2t^2$$

$$= 1.5 + 4(10) - 0.2(10)^2$$

$$= 1.5 + 40 - 20$$

$= 21.5$ [1 mark]

Add on height of cliff = 20 metres

The height of the ball above Ruby was 41.5 metres and it took 10 seconds to reach this height. [1 mark]

3.9 Review

3.9 Exercise

Technology free: short answer

1 **a** $x^2 - 4x - 21 = 0$

$$(x - 7)(x + 3) = 0$$

$$x - 7 = 0 \text{ or } x + 3 = 0$$

$$x = 7, x = -3$$

b $10x^2 + 37x + 7 = 0$

$$(5x + 1)(2x + 7) = 0$$

$$5x + 1 = 0 \text{ or } 2x + 7 = 0$$

$$5x = -1 \text{ or } 2x = -7$$

$$x = -\frac{1}{5}, x = -\frac{7}{2}$$

c $(x^2 + 4)^2 - 7(x^2 + 4) - 8 = 0$

Let $a = x^2 + 4$.

$$\therefore a^2 - 7a - 8 = 0$$

$$\therefore (a - 8)(a + 1) = 0$$

$$\therefore a = 8 \text{ or } a = -1$$

$$\therefore x^2 + 4 = 8 \text{ or } x^2 + 4 = -1$$

$$\therefore x^2 = 4 \text{ or } x^2 = -5 \text{(reject)}$$

$$\therefore x = \pm 2$$

d $2x^2 = 3x(x - 2) + 1$

$$\therefore 2x^2 = 3x^2 - 6x + 1$$

$$\therefore x^2 - 6x + 1 = 0$$

$$\therefore (x^2 - 6x + 9) - 9 + 1 = 0$$

$$\therefore (x - 3)^2 = 8$$

$$\therefore x - 3 = \pm\sqrt{8}$$

$$\therefore x = 3 \pm 2\sqrt{2}$$

e $x = \frac{12}{x - 2} - 2$

$$\therefore x + 2 = \frac{12}{x - 2}$$

$$\therefore (x - 2)(x + 2) = 12$$

$$\therefore x^2 - 4 = 12$$

$$\therefore x^2 = 16$$

$$\therefore x = \pm 4$$

f $3 + \sqrt{x} = 2x$

Let $a = \sqrt{x}$.

$$\therefore 3 + a = 2a^2$$

$$\therefore 2a^2 - a - 3 = 0$$

$$\therefore (2a - 3)(a + 1) = 0$$

$$\therefore a = \frac{3}{2} \text{ or } a = -1$$

$$\therefore \sqrt{x} = \frac{3}{2} \text{ or } \sqrt{x} = -1 \text{(reject)}$$

$$\therefore x = \left(\frac{3}{2}\right)^2$$

$$\therefore x = \frac{9}{4}$$

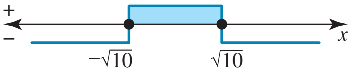
2 **a** $2x^2 - 5x - 3 > 0$

$$\therefore (2x + 1)(x - 3) > 0$$

$$\therefore x < -\frac{1}{2} \text{ or } x > 3$$

b $10 - x^2 \geq 0$

$$\therefore \left(\sqrt{10} - x\right)\left(\sqrt{10} + x\right) \geq 0$$

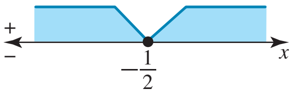


$$\therefore -\sqrt{10} \leq x \leq \sqrt{10}$$

c $20x^2 + 20x + 5 \geq 0$

$$\therefore 5(4x^2 + 4x + 1) \geq 0$$

$$\therefore 5(2x + 1)^2 \geq 0$$



$x \in R$

3 **a** $y = 2(x - 3)(x + 1)$

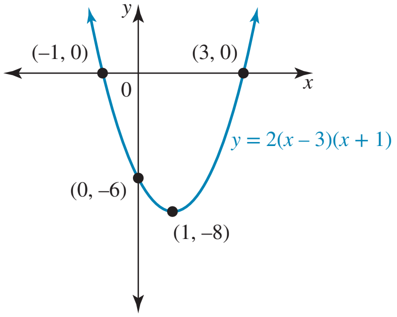
x-intercepts: $(3, 0),\ (-1, 0)$

TP: $x = \frac{3 + (-1)}{2} \Rightarrow x = 1,\ y = 2(-2)(2) \Rightarrow y = -8$

Min TP $(1, -8)$

y-intercept: $y = 2(-3)(1) \Rightarrow y = -6$

y-intercept $(0, -6)$



b $y = 1 - (x+2)^2$

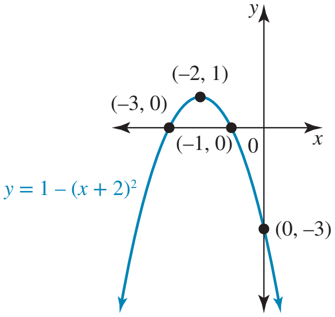
Max TP $(-2, 1)$

y-intercept $(0, -3)$

x-intercepts: $0 = 1 - (x+2)^2$

$$\therefore (x+2)^2 = 1$$
$$\therefore x + 2 = \pm 1$$

$\therefore x = -3,\ x = -1(-3, 0),\ (-1, 0)$



c $y = x^2 + x + 9$

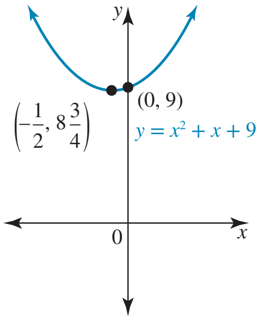
y-intercept $(0, 9)$

TP: $x = -\dfrac{1}{2 \times 1} \Rightarrow x = -\dfrac{1}{2}$

$$y = \frac{1}{4} - \frac{1}{2} + 9$$
$$= 8\frac{3}{4}$$

Min TP $\left(-\dfrac{1}{2}, 8\dfrac{3}{4}\right)$

No x-intercepts



4 a $-x^2 + 20x + 24 = -\left[x^2 - 20x - 24\right]$

$$= -\left[\left(x^2 - 20x + 100\right) - 100 - 24\right]$$
$$= -\left[(x-10)^2 - 124\right]$$
$$= -\left(x - 10 - \sqrt{124}\right)\left(x - 10 + \sqrt{124}\right)$$
$$= -\left(x - 10 - 2\sqrt{31}\right)\left(x - 10 + 2\sqrt{31}\right)$$

b $4x^2 - 2x - 9 = 4\left[x^2 - \dfrac{1}{2}x - \dfrac{9}{4}\right]$

$$= 4\left[\left(x^2 - \frac{1}{2}x + \frac{1}{6}\right) - \frac{1}{16} - \frac{9}{4}\right]$$
$$= 4\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{36}{16}\right]$$
$$= 4\left[\left(x - \frac{1}{4}\right)^2 - \frac{37}{16}\right]$$
$$= 4\left(x - \frac{1}{4} - \sqrt{\frac{37}{16}}\right)\left(x - \frac{1}{4} + \sqrt{\frac{37}{16}}\right)$$
$$= 4\left(x - \frac{1}{4} - \frac{\sqrt{37}}{4}\right)\left(x - \frac{1}{4} + \frac{\sqrt{37}}{4}\right)$$
$$= 4\left(x - \frac{1 + \sqrt{37}}{4}\right)\left(x - \frac{1 - \sqrt{37}}{4}\right)$$

5 a $5x^2 + 8x - 2 = 0$

$\Delta = b^2 - 4ac,\ a = 5, b = 8, c = -2$

$= 8^2 - 4(5)(-2)$

$= 64 + 40$

$\Delta = 104$

Since $\Delta > 0$, there are two solutions.

These are both irrational, since the discriminant is not a perfect square.

b $kx^2 - 4x(k+2) + 36 = 0$

No real roots if $\Delta < 0$

$\Delta = (-4(k+2))^2 - 4 \times k \times 36$

$= 16(k+2)^2 - 144k$

$= 16\left[k^2 + 4k + 4 - 9k\right]$

$= 16\left(k^2 - 5k + 4\right)$

$= 16(k-1)(k-4)$

For no real roots, $16(k-1)(k-4) < 0$.

$\therefore 1 < k < 4$

6 a $y = x^2 + 2x$ [1]

$y = x + 2$ [2]

At intersection, $x^2 + 2x = x + 2$.

$\therefore x^2 + x - 2 = 0$

$\therefore (x+2)(x-1) = 0$

$\therefore x = -2, x = 1$

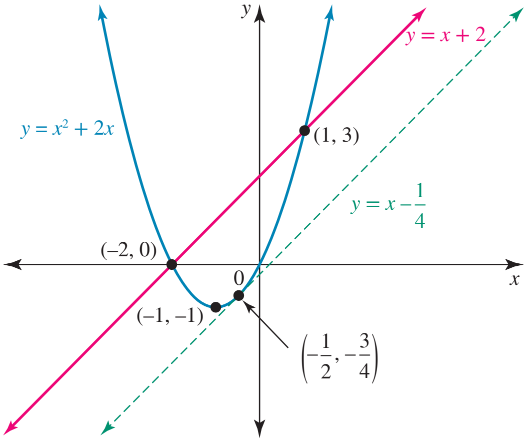
In equation [2], when $x = -2, y = 0$, and when $x = 1, y = 3$.

The points of intersection are $(-2, 0), (1, 3)$.

b $y = x + 2$ has axial intercepts $(0, 2)$ and $(-2, 0)$.

$y = x^2 + 2x$ or $y = x(x+2)$ or $y = (x+1)^2 - 1$ has axial intercepts $(0, 0)$ and $(-2, 0)$, and a minimum turning point $(-1, -1)$.

Both graphs contain the intersection points $(-2, 0), (1, 3)$



c The region enclosed between the parabola and the line lies under the line and above the parabola. It is described by the inequations $y \le x + 2$ and $y \ge x^2 + 2x$. It is the region $\{(x, y) : y \le x + 2\} \cap \left\{(x, y) : y \ge x^2 + 2x\right\}$.

d i For $y = x^2 + 2x$ and $y = x + k$ to intersect,

$x^2 + 2x = x + k$.

$\therefore x^2 + x - k = 0$

If the line is to be a tangent, there should be one point of intersection, so $\Delta = 0$.

$$\therefore 1^2 - 4 \times 1 \times (-k) = 0$$
$$\therefore 1 + 4k = 0$$
$$\therefore k = -\frac{1}{4}$$

ii With $k = -\frac{1}{4}$, $x^2 + x - k = 0$ becomes

$$x^2 + x + \frac{1}{4} = 0$$
$$\therefore \left(x + \frac{1}{2}\right)^2 = 0$$
$$\therefore x = -\frac{1}{2}$$

Substitute $x = -\frac{1}{2}$ and $k = -\frac{1}{4}$ in $y = x + k$.

$$\therefore y = -\frac{1}{2} - \frac{1}{4}$$
$$\therefore y = -\frac{3}{4}$$

The point of contact is $\left(-\frac{1}{2}, -\frac{3}{4}\right)$.

The tangent is sketched on the diagram in part **b**.

Technology active: multiple choice

7 $(x - 2)(x + 1) = 4$

$$\therefore x^2 - x - 2 = 4$$
$$\therefore x^2 - x - 6 = 0$$
$$\therefore (x - 3)(x + 2) = 0$$
$$\therefore x = 3,\ x = -2$$

The correct answer is **D**.

8 $-5x^2 + 8x + 3 = 0$

$$x = \frac{-8 \pm \sqrt{64 - 4 \times (-5) \times 3}}{-10}$$
$$= \frac{-8 \pm \sqrt{124}}{-10}$$
$$\simeq -0.31,\ 1.91$$

The correct answer is **C**.

9 As the parabola has no x intercepts, its discriminant is negative Its shape is concave up, so $a > 0$.

The correct answer is **B**.

10 Using $y = a\,(x - h)^2 + k$,

$h = 4$ and $k = 3$.

Therefore, $y = (x + 4)^2 + 3$.

The correct answer is **C**.

11 $x^2 + 4x - 6 = (x^2 + 4x + 4) - 4 - 6$

$$= (x + 2)^2 - 10$$

$\therefore b = 2,\ c = -10$

The correct answer is **A**.

12 x-intercepts at $x = -6$ and $x = 4$ mean the equation is of the form $y = a(x + 6)(x - 4)$.

Substitute the given point $(3, -4.5)$.

$$\therefore\ -4.5 = a(9)(-1)$$
$$\therefore\ -9a = -4.5$$
$$\therefore a = 0.5$$

The equation is $y = 0.5(x + 6)(x - 4)$.

Expanding,

$$y = 0.5(x^2 + 2x - 24)$$
$$= 0.5x^2 + x - 12$$

The correct answer is **B**.

13 $x^2 < 4x$

$$\therefore x^2 - 4x < 0$$
$$\therefore x(x - 4) < 0$$

$\therefore 0 < x < 4$

The correct answer is **C**.

14 Since the graph touches the x-axis at $x = -6$, $(-6, 0)$ is its turning point.

Its equation is of the form $y = a(x + 6)^2$.

The point $(0, -10)$ lies on the graph

$$\therefore\ -10 = a(36)$$
$$\therefore a = -\frac{5}{18}$$

The equation is $y = -\frac{5}{18}(x + 6)^2$.

The correct answer is **E**.

15 $y = 3x^2 - 10x + 2$ [1]

$2x - y = 1$ [2]

From equation [2], $y = 2x - 1$.

Substitute in equation [1].

$$\therefore 2x - 1 = 3x^2 - 10x + 2$$
$$\therefore 3x^2 - 12x + 3 = 0$$

This equation determines the number of intersections.

The correct answer is **B**.

16 $4 - 2x - x^2$

Completing the square,

$$= -(x^2 + 2x - 4)$$
$$= -\left((x^2 + 2x + 1) - 1 - 4\right)$$
$$= -\left((x + 1)^2 - 5\right)$$
$$= -(x + 1)^2 + 5$$

The greatest value is 5.

The correct answer is **A**.

Technology active: extended response

17 a $h = -\frac{1}{35}(x^2 - 60x - 700)$

When $x = 0$, $h = -\frac{1}{35}(-700)$.

$\therefore h = 20$ and the point S is $(0, 20)$.

Therefore, S is 20 metres above O.

b When $h = 0$,

$$0 = -\frac{1}{35}(x^2 - 60x - 700)$$
$$\therefore x^2 - 60x - 700 = 0$$
$$\therefore (x - 70)(x + 10) = 0$$
$$\therefore x = 70 \text{ or } x = -10 \text{ (reject)}$$
$$\therefore x = 70$$

The Canadian skier jumps 70 metres.

c The turning point is $(30, 35)$, so the equation of the path is of the form $h = a(x - 30)^2 + 35$

Substitute the point S $(0, 20)$.

$$\therefore 20 = a(-30)^2 + 35$$
$$\therefore -15 = 900a$$
$$\therefore a = -\frac{15}{900}$$
$$\therefore a = -\frac{1}{60}$$

The path of the Japanese competitor is

$h = -\frac{1}{60}(x - 30)^2 + 35$.

d To find how far the Japanese skier jumps, let $h = 0$.

$$\therefore 0 = -\frac{1}{60}(x - 30)^2 + 35$$
$$\therefore \frac{1}{60}(x - 30)^2 = 35$$
$$\therefore (x - 30)^2 = 2100$$
$$\therefore x - 30 = \pm 10\sqrt{21}$$
$$\therefore x = 30 \pm 10\sqrt{21}$$

Hence, $x \simeq -15.8$ (reject) or 75.83.

The Canadian competitor jumped 70 metres and the Japanese competitor jumped 75.83 metres, so the Japanese competitor wins the event.

18 **a** The arch of the bridge has the equation $y = 2.5x - 0.3125x^2$. The span, OB, is the length of the intercept this curve cuts off on the x-axis.

Let $y = 0$.

$$\therefore 0 = 2.5x - 0.3125x^2$$
$$\therefore 0 = x(2.5 - 0.3125x)$$
$$\therefore x = 0 \text{ or } x = \frac{2.5}{0.3125}$$
$$\therefore x = 0 \text{ or } x = 8$$

O $(0, 0)$ and B $(8, 0)$ are 8 units apart.

The span of the bridge is 8 metres.

b Point A is the turning point of the curve. The x-coordinate of A is midway between the x-coordinates of O and N, so it is $x = 4$.

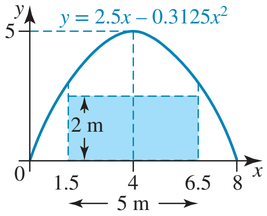
When $x = 4$,

$$y = 2.5(4) - 0.3125(4)^2$$
$$= 5$$

Therefore, A has coordinates $(4, 5)$.

The point A is 5 metres above the road.

c



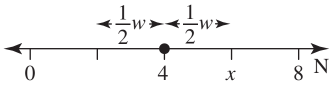
If the caravan is 5 metres wide, the height above the road when $x = 4 - 2.5$ or $x = 4 + 2.5$ would need to be greater than 2 metres, the height of the caravan, for the caravan to fit under the bridge.

When $x = 6.5$,

$$y = 2.5(6.5) - 0.3125(6.5)^2$$
$$\simeq 3.05$$

There is room for the caravan to fit under the bridge.

d



By considering the distances along the span ON,

$$\frac{1}{2}w = x - 4.$$
$$\therefore w = 2(x - 4)$$

e When $y = 3.2$,

$$3.2 = 2.5x - 0.3125x^2$$
$$\therefore 3.125x^2 - 25x + 32 = 0$$

Multiply both sides by 8.

$$\therefore 25x^2 - 200x + 256 = 0$$
$$\therefore (5x - 32)(5x - 8) = 0$$
$$\therefore x = \frac{32}{5} \text{ or } x = \frac{8}{5}$$
$$\therefore x = 6.4 \text{ or } x = 1.6$$

However, $x > 4$, so reject $x = 1.6$.

$$\therefore x = 6.4$$

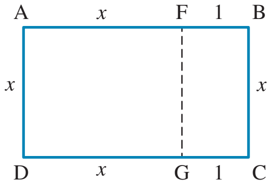
The x-coordinate of P is $x = 6.4$.

f With $x = 6.4$, $w = 2(6.4 - 4)$.

$$\therefore w = 4.8$$

The width of the caravan is 5 metres, which exceeds 4.8 metres, so under the safety restrictions, the caravan would not be permitted to be towed under the bridge.

19 **a**



b The length is $x + 1$ units, which is one more than the width. Therefore, the width is x units.

c The area of rectangle AFGD is x^2 square units.

The area of rectangle FBCG is $1 \times x = x$ square units.

$$\therefore x^2 = x + 1$$
$$\therefore x^2 - x - 1 = 0$$

d Solving $x^2 - x - 1 = 0$,

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

Since $x > 0$, reject the negative square root.

$$\therefore x = \frac{1 + \sqrt{5}}{2}$$

e $\phi = \frac{1 + \sqrt{5}}{2}$

$$\frac{1}{\phi} = \frac{2}{1 + \sqrt{5}}$$
$$= \frac{2}{1 + \sqrt{5}} \times \frac{1 - \sqrt{5}}{1 - \sqrt{5}}$$

$$= \frac{2\left(1-\sqrt{5}\right)}{1-5}$$

$$= \frac{2\left(1-\sqrt{5}\right)}{-4}$$

$$= \frac{1-\sqrt{5}}{-2}$$

$$= -\left(\frac{1-\sqrt{5}}{2}\right)$$

$x^2 - x - 1 = 0$ has two roots: the positive root is $x = \frac{1+\sqrt{5}}{2}$ and the negative root is $x = \frac{1-\sqrt{5}}{2}$.

Therefore, $\frac{1}{\phi}$ is the negative of the negative root of $x^2 - x - 1 = 0$.

f $\phi - 1 = \frac{1+\sqrt{5}}{2} - 1$

$$\therefore \phi - 1 = \frac{1+\sqrt{5}-2}{2}$$

$$\therefore \phi - 1 = \frac{\sqrt{5}-1}{2}$$

Also,

$$\frac{1}{\phi} = -\frac{1-\sqrt{5}}{2}$$

$$\therefore \frac{1}{\phi} = \frac{\sqrt{5}-1}{2}$$

$$\therefore \frac{1}{\phi} = \phi - 1$$

As $x = \phi$ is a root of $x^2 - x - 1 = 0$,

$$\phi^2 - \phi - 1 = 0$$

$$\therefore \phi(\phi - 1) = 1$$

$$\therefore \phi - 1 = \frac{1}{\phi}$$

20 a As the horizontal speed is 28 m/s, in 1 second the ball travels 28 metres horizontally.

The turning point of the path of the ball is (28, 4.9).

b Let the equation be $y = a(x-h)^2 + k$.

The turning point is (28, 4.9).

$$\therefore a = -\frac{4.9}{28 \times 28}$$

$$\therefore a = -\frac{0.1}{4 \times 4}$$

$$\therefore a = -\frac{1}{160}$$

The equation of the path of the ball is

$$y = -\frac{1}{160}(x-28)^2 + 4.9.$$

c Calculate the horizontal distance the ball has travelled when its height is 1.3 metres.

Let $y = 1.3$.

$$\therefore 1.3 = -\frac{1}{160}(x-28)^2 + 4.9$$

$$\therefore \frac{1}{160}(x-28)^2 = 3.6$$

$$\therefore (x-28)^2 = 3.6 \times 160$$

$$= 36 \times 16$$

$$\therefore x - 28 = \pm 6 \times 4$$

$$\therefore x = 4 \text{ or } x = 52$$

The ball is caught after it reaches its maximum height, so reject $x = 4$.

$$\therefore x = 52$$

The ball travels a horizontal distance of 52 metres to reach the position where the ball is caught. At a horizontal speed of 28 m/s, this would take $\frac{52}{28} = \frac{13}{7}$ seconds.

It takes the fielder $\frac{13}{7}$ seconds to reach the ball.

d Initially the fielder was 65 metres from where the ball was hit. The fielder catches the ball at the position $x = 52$.

Thus, the fielder runs a distance of 13 metres in $\frac{13}{7}$ seconds.

The uniform speed of the fielder is $\frac{13}{\frac{13}{7}} = 7$ m/s.

3.9 Exam questions

1 Let $\left(x + \frac{1}{x}\right) = a$.

$$a^2 + 4a + 4 = 0$$

$$(a+2)^2 = 0$$

$$a = -2 \quad \text{[1 mark]}$$

Substituting back,

$$x + \frac{1}{x} = -2$$

$$x^2 + 1 = -2x$$

$$x^2 + 2x + 1 = 0$$

$$(x+1)^2 = 0$$

$$x = -1 \quad \text{[1 mark]}$$

2 $b^2 - 4ac > 0$ indicates there are two real roots.

The correct answer is **B**.

3 $\Delta = b^2 - 4ac$

For $-4x^2 - x - 3$, $a = -4$, $b = -1$, $c = -3$.

$\Delta = (-1)^2 - 4(-4)(-3)$

$\Delta = -47$

Δ is negative, so there is no x-intercept.

The correct answer is **A**.

4 Options are in the form $y = a(x-h)^2 + k$.

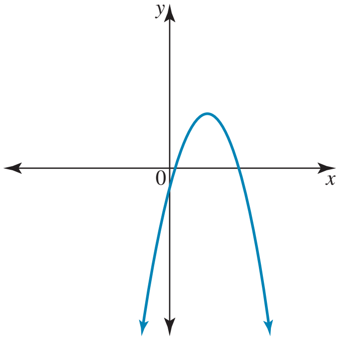
The parabola is inverted, so a is negative.

The vertex has positive values for x and y.

For positive values of the x-intercepts,

$y = -(x-2)^2 + 3$ is the only valid option.

The correct answer is **D**.



5
$$x^2 - 8x + 12 = kx + 4$$
$$x^2 - x(8+k) + 8 = 0$$

$\Delta = b^2 - 4ac$

$a = 1, b = (-8 - k), c = 8$

$\Delta = (-8 - k)^2 - 4(1)(8)$

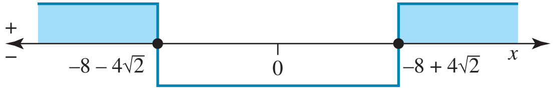
$\quad = (8 + k)^2 - 32$ [1 mark]

$\Delta \geq 0$ for one or two intersections [1 mark]

$\left(8 + k - \sqrt{32}\right)\left(8 + k + \sqrt{32}\right) = 0$

$\left(8 + k - 4\sqrt{2}\right)\left(8 + k + 4\sqrt{2}\right) = 0$

The zeros are $-8 + 4\sqrt{2}, \ -8 - 4\sqrt{2}$.



For at least one intersection,

$k \in \left(-\infty, -8 - 4\sqrt{2}\right] \cup \left[-8 + 4\sqrt{2}, \infty\right)$ [1 mark]

$\left(\text{Also accept } k \in \left(-\infty, -8 - \sqrt{32}\right] cup \left[-8 + \sqrt{32}, \infty\right)\right)$

Topic 4 — Cubic polynomials

4.2 Polynomials

4.2 Exercise

1 A and C are polynomials, B is not a polynomial due to the term $-\frac{2}{x}$.

For A: the degree is 5 (a quintic), the leading term coefficient is 4 and the constant term is 12. The coefficients are integers, so A is a polynomial over Z, the set of integers.

For C: the degree is 2, the leading term coefficient is -0.2 and the constant term is 5.6. The coefficients are rational numbers, so C is a polynomial over Q, the set of rationals.

2 **a** $7x^4 + 3x^2 + 5$ is a polynomial of degree 4.

b $9 - \frac{5}{2}x - 4x^2 + x^3$ is a polynomial of degree 3.

c $-9x^3 + 7x^2 + 11\sqrt{x} - \sqrt{5}$ is not a polynomial due to the presence of the $\sqrt{x}$ term. The algebraic expression can be written as $-9x^3 + 7x^2 + 11x^{\frac{1}{2}} - \sqrt{5}$, giving a power of x that is not a natural number.

Note that the $\sqrt{5}$ term can appear in a polynomial, so that is not the reason why **c** is not a polynomial.

d $\frac{6}{x^2} + 6x^2 + \frac{x}{2} - \frac{2}{x}$ is not a polynomial due to the $\frac{6}{x^2}$ term and the $\frac{2}{x}$ term. The algebraic expression can be written as $6x^{-2} + 6x^2 + \frac{x}{2} - 2x^{-1}$, giving powers of x that are not natural numbers.

3 **a** The polynomials are A, B, D and F.

A: $3x^5 + 7x^4 - \frac{x^3}{6} + x^2 - 8x + 12$

B: $9 - 5x^4 + 7x^2 - \sqrt{5}x + x^3 = 9 - \sqrt{5}x + 7x^2 + x^3 - 5x^4$

D: $2x^2(4x - 9x^2) = 8x^3 - 18x^4$

F: $\left(4x^2 + 3 + 7x^3\right)^2$

$$\begin{aligned}
&= ((4x^2 + 3) + 7x^3)^2 \\
&= (4x^2 + 3)^2 + 2(4x^2 + 3)(7x^3) + (7x^3)^2 \\
&= 16x^4 + 24x^2 + 9 + 14x^3(4x^2 + 3) + 49x^6 \\
&= 16x^4 + 24x^2 + 9 + 56x^5 + 42x^3 + 49x^6 \\
&= 49x^6 + 56x^5 + 16x^4 + 42x^3 + 24x^2 + 9
\end{aligned}$$

	Degree	**Type of coefficient**	**Leading term**	**Constant term**
A	5	Q	$3x^5$	12
B	4	R	$-5x^4$	9
D	4	Z	$-18x^4$	0
F	6	N	$49x^6$	9

b C: $\sqrt{4x^5} - \sqrt{5}x^3 + \sqrt{3}x - 1$ is not a polynomial due to the $\sqrt{4x^5} = 2x^{\frac{5}{2}}$ term: $\frac{5}{2} \notin N$.

E: $\frac{x^6}{10} - \frac{2x^5}{7} + \frac{5}{3x^2} - \frac{7x}{5} + \frac{4}{9}$ is not a polynomial due to the $\frac{5}{3x^2} = \frac{5}{3}x^{-2}$ term: $-2 \notin N$.

4 Many answers are possible; however, they must contain $y^7 - \sqrt{2}y^2 + 4$ and the fourth term chosen must have the power of y as one of the whole numbers 1, 3, 4, 5 or 6. For example, one answer could be $y^7 + 2y^5 - \sqrt{2}y^2 + 4$.

5 **a** $p(x) = -x^3 + 2x^2 + 5x - 1$

$$\begin{aligned}
p(1) &= -(1)^3 + 2(1)^2 + 5(1) - 1 \\
&= -1 + 2 + 5 - 1 \\
&= 5
\end{aligned}$$

b $p(x) = 2x^3 - 4x^2 + 3x - 7$

$$\begin{aligned}
p(-2) &= 2(-2)^3 - 4(-2)^2 + 3(-2) - 7 \\
&= 2(-8) - 4(4) - 6 - 7 \\
&= -16 - 16 - 6 - 7 \\
&= -45
\end{aligned}$$

c $p(x) = 3x^3 - x^2 + 5$

$$\begin{aligned}
p(3) &= 3(3)^3 - (3)^2 + 5 \\
&= 3(27) - 9 + 5 \\
&= 81 - 9 + 5 \\
&= 77
\end{aligned}$$

$$\begin{aligned}
p(-x) &= 3(-x)^3 - (-x)^2 + 5 \\
&= 3(-x^3) - (x^2) + 5 \\
&= -3x^3 - x^2 + 5
\end{aligned}$$

d $p(x) = x^3 + 4x^2 - 2x + 5$

$$\begin{aligned}
p\,(-1) &= (-1)^3 + 4\,(-1)^2 - 2\,(-1) + 5 \\
&= -1 + 4 + 2 + 5 \\
&= 10
\end{aligned}$$

$$\begin{aligned}
p\,(2a) &= (2a)^3 + 4\,(2a)^2 - 2\,(2a) + 5 \\
&= 8a^3 + 16a^2 - 4a + 5
\end{aligned}$$

6 $p(x) = 2x^3 + 3x^2 + x - 6$

a
$$\begin{aligned}
p(3) &= 2(3)^3 + 3(3)^2 + (3) - 6 \\
&= 54 + 27 + 3 - 6 \\
&= 78
\end{aligned}$$

b
$$\begin{aligned}
p(-2) &= 2(-2)^3 + 3(-2)^2 + (-2) - 6 \\
&= -16 + 12 - 2 - 6 \\
&= -12
\end{aligned}$$

c
$$\begin{aligned}
p(1) &= 2(1)^3 + 3(1)^2 + (1) - 6 \\
&= 2 + 3 + 1 - 6 \\
&= 0
\end{aligned}$$

d $p(0) = -6$

e
$$\begin{aligned}
p\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 6 \\
&= -\frac{1}{4} + \frac{3}{4} - \frac{1}{2} - 6 \\
&= -6
\end{aligned}$$

f
$$\begin{aligned}
p(0.1) &= 2(0.1)^3 + 3(0.1)^2 + (0.1) - 6 \\
&= 0.002 + 0.03 + 0.1 - 6 \\
&= -5.868
\end{aligned}$$

7 $p(x) = x^2 - 7x + 2$

a
$$\begin{aligned}
p(a) - p(-a) &= \left((a)^2 - 7(a) + 2\right) - \left((-a)^2 - 7(-a) + 2\right) \\
&= \left(a^2 - 7a + 2\right) - \left(a^2 + 7a + 2\right) \\
&= -14a
\end{aligned}$$

b $p(1+h) = (1+h)^2 - 7(1+h) + 2$
$= 1 + 2h + h^2 - 7 - 7h + 2$
$= h^2 - 5h - 4$

c $p(x+h) - p(x)$
$= \left[(x+h)^2 - 7(x+h) + 2\right] - \left[x^2 - 7x + 2\right]$
$= \left[x^2 + 2xh + h^2 - 7x - 7h + 2\right] - x^2 + 7x - 2$
$= x^2 + 2xh + h^2 - 7x - 7h + 2 - x^2 + 7x - 2$
$= 2xh + h^2 - 7h$

8 a $p(x) = 7x^3 - 8x^2 - 4x - 1$
$p(2) = 7(2)^3 - 8(2)^2 - 4(2) - 1$
$= 56 - 32 - 8 - 1$
$= 15$

b $p(x) = 2x^2 + kx + 12$
$p(-3) = 0 \Rightarrow 0 = 2(-3)^2 + k(-3) + 12$
$\therefore 0 = 18 - 3k + 12$
$0 = 30 - 3k$
$3k = 30$
$k = 10$

9 a $p(x) = ax^2 + 9x + 2$
$p(1) = a(1)^2 + 9(1) + 2$
$= a + 11$
Since $p(1) = 3$,
$a + 11 = 3$
$a = -8$

b $p(x) = -5x^2 + bx - 18$
$p(3) = -5(3)^2 + b(3) - 18$
$= -45 + 3b - 18$
$= 3b - 63$
Since $p(3) = 0$,
$3b - 63 = 0$
$3b = 63$
$b = 21$

c $p(x) = -2x^3 + 3x^2 + kx - 10$
$p(-1) = -2(-1)^3 + 3(-1)^2 + k(-1) - 10$
$= 2 + 3 - k - 10$
$= -k - 5$
Since $p(-1) = -7$,
$-k - 5 = -7$
$-k = -2$
$k = 2$

d $p(x) = x^3 - 6x^2 + 9x + m$
$p(0) = m$
$p(1) = 1 - 6 + 9 + m$
$= m + 4$
Since $p(0) = 2p(1)$,
$m = 2(m + 4)$
$m = 2m + 8$
$0 = m + 8$
$m = -8$

e $p(x) = -2x^3 + 9x + m$
$p(1) = -2(1)^3 + 9(1) + m$
$p(1) = 7 + m$
$p(-1) = -2(-1)^3 + 9(-1) + m$
$p(-1) = -7 + m$
$\therefore p(1) = 2p(-1) \Rightarrow 7 + m = 2(-7 + m)$
$7 + m = -14 + 2m$
$21 = m$
$m = 21$

f $q(x) = -x^2 + bx + c$
$q(0) = 5 \Rightarrow c = 5$
$q(5) = 0$
$\Rightarrow -(5)^2 + b(5) + c = 0$
Substitute $c = 5$.
$\therefore -25 + 5b + 5 = 0$
$\therefore 5b = 20$
$\therefore b = 4$
Answer: $b = 4, c = 5$

10 $(2x+1)(x-5) \equiv a(x+1)^2 + b(x+1) + c$
Expanding,
$2x^2 - 9x - 5 = ax^2 + 2ax + a + bx + b + c$
$2x^2 - 9x - 5 = ax^2 + (2a+b)x + (a+b+c)$
$\therefore 2 = a, \quad -9 = 2a + b, \quad -5 = a + b + c$
$a = 2, \quad -9 = 4 + b, \quad -5 = 2 + b + c$
$\therefore b = -13 \quad -5 = 2 - 13 + c$
$\therefore c = 6$

11 a $x^2 + 10x + 6 \equiv x(x+a) + b$
$x^2 + 10x + 6 \equiv x^2 + ax + b$
Equating coefficients of like terms gives $a = 10, b = 6$.

b Let $8x - 6 = ax + b(x+3)$.
$8x - 6 = ax + bx + 3b$
$= x(a+b) + 3b$
Equate coefficients of like terms:
$8 = a + b$ [1]
$-6 = 3b$ [2]
From equation [2], $b = -2$.
Substitute $b = -2$ in equation [1].
$8 = a - 2$
$a = 10$
Therefore, $8x - 6 = 10x - 2(x+3)$.

c $6x^2 + 19x - 20 = (ax+b)(x+4)$
$= ax^2 + 4ax + bx + 4b$
$= ax^2 + x(4a+b) + 4b$
Equate coefficients of like terms:
$6 = a$ [1]
$19 = 4a + b$ [2]
$-20 = 4b$ [3]
From equation [1], $a = 6$.
From equation [3], $b = -5$.
Check in equation [2]: $4a + b = 4(6) - 5 = 19$.
Therefore, $6x^2 + 19x - 20 = (6x-5)(x+4)$.

d $x^2 - 8x = a + b(x+1) + c(x+1)^2$
$= a + bx + b + c(x^2 + 2x + 1)$
$= a + bx + b + cx^2 + 2cx + c$
$= cx^2 + x(b + 2c) + a + b + c$
Equate coefficients of like terms:
$1 = c$ [1]
$-8 = b + 2c$ [2]
$0 = a + b + c$ [3]

Substitute $c = 1$ in equation [2].

$-8 = b + 2$

$b = -10$

Substitute $c = 1$ and $b = -10$ in equation [3].

$0 = a - 10 + 1$

$a = 9$

Therefore, $a = 9, b = -10, c = 1$.

12 Let $(x + 2)^3 = px^2(x + 1) + qx(x + 2) + r(x + 3) + t$.

Expanding,

$$x^3 + 3x^2(2) + 3x(2)^2 + (2)^3 = px^3 + px^2 + qx^2 + 2qx + rx + 3r + t$$

$$x^3 + 6x^2 + 12x + 8 = px^3 + (p + q)x^2 + (2q + r)x + (3r + t)$$

$$\begin{array}{llll} \therefore 1 = p & 6 = p + q, & 12 = 2q + r, & 8 = 3r + t \\ p = 1, & 6 = 1 + q, & 12 = 2q + r, & 8 = 6 + t \\ p = 1 & q = 5 & 12 = 10 + r & t = 2 \\ & & r = 2 & \end{array}$$

Therefore, $(x + 2)^3 = x^2(x + 1) + 5x(x + 2) + 2(x + 3) + 2$.

13 **a** $3x^2 + 4x - 7 = a(x + 1)^2 + b(x + 1) + c$

Expand the right-hand side.

$\therefore 3x^2 + 4x - 7 = a(x^2 + 2x + 1) + bx + b + c$

$= ax^2 + 2ax + bx + a + b + c$

$\therefore 3x^2 + 4x - 7 = ax^2 + (2a + b)x + (a + b + c)$

Equate coefficients of like terms:

$x^2 : 3 = a$ [1]

$x : 4 = 2a + b$ [2]

Constant$: -7 = a + b + c$ [3]

Substitute $a = 3$ in equation [2].

$\therefore 4 = 6 + b$

$\therefore b = -2$

Substitute $a = 3, b = -2$ in equation [3].

$\therefore\ -7 = 3 - 2 + c$

$\therefore c = -8$

Answer: $a = 3, b = -2, c = -8$

b $x^3 + mx^2 + nx + p \equiv (x - 2)(x + 3)(x - 4)$

Expand:

$\therefore x^3 + mx^2 + nx + p = (x - 2)(x^2 - x - 12)$

$= x^3 - x^2 - 12x - 2x^2 + 2x + 24$

$\therefore x^3 + mx^2 + nx + p = x^3 - 3x^2 - 10x + 24$

Equate coefficients of like terms:

$x^2 : m = -3$

$x : n = -10$

Constant$: p = 24$

Answer: $m = -3, n = -10, p = 24$

c $x^2 - 14x + 8 \equiv a(x - b)^2 + c$

$\therefore x^2 - 14x + 8 = a(x^2 - 2xb + b^2) + c$

$= ax^2 - 2abx + ab^2 + c$

Equate coefficients of like terms:

$x^2 : 1 = a$ [1]

$x : -14 = -2ab$ [2]

Constant$: 8 = ab^2 + c$ [3]

Substitute $a = 1$ in equation [2].

$\therefore\ -14 = -2b$

$\therefore b = 7$

Substitute $a = 1, b = 7$ in equation [3].

$\therefore 8 = 49 + c$

$\therefore c = -41$

$\therefore a = 1,\ b = 7,\ c = -41$

$\therefore x^2 - 14x + 8 = (x - 7)^2 - 41$

d Let $4x^3 + 2x^2 - 7x + 1 = ax^2(x + 1) + bx(x + 1) + c(x + 1) + d$.

$\therefore 4x^3 + 2x^2 - 7x + 1 = ax^3 + ax^2 + bx^2 + bx + cx + c + d$

$= ax^3 + (a + b)x^2 + (b + c)x + (c + d)$

Equating coefficients of like terms:

$x^3 : 4 = a$ [1]

$x^2 : 2 = a + b$ [2]

$x : -7 = b + c$ [3]

Constant$: 1 = c + d$ [4]

Substitute $a = 4$ in equation [2].

$\therefore 2 = 4 + b$

$\therefore b = -2$

Substitute $b = -2$ in equation [3].

$\therefore -7 = -2 + c$

$\therefore c = -5$

Substitute $c = -5$ in equation [4].

$\therefore 1 = -5 + d$

$\therefore d = 6$

Hence, $4x^3 + 2x^2 - 7x + 1$

$= 4x^2(x + 1) - 2x(x + 1) - 5(x + 1) + 6$

14 $p(x) + 2q(x)$

$= 4x^3 - ax^2 + 8 + 2(3x^2 + bx - 7)$

$= 4x^3 + (6 - a)x^2 + 2bx - 6$

Therefore, $4x^3 + (6 - a)x^2 + 2bx - 6 = 4x^3 + x^2 - 8x - 6$.

Equating coefficients:

$(x^2)\ 6 - a = 1$ and $(x)\ 2b = -8$

Hence, $a = 5$ and $b = -4$.

15 **a** $p(x) = 2x^2 - 7x - 11$ and $q(x) = 3x^4 + 2x^2 + 1$

i $q(x) - p(x)$

$= 3x^4 + 2x^2 + 1 - (2x^2 - 7x - 11)$

$= 3x^4 + 7x + 12$

ii $3p(x) + 2q(x)$

$= 3(2x^2 - 7x - 11) + 2(3x^4 + 2x^2 + 1)$

$= 6x^2 - 21x - 33 + 6x^4 + 4x^2 + 2$

$= 6x^4 + 10x^2 - 21x - 31$

iii $p(x)q(x)$

$= (2x^2 - 7x - 11)(3x^4 + 2x^2 + 1)$

$= 6x^6 + 4x^4 + 2x^2 - 21x^5 - 14x^3 - 7x - 33x^4$

$- 22x^2 - 11$

$= 6x^6 - 21x^5 - 29x^4 - 14x^3 - 20x^2 - 7x - 11$

b $p(x)$ has degree m, $q(x)$ has degree n and $m > n$.

i The leading term of $p(x) + q(x)$ must be the x^m term, so the degree is m.

ii Similarly, the degree of $p(x) - q(x)$ is m.

iii The leading term of $p(x)q(x)$ must be the $x^m \times x^n = x^{m+n}$ term, so the degree is $m + n$.

16 **a** $\dfrac{x - 12}{x + 3}$

$= \dfrac{(x + 3) - 3 - 12}{x + 3}$

$= \dfrac{(x + 3) - 15}{x + 3}$

$= \dfrac{x + 3}{x + 3} - \dfrac{15}{x + 3}$

$= 1 - \dfrac{15}{x + 3}$

The quotient is 1 and the remainder is -15.

b $\dfrac{4x+7}{2x+1}$

$= \dfrac{2(2x+1)+5}{2x+1}$

$= 2 + \dfrac{5}{2x+1}$

17 a $\dfrac{x+5}{x+1}$

$= \dfrac{(x+1)+4}{x+1}$

$= \dfrac{(x+1)}{x+1} + \dfrac{4}{x+1}$

$= 1 + \dfrac{4}{x+1}$

The remainder is 4.

b $\dfrac{2x-3}{x+4}$

$= \dfrac{2(x+4)-8-3}{x+4}$

$= \dfrac{2(x+4)-11}{x+4}$

$= \dfrac{2(x+4)}{x+4} + \dfrac{-11}{x+4}$

$= 2 - \dfrac{11}{x+4}$

The remainder is −11.

c $\dfrac{4x+11}{4x+1}$

$= \dfrac{(4x+1)-1+11}{4x+1}$

$= \dfrac{(4x+1)}{4x+1} + \dfrac{10}{4x+1}$

$= 1 + \dfrac{10}{4x+1}$

The remainder is 10.

d $\dfrac{6x+13}{2x-3}$

$= \dfrac{3(2x-3)+9+13}{2x-3}$

$= \dfrac{3(2x-3)}{2x-3} + \dfrac{22}{2x-3}$

$= 3 + \dfrac{22}{2x-3}$

The remainder is 22.

18 a The long division method gives:

$$\begin{array}{r}
2x^2 - x + 6 \\
x-2\overline{\smash{)}\,2x^3 - 5x^2 + 8x + 6} \\
\underline{2x^3 - 4x^2 \phantom{{}+8x+6}} \\
-x^2 + 8x + 6 \\
\underline{-x^2 + 2x \phantom{{}+6}} \\
6x + 6 \\
\underline{6x - 12} \\
\underline{18}
\end{array}$$

The remainder is 18.

Alternatively,

$\dfrac{2x^3-5x^2+8x+6}{x-2}$

$= \dfrac{2x^2(x-2)-x(x-2)+6(x-2)+18}{x-2}$

$= 2x^2 - x + 6 + \dfrac{18}{x-2}$

The quotient is $2x^2 - x + 6$ and the remainder is 18.

b

$$\begin{array}{r}
-x^2 - x - 1 \\
-x+1\overline{\smash{)}\,x^3 + 0x^2 + 0x + 10} \\
\underline{(x^3 - x^2) \phantom{{}+0x+10}} \\
x^2 + 0x + 10 \\
\underline{(x^2 - x) \phantom{{}+10}} \\
x + 10 \\
\underline{(x - 1)} \\
\underline{11}
\end{array}$$

$\therefore \dfrac{x^3+10}{1-x} = -x^2 - x - 1 + \dfrac{11}{1-x}.$

The remainder is 11.

19 a $\dfrac{x+7}{x-2}$

$= \dfrac{(x-2)+9}{x-2}$

$= \dfrac{x-2}{x-2} + \dfrac{9}{x-2}$

$= 1 + \dfrac{9}{x-2}$

Quotient 1, remainder 9

b $\dfrac{8x+5}{2x+1}$

$= \dfrac{4(2x+1)+1}{2x+1}$

$= \dfrac{4(2x+1)}{2x+1} + \dfrac{1}{2x+1}$

$= 4 + \dfrac{1}{2x+1}$

Quotient 4, remainder 1

c $\dfrac{x^2+6x-17}{x-1}$

$= \dfrac{x(x-1)+7x-17}{x-1}$

$= \dfrac{x(x-1)+7(x-1)-10}{x-1}$

$= \dfrac{x(x-1)}{x-1} + \dfrac{7(x-1)}{x-1} + \dfrac{-10}{x-1}$

$= x + 7 - \dfrac{10}{x-1}$

Quotient $x+7$, remainder −10

d $\dfrac{2x^2-8x+3}{x+2}$

$= \dfrac{2x(x+2)-12(x+2)+27}{x+2}$

$= \dfrac{2x(x+2)-12x+3}{x+2}$

$$= \frac{2x(x+2)}{x+2} + \frac{-12(x+2)}{x+2} + \frac{27}{x+2}$$

$$= 2x - 12 + \frac{27}{x+2}$$

Quotient $2x - 12$, remainder 27

e $\dfrac{x^3 + 2x^2 - 3x + 5}{x - 3}$

$$= \frac{x^2(x-3) + 5x^2 - 3x + 5}{x-3}$$

$$= \frac{x^2(x-3) + 5x(x-3) + 12x + 5}{x-3}$$

$$= \frac{x^2(x-3) + 5x(x-3) + 12(x-3) + 41}{x-3}$$

$$= \frac{x^2(x-3)}{x-3} + \frac{5x(x-3)}{x-3} + \frac{12(x-3)}{x-3} + \frac{41}{x-3}$$

$$= x^2 + 5x + 12 + \frac{41}{x-3}$$

Quotient $x^2 + 5x + 12$, remainder 41

f $\dfrac{x^3 - 8x^2 + 9x - 2}{x - 1}$

$$= \frac{x^2(x-1) - 7x^2 + 9x - 2}{x-1}$$

$$= \frac{x^2(x-1) - 7x(x-1) + 2x - 2}{x-1}$$

$$= \frac{x^2(x-1) - 7x(x-1) + 2(x-1)}{x-1}$$

$$= \frac{x^2(x-1)}{x-1} + \frac{-7x(x-1)}{x-1} + \frac{2(x-1)}{x-1}$$

$$= x^2 - 7x + 2$$

Quotient $x^2 - 7x + 2$, remainder 0

20 a

$$\begin{array}{r}
x^2 + x + 5 \\
x - 2\,\overline{\smash{)}\,x^3 - x^2 + 3x - 5} \\
-(x^3 - 2x^2) \\
\hline
x^2 + 3x \\
-(x^2 - 2x) \\
\hline
5x - 5 \\
5x - 10 \\
\hline
5 \\
\hline
\end{array}$$

The quotient is $x^2 + x + 5$.

The remainder is 5.

b

$$\begin{array}{r}
x^2 + 0x + 2 \\
3x - 1\,\overline{\smash{)}\,3x^3 - x^2 + 6x - 5} \\
-(3x^3 - x^2) \\
\hline
6x - 5 \\
-(6x - 2) \\
\hline
-3 \\
\hline
\end{array}$$

The quotient is $x^2 + 2$.

The remainder is -3.

c

$$\begin{array}{r}
3x^2 + 9x + 32 \\
2x - 7\,\overline{\smash{)}\,6x^3 - 3x^2 + x + 1} \\
-(6x^3 - 21x^2) \\
\hline
18x^2 + x \\
-(18x^2 - 63x) \\
\hline
64x + 1 \\
-(64x - 224) \\
\hline
225 \\
\hline
\end{array}$$

The quotient is $3x^2 + 9x + 32$.

The remainder is 225.

d

$$\begin{array}{r}
3x^2 - 7x + 11 \\
2x + 3\,\overline{\smash{)}\,6x^3 - 5x^2 + x + 3} \\
-(6x^3 + 9x^2) \\
\hline
-14x^2 + x \\
-(-14x^2 - 21x) \\
\hline
22x + 3 \\
-(22x + 33) \\
\hline
-30 \\
\hline
\end{array}$$

The quotient is $3x^2 - 7x + 11$.

The remainder is -30.

21 $3x^2 - 6x + 5 \equiv ax(x+2) + b(x+2) + c$

$\therefore 3x^2 - 6x + 5 = ax^2 + 2ax + bx + 2b + c$

$\therefore 3x^2 - 6x + 5 = ax^2 + x(2a + b) + 2b + c$

Equate coefficients of like terms:

$x^2: 3 = a$ [1]

$x: -6 = 2a + b$ [2]

Constant: $5 = 2b + c$ [3]

Substitute equation [1] in equation [2].

$\therefore -6 = 2 \times 3 + b$

$\therefore b = -12$

Substitute $b = -12$ in equation [3].

$\therefore 5 = 2 \times -12 + c$

$\therefore c = 29$

$\therefore a = 3, b = -12, c = 29$

$\therefore 3x^2 - 6x + 5 \equiv 3x(x+2) - 12(x+2) + 29$

$\dfrac{3x^2 - 6x + 5}{x+2}$

$$= \frac{3x(x+2) - 12(x+2) + 29}{x+2}$$

$$= \frac{3x(x+2)}{x+2} - \frac{12(x+2)}{x+2} + \frac{29}{x+2}$$

$$= 3x - 12 + \frac{29}{x+2}$$

$q(x) = 3x - 12,\ r = 29$

22 $p(x)q(x)$

$= (x^4 + 3x^2 - 7x + 2)(x^3 + x + 1)$

$= x^4(x^3 + x + 1) + 3x^2(x^3 + x + 1) - 7x(x^3 + x + 1)$
$\quad + 2(x^3 + x + 1)$

$= x^7 + x^5 + x^4 + 3x^5 + 3x^3 + 3x^2 - 7x^4 - 7x^2 - 7x$
$\quad + 2x^3 + 2x + 2$

$= x^7 + 4x^5 - 6x^4 + 5x^3 - 4x^2 - 5x + 2$

The degree is 7.

23 Divisor $(x - 1)^2 = x^2 - 2x + 1$

The long division method would give:

$$\begin{array}{r} x^2 - x + 3 \\ x^2 - 2x + 1\overline{\big)\, x^4 - 3x^3 + 6x^2 - 7x + 3} \\ x^4 - 2x^3 + x^2 \\ \hline -x^3 + 5x^2 - 7x + 3 \\ -x^3 + 2x^2 - x \\ \hline 3x^2 - 6x + 3 \\ 3x^2 - 6x + 3 \\ \hline 0 \\ \hline \end{array}$$

Quotient $= x^2 - x + 3$,
Remainder $= 0$

Alternatively,

$$\frac{x^4 - 3x^3 + 6x^2 - 7x + 3}{(x - 1)^2}$$
$$= \frac{x^4 - 3x^3 + 6x^2 - 7x + 3}{x^2 - 2x + 1}$$
$$= \frac{x^2(x^2 - 2x + 1) - x(x^2 - 2x + 1) + 3(x^2 - 2x + 1)}{x^2 - 2x + 1}$$
$$= x^2 - x + 3$$

24 **a** $(8x^3 + 6x^2 - 5x + 15)$ divided by $(1 + 2x)$
$\Rightarrow (8x^3 + 6x^2 - 5x + 15)$ divided by $(2x + 1)$

$$\begin{array}{r} 4x^2 + x - 3 \\ 2x + 1\overline{\big)\, 8x^3 + 6x^2 - 5x + 15} \\ 8x^3 + 4x^2 \\ \hline 2x^2 - 5x + 15 \\ 2x^2 + x \\ \hline -6x + 15 \\ -6x - 3 \\ \hline 18 \\ \hline \end{array}$$

$$\therefore \frac{8x^3 + 6x^2 - 5x + 15}{2x + 1} = 4x^2 + x - 3 + \frac{18}{2x + 1}$$

b $(4x^3 + x + 5)$ divided by $(2x - 3)$

$$\begin{array}{r} 2x^2 + 3x + 5 \\ 2x - 3\overline{\big)\, 4x^3 + 0x^2 + x + 5} \\ 4x^3 - 6x^2 \\ \hline 6x^2 + x + 5 \\ 6x^2 - 9x \\ \hline 10x + 5 \\ 10x - 15 \\ \hline 20 \\ \hline \end{array}$$

$$\therefore \frac{4x^3 + x + 5}{2x - 3} = 2x^2 + 3x + 5 + \frac{20}{2x - 3}$$

c $(x^3 + 6x^2 + 6x - 12) \div (x + 6)$

$$\begin{array}{r} x^2 + 6 \\ x + 6\overline{\big)\, x^3 + 6x^2 + 6x - 12} \\ x^3 + 6x^2 \\ \hline 6x - 12 \\ 6x + 36 \\ \hline -48 \\ \hline \end{array}$$

$$\therefore \frac{x^3 + 6x^2 + 6x - 12}{x + 6} = x^2 + 6 - \frac{48}{x + 6}$$

d $(2 + x^3) \div (x + 1) \Rightarrow (x^3 + 2) \div (x + 1)$

$$\begin{array}{r} x^2 - x + 1 \\ x + 1\overline{\big)\, x^3 + 0x^2 + 0x + 2} \\ x^3 + x^2 \\ \hline -x^2 + 0x + 2 \\ -x^3 - x \\ \hline x + 2 \\ x + 1 \\ \hline 1 \\ \hline \end{array}$$

$$\therefore \frac{x^3 + 2}{x + 1} = x^2 - x + 1 + \frac{1}{x + 1}$$

e $\dfrac{x^4 + x^3 - x^2 + 2x + 5}{x^2 - 1}$

$$\begin{array}{r} x^2 + x \\ x^2 - 1\overline{\big)\, x^4 + 3x^3 - x^2 + 2x + 5} \\ x^4 - x^2 \\ \hline x^3 + 0x^2 + 2x + 5 \\ x^3 - x \\ \hline 3x + 5 \\ \hline \end{array}$$

$$\therefore \frac{x^4 + x^3 - x^2 + 2x + 5}{x^2 - 1} = x^2 + x + \frac{3x + 5}{x^2 - 1}$$

f $\dfrac{x(7 - 2x^2)}{(x + 2)(x - 3)} = \dfrac{-2x^3 + 7x}{x^2 - x - 6}$

$$\begin{array}{r} -2x - 2 \\ x^2 - x - 6\overline{\big)\, -2x^3 + 0x^2 + 7x + 0} \\ -2x^3 + 2x^2 + 12x \\ \hline -2x^2 - 5x + 0 \\ -2x^2 + 5x + 12 \\ \hline -7x - 12 \\ \hline \end{array}$$

$$\therefore \frac{-2x^3 + 7x}{x^2 - x - 6} = -2x - 2 - \frac{7x + 12}{(x + 2)(x - 3)}$$

25 **a** $p(x)=x^3-3x^2+cx-2,\ q(x)=ax^3+bx^2+3x-2a$

Given $2p(x)-q(x)=5(x^3-x^2+x+d)$

$$\begin{aligned}&\therefore 2(x^3-3x^2+cx-2)-(ax^3+bx^2+3x-2a)\\&\qquad=5(x^3-x^2+x+d)\\&\therefore 2x^3-6x^2+2cx-4-ax^3-bx^2-3x+2a\\&\qquad=5x^3-5x^2+5x+5d\\&\therefore (2-a)x^3+(-6-b)x^2+(2c-3)x+(-4+2a)\\&\qquad=5x^3-5x^2+5x+5d\end{aligned}$$

Equate coefficients of like terms:

$x^3: 2-a=5$ [1]

$x^2: -6-b=-5$ [2]

$x: 2c-3=5$ [3]

Constant: $-4+2a=5d$ [4]

Equation [1] $\Rightarrow a=-3$, equation [2] $\Rightarrow b=-1$, equation [3] $\Rightarrow c=4$.

Substitute $a=-3$ in equation [4].

$\therefore -4-6=5d$

$\therefore d=-2$

Answer: $a=-3,\ b=-1,\ c=4,\ d=-2$

b **i** Let $4x^4+12x^3+13x^2+6x+1=\left(ax^2+bx+c\right)^2$.

$$\begin{aligned}&\therefore 4x^4+12x^3+13x^2+6x+1\\&=\left(ax^2+(bx+c)\right)^2\\&=(ax^2)^2+2ax^2(bx+c)+(bx+c)^2\\&=a^2x^4+2abx^3+2acx^2+b^2x^2+2bcx+c^2\\&=a^2x^4+2abx^3+(2ac+b^2)x^2+2bcx+c^2\end{aligned}$$

Equate coefficients of like terms:

$x^4: 4=a^2$ [1]

$x^3: 12=2ab$ [2]

$x^2: 13=2ac+b^2$ [3]

$x: 6=2bc$ [4]

Constant: $1=c^2$ [5]

Since $a>0$, equation [1] $\Rightarrow a=2$

Substitute $a=2$ in equation [2].

$\therefore 12=2\times 2\times b$

$\therefore b=3$

Substitute $b=3$ in equation [4].

$\therefore 6=2\times 3\times c$

$\therefore c=1$

Check $a=2,\ b=3,\ c=1$ satisfies the remaining two equations.

Equation [3]: $2ac+b^2=2(2)(1)+(3)^2=13$ as required.

Equation [5]: $c^2=(1)^2=1$ as required.

$\therefore 4x^4+12x^3+13x^2+6x+1=\left(2x^2+3x+1\right)^2$

ii Hence, $2x^2+3x+1$ is a square root of $4x^4+12x^3+13x^2+6x+1$ as

$$\begin{aligned}\sqrt{4x^4+12x^3+13x^2+6x+1}&=\sqrt{\left(2x^2+3x+1\right)^2}\\&=2x^2+3x+1\end{aligned}$$

26 **a** $p(x)=x^4+kx^2+n^2,\ q(x)=x^2+mx+n$

$$\begin{aligned}&p(x)q(x)\\&=\left(x^4+kx^2+n^2\right)\left(x^2+mx+n\right)\\&=x^6+mx^5+nx^4+kx^4+kmx^3+knx^2+n^2x^2+n^2mx+n^3\\&=x^6+mx^5+(n+k)x^4+kmx^3+(kn+n^2)x^2+n^2mx+n^3\end{aligned}$$

Since

$p(x)q(x)=x^6-5x^5-7x^4+65x^3-42x^2-180x+216$ (given), equating coefficients of like terms gives:

$x^5: m=-5$ [1]

$x^4: n+k=-7$ [2]

$x^3: km=65$ [3]

$x^2: kn+n^2=-42$ [4]

$x: n^2m=-180$ [5]

Constant: $n^3=216$ [6]

Substitute $m=-5$ in equation [3].

$\therefore -5k=65$

$\therefore k=-13$

Substitute $k=-13$ in equation [2].

$\therefore n-13=-7$

$\therefore n=6$

Check values $m=-5,\ k=-13,\ n=6$ in the remaining equations.

$$\begin{aligned}\text{Equation [4]: } kn+n^2&=-13\times 6+6^2\\&=-78+36\\&=-42\end{aligned}$$

Equation [5]: $n^2m=36\times -5=-180$ as required.

Equation [6]: $n^3=6^3=216$ as required.

Answer: $m=-5, k=-13, n=6$

b $p(x)q(x)=\left(x^4-13x^2+36\right)\left(x^2-5x+6\right)$

$$\begin{aligned}p(x)&=x^4-13x^2+36\\&=(x^2-4)(x^2-9)\\&=(x-2)(x+2)(x-3)(x+3)\\q(x)&=x^2-5x+6\\&=(x-2)(x-3)\end{aligned}$$

$\therefore p(x)q(x)=(x-2)^2(x-3)^2(x+2)(x+3)$

27 **a** Using CAS technology,

$$\begin{aligned}&\frac{4x^3-7x^2+5x+2}{2x+3}\\&=2x^2-\frac{13}{2}x-\frac{139}{4(2x+3)}+\frac{49}{4}.\end{aligned}$$

b The remainder is $-\frac{139}{4}$, quotient is $2x^2-\frac{13}{2}x+\frac{49}{4}$

c The dividend is $4x^3-7x^2+5x+2$.

$$\begin{aligned}\text{If } x=-\frac{3}{2},\ &4\left(-\frac{3}{2}\right)^3-7\left(-\frac{3}{2}\right)^2+5\left(-\frac{3}{2}\right)+2\\&=-\frac{139}{4}\end{aligned}$$

d The divisor is $2x+3$.

If $x=-\frac{3}{2},\ 2\left(-\frac{3}{2}\right)+3=0$

28 **a** Define using CAS technology.

b Use CAS technology to obtain $\frac{349}{9}$.

c Use CAS technology to obtain $24n^3+24n^2-24(n^3-an+6)-16n-10$ or $24n^2+24an-16n-154$.

d Use CAS technology to obtain $a=9$.

4.2 Exam questions

1

$$\begin{array}{r}
x^2-x-5\\
x-1\overline{)\,x^3-2x^2-4x+2}\\
\underline{x^3-x^2}\\
-x^2-4x\\
\underline{-x^2+x}\\
-5x+2\\
\underline{-5x+5}\\
-3
\end{array}$$

$\therefore x^2-x-5-\dfrac{3}{x-1}$

$\therefore$ quotient $= x^2-x-5$, remainder $= -3$ [1 mark]

2 A polynomial is an expression that contains positive whole number powers of x.

$\sqrt{x} = x^{\frac{1}{2}}$ – the power is not a positive whole number.

$\therefore -3x^3 = x^2 - 2\sqrt{x}$ is not a polynomical.

The correct answer is **C**.

3 $5x + 6 - 3x^2 - x^3 - 5$

Rearrange the polynomial in ascending powers and simplify:

$-x^3 - 3x^2 + 5x + 1$

The highest power is 3.

$\therefore$ degree is 3.

The leading term is $-x^3$.

$\therefore$ coefficient is -1.

The constant term $= 6 - 5 = 1$.

$\therefore 3, -1, 1$

The correct answer is **D**.

4.3 The remainder and factor theorems

4.3 Exercise

1 **a** Let $p(x) = 3x^2 + 8x - 5$.

$p(x)$ is divided by $(x-1)$, so the remainder is $p(1)$.

$$\begin{aligned}p(1) &= 3(1)^2 + 8(1) - 5\\ &= 3 + 8 - 5\\ &= 6\end{aligned}$$

The remainder is 6.

b Let $p(x) = -x^3 + 7x^2 + 2x - 12$.

$p(x)$ is divided by $(x+1)$, so the remainder is $p(-1)$.

$$\begin{aligned}p(-1) &= -(-1)^3 + 7(-1)^2 + 2(-1) - 12\\ &= 1 + 7 - 2 - 12\\ &= -6\end{aligned}$$

The remainder is -6.

2 $p(x) = x^3 + 4x^2 - 3x + 5$

a Remainder $= p(-2)$

$$\begin{aligned}p(-2) &= (-2)^3 + 4(-2)^2 - 3(-2) + 5\\ &= -8 + 16 + 6 + 5\\ &= 19\end{aligned}$$

The remainder is 19.

b Remainder $= p\left(\dfrac{1}{2}\right)$

$$\begin{aligned}p\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 5\\ &= \frac{1}{8} + 1 - \frac{3}{2} + 5\\ &= 6 - \frac{11}{8}\\ &= \frac{37}{8}\end{aligned}$$

The remainder is $\dfrac{37}{8}$.

3 $p(x)$ is divided by $(2x+9)$.

Let $2x + 9 = 0$.

$2x = -9$

$x = -\dfrac{9}{2}$

The remainder is $p\left(-\dfrac{9}{2}\right)$.

The correct answer is **C**.

4 **a** Let $p(x) = x^3 - 4x^2 - 5x + 3$.

Remainder $= p(1)$ when $p(x)$ is divided by $(x-1)$

$p(1) = 1 - 4 - 5 + 3 = -5$

The remainder is -5.

b Let $p(x) = 6x^3 + 7x^2 + x + 2$

Remainder $= p(-1)$ when $p(x)$ is divided by $(x+1)$

$p(-1) = -6 + 7 - 1 + 2 = 2$

The remainder is 2.

c Let $p(x) = -2x^3 + 2x^2 - x - 1$

Remainder $= p(4)$ when $p(x)$ is divided by $(x-4)$

$$\begin{aligned}p(4) &= -2(4)^3 + 2(4)^2 - (4) - 1\\ &= -128 + 32 - 4 - 1\\ &= -101\end{aligned}$$

The remainder is -101.

d Let $p(x) = x^3 + x^2 + x - 10$.

Remainder $= p\left(-\dfrac{1}{2}\right)$ when $p(x)$ is divided by $(2x+1)$

$$\begin{aligned}p\left(-\frac{1}{2}\right) &= \left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 10\\ &= -\frac{1}{8} + \frac{1}{4} - \frac{1}{2} - 10\\ &= -10\frac{3}{8}\end{aligned}$$

The remainder is $-10\dfrac{3}{8}$.

e Let $p(x) = 27x^3 - 9x^2 - 9x + 2$.

Remainder $= p\left(\dfrac{2}{3}\right)$ when $p(x)$ is divided by $(3x-2)$

$$\begin{aligned}p\left(\frac{2}{3}\right) &= 27\left(\frac{2}{3}\right)^3 - 9\left(\frac{2}{3}\right)^2 - 9\left(\frac{2}{3}\right) + 2\\ &= 27 \times \frac{8}{27} - 9 \times \frac{4}{9} - 9 \times \frac{2}{3} + 2\\ &= 8 - 4 - 6 + 2\\ &= 0\end{aligned}$$

The remainder is 0.

f Let $p(x) = 4x^4 - 5x^3 + 2x^2 - 7x + 8$.

Remainder $= p(2)$ when $p(x)$ is divided by $(x - 2)$

$p(2) = 4(2)^4 - 5(2)^3 + 2(2)^2 - 7(2) + 8$

$= 64 - 40 + 8 - 14 + 8$

$= 26$

The remainder is 26.

5 a Let $p(x) = ax^2 - 4x - 9$.

$p(x)$ is divided by $(x - 3)$, so the remainder is $p(3)$.

$p(3) = a(3)^2 - 4(3) - 9$

$= 9a - 12 - 9$

$= 9a - 21$

Since the remainder is 15,

$9a - 21 = 15$

$9a = 36$

$a = 4$

b Let $p(x) = x^3 + x^2 + kx + 5$.

$p(x)$ is divided by $(x + 2)$, so the remainder is $p(-2)$.

$p(-2) = (-2)^3 + (-2)^2 + k(-2) + 5$

$= -8 + 4 - 2k + 5$

$= 1 - 2k$

Since the remainder is -5,

$1 - 2k = -5$

$-2k = -6$

$k = 3$

c Let $p(x) = x^3 - kx^2 + 4x + 8$.

Remainder = 29 when $p(x)$ is divided by $(x - 3)$

$\Rightarrow p(3) = 29$

$\therefore (3)^3 - k(3)^2 + 4(3) + 8 = 29$

$\therefore 47 - 9k = 29$

$\therefore 18 = 9k$

$\therefore k = 2$

6 a $q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$

$q(2) = 4(2)^4 + 4(2)^3 - 25(2)^2 - (2) + 6$

$= 64 + 32 - 100 - 2 + 6$

$= 0$

Therefore, $(x - 2)$ is a factor.

b Let $p(x) = 3x^3 + ax^2 + bx - 2$.

Remainder of -22 when divided by $(x + 1) \Rightarrow p(-1) = -22$

$\therefore 3(-1)^3 + a(-1)^2 + b(-1) - 2 = -22$

$\therefore a - b - 5 = -22$

$\therefore a - b = -17$

Exactly divisible by $(x - 1) \Rightarrow p(1) = 0$

$\therefore 3 + a + b - 2 = 0$

$\therefore a + b = -1$

Solve the simultaneous equations

$a - b = -17$

$a + b = -1$

by adding and by subtracting

$2a = -18$ $\qquad -2b = -16$

$\therefore a = -9$ $\qquad \therefore b = 8$

Therefore, $p(x) = 3x^3 - 9x^2 + 8x - 2$.

7 a $p(x) = x^3 - 2x^2 + ax + 7$

Remainder $= p(-2)$ when $p(x)$ is divided by $(x + 2)$

$\therefore p(-2) = 11$

$\therefore (-2)^3 - 2(-2)^2 + a(-2) + 7 = 11$

$\therefore -8 - 8 - 2a + 7 = 11$

$\therefore -2a - 9 = 11$

$\therefore -2a = 20$

$\therefore a = -10$

b $p(x) = 4 - x^2 + 5x^3 - bx^4$

Remainder $= p(1)$ when $p(x)$ is divided by $(x - 1)$

Remainder = 0 since $p(x)$ is exactly divisible by $(x - 1)$

$\therefore p(1) = 0$

$\therefore 4 - 1 + 5 - b = 0$

$\therefore 8 - b = 0$

$\therefore b = 8$

c Let $p(x) = 2x^3 + cx^2 + 5x + 8$.

Remainder $= p\left(\frac{1}{2}\right)$ when $p(x)$ is divided by $(2x - 1)$

$\therefore p\left(\frac{1}{2}\right) = 6$

$\therefore 2\left(\frac{1}{2}\right)^3 + c\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) + 8 = 6$

$\therefore \frac{1}{4} + \frac{c}{4} + \frac{5}{2} + 8 = 6$

$\therefore 1 + c + 10 + 32 = 24$

$\therefore c = -19$

d Let $p(x) = x^3 + 3x^2 - 4x + d$ and $q(x) = x^4 - 9x^2 - 7$.

Same remainder when divided by $(x + 3)$

$\therefore p(-3) = q(-3)$

$\therefore (-3)^3 + 3(-3)^2 - 4(-3) + d = (-3)^4 - 9(-3)^2 - 7$

$\therefore -27 + 27 + 12 + d = 81 - 81 - 7$

$\therefore 12 + d = -7$

$\therefore d = -19$

8 Let $p(x) = 12x^2 - 4x + a$.

Since $(2x + a)$ is a factor, $p\left(-\frac{a}{2}\right) = 0$.

$\therefore 12\left(-\frac{a}{2}\right)^2 - 4\left(-\frac{a}{2}\right) + a = 0$

$\therefore 3a^2 + 2a + a = 0$

$\therefore 3a^2 + 3a = 0$

$\therefore 3a(a + 1) = 0$

$\therefore a = 0, a = -1$

9 a $q(x) = ax^3 + 4x^2 + bx + 1$

Remainder: $q(2) = 39$

$\therefore a(2)^3 + 4(2)^2 + b(2) + 1 = 39$

$\therefore 8a + 2b + 17 = 39$

$\therefore 8a + 2b = 22$

$\therefore 4a + b = 11$ [1]

$q(-1) = 0$ since $(x + 1)$ is a factor.

$\therefore a(-1)^3 + 4(-1)^2 + b(-1) + 1 = 0$

$\therefore -a - b + 5 = 0$

$\therefore a + b = 5$ [2]

Solving,

$4a + b = 11$ [1]

$a + b = 5$ [2]

Subtract equation [2] from equation [1]:

$\therefore 3a = 6$

$\therefore a = 2$

Equation [2] $\Rightarrow b = 3$

Answer: $a = 2, b = 3$

b $p(x) = \frac{1}{3}x^3 + mx^2 + nx + 2$

Remainders: $p(3) = p(-3)$

$$\therefore \frac{1}{3}(3)^3 + m(3)^2 + n(3) + 2 = \frac{1}{3}(-3)^3 + m(-3)^2 + n(-3) + 2$$

$$\therefore 9 + 9m + 3n + 2 = -9 + 9m - 3n + 2$$

$$\therefore 9 + 3n = -9 - 3n$$

$$\therefore 6n = -18$$

$$\therefore n = -3$$

$\therefore p(x) = \frac{1}{3}x^3 + mx^2 - 3x + 2$

$p(3) = 3p(1)$

$$\therefore 9 + 9m - 9 + 2 = 3\left(\frac{1}{3} + m - 3 + 2\right)$$

$$\therefore 9m + 2 = 1 + 3m - 9 + 6$$

$$\therefore 6m = -4$$

$$\therefore m = -\frac{2}{3}$$

Answer: $m = -\frac{2}{3}, n = -3$

10 a Let $p(x) = 3x^3 + 11x^2 - 6x - 8$.

If $x + 4$ is a factor, $p(-4) = 0$.

$$\begin{aligned} p(x) &= 3x^3 + 11x^2 - 6x - 8 \\ p(-4) &= 3(-4)^3 + 11(-4)^2 - 6(-4) - 8 \\ &= 3(-64) + 11(16) + 24 - 8 \\ &= -192 + 176 + 24 - 8 \\ &= 0 \end{aligned}$$

Therefore, $x + 4$ is a factor of $3x^3 + 11x^2 - 6x - 8$.

b Let $p(x) = -x^3 + 6x^2 + x - 30$

If $x - 5$ is a factor, $p(5) = 0$.

$$\begin{aligned} p(x) &= -x^3 + 6x^2 + x - 30 \\ p(5) &= -(5)^3 + 6(5)^2 + 5 - 30 \\ &= -125 + 6(25) + 5 - 30 \\ &= -125 + 150 + 5 - 30 \\ &= 0 \end{aligned}$$

Therefore, $x - 5$ is a factor of $-x^3 + 6x^2 + x - 30$.

c Let $p(x) = 6x^3 + 7x^2 - 9x + 2$.

If $2x - 1$ is a factor, $p\left(\frac{1}{2}\right) = 0$.

$$\begin{aligned} p(x) &= 6x^3 + 7x^2 - 9x + 2 \\ p\left(\frac{1}{2}\right) &= 6\left(\frac{1}{2}\right)^3 + 7\left(\frac{1}{2}\right)^2 - 9\left(\frac{1}{2}\right) + 2 \\ &= 6\left(\frac{1}{8}\right) + 7\left(\frac{1}{4}\right) - \frac{9}{2} + 2 \\ &= \frac{6}{8} + \frac{7}{4} - \frac{9}{2} + 2 \\ &= \frac{3}{4} + \frac{7}{4} - \frac{18}{4} + \frac{8}{4} \\ &= 0 \end{aligned}$$

Therefore, $2x - 1$ is a factor of $6x^3 + 7x^2 - 9x + 2$.

d Let $p(x) = 2x^3 + 13x^2 + 5x - 6$.

If $x - 1$ is a factor, $p(1) = 0$, so if $p(1) \neq 0$, then $x - 1$ is not a factor.

$$\begin{aligned} p(1) &= 2(1)^3 + 13(1)^2 + 5(1) - 6 \\ &= 2 + 13 + 5 - 6 \\ &= 14 \\ &\neq 0 \end{aligned}$$

Therefore, $x - 1$ is not a factor of $2x^3 + 13x^2 + 5x - 6$.

11 a Let $p(x) = x^3 - 13x + a$.

As $x + 3$ is a factor, $p(-3) = 0$

$0 = (-3)^3 - 13(-3) + a$

$0 = -27 + 39 + a$

$0 = 12 + a$

$a = -12$

b Let $p(x) = 4x^3 + kx^2 - 9x + 10$.

As $2x - 5$ is a factor, $p\left(\frac{5}{2}\right) = 0$,

$$4\left(\frac{5}{2}\right)^3 + k\left(\frac{5}{2}\right)^2 - 9\left(\frac{5}{2}\right) + 10 = 0$$

$$4\left(\frac{125}{8}\right) + k\left(\frac{25}{4}\right) - \frac{45}{2} + 10 = 0$$

$$\frac{125}{2} + \frac{25k}{4} - \frac{45}{2} + 10 = 0$$

$$\frac{80}{2} + \frac{25k}{4} + 10 = 0$$

$$50 + \frac{25k}{4} = 0$$

$$\frac{25k}{4} = -50$$

$$k = -50 \times \frac{4}{25}$$

$$k = -8$$

12 a
$$\begin{aligned} p(x) &= x^3 - x^2 - 10x - 8 \\ \therefore p(x) &= (x - 4)(x^2 + 3x + 2) \\ \therefore p(x) &= (x - 4)(x + 1)(x + 2) \end{aligned}$$

b
$$\begin{aligned} p(x) &= 3x^3 + 40x^2 + 49x + 12 \\ &= (x + 12)(3x^2 + 4x + 1) \\ \therefore p(x) &= (x + 12)(3x + 1)(x + 1) \end{aligned}$$

c
$$\begin{aligned} p(x) &= 20x^3 + 44x^2 + 23x + 3 \\ &= (5x + 1)(4x^2 + 8x + 3) \\ \therefore p(x) &= (5x + 1)(2x + 3)(2x + 1) \end{aligned}$$

d
$$\begin{aligned} p(x) &= -16x^3 + 12x^2 + 100x - 75 \\ &= (4x - 3)(-4x^2 + 0x + 25) \\ &= (4x - 3)(25 - 4x^2) \\ \therefore p(x) &= (4x - 3)(5 - 2x)(5 + 2x) \end{aligned}$$

e
$$\begin{aligned} p(x) &= 9x^3 - 75x^2 + 175x - 125 \\ \therefore p(x) &= (3x - 5)(3x^2 - 20x + 25) \\ &= (3x - 5)(3x - 5)(x - 5) \\ \therefore p(x) &= (3x - 5)^2(x - 5) \end{aligned}$$

f
$$\begin{aligned} p(x) &= -8x^3 + 59x^2 - 138x + 99 \\ \therefore p(x) &= (8x - 11)(x - 3)(ax + b) \\ &= (8x^2 - 35x + 33)(ax + b) \\ &= (8x^2 - 35x + 33)(-x + 3) \\ &= (8x - 11)(x - 3)\,(-(x - 3)) \\ \therefore p(x) &= -(x - 3)^2(8x - 11) \end{aligned}$$

13 a $p(x) = x^3 + 3x^2 - 13x - 15$

$p(-1) = -1 + 3 + 13 - 15 = 0 \Rightarrow (x + 1)$ is a factor.

$$\begin{aligned} x^3 + 3x^2 - 13x - 15 &= (x + 1)(x^2 + bx - 15) \\ &= (x + 1)(x^2 + 2x - 15) \\ &= (x + 1)(x + 5)(x - 3) \end{aligned}$$

b $p(x) = 12x^3 + 41x^2 + 43x + 14$

Since $(x + 1)$ and $(3x + 2)$ are factors, then $(x + 1)\,(3x + 2)$ is a quadratic factor.

$$\begin{aligned} p(x) &= 12x^3 + 41x^2 + 43x + 14 \\ &= (x + 1)(3x + 2)(ax + b) \\ &= (3x^2 + 5x + 2)(ax + b) \\ &= (3x^2 + 5x + 2)(4x + 7) \end{aligned}$$

$\therefore p(x) = (x + 1)(3x + 2)(4x + 7)$

14 a Let $p(x) = x^3 + 5x^2 + 2x - 8$.

$p(1) = 1 + 5 + 2 - 8 = 0$

$\therefore (x - 1)$ is a factor.

$$\begin{aligned} \therefore x^3 + 5x^2 + 2x - 8 &= (x - 1)(x^2 + 6x + 8) \\ &= (x - 1)(x + 2)(x + 4) \end{aligned}$$

b Let $p(x) = x^3 + 10x^2 + 31x + 30$.

$$\begin{aligned} p(-2) &= (-2)^3 + 10(-2)^2 + 31(-2) + 30 \\ &= -8 + 40 - 62 + 30 \\ &= 0 \end{aligned}$$

$\therefore (x + 2)$ is a factor.

$$\begin{aligned} \therefore x^3 + 10x^2 + 31x + 30 &= (x + 2)(x^2 + 8x + 15) \\ &= (x + 2)(x + 3)(x + 5) \end{aligned}$$

c Let $p(x) = 2x^3 - 13x^2 + 13x + 10$.

$$\begin{aligned} p(2) &= 2(2)^3 - 13(2)^2 + 13(2) + 10 \\ &= 16 - 52 + 26 + 10 \\ &= 0 \end{aligned}$$

$\therefore (x - 2)$ is a factor.

$$\begin{aligned} \therefore 2x^3 - 13x^2 + 13x + 10 &= (x - 2)(2x^2 - 9x - 5) \\ &= (x - 2)(2x + 1)(x - 5) \end{aligned}$$

d Let $p(x) = -18x^3 + 9x^2 + 23x - 4$.

$$\begin{aligned} p(-1) &= -18(-1)^3 + 9(-1)^2 + 23(-1) - 4 \\ &= 18 + 9 - 23 - 4 \\ &= 0 \end{aligned}$$

$\therefore (x + 1)$ is a factor.

$$\begin{aligned} \therefore\ -18x^3 + 9x^2 + 23x - 4 &= (x + 1)(-18x^2 + 27x - 4) \\ &= (x + 1)(3x - 4)(-6x + 1) \\ &= (x + 1)(3x - 4)(1 - 6x) \end{aligned}$$

e Let $p(x) = x^3 - 7x + 6$.

$p(1) = 1 - 7 + 6 = 0$

$\therefore (x - 1)$ is a factor.

$\therefore x^3 + 0x^2 - 7x + 6 = (x - 1)(x^2 + x - 6)$

$\therefore x^3 - 7x + 6 = (x - 1)(x + 3)(x - 2)$

f $x^3 + x^2 - 49x - 49$

$$\begin{aligned} &= x^2(x + 1) - 49(x + 1) \\ &= (x + 1)(x^2 - 49) \\ &= (x + 1)(x - 7)(x + 7) \end{aligned}$$

15 $6x^3 + 13x^2 = 2 - x$

$\therefore 6x^3 + 13x^2 + x - 2 = 0$

Let $p(x) = 6x^3 + 13x^2 + x - 2$.

$p(-1) = -6 + 13 - 1 - 2 \neq 0$

$p(-2) = -48 + 52 - 2 - 2 = 0$

$\therefore (x + 2)$ is a factor.

$$\begin{aligned} &6x^3 + 13x^2 + x - 2 \\ &= (x + 2)(6x^2 + bx - 1) \\ &= (x + 2)(6x^2 + x - 1) \\ &= (x + 2)(3x - 1)(2x + 1) \end{aligned}$$

For $6x^3 + 13x^2 + x - 2 = 0$,

$(x + 2)(3x - 1)(2x + 1) = 0$

$\therefore x = -2,\ x = \dfrac{1}{3},\ x = -\dfrac{1}{2}$

16 a $(x + 4)(x - 3)(x + 5) = 0$

$\therefore x = -4, x = 3, x = -5$

b $2(x - 7)(3x + 5)(x - 9) = 0$

$\therefore (x - 7)(3x + 5)(x - 9) = 0$

$\therefore x = 7, x = -\dfrac{5}{3}, x = 9$

c $x^3 - 13x^2 + 34x + 48 = 0$

Let $p(x) = x^3 - 13x^2 + 34x + 48$.

$p(-1) = -1 - 13 - 34 + 48 = 0$

$\therefore (x + 1)$ is a factor.

$\therefore x^3 - 13x^2 + 34x + 48 = 0$

$\Rightarrow (x + 1)(x^2 - 14x + 48) = 0$

$\therefore (x + 1)(x - 6)(x - 8) = 0$

$\therefore x = -1,\ x = 6,\ x = 8$

d $2x^3 + 7x^2 = 9$

$\therefore 2x^3 + 7x^2 - 9 = 0$

Let $p(x) = 2x^3 + 7x^2 - 9$.

$p(1) = 2 + 7 - 9 = 0$

$\therefore (x - 1)$ is a factor.

$\therefore 2x^3 + 7x^2 - 9 = 0$

$\Rightarrow (x - 1)(2x^2 + 9x + 9) = 0$

$\therefore (x - 1)(2x + 3)(x + 3) = 0$

$\therefore x = 1,\ x = -\dfrac{3}{2},\ x = -3$

e $3x^2(3x + 1) = 4(2x + 1)$

$\therefore 9x^3 + 3x^2 = 8x + 4$

$\therefore 9x^3 + 3x^2 - 8x - 4 = 0$

Let $p(x) = 9x^3 + 3x^2 - 8x - 4$.

$p(1) = 9 + 3 - 8 - 4 = 0$

$\therefore (x - 1)$ is a factor.

$\therefore 9x^3 + 3x^2 - 8x - 4 = 0$

$\Rightarrow (x - 1)(9x^2 + 12x + 4) = 0$

$\therefore (x - 1)(3x + 2)^2 = 0$

$\therefore x = 1,\ x = -\dfrac{2}{3}$

f $2x^4 + 3x^3 - 8x^2 - 12x = 0$

$\therefore x(2x^3 + 3x^2 - 8x - 12) = 0$

$\therefore x\left[x^2(2x + 3) - 4(x + 3)\right] = 0$

$\therefore x(2x + 3)(x^2 - 4) = 0$

$\therefore x(2x + 3)(x - 2)(x + 2) = 0$

$\therefore x = 0, x = -\dfrac{3}{2}, x = 2, x = -2$

17 a Use the Null Factor Law.

$(2x - 1)\,(3x + 4)\,(x + 1) = 0$

$2x - 1 = 0, \quad 3x + 4 = 0, \quad x + 1 = 0$

$x = \dfrac{1}{2} \qquad x = -\dfrac{4}{3} \qquad x = -1$

b Group '2 and 2'.

$$\begin{aligned} 2x^3 - x^2 - 6x + 3 &= 0 \\ x^2(2x-1) - 3(2x-1) &= 0 \\ (2x-1)(x^2-3) &= 0 \\ (2x-1)\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right) &= 0 \\ x &= \frac{1}{2}, \pm\sqrt{3} \end{aligned}$$

c Isolate the perfect cube and take the cube root of both sides.

$$\begin{aligned} 8 - (x-5)^3 &= 0 \\ (x-5)^3 &= 8 \\ x - 5 &= \sqrt[3]{8} \\ x - 5 &= 2 \\ x &= 7 \end{aligned}$$

d Use the factor theorem, since no other technique works.

$p(x) = -x^3 + 2x^2 + 13x + 10$

$p(1) \neq 0$

$$\begin{aligned} p(-1) &= -(-1)^3 + 2(-1)^2 + 13(-1) + 10 \\ &= 1 + 2 - 13 + 10 \\ &= 0 \end{aligned}$$

Therefore, $x + 1$ is a factor.

$$\begin{aligned} -x^3 + 2x^2 + 13x + 10 &= (x+1)(ax^2 + bx + c) \\ &= (x+1)(-x^2 + bx + 10) \end{aligned}$$

Calculate b by equating coefficients of x^2.

$2 = -1 + b$

$b = 3$

$$\begin{aligned} -x^3 + 2x^2 + 13x + 10 &= (x+1)(-x^2 + 3x + 10) \\ &= -(x+1)(x^2 - 3x - 10) \\ &= -(x+1)(x-5)(x+2) \end{aligned}$$

The equation is:

$$\begin{aligned} -(x+1)(x-5)(x+2) &= 0 \\ x &= -1, x = 5, x = -2 \end{aligned}$$

e Group '2 and 2'.

$$\begin{aligned} x^3 + 3x^2 + 2x + 6 &= 0 \\ x^2(x+3) + 2(x+3) &= 0 \\ (x+3)(x^2+2) &= 0 \end{aligned}$$

$$\begin{aligned} x + 3 &= 0 \text{ or } x^2 + 2 = 0 \\ x &= -3 \text{ or } x^2 = -2 \end{aligned}$$

No real solutions to $x^2 + 2 = 0$

$x = -3$

f Use the factor theorem.

$p(x) = 6x^3 - 11x^2 - 3x + 2$

$p(1) \neq 0$

$p(-1) \neq 0$

$$\begin{aligned} p(2) &= 6(2)^3 - 11(2)^2 - 3(2) + 2 \\ &= 48 - 44 - 6 + 2 \\ &= 0 \end{aligned}$$

Therefore, $x - 2$ is a factor.

$$\begin{aligned} 6x^3 - 11x^2 - 3x + 2 &= (x-2)(6x^2 + bx - 1) \\ &= (x-2)(6x^2 + x - 1) \\ &= (x-2)(2x+1)(3x-1) \end{aligned}$$

Solve the equation.

$$\begin{aligned} 6x^3 - 11x^2 - 3x + 2 &= 0 \\ (x-2)(2x+1)(3x-1) &= 0 \\ x &= 2, -\frac{1}{2}, \frac{1}{3} \end{aligned}$$

18 $p(x) = 12x^3 + 8x^2 - 3x - 2$

The zeros are of the form $\frac{p}{q}$, where p is a factor of 2 and q is a factor of 12.

Try $p = 1$ and $q = 2$.

$$\begin{aligned} p\left(\frac{1}{2}\right) &= 12\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) - 2 \\ &= \frac{12}{8} + 2 - \frac{3}{2} - 2 \\ &= 0 \end{aligned}$$

$\therefore$ $(2x - 1)$ is a factor.

Hence,

$$\begin{aligned} 12x^3 + 8x^2 - 3x - 2 &= (2x-1)(6x^2 + bx + 2) \\ &= (2x-1)(6x^2 + 7x + 2) \\ &= (2x-1)(2x+1)(3x+2) \end{aligned}$$

The three linear factors are $(2x - 1)$, $(2x + 1)$ and $(3x + 2)$.

19 **a** Zero $x = 5 \Rightarrow (x - 5)$ is a factor of the polynomial.

Zero $x = 9 \Rightarrow (x - 9)$ is a factor of the polynomial.

Zero $x = -2 \Rightarrow (x + 2)$ is a factor of the polynomial.

i Therefore, the degree 3 monic polynomial is $(x - 5)(x - 9)(x + 2)$ in factorised form.

ii Expanding, $(x - 5)(x - 9)(x + 2)$

$$\begin{aligned} &= (x-5)(x^2 - 7x - 18) \\ &= x^3 - 7x^2 - 18x - 5x^2 + 35x + 90 \\ &= x^3 - 12x^2 + 17x + 90 \end{aligned}$$

b Zero $x = -4 \Rightarrow (x + 4)$ is a factor of the polynomial.

Zero $x = -1 \Rightarrow (x + 1)$ is a factor of the polynomial.

Zero $x = \frac{1}{2} \Rightarrow \left(x - \frac{1}{2}\right)$ is a factor of the polynomial.

i Therefore, in factorised form, the degree 3 polynomial with leading coefficient -2 is:

$$-2(x+4)(x+1)\left(x - \frac{1}{2}\right) = (x+4)(x+1)(1-2x)$$

ii Expanding,

$$\begin{aligned} &(x+4)(x+1)(1-2x) \\ &= (x^2 + 5x + 4)(1 - 2x) \\ &= x^2 - 2x^3 + 5x - 10x^2 + 4 - 8x \\ &= -2x^3 - 9x^2 - 3x + 4 \end{aligned}$$

20 **a** Let $p(x) = 24x^3 + 34x^2 + x - 5$.

As the zeros are not integers, they must be of the form $\frac{p}{q}$, where p is a factor of 5 and q is a factor 24.

$$\begin{aligned} p\left(\frac{1}{2}\right) &= 24\left(\frac{1}{2}\right)^3 + 34\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) - 5 \\ &= 3 + \frac{17}{2} + \frac{1}{2} - 5 \\ &\neq 0 \end{aligned}$$

$$\begin{aligned} p\left(-\frac{1}{2}\right) &= -3 + \frac{17}{2} - \frac{1}{2} - 5 \\ &= 0 \end{aligned}$$

$\therefore$ $(2x + 1)$ is a factor.

Hence,

$$\begin{aligned} 24x^3 + 34x^2 + x - 5 &= (2x+1)(12x^2 + bx - 5) \\ &= (2x+1)(12x^2 + 11x - 5) \\ &= (2x+1)(3x-1)(4x+5) \end{aligned}$$

b $p(x) = 8x^3 + mx^2 + 13x + 5$

i A zero of $\frac{5}{2}$ means that $(2x - 5)$ is a factor.

ii When $p\left(\dfrac{5}{2}\right) = 0$,

$$8\left(\frac{5}{2}\right)^3 + m\left(\frac{5}{2}\right)^2 + 13\left(\frac{5}{2}\right) + 5 = 0$$

$$125 + \frac{25m}{4} + \frac{65}{2} + 5 = 0$$

$$m = -26$$

iii $\therefore 8x^3 - mx^2 + 13x + 5 = (2x - 5)(4x^2 + bx - 1)$

Equate coefficients of x:

$13 = -2 - 5b$

$\therefore 5b = -15$

$\therefore b = -3$

Hence,

$$\begin{aligned} 8x^3 + mx^2 + 13x + 5 &= (2x - 5)(4x^2 - 3x - 1) \\ &= (2x - 5)(4x + 1)(x - 1) \end{aligned}$$

21 a i $p(x) = x^3 - 12x^2 + 48x - 64$

$$\begin{aligned} p(4) &= 64 - 12(16) + 48(4) - 64 \\ &= 64 - 192 + 193 - 64 \\ &= 0 \end{aligned}$$

$\therefore (x - 4)$ is a factor.

Hence,

$$\begin{aligned} x^3 - 12x^2 + 48x - 64 &= (x - 4)(x^2 + bx + 16) \\ &= (x - 4)(x^2 - 8x + 16) \\ &= (x - 4)(x - 4)^2 \\ &= (x - 4)^3 \end{aligned}$$

$$\therefore P(x) = (x - 4)^3$$

$$\begin{aligned} q(x) &= x^3 - 64 \\ q(x) &= x^3 - 4^3 \\ &= (x - 4)(x^2 + 4x + 16) \end{aligned}$$

$\therefore q(x) = (x - 4)(x^2 + 4x + 16)$

ii $\dfrac{p(x)}{q(x)}$

$$\begin{aligned} &= \frac{(x-4)^3}{(x-4)(x^2+4x+16)} \\ &= \frac{(x-4)^2}{x^2+4x+16} \\ &= \frac{x^2-8x+16}{x^2+4x+16} \\ &= \frac{\left(x^2+4x+16\right) - 12x}{x^2+4x+16} \\ &= \frac{x^2+4x+16}{x^2+4x+16} - \frac{12x}{x^2+4x+16} \\ &= 1 - \frac{12x}{x^2+4x+16} \end{aligned}$$

b $p(x) = ax^3 + bx^2 + cx + d$

$p(0) = 9 \Rightarrow d = 9$

Given factors $\Rightarrow \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) = x^2 - 3$ is a quadratic factor.

$\therefore x^3 + bx^2 + cx + 9 = (x^2 - 3)(x - 3)$

The third factor is $(x - 3)$.

Expanding,

$$x^3 + bx^2 + cx + 9 = x^3 - 3x^2 - 3x + 9$$

$$\therefore b = -3 = c$$

22 a $p(x) = x^3 + 6x^2 - 7x - 18$

$$\begin{aligned} p(2) &= (2)^3 + 6(2)^2 - 7(2) - 18 \\ &= 8 + 24 - 14 - 18 \\ &= 0 \end{aligned}$$

Since $p(2) = 0$, $(x - 2)$ is a factor of $p(x)$.

$$\begin{aligned} &\therefore x^3 + 6x^2 - 7x - 18 \\ &= (x - 2)(x^2 + 8x + 9) \\ &= (x - 2)\left[(x^2 + 8x + 16) - 16 + 9\right] \\ &= (x - 2)\left[(x + 4)^2 - 7\right] \\ &= (x - 2)\left(x + 4 + \sqrt{7}\right)\left(x + 4 - \sqrt{7}\right) \end{aligned}$$

b Let $p(x) = 3x^3 + 5x^2 + 10x - 4$.

If $(3x - 1)$ is a factor, then $p\left(\dfrac{1}{3}\right) = 0$.

$$\begin{aligned} p\left(\frac{1}{3}\right) &= 3\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^2 + 10\left(\frac{1}{3}\right) - 4 \\ &= \frac{1}{9} + \frac{5}{9} + \frac{10}{3} - 4 \\ &= \frac{2}{3} + \frac{10}{3} - 4 \\ &= 0 \end{aligned}$$

$\therefore (3x - 1)$ is a factor.

$\therefore 3x^3 + 5x^2 + 10x - 4 = (3x - 1)(x^2 + 2x + 4)$.

Consider the quadratic factor $x^2 + 2x + 4$.

$$\begin{aligned} \Delta &= b^2 - 4ac, \quad a = 1, b = 2, c = 4 \\ &= 2^2 - 4 \times 1 \times 4 \\ &= -12 \end{aligned}$$

Since $\Delta < 0$, the quadratic has no real linear factors.

Hence, $(3x - 1)$ is the only real linear factor of $P(x) = 3x^3 + 5x^2 + 10x - 4$.

c Let $p(x) = 2x^3 - 21x^2 + 60x - 25$.

Since $2x^2 - 11x + 5 = (2x - 1)(x - 5)$, $2x^2 - 11x + 5$ will be a factor of $p(x)$ if both $(2x - 1)$ and $(x - 5)$ are factors.

$$\begin{aligned} p\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 - 21\left(\frac{1}{2}\right)^2 + 60\left(\frac{1}{2}\right) - 25 \\ &= \frac{1}{4} - \frac{21}{4} + 30 - 25 \\ &= -5 + 30 - 25 \\ &= 0 \end{aligned}$$

$\therefore (2x - 1)$ is a factor.

$$\begin{aligned} p(5) &= 2(5)^3 - 21(5)^2 + 60(5) - 25 \\ &= 250 - 525 + 300 - 25 \\ &= 0 \end{aligned}$$

$\therefore (x - 5)$ is a factor.

Hence, $2x^2 - 11x + 5$ is a factor.

$$\therefore 2x^3 - 21x^2 + 60x - 25 = 0$$

$$\Rightarrow (2x^2 - 11x + 5)(x - 5) = 0$$

$$\therefore (2x - 1)(x - 5)^2 = 0$$

$$\therefore x = \frac{1}{2}, x = 5$$

23 a $p(x) = 5x^3 + kx^2 - 20x - 36$

As $(x^2 - 4)$ is a factor,

$$\begin{aligned} 5x^3 + kx^2 - 20x - 36 &= (x^2 - 4)(ax + b) \\ &= (x^2 - 4)(5x + 9) \\ &= (x - 2)(x + 2)(5x + 9) \end{aligned}$$

Expanding the factorised form,
$5x^3 + kx^2 - 20x - 36 = (x^2 - 4)(5x + 9)$
$= 5x^3 + 9x^2 - 20x - 36$
Equating coefficients of x^2: $k = 9$

b $ax^2 - 5ax + 4(2a - 1) = 0$
Since $x = a$ is a solution, substitute $x = a$ in the equation.
$\therefore a(a)^2 - 5a(a) + 4(2a - 1) = 0$
$\therefore a^3 - 5a^2 + 8a - 4 = 0$
Let $p(a) = a^3 - 5a^2 + 8a - 4$.
$p(1) = 1 - 5 + 8 - 4 = 0$
$\therefore (a - 1)$ is a factor of $p(a)$.
$\therefore a^3 - 5a^2 + 8a - 4 = 0$
$\Rightarrow (a - 1)(a^2 - 4a + 4) = 0$
$\therefore (a - 1)(a - 2)^2 = 0$
$\therefore a = 1, a = 2$

c $p(x) = x^3 + ax^2 + bx - 3$ and $q(x) = x^3 + bx^2 + 3ax - 9$
Since $(x + a)$ is a common factor, $p(-a) = 0$ and $q(-a) = 0$.
$p(-a) = 0$
$\therefore -a^3 + a^3 - ba - 3 = 0$
$\therefore ab = -3 \quad [1]$
$q(-a) = 0$
$\therefore -a^3 + ba^2 - 3a^2 - 9 = 0$
$\therefore a^3 - a^2b + 3a^2 + 9 = 0 \quad [2]$
Solve the simultaneous equations:
$ab = -3 \quad [1]$
$a^3 - a(ab) + 3a^2 + 9 = 0 \quad [2]$
Substitute equation [1] in equation [2]
$\therefore a^3 - a(-3) + 3a^2 + 9 = 0$
$\therefore a^3 + 3a^2 + 3a + 9 = 0$
$\therefore a^2(a + 3) + 3(a + 3) = 0$
$\therefore (a + 3)(a^2 + 3) = 0$
$\therefore a = -3$ or $a^2 = -3$ (reject)
$\therefore a = -3$
Substitute $a = -3$ in equation [1]
$\therefore \ -3b = -3$
$\therefore b = 1$
With $a = -3, b = 1$,
$p(x) = x^3 - 3x^2 + x - 3$
$= x^2(x - 3) + (x - 3)$
$= (x - 3)(x^2 + 1)$
$q(x) = x^3 + x^2 - 9x - 9$
$= x^2(x + 1) - 9(x + 1)$
$= (x + 1)(x^2 - 9)$
$= (x - 3)(x + 3)(x + 1)$
$(x + a) = (x - 3)$ is a factor of each polynomial.

d $p(x) = x^3 + px^2 + 15x + a^2$
Since $(x + a)^2 = x^2 + 2ax + a^2$ is a factor,
$x^3 + px^2 + 15x + a^2 = (x^2 + 2ax + a^2)(x + 1)$
Expanding the factorised form,
$x^3 + px^2 + 15x + a^2$
$= (x^2 + 2ax + a^2)(x + 1)$
$= x^3 + x^2 + 2ax^2 + 2ax + a^2x + a^2$
$= x^3 + (1 + 2a)x^2 + (2a + a^2)x + a^2$
Equating coefficients of like terms,
$x^2: p = 1 + 2a \quad [1]$
$x: 15 = 2a + a^2 \quad [2]$
Solving equation [2],
$a^2 + 2a - 15 = 0$
$\therefore (a + 5)(a - 3) = 0$
$\therefore a = -5$ or $a = 3$
Substitute $a = -5$ in equation [1].
$\therefore p = 1 + 2 \times -5$
$\therefore p = -9$
Substitute $a = 3$ in equation [1].
$\therefore p = 1 + 2 \times 3$
$\therefore p = 7$
There are two possible polynomials, one for which $a = -5, p = -9$, and one for which $a = 3, p = 7$.
$x^3 + px^2 + 15x + a^2 = 0$
If $a = -5, p = -9$, then $x^3 - 9x^2 + 15x + 25 = 0$.
As $(x + a)^2 = (x - 5)^2$ is a factor of the polynomial,
$(x^2 - 10x + 25)(x + 1) = 0$
$\therefore (x - 5)^2(x + 1) = 0$
$\therefore x = 5, x = -1$
If $a = 3,\ p = 7$, then $x^3 + 7x^2 + 15x + 9 = 0$.
As $(x + a)^2 = (x + 3)^2$ is a factor of the polynomial,
$(x^2 + 6x + 9)(x + 1) = 0$
$\therefore (x + 3)^2(x + 1) = 0$
$\therefore x = -3, x = -1$

24 Let $p(x) = 9 + 19x - 2x^2 - 7x^3$.
The divisor $\left(x - \sqrt{2} + 1\right)$ is zero when $x = \sqrt{2} - 1$.
Hence, the remainder is $p\left(\sqrt{2} - 1\right)$.
$p\left(\sqrt{2} - 1\right) = 9 + 19\left(\sqrt{2} - 1\right) - 2\left(\sqrt{2} - 1\right)^2 - 7\left(\sqrt{2} - 1\right)^3$
Using CAS technology gives the remainder as $-12\sqrt{2} + 33$.

25 $10x^3 - 5x^2 + 21x + 12 = 0$
Use CAS technology to obtain one solution: $x = -0.4696$, correct to 4 decimal places.

4.3 Exam questions

1 Let $p(x) = 2x^3 - 7x^2 + x - 2$.
Remainder $= p(1)$
$p(1) = 2(1)^3 - 7(1)^2 + 1 - 2$
$= -6$
The correct answer is **B**.

2 $p(x) = x^3 - kx + 3$
If $(x + 3)$ is factor of $p(x)$, $p(-3) = 0$.
$p(-3) = (-3)^3 - (-3)k + 3$
$0 = -27 + 3k + 3$
$0 = -24 + 3k$
$24 = 3k$
$\therefore k = 8$
The correct answer is **A**.

3 $p(x) = ax^3 + bx + 20$
$p(x)$ is exactly divisible by $(x - 1)$, so $p(1) = 0$.
$a + b + 20 = 0$
$a + b = -20 \quad [1]$

The remainder when divided by $(x-2)$ is -14, so $p(2) = -14$.

$$\begin{aligned} 8a + 2b + 20 &= -14 \\ 8a + 2b &= -34 \\ 4a + b &= -17 \quad [2] \end{aligned}$$ [1 mark]

Subtract [1] from [2]:

$$\begin{aligned} 3a &= 3 \\ a &= 1 \end{aligned}$$

Substitute $a = 1$ into [1]:

$$\begin{aligned} 1 + b &= -20 \\ b &= -21 \end{aligned}$$

$\therefore p(x) = x^3 - 21x + 20$ [1 mark]

4.4 Graphs of cubic polynomials

4.4 Exercise

1 **a** $y = (x-7)^3$

This is the graph obtained when $y = x^3$ is translated 7 units to the right.

The point of inflection is $(7, 0)$.

Alternatively, since $y = a(x-h)^3 + k$ has point of inflection (h, k), then comparing $y = (x-7)^3$ with this gives its point of inflection as $(7, 0)$.

b $y = x^3 - 7$

This is the graph obtained when $y = x^3$ is translated 7 units vertically downwards.

The point of inflection is $(0, -7)$.

Alternatively, compare with $y = a(x-h)^3 + k$, which has point of inflection (h, k).

$y = x^3 - 7$ has point of inflection $(0, -7)$.

c $y = -7x^3$

This is the graph obtained when $y = -x^3$ is dilated by a factor of 7 units in the y direction.

The point of inflection is $(0, 0)$.

Alternatively, compare with $y = a(x-h)^3 + k$, which has point of inflection (h, k).

$y = -7x^3$ has point of inflection $(0, 0)$.

d $y = 2 - (x-2)^3$

Rearrange the equation.

$y = -(x-2)^3 + 2$

Compare with $y = a(x-h)^3 + k$, which has point of inflection (h, k).

The point of inflection is $(2, 2)$.

e $y = \frac{1}{6}(x+5)^3 - 8$

Compare with $y = a(x-h)^3 + k$, which has point of inflection (h, k).

The point of inflection is $(-5, -8)$.

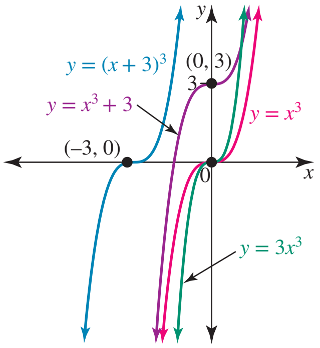
f $y = -\frac{1}{2}(2x-1)^3 + 5$

$y = -\frac{1}{2}\left(2\left(x - \frac{1}{2}\right)\right)^3 + 5$

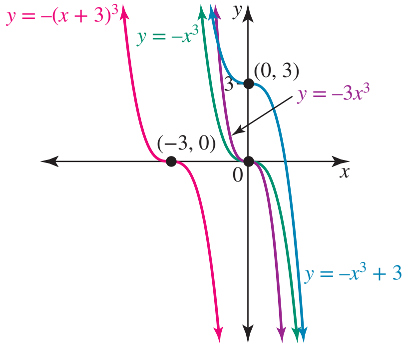
Compare with $y = a(x-h)^3 + k$, which has point of inflection (h, k).

The point of inflection is $\left(\frac{1}{2}, 5\right)$.

2 **a** Graphs of $y = x^3$, $y = 3x^3$, $y = x^3 + 3$, $y = (x+3)^3$



b Graphs of $y = -x^3$, $y = -3x^3$, $y = -x^3 + 3$, $y = -(x+3)^3$



3 **a** $y = (x-1)^3 - 8$

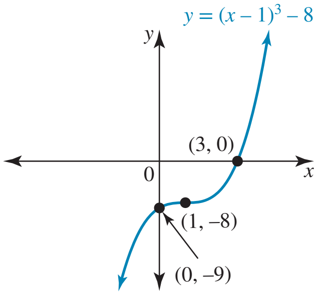
Point of inflection $(1, -8)$

y-intercept $(0, -9)$

x-intercept: let $y = 0$.

$$\begin{aligned} \therefore (x-1)^3 - 8 &= 0 \\ \therefore (x-1)^3 &= 8 \\ \therefore x - 1 &= 2 \\ \therefore x &= 3 \end{aligned}$$

$\Rightarrow (3, 0)$



b $y = 1 - \frac{1}{36}(x+6)^3$

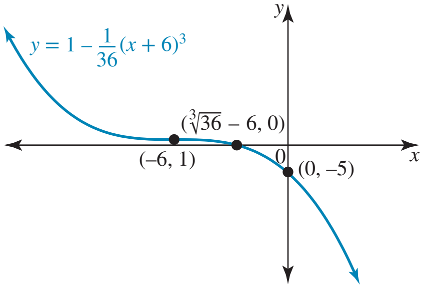
Point of inflection $(-6, 1)$

y-intercept $(0, -5)$

x-intercept: let $y = 0$.

$$\begin{aligned} \therefore 0 &= 1 - \frac{1}{36}(x+6)^3 \\ \therefore (x+6)^3 &= 36 \\ \therefore x + 6 &= \sqrt[3]{36} \\ \therefore x &= \sqrt[3]{36} - 6 \end{aligned}$$

$\Rightarrow (\sqrt[3]{36} - 6, 0) \simeq (-2.7, 0)$



4 **a** $y = -x^3 + 1$

The graph is a negative cubic.

The point of inflection is $(0, 1)$, which is also the y-intercept.

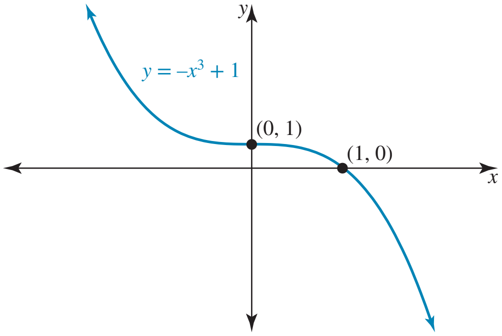
x-intercept: let $y = 0$.

$0 = -x^3 + 1$

$x^3 = 1$

$x = 1$

The x-intercept is $(1, 0)$.



b $y = 2\,(3x - 2)^3$

The graph is a positive cubic.

Point of inflection:

$y = 2\,(3x - 2)^3$

$y = 2\left(3\left(x - \frac{2}{3}\right)\right)^3$

The point of inflection is $\left(\frac{2}{3}, 0\right)$.

This is also the x-intercept.

y-intercept: let $x = 0$.

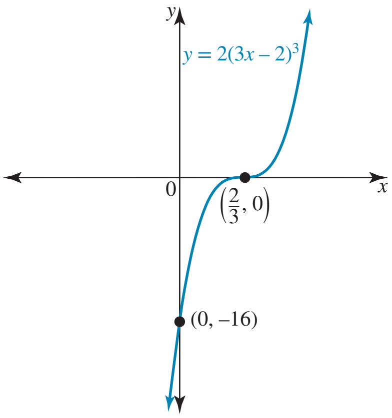
$y = 2\,(3(0) - 2)^3$

$y = 2(-2)^3$

$y = 2(-8)$

$y = -16$

The y-intercept is $(-16, 0)$.



c $y = 2(x + 3)^3 - 16$

Point of inflection: $(-3, -16)$

y-intercept: let $x = 0$.

$y = 2(0 + 3)^3 - 16$

$= 54 - 16$

$= 38$

$(0, 38)$ is the y-intercept.

x-intercept: let $y = 0$.

$2(x + 3)^3 - 16 = 0$

$2(x + 3)^3 = 16$

$(x + 3)^3 = 8$

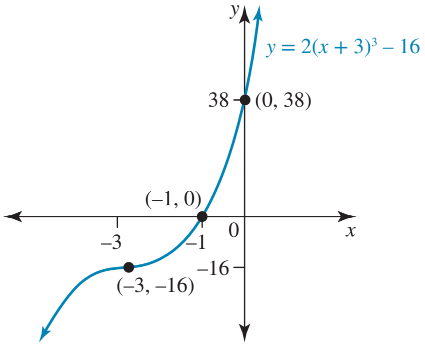
$x + 3 = \sqrt[3]{8}$

$x + 3 = 2$

$x = -1$

$(-1, 0)$ is the x-intercept.

Positive cubic graph.



d $y = (3 - x)^3 + 1$

Rearranging,

$y = (-(x - 3))^3 + 1$

The point of inflection is $(3, 1)$.

y-intercept: let $x = 0$.

$y = (3 - 0)^3 + 1$

$= 27 + 1$

$= 28$

$(0, 28)$ is the y-intercept.

x-intercept: let $y = 0$.

$(3 - x)^3 + 1 = 0$

$(3 - x)^3 = -1$

$3 - x = \sqrt[3]{-1}$

$3 - x = -1$

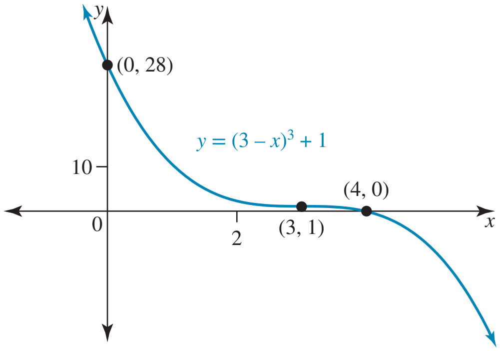
$-x = -4$

$x = 4$

$(4, 0)$ is the x-intercept.

The graph is a negative cubic.

(Note that $y = (-(x - 3))^3 + 1$ is the same as $y = -(x - 3)^3 + 1$.)



5 **a** $y = (x + 4)^3 - 27$

POI: $(-4, -27)$

y-intercept: let $x = 0$.

$\therefore y = (4^3) - 27 = 37$

$(0, 37)$

x-intercept: let $y = 0$.

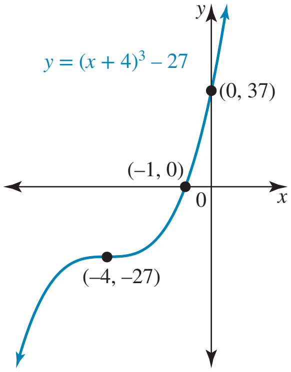
$\therefore (x + 4)^3 - 27 = 0$

$\therefore (x + 4)^3 = 27$

$\therefore x + 4 = 3$

$\therefore x = -1$

$(-1, 0)$



b $y = 2(x - 1)^3 + 10$

POI: $(1, 10)$

y-intercept: let $x = 0$, $y = 2(-1)^3 + 10 = 8 \Rightarrow (0, 8)$

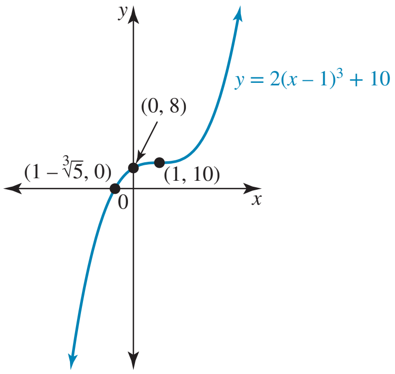
x-intercept: let $y = 0$.

$2(x - 1)^3 + 10 = 0$

$\therefore (x - 1)^3 = -5$

$\therefore x = 1 - \sqrt[3]{5} \simeq -0.7$

$(1 - \sqrt[3]{5}, 0)$



c $y = 27 + 2(x - 3)^3$

POI: $(3, 27)$

y-intercept: let $x = 0$, $y = 27 + 2(-3)^3 = -27 \Rightarrow (0, -27)$

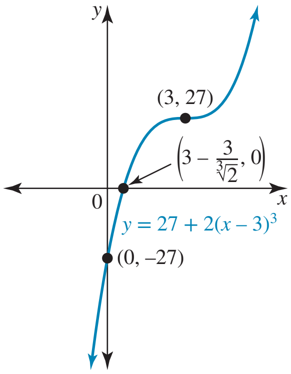
x-intercept: let $y = 0$.

$2(x - 3)^3 + 27 = 0$

$\therefore (x - 3)^3 = -\dfrac{27}{2}$

$\therefore x = 3 - \dfrac{3}{\sqrt[3]{2}} \simeq 0.6$

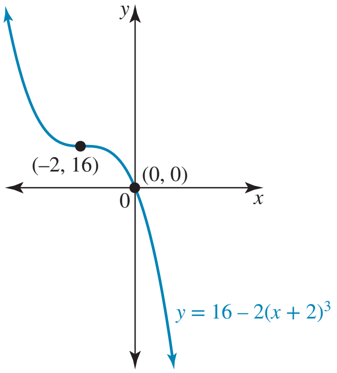
$$\left(3 - \frac{3}{\sqrt[3]{2}}, 0\right)$$



d $y = 16 - 2(x + 2)^3$

POI: $(-2, 16)$

y-intercept: let $x = 0$, $y = 16 - 2(2)^3 = 0$, so the graph passes through the origin.



e $y = -\dfrac{3}{4}(3x + 4)^3$

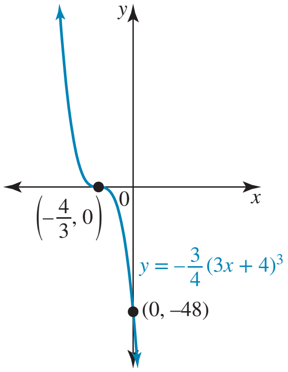
POI: $3x + 4 = 0 \Rightarrow x = -\dfrac{4}{3}$, so POI and x-intercept is

$$\left(-\frac{4}{3}, 0\right)$$

y-intercept: let $x = 0$.

$y = -\dfrac{3}{4}(4)^3$

$= -48 \Rightarrow (0, -48)$



f $y = 9 + \dfrac{x^3}{3}$

POI: $(0, 9)$

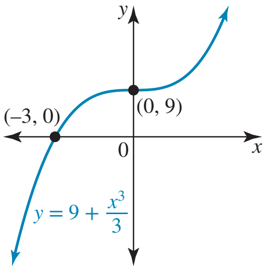
x-intercept: let $y = 0$.

$\dfrac{x^3}{3} + 9 = 0$

$\therefore x^3 = -27$

$\therefore x = -3$

$(-3, 0)$



6 **a** $y = \left(\frac{x}{2} - 3\right)^3$

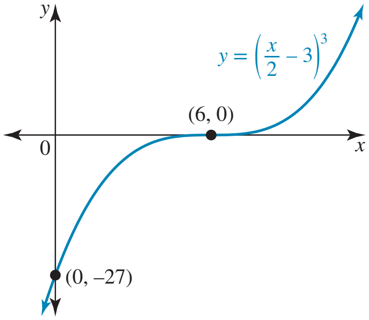
Point of inflection and x-intercept:

$\frac{x}{2} - 3 = 0$

$\therefore x = 6$

$\Rightarrow (6, 0)$

y-intercept $(0, -27)$



b $y = 2x^3 - 2$

Point of inflection and y-intercept $(0, -2)$

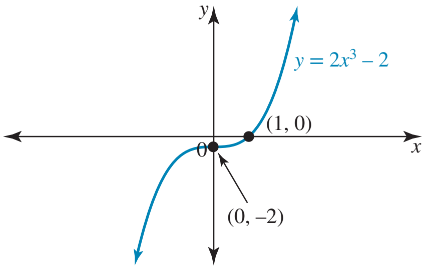
x-intercept: let $y = 0$.

$\therefore 2x^3 - 2 = 0$

$\therefore x^3 = 1$

$\therefore x = 1$

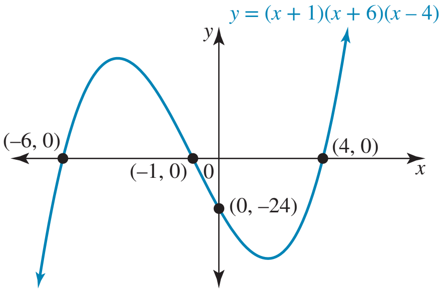
$\Rightarrow (1, 0)$



7 **a** $y = (x + 1)(x + 6)(x - 4)$

x-intercepts at $x = -1, x = -6, x = 4$

y-intercept at $y = (1)(6)(-4) = -24$



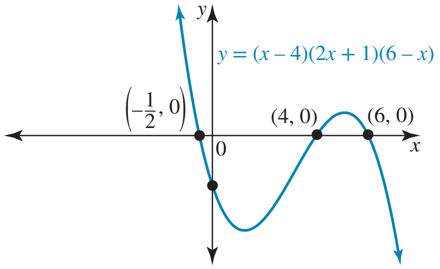
b $y = (x - 4)(2x + 1)(6 - x)$

x-intercepts occur when

$x - 4 = 0,\ 2x + 1 = 0, 6 - x = 0.$

$\therefore x = 4, -0.5, 6$

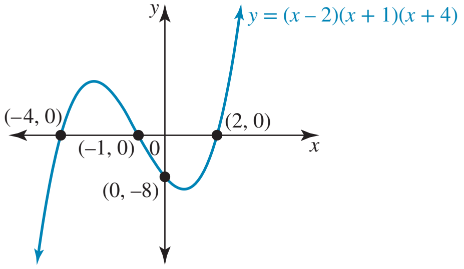
y-intercept at $y = (-4)(1)(6) = -24$



8 **a** $y = (x - 2)(x + 1)(x + 4)$

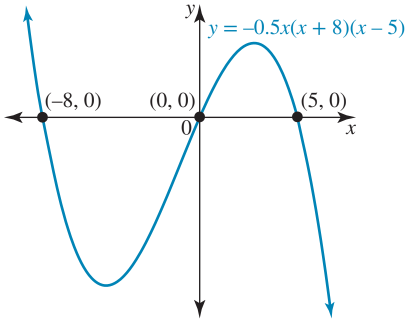
x-intercepts occur at $x = 2, x = -1, x = -4.$

y-intercept: let $x = 0$, $y = (-2)(1)(4) = -8 \Rightarrow (0, -8)$



b $y = -0.5x(x + 8)(x - 5)$

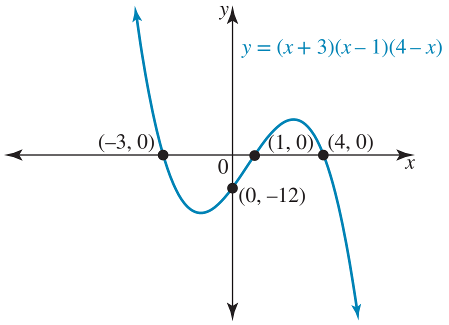
x-intercepts occur at $x = 0, x = -8, x = 5.$



c $y = (x + 3)(x - 1)(4 - x)$

x-intercepts occur at $x = -3, x = 1, x = 4.$

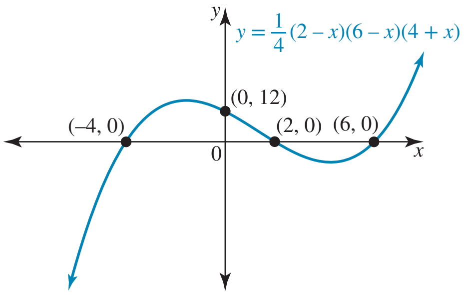
y-intercept: let $x = 0$, $y = (3)(-1)(4) = -12 \Rightarrow (0, -12)$



d $y = \frac{1}{4}(2 - x)(6 - x)(4 + x)$

x-intercepts occur when $x = 2, x = 6, x = -4.$

y-intercept: let $x = 0$, $y = \frac{1}{4}(2)(6)(4) = 12 \Rightarrow (0, 12)$



9 Factorise $y = 3x(x^2 - 4)$.

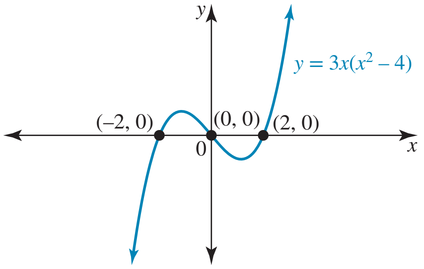
$y = 3x(x^2 - 4)$

$\therefore y = x(x-2)(x+2)$

x-intercepts at $x = 0, x = 2, x = -2$

y-intercept $(0, 0)$

Shape: positive x^3 shape



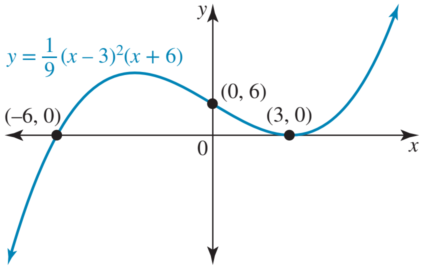
10 a $y = \frac{1}{9}(x-3)^2(x+6)$

x-intercepts: $x = 3$ (touch), $x = -6$ (cut)

Turning point at $(3, 0)$

y-intercept: when $x = 0$,

$$y = \frac{1}{9}(-3)^2(6)$$
$$= 6$$
$$\Rightarrow (0, 6)$$



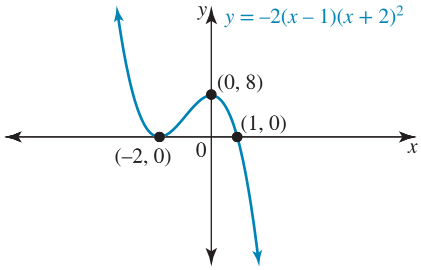
b $y = -2(x-1)(x+2)^2$

x-intercepts: $x = 1$ (cut), $x = -2$ (touch)

Turning point at $(-2, 0)$

y-intercept: when $x = 0$,

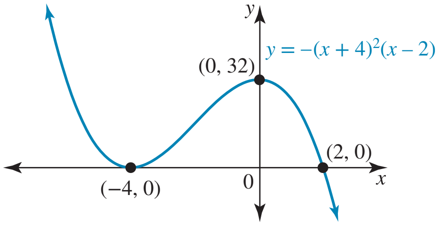
$$y = -2(-1)(2)^2$$
$$= 8$$
$$\Rightarrow (0, 8)$$



11 a $y = -(x+4)^2(x-2)$

x-intercepts occur when $x = -4$ (touch), $x = 2$ (cut).

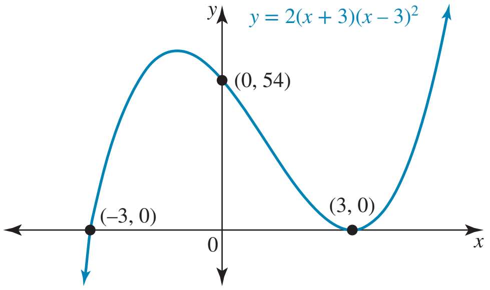
y-intercept: let $x = 0$, $y = -(4)^2(-2) = 32 \Rightarrow (0, 32)$



b $y = 2(x+3)(x-3)^2$

x-intercepts occur when $x = -3$ (cut), $x = 3$ (touch).

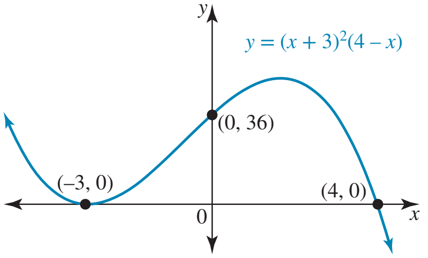
y-intercept: let $x = 0$, $y = 2(3)(-3)^2 = 54 \Rightarrow (0, 54)$



c $y = (x+3)^2(4-x)$

x-intercepts occur when $x = -3$ (touch), $x = 4$ (cut).

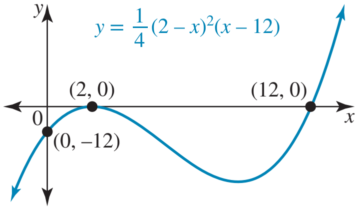
y-intercept: let $x = 0$, $y = (3)^2(4) = 36 \Rightarrow (0, 36)$



d $y = \frac{1}{4}(2-x)^2(x-12)$

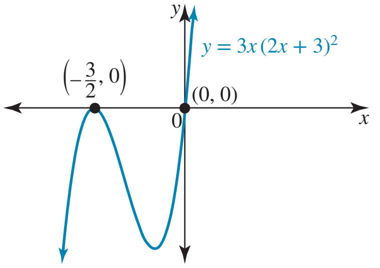
x-intercepts occur when $x = 2$ (touch), $x = 12$ (cut).

y-intercept: let $x = 0$, $y = \frac{1}{4}(2)^2(-12) = -12 \Rightarrow (0, -12)$



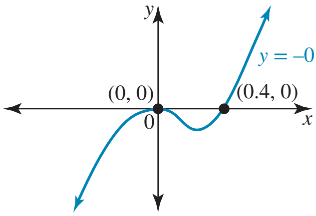
e $y = 3x(2x+3)^2$

x-intercepts occur when $x = 0$ (cut), $x = -\frac{3}{2}$ (touch).



f $y = -0.25x^2(2 - 5x)$

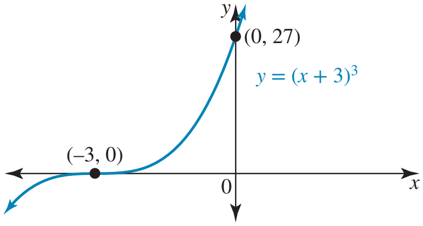
x-intercepts occur when $x = 0$ (touch), $x = \frac{2}{5}$ (cut).



12 a $y = (x + 3)^3$

POI and x-intercept $(-3, 0)$

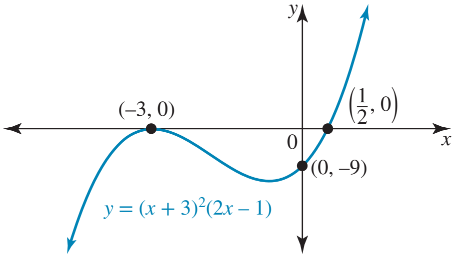
y-intercept $(0, 27)$



b $y = (x + 3)^2(2x - 1)$

x-intercepts at $x = -3$ (touch), $x = \frac{1}{2}$ (cut)

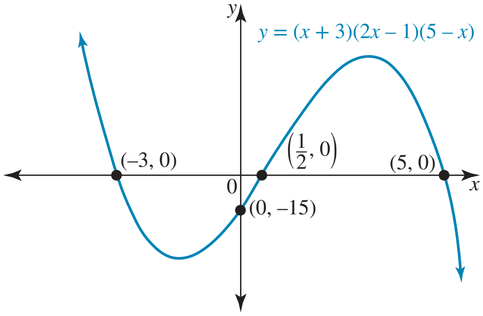
y-intercept at $y = (3)^2(-1) = -9$



c $y = (x + 3)(2x - 1)(5 - x)$

x-intercepts at $x = -3, x = \frac{1}{2}, x = 5$

y-intercept at $y = (3)(-1)(5) = -15$



d $2(y - 1) = (1 - 2x)^3$

$$\therefore y - 1 = \frac{1}{2}(1 - 2x)^3$$

$$\therefore y = \frac{1}{2}(1 - 2x)^3 + 1$$

POI: when $1 - 2x = 0, x = \frac{1}{2} \Rightarrow \left(\frac{1}{2}, 1\right)$

x-intercept: let $y = 0$.

$$\therefore 2(-1) = (1 - 2x)^3$$

$$\therefore (1 - 2x)^3 = -2$$

$$\therefore 1 - 2x = -\sqrt[3]{2}$$

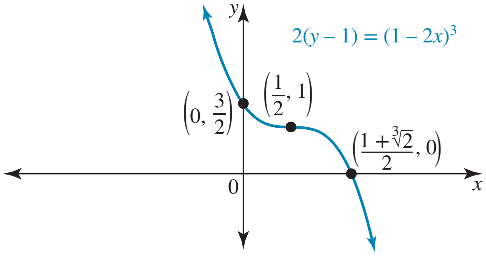
$$\therefore 1 + \sqrt[3]{2} = 2x$$

$$\therefore x = \frac{1 + \sqrt[3]{2}}{2} \simeq 1.1$$

$$\left(\frac{1 + \sqrt[3]{2}}{2}, 0\right)$$

y-intercept: let $x = 0$.

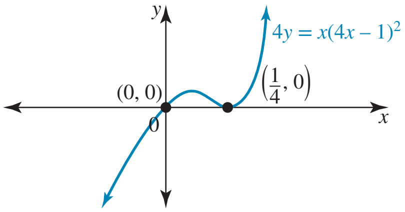
$y = \frac{1}{2}(1)^3 + 1 = \frac{3}{2} \Rightarrow \left(0, \frac{3}{2}\right)$



e $4y = x(4x - 1)^2$

$$\therefore y = \frac{1}{4}x(4x - 1)^2$$

x-intercepts at $x = 0$ (cut), $x = \frac{1}{4}$ (touch)



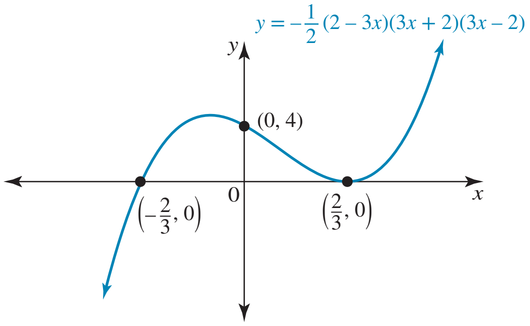
f $y = -\frac{1}{2}(2 - 3x)(3x + 2)(3x - 2)$

$$\therefore y = -\frac{1}{2} \times -1(3x - 2)(3x + 2)(3x - 2)$$

$$\therefore y = \frac{1}{2}(3x - 2)^2(3x + 2)$$

x-intercepts when $x = \frac{2}{3}$ (touch), $x = -\frac{2}{3}$ (cut)

y-intercept at $y = \frac{1}{2}(-2)^2(2) = 4$



13 $y = x^3 - 3x^2 - 10x + 24$

y-intercept: $(0, 24)$

x-intercepts: let $p(x) = x^3 - 3x^2 - 10x + 24$.

$p(1) \neq 0$

$p(2) = 8 - 12 - 20 + 24 = 0$

$\therefore (x - 2)$ is a factor.

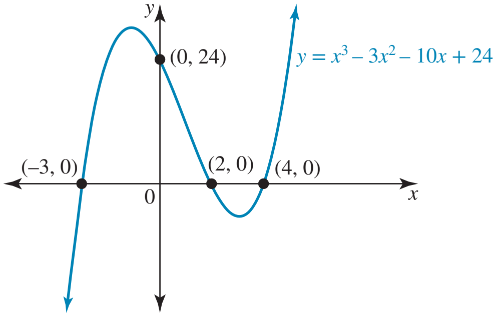
$\therefore x^3 - 3x^2 - 10x + 24$

$= (x - 2)(x^2 + bx - 12)$

$= (x - 2)(x^2 - x - 12)$

$= (x - 2)(x - 4)(x + 3)$

Therefore, the x-intercepts are at $x = 2, x = 4, x = -3$ (all cuts).



14 **a** $y = -x^3 - 3x^2 + 16x + 48$

y-intercept: let $x = 0$.

$y = -(0)^3 - 3(0)^2 + 16(0) + 48$

$= 48$

y-intercept $(0, 48)$

x-intercepts: let $y = 0$.

$-x^3 - 3x^2 + 16x + 48 = 0$

Factorise the equation.

$-\left(x^3 + 3x^2 - 16x - 48\right) = 0$

$x^3 + 3x^2 - 16x - 48 = 0$

$x^2(x + 3) - 16(x + 3) = 0$

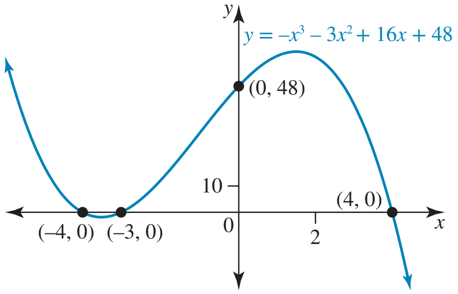
$(x + 3)(x^2 - 16) = 0$

$(x + 3)(x + 4)(x - 4) = 0$

$\therefore x = -3, -4, 4$

The x-intercepts are $(-3, 0), (-4, 0), (4, 0)$.

The equation of the graph can be expressed as $y = -(x + 3)(x + 4)(x - 4) = 0$. It is a negative cubic.



b $2x^3 + x^2 - 13x + 6$

Let $P(x) = 2x^3 + x^2 - 13x + 6$.

$P(1) = 2 + 1 - 13 + 6 \neq 0$

$P(2) = 16 + 4 - 26 + 6 = 0$

Therefore, $(x - 2)$ is a factor.

$2x^3 + x^2 - 13x + 6 = (x - 2)(2x^2 + bx - 3)$

$= (x - 2)(2x^2 + 5x - 3)$

$= (x - 2)(2x - 1)(x + 3)$

x-intercepts: let $y = 0$.

$(x - 2)(2x - 1)(x + 3) = 0$

$x = 2, \frac{1}{2}, -3$ are the places where the graph intersects the x-axis.

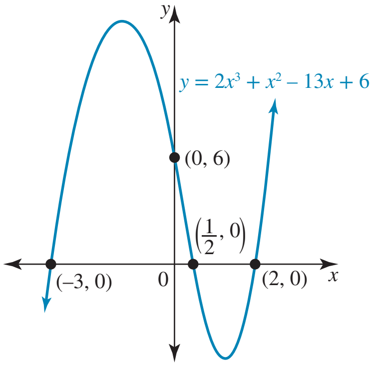
y-intercept: let $x = 0$.

$y = 2x^3 + x^2 - 13x + 6$

$= 2(0)^3 + (0)^2 - 13(0) + 6$

$= 6$

The graph intersects y-axis at $y = 6$.



c $y = x^3 + 5x^2 - x - 5$

y-intercept: when $x = 0, y = -5$.

$(0, -5)$ is the y-intercept.

x-intercepts: let $y = 0$.

$x^3 + 5x^2 - x - 5 = 0$

Factorise by grouping.

$x^2(x + 5) - (x + 5) = 0$

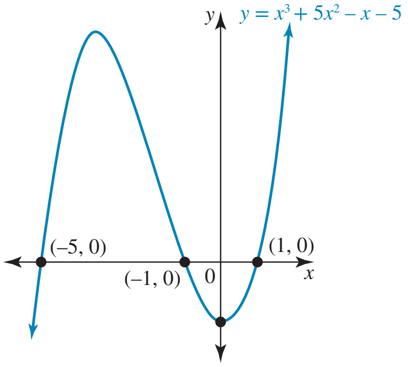
$(x + 5)(x^2 - 1) = 0$

$(x + 5)(x + 1)(x - 1) = 0$

$x = -5, -1, 1$

The x-intercepts are $(-5, 0), (-1, 0), (1, 0)$.

The graph is a positive cubic.



d $-x^3 - 5x^2 - 3x + 9$

Let $p(x) = -x^3 - 5x^2 - 3x + 9$.

$p(1) = -1 - 5 - 3 + 9$

$= 0$

$(x - 1)$ is a factor.

$-x^3 - 5x^2 - 3x + 9 = (x - 1)(-x^2 + bx - 9)$

$= (x - 1)(-x^2 - 6x - 9)$

$= -(x - 1)(x^2 + 6x + 9)$

$= -(x - 1)(x + 3)^2$

$y = -x^3 - 5x^2 - 3x + 9$

y-intercept: when $x = 0, y = 9$.

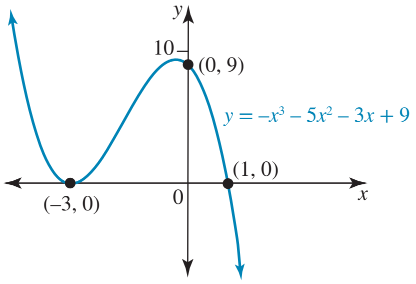
x-intercepts: when $y = 0$,

$-(x - 1)(x + 3)^2 = 0$

$x = 1, x = -3$

The graph cuts the x-axis at $(1, 0)$ and touches the x-axis at a turning point $(-3, 0)$.

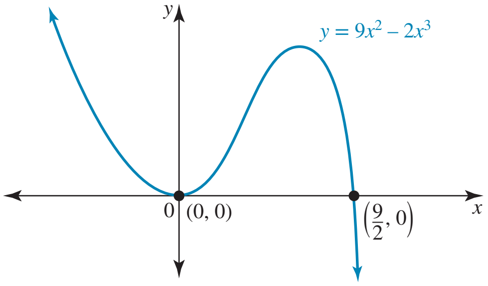
The graph is a negative cubic.



15 a $y = 9x^2 - 2x^3$

$\therefore y = x^2(9 - 2x)$

x-intercepts when $x = 0$ (touch), $x = \frac{9}{2}$ (cut)

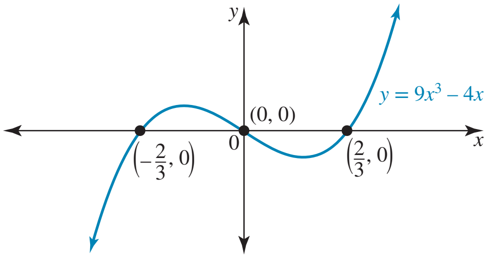


b $y = 9x^3 - 4x$

$\therefore y = x(9x^2 - 4)$

$\therefore y = x(3x - 2)(3x + 2)$

x-intercepts when $x = 0, x = \pm\frac{2}{3}$



c $y = -9x^3 - 9x^2 + 9x + 9$

$\therefore y = -9(x^3 + x^2 - x - 1)$

$= -9\left[x^2(x + 1) - (x + 1)\right]$

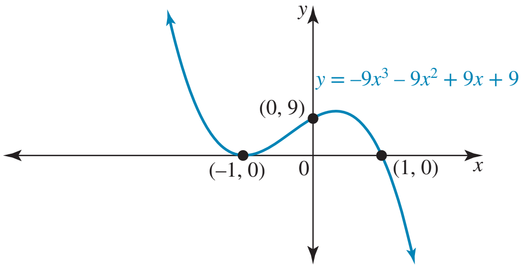
$= -9(x + 1)(x^2 - 1)$

$= -9(x + 1)(x + 1)(x - 1)$

$\therefore y = -9(x + 1)^2(x - 1)$

x-intercepts when $x = -1$ (touch), $x = 1$ (cut)

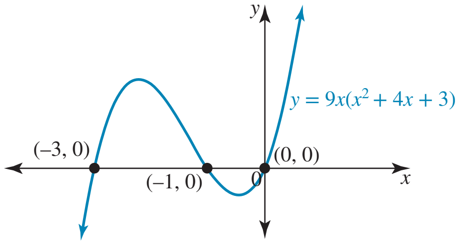
y-intercept $(0, 9)$



d $y = 9x(x^2 + 4x + 3)$

$\therefore y = 9x(x + 3)(x + 1)$

x-intercepts when $x = 0, x = -3, x = -1$



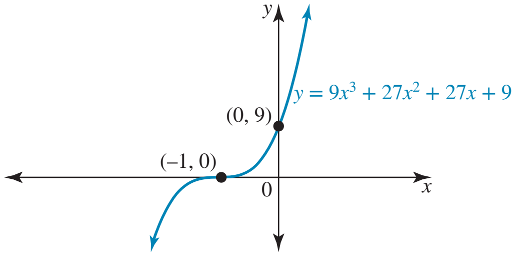
e $y = 9x^3 + 27x^2 + 27x + 9$

$\therefore y = 9(x^3 + 3x^2 + 3x + 1)$

$\therefore y = 9(x + 1)^3$

POI $(-1, 0)$

y-intercept $(0, 9)$



f $y = 9x^2 - 3x^3 + x - 3$

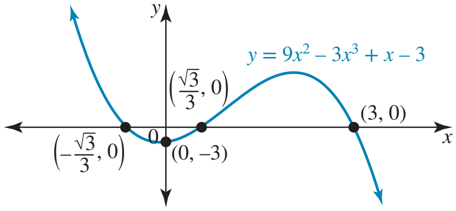
$\therefore y = 3x^2(3 - x) - (3 - x)$

$= (3 - x)(3x^2 - 1)$

$\therefore y = (3 - x)\left(\sqrt{3}x - 1\right)\left(\sqrt{3}x + 1\right)$

x-intercepts when $x = 3, x = \frac{1}{\sqrt{3}}, x = -\frac{1}{\sqrt{3}}$

y-intercept: $(0, -3)$



16 a $y = 2x^3 - 3x^2 - 17x - 12$

x-intercepts when $2x^3 - 3x^2 - 17x - 12 = 0$

Let $p(x) = 2x^3 - 3x^2 - 17x - 12$.

$p(-1) = -2 - 3 + 17 - 12 = 0$

$\therefore (x + 1)$ is a factor.

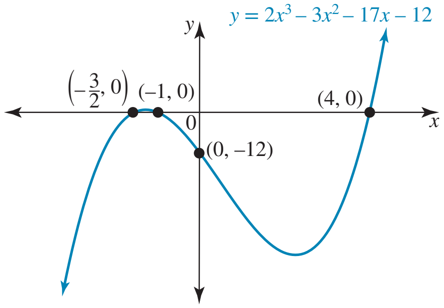
$\therefore 2x^3 - 3x^2 - 17x - 12 = (x + 1)(2x^2 - 5x - 12)$

$= (x + 1)(2x + 3)(x - 4)$

$\therefore (x + 1)(2x + 3)(x - 4) = 0$

$\therefore x = -1, -\frac{3}{2}, 4$

y-intercept $(0, -12)$



b $y = 6 - 55x + 57x^2 - 8x^3$

x-intercepts when $p(x) = 6 - 55x + 57x^2 - 8x^3 = 0$

$p(1) = 6 - 55 + 57 - 8 = 0$

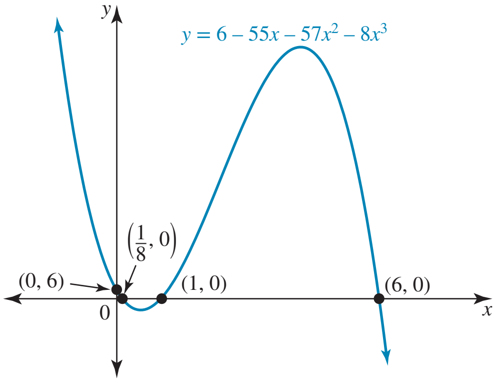
$\therefore (x - 1)$ is a factor.

$\therefore (x - 1)(-8x^2 + 49x - 6) = 0$

$\therefore (x - 1)(-8x + 1)(x - 6) = 0$

$$\therefore x = 1,\ x = \frac{1}{8},\ x = 6$$

y-intercept $(0, 6)$



c $y = 6x^3 - 13x^2 - 59x - 18$

x-intercepts when $p(x) = 6x^3 - 13x^2 - 59x - 18 = 0$

$$\begin{aligned} p(-2) &= 6 \times -8 - 13 \times 4 + 59 \times 2 - 18 \\ &= -48 - 52 + 118 - 18 \\ &= 0 \end{aligned}$$

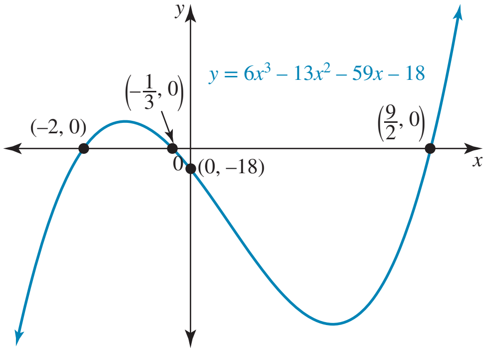
$\therefore (x + 2)$ is a factor.

$\therefore (x + 2)(6x^2 - 25x - 9) = 0$

$\therefore (x + 2)(3x + 1)(2x - 9) = 0$

$$\therefore x = -2,\ x = -\frac{1}{3},\ x = \frac{9}{2}$$

y-intercept $(0, -18)$



d $y = x^3 - 17x + 4$

x-intercepts when $p(x) = x^3 - 17x + 4 = 0$

$p(4) = 64 - 68 + 4 = 0$

$\therefore (x - 4)$ is a factor.

$$\therefore (x - 4)(x^2 + 4x - 1) = 0$$

$$\therefore (x - 4)\left[(x^2 + 4x + 4) - 4 - 1\right] = 0$$

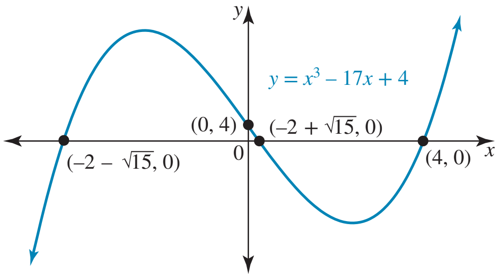
$$\therefore (x - 4)\left[(x + 2)^2 - 5\right] = 0$$

$$\therefore (x - 4)\left(x + 2 - \sqrt{5}\right)\left(x + 2 + \sqrt{5}\right) = 0$$

$$\therefore x = 4,\ x = -2 + \sqrt{5},\ x = -2 - \sqrt{5}$$

$$\therefore x = 4, x \simeq 0.2, x \simeq -4.2$$

y-intercept $(0, 4)$



e $y = -5x^3 - 7x^2 + 10x + 14$

x-intercepts when $-5x^3 - 7x^2 + 10x + 14 = 0$

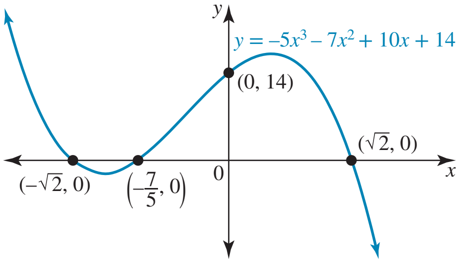
$$\therefore -x^2(5x + 7) + 2(5x + 7) = 0$$

$$\therefore (5x + 7)(2 - x^2) = 0$$

$$\therefore (5x + 7)\left(\sqrt{2} - x\right)\left(\sqrt{2} + x\right) = 0$$

$$\therefore x = -\frac{7}{5}, x = \sqrt{2}, x = -\sqrt{2}$$

y-intercept $(0, 14)$



f $y = -\frac{1}{2}x^3 + 14x - 24$

x-intercepts when $-\frac{1}{2}x^3 + 14x - 24 = 0$

Multiply by -2.

$\therefore p(x) = x^3 - 28x + 48 = 0$

$p(2) = 8 - 56 + 48 = 0$

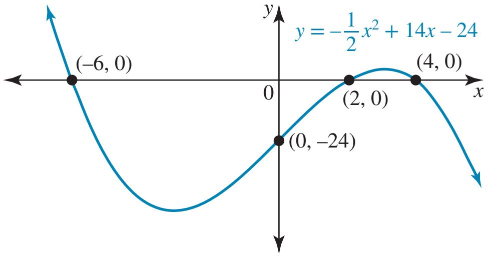
$\therefore (x - 2)$ is a factor.

$\therefore (x - 2)(x^2 + 2x - 24) = 0$

$\therefore (x - 2)(x + 6)(x - 4) = 0$

$\therefore x = 2,\ x = -6,\ x = 4$

y-intercept $(0, -24)$



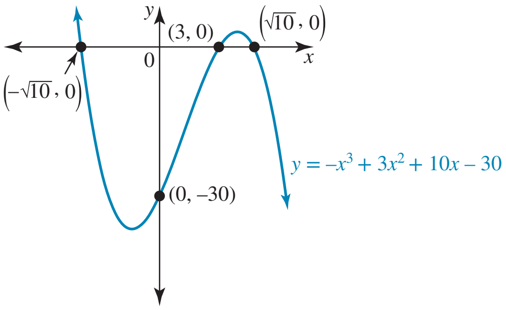
17 a $y = -x^3 + 3x^2 + 10x - 30$

y-intercept: $(0, -30)$

x-intercepts: factorise by grouping ‘2 and 2’.

$$\begin{aligned} y &= -x^3 + 3x^2 + 10x - 30 \\ &= -x^2(x - 3) + 10(x - 3) \\ &= (x - 3)(-x^2 + 10) \\ &= -(x - 3)\left(x - \sqrt{10}\right)\left(x + \sqrt{10}\right) \end{aligned}$$

Therefore, x-intercepts are at $x = 3, x = \pm\sqrt{10}$ (all cuts).



b $y = x^3 + 3x^2 + 3x + 2$

$\therefore y = x^3 + 3x^2 + 3x + 1 + 1$

$\therefore y = (x^3 + 3x^2 + 3x + 1) + 1$

$\therefore y = (x + 1)^3 + 1$

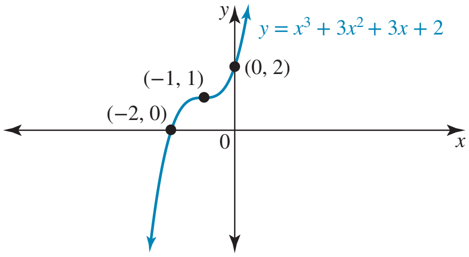
The stationary point of inflection has coordinates $(-1, 1)$.

y-intercept $(0, 2)$

x-intercept: let $y = 0$.

$$\begin{aligned}\therefore (x + 1)^3 + 1 &= 0\\ \therefore (x + 1)^3 &= -1\\ \therefore x + 1 &= -1\\ \therefore x &= -2\end{aligned}$$

$(-2, 0)$



18 $P(x) = 30x^3 + kx^2 + 1$

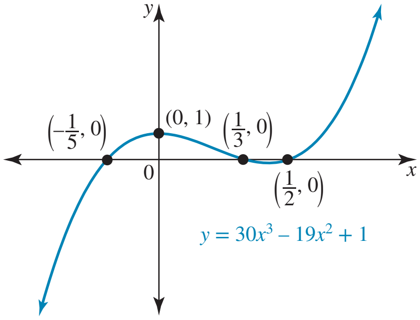
a Since $(3x - 1)$ is a factor, $p\left(\frac{1}{3}\right) = 0$.

$$\begin{aligned}\therefore 30\left(\frac{1}{3}\right)^3 + k\left(\frac{1}{3}\right)^2 + 1 &= 0\\ \therefore \frac{30}{27} + \frac{k}{9} &= -1\\ \therefore 30 + 3k &= -27\\ \therefore 3k &= -57\\ \therefore k &= -19\end{aligned}$$

b $$\begin{aligned}p(x) &= 30x^3 - 19x^2 + 1\\ &= (3x - 1)(10x^2 - 3x - 1)\\ &= (3x - 1)(5x + 1)(2x - 1)\end{aligned}$$

c $p(x) = 0 \Rightarrow x = \frac{1}{3}, x = -\frac{1}{5}, x = \frac{1}{2}$

d y-intercept $(0, 1)$



e If $x = -1$,

$$\begin{aligned}y &= 30(-1)^3 - 19(-1)^2 + 1\\ &= -48\\ &\neq -40\end{aligned}$$

The point $(-1, 40)$ does not lie on the graph.

19 $y = x^3 - 5x^2 + 11x - 7$

a Let $p(x) = x^3 - 5x^2 + 11x - 7$.

$p(1) = 1 - 5 + 11 - 7 = 0$

$\therefore (x - 1)$ is a factor.

$\therefore x^3 - 5x^2 + 11x - 7 = (x - 1)(x^2 - 4x + 7)$

Consider $x^2 - 4x + 7$.

$$\begin{aligned}\Delta &= b^2 - 4ac, \quad a = 1, b = -4, c = 7\\ &= 16 - 28\\ &= -12\end{aligned}$$

$\therefore \Delta < 0$

No real linear factors for the quadratic term

$\therefore y = x^3 - 5x^2 + 11x - 7$ has only one linear factor and therefore the graph has only one x-intercept at $x = 1$.

b Let $x^3 - 5x^2 + 11x - 7 \equiv a(x - b)^3 + c$

$\therefore x^3 - 5x^2 + 11x - 7 = ax^3 - 3ax^2b + 3ab^2x - ab^3 + c$

Equating coefficients of like terms,

$x^3: 1 = a$

$x^2: -5 = -3ab$

$\therefore -5 = -3(1)b$

$\therefore b = \frac{5}{3}$

Check the coefficient of x:

$x: 11 = 3ab^2$

$$\begin{aligned}3ab^2 &= 3(1)\left(\frac{5}{3}\right)^2\\ &= \frac{25}{3}\\ &\neq 11\end{aligned}$$

Hence, $x^3 - 5x^2 + 11x - 7$ cannot be expressed in the form $a(x - b)^3 + c$.

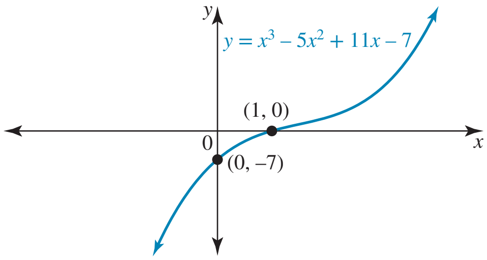
c The coefficient of x^3 is positive; therefore, as $x \to \infty$, the y-values of the graph also $\to \infty$.

d No POI, no turning points

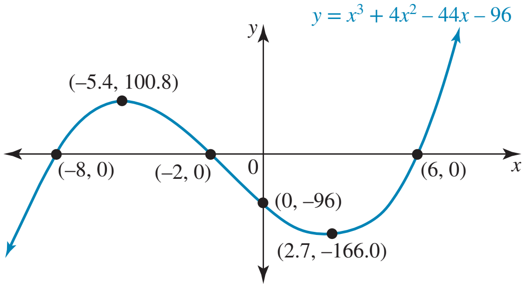
y-intercept $(0, -7)$

One x-intercept $(1, 0)$

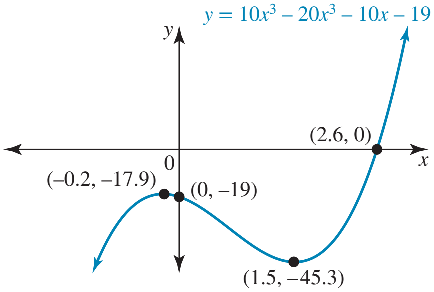
As $x \to \pm\infty, y \to \pm\infty$



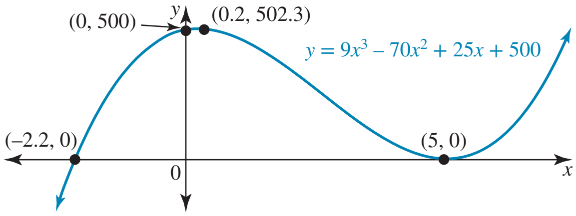
20 a i $y = x^3 + 4x^2 - 44x - 96$



ii $y = 10x^3 - 20x^2 - 10x - 19$



iii $y = 9x^3 - 70x^2 + 25x + 500$



b The maximum turning point $(-5.4, 100.8)$ lies between the x-intercepts of $(-8, 0)$ and $(-2, 0)$. The midpoint of the interval where $x \in [-8, -2]$ is $x = -5$, but for the turning point, $x_{tp} = -5.4$. The turning point is not symmetrically placed between the two intercepts.

Similarly, for the minimum turning point $(2.7, -166.0)$, $2.7 \neq \frac{-2+6}{2}$, that is, $2.7 \neq 2$.

Neither turning point is placed halfway between its adjoining x-intercepts.

4.4 Exam questions

1 $-a$, since the cubic graph slopes downwards.

x-intercept at -2 and turning point at $x = 4$ indicate that $(x + 2)$ is a factor and $(x - 4)$ is a factor with multiplicity of 2 at $x = 4$.

$\therefore y = -a\,(x + 2)\,(x - 4)^2$

The correct answer is **C**.

2 The graph will have a positive cubic shape.

x-intercepts at $y = 0$:

$(x + 4)\,(x + 1)\,(x - 3) = 0$

$x = -4,\ -1,\ 3$

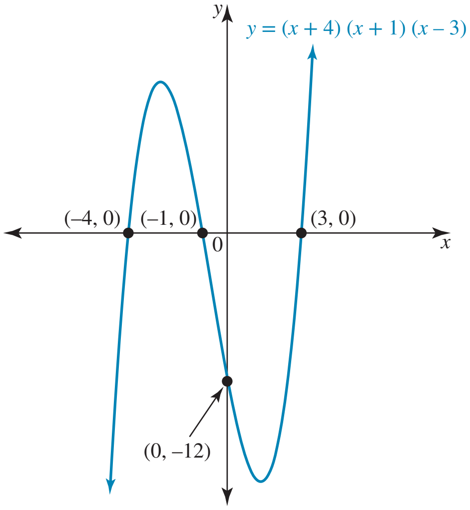
$\therefore$ x-intercepts are $(-4, 0),\ (-1, 0),\ (3, 0)$. [1 mark]

y-intercepts at $x = 0$:

$y = (4)\,(1)\,(-3)$

$y = -12$

$\therefore$ y-intercept is $(0, -12)$. [1 mark]



Award 1 mark for correct graph.

3 $y = -2(3 - x)^3 - 1$

$= -2(-(-3 + x))^3 - 1$

$= -2 \times -1(x - 3)^3 - 1$

$= 2(x - 3)^3 - 1$

The function is a positive cubic, since the coefficient of the leading term is positive. The statement that the function is a negative cubic is false.

The correct answer is **E**.

4.5 Equations of cubic polynomials

4.5 Exercise

1 a Inflection at $(3, -7)$

Therefore, the equation is $y = a(x - 3)^3 - 7$.

Substitute $(10, 0)$.

$\therefore 0 = a(7)^3 - 7$

$\therefore a = \frac{7}{7^3}$

$\therefore a = \frac{1}{49}$

The equation is $y = \frac{1}{49}(x - 3)^3 - 7$.

b Information: cuts at x-intercepts at $x = -5, 0, 4$

Therefore, the equation is:

$y = a(x + 5)(x)(x - 4)$

$\therefore y = ax(x + 5)(x - 4)$

Substitute $(2, -7)$.

$-7 = a(2)(7)(-2)$

$\therefore a = \frac{1}{4}$

The equation is $y = 0.25x(x + 5)(x - 4)$.

c Information: x-intercepts at $x = -2, 3$ turning point at $x = -2$

Therefore, the equation is $y = a(x + 2)^2(x - 3)$.

Substitute $(0, 12)$.

$\therefore 12 = a(2)^2(-3)$

$\therefore 12 = -12a$

$\therefore a = -1$

The equation is $y = -(x + 2)^2(x - 3)$.

2 **a** $y = a(x - h)^3 + k$

The stationary point of inflection is $(3, 9)$.

Therefore, $y = a(x - 3)^3 + 9$.

Substitute $(0, 0)$.

$0 = a(0 - 3)^3 + 9$

$0 = -27a + 9$

$27a = 9$

$a = \dfrac{9}{27}$

$a = \dfrac{1}{3}$

The equation is $y = \dfrac{1}{3}(x - 3)^3 + 9$.

b $y = a(x - h)^3 + k$

The stationary point of inflection is $(-2, 2)$.

Therefore, $y = a(x + 2)^3 + 2$.

Substitute $(0, 10)$.

$10 = a(0 + 2)^3 + 2$

$10 = 8a + 2$

$8 = 8a$

$a = 1$

The equation is $y = (x + 2)^3 + 2$.

c $y = a(x - h)^3 + k$

The stationary point of inflection is $(0, 4)$.

Therefore,

$y = a(x + 0)^3 + 4$

$y = ax^3 + 4$

Substitute $\left(\sqrt[3]{2}, 0\right)$.

$0 = a\left(\sqrt[3]{2}\right)^3 + 4$

$0 = 2a + 4$

$2a = -4$

$a = -2$

The equation is $y = -2x^3 + 4$.

d If the graph of $y = x^3$ is translated 5 units to the left and 4 units upwards, its equation becomes $y = (x + 5)^3 + 4$.

e If the graph $y = x^3$ is reflected in the x-axis, translated 2 units to the right and translated downwards 1 unit, its equation becomes $y = -(x - 2)^3 - 1$.

f $y = a(x - h)^3 + k$

The stationary point of inflection is $(3, -1)$.

Therefore, $y = a(x - 3)^3 - 1$.

From the diagram, the graph passes through $(0, 26)$.

Substitute $(0, 26)$.

$26 = a(0 - 3)^3 - 1$

$26 = -27a - 1$

$27 = -27a$

$a = -1$

The equation is $y = -(x - 3)^3 - 1$.

3 **a** The graph cuts the x-axis at $x = -6, x = -1, x = 2$.

The equation has factors $(x + 6), (x + 1), (x - 2)$.

Let the equation be $y = a(x + 6)(x + 1)(x - 2)$.

Substitute the point $(0, 12)$.

$12 = a(0 + 6)(0 + 1)(0 - 2)$

$= a(6)(1)(-2)$

$= -12a$

$a = -1$

The equation is $y = -(x + 6)(x + 1)(x - 2)$.

b The graph touches the x-axis at $x = -3$ and cuts the x-axis at $x = 5$.

The equation has factors $(x + 3)^2$ and $(x - 5)$.

Let the equation be $y = a(x + 3)^2(x - 5)$.

Substitute the point $(0, -45)$.

$-45 = a(0 + 3)^2(0 - 5)$

$= a(9)(-5)$

$= -45a$

$a = 1$

The equation is $y = (x + 3)^2(x - 5)$.

c The graph cuts the x-axis at $x = -2, x = 2, x = 3$.

The equation has factors $(x + 2), (x - 2), (x - 3)$.

Let the equation be $y = a(x + 2)(x - 2)(x - 3)$.

Substitute the point $(0, -12)$.

$-12 = a(0 + 2)(0 - 2)(0 - 3)$

$= a(2)(-2)(-3)$

$= 12a$

$a = -1$

The equation is $y = -(x + 2)(x - 2)(x - 3)$.

d The graph cuts the x-axis at $x = -\sqrt{5}, x = -\dfrac{1}{2}, x = \sqrt{5}$.

$x = -\dfrac{1}{2} \Rightarrow (2x + 1)$ is a factor.

Let the equation be $y = a\,(2x + 1)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$.

Substitute the point $(0, -5)$.

$-5 = a\,(2(0) + 1)\left(0 + \sqrt{5}\right)\left(0 - \sqrt{5}\right)$

$= a(1)\left(\sqrt{5}\right)\left(-\sqrt{5}\right)$

$= a\,(-5)$

$-5 = -5a$

$a = 1$

The equation is $y = (2x + 1)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$.

This can be simplified as follows:

$y = (2x + 1)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$

$= (2x + 1)\left(x^2 - \left(\sqrt{5}\right)^2\right)$

$= (2x + 1)(x^2 - 5)$

Expanding,

$y = (2x + 1)(x^2 - 5)$

$y = 2x^2 + x^2 - 10x - 5$

e Since the graph touches the x-axis at the origin and cuts the x-axis at $(2, 0)$,

$(x - 0)^2$ and $(x - 2)$ must be factors of its equation.

Let the equation be $y = ax^2(x - 2)$.

Substitute the point $(-1, 12)$.

$12 = a(-1)^2(-1-2)$
$= a(1)(-3)$
$12 = -3a$
$a = -4$

The equation is $y = -4x^2(x-2)$.

f The graph cuts the x-axis at $x = -5, x = \frac{1}{2}, x = 8$.

$x = \frac{1}{2} \Rightarrow (2x-1)$ is a factor.

Let the equation be $y = a\,(2x-1)\,(x+5)\,(x-8)$.

Substitute the point $(0, 10)$.

$10 = a\,(2(0)-1)\,(0+5)\,(0-8)$
$= a(1)(5)(8)$
$= 40a$
$a = \frac{10}{40}$
$a = \frac{1}{4}$

The equation is $y = \frac{1}{4}\,(2x-1)\,(x+5)\,(x-8)$.

4 a x-intercepts occur at $x = -8,\ x = -4,\ x = -1$.

Let the equation of the graph be $y = a(x+8)(x+4)(x+1)$.

Substitute the y-intercept $(0,\ 16)$.

$\therefore 16 = a(8)(4)(1)$
$\therefore 16 = 32a$
$\therefore a = \frac{1}{2}$

The equation is $y = \frac{1}{2}(x+8)(x+4)(x+1)$.

b x-intercepts occur at $x = 0$ (touch) and $x = 5$ (cut).

Let the equation of the graph be $y = ax^2(x-5)$.

Substitute the given point $(2, 24)$.

$\therefore 24 = a(2)^2(-3)$
$\therefore 24 = -12a$
$\therefore a = -2$

The equation is $y = -2x^2(x-5)$.

c Point of inflection $(1, -3)$

Let the equation of the graph be $y = a(x-1)^3 - 3$.

Substitute the origin $(0, 0)$.

$\therefore 0 = a(-1)^3 - 3$
$\therefore 0 = -a - 3$
$\therefore a = -3$

The equation is $y = -3(x-1)^3 - 3$.

d x-intercepts occur at $x = 1$ (cut) and $x = 5$ (touch).

Let the equation of the graph be $y = a(x-1)(x-5)^3$.

Substitute the y-intercept $(0, -20)$.

$\therefore\ -20 = a(-1)(-5)^2$
$\therefore\ -20 = -25a$
$\therefore a = \frac{4}{5}$

The equation is $y = \frac{4}{5}(x-1)(x-5)^2$.

5 a POI $(-6, -7)$

The equation is $y = -2(x+6)^3 - 7$.

b Under the translations, $y = x^3 \to y = (x-2)^3 - 4$.

y-intercept: let $x = 0$.

$y = (-2)^3 - 4$
$= -12$

The y-intercept is $(0, -12)$.

c POI $(-5, 2)$

Let the equation be $y = a(x+5)^3 + 2$.

Substitute the point $(0, -23)$.

$\therefore\ -23 = a(5)^3 + 2$
$\therefore\ -23 = 125a + 2$
$\therefore 125a = -25$
$\therefore a = -\frac{1}{5}$

The equation is $y = -\frac{1}{5}(x+5)^3 + 2$.

x-intercept: let $y = 0$.

$\therefore 0 = -\frac{1}{5}(x+5)^3 + 2$
$\therefore \frac{1}{5}(x+5)^3 = 2$
$\therefore (x+5)^3 = 10$
$\therefore x = -5 + \sqrt[3]{10}$

The x-intercept is $\left(\sqrt[3]{10} - 5, 0\right)$.

d $y = ax^3 + b$

Point $(1, 3) \Rightarrow 3 = a + b$ [1]
Point $(-2, 39) \Rightarrow 39 = -8a + b$ [2]

Subtract equation [2] from equation [1]:

$\therefore\ -36 = 9a$
$\therefore a = -4$

Substitute $a = -4$ in equation [1]

$\therefore 3 = -4 + b$
$\therefore b = 7$

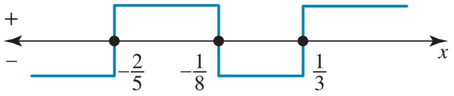
The equation is $y = -4x^3 + 7$, so the point of inflection is $(0, 7)$.

6 a To solve the inequation $(5x+2)(3x-1)(1+8x) \geq 0$ from the given graph, look to see where the graph lies above the x-axis.

Then describe the x-values of these intervals.

The solution is $-\frac{2}{5} \leq x \leq -\frac{1}{8}$ or $x \geq \frac{1}{3}$.

b The sign diagram is the ‘squashed’ graph.



c To solve the inequation $(5x+2)\,(3x-1)\,(1+8x) < 0$, identify the section of the sign diagram that lies below the x-axis.

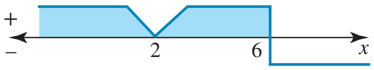
The solution is $x < -\frac{2}{5}$ or $-\frac{1}{8} < x < \frac{1}{3}$.

7 a $(x-2)^2(6-x) > 0$

Zeros: $x = 2$ (multiplicity 2), $x = 6$

Negative coefficient of x^3

Sign diagram:



Answer $x < 6, x \neq 2$

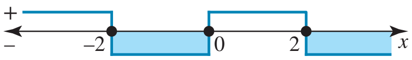
b $4x \leq x^3$

$\therefore 4x - x^3 \leq 0$
$\therefore x(4 - x^2) \leq 0$
$\therefore x(2-x)(2+x) \leq 0$

Zeros: $x = 0, x = 2, x = -2$

Negative coefficient of x^3

Sign diagram:



Answer $\{x: -2 \le x \le 0\} \cup \{x: x \ge 2\}$

c $2(x+4)^3 - 16 < 0$

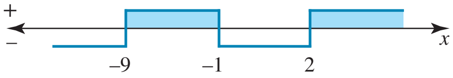
$\therefore (x+4)^3 < 8$

$\therefore x + 4 < 2$

$\therefore x < -2$

8 a $(x-2)(x+1)(x+9) \ge 0$

Zeros are $x = 2, x = -1, x = -9$; the coefficient of x^3 is positive.

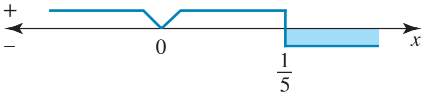


$\therefore\ -9 \le x \le -1$ or $x \ge 2$

b $x^2 - 5x^3 < 0$

$\therefore x^2(1-5x) < 0$

Zeros are $x = 0$ (multiplicity 2), $x = \frac{1}{5}$; the coefficient of x^3 is negative.



Answer: $x > \frac{1}{5}$

c $8(x-2)^3 - 1 > 0$

$\therefore (x-2)^3 > \frac{1}{8}$

$\therefore x - 2 > \frac{1}{2}$

$\therefore x > 2.5$

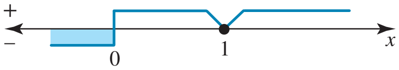
d $x^3 + x \le 2x^2$

$\therefore x^3 - 2x^2 + x \le 0$

$\therefore x(x^2 - 2x + 1) \le 0$

$\therefore x(x-1)^2 \le 0$

Zeros: $x = 0$ (cut), $x = 1$ (touch); the coefficient of x^3 is positive.



Answer: $x \le 0$ or $x = 1$

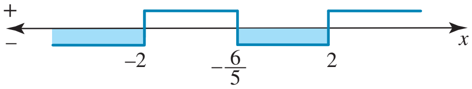
e $5x^3 + 6x^2 - 20x - 24 < 0$

$\therefore x^2(5x+6) - 4(5x+6) < 0$

$\therefore (5x+6)(x^2-4) < 0$

$\therefore (5x+6)(x-2)(x+2) < 0$

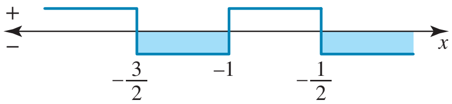
Zeros are $x = -\frac{6}{5}, x = 2, x = -2$; the coefficient of x^3 is positive.



Answer: $x < -2$ or $-\frac{6}{5} < x < 2$

f
$$2(x+1) - 8(x+1)^3 < 0$$
$$\therefore 2(x+1)\left[1 - 4(x+1)^2\right] < 0$$
$$\therefore 2(x+1)\,(1 - 2(x+1))\,(1 + 2(x+1)) < 0$$
$$\therefore 2(x+1)(-2x-1)(2x+3) < 0$$

Zero are $x = -1, x = -\frac{1}{2}, x = -\frac{3}{2}$; the coefficient of x^3 is negative.



Answer: $-\frac{3}{2} < x < -1$ or $x > -\frac{1}{2}$

9 $\{x: 3x^3 + 7 > 7x^2 + 3x\}$

$3x^3 + 7 > 7x^2 + 3x$

$\therefore 3x^3 - 7x^2 - 3x + 7 > 0$

$\therefore x^2(3x-7) - (3x-7) > 0$

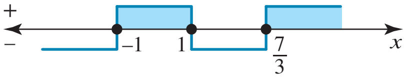
$\therefore (3x-7)(x^2-1) > 0$

$\therefore (3x-7)(x-1)(x+1) > 0$

Zeros: $x = \frac{7}{3}, x = 1, x = -1$

Positive cubic

Sign diagram:



Answer $\{x: -1 < x < 1\} \cup \left\{x: x > \frac{7}{3}\right\}$

10 a x-intercepts occur at $x = a$ (touch) and $x = b$ (cut).

Monic polynomial ⇒ the coefficient of x^3 is 1. After reflection in the x-axis, the coefficient of x^3 becomes −1.

The shape shows the coefficient is negative.

The equation is $y = -(x-a)^2(x-b)$.

b $\{x: P(x) \ge 0\}$ is $\{x: x \le b\}$.

c For both x-intercepts to be negative, the graph needs to be horizontally translated more than b units to the left.

d For both x-intercepts to be positive, the graph needs to be horizontally translated more than $|a|$ units to the right; that is, more than $-a$ units to the right.

11 The points of intersection of $y = (x+2)(x-1)^2$ and $y = -3x$ are found by solving $(x+2)(x-1)^2 = -3x$.

Expanding,

$(x+2)(x^2 - 2x + 1) = -3x$

$\therefore x^3 - 3x + 2 = -3x$

$\therefore x^3 + 2 = 0$

$\therefore x^3 = -2$

$\therefore x = -\sqrt[3]{2}$

$\therefore y = 3\sqrt[3]{2}$

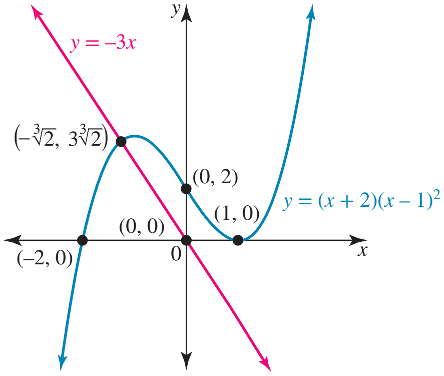
The point of intersection is $\left(-\sqrt[3]{2}, 3\sqrt[3]{2}\right)$.

Graphs: cubic $y = (x+2)(x-1)^2$

x-intercepts at $x = -2$ (cut), $x = 1$ (touch)

y-intercept $(0, 2)$

Linear $y = -3x$ passes through $(0, 0), (-1, 3)$.



12 At the intersection of $y = 4 - x^2$ and $y = 4x - x^3$,

$$4 - x^2 = 4x - x^3$$

$$\therefore x^3 - x^2 - 4x + 4 = 0$$

$$\therefore x^2(x-1) - 4(x-1) = 0$$

$$\therefore (x-1)(x^2-4) = 0$$

$$\therefore (x-1)(x-2)(x+2) = 0$$

$$\therefore x = 1,\ x = 2,\ x = -2$$

Substitute in $y = 4 - x^2$.

The points of intersection are $(1, 3), (2, 0), (-2, 0)$

Graphs: cubic $y = 4x - x^3$

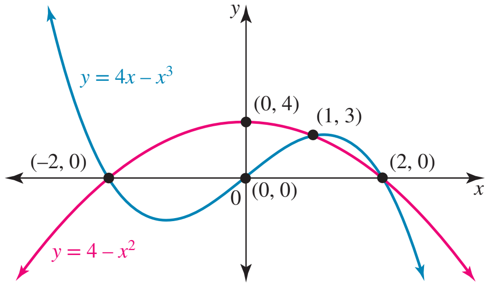
$\therefore y = x(4 - x^2)$

$\therefore y = x(x-2)(x+2)$

x-intercepts at $x = 0, x = \pm 2$

Quadratic $y = 4 - x^2$

Maximum turning point $(0, 4)$, x-intercepts at $x = \pm 2$



13 a i $y = 2x^3$ and $y = x^2$

At intersection, $2x^3 = x^2$.

$$\therefore 2x^3 - x^2 = 0$$

$$\therefore x^2(2x-1) = 0$$

$$\therefore x = 0,\ x = \frac{1}{2}$$

Substituting $x = 0$ in $y = x^2$ gives $y = 0$.

Substituting $x = \frac{1}{2}$ in $y = x^2$ gives $y = \frac{1}{4}$.

The points of intersection are $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$.

ii $y = 2x^3$ and $y = x - 1$

At intersection, $2x^3 = x - 1$.

$\therefore 2x^3 - x + 1 = 0$

Let $p(x) = 2x^3 - x + 1$.

$p(-1) = -2 + 1 + 1 = 0$

$\therefore (x + 1)$ is a factor.

$\therefore 2x^3 - x + 1 = (x+1)(2x^2 - 2x + 1)$

Consider the quadratic factor $2x^2 - 2x + 1$.

$$\Delta = (-2)^2 - 4(2)(1)$$
$$= 4 - 8$$
$$< 0$$

There are no real linear factors of this quadratic.

$$\therefore 2x^3 - x + 1 = 0 \Rightarrow (x+1)(2x^2 - 2x + 1) = 0$$
$$\therefore x = -1$$

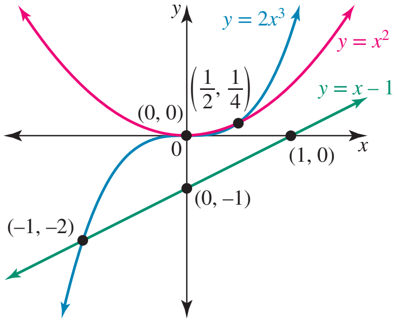
Substituting $x = -1$ in $y = x - 1$ gives $y = -2$.

The point of intersection is $(-1, -2)$.

b The cubic graph of $y = 2x^3$ has POI $(0, 0)$ and passes through $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $(-1, -2)$.

The parabola graph of $y = x^2$ has a minimum turning point at $(0, 0)$ and passes through $\left(\frac{1}{2}, \frac{1}{4}\right)$.

The linear graph of $y = x - 1$ has x-intercept $(1, 0)$ and y-intercept $(0, -1)$, and passes through $(-1, -2)$.



c $2x^3 - x^2 \leq 0$

$\therefore x^2(2x - 1) \leq 0$

Since $x^2 \geq 0$, either $x = 0$ or $(2x - 1) \leq 0$.

$\therefore x = 0$ or $x \leq 0.5$

$\therefore x \leq 0.5$

(Alternatively, draw a sign diagram to solve the inequation.)

Since $2x^3 - x^2 \leq 0 \Rightarrow 2x^3 \leq x^2$, the inequation can be solved by considering where the graph of $y = 2x^3$ lies on or below the graph of $y = x^2$. From the diagram this occurs when $x \leq 0.5$, so $x \leq 0.5$ is the solution to the inequation.

14 a $x^3 + 2x - 5 = 0$

Rearranging, $x^3 = -2x + 5$.

The solutions to the equation are the x-coordinates of the points of intersection of the graphs of $y = x^3$ and $y = -2x + 5$.

Since the line $y = -2x + 5$ has a negative gradient, it will intersect the graph of the cubic $y = x^3$ exactly once.

The equation $x^3 + 2x - 5 = 0$ has one solution.

b $x^3 + 3x^2 - 4x = 0$

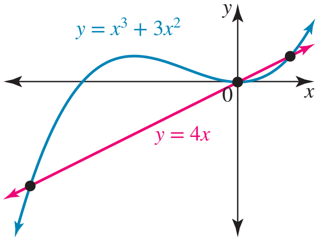
Rearranging, the equation can be written as $x^3 + 3x^2 = 4x$.

The solutions to the equation are the x-coordinates of the points of intersection of the graphs of $y = x^3 + 3x^2$ and $y = 4x$.

The cubic graph: $y = x^3 + 3x^2 \Rightarrow y = x^2(x + 3)$ touches the x-axis at $x = 0$ and cuts the x-axis at $x = -3$.

The linear graph: $y = 4x$ passes through the origin with a positive gradient. A second point on the graph is $(1, 4)$.

For the cubic graph, when $x = 1, y = 4$, so both graphs pass through the origin and the point $(1, 4)$.

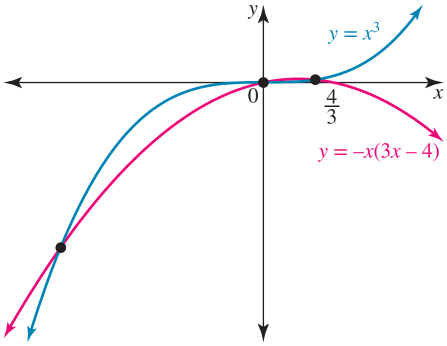


As there are 3 points of intersection, there are 3 solutions to the equation $x^3 + 3x^2 - 4x = 0$.

c $x^3 + 3x^2 - 4x = 0$

One way to interpret the equation as the intersection of a cubic and a quadratic is to rearrange the equation to the form $x^3 = -3x^2 + 4x$.

Graphing $y = x^3$ and $y = -3x^2 + 4x = -x(3x - 4)$ shows there are 3 points of intersection.



There are 3 solutions to the equation $x^3 + 3x^2 - 4x = 0$.

d $x^3 + 3x^2 - 4x = 0$

$\therefore x(x^2 + 3x - 4) = 0$

$\therefore x(x + 4)(x - 1) = 0$

$\therefore x = 0,\ x = -4,\ x = 1$

15 a At the intersection of $y = x^3$ with $y = 3x + 2$,

$x^3 = 3x + 2$

$\therefore x^3 - 3x - 2 = 0$

Let $p(x) = x^3 - 3x - 2$.

$p(-1) = -1 + 3 - 2 = 0$

$\therefore (x + 1)$ is a factor.

$$\begin{aligned} x^3 - 3x - 2 &= (x + 1)(x^2 - x - 2) \\ &= (x + 1)(x + 1)(x - 2) \\ &= (x + 1)^2(x - 2) \end{aligned}$$

$\therefore x^3 - 3x - 2 = 0 \Rightarrow (x + 1)^2(x - 2) = 0$

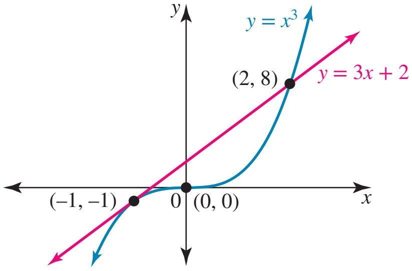
$\therefore x = -1,\ x = 2$

Since $x = -1$ is a root of multiplicity 2, the two graphs touch at $x = -1$.

When $x = -1$, $y = x^3 = -1$. The line is a tangent to the cubic curve at the point $(-1, -1)$.

b When $x = 2$, $y = x^3 = 8$, The line cuts the curve at $(2, 8)$.

c



d $y = mx + 2$ is a line through $(0, 2)$ as is $y = 3x + 2$. Hence, if $m = 3$, there will be two points of intersection.

For positive gradient, if the line through $(0, 2)$ is steeper than $y = 3x + 2$, it will no longer be a tangent to the cubic graph, so there will be three points of intersection.

However, if the line is less steep than $y = 3x + 2$, there would only be one point of intersection.

For negative gradient, the line through $(0, 2)$ would only have one point of intersection with the cubic graph.

For zero gradient, the line through $(0, 2)$ is horizontal and would only intersect the cubic graph once.

Answer: One point of intersection if $m < 3$, two points of intersection if $m = 3$ and three points of intersection if $m > 3$.

16 Let $y = ax^3 + bx^2 + cx + d$.

$(0, 3), (1, 4), (-1, 8), (-2, 7)$

$(0, 3) \Rightarrow 3 = d$

$\therefore y = ax^3 + bx^2 + cx + 3$

Substitute other points to form simultaneous equations:

$(1, 4) \Rightarrow 4 = a + b + c + 3$

$\therefore a + b + c = 1 \quad [1]$

$(-1, 8) \Rightarrow 8 = -a + b - c + 3$

$\therefore -a + b - c = 5 \quad [2]$

$(-2, 7) \Rightarrow 7 = -8a + 4b - 2c + 3$

$\therefore -8a + 4b - 2c = 4$

$\therefore 4a - 2b + c = -2 \quad [3]$

[1] + [2]

$2b = 6$

$\therefore b = 3$

[2] + [3]

$3a - b = 3$

$\therefore 3a - 3 = 3$

$\therefore a = 2$

Substitute $a = 2, b = 3$ in [1]:

$2 + 3 + c = 1$

$\therefore c = -4$

The equation is $y = 2x^3 + 3x^2 - 4x + 3$.

17 $y = x^3 + ax^2 + bx + 9$

a Turning point and x-intercept $(3, 0)$, $\Rightarrow (x - 3)^2$ is a factor.

b
$$\begin{aligned} x^3 + ax^2 + bx + 9 &= (x - 3)^2(cx + d) \\ &= (x^2 - 6x + 9)(cx + d) \\ &= (x^2 - 6x + 9)(x + 1) \end{aligned}$$

$\therefore x^3 + ax^2 + bx + 9 = (x - 3)^2(x + 1)$

The other x-intercept is $(-1, 0)$.

c Expanding,

$$\begin{aligned} x^3 + ax^2 + bx + 9 &= (x - 3)^2(x + 1) \\ &= (x^2 - 6x + 9)(x + 1) \\ &= x^3 + x^2 - 6x^2 - 6x + 9x + 9 \\ &= x^3 - 5x^2 + 3x + 9 \end{aligned}$$

Equating coefficients of x^2: $a = -5$

Equating coefficients of x: $b = 3$

18 $y = (x + a)^3 + b$

a Substitute the given points into the equation.

$(0, 0) \Rightarrow 0 = a^3 + b \quad [1]$

$(1, 7) \Rightarrow 7 = (1 + a)^3 + b \quad [2]$

$(2, 26) \Rightarrow 26 = (2 + a)^3 + b \quad [3]$

From equation [1], $b = -a^3$. Substitute this in each of the other two equations.

Equation [2]:

$7 = (1 + a)^3 - a^3$

$\therefore 7 = 1 + 3a + 3a^2 + a^3 - a^3$

$\therefore 6 = 3a + 3a^2$

$\therefore a^2 + a - 2 = 0$

$\therefore (a + 2)(a - 1) = 0$

$\therefore a = -2,\ a = 1$

Equation [3]:

$26 = (2 + a)^3 - a^3$

$\therefore 26 = 8 + 12a + 6a^2 + a^3 - a^3$

$\therefore 18 = 12a + 6a^2$

$\therefore a^2 + 2a - 3 = 0$

$\therefore (a + 3)(a - 1) = 0$

$\therefore a = -3,\ a = 1$

The consistent value for a is $a = 1$.

If $a = 1$, then $b = -a^3 = -1$.

Answer: $a = 1$ and $b = -1$

b The graph has the equation $y = (x+1)^3 - 1$.

At the intersection with the line $y = x$,

$$(x+1)^3 - 1 = x$$

$$\therefore x^3 + 3x^2 + 3x + 1 - 1 - x = 0$$

$$\therefore x^3 + 3x^2 + 2x = 0$$

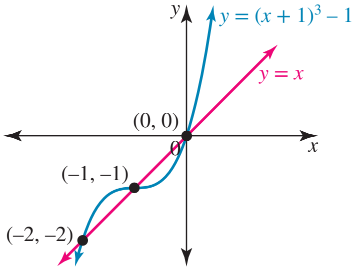
$$\therefore x(x^2 + 3x + 2) = 0$$

$$\therefore x(x+1)(x+2) = 0$$

$$\therefore x = 0, x = -1, x = -2$$

Since the points lie on $y = x$, the points of intersection are $(0, 0), (-1, -1), (-2, -2)$.

c $y = (x+1)^3 - 1$ has POI $(-1, -1)$, and its graph and the graph of $y = x$ must intersect at $(0, 0), (-1, -1), (-2, -2)$.



d $x^3 + 3x^2 + 2x > 0$

From part **b**, when the graph of $y = (x+1)^3 - 1$ intersects with the line $y = x$, $(x+1)^3 - 1 = x \Rightarrow x^3 + 3x^2 + 2x = 0$.

Therefore, $x^3 + 3x^2 + 2x > 0 \Rightarrow (x+1)^3 - 1 > x \Rightarrow$ the graph of $y = (x+1)^3 - 1$ lies above the line $y = x$.

Using the diagram in part **c**, this occurs for $-2 < x < -1$ and for $x > 0$.

The solution set for the inequation is $\{x : -2 < x < -1\} \cup \{x : x > 0\}$.

19 a The given information about the cubic polynomial is the three points $(1, 0), (2, 0), (0, 12)$. However, the equation $y = ax^3 + bx^2 + cx + d$ contains 4 unknowns, so three points are insufficient to completely determine the equation.

b $y = ax^3 + bx^2 + cx + d$

Substitute the given y-intercept $(0, 12)$.

$\therefore 12 = d$

$\therefore y = ax^3 + bx^2 + cx + 12$

Substitute the given x-intercepts.

$(1, 0) \Rightarrow 0 = a + b + c + 12$

$\therefore a + b + c = -12 \qquad [1]$

$(2, 0) \Rightarrow 0 = a(8) + b(4) + c(2) + 12$

$\therefore 8a + 4b + 2c = -12$

$\therefore 4a + 2b + c = -6 \qquad [2]$

Subtract equation [1] from equation [2]:

$\therefore 3a + b = 6$

$\therefore b = 6 - 3a$

Substitute $b = 6 - 3a$ in equation [1]:

$\therefore a + 6 - 3a + c = -12$

$\therefore c = 2a - 18$

Hence, $y = ax^3 + (6 - 3a)x^2 + (2a - 18)x + 12$, as required.

c First curve: $a = 1 \Rightarrow y = x^3 + 3x^2 - 16x + 12$

As $(1, 0), (2, 0)$ are x-intercepts, $(x - 1)$ and $(x - 2)$ are factors.

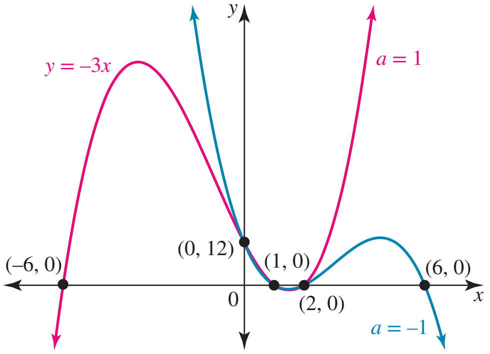
As $(x - 1)(x - 2) = x^2 - 3x + 2$, then $x^3 + 3x^2 - 16x + 12 = (x^2 - 3x + 2)(x + 6)$. The third x-intercept is $(-6, 0)$.

Second curve: $a = -1 \Rightarrow y = -x^3 + 9x^2 - 20x + 12$

$(1, 0), (2, 0)$ are also x-intercepts for the second curve, so it has $(x - 1)$ and $(x - 2)$ as factors.

$\therefore -x^3 + 9x^2 - 20x + 12 = (x^2 - 3x + 2)(-x + 6)$

The third x-intercept is $(6, 0)$.



d If the coefficient of x^2 is 15, then $6 - 3a = 15$.

$\therefore 3a = -9$

$\therefore a = -3$

$\therefore y = -3x^3 + 15x^2 - 24x + 12$

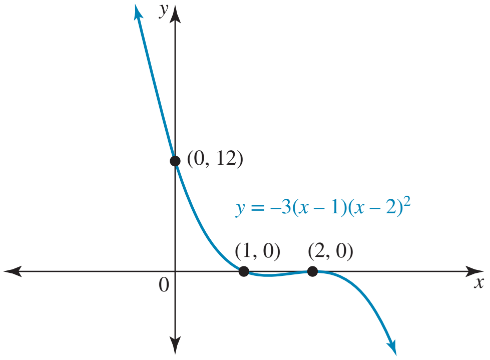
$= (x^2 - 3x + 2)(-3x + 6)$

$= -3(x^2 - 3x + 2)(x - 2)$

$= -3(x - 1)(x - 2)(x - 2)$

$\therefore y = -3(x - 1)(x - 2)^2$

The x-intercepts are $(1, 0)$ (cut) and $(2, 0)$ (touch).



20 a Use the Graph & Tab menu to draw the two graphs; then obtain the points of intersection from Analysis → G-Solve → Intersect.

The graphs of $y = (x+1)^3$ and $y = 4x + 3$ intersect at $(-3.11, -9.46), (-0.75, 0.02), (0.86, 6.44)$.

b At intersection, $(x+1)^3 = 4x + 3$, so the x-coordinates of the points of intersection are the solutions to this equation.

Tap Interactive → Transformation → Expand (or expand by hand) to obtain the equation in the form $x^3 + 3x^2 - x - 2 = 0$.

c The x-coordinates of the points of intersection of $y = (x+1)^3$ and $y = 4x + 3$ represent the x-intercepts of the graph of $y = x^3 + 3x^2 - x - 2$

4.5 Exam questions

1 The graph is a cubic shape of the form $y = a(x - h)^3 + k$.

The point of inflection is $(-1, -5)$.

$\therefore y = a(x + 1)^3 - 5$

y-intercept $(0, -3)$

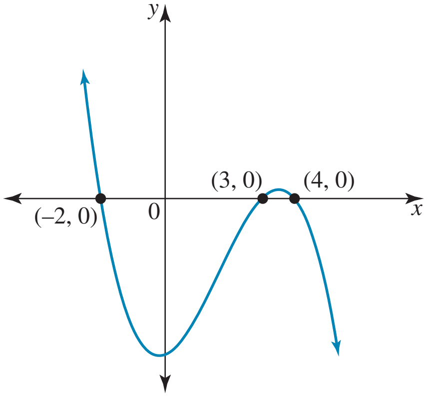
$-3 = a(x+1)^3 - 5$

$a = 2$

The equation is $y = 2(x+1)^3 - 5$.

The correct answer is **B**.

2 The graph corresponding to the sign diagram is shown below.



It is a negative cubic graph with x-intercepts at $(-2, 0), (3, 0), (4, 0)$.

Therefore, the equation is $y = -(x+2)(x-3)(x-4)$.

The correct answer is **D**.

3 $y = a(x-b)(x-c)(x-d)$

x-intercepts $(-2, 0), (0, 0), (1, 0)$

$y = ax(x-1)(x+2)$ [1 mark]

Find a using $(2, 16)$:

$16 = 2a(1)(4)$

$16 = 8a$

$a = 2$ [1 mark]

$\therefore y = 2x(x-1)(x+2)$

$y = 2x\left(x^2 + x - 2\right)$

$y = 2x^3 + 2x^2 - 4x$ [1 mark]

4.6 Cubic models and applications

4.6 Exercise

1 a The sum of the edges is 6 m.

$8x + 4h = 6$

$\therefore 2h = 3 - 4x$

$\therefore h = \dfrac{3-4x}{2}$

b $V = x^2h$

$\therefore V = x^2\left(\dfrac{3-4x}{2}\right)$

$\therefore V = \dfrac{3x^2 - 4x^3}{2}$

$\therefore V = 1.5x^2 - 2x^3$

c $x > 0, h > 0$

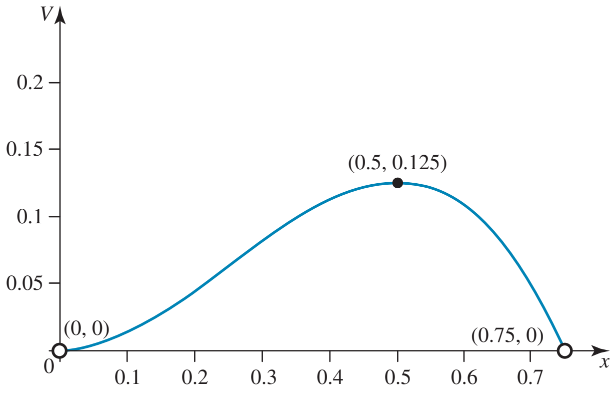
$\therefore \dfrac{3-4x}{2} > 0$

$\therefore 3 - 4x > 0$

$\therefore x < \dfrac{3}{4}$

Therefore, the restriction on the domain is $0 < x < \dfrac{3}{4}$.

d $V = x^2(1.5 - 2x)$. The graph has y-intercepts at $x = 0$ (touch) and $x = 0.75$ (cut). It has the shape of a negative cubic.



e The greatest volume occurs when $x = 0.5$ and therefore $h = \dfrac{1}{2}$.

Therefore, the container with greatest volume is a cube of edge 0.5 m.

2 a $l = 20 - 2x, w = 12 - 2x$ and $h = x$

Since volume is $V = lwh$, $V = (20 - 2x)(12 - 2x)x$

b $x > 0$

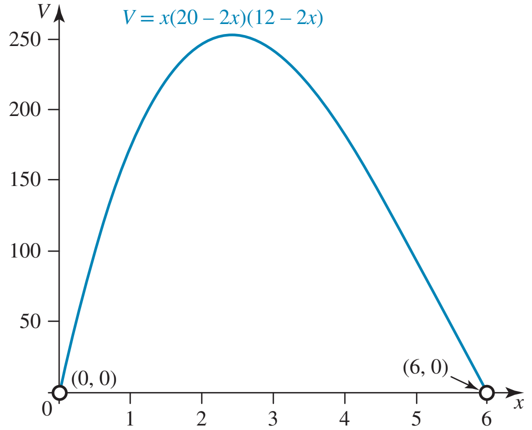
$l > 0 \Rightarrow 20 - 2x \geq 0$ $w > 0 \Rightarrow 12 - 2x > 0$

$\therefore x < 10$ $\therefore x < 6$

Therefore, to satisfy all three conditions, $0 < x < 6$.

c $V = (20 - 2x)(12 - 2x)x, 0 < x < 6$

x-intercepts at $x = 10, x = 6, x = 0$, but since $0 < x < 6$, the graph won't reach $x = 10$; its shape is a positive cubic.



d Turning points at $x = 2.43$ and $x = 8.24$. The first turning point is a maximum, and the graph doesn't reach the second turning point (which would be a minimum) due to the domain restriction.

Therefore, the greatest volume occurs when $x = 2.43$.

$l = 20 - 2(2.43)$ $w = 12 - 2(2.43)$

$= 15.14$ $= 7.14$

The box has length 15.14 cm, width 7.14 cm, height 2.43 cm.

The greatest volume is $15.14 \times 7.14 \times 2.43 = 263\text{ cm}^3$ to the nearest whole number.

3 a Cost model: $C = x^3 + 100x + 2000$

Consider the case when 5 sculptures are produced.

$C(5) = 5^3 + 100(5) + 2000$

$= 2625$

It costs the artist \$2625 to produce 5 sculptures. The artist earns $500 \times 5 = 2500$ dollars from the sale of the 5 sculptures.

This results in a loss of \$$(2625 - 2500) = \$125$.

Consider the case when 6 sculptures are produced.

$$C(6) = 6^3 + 100(6) + 2000$$
$$= 2816$$

It costs the artist \$2816 to produce 6 sculptures. The artist earns $500 \times 6 = 3000$ dollars from the sale of the 6 sculptures.

This results in a profit of \$$(3000 - 2816) = \$184$.

b The artist earns \$500x from the sale of x sculptures and the cost of production is given by $C = x^3 + 100x + 2000$.

The profit model is $P = 500x - (x^3 + 100x + 2000)$

$\therefore P = -x^3 + 400x - 2000$ as required.

c As $x \to \infty, P \to -\infty$. Thus, for large numbers of sculptures, the cost of production outweighs the revenue from their sales.

d i If 16 sculptures are produced,

$$P(16) = -(16)^3 + 400(16) - 2000$$
$$= 304$$

A profit of \$304 is made.

ii If 17 sculptures are produced,

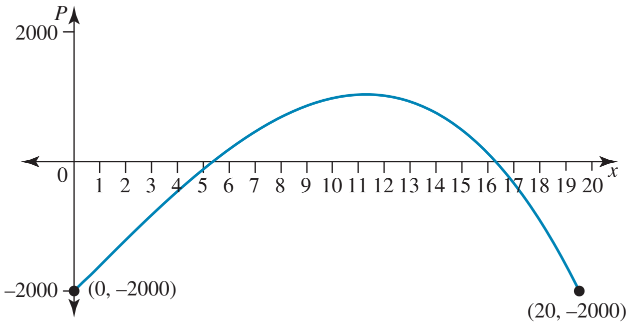
$$P(17) = -(17)^3 + 400(17) - 2000$$
$$= -113$$

A loss of \$113 is made.

e As $p(5) < 0$ and $p(6) > 0$, the graph of the profit model will have an $x =$ intercept in the interval $x \in (5, 6)$.

As $p(16) > 0$ and $p(17) < 0$, the graph of the profit model will have another x-intercept in the interval $x \in (16, 17)$.

$$\text{End points}: \ p(0) = -2000 \text{ and } p(20)$$
$$= -(20)^3 + 400(20) - 2000$$
$$= -2000.$$



f For a profit, $p > 0$. A profit is made if between 6 and 16 sculptures are produced.

4 $N = 54 + 23t + t^3$

a When $t = 0, N = 54$.

At 9 am there were initially 54 bacteria.

b When the number has doubled, $N = 108$.

$\therefore 54 + 23t + t^3 = 108$

$\therefore t^3 + 23t - 54 = 0$

Let $p(t) = t^3 + 23t - 54$.

$p(2) = 8 + 46 - 54 = 0 \Rightarrow (t - 2)$ is a factor.

$\therefore t^3 + 23t - 54 = (t - 2)(t^2 + 2t + 27) = 0$

$\therefore t = 2 \text{ or } t^2 + 2t + 27 = 0$

Consider $t^2 + 2t + 27 = 0$

$$\Delta = (2)^2 - 4 \times 1 \times 27$$
$$= -104$$

There are no real solutions.

$\therefore t = 2$.

The bacteria double their initial number after 2 hours.

c At 1 pm, $t = 4$ since time is measured from 9 am.

$$N = 54 + 23 \times 4 + 4^3$$
$$= 210$$

There are 210 bacteria at 1 pm.

d When $N = 750$,

$54 + 23t + t^3 = 750$

$\therefore t^3 + 23t - 696 = 0$

Let $p(t) = t^3 + 23t - 696$.

Trial and error using factors of 696:

$p(4) \neq 0, p(6) \neq 0, P(8) = 0$

$\therefore (t - 8)$ is a factor

$$t^3 + 23t - 696 = (t - 8)(t^2 + 8t + 87)$$

$\therefore (t - 8)(t^2 + 8t + 87) = 0$

$\therefore t = 8 \text{ or } t^2 + 8t + 87 = 0$

Consider $t^2 + 8t + 87 = 0$.

$\Delta = 64 - 4 \times 1 \times 87$

$\therefore \Delta < 0$

No real solutions

$\therefore t = 8$

The number of bacteria reaches 750 after 8 hours after 9 am. The time is 5 pm.

5 $y = ax^2(x - b)$

a Given information $\Rightarrow (6, 0)$ is an x-intercept and $(4, 1)$ is the maximum turning point.

The curve $y = ax^2(x - b)$ has x-intercepts at $x = 0$ and $x = b$, which means that $b = 6$.

Substitute the point $(4, 1)$ in $y = ax^2(x - 6)$.

$\therefore 1 = a(4)^2(4 - 6)$

$\therefore 1 = -32a$

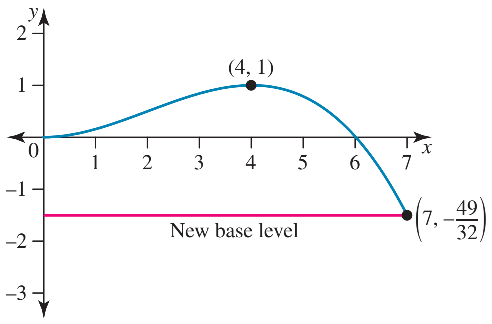
$\therefore a = -\dfrac{1}{32}$

The equation of the bounding curve is $y = -\dfrac{1}{32}x^2(x - 6)$.

b If $x = 7$, then

$$y = -\frac{1}{32}(7)^2(1)$$
$$= -\frac{49}{32}$$

The new base level ends at $\left(7, -\dfrac{49}{32}\right)$.



The greatest height will now be $\dfrac{49}{32} + 1 = \dfrac{81}{32}$ km above the base level.

6 Let the number be x with $x \in Z$.

It is required that $(x+5)^2 - (x+1)^3 > 22$.

$$\therefore x^2 + 10x + 25 - (x^3 + 3x^2 + 3x + 1) > 22$$
$$\therefore -x^3 - 2x^2 + 7x + 24 > 22$$
$$\therefore -x^3 - 2x^2 + 7x + 2 > 0$$
$$\therefore x^3 + 2x^2 - 7x - 2 < 0$$

Let $p(x) = x^3 + 2x^2 - 7x - 2$.

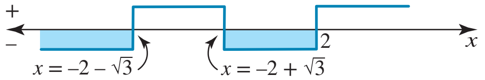
$p(2) = 8 + 8 - 14 - 2 = 0$

$\therefore (x-2)$ is a factor.

$$\begin{aligned}\therefore x^3 + 2x^2 - 7x - 2 &= (x-2)(x^2+4x+1)\\ &= (x-2)\left[(x^2+4x+4)-4+1\right]\\ &= (x-2)\left[(x+2)^2-3\right]\\ &= (x-2)\left(x+2+\sqrt{3}\right)\left(x+2-\sqrt{3}\right)\end{aligned}$$

$$x^3 + 2x^2 - 7x - 2 < 0$$
$$\Rightarrow (x-2)\left(x+2+\sqrt{3}\right)\left(x+2-\sqrt{3}\right) < 0$$

Zeros are $x = 2, x = -2-\sqrt{3}, x = -2+\sqrt{3}$; the coefficient of x^3 is positive.



$\therefore x < -2-\sqrt{3}$ or $-2+\sqrt{3} < x < 2$

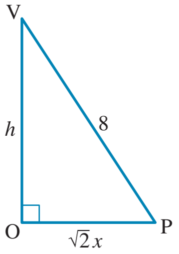
However, $x \in Z$. The solution intervals are approximately $x < -3.732$ or $-0.268 < x < 2$. The smallest positive integer that lies in the solution interval is 1 and the largest negative integer is −4.

7 **a** Using Pythagoras' theorem in the right-angled triangle MNP with l metres the length of the diagonal MP:

$$\begin{aligned} l^2 &= (2x)^2 + (2x)^2\\ &= 8x^2\end{aligned}$$
$$\therefore l = 2\sqrt{2}x$$

b Consider the right-angled triangle VOP.

Since $\text{OP} = \frac{1}{2}l$, $\text{OP} = \sqrt{2}x$.



Using Pythagoras' theorem,

$$h^2 + \left(\sqrt{2}x\right)^2 = 8^2$$
$$\therefore 2x^2 = 64 - h^2$$

c The area of the square base is given by $A = (2x)(2x) = 4x^2$.

Substitute $2x^2 = 64 - h^2$.

$$\therefore A = 2 \times (64 - h^2)$$
$$\therefore A = 128 - 2h^2$$

Volume: $V = \frac{1}{3}Ah$

$$\therefore V = \frac{1}{3}(128 - 2h^2)h$$
$$\therefore V = \frac{1}{3}(128h - 2h^3)$$

d **i** If $h = 3$, $V = \frac{1}{3}(128 \times 3 - 2 \times 27) = 110$.

The volume is 110 m^3.

ii Since the volume cannot be negative or zero,

$$\frac{1}{3}(128h - 2h^3) > 0$$
$$\therefore 128h - 2h^3 > 0$$
$$\therefore 2h(64 - h^2) > 0$$

Since the height cannot be negative, $64 - h^2 > 0$.

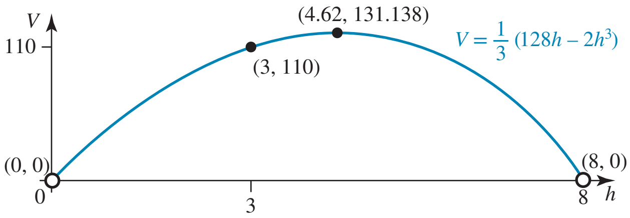
$$\therefore h^2 < 64$$
$$\therefore 0 < h < 8$$

Note: For practical purposes, the allowable values for the height would need further restriction.

e $V = \frac{1}{3}h(64 - h^2),\ 0 < h < 8$

$$\therefore V = \frac{1}{3}h(8-h)(8+h)$$

Horizontal axis intercepts at $h = 0, h = 8, h = -8$ (not applicable); negative cubic shape



The maximum turning point is (4.6188, 131.379 26), so the height for which the volume is greatest is 4.62 metres.

f We know the relationship between the height, h, and the length of the base, $2x$, is given by $2x^2 = 64 - h^2$.

Substitute $h = 4.6188$.

$$\therefore 2x^2 = 64 - (4.6188)^2$$

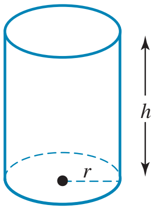
Solve for x using CAS technology.

$\therefore x = 4.6188$ (rejecting the negative solution).

The maximum volume occurs when $h = x$.

The length of the base is $2x$. Thus, the greatest volume occurs when the height is half the length of the base.

8



a The surface area of the cylinder open at the top is given by $SA = 2\pi rh + \pi r^2$.

$$\therefore 400\pi = 2\pi rh + \pi r^2$$
$$\therefore 400 = 2rh + r^2$$
$$\therefore 400 - r^2 = 2rh$$
$$\therefore h = \frac{400 - r^2}{2r}$$

b The volume formula for a cylinder is $V = \pi r^2 h$.

Substitute $h = \frac{400 - r^2}{2r}$ from part **a**:

$$\therefore V = \pi r^2\left(\frac{400 - r^2}{2r}\right)$$
$$\therefore V = \frac{\pi r^2(400 - r^2)}{2r}$$
$$\therefore V = \frac{\pi r(400 - r^2)}{2}$$

$\therefore V = \dfrac{400\pi r}{2} - \dfrac{\pi r^3}{2}$

$\therefore V = 200\pi r - \dfrac{1}{2}\pi r^3$

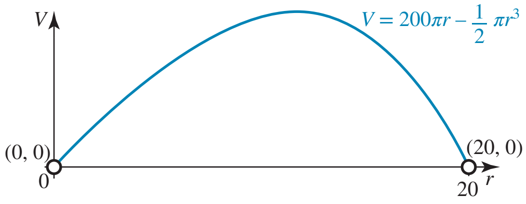
c $r > 0, h > 0, V > 0$

$$\frac{\pi r(400 - r^2)}{2} > 0$$
$$\therefore r(400 - r^2) > 0$$
$$\therefore 400 - r^2 > 0$$
$$\therefore r^2 < 400$$
$$\therefore 0 < r < 20$$

d $V = \dfrac{\pi r(400 - r^2)}{2}, 0 < r < 20$

$\therefore V = \dfrac{\pi}{2}r(20 - r)(20 + r)$

Horizontal axis intercepts occur at $r = 0, r = 20, r = -20$ (not applicable); negative cubic shape.



e $V = 396\pi$

$$\therefore 396\pi = 200\pi r - \frac{1}{2}\pi r^3$$
$$\therefore 396 = 200r - \frac{1}{2}r^3$$
$$\therefore 792 = 400r - r^3$$

$\therefore r^3 - 400r + 792 = 0$

Let $p(r) = r^3 - 400r + 792$.

Testing factors of 792, $p(2) = 8 - 800 + 792 = 0$.

$\therefore (r - 2)$ is a factor.

$\therefore r^3 - 400r + 792 = (r - 2)(r^2 + 2r - 396)$

Hence,

$$(r - 2)(r^2 + 2r - 396) = 0$$
$$\therefore r = 2 \text{ or } r^2 + 2r - 396 = 0$$
$$\therefore \left(r^2 + 2r + 1\right) - 1 - 396 = 0$$
$$\therefore (r + 1)^2 - 397 = 0$$
$$\therefore (r + 1)^2 = 397$$
$$\therefore r + 1 = \pm\sqrt{397}$$
$$\therefore r = -\sqrt{397} - 1, r = \sqrt{397} - 1$$

As $0 \le r \le 20$, $r = 2, r = \sqrt{397} - 1 \simeq 18.925$.

Height: $h = \dfrac{400 - r^2}{2r}$

If $r = 2$, $h = \dfrac{400 - 4}{4} = 99$.

If $r = 18.925$, $h = \dfrac{400 - 18.925^2}{2 \times 18.925} = 1.10$.

Both the container with height 99 cm and base radius 2 cm and the container with height 1.1 cm and base radius 18.9 cm have a volume of 396π cm^3.

f Use CAS technology to sketch $y = 200\pi x - \dfrac{1}{2}\pi x^3$ and obtain the coordinates of the maximum turning point.

The maximum turning point is $(11.547\,885, 4836.7983)$, so the maximum volume is 4836.8 cm^3.

For the maximum volume, $r = 11.547\,885$. Substitute this into the relationship between the height, h, and the radius, r, which is $h = \dfrac{400 - r^2}{2r}$.

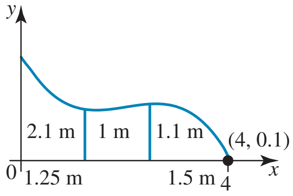
$\therefore h = \dfrac{400 - (11.547\,885)^2}{2 \times 11.547\,885}$

$\therefore h = 11.5$ to 1 decimal place.

The dimensions of the cylinder with greatest volume are a base radius of 11.5 cm and a height of 11.5 cm, to 1 decimal place.

The maximum volume is $\dfrac{8000\sqrt{3}\pi}{9}$ cm^3 or approximately 4837 cm^3.

9



a Known points on the curve are $(0, 2.1), (1.25, 1), (2.5, 1.1), (4, 0.1)$.

b $h = ax^3 + bx^2 + cx + d$

Substitute $(0, 2.1)$.

$\therefore 2.1 = d$

c $h = ax^3 + bx^2 + cx + 2.1$

Substitute $(1.25, 1)$.

$$\therefore\ 1 = a(1.25)^3 + b(1.25)^2 + c(1.25) + 2.1$$
$$\therefore\ \left(\frac{5}{4}\right)^3 a + \left(\frac{5}{4}\right)^2 b + \left(\frac{5}{4}\right) c = -1.1$$
$$\therefore\ \frac{125}{64}a + \frac{25}{16}b + \frac{5}{4}c = -1.1$$
$$\therefore\ 125a + 100b + 80c = -70.4 \qquad [1]$$

Substitute $(2.5, 1.1)$.

$$\therefore\ 1.1 = a\left(\frac{5}{2}\right)^3 + b\left(\frac{5}{2}\right)^2 + c\left(\frac{5}{2}\right) + 2.1$$
$$\therefore\ \frac{125}{8}a + \frac{25}{4}b + \frac{5}{2}c = -1$$
$$\therefore\ 125a + 50b + 20c = -8 \qquad [2]$$

Substitute $(4, 0.1)$.

$$\therefore\ 0.1 = a(4)^3 + b(4)^2 + c(4) + 2.1$$
$$\therefore\ 64a + 16b + 4c = -2 \qquad [3]$$

The system of simultaneous equations is:

$$125a + 100b + 80c = -70.4 \qquad [1]$$
$$125a + 50b + 20c = -8 \qquad [2]$$
$$64a + 16b + 4c = -2 \qquad [3]$$

d The system of simultaneous equations can be solved using CAS technology. This gives $a = -0.164$, $b = 1$, $c = -1.872$.

The equation of the slide is $y = ax^3 + bx^2 + cx + d$. It is already known that $d = 2.1$.

The equation is $y = -0.164x^3 + x^2 - 1.872x + 2.1$.

10 $T(t) = -0.000\,05(t - 6)^3 + 9.85$

a Time is measured from 1994. Therefore, $t = 3$ for 1997.

$$T(3) = -0.000\,05(-3)^3 + 9.85$$
$$= 9.851\,35$$

The model predicts 9.85 seconds, to 2 decimal places; the actual time was 9.86 seconds.

For 2014, $t = 20$.

$$T(20) = -0.000\,05(14)^3 + 9.85$$
$$= 9.7128$$

The model predicts 9.71 seconds, to 2 decimal places; the actual time was 9.72 seconds.

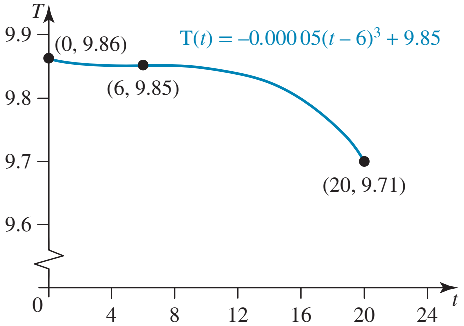
The predictions and the actual times agree to 1 decimal place.

b $T(t) = -0.000\,05(t-6)^3 + 9.85$ for $t \in [0, 20]$

POI $(6, 9.85)$

End points: If $t = 0, T = -0.000\,05(-6)^3 + 9.85 = 9.8608$.

Left end point $(0, 9.86)$, right end point $(20, 9.71)$



c For 2022, $t = 28$.

$$T(28) = -0.000\,05(22)^3 + 9.85$$
$$= 9.3176$$

The model predicts 9.32 seconds, which seems unlikely, although not impossible. The graph shows the time taken starts to decrease quite steeply after 2014, so its predictions are probably not accurate.

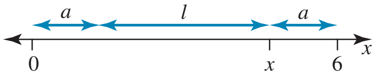
11 a $y = 9 - (x-3)^2$

x-intercepts: let $y = 0$.

$$\therefore\ 0 = 9 - (x-3)^2$$
$$\therefore\ (x-3)^2 = 9$$
$$\therefore\ x - 3 = \pm 3$$
$$\therefore\ x = 0, x = 6$$

The x-intercepts are $(0, 0)$ and $(6, 0)$.

b Length:



$$l = x - a$$
$$= x - (6 - x)$$
$$= 2x - 6$$

Width:

$$w = y$$
$$= 9 - (x-3)^2$$

c Area is given by $A = lw$.

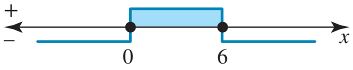
$$\therefore\ A = (2x-6)\left(9 - (x-3)^2\right)$$
$$= (2x-6)\left(9 - (x^2 - 6x + 9)\right)$$
$$= (2x-6)(9 - x^2 + 6x - 9)$$
$$= (2x-6)(-x^2 + 6x)$$
$$= -2x^3 + 12x^2 + 6x^2 - 36x$$
$$\therefore\ A = -2x^3 + 18x^2 - 36x$$

d $l > 0 \Rightarrow 2x - 6 > 0$

$\therefore x > 3$

$w > 0 \Rightarrow 9 - (x-3)^2 > 0$

$$\therefore\ -x^2 + 6x > 0$$
$$\therefore\ -x(x-6) > 0$$



$\therefore 0 < x < 6$

For both the length and width to be non-negative, the model is valid for $3 < x < 6$.

e Let $A = 16$.

$$\therefore\ 16 = -2x^3 + 18x^2 - 36x$$
$$\therefore\ 2x^3 - 18x^2 + 36x + 16 = 0$$
$$\therefore\ x^3 - 9x^2 + 18x + 8 = 0$$

Let $p(x) = x^3 - 9x^2 + 18x + 8$.

$p(4) = 64 - 144 + 72 + 8 = 0$

$\therefore\ (x-4)$ is a factor.

$\therefore\ x^3 - 9x^2 + 18x + 8 = (x-4)(x^2 - 5x - 2)$

$P(x) = 0 \Rightarrow (x-4)(x^2 - 5x - 2) = 0$

$$\therefore\ x = 4 \text{ or } x^2 - 5x - 2 = 0$$

$$\therefore\ x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times -2}}{2}$$
$$= \frac{5 \pm \sqrt{33}}{2}$$

$$\therefore x = 4, \frac{5 + \sqrt{33}}{2}, \frac{5 - \sqrt{33}}{2}$$

Since $3 \le x \le 6$, reject $x = \dfrac{5 - \sqrt{33}}{2}$.

Answer: $x = 4, x = \dfrac{5 + \sqrt{33}}{2}$

f Sketch the area graph $A = -2x^3 + 18x^2 - 36x$ and obtain the coordinates of the maximum turning point by using CAS technology.

The maximum turning point is $(4.732\,050\,7, 20.784\,609)$, so the area is greatest when $x = 4.732\,050\,7$.

Length, $l = 2x - 6$

$\therefore\ l = 2 \times 4.732\,050\,7 - 6$

$\therefore\ l = 3.464$

Width, $w = -x^2 + 6x$

$\therefore\ w = -(4.732\,050\,7)^2 + 6 \times 4.732\,050\,7$

$\therefore\ w = 6.000$

To 3 decimal places, the length and width of the rectangle that has the greatest area are 3.464 units and 6.000 units respectively.

12 a A cubic graph can have up to 2 turning points and up to 3 x-intercepts.

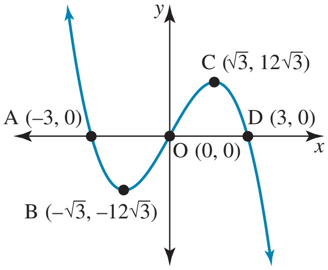
b Given the x-intercepts are A $(-3, 0)$, O $(0, 0)$, D $(3, 0)$, the equation of the path is $y = ax(x+3)(x-3)$.

Substitute the point C $\left(\sqrt{3}, 12\sqrt{3}\right)$.

$$\therefore\ 12\sqrt{3} = a\left(\sqrt{3}\right)\left(\sqrt{3} + 3\right)\left(\sqrt{3} - 3\right)$$
$$\therefore\ 12\sqrt{3} = a\sqrt{3}\left(\left(\sqrt{3}\right)^2 - 3^2\right)$$
$$\therefore\ 12\sqrt{3} = a\sqrt{3}(3 - 9)$$
$$\therefore\ 12\sqrt{3} = -6\sqrt{3}a$$
$$\therefore\ a = -2$$

The equation of the path is $y = -2x(x+3)(x-3)$ or $y = -2x\left(x^2-9\right)$.

c Points A, O and D are x-intercepts; points B and C are turning points; the shape is of a negative cubic. The value of $12\sqrt{3} \simeq 20.8$.



d Consider the gradient of the straight line through B $\left(-\sqrt{3}, -12\sqrt{3}\right)$ and C $\left(\sqrt{3}, 12\sqrt{3}\right)$.

$$\begin{aligned} m_{\text{BC}} &= \frac{12\sqrt{3} - \left(-12\sqrt{3}\right)}{\sqrt{3} - \left(-\sqrt{3}\right)} \\ &= \frac{24\sqrt{3}}{2\sqrt{3}} \\ &= 12 \end{aligned}$$

Consider the gradient of the line through O $(0, 0)$ and C $\left(\sqrt{3}, 12\sqrt{3}\right)$:

$$\begin{aligned} m_{\text{OC}} &= \frac{12\sqrt{3}}{\sqrt{3}} \\ &= 12 \end{aligned}$$

Since $m_{\text{BC}} = m_{\text{OC}}$ and the point C is common, then the three points B, O and C are collinear. Therefore, a straight line through B and C will pass through O.

Equation of BC:

$$y - 12\sqrt{3} = 12(x - \sqrt{3})$$
$$\therefore\ y = 12x$$

e Let the equation be $y = a(x-h)^3 + k$.

POI at $(0, 0)$: $\therefore y = ax^3$

Substitute the point C $\left(\sqrt{3}, 12\sqrt{3}\right)$.

$$\therefore\ 12\sqrt{3} = a\left(\sqrt{3}\right)^3$$
$$\therefore\ 12\sqrt{3} = 3\sqrt{3}a$$
$$\therefore\ a = 4$$

The equation of the path is $y = 4x^3$.

4.6 Exam questions

1 Volume of cylinder = area of circular end × length

$$\begin{aligned} \text{Volume} &= \pi r^2 \times l \\ &= \pi(2x)^2 \times 3x \\ &= 12\pi x^3 \end{aligned}$$

The correct answer is **B**.

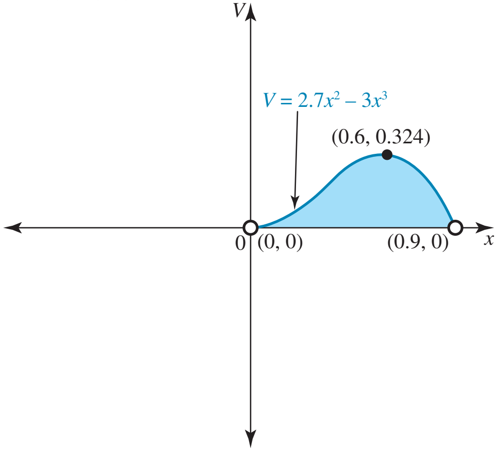
2

$$\begin{aligned} \text{Volume} &= \text{length} \times \text{width} \times \text{height} \\ 60 &= (x+2)(x+2)(x) \\ x\left(x^2+4x+4\right) &= 60 \\ x^3+4x^2+4x &= 60 \\ x^3+4x^2+4x-60 &= 0 \end{aligned}$$

The correct answer is **C**.

3 $V > 0$ and $x > 0$

$$2.7x^2 - 3x^3 > 0$$
$$-3x^2(x - 0.9) > 0$$
$$\therefore 0 < x < 0.9$$



The correct answer is **D**.

4.7 Review

4.7 Exercise

Technology free: short answer

1 $p(x) = x^3 + 5x^2 + 3x - 9$

$p(1) = 1 + 5 + 3 - 9 = 0$

$\therefore (x-1)$ is a factor.

$$\begin{aligned} \therefore\ p(x) &= x^3 + 5x^2 + 3x - 9 \\ &= (x-1)(x^2 + 6x + 9) \end{aligned}$$

$\therefore\ p(x) = (x-1)(x+3)^2$

2 $p(x) = x^3 - ax^2 + bx - 3$

$$p(1) = 2$$
$$\Rightarrow 2 = 1 - a + b - 3$$

$\therefore\ -a + b = 4 \quad [1]$

$$p(-1) = -4$$
$$\Rightarrow -4 = -1 - a - b - 3$$

$\therefore\ a + b = 0 \quad [2]$

Add equations [1] and [2]:

$\therefore 2b = 4$

$\therefore b = 2$

Substitute $b = 2$ in equation [2]:

$\therefore a + 2 = 0$

$\therefore a = -2$

Answer: $a = -2, b = 2$

3

$$\begin{aligned} &\frac{2x^3 - 3x^2 + x - 1}{x+2} \\ &= \frac{2x^2(x+2) - 4x^2 - 3x^2 + x - 1}{x+2} \\ &= \frac{2x^2(x+2) - 7x^2 + x - 1}{x+2} \\ &= \frac{2x^2(x+2) - 7x(x+2) + 14x + x - 1}{x+2} \\ &= \frac{2x^2(x+2) - 7x(x+2) + 15x - 1}{x+2} \end{aligned}$$

$$= \frac{2x^2(x+2) - 7x(x+2) + 15(x+2) - 30 - 1}{x+2}$$

$$= \frac{2x^2(x+2) - 7x(x+2) + 15(x+2) - 31}{x+2}$$

$$= 2x^2 - 7x + 15 - \frac{31}{x+2}$$

The quotient is $2x^2 - 7x + 15$ and the remainder is -31.

4 a $y = 8 - (x+3)^3$

$\therefore y = -(x+3)^3 + 8$

POI $(-3, 8)$

y-intercept: let $x = 0$.

$\therefore y = -(3)^3 + 8$

$= -27 + 8$

$= -19$

$(0, -19)$

x-intercept: let $y = 0$.

$\therefore 0 = -(x+3)^3 + 8$

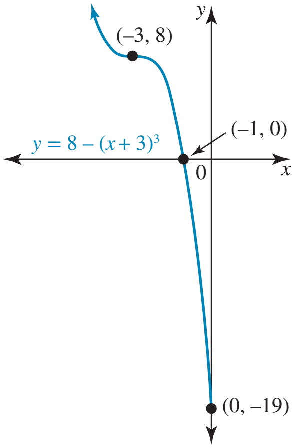
$\therefore (x+3)^3 = 8$

$\therefore x + 3 = \sqrt[3]{8}$

$\therefore x + 3 = 2$

$\therefore x = -1$

$(-1, 0)$



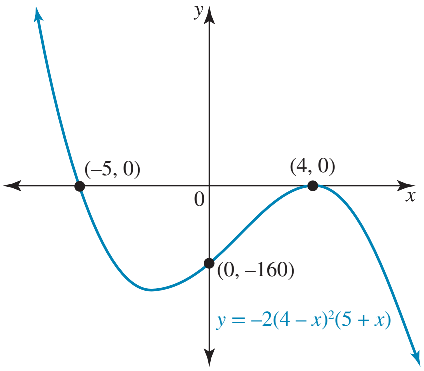
b $y = -2(4-x)^2(5+x)$

x-intercepts occur at $x = 4$ (touch) and $x = -5$.

y-intercept: let $x = 0$.

$\therefore y = -2(4)^2(5) = -160$

$(0, -160)$



c $y = (8x-3)^3$

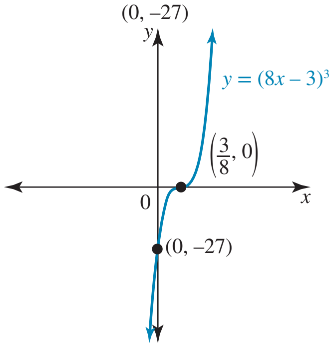
Stationary point of inflection when $8x - 3 = 0 \Rightarrow x = \dfrac{3}{8}$

POI $\left(\dfrac{3}{8}, 0\right)$

y-intercept: let $x = 0$.

$\therefore y = (-3)^3 = -27$

$(0, -27)$



d $y = 2x^3 - x$

$\therefore y = x(2x^2 - 1)$

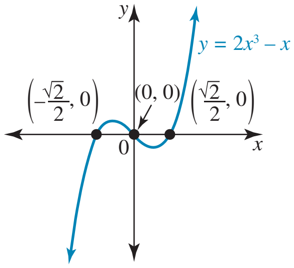
$\therefore y = x(\sqrt{2}x + 1)(\sqrt{2}x - 1)$

x-intercepts occur when $x = 0$, $\sqrt{2}x + 1 = 0$ and $\sqrt{2}x - 1 = 0$.

$\therefore x = 0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}$

$\therefore x = 0, -\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}$

y-intercept at the origin



5 $y = -x^3 + 6x^2 - 11x + 6$

y-intercept: $(0, 6)$

x-intercepts: let $y = 0$.

$\therefore 0 = -x^3 + 6x^2 - 11x + 6$

$\therefore x^3 - 6x^2 + 11x - 6 = 0$

Let $p(x) = x^3 - 6x^2 + 11x - 6$.

$p(1) = 1 - 6 + 11 - 6 = 0$

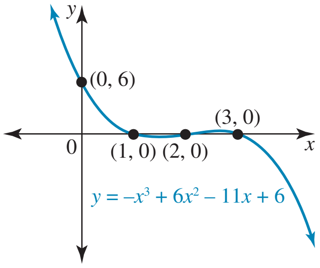
$\therefore (x-1)$ is a factor.

$\therefore p(x) = x^3 - 6x^2 + 11x - 6$

$= (x-1)(x^2 - 5x + 6)$

$= (x-1)(x-3)(x-2)$

The x-intercepts are $(1, 0)$, $(3, 0)$, $(2, 0)$.



$x^3 - 6x^2 + 11x - 6 < 0$

$\therefore -x^3 + 6x^2 - 11x + 6 > 0$

$\therefore x < 1$ or $2 < x < 3$

6 $y = 4x - x^3$ and $y = -2x$

At intersection,

$$4x - x^3 = -2x$$
$$\therefore x^3 - 6x = 0$$
$$\therefore x(x^2 - 6) = 0$$
$$\therefore x\left(x - \sqrt{6}\right)\left(x + \sqrt{6}\right) = 0$$
$$\therefore x = 0, x = \sqrt{6}, x = -\sqrt{6}$$

Substitute the x-values in $y = -2x$.

$\therefore x = 0, y = 0;\ x = \sqrt{6}, y = -2\sqrt{6};\ x = -\sqrt{6}, y = 2\sqrt{6}$

The points of intersection are

$(0, 0), \left(\sqrt{6}, -2\sqrt{6}\right), \left(-\sqrt{6}, 2\sqrt{6}\right)$.

Technology active: multiple choice

7 **A** is not a polynomial since there are terms in x with negative indices.

B is not a polynomial since there will be terms in x with fractional indices as $\sqrt{x} = x^{\frac{1}{2}}$.

C is not a polynomial since there is a term in x with a negative index.

$$\frac{4x - 7x^5}{x^3} = \frac{4x}{x^3} - \frac{7x^5}{x^3}$$
$$= 4x^{-2} - 7x^2$$

D is not a polynomial since the indices contain x terms.

The correct answer is **E**.

8 $x^3 - 2x^2 - 3x + 10 \equiv (x + 2)(ax^2 + bx + c)$

Completing the factorisation,

$x^3 - 2x^2 - 3x + 10 = (x + 2)(x^2 - 4x + 5)$.

$\therefore a = 1, b = -4, c = 5$

Alternatively, expand and equate coefficients of like terms.

The correct answer is **D**.

9 When $3x + 6 = 0$, $x = -2$. The remainder is $P(-2)$.

The correct answer is **E**.

10 $p(x) = 3 + kx - 5x^2 + 2x^3$

$p(-1) = 8$

$\Rightarrow 8 = 3 + k(-1) - 5(-1)^2 + 2(-1)^3$

$\therefore 8 = 3 - k - 5 - 2$

$\therefore k = -12$

The correct answer is **E**.

11 x-intercepts occur at:

$x = -2 \Rightarrow (x + 2)$ is a factor.

$x = 3$ (touch) $\Rightarrow (x - 3)^2$ is a factor.

The shape is of a negative cubic.

A possible equation is $y = -(x + 2)(x - 3)^2$

The correct answer is **C**.

12 $y = 2x^3$ has a stationary point of inflection at $(0, 0)$. If the graph is translated 2 units to the right and 3 units down, the point of inflection becomes $(2, -3)$ and therefore the image has the equation $y = 2(x - 2)^3 - 3$,

The correct answer is **A**.

13 $\dfrac{3 - 4x}{x + 1}$

$$= \frac{-4x + 3}{x + 1}$$
$$= \frac{-4(x + 1) + 4 + 3}{x + 1}$$
$$= \frac{-4(x + 1)}{(x + 1)} + \frac{7}{x + 1}$$
$$= -4 + \frac{7}{x + 1}$$

The correct answer is **A**.

14 $(x + 4)^3 > -1$

Take the cube root of both sides.

$\therefore (x + 4) > \sqrt[3]{-1}$

$\therefore x + 4 > -1$

$\therefore x > -5$

The correct answer is **C**.

15 $2x^3 = 14x^2 + 16x$

$\therefore 2x^3 - 14x^2 - 16x = 0$

$\therefore 2x(x^2 - 7x - 8) = 0$

$\therefore 2x(x - 8)(x + 1) = 0$

$\therefore x = 0, x = 8, x = -1$

The correct answer is **E**.

16 x-intercepts occur at:

$x = a \Rightarrow (x - a)$ is a factor.

$x = b \Rightarrow (x - b)$ is a factor.

$x = 4.5 = \dfrac{9}{2} \Rightarrow (2x - 9)$ is a factor.

Since the graph starts below the x-axis, the coefficient of x^3 must be positive.

A possible equation is $y = (x - a)(x - b)(2x - 9)$.

The correct answer is **C**.

Technology active: extended response

17 $p(x) = 8x^3 - 34x^2 + 33x - 9$

a $p(3) = 8(3)^3 - 34(3)^2 + 33(3) - 9$

$= 216 - 306 + 99 - 9$

$= 0$

Since $p(3) = 0$, then $(x - 3)$ is a factor.

b $\therefore p(x) = 8x^3 - 34x^2 + 33x - 9$

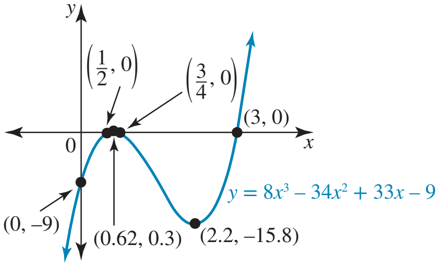
$= (x - 3)(8x^2 - 10x + 3)$

$\therefore p(x) = (x - 3)(4x - 3)(2x - 1)$

c x-intercepts: $(3, 0), \left(\dfrac{3}{4}, 0\right), \left(\dfrac{1}{2}, 0\right)$

y-intercept $(0, -9)$

Given turning points at $(0.62, 0.3), (2.2, -15.8)$



d $\{x : p(x) \geq 0\} = \left\{x : \dfrac{1}{2} \leq x \leq \dfrac{3}{4}\right\} \cup \{x : x \geq 3\}$

e $p(x) = -9$

$\therefore 8x^3 - 34x^2 + 33x - 9 = -9$

$\therefore 8x^3 - 34x^2 + 33x = 0$

$\therefore x(8x^2 - 34x + 33) = 0$

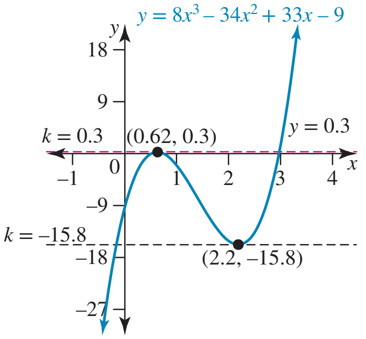
$\therefore x(4x - 11)(2x - 3) = 0$

$\therefore x = 0,\ 4x - 11 = 0,\ 2x - 3 = 0$

$\therefore x = 0,\ x = \dfrac{11}{4},\ x = \dfrac{3}{2}$

The solution set is $\left\{x : x = 0, \dfrac{3}{2}, \dfrac{11}{4}\right\}$.

f The horizontal line $y = k$ intersects the graph in two places if it is a tangent to the curve at its turning points; there will be three intersections if the line lies between the turning points; and there will be only one intersection if the line is higher than the maximum turning point or lower than the minimum turning point.



i Three intersections if $-15.8 < k < 0.3$

ii Two intersections if $k = -15.8$ or $k = 0.3$

iii One intersection if $k < -15.8$ or $k > 0.3$

18 Revenue: $r(x) = 6(2x^2 + 10x + 3)$; Cost $c(x) = x(6x^2 - x + 1)$

a $r(x)$ is a degree 2 polynomial and $c(x)$ is a degree 3 polynomial.

b If 1000 items are sold, then $x = 1$.

$r(1) = 6(2 + 10 + 3)$ and $c(1) = 1(6 - 1 + 1)$

$= 90$ $= 6$

The revenue is \$90 and the cost is \$6, so a profit of \$84 is made.

c The profit is revenue – cost.

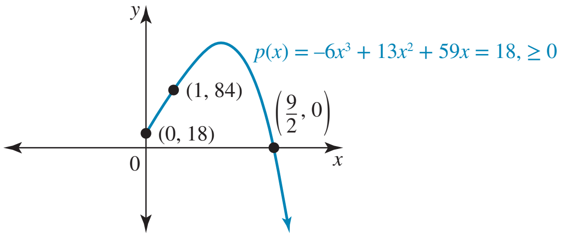
$$\begin{aligned}\therefore p(x) &= r(x) - c(x)\\ &= 6(2x^2 + 10x + 3) - x(6x^2 - x + 1)\\ &= 12x^2 + 60x + 18 - 6x^3 + x^2 - x\\ \therefore p(x) &= -6x^3 + 13x^2 + 59x + 18\end{aligned}$$

d x-intercept at $x = -2 \Rightarrow (x + 2)$ is a factor.

$$\begin{aligned}\therefore p(x) &= -6x^3 + 13x^2 + 59x + 18\\ &= (x + 2)(-6x^2 + 25x + 9)\\ \therefore p(x) &= (x + 2)(-2x + 9)(3x + 1)\end{aligned}$$

The other x-intercepts are at $x = \frac{9}{2}, x = -\frac{1}{3}$ and the y-intercept is at $y = 18$. From part **b**, the point $(1, 84)$ lies on the graph.

Since the number of items sold cannot be negative, the graph of the profit can only be drawn with the restriction that $x \geq 0$.



e From the graph it can be seen that a loss occurs for $x > \frac{9}{2}$.

If $x = \frac{9}{2} = 4.5$, then 4500 items are sold. The number of items must be a whole number, so the least number manufactured that results in a loss is 4501. The least value of d is 4501.

19 **a** A $(1, 20)$ and B $(5, 12)$

$$\begin{aligned}m_{AB} &= \frac{12 - 20}{5 - 1}\\ &= \frac{-8}{4}\\ &= -2\end{aligned}$$

Equation of line AB:

$$\begin{aligned}y - 12 &= -2(x - 5)\\ \therefore y &= -2x + 10 + 12\\ \therefore y &= -2x + 22\end{aligned}$$

b $y = a(2x - 1)(x - 6)(x + b),\ 0 \leq x \leq 8$

Substitute point A $(1, 20)$:

$$\begin{aligned}\therefore 20 &= a(2 - 1)(1 - 6)(1 + b)\\ \therefore 20 &= -5a(1 + b)\end{aligned}$$

$\therefore a(1 + b) = -4$ [1]

Substitute point B $(5, 12)$:

$$\begin{aligned}\therefore 12 &= a(10 - 1)(5 - 6)(5 + b)\\ \therefore 12 &= -9a(5 + b)\end{aligned}$$

$\therefore 3a(5 + b) = -4$ [2]

Divide equation [2] by equation [1]:

$$\begin{aligned}\therefore \frac{3a(5 + b)}{a(1 + b)} &= \frac{-4}{-4}\\ \therefore \frac{3(5 + b)}{1 + b} &= 1,\ a \neq 0\\ \therefore 15 + 3b &= 1 + b\\ \therefore 2b &= -14\\ \therefore b &= -7\end{aligned}$$

Substitute $b = -7$ in equation [1]:

$$\begin{aligned}\therefore a(1 - 7) &= -4\\ \therefore -6 &= -4\\ \therefore a &= \frac{4}{6}\\ \therefore a &= \frac{2}{3}\end{aligned}$$

c The scenic route has the equation

$y = \frac{2}{3}(2x - 1)(x - 6)(x - 7),\ 0 \leq x \leq 8.$

End points: let $x = 0$.

$$\begin{aligned}y &= \frac{2}{3}(-1)(-6)(-7)\\ &= -\frac{2}{3} \times 42\\ &= -28\end{aligned}$$

$(0, -28)$

Let $x = 8$.

$$\begin{aligned}y &= \frac{2}{3}(16 - 1)(8 - 6)(8 - 7)\\ &= \frac{2}{3} \times 15 \times 2 \times 1\\ &= 20\end{aligned}$$

$(8, 20)$

The scenic route starts at $(0, -28)$ and finishes at $(8, 20)$.

d $y = \frac{2}{3}(2x - 1)(x - 6)(x - 7),\ 0 \leq x \leq 8$

x-intercepts occur at $x = \frac{1}{2}, x = 6, x = 7$.

End points: $(0, -28)$ and $(8, 20)$

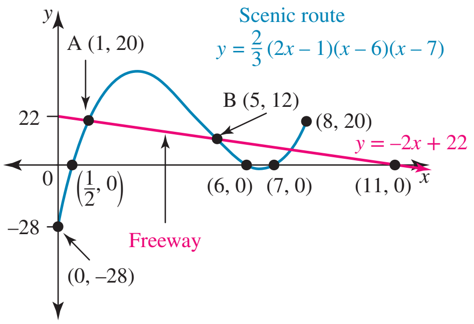
Other points: A $(1, 20)$ and B $(5, 12)$

$y = -2x + 22$

y-intercept: $(0, 22)$

x-intercept when $-2x + 22 = 0 \Rightarrow x = 11$

The freeway also passes through the points A $(1, 20)$ and B $(5, 12)$.



e At the intersection of the two roads,

$$\frac{2}{3}(2x-1)(x-6)(x-7) = -2x+22$$

$$\therefore (2x-1)(x-6)(x-7) = 2(-x+11) \times \frac{3}{2}$$

$$\therefore (2x-1)(x-6)(x-7) = -3x+33$$

$$\therefore (2x-1)(x^2-13x+42) = -3x+33$$

$$\therefore 2x^3 - 27x^2 + 97x - 42 = -3x+33$$

$$\therefore 2x^3 - 27x^2 + 100x - 75 = 0$$

Since points A and B lie on both roads, $x = 1$ and $x = 5$ are solutions, which means $(x - 1)$ and $(x - 5)$ are both factors.
$(x-1)(x-5) = x^2 - 6x + 5$, so $(x^2 - 6x + 5)$ is a quadratic factor.
$\therefore 2x^3 - 27x^2 + 100x - 75 = (x^2 - 6x + 5)(2x - 15)$
The equation becomes

$$(x^2 - 6x + 5)(2x - 15) = 0$$

$$\therefore (x-1)(x-5)(2x-15) = 0$$

$$\therefore x = 1, x = 5, x = \frac{15}{2}$$

Substitute $x = \frac{15}{2}$ in $y = -2x + 22$:

$$y = -2 \times \frac{15}{2} + 22$$

$$= -15 + 22$$

$$= 7$$

The three points of intersection are $(1, 20)$, $(5, 12)$ and $\left(\frac{15}{2}, 7\right)$.

f The closest point to O cannot be judged from the graph because the axes have different scales. The formula for the distance between two points, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, can be used.

Distance OA:

$$d_{\text{OA}} = \sqrt{(1-0)^2 + (20-0)^2}$$

$$= \sqrt{401}$$

$$\simeq 20.02$$

Distance OB:

$$d_{\text{OB}} = \sqrt{(5-0)^2 + (12-0)^2}$$

$$= \sqrt{25 + 144}$$

$$= \sqrt{169}$$

$$= 13$$

Distance OC:

$$d_{\text{OC}} = \sqrt{\left(\frac{15}{2} - 0\right)^2 + (7-0)^2}$$

$$= \sqrt{\frac{225}{4} + 49}$$

$$= \sqrt{\frac{225 + 196}{4}}$$

$$d_{\text{OC}} = \sqrt{\frac{421}{4}}$$

$$= \frac{\sqrt{421}}{2}$$

$$\simeq 10.26$$

The closest point to O is $\left(\frac{15}{2}, 7\right)$.

20 **a** Using Pythagoras' theorem, $13^2 = h^2 + r^2$.

If $r = \frac{13\sqrt{6}}{3}$,

$$169 = h^2 + \left(\frac{13\sqrt{6}}{3}\right)^2$$

$$\therefore 169 = h^2 + \frac{169 \times 6}{9}$$

$$\therefore 169 = h^2 + \frac{2}{3} \times 169$$

$$\therefore h^2 = 169 - \frac{2}{3} \times 169$$

$$\therefore h^2 = \frac{1}{3} \times 169$$

$$\therefore h = \frac{13}{\sqrt{3}} = \frac{13\sqrt{3}}{3},\ h > 0$$

The height of the cone is $\frac{13\sqrt{3}}{3}$ metres.

b Volume, $V = \frac{1}{3}\pi r^2 h$
$13^2 = h^2 + r^2$
$\therefore r^2 = 169 - h^2$
Substitute this in the volume formula:

$$\therefore V = \frac{1}{3}\pi(169 - h^2)h$$

$$\therefore V = \frac{1}{3}\pi h(169 - h^2)$$

c $h > 0$ and $r > 0$

$$\therefore 169 - h^2 > 0$$

$$\therefore h^2 < 169$$

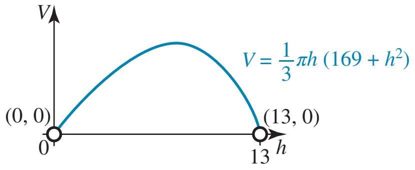
$$\therefore 0 < h < 13$$

The restriction is $0 < h < 13$.

$$V = \frac{1}{3}\pi h(169 - h^2)$$

$$= \frac{1}{3}\pi h(13 - h)(13 + h)$$

When $V = 0$, $h = 0$ or $h = 13$, but $h = -13$ is not applicable.
The shape of the graph is part of that of a negative cubic.



d $V = \frac{1}{3}\pi h(169 - h^2)$
If $h = 7$,

$$V = \frac{1}{3}\pi(7)(169 - 49)$$

$$= \frac{1}{3}\pi \times 7 \times 120$$

$$= 280\pi$$

If $h = 8$,

$$\begin{aligned} V &= \frac{1}{3}\pi(8)(169-64) \\ &= \frac{1}{3}\pi \times 8 \times 105 \\ &= 280\pi \end{aligned}$$

If $h = 9$,

$$\begin{aligned} V &= \frac{1}{3}\pi(9)(169-81) \\ &= \frac{1}{3}\pi \times 9 \times 88 \\ &= 264\pi \end{aligned}$$

Since $V(7) = V(8)$, the points where $h = 7$ and $h = 8$ lie on either side of the turning point.

$V(9) < V(8)$, so the greatest volume occurs when $7 < h < 8$. Comparing this with $a < h < a + 1$, then $a = 7$.

e **i** The midpoint of the interval $[7, 8]$ is 7.5.

Substitute $h = 7.5$ in the relationship $r^2 = 169 - h^2$.

$$\begin{aligned} \therefore r^2 &= 169 - \left(\frac{15}{2}\right)^2 \\ &= \frac{676}{4} - \frac{225}{4} \\ &= \frac{451}{4} \\ \therefore r &= \frac{\sqrt{451}}{2} \simeq 10.62 \end{aligned}$$

ii Substitute $h = \frac{15}{2}$ and $r^2 = \frac{451}{4}$ into $V = \frac{1}{3}\pi r^2 h$.

$$\begin{aligned} V &= \frac{1}{\cancel{3}_1}\pi \times \frac{451}{4} \times \frac{\cancel{15}^5}{2} \\ &= \frac{2255}{8}\pi \\ &\simeq 886 \end{aligned}$$

To the nearest whole number, an estimate of the greatest volume is 886 m^3.

f **i** The rest of the working remains and follows on.

Substitute $r = \sqrt{2}h$ in the relationship $r^2 = 169 - h^2$.

$$\begin{aligned} \therefore \left(\sqrt{2}h\right)^2 &= 169 - h^2 \\ \therefore 2h^2 &= 169 - h^2 \\ \therefore 3h^2 &= 169 \\ \therefore h^2 &= \frac{169}{3} \\ \therefore h &= \frac{13}{\sqrt{3}} = \frac{13\sqrt{3}}{3},\ h > 0 \end{aligned}$$

When $h = \frac{13\sqrt{3}}{3}$,

$$\begin{aligned} r &= \sqrt{2} \times \frac{13\sqrt{3}}{3} \\ &= \frac{13\sqrt{6}}{3} \end{aligned}$$

The greatest volume occurs when the height is $\frac{13\sqrt{3}}{3}$ metres and the radius is $\frac{13\sqrt{6}}{3}$ metres.

ii Greatest volume:

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{13\sqrt{6}}{3}\right)^2\left(\frac{13\sqrt{3}}{3}\right) \\ &= \frac{1}{\cancel{3}}\pi \times \frac{169 \times \cancel{6}^2}{9} \times \frac{13 \times \sqrt{3}}{3} \\ &= \frac{13^2 \times 2 \times 13 \times \sqrt{3}}{27}\pi \\ &= \frac{2\sqrt{3} \times 13^3}{27}\pi \\ &\simeq 886 \end{aligned}$$

To the nearest whole number, the greatest volume is 886 m^3, which agrees with the estimate given in part **e**.

4.7 Exam questions

1 $4x^2 - 3x + 1 = ax(x-2) + b(x-2) + c$

$4x^2 - 3x + 1 = ax^2 - 2ax + bx - 2b + c$ [1 mark]

Equate coefficients:

$$\begin{aligned} a &= 4 \\ -2a + b &= -3 \\ -2(4) + b &= -3 \\ \therefore b &= 5 \\ -2b + c &= 1 \\ -2(5) + b &= 1 \\ \therefore c &= 11 \\ \therefore b = 4, b &= 5, c = 11 \end{aligned}$$

[1 mark]

$4x^2 - 3x + 1 = 4x(x-2) + 5(x-2) + 11$

$\frac{4x^2 - 3x + 1}{x - 2} = 4x + 5 + \frac{11}{x-2}$ [1 mark]

2 $6x^3 - 17x^2 = 5x - 6$

$6x^2 - 17x^2 - 5x + 6 = 0$

Let $p(x) = 6x^3 - 17x^2 - 5x + 6$.

$p(3) = 0$, $\therefore (x - 3)$ is a factor. [1 mark]

$$\begin{array}{r} 6x^2 + x - 2 \\ x - 3\overline{\big)\,6x^3 - 17x^2 - 5x + 6} \\ \underline{6x^3 - 18x^2} \phantom{{}-5x+6} \\ x^2 - 5x \phantom{{}+6} \\ \underline{x^2 - 3x} \phantom{{}+6} \\ -2x + 6 \\ \underline{-2x + 6} \\ 0 \end{array}$$

$p(x) = 6x^3 - 17x^2 - 5x + 6 = 0$

$0 = (x-3)\left(6x^2 + x - 2\right)$ [1 mark]

Now factorise the quadratic factor to fully factorise the cubic.

$0 = (x-3)(2x-1)(3x+2)$ [1 mark]

$\therefore x = 3, \frac{1}{2}, -\frac{2}{3}$ [1 mark]

3 **a** $p(x) = 3x^3 + kx^2 + 4$

$(3x + 2)$ is a factor. $\therefore P\left(-\frac{2}{3}\right) = 0$.

Substitute into $p(x)$:

$$0 = 3\left(-\frac{2}{3}\right)^3 + k\left(-\frac{2}{3}\right)^2 + 4$$

$0 = -\frac{8}{9} + \frac{4k}{9} + \frac{36}{9}$

$0 = \frac{4k + 28}{9}$

$0 = 4k + 28$

$4k = -28$

$k = -7$ [1 mark]

$p(x) = 3x^3 - 7x^2 + 4$

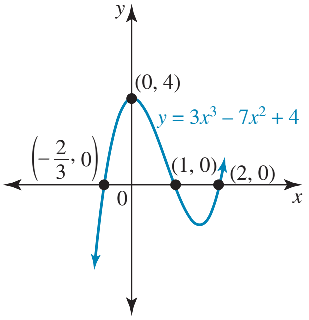
$p(x) = (3x + 2)\left(x^2 - 3x + 2\right)$

$p(x) = (3x + 2)(x - 1)(x - 2)$ [1 mark]

b $p(x) = (3x + 2)(x - 1)(x - 2)$

x-intercepts at $\left(-\frac{2}{3}, 0\right), (1, 0), (2, 0)$

y-intercepts at $(0, 4)$ [1 mark]



[1 mark]

4 $y = ax^3 + bx^2 + cx + d$

$a = 1, \; d = 36$

$y = x^3 + bx^2 + cx + 36$ [1 mark]

Using $(-1, 42)$:

$42 = -1 + b(-1)^2 + c(1) + 36$

$7 = b - c$ [1] [1 mark]

Using $(1, 20)$:

$20 = 1 + b(1)^2 + c(1) + 36$

$-17 = b + c$ [2] [1 mark]

Add [1] and [2]:

$2b = -10$

$b = -5$

Substitute $b = -5$ into [2]:

$-17 = 5 + c$

$c = -12$

Therefore, the equation is $y = x^3 - 5x^2 - 12x + 36$. [1 mark]

5 $2x + y = 18$

$y = 18 - 2x$ [1 mark]

Volume of the triangular box $= \frac{1}{2}$length × width × height

$V = \frac{1}{2}(x)(x)(y)$

$V = \frac{1}{2}x^2(8 - 2x)$

$V = 9x^2 - x^3$ [1 mark]

Restriction of volume:

$V > 0$ and $x > 0$

$9x^2 - x^3 > 0$

$x^2(9 - x) > 0$

Therefore, as volume and length must be positive,

$0 < x < 9.$ [1 mark]

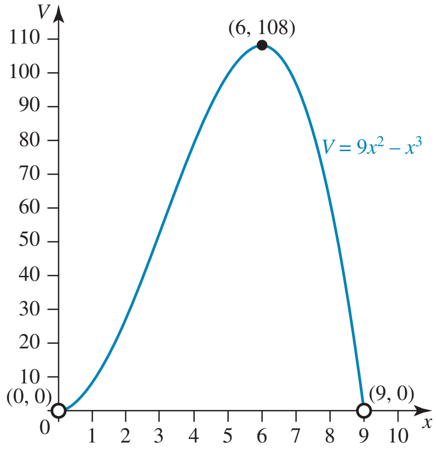
$V = 9x^2 - x^3$

This is a cubic function and, when graphed, the intercepts are $(0, 0)$ and $(9, 0)$.

Unlike a quadratic function, the maximum value does not occur halfway between the intercepts.

Using CAS, we can find the maximum TP.

TP = (6, 108)



[1 mark]

Topic 5 — Quartic polynomials

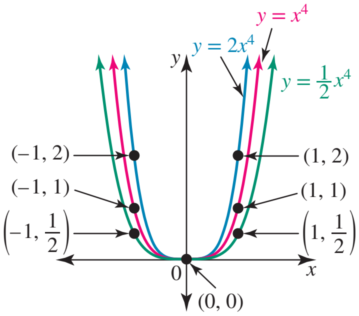
5.2 Quartic polynomials

5.2 Exercise

1 **a** $y = x^4, y = 2x^4, y = \frac{1}{2}x^4$

All three graphs have a minimum turning point at the origin.

The points $(\pm 1, 1)$ lie on $y = x^4$, the points $(\pm 1, 2)$ lie on $y = 2x^4$, and the points $\left(\pm 1, \frac{1}{2}\right)$ lie on $y = \frac{1}{2}x^4$.



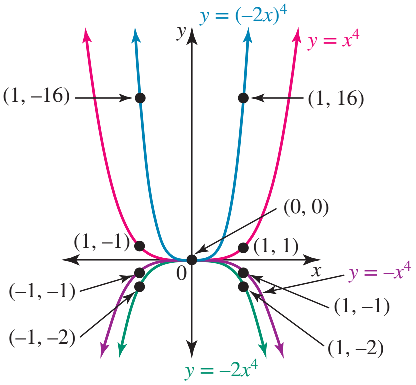
b $y = x^4, y = -x^4, y = -2x^4, y = (-2x)^4$

The points $(0, 0), (-1, -1), (1, -1)$ lie on $y = -x^4$.

The points $(0, 0), (-1, -2), (1, -2)$ lie on $y = -2x^4$.

$$\begin{aligned}(-2x)^4 &= (-2)^4x^4\\ &= 16x^4\end{aligned}$$

Therefore, the points $(0, 0), (-1, 16), (1, 16)$ lie on $y = (-2x)^4$.

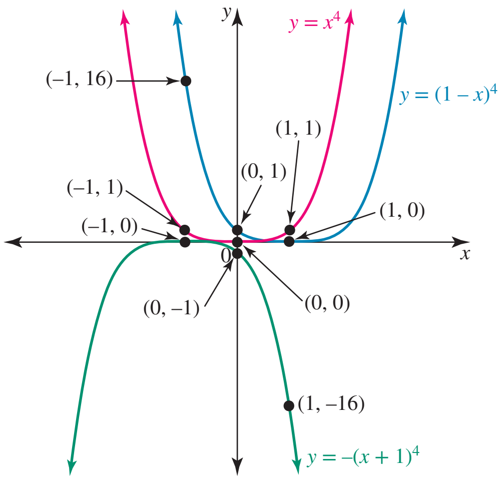


c $y = x^4, y = -(x + 1)^4, y = (1 - x)^4$

The points $(-1, 0), (0, -1), (1, -16)$ lie on $y = -(x + 1)^4$.

The points $(-1, 16), (0, 1), (1, 0)$ lie on $y = (1 - x)^4$.

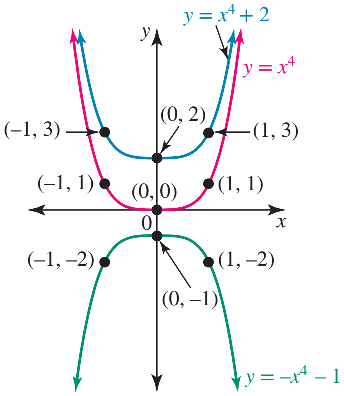
As $(1 - x)^4 = (x - 1)^4$, $y = (1 - x)^4$ is the same as $y = (x - 1)^4$.



d $y = x^4, y = x^4 + 2, y = -x^4 - 1$

The points $(-1, 3), (0, 2), (1, 3)$ lie on $y = x^4 + 2$.

The points $(-1, -2), (0, -1), (1, -2)$ lie on $y = -x^4 - 1$.



2 **a** $y = (x - 2)^4 - 1$

Minimum turning point $(2, -1)$

y-intercept: let $x = 0$.

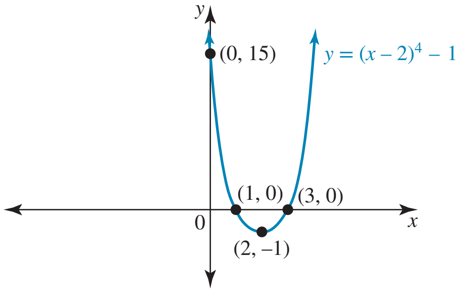
$$\begin{aligned}\therefore y &= (-2)^4 - 1\\ &= 16 - 1\\ &= 15\end{aligned}$$

$(0, 15)$

x-intercepts: let $y = 0$.

$$\begin{aligned}\therefore (x - 2)^4 - 1 &= 0\\ \therefore (x - 2)^4 &= 1\\ \therefore x - 2 &= \pm 1\\ \therefore x &= 1 \text{ or } x = 3\end{aligned}$$

$(1, 0), (3, 0)$



b $y = -(2x+1)^4$

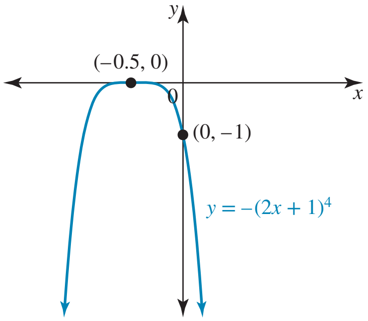
x-intercept and maximum turning point $\left(-\frac{1}{2}, 0\right)$

y-intercept: let $x = 0$.

$\therefore y = -(1)^4$

$\therefore y = -1$

$(0, -1)$



3 a $y = \frac{1}{8}(x+2)^4 - 2$

The equation is in the form $y = a(x-h)^4 + k$.

The turning point is $(-2, -2)$ and it is a minimum turning point, since $a = \frac{1}{8} > 0$.

y-intercept: let $x = 0$.

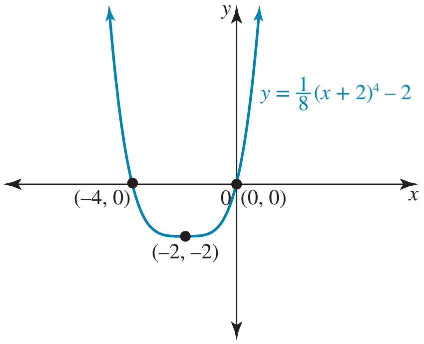
$$\begin{aligned} y &= \frac{1}{8}(0+2)^4 - 2 \\ &= \frac{1}{8}(16) - 2 \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

$(0, 0)$ is the y-intercept and one of the x-intercepts.

x-intercept: let $y = 0$.

$$\begin{aligned} \frac{1}{8}(x+2)^4 - 2 &= 0 \\ \frac{1}{8}(x+2)^4 &= 2 \\ (x+2)^4 &= 16 \\ (x+2) &= \pm\sqrt[4]{16} \\ x+2 &= \pm 2 \\ x &= 0, -4 \end{aligned}$$

The x-intercepts are $(-4, 0)$ and $(0, 0)$. (Note that this could have been deduced using the axis of symmetry, $x = -2$.)



b i The minimum turning point of $y = x^4$ is $(0, 0)$. The reflection turns the graph upside down so the turning point becomes a maximum. The translations move the turning point to $(1, -1)$.

There is a maximum turning point at $(1, -1)$.

ii $y = a(x-h)^4 + k$

The equation is $y = -(x-1)^4 - 1$.

y-intercept: let $x = 0$.

$$\begin{aligned} y &= -(0-1)^4 - 1 \\ &= -(1) - 1 \\ &= -2 \end{aligned}$$

$(0, -2)$ is the y-intercept.

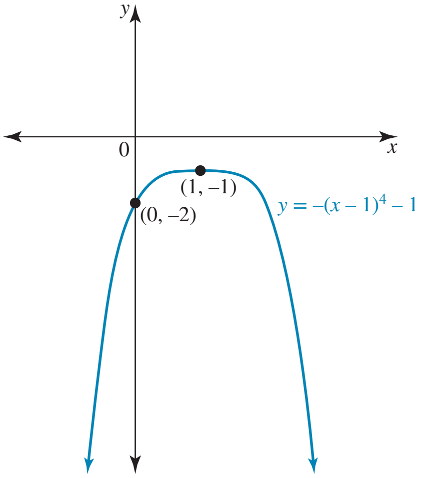
As the graph is concave down and the y-intercept is lower than the maximum turning point, there will not be any x-intercepts.

Alternative method:

x-intercept: let $y = 0$.

$$\begin{aligned} -(x-1)^4 - 1 &= 0 \\ -(x-1)^4 &= 1 \\ (x-1)^4 &= -1 \\ x - 1 &= \pm\sqrt[4]{-1} \end{aligned}$$

As the fourth root of a negative number is not real, there are no x-intercepts.



c $y = a(x-h)^4 + k$

Substitute the turning point $(4, 0)$.

$y = a(x-4)^4 + 0$

$y = a(x-4)^4$

Substitute the y-intercept $(0, 64)$.

$64 = a(0-4)^4$

$64 = a(16 \times 16)$

$$\begin{aligned} a &= \frac{64}{16 \times 16} \\ &= \frac{4}{16} \\ &= \frac{1}{4} \end{aligned}$$

The equation is $y = \frac{1}{4}(x-4)^4$.

4 a $y = (x-1)^4 - 16$

Minimum turning point $(1, -16)$

y-intercept: let $x = 0$.

$\therefore y = (-1)^4 - 16$

$\therefore y = -15$

$(0, -15)$

x-intercepts: let $y = 0$.

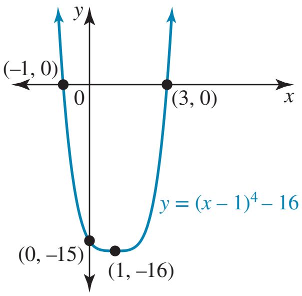
$\therefore (x-1)^4 - 16 = 0$

$\therefore (x-1)^4 = 16$

$\therefore x - 1 = \pm 2$

$\therefore x = -1, x = 3$

$(-1, 0), (3, 0)$



b $y = \frac{1}{9}(x+3)^4 + 12$

Minimum turning point $(-3, 12)$

y-intercept: let $x = 0$.

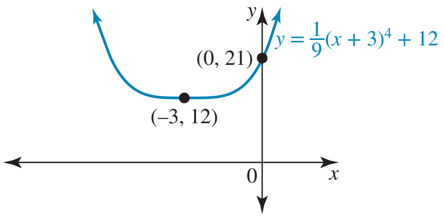
$\therefore y = \frac{1}{9}(3)^4 + 12$

$\therefore y = 9 + 12$

$\therefore y = 21$

$(0, 21)$

There are no x-intercepts.



c $y = 250 - 0.4(x+5)^4$

Maximum turning point $(-5, 250)$

y-intercept: let $x = 0$.

$\therefore y = 250 - 0.4(5)^4$

$\therefore y = 250 - 250$

$\therefore y = 0$

$(0, 0)$

x-intercepts: let $y = 0$.

$\therefore 250 - 0.4(x+5)^4 = 0$

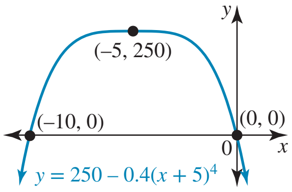
$\therefore (x+5)^4 = \frac{250}{0.4}$

$\therefore (x+5)^4 = 625$

$\therefore x + 5 = \pm 5$

$\therefore x = -10, x = 0$

$(-10, 0), (0, 0)$



d $y = -(6(x-2)^4 + 11)$

$\therefore y = -6(x-2)^4 - 11$

Maximum turning point $(2, -11)$

y-intercept: let $x = 0$.

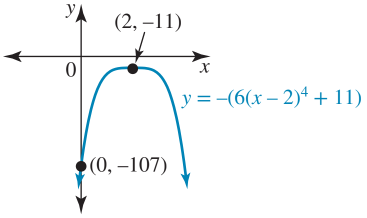
$\therefore y = -6(-2)^4 - 11$

$\therefore y = -6 \times 16 - 11$

$\therefore y = -107$

$(0, -107)$

There are no x-intercepts.



5 **a** A quartic graph with the same shape as $y = \frac{2}{3}x^4$ but whose turning point has the coordinates $(-9, -10)$ would have the equation $y = \frac{2}{3}(x+9)^4 - 10$.

b $y = a(x+b)^4 + c$

Turning point $(-3, -8)$

$\therefore y = a(x+3)^4 - 8$

Substitute the point $(-4, -2)$.

$\therefore\ -2 = a(-4+3)^4 - 8$

$\therefore\ -2 = a(-1)^4 - 8$

$\therefore\ -2 = a - 8$

$\therefore a = 6$

The equation is $y = 6(x+3)^4 - 8$

c $y = (ax+b)^4$ where $a > 0$ and $b < 0$.

Substitute the point $(0, 16)$.

$\therefore 16 = b^4$

$\therefore b = \pm 2$

Hence, $b = -2$ since $b < 0$.

$\therefore y = (ax-2)^4$

Substitute the point $(2, 256)$.

$\therefore 256 = (2a-2)^4$

$\therefore 2a - 2 = \pm 4$

$\therefore 2a = -2, \text{or } 2a = 6$

$\therefore a = -1, a = 3$

Hence, $a = 3$ since $a > 0$.

The equation is $y = (3x-2)^4$.

d $y = a(x-h)^4 + k$

From the graph the x-intercepts are $(-110, 0)$ and $(-90, 0)$, so the axis of symmetry has the equation $x = -100$.

The maximum turning point must be at $(-100, 10\,000)$.

The equation of the graph becomes

$y = a(x+100)^4 + 10\,000$.

Substitute the point $(-90, 0)$.

$\therefore 0 = a(10)^4 + 10\,000$

$\therefore a(10\,000) = -10\,000$

$\therefore a = -1$

The equation is $y = -(x+100)^4 + 10\,000$.

6 **a** $y = -(x+2)(x-3)(x-4)(x+5)$

x-intercepts: let $y = 0$.

$0 = -(x+2)(x-3)(x-4)(x+5)$

$(x+2) = 0, (x-3) = 0, (x-4) = 0, (x+5) = 0$

$x = -2, x = 3, x = 4, x = -5$

There are four x-intercepts: $(-2, 0)$, $(3, 0)$, $(4, 0)$, $(-5, 0)$.

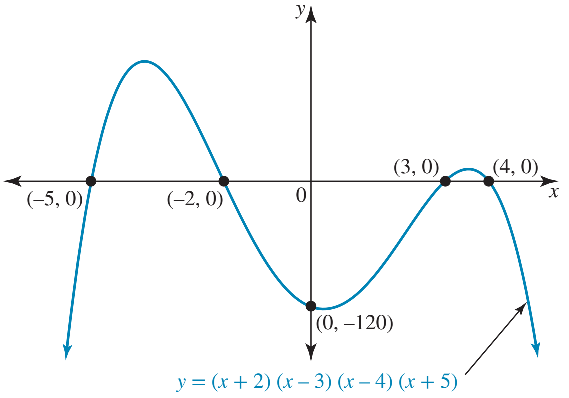
The graph cuts the x-axis at each intercept.

y-intercept: let $x = 0$.

$$\begin{aligned} y &= -(0+2)(0-3)(0-4)(0+5) \\ &= -(2)(-3)(-4)(5) \\ &= -120 \end{aligned}$$

The y-intercept is $(0, -120)$.

The leading coefficient is negative, so the graph starts below the x-axis.



b **i** The graph cuts the x-axis at $x = -1 \Rightarrow (x+1)$ is a factor.

The graph touches the x-axis at $x = 1 \Rightarrow (x-1)^2$ is a factor.

The graph cuts the x-axis at $x = 3 \Rightarrow (x-3)$ is a factor.

ii The equation is of the form $y = a(x+1)(x-1)^2(x-3)$.

iii Substitute the y-intercept $(0, -6)$ in the equation to determine the value of a.

$$\begin{aligned} -6 &= a(0+1)(0-1)^2(0-3) \\ &= a(1)(-1)^2(-3) \\ &= -3a \\ a &= 2 \end{aligned}$$

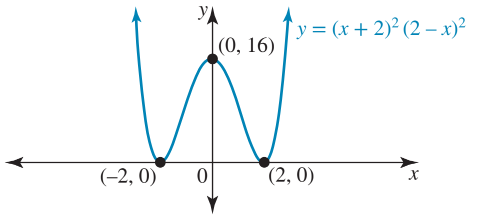
The equation of the graph is $y = 2(x+1)(x-1)^2(x-3)$.

7 $y = (x+2)^2(2-x)^2$

x-intercepts at $x = -2$ (touch) and $x = 2$ (touch)

y-intercept at $y = 16$

The leading term gives a positive x^4 shape.



8 The x-intercepts indicate the linear factors of the polynomial. As the graph cuts the x-axis at each of $x = -4, x = 0, x = 2, x = 5$, the equation of the graph is of the form $y = a(x+4)x(x-2)(x-5)$.

Substitute the given point $(-3, -30)$

$\therefore\ -30 = a(1)(-3)(-5)(-8)$

$\therefore\ -30 = -120a$

$\therefore a = \frac{1}{4}$

The equation of the given graph is

$y = \frac{1}{4}x(x+4)(x-2)(x-5)$.

9 **a** $y = (x+8)(x+3)(x-4)(x-10)$

x-intercepts: let $y = 0$.

$\therefore (x+8)(x+3)(x-4)(x-10) = 0$

$\therefore x = -8, x = -3, x = 4, x = 10$

$(-8, 0), (-3, 0), (4, 0), (10, 0)$ (all cuts)

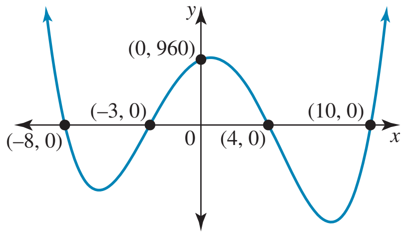
y-intercept: let $x = 0$.

$\therefore y = (8)(3)(-4)(-10)$

$\therefore y = 960$

$(0, 960)$

Shape: positive fourth degree polynomial



b $y = -\frac{1}{100}(x+3)(x-2)(2x-15)(3x-10)$

x-intercepts: let $y = 0$.

$\therefore -\frac{1}{100}(x+3)(x-2)(2x-15)(3x-10) = 0$

$\therefore x = -3, x = 2, x = \frac{15}{2}, x = \frac{10}{3}$

$(-3, 0), (2, 0), \left(\frac{15}{2}, 0\right), \left(\frac{10}{3}, 0\right)$ (all cuts)

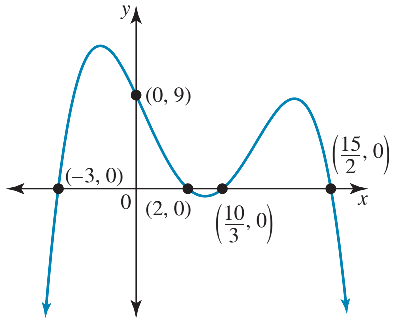
y-intercept: let $x = 0$.

$\therefore y = -\frac{1}{100}(3)(-2)(-15)(-10)$

$\therefore y = 9$

$(0, 9)$

Shape: $-\frac{1}{100}(x)(x)(2x)(3x)$ shows a negative fourth degree polynomial.



c $y = -2(x+7)(x-1)^2(2x-5)$

x-intercepts: let $y = 0$.

$\therefore\ -2(x+7)(x-1)^2(2x-5) = 0$

$\therefore x = -7, x = 1, x = 2.5$

$(-7, 0), (2.5, 0)$ and turning point $(1, 0)$

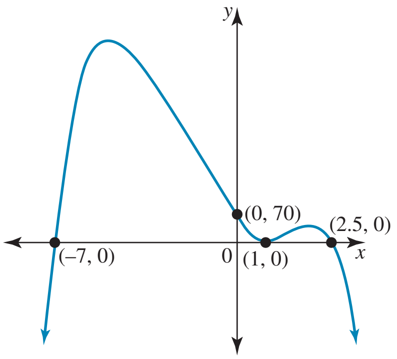
y-intercept: let $x = 0$.

$\therefore y = -2(7)(-1)^2(-5)$

$\therefore y = 70$

$(0, 70)$

Shape: $-2(x)(x)^2(2x)$ shows a negative fourth degree polynomial.



d $y = \frac{2}{3}x^2(4x - 15)^2$

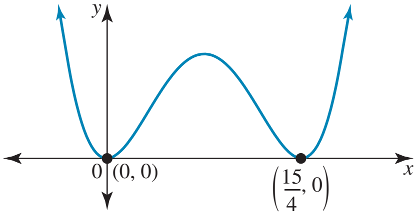
x-intercepts: let $y = 0$.

$\therefore \frac{2}{3}x^2(4x - 15)^2 = 0$

$\therefore x = 0, x = \frac{15}{4}$

$(0, 0), \left(\frac{15}{4}, 0\right)$ (both turning points)

Shape: positive fourth degree polynomial



e $y = 3(1 + x)^3(4 - x)$

x-intercepts: let $y = 0$.

$\therefore y = 3(1 + x)^3(4 - x) = 0$

$\therefore x = -1, \ x = 4$

Stationary point of inflection $(-1, 0)$ and other x-intercept $(4, 0)$ (cut)

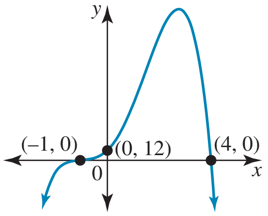
y-intercept: let $x = 0$.

$\therefore y = 3(1)^3(4)$

$\therefore y = 12$

$(0, 12)$

Shape: $3(x^3)(-x)$ shows a negative fourth degree polynomial.



f $y = (3x + 10)(3x - 10)^3$

x-intercepts: let $y = 0$.

$\therefore y = (3x + 10)(3x - 10)^3 = 0$

$\therefore x = -\frac{10}{3}, \ x = \frac{10}{3}$

Stationary point of inflection $\left(\frac{10}{3}, 0\right)$ and other x-intercept $\left(-\frac{10}{3}, 0\right)$ (cut)

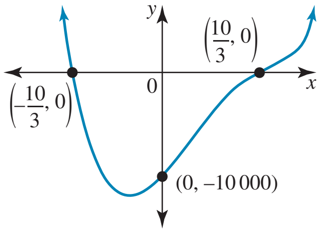
y-intercept: let $x = 0$.

$\therefore y = (10)(-10)^3$

$\therefore y = -10\,000$

$(0, -10\,000)$

Shape: $(3x)(3x)^3$ shows a positive fourth degree polynomial.



10 a The graph cuts the x-axis at $x = -6, x = -5, x = -3$ and $x = 4$. These values identify the linear factors, so the equation of the graph must be of the form $y = a(x + 6)(x + 5)(x + 3)(x - 4)$.

Substitute the point $(0, 5)$.

$\therefore 5 = a(6)(5)(3)(-4)$

$\therefore a = -\frac{1}{72}$

The equation is $y = -\frac{1}{72}(x + 6)(x + 5)(x + 3)(x - 4)$

b The x-intercepts of the graph occur at:

$x = -2$ (touch) $\Rightarrow (x + 2)^2$ is a factor.

$x = 0 \Rightarrow x$ is a factor.

$x = 4 \Rightarrow (x - 4)$ is a factor.

The equation is of the form $y = a(x + 2)^2x(x - 4)$.

Substitute the point $(3, 75)$.

$\therefore 75 = a(3 + 2)^2(3)(3 - 4)$

$\therefore 75 = a(25)(3)(-1)$

$\therefore 75 = -75a$

$\therefore a = -1$

The equation is $y = -x(x - 4)(x + 2)^2$.

c The x-intercepts of the graph occur at:

$x = -6 \Rightarrow (x + 6)$ is a factor.

$x = 0$ (saddle cut) $\Rightarrow x^3$ is a factor.

The equation is of the form $y = a(x + 6)x^3$.

Substitute the point $(-3, -54)$

$\therefore \ -54 = a(-3 + 6)(-3)^3$

$\therefore \ -54 = a(3)(-27)$

$\therefore \ -54 = -81a$

$\therefore a = \frac{54}{81}$

$\therefore a = \frac{2}{3}$

The equation is $y = \frac{2}{3}x^3(x + 6)$.

d The x-intercepts of the graph occur at:

$x = -1.5$ (touch) $\Rightarrow (x + 1.5)^2$ is a factor.

$x = 0.8$ (touch) $\Rightarrow (x - 0.8)^2$ is a factor.

The equation is of the form $y = a(x + 1.5)^2(x - 0.8)^2$.

Substitute the point $(0, 54)$.

$\therefore 54 = a(1.5)^2(-0.8)^2$

Using fractions rather than decimals,

$$54 = a\left(\frac{3}{2}\right)^2\left(-\frac{4}{5}\right)^2$$

$$\therefore 54 = a \times \frac{9 \times 16}{4 \times 25}$$

$$\therefore 54 = a \times \frac{36}{25}$$

$$\therefore a = \frac{54 \times 25}{36}$$

$$\therefore a = \frac{3 \times 25}{2}$$

$$\therefore a = \frac{75}{2}$$

The equation becomes

$$y = \frac{75}{2}\left(x + \frac{3}{2}\right)^2\left(x - \frac{4}{5}\right)^2$$

$$= \frac{75}{2} \times \frac{1}{4}(2x+3)^2 \times \frac{1}{25}(5x-4)^2$$

$$= \frac{3}{8}(2x+3)^2(5x-4)^2$$

$$\therefore y = \frac{3}{8}(2x+3)^2(5x-4)^2$$

11 $p(x) = 3x^4 + ax^3 + 2x^2 - 5x + 12$

$p(-2)$

$= 3(-2)^4 + a(-2)^3 + 2(-2)^2 - 5(-2) + 12$

$= 3(16) + a(-8) + 2(4) + 10 + 12$

$= 48 - 8a + 8 + 10 + 12$

$= 78 - 8a$

Since $p(-2) = 14$,

$14 = 78 - 8a$

$8a = 78 - 14$

$8a = 64$

$a = 8$

12 **a** $x(3x+1)(x-5)(4x+3) = 0$

Using the Null Factor Law,

$x = 0, 3x + 1 = 0, x - 5 = 0, 4x + 3 = 0$

$x = 0,\ 5,\ -\frac{1}{3},\ -\frac{3}{4}$

b $(x-2)^2(5x+3)(4-x) = 0$

Using the Null Factor Law,

$x - 2 = 0, 5x + 3 = 0, 4 - x = 0$

$x = 2, 4, -\frac{3}{5}$

c $(x+2)^2(2x-1)^2 = 0$

Using the Null Factor Law,

$x + 2 = 0, 2x - 1 = 0$

$x = -2, \frac{1}{2}$

d $(x+4)(2-x)^3 = 0$

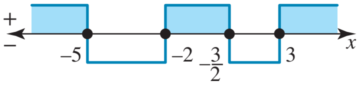
Using the Null Factor Law,

$x + 4 = 0, 2 - x = 0$

$x = -4, 2$

13 $p(x) = (x+2)(x-3)(x+5)(2x+3)$

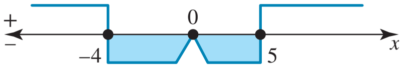
The leading coefficient is positive, so the sign diagram is:



For $p(x) \geq 0$: $x \leq -5$ or $-2 \leq x \leq -\frac{3}{2}$ or $x \geq 3$

14 **a** $p(x) = x^2(x+4)(x-5)$

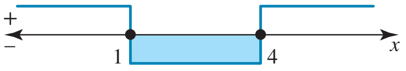
The leading coefficient is positive, so the sign diagram is:



For $p(x) < 0$: $-4 < x < 0$ or $0 < x < 5$

b $p(x) = (x-4)(x-1)^3$

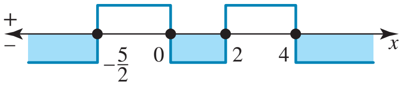
The leading coefficient is positive, so the sign diagram is:



For $p(x) < 0$: $1 < x < 4$

c $p(x) = x(x-2)(4-x)(2x+5)$

The leading coefficient is negative, so the sign diagram is:



For $p(x) < 0$: $x < -\frac{5}{2}$ or $0 < x < 2$ or $x > 4$

d $p(x) = (x+3)^2(3-x)^2$

The leading coefficient is positive, so the sign diagram is:

Since $p(x) \geq 0$, $p(x) < 0$ has no solutions.

15 **a** $p(x) = (x+5)(x-2)\left(x^2+1\right)$

The zeros of $p(x)$ are $x = -5, 2$

as $\left(x^2+1\right) > 0$ for all x.

The leading coefficient is positive, so the sign diagram is:

For $p(x) \geq 0$: $x \leq -5$ or $x \geq 2$

b $p(x) = (x-4)(6-x)\left(x^2-4x-6\right)$

The zeros of $p(x)$ are $x = 4, 6,$

and for

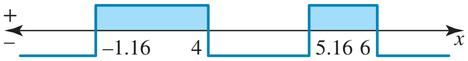
$x^2 - 4x - 6 = 0,$

the quadratic formula gives:

$x = 2 \pm \sqrt{10}$

$x \cong -1.16$ or 5.16

The leading coefficient is negative, so the sign diagram is:



For $p(x) \geq 0$: $-1.16 \leq x \leq 4$ or $5.16 \leq x \leq 6$

c $p(x) = \left(x^2+3x-5\right)\left(x^2-9\right)$

$p(x) = \left(x^2+3x-5\right)(x-3)(x+3)$

The zeros of $p(x)$ are $x = -3, 3,$

and for

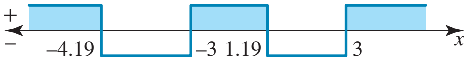
$x^2 + 3x - 6 = 0,$

the quadratic formula gives:

$$x = \frac{-3 \pm \sqrt{29}}{2}$$

$x \cong -4.19$ or 1.19

The leading coefficient is positive, so the sign diagram is:



For $p(x) \geq 0$: $x \leq -4.19$ or $-3 \leq x \leq 1.19$ or $x \geq 3$

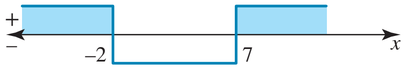
16 **a** $(x-7)(x+2)\left(x^2+1\right)>0$

Using the Null Factor Law,

the zeros are $x=-2,\ 7$

as $\left(x^2+1\right)>0$ for all x.

The leading coefficient is positive, so the sign diagram is:



For $(x-7)(x+2)\left(x^2+1\right)>0$,

$x<-2$ or $x>7$.

b $(x+3)(5-x)\left(x^2-x-7\right)\leq 0$

Using the Null Factor Law,

the zeros are $x=-3,\ 5$,

and for

$x^2-x-7=0$,

the quadratic formula gives:

$$x=\frac{1\pm\sqrt{29}}{2}$$

$x\cong-2.19$ or 3.19

The leading coefficient is negative, so the sign diagram is:

For $(x+3)(5-x)\left(x^2-x-7\right)\leq 0$

$x\leq-3$ or $-2.19\leq x\leq 3.19$ or $x\geq 5$

c $\left(3x^2-8x+2\right)\left(4x^2-25\right)>0$

$\left(3x^2-8x+2\right)(2x-5)(2x+5)>0$

Using the Null Factor Law,

the zeros are $x=-\frac{5}{2},\frac{5}{2}$,

and for

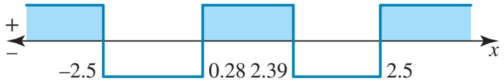
$3x^2-8x+2=0$,

the quadratic formula gives:

$$x=\frac{4\pm\sqrt{10}}{3}$$

$x\cong 0.28$ or 2.39

The leading coefficient is positive, so the sign diagram is:



For $\left(3x^2-8x+2\right)\left(4x^2-25\right)>0$

$x<-2.5$ or $0.28<x<2.39$ or $x>2.5$

d $\left(x^2-2x+8\right)\left(2x^2-3x+5\right)>0$

Using the Null Factor Law, either

$\left(x^2-2x+8\right)=0$ or $\left(2x^2-3x+5\right)=0$

For $x^2-2x+8=0$:

$\Delta=(-2)^2-4\times1\times8=-28$

No zeros and $x^2-2x+8>0$ for all x.

For $2x^2-3x+5=0$:

$\Delta=(-3)^2-4\times2\times5=-31$

No zeros and $2x^2-3x+5>0$ for all x.

$\left(x^2-2x+8\right)\left(2x^2-3x+5\right)>0$

$x\in R$

17 $(x+2)^4-13(x+2)^2-48=0$.

Let $a=(x+2)^2$.

$\therefore a^2-13a-48=0$

$\therefore (a-16)(a+3)=0$

$\therefore a=16$ or $a=-3$

$\therefore (x+2)^2=16 \qquad (x+2)^2=-3$

$x+2=\pm4 \qquad (x+2)^2\geq 0$

$\therefore x=2$ or -6

18 **a** $8x^4+72x^3=0$

Take out common factors:

$8x^3(x+9)=0$

$x=0, x=-9$

b

$$\begin{aligned}4x^4&=16x^2\\4x^4-16x^2&=0\\4x^2(x^2-4)&=0\end{aligned}$$

$4x^2(x-2)(x+2)=0$

$x=0,2,-2$

c $(x+3)^4+2(x+3)^2-15=0$

Let $a=(x+3)^2$.

$a^2+2a-15=0$

$(a+5)(a-3)=0$

$a=-5, a=3$

Substitute back:

$(x+3)^2=-5$ or $(x+3)^2=3$

Reject $(x+3)^2=-5$ because the perfect square equals a negative number.

Therefore,

$(x+3)^2=3$

$x+3=\pm\sqrt{3}$

$x=-3\pm\sqrt{3}$

d $9(x-1)^4-49(x-1)^2=0$

Factorise:

$(x-1)^2\left[9(x-1)^2-49\right]=0$

Using difference of two squares:

$(x-1)^2(3(x-1)-7)(3(x-1)+7)=0$

Simplify:

$(x-1)^2(3x-10)(3x+4)=0$

Use the Null Factor Law:

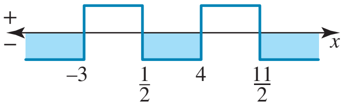
$x-1=0, 3x-10=0, 3x+4=0$

$x=1, \frac{10}{3}$ or $-\frac{4}{3}$

19 **a** $(x+3)(2x-1)(4-x)(2x-11)<0$

Zeros: $x=-3, x=\frac{1}{2}, x=4, x=\frac{11}{2}$ (all cuts)

Leading term: $(x)(2x)(-x)(2x)=-4x^4$



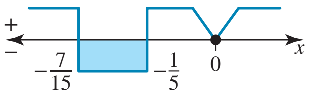
Answer: $x<-3$ or $0.5<x<4$ or $x>5.5$

b $300x^4+200x^3+28x^2\leq 0$

$\therefore 4x^2\left(75x^2+50x+7\right)\leq 0$

$\therefore 4x^2(15x+7)(5x+1)\leq 0$

Zeros: $x = 0$ (touch), $x = -\frac{7}{15}$ and $x = -\frac{1}{5}$



Answer: $-\frac{7}{15} \le x \le -\frac{1}{5}$ or $x = 0$

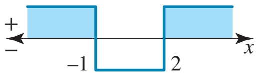
c $x^3(x+1) - 8(x+1) > 0$

$(x+1)\left(x^3 - 8\right) > 0$

$(x+1)(x-2)\left(x^2 + 2x + 4\right) > 0$

Zeros: $x = -1$ and $x = 2$

The leading coefficient is positive.



Answer: $x < -1$ or $x > 2$

d $20(2x-1)^4 - 8(1-2x)^3 \ge 0$

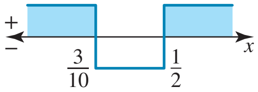
$\therefore 4(2x-1)^3(5(2x-1)+2) \ge 0$

$\therefore 4(2x-1)^3(10x-3) \ge 0$

Zeros: $x = \frac{1}{2}$ (multiplicity 3) and $x = \frac{3}{10}$

A zero of multiplicity 3 is just a ‘cut’ on a sign diagram.

The leading term in $4(2x)^3(10x)$ is positive.



Answer: $x \le \frac{3}{10}$ or $x \ge \frac{1}{2}$

20 a $y = a(x-b)^4 + c$

Turning point $(-2, 4) \Rightarrow y = a(x+2)^4 + 4$

Point $(0, 0) \Rightarrow 0 = a(2)^4 + 4$

$\therefore a = -\frac{4}{16}$

$\therefore a = -\frac{1}{4}$

Hence, the equation is $y = -\frac{1}{4}(x+2)^4 + 4$.

b x-intercepts: let $y = 0$.

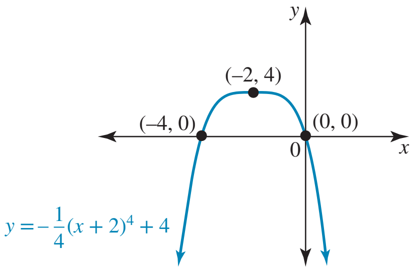
$0 = -\frac{1}{4}(x+2)^4 + 4$

$\therefore (x+2)^4 = 16$

$\therefore x + 2 = \pm\sqrt[4]{16}$

$\therefore x + 2 = \pm 2$

$\therefore x = -4, x = 0$



$\{x : -\frac{1}{4}(x+2)^4 + 4 > 0\} = \{x : -4 < x < 0\}$

21 a $y = ax^4 + k$

Substitute the point $(-1, 1)$.

$\therefore 1 = a(-1)^4 + k$

$\therefore 1 = a + k \qquad [1]$

Substitute the point $\left(\frac{1}{2}, \frac{3}{8}\right)$.

$\therefore \frac{3}{8} = a\left(\frac{1}{2}\right)^4 + k$

$\therefore \frac{3}{8} = \frac{1}{16}a + k \qquad [2]$

Subtract equation [2] from equation [1].

$\therefore \frac{5}{8} = \frac{15}{16}a$

$\therefore a = \frac{5}{8} \times \frac{16}{15}$

$\therefore a = \frac{2}{3}$

Substitute $a = \frac{2}{3}$ in equation [1].

$\therefore 1 = \frac{2}{3} + k$

$\therefore k = \frac{1}{3}$

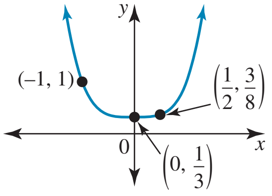
$a = \frac{2}{3}, k = \frac{1}{3}$

b The equation of the curve is $y = \frac{2}{3}x^4 + \frac{1}{3}$.

Hence, the minimum turning point is $\left(0, \frac{1}{3}\right)$.

c The axis of symmetry is $x = 0$.

d There are no x-intercepts and the turning point is the y-intercept.



22 $y = a(x+b)^4 + c$

a As the line joining the points $(-2, 3)$ and $(4, 3)$ is horizontal, the axis of symmetry passes through their midpoint.

$x = \frac{-2+4}{2}$

$= 1$

The axis of symmetry has the equation $x = 1$.

b The maximum turning point is $(1, 10)$.

c The equation is of the form $y = a(x-1)^4 + 10$.

Substitute the point $(4, 3)$.

$\therefore 3 = a(3)^4 + 10$

$\therefore 81a = -7$

$\therefore a = -\frac{7}{81}$

The equation is $y = -\frac{7}{81}(x-1)^4 + 10$.

d y-intercept: let $x = 0$.

$\therefore y = -\frac{7}{81}(-1)^4 + 10$

$\therefore y = -\frac{7}{81} + 10$

$\therefore y = \frac{-7 + 810}{81}$

$\therefore y = \frac{803}{81}$

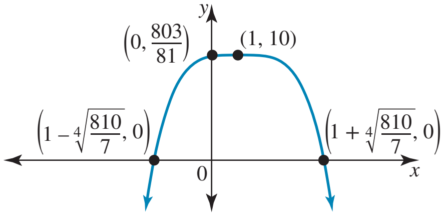
The y-intercept is $\left(0, \frac{803}{81}\right)$.

e x-intercepts: let $y = 0$.

$$\therefore 0 = -\frac{7}{81}(x-1)^4 + 10$$
$$\therefore 7(x-1)^4 = 810$$
$$\therefore (x-1)^4 = \frac{810}{7}$$
$$\therefore x - 1 = \pm\sqrt[4]{\frac{810}{7}}$$
$$\therefore x = 1 \pm \sqrt[4]{\frac{810}{7}}$$

The exact x-intercepts are $\left(1 - \sqrt[4]{\frac{810}{7}}, 0\right)$ and $\left(1 + \sqrt[4]{\frac{810}{7}}, 0\right)$.

f



23 a $-x^4 + 18x^2 - 81$.

Let $a = x^2$.

$\therefore \; -a^2 + 18a - 81$

$= -(a^2 - 18a + 81)$

$= -(a - 9)^2$

Substitute back for x:

$= -(x^2 - 9)^2$

$= -((x-3)(x+3))^2$

$= -(x-3)^2(x+3)^2$

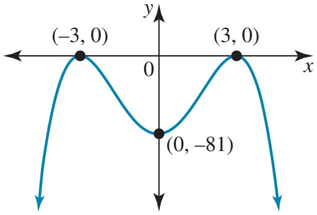
$\therefore \; -x^4 + 18x^2 - 81 = -(x-3)^2(x+3)^2$

b $y = -x^4 + 18x^2 - 81 \Rightarrow y = -(x-3)^2(x+3)^2$

x-intercepts occur at $x = \pm 3$ (both touch)

y-intercept $(0, -81)$

Shape: negative fourth degree polynomial



c As no part of the graph lies above the x-axis, $\{x : -x^4 + 18x^2 - 81 > 0\} = \varnothing$.

d $\{x : x^4 - 18x^2 + 81 > 0\}$

If $x^4 - 18x^2 + 81 > 0$, then $-1 \times \left(x^4 - 18x^2 + 81\right) < 0$.

$\therefore \; -x^4 + 18x^2 - 81 < 0$

As the graph in part **b** lies below the x-axis at all places except for the x-intercepts at $x = \pm 3$, $\{x : x^4 - 18x^2 + 81 > 0\} = R \setminus \{-3, 3\}$.

24 a At the intersection of the graphs of $y = x^4$ and $y = 2x^3$,

$x^4 = 2x^3$,

$\therefore x^4 - 2x^3 = 0$

$\therefore x^3(x - 2) = 0$

$\therefore x = 0, x = 2$

When $x = 0, y = 0 \Rightarrow (0, 0)$

When $x = 2, y = 16 \Rightarrow (2, 16)$

The point P is $(2, 16)$.

b Given the point $(2, 16)$ lies on $y = ax^2$,

$16 = a(2)^2$

$\therefore 4a = 16$

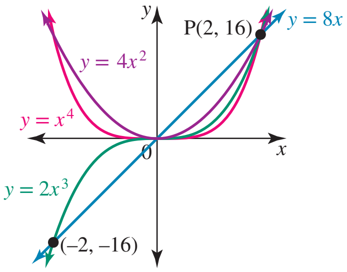
$\therefore a = 4$

Given the point $(2, 16)$ lies on $y = mx$,

$16 = m(2)$

$\therefore m = 8$

c The parabola has the equation $y = 4x^2$ and the line has the equation $y = 8x$.



d i The graphs of $y = nx^3$ and $y = x^4$ intersect when

$nx^3 = x^4$

$\therefore nx^3 - x^4 = 0$

$\therefore x^3(n - x) = 0$

$\therefore x = 0, x = n$

When $x = 0, y = 0 \Rightarrow (0, 0)$

When $x = n, y = n^4 \Rightarrow \left(n, n^4\right)$

The graphs intersect at $(0, 0)$ and (n, n^4).

ii Substitute the point $\left(n, n^4\right)$ in $y = ax^2$.

$\therefore n^4 = an^2$

$\therefore a = n^2$

Substitute the point $\left(n, n^4\right)$ in $y = mx$.

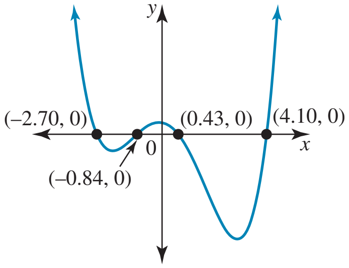
$\therefore n^4 = mn$

$\therefore m = n^3$

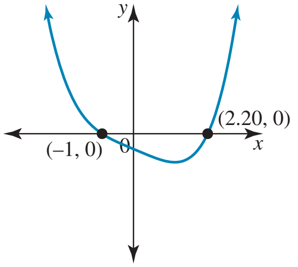
25 a Sketch the graph on the graphing screen and use the Analysis tools to obtain the required points.

The x-intercepts are $(-2.70, 0)$, $(-0.84, 0)$, $(0.43, 0)$, $(4.10, 0)$

The minimum turning points are $(-2, -12)$, $(2.92, -62.19)$ and the maximum turning point is $(-0.17, 4.34)$.



b $y = x^4 - 7x - 8$



There is one minimum turning point at $(1.21, -14.33)$ and the x-intercepts are $(-1, 0)$, $(2.20, 0)$.

5.2 Exam questions

1 $y = ax^4 + bx^3 + cx^2 + dx + e$ with $a > 0$
As $x \to \pm\infty$, $y \to \infty$.
Option **E** does not satisfy this; however, all other graphs do.
The correct answer is **E**.

2 $f(x) = -(x-a)(x-b)(x-c)^2$
The shape is a negative quartic.
$\therefore -x^4$ term
The intercepts are at a, b and c. (*Note*: a and b are positive but represent negative values due to their placement on the x-axis.)
The intercept at c has multiplicity 2.
The correct answer is **A**.

3 $y = -(x+1)^4 + 16$
The equation is of the form $y = a(x-h)^4 + k$.
$a = -1,\ h = -1,\ k = 16$ [1 mark]
The turning point is $(-1, 16)$.
y-intercept ($x = 0$):

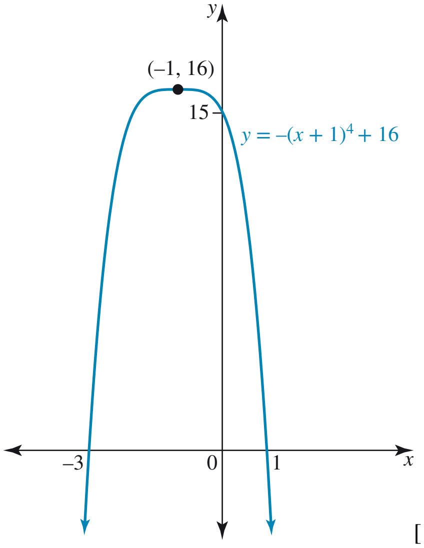
$$y = -(1)^4 + 16$$
$$= 15$$
$$\therefore (0, 15)$$

[1 mark]

x-intercept(s) ($y = 0$):

$$-(x+1)^4 + 16 = 0$$
$$(x+1)^4 = 16$$
$$x + 1 = \sqrt[4]{16}$$
$$x + 1 = \pm 2$$
$$x = 1, -3$$
$$\therefore (1, 0), (-3, 0)$$

[1 mark]

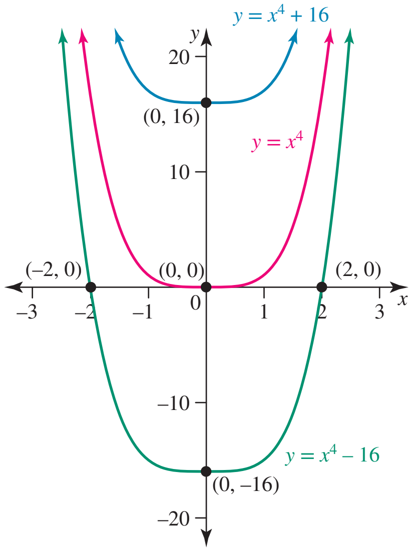


[1 mark]

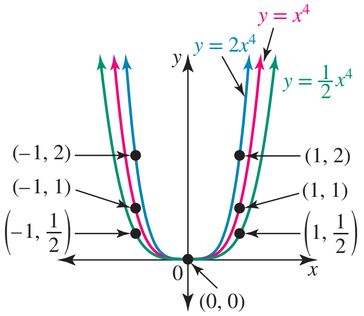
5.3 Families of polynomials

5.3 Exercise

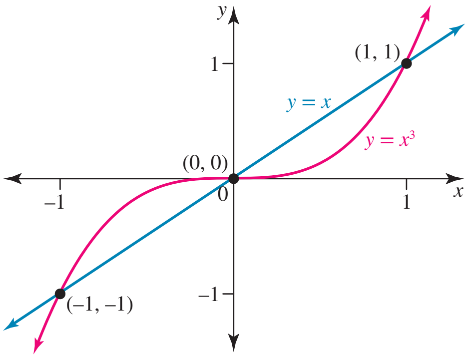
1 $y = x^4$, $y = x^4 + 16$ and $y = x^4 - 16$
All three graphs have a minimum turning point.
The turning points are $(0, 0)$, $(0, 16)$ and $(0, -16)$, respectively.



2 $y = x^4, y = 2x^4, y = \frac{1}{2}x^4$
All three graphs have a minimum turning point at the origin.
The points $(\pm 1, 1)$ lie on $y = x^4$, the points $(\pm 1, 2)$ lie on $y = 2x^4$, and the points $\left(\pm 1, \frac{1}{2}\right)$ lie on $y = \frac{1}{2}x^4$.



3 a $y = x^3$ and $y = x$
$y = x^3$ is a cubic with a stationary point of inflection at $(0, 0)$ passing through the points $(1, 1)$ and $(-1, -1)$.
$y = x$ is a straight line with a positive gradient passing through the points $(0, 0)$, $(1, 1)$ and $(-1, -1)$.
The graphs intersect at the points where $x = -1, 0, 1$.



b For $\{x : x^3 \le x\}$, the cubic is below the straight line.

$\therefore x \le -1$ or $0 \le x \le 1$

4 **a** $p(x) = a(x-h)^3$

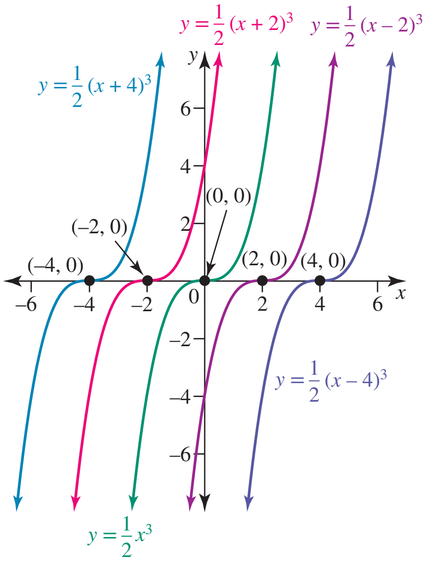
i If $a > 0$, the cubic polynomials have a stationary point of inflection at $(h, 0)$.

As $x \to +\infty, y \to +\infty$

and as

$x \to -\infty, y \to -\infty$.

ii



b If $a = -\frac{1}{2}$, the cubic polynomials would still have points of inflection at $(h, 0)$ but would be reflected in the x-axis, meaning that

as $x \to +\infty, y \to -\infty$

and

as $x \to -\infty, y \to +\infty$.

5 $p(x) = (x-3)^4 + c$

a The minimum turning point is at $(3, c)$

i $c > 0$

The minimum turning point is above the x-axis, so the curve is always positive, with no x-axis intercepts.

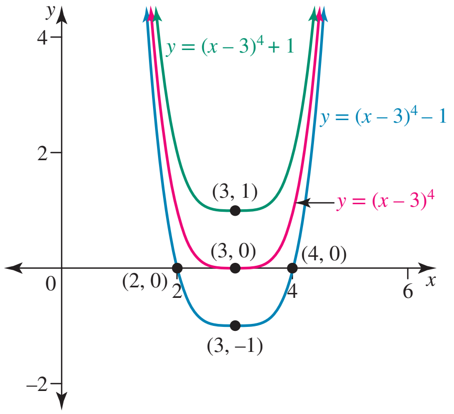
ii $c = 0$

The minimum turning point is on the x-axis, so the curve touches the x-axis at $x = 3$.

iii $c < 0$

The minimum turning point is below the x-axis, so the curve has two x-axis intercepts.

b



6 $y = a(x-b)^4 + c$

a Minimum turning point at $(-1, -12)$

So $y = a(x+1)^4 - 12$.

The graph passes through $(0, 0)$, so substitute:

$0 = a(0+1)^4 - 12$

$a = 12$

$\therefore y = 12(x+1)^4 - 12$

b The axis of symmetry passes through the turning point.

$\therefore x = -1$

c For x-intercept(s), let $y = 0$.

$12(x+1)^4 - 12 = 0$

$(x+1)^4 = 1$

$x + 1 = \pm 1$

$x = 0$ or -2

$\therefore (-2, 0)$ is the other x-intercept.

7 $p(x) = a(2-x)(2+x)^3$

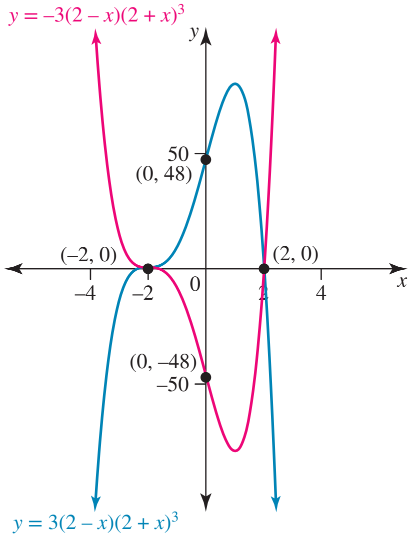
a The quartic cuts the x-axis at $(2, 0)$ and has a stationary point of inflection at $(-2, 0)$

If $a = 3$, it is a negative x^4,

so as $x \to +\infty, y \to -\infty$

If $a = -3$, it is a positive x^4,

so as $x \to +\infty, y \to +\infty$



b The curves are the same shape with x-intercepts at $(2, 0)$ and $(-2, 0)$

One curve is the reflection of the other over the x-axis, so the maximum turning point of one curve is a minimum turning point of the other curve.

8 a The section of the graph shown cuts the x-axis at $x = -1$ and at $x = 1$, and touches the x-axis at $x = 0$.

A quartic graph with these features would have an equation of the form $y = a(x+1)(x-0)^2(x-1)$, that is $y = ax^2(x+1)(x-1)$.

For a monic polynomial, $a = 1$.

The equation is $y = x^2(x+1)(x-1)$.

b i The family of quartic polynomials is not necessarily monic.

$\therefore y = ax^2(x+1)(x-1)$

ii If the graph passes through the point $(-2, 6)$, substitute and solve for a.

$6 = a(-2)^2(-2+1)(-2-1)$

$6 = a \times 4 \times (-1) \times (-3)$

$6 = 12a$

$a = \dfrac{1}{2}$

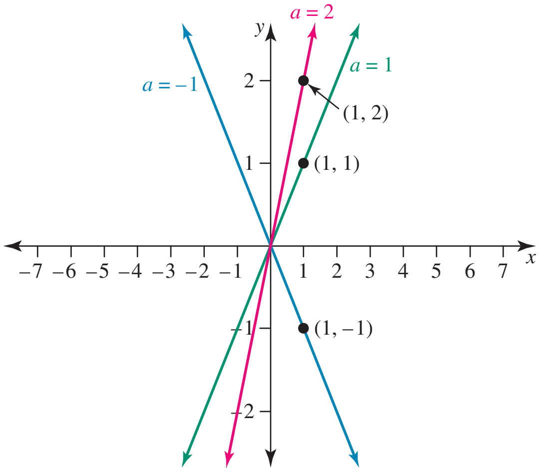
$\therefore y = \dfrac{1}{2}x^2(x+1)(x-1)$

9 a $y = ax$

If $a = 1$, the equation becomes $y = x$.

If $a = 2$, the equation becomes $y = 2x$.

If $a = -1$, the equation becomes $y = -x$.



b Substitute $(-4, -10)$ in $y = ax$.

$-10 = a(-4)$

$a = \dfrac{5}{2}$

10 a $y = -x^2 + k$

Substitute $(-5, 30)$

$30 = -(-5)^2 + k$

$30 = -25 + k$

$k = 55$

The equation of the required parabola is $y = -x^2 + 55$.

b The turning point has coordinates $(0, k)$. As the parabola is concave down, this turning point is a maximum.

c For a concave down parabola to lie below the x-axis, its maximum turning point should have a negative y-coordinate. Hence, if $k < 0$, the graph lies below the x-axis.

11 $y = x^4 - mx^3$

a x-intercepts: let $y = 0$.

$\therefore x^4 - mx^3 = 0$

$\therefore x^3(x - m) = 0$

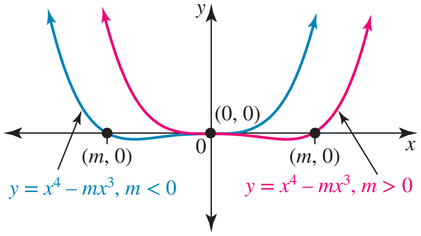
$\therefore x = 0, x = m$

$(0, 0), (m, 0)$

b At the origin there is a stationary point of inflection.

If $m < 0$, the point $(m, 0)$ lies to the left of the origin, and if $m > 0$, it will lie to the right of the origin.

The shape of the graph would be one of the forms shown.



If $m < 0$, the stationary point of inflection at the origin increases from negative to positive as x increases, and if $m > 0$, it decreases from positive to negative.

c $y = x^4 - mx^3$

Substitute the point $(-1, -16)$.

$\therefore -16 = (-1)^4 - m(-1)^3$

$\therefore -16 = 1 + m$

$\therefore m = -17$

The required curve has equation $y = x^4 + 17x^3$.

12 a The equation of a line is $y - y_1 = m(x - x_1)$.

The lines pass through the point $(2, 3)$.

$\therefore y - 3 = m(x - 2)$

$\therefore y = mx - 2m + 3$

b Substitute $(0, 0)$.

$\therefore 0 = m(0) - 2m + 3$

$\therefore 0 = -2m + 3$

$\therefore m = \dfrac{3}{2}$

Substitute $m = \dfrac{3}{2}$ in $y = mx - 2m + 3$.

$y = \dfrac{3}{2}x - 2 \times \dfrac{3}{2} + 3$

$\therefore y = \dfrac{3}{2}x$

13 a $y = ax^2 + bx$

All of the parabolas pass through the origin.

b The point $(2, 6)$ must lie on the parabola.

$\therefore 6 = a(2)^2 + b(2)$

$\therefore 6 = 4a + 2b$

$\therefore 2a + b = 3$

$\therefore b = 3 - 2a$

The equation of the parabola is $y = ax^2 + (3 - 2a)x,\ a \neq 0$.

14 $y = mx^2 - x^4, m > 0$

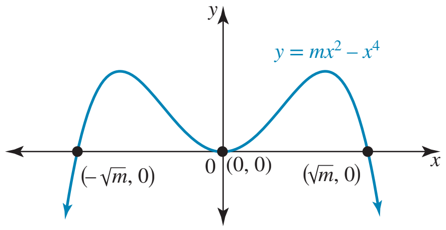
$\therefore y = x^2\left(m - x^2\right)$

Since $m > 0, \sqrt{m}$ is real.

$\therefore y = x^2\left(\sqrt{m} - x\right)\left(\sqrt{m} + x\right)$

The x-intercepts are $(0, 0)$, which is a turning point, and $\left(\sqrt{m}, 0\right)$ and $\left(-\sqrt{m}, 0\right)$.

Shape: negative fourth degree polynomial



15 $y = a(x + b)^3 + c$

Stationary point of inflection at $(-1, 7)$

so $y = a(x + 1)^3 + 7$

The curve passes through the point $(-2, -20)$, so

$-20 = a(-2 + 1)^3 + 7$

$-20 = a(-1)^3 + 7$

$-20 = -a + 7$

$a = 27$

$\therefore y = 27(x + 1)^3 + 7$

16 Turning point at $(-2, 0) \Rightarrow (x - 2)^2$ is a factor.

x-intercept $(4, 0) \Rightarrow (x - 4)$ is a factor.

As the polynomial has degree 4, there is one other linear factor.

Let this factor be $(ax + b)$.

$\therefore y = (x - 2)^2(x - 4)(ax + b)$

However, the polynomial is monic, so the coefficient of x^4 must be 1.

$\therefore a = 1$

$\therefore y = (x - 2)^2(x - 4)(x + b)$

Substitute $(0, 48)$.

$\therefore 48 = (-2)^2(-4)(b)$

$\therefore -48 = -16b$

$\therefore b = 3$

The equation is $y = (x + 2)^2(x - 4)(x - 3)$ and the other x-intercept is $(3, 0)$.

17 $y = a(x - 3)^2 + 5 - 4a, a \in R \setminus \{0\}$.

a Let $x = 1$.

$\therefore y = a(-2)^2 + 5 - 4a$

$\therefore y = 4a + 5 - 4a$

$\therefore y = 5$

Therefore, the point $(1, 5)$ lies on every member of the family regardless of the value of a.

b The turning point is $(3, 5 - 4a)$. If this lies on the x-axis, then $5 - 4a = 0$.

This means $a = \dfrac{5}{4}$ for the turning point to lie on the x-axis.

c The turning point is $(3, 5 - 4a)$ and could be a maximum or a minimum.

If the turning point is a maximum, $a < 0$, and there will be no x-intercepts if $5 - 4a < 0$.

However, if $a < 0$, then $-4a > 0$, so $5 - 4a$ could never be negative.

If the turning point is a minimum, $a > 0$, and there will be no x-intercepts if $5 - 4a > 0$.

$5 - 4a > 0$

$\therefore 5 > 4a$

$\therefore a < \dfrac{5}{4}$

Hence, if $0 < a < \dfrac{5}{4}$, the parabolas will have no x-intercepts.

18 **a** The family for which $y = k$ is a set of horizontal lines.

The family for which $y = x^2 + bx + 10$ is the set of concave up parabolas with y-intercept $(0, 10)$.

b $y = x^2 + bx + 10$ for $b = -7$ becomes $y = x^2 - 7x + 10$.

$\therefore y = (x - 2)(x - 5)$

x-intercepts: $(2, 0), (5, 0)$

Axis of symmetry: $x = \dfrac{2 + 5}{2} = 3.5$

Turning point: let $x = 3.5$.

$\therefore y = (1.5)(-1.5)$

$\therefore y = -2.25$

Minimum turning point $(3.5, -2.25)$.

For $b = 7$

$y = x^2 + 7x + 10$

$= (x + 2)(x + 5)$

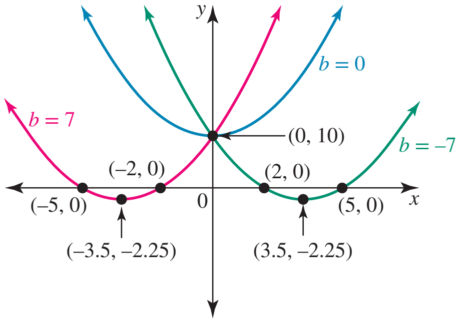
Hence,

x-intercepts: $(-2, 0), (-5, 0)$

Minimum turning point $(-3.5, -2.25)$.

For $b = 0, y = x^2 + 10$

No x-intercepts and minimum turning point $(0, 10)$



c **i** A horizontal line intersects $y = x^2 + 7x + 10$ once when it is a tangent at the turning point $(-3.5, -2.25)$, so $k = -2.25$.

ii Any horizontal line for which $k > -2.25$ will intersect the parabola twice.

iii Any horizontal line for which $k < -2.25$ will not intersect the parabola.

d At the intersection of $y = 7$ and $y = x^2 + bx + 10$,

$x^2 + bx + 10 = 7$

$\therefore x^2 + bx + 3 = 0$

$\Delta = b^2 - 4 \times 1 \times 3$

$\therefore \Delta = b^2 - 12$

i For one intersection, $\Delta = 0$

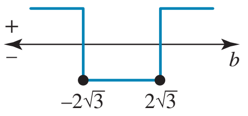
$\therefore b^2 - 12 = 0$

$\therefore b = \pm\sqrt{12}$

$\therefore b = \pm 2\sqrt{3}$

ii and **iii**

Consider the sign diagram of the discriminant.



For two intersections, $\Delta > 0$.

$\therefore b < -2\sqrt{3}$ or $b > 2\sqrt{3}$

For no intersections, $\Delta < 0$.

$\therefore -2\sqrt{3} < b < 2\sqrt{3}$

19 $y = ax^3 + (3 - 2a)x^2 + (3a + 1)x - 4 - 2a$, where $a \in R\backslash\{0\}$.

a Let $x = 1$.

$\therefore y = a + (3 - 2a) + (3a + 1) - 4 - 2a$

$\therefore y = (a - 2a + 3a - 2a) + (3 + 1 - 4)$

$\therefore y = 0$

Therefore, the point $(1, 0)$ is common to all this family.

b Substitute $(0, 0)$ in the curve's equation.

$\therefore 0 = 0 + 0 + 0 - 4 - 2a$

$\therefore 2a = -4$

$\therefore a = -2$

The equation of the curve with $a = -2$ becomes

$y = -2x^3 + (3 + 4)x^3 + (-6 + 1)x$

$\therefore y = -2x^3 + 7x^2 - 5x$

For the graph, it is known the origin and $(1, 0)$ are x-intercepts. There is one more to obtain.

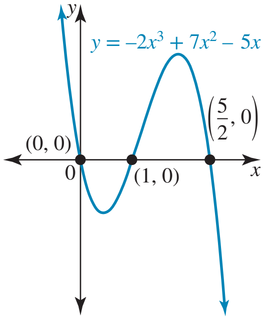
$y = -2x^3 + 7x^2 - 5x$

$= -x(2x^2 - 7x + 5)$

$= -x(x - 1)(2x - 5)$

The other x-intercept is $\left(\frac{5}{2}, 0\right)$.

The shape of the graph is for a negative third degree polynomial.



c Substitute the point $(-1, -10)$ into the curve's equation to calculate a.

$-10 = -a + (3 - 2a)(1) + (3a + 1)(-1) - 4 - 2a$

$\therefore -10 = (-a - 2a - 3a - 2a) + (3 - 1 - 4)$

$\therefore -10 = -8a - 2$

$\therefore -8 = -8a$

$\therefore a = 1$

With $a = 1$, the equation becomes

$y = x^3 + (3 - 2)x^2 + (3 + 1)x - 4 - 2$

$\therefore y = x^3 + x^2 + 4x - 6$

As $(1, 0)$ is an x-intercept, $(x - 1)$ is a factor

$\therefore x^3 + x^2 + 4x - 6 = (x - 1)(x^2 + bx + 6)$

$= (x - 1)(x^2 + 2x + 6)$

Consider the quadratic factor $x^2 + 2x + 6$.

Its discriminant is $\Delta = 4 - 4 \times 1 \times 6 = -20$

Since $\Delta < 0$, there are no real solutions to $x^2 + 2x + 6 = 0$

Hence, the cubic graph has exactly one x-intercept.

d At the intersection of

$y = ax^3 + (3 - 2a)x^2 + (3a + 1)x - 4 - 2a$ and

$y = (a - 1)x^3 - 2ax^2 + (3a - 2)x - 2a - 5$,

$ax^3 + (3 - 2a)x^2 + (3a + 1)x - 4 - 2a$

$= (a - 1)x^3 - 2ax^2 + (3a - 2)x - 2a - 5$

$\therefore x^3(a - (a - 1)) + x^2((3 - 2a) + 2a)$

$+ x((3a + 1) - (3a - 2)) - 4 - 2a + 2a + 5 = 0$

$\therefore x^3 + 3x^2 + 3x + 1 = 0$

The LHS is the expansion of $(x + 1)^3$.

$\therefore (x + 1)^3 = 0$

$\therefore x + 1 = 0$

$\therefore x = -1$

Substitute $x = -1$ in

$y = ax^3 + (3 - 2a)x^2 + (3a + 1)x - 4 - 2a$

$\therefore y = -a + (3 - 2a) - (3a + 1) - 4 - 2a$

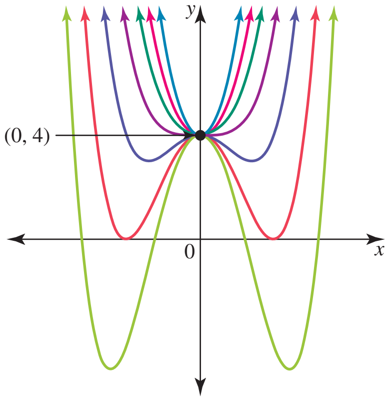
$\therefore y = -8a - 2$

The two sets of curves intersect at $(-1, -8a - 2)$.

Hence, no matter what the value of a is, the point of intersection lies on the vertical line with equation $x = -1$.

20 $y = x^4 + ax^2 + 4, a \in R$ for $a = -6, -4, -2, 0, 2, 4, 6$

Using the graphing screen, the graphs can be seen to resemble those shown in the diagram.



If $a = -4$, $y = x^4 - 4x^2 + 4$. This can be factorised as follows:

$y = (x^2 - 2)^2$

$= \left(\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)\right)^2$

$= \left(x - \sqrt{2}\right)^2\left(x + \sqrt{2}\right)^2$

Hence, the turning points of the graph lie on the x-axis and the polynomial equation $x^4 - 4x^2 + 4 = 0$ has two roots.

As the family of graphs shows, there will be:

a four roots when $a < -4$

b two roots when $a = -4$

c no roots when $a > -4$.

5.3 Exam questions

1 'The greater the value of a, the wider the graph' is incorrect. It should read, 'The greater the value of a, the narrower the graph.'

The correct answer is **E**.

2 'Each graph has a stationary point of inflection at $(0, 0)$' is incorrect. It should read, 'Each graph has a minimum turning point at $(0, 0)$.'

The correct answer is **C**.

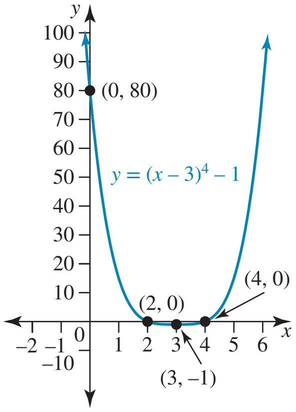
3 The equation is in turning point form, giving the turning point as $(3, -1)$. [1 mark]

x-intercept $(y = 0)$:

$$(x-3)^4 - 1 = 0$$
$$(x-3)^4 = 1$$
$$x - 3 = \pm 1$$
$$x = 2,\ x = 4$$
$$\therefore x\text{-intercept at } (2, 0), (4, 0) \quad [1 \text{ mark}]$$

y-intercept $(x = 0)$:

$$y = (-3)^4 - 1$$
$$y = 80$$
$$\therefore y\text{-intercept at } (0,\ 80)$$



[1 mark]

5.4 Numerical approximations to roots of polynomial equations

5.4 Exercise

1 $p(b) > 0, p(c) < 0$ is the required answer, since it shows the polynomial changes sign over the interval $x \in (b, c)$.

The correct answer is **C**.

2 a $x^3 + 7x - 14 = 0$

Let $p(x) = x^3 + 7x - 14$.

$$p(-2) = (-2)^3 + 7(-2) - 14$$
$$= -8 - 14 - 14$$
$$= -36$$

$$p(-1) = (-1)^3 + 7(-1) - 14$$
$$= -1 - 7 - 14$$
$$= -22$$

Since both $p(-2) < 0$ and $p(-1) < 0$, the solution to the equation does not lie between $x = -2$ and $x = -1$.

b $p(1) = (1)^3 + 7(1) - 14$
$$= -6$$

$$p(2) = (2)^3 + 7(2) - 14$$
$$= 8 + 14 - 14$$
$$= 8$$

Since $p(1) < 0$ and $p(2) > 0$, the solution to the equation lies between $x = 1$ and $x = 2$.

c The midpoint of the interval $[1, 2]$ is 1.5 or $\frac{3}{2}$.

Test the sign of $p\left(\frac{3}{2}\right)$.

$$p\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 + 7\left(\frac{3}{2}\right) - 14$$
$$= \frac{27}{8} + \frac{21}{2} - 14$$
$$= \frac{27}{8} + \frac{84}{8} - \frac{112}{8}$$
$$= \frac{111}{8} - \frac{112}{8}$$
$$= -\frac{1}{8}$$

Hence, $p(1) < 0, p\left(\frac{3}{2}\right) < 0$ and $p(2) > 0$.

A narrower interval in which the solution lies is $\left[\frac{3}{2}, 2\right]$.

3 a $p(x) = 3x^2 - 3x - 1$

$$p(-2) = 3(-2)^2 - 3(-2) - 1$$
$$= 12 + 6 - 1$$
$$= 17$$

$$p(0) = -1$$

Since $p(-2) > 0$ and $p(0) < 0$, the negative solution must lie in the interval $[-2, 0]$.

b The midpoint of the interval $[-2, 0]$ is $x = -1$.

$$p(-1) = 3(-1)^2 - 3(-1) - 1$$
$$= 3 + 3 - 1$$
$$= 5$$

$$p(-1) > 0$$

As $p(-2) > 0$ and $p(0) < 0$, a narrower interval in which the solution lies is $[-1, 0]$.

The midpoint of the interval $[-1, 0]$ is $x = -\frac{1}{2}$.

Using decimals,

$$p(-0.5) = 3(-0.5)^2 - 3(-0.5) - 1$$
$$= 3(0.25) + 1.5 - 1$$
$$= 0.75 + 0.5$$
$$= 1.25$$
$$> 0$$

As $p(-1) > 0$, $p(-0.5) > 0$ and $p(0) < 0$, a narrower interval in which the solution lies is $[-0.5, 0]$.

c The midpoint of the interval $[-0.5, 0]$ is an estimate of the negative solution to the equation.

Hence, $x = -0.25$ is an estimate.

4 a $p(x) = x^2 - 12x + 1$

$$p(10) = 100 - 120 + 1$$
$$= -19$$
$$< 0$$

$$p(12) = 144 - 144 + 1$$
$$= 1$$
$$> 0$$

Therefore, there is a zero of the polynomial between $x = 10$ and $x = 12$.

b $p(x) = -2x^3 + 8x + 3$

$$p(-2) = -2(-8) + 8(-2) + 3$$
$$= 3$$
$$> 0$$

$$p(-1) = -2(-1) + 8(-1) + 3$$
$$= -3$$
$$< 0$$

Therefore, there is a zero of the polynomial between $x = -2$ and $x = -1$.

c $p(x) = x^4 + 9x^3 - 2x + 1$

$$\begin{aligned} p(-2) &= (-2)^4 + 9(-2)^3 - 2(-2) + 1 \\ &= 16 - 72 + 4 + 1 \\ &= -51 \\ &< 0 \end{aligned}$$

$$\begin{aligned} p(1) &= 1 + 9 - 2 + 1 \\ &= 9 \\ &> 0 \end{aligned}$$

Therefore, there is a zero of the polynomial between $x = -2$ and $x = 1$.

5 a The initial interval is $[10, 12]$.

The midpoint of this interval is $x = 11$.

$$\begin{aligned} p(11) &= 121 - 132 + 1 \\ &= -10 \\ &< 0 \end{aligned}$$

Since $p(10) < 0, p(12) > 0$, the root lies in the interval $[11, 12]$.

The midpoint of this interval $[11, 12]$ is $x = 11.5$

$p(11.5) = -4.75 < 0$

The root lies in the interval $[11.5, 12]$.

An estimate is the midpoint of this interval.

$$\begin{aligned} x &= 0.5(11.5 + 12) \\ &= 11.75 \end{aligned}$$

An estimate of the root is $x = 11.75$.

b The initial interval is $[-2, -1]$.

The midpoint of this interval is $x = -1.5$.

$p(-1.5) = -2.25 < 0$

Since $p(-2) > 0, p(-1) < 0$, the root lies in the interval $[-2, -1.5]$.

The midpoint of the interval $[-2, -1.5]$ is $x = -1.75$

$p(-1.75) = -0.281\,25 < 0$

The root lies in the interval $[-2, -1.75]$.

An estimate is the midpoint of this interval.

$$\begin{aligned} x &= 0.5(-2 - 1.75) \\ &= -1.875 \end{aligned}$$

An estimate of the root is $x = -1.875$.

c The initial interval is $[-2, 1]$.

The midpoint of this interval is $x = -0.5$.

$p(-0.5) = 0.9375 > 0$

Since $p(-2) < 0, p(1) > 0$, the root lies in the interval $[-2, -0.5]$.

The midpoint of the interval $[-2, -0.5]$ is $x = -1.25$.

$p(-1.25) = -34.355... < 0$

The root lies in the interval $[-1.25, -0.5]$.

An estimate of the root is the midpoint of this interval, $x = -0.875$.

6 a $p(x) = x^3 + 3x^2 - 7x - 4$

$$\begin{aligned} p(1) &= 1 + 3 - 7 - 4 \\ &= -7 \\ &< 0 \end{aligned}$$

$$\begin{aligned} p(2) &= 8 + 12 - 14 - 4 \\ &= 2 \\ &> 0 \end{aligned}$$

Therefore, $p(x) = 0$ for some $x \in [1, 2]$.

Therefore, the equation $x^3 + 3x^2 - 7x - 4 = 0$ has a root that lies between $x = 1$ and $x = 2$.

b Since $p(2)$ is closer to zero than $p(1)$, a first estimate of the root is $x = 2$.

c First iteration: the midpoint of the interval $[1, 2]$ is $x = 1.5$. This is a second estimate of the root.

Second iteration:

$$\begin{aligned} p(1.5) &= \left(\frac{3}{2}\right)^3 + 3\left(\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) - 4 \\ &= \frac{27}{8} + \frac{27}{4} - \frac{21}{2} - 4 \\ &= \frac{27 + 54 - 84 - 32}{8} \\ &= -\frac{35}{8} \\ &< 0 \end{aligned}$$

The root lies in the interval $[1.5, 2]$.

The midpoint of this interval is $x = 1.75$. This is a third estimate of the root.

d Using a calculator gives the following.

Midpoint	**Value of $p(x)$**	**New interval**
$x = 1.75$	$-1.703\,125$	$[1.75, 2]$
$x = 1.875$	$0.013\,671\,875 < 0.0$	

An estimate of the solution to the equation is $x = 1.875$.

7 $5x^2 - 26x + 24 = 0$

a The root is in the interval $1 \le x \le 2$.

Let $p(x) = 5x^2 - 26x + 24$.

$p(1) = 3 > 0$ and $p(2) = -8 < 0$.

The midpoint of the interval is $x = 1.5$.

$$\begin{aligned} p(1.5) &= 5(2.25) - 26(1.5) + 24 \\ &= -3.75 \\ &< 0 \end{aligned}$$

The root lies in the interval $[1, 1.5]$.

Continuing the method:

Midpoint	**Value of $p(x)$ at midpoint**	**New interval**
		$[1, 1.5]$
$x = 1.25$	$-0.6875 < 0$	$[1, 1.25]$
$x = 1.125$	$1.078\,125 > 0$	$[1.125, 1.25]$
$x = 1.1875$	$0.175\,78... > 0$	$[1.1875, 1.25]$
$x = 1.21875$		

The last 2 estimates have the same value to 1 decimal place. The method of bisection estimates the root to be $x = 1.2$ to 1 decimal place.

b $5x^2 - 26x + 24 = 0$

The equation factorises.

$(5x - 6)(x - 4) = 0$

$\therefore x = \frac{6}{5}, x = 4$

$\therefore x = 1.2, x = 4$

The other root of the equation is $x = 4$.

c $x = 1.2$ is in fact an exact root of the equation. The method of bisection was very slow to converge towards the vicinity of this value.

8 **a** $y = x^4 - 3$

x	-2	-1	0	1	2
y	13	-2	-3	-2	13

b Let $y = 0$.

$\therefore x^4 - 3 = 0$

$\therefore x^4 = 3$

$\therefore x = \pm\sqrt[4]{3}$

Hence, an interval in which $\sqrt[4]{3}$ lies is $[1, 2]$.

c Use the method of bisection with initial interval $x \in [1, 2]$.

Midpoint	***y*-value at midpoint**	**New interval**
		$[1, 2]$
$x = 1.5$	$2.0625 > 0$	$[1, 1.5]$
$x = 1.25$	$-0.558... < 0$	$[1.25, 1.5]$
$x = 1.375$	$0.574... > 0$	$[1.25, 1.375]$
$x = 1.3125$	$-0.032... < 0$	$[1.3125.1.375]$
$x = 1.343\,75$	$0.260... > 0$	$[1.3125, 1.343\,75]$
$x = 1.328\,125$	$0.1113... > 0$	$[1.3125, 1.328\,125]$
$x = 1.320\,3125$	$0.0388... > 0$	$[1.3125, 1.320\,3125]$
$x = 1.316\,406\,25$		

The last two estimates are the same value correct to 2 decimal places.

An estimate of the value of $\sqrt[4]{3}$ is 1.32.

9 **a** $y = x^4 - 2x - 12$

x	-3	-2	-1	0	1	2	3
y	75	8	-9	-12	-13	0	63

b An exact solution to $x^4 - 2x - 12 = 0$ is $x = 2$.

c The other root lies in the interval $[-2, -1]$, since the graph changes position from above the x-axis to below the axis between the end points of this interval.

The midpoint of the interval is $x = \frac{1}{2}((-2) + (-1)) = -1.5$.

When $x = -1.5$, $y = -3.9375$. The root lies between $x = -2$ and $x = -1.5$.

Midpoint	***y*-value**	**New interval**
$x = -1.5$	$-1.703\,125$	$[-2, -1.5]$
$x = -1.75$	$0.878\,906\,25$	$[-1.75, -1.5]$
$x = -1.625$	$-1.777...$	$[-1.75, -1.625]$
$x = -1.6875$	$-0.516...$	$[-1.75, -1.6875]$
$x = -1.718\,75$	$0.164...$	$[-1.718\,75, -1.6875]$
$x = -1.703\,125$		

To 1 decimal place, $x = -1.7$ is a root of the equation.

10 $p(x) = x^3 + 5x - 2 = 0$

a Using trial and error,

$p(0) = -2$ and $p(1) = 4$.

As they have opposite signs, there is a root of the equation in the interval $[0, 1]$.

b The midpoint of the interval is $x = 0.5$.

$p(0.5) = 0.625 > 0$

The root lies in the interval $[0, 0.5]$.

Continuing the method until the value of $p(x)$ differs from zero by less than 0.05.

Midpoint	**Value of $p(x)$ at midpoint**	**New interval**
		$[0, 0.5]$
$x = 0.25$	$-0.734\,375 < 0$	$[0.25, 0.5]$
$x = 0.375$	$-0.1 < -0.072... < 0.1$	

The root is $x = 0.375$ to the required accuracy.

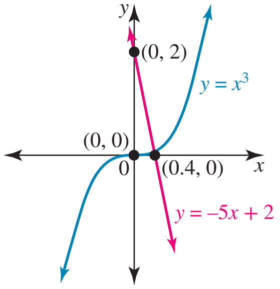
c $x^3 + 5x - 2 = 0$

$\therefore x^3 = -5x + 2$

The intersection of the graphs of $y = x^3$ and $y = -5x + 2$ will allow the solutions to the equation to be found.

d The cubic graph of $y = x^3$ has a stationary point of inflection at the origin and contains the points $(-1, -1)$ and $(1, 1)$.

The line $y = -5x + 2$ has y-intercept $(0, 2)$ and x-intercept $(0.4, 0)$.



There is one point of intersection and therefore the equation $x^3 + 5x - 2 = 0$ has only one root.

The graphs intersect slightly before $x = 0.4$, so the value $x = 0.375$ is supported by the diagram.

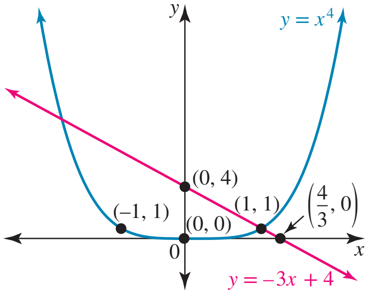
11 $x^4 + 3x - 4 = 0$

$\therefore x^4 = -3x + 4$

The solutions to the equation can be obtained from the intersection of the graphs of $y = x^4$ and the line $y = -3x + 4$.

The quartic graph has a minimum turning point at the origin and passes through the points $(\pm 1, 1)$.

The line has y-intercept $(0, 4)$ and x-intercept $\left(\frac{4}{3}, 0\right)$.



Estimating from the graph, the points of intersection have x-coordinates of approximately $x = -1.75$ and exactly $x = 1$.

To check, substitute $x = 1$ in $x^4 + 3x - 4 = 0$.

$$\begin{aligned} \text{LHS} &= 1^4 + 3(1) - 4 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

The equation has an approximate solution of $x = -1.75$ and an exact solution of $x = 1$.

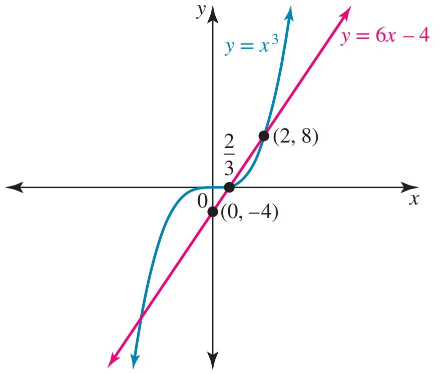
12 $x^3 - 6x + 4 = 0$

$\therefore x^3 = 6x - 4$

The solutions to the equation can be obtained from the intersection of the graphs of $y = x^3$ and the line $y = 6x - 4$.

The cubic graph has a stationary point of inflection at the origin and passes through the points $(-1, -1)$ and $(1, 1)$ and $(2, 8)$ and $(-2, -8)$.

The line has y-intercept $(0, -4)$ and x-intercept $\left(\frac{2}{3}, 0\right)$. It also passes through $(2, 8)$.



Estimating from the graph, the points of intersection have x-coordinates of $x = 2$ and approximately $x = -2.7$ and $x = 0.7$.

The equation $x^3 - 6x + 4 = 0$ has an exact solution of $x = 2$ and approximate solutions of $x = -2.7$ and $x = 0.7$.

13 **a** At the intersection points of $y = 3x - 2$ and $y = x^3$,

$$x^3 = 3x - 2$$

$$\therefore x^3 - 3x + 2 = 0$$

The x-coordinates of the points A and B are solutions of $x^3 - 3x + 2 = 0$.

b As one solution occurs when the line touches the point A, the polynomial $p(x) = x^3 - 3x + 2$ has a linear factor of multiplicity 2. The other solution occurs when the line cuts the curve at the point B, so the polynomial has a second linear factor of multiplicity 1.

There are two factors, one of multiplicity 2 and one of multiplicity 1.

c $p(x) = x^3 - 3x + 2$

$p(1) = 1 - 3 + 2 = 0$

$\therefore (x - 1)$ is a factor.

$$\begin{aligned} x^3 - 3x + 2 &= (x-1)\left(x^2 + bx - 2\right) \\ &= (x-1)\left(x^2 + x - 2\right) \\ &= (x-1)(x+2)(x-1) \\ &= (x-1)^2(x+2) \end{aligned}$$

Thus, the solutions of the equation $x^3 - 3x + 2 = 0$ are $x = 1, x = -2$.

The point A has $x = 1$ and point B has $x = -2$.

Substitute in $y = 3x - 2$

For A, $y = 1$ and for B, $y = -8$.

A is the point $(1, 1)$ and B is the point $(-2, -8)$

d The equation $x^3 - 3x + 1 = 0$ can be solved by the intersection of $y = x^3$ and $y = 3x - 1$, since $x^3 - 3x + 1 = 0$ rearranges to $x^3 = 3x - 1$.

The line $y = 3x - 1$ is parallel to the line in the diagram, but it has a higher y-intercept of $(0, -1)$. This means the line will cut the cubic curve in 3 places.

The equation $x^3 - 3x + 1 = 0$ has three solutions.

14 **a** $y = -x(x + 2)(x - 3)$

The graph cuts the x-axis at $x = 0, x = -2, x = 3$, that is at $x = -2, x = 0, x = 3$. Between successive pairs of these values, the graph must have a turning point.

As the graph is of a cubic polynomial with a negative leading coefficient, the first turning point is a minimum and the second is a maximum.

Hence, the maximum turning point must lie in the interval for which $x \in [0, 3]$.

b Construct a table of values for the interval $x \in [0, 3]$.

x	0	0.5	1	1.5	2	2.5	3
y	0	3.125	6	7.875	8	5.625	0

The maximum turning point is near $(2, 8)$. Zoom in on this point.

x	1.6	1.7	1.8	1.9	2	2.1	2.2
y	8.064	8.177	8.208	8.151	8		

An estimate of the maximum turning point is $(1.8, 8.208)$.

15 **a** $y = (x + 4)(x - 2)(x - 6)$

This positive cubic graph has x-intercepts when $x = -4, x = 2$ and $x = 6$. There is a maximum turning point between $x = -4$ and $x = 2$, and a minimum turning point between $x = 2$ and $x = 6$.

For the maximum turning point, construct a table of values between $x = -4$ and $x = 2$.

x	−4	−3	−2	−1	0	1	2
y	0	45	64	63	48	25	0

The maximum turning point is near $(-2, 64)$. Zoom in around this point.

See the table at the bottom of the page.*

An estimate of the position of the maximum turning point is $(-1.6, 65.664)$.

b $y = x(2x + 5)(2x + 1)$

x-intercepts when $x = 0, x = -\frac{5}{2}, x = -\frac{1}{2}$.

The shape is of a positive cubic, so there is a maximum turning point between $x = -2.5$ and $x = -0.5$ and a minimum turning point between $x = -0.5$ and $x = 0$.

For the minimum turning point, construct a table of values between $x = -0.5$ and $x = 0$.

x	−0.5	−0.4	−0.3	−0.2	−0.1	0
y	0	−0.336	−0.528	−0.552	−0.384	0

The minimum turning point's estimated position is $(-0.2, -0.552)$

c $y = x^2 - x^4$

x-intercepts: let $y = 0$.

$\therefore y = x^2(1 - x^2)$

$\therefore y = x^2(1 - x)(1 + x)$

$x = 0, x = 1, x = -1$

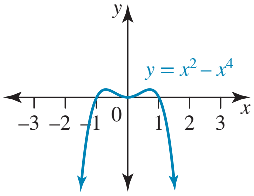
Due to the multiplicity of the factor, there is a turning point at $(0, 0)$.

The shape of the graph is of a negative fourth degree.

***15 a**

x	−2.1	−2	−1.9	−1.8	−1.7	−1.6	−1.5
y	63.099	64	64.701	65.208	65.527	65.664	65.625

Therefore, there is a maximum turning point between $x = -1$ and $x = 0$, $(0, 0)$ is a minimum turning point, and there is a second maximum turning point between $x = 0$ and $x = 1$. The two maximum turning points are symmetric about the y-axis.



Maximum turning point between $x = 0$ and $x = 1$:

x	0	0.2	0.4	0.6	0.8	1
y	0	0.0384	0.1344	0.2304	0.2304	0

The maximum turning point lies between $x = 0.6$ and $x = 0.8$. Therefore, to 1 decimal place, its x-coordinate must be $x = 0.7$. When $x = 0.7$, $y = 0.2499$

There are maximum turning points at approximately $(0.7, 0.2499)$ and $(-0.7, 0.2499)$, and there is a minimum turning point exactly at $(0, 0)$. *Note*: Since the minimum turning point is exact, we shall choose not to write it as $(0.0, 0)$.

16 $y = 2x^3 - x^2 - 15x + 9.$

a The y-intercept is $(0, 9)$.

b Let $y = 9$.

$$\therefore 9 = 2x^3 - x^2 - 15x + 9$$
$$\therefore 2x^3 - x^2 - 15x = 0$$
$$\therefore x\left(2x^2 - x - 15\right) = 0$$
$$\therefore x(2x + 5)(x - 3) = 0$$
$$x = 0, x = \frac{-5}{2}$$
$$x = 3$$

The two other points that have the same y-coordinate as the y-intercept are $\left(-\frac{5}{2}, 9\right)$, $(3, 9)$.

c There must be a turning point between $x = -\frac{5}{2}$ and $x = 0$, and a second turning point between $x = 0$ and $x = 3$.

The graph is of a positive cubic, so the first turning point is the maximum turning point. This must lie in the interval for which $-\frac{5}{2} \le x \le 0$.

d $-\frac{5}{3} \simeq -1.67$

x	$-\frac{5}{3}$	−1.5	−1	−0.5	0
y	9	22.5	21	16	9

The turning point is near $(-1.5, 22.5)$. Zoom in around this point.

x	−1.6	−1.5	−1.4	−1.3
y	22.248	22.5	22.552	22.416

An estimate of the maximum turning point is $(-1.4, 22.552)$.

17 a $p(x) = x^3 - 3x^2 - 4x + 9$

$p(0) = 9$

$$\therefore x^3 - 3x^2 - 4x + 9 = 9$$
$$\therefore x^3 - 3x^2 - 4x = 0$$
$$\therefore x(x^2 - 3x - 4) = 0$$
$$\therefore x(x - 4)(x + 1) = 0$$
$$\therefore x = -1, x = 0, x = 4$$

The shape of the graph is a positive cubic, so there is a maximum turning point in the interval $x \in [-1, 0]$ and a minimum turning point in the interval $x \in [0, 4]$.

b $p(x) = x^3 - 12x + 18$

$p(0) = 18$

$$\therefore x^3 - 12x + 18 = 18$$
$$\therefore x^3 - 12x = 0$$
$$\therefore x(x^2 - 12) = 0$$
$$\therefore x(x + \sqrt{12})(x - \sqrt{12}) = 0$$
$$\therefore x(x + 2\sqrt{3})(x - 2\sqrt{3}) = 0$$
$$\therefore x = -2\sqrt{3}, x = 0, x = 2\sqrt{3}$$

The shape of graph is a positive cubic, so there is a maximum turning point in the interval $x \in [-2\sqrt{3}, 0]$ and a minimum turning point in the interval $x \in [0, 2\sqrt{3}]$.

c $p(x) = -2x^3 + 10x^2 - 8x + 1$

$p(0) = 1$

$$\therefore -2x^3 + 10x^2 - 8x + 1 = 1$$
$$\therefore -2x^3 + 10x^2 - 8x = 0$$
$$\therefore -2x(x^2 - 5x + 4) = 0$$
$$\therefore -2x(x - 1)(x - 4) = 0$$
$$\therefore x = 0, x = 1, x = 4$$

The shape of the graph is a negative cubic, so there is a minimum turning point in the interval $x \in [0, 1]$ and a maximum turning point in the interval $x \in [1, 4]$.

d $p(x) = x^3 + x^2 + 7$

$p(0) = 7$

$$\therefore x^3 + x^2 + 7 = 7$$
$$\therefore x^3 + x^2 = 0$$
$$\therefore x^2(x + 1) = 0$$
$$\therefore x = -1, x = 0$$

The shape of the graph is a positive cubic, so there is a maximum turning point in the interval $x \in [-1, 0]$. The multiplicity of the factor indicates that the line $y = 7$ touches the graph when $x = 0$, so there is a turning point at the point $(0, 7)$. This must be a minimum turning point.

18 $y = -x^3 + 7x^2 - 3x - 4, x \ge 0$

a The number of containers is $10x$, so for 10 containers, $x = 1$, and for 20 containers, $x = 2$.

When $x = 1$, $y = -1 + 7 - 3 - 4 = -1 < 0$, so no profit is made.

When $x = 2$, $y = -8 + 28 - 6 - 4 = 4 > 0$, so a profit is made.

A profit is first made for $x \in [1, 2]$, indicating that the number of containers sold was between 10 and 20.

b The midpoint of the interval $[1, 2]$ is $x = 1.5$.

When $x = 1.5$, $y = 3.875 > 0$

A profit is first made for $x \in [1, 1.5]$.

The midpoint of the interval $[1, 1.5]$ is $x = 1.25$.

When $x = 1.25$, $y = 1.234375 > 0$

A profit is first made for $x \in [1, 1.25]$.

c The number of containers is $10x$ and must be a whole number. For $x \in [1, 1.25]$, the number of containers lies between 10 and 12, so either $x = 1.1$ or $x = 1.2$.

x	1	1.1	1.2	1.25
y	< 0	-0.161	0.752	> 0

A profit is first made when 12 containers are sold.

d The graph shows that the maximum turning point lies in the interval $x \in [3, 6]$.

e Test values of y for $x \in [3, 6]$

x	4	4.5	5	5.5
y	32	33.125	31	24.875

The maximum turning point is near $(4.5, 33.125)$

Zoom in around this point.

x	4.2	4.3	4.4	4.5	4.6
y	32.792	33.023	33.136	33.125	32.984

The maximum turning point is closest to $(4.4, 33.136)$.

Selling 44 containers will give the maximum profit of \$331 to the nearest dollar.

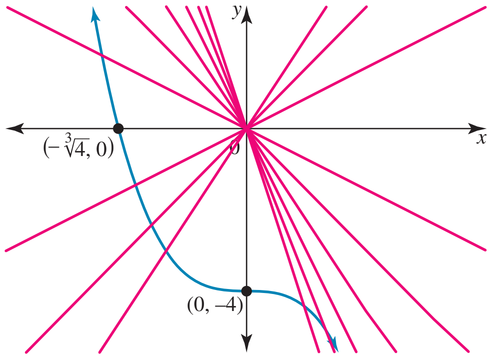
f The value $x = 6.5$ is an estimate from the graph of when $y = 0$.

Test around this value.

x	6.4	6.5
y	1.376	-2.375

When there are 65 or more containers, no profit will be made.

19 a $y = -x^3 - 4$ and the family of lines $y = ax$. The values of a shown in the diagram are integer values from -6 to 3.



b $x^3 + ax + 4 = 0$

Rearranging this equation to $ax = -x^3 - 4$ shows the solutions are the x-coordinates of the points of intersection of the graphs in part **a**.

The line $y = -5x$ is the first to intersect the cubic graph in three places for integer values of a. The largest integer is $a = -5$.

c The lines with positive gradients, i.e. the lines for which $a > 0$, intersect the cubic once. There is one solution to the equation $x^3 + ax + 4 = 0$.

d $x^3 + x + 4 = 0$

Here $a = 1$, so there will be one solution that will have a negative value.

Define the polynomial $p(x) = x^3 + x + 4$.

$p(-2) = -6 < 0$

$p(-1) = 2 > 0$

So an initial estimate is that the solution lies in the interval $[-2, -1]$. Proceed with the method of bisection from there (or program the calculator to carry out the procedure).

Midpoint	**Value of $p(x)$ at midpoint**	**New interval**
		$[-2, -1]$
$x = -1.5$	< 0	$[-1.5, -1]$
$x = -1.25$	> 0	$[-1.5, -1, 25]$
$x = -1.375$	> 0	$[-1.5, -1.375]$
$x = -1.4375$	< 0	$[-1.4375, -1.375]$
$x = -1.406\,25$	< 0	$[-1.406\,25, -1.375]$
$x = -1.390\,625$	< 0	$[-1.390\,625, -1.375]$
$x = -1.382\,812\,5$	< 0	$[-1.382\,812\,5, -1.375]$
$x = -1.378\,906\,25$		

The last two values agree at an accuracy of 2 decimal places.

To 2 decimal places, the solution is $x = -1.38$.

20 a The dimensions in cm of the box are length $18 - 2x$, width $14 - 2x$ and height x.

Let the volume be V cubic cm.

$V = l \times w \times h$

$\therefore V = x(18 - 2x)(14 - 2x)$

b The graph of the cubic polynomial $y = x(18 - 2x)(14 - 2x)$ would have x-intercepts at $x = 0, x = 7, x = 9$. Between $x = 0$ and $x = 7$ there would be a maximum turning point, and between $x = 7$ and $x = 9$ there would be a minimum turning point.

However, since neither V nor x can be negative in this practical model, the graph of $V = x(18 - 2x)(14 - 2x)$ is defined for $0 < x < 7$.

The volume is greatest within the interval between $x = 0$ and $x = 7$.

c Using CAS technology, set up the rule $V = x(18 - 2x)(14 - 2x)$ to be evaluated over $[0, 7]$.

$x = 2.6049$ gives the greatest volume.

The side length of the square to be cut out is 2.605 cm to 3 decimal places.

5.4 Exam questions

1 The graph is a positive odd cubic polynomial (x^3) with x-intercepts at $x = -5, 0, 4$ and two turning points between these points. The first turning point must be a maximum and the second one must be a minimum in order to satisfy the long-term behaviour requirements of a positive cubic polynomial.

The correct answer is **A**.

2 a $p(x) = x^3 - 2x^2 + 8x - 5$

$p(0) = -5$

$p(0) < 0$

$p(1) = 1 - 2 + 8 - 5$

$= 2$

$p(1) > 0$

$\therefore p(0) < 0, p(1) > 0$

$\Rightarrow$ The root is between $x = 0$ and $x = 1$. [1 mark]

b The root is closer to $p(1)$, so a first estimate of the root is $x = 1$. [1 mark]

Take the midpoint of $x = 0$ and $x = 1$. $\therefore x = \frac{1}{2}$

$$p\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right) - 5$$

$$\begin{aligned} p\left(\frac{1}{2}\right) &= \frac{1}{8} - \frac{1}{2} + 4 - 5 \\ &= -\frac{3}{8} - 1 \\ &= -1\frac{3}{8} \end{aligned}$$

The root is closer to $p\left(\frac{1}{2}\right)$, so a more accurate estimate of the root is $x = \frac{1}{2}$. [1 mark]

3 $y = (x+2)\left(x^2 - 5x + 4\right)$

$y = (x+2)(x-1)(x-4)$

There are three x-intercepts at $x = -2$, $x = 1$ and $x = 4$. Therefore, there are two turning points. The first turning point must be a maximum and the second one must be a minimum in order to satisfy the long-term behaviour requirements of a positive odd cubic polynomial.

The correct answer is **C**.

5.5 Review

5.5 Exercise

Technology free: short answer

1 a $y = 16 - (x+3)^4$

Maximum turning point: $(-3, 16)$

y-intercept: let $x = 0$.

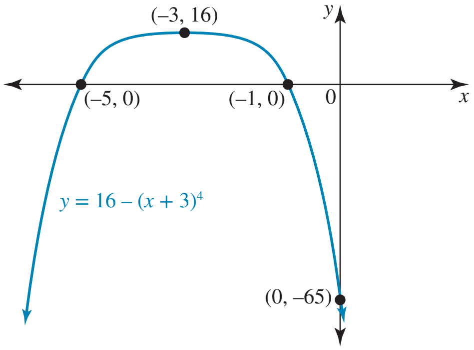
$$\begin{aligned} \therefore y &= 16 - (3)^4 \\ &= -65 \end{aligned}$$

$(0, -65)$

x-intercepts: let $y = 0$.

$$\begin{aligned} \therefore 0 &= 16 - (x+3)^4 \\ \therefore (x+3)^4 &= 16 \\ \therefore x + 3 &= \pm\sqrt[4]{16} \\ \therefore x + 3 &= \pm 2 \\ \therefore x &= -2 - 3 \text{ or } x = 2 - 3 \\ \therefore x &= -5 \text{ or } x = -1 \end{aligned}$$

$(-5, 0), (-1, 0)$



b $y = \frac{1}{18}(x+2)(x-3)^3$

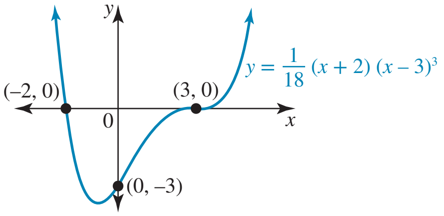
The graph cuts the x-axis at $x = -2$ and in the saddle at $x = 3$.

y-intercept: let $x = 0$.

$$\therefore y = \frac{1}{18} \times 2 \times -27$$

$$\therefore y = -3$$

$(0, -3)$



c $y = -x^4 + 8x^2 - 16$

y-intercept $(0, -16)$

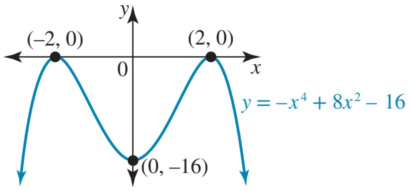
Let $a = x^2$.

$$\begin{aligned} y &= -a^2 + 8a - 16 \\ &= -(a^2 - 8a + 16) \\ &= -(a-4)^2 \\ &= -\left(x^2 - 4\right)^2 \\ &= -((x+2)(x-2))^2 \end{aligned}$$

$\therefore y = -(x+2)^2(x-2)^2$

There are turning points at the x-intercepts $(-2, 0)$ and $(2, 0)$.

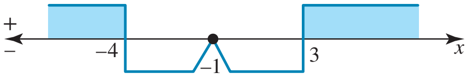
The shape of the graph is a negative quartic. There is a third turning point, which due to the symmetry of the turning points on the x-axis must be at $x = 0$. The y-intercept $(0, -16)$ is also a turning point.



2 a $(x+4)(x+1)^2(x-3) \geq 0$

Zeros: $x = -4$, $x = -1$ (touch), $x = 3$

Shape: positive fourth degree



Answer: $x \leq -4$ or $x = -1$ or $x \geq 3$

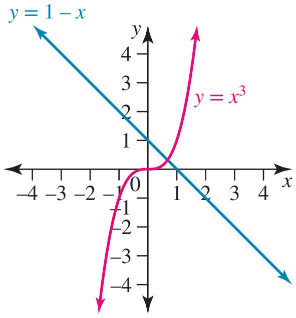
b $(x-5)^3(3x+7) < 0$

Zeros: $x = 5, x = -\frac{7}{3}$

Shape: positive fourth degree

Answer: $-\frac{7}{3} < x < 5$

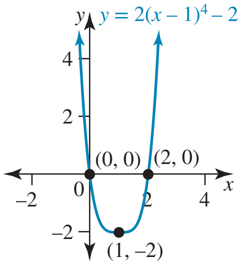
3 a



From the graph, the solution lies in [0, 1].

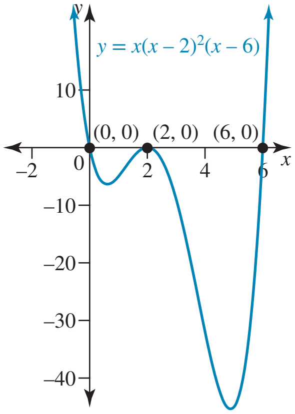
b Let $p(x) = 1 - x - x^3$.
$p(0) = 1$
$p(1) = -1$
$p(0.5) = 0.375$
The solution lies in the interval [0.5, 1].

4 a



For $2\,(x - 1)^4 - 2 < 0$, the graph must lie below the x-axis.
Hence, $0 < x < 2$.

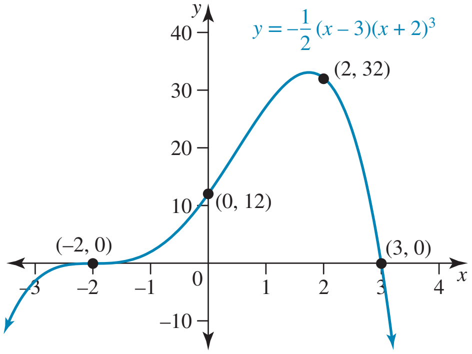
b



For $x\ \ (x - 2)^2\,(x - 6) \geq 0$, the graph must lie above or on the x-axis.
Hence, $x \leq 0$ or $x = 2$ or $x \geq 6$

5 a Intersects the x-axis at $x = 3 \Rightarrow (x - 3)$ is a factor.
Stationary point of inflection at $(-2, 0) \Rightarrow (x + 2)^3$ is a factor.
Hence, $p(x) = a(x - 3)\,(x + 2)^3$.

b If the graph passes through the point (2, 32), then $p\,(2) = 32$
$p\,(2) = a\,(2 - 3)\,(2 + 2)^3$.
$32 = a\,(-1)\,64$
$a = -\dfrac{1}{2}$
Hence, the graph is $y = -\dfrac{1}{2}\,(x - 3)\,(x + 2)^3$.



6 a $y = 2x^3 + mx^2 - 4x + 5$
Substitute the point $(-2, 9)$.
$9 = 2(-2)^3 + m(-2)^2 - 4(-2) + 5$
$\therefore -3 + 4m = 9$
$\therefore 4m = 12$
$\therefore m = 3$
The equation of this curve is $y = 2x^3 + 3x^2 - 4x + 5$.

b At the intersection with $y = 2x^3 + 5$,
$2x^3 + mx^2 - 4x + 5 = 2x^3 + 5$
$\therefore mx^2 - 4x = 0$
$\therefore x(mx - 4) = 0$
$\therefore x = 0, x = \dfrac{4}{m}$
Substitute $x = 0$ in $y = 2x^3 + 5$.
One point of intersection is (0, 5).
Substitute $x = \dfrac{4}{m}$ in $y = 2x^3 + 5$.
$\therefore y = \dfrac{128}{m^3} + 5$
The other point of intersection is $\left(\dfrac{4}{m}, \dfrac{128}{m^3} + 5\right)$, $m \neq 0$.

Technology active: multiple choice

7 $y = \dfrac{1}{2}(x + 6)^4 - 3$
Compare the equation with the general form $y = a(x - h)^4 + k$ where the turning point is (h, k). If $a > 0$, the turning point is a minimum, and if $a < 0$, it is a maximum.
Therefore, the turning point is a minimum and its coordinates are $(-6, -3)$.
The correct answer is **E**.

8 The graph cuts the x-axis at $x = -5$, so $(x + 5)$ is a factor.
It touches the x-axis at $x = -2$, so $(x + 2)^2$ is a factor.
It again cuts the x-axis at $x = 3$, so $(x - 3)$ is a factor.
The shape is of a negative quartic, so the equation is
$y = -(x + 5)(x + 2)^2(x - 3)$
This can be written with the '–' absorbed into the $(x - 3)$ bracket: $(x - 3) = -(3 - x)$
The equation would then be written as
$y = (x + 5)(x + 2)^2(3 - x)$.
The correct answer is **C**.

9 $9x^2 - 81x^4 = 0$
$\therefore 9x^2(1 - 9x^2) = 0$
$\therefore 9x^2(1 - 3x)(1 + 3x) = 0$
$\therefore x = 0, x = \dfrac{1}{3}, x = -\dfrac{1}{3}$.
The correct answer is **E**.

10 The curve $y = (5 - x)^4 + 25$ has a minimum turning point at (5, 25), so it is always above the x-axis. Hence, $(5 - x)^4 + 25 = 0$ has no solutions.
The correct answer is **A**.

11 $y = a(x + b)^4 + c$
Turning point $(0, -7) \Rightarrow y = a(x - 0)^4 - 7$
$\therefore y = ax^4 - 7$
Substitute the point $(-1, -10)$.
$\therefore\ -10 = a(-1)^4 - 7$
$\therefore\ -10 = a - 7$
$\therefore a = -3$
The equation is $y = -3x^4 - 7$; hence, $a = -3, b = 0, c = -7$.
$\therefore a + b + c = -10$
The correct answer is **D**.

12 The graphs of $y = x^4$ and $y = x^3$ intersect when $x^4 = x^3$

$\therefore x^4 - x^3 = 0$

$\therefore x^3(x - 1) = 0$

$\therefore x = 0, x = 1$

Substitute into $y = x^4$.

When $x = 0$, $y = 0$, and when $x = 1$, $y = 1$.

The points of intersection are $(0, 0)$ and $(1, 1)$.

The correct answer is **C**.

13 The graph has a stationary point of inflection at $x = -4$

$\Rightarrow (x + 4)^3$ is the factor of multiplicity 3.

The graph cuts the x-axis at $x = 4 \Rightarrow (x - 4)$ is a factor.

Hence, the degree of the curve is 4, and it has 2 x-intercepts.

The correct answer is **B**.

14 $y = 0.1\,(2x + 5)^3$.

If n is odd, the graph of $y = a(x - h)^n + k$ has a stationary point of inflection at (h, k).

Let $2x + 5 = 0$.

$\therefore x = -\dfrac{5}{2} = -2.5$

The graph has a stationary point of inflection at $(-2.5, 0)$.

The correct answer is **E**.

15 $6x^3 - 7x + 5 = 0$

Let $p(x) = 6x^3 - 7x + 5$.

Test the values of the polynomial until the sign changes.

$p(-3) = 6 \times -27 + 21 + 5 < 0$

$p(-2) = 6 \times -8 + 14 + 5 < 0$

$p(-1) = 6 \times -1 + 7 + 5 > 0$

There is a solution to $6x^3 - 7x + 5 = 0$ for which $-2 \le x \le -1$.

The correct answer is **B**.

16 At the intersection of $y = \frac{1}{2}x^3$ and $y = 2 - x^2$, $\frac{1}{2}x^3 = 2 - x^2$.

$\therefore x^3 = 2(2 - x^2)$

$\therefore x^3 = 4 - 2x^2$

$\therefore x^3 + 2x^2 - 4 = 0$

The correct answer is **B**.

Technology active: extended response

17 **a** $y = ax^2 + b$

Substitute $(0, 16)$: $16 = b$

Substitute $(2, 0)$: $0 = 4a + b$

$0 = 4a + 16$

$a = -4$

$a = -4, b = 16$ and curve: $y = -4x^2 + 16$

b Area of rectangle: $A = xy$

$A = x(-4x^2 + 16)$

$A = 16x - 4x^3$

c If the area, A, is 12, then $12 = 16x - 4x^3$.

$4x^3 - 16x + 12 = 0$

Let $p\,(x) = 4x^3 - 16x + 12$.

$p\,(1) = 4 - 16 + 12 = 0$

Hence, $x = 1$ is a solution and $(x - 1)$ is a factor of $p(x)$.

$4x^3 - 16x + 12 = 0$

$x^3 - 4x + 3 = 0$

$(x - 1)(x^2 + x - 3) = 0$

d **i** Let $q(x) = x^2 + x - 3$.

$q(1.25) = -0.1875 < 0$

$q(1.5) = 0.75 > 0$

Since $q(x)$ changes sign, there is a root in the interval $x \in [1.25, 1.5]$.

ii $q(1.375) = 0.265\,625 > 0$, giving the interval $\in [1.25, 1.375]$.

$q(1.3125) = 0.035\,156 > 0$, giving the interval $\in [1.25, 1.3125]$.

$\beta = 1.3125$

e The height of the tunnel is the y-value of the curve.

$y = -4x^2 + 16$

When $x = 1, y = 12$ metres.

When $x = \beta,\ y = 9.0938$ metres.

The height of the tunnel is smaller when $x = \beta$.

18 **a** $p(x) = x^3 + 3x^2 + 2x + 5$

If $p(x) = 5$, then $5 = x^3 + 3x^2 + 2x + 5$.

$\therefore x^3 + 3x^2 + 2x = 0$

$\therefore x(x^2 + 3x + 2) = 0$

$\therefore x(x + 2)(x + 1) = 0$

$\therefore x = -2, x = -1, x = 0$

The graph of $y = p(x)$ is a positive cubic. Therefore, there is a maximum turning point for $x \in [-2, -1]$ and a minimum turning point for $x \in [-1, 0]$.

b Maximum turning point

x	−2	−1.8	−1.6	−1.4	−1.2	−1
y	5	5.288	5.384	5.336	5.192	5

The turning point is near $(-1.6, 5.384)$. Zooming in around this point:

x	−1.7	−1.6	−1.5
y	5.357	5.384	5.336

x	−1.61	−1.6	−1.59
y	5.383	5.384	5.3846

The maximum turning point is approximately $(-1.58, 5.38)$ to 2 decimal places.

Minimum turning point

x	−1	−0.8	−0.6	−0.4	−0.2	0
y	5	4.808	4.664	4.616	4.712	5

The minimum turning point is near $(-0.4, 4.616)$.

Zooming in around this point:

x	−0.5	−0.4	−0.3
y	4.625	4.616	4.643

x	−0.43	−0.42	−0.41	−0.4	−0.39
y	4.6152	4.6151	4.6154	4.616	4.617

The minimum turning point is approximately $(-0.42, 4.62)$ to 2 decimal places.

c Both of the turning points lie above the x-axis, so the cubic graph can only go through the x-axis once.

d Since the turning points are both to the left of the y-axis, the x-intercept must be negative and have a value smaller than -1.58, the x-coordinate of the maximum turning point.

$p(x) = x^3 + 3x^2 + 2x + 5$

$p(-2) = -8 + 12 - 4 + 5$

$= 5$

> 0

$$\begin{aligned} p(-3) &= -27 + 27 - 6 + 5 \\ &= -1 \\ &< 0 \end{aligned}$$

The x-intercept lies in the interval $[-3, -2]$.

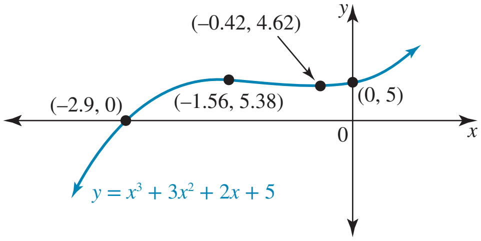
Use the method of bisection to generate three narrower intervals.

Midpoint	**Value of $p(x)$**	**New interval**
		$[-3, -2]$
$x = -2.5$	$3.125 > 0$	$[-3, -2.5]$
$x = -2.75$	$1.390\,625 > 0$	$[-3, -2.75]$
$x = -2.875$	$0.283... > 0$	$[-3, -2.875]$

e The midpoint of $[-3, -2.875]$ is $x = -2.9375$ and this is an estimate of the value of the x-intercept of the graph.

To 1 decimal place, the x-intercept is near $(-2.9, 0)$.

f x-intercept $(-2.94, 0)$, y-intercept $(0, 5)$, turning points $(-1.58, 5.38)$ and $(-0.42, 4.62)$



19 a i $y = (a - 2)x^3 - x^2 + (2 - a)x + 1$

If the coefficient of the x^3 term is zero, the rule will represent a parabola.

$a - 2 = 0$ when $a = 2$

If $a = 2$, the rule represents a parabola.

ii The equation of the parabola is $y = -x^2 + 1$,

This has a maximum turning point at $(0, 1)$.

b For $a = 3$, the rule becomes $y = x^3 - x^2 - x + 1$.

y-intercepts $(0, 1)$

x-intercepts: let $y = 0$.

$$\begin{aligned} \therefore x^3 - x^2 - x + 1 &= 0 \\ \therefore x^2(x - 1) - (x - 1) &= 0 \\ \therefore (x - 1)(x^2 - 1) &= 0 \\ \therefore (x - 1)(x - 1)(x + 1) &= 0 \\ \therefore (x - 1)^2(x + 1) &= 0 \end{aligned}$$

$\therefore x = 1, x = -1$

The x-intercepts are $(-1, 0)$ and $(1, 0)$.

c For $a = 1$, the rule becomes $y = -x^3 - x^2 + x + 1$.

y-intercept $(0, 1)$

x-intercepts: let $y = 0$.

$$\begin{aligned} \therefore -x^3 - x^2 + x + 1 &= 0 \\ \therefore -x^2(x + 1) + (x + 1) &= 0 \\ \therefore (x + 1)(-x^2 + 1) &= 0 \\ \therefore -(x + 1)(x^2 - 1) &= 0 \\ \therefore -(x + 1)(x + 1)(x - 1) &= 0 \\ \therefore -(x + 1)^2(x - 1) &= 0 \end{aligned}$$

$\therefore x = 1, \ x = -1$

The x-intercepts are $(-1, 0)$ and $(1, 0)$.

d $y = (a - 2)x^3 - x^2 + (2 - a)x + 1$

y-intercept: let $x = 0$.

$\therefore y = 1$

All the curves have the same y-intercept, $(0, 1)$.

x-intercepts: As the previous two curves in parts **b** and **c** had the same two x-intercepts, $(-1, 0)$, and $(1, 0)$, one method is to test if these are x-intercepts for all the curves. However, as the third x-intercept needs to be specified, let $y = 0$.

$\therefore 0 = (a - 2)x^3 - x^2 + (2 - a)x + 1$

$\therefore 0 = (a - 2)x^3 - x^2 - (a - 2)x + 1$

$\therefore 0 = x^2[(a - 2)x - 1] - [(a - 2)x - 1]$

$\therefore 0 = [(a - 2)x - 1](x^2 - 1)$

$\therefore 0 = [(a - 2)x - 1](x + 1)(x - 1)$

$\therefore x = -1, x = 1$ or $(a - 2)x - 1 = 0$

$\therefore x = -1, x = 1$ or $x = \dfrac{1}{a - 2}$

All the curves share $(-1, 0)$ and $(1, 0)$ as x-intercepts.

The possible third x-intercept is $\left(\dfrac{1}{a - 2}, 0\right), a \neq 2$.

If $a = 1$ this becomes $(-1, 0)$, and if $a = 3$ it becomes $(1, 0)$.

e If there is an x-intercept at $\left(\dfrac{1}{2}, 0\right)$, then $\dfrac{1}{a - 2} = \dfrac{1}{2}$, so $a = 4$.

Alternatively, substitute the point $\left(\dfrac{1}{2}, 0\right)$ in $y = (a - 2)x^3 - x^2 + (2 - a)x + 1$.

20 a $y = x^4 + ax^3 + bx^2 + cx + d$

As the curve passes through the origin, when $x = 0$, $y = 0$.

$\therefore d = 0$

b At the intersection of $y = x^4 + ax^3 + bx^2 + cx + d$ and $y = -2x$,

$x^4 + ax^3 + bx^2 + cx + d = -2x$.

Since $d = 0$, $x^4 + ax^3 + bx^2 + cx = -2x$.

$\therefore x^4 + ax^3 + bx^2 + cx + 2x = 0$

$\therefore x^4 + ax^3 + bx^2 + (c + 2)x = 0$

c Taking out the common factor, the equation becomes $x(x^3 + ax^2 + bx + (c + 2)) = 0$.

$\therefore x = 0$ or $x^3 + ax^2 + bx + (c + 2) = 0$

As the line touches the curve at $x = -3$, $(x + 3)^2$ is a factor of this equation.

The line cuts the curve at $x = -6$, so $(x + 6)$ is a factor.

The line also cuts the curve at $x = 0$, so x is a factor as is already shown.

The factors of the cubic equation

$x^3 + ax^2 + bx + (c + 2) = 0$

must be $(x + 3)^2$ and $(x + 6)$.

$$\begin{aligned} (x + 3)^2(x + 6) &= (x^2 + 6x + 9)(x + 6) \\ &= x^3 + 6x^2 + 6x^2 + 36x + 9x + 54 \\ &= x^3 + 12x^2 + 45x + 54 \end{aligned}$$

$\therefore x^3 + ax^2 + bx + (c + 2) = x^3 + 12x^2 + 45x + 54$

Equating coefficients of like terms,

$a = 12, b = 45, c + 2 = 54$

$\therefore a = 12, b = 45, c = 52$

d i The rule for the quartic polynomial $y = x^4 + ax^3 + bx^2 + cx + d$ shown in the diagram is $y = x^4 + 12x^3 + 45x^2 + 52x$.

Let $x = -4$.

$$\begin{aligned} y &= (-4)^4 + 12(-4)^3 + 45(-4)^2 + 52(-4) \\ &= 256 - 768 + 720 - 208 \\ &= 976 - 976 \\ &= 0 \end{aligned}$$

There is an x-intercept at $x = -4$.

ii $(x+4)$ is a factor and so is x.

$$\begin{aligned} y &= x^4 + 12x^3 + 45x^2 + 52x \\ &= x(x^3 + 12x^2 + 45x + 52) \\ &= x((x+4)(x^2 + nx + 13)) \\ &= x((x+4)(x^2 + 8x + 13)) \\ &= x(x+4)(x^2 + 8x + 13) \end{aligned}$$

Let $y = 0$.

$\therefore x = 0, x = -4 \text{ or } x^2 + 8x + 13 = 0$

Solving the quadratic equation by completing the square gives

$$\begin{aligned} (x^2 + 8x + 16) - 16 + 13 &= 0 \\ \therefore (x+4)^2 &= 3 \\ \therefore x + 4 &= \pm\sqrt{3} \\ \therefore x &= -4 \pm \sqrt{3} \end{aligned}$$

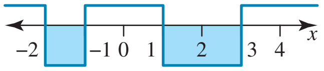
The x-intercepts apart from $(-4, 0)$, are $(0, 0)$, $\left(-4 - \sqrt{3}, 0\right)$ and $\left(-4 + \sqrt{3}, 0\right)$.

5.5 Exam questions

1 $x^4 - x^3 - 7x^2 + x + 6$

$= (x+1)(x-3)(x+2)(x-1)$ [1 mark]

Use a sign diagram to solve the inequation.



$x^4 - x^3 - 7x^2 + x + 6 < 0$

$\{x : -2 < x < -1\} \cup \{x : 1 < x < 3\}$ [1 mark]

2 The graphs have one turning point at (h, k).

The correct answer is **C**.

3 The graph:

cuts the axis at $x = -4$

touches the axis at $x = 3$

cuts the axis at $x = 6$. [1 mark]

$y = a(x+4)(x-3)^2(x-6)$

The degree is 4. [1 mark]

Work out a using point $(2, 18)$.

$18 = a(6)(-1)^2(-4)$

$18 = -24a$

$a = -\dfrac{3}{4}$ [1 mark]

$\therefore y = -\dfrac{3}{4}(x+4)(x-3)^2(x-6)$ [1 mark]

4 $y = 2x^3 - x^2 - 10x + 7$

y-intercept $(x = 0)$

$y = 7$ [1 mark]

Other points on the graph with $y = 7$:

$$\begin{aligned} 7 &= 2x^3 - x^2 - 10x + 7 \\ 2x^3 - x^2 - 10x &= 0 \\ x\left(2x^2 - x - 10\right) &= 0 \\ x(x+2)(2x-5) &= 0 \end{aligned}$$

$\therefore y = 7 \text{ when } x = -2, 0, 2.5.$ [1 mark]

The graph is a positive odd cubic with two turning points. The second turning point is a minimum turning point and lies somewhere between (0, 7) and (2.5, 7). [1 mark]

5 The curve is a negative quartic, so as $x \to \pm\infty$, $y \to -\infty$.

The curve starts in quadrant 3 and ends in quadrant 4.

The correct answer is **D**.

Topic 6 — Functions and relations

6.2 Functions and relations

6.2 Exercise

1 a $\{(-4, 7), (0, 10), (3, 5)\}$

The domain is the set of x-coordinates of the ordered pairs.

The domain is $\{-4, 0, 3\}$.

The range is the set of y-coordinates of the ordered pairs.

The range is $\{5, 7, 10\}$.

b In interval notation, $\{x: -10 \le x < 4\}$ is written as $x \in [-10, 4]$.

c The graph has open end points. Reading from left to right, the x-values lie in the open interval between $x = 2$ and $x = 7$, so the domain is $(2, 7)$.

Reading from bottom to top, the y-values lie in the open interval between $y = \frac{1}{7}$ and $y = \frac{1}{2}$, so the range is $\left(\frac{1}{7}, \frac{1}{2}\right)$.

d Both end points are closed. Reading from left to right, the x-values lie in the closed interval between $x = -2$ and $x = 6$, so the domain is $[-2, 6]$.

Reading from bottom to top, the y-values lie in the closed interval between $y = -3$ and $y = 3$, so the range is $[-3, 3]$.

2 a $y = 2x, -1 < x \le 3$

End points: when $x = -1, y = -2 \Rightarrow (-1, -2)$ is an open end point.

When $x = 3, y = 6 \Rightarrow (3, 6)$ is a closed end point.

The graph is the line segment joining these two points.

The domain is $(-1, 3]$ and the range is $(-2, 6]$.

b $y = \frac{2}{3}x, x > 3$

End point: when $x = 3, y = 2 \Rightarrow (3, 2)$ is an open end point.

This is the only end point.

As the line has a positive gradient, $y \to \infty$ as $x \to \infty$.

The domain is $(3, \infty)$ and the range is $(2, \infty)$.

c $y = -3x, -5 \le x \le 5$

End points: when $x = -5, y = 15 \Rightarrow (-5, 15)$ is a closed end point.

When $x = 5, y = -15 \Rightarrow (-15, 5)$ is a closed end point.

The graph is the line segment joining these two points.

The domain is $[-5, 5]$ and the range is $[-15, 15]$.

d $y = x^2, x \in R$

The graph is a parabola with minimum turning point at $(0, 0)$.

The domain is R, which in interval notation is $(-\infty, \infty)$, and the range is $[0, \infty)$.

e $y = x^2, x \in R^+$

The equation represents one half of the parabola $y = x^2$, the section of the graph for which $x > 0$.

Open end point at $(0, 0)$

The domain is R^+, which in interval notation is $(0, \infty)$. The range is $(0, \infty)$.

f $y = x^2, x \in (-2, 2)$

End points: when $x = -2, y = (-2)^2 = 4$, so there is an open end point at $(-2, 4)$.

When $x = 2, y = (2)^2 = 4$, so there is an open end point at $(2, 4)$.

The domain is $(-2, 2)$.

The graph would still include its minimum turning point at $(0, 0)$, since $x = 0$ lies in the domain interval $x \in (-2, 2)$.

Therefore, the range is $[0, 4)$.

3 a $\{(4, 4), (3, 0), (2, 3), (0, -1)\}$

The domain is $\{0, 2, 3, 4\}$ and the range is $\{-1, 0, 3, 4\}$.

This is a function, since each x-coordinate is used exactly once.

b The domain is $[-2, \infty)$ and the range is R. This is not a function, since a vertical line cuts the graph more than once.

c The domain is $[0, 3]$ and the range is $[0, 4]$. This is a function, since a vertical line cuts the graph exactly once.

d $\{(x, y): y = 4 - x^2\}$

This is the parabola $y = 4 - x^2$, with maximum turning point at $(0, 4)$.

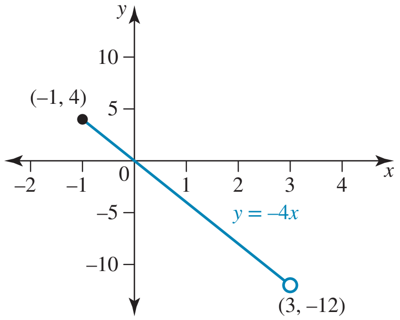
The maximal domain is R and the range is $(-\infty, 4]$. It is a function, since a vertical line would cut its graph exactly once.

4 $y = -4x, x \in [-1, 3)$

End points $x = -1, y = 4 \Rightarrow (-1, 4)$ (closed) and

$x = 3, y = -12 \Rightarrow (3, -12)$ (open)

Domain $[-1, 3)$, range $(-12, 4]$



5 $\{(x.y): y = 2x - 1, -2 < x \le 6\}$

a $y = 2x - 1$ is a straight line.

y-intercept: when $x = 0, y = -1$.

x-intercept: let $y = 0$.

$$2x - 1 = 0$$
$$x = \frac{1}{2}$$

Axis intercepts are $\left(\frac{1}{2}, 0\right)$ and $(0, -1)$.

End points: let $x = -2$.

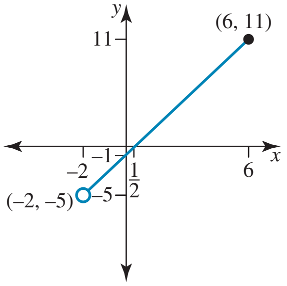
$y = 2(-2) - 1 = -5$

$(-2, -5)$ is an open end point.

Let $x = 6$.

$y = 2(6) - 1 = 11$

$(6, 11)$ is a closed end point.



b The domain is $(-2, 6]$ and the range is $(-5, 11]$.

c As a vertical line would cut the graph exactly once, this relation is a function.

6 $\left\{(x.y): y = (x-2)^3 + 4, -2 \le x \le 4\right\}$

a The graph is of a cubic polynomial.

Stationary point of inflection: $(2, 4)$.

y-intercept: when $x = 0$, $y = (-2)^3 + 4 = -4$ $(0, -4)$.

x-intercept: let $y = 0$.

$$\begin{aligned}(x-2)^3 + 4 &= 0\\ (x-2)^3 &= -4\\ x - 2 &= \sqrt[3]{-4}\\ x - 2 &= -\sqrt[3]{4}\\ x &= 2 - \sqrt[3]{4}\end{aligned}$$

$\left(2 - \sqrt[3]{4}, 0\right)$

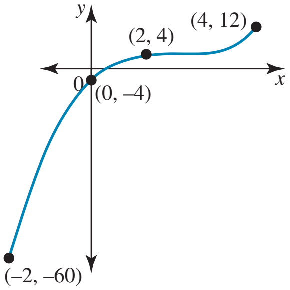
End points: when $x = -2$,

$$\begin{aligned}y &= (-2-2)^3 + 4\\ &= -64 + 4\\ &= -60\end{aligned}$$

When $x = 4$,

$$\begin{aligned}y &= (4-2)^3 + 4\\ &= 8 + 4\\ &= 12\end{aligned}$$

The closed end points are $(-2, \ -60)$ and $(4, 12)$.

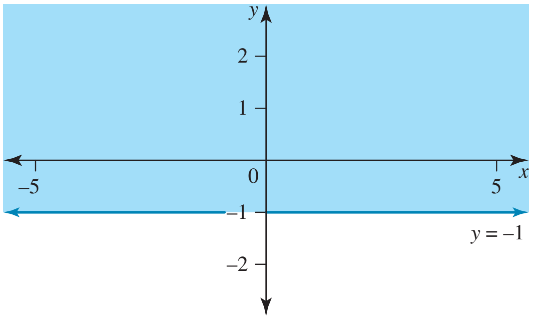


b The domain is $[-2, 4]$ and the range is $[-60, 12]$.

c As a vertical line would cut the graph exactly once, this relation is a function.

7 $\{(x.y): y \ge -1\}$

a This relation is the region that lies on and above the horizontal line $y = -1$.



b The domain is R and the range is $[-1, \infty)$.

c The relation is not a function as a vertical line cuts the graph at more than one place.

8 **a** $y = 8(x+1)^3 - 1$

This is a cubic with a stationary point of inflection at $(-1, \ -1)$. It has a one-to-one correspondence, since a horizontal line would cut the graph exactly once and a vertical line likewise. It is a function, since the vertical line cuts the graph once.

b Considering the cuts made by a horizontal and then a vertical line, the graph has many-to-many correspondence and is therefore not a function.

9 **i** Domain refers to x-values and range refers to y-values.

a Domain $[0, 5]$, range $[0, 15]$

b Domain $[-4, 2) \cup (2, \infty)$, range $(-\infty, 10)$

c Domain $[-3, 6]$, range $[0, 8]$

d Domain $[-2, 2]$, range $[-4, 4]$

e Domain $\{3\}$, range R

f Domain R, range R

ii The type of correspondence is determined by the number of intersections of a horizontal line to the number of intersections of a vertical line with the graph given.

The relation in part **a** has one-to-one correspondence; part **b** has many-to-one correspondence; part **c** has many-to-one correspondence; part **d** has one-to-many correspondence; part **e** has one-to many correspondence; part **f** has many-to-one correspondence.

iii A vertical line would cut the graphs in parts **d** and **e** in more than one place, so these are not the graphs of functions.

10 **a** $\{(-11, 2), \ (-3, 8), \ (-1, 0), \ (5, 2)\}$

Domain $\{-11, \ -3, \ -1, 5\}$, range $\{0, 2, 8\}$

Both $x = -11$ and $x = 5$ are mapped to $y = 2$, so there is a many-to-one correspondence. Each x-value is mapped to a unique y-value, so the relation is a function.

b $\{(20, 6), \ (20, 20), \ (50, 10), \ (60, 10)\}$

Domain $\{20, 50, 60\}$, range $\{6, 10, 20\}$

Since $x = 20$ is mapped to both $y = 6$ and $y = 20$, the relation is not a function.

Also, more than one x-value is mapped to $y = 10$, so the type of correspondence is many-to-many.

c $\{(-14, \ -7), \ (0, 0), \ (0, 2), \ (14, 7)\}$

Domain $\{-14, 0, 14\}$, range $\{-7, 0, 2, 7\}$

Since $x = 0$ is mapped to more than one y-value, the relation is not a function. The type of correspondence is one-to-many.

d $\left\{(x, y): y = 2(x-16)^3 + 13\right\}$

This is a cubic polynomial function with a stationary point of inflection at $(16, 13)$. Its domain is R and its range is R. It has a one-to-one correspondence.

e $\left\{(x, y): y = 4 - (x-6)^2\right\}$

This is a concave down quadratic function with a maximum turning points at $(6, 4)$. Its domain is R and its range is $(-\infty, 4]$. It has a many-to-one correspondence.

f $\left\{(x, y): y = 3x^2(x-5)^2\right\}$

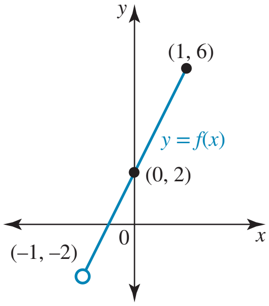
This is a positive fourth degree polynomial function with minimum turning points at $(0, 0)$ and $(5, 0)$. Its domain is R and its range is $R^+ \cup \{0\}$. Its correspondence is many-to-one.

11 **a** $f(x) = 4x + 2, x \in (-1, 1]$

Linear function

End points: $f(-1) = -4 + 2 = -2 \Rightarrow (-1, \ -2)$ open, $f(1) = 4 + 2 = 6 \Rightarrow (1, 6)$ closed

y-intercept $(0, 2)$ and x-intercept $\left(-\frac{1}{2}, 0\right)$



The domain is $(-1,\ 1]$ and the range is $(-2,\ 6]$

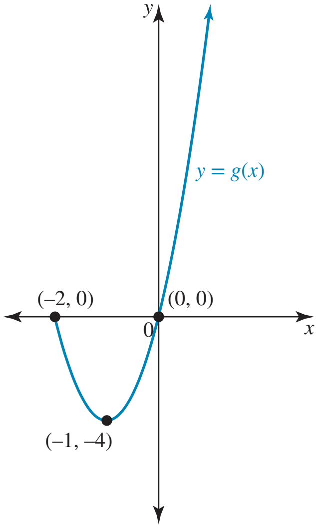
b $g(x) = 4x(x+2), x \geq -2$

The graph of $y = g(x)$ is a concave up parabola.

x-intercepts and closed end point $(-2, 0)$, x- and y-intercept $(0, 0)$

The x-coordinate of the turning point occurs midway between the x-intercepts.

When $x = -1$, $y = -4(1) = -4 \Rightarrow (-1,\ -4)$ is the minimum turning point.



Domain $[-2,\ \infty)$, range $[-4,\ \infty)$

c $h(x) = 4 - x^3, x \in R^+$

The graph of $y = h(x)$ is a cubic function with a stationary point of inflection at $(0, 4)$. Since $x \in R^+$, this point is open.

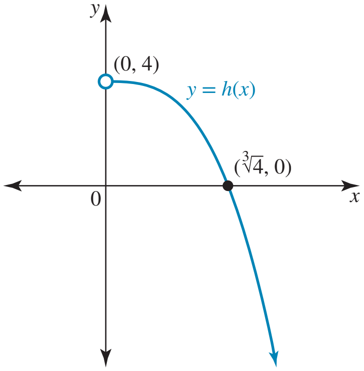
x-intercept: let $y = 0$.

$\therefore 4 - x^3 = 0$

$\therefore x^3 = 4$

$\therefore x = \sqrt[3]{4}$

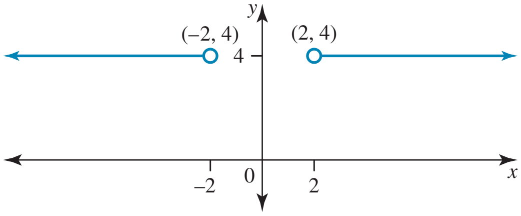
$\left(\sqrt[3]{4}, 0\right)$



Domain R^+, range $(-\infty, 4)$

d $y = 4, x \in R\backslash[-2, 2]$

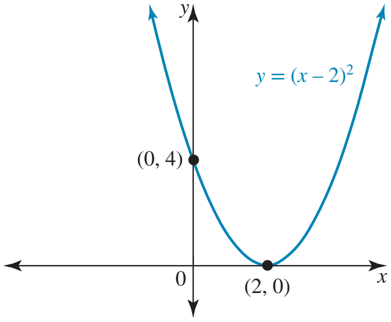
The graph is a horizontal line drawn for $x < -2$ and $x > -2$ with open end points at $(-2, 4)$ and $(2, 4)$.



Domain $R\backslash[-2, 2]$, range $\{4\}$

12 **a** $y = (x-2)^2$

Minimum turning point at $(2, 0)$,
y-intercept at $(0, 4)$



Domain R, range $R^+ \cup \{0\}$, many-to-one correspondence

b A possible answer is $[2, \infty)$.

13 **a** $f(x) = ax + b$

$f(2) = 7 \Rightarrow 7 = 2a + b$ [1]

$f(3) = 9 \Rightarrow 9 = 3a + b$ [2]

[2] − [1]:

$2 = a$

$\therefore b = 3$

$\Rightarrow f(x) = 2x + 3$

b $f(x) = 2x + 3$

$\therefore f(0) = 3$

c $f(x) = 0$

$\therefore 2x + 3 = 0$

$\therefore x = -\frac{3}{2}$

d The image of -3 is $f(-3)$.

$f(-3) = 2(-3) + 3$

$= -3$

e $g: (-\infty, 0] \rightarrow R, g(x) = 2x + 3$

14 $f(x) = 1 - 4x$

a **i** $f(0) = 1 - 4(0) = 1$

ii $f(-5) = 1 - 4(-5)$

$= 21$

iii $f\left(\frac{1}{8}\right)$

$= 1 - 4 \times \frac{1}{8}$

$= \frac{1}{2}$

b **i** $f(x) = 0$

$1 - 4x = 0$

$1 = 4x$

$x = \frac{1}{4}$

ii $f(x) = -1$

$1 - 4x = -1$

$2 = 4x$

$x = \frac{1}{2}$

iii $f(x) = x$

$1 - 4x = x$

$1 = 5x$

$x = \frac{1}{5}$

c i The image of -11 is $f(-11)$.

$f(-11) = 1 - 4(-11)$

$= 45$

ii The image of 1.5 is $f(1.5)$.

$f(1.5) = 1 - 4(1.5)$

$= -5$

iii The image of $1 - \sqrt{2}$ is $f\left(1 - \sqrt{2}\right)$.

$f\left(1 - \sqrt{2}\right) = 1 - 4\left(1 - \sqrt{2}\right)$

$= 1 - 4 + 4\sqrt{2}$

$= 4\sqrt{2} - 3$

d $f(x) = g(x)$, where $g(x) = x^2 + 5$

$1 - 4x = x^2 + 5$

$0 = x^2 + 4x + 4$

$(x + 2)^2 = 0$

$x = -2$

15 $f(x) = x^2 + 2x - 3$

a i $f(-2) = (-2)^2 + 2(-2) - 3$

$= 4 - 4 - 3$

$f(-2) = -3$

ii $f(9) = (9)^2 + 2(9) - 3$

$= 81 + 18 - 3$

$f(9) = 96$

b i $f(2a) = (2a)^2 + 2(2a) - 3$

$= 4a^2 + 4a - 3$

ii $f(1 - a) = (1 - a)^2 + 2(1 - a) - 3$

$= 1 - 2a + a^2 + 2 - 2a - 3$

$f(1 - a) = a^2 - 4a$

c $f(x + h) = (x + h)^2 + 2(x + h) - 3$

$= x^2 + 2xh + h^2 + 2x + 2h - 3$

$f(x) = x^2 + 2x - 3$

$\therefore f(x + h) - f(x)$

$= (x^2 + 2xh + h^2 + 2x + 2h - 3) - (x^2 + 2x - 3)$

$= \cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - 3 - \cancel{x^2} - \cancel{2x} + 3$

$= 2xh + h^2 + 2h$

d $f(x) > 0$

$\therefore x^2 + 2x - 3 > 0$

$\therefore (x + 3)(x - 1) > 0$

Zeros $x = -3, x = 1$

The solution set is $\{x : x < -3\} \cup \{x : x > 1\}$.

e $f(x) = 12$

$\therefore x^2 + 2x - 3 = 12$

$\therefore x^2 + 2x - 15 = 0$

$\therefore (x + 5)(x - 3) = 0$

$\therefore x = -5, x = 3$

f $f(x) = 1 - x$

$\therefore x^2 + 2x - 3 = 1 - x$

$\therefore x^2 + 3x - 4 = 0$

$\therefore (x + 4)(x - 1) = 0$

$\therefore x = -4, x = 1$

16 $f: R \rightarrow R, f(x) = x^3 - x^2$

a The image of 2 is $f(2)$.

$f(2) = 2^3 - 2^2$

$= 8 - 4$

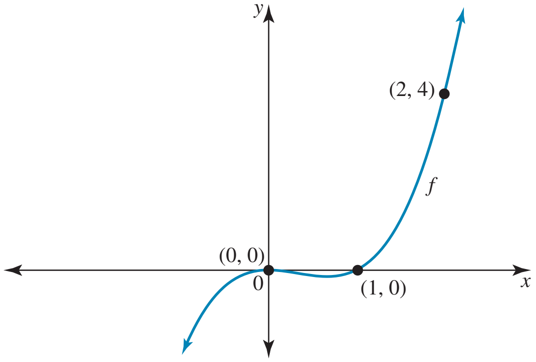
$= 4$

The image of 2 is 4.

b $f(x) = x^3 - x^2$

$= x^2(x - 1)$

x-intercepts occur at $x = 0$ (touch) and $x = 1$ (cut).



c Domain R, range R

d Many-to-one correspondence

e Many answers are possible. One answer is to restrict the domain to $(1, \infty)$; another is to restrict the domain to R^-.

f $f(x) = 4$

$\therefore x^3 - x^2 = 4$

$\therefore x^3 - x^2 - 4 = 0$

From part **a**, $f(2) = 4$, so $x = 2$ is a solution. Therefore, $(x - 2)$ is a factor.

$\therefore (x - 2)(x^2 + x + 2) = 0$

$\therefore x = 2$ or $x^2 + x + 2 = 0$

Consider $x^2 + x + 2 = 0$.

$\Delta = (1)^2 - 4 \times 1 \times 2$

$= -7$

$\therefore \Delta < 0$

There are no real solutions for $x^2 + x + 2 = 0$.

The only solution is $x = 2$.

Alternatively, the horizontal line $y = 4$ intersects the graph of $y = f(x)$ exactly once at the point $(2, 4)$, so $x = 2$ is the only solution to $f(x) = 4$.

Answer: $\{x: x = 2\}$

17 $y = x^2 - 6x + 10, 0 \leq x < 7$

$f: [0, 7) \rightarrow R, f(x) = x^2 - 6x + 10$

Domain $[0, 7)$

For the range, a sketch of the graph can be used, or the range can be deduced from the position and type of the turning point of a parabola and the end points of the graph.

$y = x^2 - 6x + 10$

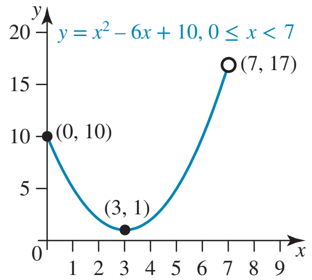
$\therefore y = (x^2 - 6x + 9) - 9 + 10$

$\therefore y = (x - 3)^2 + 1$

Minimum turning point at $(3, 1)$

End points: $x = 0, y = 10 \Rightarrow (0, 10)$ (closed) and $x = 7, y = 17 \Rightarrow (7, 17)$ (open)

Therefore, the range is $[1, 17)$. This is confirmed by the graph.



18 $A = \{(1, 2), (2, 3), (3, 2), (k, 3)\}$

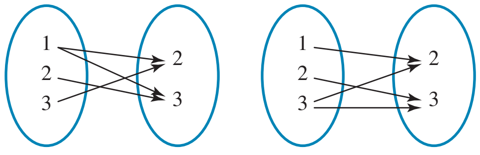
a If $k = 1$, then A would contain the two points $(1, 2)$ and $(1, 3)$. This has one x-value mapped to more than one y-value, so A would not be a function if $k = 1$.

Similarly, if $k = 3$, then $x = 3$ would be mapped to more than one y-value, so A would not be a function.

Answer: $k = 1, k = 3$, with the relation having a many-to-many correspondence for either value of k.

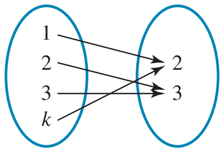
b A will be a function if $k \in R \backslash \{1, 3\}$. As A contains the points $(1, 2)$ and $(3, 2)$, it has a many-to-one correspondence.

c Mapping diagrams for $k = 1, k = 3$:



$k = 1$ $k = 3$

d Mapping diagram for $k \in R| \{1, 3\}$:



19 **a** $y = x^2, x \in Z^+$ is the rule for the set of points $\{(1, 1), (2, 4), (3, 9), \ldots\}$

It is the function $f: Z^+ \to R, f(x) = x^2$ with range $\{1, 4, 9, 16 \ldots\}$.

b $2x + 3y = 6$ is the equation of a straight line.

Rearranging,

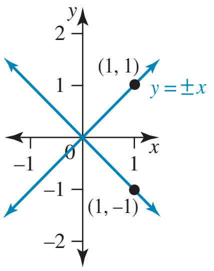
$3y = 6 - 2x$

$\therefore y = -\frac{2}{3}x + 2$

It is the function $f: R \to R, f(x) = -\frac{2}{3}x + 2$ with range R.

c $y = \pm x$

Let $x = 1$. $\therefore y = \pm 1$. The points $(1, -1)$ and $(1, 1)$ are part of this relation. It is not a function as the x-values are not mapped to a unique y-value.



The range is R.

d $\{(x, 5), x \in R\}$ is the set of points whose y-value is always equal to 5. It is the set of points that lie on the horizontal line $y = 5$.

This is a function $f: R \to R, f(x) = 5$ with range $\{5\}$.

e $\{(-1, y), y \in R\}$ is the set of points whose x-value is always equal to -1. It is the set of points that lie on the vertical line $x = -1$. This is not a function. The relation has range R.

f $y = -x^3, x \in R^+$ is part of the cubic polynomial function with a point of inflection at the origin. Since $x \in R^+$, the values of the function are always negative.

It is the function $f: R^+ \to R, f(x) = -x^3$ with range R^-.

20 $f(x) = a + bx + cx^2$ and $g(x) = f(x - 1)$

a The function f is a quadratic polynomial.

Given: $f(-2) = 0 \Rightarrow (x + 2)$ is a factor.

Given: $f(5) = 0 \Rightarrow (x - 5)$ is a factor.

Let $f(x) = a(x + 2)(x - 5)$.

Given: $f(2) = 3 \Rightarrow 3 = a(4)(-3)$

$\therefore 3 = -12a$

$\therefore a = -\frac{3}{12}$

$\therefore a = -\frac{1}{4}$

$\therefore f(x) = -\frac{1}{4}(x + 2)(x - 5)$

Expanding,

$$\begin{aligned} f(x) &= -\frac{1}{4}\left(x^2 - 3x - 10\right) \\ &= -\frac{1}{4}x^2 + \frac{3}{4}x + \frac{10}{4} \\ &= \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2 \end{aligned}$$

$\therefore f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$ is the rule for the function f.

b As $g(x) = f(x - 1)$, then $g(x) = \frac{5}{2} + \frac{3}{4}(x - 1) - \frac{1}{4}(x - 1)^2$.

$$\begin{aligned} \therefore g(x) &= \frac{5}{2} + \frac{3}{4}x - \frac{3}{4} - \frac{1}{4}\left(x^2 - 2x + 1\right) \\ &= \frac{7}{4} + \frac{3}{4}x - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{4} \\ &= \frac{6}{4} + \frac{5}{4}x - \frac{1}{4}x^2 \end{aligned}$$

$\therefore g(x) = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$

c $f(x) = g(x)$

$\therefore \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2 = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$

$\therefore \frac{5}{2} + \frac{3}{4}x = \frac{3}{2} + \frac{5}{4}x$

$$\therefore \frac{5}{2} - \frac{3}{2} = \frac{5}{4}x - \frac{3}{4}x$$

$$\therefore 1 = \frac{1}{2}x$$

$$\therefore x = 2$$

d $f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$ or $f(x) = -\frac{1}{4}(x+2)(x-5)$

x-intercepts: $(-2, 0), (5, 0)$

y-intercept: $\left(0, \frac{5}{2}\right)$

The maximum turning point occurs at $x = \frac{-2+5}{2} = \frac{3}{2}$.

$$\begin{aligned}\therefore y &= -\frac{1}{4}\left(\frac{3}{2}+2\right)\left(\frac{3}{2}-5\right)\\ &= -\frac{1}{4} \times \frac{7}{2} \times \frac{-7}{2}\\ &= \frac{49}{16}\end{aligned}$$

$\left(\frac{3}{2}, \frac{49}{16}\right)$

$$\begin{aligned}g(x) &= \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2\\ &= -\frac{1}{4}(x^2 - 5x - 6)\\ &= -\frac{1}{4}(x-6)(x+1)\end{aligned}$$

x-intercepts: $(-1, 0), (6, 0)$

y-intercept: $\left(0, \frac{3}{2}\right)$

The maximum turning point occurs at $x = \frac{-1+6}{2} = \frac{5}{2}$.

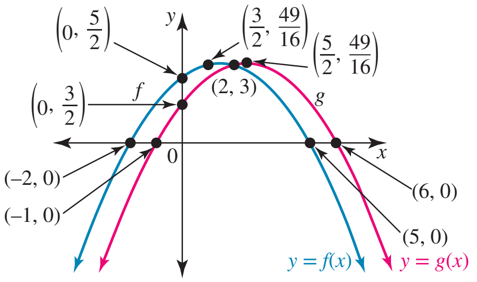
$$\begin{aligned}\therefore y &= -\frac{1}{4}\left(\frac{5}{2}-6\right)\left(\frac{5}{2}+1\right)\\ &= -\frac{1}{4} \times \frac{-7}{2} \times \frac{7}{2}\\ &= \frac{49}{16}\end{aligned}$$

$\left(\frac{5}{2}, \frac{49}{16}\right)$

The point of intersection of the two graphs occurs when $x = 2$.

$$\begin{aligned}f(2) &= -\frac{1}{4}(2+2)(2-5)\\ &= -\frac{1}{4} \times -12\\ &= 3\end{aligned}$$

$(2, 3)$



The graph of function g has the same shape as the graph of function f, but g has been horizontally translated 1 unit to the right.

21 a $x(t) = 4 + 5t, t \in [0, 5]$ is a linear function with domain $[0, 5]$.

End points of the range: $x(0) = 4$ and $x(5) = 4 + 25 = 29$

The range is $[4, 29]$.

The distance travelled is $29 - 4 = 25$ units.

b $h(t) = 10t - 5t^2$

At ground level, $h(t) = 0$.

$\therefore 0 = 10t - 5t^2$

$\therefore 0 = 5t(2 - t)$

$\therefore t = 0, t = 2$

It takes the hat 2 seconds to return to the ground.

The domain is $[0, 2]$.

For the range, the turning point is required.

The maximum turning point occurs when $t = 1$.

$h(1) = 10 - 5 = 5$

The turning point is $(1, 5)$.

The range is $[0, 5]$.

c $l(t) = 0.5 + 0.2t^3, 0 \le t \le 2$

i The domain is $[0, 2]$.

$l(0) = 0.5$, $l(2) = 0.5 + 0.2 \times 8 = 2.1$, so the range of the cubic function is $[0.5, 2.1]$.

ii At the end of the two weeks, the leaf is 2.1 units in length.

Find t when $l(t) = 0.5 \times 2.1$.

$\therefore 0.5 + 0.2t^3 = 0.5 \times 2.1$

Multiply both sides by 2.

$$\begin{aligned}\therefore 1 + 0.4t^3 &= 2.1\\ \therefore 0.4t^3 &= 1.1\\ \therefore t^3 &= \frac{11}{4}\\ \therefore t &= \sqrt[3]{2.75}\\ \therefore t &\simeq 1.4\end{aligned}$$

It took approximately 1.4 weeks for the leaf to reach half its final length.

22 a Use CAS technology to solve.

$$\left.\begin{aligned}f(3) &= -25\\ f(5) &= 49\\ f(7) &= 243\end{aligned}\right|_{l,m,n}$$

This gives $l = 0, m = -12, n = -16$.

$\therefore f(x) = x^3 - 12x - 16$

b Redefine $f(x) = x^3 - 12x - 16$.

The image of 1.2 is $f(1.2) = -28.672$.

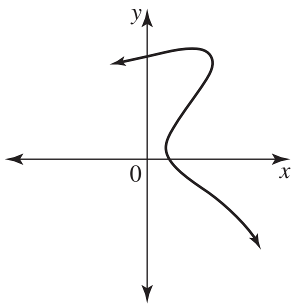
c Solve $f(x) = 20$ using CAS technology to obtain $x = 4.477$ correct to 3 decimal places.

d $f\colon [0, \infty] \rightarrow R, f(x) = x^3 - 12x - 16$

The range is $[-32, \infty]$.

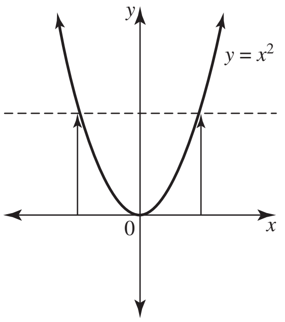
6.2 Exam questions

1

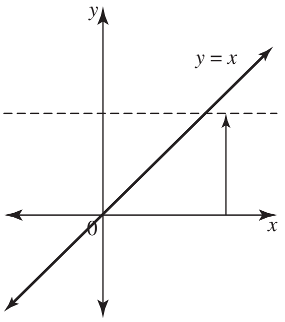


A vertical line crosses the graph in three places. Therefore, the graph is a relation but not a function. A vertical line should cross the graph of a function a maximum of one time. The correct answer is **D**.

2 The horizontal line test is used to determine the type of correspondence.



Many-to-one correspondence



One-to-one correspondence

The correct answer is **B**.

3 **a** $f(x) = 2x^3 - 3x^2$

$f(-1) = 2(-1)^3 - 3(-1)^2$

$= -5$ [1 mark]

b Calculate intercepts.

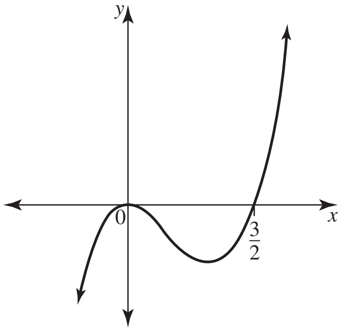
x-intercepts $(y = 0)$:

$2x^3 - 3x^2 = 0$

$x^2(2x - 3) = 0$

$x = 0, \frac{3}{2}$

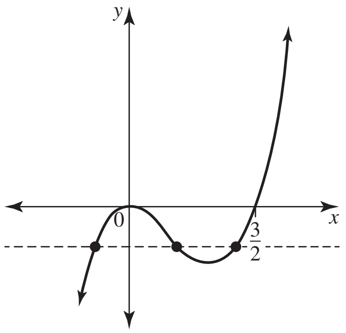
Turning point at $x = 0$ (repeated factor) [1 mark]



[1 mark]

c Domain = R, range = R [1 mark]

d The horizontal line test shows many-to-one correspondence. [1 mark]



6.3 The rectangular hyperbola and the truncus

6.3 Exercise

1 **a** $y = \frac{1}{x + 5} + 2$

Since $x + 5 = 0$ when $x = -5$, the asymptotes have the equations $x = -5$ and $y = 2$.

b $y = \frac{8}{x} - 3$

The asymptotes have the equations $x = 0$ and $y = -3$.

c $y = \frac{-3}{4x}$

Since $4x = 0$ when $x = 0$, the asymptotes have the equations $x = 0$ and $y = 0$.

d $y = \frac{-3}{14 + x} - \frac{3}{4}$

Since $14 + x = 0$ when $x = -14$, the asymptotes have the equations $x = -14$ and $y = -\frac{3}{4}$.

2 **a** $y = \frac{1}{x - 6} + 1$

Compare with the form $y = \frac{a}{x - h} + k$.

i The asymptotes have equations $x = 6$ and $y = 1$.

ii The domain is $R\backslash\{6\}$.

iii The range is $R\backslash\{1\}$.

b $y = \frac{1}{3 + x} - 7$

$y = \frac{1}{x + 3} - 7$

Compare with the form $y = \frac{a}{x - h} + k$.

i The asymptotes have equations $x = -3$ and $y = -7$.

ii The domain is $R\backslash\{-3\}$.

iii The range is $R\backslash\{-7\}$.

c $y = 4 + \frac{8}{2 - x}$

$$y = \frac{8}{2 - x} + 4$$

$$= -\frac{8}{x - 2} + 4$$

$$= \frac{-8}{x - 2} + 4$$

Compare with the form $y = \frac{a}{x - h} + k$.

i The asymptotes have equations $x = 2$ and $y = 4$.

ii The domain is $R\backslash\{2\}$.

iii The range is $R\backslash\{4\}$.

d $y = \dfrac{-5}{4(x-1)} - 3$

$y = \dfrac{-\frac{5}{4}}{x-1} - 3$

Compare with the form $y = \dfrac{a}{x-h} + k$.

i The asymptotes have equations $x = 1$ and $y = -3$.

ii The domain is $R\backslash\{1\}$.

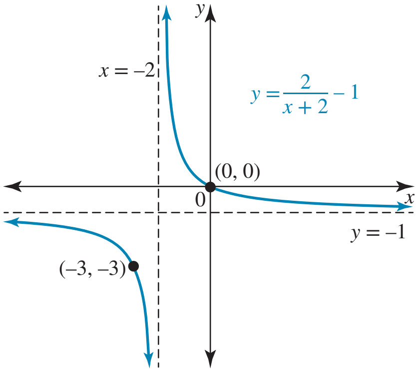
iii The range is $R\backslash\{-3\}$.

3 **a** $y = \dfrac{2}{x+2} - 1$

Vertical asymptote $x = -2$

Horizontal asymptote $y = -1$

y-intercept $(0, 0)$



Domain $R\backslash\{-2\}$, range $R\backslash\{-1\}$

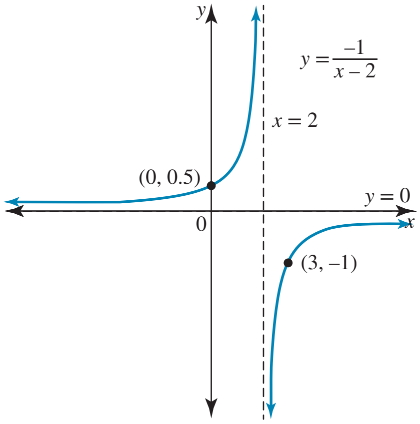
b $y = \dfrac{-1}{x-2}$

Vertical asymptote $x = 2$

Horizontal asymptote $y = 0$

y-intercept $\left(0, \dfrac{1}{2}\right)$

Domain $R\backslash\{2\}$, range $R\backslash\{0\}$



4 $y = \dfrac{1}{x-1} + 5$

a The asymptotes have equations $x = 1$ and $y = 5$.

b y-intercept: let $x = 0$.

$y = \dfrac{1}{0-1} + 5$

$= 4$

$(0, 4)$

c x-intercept: let $y = 0$.

$\dfrac{1}{x-1} + 5 = 0$

$\dfrac{1}{x-1} = -5$

$1 = -5(x-1)$

$-5x + 5 = 1$

$-5x = -4$

$x = \dfrac{4}{5}$

$\left(\dfrac{4}{5}, 0\right)$

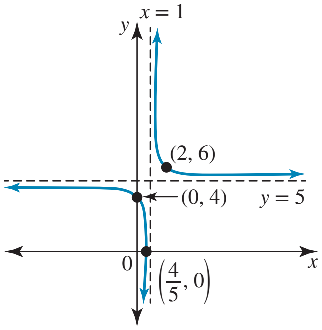
d Let $x = 2$ in $y = \dfrac{1}{x-1} + 5$.

$y = \dfrac{1}{2-1} + 5$

$= 1 + 5$

$= 6$

The point $(2, 6)$ lies on the graph.



5 $y = 5 - \dfrac{6}{x+3}$

a $y = -\dfrac{6}{x+3} + 5$

The asymptotes have equations $x = -3$ and $y = 5$.

b y-intercept: let $x = 0$.

$y = 5 - \dfrac{6}{0+3}$

$= 5 - 2$

$= 3$

$(0, 3)$

c x-intercept: let $y = 0$.

$5 - \dfrac{6}{x+3} = 0$

$5 = \dfrac{6}{x+3}$

$5(x+3) = 6$

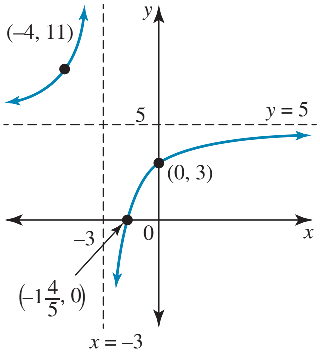
$5x + 15 = 6$

$5x = -9$

$x = -\dfrac{9}{5}$

$\left(-\dfrac{9}{5}, 0\right)$

d



Point on left-hand branch: let $x = -4$.

$$y = 5 - \frac{6}{-4+3}$$
$$= 5 + 6$$
$$= 11$$

$(-4, 11)$

6 a $y = \frac{1}{x+1} - 3$

Asymptotes: $x = -1, y = -3$

y-intercept: let $x = 0, y = \frac{1}{1} - 3 = -2.\ (0,\ -2)$

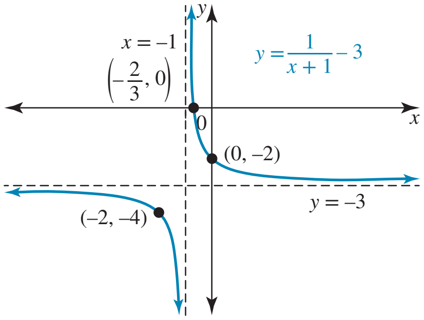
x-intercept: let $y = 0$.

$$0 = \frac{1}{x+1} - 3$$
$$\therefore 3 = \frac{1}{x+1}$$
$$\therefore 3(x+1) = 1$$
$$\therefore x + 1 = \frac{1}{3}$$
$$\therefore x = -\frac{2}{3}$$

$\left(-\frac{2}{3},\ 0\right)$

Domain $R\backslash\{-1\}$, range $R\backslash\{-3\}$

Point: when $x = -2,\ y = \frac{1}{-1} - 3 = -4.\ (-2, -4)$



b $y = 4 - \frac{3}{x-3}$ or $y = -\frac{3}{x-3} + 4$

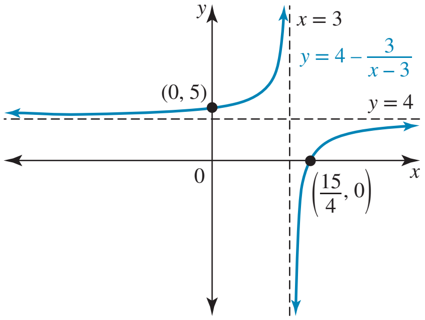
Asymptotes: $x = 3, y = 4$

y-intercept: let $x = 0,\ y = 4 - \frac{3}{-3} = 5.\ (0, 5)$

x-intercept: let $y = 0$

$$0 = 4 - \frac{3}{x-3}$$
$$\therefore \frac{3}{x-3} = 4$$
$$\therefore \frac{3}{4} = x - 3$$
$$\therefore x = \frac{15}{4}\quad \left(\frac{15}{4}, 0\right)$$

Domain $R\backslash\{3\}$, range $R\backslash\{4\}$



c $y = -\frac{5}{3+x}$

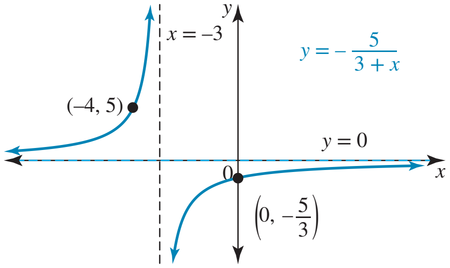
Asymptotes: $x = -3, y = 0$

y-intercept: let $x = 0,\ y = -\frac{5}{3}.\ \left(0,\ -\frac{5}{3}\right)$

No x-intercept

Domain $R\backslash\{-3\}$, range $R\backslash\{0\}$

Point: let $x = -4, y = -\frac{5}{-1} = 5\ \ (-4, 5)$



d $y = -\left(1 + \frac{5}{2-x}\right)$

$$\therefore y = -1 - \frac{5}{2-x}$$
$$\therefore y = -1 + \frac{5}{x-2}$$

Asymptotes: $x = 2, y = -1$

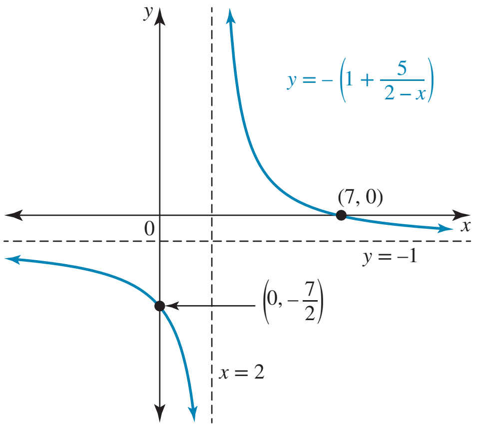
y-intercept: let $x = 0, y = -1 + \frac{5}{-2} = -\frac{7}{2}.\ \left(0,\ -\frac{7}{2}\right)$

x-intercept: let $y = 0$

$$0 = -1 + \frac{5}{x-2}$$
$$\therefore 1 = \frac{5}{x-2}$$
$$\therefore x - 2 = 5$$
$$\therefore x = 7$$

$(7, 0)$

Domain $R\backslash\{2\}$, range $R\backslash\{-1\}$



7 $f: R\backslash\left\{\frac{1}{2}\right\} \to R, f(x) = 4 - \frac{3}{1-2x}$

Vertical asymptote when $1 - 2x = 0$

Therefore, the vertical asymptote has equation $x = \frac{1}{2}$.

The horizontal asymptote is $y = 4$.

The equation $y = 4 - \frac{3}{1-2x}$ can be expressed as

$y = -\frac{3}{-2\left(x - \frac{1}{2}\right)} + 4$.

So, $y = \frac{3}{2\left(x - \frac{1}{2}\right)} + 4 \Rightarrow y = \frac{\frac{3}{2}}{x - \frac{1}{2}} + 4$.

This means the hyperbola lies in quadrants 1 and 3 (quadrants as defined by the asymptotes).

8 **a** $y = \frac{6x}{3x+2}$; improper form, so divide to obtain proper form.

$y = \frac{2(3x+2) - 4}{3x+2}$

$\therefore y = 2 - \frac{4}{3x+2}$

Vertical asymptote $x = -\frac{2}{3}$, horizontal asymptote $y = 2$

b The asymptotes are shown as $x = 4, y = \frac{1}{2}$.

The equation becomes $y = \frac{a}{x-4} + \frac{1}{2}$.

Substitute the point $(6, 0)$.

$\therefore 0 = \frac{a}{2} + \frac{1}{2}$

$\therefore a = -1$

Therefore, the equation is $y = \frac{-1}{x-4} + \frac{1}{2}$.

9 **a** $y = \frac{a}{x-h} + k$

The asymptotes are $x = -4$ and $y = 2$.

Therefore, $y = \frac{a}{x+4} + 2$.

Substitute the point $(0, 8)$.

$8 = \frac{a}{0+4} + 2$

$6 = \frac{a}{4}$

$a = 24$

The equation is $y = \frac{24}{x+4} + 2$.

b Let the equation be $y = \frac{a}{x-h} + k$.

The asymptotes are $x = 0$ and $y = 5$, so the equation becomes $y = \frac{a}{x-0} + 5$.

$\therefore y = \frac{a}{x} + 5$

Substitute the point $(2, -1)$.

$-1 = \frac{a}{2} + 5$

$-6 = \frac{a}{2}$

$a = -12$

The equation is $y = -\frac{12}{x} + 5$.

c Let the equation be $y = \frac{a}{x-h} + k$

The asymptotes are $x = 5$ and $y = 2$, so the equation becomes $y = \frac{a}{x-5} + 2$.

Substitute one of the given points.

y-intercept at $y = 4$, when $x = 0$:

$y = \frac{a}{x-5} + 2$

$4 = \frac{a}{-5} + 2$

$2 = \frac{a}{-5}$

$a = -10$

$\therefore y = 2 - \frac{10}{x-5}$

d **i** The domain is $(-3, 3)\backslash\{0\}$.

The range can be expressed as $\left(-\infty, \frac{8}{3}\right) \cup \left(\frac{10}{3}, \infty\right)$

or $R\backslash\left[\frac{8}{3}, \frac{10}{3}\right]$.

ii The asymptotes are $x = 0$ and $y = 3$.

The equation $y = \frac{a}{x-h} + k$ becomes $y = \frac{a}{x} + 3$.

Substitute the point $\left(3, 3\frac{1}{3}\right)$.

$3\frac{1}{3} = \frac{a}{3} + 3$

$\frac{1}{3} = \frac{a}{3}$

$a = 1$

The equation is $y = \frac{1}{x} + 3, -3 < x < 3, x \neq 0$.

10 **a** $\frac{11-3x}{4-x} = a - \frac{b}{4-x}$

$\therefore \frac{11-3x}{4-x} = \frac{a(4-x) - b}{4-x}$

$\therefore \frac{11-3x}{4-x} = \frac{4a - ax - b}{4-x}$

$\therefore 11 - 3x = -ax + 4a - b$

Equating coefficients of like terms:

x: $-3 = -a$

$\therefore a = 3$

Constant: $11 = 4a - b$

Substitute $a = 3$.

$\therefore 11 = 12 - b$

$\therefore b = 1$

Answer: $a = 3, b = 1$

b $y = \frac{11-3x}{4-x} \Rightarrow y = 3 - \frac{1}{4-x}$

Asymptotes: $x = 4, y = 3$

y-intercept: let $x = 0,\ y = 3 - \frac{1}{4} = \frac{11}{4}$. $\left(0, \frac{11}{4}\right)$

x-intercept: let $y = 0$ in $y = \frac{11 - 3x}{4 - x}$.

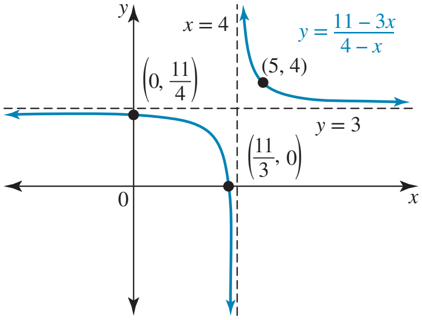
$\therefore 0 = \frac{11 - 3x}{4 - x}$

$\therefore 0 = 11 - 3x$

$\therefore x = \frac{11}{3}$

$\left(\frac{11}{3}, 0\right)$

Point: let $x = 5,\ y = 3 - \frac{1}{-1} = 4$. $(5, 4)$



c The y-values of the points on the graph are positive when $x < \frac{11}{3}$ or $x > 4$, so $\frac{11 - 3x}{4 - x} > 0$ when $x < \frac{11}{3}$ or $x > 4$.

11 a $y = \frac{x}{4x + 1}$

Using the division algorithm,

$$\begin{array}{r|l} & \frac{1}{4} \\ \hline 4x+1 & x+0 \\ & x+\frac{1}{4} \\ \hline & -\frac{1}{4} \\ \hline \end{array}$$

$\therefore \frac{x}{4x + 1} = \frac{1}{4} - \frac{\frac{1}{4}}{4x + 1}$

$\therefore y = \frac{-1}{4(4x + 1)} + \frac{1}{4}$

$\therefore y = \frac{-1}{16x + 4} + \frac{1}{4}$

This is in the form $y = \frac{a}{bx + c} + d$ with $a = -1, b = 16, c = 4, d = \frac{1}{4}$.

Since $16x + 4 = 0$ when $x = -\frac{1}{4}$, the equations of the asymptotes are $x = -\frac{1}{4}, y = \frac{1}{4}$.

b $(x - 4)(y + 2) = 4$

$\therefore (y + 2) = \frac{4}{(x - 4)}$

$\therefore y = \frac{4}{x - 4} - 2$

The equations of the asymptotes are $x = 4, y = -2$.

c $y = \frac{1 + 2x}{x}$

$\therefore y = \frac{1}{x} + \frac{2x}{x}$

$\therefore y = \frac{1}{x} + 2$

The equations of the asymptotes are $x = 0, y = 2$.

d $2xy + 3y + 2 = 0$

$\therefore y(2x + 3) + 2 = 0$

$\therefore y(2x + 3) = -2$

$\therefore y = \frac{-2}{2x + 3}$

Since $2x + 3 = 0$ when $x = -\frac{3}{2}$, the equations of the asymptotes are $x = -\frac{3}{2}, y = 0$.

12 $xy - 4y + 1 = 0$ needs to be expressed in standard hyperbola form.

$xy - 4y + 1 = 0$

$\therefore xy - 4y = -1$

$\therefore y(x - 4) = -1$

$\therefore y = \frac{-1}{x - 4}$

The asymptotes have equations $x = 4, y = 0$, so domain is $R\backslash\{4\}$ and range is $R\backslash\{0\}$.

13 Let the equation be $y = \frac{a}{(x - h)^2} + k$.

The asymptotes are $x = 7$ and $y = 0$.

The equation is of the form $y = \frac{a}{(x - 7)^2} + 0 \Rightarrow y = \frac{a}{(x - 7)^2}$.

C is the correct answer, since $y = \frac{-3}{(7 - x)^2}$ can be written as $y = \frac{-3}{(x - 7)^2}$.

14 a The domain is $R\backslash\{-3\}$ and the range is $(5, \infty)$.

b Let the equation be $y = \frac{a}{(x - h)^2} + k$.

The asymptotes are $x = -3$ and $y = 5$.

The equation becomes $y = \frac{a}{(x + 3)^2} + 5$.

Substitute the point $\left(0, 7\frac{2}{9}\right)$.

$7\frac{2}{9} = \frac{a}{(0 + 3)^2} + 5$

$2\frac{2}{9} = \frac{a}{9}$

$\frac{20}{9} = \frac{a}{9}$

$a = 20$

The equation is $y = \frac{20}{(x + 3)^2} + 5$.

15 a $R \propto \frac{1}{I} \Rightarrow R = \frac{k}{I}$

$I = 0.6, R = 400$

$\therefore 400 = \frac{k}{0.6}$

$\therefore k = 400 \times 0.6$

$\therefore k = 240$

Hence, $R = \frac{240}{I}$.

b Current is increased by 20% ⇒ $I = 0.6 \times 1.20 = 0.72$

If $I = 0.72$,

$$R = \frac{240}{0.72}$$
$$= \frac{1000}{3}$$

The resistance is $333\frac{1}{3}$ ohms.

16 a $y = \dfrac{1}{(x-3)^2} - 1$

Asymptotes: $x = 3, y = -1$

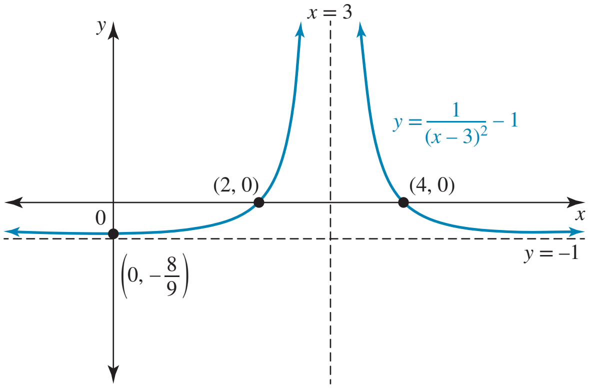
y-intercept: let $x = 0$.

$$\therefore y = \frac{1}{(-3)^2} - 1$$
$$\therefore y = -\frac{8}{9}$$
$$\left(0, -\frac{8}{9}\right)$$

x-intercepts: let $y = 0$.

$$\therefore \frac{1}{(x-3)^2} - 1 = 0$$
$$\therefore \frac{1}{(x-3)^2} = 1$$
$$\therefore (x-3)^2 = 1$$
$$\therefore x - 3 = \pm 1$$
$$\therefore x = 2, x = 4$$

$(2, 0), (4, 0)$



Domain $R\backslash\{3\}$, range $(-1, \infty)$

b $y = \dfrac{-8}{(x+2)^2} - 4$

Asymptotes: $x = -2, y = -4$

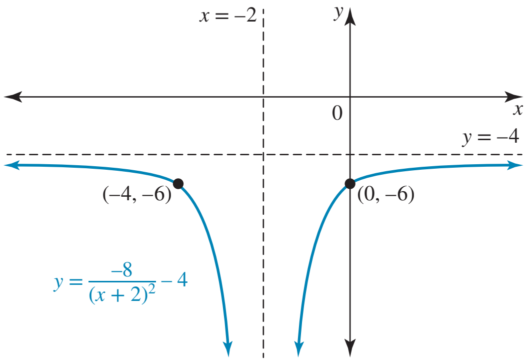
y-intercept: let $x = 0$.

$$\therefore y = \frac{-8}{(2)^2} - 4$$
$$\therefore y = -6$$

$(0, -6)$

x-intercepts: As $a < 0$ and horizontal asymptote is $y = -4$, there are no x-intercepts.

Point: By symmetry, the point $(-4, -6)$ lies on the graph.



Domain $R\backslash\{-2\}$, range $(-\infty, -4)$

17 a $y = \dfrac{4}{3x^2} - 1$

i The asymptotes are $x = 0$ and $y = -1$.

ii The domain is $R\backslash\{0\}$.

Since $a > 0$ in the equation, the graph lies above its horizontal asymptote.

The range is $(-1, \infty)$.

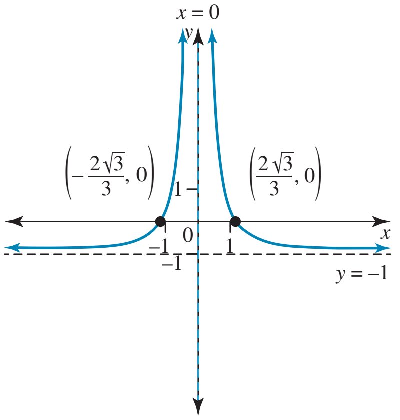
iii As $x = 0$ is an asymptote, there is no y-intercept.

x-intercepts: let $y = 0$.

$$\frac{4}{3x^2} - 1 = 0$$
$$\frac{4}{3x^2} = 1$$
$$4 = 3x^2$$
$$x^2 = \frac{4}{3}$$
$$x = \pm\frac{2}{\sqrt{3}}$$
$$x = \pm\frac{2\sqrt{3}}{3}$$

$\left(\dfrac{2\sqrt{3}}{3}, 0\right)$ and $\left(-\dfrac{2\sqrt{3}}{3}, 0\right)$

iv



b $y = \dfrac{-2}{(x-1)^2} + 4$

i The asymptotes are $x = 1$ and $y = 4$.

ii The domain is $R\backslash\{1\}$,

Since $a < 0$, the graph lies below its horizontal asymptote.

The range is $(-\infty, 4)$.

iii y-intercept: let $x = 0$.

$$y = \frac{-2}{(0-1)^2} + 4$$
$$= -2 + 4$$
$$= 2$$

$(0, 2)$

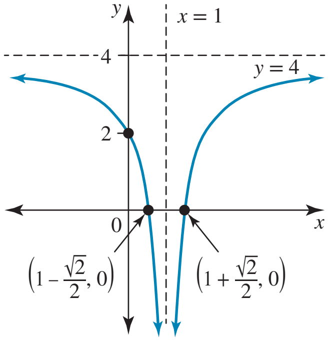
x-intercepts: let $y = 0$.

$$0 = \frac{-2}{(x-1)^2} + 4$$
$$-4 = \frac{-2}{(x-1)^2}$$
$$-4(x-1)^2 = -2$$
$$(x-1)^2 = \frac{-2}{-4}$$
$$(x-1)^2 = \frac{1}{2}$$
$$x - 1 = \pm\sqrt{\frac{1}{2}}$$
$$x = 1 \pm \frac{1}{\sqrt{2}}$$
$$x = 1 \pm \frac{\sqrt{2}}{2}$$

The x-intercepts are $\left(1 - \frac{\sqrt{2}}{2}, 0\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, 0\right)$.

The y-intercept is $(0, 2)$.

iv



18 **a** $y = \dfrac{12}{(x-2)^2} + 5$

Asymptotes: $x = 2, y = 5$

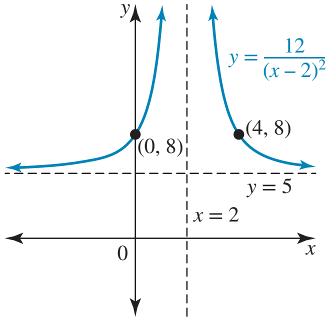
y-intercept: let $x = 0$.

$$\therefore y = \frac{12}{(-2)^2} + 5$$
$$\therefore y = 8$$

$(0, 8)$

There are no x-intercepts.

Point: By symmetry, the point $(4, 8)$ lies on the graph.



Domain $R \setminus \{2\}$, range $(5, \infty)$

b $y = \dfrac{-24}{(x+2)^2} + 6$

Asymptotes: $x = -2, y = 6$

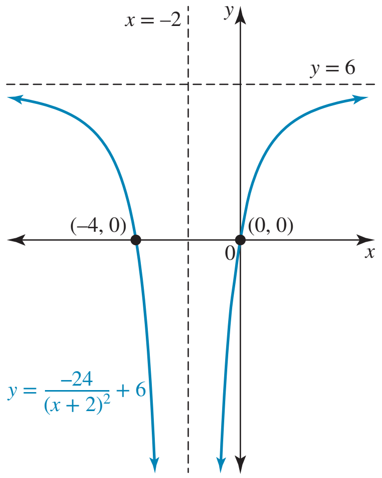
y-intercept: let $x = 0$.

$$\therefore y = \frac{-24}{(2)^2} + 6$$
$$\therefore y = 0$$

The graph passes through the origin.

The other x-intercept, by symmetry with the asymptote $x = -2$, must be $(-4, 0)$.

Domain $R \setminus \{-2\}$, range $(-\infty, 6)$



c $y = 7 - \dfrac{1}{7x^2}$

Asymptotes: $x = 0, y = 7$

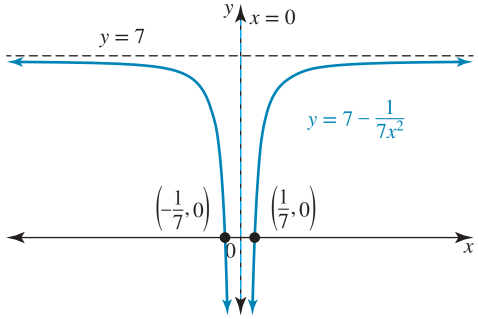
No y-intercept

x-intercepts: let $y = 0$.

$$\therefore 7 - \frac{1}{7x^2} = 0$$
$$\therefore \frac{1}{7x^2} = 7$$
$$\therefore 1 = 49x^2$$
$$\therefore x^2 = \frac{1}{49}$$
$$\therefore x = \pm\frac{1}{7}$$

$\left(-\frac{1}{7}, 0\right), \left(\frac{1}{7}, 0\right)$

Domain $R \setminus \{0\}$, range $(-\infty, 7)$



d $y = \dfrac{4}{(2x-1)^2}$

Asymptotes: $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ and the horizontal asymptote is $y = 0$.

No x-intercepts

y-intercept: let $x = 0$.

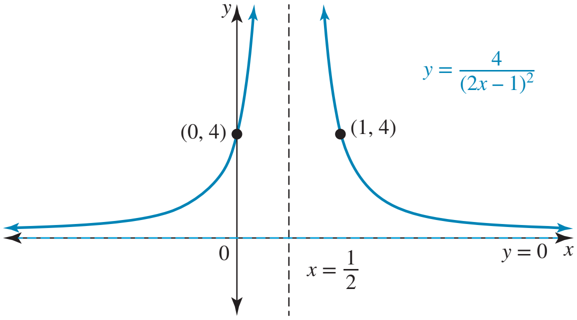
$\therefore y = \dfrac{4}{(-1)^2}$

$\therefore y = 4$

$(0, 4)$

By symmetry with the asymptote $x = \dfrac{1}{2}$, the point $(1, 4)$ is on the graph.

Domain $R\backslash\left\{\dfrac{1}{2}\right\}$, range R^+



e $y = -2 - \dfrac{1}{(2-x)^2}$

Asymptotes: $(2 - x)^2 = 0 \Rightarrow x = 2$, horizontal asymptote $y = -2$

y-intercept: let $x = 0$.

$\therefore y = -2 - \dfrac{1}{(2)^2}$

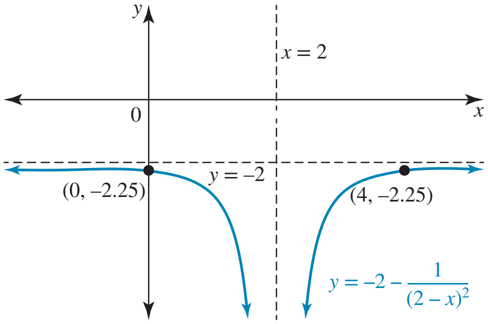
$\therefore y = -\dfrac{9}{4}$

$\left(0, -\dfrac{9}{4}\right)$

No x-intercepts

The point $\left(4, -\dfrac{9}{4}\right)$ is symmetric to the vertical asymptote with the y-intercept.

Domain $R\backslash\{2\}$, range $(-\infty, -2)$



f $y = \dfrac{x^2 + 2}{x^2}$

$\therefore y = \dfrac{x^2}{x^2} + \dfrac{2}{x^2}$

$\therefore y = 1 + \dfrac{2}{x^2}$

Asymptotes: $x = 0, y = 1$

No intercepts with the axes

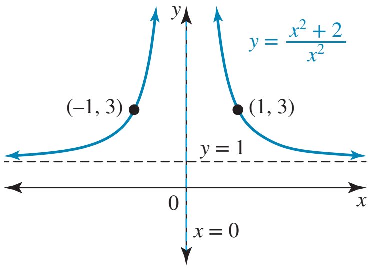
Domain $R\backslash\{0\}$, range $(1, \infty)$

Points: let $x = 1$.

$\therefore y = 3$

$(1, 3)$

By symmetry the point $(-1, 3)$ is also on the graph.



19 $y = \dfrac{3}{2(1 - 5x)^2}$

$\therefore y = \dfrac{\frac{3}{2}}{(1 - 5x)^2}$

The vertical asymptote occurs when $(1 - 5x)^2 = 0$.

$\therefore 1 - 5x = 0$

$\therefore x = \dfrac{1}{5}$

The horizontal asymptote is $y = 0$.

The domain is $R\backslash\left\{\dfrac{1}{5}\right\}$.

$a = \dfrac{3}{2} > 0$, so the range is $(0, \infty)$.

20 **a** Let the equation be $y = \dfrac{a}{x - h} + k$.

Vertical asymptote at $x = 3 \Rightarrow h = 3$

Horizontal asymptote at $y = 1 \Rightarrow k = 1$

$\therefore y = \dfrac{a}{x - 3} + 1$

Substitute the known point $(1, 0)$.

$\therefore 0 = \dfrac{a}{1 - 3} + 1$

$\therefore 0 = \dfrac{a}{-2} + 1$

$\therefore \dfrac{a}{2} = 1$

$\therefore a = 2$

The equation is $y = \dfrac{2}{x - 3} + 1$.

b Let the equation be $y = \dfrac{a}{x - h} + k$.

Vertical asymptote at $x = -3 \Rightarrow h = -3$

Horizontal asymptote at $y = 1 \Rightarrow k = 1$

$\therefore y = \dfrac{a}{x + 3} + 1$

Substitute the known point $(-5, 1.75)$.

$\therefore 1.75 = \dfrac{a}{-5 + 3} + 1$

$\therefore 0.75 = \dfrac{a}{-2}$

$\therefore a = -1.50$

The equation is $y = \dfrac{-1.5}{x + 3} + 1$ or $y = \dfrac{-3}{2(x + 3)} + 1$.

c Let the equation be $y = \dfrac{a}{(x - h)^2} + k$

Vertical asymptote at $x = 0 \Rightarrow h = 0$

Horizontal asymptote at $y = -2 \Rightarrow k = -2$

$\therefore y = \dfrac{a}{x^2} - 2$

Substitute the point $(1, 1)$.

$\therefore 1 = a - 2$

$\therefore a = 3$

The equation is $y = \dfrac{3}{x^2} - 2$.

d Let the equation be $y = \dfrac{a}{(x-h)^2} + k$.

Vertical asymptote at $x = -3 \Rightarrow h = -3$

Horizontal asymptote at $y = 2 \Rightarrow k = 2$

$\therefore y = \dfrac{a}{(x+3)^2} + 2$

Substitute the point $(0, 1)$.

$\therefore 1 = \dfrac{a}{9} + 2$

$\therefore \dfrac{a}{9} = -1$

$\therefore a = -9$

The equation is $y = \dfrac{-9}{(x+3)^2} + 2$.

e The graph with the same shape as $y = \dfrac{4}{x^2}$ and vertical asymptote $x = -2$ has the equation $y = \dfrac{4}{(x+2)^2}$. The truncus function is $f\colon R\backslash\{-2\} \to R, f(x) = \dfrac{4}{(x+2)^2}$.

f The hyperbola has a vertical asymptote $x = \dfrac{1}{4}$ and a horizontal asymptote $y = -\dfrac{1}{2}$. It passes through the point $(1, 0)$.

i The equation is of the form $y = \dfrac{a}{x - \frac{1}{4}} - \dfrac{1}{2}$.

Substitute the point $(1, 0)$.

$\therefore 0 = \dfrac{a}{1 - \frac{1}{4}} - \dfrac{1}{2}$

$\therefore \dfrac{1}{2} = \dfrac{a}{\frac{3}{4}}$

$\therefore a = \dfrac{1}{2} \times \dfrac{3}{4}$

$\therefore a = \dfrac{3}{8}$

The equation is $y = \dfrac{\frac{3}{8}}{x - \frac{1}{4}} - \dfrac{1}{2}$.

$$\begin{aligned}
\therefore y &= \frac{3}{8(x - \frac{1}{4})} - \frac{1}{2} \\
&= \frac{3}{8x - 2} - \frac{1}{2} \\
&= \frac{3}{2(4x-1)} - \frac{1}{2} \\
&= \frac{3 - (4x-1)}{2(4x-1)} \\
&= \frac{3 - 4x + 1}{2(4x-1)} \\
&= \frac{4 - 4x}{2(4x-1)} \\
&= \frac{2 - 2x}{4x - 1}
\end{aligned}$$

$\therefore y = \dfrac{-2x + 2}{4x - 1}$

The equation is in the form $y = \dfrac{ax + b}{cx + d}$ with $a = -2, b = 2, c = 4, d = -1$.

ii As the vertical asymptote is $x = \dfrac{1}{4}$, the domain is $R\backslash\left\{\dfrac{1}{4}\right\}$.

The hyperbola function can be expressed as the mapping $f\colon R\backslash\left\{\dfrac{1}{4}\right\} \to R, f(x) = \dfrac{-2x+2}{4x-1}$.

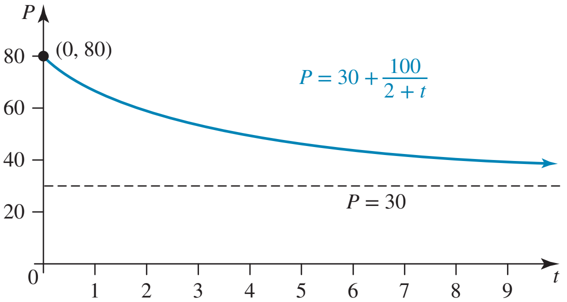
21 a $P = 30 + \dfrac{100}{2+t}$

$t = 0 \Rightarrow P = 80$

$t = 2 \Rightarrow P = 55$

Therefore, the herd has reduced by 25 cattle after the first 2 years.

b Asymptote $t = -2$ (not applicable), $P = 30$



Domain $\{t\colon t \geq 0\}$ Range $(30, 80]$

c The number of cattle will never go below 30.

22 For inverse proportion, xy is constant.

This is true for the second table **b** where $xy = 7.2$.

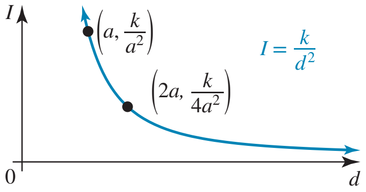
$\therefore y = \dfrac{7.2}{x}$ is the rule.

If $y = 6.4$, then $x = \dfrac{7.2}{6.4}$.

Therefore, $x = \dfrac{9}{8} = 1.125$.

If $x = 8, y = \dfrac{7.2}{8} = 0.9$.

23 a The inverse proportion relationship $I = \dfrac{k}{d^2}$ is the branch of a truncus for $d > 0$.



b To consider the effect on the intensity when the distance from the transmitter is doubled:

Let $d = a$, then $I = \dfrac{k}{a^2}$.

Let $d = 2a$, then $I = \dfrac{k}{(2a)^2} \Rightarrow I = \dfrac{k}{4a^2}$.

The intensity is one quarter of what it was before the distance was doubled. This means doubling the distance has reduced the intensity by 75%.

24 a Since time $= \dfrac{\text{distance}}{\text{speed}}$, $t = \dfrac{\text{distance}}{v}$, so the constant of proportionality is the distance travelled. Therefore, $k = 180$.

b The relationship is $t = \dfrac{180}{v}$.

This represents a hyperbola with independent variable v and dependent variable t.

Asymptotes: $v = 0, t = 0$

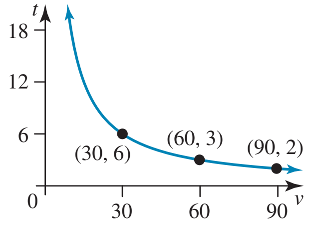
Points:

Let $v = 30, t = \dfrac{180}{30} = 6$. $(30, 6)$

Let $v = 60, t = \frac{180}{60} = 3$. $(60, 3)$

Let $v = 90, t = \frac{180}{90} = 2$. $(90, 2)$

Only the first quadrant branch is applicable, since neither time nor speed can be negative.



c Let $t = 2\frac{1}{4} = \frac{9}{4}$.

$$\therefore \frac{9}{4} = \frac{180}{v}$$

$$\therefore \frac{9}{4}v = 180$$

$$\therefore v = \cancel{180}^{20} \times \frac{4}{\cancel{9}_1}$$

$$\therefore v = 80$$

The speed should be 80 km/h.

25 $h = 25 - \frac{100}{(t+2)^2}, t \geq 0$

a Let $h = 12.5$.

$$\therefore 12.5 = 25 - \frac{100}{(t+2)^2}$$

$$\therefore \frac{100}{(t+2)^2} = 12.5$$

$$\therefore 100 = 12.5(t+2)^2$$

$$\therefore (t+2)^2 = \frac{100}{12.5}$$

$$\therefore (t+2)^2 = 8$$

$$\therefore t + 2 = \pm\sqrt{8}$$

$$\therefore t = -2 \pm \sqrt{8}$$

$t > 0$, so $t = -2 + \sqrt{8}$.

The time to reach the height is $0.8284 \times 60 \simeq 50$ seconds.

b As $t \to \infty$, the graph approaches its horizontal asymptote, $h = 25$. The limiting altitude is 25 metres above the ground.

26 **a** $xy = 2$ and $y = \frac{x^2}{4}$

At intersection, $x\left(\frac{x^2}{4}\right) = 2$.

$$\therefore \frac{x^3}{4} = 2$$

$$\therefore x^3 = 8$$

$$\therefore x = 2$$

Substitute $x = 2$ in $xy = 2$.

$$\therefore 2y = 2$$

$$\therefore y = 1$$

The point of intersection is $(2, 1)$.

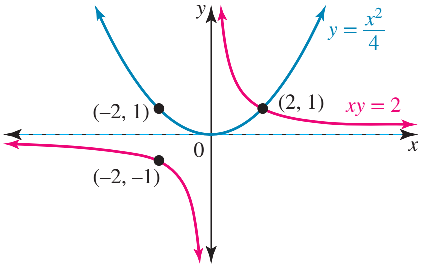
b $xy = 2$ is the hyperbola $y = \frac{2}{x}$.

Asymptotes: $x = 0, y = 0$

Points: $(2, 1)$ and when $x = -2, y = -1$, so a second point is $(-2, -1)$.

$y = \frac{x^2}{4}$ is a parabola with minimum turning point at $(0, 0)$.

Points: $(2, 1)$ and when $x = -2, y = \frac{(-2)^2}{4} = 1$, so another point is $(-2, 1)$.



c $y = \frac{4}{x^2}$ and $y = \frac{x^2}{4}$

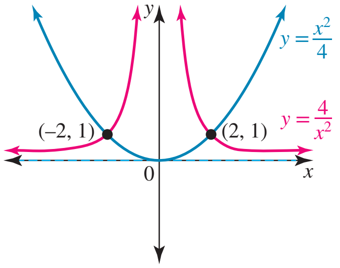
At intersection, $\frac{x^2}{4} = \frac{4}{x^2}$.

$$\therefore x^4 = 16$$

$$\therefore x = \pm 2$$

When $x = \pm 2, y = 1$, so the points of intersection are $(-2, 1)$ and $(2, 1)$.

$y = \frac{4}{x^2}$ is a truncus with asymptotes $x = 0, y = 0$.



d $y = \frac{x^2}{a}$ and $y = \frac{a}{x^2}$

At intersection,

$$\frac{x^2}{a} = \frac{a}{x^2}$$

$$\therefore x^4 = a^2$$

$$\therefore x = \pm\sqrt[4]{a^2}$$

$$\therefore x = \pm a^{\frac{1}{2}}$$

$$\therefore x = \pm\sqrt{a}$$

Substitute $x = \pm\sqrt{a}$ in $y = \frac{x^2}{a}$.

$$\therefore y = \frac{(\pm\sqrt{a})^2}{a}$$

$$\therefore y = \frac{a}{a}$$

$$\therefore y = 1$$

The points of intersection are $\left(-\sqrt{a}, 1\right)$ and $\left(\sqrt{a}, 1\right)$.

27 $N: R^+ \cup \{0\} \to R, N(t) = \frac{at+b}{t+2}$

a $N(t) = \frac{at+b}{t+2}$

$$N(0) = 20$$

$$\Rightarrow 20 = \frac{b}{2}$$

$$\therefore b = 40$$

$$N(2) = 240$$

$$\Rightarrow 240 = \frac{2a+b}{4}$$

Substitute $b = 40$.

$$\therefore 240 = \frac{2a + 40}{4}$$
$$\therefore 960 = 2a + 40$$
$$\therefore 2a = 920$$
$$\therefore a = 460$$

Answer: $a = 460, b = 40$

b The function rule is $N(t) = \dfrac{460t + 40}{t + 2}$.

When $N = 400$,

$$400 = \frac{460t + 40}{t + 2}$$
$$\therefore 400(t + 2) = 460t + 40$$
$$\therefore 400t + 800 = 460t + 40$$
$$\therefore 800 - 40 = 460t - 400t$$
$$\therefore 760 = 60t$$
$$\therefore t = \frac{760}{60}$$
$$\therefore t = \frac{38}{3}$$

The time taken is $12\frac{2}{3}$ years, which is 12 years and 8 months.

c
$$N(t) = \frac{460t + 40}{t + 2}$$
$$\therefore N(t + 1) = \frac{460(t + 1) + 40}{(t + 1) + 2}$$
$$= \frac{460t + 500}{t + 3}$$

$N(t + 1) - N(t)$

$$= \frac{460t + 500}{t + 3} - \frac{460t + 40}{t + 2}$$
$$= \frac{(460t + 500)(t + 2) - (460t + 40)(t + 3)}{(t + 3)(t + 2)}$$
$$= \frac{[460t(t + 2) - 460t(t + 3)] + [500(t + 2) - 40(t + 3)]}{(t + 2)(t + 3)}$$
$$= \frac{[-460t] + [460t + 1000 - 120]}{(t + 2)(t + 3)}$$
$$= \frac{880}{(t + 2)(t + 3)}$$

as required.

d The change in population during the 12th year is $N(13) - N(12)$.

From part **c**, $N(t + 1) - N(t) = \dfrac{880}{(t + 2)(t + 3)}$.

Let $t = 12$.

$$\therefore N(13) - N(12) = \frac{880}{(14)(15)}$$
$$= \frac{440}{7 \times 15}$$
$$= \frac{88}{7 \times 3}$$
$$= \frac{88}{21}$$

The population increased by $\frac{88}{21} \simeq 4$ insects during the 12th year.

The change in population during the 14th year is $N(15) - N(14)$.

Let $t = 14$.

$$\therefore N(15) - N(14) = \frac{880}{(16)(17)}$$
$$= \frac{110}{2 \times 17}$$
$$= \frac{55}{17} \simeq 3$$

During the 14th year the population changed by approximately 3 insects, so the growth in population is slowing.

e Let $N = 500$.

$$\therefore 500 = \frac{460t + 40}{t + 2}$$
$$\therefore 500(t + 2) = 460t + 40$$
$$\therefore 500t + 1000 = 460t + 40$$
$$\therefore 40t = 40 - 1000$$
$$\therefore 40t = -960$$
$$\therefore t = -24$$

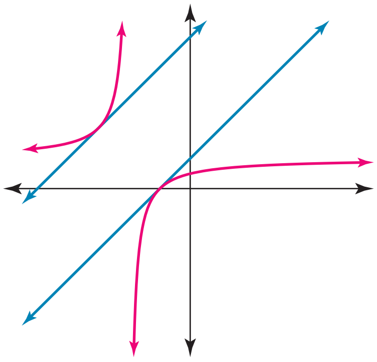
However, $t \in R^+ \cup \{0\}$, so there is no value of t for which $N = 500$. The population of insects will never reach 500.

f
$$N = \frac{460t + 40}{t + 2}$$
$$= \frac{460(t + 2) - 920 + 40}{t + 2}$$
$$= \frac{460(t + 2)}{t + 2} - \frac{880}{t + 2}$$
$$= 460 - \frac{880}{t + 2}$$

The function N is a hyperbola with horizontal asymptote $N = 460$. This means that as $t \to \infty, N \to 460$, so the population can never exceed 460 insects according to this model.

28 a Use CAS technology to obtain that there are 2 points of intersection.

b Using CAS technology:



Thus:

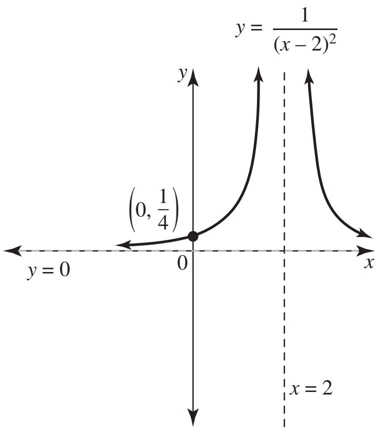
one intersection if $k = 1$ or $k = 5$;

two intersections if $k < 1$ or $k > 5$;

no intersection if $1 < k < 5$.

6.3 Exam questions

1 a



[1 mark]

$\frac{1}{(x-2)^2}$ is a truncus with asymptotes at $y = 0, x = 2$.

y-intercepts: when $x = 0$,

$y = \frac{1}{(0-2)^2} = \frac{1}{4}$. [1 mark]

b Domain $R\backslash(2)$, range $(0, \infty)$ [1 mark]

A horizontal line test would show many-to-one correspondence. [1 mark]

2 The graph is a basic hyperbola $\left(y = \frac{1}{x}\right)$ shape but reflected, hence negative, with asymptotes $x = -4, y = +1$.

2 is a dilation factor.

The correct answer is **D**.

3 $y = \frac{a}{(x-h)} + k$ is a rectangular hyperbola shape with asymptotes with equations $y = k, x = h$.

$y = \frac{1}{(x-0)} - 3$

Asymptotes occur at $x = 0, y = -3$.

The correct answer is **D**.

6.4 The square root function

6.4 Exercise

1 a $y = \sqrt{x-9}$

i The term under the square root cannot be negative.

$x - 9 \geq 0$

$x \geq 9$

The domain is $[9, \infty)$.

ii The end point is $(9, 0)$.

iii This is the positive square root, so the range is $[0, \infty)$.

b $y = \sqrt{4-x}$

i The term under the square root cannot be negative.

$4 - x \geq 0$

$-x \geq -4$

$x \leq 4$

The domain is $(-\infty, 4]$.

ii The end point is $(4, 0)$.

iii This is the positive square root, so the range is $[0, \infty)$.

c $y = -\sqrt{x+3}$

i The term under the square root cannot be negative.

$x + 3 \geq 0$

$x \geq -3$

The domain is $[-3, \infty)$.

ii The end point is $(-3, 0)$.

iii This is the negative square root, so the range is $(-\infty, 0]$.

d $y = 3 + \sqrt{-x}$

i The term under the square root cannot be negative.

$-x \geq 0$

$x \leq 0$

The domain is $(-\infty, 0]$.

ii The end point is $(0, 3)$.

iii This is the positive square root, so the range is $[3, \infty)$.

e $y = \sqrt{3x-6} - 7$

i The term under the square root cannot be negative.

$3x - 6 \geq 0$

$3x \geq 6$

$x \geq 2$

The domain is $[2, \infty)$.

ii $y = \sqrt{3(x-2)} - 7$

The end point is $(2, -7)$.

iii This is the positive square root, so the range is $[-7, \infty)$.

f $y = 4 - \sqrt{1-2x}$

i The term under the square root cannot be negative.

$1 - 2x \geq 0$

$-2x \geq -1$

$x \leq \frac{1}{2}$

The domain is $\left(-\infty, \frac{1}{2}\right]$.

ii $y = -\sqrt{1-2x} + 4$

The end point is $\left(\frac{1}{2}, 4\right)$.

iii This is the negative square root, so the range is $(-\infty, 4]$.

2 a $y = \sqrt{x-1} - 3$

End point $(1, -3)$; no y-intercept since the domain is $[1, \infty)$.

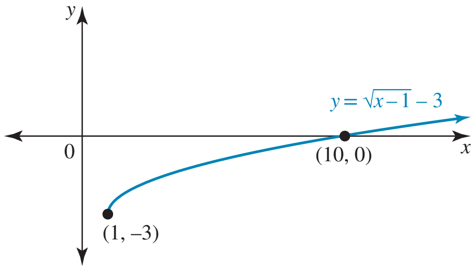
x-intercept when $\sqrt{x-1} - 3 = 0$

$\therefore \sqrt{x-1} = 3$

$\therefore x - 1 = 9$

$\therefore x = 10$

$(10, 0)$



Range $[-3, \infty)$

b i $f(x) = -\sqrt{2x+4}$

Maximal domain: $2x + 4 \geq 0$

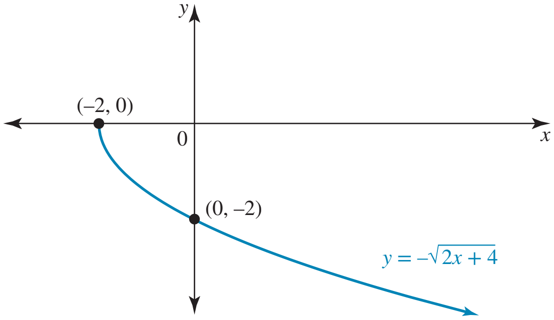
Therefore, the domain is $[-2, \infty)$.

$\therefore x \geq -2$

ii End point: when $2x + 4 = 0$

Therefore, the end point is $(-2, 0)$.

y-intercept $(0, -\sqrt{4}) = (0, -2)$



3 a $y = \sqrt{x+4} - 1$

End point: $(-4, -1)$

y-intercept: let $x = 0$.

$y = \sqrt{4} - 1$

$y = 1$

$(0, 1)$

x-intercept: let $y = 0$.

$\sqrt{x+4} - 1 = 0$

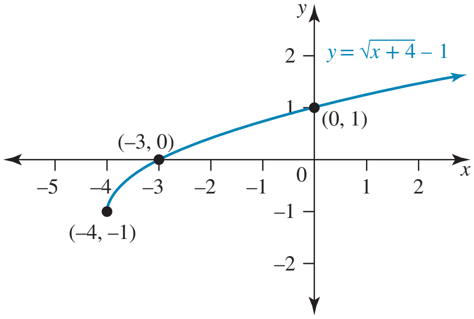
$\sqrt{x+4} = 1$

Square both sides.

$x + 4 = 1$

$x = -3$

$(-3, 0)$



b $y = 3 - \sqrt{3x}$

$y = -\sqrt{3x} + 3$

End point: $(0, 3)$, which is also the y-intercept.

x-intercept: let $y = 0$.

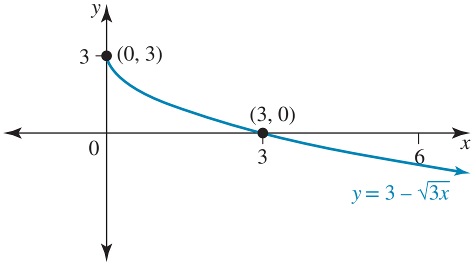
$3 - \sqrt{3x} = 0$

$3 = \sqrt{3x}$

$9 = 3x$

$x = 3$

$(3, 0)$



c $y = -\sqrt{6-x}$

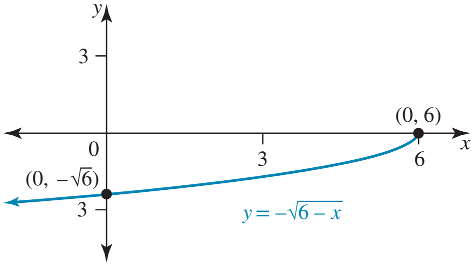
$y = -\sqrt{-(x-6)}$

End point: $(6, 0)$, which is also the x-intercept.

y-intercept: let $x = 0$.

$y = -\sqrt{6}$

$\left(0, -\sqrt{6}\right)$



d $y = \sqrt{9 - 2x} + 1$

End point:

$9 - 2x = 0$

$-2x = -9$

$x = \dfrac{9}{2}$

The end point is $\left(\dfrac{9}{2}, 1\right)$.

y-intercept: let $x = 0$.

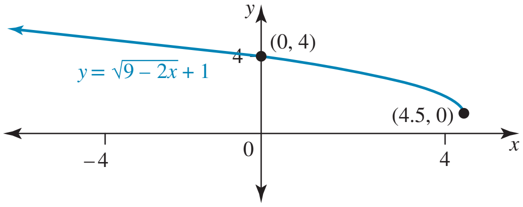
$y = \sqrt{9} + 1$

$y = 4$

$(0, 4)$

The square root is a positive one, so the range is $[1, \infty)$.

This means there cannot be an x-intercept.



4 a $y = \sqrt{x+3} - 2$

End point: $(-3, -2)$

Domain:

$x + 3 \geq 0$

$\therefore x \geq -3$

The domain is $[-3, \infty)$.

The rule contains a positive square root, so the graph rises from the end point, giving a range of $[-2, \infty)$.

y-intercept: let $x = 0$.

$\therefore y = \sqrt{3} - 2 \quad \left(0, \sqrt{3} - 2\right)$

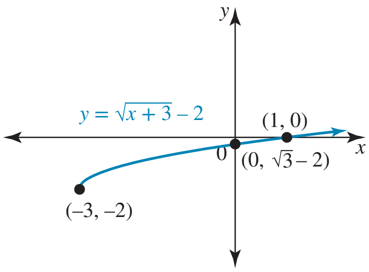
x-intercept: let $y = 0$.

$\therefore \sqrt{x+3} - 2 = 0$

$\therefore \sqrt{x+3} = 2$

$\therefore x + 3 = 4$

$\therefore x = 1 \quad (1, 0)$



b $y = 5 - \sqrt{5x}$

End point: $(0, 5)$

x-intercept: let $y = 0$.

$\therefore 0 = 5 - \sqrt{5x}$

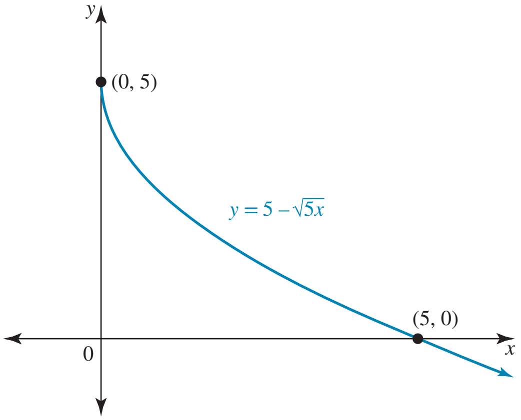
$\therefore \sqrt{5x} = 5$

$\therefore 5x = 25$

$\therefore x = 5 \qquad (5, 0)$

Domain: $5x \geq 0 \Rightarrow x \geq 0 \quad [0, \infty)$

Range: The rule contains a negative square root, so the graph falls from the end point, giving a range of $(-\infty, 5]$.



c $y = 2\sqrt{9 - x} + 4$

$\therefore y = 2\sqrt{-(x - 9)} + 4$

End point: $(9, 4)$

Domain: $9 - x \geq 0$

$\therefore 9 \geq x$

$\therefore x \leq 9 \qquad (-\infty, 9]$

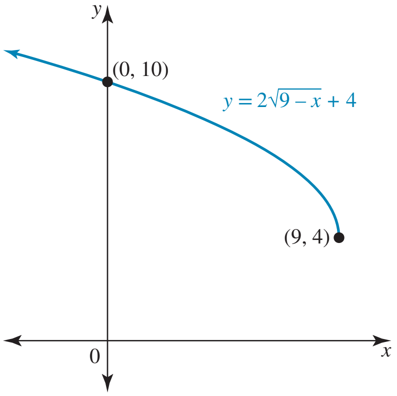
Range: The rule contains a positive square root, so the graph lies above its end point, giving a range of $[4, \infty)$.

y-intercept: let $x = 0$.

$\therefore y = 2\sqrt{9} + 4$

$\therefore y = 10 \qquad (0, 10)$

No x-intercept



d $y = \sqrt{49 - 7x}$

$\therefore y = \sqrt{-7(x - 7)}$

End point: $(7, 0)$

y-intercept: let $x = 0$.

$\therefore y = \sqrt{49}$

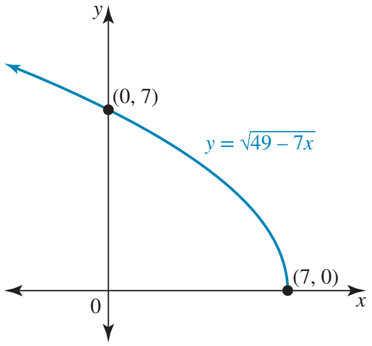
$\therefore y = 7 \qquad (0, 7)$

Domain: $49 - 7x \geq 0$

$\therefore 49 \geq 7x$

$\therefore x \leq 7 \qquad (-\infty, 7]$

Range: The rule contains a positive square root, so the graph lies above its end point, giving a range of $[0, \infty)$.



e $y = 2 + \sqrt{x + 4}$

End point on both: $(-4, 2)$

$x + 4 \geq 0$

$\therefore x \geq -4$

The domain is $[-4, \infty)$ and the range is $[2, \infty)$.

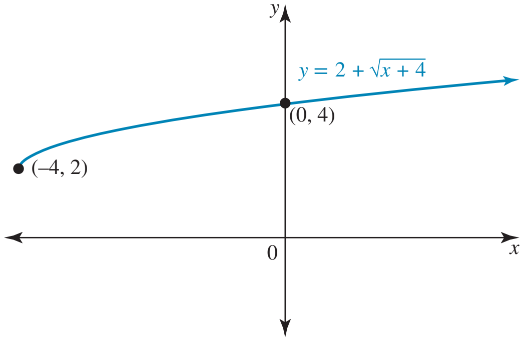
y-intercepts: let $x = 0$.

$\therefore y = 2 + \sqrt{4}$

$\therefore y = 2 + 2$

$\therefore y = 4$

$(0, 4)$



f $y + 1 + \sqrt{-2x + 3} = 0$

$$\therefore y = -\sqrt{-2\left(x - \frac{3}{2}\right)} - 1$$

End point: $\left(\frac{3}{2}, -1\right)$

y-intercept: let $x = 0$.

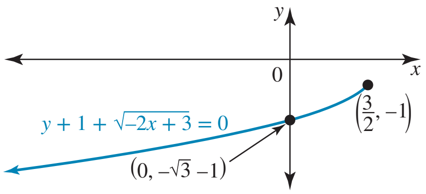
$\therefore y = -\sqrt{3} - 1 \quad \left(0, -\sqrt{3} - 1\right)$

Domain: $-2x + 3 \geq 0$

$\therefore 3 \geq 2x$

$\therefore x \leq \frac{3}{2} \quad \left(-\infty, \frac{3}{2}\right]$

Range: The rule contains a negative square root, so the graph lies below the end point, giving a range of $(-\infty, -1]$.



5 The end point is given as $(-2, 1)$ and the domain is $[-2, \infty)$, so let the equation be $y = a\sqrt{x-h} + k$.
Substitute the end point, so the equation becomes
$y = a\sqrt{x+2} + 1$
Substitute the point $(0, 3)$.

$$\therefore 3 = a\sqrt{2} + 1$$
$$\therefore \sqrt{2}a = 2$$
$$\therefore a = \frac{2}{\sqrt{2}}$$
$$\therefore a = \sqrt{2}$$

Therefore, the equation is
$y = \sqrt{2}\sqrt{x+2} + 1 \Rightarrow y = \sqrt{2(x+2)} + 1$.

6 As the graph opens to the right from its end point, let the equation be $y = a\sqrt{x-h} + k$.
The end point is $(-2, 2)$, so the equation becomes
$y = a\sqrt{x+2} + 2$.
Substitute the known point $(1, -1)$.

$$-1 = a\sqrt{1+2} + 2$$
$$-3 = a\sqrt{3}$$
$$\frac{-3}{\sqrt{3}} = a$$
$$a = -\sqrt{3}$$
$$\therefore y = -\sqrt{3}\sqrt{x+2} + 2$$
$$\Rightarrow y = -\sqrt{3\,(x+2)} + 2$$

7 Given that the end point is $(4, -1)$ and the point $(0, 9)$ lies on the function, the domain of the function must be $(-\infty, 4]$.
Let the equation be $y = a\sqrt{-(x-h)} + k$.
Substituting the end point, $y = a\sqrt{-(x-4)} - 1$.
Substitute the point $(0, 9)$.

$$\therefore 9 = a\sqrt{4} - 1$$
$$\therefore 10 = 2a$$
$$\therefore a = 5$$

The function has the equation
$y = 5\sqrt{-(x-4)} - 1 \Rightarrow y = 5\sqrt{4-x} - 1$.
x-intercept: let $y = 0$.

$$\therefore 0 = 5\sqrt{4-x} - 1$$
$$\therefore 1 = 5\sqrt{4-x}$$
$$\therefore \sqrt{4-x} = \frac{1}{5}$$
$$\therefore 4 - x = \frac{1}{25}$$
$$\therefore x = 4 - \frac{1}{25}$$
$$\therefore x = \frac{100-1}{25}$$
$$\therefore x = \frac{99}{25}$$

The graph of the function cuts the x-axis at the point $\left(\frac{99}{25}, 0\right)$.

8 The required function has the same shape as $y = \sqrt{-x}$ but with end point $(4, -4)$.
Its equation is

$$y = \sqrt{-(x-4)} - 4$$
$$\therefore y = \sqrt{4-x} - 4$$

9 $y = a\sqrt{x-h} + k$
Substitute the coordinates of the end point $(3, 2)$.
$y = a\sqrt{x-3} + 2$
Substitute the point $(4, 6)$ to calculate the value of a.

$$6 = a\sqrt{4-3} + 2$$
$$6 = a + 2$$
$$6 = 4$$

The equation is $y = 4\sqrt{x-3} + 2$.

10 $y = a\sqrt{x-h} + k$
As the graph shows the end point is $(-4, -1)$, the equation becomes
$y = a\sqrt{x+4} - 1$.
The graph passes through the origin,
Substitute $(0, 0)$ to calculate a.

$$0 = a\sqrt{0+4} - 1$$
$$0 = 2a - 1$$
$$a = \frac{1}{2}$$

The equation is $y = \frac{1}{2}\sqrt{x+4} - 1$.

11 $y = \sqrt{a(x-b)} + c$
As the graph shows the end point is $(2, 1)$, the equation becomes
$y = \sqrt{a(x-2)} + 1$.
The graph passes through the point $(0, 3)$.
Substitute $(0, 3)$ to calculate a.

$$3 = \sqrt{a(0-2)} + 1$$
$$3 = \sqrt{-2a} + 1$$
$$\sqrt{-2a} = 2$$

Square both sides (and note that a must be negative).

$$-2a = 4$$
$$a = -2$$

The equation is $y = \sqrt{-2(x-2)} + 1$, where $a = -2, b = 2, c = 1$.
The equation can also be expressed as $y = \sqrt{-2\,(x-2)} + 1$ or $y = \sqrt{4-2x} + 1$.

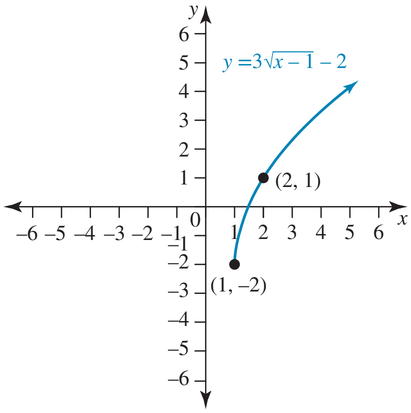
12 $S = \left\{(x, y) : y + 2 = 3\sqrt{x-1}\right\}$.

a $y + 2 = 3\sqrt{x-1}$
Express as a function with y as the subject.
$y \;=\; 3\sqrt{(x-1)} - 2$
End point: $(1, -2)$
For the x-intercept, let $y = 0$.

$$2 = 3\sqrt{x-1}$$
$$4 = 9\,(x-1)$$
$$\frac{4}{9} = x - 1$$
$$x = \frac{13}{9}$$

The x-intercept is $\left(\frac{13}{9}, 0\right)$.

b



c The related square root function would be the lower branch that completes a sideways parabola. It has the same end point but with equation
$y = -2 - 3\sqrt{x-1}$ or $y + 2 = -3\sqrt{x-1}$.

13 $t \propto \sqrt{h}$

$\therefore t = k\sqrt{h}$, where k is the constant of proportionality.

Given $h = 19.6, t = 2$, then $2 = k\sqrt{19.6}$.

$$\therefore k = \frac{2}{\sqrt{19.6}}$$

$$\therefore k = \frac{2}{2\sqrt{4.9}}$$

$$\therefore k = \frac{1}{\sqrt{4.9}}$$

Hence, the rule is $t = \frac{1}{\sqrt{4.9}}\sqrt{h}$ or $t = \sqrt{\frac{h}{4.9}}$.

14 $S = \{(x, y) : (y-2)^2 = 1 - x\}$

a $(y-2)^2 = 1 - x$

Take the square root of both sides.

$\therefore y - 2 = \pm\sqrt{1-x}$

$\therefore y = 2 \pm \sqrt{1-x}$

b The relation S is also described by $(y-2)^2 = -(x-1)$. Its vertex is $(1, 2)$ and as the coefficient of x is negative, the graph opens to the left with domain is $(-\infty, 1]$.

The two functions that together form the relation are the upper half and the lower half square root functions:

$f: (-\infty, 1] \to R, f(x) = 2 + \sqrt{1-x}$ and

$g: (-\infty, 1] \to R, g(x) = 2 - \sqrt{1-x}$.

c Each function has end point $(1, 2)$.

Graph of f:

y-intercept: let $x = 0$.

$f(0) = 2 + \sqrt{1}$

$= 3 \qquad (0, 3)$

Graph of g:

y-intercept: let $x = 0$.

$g(0) = 2 - \sqrt{1}$

$= 1 \qquad (0, 1)$

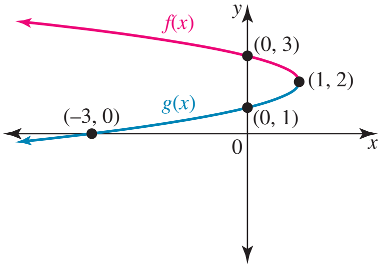
x-intercept: let $y = 0$.

$\therefore 0 = 2 - \sqrt{1-x}$

$\therefore \sqrt{1-x} = 2$

$\therefore 1 - x = 2^2$

$\therefore x = -3 \qquad (-3, 0)$



d The image of -8 under f is $f(-8)$.

$f(-8) = 2 + \sqrt{9}$

$= 5$

The image of -8 under g is $g(-8)$.

$g(-8) = 2 - \sqrt{9}$

$= -1$

15 $y = a + \sqrt{b(x-c)}$

$y = \sqrt{b(x-c)} + a$

The end point is $(2, 5)$, so $c = 2$ and $a = 5$.

$y = \sqrt{b(x-2)} + 5$

Substitute the point $(-10.5, 0)$.

$0 = \sqrt{b(-10.5-2)} + 5$

$5 = \sqrt{b(-12.5)}$

$25 = -12.5b$

$b = -2$

$\therefore a = 5, b = -2, c = 2$

Equation: $y = 5 - \sqrt{-2(x-2)}$ or $y = 5 - \sqrt{4-2x}$

Domain: $(-\infty, 2]$

Range: $(-\infty, 5]$

16 $f: [0, \infty) \to R, f(x) = \sqrt{mx} + n$

a From the given information,

$f(1) = 1 \Rightarrow 1 = \sqrt{m} + n \quad [1]$

$f(4) = 4 \Rightarrow 4 = \sqrt{4m} + n \quad [2]$

Subtract equation [1] from equation [2]:

$\therefore 3 = \sqrt{4m} - \sqrt{m}$

$\therefore 3 = 2\sqrt{m} - \sqrt{m}$

$\therefore \sqrt{m} = 3$

$\therefore m = 9$

Substitute $m = 9$ in equation [1]:

$\therefore 1 = \sqrt{9} + n$

$\therefore 3 + n = 1$

$\therefore n = -2$

Answer: $m = 9, n = -2$

b $f(x) = \sqrt{9x} - 2$

$\therefore f(x) = 3\sqrt{x} - 2$

End point: $(0, -2)$

x-intercept: let $y = 0$. $\left(\frac{4}{9}, 0\right)$

Let $f(x) = 0$.

$\therefore 0 = 3\sqrt{x} - 2$

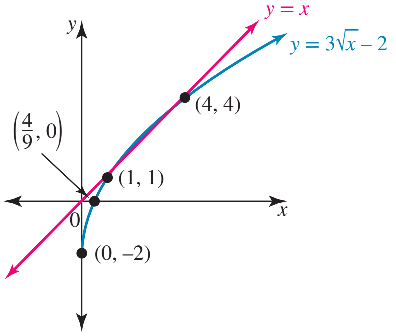
$\therefore 2 = 3\sqrt{x}$

$$\therefore \sqrt{x} = \frac{2}{3}$$

$$\therefore x = \frac{4}{9}$$

The line $y = x$ must pass through the origin and the points $(1, 1)$ and $(4, 4)$.

c



The square root function lies above the straight line for $1 < x < 4$.

$\therefore \{x: f(x) > x\} = \{x: 1 < x < 4\}$

6.4 Exam questions

1 $f(x) = a\sqrt{x-h} + k$, vertex $= (h, k)$

$\sqrt{}$ shape reflected in the y-axis; the end point is $(3, 1)$.

The correct answer is **B**.

2 The end point is $(-4, -3)$. [1 mark]

y-intercepts $(x = 0)$

$y = 3\sqrt{4} - 3$

$y = 3 \Rightarrow (0, 3)$ [1 mark]

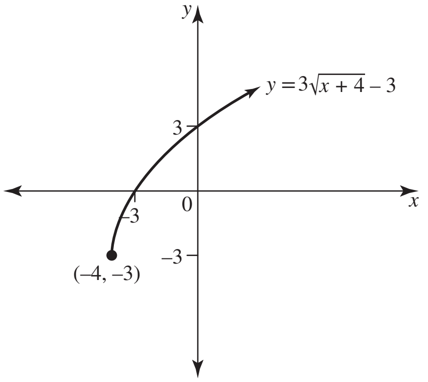
x-intercepts $(y = 0)$

$0 = 3\sqrt{x+4} - 3$

$3 = 3\sqrt{x+4}$

$1 = x + 4$

$x = -3 \Rightarrow (-3, 0)$ [1 mark]



[1 mark]

Domain $[-4, \infty)$, range $[-3, \infty)$ [1 mark]

3 $y - 2 = \sqrt{3-x}$

$y = \sqrt{-(x-3)} + 2$

$\therefore$ End point: $(3, 2)$.

The correct answer is **C**.

6.5 Other functions and relations

6.5 Exercise

1 a $y = \sqrt{3-2x}$

Require $3 - 2x \geq 0$.

$3 \geq 2x$

$\therefore x \leq \dfrac{3}{2}$

b $y = \dfrac{1}{1-2x}$

Require $1 - 2x \neq 0$.

$2x \neq 1$

$x \neq \dfrac{1}{2}$

$\therefore R \backslash \left\{\dfrac{1}{2}\right\}$

c $y = 2x^2 - 3x + 1$

No restrictions on parabolas

Therefore, the maximal domain is R.

2 a $y = \dfrac{1}{16-x^2}$

Require $16 - x^2 \neq 0 \Rightarrow x \neq \pm 4$.

Domain $R \backslash \{\pm 4\}$

b $y = \dfrac{2-x}{x^2+3}$

Require $x^2 + 3 \neq 0$, which holds for any real x.

Domain R

c $y = \sqrt{x^2 - 4}$

Require $x^2 - 4 \geq 0$.

$\therefore (x+2)(x-2) \geq 0$

$\therefore x \leq -2$ or $x \geq 2$

The domain is $(-\infty, -2] \cup [2, \infty)$ or $R \backslash (-2, 2)$.

d $y = \dfrac{1}{\sqrt{4-x}}$

Require $4 - x > 0$.

$\therefore -x > -4$

$\therefore x < 4$

Domain $(-\infty, 4)$

e $y = \sqrt{x} + \sqrt{2-x}$

Require $x \geq 0$ and $2 - x \geq 0$.

$\therefore x \geq 0$ and $x \leq 2$

$\therefore 0 \leq x \leq 2$

Domain $[0, 2]$

f $y = \dfrac{\sqrt{x^2+2}}{x^2+8}$

Require $x^2 + 2 \geq 0$ and $x^2 + 8 \neq 0$.

Both statements hold for $x \in R$, so the domain is R.

3 a $f(x) = \dfrac{1}{x^2+5x+4}$

The denominator will be zero when

$x^2 + 5x + 4 = 0$

$\therefore (x+4)(x+1) = 0$

$\therefore x = -4, x = -1$

These values must be excluded from the domain. Therefore, the maximal domain of the function f is $R \backslash \{-4, -1\}$.

b $g(x) = \sqrt{x+3}$

The domain of this square root function requires $x + 3 \geq 0$. Therefore, $x \geq -3$ and the maximal domain of the function g is $[-3, \infty)$.

c $h(x) = f(x) + g(x)$. The domain of h is the intersection of $R \backslash \{-4, -1\}$ and $[-3, \infty)$.

Therefore, the maximal domain is $[-3, \infty) \backslash \{-1\}$.

4 $S = \{(2, 9), (3, 10), (4, 11), (5, 12)\}$

a The domain is $\{2, 3, 4, 5\}$ and the range is $\{9, 10, 11, 12\}$.

b Interchange the coordinates to obtain the inverse.

The inverse is the set of points $\{(9, 2), (10, 3), (11, 4), (12, 5)\}$.

The domain is $\{9, 10, 11, 12\}$ and the range is $\{2, 3, 4, 5\}$.

c Each y-value is formed by adding 7 to its x-value.

The rule for the points in set S is $y = x + 7$.

d In the inverse set, each y-value is formed by subtracting 7 from its x-value.

The rule for the points in the inverse set is $y = x - 7$.

5 $f: (-\infty, 2] \to R, f(x) = 6 - 3x$

a The domain is $(-\infty, 2]$. The end point is $(2, 0)$ and as the line has a negative gradient for the given domain, the range is $[0, \infty)$.

b Interchanging, the domain of the inverse is $[0, \infty)$ and the range is $(-\infty, 2]$

c Function: $y = 6 - 3x$

Inverse: $x = 6 - 3y$

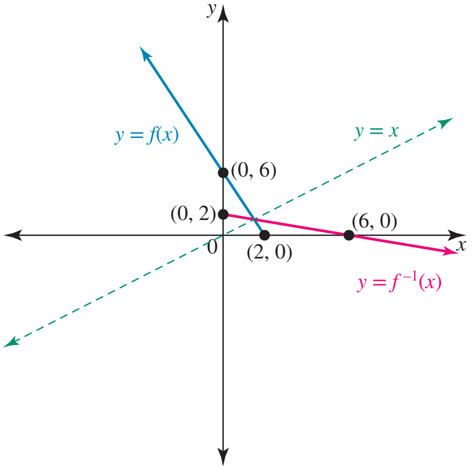
Making y the subject,

$3y = 6 - x$

$\therefore y = \dfrac{6 - x}{3}$

Hence, $f^{-1}: [0, \infty) \to R, f^{-1}(x) = \dfrac{6 - x}{3}$.

d f: points $(2, 0)$, $(0, 6)$ Inverse: points $(0, 2)$, $(6, 0)$



6 a $y = \dfrac{x + 3}{4}$

i The linear function has domain R and range R.

ii Function: $y = \dfrac{x + 3}{4}$

Inverse: interchange x- and y-coordinates.

$x = \dfrac{y + 3}{4}$

$4x = y + 3$

$y = 4x - 3$

The rule for the inverse is $y = 4x - 3$.

iii The inverse has domain R and range R.

b $y = 2 - \dfrac{5}{x}$

i This is a hyperbola with asymptotes at $x = 0$ and $y = 2$. The domain is $R\backslash\{0\}$ and the range is $R\backslash\{2\}$.

ii Function: $y = 2 - \dfrac{5}{x}$

Inverse: interchange x- and y-coordinates.

$x = 2 - \dfrac{5}{y}$

$x - 2 = -\dfrac{5}{y}$

$y(x - 2) = -5$

$y = -\dfrac{5}{x - 2}$

The rule for the inverse is $y = -\dfrac{5}{x - 2}$.

iii The inverse is a hyperbola with asymptotes at $x = 2$ and $y = 0$.

Its domain is $R\backslash\{2\}$ and its range is $R\backslash\{0\}$.

These could have also been obtained by interchanging the domain and range of the original hyperbola.

c $y = 10 - 2x, x \in [-1, 2]$

i End points: let $x = -1$.

$y = 10 - 2(-1)$

$y = 12$

$(-1, 12)$ is a closed end point of the line segment.

Let $x = 2$.

$y = 10 - 2(2)$

$y = 6$

$(2, 6)$ is a closed end point of the line segment.

The domain is $[-1, 2]$ and the range is $[6, 12]$.

ii Function: $y = 10 - 2x, x \in [-1, 2], y \in [6, 12]$

Inverse: interchange x- and y-coordinates.

$x = 10 - 2y, y \in [-1, 2], x \in [6, 12]$

$2y = 10 - x$

$y = 5 - \dfrac{x}{2}$

The inverse is $y = 5 - \dfrac{x}{2}, x \in [6, 12]$.

iii Domains and ranges are interchanged. The inverse has domain $[6, 12]$ and range $[-1, 2]$.

d $y = \sqrt{x - 2}$

i The term under the square root cannot be negative.

$x - 2 \geq 0$

$x \geq 2$

The domain is $[2, \infty)$.

The square root function has end point at $(2, 0)$ and a positive square root, so the range is $[0, \infty)$.

ii Function: $y = \sqrt{x - 2}$

Inverse: interchange x- and y-coordinates.

$x = \sqrt{y - 2}$

$x^2 = y - 2$

$y = x^2 + 2$

As the range of the square root function become the domain of the inverse, $x \in [0, \infty)$.

The inverse is $y = x^2 + 2, x \in [0, \infty)$ or $y = x^2 + 2, x \geq 0$.

iii The domain of the inverse is $[0, \infty)$ and the range is $[2, \infty)$.

7 $f: R \to R, f(x) = 3x - 2$

a Let $y = 3x - 2$.

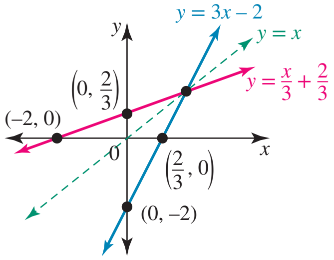
y-intercept: when $x = 0$, $y = -2$.

$(0, -2)$ is the y-intercept.

x-intercept: let $y = 0$.

$$3x - 2 = 0$$
$$3x = 2$$
$$x = \frac{2}{3}$$

$\left(\frac{2}{3}, 0\right)$ is the x-intercept.



b The inverse is the line through the points $\left(0, \frac{2}{3}\right)$ and $(-2, 0)$. It is shown on the same diagram as part **a** as the reflection of $y = 3x - 2$ in the line $y = x$.

c $y = 3x - 2$ and its inverse intersect on the line $y = x$.

At this intersection,

$$3x - 2 = x$$
$$2x = 2$$
$$x = 1$$

Since $y = x$, when $x = 1, y = 1$.

The point of intersection is $(1, 1)$.

d Function f: let $y = 3x - 2$.

Inverse: interchange x- and y-coordinates.

$$x = 3y - 2$$
$$3y = x + 2$$
$$y = \frac{x}{3} + \frac{2}{3}$$

The function f has domain R and range R.

Therefore, the inverse function has domain R and range R.

As a mapping, the inverse can be written as

$f^{-1}: R \rightarrow R, f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$.

8 **a** Given relation: $4x - 8y = 1$

Inverse: $4y - 8x = 1$

b Given relation: $y = -\frac{2}{3}x - 4$

Inverse: $x = -\frac{2}{3}y - 4$

$\therefore \frac{2}{3}y = -4 - x$

$\therefore 2y = -3x - 12$

$\therefore y = -\frac{3x}{2} - 6$

c Given relation: $y = 4x^2$

Inverse: $x = 4y^2$

$\therefore y^2 = \frac{x}{4}$ or $y = \pm\frac{\sqrt{x}}{2}$

d Given relation: $y = \sqrt{2x + 1}$

Inverse: $x = \sqrt{2y + 1}$, which requires that $x \geq 0$.

$\therefore x^2 = 2y + 1$

$\therefore 2y = x^2 - 1$

$\therefore y = \frac{x^2 - 1}{2}, \ x \geq 0$

9 $f(x) = \frac{1}{x - 2}$

a Domain: $x - 2 \neq 0 \Rightarrow x \neq 2$

The maximal domain of the function is $R \backslash \{2\}$. Its asymptotes have equations $x = 2$ and $y = 0$.

b Function: $y = \frac{1}{x - 2}$

Inverse: $x = \frac{1}{y - 2}$

$\therefore y - 2 = \frac{1}{x}$

$\therefore y = \frac{1}{x} + 2$

This is another hyperbola so it is a function.

The rule for the inverse is $f^{-1}(x) = \frac{1}{x} + 2$.

c The asymptotes of the inverse function have equations $x = 0$ and $y = 2$.

d $y = \frac{1}{x - a} + b$

Since this hyperbola has asymptotes with equations $x = a, y = b$, its inverse has asymptotes with equations $y = a, x = b$.

The equation of the hyperbola that is the inverse of $y = \frac{1}{x - a} + b$ is $y = \frac{1}{x - b} + a$.

10 f: $[-2, 4) \rightarrow R, f(x) = 4 - 2x$

a $d_f = [-2, 4) = r_{f^{-1}}$

End points for f: closed at $x = -2$, open at $x = 4$

$f(-2) = 4 + 4 \quad$ and $\quad f(4) = 4 - 8$

$\quad = 8 \qquad\qquad\qquad = -4$

$r_f = (-4, 8] = d_{f^{-1}}$

The domain of f^{-1} is $(-4, 8]$ and the range is $[-2, 4)$.

b Function f: $y = 4 - 2x$

Inverse function f^{-1}: $x = 4 - 2y$

$\therefore 2y = 4 - x$

$\therefore y = 2 - \frac{x}{2}$

$\therefore f^{-1}(x) = -\frac{x}{2} + 2, x \in (-4, 8]$

As a mapping, $f^{-1}: (-4, 8] \rightarrow R, f^{-1}(x) = -\frac{x}{2} + 2$.

c f: $y = 4 - 2x$

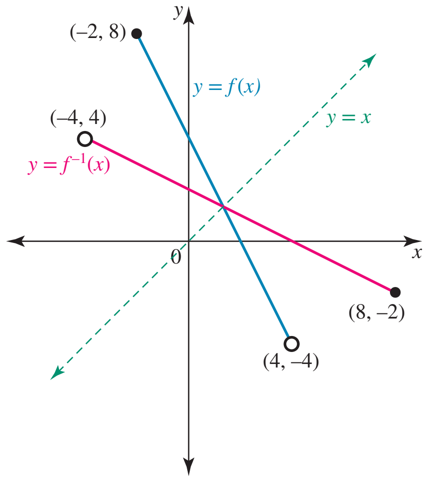
End points $(-2, 8)$ closed, $(4, -4)$ open

y-intercept $(0, 4)$, x-intercept $(2, 0)$

f^{-1}: $y = 2 - \frac{x}{2}$

End points: $(8, -2)$ closed, $(-4, 4)$ open

y-intercept $(0, 2)$, x-intercept $(4, 0)$



d $f(x) = f^{-1}(x)$ at their point of intersection, which is also their point of intersection with the line $y = x$.
Solving $4 - 2x = x$ gives $4 = 3x$.

$\therefore x = \frac{4}{3}$ when $f(x) = f^{-1}(x) = x$.

11 a $y = (x+4)^2 + 2$ is a quadratic polynomial function, so its domain is R.
Over this domain the quadratic function has a many-to-one correspondence, so its inverse will not be a function. Only one-to-one functions have inverses that are functions.

b $f: [-4, 0] \to R, f(x) = (x+4)^2 + 2$

$d_f = [-4, 0]$

$$\begin{aligned} f(-4) &= (0)^2 + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} f(0) &= (4)^2 + 2 \\ &= 18 \end{aligned}$$

$d_f = [-4, 0]$, so the inverse has range $[-4, 0]$.
$r_f = [2, 18]$, so the inverse has domain $[2, 18]$.
The turning point of the parabola occurs at $(-4, 2)$, so over the interval $[-4, 0]$ the function has one-to-one correspondence. Its inverse will be a function.

c Rule for the inverse:
Let $y = f(x)$.
Function: $y = (x+4)^2 + 2$
Inverse: interchange x- and y-coordinates.

$$\begin{aligned} x &= (y+4)^2 + 2 \\ x - 2 &= (y+4)^2 \\ y + 4 &= \pm\sqrt{x-2} \\ y &= -4 \pm \sqrt{x-2} \end{aligned}$$

The positive square root is required since the range of the inverse is $[-4, 0]$.

$$\begin{aligned} y &= -4 + \sqrt{x-2} \\ y &= \sqrt{x-2} - 4 \end{aligned}$$

The rule for the inverse function is $f^{-1}(x) = \sqrt{x-2} - 4$.

12 a $g: R \to R, g(x) = (x-1)^3 + 2$ is a cubic polynomial function with a stationary point of inflection at $(1, 2)$. It has a one-to-one correspondence. Therefore, its inverse will be a function.

b The domain of g is R and the range of g is R.
The inverse of g has domain R and range R.
Function : let $y = g(x)$.

$$y = (x-1)^3 + 2$$

Inverse: interchange x- and y-coordinates.

$$\begin{aligned} x &= (y-1)^3 + 2 \\ (y-1)^3 &= x - 2 \\ y - 1 &= \sqrt[3]{x-2} \\ y &= \sqrt[3]{x-2} + 1 \end{aligned}$$

$g^{-1}: R \to R, g^{-1}(x) = \sqrt[3]{x-2} + 1$, domain R and range R.

13 $y = 6x - x^2$

a This is a quadratic polynomial function, so the implied domain is R.
$y = -x^2 + 6x$
The axis of symmetry gives the x-coordinate of the turning point.

$$\begin{aligned} x &= -\frac{b}{2a}, a = -1, b = 6 \\ x &= -\frac{6}{-2} \\ x &= 3 \end{aligned}$$

When $x = 3$,

$$\begin{aligned} y &= -(3)^2 + 6(3) \\ &= 9 \end{aligned}$$

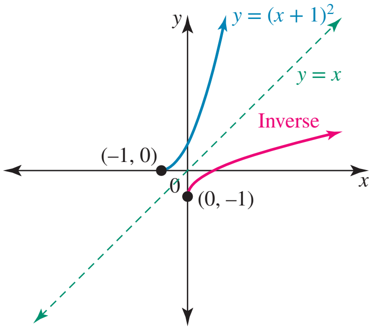
The turning point is $(3, 9)$.
The graph is concave down, so the range is $(-\infty, 9]$.

b Over the maximal domain the function is many-to-one. For its inverse to be a function, the parabola needs to be one-to-one.
As the turning point is at $(3, 9)$, the function will be one-to-one over the interval $x \in (-\infty, 3)$.
The greatest value of k is $k = 3$.

14 $y = (x+1)^2$

a Inverse: $x = (y+1)^2$
$\therefore (y+1)^2 = x$ or $y = \pm\sqrt{x} - 1$. This is two square root functions, so it fails the vertical line test (it has a one-to-many correspondence); therefore, it is not a function.

b The domain restriction $x \in [-1, \infty)$ makes the parabola $y = (x+1)^2$ one-to-one, so its inverse will be a function.
Turning point and end point $(-1, 0)$ (closed), y-intercept $(0, 1)$
For the inverse, end point $(0, -1)$ (closed), x-intercept $(1, 0)$

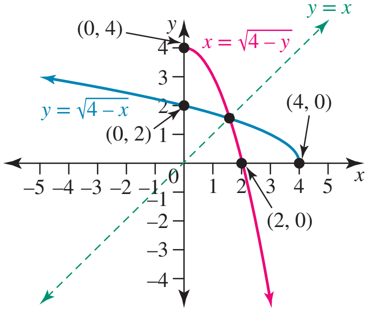


c Function: $y = (x+1)^2, x \in [-1, \infty), y \geq 0$
Inverse: $x = (y+1)^2, y \in [-1, \infty), x \geq 0$
$\therefore y + 1 = \pm\sqrt{x}$
Since the range requires $y \in [-1, \infty)$, take the positive square root (as the diagram shows).

$$\begin{aligned} \therefore y + 1 &= \sqrt{x} \\ \therefore y &= \sqrt{x} - 1 \end{aligned}$$

d It can be seen from the graph in part **b** that the functions do not intersect.

15 a For the square root graph $y = \sqrt{4-x}$,
$4 - x \geq 0$
$\therefore x \leq 0$
The end point of the graph is $(4, 0)$
y-intercept: when $x = 0$, $y = 2$.
For the inverse, interchange x- and y-coordinates:
$x = \sqrt{4-y}$ where $x \geq 0$
$x^2 = 4 - y$
$y = 4 - x^2, x \geq 0$



b The inverse function is a reflection over the line $y = x$, so the domain of the inverse function needs to be restricted to $x \geq 0$.

16 $f: [a, \infty) \to R, f(x) = x^2 - 4x + 9$

For f^{-1} to exist the parabola must be one-to-one.

$f(x) = x^2 - 4x + 9$

Completing the square to obtain its turning point,

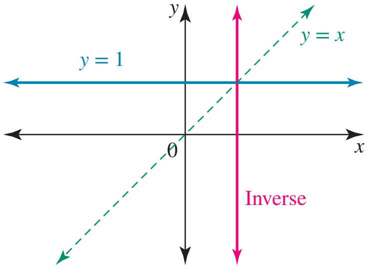
$f(x) = (x^2 - 4x + 4) - 4 + 9$

$\therefore f(x) = (x - 2)^2 + 5$

Turning point at $(2, 5)$, so on the domain $[2, \infty)$, the function is one-to-one.

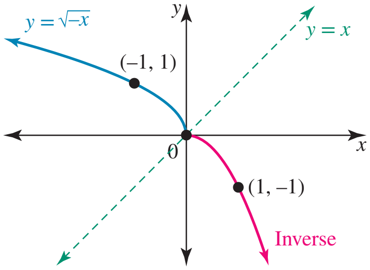
Therefore, the smallest value of a for f^{-1} to exist is $a = 2$.

17 **a** $y = 1$ represents the horizontal line through $(0, 1)$. Its inverse, $x = 1$ represents the vertical line through $(1, 0)$.



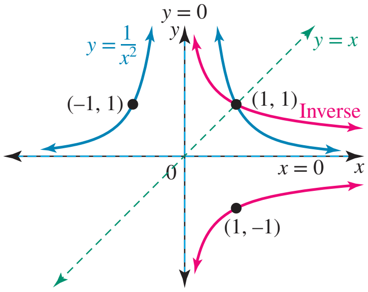
b $y = \sqrt{-x}$ is the square root function with end point $(0, 0)$ and domain for which $x \in (-\infty, 0]$. The point $(-1, 1)$ lies on the graph.

Its inverse has end point $(0, 0)$ and range for which $y \in (-\infty, 0]$. The point $(1, -1)$ lies on its graph.



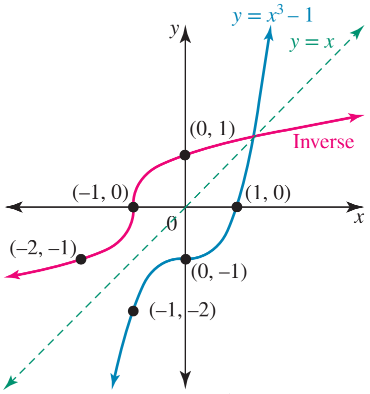
c $y = \dfrac{1}{x^2}$ is a truncus with asymptotes $x = 0, y = 0$. The points $(-1, 1)$ and $(1, 1)$ lie on its graph.

Its inverse will have asymptotes $y = 0, x = 0$ and contain the points $(1, -1)$ and $(1, 1)$.



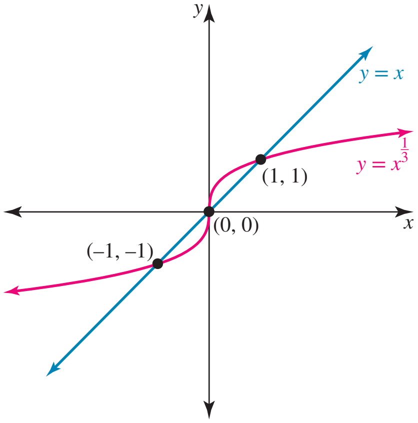
d $y = x^3 - 1$ has a stationary point of inflection at $(0, -1)$ and cuts the x-axis at $(1, 0)$.

Its inverse has a point of inflection at $(-1, 0)$ and cuts the y-axis at $(0, 1)$.



18 $y = x^n$ for $n = 1$ and $n = \dfrac{1}{3}$ are the functions $y = x$ and $y = x^{\frac{1}{3}}$.

Both graphs contain the points $(-1, -1)$, $(0, 0)$ and $(1, 1)$.



Since the cube root of numbers greater than 1 are smaller than the number, the cube root graph lies below the straight line when $x > 1$. The cube roots of numbers between 0 and 1 are larger than the number, so the cube root graph lies above the straight line between $x = 0$ and $x = 1$.

$\therefore \{x : x^{\frac{1}{3}} > x\} = \{x : 0 < x < 1\} \cup \{x : x < -1\}$

19 **a** $y = x^n$ for $n = -1$ and $n = \dfrac{1}{3}$ are the functions $y = x^{-1}$ and $y = x^{\frac{1}{3}}$.

The graph of $y = x^{-1}$ is the hyperbola $y = \dfrac{1}{x}$ and the graph of $y = x^{\frac{1}{3}}$ is the cube root function $y = \sqrt[3]{x}$

At the intersection of the two graphs, $\sqrt[3]{x} = \dfrac{1}{x}$.

Cube both sides of the equation.

$\therefore x = \left(\dfrac{1}{x}\right)^3$

$\therefore x = \dfrac{1}{x^3}$

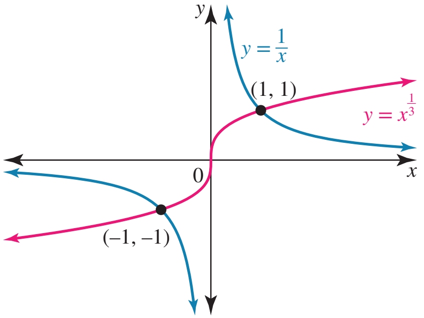
$\therefore x^4 = 1$

$\therefore x = \pm 1$

$\therefore y = \pm 1$

The points of intersection are $(-1, -1)$ and $(1, 1)$.

b Sketching the graphs:



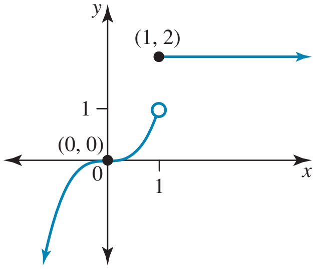
20 $f(x) = \begin{cases} x^3, & x < 1 \\ 2, & x \geq 1 \end{cases}$

a **i** $f(-2) = (-2)^3 \therefore f(-2) = -8$

ii $f(1) = 2$

iii $f(2) = 2$

b



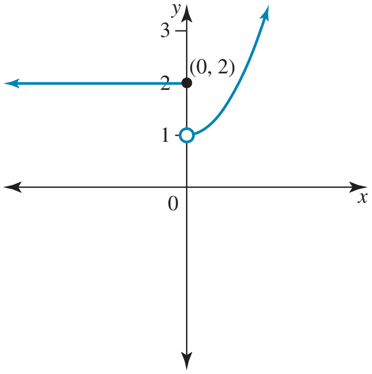
Domain R, range $(-\infty, 1) \cup \{2\}$

c Not continuous at $x = 1$

21 **a** $y = \begin{cases} 2, & x \leq 0 \\ 1 + x^2, & x > 0 \end{cases}$

If $x \leq 0$, $y = 2$. This is a horizontal line with end point $(0, 2)$, which is closed.

If $x > 0$, $y = 1 + x^2$. This is a concave up parabola with minimum turning point $(0, 1)$, which is open.



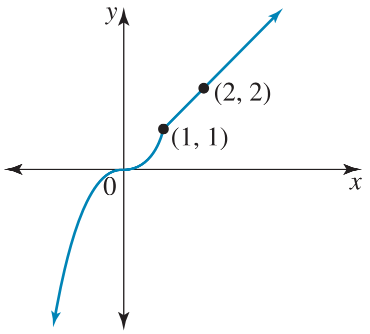
Domain R, range $(1, \infty)$

b $y = \begin{cases} x^3, & x < 1 \\ x, & x \geq 1 \end{cases}$

For $x < 1$, $y = x^3$ has a stationary point of inflection at $(0, 0)$ and an open end point at $(1, 1)$.

For $x \geq 1$, $y = x$ has a closed end point at $(1, 1)$ and also passes through the point $(2, 2)$.

As the end point of both branches has the same coordinates, the graph is continuous at $(1, 1)$, so no open or closed shading is given at this point.



Domain R, range R

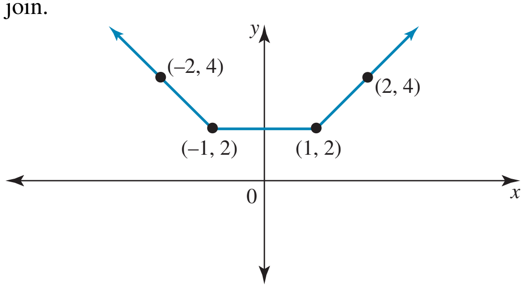
c $y = \begin{cases} -2x, & x < -1 \\ 2, & -1 \leq x \leq 1 \\ 2x, & x > 1 \end{cases}$

For $x < -1$, $y = -2x$. This line has end point $(-1, 2)$ and passes through $(-2, 4)$.

For $-1 \leq x \leq 1$, $y = 2$. This horizontal line has end points $(-1, 2)$ and $(1, 2)$.

For $x > 1$, $y = 2x$. This line has end point $(1, 2)$ and passes through $(2, 4)$.

The graph is continuous at the points where the branches join.

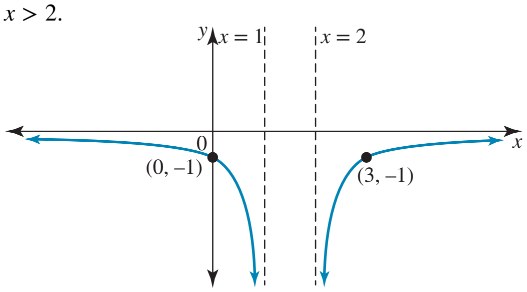


Domain R, range $[2, \infty)$

d $y = \begin{cases} \dfrac{1}{x-1}, & x < 1 \\ \dfrac{1}{2-x}, & x > 2 \end{cases}$

For $x < 1$, $y = \dfrac{1}{x-1}$. This hyperbola has asymptotes $x = 1$, $y = 0$ and contains the point $(0, -1)$. Only one branch of the hyperbola lies in the domain restriction that $x < 1$.

For $x > 2$, $y = \dfrac{1}{2-x}$. This hyperbola has asymptotes $x = 2$, $y = 0$ and contains the point $(3, -1)$. Only one branch of the hyperbola lies in the domain restriction that $x > 2$.

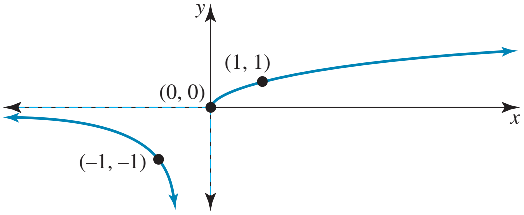


Domain $R \backslash [1, 2]$, range R^-

e $y = \begin{cases} x^{\frac{1}{3}} & x \geq 0 \\ -x^{-2} & x < 0 \end{cases}$

For $x \geq 0$, $y = x^{\frac{1}{3}}$. This cube root function contains the points $(0, 0)$ and $(1, 1)$.

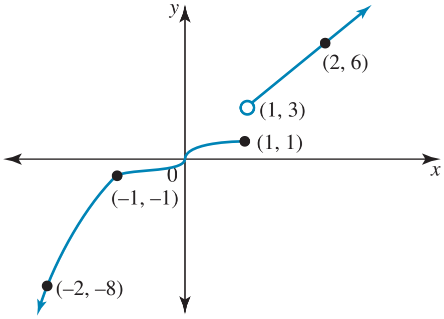
For $x < 0$, $y = -x^{-2} \Rightarrow y = \dfrac{-1}{x^2}$. This is a branch of the truncus that has asymptotes $x = 0$, $y = 0$ and contains the point $(-1, -1)$.



Domain R and range R

f $y = \begin{cases} x^3, & x < -1 \\ \sqrt[3]{x}, & -1 \leq x \leq 1 \\ 3x, & x > 1 \end{cases}$

For $x < -1$, $y = x^3$. This is the left-hand half of the cubic function. The graph approaches the point $(-1, -1)$.
For $-1 \leq x \leq 1$, $y = \sqrt[3]{x}$. This is the cube root function with a point of inflection at $(0, 0)$ and end points $(-1, -1)$ and $(1, 1)$.
For $x > 1$, $y = 3x$ is the straight line joining the points $(1, 3)$ and $(2, 6)$.
The point $(1, 3)$ is open, the point $(1, 1)$ is closed, and the cubic and cube root graphs join at $(-1, -1)$.



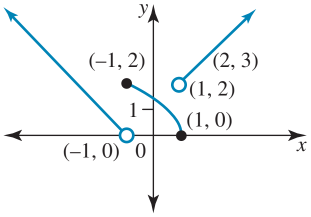
Domain R and range $R\backslash(1, 3]$

22 a $f(x) = \begin{cases} -x - 1, & x < -1 \\ \sqrt{2(1-x)}, & -1 \leq x \leq 1 \\ x + 1, & x > 1 \end{cases}$

i $f(0) = \sqrt{2(1-0)} = \sqrt{2}$
ii $f(3) = 3 + 1 = 4$
iii $f(-2) = -(-2) - 1 = 1$
iv $f(1) = \sqrt{2(1-1)} = 0$

b The two branches around $x = 1$ are $y = \sqrt{2\,(1-x)}, -1 \leq x \leq 1$ and $y = x + 1, x > 1$.
For $y = \sqrt{2(1-x)}$, there is a closed end point $(1, 0)$, but for $y = x + 1$ there is an open end point $(1, 2)$. The two branches do not join. Hence, the function is not continuous at $x = 1$ as there will be a break in its graph.

c For $x < -1$, $y = x - 1$.
This 1 line has an open end point at $(-1, 0)$ and passes through the point $(-4, 3)$.
For $-1 \leq x \leq 1$, $y = \sqrt{2(1-x)}$.
This is the square root function with the closed end point at $(1, 0)$.
The other closed end point is at $(-1, 2)$.
For $x > 1$, $y = x + 1$.
This line has an open end point at $(1, 2)$ and passes through the point $(2, 3)$.
The function has a many-to-one correspondence.



23 a The left branch is a line through $(-1, 0)$ and $(0, 1)$.
$m = \dfrac{1-0}{0+1} = 1$
$\therefore y = x + 1$
The right branch is a line through $(1, 0)$ and $(0, 1)$.
$m = -1$
$\therefore y = -x + 1$
The rule for the hybrid function is $y = \begin{cases} x + 1, & x \leq 0 \\ -x + 1, & x > 0 \end{cases}$.

b For $x < 2$, the left branch is the horizontal line $y = 3$.
For $x \geq 2$, the right branch is the line through $(2, 0)$ and $(4, 6)$.
$m = \dfrac{6-0}{4-2} = 3$
$\therefore y - 0 = 3(x - 2)$
$\therefore y = 3x - 6$
The rule for the hybrid function is $y = \begin{cases} 3, & x < 2 \\ 3x - 6, & x \geq 2 \end{cases}$.

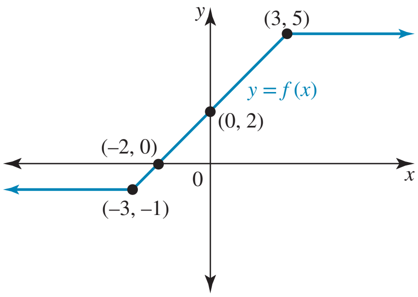
24 a i $f(x) = \begin{cases} 4x + a, & x < 1 \\ \dfrac{2}{x}, & 1 \leq x \leq 4 \end{cases}$

To be continuous the two branches must join at $x = 1$.
$\therefore 4(1) + a = \dfrac{2}{(1)}$
$\therefore 4 + a = 2$
$\therefore a = -2$

ii $x = 0$ lies in the domain for which the rule is the linear function $y = 4x + a$ so the graph will be continuous at this point.

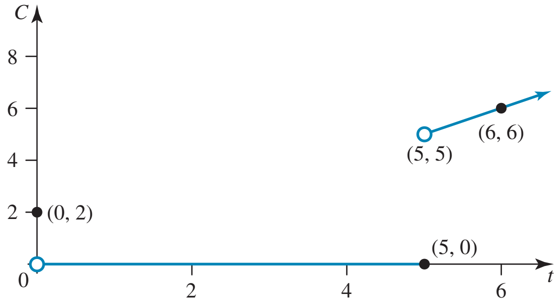
b $f(x) = \begin{cases} a, & x \in (-\infty, -3] \\ x + 2, & x \in (-3, 3) \\ b, & x \in [3, \infty) \end{cases}$

To be continuous the branches must join.
At the join where $x = -3$, $a = -3 + 2$.
$\therefore a = -1$
At the join where $x = 3$, $3 + 2 = b$.
$\therefore b = 5$
With $a = -1$ and $b = 5$, the rule for the function becomes
$f(x) = \begin{cases} -1, & x \in (-\infty, -3] \\ x + 2, & x \in (-3, 3) \\ 5, & x \in [3, \infty) \end{cases}$
For $x \in (-\infty, -3]$, $y = -1$, $y = -1$ is a horizontal line with end point $(-3, -1)$.
For $x \in (-3, 3)$, $y = x + 2$ is a line with y-intercept $(0, 2)$, x-intercept $(-2, 0)$ and end points $(-3, -1)$ and $(3, 5)$.
For $x \in [3, \infty)$, $y = 5$ is a horizontal line with end point $(3, 5)$.



25 Let the time in the shower be t minutes and the dollar amount of the fine be C.

The rule is $C = \begin{cases} 2, & t = 0 \\ 0, & 0 < t \leq 5 \\ t, & t > 5 \end{cases}$.



26 $g: D \to R, g(x) = x^2 + 8x - 9$

a g^{-1} exists for a domain where g has a one-to-one correspondence.

$$\begin{aligned} g(x) &= x^2 + 8x - 9 \\ &= (x^2 + 8x + 16) - 16 - 9 \\ &= (x + 4)^2 - 25 \end{aligned}$$

The turning point of the parabola g is the point $(-4, -25)$.

A possible domain over which g is one-to-one is $D = [-4, \infty)$. Another possibility is $D = (-\infty, -4]$.

b For the domain $D = R^+$, the x-value of the turning point does not lie in the domain.

$g(0) = -9$, so the point $(0, -9)$ is an open end point of the graph of g.

The range of g is $(-9, \infty)$.

Function g: $y = (x + 4)^2 - 25$, $d_g = R^+, r_g = (-9, \infty)$

Inverse function g^{-1}: $x = (y + 4)^2 - 25$,

$d_{g^{-1}} = (-9, \infty), r_{g^{-1}} = R^+$

$\therefore (y + 4)^2 = x + 25$

$\therefore y + 4 = \pm\sqrt{x + 25}$

Due to the range, the positive square root is required.

$\therefore y + 4 = \sqrt{x + 25}$

$\therefore y = \sqrt{x + 25} - 4$

$\therefore g^{-1}(x) = \sqrt{x + 25} - 4$

Hence, $g^{-1}: (-9, \infty) \to R, g^{-1}(x) = \sqrt{x + 25} - 4$.

The range of g^{-1} is R^+.

c **i** Graph of g: $y = x^2 + 8x - 9 \Rightarrow y = (x + 4)^2 - 25$

Domain R^+, open end point $(0, -9)$

x-intercept: let $y = 0$.

$\therefore 0 = x^2 + 8x - 9$

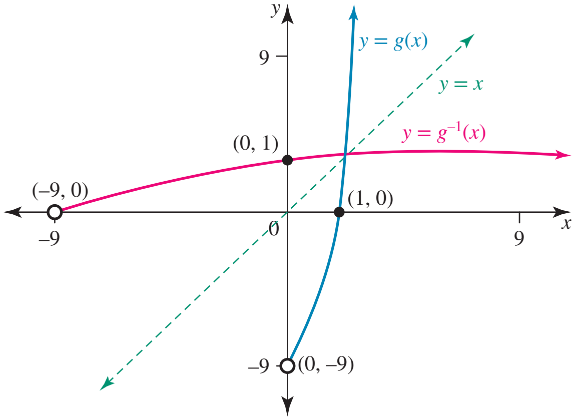
$\therefore (x + 9)(x - 1) = 0$

$\therefore x = -9, x = 1$

$x = -9$ does not lie in the domain so the x-intercept is $(1, 0)$.

Graph of g^{-1}: $y = \sqrt{x + 25} - 4$

Domain $(-9, \infty)$, open end point $(-9, 0)$, y-intercept $(0, 1)$



ii At the point of intersection, $g(x) = g^{-1}(x) = x$.

Solving $g(x) = x, x > 0$:

$$x^2 + 8x - 9 = x$$

$$\therefore x^2 + 7x - 9 = 0$$

$$\therefore x = \frac{-7 \pm \sqrt{49 - 4 \times 1 \times -9}}{2}$$

$$\therefore x = \frac{-7 \pm \sqrt{85}}{2}$$

Since $x > 0$, reject the negative square root.

$$x = \frac{-7 + \sqrt{85}}{2}$$

The point of intersection is $\left(\frac{-7 + \sqrt{85}}{2}, \frac{-7 + \sqrt{85}}{2}\right)$.

d Require the value of x so that $g^{-1}(x) = x$. This is the value from part **c**; $\frac{-7 + \sqrt{85}}{2}$ is its own image under g^{-1}.

27 $f(x) = \begin{cases} \sqrt{x - 4} + 2, & x > 4 \\ 2, & 1 \leq x \leq 4 \\ \sqrt{1 - x} + 2, & x < 1 \end{cases}$

a $f(0) = \sqrt{1 - 0} + 2 = 3$

$f(5) = \sqrt{5 - 4} + 2 = 3$

b For $x < 1, y = \sqrt{1 - x} + 2 \Rightarrow y = \sqrt{-(x - 1)} + 2$

Square root function with end point: $(1, 2)$.

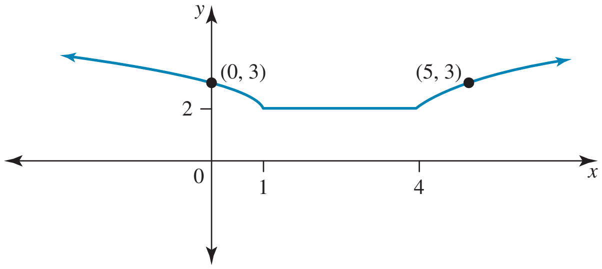
y-intercept: $(0, 3)$, no x-intercept

For $1 \leq x \leq 4$, $y = 2$ is a horizontal line with end points $(1, 2), (4, 2)$,

For $x > 4$, $y = \sqrt{x - 4} + 2$.

Square root function with end point $(4, 2)$ and passing through $(5, 3)$

As the branches join, there are no points of discontinuity.



Domain R, range $[2, \infty)$

c The horizontal line $y = 8$ would intersect the graph at a point on each of its left and right branches, but it would not intersect with the middle branch. Therefore, $f(x) = 8$ when either $\sqrt{1-x}+2 = 8,\ x < 1$ or $\sqrt{x-4}+2 = 8,\ x > 4$.

Solving $\sqrt{1-x}+2 = 8,\ x < 1$:

$\sqrt{1-x} = 6$

$\therefore 1-x = 36$

$\therefore x = -35$

Solving $\sqrt{x-4}+2 = 8,\ x > 4$:

$\sqrt{x-4} = 6$

$\therefore x-4 = 36$

$\therefore x = 40$

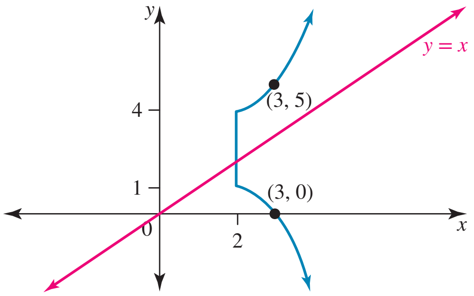
Therefore, $f(x) = 8$ when $x = -35$ or $x = 40$.

d The function has a many-to-one correspondence so its inverse is not a function since its correspondence is one-to-many.

The function contains the points $(0,3), (1,2), (4,2), (5,3)$, so its inverse contains the points $(3,0), (2,1), (2,4), (3,5)$. The graph of the inverse is the reflection of the function in the line $y = x$, so the middle branch is $x = 2, 1 \le y \le 4$.

The function has the rule $y = \begin{cases} \sqrt{x-4}+2, & x > 4, y > 2 \\ 2, & 1 \le x \le 4 \\ \sqrt{1-x}+2, & x < 1, y > 2 \end{cases}$

The inverse has the rule $x = \begin{cases} \sqrt{y-4}+2, & y > 4, x > 2 \\ 2, & 1 \le y \le 4 \\ \sqrt{1-y}+2, & y < 1, x > 2 \end{cases}$



28 **a** Using CAS technology,
$\{y = 0.5\left((4x+33)^{0.5}-5\right), y = -0.5\left((4x+33)^{0.5}+5\right)\}$.
The rule for the inverse is $y = \pm 0.5\sqrt{4x+33} - 2.5$.

b Using CAS technology, the points of intersection correct to 2 decimal places are $(-5.24, -0.76)$, $(0.45, 0.45)$, $(-4.45, -4.45)$ and $(-0.76, -5.24)$.

6.5 Exam questions

1 For the function $f(x) = 3 + \dfrac{1}{2x-1}$ to exist

$2x - 1 \ne 0$

$2x \ne 1$

$x \ne \dfrac{1}{2}$

$\therefore x$ can be any value except $\dfrac{1}{2}$.

$\therefore R \backslash \left\{\dfrac{1}{2}\right\}$

The correct answer is **B**.

2 Original function : $y = x - 2$

Interchanging x and $y \Rightarrow x = y - 2$

$\therefore y = x + 2$

$\therefore f^{-1}(x) = x + 2$

Domain of $f(x)$ $= (2, \infty)$, range of $f(x) = R^+$

Domain of $f^{-1}(x)$ $= R^+$, range $= (2, \infty)$

$\therefore f^{-1}: R^+ \to R$, where $f^{-1}(x) = x + 2$

The correct answer is **C**.

3 To ensure that the inverse exists as a function, the domain of the original function is restricted, so that $f(x)$ is a one-to-one function.

For $x^2 + 4x - 5$, a turning point exists at $x = -\dfrac{4}{2} = -2$.

To ensure one-to-one correspondence, $a \le -2$.

$\therefore$ if $a \le -2$ the inverse function exists.

The correct answer is **D**.

6.6 Transformations of functions

6.6 Exercise

1 **a** $y = x^2 \to y = 3x^2$ under a dilation of factor 3 from the x-axis.

b $y = x^2 \to y = -x^2$ under a reflection in the x-axis.

c $y = x^2 \to y = x^2 + 5$ under a vertical translation of 5 units upwards.

d $y = x^2 \to y = (x+5)^2$ under a horizontal translation of 5 units to the left.

2 **a** $y = x^2$

Translation of 3 units upwards

$y = x^2 \to y = x^2 + 3$

and then translation 2 units to the left

$y = x^2 + 3 \to y = (x+2)^2 + 3$

Under the translations, $y = x^2 \to y = (x+2)^2 + 3$.

b $y = \dfrac{1}{x}$

dilation factor 2 parallel to the y-axis

$y = \dfrac{1}{x} \to y = \dfrac{2}{x}$

and then a horizontal translation to the right of 5 units

$y = \dfrac{2}{x} \to y = \dfrac{2}{x-5}$

Under the transformations, $y = \dfrac{1}{x} \to y = \dfrac{2}{x-5}$.

c $y = \sqrt{x}$

reflected in the x-axis

$y = \sqrt{x} \to y = -\sqrt{x}$

and then dilated by a factor $\dfrac{1}{2}$ from the x-axis (parallel to the y-axis)

$y = -\sqrt{x} \to y = -\dfrac{1}{2}\sqrt{x}$

Under the transformations, $y = \sqrt{x} \to y = -\dfrac{1}{2}\sqrt{x}$.

d $y = x^4$

moved downwards 4 units

$y = x^4 \to y = x^4 - 4$

and then reflected in the x-axis

$y = x^4 - 4 \to y = -\left(x^4 - 4\right)$

Under the transformations, $y = x^4 - 4 \to y = -x^4 + 4$ or $y = 4 - x^4$.

3 **a** $y = \sqrt{x}$ to $y = \sqrt{x+4} - 5$

Horizontal translation of 4 units to the left and a vertical translation of 5 units downwards are the required transformations.

b $y = \dfrac{1}{x}$ into $y = \dfrac{3}{x-2} + 4$

Dilation of factor 3 from the x-axis (parallel to y-axis) followed by a horizontal translation of 2 units to the right and a vertical translation of 4 units upwards

c $y = x^3$ into $y = 2 - (x-3)^3$.

$y = -(x-3)^3 + 2$

Reflection in the x-axis followed by a horizontal translation of 3 units to the right and a vertical translation of 2 units upwards

d $y = \dfrac{1}{x^2}$ into $y = \dfrac{-3}{2(x+1)^2} + 4$.

$$y = \frac{-\frac{3}{2}}{(x+1)^2} + 4$$

There are four transformations required:

reflection in the x-axis and dilation by factor $\dfrac{3}{2}$ from the x-axis (parallel to the y-axis), followed by translations of 1 unit horizontally to the left and 4 units vertically upwards.

e $y = f(x)$ into $y = f(x+1) + 2$.

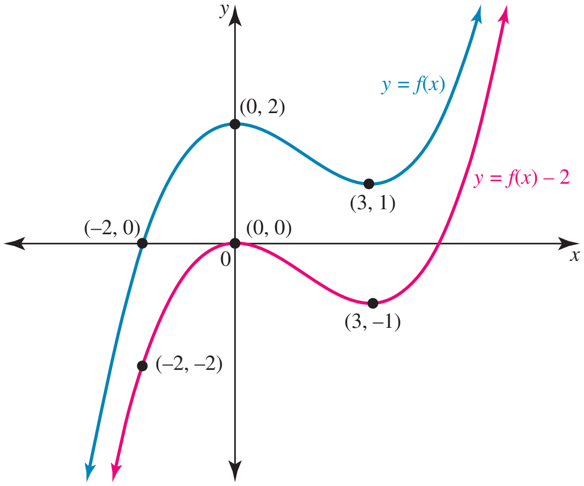
The function has undergone a horizontal translation of 1 unit to the left and a vertical translation of 2 units upwards.

f $y = f(x)$ into $y = -4f(x)$.

Reflection in the x-axis and dilation by factor 4 from the x-axis (parallel to y-axis)

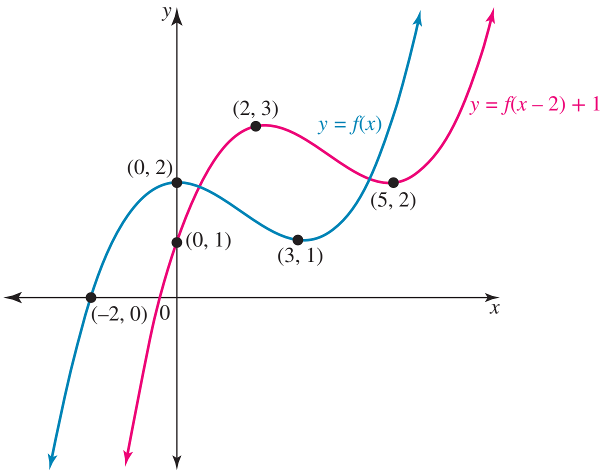
4 **a** $y = f(x) - 2$ is a vertical translation down of 2 units of the graph of $y = f(x)$.

$(-2, 0)$	→	$(-2, -2)$
$(0, 2)$	→	$(0, 0)$
$(3, 1)$	→	$(3, -1)$



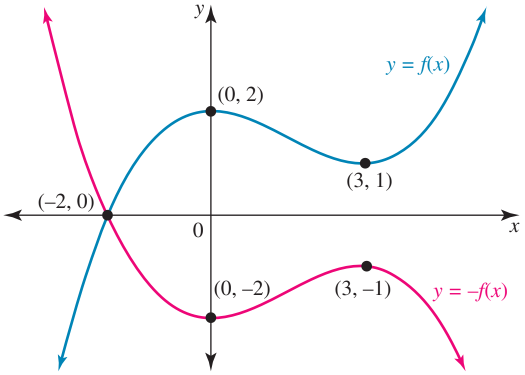
b $y = f(x-2) + 1$ is a horizontal translation 2 units to right, vertical translation 1 unit up of $y = f(x)$,

$(-2, 0)$	→	$(0, 1)$
$(0, 2)$	→	$(2, 3)$
$(3, 1)$	→	$(5, 2)$



c $y = -f(x)$ reflection in the x-axis of $y = f(x)$.

$(-2, 0)$	→	$(-2, 0)$
$(0, 2)$	→	$(0, -2)$
$(3, 1)$	→	$(3, -1)$



5 **a** $y = (x-1)^2$: reflect in the x-axis then vertically translate up 3 units.

$y = (x-1)^2 \rightarrow y = -(x-1)^2 \rightarrow y = -(x-1)^2 + 3$

Therefore, the image has equation $y = -(x-1)^2 + 3$.

b Vertically translate upwards by 3 units then reflect in the x-axis

$y = (x-1)^2 \rightarrow y = (x-1)^2 + 3 \rightarrow y = -\left((x-1)^2 + 3\right)$

Therefore, the image has equation $y = -(x-1)^2 - 3$, which is not the same as in part **a**.

6 **a** $y = x^3 \rightarrow y = \left(\dfrac{x}{3}\right)^3$ under a dilation of factor 3 from the y-axis.

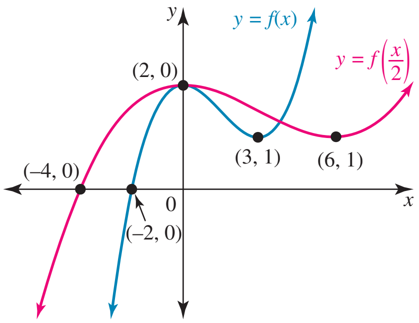
b $y = x^3 \rightarrow y = (8x)^3 + 1$ under a dilation of factor $\dfrac{1}{2}$ from the y-axis followed by a vertical translation of 1 unit upwards.

c $y = x^3 \rightarrow y = (x-4)^3 - 4$ under a horizontal translation of 4 units to the right and a vertical translation of 4 units downwards.

d $y = x^3 \rightarrow y = -2(x+1)^3$ under a reflection in the x-axis, a dilation of factor 2 from the x-axis and a horizontal translation of 1 unit to the left.

7 **a** $y = f\left(\dfrac{x}{2}\right)$: dilation of $y = f(x)$ by factor 2 from the y-axis

$(-2, 0)$	→	$(-4, 0)$
$(0, 2)$	→	$(0, 2)$
$(3, 1)$	→	$(6, 1)$

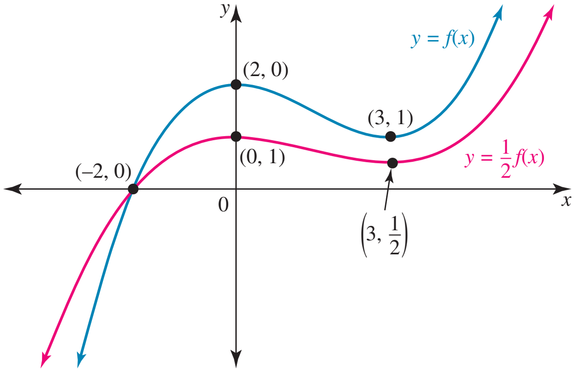


b $y = \frac{1}{2}f(x)$: dilation of $y = f(x)$ by factor $\frac{1}{2}$ from x-axis

$(-2, 0) \quad \rightarrow \quad (-2, 0)$

$(0, 2) \quad \rightarrow \quad (0, 1)$

$(3, 1) \quad \rightarrow \quad \left(3, \frac{1}{2}\right)$



8 a i Under a dilation of factor 2 from the x-axis, $y = \sqrt{x} \rightarrow y = 2\sqrt{x}$.

ii Under a dilation of factor 2 from the y-axis,

$y = \sqrt{x} \rightarrow y = \sqrt{\frac{x}{2}}$.

iii Under a reflection in the x-axis, $y = \sqrt{x} \rightarrow y = -\sqrt{x}$; and, after a translation of 2 units vertically upwards, $y = -\sqrt{x} \rightarrow y = -\sqrt{x} + 2$.

The image has equation $y = -\sqrt{x} + 2$.

iv Under a vertical translation of 2 units upwards, $y = \sqrt{x} \rightarrow y = \sqrt{x} + 2$. Then, after a reflection in the x-axis, $y = \sqrt{x} + 2 \rightarrow y = -\left(\sqrt{x} + 2\right)$.

The image has equation $y = -\sqrt{x} - 2$.

v Under a reflection in the y-axis, $y = \sqrt{x} \rightarrow y = \sqrt{(-x)}$; and then after a translation of 2 units to the right, $y = \sqrt{(-x)} \rightarrow \sqrt{(-(x-2))}$.

The equation of the image is $y = \sqrt{2 - x}$.

vi Under a translation of 2 units to the right, $y = \sqrt{x} \rightarrow y = \sqrt{(x-2)}$. Then, after a reflection in the y-axis, $y = \sqrt{x-2} \rightarrow \sqrt{(-x) - 2}$.

The equation of the image is $y = \sqrt{-x - 2}$.

b i Under a dilation of factor 2 from the x-axis, $y = x^4 \rightarrow y = 2x^4$.

ii Under a dilation of factor 2 from the y-axis,

$y = x^4 \rightarrow y = \left(\frac{x}{2}\right)^4$.

The equation of the image is $y = \frac{x^4}{16}$.

iii Under a reflection in the x-axis, $y = x^4 \rightarrow y = -x^4$; and, after a translation of 2 units vertically upwards, $y = -x^4 \rightarrow y = -x^4 + 2$.

The image has equation $y = -x^4 + 2$.

iv Under a vertical translation of 2 units upwards, $y = x^4 \rightarrow y = x^4 + 2$. Then, after a reflection in the x-axis, $y = x^4 + 2 \rightarrow y = -\left(x^4 + 2\right)$.

The image has equation $y = -x^4 - 2$.

v Under a reflection in the y-axis, $y = x^4 \rightarrow y = (-x)^4$; and then after a translation of 2 units to the right, $y = (-x)^4 \rightarrow y = (-(x-2))^4$.

As $y = (-(x-2))^4$ is equivalent to $y = (x-2)^4$, the equation of the image is $y = (x-2)^4$.

vi Under a translation of 2 units to the right, $y = x^4 \rightarrow y = (x-2)^4$. Then, after a reflection in the y-axis, $y = (x-2)^4 \rightarrow y = ((-x) - 2)^4$.

$$\begin{aligned} y &= ((-x) - 2)^4 \\ &= (-x - 2)^4 \\ &= (-(x+2))^4 \\ &= (x+2)^4 \end{aligned}$$

The equation of the image is $y = (x+2)^4$.

9 a i $y = x^4 \rightarrow y = 5(x+3)^4$

Dilation of factor 5 from the x-axis (or parallel to the y-axis) followed by a horizontal translation of 3 units to the left

ii $y = x^4 \rightarrow y = 8 - 7(x-6)^4$; that is, $y = -7(x-6)^4 + 8$

Dilation of factor 7 from the x-axis (or parallel to the y-axis) and a reflection in the x-axis, followed by a horizontal translation of 6 units to the right and a vertical translation of 8 units upwards

b $y = f(x) \rightarrow y = 3f(x+4) - 15$

Dilation of factor 3 from the x-axis (or parallel to the y-axis) followed by a horizontal translation of 4 units to the left and a vertical translation of 15 units downwards

10 a $y = \frac{1}{x^2} \rightarrow y = -\frac{1}{x^2} + 2$

Reflection in the x-axis followed by a vertical translation of 2 units upwards

b $y = \frac{1}{x^2} \rightarrow y = \frac{3}{(x+1)^2} - 5$

Dilation of factor 3 from the x-axis (or parallel to the y-axis) followed by a horizontal translation of 1 unit to the left and a vertical translation of 5 units downwards

11 Key points on the given graph are $(-1, 0), (0, -1), (1, -2), (2, 0)$.

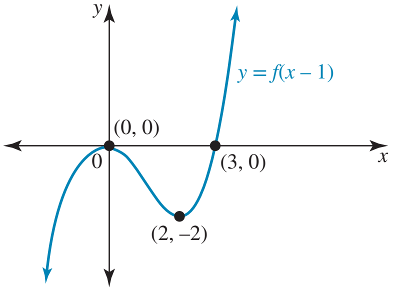
a The graph of $y = f(x-1)$ is obtained from a horizontal translation 1 unit to the right of the given graph.

$(-1, 0) \quad \rightarrow \quad (0, 0)$

$(0, -1) \quad \rightarrow \quad (1, -1)$

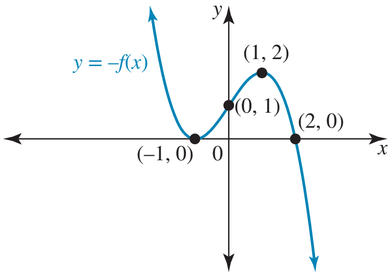
$(1, -2) \quad \rightarrow \quad (2, -2)$

$(2, 0) \quad \rightarrow \quad (3, 0)$



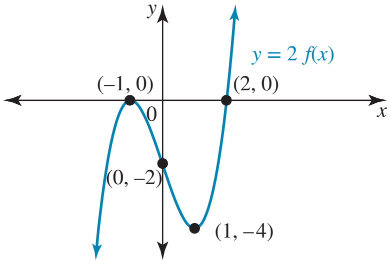
b The graph of $y = -f(x)$ is obtained by a reflection in the x-axis of the given graph.

$(-1, 0)$	→	$(-1, 0)$
$(0, -1)$	→	$(0, 1)$
$(1, -2)$	→	$(1, 2)$
$(2, 0)$	→	$(2, 0)$



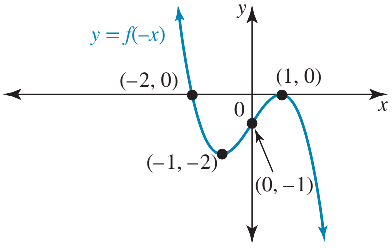
c The graph of $y = 2f(x)$ is obtained by a dilation of factor 2 from the x-axis.

$(-1, 0)$	→	$(-1, 0)$
$(0, -1)$	→	$(0, -2)$
$(1, -2)$	→	$(1, -4)$
$(2, 0)$	→	$(2, 0)$



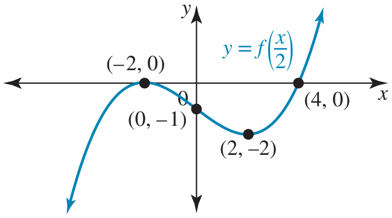
d The graph of $y = f(-x)$ is obtained by a reflection in the y-axis.

$(-1, 0)$	→	$(1, 0)$
$(0, -1)$	→	$(0, -1)$
$(1, -2)$	→	$(-1, -2)$
$(2, 0)$	→	$(-2, 0)$



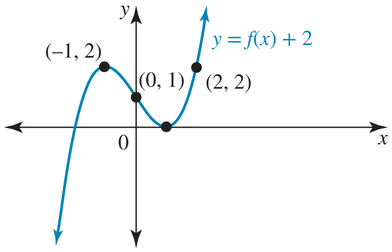
e The graph of $y = f\left(\dfrac{x}{2}\right)$ is obtained by a dilation of factor 2 from the y-axis.

$(-1, 0)$	→	$(-2, 0)$
$(0, -1)$	→	$(0, -1)$
$(1, -2)$	→	$(2, -2)$
$(2, 0)$	→	$(4, 0)$



f The graph of $y = f(x) + 2$ is obtained by a vertical translation of 2 units upwards.

$(-1, 0)$	→	$(-1, 2)$
$(0, -1)$	→	$(0, 1)$
$(1, -2)$	→	$(1, 0)$
$(2, 0)$	→	$(2, 2)$



12 a $y = f(x)$ has image $y = 2f(-x) + 1$.

The three transformations required are:

dilation of factor 2 from the x-axis (parallel to the y-axis), a reflection in the y-axis, and a vertical translation of 1 unit upwards.

b $y = f(x)$ if its image is $y = -f(3x)$.

The two transformations are:

reflection in the x-axis and dilation of factor $\dfrac{1}{3}$ from the y-axis.

13 a Dilation of factor 2 from the x-axis, followed by horizontal translation of 3 units to the left

$$y = \frac{1}{x} \rightarrow y = \frac{2}{x} \rightarrow y = \frac{2}{(x+3)}$$

Therefore, the equation of the image is $y = \dfrac{2}{(x+3)}$.

b Undoing the transformations requires the image to undergo a horizontal translation of 3 units to the right, followed by dilation of factor $\dfrac{1}{2}$ from the x-axis.

$$y = \frac{2}{(x+3)} \rightarrow y = \frac{2}{(x)} \rightarrow y = \frac{1}{x}$$

14 a Under a translation of 2 units to the left and 4 units downwards, $(3, -4) \rightarrow (1, -8)$.

b Under a reflection in the y-axis, $(3, -4) \rightarrow (-3, -4)$. Then under a reflection in the x-axis, $(-3, -4) \rightarrow (-3, 4)$.

The image is $(-3, 4)$.

c Under a dilation of factor $\dfrac{1}{5}$ from the x-axis, acting in the y direction, $(3, -4) \rightarrow \left(3, -4 \times \dfrac{1}{5}\right)$.

The image is $\left(3, -\dfrac{4}{5}\right)$.

d Under a dilation of factor $\dfrac{1}{5}$ from the y-axis, acting in the x direction, $(3, -4) \rightarrow \left(3 \times \dfrac{1}{5}, -4\right)$.

The image is $\left(\dfrac{3}{5}, -4\right)$.

e Under a reflection in the line $y = x$, coordinates are interchanged so $(3, -4) \rightarrow (-4, 3)$.

15 a i Under a dilation of factor 3 from the y-axis followed by a reflection in the y-axis,

$y = \frac{1}{x} \to y = 3 \times \frac{1}{x} = \frac{3}{x} \to y = -\frac{3}{x}$

The equation of the image is $y = -\frac{3}{x}$.

ii Under a reflection in the x-axis followed by a dilation of factor 3 from the x-axis,

$y = \frac{1}{x} \to y = \frac{-1}{x} \to y = 3 \times \frac{-1}{x} = -\frac{3}{x}$

When the order of the transformations is reversed the same image is obtained. The image has the equation $y = -\frac{3}{x}$.

b i Under a dilation of factor 3 from the x-axis followed by a vertical translation of 6 units upwards,

$y = \frac{1}{x^2} \to y = 3 \times \left(\frac{1}{x^2}\right) = \frac{3}{x^2} \to y = \frac{3}{x^2} + 6.$

The image has equation $y = \frac{3}{x^2} + 6$.

ii Under a vertical translation of 6 units upwards followed by a dilation of factor 3 from the x-axis,

$y = \frac{1}{x^2} \to y = \frac{1}{x^2 + 6} \to y = 3 \times \frac{1}{x^2 + 6} = \frac{3}{x^2 + 18}$

The image has the equation $y = \frac{3}{x^2} + 18$.

c $y = f(x) \to y = -2f(x+1)$ under a reflection in the x-axis, dilation of factor 2 from the x-axis followed by a horizontal translation 1 unit to the left.

The asymptotes of $y = f(x) = \frac{1}{x^2}$ are $x = 0$ and $y = 0$.

Under the transformations, $x = 0 \to x = 0 \to x = -1$ and $y = 0 \to y = 0 \to y = 0$.

The equations of the asymptotes of the image are $x = -1, y = 0$.

Alternatively, if $f(x) = \frac{1}{x^2}$ then $-2f(x+1) = \frac{-2}{(x+1)^2}$.

The asymptotes of $y = \frac{-2}{(x+1)^2}$ are $x = -1, y = 0$.

16 a $y = f(x) \to y = 2f(x+3)$ under a dilation of factor 2 from the x-axis followed by a horizontal translation of 3 units to the left.

b $y = f(x) \to y = 6f(x-2) + 1$ under a dilation of factor 6 from the x-axis followed by a horizontal translation of 2 units to the right and a vertical translation of 1 unit upwards.

c $y = f(x) \to y = -2f(x) + 3$ under a dilation of factor 2 from the x-axis followed by a reflection in the x-axis and a vertical translation of 3 units upwards.

17 a Under a dilation of factor $\frac{1}{3}$ from the x-axis followed by a horizontal translation 3 units to the left,

$y = \frac{1}{x^2} \to y = \frac{1}{3x^2} \to y = \frac{1}{3(x+3)^2}.$

The equation of the image is $y = \frac{1}{3(x+3)^2}$.

b Under a reflection in the x-axis followed by a horizontal translation 1 unit to the right,

$y = \frac{1}{x} \to y = \frac{-1}{x} \to y = \frac{-1}{(x-1)}.$

The equation of the image is $y = \frac{1}{1-x}$.

c Under a horizontal translation 1 unit to the right followed by a dilation of factor 2 from the x-axis,

$y = \sqrt[3]{x} \to y = \sqrt[3]{x-1} \to y = 2\sqrt[3]{x-1}.$

The equation of the image is $y = 2\sqrt[3]{x-1}$.

18 a $g: R \to R, g(x) = x^2 - 4$

Under a reflection in the y-axis, $y = g(x) \to y = g(-x)$.

Therefore, $y = x^2 - 4 \to y = (-x)^2 - 4 = x^2 - 4$.

Hence, the function g is its own image under this reflection. It is symmetric about the y-axis.

b $f: R \to R, f(x) = x^{\frac{1}{3}}$

Under a reflection in the x-axis, $y = f(x) \to y = -f(x)$.

Therefore, $y = x^{\frac{1}{3}} \to y = -x^{\frac{1}{3}}$.

Under a reflection in the y-axis, $y = f(x) \to y = f(-x)$.

Therefore, $y = x^{\frac{1}{3}} \to y = (-x)^{\frac{1}{3}}$

$$\begin{aligned} y &= (-x)^{\frac{1}{3}} \\ &= (-1)^{\frac{1}{3}} x^{\frac{1}{3}} \\ &= -x^{\frac{1}{3}} \end{aligned}$$

The image under reflection in either axis is the same, $y = -x^{\frac{1}{3}}$.

c $y = (x-2)^2 + 5$ has a minimum turning point with coordinates $(2, 5)$.

Under a reflection in the x-axis, $(2, 5) \to (2, -5)$ and becomes a maximum turning point.

Under a reflection in the y-axis, $(2, -5) \to (-2, -5)$ and stays as a maximum turning point.

If the reflections are reversed, under reflection in the y-axis, $(2, 5) \to (-2, 5)$ and stays as a minimum turning point. Then, under a reflection in the x-axis, $(-2, 5) \to (-2, -5)$ and becomes a maximum turning point.

The image has a maximum turning point with coordinates $(-2, -5)$.

d The transformations applied to $y = f(x)$ are dilation of factor 2 from the x-axis, vertical translation 1 unit upwards and reflection in the x-axis.

$\therefore y = f(x) \to y = 2f(x) \to y = 2f(x) + 1 \to y = -2f(x) - 1.$

Hence, $-2f(x) - 1 = 6(x-2)^3 - 1$, since the image has the equation $y = 6(x-2)^3 - 1$.

$$\begin{aligned} \therefore\ -2f(x) &= 6(x-2)^3 \\ \therefore f(x) &= -3(x-2)^3 \end{aligned}$$

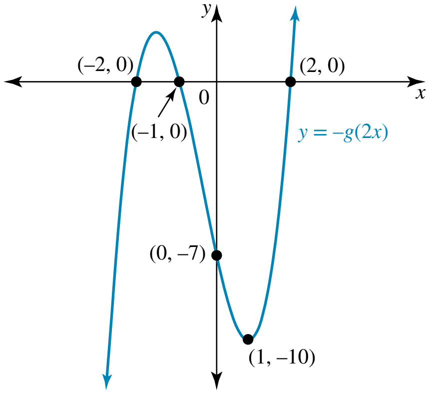
Alternatively, apply the inverse transformations to the image, which are: reflection in the x-axis, vertical translation 1 unit downwards and dilation of factor $\frac{1}{2}$ from the x-axis.

$$\begin{aligned} & y = 6(x-2)^3 - 1 \\ \to\ & y = -6(x-2)^3 + 1 \\ \to\ & y = -6(x-2)^3 \\ \to\ & y = \frac{1}{2} \times -6(x-2)^3 \\ \therefore\ & y = -3(x-2)^3 \end{aligned}$$

19 **a** The graph of $y = -g(2x)$ is obtained by reflection in the x-axis and dilation of factor $\frac{1}{2}$ from the y-axis.

Images of key points:

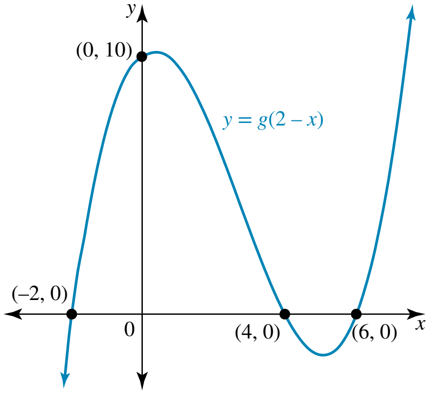
$(-4, 0)$	$\rightarrow$	$(-4, 0)$	$\rightarrow$	$(-2, 0)$
$(-2, 0)$	$\rightarrow$	$(-2, 0)$	$\rightarrow$	$(-1, 0)$
$(0, 7)$	$\rightarrow$	$(0, -7)$	$\rightarrow$	$(0, -7)$
$(2, 10)$	$\rightarrow$	$(2, -10)$	$\rightarrow$	$(1, -10)$
$(4, 0)$	$\rightarrow$	$(4, 0)$	$\rightarrow$	$(2, 0)$



b $y = g(2 - x) \Rightarrow y = g\left(-(x - 2)\right)$

The graph of this function is obtained by reflection in the y-axis followed by a horizontal translation of 2 units to the right.

$(-4, 0)$	$\rightarrow$	$(4, 0)$	$\rightarrow$	$(4, 0)$
$(-2, 0)$	$\rightarrow$	$(2, 0)$	$\rightarrow$	$(4, 0)$
$(0, 7)$	$\rightarrow$	$(0, 7)$	$\rightarrow$	$(2, 7)$
$(2, 10)$	$\rightarrow$	$(-2, 10)$	$\rightarrow$	$(0, 10)$
$(4, 0)$	$\rightarrow$	$(-4, 0)$	$\rightarrow$	$(-2, 0)$



c If the graph of $y = g(x)$ is shifted more than 4 units horizontally to the left, all three of its x-intercepts will be negative.

$y = g(x + h)$ is a horizontal translation of h units to the left.

The required values of h are $h > 4$.

d The given graph has x-intercepts at $x = -4, x = -2$ and $x = 4$. Therefore, $(x + 4)(x + 2)(x - 4)$ are factors.

Let the equation be $y = a(x + 4)(x + 2)(x - 4)$.

Substitute the point $(0, 7)$.

$\therefore 7 = a(4)(2)(-4)$

$\therefore 7 = -32a$

$\therefore a = -\frac{7}{32}$

The equation is $g(x) = -\frac{7}{32}(x + 4)(x + 2)(x - 4)$.

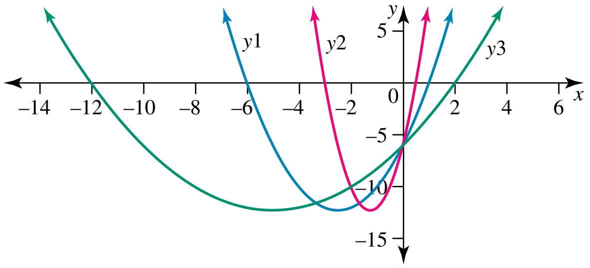
$\therefore g(2x) = -\frac{7}{32}(2x + 4)(2x + 2)(2x - 4)$

$= -\frac{7}{32} \times 2(x + 2) \times 2(x + 1) \times 2(x - 2)$

$\therefore g(2x) = -\frac{7}{4}(x + 2)(x + 1)(x - 2)$

20 Using CAS technology:

All three are parabolas with y_2 and y_3 dilations of the parabola y_1; y_2 is a dilation of factor $\frac{1}{2}$ from the y-axis and y_3 is a dilation of factor 2 from the y-axis.



6.6 Exam questions

1 $(x, y) \rightarrow (ax, y)$ when dilated by a factor of $\frac{1}{3}$ parallel to the x-axis

$\therefore (-6, 3) \rightarrow \left(-6 \times \frac{1}{3}, 3\right)$

$\therefore (-6, 3) \rightarrow (-2, 3)$

The correct answer is **A**.

2 $y = \sqrt{x}$

Dilated by a factor of 2 parallel to the x-axis:

$y = 2\sqrt{x}$ [1 mark]

Reflected in the x-axis:

$y = -2\sqrt{x}$ [1 mark]

Translated 3 units to the left:

$y = -2\sqrt{(x + 3)}$ [1 mark]

3 $y = -3 - 4f(x - 1)$

$y = -4f(x - 1) - 3$ [1 mark]

Dilation of factor 4 from the x-axis, followed by a reflection in the x-axis [1 mark]

Then a horizontal translation 1 unit to the right and a vertical translation 3 units downwards [1 mark]

6.7 Review

6.7 Exercise

Technology free: short answer

1 **a** $y = -\sqrt{4 - x}$

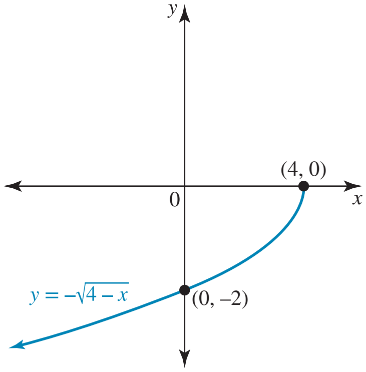
Square root function with end point $(4, 0)$.

Domain: $4 - x \geq 0 \Rightarrow x \leq 4$. Therefore, the domain is $(-\infty, 4]$.

y-intercept: let $x = 0$.

$\therefore y = -\sqrt{4} = -2 \; (0, -2)$

The range is $(-\infty, 0]$.



b $y = 4 + \sqrt[3]{x}$

The graph required is a vertical translation upwards of 4 units of the graph of the cube root function.

Point of inflection $(0, 0) \rightarrow (0, 4)$

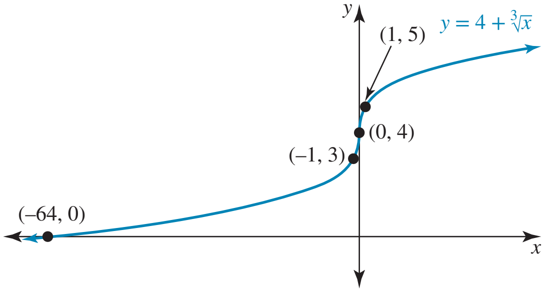
Points $(-1, -1) \rightarrow (-1, 3)$ and $(1, 1) \rightarrow (1, 5)$.

x-intercept: let $y = 0$.

$$0 = 4 + \sqrt[3]{x}$$
$$\therefore \sqrt[3]{x} = -4$$
$$\therefore x = (-4)^3$$
$$\therefore x = -64$$

$(-64, 0)$

The domain is R and the range is R.



c $y = \dfrac{4}{(4-x)^2} + 4$

Truncus with asymptotes $x = 4, y = 4$

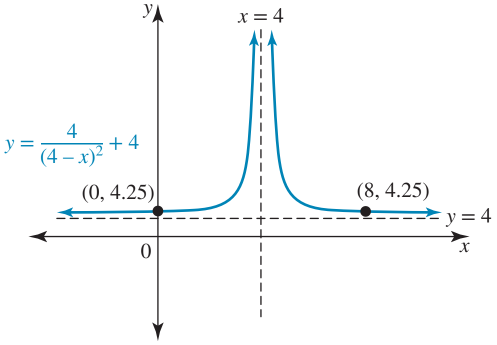
Domain $R \backslash \{4\}$, range $(4, \infty)$

y-intercept: let $x = 0$.

$$\therefore y = \frac{4}{4^2} + 4 = 4\frac{1}{4} \qquad \left(0, 4\frac{1}{4}\right)$$

By symmetry with the vertical asymptote, the point $\left(8, 4\frac{1}{4}\right)$ lies on the graph.

No x-intercept



d $y = \dfrac{6x+5}{3x+1}$ is expressed in improper rational function form.

$$\frac{6x+5}{3x+1} = \frac{2(3x+1)-2+5}{3x+1}$$
$$= \frac{2(3x+1)}{3x+1} + \frac{3}{3x+1}$$
$$= 2 + \frac{3}{3x+1}$$

$\therefore y = \dfrac{6x+5}{3x+1}$ in proper rational function form is $y = 2 + \dfrac{3}{3x+1}$.

$y = \dfrac{3}{3x+1} + 2$ has a vertical asymptote when

$$3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$$

The equations of the asymptotes are $x = -\dfrac{1}{3}$ and $y = 2$.

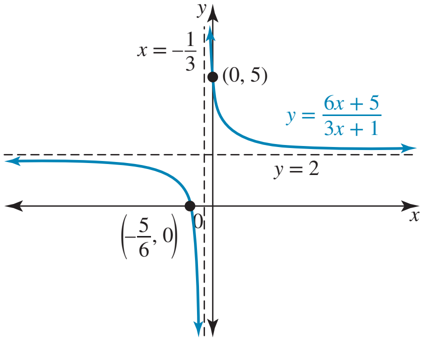
Domain $R \backslash \left\{-\dfrac{1}{3}\right\}$, range $R \backslash \{2\}$

y-intercept: let $x = 0$ in $y = \dfrac{6x+5}{3x+1}$.

$$\therefore y = \frac{5}{1} = 5 \qquad (0, 5)$$

x-intercept: let $y = 0$ in $y = \dfrac{6x+5}{3x+1}$.

$$\therefore 0 = \frac{6x+5}{3x+1}$$
$$\therefore 0 = 6x + 5$$
$$\therefore x = -\frac{5}{6} \qquad \left(-\frac{5}{6}, 0\right)$$



2 Reduce $\dfrac{3x+2}{x+3}$ to proper form.

$$\begin{array}{r|l} & \;\;3 \\ x+3 & 3x+2 \\ & \underline{3x+9} \\ & \;\;-7 \end{array}$$

$$y = \frac{3x+2}{x+3}$$
$$= 3 - \frac{7}{x+3}$$

Asymptotes:

$x = -3$

$y = 3$

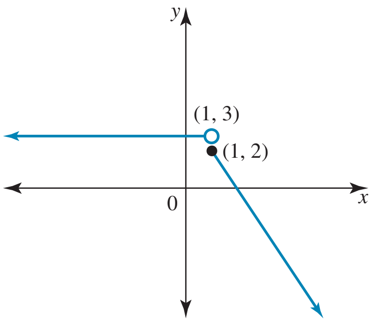
3 $f(x) = \begin{cases} 3, & x < 1 \\ 4 - 2x, & x \geq 1 \end{cases}$

a $f(0) = 3, f(1) = 4 - 2 = 2$ and $f(2) = 4 - 4 = 0$.

$\therefore f(0) + f(1) + f(2) = 3 + 2 + 0 = 5$

b For $x < 1$, $y = 3$ is a horizontal line with open end point $(1, 3)$.

For $x \geq 1$, $y = 4 - 2x$ is a line with closed end point $(1, 2)$ and passing through $(2, 0)$.



c The domain is R and the range is $(-\infty, 2] \cup \{3\}$.

d The type of correspondence is many-to-one.

4 $f: [-2, 4) \rightarrow R, f(x) = 1 - \dfrac{x}{2}$

a The domain is $[-2, 4)$.

End points: $f(-2) = 1 - \dfrac{-2}{2} = 2$, so $(-2, 2)$ is a closed end point.

$f(4) = 1 - \dfrac{4}{2} = -1$ so $(4, -1)$ is an open end point.

Therefore, the range of this linear function is $(-1, 2]$.

b Function f: $y = 1 - \dfrac{x}{2}$

Inverse function f^{-1}: $x = 1 - \dfrac{y}{2}$

$\therefore \dfrac{y}{2} = 1 - x$

$\therefore y = 2 - 2x$

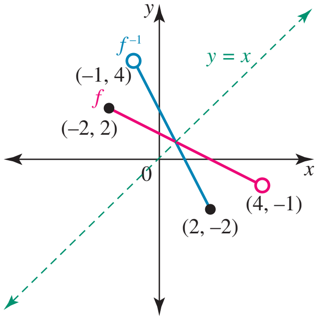
The rule for the inverse function is $f^{-1}(x) = 2 - 2x$. Its domain is $(-1, 2]$ and its range is $[-2, 4)$.

c From part **a**, the end points of $y = f(x)$ are $(-2, 2)$ (closed) and $(4, -1)$ (open).

The end points of the inverse are $(2, \ -2)$ (closed) and $(-1, 4)$ (open).

The axis intercepts of $y = f(x) = 1 - \dfrac{x}{2}$ are $(0, 1)$ and $(2, 0)$.

The axis intercepts for the inverse function are $(1, 0)$ and $(0, 2)$.



d The two graphs intersect on $y = x$.

At intersection, $2 - 2x = x$

$\therefore 3x = 2$

$\therefore x = \dfrac{2}{3}$

The point of intersection is $\left(\dfrac{2}{3}, \dfrac{2}{3}\right)$.

5 a The square root function has end point $(-2, 3)$ and its domain has $x \geq -2$.

Let the equation be $y = a\sqrt{x - h} + k$.

Substituting the end point, $y = a\sqrt{x + 2} + 3$.

The point $(0, 5) \Rightarrow 5 = a\sqrt{2} + 3$

$\therefore \sqrt{2}a = 2$

$\therefore a = \dfrac{2}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$\therefore a = \sqrt{2}$

The equation is $y = \sqrt{2}\sqrt{x + 2} + 3 \Rightarrow y = \sqrt{2x + 4} + 3$.

b For $x < -2$, the left branch is a straight line through the points $(-8, 0)$ and $(-2, 6)$.

$m = \dfrac{6 - 0}{-2 + 8} = 1$

$\therefore y - 0 = 1(x + 8)$

$\therefore y = x + 8$

For $x > -2$, the right branch is a straight line through the points $(-2, 6)$ and $(0, 4)$.

$m = \dfrac{6 - 4}{-2 - 0} = -1$

$\therefore y = -x + 4$

The rule for the hybrid function is $y = \begin{cases} x + 8, x \leq -2 \\ 4 - x, x > -2 \end{cases}$.

c Let the equation be $y = \dfrac{a}{x - h} + k$.

Horizontal asymptote $y = 2$, vertical asymptote $x = -2$

$\therefore y = \dfrac{a}{x + 2} + 2$

Substitute the known point $(0, 1)$.

$\therefore 1 = \dfrac{a}{2} + 2$

$\therefore \dfrac{a}{2} = -1$

$\therefore a = -2$

The equation is $y = \dfrac{-2}{x + 2} + 2$.

6 a Function: $y = 1 - 8x^3$

Inverse: $x = 1 - 8y^3$

$\therefore 8y^3 = 1 - x$

$\therefore y^3 = \dfrac{1 - x}{8}$

$\therefore y = \left(\dfrac{1 - x}{8}\right)^{\frac{1}{3}}$

$\therefore y = \dfrac{1}{2}(1 - x)^{\frac{1}{3}}$

b Function: $y = (x + 1)^{\frac{1}{3}} + 5$

Inverse: $x = (y + 1)^{\frac{1}{3}} + 5$

$\therefore (y + 1)^{\frac{1}{3}} = x - 5$

$\therefore y + 1 = (x - 5)^3$

$\therefore y = (x - 5)^3 - 1$

i $f: (-\infty, a] \rightarrow R, f(x) = (x - 2)^2 + 6$

For the inverse to be a function, f must have one-to-one correspondence.

As its turning point is $(2, 6)$, the largest value of a for which f is one-to-one is $a = 2$.

ii With $a = 2, \sim \ d_f = (-\infty, \ 2] = r_{f^{-1}}$ and $r_f = [6, \infty] = d_{f^{-1}}$.

Let $y = f(x)$.

Function: $y = (x - 2)^2 + 6$

Inverse: $x = (y - 2)^2 + 6$

$\therefore (y-2)^2 = x-6$

$\therefore y-2 = -\sqrt{x-6}$, as $r_{f^{-1}} = (-\infty, 2]$

$\therefore y = 2-\sqrt{x-6}$

$\therefore f^{-1}(x) = 2-\sqrt{x-6}$

$f^{-1}: [6, \infty) \to R, f^{-1}(x) = 2-\sqrt{x-6}.$

Technology active: multiple choice

7 The equation $y = \pm\sqrt{x}$ gives two values for every x-value, so it is not a function.
The correct answer is **E**.

8 $f\colon [-2, \infty] \to R, f(x) = 3[(x+2)^2 + 5]$
The rule for the parabola function is $y = 3(x+2)^2 + 15$. Its minimum turning point is $(-2, 15)$ and this lies in the domain of the function. Therefore, the range is $[15, \infty)$.
The correct answer is **D**.

9 $f\colon [-5, 0) \to R, f(x) = x^2 - 16$
$x = -5, f(-5) = (-5)^2 - 16 = 9$
$x = 0, f(0) = -16$
Always consider the turning points. In this case, the TP $= (0, -16)$, which is one of the end points.
The range is $(-16, 9]$. (Note the correct use of ( ] to match the domain.)
The correct answer is **C**.

10 $y = \dfrac{-3}{(x+2)^2} + 4$ has asymptotes with equations $x = -2, y = 4$.
The correct answer is **A**.

11 $y = c - \sqrt{a(x-b)}$
End point $y = 5 - \sqrt{a(x+2)}$
Substitute the point $(6, 1)$.

$1 = 5 - \sqrt{a(6+2)}$

$\sqrt{8a} = 4$

$8a = 16$

$\therefore a = 2$

The equation is $y = 5 - \sqrt{2(x+2)}$.
Therefore, $a = 2, b = -2, c = 5$.
The correct answer is **D**.

12 $f\colon [-2, 4] \to R, f(x) = ax + b$

$f(0) = 1 \quad \Rightarrow \quad 1 = b$

$f(1) = 0 \quad \Rightarrow \quad 0 = a + b$

$\therefore 0 = a + 1$

$\therefore a = -1$

$\therefore f(x) = -x + 1$
The image of -2 is $f(-2)$
$f(-2) = -(-2) + 1$
$= 3$
The correct answer is **D**.

13 $(x, y) \to (x, ay)$ when dilated by a factor of a from the x-axis or parallel to the y-axis.

$\therefore (4, 3) \to \left(4, -3 \times \dfrac{1}{3}\right)$

$\therefore (4, 3) \to \left(4, \dfrac{-3}{4}\right)$

The correct answer is **E**.

14 $y = \dfrac{x+6}{\sqrt{x-2}}$
The denominator cannot be zero and the term under the square root cannot be negative.
Therefore, the domain requirement is for $x - 2 > 0 \Rightarrow x > 2$.
The maximal domain is $(2, \infty)$.
The correct answer is **E**.

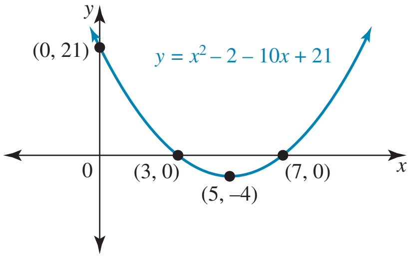
15 Under a translation of 4 units upwards followed by a reflection in the x-axis,
$y = \sqrt{x} \to y = \sqrt{x} + 4 \to y = -\left(\sqrt{x} + 4\right)$.
The equation of the image is $y = -\sqrt{x} - 4$.
The correct answer is **C**.

16 Under the transformation, $(-2, b) \to (1, b)$, $(-4, 0) \to (2, 0)$ and $(2, 0) \to (-1, 0)$. The graph in diagram (i) has been reflected in the y-axis and dilated by a factor of $\dfrac{1}{2}$ from the y-axis.
Under these transformations,
$y = f(x) \to y = f(-x) \to y = f(-2x)$.
The correct answer is **A**.

Technology active: extended response

17 **a** The graph of $y = f(x)$ where $f(x) = x^2 - 10x + 21$ is a concave up parabola.
y-intercept: $(0, 21)$
x-intercepts: let $y = 0$.
$\therefore x^2 - 10x + 21 = 0$
$\therefore (x-3)(x-7) = 0$
$\therefore x = 3, x = 7$
$(3, 0), (7, 0)$

Axis of symmetry: $x = \dfrac{3+7}{2} = 5$
$f(5) = 25 - 50 + 21 = -4$
Minimum turning point $(5, -4)$



b The minimum turning point of $y = f(x) = x^2 - 10x + 21$ is $(5, -4)$. Under a vertical translation of more than 4 units upwards, the turning point will lie above the axis.
Therefore, the graph of $y = f(x) + k$ will not intersect the x-axis if $k > 4$.

c The graph of $y = f(x)$ has x-intercepts at $x = 3$ and $x = 7$.
The roots of the equation $f(x - h) = 0$ are the x-intercepts of the graph of $y = f(x - h)$. For these roots to be negative, the graph of $y = f(x)$ needs to be shifted horizontally to the left by more than 7 units.
The roots will be negative for $h < -7$.

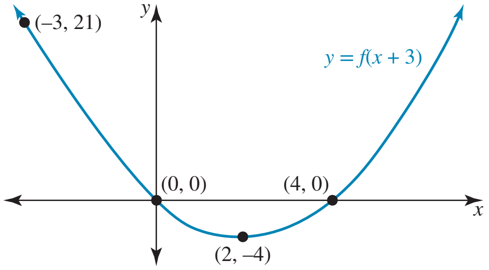
d **i** $y = f(x + 3)$ is obtained by a horizontal translation of 3 units to the left of the graph of $y = f(x)$.

$(3, 0) \quad \to \quad (0, 0)$

$(7, 0) \quad \to \quad (4, 0)$

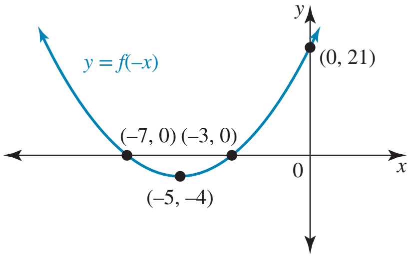
$(5, -4) \quad \to \quad (2, -4)$

$(0, 21) \quad \to \quad (-3, 21)$



ii $y = f(-x)$ is obtained by a reflection of $y = f(x)$ in the y-axis.

$(3, 0)$	$\rightarrow$	$(-3, 0)$
$(7, 0)$	$\rightarrow$	$(-7, 0)$
$(5, -4)$	$\rightarrow$	$(-5, -4)$
$(0, 21)$	$\rightarrow$	$(0, 21)$



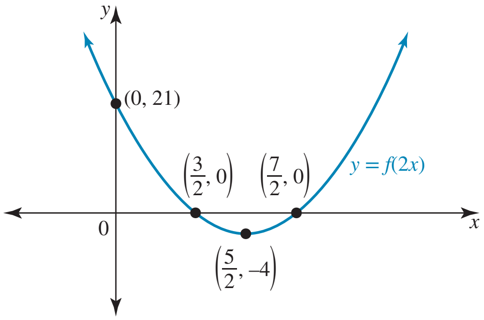
iii $y = f(2x)$ is obtained from $y = f(x)$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

$$(3, 0) = \left(\frac{3}{2}, 0\right)$$

$$(7, 0) = \left(\frac{7}{2}, 0\right)$$

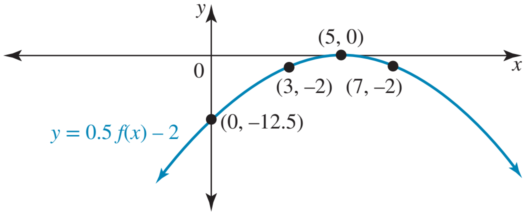
$$(5,\ -4) = \left(\frac{5}{2},\ -4\right)$$

$$(0, 21) = (0, 21)$$



iv $y = -0.5f(x) - 2$ is obtained from $y = f(x)$ by reflection in the x-axis, dilation of factor 0.5 from the x-axis and vertical translation of 2 units downwards.

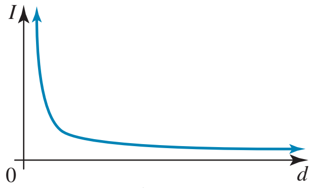
$(3, 0)$	$\rightarrow$	$(3, 0)$	$\rightarrow$	$(3, 0)$	$\rightarrow$	$(3, -2)$
$(7, 0)$	$\rightarrow$	$(7, 0)$	$\rightarrow$	$(7, 0)$	$\rightarrow$	$(7, -2)$
$(5, -4)$	$\rightarrow$	$(5, 4)$	$\rightarrow$	$(5, 2)$	$\rightarrow$	$(5, 0)$
$(0, 21)$	$\rightarrow$	$(0, -21)$	$\rightarrow$	$(0, -10.5)$	$\rightarrow$	$(0, -12.5)$



18 a $I \propto \frac{1}{d}$

$\therefore I = \frac{k}{d}$ where k is the constant of proportionality.

The graph of I against d is the branch of a hyperbola in the first quadrant.



b $(y + 2) \propto \frac{1}{(x + 2)^2}$

$$\therefore (y + 2) = \frac{k}{(x + 2)^2}$$

$\therefore (y + 2)(x + 2)^2 = k$

Given $x = 4, y = -0.5$,

$(2 - 0.5)(4 + 2)^2 = k$

$$\therefore k = 54$$

$\therefore (y + 2)(x + 2)^2 = 54$

$$\therefore y + 2 = \frac{54}{(x + 2)^2}$$

$$\therefore y = \frac{54}{(x + 2)^2} - 2$$

Substitute $x = a, y = a$

$$\therefore a = \frac{54}{(a + 2)^2} - 2$$

$$\therefore a + 2 = \frac{54}{(a + 2)^2}$$

$\therefore (a + 2)(a + 2)^2 = 54$

$$\therefore (a + 2)^3 = 54$$

$$\therefore a + 2 = \sqrt[3]{54}$$

$$\therefore a = \sqrt[3]{54} - 2$$

c i $h \propto \frac{1}{r^2}$

$h = \frac{k}{r^2}$

Substitute $h = 20, r = 3.5$.

$20 = \frac{k}{3.5^2}$

$k = 20 \times 3.5^2$

$\therefore k = 245$

$\therefore h = \frac{245}{r^2}$

Substitute $r = 7$.

$h = \frac{245}{7^2}$

$\therefore h = 5$

Height $= 5$ cm

ii Let $r = 1$: $h = 245$

Decrease r by 25%: $r = 0.75$

Substitute:

$h = \frac{245}{0.75^2}$

$\therefore h = 435.555\,555\,56$

The height has increased from 245 to 435.555 555 6

Increase in height = 190.555 555 56

$$\text{Percentage change in height} = \frac{190.555\,555\,6}{245} \times 100\%$$
$$= 77.777\,78\%$$

Height increased by 77.8% (correct to 1 decimal place).

19 **a** $y = \dfrac{2}{x-2} - 1, x > 2$

One branch of the hyperbola with asymptotes $x = 2$ and $y = -1$

x-intercept: let $y = 0$.

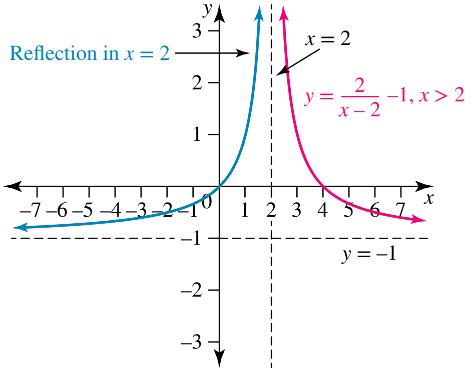
$$\therefore 0 = \frac{2}{x-2} - 1$$

$$\therefore 1 = \frac{2}{x-2}$$

$$\therefore x - 2 = 2$$

$$\therefore x = 4 \quad (4, 0)$$

Sketch this and its reflection in the line $x = 2$.



Reflecting the graph of $y = \dfrac{2}{x-2} - 1, x > 2$ in the y-axis followed by a translation of 4 units to the right would give the required image.

Alternatively, translate the graph of $y = \dfrac{2}{x-2} - 1, x > 2$ 4 units to the left and then reflect in the y-axis.

The image has the same asymptotes as $y = \dfrac{2}{x-2} - 1, x > 2$.

Let its equation be $y = \dfrac{a}{x-2} - 1, x < 2$.

Substitute $(0, 0)$.

$$\therefore 0 = \frac{a}{-2} - 1$$

$$\therefore a = -2$$

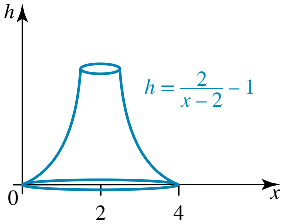
The equation of the image is $y = \dfrac{-2}{x-2} - 1, x < 2$.

b $h = \dfrac{2}{x-2} - 1, 2 < x \le 4$ This is part of the curve in part **a**.

When $x = 4$, $h = \dfrac{2}{2} - 1 = 0$.

Its reflection in the line $x = 2$ would have the equation $h = \dfrac{-2}{x-2} - 1, 0 \le x < 2$ (from part **a**).

The container is sketched in the diagram.



The diameter of the circular base is 4 cm.

c Diameter of the top circular surface is 1 cm, so the radius is 0.5 cm.

Substitute $x = 2.5$ in the equation $h = \dfrac{2}{x-2} - 1$ to obtain the height of the container.

$$\therefore h = \frac{2}{2.5-2} - 1$$
$$= \frac{2}{0.5} - 1$$
$$= 3$$

The container has a height of 3 cm.

d When $h = 1.5$,

$$1.5 = \frac{2}{x-2} - 1$$

$$\therefore 2.5 = \frac{2}{x-2}$$

$$\therefore x - 2 = \frac{2}{2.5}$$

$$\therefore x = 0.8 + 2$$

$$\therefore x = 2.8$$

The radius of the cross section is $2.8 - 2 = 0.8$ cm.

The circular surface area is calculated from $A = \pi r^2$

$$\therefore A = \pi(0.8)^2$$

$$\therefore A = 0.64\pi$$

The surface area is 0.64π sq cm.

20 **a** **i** The gradient of the line $x + y = 2$ is -1.

The angle of inclination of this line with the horizontal is given by $m = \tan\theta$. As $m < 0$, the angle is obtuse.

$$\therefore \tan(\theta°) = -1$$

$$\therefore \theta° = 180° - 45°$$
$$= 135°$$

The angle of arrival is the supplementary angle of 45°.

As the angle of departure is equal to the angle of arrival, the angle of departure is 45°.

ii The incoming line $x + y = 2$ meets the x-axis at $x = 2$.

The departing line starts from $(2, 0)$ at an inclination of 45° to the horizontal.

Gradient of departing line: $m = \tan\theta$

$$\therefore m = \tan(45°)$$

$$\therefore m = 1$$

Equation of departing line:

$$y - 0 = 1(x - 2)$$

$$\therefore y = x - 2$$

iii $y = \begin{cases} 2 - x, & x < 2 \\ x - 2, & x \ge 2 \end{cases}$

b **i** As a vertical line can cut the graph in more than one place, this section of the path of the ray of light is not a function. Its correspondence is one-to-many.

ii Incoming ray:

For this section of the path, the incoming ray has the equation $y = x - 2$ and is inclined at 45° to the horizontal.

When $x = 4$, the incoming ray meets the vertical at $y = 4 - 2 = 2$; that is, at the point $(4, 2)$.

Departing ray:

The angle of arrival of the incoming line with the vertical must also be 45°, making the angle of departure 45°.

The departing ray is inclined at 135° with the horizontal.

This makes its gradient $m = \tan(135°) = -1$.

The equation of the departing ray is

$$y - 2 = -1(x - 4)$$

$$\therefore y = -x + 6$$

Thus, for this section:

The incoming ray has equation $y = x - 2$ or $x = y + 2$ and runs between the points $(2, 0)$ and $(4, 2)$; the departing ray has equation $y = -x + 6$ or $x = 6 - y$ and runs from the point $(4, 2)$.

The hybrid rule for this section is

$$x = \begin{cases} y + 2, & 0 < y < 2 \\ 6 - y, & y \geq 2 \end{cases}.$$

c After first reflection from the line $x = 4$, the line $y = -x + 6$ meets the vertical line $x = 0$ at the point $(0, 6)$ and is reflected, so the departing line from $(0, 6)$ has gradient 1. The departing line has the equation $y = x + 6$.

Second reflection: The line $y = x + 6$ meets the vertical line $x = 4$ at the point $(4, 10)$ and is reflected, so the next departing line from $(4, 10)$ has gradient -1. This departing line has the equation

$$y - 10 = -(x - 4)$$
$$\therefore y = 14 - x$$

Third reflection: The line $y = 14 - x$ meets the vertical line $x = 0$ at the point $(0, 14)$ and is reflected, so the departing line from $(0, 14)$ has gradient 1. The departing line has the equation $y = x + 14$. This line meets the vertical line $x = 4$ at the point $(4, 18)$ and is reflected so the next departing line from $(4, 18)$ has gradient -1. This departing line has the equation

$$y - 18 = -(x - 4)$$
$$\therefore y = 22 - x$$

The pattern for the lines being reflected from $x = 4$:
$y = 6 - x, y = 14 - x, y = 22 - x, \ldots$

The y-intercepts are increasing by 8.

Immediately after its fourth reflection on the line $x = 4$, the light ray will have the equation $y = 30 - x$. This line will run from $(4, 26)$ to $(0, 30)$.

The required equation is $y = 30 - x, 0 \leq x \leq 4$.

6.7 Exam questions

1
$$\begin{aligned} f(x) &= x^2 - 6x \\ f(1 - x) &= (1 - x)^2 - 6(1 - x) \\ &= 1 - 2x + x^2 - 6 + 6x \\ &= x^2 + 4x - 5 \end{aligned}$$
[1 mark]

$$\begin{aligned} f(x) &= g(x) \\ x^2 - 6x &= x^2 + 4x - 5 \\ 10x &= -5 \\ x &= \frac{1}{2} \end{aligned}$$
[1 mark]

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) \\ &= \frac{1}{4} - 3 \\ &= -2\frac{3}{4} \end{aligned}$$

$f(x) = g(x), \left(\frac{1}{2}, -2\frac{3}{4}\right)$ [1 mark]

2 $y = \dfrac{a}{(x - h)^2} + k$ is a truncus shape with asymptotes with equations $y = k, x = h$.

Horizontal asymptote at $y = 2$, vertical asymptote at $x = 4$

The graph is reflected in the x-axis, so $y < 2$.

The correct answer is **C**.

3 $y = -1 + \sqrt{4 + x}$

End point $= (-4, -1)$ [1 mark]

x-intercept $(y = 0)$:

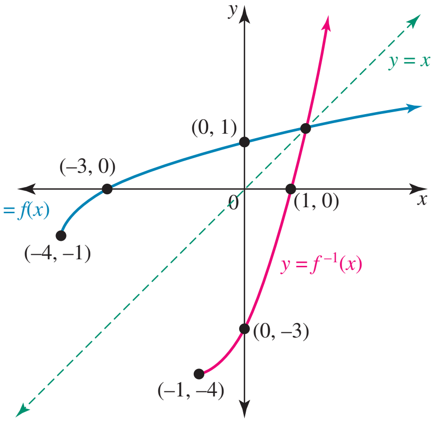
$$0 = -1 + \sqrt{4 + x}$$
$$1 = \sqrt{4 + x}$$
$$1 = 4 + x$$
$$x = -3$$

$\therefore (-3, 0)$ [1 mark]

y-intercept $(x = 0)$:

$$y = -1 + \sqrt{4 + 0}$$
$$y = 1$$

$\therefore (0, 1)$ [1 mark]



[1 mark]

Reflect each point in the line $y = x$ to obtain the graph of the inverse function. [1 mark]

$\therefore x$-intercept $= (1, 0)$, y-intercept $= (0, -3)$ and end point $= (-1, -4)$.

4 $f(x) = \begin{cases} x^2, & \text{if } x < 1 \\ 3 - x, & \text{if } x \geq 1 \end{cases}$

a $f(-1)$

Since $x = -1$ lies in the domain section $x < 1$, use the rule $f(x) = x^2$.

$\therefore f(-1) = 1$

$f(-1)$

Since $x = 1$ and $x = 2$ lie in the domain section $x \underline{>} 1$, use the rule $f(x) = 3 - x$.

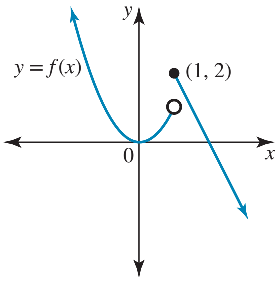
$\therefore f(2) = 1$ and $f(1) = 2$ [1 mark]

Since $x = 2$ lies in the domain section $x \geq 1$, use the rule $f(x) = 3 - x$.

$\therefore f(2) = 1$

b Sketch $y = x^2, x < 1$ (parabola open end point $(1, 1)$).

Sketch $y = 3 - x, x \underline{>} 1$ (straight line, closed end point $(1, 2)$).



[1 mark]

Domain R, range R [1 mark]

The function is not continuous at $x = 1$ because there is a break in the graph.

5 $f(x)$ reflected in the y-axis: $f(-x)$

$f(x)$ reflected in the x-axis: $-f(x)$

Translated 2 units parallel to the y-axis in the positive direction indicates vertical translation +2 units upwards.

$\therefore -f(-x) + 2$

The correct answer is **A**.

Topic 7 — Probability

7.2 Probability review

7.2 Exercise

1 a $\Pr(G) = \dfrac{5}{5+6+4+8}$

$= \dfrac{5}{23}$

b Using the addition formula:

$\Pr(O \text{ or } B) = \Pr(O \cup B)$

$= \Pr(O) + \Pr(B) - \Pr(O \cap B)$

$= \dfrac{4}{23} + \dfrac{8}{23} - 0$

$= \dfrac{12}{23}$

c Using the rule for complementary events:

$\Pr(B') = 1 - \Pr(B)$

$= 1 - \dfrac{8}{23}$

$= \dfrac{15}{23}$

d There are no black counters; therefore, the probability of selecting a black counter is zero.

$\Pr(Bl) = \dfrac{0}{5+6+4+8}$

$= 0$

2 The sample space is $\{1, 2, 3, 4, 5, 6\}$ and each outcome has an equal chance of appearing uppermost.

a $\Pr(A) = \dfrac{n(A)}{n(\xi)}$

$\Pr(4) = \dfrac{1}{6}$

b $\Pr(A') = 1 - \Pr(A)$

$\Pr(\text{not } 4) = 1 - \Pr(4)$

$\Pr(\text{not } 4) = 1 - \dfrac{1}{6}$

$\Pr(\text{not } 4) = \dfrac{5}{6}$

c $\Pr(A) = \dfrac{n(A)}{n(\xi)}$

$\Pr(\text{even}) = \dfrac{3}{6}$

$\Pr(\text{even}) = \dfrac{1}{2}$

d $\Pr(A) = \dfrac{n(A)}{n(\xi)}$

The numbers 1, 2, 3, 4 are smaller than 5.

$\Pr(< 5) = \dfrac{4}{6}$

$\Pr(< 5) = \dfrac{2}{3}$

e $\Pr(A) = \dfrac{n(A)}{n(\xi)}$

The numbers 5, 6 are at least 5.

$\Pr(\geq 5) = \dfrac{2}{6}$

$\Pr(\geq 5) = \dfrac{1}{3}$

f $\Pr(A) = \dfrac{n(A)}{n(\xi)}$

There are no numbers in the sample space that are greater than 12, so the probability is zero.

3 The 26 letters of the alphabet form the elements of the sample space.

a $\Pr(A) = \dfrac{n(A)}{n(\xi)}$

$\Pr(Q) = \dfrac{1}{26}$

b There are 5 vowels, A, E, I, O, U.

$\Pr(A) = \dfrac{n(A)}{n(\xi)}$

$\Pr(\text{a vowel}) = \dfrac{5}{26}.$

c $\Pr(A) = \dfrac{n(A)}{n(\xi)}$

$\Pr(X, Y, Z) = \dfrac{3}{26}$

d There are 25 letters other than D.

$\Pr(A) = \dfrac{n(A)}{n(\xi)}$

$\Pr(D') = \dfrac{25}{26}$

e As every letter in the alphabet is either a vowel or a consonant, the chosen letter is certain to be one of these. The probability is 1.

f The word PROBABILITY contains 9 distinct letters, P, R, O, B, A, I, L, T, Y.

The probability of choosing one of these 9 letters is $\dfrac{9}{26}$.

4 a $\Pr(1\text{st}) = \dfrac{10}{2000}$

$= \dfrac{1}{200}$

b For Josephine to just win the second prize, she must have not won first prize. Then, since one ticket was already drawn for first prize, the solution space would decrease by 1 to 1999.

$\Pr(2\text{nd}) = \Pr(1\text{st}') \times \Pr(2\text{nd})$

$= (1 - \Pr(1\text{st})) \times \dfrac{10}{2000 - 1}$

$= \left(1 - \dfrac{1}{200}\right) \times \dfrac{10}{1999}$

$= \dfrac{199}{200} \times \dfrac{10}{1999}$

$= \dfrac{199}{39\,980}$

c If Josephine wins the first prize, then the solution space is reduced by 1 and her number of tickets is reduced by 1 for her to win the second prize as well. Then the solution space and her number of tickets is further reduced by 1 for her to also win the third prize.

$$\begin{aligned}\Pr(\text{all three}) &= \frac{10}{2000} \times \frac{9}{2000-1} \times \frac{8}{2000-2}\\ &= \frac{1}{200} \times \frac{9}{1999} \times \frac{8}{1998}\\ &= \frac{1}{11\,094\,450}\end{aligned}$$

5 a Using the rule for complementary events:

$$\begin{aligned}\Pr(G') &= 1 - \Pr(G)\\ &= 1 - 0\\ &= 1\end{aligned}$$

Since there are no green cards in a standard pack of 52 playing cards, the probability of not drawing a green is 1.

b $\Pr(R) = \frac{26}{52} = \frac{1}{2}$

c $\Pr(H) = \frac{13}{52} = \frac{1}{4}$

d Using the addition formula:

$$\begin{aligned}\Pr(10 \text{ or } R) &= \Pr(10 \cup R)\\ &= \Pr(10) + \Pr(\text{red}) - \Pr(10 \cap \text{red})\\ &= \frac{4}{52} + \frac{26}{52} - \frac{2}{52}\\ &= \frac{28}{52}\\ &= \frac{7}{13}\end{aligned}$$

e Using the rule for complimentary events:

$$\begin{aligned}\Pr(A') &= 1 - \Pr(A)\\ &= \frac{52-4}{52}\\ &= \frac{48}{52}\\ &= \frac{12}{13}\end{aligned}$$

6 a $\xi = \{1, 2, 3, 4, 5, 6, 7, 8\}$

As $\Pr(6) = \frac{9}{16}$, the probability of not obtaining 6 is $1 - \frac{9}{16} = \frac{7}{16}$.

Since each of the numbers 1, 2, 3, 4, 5, 7, and 8 are equiprobable, the probability of each number is $\frac{7}{16} \times \frac{1}{7} = \frac{1}{16}$.

Hence, $\Pr(1) = \frac{1}{16}$.

b $A = \{2, 3, 5, 7\}$

$$\begin{aligned}\text{So, } \Pr(A) &= \Pr(2) + \Pr(3) + \Pr(5) + \Pr(7)\\ &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\\ &= \frac{1}{4}\end{aligned}$$

Using the rule for complementary events:

$$\begin{aligned}\Pr(A') &= 1 - \Pr(A)\\ &= 1 - \frac{1}{4}\\ &= \frac{3}{4}\end{aligned}$$

7 a Total possible outcomes is 16:

RR, RG, RB, RY
GG, GR, GB, GY
YY, YR, YG, YB
BB, BR, BG, BY

$$\begin{aligned}\Pr(\text{same colour}) &= \Pr(RR) + \Pr(GG) + \Pr(YY) + \Pr(BB)\\ &= \frac{4}{16}\\ &= \frac{1}{4}\end{aligned}$$

b

$$\begin{aligned}\Pr(R \text{ and } Y) &= \Pr(RY) + \Pr(YR)\\ &= \frac{2}{16}\\ &= \frac{1}{8}\end{aligned}$$

c Using the rule for complementary events:

$$\begin{aligned}\Pr(G') &= 1 - \Pr(G)\\ &= 1 - (\Pr(RG) + \Pr(GG) + \Pr(GR) + \Pr(GB) + \Pr(GY)\\ &\quad + \Pr(YG) + \Pr(BG))\\ &= 1 - \frac{7}{16}\\ &= \frac{9}{16}\end{aligned}$$

8 a i

$$\begin{aligned}\Pr(G \text{ or } R) &= \Pr(G) + \Pr(R)\\ &= \frac{9}{20} + \frac{6}{20}\\ &= \frac{3}{4}\end{aligned}$$

ii Using the rule for complementary events,

$$\begin{aligned}\Pr(R') &= 1 - \Pr(R)\\ &= 1 - \frac{6}{20}\\ &= \frac{7}{10}\end{aligned}$$

iii Using the rule for complementary events,

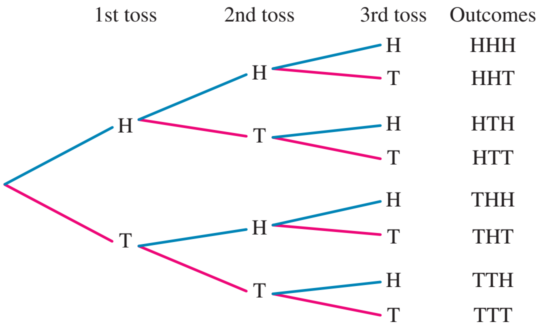
$$\begin{aligned}\Pr((G \text{ or } R)') &= 1 - \Pr(G \text{ or } R)\\ &= 1 - \frac{3}{4}\\ &= \frac{1}{4}\end{aligned}$$

b Let n be the number of additional red balls.

$$\begin{aligned}\frac{n(R)}{n(\text{total})} &= \Pr(R)\\ \frac{6+n}{20+n} &= \frac{1}{2}\\ 6 + n &= \frac{1}{2} \times (20 + n)\\ 6 + n &= 10 + \frac{n}{2}\\ \frac{n}{2} &= 4\\ n &= 8\end{aligned}$$

Therefore, an additional 8 red balls must be added.

9 a



b Obtaining at least one Head is the complementary event of obtaining no heads.

Therefore, using the rule for complementary events, the probability of obtaining at least one Head is

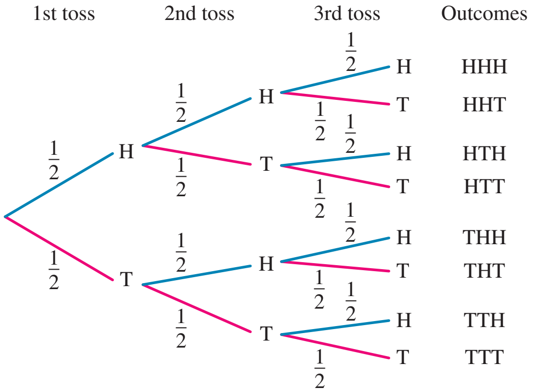
$1 - \Pr(TTT) = 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{7}{8}$.

c $\Pr(HHT) + \Pr(HTH) + \Pr(HTT) + \Pr(THH) + \Pr(THT) + \Pr(TTH)$

$$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$$

10 a



$$\Pr(2H \text{ and } 1T) = \Pr(HHT) + \Pr(HTH) + \Pr(THH)$$
$$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$$
$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$
$$= \frac{3}{8}$$

b $\Pr(3H \text{ or } 3T) = \Pr(HHH) + \Pr(TTT)$

$$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$$
$$= \frac{1}{8} + \frac{1}{8}$$
$$= \frac{2}{8}$$
$$= \frac{1}{4}$$

c $\Pr(H \text{ on first toss}) = \Pr(HHH) + \Pr(HHT) + \Pr(HTH) + \Pr(HTT)$

$$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$$
$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2}$$

Alternatively, you can see straight away from the tree diagram and the outcomes that half of the outcomes start with H (and the tree diagram splits into two after the first toss).

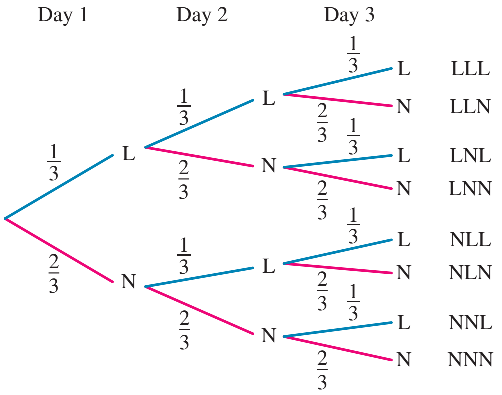
d $\Pr(H \geq 1) = 1 - \Pr(\text{no } H)$

$$= 1 - \Pr(TTT)$$
$$= 1 - \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$$
$$= 1 - \frac{1}{8}$$
$$= \frac{7}{8}$$

e $\Pr(T \leq 1) = \Pr(1T) + \Pr(0T)$

$$= \Pr(HHT) + \Pr(HTH) + \Pr(THH) + \Pr(HHH)$$
$$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$$
$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2}$$

11 a



$$\Pr(\text{late on 1 day}) = \Pr(LNN) + \Pr(NLN) + \Pr(NNL)$$
$$= \left(\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}\right) + \left(\frac{2}{3} \times \frac{1}{3} \times \frac{2}{3}\right) + \left(\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}\right)$$
$$= \frac{4}{27} + \frac{4}{27} + \frac{4}{27}$$
$$= \frac{12}{27}$$
$$= \frac{4}{9}$$

b $\Pr(\text{late at least 2 days}) = \Pr(\text{LLL}) + \Pr(\text{LLN}) + \Pr(\text{LNL}) + \Pr(\text{NLL})$

$$\begin{aligned} &= \left(\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\right) + \left(\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}\right) \\ &\quad + \left(\frac{1}{3}\times\frac{2}{3}\times\frac{1}{3}\right) + \left(\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}\right) \\ &= \frac{1}{27} + \frac{2}{27} + \frac{2}{27} + \frac{2}{27} \\ &= \frac{7}{27} \end{aligned}$$

c $\Pr(\text{on time on last day}) = \Pr(\text{LLN}) + \Pr(\text{LNN}) + \Pr(\text{NLN}) + \Pr(\text{NNN})$

$$\begin{aligned} &= \left(\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}\right) + \left(\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}\right) \\ &\quad + \left(\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}\right) + \left(\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\right) \\ &= \frac{2}{27} + \frac{4}{27} + \frac{4}{27} + \frac{8}{27} \\ &= \frac{18}{27} \\ &= \frac{2}{3} \end{aligned}$$

d $\Pr(\text{on time on all days}) = \Pr(\text{NNN})$

$$\begin{aligned} &= \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3} \\ &= \frac{8}{27} \end{aligned}$$

12 a $\Pr(L) = \dfrac{178}{800}$

$$= \frac{89}{400}$$

b $\Pr(\text{not employed}) = \Pr(\text{unemployed})$

$$= \frac{53}{800}$$

c Using the rule for complimentary events:

$$\begin{aligned} \Pr(E') &= 1 - \Pr(E) \\ &= 1 - \frac{128}{800} \\ &= 1 - \frac{4}{25} \\ &= \frac{21}{25} \end{aligned}$$

d $\Pr(T \text{ or } L) = \Pr(T) + \Pr(L)$

$$\begin{aligned} &= \frac{261}{800} + \frac{178}{800} \\ &= \frac{439}{800} \end{aligned}$$

13 a $\Pr(F) = \dfrac{26+20}{73+26+81+20}$

$$\begin{aligned} &= \frac{46}{200} \\ &= \frac{23}{100} \end{aligned}$$

b $\Pr(F \text{ and } P) = \dfrac{81}{200}$

14 a From the Venn diagram, $n(B) = 2 + 3 = 5$, so 5 students had an umbrella.

b i $\Pr(B) = \dfrac{n(B)}{n(\xi)}$

From the Venn diagram, $n(B) = 5$ and $n(\xi) = 20$.

$$\Pr(B) = \frac{5}{20}$$

$$\Pr(B) = \frac{1}{4}$$

The probability the student had an umbrella is $\dfrac{1}{4}$.

ii From the Venn diagram, $n(C \cap B')$, the number of students who had a raincoat but not an umbrella is 4.

The required probability is $\dfrac{4}{20} = \dfrac{1}{5}$.

iii From the Venn diagram, the number of students who had neither an umbrella nor a raincoat is $n(C' \cap B') = 11$.

The required probability is $\dfrac{11}{20}$.

c The addition formula states that for any two events A and B, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

Therefore, $\Pr(B' \cup C) = \Pr(B') + \Pr(C) - \Pr(B' \cap C)$.

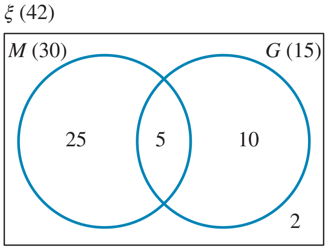
From the Venn diagram,

$$\begin{aligned} n(B) &= 2 + 3 = 5 \\ \therefore n(B') &= 20 - 5 = 15 \\ n(C) &= 3 + 4 = 7 \\ n(B' \cap C) &= 4 \end{aligned}$$

Use these values to form the probabilities in the addition formula.

$$\begin{aligned} \Pr(B' \cup C) &= \Pr(B') + \Pr(C) - \Pr(B' \cap C) \\ &= \frac{15}{20} + \frac{7}{20} - \frac{4}{20} \\ &= \frac{18}{20} \\ \Pr(B' \cup C) &= \frac{9}{10} \end{aligned}$$

15 a Given $n(\xi) = 42$, $n(M) = 30$, $n(G) = 15$ and $n(G \cap M') = 10$



b $\Pr(M \cap G') = \dfrac{n(M \cap G')}{n(\xi)}$

$$\therefore \Pr(M \cap G') = \frac{25}{42}$$

The probability that the randomly chosen student studies Mathematical Methods but not Geography is $\dfrac{25}{42}$.

c $\Pr(M' \cap G') = \dfrac{n(M' \cap G')}{n(\xi)}$

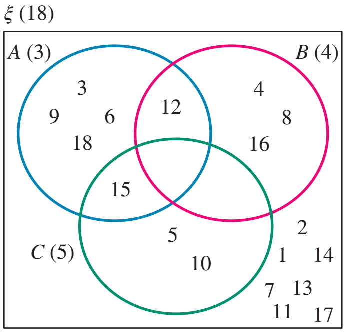
$$\begin{aligned} \therefore \Pr(M' \cap G') &= \frac{2}{42} \\ &= \frac{1}{21} \end{aligned}$$

The probability that the randomly chosen student studies neither Mathematical Methods nor Geography is $\dfrac{1}{21}$.

d $$\begin{aligned}\Pr(M\cap G')+\Pr(G\cap M')&=\frac{25}{42}+\frac{10}{42}\\&=\frac{35}{42}=\frac{5}{6}\end{aligned}$$

The probability that the randomly chosen student studies only one of the subjects Mathematical Methods or Geography is $\frac{35}{42}=\frac{5}{6}$.

16 a Given $\xi=\{1,2,3,...,18\}$, $A=\{3,6,9,12,15,18\}$, $B=\{4,8,12,16\}$ and $C=\{5,10,15\}$



b Mutually exclusive means that the events cannot occur simultaneously, i.e. the intersection is empty.
From the Venn diagram, the events that are mutually exclusive are B and C.

c From the Venn diagram,

$$\begin{aligned}\Pr(A)&=\frac{n(A)}{n(\xi)}\\&=\frac{6}{18}\\&=\frac{1}{3}\end{aligned}$$

d i $$\begin{aligned}\Pr(A\cup C)&=\frac{n(A\cup C)}{n(\xi)}\\&=\frac{8}{18}\\&=\frac{4}{9}\end{aligned}$$

ii $$\begin{aligned}\Pr(A\cap B')&=\frac{n(A\cap B')}{n(\xi)}\\&=\frac{5}{18}\end{aligned}$$

iii From the Venn diagram,

$$\begin{aligned}\Pr((A\cup B\cup C)')&=\frac{n((A\cup B\cup C)')}{n(\xi)}\\&=\frac{7}{18}\end{aligned}$$

17 a Given $\Pr(A)=0.65$, $\Pr(B)=0.5$, $\Pr(A'\cap B')=0.2$ and also $\Pr(\xi)=1$.

	B	B'	
A			0.65
A'		0.2	
	0.5		1

$\Pr(A')=1-0.65=0.35$ and $\Pr(B')=1-0.5=0.5$

	B	B'	
A			0.65
A'		0.2	0.35
	0.5	0.5	1

For the second row, $0.15+0.2=0.35$.
For the second column, $0.3+0.2=0.5$.
For the first row, $0.35+0.3=0.65$.

	B	B'	
A	0.35	0.3	0.65
A'	0.15	0.2	0.35
	0.5	0.5	1

b Using the addition formula,
$\Pr(B'\cup A)=\Pr(B')+\Pr(A)-\Pr(B'\cap A)$.
From the probability table, $\Pr(B'\cap A)=0.3$.
$\therefore \Pr(B'\cup A)=0.5+0.65-0.3$
$\therefore \Pr(B'\cup A)=0.85$

18 a Given $\Pr(A')=0.42$, $\Pr(B)=0.55$, and also $\Pr(\xi)=1$.

	B	B'	
A			
A'			0.42
	0.55		1

$\Pr(A)=1-0.42=0.58$ and $\Pr(B')=1-0.55=0.45$
Using the addition formula,
$\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$
$\therefore\ \Pr(A\cap B)=\Pr(A)+\Pr(B)-\Pr(A\cup B)$
Given $\Pr(A\cup B)=0.75$,
$\therefore\ \Pr(A\cap B)=0.58+0.55-0.75$
$\therefore\ \Pr(A\cap B)=0.38$

	B	B'	
A	0.38		0.58
A'			0.42
	0.55	0.45	1

For the first row, $0.38+0.2=0.58$.
For the first column, $0.38+0.17=0.55$.
For the second row, $0.17+0.25=0.42$.

	B	B'	
A	0.38	0.2	0.58
A'	0.17	0.25	0.42
	0.55	0.45	1

b From the probability table, $\Pr(A'\cap B')=0.25$.

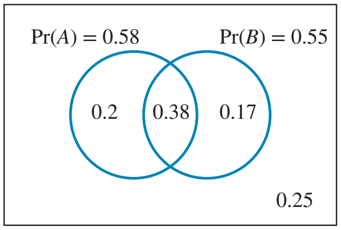
c From the probability table,
$\Pr(A\cup B)=0.75$
$\Pr(A\cup B)'=0.25$
$\Pr(A'\cup B')=0.25$
$\Rightarrow \Pr(A\cup B)'=\Pr(A'\cap B')$

d From the probability table, $\Pr(A\cap B)=0.38$.

$$\begin{aligned}1-\Pr(A'\cup B')&=1-[\Pr(A')+\Pr(B')-\Pr(A'\cap B')]\\&=1-0.42-0.45+0.25\\&=0.38\end{aligned}$$

So, $\Pr(A\cap B)=1-\Pr(A'\cup B')$.

e $\Pr(\xi) = 1$



19 a

1st roll

2nd roll	**1**	**2**	**3**	**4**	**5**	**6**
1	2	2	3	4	5	6
2	2	4	3	4	5	6
3	3	3	6	4	5	6
4	4	4	4	8	5	6
5	5	5	5	5	10	6
6	6	6	6	6	6	12

Since the size of the sample space is 36 and 5 appears 8 times in the table:

$$\begin{aligned}\Pr(5) &= \frac{8}{36}\\ &= \frac{2}{9}\end{aligned}$$

b Since the size of the sample space is 36 and 10 only occurs once:

$$\Pr(10) = \frac{1}{36}$$

c Count how many times a number greater than 5 appears in the table (i.e. numbers 6, 8, 10, 12) and divide by the total amount of numbers in the table.

$$\begin{aligned}\Pr(x > 5) &= \Pr(x = 6) + \Pr(x = 8) + \Pr(x = 10) + \Pr(x = 12)\\ &= \frac{14}{36}\\ &= \frac{7}{18}\end{aligned}$$

d We can see from the table above that 7 does not appear in the table.

Therefore, $\Pr(7) = 0$

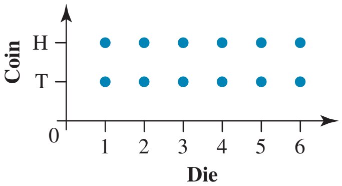
e Using the addition formula:

$$\begin{aligned}\Pr(2 \text{ digits or } x > 6) &= \Pr(2 \text{ digits} \cup x > 6)\\ &= \Pr(2 \text{ digits}) + \Pr(x > 6)\\ &\quad - \Pr(2 \text{ digits} \cap x > 6)\\ &= (\Pr(10) + \Pr(12)) + \Pr(x > 6)\\ &\quad -(\Pr(10) + \Pr(12))\\ &= \frac{1}{36} + \frac{1}{36} + \frac{3}{36} - \left(\frac{1}{36} + \frac{1}{36}\right)\\ &= \frac{3}{36}\\ &= \frac{1}{12}\end{aligned}$$

f Using the rule for complementary events:

$$\begin{aligned}\Pr(9') &= 1 - \Pr(9)\\ &= 1 - 0\\ &= 1\end{aligned}$$

20 a i



ii There are 12 equally likely outcomes in the sample space.

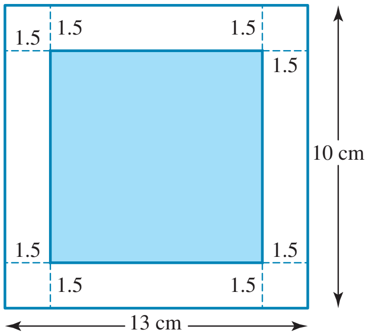
Only one of these outcomes is a 6 on the die and a Head on the coin.

The probability of obtaining a 6 on the die and a Head on the coin is $\frac{1}{12}$.

iii This event occurs for the three outcomes $2T$, $4T$ and $6T$.

So the probability of the event of obtaining an even number together with a tail on the coin is $\frac{3}{12} = \frac{1}{4}$.

b To land completely inside the rectangle, the area in which the centre of the coin may land is a rectangle of edge length $13 - 2 \times 1.5 = 10$ cm and edge width $10 - 2 \times 1.5 = 7$.



Area that the coin could land in for the coin to be completely inside the area is $10 \times 7 = 70$ cm^2.

Total area that the coin could land in is $13 \times 10 = 130$ cm^2.

Therefore, the probability the coin lands completely inside the cardboard area is $\frac{70}{130} = \frac{7}{13}$.

21 a Count the number of times 1 Head was obtained and divide by the total number of trials, 40.

$$\begin{aligned}\Pr(\text{one female}) &= \Pr(1)\\ &= \frac{14}{40}\\ &= \frac{7}{20}\end{aligned}$$

b

$$\begin{aligned}\Pr(\text{more than one female}) &= \Pr(2) + \Pr(3)\\ &= \frac{10}{40} + \frac{6}{40}\\ &= \frac{16}{40}\\ &= \frac{2}{5}\end{aligned}$$

22 a One way to generate the random numbers that lie between 1 and 6 is by using $|\text{int}(6 * r\,\text{and}(\,) + 1)$. Press Enter twenty times to generate 20 of these numbers.

The numbers obtained will vary.

b The results of one simulation were
3244 5556 4335 6253 3213.

Examining the data, there are only 2 sixes among the 20 random numbers.

An estimate of the chance of obtaining a six is $\frac{2}{20} = \frac{1}{10}$.

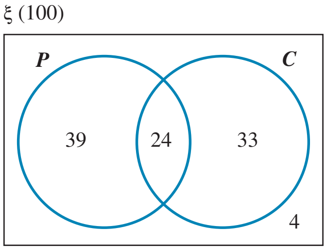
c The probability of a six is $\frac{1}{6}$. This represents the long-term proportion of sixes obtained from many trials.

So, to improve the estimate obtained from 20 trials, the number of trials should be increased from 20 to a much larger number.

23 a To find the number of students who study both chemistry and physics:

$63 + 57 + 4 = 124$. But there are only 100 students asked. Therefore, $124 - 100 = 24$.

Therefore, 24 students study both chemistry and physics.



$$\begin{aligned}\Pr(P \text{ or } C \text{ but not both}) &= \frac{39}{100} + \frac{33}{100}\\ &= \frac{72}{100}\\ &= \frac{18}{25}\end{aligned}$$

b $$\begin{aligned}\Pr(\text{both}) &= \Pr(P \cap C)\\ &= \frac{24}{100}\\ &= \frac{6}{25}\end{aligned}$$

c $\frac{6}{25} \times \frac{1200}{1} = 288$

288 students are likely to study both physics and chemistry.

d $$\begin{aligned}\Pr(\text{both students study both}) &= \Pr(P \cap C) \times \Pr(P \cap C)\\ &= \frac{24}{100} \times \frac{24}{100}\\ &= \frac{6}{25} \times \frac{6}{25}\\ &= \frac{36}{625}\\ &= 0.0576\end{aligned}$$

e Let A = the first student.

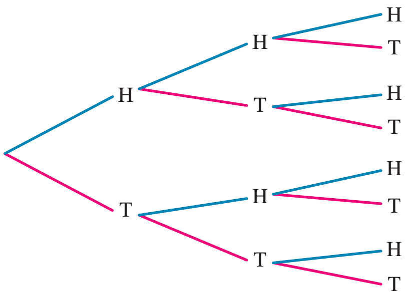
Let B = the second student.

$$\begin{aligned}\Pr(\text{each student studies just one}) &= \Pr(A \rightarrow (P \cap C') \cup (P' \cap C)) \times \Pr(B \rightarrow (P \cap C') \cup (P' \cap C))\\ &= \left(\frac{39}{100} + \frac{33}{100}\right) \times \left(\frac{39}{100} + \frac{33}{100}\right)\\ &= \frac{324}{625}\\ &= 0.5184\end{aligned}$$

f $$\begin{aligned}\Pr(\text{one student studies neither}) &= \Pr(A \text{ student neither or } B \text{ studies neither})\\ &= \Pr((A \rightarrow N) \cup (B \rightarrow N))\\ &= \Pr(A \rightarrow N) + \Pr(B \rightarrow N) - \Pr((A \rightarrow N) \cap (B \rightarrow N))\\ &= \frac{4}{100} + \frac{4}{100} - \left(\frac{4}{100} \times \frac{4}{100}\right)\\ &= \frac{8}{100} - \frac{1}{625}\\ &= \frac{49}{625}\\ &= 0.0784\end{aligned}$$

7.2 Exam questions

1 a



Total number of outcomes = 8. [1 mark]

b There is only one outcome for HHT.

$\therefore \Pr(\text{HHT}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{8}$ [1 mark]

c $\Pr(\text{HHH}) + \Pr(\text{TTT}) = \frac{1}{8} + \frac{1}{8}$

$= \frac{1}{4}$ [1 mark]

2 Completed probability table:

	B	B''	
A	0.2	0.2	0.4
A''	0.4	0.2	0.6
	0.6	0.4	1

Read the other results from the completed probability table.

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

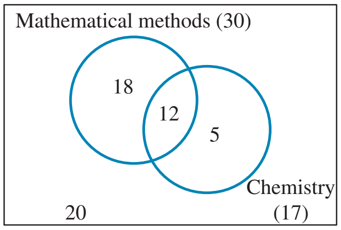
$\Pr(A \cup B) = 0.6 + 0.4 - 0.2$

$= 0.8$

$\therefore \Pr(A \cup B) \neq 0.9$

The correct answer is **C**.

3 a $\xi = 55$ [1 mark]



b From the diagram, studying Mathematical Methods but not Chemistry: $n = 18$.

$\Pr(\text{MM not C}) = \frac{18}{55}$ [1 mark]

c $\Pr(\text{not MM not C}) = \frac{20}{55}$

$= \frac{4}{11}$ [1 mark]

7.3 Conditional probability

7.3 Exercise

1 The bag has 3 red balls, 4 purple balls and 2 yellow balls, giving a total of 9 balls.

a When one yellow ball is removed, the bag contains 3 red balls, 4 purple balls and 1 yellow ball, giving a total of 8 balls.

$\Pr(R_2|Y_1) = \frac{3}{8}$.

b When one yellow ball is removed, the bag contains 3 red balls, 4 purple balls and 1 yellow ball, giving a total of 8 balls.

$\Pr(Y_2|Y_1) = \frac{1}{8}$

c When one red ball is removed, the bag contains 2 red balls, 4 purple balls and 2 yellow ball, giving a total of 8 balls.

$\Pr(P_2|R_1) = \frac{4}{8} = \frac{1}{2}$

d The first ball removed is not red, so there are still 3 red balls among the 8 balls that remain in the bag.

$\Pr(R_2|R'_1) = \frac{3}{8}$

2 13 students study Art (A), 16 study Biology (B), and 9 study both Art and Biology ($A \cap B$).

$n(A) = 13,\ n(B) = 16,\ n(A \cap B) = 9$ and $n(\xi) = 20$.

a $\Pr(A) = \frac{13}{20}$ is a correct statement, since

$\Pr(A) = \frac{n(A)}{n(\xi)} = \frac{13}{20}$. True

b–d $\Pr(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{9}{16}$, so statements **b** and **c** are false and **d** is true.

e–f $\Pr(B|A) = \frac{n(B \cap A)}{n(A)}$

$= \frac{n(A \cap B)}{n(A)}$

$= \frac{9}{13}$

Hence, statement **e** is true and statement **f** is false.

3 a $\Pr(C|D) = \frac{3}{5}$ and $\Pr(D) = \frac{1}{4}$

$\Pr(C|D) = \frac{\Pr(C \cap D)}{\Pr(D)}$

Rearranging, $\Pr(C \cap D) = \Pr(C|D) \times \Pr(D)$.

Substitute the given probabilities to calculate $\Pr(C \cap D)$.

$\Pr(C \cap D) = \Pr(C|D) \times \Pr(D)$

$= \frac{3}{5} \times \frac{1}{4}$

$= \frac{3}{20}$

b $\Pr(N|M) = 0.375$, $\Pr(M|N) = 0.6$ and $\Pr(M) = 0.8$

First consider $\Pr(N|M)$.

$\Pr(N|M) = \frac{\Pr(N \cap M)}{\Pr(M)}$

Therefore, $0.375 = \frac{\Pr(N \cap M)}{0.8}$.

Rearranging,

$M \cap N = N \cap M = 0.375 \times 0.8$

$\Pr(\cap M) = 0.3000$

$\Pr(M \cap N) = 0.3$

Now consider $\Pr(M|N) = 0.6$.

$\frac{\Pr(M \cap N)}{\Pr(N)} = 0.6$

Since $M \cap N = N \cap M$, $\Pr(M \cap N) = 0.3$.

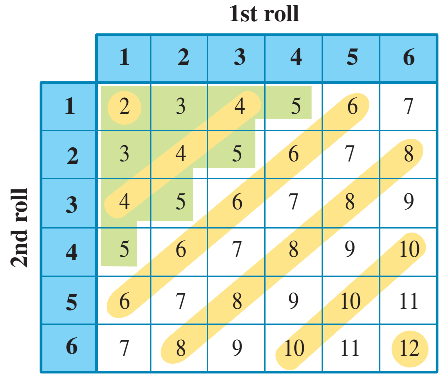
Therefore,

$$\frac{0.3}{\Pr(N)} = 0.6$$

$$\frac{0.3}{0.6} = \Pr(N)$$

$$\Pr(N) = 0.5$$

4 Make a table for the sum of the two numbers rolled:



1st roll

2nd roll	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

🟨 = Even sums circled

🟩 = Less than 6

Let x = the sum of the two numbers

$$\Pr(x < 6|x'' \text{ is even}) = \frac{\Pr(x < 6 \cap x \text{ is even})}{\Pr(x \text{ is even})}$$

$$= \frac{\Pr(x = 2) + \Pr(x = 4)}{\Pr(x = 2) + \Pr(x = 4) + \Pr(x = 6) + \Pr(x = 8) + \Pr(x = 10) + \Pr(x = 12)}$$

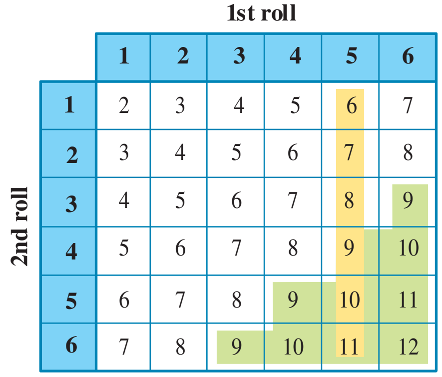
$$= \frac{\frac{4}{36}}{\frac{18}{36}}$$

$$= \frac{4}{36} \times \frac{36}{18}$$

$$= \frac{4}{18}$$

$$= \frac{2}{9}$$

5 Make a table for the sum of the two numbers rolled:



1st roll

2nd roll	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

🟩 = Sum greater than 8

🟨 = 5 on first dice

Let x = the sum of the two numbers.

$$\begin{aligned}
\Pr(x > 8|5 \text{ on first roll}) &= \frac{\Pr(x > 8 \cap 5 \text{ on first roll})}{\Pr(5 \text{ on first roll})}\\
&= \frac{\Pr(5,4) + \Pr(5,5) + \Pr(5,6)}{\Pr(5,1) + \Pr(5,2) + \Pr(5,3) + \Pr(5,4) + \Pr(5,5) + \Pr(5,6)}\\
&= \frac{\frac{3}{36}}{\frac{6}{36}}\\
&= \frac{3}{36} \times \frac{36}{6}\\
&= \frac{3}{6}\\
&= \frac{1}{2}
\end{aligned}$$

6 **a** $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$

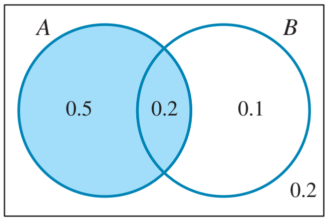
$$\begin{aligned}
&= 0.7 + 0.3 - 0.8\\
&= 1 - 0.8\\
&= 0.2
\end{aligned}$$

b
$$\begin{aligned}
\Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\\
&= \frac{0.2}{0.3}\\
&= \frac{2}{3}
\end{aligned}$$

c
$$\begin{aligned}
\Pr(B|A) &= \frac{\Pr(B \cap A)}{\Pr(A)}\\
&= \frac{\Pr(A \cap B)}{\Pr(A)}\\
&= \frac{0.2}{0.7}\\
&= \frac{2}{7}
\end{aligned}$$

d Using the rule for complimentary events:

$$\begin{aligned}
\Pr(B') &= 1 - \Pr(B)\\
&= 1 - 0.3\\
&= 0.7
\end{aligned}$$



$\Pr(A \cap B') = 0.5$

🟦 = A' ☐ = B

$$\begin{aligned}
\Pr(A|B') &= \frac{\Pr(A \cap B')}{\Pr(B')}\\
&= \frac{0.5}{0.7}\\
&= \frac{5}{7}
\end{aligned}$$

7 a Using the addition formula:

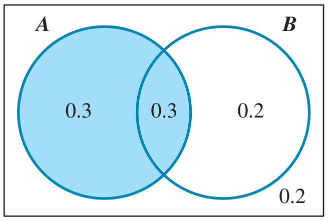
$$\begin{aligned}\Pr(A \cap B) &= \Pr(A) + \Pr(B) - \Pr(A \cup B)\\ &= 0.6 + 0.5 - 0.8\\ &= 1.1 - 0.8\\ &= 0.3\end{aligned}$$

b $$\begin{aligned}\Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\\ &= \frac{0.3}{0.5}\\ &= \frac{3}{5}\end{aligned}$$

c $$\begin{aligned}\Pr(B|A) &= \frac{\Pr(B \cap A)}{\Pr(A)}\\ &= \frac{\Pr(A \cap B)}{\Pr(A)}\\ &= \frac{0.3}{0.6}\\ &= \frac{3}{6}\\ &= \frac{1}{2}\end{aligned}$$

d Using the rule for complimentary events:

$$\begin{aligned}\Pr(B') &= 1 - \Pr(B)\\ &= 1 - 0.5\\ &= 0.5\end{aligned}$$



$\Pr(A \cap B') = 0.3$

🟦 $= A$ □ $= B'$

$$\begin{aligned}\Pr(A|B') &= \frac{\Pr(A \cap B')}{\Pr(B')}\\ &= \frac{0.3}{0.5}\\ &= \frac{3}{5}\end{aligned}$$

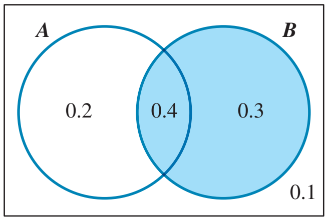
8 a Using the addition formula:

$$\begin{aligned}\Pr(A \cap B) &= \Pr(A) + \Pr(B) - \Pr(A \cup B)\\ \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= 0.6 + 0.7 - 0.4\\ &= 1.3 - 0.4\\ &= 0.9\end{aligned}$$

b $$\begin{aligned}\Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\\ &= \frac{0.4}{0.7}\\ &= \frac{4}{7}\end{aligned}$$

c Using the rule for complimentary events:

$$\begin{aligned}\Pr(A') &= 1 - \Pr(A)\\ &= 1 - 0.6\\ &= 0.4\end{aligned}$$



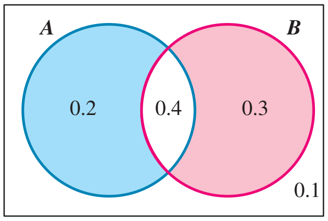
$\Pr(B \cap A') = 0.3$

□ $= A'$ 🟦 $= B$

$$\begin{aligned}\Pr(B|A') &= \frac{\Pr(B \cap A')}{\Pr(A')}\\ &= \frac{0.3}{0.4}\\ &= \frac{3}{4}\end{aligned}$$

d Using the rule for complimentary events:

$$\begin{aligned}\Pr(B') &= 1 - \Pr(B)\\ &= 1 - 0.7\\ &= 0.3\end{aligned}$$



$\Pr(A' \cap B') = 0.1$

🟦 $= A'$ 🟥 $= B'$

$$\begin{aligned}\Pr(A'|B') &= \frac{\Pr(A' \cap B')}{\Pr(B')}\\ &= \frac{0.1}{0.3}\\ &= \frac{1}{3}\end{aligned}$$

9 a From reading the table,

$$\begin{aligned}\Pr(B' \cap C') &= \frac{n(B' \cap C')}{n(\xi)}\\ &= \frac{20}{100}\\ &= \frac{1}{5}\end{aligned}$$

b $$\begin{aligned}\Pr(B'|C') &= \frac{n(B' \cap C')}{n(C')}\\ &= \frac{20}{36}\\ &= \frac{5}{9}\end{aligned}$$

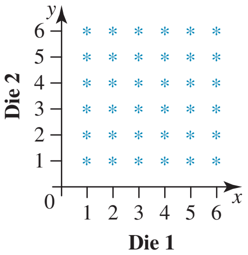
c $$\begin{aligned}\Pr(C|B) &= \frac{n(C \cap B)}{n(B)}\\ &= \frac{28}{44}\\ &= \frac{7}{11}\end{aligned}$$

d $\Pr(B) = \dfrac{n(B)}{n(\xi)}$

$= \dfrac{44}{100}$

$= \dfrac{11}{25}$

10 a



Let A be the event of obtaining a sum of 8.

$\therefore A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$. There are 36 possible outcomes.

$\Pr(A) = \dfrac{n(A)}{n(\xi)}$

$= \dfrac{5}{36}$

b Let B be the event of obtaining two numbers that are the same. $\therefore B = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.

$\Pr(A|B') = \dfrac{n(A \cap B')}{n(B')}$

$= \dfrac{4}{30}$

$= \dfrac{2}{15}$

c Obtaining the sum of 8 but the numbers are not the same is $A \cap B'$.

$\Pr(A \cap B') = \dfrac{n(A \cap B')}{n(\xi)}$

$= \dfrac{4}{36}$

$= \dfrac{1}{9}$

d $\Pr(B'|A) = \dfrac{n(B' \cap A)}{n(A)}$

$= \dfrac{4}{5}$

11 a Given: $\Pr(A') = 0.6$, $\Pr(B|A) = 0.3$ and $\Pr(B) = 0.5$

For complementary events,

$\Pr(A) = 1 - \Pr(A')$

$\therefore \Pr(A) = 1 - 0.6$

$\therefore \Pr(A) = 0.4$

$\Pr(B|A) = \dfrac{\Pr(B \cap A)}{\Pr(A)}$

$\therefore 0.3 = \dfrac{\Pr(B \cap A)}{0.4}$

$\therefore \Pr(B \cap A) = 0.3 \times 0.4$

$\therefore \Pr(A \cap B) = \Pr(B \cap A) = 0.12$

$\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

$= \dfrac{0.12}{0.5}$

$= \dfrac{6}{25}$

b Let G be the event a green ribbon is chosen.

$\Pr(G) = \dfrac{n(G)}{n(\xi)}$

For the first pick, $n(G) = 8$ and $n(\xi) = 12$; therefore, $\Pr(G) = \dfrac{8}{12}$.

For the second pick, if one green ribbon has been removed, $n(G) = 7$ and $n(\xi) = 11$; therefore, $\Pr(G) = \dfrac{7}{11}$.

For the third pick, if two green ribbons have been removed, $n(G) = 6$ and $n(\xi) = 10$; therefore, $\Pr(G) = \dfrac{6}{10}$.

$\Pr(G \cap G \cap G) = \dfrac{8}{12} \times \dfrac{7}{11} \times \dfrac{6}{10}$

$= \dfrac{14}{55}$

12 a Given $\Pr(A) = 0.61$, $\Pr(B) = 0.56$ and $\Pr(A \cup B) = 0.81$,

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\therefore \Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$

$= 0.61 + 0.56 - 0.81$

$= 0.36$

$\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

$= \dfrac{0.36}{0.56}$

$= \dfrac{9}{14}$

b $\Pr(A|A \cup B) = \dfrac{\Pr(A \cap (A \cup B))}{\Pr(A \cup B)}$

$= \dfrac{\Pr(A)}{\Pr(A \cup B)}$

$= \dfrac{0.61}{0.81}$

$= \dfrac{61}{81}$

c $\Pr(A|A \cap B) = \dfrac{\Pr(A \cap (A \cap B))}{\Pr(A \cap B)}$

$= \dfrac{\Pr(A \cap B)}{\Pr(A \cap B)}$

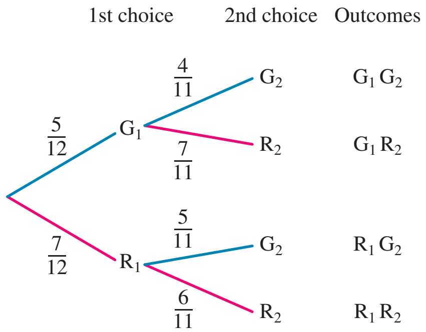
$= 1$

If $(A \cap B)$ has occurred, then A is certain to have occurred.

13 a If a red jube has already been chosen first, there remains in the box 6 red jubes and 5 green jubes.

$\therefore \Pr(G_2|R_1) = \dfrac{5}{11}$

b



c $\Pr(G_1 \cap R_2) = \Pr(G_1) \times \Pr(R_2|G_1)$

$$= \frac{5}{12} \times \frac{7}{11}$$

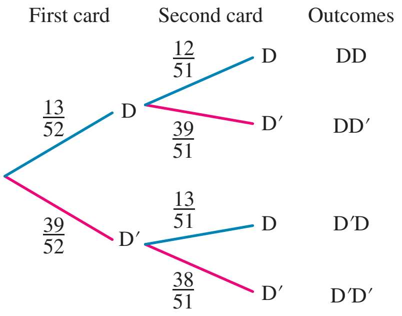
$$= \frac{35}{132}$$

d $\Pr((G_1 \cap G_2) \cup (R_1 \cap R_2)) = \Pr(G_1 \cap G_2) + \Pr(R_1 \cap R_2)$

$$= \frac{5}{12} \times \frac{4}{11} + \frac{7}{12} \times \frac{6}{11}$$

$$= \frac{31}{66}$$

14 a



If one diamond is selected then the number of cards and the number of diamonds decreases by one for the second card.

$$\Pr(DD) = \frac{13}{52} \times \frac{12}{51}$$

$$= \frac{1}{17}$$

b $\Pr(\text{at least one diamond}) = \Pr(D \geq 1)$

$$= 1 - \Pr(D = 0)$$

$$= 1 - \Pr(D'D')$$

$$= 1 - \left(\frac{39}{52} \times \frac{38}{51}\right)$$

$$= 1 - \frac{19}{34}$$

$$= \frac{15}{34}$$

c $\Pr(DDI|D \geq 1) = \Pr(DD \cap\ D \geq 1)$

$$= \frac{\Pr(DD)}{\Pr(D \geq 1)}$$

$$= \frac{1}{\frac{7}{\frac{15}{34}}}$$

$$= \frac{1}{17} \times \frac{34}{15}$$

$$= \frac{2}{15}$$

d $\Pr(DD|\text{first card } D) = \dfrac{\Pr(DD \cap \text{first card } D)}{\Pr(\text{first card } D)}$

$$= \frac{\Pr(DD)}{\Pr(DD) + \Pr(DD')}$$

$$= \frac{\frac{1}{17}}{\frac{1}{17} + \left(\frac{13}{52} \times \frac{39}{51}\right)}$$

$$= \frac{\frac{1}{17}}{\frac{1}{17} + \frac{13}{68}}$$

$$= \frac{\frac{1}{17}}{\frac{17}{68}}$$

$$= \frac{1}{17} \times \frac{68}{17}$$

$$= \frac{68}{289}$$

$$= \frac{4}{17}$$

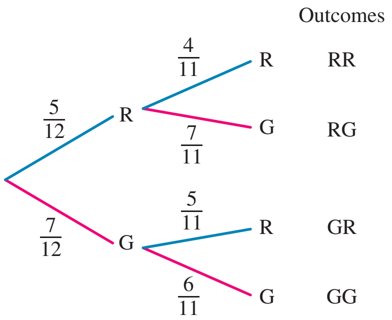
Alternatively,

$$\Pr(D1) = \frac{13}{52} = \frac{1}{4}$$

$$\therefore Pr(DD|D1) = \frac{1}{17} \div \frac{1}{4}$$

$$= \frac{4}{17}$$

15 a



$$\Pr(GG) = \frac{7}{12} \times \frac{6}{11}$$

$$= \frac{42}{132}$$

$$= \frac{7}{22}$$

b $\Pr(G \geq 1) = 1 - \Pr(RR)$

$$= 1 - \left(\frac{5}{12} \times \frac{4}{11}\right)$$

$$= 1 - \frac{20}{132}$$

$$= \frac{112}{132}$$

$$= \frac{28}{33}$$

c $\Pr(GG|G \geq 1) = \dfrac{\Pr(GG \cap G \geq 1)}{\Pr(G \geq 1)}$

$$= \frac{\Pr(GG)}{\Pr(G \geq 1)}$$

$$= \frac{\frac{7}{22}}{\frac{28}{33}}$$

$$= \frac{7}{22} \times \frac{33}{28}$$

$$= \frac{1}{2} \times \frac{3}{4}$$

$$= \frac{3}{8}$$

d $\Pr(G \text{ first}|\text{different colours}) = \dfrac{\Pr(G \text{ first} \cap \text{different colours})}{\Pr(\text{different colours})}$

$$= \frac{\Pr(GR)}{\Pr(GR) + \Pr(RG)}$$

$$= \frac{\frac{7}{12} \times \frac{5}{11}}{\left(\frac{7}{12} \times \frac{5}{11}\right) + \left(\frac{5}{12} \times \frac{7}{11}\right)}$$

$$= \frac{\frac{35}{132}}{\frac{35}{132} + \frac{35}{132}}$$

$$= \frac{\frac{35}{132}}{2\left(\frac{35}{132}\right)}$$

$$= \frac{1}{2}$$

16 **a** Let A = has the disease.
Let B = positive test result.

	A	A′	
B	23	7	30
B′	4	66	70
	27	73	100

$\Pr(A') = \frac{73}{100}$ or 0.73

b $\Pr(B \cap A') = \frac{7}{100}$

c $$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{\frac{23}{100}}{\frac{30}{100}}$$

$$= \frac{23}{100} \times \frac{100}{30}$$

$$= \frac{23}{30}$$

d $$\Pr(A'|B') = \frac{\Pr(A' \cap B')}{\Pr(B')}$$

$$= \frac{\frac{66}{100}}{\frac{70}{100}}$$

$$= \frac{66}{100} \times \frac{100}{70}$$

$$= \frac{66}{70}$$

$$= \frac{33}{35}$$

17 **a** $\Pr(S') = \Pr(FS') + \Pr(MS')$

$$= \frac{224}{500} + \frac{203}{500}$$

$$= \frac{427}{500}$$

b $\Pr(M) = \Pr(MS) + \Pr(MS')$

$$= \frac{32}{500} + \frac{203}{500}$$

$$= \frac{235}{500}$$

$$= \frac{47}{100}$$

c $$\Pr(F|S') = \frac{\Pr(F \cap S')}{\Pr(S')}$$

$$= \frac{\frac{224}{500}}{\frac{427}{500}}$$

$$= \frac{224}{500} \times \frac{500}{427}$$

$$= \frac{224}{427}$$

$$= \frac{32}{61}$$

18 **a** $\Pr(O) = \Pr(OH) + \Pr(OH')$

$$= \frac{82}{1000} + \frac{185}{1000}$$

$$= \frac{267}{1000}$$

b $\Pr(H) = \Pr(OH) + \Pr(O'H)$

$$= \frac{82}{1000} + \frac{175}{1000}$$

$$= \frac{257}{1000}$$

c $$\Pr(H|O) = \frac{\Pr(H \cap O)}{\Pr(O)}$$

$$= \frac{\frac{82}{1000}}{\frac{267}{1000}}$$

$$= \frac{82}{1000} \times \frac{1000}{267}$$

$$= \frac{82}{267}$$

d Using the rule for complementary events:

$$\Pr(H') = 1 - \frac{257}{1000}$$

$$= \frac{743}{1000}$$

$$\Pr(O|H') = \frac{\Pr(O \cap H')}{\Pr(H')}$$

$$= \frac{\frac{185}{1000}}{\frac{743}{1000}}$$

$$= \frac{185}{1000} \times \frac{1000}{743}$$

$$= \frac{185}{743}$$

Alternatively,

$$\Pr(O|H') = \frac{\Pr(O \cap H')}{\Pr(H')} = \frac{185}{743}$$

19 **a** $\Pr(R) = \Pr(RA) + \Pr(RB)$

$$= \frac{25}{400} + \frac{47}{400}$$

$$= \frac{72}{400}$$

$$= \frac{9}{50}$$

b $\Pr(A) = \Pr(RA) + \Pr(R'A)$

$$= \frac{25}{400} + \frac{143}{400}$$
$$= \frac{168}{400}$$
$$= \frac{21}{50}$$

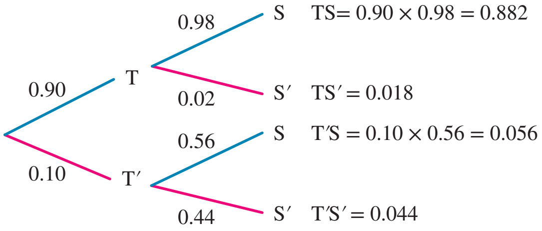
c $\Pr(B/R) = \dfrac{\Pr(B \cap R)}{\Pr(R)}$

$$= \frac{n(B \cap R)}{n(R)}$$
$$= \frac{47}{72}$$

d $\Pr(R'|A) = \dfrac{\Pr(R' \cap A)}{\Pr(A)}$

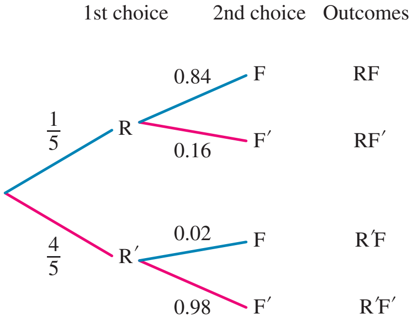
$$= \frac{\frac{143}{400}}{\frac{21}{50}}$$
$$= \frac{143}{400} \times \frac{50}{21}$$
$$= \frac{143}{168}$$

20 **a** Let T = the bus being on time and T'= the bus being late. Let S = Rodney gets to school on time and S'= Rodney gets to school late.



b Pr(Rodney will arrive to school on time)
= 0.882 + 0.056 = 0.938

21 **a**



$\Pr(RF) = 0.2 \times 0.84$
$= 0.168$

b $\Pr(F|R) = \dfrac{\Pr(F \cap R)}{\Pr(R)}$

$$= \frac{\Pr(RF)}{\Pr(RF) + \Pr(RF')}$$
$$= \frac{0.2 \times 0.84}{0.2 \times 0.84 + 0.2 \times 0.16}$$
$$= \frac{0.168}{0.2}$$

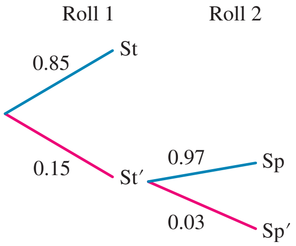
= 0.84 (as shown on the probability tree)

c Incy Wincy makes it to the top, so therefore he does not fall. Therefore, we need to find the probability it is raining given he does not fall.

$\Pr(R|F') = \dfrac{\Pr(R \cap F')}{\Pr(F')}$

$$= \frac{\Pr(R \cap F')}{\Pr(RF') + \Pr(R'F')}$$
$$= \frac{0.2 \times 0.16}{0.2 \times 0.16 + 0.8 \times 0.98}$$
$$= \frac{0.032}{0.032 + 0.784}$$
$$= \frac{0.032}{0.816}$$
$$= \frac{2}{51}$$
$$= 0.0392$$

22 **a**



The tree diagram for each frame is the same, since Richard's probability for obtaining a strike and spare remains the same throughout the club championship. Therefore, frame 2 is not dependent on the previous frame, frame 1.

Pr(both strikes) = 0.85 × 0.85
= 0.7225

b Pr(all down in one frame) = 0.85 + 0.15 × 0.97
= 0.9955

Pr(all down) = 1 − Pr((all down)′)
= [1 − Pr(St′Sp′)] × [1 − Pr(St′Sp′)]
= [1 − (0.15 × 0.03)] × [1 − (0.15 × 0.03)]

Pr(all down in both frame) = 0.9955 × 0.9955
= 0.9910

c $\Pr(\text{1st St}|\text{2nd St}) = \dfrac{\Pr(\text{1st St} \cap \text{2nd St})}{\Pr(\text{2nd St})}$

$$= \frac{\Pr(\text{St St})}{\Pr(\text{2nd St})}$$
$$= \frac{0.85 \times 0.85}{1 \times 0.85}$$
$$= \frac{0.85 \times 0.85}{1 \times 0.85}$$
$$= 0.85$$

7.3 Exam questions

1 $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$
$= 0.7 + 0.3 - 0.8$
$= 0.2$

The correct answer is **D**.

2 $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$$\begin{aligned}\Pr(A \cap B) &= \Pr(A) + \Pr(B) - \Pr(A \cup B)\\ &= 0.7 + 0.3 - 0.8\\ &= 0.2\end{aligned}$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

$$\begin{aligned}\Pr(B|A) &= \frac{0.2}{0.7}\\ &= \frac{2}{7}\end{aligned}$$

The correct answer is **A**.

3

$$\begin{aligned}n\,(\text{Total}) &= 50\\ n\,(\text{class 1 + passed}) &= 15\\ \Pr(\text{class 1}) &= \frac{15}{50}\\ &= \frac{3}{10}\end{aligned}$$

The correct answer is **B**.

7.4 Independence

7.4 Exercise

1 If two events A and B are independent:

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$$\begin{aligned}\text{LHS} &= \Pr(A \cap B)\\ &= \Pr(A) + \Pr(B) - \Pr(A \cup B)\\ &= 0.7 + 0.8 - 0.94\\ &= 1.5 - 0.94\\ &= 0.56\\ \text{RHS} &= \Pr(A) \times \Pr(B)\\ &= 0.7 \times 0.8\\ &= 0.56\\ \text{LHS} &= \text{RHS}\end{aligned}$$

Therefore, since $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, the events A and B are independent.

2 If two events A and B are independent:

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$$\begin{aligned}\text{LHS} &= \Pr(A \cap B)\\ &= \Pr(A) + \Pr(B) - \Pr(A \cup B)\\ &= 0.75 + 0.64 - 0.91\\ &= 1.39 - 0.91\\ &= 0.48\\ \text{RHS} &= \Pr(A) \times \Pr(B)\\ &= 0.75 \times 0.64\\ &= 0.48\\ \text{LHS} &= \text{RHS}\end{aligned}$$

Therefore, since $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, the events A and B are independent.

3 a $\Pr(A) = 0.3$, $\Pr(B) = p$ and $\Pr(A \cap B) = 0.12$

Since events A and B are independent,

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$\therefore 0.12 = 0.3 \times p$

Solve the equation for p

$$\begin{aligned}0.3p &= 0.12\\ p &= \frac{0.12}{0.3}\\ p &= 0.4\end{aligned}$$

b $\Pr(A) = p$, $\Pr(B) = 2p$ and $\Pr(A \cap B) = 0.72$

Since events A and B are independent,

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$$\begin{aligned}0.72 &= p \times 2p\\ 2p^2 &= 0.72\\ p^2 &= 0.36\\ p &= \sqrt{0.36},\ p > 0\\ p &= 0.6\end{aligned}$$

c $\Pr(A) = q$, $\Pr(B) = \frac{1}{5}$ and $\Pr(A \cup B) = \frac{4}{5}$

Since events A and B are independent,

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$\Pr(A \cap B) = q \times \frac{1}{5}$

$\Pr(A \cap B) = \frac{q}{5}$

Using the addition formula,

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$$\begin{aligned}\frac{4}{5} &= q + \frac{1}{5} - \frac{q}{5}\\ 4 &= 5q + 1 - q\\ 3 &= 4q\\ q &= \frac{3}{4}\end{aligned}$$

4 a $\xi = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$

$A = \{\text{TH}, \text{TT}\}$

$B = \{\text{HT}, \text{TH}\}$

$C = \{\text{HH}, \text{HT}, \text{TH}\}$.

b A and B are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$\Pr(A) = \frac{2}{4}$ and $\Pr(B) = \frac{2}{4}$.

Since $A \cap B = \{TH\}$, $\Pr(A \cap B) = \frac{1}{4}$.

Substitute values into the formula $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$$\begin{aligned}\text{LHS} &= \frac{1}{4}\\ \text{RHS} &= \frac{2}{4} \times \frac{2}{4}\\ &= \frac{1}{2} \times \frac{1}{2}\\ &= \frac{1}{4}\end{aligned}$$

Since, the events A and B are independent.

c B and C are independent if $\Pr(B \cap C) = \Pr(B)\Pr(C)$.

$\Pr(B) = \frac{2}{4}$ and $\Pr(C) = \frac{3}{4}$.

Since $B \cap C = \{HT, TH\}$, $\Pr(B \cap C) = \frac{1}{2}$.

Substitute values into the formula $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$$\begin{aligned}\text{LHS} &= \frac{1}{2}\\ \text{RHS} &= \frac{2}{4}\times\frac{3}{4}\\ &= \frac{1}{2}\times\frac{3}{4}\\ &= \frac{3}{8}\end{aligned}$$

Since LHS $\neq$ RHS, the events B and C are not independent.

d $\Pr(B \cup A) = \Pr(B) + \Pr(A) - \Pr(B \cap A)$

Since B and A are independent, $\Pr(B \cap A) = \Pr(B)\Pr(A)$

$\therefore \Pr(B \cup A) = \Pr(B) + \Pr(A) - \Pr(B) \times \Pr(A)$

$\therefore \Pr(B \cup A) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2}$

$\therefore \Pr(B \cup A) = \frac{3}{4}$

5 History: $\Pr(H) = \frac{8}{10}$, Chemistry: $\Pr(C) = \frac{7}{10}$.

a Pr(passing both tests) is $\Pr(H \cap C)$.

As H and C are independent, $\Pr(H \cap C) = Pr(H) \times \Pr(C)$

$$\begin{aligned}\Pr(H \cap C) &= \frac{8}{10}\times\frac{7}{10}\\ &= \frac{56}{100}\\ &= 0.56\end{aligned}$$

The probability that the student passes both tests is 0.56.

b Passes Chemistry but not History

$\Pr(C \cap H') = \Pr(C) \times \Pr(H')$ since the events are independent.

$$\begin{aligned}\Pr(C \cap H') &= \Pr(C) \times (1 - \Pr(H))\\ &= \frac{7}{10}\times\left(1-\frac{8}{10}\right)\\ &= \frac{7}{10}\times\frac{2}{10}\\ &= \frac{14}{100}\\ &= 0.14\end{aligned}$$

The probability that the student passes the Chemistry test but not the History test is 0.14.

c Does not pass either test:

Pr(not passing both tests) is $\Pr(H' \cap C')$.

$$\begin{aligned}\Pr(H' \cap C') &= \Pr(H') \times \Pr(C')\\ &= \left(1-\frac{8}{10}\right)\times\left(1-\frac{7}{10}\right)\\ &= \frac{2}{10}\times\frac{3}{10}\\ &= \frac{6}{100}\\ &= 0.06\end{aligned}$$

The probability the student does not pass either test is 0.06.

d Passes at least one test:

Pr(passing at least one test) = 1− Pr(failing both tests).

From part **c**, $\Pr(H' \cap C') = 0.06$.

The probability of passing at least one test $1 - 0.06 = 0.94$.

6 The probability a patient is cured is $\frac{4}{5}$.

Consider a cure as a success, S, and not cured as a failure, F.

$\Pr(S) = \frac{4}{5}$ and $\Pr(F) = 1 - \frac{4}{5} = \frac{1}{5}$

a The event ‘all three patients are cured’ can be written as SSS.

$$\begin{aligned}\Pr(SSS) &= \Pr(S) \times \Pr(S) \times \Pr(S)\\ &= \frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\\ &= \frac{64}{125}\end{aligned}$$

b The event ‘none of the patients is cured’ can be written as FFF.

$$\begin{aligned}\Pr(FFF) &= \Pr(F) \times \Pr(F) \times \Pr(F)\\ &= \frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}\\ &= \frac{1}{125}\end{aligned}$$

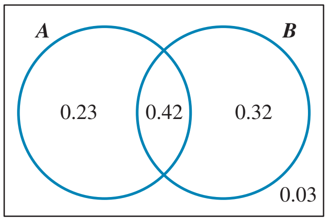
c The event ‘the first patient is cured, the second patient is not cured, the third patient is cured’ can be written as SFS.

$$\begin{aligned}\Pr(SFS) &= \Pr(S) \times \Pr(F) \times \Pr(S)\\ &= \frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}\\ &= \frac{16}{125}\end{aligned}$$

7
$$\begin{aligned}\Pr(A) &= 0.65\\ \Pr(B) &= 0.74\end{aligned}$$

Pr(at least one) = 0.97

Since at least one of the cars is used 97% of the time, then none of the cars are used 3% of the time. Therefore, when forming a Venn diagram for this situation, 0.03 lies outside the circles, and the rest of the 0.97 is distributed with A, B and $A \cap B$.



$$\begin{aligned}\Pr(A) + \Pr(B) &= 0.65 + 0.74\\ &= 1.39\\ \Pr(A \cap B) &= 1.39 - 0.97\\ &= 0.42\\ \Pr(A) \times \Pr(B) &= 0.65 \times 0.74\\ &= 0.481\\ \Pr(A \cap B) &\neq \Pr(A) \times \Pr(B)\end{aligned}$$

Therefore, since $\Pr(A \cap B) \neq \Pr(A) \times \Pr(B)$, the cars A and B are not used independently.

8 **a** A is the event the same number is obtained on each die.

$\therefore A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$.

B is the event the sum of the numbers on each die exceeds 8.

$\therefore B = \{(3,6),\ (4,5),\ (4,6),\ (5,4),\ (5,5),\ (5,6),\ (6,3),\ (6,4),\ (6,5),\ (6,6)\}$.

A and B are mutually exclusive if $n(A \cap B) = 0$.

Since $A \cap B = \{(5,5),(6,6)\}$, A and B are not mutually exclusive.

b A and B are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$\Pr(A) = \frac{6}{36}$ and $\Pr(B) = \frac{10}{36}$.

Since $A \cap B = \{(5,5),(6,6)\}$, $\Pr(A \cap B) = \frac{2}{36}$.

Substitute values into the formula $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$\text{LHS} = \frac{2}{36}$

$$\begin{aligned}\text{RHS} &= \frac{6}{36} \times \frac{10}{36}\\ &= \frac{1}{6} \times \frac{5}{18}\\ &= \frac{5}{108}\end{aligned}$$

Since LHS $\neq$ RHS, the events A and B are not independent.

c i C is the event the sum of the two numbers equals 8.

$\therefore C = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.

B and C are mutually exclusive if $n(B \cap C) = 0$.

Since there are no outcomes common to B and C, they are mutually exclusive.

Note that B is the event the sum of the numbers on each die *exceeds* 8 and C is the event the sum of the two numbers *equals* 8. Hence, they cannot occur simultaneously, so they must be mutually exclusive.

ii B and C are independent if $\Pr(B \cap C) = \Pr(B)\Pr(C)$.

From **i**, $\Pr(B \cap C) = 0$.

Therefore, assuming that neither $\Pr(B)$ nor $\Pr(C)$ are zero, $\Pr(B \cap C) \neq \Pr(B)\Pr(C)$

Since LHS $\neq$ RHS, the events A and B are not independent.

Note that mutually exclusive events cannot be independent, and vice versa.

9 a Let A be the event that Bree sticks to the diet, B be the event that Seiko sticks to the diet and C be the event that Hadiyah sticks to the diet.

$\Pr(A) = 0.4$, $\Pr(B) = 0.9$ and $\Pr(C) = 0.6$

Since the events are independent,

$$\begin{aligned}\Pr(A \cap B \cap C) &= \Pr(A) \times \Pr(B) \times \Pr(C)\\ &= 0.4 \times 0.9 \times 0.6\\ &= 0.216\end{aligned}$$

The probability all three stick to the diet is 0.216.

b $$\begin{aligned}\Pr(A \cap B' \cap C) &= \Pr(A) \times \Pr(B') \times \Pr(C)\\ &= 0.4 \times (1 - 0.9) \times 0.6\\ &= 0.4 \times 0.1 \times 0.6\\ &= 0.024\end{aligned}$$

The probability only Bree and Hadiyah stick to the diet is 0.024.

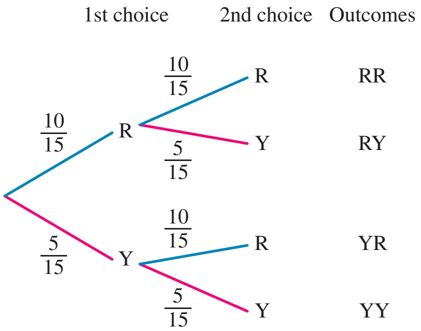
c Using the rule for complementary events,

Pr(at least one does not stick to the diet)

$$\begin{aligned}&= 1 - \Pr(\text{all stick to the diet})\\ &= 1 - \Pr(A \cap B \cap C)\\ &= 1 - 0.4 \times 0.9 \times 0.6\\ &= 1 - 0.216\\ &= 0.784\end{aligned}$$

The probability that at least one person does not stick to the diet is 0.784.

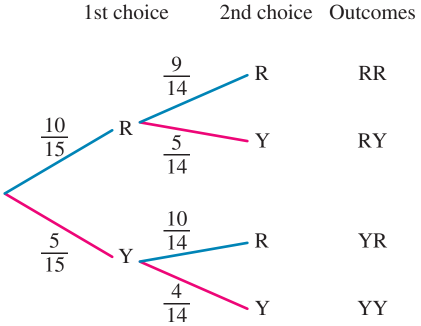
10 a



Let A be the event that one block of each colour is obtained (*with* replacement). $\therefore A = \{RY, YR\}$

$$\begin{aligned}\Pr(A) &= \Pr(RY) + \Pr(YR)\\ &= \frac{10}{15} \times \frac{5}{15} + \frac{5}{15} \times \frac{10}{15}\\ &= \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3}\\ &= \frac{4}{9}\end{aligned}$$

b



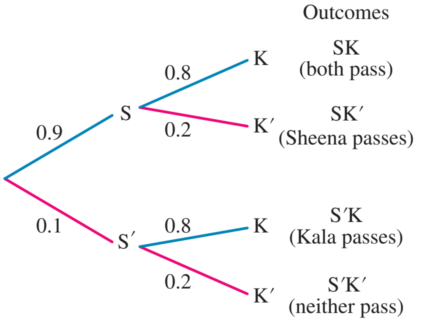
Let B be the event that one block of each colour is obtained (*without* replacement). $\therefore B = \{RY, YR\}$

$$\begin{aligned}\Pr(B) &= \Pr(RY) + \Pr(YR)\\ &= \frac{10}{15} \times \frac{5}{14} + \frac{5}{15} \times \frac{10}{14}\\ &= \frac{50}{210} + \frac{50}{210}\\ &= \frac{10}{21}\end{aligned}$$

c Let C be the event that three blocks of the same colour are obtained (*with* replacement). $\therefore C = \{RRR, YYY\}$

$$\begin{aligned}\Pr(C) &= \Pr(RRR) + \Pr(YYY)\\ &= \frac{10}{15} \times \frac{10}{15} \times \frac{10}{15} + \frac{5}{15} \times \frac{5}{15} \times \frac{5}{15}\\ &= \frac{1000}{3375} + \frac{125}{3375}\\ &= \frac{1}{3}\end{aligned}$$

11 a



$$\begin{aligned}\Pr(\text{both pass}) &= 0.9 \times 0.8\\ &= 0.72\end{aligned}$$

b $$\begin{aligned}\Pr(\text{at least one passes}) &= 1 - \Pr(\text{neither pass})\\ &= 1 - 0.1 \times 0.2\\ &= 1 - 0.02\\ &= 0.98\end{aligned}$$

c $$\begin{aligned}\Pr(\text{only one passes}|\text{S passes}) &= \frac{\Pr(\text{only one passes} \cap \text{S passes})}{\Pr(\text{S passes})}\\ &= \frac{\Pr(SK')}{\Pr(SK') + \Pr(SK)}\\ &= \frac{0.9 \times 0.2}{0.9 \times 0.2 + 0.9 \times 0.8}\\ &= \frac{0.18}{0.18 + 0.72}\\ &= \frac{0.18}{0.9}\\ &= \frac{18}{90}\\ &= \frac{1}{5} \text{ or } 0.2\end{aligned}$$

12 a $$\begin{aligned}\Pr(\text{age} < 25) &= \frac{8 + 30 + 7}{200}\\ &= \frac{45}{200}\\ &= \frac{9}{40}\end{aligned}$$

b $$\begin{aligned}\Pr(\text{at least one violation}) &= 1 - \Pr(\text{zero violations})\\ &= 1 - \frac{8 + 47 + 45 + 20}{200}\\ &= 1 - \frac{120}{200}\\ &= \frac{80}{200}\\ &= \frac{2}{5}\end{aligned}$$

c $$\begin{aligned}\Pr(\text{only one violation}|\text{at least one violation}) &= \frac{\Pr(\text{only one violation} \cap \text{at least one violation})}{\Pr(\text{at least one violation})}\\ &= \frac{\Pr(\text{only one violation})}{\Pr(\text{at least one violation})}\\ &= \frac{\frac{30+15+18+5}{200}}{\frac{2}{5}}\\ &= \frac{68}{200} \times \frac{5}{2}\\ &= \frac{17}{20}\end{aligned}$$

d *Note:* If the person is 38, they are in the 25−45 age group range.
Therefore, look in the 25−45 row and 0 violations column.

$$\Pr(\text{38 and no violations}) = \frac{47}{200}$$

e $$\begin{aligned}\Pr(\text{age} < 25|\text{2 violations}) &= \frac{\Pr(x < 25 \cap \text{2 violations})}{\Pr(\text{2 violations})}\\ &= \frac{\frac{7}{200}}{\frac{7+2+3}{200}}\\ &= \frac{7}{200} \times \frac{200}{12}\\ &= \frac{7}{12}\end{aligned}$$

13 **a** Since A and B are independent,

$\Pr(A|B) = \Pr(A)$.

Given $\Pr(A|B) = \frac{4}{5}$

$\therefore \Pr(A) = \frac{4}{5}$

b $\Pr(B|A) = \Pr(B)$

Since A and B are independent:

Given $\Pr(B) = \frac{2}{3}$

$\therefore \Pr(B|A) = \frac{2}{3}$

c Using the formula for independence:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$
$$\Pr(A \cap B) = \frac{4}{5} \times \frac{2}{3}$$
$$\Pr(A \cap B) = \frac{8}{15}$$

d Using the addition formula:

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= \frac{4}{5} + \frac{2}{3} - \frac{8}{15}\\ &= \frac{12}{15} + \frac{10}{15} - \frac{8}{15}\\ &= \frac{22}{15} - \frac{8}{15}\\ &= \frac{14}{15}\end{aligned}$$

14 **a** Using the formula for conditional probability:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$
$$\frac{1}{3} = \frac{\Pr(A \cap B)}{\frac{3}{5}}$$
$$\frac{1}{3} = \Pr(A \cap B) \times \frac{5}{3}$$
$$\Pr(A \cap B) = \frac{1}{3} \times \frac{3}{5}$$
$$\Pr(A \cap B) = \frac{1}{5}$$

b Using the addition formula:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$\begin{aligned}\Pr(A) &= \Pr(A \cup B) - \Pr(B) + \Pr(A \cap B)\\ &= \frac{23}{30} - \frac{3}{5} + \frac{1}{5}\\ &= \frac{23}{30} - \frac{2}{5}\\ &= \frac{23}{30} - \frac{12}{30}\\ &= \frac{11}{30}\end{aligned}$$

c If two events A and B are independent:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$
$$\text{LHS} = \frac{1}{5}$$
$$\begin{aligned}\text{RHS} &= \Pr(A) \times \Pr(B)\\ &= \frac{11}{30} \times \frac{3}{5}\\ &= \frac{11}{50}\end{aligned}$$
$$\text{LHS} \neq \text{RHS}$$

Therefore, since $\Pr(A \cap B) \neq \Pr(A) \times \Pr(B)$, the events A and B are not independent.

15 Given events A and B are independent,

$$\begin{aligned}\Pr(A|B') &= \frac{\Pr(A \cap B')}{\Pr(B')} = 0.6\\ &= \frac{\Pr(A) \times \Pr(B')}{\Pr(B')} = 0.6\\ &= \Pr(A) = 0.6\end{aligned}$$

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ 0.8 &= \Pr(A) + \Pr(B) - \Pr(A) \times \Pr(B)\\ 0.8 &= 0.6 + x - 0.6 \times x\\ 0.5 &= x\end{aligned}$$

$\Rightarrow \Pr(B) = 0.5$

16 Let Y = age 15 to 30

Let T = TV

If two events Y and T are independent:

$$\Pr(Y \cap T) = \Pr(Y) \times \Pr(T)$$
$$\begin{aligned}\text{LHS} &= \frac{95}{600}\\ &= \frac{19}{120}\end{aligned}$$
$$\begin{aligned}\text{RHS} &= \Pr(Y) \times \Pr(T)\\ &= \frac{290}{600} \times \frac{270}{600}\\ &= \frac{29}{60} \times \frac{9}{20}\\ &= \frac{87}{400}\end{aligned}$$
$$\text{LHS} \neq \text{RHS}$$

Therefore, since $\Pr(Y \cap T) \neq \Pr(Y) \times \Pr(T)$, the events P and T are not independent.

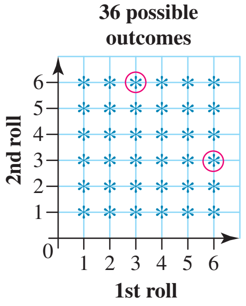
17 If two events P and T are independent:

$$\Pr(P \cap T) = \Pr(P) \times \Pr(T)$$
$$\begin{aligned}\text{LHS} &= \frac{34}{100}\\ &= \frac{17}{50}\end{aligned}$$
$$\begin{aligned}\text{RHS} &= \Pr(P) \times \Pr(T)\\ &= \frac{58}{100} \times \frac{50}{100}\\ &= \frac{29}{50} \times \frac{1}{2}\\ &= \frac{29}{100}\end{aligned}$$
$$\text{LHS} \neq \text{RHS}$$

Therefore, since $\Pr(P \cap T) \neq \Pr(P) \times \Pr(T)$, the events P and T are not independent.

18 **a** $\Pr(A) = \frac{1}{6}$

$\Pr(B) = \frac{1}{6}$



$$\Pr(A \cap B) = \frac{2}{36}$$
$$= \frac{1}{18}$$

If two events A and B are independent:

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$\text{LHS} = \dfrac{1}{18}$

$$\text{RHS} = \Pr(A) \times \Pr(B)$$
$$= \frac{1}{6} \times \frac{1}{6}$$
$$\frac{1}{36}$$

$\text{LHS} \neq \text{RHS}$

Therefore, since $\Pr(A \cap B) \neq \Pr(A) \times \Pr(B)$, the events A and B are not independent.

b Similarly, B and C are not independent.

c Similarly, A and C are not independent.

7.4 Exam questions

1 $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
$$= 0.4 \times 0.5$$
$$= 0.2$$
$$\Pr = (A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 0.4 + 0.5 - 0.2$$
$$= 0.7$$

The correct answer is **D**.

2 Independent events:
$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$
$$= 0.7 \times 0.8$$
$$= 0.56$$

The correct answer is **C**.

3 a $\Pr(B) = 0.8, \Pr(S) = 0.7, \Pr(F) = 0.4$

$\Pr(S \cap B \cap F) = \Pr(S) \times \Pr(B) \times \Pr(F)$

$\Pr(S \cap B \cap F) = 0.7 \times 0.8 \times 0.4$

$= 0.224$ [1 mark]

b Calculate:

$1 - \Pr(\text{none swims under 1 minute})$

$\Pr(B') = 0.2, \Pr(S') = 0.3, \Pr(F') = 0.6$

$\Pr(S' \cap B' \cap F') = \Pr(S') \times \Pr(B') \times \Pr(F')$

$\Pr(S' \cap B' \cap F') = 0.2 \times 0.3 \times 0.6$

$= 0.036$ [1 mark]

Probability that at least one swimmer will break 1 minute:

$1 - \Pr(S' \cap B' \cap F') = 1 - 0.036$

$= 0.964$ [1 mark]

7.5 Counting techniques

7.5 Exercise

1 Using the multiplication principle, three students can be seated in a row in $3 \times 2 \times 1 = 6$ ways.

2 Of the five digits 4, 5, 6, 7, 8, three are even. One of these must be chosen as the last digit of the two-digit number required to be formed.

	3

There are four remaining digits from which the first digit can be chosen.

4	3

Using the multiplication principle, the number of even two-digit numbers that can be formed is $4 \times 3 = 12$.

3 a SAGE contains four letters.

Four letters can be arranged in $4 \times 3 \times 2 \times 1$ ways or 4! ways.

The number of possible arrangements is 24.

b THYME contains five letters.

As the letter T is required to be in the first position, there is only one choice for the first position.

1				

There are then 4 choices for the second position, 3 choices for the third position, 2 choices for the fourth position and 1 choice for the fifth position.

1	4	3	2	1

Using the multiplication principle, the number of arrangements is $1 \times 4 \times 3 \times 2 \times 1 = 24$.

4 Form two- or three-digit numbers from $\{9, 8, 5, 3\}$ without repetition.

Consider the two-digit numbers.

There are 4 choices for the first position, which then leaves 3 choices for the second position.

4	3

Using the multiplication principle, the number of possible two-digit numbers is $4 \times 3 = 12$.

Consider the three-digit numbers.

4	3	2

There are 4 choices for the first position, which leaves 3 choices for the second position, which leaves 2 choices for the third position.

Using the multiplication principle, the number of possible three-digit numbers is $4 \times 3 \times 2 = 24$.

Then, using the addition principle, the number of two- or three-digit numbers that can be formed is $12 + 24 = 36$

5 Bendigo (B), Castlemaine (C), Newstead (N), Lockwood South (L), Maldon (M)

The driving routes are $B \rightarrow C \rightarrow M$ or $B \rightarrow N \rightarrow M$ or $B \rightarrow L \rightarrow M$.

There are 3 driving routes.

There is 1 train and bus route $B \rightarrow C \rightarrow M$.

By the addition principle, there are $3 + 1 = 4$ ways to travel from $B \rightarrow M$.

This means there are 4 ways to return $M \rightarrow B$.

By the multiplication principle, there are $4 \times 4 = 16$ ways to go from Bendigo to Maldon and back to Bendigo.

6 a One bib can be selected from ten bibs in ${}^{10}C_1 = 10$ ways.

One body suit can be selected from twelve body suits in ${}^{12}C_1 = 12$ ways.

Using the multiplication principle, there are $10 \times 12 = 120$ different combinations of bib and body suit she can wear.

b For the first leg of her trip, Christine has 2 options, the motorway or the highway.

For the second leg of her trip, Christine has 3 options of routes through the suburban streets.

Using the multiplication principle, there are $2 \times 3 = 6$ different routes she can take. Therefore, if she wishes to take a different route to work each day, she will be able to take a different route on 6 days before she must use a route already travelled.

c For the first characteristic, Abdul has 2 options, manual or automatic.

For the second characteristic, Abdul has 5 options of exterior colour.

For the third characteristic, Abdul has 2 options, leather or vinyl seats.

For the fourth characteristic, Abdul has 3 options of interior colour.

For the fifth characteristic, Abdul has 2 options, seat heating or not.

For the sixth characteristic, Abdul has 2 options, self parking or not.

Using the multiplication principle, there are $2 \times 5 \times 2 \times 3 \times 2 \times 2 = 240$ different combinations Abdul can choose from.

d Using the multiplication principle, there are $3 \times 2 \times 7 \times 5 = 210$ different combinations of clothes possible.

e Using the multiplication principle, there are $6 \times 52 = 312$ different starting combinations.

f Using the multiplication principle, there are $3 \times 6 \times 12 = 216$ different trips possible.

7 a There are five choices for the first digit, leaving four choices for the second digit, three choices for the third digit and then two choices for the fourth digit.

5	4	3	2

Using the multiplication principle, there are $5 \times 4 \times 3 \times 2 = 120$ possible four-digit numbers that could be formed.

b At least three-digit numbers means either three-digit, four-digit or five-digit numbers are to be counted.

For three digit numbers:

5	4	3

For four-digit numbers:

5	4	3	2

For five-digit numbers:

5	4	3	2	1

There are $5 \times 4 \times 3 = 60$ three-digit numbers, $5 \times 4 \times 3 \times 2 = 120$ four-digit numbers and $5 \times 4 \times 3 \times 2 \times 1 = 120$ five-digit numbers. Using the addition principle, there are $60 + 120 + 120 = 300$ possible three-, four- or five-digit numbers.

c For the number to be even its last digit must be even, so the number must end in 6. This means there is one choice for the last digit.

				1

Once the last digit has been formed, there are four choices for the first digit, then three choices for the second digit, two choices for the third digit and one choice for the fourth digit.

4	3	2	1	1

Using the multiplication principle, there are $4 \times 3 \times 2 \times 1 \times 1 = 24$ even five-digit numbers possible.

d The sample space is the set of five-digit numbers. From part **b**, $n(\xi) = 120$.

Let A be the event the five-digit number is even. From part **c**, $n(A) = 24$.

$$\begin{aligned}\Pr(A) &= \frac{n(A)}{n(\xi)}\\ &= \frac{24}{120}\\ &= \frac{1}{5}.\end{aligned}$$

The probability the five-digit number is even is $\frac{1}{5}$.

8 a There are 26 letters in the English alphabet and 10 digits from 0 to 9. Repetition of letters and digits is allowed.

26	26	10	10	26

Using the multiplication principle, there are $26 \times 26 \times 10 \times 110 \times 26 = 1\,757\,600$ possible number plates that could be formed.

b There are 7 different letters, from which 5 letter words are formed. Repetitions are allowed.

7	7	7	7	7

Using the multiplication principle, there are $7 \times 7 \times 7 \times 7 \times 7 = 16\,807$ possible words that could be formed.

c For each single roll there are 6 possible outcomes.

Using the multiplication principle, for a die rolled three times there are $6 \times 6 \times 6 = 216$ possible outcomes.

d There are 6 different digits from which 3-digit numbers are formed. Repetitions are allowed.

6	6	6

Using the multiplication principle, there are $6 \times 6 \times 6 = 216$ possible 3-digit numbers that could be formed.

e Three rooms can be selected from the four available in ${}^4C_3 = 4$ ways.

The three selected rooms can be arranged among the three friends in $3! = 3 \times 2 \times 1 = 6$ ways.

Using the multiplication principle, there are $4 \times 6 = 24$ possible ways can the rooms be allocated.

9 a If there are no restrictions, eight people can be arranged in a row in $8! = 40\,320$ ways.

b If the boys and girls are to alternate, assuming the row begins with a boy, there are 4 choices of boy for the first position, 4 choices of girl for the second position, 3 choices of boy for the third position, 3 choices of girl for the fourth position and so on.

4	4	3	3	2	2	1	1

If the row begins with a girl, since there are an equal number of boys and girls, the box table will be identical.

Using the multiplication principle, there are $4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 576$ ways that 4 boys and 4 girls can be arranged in a row, if a boy is first. By the same rule there will be 576 ways if a girl goes first.

Therefore, using the addition principle, there is a total of $576 + 576 = 1152$ ways that 4 boys and 4 girls can be arranged in a row if the boys and girls are to alternate.

c If the end seats must be occupied by a girl, there are 4 choices of girl for the first position and 3 choices of girl for the last position. Hence, for the remaining seats there are a remainder of 4 boys and 2 girls = 6 people left.

4	6	5	4	3	2	1	3

Using the multiplication principle, there are $4 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 = 8640$ ways that 4 boys and 4 girls can be arranged in a row, if the end seats must be occupied by a girl.

d If the brother and sister *must not* sit together, this is the complementary event of the brother and sister *must* sit together. Find the number of ways for the latter event first.

Treat the brother and sister as one unit. Now there are 7 groups, which will arrange in 7! ways.

The unit (brother and sister) can internally rearrange in 2! ways.

Hence, the number of arrangements

$= 7! \times 2!$

$= 5040 \times 2$

$= 10\,080$

From **a**, the total number of ways 8 people can be arranged is 40 320.

Number of ways if the brother and sister must not sit together

$= 40\,320 - 10\,080$

$= 30\,240$

e If the girls must sit together, treat the girls as one unit. Now there are 5 groups, which will arrange in 5! ways.

The unit (the girls) can internally rearrange in 4! ways.

Hence, the number of arrangements

$= 5! \times 4!$

$= 120 \times 24$

$= 2880$

10 a There are ten digits. There are no repetitions allowed, and the number cannot start with 0. Therefore, there are 9 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit.

9	9	8

Using the multiplication principle, there are $9 \times 9 \times 8 = 648$ possible three-digit numbers that could be formed.

b For the number to be even its last digit must be even, so the number must end in a 0, 2, 4, 6 or 8. This means there are 5 choices for the last digit.

Once the last digit has been formed, if the last digit was 0, there are 9 choices for the first digit, then 8 choices for the second digit.

9	8	1

Therefore there are $9 \times 8 \times 1 = 72$ numbers possible.

If the last digit was not 0, there are 8 choices for the first digit, then 8 choices for the second digit (because the first digit cannot be 0).

8	8	4

Therefore, there are $8 \times 8 \times 4 = 256$ numbers possible.

Using the addition principle, there are $72 + 256 = 328$ even 3-digit numbers that could be formed.

c For the number to be less than 400 its first digit must be less than 4, so the number must start in a 1, 2 or 3 (remembering the number cannot start with 0). This means there are 3 choices for the first digit.

3		

Once the first digit has been formed, there are 9 choices for the second digit, then 8 choices for the last digit.

3	9	8

Using the multiplication principle, there are $3 \times 9 \times 8 = 216$ even 3-digit numbers that could be formed.

d For the number to made up of odd digits only, each digit may only be a 1, 3, 5, 7 or a 9. Therefore, there are 5 choices for the first digit, 4 choices for the second digit and 3 choices for the third digit.

5	4	3

Using the multiplication principle, there are $5 \times 4 \times 3 = 60$ possible three-digit numbers that could be formed.

11 a Six objects can be arranged in 6! ways. So the six girls can be seated in $6! = 720$ ways.

b Treat Agnes and Betty as one unit (AB) that can internally arrange in $2! = 2$ ways.

There are 4 girls plus the unit (AB) making 5 objects to arrange.

Five objects can be arranged in 5! ways.

As the unit (AB) can internally arrange in 2! ways, the number of seating arrangements that have Agnes and Betty sat together is $5! \times 2! = 120 \times 2 = 240$ ways.

c Let T be the event that Agnes and Betty are sat together.

$$\Pr(T) = \frac{n(T)}{n(\xi)}$$

From part **a**, $n(\xi) = 6! = 720$

From part **b**, $n(T) = 5! \times 2! = 240$

Therefore,

$$\begin{aligned}\Pr(T) &= \frac{5! \times 2!}{6!} \quad \text{or} \quad \Pr(T) = \frac{240}{720}\\ &= \frac{5! \times 2!}{6 \times 5!} \qquad\qquad\quad = \frac{24}{72}\\ &= \frac{2}{6} \qquad\qquad\qquad\quad\; = \frac{1}{3}\\ &= \frac{1}{3}\end{aligned}$$

12 The shelf has 8 books in total. Of these, there are 3 hardbacks and 5 paperbacks.

a Choose 2 hardbacks from the 3 hardbacks and 3 paperbacks from the 5 paperbacks.

This can be done in ${}^3C_2 \times {}^5C_3$ ways.

${}^3C_2 \times {}^5C_3 = 3 \times 10 = 30.$

The selection of books can be chosen in 30 ways.

b Any set of 5 books can be selected from this group of 8 books on the shelf in ${}^8C_5$ ways.

Use a calculator or the formula to evaluate ${}^8C_5$.

Using the formula,

$$\begin{aligned} {}^{8}C_5 &= \frac{8!}{(8-5)! \times 5!} \\ &= \frac{8!}{3! \times 5!} \\ &= \frac{8 \times 7 \times 6 \times 5!}{6 \times 5!} \\ &= 8 \times 7 \\ &= 56 \end{aligned}$$

There are 56 ways to select a set of 5 books.

c Let B be the event of selecting 2 hardbacks and 3 paperbacks.

$$\Pr(B) = \frac{n(B)}{n(\xi)}$$

From part **a**, $n(B) = 30$ and from part **b**, $n(\xi) = 56$.

Therefore,

$$\begin{aligned} \Pr(B) &= \frac{30}{56} \\ &= \frac{15}{28} \end{aligned}$$

The probability of choosing 2 hardbacks and 3 paperbacks when selecting 5 books from the shelf is $\frac{15}{28}$.

13 a There are 26 letters in the English alphabet and 10 digits from 0 to 9. Repetition of letters and digits is allowed.

26	26	10	10	10	26

Using the multiplication principle, there are $26 \times 26 \times 10 \times 10 \times 10 \times 26 = 17\,576\,000$ possible number plates that could be formed.

b If the letter X is used exactly once, there will be one choice for that position and 25 choices for the other two positions where letters are used.

For an X in the first position:

1	25	10	10	10	25

For an X in the second position:

25	1	10	10	10	25

For an X in the sixth position:

25	25	10	10	10	1

Using the multiplication principle, there are $1 \times 25 \times 10 \times 10 \times 10 \times 25 = 625\,000$ number plates that use the letter X exactly once (in the first position), $25 \times 1 \times 10 \times 10 \times 10 \times 25 = 625\,000$ (in the second position) and $25 \times 25 \times 10 \times 10 \times 10 \times 1 = 625\,000$ (in the sixth position). Using the addition principle there are $625\,000 + 625\,000 + 625\,000 = 1\,875\,000$ possible number plates that use the letter X exactly once.

c Using the multiplication principle, there are $26 \times 1 \times 10 \times 1 \times 1 \times 25 = 6500$ possible number plates that could be formed.

$$\begin{aligned} \Pr(\text{first 2 letters identical}) &= \frac{6500}{17\,576\,000} \\ &= \frac{1}{2704} \end{aligned}$$

14 a Six people can arrange in a straight line in 6! ways.

Since $6! = 6 \times 5!$ and $5! = 120$,

$$\begin{aligned} 6! &= 6 \times 120 \\ &= 720 \end{aligned}$$

There are 720 ways in which the students can form the queue.

b For circular arrangements, 6 people can be arranged in $(6-1)! = 5!$ ways.

Since $5! = 120$, there are 120 different arrangements in which the six students may be seated.

c There are three prizes. Each prize can be awarded to any one of the six students.

6	6	6

The total number of ways the prizes can be awarded is $6 \times 6 \times 6$.

$\therefore n(\xi) = 6 \times 6 \times 6$

Let A be the event that the same student receives all three prizes. There are six choices for that student.

$\therefore n(A) = 6$

$$\begin{aligned} \Pr(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{6}{6 \times 6 \times 6} \\ &= \frac{1}{36} \end{aligned}$$

The probability that one student receives all three prizes is $\frac{1}{36}$.

15 a Number of ways 7 men can be selected from a group of 15 men

$$\begin{aligned} &= {}^{15}C_7 \\ &= \frac{15!}{7!\,(15-7)!} \\ &= 6435 \end{aligned}$$

b Number of 5-card hands that can be dealt from a standard pack of 52 cards

$$\begin{aligned} &= {}^{52}C_5 \\ &= \frac{52!}{5!\,(52-5)!} \\ &= 2\,598\,960 \end{aligned}$$

c If a 5-card hand is to contain all 4 aces, this leaves $52 - 4 = 48$ choices for the remaining 1 card. Therefore, the number of 5-card hands that contain all 4 aces that can be dealt from a standard pack of 52 cards = 48

d Number of ways 3 prime numbers be selected from the set containing the first 10 prime numbers

$$\begin{aligned} &= {}^{10}C_3 \\ &= \frac{10!}{3!\,(10-3)!} \\ &= 120 \end{aligned}$$

16 a There are 18 people in total from whom 8 people are to be chosen. This can be done in ${}^{18}C_8$ ways.

$$\begin{aligned} {}^{18}C_8 &= \frac{18!}{8! \times (18-8)!} \\ &= \frac{18!}{8! \times 10!} \\ &= 43\,758 \end{aligned}$$

There are 43 758 possible committees.

b The 5 men can be chosen from the 8 men available in ${}^{8}C_5$ ways.

The 3 women can be chosen from the 10 women available in ${}^{10}C_3$ ways.

The total number of committees that contain 5 men and 3 women is ${}^{8}C_5 \times {}^{10}C_3$.

$$\begin{aligned}{}^{8}C_{5} \times {}^{10}C_{3} &= \frac{8!}{5! \times 3!} \times \frac{10!}{3! \times 7!} \\ &= \frac{8 \times 7 \times 6}{3!} \times \frac{10 \times 9 \times 8}{3!} \\ &= 56 \times 120 \\ &= 6720\end{aligned}$$

There are 6720 committees possible with the given restriction.

c As there are 8 men available, at least 6 men means either 6, 7 or 8 men.

The panels of people that satisfy this restriction have either 6 men and 2 women, 7 men and 1 woman, or 8 men and no women.

6 men and 2 women are chosen in ${}^{8}C_{6} \times {}^{10}C_{2}$ ways.

7 men and 1 woman are chosen in ${}^{8}C_{7} \times {}^{10}C_{1}$ ways.

8 men and 0 women are chosen in ${}^{8}C_{8} \times {}^{10}C_{0}$ ways.

The number of committees with at least 6 men is ${}^{8}C_{6} \times {}^{10}C_{2} + {}^{8}C_{7} \times {}^{10}C_{1} + {}^{8}C_{8} \times {}^{10}C_{0}$.

$$\begin{aligned}&{}^{6}C_{4} \times {}^{8}C_{1} + {}^{6}C_{5} \times {}^{8}C_{0}{}^{8}C_{6} \times {}^{10}C_{2} + {}^{8}C_{7} \times {}^{10}C_{1} + {}^{8}C_{8} \times {}^{10}C_{0} \\ &= 1260 + 80 + 1 \\ &= 1341\end{aligned}$$

There are 1341 committees with at least 6 men.

d The total number of panels of 8 people is ${}^{18}C_{8} = 43\,758$ from part **a**.

If two particular men cannot both be included, this is the complementary event of the two men both being included. In this case, the other 6 panel members need to be selected from the remaining 16 people to form the panel of 8. This can be done in ${}^{16}C_{6}$ ways.

$$\begin{aligned}{}^{16}C_{6} &= \frac{16!}{6! \times (16-6)!} \\ &= \frac{16!}{6! \times 10!} \\ &= 8008\end{aligned}$$

Therefore, the number of panels that can be formed if two particular men cannot both be included

$= 43\,758 - 8008 = 35\,750$.

e If a particular man and woman *must* be included on the panel, then the other 6 panel members need to be selected from the remaining 16 people to form the panel of 8. This can be done in ${}^{16}C_{6}$ ways.

$$\begin{aligned}{}^{16}C_{6} &= \frac{16!}{6! \times (16-6)!} \\ &= \frac{16!}{6! \times 10!} \\ &= 8008\end{aligned}$$

There are 8008 committees possible with the given restriction.

17 a For circular arrangements, 4 people can be arranged in $(4-1)! = 3!$ ways.

Since $3! = 6$, there are 6 different arrangements in which the four boys may be seated.

b There are 4 possible seats each girl can sit in. There are $4 \times 3 \times 2 \times 1$ ways that the four girls may be seated. So, 24 ways in total.

18 a Treat the letters, Q and U, as one unit.

Now there are eight groups to arrange:

(QU), E, A, T, I, O, N, S.

These arrange in 8! ways.

The unit (QU) can internally rearrange in 2! ways.

Hence, the total number of arrangements

$= 8! \times 2!$

$= 80\,640$

b The number of arrangements with the letters Q and U separated is equal to the total number of arrangements minus the number of arrangements with the vowels together.

The nine letters of the word EQUATIONS can be arranged in $9! = 362\,880$ ways.

From part **a**, there are 80 640 arrangements with the letters together.

Therefore, there are $362\,880 - 80\,640 = 282\,240$ arrangements in which the two letters are separated.

c The word SIMULTANEOUS contains 12 letters of which there are 2 S's and 2 U's.

The number of arrangements of the word SIMULTANEOUS is $\dfrac{12!}{2! \times 2!} = 119\,750\,400$.

d As there are 119 750 400 total arrangements of the word SIMULTANEOUS, $n(\xi) = 119\,750\,400$ or $\dfrac{12!}{2 \times 2!}$.

For the letters U to be together, treat these two letters as one unit. This creates eleven groups, (UU), S, I, M, L, T, A, N, E, O, S, of which two are identical S's.

The eleven groups arrange in $\dfrac{11!}{2!}$ ways. As the unit (UU) contains two identical letters, there are no distinct internal rearrangements of this unit that need to be taken into account.

Hence, the number of elements in the event is $\dfrac{11!}{2!}$.

The probability that the U's are together

$$\begin{aligned}&= \frac{\text{number of arrangements with the U's together}}{\text{total number of arrangements}} \\ &= \frac{11!}{2!} \div \frac{12!}{2! \times 2!} \\ &= \frac{11!}{2!} \times \frac{2! \times 2!}{12 \times 11!} \\ &= \frac{2}{12} \\ &= \frac{1}{6}\end{aligned}$$

19 a There are 14 students in total from whom 5 students are to be chosen. This can be done in ${}^{14}C_{5}$ ways.

$$\begin{aligned}{}^{14}C_{5} &= \frac{14!}{5! \times (14-5)!} \\ &= \frac{14!}{5! \times 9!} \\ &= \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9!}{5! \times 9!} \\ &= \frac{14 \times 13 \times 12 \times 11 \times 10}{120} \\ &= 2002\end{aligned}$$

There are 2002 possible committees.

b The 2 boys can be chosen from the 6 boys available in ${}^{6}C_{2}$ ways.

The 3 girls can be chosen from the 8 girls available in ${}^{8}C_{3}$ ways.

The total number of committees that contain two boys and three girls is

${}^{6}C_{2} \times {}^{8}C_{3}$.

$$\begin{aligned}{}^6C_2 \times {}^8C_3 &= \frac{6!}{2!\times 4!} \times \frac{8!}{3!\times 5!}\\ &= \frac{6\times 5\times 4!}{2!\times 4!} \times \frac{8\times 7\times 6\times 5!}{3!\times 5!}\\ &= 15\times 56\\ &= 840\end{aligned}$$

There are 840 committees possible with the given restriction.

c As there are six boys available, at least four boys means either four or five boys.

The committees of five students that satisfy this restriction have either 4 boys and 1 girl or they have 5 boys and no girls.

4 boys and 1 girl are chosen in ${}^6C_4 \times {}^8C_1$ ways.

5 boys and no girls are chosen in ${}^6C_5 \times {}^8C_0$ ways.

The number of committees with at least four boys is ${}^6C_4 \times {}^8C_1 + {}^6C_5 \times {}^8C_0$.

$$\begin{aligned}{}^6C_4 \times {}^8C_1 +{}^6 C_5 \times {}^8C_0 &= 15\times 8 + 6\times 1\\ &= 126\end{aligned}$$

There are 126 committees with at least four boys.

d The total number of committees of five students is ${}^{14}C_5 = 2002$ from part **a**.

Each committee must have five students. If neither the oldest nor youngest student are placed on the committee, then 5 students need to be selected from the remaining 12 students to form the committee of five. This can be done in ${}^{12}C_5$ ways.

Let A be the event that neither the oldest nor youngest are on the committee.

$$\begin{aligned}\Pr(A) &= \frac{n(A)}{n(\xi)}\\ &= \frac{{}^{12}C_5}{{}^{14}C_5}\end{aligned}$$

Hence,

$$\begin{aligned}\Pr(A) &= \frac{12!}{5!\times 7!} \div \frac{14!}{5!\times 9!}\\ &= \frac{12!}{5!\times 7!} \times \frac{5!\times 9!}{14!}\\ &= \frac{1}{1} \times \frac{9\times 8}{14\times 13}\\ &= \frac{72}{182}\\ &= \frac{36}{91}\end{aligned}$$

The probability of the committee containing neither the youngest nor oldest student is $\frac{36}{91}$.

20 a The word *bananas* contains 7 letters of which there are 3 *a*'s and 2 *n*'s.

Number of words that can be formed, given all letters are used

$$\begin{aligned}&= \frac{7!}{3!\times 2!}\\ &= 420\end{aligned}$$

b *BANANAS*

Require 4-letter words including at least one *A*.

Possibilities: one *A*, two *A*'s, three *A*'s.

Exactly one *A*: Other three letters come from *BNNS*.

ABNN arrange in $\frac{4!}{2!} = 12$ ways

ABNS arrange in $4! = 24$ ways

ANNS arrange in $\frac{4!}{2!} = 12$ ways

Total of 48 arrangements with one *A*.

Exactly two ***A*****'s**: Other two letters come from *BNNS*

AABN arrange in $\frac{4!}{2!} = 12$ ways

AABS arrange $\frac{4!}{2!} = 12$ ways

AANS arrange $\frac{4!}{2!} = 12$ ways

AANN arrange $\frac{4!}{2!\times 2!} = 6$ ways.

Total of 42 arrangements with two *A*'s.

All 3 ***A*****'s**: Other one letter comes from *BNNS*

AAAB, AAAN, AAAS each arrange $\frac{4!}{3!} = 4$ ways.

Total of 12 arrangements with all three *A*'s.

Thus, the total number of arrangements of the letters of *BANANAS* that include at least one *A* is $48 + 42 + 12 = 102$.

c To form a 4-letter word using all different letters, exclude any repeated letter. Hence, the available letters are *b, a, n* and *s*.

Number of words that can be formed $= 4! = 24$

21 a Using the rule for number of arrangements, some of which are identical, number of arrangements of mugs

$$\begin{aligned}&= \frac{12!}{4!\,3!\times 5!}\\ &= 27\,720\end{aligned}$$

b If the 12 mugs from part **a** are to be arranged in 2 rows of 6 and the green ones must be on the front row,

$$\begin{aligned}&\left(\frac{6!}{4!\times 2!} + \frac{6!}{3!\times 3!}\right) \times \frac{6!}{5!\times 1!}\\ &= 210\end{aligned}$$

22 a For circular arrangements, 9 people can be arranged in $(9-1)! = 8! = 40\,320$ ways.

b If the men can only be seated in pairs, treat each pair as one unit. Now there are 6 groups, which will arrange in $(6-1)! = 5!$ ways.

The 6 men can be arranged into 3 pairs in $\frac{{}^6C_2 \times {}^4C_2 \times {}^2C_2}{3!} = 15$ ways.

The units (the pairs) can each internally rearrange in 2! ways.

Hence, the number of arrangements

$$\begin{aligned}&= 5!\times 15\times 2!\times 2!\times 2!\\ &= 120\times 30\times 4\\ &= 14\,400\end{aligned}$$

23 a There are 12 different numbers in S.

The total number of subsets of S is equal to the number of ways 0 numbers can be selected from S plus the number of ways 1 number can be selected from S plus the number of ways 2 numbers can be selected from S, and so on.

Hence,

Number of subsets of S

$$\begin{aligned}&= {}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 + {}^{12}C_4 + {}^{12}C_5 + {}^{12}C_6 + {}^{12}C_7\\ &\quad + {}^{12}C_8 + {}^{12}C_9 + {}^{12}C_{10} + {}^{12}C_{11} + {}^{12}C_{12}\\ &= 4096\end{aligned}$$

b To determine the number of subsets whose elements are all even numbers, only regard the even numbers in the subset, i.e. $S_{\text{even}} = \{2, 4, 8, 10, 14\}$. There are 5 different numbers in S_{even}.

Hence,

Number of subsets of S_{even}

$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$
$= 31$

Note that the empty set found by ${}^5C_0$ is not included, since an empty set does not fit the condition.

Therefore, there are 31 subsets whose elements are all even numbers.

c The total number of subsets of S is 4096 from part **a**.

Let C be the event that a subset selected at random will contain only even numbers.

$$\Pr(C) = \frac{n(C)}{n(\xi)} = \frac{31}{4096}$$

The probability that a subset selected at random will contain only even numbers is $\frac{31}{4096}$.

d The total number of subsets of S is 4096 from part **a**.

The number of subsets containing at least 3 elements will be the total number of subsets less those containing 0, 1 or 2 elements.

Hence, the number of subsets containing at least 3 elements

$= 4096 - \left({}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2\right)$
$= 4017$

Let D be the event that a subset selected at random will contain at least 3 elements.

$$\Pr(D) = \frac{n(D)}{n(\xi)} = \frac{4017}{4096}$$

The probability that a subset selected at random will contain only even numbers is $\frac{4017}{4096}$.

e The total number of subsets of S is 4096 from part **a**.

To determine the number of subsets whose elements are all prime numbers, only regard the prime numbers in the subset, i.e. $S_{\text{prime}} = \{2, 3, 5, 7, 11, 13, 17\}$.

There are 7 different numbers in S_{prime}.

Hence, the number of subsets of S_{prime} that contain exactly 3 elements

$= {}^7C_3$
$= 35$

Therefore, there are 35 subsets of S that contain exactly 3 elements, all of which are prime numbers.

Let E be the event that a subset selected at random will contain exactly 3 elements, all of which are prime numbers.

$$\Pr(E) = \frac{n(E)}{n(\xi)} = \frac{35}{4096}$$

The probability that a subset selected at random will contain exactly 3 elements, all of which are prime numbers, is $\frac{35}{4096}$.

24 **a** The words 'PARALLEL LINES' contain 13 letters, of which there are 2 A's, 4 L's and 2 E's.

The number of arrangements in a row of the words PARALLEL LINES equals $\frac{13!}{2! \times 4! \times 2!}$.

$$\frac{13!}{2! \times 4! \times 2!} = \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{2! \times 4! \times 2!} = 64\,864\,800$$

There are 64 864 800 arrangements in a row.

b For circular arrangements, the 13 letters, of which there are 2 A's, 4 L's and 2 E's, can be arranged in $\frac{(13-1)!}{2! \times 4! \times 2!}$ ways.

$$\frac{(13-1)!}{2! \times 4! \times 2!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{2! \times 4! \times 2!} = 4\,989\,600$$

There are 4 989 600 arrangements in a circle.

c There are five vowels in the words PARALLEL LINES. Treat these letters, A, A, E, I and E as one unit.

Now there are nine groups to arrange:

(AAEIE), P, R, L, L, L, L, N, S.

These arrange in 9! ways.

The unit (AAEIE) can internally rearrange in 5! ways.

Hence, the total number of arrangements

$= (9! \times 5!) \div (2! \times 4! \times 2!)$
$= 9 \times 8 \times 7 \times 6 \times 5! \div (2! \times 4! \times 2!)$
$= 3024 \times 120 \div (2! \times 4! \times 2!)$
$= 362\,880 \div (2! \times 4! \times 2!)$
$= 453\,600$

25 There are 17 players in total from whom 11 players are to be chosen. This can be done in ${}^{17}C_{11}$ ways.

$$\begin{aligned} {}^{17}C_{11} &= \frac{17!}{11! \times (17-11)!} \\ &= \frac{17!}{11! \times 6!} \\ &= \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11!}{11! \times 6!} \\ &= \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12}{720} \\ &= 12\,376 \end{aligned}$$

There are 12 376 possible teams.

The 1 wicketkeeper can be chosen from the 3 wicketkeepers available in ${}^3C_1$ ways.

The 4 bowlers can be chosen from the 6 bowlers available in ${}^6C_4$ ways.

The 6 batsmen can be chosen from the 8 batsmen available in ${}^8C_6$ ways.

The total number of teams which contain one wicketkeeper, four bowlers and six batsmen is

${}^3C_1 \times {}^6C_4 \times 8{}^8C_6$.

$$\begin{aligned} {}^3C_1 \times {}^6C_4 \times {}^8C_6 &= \frac{3!}{1! \times 2!} \times \frac{6!}{4! \times 2!} \times \frac{8!}{6! \times 2!} \\ &= \frac{3 \times 2!}{2!} \times \frac{6 \times 5 \times 4!}{4! \times 2!} \times \frac{8 \times 7 \times 6!}{6! \times 2!} \\ &= 3 \times 15 \times 28 \end{aligned}$$

There are 1260 teams possible with the given restriction. Let A be the event that the team chosen consists of one wicketkeeper, four bowlers and six batsmen.

$$\Pr(A) = \frac{n(A)}{n(\xi)}$$
$$= \frac{1260}{12\,367}$$
$$= \frac{45}{442}$$

The probability that the team chosen consists of one wicketkeeper, four bowlers and six batsmen is $\frac{45}{442}$.

26 **a** There are 56 players in total from whom 7 members are to be selected. This can be done in $^{56}\text{C}_7$ ways.

$$^{56}\text{C}_7 = \frac{56!}{7! \times (56-7)!}$$
$$= \frac{56!}{7! \times 49!}$$
$$= \frac{56 \times 55 \times 54 \times 53 \times 52 \times 51 \times 50}{7!}$$
$$= 231\,917\,400$$

There are 231 917 400 possible teams.

The 7 members can be selected from the 17 squash players available in $^{17}\text{C}_7 = 19\,448$ ways.

The total number of committees that contain 7 squash players is 19 448.

Let A be the event that the committee contains 7 squash players.

$$\Pr(A) = \frac{n(A)}{n(\xi)}$$
$$= \frac{19\,448}{231\,917\,400}$$
$$= \frac{1}{11\,925}$$

The probability the committee consists of 7 squash players is $\frac{1}{11\,925}$.

b As there are 21 tennis players available, at least 5 tennis players means either 5, 6 or 7 tennis players. The committees of 7 members that satisfy this restriction have the other members chosen from the remaining 35 squash or badminton players.

5 tennis players and 2 squash or badminton players are chosen in $^{21}\text{C}_5 \times {}^{35}\text{C}_2$ ways.

tennis players and 1 squash or badminton player are chosen in $^{21}\text{C}_6 \times {}^{35}\text{C}_1$ ways.

7 tennis players and no squash or badminton players are chosen in $^{21}\text{C}_7 \times {}^{35}\text{C}_0$ ways.

The number of committees with at least 5 tennis players is $^{21}\text{C}_5 \times {}^{35}\text{C}_2 + {}^{21}\text{C}_6 \times {}^{35}\text{C}_1 + {}^{21}\text{C}_7 \times {}^{35}\text{C}_0$.

$$^{21}\text{C}_5 \times {}^{35}\text{C}_2 + {}^{21}\text{C}_6 \times {}^{35}\text{C}_1 + {}^{21}\text{C}_7 \times {}^{35}\text{C}_0$$
$$= 12\,107\,655 + 1\,899\,240 + 116\,280$$
$$= 14\,123\,175$$

There are 14 123 175 committees with at least 5 tennis players.

Let B be the event that the committee contains at least 5 tennis players.

$$\Pr(B) = \frac{n(B)}{n(\xi)}$$
$$= \frac{14\,123\,175}{231\,917\,400}$$
$$= \frac{19}{312}$$

The probability the committee contains at least 5 tennis players is $\frac{19}{312}$.

c The committee is to have at least one representative from each sport.

To insure this, first choose one player from each sport, then fill the 4 other positions from the $56 - 3 = 53$ remaining players.

This can be done in $21 \times 17 \times 18 \times 53 \times 52 \times 51 \times 50$ ways. However, this needs to be divided by 7! to eliminate counting the same committee more than once. (e.g. $T_1S_1B_1T_2S_2B_2T_3$ is the same as $T_2S_1B_1T_3S_2B_2T_1$.)

The number of committees with at least one representative from each sport is

$$\frac{21 \times 17 \times 18 \times 53 \times 52 \times 51 \times 50}{7!} = 8\,960\,445.$$

In part **a** it was calculated that the total number of possible committees of 7 was 231 917 400. This forms the sample space number for the probability calculation.

Let C be the event that the committee has at least one representative from each sport.

$$\Pr(C) = \frac{n(C)}{n(\xi)}$$
$$= \frac{8\,960\,445}{231\,917\,400}$$
$$\approx 0.0386$$

The probability that the committee contains at least one representative from each sport is $\frac{8\,960\,445}{231\,917\,400} \approx 0.0386$.

d To find the probability that the committee contains exactly 3 badminton players, given that it contains at least 1 badminton player, first find the number of committees that contain at least 1 badminton player.

If the committee contains no badminton players, there are 38 remaining players from which 7 members must be chosen. Hence, the number of combinations $= {}^{38}\text{C}_7 = 12\,620\,256$.

Therefore, the number of committees which contain at least 1 badminton player

$$= 231\,917\,400 - 12\,620\,256$$
$$= 219\,297\,144$$

For a committee that contains exactly 3 badminton players, the 3 badminton players and 4 tennis or squash players are chosen in $^{18}\text{C}_3 \times {}^{38}\text{C}_4$ ways.

$$^{18}\text{C}_3 \times {}^{38}\text{C}_4 = 816 + 73\,815$$
$$= 74\,631$$

There are 74 631 committees with exactly 3 badminton players.

Let D_1 be the event that the committee contains exactly 3 badminton players.

Let D_2 be the event that the committee contains at least 1 badminton player.

$$\Pr(D_1|D_2) = \frac{n(D_1 \cap D_2)}{n(D_2)}$$
$$= \frac{74\,631}{219\,297\,144}$$
$$= \frac{24\,877}{73\,099\,048}$$

The probability the committee contains exactly 3 badminton players, given that it contains at least 1 badminton player, is $\frac{24\,877}{73\,099\,048} \simeq 0.2747$.

27 **a** First find the total possible number of 3-digit numbers.

There are ten digits. There are repetitions allowed, but the number cannot start with 0.

Therefore, there are 9 choices for the first digit, 10 choices for the second digit and 10 choices for the third digit.

9	10	10

Using the multiplication principle, there are $9 \times 10 \times 10 = 900$ possible three-digit numbers that could be formed.

Of the ten digits, $\{2, 3, 5, 7\}$ are prime numbers.

To form a three-digit number where all the digits are primes, there are 4 choices for each digit.

4	4	4

Using the multiplication principle, there are $4 \times 4 \times 4 = 64$ possible three-digit numbers that have each digit a prime number.

Let A be the event that all three digits of the number selected are prime numbers.

$$\begin{aligned}\Pr(A) &= \frac{n(A)}{n(\xi)}\\ &= \frac{64}{900}\\ &= \frac{16}{225}\end{aligned}$$

The probability that all digits are primes for the three-digit number selected is $\frac{16}{225}$.

b The number chosen consists of a single repeated digit.

There are ten digits and the number cannot start with 0. Therefore, there are 9 choices for the first digit. Once that digit is chosen, the same digit needs to be used for both the second and third digits.

9	1	1

Using the multiplication principle, there are $9 \times 1 \times 1 = 9$ possible three-digit numbers that consist of a single repeated digit.

Let B be the event that the number chosen consists of a single repeated digit.

$$\begin{aligned}\Pr(B) &= \frac{n(B)}{n(\xi)}\\ &= \frac{9}{900}\\ &= \frac{1}{100}\end{aligned}$$

The probability that the number chosen consists of a single repeated digit is $\frac{1}{100}$.

c The perfect squares in the first ten digits are $\{0, 1, 4, 9\}$.

To form a three-digit number where all the digits are perfect squares, there are 3 choices for the first digit since it cannot be 0, and 4 choices for each of the second and third digits.

3	4	4

Using the multiplication principle, there are $3 \times 4 \times 4 = 48$ possible three-digit numbers in which the digits are perfect squares.

Let C be the event that the digits of the number chosen are perfect squares.

$$\begin{aligned}\Pr(C) &= \frac{n(C)}{n(\xi)}\\ &= \frac{48}{900}\\ &= \frac{4}{75}\end{aligned}$$

The probability that the digits of the number chosen are perfect squares is $\frac{4}{75}$.

d There are ten digits to choose from, but repetition is not allowed and the number cannot start with 0.

Therefore, there are 9 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit.

9	9	8

Using the multiplication principle, there are $9 \times 9 \times 8 = 648$ possible three-digit numbers with no repeated digits that could be formed.

Let D be the event that the number chosen has no repeated digits.

$$\begin{aligned}\Pr(D) &= \frac{n(D)}{n(\xi)}\\ &= \frac{648}{900}\\ &= \frac{18}{25}\end{aligned}$$

The probability that the number chosen has no repeated digits is $\frac{18}{25}$.

e There are 101 numbers that lie between 300 and 400, including both ends, 300 and 400.

Using complementary events, the number of three-digit numbers > 200 is the total number of three-digit numbers minus the number of three-digit numbers ≤ 200.

There are 101 numbers that lie between 100 and 200, including both ends 100 and 200.

Therefore, the number of three-digit numbers > 200 equals $900 - 101 = 799$.

Let E be the event that the number lies between 300 and 400.

Let F be the event that the number is greater than 200.

The formula for conditional probabilities gives:

$$\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$

Since E is a subset of F, $E \cap F = E$.

$$\begin{aligned}\Pr(E|F) &= \frac{\Pr(E)}{\Pr(F)}\\ &= \frac{n(E)}{n(F)}\\ &= \frac{101}{799}\end{aligned}$$

The probability that the number chosen lies between 300 and 400, given that the number is greater than 200, is $\frac{101}{799}$.

7.5 Exam questions

1 Sam is not counted in calculations because we know he must be first. The other appointments can then be arranged in any order.

$4! = 4 \times 3 \times 2 \times 1$

$\quad = 24$

The correct answer is **C**.

2 Count the IE as one unit.

Therefore, there are 5! ways to arrange the letters.

The IE can also be EI.

$5! \times 2 = 5 \times 4 \times 3 \times 2 \times 1 \times 2$

$\quad = 240$

The correct answer is **C**.

3 a $^{16}\mathrm{C}_4 = \dfrac{16!}{4! \times 12!}$

$= \dfrac{16 \times 15 \times 14 \times 13 \times \cancel{12!}}{4! \times \cancel{12!}}$

$= \dfrac{43\,680}{24}$

$= 1820$ [1 mark]

b 3 females (and 1 male) or 4 females

$^9\mathrm{C}_3 \times {}^7\mathrm{C}_1 + {}^9\mathrm{C}_4 = \dfrac{9!}{3! \times 6!} \times 7 + \dfrac{9!}{4! \times 5}$ [1 mark]

$= 588 + 126$

$= 714$

c Calculations from parts **a** and **b**:

$\Pr(\text{only 1 male}) = \dfrac{\Pr(\text{3 females + 1 male})}{\text{total number of panels}}$ [1 mark]

$= \dfrac{588}{1820}$

$= 0.3231$ [1 mark]

7.6 Binomial coefficients and Pascal's triangle

7.6 Exercise

1 a The events 'H on the first toss' and 'T on the second toss' are independent since $\Pr(T|H) = \Pr(T)$ or $\Pr(H_1 \cap T_2) = \Pr(H_1) \times \Pr(T_2)$.

b i $\Pr(TT) = q \times q = q^2$

ii $\Pr(HH) = p \times p = p^2$

iii $\Pr(TH) = \Pr(T) \times \Pr(H)$

$= q \times p$

$= qp$

iv $\Pr(TH \text{ or } HT)$

$= \Pr(TH) + \Pr(HT)$

$= \Pr(T) \times \Pr(H) + \Pr(H) \times \Pr(T)$

$= q \times p + p \times q$

$= 2pq$

c $(p + q)^2 = p^2 + 2pq + q^2$

d i $\Pr(HH) = p^2$ is the first term in the expansion $p^2 + 2pq + q^2$.

ii $\Pr(TT) = q^2$, the third term in the expansion $p^2 + 2pq + q^2$.

iii Pr(one Head and one Tail) is $\Pr(TH \text{ or } HT) = 2pq$, the second term in the expansion $p^2 + 2pq + q^2$.

2 a The probability that the drug will be effective is 0.7. Using probabilities for complementary events, the probability that the drug will not be effective $= 1 - 0.7 = 0.3$.

b $(0.7 + 0.3)^3 = 0.7^3 + \dbinom{3}{1}(0.7)^2(0.3)$

$+ \dbinom{3}{2}(0.7)(0.3)^2 + 0.3^3$

i Let R be the event that the drug gives pain relief.

The complement R' is the event that the drug does not give pain relief.

The probability that the drug does not give pain relief to exactly one of the patients is

$\Pr(RRR' \text{ or } RR'R \text{ or } R'RR)$

$= 0.7 \times 0.7 \times 0.3 + 0.7 \times 0.3 \times 0.7 + 0.3 \times 0.7 \times 0.7$

$= 0.7^2 \times 0.3 + 0.7^2 \times 0.3 + 0.7^2 \times 0.3$

$= 3 \times 0.7^2 \times 0.3$

Since $3 = \dbinom{3}{1}$,

$\Pr(RRR' \text{ or } RR'R \text{ or } R'RR) = \dbinom{3}{1}(0.7)^2(0.3)$.

The second term in the given expansion, $\dbinom{3}{1}(0.7)^2(0.3)$, is the probability that the drug gives pain relief to two patients but does not give pain relief to exactly one of the patients.

ii The probability that the drug is effective for all three patients is $(0.7)^3$, the first term in the given expansion.

The third term represents the probability that drug is effective for only one of the three patients. This means the drug is ineffective for two of the three patients.

There are 3 patients, so there are $^3\mathrm{C}_2 = \dbinom{3}{2}$ ways to choose the two patients for whom the drug does not work.

$\Pr(\text{1 effective, 2 ineffective}) = \dbinom{3}{2}(0.7)(0.3)^2$

The fourth term in the given expansion, $(0.3)^3$, is the probability that the drug is ineffective for all three patients.

3 a The word *egg* contains 3 letters, two of which are identical.

The number of ways of arranging 3 letters, two of which are identical, is $\dfrac{3!}{2!}$.

$\dfrac{3!}{2!} = \dfrac{6}{2} = 3.$

There are 3 different arrangements of the word *egg*.

b i $(e + g)^3 = e^3 + 3e^2g + 3eg^2 + g^3$

ii The coefficient of the term g^2e or eg^2 in the expansion is 3.

c Each term in the expansion contains an arrangement of three symbols that is some combination of e or g.

The first term is $eee = e^3$. There is only one way to arrange the letters *eee* so the coefficient of the first term is 1.

The second term $e^2g = eeg$. The three letters *eeg* can be arranged in $\dfrac{3!}{2!} = 3$ ways. The coefficient of the second term is 3.

The coefficient of the third term is 3, as is the number of arrangements of $eg^2 = egg$ shown in part a. So the two answers agree.

4 a The word *abba* contains 4 letters, two of one type (a) and two of another type (b).

The number of arrangements of the 4 letters is $\frac{4!}{2!\times 2!}$,

$$\begin{aligned}\frac{4!}{2!\times 2!} &= \frac{4\times 3\times 2!}{2\times 1\times 2!}\\ &= \frac{12}{2}\\ &= 6\end{aligned}$$

There are 6 possible arrangements.

b $(a+b)^4 = a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + b^4$

The coefficient of the a^2b^2 term is $\binom{4}{2}$.

$$\begin{aligned}\binom{4}{2} &= \frac{4!}{(4-2)!\times 2!}\\ &= \frac{4!}{2!\times 2!}\\ &= 6\end{aligned}$$

This is the same as part a because there are 6 ways to arrange $a^2b^2 = aabb$. It is the same as arranging the word *abba*.

5 a A die is rolled four times.

S is the event of obtaining a six in a single roll of the die.

$\Pr(S) = \frac{1}{6}$.

F is the event of not obtaining a six in a single roll of the die.

Using the formula for complementary events,

$$\begin{aligned}\Pr(F) &= 1 - \Pr(S)\\ &= 1 - \frac{1}{6}\\ &= \frac{5}{6}\end{aligned}$$

b i The outcome '*SSFS*' represents obtaining a six on the first roll, another six on the second roll, not getting a six on the third roll, and on the fourth roll of the die obtaining another six.

ii There are $\frac{4!}{3!} = 4$ ways of arranging three Ss and one F.

The possible arrangements are *SSSF, SSFS, SFSS, FSSS*.

iii The probability of obtaining exactly three sixes in four rolls of the die is the same as calculating the probability of '*SSSF, SSFS, SFSS, FSSS*'.

$$\Pr(SSSF) = \frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6} = \left(\frac{1}{6}\right)^3\times\frac{5}{6}$$

$$\Pr(SSFS) = \frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6} = \left(\frac{1}{6}\right)^3\times\frac{5}{6}$$

$$\Pr(SFSS) = \frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6} = \left(\frac{1}{6}\right)^3\times\frac{5}{6}$$

$$\Pr(FSSS) = \frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6} = \left(\frac{1}{6}\right)^3\times\frac{5}{6}$$

$$\begin{aligned}\Pr(3\text{ sixes, 1 not a six}) &= 4\times\left(\frac{1}{6}\right)^3\times\frac{5}{6}\\ &= \frac{4\times 5}{36\times 36}\\ &= \frac{5}{9\times 36}\\ &= \frac{5}{324}\end{aligned}$$

c i $(q+p)^4 = q^4 + \binom{4}{1}q^3p + \binom{4}{2}q^2p^2 + \binom{4}{3}qp^3 + p^4$

ii $p = \frac{1}{6}, q = \frac{5}{6}$ from part **a** and $\binom{4}{3} = 4$

The term $\begin{pmatrix}4\\3\end{pmatrix}qp^3 = 4 \times \frac{5}{6} \times \left(\frac{1}{6}\right)^3$ is what was calculated as the probability of three sixes in four rolls of the die in part **biii**.

6 $0.5^4 + {}^4C_1(0.5)^3\,(0.5) + {}^4C_2(0.5)^2(0.5)^2 + {}^4C_3\,(0.5)\,(0.5)^3.$
$+0.5^4$ is the expansion of $(0.5 + 0.5)^4$
Since $0.5 + 0.5 = 1$, $(0.5 + 0.5)^4 = (1)^4 = 1$.

7 a For $(a+b)^4$, $n = 4,\ a = a,\ b = b$
Using the rule for binomial expansion,

$$(a+b)^4 = \begin{pmatrix}4\\0\end{pmatrix}a^4 + \begin{pmatrix}4\\1\end{pmatrix}a^3b + \begin{pmatrix}4\\2\end{pmatrix}a^2b^2 + \begin{pmatrix}4\\3\end{pmatrix}ab^3 + \begin{pmatrix}4\\4\end{pmatrix}b^4$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

b For $(2+x)^4$, $n = 4,\ a = 2,\ b = x$
Using the rule for binomial expansion,

$$(2+x)^4 = \begin{pmatrix}4\\0\end{pmatrix}2^4 + \begin{pmatrix}4\\1\end{pmatrix}2^3x + \begin{pmatrix}4\\2\end{pmatrix}2^2x^2 + \begin{pmatrix}4\\3\end{pmatrix}2x^3 + \begin{pmatrix}4\\4\end{pmatrix}x^4$$

$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

c For $(t-2)^3$, $n = 3,\ a = t,\ b = -2$
Using the rule for binomial expansion,

$$(t-2)^3 = \begin{pmatrix}3\\0\end{pmatrix}t^3 + \begin{pmatrix}3\\1\end{pmatrix}t^2(-2) + \begin{pmatrix}3\\2\end{pmatrix}t(-2)^2 + \begin{pmatrix}3\\3\end{pmatrix}(-2)^3$$

$$= t^3 - 6t^2 + 12t - 8$$

8 a For $(x+y)^3$, $n = 3$, $a = x$, $b = y$.

$$(x+y)^3$$
$$= \begin{pmatrix}3\\0\end{pmatrix}x^3 + \begin{pmatrix}3\\1\end{pmatrix}x^2y + \begin{pmatrix}3\\2\end{pmatrix}xy^2 + \begin{pmatrix}3\\3\end{pmatrix}y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

b For $(a+2)^4$, $n = 4$, $a = a$, $b = 2$.

$$(a+2)^4$$
$$= \begin{pmatrix}4\\0\end{pmatrix}a^4 + \begin{pmatrix}4\\1\end{pmatrix}a^3(2) + \begin{pmatrix}4\\2\end{pmatrix}a^2(2)^2 + \begin{pmatrix}4\\3\end{pmatrix}a(2)^3 + \begin{pmatrix}4\\4\end{pmatrix}(2)^4$$
$$= a^4 + 8a^3 + 24a^2 + 32a + 16$$

c For $(m-3)^4$, $n = 4$, $a = m$, $b = -3$.

$$(m-3)^4$$
$$= \begin{pmatrix}4\\0\end{pmatrix}m^4 + \begin{pmatrix}4\\1\end{pmatrix}m^3(-3) + \begin{pmatrix}4\\2\end{pmatrix}m^2(-3)^2 + \begin{pmatrix}4\\3\end{pmatrix}m(-3)^3 + \begin{pmatrix}4\\4\end{pmatrix}(-3)^4$$
$$= m^4 - 12m^3 + 54m^2 - 108m + 81$$

d For $(2-x)^5$, $n = 5$, $a = 2$, $b = -x$.

$$(2-x)^5$$
$$= \begin{pmatrix}5\\0\end{pmatrix}(2)^5 + \begin{pmatrix}5\\1\end{pmatrix}(2)^4(-x) + \begin{pmatrix}5\\2\end{pmatrix}(2)^3(-x)^2 + \begin{pmatrix}5\\3\end{pmatrix}(2)^2(-x)^3 + \begin{pmatrix}5\\4\end{pmatrix}(2)(-x)^4 + \begin{pmatrix}5\\5\end{pmatrix}(-x)^5$$
$$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$

9 a For $(m+3b)^2$, $n = 2,\ a = m,\ b = 3b$
Using the rule for binomial expansion,

$$(m+3b)^2 = \begin{pmatrix}2\\0\end{pmatrix}m^2 + \begin{pmatrix}2\\1\end{pmatrix}(m)(3b) + \begin{pmatrix}2\\2\end{pmatrix}(3b)^2$$

$$= m^2 + 6bm + 9b^2$$

b For $(2d-x)^4$, $n = 4,\ a = 2d,\ b = -x$
Using the rule for binomial expansion,

$$(2d-x)^4 = \begin{pmatrix}4\\0\end{pmatrix}(2d)^4 + \begin{pmatrix}4\\1\end{pmatrix}(2d)^3(-x) + \begin{pmatrix}4\\2\end{pmatrix}(2d)^2(-x)^2 + \begin{pmatrix}4\\3\end{pmatrix}(2d)(-x)^3 + \begin{pmatrix}4\\4\end{pmatrix}(-x)^4$$

$$= 16d^4 - 32d^3x + 24d^2x^2 - 8dx^3 + x^4$$

c For $\left(h+\frac{2}{h}\right)^3$, $n = 3,\ a = h,\ b = \frac{2}{h}$

Using the rule for binomial expansion,

$$\begin{aligned}\left(h+\frac{2}{h}\right)^3 &= \binom{3}{0}h^3 + \binom{3}{1}h^2\left(\frac{2}{h}\right) + \binom{3}{2}h\left(\frac{2}{h}\right)^2 + \binom{3}{3}\left(\frac{2}{h}\right)^3 \\ &= h^3 + 3h^2 \times \frac{2}{h} + 3h \times \frac{4}{h^2} + \frac{8}{h^3} \\ &= h^3 + 6h + \frac{12}{h} + \frac{8}{h^3}\end{aligned}$$

10 a $(0.5+0.5)^4 = 0.5^4 + {}^4C_1(0.5)^3\,(0.5) + {}^4C_2(0.5)^2(0.5)^2$
$+{}^4C_3\,(0.5)\,(0.5)^3 + 0.5^4$

i Pr(all 4 children are girls) $= 0.5^4$

ii Pr(all 4 children are boys) $= 0.5^4$

iii There are ${}^4C_2 \times {}^2C_2$ of selecting 2 boys and 2 girls from 4 children.
Since ${}^2C_2 = 1$, we can say there are ${}^4C_2$ of selecting 2 boys and 2 girls from 4 children.
The term that gives the probability of this event is ${}^4C_2(0.5)^2(0.5)^2$.

iv For there to be only one boy in the family, the boy can be selected in ${}^4C_1$ ways, leaving the rest of the children to be girls.
One term that gives this probability is ${}^4C_1(0.5)(0.5)^3$.
However, in this case where the probability of a boy and the probability of a girl are both the same value, 0.5, the term ${}^4C_3(0.5)^3(0.5)$ would give the same answer as ${}^4C_1(0.5)(0.5)^3$.

b Using complementary events,
Pr(at least one of the children is a girl) = 1− Pr(no girls).
The probability of no girls is the same as the probability of all boys.
Pr(at least one of the children is a girl) $= 1 - (0.5)^4$

$$\begin{aligned}1-(0.5)^4 &= 1 - \left(\frac{1}{2}\right)^4 \\ &= 1 - \frac{1}{16} \\ &= \frac{15}{16}\end{aligned}$$

The probability of at least one girl is $\frac{15}{16}$.

11 a For the terms in the expansion of $(p+q)^n, n \in N$ to represent the sample space probabilities of a set of independent trials, p and q.
This is because the total sum of the probabilities must equal 1. Hence, $p + q = 1$.
Also, every probability must lie between 0 and 1, so $0 \le p \le 1$ and $0 \le q \le 1$.
The relationship between p and q is $p + q = 1$, $0 \le p \le 1$ and $0 \le q \le 1$.

b There are two outcomes for each trial, one that has probability p and one that has probability q.

c The number of trials (such as the number of times a coin is tossed) is the value of n.

d The expansion of $(p+q)^n$ contains $(n+1)$ terms. Therefore, there are $(n+1)$ elements in the sample space.

12 a There are a total of 8 biscuits, of which 5 are chocolate and 3 are jam-centred.
Case A: Three biscuits are chosen with replacement.
Since there are three trials, $n = 3$.
The chance of selecting a chocolate biscuit on each trial is $\frac{5}{8}$ and the chance of selecting a jam-centred biscuit on each trial is $\frac{3}{8}$.
Hence, we can say $p = \frac{5}{8}, q = \frac{3}{8}$.

b Case B: Three biscuits are chosen without replacement.
As the three trials in Case B are not independent, because the sampling is without replacement, the sample space probabilities are not the terms in the expansion of $(p+q)^n$.

13 $(p+q)^6$ in sigma notation can be written as $\sum_{r=0}^{6}\binom{6}{r}p^{6-r}q^r$.

A die is rolled six times. If p is the probability of obtaining a five in a single trial and q is the probability of not obtaining a five in a single trial, then $p = \frac{1}{6}, q = \frac{5}{6}$.
As the die is rolled six times, if exactly 2 fives are obtained, then 4 non-fives are obtained.
To find this probability we need the term in p^2q^4.

Consider $\sum_{r=0}^{6}\binom{6}{r}p^{6-r}q^{r}$. The general term is $\binom{6}{r}p^{6-r}q^{r}$.

For the term in p^2q^4, $6 - r = 2$ and $r = 4$.

Substitute $r = 4$ in the term $\binom{6}{r}p^{6-r}q^{r}$.

$$\begin{aligned}
&\binom{6}{4}p^2q^4\\
&= \frac{6!}{(6-4)!\times -4!}p^2q^4\\
&= \frac{6\times 5\times -4!}{2!\times -4!}p^2q^4\\
&= 15p^2q^4
\end{aligned}$$

Substitute $p = \frac{1}{6}, q = \frac{5}{6}$ to calculate the probability of exactly two fives.

$$15p^2q^2 = 15\times\left(\frac{1}{6}\right)^2\times\left(\frac{5}{6}\right)^4$$

14 a
$$\begin{aligned}
(2w-3)^5 &= \sum_{r=0}^{5}\binom{5}{r}(2w)^{5-r}(-3)^r\\
&= \sum_{r=0}^{5}\binom{5}{r}2^{5-r}(-3)^r w^{5-r}
\end{aligned}$$

For the term in w^3, $r = 2$.

$$\begin{aligned}
\binom{5}{r}2^{5-r}(-3)^r w^{5-r} &= \binom{5}{2}2^{5-2}(-3)^2 w^{5-2}\\
&= 80\times 9w^3\\
&= 720w^3
\end{aligned}$$

Therefore, the term is $720w^3$.

b
$$\begin{aligned}
\left(3-\frac{1}{b}\right)^7 &= \sum_{r=0}^{7}\binom{7}{r}3^{7-r}\left(-\frac{1}{b}\right)^r\\
&= \sum_{r=0}^{7}(-1)^r\binom{7}{r}3^{7-r}\frac{1}{b^r}
\end{aligned}$$

For the term in b^{-4}, $r = 4$.

$$\begin{aligned}
(-1)^r\binom{7}{r}3^{7-r}\frac{1}{b^r} &= (-1)^4\binom{7}{4}3^{7-4}\frac{1}{b^4}\\
&= 35\times 27\times\frac{1}{b^4}\\
&= \frac{945}{b^4}
\end{aligned}$$

Therefore the term is $\frac{945}{b^4}$.

c
$$\begin{aligned}
\left(y-\frac{3}{y}\right)^4 &= \sum_{r=0}^{4}\binom{4}{r}y^{4-r}\left(-\frac{3}{y}\right)^r\\
&= \sum_{r=0}^{4}(-1)^r\binom{4}{r}3^r y^{4-2r}
\end{aligned}$$

For the constant term, $4 - 2r = 0$.

$\rightarrow r = 2$

$$\begin{aligned}
(-1)^r\binom{4}{r}3^r y^{4-2r} &= (-1)^2\binom{4}{2}3^2 y^{4-4}\\
&= 1\times 6\times 9\\
&= 54
\end{aligned}$$

Therefore, the constant term is 54.

15 a
$$\begin{aligned}
\left(x+\frac{1}{2x}\right)^4 &= \sum_{r=0}^{4}\binom{4}{r}x^{4-r}\left(\frac{1}{2x}\right)^r\\
&= \sum_{r=0}^{4}\binom{4}{r}2^{-r}x^{4-r}x^{-r}\\
&= \sum_{r=0}^{4}\binom{4}{r}2^{-r}x^{4-2r}
\end{aligned}$$

For the term independent of x,

$4 - 2r = 0$

$\rightarrow r = 2$

$$\begin{aligned}
\binom{4}{r}2^{-r}x^{4-2r} &= \binom{4}{2}2^{-2}\\
&= 6\times\frac{1}{4}\\
&= \frac{3}{2}
\end{aligned}$$

The term $\frac{3}{2}$ is independent of x.

b
$$\begin{aligned}
\left(2m-\frac{1}{3m}\right)^6 &= \sum_{r=0}^{6}\binom{6}{r}(2m)^{6-r}\left(\frac{1}{3m}\right)^r\\
&= \sum_{r=0}^{6}\binom{6}{r}2^{6-r}m^{6-r}3^{-r}m^{-r}\\
&= \sum_{r=0}^{4}\binom{4}{r}2^{6-r}3^{-r}m^{6-2r}
\end{aligned}$$

For the term in m^2, we need to make the power of m equal to 2.

$6 - 2r = 2$

$\rightarrow r = 2$

$$\begin{aligned}
\binom{6}{r}2^{6-r}3^{-r}m^{6-2r} &= \binom{6}{2}2^4 3^{-2}m^2\\
&= 15\times 16\times\frac{1}{9}\times m^2\\
&= \frac{80}{3}m^2
\end{aligned}$$

16 a
$$\begin{aligned}
(2b+3d)^5 &= \sum_{r=0}^{5}\binom{5}{r}(2b)^{5-r}(3d)^r\\
&= \sum_{r=0}^{5}\binom{5}{r}2^{5-r}3^r b^{5-r}d^r
\end{aligned}$$

For the term in d^5, $r = 5$.

$$\begin{aligned}
\binom{5}{r}2^{5-r}3^r b^{5-r}d^r &= \binom{5}{5}2^{5-5}3^5 b^{5-5}d^5\\
&= 1\times 1\times 243\times 1\times d^5\\
&= 243d^5
\end{aligned}$$

Therefore, the term is $243d^5$.

b
$$\begin{aligned}
(3x-5y)^5 &= \sum_{r=0}^{5}\binom{5}{r}(3x)^{5-r}(-5y)^r\\
&= \sum_{r=0}^{5}\binom{5}{r}3^{5-r}(-5)^r x^{5-r}y^r
\end{aligned}$$

For the coefficient of the term x^2y^3, $5 - r = 2$.

$\rightarrow r = 3$

$$\begin{aligned}
\binom{5}{r}3^{5-r}(-5)^r x^{5-r}y^r &= \binom{5}{3}3^{5-3}(-5)^3 x^{5-3}y^3\\
&= 10\times 9\times -125\times x^2\times y^3\\
&= -11\,250x^2y^3
\end{aligned}$$

Therefore, the coefficient of the term x^2y^3 is −11 250.

17 a For the 4th coefficient in the 7th row of Pascal's triangle, $n = 7$ and $r = 3$.

The 4th coefficient in the 7th row is ${}^7\text{C}_3$.

b
$$\begin{aligned}\binom{9}{6} &= \frac{9!}{(9-6)!\,6!}\\ &= \frac{9\times 8\times 7}{3!}\\ &= \frac{504}{6}\\ &= 84\end{aligned}$$
$${}^9\text{C}_6 = 84$$

c In a single toss of the biased coin, $\Pr(H) = 0.55$ and $\Pr(T) = 1 - 0.55 = 0.45$.

The event of obtaining 12 Heads in 18 tosses of the coin means obtaining 12 Heads and 6 Tails.

The number of ordered selections of 18 objects in which 12 are alike of one type (H) and 6 are alike of another type (T) is $\binom{18}{12}$.

The probability of obtaining 12 Heads in 18 tosses of the coin is given by the expression $\binom{18}{12}(0.55)^{12}(0.45)^6$.

18 a
$$\begin{aligned}\binom{22}{8} &= \frac{22!}{(22-8)!\,8!}\\ &= \frac{22!}{14!\,8!}\\ &= \frac{22!}{8!\,14!}\\ &= \frac{22!}{(22-14)!\,14!}\\ &= \binom{22}{14}\end{aligned}$$

b
$$\binom{15}{7} + \binom{15}{6} = \binom{16}{7}$$

Evaluate LHS:
$$\binom{15}{7} + \binom{15}{6} = \binom{16}{7}$$
$$\begin{aligned}\binom{15}{7} + \binom{15}{6} &= \frac{15!}{(15-7)!\,7!} + \frac{15!}{(15-6)!\,6!}\\ &= \frac{15!}{8!\,7!} + \frac{15!}{9!\,6!}\\ &= \frac{15\times 14\times 13\times 12\times 11\times 10\times 9}{7!}\\ &\quad + \frac{15\times 14\times 13\times 12\times 11\times 10}{6!}\\ &= 6435 + 5005\\ &= 11\,440\end{aligned}$$

Evaluate RHS:
$$\begin{aligned}\binom{16}{7} &= \frac{16!}{(16-7)!\,7!}\\ &= \frac{16!}{9!\,7!}\\ &= \frac{16\times 15\times 14\times 13\times 12\times 11\times 10}{7!}\\ &= 11\,440\end{aligned}$$
$$\text{LHS} = \text{RHS}$$

c Given ${}^n\text{C}_r +{}^n\text{C}_{r+1} = {}^{n+1}r + 1$,

Use the formula to express the LHS:
$$\begin{aligned}{}^n\text{C}_r +{}^n\text{C}_{r+1} &= \frac{n!}{(n-r)\,!\,r!} + \frac{n!}{(n-r-1)\,!\,(r+1)\,!}\\ &= n!\left[\frac{1}{(n-r)\,!\,r!} + \frac{1}{(n-r-1)\,!\,(r+1)\,!}\right]\\ &= n!\left[\frac{r+1+n-r}{(n-r)\,!\,(r+1)\,!}\right]\\ &= n!\left[\frac{n+1}{(n-r)\,!\,(r+1)\,!}\right]\\ &= n!\,\frac{(n+1)\,!}{(n-r)\,!\,(r+1)\,!}\\ &= {}^{n+1}\text{C}_{r+1}\end{aligned}$$

19 Given $\left(2-\sqrt{5}\right)^4 = a + b\sqrt{5}$,

for $\left(2-\sqrt{5}\right)^4$, $n = 4$, $a = 2$, $b = -\sqrt{5}$.
$$\begin{aligned}\left(2-\sqrt{5}\right)^4 &= \binom{4}{0}2^4 + \binom{4}{1}2^3\left(-\sqrt{5}\right) + \binom{4}{2}2^2\left(-\sqrt{5}\right)^2\\ &\quad + \binom{4}{3}2\left(-\sqrt{5}\right)^3 + \binom{4}{4}\left(-\sqrt{5}\right)^4\\ &= 16 - 32\sqrt{5} + 120 - 40\sqrt{5} + 25\\ &= 161 - 72\sqrt{5}\end{aligned}$$
$$a = 161, \text{ and } b = -72$$

20 $1 - 6m + 15m^2 - 20m^3 + 15m^4 - 6m^5 + m^6 = (a-m)^6$
$$\begin{aligned}(a-m)^6 &= \binom{6}{0}a^6 + \binom{6}{1}a^5(-m) + \binom{6}{2}a^4(-m)^2\\ &\quad + \binom{6}{3}a^3(-m)^3 + \binom{6}{4}a^2(-m)^4\\ &\quad + \binom{6}{5}a(-m)^5 + \binom{6}{6}(-m)^6\end{aligned}$$
$$= a^2 - 6a^5m + 15a^4m^2 - 20a^3m^3 + 15a^2m^4 + 6am^5 + m^6$$
$$\therefore a^6 - 6a^5m + 15a^4m^2 - 20a^3m^3 + 15a^2m^4 + 6am^5 + m^6$$
$$= 1 - 6m + 15m^2 - 20m^3 + 15m^4 - 6m^5 + m^6$$

Equating equivalent terms (first term):
$$a^6 = 1$$
$$a = \pm 1$$

Check if positive or negative by equating equivalent terms (second term):
$$\begin{aligned}-6a^5m &= -6m\\ a^5 &= 1\\ a &= 1\end{aligned}$$
$$\therefore 1 - 6m + 15m^2 - 20m^3 + 15m^4 - 6m^5 + m^6 = (1-m)^6$$

21 $(1-x)^4 - 4(1-x)^3 + 6(1-x)^2 - 4(1-x) + 1$ is the expansion of $((1-x)-1)^4 = (-x)^4 = x^4$.

22
$$\begin{aligned}(1+kx)^n &= \sum_{r=0}^{n}\binom{n}{r}1^{n-r}(kx)^r\\ &= \sum_{r=0}^{n}\binom{n}{r}1^{n-r}k^rx^r\end{aligned}$$

For the 1*st* term, $r = 0$.

$$\binom{n}{r}1^{n-r}k^rx^r = \binom{n}{0}1^{n-r}k^0x^0$$

$$= \binom{n}{0}1^n$$

$$= 1^n$$

The first term in the expansion is 1.

$1^n = 1$

This is unhelpful because 1 raised to any power is still 1.

For the second term, $r = 1$.

$$\binom{n}{r}1^{n-1}k^rx^r = \binom{n}{1}1^{n-1}k^1x^1$$

$$= \binom{n}{1}1^{n-1}kx$$

The second term in the expansion is $2x$.

$$\binom{n}{2}1^{n-1}kx = 2x \to [1]$$

For the third term, $r = 2$.

$$n \times 1 \times kx = 2x$$

$$nk = 2$$

$$k = \frac{2}{n}$$

$$\binom{n}{r}1^{n-r}k^r - x^r = \binom{n}{2}$$

$$= \frac{n!}{2!\,(n-2)!} \times 1 \times k^2 \times x^2$$

$$= \frac{n \times (n-1) \times \cancel{(n-2)}!}{2!\,\cancel{(n-2)}!}k^2x^2$$

$$= \frac{n \times (n-1)}{2}k^2x^2$$

The third term in the expansion is $\frac{3}{2}x^2$.

$$\frac{n \times (n-1)}{2}k^2\,x^2 = 3 \quad \to [2]$$

$$n \times (n-1) \times k^2 = 3$$

$$k^2(n^2 - n) = 3$$

Substitute [1] in [2]:

$$\left(\frac{2}{n}\right)^2(n^2 - n) = 3$$

$$\frac{4}{n^2}(n^2 - n) = 3$$

$$4 - \frac{4}{n} = 3$$

$$1 = \frac{4}{n}$$

$$n = 4$$

Substitute in [1]:

$$k = \frac{2}{4}$$

$$k = \frac{1}{2}$$

$$\therefore n = 4 \quad \text{and} \quad k = \frac{1}{2}$$

23 $$\left(1 + \frac{x}{2}\right)^n = \sum_{r=0}^{n}\binom{n}{r}1^{n-r}\left(\frac{x}{2}\right)^r$$

$$= \sum_{r=0}^{n}\binom{n}{r}\frac{1}{2^r}x^r$$

For the term in x^3, $r = 3$.

$$\binom{n}{r}\frac{1}{2^r}x^r = \binom{n}{3}\frac{1}{2^3}$$

$$= \frac{n!}{3!\,(n-3)!} \times \frac{1}{8}x^3$$

$$= \frac{n \times (n-1) \times (n-2) \times \cancel{(n-3)}!}{3!\,\cancel{(n-3)}! \times 8}x^3$$

$$= \frac{n \times (n-1)(n-2) \times (n-2)x^3}{48}$$

Given that the coefficient of x^3 is 70,

$$\frac{n(n-1)(n-2)}{48} = 70$$

$$n(n-1)(n-2) = 3360$$

$$(n^2 - n)(n-2) = 3360$$

$$n^3 - 2n^2 - n^2 + 2n = 3360$$

$$n^3 - 2n^2 - n^2 + 2n - 3360 = 0$$

$$n^3 - 3n^2 + 2n - 3360 = 0$$

$$(n-16)(n^2 + 13n + 210) = 0$$

$$\therefore n = 16$$

24 Let the three consecutive terms in Pascal's triangle be ${}^n\text{C}_{r-1}, {}^n\text{C}_r, {}^n\text{C}_{r+1}$.

The three terms are in the ratio 13 : 8 : 4.

Hence, ${}^n\text{C}_{r-1} : {}^n\text{C}_r = 13 : 8$ and ${}^n\text{C}_r : {}^n\text{C}_{r+1} = 8 : 4 = 2 : 1$.

Consider ${}^n\text{C}_{r-1} : {}^n\text{C}_r = 13 : 8$.

$$\frac{{}^n\text{C}_{r-1}}{{}^n\text{C}_r} = \frac{13}{8}$$

$$\frac{n!}{(n-r+1)!\,(r-1)!} \times \frac{(n-r)!\,r!}{n!} = \frac{13}{8}$$

$$\frac{1}{n-r+1} \times \frac{r}{1} = \frac{13}{8}$$

$$8r = 13(n-r+1)$$

$$21r = 13n + 13 \to [1]$$

Consider ${}^n\text{C}_r : {}^n\text{C}_{r+1} = 2 : 1$

$$\frac{{}^n\text{C}_r}{{}^n\text{C}_{r+1}} = \frac{2}{1}$$

$$\frac{n!}{(n-r)!\,r!} \times \frac{(n-r-1)!\,(r+1)!}{n!} = \frac{2}{1}$$

$$\frac{1}{n-r} \times \frac{r+1}{1} = 2$$

$$r + 1 = 2(n-r)$$

$$3r = 2n - 1 \to [2]$$

Substitute [2] in [1]:

$$7(2n-1) = 13n + 13$$

$$n = 20$$

Substitute $n = 20$ in [2]:

$$3r = 2(20) - 1$$

$$3r = 39$$

$$r = 3$$

The three terms are ${}^n\text{C}_{r-1}, {}^n\text{C}_r, \text{C}_{r+1} = {}^{20}\text{C}_{12}, {}^{20}\text{C}_{13}, {}^{20}\text{C}_{14}$

Using a calculator, these terms are

125 970, 77 520, 38 760 (or ${}^{20}\text{C}_{12}, {}^{20}\text{C}_{13}, {}^{20}\text{C}_{14}$).

7.6 Exam questions

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

1 $$\left(2x + \frac{3}{x^3}\right)^4 = \sum_{r=0}^{4}\binom{4}{r}(2x)^{4-r}\left(\frac{3}{x^3}\right)^r \qquad \text{[1 mark]}$$

$$= \sum_{r=0}^{4}\binom{4}{r}2^{4-r}x^{4-r}3^{2r}x^{-3r}$$

Power of $x = 0$ for the term independent of x

$4 - r - 3r = 0$ [1 mark]

$r = 1$ (2nd term)

$\therefore$ 2nd term $= \binom{4}{1} 2^3x^3 3^1 x^{-3}$ [1 mark]

$= 4 \times 8 \times 3$

$= 96$

2 $\binom{7}{5} = \binom{7}{2}$

$= \dfrac{7 \times 6 \times \cancel{5!}}{2 \times \cancel{5!}}$

$= 21$

The correct answer is **C**.

3 The values of n and r are required. Remember, numbering starts at 0.

$n = 14$

$r = 9 - 1$

$= 8$

$\therefore {}^nC_r = {}^{14}C_8$

The correct answer is **B**.

7.7 Review

7.7 Exercise

Technology free: short answer

1 a Given $\Pr(A) = 0.4$, $\Pr(A \cup B) = 0.58$ and $\Pr(B|A) = 0.3$, using the formula for conditional probability,

$\Pr(B|A) = \dfrac{\Pr(B \cap A)}{\Pr(A)}$

$0.3 = \dfrac{\Pr(B \cap A)}{0.4}$

$\Pr(B \cap A) = 0.12$

Note that $\Pr(B \cap A) = \Pr(A \cap B)$.

Using the addition formula,

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$0.58 = 0.4 + \Pr(B) - 0.12$

$\Pr(B) = 0.3$

b A and B are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$\Pr(A) = 0.4$ and $\Pr(B) = 0.3$.

$\Pr(A \cap B) = 0.12$.

Substitute values into the formula $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

LHS $= 0.12$

RHS $= 0.4 \times 0.3$

$= 0.12$

Since LHS = RHS, the events A and B are independent.

2 6 red and 7 green balls gives a total of 13 balls.

a Require RG or GR.

Selecting without replacement,

$\Pr(RG\text{or}GR) = \dfrac{6}{13} \times \dfrac{7}{12} + \dfrac{7}{13} \times \dfrac{6}{12}$

$= \dfrac{1}{13} \times \dfrac{7}{2} + \dfrac{7}{13} \times \dfrac{1}{2}$

$= \dfrac{7}{26} + \dfrac{7}{26}$

$= \dfrac{14}{26}$

$= \dfrac{7}{13}$

b At least one green ball:

Pr(at least one green ball) = 1− Pr(no green balls)

No green balls ⇒ both balls are red.

Pr(at least one green ball)

$= 1 - \Pr(RR)$

$= 1 - \dfrac{6}{13} \times \dfrac{5}{12}$

$= 1 - \dfrac{1}{13} \times \dfrac{5}{2}$

$= 1 - \dfrac{5}{26}$

$= \dfrac{21}{26}$

c Pr(one ball of each colour is chosen given at least one green ball is chosen)

This is the conditional probability $\Pr(RG \text{ or } GR \mid \geq 1G)$.

$\Pr(RG \text{ or } GR| \geq 1G)$

$= \dfrac{\Pr(RG \text{ or GR})}{\Pr(\geq 1G)}$

$= \dfrac{7}{13} \div \dfrac{21}{26}$

$= \dfrac{7}{13} \times \dfrac{26}{21}$

$= \dfrac{2}{3}$

3 a $\left(1 + \dfrac{x}{2}\right)^7 = \displaystyle\sum_{r=0}^{7} \binom{7}{r} 1^{7-r} \left(\dfrac{x}{2}\right)^r$

$= \displaystyle\sum_{r=0}^{7} \binom{7}{r} \dfrac{1}{2^r} x^r$

For the term in x^5, $r = 5$.

$\binom{7}{r} \dfrac{1}{2^r} x^r = \binom{7}{5} \dfrac{1}{2^5} x^5$

$= 21 \times \dfrac{1}{32} \times x^5$

$= \dfrac{21}{32} x^5$

Therefore, the coefficient of x^5 is $\dfrac{21}{32}$.

b For $\left(3 - \sqrt{2}\right)^3$, $n = 3$, $a = 3$, $b = -\sqrt{2}$.

$\left(3 - \sqrt{2}\right)^3$

$= \binom{3}{0} 3^3 + \binom{3}{1} 3^2 \left(-\sqrt{2}\right) + \binom{3}{2} 3 \left(-\sqrt{2}\right)^2 + \binom{3}{3} \left(-\sqrt{2}\right)^3$

$= 27 - 27\sqrt{2} + 18 - 2\sqrt{2}$

$= 45 - 29\sqrt{2}$

c $\left(x^2 - \dfrac{3}{x}\right)^4$

For $\left(x^2 - \dfrac{3}{x}\right)^4$, $n = 4$, $a = x$, $b = -\dfrac{3}{x}$.

$$\left(x^2 - \frac{3}{x}\right)^4$$
$$= \binom{4}{0}\left(x^2\right)^4 + \binom{4}{1}\left(x^2\right)^3\left(-\frac{3}{x}\right) + \binom{4}{2}\left(x^2\right)^2\left(-\frac{3}{x}\right)^2$$
$$+ \binom{4}{3}\left(x^2\right)\left(-\frac{3}{x}\right)^3 + \binom{4}{4}\left(-\frac{3}{x}\right)^4$$
$$= x^8 - 12x^5 + 54x^2 - \frac{108}{x} + \frac{81}{x^4}$$
$$= x^8 - 12x^5 + 54x^2 - 108x^{-1} + 81x^{-4}$$

d $$\left(\frac{3}{y^2} - y\right)^{12} = \sum_{r=0}^{12}\binom{12}{r}\left(\frac{3}{y^2}\right)^{12-r}(-y)^r$$
$$= \sum_{r=0}^{12}(-1)^r\binom{12}{r}3^{12-r}y^{-2(12-r)}y^r$$
$$= \sum_{r=0}^{12}(-1)^r\binom{12}{r}3^{12-r}y^{-24+3r}$$

For the term independent of y, $-24 + 3r = 0$.

$\rightarrow r = 8$

$$(-1)^r\binom{12}{r}3^{12-r}y^{-24+3r} = (-1)^8\binom{12}{8}3^{12-8}y^{-24+3\times 8}$$
$$= 1 \times 495 \times 81$$
$$= 40\ 095$$

Therefore, the term independent of y is 40 095.

4 If $\Pr(F) = 0.7$, $\Pr(F') = 0.3$.

75% of female students graduate; therefore,

$\Pr(F \cap G) = 0.7 \times 0.75$

$= 0.525$

85% of male students graduate; therefore,

$\Pr(F' \cap G) = 0.3 \times 0.85$

$= 0.255$

Also, $\Pr(\xi) = 1$.

	G	G'	
F	0.525		0.7
F'	0.255		0.3
			1

Fill in the remaining gaps using arithmetic:

	G	G'	
F	0.525	0.175	0.7
F'	0.255	0.045	0.3
	0.78	0.22	1

$$\Pr(F'|G) = \frac{\Pr(F' \cap G)}{\Pr(G)}$$
$$= \frac{0.255}{0.78}$$
$$= \frac{17}{52}$$

5 **a** Given $n(L) = 20$:

For the first row, $18 + 67 = 85$.

For the first column, $18 + 2 = 20$.

For the third row, $20 + 80 = 100$.

For the third column, $85 + 15 = 100$.

For the second row, $2 + 13 = 15$.

	Liars (L)	Honest people (H)	Totals
Correctly tested (C)	18	67	85
Incorrectly tested (I)	2	13	15
Totals	20	80	100

b Using the values from the table,

$$\Pr(I) = \frac{n(I)}{n(\xi)}$$
$$= \frac{15}{100}$$
$$= \frac{3}{20}$$

c Using the values from the table,

$$\Pr(H \cap C) = \frac{n(H \cap C)}{n(\xi)}$$
$$= \frac{67}{100}$$

d Using the values from the table and the rule for conditional probability,

$$\Pr(C|H) = \frac{n(C \cap H)}{n(H)}$$
$$= \frac{67}{80}$$

6 **a** A group of 5 people consisting of 2 girls and 3 boys.

i 5 people can be arranged in a row in 5! ways.

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

There are 120 ways to arrange 2 girls and 3 boys in a row.

ii First place the girls on the ends. There are two choices for the girl on the first end and one choice for the other end.

2				1

Then arrange the 3 boys in the three spaces between the girls.

2	3	2	1	1

Using the multiplication principle, the number of arrangements with the girls on each end is

$2 \times 1 \times 3 \times 2 \times 1 = 12$.

iii Treat the two girls as one unit that can be internally arranged in 2! ways

There are now 3 boys and one unit of girls, making a total of 4 units.

4 units can be arranged in 4! ways.

The total number of arrangements is $4! \times 2!$.

$4! \times 2! = 4 \times 3 \times 2 \times 1 \times 2 = 48$

There are 48 arrangements with the girls together.

b 2 girls can be chosen from a group of 6 girls in ${}^6C_2$ ways.

$${}^6C_2 = \frac{6!}{(6-2)! \times 2!}$$
$$= \frac{6!}{4! \times 2!}$$
$$= \frac{6 \times 5 \times 4!}{4! \times 2 \times 1}$$
$$= 15$$

3 boys be chosen from a group of 5 boys in ${}^5C_3$ ways.

$$\begin{aligned} {}^5C_3 &= \frac{5!}{(5-3)!\times 3!} \\ &= \frac{5!}{2!\times 3!} \\ &= \frac{5\times 4\times 3!}{2\times 1\times 3!} \\ &= 10 \end{aligned}$$

The number of ways of choosing 2 girls and 3 boys from a group of 6 girls and 5 boys is $15 \times 10 = 150$ ways.

c The probability of a girl is 0.52.

Therefore, the probability of a boy is $1 - 0.52 = 0.48$.

The probability of having 2 girls and 3 boys in a group of 5 babies is one of the terms in the expansion of $(0.52 + 0.48)^5$.

The term required is $\binom{5}{2}(0.52)^2(0.48)^3$.

Technology active: multiple choice

7 B and C are mutually exclusive. This is because: B is the set of positive prime numbers less than 50, and C is the set of positive multiples of 10 less than or equal to 50. For C, the number 10 is not a prime number, and any multiple of 10 is also not a prime number. Therefore, there are no elements common to subsets B and C. Hence they are mutually exclusive.

The correct answer is **D**.

8 For Ash scoring a goal, $\Pr(A) = \frac{2}{3}$, so the probability Ash does not score a goal is $\Pr(A') = 1 - \frac{2}{3} = \frac{1}{3}$.

For Ben scoring a goal, $\Pr(B) = \frac{2}{5}$, so the probability Ben does not score a goal is $\Pr(B') = 1 - \frac{2}{5} = \frac{3}{5}$.

The probability that neither scores a goal is

$$\begin{aligned} &\Pr(A') \times \Pr(B') \\ &= \frac{1}{3} \times \frac{3}{5} \\ &= \frac{1}{5} \end{aligned}$$

The correct answer is **A**.

9 Using the multiplication principle, the number of choices of adventure $= 3 \times 4 = 12$.

The correct answer is **C**.

10
$$\begin{aligned} \Pr(G) &= \frac{1}{2} \times \frac{2}{8} + \frac{1}{2} \times \frac{4}{4} \\ &= \frac{1}{8} + \frac{1}{2} \\ &= \frac{5}{8} \end{aligned}$$

The correct answer is **A**.

11
$$\begin{aligned} &\Pr(\geq 3) = \Pr(3) + \Pr(4) + \Pr(5) + \Pr(6) \\ &= \frac{n(3) + n(4) + n(5) + n(6)}{n(\xi)} \\ &= \frac{13 + 5 + 2 + 1}{40} \\ &= \frac{21}{40} \end{aligned}$$

The correct answer is **A**.

12 8 points can be arranged in a row in $8! = 40\,320$ ways.

The correct answer is **E**.

13 There are 7 people. If 5 are in favour of the proposal, then 2 are not in favour.

The probability of being in favour is 0.62, so the probability of not being in favour is$1 - 0.62 = 0.38$.

The five people in favour can be chosen from the 7 people in ${}^7C_5$ ways.

The required probability that 5 are in favour of the proposal is ${}^7C_5\,(0.62)^5\,(0.38)^2$.

The correct answer is **B**.

14 This is a case where order does matter. The number of arrangements of 14 objects taken 3 at a time is ${}^{14}P_3$.

The correct answer is **A**.

15 The word *emergency* has 9 letters, including 3 *e*'s.

The number of words that can be formed, given all letters are used

$$\begin{aligned} &= \frac{9!}{3!} \\ &= 60\,480 \end{aligned}$$

The correct answer is **C**.

16 The 3 men can be chosen from the 7 men available in ${}^7C_3$ ways.

The 4 women can be chosen from the 6 women available in ${}^6C_4$ ways.

The total number of committees that contain 3 men and 4 women is ${}^7C_3 \times {}^6C_4$.

$$\begin{aligned} {}^7C_3 \times {}^6C_4 &= \frac{7!}{3! \times 4!} \times \frac{6!}{4! \times 2!} \\ &= \frac{7\times 6\times 5}{3!} \times \frac{6\times 5}{2!} \\ &= 35 \times 15 \\ &= 525 \end{aligned}$$

There are 525 committees possible with the given restriction.

The correct answer is **D**.

Technology active: extended response

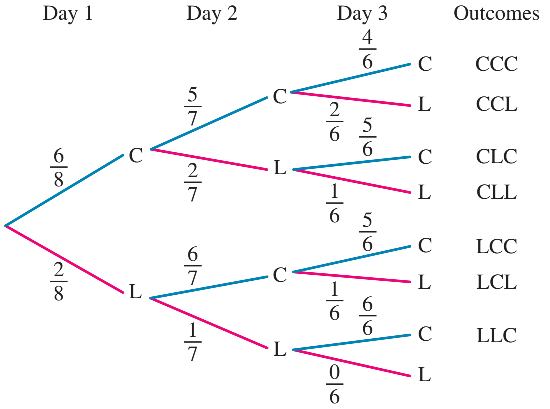
17 **a** Sample space = {C, LC, LLC, LLLC}

b Since there are 7 tins of chickpeas and 3 tins of lentils, the minimum number to ensure Jo opens a tin of chickpeas is 4.

c If the chickpeas are in the third tin Jo opens, this means the previous two must have contained lentils.

$$\begin{aligned} \Pr(LLC) &= \frac{3}{10} \times \frac{2}{9} \times \frac{7}{8} \\ &= \frac{7}{120} \end{aligned}$$

d



e Let A be the probability that Jo opens at least one tin of each over the next 3 nights.

The complement of A would be opening three tins the same over the next 3 nights.

$A' = \{CCC, LLL\}$

$$\begin{aligned}\Pr(A') &= \Pr(CCC) + \Pr(LLL)\\ &= \frac{6}{8}\times\frac{5}{7}\times\frac{4}{6} + \frac{2}{8}\times\frac{1}{7}\times\frac{0}{6}\\ &= \frac{5}{14}\end{aligned}$$

Using the rule for complementary events,

$$\begin{aligned}\Pr(A) &= 1 - \Pr(A')\\ &= 1 - \frac{5}{14}\\ &= \frac{9}{14}\end{aligned}$$

Therefore, the probability that he opens at least one tin of each over the next 3 nights is $\frac{9}{14}$.

f $\Pr(3\text{rd } L \mid 1\text{st } C) = \dfrac{\Pr(3\text{rd } L \cap 1\text{st } C)}{\Pr(1\text{st } C)}$

$$\begin{aligned}&= \frac{\Pr(CCL) + \Pr(CLL)}{\Pr(CCC) + \Pr(CCL) + \Pr(CLC) + \Pr(CLL)}\\ &= \frac{\tfrac{6}{8}\times\tfrac{5}{7}\times\tfrac{2}{6} + \tfrac{6}{8}\times\tfrac{2}{7}\times\tfrac{1}{6}}{\tfrac{6}{8}\times\tfrac{5}{7}\times\tfrac{4}{6} + \tfrac{6}{8}\times\tfrac{5}{7}\times\tfrac{2}{6} + \tfrac{6}{8}\times\tfrac{2}{7}\times\tfrac{5}{6} + \tfrac{6}{8}\times\tfrac{2}{7}\times\tfrac{1}{6}}\\ &= \frac{\tfrac{3}{14}}{\tfrac{3}{4}}\\ &= \frac{2}{7}\end{aligned}$$

18 **a** ${}^{7}C_2 \times {}^{6}C_2 = 315$

b ${}^{2}C_2 \times {}^{6}C_2 = 15$

c $\frac{15}{315} = \frac{1}{21}$

d $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

19 **a** Pr(Aino misses) $= 1 - 20 \times 0.045 - 0.005 - 0.006 = 0.089$

b Aino misses on the first throw but hits on the second. That is, $0.089 \times 0.911 = 0.0811$

c Aino buys an item for \$900 on the first throw means that she scores a 9. Pr(9) $= 0.045$

d If the item costs more than \$1900, then she had to score a 20 or an inner bull or outer bull.

$$\begin{aligned}\Pr(20 \text{ or inner or outer}) &= 0.045 + 0.005 + 0.006\\ &= 0.056\end{aligned}$$

e Bryan wins with first shot if B scores n and A scores $< n$ with n ranging from 2 to 20.

or B scores any number 1 to 20 and A misses the target

or B scores inner bull

or B scores outer bull and A does not score inner bull.

If that is a correct interpretation, then:

$$\begin{aligned}\Pr(\text{B wins with first shot}) = 0.046 \times (1 \times 0.045 + 2 \times 0.045\\ + 3 \times 0.045 + \ldots 19 \times 0.045) + 20 \times 0.046 \times 0.089\\ + 0.004 + 0.003 \times (1 - 0.0045)\end{aligned}$$

Pr(B wins on first shot)

$$\begin{aligned}&= 0.046 \times (190 \times 0.045) + 20 \times 0.046 \times 0.089\\ &\quad + 0.004 + 0.003 \times 0.995\\ &= 0.4822\end{aligned}$$

f $$\begin{aligned}\Pr(\text{Aino beats Bryan's score}) &= 5 \times 0.045 + 0.005 + 0.006\\ &= 0.236\end{aligned}$$

20 **a** 5 teams: A, B, C, D, E

Therefore, the games are:

$A \times B, A \times B, A \times C, A \times C, A \times D, A \times D, A \times E, A \times E,$
$B \times C, B \times C, B \times D, B \times D, B \times E, B \times E,$
$C \times D, C \times D, C \times E, C \times E,$
$D \times E, D \times E$

So, in total there must be 20 games for every team to play each other team twice.

b If there are n teams in the competition, then there must be $n(n-1)$ games played. Each team plays $n-1$ games; therefore, $n(n-1)$ gives the total number of games. (Usually when working this out we need to divide the result by two since each game would be counted twice, e.g. $A \times B$ and $B \times A$. However, since every team plays the same team twice, this can be ignored for this situation.)

c Each team must play each other team twice. Since there are $16 - 1 = 15$ teams to play, each team must play $15 \times 2 = 30$ games.

d $n = 16$

$$\begin{aligned}\text{Total number of games} &= n(n-1)\\ &= 16(16-1)\\ &= 16 \times 15\\ &= 240\end{aligned}$$

Therefore, there are a total of 240 games played if there are 16 teams in the competition.

7.7 Exam questions

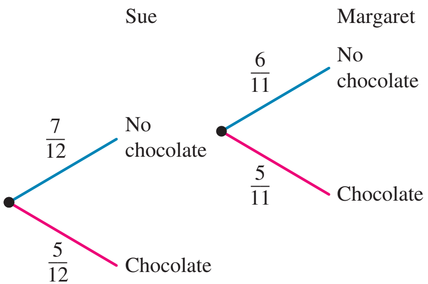
1 If the events A and B are mutually exclusive, then they cannot occur simultaneously.

For mutually exclusive events, $n(A \cap B) = 0$ and, therefore, $\Pr(A \cap B) = 0$.

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B)\\ \therefore \Pr(A \cup B) &= 0.6 + 0.3\\ &= 0.9\end{aligned}$$

The correct answer is **C**.

2



[1 mark]

No chocolate surprise for Sue:

$\Pr(\text{no C}) = \frac{7}{12}$

Chocolate surprise for Margaret:

$\Pr(\text{C}) = \frac{5}{11}$ [1 mark]

$$\begin{aligned}\Pr(\text{no C} \cap \text{C}) &= \frac{7}{12} \times \frac{5}{11}\\ &= \frac{35}{132}\end{aligned}$$ [1 mark]

3 It could have been red then blue, or blue then red.

$$\begin{aligned}\Pr(\text{red}) &= \frac{6}{20}\\ \Pr(\text{blue}) &= \frac{9}{20}\\ \Pr(A \cap B) &= \Pr(A) \times \Pr(B)\\ \Pr(\text{red} \cap \text{blue}) &= \frac{6}{20} \times \frac{9}{20} \times 2\\ &= 0.27\end{aligned}$$

The correct answer is **B**.

4 **a** The colours can be arranged in 2 ways: RGB or BGR.
[1 mark]

$2 \times {}^8C_1 \times {}^5C_1 \times {}^7C_1 = 2 \times 8 \times 7 \times 5$ [1 mark]

$= 560$

b Total of 20 bottles

The total number of ways of choosing three bottles from 20 available is ${}^{20}C_3$.

The number of ways of selecting three green bottles is ${}^7C_3$.

$$\begin{aligned}\Pr(3G) &= \frac{{}^7C_3}{{}^{20}C_3}\\ &= \frac{7!}{3!\,4!} \div \frac{20!}{3!\,17!} \quad \text{[1 mark]}\\ &= \frac{7 \times 6 \times 5 \times \cancel{4!}}{\cancel{3!} \times \cancel{4!}} \times \frac{\cancel{3!} \times \cancel{17!}}{20 \times 19 \times 18 \times \cancel{17}}\\ &= \frac{7 \times 6 \times 5}{20 \times 19 \times 18}\\ &= 0.0307 \quad \text{[1 mark]}\end{aligned}$$

5 Four digits:

$4 \times 3 \times 2 \times 1 = 24$

The correct answer is **B**.

Topic 8 — Trigonometric functions

8.2 Trigonometric ratios

8.2 Exercise

1 **a** $\sin(50°) = \dfrac{h}{10}$

$\therefore h = 10\sin(50°)$

$\therefore h \simeq 7.66$

b Recognising the ‘3, 4, 5’ Pythagorean triad gives

$\tan(a°) = \dfrac{5}{2}$

$\therefore a° = \tan^{-1}(2.5)$

$\therefore a° \simeq 68.20$

Hence, $a \simeq 68.20$.

2 **a** $\cos(27°) = \dfrac{x}{8}$

$\therefore x = 8 \times \cos(27°)$

$\therefore x = 7.13$

$\sin(27°) = \dfrac{y}{8}$

$\therefore y = 8 \times \sin(27°)$

$\therefore y = 3.63$

b $\tan(37°) = \dfrac{10}{x}$

$\therefore x = \dfrac{10}{\tan(37°)}$

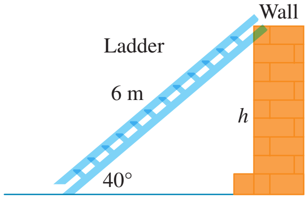
$\therefore x = 13.27$

$\sin(37°) = \dfrac{10}{h}$

$\therefore h = \dfrac{10}{\sin(37°)}$

$\therefore h = 16.62$

3



Let the required height be h metres.

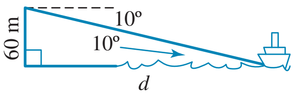
$\sin(40°) = \dfrac{h}{6}$

$\therefore h = 6 \times \sin(40°)$

$\therefore h = 3.86$

Correct to 2 decimal places, the ladder reaches 3.86 metres up the wall.

4



Let the distance of the boat from the base of the cliff be d m.

$\tan(10°) = \dfrac{60}{d}$

$\therefore d = \dfrac{60}{\tan(10°)}$

$\therefore d = 340$

To the nearest metre, the boat is 340 metres from the base of the cliff.

5 **a** $\cos(\theta) = \dfrac{2}{5}$

$\cos(\theta) = 0.4$

$\therefore \theta = \cos^{-1}(0.4)$

$\therefore \theta = 66.42°$

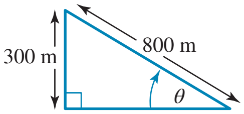
b $\tan(\theta) = \dfrac{10}{8}$

$\tan(\theta) = 1.25$

$\therefore \theta = \tan^{-1}(1.25)$

$\therefore \theta = 51.34°$

6



Let the angle of elevation be θ.

$\sin(\theta) = \dfrac{300}{800}$

$\sin(\theta) = \dfrac{3}{8}$

$\therefore \theta = \sin^{-1}\left(\dfrac{3}{8}\right)$

$\therefore \theta = 22°$

The angle of elevation of the cable is 22° to the nearest degree.

7 **a** $\sin(45°) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

b $\tan(30°) = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

c $\cos(60°) = \dfrac{1}{2}$

d $\tan(45°) + \cos(30°) - \sin(60°)$

$\tan(45°) = 1, \cos(30°) = \dfrac{\sqrt{3}}{2}$ and $\sin(60°) = \dfrac{\sqrt{3}}{2}$

Therefore,

$\tan(45°) + \cos(30°) - \sin(60°)$

$= 1 + \dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2}$

$= 1$

8 Let the ladder reach h metres up the wall.

$\sin(60°) = \dfrac{h}{3}$

$h = 3 \times \sin(60°)$

$h = 3 \times \dfrac{\sqrt{3}}{2}$

$h = \dfrac{3\sqrt{3}}{2}$

The ladder reaches $\dfrac{3\sqrt{3}}{2}$ metres up the wall.

9 Let the broom reach h metres up the wall.

$\cos(30°) = \dfrac{h}{1}$

$h = 1 \times \cos(30°)$

$h = \cos(30°)$

$h = \dfrac{\sqrt{3}}{2}$

The broom reaches $\dfrac{\sqrt{3}}{2}$ metres up the wall.

10 Let x metres be the required distance.

$$\cos(45°) = \frac{x}{4}$$
$$\therefore x = 4\cos(45°)$$
$$= 4 \times \frac{\sqrt{2}}{2}$$
$$= 2\sqrt{2}$$

The foot of the ladder is $2\sqrt{2}$ metres from the fence.

11 $$\frac{\cos(30°)\sin(45°)}{\tan(45°)+\tan(60°)}$$
$$= \frac{\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}}{1+\sqrt{3}}$$
$$= \frac{\sqrt{6}}{4\left(\sqrt{3}+1\right)}$$

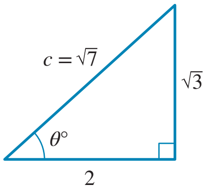
Rationalising the denominator,

$$= \frac{\sqrt{6}}{4\left(\sqrt{3}+1\right)} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$
$$= \frac{\sqrt{6}\left(\sqrt{3}-1\right)}{4(3-1)}$$
$$= \frac{3\sqrt{2}-\sqrt{6}}{8}$$

12 $$\frac{\sin(30°)\cos(45°)}{\tan(60°)}$$
$$= \frac{1}{2}\times\frac{\sqrt{2}}{2}\div\frac{\sqrt{3}}{1}$$
$$= \frac{\sqrt{2}}{4}\times\frac{1}{\sqrt{3}}$$
$$= \frac{\sqrt{2}}{4\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{6}}{12}$$

13 **a** $\tan(\theta°) = \frac{\sqrt{3}}{2}$

Draw a right-angled triangle containing the angle $\theta°$. Label the sides opposite and adjacent to the angle in the ratio $\sqrt{3}$ to 2.



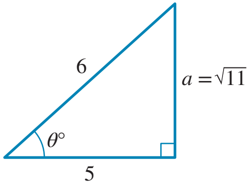
Using Pythagoras' theorem,

$$c^2 = 2^2 + \left(\sqrt{3}\right)^2$$
$$\therefore c^2 = 4 + 3$$
$$\therefore c = \sqrt{7} \quad (c > 0)$$

$\sin(\theta°)$ is the ratio of the opposite side to the hypotenuse.

$$\sin(\theta°) = \frac{\sqrt{3}}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$
$$\therefore \sin(\theta°) = \frac{\sqrt{21}}{7}$$

b $\cos(\theta°) = \frac{5}{6}$

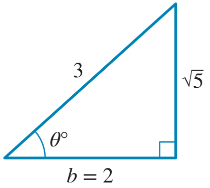


Using Pythagoras' theorem,

$$a^2 + 5^2 = 6^2$$
$$\therefore a^2 + 25 = 36$$
$$\therefore a^2 = 11$$
$$\therefore a = \sqrt{11}$$

Hence, $\tan(\theta°) = \frac{\sqrt{11}}{5}$.

c $\sin(\theta°) = \frac{\sqrt{5}}{3}$



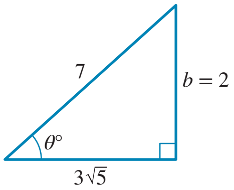
Using Pythagoras' theorem,

$$b^2 + \left(\sqrt{5}\right)^2 = 3^2$$
$$\therefore b^2 + 5 = 9$$
$$\therefore b^2 = 4$$
$$\therefore b = 2$$

Hence, $\cos(\theta°) = \frac{2}{3}$.

14 $\cos(\theta°) = \frac{3\sqrt{5}}{7}$

Construct a right-angled triangle with hypotenuse 7 and adjacent side to θ of $3\sqrt{5}$.

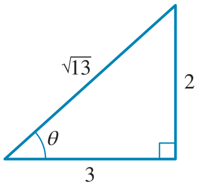


Let the opposite side be b.
Using Pythagoras' theorem.

$$b^2 + \left(3\sqrt{5}\right)^2 = 7^2$$
$$\therefore b^2 + 45 = 49$$
$$\therefore b^2 = 4$$
$$\therefore b = 2$$

Hence, $\sin(\theta°) = \frac{2}{7}$ and $\tan(\theta°) = \frac{2}{3\sqrt{5}}$ or $\frac{2\sqrt{5}}{15}$.

15 **a** If $\tan(a°) = \frac{2}{3}$, using Pythagoras' theorem, a triangle with sides 2, 3 and hypotenuse $\sqrt{13}$ gives $\cos(a°) = \frac{3}{\sqrt{13}}$.



b Let the horizontal run be x cm.

$$\cos(a°) = \frac{x}{26}$$
$$\therefore x = 26\cos(a°)$$
$$= 26 \times \frac{3}{\sqrt{13}}$$
$$= \frac{26 \times 3\sqrt{13}}{13}$$
$$= 6\sqrt{13}$$

Therefore, the horizontal run is $6\sqrt{13}$ cm.

16 Since $\sin(\theta°) = \frac{3}{5}$, the ratio of the opposite side to the hypotenuse is 3 : 5. The set of numbers '3, 4, 5' are a Pythagorean triple, so the sides of a triangle for which $\sin(\theta°) = \frac{3}{5}$ must be in the ratio 3 : 4 : 5.

Since the longest side is the hypotenuse, for the given triangle the hypotenuse is 60 cm. This is a factor of 12 times 5, so the other sides of the triangle must be $3 \times 12 = 36$ cm and $4 \times 12 = 48$ cm.

The shortest side is 36 cm.

17 The angle of 30° is the included angle between two sides each of length 7 m.

Using the area formula Area $= \frac{1}{2}ab\sin C$ with $a = 7,\ b = 7,\ C = 30°$ gives

$$\text{Area} = \frac{1}{2} \times 7 \times 7 \times \sin(30°)$$
$$= \frac{49}{2} \times \frac{1}{2}$$
$$= \frac{49}{4}$$
$$= 12.25$$

The area of the front of the house is 12.25 m^2.

18 **a** The angle A is the included angle between the sides b and c.

Using the area formula Area $= \frac{1}{2}bc\sin A$ with $b = 11, c = 8, A = 60°$ gives

$$\text{Area} = \frac{1}{2} \times 11 \times 8 \times \sin(60°)$$
$$= 44 \times \frac{\sqrt{3}}{2}$$
$$= 22\sqrt{3}$$

The area of triangle ABC is $22\sqrt{3}$ square units.

b The angle B is the included angle between the sides a and c.

Using the area formula Area $= \frac{1}{2}ac\sin B$ with $a = 4\sqrt{2}, c = 5, B = 45°$ gives

$$\text{Area} = \frac{1}{2} \times 4\sqrt{2} \times 5 \times \sin(45°)$$
$$= 10\sqrt{2} \times \frac{\sqrt{2}}{2}$$
$$= 10$$

The area of triangle ABC is 10 square units.

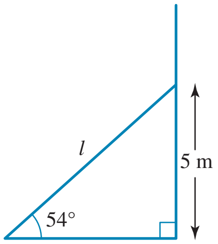
19 $a = 10, b = 6\sqrt{2}, c = 2\sqrt{13}$ cm and $C = 45°$.

Area is $A = \frac{1}{2}\ ab\ \sin(C)$

$$\therefore A = \frac{1}{2} \times 10 \times 6\sqrt{2} \times \sin(45°)$$
$$= 30\sqrt{2} \times \frac{\sqrt{2}}{2}$$
$$\therefore A = 30$$

The area is 30 cm^2.

20 **a** Let the length of the ladder be l metres.



$$\sin(54°) = \frac{5}{l}$$
$$\therefore l = \frac{5}{\sin(54°)}$$
$$\therefore\ l = 6.1803$$

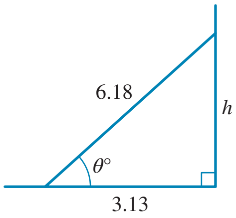
The ladder is 6.18 metres in length.

b Let the initial distance of the ladder from the pole be x metres.

$$\tan(54°) = \frac{5}{x}$$
$$\therefore x = \frac{5}{\tan(54°)}$$
$$\therefore x \simeq 3.6327$$

Initially the ladder is 3.6327 metres from the pole. Moving the ladder 0.5 metres closer to the pole reduces this distance to 3.1327 metres from the pole.

Let the new inclination to the ground be $\theta°$ and let the new height the ladder reaches up the pole be h metres.



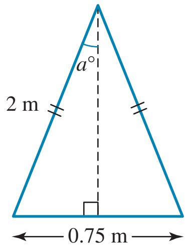
$$\cos(\theta°) = \frac{3.1327}{6.1803}$$
$$\therefore \theta° = \cos^{-1}\left(\frac{3.1327}{6.1803}\right)$$
$$\therefore \theta° \simeq 59.54°$$

The new inclination to the ground is 59.5°.

$$\tan(59.54°) = \frac{h}{3.1327}$$
$$\therefore h = 3.1327\tan(59.54°)$$
$$\therefore h \simeq 5.33$$

The new height the ladder reaches up the pole is 5.3 metres.

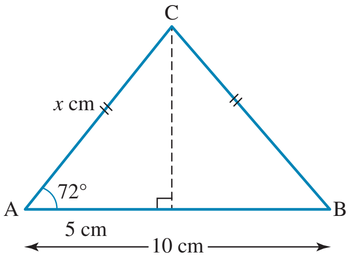
21 Let the angle between the legs be $2a°$.



$$\sin(a°) = \frac{0.75 \div 2}{2}$$
$$\therefore\ \sin(a°) = \frac{0.375}{2}$$
$$\therefore a° = \sin^{-1}(0.1875)$$
$$\therefore 2a° = 2\sin^{-1}(0.1875)$$
$$\simeq 21.6°$$

The angle between the legs of the ladder is approximately 21.6°.

22 Divide the isosceles triangle in half to create a right-angled triangle.



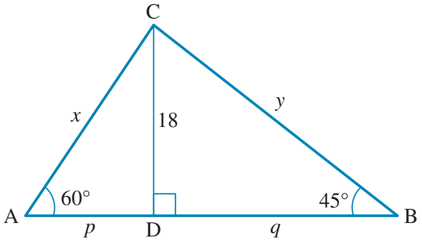
$$\cos(72°) = \frac{5}{x}$$

$$\therefore x = \frac{5}{\cos(72°)}$$

$$\therefore x \simeq 16.18$$

The equal sides are approximately 16.18 cm in length.
The equal angles $\angle CBA = \angle CAB = 72°$ and $\angle ACB = 180° - 2 \times 72° = 36°$

23



Consider triangle ACD.

$$\tan(60°) = \frac{18}{p}$$

$$\therefore p = \frac{18}{\tan(60°)}$$

$$\therefore p = \frac{18}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\therefore p = 6\sqrt{3}$$

Also,

$$\cos(60°) = \frac{p}{x}$$

$$\therefore \cos(60°) = \frac{6\sqrt{3}}{x}$$

$$\therefore x = \frac{6\sqrt{3}}{\cos(60°)}$$

$$\therefore x = 6\sqrt{3} \div \frac{1}{2}$$

$$\therefore x = 12\sqrt{3}$$

Consider triangle BCD.
Since $\angle BCD = 45°$ (angle sum of a triangle is 180°), triangle BCD is isosceles.
$\therefore q = 18$
Also,

$$\cos(45°) = \frac{q}{y}$$

$$\therefore \cos(45°) = \frac{18}{y}$$

$$\therefore y = \frac{18}{\cos(45°)}$$

$$\therefore y = 18 \div \frac{1}{\sqrt{2}}$$

$$\therefore y = 18\sqrt{2}$$

The sides have lengths: $AC = 12\sqrt{3}$ cm, $BC = 18\sqrt{2}$ cm and $AB = \left(6\sqrt{3} + 18\right)$ cm.

24 **a** In the isosceles triangle, the 20° angle is included between the two equal sides of 5 cm.
The area of the triangle is

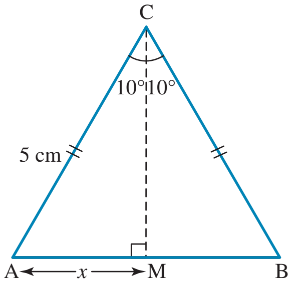
$$A = \frac{1}{2} \times 5 \times 5 \times \sin(20°)$$

$$= 12.5 \times \sin(20°)$$

$$\therefore A = 4.275$$

The area is 4.275 cm^2 correct to 3 decimal places.

b Divide the isosceles triangle into two right-angled triangles by joining *C* to the midpoint *M* of the side *AB*.



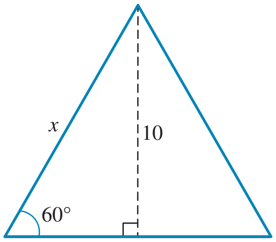
CM bisects the angle ACB and the side AB.
Let AM have length *x* metres, so *AB* has length 2*x* metres.

$$\sin(10°) = \frac{x}{5}$$

$$\therefore x = 5\sin(10°)$$

The third side, AB has length $10\sin(10°) \simeq 1.736$ cm.

25 The angles in an equilateral triangle are each 60° and the sides are equal in length. Let the length of a side be *x* cm.



$$\sin(60°) = \frac{10}{x}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{10}{x}$$

$$\therefore \sqrt{3}x = 20$$

$$\therefore x = \frac{20}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\therefore x = \frac{20\sqrt{3}}{3}$$

The perimeter is $3x = 20\sqrt{3}$ cm.
The base and height of the triangle are known.

$$A = \frac{1}{2}bh$$

$$\therefore A = \frac{1}{2} \times \frac{20\sqrt{3}}{3} \times 10$$

$$\therefore A = \frac{100\sqrt{3}}{3}$$

The area is $\frac{100\sqrt{3}}{3}$ cm^2.

8.2 Exam questions

1 **a** $\cos(22.5°) = \dfrac{\text{BD}}{3\sqrt{2}}$

$\therefore \text{BD} = 3.92 \text{ cm}$ [1 mark]

b $\text{Area of } \Delta\, \text{ABC} = \dfrac{1}{2} \times \text{BA} \times \text{BC} \times \sin(45°)$ [1 mark]

$$= \frac{1}{2} \times 3\sqrt{2} \times 3\sqrt{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{9\sqrt{2}}{2} \text{ cm}^2$$ [1 mark]

2 $\tan(30°) = \dfrac{1}{\sqrt{3}}$

From the diagram:

$\tan(30°) = \dfrac{x}{4}$

$\tan(30°) = \dfrac{x}{4}$

$\dfrac{1}{\sqrt{3}} = \dfrac{x}{4}$

$\therefore x = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$

The correct answer is **C**.

3 $\sin(60°) = \dfrac{\sqrt{3}}{2}, \cos(60°) = \dfrac{1}{2}$

$4\sin(60°)\cos(60°) = 4 \times \dfrac{\sqrt{3}}{2} \times \dfrac{1}{2}$

$= \sqrt{3}$

The correct answer is **B**.

8.3 Circular measure

8.3 Exercise

1 **a** To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.

$30° = \cancel{30}^1 \times \dfrac{\pi}{\cancel{180}_6}$ $\quad 45° = \cancel{45}^1 \times \dfrac{\pi}{\cancel{180}_4}$

$= \dfrac{\pi}{6}$ $\quad = \dfrac{\pi}{4}$

$60° = \cancel{60}^1 \times \dfrac{\pi}{\cancel{180}_3}$

$= \dfrac{\pi}{3}$

Degrees	30°	45°	60°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

b $0° = 0^c$, $180° = \pi^c$

$\therefore 90° = \dfrac{1}{2}\pi^c = \dfrac{\pi}{2}$ and $360° = 2 \times \pi^c = 2\pi$.

$270° = 180° + 90°$

$= \pi + \dfrac{\pi}{2}$

$= \dfrac{3\pi}{2}$

Degrees	0°	90°	180°	270°	360°
Radians	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

2 To convert radians to degrees, multiply by $\dfrac{180}{\pi}$.

a $\dfrac{\pi^c}{5}$

$= \dfrac{\pi}{5} \times \dfrac{180°}{\pi}$

$= 36°$

b $\dfrac{2\pi^c}{3}$

$= \dfrac{2\pi}{3} \times \dfrac{180°}{\pi}$

$= 2 \times 60°$

$= 120°$

c $\dfrac{5\pi}{12}$

$= \dfrac{5\pi}{12} \times \dfrac{180°}{\pi}$

$= 5 \times 15°$

$= 75°$

d $\dfrac{11\pi}{6}$

$= \dfrac{11\pi}{6} \times \dfrac{180°}{\pi}$

$= 11 \times 30°$

$= 330°$

e $\dfrac{7\pi}{9}$

$= \dfrac{7\pi}{9} \times \dfrac{180°}{\pi}$

$= 7 \times 20°$

$= 140°$

f $\dfrac{9\pi}{2}$

$= \dfrac{9\pi}{2} \times \dfrac{180°}{\pi}$

$= 9 \times 90°$

$= 810°$

3 **a** $40°$

$= 40 \times \dfrac{\pi}{180}$

$= \dfrac{2\pi}{9}$

b $150°$

$= 150 \times \dfrac{\pi}{180}$

$= \dfrac{15\pi}{18}$

$= \dfrac{5\pi}{6}$

c $225°$
$= 225 \times \dfrac{\pi}{180}$
$= \dfrac{45\pi}{36}$
$= \dfrac{5\pi}{4}$

d $300°$
$= 300 \times \dfrac{\pi}{180}$
$= \dfrac{30\pi}{18}$
$= \dfrac{5\pi}{3}$

e $315°$
$= 315 \times \dfrac{\pi}{180}$
$= \dfrac{63\pi}{36}$
$= \dfrac{7\pi}{4}$

f $720°$
$= 720 \times \dfrac{\pi}{180}$
$= \dfrac{72\pi}{18}$
$= \dfrac{8\pi}{2}$
$= 4\pi$
or, $720° = 2 \times 360° = 2 \times 2\pi = 4\pi$

4 **a** $60° = 60 \times \dfrac{\pi}{180}$
$= \dfrac{\pi}{3}$

b $\dfrac{3\pi^c}{4} = \dfrac{3\pi}{4} \times \dfrac{180°}{\pi}$
$= 135°$

c $\dfrac{\pi}{6} = \dfrac{\cancel{\pi}}{6} \times \dfrac{180°}{\cancel{\pi}}$
$= 30°$
$\therefore \tan\left(\dfrac{\pi}{6}\right) = \tan(30°)$
$= \dfrac{\sqrt{3}}{3}$

5 Arc length formula: $l = r\theta$ with $r = 5, \theta = 2$
$\therefore l = 5 \times 2$
$\therefore l = 10$
The arc length is 10 cm.

6 Arc length formula: $l = r\theta$ with $l = 6, \theta = 0.5$
$\therefore 6 = r \times 0.5$
$\therefore r = \dfrac{6}{0.5}$
$\therefore r = 12$
The radius is 12 mm.

7 **a** To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.
$36° = 36 \times \dfrac{\pi^c}{180}$
$= 2 \times \dfrac{\pi^c}{10}$
$= \dfrac{\pi^c}{5}$

b Arc length formula: $l = r\theta$ with $r = 7, \theta = \dfrac{\pi}{5}$
$\therefore l = 7 \times \dfrac{\pi}{5}$
$\therefore l = \dfrac{7\pi}{5}$
The arc length is $\dfrac{7\pi}{5}$ cm.

8 Arc length formula: $l = r\theta$ with $l = \dfrac{3\pi}{4}, r = 6$

a $\therefore \dfrac{3\pi}{4} = 6 \times \theta$
$\therefore \theta = \dfrac{3\pi}{4} \div 6$
$\therefore \theta = \dfrac{3\pi}{4} \times \dfrac{1}{6}$
$\therefore \theta = \dfrac{\pi}{4} \times \dfrac{1}{2}$
$\therefore \theta = \dfrac{\pi}{8}$
The arc subtends an angle of $\dfrac{\pi}{8}$ radians at the centre of the circle.

b To convert radians to degrees, multiply by $\dfrac{180}{\pi}$
$\dfrac{\pi}{8} = \dfrac{\pi}{8} \times \dfrac{180°}{\pi}$
$= 22.5°$
The arc subtends an angle of 22.5° at the centre of the circle.

9 **a** $l = r\theta,\ r = 12,\ \theta = 150 \times \dfrac{\pi}{180}$
$\therefore l = 12 \times \dfrac{15\pi}{18}$
$= 12 \times \dfrac{5\pi}{6}$
$= 10\pi$
The arc length is 10π cm.

b As the chord or minor arc subtends an angle of 60° at the centre, the major arc subtends an angle of $(360° - 60°) = 300°$ at the centre.
$l = r\theta,\ r = 3,\ \theta = 300 \times \dfrac{\pi}{180}$
$\therefore l = 3 \times \dfrac{30\pi}{18}$
$= 3 \times \dfrac{5\pi}{3}$
$= 5\pi$
The arc length is 5π cm.

10 **a** **i** $3°$
$= 3 \times \dfrac{\pi}{180}$
$= \dfrac{\pi}{60}$
$\simeq 0.052$

ii $112°15' = 112.25°$
$= 112.25 \times \dfrac{\pi}{180}$
$\simeq 1.959$

iii $215.36°$
$= 215.36 \times \dfrac{\pi}{180}$
$\simeq 3.759$

b **i** 3^c

$= 3 \times \dfrac{180°}{\pi}$

$\simeq 171.887°$

ii 2.3π

$= 2.3\pi \times \dfrac{180°}{\pi}$

$= 2.3 \times 180°$

$= 414°$

c $\left\{1.5^c,\ 50°,\ \dfrac{\pi^c}{7}\right\}$

Converting radians to degrees,

$1.5^c = 1.5 \times \dfrac{180°}{\pi}$

$\simeq 85.9°$

$\dfrac{\pi}{7}^c = \dfrac{\pi}{7} \times \dfrac{180°}{\pi}$

$= \dfrac{180°}{7}$

$\simeq 25.7°$

In order from smallest to largest, $\left\{\dfrac{\pi}{7}^c, 50°,\ 1.5^c\right\}$.

11 $145°12' = 145.2°$

$145.2° = 145.2 \times \dfrac{\pi}{180}$

$= \dfrac{726}{5} \times \dfrac{\pi}{180}$

$= \dfrac{121\pi}{150}$

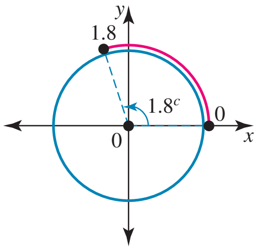
$\simeq 2.53$

12 **a** 1.8^c

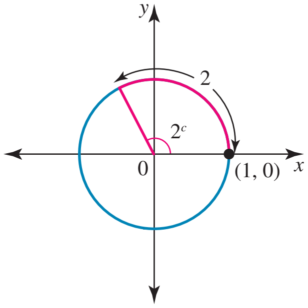
$1.8^c = 1.8 \times \dfrac{180°}{\pi}$

$\simeq 103.1°$

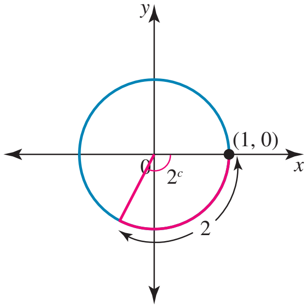
b



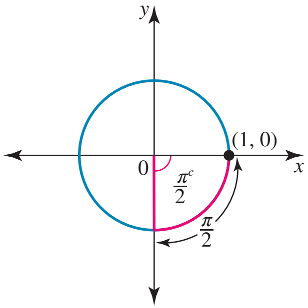
13 **a**



b



c



14 **a** $l = r\theta,\ r = 8,\ \theta = 75 \times \dfrac{\pi}{180}$

$l = 8 \times 75 \times \dfrac{\pi}{180}$

$= 8 \times \dfrac{15\pi}{36}$

$= \dfrac{30\pi}{9}$

The arc length is $\dfrac{30\pi}{9} \simeq 10.47$ cm correct to 2 decimal places.

b Calculate the angle at the centre.

$l = r\theta$, where $l = 12\pi,\ r = 10$

$12\pi = 10\theta$

$\therefore \theta = 1.2\pi$

This angle is in radian measure, so it needs to be converted to degrees.

In degrees,

$1.2\pi = 1.2\pi \times \dfrac{180°}{\pi}$

$= 216°$

Therefore, the angle at the centre of the circle subtended by the arc is 216°.

15 The rope length is the radius of 2.5 metres; the arc length of 75 cm is 0.75 metres.

$l = r\theta,\ l = 0.75,\ r = 2.5$

$\therefore 0.75 = 2.5 \times \theta$

$\therefore \theta = \dfrac{0.75}{2.5}$

$\therefore \theta = 0.3$

Convert the radians to degrees.

$0.3^c = 0.3 \times \dfrac{180°}{\pi}$

$= \dfrac{54°}{\pi}$

$\simeq 17.2°$

The ball swings through an angle of 17.2°.

16 The speed is 2 m/s, so in 5 seconds, the point travels a distance of 10 metres around the circumference of the wheel. The radius of the wheel is 3 metres.

$l = r\theta,\ r = 3,\ l = 10$

$\therefore 10 = 3\theta$

$\therefore \theta = \dfrac{10}{3}$

In degrees, $\dfrac{10}{3}$ radians is

$\left(\dfrac{10}{3} \times \dfrac{180}{\pi}\right)^{\circ}$

$= \dfrac{600°}{\pi}$

$\simeq 191°$

The angle of rotation is 191° or $\dfrac{10}{3}^c$.

17 Every 60 minutes, the minute hand rotates through 360°. In 1 minute it rotates through 6°.
In the 5 minutes between 9:50 am and 9:50 am, the minute hand will rotate through 30°.
Arc length:

$$l = r\theta,\ r = 11,\ \theta = 30 \times \frac{\pi}{180}$$

$$\begin{aligned}\therefore l &= 11 \times \frac{\pi}{6}\\ &= \frac{11\pi}{6}\\ &\simeq 5.76\end{aligned}$$

The arc length is 5.76 mm.

18 **a** The angle has no degree sign, so it must be assumed to be in radians. Ensure the calculator is on Rad mode.
$\tan(1.2) = \tan(1.2^c)$
$\therefore \tan(1.2) = 2.572$ correct to 3 decimal places.

b Using degree mode on the calculator,
$\tan(1.2°) = 0.021$ correct to 3 decimal places.

19 **a** **i** $\tan(1) = \tan(1^c) = 1.557$

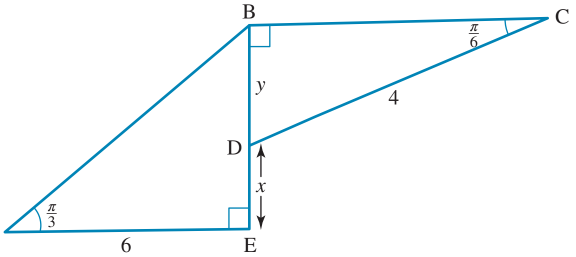
ii $\cos\left(\frac{2\pi}{7}\right) = \cos\left(\frac{2\pi^c}{7}\right) = 0.623$

iii $\sin(1.46°) = 0.025$

b As $\frac{\pi}{6} = 30°, \frac{\pi}{4} = 45°, \frac{\pi}{3} = 60°$ then $\sin\left(\frac{\pi}{6}\right) = \sin(30°)$ and so on.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

20



In triangle BDC,

$$\sin\left(\frac{\pi}{6}\right) = \frac{y}{4}$$

$$\begin{aligned}\therefore y &= 4\sin\left(\frac{\pi}{6}\right)\\ &= 4 \times \frac{1}{2}\\ &= 2\end{aligned}$$

In triangle AEB,

$$\begin{aligned}\tan\left(\frac{\pi}{3}\right) &= \frac{y + x}{6}\\ &= \frac{2 + x}{6}\end{aligned}$$

$$\therefore 2 + x = 6\tan\left(\frac{\pi}{3}\right)$$

$$\therefore x = 6 \times \sqrt{3} - 2$$

$$\therefore x = 6\sqrt{3} - 2$$

8.3 Exam questions

1 $300° \times \frac{\pi}{180°} = \frac{5\pi}{3}$
The correct answer is **E**.

2 $\frac{7\pi}{6} \times \frac{180°}{\pi} = 210°$
The correct answer is **B**.

3 $l = r\theta$

$$\therefore \theta = \frac{l}{r}$$

$$\begin{aligned}\theta &= \frac{3.5}{5}\\ &= 0.7\end{aligned}$$

Convert to degrees:

$$0.7 \times \frac{180°}{\pi} \cong 40°$$

The correct answer is **D**.

8.4 Unit circle definitions

8.4 Exercise

1 **a** Angles in quadrant 1 lie between 0° and 90°.
Since $24° \in (0°, 90°)$, 24° lies in quadrant 1.

b Angles in quadrant 3 lie between 180° and 270°.
As $240° \in (180°, 270°)$, 240° lies in quadrant 3.

c Angles in quadrant 2 lie between 90° and 180°.
As $123° \in (90°, 180°)$, 123° lies in quadrant 2.

d $365° = 360° + 5°$.
Therefore, the angle 365° lies in the same position as the angle 5°.
Since $5° \in (0°, 90°)$, 5° lies in quadrant 1.
Therefore, 365° also lies in quadrant 1.

e For negative rotations, angles that lie between −90° and 0° are in quadrant 4.
Since $-50° \in (-90°, 0°)$, −50° lies in quadrant 4.

f For negative rotations, angles that lie between −180° and −90° and 0° are in quadrant 3.
Since $-120° \in (-180°, -90°)$, −120° lies in quadrant 3.

2 $\left\{\frac{\pi}{3}, \frac{3\pi}{4}, \frac{7\pi}{6}, -\frac{\pi}{3}, -\frac{5\pi}{4}, -\frac{11\pi}{6}\right\}$

a Positive angles in the first quadrant lie between 0 and $\frac{\pi}{2}$.
Hence, $\frac{\pi}{3}$ lies in the first quadrant.
Negative angles in the first quadrant lie between $-\frac{3\pi}{2}$ and -2π, i.e. between $-\frac{9\pi}{6}$ and $-\frac{12\pi}{6}$. Hence, $-\frac{11\pi}{6}$ also lies in the first quadrant.

b Positive angles in the second quadrant lie between $\frac{\pi}{2}$ and π, i.e. between $\frac{2\pi}{4}$ and $\frac{4\pi}{4}$.
Hence, $\frac{3\pi}{4}$ lies in the second quadrant.
Negative angles in the second quadrant lie between $-\pi$ and $-\frac{3\pi}{2}$ i.e. between $-\frac{4\pi}{4}$ and $-\frac{6\pi}{4}$. Hence, $-\frac{5\pi}{4}$ also lies in the second quadrant.

c Positive angles in the third quadrant lie between π and $\frac{3\pi}{2}$, i.e. between $\frac{6\pi}{6}$ and $\frac{9\pi}{6}$.

Hence, $\frac{7\pi}{6}$ lies in the third quadrant.

d The remaining angle, $-\frac{\pi}{3}$ lies in the fourth first quadrant due to the negative rotation. Negative angles in the fourth quadrant lie between $-\frac{\pi}{2}$ and 0 i.e. between $-\frac{3\pi}{6}$ and 0.

Hence, $-\frac{\pi}{3} = -\frac{2\pi}{6}$ lies in the fourth quadrant.

3 a i The trigonometric point θ lies on the boundary between quadrants 1 and 2.

The coordinates of this point are $(0, 1)$.

ii $\sin(\theta)$ is the y coordinate of the point $(0, 1)$.

Hence, $\sin(\theta) = 1$.

b i The trigonometric point α lies on the boundary between quadrants 2 and 3.

The coordinates of this point are $(-1, 0)$.

ii $\cos(\alpha)$ is the x coordinate of the point $(-1, 0)$.

Hence, $\cos(\alpha) = -1$.

c i The trigonometric point β lies on the boundary between quadrants 3 and 4.

The coordinates of this point are $(0, -1)$.

ii $\tan(\beta) = \frac{y}{x}$ where $x = 0, y = -1$

$\therefore \tan(\beta) = \frac{-1}{0}$, which is undefined.

d i The trigonometric point v lies on the boundary between quadrants 4 and 1.

The coordinates of this point are $(1, 0)$.

ii The point $(1, 0)$ has $x = 1, y = 0$.

$\sin(v) = y \Rightarrow \sin(v) = 0$

$\cos(v) = x \Rightarrow \cos(v) = 1$

$\tan(v) = \frac{y}{x} = \frac{0}{1} \therefore \tan(v) = 0$

4 a Since $585° = 360° + 225°$, the end ray of 585° lies in the same quadrant as that of 225°, which is quadrant 3.

b $\frac{11\pi}{12} = \frac{12\pi}{12} - \frac{\pi}{12} = \pi - \frac{\pi}{12}$, so it lies in quadrant 2.

c $-18\pi = 9 \times (-2\pi)$, so it lies on the boundary between quadrants 1 and 4.

d $\frac{7\pi}{4} = \frac{8\pi}{4} - \frac{\pi}{4} = 2\pi - \frac{\pi}{4}$, so it lies in quadrant 4.

5 a To reach the boundary between quadrants 1 and 2, a rotation of $\frac{\pi}{2}$ would be needed from the point $(1, 0)$.

Therefore, the trigonometric point is $P\left[\frac{\pi}{2}\right]$.

b

120°	−400° $= -360° - 40°$	$\frac{4\pi}{3} = 1\frac{1}{3\pi}$	$\frac{\pi}{4}$
Quadrant 2	Quadrant 4	Quadrant 3	Quadrant 1

c Rotating clockwise $(360 - 120)° = 240°$ gives $Q[-240°]$; one full revolution of 360° plus another 120° gives $R[480°]$.

6 An anticlockwise rotation of $\frac{3\pi}{2}$ or a clockwise rotation of $\frac{\pi}{2}$ would reach the boundary between quadrants 3 and 4. The point could be $P\left[\frac{3\pi}{2}\right]$ or $P\left[-\frac{\pi}{2}\right]$.

7 a $P\left[\frac{\pi}{6}\right], \theta = \frac{\pi}{6}$

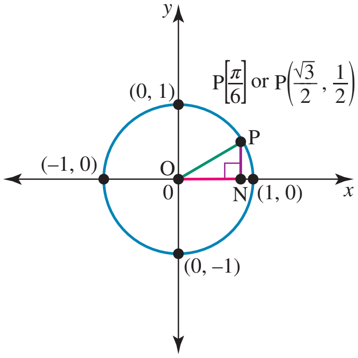
Cartesian coordinates:

$x = \cos(\theta)$ $\quad$ $y = \sin(\theta)$

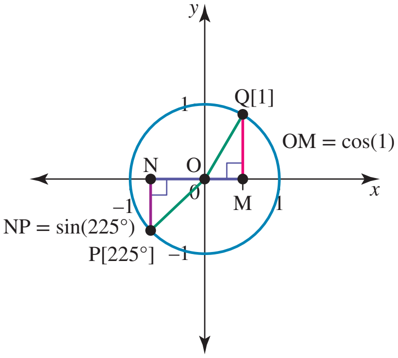
$= \cos\left(\frac{\pi}{6}\right)$ and $= \sin\left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2}$ $\quad$ $= \frac{1}{2}$

Therefore, P is the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



b



c The point $P\left[-\frac{\pi}{2}\right]$ is the point $(0, -1)$.

$x = \cos(\theta), \theta = -\frac{\pi}{2}, x = 0, \therefore \cos\left(-\frac{\pi}{2}\right) = 0$

$y = \sin(\theta), \theta = -\frac{\pi}{2}, y = -1, \therefore \sin\left(-\frac{\pi}{2}\right) = -1$

d $f(\theta) = \sin(\theta)$

$\therefore f(0) = \sin(0)$

The trigonometric point [0] has Cartesian coordinates $(1, 0)$. Its y-coordinate gives the value of $\sin(0)$.

$\therefore \sin(0) = 0$

$\therefore f(0) = 0$

8 a $f(t) = \sin(t)$

$f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right)$

$\sin\left(\frac{3\pi}{2}\right)$ is the y-coordinate of the trigonometric point $\left[\frac{3\pi}{2}\right]$.

As this trigonometric point lies on the boundary between quadrants 3 and 4, its Cartesian coordinates are $(0, -1)$.

Hence, $\sin\left(\frac{3\pi}{2}\right) = -1$.

$\therefore f\left(\frac{3\pi}{2}\right) = -1$.

b $g(t) = \cos(t)$

$g(4\pi) = \cos(4\pi)$

$\cos(4\pi)$ is the x-coordinate of the trigonometric point $[4\pi]$.

This trigonometric point lies in the same position as $[2\pi]$ at the Cartesian point $(1, 0)$.
Hence, $\cos(4\pi) = 1$.
$\therefore g(4\pi) = 1$

c $h(t) = \tan(t)$
$h(-\pi) = \tan(-\pi)$
$\tan(-\pi) = \dfrac{y}{x}$ where (x, y) are the coordinates of the trigonometric point $[-\pi]$.
This point lies on the boundary between the second and the third quadrants. Its Cartesian coordinates are $(-1, 0)$.
$\therefore \tan(-\pi) = \dfrac{y}{x} = \dfrac{0}{1} = 0$
Hence, $h(-\pi) = 0$.

d $k(t) = \sin(t) + \cos(t)$
$k(6.5\pi) = \sin(6.5\pi) + \cos(6.5\pi)$
Since

$$\begin{aligned}6.5\pi &= 6\pi + 0.5\pi\\ &= 6\pi + \frac{\pi}{2}\end{aligned}$$

the trigonometric point $[6.5\pi]$ has the same position as $\left[\dfrac{\pi}{2}\right]$. Its Cartesian coordinates are $(0, 1)$.
$\sin(6.5\pi) = y = 1$ and $\cos(6.5\pi) = x = 0$
Hence, $k(6.5\pi) = 1 + 0 = 1$.

9 a Since $\sin(\theta)$ is the y-coordinate, $\sin(\theta)$ is positive in the first and second quadrants.

b $\cos(\theta)$ is the x-coordinate, so $\cos(\theta)$ is positive in the first and fourth quadrants.

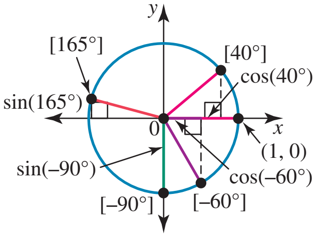
10 a $P\left[\dfrac{\pi}{4}\right]$
$x = \cos\theta$ and $y = \sin\theta$ where $\theta = \dfrac{\pi}{4}$.

$$\therefore x = \cos\left(\frac{\pi}{4}\right) \text{ and } y = \sin\left(\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2} \qquad\qquad = \frac{\sqrt{2}}{2}$$

P has Cartesian coordinates $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

b The point P $(0, -1)$ lies on the boundary between quadrants 3 and 4.
Therefore, P could be the trigonometric point $\left[\dfrac{3\pi}{2}\right]$ or the trigonometric point $\left[-\dfrac{\pi}{2}\right]$.

11



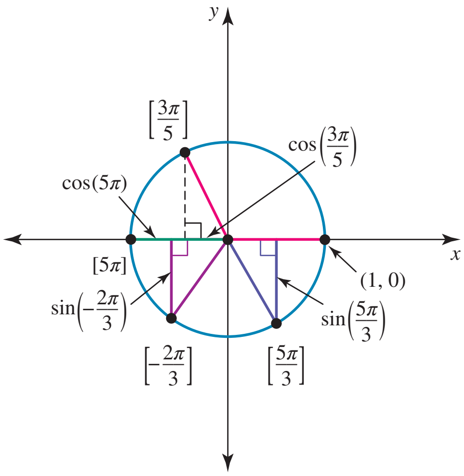
a $\cos(40°)$ is the x-coordinate of the trigonometric point $[40°]$, which lies in the first quadrant.

b $\sin(165°)$ is the y-coordinate of the trigonometric point $[165°]$, which lies in the second quadrant.

c $\cos(-60°)$ is the x-coordinate of the trigonometric point $[-60°]$, which lies in the fourth quadrant.

d $\sin(-90°)$ is the y-coordinate of the trigonometric point $[-90°]$, which lies on the boundary between the third and fourth quadrants.

12



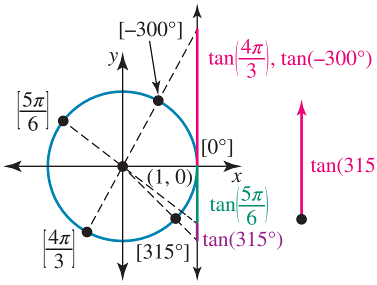
a $\sin\left(\dfrac{5\pi}{3}\right)$ is the y-coordinate of the trigonometric point $\left[\dfrac{5\pi}{3}\right]$, which lies in the fourth quadrant.

b $\cos\left(\dfrac{3\pi}{5}\right)$ is the x-coordinate of the trigonometric point $\left[\dfrac{3\pi}{5}\right]$, which lies in the second quadrant.

c $\cos(5\pi)$ is the x-coordinate of the trigonometric point $[5\pi]$, which lies on the boundary between the second and third quadrants.

d $\sin\left(-\dfrac{2\pi}{3}\right)$ is the y-coordinate of the trigonometric point $\left[-\dfrac{2\pi}{3}\right]$, which lies in the third quadrant.

13



The tangent to the unit circle at the point $(1, 0)$ is drawn.

a $\tan(315°)$ is the length of the intercept cut off on the tangent by the extended radius through the trigonometric point $[315°]$ in the fourth quadrant.

b $\tan\left(\dfrac{5\pi}{6}\right)$ is the length of the intercept cut off on the tangent by the extended radius from the trigonometric point $\left[\dfrac{5\pi}{6}\right]$ in the second quadrant.

c $\tan\left(\dfrac{4\pi}{3}\right)$ is the length of the intercept cut off on the tangent by the extended radius from the trigonometric point $\left[\dfrac{4\pi}{3}\right]$ in the third quadrant.

d $\tan(-300°)$ is the length of the intercept cut off on the tangent by the extended radius through the trigonometric point $[-300°]$ in the first quadrant. This is the same value as $\tan\left(\dfrac{4\pi}{3}\right)$.

14 a P[θ] is the Cartesian point $P(-0.8, 0.6)$.

Since $x < 0, y > 0$ the point lies in quadrant 2.

Check the point lies on a unit circle by showing $x^2 + y^2 = 1$.

$$\begin{aligned} x^2 + y^2 &= (-0.8)^2 + (0.6)^2 \\ &= 0.64 + 0.36 \\ &= 1 \end{aligned}$$

$\sin(\theta) = y$-coordinate of P

$\therefore \sin(\theta) = 0.6$

$\cos(\theta) = x$-coordinate of P

$\therefore \cos(\theta) = -0.8$

$$\tan(\theta) = \frac{y}{x}$$

$$\begin{aligned} \therefore \tan(\theta) &= \frac{0.6}{-0.8} \\ &= -\frac{3}{4} \end{aligned}$$

b Q[θ] is the trigonometric point $Q\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Since $x > 0, y < 0$, Q lies in quadrant 4.

Check Q lies on a unit circle.

$$\begin{aligned} x^2 + y^2 &= \left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 \\ &= \frac{2}{4} + \frac{2}{4} \\ &= 1 \end{aligned}$$

$$\sin(\theta) = y = -\frac{\sqrt{2}}{2}$$

$$\cos(\theta) = x = \frac{\sqrt{2}}{2}$$

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= -1 \end{aligned}$$

c R[θ] is the trigonometric point $R\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

Since $x > 0, y > 0$, R lies in quadrant 1.

Check R lies on a unit circle.

$$\begin{aligned} x^2 + y^2 &= \left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 \\ &= \frac{4}{5} + \frac{1}{5} \\ &= 1 \end{aligned}$$

$$\sin(\theta) = \frac{1}{\sqrt{5}}$$

$$\cos(\theta) = \frac{2}{\sqrt{5}}$$

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \frac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} \\ &= \frac{1}{2} \end{aligned}$$

d S[θ] is the Cartesian point S(0, 1). It lies on the boundary between the first and second quadrants.

$\sin(\theta) = y = 1$

$\cos(\theta) = x = 0$

$$\tan(\theta) = \frac{y}{x}$$

$\therefore \tan(\theta) = \frac{1}{0}$ which is undefined.

15 a cos(0) is the x-coordinate of the trigonometric point [0], which has Cartesian coordinates (1, 0).

$\therefore \cos(0) = 1$

b $\sin\left(\frac{\pi}{2}\right)$ is the y-coordinate of the trigonometric point $\left[\frac{\pi}{2}\right]$, which has Cartesian coordinates (0, 1).

$\therefore \sin\left(\frac{\pi}{2}\right) = 1$

c $\tan(\pi)$ is the ratio of the y-coordinate to the x-coordinate of the trigonometric point $[\pi]$ which has Cartesian coordinates $(-1, 0)$.

$$\begin{aligned} \therefore \tan(\pi) &= \frac{y}{x} \\ &= \frac{0}{-1} \\ &= 0 \end{aligned}$$

d $\cos\left(\frac{3\pi}{2}\right)$ is the x-coordinate of the trigonometric point $\left[\frac{3\pi}{2}\right]$, which has Cartesian coordinates $(0, -1)$.

$\therefore \cos\left(\frac{3\pi}{2}\right) = 0$

e $\sin(2\pi)$ is the y-coordinate of the trigonometric point $[2\pi]$, which has Cartesian coordinates (1, 0).

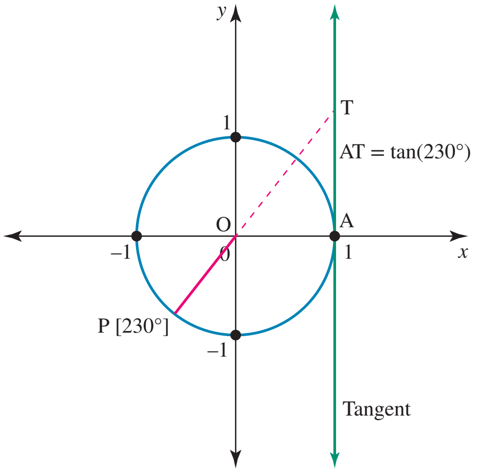
$\therefore \sin(2\pi) = 0$

f $\tan(-11\pi)$ is the ratio of the y-coordinate to the x-coordinate of the trigonometric point $[-11\pi]$.

Since $-11\pi = -10\pi - \pi$, the trigonometric point has the same position as $[-\pi]$, which has Cartesian coordinates $(-1, 0)$.

$$\begin{aligned} \therefore \tan(-11\pi) &= \frac{y}{x} \\ &= \frac{0}{-1} \\ &= 0 \end{aligned}$$

16 a $\tan(230°) = 1.192$



b P $[2\pi]$ is the point $(1, 0)$.

$$\tan(\theta) = \frac{y}{x}, \theta = 2\pi, x = 1, y = 0$$

$$\therefore \tan(2\pi) = \frac{0}{1}$$

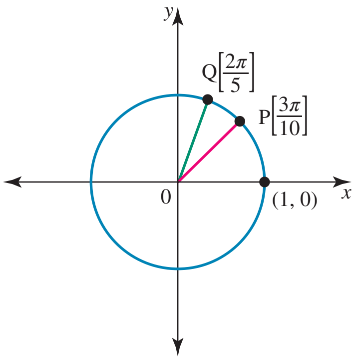
$$\therefore \tan(2\pi) = 0$$

17 $\left\{\tan(-3\pi), \tan\left(\frac{5\pi}{2}\right), \tan(-90°), \tan\left(\frac{3\pi}{4}\right), \tan(780°)\right\}$

a Neither $\tan\left(\frac{5\pi}{2}\right)$ nor $\tan(-90°)$ are defined, since the ray forming each is parallel to the vertical tangent.

b $\tan\left(\frac{3\pi}{4}\right)$ will be negative since $\frac{3\pi}{4}$ lies in the second quadrant, so the extended ray forming it would intersect the tangent below the x axis.

18 a As $\frac{\pi}{2} = \frac{5\pi}{10}$, and $\frac{3\pi}{10} < \frac{\pi}{2}$ and $\frac{2\pi}{5} = \frac{4\pi}{10} < \frac{\pi}{2}$, the points P and Q lie in quadrant 1.



b $\angle QOP = \frac{4\pi}{10} - \frac{3\pi}{10} = \frac{\pi^c}{10}$

c Using a clockwise rotation from $(1, 0)$ to reach point Q would require a rotation of $-\frac{3\pi}{2} - \frac{\pi}{10} = -\frac{16\pi}{10}$. Q could be described as the trigonometric point $\left[-\frac{8\pi}{5}\right]$.

To reach point P a further rotation from Q of $-\frac{\pi}{10}$ would be required, making the rotation from $(1, 0)$ to reach point P of $-\frac{16\pi}{10} - \frac{\pi}{10} = -\frac{17\pi}{10}$. P could be described as the trigonometric point $\left[-\frac{17\pi}{10}\right]$.

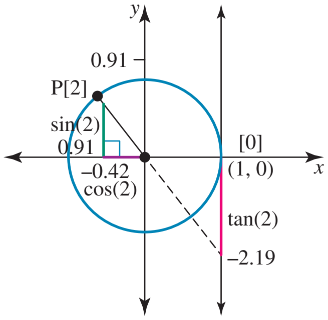
Other answers can be formed by adding multiples of -2π.

d Rotating anticlockwise a complete revolution of 2π and then a further $\frac{3\pi}{10}$ would reach point P. Since $2\pi + \frac{3\pi}{10} = \frac{23\pi}{10}$, P could be described as the trigonometric point $\left[\frac{23\pi}{10}\right]$.

Similarly, $2\pi + \frac{2\pi}{5} = \frac{12\pi}{5}$, so Q could be described as the trigonometric point $\left[\frac{12\pi}{5}\right]$.

Other answers can be formed by adding multiples of 2π.

19 a



b P has the Cartesian coordinates $(-0.42, 0.91)$.

20 a $\cos^2\left(\frac{7\pi}{6}\right) + \sin^2\left(\frac{7\pi}{6}\right)$

In the Main menu using TRIG in the mth keyboard, key in $(\cos(7\pi/6))^2 + (\sin(7\pi/6))^2$, with the calculator set on Rad and Standard modes.

The value is 1.

b The value of $\cos(7\pi/6) + \sin(7\pi/6)$ is $\frac{-\sqrt{3}}{2} - \frac{1}{2}$.

c $(\sin(7/6))^2 + (\cos(7/6))^2$

$$= \left(\sin\left(\frac{7}{6}\right)\right)^2 + \left(\cos\left(\frac{7}{6}\right)\right)^2$$

Switch to Decimal mode to obtain the value 1.

d $\sin^2(60°) + \cos^2(60°)$

In Standard and Deg modes the value is 1. (In fact, it will be the value 1 even if Rad mode is used.)

e $\sin^2(t) + \cos^2(t) = 1$

$\sin(t) = y$-coordinate of trigonometric point $[t]$ and $\cos(t) = x$-coordinate of point $[t]$.

Point $[t]$ lies on the unit circle $x^2 + y^2 = 1$.

$$\therefore (\cos(t))^2 + (\sin(t))^2 = 1$$

$$\therefore \cos^2(t) + \sin^2(t) = 1$$

$$\therefore \sin^2(t) + \cos^2(t) = 1$$

8.4 Exam questions

1 $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$

∴ third quadrant

The correct answer is **C**.

2 Delete multiples of 2π to find the position of $\frac{-11\pi}{2}$.

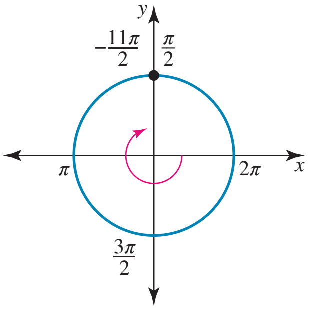
$$\frac{-11\pi}{2} = -\left(\frac{8\pi}{2} + \frac{3\pi}{2}\right)$$

$$= -\left(4\pi + \frac{3\pi}{2}\right)$$

$$= -\frac{3\pi}{2}$$

The angle $-\frac{3\pi}{2}$ is in the same position as the angle $\frac{\pi}{2}$.

Therefore, it is exactly between the first and second quadrants.



The correct answer is **A**.

3 $\sin(x) < 0$, so the y-coordinate is negative.
$\cos(x) < 0$, so the x-coordinate is negative.
$\therefore$ the third quadrant is required.
$\therefore \pi < x < \frac{3\pi}{2}$
The correct answer is **C**.

8.5 Symmetry properties

8.5 Exercise

1 **a** $\sin(120°)$
$= \sin(180° - 60°)$
$= \sin(60°)$
$= \frac{\sqrt{3}}{2}$

b $\tan(210°)$
$= \tan(180° + 30°)$
$= \tan(30°)$
$= \frac{\sqrt{3}}{3}$

c $\cos(135°)$
$= \cos(180° - 45°)$
$= -\cos(45°)$
$= -\frac{\sqrt{2}}{2}$

d $\cos(300°)$
$= \cos(360° - 60°)$
$= \cos(60°)$
$= \frac{1}{2}$

e $\tan(150°)$
$= \tan(180° - 30°)$
$= -\tan(30°)$
$= -\frac{\sqrt{3}}{3}$

f $\sin(315°)$
$= \sin(360° - 45°)$
$= -\sin(45°)$
$= -\frac{\sqrt{2}}{2}$

2 **a** $\cos(150°)$
$= \cos(180° - 30°)$
$= -\cos(30°)$
$= -\frac{\sqrt{3}}{2}$

b $\sin(240°)$
$= \sin(180° + 60°)$
$= -\sin(60°)$
$= -\frac{\sqrt{3}}{2}$

c $\tan(330°)$
$= \tan(360° - 30°)$
$= -\tan(30°)$
$= -\frac{\sqrt{3}}{3}$

d $\tan(-45°)$
$= -\tan(45°)$
$= -1$

e $\sin(-30°)$
$= -\sin(30°)$
$= -\frac{1}{2}$

f $\cos(-60°)$
$= \cos(60°)$
$= \frac{1}{2}$

3 **a** $\tan(420°)$
$= \tan(360° + 60°)$
$= \tan(60°)$
$= \sqrt{3}$

b $\sin(405°)$
$= \sin(360° + 45°)$
$= \sin(45°)$
$= \frac{\sqrt{2}}{2}$

c $\cos(480°)$
$= \cos(360° + 120°)$
$= \cos(120°)$
$= \cos(180° - 60°)$
$= -\cos(60°)$
$= -\frac{1}{2}$

d $\sin(765°)$
$= \sin(720° + 45°)$
$= \sin(45°)$
$= \frac{\sqrt{2}}{2}$

e $\cos(-510°)$
$= \cos(510°)$
$= \cos(360° + 150°)$
$= \cos(150°)$
$= \cos(180° - 30°)$
$= -\cos(30°)$
$= -\frac{\sqrt{3}}{2}$

f $\tan(-585°)$
$= -\tan(585°)$
$= -\tan(360° + 225°)$
$= -\tan(225°)$
$= -\tan(180° + 45°)$
$= -\tan(45°)$
$= -1$

4 **a** Cosine is negative in the second quadrant and $120° = 180° - 60°$.
$\cos(120°) = -\cos(60°)$
$= -\frac{1}{2}$

b Tangent is positive in the third quadrant and $225° = 180° + 45°$.
$\tan(225°) = \tan(45°)$
$= 1$

c Sine is negative in the fourth quadrant and $330° = 360° - 30°$.
$\sin(330°) = -\sin(30°)$
$= -\frac{1}{2}$

d Tangent is negative in the fourth quadrant.

$\tan(-60°) = -\tan(60°)$

$= -\sqrt{3}$

e Cosine is positive in the first quadrant.

$\cos(-315°) = \cos(45°)$

$= \dfrac{\sqrt{2}}{2}$

f $\sin(510°)$

$= \sin(360° + 150°)$

$= \sin(150°)$

$= \sin(30°)$

$= \dfrac{1}{2}$

5 **a** $\tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$

b $\tan\left(\dfrac{5\pi}{3}\right)$

$= \tan\left(2\pi - \dfrac{\pi}{3}\right)$

$= -\tan\left(\dfrac{\pi}{3}\right)$

$= -\sqrt{3}$

c $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

d $\sin\left(\dfrac{5\pi}{6}\right)$

$= \sin\left(\pi - \dfrac{\pi}{6}\right)$

$= \sin\left(\dfrac{\pi}{6}\right)$

$= \dfrac{1}{2}$

e $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

f $\cos\left(\dfrac{5\pi}{4}\right)$

$= \cos\left(\pi + \dfrac{\pi}{4}\right)$

$= -\cos\left(\dfrac{\pi}{4}\right)$

$= -\dfrac{\sqrt{2}}{2}$

6 **a** $\cos\left(-\dfrac{5\pi}{6}\right)$

$= \cos\left(\dfrac{5\pi}{6}\right)$

$= \cos\left(\pi - \dfrac{\pi}{6}\right)$

$= -\cos\left(\dfrac{\pi}{6}\right)$

$= -\dfrac{\sqrt{3}}{2}$

b $\sin\left(-\dfrac{4\pi}{3}\right)$

$= -\sin\left(\dfrac{4\pi}{3}\right)$

$= -\sin\left(\pi + \dfrac{\pi}{3}\right)$

$= -\left(-\sin\left(\dfrac{\pi}{3}\right)\right)$

$= \sin\left(\dfrac{\pi}{3}\right)$

$= \dfrac{\sqrt{3}}{2}$

c $\tan\left(-\dfrac{3\pi}{4}\right)$

$= -\tan\left(\dfrac{3\pi}{4}\right)$

$= -\tan\left(\pi - \dfrac{\pi}{4}\right)$

$= -\left(-\tan\left(\dfrac{\pi}{4}\right)\right)$

$= \tan\left(\dfrac{\pi}{4}\right)$

$= 1$

d $\tan\left(\dfrac{13\pi}{6}\right)$

$= \tan\left(2\pi + \dfrac{\pi}{6}\right)$

$= \tan\left(\dfrac{\pi}{6}\right)$

$= \dfrac{\sqrt{3}}{3}$

e $\sin\left(\dfrac{17\pi}{4}\right)$

$= \sin\left(4\pi + \dfrac{\pi}{4}\right)$

$= \sin\left(\dfrac{\pi}{4}\right)$

$= \dfrac{\sqrt{2}}{2}$

f $\cos\left(\dfrac{8\pi}{3}\right)$

$= \cos\left(2\pi + \dfrac{2\pi}{3}\right)$

$= \cos\left(\dfrac{2\pi}{3}\right)$

$= \cos\left(\pi - \dfrac{\pi}{3}\right)$

$= -\cos\left(\dfrac{\pi}{3}\right)$

$= -\dfrac{1}{2}$

7 **a** Sine is positive in the second quadrant and $\dfrac{3\pi}{4} = \pi - \dfrac{\pi}{4}$.

$\sin\left(\dfrac{3\pi}{4}\right) = \sin\left(\dfrac{\pi}{4}\right)$

$= \dfrac{\sqrt{2}}{2}$

b Tangent is negative in the second quadrant and $\dfrac{2\pi}{3} = \pi - \dfrac{\pi}{3}$.

$\tan\left(\dfrac{2\pi}{3}\right) = -\tan\left(\dfrac{\pi}{3}\right)$

$= -\sqrt{3}$

c Cosine is negative in the second quadrant and $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$.

$$\begin{aligned}\cos\left(\frac{5\pi}{6}\right) &= -\cos\left(\frac{\pi}{6}\right)\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

d Cosine is negative in the third quadrant and $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$.

$$\begin{aligned}\cos\left(\frac{4\pi}{3}\right) &= -\cos\left(\frac{\pi}{3}\right)\\ &= -\frac{1}{2}\end{aligned}$$

e Tangent is positive in the third quadrant and $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$.

$$\begin{aligned}\tan\left(\frac{7\pi}{6}\right) &= \tan\left(\frac{\pi}{6}\right)\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

f Sine is negative in the fourth quadrant and $\frac{11\pi}{6} = 2\pi - \frac{\pi}{6}$.

$$\begin{aligned}\sin\left(\frac{11\pi}{6}\right) &= -\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{1}{2}\end{aligned}$$

8 **a** $\cos\left(-\frac{\pi}{4}\right)$

$$\begin{aligned}&= \cos\left(\frac{\pi}{4}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

b $\sin\left(-\frac{\pi}{3}\right)$

$$\begin{aligned}&= -\sin\left(\frac{\pi}{3}\right)\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

c Tangent is positive in the third quadrant and

$-\frac{5\pi}{6} = -\pi + \frac{\pi}{6}$.

$$\begin{aligned}\tan\left(-\frac{5\pi}{6}\right) &= \tan\left(\frac{\pi}{6}\right)\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

d $\sin\left(\frac{8\pi}{3}\right)$

$$\begin{aligned}&= \sin\left(2\pi + \frac{2\pi}{3}\right)\\ &= \sin\left(\frac{2\pi}{3}\right)\\ &= \sin\left(\frac{\pi}{3}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

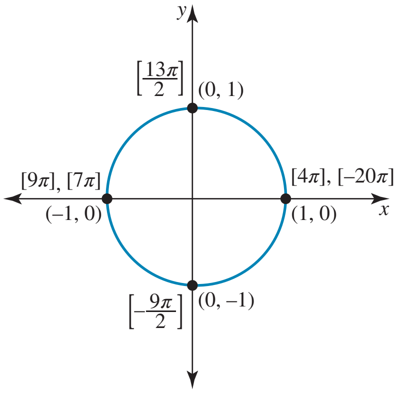
e $\cos\left(\frac{9\pi}{4}\right)$

$$\begin{aligned}&= \cos\left(2\pi + \frac{\pi}{4}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

f $\tan\left(\frac{23\pi}{6}\right)$

$$\begin{aligned}&= \tan\left(4\pi - \frac{\pi}{6}\right)\\ &= -\tan\left(\frac{\pi}{6}\right)\\ &= -\frac{\sqrt{3}}{3}\end{aligned}$$

9



a $\cos(4\pi)$ is the x-coordinate of the Cartesian point $(1, 0)$.

$\therefore \cos(4\pi) = 1$

b $\tan(9\pi) = \frac{y}{x}$ for the point $(-1, 0)$

$\therefore \tan(9\pi) = \frac{0}{-1} = 0$

c $\sin(7\pi)$ is the y-coordinate of the Cartesian point $(-1, 0)$.

$\therefore \sin(7\pi) = 0$

d $\sin\left(\frac{13\pi}{2}\right)$ is the y-coordinate of the Cartesian point (0, 1).

$\therefore \sin\left(\frac{13\pi}{2}\right) = 1$

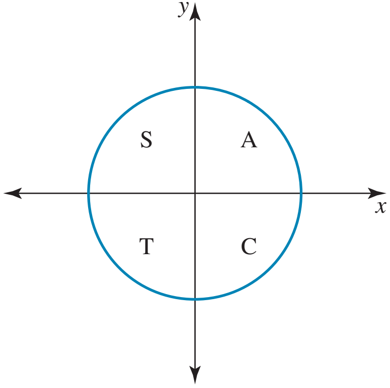
e $\cos\left(-\frac{9\pi}{2}\right)$ is the x-coordinate of the Cartesian point $(0, -1)$.

$\therefore \cos\left(-\frac{9\pi}{2}\right) = 0$

f $\tan(-20\pi) = \frac{y}{x}$ for the point $(1, 0)$.

$\therefore \tan(-20\pi) = \frac{0}{1} = 0$

10 Consider the ‘ASTC’ diagram.



a $\cos(\theta) > 0, \sin(\theta) < 0$ in quadrant 4.

b $\tan(\theta) > 0, \cos(\theta) > 0$ in quadrant 1.

c $\sin(\theta) > 0, \cos(\theta) < 0$ in quadrant 2.

d $\cos(\theta) = 0$ at the points on the unit circle that have $x = 0$. This occurs at the boundary between quadrants 1 and 2, and at the boundary between quadrants 3 and 4.

e $\cos(\theta) = 0, \sin(\theta) > 0$ when $x = 0, y > 0$. This occurs at the boundary between quadrants 1 and 2.

f $\sin(\theta) = 0, \cos(\theta) < 0$ when $y = 0, x < 0$. This occurs at the boundary between quadrants 2 and 3.

11 a $\cos(\theta)$ is negative and $\tan(\theta)$ is positive in the third quadrant.

b
$$\begin{aligned} f(t) &= \tan(t) \\ \therefore f(4\pi) &= \tan(4\pi) \\ &= 0 \end{aligned}$$

12 a Third quadrant, base $\frac{\pi}{3}$ since $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$

$$\begin{aligned} &\sin\left(\frac{4\pi}{3}\right) \\ &= -\sin\left(\frac{\pi}{3}\right) \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

b
$$\begin{aligned} &\tan\left(\frac{5\pi}{6}\right) \\ &= \tan\left(\pi - \frac{\pi}{6}\right) \\ &= -\tan\left(\frac{\pi}{6}\right) \\ &= -\frac{\sqrt{3}}{3} \end{aligned}$$

c $\cos(-30°)$, fourth quadrant, base 30°

$$\begin{aligned} &\cos(-30°) \\ &= \cos(30°) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

13 $-\frac{5\pi}{4}$ lies in second quadrant with base $\frac{\pi}{4}$ since $-\frac{5\pi}{4} = -\pi - \frac{\pi}{4}$. Only sine is positive in the second quadrant.

$$\begin{array}{lclcl} \sin\left(-\frac{5\pi}{4}\right) & \text{and} & \cos\left(-\frac{5\pi}{4}\right) & \text{and} & \tan\left(-\frac{5\pi}{4}\right) \\ = \sin\left(\frac{\pi}{4}\right) & & = -\cos\left(\frac{\pi}{4}\right) & & = -\tan\left(\frac{\pi}{4}\right) \\ = \frac{\sqrt{2}}{2} & & = -\frac{\sqrt{2}}{2} & & = -1 \end{array}$$

14 $\cos(\theta) = 0.2$

a $\cos(\pi - \theta) = -\cos(\theta) = -0.2$

b $\cos(\pi + \theta) = \cos(\pi - \theta) = -0.2$

c $\cos(-\theta) = -\cos(\theta) = -0.2$

d $\cos(2\pi + \theta) = \cos(\theta) = 0.2$

15 $\sin(t) = 0.9$ and $\tan(x) = 4$

a $\tan(-x) = -\tan(x) = -4$

b $\sin(\pi - t) = \sin(t) = 0.9$

c $\tan(2\pi - x) = -\tan(x) = -4$

d
$$\begin{aligned} &\sin(-t) + \tan(\pi + x) \\ &= -\sin(t) + \tan(x) \\ &= -0.9 + 4 \\ &= 3.1 \end{aligned}$$

16 Given $\cos(\theta) = 0.91$, $\sin(t) = 0.43$ and $\tan(x) = 0.47$.

a
$$\begin{aligned} \cos(\pi + \theta) &= -\cos(\theta) \\ &= -0.91 \end{aligned}$$

b
$$\begin{aligned} \sin(\pi - t) &= \sin(t) \\ &= 0.43 \end{aligned}$$

c
$$\begin{aligned} \tan(2\pi - x) &= -\tan(x) \\ &= -0.47 \end{aligned}$$

d
$$\begin{aligned} \cos(-\theta) &= \cos(\theta) \\ &= 0.91 \end{aligned}$$

e
$$\begin{aligned} \sin(-t) &= -\sin(t) \\ &= -0.43 \end{aligned}$$

f
$$\begin{aligned} \tan(2\pi + x) &= \tan(x) \\ &= 0.47 \end{aligned}$$

17 Given $\sin(\theta) = p$

a
$$\begin{aligned} \sin(2\pi - \theta) &= -\sin(\theta) \\ &= -p \end{aligned}$$

b
$$\begin{aligned} \sin(3\pi - \theta) &= \sin(\theta) \\ &= p \end{aligned}$$

c
$$\begin{aligned} \sin(-\pi + \theta) &= -\sin(\theta) \\ &= -p \end{aligned}$$

d
$$\begin{aligned} \sin(\theta + 4\pi) &= \sin(\theta) \\ &= p \end{aligned}$$

18 a [75°]. Symmetric points are found from 180° ± 75°, 360° ± 75°.

Therefore, second quadrant [105°], third quadrant [255°], fourth quadrant [285°].

Cosine is positive in the fourth quadrant, so $\cos(285°) = \cos(75°)$. (The first and fourth quadrant points have the same x-coordinates.) The trigonometric point is [285°].

b $\frac{6\pi}{7} = \pi - \frac{\pi}{7}$

$$\begin{aligned} &\therefore \tan\left(\frac{6\pi}{7}\right) \\ &= \tan\left(\pi - \frac{\pi}{7}\right) \\ &= -\tan\left(\frac{\pi}{7}\right) \end{aligned}$$

c
$$\begin{aligned} \sin(\theta) &= 0.8 \\ \sin(\pi - \theta) &= \sin(\theta) \\ &= 0.8 \\ \sin(2\pi - \theta) &= -\sin(\theta) \\ &= -0.8 \end{aligned}$$

d i
$$\begin{aligned} \cos\left(\frac{5\pi}{4}\right) &= \cos\left(\pi + \frac{\pi}{4}\right) \\ &= -\cos\left(\frac{\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{2} \end{aligned}$$

ii
$$\begin{aligned} \sin\left(\frac{25\pi}{6}\right) &= \sin\left(4\pi + \frac{\pi}{6}\right) \\ &= \sin\left(2\pi + \frac{\pi}{6}\right) \\ &= \sin\left(\frac{\pi}{6}\right) \\ &= \frac{1}{2} \end{aligned}$$

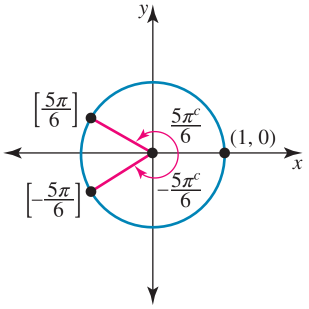
19 a Statement: $\sin^2\left(\frac{5\pi}{4}\right) + \cos^2\left(\frac{5\pi}{4}\right) = 1$

$$\begin{array}{ll} \sin\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) \text{ and } & \cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) \\ \qquad = -\frac{\sqrt{2}}{2} & \qquad = -\frac{\sqrt{2}}{2} \end{array}$$

Substitute these values into the left-hand side of the statement.

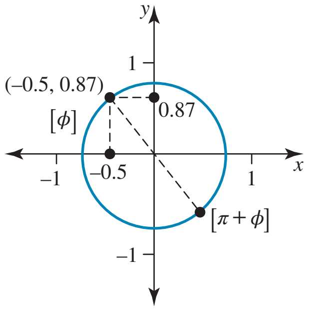
$$\begin{aligned}\text{LHS} &= \left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2\\ &= \frac{2}{4} + \frac{2}{4}\\ &= 1\\ &= \text{RHS}\end{aligned}$$

b Plot the trigonometric points $\left[\frac{5\pi}{6}\right]$ and $\left[-\frac{5\pi}{6}\right]$ on a unit circle diagram.



The two points are symmetric relative to the x-axis, since $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$ and $-\frac{5\pi}{6} = -\pi + \frac{\pi}{6}$. Their x-values are the same, so $\cos\left(\frac{5\pi}{6}\right) = \cos\left(-\frac{5\pi}{6}\right)$.

c The point $[\phi]$ is the Cartesian point $(-0.5, 0.87)$.



Comparing the y-coordinates of $[\phi]$ and $[\pi + \phi]$,

$$\begin{aligned}\sin(\pi + \phi) &= -\sin(\phi)\\ &= -0.87\end{aligned}$$

Comparing the x-coordinates of $[\phi]$ and $[\pi + \phi]$,

$$\begin{aligned}\cos(\pi + \phi) &= -\cos(\phi)\\ &= -(-0.5)\\ &= 0.5\end{aligned}$$

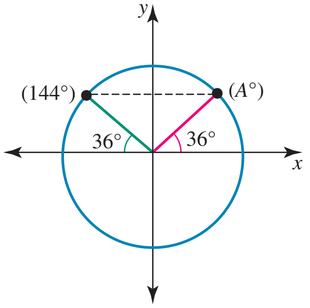
$$\begin{aligned}\tan(\pi + \phi) &= \frac{\sin(\pi + \phi)}{\cos(\pi + \phi)}\\ &= \frac{-0.87}{0.5}\\ &= -1.74\end{aligned}$$

d $\sin(-\pi + t) + \sin(-3\pi - t) + \sin(t + 6\pi)$

$$\begin{aligned}&= -\sin(t) + \sin(t) + \sin(t)\\ &= \sin(t)\end{aligned}$$

e If $\sin(A°) = \sin(144°)$, then the points $[A°]$ and $[144°]$ have the same y-coordinates.

The point $[144°]$ is in the second quadrant.

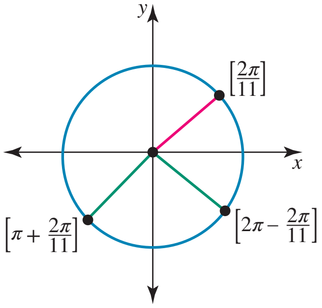


As $144° = 180° - 36°$, the point $[36°]$ has the same y-value as the point $[144°]$. Hence, one value for $A°$ is $A° = 36°$.

Another value could be $-360° + 36° = -324°$.

Other values are possible, including other values for $[144°]$ such as $[-216°]$, so $A° = -216°$.

f If $\sin(B) = -\sin\left(\frac{2\pi}{11}\right)$, then the y-value of the point $[B]$ has the opposite sign to that of the point $\left[\frac{2\pi}{11}\right]$. Since $\left[\frac{2\pi}{11}\right]$ lies in the first quadrant, its y-value is positive, so the y-value of the point $[B]$ is negative.



The point $[B]$ is in either the third or the fourth quadrant.

Possible values are:

$$\begin{aligned}B &= \pi + \frac{2\pi}{11}\\ &= \frac{13\pi}{11}\end{aligned} \quad \text{and} \quad \begin{aligned}B &= 2\pi - \frac{2\pi}{11}\\ &= \frac{20\pi}{11}\end{aligned}$$

For a third value, $B = -\frac{2\pi}{11}$. However, there are many answers possible for $B = -\frac{2\pi}{11}$.

20 $Q[\theta]$ where $\tan(\theta) = 5$.

a Since $\tan(\theta) > 0$, the point Q could lie in either the first or the third quadrant.

b For quadrant 1,

$$\begin{aligned}\theta &= \tan^{-1}(5)\\ &\simeq 1.3734\end{aligned}$$

For quadrant 3,

$$\begin{aligned}\theta &= \pi + \tan^{-1}(5)\\ &\simeq 4.5150\end{aligned}$$

c For quadrant 1, with $\theta = \tan^{-1}(5)$,

$$x = \cos(\theta)$$

$$\therefore x = \cos(\tan^{-1}(5))$$

Evaluate this in Standard and Rad modes to obtain

$$x = \frac{\sqrt{26}}{26}.$$

$y = \sin(\theta)$

$\therefore y = \sin\left(\tan^{-1}(5)\right)$

$\therefore y = \dfrac{5\sqrt{26}}{26}$

The first quadrant point has Cartesian coordinates $\left(\dfrac{\sqrt{26}}{26}, \dfrac{5\sqrt{26}}{26}\right)$.

For the point in the third quadrant, $x < 0$ and $y < 0$, so the point must have Cartesian coordinates $\left(-\dfrac{\sqrt{26}}{26}, -\dfrac{5\sqrt{26}}{26}\right)$.

For this third quadrant point Q $[\theta]$, $\cos(\theta) = -\dfrac{\sqrt{26}}{26}$ and $\sin(\theta) = -\dfrac{5\sqrt{26}}{26}$.

Alternatively, given $\tan(\theta) = 5 = \dfrac{5}{1}$, draw a right-angled triangle with opposite side 5 and adjacent side 1 relative to angle θ.

Use Pythagoras' theorem to calculate the hypotenuse.

$c^2 = 5^2 + 1^2$

$c^2 = 26$

$c = \sqrt{26}$

Hence, for the first quadrant, $\cos(\theta) = \dfrac{1}{\sqrt{26}}$ and $\sin(\theta) = \dfrac{5}{\sqrt{26}}$. The first quadrant point has Cartesian coordinates $\left(\dfrac{1}{\sqrt{26}}, \dfrac{5}{\sqrt{26}}\right)$.

For the third quadrant, both cosine and sine are negative. Hence,

$\cos(\theta) = -\dfrac{1}{\sqrt{26}}$ and $\sin(\theta) = -\dfrac{5}{\sqrt{26}}$. The third quadrant point has Cartesian coordinates $\left(-\dfrac{1}{\sqrt{26}}, -\dfrac{5}{\sqrt{26}}\right)$.

8.5 Exam questions

1 $$\begin{aligned}\cos\left(\frac{3\pi}{4}\right) &= \cos\left(\pi - \frac{\pi}{4}\right)\\ &= -\cos\left(\frac{\pi}{4}\right)\\ &= -\frac{\sqrt{2}}{2}\end{aligned}$$

The correct answer is **B**.

2 $\sin(\theta)$ is negative. $\therefore \theta$ is in the third or fourth quadrant.

The basic angle for which $\sin(\theta) = \dfrac{1}{\sqrt{2}}$ is 45.

Third quadrant:

$\theta = 180° + 45°$

$= 225°$

Fourth Quadrant:

$\theta = 360° - 45°$

$= 315°$

$\therefore \theta = 225°, 315°$

The correct answer is **D**.

3 $4\sin\left(\dfrac{5\pi}{6}\right) - 2\cos\left(\dfrac{\pi}{3}\right) + \tan\left(\dfrac{5\pi}{4}\right) + \sin\left(\dfrac{7\pi}{3}\right)$

$= 4\sin\left(\dfrac{\pi}{6}\right) - 2\cos\left(\dfrac{\pi}{3}\right) + \tan\left(\dfrac{\pi}{4}\right) + \sin\left(\dfrac{\pi}{3}\right)$

$= 4 \times \dfrac{1}{2} - 2 \times \dfrac{1}{2} + 1 + \dfrac{\sqrt{3}}{2}$ [1 mark]

$= 2 - 1 + 1 + \dfrac{\sqrt{3}}{2}$

$= 2 + \dfrac{\sqrt{3}}{2}$ [1 mark]

$= \dfrac{4+\sqrt{3}}{2}$

8.6 Graphs of the sine and cosine functions

8.6 Exercise

1 **a** The domain is $[0, 4\pi]$ and the range is $[-1, 1]$.

b The graph starts at the equilibrium position and rises, which is the shape of $y = \sin(x)$.

c The x-coordinates of the turning points lie midway between the x-intercepts.

Hence, the coordinates are $\left(\dfrac{\pi}{2}, 1\right), \left(\dfrac{3\pi}{2}, -1\right), \left(\dfrac{5\pi}{2}, 1\right), \left(\dfrac{7\pi}{2}, -1\right)$.

d The graph completes one cycle in 2π units, so its period is 2π.

The graph rises and falls 1 unit from its equilibrium position, so its amplitude is 1.

e The graph oscillates about the x-axis. Hence, its mean or equilibrium position is the x-axis. The equation of the mean position is $y = 0$.

f $f(x) > 0$ where the graph lies above the x-axis. This occurs for $x \in (0, \pi) \cup (2\pi, 3\pi)$.

2 **a** The domain is $[-2\pi, 2\pi]$ and the range is $[-1, 1]$.

b The graph starts at a maximum and falls towards the equilibrium position, which is the shape of $y = \cos(x)$.

c There are two minimum turning points on the graph. They have coordinates $(-\pi, -1)$ and $(\pi, -1)$.

d The graph completes one cycle in 2π units, so its period is 2π.

The graph rises and falls 1 unit from its equilibrium position, so its amplitude is 1.

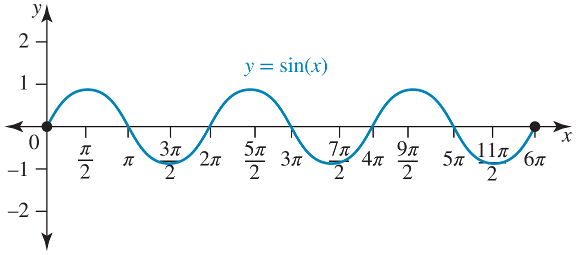
The graph oscillates about the x axis. Hence, its mean, or equilibrium, position is the x axis. The equation of the mean position is $y = 0$.

e The x intercepts lie midway between the x coordinates of successive turning points.

Hence, the coordinates are $\left(-\dfrac{3\pi}{2}, 0\right), \left(-\dfrac{\pi}{2}, 0\right), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{2}, 0\right)$.

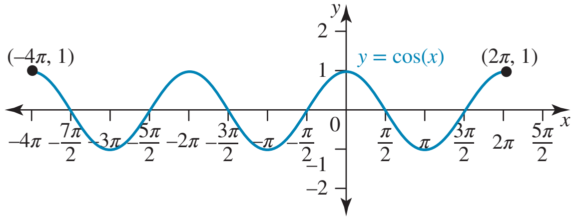
f $g(x) < 0$ where the graph lies below the x-axis. This occurs for $x \in \left(-\dfrac{3\pi}{2}, -\dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

3 a The graph of $y = \sin(x),\ 0 \le x \le 6\pi$ covers three cycles. Sketch one cycle over $[0, 2\pi]$ and extend the pattern.



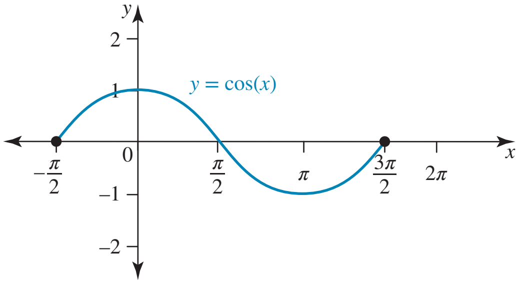
b $y = \cos(x), -4\pi \le x \le 2\pi$

Sketch one cycle of the cosine graph over $[0, 2\pi]$ and extend the pattern to the left of the origin for two cycles.



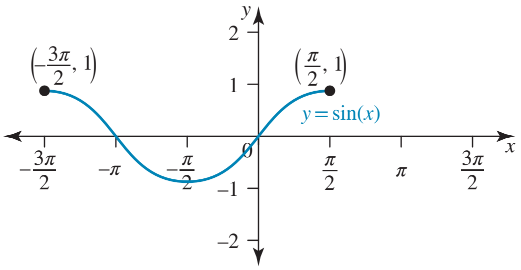
c $y = \cos(x), -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Draw part of the cycle of the basic cosine graph but stop it at $x = \frac{3\pi}{2}$. Extend the graph back to its equilibrium position at $x = -\frac{\pi}{2}$.

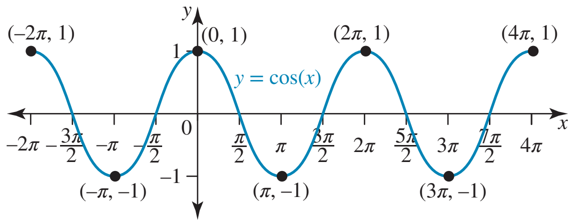


d $y = \sin(x), -\frac{3\pi}{2} \le x \le \frac{\pi}{2}$

One cycle is required starting at a maximum point at $x = -\frac{3\pi}{2}$ and finishing at a maximum point at $x = \frac{\pi}{2}$.



4 To sketch $y = \cos(x)$ over the domain $[-2\pi, 4\pi]$, sketch the basic graph on the domain $[0, 2\pi]$ and continue the pattern.



The graph shows three cycles of the cosine function.

5 a $f : [-4\pi, 0] \to R, f(x) = \sin(x)$

In every cycle of the sine graph, there is one maximum turning point. The domain $[-4\pi, 0]$ covers two cycles. Therefore, the function has 2 maximum turning points.

b $f : [0, 14\pi] \to R, f(x) = \cos(x)$

In every cycle of the cosine graph, there is one minimum turning point. The domain $[0, 14\pi]$ covers seven cycles. Therefore, the function has 7 minimum turning points.

6 a $y = \cos(x),\ 0 \le x \le \frac{7\pi}{2}$

Over one period of 2π, the cosine graph has two x-intercepts. The domain interval $\left[0, \frac{7\pi}{2}\right]$ covers $1\frac{3}{4}$ cycles, ending at equilibrium, which is the x-axis. So the graph has $2 + 2 = 4$ x intercepts.

b Over one period of 2π, the sine graph has three x-intercepts but as the graph shape starts and stops at the equilibrium position, the x-intercept at the end of one cycle is also the x-intercept for the start of the next cycle.

Over the domain $[-2\pi, 4\pi]$ there will be three cycles with $3 + 2 + 2 = 7$ x-intercepts.

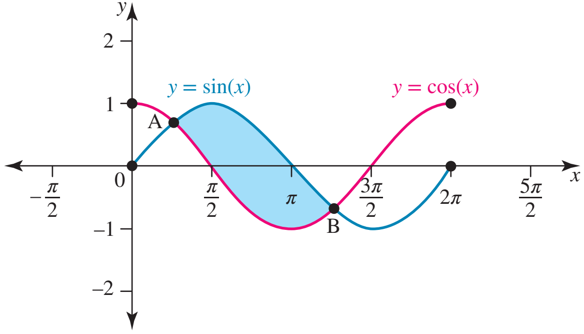
c $y = \sin(x),\ 0 \le x \le 20\pi$

Over the domain $[0, 20\pi]$ there will be ten cycles with $3 + 9 \times 2 = 21$ x-intercepts.

d $y = \cos(x),\ \pi \le x \le 4\pi$

Over the domain $[\pi, 4\pi]$ there will be $1\frac{1}{2}$ cycles with $2 + 1 = 3$ x-intercepts.

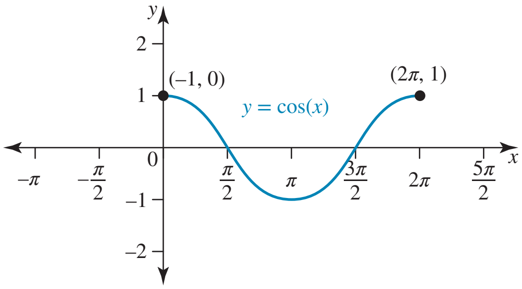
7 Over the domain $[0, 2\pi]$ one cycle of each of the graphs of $y = \cos(x)$ and $y = \sin(x)$ is required.



The graphs intersect at the points A and B. The region below the sine graph and above the cosine graph between these points is the required region,

$\{(x, y) : \sin(x) \ge \cos(x), x \in [0, 2\pi]\}$.

8 a $y = \cos(x),\ 0 \le x \le 2\pi$



$\cos(x) < 0$ when the graph lies below the x-axis. Hence, $\cos(x) < 0$ for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

b If x is in either the second or third quadrants, then $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and $y = \cos(x) < 0$.

In the first quadrant, $0 < x < \frac{\pi}{2}$ and the graph lies above the x-axis showing $\cos(x) > 0$; in the fourth quadrant $\frac{3\pi}{2} < x < 2\pi$ and $\cos(x) > 0$.
Cosine is negative in the second and third quadrants and positive in the first and fourth quadrants. The graph is illustrating what the ‘CAST’ diagram said about the sign of cosine.

9 $f: [0, a] \rightarrow R, f(x) = \cos(x)$
As the domain starts at $x = 0$ and every interval of length 2π has two intersections with the x-axis, for 10 such intersections the graph must cover between $4\frac{3}{4}$ and 5 cycles. The smallest value for a is $a = \frac{19\pi}{2}$.

10 **a** $\sin(x) \approx x$

Hence, $\sin\left(\frac{\pi}{12}\right) \approx \frac{\pi}{12}$

b $\sin(2°30') = \sin(2.5°)$

Convert 2.5° to radian measure by multiplying by $\frac{\pi}{180}$.

$$\begin{aligned} 2.5° &= 2.5 \times \frac{\pi}{180} \\ &= \frac{5}{2} \times \frac{\pi}{180} \\ &= \frac{\pi}{72} \end{aligned}$$

Hence, using $\sin(x) \approx x$, $\sin\left(\frac{\pi}{72}\right) \approx \frac{\pi}{72}$.

Therefore, $\sin(2.5°) \approx \frac{\pi}{72}$.

11 **a** $\sin(x) - 11x + 4 = 0$

Rearrange the equation to $\sin(x) = 11x - 4$.
The number of solutions to the equation can be determined by sketching the graph of $y = \sin(x)$ and the graph of the straight line $y = 11x - 4$.

b As there is only one point of intersection, there is only one solution to the equation.

c Using the linear approximation $\sin(x) \approx x$, the equation $\sin(x) - 11x + 4 = 0$ becomes $x - 11x + 4 = 0$.
Solving,

$$\begin{aligned} -10x + 4 &= 0 \\ 10x &= 4 \\ x &= \frac{4}{10} \\ x &= 0.4 \end{aligned}$$

An estimate of the solution is $x = 0.4$.

12 $\sin(x) = x$ for small values of x.

a
$$\begin{aligned} 1° &= \frac{\pi^c}{180} \\ \sin(1°) &= \sin\left(\frac{\pi^c}{180}\right) \\ &= \sin\left(\frac{\pi}{180}\right) \\ &= \frac{\pi}{180} \end{aligned}$$

b $\sin\left(\frac{\pi}{9}\right) \simeq \frac{\pi}{9}$

c
$$\begin{aligned} -2° &= -2 \times \frac{\pi^c}{180} \\ \therefore \sin(-2°) &= \sin\left(-\frac{\pi}{90}\right) \\ &\simeq -\frac{\pi}{90} \end{aligned}$$

d Using the approximation, $\sin\left(\frac{\pi}{6}\right) \simeq \frac{\pi}{6}$.

$\therefore \sin\left(\frac{\pi}{6}\right) \simeq 0.5236$ correct to 4 decimal places.

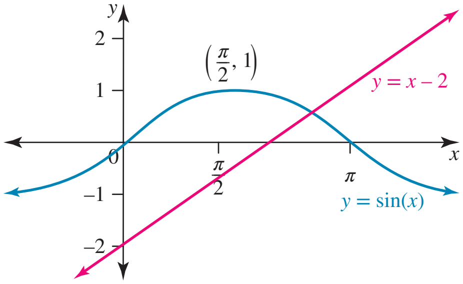
The exact value of $\sin\left(\frac{\pi}{6}\right)$ is $\frac{1}{2} = 0.5$.

The discrepancy is due to the approximation $\sin(x) \simeq x$ becoming less accurate as the size of x increases.

13 $\cos(x) + 5x - 2 = 0$
Substitute 1 for $\cos(x)$.

$$\begin{aligned} \therefore 1 + 5x - 2 &= 0 \\ 5x &= 1 \\ x &= 0.2 \end{aligned}$$

14 **a** The graph of $y = \sin(x)$ has x-intercepts at the origin and at $(\pi, 0) \simeq (-3.14, 0)$ as well others. The line $y = x - 2$ has intercepts with the axes at $(0, -2)$ and $(2, 0)$.



There is only one point of intersection of the two graphs. Therefore, the equation $\sin(x) = x - 2$ has only one solution.

b From the graph, the point of intersection lies in the interval between $x = 2$ and $x = 3$ (answers could vary).

c Let $f(x) = \sin(x) - x + 2$
From the graph, the sine graph is higher than the line at $x = 2$, so $f(2) > 0$.
However, the sine graph lies below the line at $x = 3$, so $f(3) < 0$.
The midpoint of $[2, 3]$ is $x = 2.5$.

$$\begin{aligned} f(2.5) &= \sin(2.5) - 2.5 + 2 \\ &= \sin(2.5^c) - 0.5 \\ &= 0.098.. \\ &> 0 \end{aligned}$$

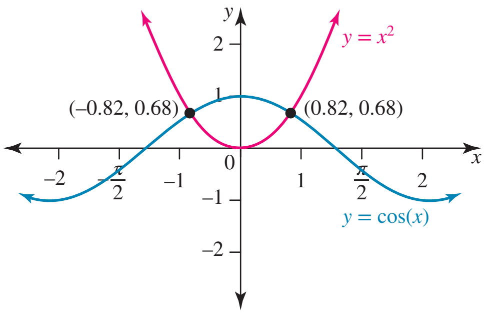
The root lies in the interval $[2.5, 3]$.
The midpoint of $[2.5, 3]$ is $x = 2.75$

$$\begin{aligned} f(2.75) &= \sin(2.75) - 2.75 + 2 \\ &= -0.368.. \\ &< 0 \end{aligned}$$

The root lies in the interval $[2.5, 2.75]$,
The midpoint of $[2.5, 2.75]$ gives an estimate of the root.
An estimate is $x = 2.625$.

15 **a** The number of solutions to the equation $\cos(x) - x^2 = 0$ is determined by the number of intersections of the graphs of $y = \cos(x)$ and $y = x^2$. At the intersection of these graphs, $\cos(x) = x^2$ and therefore $\cos(x) - x^2 = 0$.

b The graph of $y = x^2$ contains the points $(0, 0), (\pm 1, 1)$.
The graph of $y = \cos(x)$ contains the point $(0, 1)$ and among its x-intercepts are $\left(\pm\frac{\pi}{2}, 0\right) \simeq (\pm 1.57, 0)$.



As the graphs intersect twice, there are two roots to the equation $\cos(x) - x^2 = 0$.

c If the graph of $y = x^2$ is vertically translated upwards by one unit, then it will meet the cosine graph at the point (0, 1). Therefore, if $k = 1$, the equation $\cos(x) = x^2 + k$ will have exactly one solution.

16 a $\sin(1.8°)$

Converting the angle to radian measure,

$$1.8° = 1.8 \times \frac{\pi^c}{180}$$

$$\therefore 1.8° = \frac{\pi^c}{100}$$

$$\sin(1.8°) = \sin\left(\frac{\pi^c}{100}\right).$$

For small x, $\sin(x) \simeq x$. As $\frac{\pi}{100}$ is small,

$$\sin\left(\frac{\pi^c}{100}\right) \simeq \frac{\pi}{100}.$$

$$\therefore \sin(1.8°) \simeq 0.01\pi.$$

From a calculator, $\sin(1.8°) \simeq 0.03141$ and $0.01\pi \simeq 0.03142$, so the two values are the same correct to 4 decimal places.

b $\cos(x) - 10.5x^2 = 0$

Let $f(x) = \cos(x) - 10.5x^2$.

$$f(0) = \cos(0) - 10.5\,(0)^2$$
$$= 1$$
$$> 0$$

$$f(0.4) = \cos(0.4) - 10.5\,(0.4)^2$$
$$= -0.7589..$$
$$< 0$$

Therefore, the equation $\cos(x) - 10.5x^2 = 0$ has a root for which $0 \le x \le 0.4$.

c As the root is small, let $\cos(x) \simeq 1$.

$$1 - 10.5x^2 = 0$$
$$1 = 10.5x^2$$
$$x^2 = \frac{2}{21}$$
$$\therefore x = \pm\frac{2}{\sqrt{21}}$$

The positive root is $x = 0.31$, correct to 2 decimal places.

17 a As 0.05 is small, the approximation $\cos(x) \simeq 1$ can be used to calculate the value of

$\cos(0.05) \approx 1$

The calculator gives

$\cos(0.05) = 0.99875$, so the approximation is valid.

b As 5 is not a small number close to zero, the quadratic approximation for $\cos(5)$ is not applicable.

18 a Let $f(x) = 4x\sin(x) - 1$.

$$f(0) = -1$$
$$< 0$$

$$f(0.6) = 4(0.6)\sin(0.6) - 1$$
$$= 0.355..$$
$$> 0$$

Therefore, there is a solution to the equation for which $0 \le x \le 0.6$.

Let $\sin(x) = x$.

$$\therefore 4x(x) - 1 = 0$$
$$\therefore 4x^2 = 1$$
$$\therefore x^2 = \frac{1}{4}$$
$$\therefore x = \pm\frac{1}{2}$$

An estimate of the solution for which $0 \le x \le 0.6$ is $x = 0.5$.

b Rearranging the equation,

$$4x\sin(x) - 1 = 0$$
$$\therefore 4x\sin(x) = 1$$
$$\therefore \sin(x) = \frac{1}{4x}$$

The intersection of the graph of $y = \frac{1}{4x}$ with the graph of $y = \sin(x)$ determines the number of solutions.

The function $y = \frac{1}{4x}$ is a hyperbola with the x-axis as its horizontal asymptote. For positive values of x, as $x \to \infty, y \to 0$, so the hyperbola will intersect with the sine graph in an infinite number of points.

c The linear approximation only applies for small values of x around zero. The other positive solutions will all be greater than 1.

d Consider one cycle of $y = \sin(x)$ together with $y = \frac{1}{4x}$ for $x > 0$.

The sine graph remains above the x-axis for $0 < x < \pi$ and the hyperbola remains above the x-axis for $x > 0$.

If x is very close to zero, the hyperbola has a large y-value and the sine graph has a y-value close to zero. The hyperbola is above the sine graph.

If $x = \frac{\pi}{2}$, the hyperbola has a y-value of $y = \frac{1}{2\pi} \simeq 0.16$, and the sine graph has $y = 1$. The hyperbola is now below the sine graph, so it has crossed the sine graph between $0 < x < \frac{\pi}{2}$. As the hyperbola is approaching the x-axis, it must intersect the sine graph again between $\frac{\pi}{2} < x < \pi$.

For $\pi < x < 2\pi$, the sine graph is below the x-axis, so the hyperbola will not intersect it.

A similar analysis can be made for $x < 0$ with the hyperbola intersecting the sine graph when it lies below the x-axis.

Hence, the hyperbola will intersect the sine graph twice over a period.

Over the domain $[-4\pi, 4\pi]$ the sine function covers 4 periods, so the hyperbola will intersect the sine curve $4 \times 2 = 8$ times.

19 a i $\sin(x) = 1 - x^2$ when $x = -1.4096$ or $x = 0.6367$ using equation solver.

ii Using the linear approximation, $x = 1 - x^2$. This equation has solutions $x = 0.618$ or $x = -1.618$.

For the solution closer to zero, the linear approximation gives $x = 0.618$ compared to $x = 0.6367$ from CAS. They agree at 1 decimal place accuracy.

b Comparing the values of $\sin(x)$ and x for $0 \le x \le 0.7$, some of the values are shown in the table.

$\boldsymbol{x}$	0	0.1	0.5	0.6	0.7
$\mathbf{sin}(\boldsymbol{x})$	0	0.9983	0.47943	0.56464	0.64422

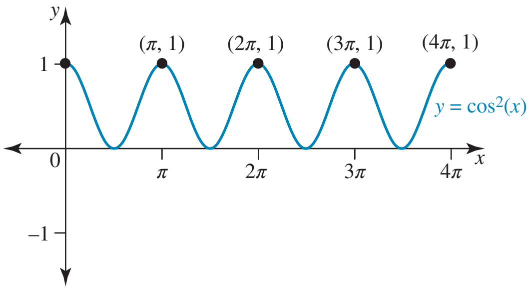
At $x = 0$, $\sin(x) = x$. The values of each are close until $x = 0.7$ where they no longer agree to 1 decimal place.

The approximation seems good for $0 \le x \le 0.5$ and reasonable for $x = 0.6$.

As $\sin(-x) = -\sin(x)$, a similar degree of closeness is obtained for negative values.

The approximation is 'reasonable' for $-0.6 \le x \le 0.6$.

20 $y = \cos^2(x)$ for $x \in [0, 4\pi]$



Where $\cos(x) < 0$, its value squared will be positive. The period of the cosine graph is 2π, so the period of the graph of $y = \cos^2(x)$ is π.

8.6 Exam questions

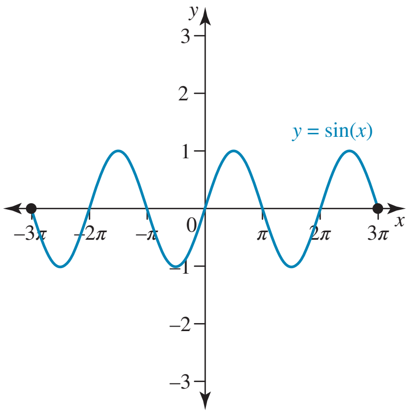
1 $y = \sin(x)$ passes through $(0, 0)$.

$y = \cos(x)$ passes through $\left(\frac{\pi}{2}, 0\right)$ and $(0, 1)$.

$\therefore$ option D is true for $f(x) = \sin(x)$ but false for $g(x) = \cos(x)$.

The correct answer is **D**.

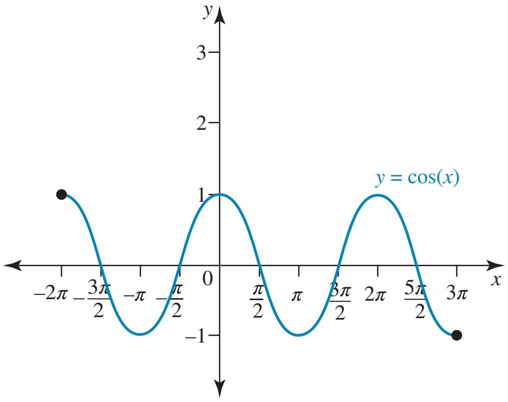
2



The graph of $y = \sin(x)$ has x−intercepts at $0, \ \pm\pi, \pm 2\pi, \pm 3\pi$.

The correct answer is **A**.

3



The graph shows solution $\pm\pi, 3\pi$.

The correct answer is **E**.

8.7 Review

8.7 Exercise

Technology free: short answer

1 a $\frac{11\pi^c}{9} = \frac{11\pi}{9} \times \frac{180°}{\pi}$

$\therefore \frac{11\pi^c}{9} = 220°$

b $-3.5\pi^c = -\frac{7\pi}{2} \times \frac{180°}{\pi}$

$\therefore -3.5\pi^c = -630°$

2 $\cos\left(\frac{11\pi}{6}\right) - \tan\left(\frac{11\pi}{3}\right) + \sin\left(-\frac{11\pi}{4}\right)$

Calculating each value,

$$\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2}$$

$$\tan\left(\frac{11\pi}{3}\right) = \tan\left(2\pi + \frac{5\pi}{3}\right)$$
$$= \tan\left(\frac{5\pi}{3}\right)$$
$$= -\tan\left(\frac{\pi}{3}\right)$$
$$= -\sqrt{3}$$

$$\sin\left(-\frac{11\pi}{4}\right) = -\sin\left(\frac{11\pi}{4}\right)$$
$$= -\sin\left(2\pi + \frac{3\pi}{4}\right)$$
$$= -\sin\left(\frac{3\pi}{4}\right)$$
$$= -\sin\left(\frac{\pi}{4}\right)$$
$$= -\frac{\sqrt{2}}{2}$$

Hence,

$\cos\left(\frac{11\pi}{6}\right) - \tan\left(\frac{11\pi}{3}\right) + \sin\left(-\frac{11\pi}{4}\right).$

$$= \frac{\sqrt{3}}{2} - \left(-\sqrt{3}\right) + \left(-\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} + \sqrt{3} - \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{3} + 2\sqrt{3} - \sqrt{2}}{2}$$

$$= \frac{3\sqrt{3} - \sqrt{2}}{2}$$

3 $\cos(t) = 0.6$

Using symmetry properties,

a $\cos(-t) = \cos(t)$

$\therefore \cos(-t) = 0.6$

b $\cos(\pi + t) = -\cos(t)$

$\therefore \cos(\pi + t) = -0.6$

c $\cos(3\pi - t) = \cos(\pi - t)$

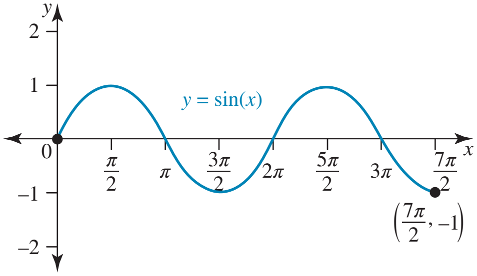
As $\cos(\pi - t) = -\cos(t)$, $\cos(3\pi - t) = -0.6$.

d $\cos(-2\pi + t) = \cos(t)$

$\therefore \cos(-2\pi + t) = 0.6$

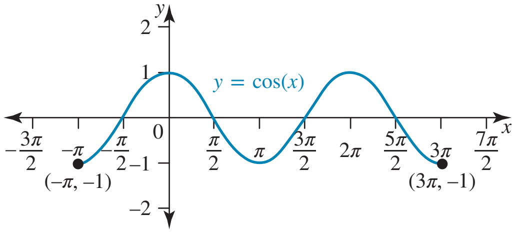
4 **a** $y = \sin(x),\ 0 \le x \le \frac{7\pi}{2}$

Draw one cycle of $y = \sin(x)$ for $[0, 2\pi]$ and extend the pattern to $x = \frac{7\pi}{2} = 2\pi + \frac{3\pi}{2}$.



b $y = \cos(x), -\pi \le x \le 3\pi$

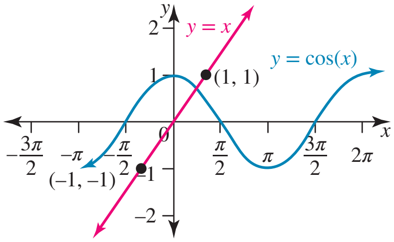
Draw one cycle of $y = \cos(x)$ for $[0, 2\pi]$ and extend the pattern both to the left and to the right for half a cycle.



5 **a** $\cos(x) - x = 0$

$\therefore \cos(x) = x$

Sketch the graphs of $y = \cos(x)$ and $y = x$.



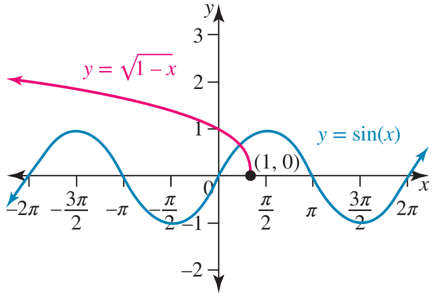
There is one point of intersection so the equation has one solution.

b $\sin(x) - \sqrt{1 - x} = 0$

$\therefore \sin(x) = \sqrt{1 - x}$

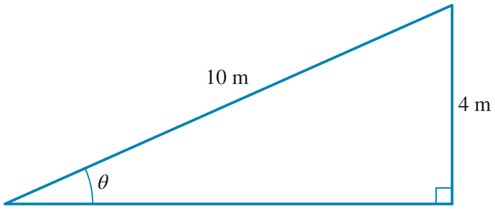
Sketch the graphs of $y = \sin(x)$ and the square root function $y = \sqrt{1 - x}$.

The square root function has end point $(1, 0)$ and passes through the point $(0, 1)$.



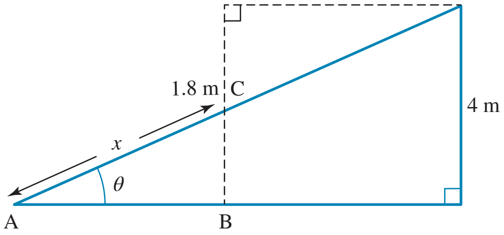
The graphs intersect once, so there is one solution to the equation.

6 **a**



$\sin(\theta) = \frac{4}{10}$

b



Let the distance the person climbs up the ladder be x.

Consider the right-angled triangle ABC.

$BC = 4 - 1.8 = 2.2$ metres, $\angle CAB = \theta$ and $\angle ABC = 90°$.

$\sin(\theta) = \frac{2.2}{x}$

$\therefore x = \frac{2.2}{\sin(\theta)}$

From part **a**, the exact value for $\sin(\theta)$ is 0.4.

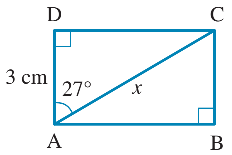
$\therefore x = \frac{2.2}{0.4}$

$\therefore x = 5.5$

The person needs to climb 5.5 metres up the ladder.

Technology active: multiple choice

7



Let $AC = x$ cm.

In triangle ADC,

$\cos(27°) = \frac{3}{x}$

$\therefore x = \frac{3}{\cos(27°)}$

$\therefore x = 3.37$

The diagonal AC is of length 3.37 cm.

The correct answer is **C**.

8 $\sin(45°) + \tan(30°)\cos(60°)$

$= \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{3} \times \dfrac{1}{2}$

$= \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{6}$

$= \dfrac{3\sqrt{2} + \sqrt{3}}{6}$

The correct answer is **C**.

9 $100°$

$= 100 \times \dfrac{\pi}{180}$

$= \dfrac{10\pi}{18}$

$= \dfrac{5\pi}{9}$

The correct answer is **A**.

10 $l = r\theta,\ r = 3,\ \theta = 30° = \dfrac{\pi}{6}$

$= 3 \times \dfrac{\pi}{6}$

$= \dfrac{\pi}{2}$

Arc length is $\dfrac{\pi}{2}$ cm.

The correct answer is **A**.

11 $\theta = \pi - \dfrac{\pi}{5}$

$\therefore \theta = \dfrac{4\pi}{5}$

The correct answer is **B**.

12 $\sin(\theta)$ is the y-coordinate of P, so its value is the length of the line segment NP.

The correct answer is **D**.

13 $\sin(\theta) < 0$ and $\cos(\theta) > 0$ in quadrant 4.

The correct answer is **D**.

14 $\tan(330°)$

$= -\tan(30°)$

$= -\dfrac{\sqrt{3}}{3}$

The correct answer is **E**.

15 $\cos(-5\pi)$ is the x-coordinate of the trigonometric point $[-5\pi]$.

A clockwise rotation of 5π from the point $(1, 0)$ in a unit circle ends at the point $(-1, 0)$.

$\therefore \cos(-5\pi) = -1$

The correct answer is **B**.

16 $\sin(4.5°)$

$4.5° = 4.5 \times \dfrac{\pi^c}{180}$

$= \dfrac{\pi^c}{40}$

$\therefore \sin(4.5°) = \sin\left(\dfrac{\pi}{40}\right)$

For small values of x, $\sin(x) \simeq x$.

$\therefore \sin\left(\dfrac{\pi}{40}\right) \simeq \dfrac{\pi}{40}$

$\therefore \sin(4.5°) \simeq \dfrac{\pi}{40}$

The correct answer is **C**.

Technology active: extended response

17 a Given $AD = 3\,\text{cm}$, $BC = 8\,\text{cm}$ and $AB = 13\,\text{cm}$

Consider triangle DCE.

$DE = AC = 13\,\text{cm},\ \angle DCE = 90°$

$EC = BC - BE$

$= 8 - 3$

$= 5$

Since $\{5, 12, 13\}$ is a Pythagorean triple, $DC = 12\,\text{cm}$.

b In triangle CED,

$\cos(E) = \dfrac{5}{13}$

$E = \cos^{-1}\left(\dfrac{5}{13}\right)$

This can be evaluated in radians by using the calculator set on radian mode.

$\therefore E = 1.176$

Angle CED is 1.176^c.

c $\angle ABE = \angle CED$ (corresponding angles on parallel lines)

$\therefore \angle ABE = 1.176^c$

$\angle ABE + \angle EBF = \pi$ (supplementary angles)

$\angle EBF = \pi - 1.176^c$

$= 1.966^c$

d $\angle GAD = \angle ABE$ (corresponding angles)

$\therefore \angle GAD = 1.176^c$

e Consider the arc CF.

$l = r\theta,\ r = 8,\ \theta = 1.966$

$l = 8 \times 1.966$

$= 15.725$

Consider the arc GD.

$l = r\theta,\ r = 3,\ \theta = 1.176$

$l = 3 \times 1.176$

$= 3.528$

The length of the belt is

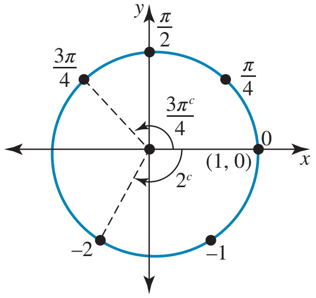
$2(GD + DC + CF)$

$= 2(3.528 + 12 + 15.725)$

$\simeq 62.5$

The length of the belt is 62.5 cm.

18 a



b $P\left[\dfrac{3\pi}{4}\right]$ has Cartesian coordinates (x, y) where

$x = \cos\left(\dfrac{3\pi}{4}\right)$ and $y = \sin\left(\dfrac{3\pi}{4}\right)$

$= -\cos\left(\dfrac{\pi}{4}\right) \qquad = \sin\left(\dfrac{\pi}{4}\right)$

$= -\dfrac{\sqrt{2}}{2} \qquad = \dfrac{\sqrt{2}}{2}$

The Cartesian coordinates of point P are $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

c Q $[-2]$ has Cartesian coordinates (x, y) where

$x = \cos(-2)$ and $y = \sin(-2)$

$= \cos(-2^c) \qquad = \sin(-2^c)$

$= -0.42 \qquad = -0.91$

To 2 decimal places the Cartesian coordinates of Q are $(-0.42, -0.91)$.

d Point R [θ] is in the first quadrant. P is the second quadrant symmetric point to R.

$$\therefore \pi - \theta = \frac{3\pi}{4}$$

$$\therefore \theta = \frac{\pi}{4}$$

e i The initial ray with endpoint $(1, 0)$ is rotated anticlockwise $\frac{3\pi}{4}^{c}$ to reach P and 2^{c} clockwise to reach Q.

The angle POQ must be equal to $2\pi - \left(\frac{3\pi}{4} + 2\right)$ radians.

Angle POQ equals $\left(\frac{5\pi}{4} - 2\right)$ radians.

ii Convert $\left(\frac{5\pi}{4} - 2\right)$ radians to degrees,

$$\left(\frac{5\pi}{4} - 2\right)^{c} = \left(\frac{5\pi}{4} - 2\right) \times \frac{180°}{\pi}$$

$$= \left(225 - \frac{360}{\pi}\right)°$$

$$\simeq 110.41°$$

Angle POQ equals 110.41°, correct to 2 decimal places.

f Many answers are possible.

The number $-2\pi - 2$ would be mapped to the same position as the number -2.

The number $2\pi + \frac{3\pi}{4} = \frac{11\pi}{4}$ would be mapped to the same position as the number $\frac{3\pi}{4}$.

19 a Each angle in an equilateral triangle is 60°.

The side length is $\sqrt{12}$ km.

Using the rule $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle,

$$A = \frac{1}{2} \times \sqrt{12} \times \sqrt{12} \times \sin(60°)$$

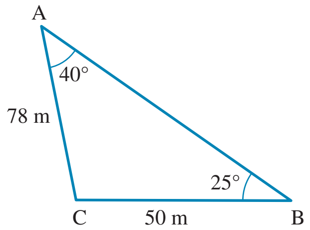
$$= 6\sin(60°)$$

$$= 6 \times \frac{\sqrt{3}}{2}$$

$$= 3\sqrt{3}$$

The area is $3\sqrt{3}$ km^2.

b



Angle C is equal to $180° - (40° + 25°) = 115°$

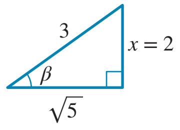
$$A = \frac{1}{2}ab\sin(C)$$

$$= \frac{1}{2} \times 50 \times 78 \times \sin(115°)$$

$$= 1767$$

The area is 1767 m^2 to the nearest whole number.

c i $\cos\beta = \frac{\sqrt{5}}{3}$



The right-angled triangle has the ratio of the adjacent side to the hypotenuse as $\sqrt{5}:3$.

Using Pythagoras' theorem,

$$x^2 + \left(\sqrt{5}\right)^2 = 3^2$$

$$\therefore x^2 + 5 = 9$$

$$\therefore x^2 = 4$$

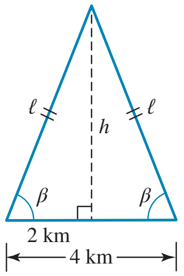
$$\therefore x = 2$$

$$\tan\beta = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$= \frac{2}{\sqrt{5}}$$

$$\therefore \tan(\beta) = \frac{2\sqrt{5}}{5}$$

ii



With the side of 4 km acting as the base of the isosceles triangle, let the height of the triangle be h km.

$$\tan(\beta) = \frac{h}{2}$$

$$\therefore h = 2\tan(\beta)$$

Substitute the exact value of $\tan(\beta)$

$$\therefore h = 2 \times \frac{2\sqrt{5}}{5}$$

$$\therefore h = \frac{4\sqrt{5}}{5}$$

For a triangle, Area $= \frac{1}{2} \times \text{base} \times \text{height}$

$$\therefore A = \frac{1}{2} \times 4 \times \frac{4\sqrt{5}}{5}$$

$$\therefore A = \frac{8\sqrt{5}}{5}$$

The area is $\frac{8\sqrt{5}}{5}$ km^2

iii The third angle of the triangle is $180° - 2\beta$.

iv Let the equal sides have length l km.

Using Pythagoras' theorem, the equal sides have lengths satisfying

$$l^2 = 2^2 + h^2$$

$$= 4 + \left(\frac{4\sqrt{5}}{5}\right)^2$$

$$= 4 + \frac{16}{5}$$

$$= \frac{36}{5}$$

$\therefore l = \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$ (negative square root is not applicable)

The area of the triangle is equal to
$\frac{1}{2} \times l \times l \times \sin(180° - 2\beta)$.

$\therefore \frac{1}{2} \times l \times l \times \sin(180° - 2\beta) = \frac{8\sqrt{5}}{5}$

Using the symmetry property, $\sin(180° - 2\beta) = \sin(2\beta)$

$$\therefore \frac{1}{2} \times l^2 \times \sin(2\beta) = \frac{8\sqrt{5}}{5}$$

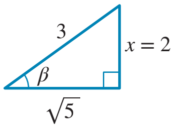
$$\therefore \frac{1}{2} \times \frac{36}{5} \times \sin(2\beta) = \frac{8\sqrt{5}}{5}$$

$$\therefore 18\sin(2\beta) = 8\sqrt{5}$$

$$\therefore \sin(2\beta) = \frac{8\sqrt{5}}{18}$$

$$\therefore \sin(2\beta) = \frac{4\sqrt{5}}{9}$$

v Referring to the triangle drawn in part **i**,



$\cos(\beta) = \frac{\sqrt{5}}{3}$ and $\sin(\beta) = \frac{2}{3}$.

From part **iv**, $\sin(2\beta) = \frac{4\sqrt{5}}{9}$

Substitute these values in each side of
$\sin(2\beta) = 2\sin(\beta)\cos(\beta)$

$$\begin{aligned}\text{RHS} &= 2 \times \frac{2}{3} \times \frac{\sqrt{5}}{3} \\ &= \frac{4\sqrt{5}}{9} \\ &= \text{LHS}\end{aligned}$$

The result is verified.

20 $y = 2\sin(x) + x - 2$.

a Let $f(x) = 2\sin(x) + x - 2$

$$\begin{aligned}f(0) &= 2\sin(0) + (0) - 2 \\ &= -2 \\ &< 0\end{aligned}$$

$$\begin{aligned}f(0.7) &= 2\sin(0.7) + (0.7) - 2 \\ &= -0.01156.. \\ &< 0\end{aligned}$$

$f(x)$ has not changed its sign. Since the sign is negative, the graph is still below the x-axis between $x = 0$ and $x = 0.7$.

b Use trial and error and continue testing the sign of the function

$$\begin{aligned}f(0.8) &= 2\sin(0.8) + (0.8) - 2 \\ &= 0.234.. \\ &> 0\end{aligned}$$

The graph is above the x-axis when $x = 0.8$.
The graph crosses the x-axis in the interval $[0.7,\ 0.8]$.
$\therefore a = 0.7$

c When the graph crosses the x-axis, its y-coordinate is zero.
And when $y = 0$, $0 = 2\sin(x) + x - 2$.
Given $x = \phi$ is the root of this equation, $x = \phi$ is where the graph crosses the x-axis.
Therefore, ϕ lies in the interval $[0.7,\ 0.8]$.

d The first estimate for ϕ is the midpoint of the interval $[0.7, 0.8]$, $x = 0.75$.

$$\begin{aligned}f(0.75) &= 2\sin(0.75) + (0.75) - 2 \\ &= 0.11.. \\ &> 0\end{aligned}$$

The root lies in the interval $[0.7, 0.75]$.
The second estimate for ϕ is the midpoint of this interval, $x = 0.725$.

$$\begin{aligned}f(0.725) &= 0.05.. \\ &> 0\end{aligned}$$

The root lies in $[0.7, 0.725]$.
The third estimate is $x = 0.7125$.
After three iterations, $\phi = 0.7125$.

e $2\sin(x) + x - 2 = 0$
When x is small, $\sin(x) \simeq x$.
Replacing $\sin(x)$ by x, the equation becomes

$$\begin{aligned}2x + x - 2 &= 0 \\ \therefore 3x &= 2 \\ \therefore x &= \frac{2}{3}\end{aligned}$$

The estimate for ϕ is $\phi = \frac{2}{3}$.
This is not as accurate as the value obtained using the method of bisection, since it has been established that $\phi \in [0.7, 0.8]$.

f i Rearranging the equation $2\sin(x) + x - 2 = 0$,

$$2\sin(x) = -x + 2$$
$$\therefore \sin(x) = -\frac{1}{2}x + 1$$

The graph of $y = -\frac{1}{2}x + 1$ needs to be graphed together with $y = \sin(x)$.

ii The x-coordinate of the point of intersection of the graphs of $y = -\frac{1}{2}x + 1$ and $y = \sin(x)$ is the root of the equation $2\sin(x) + x - 2 = 0$.
Therefore, the x-coordinate of the point of intersection is $x = \phi$.
When $x = \phi$, $y = -\frac{1}{2}\phi + 1$, so the coordinates of the point of intersection are $\left(\phi, -\frac{1}{2}\phi + 1\right)$.
The coordinates could also be $(\phi, \sin(\phi))$, but the y-coordinate is not an algebraic expression.

8.7 Exam questions

1 $\dfrac{\tan(30°)\sin(60°)}{\cos(60°)\sin(30°) - \tan(45°)} + \dfrac{\sin(45°)\cos(45°)}{\tan(45°)}$

$= \dfrac{\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2}}{\frac{1}{2} \times \frac{1}{2} - 1} + \dfrac{\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}}{1}$ [1 mark]

$= \dfrac{\frac{1}{2}}{-\frac{3}{4}} + \dfrac{1}{2}$ [1 mark]

$= -\dfrac{2}{3} + \dfrac{1}{2}$ [1 mark]

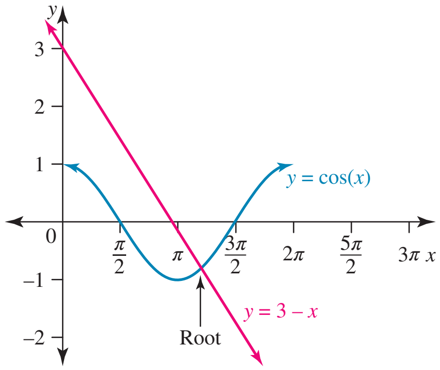
$= -\dfrac{1}{6}$ [1 mark]

2 $\tan(45°) + \tan(45°) = 1 + 1$
$= 2$
$\tan(90°)$ is undefined.
The correct answer is **D**.

3 $\pi + \theta$ is in the third quadrant.
$\sin(\pi + \theta) = -\sin(\theta)$ as $\sin(\theta)$ is negative in the third quadrant.
The correct answer is **B**.

4 $\sin(\theta) = 0.61,\ \sin(\pi + \theta) = -\sin(\theta) = -0.61$ [1 mark]
$\cos(t) = 0.48,\ \cos(\pi - t) = -\cos(t) = -0.48$ [1 mark]
$\tan(x) = 1.6, \tan(2\pi + x) = \tan(x) = 1.6$ [1 mark]
$\cos(-t) = \cos(t) = 0.48$ [1 mark]

5 $y = \cos(x)$
One cycle of the graph has domain $[0, 2\pi]$.
The axis intercepts are $(0, 1), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$.
$y = 3 - x$
The axis intercepts are $(0, 3)$ and $(3, 0)$.



Award 1 mark for correctly drawing each of the two graphs.
The two graphs intersect at one point only, so the equation $\cos(x) = 3 - x$ has only one root. [1 mark]
From the graph, it can be seen that the root lies between π and $\frac{3\pi}{2}$
$\therefore$ the interval in which the root lies is $\left[\pi, \frac{3\pi}{2}\right]$. [1 mark]

Topic 9 — Trigonometric functions and applications

9.2 Trigonometric equations

9.2 Exercise

1 $0 \le x \le 2\pi$

a $\cos(x) = \frac{1}{\sqrt{2}}$

Cosine is positive in quadrants 1 and 4; since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, the base for the solutions is $\frac{\pi}{4}$.

$\therefore x = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{7\pi}{4}$

b $\sin(x) = -\frac{1}{\sqrt{2}}$

Sine is negative in quadrants 3 and 4; since $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, the base for the solutions is $\frac{\pi}{4}$.

$\therefore x = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{5\pi}{4}, \frac{7\pi}{4}$

c $\tan(x) = -\frac{1}{\sqrt{3}}$

Tangent is negative in quadrants 2 and 4; since $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, the base for the solutions is $\frac{\pi}{6}$.

$\therefore x = \pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore x = \frac{5\pi}{6}, \frac{11\pi}{6}$

d $2\sqrt{3}\cos(x) + 3 = 0$

$\therefore 2\sqrt{3}\cos(x) = -3$

$\therefore \cos(x) = -\frac{3}{2\sqrt{3}}$

$\therefore \cos(x) = -\frac{\sqrt{3}}{2}$

Quadrants 2 and 3, base $\frac{\pi}{6}$

$\therefore x = \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}$

$\therefore x = \frac{5\pi}{6}, \frac{7\pi}{6}$

e $4 - 8\sin(x) = 0$

$\therefore 4 = 8\sin(x)$

$\therefore \sin(x) = \frac{4}{8}$

$\therefore \sin(x) = \frac{1}{2}$

Quadrants 1 and 2, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

f $2\sqrt{2}\tan(x) = \sqrt{24}$

$\therefore 2\sqrt{2}\tan(x) = 2\sqrt{6}$

$\therefore \tan(x) = \frac{\sqrt{6}}{\sqrt{2}}$

$\therefore \tan(x) = \sqrt{3}$

Quadrants 1 and 3, base $\frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$

2 a $\sin(x) = \frac{1}{2}, 0 \le x \le 2\pi$

Quadrants 1 and 2, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

b $\sqrt{3} - 2\cos(x) = 0, 0 \le x \le 2\pi$

Rearranging the equation gives $\cos(x) = \frac{\sqrt{3}}{2}$.

Quadrants 1 and 4, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{11\pi}{6}$

c $4 + 4\tan(x) = 0, -2\pi \le x \le 2\pi$

Rearranging the equation gives $\tan(x) = -1$.

Quadrants 2 and 4, base $\frac{\pi}{4}$, one positive and one negative rotation

$\therefore x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$ or $-\frac{\pi}{4}, -\pi - \frac{\pi}{4}$

$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}, -\frac{\pi}{4}, -\frac{5\pi}{4}$

3 $\theta \in [-2\pi, 2\pi]$

a $\tan(\theta) = 1$

Tangent is positive in first and third quadrants.

Since $\tan\left(\frac{\pi}{4}\right) = 1$, $\frac{\pi}{4}$ is the base.

Positive solutions from one anticlockwise rotation:

$\theta = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

Negative solutions from one clockwise rotation:

$\theta = -\pi + \frac{\pi}{4}, -2\pi + \frac{\pi}{4}$

$\therefore \theta = -\frac{3\pi}{4}, -\frac{7\pi}{4}$

Therefore, $\theta = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$.

b $\cos(\theta) = -0.5$

$\therefore \cos(\theta) = -\frac{1}{2}$

Cosine is negative in quadrants 2 and 3.

Since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\frac{\pi}{3}$ is the base.

Positive solutions from one anticlockwise rotation:

$\theta = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Negative solutions from one clockwise rotation”

$\theta = -\pi + \frac{\pi}{3}, -\pi - \frac{\pi}{3}$

$\therefore \theta = -\frac{2\pi}{3}, -\frac{4\pi}{3}$

Therefore, $\theta = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$.

c $1 + 2\sin(\theta) = 0$

$\therefore 2\sin(\theta) = -1$

$\therefore \sin(\theta) = -\frac{1}{2}$

Sine is negative in quadrants 3 and 4.

Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\frac{\pi}{6}$ is the base.

Positive solutions formed by one anticlockwise rotation:

$\theta = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Negative solutions formed by one clockwise rotation:

$\theta = -\frac{\pi}{6}, -\pi + \frac{\pi}{6}$

$\therefore \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}$

Therefore, $\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

4 a $3\tan(x) + 3\sqrt{3} = 0, x \in [0, 3\pi]$.

Rearrange the equation:

$3\tan(x) = -3\sqrt{3}$

$\tan(x) = -\frac{3\sqrt{3}}{3}$

$\tan(x) = -\sqrt{3}$

Tangent is negative in quadrants 2 and 4.

Since $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, the base for the solutions is $\frac{\pi}{3}$.

Solutions generated by $1\frac{1}{2}$ anticlockwise rotations:

$x = \pi - \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 3\pi - \frac{\pi}{3}$

$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$

b $10\sin(t) - 3 = 2, 0 \leq t \leq 4\pi$

$10\sin(t) = 5$

$\sin(t) = \frac{5}{10}$

$\sin(t) = \frac{1}{2}$

Sine is positive in quadrants 1 and 2.

Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, the base for the solutions is $\frac{\pi}{6}$.

Solutions generated by two anticlockwise rotations:

$t = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$

$\therefore t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

c $4\sqrt{2}\cos(v) = \sqrt{2}\cos(v) + 3, -\pi \leq v \leq 5\pi$

Rearrange the equation:

$4\sqrt{2}\cos(v) - \sqrt{2}\cos(v) = 3$

$3\sqrt{2}\cos(v) = 3$

$\cos(v) = \frac{3}{3\sqrt{2}}$

$\cos(v) = \frac{1}{\sqrt{2}}$

Cosine is positive in quadrants 1 and 4.

Since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\frac{\pi}{4}$ is the base for the solutions.

Positive solutions are generated by $2\frac{1}{2}$ anticlockwise rotations:

$v = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 4\pi - \frac{\pi}{4}, 4\pi + \frac{\pi}{4}$

$\therefore v = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}$

One negative solution, $v = -\frac{\pi}{4}$, is generated by $\frac{1}{2}$ a clockwise rotation.

$\therefore v = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}$

5 $\theta \in [0°, 360°]$

a $\cos(\theta) = \frac{\sqrt{2}}{2}$

Cosine is positive in quadrants 1 and 4.

Since $\cos(45°) = \frac{\sqrt{2}}{2}$, 45° is the base for the solutions.

$\theta = 45°, 360° - 45°$

$\theta = 45°, 315°$

b $\sin(\theta) = -\frac{1}{2}$

Sine is negative in quadrants 3 and 4.

Since $\sin(30°) = \frac{1}{2}$, 30° is the base for the solutions.

$\theta = 180° + 30°, 360° - 30°$

$\theta = 210°, 330°$

c $\tan(\theta) = \sqrt{3}$

Tangent is positive in quadrants 1 and 3.

Since $\tan(60°) = \sqrt{3}$, 60° is the base for the solutions.

$\theta = 60°, 180° + 60°$

$\theta = 60°, 240°$

d $6\tan(\theta) = -6$

Divide both sides of the equation by 6:

$\therefore \tan(\theta) = -1$

Tangent is negative in quadrants 2 and 4.

Since $\tan(45°) = 1$, 45° is the base for the solutions.

$\theta = 180° - 45°, 360° - 45°$

$\theta = 135°, 315°$

e $6\sin(\theta) - 3\sqrt{2} = 0$

Rearrange the equation:

$6\sin(\theta) = 3\sqrt{2}$

$\sin(\theta) = \frac{3\sqrt{2}}{6}$

$\sin(\theta) = \frac{\sqrt{2}}{2}$

Sine is positive in quadrants 1 and 2.

Since $\sin(45°) = \dfrac{\sqrt{2}}{2}$, 45° is the base for the solutions.

$\theta = 45°, 180° - 45°$

$\theta = 45°, 135°$

f $-\sqrt{3} - 2\cos(\theta) = 0.$

Rearrange the equation:

$-\sqrt{3} = 2\cos(\theta)$

$\cos(\theta) = -\dfrac{\sqrt{3}}{2}$

Cosine is negative in quadrants 2 and 3.

Since $\cos(30°) = \dfrac{\sqrt{3}}{2}$, 30° is the base for the solutions.

$\theta = 180° - 30°, 180° + 30°$

$\theta = 150°, 210°$

6 $\alpha \in [-360°, 360°]$

a $\tan(\alpha) = 1$

Tangent is positive in quadrants 1 and 3.

Since $\tan(45°) = 1$, 45° is the base for the solutions.

Positive solutions are generated by one anticlockwise rotation and negative solutions by one clockwise rotation.

$\alpha = 45°, 180° + 45°$ and $\alpha = -180° + 45°, -360° + 45°$

Therefore, $\alpha = 45°, 225°$ and $\alpha = -135°, -315°$.

$\therefore \alpha = -315°, -135°, 45°, 225°$

b $\cos(\alpha) = 0$

$\cos(\alpha) = 0$ at the boundary points $(0, 1)$ and $(0, -1)$, since the x-coordinate of these points is $x = 0$.

Positive solutions are generated by one anticlockwise rotation and negative solutions by one clockwise rotation.

$\alpha = 90°, 270°$ and $\alpha = -90°, -270°$.

$\therefore \alpha = -270°, -90°, 90°, 270°$

c $5\sin(\alpha) + 5 = 0$

Rearrange the equation:

$5\sin(\alpha) = -5$

$\sin(\alpha) = -1$

$\sin(\alpha) = -1$ at the boundary point $(0, -1)$, since the y-coordinate of this point is $y = -1$.

The positive solution is generated by one anticlockwise rotation and the negative solution by one clockwise rotation.

$\alpha = -90°, 270°$.

d $2\tan(\alpha) + 7 = 7$

Rearrange the equation:

$2\tan(\alpha) = 0$

$\tan(\alpha) = 0$

$\tan(\alpha) = 0$ at the boundary points $(1, 0), (-1, 0)$ since $\dfrac{y}{x} = 0$ at these points.

Positive solutions are generated by one anticlockwise rotation and negative solutions by one clockwise rotation.

$\alpha = 0°, 180°, 360°$ and $-180°, -360°$

$\therefore \alpha = -360° - 180°, 0°, 180°, 360°$

7 a $1 - \sin(x) = 0, -4\pi \le x \le 4\pi$

$\therefore \sin(x) = 1$

Boundary of first and second quadrants where $\sin\left(\dfrac{\pi}{2}\right) = 1$

For two positive and two negative rotations,

$\therefore x = \dfrac{\pi}{2}, 2\pi + \dfrac{\pi}{2}$ or $-\dfrac{3\pi}{2}, -2\pi - \dfrac{3\pi}{2}$

$\therefore x = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, -\dfrac{3\pi}{2}, -\dfrac{7\pi}{2}$

b $\tan(x) = 0.75, 0 \le x \le 4\pi$

Quadrants 1 and 3. The base is $\tan^{-1}(0.75) = 0.644$ using radian mode on calculator

Two complete positive rotations give

$x = 0.644, \pi + 0.644$ or $2\pi + 0.644, 3\pi + 0.644$

$= 0.64, 3.79, 6.93, 10.07$

c $4\cos(x°) + 1 = 0, -180° \le x° \le 180°$

$\therefore \cos(x°) = -\dfrac{1}{4}$

Quadrants 2 and 3, base $\cos^{-1}\left(\dfrac{1}{4}\right) = 75.52°$ using degree mode

$\therefore x° = -180° + 75.52°$ or $180° - 75.52°$

$\therefore x° = \pm 104.5°$

Hence, $x = \pm 104.5$

8 $\cos(\theta) = -\dfrac{1}{2}, -180° \le \theta \le 540°$

a Solutions lie in quadrants 2 and 3.

The negative rotation 0° to −180° picks up one solution in quadrant 3, and the positive rotation 0° to 180° picks up one solution in quadrant 2.

Therefore, there are 2 solutions

b The base is 60°.

$\therefore \theta = -180° + 60°$ or $180° - 60°$

$\therefore \theta = -120°$ or $120°$

9 a $2 + 3\cos(\theta) = 0,\ 0 \le \theta \le 2\pi$

$\therefore \cos(\theta) = -\dfrac{2}{3}$

Quadrants 2 and 3; the base is $\cos^{-1}\left(\dfrac{2}{3}\right) = 0.841.$

$\therefore \theta = \pi - 0.841, \pi + 0.841$

$\therefore \theta = 2.30, 3.98$

b $\tan(\theta) = \dfrac{1}{\sqrt{2}},\ -2\pi \le \theta \le 3\pi$

Quadrants 1 and 3; the base is $\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 0.615.$

Solutions are generated by one and a half anticlockwise rotations and one clockwise rotation.

$\therefore \theta = 0.615, \pi + 0.615, 2\pi + 0.615$ or

$\theta = -\pi + 0.615, -2\pi + 0.615$

$\therefore \theta = 0.62, 3.76, 6.90$ or $\theta = -2.53, -5.67$

c $5\sin(\theta°) + 4 = 0,\ -270° \le \theta° \le 270°$

$\therefore \sin(\theta°) = -\dfrac{4}{5}$

Quadrants 3 and 4; the base in degrees is

$\sin^{-1}\left(\dfrac{4}{5}\right) = 53.130°.$

Solutions are generated by rotating 0° → 270° anticlockwise and 0° → −270° clockwise.

$\therefore \theta° = 180° + 53.13°$ or $\theta° = -53.13°, -180° + 53.13°$

$\therefore \theta° = 233.13°$ or $\theta° = -53.13°, -126.87°$

$\therefore \theta = -126.87, -53.13, 233.13$

d $\cos^2(\theta°) = 0.04,\ 0° \le \theta° \le 360°$

$\therefore\ \cos(\theta°) = \pm\sqrt{0.04}$

$\therefore\ \cos(\theta°) = \pm 0.2$

Quadrants 1 and 4 and quadrants 2 and 3. The base, in degrees, is $\cos^{-1}(0.2) = 78.463°$.

$\therefore \theta° = 78.463°,\ 180° - 78.463°,\ 180° + 78.463°,$
$\quad 360° - 78.463°$

$\therefore \theta° = 78.46°,\ 101.54°,\ 258.46°,\ 281.54°$

$\therefore \theta = 78.46,\ 101.54,\ 258.46,\ 281.54$

10 $t \in [-\pi, 4\pi]$. Solutions are generated by two complete anticlockwise rotations and half a clockwise rotation.

a $\tan(t) = 0$ at the boundary points $(1, 0), (-1, 0)$ since $\dfrac{y}{x} = 0$ for these points.

$\therefore t = 0, \pi, 2\pi, 3\pi, 4\pi$ or $t = -\pi$

b $\cos(t) = 0$ at the boundary points $(0, 1), (0, -1)$ since $x = 0$ for these points.

$\therefore t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$ or $t = -\dfrac{\pi}{2}$

c $\sin(t) = -1$ at the boundary point $(0, -1)$ since $y = -1$ at this point.

$\therefore t = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$ or $t = -\dfrac{\pi}{2}$

d $\cos(t) = 1$ at the boundary point $(1, 0)$ since $x = 1$ at this point.

$\therefore t = 0, 2\pi, 4\pi$

e $\sin(t) = 1$ at the boundary point $(0, 1)$ since $y = 1$ at this point.

$\therefore t = \dfrac{\pi}{2}, \dfrac{5\pi}{2}$

f $\tan(t) = 1$. This is not a boundary value.

Quadrants 1 and 3, base $\dfrac{\pi}{4}$

$\therefore t = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi + \dfrac{\pi}{4}, 3\pi + \dfrac{\pi}{4}$ or $t = -\pi + \dfrac{\pi}{4}$

$\therefore t = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$ or $t = -\dfrac{3\pi}{4}$

11 $f : [0, 2] \to R, f(x) = \cos(\pi x)$

a $f(0) = \cos(0)$
$\quad = 1$

b $f(x) = 0$

$\therefore\ \cos(\pi x) = 0$

Boundary value between first and second quadrants and between third and fourth quadrants

$\therefore (\pi x) = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$

Dividing by π gives $x = \dfrac{1}{2}, \dfrac{3}{2}$.

The solution set is $\left\{\dfrac{1}{2}, \dfrac{3}{2}\right\}$.

12 **a** $\sqrt{3}\sin(x) = 3\cos(x)$

$\therefore\ \dfrac{\sqrt{3}\sin(x)}{\cos(x)} = 3$

$\therefore \sqrt{3}\tan(x) = 3$

$\therefore\ \tan(x) = \dfrac{3}{\sqrt{3}}$

$\therefore\ \tan(x) = \sqrt{3}$

Quadrants 1 and 3, base $\dfrac{\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}$

b $\sin^2(x) - 5\sin(x) + 4 = 0$

Let $a = \sin(x)$

$\therefore a^2 - 5a + 4 = 0$

Solving this quadratic equation gives

$(a - 4)(a - 1) = 0$

$\therefore a = 4$ or $a = 1$

Hence, $\sin(x) = 4$ or $\sin(x) = 1$

Reject $\sin(x) = 4$ since $-1 \le \sin(x) \le 1$

Therefore, $\sin(x) = 1$

Boundary between first and second quadrants

$\therefore x = \dfrac{\pi}{2}$

13 $\cos^2(x) = \dfrac{3}{4}, 0 \le x \le 2\pi$

Taking square roots of both sides gives

$\cos(x) = \pm\sqrt{\dfrac{3}{4}}$

$\therefore\ \cos(x) = \pm\dfrac{\sqrt{3}}{2}$

There is a solution in each quadrant since cosine can be positive or negative. The base is $\dfrac{\pi}{6}$.

$\therefore x = \dfrac{\pi}{6}, \pi - \dfrac{\pi}{6}, \pi + \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

$\therefore x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

14 $0 \le x \le 2\pi$

a $\sin(x) = \sqrt{3}\cos(x)$

$\therefore\ \dfrac{\sin(x)}{\cos(x)} = \sqrt{3}$

$\therefore\ \tan(x) = \sqrt{3}$

Quadrants 1 and 3, base $\dfrac{\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}$

b $\sin(x) = -\dfrac{\cos(x)}{\sqrt{3}}$

$\therefore\ \dfrac{\sin(x)}{\cos(x)} = -\dfrac{1}{\sqrt{3}}$

$\therefore\ \tan(x) = -\dfrac{1}{\sqrt{3}}$

Quadrants 2 and 4, base $\dfrac{\pi}{6}$

$\therefore x = \pi - \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

$\therefore x = \dfrac{5\pi}{6}, \dfrac{11\pi}{6}$

c $\sin(2x) + \cos(2x) = 0$

$\therefore\ \sin(2x) = -\cos(2x)$

$\therefore\ \dfrac{\sin(2x)}{\cos(2x)} = -1$

$\therefore\ \tan(2x) = -1$

Quadrants 2 and 4, base $\dfrac{\pi}{4}$.

Since $0 \le x \le 2\pi$, $0 \le 2x \le 4\pi$.

$\therefore 2x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 4\pi - \frac{\pi}{4}$

$\therefore 2x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$

$\therefore x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

d $\frac{3\sin(x)}{8} = \frac{\cos(x)}{2}$

$\therefore \sin(x) = \frac{\cos(x)}{2} \times \frac{8}{3}$

$\therefore \sin(x) = \frac{4\cos(x)}{3}$

$\therefore \frac{\sin(x)}{\cos(x)} = \frac{4}{3}$

$\therefore \tan(x) = \frac{4}{3}$

Quadrants 1 and 3, base $\tan^{-1}\left(\frac{4}{3}\right) = 0.927$

$\therefore x = 0.927, \pi + 0.927$

$\therefore x = \tan^{-1}\left(\frac{4}{3}\right), \pi + \tan^{-1}\left(\frac{4}{3}\right)$

e $\sin^2(x) = \cos^2(x)$

$\therefore \frac{\sin^2(x)}{\cos^2(x)} = 1$

$\therefore \left(\frac{\sin(x)}{\cos(x)}\right)^2 = 1$

$\therefore (\tan(x))^2 = 1$

$\therefore \tan(x) = \pm 1$

Quadrants 1, 2, 3 and 4, base $\frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

f $\cos(x)(\cos(x) - \sin(x)) = 0$

Using the Null Factor Law,

$\cos(x) = 0$ or $\cos(x) - \sin(x) = 0$

If $\cos(x) = 0$, boundary solutions occur at points $(0, 1)$ and $(0, -1)$.

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$

If $\cos(x) - \sin(x) = 0$, then

$\cos(x) = \sin(x)$

$\therefore 1 = \frac{\sin(x)}{\cos(x)}$

$\therefore \tan(x) = 1$

Quadrants 1 and 3, base $\frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

The answers are $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$.

15 $0 \le x \le 2\pi$

a $\sin^2(x) = \frac{1}{2}$

$\therefore \sin(x) = \pm\frac{1}{\sqrt{2}}$

Quadrants 1, 2, 3 and 4, base $\frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b $2\cos^2(x) + 3\cos(x) = 0$

$\cos(x)$ is a common factor.

$\therefore \cos(x)[2\cos(x) + 3] = 0$

$\therefore \cos(x) = 0$ or $2\cos(x) + 3 = 0$

$\therefore \cos(x) = 0$ or $\cos(x) = -\frac{3}{2}$

Reject $\cos(x) = -\frac{3}{2}$ since $-1 \le \cos(x) \le 1$

$\therefore \cos(x) = 0$

Boundary solutions occur at points $(0, 1)$ and $(0, -1)$.

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$

c $2\sin^2(x) - \sin(x) - 1 = 0$

Let $a = \sin(x)$.

$\therefore 2a^2 - a - 1 = 0$

$\therefore (2a + 1)(a - 1) = 0$

$\therefore a = -\frac{1}{2}, a = 1$

$\therefore \sin(x) = -\frac{1}{2}$ or $\sin(x) = 1$

For $\sin(x) = -\frac{1}{2}$, solutions lie in quadrants 3 and 4 with base $\frac{\pi}{6}$.

$\therefore x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6}$

For $\sin(x) = 1$, the boundary solution occurs at $(0, 1)$.

$\therefore x = \frac{\pi}{2}$

The answers are $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

d $\tan^2(x) + 2\tan(x) - 3 = 0$

Let $a = \tan(x)$.

$\therefore a^2 + 2a - 3 = 0$

$\therefore (a + 3)(a - 1) = 0$

$\therefore a = -3, a = 1$

$\therefore \tan(x) = -3$ or $\tan(x) = 1$

For $\tan(x) = -3$, solutions lie in quadrants 2 and 4 with base $\tan^{-1}(3) = 1.25$.

$\therefore x = \pi - 1.25, 2\pi - 1.25$

$\therefore x = 1.89, 5.03$

For $\tan(x) = 1$: quadrants 1 and 3, base $\frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

The answers are $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

e $\sin^2(x) + 2\sin(x) + 1 = 0$

$\therefore (\sin(x) + 1)^2 = 0$

$\therefore \sin(x) + 1 = 0$

$\therefore \sin(x) = -1$

Boundary solution at $(0, -1)$

$\therefore x = \frac{3\pi}{2}$

f $\cos^2(x) - 9 = 0$

$\therefore\ (\cos(x) - 3)(\cos(x) + 3) = 0$

$\therefore\ \cos(x) = 3 \text{ or } \cos(x) = -3$

There are no solutions possible as $\cos(x) \in [-1, 1]$.

16 **a** $\sin(2x) = \dfrac{1}{\sqrt{2}}, 0 \le x \le 2\pi$

Since $0 \le x \le 2\pi$, $0 \le 2x \le 4\pi$

Sine is positive in quadrants 1 and 2; the base is $\dfrac{\pi}{4}$.

$\therefore 2x = \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, 2\pi + \dfrac{\pi}{4}, 3\pi - \dfrac{\pi}{4}$

$\therefore 2x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}$

$\therefore x = \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{9\pi}{8}, \dfrac{11\pi}{8}$

b $\cos\left(2x + \dfrac{\pi}{6}\right) = 0, 0 \le x \le \dfrac{3\pi}{2}$

Let $\theta = 2x + \dfrac{\pi}{6}$.

Since $0 \le x \le \dfrac{3\pi}{2}, \dfrac{\pi}{6} \le \theta \le 3\pi + \dfrac{\pi}{6}$.

$\therefore\ \cos(\theta) = 0, \dfrac{\pi}{6} \le \theta \le \dfrac{19\pi}{6}$

The boundary value is at points $(0, -1)$ and $(0, 1)$, since $\cos\left(\dfrac{\pi}{2}\right) = 0 = \cos\left(\dfrac{3\pi}{2}\right)$.

As $\dfrac{\pi}{6} \le \theta \le \dfrac{19\pi}{6}$, solutions for θ are:

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$

$\therefore 2x + \dfrac{\pi}{6} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$

$\therefore 2x = \dfrac{2\pi}{6}, \dfrac{8\pi}{6}, \dfrac{14\pi}{6}$

$\therefore x = \dfrac{\pi}{6}, \dfrac{4\pi}{6}, \dfrac{7\pi}{6}$

$\therefore x = \dfrac{\pi}{6}, \dfrac{2\pi}{3}, \dfrac{7\pi}{6}$

17 $\sin\left(\dfrac{x}{2}\right) = \sqrt{3}\cos\left(\dfrac{x}{2}\right), 0 \le x \le 2\pi$

$\therefore\ \dfrac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \sqrt{3}$

$\therefore\ \tan\left(\dfrac{x}{2}\right) = \sqrt{3}$

As $0 \le x \le 2\pi, 0 \le \dfrac{x}{2} \le \pi$.

Tangent is positive in quadrants 1 and 3. The base is $\dfrac{\pi}{3}$.

Since $0 \le \dfrac{x}{2} \le \pi$, only the first quadrant value is reached.

$\therefore\ \dfrac{x}{2} = \dfrac{\pi}{3}$

$\therefore x = \dfrac{2\pi}{3}$

18 $0 \le \theta \le 2\pi$

a $\sqrt{3}\tan(3\theta) + 1 = 0$

$\therefore\ \tan(3\theta) = -\dfrac{1}{\sqrt{3}},\ 0 \le 3\theta \le 6\pi$

Quadrants 2 and 4, base $\dfrac{\pi}{6}$

$\therefore 3\theta = \pi - \dfrac{\pi}{6},\ 2\pi - \dfrac{\pi}{6},\ 3\pi - \dfrac{\pi}{6},\ 4\pi - \dfrac{\pi}{6},\ 5\pi - \dfrac{\pi}{6},\ 6\pi - \dfrac{\pi}{6}$

$\therefore 3\theta = \dfrac{5\pi}{6},\ \dfrac{11\pi}{6},\ \dfrac{17\pi}{6},\ \dfrac{23\pi}{6},\ \dfrac{29\pi}{6},\ \dfrac{35\pi}{6}$

$\therefore \theta = \dfrac{5\pi}{18},\ \dfrac{11\pi}{18},\ \dfrac{17\pi}{18},\ \dfrac{23\pi}{18},\ \dfrac{29\pi}{18},\ \dfrac{35\pi}{18}$

b $2\sqrt{3}\sin\left(\dfrac{3\theta}{2}\right) - 3 = 0$

$\therefore 2\sqrt{3}\sin\left(\dfrac{3\theta}{2}\right) = 3$

$\therefore\ \sin\left(\dfrac{3\theta}{2}\right) = \dfrac{3}{2\sqrt{3}}$

$\therefore\ \sin\left(\dfrac{3\theta}{2}\right) = \dfrac{\sqrt{3}}{2},\ 0 \le \dfrac{3\theta}{2} \le 3\pi$

Quadrants 1 and 2, base $\dfrac{\pi}{3}$

$\therefore\ \dfrac{30}{2} = \dfrac{\pi}{3}, \pi - \dfrac{\pi}{3}, 2\pi + \dfrac{\pi}{3}, 3\pi - \dfrac{\pi}{3}$

$\therefore\ \dfrac{3\theta}{2} = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{7\pi}{3}, \dfrac{8\pi}{3}$

$\therefore 3\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{14\pi}{3}, \dfrac{16\pi}{3}$

$\therefore \theta = \dfrac{2\pi}{9}, \dfrac{4\pi}{9}, \dfrac{14\pi}{9}, \dfrac{16\pi}{9}$

c $4\cos^2(-\theta) = 2$

As $\cos(-\theta) = \cos(\theta)$, the equation becomes $4\cos^2(\theta) = 2$.

$\therefore\ \cos^2(\theta) = \dfrac{1}{2}$

$\therefore\ \cos(\theta) = \pm\dfrac{1}{\sqrt{2}}$

Quadrants 1, 2, 3 and 4, base $\dfrac{\pi}{4}$

$\therefore \theta = \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$

$\therefore \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

d $\sin\left(2\theta + \dfrac{\pi}{4}\right) = 0$

Boundary solutions at $(1, 0)$ and $(-1, 0)$

As $0 \le \theta \le 2\pi$, $0 \le 2\theta \le 4\pi$

$\therefore 0 + \dfrac{\pi}{4} \le 2\theta + \dfrac{\pi}{4} \le 4\pi + \dfrac{\pi}{4}$

$\therefore\ \dfrac{\pi}{4} \le 2\theta + \dfrac{\pi}{4} \le 4\pi + \dfrac{\pi}{4}$

Solutions in this interval are:

$2\theta + \dfrac{\pi}{4} = \pi, 2\pi, 3\pi, 4\pi$

$\therefore 2\theta = \pi - \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}, 3\pi - \dfrac{\pi}{4}, 4\pi - \dfrac{\pi}{4}$

$= \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \dfrac{15\pi}{4}$

$\therefore \theta = \dfrac{3\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}, \dfrac{15\pi}{8}$

19 **a** $f : [0, 2\pi] \to R, f(x) = a\sin(x)$

$f\left(\dfrac{\pi}{6}\right) = 4$

$\therefore a\sin\left(\dfrac{\pi}{6}\right) = 4$

$\therefore a \times \dfrac{1}{2} = 4$

$\therefore a = 8$

b $f(x) = 8\sin(x)$

i $f(x) = 3$

$\therefore 8\sin(x) = 3$

$\therefore \sin(x) = \frac{3}{8}$

Quadrants 1 and 2, base $\sin^{-1}\left(\frac{3}{8}\right) = 0.384$

$\therefore x = 0.384, \pi - 0.384$

$\therefore x = 0.38, 2.76$

ii $f(x) = 8$

$\therefore 8\sin(x) = 8$

$\therefore \sin(x) = 1$

Boundary value at point (0, 1)

$\therefore x = \frac{\pi}{2}$

Correct to 2 decimal places, $x = 1.57$.

iii $f(x) = 10$

$\therefore 8\sin(x) = 10$

$\therefore \sin(x) = \frac{10}{8}$

$\therefore \sin(x) = \frac{5}{4} > 1$

Since $-1 \leq \sin(x) \leq 1$, there is no solution.

20 a Set the calculator to Standard and Deg modes and enter sin(75).

The value given, $\frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$, is the exact value of sin(75°).

b $\sin(A°) = \frac{\sqrt{6}+\sqrt{2}}{4}, -270 \leq A \leq 270$

As the question says 'Hence, solve', the result in part **a** must be used in solving the equation.

Expressing $\sqrt{6}+\sqrt{2} = \sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)$, the equation becomes

$\sin(A°) = \frac{\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)}{4}$.

From part **a**, $A° = 75°$ is a solution and forms the base for other solutions.

Solutions lie in quadrants 1 and 2, since $\sin(A°) > 0$.

$\therefore A° = 75°, 180° - 75° \text{ or } A° = -180° - 75°$

$\therefore A° = 75°, 105° \text{ or } -255°$

$\therefore A° = -255°, 75°, 105°$

9.2 Exam questions

1 Sine is positive in quadrants 1 and 2.
The correct answer is **A**.

2 Sine is negative in quadrants 3 and 4.
Between 0° and −180°, there are 2 solutions.
Between 0° and 720°, there are 4 solutions.
The total number of solutions is 6.
The correct answer is **E**.

3 $2\cos(x) = -\sqrt{3}$

$\cos(x) = -\frac{\sqrt{3}}{2}$

Cosine is negative in quadrants 2 and 3. [1 mark]

The base is $\frac{\pi}{6}$.

Since $x \in [0, 2\pi]$, there will be two solutions. [1 mark]

$\therefore x = \pi - \frac{\pi}{6} \text{ or } x = \pi + \frac{\pi}{6}$

$\therefore x = \frac{5\pi}{6} \text{ or } \frac{7\pi}{6}$ [1 mark]

9.3 Transformations of sine and cosine graphs

9.3 Exercise

1 a $y = 6\cos(2x)$ has amplitude 6 and period $\frac{2\pi}{2} = \pi$.

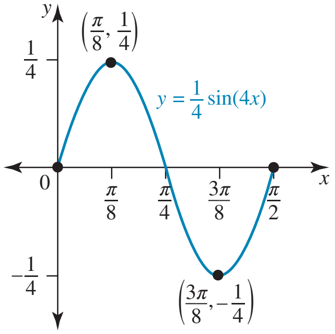
b $y = -7\cos\left(\frac{x}{2}\right)$ has amplitude 7 and period $\frac{2\pi}{\frac{1}{2}} = 4\pi$.

c The graph has an amplitude of 2 and a period of 4π.

2 a $y = \frac{1}{4}\sin(4x)$

Amplitude $\frac{1}{4}$, range $\left[-\frac{1}{4}, \frac{1}{4}\right]$, period $\frac{2\pi}{4} = \frac{\pi}{2}$

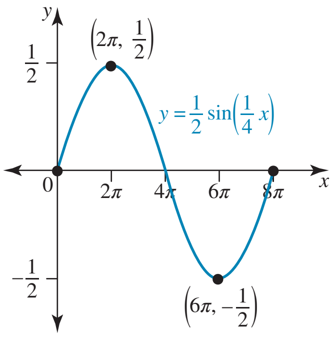
For one cycle, scale the domain $\left[0, \frac{\pi}{2}\right]$ into quarters, so each horizontal scale unit is $\frac{\pi}{8}$.



b $y = \frac{1}{2}\sin\left(\frac{1}{4}x\right)$

Amplitude $\frac{1}{2}$, range $\left[-\frac{1}{2}, \frac{1}{2}\right]$, period $\frac{2\pi}{\frac{1}{4}} = 2\pi \times \frac{4}{1} = 8\pi$

For one cycle, scale the domain $[0, 8\pi]$ into quarters, so each horizontal scale unit is 2π.

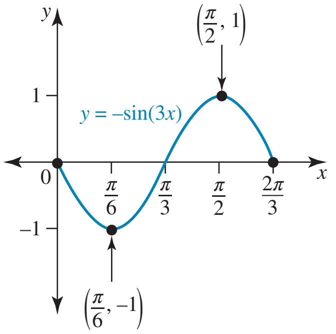


c $y = -\sin(3x)$

Amplitude 1, range $[-1, 1]$, period $\frac{2\pi}{3}$

The graph is inverted.

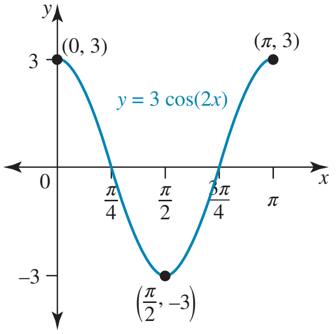
For one cycle, scale the domain $\left[0, \frac{2\pi}{3}\right]$ into quarters, so each horizontal scale unit is $\frac{2\pi}{12} = \frac{\pi}{6}$.



d $y = 3\cos(2x)$

Amplitude 3, range $[-3, 3]$, period $\frac{2\pi}{2} = \pi$

For one cycle, scale the domain $[0, \pi]$ into quarters, so each horizontal scale unit is $\frac{\pi}{4}$.

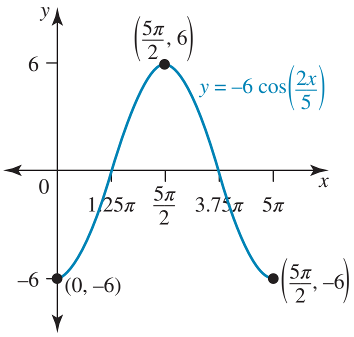


e $y = -6\cos\left(\frac{2x}{5}\right)$

Amplitude 6, range $[-6, 6]$, period $\frac{2\pi}{\frac{2}{5}} = 2\pi \times \frac{5}{2} = 5\pi$

The graph is inverted.

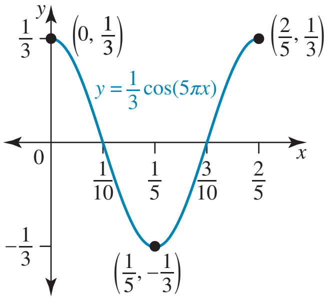
For one cycle, scale the domain $[0, 5\pi]$ into quarters, so each horizontal scale unit is $\frac{5\pi}{4} = 1.25\pi$.



f $y = \frac{1}{3}\cos(5\pi x)$

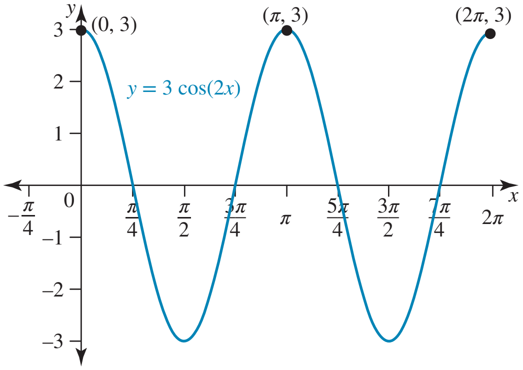
Amplitude $\frac{1}{3}$, range $\left[-\frac{1}{3}, \frac{1}{3}\right]$, period $\frac{2\pi}{5\pi} = \frac{2}{5}$

For one cycle, scale the domain $[0, 0.4]$ into quarters, so each horizontal scale unit is $0.1 = \frac{1}{10}$.



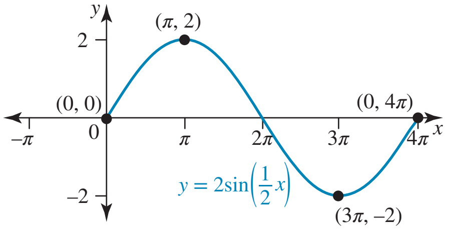
3 a $y = 3\cos(2x), 0 \le x \le 2\pi$

Period $\frac{2\pi}{2} = \pi$, amplitude 3



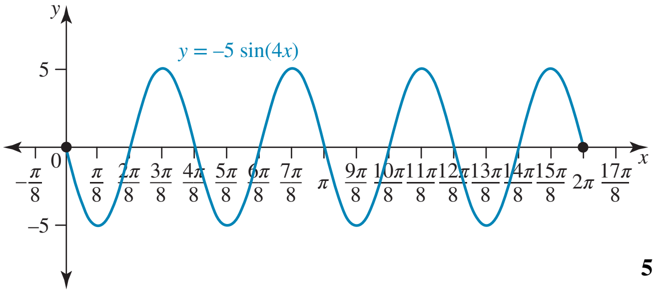
b $y = 2\sin\left(\frac{1}{2}x\right), 0 \le x \le 4\pi$

Period $\frac{2\pi}{\frac{1}{2}} = 4\pi$, amplitude 2



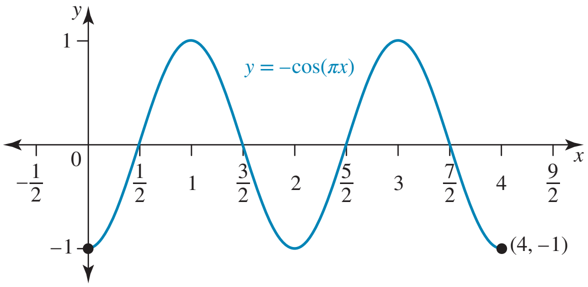
c $y = -5\sin(4x),\ 0 \le x \le 2\pi$

Period $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$, amplitude 5, inverted



d $y = -\cos(\pi x),\ 0 \le x \le 4$

Period $\dfrac{2\pi}{\pi} = 2$, amplitude 1, inverted



4 a $y = 2\sin(x) + 1, 0 \le x \le 2\pi$

Period 2π, amplitude 2, equilibrium position $y = 1$, oscillates between $[-1, 3]$

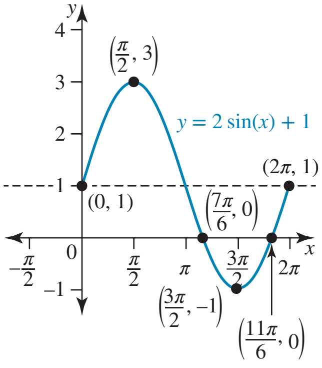
x-intercepts: $2\sin(x) + 1 = 0$

$\therefore\ \sin(x) = -\dfrac{1}{2}$

Quadrants 3 and 4, base $\dfrac{\pi}{6}$

$\therefore x = \pi + \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

$\therefore x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

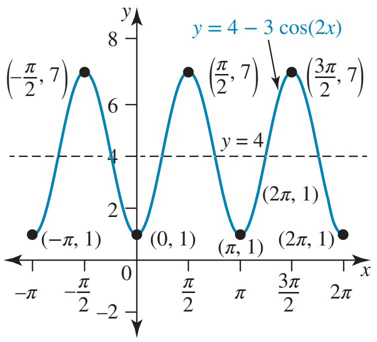


b $y = 4 - 3\cos(2x), -\pi \le x \le 2\pi$

$\therefore y = -3\cos(2x) + 4$

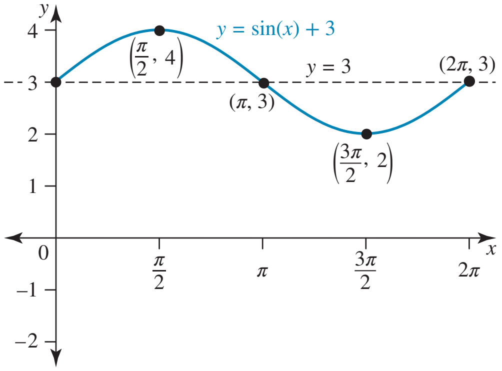
Period $\dfrac{2\pi}{2} = \pi$, amplitude 3, inverted, equilibrium $y = 4$, range $[4 - 3, 4 + 3] = [1, 7]$, no x-intercepts

Domain $[-\pi, 2\pi]$



5 a $y = \sin(x) + 3, 0 \le x \le 2\pi$

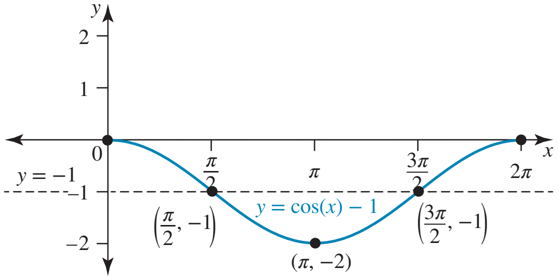
Period 2π, amplitude 1, equilibrium or mean position $y = 3$



The range is $[2, 4]$.

b $y = \cos(x) - 1, 0 \le x \le 2\pi$

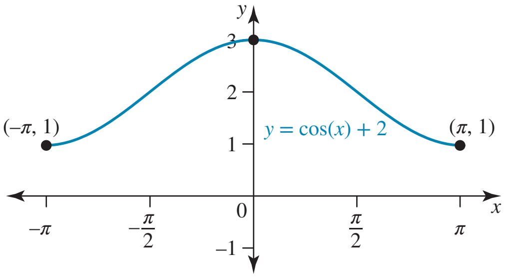
Period 2π, amplitude 1, equilibrium or mean position $y = -1$



The range is $[-2, 0]$.

c $y = \cos(x) + 2, -\pi \le x \le \pi$

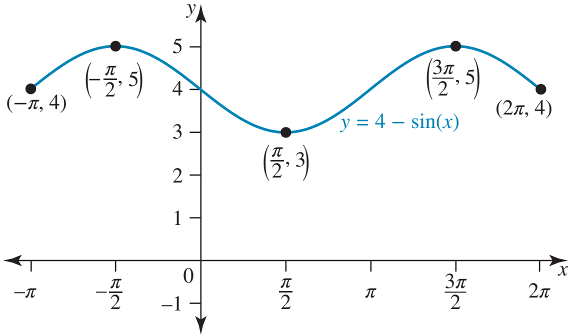
Period 2π, amplitude 1, equilibrium $y = 2$, range $[2 - 1, 2 + 1] = [1, 3]$



d $y = 4 - \sin(x), -\pi \le x \le 2\pi$

$\therefore y = -\sin(x) + 4$

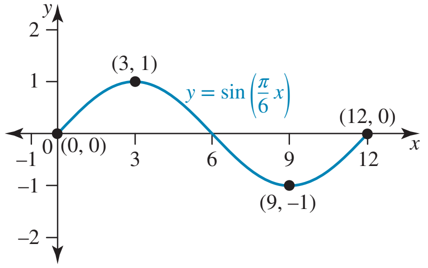
Period 2π, amplitude 1, inverted, equilibrium $y = 4$, range $[4 - 1, 4 + 1] = [3, 5]$



6 $f: [0, 12] \to R, f(x) = \sin\left(\frac{\pi x}{6}\right)$

Period $\frac{2\pi}{\frac{\pi}{6}} = 12$, amplitude 1, equilibrium $y = 0$, range $[-1, 1]$, domain $[0, 12]$

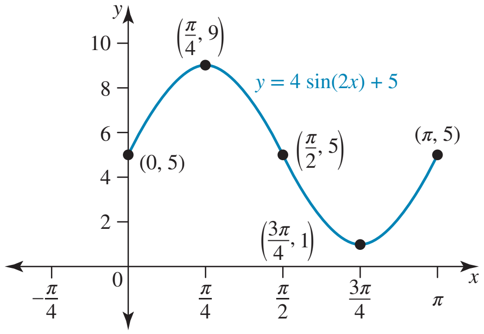
Scale on x-axis $x = 0, 3, 6, 9, 12$ to give the five key points



7 a $y = 4\sin(2x) + 5$

Period $\frac{2\pi}{2} = \pi$, mean position $y = 5$, amplitude 4, range $[5 - 4, 5 + 4] = [1, 9]$

Since the range is $[1, 9]$, there are no x-intercepts.

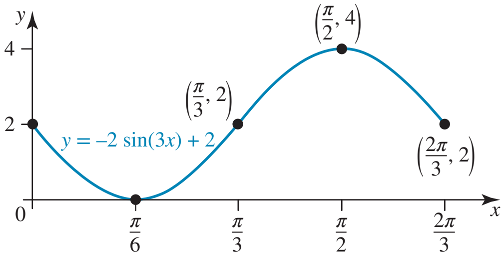


b $y = -2\sin(3x) + 2$

Period $\frac{2\pi}{3}$, mean position $y = 2$, amplitude 2, inverted, range $[2 - 2, 2 + 2] = [0, 4]$

x-intercept: let $y = 0$.

$$0 = -2\sin(3x) + 2$$
$$2\sin(3x) = 2$$
$$\sin(3x) = 1$$
$$3x = \frac{\pi}{2}$$
$$x = \frac{\pi}{6}$$

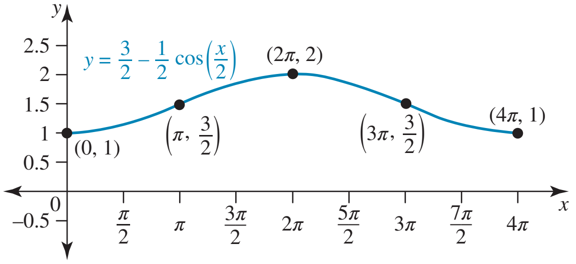


c $y = \frac{3}{2} - \frac{1}{2}\cos\left(\frac{x}{2}\right)$

$y = -\frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{3}{2}$

Period $\frac{2\pi}{\frac{1}{2}} = 4\pi$, mean position $y = \frac{3}{2}$, amplitude $\frac{1}{2}$, inverted,

range $\left[\frac{3}{2} - \frac{1}{2}, \frac{3}{2} + \frac{1}{2}\right] = [1, 2]$. There are no x-intercepts.



d $y = 2\cos(\pi x) - \sqrt{3}$

Period $\frac{2\pi}{\pi} = 2$, mean position $y = -\sqrt{3}$, amplitude 2,

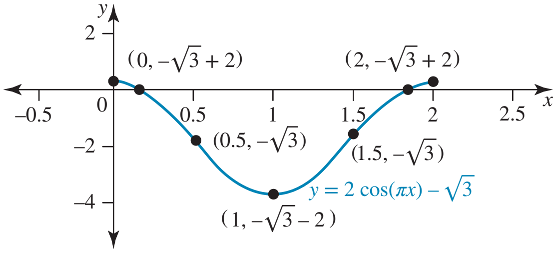
range $\left[-\sqrt{3} - 2, -\sqrt{3} + 2\right]$

As $-\sqrt{3} - 2 < 0$ and $-\sqrt{3} + 2 > 0$, there are x-intercepts.

There will be 2 x-intercepts in one cycle of the graph.

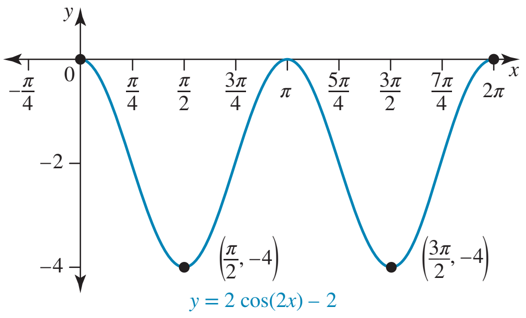
x-intercepts: let $y = 0$.

$$0 = 2\cos(\pi x) - \sqrt{3}$$
$$2\cos(\pi x) = \sqrt{3}$$
$$\cos(\pi x) = \frac{\sqrt{3}}{2}$$
$$\pi x = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$
$$\pi x = \frac{\pi}{6}, \frac{11\pi}{6}$$
$$x = \frac{1}{6}, \frac{11}{6}$$



8 a $y = 2\cos(2x) - 2, 0 \le x \le 2\pi$

Amplitude 2, period $\frac{2\pi}{2} = \pi$, equilibrium $y = -2$, range $[-2 - 2, -2 + 2] = [-4, 0]$



b $y = 2\sin(x) + \sqrt{3}, 0 \le x \le 2\pi$

Period 2π, amplitude 2, equilibrium $y = \sqrt{3}$, range $\left[\sqrt{3} - 2, \sqrt{3} + 2\right]$

Since $\sqrt{3} - 2 < 0$, there will be x-intercepts.

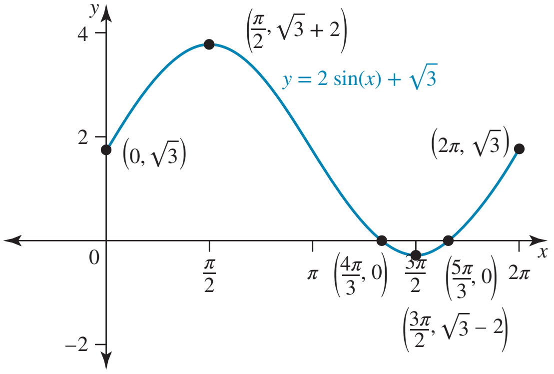
Let $y = 0$.

$$\therefore 2\sin(x) + \sqrt{3} = 0$$

$$\therefore \sin(x) = -\frac{\sqrt{3}}{2}$$

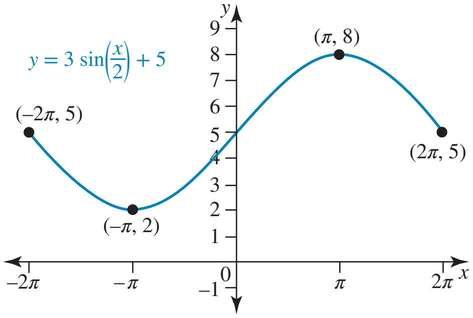
$$\therefore x = \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$$

$$\therefore x = \frac{4\pi}{3}, \frac{5\pi}{3}$$



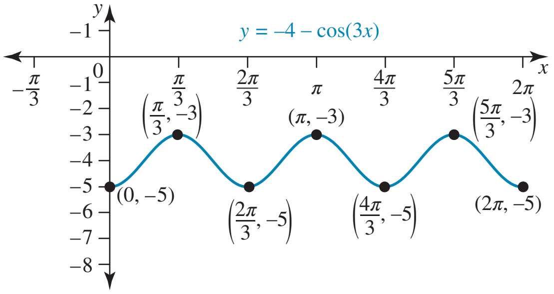
c $y = 3\sin\left(\frac{x}{2}\right) + 5, -2\pi \le x \le 2\pi$

Amplitude 3, period $\frac{2\pi}{\frac{1}{2}} = 4\pi$, equilibrium $y = 5$, range $[5 - 3, 5 + 3] = [2, 8]$, no x-intercepts



d $y = -4 - \cos(3x), 0 \le x \le 2\pi$

Amplitude 1, inverted, period $\frac{2\pi}{3}$, equilibrium $y = -4$, range $[-4 - 1, -4 + 1] = [-5, -3]$, no x-intercepts-



e $y = 1 - 2\sin(2x), -\pi \le x \le 2\pi$

Amplitude 2, inverted, period $\frac{2\pi}{2} = \pi$, equilibrium $y = 1$, range $[1 - 2, 1 + 2] = [-1, 3]$

x-intercepts: let $y = 0$

$$\therefore 1 - 2\sin(2x) = 0$$

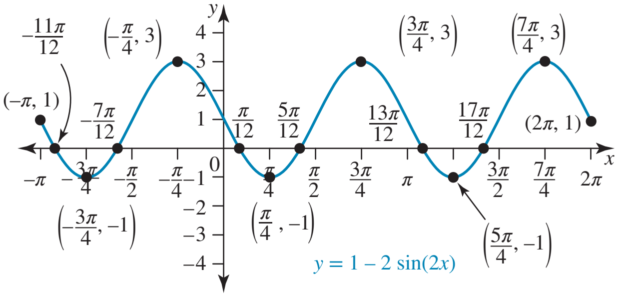
$$\therefore \sin(2x) = \frac{1}{2}, -2\pi \le 2x \le 4\pi$$

$$\therefore 2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6} \text{ or}$$

$$x = -\pi - \frac{\pi}{6}, -2\pi + \frac{\pi}{6}$$

$$\therefore 2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{-7\pi}{6}, \frac{-11\pi}{6}$$

$$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, -\frac{7\pi}{12}, -\frac{11\pi}{12}$$



f $y = 2[1 - 3\cos(x°)], 0 \le x \le 360$

$\therefore y = 2 - 6\cos(x°)$

Amplitude 6, inverted, period 360°, equilibrium $y = 2$, range $[2 - 6, 2 + 6] = [-4, 8]$

x-intercepts: let $y = 0$

$$\therefore 2 - 6\cos(x°) = 0$$

$$\therefore \cos(x°) = \frac{2}{6}$$

$$\therefore \cos(x°) = \frac{1}{3}$$

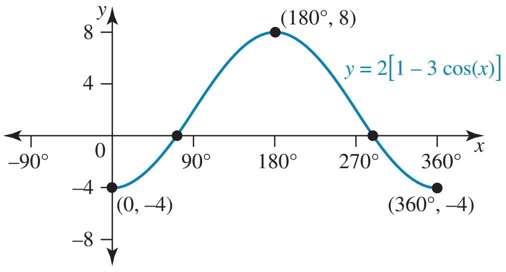
The base, in degrees, is $\cos^{-1}\left(\frac{1}{3}\right) = 70.53°$

$\therefore x° = 70.53°, 360° - 70.53°$

$\therefore x° = 70.53°, 289.47°$ or

$x° = \cos^{-1}\left(\frac{1}{3}\right), 360° - \cos^{-1}\left(\frac{1}{3}\right)$

without a calculator



9 a $f : R \to R, f(x) = 3 + 2\sin(5x)$

The function has amplitude 2 and equilibrium position $y = 3$.

Its range is $[3 - 2, 3 + 2] = [1, 5]$.

b $f : [0, 2\pi] \to R, f(x) = 10\cos(2x) - 4$

The function has amplitude 10 and equilibrium position $y = -4$.

As its period is $\frac{2\pi}{2} = \pi$, the function will cover its maximal

range over the interval $x \in [0, 2\pi]$.
Its range is $[-4 - 10, -4 + 10] = [-14, 6]$.
Its minimum value is -14.
Alternatively, replacing $\cos(2x)$ by its minimum value of -1,

$$f_{\min}(x) = 10 \times (-1) - 4$$
$$= -14$$

c $f : [0, 2\pi] \to R, f(x) = 56 - 12\sin(x)$
If $\sin(x)$ is replaced by its most negative value, the greatest value of the function will occur.

$$f_{\max}(x) = 56 - 12 \times (-1)$$
$$= 56 + 12$$
$$= 68$$

The maximum occurs when $\sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$.
Alternatively, the range of the function is $[56 - 12, 56 + 12] = [44, 68]$, so the maximum value is 68.
When $f(x) = 68$,

$$56 - 12\sin(x) = 68$$
$$\therefore -12 = 12\sin(x)$$
$$\therefore \sin(x) = -1$$
$$\therefore x = \frac{3\pi}{2}$$

d **i** $\sin(x) \to 3 + 2\sin(5x)$ under the transformations: dilation of factor $\frac{1}{5}$ from the y-axis and dilation of factor 2 from the x-axis followed by a vertical translation of 3 units upwards.

ii $\cos(x) \to 10\cos(2x) - 4$ under the transformations: dilation of factor $\frac{1}{2}$ from the y-axis and dilation of factor 10 from the x-axis, followed by a vertical translation of 4 units downwards.

iii $\sin(x) \to 56 - 12\sin(x)$ under the transformations: dilation of factor 12 from the x-axis, reflection in the x-axis, vertical translation of 56 units upwards.

10 $y = a\sin(nx)$
Observe from the given graph that the amplitude is 3 so $a = 3$.
The graph covers 2 cycles over the domain $[0, 2\pi]$, so the period is π.

$$\therefore \frac{2\pi}{n} = \pi$$
$$\therefore 2\pi = n\pi$$
$$\therefore n = 2$$

The equation of the given graph is $y = 3\sin(2x), 0 \le x \le 2\pi$.

11 $y = a\cos(nx)$
Observe from the given graph that the amplitude is 0.5. Since the graph is inverted, $a = -0.5$.
The graph covers one cycle over the domain $[0, 3\pi]$, so the period is 3π.

$$\therefore \frac{2\pi}{n} = 3\pi$$
$$\therefore 2\pi = 3n\pi$$
$$\therefore n = \frac{2}{3}$$

The equation of the given graph is
$y = -0.5\cos\left(\frac{2x}{3}\right), 0 \le x \le 3\pi$.

12 $y = a\cos(nx)$
Observe from the given graph that the amplitude is 4.5, so $a = 4.5$.
The graph covers one cycle over the domain $\left[0, \frac{3\pi}{2}\right]$, so the period is $\frac{3\pi}{2}$.

$$\therefore \frac{2\pi}{n} = \frac{3\pi}{2}$$
$$\therefore 4\pi = 3n\pi$$
$$\therefore n = \frac{4}{3}$$

The equation of the given graph is
$y = 4.5\cos\left(\frac{4}{3}x\right), 0 \le x \le 2\pi$.

13 **a** The given graph shows an upright sine function.
The equation is of the form $y = a\sin(nx)$.
The amplitude of the graph is 4, so $a = 4$.
Three cycles of the function are covered over the domain $[0, 2\pi]$, so the period is $\frac{2\pi}{3}$.

$$\therefore \frac{2\pi}{n} = \frac{2\pi}{3}$$
$$\therefore 6\pi = 2n\pi$$
$$\therefore n = 3$$

The equation is $y = 4\sin(3x)$.

b The given graph shows an upright cosine function.
The equation is of the form $y = a\cos(nx)$.
The amplitude of the graph is 6, so $a = 6$.
The graph completes one complete cycle over the domain $[0, 4\pi]$, so the period is 4π.

$$\therefore \frac{2\pi}{n} = 4\pi$$
$$\therefore 2\pi = 4n\pi$$
$$\therefore n = \frac{1}{2}$$

The equation is $y = 6\cos\left(\frac{x}{2}\right)$.

c The given graph shows an inverted sine function.
The equation is of the form $y = a\sin(nx)$.
The amplitude of the graph is 4, so $a = -4$.
Two cycles of the function are shown with one cycle completed over the domain $[0, \pi]$, so the period is π.

$$\therefore \frac{2\pi}{n} = \pi$$
$$\therefore 2\pi = n\pi$$
$$\therefore n = 2$$

The equation is $y = -4\sin(2x)$.

14 **a** The given graph is an upright sine function.
The equation is of the form $y = a\sin(nx) + c$.
The range of the graph is $[1, 5]$, so the graph is oscillating about $y = \frac{1+5}{2} = 3$. The mean or equilibrium position is $y = 3$, so $b = 3$.
The amplitude is $\frac{5-1}{2} = 2$, so $a = 2$.
The period of the graph is 2π.

$$\therefore \frac{2\pi}{n} = 2\pi$$
$$\therefore 2\pi = 2n\pi$$
$$\therefore n = 1$$

The equation is $y = 2\sin(x) + 3$.

b The given graph is an inverted sine function.
The equation is of the form $y = a\sin(nx) + c$.

The range of the graph is $[0, 6]$, so the graph is oscillating about $y = \frac{0+6}{2} = 3$. The mean or equilibrium position is $y = 3$, so $c = 3$.

The amplitude is $\frac{6-0}{2} = 3$, so $a = -3$.

The period of the graph is 4π.

$$\therefore \frac{2\pi}{n} = 4\pi$$
$$\therefore 2\pi = 4n\pi$$
$$\therefore n = \frac{1}{2}$$

The equation is $y = -3\sin\left(\frac{x}{2}\right) + 3$.

c The given graph is an upright cosine function.

The equation is of the form $y = a\cos(nx) + c$.

The range of the graph is $[-7, -1]$, so the graph is oscillating about $y = \frac{-7-1}{2} = -4$. The mean or equilibrium position is $y = -4$, so $c = -4$.

The amplitude is $\frac{-1-(-7)}{2} = 3$, so $a = 3$.

The period of the graph is 3π.

$$\therefore \frac{2\pi}{n} = 3\pi$$
$$\therefore 2\pi = 3n\pi$$
$$\therefore n = \frac{2}{3}$$

The equation is $y = 3\cos\left(\frac{2x}{3}\right) - 4$.

d The given graph is an inverted cosine function.

The equation is of the form $y = a\cos(nx) + c$.

The range of the graph is $[-6, 4]$ so the graph is oscillating about $y = \frac{-6+4}{2} = -1$. The mean or equilibrium position is $y = -1$, so $c = -1$.

The amplitude is $\frac{4-(-6)}{2} = 5$, and as the graph is inverted, $a = -5$.

The period of the graph is π.

$$\therefore \frac{2\pi}{n} = \pi$$
$$\therefore 2\pi = n\pi$$
$$\therefore n = 2$$

The equation is $y = -5\cos(2x) - 1$.

15 Range is $[2, 8]$, so the equilibrium is $y = 5$ and the amplitude is 3. The period is π.

Let the equation be $y = a\sin(nx) + c$ with $a = 3, c = 5$.

$\therefore y = 3\sin(nx) + 5$

Period: $\frac{2\pi}{n} = \pi \Rightarrow n = 2$

Therefore, a possible equation is $y = 3\sin(2x) + 5$.

16 a Consider the given graph as an inverted sine graph with equilibrium position $y = 0$. Let the equation be $y = -a\sin(nx)$.

The maximum point $(3\pi, 3)$ shows the amplitude is 3.

$\therefore y = -3\sin(nx)$

Between the points $(0, 0)$ and $(6\pi, 0)$, the graph completes $1\frac{1}{2}$ cycles. Therefore, its period is $\frac{2}{3} \times 6\pi = 4\pi$.

$$\therefore \frac{2\pi}{n} = 4\pi$$
$$\therefore \frac{2\pi}{4\pi} = n$$
$$\therefore n = \frac{1}{2}$$

A possible equation is $y = -3\sin\left(\frac{x}{2}\right)$.

b Consider the given graph as a cosine graph with equilibrium position $y = 0$. Let the equation be $y = a\cos(nx)$.

The minimum point $\left(\frac{\pi}{3}, -4\right)$ shows the amplitude is 4.

$\therefore y = 4\cos(nx)$

Between the points $(0, 4)$ and $\left(\frac{\pi}{3}, -4\right)$ the graph completes $\frac{1}{2}$ a cycle. Therefore, its period is $2 \times \frac{\pi}{3} = \frac{2\pi}{3}$.

$$\therefore \frac{2\pi}{n} = \frac{2\pi}{3}$$
$$\therefore n = 3$$

A possible equation is $y = 4\cos(3x)$.

c The range of the graph is $[2, 10]$.

Hence, its equilibrium position is $y = \frac{2+10}{2} = 6$ and its amplitude is $10 - 6 = 4$.

Consider the graph as an inverted cosine with equation $y = -a\cos(nx) + c$.

$\therefore y = -4\cos(nx) + 6$

The period of the graph is 2π.

$$\therefore \frac{2\pi}{n} = 2\pi$$
$$\therefore n = 1$$

A possible equation is $y = -4\cos(x) + 6$.

17 Consider the graph as an inverted sine graph with amplitude 2. Its equation could be $y = -2\sin(nx)$.

The period is 3π, so $\frac{2\pi}{n} = 3\pi$

Therefore, $n = \frac{2}{3}$ and a possible equation is

$y = -2\sin\left(\frac{2x}{3}\right)$.

18 $f(x) = a\sin(bx) + c$

a Since $a > 0$ and the function has a maximum at $x = \frac{\pi}{6}$,

$$\text{period} = 4 \times \frac{\pi}{6}$$
$$= \frac{2\pi}{3}$$

b The constants a, b and c are positive.

Period $\frac{2\pi}{3}$

$$\therefore \frac{2\pi}{b} = \frac{2\pi}{3}$$
$$\therefore b = 3$$

The range is $[5, 9]$, so the equilibrium position is $y = \frac{5+9}{2} = 7$.

$\therefore c = 7$

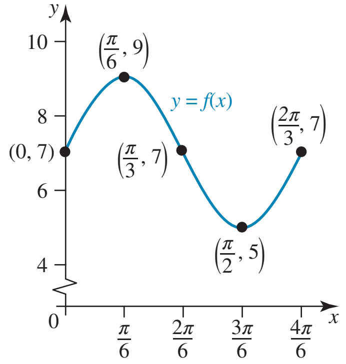
The amplitude is $9 - 7 = 2$.

Since $a > 0$, $a = 2$.

The correct answer is $a = 2, b = 3, c = 7$.

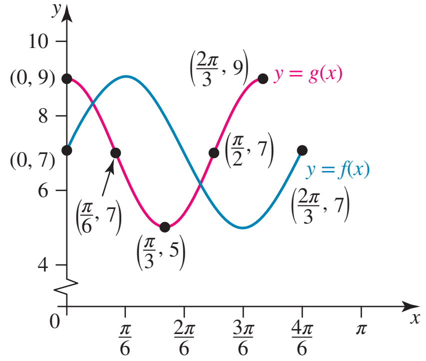
c The rule for the function is $f(x) = 2\sin(3x) + 7$.

For one cycle, the domain $D = \left[0, \frac{2\pi}{3}\right]$.



d $g(x) = a\cos(bx) + c, x \in D$

$\therefore g(x) = 2\cos(3x) + 7, x \in \left[0, \frac{2\pi}{3}\right]$.



e At intersection,

$$2\sin(3x) + 7 = 2\cos(3x) + 7, x \in \left[0, \frac{2\pi}{3}\right]$$

$$\therefore 2\sin(3x) = 2\cos(3x)$$

$$\therefore \sin(3x) = \cos(3x)$$

$$\therefore \frac{\sin(3x)}{\cos(3x)} = 1$$

$$\therefore \tan(3x) = 1$$

Quadrants 1 and 3, base $\frac{\pi}{4}$. The diagram shows there are 2 points of intersection.

$$\therefore 3x = \frac{\pi}{4}, \pi + \frac{\pi}{4}$$

$$\therefore 3x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}$$

If $x = \frac{\pi}{12}$,

$$f\left(\frac{\pi}{12}\right) = 2\sin\left(\frac{\pi}{4}\right) + 7$$

$$= 2 \times \frac{\sqrt{2}}{2} + 7$$

$$= \sqrt{2} + 7$$

If $x = \frac{5\pi}{12}$,

$$f\left(\frac{5\pi}{12}\right) = 2\sin\left(\frac{5\pi}{4}\right) + 7$$

$$= -2\sin\left(\frac{\pi}{4}\right) + 7$$

$$= -2 \times \frac{\sqrt{2}}{2} + 7$$

$$= -\sqrt{2} + 7$$

The points of intersection are

$\left(\frac{\pi}{12}, 7 + \sqrt{2}\right), \left(\frac{5\pi}{12}, 7 - \sqrt{2}\right)$.

f From the diagram, $f(x) \geq g(x)$ for $\frac{\pi}{12} \leq x \leq \frac{5\pi}{12}$.

19 Under the vertical translation of 3 units up followed by the reflection in the x-axis followed by a dilation of factor 3 from the y-axis:

$$y = \sin(x)$$

$$\rightarrow y = \sin(x) + 3$$

$$\rightarrow y = -[\sin(x) + 3] = -\sin(x) - 3$$

$$\rightarrow y = -\sin\left(\frac{x}{3}\right) - 3$$

The final image has the equation $y = -\sin\left(\frac{x}{3}\right) - 3$.

The range of the final image is $[-3 - 1, -3 + 1] = [-4, -2]$.

As the range of $y = \sin(x)$ is $[-1, 1]$, the two graphs do not intersect.

20 $f: [0, 4] \rightarrow R, f(x) = a - 20\sin\left(\frac{\pi x}{3}\right)$

a Given $f(4) = 10\left(\sqrt{3} + 1\right)$

$$\therefore a - 20\sin\left(\frac{4\pi}{3}\right) = 10\left(\sqrt{3} + 1\right)$$

$$\therefore a - 20\left(-\sin\left(\frac{\pi}{3}\right)\right) = 10\left(\sqrt{3} + 1\right)$$

$$\therefore a + 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3} + 10$$

$$\therefore a + 10\sqrt{3} = 10\sqrt{3} + 10$$

$$\therefore a = 10$$

b $f(x) = 10 - 20\sin\left(\frac{\pi x}{3}\right)$

Let $f(x) = 0$.

$$\therefore 10 - 20\sin\left(\frac{\pi x}{3}\right) = 0$$

$$\therefore 10 = 20\sin\left(\frac{\pi x}{3}\right)$$

$$\therefore \sin\left(\frac{\pi x}{3}\right) = \frac{1}{2}$$

Quadrants 1 and 2, base $\frac{\pi}{6}$

As $x \in [0, 4], \frac{\pi x}{3} \in \left[0, \frac{4\pi}{3}\right]$

$$\therefore \frac{\pi x}{3} = \frac{\pi}{6}, \pi - \frac{\pi}{6}$$

$$= \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore \pi x = \frac{3\pi}{6}, \frac{15\pi}{6}$$

$$\therefore x = \frac{1}{2}, \frac{5}{2}$$

c $f(x) = 10 - 20\sin\left(\frac{\pi x}{3}\right), 0 \leq x \leq 4$

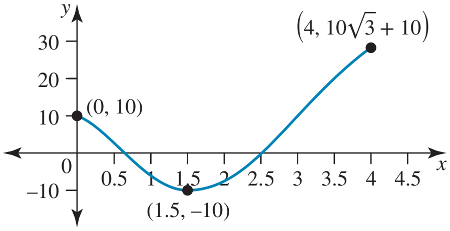
Amplitude 20, inverted, equilibrium $y = 10$

Period $\frac{2\pi}{\frac{\pi}{3}}$

$$= 2\pi \times \frac{3}{\pi}$$

$$= 6$$

The domain is $[0, 4]$. Since the period is 6, this is not a full cycle of the graph.
End points: let $x = 0$.
$f(0) = 10 - 20\sin(0)$
$= 10$
$(0, 10)$
If $x = 4$, then $f(4) = 10\sqrt{3} + 10$ (given), so the end point is $\left(4, 10\sqrt{3} + 10\right)$.
x-intercepts occur when $x = \frac{1}{2}, \frac{5}{2}$.
Therefore, the graph will reach its minimum point at
$x = \frac{\frac{1}{2} + \frac{5}{2}}{2} = \frac{3}{2}$.
The minimum point is $\left(\frac{3}{2}, 10 - 20\right) = \left(\frac{3}{2}, -10\right)$. Its maximum point is its end point $\left(4, 10\sqrt{3} + 10\right)$.
The range is $\left[-10, 10\sqrt{3} + 10\right]$.



9.3 Exam questions

1 The coefficient 2 is the amplitude.
The correct answer is **D**.

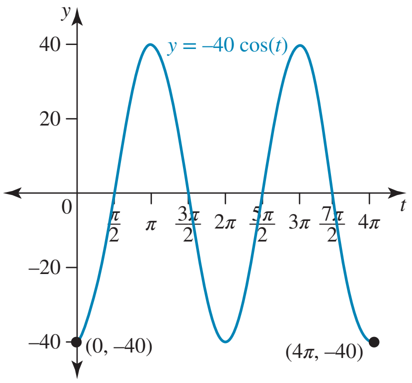
2 Amplitude = 2
Vertical translation 1 unit downwards
$\therefore k = -1$
Instead of the graph oscillating between 2 and −2, it oscillates between −3 and 1.
The correct answer is **C**.

3 Period $= \frac{2\pi}{n}$
$= \frac{2\pi}{\frac{3}{2}}$
$= \frac{4\pi}{3}$
The correct answer is **B**.

9.4 Applications of sine and cosine functions

9.4 Exercise

1 a $y = -40\cos(t)$
Period 2π, amplitude 40, inverted, range $[-40, 40]$, domain for two cycles $[0, 4\pi]$



b The greatest distance below rest, or equilibrium position, is 40 cm.

c The yo-yo is at its rest, or equilibrium position when $y = 0$.
The times are $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ in seconds.

d Let $y = 20$
$\therefore 20 = -40\cos(t)$
$\therefore \cos(t) = -\frac{1}{2}$
For the first positive solution,
$t = \pi - \frac{\pi}{3}$
$\therefore t = \frac{2\pi}{3}$
The yo-yo first reaches the height after $\frac{2\pi}{3} \simeq 2.1$ seconds.

2 The period is 12 hours, range $[20, 36]$, amplitude
$\frac{36 - 20}{2} = 8$, so the equilibrium is $T = 28$.
The equation is $T = -8\cos(nt) + 28$, where $\frac{2\pi}{n} = 12$.
Therefore, $n = \frac{\pi}{6}$.
The equation is $T = 28 - 8\cos\left(\frac{\pi}{6}t\right)$.
Let $T = 30$
$\therefore 30 = 28 - 8\cos\left(\frac{\pi}{6}t\right)$
$\therefore \cos\left(\frac{\pi}{6}t\right) = -\frac{1}{4}$
Second and third quadrants; the base is $\cos^{-1}\left(\frac{1}{4}\right) = 1.32$.
$\frac{\pi}{6}t = \pi - 1.32$ or $\pi + 1.32$
$\therefore t = 3.48$ or $t = 8.52$
Since time is measured from 8:00 am, the temperature exceeds 30° between 11:29 am and 4:31 pm.

3 a Since the range is $[0, 10]$, the equilibrium position is $I = 5$ and the amplitude is 5.
Let the equation be $I = a\sin(nt) + k$
$\therefore I = 5\sin(nt) + 5$
The period is 4 weeks.
$\therefore \frac{2\pi}{n} = 4$
$\therefore n = \frac{2\pi}{4}$
$\therefore n = \frac{\pi}{2}$
The equation is $I = 5\sin\left(\frac{\pi}{2}t\right) + 5$.

b Let $I = 6$

$$\therefore 6 = 5\sin\left(\frac{\pi}{2}t\right) + 5$$

$$\therefore 1 = 5\sin\left(\frac{\pi}{2}t\right)$$

$$\therefore \sin\left(\frac{\pi}{2}t\right) = \frac{1}{5}$$

Quadrants 1 and 2, base $\sin^{-1}\left(\frac{1}{5}\right) \simeq 0.20$

In one cycle, the solutions are:

$$\therefore \frac{\pi}{2}t = 0.20, \pi - 0.20$$

$$\therefore t = \frac{2}{\pi} \times 0.20, \frac{2}{\pi} \times (\pi - 0.20)$$

$$\therefore t = 0.128, 1.872$$

From the graph, $I \geq 6$ for $0.128 \leq t \leq 1.872$.

The length of the interval is $1.872 - 0.128 = 1.744$ weeks.

The percentage of the 4-week cycle is

$$\frac{1.744}{4} \times 100 = 43.6\%.$$

For 44% of the 4-week cycle, the person experiences a high level of happiness.

4 a $T = 19 - 3\sin\left(\frac{\pi}{12}t\right)$

At midnight, $t = 0$

Therefore, at midnight, $T = 19 - 3\sin(0) \Rightarrow T = 19$.

The temperature was 19° at midnight.

b Maximum temperature when $\sin\left(\frac{\pi}{12}t\right) = -1$

$$\therefore T_{\max} = 19 - 3 \times (-1)$$

$$\therefore T_{\max} = 22$$

The maximum temperature was 22°.

The maximum occurred when $\sin\left(\frac{\pi}{12}t\right) = -1$

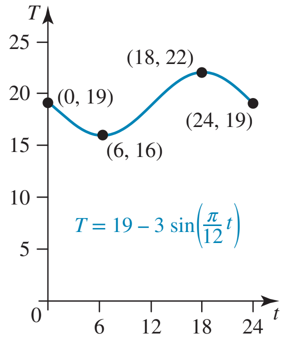
$$\therefore \frac{\pi}{12}t = \frac{3\pi}{2}$$

$$\therefore t = 18$$

The temperature reached its maximum of 22° at 6:00 pm.

c Since the amplitude was 3 and the equilibrium occurred at $T = 19$, the range of temperature was given by 19 ± 3 degrees. Therefore, the temperature varied over the interval 16° to 22°.

d Period $2\pi \div \frac{\pi}{12} = 24$ hours



e For the temperature to be below k for 3 hours, the interval must lie between $t = 6 - \frac{3}{2}$ and $t = 6 + \frac{3}{2}$; that is, $t = 4.5$ to $t = 7.5$.

When $t = 4.5$,

$$T = 19 - 3\sin\left(\frac{\pi}{12} \times \frac{9}{2}\right)$$

$$= 19 - 3\sin\left(\frac{3\pi}{8}\right)$$

$$\simeq 16.2$$

Therefore, $k = 16.2$.

5 a $h = a\sin\left(\frac{\pi}{5}x\right) + b$

Refer to the diagram given in the question.

The equilibrium position is $h = 4.5$, so $b = 4.5$.

The amplitude is $7 - 4.5 = 2.5$, so $a = 2.5$

The equation is $h = 2.5\sin\left(\frac{\pi}{5}x\right) + 4.5$.

b The base is one period in length.

The period is

$$2\pi \div \frac{\pi}{5}$$

$$= 2\pi \times \frac{5}{\pi}$$

$$= 10$$

The length of the base is 10 cm.

c The highest point on the curve occurs at a quarter of the cycle.

Hence, the highest point is $\left(\frac{10}{4}, 7\right) = (2.5, 7)$.

The x-coordinate of the centre of the circle is therefore $x = 2.5$.

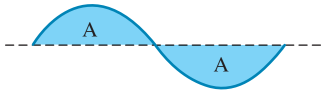
The y-coordinate of the centre of the circle is

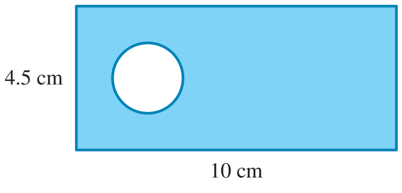
$$y = \frac{2 + 4.5}{2} = 3.25.$$

The centre of the circle is the point $(2.5, 3.25)$.

d The radius of the circle is $\frac{4.5 - 2}{2} = 1.25$.

Due to the symmetry of the sine curve, the area above the equilibrium position is equal to the area below the equilibrium position. The total area under the sine curve is that of a rectangle with dimensions 10 cm by 4.5 cm.





The required area is the rectangular area less the area of the circle.

Area, in sq cm, is $10 \times 4.5 - \pi(1.25)^2 \simeq 40.1$.

The shaded area is 40.1 sq cm.

6 $T = 30 - \cos\left(\frac{\pi}{12}t\right)$.

a The amplitude is 1 and the equilibrium is $T = 30$, so the range of the temperature in the incubator is $[29, 31]$, units being °C.

b As the graph is inverted, the cosine function will reach its maximum value after $\frac{1}{2}$ of its period.
The period, in minutes, is

$2\pi \div \frac{\pi}{12}$

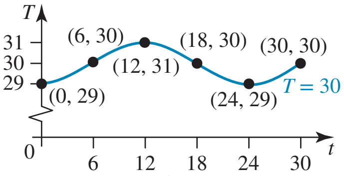
$= 2\pi \times \frac{12}{\pi}$

$= 24$

The maximum temperature is reached after $\frac{1}{2} \times 24 = 12$ minutes.

c $T = 30 - \cos\left(\frac{\pi}{12}t\right), t \in (0, 30)$

As the period is 24 minutes, the graph has $1\frac{1}{4}$ cycles.



d In 30 minutes, $1\frac{1}{4}$ cycles are completed, so in 60 minutes, $2\frac{1}{2}$ cycles are completed.

e In 1 hour, $2\frac{1}{2}$ cycles are completed; in 2 hours, 5 cycles are completed and the temperature is 29°. Thus in 2.5 hours, $5 + 1\frac{1}{4} = 6\frac{1}{4}$ cycles will be completed and the temperature will be 30° as shown by the graph in part **c**.

f The graph can be considered to be a sine function with a horizontal translation of 6 to the right. A possible equation is $T = \sin\left(\frac{\pi}{12}(t-6)\right) + 30$.

7 $T = 19 + 6\sin\left(\frac{\pi t}{6}\right)$ where t is the time in hours since 10:00 am.

a i As for any sine function, $-1 \le \sin\left(\frac{\pi t}{6}\right) \le 1$.

$\therefore T_{\max} = 19 + 6 \times 1$

$= 25$

The maximum temperature is 25°.

The maximum temperature occurs when $\sin\left(\frac{\pi t}{6}\right) = 1$

$\therefore \frac{\pi t}{6} = \frac{\pi}{2}$

$\therefore t = 3$

The maximum temperature occurs at 1:00 pm.

ii The minimum temperature occurs when $\sin\left(\frac{\pi t}{6}\right) = -1$.

$\therefore \frac{\pi t}{6} = \frac{3\pi}{2}$

$\therefore t = 9$

$T_{\min} = 19 + 6 \times (-1)$

$= 13°$

The minimum temperature of 13° occurs at 7:00 pm.

b i At 11:30 am, $t = 1.5$

$\therefore T = 19 + 6\sin\left(\frac{1.5\pi}{6}\right)$

$\therefore T = 19 + 6\sin\left(\frac{\pi}{4}\right)$

$= 19 + 6 \times \frac{\sqrt{2}}{2}$

$= 19 + 3\sqrt{2}$

$\therefore T = 23.2$

The temperature at 11:30 am is 23.2°.

ii At 7:30 pm, $t = 9.5$

$\therefore T = 19 + 6\sin\left(\frac{9.5\pi}{6}\right)$

$\therefore T = 19 + 6\sin\left(\frac{19\pi}{12}\right)$

$\therefore T = 13.2$

The temperature at 7:30 pm is 13.2°.

c $T = 19 + 6\sin\left(\frac{\pi t}{6}\right), t \in [0, 9.5]$.

Amplitude 6, equilibrium $T = 19$

The period is $2\pi \div \frac{\pi}{6} = 12$, so for the domain specified the graph will not cover a full cycle.

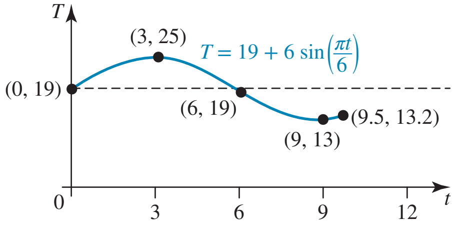
Right end point (9.5, 13.2), maximum point (3, 25), minimum point (9, 13).

Left end point: let $t = 0$

$\therefore T = 19 + 6\sin(0)$

$\therefore T = 19$

(0, 19)



d Let $T = 24$

$\therefore 24 = 19 + 6\sin\left(\frac{\pi t}{6}\right)$

$\therefore 5 = 6\sin\left(\frac{\pi t}{6}\right)$

$\therefore \sin\left(\frac{\pi t}{6}\right) = \frac{5}{6}$

Quadrants 1 and 2, base $\sin^{-1}\left(\frac{5}{6}\right) \simeq 0.99$

$\therefore \frac{\pi t}{6} = 0.99, \pi - 0.99$

$\therefore t = \frac{6}{\pi} \times 0.99, \frac{6}{\pi} \times (\pi - 0.99)$

$\therefore t = 1.88, 4.12$

The air conditioner is switched on at $t = 1.88$ and switched off 2.24 hours later at $t = 4.12$.

8 $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$

a When $t = 0$,

$h = 10 - 8.5\cos(0)$

$= 10 - 8.5 \times 1$

$= 1.5$

Initially the carriage is 1.5 metres above the ground.

b After 1 minute, $t = 60$,

$h = 10 - 8.5\cos(\pi)$

$= 10 - 8.5 \times (-1)$

$= 18.5$

After 1 minute the carriage is 18.5 metres above the ground.

c The period is the time to complete one revolution.

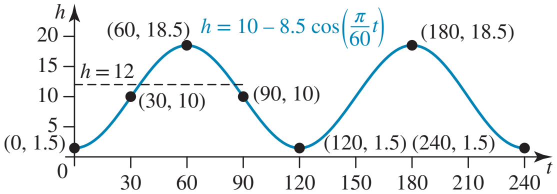
$2\pi \div \frac{\pi}{60} = 2\pi \times \frac{60}{\pi}$

$= 120$

The period is 120 seconds or 2 minutes.
In 4 minutes two revolutions will be completed.

d $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right), t \in [0, 240]$

Amplitude 8.5, inverted, equilibrium $h = 10$, range $[1.5, 18.5]$, period 120, 2 cycles



e Let $h = 12$

$$\therefore 12 = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$$

$$\therefore 2 = 8.5\cos\left(\frac{\pi}{60}t\right)$$

$$\therefore \cos\left(\frac{\pi}{60}t\right) = \frac{2}{8.5}$$

$$\therefore \cos\left(\frac{\pi}{60}t\right) = \frac{4}{17}$$

Quadrants 2 and 3, base $\cos^{-1}\left(\frac{4}{17}\right)$

$$\therefore \frac{\pi}{60}t = \pi - \cos^{-1}\left(\frac{4}{17}\right), \pi + \cos^{-1}\left(\frac{4}{17}\right)$$

$$\therefore t = \frac{60}{\pi}\left[\pi - \cos^{-1}\left(\frac{4}{17}\right)\right], \frac{60}{\pi}\left[\pi + \cos^{-1}\left(\frac{4}{17}\right)\right]$$

The graph shows that the carriage is higher than 12 metres above the ground for the time interval between these two values for t.

The time, in seconds, is

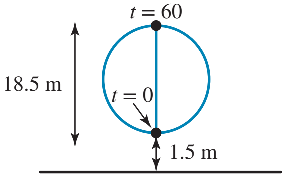
$$\frac{60}{\pi}\left[\pi + \cos^{-1}\left(\frac{4}{17}\right)\right] - \frac{60}{\pi}\left[\pi - \cos^{-1}\left(\frac{4}{17}\right)\right]$$

$$= 2 \times \frac{60}{\pi} \times \cos^{-1}\left(\frac{4}{17}\right)$$

$$= 51$$

The time is 51 seconds.

f The highest height above the ground is 18.5 metres and the lowest height is 1.5 metres.



The diameter of the circle is $18.5 - 1.5 = 17$ metres.
Therefore, the length of a radial spoke is 8.5 metres.

9 $p = 3\sin(n\pi t) + 5$

a p measures the distance of the water from the sunbather. From the equation, the amplitude is 3 and the equilibrium is $p = 5$, so the range of values for p are $p \in [2, 8]$.

The closest distance the water reaches to the sunbather is 2 metres.

b In 1 hour or 60 minutes, 40 cycles of the sine function are completed.

Therefore, one cycle is completed in $\frac{60}{40} = \frac{3}{2}$ minutes.

The period is $\frac{3}{2}$ minutes.

$$\therefore \frac{2\pi}{n\pi} = \frac{3}{2}$$

$$\therefore \frac{2}{n} = \frac{3}{2}$$

$$\therefore n = \frac{4}{3}$$

c The second model has equation $p = a\sin(4\pi t) + 5$.
Its range, assuming $a > 0$, is $[5 - a, 5 + a]$.
As the water just reaches the sunbather, $5 - a = 0$.
Therefore, $a = 5$.

The period is $\frac{2\pi}{4\pi} = \frac{1}{2}$ minute, so in 30 minutes the function completes 60 cycles, reaching P once every cycle.
The water reaches the sunbather 60 times in half an hour.

d For the first model $p = 3\sin(n\pi t) + 5$ where $n = \frac{4}{3}$, one cycle was completed in $\frac{3}{2}$ minutes.

In 1 minute, $\frac{2}{3}$ of a cycle would be completed, so the number of waves per minute is $\frac{2}{3}$.

For the second model, $p = a\sin(4\pi t) + 5$, one cycle is completed in $\frac{1}{2}$ minute, so two cycles are completed in 1 minute. The number of waves per minute is 2.
Therefore, the second model, $p = a\sin(4\pi t) + 5$, has the greater number of waves per minute.

10 $x = a\sin(bt)$

a The given information is that the amplitude is 20 cm and the can initially moves downwards from the mean position of $x = 0$.

$\therefore a = -20$

The time interval between the lowest point and the following highest point is half a period.
Therefore, the period is $2 \times 1.5 = 3$ seconds.

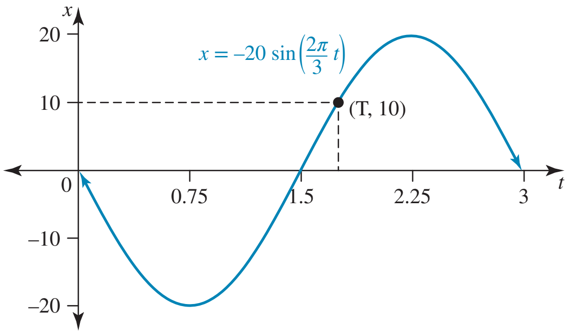
$$\therefore \frac{2\pi}{b} = 3$$

$$\therefore b = \frac{2\pi}{3}$$

The equation for the vertical displacement is

$x = -20\sin\left(\frac{2\pi}{3}t\right)$.

b



c The amplitude is 20, so the displacement must be 10.
Let $x = 10$

$$\therefore 10 = -20\sin\left(\frac{2\pi}{3}t\right)$$

$$\therefore \sin\left(\frac{2\pi}{3}t\right) = -\frac{1}{2}$$

For the shortest time, we require the solution in quadrant 3.
The base is $\frac{\pi}{6}$

$\therefore \frac{2\pi}{3}t = \pi + \frac{\pi}{6}$

$\therefore \frac{2\pi}{3}t = \frac{7\pi}{6}$

$\therefore t = \frac{7\pi}{6} \times \frac{3}{2\pi}$

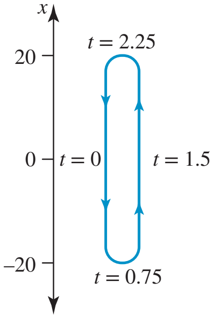
$\therefore t = \frac{7}{4}$

$\therefore T = \frac{7}{4}$

The shortest time is $\frac{7}{4}$ seconds.

d The can falls 20 centimetres, rises 20 centimetres to equilibrium, rises 20 centimetres to reach its greatest displacement and then falls 20 centimetres back to equilibrium.

The total distance moved is 80 centimetres.



11 a $I = 4\sin(t) - 3\cos(t)$

Sketch the graph in the Graph&Tab menu over $[0, 4\pi]$ and use the Analysis tools to obtain the coordinates of the first maximum point a (2.214..., 5). The y-coordinate is the value of the intensity.

The maximum intensity is 5 units.

b Tap Analysis → G-Solve → Root to obtain the first value of t for which $I = 0$ as $t = 0.64$

c The amplitude of the graph is 2 and its period is 2π. This is confirmed by observations of the graph. To consider the graph as that of a sine function, the horizontal translation would be 0.64 units to the right.

$\therefore I = 5\sin(t - 0.64)$ is the same curve as $I = 4\sin(t) - 3\cos(t)$.

d In the form $I = a\cos(t + b)$, the horizontal translation would be 2.214 …

To 2 decimal places, $I = 5\cos(t - 2.21)$ is the same curve as $I = 4\sin(t) - 3\cos(t)$.

12 a Counting the number of peaks on the given graph, including the ones at the two ends, gives 13 teeth.

b Sketch the graph in Graph&Tab menu and obtain the coordinates of each peak. Test to see how far apart each is.

***x*-coordinate of maximum points**	**Difference between *x*-values of successive maximums**
$x_1 = 1.0541$	
	$x_2 - x_1 = 1.0472$
$x_2 = 2.1013$	
	$x_3 - x_2 = 1.0472$
$x_3 = 3.1485$	
	$x_4 - x_3 = 1.0472$
$x_4 = 4.1957$	0

The x-values of the maximum points appear to be the same distance of 1.0472 cm apart.

The cos $(6x)$ term has period $\frac{2\pi}{6} = \frac{\pi}{3} = 1.0472$.

Successive peaks are $\frac{\pi}{3}$ cm apart.

c The greatest width is the y-value of the 13th peak. This can be calculated in exact form from the equation.

Let $x = 4\pi$.

$\therefore y = 4\pi + 4 + 4\cos(24\pi)$

$= 4\pi + 4 + 4 \times 1$

$= 4\pi + 8$

The greatest width of the saw is $(4\pi + 8)$ cm.

As a decimal value of 20.566 cm, the greatest width could be calculated by tapping Analysis → G-Solve → y-Cal for $x = 4\pi$.

d $-1 \le \cos(6x) \le 1$

$-4 \le 4\cos(6x) \le 4$

$\therefore -4 + (x + 4) \le 4\cos(6x) + (x + 4) \le 4 + (x + 4)$

$\therefore x \le y \le x + 8$

The line $y = x + 8$ will touch the teeth. This can be confirmed by sketching this line with that of $y = x + 4 + 4\cos(6x), 0 \le x \le 4\pi$

9.4 Exam questions

1 $y = 7 - 4\cos(3x)$

Minimum value:

largest positive value of $\cos(3x) = 1$

$y = 7 - 4\cos(3x)$

$y = 7 - 4 \times (1)$

$y = 3$

Maximum value: largest negative value of $(3x) = -1$

$y = 7 - 4\cos(3x)$

$y = 7 - 4 \times (-1)$

$y = 11$

∴ the minimum value is 3 and the maximum value is 11.

The correct answer is **B**.

2 The minimum temperature occurs when $\sin\left(\frac{\pi}{12}t\right) = 1$ (because the graph is reflected).

Minimum temperature: $20 - 4 \times 1 = 16$°C

$\sin\left(\frac{\pi}{12}t\right) = 1$

Solving:

$\frac{\pi}{12}t = \frac{\pi}{2}$

$\therefore t = 6$

The minimum temperature was 16°C and it occurred at 6:00 am.

The correct answer is **B**.

3 a Initial population when $t = 0$:

$p = 1000 + 500\sin(0)$

$= 1000$ [1 mark]

b Highest population when $\sin\left(\frac{\pi t}{4}\right) = 1$:

$p = 1000 + 500 \times (1)$

$= 1500$

∴ the highest population is 1500. [1 mark]

Lowest population when $\sin\left(\frac{\pi t}{4}\right) = -1$:

$$\begin{aligned} p &= 1000 + 500 \times (-1) \\ &= 500 \end{aligned}$$ [1 mark]

∴ the lowest population is 500.

c At the end of 2 years (24 months):

$$\begin{aligned} p &= 1000 + 500 \sin\left(\frac{\pi t}{4}\right) \\ &= 1000 + 500 \sin\left(\frac{\pi \times 24}{4}\right) \\ &= 1000 + 500 \sin(6\pi) \end{aligned}$$

As $\sin(6\pi) = 0$

$p = 1000$

∴ population after 2 years is 1000. [1 mark]

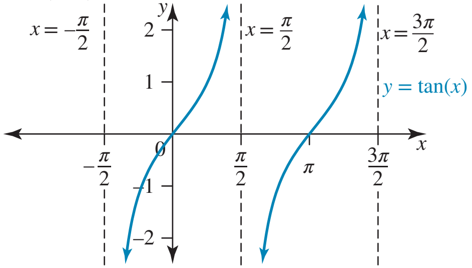
9.5 The tangent function

9.5 Exercise

1 $y = \tan(x), x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Asymptotes: $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

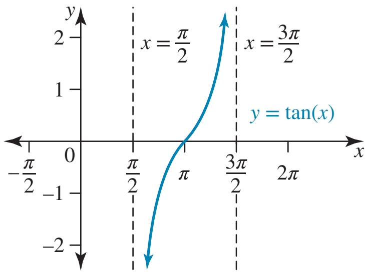
x-intercepts are midway between the asymptotes at the origin and $(\pi, 0)$.



2 a $y = \tan(x), x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Asymptotes: $x = \frac{\pi}{2}, \frac{3\pi}{2}$

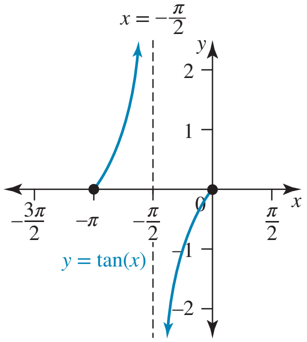
x-intercept: $(\pi, 0)$



b $y = \tan(x), x \in [-\pi, 0]$

Asymptote: $x = -\frac{\pi}{2}$

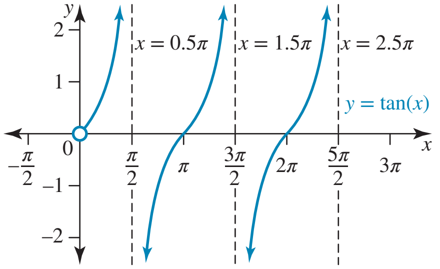
x-intercepts and end points: $(-\pi, 0), (0, 0)$



c $y = \tan(x), x \in \left(0, \frac{5\pi}{2}\right)$

Asymptotes: $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

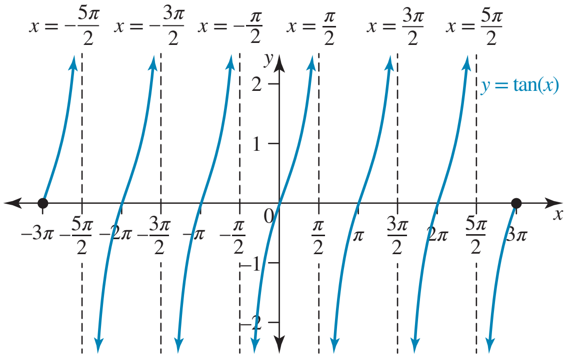
x-intercepts: $(\pi, 0), (2\pi, 0)$, open point at $(0, 0)$.



3 $y = \tan(x), x \in [-3\pi, 3\pi]$

Asymptotes: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$

x-intercepts midway between successive pairs of asymptotes: $(\pm\pi, 0), (\pm 2\pi, 0), (\pm 3\pi, 0)$ and $(0, 0)$.

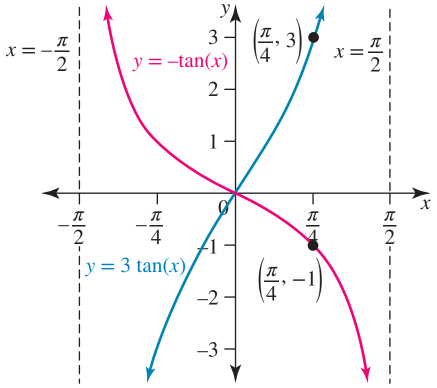


The domain of the graph is $[-3\pi, 3\pi] \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}\right\}$ and the range is R.

4 a When $x = \frac{\pi}{4}$,

$$\begin{aligned} y &= -\tan(x) & y &= 3\tan(x) \\ &= -\tan\left(\frac{\pi}{4}\right) \text{ and } & &= 3\tan\left(\frac{\pi}{4}\right) \\ &= -1 & &= 3 \end{aligned}$$

Both graphs have asymptotes with equations $x = \pm\frac{\pi}{2}$, period π and contain the point $(0, 0)$.



b $y = \tan(x) + \sqrt{3}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Vertical translation of $\sqrt{3}$ upwards does not affect the asymptotes at $x = \pm\frac{\pi}{2}$.

x-intercept: let $y = 0$

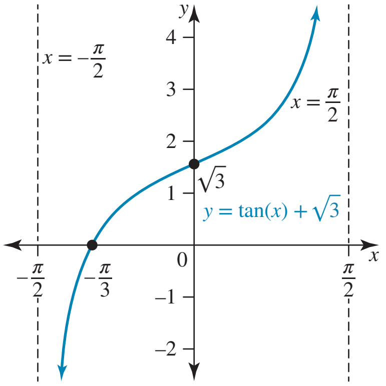
∴ $\tan(x) + \sqrt{3} = 0$

$\therefore \tan(x) = -\sqrt{3},\ x \in \left(-\frac{\pi}{2}, \quad \frac{\pi}{2}\right)$

$\therefore x = -\frac{\pi}{3}$

$\left(-\frac{\pi}{3}, 0\right)$ is the intercept.

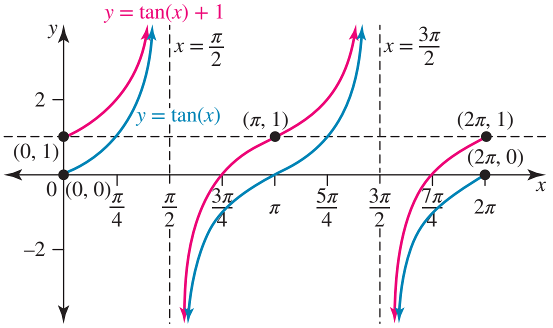
When $x = 0$, $y = \sqrt{3}$.



5 $y = \tan(x) + 1$ is obtained from $y = \tan(x)$ by a vertical translation upwards of 1 unit. For both graphs, the equations of the asymptotes are $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$ and the period is π.
x-intercepts for $y = \tan(x)$ occur at $x = 0, \pi, 2\pi$.
x-intercepts for $y = \tan(x) + 1$ occur when $\tan(x) + 1 = 0$
$\therefore \tan(x) = -1$

$\therefore x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}$



6 a $y = 4\tan(x), x \in [0, 2\pi]$

Period π, asymptotes $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$

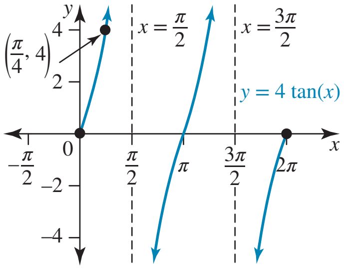
Point: let $x = \frac{\pi}{4}$

$\therefore y = 4\tan\left(\frac{\pi}{4}\right)$

$= 4 \times 1$

$= 4$

$\left(\frac{\pi}{4}, 4\right)$



b $y = -0.5\tan(x), x \in [0, 2\pi]$

Period π, asymptotes $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$, inverted graph

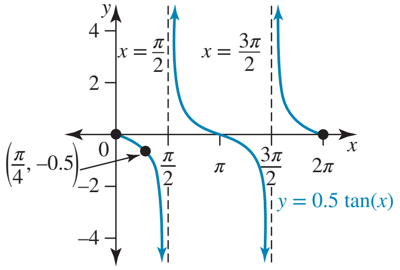
Point: let $x = \frac{\pi}{4}$

$\therefore y = -0.5\tan\left(\frac{\pi}{4}\right)$

$= -0.5 \times 1$

$= -0.5$

$\left(\frac{\pi}{4}, -0.5\right)$



c $y = -\tan(x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

Period π, asymptotes $x = \pm\frac{\pi}{2}$, x-intercept $(0, 0)$, reflected in x-axis

$y = 1 - \tan(x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

Vertical translation up 1 unit from $y = -\tan(x)$

Period π, asymptotes $x = \pm\frac{\pi}{2}$, passes through $(0, 1)$

x-intercept: let $y = 0$

$\therefore 0 = 1 - \tan(x)$

$\therefore \tan(x) = 1$

$\therefore x = \frac{\pi}{4}$

$\left(\frac{\pi}{4}, 0\right)$

$y = -1 - \tan(x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

Vertical translation down 1 unit from $y = -\tan(x)$

Period π, asymptotes $x = \pm\frac{\pi}{2}$, passes through $(0, -1)$

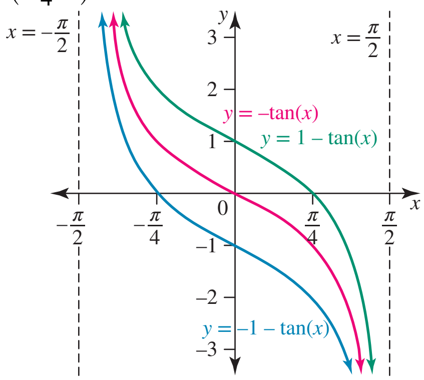
x-intercept: let $y = 0$

$\therefore 0 = -1 - \tan(x)$

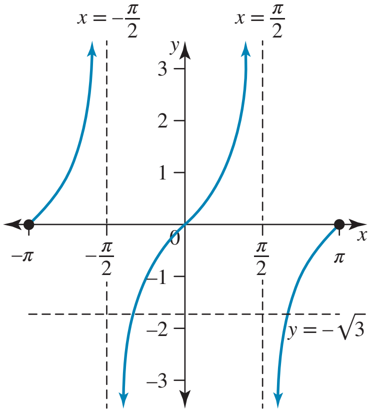
$\therefore \tan(x) = -1$

$\therefore x = -\frac{\pi}{4}$

$\left(-\frac{\pi}{4}, 0\right)$



7 **a** $y = \tan(x);\ \pi \le x \le \pi$



b Let $y = -\sqrt{3}$

$\therefore\ \tan(x) = -\sqrt{3}, -\pi \le x \le \pi$

Solutions lie in quadrant 2 with a positive rotation and quadrant 4 with a negative rotation. Base $\frac{\pi}{3}$

$\therefore x = -\frac{\pi}{3}, \pi - \frac{\pi}{3}$

$\therefore x = -\frac{\pi}{3}, \frac{2\pi}{3}$

c $\tan(x) + \sqrt{3} < 0$

$\therefore\ \tan(x) < -\sqrt{3}$

Reading from the graph, $-\frac{\pi}{2} < x < -\frac{\pi}{3}$ or $\frac{\pi}{2} < x < \frac{2\pi}{3}$

8 **a** $y = \tan(6x)$

The period is $\frac{\pi}{6}$.

b $y = 5\tan\left(\frac{x}{4}\right)$

The period is $\pi \div \frac{1}{4} = 4\pi$.

c $y = -2\tan\left(\frac{3x}{2}\right) + 5$

The period is $\pi \div \frac{3}{2} = \frac{2\pi}{3}$.

d $y = \tan(\pi x)$

The period is $\frac{\pi}{\pi} = 1$.

9 **a** $y = \tan(5x)$

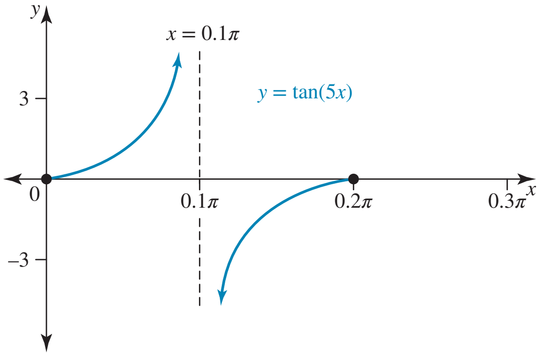
The first positive asymptote occurs when

$5x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{10}$.

b The period of $y = \tan(5x)$ is $\frac{\pi}{5}$.

When $x = 0$, $y = \tan(0) = 0$.

Sketch the graph over $\left[0, \frac{\pi}{5}\right]$.



10 $y = -\frac{1}{4}\tan\left(\frac{\pi x}{2}\right)$

a The period is $\frac{\pi}{n}$ where $n = \frac{\pi}{2}$.

Period $= \pi \times \frac{2}{\pi} = 2$

b The first positive asymptote occurs when $\frac{\pi x}{2} = \frac{\pi}{2} \Rightarrow x = 1$.

c Since the period is 2, two cycles will be covered over the domain $x \in [0, 4]$.

There will be another asymptote at $x = 1 + 2 \Rightarrow x = 3$.

x-intercepts: let $y = 0$

$\therefore\ -\frac{1}{4}\tan\left(\frac{\pi x}{2}\right) = 0, x \in [0, 4]$

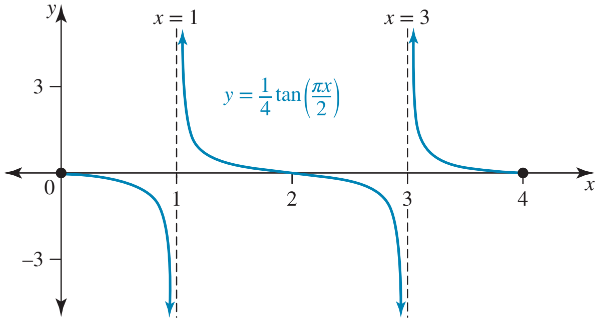
$\therefore\ \tan\left(\frac{\pi x}{2}\right) = 0, \frac{\pi x}{2} \in [0, 2\pi]$

$\therefore\ \frac{\pi x}{2} = 0, \pi, 2\pi$

$\therefore\ \frac{x}{2} = 0, 1, 2$

$\therefore x = 0, 2, 4$

The graph of $y = -\frac{1}{4}\tan\left(\frac{\pi x}{2}\right)$ is a reflection in the x-axis.



11 $y = \tan(3x)$ has period $\frac{\pi}{3}$.

Asymptotes occur when $3x = \frac{\pi}{2}, \frac{3\pi}{2}$

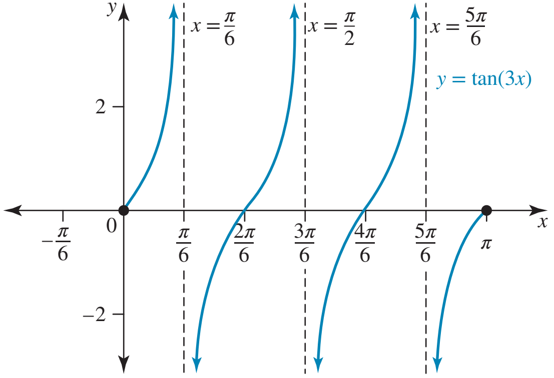
$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}$

Adding a period gives another asymptote at $x = \frac{\pi}{2} + \frac{\pi}{3}$ (others are outside the given domain).

$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ are the equations of the asymptotes.

x-intercepts lie midway between the asymptotes at $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$.

Scale the horizontal axis in multiples of $\frac{\pi}{6}$.



12 The period of $y = -2\tan\left(\frac{x}{2}\right)$ is $\pi \div \frac{1}{2} = 2\pi$, There will only be one asymptote, since the domain allows only one period.

Asymptote when $\frac{x}{2} = \frac{\pi}{2}$; therefore, $x = \pi$.

The graph is inverted.

Point to illustrate the dilation factor of 2:

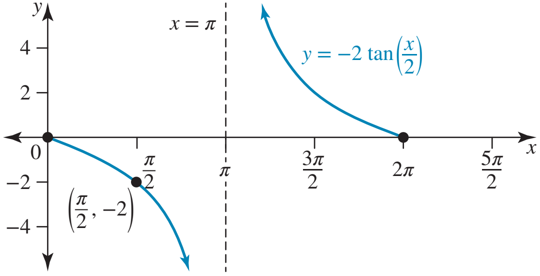
When $x = \frac{\pi}{2}$,

$$y = -2\tan\left(\frac{\pi}{4}\right)$$
$$= -2$$

Point $\left(\frac{\pi}{2}, -2\right)$

x-intercepts: If $x = 0$ or $2\pi, y = 0$.

Scale the horizontal axis in multiples of $\frac{\pi}{2}$.



13 **a** $y = \tan(4x),\ 0 \le x \le \pi$

The period is $\frac{\pi}{4}$.

An asymptote occurs when $4x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{8}$, and others are a period apart.

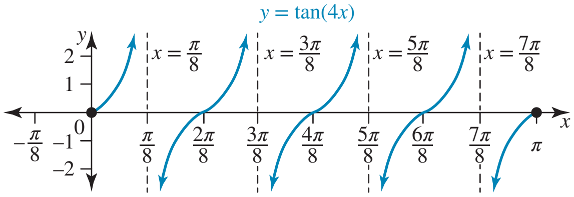
For $0 \le x \le \pi$, the equations of the asymptotes are:

$x = \frac{\pi}{8}, x = \frac{3\pi}{8}, x = \frac{5\pi}{8}, x = \frac{7\pi}{8}$.

x-intercepts occur midway between the asymptotes at

$x = 0, x = \frac{2\pi}{8}, x = \frac{4\pi}{8}, x = \frac{6\pi}{8}, x = \pi$.

$\therefore x = 0, x = \frac{\pi}{4}, x = \frac{\pi}{2}, x = \frac{3\pi}{4}, x = \pi$



b $y = 2\tan(2x),\ 0 \le x \le \pi$

Period $\frac{\pi}{2}$

Asymptotes: when $2x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{4}$

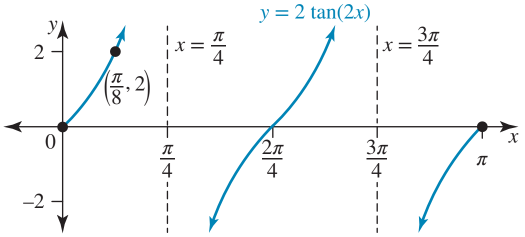
$x = \frac{\pi}{4}, x = \frac{3\pi}{4}$ are the equations of the asymptotes.

x-intercepts at $x = 0, x = \frac{\pi}{2}, x = \pi$

Point: let $x = \frac{\pi}{8}$

$$\therefore y = 2\tan\left(\frac{\pi}{4}\right)$$
$$= 2$$

$\left(\frac{\pi}{8}, 2\right)$



c $y = -\tan\left(\frac{x}{3}\right), 0 \le x \le \pi$

The period is $\pi \div \frac{1}{3} = 3\pi$ and the graph is inverted.

Asymptotes: when $\frac{x}{3} = \frac{\pi}{2} \Rightarrow x = \frac{3\pi}{2}$, which is outside the interval allowed.

Subtracting a period, $x = -\frac{3\pi}{2}$, which is also outside the interval allowed.

There are no asymptotes for $0 \le x \le \pi$.

x-intercepts: let $y = 0$

$$\therefore 0 = -\tan\left(\frac{x}{3}\right)$$

$$\therefore \tan\left(\frac{x}{3}\right) = 0$$

$$\therefore \frac{x}{3} = -\pi, 0, \pi...$$

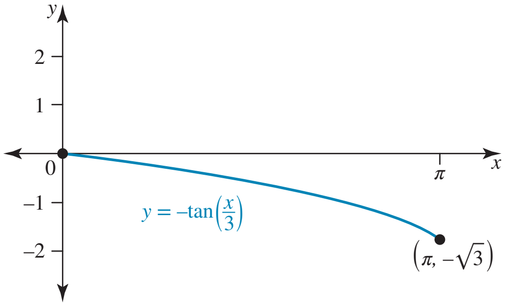
$$\therefore x = 0$$

For $0 \le x \le \pi$, there is only one x-intercept at $(0, 0)$.

End point: let $x = \pi$

$$\therefore y = -\tan\left(\frac{\pi}{3}\right)$$
$$= -\sqrt{3}$$

$\left(\pi, -\sqrt{3}\right)$



14 $y = a\tan(nx)$

The first positive asymptote occurs when $nx = \frac{\pi}{2}$.

Given this asymptote has equation $x = \frac{\pi}{8}$,

$$n \times \frac{\pi}{8} = \frac{\pi}{2}$$

$$\therefore \frac{n}{8} = \frac{1}{2}$$

$$\therefore n = 4$$

Hence, $y = a\tan(4x)$

The point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ lies on the graph.

$$\therefore \frac{\sqrt{3}}{2} = a\tan\left(4 \times \frac{\pi}{3}\right)$$

$$\therefore \frac{\sqrt{3}}{2} = a\tan\left(\frac{4\pi}{3}\right)$$

$$\therefore \frac{\sqrt{3}}{2} = a\tan\left(\frac{\pi}{3}\right)$$

$\therefore \frac{\sqrt{3}}{2} = a \times \sqrt{3}$

$\therefore a = \frac{1}{2}$

The equation is $y = \frac{1}{2}\tan(4x)$.

15 $y = a\tan(nx)$

a The first negative asymptote occurs when $nx = -\frac{\pi}{2}$.

Given the equation of this asymptote is $x = -\frac{3\pi}{4}$.

$n \times \frac{-3\pi}{4} = -\frac{\pi}{2}$

$\therefore n = -\frac{\pi}{2} \times \frac{4}{-3\pi}$

$\therefore n = \frac{2}{3}$

Hence, $y = a\tan\left(\frac{2}{3}x\right)$

Substitute the point $\left(-\frac{\pi}{4}, 3\right)$

$\therefore 3 = a\tan\left(\frac{2}{3} \times -\frac{\pi}{4}\right)$

$\therefore 3 = a\tan\left(-\frac{\pi}{6}\right)$

$\therefore 3 = a \times -\tan\left(\frac{\pi}{6}\right)$

$\therefore 3 = -a \times \frac{1}{\sqrt{3}}$

$\therefore a = -3\sqrt{3}$

The equation is $y = -3\sqrt{3}\tan\left(\frac{2}{3}x\right)$.

b The period of $y = -3\sqrt{3}\tan\left(\frac{2}{3}x\right)$ is $\frac{\pi}{n}$.

$\frac{\pi}{n} = \pi \div \frac{2}{3}$

$= \frac{3\pi}{2}$

The period is $\frac{3\pi}{2}$.

16 $y = \tan(bx), 0 \le x \le \frac{3\pi}{2}$

a There are 4 cycles shown.

b If 4 cycles are completed over an interval of $\frac{3\pi}{2}$ units, then one cycle is completed over an interval of length $\frac{3\pi}{8}$ units.

Therefore, the period of the graph is $\frac{3\pi}{8}$.

c From the equation, the period is $\frac{\pi}{b}$.

$\therefore \frac{\pi}{b} = \frac{3\pi}{8}$

$\therefore \frac{1}{b} = \frac{3}{8}$

$\therefore b = \frac{8}{3}$

d An asymptote occurs when $bx = \frac{\pi}{2}$

$\therefore \frac{8}{3}x = \frac{\pi}{2}$

$\therefore x = \frac{\pi}{2} \times \frac{3}{8}$

$\therefore x = \frac{3\pi}{16}$

Other asymptotes are a period apart. Since the period is $\frac{3\pi}{8} = \frac{6\pi}{16}$, the equations of the asymptotes are $x = \frac{3\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16}, \frac{21\pi}{16}$.

17 $f(x) = a - \tan(cx)$

The function is undefined at its asymptote positions.

There is an asymptote when $cx = -\frac{\pi}{2} \Rightarrow x = -\frac{\pi}{2c}$. This is the first negative value for which the function is undefined.

$\therefore -\frac{\pi}{2c} = -\frac{6}{5}$

$\therefore 5\pi = 12c$

$\therefore c = \frac{5\pi}{12}$

$\therefore f(x) = a - \tan\left(\frac{5\pi}{12}x\right)$

Given $f(-12) = -2$,

$a - \tan\left(\frac{5\pi}{12} \times -12\right) = -2$

$\therefore a - \tan(-5\pi) = -2$

$\therefore a - 0 = -2$

$\therefore a = -2$

Answer: $a = -2, c = \frac{5\pi}{12}$.

18 **a** $y = 3\tan(x) + 2, 0 \le x \le \pi$

The period is π.

Asymptote: $x = \frac{\pi}{2}$

y-intercept is $(0, 2)$ since there is a vertical translation 2 units upwards.

x-intercepts: let $y = 0$

$\therefore 3\tan(x) + 2 = 0$

$\therefore \tan(x) = -\frac{2}{3}$

As $x \in [0, \pi]$, the solution is in the second quadrant.

$\therefore x = \pi - \tan^{-1}\left(\frac{2}{3}\right)$

$\therefore x \simeq 2.55$

Point: let $x = \frac{\pi}{4}$

$\therefore y = 3\tan\left(\frac{\pi}{4}\right) + 2$

$\therefore y = 5$

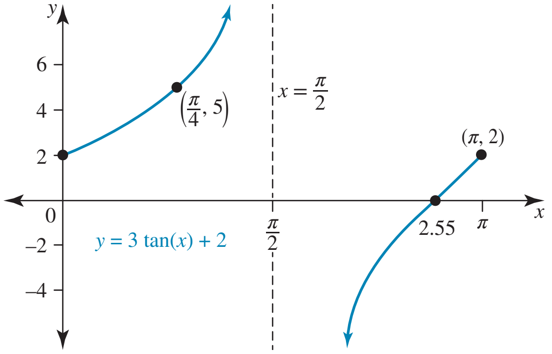
$\left(\frac{\pi}{4}, 5\right)$

End point: let $x = \pi$

$\therefore y = 3\tan(\pi) + 2$

$\therefore y = 2$

$(\pi, 2)$



b $y = 3(1 - \tan(x)), 0 \le x \le \pi$

$\therefore y = 3 - 3\tan(x)$

Period π, asymptote equation $x = \dfrac{\pi}{2}$

The graph is reflected in the x-axis and has a vertical translation 3 upwards, so the y-intercept is $(0, 3)$.

x-intercepts: let $y = 0$

$\therefore 0 = 3 - 3\tan(x)$

$\therefore \tan(x) = 1$

$\therefore x = \dfrac{\pi}{4}$

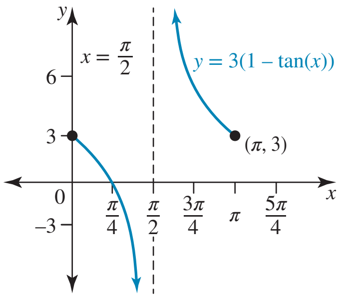
$\left(\dfrac{\pi}{4}, 0\right)$

End point: let $x = \pi$

$\therefore y = 3 - 3\tan(\pi)$

$\therefore y = 3$

$(\pi, 3)$



c $y = \tan(3x) - 1, 0 \le x \le \pi$

The period is $\dfrac{\pi}{3}$.

Asymptotes: when $3x = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{6}$

For $0 \le x \le \pi$, the equations of the asymptotes are $x = \dfrac{\pi}{6}, x = \dfrac{3\pi}{6} = \dfrac{\pi}{2}, x = \dfrac{5\pi}{6}$.

Vertical translation of 1 unit downwards, so the y-intercept is $(0, -1)$.

x-intercepts: let $y = 0$

$\therefore \tan(3x) - 1 = 0$

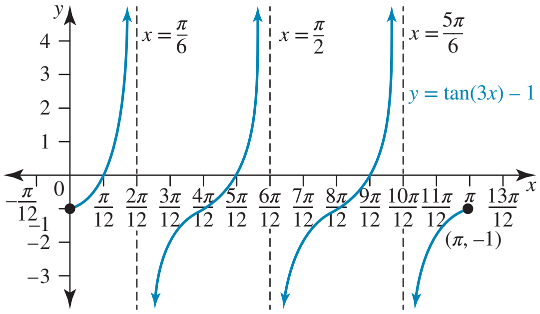
$\therefore \tan(3x) = 1, 0 \le 3x \le 3\pi$

$$\begin{aligned}\therefore 3x &= \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}\\ &= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}\\ \therefore x &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}\end{aligned}$$

$y = \tan(3\pi) - 1$

$= -1$

$(\pi, -1)$



19 $f: [-3, 6]\backslash D \to R, f(x) = \tan\left(\dfrac{\pi x}{3}\right)$

a $f(0) = \tan(0) = 0$

b The function rule is $f(x) = \tan\left(\dfrac{\pi}{3}x\right)$.

The function is not defined at its asymptotes. The asymptotes occur when

$\dfrac{\pi}{3}x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\therefore x = \dfrac{3}{2}, \dfrac{9}{2}$

Subtracting the period, another asymptote occurs when $x = -\dfrac{3}{2}$.

The function is not defined for $x = -\dfrac{3}{2}, \dfrac{3}{2}, \dfrac{9}{2}$, so $D = \left\{-\dfrac{3}{2}, \dfrac{3}{2}, \dfrac{9}{2}\right\}$.

c The period and asymptotes are known.

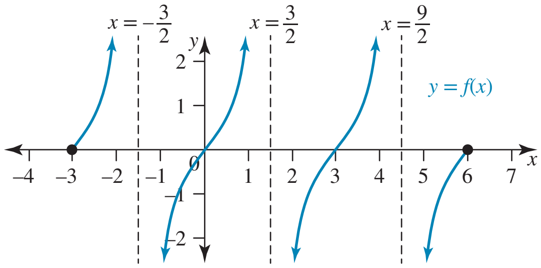
x-intercepts: let $y = 0$

$\therefore \tan\left(\dfrac{\pi}{3}x\right) = 0$

As $-3 \le x \le 6, -\pi \le \dfrac{\pi}{3}x \le 2\pi$.

$\therefore \dfrac{\pi}{3}x = -\pi, 0, \pi, 2\pi$

$\therefore x = -3, 0, 3, 6$



d $f(x) = 1$

$\therefore \tan\left(\dfrac{\pi}{3}x\right) = 1, -\pi \le \dfrac{\pi}{3}x \le 2\pi$

Quadrants 1 and 3, base $\dfrac{\pi}{4}$

$\therefore \dfrac{\pi}{3}x = -\pi + \dfrac{\pi}{4}, \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}$

$\therefore \dfrac{\pi}{3}x = -\dfrac{3\pi}{4}, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$\therefore x = -\dfrac{9}{4}, \dfrac{3}{4}, \dfrac{15}{4}$

20 $h = 12\tan\left(\dfrac{\pi}{144}t\right), 0 \le t \le 48$

a When $t = 24$,

$$\begin{aligned}h &= 12\tan\left(\frac{\pi}{144} \times 24\right)\\ &= 12\tan\left(\frac{\pi}{6}\right)\\ &= 12 \times \frac{\sqrt{3}}{3}\\ &= 4\sqrt{3}\end{aligned}$$

The wave is $4\sqrt{3} \simeq 6.93$ metres high.

b When $h = 12$,

$12 = 12\tan\left(\dfrac{\pi}{144}t\right)$

$$\therefore \tan\left(\frac{\pi}{144}t\right) = 1$$

$$\therefore \frac{\pi}{144}t = \frac{\pi}{4}$$

$$\therefore t = \frac{\pi}{4} \times \frac{144}{\pi}$$

$$\therefore t = 36$$

The wave reaches a height of 12 metres after 36 minutes.

c When $t = 48$,

$$h = 12\tan\left(\frac{\pi}{144} \times 48\right)$$

$$= 12\tan\left(\frac{\pi}{3}\right)$$

$$= 12 \times \sqrt{3}$$

$$= 12\sqrt{3}$$

The peak height is $12\sqrt{3} \simeq 20.78$ metres. This is lower than the peak height of 40.5 metres reached by the Japanese tsunami by approximately 19.72 metres.

d Graph of $h = 12\tan\left(\frac{\pi}{144}t\right), 0 \le t \le 48$

The period is

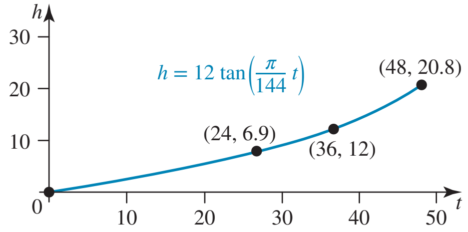
$$\pi \div \frac{\pi}{144}$$

$$= \pi \times \frac{144}{\pi}$$

$$= 144$$

The asymptote at $t = 72$ lies outside the domain specified.

The end points are $(0, 0)$ and $\left(48, 12\sqrt{3}\right)$.



9.5 Exam questions

1 The range is R, so the statement that the range is $[-1, 1]$ is incorrect.

The correct answer is **D**.

2 For $y = \tan(4x)$, $n = 4$

The period is $\frac{\pi}{n}$

$\therefore$ the period is $\frac{\pi}{4}$.

An asymptote occurs when:

$$4x = \frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{8}$$

The other asymptotes in the domain are formed by adding multiples of the period to $x = \frac{\pi}{8}$.

$$x = \frac{\pi}{8} + \frac{\pi}{4}$$

$$= \frac{3\pi}{8}$$

The equations of the asymptotes in the given domain are $x = \frac{\pi}{8}, \frac{3\pi}{8}$.

The correct answer is **A**.

3 $y = 2\tan(x),\ 0 \le x \le 2\pi$

The period is π. [1 mark]

Asymptotes:

An asymptote occurs when $x = \frac{\pi}{2}$.

For the domain $[0, 2\pi]$ and period, other asymptotes occur when

$$x = \frac{\pi}{2} + \pi$$

$$= \frac{3\pi}{2}$$

[1 mark]

x-intercepts:

$2\ \tan(x) = 0$

$x = 0,\ \pi,\ 2\pi$ [1 mark]

A dilation of factor 2 from the x-axis makes the graph of $y = 2\tan(x)$ narrower than $y = \tan(x)$.

When

$$x = \frac{\pi}{4},$$

$$y = 2\ \tan\left(\frac{\pi}{4}\right)$$

$$= 2$$

$$\Rightarrow \left(\frac{\pi}{4}, 2\right)$$

End points:

When $x = 0$,

$$y = 2\tan(0)$$

$$= 0$$

$$\Rightarrow (0, 0)$$

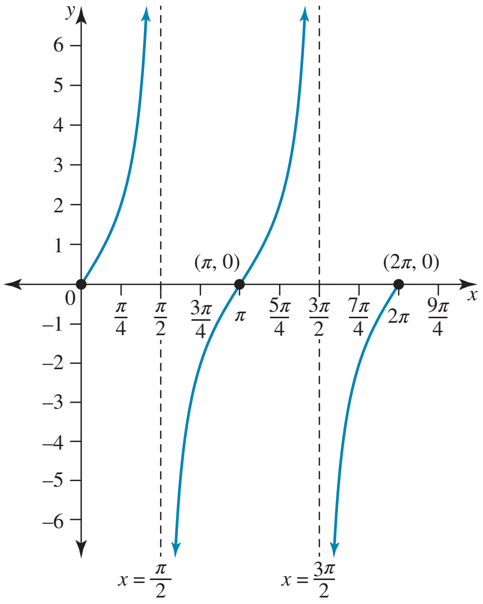
when $x = 2\pi$,

$$y = 2\tan(2\pi)$$

$$= 0$$

$$\Rightarrow (2\pi, 0)$$

[1 mark]



[1 mark]

9.6 Trigonometric identities and properties

9.6 Exercise

1 **a** $\sin^2(45°)$

$$= (\sin(45°))^2$$

$= \left(\dfrac{1}{\sqrt{2}}\right)^2$

$= \dfrac{1}{2}$

b $\cos^2\left(\dfrac{\pi}{3}\right)$

$= \left(\cos\left(\dfrac{\pi}{3}\right)\right)^2$

$= \left(\dfrac{1}{2}\right)^2$

$= \dfrac{1}{4}$

c $\tan^2\left(\dfrac{\pi}{6}\right)$

$= \left(\tan\left(\dfrac{\pi}{6}\right)\right)^2$

$= \left(\dfrac{1}{\sqrt{3}}\right)^2$

$= \dfrac{1}{3}$

d $\cos^2(\pi)$

$= (\cos(\pi))^2$

$= (-1)^2$

$= 1$

2 a The Pythagorean identity is $\sin^2(\theta) + \cos^2(\theta) = 1$.

Let $\theta = 2$.

Then $\sin^2(2) + \cos^2(2) = 1$.

b $\dfrac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$

Let $\theta = 60°$.

Then $\dfrac{\sin(60°)}{\cos(60°)} = \tan(60°)$.

Since $\tan(60°) = \sqrt{3}$,

$\dfrac{\sin(60°)}{\cos(60°)} = \sqrt{3}$.

c $1 - \sin^2(\theta) = \cos^2(\theta)$

Let $\theta = \dfrac{\pi}{2}$.

$1 - \sin^2\left(\dfrac{\pi}{2}\right) = \cos^2\left(\dfrac{\pi}{2}\right)$.

Evaluating,

$\cos^2\left(\dfrac{\pi}{2}\right) = \left(\cos\left(\dfrac{\pi}{2}\right)\right)^2$

$= (0)^2$

$= 0$

d $2\cos^2(D) + 2\sin^2(D)$

Take out the common factor.

$2\cos^2(D) + 2\sin^2(D)$

$= 2\left(\cos^2(D) + \sin^2(D)\right)$

$= 2(1)$

$= 2$

3 a $2 - 2\sin^2(\theta)$

$= 2\left(1 - \sin^2(\theta)\right)$

$= 2\left(\cos^2(\theta)\right)$

$= 2\cos^2(\theta)$

b $\dfrac{\sin^3(A) + \sin(A)\cos^2(A)}{\cos^3(A) + \cos(A)\sin^2(A)} = \tan(A)$

$\text{LHS} = \dfrac{\sin^3(A) + \sin(A)\cos^2(A)}{\cos^3(A) + \cos(A)\sin^2(A)}$

$= \dfrac{\sin(A)\left(\sin^2(A) + \cos^2(A)\right)}{\cos(A)\left(\cos^2(A) + \sin^2(A)\right)}$

$= \dfrac{\sin(A)(1)}{\cos(A)(1)}$

$= \dfrac{\sin(A)}{\cos(A)}$

$= \tan(A)$

$= \text{RHS}$

4 $\tan(u) + \dfrac{1}{\tan(u)}$

Replace $\tan(u)$ with $\dfrac{\sin(u)}{\cos(u)}$.

$\therefore\ \tan(u) + \dfrac{1}{\tan(u)} = \dfrac{\sin(u)}{\cos(u)} + \dfrac{\cos(u)}{\sin(u)}$

$= \dfrac{\sin(u) \times \sin(u)}{\cos(u)\sin(u)} + \dfrac{\cos(u) \times \cos(u)}{\sin(u)\cos(u)}$

$= \dfrac{\sin^2(u) + \cos^2(u)}{\cos(u)\sin(u)}$

$= \dfrac{1}{\cos(u)\sin(u)}$

5 a $4 - 4\cos^2(\theta)$

$= 4\left(1 - \cos^2(\theta)\right)$

$= 4\sin^2(\theta)$

b $\dfrac{2\sin(\alpha)}{\cos(\alpha)}$

$= 2 \times \dfrac{\sin(\alpha)}{\cos(\alpha)}$

$= 2\tan(\alpha)$

c $8\cos^2(\beta) + 8\sin^2(\beta)$

$= 8\left[\cos^2(\beta) + \sin^2(\beta)\right]$

$= 8 \times 1$

$= 8$

d $(1 - \sin(A))(1 + \sin(A))$

Expand the difference of two squares

$= 1^2 - (\sin(A))^2$

$= 1 - \sin^2(A)$

$= \cos^2(A)$

6 a $\tan^2(\theta) + 1 = \dfrac{1}{\cos^2(\theta)}$

$\text{LHS} = \tan^2(\theta) + 1$

$= \left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)^2 + 1$

$= \dfrac{\sin^2(\theta)}{\cos^2(\theta)} + 1$

$= \dfrac{\sin^2(\theta)}{\cos^2(\theta)} + \dfrac{\cos^2(\theta)}{\cos^2(\theta)}$

$= \dfrac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)}$

$= \dfrac{1}{\cos^2(\theta)}$

$= \text{RHS}$

b $\cos^3(\theta) + \cos(\theta)\sin^2(\theta) = \cos(\theta)$

$$\begin{aligned}\text{LHS} &= \cos^3(\theta) + \cos(\theta)\sin^2(\theta)\\ &= \cos(\theta)\left[\cos^2(\theta) + \sin^2(\theta)\right]\\ &= \cos(\theta) \times 1\\ &= \cos(\theta)\\ &= \text{RHS}\end{aligned}$$

c $\dfrac{1}{1-\cos(\theta)} + \dfrac{1}{1+\cos(\theta)} = \dfrac{2}{\sin^2(\theta)}$

$$\begin{aligned}\text{LHS} &= \frac{1}{1-\cos(\theta)} + \frac{1}{1+\cos(\theta)}\\ &= \frac{1+\cos(\theta)+1-\cos(\theta)}{(1-\cos(\theta))(1+\cos(\theta))}\\ &= \frac{2}{1-\cos^2(\theta)}\\ &= \frac{2}{\sin^2(\theta)}\\ &= \text{RHS}\end{aligned}$$

d $(\sin(\theta)+\cos(\theta))^2 + (\sin(\theta)-\cos(\theta))^2 = 2$

$$\begin{aligned}\text{LHS} &= (\sin(\theta)+\cos(\theta))^2 + (\sin(\theta)-\cos(\theta))^2\\ &= \sin^2(\theta) + \cancel{2\sin(\theta)\cos(\theta)} + \cos^2(\theta) + \sin^2(\theta)\\ &\quad -\cancel{2\sin(\theta)\cos(\theta)} + \cos^2(\theta)\\ &= 2(\sin^2(\theta)+\cos^2(\theta))\\ &= 2\\ &= \text{RHS}\end{aligned}$$

7 $\cos(x) = -\dfrac{2}{7}, \dfrac{\pi}{2} \le x \le \pi$, second quadrant

The first quadrant triangle has hypotenuse 7 and adjacent side 2.

The opposite side is $\sqrt{7^2 - 2^2} = \sqrt{45}$.

In the second quadrant, $\sin(x) = \dfrac{3\sqrt{5}}{7}$ and $\tan(x) = -\dfrac{3\sqrt{5}}{2}$.

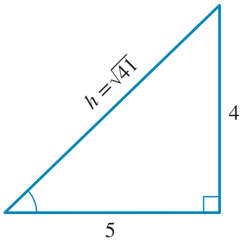
8 $\tan(x) = -3, \pi \le x \le 2\pi$. Since tan is negative, the fourth quadrant is required.

The first quadrant triangle has opposite side 3 and adjacent side 1.

The hypotenuse is $\sqrt{3^2 + 1^2} = \sqrt{10}$.

In the fourth quadrant, $\sin(x) = -\dfrac{3}{\sqrt{10}}$ and $\cos(x) = \dfrac{1}{\sqrt{10}}$.

9 **a** $\tan(x) = -\dfrac{4}{5}, \dfrac{3\pi}{2} \le x \le 2\pi$



In quadrant 1,

$$\begin{aligned}h^2 &= 4^2 + 5^2\\ &= 41\end{aligned}$$

$\therefore h = \sqrt{41}$

In quadrant 4,

$\sin(x) = -\dfrac{4}{\sqrt{41}}, \cos(x) = \dfrac{5}{\sqrt{41}}$.

b $\sin(x) = \dfrac{\sqrt{3}}{2}, \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}$

Quadrant 2

As this is an exact value, $x = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$. (Alternatively, use the method of part **a**.)

i $\cos(x)$

$$\begin{aligned}&= \cos\left(\frac{2\pi}{3}\right)\\ &= -\cos\left(\frac{\pi}{3}\right)\\ &= -\frac{1}{2}\end{aligned}$$

ii $1 + \tan^2(x)$

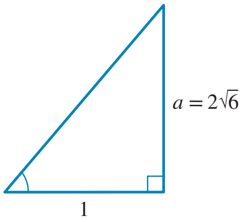
$$\begin{aligned}\tan(x) &= \tan\left(\frac{2\pi}{3}\right)\\ &= -\tan\left(\frac{\pi}{3}\right)\\ &= -\sqrt{3}\end{aligned}$$

$\therefore 1 + \tan^2(x) = 1 + 3 = 4$

c $\cos(x) = -0.2, \pi \le x \le \dfrac{3\pi}{2}$

$\therefore \cos(x) = -\dfrac{1}{5}$

Quadrant 3



In quadrant 1,

$a^2 + 1^2 = 5^2$

$\therefore a^2 = 24$

$\therefore a = 2\sqrt{6}$

i $\tan(x) = 2\sqrt{6}$

ii $1 - \sin^2(x)$

$$\begin{aligned}&= 1 - (\sin(x))^2\\ &= 1 - \left(-\frac{2\sqrt{6}}{5}\right)^2\\ &= 1 - \frac{24}{25}\\ &= \frac{1}{25}\end{aligned}$$

iii $\sin(x)\tan(x)$

$$\begin{aligned}&= -\frac{2\sqrt{6}}{5} \times 2\sqrt{6}\\ &= -\frac{24}{5}\end{aligned}$$

10 **a** **i** $\cos(x) = \dfrac{2\sqrt{3}}{5}, 0 \le x \le \dfrac{\pi}{2}$

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1\\ \therefore \sin^2(x) + \left(\frac{2\sqrt{3}}{5}\right)^2 &= 1\\ \therefore \sin^2(x) + \frac{12}{25} &= 1\\ \therefore \sin^2(x) &= \frac{13}{25}\end{aligned}$$

As x lies in the first quadrant, $\sin(x) > 0$

$\therefore \sin(x) = \frac{\sqrt{13}}{5}$

ii $\cos(x) = \frac{2\sqrt{3}}{5}, \frac{3\pi}{2} \le x \le 2\pi$

As x lies in the fourth quadrant, $\sin(x) < 0$

$\therefore \sin(x) = -\frac{\sqrt{13}}{5}$

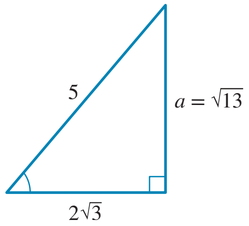
b $\tan(x) = \frac{\sin(x)}{\cos(x)}$

For x in the first quadrant,

$$\begin{aligned}\tan(x) &= \frac{\sqrt{13}}{5} \div \frac{2\sqrt{3}}{5}\\ &= \frac{\sqrt{13}}{5} \times \frac{5}{2\sqrt{3}}\\ &= \frac{\sqrt{13}}{2\sqrt{3}}\\ &= \frac{\sqrt{39}}{6}\end{aligned}$$

For x in the fourth quadrant, $\tan(x) = -\frac{\sqrt{39}}{6}$.

c



In quadrant 1,

$$\begin{aligned}a^2 + \left(2\sqrt{3}\right)^2 &= 5^2\\ \therefore a^2 &= 25 - 12\\ \therefore a &= \sqrt{13}\end{aligned}$$

$\therefore \sin(x) = \frac{\sqrt{13}}{5}, \tan(x) = \frac{\sqrt{13}}{2\sqrt{3}} = \frac{\sqrt{39}}{6}$. This agrees with the answers in part **a**.

11 $\sin(\theta) = 0.8, 0 \le \theta \le \frac{\pi}{2}$

a $\sin(\pi + \theta)$

$= -\sin(\theta)$

$= -0.8$

b $\cos\left(\frac{\pi}{2} - \theta\right)$

$= \sin(\theta)$

$= 0.8$

c $\cos(\theta)$

Since $\cos^2(\theta) + \sin^2(\theta) = 1$,

$$\begin{aligned}\cos^2(\theta) &= 1 - \sin^2(\theta)\\ &= 1 - (0.8)^2\\ &= 1 - 0.64\\ &= 0.36\end{aligned}$$

As $0 \le \theta \le \frac{\pi}{2}, \cos(\theta) > 0$

$\therefore \cos(\theta) = \sqrt{0.36}$

$\therefore \cos(\theta) = 0.6$

d $\tan(\pi + \theta) = \frac{4}{3}$

12 a $\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta)$

Let $\theta = \frac{\pi}{6}$

$$\begin{aligned}\text{LHS} &= \sin\left(\frac{\pi}{2} + \theta\right)\\ &= \sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right)\\ &= \sin\left(\frac{4\pi}{6}\right)\\ &= \sin\left(\frac{2\pi}{3}\right)\\ &= \sin\left(\frac{\pi}{3}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \cos\left(\frac{\pi}{6}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$\therefore$ LHS = RHS, so the result is verified.

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$ for $\theta = \frac{\pi}{6}$

$$\begin{aligned}\text{LHS} &= \cos\left(\frac{\pi}{2} + \theta\right)\\ &= \cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right)\\ &= \cos\left(\frac{2\pi}{3}\right)\\ &= -\cos\left(\frac{\pi}{3}\right)\\ &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= -\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{1}{2}\end{aligned}$$

$\therefore \text{LHS} = \text{RHS} = -\frac{1}{2}$

c $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos(\theta)$

Let $\theta = \frac{\pi}{3}$

$$\begin{aligned}\text{LHS} &= \sin\left(\frac{3\pi}{2} - \theta\right)\\ &= \sin\left(\frac{3\pi}{2} - \frac{\pi}{3}\right)\\ &= \sin\left(\frac{7\pi}{6}\right)\\ &= -\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= -\cos\left(\frac{\pi}{3}\right)\\ &= -\frac{1}{2}\end{aligned}$$

$\therefore \text{LHS} = \text{RHS} = -\frac{1}{2}$

d $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin(\theta)$

Let $\theta=\frac{\pi}{4}$

$$\begin{aligned}\text{LHS}&=\cos\left(\frac{3\pi}{2}+\theta\right)\\&=\cos\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)\\&=\cos\left(\frac{7\pi}{4}\right)\\&=\cos\left(\frac{\pi}{4}\right)\\&=\frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\text{RHS}&=\sin\left(\frac{\pi}{4}\right)\\&=\frac{\sqrt{2}}{2}\end{aligned}$$

$\therefore \text{LHS}=\text{RHS}=\frac{\sqrt{2}}{2}$

13 a $\cos\left(\frac{3\pi}{2}-\theta\right)$ is in the third quadrant so cosine is negative.

The base is $\frac{\pi}{2}-\theta$.

Therefore,

$$\begin{aligned}\cos\left(\frac{3\pi}{2}-\theta\right)&=-\cos\left(\frac{\pi}{2}-\theta\right)\\&=-\sin(\theta)\end{aligned}$$

b Use the complement.

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right)&=\cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right)\\&=\cos\left(\frac{5\pi}{12}\right)\end{aligned}$$

Therefore, $t=\frac{5\pi}{12}$.

14 $\sin\left(\frac{\pi}{2}+\theta\right)+\sin(\pi+\theta)$

$\sin\left(\frac{\pi}{2}+\theta\right)=\sin\left(\frac{\pi}{2}-\theta\right)$ using symmetry property of second quadrant

$=\cos(\theta)$ using complementary property

$\sin(\pi+\theta)=-\sin(\theta)$ using symmetry property of third quadrant.

Therefore, $\sin\left(\frac{\pi}{2}+\theta\right)+\sin(\pi+\theta)=\cos(\theta)-\sin(\theta)$.

15 a $\cos\left(\frac{3\pi}{2}+\theta\right)$ fourth quadrant.

$=\cos\left(\frac{\pi}{2}-\theta\right)$

$=\sin(\theta)$

b $\cos\left(\frac{3\pi}{2}-\theta\right)$ third quadrant

$=-\cos\left(\frac{\pi}{2}-\theta\right)$

$=-\sin(\theta)$

c $\sin\left(\frac{\pi}{2}+\theta\right)$ second quadrant

$=\sin\left(\frac{\pi}{2}-\theta\right)$

$=\cos(\theta)$

d $\sin\left(\frac{5\pi}{2}-\theta\right)$

$=\sin\left(2\pi+\left(\frac{\pi}{2}-\theta\right)\right)$

$=\sin\left(\frac{\pi}{2}-\theta\right)$

$=\cos(\theta)$

16 a $\frac{\cos(90°-a°)}{\cos(a°)}$

$=\frac{\sin(a°)}{\cos(a°)}$

$=\tan(a°)$

b $1-\sin^2\left(\frac{\pi}{2}-\theta\right)$

$=1-\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]^2$

$=1-\left[\cos(\theta)\right]^2$

$=1-\cos^2(\theta)$

$=\sin^2(\theta)$

c $\cos(\pi-x)+\sin\left(x+\frac{\pi}{2}\right)$

$=-\cos(x)+\sin\left(\frac{\pi}{2}+x\right)$

$=-\cos(x)+\cos(x)$

$=0$

d $\frac{1-\cos^2\left(\frac{\pi}{2}-\theta\right)}{\cos^2(\theta)}$

$=\frac{1-\left[\cos\left(\frac{\pi}{2}-\theta\right)\right]^2}{\cos^2(\theta)}$

$=\frac{[1-\sin(\theta)]^2}{\cos^2(\theta)}$

$=\frac{1-\sin^2(\theta)}{\cos^2(\theta)}$

$=\frac{\cos^2(\theta)}{\cos^2(\theta)}$

$=1$

17 a $\cos(a°)=\sin(27°)$

$\therefore \cos(a°)=\cos(90°-27°)$

$\therefore \cos(a°)=\cos(63°)$

$\therefore a=63°$

b $\sin(b)=\cos\left(\frac{\pi}{9}\right)$

$\therefore \sin(b)=\sin\left(\frac{\pi}{2}-\frac{\pi}{9}\right)$

$\therefore \sin(b)=\sin\left(\frac{7\pi}{18}\right)$

$\therefore b=\frac{7\pi}{18}$

c $\sin(c)=\cos(0.35)$

$\therefore \sin(c)=\sin\left(\frac{\pi}{2}-0.35\right)$

$\therefore c=\frac{\pi}{2}-0.35$

$\therefore c=1.22$

d $\cos(d)=\sin\left(\frac{\pi}{4}\right)$

$\therefore \cos(d)=\cos\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$

$\therefore d=\frac{\pi}{4}$

Alternatively,

$\cos(d) = \sin\left(\frac{\pi}{4}\right)$

$\therefore \cos(d) = \cos\left(\frac{3\pi}{2} + \frac{\pi}{4}\right)$

$\therefore d = \frac{7\pi}{4}$

18 a $\cos^2(x) + 3\sin(x) - 1 = 0, 0 \le x \le 2\pi$

As $\cos^2(x) = 1 - \sin^2(x)$, substitute this in the equation.

$\therefore 1 - \sin^2(x) + 3\sin(x) - 1 = 0$

$\therefore -\sin^2(x) + 3\sin(x) = 0$

$\therefore \sin(x)\,[-\sin(x) + 3] = 0$

$\therefore \sin(x) = 0$ or $\sin(x) = 3$

Reject $\sin(x) = 3$ since $\sin(x) \in [-1, 1]$.

$\therefore \sin(x) = 0$ Boundary solutions at $(1, 0)$ and $(-1, 0)$

$\therefore x = 0, \pi, 2\pi$

b $3\cos^2(x) + 6\sin(x) + 3\sin^2(x) = 0, 0 \le x \le 2\pi$

$\therefore 3\left[\cos^2(x) + \sin^2(x)\right] + 6\sin(x) = 0$

$\therefore 3\,[1] + 6\sin(x) = 0$

$\therefore 6\sin(x) = -3$

$\therefore \sin(x) = -\frac{1}{2}$

Quadrants 3 and 4, base $\frac{\pi}{6}$

$\therefore x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6}$

c $\frac{1 - \cos(x)}{1 + \sin(x)} \times \frac{1 + \cos(x)}{1 - \sin(x)} = 3, 0 \le x \le 2\pi$

$\therefore \frac{(1 - \cos(x))(1 + \cos(x))}{(1 + \sin(x))(1 - \sin(x))} = 3$

$\therefore \frac{1 - \cos^2(x)}{1 - \sin^2(x)} = 3$

$\therefore \frac{\sin^2(x)}{\cos^2(x)} = 3$

$\therefore \tan^2(x) = 3$

$\therefore \tan(x) = \pm\sqrt{3}$

Quadrants 1, 2, 3 and 4, base $\frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

d $\sin^2(x) = \cos(x) + 1$

$\therefore 1 - \cos^2(x) = \cos(x) + 1$

$\therefore -\cos^2(x) = \cos(x)$

$\therefore \cos(x) + \cos^2(x) = 0$

$\therefore \cos(x)\,(1 + \cos(x)) = 0$

$\therefore \cos(x) = 0$ or $\cos(x) = -1$

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $x = \pi$

$\therefore x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

19 a $1 - \cos^2(40°) = \cos^2(50°)$

$\text{LHS} = 1 - \cos^2(40°)$

$= \sin^2(40°)$

$= [\sin(40°)]^2$

$= [\cos(50°)]^2$

$= \cos^2(50°)$

$= \text{RHS}$

b $\frac{\cos^3(x) - \sin^3(x)}{\cos(x) - \sin(x)} = 1 + \sin(x)\cos(x)$

Recall that the factors of the difference of two cubes are $a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$.

$\text{LHS} = \frac{\cancel{[\cos(x) - \sin(x)]}\left[\cos^2(x) + \cos(x)\sin(x) + \sin^2(x)\right]}{\cancel{\cos(x) - \sin(x)}}$

$= \cos^2(x) + \cos(x)\sin(x) + \sin^2(x)$

$= \left(\cos^2(x) + \sin^2(x)\right) + \cos(x)\sin(x)$

$= 1 + \sin(x)\cos(x)$

$= \text{RHS}$

c $\frac{\sin^3(\theta) + \cos\left(\frac{\pi}{2} - \theta\right)\cos^2(\theta)}{\cos(\theta)} = \tan(\theta)$

$\text{LHS} = \frac{\sin^3(\theta) + \cos\left(\frac{\pi}{2} - \theta\right)\cos^2(\theta)}{\cos(\theta)}$

$= \frac{\sin^3(\theta) + \sin(\theta)\cos^2(\theta)}{\cos(\theta)}$

$= \frac{\sin(\theta)\left[\sin^2(\theta) + \cos^2(\theta)\right]}{\cos(\theta)}$

$= \frac{\sin(\theta) \times 1}{\cos(\theta)}$

$= \frac{\sin(\theta)}{\cos(\theta)}$

$= \tan(\theta)$

$= \text{RHS}$

d $\cos(2\pi - A)\cos(2\pi + A) - \cos\left(\frac{3\pi}{2} - A\right)$

$\cos\left(\frac{3\pi}{2} + A\right) = 1$

$\text{LHS} = \cos(2\pi - A)\cos(2\pi + A)$

$- \cos\left(\frac{3\pi}{2} - A\right)\cos\left(\frac{3\pi}{2} + A\right)$

$= \cos(A)\cos(A) - (-\sin(A))(\sin(A))$

$= \cos^2(A) + \sin^2(A)$

$= 1$

$= \text{RHS}$

20 a Using CAS technology gives the result that $\sin(5.5\pi - x) \div \cos\left(x - \frac{\pi}{2}\right) = \frac{1}{\tan(x)}$.

b Using CAS technology gives the result $\sin(3x)\cos(2x) - 2\cos^2(x)\sin(3x) = -\sin(3x)$.

9.6 Exam questions

1 The relationship $\cos^2(\theta) + \sin^2(\theta) = 1$ is known as the Pythagorean identity. It is a true statement for any value of θ.

$\cos^2(x) = 1 - \sin^2(x)$

$\cos(x) = \sqrt{1 - \sin^2(x)}$

The correct answer is **D**.

2 Since $\sin(\theta) = \frac{3}{5}$, the hypotenuse of a right-angled triangle is 5 and the side opposite θ is 3. The remaining side is 4, since 3, 4, 5 form a right-angled triangle; $\therefore \cos(\theta) = \frac{4}{5}$.

$\cos(\theta) = \frac{4}{5}, \left(0 < \theta < \frac{\pi}{2}\right)$

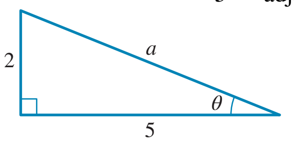
The correct answer is **A**.

3 $\tan(x) = -\frac{2}{5}, 0 \le x \le \pi$

The sign of $\tan(x)$ indicates either quadrant 2 or 4.

The condition that $0 \le x \le \pi$ makes it quadrant 2. [1 mark]

Quadrant 1: $\tan(x) = \frac{2}{5} = \frac{\text{opposite}}{\text{adjacent}}$



[1 mark]

Using Pythagoras' theorem,

$2^2 + 5^2 = a^2$

$\therefore a = \sqrt{29}$ (positive root required) [1 mark]

In quadrant 2, sine is positive and cosine is negative.

$\sin(x) = +\frac{2}{\sqrt{29}}$ and $\cos(x) = -\frac{5}{\sqrt{29}}$ [1 mark]

9.7 Review

9.7 Exercise

Technology free: short answer

1 a $\sqrt{6}\cos(x) = -\sqrt{3}, 0 \le x \le 2\pi$

$\therefore \cos(x) = -\frac{\sqrt{3}}{\sqrt{6}}$

$\therefore \cos(x) = -\frac{1}{\sqrt{2}}$

Quadrants 2 and 3, base $\frac{\pi}{4}$

$\therefore x = \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$\therefore x = \frac{3\pi}{4}, \frac{5\pi}{4}$

b $2 - 2\cos(x) = 0, 0 \le x \le 4\pi$

$\therefore 2 = 2\cos(x)$

$\therefore \cos(x) = 1$

Boundary value at the Cartesian point $(1, 0)$, two positive rotations

$\therefore x = 0, 2\pi, 4\pi$

c $2\sin(x) = \sqrt{3}, -2\pi \le x \le 2\pi$

$\therefore \sin(x) = \frac{\sqrt{3}}{2}$

Quadrants 1 and 2, base $\frac{\pi}{3}$, one negative rotation and one positive rotation

$\therefore x = -\pi - \frac{\pi}{3}, -2\pi + \frac{\pi}{3}$ or $x = \frac{\pi}{3}, \pi - \frac{\pi}{3}$

$\therefore x = -\frac{4\pi}{3}, -\frac{5\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

d $\sqrt{5}\sin(x) = \sqrt{5}\cos(x), 0 \le x \le 2\pi$

$\therefore \frac{\sin(x)}{\cos(x)} = \frac{\sqrt{5}}{\sqrt{5}}$

$\therefore \tan(x) = 1$

Quadrants 1 and 3, base $\frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

2 a $8\sin(3x) + 4\sqrt{2} = 0, -\pi \le x \le \pi$

$\therefore 8\sin(3x) = -4\sqrt{2}$

$\therefore \sin(3x) = -\frac{4\sqrt{2}}{8}$

$\therefore \sin(3x) = -\frac{\sqrt{2}}{2}$

As $-\pi \le x \le \pi$ then $-3\pi \le 3x \le 3\pi$.

Quadrants 3 and 4, base $\frac{\pi}{4}$

The negative solutions are:

$\therefore 3x = -\frac{\pi}{4}, -\pi + \frac{\pi}{4}, -2\pi - \frac{\pi}{4}, -3\pi + \frac{\pi}{4}$

$\therefore 3x = -\frac{\pi}{4}, -\frac{3\pi}{4}, -\frac{9\pi}{4}, -\frac{11\pi}{4}$

$\therefore x = -\frac{\pi}{12}, -\frac{3\pi}{12}, -\frac{9\pi}{12}, -\frac{11\pi}{12}$

$\therefore x = -\frac{\pi}{12}, -\frac{\pi}{4}, -\frac{3\pi}{4}, -\frac{11\pi}{12}$

The positive solutions are:

$\therefore 3x = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\therefore 3x = \frac{5\pi}{4}, \frac{7\pi}{4}$

$\therefore x = \frac{5\pi}{12}, \frac{7\pi}{12}$

The solutions are $x = -\frac{11\pi}{12}, -\frac{3\pi}{4}, -\frac{\pi}{4}, -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$.

b $\tan(2x^\circ) = -2, 0 \le x \le 270$

Quadrants 2 and 4, base, in degrees, $\tan^{-1}(2) \ne 63.43^\circ$

Since $0 \le x \le 270$, $0 \le 2x \le 540$. Solutions are generated by one and a half rotations.

$\therefore 2x^\circ = 180^\circ - 63.43^\circ, 360^\circ - 63.43^\circ, 540^\circ - 63.43^\circ$

$\therefore 2x^\circ = 116.57^\circ, 296.57^\circ, 476.57^\circ$

$\therefore x^\circ = 58.28^\circ, 148.28^\circ, 238.28^\circ$

$\therefore x = 58.28, 148.28, 238.28$

c $2\sin^2(x) - \sin(x) - 3 = 0, 0 \le x \le 2\pi$

$\therefore (2\sin(x) - 3)(\sin(x) + 1) = 0$

$\therefore \sin(x) = \frac{3}{2}$ or $\sin(x) = -1$

Reject $\sin(x) = \frac{3}{2}$ since $-1 \le \sin(x) \le 1$.

$\therefore \sin(x) = -1$

Boundary value at the point $(0, -1)$

$\therefore x = \frac{3\pi}{2}$

d $2\sin^2(x) - 3\cos(x) - 3 = 0, 0 \le x \le 2\pi$

Substitute $1 - \cos^2(x)$ for $\sin^2(x)$.

$\therefore 2(1 - \cos^2(x)) - 3\cos(x) - 3 = 0$

$\therefore 2 - 2\cos^2(x) - 3\cos(x) - 3 = 0$

$\therefore 2\cos^2(x) + 3\cos(x) + 1 = 0$

$\therefore (2\cos(x) + 1)(\cos(x) + 1) = 0$

$\therefore \cos(x) = -\frac{1}{2}$ or $\cos(x) = -1$

$\therefore x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$ or $x = \pi$

$\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$

3 $\cos(\theta) = p$

a $\sin\left(\frac{\pi}{2} - \theta\right)$

$= \cos(\theta)$

$= p$

b $\sin\left(\frac{3\pi}{2} + \theta\right)$ Fourth quadrant, complementary property

$= -\cos(\theta)$

$= -p$

c $\sin^2(\theta)$

$= 1 - \cos^2(\theta)$

$= 1 - p^2$

d $\cos(5\pi - \theta)$ The period of cosine is 2π

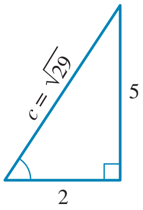
$= \cos(\pi - \theta)$ Second quadrant, symmetry property

$= -\cos(\theta)$

$= -p$

4 **a** $\tan(x) = \frac{5}{2}, x \in \left(\pi, \frac{3\pi}{2}\right)$

Third quadrant



In the first quadrant,

$c^2 = 5^2 + 2^2$

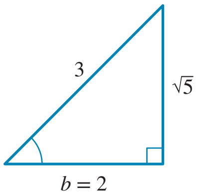
$= 29$

$\therefore c = \sqrt{29}$

In the third quadrant, $\cos(x) = -\frac{2}{\sqrt{29}}$ and $\sin(x) = -\frac{5}{\sqrt{29}}$.

b $\sin(y) = -\frac{\sqrt{5}}{3}, y \in \left(\frac{3\pi}{2}, 2\pi\right)$

Fourth quadrant



In the first quadrant,

$b^2 + \left(\sqrt{5}\right)^2 = 3^2$

$\therefore b^2 + 5 = 9$

$\therefore b^2 = 4$

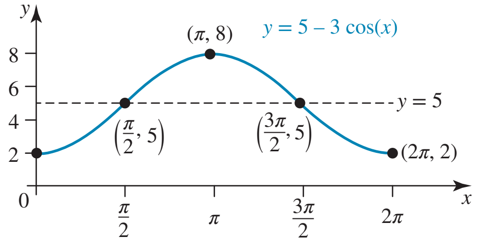
$\therefore b^2 = 2$

In the fourth quadrant, $\cos(y) = \frac{2}{3}$ and $\tan(y) = -\frac{\sqrt{5}}{2}$.

5 **a** $y = 5 - 3\cos(x), 0 \le x \le 2\pi$

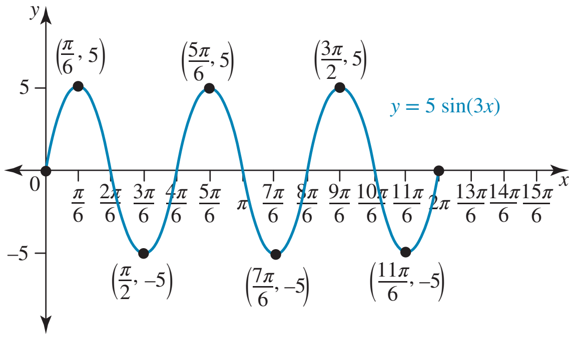
Amplitude 3, inverted, equilibrium $y = 5$, period 2π

The range is $[5 - 3, 5 + 3] = [2, 8]$, no x-intercepts.



b $y = 5\sin(3x), 0 \le x \le 2\pi$

Amplitude 5, equilibrium $y = 0$, range $[-5, 5]$, period $\frac{2\pi}{3}$



c $y = -4\tan(x), 0 \le x \le 2\pi$

Period π

Asymptotes: $x = \frac{\pi}{2}, \frac{3\pi}{2}$

x-intercepts occur midway between successive pairs of asymptotes at $x = 0, \pi, 2\pi$.

The graph is reflected in the x-axis.

Point: let $x = \frac{\pi}{4}$

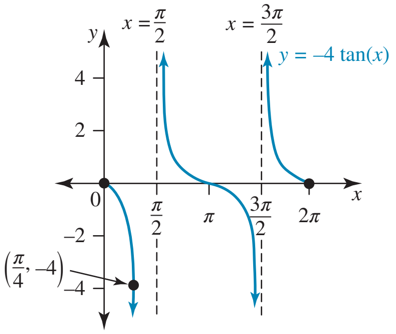
$y = -4\tan\left(\frac{\pi}{4}\right)$

$= -4 \times 1$

$= -4$

$\left(\frac{\pi}{4}, -4\right)$

The range is R.



d $y = 1 + 2\cos\left(\frac{x}{2}\right), -2\pi \le x \le 3\pi$

Amplitude 2, equilibrium $y = 1$, range $[1 - 2, 1 + 2] = [-1, 3]$

Period $2\pi \div \frac{1}{2} = 2\pi \times \frac{2}{1} = 4\pi$

End points: when $x = -2\pi$,

$y = 1 + 2\cos(-\pi)$

$= 1 + 2 \times -1$

$= -1$

$(-2\pi, -1)$

When $x = 3\pi$,

$y = 1 + 2\cos\left(\frac{3\pi}{2}\right)$

$= 1 + 2 \times 0$

$= 1$

$(3\pi, 1)$

x-intercepts: let $y = 0$.

$\therefore 1 + 2\cos\left(\frac{x}{2}\right) = 0$

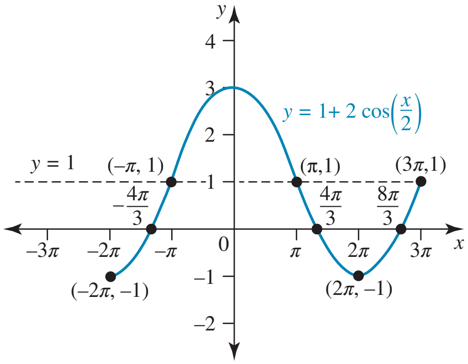
$\therefore \cos\left(\frac{x}{2}\right) = -\frac{1}{2}, -\pi \le \frac{x}{2} \le \frac{3\pi}{2}$

Quadrants 2 and 3, base $\frac{\pi}{3}$

$\therefore \frac{x}{2} = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$ or $\frac{x}{2} = -\pi + \frac{\pi}{3}$

$\therefore \frac{x}{2} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{-2\pi}{3}$

$\therefore x = \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{-4\pi}{3}$



6 a $\sin^2(4x) + \cos^2(4x) = 1$

b $\cos^2(A)(1 + 2\tan(A)) + \sin^2(A) = (\cos(A) + \sin(A))^2$

LHS $= \cos^2(A) + \cos^2(A) \times 2\tan(A) + \sin^2(A)$

$= \cos^2(A) + \cos^2(A) \times \frac{2\sin(A)}{\cos(A)} + \sin^2(A)$

$= \cos^2(A) + \cos(A) \times 2\sin(A) + \sin^2(A)$

$= \cos^2(A) + 2\cos(A)\sin(A) + \sin^2(A)$

$= (\cos(A) + \sin(A))^2$

$=$ RHS

c $(1 + \cos(\theta))\left(1 - \sin\left(\theta + \frac{\pi}{2}\right)\right)$

$= (1 + \cos(\theta))\left(1 - \sin\left(\frac{\pi}{2} + \theta\right)\right)$

$= (1 + \cos(\theta))(1 - \cos(\theta))$

$= 1 - \cos^2(\theta)$

$= \sin^2(\theta)$

Technology active: multiple choice

7 As $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$, the equation $f(x) = 5$ will only have solutions if f is the tangent function as $\tan(x) \in R$.
The correct answer is **C**.

8 The equation $\cos(x) = 0, -\frac{\pi}{2} \le x \le \frac{5\pi}{2}$ has solutions at the x-intercepts of the graph of $y = \cos(x)$. The x-intercepts occur at every odd multiple of $\frac{\pi}{2}$. Over the given interval, there would be four x-intercepts, so the equation has four solutions.
The correct answer is **D**.

9 $\sin(x) = \frac{1}{2}, 0 \le x \le 2\pi$

Quadrants 1 and 2, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

The correct answer is **A**.

10 $y = -4\cos\left(\frac{x}{4}\right)$

Amplitude 4, period 8π

The correct answer is **E**.

11 The period of the graph is π, its equilibrium position is $y = 0$ and its amplitude is 3.

Let the equation be that of the inverted sine function.

$y = -a\sin(nx)$

$\therefore y = -3\sin(nx)$

Period $\frac{2\pi}{n} = \pi \Rightarrow n = 2$

$\therefore y = -3\sin(2x)$

The correct answer is **C**.

12 $y = 2 + 4\cos\left(\frac{3x}{2}\right)$

The equilibrium is $y = 2$ and the amplitude is 4.

Therefore, the range is $[2 - 4, 2 + 4] = [-2, 6]$

The correct answer is **C**.

13 The period of the graph of $y = 2\tan(3x)$ is $\frac{\pi}{n}, n = 3$. The period is $\frac{\pi}{3}$.

The correct answer is **A**.

14 The vertical asymptotes of the graph of $y = \tan(x)$ occur at odd multiples of $\frac{\pi}{2}$. Over the interval $x \in (-\pi, 2\pi)$ there are three odd multiples of $\frac{\pi}{2}$, so there would be three vertical asymptotes.

The correct answer is **C**.

15 $\sqrt{9 - 9\sin^2(\theta)}, 0 \le \theta \le \frac{\pi}{2}$.

$\sqrt{9 - 9\sin^2(\theta)} = \sqrt{9(1 - \sin^2(\theta))}$

$= \sqrt{9\cos^2(\theta)}$

$= 3\cos(\theta)$

The correct answer is **D**.

16 $\cos\left(\frac{\pi}{2} + x\right)$

Second quadrant, complementary property

$\cos\left(\frac{\pi}{2} + x\right) = -\sin(x)$

The correct answer is **B**.

Technology active: extended response

17 $f: [-12, 18] \to R, f(x) = 10\sin\left(\frac{\pi x}{6}\right) + 3$

a i Maximum value: $f_{\max}(x) = 10 \times 1 + 3 = 13$

Minimum value: $f_{\min}(x) = 10 \times -1 + 3 = -7$

ii Maximum occurs when $\sin\left(\frac{\pi x}{6}\right) = 1$

$\therefore \frac{\pi x}{6} = \frac{\pi}{2}$

$\therefore x = 3$

Minimum occurs when $\sin\left(\frac{\pi x}{6}\right) = -1$

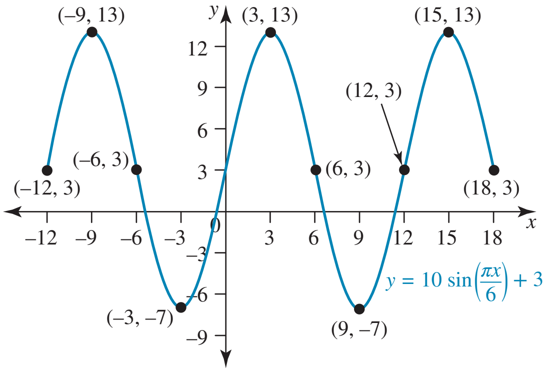
$$\therefore \frac{\pi x}{6} = \frac{3\pi}{2}$$
$$\therefore x = 9$$

b $f(-12) = 10\sin\left(\frac{\pi(-12)}{6}\right) + 3$
$= 10\sin(-2\pi) + 3$
$= 10 \times 0 + 3$
$= 3$

$f(18) = 10\sin\left(\frac{\pi(18)}{6}\right) + 3$
$= 10\sin(3\pi) + 3$
$= 10\sin(\pi) + 3$
$= 10 \times 0 + 3$
$= 3$

c $f(x) = 10\sin\left(\frac{\pi x}{6}\right) + 3$

Amplitude 10, equilibrium $y = 3$, period $2\pi \div \frac{\pi}{6} = 12$



d The graph shows there are 4 intercepts with the x-axis, which means there are 4 solutions to the equation $f(x) = 0$: two positive solutions and two negative solutions.

When $f(x) = 0$,

$10\sin\left(\frac{\pi x}{6}\right) + 3 = 0$

$\therefore 10\sin\left(\frac{\pi x}{6}\right) = -3$

$\therefore \sin\left(\frac{\pi x}{6}\right) = -0.3$

Quadrants 3 and 4, base $\sin^{-1}(0.3) \simeq 0.30$

$\therefore \frac{\pi x}{6} = \pi + 0.30,\ 2\pi - 0.30 \text{ or } -0.30, -\pi + 0.30$

$\therefore x = \frac{6}{\pi} \times (\pi + 0.30),\ \frac{6}{\pi} \times (2\pi - 0.30),$
$\frac{6}{\pi} \times -0.30,\ \frac{6}{\pi} \times (-\pi + 0.30)$

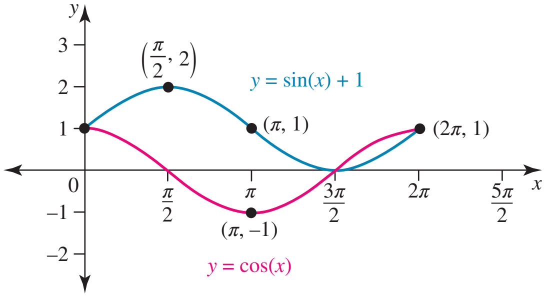
$\therefore x = 6.6,\ 11.4,\ -0.6,\ -5.4$

$\therefore x = -5.4,\ -0.6,\ 6.6,\ 11.4$

18 a $\cos(x) = \sin(x) + 1$

$y = \cos(x)$ has period 2π and range $[-1, 1]$.

$y = \sin(x) + 1$ has period 2π and as it oscillates about $y = 1$, its range is $[0, 2]$.



As the graphs intersect at the three points $(0, 1), \left(\frac{3\pi}{2}, 0\right), (2\pi, 1)$, the solutions to the equation $\cos(x) = \sin(x) + 1$ over the given domain are $x = 0, \frac{3\pi}{2}, 2\pi$.

b i $\cos(x) = \tan(x), x \in [0, 2\pi]$

Sketch the graphs of $y = \cos(x)$ and $y = \tan(x)$ for $x \in [0, 2\pi]$.

$y = \tan(x)$ has period π, asymptotes at $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and x-intercepts when $x = 0, \pi, 2\pi$.

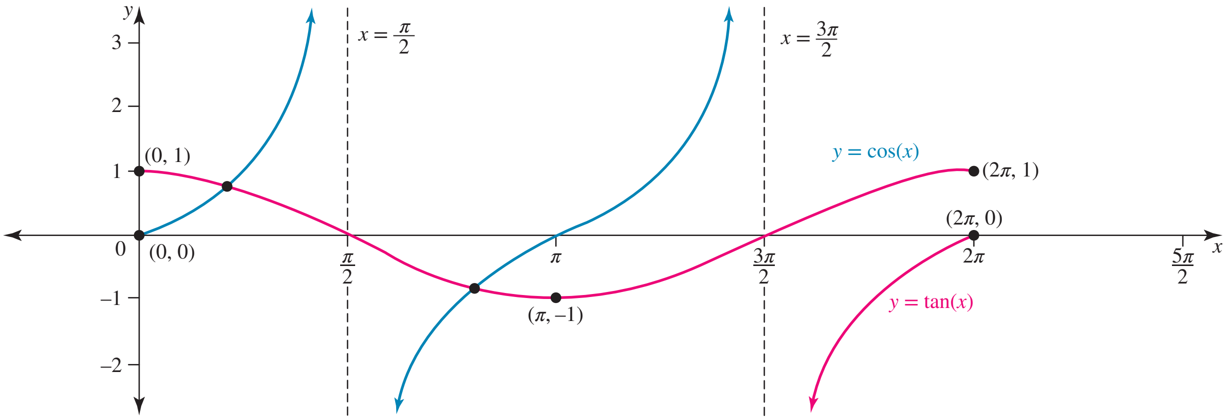
When $x = \frac{\pi}{4}, y = \tan\left(\frac{\pi}{4}\right) = 1 \Rightarrow \left(\frac{\pi}{4}, 1\right)$ lies on the graph.

See the image at the bottom of the page.*

There are two intersection points, so there are two solutions to the equation $\cos(x) = \tan(x)$ over the given interval.

To show one root lies in the interval $0.6 \le x \le 0.8$:

***18 b i**



When $x = 0.6$, $\cos(0.6) \simeq 0.825$ and $\tan(0.6) \simeq 0.684$.
The graph of $y = \cos(x)$ lies above the graph of $y = \tan(x)$.

When $x = 0.8$, $\cos(0.8) \simeq 0.697$ and $\tan(0.8) \simeq 1.03$.
The graph of $y = \cos(x)$ lies below the graph of $y = \tan(x)$.

Therefore, at some value of x between 0.6 and 0.8, the graphs have crossed each other.

There is a root that lies between 0.6 and 0.8.

ii $\cos(x) = \tan(x)$

$$\therefore \cos(x) = \frac{\sin(x)}{\cos(x)}$$

$$\therefore \cos^2(x) = \sin(x)$$

$$\therefore 1 - \sin^2(x) = \sin(x)$$

$$\therefore \sin^2(x) + \sin(x) - 1 = 0$$

Using the formula for quadratic equations,

$$\sin(x) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a = 1, b = 1, c = -1$$

$$\therefore \sin(x) = \frac{-1 \pm \sqrt{5}}{2}$$

As $0 < x < \frac{\pi}{2}$, $\sin(x) > 0$

$$\therefore \sin(x) = \frac{-1 + \sqrt{5}}{2}$$

$$\therefore \sin(x) = \frac{1}{2}\left(\sqrt{5} - 1\right)$$

To 3 decimal places, $x = 0.618$.

19 a The graph covers 5 cycles in 5 seconds, so the period is 1 second.

b The range of the graph is [40, 100], so the amplitude is $\frac{100 - 40}{2} = 30$ decibels.

c The equation is $d = a\cos(bt) + c$.

As the graph is of an inverted cosine function, $a = -30$.

The period is $\frac{2\pi}{b} = 1$, so $b = 2\pi$.

The equilibrium position is $d = \frac{40 + 100}{2} = 70$, so $c = 70$.

Hence, $a = -30, b = 2\pi, c = 70$ and $d = -30\cos(2\pi t) + 70$.

d When $d = 50$,

$$50 = -30\cos(2\pi t) + 70$$

$$\therefore 30\cos(2\pi t) = 20$$

$$\therefore \cos(2\pi t) = \frac{2}{3}$$

$$\therefore 2\pi t = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\therefore t = \frac{1}{2\pi} \times \cos^{-1}\left(\frac{2}{3}\right)$$

$$\therefore t \simeq 0.133\,86$$

When $d = 60$,

$$60 = -30\cos(2\pi t) + 70$$

$$\therefore 30\cos(2\pi t) = 10$$

$$\therefore \cos(2\pi t) = \frac{1}{3}$$

$$\therefore 2\pi t = \cos^{-1}\left(\frac{1}{3}\right)$$

$$\therefore t = \frac{1}{2\pi} \times \cos^{-1}\left(\frac{1}{3}\right)$$

$$\therefore t \simeq 0.195\,91$$

The sound level was between the normal speech range for an interval of 0.195 91 − 0.133 86 = 0.062 05 seconds as the sound level increased. By symmetry, there was another interval of 0.062 05 seconds as the sound level decreased where the level was in the normal speech range.

The percentage of time during the first second when the sound level is in the normal speech range is

$$\frac{2 \times 0.062\ 05}{1} \times 100\% = 12.41\%, \text{ or approximately } 12\%.$$

e The sound reaches its maximum once per cycle of 1 second.
In 1 minute, it will reach its maximum 60 times.
In 10 minutes, the sound is at its greatest level 600 times.

f Consider one cycle. Let $d = 80$.

$$\therefore 80 = -30\cos(2\pi t) + 70$$

$$\therefore 30\cos(2\pi t) = -10$$

$$\therefore \cos(2\pi t) = -\frac{1}{3}$$

Quadrants 2 and 3, base $\cos^{-1}\left(\frac{1}{3}\right)$

$$\therefore 2\pi t = \pi - \cos^{-1}\left(\frac{1}{3}\right), \pi + \cos^{-1}\left(\frac{1}{3}\right)$$

$$\therefore t = \frac{1}{2\pi} \times \left(\pi - \cos^{-1}\left(\frac{1}{3}\right)\right), \frac{1}{2\pi} \times \left(\pi + \cos^{-1}\left(\frac{1}{3}\right)\right)$$

$$\therefore t = 0.3041,\ 0.6959$$

The sound level is potentially damaging for an interval of 0.6959 − 0.3041 = 0.3918 seconds per cycle. This is the same for every cycle, so the percentage of time the sound level is above 80 decibels is 39.18% or 39.2%.

20 a $y = a\tan(bx)$, $x \in [0, 12]$.

A quarter of the period is 12, so the period is 48.

$$\therefore \frac{\pi}{b} = 48$$

$$\therefore b = \frac{\pi}{48}$$

$$\therefore y = a\tan\left(\frac{\pi}{48}x\right)$$

The point (12, 15) lies on the curve.

$$\therefore 15 = a\tan\left(\frac{\pi}{48} \times 12\right)$$

$$\therefore 15 = a\tan\left(\frac{\pi}{4}\right)$$

$$\therefore 15 = a \times 1$$

$$\therefore a = 15$$

Hence, $a = 15, b = \frac{\pi}{48}$ and $y = 15\tan\left(\frac{\pi}{48}x\right)$.

b As the point (12, 15) is the end point of the horizontal section, the equation of that part of the tree stump for which $x \in (12, 36)$ is $y = 15$.

c By reflecting the graph of $y = 15\tan\left(\frac{\pi}{48}x\right)$ in the y-axis and then translating it horizontally 48 units to the right, the graph of $y = c\tan(dx)$ is obtained.

Under the reflection and translation, $y = 15\tan\left(\frac{\pi}{48}x\right)$

$\rightarrow y = 15\tan\left(\frac{\pi}{48}(-x)\right)$

$\rightarrow y = 15\tan\left(\frac{\pi}{48}(-(x-48))\right)$

The end points $(12, 15) \rightarrow (-12, 15) \rightarrow (36, 15)$ and $(0, 0) \rightarrow (0, 0) \rightarrow (48, 0)$.

Using symmetry properties, the image can be expressed as

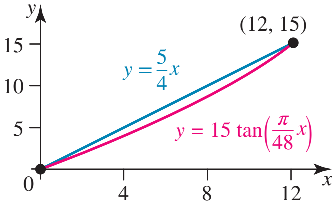
$$\begin{aligned} y &= 15\tan\left(\frac{\pi}{48}(-(x-48))\right) \\ &= 15\tan\left(-\frac{\pi}{48}x + \pi\right) \\ &= 15\tan\left(\pi - \frac{\pi}{48}x\right) \\ &= -15\tan\left(\frac{\pi}{48}x\right) \end{aligned}$$

For $x \in [36, 48]$, the tree stump has the equation $y = -15\tan\left(\frac{\pi}{48}x\right)$. This is in the form $y = c\tan(dx)$ with $c = -15$ and $d = \frac{\pi}{48}$.

d As a hybrid function, the tree stump has the rule:

$$y = \begin{cases} 15\tan\left(\frac{\pi}{48}x\right), & 0 \le x \le 12 \\ 15, & 12 < x < 36 \\ -15\tan\frac{\pi}{48}x, & 36 \le x \le 48 \end{cases}$$

e i The line segment joining the points $(0, 0)$ and $(12, 15)$ has gradient $m = \frac{15}{12} = \frac{5}{4}$ and equation $y = \frac{5}{4}x$, for $x \in [0, 12]$.



Under the reflection in the y-axis and horizontal translation 48 units to the right,

$y = \frac{5}{4}x \rightarrow y = \frac{5}{4}(-x) = -\frac{5}{4}x \rightarrow y = -\frac{5}{4}(x-48)$.

$y = -\frac{5}{4}(x-48) \Rightarrow y = -\frac{5}{4}x + 60$

ii The equation of the line segment for which $x \in [36, 48]$ is $y = -\frac{5}{4}x + 60$.

f Using the line segments, the area is enclosed by a trapezium.

See the image at the bottom of the page.*

The area of the trapezium is

$$\begin{aligned} A &= \frac{1}{2}h(a+b), h = 15, a = 48, b = 24 \\ &= \frac{1}{2} \times 15 \times (48 + 24) \\ &= \frac{1}{2} \times 15 \times 72 \\ &= 540 \end{aligned}$$

An estimate of the cross-sectional area of the tree stump is 540 sq cm.

As the tangent sections lie below the oblique lines, this area estimate is an overestimate of the actual area.

9.7 Exam questions

1 $\sin(2x) = \frac{1}{2}, 0 \le x \le 2\pi$

Multiply the end points of the domain of x by 2.

$\therefore \sin(2x) = \frac{1}{2}, 0 \le 2x \le 4\pi$ [1 mark]

Sine is positive in quadrants 1 and 2.

The base is $\frac{\pi}{6}$ [1 mark]

As $2x \in [0, 4\pi]$, each of the two revolutions will generate 2 solutions, giving a total of 4 values for $2x$.

$\therefore 2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$

$\therefore 2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

[1 mark]

Divide each of the solutions by 2 to obtain the solutions for x within the original domain of $0 \le x \le 2\pi$.

$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ [1 mark]

2 Divide both sides by $\cos(x)$.

$\frac{\sin(x)}{\cos(x)} = \sqrt{3}$

$\therefore \tan(x) = \sqrt{3}$

Tangent is positive in quadrants 1 and 3.

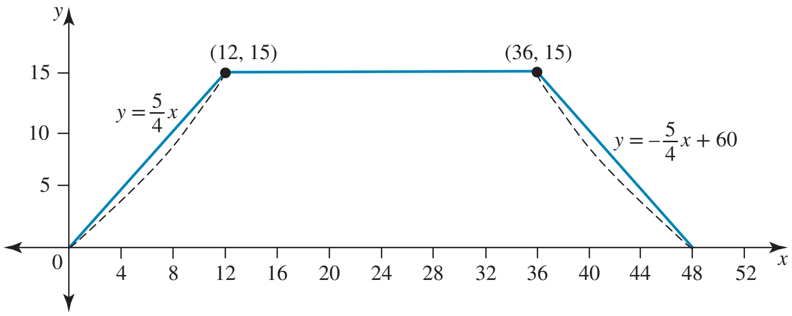
The base is $\frac{\pi}{3}$.

$\therefore x = \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$

The correct answer is **D**.

***20 f**



3 $y = -3\cos(5x) + 7$

$a = 3,\ n = 5,\ c = 7$

Period:

$\frac{2\pi}{n} = \frac{2\pi}{5}$ [1 mark]

Amplitude:

$a = 3$ and -3 (negative indicates that the graph is inverted). [1 mark]

Range:

The graph oscillates between $y = 7 - 3$ and $y = 7 + 3$.

$\therefore$ range is [4, 10]. [1 mark]

4
$$\begin{aligned}\text{Amplitude} &= \frac{\max - \min}{2}\\ &= \frac{19-8}{2}\\ &= 5.5\\ &\therefore r = 5.5\\ c &= \text{mean position}\\ &= \min + \text{amplitude}\\ &= 8 + 5.5\\ &= 13.5\end{aligned}$$

The correct answer is **A**.

5 **a** $t = 0$

$$\begin{aligned}h &= 80 - 25\cos\left(\frac{\pi}{2}t\right)\\ &= 80 - 25\cos(0)\\ &= 80 - 25 \times 1\\ &= 55 \text{ metres}\end{aligned}$$ [1 mark]

b
$$\begin{aligned}\text{Period} &= \frac{2\pi}{n}\\ &= \frac{2\pi}{\frac{\pi}{2}}\\ &= 4\end{aligned}$$

Two minutes = 120 seconds

Therefore, the rotor blades complete 30 revolutions in 2 minutes. [1 mark]

c Area covered by rotor blades:

Length of rotor blades = 25 m

Area = $\pi(25)^2$

The area is 1963.50 m^2. [1 mark]

Topic 10 — Exponential functions and logarithms

10.2 Indices as exponents

10.2 Exercise

1 a i $\sqrt{a^3b^4} = \left(a^3b^4\right)^{\frac{1}{2}}$

$= a^{\frac{3}{2}}b^2$

ii $\sqrt{\dfrac{a^5}{b^{-4}}} = \left(\dfrac{a^5}{b^{-4}}\right)^{\frac{1}{2}}$

$= \dfrac{a^{\frac{5}{2}}}{b^{-2}}$

$= a^{\frac{5}{2}}b^2$

iii $\sqrt[3]{a^2b} = \left(a^2b\right)^{\frac{1}{3}}$

$= a^{\frac{2}{3}}b^{\frac{1}{3}}$

b i $a^{\frac{1}{2}} \div b^{\frac{3}{2}} = \sqrt{a} \div \sqrt{b^3}$

$= \sqrt{\dfrac{a}{b^3}}$

ii $2^{\frac{5}{2}} = \sqrt{2^5}$

$= \sqrt{32}$

iii $3^{-\frac{2}{5}} = \sqrt[5]{3^{-2}}$

$= \sqrt[5]{\dfrac{1}{3^2}}$

$= \sqrt[5]{\dfrac{1}{9}}$ or $\dfrac{1}{\sqrt[5]{9}}$

2 a $4^{\frac{3}{2}} = \sqrt{4^3}$

$= \sqrt{64}$

$= 8$

b $3^{-1} + 5^0 - 2^2 \times 9^{-\frac{1}{2}} = \dfrac{1}{3} + 1 - 4 \times \dfrac{1}{\sqrt{9}}$

$= \dfrac{4}{3} - \dfrac{4}{3}$

$= 0$

c $2^3 \times \left(\dfrac{4}{9}\right)^{-\frac{1}{2}} = 8 \times \dfrac{4^{-\frac{1}{2}}}{9^{-\frac{1}{2}}}$

$= 8 \times \dfrac{3}{2}$

$= \dfrac{24}{2}$

$= 12$

d $\dfrac{15 \times 5^{\frac{3}{2}}}{125^{\frac{1}{2}}} = \dfrac{15 \times \sqrt{5^3}}{\sqrt{125}}$

$= \dfrac{15 \times 5\sqrt{5}}{5\sqrt{5}}$

$= 15$

3 a $a^{3-3n} \times \left(a^2\right)^{2n} \div \left(a^3\right)^{n+2}$

$= a^{3-3n} \times a^{4n} \div a^{3n+6}$

$= a^{-2n-3}$

$= \dfrac{1}{a^{2n+3}}$

b $\dfrac{\left(x^4 y\right)^3 \times \left(9xy^2\right)^2}{(2y)^0 \times (3x^2)^3}$

$= \dfrac{x^{12}y^3 \times 9^2x^2y^4}{1 \times 3^3x^6}$

$= \dfrac{81x^{14}y^7}{27x^6}$

$= 3x^8y^7$

c $\dfrac{(a^3b^2)^2 \times (4ab^3)^3}{8\left(a^3b^9\right)^2}$

$= \dfrac{a^6b^4 \times 64a^3b^9}{8a^6b^{18}}$

$= \dfrac{64a^9b^{13}}{8a^6b^{18}}$

$= \dfrac{8a^3}{b^5}$

d $\dfrac{2\left(x^2y\right)^4 \times \left(2x^{\frac{1}{3}}y^2\right)^3}{y^3\left(2x^{-1}\right)^5}$

$= \dfrac{2x^8y^4 \times 2^3x^1y^6}{y^3 \times 2^5x^{-5}}$

$= \dfrac{2^4x^9y^{10}}{2^5x^{-5}y^3}$

$= \dfrac{x^{14}y^7}{2}$

e $\dfrac{81^2 \times 27^{-\frac{2}{3}} \times \sqrt{3}}{243^{\frac{3}{5}}}$

$= \dfrac{\left(3^4\right)^2 \times \left(3^3\right)^{-\frac{2}{3}} \times 3^{\frac{1}{2}}}{\left(3^5\right)^{\frac{3}{5}}}$

$= \dfrac{3^8 \times 3^{-2} \times 3^{\frac{1}{2}}}{3^3}$

$= \dfrac{3^6 \times 3^{\frac{1}{2}}}{3^3}$

$= 3^3 \times 3^{\frac{1}{2}}$

$= 3^{\frac{7}{2}}$

f $\dfrac{\left(a^nb^{3n}\right)^{-2} \times \left(a^{3n}b^{\frac{3}{2}}\right)^{\frac{1}{3}}}{\left(a^{-4n}b\right)^{-\frac{3}{2}}}$

$= \dfrac{a^{-2n}b^{-6n} \times a^nb^{\frac{1}{2}}}{a^{6n}b^{-\frac{3}{2}}}$

$= \dfrac{a^{-n}b^{-6n} \times b^{\frac{1}{2}}}{a^{6n}b^{-\frac{3}{2}}}$

$$= \frac{b^{\frac{1}{2}}b^{\frac{3}{2}}}{a^{7n}b^{6n}}$$

$$= \frac{b^2}{a^{7n}b^{6n}}$$

$$= \frac{1}{a^{7n}b^{6n-2}}$$

4 a $$\frac{2^{1-n} \times 8^{1+2n}}{16^{1-n}} = \frac{2^{1-n} \times 2^{3(1+2n)}}{2^{4(1-n)}}$$

$$= \frac{2^{4+5n}}{2^{4-4n}}$$

$$= 2^{9n}$$

b $$\left(9a^3b^{-4}\right)^{\frac{1}{2}} \times 2\left(a^{\frac{1}{2}}b^{-2}\right)^{-2} = 3a^{\frac{3}{2}}b^{-2} \times 2 \times a^{-1}b^4$$

$$= 6a^{\frac{1}{2}}b^2$$

c $$27^{-\frac{2}{3}} + \left(\frac{49}{81}\right)^{\frac{1}{2}} = \frac{1}{27^{\frac{2}{3}}} + \left(\frac{49}{81}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\left(\sqrt[3]{27}\right)^2} + \frac{\sqrt{49}}{\sqrt{81}}$$

$$= \frac{1}{(3)^2} + \frac{7}{9}$$

$$= \frac{1}{9} + \frac{7}{9}$$

$$= \frac{8}{9}$$

5 $$\frac{20p^5}{m^3q^{-2}} \div \frac{5\left(p^2q^{-3}\right)^2}{-4m^{-1}} = \frac{20p^5}{m^3q^{-2}} \times \frac{-4m^{-1}}{5p^4q^{-6}}$$

$$= \frac{{}^{4}\cancel{20}p^5q^2}{m^3} \times \frac{-4q^6}{\cancel{5}p^4m}$$

$$= \frac{-16p^5q^8}{m^4p^4}$$

$$= \frac{-16pq^8}{m^4}$$

6 a $$\frac{3\left(x^2y^{-2}\right)^3}{\left(3x^4y^2\right)^{-1}} = \frac{3x^6y^{-6}}{3^{-1}x^{-4}y^{-2}}$$

$$= 3^2x^{10}y^{-4}$$

$$= \frac{9x^{10}}{y^4}$$

b $$\frac{2a^{\frac{2}{3}}b^{-3}}{3a^{\frac{1}{3}}b^{-1}} \times \frac{3^2 \times 2 \times (ab)^2}{\left(-8a^2\right)^2 b^2} = \frac{2a^{\frac{1}{3}}b^{-2}}{3} \times \frac{18a^2b^2}{64a^4b^2}$$

$$= \frac{2a^{\frac{1}{3}}}{3b^2} \times \frac{9}{32a^2}$$

$$= \frac{a^{\frac{1}{3}}}{b^2} \times \frac{3}{16a^2}$$

$$= \frac{3}{16a^{\frac{5}{3}}b^2}$$

c $$\frac{\left(2mn^{-2}\right)^{-2}}{m^{-1}n} \div \frac{10n^4m^{-1}}{3\left(m^2n\right)^{\frac{3}{2}}} = \frac{2^{-2}m^{-2}n^4}{m^{-1}n} \times \frac{3m^3n^{\frac{3}{2}}}{10n^4m^{-1}}$$

$$= \frac{n^3}{2^2m} \times \frac{3m^4}{10n^{\frac{5}{2}}}$$

$$= \frac{3n^3m^4}{40mn^{\frac{5}{2}}}$$

$$= \frac{3m^3n^{\frac{1}{2}}}{40}$$

d $$\frac{4m^2n^{-2} \times -2\left(m^2n^{\frac{3}{2}}\right)^2}{\left(-3m^3n^{-2}\right)^2} = \frac{4m^2n^{-2} \times -2m^4n^3}{9m^6n^{-4}}$$

$$= \frac{-8m^6n}{9m^6n^{-4}}$$

$$= \frac{-8n^5}{9}$$

e $$\frac{m^{-1} - n^{-1}}{m^2 - n^2} = \left(\frac{1}{m} - \frac{1}{n}\right) \div \left(m^2 - n^2\right)$$

$$= \left(\frac{n-m}{mn}\right) \times \frac{1}{m^2 - n^2}$$

$$= \frac{n-m}{mn} \times \frac{1}{(m-n)(m+n)}$$

$$= \frac{-(m-n)}{mn} \times \frac{1}{(m-n)(m+n)}$$

$$= \frac{-1}{mn(m+n)}$$

f $$\sqrt{4x-1} - 2x(4x-1)^{-\frac{1}{2}} = (4x-1)^{\frac{1}{2}} - \frac{2x}{(4x-1)^{\frac{1}{2}}}$$

$$= \frac{(4x-1)^{\frac{1}{2}}(4x-1)^{\frac{1}{2}} - 2x}{(4x-1)^{\frac{1}{2}}}$$

$$= \frac{(4x-1) - 2x}{(4x-1)^{\frac{1}{2}}}$$

$$= \frac{2x-1}{(4x-1)^{\frac{1}{2}}}$$

7 a $$\frac{32 \times 4^{3x}}{16^x} = \frac{2^5 \times \left(2^2\right)^{3x}}{\left(2^4\right)^x}$$

$$= \frac{2^5 \times 2^{6x}}{2^{4x}}$$

$$= \frac{2^{5+6x}}{2^{4x}}$$

$$= 2^{5+6x-4x}$$

$$= 2^{5+2x}$$

b $$\frac{3^{1+n} \times 81^{n-2}}{243^n} = \frac{3^{1+n} \times \left(3^4\right)^{n-2}}{\left(3^5\right)^n}$$

$$= \frac{3^{1+n} \times 3^{4n-8}}{3^{5n}}$$

$$= \frac{3^{5n-7}}{3^{5n}}$$

$$= 3^{-7}$$

c $$\begin{aligned} 0.001 \times \sqrt[3]{10} \times 100^{\frac{5}{2}} \times (0.1)^{-\frac{2}{3}} &= \frac{1}{1000} \times 10^{\frac{1}{3}} \\ &\quad \times \left(10^2\right)^{\frac{5}{2}} \times \left(\frac{1}{10}\right)^{-\frac{2}{3}} \\ &= \frac{1}{10^3} \times 10^{\frac{1}{3}} \times 10^5 \\ &\quad \times \left(10^{-1}\right)^{-\frac{2}{3}} \\ &= 10^{-3} \times 10^{\frac{1}{3}} \times 10^5 \times 10^{\frac{2}{3}} \\ &= 10^{-3+\frac{1}{3}+5+\frac{2}{3}} \\ &= 10^{\frac{-9+1+15+2}{3}} \\ &= 10^{\frac{9}{3}} \\ &= 10^3 \end{aligned}$$

d $$\begin{aligned} \frac{5^{n+1} - 5^n}{4} &= \frac{5^n \times 5^1 - 5^n}{4} \\ &= \frac{5^n(5-1)}{4} \\ &= 5^n \end{aligned}$$

8 a $2^{3x} = 16$

$\therefore 2^{3x} = 2^4$

$\therefore 3x = 4$

$\therefore x = \frac{4}{3}$

b $7^{3x} = 49^{1+x}$

$\therefore 7^{3x} = \left(7^2\right)^{1+x}$

$\therefore 7^{3x} = 7^{2+2x}$

$\therefore 3x = 2 + 2x$

$\therefore x = 2$

c $3^{1-5x} = 1$

$\therefore 3^{1-5x} = 3^0$

$\therefore 1 - 5x = 0$

$\therefore 1 = 5x$

$\therefore x = \frac{1}{5}$

d $2^{x-4} = 8^{4-x}$

$\therefore 2^{x-4} = \left(2^3\right)^{4-x}$

$\therefore 2^{x-4} = 2^{12-3x}$

$\therefore x - 4 = 12 - 3x$

$\therefore 4x = 16$

$\therefore x = 4$

e $25^{3-x} = \frac{125^{2x}}{5^{x+2}}$

$\therefore \left(5^2\right)^{3-x} = \frac{\left(5^3\right)^{2x}}{5^{x+2}}$

$\therefore 5^{6-2x} = \frac{5^{6x}}{5^{x+2}}$

$\therefore 5^{6-2x} = 5^{6x-(x+2)}$

$\therefore 5^{6-2x} = 5^{5x-2}$

$\therefore 6 - 2x = 5x - 2$

$\therefore 8 = 7x$

$\therefore x = \frac{8}{7}$

f $3^{x+5} - \left(\frac{1}{3}\right)^{1-2x} = 0$

$\therefore 3^{x+5} = \left(\frac{1}{3}\right)^{1-2x}$

$\therefore 3^{x+5} = \left(3^{-1}\right)^{1-2x}$

$\therefore 3^{x+5} = 3^{-1+2x}$

$\therefore x + 5 = -1 + 2x$

$\therefore 5 + 1 = 2x - x$

$\therefore x = 6$

9 $$\frac{2^{5x-3} \times 8^{9-2x}}{4^x} = 1$$

$$\frac{2^{5x-3} \times 2^{3(9-2x)}}{2^{2x}} = 1$$

$$\frac{2^{24-x}}{2^{2x}} = 1$$

$2^{24-3x} = 1$

$2^{24-3x} = 2^0$

Equating indices:

$24 - 3x = 0$

$\therefore x = 8$

10 a $2 \times 5^x + 5^x < 75$

Adding:

$3 \times 5^x < 75$

$5^x < \frac{75}{3}$

$5^x < 25$

$5^x < 5^2$

Hence, the indices give $x < 2$.

b $\left(\frac{1}{9}\right)^{2x-3} > \left(\frac{1}{9}\right)^{7-x}$

$9^{-(2x-3)} > 9^{-(7-x)}$

$9^{-2x+3} > 9^{-7+x}$

$-2x + 3 > -7 + x$

$10 > 3x$

$x < \frac{10}{3}$

11 a $2^{2x} \times 8^{2-x} \times 16^{-\frac{3x}{2}} = \frac{2}{4^x}$

$\therefore 2^{2x} \times (2^3)^{2-x} \times \left(2^4\right)^{-\frac{3x}{2}} = \frac{2}{(2^2)^x}$

$\therefore 2^{2x} \times 2^{6-3x} \times 2^{-6x} = \frac{2^1}{2^{2x}}$

$\therefore 2^{-7x+6} = 2^{1-2x}$

Equating indices,

$-7x + 6 = 1 - 2x$

$\therefore 5 = 5x$

$\therefore x = 1$

b $25^{3x-3} \leq 125^{4+x}$

$\therefore \left(5^2\right)^{3x-3} \leq \left(5^3\right)^{4+x}$

$\therefore 5^{6x-6} \leq 5^{12+3x}$

As the base is greater than 1,

$6x - 6 \le 12 + 3x$

$\therefore 3x \le 18$

$\therefore x \le 6$

c $9^x \div 27^{1-x} = \sqrt{3}$

$\therefore \left(3^2\right)^x \div \left(3^3\right)^{1-x} = 3^{\frac{1}{2}}$

$\therefore 3^{2x} \div 3^{3-3x} = 3^{\frac{1}{2}}$

$\therefore 3^{5x-3} = 3^{\frac{1}{2}}$

$\therefore 5x - 3 = \frac{1}{2}$

$\therefore 5x = \frac{7}{2}$

$\therefore x = \frac{7}{10}$

d $\left(\frac{2}{3}\right)^{3-2x} > \left(\frac{27}{8}\right)^{-\frac{1}{3}} \times \frac{1}{\sqrt{2\frac{1}{4}}}$

$\therefore \left(\frac{2}{3}\right)^{3-2x} > \left(\frac{8}{27}\right)^{\frac{1}{3}} \times \frac{1}{\sqrt{\frac{9}{4}}}$

$\therefore \left(\frac{2}{3}\right)^{3-2x} > \sqrt[3]{\frac{8}{27}} \times \frac{1}{\frac{3}{2}}$

$\therefore \left(\frac{2}{3}\right)^{3-2x} > \frac{2}{3} \times \frac{2}{3}$

$\therefore \left(\frac{2}{3}\right)^{3-2x} > \left(\frac{2}{3}\right)^{2}$

As the base is less than 1,

$3 - 2x < 2$

$\therefore -2x < -1$

$\therefore x > \frac{1}{2}$

e $4^{5x} + 4^{5x} = \frac{8}{2^{4x-5}}$

$\therefore 2 \times 4^{5x} = \frac{2^3}{2^{4x-5}}$

$\therefore 2 \times \left(2^2\right)^{5x} = 2^{8-4x}$

$\therefore 2 \times 2^{10x} = 2^{8-4x}$

$\therefore 2^{1+10x} = 2^{8-4x}$

$\therefore 1 + 10x = 8 - 4x$

$\therefore 14x = 7$

$\therefore x = \frac{7}{14}$

$\therefore x = \frac{1}{2}$

f $5^{\frac{2x}{3}} \times 5^{\frac{3x}{2}} = 25^{x+4}$

$\therefore 5^{\frac{2x}{3}+\frac{3x}{2}} = \left(5^2\right)^{x+4}$

$\therefore 5^{\frac{4x+9x}{6}} = 5^{2x+8}$

$\therefore \frac{13x}{6} = 2x + 8$

$\therefore 13x = 12x + 48$

$\therefore x = 48$

12 **a** $2^{2x} - 9 \times 2^x + 8 = 0$

$(2^x)^2 - 9 \times (2^x) + 8 = 0$

Let $a = 2^x$.

$\therefore a^2 - 9a + 8 = 0$

$\therefore (a-1)(a-8) = 0$

$\therefore a = 1 \text{ or } a = 8$

$\therefore 2^x = 1 \text{ or } 2^x = 8$

$\therefore x = 0 \text{ or } x = 3$

b $3^{2x} + 6 \times 3^x - 27 = 0$

$(3^x)^2 + 6 \times (3^x) - 27 = 0$

Let $a = 3^x$.

$\therefore a^2 + 6a - 27 = 0$

$\therefore (a-3)(a+9) = 0$

$\therefore a = 3 \text{ or } a = -9$

$\therefore 3^x = 3 \text{ or } 3^x = -9$

$\therefore 3^x = 3 \text{ (as } 3^x > 0)$

$\therefore x = 1$

c $4^{2x} + 20 \times 4^x + 64 = 0$

$(4^x)^2 + 20 \times (4^x) + 64 = 0$

Let $a = 4^x$.

$\therefore a^2 + 20a + 64 = 0$

$\therefore (a+4)(a+16) = 0$

$\therefore a = -4 \text{ or } a = -16$

$\therefore 4^x = -4 \text{ or } 4^x = -16$

Since $4^x > 0$, there are no real solutions.

d $5^{2x} - 2 \times 5^x + 1 = 0$

$(5^x)^2 - 2 \times (5^x) + 1 = 0$

Let $a = 5^x$.

$\therefore a^2 - 2a + 1 = 0$

$\therefore (a-1)^2 = 0$

$\therefore a = 1$

$\therefore 5^x = 1$

$\therefore x = 0$

13 $30 \times 10^{2x} + 17 \times 10^x - 2 = 0$

Let $a = 10^x$.

$30a^2 + 17a - 2 = 0$

$(10a - 1)(3a + 2) = 0$

$a = \frac{1}{10}, \ a = -\frac{2}{3}$

$\therefore 10^x = \frac{1}{10}$ or $10^x = -\frac{2}{3}$, for which there is no real solution

$10^x = \frac{1}{10}$

$10^x = 10^{-1}$

$x = -1$

14 $2^x - 48 \times 2^{-x} = 13$

$\therefore 2^x - \frac{48}{2^x} = 13$

Let $a = 2^x$.

$a - \frac{48}{a} = 13$

$a^2 - 48 = 13a$

$a^2 - 13a - 48 = 0$

$(a - 16)(a + 3) = 0$

$a = 16, \ a = -3$

$\therefore 2^x = 16, \ 2^x = -3$

Reject $2^x = -3$ since there are no real solutions.

$2^x = 16$
$2^x = 2^4$
$x = 4$

15 **a** $3^{2x} - 10 \times 3^x + 9 = 0$
Let $a = 3^x$.
$\therefore a^2 - 10a + 9 = 0$
$\therefore (a-1)(a-9) = 0$
$\therefore a = 1 \text{ or } a = 9$
$\therefore 3^x = 1 \text{ or } 3^x = 9$
$\therefore x = 0 \text{ or } x = 2$

b $24 \times 2^{2x} + 61 \times 2^x = 2^3$
Let $a = 2^x$.
$\therefore 24a^2 + 61a = 8$
$\therefore 24a^2 + 61a - 8 = 0$
$\therefore (8a-1)(3a+8) = 0$
$\therefore a = \frac{1}{8} \text{ or } a = -\frac{8}{3}$
$\therefore 2^x = \frac{1}{8} \text{ or } 2^x = -\frac{8}{3}$
Reject $2^x = -\frac{8}{3}$ since $2^x > 0$
$\therefore 2^x = \frac{1}{8}$
$\therefore 2^x = 2^{-3}$
$\therefore x = -3$

c $25^x + 5^{2+x} - 150 = 0$
$\therefore 5^{2x} + 5^2 \times 5^x - 150 = 0$
Let $a = 5^x$.
$\therefore a^2 + 25a - 150 = 0$
$\therefore (a-5)(a+30) = 0$
$\therefore a = 5 \text{ or } a = -30$
$\therefore 5^x = 5 \text{ or } 5^x = -30$
$\therefore 5^x = 5, \ (5^x > 0)$
$\therefore x = 1$

d $(2^x + 2^{-x})^2 = 4$
$(2^x + 2^{-x})^2 = 4$
$\therefore 2^x + 2^{-x} = \pm\sqrt{4}$
$\therefore 2^x + 2^{-x} = \pm 2$
However, $2^x + 2^{-x} > 0$.
$\therefore 2^x + 2^{-x} = 2$
$\therefore 2^x + \frac{1}{2^x} = 2$
Let $a = 2^x$.
$\therefore a + \frac{1}{a} = 2$
$\therefore a^2 + 1 = 2a$
$\therefore a^2 - 2a + 1 = 0$
$\therefore (a-1)^2 = 0$
$\therefore a = 1$
$\therefore 2^x = 1$
$\therefore x = 0$

e $10^x - 10^{2-x} = 99$
$\therefore 10^x - \frac{10^2}{10^x} = 99$
Let $a = 10^x$.
$\therefore a - \frac{100}{a} = 99$
$\therefore a^2 - 100 = 99a$
$\therefore a^2 - 99a - 100 = 0$
$\therefore (a-100)(a+1) = 0$
$\therefore a = 100 \text{ or } a = -1$
$\therefore 10^x = 100 \text{ or } 10^x = -1$
$\therefore 10^x = 100 \ (10^x > 0)$
$\therefore 10^x = 10^2$
$\therefore x = 2$

f $2^{3x} + 3 \times 2^{2x-1} - 2^x = 0$
$\therefore 2^{3x} + 3 \times \frac{2^{2x}}{2^1} - 2^x = 0$
Let $a = 2^x$.
$\therefore a^3 + 3 \times \frac{a^2}{2} - a = 0$
$\therefore 2a^3 + 3a^2 - 2a = 0$
$\therefore a(2a^2 + 3a - 2) = 0$
$\therefore a(2a-1)(a+2) = 0$
$\therefore a = 0 \text{ or } a = \frac{1}{2} \text{ or } a = -2$
$\therefore 2^x = 0 \text{ or } 2^x = \frac{1}{2} \text{ or } 2^x = -2$
$\therefore 2^x = \frac{1}{2} \ (2^x > 0)$
$\therefore 2^x = 2^{-1}$
$\therefore x = -1$

16 **a** **i** $1\,409\,000 = 1.409 \times 10^6$ and it contains 4 significant figures.
ii $0.000\,130\,6 = 1.306 \times 10^{-4}$ and it contains 4 significant figures.
b **i** $3.04 \times 10^5 = 304\,000$
ii $5.803 \times 10^{-2} = 0.058\,03$

17 $\left(4 \times 10^6\right)^2 \times (5 \times 10^{-3}) = 16 \times 10^{12} \times 5 \times 10^{-3}$
$= 16 \times 5 \times 10^{12} \times 10^{-3}$
$= 80 \times 10^9$
$= 8.0 \times 10^1 \times 10^9$
$= 8 \times 10^{10}$

18 **a** **i** $-0.000\,000\,050\,6 = -5.06 \times 10^{-8}$
ii The diameter is $2 \times 6370 = 12\,740$ km.
In scientific notation, the diameter is 1.274×10^4 km.
iii $3.2 \times 10^4 \times 5 \times 10^{-2}$
$= (3.2 \times 5) \times \left(10^4 \times 10^{-2}\right)$
$= 16 \times 10^2$
$= 1.6 \times 10^3$
iv 16,878.7 km is equal to $1.687\,87 \times 10^4$ km
b **i** $6.3 \times 10^{-4} + 6.3 \times 10^4 = 0.000\,63 + 63\,000$
$= 63\,000.000\,63$
ii $\left(1.44 \times 10^6\right)^{\frac{1}{2}} = (1.44)^{\frac{1}{2}} \times \left(10^6\right)^{\frac{1}{2}}$
$= \sqrt{1.44} \times 10^3$
$= 1.2 \times 10^3$
$= 1200$

c i 60 589

$= 6.0589 \times 10^4$

$\simeq 6.1 \times 10^4$

$\therefore 60\,589 \simeq 61\,000$

Correct to 2 significant figures, 61 000 people attended the match.

ii $1.994 \times 10^{-2} \simeq 2.0 \times 10^{-2}$

The probability, correct to 2 significant figures, is 0.020.

iii $-0.006\,34$

$= -6.34 \times 10^{-3}$

$\simeq -6.3 \times 10^{-3}$

Correct to 2 significant figures, $x = -0.0063$.

iv 26 597 696

$= 2.659\,769\,6 \times 10^7$

$\simeq 2.7 \times 10^7$

Correct to 2 significant figures, the distance flown is 27 000 000 km.

19 a Consider the system of equations:

$5^{2x-y} = \frac{1}{125}$ [1]

$10^{2y-6x} = 0.01$ [2]

From equation [1],

$5^{2x-y} = 5^{-3}$

$\therefore 2x - y = -3$ [3]

From equation [2],

$10^{2y-6x} = \frac{1}{100}$

$\therefore 10^{2y-6x} = 10^{-2}$

$\therefore 2y - 6x = -2$

$\therefore -3x + y = -1$ [4]

Consider the simultaneous equations [3] and [4].

$2x - y = -3$ [3]

$-3x + y = -1$ [4]

Add equations [3] and [4].

$\therefore -x = -4$

$\therefore x = 4$

Substitute $x = 4$ in equation [4].

$\therefore -12 + y = -1$

$\therefore y = 11$

Answer: $x = 4, y = 11$

b Consider the system of equations:

$a \times 2^{k-1} = 40$ [1]

$a \times 2^{2k-2} = 10$ [2]

Divide equation [1] by equation [2].

$$\therefore \frac{\cancel{a} \times 2^{k-1}}{\cancel{a} \times 2^{2k-2}} = \frac{40}{10}$$

$\therefore 2^{k-1-2k+2} = 4$

$\therefore 2^{-k+1} = 2^2$

$\therefore -k + 1 = 2$

$\therefore k = -1$

Substitute $k = -1$ in equation [1].

$\therefore a \times 2^{-2} = 40$

$\therefore a \times \frac{1}{4} = 40$

$\therefore a = 160$

Answer: $a = 160,\ k = -1$

20 a Use CAS technology to obtain 9.872 541 804E − 4 .

Correct to 4 significant figures, $5^{-4.3} = 9.873 \times 10^{-4}$.

b $22.9 \div 1.3\text{E}2.\text{E}$. The value returned is 0.1762 correct to 4 significant figures.

The notation 1.3E2 stands for 1.3×10^2.

c $5.04 \times 10^{-6} \div \left(3 \times 10^9\right)$ could be entered as $5.04\text{E} - 6 \div 3\text{E}9$ (if desired) to return the answer $1.68\text{E} - 15$.

In standard form, $5.04 \times 10^{-6} \div \left(3 \times 10^9\right) = 1.68 \times 10^{-15}$.

d $5.04 \times 10^{-6} \div \left(3.2 \times 10^{4.2}\right)$. The E notation is for standard form where the power of 10 is always an integer, so it cannot be used for $10^{4.2}$ since 4.2 is not an integer.

$5.04 \times 10^{-6} \div \left(3.2 \times 10^{4.2}\right) = 9.937\,578\,176\text{E} - 11$

In standard form correct to 4 significant figures,

$5.04 \times 10^{-6} \div \left(3.2 \times 10^{4.2}\right) = 9.938 \times 10^{-11}$.

10.2 Exam questions

1
$$3^{-4} \times \left(\frac{8}{27}\right)^{-\frac{1}{3}} = 3^{-4} \times \frac{2^{3\times\frac{1}{3}}}{3^{3\times\frac{1}{3}}}$$
$$= 3^{-4} \times \frac{2^{-1}}{3^{-1}}$$
$$= 3^{-3} \times 2^{-1}$$
$$= \frac{1}{27} \times \frac{1}{2}$$
$$= \frac{1}{54}$$

The correct answer is **D**.

2 $27^{3-2x} = 729^x$

$3^{3(3-2x)} = 3^{6x}$

$9 - 6x = 6x$

$9 = 12x$

$x = \frac{9}{12}$

$= \frac{3}{4}$

The correct answer is **B**.

3 $27^x \div 9^{1-x} = 3\sqrt{3}$

$3^{3x} \div 3^{2(1-x)} = 3 \times 3^{\frac{1}{2}}$

$3^{3x-2(1-x)} = 3^{\frac{3}{2}}$ [1 mark]

$5x - 2 = \frac{3}{2}$

$5x = \frac{7}{2}$

$x = \frac{7}{10}$ [1 mark]

10.3 Indices as logarithms

10.3 Exercise

1 a $5^4 = 625 \Rightarrow 4 = \log_5(625)$

b $\log_{36}(6) = \frac{1}{2} \Rightarrow 6 = 36^{\frac{1}{2}}$

c $10^x = 8.52$

$x = \log_{10}(8.52)$

≈ 0.93

d $\log_3(x) = -1$

$$\begin{aligned} x &= 3^{-1} \\ x &= \frac{1}{3} \end{aligned}$$

2 **a** $\log_e(5) \approx 1.609$; the index statement is $5 = e^{1.609}$.

b $10^{3.5} \approx 3162$; the log statement is $3.5 = \log_{10}(3162)$

3 **a**

$$\begin{aligned} \log_{12}(3) + \log_{12}(4) &= \log_{12}(3 \times 4) \\ &= \log_{12}(12) \\ &= 1 \end{aligned}$$

b

$$\begin{aligned} \log_2(192) - \log_2(12) &= \log_2\left(\frac{192}{12}\right) \\ &= \log_2(16) \\ &= \log_2\left(2^4\right) \\ &= 4\log_2(2) \\ &= 4 \end{aligned}$$

c

$$\begin{aligned} \log_3(3a^3) - 2\log_3\left(a^{\frac{3}{2}}\right) &= \log_3(3) + \log_3\left(a^3\right) \\ &\quad -2 \times \frac{3}{2}\log_3(a) \\ &= 1 + 3\log_3(a) - 3\log_3(a) \\ &= 1 \end{aligned}$$

d

$$\begin{aligned} \frac{\log_a(8)}{\log_a(4)} &= \frac{\log_a\left(2^3\right)}{\log_a\left(2^2\right)} \\ &= \frac{3\cancel{\log_a(2)}}{2\cancel{\log_a(2)}} \\ &= \frac{3}{2} \end{aligned}$$

4 $\log_a(2) = 0.3$; $\log_a(5) = 0.7$

a

$$\begin{aligned} \log_a(0.5) &= \log_a\left(\frac{1}{2}\right) \\ &= \log_a\left(2^{-1}\right) \\ &= -\log_a(2) \\ &= -0.3 \end{aligned}$$

b

$$\begin{aligned} \log_a(2.5) &= \log_a\left(\frac{5}{2}\right) \\ &= \log_a(5) - \log_a(2) \\ &= 0.7 - 0.3 \\ &= 0.4 \end{aligned}$$

c

$$\begin{aligned} \log_a(20) &= \log_a\left(2^2 \times 5\right) \\ &= \log_a\left(2^2\right) + \log_a(5) \\ &= 2\log_a(2) + \log_a(5) \\ &= 2 \times 0.3 + 0.7 \\ &= 1.3 \end{aligned}$$

5 **a** $\log_5(5 \div 5)$

$$\begin{aligned} &= \log_5(1) \\ &= 0 \end{aligned}$$

b $\log_{10}(5) + \log_{10}(2)$

$$\begin{aligned} &= \log_{10}(5 \times 2) \\ &= \log_{10}(10) \\ &= 1 \end{aligned}$$

c $\log_3\left(\frac{1}{3}\right)$

$$\begin{aligned} &= \log_3(1) - \log_3(3) \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

d $\log_2(32)$

$$\begin{aligned} &= \log_2\left(2^5\right) \\ &= 5\log_2(2) \\ &= 5 \times 1 \\ &= 5 \end{aligned}$$

e $\log_4\left(\frac{7}{32}\right) - \log_4(14)$

$$\begin{aligned} &= \log_4\left(\frac{7}{32} \div 14\right) \\ &= \log_4\left(\frac{7}{32} \times \frac{1}{14}\right) \\ &= \log_4\left(\frac{1}{64}\right) \\ &= \log_4\left(4^{-3}\right) \\ &= -3\log_4(4) \\ &= -3 \times 1 \\ &= -3 \end{aligned}$$

f $\log_6(9) + \log_6(8) - \log_6(2)$

$$\begin{aligned} &= \log_6(9 \times 8 \div 2) \\ &= \log_6(36) \\ &= \log_6\left(6^2\right) \\ &= 2\log_6(6) \\ &= 2 \times 1 \\ &= 2 \end{aligned}$$

6 **a** $\log_3\left(x^5\right) - \log_3\left(x^2\right)$

$$\begin{aligned} &= 3\log_3(x) - 2\log_3(x) \\ &= \log_3(x) \end{aligned}$$

b $\log_a\left(2x^5\right) + \log_a\left(\frac{x}{2}\right)$

$$\begin{aligned} &= \log_a\left(2x^5 \times \frac{x}{2}\right) \\ &= \log_a(x^6) \\ &= 6\log_a(x) \end{aligned}$$

c $-\frac{1}{2}\log_{10}(a) + \log_{10}\left(\sqrt{a}\right)$

$$\begin{aligned} &= \log_{10}\left(a^{-\frac{1}{2}}\right) + \log_{10}\left(\sqrt{a}\right) \\ &= \log_{10}\left(a^{-\frac{1}{2}} \times \sqrt{a}\right) \\ &= \log_{10}\left(\frac{1}{\sqrt{a}} \times \sqrt{a}\right) \\ &= \log_{10}(1) \\ &= 0 \end{aligned}$$

d $\frac{1}{2}\log_b\left(16a^4\right) - \frac{1}{3}\log_b\left(8a^3\right)$

$$\begin{aligned} &= \log_b\left(\left(16a^4\right)^{\frac{1}{2}}\right) - \log_b\left(\left(8a^3\right)^{\frac{1}{3}}\right) \\ &= \log_b\left(4a^2\right) - \log_b(2a) \\ &= \log_b\left(\frac{4a^2}{2a}\right) \\ &= \log_b(2a) \end{aligned}$$

e $3\log_{10}(x) - 2\log_{10}\left(x^3\right) + \frac{1}{2}\log_{10}\left(x^5\right)$

$= \log_{10}\left(x^3\right) - \log_{10}\left(\left(x^3\right)^2\right) + \log_{10}\left(\left(x^5\right)^{\frac{1}{2}}\right)$

$= \log_{10}\left(x^3\right) - \log_{10}\left(x^6\right) + \log_{10}\left(x^{\frac{5}{2}}\right)$

$= \log_{10}\left(\dfrac{x^3 \times x^{\frac{5}{2}}}{x^6}\right)$

$= \log_{10}\left(x^{3+\frac{5}{2}-6}\right)$

$= \log_{10}\left(x^{-\frac{1}{2}}\right)$

$= -\frac{1}{2}\log_{10}(x)$

f $\dfrac{\log_3\left(x^6\right)}{\log_3\left(x^2\right)}$

$= \dfrac{6\log_3(x)}{2\log_3(x)}$

$= \dfrac{6}{2}$

$= 3$

7 **a** $\log_9(3) + \log_9(27) = \log_9(3 \times 27)$

$= \log_9(81)$

$= \log_9\left(9^2\right)$

$= 2\log_9(9)$

$= 2 \times 1$

$= 2$

b $\log_9(3) - \log_9(27) = \log_9(3 \div 27)$

$= \log_9\left(\dfrac{1}{9}\right)$

$= \log_9\left(9^{-1}\right)$

$= -\log_9(9)$

$= -1$

c $2\log_2(4) + \log_2(6) - \log_2(12) = \log_2\left(4^2\right) + \log_2(6)$

$- \log_2(12)$

$= \log_2\left(\dfrac{16 \times 6}{12}\right)$

$= \log_2(8)$

$= \log_2\left(2^3\right)$

$= 3\log_2(2)$

$= 3$

d $\log_5\left(\log_3(3)\right) = \log_5(1)$

$= 0$

e $\log_{11}\left(\dfrac{7}{3}\right) + 2\log_{11}\left(\dfrac{1}{3}\right) - \log_{11}\left(\dfrac{11}{3}\right) - \log_{11}\left(\dfrac{7}{9}\right)$

$= \left[\log_{11}(7) - \log_{11}(3)\right] + 2\left[\log_{11}(1) - \log_{11}(3)\right]$

$- \left[\log_{11}(11) - \log_{11}(3)\right] - \left[\log_{11}(7) - \log_{11}(9)\right]$

$= \cancel{\log_{11}(7)} - \log_{11}(3) + 2\left[0 - \log_{11}(3)\right] - \left[1 - \log_{11}(3)\right]$

$-\cancel{\log_{11}(7)} + \log_{11}(9)$

$= -\cancel{\log_{11}(3)} - 2\log_{11}(3) - 1 + \cancel{\log_{11}(3)} + \log_{11}\left(3^2\right)$

$= -2\log_{11}(3) - 1 + 2\log_{11}(3)$

$= -1$

f $\log_2\left(\dfrac{\sqrt{x} \times x^{\frac{3}{2}}}{x^2}\right) = \log_2\left(\dfrac{x^{\frac{1}{2}} \times x^{\frac{3}{2}}}{x^2}\right)$

$= \log_2\left(\dfrac{x^2}{x^2}\right)$

$= \log_2(1)$

$= 0$

8 **a** **i** $2^5 = 32 \Leftrightarrow 5 = \log_2(32)$

ii $4^{\frac{1}{3}} = 8 \Leftrightarrow \dfrac{3}{2} = \log_4(8)$

iii $10^{-3} = 0.001 \Leftrightarrow -3 = \log_{10}(0.001)$

b **i** $\log_2(16) = 4 \Leftrightarrow 2^4 = l6$

ii $\log_9(3) = \dfrac{1}{2} \Leftrightarrow 9^{\frac{1}{2}} = 3$

iii $\log_{10}(0.1) = -1 \Leftrightarrow 10^{-1} = 0.1$

9 **a** $\log_{10}(2) + \log_{10}(7)$

$= \log_{10}(2 \times 7)$

$= \log_{10}(14)$

b $\log_5(4) + \log_5(11)$

$= \log_5(4 \times 11)$

$= \log_5(44)$

c $\log_3(20) - \log_3(2)$

$= \log_3(20 \div 2)$

$= \log_3(10)$

d $\log_{10}(32) - \log_{10}(4)$

$= \log_{10}(32 \div 4)$

$= \log_{10}(8)$

e $2\log_{10}(5)$

$= \log_{10}(5^2)$

$= \log_{10}(25)$

f $-3\log_5(2)$

$= \log_5(2^{-3})$

$= \log_5\left(\dfrac{1}{2^3}\right)$

$= \log_5\left(\dfrac{1}{8}\right)$

10 **a** $\log_7(7) = 1$ since

$\log_a(a) = 1$

b $\log_6(3 \div 3)$

$= \log_6(1)$

$= 0$

c $\log_2(8)$

$= \log_2\left(2^3\right)$

$= 3\log_2(2)$

$= 3 \times 1$

$= 3$

d $\log_{10}\left(1 + 3^2\right)$

$= \log_{10}(1 + 9)$

$= \log_{10}(10)$

$= 1$

e $\log_9(3)$

$= \log_9\left(\sqrt{9}\right)$

$= \log_9\left(9^{\frac{1}{2}}\right)$

$= \frac{1}{2}\log_9(9)$

$= \frac{1}{2} \times 1$

$= \frac{1}{2}$

f $2\log_2\left(\frac{1}{4}\right)$

$= 2\log_2\left(\frac{1}{2^2}\right)$

$= 2\log_2(2^{-2})$

$= 2 \times -2\log_2(2)$

$= -4\log_2(2)$

$= -4 \times 1$

$= -4$

11 a $x = \log_2\left(\frac{1}{8}\right)$

$\therefore 2^x = \frac{1}{8}$

$\therefore 2^x = 2^{-3}$

$\therefore x = -3$

b $\log_{25}(x) = -0.5$

$\therefore 25^{-0.5} = x$

$\therefore x = \frac{1}{\sqrt{25}}$

$\therefore x = \frac{1}{5}$

c $10^{2x} = 4$

$\therefore 2x = \log_{10}(4)$

$\therefore x = \frac{1}{2}\log_{10}(4)$

$\therefore x = \frac{1}{2}\log_{10}\left(2^2\right)$

$\therefore x = \frac{1}{2} \times 2\log_{10}(2)$

$\therefore x = \log_{10}(2)$

$\therefore x \simeq 0.30$

d $3 = e^{-x}$

$\therefore -x = \log_e(3)$

$\therefore x = -\log_e(3)$

$\therefore x \simeq -1.10$

e $\log_x(125) = 3$

$\therefore x^3 = 125$

$\therefore x = 5$

f $\log_x(25) = -2$

$\therefore x^{-2} = 25$

$\therefore \frac{1}{x^2} = 25$

$\therefore x^2 = \frac{1}{25}$

$\therefore x = \pm\frac{1}{5}$

However, the base of the logarithm is an element of the set $R^+ \setminus \{1\}$, so reject $x = -\frac{1}{5}$.

$\therefore x = \frac{1}{5}$

12 a $7^x = 15 \Rightarrow x = \log_7(15)$

$\log_7(15) = \frac{\log_{10}(15)}{\log_{10}(7)}$

$\therefore x \approx 1.392$

b $3^{2x+5} = 4^x$

Take logs in base 10 of both sides:

$$\log\left(3^{2x+5}\right) = \log\left(4^x\right)$$
$$(2x+5)\log(3) = x\log(4)$$
$$2x\log(3) + 5\log(3) = x\log(4)$$
$$5\log(3) = x\log(4) - 2x\log(3)$$
$$5\log(3) = x(\log(4) - 2\log(3))$$
$$x = \frac{5\log(3)}{\log(4) - 2\log(3)}$$

Correct to 3 decimal places, $x = -6.774$.

13 a $\log_5(x-1) = 2$

Convert to index form.

$x - 1 = 5^2$

$\therefore x = 25 + 1$

$\therefore x = 26$

b $\log_2(2x+1) = -1$

Convert to index form.

$2x + 1 = 2^{-1}$

$\therefore 2x = \frac{1}{2} - 1$

$\therefore 2x = -\frac{1}{2}$

$\therefore x = -\frac{1}{4}$

c $\log_x\left(\frac{1}{49}\right) = -2$

Convert to index form.

$\frac{1}{49} = x^{-2}$

$\therefore \frac{1}{49} = \frac{1}{x^2}$

$\therefore x^2 = 49$

$\therefore x = \pm 7$

However, $x > 0$, so reject $x = -7$.

$\therefore x = 7$

d $\log_x(36) - \log_x(4) = 2$

$\therefore \log_x(36 \div 4) = 2$

$\therefore \log_x(9) = 2$

$\therefore 9 = x^2$

$\therefore x = \pm 3$

However, $x > 0$, so reject $x = -3$.

$\therefore x = 3$

14 a $\log_{10}(x+5) = \log_{10}(2) + 3\log_{10}(3)$

$\therefore \log_{10}(x+5) = \log_{10}(2) + \log_{10}\left(3^3\right)$

$\therefore \log_{10}(x+5) = \log_{10}(2 \times 27)$

$\therefore \log_{10}(x+5) = \log_{10}(54)$

$\therefore x + 5 = 54$

$\therefore x = 49$

b $\log_4(2x)+\log_4(5)=3$

$$\begin{aligned}
\therefore \log_4(2x\times 5)&=3\\
\therefore \log_4(10x)&=3\\
\therefore 10x&=4^3\\
\therefore 10x&=64\\
\therefore x&=6.4
\end{aligned}$$

c $\log_{10}(x+1)=\log_{10}(x)+1$

Collect log terms together.

$$\begin{aligned}
\therefore \log_{10}(x+1)-\log_{10}(x)&=1\\
\therefore \log_{10}\left(\frac{x+1}{x}\right)&=1\\
\therefore \frac{x+1}{x}&=10^1\\
\therefore \frac{x+1}{x}&=10\\
\therefore x+1&=10x\\
\therefore 1&=9x\\
\therefore x&=\frac{1}{9}
\end{aligned}$$

d $2\log_6(3x)+3\log_6(4)-2\log_6(12)=2$

$$\begin{aligned}
2\log_6(3x)+3\log_6(4)-2\log_6(12)&=2\\
\therefore \log_6\left((3x)^2\right)+\log_6\left(4^3\right)-\log_6\left(12^2\right)&=2\\
\therefore \log_6\left(9x^2\right)+\log_6(64)-\log_6(144)&=2\\
\therefore \log_6\left(\frac{9x^2\times 64}{144}\right)&=2\\
\therefore \log_6\left(\frac{x^2\times 64}{16}\right)&=2\\
\therefore \log_6\left(4x^2\right)&=2\\
\therefore 4x^2&=6^2\\
\therefore 4x^2&=36\\
\therefore x^2&=9\\
\therefore x&=\pm 3
\end{aligned}$$

However, $x=-3$ is inadmissible, since it would create log of a negative number if substituted into the first term $2\log_6(3x)$.

$\therefore x=3$

15 $\log_2(3)-\log_2(2)=\log_2(x)+\log_2(5)$

$$\begin{aligned}
\log_2\left(\frac{3}{2}\right)&=\log_2(5x)\\
\frac{3}{2}&=5x\\
x&=\frac{3}{10}
\end{aligned}$$

16 $\log_3(x)+\log_3(2x+1)=1$

$$\begin{aligned}
\log_3(x(2x+1))&=1\\
\log_3\left(2x^2+x\right)&=1\\
2x^2+x&=3^1\\
2x^2+x&=3\\
2x^2+x-3&=0\\
(2x+3)(x-1)&=0\\
x&=-1.5,\ x=1
\end{aligned}$$

Check in $\log_3(x)+\log_3(2x+1)=1$.

If $x=-1.5$,

LHS $=\log_3(-1.5)+\log_3(-2)$ is not admissible; therefore, reject $x=-1.5$.

If $x=1$,

$$\begin{aligned}
\text{LHS}&=\log_3(1)+\log_3(3)\\
&=\log_3(3)\\
&=1\\
&=\text{RHS}
\end{aligned}$$

Therefore, $x=1$.

17 $\log_6(x)-\log_6(x-1)=2$

$$\begin{aligned}
\log_6\left(\frac{x}{x-1}\right)&=2\\
\frac{x}{x-1}&=6^2\\
\frac{x}{x-1}&=36\\
x&=36(x-1)\\
35x&=36\\
x&=\frac{36}{35}
\end{aligned}$$

Check in $\log_6(x)-\log_6(x-1)=2$.

$$\begin{aligned}
\text{LHS}&=\log_6\left(\frac{36}{35}\right)-\log_6\left(\frac{1}{35}\right)\\
&=\log_6\left(\frac{36}{35}\div\frac{1}{35}\right)\\
&=\log_6(36)\\
&=2\\
&=\text{RHS}
\end{aligned}$$

Therefore, $x=\frac{36}{35}$.

18 **a** $\log_2(2x+1)+\log_2(2x-1)=3\log_2(3)$

$$\begin{aligned}
\therefore \log_2((2x+1)(2x-1))&=\log_2\left(3^3\right)\\
\therefore \log_2\left(4x^2-1\right)&=\log_2\left(3^3\right)\\
\therefore 4x^2-1&=27\\
\therefore 4x^2&=28\\
\therefore x^2&=7\\
\therefore x&=\pm\sqrt{7}
\end{aligned}$$

Reject $x=-\sqrt{7}$ since it creates logarithms of negative numbers in the original equation

$\therefore x=\sqrt{7}$

b $\log_3(2x)+\log_3(4)=\log_3(x+12)-\log_3(2)$

$$\begin{aligned}
\therefore \log_3(2x\times 4)&=\log_3\left(\frac{x+12}{2}\right)\\
\therefore \log_3(8x)&=\log_3\left(\frac{x+12}{2}\right)\\
\therefore 8x&=\frac{x+12}{2}\\
\therefore 16x&=x+12\\
\therefore 15x&=12\\
\therefore x&=\frac{12}{15}\\
\therefore x&=\frac{4}{5}
\end{aligned}$$

c $\log_2(2x+12)-\log_2(3x)=4$

$$\begin{aligned}
\therefore \log_2\left(\frac{2x+12}{3x}\right)&=4\\
\therefore 2^4&=\frac{2x+12}{3x}
\end{aligned}$$

$\therefore 16 \times 3x = 2x + 12$
$\therefore 46x = 12$
$\therefore x = \frac{12}{46}$
$\therefore x = \frac{6}{23}$

d $\log_2(x) + \log_2(2 - 2x) = -1$
$\therefore \log_2(x(2 - 2x)) = -1$
$\therefore \log_2\left(2x - 2x^2\right) = -1$
$\therefore 2^{-1} = 2x - 2x^2$
$\therefore \frac{1}{2} = 2x - 2x^2$
$\therefore 1 = 4x - 4x^2$
$\therefore 4x^2 - 4x + 1 = 0$
$\therefore (2x - 1)^2 = 0$
$\therefore x = \frac{1}{2}$

e $\left(\log_{10}(x) + 3\right)\left(2\log_4(x) - 3\right) = 0$
Using the Null Factor Law,
$\log_{10}(x) + 3 = 0$ or $2\log_4(x) - 3 = 0$
$\therefore \log_{10}(x) = -3$ or $\log_4(x) = \frac{3}{2}$
$\therefore x = 10^{-3}$ or $x = 4^{\frac{3}{2}}$
$\therefore x = 0.001$ or $x = 8$

f $2\log_3(x) - 1 = \log_3(2x - 3)$
$\therefore \log_3\left(x^2\right) - \log_3(2x - 3) = 1$
$\therefore \log_3\left(\frac{x^2}{2x - 3}\right) = 1$
$\therefore \frac{x^2}{2x - 3} = 3^1$
$\therefore x^2 = 6x - 9$
$\therefore x^2 - 6x + 9 = 0$
$\therefore (x - 3)^2 = 0$
$\therefore x = 3$

19 a $3^{x+2} = 7$
Express in logarithm form.
$\therefore x + 2 = \log_3(7)$
$\therefore x = \log_3(7) - 2$

b $2 \times 6^{3x-5} = 10$
Divide both sides by 2 so the equation is in index form.
$\therefore 6^{3x-5} = 5$
Convert to logarithm form.
$\therefore 3x - 5 = \log_6(5)$
$\therefore 3x = \log_6(5) + 5$
$\therefore x = \frac{1}{3}\log_6(5) + \frac{5}{3}$

c $2^{3-x} = 5^x$
Take logs in base 10 of both sides.
$\therefore \log_{10}\left(2^{3-x}\right) = \log_{10}(5^x)$
$\therefore (3 - x)\log(2) = x\log(5)$
$\therefore 3\log(2) - x\log(2) = x\log_6(5)$
$\therefore 3\log(2) = x\log(5) + x\log(2)$
$\therefore 3\log(2) = x\left(\log_6(5) + \log(2)\right)$
$\therefore 3\log(2) = x(\log_{10}(5 \times 2))$
$\therefore 3\log(2) = x\log_{10}(10)$
$\therefore 3\log(2) = x \times 1$
$\therefore x = 3\log(2)$

20 Given $\log_a(3) = p$ and $\log_a(5) = q$.

a $\log_a(15) = \log_a(3 \times 5)$
$= \log_a(3) + \log_a(5)$
$= p + q$

b $\log_a(125) = \log_a\left(5^3\right)$
$= 3\log_a(5)$
$= 3q$

c $\log_a(45) = \log_a(9 \times 5)$
$= \log_a\left(3^2 \times 5\right)$
$= \log_a\left(3^2\right) + \log_a(5)$
$= 2\log_a(3) + \log_a(5)$
$= 2p + q$

d $\log_a(0.6) = \log_a\left(\frac{3}{5}\right)$
$= \log_a(3) - \log_a(5)$
$= p - q$

e $\log_a\left(\frac{25}{81}\right) = \log_a(25) - \log_a(81)$
$= \log_a\left(5^2\right) - \log_a\left(3^4\right)$
$= 2\log_a(5) - 4\log_a(3)$
$= 2q - 4p$

f $\log_a\left(\sqrt{5}\right) \times \log_a\left(\sqrt{27}\right) = \log_a\left(5^{\frac{1}{2}}\right) \times \log_a\left(\sqrt{3^3}\right)$
$= \log_a\left(5^{\frac{1}{2}}\right) \times \log_a\left(3^{\frac{3}{2}}\right)$
$= \frac{1}{2}\log_a(5) \times \frac{3}{2}\log_a(3)$
$= \frac{3}{4}\log_a(5) \times \log_a(3)$
$= \frac{3}{4}qp$

21 a $2^{\log_5(x)} = 8$
$\therefore 2^{\log_5(x)} = 2^3$
$\therefore \log_5(x) = 3$
$\therefore 5^3 = x$
$\therefore x = 125$

b $2^{\log_2(x)} = 7$
$\therefore \log_2(x) = \log_2(7)$
$\therefore x = 7$

22 a $\log_{10}(y) = \log_{10}(x) + 2$
$\therefore \log_{10}(y) - \log_{10}(x) = 2$
$\therefore \log_{10}\left(\frac{y}{x}\right) = 2$
$\therefore 10^2 = \frac{y}{x}$
$\therefore y = 100x$

b $x = 10^{y-2}$
$\therefore y - 2 = \log_{10}(x)$
$\therefore y = \log_{10}(x) + 2$

c $\log_{10}\left(10^{3xy}\right) = 3$
$\therefore 3xy\log_{10}(10) = 3$
$\therefore 3xy = 3$
$\therefore y = \frac{1}{x}$

d $10^{3\log_{10}(y)} = xy$

$\therefore 3\log_{10}(y) = \log_{10} xy$

$\therefore \log_{10}\left(y^3\right) = \log_{10}(xy)$

$\therefore y^3 = xy$

$\therefore y^3 - xy = 0$

$\therefore y\left(y^2 - x\right) = 0$

$\therefore y = 0$ (reject) or $y^2 = x$

Both x and y must be positive.

$\therefore y = \sqrt{x}, x > 0$

23 **a** $2^{2x} - 14 \times 2^x + 45 = 0$

Let $a = 2^x$.

$\therefore a^2 - 14a + 45 = 0$

$\therefore (a-5)(a-9) = 0$

$\therefore a = 5 \text{ or } a = 9$

$\therefore 2^x = 5 \text{ or } 2^x = 9$

$\therefore x = \log_2(5) \text{ or } x = \log_2(9)$

b $5^{-x} - 5^x = 4$

$\therefore \frac{1}{5^x} - 5^x = 4$

Let $a = 5^x$.

$\therefore \frac{1}{a} - a = 4$

$\therefore 1 - a^2 = 4a$

$\therefore a^2 + 4a - 1 = 0$

Completing the square,

$\left(a^2 + 4a + 4\right) - 4 - 1 = 0$

$\therefore (a+2)^2 = 5$

$\therefore a + 2 = \pm\sqrt{5}$

$\therefore a = \sqrt{5} - 2 \text{ or } a = -\sqrt{5} - 2$

$\therefore 5^x = \sqrt{5} - 2 \text{ or } 5^x = -\sqrt{5} - 2$

As $5^x > 0$, reject

$5^x = -\sqrt{5} - 2$

$\therefore 5^x = \sqrt{5} - 2$

$\therefore x = \log_5\left(\sqrt{5} - 2\right)$

c $9^{2x} - 3^{1+2x} + 2 = 0$

$\therefore 9^{2x} - 3^1 \times 3^{2x} + 2 = 0$

$\therefore 9^{2x} - 3 \times 9^x + 2 = 0$

Let $a = 9^x$.

$\therefore a^2 - 3a + 2 = 0$

$\therefore (a-1)(a-2) = 0$

$\therefore a = 1 \text{ or } a = 2$

$\therefore 9^x = 1 \text{ or } 9^x = 2$

$\therefore x = 0 \text{ or } x = \log_9(2)$

d $\log_a\left(x^3\right) + \log_a\left(x^2\right) - 4\log_a(2) = \log_a(x)$

$\therefore \log_a\left(x^3\right) + \log_a\left(x^2\right) - \log_a(x) = 4\log_a(2)$

$\therefore \log_a\left(\frac{x^3 \times x^2}{x}\right) = \log_a\left(2^4\right)$

$\therefore \log_a\left(x^4\right) = \log_a\left(2^4\right)$

$\therefore 4\log_a(x) = 4\log_a(2)$

$\therefore \log_a(x) = \log_a(2)$

$\therefore x = 2$

24 **a** $11^x = 18$

$\therefore x = \log_{11}(18)$ is the exact solution.

Changing the base to 10,

$\log_{11}(18) = \frac{\log_{10}(18)}{\log_{10}(11)}$

$\simeq 1.205$

$\therefore x \simeq 1.205$

b $5^{-x} = 8$

$\therefore -x = \log_5(8)$

$\therefore x = -\log_5(8)$

Hence,

$x = -\frac{\log_{10}(8)}{\log_{10}(5)}$

$\therefore x \simeq -1.292$

c $7^{2x} = 3$

$\therefore 2x = \log_7(3)$

$\therefore x = \frac{1}{2}\log_7(3)$

Hence,

$x = \frac{1}{2} \times \frac{\log_{10}(3)}{\log_{10}(7)}$

$\therefore x \simeq 0.2823$

25 **a** $3^x \leq 10$

$\therefore x \leq \log_3(10)$

$\therefore x \leq \frac{\log_{10}(10)}{\log_{10}(3)}$

$\therefore x \leq \frac{1}{\log_{10}(3)}$

$\therefore x \leq 2.096$

b $5^{-x} > 0.4$

$\therefore -x > \log_5(0.4)$

$\therefore x < -\log_5(0.4)$

$\therefore x < -\frac{\log_{10}(0.4)}{\log_{10}(5)}$

$\therefore x < \frac{1}{\log_{10}(3)}$

$\therefore x < 0.5693$

10.3 Exam questions

1 $\log_6(42) - \log_6(7) = \log_6\frac{42}{7}$

$= \log_6 6$

$= 1$

The correct answer is **D**.

2 The statements $n = a^x$ and $x = 1\log_a(n)$ are equivalent.

$\therefore y = a^x \Rightarrow x = \log_a(y)$

The correct answer is **C**.

3 $\log_{12}(x) + \log_{12}(x+1) = 1$

$\log_{12}(x(x+1)) = 1$

$\log_{12}(x(x+1)) = \log_{12} 12$ [1 mark]

$x^2 + x = 12$

$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$\therefore x = -4,\ x = 3$ [1 mark]

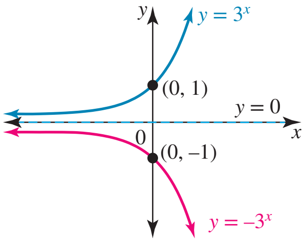
As $x > 0$, $x = -4$ is not valid solution.

$\therefore x = 3$ [1 mark]

10.4 Graphs of exponential functions

10.4 Exercise

1 a



For $y = 3^x$, the range is R^+, and for $y = -3^x$, the range is R^-. The asymptote is $y = 0$ for both graphs.

b The graph has a 'decay' shape, so $y = a^{-x}$. The point $(-1, 3)$ lies on the graph.

$\therefore 3 = a^1$

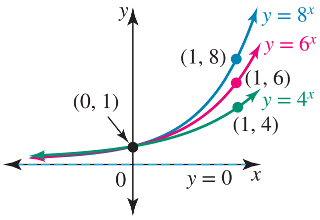
$\therefore a = 3$

The equation is $y = 3^{-x}$.

2 a i $y = 4^x, y = 6^x$ and $y = 8^x$.

Each graph passes through the point $(0, 1)$ and the line $y = 0$ is the asymptote for each graph.

If $x = 1$, the graphs of $y = 4^x, y = 6^x$ and $y = 8^x$ pass through $(1, 4), (1, 6)$ and $(1, 8)$, respectively.

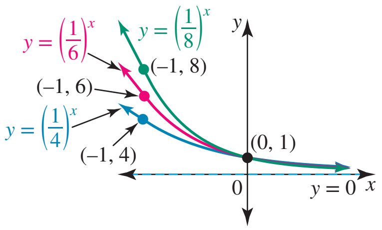


ii For $x > 0$, as the base increases, the steepness of the graph increases.

b i $y = \left(\frac{1}{4}\right)^x, y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$.

Each graph passes through the point $(0, 1)$ and the line $y = 0$ is the asymptote for each graph.

If $x = -1$, the graphs of $y = \left(\frac{1}{4}\right)^x, y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$ pass through $(-1, 4), (-1, 6)$ and $(-1, 8)$ respectively.

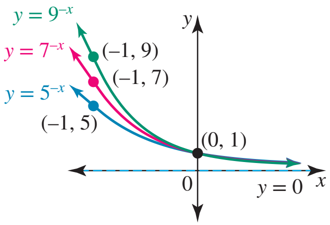


ii The rules for the graphs can be expressed as $y = 4^{-x}, y = 6^{-x}$ and $y = 8^{-x}$.

3 $y = 5^{-x}, y = 7^{-x}$ and $y = 9^{-x}$.

Each graph passes through the point $(0, 1)$ and the line $y = 0$ is the asymptote for each graph

If $x = -1$, the graphs of $y = 5^{-x}, y = 7^{-x}$ and $y = 9^{-x}$ pass through $(-1, 5), (-1, 7)$ and $(-1, 9)$ respectively.



As the base increases, the decrease of the graph is more steep for $x < 0$.

4 a $y = 4^x - 2$

asymptote: $y = -2$

y-intercept $x = 0, y = 1 - 2 \Rightarrow (0, -1)$

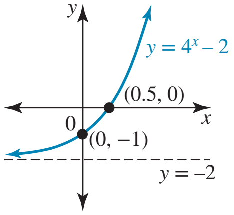
x-intercept: when $y = 0$,

$4^x - 2 = 0$

$4^x = 2$

$4^x = 4^{\frac{1}{2}}$

$x = \frac{1}{2}$



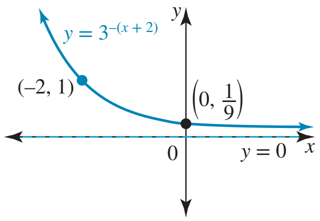
Range is $(-2, \infty)$.

b $y = 3^{-(x+2)}$

asymptote: $y = 0$

y-intercept $x = 0, y = 3^{-2} \Rightarrow \left(0, \frac{1}{9}\right)$

Point: when $x = -2, y = 1$.



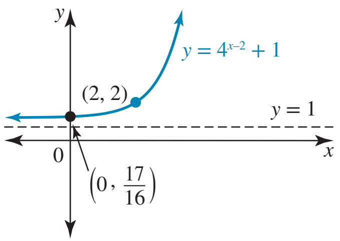
The range is R^+.

5 $y = 4^{x-2} + 1$

asymptote: $y = 1$

y-intercept $x = 0, y = 4^{-2} + 1 \Rightarrow \left(0, 1\frac{1}{16}\right)$

Point: when $x = 2, y = 2$



The range is $(1, \infty)$.

6 Any function of the form $y = a \times n^x + k$ has a horizontal asymptote $y = k$.

a $y = 3^{2x} - 1$

i The horizontal asymptote has the equation $y = -1$.

ii y-intercept: let $x = 0$.

$y = 3^{2(0)} - 1$

$= 3^0 - 1$

$= 1 - 1$

$= 0$

$(0, 0)$

iii As the y-intercept lies above the asymptote, the range is $(-1, \infty)$.

b $y = 2^{-x} + 3$

i The horizontal asymptote has the equation $y = 3$.

ii y-intercept: let $x = 0$.

$y = 2^{-(0)} + 3$

$= 1 + 3$

$= 4$

$(0, 4)$

iii As the y-intercept lies above the asymptote, the range is $(3, \infty)$.

c $y = 2 - 5^{1-x}$

$\therefore y = -5^{1-x} + 2$

i The horizontal asymptote has the equation $y = 2$.

ii y-intercept: let $x = 0$.

$y = -5^{(1-0)} + 2$

$= -5^1 + 2$

$= -3$

$(0, -3)$

iii As the y-intercept lies above the asymptote, the range is $(-1, \infty)$.

d $y = 10 \times 2^{3x}$

i The equation of the asymptote is $y = 0$.

ii y-intercept: let $x = 0$.

$y = 10 \times 2^{(0)}$

$= 10 \times 1$

$= 10$

$(0, 10)$,

iii As the y-intercept lies above the asymptote, the range is $(0, \infty)$ or R^+.

7 a $y = 2^{\frac{x}{2}} - 128$

y-intercept: let $x = 0$.

$y = 2^{\frac{0}{2}} - 128$

$= 2^0 - 128$

$= 1 - 128$

$= -127$

$(0, -127)$

x-intercept: let $y = 0$.

$0 = 2^{\frac{x}{2}} - 128$

$\therefore 2^{\frac{x}{2}} = 128$

$\therefore 2^{\frac{x}{2}} = 2^7$

$\therefore \frac{x}{2} = 7$

$\therefore x = 14$

$(14, 0)$

b $y = 5^{-2x} - 10$

y-intercept: let $x = 0$.

$y = 5^{-2(0)} - 10$

$= 5^0 - 10$

$= 1 - 10$

$= -9$

$(0, -9)$

x-intercept: let $y = 0$.

$0 = 5^{-2x} - 10$

$\therefore 5^{-2x} = 10$

$\therefore -2x = \log_5(10)$

$\therefore x = -\frac{1}{2}\log_5(10)$

$\left(-\frac{1}{2}\log_5(10), 0\right)$

8 a $y = 5^{-x} + 1$

Asymptote: $y = 1$

y-intercept: let $x = 0$.

$\therefore y = 5^0 + 1$

$\therefore y = 2$

$(0, 2)$

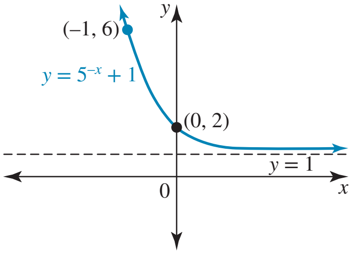
No x-intercept

Point: let $x = -1$.

$\therefore y = 5^1 + 1$

$\therefore y = 6$

$(-1, 6)$



b $y = 1 - 4^x$

$\therefore y = -4^x + 1$

Asymptote: $y = 1$

y-intercept: let $x = 0$.

$\therefore y = -4^0 + 1$

$\therefore y = 0$

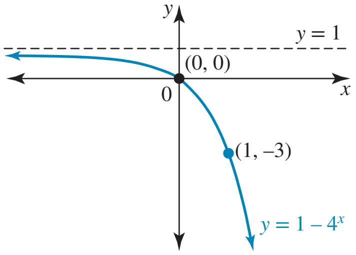
$(0, 0)$

Point: let $x = 1$.

$\therefore y = -4^1 + 1$

$\therefore y = -3$

$(1, -3)$



c $y = 10^x - 2$

Asymptote: $y = -2$

y-intercept: let $x = 0$.

$\therefore y = 10^0 - 2$

$\therefore y = -1$

$(0, -1)$

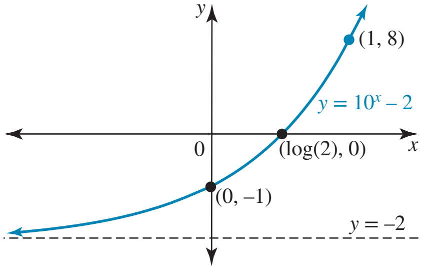
x-intercept: let $y = 0$.

$\therefore 0 = 10^x - 2$

$\therefore 10^x = 2$

$\therefore x = \log_{10}(2)$

$(\log_{10}(2), 0)$



d $y = 6.25 - (2.5)^{-x}$

Asymptote: $y = 6.25$

y-intercept: let $x = 0$.

$\therefore y = 6.25 - 1$

$\therefore y = 5.25$

$(0, 5.25)$

x-intercept: let $y = 0$.

$\therefore 0 = 6.25 - (2.5)^{-x}$

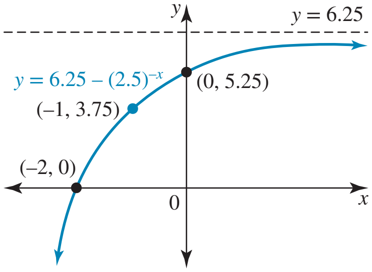
$\therefore (2.5)^{-x} = 6.25$

$\therefore (2.5)^{-x} = (2.5)^2$

$\therefore -x = 2$

$\therefore x = -2$

$(-2, 0)$



9 a $y = 2^{x-2}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 2^{-2}$

$\therefore y = \frac{1}{4}$

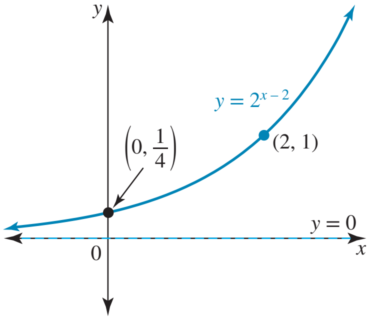
$\left(0, \frac{1}{4}\right)$

Point: let $x = 2$.

$\therefore y = 2^0$

$\therefore y = 1$

$(2, 1)$



b $y = -3^{x+2}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = -3^2$

$\therefore y = -9$

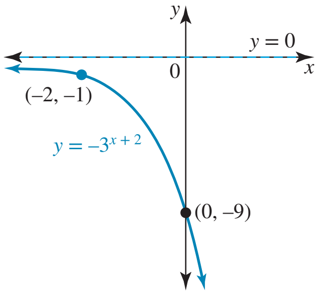
$(0, -9)$

Point: let $x = -2$.

$\therefore y = -3^0$

$\therefore y = -1$

$(-2, -1)$



c $y = 4^{x-0.5}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 4^{-0.5}$

$\therefore y = \frac{1}{\sqrt{4}}$

$\therefore y = \frac{1}{2}$

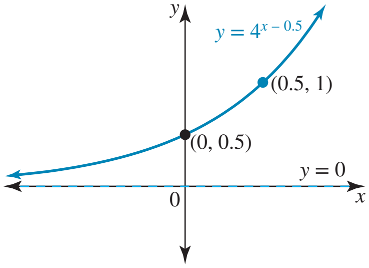
$\left(0, \frac{1}{2}\right)$

Point: let $x = 0.5$.

$\therefore y = 4^0$

$\therefore y = 1$

$(0.5, 1)$



d $y = 7^{1-x}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 7^1$

$\therefore y = 7$

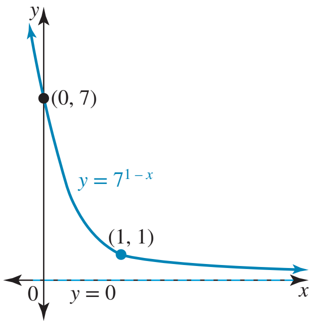
$(0, 7)$

Point: let $x = 1$.

$\therefore y = 7^0$

$\therefore y = 1$

$(1, 1)$

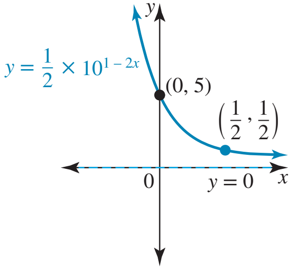


10 **a** $y = \frac{1}{2} \times 10^{1-2x}$

Asymptote: $y = 0$

y-intercept: $x = 0, y = \frac{1}{2} \times 10 \Rightarrow (0, 5)$

Point: when $x = \frac{1}{2}, y = \frac{1}{2}$.



The range is R^+.

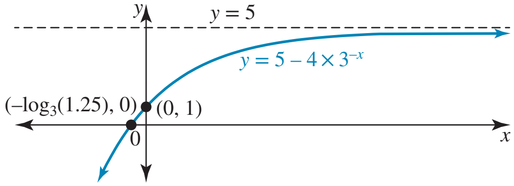
b $y = 5 - 4 \times 3^{-x}$

Asymptote: $y = 5$

y-intercept: $x = 0, y = 5 - 4 \Rightarrow (0, 1)$

x-intercept: when $y = 0$,

$$5 - 4 \times 3^{-x} = 0$$
$$4 \times 3^{-x} = 5$$
$$3^{-x} = \frac{5}{4}$$
$$-x = \log_3\left(\frac{5}{4}\right)$$
$$-x = \frac{\log_{10}\left(\frac{5}{4}\right)}{\log_{10}(3)}$$
$$x \approx -0.203$$



The range is $(-\infty, 5)$.

11 **a** $y = 3 \times 2^x$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 3 \times 1$

$\therefore y = 3$

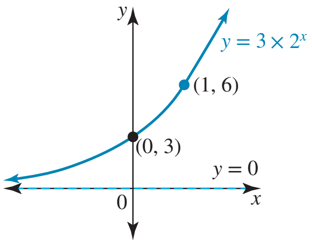
$(0, 3)$

Point: let $x = 1$.

$\therefore y = 3 \times 2^1$

$\therefore y = 6$

$(1, 6)$



b $y = 2^{\frac{3x}{4}}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 2^0$

$\therefore y = 1$

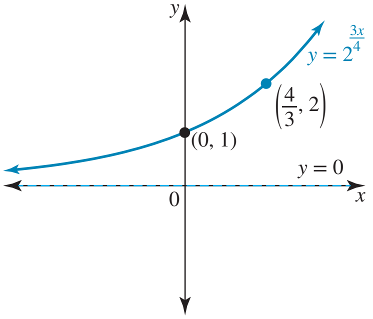
$(0, 1)$

Point: let $x = \frac{4}{3}$.

$\therefore y = 2^1$

$\therefore y = 2$

$\left(\frac{4}{3}, 2\right)$



c $y = -3 \times 2^{-3x}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = -3 \times 2^0$

$\therefore y = -3$

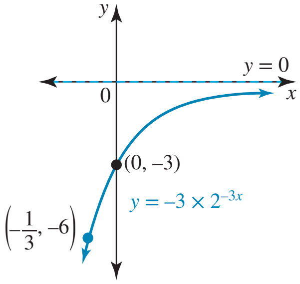
$(0, -3)$

Point: let $x = -\frac{1}{3}$.

$\therefore y = -3 \times 2^1$

$\therefore y = -6$

$\left(-\frac{1}{3}, -6\right)$



d $y = 1.5 \times 10^{-\frac{x}{2}}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 1.5 \times 10^0$

$\therefore y = 1.5$

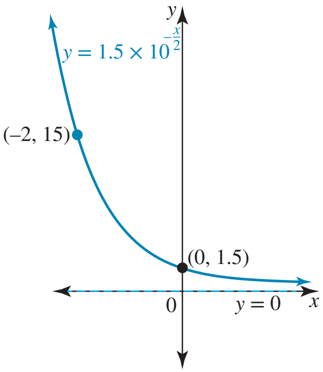
$(0, 1.5)$

Point: let $x = -2$.

$\therefore y = 1.5 \times 10^1$

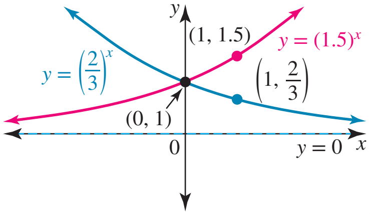
$\therefore y = 15$

$(-2, 15)$



12 $y = (1.5)^x$ $y = \left(\frac{2}{3}\right)^x$

Asymptote: $y = 0$	Asymptote: $y = 0$
y-intercept: if $x = 0, y = 1$	y-intercept: if $x = 0, y = 1$
Point: $x = 1, y = 1.5$	Point: $x = 1, y = \frac{2}{3}$



Note that $1.5 = \frac{3}{2}, y = (1.5)^x = \left(\frac{3}{2}\right)^x$, and since $\left(\frac{2}{3}\right) = \left(\frac{3}{2}\right)^{-1}, y = \left(\frac{2}{3}\right)^x = \left(\frac{3}{2}\right)^{-x}$

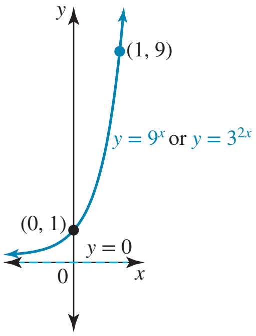
13 **a** $y = 3^{2x}$ and $y = 9^x$ are identical equations since:

$y = 9^x$

$= \left(3^2\right)^x$

$= 3^{2x}$

The graph has an asymptote when $y = 0$ and passes through the points $(0, 1)$ and $(1, 9)$.



b **i** $y = 2 \times 4^{0.5x}$

$\therefore y = 2 \times \left(2^2\right)^{0.5x}$

$\therefore y = 2 \times 2^x$

$\therefore y = 2^{1+x}$

$\therefore y = 2^{x+1}$

ii $y = 2^{x+1}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 2^1$

$\therefore y = 2$

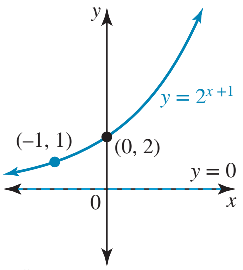
$(0, 2)$

Point: let $x = -1$.

$\therefore y = 2^0$

$\therefore y = 1$

$(-1, 1)$



14 **a** $y = 2^x + k$

y-intercept at $y = 7$ means the point $(0, 7)$ lies on the function.

$\therefore 7 = 2^0 + k$

$\therefore 7 = 1 + k$

$\therefore k = 6$

The rule is $y = 2^x + 6$. The domain is R.

As the asymptote $y = 6$ lies below the point $(0, 7)$, the range is $(6, \infty)$.

b $y = 5^{ax} + b$

Substitute the point $(0, -4)$.

$\therefore -4 = 5^{a(0)} + b$

$\therefore -4 = 5^0 + b$

$\therefore -4 = 1 + b$

$\therefore b = -5$

Substitute the point $(2, 0)$ in $y = 5^{ax} - 5$.

$\therefore 0 = 5^{a(2)} - 5$

$\therefore 5 = 5^{2a}$

$\therefore 2a = 1$

$\therefore a = \frac{1}{2}$

The rule is $y = 5^{\frac{1}{2}x} - 5$. The domain is R.

As the given points lie above the asymptote $y = -5$, the range is $(-5, \infty)$.

c $y = b - a \times 3^{-x}$

Rearranging, $y = -a \times 3^{-x} + b$.

The asymptote is $y = b$. Given the asymptote is $y = 10, b = 10$.

Substitute the point $(0, 0)$ in $y = -a \times 3^{-x} + 10$.

$\therefore 0 = -a \times 3^{-(0)} + 10$

$\therefore 0 = -a \times 1 + 10$

$\therefore a = 10$

$\therefore y = -10 \times 3^{-x} + 10$

The rule is $y = 10 - 10 \times 3^{-x}$. The domain is R.

As the given point (the origin) lies below the asymptote, the range is $(-\infty, 10)$.

15 $y = a.3^x + b$

The asymptote is at $y = 2$, so $b = 2$.

$y = a.3^x + 2$

The graph passes through the origin.

$\Rightarrow 0 = a + 2$

$\therefore a = -2$

Therefore, the equation is $y = -2 \times 3^x + 2$ and $a = -2$, $b = 2$.

16 **a** $y = a.10^x + b$

The asymptote equation is $y = 3$, so $b = 3$.

$\therefore y = a.10^x + 3$

Substitute the point $(0, 5)$

$\therefore 5 = a.10^0 + 3$

$\therefore 5 = a + 3$

$\therefore a = 2$

The rule for the graph is $y = 2 \times 10^x + 3$.

b $y = a.3^{kx}$

Point $(1, 36) \Rightarrow 36 = a.3^k \ldots$ [1]

Point $(0, 4) \Rightarrow 4 = a.3^0 \Rightarrow a = 4$

Substitute $a = 4$ in equation [1]:

$\therefore 36 = 4 \times 3^k$

$\therefore 3^k = 9$

$\therefore k = 2$

Hence, the rule is $y = 4 \times 3^{2x}$ and the asymptote equation is $y = 0$.

c $y = a - 2 \times 3^{b-x}$

Asymptote $y = 6 \Rightarrow a = 6$

$\therefore y = 6 - 2 \times 3^{b-x}$

Substitute the point $(0, 0)$.

$\therefore 0 = 6 - 2 \times 3^b$

$\therefore 2 \times 3^b = 6$

$\therefore 3^b = 3$

$\therefore b = 1$

The rule is $y = 6 - 2 \times 3^{1-x}$.

d $y = 6 - 2 \times 3^{1-x}$

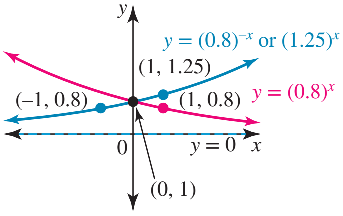
$\therefore y = 6 - 2 \times 3^1 \times 3^{-x}$

$\therefore y = 6 - 6 \times 3^{-x}$

17 $y = (0.8)^x$, $y = (1.25)^x$ and $y = (0.8)^{-x}$

Each graph passes through the point $(0, 1)$ and the line $y = 0$ is the asymptote for each graph

The point $(1, 0.8)$ lies on $y = (0.8)^x$, the point $(1, 1.25)$ lies on $y = (1.25)^x$ and the point $(-1, 0.8)$ lies on $y = (0.8)^{-x}$.



The graphs of $y = (0.8)^{-x}$ and $y = (1.25)^x$ are the same and the graph of $y = (0.8)^{-x}$ is the reflection in the y-axis of the graph of $y = (0.8)^x$. This is because:

$y = (0.8)^{-x}$

$= \left(\frac{4}{5}\right)^{-x}$

$= \left(\frac{5}{4}\right)^{x}$

$y = (1.25)^x$

$= \left(\frac{5}{4}\right)^{x}$

$y = (0.8)^x$

$= \left(\frac{4}{5}\right)^{x}$ or $\left(\frac{5}{4}\right)^{-x}$

18 **a** $y = 2 \times 10^{2x} - 20$

Asymptote: $y = -20$

y-intercept: let $x = 0$.

$\therefore y = 2 \times 10^0 - 20$

$\therefore y = -18$

$(0, -18)$

Range $(-20, \infty)$

x-intercept: let $y = 0$.

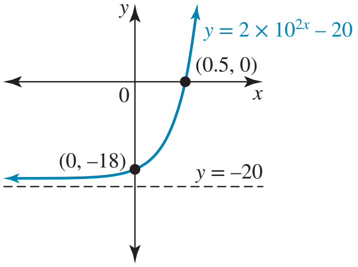
$\therefore 0 = 2 \times 10^{2x} - 20$

$\therefore 10^{2x} = 10$

$\therefore 2x = 1$

$\therefore x = \frac{1}{2}$

$(0.5, 0)$



b $y = 5 \times 2^{1-x} - 1$

Asymptote: $y = -1$

y-intercept: let $x = 0$.

$\therefore y = 5 \times 2^1 - 1$

$\therefore y = 9$

$(0, 9)$

Range $(-1, \infty)$

x-intercept: let $y = 0$.

$\therefore 0 = 5 \times 2^{1-x} - 1$

$\therefore 5 \times 2^{1-x} = 1$

$\therefore 2^{1-x} = \frac{1}{5}$

$\therefore \log\left(2^{1-x}\right) = \log(0.2)$

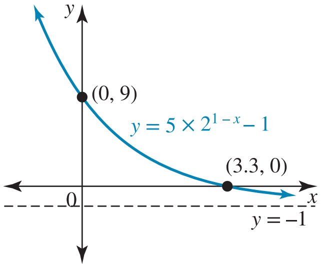
$\therefore (1 - x)\log(2) = \log(0.2)$

$\therefore 1 - x = \frac{\log(0.2)}{\log(2)}$

$\therefore 1 - \frac{\log(0.2)}{\log(2)} = x$

$\therefore x \simeq 3.3$

$(3.3, 0)$



c $y = 3 - 2\left(\frac{2}{3}\right)^x$

Asymptote: $y = 3$

y-intercept: let $x = 0$.

$\therefore y = 3 - 2 \times 1$

$\therefore y = 1$

$(0, 1)$

Range $(-\infty, 3)$

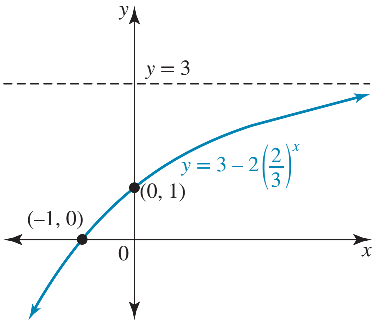
x-intercept: let $y = 0$.

$$\therefore 0 = 3 - 2\left(\frac{2}{3}\right)^x$$

$$\therefore \left(\frac{2}{3}\right)^x = \frac{3}{2}$$

$$\therefore x = -1$$

$(-1, 0)$



d $y = 2\,(3.5)^{x+1} - 7$

Asymptote: $y = -7$

y-intercept: let $x = 0$.

$\therefore y = 2 \times (3.5)^1 - 7$

$\therefore y = 0$

$(0, 0)$

Range $(-7, \infty)$

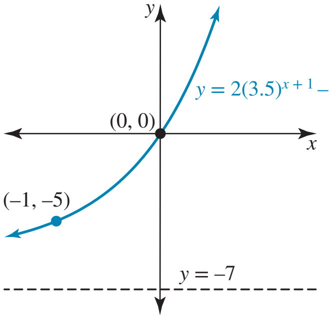
x-intercept: $(0, 0)$

Point: let $x = -1$.

$\therefore y = 2 \times 1 - 7$

$\therefore y = -5$

$(-1, -5)$



e $y = 8 - 4 \times 5^{2x-1}$

Asymptote: $y = 8$

y-intercept: let $x = 0$.

$\therefore y = 8 - 4 \times 5^{-1}$

$\therefore y = 8 - \frac{4}{5}$

$\therefore y = \frac{36}{5}$

$(0, 7.2)$

Range $(-\infty, 8)$

x-intercept: let $y = 0$.

$$\therefore 0 = 8 - 4 \times 5^{2x-1}$$

$$\therefore 4 \times 5^{2x-1} = 8$$

$$\therefore 5^{2x-1} = 2$$

$$\therefore \log\left(5^{2x-1}\right) = \log(2)$$

$$\therefore (2x - 1)\log(5) = \log(2)$$

$$\therefore 2x\log(5) - \log(5) = \log(2)$$

$$\therefore 2x\log(5) = \log(2) + \log(5)$$

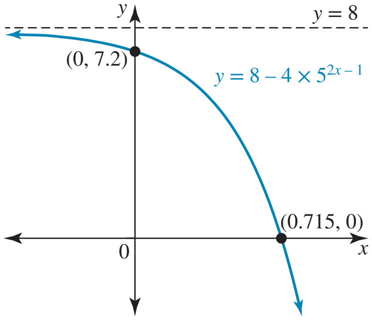
$$= \log(10)$$

$$= 1$$

$$\therefore x = \frac{1}{2\log(5)}$$

$\therefore x \simeq 0.7$

$(0.7, 0)$



f $y = -2 \times 10^{3x-1} - 4$

Asymptote: $y = -4$

y-intercept: let $x = 0$.

$\therefore y = -2 \times 10^{-1} - 4$

$\therefore y = -0.2 - 4$

$= -4.2$

$(0, -4.2)$

Range $(-\infty, -4)$

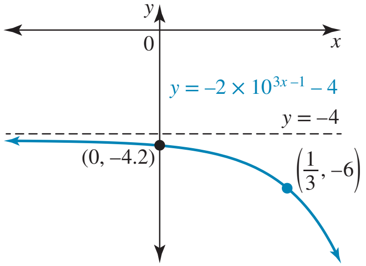
No x-intercept

Point: let $x = \frac{1}{3}$.

$\therefore y = -2 \times 1 - 4$

$\therefore y = -6$

$\left(\frac{1}{3}, -6\right)$



19 $f : R \to R, f(x) = 3 - 6 \times 2^{\frac{x-1}{2}}$

a i $f(1) = 3 - 6 \times 2^0$

$\therefore f(1) = -3$

ii $f(0) = 3 - 6 \times 2^{-\frac{1}{2}}$

$$= +3 - \frac{6}{\sqrt{2}}$$

$$= +3 - \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 3 - 3\sqrt{2}$$

b i $f(x) = -9$

$$\therefore 3 - 6 \times 2^{\frac{x-1}{2}} = -9$$

$$\therefore 12 = 6 \times 2^{\frac{x-1}{2}}$$

$$\therefore 2^{\frac{x-1}{2}} = 2$$

$$\therefore \frac{x-1}{2} = 1$$
$$\therefore x - 1 = 2$$
$$\therefore x = 3$$

ii $f(x) = 0$

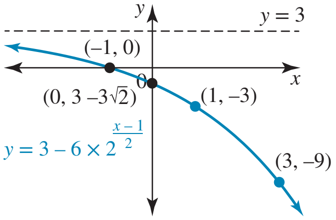
$$\therefore 3 - 6 \times 2^{\frac{x-1}{2}} = 0$$
$$\therefore 3 = 6 \times 2^{\frac{x-1}{2}}$$
$$\therefore 2^{\frac{x-1}{2}} = \frac{1}{2}$$
$$\therefore \frac{x-1}{2} = -1$$
$$\therefore x - 1 = -2$$
$$\therefore x = -1$$

iii $f(x) = 9$

$$\therefore 3 - 6 \times 2^{\frac{x-1}{2}} = 9$$
$$\therefore -6 = 6 \times 2^{\frac{x-1}{2}}$$
$$\therefore 2^{\frac{x-1}{2}} = -1$$

As $2^{\frac{x-1}{2}} > 0$, there are no values of x for which $f(x) = 9$.

c From previous calculations, the points $(1, -3)$, $\left(0, 3 - 3\sqrt{2}\right)$, $(3, -9)$ and $(-1, 0)$ lie on the function's graph. The equation of the asymptote is $y = 3$; the range is $(-\infty, 3)$.



d Let $f(x) = -1$.

$$\therefore -1 = 3 - 6 \times 2^{\frac{x-1}{2}}$$
$$\therefore 6 \times 2^{\frac{x-1}{2}} = 4$$
$$\therefore 2^{\frac{x-1}{2}} = \frac{2}{3}$$
$$\therefore \log\left(2^{\frac{x-1}{2}}\right) = \log\left(\frac{2}{3}\right)$$
$$\therefore \frac{x-1}{2}\log(2) = \log\left(\frac{2}{3}\right)$$
$$\therefore x - 1 = \frac{2\log\left(\frac{2}{3}\right)}{\log(2)}$$
$$\therefore x = \frac{2\log\left(\frac{2}{3}\right)}{\log(2)} + 1$$
$$\therefore x \simeq -0.17$$

Using the graph, $f(x) \geq -1$ for $x \leq -0.17$.

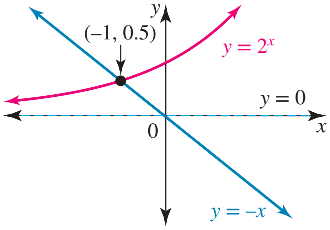
20 a Let $y_1 = 2^x$ and $y_2 = -x$.

When $x = 0$, $y_1 = 1$ and $y_2 = 0$; $y_1 > y_2$.

When $x = -\frac{1}{2}$, $y_1 = \frac{1}{\sqrt{2}} \simeq 0.7$ and $y_2 = 0.5$, $y_1 > y_2$.

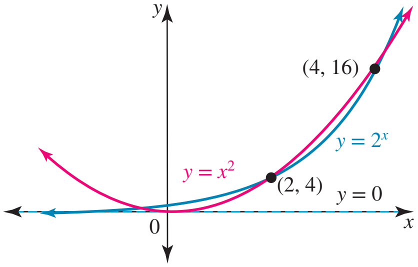
When $x = -1$, $y_1 = \frac{1}{2} = 0.5$ and $y_2 = 1$, $y_1 < y_2$.

Therefore, $y_1 = y_2$ for some $x \in (-1, -0.5)$.



There is one point of intersection for which $x \in (-1, -0.5)$.

b $y = 2^x$ and $y = x^2$ both contain the points $(2, 4)$ and $(4, 16)$.



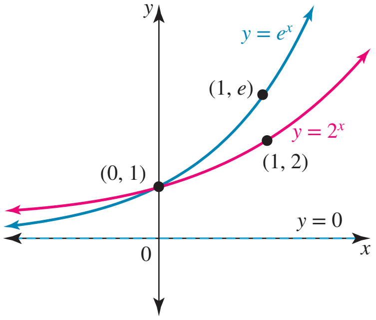
From the diagram there is a point of intersection for $x \in (-1, 0)$, so there are three points of intersection.

c $y = e^x$ and $y = 2^x$ both contain the point $(0, 1)$.

As $e > 2$, the graph of $y = e^x$ should be steeper than that of $y = 2^x$, for $x > 0$.

$y = e^x$ contains the point $(1, e)$, approximately $(1, 2.7)$ and $y = 2^x$ contains the point $(1, 2)$.

There will only one point of intersection, as confirmed by the diagram.



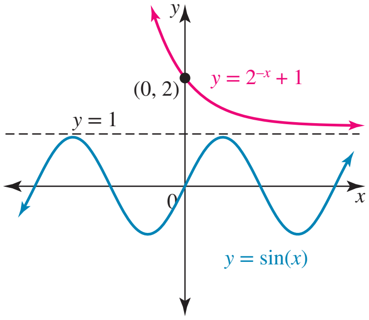
d $y = 2^{-x} + 1$ and $y = \sin(x)$

Consider $y = 2^{-x} + 1$:

The asymptote is $y = 1$ and the y-intercept is $(0, 2)$, so the range is $(1, \infty)$.

As the range of $y = \sin(x)$ is $[-1, 1]$, there will not be any intersections of the two graphs.

This is confirmed by the diagram.



e $y = 3 \times 2^x$ and $y = 6^x$.

At intersection, $3 \times 2^x = 6^x$,

$$\therefore 3 = \frac{6^x}{2^x}$$

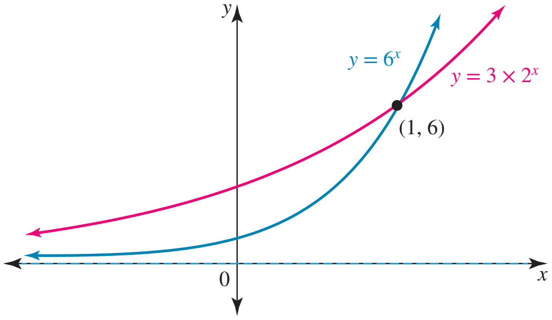
$$\therefore 3 = \left(\frac{6}{2}\right)^x$$

$$\therefore 3 = 3^x$$

$$\therefore x = 1$$

There is one point of intersection, $(1, 6)$.

Both $y = 3 \times 2^x$ and $y = 6^x$ have asymptote $y = 0$; they have y-intercepts of $(0, 3)$ and $(0, 1)$, respectively.



f $y = 2^{2x-1}$ and $y = \frac{1}{2} \times 16^{\frac{x}{2}}$

Consider $y = \frac{1}{2} \times 16^{\frac{x}{2}}$. This can be expressed as

$$\begin{aligned} y &= \frac{1}{2} \times \left(2^4\right)^{\frac{x}{2}} \\ &= 2^{-1} \times 2^{2x} \\ &= 2^{2x-1} \end{aligned}$$

The two curves are identical and therefore have an infinite number of intersections. The coordinates of the points of intersection are of the form $\left(t, 2^{2t-1}\right)$, $t \in R$.

10.4 Exam questions

1 For $x \in R, y > 0$

$\therefore$ range R^+

The correct answer is **E**.

2 $y = a^{-x}$ where $a > 1$:

- horizontal asymptote with equation, $y = 0$
- y-intercept $(0, 1)$
- shape of ‘exponential decay’
- domain R
- range R^+
- one-to-one correspondence

$y = 10^{-x}$

The correct answer is **B**.

3 Intercepts:

y-intercept $(x = 0)$

$$\begin{aligned} y &= 9 \times 3^{3x-2} \\ &= 9 \times 3^{0-2} \\ &= 9 \times 3^{-2} \\ &= 9 \times \frac{1}{9} \\ &= 1 \end{aligned}$$

y-intercept is $(0, 1)$ [1 mark]

x-intercept $(y = 0)$

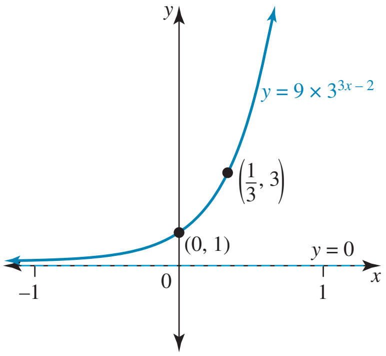
No intercept; the x-axis is an asymptote.

The second point can be found by substituting $x = \frac{1}{3}$.

$$\begin{aligned} y &= 9 \times 3^{1-2} \\ &= 9 \times 3^{-1} \\ &= 9 \times \frac{1}{3} \\ &= 3 \end{aligned}$$

$\left(\frac{1}{3}, 3\right)$ [1 mark]

The range is R^+. [1 mark]



[1 mark]

10.5 Applications of exponential functions

10.5 Exercise

1 $V = V_0 \times 2^{-kt}$

a When $t = 0$, $V = V_0$, so V_0 is the purchase price.

When $t = 5$, $V = \frac{1}{2} V_0$.

$$\therefore \frac{1}{2} V_0 = V_0 \times 2^{-5k}$$

$$\therefore \frac{1}{2} = 2^{-5k}$$

$$\therefore 2^{-1} = 2^{-5k}$$

$$\therefore -1 = -5k$$

$$\therefore k = \frac{1}{5} \text{ or } 0.2$$

b The model is $V = V_0 \times 2^{-0.2t}$.

When 75% of the purchase price is lost, 25% remains.

Let $V = 0.25 V_0$.

$$\therefore 0.25 V_0 = V_0 \times 2^{-0.2t}$$

$$\therefore \frac{1}{4} = 2^{-0.2t}$$

$$\therefore 2^{-2} = 2^{-0.2t}$$

$$\therefore -2 = -0.2t$$

$\therefore t = \dfrac{2}{0.2}$

$\therefore t = 10$

It takes 10 years for the value of the car to lose 75% of its purchase price.

2 $D = 42 \times 2^{\frac{t}{16}}$

a When $t = 0$, $D = 42$. The initial average number of daily emails was 42 emails per day.

b When $D = 84$:

$$84 = 42 \times 2^{\frac{t}{16}}$$

$$\frac{84}{42} = 2^{\frac{t}{16}}$$

$$2 = 2^{\frac{t}{16}}$$

$$1 = \frac{t}{16}$$

$$t = 16$$

After 16 weeks the average number of daily emails is predicted to double.

3 $N = 30 \times 2^{0.072t}$

a When $t = 0$, $N = 30$, so there were initially 30 drosophilae.

b When $t = 5$,

$$N = 30 \times 2^{0.072 \times 5}$$

$$= 30 \times 2^{0.36}$$

$$\simeq 38.503$$

After 5 days there were approximately 39 drosophilae.

c When $N = 60$,

$$60 = 30 \times 2^{0.072t}$$

$$\therefore 2 = 2^{0.072t}$$

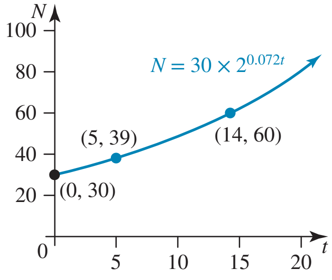
$$\therefore 1 = 0.072t$$

$$\therefore t = \frac{1}{0.072}$$

$$\therefore t \simeq 13.9$$

The population doubles after 14 days.

d



e Let $N = 100$.

$$\therefore 100 = 30 \times 2^{0.072t}$$

$$\therefore \frac{10}{3} = 2^{0.072t}$$

$$\therefore \log\left(\frac{10}{3}\right) = \log\left(2^{0.072t}\right)$$

$$\therefore \log\left(\frac{10}{3}\right) = 0.072t \log(2)$$

$$\therefore t = \frac{\log\left(\frac{10}{3}\right)}{0.072 \log(2)}$$

$$\therefore t \simeq 24.12$$

Using the graph, for N to exceed 100, $t > 24.12$.

The population first exceeds 100 after 25 days.

4 $Q(t) = Q_0 \times 1.7^{-kt}$

a When $t = 0$, $Q(0) = Q_0 \times 1 = Q_0$, so Q_0 is the initial amount of the substance.

b $Q(300) = \dfrac{1}{2}Q_0$

$$\frac{1}{2}Q_0 = Q_0 \times 1.7^{-300k}$$

$$0.5 = 1.7^{-300k}$$

$$\log(0.5) = -300k \log(1.7)$$

$$k = -\frac{\log(0.5)}{300 \log(1.7)}$$

$$k \approx 0.004$$

c $Q_0 = 250$, $Q = 250 \times 1.7^{-0.004t}$

When $t = 10$,

$$Q = 250 \times 1.7^{-0.04}$$

$$\approx 244.7$$

The amount that has decayed is:

$250 - 244.7 = 5.3$

Therefore, 5.3 kg has decayed.

5 $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

a **i** $P = 2000$, $r = 0.03$ and $n = 12$

$$\therefore A = 2000\left(1 + \frac{0.03}{12}\right)^{12t}$$

$$\therefore A = 2000\,(1 + 0.0025)^{12t}$$

$$\therefore A = 2000\,(1.0025)^{12t}$$

ii For 6 months, $t = \dfrac{1}{2}$

$$A = 2000\,(1.0025)^{6}$$

$$\simeq 2030.19$$

The investment is worth \$2030.19 after 6 months.

iii Let $A = 2500$.

$$\therefore 2500 = 2000\,(1.0025)^{12t}$$

$$\therefore 1.0025^{12t} = \frac{2500}{2000}$$

$$\therefore 1.0025^{12t} = 1.25$$

$$\therefore \log\left(1.0025^{12t}\right) = \log(1.25)$$

$$\therefore 12t \log(1.0025) = \log(1.25)$$

$$\therefore t = \frac{\log(1.25)}{12 \log(1.0025)}$$

$$\therefore t \simeq 7.45$$

It takes 7.45 years for the investment to reach \$2500.

b Let $A = 2500$ and $t = 4$.

$$\therefore 2500 = 2000\left(1 + \frac{r}{12}\right)^{48}$$

$$\therefore \left(1 + \frac{r}{12}\right)^{48} = \frac{2500}{2000}$$

$$\therefore \left(1 + \frac{r}{12}\right)^{48} = 1.25$$

$$\therefore \left(1 + \frac{r}{12}\right) = \sqrt[48]{1.25}$$

$$\therefore \frac{r}{12} = 1.25^{\frac{1}{48}} - 1$$

$$\therefore r = 12\left(1.25^{\frac{1}{48}} - 1\right)$$

$$\therefore r \simeq 0.056$$

The interest rate would need to be 5.6% p.a. to achieve the goal.

6 $T = 85 \times 3^{-0.008t}$

a When $t = 0$, $T = 85$, so the initial temperature is 85 degrees.

Let $t = 10$.

$\therefore T = 85 \times 3^{-0.08}$

$\therefore T \simeq 77.85$

After 10 minutes the temperature is approximately 78 degrees, so the coffee has cooled by 7 degrees.

b Let $T = 65$.

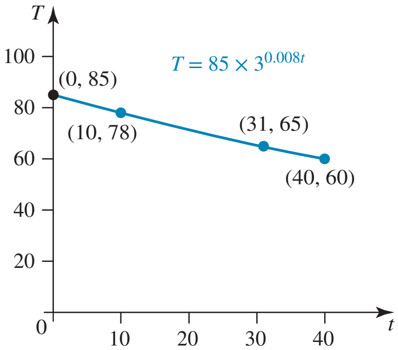
$$\begin{aligned}\therefore 65 &= 85 \times 3^{-0.008t}\\ \therefore 3^{-0.008t} &= \frac{65}{85}\\ &= \frac{13}{17}\\ \therefore \log\left(3^{-0.008t}\right) &= \log\left(\frac{13}{17}\right)\\ \therefore -0.008t\log(3) &= \log\left(\frac{13}{17}\right)\\ \therefore t &= \frac{\log\left(\frac{13}{17}\right)}{-0.008\log(3)}\\ \therefore t &\simeq 30.5\end{aligned}$$

It takes just over half an hour to cool to 65 degrees.

c The graph of $T = 85 \times 3^{-0.008t}$ passes through the three points $(0, 85), (10, 78), (31, 65)$.

When $t = 40$,

$$\begin{aligned}T &= 85 \times 3^{-0.008 \times 40}\\ &= 85 \times 3^{-0.32}\\ &\simeq 59.8\end{aligned}$$



d The asymptote for the graph of $T = 85 \times 3^{-0.008t}$ is $T = 0$. This model therefore predicts that the temperature will approach zero degrees. This makes the model unrealistic for most places in Australia.

7 $T = a \times 3^{-0.13t} + 25$

a When $t = 0, T = 95$.

$\therefore 95 = a \times 1 + 25$

$\therefore a = 70$

b The model is $T = 70 \times 3^{-0.13t} + 25$.

Let $t = 2$.

$$\begin{aligned}\therefore T &= 70 \times 3^{-0.13 \times 2} + 25\\ &= 70 \times 3^{-0.26} + 25\\ \therefore T &\simeq 77.6\end{aligned}$$

After 2 minutes, the pie has cooled to 77.6 degrees.

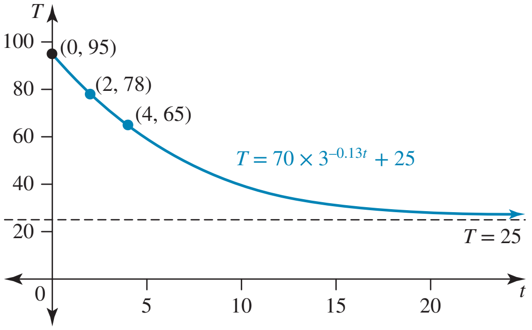
c Let $T = 65$.

$$\begin{aligned}\therefore 65 &= 70 \times 3^{-0.13t} + 25\\ \therefore 40 &= 70 \times 3^{-0.13t}\\ \therefore 3^{-0.13t} &= \frac{4}{7}\\ \therefore \log\left(3^{-0.13t}\right) &= \log\left(\frac{4}{7}\right)\\ \therefore -0.13t\log(3) &= \log\left(\frac{4}{7}\right)\\ \therefore t &= \frac{\log\left(\frac{4}{7}\right)}{-0.13\log(3)}\\ \therefore t &\simeq 3.9\end{aligned}$$

It takes 4 minutes for the pie to cool to 65 degrees.

d The graph of $T = 70 \times 3^{-0.13t} + 25$ contains the points $(0, 95), (2, 77.6), (3.9, 65)$.

Its asymptote is $T = 25$. This means that the model predicts the temperature of the pie will approach 25 degrees. In the long term, the temperature of the pie will not fall below 25 degrees.



8 $P = P_0 \times 10^{-kh}$

a When $h = 0$, $P = P_0$, so P_0 is the barometric pressure at sea level.

For Mt Everest, $P = \frac{1}{3}P_0$ and $h = 8.848$ (measured in km).

$$\begin{aligned}\therefore \frac{1}{3}P_0 &= P_0 \times 10^{-8.848k}\\ \therefore \frac{1}{3} &= 10^{-8.848k}\\ \therefore \log\left(\frac{1}{3}\right) &= -8.848k\\ \therefore k &= \log\left(\frac{1}{3}\right) \div (-8.848)\\ \therefore k &\simeq 0.054\end{aligned}$$

b The model is $P = P_0 \times 10^{-0.054h}$.

Mt Kilimanjaro: $h = 5.895,\ P = 48.68$

$\therefore 48.68 = P_0 \times 10^{-0.054 \times 5.895}$

$\therefore P_0 = 48.68 \times 10^{0.054 \times 5.895}$

$\therefore P_0 \simeq 101.317$

c Use the model now as $P = 101.317 \times 10^{-0.054h}$.

For Mont Blanc, $h = 4.810$

$\therefore P = 101.317 \times 10^{-0.054 \times 4.810}$

$\therefore P \simeq 55.71$

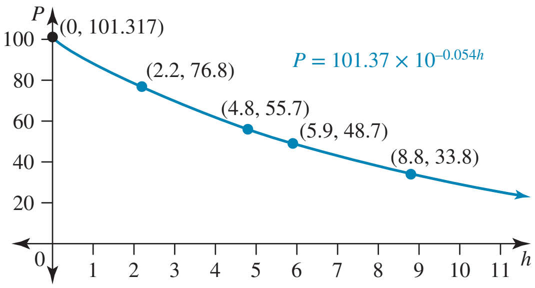
For Mt Kosciuszko, $h = 2.228$

$\therefore P = 101.317 \times 10^{-0.054 \times 2.228}$

$\therefore P \simeq 76.80$

The atmospheric pressure is 55.71 kilopascals at the summit of Mont Blanc and 76.80 kilopascals at the summit of Mt Kosciuszko.

d



9 a $D = D_0 \times 10^{kt}$

In 1991: $t = 15, D = 15 \Rightarrow 15 = D_0 \times 10^{15k}$ [1]

In 1994: $t = 18, D = 75 \Rightarrow 75 = D_0 \times 10^{18k}$ [2]

b Divide equation [2] by equation [1].

$$\therefore \frac{75}{15} = \frac{D_0 \times 10^{18k}}{D_0 \times 10^{15k}}$$
$$\therefore 5 = 10^{3k}$$
$$\therefore 3k = \log(5)$$
$$\therefore k = \frac{1}{3}\log(5)$$

Substitute $k = \frac{1}{3}\log(5)$ in equation [1].

$$\therefore 15 = D_0 \times 10^{15 \times \frac{1}{3}\log(5)}$$
$$\therefore 15 = D_0 \times 10^{5\log(5)}$$
$$\therefore 15 = D_0 \times 10^{\log\left(5^5\right)}$$
$$\therefore 15 = D_0 \times 5^5$$
$$\therefore D_0 = \frac{15}{5^5}$$
$$\therefore D_0 = 3 \times 5 \times 5^{-5}$$
$$\therefore D_0 = 3 \times 5^{-4}$$

Correct to 3 decimal places, $k = \frac{1}{3}\log(5) = 0.233$ and $D_0 = 3 \times 5^{-4} = 0.005$.

c The model is $D = 0.005 \times 10^{0.233t}$.

1996: let $t = 20$.

$$\therefore D = 0.005 \times 10^{0.233 \times 20}$$
$$\therefore D \simeq 228.54$$

In 1996, the density was 229 birds per square kilometre.

d The new model is given by $D = 230 \times 10^{-\frac{t}{3}} + b$. This model has an asymptote when $D = b$.

When $t = 4, D = 40$.

$$\therefore 40 = 30 \times 10^{-\frac{4}{3}} + b$$
$$\therefore b = 40 - 30 \times 10^{-\frac{4}{3}}$$
$$\therefore b \simeq 38.6$$

The asymptote is approximately $D = 39$, so the density cannot be expected to fall below 39 birds per square kilometre.

10 $C = C_0 \times \left(\frac{1}{2}\right)^{kt}$

a A half-life of 5730 years means that $C = \frac{1}{2}C_0$ when $t = 5730$.

$$\therefore \frac{1}{2}C_0 = C_0 \times \left(\frac{1}{2}\right)^{5730k}$$
$$\therefore \frac{1}{2} = \left(\frac{1}{2}\right)^{5730k}$$
$$\therefore 1 = 5730k$$
$$\therefore k = \frac{1}{5730}$$

b The model is $C = C_0 \times \left(\frac{1}{2}\right)^{\frac{t}{5730}}$.

Let $C = 0.83C_0$.

$$\therefore 0.83C_0 = C_0 \times \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$
$$\therefore 0.83 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$
$$\therefore \log(0.83) = \log\left(\left(\frac{1}{2}\right)^{\frac{t}{5730}}\right)$$
$$\therefore \frac{t}{5730}\log\left(\frac{1}{2}\right) = \log(0.83)$$
$$\therefore t = \frac{5730\log(0.83)}{\log(0.5)}$$
$$\therefore t \simeq 1540$$

The bones are estimated to be 1540 years old.

11 a $P = P_0 \times 10^{kt}$

When $t = 15$, $P = 3.9 \Rightarrow 13.9 = P_0 \times 10^{15k}$

When $t = 30$, $P = 17.1 \Rightarrow 17.1 = P_0 \times 10^{30k}$

The pair of simultaneous equations is:

$13.9 = P_0 \times 0^{15k}$

$17.1 = P_0 \times 10^{30k}$

b Solve the equations simultaneously using a CAS calculator:

$P_0 = 11.298\,83$ and $k = 0.005\,998\,75$

Give the results to the required decimal places:

$P_0 \approx 11.3$ and $k \approx 0.006$

c $P = 11.3 \times 10^{0.006t}$

For the population in 1960, $t = 0$

$\therefore P = 11.3$ million

To double, $P = 22.6$ million

$22.6 = 11.3 \times 10^{0.006t}$

Using the solve menu on the CAS calculator,

$t = 50.171\,666$

The population doubled after approximately 50 years.

d For the year 2030, $t = 70$:

$P = 11.3 \times 10^{0.006 \times 70}$

$P = 29.722\,028$

The model predicts the population will be 29.7 million, so this model supports the claim that the population will exceed 28 million by 2030.

12 $P(t) = (200t + 16) \times 2.7^{-t}$

a i When $t = 0$,

$$P(0) = (16) \times 1$$
$$= 16$$

Stephan's pain level is 16.

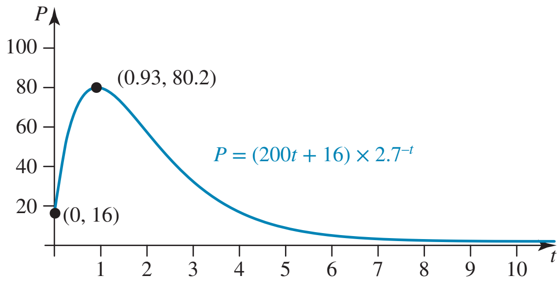
ii 15 seconds equals $\frac{15}{60} = \frac{1}{4}$ minutes.

When $t = \frac{1}{4}$,

$$P\left(\frac{1}{4}\right) = \left(200 \times \frac{1}{4} + 16\right) \times 2.7^{-\frac{1}{4}}$$
$$\simeq 51.5$$

After 15 seconds, Stephan's pain level is 51.5.

b The graph obtained should be similar to that shown.



i Use the Analysis tools to obtain the maximum turning point as: (0.926 794, 80.202 012).
Hence, the maximum pain measure is approximately 80.20.

ii It takes approximately 0.93 minutes or 55.61 seconds for the injection to start to reduce the pain.

iii Using y-Cal, when $x = 5, y = 7.080\,678\,6$, and when $x = 10, y = 0.097\,915$.
Hence, the pain level is 7.08 after 5 minutes and 0.10 after 10 minutes.

c i $P(t) = (100(t - 10) + a) \times 2.7^{-(t-10)}$

$P\,(10)$ is known to equal 0.10.

$\therefore 0.10 = (100(10 - 10) + a) \times 2.7^{-(0)}$

$\therefore 0.10 = a \times 1$

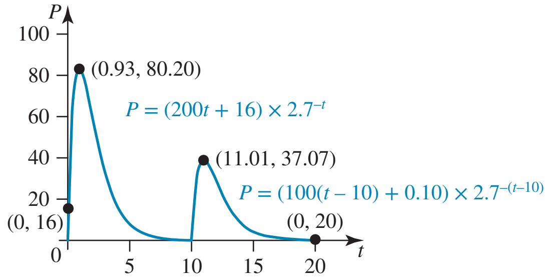
$\therefore a = 0.10$

ii Enter the hybrid function as:

$y1 = (200x + 16) \times 2.7^{-x} 10 \le x \le 10$

$y2 = (100(x - 10) + 0.10) \times 2.7^{-(x-10)} 110 \le x \le 20$

The graph obtained should be a similar shape to that shown. Use the Analysis tools to obtain the coordinates of the second maximum as approximately (11.01, 37.07).



10.5 Exam questions

1 $V = V_0 \times 3^{-0.1t}$

$t = 0,\ V = V_0$ the purchase value

When $\frac{2}{3}$ of the value is lost, $\frac{1}{3}$ remains.

A third of the purchase value is $\frac{V_0}{3}$.

$$\frac{V_0}{3} = V_0 \times 3^{-0.1t}$$

$$\frac{1}{3} = 3^{-0.1t}$$

$$3^{-1} = 3^{-0.1t}$$

$$-1 = -0.1t$$

$$t = 10 \text{ years}$$

The correct answer is **C**.

2 $t = 0$

$Q(t) = 15 \times 5^0$

$= 15$

The correct answer is **B**.

3 $T = 90 \times 3^{-0.01t}$

$= 90 \times 3^{-0.01 \times 30}$

$= 64.73\ °\text{C}$

The initial temperature is 90 °C.

Drop in temperature $= 90\ °\text{C} - 64.73°\text{C}$

$= 25.27\ °\text{C}$

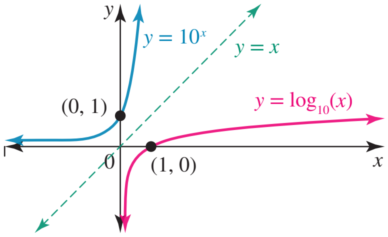
The correct answer is **B**.

10.6 Inverses of exponential functions

10.6 Exercise

1 a $y = \log_{10}(x)$

The inverse is $x = \log_{10}(y)$, which is the exponential function $y = 10^x$.



b The points (10, 1), (100, 2), (1000, 3) lie on the logarithm graph.

With $m = 1000, n = 100$, the logarithm law is

$\log_{10}(1000) - \log_{10}(100) = \log_{10}\left(\frac{1000}{100}\right)$.

$\therefore \log_{10}(1000) - \log_{10}(100) = \log_{10}(10)$

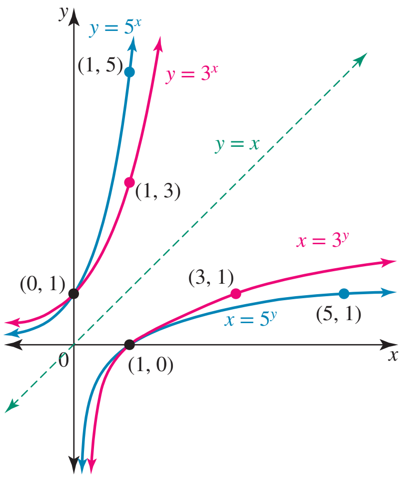
This means the difference between the y coordinates of the points (1000, 3) and (100, 2) should equal the y coordinate of the point (10, 1).

This does hold since $3 - 2 = 1$.

2 a $y = 3^x$ has an asymptote $y = 0$ and passes through the points (0, 1) and (1, 3). Its inverse will have an asymptote $x = 0$ and pass through the points (1, 0) and (3, 1).

$y = 5^x$ has an asymptote $y = 0$ and passes through the points (0, 1) and (1, 5). Its inverse will have an asymptote $x = 0$ and pass through the points (1, 0) and (5, 1).

The inverse is a reflection of the exponential graph in the line $y = x$.

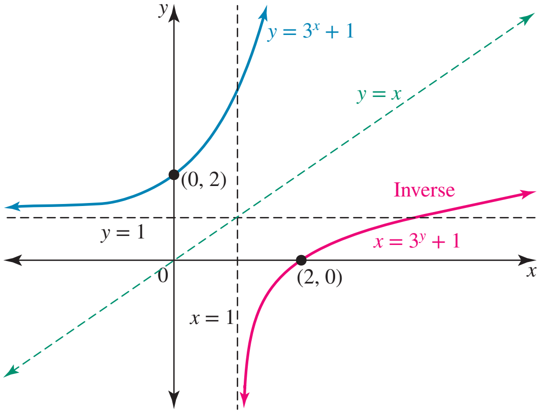


b The inverse of $y = 3^x$ is $x = 3^y$. Therefore, the inverse is $y = \log_3(x)$.

The inverse of $y = 5^x$ is $x = 5^y$. Therefore, the inverse is $y = \log_5(x)$.

c The larger the base of the logarithmic function, the more slowly its graph increases for $x > 1$.

3 **a** $y = 3^x + 1$ is a vertical translation up one unit of $y = 3^x$. The asymptote is $y = 1$ and it passes through $(0, 2)$. Its inverse has asymptote $x = 1$ and passes through $(2, 0)$.



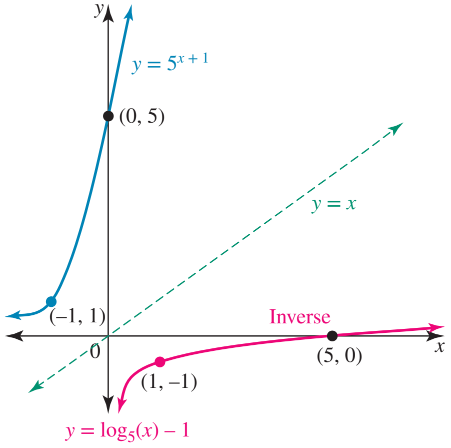
b The inverse of $y = 3^x + 1$ has the equation $x = 3^y + 1$.

$\therefore x - 1 = 3^y$

$\therefore y = \log_3(x - 1)$

4 $y = 5^{x+1}$ is a horizontal translation one unit to the left of $y = 5^x$. Its asymptote is $y = 0$ and it passes through $(-1, 1)$ and $(0, 5)$.

Its inverse has asymptote is $x = 0$ and it passes through $(1, -1)$ and $(5, 0)$.



Rule for the inverse:

$x = 5^{y+1}$

$\therefore \log_5(x) = y + 1$

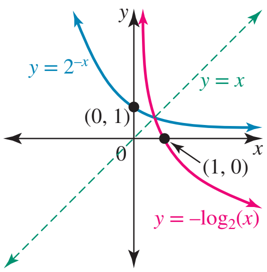
$\therefore y = \log_5(x) - 1$

5 $y = 2^{-x}$

Its inverse is $x = 2^{-y}$, which rearranges to

$-y = \log_2(x)$

$\therefore y = -\log_2(x)$



6 $f : R \to R, f(x) = 4 - 2^{3x}$

a The domain of the inverse is the range of f.

f:asymptote at $y = 4$, y-intercept $(0, 3)$. Therefore, the range is $(-\infty, 4)$.

The domain of the inverse is $(-\infty, 4)$.

b Let $y = f(x)$.

f: $y = 4 - 2^{3x}$

Inverse: $x = 4 - 2^{3y}$

$\therefore 2^{3y} = 4 - x$

$\therefore 3y = \log_2(4 - x)$

$\therefore y = \frac{1}{3}\log_2(4 - x)$

$\therefore f^{-1}(x) = \frac{1}{3}\log_2(4 - x)$

As a mapping, $f^{-1} : (-\infty, 4) \to R$, $f^{-1}(x) = \frac{1}{3}\log_2(4 - x)$

c x-intercept: $4 - 2^{3x} = 0$

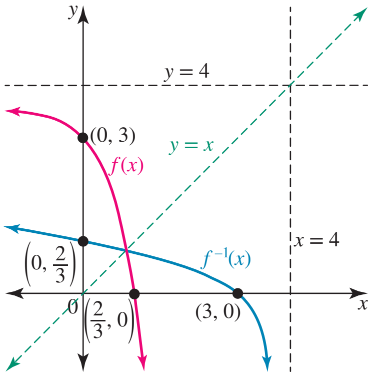
$\therefore 4 = 2^{3x}$

$\therefore 2^2 = 2^{3x}$

$\therefore 3x = 2$

$\therefore x = \frac{2}{3}$

Inverse: asymptote at $x = 4$, x-intercept $(3, 0)$, y-intercept $\left(0, \frac{2}{3}\right)$, domain $(-\infty, 4)$



7 $f: R \to R, f(x) = 8 - 2 \times 3^{2x}$

a $r_{f^{-1}} = d_f = R$.

The function has an asymptote at $y = 8$ and a y-intercept at $(0, 6)$.

Therefore, $r_f = d_{f^{-1}} = (-\infty, 8)$.

b $f(0) = 6$

c As the point $(0, 6)$ is on f, the point $(6, 0)$ is on f^{-1}. This is the x-intercept of the graph of f^{-1}.

d Function f: $y = 8 - 2 \times 3^{2x}$

Inverse function f^{-1}: $x = 8 - 2 \times 3^{2y}$

$$\therefore 2 \times 3^{2y} = 8 - x$$
$$\therefore 3^{2y} = \frac{8-x}{2}$$
$$\therefore 2y = \log_3\left(\frac{8-x}{2}\right)$$
$$\therefore y = \frac{1}{2}\log_3\left(\frac{8-x}{2}\right)$$

The rule for f^{-1} is $f^{-1}(x) = \frac{1}{2}\log_3\left(\frac{8-x}{2}\right)$.

As a mapping, the inverse function is

$f^{-1}: (-\infty, 8) \to R, f^{-1}(x) = \frac{1}{2}\log_3\left(\frac{8-x}{2}\right)$.

8 a $\log_6\left(2^{2x} \times 9^x\right) = \log_6\left(2^{2x} \times 3^{2x}\right)$
$= \log_6\left((2 \times 3)^{2x}\right)$
$= \log_6\left(6^{2x}\right)$
$= 2x$

b $2^{-3\log_2(10)} = 2^{\log_2(10)^{-3}}$
$= (10)^{-3}$
$= \frac{1}{1000}$
$= 0.001$

9 a $y = \log_{10}(x-1)$

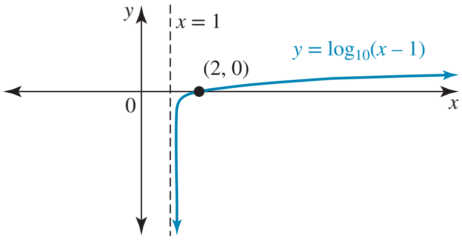
Horizontal translation 1 unit to the right gives the asymptote of $x = 1$ and domain $(1, \infty)$. There will not be a y-intercept.

x-intercept: when $y = 0$,

$\log_{10}(x-1) = 0$

$\therefore x - 1 = 10^0$

$\therefore x = 2$



b $y = \log_5(x) - 1$

Vertical translation of 1 unit downwards. This does not affect the asymptote or the domain.

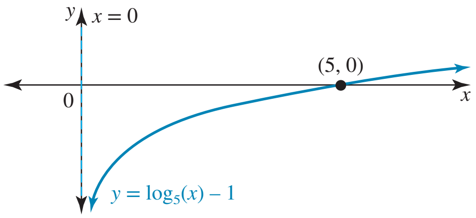
Asymptote: $x = 0$, no y-intercept, domain: R^+

x-intercept: $\log_5(x) - 1 = 0$

$\therefore \log_5(x) = 1$

$\therefore x = 5^1$

$\therefore x = 5$



c i The asymptote occurs when $x + b = 0 \Rightarrow x = -b$. The graph shows the asymptote is $x = -1$

and so $-b = -1 \Rightarrow b = 1$

Alternatively, as the asymptote is $x = -1$, there is a horizontal translation of 1 unit to the left so $b = 1$.

ii The domain of the inverse is the range of the given graph, so domain is R.

The range of the inverse is the domain of the given graph, so range is $(-1, \infty)$.

The rule for the inverse:

Function: $y = -\log_2(x+1)$

Inverse: $x = -\log_2(y+1)$

$\therefore -x = \log_2(y+1)$

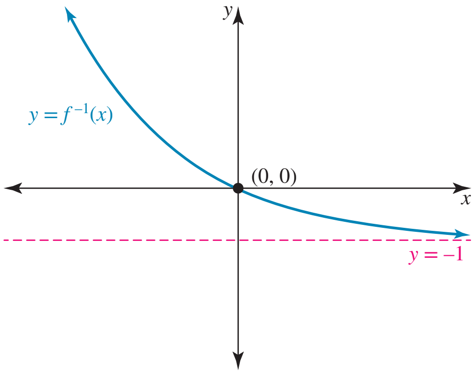
$\therefore 2^{-x} = y + 1$

$\therefore y = 2^{-x} - 1$

The rule for the inverse function is given by $f^{-1}(x) = 2^{-x} - 1$.

iii Deduce the features of the inverse from the given graph.

Asymptote $x = -1 \Rightarrow y = -1$, point $(0, 0) \Rightarrow (0, 0)$



10 a $y = \log_{10}(x+6)$

Domain: $x + 6 > 0$

$\therefore x > -6$

The domain is $(-6, \infty)$.

Asymptote: $x + 6 = 0$

$\therefore x = -6$ is the equation of the asymptote.

b $y = \log_{10}(x-6)$

Domain: $x - 6 > 0$

$\therefore x > 6$

The domain is $(6, \infty)$.

Asymptote: $x - 6 = 0$

$\therefore x = 6$

c $y = \log_{10}(6-x)$

Domain: $6 - x > 0$

$\therefore -x > -6$

$\therefore x < 6$

The domain is $(-\infty, 6)$.

Asymptote: $6 - x = 0$

$\therefore x = 6$

d $y = \log_{10}(6x)$

Domain: $6x > 0$

$\therefore x > 0$

The domain is $(0, \infty)$ or R^+.

Asymptote: $6x = 0$

$\therefore x = 0$

e $y = 3\log_5(-x)$

Domain: $-x > 0$

$\therefore x < 0$

The domain is $(-\infty, 0)$ or R^-.

Asymptote: $-x = 0$

$\therefore x = 0$

f $y = -\log_4(2x-3)+1$

Domain: $2x-3>0$

$\therefore 2x>3$

$\therefore x<\frac{3}{2}$

Domain is $\left(-\infty, \frac{3}{2}\right)$

Asymptote: $2x-3=0$

$\therefore x=\frac{3}{2}$

11 $y=\log_3(x+9)$

a Domain: $x+9>0 \Rightarrow x>-9$

The domain is $(-9, \infty)$.

b Asymptote: $x+9=0 \Rightarrow x=-9$.

c y-intercept: let $x=0$.

$\therefore y=\log_3(9)$

$\therefore y=\log_3(3^2)$

$\therefore y=2\log_3(3)$

$\therefore y=2\times 1$

$\therefore y=2$

The y-intercept is $(0, 2)$.

d x-intercept: let $y=0$.

$\therefore 0=\log_3(x+9)$

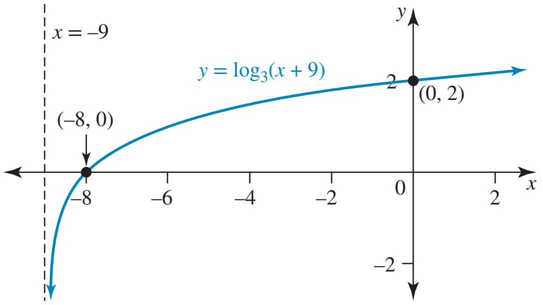
$\therefore 3^0=x+9$

$\therefore 1=x+9$

$\therefore x=-8$

The x-intercept is $(-8, 0)$.

e



f The range is R.

12 $y=\log_{10}(x-5)$

a Domain: $x-5>0 \Rightarrow x>5$

The domain is $(5, \infty)$.

b Asymptote: $x-5=0 \Rightarrow x=5$

c Since $x=0$ is not in the domain, there is no y-intercept.

x-intercept: let $y=0$.

$\therefore 0=\log_5(x-5)$

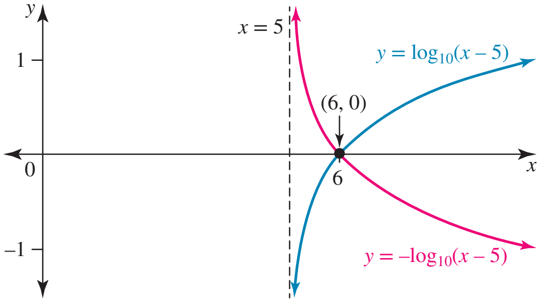
$\therefore 5^0=x-5$

$\therefore 1=x-5$

$\therefore x=6$

The x-intercept is $(6, 0)$.

d and **e**



The graph of $y=-\log_{10}(x-5)$ is the reflection in the x-axis of $y=\log_{10}(x-5)$. Both graphs have range R.

13 **a** $y=\log_5(x)-2$

Domain: $x>0 \Rightarrow$ domain is R^+

Asymptote: $x=0$

No y-intercept

Vertical translation 2 units down

Range: R

x-intercept: let $y=0$.

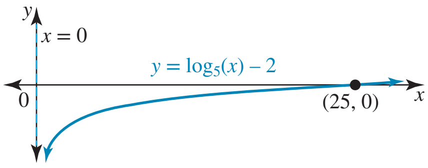
$\therefore 0=\log_5(x)-2$

$\therefore \log_5(x)=2$

$\therefore x=5^2$

$\therefore x=25$

$(25, 0)$



b $y=\log_5(x-2)$

Domain: $x-2>0 \Rightarrow x>2$

The domain is $(2, \infty)$.

Asymptote: $x=2$

No y-intercept

Horizontal translation 2 units right

Range: R

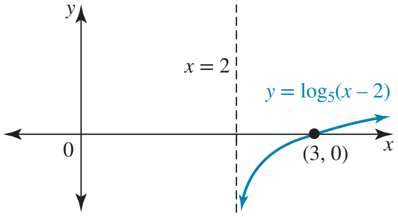
x-intercept: let $y=0$.

$\therefore 0=\log_5(x-2)$

$\therefore x-2=5^0$

$\therefore x=3$

$(3, 0)$



c $y=\log_{10}(x)+1$

Domain: $x>0 \Rightarrow$ domain R^+

Asymptote: $x=0$

No y-intercept

Vertical translation 1 unit up

Range: R

x-intercept: let $y=0$.

$\therefore 0=\log_{10}(x)+1$

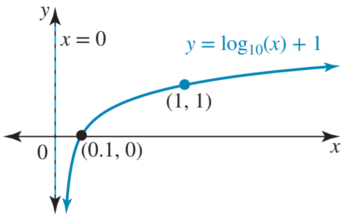
$\therefore \log_{10}(x)=-1$

$\therefore x=10^{-1}$

$\therefore x=0.1$

$(0.1, 0)$

Point: If $x=1$, then $y=1$. $(1, 1)$ is on the graph.



d $y = \log_{10}(x+1)$

Domain: $x + 1 > 0 \Rightarrow x > -1$

The domain is $(-1, \infty)$.

Asymptote: $x = -1$

y-intercept: let $x = 0$.

$\therefore y = \log_{10}(1)$

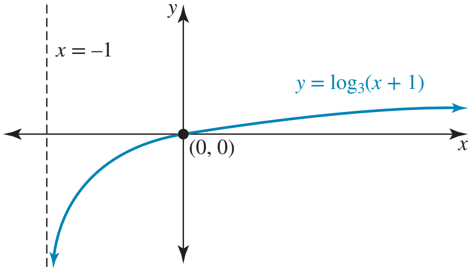
$\therefore y = 0$

$(0, 0)$

This is the x-intercept also.

Horizontal translation 1 unit left

Range: R



e $y = \log_3(4-x)$

Domain: $4 - x > 0$

$\therefore 4 > x$

$\therefore x < 4$

The domain is $(-\infty, 4)$.

Asymptote: $x = 4$

y-intercept: let $x = 0$.

$\therefore y = \log_3(4)$

$\left(0, \log_3(4)\right)$

Reflection in the y-axis, then horizontal translation 4 units right

Range: R

x-intercept: let $y = 0$.

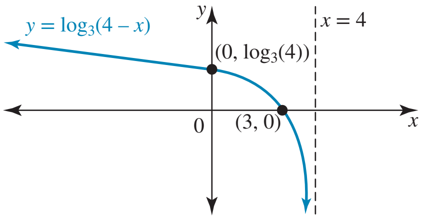
$\therefore 0 = \log_3(4-x)$

$\therefore 4 - x = 3^0$

$\therefore 4 - x = 1$

$\therefore x = 3$

$(3, 0)$



f $y = -\log_2(x+4)$

Domain: $x + 4 > 0 \Rightarrow x > -4$

The domain is $(-4, \infty)$.

Asymptote: $x = -4$

y-intercept: let $x = 0$.

$\therefore y = -\log_2(4)$

$\therefore y = -\log_2\left(2^2\right)$

$\therefore y = -2$

$(0, -2)$

Reflection in the x-axis, then horizontal translation 4 units left

Range: R

x-intercept: let $y = 0$.

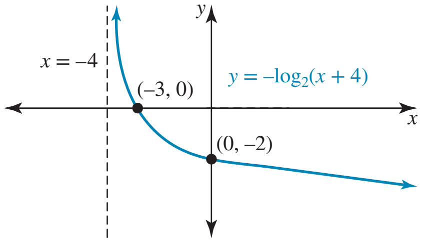
$\therefore 0 = \log_2(x+4)$

$\therefore x + 4 = 2^0$

$\therefore x + 4 = 1$

$\therefore x = -3$

$(-3, 0)$



14 $y = \log_2(2x)$

$\therefore y = \log_2(x) + \log_2(2)$

$y = \log_2(x) + 1$

Therefore, $y = \log_2(x) \to y = \log_2(x) + 1$ under a vertical translation of 1 unit upwards

15 $5^{x\log_5(2)-\log_5(3)}$

$$\begin{aligned} 5^{x\log_5(2)-\log_5(3)} &= 5^{\log_5(2^x)-\log_5(3)} \\ &= 5^{\log_5\left(\frac{2^x}{3}\right)} \\ &= \frac{2^x}{3} \\ &= \frac{1}{3} \times 2^x \end{aligned}$$

16 a i $3^{\log_3(8)} = 8$

ii $$\begin{aligned} 10^{\log_{10}(2)+\log_{10}(3)} &= 10^{\log_{10}(2\times 3)} \\ &= 10^{\log_{10}(6)} \\ &= 6 \end{aligned}$$

iii $$\begin{aligned} 5^{-\log_5(2)} &= 5^{\log_5\left(2^{-1}\right)} \\ &= 2^{-1} \\ &= \frac{1}{2} \end{aligned}$$

iv $$\begin{aligned} 6^{\frac{1}{2}\log_6(25)} &= 6^{\log_6\left(25^{\frac{1}{2}}\right)} \\ &= 25^{\frac{1}{2}} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

b i $3^{\log_3(x)} = x$

ii $$\begin{aligned} 2^{3\log_2(x)} &= 2^{\log_2\left(x^3\right)} \\ &= x^3 \end{aligned}$$

iii $$\begin{aligned} \log_2(2^x) + \log_3(9^x) &= \log_2(2^x) + \log_3\left(3^{2x}\right) \\ &= x + 2x \\ &= 3x \end{aligned}$$

iv $\log_6\left(\frac{6^{x+1}-6^x}{5}\right) = \log_6\left(\frac{6^x(6^1-1)}{5}\right)$

$= \log_6\left(\frac{6^x(5)}{5}\right)$

$= \log_6(6^x)$

$= x$

17 a $y = a\log_7(bx)$

Substitute the given points.

Point $(2, 0) \Rightarrow 0 = a\log_7(2b)$

$\therefore 0 = \log_7(2b)$

$\therefore 2b = 7^0$

$\therefore 2b = 1$

$\therefore b = \frac{1}{2}$

The equation is now $y = a\log_7\left(\frac{x}{2}\right)$.

Point $(14, 14) \Rightarrow 14 = a\log_7(7)$

$\therefore 14 = a \times 1$

$\therefore a = 14$

The equation is $y = 14\log_7\left(\frac{x}{2}\right)$.

b i $y = a\log_3(x) + b$

Point $(1, 4) \Rightarrow 4 = a\log_3(1) + b$

$\therefore 4 = a \times 0 + b$

$\therefore b = 4$

Hence, $y = a\log_3(x) + 4$.

Point $\left(\frac{1}{3}, 8\right) \Rightarrow 8 = a\log_3\left(\frac{1}{3}\right) + 4$

$\therefore 4 = a\log_3\left(3^{-1}\right)$

$\therefore 4 = -a\log_3(3)$

$\therefore 4 = -a$

$\therefore a = -4$

The equation is $y = -4\log_3(x) + 4$.

ii Let the inverse cut the y-axis at the point $(0, k)$. Then the function $y = -4\log_3(x) + 4$ cuts the x-axis at $(k, 0)$.

$\therefore 0 = -4\log_3(k) + 4$

$\therefore 4\log_3(k) = 4$

$\therefore \log_3(k) = 1$

$\therefore k = 3^1$

$\therefore k = 3$

Therefore, the inverse function would cut the y-axis at the point $(0, 3)$.

c i $y = a\log_2(x - b) + c$

From the diagram the asymptote is $x = -2$ and from the equation the asymptote is $x = b$.

Therefore, $b = -2$

$\therefore y = a\log_2(x + 2) + c$

Substitute the x- and y-intercepts shown on the diagram.

$(-1.5, 0) \Rightarrow 0 = a\log_2(-1.5 + 2) + c$

$\therefore 0 = a\log_2(0.5) + c$

$\therefore 0 = a\log_2\left(2^{-1}\right) + c$

$\therefore 0 = -a\log_2(2) + c$

$\therefore 0 = -a + c$

$\therefore a = c \ldots.(1)$

$(0, -2) \Rightarrow -2 = a\log_2(2) + c$

$\therefore\ -2 = a + c \ldots.(2)$

Substitute equation (1) in equation (2).

$\therefore\ -2 = c + c$

$\therefore c = -1$

$\therefore a = -1$

The equation of the graph is $y = -\log_2(x + 2) - 1$.

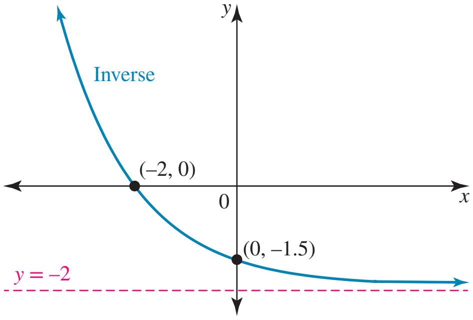
ii The graph of the inverse function has an asymptote $y = -2$ and axis intercepts $(-2, 0)$ and $(0, -1.5)$.

Its rule is $x = -\log_2(y + 2) - 1$

$\therefore\ \log_2(y + 2) = -x - 1$

$\therefore y + 2 = 2^{-x-1}$

$\therefore y = 2^{-(x+1)} - 2$



18 i $df\colon 4x + 9 > 0 \Rightarrow x > \frac{-9}{4}$

$\therefore df = \left(-\frac{9}{4}, \infty\right)$

$\therefore dg = (-\infty, 20)$

$dg\colon\ 2 - 0.1\,x > 0 \Rightarrow x < 20$

ii $x = -\frac{9}{4}, x = 20$

iii $f\colon (-2, 0), (0, 2);\ g : (10, 0), \left(0, \frac{1}{2}\right)$

iv Sketch using CAS technology.

19 a $y = 2^{-\frac{x}{3}}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 1 \quad (0, 1)$.

Point: let $x = -3$.

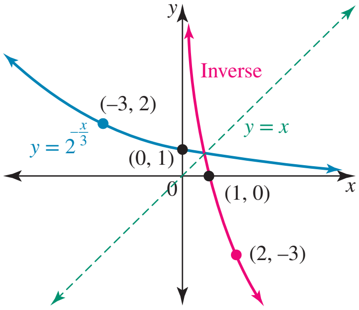
$\therefore y = 2 \quad (-3, 2)$

Inverse: asymptote $x = 0$, x-intercept $(1, 0)$, point $(2, -3)$.

Rule for the inverse: $x = 2^{-\frac{y}{3}}$

$\therefore\ \log_2(x) = -\frac{y}{3}$

$\therefore y = -3\log_2(x)$



b $y = \frac{1}{2} \times 8^x - 1$

Asymptote: $y = -1$

y-intercept: let $x = 0$.

$\therefore y = \frac{1}{2} - 1$

$\therefore y = -\frac{1}{2}.$

$\left(0, -\frac{1}{2}\right)$

x-intercept: let $y = 0$.

$\therefore 0 = \frac{1}{2} \times 8^x - 1$

$\therefore 8^x = 2$

$\therefore 2^{3x} = 2^1$

$\therefore 3x = 1$

$\therefore x = \frac{1}{3}$

$\left(\frac{1}{3}, 0\right)$

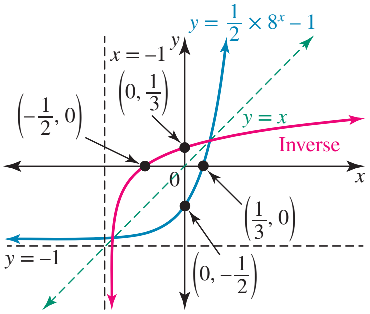
Inverse: asymptote $x = -1$, x-intercept $\left(-\frac{1}{2}, 0\right)$, y-intercept $\left(0, \frac{1}{3}\right)$

Rule for the inverse: $x = \frac{1}{2} \times 8^y - 1$

$\therefore \frac{1}{2} \times 8^y = x + 1$

$\therefore 8^y = 2x + 2$

$\therefore y = \log_8 (2x + 2)$



c $y = 2 - 4^x$

Asymptote: $y = 2$

y-intercept: let $x = 0$.

$\therefore y = 2 - 1$

$\therefore y = 1.$

$(0, 1)$

x-intercept: let $y = 0$.

$\therefore 0 = 2 - 4^x$

$\therefore 4^x = 2$

$\therefore 2^{2x} = 2^1$

$\therefore 2x = 1$

$\therefore x = \frac{1}{2}$

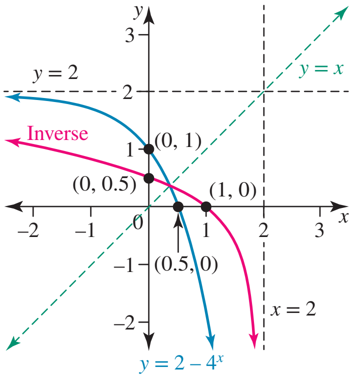
$\left(\frac{1}{2}, 0\right)$

Inverse: asymptote $x = 2$, x-intercept $(1, 0)$, y-intercept $\left(0, \frac{1}{2}\right)$

Rule for the inverse: $x = 2 - 4^y$

$\therefore 4^y = 2 - x$

$\therefore y = \log_4 (2 - x)$



d $y = 3^{x+1} + 3$

Asymptote: $y = 3$

y-intercept: let $x = 0$.

$\therefore y = 3 + 3$

$\therefore y = 6.$

$(0, 6)$

No x-intercept

Point: let $x = -1$.

$\therefore y = 1 + 3$

$\therefore y = 4$

$(-1, 4)$

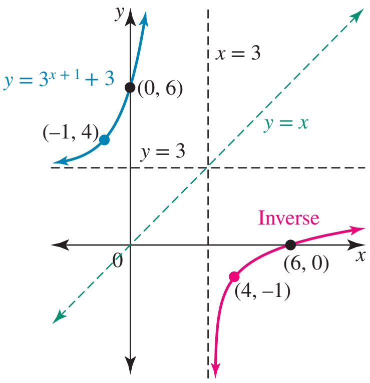
Inverse: asymptote $x = 3$, x-intercept $(6, 0)$, no y-intercept, point $(4, -1)$.

Rule for the inverse: $x = 3^{y+1} + 3$

$\therefore 3^{y+1} = x - 3$

$\therefore y + 1 = \log_3 (x - 3)$

$\therefore y = \log_3 (x - 3) - 1$



e $y = -10^{-2x}$

Asymptote: $y = 0$

No x-intercept

y-intercept: let $x = 0$.

$\therefore y = -1$

$(0, -1)$

Point: let $x = -\frac{1}{2}$.

$\therefore y = -10$

$\left(-\frac{1}{2}, -10\right)$

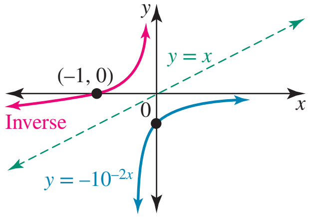
Inverse: asymptote $x = 0$, x-intercept $(-1, 0)$, no y-intercept, point $\left(-10, -\frac{1}{2}\right)$.

Rule for the inverse: $x = -10^{-2y}$

$\therefore 10^{-2y} = -x$

$\therefore \ -2y = \log_{10}(-x)$

$\therefore y = -\frac{1}{2}\log_{10}(-x)$



f $y = 2^{1-x}$

Asymptote: $y = 0$

No x-intercept

y-intercept: let $x = 0$.

$\therefore y = 2$

$(0, 2)$

Point: let $x = 1$.

$\therefore y = 1$

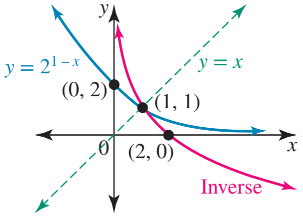
$(1, 1)$

Inverse: asymptote $x = 0$, x-intercept $(2, 0)$, no y-intercept, point $(1, 1)$.

Rule for the inverse: $x = 2^{1-y}$

$\therefore 1 - y = \log_2(x)$

$\therefore y = 1 - \log_2(x)$



20 a $y = 2^{ax+b} + c$

From the diagram, the asymptote is $y = -4$, so $c = -4$.

$\therefore y = 2^{ax+b} - 4$

Substitute the known points on the curve.

$(0, -2) \Rightarrow -2 = 2^b - 4$

$\therefore 2^b = 2$

$\therefore b = 1$

$\therefore y = 2^{ax+1} - 4$

$\left(\frac{1}{2}, 0\right) \Rightarrow 0 = 2^{\frac{1}{2}a+1} - 4$

$\therefore 2^{\frac{1}{2}a+1} = 4$

$\therefore 2^{\frac{1}{2}a+1} = 2^2$

$\therefore \frac{1}{2}a + 1 = 2$

$\therefore a = 2$

The rule for the function is $y = 2^{2x+1} - 4$.

b Inverse function rule:

$x = 2^{2y+1} - 4$

$\therefore 2^{2y+1} = x + 4$

$\therefore 2y + 1 = \log_2(x + 4)$

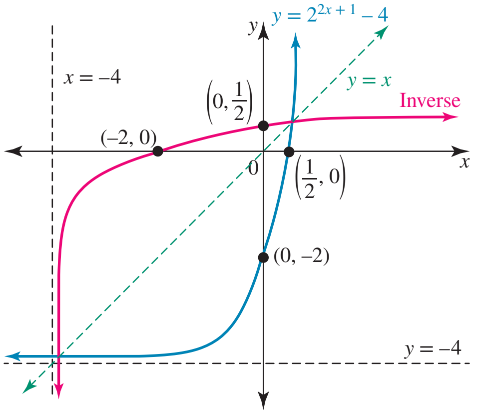
$\therefore 2y = \log_2(x + 4) - 1$

$\therefore y = \frac{1}{2}\log_2(x + 4) - \frac{1}{2}$

c Since $y = -4$ is an asymptote for the exponential function, $x = -4$ is the asymptote for the inverse function.

The points $\left(\frac{1}{2}, 0\right)$ and $(0, -2)$ are on the exponential function, so the y-intercept of the inverse is $\left(0, \frac{1}{2}\right)$ and the x-intercept is $(-2, 0)$.

d As the exponential graph intersects the line $y = x$ twice, the graph of the inverse must intersect the exponential at these points, giving two points of intersection between the graphs.



e $y = 2^{2x+1} - 4$

Substitute the point $(\log_2(3), k)$.

$\therefore k = 2^{2\log_2(3)+1} - 4$

$\therefore k = 2^{2\log_2(3)} \times 2^1 - 4$

$\therefore k = 2^{\log_2\left(3^2\right)} \times 2^1 - 4$

$\therefore k = 3^2 \times 2 - 4$

$\therefore k = 14$

f The inverse has equation $y = \frac{1}{2}\log_2(x + 4) - \frac{1}{2}$.

Substitute the point $(14, \log_2(3))$

$\text{LHS} = \log_2(3)$

$$\begin{aligned}\text{RHS} &= \frac{1}{2}\log_2(14 + 4) - \frac{1}{2}\\ &= \frac{1}{2}\log_2(18) - \frac{1}{2}\\ &= \frac{1}{2}\log_2\left(3^2 \times 2\right) - \frac{1}{2}\\ &= \frac{1}{2}\left[\log_2\left(3^2\right) + \log_2(2) - 1\right]\\ &= \frac{1}{2}\left[2\log_2(3) + 1 - 1\right]\\ &= \log_2(3)\\ &= \text{LHS}\end{aligned}$$

The point $(14, \log_2(3))$ lies on the inverse function.

10.6 Exam questions

1 Original function: $y = a^x$, y-intercept $(0, 1)$, no x-intercept

Inverse function: $y = \log_a x$, x-intercept $(1, 0)$, no y-intercept

y-intercept at $(0, 1)$ is for the inverse function.

The correct answer is **C**.

2 $f(x) = 4 \times 2^{-x} + 5$

y-intercept $(0, 9)$

Asymptote at $y = 5$

As is the y-intercept lies above the asymptote, the range is $(5.\infty)$,

The domain of the inverse function is the range of the given function,

$\therefore$ the domain of the inverse is $(5, \infty)$,

The correct answer is **B.**

3 $y = \log_2(x + 4)$

Horizontal translation 4 units to the left.

Asymptote at $x = -4$ [1 mark]

Domain $(x : x > -4)$

y-intercept $(x = 0)$

$y = \log_2(0 + 4)$

$y = \log_2 (2)^2$

$= 2\log_2(2)$

$= 2$

$(0, 2)$ [1 mark]

x-intercept $(y = 0)$

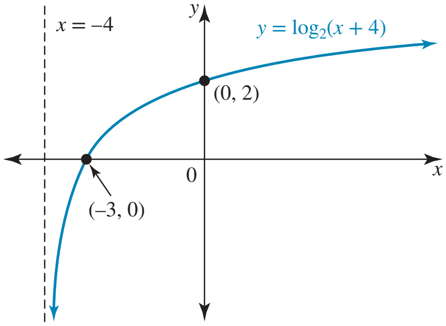
$0 = \log_2(x + 4)$

$\therefore x + 4 = 2^0$

$x = -3$

$(-3, 0)$ [1 mark]

Check: the point $(1, 0) \to (-3, 0)$ under the horizontal translation. [1 mark]



10.7 Review

10.7 Exercise

Technology free: short answer

1 a $\dfrac{\left(3a^2b^{\frac{1}{2}}\right)^{-2} \times 2\left(a^{-3}b\right)^{-1}}{\left(4a^{-4}\right)^{\frac{1}{2}}}$

$= \dfrac{3^{-2}a^{-4}b^{-1} \times 2a^3b^{-1}}{4^{\frac{1}{2}}a^{-2}}$

$= \dfrac{3^{-2} \times \cancel{2} \times a^{-1}b^{-2}}{\cancel{2}a^{-2}}$

$= \dfrac{ab^{-2}}{3^2}$

$= \dfrac{a}{9b^2}$

b $\dfrac{2^{n+3} - 2^n}{14}$

$= \dfrac{2^n \times 2^3 - 2^n}{14}$

$= \dfrac{2^n\left(2^3 - 1\right)}{14}$

$= \dfrac{2^n(7)}{14}$

$= \dfrac{2^n}{2}$

$= 2^{n-1}$

c $\dfrac{a - a^{-1}}{a + 1}$

$= \left(a - \dfrac{1}{a}\right) \div (a + 1)$

$= \left(\dfrac{a^2 - 1}{a}\right) \times \dfrac{1}{a + 1}$

$= \dfrac{\cancel{(a + 1)}(a - 1)}{a} \times \dfrac{1}{\cancel{a + 1}}$

$= \dfrac{a - 1}{a}$

d $18^{\frac{3}{4}} \times \left(\dfrac{2}{9}\right)^{-2} \div \sqrt{6}$

$= 18^{\frac{3}{4}} \times \left(\dfrac{9}{2}\right)^2 \times \dfrac{1}{\sqrt{6}}$

$= \left(3^2 \times 2\right)^{\frac{3}{4}} \times \left(\dfrac{3^2}{2}\right)^2 \times \dfrac{1}{(3 \times 2)^{\frac{1}{2}}}$

$= 3^{\frac{3}{2}} \times 2^{\frac{3}{4}} \times \dfrac{3^4}{2^2} \times \dfrac{1}{3^{\frac{1}{2}} \times 2^{\frac{1}{2}}}$

$= \dfrac{3^{\frac{3}{2}} \times 2^{\frac{3}{4}} \times 3^4}{2^2 \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}}}$

$= \dfrac{3^{\frac{11}{2}} \times 2^{\frac{3}{4}}}{2^{\frac{5}{2}} \times 3^{\frac{1}{2}}}$

$= \dfrac{3^{\frac{10}{2}}}{2^{\frac{7}{4}}}$

$= \dfrac{3^5}{2^{\frac{7}{4}}}$

2 a $\log_6(9) - \log_6\left(\dfrac{1}{4}\right)$

$= \log_6\left(9 \div \dfrac{1}{4}\right)$

$= \log_6(9 \times 4)$

$= \log_6(36)$

$= \log_6\left(6^2\right)$

$= 2\log_6(6)$

$= 2$

b $2\log_a(4) + 0.5\log_a(16) - 6\log_a(2)$

$= \log_a\left(4^2\right) + \log_a\left(16^{0.5}\right) - \log_a\left(2^6\right)$

$= \log_a(16) + \log_a(4) - \log_a(64)$

$= \log_a\left(\dfrac{16 \times 4}{64}\right)$

$= \log_a(1)$

$= 0$

c $\dfrac{\log_a(27)}{\log_a(3)}$

$= \dfrac{\log_a\left(3^3\right)}{\log_a(3)}$

$= \dfrac{3\log_a(3)}{\log_a(3)}$

$= 3$

d $-\log_{10}(5) \times \log_9(3) - \log_{10}\left(\sqrt{2}\right)$

$= -\log_{10}(5) \times \log_9\left(9^{\frac{1}{2}}\right) - \log_{10}\left(2^{\frac{1}{2}}\right)$

$= -\log_{10}(5) \times \dfrac{1}{2}\log_9(9) - \dfrac{1}{2}\log_{10}(2)$

$= -\log_{10}(5) \times \dfrac{1}{2} \times 1 - \dfrac{1}{2}\log_{10}(2)$

$= -\dfrac{1}{2}\left[\log_{10}(5) + \log_{10}(2)\right]$

$= -\dfrac{1}{2}\log_{10}(10)$

$= -\dfrac{1}{2}$

3 a $3^{1-7x} = 81^{x-2} \times 9^{2x}$

$\therefore 3^{1-7x} = \left(3^4\right)^{x-2} \times \left(3^2\right)^{2x}$

$\therefore 3^{1-7x} = 3^{4x-8} \times 3^{4x}$

$\therefore 3^{1-7x} = 3^{8x-8}$

$\therefore 1 - 7x = 8x - 8$

$\therefore 9 = 15x$

$\therefore x = \dfrac{9}{15}$

$\therefore x = \dfrac{3}{5}$

b $2^{2x} - 6 \times 2^x - 16 = 0$

Let $a = 2^x$.

$\therefore a^2 - 6a - 16 = 0$

$\therefore (a - 8)(a + 2) = 0$

$\therefore a = 8 \text{ or } a = -2$

$\therefore 2^x = 8 \text{ or } 2^x = -2$

Reject $2^x = -2$ since $2^x > 0$.

$\therefore 2^x = 8$

$\therefore x = 3$

c $\log_5(x + 2) + \log_5(x - 2) = 1$

$\therefore \log_5((x + 2)(x - 2)) = 1$

$\therefore \log_5\left(x^2 - 4\right) = 1$

$\therefore x^2 - 4 = 5^1$

$\therefore x^2 = 9$

$\therefore x = \pm 3$

Reject $x = -3$ since it is not in the allowable domain, which requires both $x + 2 > 0$ and $x - 2 > 0$.

$\therefore x = 3$

d $2\log_{10}(x) - \log_{10}(101x - 10) = -1$

$\therefore \log_{10}\left(x^2\right) - \log_{10}(101x - 10) = -1$

$\therefore \log_{10}\left(\dfrac{x^2}{101x - 10}\right) = -1$

$\therefore \dfrac{x^2}{101x - 10} = 10^{-1}$

$\therefore \dfrac{x^2}{101x - 10} = \dfrac{1}{10}$

$\therefore 10x^2 = 101x - 10$

$\therefore 10x^2 - 101x + 10 = 0$

$\therefore (10x - 1)(x - 10) = 0$

$\therefore x = \dfrac{1}{10} \text{ or } x = 10$

Both values satisfy the domain check when substituted into the original equation.

The solutions are $x = \dfrac{1}{10}$, $x = 10$.

4 a $3^{1-x} = 7$

$\therefore 1 - x = \log_3(7)$

$\therefore x = 1 - \log_3(7)$

Alternatively, convert the logarithm to base 10.

$\therefore x = 1 - \dfrac{\log_{10}(7)}{\log_{10}(3)}$

b $5^{x-1} = 2^{x+1}$

$\therefore \log\left(5^{x-1}\right) = \log\left(2^{x+1}\right)$

$\therefore (x - 1)\log(5) = (x + 1)\log(2)$

$\therefore x\log(5) - \log(5) = x\log(2) + \log(2)$

$\therefore x\log(5) - x\log(2) = \log(5) + \log(2)$

$\therefore x\left[\log(5) - \log(2)\right] = \log(5) + \log(2)$

$\therefore x = \dfrac{\log(5) + \log(2)}{\log(5) - \log(2)} = \dfrac{1}{\log(2.5)}$

c $0.2^x < 3$

$\therefore \log\left(0.2^x\right) < \log(3)$

$\therefore x\log(0.2) < \log(3)$

As $\log(0.2) < 0$,

$x > \dfrac{\log(3)}{\log(0.2)}$

d $10^x = 12 \times 10^{-x} + 4$

$\therefore 10^x = 12 \times \dfrac{1}{10^x} + 4$

Let $a = 10^x$.

$\therefore a = 12 \times \dfrac{1}{a} + 4$

$\therefore a^2 = 12 + 4a$

$\therefore a^2 - 4a - 12 = 0$

$\therefore (a - 6)(a + 2) = 0$

$\therefore a = 6 \text{ or } a = -2$

$\therefore 10^x = 6 \text{ or } 10^x = -2$

Reject $10^x = -2$.

$\therefore 10^x = 6$

$\therefore x = \log_{10}(6)$

5 a $y = 2^{3x} - 1$

Asymptote: $y = -1$

y-intercept: let $x = 0$.

$\therefore y = 1 - 1$

$\therefore y = 0$

$(0, 0)$

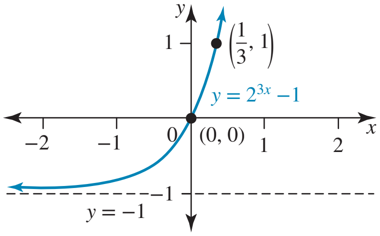
Range $(-1, \infty)$, domain R

Point: let $x = \dfrac{1}{3}$.

$\therefore y = 2^1 - 1$

$\therefore y = 1$

$\left(\dfrac{1}{3}, 1\right)$



b $y = -2 \times 3^{(x-1)}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = -2 \times 3^{-1}$

$\therefore y = -\frac{2}{3}$

$\left(0, -\frac{2}{3}\right)$

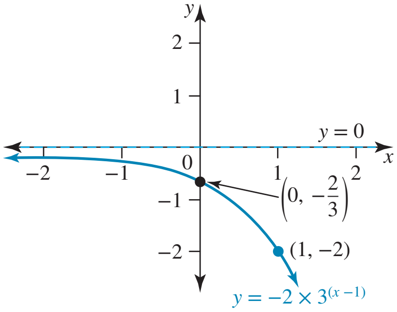
Range R^-, domain R

Point: let $x = 1$.

$\therefore y = -2 \times 1$

$\therefore y = -2$

$(1, -2)$



c $y = 5 - 5^{-x}$

Asymptote: $y = 5$

y-intercept: let $x = 0$.

$\therefore y = 5 - 1$

$\therefore y = 4$

$(0, 4)$

Range $(-\infty, 5)$, domain R

x-intercept: let $y = 0$.

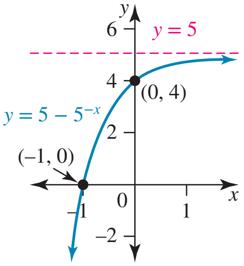
$\therefore 0 = 5 - 5^{-x}$

$\therefore 5^{-x} = 5$

$\therefore -x = 1$

$\therefore x = -1$

$(-1, 0)$



d $y = 3 \times \left(\frac{2}{5}\right)^{x+1}$

Asymptote: $y = 0$

y-intercept: let $x = 0$.

$\therefore y = 3 \times \left(\frac{2}{5}\right)^1$

$\therefore y = \frac{6}{5}$

$\left(0, \frac{6}{5}\right)$

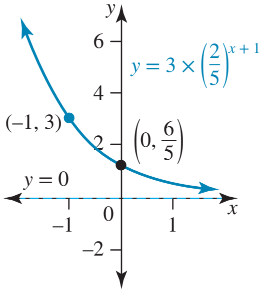
Range R^+, domain R

Point: let $x = -1$.

$\therefore y = 3 \times 1$

$\therefore y = 3$

$(-1, 3)$



6 **a** $y = \log_3(x) \rightarrow y = -\log_3(x + 3)$ under a reflection in the x-axis and a horizontal translation 3 units to the left.

Asymptote $x = -3$.

b The inverse of $y = -\log_3(x + 3)$ has the rule:

$x = -\log_3(y + 3)$

$\therefore -x = \log_3(y + 3)$

$\therefore y + 3 = 3^{-x}$

$\therefore y = 3^{-x} - 3$

The domain is R.

The domain of $y = -\log_3(x + 3)$ requires that $x + 3 > 0 \Rightarrow x > -3$. This means the range of the inverse is $(-3, \infty)$.

For $y = -\log_3(x + 3)$:

Asymptote: $x = -3$

y-intercept: let $x = 0$.

$\therefore y = -\log_3(3)$

$\therefore y = -1$

$(0, -1)$

x-intercept: let $y = 0$.

$\therefore 0 = -\log_3(x + 3)$

$\therefore \log_3(x + 3) = 0$

$\therefore x + 3 = 3^0$

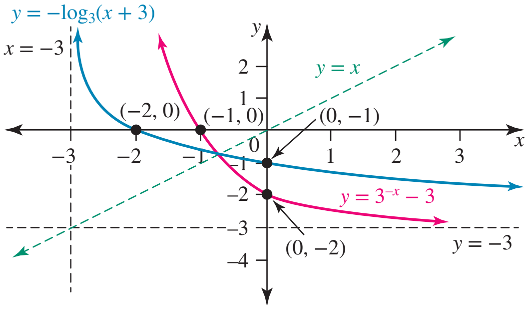
$\therefore x + 3 = 1$

$\therefore x = -2$

$(-2, 0)$

For the inverse $y = 3^{-x} - 3$:

asymptote $y = -3$, x-intercept $(-1, 0)$ and y-intercept $(0, -2)$



c $y = 5 \times 7^{1-x}$ can be written as $y = 5 \times 7^{-(x-1)}$.

The transformations for $y = 7^x \rightarrow y = 5 \times 7^{-(x-1)}$ are a dilation of factor 5 from the x-axis, a reflection in the y-axis and then a horizontal translation 1 unit to the right.

The equation of the asymptote is $y = 0$.

d $f: R \rightarrow R, f(x) = 5 \times 7^{1-x}$

The rule for f is $y = 5 \times 7^{1-x}$.

The rule for the inverse function f^{-1} is $x = 5 \times 7^{1-y}$.

$$\therefore \frac{x}{5} = 7^{1-y}$$

$$\therefore 1 - y = \log_7\left(\frac{x}{5}\right)$$

$$\therefore y = 1 - \log_7\left(\frac{x}{5}\right)$$

The inverse function is

$f^{-1}: R^+ \rightarrow R, f^{-1}(x) = 1 - \log_7\left(\frac{x}{5}\right)$.

Technology active: multiple choice

7 $\dfrac{(2^n)^2 \times 2^{n-2}}{4}$

$= \dfrac{2^{2n} \times 2^{n-2}}{2^2}$

$= \dfrac{2^{3n-2}}{2^2}$

$= 2^{3n-4}$

The correct answer is **B**.

8 $5^{2x+3} = 125^x$

$\therefore 5^{2x+3} = (5^3)^x$

$\therefore 5^{2x+3} = 5^{3x}$

$\therefore 2x + 3 = 3x$

$\therefore x = 3$

The correct answer is **D**.

9 $(3.2 \times 10^{-2}) \times (5 \times 10^5)$

$= (3.2 \times 5) \times (10^{-2} \times 10^5)$

$= 16 \times 10^3$

$= 1.6 \times 10^4$

The correct answer is **E**.

10 By the year 2020, the population is expected to be 4 485 211 + 942 000 = 5 427 211. To 2 significant figures, the expected population is 5 400 000.

The correct answer is **C**.

11 $3^5 = 243$ in logarithmic form is $5 = \log_3(243)$.

The correct answer is **D**.

12 $\log_{64}(x) = -\dfrac{2}{3}$

$\therefore x = 64^{-\frac{2}{3}}$

$= (4^3)^{-\frac{2}{3}}$

$= 4^{-2}$

$\therefore x = \dfrac{1}{16}$

The correct answer is **C**.

13 $\log_5(35) = p$

$\therefore \log_5(7 \times 5) = p$

$\therefore \log_5(7) + \log_5(5) = p$

$\therefore \log_5(7) + 1 = p$

$\therefore \log_5(7) = p - 1$

The correct answer is **A**.

14 $\sqrt[3]{5} + \sqrt[3]{625}$

$= \sqrt[3]{5} + \sqrt[3]{25^2}$

$= 5^{\frac{1}{3}} + (25)^{\frac{2}{3}}$

$= 5^{\frac{1}{3}} + 5^{\frac{4}{3}}$

$= 5^{\frac{1}{3}}(1 + 5^1)$

$= 6 \times 5^{\frac{1}{3}}$

$= 6 \times \sqrt[3]{5}$

or $\sqrt[3]{5} + \sqrt[3]{625}$

$= \sqrt[3]{5} + \sqrt[3]{25 \times 25}$

$= \sqrt[3]{5} + \sqrt[3]{5^3 \times 5}$

$= \sqrt[3]{5} + 5 \times \sqrt[3]{5}$

$= 6 \times \sqrt[3]{5}$

The correct answer is **B**.

15 The graph has a horizontal asymptote $y = 1$, so options **A** and **B** can be eliminated.

Of the remaining options **C**, **D** and **E**, the y-intercept $(0, 2)$ is only satisfied by option **E**.

The correct answer is **E**.

16 $2^{-3\log_2(x)}$

$= 2^{\log_2(x^{-3})}$

$= x^{-3}$

The correct answer is **C**.

Technology active: extended response

17 $f: R \rightarrow R, f(x) = 2^{x+b} + c$

a Let $f(x) = 2^{x+b} + c$.

Asymptote at $y = 2 \Rightarrow c = 2$

Point $(1, 3)$ lies on the graph, so $f(1) = 3$

$\therefore 3 = 2^{1+b} + 2$

$\therefore 2^{1+b} = 1$

$\therefore 1 + b = 0$

$\therefore b = -1$

Hence, $b = -1$, $c = 2$ and the function's rule is $f(x) = 2^{x-1} + 2$.

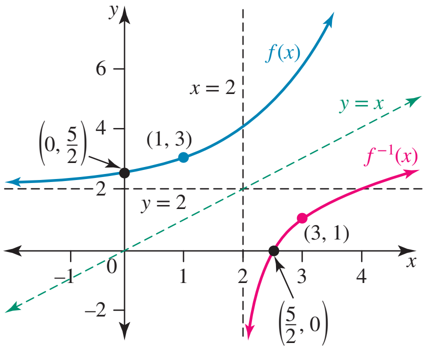
b $f(x) = 2^{x-1} + 2$

$= 2^x \times 2^{-1} + 2$

$= \dfrac{1}{2} \times 2^x + 2$

The rule can be expressed as $f(x) = \dfrac{1}{2} \times 2^x + 2$, which is in the form $f(x) = a \times 2^x + c$ with $a = \dfrac{1}{2}$, $c = 2$.

c The inverse function will have asymptote $x = 2$ and contain the point $(3, 1)$. The domain of the inverse is the same set as the range of f, so the domain of the inverse is $(2, \infty)$.



d For the function rule in the form $y = 2^{x-1} + 2$, the rule for the inverse is

$$\begin{aligned} x &= 2^{y-1} + 2 \\ \therefore 2^{y-1} &= x - 2 \\ \therefore y - 1 &= \log_2(x-2) \\ \therefore y &= \log_2(x-2) + 1 \end{aligned}$$

For the function rule in the form $y = \frac{1}{2} \times 2^x - 2$, the rule for the inverse becomes

$$\begin{aligned} x &= \frac{1}{2} \times 2^y + 2 \\ \therefore \frac{1}{2} \times 2^y &= x - 2 \\ \therefore 2^y &= 2x - 4 \\ \therefore y &= \log_2(2x-4) \end{aligned}$$

To show that $\log_2(2x-4) = \log_2(x-2) + 1$ using logarithm laws:

$$\begin{aligned} \text{LHS} &= \log_2(2x-4) \\ &= \log_2(2(x-2)) \\ &= \log_2(2) + \log_2(x-2) \\ &= 1 + \log_2(x-2) \\ &= \log_2(x-2) + 1 \\ &= \text{RHS} \end{aligned}$$

The rule for the inverse function is $f^{-1}(x) = \log_2(2x-4)$ or $f^{-1}(x) = \log_2(x-2) + 1$.

e **i** $f(x) = 6$

$$\begin{aligned} \therefore 2^{x-1} + 2 &= 6 \\ \therefore 2^{x-1} &= 4 \\ \therefore 2^{x-1} &= 2^2 \\ \therefore x - 1 &= 2 \\ \therefore x &= 3 \end{aligned}$$

Point A has coordinates (3, 6).

ii $f^{-1}(x) = 6$

$$\begin{aligned} \therefore \log_2(x-2) + 1 &= 6 \\ \therefore \log_2(x-2) &= 5 \\ \therefore x - 2 &= 2^5 \\ \therefore x - 2 &= 32 \\ \therefore x &= 34 \end{aligned}$$

Point B has coordinates (34, 6).

iii $f^{-1}(x) = 1$

As an alternative to the method used in part **ii**, if $f^{-1}(x) = 1$, then $x = f(1)$.

$$\begin{aligned} f(1) &= 2^0 + 2 \\ &= 3 \end{aligned}$$

Point P has coordinates (3, 1).

f A (3, 6) lies 5 units vertically above P(3, 1).

B (34, 6) lies 31 units horizontally from A (3, 6).

The sides AP and AB are perpendicular and their lengths give the base and height of triangle ABP

Area of triangle ABP:

$$\begin{aligned} A_{\triangle} &= \frac{1}{2} \times 31 \times 5 \\ &= \frac{155}{2} \\ &= 77.5 \end{aligned}$$

The area is 77.5 square units.

18 **a** $T = a \times \left(\frac{16}{5}\right)^{-kt}$

At 10 am, $t = 0$ and $T = 100$

$$\therefore 100 = a \times \left(\frac{16}{5}\right)^0$$

$$\therefore a = 100$$

b At 10 : 12 am, $t = 12$, $T = 75$

$$\begin{aligned} \therefore 75 &= 100 \times \left(\frac{16}{5}\right)^{-12k} \\ \therefore 0.75 &= \left(\frac{16}{5}\right)^{-12k} \\ \therefore (3.2)^{-12k} &= 0.75 \\ \therefore -12k &= \log_{3.2}(0.75) \\ \therefore k &= -\frac{1}{12} \times \frac{\log(0.75)}{\log(3.2)} \\ \therefore k &\simeq 0.02 \end{aligned}$$

c The temperature model is $T = 100 \times \left(\frac{16}{5}\right)^{-0.02t}$

At 10 : 30 am, $t = 30$

$$\begin{aligned} \therefore T &= 100 \times \left(\frac{16}{5}\right)^{-0.02 \times 30} \\ \therefore T &= 100 \times \left(\frac{16}{5}\right)^{-0.6} \\ \therefore T &\simeq 49.8 \end{aligned}$$

To the nearest degree, the temperature of the water is 50 degrees.

d $T = 50 \times 2^{\frac{t}{9}}$

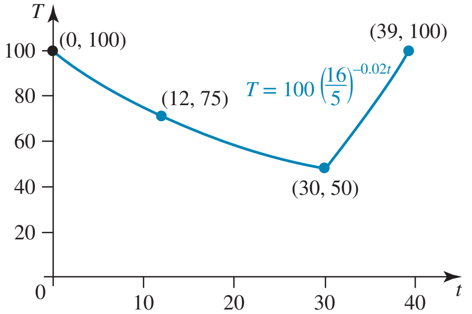
Let $T = 100$.

$$\begin{aligned} \therefore 100 &= 50 \times 2^{\frac{t}{9}} \\ \therefore 2 &= 2^{\frac{t}{9}} \\ \therefore \frac{t}{9} &= 1 \\ \therefore t &= 9 \end{aligned}$$

The water reaches boiling point 9 minutes after 10:30 am at 10:39 am.

e Cooling period: $T = 100 \times \left(\frac{16}{5}\right)^{-0.02t}$, $0 \le t \le 30$ are (0, 100) and (30, 50).

Heating period: $T = 50 \times 2^{\frac{t}{9}}$, $30 < t \le 39$. The end points are (30, 50) and (39, 100)



f The temperature falls to 75 degrees at 10 : 12 am. After 18 minutes of further cooling, the water starts to be reheated. We need to find when it again reaches 75 degrees as it heats up.

$T = 50 \times 2^{\frac{t}{9}}$

Let $T = 75$.

$$\begin{aligned} \therefore 75 &= 50 \times 2^{\frac{t}{9}} \\ \therefore 1.5 &= 2^{\frac{t}{9}} \end{aligned}$$

$$\therefore \frac{t}{9} = \log_2(1.5)$$

$$\therefore t = 9 \times \frac{\log(1.5)}{\log(2)}$$

$$\therefore t \simeq 5.26$$

The temperature is again at 75 degrees 5.3 minutes after 10 : 30 am. The length of time that the temperature is below 75 degrees is 18 + 5.3 = 23.3 minutes.

19 **a** $T(n) = a \times 2^{kn}$

Given $T(2.4) = 2T(1.2)$

$$\therefore a \times 2^{2.4k} = 2 \times a \times 2^{1.2k}$$

$$\therefore 2^{2.4k} = 2 \times 2^{1.2k}$$

$$\therefore 2^{2.4k} = 2^{1.2k+1}$$

$$\therefore 2.4k = 1.2k + 1$$

$$\therefore 1.2k = 1$$

$$\therefore \frac{6}{5}k = 1$$

$$\therefore k = \frac{5}{6}$$

b The time model is $T(n) = a \times 2^{\frac{5n}{6}}$.

Initially, $T = 20$ with $n = 1.2$.

$T(1.2) = 20$

$$\therefore 20 = a \times 2^{\frac{5}{6}\times 1.2}$$

$$\therefore 20 = a \times 2^1$$

$$\therefore a = 10$$

The model is determined as $T(n) = 10 \times 2^{\frac{5n}{6}}$.

Let $n = 3 \times 1.2 = 3.6$.

$$\begin{aligned} T(3.6) &= 10 \times 2^{\frac{5}{6}\times 1.2 \times 3} \\ &= 10 \times 2^3 \\ &= 80 \end{aligned}$$

When the population trebles, it takes 80 minutes to drive to work.

c $Q = 0.5 - 0.27 \times 3^{-0.2t}$

i When $t = 0$,

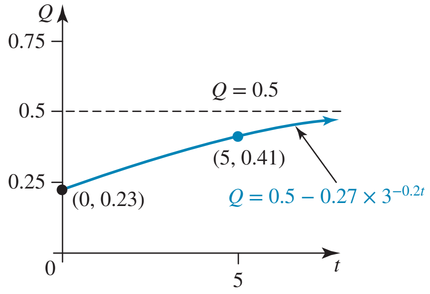
$$\begin{aligned} Q &= 0.5 - 0.27 \times 3^0 \\ &= 0.23 \end{aligned}$$

When $t = 5$,

$$\begin{aligned} Q &= 0.5 - 0.27 \times 3^{-0.2\times 5} \\ &= 0.5 - 0.27 \times 3^{-1} \\ &= 0.5 - 0.27 \times \frac{1}{3} \\ &= 0.5 - 0.09 \\ &= 0.41 \end{aligned}$$

The amount of pollutant has increased by $(0.41 - 0.23) = 0.18$ grams per litre.

ii Asymptote $Q = 0.5$, point $(5, 0.41)$ and end point $(0, 0.23)$



iii Let $Q = 2 \times 0.23 = 0.46$.

$$\therefore 0.46 = 0.5 - 0.27 \times 3^{-0.2t}$$

$$\therefore 0.27 \times 3^{-0.2t} = 0.04$$

$$\therefore 3^{-0.2t} = \frac{4}{27}$$

$$\therefore -0.2t = \log_3\left(\frac{4}{27}\right)$$

$$\therefore t = -5 \times \frac{\log\left(\frac{4}{27}\right)}{\log(3)}$$

$$\therefore t \simeq 8.69$$

iv The amount of pollutant per litre doubles after approximately 8.7 years.

The asymptote shows that as $t \to \infty, Q \to 0.5$. The model predicts that the pollutant level will approach 0.5 grams per litre in the long term.

d **i** $L = 10\log_{10}\left(I \times 10^{12}\right)$

Let $L = 70$.

$$\therefore 70 = 10\log_{10}\left(I \times 10^{12}\right)$$

$$\therefore \log_{10}\left(I \times 10^{12}\right) = 7$$

$$\therefore I \times 10^{12} = 10^7$$

$$\therefore I = 10^{-5}$$

The intensity of sound produced by each guitar is 1×10^{-5} watts per square metre.

ii The combined intensity when two guitars are played is $2 \times \left(1 \times 10^{-5}\right) = 2 \times 10^{-5}$ watts per square metre

Let $I = 2 \times 10^{-5}$.

$$\therefore L = 10\log_{10}\left(2 \times 10^{-5} \times 10^{12}\right)$$

$$\therefore L = 10\log_{10}\left(2 \times 10^{7}\right)$$

$$\therefore L = 73$$

The decibel reading when both guitars are played is 73 dB.

20 **a** **i** $(x+2)^{\frac{3}{2}} - (x+2)^{\frac{1}{2}}$

$$\begin{aligned} &= (x+2)(x+2)^{\frac{1}{2}} - (x+2)^{\frac{1}{2}} \\ &= (x+2)^{\frac{1}{2}}[(x+2) - 1] \\ &= (x+2)^{\frac{1}{2}}(x+1) \end{aligned}$$

ii $4x\left(4x^2+1\right)^{-\frac{3}{2}}(2x-1)^{-1} - 2(2x-1)^{-2}\left(4x^2+1\right)^{-\frac{1}{2}}$

$$\begin{aligned} &= \left(4x^2+1\right)^{-\frac{3}{2}}(2x-1)^{-2}\left[4x(2x-1)^1 - 2\left(4x^2+1\right)^1\right] \\ &= \left(4x^2+1\right)^{-\frac{3}{2}}(2x-1)^{-2}\left[8x^2 - 4x - 8x^2 - 2\right] \\ &= \left(4x^2+1\right)^{-\frac{3}{2}}(2x-1)^{-2}(-4x-2) \\ &= \left(4x^2+1\right)^{-\frac{3}{2}}(2x-1)^{-2}(-2)(2x+1) \\ &= -2\left(4x^2+1\right)^{-\frac{3}{2}}(2x-1)^{-2}(2x+1) \end{aligned}$$

If $4x\left(4x^2+1\right)^{-\frac{3}{2}}(2x-1)^{-1}$
$-2(2x-1)^{-2}\left(4x^2+1\right)^{-\frac{1}{2}} = 0$, then
$-2\left(4x^2+1\right)^{-\frac{3}{2}}(2x-1)^{-2}(2x+1) = 0$

$$\therefore \frac{-2(2x+1)}{\left(4x^2+1\right)^{\frac{3}{2}}(2x-1)^2} = 0$$

$$\therefore -2(2x+1) = 0$$

$$\therefore 2x + 1 = 0$$

$$\therefore x = -\frac{1}{2}$$

b i $\dfrac{a^{-3}-b^{-3}}{a^{-2}-b^{-2}} \div \dfrac{1}{a^{-1}+b^{-1}}$

$$= \frac{a^{-3}-b^{-3}}{a^{-2}-b^{-2}} \times \frac{a^{-1}+b^{-1}}{1}$$

$$= \left(\frac{1}{a^3}-\frac{1}{b^3}\right) \div \left(\frac{1}{a^2}-\frac{1}{b^2}\right) \times \left(\frac{1}{a}+\frac{1}{b}\right)$$

$$= \left(\frac{b^3-a^3}{a^3b^3}\right) \div \left(\frac{b^2-a^2}{a^2b^2}\right) \times \left(\frac{b+a}{ab}\right)$$

$$= \left(\frac{b^3-a^3}{a^3b^3}\right) \times \left(\frac{a^2b^2}{b^2-a^2}\right) \times \left(\frac{b+a}{ab}\right)$$

$$= \left(\frac{\cancel{(b-a)}(b^2+ba+a^2)}{a^3b^3ab}\right) \times \left(\frac{a^2b^{21}}{\cancel{(b-a)}\,\cancel{(b+a)}}\right)$$

$$\times \left(\frac{\cancel{b+a}^{1}}{ab}\right)$$

$$= \frac{\left(b^2+ba+a^2\right)}{ab} \times 1 \times \frac{1}{ab}$$

$$= \frac{b^2+ba+a^2}{a^2b^2}$$

ii $\left[(m+n)^{\frac{1}{2}}+m^{\frac{1}{2}}-n^{\frac{1}{2}}\right]\left[(m+n)^{\frac{1}{2}}-m^{\frac{1}{2}}+n^{\frac{1}{2}}\right]$

$$= \left[(m+n)^{\frac{1}{2}}+\left(m^{\frac{1}{2}}-n^{\frac{1}{2}}\right)\right]$$

$$\left[(m+n)^{\frac{1}{2}}-\left(m^{\frac{1}{2}}-n^{\frac{1}{2}}\right)\right]$$

$$= \left((m+n)^{\frac{1}{2}}\right)^2-\left(m^{\frac{1}{2}}-n^{\frac{1}{2}}\right)^2$$

$$= (m+n)^1-\left[\left(m^{\frac{1}{2}}\right)^2-2m^{\frac{1}{2}}n^{\frac{1}{2}}+\left(n^{\frac{1}{2}}\right)^2\right]$$

$$= m+n-m+2m^{\frac{1}{2}}n^{\frac{1}{2}}-n$$

$$= 2m^{\frac{1}{2}}n^{\frac{1}{2}} \text{ or } 2\sqrt{mn}$$

If $m = \log_a(b)$ and $n = \log_b(a)$, then $mn = \log_a(b) \times \log_b(a)$. Using the change of base law,

$$mn = \frac{\log_{10}(b)}{\log_{10}(a)} \times \frac{\log_{10}(a)}{\log_{10}(b)}$$

$$= 1$$

$$\therefore 2\sqrt{mn} = 2\sqrt{1} = 2$$

c i Show that $\log(n!) = \log(2)+\log(3)+\ldots+\log(n)$, $n \in N$

$\text{LHS} = \log(n!)$

$$= \log(n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1)$$

$$= \log(n)+\log(n-1)+\log(n-2)+\ldots+\log(3)$$
$$+\log(2)+\log(1)$$

$$= \log(n)+\log(n-1)+\log(n-2)+\ldots+\log(3)$$
$$+\log(2)+0$$

$$= \log(2)+\log(3)+\ldots+\log(n)$$

$$= \text{RHS}$$

ii $\log(10!)-\log(9!)$

Using the result in part **c i**,

$\log(10!)-\log(9!)$

$$= \left[\log(2)+\log(3)+\ldots+\log(10)\right]$$
$$-\left[\log(2)+\log(3)+\ldots+\log(9)\right]$$

$$= \log(10)$$

$$= 1$$

d $p(x) = \dfrac{x}{\log_e(x)}$

$$p(10) = \frac{10}{\log_e(10)}$$

$\therefore p(10) \simeq 4$

The primes less than 10 are 2, 3, 5 and 7, so there are 4 prime numbers less than 10. This agrees with the estimate $p(10)$.

$$p(30) = \frac{30}{\log_e(30)}$$

$\therefore p(30) \simeq 9$

The primes less than 30 are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29, so there are 10 prime numbers less than 30. This differs from the estimate $p(30)$ by one.

10.7 Exam questions

1 $52.2 \times 10^{-4} \div \left(5 \times 10^8\right) = \dfrac{52.2 \times 10^{-4}}{5 \times 10^8}$

$$= 10.44 \times 10^{-12}$$

$$= 1.044 \times 10^{-11}$$

The correct answer is **D**.

2 $\log_{10}(2x-3) = 0$

First law $\log_a(1) = 0$

$$\therefore \log_{10}(2x-3) = \log_{10} 1$$

$$2x-3 = 1$$

$$2x = 4$$

$$x = 2$$

The correct answer is **C**.

3 y-intercept $(x = 0)$

$y = -4 \times 5^0$

$y = -4$

$(0, 4)$

Decay shaped graph

Asymptote $y = 0$

$\therefore$ option E

The correct answer is **E**.

4 $y = a\log_9(bx)$

$9^y = (bx)^a$ [1 mark]

Point $(3, 0)$

$9^0 = b^a3^a$

$b^a = \dfrac{1}{3^a}$ [1 mark]

Point $(27, 4)$

$9^4 = 27^a3^a$

Substitute $b^a = \dfrac{1}{3^a} = 3^{-a}$

$$9^4 = 27^a \times 3^{-a}$$

$$\left(3^2\right)^4 = 3^{-a} \times \left(3^3\right)^a$$

$$3^8 = 3^{-a} \times 3^{3a}$$

$$3^8 = 3^{2a}$$

$$\therefore a = 4$$

Substitute $a = 4$ to find b.

$$\begin{aligned} 4 &= 4\log_9(27b) \\ 1 &= \log_9(27b) \\ 9 &= 27b \\ \frac{1}{3} &= b \end{aligned}$$ [1 mark]

$\therefore a = 4,\ b = \dfrac{1}{3},\ y = 4\log_9\left(\dfrac{1}{3}x\right)$ [1 mark]

5 $$\begin{aligned} \log_{20}\left(16^x \times 5^{2x}\right) &= \log_{20}\left(\left(4^2\right)^x \times 5^{2x}\right) \\ &= \log_{20}\left(4^{2x} \times 5^{2x}\right) \\ &= \log_{20}\left(20^{2x}\right) \\ &= 2x \end{aligned}$$

The correct answer is **C**.

Topic 11 — Introduction to differential calculus

11.2 Rates of change

11.2 Exercise

1 **a** The temperature is 15° at 7 am and 29° by 2 pm, 7 hours later.

Average rate of change of the temperature

$= \frac{\text{change in temperature}}{\text{time}}$

$= \frac{29-15}{7}$

$= \frac{14}{7}$

$= 2$

The average rate of change of the temperature is 2°/h.

b If the temperature increases at the steady rate of 2° /h, then every hour the temperature increases by 2°. At 8 am, the temperature is 17° and at 9 am the temperature would be 19°.

2 **a** Kate must produce 30 dresses in 5 days.

Her average rate of production must be $\frac{30}{5} = 6$ dresses per day.

b As Kate works from 9 am to 12 noon, she works for 3 hours each day.

Therefore, over 5 days, she works 15 hours.

Her average rate of production must be $\frac{30}{15} = 2$ dresses per hour.

c Distance travelled is 25 km, time taken is 30 minutes or $\frac{1}{2}$ an hour.

Average speed

$= \frac{\text{distance}}{\text{time}}$

$= \frac{25}{\frac{1}{2}}$

$= 25 \times \frac{2}{1}$

$= 50$

Kate's average speed is 50 km/h.

3 A (2, 6) and B (−7, 12)

a Average rate of change

$= \frac{12-6}{-7-2}$

$= \frac{6}{-9}$

$= -\frac{2}{3}$

b The average rate of change is the gradient of the line AB. Hence, the gradient of the line is $-\frac{2}{3}$.

c Only linear functions have a constant rate of change.

d The function with rule $y = 12 - 3x$ is a linear function with gradient −3. Its rate of change is constant, so the instantaneous rate of change at A is −3.

4 **a** The average rate of change between points P and R measures the gradient of the line segment joining the two points.

b The line segment PQ has a negative gradient so the average rate of change between points P and Q is negative.

c If the average rate of change between the points R and S is zero, then RS is horizontal.

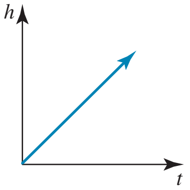
Point S is the reflection of point R in the y-axis. If R has coordinates (a, b), then S has coordinates $(-a, b)$.

d The instantaneous rate of change of a function at a point is measured by the gradient of the tangent to the curve drawn at that point.

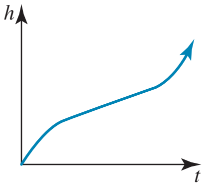
e The tangent to the curve at point P would have a negative gradient. Therefore, the instantaneous rate of change of the function at P is negative.

f The tangents drawn at both Q and R will have positive gradients. Of these, the tangent drawn at R would be the steeper. Of the points P, Q and R, the instantaneous rate of change is greatest at point R.

5 The depth of petrol in the cylindrical container will rise at a steady rate. The graph of h versus t will be linear.



The depth of petrol in the jerry can container will rise, but more slowly as the container widens. It will then rise steadily once the petrol level reaches the uniform cross-section of the container.



6 $f(x) = x^2 + 3, x \in [1, 4]$

Average rate of change

$= \frac{f(4) - f(1)}{4-1}$

$= \frac{19-4}{3}$

$= 5$

7 1 hour 20 minutes is $1\frac{1}{3}$ hours.

speed $= \frac{\text{distance}}{\text{time}}$

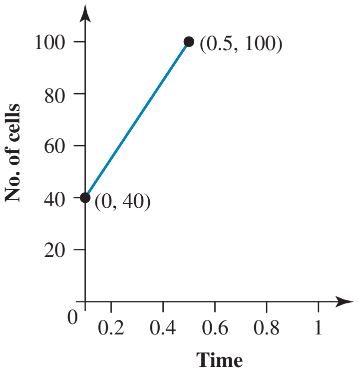
$= \frac{48}{1\frac{1}{3}}$

$= 48 \times \frac{3}{4}$

$= 36$

The speed of the car is 36 km/h.

8 **a** The cells multiply steadily, so it's a linear relationship.



The rate of growth of the bacteria equals the gradient of the line.

$$\text{Rate} = \frac{100 - 40}{0.5 - 0}$$
$$= \frac{60}{0.5}$$
$$= 120$$

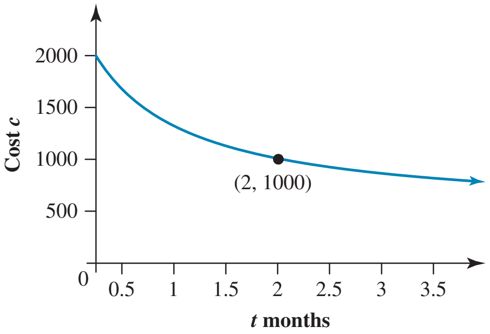
Therefore, the bacteria are growing at 120 bacterial cells per hour.

b The cost of the department is given by $c = \dfrac{4000}{t + 2}$.

Hyperbola shape on domain where $t \geq 0$

$t = 0 \Rightarrow c = 2000$

$t = 2 \Rightarrow c = 1000$



Average rate of change of costs

$$= \frac{\text{change in cost}}{\text{change in time}}$$
$$= \frac{2000 - 1000}{0 - 2}$$
$$= -500$$

Therefore, the costs have decreased at an average rate of \$500 per month over the first two months.

9 Refer to the diagram given in the question.

a Student II maintained the same level of interest for the semester, so for this student there was a zero rate of change in interest level.

b The level of interest of Student I steadily increased during the semester, so for this student there was a constant positive rate of change in interest level.

c The level of interest of Student III initially increased slowly but as the semester progressed this student's interest level grew more quickly, reaching a high level.

After an initially brief growth in interest, the level of interest of Student IV started to decline. However, this loss of interest eventually slowed and thereafter the student's interest level grew again and quickly reached a high level.

d Answers will vary.

10 **a** $f(x) = 2x - x^2, \ x \in [-2, 6]$

$$f(-2) = 2(-2) - (-2)^2$$
$$= -8$$
$$f(6) = 2(6) - (6)^2$$
$$= -24$$

Average rate of change:

$$\frac{f(6) - f(-2)}{6 - (-2)} = \frac{-24 - (-8)}{8}$$
$$= \frac{-16}{8}$$
$$= -2$$

b $f(x) = 2 + 3x, \ x \in [12, 16]$

$$f(12) = 2 + 3(12)$$
$$= 38$$
$$f(16) = 2 + 3(16)$$
$$= 50$$

Average rate of change:

$$\frac{f(16) - f(12)}{16 - (12)} = \frac{50 - 38}{4}$$
$$= 3$$

(which is the gradient of the line).

c $f(t) = t^2 + 3t - 1, \ t \in [1, \ 3]$

$$f(1) = 1 + 3 - 1$$
$$= 3$$
$$f(3) = 9 + 9 - 1$$
$$= 17$$

Average rate of change:

$$\frac{f(3) - f(1)}{3 - 1} = \frac{17 - 3}{2}$$
$$= 7$$

d $f(t) = t^3 - t, \ t \in [-1, \ 1]$

$$f(-1) = (-1)^3 - (-1)$$
$$= -1 + 1$$
$$= 0$$
$$f(1) = 1 - 1$$
$$= 0$$

Average rate of change:

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{0 - 0}{2}$$
$$= 0$$

11 **a** 20 minutes is $\frac{1}{3}$ of an hour. In this time, \$200 is received.

The plumber's hourly rate of pay was \$600 per hour.

b 15 minutes is $\frac{1}{4}$ of an hour. In this time, \$180 is received.

The surgeon's hourly rate of pay was \$720 per hour.

c In 52 hours, \$1820 was received.

The teacher's hourly rate of pay was $\$\left(\dfrac{1820}{52}\right) = \$35$ per hour.

12 **a** There is a 9-month interval between the months of November and August of the following year. Over this interval, the Australian dollar rose in price by $(0.83 - 0.67) = 0.16$ euros.

The average rate of change of the dollar over this time period is $\dfrac{0.16}{9} \simeq 0.018$ euros per month.

b Over 3 years the value of the investment rose from \$1000 to \$1150 at a steady rate.

The rate of change is $\frac{150}{3} = 50$ dollars per year.

The percentage rate of interest: $\frac{50}{1000} \times 100 = 5$

Therefore, the investment earns 5% per annum interest and $r = 5$.

13 a $x = t^3 + 4t^2 + 3t$ over the interval $[0, 2]$

$t = 0 \Rightarrow x = 0$

$t = 2 \Rightarrow x = 30$

Average rate of change

$= \frac{30 - 0}{2 - 0}$

$= 15$

Therefore, the average speed over the interval $t \in [0, 2]$ is 15 m/s.

Over the interval $[0.9, 1.1]$,

$t = 0.9 \Rightarrow x = (0.9)^3 + 4(0.9)^2 + 3(0.9)$

$\therefore t = 0.9,\ x = 6.669$

$t = 1.1 \Rightarrow x = (1.1)^3 + 4(1.1)^2 + 3(1.1)$

$\therefore t = 1.1,\ x = 9.471$

Average rate of change

$= \frac{9.471 - 6.669}{1.1 - 0.9}$

$= \frac{2.802}{0.2}$

$= 14.01$

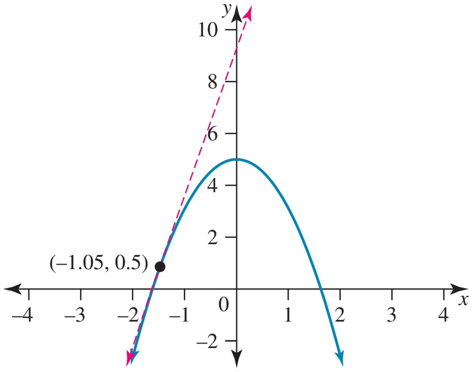
Therefore, the average speed over the interval is $t \in [0.9, 1.1]$ is 14.01 m/s.

b The speed at $t = 1$ is better estimated using the interval $[0.9, 1.1]$, because these end points give to closer points to $t = 1$ than do the end points of the interval $[0, 2]$.

Therefore, the better estimate for the instantaneous speed at $t = 1$ is 14.01 m/s.

14 Answers will vary.

a



Point Q $(-1.5, 0.5)$ lies on the tangent and another point is estimated to be $(0, 9.5)$.

Gradient of tangent:

$m = \frac{9.5 - 0.5}{0 - (-1.5)}$

$= \frac{9}{1.5}$

$= 6$

Therefore, the gradient of the curve at point Q is estimated to be 6.

b Choose a point with $x = -1.4$ close to Q.

y-coordinate: $5 - 2(-1.4)^2 = 1.08$

Points $(-1.4, 1.08)$, $(-1.5, 0.5)$

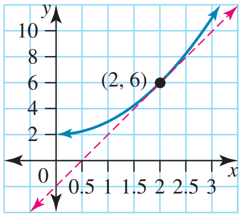
$m = \frac{1.08 - 0.5}{-1.4 - (-1.5)}$

$= \frac{0.58}{0.1}$

$= 5.8$

Therefore, the gradient of the curve at point Q is estimated to be 5.8.

15 a Construct a tangent line to the curve at the point $(2, 6)$ and estimate a second point that lies on the tangent line.

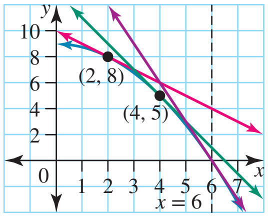


Answers will vary, but suppose a second point is $(0, -2)$.

Gradient of tangent is $\frac{6 - (-2)}{2 - 0} = 4$.

An estimate of the gradient of the curve at the point on the curve where $x = 2$ is 4.

b Construct tangent lines to the curve at the points $(2, 8)$ and $(4, 5)$ and estimate a second point that lies on each of these lines.



The tangent at $(2, 8)$ passes through $(0, 10)$ (answers will vary).

Its gradient is $\frac{8 - 10}{2 - 0} = -1$.

An estimate of the gradient of the curve at the point on the curve where $x = 2$ is -1.

The tangent at $(4, 5)$ passes through $(6.5, 0)$ (answers will vary).

Its gradient is $\frac{5 - 0}{4 - 6.5} = -\frac{5}{2.5} = -2$.

An estimate of the gradient of the curve at the point on the curve where $x = 4$ is -2.

c The average rate of change over the interval $x \in [4, 6]$ is the gradient of the line joining the end points $(4, 5)$ and $(6, 0)$.

Average rate of change is $= \frac{0 - 5}{6 - 4} = \frac{-5}{2}$.

d The gradient of the tangent at $(6, 0)$ is required to calculate the angle at which the curve cuts the x-axis. Construct a tangent line to the curve at the point $(6, 0)$ and estimate a second point that lies on the tangent line. (See the diagram in part **b**).

The tangent at $(6, 0)$ passes through $(3, 9)$ (answers will vary).

Its gradient is $\frac{9 - 0}{3 - 6} = -3$.

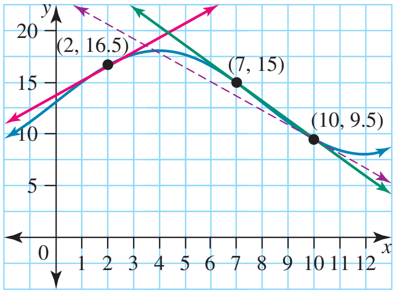
$\therefore \tan\theta = -3$

$\therefore \theta = 180° - \tan^{-1}(3)$

$\therefore \theta \simeq 108°$

The curve cuts the x-axis at an angle of approximately 108°.

16 Construct a tangent line at each of the given points and obtain the coordinates of a second point that lies on each tangent line. Answers will vary for parts **a**, **c** and **d**.



a **i** Two points estimated to be on tangent are (2, 16.5) and (4.5, 20).

$$m = \frac{20 - 16.5}{4.5 - 2}$$
$$= \frac{3.5}{2.5}$$
$$\simeq 1.4$$

At 2 pm the temperature is estimated to be changing at 1.4°C per hour.

ii The tangent at the maximum turning point where $t = 4$ is horizontal so there is zero rate of change of temperature at 4 pm.

iii Two points estimated to be on tangent are (7, 15) and (12.5, 5).

$$m = \frac{5 - 15}{12.5 - 7}$$
$$= \frac{-10}{5.5}$$
$$\simeq -1.8$$

At 7 pm the temperature is estimated to be changing at −1.8°C per hour.

iv Two points estimated to be on tangent are (10, 9.5) and (6, 15).

$$m = \frac{9.5 - 15}{10 - 6}$$
$$= \frac{-5.5}{4}$$
$$\simeq -1.4$$

At 10 pm the temperature is estimated to be changing at −1.4°C per hour.

b Judging where the tangent to the curve would be steepest with a negative gradient, suggests the temperature is decreasing most rapidly at around 8 pm. Answers may vary slightly.

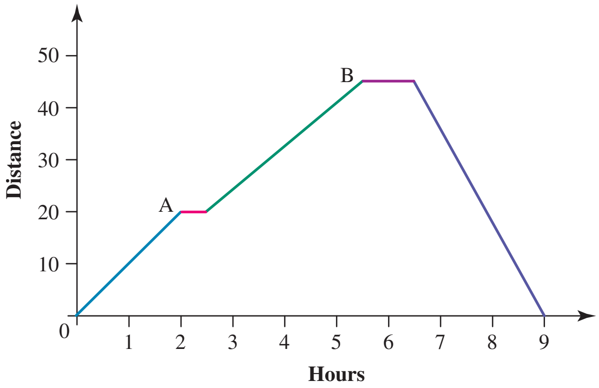
c At 1 pm, the temperature is approximately 15° and at 9:30 pm the temperature is approximately 10°.

The average rate of change of temperature over this time interval is

$$= \frac{10 - 15}{9.5 - 1}$$
$$= \frac{-5}{8.5}$$
$$= -0.59$$

The average rate of change of the temperature between 1 pm and 9:30 pm is −0.59° C/hour.

17 **a**



b From O to A, it took the cyclist 2 hours, rest period was 0.5 hours, lunch break was 1 hour and from B back to O took 2.5 hours. As the entire journey took 9 hours, the time it took for the cyclist to ride from A to B is
$9 - (2 + 0.5 + 1 + 2.5) = 3$ hours.

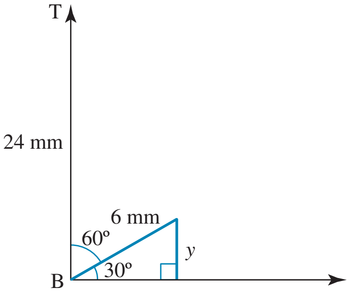
The cyclist travelled from O to A at a constant speed of 10 km/h, so A is 20 km from O. As B is 45 km from O, the distance between A and B is 25 km.

Thus, the constant speed the cyclist rode at between A and B is $\frac{25}{3} = 8\frac{1}{3}$ km/h.

c The constant speed from B back to O is $\frac{45}{2.5} = 18$ km/h.

d The total distance ridden in 9 hours is $2 \times 45 = 90$ km, giving the cyclist an average speed of $\frac{90}{9} = 10$ km/h.

18 **a** The shoot grows at 2 mm/week. In 3 weeks it will have grown to length 6 mm.

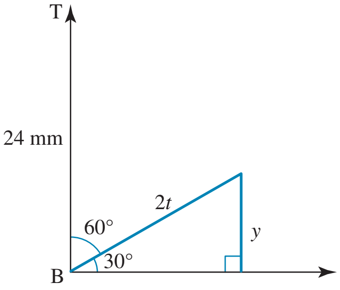


Let y be the vertical height of the tip of the shoot above the base B.

$$\sin(30°) = \frac{y}{6}$$
$$\therefore y = 6 \sin(30°)$$
$$\therefore y = 6 \times \frac{1}{2}$$
$$\therefore y = 3$$

T is 24 mm vertically above B. Therefore, the tip of the shoot is 21 mm vertically below T.

b After t weeks, the shoot growing at 2 mm/week, will be $2t$ mm in length.



$\sin(30°) = \dfrac{y}{2t}$

$\therefore y = 2t\sin(30°)$

$\therefore y = 2t \times \dfrac{1}{2}$

$\therefore y = t$

The tip of the shoot is $(24 - t)$ mm below T.

c When the tip of the shoot is at the same height as T, $(24 - t) = 0$. This will occur after 24 weeks.

19 **a** $d = \dfrac{200t}{t+1}, t \geq 0$

When $t = 0$, $d = 0$ and when $t = 4$, $d = \dfrac{800}{5} = 160$.

Average speed is $\dfrac{\text{distance travelled}}{\text{time taken}}$

Therefore, the average speed is $\dfrac{160}{4} = 40$ m/hour.

b $d = \dfrac{200t}{t+1}$

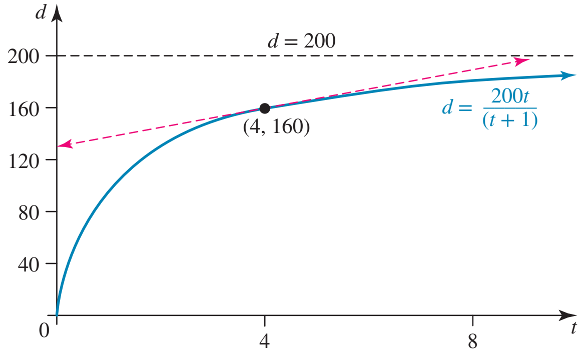
$\therefore d = \dfrac{200(t+1) - 200}{t+1}$

$\therefore d = \dfrac{200(t+1)}{t+1} - \dfrac{200}{t+1}$

$\therefore d = 200 - \dfrac{200}{t+1}$

Asymptote $d = 200$ and $t = -1$ (outside the domain)

Right end point $(0, 0)$ and point $(4, 160)$ is on graph.



The boat travels away from the jetty quickly but its speed slows to almost stationary as it nears a distance of 200 metres from the jetty. It does not travel further beyond this distance.

c Draw the tangent at $(4, 160)$. Another point on this tangent is approximately $(9, 200)$.

Its gradient is

$m = \dfrac{200 - 160}{9 - 4}$

$= \dfrac{40}{5}$

$= 8$

The instantaneous speed of the boat 4 hours after leaving the jetty is approximately 8 m/hour.

d $d = \dfrac{200t}{t+1}$

Let $t = 4.1$

$d = \dfrac{200 \times 4.1}{4.1 + 1}$

$= \dfrac{820}{5.1}$

$\simeq 160.78$

The gradient of the line segment joining $(4.1, 160.78)$ and $(4, 160)$ is

$m = \dfrac{160.78 - 160}{4.1 - 4}$

$= \dfrac{0.78}{0.1}$

$\simeq 7.8$

An estimate of the speed of the boat is 7.8 m/hour.

20 Enter the equation $y = 4 - 3x^2$ in the Graph & Tab editor. Tap Analysis → Sketch → Tangent and enter the x value -2. The tangent is sketched and its equation is given as $y = 12x + 16$. As the gradient of the tangent is 12, the gradient of $y = 4 - 3x^2$ at $x = -2$ is 12.

11.2 Exam questions

1 Average rate of growth $= \dfrac{\text{change in growth (cm)}}{\text{time (days)}}$

$= \dfrac{40 - 5}{14}$

$= 2.5$ cm/day

The correct answer is **C**.

2 $p = 200t(10 - t)$

a p-intercept, $t = 0$

$p = 0 \Rightarrow (0, 0)$

t-intercept, $p = 0$

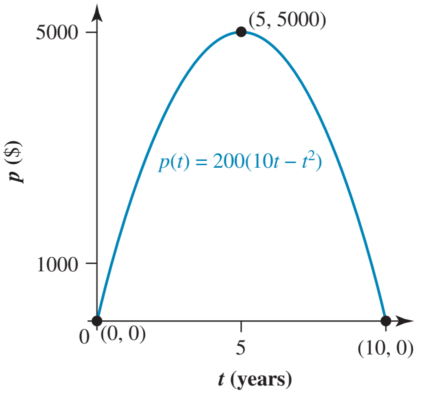
$0 = 200t(10 - t)$

$p = 0, 10 \Rightarrow (0, 0), (10, 0)$ [1 mark]

$\therefore$ T.P, $t = 5$

$\therefore p = 200 \times 5(10 - 5)$

$= 5000 \Rightarrow (5.5000)$ [1 mark]



[1 mark]

b Average rate of change over the first 3 years

$= \dfrac{p(3) - p(0)}{3 - 0}$

$= \dfrac{200 \times 21 - 0}{3}$

$= 1400$ [1 mark]

Average rate of change years 6 to 9

$= \dfrac{p(9) - p(6)}{9 - 6}$

$= \dfrac{200 \times (90 - 81) - 200(60 - 36)}{3}$

$= \dfrac{1800 - 4800}{3}$

$= -1000$ [1 mark]

In the first 3 years, the rate of change of profit was positive (\$1400 per year) but, during years 6 to 9, the rate of change of profit was in decline at −\$1000 per year. [1 mark]

3 Average rate of change is calculated by:

$$\begin{aligned}\frac{f(b)-f(a)}{b-a} &= \frac{f(5)-f(1)}{5-1}\\ &= \frac{3-25-(3-1)}{4}\\ &= -\frac{24}{4}\\ &= -6\end{aligned}$$

The correct answer is **D**.

11.3 Gradients of secants

11.3 Exercise

1 A secant is a line that cuts through a given curve at two points. A tangent is a line that touches a given curve at a single point. A chord is the line segment joining two points on a curve as its end points.

2 **a** The secant passes through the points $(1, 1)$ and $(1.5, 2.25)$.

$$\begin{aligned}m_{\text{secant}} &= \frac{2.25-1}{1.5-1}\\ &= \frac{1.25}{0.5}\\ &= 2.5\end{aligned}$$

b The chord has end points $(1, 1)$ and $(1.5, 2.25)$

$$\begin{aligned}m_{\text{chord}} &= \frac{2.25-1}{1.5-1}\\ &= \frac{1.25}{0.5}\\ &= 2.5\end{aligned}$$

c The gradients are equal. This will always be the case if the end points of the chord are the points the secant passes through. In this case the chord is part of the secant.

3 **a** $y = x^2$

Let $x = 1 + h$.

$$\begin{aligned}y &= (1+h)^2\\ &= 1 + 2h + h^2\end{aligned}$$

b $y = x^2 + 5x$

Let $x = -2 + h$.

$$\begin{aligned}y &= (-2+h)^2 + 5(-2+h)\\ &= 4 - 4h + h^2 - 10 + 5h\\ y &= h^2 + h - 6\end{aligned}$$

c $y = 4 - 3x^2$

Let $x = 3 + h$.

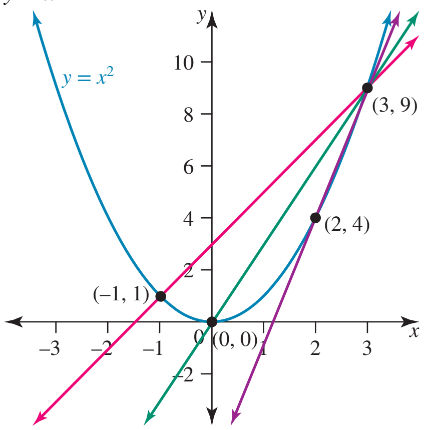
$$\begin{aligned}y &= 4 - 3(3+h)^2\\ &= 4 - 3(9 + 6h + h^2)\\ &= 4 - 27 - 18h - 3h^2\\ y &= -3h^2 - 18h - 23\end{aligned}$$

d $y = -2x^2 + 3x + 11$

Let $x = -4 + h$.

$$\begin{aligned}y &= -2(-4+h)^2 + 3(-4+h) + 11\\ &= -2(16 - 8h + h^2) - 12 + 3h + 11\\ &= -32 + 16h - 2h^2 - 12 + 3h + 11\\ y &= -2h^2 + 19h - 33\end{aligned}$$

4 **a** $y = x^2$



b The secant through $(2, 4)$ and $(3, 9)$ is the best of the three for approximating the tangent to the parabola at $(3, 9)$. To improve on this approximation, use a secant through $(3, 9)$ and a point closer to $(3, 9)$ than the point $(2, 4)$.

5 **a** $y = x^2$

Point P: Let $x = 1$

$y = 1^2 = 1$

$P\,(1, 1)$

Point Q: Let $x = 1 + h$

$y = (1+h)^2$

$Q\left(1+h,\ (1+h)^2\right)$

Gradient of secant through P and Q:

$$\begin{aligned}m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{(1+h)^2 - 1^2}{1 + h - 1}\\ &= \frac{1 + 2h + h^2 - 1}{h}\\ &= \frac{2h + h^2}{h}\\ &= \frac{h(2+h)}{h}\\ &= 2 + h,\quad h \neq 0\end{aligned}$$

b $y = x^2 + 3x$

Point P: Let $x = 2$

$y = 2^2 + 3(2) = 10$

$P\,(2, 10)$.

Point Q: Let $x = 2 + h$

$$\begin{aligned}y &= (2+h)^2 + 3(2+h)\\ &= 4 + 4h + h^2 + 6 + 3h\\ &= h^2 + 7h + 10\end{aligned}$$

$Q\left(2+h,\ h^2 + 7h + 10\right)$

Gradient of secant through P and Q:

$$\begin{aligned}&= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{h^2 + 7h + 10 - 10}{2 + h - 2}\\ &= \frac{h^2 + 7h}{h}\\ &= \frac{h(h+7)}{h}\\ &= h + 7,\quad h \neq 0\end{aligned}$$

c $y = 2x^2 - 6$

Point P: Let $x = 3$

$y = 2(3)^2 - 6 = 12$

P $(3, 12)$

Point Q: Let $x = 3 + h$

$$\begin{aligned} y &= 2(3 + h)^2 - 6 \\ &= 2(9 + 6h + h^2) - 6 \\ &= 2h^2 + 12h + 12 \end{aligned}$$

Q $\left(3 + h,\ 2h^2 + 12h + 12\right)$

Gradient of secant through P and Q:

$$\begin{aligned} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2h^2 + 12h + 12 - 12}{3 + h - 3} \\ &= \frac{2h^2 + 12h}{h} \\ &= \frac{2h(h + 6)}{h} \\ &= 2(h + 6),\ \ h \neq 0 \end{aligned}$$

d $y = x^2 - 10x + 20$

Point P: Let $x = 4$

$y = (4)^2 - 10(4) + 20 = -4$

P $(4, -4)$

Point Q: Let $x = 4 + h$

$$\begin{aligned} y &= (4 + h)^2 - 10(4 + h) + 20 \\ &= (16 + 8h + h^2) - 40 - 10h + 20 \\ &= h^2 - 2h - 4 \end{aligned}$$

Q $\left(4 + h,\ h^2 - 2h - 4\right)$

Gradient of secant through P and Q:

$$\begin{aligned} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{h^2 - 2h - 4 - (-4)}{4 + h - 4} \\ &= \frac{h^2 - 2h}{h} \\ &= \frac{h(h - 2)}{h} \\ &= h - 2,\ \ h \neq 0 \end{aligned}$$

6 $y = 2x^2 - x$

a When $x = -1$, $y = 3 \Rightarrow (-1, 3)$

When $x = -1 + h$,

$$\begin{aligned} y &= 2(-1 + h)^2 - (-1 + h) \\ &= 2 - 4h + 2h^2 + 1 - h \\ &= 2h^2 - 5h + 3 \end{aligned}$$

$\Rightarrow (-1 + h,\ 2h^2 - 5h + 3)$

Gradient of secant

$$\begin{aligned} &= \frac{\left(2h^2 - 5h + 3\right) - (3)}{(-1 + h) - (-1)} \\ &= \frac{2h^2 - 5h}{h} \\ &= \frac{h(2h - 5)}{h} \\ &= 2h - 5,\ \ h \neq 0 \end{aligned}$$

b If $h = 0.01$, the gradient of the secant is $0.02 - 5 = -4.98$.
Therefore, an estimate for the gradient of the tangent is -4.98.

c As $h \to 0,\ (2h - 5) \to -5$

$\therefore \lim\limits_{h \to 0}(2h - 5) = -5$

Therefore, the gradient of the tangent at $x = -1$ is -5.

7 a $y = 4 - x^2$

Let $x = 2 + h$

$$\begin{aligned} y &= 4 - (2 + h)^2 \\ &= 4 - \left(4 + 4h + h^2\right) \\ &= -4h - h^2 \end{aligned}$$

A is the point $\left(2 + h, -4h - h^2\right)$.

b C $(2, 0)$

$$\begin{aligned} m_{\text{AC}} &= \frac{-4h - h^2 - 0}{2 + h - 2} \\ &= \frac{-h(4 + h)}{h} \\ &= -(4 + h),\ \ h \neq 0 \\ &= -4 - h \end{aligned}$$

c $m_{\text{AC}} = -5$

$\therefore\ -4 - h = -5$

$\therefore h = 1$

For $h = 1$, the coordinates of point A $\left(2 + h, -4h - h^2\right)$ become $(3, -5)$.

d If A has coordinates $(2.1, -0.41)$, then

$2 + h = 2.1$

$\therefore h = 0.1$

Checking y:

$$\begin{aligned} -4h - h^2 &= -0.4 - 0.01 \\ &= -0.41 \end{aligned}$$

as required.

With $h = 0.1$, then $m_{AC} = -(4 + h)$ becomes $m_{AC} = -4.1$.

e Let $x = 2 - h$

$$\begin{aligned} y &= 4 - (2 - h)^2 \\ &= 4 - \left(4 - 4h + h^2\right) \\ &= 4h - h^2 \end{aligned}$$

B is the point $\left(2 - h, 4h - h^2\right)$.

$$\begin{aligned} m_{\text{BC}} &= \frac{4h - h^2 - 0}{2 - h - 2} \\ &= \frac{h(4 - h)}{-h} \\ &= -(4 - h),\ \ h \neq 0 \\ &= h - 4 \end{aligned}$$

Substitute $h = 0.1$

$\therefore m_{\text{BC}} = -3.9$

f A $\left(2 + h, -4h - h^2\right)$ and B $\left(2 - h, 4h - h^2\right)$

$$\begin{aligned} m_{\text{AB}} &= \frac{\left(4h - h^2\right) - \left(-4h - h^2\right)}{(2 - h) - (2 + h)} \\ &= \frac{8h}{-2h} \\ &= -4,\ h \neq 0 \end{aligned}$$

8 a $y = x^3 + x$

Let $x = 3 + h$.

$$\begin{aligned} y &= (3 + h)^3 + 3 + h \\ &= 3^3 + 3 \times 3^2 \times h + 3 \times 3 \times h^2 + h^3 + 3 + h \\ &= 27 + 27h + 9h^2 + h^3 + 3 + h \\ &= h^3 + 9h^2 + 28h + 30 \end{aligned}$$

The y-coordinate of D is $h^3 + 9h^2 + 28h + 30$.

b Let $x = 3$, then $y = 27 + 3 = 30$.

D $\left(3 + h, h^3 + 9h^2 + 28h + 30\right)$ and $(3, 30)$

$$\begin{aligned} m &= \frac{\left(h^3 + 9h^2 + 28h + 30\right) - 30}{(3 + h) - 3} \\ &= \frac{h^3 + 9h^2 + 28h}{h} \\ &= \frac{h\left(h^2 + 9h + 28\right)}{h} \\ &= h^2 + 9h + 28, h \neq 0 \end{aligned}$$

c When $h = 0.001$,

$$\begin{aligned} h^2 + 9h + 28 &= (0.001)^2 + 9\,(0.001) + 28 \\ &= 28.009\ 001 \end{aligned}$$

The gradient is 28.009 001.

d The answer to part **c** suggests that the gradient of the tangent to the curve $y = x^3 + x$ at the point where $x = 3$ is 28.

9 a $y = \dfrac{1}{x}$

When $x = 1$, $y = 1$, so M is the point $(1, 1)$.

When $x = 1 + h$, $y = \dfrac{1}{1 + h}$, so N is the point $\left(1 + h, \dfrac{1}{1 + h}\right)$.

b The gradient of the secant through M and N is

$$\begin{aligned} m &= \frac{\left(\frac{1}{1+h}\right) - 1}{(1 + h) - 1} \\ &= \left(\frac{1}{1 + h} - 1\right) \div h \\ &= \frac{1 - (1 + h)}{1 + h} \times \frac{1}{h} \\ &= \frac{-h}{1 + h} \times \frac{1}{h} \\ &= \frac{-1}{1 + h},\ \ h \neq 0 \end{aligned}$$

c As $h \to 0$, $\dfrac{-1}{1 + h} \to \dfrac{-1}{1 + 0} = -1$

The gradient approaches the value -1.

d Hence, the gradient of the tangent to $y = \dfrac{1}{x}$ at M is -1.

10 a $y = 1 + 3x - x^2$

When $x = 1, y = 3$

When $x = 1 - h$,

$$\begin{aligned} y &= 1 + 3(1 - h) - (1 - h)^2 \\ &= 1 + 3 - 3h - \left(1 - 2h + h^2\right) \\ &= 3 - h - h^2 \end{aligned}$$

The gradient of the secant through the points $(1, 3)$ and $\left(1 - h,\ 3 - h - h^2\right)$ is

$$\begin{aligned} m_{\text{secant}} &= \frac{\left(3 - h - h^2\right) - 3}{(1 - h) - 1} \\ &= \frac{-h - h^2}{-h} \\ &= \frac{-h(1 + h)}{-h} \\ &= 1 + h,\ \ h \neq 0 \end{aligned}$$

b The gradient of the tangent at $x = 1$ is $\lim\limits_{h \to 0} (1 + h)$.

c As $h \to 0, 1 + h \to 1 + 0 = 1$

Therefore, the gradient of the tangent is 1.

11 $y = \dfrac{1}{3}x^3 - x^2 + x + 5$

a When $x = 1$, $y = 5\dfrac{1}{3} \Rightarrow \left(1,\ \dfrac{16}{3}\right)$

When $x = 1 - h$,

$$\begin{aligned} y &= \frac{1}{3}(1 - h)^3 - (1 - h)^2 + (1 - h) + 5 \\ &= \frac{1}{3}\left(1 - 3h + 3h^2 - h^3\right) - \left(1 - 2h + h^2\right) + 1 - h + 5 \\ &= \frac{1}{3} - h + h^2 - \frac{1}{3}h^3 - 1 + 2h - h^2 + 1 - h + 5 \\ &= -\frac{1}{3}h^3 + 5\frac{1}{3} \end{aligned}$$

$\Rightarrow \left(1 - h,\ \dfrac{-h^3 + 16}{3}\right)$

Gradient of secant

$$\begin{aligned} &= \frac{\left(\frac{\left(-h^3+16\right)}{3}\right) - \frac{16}{3}}{(1 - h) - 1} \\ &= \frac{\frac{-h^3}{3}}{-h} \\ &= \frac{h^2}{3},\ \ h \neq 0 \end{aligned}$$

b As $h \to 0$, $\dfrac{h^2}{3} \to 0$

$\therefore \lim\limits_{h \to 0} \left(\dfrac{h^2}{3}\right) = 0$

Therefore, the gradient of the tangent at $x = 1$ is 0.

12 a $y = \dfrac{x^4}{4}$

When $x = 3.9$,

$y = \dfrac{(3.9)^4}{4}$

$\therefore y = 57.836\,025$

When $x = 4$,

$$\begin{aligned} y &= \frac{4^4}{4} \\ &= 64 \end{aligned}$$

The chord joins the points $(3.9, 57.836\,025)$ and $(4, 64)$.

$$\begin{aligned} m_{\text{chord}} &= \frac{64 - 57.836\,025}{4 - 3.9} \\ &= \frac{6.1639\,75}{0.1} \\ &= 61.639\,75 \end{aligned}$$

b A whole number estimate for the gradient of the tangent at $x = 4$ is 62.

c To improve on this estimate, use an end point of the chord that is closer to $(4, 64)$ than $(3.9, 57.836\,025)$.

13 a $y = (x - 2)(x + 1)(x - 3)$

When $x = 0, y = (-2)(1)(-3) = 6$

When $x = h, y = (h - 2)(h + 1)(h - 3)$

$$\begin{aligned} \therefore y &= (h - 2)(h^2 - 2h - 3) \\ &= h^3 - 2h^2 - 3h - 2h^2 + 4h + 6 \\ &= h^3 - 4h^2 + h + 6 \\ &= h^3 - 4h^2 + h + 6 \end{aligned}$$

The gradient of the secant through the points $(0, 6)$ and $\left(h, h^3 - 4h^2 + h + 6\right)$ is

$$\begin{aligned} m_{\text{secant}} &= \frac{\left(h^3 - 4h^2 + h + 6\right) - 6}{h - 0} \\ &= \frac{h^3 - 4h^2 + h}{h} \\ &= \frac{h\left(h^2 - 4h + 1\right)}{h} \\ &= h^2 - 4h + 1, \quad h \neq 0 \end{aligned}$$

as required.

b Hence, the gradient of the tangent to the curve at $x = 0$ is $\lim\limits_{h \to 0}\left(h^2 - 4h + 1\right)$.

c As $h \to 0$, $h^2 - 4h + 1 \to 0^2 - 4 \times 0 + 1 = 1$

The gradient of the tangent is 1.

14 **a** $m_{\text{secant}} = 3 + h$. This expression cannot be simplified further.

$$\lim_{h \to 0}(3 + h) = 3$$

b
$$\begin{aligned} m_{\text{secant}} &= \frac{8h - 2h^2}{h} \\ \therefore m_{\text{secant}} &= \frac{h(8 - 2h)}{h} \\ &= 8 - 2h, \quad h \neq 0 \end{aligned}$$

$$\begin{aligned} &\lim_{h \to 0}\left(\frac{8h - 2h^2}{h}\right) \\ &= \lim_{h \to 0}(8 - 2h) \\ &= 8 \end{aligned}$$

c
$$\begin{aligned} m_{\text{secant}} &= \frac{(2 + h)^3 - 8}{h} \\ \therefore m_{\text{secant}} &= \frac{\left(8 + 12h + 6h^2 + h^3\right) - 8}{h} \\ &= \frac{12h + 6h^2 + h^3}{h} \\ &= \frac{h\left(12 + 6h + h^2\right)}{h} \\ &= 12 + 6h + h^2, \quad h \neq 0 \end{aligned}$$

$$\begin{aligned} &\lim_{h \to 0}\left(\frac{(2 + h)^3 - 8}{h}\right) \\ &= \lim_{h \to 0}\left(12 + 6h + h^2\right) \\ &= 12 \end{aligned}$$

d
$$\begin{aligned} m_{\text{secant}} &= \frac{(1 + h)^4 - 1}{h} \\ \therefore m_{\text{secant}} &= \frac{\left(1 + 4h + 6h^2 + 4h^3 + h^4\right) - 1}{h} \\ &= \frac{4h + 6h^2 + 4h^3 + h^4}{h} \\ &= \frac{h\left(4 + 6h + 4h^2 + h^3\right)}{h} \\ &= 4 + 6h + 4h^2 + h^3, \quad h \neq 0 \end{aligned}$$

$$\begin{aligned} &\lim_{h \to 0}\left(\frac{(1 + h)^4 - 1}{h}\right) \\ &= \lim_{h \to 0}\left(4 + 6h + 4h^2 + h^3\right) \\ &= 4 \end{aligned}$$

15 **a**
$$\begin{aligned} m_{\text{secant}} &= \frac{\frac{1}{(4+h)} - \frac{1}{4}}{h} \\ \therefore m_{\text{secant}} &= \left(\frac{1}{(4 + h)} - \frac{1}{4}\right) \div h \\ &= \frac{4 - (4 + h)}{(4 + h)4} \times \frac{1}{h} \\ &= \frac{-h}{4(4 + h)} \times \frac{1}{h} \\ &= \frac{-1}{4(4 + h)}, \quad h \neq 0 \end{aligned}$$

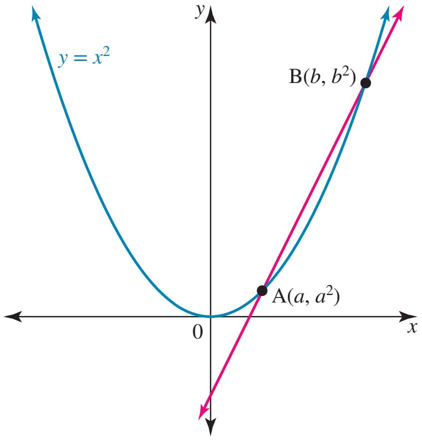
$$\begin{aligned} &\lim_{h \to 0}\left(\frac{-1}{4(4 + h)}\right) \\ &= \frac{-1}{4(4)} \\ &= \frac{-1}{16} \end{aligned}$$

b
$$\begin{aligned} m_{\text{secant}} &= \frac{h}{\sqrt{1 + h} - 1} \\ \therefore m_{\text{secant}} &= \frac{h}{\sqrt{1 + h} - 1} \times \frac{\sqrt{1 + h} + 1}{\sqrt{1 + h} + 1} \\ &= \frac{h\left(\sqrt{1 + h} + 1\right)}{\left(\sqrt{1 + h}\right)^2 - 1^2} \\ &= \frac{h\left(\sqrt{1 + h} + 1\right)}{1 + h - 1} \\ &= \frac{h\left(\sqrt{1 + h} + 1\right)}{h} \\ &= \sqrt{1 + h} + 1, \quad h \neq 0 \end{aligned}$$

$$\begin{aligned} &\lim_{h \to 0}\left(\frac{h}{\sqrt{1 + h} - 1}\right) \\ &= \lim_{h \to 0}\left(\sqrt{1 + h} + 1\right) \\ &= \sqrt{1} + 1 \\ &= 2 \end{aligned}$$

16 **a** $f(x) = x^2$

$f(a) = a^2$ and $f(b) = b^2$, so A is the point $\left(a, a^2\right)$ and B is the point $\left(b, b^2\right)$.



b The gradient of the secant through points A and B is

$$\begin{aligned} m_{\text{AB}} &= \frac{b^2-a^2}{b-a} \\ &= \frac{(b-a)(b+a)}{b-a} \\ &= a+b, a \neq b \end{aligned}$$

c i As $b \to a$, the secant AB → tangent at A.
Therefore, the gradient of the tangent at A is
$\lim\limits_{b\to a}(b+a) = 2a$.

ii As $a \to b$, the secant AB → tangent at B.
Therefore, the gradient of the tangent at B is
$\lim\limits_{a\to b}(b+a) = 2b$.

17 a $\lim\limits_{h\to 0}\left(\dfrac{(h+2)^2(h+1)-4}{h}\right)$

The limit template is in the 2D CALC section of the *m*th keyboard and *h* is in the abc section of the keyboard.

$$\lim_{h\to 0}\left(\frac{(h+2)^2(h+1)-4}{h}\right) = 8$$

b $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}+3\right) = 3$. The symbol ∞ is in the same row as π in the *m*th keyboard.

c $\lim\limits_{x\to\infty}\left(\dfrac{2x+1}{3x-2}\right) = \dfrac{2}{3}$

18 a $f(x) = \sqrt{x}$

$f(9) = \sqrt{9} = 3$ and $f(9+h) = \sqrt{9+h}$.

The gradient of the secant through the points $(9, 3)$ and $\left(9+h, \sqrt{9+h}\right)$ is

$$m_{\text{secant}} = \frac{\sqrt{9+h}-3}{9+h-9}$$

$$m_{\text{secant}} = \frac{\sqrt{9+h}-3}{h}$$

b The gradient of the tangent at the point $(9, 3)$ is

$$\lim_{h\to 0}\left(\frac{\sqrt{9+h}-3}{h}\right).$$

c Using the 2D CALC mth keyboard,

$$\lim_{h\to 0}\left(\frac{\sqrt{9+h}-3}{h}\right) = \frac{1}{6}.$$

The gradient of the tangent is $\dfrac{1}{6}$.

11.3 Exam questions

1 The correct answer is **B**.

2 a $y = x^2 - 4x$

Point $x = 6$

$$\begin{aligned} y &= 36-24 \\ &= 12 \\ &\Rightarrow (6, 12) \end{aligned}$$

Point $x = 6+h$

$$\begin{aligned} y &= (6+h)^2 - 4(6+h) \\ y &= 36 + 12h + h^2 - 24 - 4h \\ y &= h^2 + 8h + 12 \\ &\Rightarrow \left(6+h, h^2+8h+12\right) \end{aligned}$$

[1 mark]

$$\begin{aligned} \text{Gradient}: \frac{y_2-y_1}{x_2-x_1} &= \frac{h^2+8h+12-12}{6+h-6} \\ &= \frac{h^2+8h}{h} \\ &= \frac{\cancel{h}(h+8)}{\cancel{h}} \\ &= h+8,\ h \neq 0 \end{aligned}$$

[1 mark]

b When $h = 0.01$

$$\begin{aligned} \text{Gradient} &= h+8 \\ &= 8.01 \end{aligned}$$

[1 mark]

c The gradient of the tangent to the curve at the point where $x = 6$ is 8. [1 mark]

3 Point $x = 0$

$y = (x-3)(x+1)(x-4)$

$y = (-3)(1)(-4)$

$y = 12$

$\therefore (0, 12)$

Point $x = h$

$y = (h-3)(h+1)(h-4)$

$y = \left(h^2-2h-3\right)(h-4)$

$y = h^3 - 4h^2 - 2h^2 + 8h - 3h + 12$

$y = h^3 - 6h^2 + 5h + 12$

$\therefore$ point $\left(h, h^3-6h^2+5h+12\right)$ and $(0, 12)$

$$\begin{aligned} \text{Gradient} &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{h^3-6h^2+5h+12-12}{h-0} \\ &= \frac{h^3-6h^2+5h}{h} \\ &= \frac{\cancel{h}\left(h^2-6h+5\right)}{\cancel{h}},\ h \neq 0 \\ &= h^2-6h+5 \end{aligned}$$

The correct answer is **A**.

11.4 The derivative function

11.4 Exercise

1 a A symbol for the derivative of $y = f(x)$ could be $f'(x)$ or $\dfrac{dy}{dx}$.
The limit definitions, respectively, are

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ and } \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}.$$

b $f'(a)$ measures the rate of change of the function at the point where $x = a$. It measures the gradient of the tangent drawn to the curve at the point where $x = a$.

2 a i $f(x) = x^2 + 2x$

$$\begin{aligned} f(x+h) &= (x+h)^2 + 2(x+h) \\ &= x^2 + 2xh + h^2 + 2x + 2h \end{aligned}$$

ii $f(x) = 3 - 2x^2$

$$\begin{aligned} f(x+h) &= 3 - 2(x+h)^2 \\ &= 3 - 2(x^2+2xh+h^2) \\ &= 3 - 2x^2 - 4xh - 2h^2 \end{aligned}$$

b i $f(x) = 3x^2 - 7x$

$f(x+h) = 3(x+h)^2 - 7(x+h)$

$\therefore f(x+h) - f(x)$
$= 3(x+h)^2 - 7(x+h) - (3x^2 - 7x)$
$= 3(x^2 + 2xh + h^2) - 7x - 7h - 3x^2 + 7x$
$= 3x^2 + 6xh + 3h^2 - 7x - 7h - 3x^2 + 7x$
$= 6xh + 3h^2 - 7h$

ii $f(x) = 5x^2 - 3x + 2$
$f(x+h) = 5(x+h)^2 - 3(x+h) + 2$
$\therefore f(x+h) - f(x)$
$= 5(x+h)^2 - 3(x+h) + 2 - (5x^2 - 3x + 2)$
$= 5(x^2 + 2xh + h^2) - 3x - 3h + 2 - 5x^2 + 3x - 2$
$= 5x^2 + 10xh + 5h^2 - 3x - 3h + 2 - 5x^2 + 3x - 2$
$= 10xh + 5h^2 - 3h$

3 a $f(x) = x^2$
$f(x+h) = (x+h)^2$
The difference quotient $\dfrac{f(x+h) - f(x)}{h}$ becomes

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = \frac{h(2x+h)}{h} = 2x + h,\ h \neq 0$$

b $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$
$= \lim\limits_{h\to 0}(2x + h)$
$= 2x$

c At (3, 9) the value of x is 3.
$\therefore f'(3) = 2 \times 3$
$\therefore f'(3) = 6$
The gradient of the tangent at the point (3, 9) is 6.

d A near neighbouring point to (3, 9) is the point for which $x = 3 + h$ and $y = f(3 + h)$.
The gradient of the secant through these two points is

$$m_{\text{secant}} = \frac{f(3+h) - 9}{3 + h - 3} = \frac{f(3+h) - f(3)}{h}$$

since $9 = f(3)$ for this function $f(x) = x^2$.
The gradient of the tangent at (3, 9) is the limiting value that the secant's gradient approaches as the two points become closer together.
The gradient of tangent at (3, 9) is $\lim\limits_{h\to 0}\dfrac{f(3+h) - f(3)}{h}$.
Evaluating this limit,

$$\lim_{h\to 0}\frac{f(3+h) - f(3)}{h} = \lim_{h\to 0}\frac{(3+h)^2 - 9}{h} = \lim_{h\to 0}\frac{9 + 6h + h^2 - 9}{h} = \lim_{h\to 0}\frac{6h + h^2}{h} = \lim_{h\to 0}\frac{h(6+h)}{h}$$

$= \lim\limits_{h\to 0}(6 + h)$
$= 6$
$= f'(3)$

4 $f(x) = x^3$
$f(x+h) = (x+h)^3$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0}\frac{h\left(3x^2 + 3xh + h^2\right)}{h} = \lim_{h\to 0}\left(3x^2 + 3xh + h^2\right) = 3x^2$$

5 $f(x) = 2x^2 + x + 1$

a $f(x+h) = 2(x+h)^2 + (x+h) + 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^2 + (x+h) + 1 - \left(2x^2 + x + 1\right)}{h} = \lim_{h\to 0}\frac{2x^2 + 4xh + 2h^2 + x + h + 1 - 2x^2 - x - 1}{h} = \lim_{h\to 0}\frac{4xh + 2h^2 + h}{h} = \lim_{h\to 0}\frac{h(4x + 2h + 1)}{h} = \lim_{h\to 0}(4x + 2h + 1) = 4x + 1$$

b The gradient at $x = -1$ is the value of $f'(-1)$.
$f'(-1) = 4(-1) + 1$
$= -3$

c The instantaneous rate of change is the gradient of the tangent, i.e. the gradient of the curve.
Rate $= f'(0)$
$= 1$

6 a $\dfrac{f(x+h) - f(x)}{h}$ is the difference quotient representing the gradient of the secant through the points $(x, f(x))$ and $(x + h, f(x + h))$ on the curve $y = f(x)$.

b $\lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$ is the gradient of the tangent to the curve $y = f(x)$ at the point $(x, f(x))$.

c $\lim\limits_{h\to 0}\dfrac{f(3+h) - f(3)}{h}$ is the gradient of the tangent to the curve $y = f(x)$ at the point $(3, f(3))$.

d $\dfrac{f(3) - f(3-h)}{h}$ is the gradient of the secant through the points $(3, f(3))$ and $(3 - h, f(3 - h))$ on the curve $y = f(x)$.

7 a $f(x) = 3x - 2x^2$
$f(x+h) = 3(x+h) - 2(x+h)^2$

By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\left[3(x+h)-2(x+h)^2\right]-\left[3x-2x^2\right]}{h} \\ &= \lim_{h\to 0}\frac{3x+3h-2(x^2+2xh+h^2)-3x+2x^2}{h} \\ &= \lim_{h\to 0}\frac{3h-4xh-2h^2}{h} \\ &= \lim_{h\to 0}\frac{h(3-4x-2h)}{h} \\ &= \lim_{h\to 0}(3-4x-2h) \\ &= 3-4x \end{aligned}$$

b The gradient of the tangent at (0, 0) is $f'(0) = 3$.

c $y = 3x - 2x^2$

x-intercepts: let $y = 0$

$\therefore 3x - 2x^2 = 0$

$\therefore x(3 - 2x) = 0$

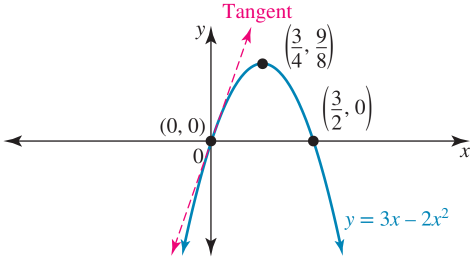
$\therefore x = 0,\ x = \frac{3}{2}$

Turning point: $x = \frac{0+\frac{3}{2}}{2} = \frac{3}{4}$

$$\begin{aligned} y &= 3\times\frac{3}{4}-2\times\left(\frac{3}{4}\right)^2 \\ &= \frac{9}{4}-\frac{9}{8} \\ &= \frac{9}{8} \end{aligned}$$

The maximum turning point is $\left(\frac{3}{4}, \frac{9}{8}\right)$.

The tangent at (0, 0) can be drawn by eye. Alternatively, it has gradient 3 and passes through the origin, so its equation is $y = 3x$.



8 a P (1, 5) and Q $\left(1+h, 3(1+h)^2+2\right)$ are on $y = 3x^2 + 2$.

The average rate of change of $y = 3x^2 + 2$ between P and Q

$$\begin{aligned} &= \frac{3(1+h)^2+2-5}{1+h-1} \\ &= \frac{3(1+2h+(h)^2)-3}{h} \\ &= \frac{3+6h+3(h)^2-3}{h} \\ &= \frac{6h+3(h)^2}{h} \\ &= \frac{h(6+3h)}{h} \\ &= 6+3h,\ h\neq 0 \end{aligned}$$

b 6

c i When $x = 2, y = 14$, so S is the point (2, 14).

When $x = 2 + h$, $y = 3(2 + h)^2 + 2$, so T is the point $\left(2+h, 3(2+h)^2+2\right)$.

ii The gradient of the secant through points S and T is

$$\begin{aligned} m_{\text{secant}} &= \frac{3(2+h)^2+2-14}{2+h-2} \\ &= \frac{3(4+4h+(h)^2)-12}{h} \\ &= \frac{12+12h+3(h)^2-12}{h} \\ &= \frac{12h+3(h)^2}{h} \\ &= \frac{h(12+3h)}{h} \\ &= 12+3h,\ h\neq 0 \end{aligned}$$

iii Gradient of tangent at S $= \lim_{h\to 0}(12+3h) = 12$.

9 Let $f(x) = 5x + 3x^4$

then $f(x + h) = 5(x + h) + 3(x + h)^4$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{5(x+h)+3(x+h)^4-(5x+3x^4)}{h} \\ &= \lim_{h\to 0}\frac{5x+5h+3(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-(5x+3x^4)}{h} \\ &= \lim_{h\to 0}\frac{5x+5h+3x^4+12x^3h+18x^2h^2+12xh^3+3h^4-5x-3x^4}{h} \\ &= \lim_{h\to 0}\frac{5h+12x^3h+18x^2h^2+12xh^3+3h^4}{h} \\ &= \lim_{h\to 0}\frac{h(5+12x^3+18x^2h+12xh^2+3h^3)}{h} \\ &= \lim_{h\to 0}(5+12x^3+18x^2h+12xh^2+3h^3) \\ &= 5+12x^3 \end{aligned}$$

Therefore, the derivative of $5x + 3x^4$ with respect to x is $5 + 12x^3$.

10 a $f(x) = 8x^2 + 2$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\left[8(x+h)^2+2\right]-\left[8x^2+2\right]}{h} \\ &= \lim_{h\to 0}\frac{8x^2+16xh+8h^2+2-8x^2-2}{h} \\ &= \lim_{h\to 0}\frac{16xh+8h^2}{h} \\ &= \lim_{h\to 0}\frac{h(16x+8h)}{h} \\ &= \lim_{h\to 0}(16x+8h) \\ &= 16x \end{aligned}$$

b $f(x) = \frac{1}{2}x^2 - 4x - 1$

$$\begin{aligned}
f'(x) &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to0}\frac{\left[\frac{1}{2}(x+h)^2 - 4(x+h) - 1\right] - \left[\frac{1}{2}x^2 - 4x - 1\right]}{h}\\
&= \lim_{h\to0}\frac{\frac{1}{2}x^2 + xh + \frac{1}{2}h^2 - 4x - 4h - 1 - \frac{1}{2}x^2 + 4x + 1}{h}\\
&= \lim_{h\to0}\frac{xh + \frac{1}{2}h^2 - 4h}{h}\\
&= \lim_{h\to0}\frac{h(x + \frac{1}{2}h - 4)}{h}\\
&= \lim_{h\to0}\,(x + \frac{1}{2}h - 4)\\
&= x - 4
\end{aligned}$$

c $f(x) = 6 - 2x$

$$\begin{aligned}
f'(x) &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to0}\frac{[6 - 2(x+h)] - [6 - 2x]}{h}\\
&= \lim_{h\to0}\frac{6 - 2x - 2h - 6 + 2x}{h}\\
&= \lim_{h\to0}\frac{-2h}{h}\\
&= \lim_{h\to0}(-2)\\
&= -2
\end{aligned}$$

d $f(x) = x^3 - 6x^2 + 2x$

$$\begin{aligned}
f'(x) &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to0}\frac{\left[(x+h)^3 - 6(x+h)^2 + 2(x+h)\right] - \left[x^3 - 6x^2 + 2x\right]}{h}\\
&= \lim_{h\to0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - 6x^2 - 12xh - 6h^2 + 2x + 2h - x^3 + 6x^2 - 2x}{h}\\
&= \lim_{h\to0}\frac{3x^2h + 3xh^2 + h^3 - 12xh - 6h^2 + 2h}{h}\\
&= \lim_{h\to0}\frac{h\left(3x^2 + 3xh + h^2 - 12x - 6h + 2\right)}{h}\\
&= \lim_{h\to0}\left(3x^2 + 3xh + h^2 - 12x - 6h + 2\right)\\
&= 3x^2 - 12x + 2
\end{aligned}$$

11 a $f(x) = (x+5)^2$

$$\begin{aligned}
f'(x) &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to0}\frac{(x+h+5)^2 - (x+5)^2}{h}\\
&= \lim_{h\to0}\frac{[(x+h+5) - (x+5)]\,[(x+h+5) + (x+5)]}{h}\\
&= \lim_{h\to0}\frac{[h][2x + h + 10]}{h}\\
&= \lim_{h\to0}(2x + h + 10)\\
&= 2x + 10
\end{aligned}$$

b $f'(-5) = 2 \times -5 + 10 = 0$

The gradient of the tangent at the point where $x = -5$ is zero. The tangent at this point is horizontal.

c At the y-intercept, $x = 0$.

$f'(0) = 10$

The gradient of the tangent at the y-intercept is 10.

d The instantaneous rate of change is measured by the gradient of the tangent.
At $(-2, 9), f'(-2) = -4 + 10 = 6$.
The instantaneous rate of change of the function at this point is 6.

12 a $f(x) = 5x^2 - 2x - 6$
Replacing every x with $(x + h)$,
$f(x + h) = 5(x + h)^2 - 2(x + h) - 6$
Replacing every x with $(x - h)$,
$f(x - h) = 5(x - h)^2 - 2(x - h) - 6$

$$\begin{aligned}
f'(x) &\approx \frac{f(x+h) - f(x-h)}{2h} \\
&\approx \frac{\left[5(x+h)^2 - 2(x+h) - 6\right] - \left[5(x-h)^2 - 2(x-h) - 6\right]}{2h} \\
&\approx \frac{\left[5x^2 + 10xh + 5h^2 - 2x - 2h - 6\right] - \left[5x^2 - 10xh + 5h^2 - 2x + 2h - 6\right]}{2h} \\
&\approx \frac{5x^2 + 10xh + 5h^2 - 2x - 2h - 6 - 5x^2 + 10xh - 5h^2 + 2x - 2h + 6}{2h} \\
&\approx \frac{20xh - 4h}{2h} \\
&\approx \frac{4h(5x - 1)}{2h} \\
&\approx 2(5x - 1)
\end{aligned}$$

$f'(x) \approx 10x - 2$

b For the point $(2,\ 10), x = 2$.
The gradient at this point is the value of $f'(2)$.
$f'(2) = 10(2) - 2$
$f'(2) = 18$
Therefore, the gradient at the point $(2,\ 10)$ is 18.

c The instantaneous rate of change at $x = -3$ is $f'(-3)$.
$f'(-3) = 10(-3) - 2$
$f'(-3) = -32$
Therefore, the instantaneous rate of change at $x = -3$ is -32.

13 a $f(x) = 8 - 3x^2$
Replacing every x with $(x + h)$,
$f(x + h) = 8 - 3(x + h)^2$
Replacing every x with $(x - h)$,
$f(x - h) = 8 - 3(x - h)^2$

$$\begin{aligned}
f'(x) &\approx \frac{f(x+h) - f(x-h)}{2h} \\
&\approx \frac{\left[8 - 3(x+h)^2\right] - \left[8 - 3(x-h)^2\right]}{2h} \\
&\approx \frac{\left[8 - 3\left(x^2 + 2xh + h^2\right)\right] - \left[8 - 3\left(x^2 - 2xh + h^2\right)\right]}{2h} \\
&\approx \frac{8 - 3x^2 - 6xh - 3h^2 - 8 + 3x^2 - 6xh + 3h^2}{2h} \\
&\approx \frac{-12xh}{2h} \\
&\approx -6x
\end{aligned}$$

$f'(x) \approx -6x$

b $f(x) = 2x^3 + 5x$
Replacing every x with $(x + h)$,
$f(x + h) = 2(x + h)^3 + 5(x + h)$
Replacing every x with $(x - h)$,
$f(x - h) = 2(x - h)^3 + 5(x - h)$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$\approx \frac{\left[2(x+h)^3 + 5\,(x+h)\right] - \left[2(x-h)^3 + 5\,(x-h)\right]}{2h}$$

$$\approx \frac{\left[2(x^3 + 3x^2h + 3xh^2 + h^3) + 5x + 5h\right] - \left[2(x^3 - 3x^2h + 3xh^2 - h^3) + 5x - 5h\right]}{2h}$$

$$\approx \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 + 5x + 5h - 2x^3 + 6x^2h - 6xh^2 + 2h^3 - 5x + 5h}{2h}$$

$$\approx \frac{12x^2h + 4h^3 + 10h}{2h}$$

$$\approx \frac{2h(6x^2 + 2h^2 + 5)}{2h}$$

$$\approx \left(6x^2 + 2h^2 + 5\right)$$

$f'(x) \approx 6x^2 + 5 + 2h^2$

c $f(x) = x^3 + 7x^2 - 4x$

Replacing every x with $(x + h)$,

$f(x+h) = (x+h)^3 + 7\,(x+h)^2 - 4\,(x+h)$

Replacing every x with $(x - h)$,

$f(x-h) = (x-h)^3 + 7\,(x-h)^2 - 4\,(x-h)$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$\approx \frac{\left[(x+h)^3 + 7(x+h)^2 - 4(x+h)\right] - \left[(x-h)^3 + 7(x-h)^2 - 4(x-h)\right]}{2h}$$

$$\approx \frac{\left[x^3 + 3x^2h + 3xh^2 + h^3 + 7x^2 + 14xh + 7h^2 - 4x - 4h\right] - \left[x^3 - 3x^2h + 3xh^2 - h^3 + 7x^2 - 14xh + 7h^2 - 4x + 4h\right]}{2h}$$

$$\approx \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 7x^2 + 14xh + 7h^2 - 4x - 4h - x^3 + 3x^2h - 3xh^2 + h^3 - 7x^2 + 14xh - 7h^2 + 4x - 4h}{2h}$$

$$\approx \frac{6x^2h + 2h^3 + 28xh - 8h}{2h}$$

$$\approx \frac{2h(3x^2 + h^2 + 14x - 4)}{2h}$$

$$\approx 3x^2 + h^2 + 14x - 4$$

$f'(x) \approx 3x^2 + 14x - 4 + h^2$

d $f(x) = (x+3)\,(x-7)$

Expand and simplify:

$f(x) = x^2 - 4x - 21$

Replacing every x with $(x + h)$,

$f(x+h) = (x+h)^2 - 4\,(x+h) - 21$

Replacing every x with $(x - h)$,

$f(x-h) = (x-h)^2 - 4\,(x-h) - 21$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$\approx \frac{\left[(x+h)^2 - 4\,(x+h) - 21\right] - \left[(x-h)^2 - 4\,(x-h) - 21\right]}{2h}$$

$$\approx \frac{\left[x^2 + 2xh + h^2 - 4x - 4h - 21\right] - \left[x^2 - 2xh + h^2 - 4x + 4h - 21\right]}{2h}$$

$$\approx \frac{x^2 + 2xh + h^2 - 4x - 4h - 21 - x^2 + 2xh - h^2 + 4x - 4h + 21}{2h}$$

$$\approx \frac{4xh - 8h}{2h}$$

$$\approx \frac{4h\,(x-2)}{2h}$$

$$\approx 2(x-2)$$

$f'(x) \approx 2x - 4$

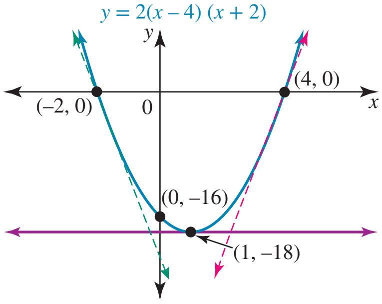
14 a $f(x) = 2\,(x-4)\,(x+2)$

$f(0) = 2\,(-4)\,(2) = -16$ so is y-intercepts $(0, -16)$.

x-intercepts occur at $x = 4,\ x = -2$.

Turning point: $x = \dfrac{-2+4}{2} = 1$, and $f(1) = 2(-3)(3) = -18$.

Minimum turning point $(1, -18)$



b $f(x) = 2(x-4)(x+2)$

Expand and simplify:

$f(x) = 2x^2 - 4x - 16$

Replacing every x with $(x + h)$,

$f(x+h) = 2(x+h)^2 - 4(x+h) - 16$

Replacing every x with $(x - h)$,

$f(x-h) = 2(x-h)^2 - 4(x-h) - 16$

$$\begin{aligned}
f'(x) &\approx \frac{f(x+h)-f(x-h)}{2h}\\
&\approx \frac{\left[2(x+h)^2 - 4(x+h) - 16\right] - \left[2(x-h)^2 - 4(x-h) - 16\right]}{2h}\\
&\approx \frac{\left[2x^2 + 4xh + 2h^2 - 4x - 4h - 16\right] - \left[2x^2 - 4xh + 2h^2 - 4x + 4h - 16\right]}{2h}\\
&\approx \frac{2x^2 + 4xh + 2h^2 - 4x - 4h - 16 - 2x^2 + 4xh - 2h^2 + 4x - 4h + 16}{2h}\\
&\approx \frac{8xh - 8h}{2h}\\
&\approx \frac{8h(x-1)}{2h}\\
&\approx 4(x-1)
\end{aligned}$$

$f'(x) \approx 4x - 4$

c $f'(1) = 0$. The tangent at the turning point $(1, -18)$ is horizontal.

d The gradients of the tangents at the x-intercepts are $f'(-2) = -12$ and $f'(4) = 12$.

The tangent lines from parts **c** and **d** shown on the diagram in part **a**.

15 **a** $f(x) = ax^2 + bx + c$

Replacing every x with $(x + h)$,

$f(x+h) = a(x+h)^2 + b(x+h) + c$

Replacing every x with $(x - h)$,

$f(x-h) = a(x-h)^2 + b(x-h) + c$

$$\begin{aligned}
f'(x) &\approx \frac{f(x+h)-f(x-h)}{2h}\\
&\approx \frac{\left[a(x+h)^2 + b(x+h) + c\right] - \left[a(x-h)^2 + b(x-h) + c\right]}{2h}\\
&\approx \frac{\left[ax^2 + 2axh + ah^2 + bx + bh + c\right] - \left[ax^2 - 2axh + ah^2 + bx - bh + c\right]}{2h}\\
&\approx \frac{ax^2 + 2axh + ah^2 + bx + bh + c - ax^2 + 2axh - ah^2 - bx + bh - c}{2h}\\
&\approx \frac{4axh + 2bh}{2h}\\
&\approx \frac{2h(2ax+b)}{2h}.\\
&\approx 2ax + b
\end{aligned}$$

$f'(x) \approx 2ax + b$

b The polynomial function $f\colon R \to R, f(x) = ax^2 + bx + c$.

Its derivative functions is $f'\colon R \to R, f'(x) = 2ax + b$.

c For $f(x) = 3x + 4x + 2, a = 3, b = 4, c = 2$

$f'(x) = 2ax + b$

$\therefore f'(x) = 6x + 4$

d f has degree 2; f' has degree 1

16 **a** $(2 + h)^3 - 8 = (2 + h)^3 - 2^3$

$= ((2 + h) - 2)\left((2 + h)^2 + (2 + h)2 + 2^2\right)$

$= (h)(4 + 4h + h^2 + 4 + 2h + 4)$

$= h(h^2 + 6h + 12)$

b $g(x) = x^3$

$$g'(x) = \lim_{h\to 0}\frac{g(x + h) - g(x)}{h}$$

Hence, $g'(2) = \lim_{h\to 0}\dfrac{g(2 + h) - g(2)}{h}$

$$\therefore g'(2) = \lim_{h\to 0}\frac{(2 + h)^3 - 8}{h}$$

c Using the factorisation from part **i**,

$$g'(2) = \lim_{h\to 0}\frac{h(h^2 + 6h + 12)}{h}$$

$$= \lim_{h\to 0}(h^2 + 6h + 12)$$

$$= 12$$

17 Use CAS technology to obtain

$\lim_{h\to 0}\dfrac{f(x + h) - f(x)}{h} = 2x.$

18 **a** Use CAS technology to obtain

$\lim_{h\to 0}\dfrac{f(x + h) - f(x)}{h} = 3ax^2 + 2bx + c.$

b Since $f'(x) = \lim_{h\to 0}\dfrac{f(x + h) - f(x)}{h}, f'(x) = 3ax^2 + 2bx + c.$

The degree of $f(x) = ax^3 + bx^2 + cx + d$ is three and the degree of its derivative is two.

11.4 Exam questions

1 For the function $y = f(x)$, its gradient function is defined as the limiting value of the difference quotient as $h \to 0$.

Gradient $= \lim_{h\to 0}\dfrac{f(x + h) - f(x)}{h}$

At $x = 3$

Gradient $= \lim_{h\to 0}\dfrac{f(3 + h) - f(3)}{h}$

The correct answer is **D**.

2 Let $f(x) = 3x^3 - 2x$

$f(x + h) = 3(x + h)^3 - 2(x + h)$

$$f'(x) = \lim_{h\to 0}\frac{f(x + h) - f(x)}{h} \quad \text{[1 mark]}$$

$$= \lim_{h\to 0}\frac{3(x + h)^3 - 2(x + h) - [3x^3 - 2x]}{h}$$

$$= \lim_{h\to 0}\frac{3(x^3 + 3x^2h + 3xh^2 + h^3) - 2(x + h) - [3x^3 - 2x]}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{3x^3} + 9x^2h + 9xh^2 + 3h^3 \cancel{-2x} - 2h - \cancel{3x^3} + \cancel{2x}}{h}$$

$$= \lim_{h\to 0}\frac{9x^2h + 9xh^2 + 3h^3 - 2h}{h} \quad \text{[1 mark]}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(9x^2 + 9xh + 3h^2 - 2), (h \neq 0)}{\cancel{h}} \quad \text{[1 mark]}$$

$$= \lim_{h\to 0} 9x^2 + 9xh + 3h^2 - 2$$

Substitute $h = 0$

$$f'(x) = 9x^2 - 2 \quad \text{[1 mark]}$$

3 $y = 2 - x^2$

Using the central difference approximation,

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h}$$

$$f'(x) \approx \frac{[2 - (x + h)^2] - [2 - (x - h)^2]}{2h} \quad \text{[1 mark]}$$

$$\approx \frac{[2 - x^2 - 2xh - h^2] - [2 - x^2 + 2xh - h^2]}{2h} \quad \text{[1 mark]}$$

$$\approx \frac{2 - x^2 - 2xh - h^2 - 2 + x^2 - 2xh + h^2}{2h}$$

$$\approx \frac{-4xh}{2h}$$

$$\approx -2x \quad \text{[1 mark]}$$

11.5 Differentiation of polynomials by rule

11.5 Exercise

1 **a** $f(x) = 16$

The derivative of a constant function is zero. If $f(x) = c$, then $f'(x) = 0$.

(Note that since $x^0 = 1$, we could write $f(x) = 16x^0$. Then using the rule for differentiation we have $f'(x) = 16 \times 0x^{0-1} = 0 \times 16x^{-1} = 0$).

b $f(x) = 21x + 9$

$f(x) = 21x^1 + 9$

$f'(x) = 21 \times 1x^{1-1} + 0$

$f'(x) = 21x^0$

$\therefore f'(x) = 21$

Note that the derivative, or gradient of a linear function is required. If $f(x) = mx + c, f'(x) = m$.

c $f(x) = 3x^2 - 12x + 2$

$f'(x) = 3 \times 2x^{2-1} - 12 + 0$

$\therefore f'(x) = 6x - 12$

d $f(x) = 0.3x^4$

$f'(x) = 0.3 \times 4x^{4-1}$

$f'(x) = 1.2x^3$

e $f(x) = -\dfrac{2}{3}x^6$

$f'(x) = -\dfrac{2}{3} \times 6x^{6-1}$

$\therefore f'(x) = -4x^5$

f $f(x) = 10x^4 - 2x^3 + 8x^2 + 7x + 1$

$f'(x) = 10 \times 4x^{4-1} - 2 \times 3x^{3-1} + 8 \times 2x^{2-1} + 7 + 0$

$\therefore f'(x) = 40x^3 - 6x^2 + 16x + 7$

2 **a** $y = -2x(5x - 4)$

$y = -10x^2 + 8x$

$\dfrac{dy}{dx} = -20x + 8$

b $y = (6x + 5)(4x + 1)$

$= 24x^2 + 6x + 20x + 5$

$y = 24x^2 + 26x + 5$

$\dfrac{dy}{dx} = 48x + 26$

c $y = \left(\frac{x^2}{2}+1\right)\left(\frac{x^2}{2}-1\right)$

$= \left(\frac{x^2}{2}\right)^2 - 1^2$

$y = \frac{x^4}{4} - 1$

$\frac{dy}{dx} = \frac{4x^3}{4}$

$= x^3$

d $y = (2x-3)\left(4x^2+x+9\right)$

$= 8x^3 + 2x^2 + 18x - 12x^2 - 3x - 27$

$y = 8x^3 - 10x^2 + 15x - 27$

$\frac{dy}{dx} = 24x^2 - 20x + 15$

e $y = (5x+4)^2$

$= 25x^2 + 40x + 16$

$\frac{dy}{dx} = 50x + 40$

f $y = 0.5x^3(2-x)^2$

$= 0.5x^3(4 - 4x + x^2)$

$y = 2x^3 - 2x^4 + 0.5x^5$

$\frac{dy}{dx} = 6x^2 - 8x^3 + 2.5x^4$

3 a $\frac{d}{dx}\left(5x^8 + \frac{1}{2}x^{12}\right) = 40x^7 + 6x^{11}$

b Let $y = 2t^3 + 4t^2 - 7t + 12$.

$\frac{dy}{dt} = 6t^2 + 8t - 7$

c $f(x) = (2x+1)(3x-4)$

$= 6x^2 - 5x - 4$

$f'(x) = 12x - 5$

d $y = \frac{4x^3 - x^5}{2x^2}$

$= \frac{4x^3}{2x^2} - \frac{x^5}{2x^2}$

$= 2x - \frac{x^3}{2}$

$\frac{dy}{dx} = 2 - \frac{3x^2}{2}$

4 a $f(x) = x^7$

$\therefore f'(x) = 7x^{7-1}$

$\therefore f'(x) = 7x^6$

b $y = 3 - 2x^3$

$\therefore \frac{dy}{dx} = 0 - 2 \times 3x^{3-1}$

$\therefore \frac{dy}{dx} = -6x^2$

c $\frac{d}{dx}\left(8x^2 + 6x - 4\right) = 16x + 6$

d $f(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + x - \frac{3}{4}$

$\therefore D_x(f) = \frac{1}{6} \times 3x^2 - \frac{1}{2} \times 2x + 1$

$\therefore D_x(f) = \frac{1}{2}x^2 - x + 1$

e $\frac{d}{du}\left(u^3 - 1.5u^2\right) = 3u^2 - 3u$

f $z = 4\left(1 + t - 3t^4\right)$

$\therefore \frac{dz}{dt} = 4\left(1 - 12t^3\right)$

5 a Let $y = (2x+7)(8-x)$.

Expand first.

$\therefore y = -2x^2 + 9x + 56$

Then differentiate.

$\therefore \frac{dy}{dx} = -4x + 9$

b Let $y = 5x(3x+4)^2$.

$\therefore y = 5x\left(9x^2 + 24x + 16\right)$

$\therefore y = 45x^3 + 120x^2 + 80x$

$\frac{dy}{dx} = 135x^2 + 240x + 80$

c Let $y = (x-2)^4$.

$\therefore y = x^4 - 4x^3(2) + 6x^2(2)^2 - 4x(2)^3 + (2)^4$

$\therefore y = x^4 - 8x^3 + 24x^2 - 32x + 16$

$\frac{dy}{dx} = 4x^3 - 24x^2 + 48x - 32$

d Let $y = 250\left(3x + 5x^2 - 17x^3\right)$.

$\therefore \frac{dy}{dx} = 250\left(3 + 10x - 51x^2\right)$

e Let $y = \frac{3x^2 + 10x}{x}$.

$\therefore y = \frac{3x^2}{x} + \frac{10x}{x}$

$\therefore y = 3x + 10, x \neq 0$

$\frac{dy}{dx} = 3$

f Let $y = \frac{4x^3 - 3x^2 + 10x}{2x}$.

$\therefore y = \frac{4x^3}{2x} - \frac{3x^2}{2x} + \frac{10x}{2x}$

$\therefore y = 2x^2 - \frac{3}{2}x + 5,\ x \neq 0$

$\frac{dy}{dx} = 4x - \frac{3}{2}$

6 $z = \frac{1}{420}\left(3x^5 - 3.5x^4 + x^3 - 2x^2 + 12x - 99\right)$

$\frac{dz}{dx} = \frac{1}{420} \times \frac{d}{dx}\left(3x^5 - 3.5x^4 + x^3 - 2x^2 + 12x - 99\right)$

$= \frac{1}{420}\left(15x^4 - 14x^3 + 3x^2 - 4x + 12\right)$

7 a $f(x) = \left(3x^4\right)^2$

$f(x) = 9x^8$

$f'(x) = 72x^7$

Substitute $x = -1$.

$f'(-1) = 72(-1)^7$

$f'(-1) = -72$

b $g(x) = \frac{2x^3 + x^2}{x},\ x \neq 0$

$g(x) = \frac{2x^3}{x} + \frac{x^2}{x}$

$= 2x^2 + x$

$g'(x) = 4x + 1$

$g'(2) = 4(2) + 1$

$= 9$

c $y = -\frac{1}{3}x^3 + 4x^2 + 8$

$\frac{dy}{dx} = -x^2 + 8x$

At the point $(6, 80)$, $x = 6$.

$$\begin{aligned}\frac{dy}{dx} &= -(6)^2 + 8(6)\\ &= -36 + 48\\ &= 12\end{aligned}$$

The gradient of the tangent is 12.

d $h = 10 + 20t - 5t^2$

$\frac{dh}{dt} = 20 - 10t$

Let $\frac{dh}{dt} = 0$.

$$\begin{aligned}\therefore 20 - 10t &= 0\\ \therefore 20 &= 10t\\ \therefore t &= 2\end{aligned}$$

When $t = 2$,

$$\begin{aligned}h &= 10 + 20(2) - 5(2)^2\\ &= 10 + 40 - 20\\ &= 30\end{aligned}$$

The point where $\frac{dh}{dt} = 0$ is $(2, 30)$.

8 a i $f(x) = 3x^2 - 2x$

$f'(x) = 6x - 2$

The (instantaneous) rate of change at $x = -1$ is $f'(-1)$.

$$\begin{aligned}f'(-1) &= 6(-1) - 2\\ &= -8\end{aligned}$$

The rate of change is -8.

ii The average rate of change over $x \in [-1, 0]$ is the gradient of the chord joining the end points of the interval: $\frac{f(-1) - f(0)}{-1 - 0}$.

$f(x) = 3x^2 - 2x$

$f(0) = 0$ and $f(-1) = 3(-1)^2 - 2(-1) = 5$

Average rate of change

$$\begin{aligned}&= \frac{f(-1) - f(0)}{-1 - 0}\\ &= \frac{5 - 0}{-1 - 0}\\ &= -5\end{aligned}$$

b i $g(x) = x^2\left(2 + x - x^2\right)$

$g(x) = 2x^2 + x^3 - x^4$

$\therefore g'(x) = 4x + 3x^2 - 4x^3$

The rate of change at the point $(1, 2)$ is $g'(1)$.

$g'(1) = 4(1) + 3(1)^2 - 4(1)^3 = 3$

The rate of change is 3.

ii The average rate of change over the interval $x \in [0, 2]$ is given by the gradient of the secant through the end points of the interval.

$g(x) = 2x^2 + x^3 - x^4$

$g(0) = 0$ and $g(2) = 2(4) + 8 - 16 = 0$.

Hence, over the interval between the points $(0, 0)$ and $(2, 0)$ the average rate of change is zero.

9 $y = \frac{2x^3}{3} - x^2 + 3x - 1$

a $\frac{dy}{dx} = 2x^2 - 2x + 3$

At the point $(6, 125)$,

$$\begin{aligned}\frac{dy}{dx} &= 2(6)^2 - 2(6) + 3\\ &= 63\end{aligned}$$

Therefore, the rate of change is 63.

b $y = \frac{2x^3}{3} - x^2 + 3x - 1$ over $x \in [0, 6]$

When $x = 0, y = -1 \Rightarrow (0, -1)$ and when $x = 6, y = 125 \Rightarrow (6, 125)$

Average rate of change is $\frac{125 - (-1)}{6 - 0} = 21$.

c Gradient 3, so $\frac{dy}{dx} = 3$

$$\begin{aligned}2x^2 - 2x + 3 &= 3\\ 2x^2 - 2x &= 0\\ 2x(x - 1) &= 0\\ x &= 0,\ x = 1\end{aligned}$$

Substitute into the equation of curve:

The points are $x = 0, y = -1 \Rightarrow (0, -1)$ and

$x = 1, y = \frac{2}{3} - 1 + 3 - 1 \Rightarrow \left(1, \frac{5}{3}\right)$

10 a $y = 3x^2 - x + 4$

$\therefore \frac{dy}{dx} = 6x - 1$

When $x = \frac{1}{2}$,

$$\begin{aligned}\frac{dy}{dx} &= 6 \times \frac{1}{2} - 1\\ &= 2\end{aligned}$$

The gradient of the tangent at the point where $x = \frac{1}{2}$ is 2.

b $f(x) = \frac{1}{6}x^3 + 2x^2 - 4x + 9$

$\therefore f'(x) = \frac{1}{6} \times 3x^2 + 4x - 4$

$\therefore f'(x) = \frac{1}{2}x^2 + 4x - 4$

At $(0, 9)$, $f'(0) = -4$.

The gradient of the tangent at the point $(0, 9)$ is -4.

c $y = (x + 6)^2 - 2$ has turning point $(-6, -2)$.

Expanding the equation,

$y = x^2 + 12x + 34$

$\therefore \frac{dy}{dx} = 2x + 12$

At the point $(-6, -2)$,

$$\begin{aligned}\frac{dy}{dx} &= 2 \times -6 + 12\\ &= 0\end{aligned}$$

The gradient of the tangent at the turning point is zero.

d $y = 2 - (x - 3)^3$ has a stationary point of inflection at $(3, 2)$.

Expanding the equation,

$$\begin{aligned}y &= 2 - \left(x^3 - 3x^2(3) + 3x(3)^2 - (3)^3\right)\\ &= 2 - x^3 + 9x^2 - 27x + 27\\ &= 29 - x^3 + 9x^2 - 27x\end{aligned}$$

$\frac{dy}{dx} = -3x^2 + 18x - 27$

At the point $(3, 2)$,

$$\begin{aligned}\frac{dy}{dx} &= -3(3)^2 + 18(3) - 27\\ &= -27 + 54 - 27\\ &= 0\end{aligned}$$

The gradient of the tangent at the stationary point of inflection is zero.

11 a $f(x) = 5x - \frac{3}{4}x^2$

$\therefore f'(x) = 5 - \frac{3}{4} \times 2x$

$\therefore f'(x) = 5 - \frac{3}{2}x$

$f'(-6) = 5 - \frac{3}{2} \times (-6)$

$= 5 + 9$

$= 14$

b $f'(0) = 5$

c $f'(x) = 0$

$\therefore 5 - \frac{3}{2}x = 0$

$\therefore 5 = \frac{3}{2}x$

$\therefore 10 = 3x$

$\therefore x = \frac{10}{3}$

The solution set is $\left\{\frac{10}{3}\right\}$.

d $f'(x) > 0$

$\therefore 5 - \frac{3}{2}x > 0$

$\therefore -\frac{3}{2}x > -5$

$\therefore -3x > -10$

$\therefore x < \frac{10}{3}$

The solution set is $\left\{x : x < \frac{10}{3}\right\}$.

e x-intercepts: let $f(x) = 0$.

$\therefore 5x - \frac{3}{4}x^2 = 0$

$\therefore 20x - 3x^2 = 0$

$\therefore x(20 - 3x) = 0$

$\therefore x = 0,\ x = \frac{20}{3}$

The gradient at $x = 0$ is $f'(0) = 5$.

Gradient at $x = \frac{20}{3}$:

$f'\left(\frac{20}{3}\right) = 5 - \frac{3}{2} \times \frac{20}{3}$

$= 5 - 10$

$= -5$

The gradients of the curve at the points where it cuts the x-axis are -5 and 5.

f If the gradient of the tangent is 11, then $f'(x) = 11$.

$\therefore 5 - \frac{3}{2}x = 11$

$\therefore -\frac{3}{2}x = 6$

$\therefore -3x = 12$

$\therefore x = -4$

Substitute $x = -4$ into the equation of the curve.

$f(-4) = 5 \times (-4) - \frac{3}{4} \times (-4)^2$

$= -20 - 12$

$= -32$

At the point $(-4, -32)$, the gradient of the tangent is 11.

12 a $f(x) = x^2 - 2x - 3$

$\therefore f'(x) = 2x - 2$

Let $f'(x) = 0$.

$\therefore 2x - 2 = 0$

$\therefore x = 1$

$f(1) = 1 - 2 - 3$

$\therefore f(1) = -4$

The gradient is zero at the point $(1, -4)$.

b The line $y = 5 - 4x$ has gradient -4.

As the tangent is parallel to this line, the tangent has gradient -4.

Let $f'(x) = -4$.

$\therefore 2x - 2 = -4$

$\therefore x = -1$

$f(-1) = 1 + 2 - 3$

$\therefore f(-1) = 0$

The tangent at the point $(-1, 0)$ is parallel to the line $y = 5 - 4x$.

c The line $x + y = 7$ has gradient -1.

As the tangent is perpendicular to this line, the tangent has gradient 1.

Let $f'(x) = 1$.

$2x - 2 = 1$

$\therefore x = \frac{3}{2}$

$f\left(\frac{3}{2}\right) = \frac{9}{4} - 3 - 3$

$\therefore f\left(\frac{3}{2}\right) = -\frac{15}{4}$

The tangent at the point $\left(\frac{3}{2}, -\frac{15}{4}\right)$ is perpendicular to the line $x + y = 7$.

d At the point $(5, 12)$,

$f'(5) = 10 - 2$

$= 8$

Require the point where $f'(x) = 4$

$\therefore 2x - 2 = 4$

$\therefore x = 3$

$f(3) = 9 - 6 - 3$

$\therefore f(3) = 0$

The slope of the tangent at the point $(3, 0)$ is half that of the tangent at the point $(5, 12)$.

13 a Refer to the graph given in the question.

i $f'(x) = 0$ at the turning points. These occur at $x = \pm\sqrt{3}$.
The solution set is $\left\{\pm\sqrt{3}\right\}$.

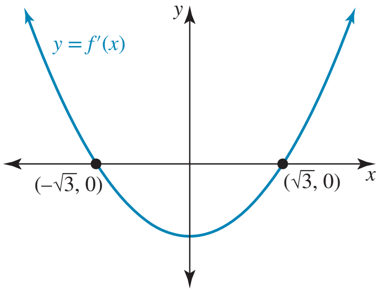
ii $f'(x) < 0$ where the tangent to the curve has a negative gradient. This occurs between the two turning points.
The solution set is $\left\{x : -\sqrt{3} < x < \sqrt{3}\right\}$.

iii $f'(x) > 0$ where the tangent to the curve has a positive gradient.
The solution set is $\left\{x : x < -\sqrt{3}\right\} \cup \left\{x : x > \sqrt{3}\right\}$.

b The graph of $y = f'(x)$ will have x-intercepts at $x = \pm\sqrt{3}$.
The graph lies below the x-axis for $\left\{x : -\sqrt{3} < x < \sqrt{3}\right\}$ and the graph lies above the x-axis for

$\left\{x : x < -\sqrt{3}\right\} \cup \left\{x : x > \sqrt{3}\right\}$.

A possible graph is shown.



14 $f(x) = (x - 1)(x + 2)$

$= x^2 + x - 2$

$f'(x) = 2x + 1$

The line $3x + 3y = 4$ has gradient -1.

Therefore, $f'(x) = -1$

$\therefore 2x + 1 = -1$

$\therefore x = -1$

When $x = -1$,

$f(x) = (-1 - 1)(-1 + 2)$

$= -2$

Therefore, the point has coordinates $(-1, -2)$.

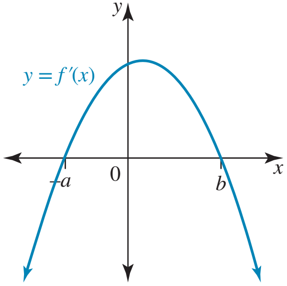
15 Turning points when $x = -a$ and $x = b \Rightarrow x = -a$ and $x = b$ are the x-intercepts of the gradient graph.

If $x < -a$, the gradient is negative.

If $-a < x < b$, the gradient is positive.

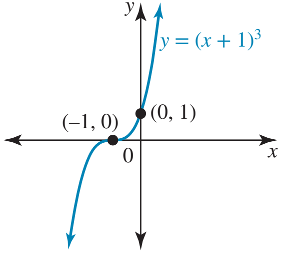
If $x > b$, the gradient is negative.

This is a cubic graph, so the gradient graph is a parabola with axis of symmetry $x = \dfrac{-a + b}{2}$.

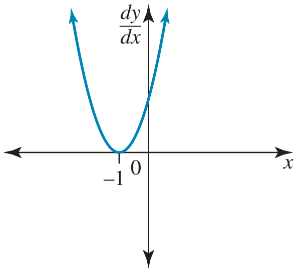


16 $y = (x + 1)^3$

Stationary point of inflection at $(-1, 0)$; y-intercept $(0, 1)$



The gradient is positive everywhere except at $x = -1$, where the gradient is 0. The gradient graph is a parabola with a turning point at its x-intercept at $x = -1$.



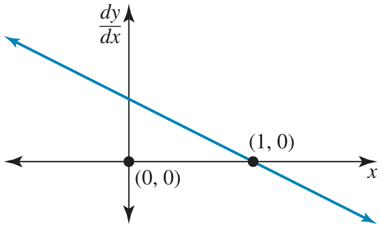
17 Refer to the graphs given in the question.

a Turning point at $(1, 4) \Rightarrow$ the gradient graph has an x-intercept when $x = 1$.

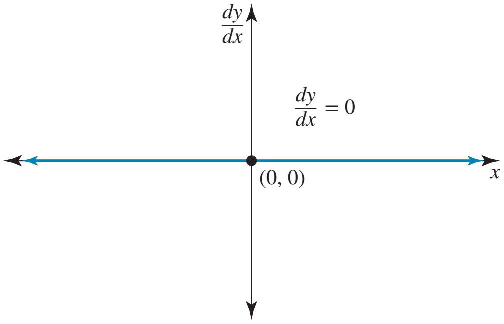
For $x < 1$, the given graph has a positive gradient $\Rightarrow$ the gradient graph lies above the x-axis.

For $x > 1$, the given graph has a negative gradient $\Rightarrow$ the gradient graph lies below the x-axis.

The given graph is degree 2, so the gradient graph is degree 1 and the graph is linear.



b The given graph is horizontal, so its gradient is zero.



c The given graph has turning points at $(0, -2)$ and $(2, 2)$.

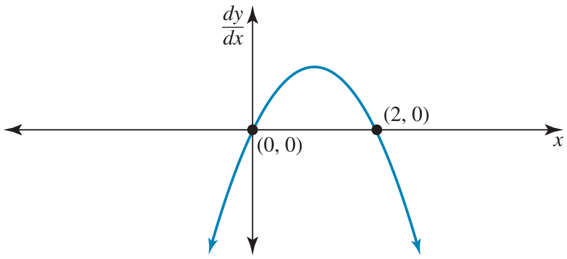
The gradient graph will have x-intercepts at $x = 0, x = 2$.

For $x < 0$, the given graph has a negative gradient $\Rightarrow$ the gradient graph lies below the x-axis.

For $0 < x < 2$, the given graph has a positive gradient $\Rightarrow$ the gradient graph lies above the x-axis.

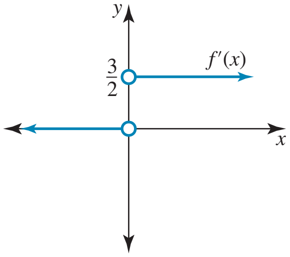
For $x > 2$, the given graph has a negative gradient $\Rightarrow$ the gradient graph lies below the x-axis.

The given graph has degree 3 $\Rightarrow$ the gradient graph has degree 2.



d For $x < 0$, the gradient is zero.

For $x > 0$, the gradient is $\dfrac{3}{2}$.



e The given graph has turning points at $\left(-\sqrt{5},-5\right)$, (0, 3) and $\left(\sqrt{5},-5\right)$. The gradient graph will have x-intercepts at $x=-\sqrt{5}, x=0, x=\sqrt{5}$.

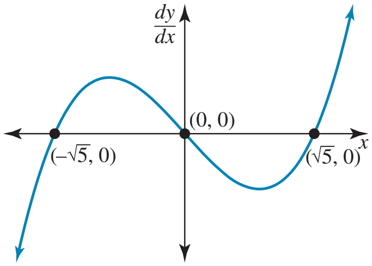
For $x<-\sqrt{5}$, the given graph has a negative gradient ⇒ the gradient graph lies below the x-axis.

For $-\sqrt{5}<x<0$, the given graph has a positive gradient ⇒ the gradient graph lies above the x-axis.

For $0<x<\sqrt{5}$, the given graph has a negative gradient ⇒ the gradient graph lies below the x-axis.

For $x>\sqrt{5}$, the given graph has a positive gradient ⇒ the gradient graph lies above the x-axis.

The given graph has degree 4 ⇒ the gradient graph has degree 3.



f The given graph has a stationary point of inflection at (0, 0) and a turning point at (3, 3). The gradient graph has x-intercepts at (0, 0) and (3, 0).

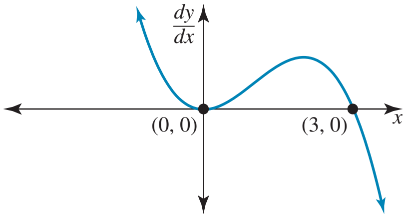
For $x<0$, the given graph has a positive gradient ⇒ the gradient graph lies above the x-axis.

For $0<x<3$, the given graph has a positive gradient ⇒ the gradient graph lies above the x-axis.

This means that the gradient graph must have a turning point at (0, 0).

For $x>3$, the given graph has a negative gradient ⇒ the gradient graph lies below the x-axis.

The given graph has degree 4 ⇒ the gradient graph has degree 3.



18 **a** $N=0.5\left(t^2+1\right)^2+2.5$

The instantaneous rate at which the population is changing is $\dfrac{dN}{dt}$.

$N=0.5\left(t^4+2t^2+1\right)+2.5$

$\therefore N=0.5t^4+t^2+3$

$\dfrac{dN}{dt}=2t^3+2t$

When $t=1$, $\dfrac{dN}{dt}=2+2=4$. The population is growing at 4 ants per hour when $t=1$.

When $t=2$,

$\dfrac{dN}{dt}=2\times 8+2\times 2$

$\therefore \dfrac{dN}{dt}=20$

The population is growing at 20 ants per hour when $t=2$.

b When $t=0$, $N=3$ and when $t=2$,

$N=0.5\times 16+4+3=15.$

Average rate of change between (0, 3) and (2, 15) is

$\dfrac{15-3}{2-0}=6$ ants/hour.

c Let $\dfrac{dN}{dt}=60$.

$\therefore 2t^3+2t=60$

$\therefore t^3+t-30=0$

Let $P(t)=t^3+t-30$.

$P(3)=27+3-30=0$

$\therefore (t-3)$ is a factor.

$\therefore t^3+t-30=(t-3)\left(t^2+3t+10\right)$

Hence, $t^3+t-30=0 \Rightarrow (t-3)\left(t^2+3t+10\right)=0$.

$\therefore t=3$ or $t^2+3t+10=0$

Consider $t^2+3t+10=0$.

$\Delta=9-4\times 1\times 10$

$=-31$

<0

There are no real solutions.

$\therefore t=3$

The population of ants is increasing at 60 ants per hour 3 hours after the pot of honey is spilt.

19 **a** $y=4x^2+kx-5$

$\dfrac{dy}{dx}=8x+k$

Given that $\dfrac{dy}{dx}=4$ when $x=-2$,

$\therefore 4=8\times -2+k$

$\therefore k=20$

b $y=ax^2+bx+c$

$\dfrac{dy}{dx}=2ax+b$

At the turning point, the tangent is horizontal, so $\dfrac{dy}{dx}=0$.

$\therefore 2ax+b=0$

$\therefore 2ax=-b$

$\therefore x=-\dfrac{b}{2a}$

c **i** $f(x)=x^3+9x^2+30x+c$

$\therefore f'(x)=3x^2+18x+30$

$=3\left(x^2+6x+10\right)$

$=3\left[\left(x^2+6x+9\right)-9+10\right]$

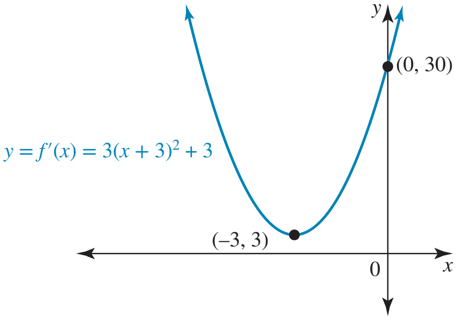
$=3\left[(x+3)^2+1\right]$

$=3(x+3)^2+3$

Since $(x+3)^2\geq 0, f'(x)\geq 3$ and is therefore always positive.

ii Let $y=f'(x)$.

$y=3(x+3)^2+3$ is a parabola with minimum turning point $(-3,3)$ and y-intercept $(0,30)$, since $f'(0)=30$.



d $r = at^2 + bt$

$\frac{dr}{dt} = 2at + b$

When $t = 1, \frac{dr}{dt} = 6$

$\therefore 6 = 2a + b$ [1]

When $t = 3, \frac{dr}{dt} = 14$

$\therefore 14 = 6a + b$ [2]

Equation [2] subtract equation [1]

$8 = 4a$

$\therefore a = 2$

Substitute $a = 2$ in equation [1]

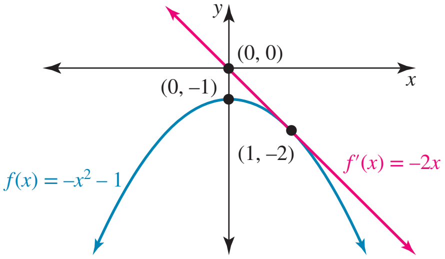
$\therefore 6 = 4 + b$

$\therefore b = 2$

Hence, $a = 2, b = 2$.

20 a $f(x) = -x^2 - 1$ is a parabola with a maximum turning point at $(0, -1)$.

$f'(x) = -2x$ is a straight line through the origin and the point $(1, -2)$.



b $f(x) = x^3 - x^2$

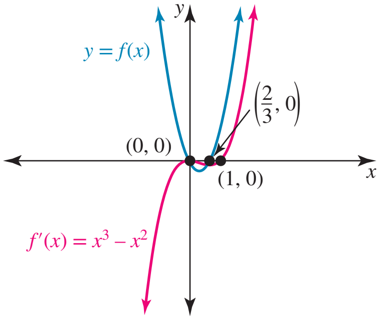
$\therefore f(x) = x^2(x - 1)$

The cubic graph touches the x-axis at $(0, 0)$ and cuts the x-axis at $(1, 0)$.

$f'(x) = 3x^2 - 2x$

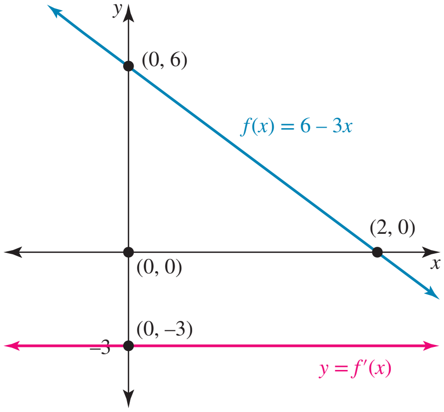
$\therefore f'(x) = x(3x - 2)$

The quadratic graph cuts the x-axis at $(0, 0)$ and $\left(\frac{2}{3}, 0\right)$.



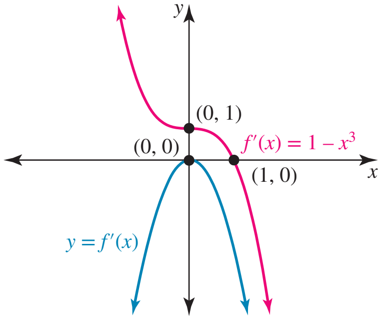
c $f(x) = 6 - 3x$ is a straight line through $(0, 6)$ and $(2, 0)$ with gradient -3.

$f'(x) = -3$ is a horizontal line through $(0, -3)$.



d $f(x) = 1 - x^3$ is a cubic graph with stationary point of inflection at $(0, 1)$ and cutting the x-axis at $(1, 0)$.

$f'(x) = -3x^2$ is a parabola with maximum turning point $(0, 0)$.



11.5 Exam questions

1 $f(x) = 2x^2 - 3x + 5$

$f'(x) = 4x - 3$

$f'(2) = 4(2) - 3$

$= 5$

The correct answer is **C**.

2 $f(x) = 3x^2 - 12x + 8$

$f'(x) = 6x - 12$

$0 = 6x - 12$

$6x = 12$

Zero gradient at $x = 2$

$f(x) = 3x^2 - 12(2) + 8$

$f(2) = 3(2)^2 - 12(2) + 8$

$= 12 - 24 + 8$

$= -4$

$\therefore (2, -4)$

The correct answer is **B**.

3 Let volume be V

$V = \frac{4\pi r^3}{3}$

$\frac{dV}{dr} = 4\pi r^2$ [1 mark]

$r = 15$

$\frac{dV}{dr} = 4\pi \times 15^2$

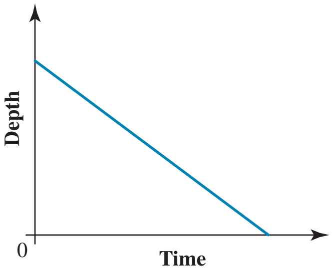
$\frac{dV}{dr} = 900\pi\ \text{cm}^3/\text{cm}$ [1 mark]

11.6 Review

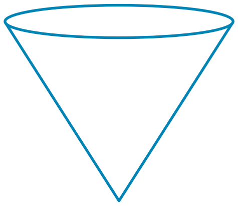
11.6 Exercise

Technology free: short answer

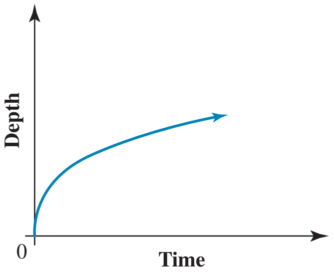
1 a The depth of water decreases steadily at 4 litres/hour. The graph of the depth versus time will be linear with a gradient of −4.



b The depth of water collecting in an inverted right cone will increase quickly at first in the narrow section of the cone but then more slowly as the water reaches the wider sections of the cone.



A possible graph of depth versus time is shown.



2 a The points (2, 1) and $(2+h, f(2+h))$ lie on $f(x)=\dfrac{x^2}{4}$.

Gradient of chord with these end points:

$$\begin{aligned} m_{\text{chord}} &= \frac{f(2+h)-1}{2+h-2} \\ &= \frac{\frac{1}{4}(2+h)^2-1}{h} \\ &= \frac{\frac{1}{4}\left(4+4h+h^2\right)-1}{h} \\ &= \frac{1+h+\frac{1}{4}h^2-1}{h} \\ &= \frac{h+\frac{1}{4}h^2}{h} \\ &= 1+\frac{1}{4}h,\quad h\neq 0 \end{aligned}$$

If $h=0.01$, $m_{\text{chord}}=1+\dfrac{0.01}{4}=1.0025$.

b As h becomes smaller and smaller and approaches zero, the chord becomes closer to the tangent at (2, 1). The gradient of the tangent measures the gradient of the curve at (2, 1). Hence, the gradient of the curve is 1.

3 a $f(x)=2x^2-3x+4$

Replacing every x with $(x+h)$,

$f(x+h)=2(x+h)^2-3(x+h)-4$

Replacing every x with $(x-h)$,

$f(x-h)=2(x-h)^2-3(x-h)-4$

$$\begin{aligned}
f'(x) &\approx \frac{f(x+h)-f(x-h)}{2h}\\
&\approx \frac{\left[2(x+h)^2-3(x+h)-4\right]-\left[2(x-h)^2-3(x-h)-4\right]}{2h}\\
&\approx \frac{\left[2x^2+4xh+2h^2-3x-3h-4\right]-\left[2x^2-4xh+2h^2-3x+3h-4\right]}{2h}\\
&\approx \frac{2x^2+4xh+2h^2-3x-3h-4-2x^2+4xh-2h^2+3x-3h+4}{2h}\\
&\approx \frac{8xh-6h}{2h}\\
&\approx \frac{2h(4x-3)}{2h}\\
&\approx 4x-3\\
f'(x) &\approx 4x-3
\end{aligned}$$

b $f(x)=x(1-x^3)$

$$\begin{aligned}
\therefore f(x) &= x-x^4\\
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\left[(x+h)-(x+h)^4\right]-\left[x-x^4\right]}{h}\\
&= \lim_{h\to 0}\frac{x+h-\left(x^4+4x^3h+6x^2h^2+4xh^3+h^4\right)-x+x^4}{h}\\
&= \lim_{h\to 0}\frac{h-4x^3h-6x^2h^2-4xh^3-h^4}{h}\\
&= \lim_{h\to 0}\frac{h\left(1-4x^3-6x^2h-4xh^2-h^3\right)}{h}\\
&= \lim_{h\to 0}\left(1-4x^3-6x^2h-4xh^2-h^3\right)\\
&= 1-4x^3
\end{aligned}$$

4 a $y=\frac{1}{3}x^2(6+x)$

$\therefore y=2x^2+\frac{1}{3}x^3$

$\therefore \frac{dy}{dx}=4x+x^2$

b $f(x)=2x^3-\frac{3x(2x+1)}{2}-7$

$\therefore f(x)=2x^3-\frac{6x^2+3x}{2}-7$

$\therefore f(x)=2x^3-\frac{6x^2}{2}-\frac{3x}{2}-7$

$\therefore f(x)=2x^3-3x^2-\frac{3}{2}x-7$

Hence, $f'(x)=6x^2-6x-\frac{3}{2}$.

c $f(x)=(4x+1)^3-(1-2x)^2$

$\therefore f(x)=(4x)^3+3(4x)^2(1)+3(4x)(1)^2+(1)^3-\left(1-4x+4x^2\right)$

$\therefore f(x)=64x^3+48x^2+12x+1-1+4x-4x^2$

$\therefore f(x)=64x^3+44x^2+16x$

Hence, $f'(x)=192x^2+88x+16$.

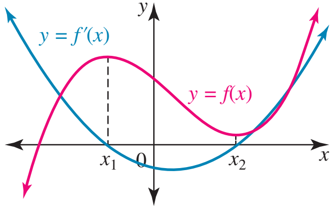
d Let $y=\frac{2x^3+4x^2-3x}{4x}$.

$\therefore y=\frac{2x^3}{4x}+\frac{4x^2}{4x}-\frac{3x}{4x}$

$\therefore y=\frac{1}{2}x^2+x-\frac{3}{4}$

$\therefore \frac{dy}{dx}=x+1$

5 If the turning points of the given graph occur when $x = x_1$ and $x = x_2$, the derivative graph will have x-intercepts at x_1 and x_2. The section of the given graph that lies between the two turning points has a negative gradient, so the derivative graph lies below the x-axis for this section of the domain.
Otherwise the given graph has a positive gradient, so the derivative graph will lie above the x-axis.
The given graph has degree 3, so the derivative graph has degree 2.



6 **a** $f(x) = 3x^3 - 4x^2 + 2x + 5$

$\therefore f'(x) = 9x^2 - 8x + 2$

$\therefore f'(2) = 9 \times 4 - 16 + 2$

$\therefore f'(2) = 22$

b Let $f'(x) = 3$.

$\therefore 9x^2 - 8x + 2 = 3$

$\therefore 9x^2 - 8x - 1 = 0$

$\therefore (9x + 1)(x - 1) = 0$

$\therefore x = -\frac{1}{9}$ or $x = 1$

Technology active: multiple choice

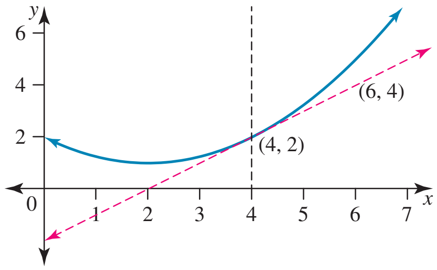
7 The graph shows that the distance walked increases at a steady rate. The rate of change, or gradient, is the speed so speed is constant.
The correct answer is **C**.

8 Reading from the given graph:
when $x = 0, y = 2$ and when $x = 6, y = 5$.
The average rate of change between $(0, 2)$ and $(6, 5)$ is

$\frac{5 - 2}{6 - 0} = \frac{1}{2}$.

The correct answer is **A**.

9 Draw a tangent line to the graph at $x = 4$ and estimate the coordinates of two points on the tangent line.



The points $(4, 2)$ and $(6, 4)$ lie on the tangent. Therefore, the gradient is $\frac{4 - 2}{6 - 4} = 1$.

The correct answer is **C**.

10 $y = x^3$
When $x = 2, y = 8$ and when $x = 2.5, y = 2.25^3 = 15.625$.
Gradient of secant through points $(2, 8)$ and $(2.5, 15.625)$

$= \frac{15.625 - 8}{2.5 - 2}$

$= \frac{7.625}{0.5}$

$= 15.25$

The correct answer is **D**.

11 $f(x) = x^2 + 2x$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(x + h)^2 + 2(x + h)\right] - \left[x^2 + 2x\right]}{h}$$

$$= \lim_{h \to 0} \frac{(x + h)^2 + 2(x + h) - (x^2 + 2x)}{h}$$

The correct answer is **E**.

12 $f(x) = 10 - 7x^2 - 3x^3$

$\therefore f'(x) = -14x - 9x^2$

$\therefore f'(-1) = 14 - 9$

$\therefore f'(-1) = 5$

The correct answer is **B**.

13 $y = \left(2x^2 - 5\right)\left(2x^2 + 5\right)$

$\therefore y = \left(2x^2\right)^2 - (5)^2$

$\therefore y = 4x^4 - 25$

$\therefore \frac{dy}{dx} = 16x^3$

The correct answer is **E**.

14 $\frac{d}{dx}\left(2\left(x^3 - 5x^2 + 1\right)\right)$

$= \frac{d}{dx}\left(2x^3 - 10x^2 + 2\right)$

$= 6x^2 - 20x$

$= 2x(3x - 10)$

The correct answer is **D**.

15 $y = \frac{1}{250}\left(1.5x^2 - 6x\right)$

The tangent is parallel to the x-axis if its gradient is zero.

$\frac{dy}{dx} = \frac{1}{250}(3x - 6)$

Let $\frac{dy}{dx} = 0$

$\therefore \frac{1}{250}(3x - 6) = 0$

$\therefore 3x - 6 = 0$

$\therefore x = 2$

The correct answer is **B**.

16 $y = 4 - x + x^2$

$\therefore y + k = 4 - (x + h) + (x + h)^2$

$\therefore k = 4 - (x + h) + (x + h)^2 - \left(4 - x + x^2\right)$

$\therefore k = 4 - x - h + x^2 + 2xh + (h)^2 - 4 + x - x^2$

$\therefore k = -h + 2xh + (h)^2$

$\therefore k = (h)^2 + 2xh - h$

The correct answer is **A**.

Technology active: extended response

17 **a** $f : R \to R, f(x) = x^3 - 4x^2 + 5x$

The rule for $f'(x)$ is $f'(x) = 3x^2 - 8x + 5$.

The derivative function is $f' : R \to R,\ f'(x) = 3x^2 - 8x + 5$.

b Let $f(x) = 0$.

$\therefore x^3 - 4x^2 + 5x = 0$

$\therefore x\left(x^2 - 4x + 5\right) = 0$

$\therefore x = 0$ or $x^2 - 4x + 5 = 0$

Consider the discriminant of $x^2 - 4x + 5 = 0$.

$\Delta = (-4)^2 - 4 \times 1 \times 5$

$= 16 - 20$

$= -4$

< 0

There are no real solutions to the quadratic equation.
This means there is only one x-intercept at $x = 0$.
Since $f'(0) = 5$, the gradient of the curve at $(0, 0)$ is 5.

c The equation of the line $3y + x = 6$ rearranges to $y = -\frac{1}{3}x + 2$. Its gradient is $-\frac{1}{3}$. The tangent perpendicular to this line would have gradient 3.

Let $f'(x) = 3$.

$\therefore 3x^2 - 8x + 5 = 3$

$\therefore 3x^2 - 8x + 2 = 0$

$$\begin{aligned}\therefore x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 3 \times 2}}{6}\\ &= \frac{8 \pm \sqrt{40}}{6}\\ &= \frac{8 \pm 2\sqrt{10}}{6}\\ \therefore x &= \frac{4 \pm \sqrt{10}}{3}\end{aligned}$$

d $$\begin{aligned}f'(x) &= 3x^2 - 8x + 5\\ &= (x - 1)(3x - 5)\\ &= (x - 1)(ax + b)\end{aligned}$$

$\therefore a = 3, b = -5$

e i Let $f'(x) = 0$.

$\therefore (x - 1)(3x - 5) = 0$

$\therefore x = 1 \text{ or } x = \frac{5}{3}$

Substitute these values into the function's equation to determine which has a corresponding y-coordinate of $\frac{50}{27}$.

$f(x) = x^3 - 4x + 5x$

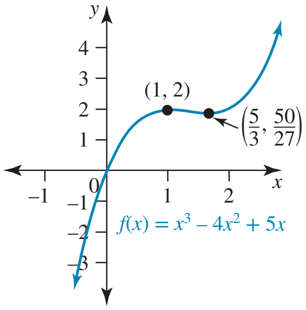
$f(1) = 1 - 4 + 5 = 2$

$$\begin{aligned}f\left(\frac{5}{3}\right) &= \frac{125}{27} - \frac{100}{9} + \frac{25}{3}\\ &= \frac{50}{27}\end{aligned}$$

Therefore, $c = \frac{5}{3}$.

ii The gradient at the point $(1, 2)$ is zero as well as the point $\left(\frac{5}{3}, \frac{50}{27}\right)$.

f The cubic function passes through the origin and has turning points at $(1, 2)$ and at $\left(\frac{5}{3}, \frac{50}{27}\right)$.



18 a $y = x(x + 3)$

$y = x^2 + 3x$

For point P:

$x = 3$

$$\begin{aligned}y &= 3^2 + 3(3)\\ &= 18\end{aligned}$$

The coordinates of P are $(3, 18)$.

For point Q:

$x = 3 + h$

$y = (3 + h)^2 + 3(3 + h)$

$y = 9 + 6h + h^2 + 9 + 3h$

$y = 18 + 9h + h^2$

The coordinates of Q are $\left(3 + h, 18 + 9h + h^2\right)$.

b Gradient of chord PQ

$$\begin{aligned}&= \frac{(18 + 9h + h^2) - 18}{(3 + h) - 3}\\ &= \frac{9h + h^2}{h}\\ &= \frac{h(9 + h)}{h}\\ &= 9 + h\end{aligned}$$

c The gradient of the tangent at P is the limiting position as Q → P.

Therefore, the gradient of the tangent at P

$$\begin{aligned}&= \lim_{h \to 0}(9 + h)\\ &= 9\end{aligned}$$

d $y = x^2 + 3x$

$\therefore \frac{dy}{dx} = 2x + 3$

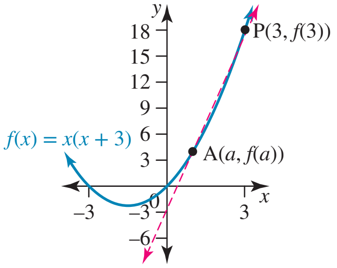
At P, $x = 3$.

When $x = 3$, $\frac{dy}{dx} = 2 \times 3 + 3 = 9$.

e Let $f(x) = x(x + 3) = x^2 + 3x$.

$$\begin{aligned}&\lim_{a \to 3} \frac{f(3) - f(a)}{3 - a}\\ &= \lim_{a \to 3} \frac{18 - (a^2 + 3a)}{3 - a}\\ &= \lim_{a \to 3} \frac{18 - (a^2 - 3a)}{3 - a}\\ &= \lim_{a \to 3} \frac{18 - 3a - a^2}{3 - a}\\ &= \lim_{a \to 3} \frac{(6 + a)(3 - a)}{3 - a}\\ &= \lim_{a \to 3}(6 + a)\\ &= 9\end{aligned}$$

Let A be the point $(a, f(a))$ on the curve $y = f(x) = x^2 + 3x$. P(3, 18) lies on this curve, so its coordinates could be expressed as $(3, f(3))$.



The line through A and P is a secant with gradient $m_{\text{secant}} = \frac{f(3) - f(a)}{3 - a}$.

As the point A is moved closer to P, $a \to 3$ and the gradient of the secant approaches the value of the gradient of the tangent at P.

$\therefore \lim\limits_{a \to 3} \dfrac{f(3) - f(a)}{3 - a} = f'(3).$

19 a i $P_N = 2\left(1 + 4n + \dfrac{n^2}{4}\right)$

In the year 2010, $n = 0$ and $P_N = 2$. The population is 2000.

In the year 2015, $n = 5$.

$P_N = 2\left(1 + 20 + \dfrac{25}{4}\right)$

$= 2(27.25)$

$= 54.5$

Population is 54 500.

The average rate of growth of the population

$= \dfrac{54\,500 - 2000}{5}$

$= \dfrac{52\,500}{5}$

$= 105\,00$

The average rate of growth is 10 500 people per year.

ii In the year 2020, $n = 10$.

$\dfrac{dP_N}{dn}$ measures the rate of growth.

$\dfrac{dP_N}{dn} = 2\left(4 + \dfrac{n}{2}\right) = 8 + n$

When $n = 10$, $\dfrac{dP_N}{dn} = 18$.

The model predicts that the population will be growing at 18 thousand people per year in the year 2020.

b i $P_W = 2\left(1 + n - \left(\dfrac{n}{4}\right)^2\right)$

When $n = 0$, $P_W = 2 = P_N$, so both towns had a population of 2000 in the year 2010.

ii $P_W = 2\left(1 + n - \dfrac{n^2}{16}\right) = 2 + 2n - \dfrac{n^2}{8}$

$\dfrac{dP_W}{dn} = 2 - \dfrac{n}{4}$

When $n = 10$,

$\dfrac{dP_W}{dn} = 2 - \dfrac{10}{4}$

$= -0.5$

The model predicts that the population will be decreasing at 500 people per year in the year 2020.

c Calculate n so that $\dfrac{dP_N}{dn} = 12\dfrac{dP_W}{dn}$.

$\therefore 8 + n = 12\left(2 - \dfrac{n}{4}\right)$

$\therefore 8 + n = 24 - 3n$

$\therefore 4n = 16$

$\therefore n = 4$

This will occur in the year 2014.

d The population ceases to grow when its rate of growth becomes zero.

$\therefore \dfrac{dP_W}{dn} = 0$

$\therefore 2 - \dfrac{n}{4} = 0$

$\therefore n = 8$

Town W ceases growing in the year 2018.

e Let $P_W = 2$

$\therefore 2 + 2n - \dfrac{n^2}{8} = 2$

$\therefore 2n - \dfrac{n^2}{8} = 0$

$\therefore 16n - n^2 = 0$

$\therefore n(16 - n) = 0$

$\therefore n = 0$ or $n = 16$

$n = 0$ corresponds to the year 2010 and $n = 16$ corresponds to the year 2026.

By 2026, the population of Town W is predicted to fall back to 2000 people.

f i The year 2020 $\Rightarrow n = 10$

$P_N = 2(1 + 40 + 25) = 132$

$P_W = 2 + 20 - \dfrac{100}{8} = 9.5$

$\therefore P_{NEW} = 132 + 9.5 = 141.5$

The initial population of NEW town is 141 500.

ii $P_{NEW} = P_N + P_W$

Using the linearity property of differentiation,

$\dfrac{d(P_{NEW})}{dn} = \dfrac{dP_N}{dn} + \dfrac{dP_W}{dn}$

The rates for town N and W have been calculated in parts **a ii** and **b ii**.

$\therefore \dfrac{d(P_{NEW})}{dn} = 18 + -0.5 = 17.5$

The initial rate at which NEW town will grow is 17 500 people per year.

20 a $y = x^4 + 2x^2$

$\therefore \dfrac{dy}{dx} = 4x^3 + 4x$

$y = x^4 + 2x^2$

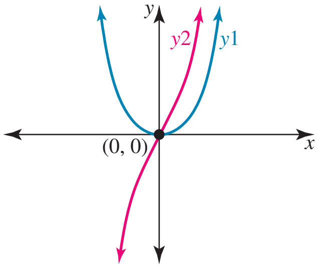
In Graph menu:

$y1 = x^4 + 2x^2$

$y2 = 4x^3 + 4x$

and sketch the graphs.

The graphs should be similar to that shown below.



b At a stationary point of inflection, the tangent to the curve is horizontal. To test whether the origin is a stationary point of inflection on the graph $y2$, use the Analysis tools to form the tangent to the curve at the origin. The tangent has a gradient of 4, not zero, and therefore the origin is not a stationary point of inflection on the derivative graph.

c If $\dfrac{dy}{dx} = y$, then the points of intersections of the graph $y1$ and the derivative graph $y2$ will give the x values for the solution of the equation. From the diagram, it can be seen that the graphs intersect at the origin. In the main menu, solving $x^4 + 2x^2 = 4x^3 + 4x$ gives a second non-zero solution of $x = 3.7511$ (correct to 4 decimal places).

d The graphs of $y = x^4 + 2x^2$ and its derivative will be parallel when the gradients of their tangents are the same.

For $y1 = x^4 + 2x^2$, the gradient of the tangent is $4x^3 + 4x$
For $y2 = 4x^3 + 4x$, the gradient of the tangent is $12x^2 + 4$
Solve $4x^3 + 4x = 12x^2 + 4$ to obtain $x = 2.7\,692\,924$
The graphs are parallel at the point where $x = 2.77$ (correct to 2 decimal places).

11.6 Exam questions

1 $y = x^3 - x$
Point D
$x = 3 + h$
$y = (3+h)^3 - (3+h)$
$y = 3^3 + 3(3)^2 h + 3(3)h^2 + h^3 - (3+h)$
$y = 27 + 27h + 9h^2 + h^3 - 3 - h$
$y = h^3 + 9h^2 + 26h + 24$
The correct answer is **C**.

2 At point $x = 2.9$,

$$\begin{aligned} y &= \frac{x^3}{3} + 1 \\ &= \frac{(2.9)^3}{3} + 1 \\ &= 9.1297 \end{aligned}$$

$\therefore (2.9, 19.1297)$ [1 mark]

At point $x = 3$,

$$\begin{aligned} y &= \frac{3^3}{3} + 1 \\ &= 10 \end{aligned}$$

$\therefore (3, 10)$

$$\begin{aligned} \text{Gradient fo chord} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{10 - 9.1297}{3 - 2.9} \\ &= 8.703 \end{aligned}$$ [1 mark]

Estimated gradient of the tangent to the curve = 9 [1 mark]
Improve the estimate by using a point closer to 3 such as $x = 2.99$. [1 mark]

3 The gradient of a secant or the average rate of change of the function between the two points is calculated using

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 - 4 - \left(2x^2 - 4\right)}{h} \\ &= \frac{2x^2 + 4xh + 2h^2 - 4 - 2x^2 + 4}{h} \\ &= \frac{4xh + 2h^2}{h} \\ &= \frac{\cancel{h}(4x + 2h)}{\cancel{h}}, \ h \neq 0 \\ &= 2(2x + h) \end{aligned}$$

The correct answer is **D**.

4 $f(x) = \dfrac{4x^3 - 3x + 15}{3}$

$$\begin{aligned} f'(x) &= \frac{12x^2 - 3}{3} \\ &= 4x^2 - 1 \end{aligned}$$

The correct answer is **A**.

5 $f(x) = 2 - x^2$
$f'(x) = -2x$ [1 mark]
Graph $y = 2 - x^2$ and $y = -2x$.

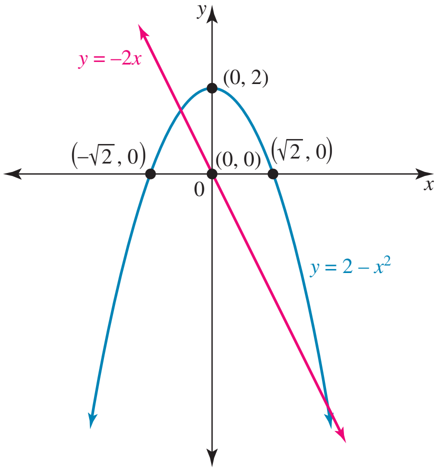
For $y = 2 - x^2$:
x-intercepts ($y = 0$):

$$\begin{aligned} 0 &= 2 - x^2 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

$\therefore \left(-\sqrt{2}, 0\right), \left(\sqrt{2}, 0\right)$

y-intercepts ($x = 0$):
$y = 2$
$\therefore (0, 2)$ [1 mark]

For $y = -2x$:
x- and y-intercepts:
$(0, 0)$
Other point:
$(1, -2)$ [1 mark]



[1 mark]

Topic 12 — Differentiation and applications

12.2 Limits, continuity and differentiability

12.2 Exercise

1 a $\lim\limits_{x\to 3}(x^2+1)$

The function is well behaved at $x = 3$, so the limit can be calculated by substitution.

$\lim\limits_{x\to 3}(x^2+1) = 3^2 + 1 = 10$

b $\lim\limits_{x\to 1}(x+1)(7-2x)$

The function is well behaved at $x = 1$, so the limit can be calculated by substitution.

$$\begin{aligned}\lim_{x\to 1}(x+1)(7-2x) &= (1+1)(7-2\times 1)\\ &= 2\times 5\\ &= 10\end{aligned}$$

c $\lim\limits_{x\to 0}(5x^3+x^2+3x+8)$

The function is well behaved at $x = 0$, so the limit can be calculated by substitution.

$\lim\limits_{x\to 0}(5x^3+x^2+3x+8) = 8.$

d $\lim\limits_{x\to -1}\dfrac{x+6}{x-4}$

The function is well behaved at $x = -1$, so the limit can be calculated by substitution.

$$\begin{aligned}\lim_{x\to -1}\frac{x+6}{x-4} &= \frac{-1+6}{-1-4}\\ &= \frac{5}{-5}\\ &= -1\end{aligned}$$

2 a $\lim\limits_{x\to 0}\dfrac{13x}{x} = \lim\limits_{x\to 0}(13) = 13$

b $\lim\limits_{x\to 0}\dfrac{2x(x+1)}{x}$

$= \lim\limits_{x\to 0}2(x+1)$

$= 2$

c $\lim\limits_{x\to 9}\dfrac{(x-9)(x+11)}{x-9}$

$= \lim\limits_{x\to 9}(x+11)$

$= 20$

d $\lim\limits_{x\to -8}\dfrac{(4x+1)(x+8)}{x+8}$

$= \lim\limits_{x\to -8}(4x+1)$

$= -32+1$

$= -31$

3 a $\lim\limits_{x\to -10}\dfrac{x^2-100}{x+10}$

$= \lim\limits_{x\to -10}\dfrac{(x+10)(x-10)}{x+10}$

$= \lim\limits_{x\to -10}(x-10)$

$= -20$

b $\lim\limits_{x\to 4}\dfrac{5x-x^2}{x}$

$= \lim\limits_{x\to 4}\dfrac{x(5-x)}{x}$

$= \lim\limits_{x\to 4}(5-x)$

$= 1$

c $\lim\limits_{x\to \frac{1}{3}}\dfrac{6x^2-5x+1}{3x-1}$

$= \lim\limits_{x\to \frac{1}{3}}\dfrac{(3x-1)(2x-1)}{3x-1}$

$= \lim\limits_{x\to \frac{1}{3}}(2x-1)$

$= \dfrac{2}{3} - 1$

$= -\dfrac{1}{3}$

d $\lim\limits_{x\to -1}\dfrac{x^3+1}{x^2-1}$

$= \lim\limits_{x\to -1}\dfrac{(x+1)(x^2-x+1)}{(x-1)(x+1)}$

$= \lim\limits_{x\to -1}\dfrac{(x^2-x+1)}{(x-1)}$

$= \dfrac{1-(-1)+1}{-2}$

$= -\dfrac{3}{2}$

4 a
$$\begin{aligned}\lim_{x\to -5}(8-3x) &= 8+15\\ &= 23\end{aligned}$$

b $\lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right)$

$= \lim\limits_{x\to 3}\left(\dfrac{(x-3)(x+3)}{x-3}\right)$

$= \lim\limits_{x\to 3}(x+3)$

$= 6$

c $\lim\limits_{x\to -3}\left(\dfrac{1}{x+3}\right)$ does not exist.

5 a $\lim\limits_{x\to 2}\left(\dfrac{x^3-8}{x-2}\right)$

$= \lim\limits_{x\to 2}\left(\dfrac{(x-2)(x^2+2x+4)}{x-2}\right)$

$= \lim\limits_{x\to 2}\left(x^2+2x+4\right)$

$= 4+4+4$

$= 12$

b $\lim\limits_{x\to 2}\left(\dfrac{1}{x+3}\right)$

$= \dfrac{1}{2+3}$

$= \dfrac{1}{5}$

6 a $\lim\limits_{x\to3}(6x-1) = 6\times3-1 = 17$

b $$\begin{aligned}&\lim_{x\to3}\frac{2x^2-6x}{x-3}\\&=\lim_{x\to3}\frac{2x(x-3)}{x-3}\\&=\lim_{x\to3}(2x)\\&=2\times3\\&=6\end{aligned}$$

c $$\begin{aligned}&\lim_{x\to1}\frac{2x^2+3x-5}{x^2-1}\\&=\lim_{x\to1}\frac{(2x+5)(x-1)}{(x+1)(x-1)}\\&=\lim_{x\to1}\frac{2x+5}{x+1}\\&=\frac{7}{2}\end{aligned}$$

d $$\begin{aligned}&\lim_{x\to0}\frac{3x-5}{2x-1}\\&=\frac{3(0)-5}{2(0)-1}\\&=5\end{aligned}$$

e $$\begin{aligned}&\lim_{x\to-4}\frac{64+x^3}{x+4}\\&=\lim_{x\to-4}\frac{(4+x)\left(16-4x+x^2\right)}{x+4}\\&=\lim_{x\to-4}\left(16-4x+x^2\right)\\&=16+16+16\\&=48\end{aligned}$$

f $$\begin{aligned}&\lim_{x\to\infty}\frac{x+1}{x}\\&=\lim_{x\to\infty}\left(\frac{x}{x}+\frac{1}{x}\right)\\&=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\\&=1+0\\&=1\end{aligned}$$

(The hyperbola graph $y = 1 + \frac{1}{x}$ has an asymptote $y = 1$, so as $x \to \infty, y \to 1$.)

7 a i $\lim\limits_{x\to1} f(x)$

Limit from the left of $x = 1$: $\lim\limits_{x\to1}(x^2) = 1$

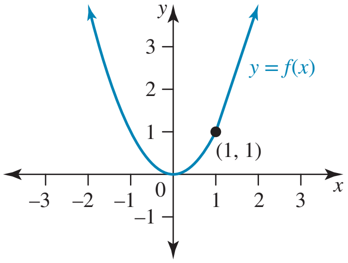
Limit from the right of $x = 1$: $\lim\limits_{x\to1}(2x-1) = 1$

Since $L^- = L^+$, $\lim\limits_{x\to1} f(x) = 1$.

ii $f(1) = 1$

The function is continuous at $x = 1$ since $f(1)$ and $\lim\limits_{x\to1} f(x)$ exist and $f(1) = \lim\limits_{x\to1} f(x)$.

$$f(x)=\begin{cases}x^2, & x<1\\1, & x=1\\2x-1, & x>1\end{cases}$$



b i $g(x) = \dfrac{x^2-x}{x}$

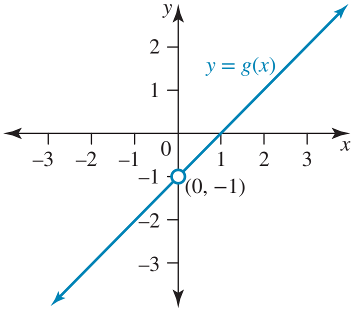
The denominator cannot be zero, so the maximal domain is $R\setminus\{0\}$.

ii $$\begin{aligned}\lim_{x\to0} g(x) &= \lim_{x\to0}\frac{x(x-1)}{x}\\&=\lim_{x\to0}(x-1)\\&=-1\end{aligned}$$

iii The function is not continuous at $x = 0$ because $g(0)$ is not defined.

$g(x) = x-1,\ x \neq 0$

iv Its graph will be the same as $y = x - 1$ with a hole at $(0, -1)$.



8 a i $f(x)=\begin{cases}x^2, & x\le2\\-2x, & x>2\end{cases}$

For $\lim\limits_{x\to2} f(x)$ to exist, the limit from the left of $x = 2$ must equal the limit from the right of $x = 2$.

$$\begin{aligned}L^- &= \lim_{x\to2^-} f(x) & L^+ &= \lim_{x\to2^+} f(x)\\&=\lim_{x\to2}\left(x^2\right) & &=\lim_{x\to2}(-2x)\\&=4 & &=-4\end{aligned}$$

Since $L^- \neq L^+$, $\lim\limits_{x\to2} f(x)$ does not exist.

ii Since $\lim\limits_{x\to2} f(x)$ does not exist for the function, the function is not continuous at $x = 2$.

b i $f(x)=\begin{cases}(x-2)^2, & x<2\\x-2, & x\ge2\end{cases}$

$$\begin{aligned}L^- &= \lim_{x\to2^-} f(x) & L^+ &= \lim_{x\to2^+} f(x)\\&=\lim_{x\to2}\left((x-2)^2\right) & &=\lim_{x\to2}(x-2)\\&=0 & &=0\end{aligned}$$

Since $L^- = L^+ = 0$, $\lim\limits_{x\to2} f(x) = 0$.

ii $\lim\limits_{x\to2} f(x) = 0$.

The value of the function at $x = 2$ is $f(2) = 2 - 2 = 0$.

$\therefore \lim\limits_{x\to2} f(x) = f(2)$.

The function is continuous at $x = 2$.

c i $f(x)=\begin{cases}-x, & x<2\\-2, & x=2\\x-4, & x>2\end{cases}$

$$\begin{aligned} L^- &= \lim_{x\to 2^-} f(x) & L^+ &= \lim_{x\to 2^+} f(x)\\ &= \lim_{x\to 2}(-x) & &= \lim_{x\to 2}(x-4)\\ &= -2 & &= -2 \end{aligned}$$

Since $L^- = L^+ = -2$, $\lim\limits_{x\to 2} f(x) = -2$.

ii $\lim\limits_{x\to 2} f(x) = -2$.

The value of the function at $x = 2$ is $f(2) = -2$.

$\therefore \lim\limits_{x\to 2} f(x) = f(2)$.

The function is continuous at $x = 2$.

d i $$\begin{aligned} f(x) &= \frac{x^2-4}{x-2}\\ &= \lim_{x\to 2} f(x)\\ &= \lim_{x\to 2}\frac{\left(x^2-4\right)}{x-2}\\ &= \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}\\ &= \lim_{x\to 2}(x+2)\\ &= 4 \end{aligned}$$

ii $\lim\limits_{x\to 2} f(x) = 4$.

However, the value of the function at $x = 2$ is not defined.

The function is not continuous at $x = 2$.

9 $$f(x) = \begin{cases} x^2-4, & x < 0\\ 4-x^2, & x \ge 0 \end{cases}$$

$$\begin{aligned} f(0) &= 4 - 0^2\\ &= 4 \end{aligned}$$

$\lim\limits_{x\to 0} f(x)$:

Limit from the left:	Limit from the right:
$L^- = \lim\limits_{x\to 0}(x^2-4)$	$L^+ = \lim\limits_{x\to 0}(4-x^2)$
$= -4$	$= 4$

Since $L^- \ne L^+$, $\lim\limits_{x\to 0} f(x)$ does not exist.

Hence the function is not continuous at $x = 0$. The branches of the function are polynomials, so they are continuous except at their end point at $x = 0$. Therefore, the function is continuous for the domain $R\setminus\{0\}$.

10 a $f(x) = x^2 + 5x + 2$ is a polynomial function, so it is continuous over R.

b $g(x) = \dfrac{4}{x+2}$ is not defined when $x = -2$, so it is not continuous for all $x \in R$.

c $h(x) = \begin{cases} x^2+5x+2, & x < 0\\ 5x+2, & x > 0 \end{cases}$ is not defined when $x = 0$, so it is not continuous for all $x \in R$.

d $$k(x) = \begin{cases} x^2+5x+2, & x < 0\\ \dfrac{4}{x+2}, & x \ge 0 \end{cases}$$

The branch of the function for $x < 0$ is a polynomial $x^2 + 5x + 2$, so it is continuous for all $x < 0$.

The branch of the function for $x \ge 0$ is $\dfrac{4}{x+2}$. Although this is not defined for $x = -2$, the value $x = -2$ is not in the domain for which this rule applies. So, for $x > 0$ it is continuous.

Now we must test whether the function is continuous at $x = 0$.

$$k(0) = \frac{4}{2} = 2$$

$$\lim_{x\to 0}\left(x^2+5x+2\right) = 2$$

$$\lim_{x\to 0}\left(\frac{4}{x+2}\right) = 2$$

$$\therefore \lim_{x\to 0} k(x) = 2 = k(0)$$

The function is continuous at $x = 0$.

Hence, the function is continuous over R.

11 $$y = \begin{cases} x+a, & x < 1\\ 4-x, & x \ge 1 \end{cases}$$

Both branches are linear polynomials, so they are each continuous. For the function to be continuous the two branches need to join at $x = 1$.

When $x = 1$, $y = 4 - 1 = 3$.

For continuity at $x = 1$, $L^- = \lim\limits_{x\to 1}(x + a) = 3$.

$$\begin{aligned} &\therefore 1 + a = 3\\ &\quad \therefore a = 2 \end{aligned}$$

12 $$y = \begin{cases} ax+b, & x < -1\\ 5, & -1 \le x \le 2\\ 2bx+a, & x > 2 \end{cases}$$

For continuity, $\lim\limits_{x\to -1}(ax + b) = 5$ and $\lim\limits_{x\to 2}(2bx + a) = 5$.

Hence,

$-a + b = 5$ [1]

$4b + a = 5$ [2]

Add the two equations together to eliminate a.

$\therefore 5b = 10$

$\therefore b = 2$

Substitute $b = 2$ in equation [1].

$\therefore -a + 2 = 5$

$\therefore a = -3$

The function will be continuous if $a = -3$, $b = 2$.

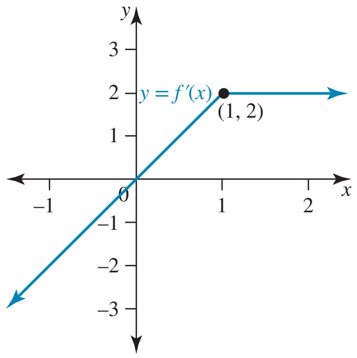
13 a As the function is continuous, test the derivative from the left and right of $x = 1$.

Derivative from the left:	Derivative from the right:
$f(x) = x^2$	$f(x) = 2x - 1$
$\therefore f'(x) = 2x$	$\therefore f'(x) = 2$
$\therefore f'(1) = 2$	$\therefore f'(1) = 2$

Since the derivatives from each side are equal, the function is differentiable at $x = 1$.

b $$f'(x) = \begin{cases} 2x, & x \le 1\\ 2, & x > 1;\ \text{domain is } R \end{cases}$$

The graph of $y = f'(x)$ is continuous.



14 Refer to the diagram given in the question.

a There are breaks in the curve at x_3 and x_5, so the function is not continuous when $x = x_3$ and $x = x_5$. The left-hand limit does not equal the right-hand limit at each place.

b The function cannot be differentiated at a point of discontinuity, so it is not differentiable at x_3 and x_5.

The function must be smoothly continuous to be differentiable. There are sharp points at x_1, x_2 and x_4 where the gradient from immediately left of the point does not equal the gradient immediately to the right. The function cannot be differentiated at these vales of x.

Overall, the function is not differentiable when $x = x_1, x_2, x_3, x_4, x_5$.

15 $f(x) = \begin{cases} 3 - 2x, & x < 0 \\ x^2 + 3, & x \geq 0 \end{cases}$

a First test continuity at $x = 0$.

$$L^- = \lim_{x \to 0}(3 - 2x) = 3$$
$$L^+ = \lim_{x \to 0}(x^2 + 3) = 3$$
$$f(0) = 0^2 + 3 = 3$$
$$\therefore \lim_{x \to 0} f(x) = f(0)$$

The function is continuous at $x = 0$.

Derivative from the left of $x = 0$:

$f(x) = 3 - 2x$

$\therefore f'(x) = -2$

$\therefore f'(0) = -2$

Derivative from the right of $x = 0$:

$f(x) = x^2 + 3$

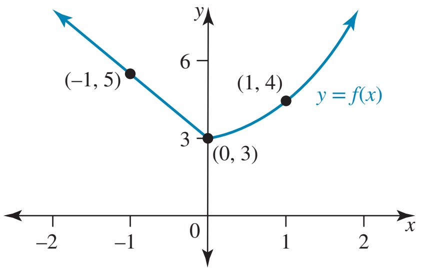
$\therefore f'(x) = 2x$

$\therefore f'(0) = 0$

Derivative from the left does not equal the derivative from the right.

The function is not differentiable at $x = 0$.

b $f(0) = 3, f(-1) = 3 + 2 = 5$ and $f(1) = 1^2 + 3 = 4$

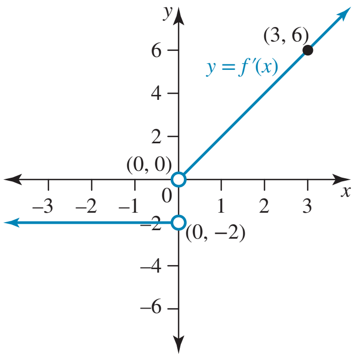


As the function is not differentiable at $x = 0$, it is not smoothly continuous at $x = 0$. Hence, the two branches do not join smoothly.

c $f'(x) = \begin{cases} -2, & x < 0 \\ 2x, & x > 0 \end{cases}$. The domain of the derivative is $R \setminus \{0\}$.

d $f'(3) = 2 \times 3 = 6$.

e



16 $f(x) = \begin{cases} ax^2, & x \leq 2 \\ 4x + b, & x > 2 \end{cases}$

To be continuous at $x = 2$, the two branches must have the same end point when $x = 2$.

Therefore, $a(2)^2 = 4(2) + b$

$\therefore 4a - b = 8 \quad [1]$

To be smoothly continuous, the derivatives of the two branches must be equal when $x = 2$.

From the left:	From the right:
$f(x) = ax^2$	$f(x) = 4x + b$
$\therefore f'(x) = 2ax$	$\therefore f'(x) = 4$
$\therefore f'(2) = 4a$	$\therefore f'(2) = 4$

Hence, $4a = 4$, giving $a = 1$.

Substitute into equation [1].

$4 - b = 8$

$\therefore b = -4$

Answer: $a = 1, b = -4$

17 **a** $f(x) = \begin{cases} 4x^2 - 5x + 2, & x \leq 1 \\ -x^3 + 3x^2, & x > 1 \end{cases}$

First test if the function is continuous at $x = 1$.

$$\begin{aligned} f(1) &= 4 - 5 + 2 \\ &= 1 \\ L^+ &= \lim_{x \to 1}\left(-x^3 + 3x^2\right) \\ &= -1 + 3 \\ &= 2 \end{aligned}$$

The function is not continuous at $x = 1$. Therefore, it is not differentiable at $x = 1$.

b $f'(x) = \begin{cases} 8x - 5, & x < 1 \\ -3x^2 + 6x, & x > 1 \end{cases}$

c $f'(x) = 0$

For $x < 1$, let $8x - 5 = 0$.

$\therefore x = \frac{5}{8}$

For $x > 1$, let $-3x^2 + 6x = 0$.

$\therefore \; -3x(x - 2) = 0$

$\therefore x = 0$ or $x = 2$

However, $x > 1$, so the only admissible value is $x = 2$.

$f'(x) = 0$ when $x = \frac{5}{8}$ or $x = 2$

d $f'(x) = \begin{cases} 8x - 5, & x < 1 \\ -3x^2 + 6x, & x > 1 \\ \text{not defined}, & x = 1 \end{cases}$

Consider the linear left branch of the gradient function:

$y = 8x - 5$ passes through $(0, -5)$ and $\left(\frac{5}{8}, 0\right)$.

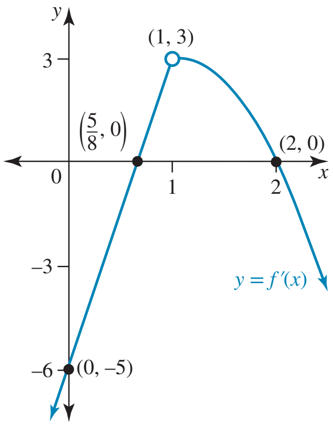
The end point $(1, 3)$ is open.

Consider the quadratic right branch of the gradient function:

$y = -3x^2 + 6x$ passes through $(2, 0)$.

$$\begin{aligned} y &= -3\left[\left(x^2 - 2x + 1\right) - 1\right] \\ &= -3(x-1)^2 + 3 \end{aligned}$$

Maximum turning point $(1, 3)$ is an open end point.



e Since $f'(x) = 0$ when $x = \frac{5}{8}$ or $x = 2$, at these points on $y = f(x)$ the gradient of the tangent is zero.

$$\begin{aligned} f\left(\frac{5}{8}\right) &= 4 \times \left(\frac{5}{8}\right)^2 - 5 \times \frac{5}{8} + 2 \\ &= \frac{25}{16} - \frac{25}{8} + 2 \\ &= \frac{-25}{16} + \frac{32}{16} \\ &= \frac{7}{16} \end{aligned}$$

$$\begin{aligned} f(2) &= -8 + 12 \\ &= 4 \end{aligned}$$

At the points $\left(\frac{5}{8}, \frac{7}{16}\right)$ and $(2, 4)$, the gradient of the tangent is zero.

f Consider the quadratic left branch of the function:

$y = 4x^2 - 5x + 2$ passes through $(0, 2)$.

Its minimum turning point is $\left(\frac{5}{8}, \frac{7}{16}\right)$ since the tangent there has zero gradient.

Its closed end point is $(1, 1)$.

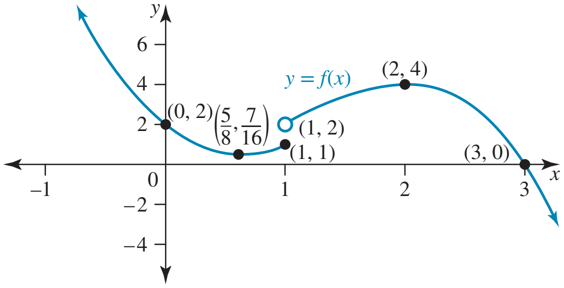
Consider the cubic right branch of the function:

$y = -x^3 + 3x^2$

$\therefore y = -x^2(x - 3)$

The graph passes through $(3, 0)$ and has an open end point at $(1, 2)$.

There is a turning point at $(2, 4)$ since the gradient of the tangent is zero at that point.



18 $y = \begin{cases} ax^2 + b, & x \le 1 \\ 4x, & 1 < x < 2 \\ cx^2 + d, & x \ge 2 \end{cases}$

For continuity at the join of the two branches around $x = 1$,

$a \times 1^2 + b = 4 \times 1$

$\therefore a + b = 4 \quad [1]$

For continuity at the join of the two branches around $x = 2$,

$4 \times 2 = c \times 2^2 + d$

$\therefore 8 = 4c + d \quad [2]$

For the joins to be smooth, the derivatives either side of each join must be equal.

For $x = 1$, $2ax = 4$ when $x = 1$.

$\therefore 2a = 4$

$\therefore a = 2$

For $x = 2$, $4 = 2cx$ when $x = 2$.

$\therefore 4 = 4c$

$\therefore c = 1$

Substitute $a = 2$ in equation [1].

$\therefore 2 + b = 4$

$\therefore b = 2$

Substitute $c = 1$ in equation [2].

$\therefore 8 = 4 + d$

$\therefore d = 4$

For the function to be differentiable over R,

$a = 2,\ b = 2,\ c = 1,\ d = 4.$

19 $f(x) = \begin{cases} x^2, & x < 2 \\ 2^x, & x \ge 2 \end{cases}$

a
$$\begin{aligned} L^- &= \lim_{x \to 2}(x^2) = 4 \\ L^+ &= \lim_{x \to 2}(2^x) = 4 \\ f(2) &= 2^2 \qquad = 4 \\ \therefore \lim_{x \to 2} f(x) &= 2 \qquad = f(2) \end{aligned}$$

Hence, the function is continuous when $x = 2$.

b The derivative from the left of $x = 2$ is

$$\begin{aligned} f'(2^-) &= \lim_{h \to 0} \frac{f(2) - f(2-h)}{2 - (2-h)} \\ &= \lim_{h \to 0} \frac{4 - (2-h)^2}{h} \end{aligned}$$

c The derivative from the right of $x = 2$ is

$$\begin{aligned} f'(2^+) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{2 + h - 2} \\ &= \lim_{h \to 0} \frac{2^{2+h} - 4}{h} \end{aligned}$$

d Using the limit template, $f'(2^-) = 4$ and $f'(2^+) = 4\ln(2)$. Since $4 \ne 4\log_e(2)$, the derivative from the left does not equal the derivative from the right. Hence, the function is not differentiable at $x = 2$.

12.2 Exam questions

1 $f(x) = \dfrac{x^2 - 9}{x - 3}, x \neq 3$

$f(x) = \dfrac{\cancel{(x-3)}\,(x+3)}{\cancel{x-3}}$

$\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} f(x+3)$

$f(3) = 3 + 3$

$= 6$

The correct answer is **C**.

2 $L^- = \lim\limits_{x\to 1^-} f(x)$

$= \lim\limits_{x\to 1}(x+2)$

$L^+ = \lim\limits_{x\to 1} f(x)$

$= \lim\limits_{x\to 1}(v - x)$

Since $L^- = L^+$ to be continuous

$x + 2 = v - x,$ when $x = 1$

$1 + 2 = v - 1$

$v = 4$

The correct answer is **A**.

3 To be continuous at $x = 1$, the two branches must have the same end point when $x = 1$.

$ax^2 + 4 = 6x + b$

$a(1)^2 + 4 = 6\,(1) + b$

$a = 2 + b$ [1] [1 mark]

To be smooth and continuous, the derivatives of the two branches must be equal when $x = 1$.

From the left:	From the right:
$f(x) = ax^2 + 4$	$f(x) = 6x + b$
$\therefore f'(x) = 2ax$	$\therefore f'(x) = 6$
$\therefore f'(1) = 2a$	$\therefore f'(1) = 6$

Hence, $2a = 6$, giving $a = 3$. [1 mark]

Substitute $a = 3$ in equation [1].

$a = 2 + b$

$3 = 2 + b$

$\therefore b = 1$ [1 mark]

$\therefore a = 3, b = 1$

12.3 Coordinate geometry applications of differentiation

12.3 Exercise

1 $y = 5x - \dfrac{1}{3}x^3$

Point: $x = 3$

$y = 15 - 9$

$= 6$

The point is $(3, 6)$.

Gradient: $\dfrac{dy}{dx} = 5 - x^2$

When $x = 3$, $\dfrac{dy}{dx} = -4$, so the gradient is -4.

Equation of tangent:

$y - 6 = -4(x - 3)$

$y = -4x + 18$

2 a $y = 2x^2 - 7x + 3$

$\dfrac{dy}{dx} = 4x - 7$

At $(0, 3)$,

$\dfrac{dy}{dx} = 4 \times 0 - 7$

$= -7$

Equation of tangent:

$y - y_1 = m(x - x_1),\ m = -7,\ (x_1, y_1) = (0, 3)$

$\therefore y - 3 = -7x$

$\therefore y = -7x + 3$

b $y = 5 - 8x - 3x^2$

Point: $(-1, 10)$

Gradient: $\dfrac{dy}{dx} = -8 - 6x$

At $(-1, 10)$, $\dfrac{dy}{dx} = -8 - 6 \times -1 = -2$

Equation of tangent: $y - 10 = -2(x + 1)$

$\therefore y = -2x + 8$

c $y = \dfrac{1}{2}x^3$

Point: $(2, 4)$

Gradient: $\dfrac{dy}{dx} = \dfrac{3}{2}x^2$

At $(2, 4)$, $\dfrac{dy}{dx} = \dfrac{3}{2} \times 2^2 = 6$

Equation of tangent: $y - 4 = 6(x - 2)$

$\therefore y = 6x - 8$

d $y = \dfrac{1}{3}x^3 - 2x^2 + 3x + 5$

Point: $(3, 5)$

Gradient: $\dfrac{dy}{dx} = x^2 - 4x + 3$

At $(3, 5)$, $\dfrac{dy}{dx} = 3^2 - 4 \times 3 + 3 = 0$, so the tangent is horizontal.

Equation of tangent: $y = 5$

3 a $y = x^2$

$\dfrac{dy}{dx} = 2x$

At the point $(3, 9)$, $\dfrac{dy}{dx} = 2(3) = 6$.

Equation of tangent:

$y - y_1 = m(x - x_1),\ m = 6,\ (x_1, y_1) = (3, 9)$

$\therefore y - 9 = 6(x - 3)$

$\therefore y - 9 = 6x - 18$

$\therefore y = 6x - 9$

b $y = -3x^2$

$\dfrac{dy}{dx} = -6x$

At the point $(1, -3)$, $\dfrac{dy}{dx} = -6(1) = -6$.

Equation of tangent:

$y - y_1 = m(x - x_1),\ m = -6,\ (x_1, y_1) = (1, -3)$

$\therefore y + 3 = -6(x - 1)$

$\therefore y + 3 = -6x + 6$

$\therefore y = -6x + 3$

c $y = x^2 + 2x + 5$

y-intercept: when $x = 0$, $y = 5$. The point is $(0, 5)$.

$\dfrac{dy}{dx} = 2x + 2$

At the point $(0, 5)$, $\frac{dy}{dx} = 2(0) + 2 = 2$.

Equation of tangent: Point $(0, 5)$, $m = 2$

$y = mx + c$

$\therefore y = 2x + 5$

d $y = 7 - 4x^2$

Point: when $x = -\frac{1}{2}$,

$$\begin{aligned} y &= 7 - 4x^2 \\ &= 7 - 4\left(-\frac{1}{2}\right)^2 \\ &= 7 - 4 \times \frac{1}{4} \\ &= 6 \end{aligned}$$

The point is $\left(-\frac{1}{2}, 6\right)$.

Gradient:

$y = 7 - 4x^2$

$\frac{dy}{dx} = -8x$

At the point $\left(-\frac{1}{2}, 6\right)$, $\frac{dy}{dx} = -8\left(-\frac{1}{2}\right) = 4$.

Equation of tangent:

$y - y_1 = m(x - x_1),\ m = 4,\ (x_1, y_1) = \left(-\frac{1}{2}, 6\right)$

$\therefore y - 6 = 4\left(x + \frac{1}{2}\right)$

$\therefore y - 6 = 4x + 2$

$\therefore y = 4x + 8$

e $y = \frac{4}{3}x^3$.

Point: when $x = -\frac{3}{2}$,

$$\begin{aligned} y &= \frac{4}{3} \times \left(-\frac{3}{2}\right)^3 \\ &= \frac{4}{3} \times \left(-\frac{27}{8}\right) \\ &= -\frac{9}{2} \end{aligned}$$

The point is $\left(-\frac{3}{2}, -\frac{9}{2}\right)$.

Gradient:

$$\begin{aligned} \frac{dy}{dx} &= \frac{4}{3} \times 3x^2 \\ &= 4x^2 \end{aligned}$$

At the point $\left(-\frac{3}{2}, -\frac{9}{2}\right)$,

$$\begin{aligned} \frac{dy}{dx} &= 4\left(-\frac{3}{2}\right)^2 \\ &= 4 \times \frac{9}{4} \\ &= 9 \end{aligned}$$

Equation of the tangent:

$y - y_1 = m(x - x_1),\ m = 9,\ (x_1, y_1) = \left(-\frac{3}{2}, -\frac{9}{2}\right)$

$\therefore y + \frac{9}{2} = 9\left(x + \frac{3}{2}\right)$

$\therefore y + \frac{9}{2} = 9x + \frac{27}{2}$

$\therefore y = 9x + \frac{27}{2} - \frac{9}{2}$

$\therefore y = 9x + 9$

f $y = -x^3 + 8$

x-intercept: let $y = 0$.

$0 = -x^3 + 8$

$\therefore x^3 = 8$

$\therefore x = 2$

The point is $(2, 0)$.

Gradient:

$y = -x^3 + 8$

$\frac{dy}{dx} = -3x^2$

At the point $(2, 0)$, $\frac{dy}{dx} = -3(2)^2 = -12$.

Equation of tangent:

$y - y_1 = m(x - x_1),\ m = -12,\ (x_1, y_1) = (2, 0)$

$\therefore y - 0 = -12(x - 2)$

$\therefore y = -12x + 24$

4 a i $y = x^2 - 3x + 9$

$\frac{dy}{dx} = 2x - 3$

Let $\frac{dy}{dx} = 3$.

$\therefore 2x - 3 = 3$

$\therefore 2x = 6$

$\therefore x = 3$

When $x = 3$, $y = (3)^2 - 3(3) + 9 = 9$.

The point is $(3, 9)$.

ii Equation of the tangent:

$y - y_1 = m(x - x_1),\ m = 3,\ (x_1, y_1) = (3, 9)$

$\therefore y - 9 = 3(x - 3)$

$\therefore y - 9 = 3x - 9$

$\therefore y = 3x$

b i $y = 6x(x + 2)$

$y = 6x^2 + 12x$

$\frac{dy}{dx} = 12x + 12$

Let $\frac{dy}{dx} = 0$.

$\therefore 12x + 12 = 0$

$\therefore 12x = -12$

$\therefore x = -1$

When $x = -1$, $y = 6(-1)(-1 + 2) = -6$.

The point is $(-1, -6)$.

ii Equation of the tangent:

As the gradient of the tangent is zero, the tangent is the horizontal line through the point $(-1, -6)$. Its equation is $y = -6$.

c $y = x^2 + x$

The tangent has the same gradient as the line $y = 5x$. Its gradient is 5.

$\frac{dy}{dx} = 2x + 1$

Let $\frac{dy}{dx}=5$.

$\therefore 2x+1=5$

$\therefore 2x=4$

$\therefore x=2$

When $x=2$, $y=(2)^2+2=6$.

The point of tangency is $(2, 6)$. and the gradient of the tangent is 5.

The equation of the tangent is

$y-6=5(x-2)$

$\therefore y-6=5x-10$

$\therefore y=5x-4$

d $y=x^3-4x$

The tangent has the same gradient as the line $y=-4x+21$. Its gradient is -4.

$\frac{dy}{dx}=3x^2-4$

Let $\frac{dy}{dx}=-4$.

$\therefore 3x^2-4=-4$

$\therefore 3x^2=0$

$\therefore x=0$

When $x=0$, $y=(0)^3-4(0)=0$.

The point of tangency is $(0, 0)$ and the gradient of the tangent is -4.

The equation of the tangent is $y=-4x$.

5 $y=4x^2+3$

Gradient: as the tangent is parallel to $y=-8x$, the gradient is -8.

Point:

$\frac{dy}{dx}=8x$

As the tangent has gradient -8,

$8x=-8$

$x=-1$

Substitute $x=-1$ into equation of curve.

$y=4(-1)^2+3$

$=7$

The point is $(-1, 7)$.

Equation of tangent:

$y-7=-8(x+1)$

$y=-8x-1$

6 $y=x^2-6x+3$

a As the tangent is parallel to $y=4x-2$, the gradient of the tangent is 4.

$\frac{dy}{dx}=2x-6$

Let $\frac{dy}{dx}=4$.

$\therefore 2x-6=4$

$\therefore 2x=10$

$\therefore x=5$

When $x=5$, $y=25-30+3=-2$

The point is $(5, -2)$.

The equation of the tangent is

$y+2=4(x-5)$

$\therefore y=4x-22$

b If the tangent is parallel to the x-axis, its gradient is zero.

Let $\frac{dy}{dx}=0$.

$\therefore 2x-6=0$

$\therefore x=3$

When $x=3$, $y=9-18+3=-6$

The tangent is the horizontal line through the point $(3, -6)$. Its equation is $y=-6$.

c The line $6y+3x-1=0$ has gradient $-\frac{3}{6}=-\frac{1}{2}$.

The tangent is perpendicular to this line, so the gradient of the tangent is 2.

Let $\frac{dy}{dx}=2$.

$\therefore 2x-6=2$

$\therefore 2x=8$

$\therefore x=4$

When $x=4$, $y=16-24+3=-5$

The point is $(4, -5)$.

Equation of tangent:

$y+5=2(x-4)$

$\therefore y=2x-13$

7 $y=x^2-2x+5$

a Point: when $x=2$, $y=4-4+5=5$

The point is $(2, 5)$.

Gradient: $y=x^2-2x+5$

$\frac{dy}{dx}=2x-2$

at $(2, 5)$, $\frac{dy}{dx}=2\times2-2$

$=2$

Equation of tangent: $y-5=2(x-2)$

$\therefore y=2x+1$

b The gradient of the line perpendicular to the tangent is $\frac{-1}{2}$.

Equation of this line: $y-5=\frac{-1}{2}(x-2)$.

$\therefore y=\frac{-1}{2}x+6$

c The tangent has gradient $m=\tan(135°)$

$\therefore m=-1$

Let $\frac{dy}{dx}=-1$

$\therefore 2x-2=-1$

$x=\frac{1}{2}$

When $x=\frac{1}{2}$, $y=\frac{17}{4}$

Therefore, the tangent is drawn to the curve at the point $\left(\frac{1}{2}, \frac{17}{4}\right)$. It has gradient -1.

The equation of the tangent is

$y-\frac{17}{4}=-1\left(x-\frac{1}{2}\right)$

$\therefore y=-x+\frac{19}{4}$

8 a $y = x(4 - x)$

The x-intercepts are $(0, 0)$ and $(4, 0)$.

$y = 4x - x^2$

$\therefore \dfrac{dy}{dx} = 4 - 2x$

At $(0, 0)$, $\dfrac{dy}{dx} = 4$ and at $(4, 0)$, $\dfrac{dy}{dx} = -4$.

Equation of the tangent at $(0, 0)$ is $y = 4x$.

Equation of the tangent at $(4, 0)$ is $y = -4(x - 4)$

$\therefore y = -4x + 16$

At the intersection of the tangent lines $y = 4x$ and $y = -4x + 16$,

$4x = -4x + 16$

$\therefore 8x = 16$

$\therefore x = 2$

Substitute $x = 2$ in $y = 4x$

$\therefore y = 8$

The tangents intersect at $(2, 8)$.

b For the curve $y = x(4 - x)$, the point A has coordinates $(4, 0)$.

The line through A perpendicular to the tangent has gradient $\dfrac{1}{4}$, and its equation is

$y = \dfrac{1}{4}(x - 4)$

$\therefore y = \dfrac{1}{4}x - 1$

This line intersects $y = x(4 - x)$ when $\dfrac{1}{4}x - 1 = x(4 - x)$.

$\therefore x - 4 = 4x(4 - x)$

$\therefore x - 4 = 16x - 4x^2$

$\therefore 4x^2 - 15x - 4 = 0$

$\therefore (4x + 1)(x - 4) = 0$

$\therefore x = -\dfrac{1}{4}$ or $x = 4$

$x = 4$ is point A, so $x = -\dfrac{1}{4}$ gives the x-coordinate of the point where $y = \dfrac{1}{4}x - 1$ again meets the parabola.

When $x = -\dfrac{1}{4}$, $y = -\dfrac{1}{16} - 1 = -\dfrac{17}{16}$

The point is $\left(-\dfrac{1}{4}, -\dfrac{17}{16}\right)$.

9 $y = (x - a)(x - b)$

x-intercepts are $(a, 0)$ and $(b, 0)$.

$y = x^2 - ax - bx + ab$

$\therefore \dfrac{dy}{dx} = 2x - a - b$

At $(a, 0)$,

$\dfrac{dy}{dx} = 2a - a - b$

$= a - b$

The equation of the tangent is $y = (a - b)(x - a)$.

At $(b, 0)$,

$\dfrac{dy}{dx} = 2b - a - b$

$= b - a$

The equation of the tangent is $y = (b - a)(x - b)$.

At the intersection of the tangents,

$(a - b)(x - a) = (b - a)(x - b)$.

For two tangents to exist, $a \neq b$.

Dividing by $(a - b)$ gives

$x - a = -(x - b)$

$\therefore x - a = -x + b$

$\therefore 2x = a + b$

$\therefore x = \dfrac{a + b}{2}$

The axis of symmetry of a parabola lies midway between its x intercepts.

Therefore, the equation of the axis of symmetry is $x = \dfrac{a + b}{2}$, which is the same as the x-coordinate of the point of intersection of the tangents drawn at the x-intercepts. Therefore, the tangents intersect at some point on the parabola's axis of symmetry.

10 a The coordinates of P are $\left(t, t^2 + 1\right)$, $t > 0$ and O is the origin.

$m_{OP} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{\left(t^2 + 1\right) - 0}{t - 0}$

$= \dfrac{t^2 + 1}{t}$

b The curve has equation $y = x^2 + 1$.

$\dfrac{dy}{dx} = 2x$

At the point P $\left(t, t^2 + 1\right)$, $t > 0$, $\dfrac{dy}{dx} = 2t$.

c The expressions for the gradient in parts **a** and **b** must be equal.

$\therefore 2t = \dfrac{t^2 + 1}{t}$

$\therefore 2t^2 = t^2 + 1$

$\therefore t^2 = 1$

$\therefore t = \pm 1$

Since $t > 0$, $t = 1$.

The coordinates of P are $\left(t, t^2 + 1\right) = (1, 2)$ since $t = 1$.

d Equation of tangent: point $(1, 2)$ and gradient $= 2(1) = 2$.

$\therefore y - 2 = 2(x - 1)$

$\therefore y = 2x$

11 a $y = x^2$ so $\dfrac{dy}{dx} = 2x$

When $x = -1$, $\dfrac{dy}{dx} = -2$

The line perpendicular to the tangent has a gradient of $\dfrac{1}{2}$ and a point $(-1, 1)$.

Its equation is:

$y - 1 = \dfrac{1}{2}(x + 1)$

$y = \dfrac{1}{2}x + \dfrac{3}{2}$

At Q,

$x^2 = \dfrac{1}{2}x + \dfrac{3}{2}$

$$2x^2 - x - 3 = 0$$
$$(2x-3)(x+1) = 0$$
$$x = \frac{3}{2}, x = -1$$

At Q, $x = \frac{3}{2}$ and therefore $y = \frac{9}{4}$, so Q is the point $\left(\frac{3}{2}, \frac{9}{4}\right)$.

b The line perpendicular to the tangent has a gradient of $\frac{1}{2}$.

Therefore, the required acute angle satisfies $\tan\theta = \frac{1}{2}$.

$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$
$$\approx 26.6°$$

To 1 decimal place, the angle of inclination with the x-axis is 26.6°.

12 a $y = x^3 + 3x^2$

$$\frac{dy}{dx} = 3x^2 + 6x$$

When $x = 1$: $\frac{dy}{dx} = 3 + 6 = 9$

Point A: $x = 1, y = 4$ giving point $A = (1, 4)$

Equation of tangent at A $(1, 4)$, gradient $m = 9$

$$y - 4 = 9(x - 1)$$
$$y - 4 = 9x - 9$$
$$y = 9x - 5$$

Solve simultaneously to find point B.

$y = x^3 + 3x^2$ (1)

$y = 9x - 5$ (2)

$$x^3 + 3x^2 = 9x - 5$$
$$x^3 + 3x^2 - 9x + 5 = 0$$

Since the tangent is at $x = 1$, $(x-1)^2$ is a factor, so the cubic equation becomes

$$(x-1)^2(x+5) = 0$$
$$\therefore x = 1 \text{ or } x = -5$$

Point B: $x = -5$

$$y = (-5)^3 + 3(-5)^2$$
$$y = -125 + 75$$
$$y = -50$$

Point B $(-5, -50)$

b Midpoint, M, of AB

$$M = \left(\frac{1+-5}{2}, \frac{4+-50}{2}\right)$$
$$M = \left(\frac{-4}{2}, \frac{-46}{2}\right)$$

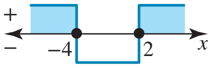
Coordinates of the midpoint are: $(-2, -23)$

13 a $f(x) = \frac{1}{3}x^3 + x^2 - 8x + 6$

$f'(x) = x^2 + 2x - 8$

Functions increasing when $f'(x) > 0$, so:

$$x^2 + 2x - 8 > 0$$
$$(x+4)(x-2) > 0$$
$$x < -4 \text{ or } x > 2$$



The interval over which the function is increasing is $(-\infty, -4) \cup (2, \infty)$.

b i $y = ax^2 + 4x + 5$

Point: $x = 1, y = a + 9 \Rightarrow (1, a+9)$

Gradient: $\frac{dy}{dx} = 2ax + 4$

When $x = 1$, $\frac{dy}{dx} = 2a + 4$.

Equation of tangent:

$$y - (a+9) = (2a+4)(x-1)$$
$$y = (2a+4)x - 2a - 4 + a + 9$$
$$= (2a+4)x + 5 - a$$

ii If the function is decreasing, the gradient of the tangent is negative.

Hence, $2a + 4 < 0$.

$\therefore a < -2$

14 a $f(x) = 3 - 7x + 4x^2$

$f'(x) = -7 + 8x$

The function is decreasing when $f'(x) < 0$.

$$\therefore -7 + 8x < 0$$
$$\therefore 8x < 7$$
$$\therefore x < \frac{7}{8}$$

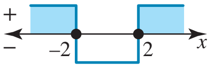
The function is decreasing over the interval $x \in \left(-\infty, \frac{7}{8}\right)$.

b $y = -12x + x^3$

$$\frac{dy}{dx} = -12 + 3x^2$$

The function is increasing when $\frac{dy}{dx} > 0$.

$$\therefore -12 + 3x^2 > 0$$
$$\therefore x^2 - 4 > 0$$
$$\therefore (x+2)(x-2) > 0$$



$\therefore x < -2$ or $x > 2$

The function is increasing over the interval $x \in (-\infty, -2) \cup (2, \infty)$.

c i $y = x^3 + 3x + 5$

$$\frac{dy}{dx} = 3x^2 + 3$$

Since $x^2 \geq 0$, $3x^2 + 3 \geq 3$.

$\therefore \frac{dy}{dx} > 0$ for any value of $x \in R$.

Therefore, any tangent will have a positive gradient.

ii When $x = -1$, $y = -1 - 3 + 5 = 1$.

At the point $(-1, 1)$, $\frac{dy}{dx} = 3 + 3 = 6$.

The equation of the tangent is

$$y - 1 = 6(x+1)$$
$$\therefore y = 6x + 7$$

15 a $f(x) = x^3 + 3$

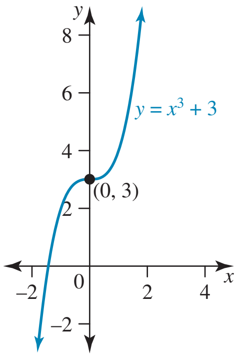
$f'(x) = 3x^2$

Increasing function when $f'(x) > 0$

$3x^2 > 0$ for all $x, x \neq 0$

Interval: $x \in R \backslash \{0\}$

b At $x = 0$, the function is stationary at the point $(0, 3)$.



16 $\frac{1}{3}x^3 + 7x - 3 = 0,\ x = 2$

Let $f(x) = \frac{1}{3}x^3 + 7x - 3$.

$=> f'(x) = x^2 + 7$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Let $x_0 = 2$.

$$\begin{aligned} f(2) &= \frac{1}{3}(2)^3 + 7(2) - 3 \\ &= \frac{8}{3} + 11 \\ &= \frac{41}{3} \end{aligned}$$

$$\begin{aligned} f'(2) &= 2^2 + 7 \\ &= 11 \end{aligned}$$

$$\begin{aligned} x_1 &= 2 - \frac{\frac{41}{3}}{11} \\ &= \frac{25}{33} \end{aligned}$$

$$\begin{aligned} f\left(\frac{25}{33}\right) &= \frac{1}{3}\left(\frac{25}{33}\right)^3 + 7\left(\frac{25}{33}\right) - 3 \\ &= 2.45 \end{aligned}$$

$$\begin{aligned} f'\left(\frac{25}{33}\right) &= \left(\frac{25}{33}\right)^2 + 7 \\ &= 7.57 \end{aligned}$$

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= \frac{25}{33} - \frac{2.45}{7.57} \\ &= 0.4339 \end{aligned}$$

Similarly, $x_3 = 0.4339$ (correct to 4 decimal places).

17 **a** Let $f(x) = -x^3 + x^2 - 3x + 5$.

$f(1) = -1 + 1 - 3 + 5 = 2 > 0$

$f(2) = -8 + 4 - 6 + 5 = -5 < 0$

Since the polynomial changes sign between $x = 1$ and $x = 2$, there is a root that lies between $x = 1$ and $x = 2$.

b Let $f(x) = -x^3 + x^2 - 3x + 5$.

$=> f'(x) = -3x^2 + 2x - 3$

Let $x_0 = 1$.

$$\begin{aligned} f(1) &= -(1)^3 + (1)^2 - 3(1) + 5 \\ &= 2 \end{aligned}$$

$$\begin{aligned} f'(1) &= -3(1)^2 + 2(1) - 3 \\ &= -4 \end{aligned}$$

$$\begin{aligned} \Rightarrow x_1 &= 1 - \frac{2}{-4} \\ &= 1.5 \end{aligned}$$

$$\begin{aligned} f(1.5) &= -(1.5)^3 + (1.5)^2 - 3(1.5) + 5 \\ &= -0.625 \end{aligned}$$

$$\begin{aligned} f'(1.5) &= -3(1.5)^2 + 2(1.5) - 3 \\ &= -6.75 \end{aligned}$$

$$\begin{aligned} \Rightarrow x_2 &= 1.5 - \frac{-0.625}{-6.75} \\ &= 1.4074 \end{aligned}$$

$$\begin{aligned} f(1.41) &= -(1.41)^3 + (1.41)^2 - 3(1.41) + 5 \\ &= -0.0451 \end{aligned}$$

$$\begin{aligned} f'(1.41) &= -3(1.41)^2 + 2(1.41) - 3 \\ &= -6.1443 \end{aligned}$$

$$\begin{aligned} \Rightarrow x_3 &= 1.41 - \frac{-0.0451}{-6.1443} \\ &= 1.4026 \end{aligned}$$

Similarly, $x_4 \simeq 1.4026$.

18 **a** $x^3 + x - 4 = 0$

Let $f(x) = x^3 + x - 4$.

$\therefore f'(x) = 3x^2 + 1$

Let $x_0 = 1$.

$$\begin{aligned} \therefore x_1 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{\left(1^3 + 1 - 4\right)}{\left(3 \times 1^2 + 1\right)} \end{aligned}$$

$\therefore x_1 = 1.5$

Continuing the iteration using

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= Ans - \frac{\left(Ans^3 + Ans - 4\right)}{\left(3 \times Ans^2 + 1\right)} \end{aligned}$$

gives the following values:

$x_2 = 1.387\,096\,774$

$x_3 = 1.378\,838\,948$

$x_4 = 1.378\,796\,701$

$x_5 = 1.378\,796\,7$

Correct to 4 decimal places, the solution is $x = 1.3788$.

b $x^3 - 8x - 10 = 0$

i Let $f(x) = x^3 - 8x - 10$.

$$\begin{aligned} f(1) &= 1 - 8 - 10 \\ &= -17 \\ &< 0 \end{aligned}$$

$$\begin{aligned} f(2) &= 8 - 16 - 10 \\ &= -18 \\ &< 0 \\ f(3) &= 27 - 24 - 10 \\ &= -7 \\ &< 0 \\ f(4) &= 64 - 32 - 10 \\ &= 22 \\ &> 0 \end{aligned}$$

There is a sign change between $x = 3$ and $x = 4$, so the solution to the equation lies between these integers.

ii Let $x_0 = 3$ be a first estimate to the solution of the equation.

$x_1 = 3 - \dfrac{f(3)}{f'(3)}$ where $f(x) = x^3 - 8x - 10$ and

$f'(x) = 3x^2 - 8$

Applying the iteration yields the values

$x_1 = 3.368\,421\,053$

$x_2 = 3.319\,585\,666$

$x_3 = 3.318\,628\,582$

$x_4 = 3.318\,628\,218$

Correct to 4 decimal places, the solution to the equation is $x = 3.3186$.

c $x^3 + 5x^2 = 10$

Rewrite the equation as $x^3 + 5x^2 - 10 = 0$ and define $f(x) = x^3 + 5x^2 - 10$.

$f'(x) = 3x^2 + 10x$

Let $x_0 = -2$.

$x_1 = -2 - \dfrac{f(-2)}{f'(-2)}$

Applying the iteration yields the values

$x_1 = -2.062\,5$

$x_2 = -1.745\,094\,386$

$x_3 = -1.755\,636\,788$

$x_4 = -1.755\,640\,076$

Correct to 4 decimal places, the solution to the equation is $x = -1.7556$.

19 a If $x^3 = 16$ then $x = \sqrt[3]{16}$.

The equation $x^3 - 16 = 0$ has the exact solution $x = \sqrt[3]{16}$.

To apply the Newton–Raphson method, let $f(x) = x^3 - 16$.

$f'(x) = 3x^2$.

As $2^3 = 8$ and $3^3 = 27$, $2 < \sqrt[3]{16} < 3$.

Let $x_0 = 2$.

$x_1 = 2 - \dfrac{f(2)}{f'(2)}$

Applying the iteration yields the values

$x_1 = 2.666\,666\,6...$

$x_2 = 2.527\,777\,7...$

$x_3 = 2.519\,866\,987$

$x_4 = 2.519\,842\,1$

$x_5 = 2.519\,842\,1$

Correct to 4 decimal places, $\sqrt[3]{16} = 2.5198$.

b $x^2 - 8x + 9 = 0$

Completing the square

$$\begin{aligned} \left(x^2 - 8x + 16\right) - 16 + 9 &= 0 \\ \therefore (x - 4)^2 &= 7 \\ \therefore x - 4 &= \pm\sqrt{7} \\ \therefore x &= 4 \pm \sqrt{7} \end{aligned}$$

Consider the larger root $4 + \sqrt{7}$.

Since $4 < 7 < 9$, then $2 < \sqrt{7} < 3$, so $6 < 4 + \sqrt{7} < 7$

Now use the Newton–Raphson method to find the larger root.

Let $f(x) = x^2 - 8x + 9$ and $x_0 = 7$.

$f'(x) = 2x - 8$

$x_1 = 7 - \dfrac{f(7)}{f'(7)}$

Applying the iteration yields the values

$x_1 = 6.666\,666\,6...$

$x_2 = 6.645\,833\,3...$

$x_3 = 6.645\,751\,311$

The root is $x = 6.6458$ correct to 4 decimal places.

$\therefore 4 + \sqrt{7} = 6.6458$

$\therefore \sqrt{7} = 2.6458$

20 $y = x^4 - 2x^3 - 5x^2 + 9x$

a $\dfrac{dy}{dx} = 4x^3 - 6x^2 - 10x + 9$

At $x = 0$, $\dfrac{dy}{dx} = 9 > 0$, so the quartic function is increasing.

At $x = 1$,

$$\begin{aligned} \frac{dy}{dx} &= 4 - 6 - 10 + 9 \\ &= -3 \\ &< 0 \end{aligned}$$

The quartic function is decreasing at $x = 1$.

b For $x \in [0, 1]$, the quartic function changes from increasing to decreasing when $4x^3 - 6x^2 - 10x + 9 = 0$.

To solve this equation using the Newton–Raphson method, let $f(x) = 4x^3 - 6x^2 - 10x + 9$.

$f'(x) = 12x^2 - 12x - 10$

Let $x_0 = 0$.

$x_1 = 0 - \dfrac{f(0)}{f'(0)}$

Applying the iteration yields the values

$x_1 = 0.9$

$x_2 = 0.7345487365$

$x_3 = 0.7347267115$

The root is $x = 0.735$ correct to 3 decimal places.

Therefore, at $x = 0.735$, the quartic function changes from increasing to decreasing.

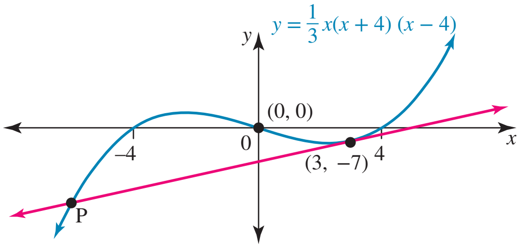
21 a $y = \dfrac{1}{3}x(x + 4)(x - 4)$

x-intercepts occur at $x = 0$, $x = -4$ and $x = 4$ (all cuts).

Point: let $x = 3$.

$$\begin{aligned} \therefore y &= \frac{1}{3} \times 3 \times 7 \times -1 \\ &= -7 \end{aligned}$$

$(3, -7)$



b Point: $(3, -7)$

Gradient:

$$y = \frac{1}{3}x(x^2 - 16)$$
$$= \frac{1}{3}x^3 - \frac{16}{3}x$$
$$\therefore \frac{dy}{dx} = x^2 - \frac{16}{3}$$

At $(3, -7)$,

$$\frac{dy}{dx} = 9 - \frac{16}{3}$$
$$= \frac{11}{3}$$

Equation of tangent:

$$y + 7 = \frac{11}{3}(x - 3)$$
$$\therefore y = \frac{11}{3}x - 11 - 7$$
$$\therefore y = \frac{11}{3}x - 18$$

c i The tangent meets the curve again when

$$\frac{1}{3}x^3 - \frac{16}{3}x = \frac{11}{3}x - 18$$
$$\therefore x^3 - 16x = 11x - 54$$
$$\therefore x^3 - 27x + 54 = 0$$

ii As the tangent line touches the cubic graph at $(3, -7)$, $x = 3$ is a solution of the equation and this solution has multiplicity 2.

$\therefore (x - 3)^2$ is a factor of the equation.

Since $(x - 3)^2 = x^2 - 6x + 9$, then

$x^3 - 27x + 54 = (x^2 - 6x + 9)(x + 6)$

The equation becomes $(x - 3)^2(x + 6) = 0$ with solutions $x = 3, x = -6$.

At P, the tangent cuts the cubic graph, at P, $x = -6$.

When $x = -6$,

$$y = \frac{1}{3} \times -6 \times -2 \times -10$$
$$\therefore y = -40$$

P has coordinates $(-6, -40)$.

d $\frac{dy}{dx} = x^2 - \frac{16}{3}$

When $x = -4$,

$$\frac{dy}{dx} = 16 - \frac{16}{3}$$
$$= \frac{32}{3}$$

When $x = 4$,

$$\frac{dy}{dx} = 16 - \frac{16}{3}$$
$$= \frac{32}{3}$$

The tangents to the curve at $x = \pm 4$ are parallel since they have the same gradient.

e i
$$y = x(x + a)(x - a)$$
$$\therefore y = x\left(x^2 - a^2\right)$$
$$\therefore y = x^3 - a^2x$$
$$\frac{dy}{dx} = 3x^2 - a^2$$

At $x = \pm a$,

$$\frac{dy}{dx} = 3 \times (\pm a)^2 - a^2$$
$$= 3a^2 - a^2$$
$$= 2a^2$$

The tangents have the same gradients and therefore the tangents are parallel.

ii Equation of tangent at $(-a, 0)$

$$y = 2a^2(x + a)$$
$$\therefore y = 2a^2x + 2a^3 \quad [1]$$

Equation of tangent at $(a, 0)$

$$y = 2a^2(x - a)$$
$$\therefore y = 2a^2x - 2a^3 \quad [2]$$

Equation of tangent at $(0, 0)$:

At $(0, 0)$, $\frac{dy}{dx} = -a^2$

Therefore, the tangent has equation $y = -a^2x$ [3]

Intersection of tangents [1] and [3]:

$$2a^2x + 2a^3 = -a^2x$$
$$\therefore 3a^2x + 2a^3 = 0$$
$$\therefore 3a^2x = -2a^3$$
$$\therefore x = -\frac{2a^3}{3a^2}$$
$$\therefore x = -\frac{2a}{3}$$

Substitute $x = -\frac{2a}{3}$ in equation [3].

$$\therefore y = -a^2 \times -\frac{2a}{3}$$
$$\therefore y = \frac{2a^3}{3}$$

The point of intersection is $\left(-\frac{2a}{3}, \frac{2a^3}{3}\right)$.

Intersection of tangents [2] and [3]:

$$2a^2x - 2a^3 = -a^2x$$
$$\therefore 3a^2x - 2a^3 = 0$$
$$\therefore 3a^2x = 2a^3$$
$$\therefore x = \frac{2a^3}{3a^2}$$
$$\therefore x = \frac{2a}{3}$$

Substitute $x = \dfrac{2a}{3}$ in equation [3]

$\therefore y = -a^2 \times \dfrac{2a}{3}$

$\therefore y = -\dfrac{2a^3}{3}$

The point of intersection is $\left(\dfrac{2a}{3}, -\dfrac{2a^3}{3}\right)$.

22 **a** $\dfrac{dy}{dx} = 2x + a$

When $x = -a$, $\dfrac{dy}{dx} = -2a + a = -a$.

When $x = -a$, $y = a^2 - a^2 + 3 = 3$.

The equation of the tangent at point $(-a, 3)$ with gradient $-a$ is

$y - 3 = -a(x + a)$

$\therefore y = -ax + 3 - a^2$

b The tangent has a y-intercept when $x = 0$.

$\therefore y = 3 - a^2$

Given the tangent cuts the y-axis at $y = -6$, then

$-6 = 3 - a^2$

$\therefore a^2 = 9$

$\therefore a = \pm 3$

Also given that the curve $y = x^2 + ax + 3$ is increasing at $x = -a$ then, its gradient at this point must be positive.

$\therefore \dfrac{dy}{dx} > 0$

$\therefore -a > 0$

$\therefore a < 0$

Hence, $a = -3$.

c The family of curves are decreasing where $\dfrac{dy}{dx} < 0$.

$\therefore 2x + a < 0$

$\therefore 2x < -a$

$\therefore x < -\dfrac{a}{2}$

For the domain $\left(-\infty, -\dfrac{a}{2}\right)$, all the curves in the family C are decreasing functions.

d Consider the case $a \neq 0$.

From part **a**, the tangent at $(-a, 3)$ has gradient $-a$. The line perpendicular to the tangent has gradient $\dfrac{1}{a}$.

The equation of the line through $(-a, 3)$ with gradient $\dfrac{1}{a}$ is

$y - 3 = \dfrac{1}{a}(x + a)$

$\therefore y - 3 = \dfrac{x}{a} + 1$

$\therefore y = \dfrac{x}{a} + 4$

When $x = 0$, $y = 4$, so all such lines pass through the point $(0, 4)$ for $a \in R \mid \{0\}$.

Consider the case $a = 0$:

The curve in the family C for which $a = 0$ is $y = x^2 + 3$

The tangent to this curve $(0, 3)$ is horizontal and has the equation $y = 3$. The line perpendicular to $y = 3$ through the point $(0, 3)$ is the vertical line with equation $x = 0$. This line, $\{(x, y) : x = 0\}$ does contain the point $(0, 4)$.

Therefore, the statement will hold if $a = 0$.

12.3 Exam questions

1 $y = 3 + 7x - 2x^2$

$\therefore \dfrac{dy}{dx} = 7 - 4x$

At the point $(2, 3)$,

$\dfrac{dy}{dx} = 7 - 4 \times 2$

$= -1$

For perpendicular lines, $m_1, m_2 = -1$.

$m_2 = -\dfrac{1}{m_1}$

The gradient of the perpendicular is 1.

The correct answer is **A**.

2 $y = x(6 - x)$

$y = 6x - x^2$

$\dfrac{dy}{dx} = 6 - 2x$ [1 mark]

x-intercepts $(y = 0)$:

$0 = x(6 - x)$

$\therefore x = 0, 6$

$\therefore$ points $(0, 0)$ and $(6, 0)$ [1 mark]

Gradient at $(0, 0)$:

$\dfrac{dy}{dx} = 6 - 2 \times 0$

$= 6$

Equation of tangent 1: $m = 6$, point $(0, 0)$

$y = mx + c$

$y = 6x$

Gradient at $(6, 0)$:

$\dfrac{dy}{dx} = 6 - 2 \times 6$

$= -6$

$y = mx + c$

$y = -6x + c$

Substituting $(6, 0)$:

$0 = -36 + c$

$c = 36$

Equation of tangent 2:

$y = -6x + 36$ [1 mark]

Solve the equations simultaneously to point of intersection.

$6x = -6x + 36$

$12x = 36$

$x = 3$

Substitute $x = 3$ into $y = 6x$.

Therefore, the point of intersection (3, 18). [1 mark]

3 $f'(x) = x^2 + 10x + 24$

$f'(x) = (x + 4)(x + 6)$ [1 mark]

For a decreasing function, $f'(x) < 0$.

$\therefore (x + 4)(x + 6) < 0$

Zeros are $x = -6, x = -4$. [1 mark]

Sign diagram of $f'(x)$

$\therefore f'(x) < 0$ for $-6 < x < -4$

The function is decreasing over the interval $x \in (-6, -4)$.

[1 mark]

12.4 Curve sketching

12.4 Exercise

1 **a** $y = 4x - x^2$

$\frac{dy}{dx} = 4 - 2x$

At a stationary point, $\frac{dy}{dx} = 0$.

$\therefore 4 - 2x = 0$

$\therefore x = 2$

When $x = 2$, $y = 4(2) - (2)^2 = 4$.

The stationary point is $(2, 4)$.

b $y = 12x^2 - 24x + 5$

$\frac{dy}{dx} = 24x - 24$

At a stationary point, $\frac{dy}{dx} = 0$.

$\therefore 24x - 24 = 0$

$\therefore x = 1$

When $x = 1$, $y = 12(1)^2 - 24(1) + 5 = -7$.

The stationary point is $(1, -7)$.

c $y = -x(6 + x)$

$y = -6x - x^2$

$\frac{dy}{dx} = -6 - 2x$

At a stationary point, $\frac{dy}{dx} = 0$.

$\therefore -6 - 2x = 0$

$\therefore x = -3$

When $x = -3$, $y = -6(-3) - (-3)^2 = 9$.

The stationary point is $(-3, 9)$.

d $y = x^3 + 1$

$\frac{dy}{dx} = 3x^2$

At a stationary point, $\frac{dy}{dx} = 0$.

$\therefore 3x^2 = 0$

$\therefore x = 0$

When $x = 0$, $y = (0)^3 + 1 = 1$.

The stationary point is $(0, 1)$.

e $y = x^3 - 3x$

$\frac{dy}{dx} = 3x^2 - 3$

At a stationary point, $\frac{dy}{dx} = 0$.

$\therefore 3x^2 - 3 = 0$

$\therefore x^2 = 1$

$\therefore x = \pm 1$

When $x = 1$, $y = (1)^3 - 3(1) = -2$,
and when $x = -1$, $y = (-1)^3 - 3(-1) = 2$.

The stationary points are $(1, -2)$ and $(-1, 2)$.

f $y = 2 - x^4$

$\frac{dy}{dx} = -4x^3$

At a stationary point, $\frac{dy}{dx} = 0$.

$\therefore -4x^3 = 0$

$\therefore x = 0$

When $x = 0$, $y = 2$.

The stationary point is $(0, 2)$.

2 At a local maximum turning point, a function changes from increasing just before the point, to stationary at the point, to decreasing just after the point. Hence, the gradient function changes from positive to zero to negative.
Option B is the correct answer.

3 $f(x) = 2x^3 - 3x^2 - 12x$

a $f'(x) = 6x^2 - 6x - 12$

$\therefore f'(2) = 6 \times 4 - 6 \times 2 - 12$

$\therefore f'(2) = 0$

b $f'(1) = 6 - 6 - 12 = -12$

$f'(3) = 54 - 18 - 12 = 24$

c Since $f'(1) < 0, f'(2) = 0$ and $f'(3) > 0$, the curve changes from decreasing to stationary to increasing about $x = 2$. The stationary point at $x = 2$ is a local minimum turning point.

$f(2) = 2(2)^3 - 3(2)^2 - 12(2)$

$= 16 - 12 - 24$

$= -20$

The minimum turning point is $(2, -20)$.

d Let $f'(x) = 0$ to obtain the other turning point.

$\therefore 6x^2 - 6x - 12 = 0$

$\therefore x^2 - x - 2 = 0$

$\therefore (x - 2)(x + 1) = 0$

$\therefore x = 2$ or $x = -1$

The other stationary point occurs when $x = -1$.

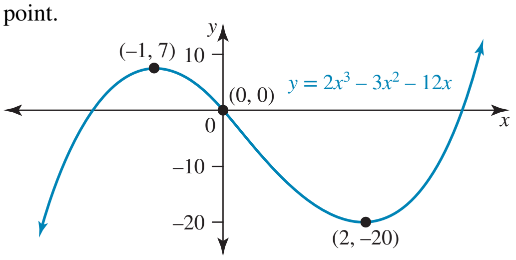
$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1)$

$= -2 - 3 + 12$

$= 7$

The coordinates of the other stationary point are $(-1, 7)$.

e $f(x) = 2x^3 - 3x^2 - 12x$. Since $f(0) = 0$, the cubic graph passes through the origin. The point $(-1, 7)$ is a maximum turning point as the point $(2 - 20)$ is a minimum turning point.



4 **a** $f(x) = x^3 + x^2 - x + 4$

$f'(x) = 3x^2 + 2x - 1$

At stationary points, $f'(x) = 0$:

$3x^2 + 2x - 1 = 0$

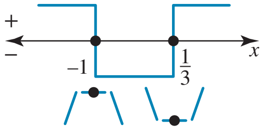
$(3x - 1)(x + 1) = 0$

$x = \frac{1}{3}, \; x = -1$

When $x = \frac{1}{3}$,

$$\begin{aligned} y &= \frac{1}{27} + \frac{1}{9} - \frac{1}{3} + 4 \\ &= \frac{-5}{27} + 4 \\ &= 3\frac{22}{27} \end{aligned}$$

When $x = -1$, $y = 5$.
For type of stationary points, draw a sign diagram of $f'(x)$:



$(-1, 5)$ is a maximum turning point and $\left(\frac{1}{3}, \frac{103}{27}\right)$ is a minimum turning point.

b $y = ax^2 + bx + c$

$(0, 5) \Rightarrow 5 = c$

$\therefore y = ax^2 + bx + 5$

$(2, -14) \Rightarrow -14 = 4a + 2b + 5$

$\therefore 4a + 2b = -19 \qquad [1]$

$\frac{dy}{dx} = 2ax + b$

Since $(2, -14)$ is a stationary point, $4a + b = 0$. [2]

[1] − [2]:

$b = -19$

$\therefore a = \frac{19}{4}$

Hence, $a = \frac{19}{4}$, $b = -19$, $c = 5$.

5 a i $y = x^2 - 8x + 10$

$\frac{dy}{dx} = 2x - 8$

At stationary points, $\frac{dy}{dx} = 0$.

$\therefore 2x - 8 = 0$

$\therefore x = 4$

When $x = 4$, $y = 16 - 32 + 10 = -6$.
The stationary point is $(4, -6)$.

ii $y = -5x^2 + 6x - 12$

$\frac{dy}{dx} = -10x + 6$

At stationary points, $\frac{dy}{dx} = 0$.

$\therefore -10x + 6 = 0$

$\therefore x = 0.6$

When $x = 0.6$, $y = -5 \times 0.36 + 6 \times 0.6 - 12 = -10.2$.
The stationary point is $(0.6, -10.2)$.

b i $y = ax^2 + bx$

$\frac{dy}{dx} = 2ax + b$

$(4, -8)$ is a stationary point.

$\therefore 2a(4) + b = 0$

$\therefore 8a + b = 0 \quad [1]$

$(4, -8)$ lies on the curve.

$\therefore -8 = a(4)^2 + b(4)$

$\therefore 16a + 4b = -8$

$\therefore 4a + b = -2 \quad [2]$

Subtract equation [2] from equation [1]:

$\therefore 4a = 2$

$\therefore a = \frac{1}{2}$

Substitute $a = \frac{1}{2}$ in equation [1]:

$\therefore 4 + b = 0$

$\therefore b = -4$

The answer is $a = \frac{1}{2}$, $b = -4$

ii The equation of the curve is $y = \frac{1}{2}x^2 - 4x$
The minimum turning point is $(4, -8)$.
x-intercepts: let $y = 0$.

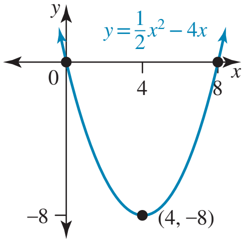
$\therefore \frac{1}{2}x^2 - 4x = 0$

$\therefore x^2 - 8x = 0$

$\therefore x(x - 8) = 0$

$\therefore x = 0, \ x = 8$

$(0, 0)$, $(8, 0)$



6 $f(x) = x^3 + 3x^2 + 8$

a $f'(x) = 3x^2 + 6x$

$f'(-2) = 3 \times 4 + 6 \times -2$

$\therefore f'(-2) = 0$

The function has a stationary point when $x = -2$.

$f(-2) = -8 + 12 + 8 = 12$

Therefore, $(-2, 12)$ is a stationary point of the function.

b The slope of the tangent of the curve around $x = -2$.

x	−3	−2	−1
f′(x)	9	0	−3
Slope	/	—	\

The slope of the tangent shows the point $(-2, 12)$ is a maximum turning point.

c Let $f'(x) = 0$

$\therefore 3x^2 + 6x = 0$

$\therefore 3x(x + 2) = 0$

$\therefore x = 0, x = -2$

Since $f(0) = 8$, the other stationary point is $(0, 8)$.

d Sign diagram of $f'(x) = 3x(x + 2)$

The sign of the gradient changes from negative to positive about $x = 0$, indicating $(0, 8)$ is a minimum turning point.

7 a $y = \frac{1}{3}x^3 + x^2 - 3x - 1$

$\frac{dy}{dx} = x^2 + 2x - 3$

At stationary points, $\frac{dy}{dx} = 0$.

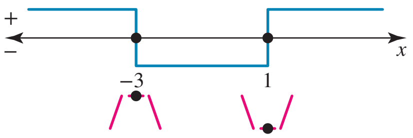
$\therefore x^2 + 2x - 3 = 0$

$\therefore (x+3)(x-1) = 0$

$\therefore x = -3,\ x = 1$

When $x = -3$, $y = \frac{1}{3} \times -27 + 9 + 9 - 1 = 8$ and when $x = 1$, $y = \frac{1}{3} + 1 - 3 - 1 = -\frac{8}{3}$.

Sign of the derivative:



Hence, $(-3,\ 8)$ is a maximum turning point and $\left(1,\ -\frac{8}{3}\right)$ is a minimum turning point.

b $y = -x^3 + 6x^2 - 12x + 18$

$\frac{dy}{dx} = -3x^2 + 12x - 12$

At stationary points, $\frac{dy}{dx} = 0$.

$\therefore\ -3x^2 + 12x - 12 = 0$

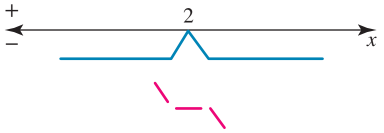
$\therefore\ -3\left(x^2 - 4x + 4\right) = 0$

$\therefore\ -3(x-2)^2 = 0$

$\therefore x = 2$

When $x = 2$, $y = -8 + 24 - 24 + 18 = 10$.

Sign of the derivative:



Hence, $(2,\ 10)$ is a stationary point of inflection.

c $y = \frac{23}{6}x(x-3)(x+3)$

$\therefore y = \frac{23}{6}x(x^2 - 9)$

$\therefore y = \frac{23}{6}\left(x^3 - 9x\right)$

$\frac{dy}{dx} = \frac{23}{6}\left(3x^2 - 9\right)$

$\therefore \frac{dy}{dx} = \frac{23}{2}\left(x^2 - 3\right)$

At stationary points, $\frac{dy}{dx} = 0$

$\therefore \frac{23}{2}(x^2 - 3) = 0$

$\therefore x^2 = 3$

$\therefore x = \pm\sqrt{3}$

Substitute $x = \sqrt{3}$ in $y = \frac{23}{6}\left(x^3 - 9x\right)$.

$\therefore y = \frac{23}{6}\left(3\sqrt{3} - 9\sqrt{3}\right)$

$= \frac{23}{6} \times -6\sqrt{3}$

$= -23\sqrt{3}$

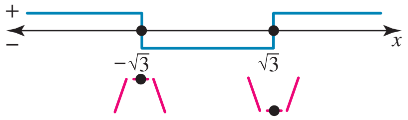
When $x = -\sqrt{3}$,

$y = \frac{23}{6}\left(-3\sqrt{3} + 9\sqrt{3}\right)$

$= \frac{23}{6} \times 6\sqrt{3}$

$= 23\sqrt{3}$

Sign of the derivative:



$\left(-\sqrt{3},\ 23\sqrt{3}\right)$ is a maximum turning point and $\left(\sqrt{3},\ -23\sqrt{3}\right)$ is a minimum turning point.

d $y = 4x^3 + 5x^2 + 7x - 10$

$\frac{dy}{dx} = 12x^2 + 10x + 7$

At stationary points, $\frac{dy}{dx} = 0$.

$\therefore 12x^2 + 10x + 7 = 0$

$\Delta = 100 - 4 \times 12 \times 7$

$= 100 - 336$

< 0

Since $\Delta < 0$, there are no real solutions to $\frac{dy}{dx} = 0$.

Therefore, there are no stationary points.

8 a $y = x^3 + x^2 + ax - 5$

$\frac{dy}{dx} = 3x^2 + 2x + a$

At $x = 1$, $\frac{dy}{dx} = 0$.

$\therefore 3(1)^2 + 2(1) + a = 0$

$\therefore 5 + a = 0$

$\therefore a = -5$

b $y = -x^4 + bx^2 + 2$

$\frac{dy}{dx} = -4x^3 + 2bx$

At $x = -2$, $\frac{dy}{dx} = 0$.

$\therefore\ -4(-2)^3 + 2b(-2) = 0$

$\therefore 32 - 4b = 0$

$\therefore b = 8$

c $y = ax^2 - 4x + c$

$\frac{dy}{dx} = 2ax - 4$

At $(1,\ -1)$, $\frac{dy}{dx} = 0$.

Substitute $x = 1$.

$\therefore 2a(1) - 4 = 0$

$\therefore 2a = 4$

$\therefore a = 2$

Hence, $y = 2x^2 - 4x + c$.

Substitute the point $(1, -1)$.

$-1 = 2(1)^2 - 4(1) + c$

$-1 = -2 + c$

$c = 1$

Answer: $a = 2,\ c = 1$

9 $y = x^3 + ax^2 + bx - 11$

a $\frac{dy}{dx} = 3x^2 + 2ax + b$

At $x = 2$ and $x = 4$, $\frac{dy}{dx} = 0$.

$$\begin{aligned} x = 2 &\Rightarrow 3(2)^2 + 2a(2) + b = 0 \\ \therefore 12 + 4a + b &= 0 \quad [1] \\ x = 4 &\Rightarrow 3(4)^2 + 2a(4) + b = 0 \\ \therefore 48 + 8a + b &= 0 \quad [2] \end{aligned}$$

$[2] - [1]$

$36 + 4a = 0$

$\therefore a = -9$

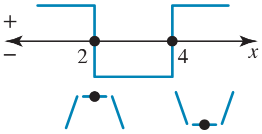
$\therefore b = 24$

b Stationary points $x = 2$ and $x = 4$ mean $(x - 2)(x - 4)$ must be factors of the gradient function.

$y = x^3 - 9x^2 + 24x - 11$

$\frac{dy}{dx} = 3x^2 - 18x + 24$

$= 3(x - 2)(x - 4)$



When $x = 2$,

$y = (2)^3 - 9(2)^2 + 24(2) - 11$

$= 9$

When $x = 4$,

$y = (4)^3 - 9(4)^2 + 24(4) - 11$

$= 5$

Therefore, $(2,\ 9)$ is a maximum turning point and $(4,\ 5)$ is a minimum turning point.

10 $y = 2x^2 - x^3$

$= x^2(2 - x)$

x-intercepts at $x = 2, x = 0$

Stationary points: when $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = 4x - 3x^2$

$= x(4 - 3x)$

$\frac{dy}{dx} = 0 \Rightarrow x = 0, x = \frac{4}{3}$

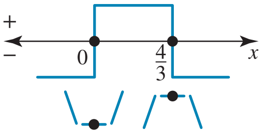
When $x = 0,\ y = 0 \Rightarrow (0,\ 0)$

When $x = \frac{4}{3}$,

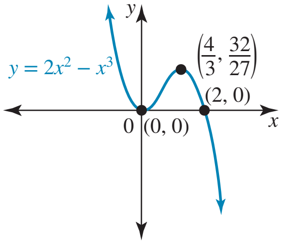
$y = 2 \times \frac{16}{9} - \frac{64}{27}$

$= \frac{32}{27} \Rightarrow \left(\frac{4}{3}, \frac{32}{27}\right)$

Sign diagram of $\frac{dy}{dx}$:



Therefore, $(0, 0)$ is a minimum turning point and $\left(\frac{4}{3}, \frac{32}{27}\right)$ is a maximum turning point.



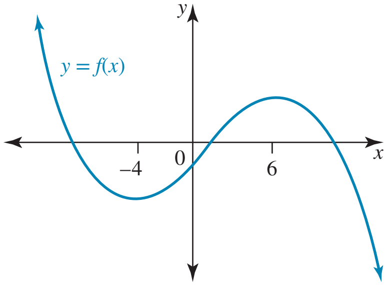
11 **a** $f'(-4) = 0$ and $f'(6) = 0 \Rightarrow$ there are stationary points at $x = -4$ and $x = 6$.

As $f'(x) < 0$ for $x < -4$ and $f'(x) > 0$ for $-4 < x < 6$, the slope of the curve changes from negative to positive about $x = -4$. Therefore, there is a minimum turning point when $x = -4$.

As $f'(x) > 0$ for $-4 < x < 6$ and $f'(x) < 0$ for $x > 6$, the slope of the curve changes from positive to negative about $x = 6$. Therefore, there is a maximum turning point when $x = 6$.

The y-coordinates of these turning points are not known.

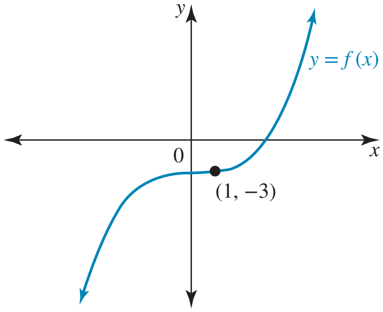
A possible graph for $y = f(x)$ is shown.



b $f'(1) = 0$ and $f(1) = -3 \Rightarrow$ there is a stationary point at $(1, -3)$.

Since $f'(x) > 0$ for both $x < 1$ and $x > 1$, the point $(1, -3)$ is a stationary point of inflection.

A possible graph for $y = f(x)$ is shown.



12 **a** $f : R \to R,\ f(x) = 2x^3 + 6x^2$

x-intercepts: let $f(x) = 0$.

$\therefore 2x^3 + 6x^2 = 0$

$\therefore 2x^2(x + 3) = 0$

$\therefore x = 0, x = -3$

The graph touches the x-axis at $(0, 0)$ and cuts the x-axis at $(-3, 0)$.

Stationary points: $f'(x) = 0$

$\therefore 6x^2 + 12x = 0$

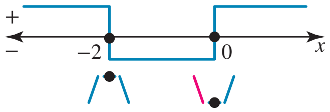
$\therefore 6x(x + 2) = 0$

$\therefore x = 0, x = -2$

When $x = -2$, $y = -16 + 24 = 8$.

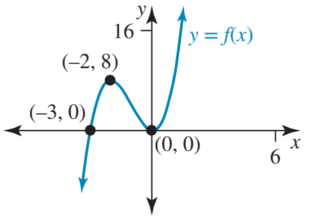
Stationary points $(0, 0)$ and $(-2, 8)$

Sign of the derivative:



Therefore, $(-2, 8)$ is a maximum turning point and $(0, 0)$ is a minimum turning point.

Graph of $y = f(x)$:



b $g : R \to R, g(x) = -x^3 + 4x^2 + 3x - 12$

The y-intercept is $(0, -12)$.

x-intercepts when $-x^3 + 4x^2 + 3x - 12 = 0$

$\therefore\ -x^2(x - 4) + 3(x - 4) = 0$

$\therefore (x - 4)(-x^2 + 3) = 0$

$\therefore x = 4 \text{ or } x^2 = 3$

$\therefore x = 4 \text{ or } x = \pm\sqrt{3}$

$(4, 0), \left(-\sqrt{3}, 0\right), \left(\sqrt{3}, 0\right)$

Stationary points when $g'(x) = 0$

$\therefore\ -3x^2 + 8x + 3 = 0$

$\therefore (-3x - 1)(x - 3) = 0$

$\therefore x = -\frac{1}{3} \text{ or } x = 3$

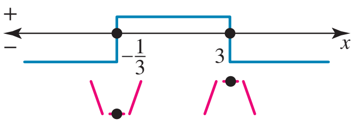
$$g\left(-\frac{1}{3}\right) = \frac{1}{27} + \frac{4}{9} - 1 - 12$$
$$= \frac{13}{27} - 13$$
$$= -12\frac{14}{27}$$

$g(3) = -27 + 36 + 9 - 12$

$= 6$

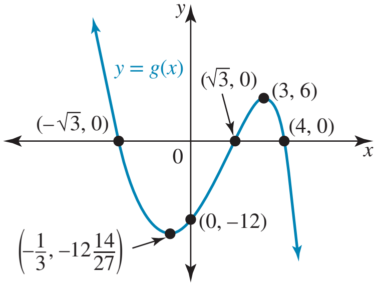
Stationary points are $\left(-\frac{1}{3}, -12\frac{14}{27}\right)$ and $(3, 6)$.

Sign of the derivative:



Therefore, $\left(-\frac{1}{3}, -12\frac{14}{27}\right)$ is a minimum turning point and $(3, 6)$ is a maximum turning point.

Graph of $y = g(x)$:



c $p : [-1, 1] \to R, p(x) = x^3 + 2x$

$p(x) = x^3 + 2x$

$= x\left(x^2 + 2\right)$

Since $x^2 + 2 \neq 0$, there is only one x-intercept at $(0, 0)$.

End points: $p(-1) = -1 - 2 = -3$ $(-1, -3)$ is a closed left end point.

$p(1) = 1 + 2 = 3$ $(1, 3)$ is a closed right end point.

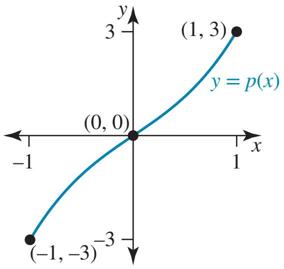
Stationary points occur when $p'(x) = 0$

$\therefore 3x^2 + 2 = 0$

$\therefore x^2 = -\frac{2}{3}$

As there are no real solutions to this equation, there are no stationary points.

$p'(x) > 0$, so the graph of $y = p(x)$ is increasing.



d $\left\{(x, y) : y = x^4 - 6x^2 + 8\right\}$

$(0, 8)$ is the y-intercept.

x-intercepts occur when $x^4 - 6x^2 + 8 = 0$

$\therefore \left(x^2 - 4\right)\left(x^2 - 2\right) = 0$

$\therefore (x + 2)(x - 2)\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right) = 0$

$\therefore x = -2, x = 2, x = -\sqrt{2}, x = \sqrt{2}$

$\left(\pm\sqrt{2}, 0\right), (\pm 2, 0)$

Stationary points occur when $\frac{dy}{dx} = 0$.

$\therefore 4x^3 - 12x = 0$

$\therefore 4x\left(x^2 - 3\right) = 0$

$\therefore 4x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right) = 0$

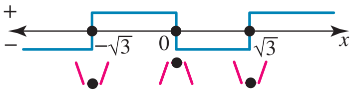
$\therefore x = 0, x = -\sqrt{3}, x = \sqrt{3}$

When $x = -\sqrt{3}$, $y = 9 - 18 + 8 = -1$.

When $x = \sqrt{3}$, $y = 9 - 18 + 8 = -1$.

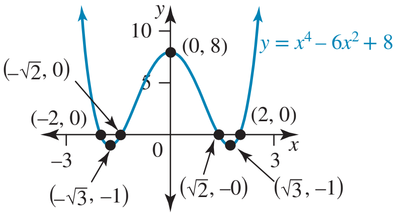
Stationary points are $(0, 8)$, $\left(-\sqrt{3}, -1\right)$ and $\left(\sqrt{3}, -1\right)$

Sign of the derivative:



Therefore, $\left(-\sqrt{3}, -1\right)$ is a minimum turning point, $(0, 8)$ is a maximum turning point and $\left(\sqrt{3}, -1\right)$ is a minimum turning point.

The graph:



13 a The greatest number of turning points a cubic function can have is 2 and the least number is 0.

b $y = 3x^3 + 6x^2 + 4x + 6$

$$\frac{dy}{dx} = 9x^2 + 12x + 4$$

Stationary points occur when $\frac{dy}{dx} = 0$.

$$\therefore 9x^2 + 12x + 4 = 0$$
$$\therefore (3x + 2)^2 = 0$$
$$\therefore x = -\frac{3}{2}$$

There is only one stationary point.

As $\frac{dy}{dx} = (3x + 2)^2$, then $\frac{dy}{dx} > 0$ for $x \in R \backslash \left\{-\frac{3}{2}\right\}$. The stationary point is a stationary point of inflection.

c $y = 3x^3 + 6x^2 + kx + 6$

$$\frac{dy}{dx} = 9x^2 + 12x + k$$

Stationary points occur when $\frac{dy}{dx} = 0$.

For the function to have no stationary points, the quadratic equation $9x^2 + 12x + k = 0$ will have no real solutions.

Therefore, its discriminant must be negative.

$$\Delta = 144 - 36k$$
$$\therefore \Delta < 0 \Rightarrow 144 - 36k < 0$$
$$\therefore 144 < 36k$$
$$\therefore k > \frac{144}{36}$$
$$\therefore k > 4$$

d For a cubic function with a positive coefficient of x^3, as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$. It is not possible for $x \to \infty, y \to \infty$ if there is exactly one stationary point that is a maximum turning point.

For there to be exactly one stationary point, the point must be a stationary point of inflection.

e The gradient function has degree 2.

Suppose a cubic function has one stationary point of inflection at $x = a$ and one maximum turning point at $x = b$. Then $(x - a)^2$ and $(x - b)$ must be factors of the gradient function. However, this would make the gradient function's degree 3, which is not possible.

Therefore, it is not possible for a cubic function to have both a stationary point of inflection and a maximum turning point.

14 a $y = 16x(x + 1)^3$

For the domain $-2 \le x \le 1$

Left end point: when $x = -2$,

$$y = 16(-2)(-2 + 1)^3$$
$$= -32 \times -1$$
$$= 32$$

Right end point: when $x = 1$,

$$y = 16(1)(1 + 1)^3$$
$$= 16 \times 8$$
$$= 128$$

End points are: $(-2, 32), (1, 128)$

b $y = 16x(x + 1)^3$

$$= 16x(x^3 + 3x^2 + 3x + 1)$$
$$= 16(x^4 + 3x^3 + 3x^2 + x)$$

$$\frac{dy}{dx} = 16(4x^3 + 9x^2 + 6x + 1)$$

At a stationary point, $\frac{dy}{dx} = 0$, so

$$16(4x^3 + 9x^2 + 6x + 1) = 0$$
$$4x^3 + 9x^2 + 6x + 1 = 0$$

Solving the cubic equation:

Substitute $x = -1$ into the cubic

$$4(-1)^3 + 9(-1)^2 + 6(-1) + 1 = 0$$

$\therefore (x + 1)$ is a factor

Using long division, the cubic equation becomes

$$(x + 1)(4x^2 + 5x + 1) = 0$$
$$(x + 1)(4x + 1)(x + 1) = 0$$
$$x = -1, -\frac{1}{4}$$

When $x = -1$,

$$y = 16(-1)(-1 + 1)^3$$
$$= 0$$

When $x = -\frac{1}{4}$,

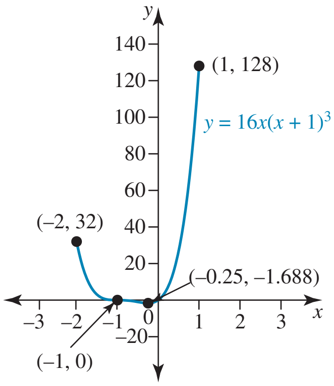
$$y = 16\left(\frac{-1}{4}\right)\left(\frac{-1}{4} + 1\right)^3$$
$$= (-4)\left(\frac{3}{4}\right)^3$$
$$= (-4)\left(\frac{27}{64}\right)$$
$$= -\frac{27}{16}$$

Stationary points: $(-1, 0)$ and $\left(-\frac{1}{4}, -\frac{27}{16}\right)$

See the figure at the bottom of the page.*

The point $(-1, 0)$ is a stationary point of inflection and the point $\left(-\frac{1}{4}, -\frac{27}{16}\right)$ is a minimum turning point.

c The curve cuts the x-axis at $(0, 0)$ and $(-1, 0)$



d The absolute maximum is at the point $(1, 128)$ and the absolute minimum at the turning point $\left(-\frac{1}{4}, -\frac{27}{16}\right)$.

Hence, the absolute maximum value is 128 and the absolute minimum value is $-\frac{27}{16}$

15 $f(x) = 4x^3 - 12x$

End point: when $x = \sqrt{3}, f(x) = 12\sqrt{3} - 12\sqrt{3} \Rightarrow \left(\sqrt{3}, 0\right)$

Stationary points: $f'(x) = 12x^2 - 12$

At stationary points, $f'(x) = 0$, so:

$$12x^2 - 12 = 0$$
$$12(x^2 - 1) = 0$$
$$12(x + 1)(x - 1) = 0$$
$$x = -1, x = 1$$

This is a positive cubic graph, so the first turning point is a maximum and the second is a minimum.

When $x = -1, f(x) = 8 \Rightarrow (-1, 8)$ is a maximum turning point.

When $x = 1, f(x) = -8 \Rightarrow (1, -8)$ is a minimum turning point.

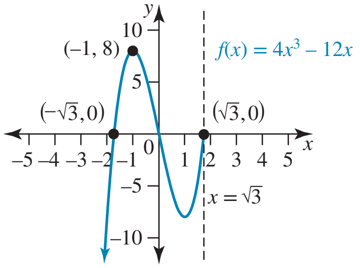
Domain: $\left(-\infty, \sqrt{3}\right)$ so as $x \to -\infty, y \to -\infty$

As global extrema occur at either a local turning point or an end point, there is no global minimum and there is a global maximum of 8 at the local maximum turning point.

This is confirmed by sketching the graph:

x-intercepts:

$$4x^3 - 12x = 0$$
$$4x(x^2 - 3) = 0$$
$$4x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right) = 0$$
$$x = 0,\ x = \pm\sqrt{3}$$



16 a $y = 4x^2 - 2x + 3,\ -1 \le x \le 1$

End points: When $x = -1, y = 4 + 2 + 3 = 9$.

Left end point $(-1, 9)$

When $x = 1, y = 4 - 2 + 3 = 5$.

Right end point $(1, 5)$

y-intercept $(0, 3)$

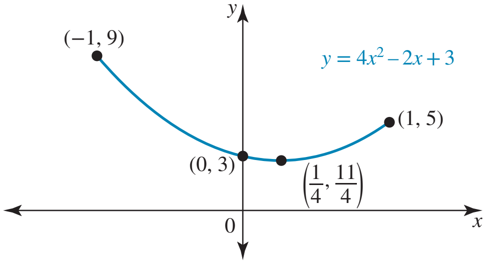
Turning point: $\frac{dy}{dx} = 8x - 2$

At the turning point, $8x - 2 = 0$.

$$\therefore x = \frac{1}{4}$$
$$\therefore y = 4 \times \frac{1}{16} - 2 \times \frac{1}{4} + 3$$
$$\therefore y = \frac{1}{4} - \frac{1}{2} + 3$$
$$\therefore y = \frac{11}{4}$$

Minimum turning point $\left(\frac{1}{4}, \frac{11}{4}\right)$

No x-intercepts



The local minimum and global minimum value is $\frac{11}{4}$.

The global maximum value is 9.

There is no local maximum.

b $y = x^3 + 2x^2,\ -3 \le x \le 3$

End points: When $x = -3, y = -27 + 18 = -9$.

Left end point $(-3, -9)$

***14 b**

x	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0
$\frac{dy}{dx}$	$16 \times (-7) < 0$	0	$16 \times \left(-\frac{1}{4}\right) < 0$	0	$16 \times (1) > 0$
Slope of the tangent	╲	—	╲	—	╱

When $x = 3$, $y = 27 + 18 = 45$. Right end point $(3, 45)$

Stationary points: $\frac{dy}{dx} = 3x^2 + 4x$

At stationary points, $3x^2 + 4x = 0$.

$\therefore x(3x + 4) = 0$

$\therefore x = 0, x = -\frac{4}{3}$

When $x = 0$, $y = 0$.

When $x = -\frac{4}{3}$,

$$\begin{aligned} y &= -\frac{64}{27} + 2 \times \frac{16}{9} \\ &= -\frac{64}{27} + \frac{32}{9} \\ &= \frac{-64 + 96}{27} \\ &= \frac{32}{27} \end{aligned}$$

The stationary points are $\left(-\frac{4}{3}, \frac{32}{27}\right)$ and $(0, 0)$.

As the function is a positive cubic, $\left(-\frac{4}{3}, \frac{32}{27}\right)$ is a maximum turning point and $(0, 0)$ is a minimum turning point.

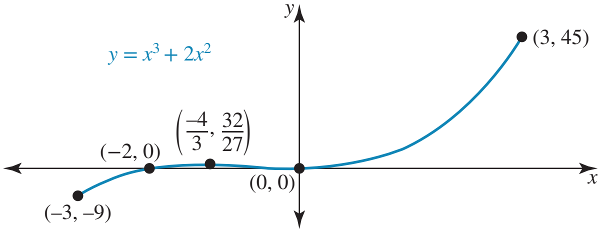
x-intercepts occur when $y = 0$.

$\therefore x^3 + 2x^2 = 0$

$\therefore x^2(x + 2) = 0$

$\therefore x = 0, x = -2$

$(0, 0), (-2, 0)$



The local maximum value is $\frac{32}{27}$.

The local minimum value is 0.

The global maximum value is 45.

The global minimum value is −9.

c $y = 3 - 2x^3$, $x \le 1$

End point: when $x = 1$, $y = 1$. Right end point $(1, 1)$

y-intercept $(0, 3)$

Stationary point of inflection at $(0, 3)$

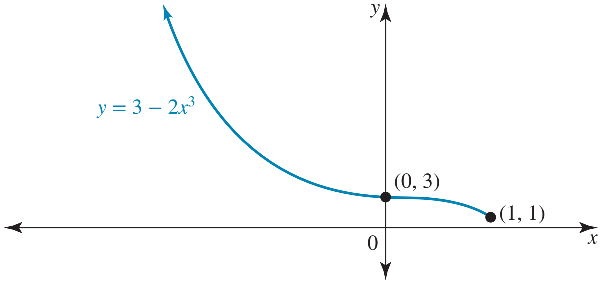
x-intercept: let $y = 0$.

$\therefore 3 - 2x^3 = 0$

$2x^3 = 3$

$\therefore x^3 = \frac{3}{2}$

Since $x^3 > 1$, then $x > 1$, so there is no x-intercept in the given domain $(-\infty, 1)$.



No local maximum or minimum

No global maximum

The global minimum value is 1.

d $f : [0, \infty] \to R$, $f(x) = x^3 + 6x^2 + 3x - 10$

Let $y = f(x)$.

End point: when $x = 0$, $y = -10$. Left end point and y-intercept $(0, -10)$

Stationary points: $f'(x) = 3x^2 + 12x + 3$

At stationary points, $3x^2 + 12x + 3 = 0$.

$\therefore 3\left(x^2 + 4x + 1\right) = 0$

$\therefore \left(x^2 + 4x + 4\right) - 4 + 1 = 0$

$\therefore (x + 2)^2 = 3$

$\therefore x = -2 \pm \sqrt{3}$

$-2 - \sqrt{3} < 0$ and $-2 + \sqrt{3} < 0$, so there are no stationary points in the given domain $[0, \infty]$

x-intercepts occur where $f(x) = x^3 + 6x^2 + 3x - 10 = 0$

$f(1) = 1 + 6 + 3 - 10 = 0$

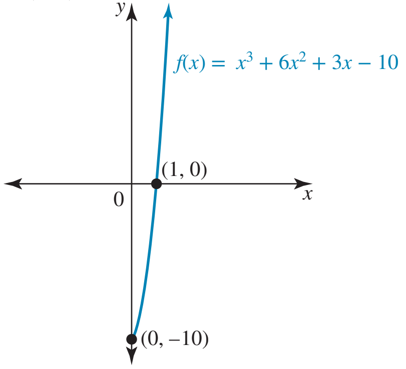
$\therefore (x - 1)$ is a factor.

$\therefore x^3 + 6x^2 + 3x - 10 = (x - 1)\left(x^2 + 7x + 10\right) = 0$

$\therefore (x - 1)(x + 2)(x + 5) = 0$

$\therefore x = 1, x = -2, x = -5$

Only $x = 1$ in the domain, so there is one x-intercept at $(1, 0)$.



No local maximum or minimum

No global maximum

The global minimum value is −10.

17 a $y = x^3 + bx^2 + cx - 26$

The point $(2, -54)$ lies on the curve.

$\therefore -54 = 8 + 4b + 2c - 26$

$\therefore -36 = 4b + 2c$

$\therefore 2b + c = -18 \quad [1]$

$\frac{dy}{dx} = 3x^2 + 2bx + c$

As $(2, -54)$ is a stationary point, $\frac{dy}{dx} = 0$ at $(2, -54)$.

$\therefore 12 + 4b + c = 0$

$\therefore 4b + c = -12 \quad [2]$

Subtract equation [1] from equation [2].

$\therefore 2b = 6$

$\therefore b = 3$

Substitute $b = 3$ in equation [1].

$\therefore 6 + c = -18$

$\therefore c = -24$

Hence, $b = 3$, $c = -24$.

b $y = x^3 + 3x^2 - 24x - 26$ and $\frac{dy}{dx} = 3x^2 + 6x - 24$

Let $\frac{dy}{dx} = 0$.

$\therefore 3x^2 + 6x - 24 = 0$

$\therefore 3\left(x^2 + 2x - 8\right) = 0$

$\therefore 3(x + 4)(x - 2) = 0$

$\therefore x = -4, x = 2$

When $x = -4$, $y = -64 + 48 + 96 - 26 = 54$.

The other stationary point is $(-4, 54)$.

c $y = x^3 + 3x^2 - 24x - 26$ has y-intercept $(0, -26)$.

x-intercepts: let $y = 0$.

$\therefore x^3 + 3x^2 - 24x - 26 = 0$

Let $P(x) = x^3 + 3x^2 - 24x - 26$.

$P(-1) = -1 + 3 + 24 - 26 = 0$

$\therefore (x + 1)$ is a factor.

$\therefore x^3 + 3x^2 - 24x - 26 = (x + 1)(x^2 + 2x - 26)$

$\therefore (x + 1)(x^2 + 2x - 26) = 0$

$\therefore x = -1$ or $x^2 + 2x - 26 = 0$

$\therefore x = -1$ or $\left(x^2 + 2x + 1\right) - 1 - 26 = 0$

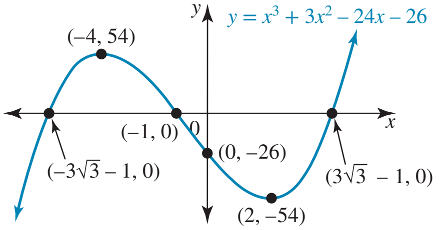
$\therefore x = -1$ or $(x + 1)^2 = 27$

$\therefore x = -1$ or $x + 1 = \pm 3\sqrt{3}$

$\therefore x = -1$ or $x = \pm 3\sqrt{3} - 1$

$(-1,\ 0), \left(-3\sqrt{3} - 1,\ 0\right), \left(3\sqrt{3} - 1,\ 0\right)$

d



18 $y = x^4 + 2x^3 - 2x - 1$

y-intercept: $(0, -1)$

x-intercepts: let $y = 0$.

$$x^4 + 2x^3 - 2x - 1 = 0$$
$$(x^4 - 1) + (2x^3 - 2x) = 0$$
$$(x^2 - 1)(x^2 + 1) + 2x(x^2 - 1) = 0$$
$$(x^2 - 1)(x^2 + 1 + 2x) = 0$$
$$(x - 1)(x + 1)(x + 1)^2 = 0$$
$$(x - 1)(x + 1)^3 = 0$$
$$x = 1,\ x = -1$$

Stationary points: $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 4x^3 + 6x^2 - 2$

$\therefore 4x^3 + 6x^2 - 2 = 0$

If $x = -1$, $\frac{dy}{dx} = 0 \Rightarrow (x + 1)$ is a factor.

$4x^3 + 6x^2 - 2$

$= 2(2x^3 + 3x^2 - 1)$

$= 2(x + 1)(2x^2 + x - 1)$

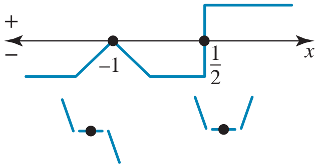
$= 2(x + 1)(2x - 1)(x + 1)$

$= 2\,(x + 1)^2\,(2x - 1)$

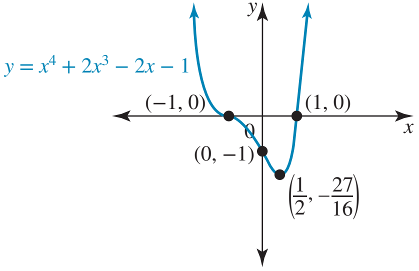
$\therefore x = -1,\ x = \frac{1}{2}$

When $x = -1$, $y = 0$; when $x = \frac{1}{2}$, $y = -\frac{27}{16}$.

For type of stationary point, draw a sign diagram of $\frac{dy}{dx} = 4(x + 1)^2(2x - 1)$:



Therefore, $(-1, 0)$ is a stationary point of inflection and $\left(\frac{1}{2}, \frac{-27}{16}\right)$ is a minimum turning point.



19 $y = xa^2 - x^3$

$\frac{dy}{dx} = a^2 - 3x^2$

At stationary points, $a^2 - 3x^2 = 0$

$\therefore a^2 = 3x^2$

$\therefore x^2 = \frac{a^2}{3}$

$\therefore x = \pm\frac{a}{\sqrt{3}}$

When $x = -\frac{a}{\sqrt{3}}$,

$y = -\frac{a}{\sqrt{3}} \times a^2 + \frac{a^3}{3\sqrt{3}}$

$= -\frac{3a^3}{3\sqrt{3}} + \frac{a^3}{3\sqrt{3}}$

$= -\frac{2a^3}{3\sqrt{3}}$

When $x = \frac{a}{\sqrt{3}}$,

$$y = \frac{a}{\sqrt{3}} \times a^2 - \frac{a^3}{3\sqrt{3}}$$
$$= \frac{3a^3}{3\sqrt{3}} - \frac{a^3}{3\sqrt{3}}$$
$$= \frac{2a^3}{3\sqrt{3}}$$

The stationary points are $\left(-\frac{a}{\sqrt{3}}, -\frac{2a^3}{3\sqrt{3}}\right)$ and $\left(\frac{a}{\sqrt{3}}, \frac{2a^3}{3\sqrt{3}}\right)$.

Let A be $\left(-\frac{a}{\sqrt{3}}, -\frac{2a^3}{3\sqrt{3}}\right)$, B be $\left(\frac{a}{\sqrt{3}}, \frac{2a^3}{3\sqrt{3}}\right)$ and C be the point $(0, 0)$.

The line through A and B will pass through C if the three points are collinear.

$$m_{BC} = \frac{\left(\frac{2a^3}{3\sqrt{3}} - 0\right)}{\left(\frac{a}{\sqrt{3}} - 0\right)}$$
$$= \frac{2a^3}{3\sqrt{3}} \div \frac{a}{\sqrt{3}}$$
$$= \frac{2a^3}{3\sqrt{3}} \times \frac{\sqrt{3}}{a}$$
$$= \frac{2a^2}{3}$$

$$m_{AC} = \frac{\left(-\frac{2a^3}{3\sqrt{3}} - 0\right)}{\left(-\frac{a}{\sqrt{3}} - 0\right)}$$
$$= -\frac{2a^3}{3\sqrt{3}} \div -\frac{a}{\sqrt{3}}$$
$$= -\frac{2a^3}{3\sqrt{3}} \times -\frac{\sqrt{3}}{a}$$
$$= \frac{2a^2}{3}$$

Since $m_{AC} = m_{BC}$ and point C is common, the three points A, B and C are collinear.

Therefore, the line joining the turning points passes through the origin. The equation of the line is $y = \frac{2a^2}{3}x$.

20 **a** $y = ax^3 + bx^2 + cx + d$

Since $(0, 0)$ lies on the curve, $d = 0$.

$\therefore y = ax^3 + bx^2 + cx$

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

The tangent at $(0, 0)$ has gradient $m = \tan(135°)$.

$\therefore m = -\tan(45°)$

$\therefore m = -1$

At $(0, 0)$, $\frac{dy}{dx} = -1$.

$\therefore -1 = c$

Hence, $c = -1, d = 0$ and $y = ax^3 + bx^2 - x$.

b The point $(2, -2)$ lies on the curve.

$\therefore -2 = 8a + 4b - 2$

$\therefore 8a + 4b = 0$

$\therefore b = -2a \quad [1]$

The point $(2, -2)$ is a stationary point, so $\frac{dy}{dx} = 0$ at this point.

$\frac{dy}{dx} = 3ax^2 + 2bx - 1$

$\therefore 12a + 4b - 1 = 0 \quad [2]$

Substitute $b = -2a$ in equation [2].

$\therefore 12a - 8a - 1 = 0$

$\therefore 4a = 1$

$\therefore a = \frac{1}{4}$

Substitute $a = \frac{1}{4}$ in equation [1].

$\therefore b = -\frac{1}{2}$

Hence, $a = \frac{1}{4}, b = -\frac{1}{2}$ and $y = \frac{1}{4}x^3 - \frac{1}{2}x^2 - x$.

c $\frac{dy}{dx} = \frac{3}{4}x^2 - x - 1$

At stationary points, $\frac{3}{4}x^2 - x - 1 = 0$

$\therefore 3x^2 - 4x - 4 = 0$

$\therefore (3x + 2)(x - 2) = 0$

$\therefore x = -\frac{2}{3}, x = 2$

The second stationary point occurs at $x = -\frac{2}{3}$.

When $x = -\frac{2}{3}$,

$$y = \frac{1}{4} \times -\frac{8}{27} - \frac{1}{2} \times \frac{4}{9} + \frac{2}{3}$$
$$= -\frac{2}{27} - \frac{2}{9} + \frac{2}{3}$$
$$= \frac{-2 - 6 + 18}{27}$$
$$= \frac{10}{27}$$

The other stationary point is $\left(-\frac{2}{3}, \frac{10}{27}\right)$.

d Since $(2, -2)$ is a stationary point, the tangent to the curve at this point is the horizontal line $y = -2$.

This line intersects $y = \frac{1}{4}x^3 - \frac{1}{2}x^2 - x$ when $\frac{1}{4}x^3 - \frac{1}{2}x^2 - x = -2$.

$\therefore x^3 - 2x^2 - 4x = -8$

$\therefore x^3 - 2x^2 - 4x + 8 = 0$

$\therefore x^2(x - 2) - 4(x - 2) = 0$

$\therefore (x - 2)\left(x^2 - 4\right) = 0$

$\therefore (x - 2)(x - 2)(x + 2) = 0$

$\therefore (x - 2)^2(x + 2) = 0$

$\therefore x = 2$ (touch) or $x = -2$ (cut)

The tangent meets the curve again when $x = -2$

When $x = -2$, $y = -\frac{1}{4} \times 8 - \frac{1}{2} \times 4 + 2 = -2$

The tangent meets the curve again at the point $(-2, -2)$.

12.4 Exam questions

1 Point (2, −16) is a stationary point, so $\frac{dy}{dx} = 0$ at this point.

$y = ax^2 + bx$

$\frac{dy}{dx} = 2ax + b$

$x = 2, \frac{dy}{dx} = 0$

$\therefore 0 = 2a(2) + b$

$\therefore b = -4a$ [1] [1 mark]

Point (2, −16) is a point on the curve.

$-16 = a(2)^2 + b$ [2]

$= 4a + 2b$ [2] [1 mark]

Substitute equation [1] into equation [2].

$-16 = 4a + 2(-4a)$

$= -4a$

$a = 4$

Substitute $a = 4$ into [1] to find b.

$b = -4(4)$

$= -16$

$\therefore a = 4, b = -16$

$\therefore y = 4x^2 - 16x$

y-intercepts $(x = 0)$:

$y = 0$

(0, 0) is a y-intercepts.

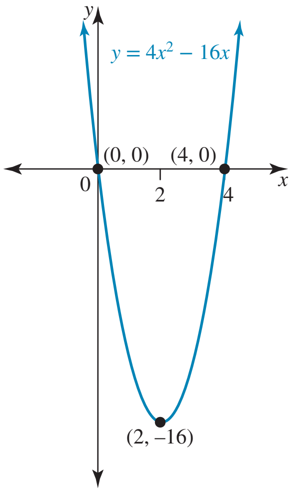
x-intercepts $(y = 0)$:

$0 = 4x^2 - 16x$

$0 = 4x(x - 4)$

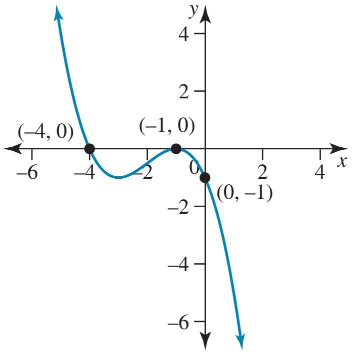
$x = 0, 4$

(0, 0) and (4, 0) are x-intercepts [1 mark]



[1 mark]

2 $f(x)$ has a y-intercept at −1 and x-intercepts at −4 and−1.
$f(x)$ is decreasing for $x < -3$ and increasing for $x \in (-3, -1)$, so $x = -3$ must be a minimum turning point. [1 mark]
$f'(-1) = 0$ so $x = -1$ is also a turning point and, since $f(x)$ is decreasing for $x > -1$, it must be a maximum turning point.
[1 mark]



[1 mark]

3 a $f(x) = -x^3 + 3x^2 - 4$

$f'(x) = -3x^2 + 6x$ [1 mark]

For stationary points, $f'(x) = 0$.

$-3x^2 + 6x = 0$

$-3x(x - 2) = 0$

$\therefore x = 0, \ x = 2$ [1 mark]

$f(0) = -4$

$\therefore (0, -4)$ is a stationary point.

$f(2) = -(2)^3 + 3(2)^2 - 4$

$f(2) = 0$

$\therefore (2, 0)$ is a turning point and also an x-intercept. [1 mark]

b End points:

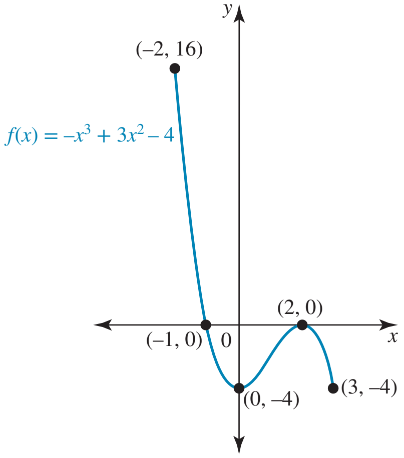
$f(-2) = -(-2)^3 + 3(-2)^2 - 4$

$= 16$

$f(3) = -(3)^3 + 3(3)^2 - 4$

$= -4$

$\therefore$ the end points are (−2, 16) and (3, −4).



[1 mark]

c The maximum value in the given domain is 16, and the minimum value is −4. [1 mark]

12.5 Optimisation problems

12.5 Exercise

1 a $M = 40x - 2x^2$

$\frac{dM}{dx} = 40 - 4x$

b As the given money function is a concave down parabola, M is greatest at its turning point.

Let $\frac{dM}{dx} = 0$.

$\therefore 40 - 4x = 0$

$\therefore x = 10$

Therefore, M is greatest when $x = 10$.

c When $x = 10$,

$M = 40(10) - 2(10)^2$

$= 400 - 200$

$= 200$

The maximum amount of money that Murray expects to earn is \$200.

2 a Since the perimeter of the rectangle is 32 cm, $2x + 2y = 32$.

$\therefore x + y = 16$

$\therefore y = 16 - x$

b The area of the rectangle is $A = xy$.

Substitute $y = 16 - x$.

$\therefore A = x(16 - x)$

$\therefore A = 16x - x^2$

As $y > 0$, $x < 16$, so the domain of the area function has the restriction that $0 < x < 16$.

c The maximum turning point of the concave down area function occurs when $\frac{dA}{dx} = 0$.

$A = 16x - x^2$

$\frac{dA}{dx} = 16 - 2x$

$\therefore 16 - 2x = 0$

$\therefore 16 = 2x$

$\therefore x = 8$

d When $x = 8$, $y = 16 - 8 = 8$.

The placemat with greatest area has dimensions 8 cm by 8 cm.

The greatest area of the placemat is 64 sq cm.

3 a The perimeters of the squares are $4x$ and $4y$.

Hence, $4x + 4y = 40$.

$\therefore x + y = 10$

$\therefore y = 10 - x$

b The sum of the two areas, $S = x^2 + y^2$.

Substitute $y = 10 - x$.

$S = x^2 + (10 - x)^2$

$= x^2 + 100 - 20x + x^2$

$= 2x^2 - 20x + 100$

c $\frac{dS}{dx} = 4x - 20$

Let $\frac{dS}{dx} = 0$.

$\therefore 4x - 20 = 0$

$\therefore x = 5$

d As the function S is a concave up quadratic, its turning point is a minimum.

When $x = 5$,

$S = x^2 + (10 - x)^2$

$= 5^2 + 5^2$

$= 50$

The minimum turning point is $(5, 50)$.

Hence, the minimum value of the sum of the areas of the two squares is 50 sq cm.

e Since the turning point is a minimum and there are no other turning points, the maximum value must occur at one of the end points for which $3 \le x \le 6$.

Let $x = 3$.

$S = x^2 + (10 - x)^2$

$= 3^2 + 7^2$

$= 58$

Let $x = 6$.

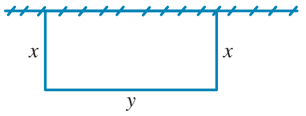
$S = x^2 + (10 - x)^2$

$= 6^2 + 4^2$

$= 52$

If $3 \le x \le 6$, then the greatest value of the sum of the areas of the two squares is 58 sq cm.

4 a Let width be x metres and length y metres.



There is an amount of 40 metres of fencing available.

$\therefore 2x + y = 40$

$\therefore y = 40 - 2x$

Area, A sq m, of the rectangular garden is

$A = xy$

$\therefore A = x(40 - 2x)$

$\therefore A = 40x - 2x^2$

b $\frac{dA}{dx} = 40 - 4x$

At the maximum area, $\frac{dA}{dx} = 0$.

$\therefore 40 - 4x = 0$

$\therefore x = 10$

When $x = 10$, $y = 40 - 20 = 20$.

The dimensions for maximum area are width 10 metres and length 20 metres.

c The maximum area is 200 sq m.

d The area function is restricted to $A = 40x - 2x^2,\ 5 \le x \le 7$, so the maximum turning point occurs outside the domain $[5, 7]$. In this case, the greatest area will occur at an end point of the domain.

$A(5) = 200 - 50 = 150$

$A(7) = 280 - 98 = 182$

The greatest area that can be enclosed is 182 sq m.

5 $C = n^3 - 10n^2 - 32n + 400,\ 5 \le n \le 10$

$\frac{dC}{dn} = 3n^2 - 20n - 32$

At stationary points, $\frac{dC}{dn} = 0$.

$\therefore 3n^2 - 20n - 32 = 0$

$\therefore (3n + 4)(n - 8) = 0$

$\therefore n = -\frac{4}{3}$ (reject) or $n = 8$.

Test the slope of the function around $n = 8$ to determine the nature of the stationary point.

See the table at the bottom of the page.*
There is a minimum turning point at $n = 8$.
As the cost function is a cubic polynomial, $n = 8$ will be the value in the domain [5, 10] for which the cost is minimised.
Therefore, 8 people should be employed in order to minimise the cost.

6 a $y = 0.0001x^2\left(625 - x^2\right)$

$\therefore y = 0.0625x^2 - 0.0001x^4$

$\dfrac{dy}{dx} = 0.1250x - 0.0004x^3$

At greatest height, $\dfrac{dy}{dx} = 0$.

$\therefore 0.1250x - 0.0004x^3 = 0$

$\therefore x\left(0.1250 - 0.0004x^2\right) = 0$

$\therefore x = 0$ (reject) or $0.1250 - 0.0004x^2 = 0$

$\therefore x^2 = \dfrac{0.1250}{0.0004}$

$\therefore x^2 = \dfrac{1250}{4}$

$\therefore x = \pm\dfrac{\sqrt{1250}}{2}$

$\therefore x = \pm\dfrac{25\sqrt{2}}{2}$

Reject the negative value.

$\therefore x = \dfrac{25\sqrt{2}}{2} \simeq 17.68$

Test the slope of the curve either side of this value
See the table at the bottom of the page.*

There is a maximum turning point at $x = \dfrac{25\sqrt{2}}{2}$

When $x = \dfrac{25\sqrt{2}}{2}$, $x^2 = \dfrac{625 \times 2}{4} = \dfrac{625}{2}$

$$\therefore y = 0.0001 \times \frac{625}{2}\left(625 - \frac{625}{2}\right)$$

$\therefore y \simeq 9.77$

The greatest height the ball reaches is 9.77 metres above the ground.

b When the ball strikes the ground, $y = 0$

$\therefore 0.001x^2\left(625 - x^2\right) = 0$

$\therefore x = 0$ or $x^2 = 625$

$\therefore x = 0, x = 25, x = -25$

Only $x = 25$ is a practical solution.
Therefore, the ball travels 25 metres horizontally before it strikes the ground.

7 a The box has length $= (20 - 2x)$ cm, width $= (12 - 2x)$ cm and height $= x$ cm.
Therefore, the volume, V cm^3, is $V = x(20 - 2x)(12 - 2x)$.

$\therefore V = 240x - 64x^2 + 4x^3$

b The greatest volume occurs when $\dfrac{dV}{dx} = 0$.

$240 - 128x + 12x^2 = 0$

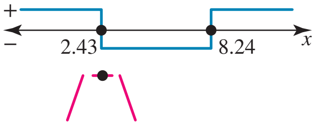
$4(3x^2 - 32x + 60) = 0$

Using the formula for solving quadratic equations gives:

$$x = \frac{32 \pm \sqrt{(32)^2 - 4(3)(60)}}{6}$$

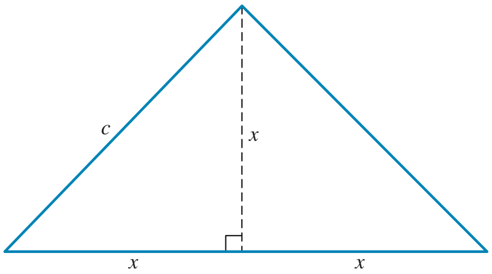
$x \approx 2.43$ or $x \approx 8.24$ (or use CAS)

Sign diagram of $\dfrac{dV}{dx}$:



The maximum volume occurs for $x = 2.43$.
(Alternatively, consider the domain which would require $x \in (0, 6)$ and the shape of the volume function's graph.)
Therefore, the box with length 15.15 cm, width 7.15 cm and height 2.43 cm has the greatest volume of 263 cm^3, to the nearest whole number.

8 a



Consider the isosceles triangle to find the lengths of its sloping sides.
Using Pythagoras' theorem, $c^2 = x^2 + x^2$, so the sloping sides have lengths $\sqrt{2}x$ cm.
Since the perimeter of the figure is 150 cm,

$2x + 2y + 2\sqrt{2}x = 150$

$y = 75 - x - \sqrt{2}x$

The area of the figure is the sum of the areas of the rectangle and the triangle, base $2x$, height x.

---

*5

n	7	8	9
$\dfrac{dc}{dn}$	$(25)(-1) = -25$	0	$(33)(1) = 33$
Slope of tangent	negative	zero	positive

*6 a

x	17	$\dfrac{25\sqrt{2}}{2}$	18
$\dfrac{dy}{dx}$	$0.125 \times 17 - 0.0004 \times 17^3$ $\simeq 0.16$	0	$0.125 \times 18 - 0.0004 \times 1^3$ $= -0.08$
Slope	positive	zero	negative

$$\begin{aligned}A &= 2xy + \frac{1}{2}(2x)x\\ &= 2x(75 - x - \sqrt{2}x) + x^2\\ &= 150x - 2x^2 - 2\sqrt{2}x^2 + x^2\\ &= 150x - \left(1 + 2\sqrt{2}\right)x^2\end{aligned}$$

b To find the maximum area, let $\frac{dA}{dx} = 0$.

$$150 - 2\left(1 + 2\sqrt{2}\right)x = 0$$

$$\begin{aligned}x &= \frac{75}{1 + 2\sqrt{2}}\\ &\approx 19.6\end{aligned}$$

As the area function is a concave down parabola, the area is greatest when $x \approx 19.6$.

Substitute $x \approx 19.6$ to find y.

$$\begin{aligned}y &= 75 - \left(1 + \sqrt{2}\right)x\\ &= 75 - \left(1 + \sqrt{2}\right) \times \frac{75}{1 + 2\sqrt{2}}\\ &\approx 27.7\end{aligned}$$

Width: $2x = 39.2$; total height: $y + x = 47.3$

Therefore, the figure has a width of 39.2 cm and total height of 47.3 cm for greatest area.

c If the width of the figure cannot exceed 30 cm, $x \le 15$, so the maximum area at $x \approx 19.6$ is not in the domain. Hence, the maximum area will occur when $x = 15$ and therefore $y = 75 - 15\left(1 + \sqrt{2}\right) \approx 38.8$.

The dimensions of the figure for maximum area are now width 30 cm and height 53.8 cm.

9 **a** TSA of cylinder is the sum of the areas of the two circular ends together with the curved surface area.

$$\begin{aligned}\therefore 200 &= 2\pi r^2 + 2\pi rh\\ \therefore \pi r^2 + \pi rh &= 100\\ \therefore \pi rh &= 100 - \pi r^2\\ \therefore h &= \frac{100 - \pi r^2}{\pi r}\end{aligned}$$

b The formula for the volume of a cylinder is $V = \pi r^2 h$

$$\therefore V = \overset{r}{\cancel{\left(\pi r^2\right)}} \times \frac{100 - \pi r^2}{\cancel{\pi r}}$$

$$\therefore V = r\left(100 - \pi r^2\right)$$

$$\therefore V = 100r - \pi r^3$$

c $\frac{dV}{dr} = 100 - 3\pi r^2$

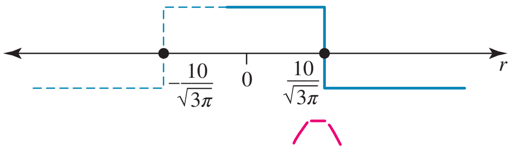
At maximum volume, $\frac{dV}{dr} = 0$

$$\therefore 100 - 3\pi r^2 = 0$$

$$\therefore r^2 = \frac{100}{3\pi}$$

$\therefore r = \frac{10}{\sqrt{3\pi}}$ (reject the negative square root)

The sign of $\frac{dV}{dr}$ is that of a concave down parabola with zeros at $r = \pm\frac{10}{\sqrt{3\pi}}$



There is a maximum turning point at $r = \frac{10}{\sqrt{3\pi}}$.

When $r = \frac{10}{\sqrt{3\pi}}$,

$$\begin{aligned}h &= \left(100 - \pi \times \frac{100}{3\pi}\right) \div \left(\pi \times \frac{10}{\sqrt{3\pi}}\right)\\ &= \left(100 - \frac{100}{3}\right) \div \frac{10\sqrt{\pi}}{\sqrt{3}}\\ &= \frac{200}{3} \times \frac{\sqrt{3}}{10\sqrt{\pi}}\\ &= \frac{20\sqrt{3}}{3\sqrt{\pi}}\\ &= \frac{20}{\sqrt{3\pi}}\\ &= 2 \times \frac{10}{\sqrt{3\pi}}\\ &= 2r\end{aligned}$$

For maximum volume, the height is equal to twice the radius, or the height is equal to the diameter of the base of the circular cylinder.

d $r = \frac{10}{\sqrt{3\pi}} \simeq 3.26$ is the value where the maximum volume occurs.

For the interval $2 \le r \le 4$, there are no other stationary points. Hence the minimum volume must occur at one of the end points $r = 2$ or $r = 4$.

$$V = 100r - \pi r^3$$

$V(2) = 200 - 8\pi \simeq 174.88$

$V(4) = 400 - 64\pi \simeq 198.94$

For the given restriction, the minimum volume is 175 cubic cm, correct to the nearest integer.

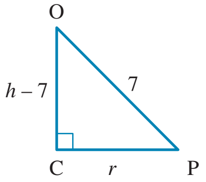
10 Refer to the diagram given in the question.

a The volume is required as a function of h only, so it is necessary to express r in terms of h.

In the diagram, $OV = OP = 7$ as theses lengths are radii of the sphere.

$$\begin{aligned}CV &= CO + OV\\ \therefore h &= CO + 7\\ \therefore CO &= h - 7\end{aligned}$$

Consider the right-angled triangle COP.



Using Pythagoras' theorem,

$$\begin{aligned}(h - 7)^2 + r^2 &= 7^2\\ \therefore r^2 &= 49 - (h - 7)^2\\ \therefore r^2 &= 49 - \left(h^2 - 14h + 49\right)\\ \therefore r^2 &= 14h - h^2\end{aligned}$$

The volume of the cone is $V = \frac{1}{3}\pi r^2 h$.

$\therefore V = \frac{1}{3}\pi\left(14h - h^2\right)h$

$\therefore V = \frac{1}{3}\pi\left(14h^2 - h^3\right)$

b $\frac{dV}{dh} = \frac{1}{3}\pi\left(28h - 3h^2\right)$

$\therefore \frac{dV}{dh} = \frac{1}{3}\pi h(28 - 3h)$

At the maximum volume, $\frac{dV}{dh} = 0$.

$\therefore \frac{1}{3}\pi h(28 - 3h) = 0$

$\therefore h = 0$ (reject) or $h = \frac{28}{3}$

Test the slope of the function about $h = \frac{28}{3} = 9\frac{1}{3}$.

See the table at the bottom of the page.*

There is a maximum turning point when $h = \frac{28}{3}$.

The volume is greatest when $h = \frac{28}{3}$.

Substitute $h = \frac{28}{3}$ in $r^2 = 49 - (h - 7)^2$

$\therefore r^2 = 49 - \left(\frac{28}{3} - \frac{21}{3}\right)^2$

$\therefore r^2 = 49 - \frac{49}{9}$

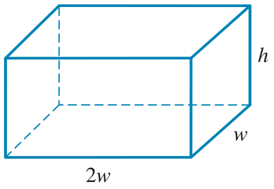
$\therefore r^2 = \frac{8}{9} \times 49$

$\therefore r = \frac{2\sqrt{2}}{3} \times 7$ (reject negative square root)

$\therefore r = \frac{14\sqrt{2}}{3}$

When the height measure, $h = \frac{28}{3}$ and the base radius measure $r = \frac{14\sqrt{2}}{3}$, the volume of the cone is greatest.

11 a The rectangular prism has length $2w$ cm, width w cm and height h cm.



The surface area is the sum of the areas of the six rectangular faces.

$SA = 2 \times (2w \times w) + 2 \times (2w \times h) + 2 \times (w \times h)$

$= 4w^2 + 4wh + 2wh$

$= 4w^2 + 6wh$

b Given the surface area is 300 cm^2, then

$4w^2 + 6wh = 300$

$\therefore 2w^2 + 3wh = 150$

$\therefore 3wh = 150 - 2w^2$

$\therefore h = \frac{150 - 2w^2}{3w}$

The volume, V, of the prism is

$V = 2w \times w \times h$

$\therefore V = 2w^2h$

$\therefore V = 2w^2\left(\frac{150 - 2w^2}{3w}\right)$

$\therefore V = \frac{2}{3}w\left(150 - 2w^2\right)$

$\therefore V = 100w - \frac{4}{3}w^3$

$\frac{dV}{dw} = 100 - 4w^2$

At maximum volume, $\frac{dV}{dw} = 0$.

$\therefore 100 - 4w^2 = 0$

$w^2 = 25$

$\therefore w = 5$ (reject negative square root)

To justify the nature of the turning point, consider the slope of the tangent either side of $w = 5$.

w	4	5	6
$\frac{dV}{dW}$	$100 - 64 = 36$	0	$100 - 144 = -44$
slope	positive	zero	negative

There is a maximum turning point when $w = 5$.

$\therefore V_{\max} = 100 \times 5 - \frac{4}{3} \times 125$

$= \frac{1500 - 500}{3}$

$= \frac{1000}{3}$

The maximum volume is $\frac{1000}{3}$ cubic cm.

c The maximum volume occurs when $w = 5$. Substitute $w = 5$ in $h = \frac{150 - 2w^2}{3w}$,

$\therefore h = \frac{150 - 50}{15}$

$\therefore h = \frac{20}{3}$

The dimensions of the rectangular prism with greatest volume are length 10 cm, width 5 cm and height $\frac{20}{3}$ cm.

d Since $5 \in [2, 6]$, the greatest volume is $\frac{1000}{3}$ cubic cm.

There is no other turning point in the interval $w \in [2, 6]$.

Test the value of $V = 100w - \frac{4}{3}w^3$ at the end points $w = 2$ and $w = 6$ to determine the minimum value over the domain $[2, 6]$.

***10 b**

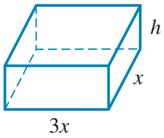
h	9	$\frac{28}{3}$	10
$\frac{dV}{dh}$	$\frac{1}{3}\pi \times 9 \times (1) = 3\pi$	0	$\frac{1}{3}\pi \times 10 \times (-2) = -\frac{20\pi}{3}$
Slope	positive	zero	negative

$$V(2) = 200 - \frac{32}{3}$$
$$= \frac{568}{3}$$
$$V(6) = 600 - 4 \times 72$$
$$= 312$$
$$= \frac{936}{3}$$

The minimum volume is $\frac{568}{3}$ cubic cm.

In cubic cm, the range of values for the volume over the restricted domain is $\left[\frac{568}{3}, \frac{1000}{3}\right]$.

12



a The total perimeter of the rectangular box is 240 cm.

$4\,(3x + x + h) = 240$

$4x + h = 60$

$h = 60 - 4x$

b Volume of rectangular box:

$V = 3x \times x \times h$

$V = 3x^2h.$

$= 3x^2(60 - 4x)$

$= 180x^2 - 12x^3$

c For maximum volume, $\frac{dV}{dx} = 0$

$360x - 36x^2 = 0$

$36x\,(10 - x) = 0$

$\therefore x = 0 \text{ or } x = 10$

Maximum volume when $x = 10$

$V = 180(10)^2 - 12(10)^3$

$= 18000 - 12000$

$= 6000$

Maximum volume: 6000 cm^3

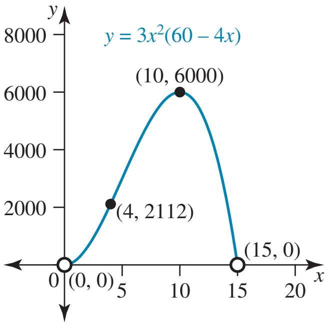
Dimensions: 30 cm by 10 cm by 20 cm

d Since lengths are positive, $x > 0$ and $h > 0$

$60 - 4x > 0$

$x < 15$

The domain of the volume is $x \in (0, 15)$



e If the length of the box is 12 cm or less:

$3x \le 12$

$0 < x \le 4$

When $x = 4$:

$V = 180(4)^2 - 12(4)^3$

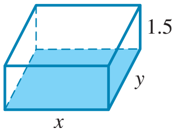
$V = 2112$ cm^3

The point on the curve (4, 2112) is shown on the diagram and will be the absolute maximum for the restricted domain $0 < x \le 4$

Maximum volume: 2112 cm^3

Dimensions: 12 cm by 4 cm by 44 cm

13



The total perimeter of the rectangular box is 100 cm.

$4\,(x + y + 1.5) = 100$

$x + y + 1.5 = 25$

$y = 23.5 - x$

For the largest bin:

$V = x \times y \times 1.5$

Substitute for y

$V = 1.5x\,(23.5 - x)$

$= 35.25x - 1.5x^2$

For largest volume: $\frac{dV}{dx} = 0$

$\frac{dV}{dx} = 35.25 - 3x$

$35.25 - 3x = 0$

$x = 11.75$

When $x = 11.75$:

$y = 23.5 - 11.75$

$= 11.75$

Maximum volume as the curve is a concave down parabola where $x = 11.75$ and $y = 11.75$.

Cost of sides $= 10\,(1.5x + 1.5x + 1.5y + 1.5y)$

$= 30x + 30y$

$= 30 \times 11.75 + 30 \times 11.75$

$= \$705$

Cost of base $= \$25\,(xy)$

$= \$25\,(11.75)\,(11.75)$

$= \$3451.56$

Total cost $= 705 + 3451.56$

$= 4156.56$

The total cost of materials for the largest bin is \$4157 (to nearest dollar).

14 Refer to the diagram given in the question.

a Let the perimeter be P metres.

$P = r + l + r$

$= 2r + l$

Since the arc length $l = r\theta$, then $P = 2r + r\theta$.

Given $P = 8$

$\therefore 2r + r\theta = 8$

$\therefore r\theta = 8 - 2r$

$\therefore \theta = \frac{8 - 2r}{r}$

b The formula for the area of a sector is $A = \frac{1}{2}r^2\theta$

$\therefore A = \frac{1}{2}r^2 \times \frac{8 - 2r}{r}$

$\therefore A = \frac{1}{2}r \times 2(4 - r)$

$\therefore A = r(4 - r)$

$\therefore A = 4r - r^2$

c $\frac{dA}{dr} = 4 - 2r$

At maximum, $\frac{dA}{dr} = 0$

$\therefore 4 - 2r = 0$

$\therefore r = 2$

As the area function is a concave down parabola, the stationary point must be a maximum.

When $r = 2$, $\theta = \frac{8-4}{2} = 2$.

For greatest area, the value of θ is 2 radians.

12.5 Exam questions

1 Surface area of cylinder $= 2\pi r^2 + 2\pi rh$

$$100 = 2\pi r^2 + 2\pi rh$$
$$2\pi rh = 100 - 2\pi r^2$$
$$h = \frac{100 - 2\pi r^2}{2\pi r}$$
$$h = \frac{50 - \pi r^2}{\pi r}$$

The correct answer is **A**.

2 Side lengths of triangle at top are x, x and $\sqrt{2}x$ (via Pythagoras' theorem).

The total perimeter is

$$200 = 2h + 2x + \sqrt{2}x$$
$$2h = 200 - 2x - \sqrt{2}x$$
$$h = \frac{200 - 2x - \sqrt{2}x}{2}$$
$$= 100 - x - \frac{\sqrt{2}}{2}x$$

Total area = area of rectangle + area of triangle

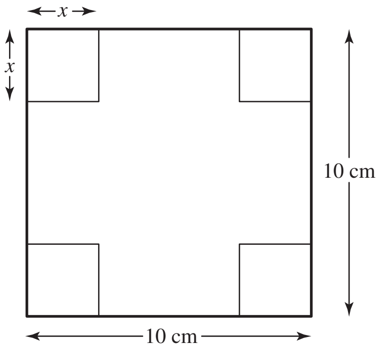
$$\text{Area} = xh + \frac{1}{2}x^2$$

Substitute $h = 100 - x - \frac{\sqrt{2}}{2}x$.

$$\text{Area} = x\left(100 - x - \frac{\sqrt{2}}{2}x\right) + \frac{1}{2}x^2$$
$$= 100x - x^2 - \frac{\sqrt{2}}{2}x^2 + \frac{1}{2}x^2$$
$$= 100x - x^2\left(1 + \frac{\sqrt{2}}{2} - \frac{1}{2}\right)$$
$$= 100x - x^2\left(\frac{1 + \sqrt{2}}{2}\right)$$

The correct answer is **C**.

3 a



[1 mark]

$$V = lwh$$
$$= (10 - 2x)(10 - 2x)(x)$$
$$= x(10 - 2x)^2$$
$$= x\left(100 - 40x + 4x^2\right)$$
$$= 100x - 40x^2 + 4x^3 \qquad \text{[1 mark]}$$

b $\frac{dV}{dx} = 100 - 80x + 12x^2$

When $\frac{dV}{dx} = 0$,

$$0 = 100 - 80x + 12x^2$$
$$0 = 25 - 20x + 3x^2$$
$$0 = (3x - 5)(x - 5) \qquad \text{[1 mark]}$$
$$x = 5, \frac{5}{3}$$

$x = 5$ is unrealistic as the lengths would be 0, so no volume.

$\therefore x = \frac{5}{3}$ is a maximum. [1 mark]

$\therefore$ length and width $= 10 - 2 \times \frac{5}{3}$

$$= \frac{20}{3}$$

Height $= \frac{5}{3}$ [1 mark]

Maximum volume:

$$V = x(10 - 2x)^2$$
$$= \frac{5}{3}\left(10 - 2 \times \frac{5}{3}\right)^2$$

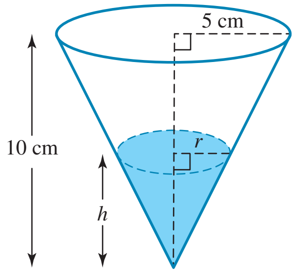
$\therefore$ maximum volume is 74.07 cm^2. [1 mark]

The maximum volume is 74 cm^3 (to the nearest whole number).

12.6 Rates of change and kinematics

12.6 Exercise

1 a



Using similar triangles,

$$\frac{r}{h} = \frac{5}{10}$$
$$\therefore r = \frac{1}{2}h$$

b The volume of water is $V = \frac{1}{3}\pi r^2 h$.

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$
$$= \frac{1}{12}\pi h^3$$

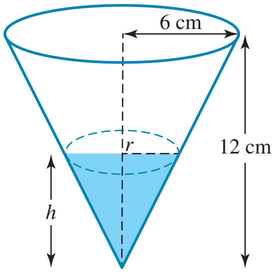
c Rate:

When $\frac{dV}{dh} = \frac{1}{4}\pi h^2$,

$h = 3$, $\frac{dV}{dh} = \frac{9\pi}{4}$

With respect to its depth, the volume is changing at the rate of $\frac{9\pi}{4}$ cm^3/cm when depth is 3 cm.

2 Consider the volume of water in the cone when the depth is h cm and the radius of the surface is r cm.



Using similar triangles,

$$\frac{r}{h} = \frac{6}{12}$$

$$\therefore r = \frac{1}{2}h$$

The volume of water is $V = \frac{1}{3}\pi r^2 h$.

$$\therefore V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

$$\therefore V = \frac{1}{12}\pi h^3$$

The rate of change of the volume with respect to the depth of water is $\frac{dV}{dh} = \frac{1}{4}\pi h^2$.

The height of the cone is 12 cm.

When $\quad h = \frac{1}{2} \times 12 = 6$

$$\begin{aligned}\frac{dV}{dh} &= \frac{1}{4}\pi \times 36\\ &= 9\pi,\end{aligned}$$

the volume is changing at 9π cm^3/cm.

3 a $C = 2\pi r$

The rate of change of C with respect to r is $\frac{dC}{dr}$.

$$\frac{dC}{dr} = 2\pi$$

b $A = \pi r^2$

i The rate of change of A with respect to r is $\frac{dA}{dr}$.

$$\frac{dA}{dr} = 2\pi r$$

ii Let $r = 3$, $\frac{dA}{dr} = 2\pi(3) = 6\pi$.

The area is changing at 6π mm^2/mm.

c i $S = 4\pi r^2$

The rate of change of S with respect to r is $\frac{dS}{dr}$.

$$\frac{dS}{dr} = 8\pi r$$

ii When $r = 5$, $\frac{dS}{dr} = 8\pi(5) = 40\pi$.

The surface area is changing at 40π cm^2/cm.

iii The rate of change of the surface area is 16π cm^2/cm.

Let $\quad \frac{dS}{dr} = 16\pi$

$$\therefore 16\pi = 8\pi \mathrm{r}$$

$$\therefore r = 2.$$

The radius is 2 cm.

d i $V = \frac{4}{3}\pi r^3$

The rate of change of V with respect to r is $\frac{dV}{dr}$.

$$\frac{dV}{dr} = 4\pi r^2$$

ii Let $\quad r = 0.5$

$$\frac{dV}{dr} = 4\pi\left(\frac{1}{2}\right)^2 = \pi.$$

The rate of change of the volume is π m^3/m.

4 a The formula for the volume, V, of a cube of side length x is $V = x^3$.

$$\frac{dV}{dx} = 3x^2$$

When $x = 2$, $\frac{dV}{dx} = 3(2)^2 = 12$.

The rate of change of the volume is 12 cm^3/cm.

b The formula for the surface area, A, of a cube of side length l is $A = 6l^2$.

$$\frac{dA}{dl} = 12l$$

When $l = 5$, $\frac{dA}{dl} = 12 \times 5 = 60$.

The rate of change of the surface area is 60 cm^2/cm.

c $V = \frac{1}{3}h^3 + 2h$

i The rate of change of the volume of water with respect to the depth of water is $\frac{dV}{dh}$.

$$\frac{dV}{dh} = h^2 + 2$$

ii When $h = 3$,

$V = \frac{1}{3}(3)^3 + 2(3) = 15$ and $\frac{dV}{dh} = 3^2 + 2 = 11$.

Therefore, when the depth of water is 3 cm, the volume is 15 cm^3 and it is changing at a rate of 11 cm^3/cm.

5 a The area of a circle is $A = \pi r^2$.

$$\therefore \frac{dA}{dr} = 2\pi r$$

When $r = 0.2$, $\frac{dA}{dr} = 0.4\pi$.

The area is changing with respect to its radius at 0.4π m^2/m at the instant its radius is 0.2 metres.

b Let the length of the edge of the cube be x mm.

The surface area is the sum of the area of its six square faces.

$$\therefore A = 6x^2$$

The rate at which the surface area changes with respect to its side length is $\frac{dA}{dx} = 12x$.

When $x = 8$, $\frac{dA}{dx} = 96$.

The surface area is changing at 96 mm^2/mm.

6 a An equilateral triangle has angles of 60°.

The area of a triangle is $A = \frac{1}{2}ab\sin C$

$$\begin{aligned}\therefore A &= \frac{1}{2} \times x \times x \times \sin(60°)\\ &= \frac{1}{2}x^2 \times \frac{\sqrt{3}}{2}\end{aligned}$$

$$\therefore A = \frac{\sqrt{3}}{4}x^2$$

b $\frac{dA}{dx} = \frac{\sqrt{3}}{2}x$

When $x = 2$, $\frac{dA}{dx} = \sqrt{3}$.

The area is changing at $\sqrt{3}$ sq cm/cm.

c When $A = 64\sqrt{3}$,

$$\frac{\sqrt{3}}{4}x^2 = 64\sqrt{3}$$
$$\therefore x^2 = 64\sqrt{3} \times \frac{4}{\sqrt{3}}$$
$$\therefore x^2 = 256$$
$$\therefore x = 16$$

(The negative square root is not applicable.)

When $x = 16$, $\frac{dA}{dx} = 8\sqrt{3}$.

The area is changing at $8\sqrt{3}$ sq cm/cm.

7 a i Refer to the diagram given in the question. The two right-angled triangles are similar because they are equiangular.

$$\therefore \frac{1.5}{9} = \frac{s}{s+x}$$
$$\therefore \frac{1}{6} = \frac{s}{s+x}$$
$$\therefore s + x = 6s$$
$$\therefore 5s = x$$
$$\therefore s = \frac{x}{5}$$

ii $\frac{ds}{dx} = \frac{1}{5}$

This measures the rate of change of the length of the shadow with respect to the distance of the person from the foot of the pole. It shows the length of the shadow changes at a constant rate of $\frac{1}{5}$ m/m.

b The volume of a cylinder is $V = \pi r^2 h$ or $V = Ah$ where A is the area of its circular base.

i As the oil leaks out, the depth of water decreases, but the area of the circular base does not change.

When the oil drum is full, $V = 0.25$ and $h = 0.75$.

$$\therefore 0.25 = A \times 0.75$$
$$\therefore A = \frac{0.25}{0.75}$$
$$\therefore A = \frac{1}{3}\text{m}^2$$

ii Hence, the volume is $V = \frac{1}{3}h$.

The rate of decrease of the volume with respect to the depth of oil is $\frac{dV}{dh} = \frac{1}{3}$.

Therefore, the volume is decreasing at $\frac{1}{3}$ m^3/m.

8 $x = 3t^2 - 6t,\ t \geq 0$

a When $t = 0$, $x = 0$ so initially the particle is at the origin.

b Velocity:

$$v = \frac{dx}{dt}$$
$$\therefore v = 6t - 6$$

The initial velocity is −6 m/s. The particle starts to move to the left.

c When $v = 0,\ 6t - 6 = 0$

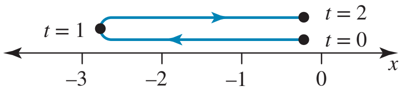
Therefore, $t = 1$ and when $t = 1$, $x = -3$.

The particle is momentarily at rest after 1 second at the position 3 m to the left of the origin.

d When $t = 2$,

$$x = 3(4) - 6(2)$$
$$= 0$$

Therefore, the particle is at the origin after 2 seconds.



The path of the particle shows the distance travelled is 3 m to the left and 3 m to the right, giving a total distance of 6 m travelled in the first 2 seconds.

e Average speed $= \frac{\text{distance}}{\text{time}}$

Therefore, average speed $\frac{6}{2} = 3$ m/s

f Average velocity $= \frac{x(2) - x(0)}{2 - 0} = 0$ m/s

9 $x = 12t + 9,\ t \geq 0$

a The rate of change of the displacement is $\frac{dx}{dt}$.

$$\frac{dx}{dt} = 12$$

b Velocity is the rate of change of displacement. Hence, from part **a**, $\frac{dx}{dt} = 12 \Rightarrow v = 12$.

The velocity is always a constant 12 m/s.

c When $t = 0, x = 9$ and when $t = 1, x = 12 + 9 = 21$.

The distance travelled is $21 - 9 = 12$ m.

d Let $x = 45$.

$$12t + 9 = 45$$
$$12t = 36$$
$$t = 3.$$

After 3 seconds, the particle is 45 metres to the right of the origin.

10 $x = t^2 - 6t - 7,\ t \geq 0$

a To find the initial position, let $t = 0$.

$$\therefore x = -7$$

Initially, the particle is 7 m to the left of the origin.

b $v = \frac{dx}{dt}$

$$\therefore v = 2t - 6$$

To obtain the initial velocity, let $t = 0$.

$$\therefore v = -6$$

The initial velocity is −6 m/s.

c The particle is momentarily at rest when $v = 0$.

$$\therefore 2t - 6 = 0$$
$$\therefore t = 3$$

After 3 seconds, the particle is momentarily at rest.

d Let $x = -7$.

$$\therefore t^2 - 6t - 7 = -7$$
$$\therefore t^2 - 6t = 0$$
$$\therefore t(t - 6) = 0$$
$$\therefore t = 0, t = 6.$$

The particle returns to its initial position after 6 seconds.

When $t = 6$,

$$v = 2t - 6$$
$$= 2(6) - 6$$
$$= 6$$

The particle returns to its initial position after 6 seconds with a velocity of 6 m/s.

11 $x = t^3 - 4t^2 - 3t + 12,\ t \geq 0$

a Velocity: $v = \dfrac{dx}{dt} = 3t^2 - 8t - 3$

Acceleration: $a = \dfrac{dv}{dt} = 6t - 8$

b Let $v = 0$.

$$3t^2 - 8t - 3 = 0$$
$$(3t + 1)(t - 3) = 0$$
$$t = -\frac{1}{3}(\text{reject, } t \geq 0), t = 3$$
$$\therefore t = 3$$

The velocity is zero after 3 seconds.

$a(3) = 6(3) - 8 = 10$

The acceleration is 10 m/s^2 at that instant.

$x(3) = 27 - 4 \times 9 - 9 + 12 = -6$

The particle is 6 m to the left of the origin at that instant.

c Average velocity over the first 3 seconds is $\dfrac{x(3) - x(0)}{3 - 0}$.

$x(3) = -6$ and $x(0) = 12$

Average velocity

$$= \frac{-6 - 12}{3}$$
$$= \frac{-18}{3}$$
$$= -6$$

The average velocity over the first 3 seconds is -6 m/s.

12 $x = 5t - 10,\ t \geq 0$

a Let $t = 0$.

$\therefore x = -10$

Initially, the particle is 10 cm to the left of the fixed origin.

Let $t = 3$.

$\therefore x = 15 - 10 = 5$

After 3 seconds, the particle is 5 cm to the right of the origin.

b The distance between the positions $x = -10$ and $x = 5$ is 15 cm.

c The velocity is the rate of change of the displacement,

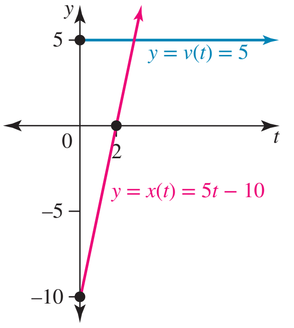
$v = \dfrac{dx}{dt}$

$\therefore v = \dfrac{d}{dt}(5t - 10)$

$\therefore v = 5$

The particle moves with a constant velocity of 5 cm/s.

d $x = 5t - 10,\ t \geq 0$



The velocity graph is the gradient graph of the displacement graph.

13 $x = 6t - t^2,\ t \geq 0$

a Velocity: $v = \dfrac{dx}{dt} = 6 - 2t$

Acceleration: $a = \dfrac{dv}{dt} = -2$

b Displacement–time graph: $x = 6t - t^2$

$\therefore x = t(6 - t)$

t-intercepts occur at $t = 0, t = 6$

Therefore, the turning point occurs at $t = 3$.

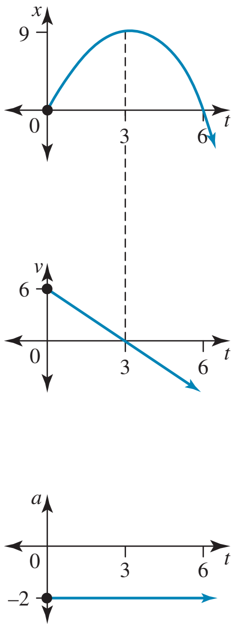
When $t = 3$, $x = 9$, so the maximum turning point is $(3, 9)$.

Velocity–time graph: $v = 6 - 2t$

Points $(0, 6)$ and $(3, 0)$.

Acceleration–time graph: $a = -2$

Horizontal line with end point $(0, -2)$.



The displacement–time graph is quadratic with maximum turning point when $t = 3$; the velocity–time graph is linear with $v = 0$ at the t-intercept of $t = 3$; the acceleration–time graph is a horizontal line, since the acceleration is constant. The acceleration is the gradient of the velocity graph.

c The velocity is zero when $t = 3$. At this time, the displacement graph is at its maximum turning point $(3, 9)$. The velocity is zero after 3 seconds when the value of x is 9. The displacement is 9 metres to the right of the origin.

d The displacement graph has a positive gradient for $0 \leq t < 3$. Over this same interval the velocity graph lies above the horizontal axis, so the velocity is positive.

14 $P(t) = 3t(200 - 2t)$

a The rate of change of the population of bacteria with respect to time is $P'(t)$.

$P(t) = 600t - 6t^2$

$P'(t) = 600 - 12t$

b When $t = 20$, $P'(20) = 600 - 12 \times 20 = 360$.

The rate of change of the population is 360 bacteria per hour.

The population is increasing as the rate of change is positive.

c When $t = 60$, $P'(60) = 600 - 12 \times 60 = -120$.

The rate of change of the population is -120 bacteria per hour.

The population is decreasing as the rate of change is negative.

d **i** Let the rate of change $P'(t) = 240$.

$240 = 600 - 12t$

$12t = 360$

$t = 30$

After 30 hours, the population is increasing at 240 bacteria per hour.

ii Let the rate of change $P'(t) = -240$

$-240 = 600 - 12t$

$12t = 840$

$t = 70.$

After 70 hours, the population is decreasing at 240 bacteria per hour.

iii Let the rate of change $P'(t) = 0$

$0 = 600 - 12t$

$12t = 600$

$t = 50.$

After 50 hours, the population is neither increasing nor decreasing.

15 $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1$

a $x(0) = 1$, so the particle is initially 1 metre to the right of the origin.

$v(t) = x'(t)$

$= -t^2 + 2t + 8$

$\therefore v(0) = 8$ so initial velocity is 8 m/s.

b The particle changes its direction of motion when its velocity is zero.

$-t^2 + 2t + 8 = 0$

$-(t-4)(t+2) = 0$

$t = 4,\ t = -2$

Since $t \geq 0$, the velocity is zero when $t = 4$.

$x(4) = -\frac{64}{3} + 16 + 32 + 1$

$= \frac{83}{3}$

The distance travelled is $\frac{83}{3} - 1 = \frac{80}{3}$ metres

$= 26\frac{2}{3}$ metres.

c $a = v'(t)$

$\therefore a(t) = -2t + 2$

When $t = 4$, $a(4) = -6$, so the acceleration is -6 m/s^2.

16 $x = \frac{1}{3}t^3 - t^2,\ t \geq 0$

a When $t = 0$, $x = 0$.

Velocity: $v = \frac{dx}{dt}$

$\therefore v = t^2 - 2t$

When $t = 0$, $v = 0$.

Therefore, the particle starts at the origin from rest.

b Let $v = 0$.

$\therefore t^2 - 2t = 0$

$\therefore t(t-2) = 0$

$\therefore t = 0,\ t = 2$

The particle is next at rest at $t = 2$.

Its position when $t = 2$ is $x = \frac{8}{3} - 4 = -\frac{4}{3}$.

The particle is next at rest after 2 seconds when it is $\frac{4}{3}$ cm to the left of the origin.

c Let $x = 0$

$\therefore \frac{1}{3}t^3 - t^2 = 0$

$\therefore t^3 - 3t^2 = 0$

$\therefore t^2(t-3) = 0$

$\therefore t = 0,\ t = 3$

The particle returns to the origin after 3 seconds.

d When $t = 3$, $v = 9 - 6 = 3$.

The particle's speed when it returns to the origin is 3 cm/s.

Acceleration $a = \frac{dv}{dt}$

$\therefore a = 2t - 2$

When $t = 3$, $a = 4$.

The acceleration of the particle when it returns to the origin is 4 cm/s^2.

e Displacement–time graph $x = \frac{1}{3}t^3 - t^2$

This is a cubic graph with t-intercepts at $t = 0, t = 3$ and a minimum turning point at $\left(2, -\frac{4}{3}\right)$, since its derivative is zero when $t = 2$.

Velocity–time graph $v = t^2 - 2t$

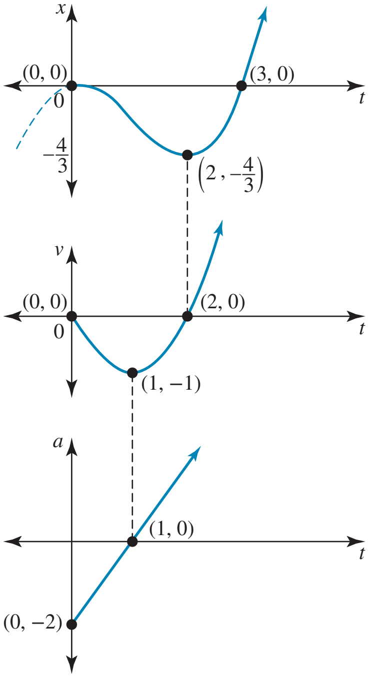
This is a quadratic graph with t-intercepts at $t = 0, t = 2$.

Turning point when $t = 1$ and therefore $v = -1$: minimum turning point $(1, -1)$

Acceleration–time graph $a = 2t - 2$

This is a linear graph containing points $(0, -2)$ and $(1, 0)$.

When $t = 2$, $a = 2$, so the point $(2, 2)$ is also on the graph.



At $t = 2$, the displacement graph reaches its most negative displacement and the velocity is zero. The acceleration is positive with value 2 cm/s^2. This represents the instant when the particle ceases to move to the left and starts to move to towards the right.

f When $t = 1$, the velocity reaches its most negative value. The particle is moving to the left since its displacement is negative, but at $t = 1$ the velocity starts to reduce to less negative values. At that instant the acceleration is zero as it changes from negative to positive acceleration. The overall effect is to start to slow the particle down until the point $t = 2$, where it will change its direction of motion and begin to move to the right under positive acceleration.

17 $h = 40t - 5t^2$

a The rate of change of the height of the ball is given $\frac{dh}{dt}$.

$$\frac{dh}{dt} = 40 - 10t$$

When $t = 2$, $\frac{dh}{dt} = 20$.

The height is changing at 20 m/s.

b The velocity is the rate of change of height.

$$\therefore v = \frac{dh}{dt} = 40 - 10t$$

When $t = 3$, $v = 10$.

The vertical velocity upwards is 10 m/s.

c Let $v = -10$.

$$\therefore 40 - 10t = -10$$
$$\therefore 50 = 10t$$
$$\therefore t = 5$$

After 5 seconds, the velocity of the ball is -10 m/s. The negative sign indicates the ball is travelling vertically downwards towards the ground.

d Let $v = 0$.

$$\therefore 40 - 10t = 0$$
$$\therefore t = 4$$

The velocity is zero after 4 seconds.

e The greatest height occurs when $\frac{dh}{dt} = v = 0$. Hence, the greatest height occurs when $t = 4$.

When $t = 4$, $h = 160 - 80 = 80$.

The greatest height the ball reaches is 80 metres above the ground.

f When the ball reaches the ground, $h = 0$.

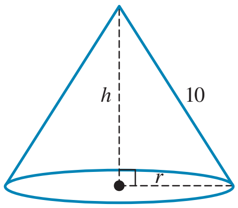
$$\therefore 40t - 5t^2 = 0$$
$$\therefore 5t(8 - t) = 0$$
$$\therefore t = 0, t = 8$$

The ball returns to the ground after 8 seconds.

When $t = 8$, $v = 40 - 80 = -40$.

The ball strikes the ground with speed 40 m/s.

18 a



Using Pythagoras' theorem:

$$h^2 + r^2 = 10^2$$
$$\therefore r^2 = 100 - h^2$$

Since $r > 0$, $r = \sqrt{100 - h^2}$

b The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

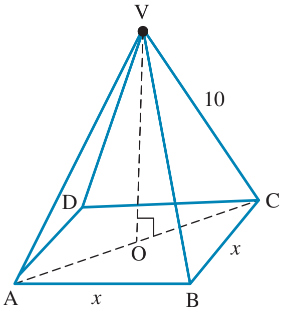
$$V = \frac{1}{3}\pi(100 - h^2)h$$
$$= \frac{1}{3}\pi(100h - h^3)$$

c Rate: $\frac{dV}{dh} = \frac{1}{3}\pi(100 - 3h^2)$

When $h = 6$, $\frac{dV}{dh} = -\frac{8\pi}{3}$.

With respect to its height, the volume is decreasing at the rate of $\frac{8\pi}{3}$ cm^3/cm when the height is 6 cm.

19



a Let the diagonal AC of the square base be of length l metres.

Using Pythagoras' theorem in the right-angled triangle ABC,

$$l^2 = x^2 + x^2$$
$$\therefore l^2 = 2x^2$$
$$\therefore l = \sqrt{2}x$$

(negative square root is not applicable)

The base diagonal is of length $\sqrt{2}x$ metres.

b Since AC has length $\sqrt{2}x$ metres, OC length $\frac{1}{2} \times \sqrt{2}x$ metres.

Consider the right-angled triangle OCV where

$OC = \frac{\sqrt{2}x}{2}$, $OV = h$ and $VC = 10$.

Using Pythagoras' theorem,

$$h^2 + \left(\frac{\sqrt{2}x}{2}\right)^2 = 10^2$$
$$\therefore h^2 + \frac{x^2}{2} = 100$$
$$\therefore 2h^2 + x^2 = 200$$
$$\therefore x^2 = 200 - 2h^2$$

The volume of air in the tent is $V = \frac{1}{3}Ah$

$$\therefore V = \frac{1}{3} \times (x^2) \times h$$

Substitute $x^2 = 200 - 2h^2$.

$$\therefore V = \frac{1}{3} \times (200 - 2h^2) \times h$$
$$\therefore V = \frac{1}{3}(200h - 2h^3)$$

c $\frac{dV}{dh} = \frac{1}{3}(200 - 6h^2)$

When $h = 2\sqrt{3}$,

$$\frac{dV}{dh} = \frac{1}{3}(200 - 6 \times 4 \times 3)$$
$$= \frac{128}{3}$$
$$= 42\frac{2}{3}$$

The volume is changing at the rate $42\frac{2}{3}$ m^3/m.

d Let $\frac{dV}{dh} = 0$.

$$\therefore \frac{1}{3}\left(200 - 6h^2\right) = 0$$
$$\therefore 200 - 6h^2 = 0$$
$$\therefore h^2 = \frac{200}{6}$$
$$\therefore h^2 = \frac{100}{3}$$
$$\therefore h = \frac{10}{\sqrt{3}}$$

(negative square root not applicable)

The greatest volume occurs when the rate of change of the volume is instantaneously zero. Hence, the value $h = \frac{10}{\sqrt{3}}$ is the height of the tent for it to hold the greatest volume of air.

20 $x(t) = 3t^2 - 24t - 27,\ t \geq 0$

a $x(2) = 12 - 48 - 27 = -63$

The particle is a distance of 63 metres from O.

b $v = x'(t)$

$$\therefore v(t) = 6t - 24$$
$$v(2) = 12 - 24$$
$$= -12$$

The speed is 12 m/s.

c Average velocity is the average rate of change of displacement.

$$= \frac{x(2) - x(0)}{2 - 0}$$
$$= \frac{-63 - (-27)}{2}$$
$$= -18$$

The average velocity over the first two seconds of motion is −18 m/s.

d Let $x = 0$.

$$\therefore 3t^2 - 24t - 27 = 0$$
$$\therefore t^2 - 8t - 9 = 0$$
$$\therefore (t - 9)(t + 1) = 0$$
$$\therefore t = 9 \text{ or } t = -1 \text{ (reject)}$$
$$\therefore t = 9$$
$$v(9) = 54 - 24$$
$$= 30$$

The particle returns to the origin after 9 seconds with a velocity of 30 m/s.

e The particle's position after 6 seconds is $x(6) = 108 - 144 - 27 = -63$.

From the earlier working it is known that $x(0) = -27$, $x(2) = -63$ and $x(9) = 0$. Initially the particle moved to the left but at some time changed direction.

To find the time when the motion changed direction, let $v = 0$.

$$\therefore 6t - 24 = 0$$
$$\therefore t = 4$$

The position of the particle when it changes direction at $t = 4$ is $x(4) = 48 - 96 - 27 = -75$.

See the figure at the bottom of the page.*

The distance travelled in the first 6 seconds of motion = (48 + 12) = 60 metres.

f Average speed $= \frac{\text{distance travelled}}{\text{time taken}}$

$$= \frac{60}{6}$$
$$= 10$$

The average speed was 10 m/s.

21 $x_P(t) = t^3 - 12t^2 + 45t - 34$

a The particle is stationary when its velocity is zero.

$$v_P = x'(t)$$
$$= 3t^2 - 24t + 45$$

Let $v_P = 0$.

$$\therefore 3t^2 - 24t + 45 = 0$$
$$\therefore t^2 - 8t + 15 = 0$$
$$\therefore (t - 3)(t - 5) = 0$$
$$\therefore t = 3, t = 5$$

The particle P is instantaneously stationary after 3 seconds and after 5 seconds.

b If $v < 0$, then $(t - 3)(t - 5) < 0$.

Therefore, $v < 0$ when $3 < t < 5$.

The velocity is negative for the time interval $t \in (3, 5)$.

c $a_P = v'(t)$

$$\therefore a_P = 6t - 24$$

If $a_P < 0$, then $6t - 24 < 0$.

$$\therefore t < 4$$

The acceleration is negative for the time interval $t \in [0, 4)$

d $x_Q(t) = -12t^2 + 54t - 44$

$v_Q(t) = -24t + 54$

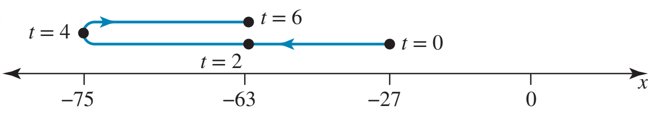
P and Q have the same velocities when $v_P = v_Q$.

$$\therefore 3t^2 - 24t + 45 = -24t + 54$$
$$\therefore 3t^2 = 9$$
$$\therefore t^2 = 3$$
$$\therefore t = \sqrt{3}$$

(negative square root not applicable)

Particles P and Q are travelling with the same velocities after $\sqrt{3}$ seconds.

***20 e**



e P and Q have the same displacements when $x_P = x_Q$.

$\therefore t^3 - 12t^2 + 45t - 34 = -12t^2 + 54t - 44$

$\therefore t^3 - 9t + 10 = 0$

By inspection, $t = 2$ is a solution and therefore $(t - 2)$ is a factor.

$\therefore t^3 - 9t + 10 = (t-2)\left(t^2 + 2t - 5\right) = 0$

$\therefore t = 2$ or $t^2 + 2t - 5 = 0$

Consider $t^2 + 2t - 5 = 0$

Completing the square,

$\left(t^2 + 2t + 1\right) - 1 - 5 = 0$

$\therefore (t+1)^2 = 6$

$\therefore t = \pm\sqrt{6} - 1$

However, $t = -\sqrt{6} - 1 < 0$, so reject this solution.

P and Q have the same displacements when $t = 2$ and $t = \sqrt{6} - 1$; that is, their displacements are equal after $\left(\sqrt{6} - 1\right)$ seconds and after 2 seconds.

22 $x = 0.25t^4 - t^3 + 1.5t^2 - t - 3.75$

a $x(0) = -3.75$ and $x(4)$ evaluates to give $x(4) = 16.25$.

We need to check whether or not there has been a change of direction between these two positions.

$v = \dfrac{dx}{dt}$

$\therefore v = t^3 - 3t^2 + 3t - 1$

$\therefore v = (t-1)^3$

The velocity is zero when $t = 1$, so the direction of motion does change between $t = 0$ and $t = 4$.

$x(1) = -4$

The particle travels $x(0) = -3.75 \rightarrow x(1) = -4 \rightarrow$ origin $\rightarrow x(4) = 16.25$.

The distance travelled is $(0.25 + 4 + 16.25) = 20.5$ metres.

b Let $x = 0$.

Solve $0.25t^4 - t^3 + 1.5t^2 - t - 3.75 = 0$ to obtain $t = -1, t = 3$.

Rejecting $t = -1$, the particle is at the origin after 3 seconds.

c $a = \dfrac{dv}{dt}$

$\therefore a = 3t^2 - 6t + 3$

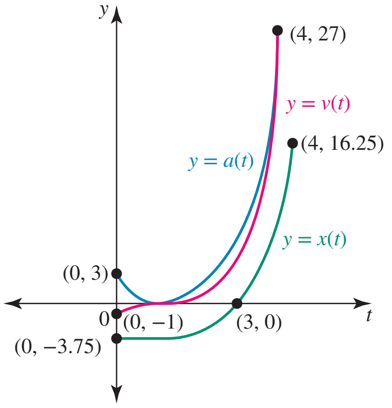
$= 3\left(t^2 - 2t + 1\right)$

$\therefore a = 3(t-1)^2$

$\therefore a \geq 0$

Therefore, the acceleration is never negative.

d The graphs obtained should be similar to those shown.



e Both the cubic velocity graph and the quadratic acceleration graph are positive for all values $t > 1$, and as t increases beyond $t = 1$, both the velocity and the acceleration continue to increase. The particle is at the origin at $t = 3$. Since its velocity and acceleration are both positive, the particle continues moving to the right with increasing velocity due to the increasing acceleration, so the particle can never return to the origin.

12.6 Exam questions

1 a $x = 2t^2 - 8t + 9, t \geq 0$

$v = 4t - 8$

Initial velocity $t = 0$

$v = -8$

The initial elocity is -8 m/s. [1 mark]

Particle at rest: $\dfrac{dx}{dt} = 0$

$0 = 4t - 8$

$t = 2$

The particle is at rest after 2 seconds. [1 mark]

Position when the particle is at rest:

$x = 2t^2 - 8t + 9$

$x = 2(2)^2 - 8(2) + 9$

$x = 1$

Therefore, the particle is momentarily at rest after 2 seconds at the position 1 metre to the right of the origin. [1 mark]

b Average velocity is the average rate of change of displacement.

For the first 3 seconds, when $t = 3$:

$x = 2t^2 - 8t + 9$

$x = 2(3)^2 - 8(3) + 9$

$x = 3$

$(t_1, x) = (0, 9), (t_2, x_2) = (3, 3)$ [1 mark]

$$\text{Average velocity} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{3-9}{3-0} = -2$$

The average velocity for the first 3 seconds is -2 m/s. [1 mark]

2 $x = 2t^2 - 15t + 23, t \geq 0$

$\dfrac{dx}{dt} = 4t - 15$

When $t = 3$,

$\dfrac{dx}{dt} = 4(3) - 15$

$= -3$

$\therefore$ speed is 3 m/s

(Speed is not concerned with the direction of motion and is never negative.)

The correct answer is **C**.

3 Area $= \pi r^2$

Rate of change:

$\dfrac{dA}{dr} = 2\pi r$

$r = 3$

$\dfrac{dA}{dr} = 6\pi \text{cm}^2/\text{cm}$

The correct answer is **D**.

12.7 Derivatives of power functions

12.7 Exercise

1 a
$$\begin{aligned} f(x) &= x^{-3}\\ \therefore f'(x) &= -3x^{-3-1}\\ \therefore f'(x) &= -3x^{-4} \end{aligned}$$

b
$$\begin{aligned} f(x) &= x^{-6}\\ \therefore f'(x) &= -4x^{-4-1}\\ \therefore f'(x) &= -4x^{-5} \end{aligned}$$

c
$$\begin{aligned} f(x) &= x^{\frac{5}{2}}\\ \therefore f'(x) &= \frac{5}{2}x^{\frac{5}{2}-1}\\ \therefore f'(x) &= \frac{5}{2}x^{\frac{3}{2}} \end{aligned}$$

d
$$\begin{aligned} f(x) &= x^{\frac{2}{3}}\\ \therefore f'(x) &= \frac{2}{3}x^{\frac{2}{3}-1}\\ \therefore f'(x) &= \frac{2}{3}x^{-\frac{1}{3}} \end{aligned}$$

e
$$\begin{aligned} f(x) &= 4x^{-2}\\ \therefore f'(x) &= 4\times(-2)x^{-2-1}\\ \therefore f'(x) &= -8x^{-3} \end{aligned}$$

f
$$\begin{aligned} f(x) &= \frac{1}{2}x^{-\frac{6}{5}}\\ \therefore f'(x) &= \frac{1}{2}\times\left(-\frac{6}{5}\right)x^{-\frac{6}{5}-1}\\ \therefore f'(x) &= -\frac{3}{5}x^{-\frac{11}{5}} \end{aligned}$$

2 a
$$\begin{aligned} y &= 4x^{-1}+5x^{-2}\\ \frac{dy}{dx} &= 4\times(-1)x^{-1-1}+5\times(-2)x^{-2-1}\\ &= -4x^{-2}-10x^{-3} \end{aligned}$$

b
$$\begin{aligned} y &= 4x^{\frac{1}{2}}-3x^{\frac{2}{3}}\\ \frac{dy}{dx} &= 4\times\frac{1}{2}x^{\frac{1}{2}-1}-3\times\frac{2}{3}x^{\frac{2}{3}-1}\\ &= 2x^{-\frac{1}{2}}-2x^{-\frac{1}{3}} \end{aligned}$$

c
$$\begin{aligned} y &= 2+8x^{-\frac{1}{2}}\\ \frac{dy}{dx} &= 8\times\frac{-1}{2}x^{-\frac{1}{2}-1}\\ &= -4x^{-\frac{3}{2}} \end{aligned}$$

d
$$\begin{aligned} y &= 0.5x^{1.8}-6x^{3.1}\\ \frac{dy}{dx} &= 0.5\times1.8x^{1.8-1}-6\times3.1x^{3.1-1}\\ &= 0.9x^{0.8}-18.6x^{2.1} \end{aligned}$$

3 a
$$\begin{aligned} y &= 1+\frac{1}{x}+\frac{1}{x^2}\\ \therefore y &= 1+x^{-1}+x^{-2}\\ \frac{dy}{dx} &= -1x^{-2}-2x^{-3}\\ &= -\frac{1}{x^2}-\frac{2}{x^3} \end{aligned}$$

b
$$\begin{aligned} y &= 5\sqrt{x}\\ \therefore y &= 5x^{\frac{1}{2}}\\ \frac{dy}{dx} &= 5\times\frac{1}{2}x^{-\frac{1}{2}}\\ &= \frac{5}{2x^{\frac{1}{2}}}\\ &= \frac{5}{2\sqrt{x}} \end{aligned}$$

c
$$\begin{aligned} y &= 3x-\sqrt[3]{x}\\ \therefore y &= 3x-x^{\frac{1}{3}}\\ \frac{dy}{dx} &= 3-\frac{1}{3}x^{-\frac{2}{3}}\\ &= 3-\frac{1}{3x^{\frac{2}{3}}} \end{aligned}$$

d
$$\begin{aligned} y &= x\sqrt{x}\\ \therefore y &= x\times x^{\frac{1}{2}}\\ \therefore y &= x^{\frac{3}{2}}\\ \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}}\\ &= \frac{3\sqrt{x}}{2} \end{aligned}$$

e
$$\begin{aligned} y &= \frac{3}{x^8}\\ y &= 3x^{-8}\\ \frac{dy}{dx} &= 3\times(-8)\,x^{-8-1}\\ &= -24x^{-9}\\ \therefore \frac{dy}{dx} &= -\frac{24}{x^9} \end{aligned}$$

f
$$\begin{aligned} y &= \frac{4}{x}-5x^3\\ y &= 4x^{-1}-5x^3\\ \frac{dy}{dx} &= -4x^{-1-1}-5\times3x^{3-1}\\ &= -4x^{-2}-15x^2\\ \therefore \frac{dy}{dx} &= -\frac{4}{x^2}-15x^2 \end{aligned}$$

4 a
$$\begin{aligned} y &= \frac{4-3x+7x^4}{x^4}\\ &= \frac{4}{x^4}-\frac{3x}{x^4}+\frac{7x^4}{x^4}\\ &= 4x^{-4}-3x^{-3}+7\\ \frac{dy}{dx} &= -16x^{-5}+9x^{-4}\\ &= -\frac{16}{x^5}+\frac{9}{x^4} \end{aligned}$$

The derivative has x terms in its denominator, so its domain is $R\backslash\{0\}$.

b i $f(x) = 4\sqrt{x} + \sqrt{2x}$

$= 4\sqrt{x} + \sqrt{2}\sqrt{x}$

$= 4x^{\frac{1}{2}} + \sqrt{2}x^{\frac{1}{2}}$

$f'(x) = 2x^{-\frac{1}{2}} + \frac{\sqrt{2}}{2}x^{-\frac{1}{2}}$

$= \frac{2}{\sqrt{x}} + \frac{\sqrt{2}}{2\sqrt{x}}$

The derivative has $\sqrt{x}$ terms in its denominator, so its domain is R^+.

ii At $x = 1$, gradient is $f'(1)$

$f'(1) = \frac{2}{1} + \frac{\sqrt{2}}{2}$

$= \frac{4+\sqrt{2}}{2}$

5 a $y = 3\sqrt{x}$

$y = 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = 3 \times \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{3}{2}x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2\sqrt{x}}$

At the point where $x = 9$,

$\frac{dy}{dx} = \frac{3}{2\sqrt{9}} = \frac{1}{2}$

The gradient of the tangent is $\frac{1}{2}$.

b $y = \sqrt[3]{x} + 10$

$y = x^{\frac{1}{3}} + 10$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

$\frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}}$

At the point $(-8, 8)$,

$\frac{dy}{dx} = \frac{1}{3(-8)^{\frac{2}{3}}}$

$= \frac{1}{3\left(\sqrt[3]{-8}\right)^2}$

$= \frac{1}{3(-2)^2}$

$= \frac{1}{12}$

The gradient of the tangent is $\frac{1}{12}$.

c $y = \frac{5}{x} - 1$

x-intercept: let $y = 0$.

$\frac{5}{x} - 1 = 0$

$\frac{5}{x} = 1$

$5 = x$

x-intercept is $(5, 0)$.

Gradient of curve:

$y = \frac{5}{x} - 1$

$y = 5x^{-1} - 1$

$\frac{dy}{dx} = -5x^{-2}$

$\frac{dy}{dx} = -\frac{5}{x^2}$

At the point $(5, 0)$,

$\frac{dy}{dx} = -\frac{5}{(5)^2} = -\frac{1}{5}$

The gradient of the tangent is $-\frac{1}{5}$.

d $y = 6 - \frac{2}{x^2}$

$y = 6 - 2x^{-2}$

$\frac{dy}{dx} = 4x^{-3}$

$\frac{dy}{dx} = \frac{4}{x^3}$

Let $\frac{dy}{dx} = \frac{1}{2}$

$\frac{1}{2} = \frac{4}{x^3}$

$x^3 = 8$

$x = \sqrt[3]{8}$

$x = 2$

Substitute $x = 2$ in $y = 6 - \frac{2}{x^2}$ to obtain the y-coordinate of the point.

$y = 6 - \frac{2}{(2)^2}$

$= 6 - \frac{1}{2}$

$= 5.5$

The required point is $(2, 5.5)$.

e $y = \frac{4}{3x}$

$y = \frac{4}{3}x^{-1}$

$\frac{dy}{dx} = -\frac{4}{3}x^{-2}$

$\frac{dy}{dx} = -\frac{4}{3x^2}$

Let $\frac{dy}{dx} = -12$

$-12 = -\frac{4}{3x^2}$

$36x^2 = 4$

$x^2 = \frac{4}{36}$

$x^2 = \frac{1}{9}$

$x = \pm\frac{1}{3}$

Substitute $x = \frac{1}{3}$ in $y = \frac{4}{3x}$.

$y = \frac{4}{3\left(\frac{1}{3}\right)}$

$y = 4$

Substitute $x = -\frac{1}{3}$ in $y = \frac{4}{3x}$.

$y = \frac{4}{3\left(-\frac{1}{3}\right)}$

$y = -4$

The curve has gradient -12 at the points $\left(\frac{1}{3}, 4\right)$ and $\left(-\frac{1}{3}, -4\right)$.

f $y = 3 - 2\sqrt{x}$

$y = 3 - 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = -x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{\sqrt{x}}$

For the tangent to be parallel to the line $y = -\frac{2}{3}x$, its gradient must be $-\frac{2}{3}$.

Let $\frac{dy}{dx} = -\frac{2}{3}$

$-\frac{2}{3} = -\frac{1}{\sqrt{x}}$

$2\sqrt{x} = 3$

$\sqrt{x} = \frac{3}{2}$

$x = \left(\frac{3}{2}\right)^2$

$x = \frac{9}{4}$

Substitute $x = \frac{9}{4}$ in $y = 3 - 2\sqrt{x}$.

$y = 3 - 2\sqrt{\frac{9}{4}}$

$= 3 - 2 \times \frac{3}{2}$

$= 0$

The required point is $\left(\frac{9}{4}, 0\right)$.

6 a $f(x) = x^2 + \frac{2}{x}$

$\therefore f(x) = x^2 + 2x^{-1}$

$f'(x) = 2x - 2x^{-2}$

$= 2x - \frac{2}{x^2}$

$\therefore f'(2) = 4 - \frac{2}{4}$

$\therefore f'(2) = 3.5$

b Let $f'(x) = 0$

$\therefore 2x - \frac{2}{x^2} = 0$

$\therefore 2x = \frac{2}{x^2}$

$\therefore x^3 = 1$

$\therefore x = 1$

$f(1) = 1 + 2 = 3$

At the point (1, 3), $f'(x) = 0$.

c Let $f'(x) = -4$

$\therefore 2x - \frac{2}{x^2} = -4$

$\therefore 2x^3 - 2 = -4x^2$

$\therefore x^3 + 2x^2 - 1 = 0$

Let $P(x) = x^3 + 2x^2 - 1$

$P(-1) = -1 + 2 - 1 = 0$

$\therefore (x + 1)$ is a factor.

$\therefore x^3 + 2x^2 - 1 = (x + 1)\left(x^2 + x - 1\right) = 0$

$\therefore x = -1$ or $x^2 + x - 1 = 0$

$\therefore x = -1$ or $x = \frac{-1 \pm \sqrt{1 + 4}}{2}$

$\therefore x = -1$ or $x = \frac{-1 \pm \sqrt{5}}{2}$

7 $h = 0.5 + \sqrt{t}$

a Let $t = 0$

$\therefore h = 0.5$

The tree was 0.5 metres when first planted.

b $h = 0.5 + t^{\frac{1}{2}}$

$\therefore \frac{dh}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$

$\therefore \frac{dh}{dt} = \frac{1}{2\sqrt{t}}$

When $t = 4$,

$\frac{dh}{dt} = \frac{1}{2\sqrt{4}}$

$= \frac{1}{4}$

After 4 years, the height is growing at 0.25 metres per year.

c Let $h = 3$

$\therefore 0.5 + \sqrt{t} = 3$

$\therefore \sqrt{t} = 2.5$

$\therefore t = 6.25$

The tree is 3 metres tall, $6\frac{1}{4}$ years after it was planted.

d $t = 0, h = 0.5$ and $t = 6.25, h = 3$

Average rate of growth

$= \frac{3 - 0.5}{6.25 - 0}$

$= \frac{2.5}{6.25}$

$= \frac{1}{2.5}$

$= 0.4$

The average rate of growth is 0.4 metres per year.

8 a $V = \frac{60t + 2}{3t}, t > 0$

$\therefore V = \frac{60t}{3t} + \frac{2}{3t}$

$\therefore V = 20 + \frac{2}{3}t^{-1}$

$\frac{dV}{dt} = -\frac{2}{3}t^{-2}$

$= -\frac{2}{3t^2}$

Since $t^2 > 0,\ -\dfrac{2}{3t^2} < 0$. Hence, $\dfrac{dV}{dt} < 0$.

b Let $t = 2$

$$\begin{aligned}\frac{dV}{dt} &= -\frac{2}{3(2)^2}\\ &= -\frac{1}{6}\end{aligned}$$

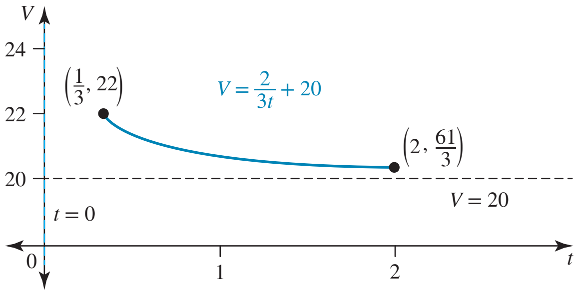
The water is evaporating at $\dfrac{1}{6}$ mL per hour.

c $V = \dfrac{2}{3t} + 20,\ t \in \left[\dfrac{1}{3}, 2\right]$

First quadrant branch of hyperbola with asymptotes $t = 0,\ V = 20$

End points: when $t = \dfrac{1}{3}$, $V = 2 + 20 = 22$. Point $\left(\dfrac{1}{3}, 22\right)$

When $t = 2$, $V = \dfrac{1}{3} + 20 = \dfrac{61}{3}$. Point $\left(2, \dfrac{61}{3}\right)$



d Gradient of chord with end points $\left(\dfrac{1}{3}, 22\right)$ and $\left(2, \dfrac{61}{3}\right)$

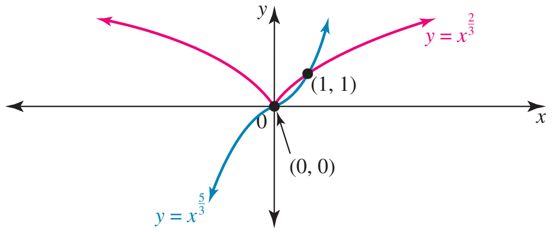
$$\begin{aligned}&= \frac{\frac{61}{3} - 22}{2 - \frac{1}{3}}\\ &= \left(\frac{61}{3} - \frac{66}{3}\right) \div \left(\frac{6}{3} - \frac{1}{3}\right)\\ &= -\frac{5}{3} \times \frac{3}{5}\\ &= -1\end{aligned}$$

This value measures the average rate of evaporation over the interval $t \in \left[\dfrac{1}{3}, 2\right]$.

9 Use CAS technology to obtain the gradient of the tangent. The gradient is 3.

10 a The graphs of $y = x^{\frac{2}{3}}$ and $y = x^{\frac{5}{3}}$ will both pass through the points (0, 0) and (1, 1).

The graphs obtained should be similar to that shown in the diagram.



b Using CAS technology, the slopes of the tangents at (1, 1) can be deduced from the equation of the tangent.

At (1, 1), $y = x^{\frac{2}{3}}$ has gradient $\dfrac{2}{3}$ and $y = x^{\frac{5}{3}}$ has gradient $\dfrac{5}{3}$, making the graph of $y = x^{\frac{5}{3}}$ steeper than that of $y = x^{\frac{2}{3}}$ at the point (1, 1).

At (0, 0), the gradient of $y = x^{\frac{5}{3}}$ is zero. The gradient of $y = x^{\frac{2}{3}}$ is undefined. The tangent to $y = x^{\frac{2}{3}}$ is vertical and the tangent to $y = x^{\frac{5}{3}}$ is horizontal.

12.7 Exam questions

1
$$\begin{aligned}y &= 4x\sqrt{x}\\ &= 4x^{\frac{3}{2}}\\ \frac{dy}{dx} &= \left(4 \times \frac{3}{2}\right) x^{\frac{3}{2}-1}\\ &= 6x^{\frac{1}{2}}\\ &= 6\sqrt{x}\end{aligned}$$

The correct answer is **A**.

2 a
$$\begin{aligned}f(x) &= 3 - 2\sqrt{x}\\ &= 3 - 2x^{\frac{1}{2}}\\ f'(x) &= -\left(2 \times \frac{1}{2}\right) x^{\frac{1}{2}-1}\\ &= -x^{-\frac{1}{2}}\\ &= -\frac{1}{\sqrt{x}}\end{aligned}$$
[1 mark]

b Graph cuts the x-axis when $y = 0$

$$\begin{aligned}y &= 3 - 2\sqrt{x}\\ 2\sqrt{x} &= 3\\ \sqrt{x} &= \frac{3}{2}\\ x &= \frac{9}{4}\end{aligned}$$
[1 mark]

$$\begin{aligned}f'\left(\frac{9}{4}\right) &= -\frac{1}{\sqrt{\frac{9}{4}}}\\ &= -\frac{1}{\frac{3}{2}}\\ &= -\frac{2}{3}\end{aligned}$$
[1 mark]

$\therefore$ gradient at $y = 0$ is $-\dfrac{2}{3}$ [1 mark]

3 a The denominator cannot equal 0 as the function is undefined.

When $x = 0$, $3x = 0$ and $\dfrac{1}{3x}$ is undefined. Therefore, the domain is $R\backslash(0)$. [1 mark]

b
$$\begin{aligned}f(x) &= 2x + \frac{1}{3x}\\ &= 2x + \frac{1}{3}x^{-1}\\ f'(x) &= 2 + \left(\frac{1}{3} \times -1\right) x^{-1-1}\\ &= 2 - \frac{1}{3}x^{-2}\\ &= 2 - \frac{1}{3x^2}, \text{ domain } R\backslash(0)\end{aligned}$$
[1 mark]

c Gradient is −1

$$\begin{aligned} f'(x) &= -1 \\ -1 &= 2 - \frac{1}{3x^2} \\ \frac{1}{3x^2} &= 2 + 1 \\ 1 &= 9x^2 \\ x^2 &= \frac{1}{9} \\ x &= \pm\frac{1}{3} \end{aligned}$$

[1 mark]

$$f(x) = 2x + \frac{1}{3x}$$

$$\begin{aligned} f\left(\frac{1}{3}\right) &= 2 \times \frac{1}{3} + \frac{1}{3 \times \frac{1}{3}} \\ &= \frac{2}{3} + 1 = \frac{5}{3} \end{aligned}$$

$$\begin{aligned} f\left(-\frac{1}{3}\right) &= 2 \times -\frac{1}{3} + \frac{1}{3 \times -\frac{1}{2}} \\ &= -\frac{2}{3} - 1 \\ &= -\frac{5}{3} \end{aligned}$$

Coordinates of the points on the curve where the tangent has a gradient of −1 are $\left(\frac{1}{3}, \frac{5}{3}\right)$ and $\left(-\frac{1}{3}, -\frac{5}{3}\right)$. [1 mark]

12.8 Review

12.8 Exercise

Technology free: short answer

1 a $\lim\limits_{x \to -6} \frac{x^3 - 36x}{x + 6}$

$$\begin{aligned} &= \lim_{x \to -6} \frac{x\left(x^2 - 36\right)}{x + 6} \\ &= \lim_{x \to -6} \frac{x(x-6)(x+6)}{x + 6} \\ &= \lim_{x \to -6} x(x - 6) \\ &= -6 \times -12 \\ &= 72 \end{aligned}$$

b $\lim\limits_{x \to 4} \frac{x + 12}{x^2 - 1}$

As the function is well behaved at $x = 4$, $\lim\limits_{x \to 4} \frac{x + 12}{x^2 - 1} = \frac{16}{15}$.

c $\lim\limits_{x \to 0} \frac{x + 9}{9x}$

$$\begin{aligned} &= \lim_{x \to 0} \left(\frac{x}{9x} + \frac{9}{9x}\right) \\ &= \lim_{x \to 0} \left(\frac{1}{9} + \frac{1}{x}\right) \end{aligned}$$

The limit does not exist.

d $f(x) = \begin{cases} \dfrac{3x + 5}{2}, & x < 1 \\ 4 - x, & x \geq 1 \end{cases}$

$\lim\limits_{x \to 1} f(x)$ will exist if $L^- = L^+$.

$$\begin{aligned} L^- &= \lim_{x \to 1^-} f(x) \\ &= \lim_{x \to 1} \frac{3x + 5}{2} \\ &= \frac{8}{2} \\ &= 4 \end{aligned}$$

$$\begin{aligned} L^+ &= \lim_{x \to 1^+} f(x) \\ &= \lim_{x \to 1} (4 - x) \\ &= 3 \end{aligned}$$

Since $L^- \neq L^+$, $\lim\limits_{x \to 1} f(x)$ does not exist.

2 $f(x) = x^3 - 5x^2 - 8x$

$f''(x) = 3x^2 - 10x - 8$

a $$\begin{aligned} f'(4) &= 3 \times 4^2 - 10 \times 4 - 8 \\ &= 48 - 40 - 8 \\ f'(4) &= 0 \end{aligned}$$

b $$\begin{aligned} f'(3) &= 3 \times 3^2 - 10 \times 3 - 8 \\ &= 27 - 30 - 8 \\ f'(3) &= -11 \\ f'(5) &= 3 \times 5^2 - 10 \times 5 - 8 \\ &= 75 - 50 - 8 \\ f'(5) &= 17 \end{aligned}$$

$\therefore f'(3) = -11$ and $f'(5) = 17$

c The function is decreasing to the left of $x = 4$ and increasing to the right of $x = 4$.

Hence, $x = 4$ is a minimum stationary point.

d For stationary points: $f'(x) = 0$

$$\begin{aligned} 3x^2 - 10x - 8 &= 0 \\ (3x + 2)(x - 4) &= 0 \\ \therefore x = 4 \text{ or } x &= -\frac{2}{3} \end{aligned}$$

The other stationary point is where $x = -\frac{2}{3}$.

3 $$\begin{aligned} f(x) &= x^3 + 9x^2 \\ f'(x) &= 3x^2 + 18x \\ \therefore f'(x) &= 3x(x + 6) \end{aligned}$$

Sign diagram of $f'(x)$ with zeros $x = -6$ and $x = 0$:

a The function is decreasing when $f'(x) < 0$.

From the sign diagram, $3x^2 + 18x < 0$ when $-6 < x < 0$

The function is decreasing over the domain subset $(-6, 0)$.

b The function is stationary when $f'(x) = 0$.

$\therefore x = -6$ or $x = 0$

The function is stationary for the domain subset $\{-6, 0\}$.

c The function is increasing when $f'(x) > 0$.

From the sign diagram, $3x^2 + 18x > 0$ when $x < -6$ or $x > 0$

The function is increasing over the domain subset $(-\infty, -6) \cup (0, \infty)$.

4 a $y = 2x^2 + 8x - 9$

$\frac{dy}{dx} = 4x + 8$

The tangent is parallel to the line $y + 4x = 1$. This line has a gradient of −4, so the tangent line has gradient $m = -4$.

Let $\frac{dy}{dx} = -4$.

$\therefore 4x + 8 = -4$

$\therefore 4x = -12$

$\therefore x = -3$

When $x = -3$, $y = 18 - 24 - 9 = -15$.

The point of contact of the tangent is $(-3, -15)$ and its gradient is $m = -4$.

The equation of the tangent is

$y + 15 = -4(x + 3)$

$\therefore y = -4x - 27$

b If the tangent is parallel to the x-axis, its gradient is zero.

Let $\frac{dy}{dx} = 0$.

$\therefore 4x + 8 = 0$

$\therefore x = -2$

When $x = -2$, $y = 8 - 16 - 9 = -17$.

The tangent is the horizontal line through the point $(-2, -17)$. Its equation is $y = -17$.

5 a $y = x^4 - 4x^3 + 16x + 16$

$\frac{dy}{dx} = 4x^3 - 12x^2 + 16$

At $x = 2$, $\frac{dy}{dx} = 32 - 48 + 16 = 0$, so there is a stationary point at $x = 2$.

Test the slope of the tangent either side of $x = 2$ to determine the nature of the stationary point.

See the table at the bottom of the page.*

The slope of the tangent shows there is a stationary point of inflection at $x = 2$.

b Given there is a stationary point at $x = -1$, test the sign of the gradient about this point.

$\frac{dy}{dx} = 4x^3 - 12x^2 + 16$

$\left.\frac{dy}{dx}\right|_{x=-2} = 4 \times (-8) - 12 \times 4 + 16$

$= -64 < 0$

$\left.\frac{dy}{dx}\right|_{x=-1} = 0$

$\left.\frac{dy}{dx}\right|_{x=0} = 16 > 0$

As the gradient changes from negative to zero to positive about $x = -1$, there is a minimum turning point at $x = -1$.

c $y = x^4 - 4x^3 + 16x + 16$

Let $x = 2$.

$y = 16 - 32 + 32 + 16 = 32$

Let $x = -1$.

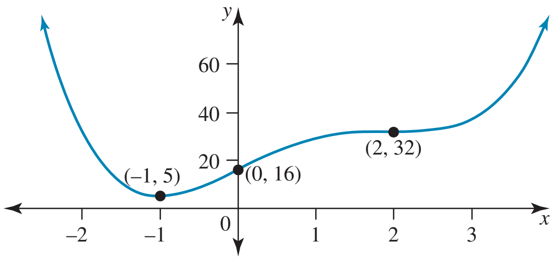
$y = 1 + 4 - 16 + 16 = 5$

Let $x = 0$.

$y = 16$

Point of inflection $(2, 32)$, minimum turning point $(-1, 5)$, y-intercept $(0, 16)$.

d The function is a quartic with a positive leading coefficient. There are no other stationary points, so there cannot be an x-intercept.



6 a The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

The rate of change of the volume with respect to its radius is $\frac{dV}{dr} = 4\pi r^2$.

When $r = 5$, $\frac{dV}{dr} = 100\pi$

The volume is changing at the rate 100π cm^3/cm.

b Let $V = 4.5\pi$.

$\therefore \frac{4}{3}\pi r^3 = 4.5\pi$

$\therefore \frac{4}{3}r^3 = \frac{9}{2}$

$\therefore r^3 = \frac{9}{2} \times \frac{3}{4}$

$\therefore r^3 = \frac{27}{8}$

$\therefore r = \frac{3}{2}$

When $r = \frac{3}{2}$,

$\frac{dV}{dr} = 4\pi \times \frac{9}{4}$

$\therefore \frac{dV}{dr} = 9\pi$

The volume is changing at the rate 9π cm^3/cm.

Technology active: multiple choice

7 Limit from the left of $x = 1$

$L^- = \lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} \left(2x^2\right)$

$= 2$

Limit from the right of $x = 1$

$L^+ = \lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} \left(2x^2\right)$

$= 2$

$\therefore$ function on is continuous

$$f(x) = \begin{cases} 2x, & x \underline{<} 1 \\ 2x^2, & x > 1 \end{cases}$$

Derivative from the right of $x = 1$

$f(x) = 2x$

$\therefore f'(x) = 2(1)$

$\therefore f'(1^-) = 2$

***5 a**

x	1	2	3
$\frac{dy}{dx}$	$4 - 12 + 16 = 8$	0	$4 \times 27 - 108 + 16 = 16$
Slope	positive	zero	positive

Derivative from the right of $x = 1$

$$f(x) = 2x^2$$
$$\therefore f'(x) = 4x$$
$$\therefore f'\left(1^+\right) = 4 \times 1$$
$$= 4$$

Since the derivative from the left does not equal the derivative from the right, the function is not differentiable at $x = 1$.

The correct answer is **C**.

8 $y = 3x^2 - 9$

$$\therefore \frac{dy}{dx} = 6x$$

At the point $(2, 3)$,

$$\therefore \frac{dy}{dx} = 6 \times 2$$
$$= 12$$

The correct answer is **B**.

9 $y = 4x - x^2$

Point: $(0, 0)$

Gradient: $\frac{dy}{dx} = 4 - 2x$

At $x = 0$, $\frac{dy}{dx} = 4$

The equation of the tangent is $y = 4x$

The correct answer is **B**.

10 $f(x) = x^2 - 8x + 7$

$f'(x) = 2x - 8$

$f'(x) < 0$ when $2x - 8 < 0$

$\therefore x < 4$

The correct answer is **D**.

11 $f(2) = -1$ and $f'(2) = 0 \Rightarrow$ point $(2, -1)$ is a stationary point.

The sign of the gradient changes from positive to zero to negative about $x = 2$.

Therefore, the point $(2, -1)$ is a local maximum turning point.

The correct answer is **C**.

12 $y = 4x\left(x^2 - 3\right)$

$\therefore y = 4x^3 - 12x$

$$\frac{dy}{dx} = 12x^2 - 12$$

At stationary points, $\frac{dy}{dx} = 0$

$\therefore 12x^2 - 12 = 0$

$\therefore x^2 = 1$

$\therefore x = \pm 1$

The correct answer is **D**.

13 $V = 2t(8 - t)$

$V = 16t - 2t^2$

Rate of change: $\frac{dV}{dt}$

$$\frac{dV}{dt} = 16 - 4t$$

When $t = 2$:

$$\frac{dV}{dt} = 16 - 4 \times 2$$
$$\frac{dV}{dt} = 8 \text{ mL/s}$$

The correct answer is **C**.

14 $x(t) = t^2 - 10t + 24, \ t \geq 0$

$$v = \frac{dx}{dt}$$

$\therefore v = 2t - 10$

Let $v = 0$.

$\therefore 2t - 10 = 0$

$\therefore t = 5$

The correct answer is **C**.

15 $x(0) = 24$ and $x(1) = 15$.

The distance travelled in the first second is 9 metres.

The correct answer is **C**.

16 Options **A** and **B** are incorrect, as it can be seen from the graph that the left limit of value 2 does not equal the right limit of value 1. The limit does not exist.

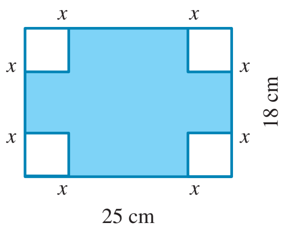
Options **C** and **E** are incorrect because the function is not continuous at $x = 2$, since the limit does not exist and it is not differentiable where it is discontinuous.

Option **D** is correct since $f(2) = 1$.

The correct answer is **D**.

Technology active: extended response

17



a The volume of a rectangular box, $V = lwh$.

For the box, its length $l = 25 - 2x$, its width $w = 18 - 2x$ and its height $h = x$.

$$\therefore V = (25 - 2x)(18 - 2x)x$$
$$= \left(450 - 86x + 4x^2\right)x$$
$$\therefore V = 4x^3 - 86x^2 + 450x$$

The domain requirements are that the length, width and height measurements cannot be negative or zero.

$\therefore 25 - 2x > 0 \Rightarrow x < \frac{25}{2}$ and $18 - 2x > 0 \Rightarrow x < 9$ and $x > 0$.

These restrictions are all satisfied if $0 < x < 9$.

The domain of the volume function is $(0, 9)$.

b $\frac{dV}{dx} = 12x^2 - 172x + 450$

At maximum volume, $\frac{dV}{dx} = 0$.

$\therefore 12x^2 - 172x + 450 = 0$

$\therefore 6x^2 - 86x + 225 = 0$

$$\therefore x = \frac{86 \pm \sqrt{86^2 - 4 \times 6 \times 225}}{12}$$
$$\therefore x = \frac{86 \pm \sqrt{1996}}{12}$$
$$\therefore x \simeq 10.9 \text{ or } x \simeq 3.4$$

Reject $x \simeq 10.9$ as it lies outside the domain.

$\therefore x \simeq 3.4$

To test the nature of the stationary point at $x \simeq 3.4$, consider the slope of the tangent in the neighbourhood of $x = 3.4$.

x	3	3.4	4
$\frac{dV}{dx}$	$108 - 516 + 450$ $= 42$	0	$192 - 688 + 450$ $= -46$
slope	positive	zero	negative

The slope of the tangent shows there is a maximum turning point when $x = 3.4$.

Squares of side length 3.4 cm need to be cut out in order for the volume of the box to be greatest.

c Let $x = 3.4$.

$\therefore \text{V}_{\text{max}} \simeq (25 - 6.8)(18 - 6.8)3.4$

$= 18.2 \times 11.2 \times 3.4$

$\simeq 693$

The maximum volume to the nearest whole number is 693 cubic cm.

d The sum of the areas of the four squares cut out is such that $4x^2 \le 36$.

$\therefore x^2 \le 9$

$\therefore x \le 3 \quad (x \ge 0)$

The domain restriction becomes $[0, 3]$ and since $x = 3.4$ lies outside this interval, the maximum volume must occur at one of the domain end points.

Since $V(0) = 0$, the maximum volume will occur when $x = 3$.

The dimensions of the box with greatest volume are: length $25 - 6 = 19$ cm, width $18 - 6 = 12$ cm and height 3 cm.

The maximum volume is $19 \times 12 \times 3 = 684$ cubic cm.

e The domain restriction is that $x \in [2.5, 6]$. The value $x = 3.4$ lies in this interval.

Testing the values of the volume at the end points and at $x = 3.4$ to determine the global maximum and minimum values over the interval $x \in [2.5, 6]$:

When $x = 2.5$,

$V = (25 - 5)(18 - 5) \times 2.5$

$= 20 \times 13 \times 2.5$

$= 650$

When $x = 3.4$, $V = 693$.

When $x = 6$,

$V = (25 - 12)(18 - 12) \times 6$

$= 13 \times 6 \times 6$

$= 468$

The greatest volume is 693 cubic cm and the least volume is 468 cubic cm.

18 **a** $y = x^4 - 3x^3 + 16x + 16$

i $\dfrac{dy}{dx} = 4x^3 - 9x^2 + 16$

At the stationary points, $4x^3 - 9x^2 + 16 = 0$, so the solutions to this cubic equation give the x-coordinate of any stationary points.

ii Let $f(x) = 4x^3 - 9x^2 + 16$

$f'(x) = 12x^2 - 18x$

Let $x_0 = -1$.

Subsequent approximations are obtained using the iteration formula

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Calculations give

$x_1 = -1.1$

$x_2 = -1.093\,764\,569$

$x_3 = -1.093\,739\,243$

$x_4 = -1.093\,739\,243$

To 4 decimal places, there is a stationary point at $x = -1.0937$.

iii Consider $f(x) = 4x^3 - 9x^2 + 16$.

$f(-2) = -32 - 36 + 16$

$= -52$

< 0

$f(-1) = -4 - 9 + 16$

$= 3$

> 0

This means the gradient of the quartic polynomial changes sign from negative to positive between $x = -2$ and $x = -1$, indicating there is a local minimum turning point where $x = -1.0937$.

b $f(x) = ax^3 + bx^2 + cx + d$

i At the origin, the gradient of the tangent is $m = \tan(45°)$. Therefore, the gradient is $m = 1$.

The origin lies on the curve:

$\therefore f(0) = 0$

$\therefore d = 0$

The gradient of the curve at the origin is 1:

$f'(x) = 3ax^2 + 2bx + c$

$\therefore f'(0) = 1$

$\therefore c = 1$

Hence, $c = 1, d = 0$ and $f(x) = ax^3 + bx^2 + x$.

ii The point $\left(1, \dfrac{5}{6}\right)$ lies on the curve:

$\therefore f(1) = \dfrac{5}{6}$

$\therefore a + b + 1 = \dfrac{5}{6}$

$\therefore a + b = -\dfrac{1}{6} \quad [1]$

At this point the tangent is perpendicular to the line $y + 2x = 0$. The gradient of this line is -2, so the gradient of the tangent is $\dfrac{1}{2}$.

$\therefore f'(1) = \dfrac{1}{2}$

$\therefore 3a + 2b + 1 = \dfrac{1}{2}$

$\therefore 3a + 2b = -\dfrac{1}{2} \quad [2]$

3× equation [1] minus equation [2]:

$\therefore b = -\dfrac{1}{2} + \dfrac{1}{2}$

$\therefore b = 0$

Substitute $b = 0$ in equation [1]:

$\therefore a = -\dfrac{1}{6}$

Hence, $a = -\dfrac{1}{6},\ b = 0$.

$f(x) = -\dfrac{1}{6}x^3 + x$

$\therefore f(x) = x - \dfrac{x^3}{6}$

iii At stationary points, $f'(x) = 0$.

$\therefore 1 - \dfrac{x^2}{2} = 0$

$\therefore x^2 = 2$

$\therefore x = \pm\sqrt{2}$

If $x = -\sqrt{2}$,

$$f\left(-\sqrt{2}\right) = -\sqrt{2} + \frac{2\sqrt{2}}{6}$$

$$= -\sqrt{2} + \frac{\sqrt{2}}{3}$$

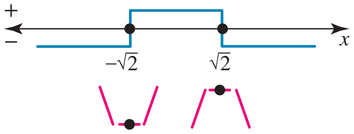
$$= -\frac{2\sqrt{2}}{3}$$

If $x = \sqrt{2}$,

$$f\left(\sqrt{2}\right) = \sqrt{2} - \frac{\sqrt{2}}{3}$$

$$= \frac{2\sqrt{2}}{3}$$

Sign of $f'(x)$



The point $\left(-\sqrt{2}, -\frac{2\sqrt{2}}{3}\right)$ is a minimum turning point and $\left(\sqrt{2}, \frac{2\sqrt{2}}{3}\right)$ is a maximum turning point.

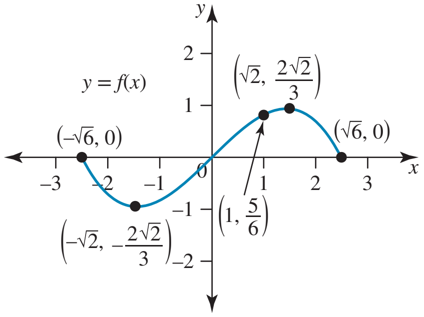
iv $f\colon \left[-\sqrt{6}, \sqrt{6}\right] \to R,\ f(x) = x - \frac{x^3}{6}$

End points: $f\left(-\sqrt{6}\right) = -\sqrt{6} + \frac{6\sqrt{6}}{6} = 0$ and $f\left(\sqrt{6}\right) = \sqrt{6} - \sqrt{6} = 0$.

End points are the x-intercepts $\left(-\sqrt{6}, 0\right)$ and $\left(\sqrt{6}, 0\right)$.

The graph passes through the origin and the point $\left(1, \frac{5}{6}\right)$.

The minimum turning point is $\left(-\sqrt{2},\ -\frac{2\sqrt{2}}{3}\right)$ and the maximum turning point is $\left(\sqrt{2}, \frac{2\sqrt{2}}{3}\right)$.



The local and global minimum value is $-\frac{2\sqrt{2}}{3}$.

The local and global maximum value is $\frac{2\sqrt{2}}{3}$.

v From part **ii**, it is known that the tangent at the point $\left(1, \frac{5}{6}\right)$ has gradient $m = \frac{1}{2}$.

Therefore, the equation of the tangent is

$$y - \frac{5}{6} = \frac{1}{2}(x - 1)$$

$$\therefore y = \frac{1}{2}x - \frac{1}{2} + \frac{5}{6}$$

$$\therefore y = \frac{1}{2}x + \frac{1}{3}$$

This tangent line meets the curve again when

$$\frac{1}{2}x + \frac{1}{3} = x - \frac{x^3}{6}$$

$$\therefore 3x + 2 = 6x - x^3$$

$$\therefore x^3 - 3x + 2 = 0$$

As the tangent touches the curve at $\left(1, \frac{5}{6}\right)$, $(x-1)^2$ is a factor.

$$\therefore x^3 - 3x + 2 = \left(x^2 - 2x + 1\right)(x + 2) = 0$$

$$\therefore (x-1)^2(x+2) = 0$$

$$\therefore x = 1, x = -2$$

When $x = -2$, $y = -1 + \frac{1}{3} = -\frac{2}{3}$.

The point P where the tangent at $\left(1, \frac{5}{6}\right)$ meets the curve again has coordinates $\left(-2, -\frac{2}{3}\right)$.

vi Consider the graph of the function in part **iv**.

Treating the graph as one cycle of $y = a\sin(nx)$, the amplitude is $\frac{2\sqrt{2}}{3}$ and the period is $2\sqrt{6}$.

$$\therefore a = \frac{2\sqrt{2}}{3}$$

and $\frac{2\pi}{n} = 2\sqrt{6}$

$$\therefore n = \frac{2\pi}{2\sqrt{6}}$$

$$\therefore n = \frac{\pi}{\sqrt{6}}$$

The sine function has the equation

$$y = \frac{2\sqrt{2}}{3}\sin\left(\frac{\pi}{\sqrt{6}}x\right).$$

19 For particle A, $x = 5t^2 - 40t - 12,\ t \geq 0$.

a Initial position: when $t = 0,\ x = -12$.

Velocity, $v = \frac{dx}{dt}$.

$$\therefore v = 10t - 40$$

Initial velocity: when $t = 0,\ v = -40$.

Initially the particle is 12 metres to the left of O. Since its initial velocity is negative, it starts moving to the left away from O.

b Let $v = 0$.

$$\therefore 10t - 40 = 0$$

$$\therefore t = 4$$

When $t = 4$,

$$x = 80 - 160 - 12$$

$$= -92$$

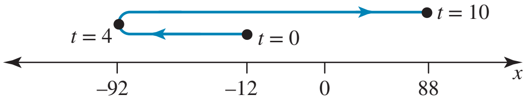
The particle has moved from $x = -12$ to $x = -92$, which is a distance of 80 metres.

c At $t = 4$ the particle changes direction and so starts to move to the right.

When $t = 10$,

$x = 500 - 400 - 12$

$= 88$



The distance travelled in the first 10 seconds is $80 + 92 + 88 = 260$ metres.

Over this time, the average speed is $\frac{260}{10} = 26$ m/s.

d For particle B, $s = t^2 + 8t - 12,\ t \geq 0$.

Initial position: when $t = 0, s = -12$. Therefore, both particles A and B start from the same initial position 12 metres to the left of O.

B's velocity: $v = \frac{ds}{dt}$

$\therefore v = 2t + 8$

When $t = 0,\ v = 8$.

Since particle B's initial velocity is positive, it starts to move to the right toward O. Hence, particles A and B start from the same initial position but they start to move in opposite directions.

e Particle A overtakes B when they are at the same position.

$$\begin{aligned} \therefore x &= s \\ \therefore 5t^2 - 40t - 12 &= t^2 + 8t - 12 \\ \therefore 4t^2 - 48t &= 0 \\ \therefore 4t(t - 12) &= 0 \\ \therefore t &= 0, t = 12 \end{aligned}$$

A overtakes B after 12 seconds.

At this time,

$s = 144 + 96 - 12$

$= 228$

The position where A overtakes B 228 metres to the right of O.

f The particles are travelling with the same speed when $v_A = v_B$ or when $v_A = -v_B$.

If $v_A = v_B$,

$10t - 40 = 2t + 8$

$\therefore 8t = 48$

$\therefore t = 6$

Position of each particle when $t = 6$:

$x = 180 - 240 - 12$

$= -72$

$s = 36 + 48 - 12$

$= 72$

If $v_A = -v_B$,

$10t - 40 = -\ (2t + 8)$

$\therefore 10t - 40 = -2t - 8$

$\therefore 12t = 32$

$\therefore t = \frac{8}{3}$

Position of each particle when $t = \frac{8}{3}$:

$x = 5 \times \frac{64}{9} - \frac{320}{3} - 12$

$= \frac{320 - 960 - 108}{9}$

$= -\frac{748}{9}$

$s = \frac{64}{9} + \frac{64}{3} - 12$

$= \frac{64 + 192 - 108}{9}$

$= \frac{148}{9}$

The two particles are travelling with the same speeds after $\frac{8}{3}$ seconds when A is $\frac{748}{9}$ metres to the left of O and B is $\frac{148}{9}$ metres to the right of O; and after 6 seconds when A is 72 metres to the left of O and B is 72 metres to the right of O.

20 a The total surface area of a closed cylinder is $2\pi rh + 2\pi r^2$.

$$\begin{aligned} \therefore 2\pi rh + 2\pi r^2 &= 308 \\ \therefore 2\pi rh &= 308 - 2\pi r^2 \\ \therefore \pi rh &= 154 - \pi r^2 \\ \therefore h &= \frac{154 - \pi r^2}{\pi r} \end{aligned}$$

b The volume of a cylinder is $\pi r^2 h$.

Substituting the expression for h in terms of r from part **a**,

$V = \pi r^2 \times \frac{154 - \pi r^2}{\pi r}$

$= r \times (154 - \pi r^2)$

$\therefore V = 154r - \pi r^3$

c i When $r = 2$, $V = 308 - 8\pi$.

When $r = 3$, $V = 462 - 27\pi$.

The average rate of change of the volume

$= \frac{V(3) - V(2)}{3 - 2}$

$= (462 - 27\pi) - (308 - 8\pi)$

$= 154 - 19\pi$

Therefore, the average rate of change of the volume is $(154 - 19\pi)$ cubic cm/cm.

ii $\frac{dV}{dr} = 154 - 3\pi r^2$

When $r = 2$, $\frac{dV}{dr} = 154 - 12\pi$

The rate of change of the volume at the instant $r = 2$ is $(154 - 12\pi)$ cubic cm/cm.

d The maximum volume occurs when $\frac{dV}{dr} = 0$.

$$\begin{aligned} \therefore 154 - 3\pi r^2 &= 0 \\ \therefore 154 &= 3\pi r^2 \\ \therefore r^2 &= \frac{154}{3\pi} \\ \therefore r &= \sqrt{\frac{154}{3\pi}} \end{aligned}$$

(negative square root is not applicable)

$\therefore r \simeq 4.04$

To check this value does give the maximum volume, consider the slope of the tangent about $r = 4.04$.

r	4	4.04	5
$\frac{dV}{dr}$	$154 - 48\pi$ $\simeq 3.2$	0	$154 - 75\pi$ $\simeq -81.6$
Slope	positive	zero	negative

There is a maximum turning point when $r = 4.04$.

Maximum volume:

$$V_{\max} = 154 \times \sqrt{\frac{154}{3\pi}} - \pi \times \left(\frac{154}{3\pi}\right)^{\frac{3}{2}}$$

$$\simeq 415$$

Height for maximum volume:

$$h = \left(154 - \pi \times \frac{154}{3\pi}\right) \div \left(\pi \times \sqrt{\frac{154}{3\pi}}\right)$$

$$= \left(\frac{2}{3} \times 154\right) \times \frac{\sqrt{3\pi}}{\pi\sqrt{154}}$$

$$= \frac{2\sqrt{154} \times \sqrt{3\pi}}{3\pi}$$

$$= 2\sqrt{\frac{154}{3\pi}}$$

$$\simeq 8.08$$

The maximum volume is 415 cm^3. The height is twice the radius, and the dimensions to 2 significant figures are height 8.1 cm and base radius 4.0 cm.

e Refer to the diagram given in the question.

The diameter of each circle is $2r$, so the vertical distance on the diagram is $2r + h + 2r = 4r + h$.

The area of the rectangle area forms the curved surface area $2\pi rh$ of the cylinder. Therefore, the width of the rectangle must be $2\pi r$.

The length and width of the rectangular sheet of cardboard are $2\pi r$ cm and $(4r + h)$ cm.

f The area of the rectangular sheet of cardboard is $A = 2\pi r(4r + h)$.

Let the cost, in dollars, of making the cylinder be C.

$C \propto A$

$\therefore C = 0.01A$

$\therefore C = 0.01 \times 2\pi r(4r + h)$

$\therefore C = 0.02\pi r(4r + h)$

Let $r = 4.0$ and $h = 8.1$.

$C = 0.08\pi \times 24.1$

$= 6.06$

The cost is \$6.06 using the dimensions to 3 significant figures.

12.8 Exam questions

1 Limit from the left of $x = 1$:

$$L^- = \lim_{x\to 1^-} f(x) = \lim_{x\to 1}\left(x^2 - 2\right) = -1$$

Limit from the right of $x = 1$:

$$L^+ = \lim_{x\to 1^+} f(x) = \lim_{x\to 1}(2x - 3) = -1$$

$\therefore$ function is continuous.

$$f(x) = \begin{cases} x^2 - 2, & x \le 1 \\ 2x - 3, & x > 1 \end{cases}$$

Derivative from the left of $x = 1$:

$$f(x) = x^2 - 2$$
$$\therefore f'(x) = 2x$$
$$\therefore f'(1^-) = 2$$

Derivative from the right of $x = 1$:

$$f(x) = 2x - 3$$
$$\therefore f'(x) = 2$$
$$\therefore f'(1^+) = 2$$

Since the derivative from the left equals the derivative from the right, the function is differentiable at $x = 1$.

The correct answer is **A**.

2 The gradient is positive where the function is increasing; that is, it slopes up towards the right.

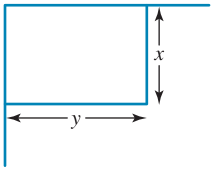
$\therefore x \in (-\infty, -3) \cup (1, \infty)$

The correct answer is **C**.

3 The gradient changes from negative to zero and back to negative, so it doesn't change from negative to positive. This means there is a stationary point of inflection at $x = 3$.

The correct answer is **D**.

4



[1 mark]

Available fencing:

$20 = x + y$

$y = 20 - x$ [1 mark]

Area:

$A = xy$

Substitute $y = 20 - x$

$A = x(20 - x)$

$= 20x - x^2$

Maximum area:

$$\frac{dA}{dx} = 20 - 2x$$

For maximum area, $\frac{dA}{dx} = 0$ [1 mark]

$0 = 20 - 2x$

$2x = 20$

$\therefore x = 10$

$\therefore y = 20 - 10 = 10$

The dimensions of the garden bed are 10×10 m and the area is $100\,\text{m}^2$. [1 mark]

5 **a** Height of the triangle, using Pythagoras' theorem:

$$h^2 + \left(\frac{x}{4}\right)^2 = x^2$$

$$h^2 + \frac{x^2}{16} = x^2$$

$$h^2 = \frac{15x^2}{16}$$

$$h = \pm\sqrt{\frac{15}{16}}x$$

$$= \frac{\sqrt{15}}{4}x, (h > 0)$$ [1 mark]

Area of a triangle is $\frac{1}{2}\times \text{base} \times \text{height}$:

$$A=\frac{1}{2}\times\left(\frac{x}{2}\right)\times\frac{\sqrt{15}}{4}x$$

$$A=\frac{\sqrt{15}}{16}x^2 \qquad \text{[1 mark]}$$

b $\frac{dA}{dx}=\frac{\sqrt{15}}{8}x$

When $x=8$.

$$\begin{aligned}\frac{dA}{dx}&=\frac{\sqrt{15}}{8}\times 8\\&=\sqrt{15}\end{aligned}$$

The rate that the area is changing when $x=8$ is $\sqrt{15}\,\text{cm}^2/\text{cm}$. [1 mark]

Topic 13 — Anti-differentiation and introduction to integral calculus

13.2 Anti-derivatives

13.2 Exercise

1 a $\frac{dy}{dx} = x^3$

$$y = \frac{x^{3+1}}{3+1} + c$$

$$y = \frac{1}{4}x^4 + c$$

b $\frac{dy}{dx} = x^6$

$$y = \frac{x^{6+1}}{6+1} + c$$

$$y = \frac{1}{7}x^7 + c$$

c $\frac{dy}{dx} = x^8$

$$y = \frac{x^{8+1}}{8+1} + c$$

$$y = \frac{1}{9}x^9 + c$$

d $\frac{dy}{dx} = 12x$

$$y = 12 \times \frac{x^{1+1}}{1+1} + c$$

$$y = 12 \times \frac{x^2}{2} + c$$

$$y = 6x^2 + c$$

e $\frac{dy}{dx} = -6x^2$

$$y = -6 \times \frac{x^{2+1}}{2+1} + c$$

$$y = -6 \times \frac{x^3}{3} + c$$

$$y = -2x^3 + c$$

f $\frac{dy}{dx} = \frac{1}{3}x^4$

$$y = \frac{1}{3} \times \frac{x^{4+1}}{4+1} + c$$

$$y = \frac{1}{3} \times \frac{x^5}{5} + c$$

$$y = \frac{1}{15}x^5 + c$$

2 a $\int x^5 dx$

$$= \frac{x^{5+1}}{5+1} + c$$

$$= \frac{1}{6}x^6 + c$$

b $\int x^7 dx$

$$= \frac{x^{7+1}}{7+1} + c$$

$$= \frac{1}{8}x^8 + c$$

c $\int 9x^8 dx$

$$= 9 \times \frac{x^{8+1}}{8+1} + c$$

$$= 9 \times \frac{x^9}{9} + c$$

$$= x^9 + c$$

d $\int 6x^{11} dx$

$$= 6 \times \frac{x^{11+1}}{11+1} + c$$

$$= 6 \times \frac{x^{12}}{12} + c$$

$$= \frac{1}{2}x^{12} + c$$

e $\int (2x+3)\,dx$

$$= \int \left(2x^1 + 3x^0\right) dx$$

$$= 2 \times \frac{x^{1+1}}{1+1} + 3 \times \frac{x^{0+1}}{0+1} + c$$

$$= 2 \times \frac{x^2}{2} + 3 \times \frac{x^1}{1} + c$$

$$= x^2 + 3x + c$$

f $\int \left(10 - x^2\right) dx$

$$= \int \left(10x^0 - x^2\right) dx$$

$$= 10 \times \frac{x^{0+1}}{0+1} - \frac{x^{2+1}}{2+1} + c$$

$$= 10 \times \frac{x^1}{1} - \frac{x^3}{3} + c$$

$$= 10x - \frac{x^3}{3} + c$$

3 a $f(x) = 10x^9$

$$F(x) = 10 \times \frac{x^{9+1}}{9+1} + c$$

$$F(x) = 10 \times \frac{x^{10}}{10} + c$$

$$F(x) = x^{10} + c$$

b $f(x) = -x^{12}$

$$F(x) = -\frac{x^{12+1}}{12+1} + c$$

$$F(x) = -\frac{x^{13}}{13} + c$$

c $f(x) = 6x^5 + 4x$

$$F(x) = 6 \times \frac{x^{5+1}}{5+1} + 4 \times \frac{x^{1+1}}{1+1} + c$$

$$F(x) = 6 \times \frac{x^6}{6} + 4 \times \frac{x^2}{2} + c$$

$$F(x) = x^6 + 2x^2 + c$$

d $f(x) = x^3 - 7x^2$

$$F(x) = \frac{x^{3+1}}{3+1} - 7 \times \frac{x^{2+1}}{2+1} + c$$

$$F(x) = \frac{x^4}{4} - 7 \times \frac{x^3}{3} + c$$

$$F(x) = \frac{x^4}{4} - \frac{7x^3}{3} + c$$

e $f(x) = 4 + x + x^2$

$$F(x) = 4 \times \frac{x^{0+1}}{0+1} + \frac{x^{1+1}}{1+1} + \frac{x^{2+1}}{2+1} + c$$

$$F(x) = 4 \times \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + c$$

$$F(x) = 4x + \frac{x^2}{2} + \frac{x^3}{3} + c$$

f $f(x) = x^3 + 9x^2 - 5x + 1$

$$F(x) = \frac{x^{3+1}}{3+1} + 9 \times \frac{x^{2+1}}{2+1} - 5 \times \frac{x^{1+1}}{1+1} + 1 \times \frac{x^{0+1}}{0+1} + c$$

$$F(x) = \frac{x^4}{4} + 9 \times \frac{x^3}{3} - 5 \times \frac{x^2}{2} + 1 \times \frac{x^1}{1} + c$$

$$F(x) = \frac{x^4}{4} + 3x^3 - \frac{5x^2}{2} + x + c$$

4 a $\frac{dy}{dx} = 12x^5$

$$y = 12 \times \frac{1}{6}x^6 + c$$

$$y = 2x^6 + c$$

b An anti-derivative of $4x^2 + 2x - 5$ equals $\frac{4}{3}x^3 + \frac{2x^2}{2} - 5x$.

Therefore, an anti-derivative could be $\frac{4}{3}x^3 + x^2 - 5x$.

c $f'(x) = (x-2)(3x+8)$

$$= 3x^2 + 8x - 6x - 16$$

$$= 3x^2 + 2x - 16$$

$$f(x) = x^3 + x^2 - 16x + c$$

5 a $\frac{dy}{dx} = 5x^9$

$$\therefore y = \frac{5x^{9+1}}{10} + c$$

$$\therefore y = \frac{1}{2}x^{10} + c$$

b $\frac{dy}{dx} = -3 + 4x^7$

$$\therefore y = -3x + \frac{4x^8}{8} + c$$

$$\therefore y = -3x + \frac{1}{2}x^8 + c$$

c $\frac{dy}{dx} = 2\left(x^2 - 6x + 7\right)$

$$\therefore \frac{dy}{dx} = 2x^2 - 12x + 14$$

$$\therefore y = \frac{2x^3}{3} - \frac{12x^2}{2} + 14x + c$$

$$\therefore y = \frac{2}{3}x^3 - 6x^2 + 14x + c$$

d $\frac{dy}{dx} = (8-x)(2x+5)$

$$= 16x + 40 - 2x^2 - 5x$$

$$\therefore \frac{dy}{dx} = -2x^2 + 11x + 40$$

$$\therefore y = -\frac{2x^3}{3} + \frac{11x^2}{2} + 40x + c$$

$$\therefore y = -\frac{2}{3}x^3 + \frac{11}{2}x^2 + 40x + c$$

6 a $f'(x) = \frac{1}{2}x^5 + \frac{7}{3}x^6$

$$\therefore f(x) = \frac{1}{2} \times \frac{x^6}{6} + \frac{7}{3} \times \frac{x^7}{7} + c$$

$$\therefore f(x) = \frac{1}{12}x^6 + \frac{1}{3}x^7 + c$$

b $f'(x) = \frac{4x^6}{3} + 5$

$$\therefore f(x) = \frac{4}{3} \times \frac{x^7}{7} + 5x + c$$

$$\therefore f(x) = \frac{4}{21}x^7 + 5x + c$$

c $f'(x) = \frac{4x^4 - 6x^8}{x^2}$

$$\therefore f'(x) = \frac{4x^4}{x^2} - \frac{6x^8}{x^2}$$

$$\therefore f'(x) = 4x^2 - 6x^6$$

$$\therefore f(x) = \frac{4x^3}{3} - \frac{6x^7}{7} + c$$

$$\therefore f(x) = \frac{4}{3}x^3 - \frac{6}{7}x^7 + c$$

d $f'(x) = \left(3 - 2x^2\right)^2$

$$\therefore f'(x) = 9 - 12x^2 + 4x^4$$

$$\therefore f(x) = 9x - \frac{12x^3}{3} + \frac{4x^5}{5} + c$$

$$\therefore f(x) = 9x - 4x^3 + \frac{4}{5}x^5 + c$$

7 a $\int \frac{3x^8}{5} dx$

$$= \frac{3}{5} \times \frac{x^9}{9} + c$$

$$= \frac{1}{15}x^9 + c$$

b $\int 2dx = 2x + c$

c $4\int \left(20x - 5x^7\right) dx$

$$= 4\left[\frac{20x^2}{2} - \frac{5x^8}{8}\right] + c$$
$$= 4\left[10x^2 - \frac{5}{8}x^8\right] + c$$
$$= 40x^2 - \frac{5}{2}x^8 + c$$

d $\int \frac{1}{100}\left(9 + 6x^2 - 5.5x^{10}\right) dx$
$$= \frac{1}{100}\int \left(9 + 6x^2 - 5.5x^{10}\right) dx$$
$$= \frac{1}{100}\left[9x + \frac{6x^3}{3} - \frac{5.5x^{11}}{11}\right] + c$$
$$= \frac{1}{100}\left[9x + 2x^3 - 0.5x^{11}\right] + c$$

8 a Let $f(x) = 2ax + b$.

The primitive function is $F(x)$ where
$$F(x) = \frac{2ax^2}{2} + bx + c$$
$$\therefore F(x) = ax^2 + bx + c$$

b Let $f(x) = 0.05x^{99}$.
$$\therefore F(x) = \frac{0.05x^{100}}{100} + c$$
$$\therefore F(x) = 0.0005x^{100} + c$$

c Let $f(x) = (2x + 1)^3$.
$$\therefore f(x) = (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3$$
$$\therefore f(x) = 8x^3 + 12x^2 + 6x + 1$$
$$\therefore F(x) = \frac{8x^4}{4} + \frac{12x^3}{3} + \frac{6x^2}{2} + x + c$$
$$\therefore F(x) = 2x^4 + 4x^3 + 3x^2 + x + c$$

An anti-derivative could be $2x^4 + 4x^3 + 3x^2 + x$.

d Let $f(x) = 7 - x\left(5x^3 - 4x - 8\right)$.
$$\therefore f(x) = 7 - 5x^4 + 4x^2 + 8x$$
$$\therefore F(x) = 7x - \frac{5x^5}{5} + \frac{4x^3}{3} + \frac{8x^2}{2} + c$$
$$\therefore F(x) = 7x - x^5 + \frac{4}{3}x^3 + 4x^2 + c$$

9 $\frac{dy}{dx} = \frac{2x^3 - 3x^2}{x},\ x \neq 0$
$$= \frac{2x^3}{x} - \frac{3x^2}{x}$$
$$= 2x^2 - 3x$$
$$y = \frac{2x^3}{3} - \frac{3x^2}{2} + c$$

10 a $f(x) = \frac{3x^2 - 2}{4}$
$$\therefore f(x) = \frac{1}{4}\left(3x^2 - 2\right)$$
$$\therefore F(x) = \frac{1}{4}\left(\frac{3x^3}{3} - 2x\right) + c$$
$$\therefore F(x) = \frac{1}{4}\left(x^3 - 2x\right) + c$$

b $f(x) = \frac{3x}{4} + \frac{2(1 - x)}{3}$
$$\therefore f(x) = \frac{3x}{4} + \frac{2}{3} - \frac{2x}{3}$$
$$= \frac{x}{12} + \frac{2}{3}$$
$$\therefore F(x) = \frac{1}{12} \times \frac{x^2}{2} + \frac{2}{3}x + c$$
$$\therefore F(x) = \frac{1}{24}x^2 + \frac{2}{3}x + c$$

c $f(x) = 0.25\left(1 + 5x^{14}\right)$
$$\therefore f(x) = \frac{1}{4} + \frac{5}{4}x^{14}$$
$$\therefore F(x) = \frac{1}{4}x + \frac{5}{4} \times \frac{x^{15}}{15} + c$$
$$\therefore F(x) = \frac{1}{4}x + \frac{1}{12}x^{15} + c$$

d $f(x) = \frac{12\left(x^5\right)^2 - (4x)^2}{3x^2}$
$$\therefore f(x) = \frac{12x^{10} - 16x^2}{3x^2}$$
$$= \frac{12x^{10}}{3x^2} - \frac{16x^2}{3x^2}$$
$$\therefore f(x) = 4x^8 - \frac{16}{3}$$
$$\therefore F(x) = \frac{4x^9}{9} - \frac{16}{3}x + c$$

11 a $\int \left(-7x^4 + 3x^2 - 6x\right) dx$
$$= -7 \times \frac{x^5}{5} + x^3 - 6 \times \frac{x^2}{2} + c$$
$$= -\frac{7x^5}{5} + x^3 - 3x^2 + c$$

b $\int \frac{5x^8 + 3x^3}{4x^2} dx$
$$= \int \left(\frac{5x^8}{4x^2} + \frac{3x^3}{4x^2}\right) dx$$
$$= \int \left(\frac{5}{4}x^6 + \frac{3}{4}x\right) dx$$
$$= \frac{5}{4} \times \frac{x^7}{7} + \frac{3}{4} \times \frac{x^2}{2} + c$$
$$= \frac{5}{28}x^7 + \frac{3}{8}x^2 + c$$

c $f(x) = 2x^4$
$$F(x) = 2 \times \frac{x^5}{5} + c$$
$$\therefore\ F(x) = \frac{2}{5}x^5 + c$$

d $\int (3x + 4)^2 dx$
$$= \int \left(9x^2 + 24x + 16\right) dx$$
$$= 9\frac{x^3}{3} + 24 \times \frac{x^2}{2} + 16x + c$$
$$= 3x^3 + 12x^2 + 16x + c$$

12 $\int 2(t^2+2)\,dt$

$= \int \left(2t^2+4\right)\,dt$

$= \dfrac{2t^3}{3} + 4t + c$

$2\int (t^2+2)\,dt$

$= 2\left(\dfrac{t^3}{3} + 2t\right) + c$

$= \dfrac{2t^3}{3} + 4t + c$

Therefore, $\int 2\left(t^2+2\right)dt = 2\int \left(t^2+2\right)dt.$

13 $\int \left(\dfrac{2x^5+7x^3-5x^2}{x^2}\right)dx$

$= \int \left(2x^3+7x-5\right)dx$

$= 2\times\dfrac{x^4}{4} + 7\times\dfrac{x^2}{2} - 5x + c$

$= \dfrac{x^4}{2} + \dfrac{7x^2}{2} - 5x + c$

14 a $f(x) = (2ax)^2 + b^3$

$\therefore f(x) = 4a^2x^2 + b^3$

$\therefore F(x) = 4a^2 \times \dfrac{x^3}{3} + b^3x + c$

$\therefore F(x) = \dfrac{4a^2x^3}{3} + b^3x + c$

15 a Let $f(x) = (x-1)(x+4)(x+1)$.

$\therefore f(x) = (x^2-1)(x+4)$

$\therefore f(x) = x^3 + 4x^2 - x - 4$

The primitive is $F(x) = \dfrac{x^4}{4} + \dfrac{4x^3}{3} - \dfrac{x^2}{2} - 4x + c.$

b The anti-derivative of $(x-4)(x+4)$ is

$\int (x-4)(x+4)\,dx$

$= \int \left(x^2-16\right)dx$

$= \dfrac{x^3}{3} - 16x + c$

c Let $f(x) = \dfrac{\left(2x^3\right)-4x^2}{2x^2}.$

$\therefore f(x) = \dfrac{8x^3}{2x^2} - \dfrac{4x^2}{2x^2}$

$= 4x - 2$

The anti-derivative is:

$F(x) = 4\times\dfrac{x^2}{2} - 2x + c$

$= 2x^2 - 2x + c$

An anti-derivative could be $2x^2 - 2x$.

d $\dfrac{d}{dx}\left(\int (4x+7)dx\right)$

$= \dfrac{d}{dx}\left(2x^2+7x+c\right)$

$= 4x + 7$

16 a An anti-derivative of x^3 can be obtained using CAS technology.

$\int x^3dx = \dfrac{x^4}{4}$. For the family of anti-derivatives, the constant of integration needs to be inserted.

b i $\dfrac{dy}{dx} = \left(x^4+1\right)^2$

$\therefore y = \int \left(x^4+1\right)^2 dx$

$\therefore y = \dfrac{x^9}{9} + \dfrac{2x^5}{5} + x$

ii $f(t) = 100\,(t-5)$

$\therefore\ F(t) = \int 100\,(t-5)\,dt$

$F(t) = 50t\,(t-10)$

$= 50t^2 - 500t$

iii $\int \left(y^5 - y^3\right)dy$

$= \dfrac{y^6}{6} - \dfrac{y^4}{4}$

iv $\int (4u+5)^2\,du$

$= \dfrac{(4u+5)^3}{12}$

13.2 Exam questions

1 If $\dfrac{dy}{dx} = x^n, n \in N,$ then $y = \dfrac{1}{n+1}x^{n+1} + c,$ where c is an arbitrary constant.

$\dfrac{dy}{dx} = 6x^{11}$

$y = \dfrac{1}{11+1}\times 6x^{11+1} + c$

$y = \dfrac{1}{2}x^{12} + c$

$y = \dfrac{x^{12}}{2} + c$

The correct answer is **E**.

2 $f(x) = \dfrac{3}{5+1}x^{5+1} - \dfrac{3}{8+1}x^{8+1} + c$

$= \dfrac{1}{2}x^6 - \dfrac{1}{3}x^9 + c$

The correct answer is **A**.

3 $\int \dfrac{7x^6-4x^5}{2x^3}dx = \int \left(\dfrac{7x^6}{2x^3} - \dfrac{4x^5}{2x^3}\right)dx$

$= \int \left(\dfrac{7x^3}{2} - 2x^2\right)dx$

$= \dfrac{7x^{3+1}}{(3+1)\times 2} - \dfrac{2}{2+1}x^{2+1} + c$

$= \dfrac{7}{8}x^4 - \dfrac{2}{3}x^3 + c$

The correct answer is **E**.

13.3 Anti-derivative functions and graphs

13.3 Exercise

1 a $f'(x) = 16x$ and $f(1) = 3$

Use anti-differentiation to obtain $f(x)$.

$f(x) = \int (16x)\,dx$

$= 8x^2 + c$

Use the given information $f(1) = 3$ to calculate the constant c.

$f(x) = 8x^2 + c$
$f(1) = 8 + c$
$\therefore 3 = 8 + c$
$\therefore c = -5$

Hence, $f(x) = 8x^2 - 5$.

b $f'(x) = 3x^2 + 4x$ and $f(0) = 8$

Use anti-differentiation to obtain $f(x)$.

$$f(x) = \int (3x^2 + 4x)dx$$
$$= x^3 + 2x^2 + c$$

Use the given information $f(0) = 8$ to calculate the constant c.

$f(0) = c$
$\therefore\ 8 = c$
$c = 8$

Hence, $f(x) = x^3 + 2x^2 + 8$.

c $f'(x) = x^2(9 - 8x)$ and $f(1) = -12$

Expanding, $f'(x) = 9x^2 - 8x^3$.

Use anti-differentiation to obtain $f(x)$.

$$f(x) = \int (9x^2 - 8x^3)\ dx$$
$$= 3x^3 - 2x^4 + c$$

Use the given information $f(1) = -12$ to calculate the constant c.

$f(x) = 3x^3 - 2x^2 + c$
$f(1) = 3 - 2 + c$
$\therefore\ -12 = 1 + c$
$\therefore c = -13$

Hence, $f(x) = 3x^3 - 2x^2 - 13$.

d $f'(x) = (5x + 1)^2$ and $f(1) = 6$

Expanding, $f'(x) = 25x^2 + 10x + 1$.

Use anti-differentiation to obtain $f(x)$.

$$f(x) = \int (25x^2 + 10x + 1)dx$$
$$= \frac{25x^3}{3} + 5x^2 + x + c$$

Use the given information $f(1) = 6$ to calculate the constant c.

$$f(x) = \frac{25x^3}{3} + 5x^2 + x + c$$
$$f(1) = \frac{25}{3} + 5 + 1 + c$$
$$\therefore 6 = \frac{25}{3} + 6 + c$$
$$\therefore c = -\frac{25}{3}$$

Hence, $f(x) = \dfrac{25x^3}{3} + 5x^2 + x - \dfrac{25}{3}$.

2 $f'(x) = -3x^2 + 4$

$\therefore f(x) = -x^3 + 4x + c$

Since the point $(-1, 2)$ lies on the function, $f(-1) = 2$.

$\therefore\ -(-1)^3 + 4(-1) + c = 2$
$\therefore 1 - 4 + c = 2$
$\therefore c = 5$

The equation is $f(x) = -x^3 + 4x + 5$

3 a $\dfrac{dy}{dx} = 4 - 5x$

$$y = 4x - \frac{5x^2}{2} + c$$

Substitute the given point $(2, 3)$.

$$\therefore 3 = 8 - \frac{5 \times 4}{2} + c$$
$\therefore 3 = -2 + c$
$\therefore c = 5$

The equation of the curve is $y = -\dfrac{5}{2}x^2 + 4x + 5$.

b $\dfrac{dy}{dx} = (x + 1)(x - 2)$

$$\frac{dy}{dx} = x^2 - x - 2$$
$$y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + c$$

Substitute the given point $(0, -5)$.

$$\therefore\ -5 = \frac{1}{3} \times (0) - \frac{1}{2} \times (0) - 2(0) + c$$
$\therefore\ -5 = c$

The equation of the curve is $y = \dfrac{1}{3}x^3 - \dfrac{1}{2}x^2 - 2x - 5$.

4 a $\dfrac{dy}{dx} = \dfrac{2x}{5} - 3$

$$\therefore y = \frac{x^2}{5} - 3x + c$$

Substitute the point $(5, 0)$.

$\therefore 0 = 5 - 15 + c$
$\therefore c = 10$

The equation is $y = \dfrac{x^2}{5} - 3x + 10$.

b Let $y = 0$.

$$\therefore \frac{x^2}{5} - 3x + 10 = 0$$
$\therefore x^2 - 15x + 50 = 0$
$\therefore (x - 5)(x - 10) = 0$
$\therefore x = 5, x = 10$

The x-intercepts are $(5, 0)$ and $(10, 0)$.

5 $\dfrac{dy}{dx} = 2x$

$$\therefore y = 2 \times \frac{x^2}{2} + c$$
$y = x^2 + c$

Substitute $y = 10$ when $x = 4$.

$10 = 4^2 + c$
$c = -6$
$y = x^2 - 6$

Substitute $x = 5$.

$y = 5^2 - 6$
$y = 19$

6 a $\dfrac{dy}{dx} = ax - 6$

Stationary point at $(-1,\ 10)$

$$\therefore\ \left.\frac{dy}{dx}\right|_{x=-1} = 0$$
$a(-1) - 6 = 0$
$a = -6$

$$\frac{dy}{dx} = -6x - 6$$
$$\therefore y = -3x^2 - 6x + c$$
Substitute $(-1,\ 10)$:
$$\therefore 10 = -3(-1)^2 - 6(-1) + c$$
$$\therefore 10 = -3 + 6 + c$$
$$\therefore c = 7$$
The equation of the curve is $y = -3x^2 - 6x + 7$.

b $f'(x) = 2x^2 - 9$
$$\therefore f(x) = 2 \times \frac{x^3}{3} - 9x + c$$
$$f(x) = \frac{2x^3}{3} - 9x + c$$
Since $f(3) = 0$:
$$\frac{2 \times 3^3}{3} - 9 \times 3 + c = 0$$
$$18 - 27 + c = 0$$
$$c = 9$$
$$\therefore f(x) = \frac{2x^3}{3} - 9x + 9$$
$$f(-3) = \frac{2\,(-3)^3}{3} - 9\,(-3) + 9$$
$$= -18 + 27 + 9$$
$$f(-3) = 18$$

7 a $\frac{dy}{dx} \propto x$
$$\therefore \frac{dy}{dx} = kx$$
Given that $\frac{dy}{dx} = -3$ at the point $(2, 5)$
$$\therefore -3 = k \times 2$$
$$\therefore k = -\frac{3}{2}$$
The constant of proportionality is $-\frac{3}{2}$.

b $\frac{dy}{dx} = -\frac{3}{2}x$
$$\therefore y = -\frac{3}{2} \times \frac{x^2}{2} + c$$
$$\therefore y = -\frac{3}{4}x^2 + c$$
Substitute the point $(2, 5)$.
$$\therefore 5 = -\frac{3}{4} \times 4 + c$$
$$\therefore c = 8$$
The equation is $y = -\frac{3}{4}x^2 + 8$.

8 $f(x) = (4 - x)\,(5 - x)$
$$= 20 - 9x + x^2$$
The primitive function is:
$$F(x) = 20x - \frac{9}{2}x^2 + \frac{x^3}{3} + c$$
The point $(1, -1)$ lies on the primitive function:
$$F(1) = 20\,(1) - \frac{9}{2}\,(1) + \frac{1}{3} + c = -1$$
$$20 - \frac{9}{2} + \frac{1}{3} + c = -1$$
$$c = -\frac{101}{6}$$
$$F(x) = 20x - \frac{9}{2}x^2 + \frac{x^3}{3} - \frac{101}{6}$$

9 a $\frac{dy}{dx} = 2x(3 - x)$
$$\therefore \frac{dy}{dx} = 6x - 2x^2$$
$$\therefore y = 3x^2 - \frac{2x^3}{3} + c$$
$$x = 3, y = 0 \Rightarrow 0 = 27 - 18 + c$$
$$\therefore c = -9$$
$$\therefore y = 3x^2 - \frac{2x^3}{3} - 9$$
Let $x = 0$.
$$\therefore y = -9$$

b $\frac{dz}{dx} = (10 - x)^2$
$$\therefore \frac{dz}{dx} = 100 - 20x + x^2$$
$$\therefore z = 100x - 10x^2 + \frac{x^3}{3} + c$$
$$x = 10,\ z = 200 \Rightarrow 200 = 1000 - 1000 + \frac{1000}{3} + c$$
$$\therefore c = 200 - \frac{1000}{3}$$
$$\therefore c = -\frac{400}{3}$$
$$\therefore z = 100x - 10x^2 + \frac{x^3}{3} - \frac{400}{3}$$
Let $x = 4$.
$$z = 400 - 160 + \frac{64}{3} - \frac{400}{3}$$
$$= 240 - \frac{336}{3}$$
$$= 240 - 112$$
$$= 128$$

10 a $\frac{dy}{dx} = a - x^3$

Stationary point at $(2,\ 9)$
$$\therefore \frac{dy}{dx} = 0 \text{ when } x = 2$$
$$a - 2^3 = 0$$
$$\therefore a = 8$$

b $\frac{dy}{dx} = 8 - x^3$
$$\therefore y = 8x - \frac{x^4}{4} + c$$
The point $(2,\ 9)$ lies on the curve
$$\therefore 9 = 8\,(2) - \frac{2^4}{4} + c$$
$$9 = 16 - 4 + c$$
$$c = -3$$
$$\therefore y = 8x - \frac{x^4}{4} - 3$$

c When $x = 1$:
$$y = 8 - \frac{1}{4} - 3 = \frac{19}{4}$$
Gradient of tangent: $\frac{dy}{dx} = 8 - 1 = 7$

Equation of tangent at the point $\left(1,\ \frac{19}{4}\right)$, $m = 7$
$$y - \frac{19}{4} = 7\,(x - 1)$$
$$y - \frac{19}{4} = 7x - 7$$

$$y = 7x - 7 + \frac{19}{4}$$

$$\therefore y = 7x - \frac{9}{4}$$

11 The graph has a gradient of $\frac{1}{2}$ and a vertical axis intercept at $(0, 2)$.

Its equation is $\frac{dy}{dx} = \frac{1}{2}x + 2$.

Therefore, the equation of the curve must be

$y = \frac{1}{4}x^2 + 2x + c$.

Since $(-2, 3)$ lies on the curve,

$$3 = \frac{1}{4} \times 4 + 2 \times (-2) + c$$

$$\therefore c = 6$$

Thus, the equation of the curve is $y = \frac{1}{4}x^2 + 2x + 6$.

From the gradient graph, the turning point of the curve must occur at $x = -4$.

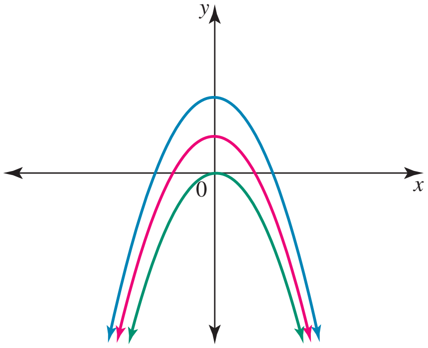
$$y = \frac{1}{4} \times 16 - 8 + 6$$

$$\therefore y = 2$$

The minimum turning point has coordinates $(-4, 2)$.

12 **a** At $x = 0$ there is a stationary point on the graph of $y = F(x)$. Since f changes sign from positive to zero to negative, the point will be a maximum turning point.

b Three possible graphs with a maximum turning point at $x = 0$ are shown.



Family of curves $y = -x^2 + c$

c The rule for the graph is of the form $y = ax$.

Substitute the point $(1, -2)$:

$$-2 = a(1)$$

$$a = -2$$

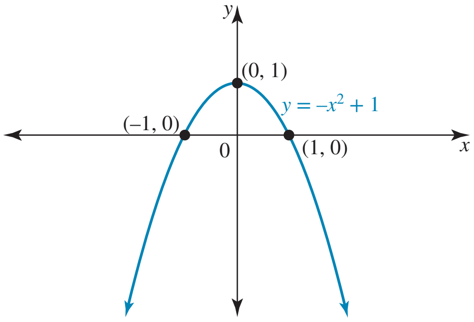
Hence, $f(x) = -2x$.

d $F(x) = -x^2 + c$

$F(0) = 1 \Rightarrow 1 = c$

$\therefore F(x) = -x^2 + 1$

Maximum turning point at $(0, 1)$ and x-intercepts at $(\pm 1, 0)$.



13 **a** For a minimum turning point the gradient, $f'(x) = 0$ and must change from negative to positive about the point.

From the graph, $f'(x) = 0$ at $x = -2$ and $x = 6$, the x-intercepts of the graph of $y = f'(x)$.

The sign of the gradient function changes from negative to positive about $x = 6$. Therefore, there is a minimum turning point at $x = 6$.

The correct answer is **E**.

b For a maximum turning point the gradient, $f'(x) = 0$ and must change from positive to negative about the point. From the graph, $f'(x) = 0$ at $x = -2$ and $x = 6$, the x-intercepts of the graph of $y = f'(x)$. The sign of the gradient function changes from positive to negative about $x = -2$. Therefore, there is a maximum turning point at $x = -2$.

The correct answer is **A**.

c **i** and **ii**. The graph of $y = f'(x)$ lies below the x-axis over the interval $x \in (-2, 6)$. This means $f'(x) < 0$ over this interval. Therefore, at $x = 4$, $f'(x) < 0$ so the gradient at the point where $x = 4$ on the curve $y = f(x)$ must be negative.

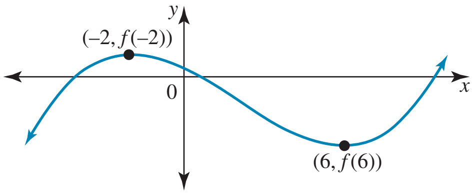
Statement **i** is false and statement **ii** is true.

iii. At both $x = 4$ and $x = 0$ on the graph of $y = f'(x)$, the y-coordinate is -6. This means the gradients at the points on the curve $y = f(x)$ where $x = 4$ and $x = 0$ will have the same negative gradient.

Statement **iii** is true.

d A possible graph of $y = f(x)$ must have a maximum turning point when $x = -2$, a minimum turning point when $x = 6$ and be the shape of a cubic function since the gradient function has a quadratic shape.

One possibility is shown.



14 **a** The x intercepts of the given gradient graph show that $\frac{dy}{dx} = 0$ when $x = -2$ and when $x = 2$. Hence, the curve with this gradient function has stationary points where $x = -2$ and $x = 2$.

Consider how the gradient changes about each point to determine the nature of the stationary points.

About $x = -2$:

For $x < -2$, the value of the gradient is negative as its graph lies below the x-axis.

For $-2 < x < 2$, the value of the gradient is positive as its graph lies above the x-axis.

Therefore, the curve with this gradient function has a minimum turning point where $x = -2$.

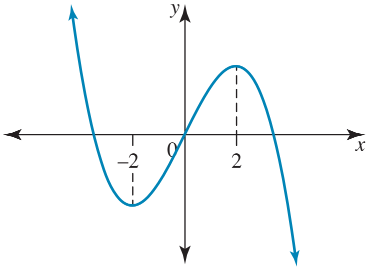
About $x = 2$:

For $-2 < x < 2$, the value of the gradient is positive

For $x > 2$, the value of the gradient is negative as its graph lies below the x-axis.

Therefore, the curve with this gradient function has a maximum turning point where $x = 2$.

b The y-coordinates of the turning points are not known. A possible graph is shown.



c The gradient graph is a parabola with a maximum turning point at (0, 4).

Let its equation be $\frac{dy}{dx} = ax^2 + 4$.

Substitute the point (2, 0).

$\therefore 0 = 4a + 4$

$\therefore a = -1$

The rule for the gradient graph is

$$\frac{dy}{dx} = -x^2 + 4$$

$$y = 4x - \frac{x^3}{3} + c$$

The family of curves with this gradient function are those given by $y = 4x - \frac{x^3}{3} + c$.

d For the curve containing the point (3, 0),

$0 = -9 + 12 + c$

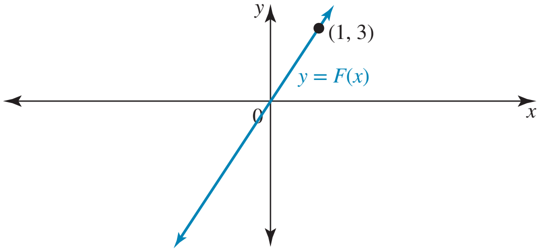
$\therefore c = -3$

The equation of this particular curve is $y = -\frac{x^3}{3} + 4x - 3$.

At (3, 0), $\frac{dy}{dx} = -9 + 4 = -5$, so the slope of the curve is −5.

15 **a** The equation of the given graph is $f(x) = 3$. Hence, the anti-derivative graph has equation of the form $F(x) = 3x + c$.

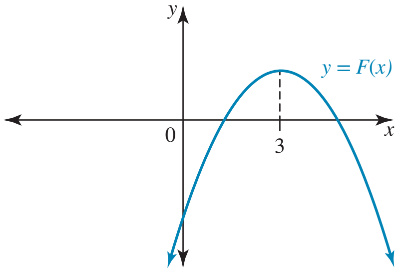
Any linear graph with gradient 3 is suitable. One such possibility is shown.



b Since $f(3) = 0$ and $f(x) > 0$ for $x < 3$ and $f(x) < 0$ for $x > 3$, the graph of $y = F(x)$ will have a maximum turning point where $x = 3$.

$y = f(x)$ is linear, so $y = F(x)$ is quadratic.

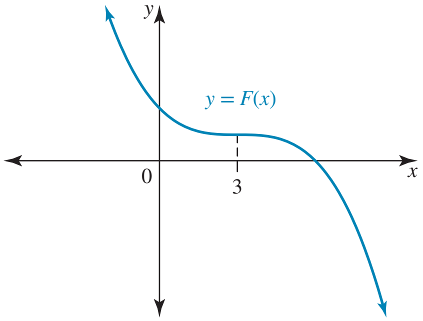
Any concave down parabola with a maximum turning point where $x = 3$ is suitable. An example is shown.



c Since $f(3) = 0$ and $f(x) < 0$ for $x < 3$ and $f(x) < 0$ for $x > 3$, the graph of $y = F(x)$ will have a stationary point of inflection where $x = 3$.

$y = f(x)$ is quadratic, so $y = F(x)$ is cubic.

Any cubic graph with a stationary point of inflection where $x = 3$ and negative gradient elsewhere is suitable. An example is shown.



d Since $f(-3) = 0, f(0) = 0$ and $f(3) = 0$, the graph of $y = F(x)$ has stationary points where $x = -3, x = 0$ and $x = 3$.

About $x = -3$:

For $x < -3$, the value of the gradient is negative as its graph lies below the x-axis.

For $-3 < x < 0$, the value of the gradient is positive as its graph lies above the x-axis.

Therefore, the curve with this gradient function has a minimum turning point where $x = -3$.

About $x = 0$:

For $-3 < x < 0$ the value of the gradient is positive

For $0 < x < 3$, the value of the gradient is negative as its graph lies below the x-axis.

Therefore, the curve with this gradient function has a maximum turning point where $x = 0$.

About $x = 3$:

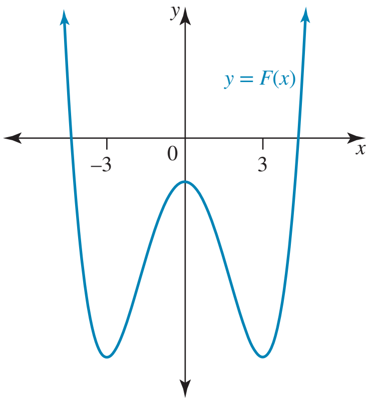
For $0 < x < 3$, the value of the gradient is negative

For $x > 3$, the value of the gradient is positive as its graph lies above the x-axis.

Therefore, the curve with this gradient function has a minimum turning point where $x = 3$.

$y = f(x)$ is a cubic graph, so $y = F(x)$ is a quartic graph.

An example of a possible graph for $y = F(x)$ is shown.



16 **a** Let the equation of the cubic function with stationary point of inflection (0, 18) be $f(x) = ax^3 + 18$.

Substitute the point (1, 9).

$\therefore 9 = a + 18$

$\therefore a = -9$

Hence, $f(x) = -9x^3 + 18$.

$$\int f(x)dx = \int \left(-9x^3 + 18\right) dx$$
$$= -\frac{9x^4}{4} + 18x + c$$

b **i** Let the anti-derivative function be $y = -\frac{9x^4}{4} + 18x + c$.

Since the function contains the point $(0, 0)$, $c = 0$.

Its equation is $y = -\frac{9x^4}{4} + 18x$.

ii Let $y = 0$.

$$\therefore -\frac{9x^4}{4} + 18x = 0$$
$$\therefore -\frac{x^4}{4} + 2x = 0$$
$$\therefore -x^4 + 8x = 0$$
$$\therefore x\left(-x^3 + 8\right) = 0$$
$$\therefore x = 0 \text{ or } x^3 = 8$$
$$\therefore x = 0 \text{ or } x = 2$$

The x-intercept other than the origin is $(2, 0)$.

The gradient of this curve is $f(x) = -9x^3 + 18$.

$f(0) = 18$ (positive slope) and $f(2) = -72 + 18 = -54$ (negative slope)

The graph is steeper at $(2, 0)$ than at $(0, 0)$.

iii At stationary points, the gradient function $f(x) = 0$.

$$\therefore -9x^3 + 18 = 0$$
$$\therefore x^3 = 2$$
$$\therefore x = \sqrt[3]{2}$$

From the graph of $y = f(x)$ given in the question,

if $x < \sqrt[3]{2}$, the gradient is positive since $y = f(x)$ lies above the x-axis.

and if $x > \sqrt[3]{2}$, the gradient is negative since $y = f(x)$ lies below the x-axis.

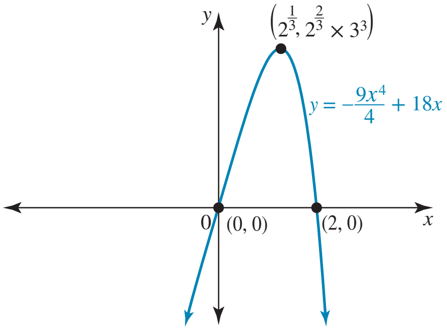
The anti-derivative function has a maximum turning point when $x = \sqrt[3]{2}$.

iv When $x = \sqrt[3]{2} = 2^{\frac{1}{3}}$,

$$y = -\frac{9}{4} \times \left(2^{\frac{1}{3}}\right)^4 + 18 \times 2^{\frac{1}{3}}$$
$$= -\frac{3^2}{2^2} \times 2^{\frac{4}{3}} + 3^2 \times 2^1 \times 2^{\frac{1}{3}}$$
$$= -3^2 \times 2^{\frac{4}{3}-2} + 3^2 \times 2^{1+\frac{1}{3}}$$
$$= -3^2 \times 2^{-\frac{2}{3}} + 3^2 \times 2^{\frac{4}{3}}$$
$$= 3^2 \times 2^{-\frac{2}{3}} \left[-1 + 2^{\frac{6}{3}}\right]$$
$$= 3^2 \times 2^{-\frac{2}{3}} \left[-1 + 2^2\right]$$
$$= 3^2 \times 2^{-\frac{2}{3}} \times 3$$
$$= 2^{-\frac{2}{3}} \times 3^3$$

The y-coordinate of the turning point is $2^p \times 3^q$ where $p = -\frac{2}{3}$ and $q = 3$.

v

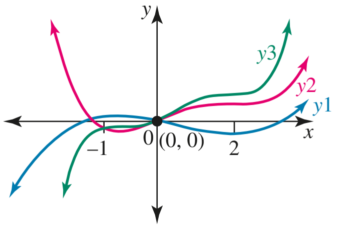


17 The shape of the graphs should be similar to those shown.

a $y1 = \int (x - 2)(x + 1)dx$

b $y2 = \int (x - 2)^2(x + 1)dx$

c $y3 = \int (x - 2)^2(x + 1)^2 dx$



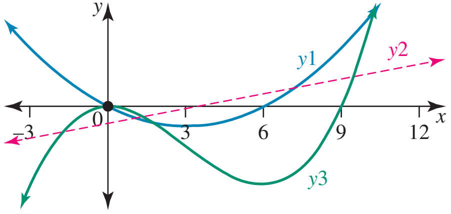
d The factor $(x - 2)$ has a zero of $x = 2$ and the factor $(x + 1)$ has a zero of $x = -1$.

The graph of $y1$ has turning points where $x = 2$ and $x = -1$; the graph of $y2$ has a stationary point of inflection at $x = 2$ and a turning point at $x = -1$; the graph of $y3$ has stationary points of inflection at $x = 2$ and at $x = -1$.

For factors of multiplicity 1, the anti-derivative graphs have turning points at the zero of each factor.

For factors of multiplicity 2, the anti-derivative graphs have stationary points of inflection at the zero of each factor.

18 Use CAS technology to obtain the graph below.



Connections between $y1$ and $y2$:

$y1$ is quadratic with x intercepts at $x = 0$ and $x = 6$. It has a minimum turning point when $x = 3$.

$y2$ is linear with x intercept at $x = 3$. Its sign changes from negative to zero to positive as it cuts through the x intercept.

This shows that $y1$ has a minimum turning point when $x = 3$.

The x-intercept of $y = f'(x)$ is connected to the turning point of $y = f(x)$; the degree of $f'(x)$ is one less than the degree of $f(x)$.

Connections between $y1$ and $y3$:

$y1$ is the gradient graph of $y3$. The x intercepts of $y1$ give the x-coordinates of the stationary points of $y3$.

The sign of $y1$ changes from positive to zero to negative at $x = 0$, showing that $y3$ has a maximum turning point when $x = 0$.

The sign of $y1$ changes from negative to zero to positive at

$x = 6$, showing that $y3$ has a minimum turning point when $x = 6$.

The turning points of $y = \int f(x)dx$ are connected to the x intercepts of $y = f(x)$; the degree of $\int f(x)dx$ is one more than the degree of $f(x)$.

(Note that the maximum turning point of $y3$ occurs at $x = 0$ but it is not necessarily at $(0, 0)$. The CAS calculator has used zero for the constant of integration).

13.3 Exam questions

1 $f'(x) = -6x^2 + 2$

$$f(x) = \int \left(-6x^2 + 2\right) dx$$

$$f(x) = -2x^3 + 2x + c$$

Point $(-1, 8)$

$$8 = -2(-1)^3 + 2(-1) + c$$
$$8 = 2 - 2 + c$$
$$c = 8$$

$$f(x) = -2x^3 + 2x + 8$$

The correct answer is **C**.

2 $\frac{dy}{dx} = 5x + b$

Stationary point $\frac{dy}{dx} = 0$, when $x = 4$

$$0 = 5 \times 4 + b$$

$\therefore b = -20$ [1 mark]

$$\frac{dy}{dx} = 5x - 20$$

$$y = \int (5x - 20)\, dx$$

$y = \frac{5x^2}{2} - 20x + c$ [1 mark]

Substitute point $(4, 6)$.

$$6 = \frac{5(4)^2}{2} - 20 \times 4 + c$$
$$6 = 40 - 80 + c$$
$$\therefore c = 46$$

The equation is $y = \frac{5x^2}{2} - 20x + 46$. [1 mark]

3 Equation of gradient function:

Slope $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3}{4 - 0} = -\frac{3}{4}$ [1 mark]

$$\frac{dy}{dx} = mx + c_1$$

$$\frac{dy}{dx} = -\frac{3}{4}x + 3$$

$$y = \int \left(-\frac{3}{4}x + 3\right) dx$$

$y = \frac{3}{8}x^2 + 3x + c_2$ [1 mark]

Point $(8, 9)$

$$9 = -\frac{3}{8}(8)^2 + 3(8) + c_2$$
$$9 = -24 + 24 + c_2$$
$$c_2 = 9$$

The equation is $y = -\frac{3}{8}x^2 + 3x + 9$. [1 mark]

13.4 Application of anti-differentiation

13.4 Exercise

1 a $v = 2t + 5$

$$x = \int v dt$$
$$x = \int (2t + 5)\, dt$$
$$x = t^2 + 5t + c$$

Given $x = 0$ when $t = 0$,

$$0 = 0 + c$$
$$c = 0$$

The position at any time t is $x = t^2 + 5t$.

b $v = 8t - 3t^2$

$$x = \int v dt$$
$$x = 4t^2 - t^3 + c$$

Given $x = -2$ when $t = 0$,

$$-2 = 0 + c$$
$$c = -2$$

The position at any time t is $x = -t^3 + 4t^2 - 2$.

c $v = 2(t - 3)^2$

$$v = 2(t^2 - 6t + 9) = 2t^2 - 12t + 18$$

i Acceleration, $a = \frac{dv}{dt}$

$$\therefore a = 4t - 12$$

The acceleration at any time t is $a = 4t - 12$.

ii $x = \int v dt$

$$x = \int (2t^2 - 12t + 18)\, dt$$
$$x = \frac{2t^3}{3} - 6t^2 + 18t + c$$

Given $x = 0$ when $t = 3$,

$$0 = 18 - 54 + 54 + c$$
$$c = -18$$

The position at any time t is

$$x = \frac{2t^3}{3} - 6t^2 + 18t - 18.$$

2 a $a = 4t$

$$v = \int (a)\, dt$$
$$v = \int (4t)\, dt$$
$$v = 2t^2 + c_1$$

When $t = 0$, $v = 0$.

$$\therefore c_1 = 0$$

The velocity at any time is $v = 2t^2$.

b $x = \int (v)\, dt$

$$x = \int 2t^2 dt$$
$$x = \frac{2}{3}t^3 + c_2$$

When $t = 0, x = 1$:

$1 = 0 + c_2$

$\therefore c_2 = 1.$

The position at any time is $x = \frac{2}{3}t^3 + 1$.

3 $v = 6 - 2t,\ t \geq 0$

a $x = \int v\,dt$

$x = \int (6 - 2t)\,dt$

$x = 6t - t^2 + c$

When $t = 1, x = 2$:

$2 = 6 - 1 + c$

$c = -3$

$x = 6t - t^2 - 3$

The position at any time is $x = 6t - t^2 - 3$.

b When $x = 6$:

$6 = 6t - t^2 - 3$

$t^2 - 6t + 9 = 0$

$(t - 3)(t - 3) = 0$

$t = 3$

The particle's position is 6 metres at 3 seconds.

4 $v = 2\left(3t^2 + 1\right), t \geq 0$

$v = 6t^2 + 2$

For position anti-differentiate the velocity.

$x = \int v\,dt$

$x = \int \left(6t^2 + 2\right)dt$

$x = 6 \times \frac{t^3}{3} + 2t + c$

$= 2t^3 + 2t + c$

When $t = 1, x = -1$.

$-1 = 2 + 2 + c$

$c = -5$

$x = 2t^3 + 2t - 5$

For acceleration, differentiate the velocity.

$a = 12t$

After t seconds, the position is $x = 2t^3 + 2t - 5$ and the acceleration is $a = 12t$.

5 **a** $v = 8t^2 - 20t - 12,\ t \geq 0$

$x =$ Since, $\int vdt$

$x = \frac{8t^3}{3} - 10t^2 - 12t + c$

Given $x = 54$ when $t = 0$,

$54 = c$

$\therefore x = \frac{8t^3}{3} - 10t^2 - 12t + 54$

b When $t = 1$,

$x = \frac{8}{3} - 10 - 12 + 54$

$= \frac{8}{3} + 32$

$= 34\frac{2}{3}$

This position is $\left(54 - 34\frac{2}{3}\right) = 19\frac{1}{3}$ metres from its initial position.

c Let $v = 0$.

$\therefore 8t^2 - 20t - 12 = 0$

$\therefore 2t^2 - 5t - 3 = 0$

$\therefore (2t + 1)(t - 3) = 0$

$\therefore t = -\frac{1}{2}$, or $t = 3$

Reject the negative value since $t \geq 0$

$\therefore t = 3$

When $t = 3$,

$x = 72 - 90 - 36 + 54$

$= 0$

When the velocity is zero, the particle is at the origin.

6 $v = -3t^3,\ t \geq 0$

Given that $v = -24$ when $x = 1$

Let $v = -24$.

$\therefore\ -24 = -3t^3$

$\therefore\ t^3 = 8$

$\therefore\ t = 2$

Hence, when $t = 2, x = 1$.

Since $x = \int vdt$, then $x = \frac{-3t^4}{4} + c$.

Substitute $t = 2, x = 1$.

$\therefore 1 = -12 + c$

$\therefore c = 13$

$\therefore x = -\frac{3t^4}{4} + 13$

Let $t = 0$.

$\therefore x = 13$

Initially the particle is 13 metres to the right of the origin.

7 **a** Acceleration: $a = 8 + 6t$

Velocity: $v = \int a\,dt$

$\therefore v = 8t + 3t^2 + c_1$

Given $v = 3$ when $t = 1$,

$\therefore 3 = 8 + 3 + c_1$

$\therefore c_1 = -8$

The velocity is $v = 3t^2 + 8t - 8$.

b Position $\int vdt$

$\therefore x = t^3 + 4t^2 - 8t + c_2$

Given $x = 2$ when $t = 1$,

$\therefore 2 = 1 + 4 - 8 + c_2$

$\therefore c_2 = 5$

The position is $x = t^3 + 4t^2 - 8t + 5$.

8 **a** $v = 3t^2 - 10t - 8$

To obtain position anti-differentiate velocity.

$\therefore x = t^3 - 5t^2 - 8t + c$

When $t = 0,\ x = 40$.

$\therefore 40 = c$

$\therefore x = t^3 - 5t^2 - 8t + 40$

When the particle is at the origin, $x = 0$.

$t^3 - 5t^2 - 8t + 40 = 0$

$t^2(t - 5) - 8(t - 5) = 0$

$(t - 5)(t^2 - 8) = 0$

$(t - 5)(t - \sqrt{8})(t + \sqrt{8}) = 0$

$t = 5, \pm\sqrt{8}$

Since $t \geq 0,\ t = 5, \sqrt{8}$.

The first time the particle is at the origin is when $t = \sqrt{8}$.

Therefore, the particle is first at the origin after $2\sqrt{2}$ seconds.

b Acceleration is the rate of change of velocity.

$a = 6t - 10$.

When velocity is 0,

$3t^2 - 10t - 8 = 0$

$(3t + 2)(t - 4) = 0$

$t = -\frac{2}{3},\ t = 4$

Since $t \geq 0,\ t = 4$.

Therefore,

$a = 6 \times 4 - 10$

$= 14$

The acceleration is 14 m/s^2 when the velocity is 0.

9 $a = 8 - 18t$

Anti-differentiate to obtain velocity.

$v = 8t - 9t^2 + c_1$

$v = 10$ when $t = 0 \Rightarrow c_1 = 10$

$\therefore v = 8t - 9t^2 + 10$

When $t = 1$, $v = 9$, so the velocity is 9 m/s.

Position:

$v = 8t - 9t^2 + 10$

$\therefore x = 4t^2 - 3t^3 + 10t + c_2$

When $t = 0, x = -2 \ \Rightarrow c_2 = -2$

$\therefore x = 4t^2 - 3t^3 + 10t - 2$

When $t = 1,\ x = 9$, so the position is 9 metres. The position of the particle is 9 metres to the right of the origin.

10 $a = 9.8$

$\therefore v = 9.8t + c_1$

$v = 0, t = 0 \Rightarrow c_1 = 0$

$\therefore v = 9.8t$

$\therefore x = 4.9t^2 + c_2$

$x = 0, t = 0 \Rightarrow c_2 = 0$

$\therefore x = 4.9t^2$

When $t = 5$,

$x = 4.9 \times 25$

$= 122.5$

Therefore, the position is 122.5 metres.

11 a Acceleration, $a = -10$

Velocity: $v = -10t + c_1$

Given $v = 20$ when $t = 0$

$\therefore 20 = c_1$

$\therefore v = -10t + 20$

Position: $x = -5t^2 + 20t + c_2$

Given $x = 0$ when $t = 0$

$\therefore 0 = c_2$

$\therefore x = -5t^2 + 20t$

Let $v = 0$.

$\therefore -10t + 20 = 0$

$\therefore t = 2$

When $t = 2$,

$x = -20 + 40$

$= 20$

The velocity is zero after 2 seconds when its position is 20 metres from the origin in the positive direction.

b The object starts from the origin.

Let $x = 0$

$\therefore -5t^2 + 20t = 0$

$\therefore -5t(t - 4) = 0$

$\therefore t = 0, t = 4$

The object returns to its starting point after 4 seconds.

12 a $v = 6 - 6t,\ t \geq 0$

Acceleration: $a = \frac{dv}{dt}$

$\therefore a = -6$

The particle moves with a constant acceleration of -6 m/s^2.

b Let $v = 0$.

$v = 0$

$\therefore 6 - 6t = 0$

$\therefore t = 1$

Position: $x = 6t - 3t^2 + c$

When $t = 0$, $x = 9$.

$\therefore 9 = c$

$\therefore x = 6t - 3t^2 + 9$

Let $t = 1$.

$x = 6 - 3 + 9$

$= 12$

The velocity is zero after 1 second when the particle is 12 metres to the right of the origin.

c Let $x = 0$.

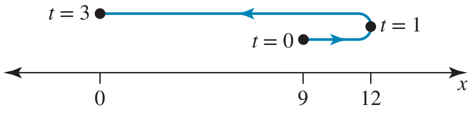
$\therefore 6t - 3t^2 + 9 = 0$

$\therefore t^2 - 2t - 3 = 0$

$\therefore (t - 3)(t + 1) = 0$

$\therefore t = 3$

The particle reaches the origin after 3 seconds.



The distance travelled is (3 + 12) = 15 metres.

d The average speed in travelling 15 metres in 3 seconds is equal to $\frac{15}{3} = 5$ m/s

e The average velocity is the average rate of change of position

$\frac{x(3) - x(0)}{3 - 0} = \frac{0 - 9}{3}$

$= -3$

The average velocity is -3 m/s.

13 $h'(t) = 0.2t$

$\therefore h(t) = 0.1t^2 + c$

When $t = 0, h = 50 \Rightarrow c = 50$

$\therefore h(t) = 0.1t^2 + 50$

After one year or 12 months,

$h(12) = 0.1 \times 144 + 50$

$= 64.4$

The rubber plant reached a height of 64.4 cm after one year.

14 a $\frac{dA}{dt} = -18t$

Anti-differentiate with respect to t.

$\therefore A = -9t^2 + c$

When $t = 0,\ A = 90$.

$\therefore 90 = c$

$\therefore A = -9t^2 + 90$

b Let $A = 0$

$\therefore -9t^2 + 90 = 0$

$\therefore t^2 = 10$

$\therefore t = \sqrt{10}$

(The negative square root is not applicable.)

Since $\sqrt{9} < \sqrt{10} < \sqrt{16}$, then $3 < \sqrt{10} < 4$. Therefore, after 4 whole days, the weed will be completely removed.

c The values of t lie in the interval $\left[0, \sqrt{10}\right]$, so the model is valid for $0 \le t \le \sqrt{10}$.

15 a $A'(t) = 4 - t$

$A'(0) = 4 > 0$, so the area increases initially.

Let the rate of change be zero.

$A'(t) = 0$

$\therefore 4 - t = 0$

$\therefore t = 4$

After 4 days the area stops increasing.

b $A(t) = 4t - \dfrac{t^2}{2} + c$

When $t = 0,\ A = 0$

$\therefore 0 = c$

$\therefore A(t) = 4t - \dfrac{t^2}{2}$

The area of the puddle is zero when $4t - \dfrac{t^2}{2} = 0$.

$\therefore 8t - t^2 = 0$

$\therefore t(8 - t) = 0$

$\therefore t = 0,\ t = 8$

After 8 days, the puddle has dried out.

Therefore, the domain is $[0, 8]$.

c The greatest area occurs when $A'(t) = 0$.

From part **a**, this occurs when $t = 4$

$A(4) = 16 - 8$

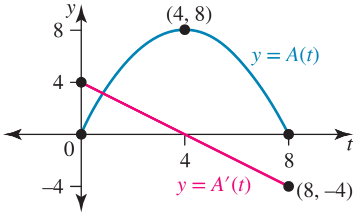
$= 8$

The greatest area of the puddle is 8 sq m.

d The domain for each graph is $[0, 8]$.

$y = A'(t) = 4 - t$ is linear with end points $(0, 4)$ and $(8, -4)$. It also contains the point $(4, 0)$.

$y = A(t) = 4t - \dfrac{t^2}{2}$ is quadratic with maximum turning point $(4, 8)$ and end points $(0, 0)$ and $(8, 0)$.



16 a $v(t) = 3(t - 2)(t - 4),\ t \ge 0$

$v(0) = 3(-2)(-4)$

$= 24$

The initial velocity is 24 m/s.

Acceleration: $a = v'(t)$

$v(t) = 3(t^2 - 6t + 8)$

$\therefore v(t) = 3t^2 - 18t + 24$

$\therefore a = 6t - 18$

$a(0) = -18$, so the initial acceleration is $-18\ \text{m/s}^2$.

b $v(t) = 3t^2 - 18t + 24$

$\therefore x(t) = t^3 - 9t^2 + 24t + c$

Since $x = 0$ when $t = 0, c = 0$.

$\therefore x(t) = t^3 - 9t^2 + 24t$

c $x(5) = 125 - 225 + 120$

$\therefore x(5) = 20$

d Since $v(t) = 3(t - 2)(t - 4)$, the velocity is zero when $t = 2$ and when $t = 4$. The particle will therefore change its direction of motion twice during the first 5 seconds of motion.

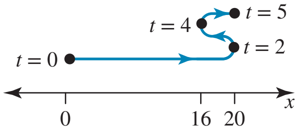
$x(0) = 0$ and $x(5) = 20$.

$x(2) = 8 - 36 + 48$

$= 20$

$x(4) = 64 - 144 + 96$

$= 16$



The distance travelled is $20 + 4 + 4 = 28$ metres.

17 $\dfrac{dV}{dt} = -0.25$

a Anti-differentiation gives $V = -0.25t + c$

When $t = 0,\ V = 4.5\pi, \Rightarrow c = 4.5\pi$

$\therefore V = -0.25t + 4.5\pi$

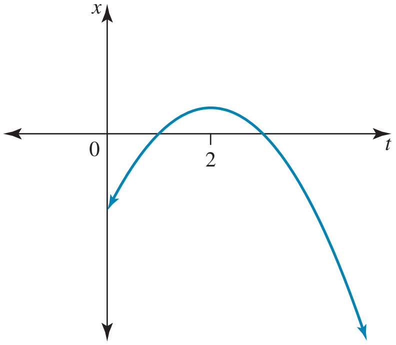
b When $V = 0$, $t = \dfrac{4.5\pi}{0.25}$.

Therefore, it takes approximately 56.5 seconds for the ice block to melt.

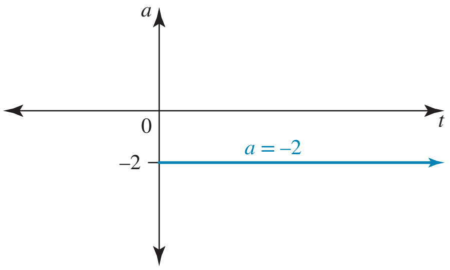
18 a The given velocity graph is the gradient graph for the position. The position graph is an anti-derivative graph. Since $v = 0$ when $t = 2$, there is a stationary point at $t = 2$ on the position graph. For increasing t, the velocity changes its sign from positive to zero to negative as it passes through $(2, 0)$. This means there is a maximum turning point at $t = 2$ on the position graph.

The velocity graph is linear so the position graph is quadratic.

An example of a possible position graph is shown.



b The acceleration graph is the gradient of the velocity graph. The gradient of the given graph is -2, so the acceleration graph is a horizontal line $a = -2, t \ge 0$.



c The given velocity graph has gradient $m = -2$ and its vertical intercept is $(0, 4)$.

Its equation is $v = -2t + 4$.

$\therefore x = -t^2 + 4t + c$

When $t = 0$, $x = -4$.

$\therefore\ -4 = c$

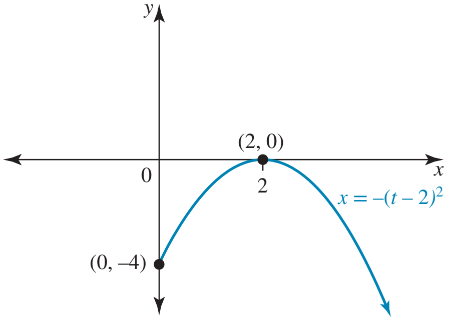
$\therefore x = -t^2 + 4t - 4$

Factorising the expression for the position

$x = -\left(t^2 - 4t + 4\right)$

$= -(t-2)^2, t \geq 0$

d There is a maximum turning point at $(2, 0)$.



19 $\dfrac{dm}{dt} = kt^2$

a After 5 days, growing at 300 microbes per day

$t = 5, \dfrac{dm}{dt} = 300$

$300 = k \times 5^2$

$k = 12$

b $\dfrac{dm}{dt} = 12t^2$

$m = 12 \times \dfrac{t^3}{3} + c$

$= 4t^3 + c$

When $t = 0$, $m = 20$:

$20 = c$

$m = 4t^3 + 20$

c For $m = 6420$:

$6420 = 4t^3 + 20$

$t = 11.696071$

It takes 12 days for the population to reach 6420.

20 **a** $v = (2t+1)^4$

Acceleration: $a = \dfrac{dv}{dt}$

$a = \dfrac{d}{dt}\left((2t+1)^4\right)$

$= 8(2t+1)^3$

b Position:

$x = \displaystyle\int (2t+1)^4 dt$

$= \dfrac{(2t+1)^5}{10}$

Hence, $x = \dfrac{(2t+1)^5}{10} + c$

When $t = 0, x = 4.2$.

$\therefore 4.2 = \dfrac{1}{10} + c$

$\therefore c = 4.1$

$\therefore x = \dfrac{(2t+1)^5}{10} + 4.1$

c Let $x = 8.4$.

$\therefore 8.4 = \dfrac{(2t+1)^5}{10} + 4.1$

Solve using CAS technology to obtain $t = 0.561$.

When $t = 0.561$,

$v = (2 \times 0.561 + 1)^4$

$= 20.266$

Correct to 2 decimal places, the time is 0.56 seconds and the velocity is 20.27 m/s (assuming the time unit is seconds).

13.4 Exam questions

1 $a = 2t + 1,\ t \geq 0$

$v = \displaystyle\int (2t+1)\, dt$

$v = t^2 + t + c$

$v = 10$ when $t = 0$

$\therefore v = t^2 + t + 10$

$t = 2$

$v = 2^2 + 2 + 10$

$v = 16$

The correct answer is **D**.

2 $a = 16,\ t \geq 0$

$v = \displaystyle\int (16)\, dt$

$v = 16t + c_1$

When $t = 0,\ v = 2$ m/s.

$\therefore v = 16t + 2$ [1 mark]

$x = \displaystyle\int (16t + 2)$

$x = 8t^2 + 2t + c_2$

When $t = 0, v = 2$ m.

$x = 8t^2 + 2t + 2$ [1 mark]

Position at $t = 3$:

$x = 8(3)^2 + 2(3) + 2$

$x = 72 + 6 + 2$

$x = 80$

The position at $t = 3$ is 80 m. [1 mark]

3 $\dfrac{dV}{dt} = -0.4t$ [1 mark]

$V = \displaystyle\int (-0.4t)\, dt$

$V = -0.2t^2 + c$

When $t = 0, V = 2880$.

$\therefore V = -0.2t^2 + 2880$ [1 mark]

When $V = 0$.

$0 = -0.2t^2 + 2880$

$0.2t^2 = 2800$

$t^2 = 14400$

$t = \pm 120$

$\therefore$ 120 seconds $(t > 0)$

$t = 2$ minutes [1 mark]

13.5 The definite integral

13.5 Exercise

1 a $\int_{1}^{4} 2x dx$

$= \left[x^2\right]_1^4$

$= \left(4^2\right) - \left(1^2\right)$

$= 16 - 1$

$= 15$

b $\int_{0}^{2} 3x^2 dx$

$= \left[x^3\right]_0^2$

$= \left(2^3\right) - \left(0^3\right)$

$= 8$

c $\int_{-5}^{-2} 7dx$

$= [7x]_{-5}^{-2}$

$= (-14) - (-35)$

$= 21$

d $\int_{-2}^{0} (5x + 4)dx$

$= \left[\frac{5x^2}{2} + 4x\right]_{-2}^{2}$

$= \left(\frac{5 \times 4}{2} + 8\right) - \left(\frac{5 \times 4}{2} - 8\right)$

$= (18) - (2)$

$= 16$

e $\int_{5}^{10} (6 - 4x)dx$

$= \left[6x - 2x^2\right]_5^{10}$

$= (60 - 200) - (30 - 50)$

$= (-140) - (-20)$

$= -120$

f $\int_{-\frac{1}{2}}^{0} (1 + 5x + 3x^2)dx$

$= \left[x + \frac{5}{2}x^2 + x^3\right]_{-\frac{1}{2}}^{0}$

$= (0) - \left(-\frac{1}{2} + \frac{5}{2} \times \frac{1}{4} - \frac{1}{8}\right)$

$= -\left(-\frac{1}{2} + \frac{5}{8} - \frac{1}{8}\right)$

$= -\left(-\frac{4}{8} + \frac{5}{8} - \frac{1}{8}\right)$

$= 0$

2 $\int_{0}^{3} (3x^2 - 2x)dx$

$= \left[x^3 - x^2\right]_0^3$

$= \left(3^3 - 3^2\right) - \left(0^3 - 0^2\right)$

$= 27 - 9 - 0$

$= 18$

3 a $2\int_{0}^{1} (x(1 - x))\, dx$

$= 2\int_{0}^{1} \left(x - x^2\right) dx$

$= 2 \times \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= 2 \times \left\{\left(\frac{1}{2} - \frac{1}{3}\right) - (0)\right\}$

$= 2 \times \frac{3 - 2}{6}$

$= \frac{1}{3}$

b $\int_{0}^{3} (3 - x)dx - \int_{3}^{4} (3 - x)dx$

$= \left[3x - \frac{x^2}{2}\right]_0^3 - \left[3x - \frac{x^2}{2}\right]_3^4$

$= \left[\left(9 - \frac{9}{2}\right) - (0)\right] - \left[(12 - 8) - \left(9 - \frac{9}{2}\right)\right]$

$= \left[\frac{9}{2}\right] - \left[4 - \frac{9}{2}\right]$

$= \frac{9}{2} - 4 + \frac{9}{2}$

$= 5$

c $\int_{-2}^{1} \left(1 - y^3\right) dy$

$= \left[y - \frac{y^4}{4}\right]_{-2}^{1}$

$= \left(1 - \frac{1}{4}\right) - (-2 - 4)$

$= \frac{3}{4} + 6$

$= 6\frac{3}{4}$

d $\int_{-2}^{-1} (t(3t + 2))\, dt$

$= \int_{-2}^{-1} \left(3t^2 + 2t\right) dt$

$= \left[t^3 + t^2\right]_{-2}^{-1}$

$= (-1 + 1) - (-8 + 4)$

$= (0) - (-4)$

$= 4$

4 a $\int_{-2}^{2}(x-2)(x+2)\,dx$

$$=\int_{-2}^{2}(x^2-4)\,dx$$
$$=\left[\frac{x^3}{3}-4x\right]_{-2}^{2}$$
$$=\left(\frac{8}{3}-8\right)-\left(\frac{-8}{3}-(-8)\right)$$
$$=\frac{8}{3}-8+\frac{8}{3}-8$$
$$=\frac{16}{3}-16$$
$$=-\frac{32}{3}$$

b $\int_{-1}^{1}x^3dx$

$$=\left[\frac{x^4}{4}\right]_{-1}^{1}$$
$$=\left(\frac{1}{4}\right)-\left(\frac{1}{4}\right)$$
$$=0$$

5 a $\int_{-2}^{2}5x^4dx$

$$=\left[x^5\right]_{-2}^{2}$$
$$=(32)-(-32)$$
$$=64$$

b $\int_{0}^{2}\left(7-2x^3\right)dx$

$$=\left[7x-\frac{x^4}{2}\right]_{0}^{2}$$
$$=(14-8)-(0)$$
$$=6$$

c $\int_{1}^{3}\left(6x^2+5x-1\right)dx$

$$=\left[2x^3+\frac{5x^2}{2}-x\right]_{1}^{3}$$
$$=\left(54+\frac{45}{2}-3\right)-\left(2+\frac{5}{2}-1\right)$$
$$=\left(51+\frac{45}{2}\right)-\left(\frac{7}{2}\right)$$
$$=51+\frac{45}{2}-\frac{7}{2}$$
$$=51+19$$
$$=70$$

d $\int_{-3}^{0}12x^2(x-1)\,dx$

$$=\int_{-3}^{0}\left(12x^3-12x^2\right)dx$$
$$=\left[3x^4-4x^3\right]_{-3}^{0}$$
$$=(0)-(243+108)$$
$$=-351$$

e $\int_{-4}^{-2}(x+4)^2\,dx$

$$=\int_{-4}^{-2}\left(x^2+8x+16\right)dx$$
$$=\left[\frac{x^3}{3}+4x^2+16x\right]_{-4}^{-2}$$
$$=\left(-\frac{8}{3}+16-32\right)-\left(-\frac{64}{3}+64-64\right)$$
$$=-\frac{8}{3}-16+\frac{64}{3}$$
$$=\frac{56}{3}-\frac{48}{3}$$
$$=\frac{8}{3}$$

f $\int_{-\frac{1}{2}}^{\frac{1}{2}}(x+1)\left(x^2-x\right)dx$

$$=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(x^3-x\right)dx$$
$$=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$
$$=\left(\frac{1}{64}-\frac{1}{8}\right)-\left(\frac{1}{64}-\frac{1}{8}\right)$$
$$=0$$

6 a $\int_{0}^{1}(ax-2)dx=7$

$$=\left[\frac{ax^2}{2}-2x\right]_{0}^{1}$$
$$\int_{0}^{1}(ax-2)dx$$
$$=\left(\frac{a}{2}-2\right)-(0)$$
$$=\frac{a}{2}-2$$
$$\therefore\frac{a}{2}-2=7$$
$$\therefore\frac{a}{2}=9$$
$$\therefore a=18$$

b $\int_{-2}^{1}(b+8x)dx = 0$

$\int_{-2}^{1}(b+8x)dx$

$= \left[bx+4x^2\right]_{-2}^{1}$

$= (b+4)-(-2b+16)$

$= b+4+2b-16$

$= 3b-12$

$\therefore 3b-12=0$

$\therefore 3b=12$

$\therefore b=4$

c $\int_{3}^{k} 2dx = 14$

$\int_{3}^{k} 2dx$

$= [2x]_3^k$

$= (2k)-(6)$

$= 2k-6$

$\therefore 2k-6=14$

$\therefore 2k=20$

$\therefore k=10$

d $\int_{-n}^{n} 9x^2dx = 48$

$\int_{-n}^{n} 9x^2dx$

$= \left[3x^3\right]_{-n}^{n}$

$= \left(3n^3\right)-\left(-3n^3\right)$

$= 6n^3$

$\therefore 6n^3=48$

$\therefore n^3=8$

$\therefore n=2$

7 a $\int_{1}^{2}(20x+15)dx = 5\int_{1}^{2}(4x+3)dx$

$\text{LHS} = \int_{1}^{2}(20x+15)dx$

$= \left[10x^2+15x\right]_1^2$

$= (40+30)-(10+15)$

$= 70-25$

$= 45$

$\text{RHS} = 5\int_{1}^{2}(4x+3)dx$

$= 5\left[2x^2+3x\right]_1^2$

$= 5\{(8+6)-(2+3)\}$

$= 5\{14-5\}$

$= 5\times 9$

$= 45$

$\therefore$ LHS = RHS

b $\int_{-1}^{2}(x^2+2)dx = \int_{-1}^{2}x^2dx + \int_{-1}^{2}2dx$

$\text{LHS} = \int_{-1}^{2}(x^2+2)dx$

$= \left[\frac{x^3}{3}+2x\right]_{-1}^{2}$

$= \left(\frac{8}{3}+4\right)-\left(-\frac{1}{3}-2\right)$

$= \left(\frac{20}{3}\right)-\left(-\frac{7}{3}\right)$

$= \frac{20}{3}+\frac{7}{3}$

$= 9$

$\text{RHS} = \int_{-1}^{2}x^2dx + \int_{-1}^{2}2dx$

$= \left[\frac{x^3}{3}\right]_{-1}^{2} + [2x]_{-1}^{2}$

$= \left[\left(\frac{8}{3}\right)-\left(-\frac{1}{3}\right)\right] + [(4)-(-2)]$

$= (3)+(6)$

$= 9$

$\therefore$ LHS = RHS

c $\int_{1}^{3}3x^2dx = \int_{1}^{3}t^2dt$

$\text{LHS} = \int_{1}^{3}3x^2dx$

$= \left[x^3\right]_1^3$

$= (27)-(1)$

$= 26$

$\text{RHS} = \int_{1}^{3}3t^2dt$

$= \left[t^3\right]_1^3$

$= (27)-(1)$

$= 26$

$\therefore$ LHS = RHS

d $\int_{a}^{a}3x^2dx = 0$

$\text{LHS} = \int_{a}^{a}3x^2dx$

$= \left[x^3\right]_a^a$

$= \left(a^3\right)-\left(a^3\right)$

$= 0$

$= \text{RHS}$

e $\int_{b}^{a}3x^2dx = -\int_{a}^{b}3x^2dx$

$\text{LHS} = \int_{b}^{a}3x^2dx$

$= \left[x^3\right]_b^a$

$= \left(a^3\right) - \left(b^3\right)$
$= a^3 - b^3$

$\text{RHS} = -\int_a^b 3x^2 dx$
$= -\left[x^3\right]_a^b$
$= -\left(b^3 - a^3\right)$
$= a^3 - b^3$
$\therefore$ LHS = RHS

f $\int_{-a}^{a} dx = 2a$

$\text{LHS} = \int_{-a}^{a} dx$
$= \int_{-a}^{a} 1\, dx$
$= [x]_{-a}^{a}$
$= (a) - (-a)$
$= 2a$
$= \text{RHS}$
$\therefore$ LHS = RHS

8 $\int_4^n 3dx = 9$

$\therefore\ [3x]_4^n = 9$
$\therefore 3n - 12 = 9$
$\therefore 3n = 21$
$\therefore n = 7$

9 a The area measure is given by $\int_{-2}^{2} (3x + 6)\, dx$.

Integration method to calculate the area:

$A = \int_{-2}^{2} (3x + 6)\, dx$
$= \left[\frac{3x^2}{2} + 6x\right]_{-2}^{2}$
$= (6 + 12) - (6 - 12)$
$= 18 + 6$
$= 24$

The area is 24 square units.

The triangular area can also be calculated with the formula $A = \frac{1}{2}bh$.

The base, $b = 4$.

Height: When $x = 2$,
$y = 3 \times 2 + 6$
$= 12$
$\therefore h = 12$
$A = \frac{1}{2} \times 4 \times 12$
$= 24$

The area is 24 square units.

b The area measure is given by $\int_{-6}^{-2} (1)\, dx$.

Integration method to calculate the area:

$A = \int_{-6}^{-2} (1)\, dx$
$= [x]_{-6}^{-2}$
$= (-2) - (-6)$
$= 4$

The area is 4 square units.

The rectangular area can also be calculated from
$A = lw,\ l = 4, w = 1$
$\therefore A = 4 \times 1$
$= 4$

The area is 4 square units.

c The area measure is given by $\int_{-2}^{0} (4 - 3x)\, dx$.

Integration method to calculate the area:

$A = \int_{-2}^{0} (4 - 3x)\, dx$
$= \left[4x - \frac{3x^2}{2}\right]_{-2}^{0}$
$= (0) - (-8 - 6)$
$= 14$

The area is 14 square units.

The trapezoidal area can also be calculated from $A = \frac{1}{2}h(a + b)$.

For the line $y = 4 - 3x$, when $x = -2$, $y = 10$ and when $x = 0$, $y = 4$.

$A = \frac{1}{2}h(a + b)$
$h = 2, a = 10, b = 4$
$\therefore A = \frac{1}{2} \times 2 \times (10 + 4)$
$= 14,$

The area is 14 square units.

10 a The base is 4 units and the height is 3 units.

$A = \frac{1}{2}bh$
$= \frac{1}{2} \times 4 \times 3$
$= 6$

The area is 6 square units.

b $\int_0^4 0.75x\, dx$

c $\int_0^4 0.75x\, dx$
$= \left[0.75 \times \frac{x^2}{2}\right]_0^4$
$= 0.75 \times 8 - 0$
$= 6$

Therefore, the area is 6 square units.

11 a $\int_1^3 \left(16 - x^2\right) dx$

b $\int_{1}^{3}(16-x^2)dx$

$= \left[16x - \frac{x^3}{3}\right]_1^3$

$= (48-9) - \left(16 - \frac{1}{3}\right)$

$= 39 - 15\frac{2}{3}$

$= 23\frac{1}{3}$

Therefore, the area is $23\frac{1}{3}$ square units.

12 **a** **i** The area measure is given by $\int_{-1}^{1}\left(x^2+1\right)dx$

ii $A = \int_{-1}^{1}\left(x^2+1\right)dx$

$= \left[\frac{x^3}{3}+x\right]_{-1}^{1}$

$= \left(\frac{1}{3}+1\right) - \left(-\frac{1}{3}-1\right)$

$= \frac{4}{3} - \left(-\frac{4}{3}\right)$

$= \frac{8}{3} = 2\frac{2}{3}$

The area is $2\frac{2}{3}$ square units.

b **i** The area measure is given by $\int_{-2}^{1}\left(1-x^3\right)dx$

ii $A = \int_{-2}^{1}\left(1-x^3\right)dx$

$= \left[x - \frac{x^4}{4}\right]_{-2}^{1}$

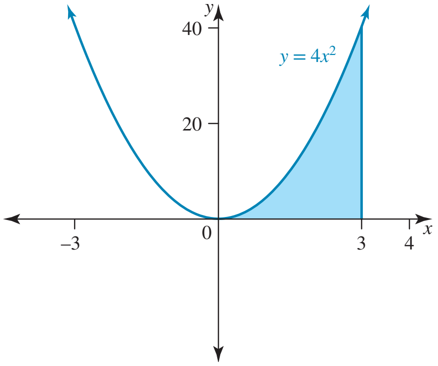
$= \left(1-\frac{1}{4}\right) - (-2-4)$

$= \frac{3}{4} - (-6)$

$= 6\frac{3}{4}$

The area is $6\frac{3}{4}$ square units.

13 **a** The area represented by $\int_{0}^{3} 4x^2dx$ is the area enclosed by the graph of $y = 4x^2$, the x-axis and $x = 0$, $x = 3$.



$A = \int_{0}^{3} 4x^2dx$

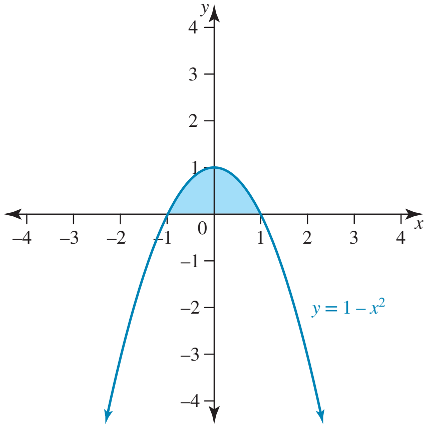
$= \left[\frac{4x^3}{3}\right]_0^3$

$= (36) - (0)$

$= 36$

The area is 36 square units.

b The area represented by $\int_{-1}^{1}\left(1-x^2\right)dx$ is the area bounded by the graph of $y = 1 - x^2$, the x-axis and $x = -1$, $x = 1$.



$A = \int_{-1}^{1}\left(1-x^2\right)dx$

$= \left[x - \frac{x^3}{3}\right]_{-1}^{1}$

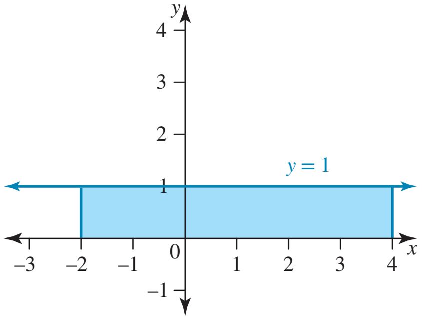
$= \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)$

$= \left(\frac{2}{3}\right) - \left(-\frac{2}{3}\right)$

$= \frac{4}{3}$

The area is $\frac{4}{3}$ square units.

c The area represented by $\int_{-2}^{4} dx$ is the rectangular area enclosed by the line $y = 1$, the x-axis and $x = -2$, $x = 4$.



The area is 6 × 1 = 6 square units.

14 **a** $v = t^3 + 2$

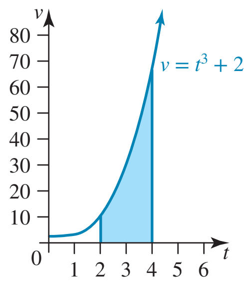
When $t = 1$, $v = 3$, so the velocity is 3 m/s.

$a = \frac{dv}{dt}$

$\therefore a = 3t^2$

When $t = 1$, $a = 3$, so the acceleration is 3 m/s^2.

b i



ii The measure of the distance equals $\int_2^4 (t^3 + 2)\,dt$.

$$\int_2^4 (t^3 + 2)\,dt$$

$$= \left[\frac{t^4}{4} + 2t\right]_2^4$$

$$= \left(\frac{4^4}{4} + 2(4)\right) - \left(\frac{2^4}{4} + 2(2)\right)$$

$$= (64 + 8) - (4 + 4)$$

$$= 64$$

The distance travelled is 64 metres

15 a $v = 3t^2 - 2t + 5$

The velocity is a quadratic function in t.

The discriminant of this quadratic is:

$$\Delta = b^2 - 4ac,\ a = 3, b = -2, c = 5$$

$$= 4 - 60$$

$$= -56$$

Since $\Delta < 0$ and $a > 0$, the velocity is always positive.

b i The distance measure is given by $\int_0^2 (3t^2 - 2t + 5)\,dt$.

$$\int_0^2 (3t^2 - 2t + 5)\,dt$$

$$= \left[t^3 - t^2 + 5t\right]_0^2$$

$$= 8 - 4 + 10 = 14$$

Therefore, the distance travelled is 14 metres.

ii $v = 3t^2 - 2t + 5$

Anti-differentiate with respect to t: $x = t^3 - t^2 + 5t + c$

When $t = 0,\ x = c$ and when

$t = 2,\ x = 8 - 4 + 10 + c = 14 + c$.

The distance travelled is:

$$x(1) - x(0)$$

$$= (14 + c) - (c)$$

$$= 14$$

Therefore, the distance travelled is 14 metres.

16 The area measure is given by $\int_{0.5}^{1.5} (1 + x)(3 - x)\,dx$

$$A = \int_{0.5}^{1.5} (1 + x)(3 - x)\,dx$$

$$= \int_{0.5}^{1.5} \left(3 + 2x - x^2\right) dx$$

$$= \left[3x + x^2 - \frac{x^3}{3}\right]_{0.5}^{1.5}$$

$$= (4.5 + 2.25 - 1.125) - \left(1.5 + 0.25 - \frac{0.125}{3}\right)$$

$$= (5.625) - \left(1.75 - \frac{1}{24}\right)$$

$$= 5.625 - 1.75 + \frac{1}{24}$$

$$= 3.875 + \frac{1}{24}$$

$$= 3 + \frac{7}{8} + \frac{1}{24}$$

$$= 3\frac{11}{12}$$

The area is $3\frac{11}{12}$ square units.

17 a $v = 10t - 5t^2,\ t \geq 0$

Position:

$$x = \int v dt$$

$$x = 5t^2 - \frac{5t^3}{3} + c$$

When $t = 0$, $x = 0$

$\therefore c = 0$

$$\therefore x = 5t^2 - \frac{5t^3}{3}$$

b Let $v = 0$.

$\therefore\ 10t - 5t^2 = 0$

$\therefore\ 5t(2 - t) = 0$

$\therefore\ t = 0$ or $t = 2$

The particle is next at rest after 2 seconds.

When $t = 2$,

$$x = 5 \times 4 - 5 \times \frac{8}{3}$$

$$= 20 - \frac{40}{3}$$

$$= \frac{20}{3} = 6\frac{2}{3}$$

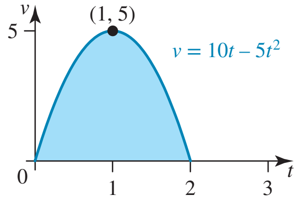
The distance the particle travels by the time it is next at rest is $6\frac{2}{3}$ units.

c $v = 10t - 5t^2$ for $t \in [0, 2]$

The concave down parabolic graph has end points $(0, 0)$ and $(2, 0)$.

Its turning point occurs when $t = 1$ and therefore $v = 5$.

The point $(1, 5)$ is a maximum turning point.



d The area measure is given by $\int_0^2 \left(10t - 5t^2\right)\ dt$.

$$\int_0^2 \left(10t - 5t^2\right)\ dt$$

$$= \left[5t^2 - \frac{5t^3}{3}\right]_0^2$$

$$= \left(20 - \frac{5}{3} \times 8\right) - (0)$$

$$= 20 - \frac{40}{3}$$

$$= \frac{20}{3} = 6\frac{2}{3}$$

The area is $6\frac{2}{3}$ square units.

e The area under the velocity–time graph measures the distance travelled in the first 2 seconds.

18 a Refer to the velocity–time graph given in the question.

The greatest speed occurs at the global maximum point $\left(4, \frac{22}{3}\right)$. Thus, the greatest speed is $7\frac{1}{3}$ km/h.

The least speed occurs at the global minimum point $(0, 2)$.

The least speed is 2 km/h.

b Acceleration is the rate of change of the velocity and is measured by the gradient of the tangent to the velocity graph. If the acceleration is zero, the tangent is horizontal. Therefore, the acceleration is zero at the turning points.

The acceleration is zero after 1 hour, after 2 hours and after 4 hours.

c $v = 2 + 8t - 7t^2 + \frac{7t^3}{3} - \frac{t^4}{4}$.

The distance the athlete runs is the area under the velocity–time graph.

$$x = \int_0^5 v dt$$

$$= \int_0^5 \left(2 + 8t - 7t^2 + \frac{7t}{3} - \frac{t^4}{4}\right) dt$$

$$= \left[2t + 4t^2 - \frac{7t^3}{3} + \frac{7}{3} + \frac{t^4}{4} - \frac{1}{4} \times \frac{t^5}{5}\right]_0^5$$

$$= \left[2t + 4t^2 - \frac{7t^3}{3} + \frac{7t^4}{12} - \frac{t^5}{20}\right]_0^5$$

$$= \left(10 + 100 - \frac{7 \times 125}{3} + \frac{7 \times 625}{12} - \frac{625 \times 5}{20}\right) - (0)$$

$$= 110 + 125\left(-\frac{7}{3} + \frac{7 \times 5}{12} - \frac{5}{4}\right)$$

$$= 110 + 125\left(\frac{-28 + 35 - 15}{12}\right)$$

$$= 110 + 125 \times \frac{-\cancel{8}^{2}}{\cancel{12}_{3}}$$

$$= 110 - \frac{250}{3}$$

$$= \frac{80}{3}$$

The distance the athlete ran during the 5-hour training period is $\frac{80}{3} = 26\frac{2}{3}$ km.

19 a $\int_{-3}^{4} \left(2 - 3x + x^2\right) dx = \frac{203}{6}$

b $\int_{4}^{-3} \left(2 - 3x + x^2\right) dx = -\frac{203}{6}$

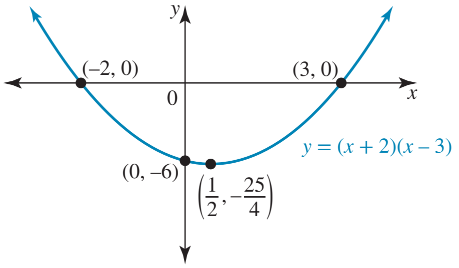
c Interchanging the terminals changes the sign of the integral.

20 a $\int_{-2}^{3} (x + 2)(x - 3) dx = -\frac{125}{6}$

b The graph of $y = (x + 2)(x - 3)$ is a concave up parabola that cuts the x-axis at $x = -2$ and $x = 3$. The y intercept is $(0, -6)$.

The turning point occurs when $x = \frac{-2 + 3}{2} = \frac{1}{2}$ and $y = \frac{5}{2} \times \frac{-5}{2} = -\frac{25}{4}$.

Minimum turning point $\left(\frac{1}{2}, -\frac{25}{4}\right)$



c The area enclosed by the graph and the x-axis lies completely below the x-axis.

$$\text{The area } A = -\int_{-2}^{3} (x + 2)(x - 3) dx$$

$$= -\left(-\frac{125}{6}\right)$$

$$= \frac{125}{6}$$

The area is $\frac{125}{6}$ square units.

d The definite integral measures the signed area. As the area lies under the x-axis, the signed area is negative and therefore the definite integral is negative.

The actual area and the value of the definite integral differ in sign.

A possible integral that would give the actual area is obtained by interchanging the terminals so that

$$A = \int_{3}^{-2} (x + 2)(x - 3) dx.$$

Another possible integral is to write

$$A = -\int_{-2}^{3} (x + 2)(x - 3) dx \text{ as}$$

$$A = \int_{-2}^{3} (x + 2)(3 - x) dx$$

13.5 Exam questions

1 a $v = t^3 - 2$

When $t = 2$

$$v = 2^3 - 2$$

$$= 6$$

The velocity is 6 m/s. [1 mark]

$$a = \frac{dv}{dt}$$

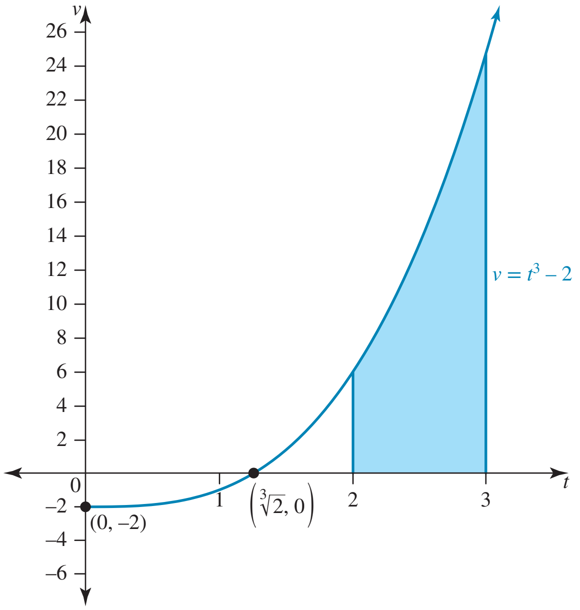
$$= 3t^2$$

$$= 3(2)^2$$

$$= 12$$

The acceleration is 12 m/s^2. [1 mark]

b



[1 mark]

c Area $= \int_2^3 \left(t^3 - 2\right) dt$

$= \left[\frac{t^4}{4} - 2t\right]_2^3$

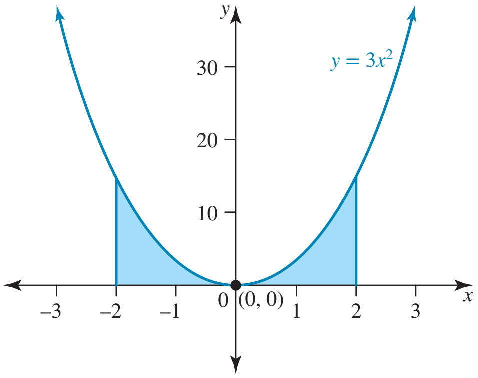
$= \left(\frac{81}{4} - 2 \times 3\right) - \left(\frac{16}{4} - 4\right)$

$= 14.25 - 0$

$= 14.25$

The distance travelled by the particle over the time interval [2, 3] is 14.25 metres. [1 mark]

2



[1 mark]

$\int_{-2}^{2} 3x^2 dx = \left[x^3\right]_{-2}^{2}$

$= (2)^3 - (-2)^3$

$= 16$

∴ the area is 16 square units. [1 mark]

3 $\int_0^3 (x+2)^2\, dx = \int_0^3 \left(x^2 + 4x + 4\right) dx$

$= \left[\frac{x^3}{3} + 2x^2 + 4x\right]_0^3$

$= (9 + 18 + 12) - (0)$

$= 39$

The correct answer is **E**.

13.6 Review

13.6 Exercise

Technology free: short answer

1 **a** $\int 2\left(1 - 5x - 3x^4\right)\, dx$

$= \int \left(2 - 10x - 6x^4\right)\, dx$

$= 2x - 5x^2 - \frac{6x^5}{5} + c$

b $\int 2x(x+3)(x-3)\, dx$

$= \int 2x\left(x^2 - 9\right)\, dx$

$= \int \left(2x^3 - 18x\right) dx$

$= \frac{2x^4}{4} - 9x^2 + c$

$= \frac{x^4}{2} - 9x^2 + c$

c $\int \frac{2x^3 - 5x^4}{8x} dx$

$= \int \left(\frac{2x^3}{8x} - \frac{5x^4}{8x}\right)\, dx$

$= \int \left(\frac{1}{4}x^2 - \frac{5}{8}x^3\right) dx$

$= \frac{1}{4} \times \frac{x^3}{3} - \frac{5}{8} \times \frac{x^4}{4} + c$

$= \frac{x^3}{12} - \frac{5x^4}{32} + c$

d $\int \left(x^2 + 1\right)^3 dx$

$= \int \left(\left(x^2\right)^3 + 3\left(x^2\right)^2 + 3\left(x^2\right) + 1\right) dx$

$= \int \left(x^6 + 3x^4 + 3x^2 + 1\right)\, dx$

$= \frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c$

2 $f'(x) = \frac{3x}{2} - 4$

$\therefore f(x) = \frac{3}{2} \times \frac{x^2}{2} - 4x + c$

$\therefore f(x) = \frac{3x^2}{4} - 4x + c$

Given $f(2) = 10$

$\therefore 10 = 3 - 8 + c$

$\therefore c = 15$

$\therefore f(x) = \frac{3x^2}{4} - 4x + 15$

$f(4) = 12 - 16 + 15$

$\therefore f(4) = 11$

3 **a** Let $\frac{dy}{dx} = 1 + \frac{1}{12}x$.

$\therefore y = x + \frac{1}{12} \times \frac{x^2}{2} + c$

$\therefore y = x + \frac{x^2}{24} + c$

The point (24, 12) lies on the curve.

$\therefore 12 = 24 + \frac{24^2}{24} + c$

$\therefore 12 = 24 + 24 + c$

$\therefore c = -36$

The equation of the curve is $y = \frac{x^2}{24} + x - 36$.

b The y intercept of the curve is the point $(0, -36)$.

At this point, $\frac{dy}{dx} = 1$.

The equation of the tangent with gradient $m = 1$ and passing through the point $(0, \ -36)$ is

$y = mx + c$

$\therefore y = x - 36$

4 **a** $v = 2t(t - 3)$

Let $v = 0$.

$\therefore 2t(t - 3) = 0$

$\therefore t = 0, t = 3$

The particle is next at rest when $t = 3$.

Acceleration: $a = \frac{dv}{dt}$

$v = 2t^2 - 6t$

$\therefore \frac{dv}{dt} = 4t - 6$

$\therefore a = 4t - 6$

At $t = 3$, $a = 6$.

The acceleration when the particle is next at rest is 6 m/s^2.

b Position: $x = \int v dt$

$v = 2t^2 - 6t$

$\therefore x = \frac{2t^3}{3} - 3t^2 + c$

When $t = 0, \ x = 3$.

$\therefore 3 = c$

$\therefore x = \frac{2t^3}{3} - 3t^2 + 3$

c Let $t = 3$.

$\therefore x = 18 - 27 + 3$

$\therefore x = -6$

The particle was initially 3 metres to the right of the origin, When it next comes to rest it is 6 metres to the left of the origin. This position is 9 metres from its starting point.

5 **a** $\int_1^3 (3 - 2x)\, dx$

$= \left[3x - x^2\right]_1^3$

$= (9 - 9) - (3 - 1)$

$= 0 - 2$

$= -2$

b $\int_0^2 (4 + x)(2 - x)\, dx$

$= \int_0^2 \left(8 - 2x - x^2\right) dx$

$= \left[8x - x^2 - \frac{x^3}{3}\right]_0^2$

$= \left(16 - 4 - \frac{8}{3}\right) - (0)$

$= 12 - \frac{8}{3}$

$= \frac{28}{3}$

6 Refer to the graph given in the question.

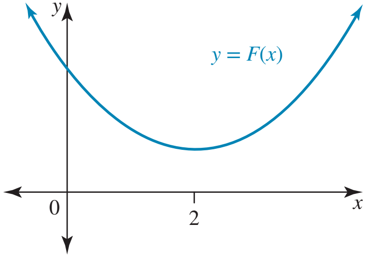
The given graph $y = f(x)$ is the gradient graph of the function $y = F(x)$.

Since $f(x) = 0$ when $x = 2$, there is a stationary point at $x = 2$ on the graph of $y = F(x)$.

For increasing x, the sign of $f(x)$ changes from negative to zero to positive about the point $x = 2$. Therefore, there is a minimum turning point at $x = 2$ on the graph.

The graph of $y = f(x)$ is linear, so the graph of $y = F(x)$ is quadratic.

One possible graph of the anti-derivative function $y = F(x)$ is shown.



Technology active: multiple choice

7 $f'(x) = x(2x + 5)$

$\therefore f'(x) = 2x^2 + 5x$

$\therefore f(x) = \frac{2x^3}{3} + \frac{5x^2}{2} + c$

The correct answer is **C**.

8 $\frac{dy}{dx} = 2x^3$

$y = 2 \times \frac{x^4}{4} + c$

$\therefore y = \frac{1}{2}x^4 + c$

The correct answer is **E**.

9 $\int (3x - 2)^2\, dx$

$= \int \left(9x^2 - 12x + 4\right) dx$

$= \ 9 \times \frac{x^3}{3} - 12 \times \frac{x^2}{2} + 4x + c$

$= 3x^3 - 6x^2 + 4x + c$

The correct answer is **E**.

10 $f(x) = \frac{3x^4 - 5x^3}{2x^2}, \ x \neq 0$

$f(x) = \frac{3}{2}x^2 - \frac{5}{2}x$

$F(x) = \frac{3}{2} \times \frac{x^3}{3} - \frac{5}{2} \times \frac{x^2}{2} + c$

$= \frac{1}{2}x^3 - \frac{5}{4}x^2 + c$

The correct answer is **D**.

11 $\frac{dy}{dx} = 4x^3$

$\therefore y = x^4 + c$

$y = 2$ when $x = 1$

$\therefore 2 = 1 + c$

$\therefore c = 1$

$\therefore y = x^4 + 1$

When $x = 2$,

$y = 16 + 1$

$= 17$

The correct answer is **C**.

12 $v = 3t^2 + 2t - 1,\ t \geq 0$

$x = t^3 + t^2 - t + c$

When $t = 0,\ x = -1$

$\therefore -1 = c$

$\therefore x = t^3 + t^2 - t - 1$

The correct answer is **B**.

13 The graph shown is the gradient graph of $y = F(x)$. The gradient is zero at $x = 0$ and $x = 4$ so there are stationary points on $y = F(x)$ when $x = 0$ and $x = 4$.

As x increases, the gradient changes sign from positive to zero to negative at $x = 0$. This indicates there is a maximum turning point on $y = F(x)$ when $x = 0$.

As x increases, the gradient changes sign from negative to zero to positive at $x = 4$. This indicates there is a minimum turning point on $y = F(x)$ when $x = 4$.

The correct answer is **D**.

14 $\displaystyle\int_0^1 4x^7\,dx$

$$= \left[\frac{4x^8}{8}\right]_0^1$$

$$= \left[\frac{x^8}{2}\right]_0^1$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

The correct answer is **A**.

15 $\displaystyle\int_1^2 \left(4 - 3x^2\right) dx$

$= \left[4x - x^3\right]_1^2$

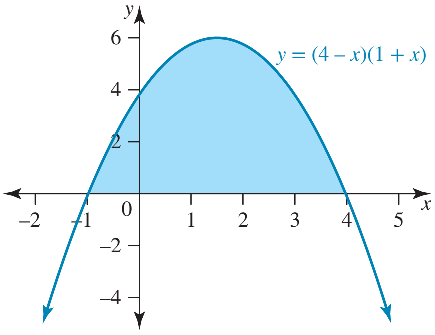
$= (8 - 8) - (4 - 1)$

$= 0 - 3$

$= -3$

The correct answer is **D**.

16 The graph of $y = (4 - x)(1 + x)$ is a concave down parabola that cuts the x-axis at $x = 4$ and $x = -1$.



The area lies above the x-axis.

The area is given by $\displaystyle\int_{-1}^{4} (4 - x)(1 + x)\,dx$.

The correct answer is **C**.

Technology active: extended response

17 **a** Acceleration, $a = 4$

$\therefore v = 4t + c_1$

When $t = 0$, $v = 3$.

$\therefore 3 = c_1$

The velocity is $v = 4t + 3$.

$x = 2t^2 + 3t + c_2$

When $t = 0$, $x = -6$.

$\therefore -6 = c_2$

The position is $x = 2t^2 + 3t - 6$.

b **i** For particle P, $v_P = 4t + 3$ and for particle Q, $v_Q = 7$.

The particles have the same velocities when

$4t + 3 = 7$.

$\therefore t = 1$

When $t = 1$,

$x_P = 2 + 3 - 6$

$= -1$

For Q's position, anti-differentiate its velocity with respect to t.

$v = 7$

$\therefore x = 7t + c$

When $t = 0$, $x_Q = 0$.

$\therefore 0 = c$

$\therefore x = 7t$

When $t = 1$, $x_Q = 7$ and $x_p = -1$. The two particles are 8 metres apart at the time when they are moving with the same velocity.

ii Let $x_P = x_Q$

$\therefore 2t^2 + 3t - 6 = 7t$

$\therefore 2t^2 - 4t - 6 = 0$

$\therefore t^2 - 2t - 3 = 0$

$\therefore (t - 3)(t + 1) = 0$

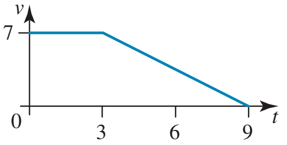
$\therefore t = 3$ or $t = -1$ (reject the negative value)

$\therefore t = 3$

When $t = 3$, $x_Q = 21 = x_P$.

Therefore, P overtakes Q after 3 seconds at the position 21 metres to the right of the origin.

c **i**



ii The distance travelled is the area under the velocity–time graph.

$$A = \frac{1}{2}h(a + b),\ h = 7,\ a = 9,\ b = 3$$

$$= \frac{1}{2} \times 7 \times 12$$

$$= 42$$

The distance Q travels is 42 metres.

d The distance particle R travels is the sum of the two triangular areas shown on the graph in the question. The area of the triangle above the horizontal axis is

$$A_1 = \frac{1}{2}bh,\ b = 6,\ h = 7$$
$$= \frac{1}{2} \times 6 \times 7$$
$$= 21$$

The area of the triangle below the horizontal axis is

$$A_2 = \frac{1}{2}bh,\ b = 6,\ h = 7$$
$$= 21$$

The total distance is 42 metres, the same distance as Q travelled.

18 **a** $f'(x) = ax + b$

At the stationary point $(-2, 6), f'(x) = 0$.

$\therefore -2a + b = 0$ [1]

$$f(x) = \frac{ax^2}{2} + bx + c$$

The point $(0, 4)$ lies on the curve.

$\therefore 4 = c$

$$\therefore f(x) = \frac{ax^2}{2} + bx + 4$$

The point $(-2, 6)$ lies on the curve.

$\therefore 6 = 2a - 2b + 4$

$\therefore 2a - 2b = 2$

$\therefore a - b = 1$ [2]

Add equations [1] and [2]:

$\therefore -a = 1$

$\therefore a = -1$

Substitute $a = -1$ in equation [2]:

$\therefore 2 + b = 0$

$\therefore b = -2$

The values are $a = -1,\ b = -2$.

b With $a = -1,\ b = -2$, the equation of the curve $y = f(x)$ is $f(x) = -\frac{1}{2}x^2 - 2x + 4$.

c $F(x)$ is the anti-derivative of $f(x)$. The curve $y = F(x)$ will have stationary points when its gradient $f(x) = 0$.

$$\therefore\ -\frac{1}{2}x^2 - 2x + 4 = 0$$
$$\therefore x^2 + 4x - 8 = 0$$
$$\therefore \left(x^2 + 4x + 4\right) - 4 - 8 = 0$$
$$\therefore (x + 2)^2 = 12$$
$$\therefore x + 2 = \pm 2\sqrt{3}$$
$$\therefore x = -2 \pm 2\sqrt{3}$$

The x-coordinates of the stationary points of $y = F(x)$ are $x = -2 \pm 2\sqrt{3}$.

d $f(x) = -\frac{1}{2}x^2 - 2x + 4$

$$\therefore F(x) = -\frac{1}{2} \times \frac{x^3}{3} - x^2 + 4x + c_1$$
$$\therefore F(x) = -\frac{x^3}{6} - x^2 + 4x + c_1$$

$F(0) = 4$

$\therefore 4 = c_1$

The equation is $F(x) = -\dfrac{x^3}{6} - x^2 + 4x + 4$.

19 **a** Given $\int f(x)dx = F(x) + c$, then $F(x)$ is an anti-derivative of $f(x)$.

i $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

$$\text{LHS} = \int_a^b f(x)dx$$
$$= [F(x)]_a^b$$
$$= F(b) - F(a)$$

$$\text{RHS} = -\int_b^a f(x)dx$$
$$= -[F(x)]_b^a$$
$$= -(F(a) - F(b))$$
$$= -F(a) + F(b)$$
$$= F(b) - F(a)$$

$\therefore$ LHS = RHS

ii $\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

$$\text{LHS} = \int_a^b f(x)dx + \int_b^c f(x)dx$$
$$= [F(x)]_a^b + [F(x)]_b^c$$
$$= F(b) - F(a) + F(c) - F(b)$$
$$= F(c) - F(a)$$

$$\text{RHS} = \int_a^c f(x)dx$$
$$= [F(x)]_a^c$$
$$= F(c) - F(a)$$

$\therefore$ LHS = RHS

iii $\displaystyle\int_a^a f(x)dx = 0$

$$\text{LHS} = \int_a^a f(x)dx$$
$$= [F(x)]_a^a$$
$$= F(a) - F(a)$$
$$= 0$$
$$= \text{RHS}$$

b $\displaystyle\int_{-1}^2 f(x)dx = 5$

i $\displaystyle\int_2^{-1} f(x)dx$

$$= -\int_{-1}^2 f(x)dx$$
$$= -5$$

ii $\displaystyle = \int_{-1}^2 f(x)dx$

$$= 5$$

$$= \int_{-1}^{2} f(x)dx$$

$$= 5$$

iii $\displaystyle\int_{-1}^{2} 3f(x)dx$

$$= 3\int_{-1}^{2} f(x)dx$$

$$= 3 \times 5$$

$$= 15$$

iv $\displaystyle\int_{-1}^{2} (1 - f(x))\,dx$

$$= \int_{-1}^{2} (1)\,dx - \int_{-1}^{2} (f(x))\,dx$$

$$= [x]_{-1}^{2} - 5$$

$$= (2) - (-1) - 5$$

$$= 2 + 1 - 5$$

$$= -2$$

20 a i The area measure is given by $\displaystyle\int_{-4}^{0} \frac{1}{2}x^2 dx$.

ii $$A = \int_{-4}^{0} \frac{1}{2}x^2 dx$$

$$= \left[\frac{1}{2} \times \frac{x^3}{3}\right]_{-4}^{0}$$

$$= \left[\frac{x^3}{6}\right]_{-4}^{0}$$

$$= (0) - \left(-\frac{64}{6}\right)$$

$$= \frac{32}{3}$$

$$\therefore A = \frac{32}{3}$$

b $y = \dfrac{1}{2}x^2$

Let $x = -4$.

$$y = \frac{1}{2} \times 16$$

$$= 8$$

The slide starts at a height of 8 metres.

c i $\dfrac{dE}{dt} = t$

$$\therefore E = \frac{t^2}{2} + c$$

When $t = 0$, $E = 4$.

$$\therefore 4 = c$$

$$\therefore E = \frac{1}{2}t^2 + 4$$

Let $t = 3$.

$$E = \frac{1}{2} \times 9 + 4$$

$$= 8.5$$

Sam's enjoyment factor as she entered the water is 8.5.

ii Let $E = \dfrac{1}{2}t^2 + c$.

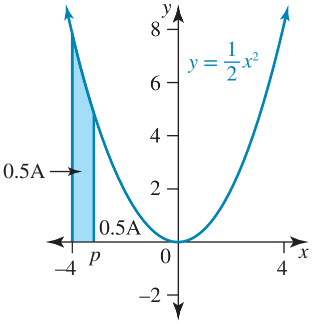
If $E = 10$ when $t = 3$, then

$$10 = \frac{9}{2} + c$$

$$\therefore c = \frac{11}{2}$$

For the enjoyment factor to be 10 at the end of the slide, the initial enjoyment factor would need to be 5.5.

d i



From part **a**, $A = \dfrac{32}{3}$.

$$\therefore \int_{-4}^{p} \frac{1}{2}x^2 dx = \frac{16}{3} = \int_{p}^{0} \frac{1}{2}x^2 dx$$

$$\int_{p}^{0} \frac{1}{2}x^2 dx = \frac{16}{3}$$

$$\therefore \left[\frac{x^3}{6}\right]_{p}^{0} = \frac{16}{3}$$

$$\therefore (0) - \left(\frac{p^3}{6}\right) = \frac{16}{3}$$

$$\therefore -\frac{p^3}{6} = \frac{16}{3}$$

$$\therefore p^3 = -\frac{16}{3} \times 6$$

$$\therefore p^3 = -32$$

$$\therefore p = -\sqrt[3]{32}$$

$$-\sqrt[3]{32} = -\left(2^5\right)^{\frac{1}{3}}$$

$$\therefore -\sqrt[3]{32} = -2^{\frac{5}{3}}$$

$$y = \frac{1}{2}x^2$$

When $x = -2^{\frac{5}{3}}$,

$$y = \frac{1}{2} \times \left(-2^{\frac{5}{3}}\right)^2$$

$$= 2^{-1} \times 2^{\frac{10}{3}}$$

$$= 2^{-1+\frac{10}{3}}$$

$$= 2^{\frac{7}{3}}$$

At the starting point for children, the slide has a height of $2^{\frac{7}{3}}$ metres, which is approximately 5 metres.

13.6 Exam questions

1 Let $f'(x) = (x-2)(2x+5)(x+2)$.

$$= \left(x^2-4\right)(2x+5)$$
$$= 2x^3+5x^2-8x-20 \qquad \text{[1 mark]}$$
$$f(x) = \int f'(x)\,dx$$
$$= \frac{1}{3+1}\times 2x^{3+1} + \frac{1}{2+1}\times 5x^{2+1} - \frac{1}{1+1}\times 8x^{1+1} - 20x + c$$
$$= \frac{1}{2}x^4 + \frac{5}{3}x^3 - 4x^2 - 20x + c \qquad \text{[1 mark]}$$

2 $\frac{dy}{dx} \propto x^2$

$\frac{dy}{dx} = kx^2$

The gradient is -18.

When $x = 3$,

$-18 = k(3)^2$

$9k = -18$

$k = -2$

The correct answer is **B**.

3 Distance is not concerned with the direction of motion. It should read: ‘Velocity is the rate of change of position with respect to time’.

The correct answer is **C**.

4 $v = 2$

$x = \int v\,dt$

$x = 2t + c$

$t = 0,\ x = -6$ (6 m left of the origin)

$x = 2t - 6$

origin, $x = 0$

$0 = 2t - 6$

$2t = 6$

$\therefore t = 3$

The correct answer is **D**.

5 $$\int_{-2}^{2} 3x^2 dx = \left[x^3\right]_{-2}^{2}$$
$$= (2)^3 - (-2)^3$$
$$= 16$$

The correct answer is **D**.