1 Functions and graphs

Topic	1	Functions and graphs
Subtopic	1.2	Linear functions

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***Source:** VCE 2013, Mathematical Methods (CAS) Exam 2, Section 1, Q2; © VCAA*

Question 1 (1 mark)

The midpoint of the line segment that joins $(1, -5)$ to $(d, 2)$ is

A. $\left(\frac{d+1}{2}, -\frac{3}{2}\right)$

B. $\left(\frac{1-d}{2}, -\frac{7}{2}\right)$

C. $\left(\frac{d-4}{2}, 0\right)$

D. $\left(0, \frac{1-d}{3}\right)$

E. $\left(\frac{5+d}{2}, 2\right)$

***Source:** VCE 2014, Mathematical Methods (CAS) Exam 2, Section 1, Q2; © VCAA*

Question 2 (1 mark)

The linear function $f: D \to R, f(x) = 4 - x$ has range $[-2, 6)$.

The domain D of the function is

A. $[-2, 6)$

B. $[-2, 2)$

C. R

D. $(-2, 6]$

E. $[-6, 2)$

Question 3 (1 mark)

The gradient of a line **perpendicular** to the line that passes through $(3, 1)$ and $(0, -5)$ is

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -6

E. -2

Source: *VCE 2016, Mathematical Methods 2, Section 1, Q1; © VCAA*

Question 4 (1 mark)

The linear function $f: D \to R,\ f(x) = 5 - x$ has range $[-4, 5)$.

The domain D is

A. $(0, 9]$

B. $(0, 1]$

C. $[5, -4)$

D. $[-9, 0)$

E. $[1, 9)$

Topic	1	Functions and graphs
Subtopic	1.3	Solving systems of equations

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Source: *VCE 2014, Mathematical Methods (CAS) Exam 2, Section 1, Q17; © VCAA*

Question 1 (1 mark)

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have **no solution** for

A. $a = 3$
B. $a = -3$
C. both $a = 3$ and $a = -3$
D. $a \in R \setminus \{3\}$
E. $a \in R \setminus [-3, 3]$

Question 2 (1 mark)

The simultaneous linear equations

$$-2x - my = -4 \text{ and}$$
$$(m - 1)x + 6y = 2(m - 1),$$

where m is a real constant, have a unique solution for

A. $m = 4$ or $m = -3$
B. $m = 4$ only
C. $m \in R \setminus (4, -3)$
D. $m \neq 4$
E. $m = -3$ only

Question 3 (1 mark)

A unique solution for solving a system of three simultaneous equations in three variables represents

A. three planes intersecting along a line.
B. three planes intersecting along a plane.
C. three planes intersecting at a point.
D. three planes having no common intersection.
E. three planes intersecting at a point, a line or a plane.

Question 4 (1 mark)

If $ax + by = (b + 1)x + (2 - a)y$, the values of a and b are

A. $a = \frac{1}{2}$ and $b = \frac{3}{2}$
B. $a = \frac{3}{2}$ and $b = \frac{1}{2}$
C. $a = \frac{2}{3}$ and $b = \frac{5}{3}$
D. $a = b + 1$ and $b = 2 - a$
E. $a = \frac{3}{2}$ and $b = \frac{5}{2}$

Question 5 (1 mark)

If $ax - 3b + 2bx + 6a - 5x = 0$, the values of a and b are

A. $a = 1$ and $b = 2$
B. $a = 1$ and $b = 1$
C. $a = 2$ and $b = 2$
D. $a = 2$ and $b = 1$
E. $a = 2$ and $b = 0$

Topic	1	Functions and graphs
Subtopic	1.4	Quadratic functions

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Source: *VCE 2019, Mathematical Methods Exam 2, Section A, Q2; © VCAA*

Question 1 (1 mark)

The set of values of k for which $x^2 + 2x - k = 0$ has two real solutions is

A. $\{-1, 1\}$
B. $(-1, \infty)$
C. $(-\infty, -1)$
D. $\{-1\}$
E. $[-1, \infty)$

Source: *VCE 2018, Mathematical Methods Exam 2, Section A, Q17; © VCAA*

Question 2 (1 mark)

The turning point of the parabola $y = x^2 - 2bx + 1$ is closest to the origin when

A. $b = 0$
B. $b = -1$ or $b = 1$
C. $b = -\frac{1}{\sqrt{2}}$ or $b = \frac{1}{\sqrt{2}}$
D. $b = \frac{1}{2}$ or $b = -\frac{1}{2}$
E. $b = \frac{1}{4}$ or $b = -\frac{1}{4}$

***Source:** VCE 2015, Mathematical Methods (CAS) Exam 2, Section 1, Q21; © VCAA*

Question 3 (1 mark)
The graphs of $y = mx + c$ and $y = ax^2$ will have no points of intersection for all values of, m, c and a such that

A. $a > 0$ and $c > 0$

B. $a > 0$ and $c < 0$

C. $a > 0$ and $c > -\frac{m^2}{4a}$

D. $a < 0$ and $c > -\frac{m^2}{4a}$

E. $m > 0$ and $c > 0$

***Source:** VCE 2017, Mathematical Methods 2, Section 1, Q7; © VCAA*

Question 4 (1 mark)
The equation $(p - 1)x^2 + 4x = 5 - p$ has no real roots when

A. $p^2 - 6p + 6 < 0$
B. $p^2 - 6p + 1 > 0$
C. $p^2 - 6p - 6 < 0$
D. $p^2 - 6p + 1 < 0$
E. $p^2 - 6p + 6 > 0$

***Source:** VCE 2015, Mathematical Methods (CAS) 2, Section 1, Q7; © VCAA*

Question 5 (1 mark)
The range of the function $f: (-1, 2] \to R, f(x) = -x^2 + 2x - 3$ is

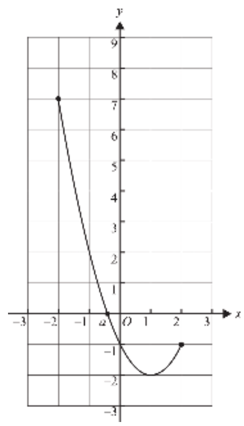
A. R
B. $(-6, -3]$
C. $(-6, -2]$
D. $[-6, -3]$
E. $[-6, -2]$

Topic 1 Subtopic 1.4 Quadratic functions

***Source:** VCE 2013, Mathematical Methods (CAS) 1, Q9.a; © VCAA*

Question 6 (1 mark)

The graph of $f(x) = (x-1)^2 - 2, x \in [-2, 2]$, is shown below. The graph intersects the x-axis where $x = a$.



Find the value of a.

Question 7 (1 mark)

The graph of $y = 3x^2 - 6x + 7$

A. crosses the x-axis at two distinct points and has an axis of symmetry at $x = 1$.

B. touches the x-axis at one point and has an axis of symmetry at $x = 1$.

C. does not cross the x-axis and has an axis of symmetry at $x = 1$.

D. crosses the x-axis at two distinct points and has an axis of symmetry at $x = -1$.

E. does not cross the x-axis and has an axis of symmetry at $x = -1$.

Question 8 (1 mark)

Consider the graph of $y = x^2 - mx + 9$. Which of the following statements is **false**?

A. If $m = \pm 6$ the graph touches the x-axis.

B. If $m > 6$ the graph crosses the x-axis twice.

C. If $m < -6$ the graph crosses the x-axis twice.

D. The graph has an axis of symmetry at $x = \frac{m}{2}$.

E. The graph has a turning point at $\left(-\frac{m}{2}, 9 - \frac{m^2}{4}\right)$.

Question 9 (1 mark)
Find the range of the function $f: [1, 3] \rightarrow R,\ f(x) = x^2 - 3x + 3$.

Question 10 (1 mark)
A quadratic graph crosses the x-axis at $x = -1$ and $x = 5$, and crosses the y-axis at $y = -5$.
The equation of the graph is
A. $y = x^2 - 4x - 5$
B. $y = x^2 + 4x - 5$
C. $y = -x^2 - 4x + 5$
D. $y = -x^2 + 4x - 5$
E. $y = x^2 - 25$

Question 11 (1 mark)
A quadratic graph crosses the x-axis at $x = -1$, and $x = 3$, and crosses the y-axis at $y = 6$.
The equation of the graph is
A. $y = -2x^2 + 4x + 6$
B. $y = -2x^2 - 4x + 6$
C. $y = 2x^2 + 4x + 6$
D. $y = 2x^2 - 4x + 6$
E. $y = -x^2 + 4x + 6$

Question 12 (1 mark)

Consider the function $f: (-2, \infty) \rightarrow R$ where $f(x) = 8 - 2(x - 2)^2$. Which of the following statements is **true**?

A. The graph crosses the y-axis at $y = 8$.
B. The domain is R.
C. The range is $(-\infty, 8]$.
D. The graph has a minimum turning point at $(2, 8)$.
E. The graph crosses the x-axis at $x = 2$.

Question 13 (1 mark)

A quadratic graph has a turning point at $(-2, 4)$ and passes through the origin.
Which of the following statements is **false**?

A. The graph crosses the x-axis at $x = -4$.
B. The axis of symmetry is about the line $x = -2$.
C. The range is $(-\infty, 4]$.
D. The equation of the curve is $y = 4 - (x + 2)^2$.
E. The vertex is a minimum turning point.

Question 14 (1 mark)

Consider the function $f: R \rightarrow R$ where $f(x) = -\left[8 - 2(x - 3)^2\right]$.
Which of the following statements is **false**?

A. The graph crosses the y-axis at $y = 10$.
B. The domain is R.
C. The range is $(-\infty, -8]$.
D. The graph has a minimum turning point at $(3, -8)$.
E. The graph crosses the x-axis at $x = 1$ and $x = 5$.

Question 15 (2 marks)

Express $y = 2x^2 + 4x + 5$ in the form $y = a(x + h)^2 + k$.

Question 16 (2 marks)

Calculate the turning point and state the nature of the turning point for the parabola with the equation $y = 2\left[(x - 3)^2 - 3\right]$.

Question 17 (2 marks)

Find the values of k for when the graph of $y = kx - 2$ intersects the graph of $y = x^2 + 4x$ at two distinct points.

Topic	1	Functions and graphs
Subtopic	1.5	Cubic functions

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Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q2; © VCAA*

Question 1 (1 mark)

Let $p(x) = x^3 - 2ax^2 + x - 1$ where $a \in R$. When p is divided by $x + 2$, the remainder is 5.

The value of a is

A. 2

B. $-\frac{7}{4}$

C. $\frac{1}{2}$

D. $-\frac{3}{2}$

E. -2

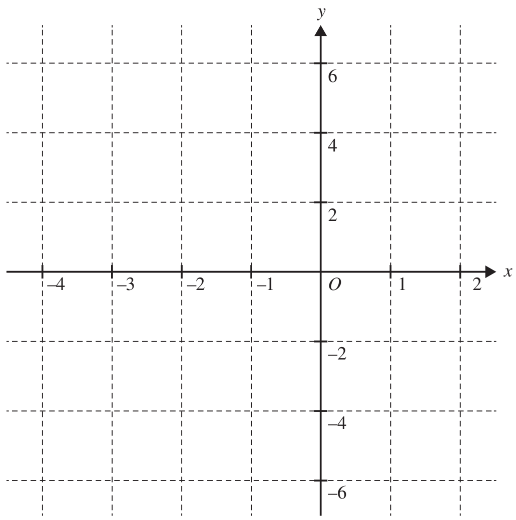
Source: *VCE 2017, Mathematical Methods Exam 1, Q3; © VCAA*

Question 2 (4 marks)

Let $f: [-3, 0] \to R, f(x) = (x+2)^2(x-1)$.

a. Show that $(x+2)^2(x-1) = x^3 + 3x^2 - 4$. **(1 mark)**

b. Sketch the graph of f on the axes below. Label the axis intercepts and any stationary points with their coordinates. **(3 marks)**



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Question 3 (1 mark)

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts.

The set of all possible values of c is

A. R

B. R^+

C. $\{0, 4\}$

D. $(0, 4)$

E. $(-\infty, 4)$

Topic 1 | Subtopic 1.5 Cubic functions

Source: *VCE 2015, Mathematical Methods (CAS) 2, Section 1, Q6; © VCAA*

Question 4 (1 mark)

For the polynomial $P(x) = x^3 - ax^2 - 4x + 4,\ P(3) = 10$, he value of a is

A. -3
B. -1
C. 1
D. 3
E. 10

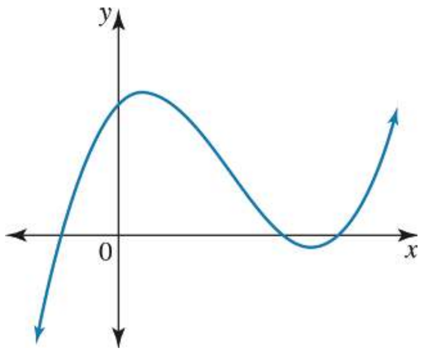
Source: *VCE 2013, Mathematical Methods (CAS) 2, Section 1, Q3; © VCAA*

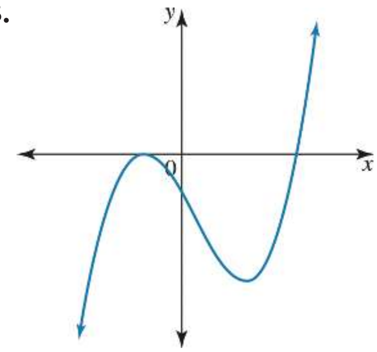
Question 5 (1 mark)

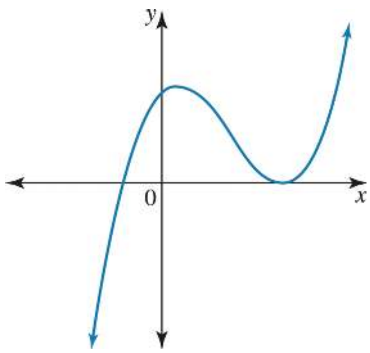
If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in R\backslash\{0\}$, then the value of a is

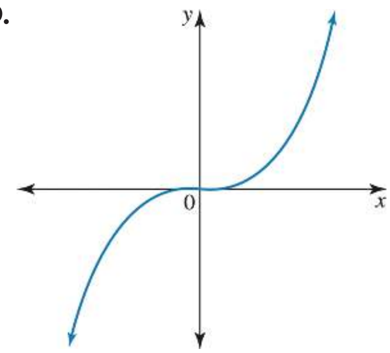
A. -4
B. -2
C. -1
D. 1
E. 2

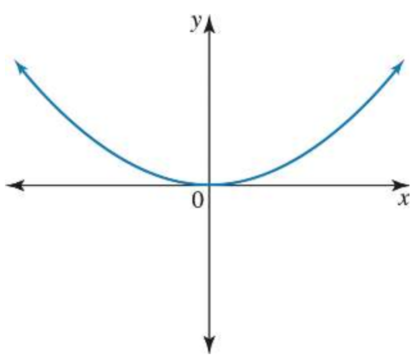
Question 6 (1 mark)
Which of the following graphs could **not** be the graph of $y = ax^3 + bx^2 + cx + d$ with $a > 0$

A.


B.


C.


D.


E.


Question 7 (1 mark)

If a and b are non-zero real numbers, and the following graph is defined on its maximal domain, then the graph of $y = ax^3 + b$ is

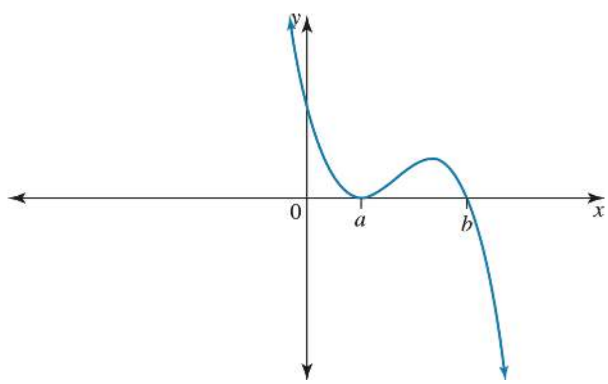
A. a one-to-one function.
B. a one-to-many relation.
C. a many-to-one function.
D. a many-to-many relation.
E. not a relation.

Question 8 (3 marks)

Sketch the cubic curve with the equation $y = x^3 - 12x$. Clearly label the coordinates of all intercepts and turning points.

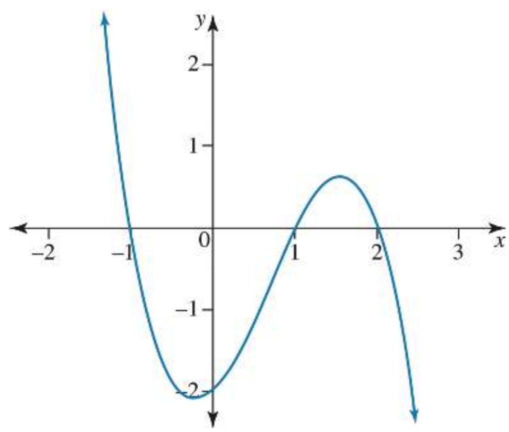
Question 9 (1 mark)

The graph shown could have the equation



A. $y = -(x-a)^2(x-b)$
B. $y = (x-a)^2(x-b)$
C. $y = (x+a)^2(x-b)$
D. $y = -(x-a)^2(x+b)$
E. $y = -x(x-a)(x-b)$

Question 10 (1 mark)
The equation for the cubic graph shown below is



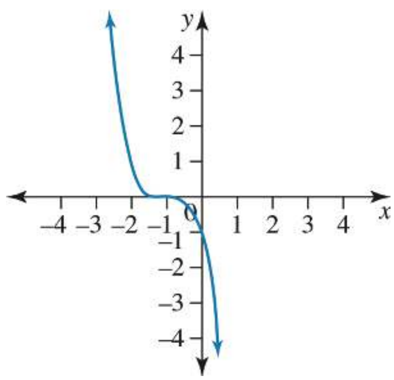
A $y = (x - 1)(x + 1)(x + 2)$
B $y = -(x - 1)(x + 1)(x + 2)$
C $y = (x - 1)(x + 1)(x - 2)$
D $y = -(x - 1)(x + 1)(x - 2)$
E $y + 2 = (x - 1)(x + 1)(x - 2)$

Question 11 (1 mark)
A cubic graph has a minimum turning point at $(-2, 0)$ and crosses the x-axis at $x = 2$.
The equation of the graph is
A. $y = -(x + 2)^2(x - 2)$
B. $y = (x + 2)^2(x - 2)$
C. $y = -(x - 2)^2(x + 2)$
D. $y = (x - 2)^2(x + 2)$
E. $y = -(x + 2)^3$

Question 12 (1 mark)
Find the number of x-intercepts for the equation $y = x^3 - x^2 - 2x - 12$

Question 13 (2 marks)
Solve $2x^3 - 8x^2 + 8x > 0$.

Question 14 (1 mark)
The equation of the graph given below is

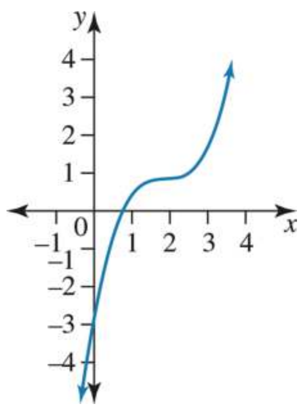


A. $y = -(x + 1)^3$
B. $y = -(x - 1)^3$
C. $y = (x - 1)^3$
D. $y = -1 - (x + 1)^3$
E. $y = (x + 1)^3 - 1$

Topic 1 Subtopic 1.5 Cubic functions

Question 15 (1 mark)

The graph below is $y = a(x + b)^3 + c$.



Then

A. $a = \frac{1}{2}, b = -2$ and $c = 1$

B. $a = \frac{1}{2}, b = -2$ and $c = -3$

C. $a = 1, b = -2$ and $c = 1$

D. $a = -1, b = 2$ and $c = 1$

E. $a = -1, b = 2$ and $c = -1$

Question 16 (1 mark)

For the graph of $y = 27 + (x + 2)^3$, which of the following statements is most **correct**?

A. The graph crosses the x-axis at $x = -2$ and crosses the y-axis at $y = 0$.

B. The graph crosses the x-axis at $x = -2$ and crosses the y-axis at $y = 27$.

C. The graph crosses the x-axis at $x = -5$ and crosses the y-axis at $y = 27$.

D. The graph crosses the x-axis at $x = -5$ and crosses the y-axis at $y = 35$.

E. The graph passes through the point $(-2, 27)$ and does not cross the y-axis.

Question 17 (3 marks)

Answer the following.

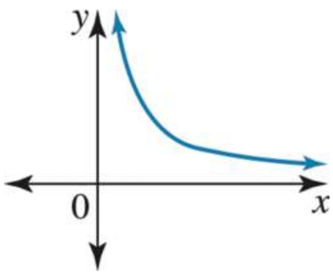
A. Sketch the graph of $y = (x + 1)^3 + 2$. **(2 marks)**

B. Find the values of x for when the function is always increasing. **(1 mark)**

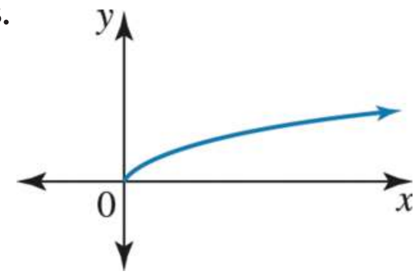
Question 18 (1 mark)

For the graph of $y = x^3$, the shape of the graph in the first quadrant will look like:

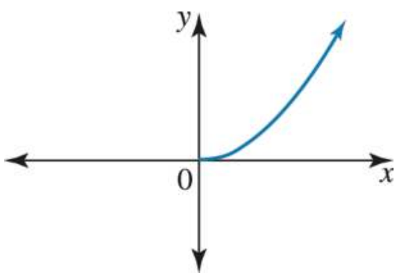
A.



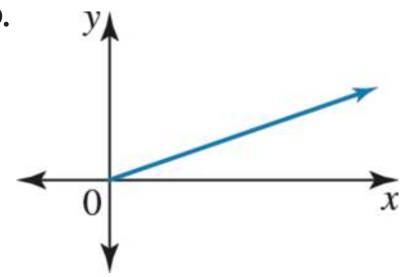
B.



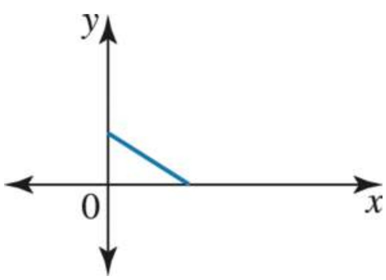
C.



D.



E.



Question 19 (2 marks)

If $x + a$ is a factor of $6x^3 + 22x^2 - 2ax$, where $a \in R \setminus \{0\}$, find the value of a .

Question 20 (1 mark)

When the polynomial $f(x) = x^3 - x^2 - kx + 2$ is divided by $(x + 1)$, the remainder is 3.
The value of k is then equal to

A. 3
B. 2
C. 1
D. 0
E. -1

Question 21 (2 marks)

If $a(x-3)x^2 + b(x-3)x + c(x-3) = 2x^3 - 3x^2 - 14x + 15$, find a, b and c.

Question 22 (2 marks)

If $(4x-1)\left(ax^2 + bx + c\right) = 12x^3 - 7x^2 - 23x + 6$, find a, b and c.

Topic	1	Functions and graphs
Subtopic	1.6	Higher degree polynomials

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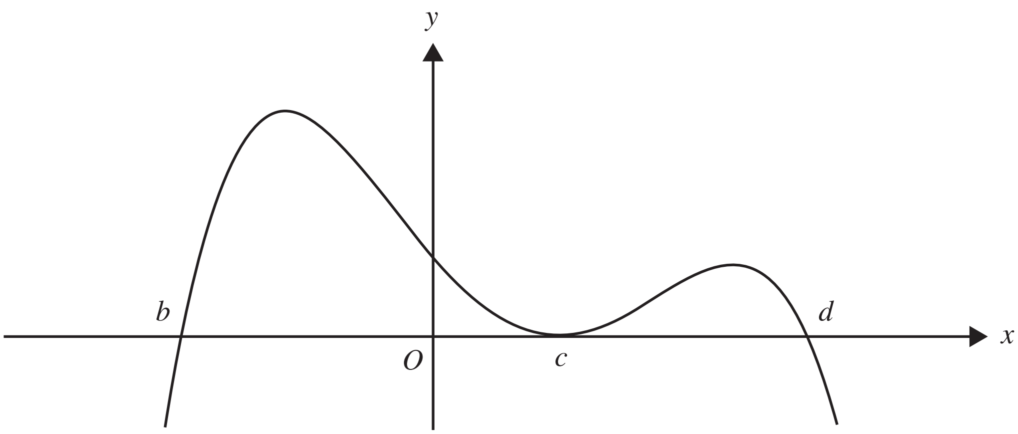
Question 1 (1 mark)



The rule for a function with the graph above could be

A. $y = -2(x + b)(x - c)^2(x - d)$

B. $y = 2(x + b)(x - c)^2(x - d)$

C. $y = -2(x - b)(x - c)^2(x - d)$

D. $y = 2(x - b)(x - c)(x - d)$

E. $y = -2(x - b)(x + c)^2(x + d)$

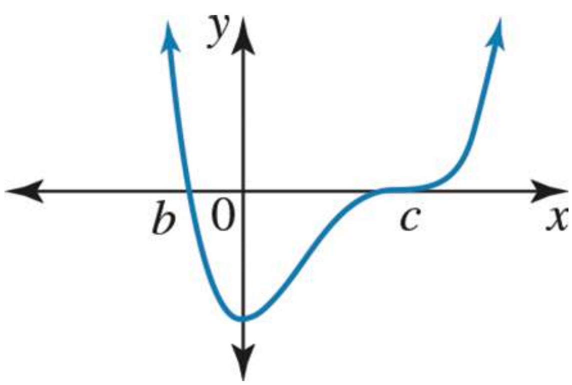
Question 2 (2 marks)

Solve $-x^4 + 7x^3 - 12x^2 \geq 0$.

Question 3 (1 mark)
A quartic graph crosses the x-axis at $x = -3, x = -1, x = 2$ and $x = 4$, and crosses the y-axis at $y = -24$. The equation of the graph is

A. $y = -(x-3)(x-1)(x+2)(x+4)$
B. $y = (x-3)(x-1)(x+2)(x+4)$
C. $y = -(x+3)(x+1)(x-2)(x-4)$
D. $y = -(x-3)(x+1)(x-2)(x-4)$
E. $y = (x+3)(x+1)(x-2)(x-4)$

Question 4 (1 mark)
The equation best represented by the graph is:

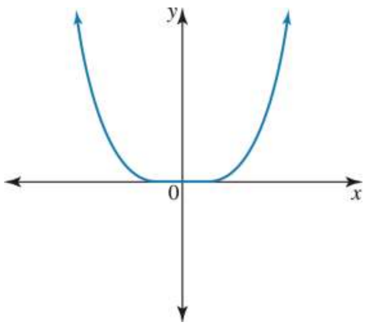


A. $y = a(x-b)^2(x-c)^2$
B. $y = a(x-b)(x-c)^3$
C. $y = ax^2(x-b)(x-c)$
D. $y = a(x-b)^3(x-c)$
E. $y = ax(x-b)(x-c)$

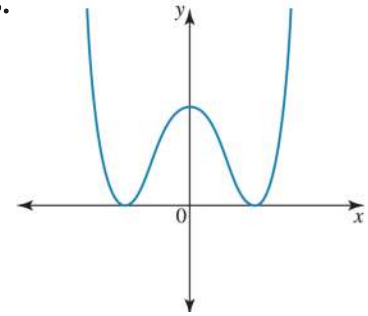
Question 5 (1 mark)

Which of the following options could **not** be the graph of $y = ax^4 + bx^3 + cx^2 + dx + e$ with $a > 0$?

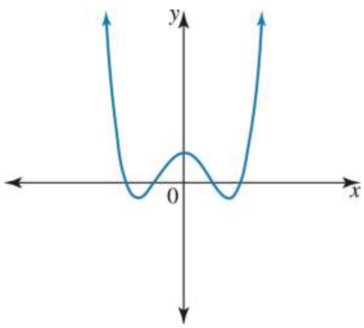
A.



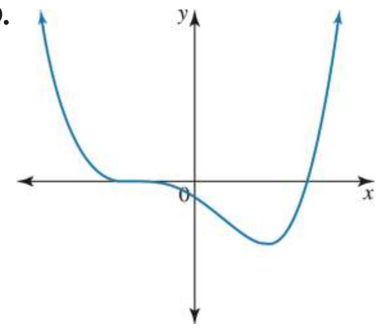
B.



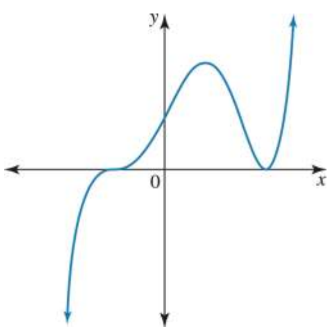
C.



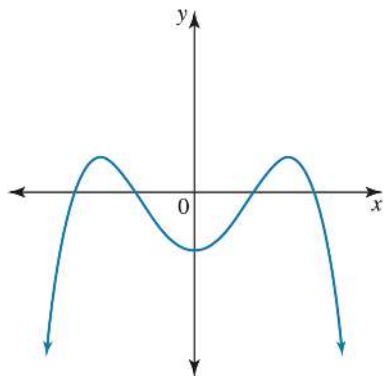
D.



E.



Question 6 (1 mark)
The graph below is of the function $y = ax^4 + bx^3 + cx^2 + dx + e$.



Which of the following options is most **correct**?
A. $a > 0, c > 0$ and $e < 0$
B. $a > 0, c < 0$ and $e > 0$
C. $a < 0, b = 0, c < 0, d = 0$ and $e > 0$
D. $a < 0, b = 0, c > 0, d = 0$ and $e < 0$
E. $a < 0, b = 0, c < 0, d = 0$ and $e < 0$

Question 7 (2 marks)
Find the number of real solutions for $\left(x^2 + a\right)(x - b)(x + c)$, where a, b, and c are three positive real numbers.

Question 8 (1 mark)
Which of the following expressions is a factor of $f(x) = x^4 - 4x^3 - x^2 + 16x - 12$?
A. $x + 1$
B. x
C. $x + 2$
D. $x + 3$
E. $x - 4$

Question 9 (1 mark)

If $f(x) = x(x+1)(x-2)^2$, then $f(x+1)$ is equal to

A. $x(x-1)(x-3)^2$

B. $x(x+1)(x-1)^2$

C. $(x+1)(x+2)(x-3)^2$

D. $(x+1)(x+2)(x-1)^2$

E. $x(x+1)(x-2)^2 + 1$

Question 10 (1 mark)

The graph of a quartic function has x-intercepts at $(a, 0)$ and $(c, 0)$ as well as a turning point and x-intercept at $(b, 0)$ where a, b and c are positive integers. If $a > b > c$, the rule for f could be

A. $f(x) = x(x-a)(x-b)(x+c)$

B. $f(x) = (x-a)(x-c)(x-b)^2$

C. $f(x) = x(x-a)(x+b)(x-c)$

D. $f(x) = (x+a)(x-c)(x+b)^2$

E. $f(x) = (x-a)(x-c)(x+b)^2$

Question 11 (1 mark)

Consider the function $f: R \to R$ where $f(x) = 2 - (x+3)^4$.

Which of the following statements is true?

A. The graph has a maximum at $(3, 2)$.

B. The graph has a minimum at $(3, 2)$.

C. The graph has a maximum at $(-3, 2)$.

D. The graph has a minimum at $(-3, 2)$.

E. The graph has a minimum at $(2, 3)$.

Question 12 (1 mark)

Consider the function $f: R \to R$ where $f(x) = 5 + 2(x - 6)^4$.

The range is

A. $[5, \infty)$
B. $(-\infty, 5]$
C. $(-\infty, 6]$
D. $[6, \infty)$
E. R

Question 13 (1 mark)

Which of the following statements is false for the graph of $y = 1 - (x + 2)^4$?

A. The domain is R and the range is $(-\infty, 1]$.
B. The graph crosses the x-axis at $(-3, 0)$ and $(-1, 0)$.
C. The graph crosses the y-axis at $(0, -15)$.
D. The line $x = -2$ is an axis of symmetry.
E. The point $(-2, 1)$ is a minimum turning point.

Question 14 (3 marks)

Answer the following.

a. Determine the turning point for $y = -\frac{1}{2}(x - 1)^4 + 1$. **(1 mark)**

b. Sketch the graph of $y = -\frac{1}{2}(x - 1)^4 + 1$ **(2 marks)**

Question 15 (1 mark)

Consider the function $f: R \rightarrow R$ where $f(x) = x^5$. Which of the following statements is false?

A. The graph passes through the point $(-2, -32)$.
B. The point (0, 0) is a stationary point of inflexion.
C. The domain and range of the graph are both R.
D. The graph crosses the x-axis at $(-1, 0)$ and $(1, 0)$.
E. The graph is not symmetrical about the x-axis.

Topic	1	Functions and graphs
Subtopic	1.7	Other algebraic functions

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Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q5; © VCAA*

Question 1 (1 mark)

The graph of function $f: D \to R, f(x) = \frac{3x+2}{5-x}$, where D is the maximal domain, has asymptotes

A. $x = -5,\ y = \frac{3}{2}$

B. $x = -3,\ y = 5$

C. $x = \frac{2}{3},\ y = -3$

D. $x = 5,\ y = 3$

E. $x = 5,\ y = -3$

Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q18; © VCAA*

Question 2 (1 mark)

Let $a \in (0, \infty)$ and $b \in R$

Consider the function $h: [-a, 0] \cup (0, a) \to R, h(x) = \frac{a}{x} + b$.

The range of h is

A. $[b-1, b+1]$

B. $(b-1, b+1)$

C. $(-\infty, b-1) \cup (b+1, \infty)$

D. $(-\infty, b-1] \cup [b+1, \infty)$

E. $[b-1, \infty)$

Source: *VCE 2018, Mathematical Methods Exam 2, Section A, Q2; © VCAA*

Question 3 (1 mark)

The maximal domain of the function f is $R\backslash\{1\}$.

A possible rule for f is

A. $f(x) = \dfrac{x^2 - 5}{x - 1}$

B. $f(x) = \dfrac{x + 4}{x - 5}$

C. $f(x) = \dfrac{x^2 + x + 4}{x^2 + 1}$

D. $f(x) = \dfrac{5 - x^2}{1 + x}$

E. $f(x) = \sqrt{x - 1}$

Source: *VCE 2021, Mathematical Methods 1, Q4; © VCAA*

Question 4 (4 marks)

Answer the following.

a. Sketch the graph of $y = 1 - \dfrac{2}{x - 2}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates. **(3 marks)**

b. Find the values of x for which $1 - \dfrac{2}{x - 2} \geq 3$. **(1 mark)**

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q2; © VCAA*

Question 5 (1 mark)

The linear function $f: D \rightarrow R, f(x) = 4 - x$ has range $[-2, 6)$.

The domain D of the function is

A. $[-2, 6)$

B. $(-2, 2]$

C. R

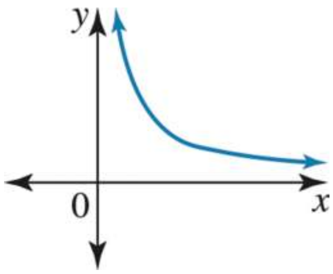
D. $(-2, 6]$

E. $[-6, 2]$

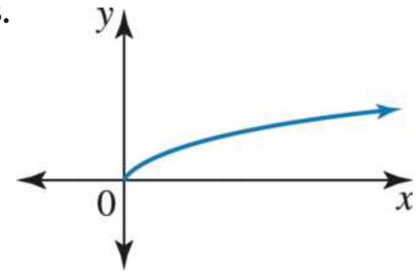
Question 6 (1 mark)

For the graph of $y = x^{-5}$, the shape of the graph in the first quadrant will look like

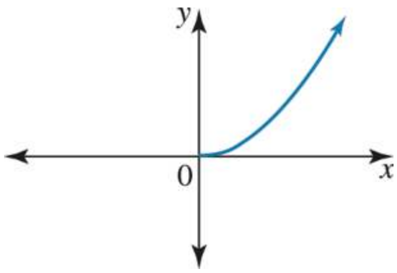
A.



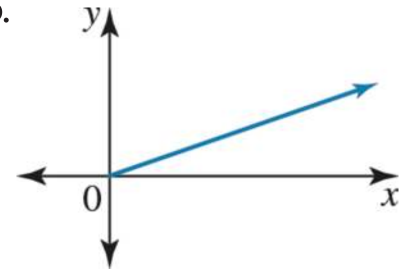
B.



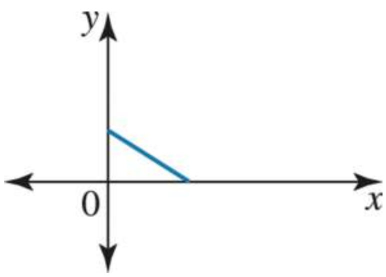
C.



D.



E.



Question 7 (3 marks)

The graph of a truncus has asymptotes at $x = -1$ and $y = 1$. Its y-intercept is -2. Find the equation of the truncus.

Question 8 (1 mark)

The graph of $y = x^{\frac{1}{4}}$ has domain

A. $x \in R$

B. $x \in R^{+} \cup \{0\}$

C. $x \in R^{-}$

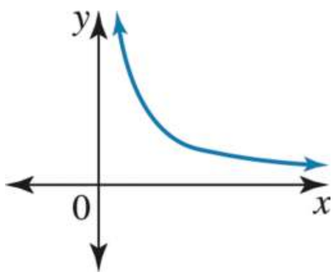
D. $x \in R|\{0\}$

E. $x \in (-1, 4)$

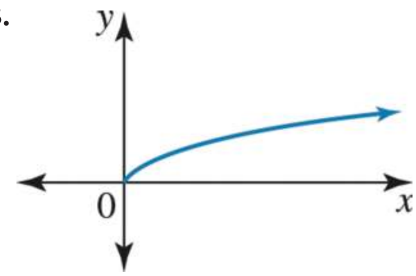
Question 9 (1 mark)

The basic shape of the graph of $y = x^{\frac{2}{3}}$ in the first quadrant will look similar to

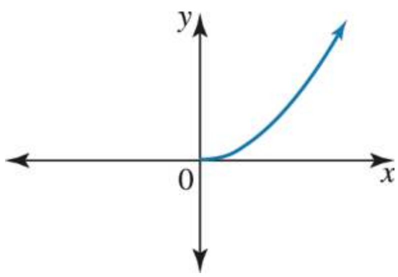
A.



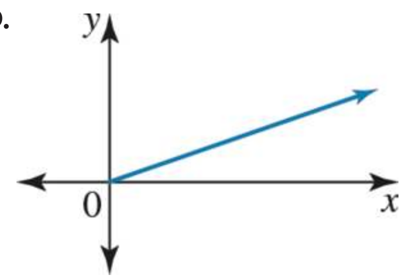
B.



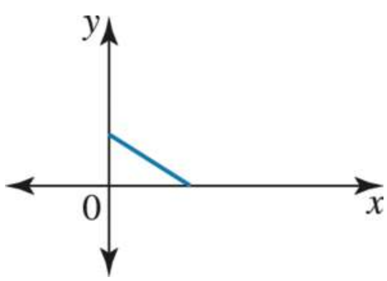
C.



D.



E.



Question 10 (1 mark)

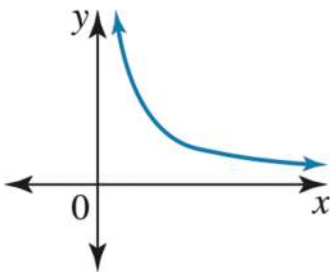
If a and b are non-zero real numbers, and the following graph is defined on its maximal domain, then the graph of $y^2 = ax + b$ is

A. a one-to-one function.
B. a one-to-many relation.
C. a many-to-one function.
D. a many-to-many relation.
E. not a relation.

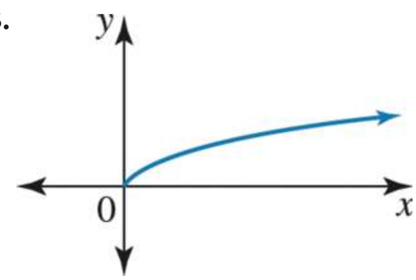
Question 11 (1 mark)

The basic shape of the graph in the first quadrant of the function $f: R^+ \rightarrow R,\ f(x) = x^{\frac{4}{7}}$. will be similar to

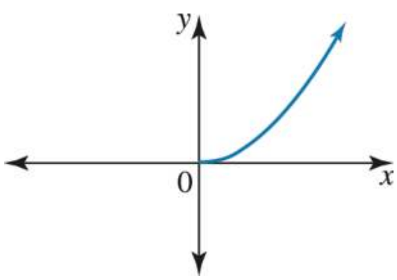
A.



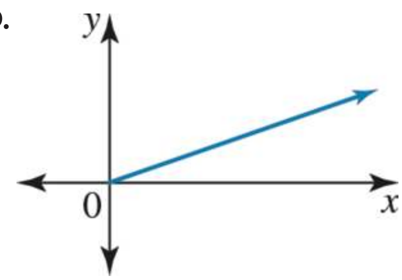
B.



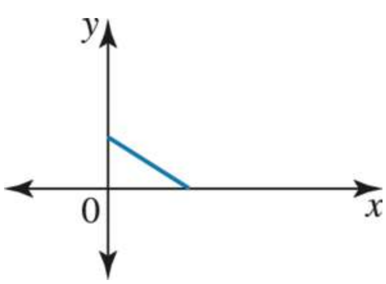
C.



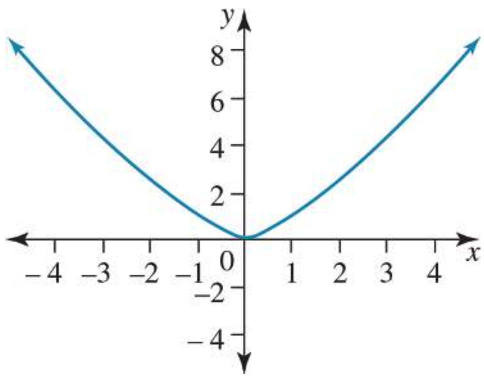
D.



E.



Question 12 (1 mark)
Consider the graph below.



The equation of the graph could be
A. $y = \sqrt[5]{x^3}$
B. $y = \sqrt[3]{x^5}$
C. $y = \sqrt{x^3}$
D. $y = \sqrt[4]{x^3}$
E. $y = \sqrt[3]{x^4}$

Question 13 (1 mark)
The graph of a one-to-one function $y = f(x)$ has a maximal domain of $[-a, \infty)$ and a range of $[0, \infty)$, where a is a positive, real constant. Which of the following options could be the equation of the graph?
A. $y = \sqrt{(x+a)^3}$
B. $y = \sqrt{(x-a)^3}$
C. $y = \sqrt[3]{(x+a)}$
D. $y = \sqrt[3]{(x-a)}$
E. $y = \sqrt[3]{(x+a)^2}$

Topic	1	Functions and graphs
Subtopic	1.8	Combinations of functions

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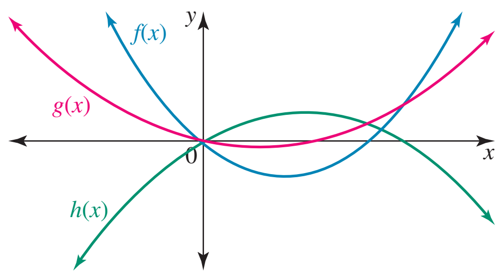
Question 1 (1 mark)

The diagram shows the graphs of three functions: $y = f(x)$, $y = g(x)$ and $y = h(x)$.
State which of the following statements is **true**.



A. $f(x) = g(x) - h(x)$
B. $g(x) = f(x) - h(x)$
C. $g(x) = h(x) - f(x)$
D. $h(x) = f(x) - g(x)$
E. $h(x) = g(x) - f(x)$

Question 2 (1 mark)

Given the functions $f(x) = \sqrt{x+3}$ and $g(x) = \sqrt{1-x}$, the graph of $y = f(x) - g(x)$ has
A. a maximal domain of $[-3, \infty)$ and a range of R.
B. a maximal domain of $[1, \infty)$ and a range of R^+.
C. a maximal domain of $(-\infty, -3] \cup [1, \infty)$ and a range of R.
D. a maximal domain of $[-3, 1]$ and a range of R^+.
E. a maximal domain of $[-3, 1]$ and a range of $[-2, 2]$.

Question 3 (1 mark)

For the function defined by $f(x) = \begin{cases} x-2, & x \geq 1 \\ -1, & -1 < x < 1 \\ x, & x \leq -2 \end{cases}$ then

A. $f(1) = -1$ and the range is R.
B. $f(1) = -1$ and the range is $[-2, 1]$.
C. $f(-1) = -1$ and the range is $R \backslash (-2, 1)$.
D. $f(-1)$ does not exist and the range is $R \backslash (-2, 1]$.
E. $f(-1)$ does not exist and the range is $R \backslash [-2, 1)$.

Source: VCE 2021, Mathematical Methods 2, Section A, Q10; © VCAA

Question 4 (1 mark)

Consider the functions $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{1-2x}$, defined over their maximal domains.
The maximal domain of the function $h = f + g$ is

A. $\left(-2, \frac{1}{2}\right)$
B. $[-2, \infty)$
C. $(-\infty, -2) \cup \left(\frac{1}{2}, \infty\right)$
D. $\left[-2, \frac{1}{2}\right]$
E. $[-2, 1]$

Source: VCE 2013, Mathematical Methods (CAS) 2, Section 1, Q5; © VCAA

Question 5 (1 mark)

If $f: (-\infty, 1) \rightarrow R,\ f(x) = 2\log_e(1-x)$ and $g: [-1, \infty] \rightarrow R,\ g(x) = 3\sqrt{x+1}$, then the maximal domain of the function $f + g$ is

A. $[-1, 1)$
B. $(1, \infty)$
C. $(-1, 1]$
D. $(-\infty, -1]$
E. R

Question 6 (2 marks)

The function $f: R\backslash\{0\} \to R, f(x) = \dfrac{x^2 + 1}{x}$ is determined by adding two functions, g and h.

Write a rule for both g and h that could be used.

Question 7 (2 marks)

For f: $[-4, 1] \to R,\ f(x) = x + 2$ and $g: [-2, \infty) \to R,\ g(x) = x^2$, find the rule $(f + g)(x)$ and state the domain.

Question 8 (1 mark)

Given the functions

f: $(-a, \infty) \to R$ where $f(x) = \log_e(x + a)$ and $g: (-\infty, a) \to R$ where $g(x) = -\log_e(a - x)$, where a is a positive real constant, if $h(x) = f(x) - g(x)$, then the function h is defined by

A. $h: R \to R$ where $h(x) = \log_e(2x)$

B. h: $(-\text{a}, \infty) \to R$ where $h(x) = \log_e(2x)$

C. $h: (\infty, a) \to R$ where $h(x) = \log_e(2x)$

D. $h: (-a, a) \to R$ where $h(x) = \log_e\left(a^2 - x^2\right)$

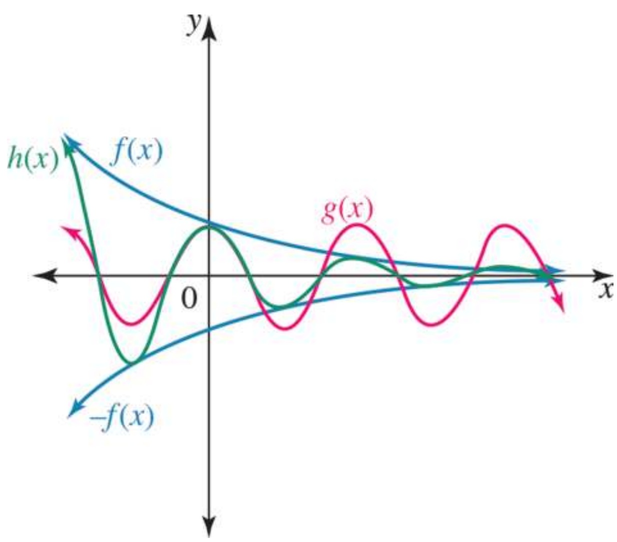
E. $h: (-a, a) \to R$ where $h(x) = \log_e\left(x^2 - a^2\right)$

Question 9 (2 marks)

For $f: R^+ \to R, f(x) = \frac{1}{x}$ and $g: (1, \infty) \to R,\ g(x) = \sqrt{x-1}$, find the rule $(f - g)(x)$ and state the domain.

Question 10 (1 mark)

The diagram below shows the graphs of three functions: $y = \pm f(x)$, $y = g(x)$ and $y = h(x)$.



Which of the following statements is true?

A. $f(x) = g(x)\,h(x)$

B. $g(x) = f(x)\,h(x)$

C. $h(x) = f(x)\,g(x)$

D. $f(x) = \dfrac{g(x)}{h(x)}$

E. $h(x) = \dfrac{f(x)}{g(x)}$

Question 11 (1 mark)

Given the functions $f:(1,\infty)\to R$ where $f(x)=3x+4$ and $g:R^+\to R$ where $g(x)=2x-5$, if $h(x)=f(x)\,g(x)$, then the function h is defined by

A. $h:(1,\infty)\to R$ where $h(x)=6x^2-20$
B. $h:R^+\to R$ where $h(x)=6x^2-20$
C. $h:R\to R$ where $h(x)=6x^2-10x-20$
D. $h:(1,\infty)\to R$ where $h(x)=6x^2-7x-20$
E. $h:(1,\infty)\to R$ where $h(x)=6x-11$

Question 12 (1 mark)

Given the functions $y=f(x)=\sqrt{x+2}$ and $y=g(x)=\dfrac{1}{\sqrt{2-x}}$, a graph of $y=f(x)\,g(x)$ has

A. a maximal domain of $[-2,\infty)$ and a range of R.
B. a maximal domain of $(2,\infty)$ and a range of R^+.
C. a maximal domain of $(-\infty,2]\cup(2,\infty)$ and a range of R.
D. a maximal domain of $[-2,2)$ and a range of $R^+\cup\{0\}$.
E. a maximal domain of $[-2,2)$ and a range of R^+.

Question 13 (2 marks)

For $f:R^+\to R,\ f(x)=\dfrac{1}{x}$ and $g:[-1,\infty]\to R,\ g(x)=\sqrt{x+1}$, find the rule $(fg)(x)$ and state the domain.

Question 14 (1 mark)

For the function defined by $f(x) = \begin{cases} x^2, & x \geq 1 \\ x, & -1 < x < 1 \\ -1, & x \leq -1 \end{cases}$ then

A. $f(1) = 1$ and the range is R.
B. $f(-1) = 1$ and the range is R.
C. $f(-1) = -1$ and the range is $[-1, \infty)$.
D. $f(1) = -1$ and the range is $[-1, \infty)$.
E. $f(-1) = 1$ and the range is $[-1, \infty)$.

Question 15 (1 mark)

For the function defined by $f(x) = \begin{cases} x-2, & x \geq 3 \\ 1, & 1 \leq x < 2 \\ x, & -1 < x < 1 \\ -1, & x \leq -1 \end{cases}$ then

A. the domain is $(-\infty, 2) \cup [3, \infty)$ and the range is $[-1, \infty)$.
B. the domain is $(-\infty, 2) \cup [3, \infty)$ and the range is R.
C. the domain is R and the range is $[-1, \infty)$
D. the domain is $(-\infty, 2) \cup [3, \infty)$ and the range is $(-1, \infty]$.
E. the domain and range are both R.

Question 16 (1 mark)

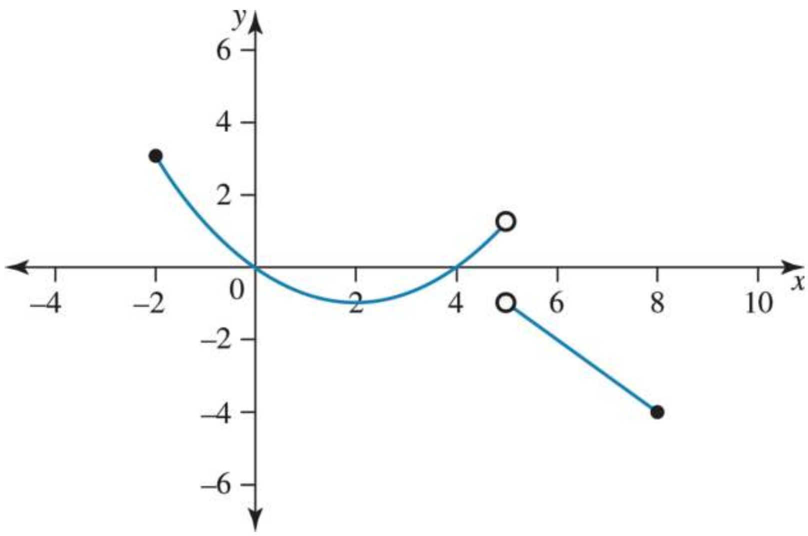
Determine the range of the function defined by

$$f(x) = \begin{cases} x+3, & -2 \leq x \leq 0 \\ -(x-2)^2 + 4, & 0 < x \leq 4 \\ x-7, & 4 < x \leq 6 \end{cases}$$

Question 17 (3 marks)

Answer the following.

A. Determine the equation of the hybrid function. **(2 marks)**



B. Is the function continuous or discontinuous? **(1 mark)**

Topic	1	Functions and graphs
Subtopic	1.9	Modelling and applications

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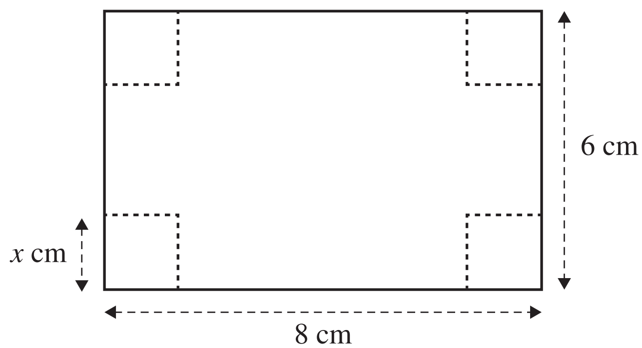
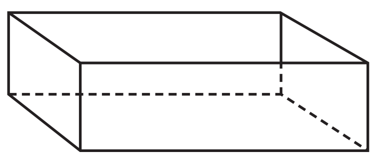
Source: *VCE 2014, Mathematical Methods (CAS) Exam 2, Section 1, Q15; © VCAA*

Question 1 (1 mark)

Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.



Zoe turns up the sides to form an open box.

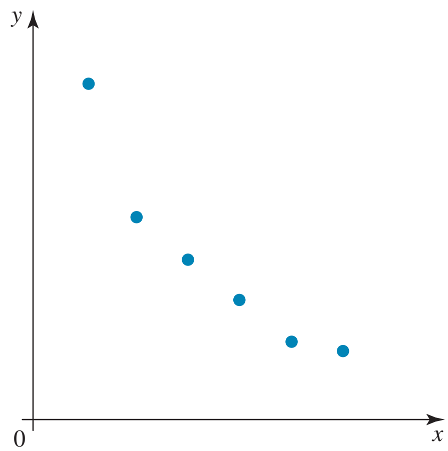


The value of x for which the volume of the box is a maximum is closest to

A. 0.8
B. 1.1
C. 1.6
D. 2.0
E. 3.6

Question 2 (1 mark)

The graph shows the relationship between two variables, x and y.



If a is a positive constant, the equation relating x and y is most likely

A. $y = a\sqrt{x}$

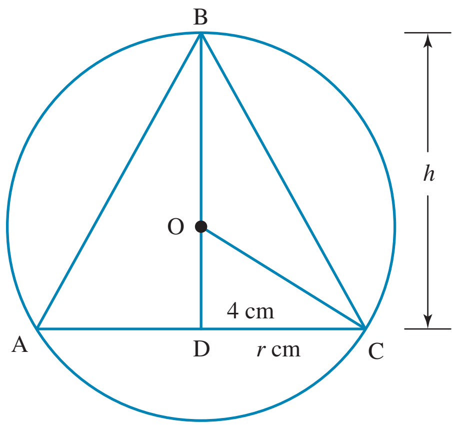
B. $y = ax^2$

C. $y = \frac{a}{x}$

D. $y = ae^x$

E. $y = a\log_e(x)$

Question 3 (6 marks)
A right circular cone is inscribed in a sphere of radius 4 cm, as shown in the cross-section.



a. Express the radius, r cm, of the cone in terms of h. **(1 mark)**

b. Write an equation expressing the volume of the cone, V cm^3, in terms of h and state any restrictions on h. **(2 marks)**

c. Sketch the graph of V versus h. **(2 marks)**

d. Use the graph to find the maximum volume for the cone to the nearest cm^3. **(1 mark)**

Topic	1	Functions and graphs
Subtopic	1.10	Review

online only

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***Source:** VCE 2018, Mathematical Methods Exam 2, Section A, Q3; © VCAA*

Question 1 (1 mark)

Consider the function $f[a, b] \to R$, $f(x) = \frac{1}{x}$, where a and b are positive real numbers. The range of f is

A. $\left[\frac{1}{a}, \frac{1}{b}\right)$

B. $\left(\frac{1}{a}, \frac{1}{b}\right]$

C. $\left[\frac{1}{b}, \frac{1}{a}\right)$

D. $\left(\frac{1}{b}, \frac{1}{a}\right]$

E. $[a, b)$

***Source:** VCE 2015, Mathematical Methods (CAS) Exam 2, Section 1, Q7; © VCAA*

Question 2 (1 mark)

The range of the function $f: (-1, 2] \to R, f(x) = -x^2 + 2x - 3$ is

A. R

B. $(-6, -3]$

C. $(-6, -2]$

D. $[-6, -3]$

E. $[-6, -2]$

***Source:** VCE 2014, Mathematical Methods (CAS) Exam 2, Section 1, Q18; © VCAA*

Question 3 (1 mark)

The graph of $y = kx - 4$ intersects the graph of $y = x^2 + 2x$ at two distinct points for

A. $k = 6$

B. $k > 6$ or $k < -2$

C. $-2 \leq k \leq 6$

D. $6 - 2\sqrt{3} \leq k \leq 6 + 2\sqrt{3}$

E. $k = -2$

Question 4 (1 mark)

The simultaneous linear equations

$$-3x + my = m - 1 \text{ and}$$
$$(m + 1)x - 10y = -8,$$

where m is a real constant, have an infinite number of solutions for

A. $m = 5$ or $m = -6$

B. $m = 5$ only

C. $m \in R \backslash \{-6, 5\}$

D. $m \neq -6$

E. $m \neq 5$

Question 5 (2 marks)

Find the value(s) of a for which the simultaneous equations $3x + ay = 5$ and $(a + 2)x + 5y = a$ have no solution.

Source: *VCE 2020, Mathematical Methods 2, Section A, Q18; © VCAA*

Question 6 (1 mark)

Let $a \in (0, \infty)$ and $b \in R$.

Consider the function $h : [-a, 0) \cup (0, a] \to R, h(x) = \frac{a}{x} + b$.

The range of h is

A. $[b - 1, b + 1]$

B. $(b - 1, b + 1)$

C. $(-\infty, b - 1) \cup (b + 1, \infty)$

D. $(-\infty, b - 1] \cup [b + 1, \infty)$

E. $[b - 1, \infty)$

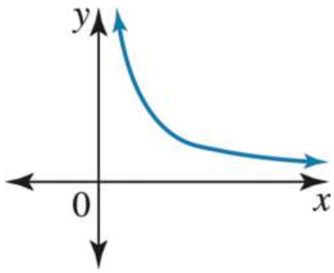
Question 7 (2 marks)

Find the set of linear factors for $ax^3 - bx$, where a and b are positive.

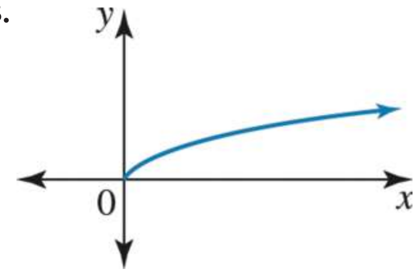
Question 8 (1 mark)

For the graph of $f(x) = x^n$, where n is $0 < n < 1$, the shape of the graph in the first quadrant will look like:

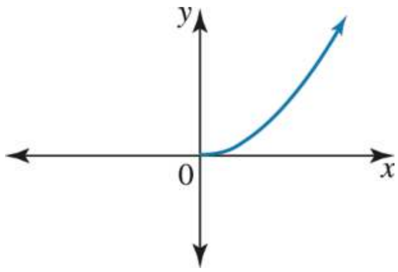
A.



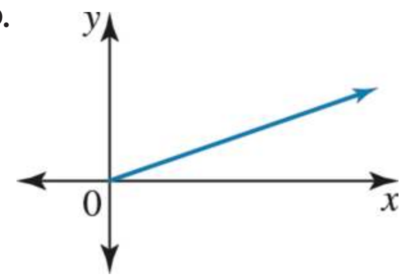
B.



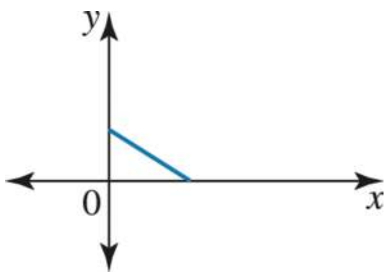
C.



D.



E.



Question 9 (1 mark)

Consider the function $f: R \rightarrow R$ where $f(x) = -x^6$. Which of the following statements is **false**?

A. $f(-x) = f(x)$ so the graph is symmetrical about the y-axis.
B. The domain is R.
C. The range is $(-\infty, 0]$.
D. The graph has a maximum at (0, 0).
E. The graph is symmetrical about the x-axis.

Question 10 (1 mark)

Consider the function $f: R \to R$ where $f(x) = x^n$ and n is even. Which of the following statements is **true**?

A. The maximal domain is $[0, \infty)$

B. The range is R.

C. The graph passes through the origin.

D. $f(-1) = -1$

E. The graph is symmetrical about the x-axis.

Question 11 (1 mark)

Consider the function $f: R \to R$ where $f(x) = x^5$. Which of the following statements is **false**?

A. The graph passes through the point $(-2, -32)$.

B. The point $(0, 0)$ is a stationary point of inflexion.

C. The domain and range of the graph are both R.

D. The graph crosses the x-axis at $(-1, 0)$ and $(1, 0)$.

E. The graph is not symmetrical about the x-axis.

Question 12 (1 mark)

For the graph of $y = \dfrac{1}{(x-5)^7}$, which of the following statements is **false**?

A. The graph of the function is a one-one function.

B. The line $x = 5$ is a vertical asymptote.

C. The line $y = 0$ is a horizontal asymptote.

D. The domain is $R \backslash \{5\}$ and the range is $R \backslash \{0\}$.

E. The graph is symmetrical about the line $x = 5$.

Question 13 (1 mark)

For the graph $y = \dfrac{1}{(x-4)^4}$, which of the following statements is **true**?

A. The line $x = -4$ is a vertical asymptote and the domain is R.
B. The line $x = 4$ is a vertical asymptote and the range is $(0, \infty)$.
C. The line $x = -4$ is a vertical asymptote and the range is $(0, \infty)$.
D. The line $y = 0$ is a horizontal asymptote and the domain is R.
E. The line $y = 0$ is a horizontal asymptote and the range is $R \setminus (4)$.

Question 14 (1 mark)

Which of the following statements is false for the graph of $y = \dfrac{1}{(x-a)^n}$ where n is a positive, even integer and a is a non-zero, real constant?

A. The line $x = a$ is a vertical asymptote and the domain is $R \setminus (a)$.
B. The line $y = 0$ is a horizontal asymptote and the range is $(0, \infty)$.
C. The graph is symmetrical about the line $x = a$.
D. The graph is a many-to-one function.
E. The graph crosses the y-axis at $\left(0, \dfrac{-1}{a^n}\right)$.

Question 15 (1 mark)

Which of the following rules does **not** describe a function?

A. $y = x^2 - 5$
B. $y = -5$
C. $x = -5$
D. $y = \sqrt{5 - x^2}$
E. $y = \dfrac{1}{x-5}$

Answers and marking guide

1.2 Linear functions

Question 1

Midpoint: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ given $(1,-5),(d,2)$

$$\text{Midpoint} = \left(\frac{1+d}{2}, \frac{-5+2}{2}\right)$$
$$= \left(\frac{d+1}{2}, -\frac{3}{2}\right)$$

The correct answer is **A**.

Question 2

$f(x) = 4 - x$

$f(a) = -2 \Rightarrow 4 - a = -2 \Rightarrow a = 6$ included

$f(b) = 6 \Rightarrow 4 - b = 6 \Rightarrow b = -2$ not included

The domain is $(-2, 6]$.

The correct answer is **D**.

Question 3

Gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$, so $m = \frac{1-(-5)}{3-0} = \frac{6}{3} = 2 = m_T$

Now $m_N m_T = -1$, so $m_N = \frac{-1}{2}$.

The correct answer is **C**.

Question 4

$f: D \to R, f(x) = 5 - x$, range $[-4, 5)$

$f(x) = -4 = 5 - x \Rightarrow x = 9$ included

$f(x) = 5 = 5 - x \Rightarrow x = 0$ not included

Domain $D = (0, 9]$

The correct answer is **A**.

1.3 Solving systems of equations

Question 1

Consider the simultaneous equations:

$ax - 3y = 5$ [1]

$3x - ay = 8 - a$ [2]

There will be no solutions when the gradients of both equations are the same and the y-intercept is different.

First, rearrange the equations to determine the gradient of each line.

$$\begin{aligned} ax - 3y &= 5 \qquad [1] \\ -3y &= -ax + 5 \\ y &= \frac{a}{3}x - \frac{5}{3} \end{aligned}$$

Gradient $= \frac{a}{3}$

$$\begin{aligned} 3x - ay &= 8 - a \qquad [2] \\ -ay &= -3x + 8 - a \\ y &= \frac{3}{a}x - \frac{8-a}{a} \end{aligned}$$

Gradient = $\frac{3}{a}$

Solve for when the gradients are equal.

$\frac{a}{3} = \frac{3}{a}$

$a^2 = 9$

$a = \pm 3$

Now test each a value to see which one(s) means that the y-intercepts are different.

$a = 3$:

$3x - 3y = 5$ [1]

$3x - 3y = 5$ [2]

When $a = 3$, the equations are the same; therefore, there would be infinitely many solutions.

$a = -3$:

$-3x - 3y = 5$ [1]

$3x + 3y = 11$ [2]

The gradients of both equations are the same; however, the y-intercepts are different.

Therefore, when $a = -3$, there are no solutions.

The correct answer is **B**.

Question 2

Consider the simultaneous equations:

$-2x - my = -4$ [1]

$(m-1)x + 6y = 2(m-1)$ [2]

There will be a unique solution provided the gradients of the two lines are not equal.

First, rearrange the equations to determine the gradient of each line.

$-2x - my = -4$ [1]

$-my = 2x - 4$

$y = -\frac{2}{m}x + \frac{4}{m}$

Gradient = $-\frac{2}{m}$

$(m-1)x + 6y = 2(m-1)$ [2]

$6y = -(m-1)x + 2(m-1)$

$y = -\frac{m-1}{6}x + \frac{m-1}{3}$

Gradient = $-\frac{m-1}{6}$

Solve for when the gradients are equal.

$-\frac{2}{m} = -\frac{m-1}{6}$

$12 = m^2 - m$

$0 = m^2 - m - 12$

$= (m-4)(m+3)$

$m = 4, -3$

Therefore, there will be a unique solution for all values other than 4 and -3.

$m \in R\backslash\{-3, 4\}$

The correct answer is **C**.

Question 3

A unique solution represents only one value for each of the three variables and will only occur at a point.

The correct answer is **C**.

Question 4
Equating coefficients and solving simultaneously using CAS technology gives:
$a = b + 1$ (equation 1)
$b = 2 - a$ (equation 2)
$a = \frac{3}{2}$ and $b = \frac{1}{2}$
The correct answer is **B**.

Question 5
Equating coefficients and solving simultaneously using CAS technology gives:
$a + 2b - 5 = 0$ (equation 1)
$-3b + 6a = 0$ (equation 2)
$a = 1$ and $b = 2$
The correct answer is **A**.

1.4 Quadratic functions

Question 1
$x^2 + 2x - k = 0$
$\Delta = (2)^2 + 4k = 4 + 4\,k > 0$
$k > -1, (-1, \infty)$
The correct answer is **B**.

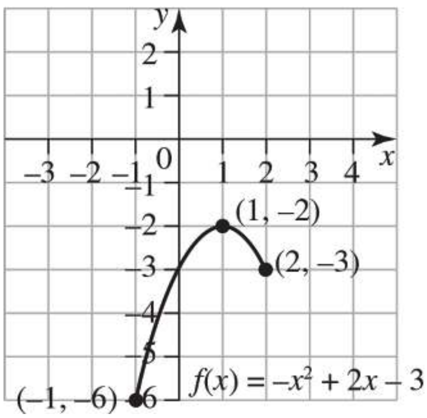
Question 2
$y = x^2 - 2bx + 1$
$y = \left(x^2 - 2bx + b^2\right) + 1 - b^2$
$y = (x - b)^2 + 1 - b^2$
The turning point is V $\left(b, 1 - b^2\right)$.
Define the distance, s, from the origin, O, to the turning point, V, in terms of b.
$d_{OV} = s(b) = \sqrt{b^2 + (1 - b^2)^2}$
Enter the function in your CAS calculator and determine which b value gives the smallest value of the square root, the distance from the origin.
When $b = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$, the distance is $\frac{3}{4}$.
The correct answer is **C**.

Question 3
$y_1 = mx + c,\ y_2 = ax^2$
For intersection points, $y_1 = y_2$:
$mx + c = ax^2 \Rightarrow ax^2 - mx - c = 0$
For no points of intersection, the discriminant, $\Delta = m^2 + 4ac < 0$.
$m^2 < -4ac \Rightarrow c > -\frac{m^2}{4a}$ if $a < 0$. Dividing by a positive does not reverse the inequality.
The correct answer is **D**.

Question 4
$(p - 1)\,x^2 + 4x = 5 - p$
$(p - 1)\,x^2 + 4x + p - 5 = 0$
$\Delta = 16 - 4\,(p - 1)\,(p - 5)$
$\Delta = 16 - 4\left(p^2 - 6p + 5\right)$
$\Delta = 16 - 4p^2 + 24p - 20$

For no real roots:
$\Delta < 0$
$-4\left(p^2 - 6p + 1\right) < 0$
$p^2 - 6p + 1 > 0$
The correct answer is **B**.

Question 5
$f: (-1, 2] \to R\, f(x) = -x^2 + 2x - 3$
Endpoint included: $f(2) = -4 + 4 - 3 = -3$
Endpoint, not included: $f(-1) = -1 - 2 - 3 = -6$
Turning points: $f'(x) = -2x + 2 = 0 \Rightarrow x = 1$
$f(1) = -1 + 2 - 3 = -2$
Range: $(-6, -2]$



The correct answer is **C**.

Question 6
$f(x) = (x - 1)^2 - 2,\ x \in [-2, 2]$
$f(a) = 0 \Rightarrow (a - 1)^2 - 2 = 0$
$(a - 1)^2 = 2$
$a - 1 = \pm\sqrt{2}$
$a = 1 \pm \sqrt{2}$ but $a < 0$
$a = 1 - \sqrt{2}$
Award 1 mark for the correct value of a.

***VCAA Assessment Report* note:**
The most common errors included not giving the answer in the form required, choosing the alternative x-value and not setting up the correct quadratic.

Question 7
$a = 3,\ b = -6,\ c = 7$
$\Delta = b^2 - 4ac = (-6)^2 - 4 \times 3 \times 7 = 36 - 84 = -48 < 0$
This indicates that the graph does not cross the x-axis.
$x = -\dfrac{b}{2a} = \dfrac{6}{6} = 1$
Therefore, $x = 1$ is the axis of symmetry.
The correct answer is **C**.

Question 8

$a = 1,\ b = -m.\ c = 9$

$\Delta = b^2 - 4ac = m^2 - 4 \times 9 = m^2 - 36$

$\Delta = 0 \Rightarrow m^2 = 36 \Rightarrow m = \pm 6$

The graph touches the x-axis; thus, A is true.

$\Delta > 0 \Rightarrow m^2 > 36 \Rightarrow m > 6$ and $m < -6$

The graph crosses the x-axis twice; therefore, both B and C are true.

$x = -\dfrac{b}{2a} = \dfrac{m}{2}$ is the axis of symmetry, so D is true.

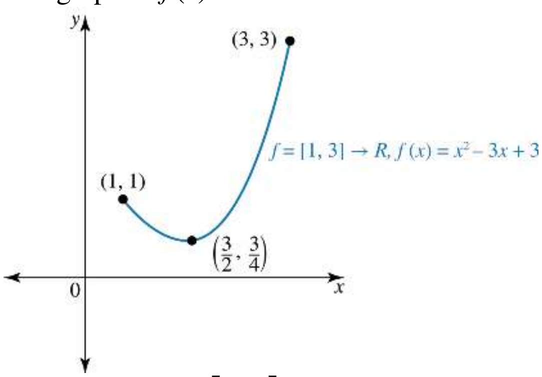
E is false, as the turning point is at $x = \dfrac{m}{2}$.

The correct answer is **E**.

Question 9

Turning point $\left(\dfrac{3}{2}, \dfrac{3}{4}\right)$

The graph of $f(x)$ looks like this.



The range is $y \in \left[\dfrac{3}{4}, 3\right]$.

Question 10

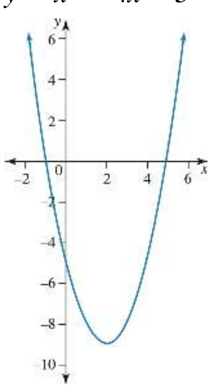
Now $y = a(x+1)(x-5)$, when $x = 0\ y = -5$

$\Rightarrow -5 = -5a$ so $a = 1$

$y = (x+1)(x-5)$

Expanding this gives

$y = x^2 - 4x - 5$



The correct answer is **A**.

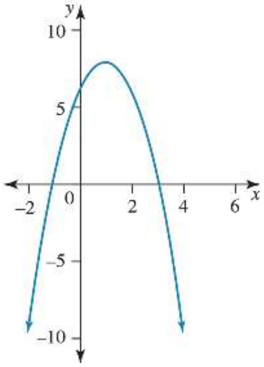
Question 11

Now $y = a(x+1)(x-3)$, when $x = 0\ y = 6$

$\Rightarrow 6 = -3a$ so $a = -2$

$y = -2(x+1)(x-3)$

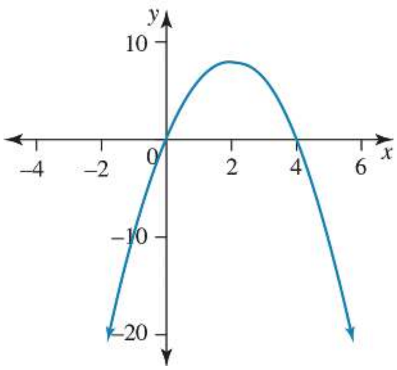
Expanding this gives
$y = -2x^2 + 4x + 6$



The correct answer is **A**.

Question 12

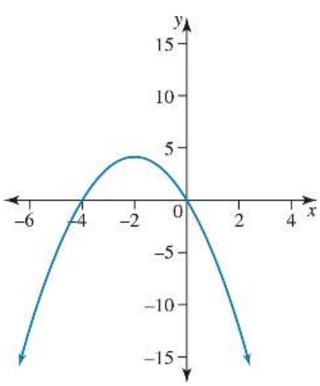
$f: (2, \infty) \to R$ where $f(x) = 8 - 2(x-2)^2$
We have a restricted domain, $(-2, \infty)$, and a range of $(-\infty, 8]$
The graph has a maximum turning point at $(2, 8)$.
The graph crosses the x-axis at $x = 0$ and $x = 4$, and crosses the y-axis at $y = 0$.



The correct answer is **C**.

Question 13

The graph crosses the x-axis at $x = -4$.
The axis of symmetry is the line $x = -2$.
The range is $(-\infty, 4]$.
The equation of the curve is $y = 4 - (x+2)^{2.}$
Therefore, A, B, C, and D are true, and E is false.
The vertex is a minimum turning point.



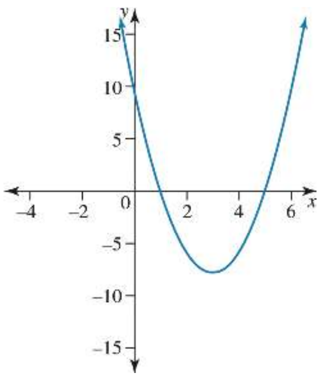
The correct answer is **E**.

Question 14

The domain of the graph is R and the range is $[-8, \infty)$.

The graph has a minimum turning point at $(3, -8)$, crosses the x-axis at $x = 1$ and $x = 5$, and crosses the y-axis at $y = 10$.

Therefore, A, B, D, and E are true, and C is false.



The correct answer is **C**.

Question 15

$$y = 2\left[x^2 + 2x + \frac{5}{2}\right]$$

$$= 2\left[(x+1)^2 + \frac{3}{2}\right] \quad \textbf{[1 mark]}$$

$$= 2(x+1)^2 + 3 \quad \textbf{[1 mark]}$$

Question 16

$y = 2(x-3)^2 - 6$

The turning point is at $(3, -6)$. **[1 mark]**

The turning point is a minimum. **[1 mark]**

Question 17

$$kx - 2 = x^2 + 4x$$

$$x^2 + x(4-k) + 2 = 0$$

$$\Delta = (4-k)^2 - 8 \quad \textbf{[1 mark]}$$

$$\Delta = 0$$

$$(4-k)^2 = 8$$

$$(4-k) = \pm 8$$

$$k = 4 - 2\sqrt{2} \text{ or } k = 4 + 2\sqrt{2}$$

For $\Delta \geq 0$,

$$k \leq 4 - 2\sqrt{2} \text{ and } k \geq 4 + 2\sqrt{2} \quad \textbf{[1 mark]}$$

1.5 Cubic functions

Question 1

$$p(x) = x^3 - 2ax^2 + x - 1$$

$$p(-2) = -8 - 8a - 2 - 1 = 5$$

$$8a = -16$$

$$a = -2$$

The correct answer is **E**.

Question 2

Let $f: [-3, 0] \rightarrow R, f(x) = (x + 2)^2(x - 1)$.

a $f(x) = (x + 2)^2(x - 1)$

$= (x^2 + 4x + 4)(x - 1)$

$= x^3 - x^2 + 4x^2 - 4x + 4x - 4$

$= x^3 + 3x^2 - 4$ **[1 mark]**

b The x-intercepts are $x = -2$ and $x = 1$. ($x = 1$ is outside the domain, but it is useful to know for the shape of the graph.)

There is a maximum turning point at $(-2, 0)$.

Check for another turning point:

$f'(x) = 3x^2 + 6x$

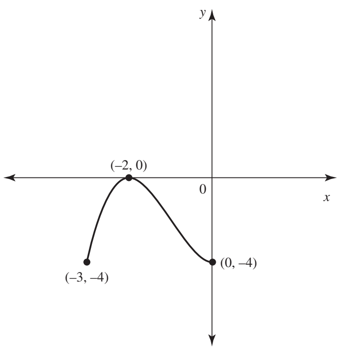
$0 = 3x(x + 2)$

$x = 0, -2$

$f(0) = (2)^2(-1)$

$= -4$

Therefore, there is a minimum turning point at $(0, -4)$.



Award 1 mark for correct end points.

Award 1 mark for the maximum turning point.

Award 1 mark for correct shape.

Question 3

First consider $f(x) = x^3 - 3x^2$.

Using CAS or calculus, there is a maximum turning point at $(0, 0)$ and a minimum turning point at $(2, -4)$.

Currently there are two x-intercepts. To obtain three distinct x-intercepts, the graph needs to be translated upwards by no more than the magnitude of the y-value of the minimum turning point.

Hence, $c \in (0, 4)$.

The correct answer is **D**.

Question 4

$P(x) = x^3 - ax^2 - 4x + 4$

$P(3) = 27 - 9a - 12 + 4 = 10$

$9a = 9$

$a = 1$

The correct answer is **C**.

Question 5

$P(x) = 7x^3 + 9x^2 - 5ax$

$x + a$ is a factor, $a \in R \backslash \{0\}$

$P(-a) = 7(-a)^3 + 9(-a)^2 + 5a^2 = 0$

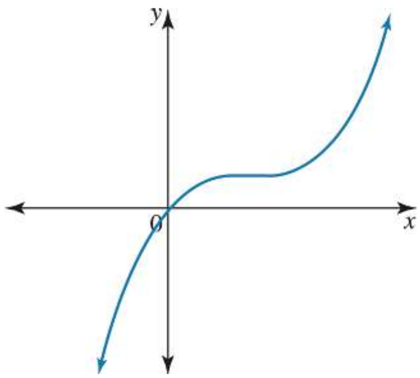
$\Rightarrow 14a^2 - 7a^3 = 7a^2(2 - a) = 0$

$\Rightarrow a = 2$

The correct answer is **E**.

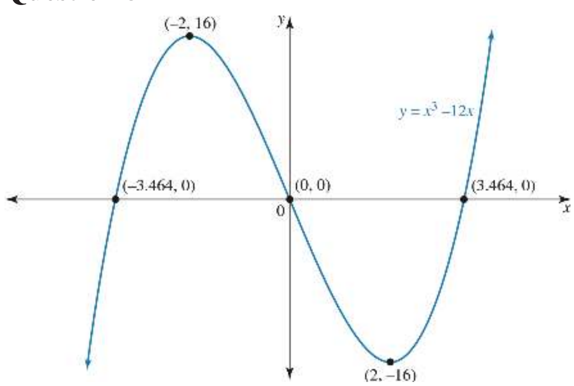
Question 6
$y = ax^3 + bx^2 + cx + d$ with $a > 0$
As $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$.
Graph E does not satisfy this, but all other graphs do.
The correct answer is **E**.

Question 7



A line parallel to the x or y-axis crosses the graph only once, so the graph is a one-to-one function.
The correct answer is **A**.

Question 8



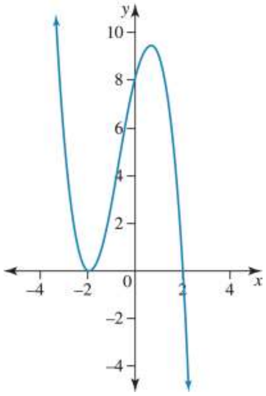
Award 1 mark for the correct curve.
Award 1 mark for the correct intercepts.
Award 1 mark for the correct turning points.

Question 9
The cubic curve is negative and has a turning point and x-intercept at the same value.
$y = -(x - a)^2 (x - b)$
The correct answer is **A**.

Question 10
Now $y = a(x + 1)(x - 1)(x - 2)$, when $x = 0 \ \ y = 0$
$\Rightarrow -2 = 2a$ so $a = -1$
$\therefore y = -(x + 1)(x - 1)(x - 2)$
The correct answer is **D**.

Question 11

$y = -(x+2)^2(x-2)$ since it has a minimum at $x = 2, a < 0$



The correct answer is **A**.

Question 12

$$(x-3)\left(x^2+2x+4\right) = 0$$
$$x = 3$$

There is one x-intercept only. **[1 mark]**

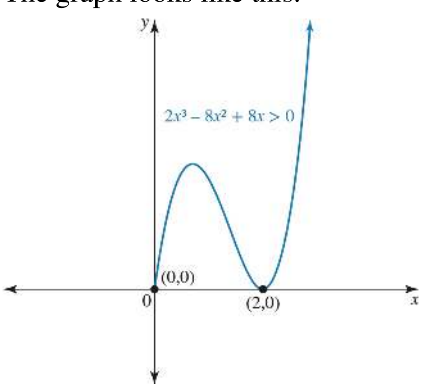
Question 13

$$2x^3 - 8x^2 + 8x > 0$$
$$\Rightarrow x^3 - 4x^2 + 4x > 0$$

Let:

$$x^3 - 4x^2 + 4x = 0$$
$$x\left(x^2 - 4x + 4\right) = 0$$
$$x(x-2)(x-2) = 0$$
$$x = 0,\ x = 2$$

The graph looks like this.



$\Rightarrow 2x^3 - 8x^2 + 8x > 0$

when $x \in (0,2) \cup (2,\infty)$

Award 1 mark for each of the two correct intervals.

Question 14

The inflection point is at $(-1,0)$ and it passes through the point $(0,-1)$.

Therefore, the equation of the graph is $y = -(x+1)^3$.

The correct answer is **A**.

Question 15

$y = a(x+b)^3 + c$

The inflection point is at $(2, 1)$ and the graph passes through the point $(0, -3)$.

Its equation is $y = a(x-2)^3 + 1$.

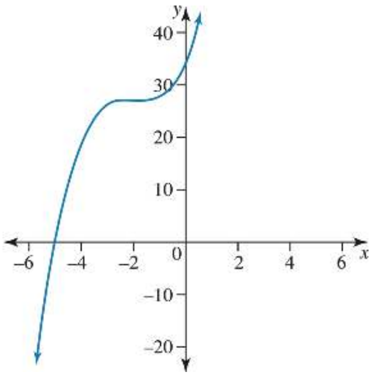
When $x = 0$, $y = -3$, so $-3 = -8a + 1 \Rightarrow 8a = 4$.

Therefore, $a = \dfrac{1}{2}$, $b = -2$ and $c = 1$.

The correct answer is **A**.

Question 16

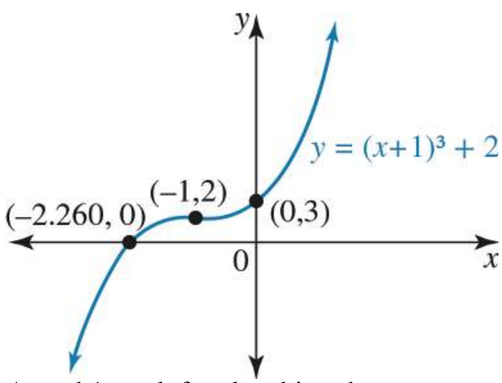
The inflection point is at $(-2, 27)$ and the graph crosses the coordinate axes at $(-5, 0)$ and $(0, 35)$.



The correct answer is **D**.

Question 17

a.

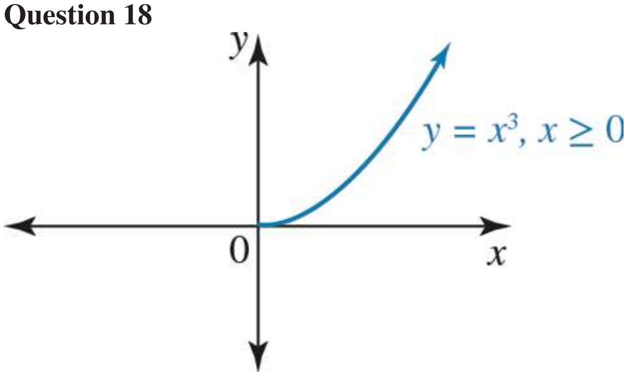


Award 1 mark for sketching the correct curve.

Award 1 mark for the correct point of inflection.

b. $x \in R$

Question 18



The correct answer is **C**.

Question 19

$$P(x) = 2x\left(3x^2 + 11x - a\right)$$
$$P(-a) = -2a\left(3a^2 - 11a + a\right)$$
$$= 0 \quad \textbf{[1 mark]}$$
$$3a(a-4) = 0$$
$$a = 4 \quad \textbf{[1 mark]}$$

Question 20

Dividing $(x + 1)$ into $f(x)$ gives a remainder of 3, then $f(-1) = 3$.

$$f(-1) = (-1)^3 - (-1)^2 + k + 2 = 3$$
$$-1 - 1 + k + 2 = 3$$
$$k = 3$$

The correct answer is **A**.

Question 21

$$2x^3 - 3x^2 - 14x + 15 = ax^3 - 3ax^2 + bx^2 - 3bx + cx - 3c$$
$$= ax^3 + x^2(b - 3a) + x(c - 3b) - 3c \quad \textbf{[1 mark]}$$

Equating coefficients gives:

$$a = 2$$
$$b - 3a = -3$$
$$b - 6 = -3$$
$$b = 3$$
$$-3c = 15$$
$$c = -5$$
$$a = 2,\ b = 3,\ c = -5 \quad \textbf{[1 mark]}$$

Question 22

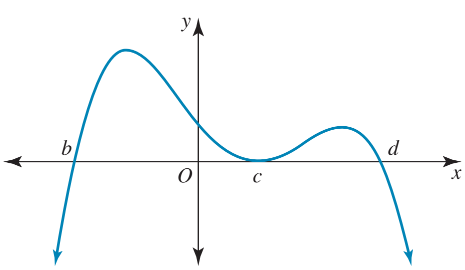
$$4ax^3 + x^3(4b - a) + x(4c - b) - c = 12x^3 - 7x^2 - 23x + 6 \qquad \textbf{[1 mark]}$$

Equating coefficients gives:

$$4a = 12$$
$$a = 3$$
$$4b - a = -7$$
$$b = -1$$
$$c = -6$$
$$a = 3,\ b = -1,\ c = -6 \quad \textbf{[1 mark]}$$

1.6 Higher degree polynomials

Question 1



The graph is a negative quartic, crosses at $x = b, x = c$ is a double root, and $x = d$. Its equation could be $y = -k(x-b)(x-c)^2(x-d)$, $k > 0$, $k = 2$

The correct answer is **C**.

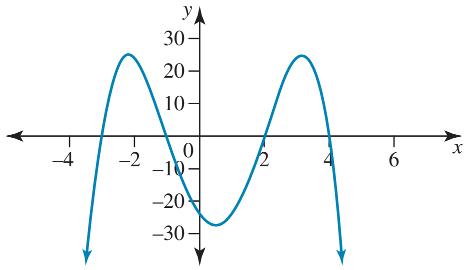
***VCAA Assessment Report* note:**
Most students chose option A, $y = -2(x+b)(x-c)^2(x-d)$, but the factor $(x+b)$ is incorrect.

Question 2

$$-x^4 + 7x^3 - 12x^2 = 0$$
$$-x^2\left(x^2 - 7x + 12\right) = 0$$
$$-x^2(x-4)(x-3) = 0$$
$$x = 0, 4, 3 \qquad \textbf{[1 mark]}$$

The graph is inverted, so when solving $-x^4 + 7x^3 - 12x^2 \ge 0$, the graph is only above the x-axis between $x = 3$ and $x = 4$, and is equal to zero at the x-intercepts.
Therefore, $x \in [3, 4] \cup \{0\}$. **[1 mark]**

Question 3
$y = a(x+3)(x+1)(x-2)(x-4)$
When $x = 0$, $y = -24$.
Now $-24 = 24a$, so $a = -1$.
$\therefore y = -(x+3)(x+1)(x-2)(x-4)$



The correct answer is **C**.

Question 4
The curve is a positive quartic curve with both an x-intercept and point of inflection at $x = c$.
$y = a(x-b)(x-c)^3$
The correct answer is **B**.

Question 5
$y = ax^4 + bx^3 + cx^2 + dx + e$ with $a > 0$
As $x \to \pm\infty$. $y \to \infty$
Option E does not satisfy this; however, all other graphs do.
The correct answer is **E**.

Question 6
The graph is an even function, and symmetrical about the y-axis,
so
$f(-x) = f(x) \Rightarrow b = d = 0$
The graph crosses the y-axis, below the x-axis.
$\Rightarrow e < 0$
as $x \to \pm\infty$ $y \to -\infty \Rightarrow a < 0$,
Consider the graph of

$$y = -(x+1)(x-1)(x+2)(x-2)$$
$$= -\left(x^2 - 1\right)\left(x^2 - 4\right)$$
$$= -\left(x^4 - 5x^2 + 4\right)$$
$$= -x^4 + 5x^2 - 4 \Rightarrow c > 0$$

The correct answer is **D**.

Question 7

$\left(x^2 + a\right)(x - b)(x + c) = 0$

$\left(x^2 + a\right) = 0$ and $(x - b)(x + c) = 0$ **[1 mark]**

$x =$ no solution and $x = b$ or $x = -c$

Two solutions, at $x = b$ or $x = -c$ **[1 mark]**

Question 8

Factorising $f(x) = x^4 - 4x^3 - x^2 + 16x - 12$

gives $f(x) = (x - 3)(x - 2)(x - 1)(x + 2)$.

The correct answer is **C**.

Question 9

$f(x + 1) = (x + 1)(x + 1 + 1)(x + 1 - 2)^2$

$= (x + 1)(x + 2)(x - 1)^2$

The correct answer is **D**.

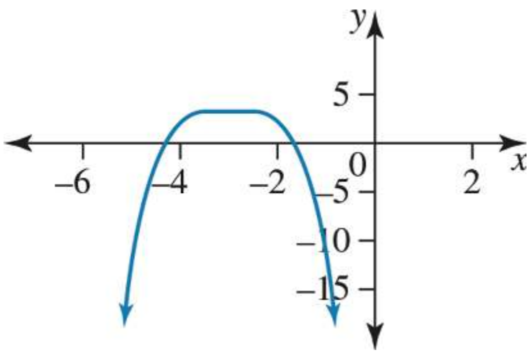
Question 10

For the quartic function, if $a > b > c, f(x) = (x - a)(x - c)(x - b)^2$.

The correct answer is **B**.

Question 11

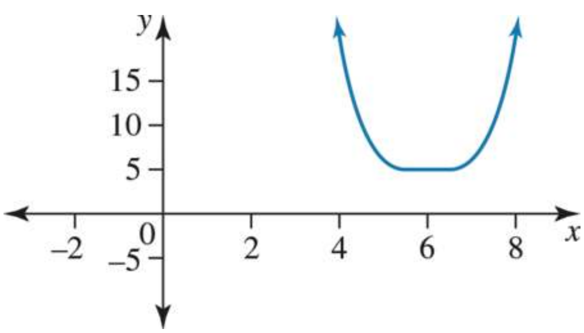
The graph has a maximum at $(-3, 2)$.



The correct answer is **C**.

Question 12

The range is $[5, \infty)$.



The correct answer is **A**.

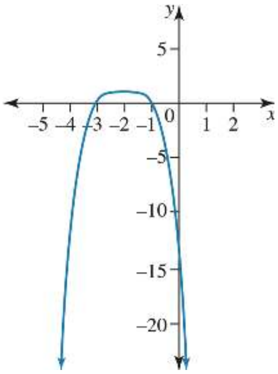
Question 13

The domain is $\boldsymbol{R}$ and the range is $(-\infty, 1]$.

The graph crosses the x-axis at $(-3, 0)$ and $(-1, 0)$, and crosses the y-axis at $(0, -15)$.

The line $x = -2$ is an axis of symmetry.

Therefore, A, B, C and D are true, and E is false.
The vertex $(-2, 1)$ is a maximum turning point.

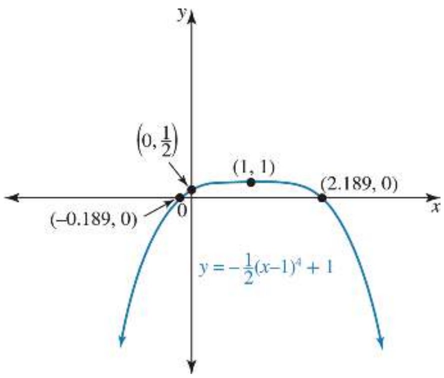


The correct answer is **E**.

Question 14

a. The turning point is at $(1, 1)$.

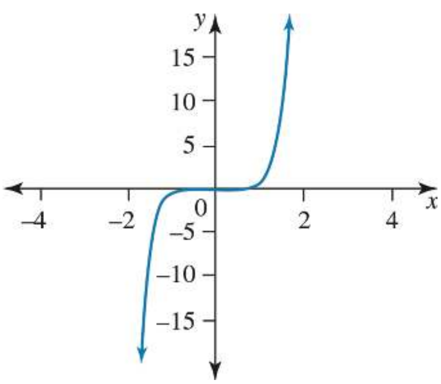
b.



Award 1 mark for the correct curve.
Award 1 mark for correct turning point and intercepts.

Question 15

The graph crosses the x-axis at $(0, 0)$. Therefore, the false statement is:
The graph crosses the x-axis at $(-1, 0)$ and $(1, 0)$.



The correct answer is **D**.

1.7 Other algebraic functions

Question 1

$f: D \to R, f(x) = \dfrac{3x + 2}{5 - x}$

$f(x) = \dfrac{3x + 2}{5 - x} = \dfrac{-3(5 - x) + 17}{5 - x}$

$f(x) = -3 + \dfrac{17}{5 - x}$

$x = 5$ is a vertical asymptote.

$y = -3$ is a horizontal asymptote.

The correct answer is **E**.

Question 2

$a \in (0, \infty),\ b \in R$

$h: [-a, \infty] \cup (0, a) \to h(x) = \dfrac{a}{x} + b$

$h(a) = b + 1,\ h(-a) = b - 1$

$x = 0$ is a vertical asymptote.

$y = b$ is a horizontal asymptote.

The correct answer is **D**.

Question 3

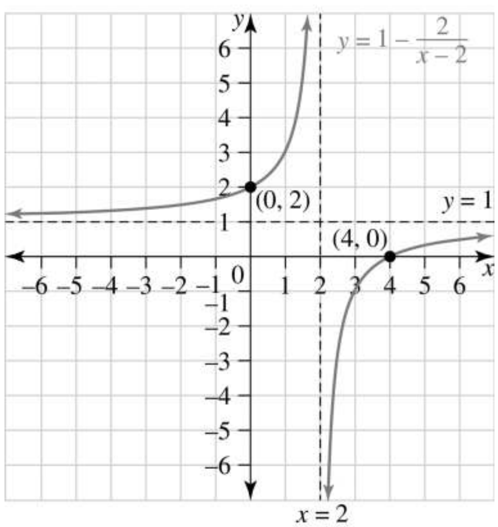
$f(x) = \dfrac{x^2 - 5}{x - 1}$ has maximal domain $R \backslash (1)$.

$x = 1$ is a vertical asymptote.

The correct answer is **A**.

Question 4

a. The sketch of $y = 1 - \dfrac{2}{x - 2}$ is shown below.



Award 1 mark for the correct graph shape.

Award 1 mark for the correct asymptotes.

Award 1 mark for the correct axial intercepts.

b. $1 - \dfrac{2}{x-2} \geq 3$

$1 - \dfrac{2}{x-2} = 3$

$\dfrac{2}{x-2} = -2$

$x - 2 = -1$

$x = 1$

$1 - \dfrac{2}{x-2} \geq 3 \quad x \in [1, 2), \qquad 1 \leq x < 2$

Award 1 mark for the final values.

Question 5

$f(x) = 4 - x$

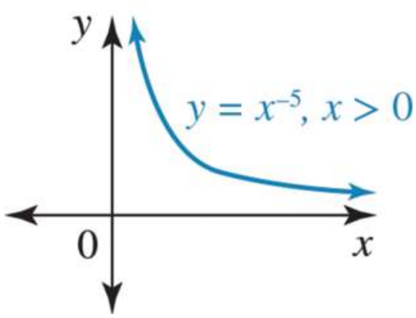
$f(a) = -2 \Rightarrow 4 - a = -2 \Rightarrow a = 6$ included

$f(b) = 6 \Rightarrow 4 - b = 6 \Rightarrow b = -2$ not included

The domain is $(-2, 6]$.

The correct answer is **D**.

Question 6



The correct answer is **A**.

Question 7

Asymptotes at $x = -1$ and $y = 1$

$$y = \frac{A}{(x+1)^2} + 1 \quad \textbf{[1 mark]}$$

y-intercept is -2

$$-2 = \frac{A}{(0+1)^2} + 1$$

$$A = -3 \qquad \textbf{[1 mark]}$$

$$y = \frac{-3}{(x+1)^2} + 1 \quad \textbf{[1 mark]}$$

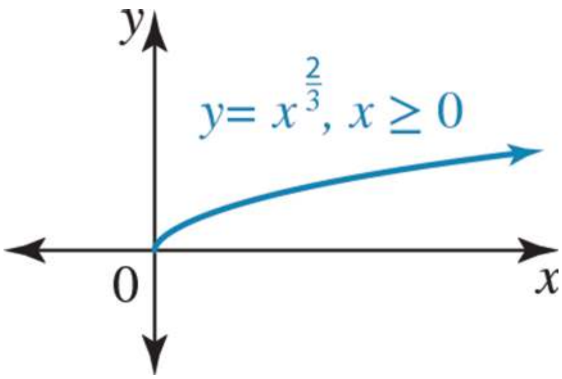
Question 8

Using CAS technology to obtain the solution:

$x \in R^+ \cup \{0\}$

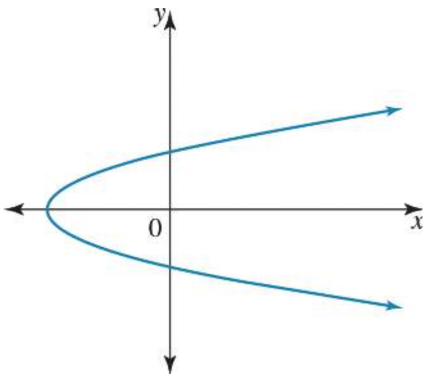
The correct answer is **B**.

Question 9
Use CAS technology to obtain the graph:



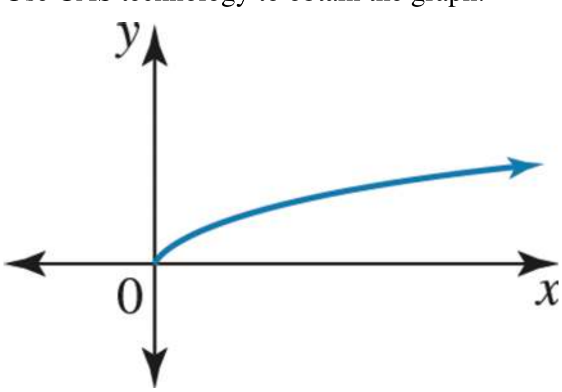
The correct answer is **B**.

Question 10



A line parallel to the *y*-axis crosses the graph twice, so the graph is a one-to-many relation (it is not a function).
The correct answer is **B**.

Question 11
Use CAS technology to obtain the graph:



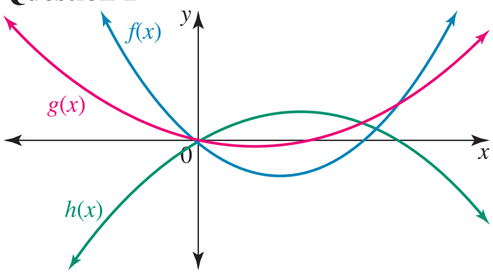
The correct answer is **B**.

Question 12
The domain is R and the range is $[0, \infty)$. The only graph satisfying this is $y = \sqrt[3]{x^4}$.
The correct answer is **E**.

Question 13
$y = \sqrt{(x + a)^3}$ is the only graph satisfying the maximal domain of $[-a, \infty)$ and a range of $[0, \infty)$.
The correct answer is **A**.

1.8 Combinations of functions

Question 1



Using addition of ordinates, $h(x) = g(x) + (-f(x)) = g(x) - f(x)$.
The correct answer is **E**.

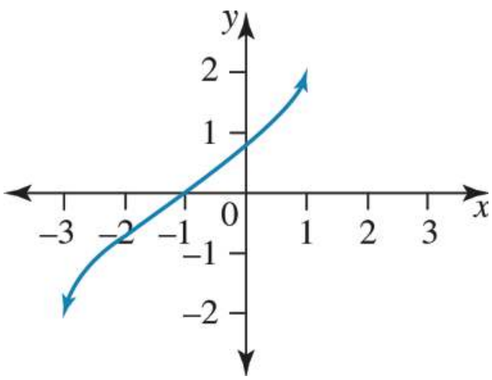
Question 2

$(-\infty, -2) \cup (-1, \infty) = R \backslash (-2, -1)$

$f(x) = \sqrt{x+3}$: $\text{dom} f = [-3, \infty]$ and $\text{ran} f = [0, \infty]$

$g(x) = \sqrt{1-x}$: $\text{dom}\, g = (-\infty, 1)$ and $\text{ran}\, g = [0, \infty]$

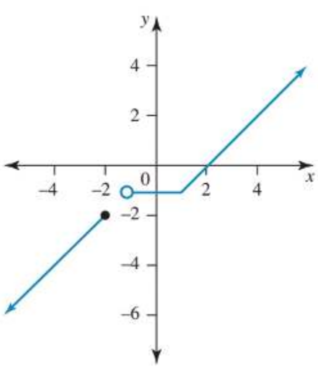
$\text{ran} f = [0, \infty]$
$\text{ran}\, g = [0, \infty]$
$y = f(x) - g(x) = \sqrt{x+3} - \sqrt{1-x}$
$\text{dom}\, f - g = \text{dom}\, f \cap g = [-3, 1]$ and $\text{ran}\, f \cap g = [-2, 2]$



The correct answer is **E**.

Question 3

$f(-1)$ does not exist and the range is $(-\infty, 2) \cup (-1, \infty) = R \backslash (-2, 1]$



The correct answer is **D**.

Question 4

$f(x) = \sqrt{x+2}$ domain $f = [-2, \infty)$

$g(x) = \sqrt{1-2x}$ domain $g = \left(-\infty, \frac{1}{2}\right)$

$h(x) = f(x) + g(x) = \sqrt{x+2} + \sqrt{1-2x}$

domain h = domain $f \cap$ domain $g = \left[-2, \frac{1}{2}\right]$

The correct answer is **D**.

Question 5

$f: (-\infty, 1) \to R,\ f(x) = 2\log_e(1-x)$

$g: [-1, \infty] \to R,\ g(x) = 3\sqrt{x+1}$

dom $f + g$ = dom $f \cap$ dom $g = [-1, 1)$

The correct answer is **A**.

Question 6

Answers will vary. One possible option is:

$g(x) = x$ [**1 mark**]

$h(x) = \frac{1}{x}$ [**1 mark**]

Question 7

$(f+g)(x) = x^2 + x + 2$ **[1 mark]**

Domain $(f+g)(x)$ = domain $f(x) \cap$ domain $g(x)$

$(f+g)(x)$ is $x \in [-2, 1]$. **[1 mark]**

Question 8

$f(x) = \log_e(x+a)$ dom $f = (-a, \infty)$ and ran $f = R$

$g(x) = -\log_e(a-x)$ dom $g = (-\infty, a)$ and ran $g = R$

$h(x) = f(x) - g(x) = \log_e(x+a) + \log_e(a-x) = \log_e((x+a)(a-x)) = \log_e(a^2 - x^2)$

dom $f - g$ = dom $f \cap\ g = (-a, a)$

The correct answer is **D**.

Question 9

$(f-g)(x) = \frac{1}{x} - \sqrt{x-1}$ **[1 mark]**

Domain $(f-g)(x)$ = domain $f(x) \cap$ domain $g(x)$

$(f-g)(x)$ is $x \in (1, \infty)$. **[1 mark]**

Question 10

The functions are like

$f(x) = e^{-x}$ and $g(x) = \cos(x)$

so $h(x) = f(x)\,g(x) = e^{-x}\cos(x)$.

The correct answer is **C**.

Question 11

$f(x) = 3x + 4$ dom $f = (1, \infty)$ and ran $f = (7, \infty)$

$g(x) = 2x - 5$ dom $g = R^+$ and ran $g = [-5, \infty]$

$h(x) = f(x)g(x) = (3x+4)(2x-5) = 6x^2 - 7x - 20$

dom $f.g$ = dom $f \cap\ g = (1, \infty)$

The correct answer is **D**.

Question 12

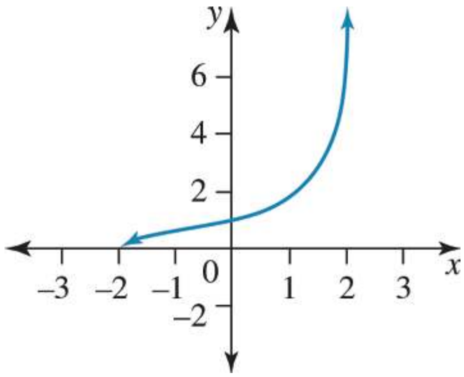
$f(x) = \sqrt{x+2}$ dom $f = [-2, \infty)$ and ran $f = [0, \infty)$

$g(x) = \dfrac{1}{\sqrt{2-x}}$ dom $g = (-\infty, 2)$ and ran $g = (0, \infty)$

$h(x) = f(x)\,g(x) = \sqrt{\dfrac{x+2}{2-x}}$

dom $f.g$ = dom $f \cap$ $g = [-2, 2)$ and ran $f.g = [0, \infty) = R^+ \cup \{0\}$

$x = 2$ is a vertical asymptote, and when $x = -2$, $y = 0$



The correct answer is **D**.

Question 13

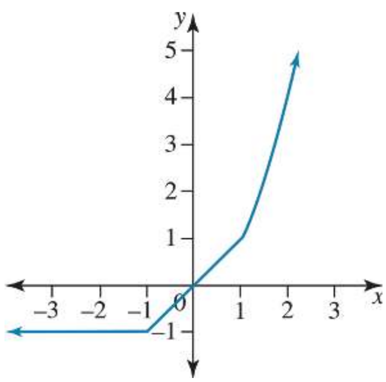
$(fg)(x) = \dfrac{\sqrt{x+1}}{x}$ **[1 mark]**

Domain $(fg)(x)$ = domain $f(x) \cap$ domain $g(x)$

$(fg)(x)$ is $x \in (0, \infty)$. **[1 mark]**

Question 14

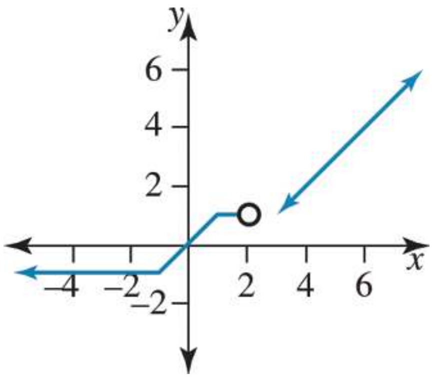
$f(-1) = -1$ and the range is $[-1, \infty)$



The correct answer is **C**.

Question 15

The domain is $(-\infty, 2) \cup [3, \infty)$ and the range is $(-1, \infty]$.



The correct answer is **A**.

Question 16

Draw the graph to find that:

Range $y \in (-3, -1] \cup [0, 4]$

Question 17

a. $f(x) = \begin{cases} \frac{1}{4}x(x-4), & -2 \le x < 5 \\ -x+4, & 5 < x \le 8 \end{cases}$

Award 1 mark for each rule with correct domain.

b. Discontinuous at $x = 5$ **[1 mark]**

1.9 Modelling and applications

Question 1

$V = lwh$

$= (8 - 2x)(6 - 2x)x$

Domain: $x \in (0, 3)$

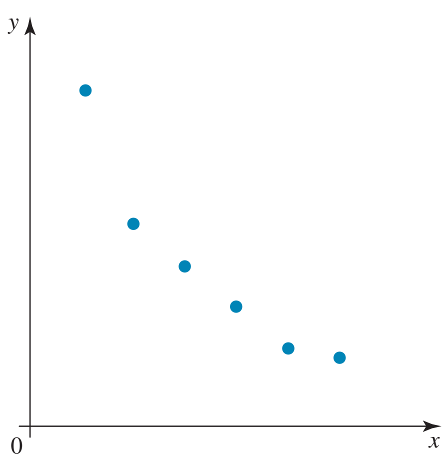
Sketch the graph and find the turning point using CAS over the domain $x \in (0, 3)$.

TP = (1.13, 24.26)

Therefore, maximum volume occurs when $x = 1.1$.

The correct answer is **B**.

Question 2



The shape of the graph is a hyperbola, $y = \frac{a}{x}$.

The correct answer is **C**.

Question 3

a. In triangle ODC, OC is of length $h - 4$ cm, $h > 4$.

Using Pythagoras' theorem

$$(h-4)^2 + r^2 = 4^2$$
$$\therefore r^2 = 16 - (h-4)^2$$
$$\therefore r = \sqrt{16-(h-4)^2}, r > 0$$
$$\therefore r = \sqrt{8h - h^2} \quad \textbf{[1 mark]}$$

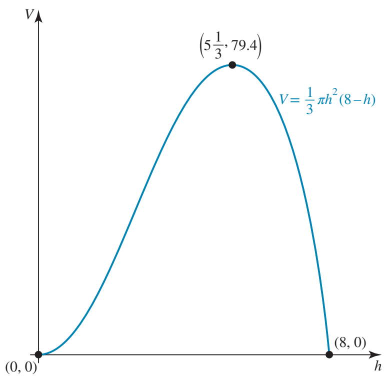
b. $V = \frac{1}{3}\pi r^2 h$

$\therefore\ V = \frac{1}{3}\pi(8h - h^2)h$

$\therefore\ V = \frac{1}{3}\pi h^2(8-h)$ **[1 mark]**

$h > 0$ and $8 - h > 0$, so the restriction on h is $0 < h < 8$. This would be seen on the graph to be the domain interval where $V > 0$. **[1 mark]**

c.



Award 1 mark for correct shape.

Award 1 mark for correct turning point and end points.

d. Using CAS, the greatest volume is 79 cm^3.

The range is $R\backslash\{5\}$. **[1 mark]**

1.10 Review

Question 1

$f: [a, b) \to R, f(x) = \frac{1}{x}$

$b > a > 0$

$f(b) = \frac{1}{b} < \frac{1}{a} = f(a)$

The end point a $x = a$ is included, but the end point at $x = b$ is not included.

The range is $\left(\frac{1}{b}, \frac{1}{a}\right]$.

The correct answer is **D**.

Question 2

$f: (-1, 2] \to R, f(x) = -x^2 + 2x - 3$

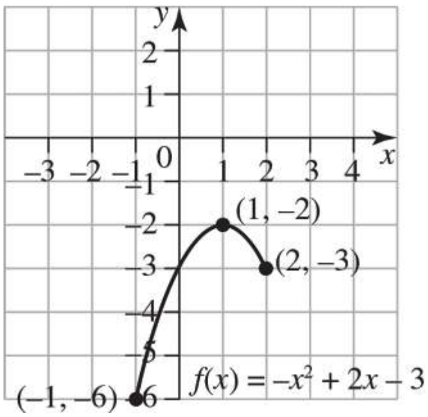
End point included: $f(2) = -4 + 4 - 3 = -3$

End point, not included: $f(-1) = -1 - 2 - 3 = -6$

Turning points: $f'(x) = -2x + 2 = 0 \Rightarrow x = 1$

$f(1) = -1 + 2 - 3 = -2$

Range: $(-6, -2]$



The correct answer is **C**.

Question 3

$y_1 = kx - 4,\ y_2 = x^2 + 2x$

$y_1 = y_2 \Rightarrow kx - 4 = x^2 + 2x$

$x^2 + (2 - k)x + 4 = 0$

$\Delta = (2 - k)^2 - 4 \times 1 \times 4$

$\quad = k^2 - 4k - 12$

$\quad = (k - 6)(k + 2)$

$\Delta > 0 \Rightarrow k > 6$ or $k < -2$

The correct answer is **B**.

Question 4

Consider the simultaneous equations:

$-3x + my = m - 1 \quad [1]$

$(m + 1)x - 10y = -8 \quad [2]$

There will be an infinite number of solutions provided the gradients and y-intercepts of the two lines are equal.

First, rearrange the equations to determine the gradient of each line.

$-3x + my = m - 1 \quad [1]$

$my = 3x + m - 1$

$y = \dfrac{3}{m}x + \dfrac{m - 1}{m}$

Gradient $= \dfrac{3}{m}$

$(m + 1)x - 10y = -8 \quad [2]$

$-10y = -(m + 1)x - 8$

$y = \dfrac{m + 1}{10}x + \dfrac{4}{5}$

Gradient $= \dfrac{m + 1}{10}$

Solve for when the gradients are equal.

$\dfrac{3}{m} = \dfrac{m + 1}{10}$

$30 = m^2 + m$

$0 = m^2 + m - 30$

$\quad = (m - 5)(m + 6)$

$m = 5, -6$

Now test each m value to see which one means that the y-intercepts are the same.

$m = 5$:

$-3x + 5y = 4$ [1]

$6x - 10y = -8$ [2]

If equation 1 is multiplied by -2, it is the same as equation 2.

Therefore, when $m = 5$, the equations are the same and there would be infinitely many solutions.

$m = -6$.

$-3x - 6y = -7$ [1]

$-5x - 10y = -8$ [2]

The gradients of both equations are the same; however, the y-intercepts are different.

So, when $m = 5$, there are an infinite number of solutions.

The correct answer is **B**.

Question 5

$$\begin{bmatrix} 3 & a \\ a+2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ a \end{bmatrix}$$

$$15 - a(a+2) = 0$$

$$15 - a^2 - 2a = 0$$

$$(a+5)(a-3) = 0$$

$a = -5, a = 3$ [**1 mark**]

if $a = -5$, $3x - 5y = 5$ and $-3x + 5y = -5$, lines are the same and there are infinite solutions.

If $a = 3$, $3x - 5y = 5$ and $5x + 5y = 3$, lines are parallel and there are no solutions.

No solutions, $a = 3$ **[1 mark]**

Question 6

$a \in (0, \infty), b \in R$

$h: [-a, \infty) \cup (0, a] \to h(x) = \dfrac{a}{x} + b$

$h(a) = b + 1, h(-a) = b - 1$

$x = 0$ is a vertical asymptote.

$y = b$ is a horizontal asymptote.

$(-\infty, b - 1] \cup [b + 1, \infty)$

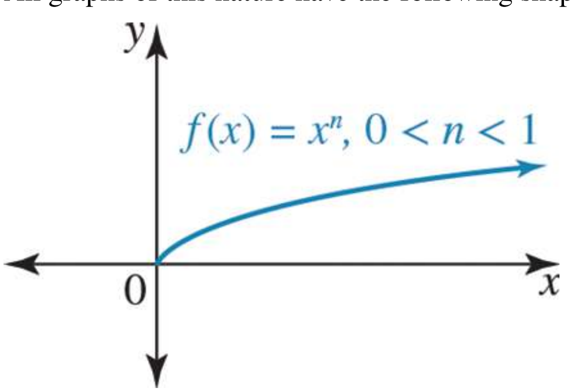
Question 7

$$x\left(ax^2 - b\right) = x\left(\left(\sqrt{a}x\right)^2 - \left(\sqrt{b}\right)^2\right) \quad \textbf{[1 mark]}$$

$$= x\left(\sqrt{a}x - \sqrt{b}\right)\left(\sqrt{a}x + \sqrt{b}\right) \quad \textbf{[1 mark]}$$

Question 8

All graphs of this nature have the following shape. This can be checked using CAS technology.

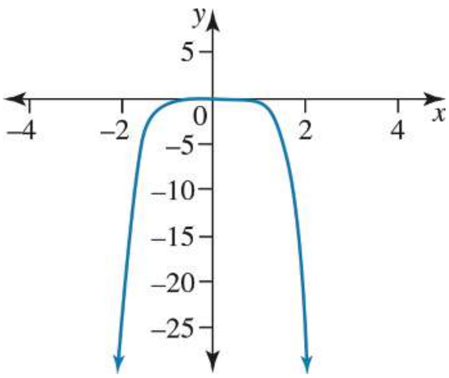


The correct answer is **B**.

Question 9

$f: R \to R$ where $f(x) = -x^6$

A, B, C, and D are all true. E is false since the graph is symmetrical about the y-axis.



The correct answer is **E**.

Question 10

$f: R \to R$

where $f(x) = x^n$ and n is even.

A, B, D and E are all false. Therefore, C is true.

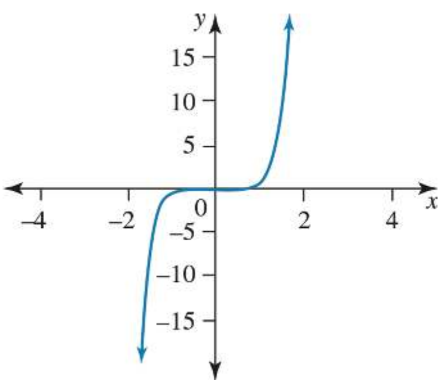
The maximal domain of the graph is R and the range is $[0, \infty)$. The graph passes through the origin, since n is even and the graph is symmetrical about the y-axis.

The correct answer is **C**.

Question 11

The graph crosses the x-axis at $(0, 0)$. Therefore, the false statement is:

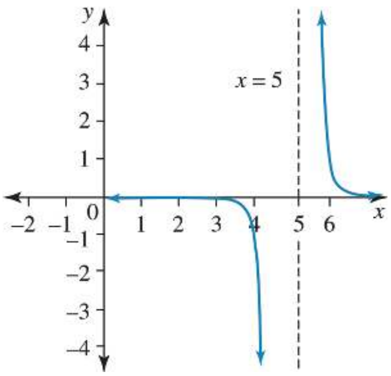
The graph crosses the x-axis at $(-1, 0)$ and $(1, 0)$.



The correct answer is **D**.

Question 12

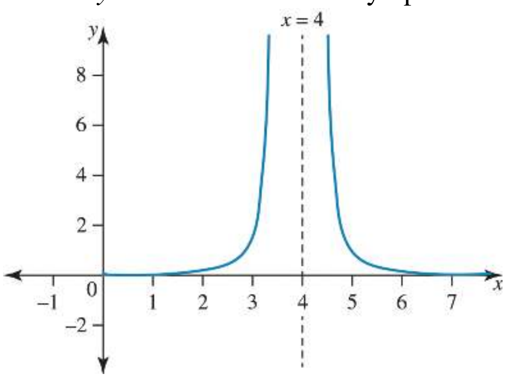
A, B, C, and D are true. E is false, as the graph is not symmetrical about the line $x = 5$.



The correct answer is **E**.

Question 13

The line $x = 4$ is a vertical asymptote, the domain is $R \setminus (4)$,
the line $y = 0$ is a horizontal asymptote and the range is $(0, \infty)$.

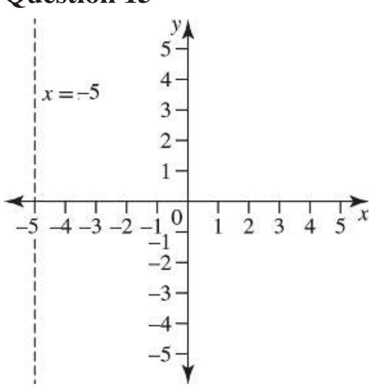


The correct answer is **B**.

Question 14

A, B, C, and D are true.
E is false: when $x = 0$,
$y = \dfrac{1}{(-a)^n} = \dfrac{1}{a^{n,}}$ since n is even.
The correct answer is **E**.

Question 15



A line parallel to the y-axis crosses the graph at infinitely many points, so the graph of $x = -5$ is not a function.
The correct answer is **C**.

2 Trigonometric (circular) functions

Topic	2	Trigonometric (circular) functions
Subtopic	2.2	Trigonometric symmetry properties

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Question 1 (1 mark)

Select the expression from the following that is equal to $\sin\left(-\frac{4\pi}{3}\right)$.

A. $\frac{\sqrt{3}}{2}$

B. $-\frac{\sqrt{3}}{2}$

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

E. $\sin\left(\frac{4\pi}{3}\right)$

Question 2 (1 mark)

Select the false statement from the following.

A. $\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)$

B. $\cos^3(\pi) + \sin^3(\pi) = 1$

C. $\cos^2(\pi) + \sin^2(\pi) = 1$

D. $\cos(\pi) + \sin(\pi) = -1$

E. $\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1$

Question 3 (1 mark)

Select the false statement from the following.

A. $\sin(\pi + \theta) + \sin(\pi - \theta) = 0$

B. $\cos(\pi + \theta) + \cos(2\pi - \theta) = 0$

C. $\tan(\pi + \theta) + \tan(2\pi - \theta) = 0$

D. $\cos(\pi + \theta) - \cos(\pi - \theta) = 0$

E. $\sin(\pi + \theta) + \sin(2\pi - \theta) = 0$

Question 4 (1 mark)

Let $\cos(x) = \frac{3}{5}$ and $\sin^2(y) = \frac{25}{169}$, where $x \in \left[\frac{3\pi}{2}, 2\pi\right]$ and $y \in \left[\frac{3\pi}{2}, 2\pi\right]$. The value of $\sin(x) + \cos(y)$ is

A. $\frac{8}{65}$

B. $-\frac{112}{65}$

C. $\frac{112}{65}$

D. $-\frac{8}{65}$

E. $\frac{64}{65}$

Question 5 (1 mark)

Which of the following statements is **false**?

A. $2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 1$

B. $\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right) = 0$

C. $\tan\left(\frac{\pi}{2}\right) - \tan\left(\frac{3\pi}{2}\right) = 0$

D. $\tan\left(\frac{\pi}{3}\right) = \frac{2\tan\left(\frac{\pi}{6}\right)}{1 - \tan^2\left(\frac{\pi}{6}\right)}$

E. $\cos\left(\frac{\pi}{6}\right) = \sqrt{\frac{1}{2} + \left(1 + \cos\left(\frac{\pi}{3}\right)\right)}$

Question 6 (1 mark)

Which of the following expressions is $\tan\left(\frac{13\pi}{8}\right)$ **not** equal to?

A. $\tan\left(\frac{5\pi}{8}\right)$

B. $-\tan\left(\frac{3\pi}{8}\right)$

C. $\frac{\sin\left(\frac{13\pi}{8}\right)}{\cos\left(\frac{13\pi}{8}\right)}$

D. $-1-\sqrt{2}$

E. $\frac{\cos\left(\frac{8}{13\pi}\right)}{\sin\left(\frac{8}{13\pi}\right)}$

Question 7 (1 mark)

Which of the following expressions is $\sin\left(\frac{13\pi}{12}\right)$ **not** equal to?

A. $-\sin\left(\frac{\pi}{12}\right)$

B. $-\sin\left(\frac{11\pi}{12}\right)$

C. $\cos\left(\frac{11\pi}{12}\right)$

D. $-\cos\left(\frac{5\pi}{12}\right)$

E. $\cos\left(\frac{7\pi}{12}\right)$

Question 8 (2 marks)

If $\cos(\theta) = -\frac{5}{13}$ for $\frac{\pi}{2} < \theta < \pi$, find the exact value of $\tan(\theta)$.

Question 9 (1 mark)

Which of the following expressions is $\sin\left(-\frac{4\pi}{3}\right)$ equal to?

A. $\frac{\sqrt{3}}{2}$

B. $-\frac{\sqrt{3}}{2}$

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

E. $\sin\left(\frac{4\pi}{3}\right)$

Question 10 (1 mark)

Express $\tan\left(\frac{5\pi}{3}\right) - \sin\left(\frac{3\pi}{4}\right)$ in the form $\frac{a+b}{c}$.

Question 11 (1 mark)

Which of the following expressions is $\cos\left(-\frac{\pi}{3}\right)$ **not** equal to?

A. $\cos\left(\frac{\pi}{3}\right)$

B. $-\cos\left(\frac{2\pi}{3}\right)$

C. $-\cos\left(\frac{4\pi}{3}\right)$

D. $-\cos\left(\frac{5\pi}{3}\right)$

E. $\frac{1}{2}$

Question 12 (1 mark)

Which of the following expressions is $\tan\left(-\frac{3\pi}{4}\right)$ **not** equal to?

A. $\tan\left(\frac{\pi}{4}\right)$

B. $\tan\left(\frac{5\pi}{4}\right)$

C. $-\tan\left(\frac{3\pi}{4}\right)$

D. $-\frac{\sin\left(-\frac{3\pi}{4}\right)}{\cos\left(-\frac{3\pi}{4}\right)}$

E. 1

Question 13 (3 marks)

If $\sin(\theta) = 0.25$, $\cos(\theta) = 0.67$ and $\tan(\theta) = 0.54$, find the values of

a. $\sin(\pi + \theta) =$ **(1 mark)**

b. $\cos(-\theta) =$ **(1 mark)**

c. $\tan(2\pi - \theta) =$ **(1 mark)**

Question 14 (1 mark)

$\sin\left(\frac{5\pi}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$ is also equivalent to

A. $\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

B. $\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

C. $\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

D. $\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

E. $\tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

Question 15 (1 mark)

Which of the following statements is **false**?

A. $2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right)$

B. $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$

C. $\tan\left(\frac{3\pi}{2}\right) = \frac{\sin\left(\frac{3\pi}{2}\right)}{\cos\left(\frac{3\pi}{2}\right)}$

D. $\frac{1}{\tan\left(\frac{\pi}{6}\right)} = \frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}$

E. $\sin\left(\frac{\pi}{6}\right) = \sqrt{\frac{1}{2}\left(1 - \cos\left(\frac{\pi}{3}\right)\right)}$

Topic	2	Trigonometric (circular) functions
Subtopic	2.3	Trigonometric equations

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Source: *VCE 2019, Mathematical Methods Exam 2, Section A, Q19; © VCAA*

Question 1 (1 mark)

Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \frac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \frac{5\pi}{4}$ in terms of α, is

A. 0
B. 2α
C. $\pi + 2\alpha$
D. $\frac{\pi}{2} + \alpha$
E. $\frac{3(\pi + \alpha)}{2}$

Source: *VCE 2017, Mathematical Methods Exam 2, Section A, Q12; © VCAA*

Question 2 (1 mark)

The sum of the solution of $\sin(2x) = \frac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$.
The value of d could be

A. 0
B. $\frac{\pi}{6}$
C. $\frac{3\pi}{4}$
D. $\frac{7\pi}{6}$
E. $\frac{3\pi}{2}$

Topic 2 Subtopic 2.3 Trigonometric equations

Source: *VCE 2014, Mathematical Methods (CAS) Exam 1, Q3; © VCAA*

Question 3 (2 marks)

Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \le x \le \pi$.

Question 4 (1 mark)

Determine the values of θ for $\cos^2(\theta) = 1$ in the interval $\theta \in [0, 2\pi]$.

Question 5 (2 marks)

Show that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ for $\theta = \frac{5\pi}{6}$.

Question 6 (1 mark)

The number of different values of x that satisfy the equation $3\sin(4x) = 1$ for $x \in [0, 2\pi]$ is

A. 1
B. 2
C. 4
D. 6
E. 8

Question 7 (1 mark)

The number of solutions of the equation $\cos(3x) = -1$ for $x \in [0, 2\pi]$ is

A. 0
B. 1
C. 2
D. 3
E. 6

Question 8 (1 mark)

The sum of solutions of the equation $\sin(4x) = 0.5$ for is $x \in \left[0, \frac{\pi}{2}\right]$ is

A. $\frac{\pi}{24}$
B. $\frac{\pi}{4}$
C. $\frac{3\pi}{24}$
D. π
E. $\frac{3\pi}{2}$

Question 9 (1 mark)

Solve the equation $\sin\left(\frac{3x}{2}\right) = \frac{1}{2}$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Question 10 (1 mark)

Solve the equation $\tan(2x) = \frac{1}{\sqrt{3}}$ for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$.

Question 11 (2 marks)

Solve $\sqrt{3}\cos(2x) = \sin(2x)$ for $x \in [0, 2\pi]$.

Question 12 (2 marks)

Find the value of a if the graphs of $y = \sin(x)$ and $y = a\cos(x)$ intersect at point $x = \dfrac{\pi}{3}$.

Question 13 (1 mark)

The sum of the solutions of the equation $\sqrt{3}\sin(2x) + \cos(2x) = 0$ over $0 \le x \le 2\pi$ is equal to

A. $\dfrac{11\pi}{3}$

B. $\dfrac{13\pi}{3}$

C. $\dfrac{14\pi}{3}$

D. 4π

E. 5π

Topic	2	Trigonometric (circular) functions
Subtopic	2.4	General solutions of trigonometric equations

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***Source:** VCE 2020, Mathematical Methods Exam 2, Section A, Q4; © VCAA*

Question 1 (1 mark)

The solutions of the equation $2\cos\left(2x - \frac{\pi}{3}\right) + 1 = 0$ are

A. $x = \frac{\pi(6k-2)}{6}$ or $x = \frac{\pi(6k-2)}{6}$, for $k \in z$

B. $x = \frac{\pi(6k-2)}{6}$ or $x = \frac{\pi(6k+5)}{6}$, for $k \in z$

C. $x = \frac{\pi(6k-1)}{6}$ or $x = \frac{\pi(6k+2)}{6}$, for $k \in z$

D. $x = \frac{\pi(6k-1)}{6}$ or $x = \frac{\pi(6k+3)}{6}$, for $k \in z$

E. $x = \pi$ or $x = \frac{\pi(6k+2)}{6}$, for $k \in z$

Question 2 (2 marks)

Find the general solution of $\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

Question 3 (1 mark)

$x = n\pi - \frac{\pi}{4},\ n \in z$ is the general solution to the equation

A. $\sin(2x) = 1$

B. $\sin(x) = 1$

C. $\sin(2x) = -1$

D. $\sin(x) = -1$

E. $\cos(x) = 1$

Question 4 (2 marks)

Find the general solution of $\sin(3x) = \cos(3x)$.

Question 5 (1 mark)

The general solution to the equation $\cos(x) = \frac{1}{\sqrt{2}}$ is

A. $x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{7\pi}{4} + 2n\pi,\ n \in Z$

B. $x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{5\pi}{4} + 2n\pi,\ n \in Z$

C. $x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{3\pi}{4} + 2n\pi,\ n \in Z$

D. $x = \frac{\pi}{4} + 2n\pi,\ n \in Z$

E. $x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{7\pi}{2} + 2n\pi,\ n \in Z$

Question 6 (1 mark)

The general solution to the equation $\tan\left(2x - \frac{\pi}{3}\right) = 1$ is

A. $x = \frac{\pi}{12} + 2n\pi,\ n \in Z$

B. $x = \left(n\pi + \frac{\pi}{24}\right), n \in Z$

C. $x = \left(\frac{12n + 7}{24}\right)\pi, n \in Z$

D. $x = \left(\frac{7n + 7}{24}\right)\pi, n \in Z$

E. $x = \left(\frac{7n\pi + \pi}{24}\right), n \in Z$

Topic	2	Trigonometric (circular) functions
Subtopic	2.5	The sine and cosine functions

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Source: *VCE 2019, Mathematical Methods Exam 2, Section A, Q1; © VCAA*

Question 1 (1 mark)

Let $f: R \to R,\ f(x) = 3\sin\left(\frac{2x}{5}\right) - 2$.

The period and range of f are respectively

A. 5π and $[-3, 3]$

B. 5π and $[-5, 1]$

C. 5π and $[-1, 5]$

D. $\frac{5\pi}{2}$ and $[-5, 1]$

E. $\frac{5\pi}{2}$ and $[-3, 3]$

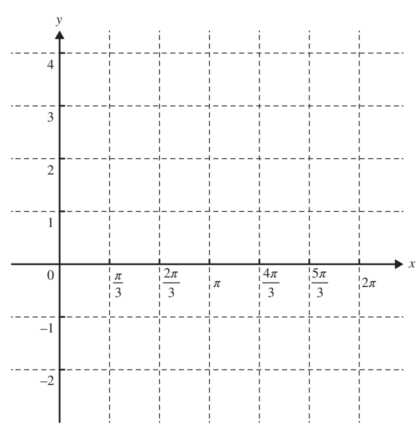
Source: *VCE 2018, Mathematical Methods Exam 1, Q3; © VCAA*

Question 2 (5 marks)

Let $f: [0, 2\pi] \to R,\ f(x) = 2\cos(x) + 1$.

a. Solve the equation $2\cos(x) + 1 = 0$ for $0 \le x \le 2\pi$. **(2 marks)**

b. Sketch the graph of the function f on the axes below. Label the end points and local minimum point with their coordinates. **(3 marks)**



Source: *VCE 2016, Mathematical Methods Exam 2, Section A, Q2; © VCAA*

Question 3 (1 mark)

Let $f: R \to R, f(\pi) = 1 - 2\cos\left(\frac{\pi x}{2}\right)$.

The period and range of this function are respectively

A. 4 and $[-2, 2]$
B. 4 and $[-1, 3]$
C. 1 and $[-1, 3]$
D. 4π and $[-1, 3]$
E. 4π and $[-2, 2]$

Question 4 (3 marks)

Find the sum of the solutions of the equation $2\sin(2x) = 1$ for $x \in [-\pi, \pi]$.

Question 5 (1 mark)

A trigonometric function is given by $f: R \to R$ where $f(x) = -4\sin\left(\frac{\pi x}{4}\right)$ where the amplitude and period respectively are

A. $-4, 4$
B. $4, \frac{\pi}{4}$
C. $-4, \frac{\pi}{4}$
D. $4, 4$
E. $4, 8$

Question 6 (1 mark)

A trigonometric function is given by $f: R \to R$ where $f(x) = 100\sin(10x)$.

Which of the following statements is **false**?

A. The domain is R and the range is $[-100, 100]$.

B. The amplitude is 100.

C. The period is $\frac{\pi}{10}$.

D. The function is a many-to-one function.

E. The graph crosses the x-axis at $x = \frac{k\pi}{10}$ where $k \in Z$.

Question 7 (1 mark)

A trigonometric function is given by $f: R \to R$ where $f(x) = -5\cos\left(\frac{\pi x}{5}\right)$.

The amplitude and period of f are respectively

A. $-5, \frac{\pi}{5}$

B. $5, \frac{\pi}{5}$

C. $-5, 10$

D. $5, 10$

E. $10, \frac{\pi}{5}$

Question 8 (1 mark)

A trigonometric function is given by $f: R \to R$ where $f(x) = -50\cos\left(\frac{x}{5}\right)$.

Which of the following statements is **false**?

A. The domain is R and the range is $[-50, 50]$.

B. The amplitude is 50.

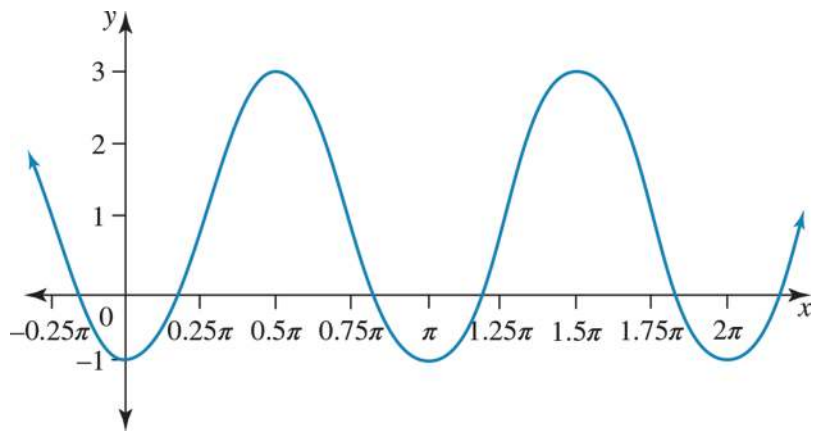
C. The period is 10π.

D. The function is a many-to-one function.

E. The graph crosses the x-axis at $x = \frac{5k\pi}{2}$ where $k \in Z$.

Question 9 (1 mark)

If the equation of the graph shown below is $y = a\cos(nx) + c$, then



A. $a = -2, n = \frac{1}{2}$ and $c = 1$

B. $a = -2, n = 2$ and $c = 1$

C. $a = 2, n = 2$ and $c = 1$

D. $a = 2, n = \frac{1}{2}$ and $c = 1$

E. $a = 2, n = \frac{1}{2}$ and $c = -1$

Question 10 (1 mark)

Let $f(x) = b\left[\frac{1}{2} + \frac{1}{2}\sin\left(\frac{\pi(4x - d)}{2d}\right)\right]$ and $g(x) = \frac{1}{20}(1 - \cos(4\pi x))$. If $f(x) = g(x)$, the values of b and d are

A. $b = \frac{1}{40}, d = \frac{1}{2}$

B. $b = \frac{1}{2}, d = \frac{1}{10}$

C. $b = \frac{1}{20}, d = \frac{1}{10}$

D. $b = \frac{1}{4}, d = \frac{1}{2}$

E. $b = \frac{1}{10}, d = \frac{1}{2}$

Question 11 (1 mark)

A sine function f has an amplitude of 2 and a period of $\frac{1}{5}$. The rule for f could be

A. $f(x) = 2\sin\left(\frac{\pi x}{10}\right)$

B. $f(x) = 2\sin(10\pi x)$

C. $f(x) = 2\sin(10x)$

D. $f(x) = \sin\left(\frac{\pi x}{10}\right)$

E. $f(x) = \sin(10\pi x)$

Question 12 (1 mark)

State the types of transformations used to transform the graph of $y = 2\sin(x)$ to the graph of

$y = 2\sin\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) + 2.$

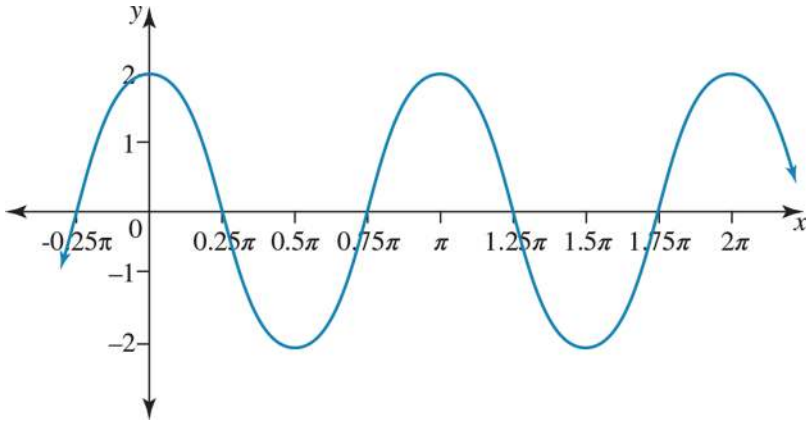
Question 13 (1 mark)

State the type of transformations to transform the graph of $y = \cos(x)$ to the graph of

$y = 3\cos\left(2\left(x + \frac{\pi}{2}\right)\right).$

Question 14 (1 mark)

The graph of $y = \cos(x)$ is transformed into the graph below. Which of the following statements describes the correct transformations?



A. A dilation by a factor of 2 parallel to the y-axis and a dilation by a factor of 2 parallel to the x-axis

B. A dilation by a factor of 2 parallel to the y-axis and a dilation by a factor of $\frac{1}{2}$ parallel to the x-axis

C. A dilation by a factor of 2 away from the y-axis and a dilation by a factor of 2 away from the x-axis

D. A dilation by a factor of $\frac{1}{2}$ away from the y-axis and a dilation by a factor of $\frac{1}{2}$ away from the x-axis

E. A dilation by a factor of $\frac{1}{2}$ parallel to the x-axis and a dilation by a factor of $\frac{1}{2}$ away from the x-axis

Question 15 (1 mark)

The graph of $y = \cos(x)$ is transformed to the graph of $y = \cos\left(2x - \frac{\pi}{3}\right)$ by

A. a dilation by a scale factor of 2 units from the y-axis, followed by a horizontal translation of $\frac{\pi}{6}$ units to the right parallel to the x-axis.

B. a dilation by a scale factor of 2 units from the x-axis, followed by a horizontal translation of $\frac{\pi}{6}$ units to the right parallel to the x-axis.

C. a dilation by a scale factor of $\frac{1}{2}$ units parallel to the x-axis, followed by a horizontal translation of $\frac{\pi}{6}$ units to the right parallel to the x-axis.

D. a dilation by a scale factor of 2 units from the y-axis, followed by a horizontal translation of $\frac{\pi}{3}$ units to the left parallel to the x-axis.

E. a dilation by a scale factor of $\frac{1}{2}$ units from the y-axis, followed by a horizontal translation of $\frac{\pi}{3}$ units to the right parallel to the x-axis.

Topic	2	Trigonometric (circular) functions
Subtopic	2.6	The tangent function

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Source: *VCE 2021, Mathematical Methods Exam 2, Section A, Q1: © VCAA*

Question 1 (1 mark)

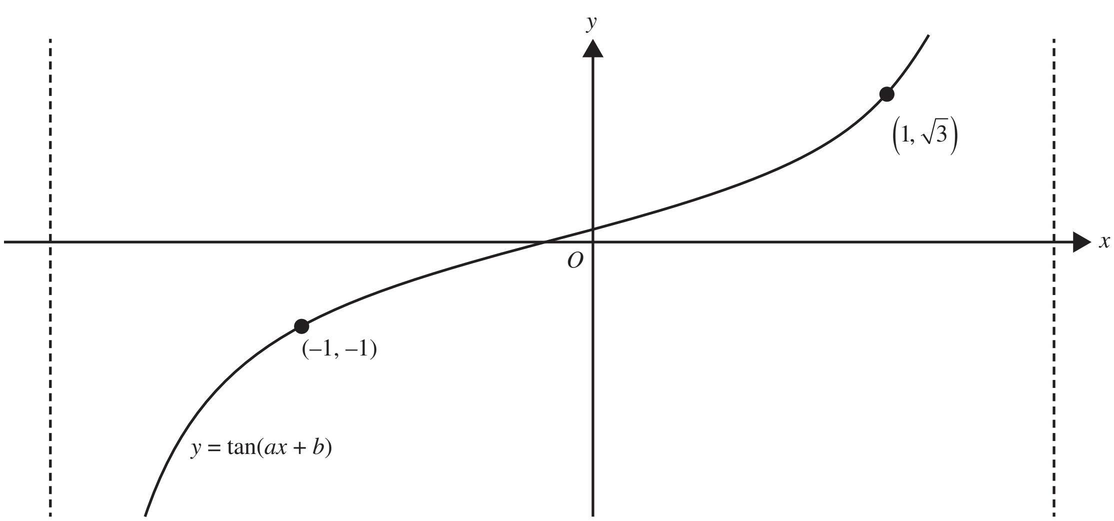
The period of the function with rule $y = \tan\left(\frac{\pi x}{2}\right)$ is

A. 1
B. 2
C. 4
D. 2π
E. 4π

Source: *VCE 2020, Mathematical Methods Exam 1, Q3; © VCAA*

Question 2 (3 marks)

Shown below is part of the graph of a period of the function of the form $y = \tan(ax + b)$.



The graph is continuous for $x \in [-1, 1]$.
Find the value of a and the value of b, where $a > 0$ and $0 < b < 1$.

Source: *VCE 2018, Mathematical Methods Exam 2, Section A, Q11; © VCAA.*

Question 3 (1 mark)

The graph of $y = \tan(ax)$, where $a \in R^+$, has a vertical asymptote $x = 3\pi$ and has exactly one x-intercept in the region $(0, 3\pi)$. The value of a is

A. $\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. 1

E. 2

Question 4 (1 mark)

The function with rule $f(x) = -3\tan(2\pi x)$ has a period of

A. $\frac{2}{\pi}$

B. 2

C. $\frac{1}{2}$

D. $\frac{1}{4}$

E. 2π

Question 5 (1 mark)

The graph of $y = \tan\left(\frac{bx}{3}\right)$, where $b \neq 0$, has a vertical asymptote at

A. $x = 0$

B. $x = \frac{3\pi}{b}$

C. $x = \frac{2\pi}{3b}$

D. $x = \frac{\pi}{2b}$

E. $x = \frac{3\pi}{2b}$

Question 6 (2 marks)

Determine the vertical asymptotes for the function $f: (-\pi, \pi) \rightarrow R, f(x) = -\tan\left(x - \frac{\pi}{3}\right)$.

Question 7 (2 marks)

Answer the following.

a. Determine the period of the function $f(x) = \tan\left(\frac{\pi}{4}\right)$. **(1 mark)**

b. Sketch the graph of the function with rule $f: [0, 2\pi] \rightarrow R, f(x) = \tan\left(\frac{\pi}{4}\right)$. **(1 mark)**

Question 8 (1 mark)

For the graph of $y = 500\tan\left(\frac{x}{5}\right)$, which of the following options is correct?

A. The domain is $[-500,\ 500]$ and the period is 10π.
B. The domain is $[-500,\ 500]$ and the period is 5π.
C. The domain is R and the period is 10π.
D. The domain is R and the period is 5π.
E. The range is R and the period is 5π.

Question 9 (1 mark)

The graph of $y = \tan(x)$ is transformed to the graph of $y = \tan\left(2x + \frac{\pi}{4}\right)$ by

A. a dilation by a scale factor of 2 units from the y-axis, followed by a horizontal translation of $\frac{\pi}{8}$ units to the left parallel to the x-axis.

B. a dilation by a scale factor of 2 units from the x-axis, followed by a horizontal translation of $\frac{\pi}{8}$ units to the right parallel to the x-axis.

C. a dilation by a scale factor of $\frac{1}{2}$ units parallel to the x-axis, followed by a horizontal translation of $\frac{\pi}{8}$ units to the left parallel to the x-axis.

D. a dilation by a scale factor of 2 units from the y-axis, followed by a horizontal translation of $\frac{\pi}{4}$ units to the left parallel to the x-axis.

E. a dilation by a scale factor of $\frac{1}{2}$ units from the y-axis, followed by a horizontal translation of $\frac{\pi}{4}$ units to the left parallel to the x-axis.

Question 10 (1 mark)

The graph of $y = \tan(x)$ is transformed to the graph of $y = 4 - 4\tan(4x)$ by

A. a reflection in the x-axis, a dilation by a scale factor of 4 units from the y-axis, followed by a dilation by a scale factor of 4 units up and away from the x-axis.

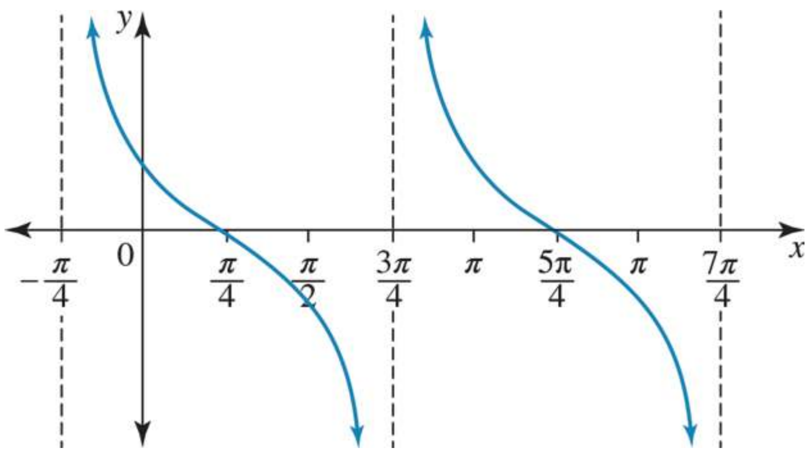
B. a reflection in the x-axis, a dilation by a scale factor of 4 units from the y-axis, followed by a dilation by a scale factor of $\frac{1}{4}$ from the x-axis.

C. a reflection in the x-axis, a dilation by a scale factor of $\frac{1}{4}$ units from the y-axis, followed by a dilation by a scale factor of 4 units up and away from the x-axis.

D. a reflection in the y-axis, a dilation by a scale factor of $\frac{1}{4}$ units from the y-axis, followed by a dilation by a scale factor of 4 units up and away from the x-axis.

E. a reflection in the y-axis, a dilation by a scale factor of 4 units from the y-axis, followed by a dilation by a scale factor of $\frac{1}{4}$ from the x-axis.

Question 11 (1 mark)
The diagram shows two cycles of a circular function.



The period of the function is:

A. $\frac{\pi}{2}$

B. $\frac{3\pi}{4}$

C. π

D. $\frac{7\pi}{4}$

E. 2

Topic	2	Trigonometric (circular) functions
Subtopic	2.7	Modelling and applications

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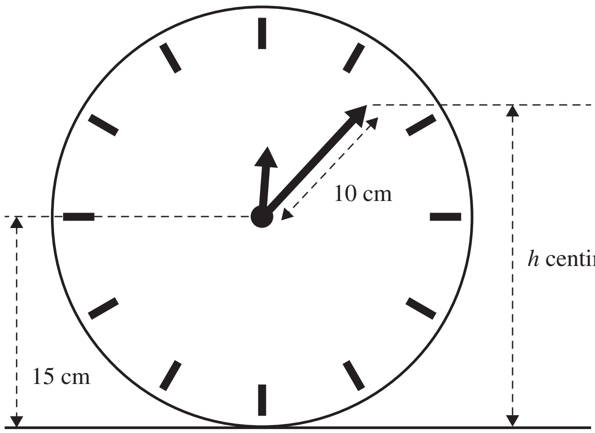
Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q12; © VCAA*

Question 1 (1 mark)

A clock has a minute hand that is 10 cm long and a clock face with a radius of 15 cm, as shown below.



At 12.00 noon, both hands of the clock point vertically upwards and the tip of the minute hand is at its maximum distance above the base of the clock face.

The height, h centimetres, of the tip of the minute hand above the base of the clock face t minutes after 12.00 noon is given by

A. $h(t) = 15 + 10\sin\left(\dfrac{\pi t}{30}\right)$

B. $h(t) = 15 - 10\sin\left(\dfrac{\pi t}{30}\right)$

C. $h(t) = 15 + 10\sin\left(\dfrac{\pi t}{60}\right)$

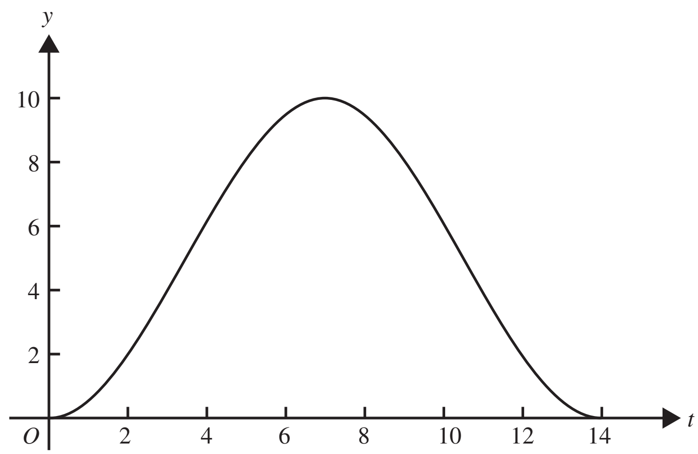
D. $h(t) = 15 + 10\cos\left(\dfrac{\pi t}{60}\right)$

E. $h(t) = 15 + 10\cos\left(\dfrac{\pi t}{30}\right)$

***Source:** VCE 2016, Mathematical Methods Exam 2, Section A, Q8; © VCAA*

Question 2 (1 mark)

The UV index, y, for a summer day in Melbourne is illustrated in the graph below, where t is the number of hours after 6 am.



The graph is most likely to be the graph of

A. $y = 5 + 5\cos\left(\frac{\pi t}{7}\right)$

B. $y = 5 - 5\cos\left(\frac{\pi t}{7}\right)$

C. $y = 5 + 5\cos\left(\frac{\pi t}{14}\right)$

D. $y = 5 - 5\cos\left(\frac{\pi t}{14}\right)$

E. $y = 5 + 5\sin\left(\frac{\pi t}{14}\right)$

Source: VCE 2015, Mathematical Methods (CAS) Exam 1 Q5; © VCAA

Question 3 (3 marks)

On any given day, the depth of water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right),\ 0 \le t \le 24$$

where h is the depth of water, in metres, and t is the time, in hours, after 6 am.

a. Find the minimum depth of the water in the river. **(1 mark)**

b. Find the values of t for which $h(t) = 10$. **(2 marks)**

Source: VCE 2014, Mathematical Methods (CAS) Exam 2, Section 2, Q1; @ VCAA

Question 4 (7 marks)

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n. **(2 marks)**

b. Find the maximum and minimum populations of wombats in this location. **(2 marks)**

c. Find $n(10)$. **(1 mark)**

d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$. **(2 marks)**

Question 5 (1 mark)

The number of birds, n, in a colony is given by the function $n(t) = 100\sin\left(\dfrac{\pi(t+12)}{12}\right) + 300$, where t is the time in months, and $t \geq 0$.

Which of the following statements is **false**?

A. The maximum number of birds is 400.
B. The minimum number of birds is 200.
C. The initial number of birds is 100.
D. After one year, the number of birds is 274.
E. The number of birds in the colony increases and decreases.

Topic	2	Trigonometric (circular) functions
Subtopic	2.8	Review

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Source: *VCE 2021, Mathematical Methods Exam 1, Q3; © VCAA*

Question 1 (5 marks)

Consider the function $g: R \rightarrow R, g(x) = 2\sin(2x)$.

a. State the range of g. **(1 mark)**

b. State the period of g. **(1 mark)**

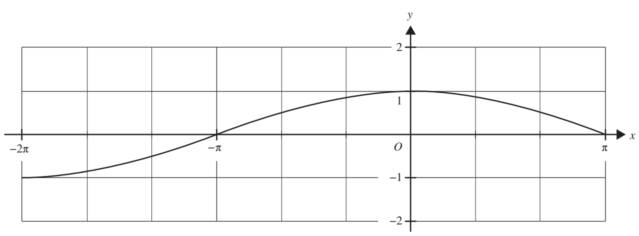
c. Solve $2\sin(2x) = \sqrt{3}$ for $x \in R$. **(3 marks)**

Source: *VCE 2019, Mathematical Methods Exam 1, Q4; © VCAA*

Question 2 (4 marks)

a. Solve $1 - \cos\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)$ for $x \in [-2\pi, \pi]$. **(2 marks)**

b. The function $f: [-2\pi, \pi] \rightarrow R, f(x) = \cos\left(\frac{x}{2}\right)$ is shown on the axes below.



Let $g: [-2\pi, \pi] \rightarrow R, g(x) = 1 - f(x)$.

Sketch the graph of g on the axes above. Label all points of intersection of the graphs of f and g, and the endpoints of g, with their coordinates. **(2 marks)**

***Source:** VCE 2018, Mathematical Methods Exam 2, Section A, Q1; © VCAA*

Question 3 (1 mark)

Let $f: R \to R,\ f(x) = 4\cos\left(\dfrac{2\pi x}{3}\right) + 1$.

The period of this function is

A. 1
B. 2
C. 3
D. 4
E. 5

***Source:** VCE 2017, Mathematical Methods Exam 1, Q6; © VCAA*

Question 4 (3 marks)

Let $(\tan(\theta) - 1)\left(\sin(\theta) - \sqrt{3}\cos(\theta)\right)\left(\sin(\theta) + \sqrt{3}\cos(\theta)\right) = 0$.

a. State all possible values of $\tan(\theta)$. **(1 mark)**

b. Hence, find all possible solutions for $(\tan(\theta) - 1)\left(\sin^2(\theta) - 3\cos^2(\theta)\right) = 0$, where $0 \le \theta \le \pi$. **(2 marks)**

Source: *VCE 2015, Mathematical Methods (CAS) Exam 2, Section 1, Q1; © VCAA*

Question 5 (1 mark)

Let $f: R \to R,\ f(x) = 2\sin(3x) - 3$.

The period and range of this function are respectively

A. period $= \dfrac{2\pi}{3}$ and range $= [-5, -1]$

B. period $= \dfrac{2\pi}{3}$ and range $= [-2, 2]$

C. period $= \dfrac{\pi}{3}$ and range $= [-1, 5]$

D. period $= 3\pi$ and range $= [-1, 5]$

E. period $= 3\pi$ and range $= [-2, 2]$

Source: *VCE 2019, Mathematical Methods 2, Section A, Q.1; © VCAA*

Question 6 (1 mark)

Let $f: R \to R, f(x) = 3\sin\left(\dfrac{2x}{5}\right) - 2$

The period and range of f are respectively

A. 5π and $[-3, 3]$

B. 5π and $[-5, 1]$

C. 5π and $[-1, 5]$

D. $\dfrac{5\pi}{2}$ and $[-5, 1]$

E. $\dfrac{5\pi}{2}$ and $[-3, 3]$

Question 7 (1 mark)

If $\sin(x) = 0.6$ for $x \in \left[\frac{\pi}{2}, \quad \pi\right]$ then $\sin(2x)$ is

A. $\frac{24}{25}$

B. $-\frac{24}{5}$

C. $\frac{6}{25}$

D. $-\frac{8}{25}$

E. $-\frac{24}{25}$

Question 8 (1 mark)

$\cos(3x)\cos(2x) + \sin(3x)\sin(2x)$ simplified is

A. $\cos(5x)$

B. $\cos(x)$

C. $\sin(5x)$

D. $\sin(x)$

E. $\cos^2(6x)$

Question 9 (2 marks)

Given that $\cos(\theta) = \frac{5}{13}$ and θ is in the fourth quadrant, find the value of $\sin(\theta)$ using the Pythagorean identity.

Question 10 (1 mark)

Which of the following statements is true?

A. $\left(2\sin\left(\frac{x}{2}\right)\right)^2 + \left(2\cos\left(\frac{x}{2}\right)\right)^2 = 1$

B. $\left(2\sin\left(\frac{x^2}{2}\right)\right)^2 + \left(2\cos\left(\frac{x^2}{2}\right)\right)^2 = 1$

C. $\left(2\sin^2\left(\frac{x}{2}\right)\right)^2 + \left(2\cos^2\left(\frac{x}{2}\right)\right)^2 = 1$

D. $\left(\sin\left(\frac{x}{2}\right)\right)^2 + \left(\cos\left(\frac{x}{2}\right)\right)^2 = 1$

E. $\sqrt{\sin^2\left(\frac{x^2}{2}\right)} + \sqrt{\cos^2\left(\frac{x^2}{2}\right)} = 1$

Question 11 (2 marks)

Find the value of $\tan(\theta)$ if $\sec(\theta) = 2$.

Question 12 (1 mark)

$\frac{\sin(A)}{1+\cos(A)} + \frac{1+\cos(A)}{\sin(A)}$ is equal to

A. $2\sin(A)$

B. $2\cos(A)$

C. 2

D. $\frac{2}{\sin(A)}$

E. $\frac{2}{\cos(A)}$

Question 13 (1 mark)

$\dfrac{1 - \tan^2(A)}{1 + \tan^2(A)}$ is equal to

A. $\cos^2(A) - \sin^2(A)$
B. $\sin^2(A) - \cos^2(A)$
C. 1
D. -1
E. $\dfrac{1 - \sin^2(A)}{1 + \sin^2(A)}$

Question 14 (1 mark)

$\cos\left(\dfrac{3\pi}{2} - \theta\right) + \sin\left(\dfrac{3\pi}{2} - \theta\right)$ is equal to

A. $\sin(\theta) - \cos(\theta)$
B. $\sin(\theta) + \cos(\theta)$
C. $\cos(\theta) - \sin(\theta)$
D. $-\sin(\theta) - \cos(\theta)$
E. 0

Question 15 (1 mark)

If $\cos(\alpha) = 0.6$, find the value of $\sin\left(\dfrac{3\pi}{2} + \alpha\right)$.

Question 16 (1 mark)

Which of the following statements is **false**?

A. $\sin\left(\frac{\pi}{2}+\theta\right)+\sin\left(\frac{3\pi}{2}+\theta\right)=0$

B. $\cos\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{3\pi}{2}-\theta\right)=0$

C. $\tan\left(\frac{\pi}{2}+\theta\right)+\tan\left(\frac{3\pi}{2}-\theta\right)=0$

D. $\sin\left(\frac{\pi}{2}-\theta\right)+\sin\left(\frac{3\pi}{2}-\theta\right)=0$

E. $\tan\left(\frac{\pi}{2}+\theta\right)+\tan\left(\frac{3\pi}{2}+\theta\right)=0$

Question 17 (1 mark)

Which of the following statements is **false**?

A. $\cos\left(\theta-\frac{\pi}{2}\right)=\sin(\theta)$

B. $\cos\left(\theta-\frac{3\pi}{2}\right)=-\sin(\theta)$

C. $\sin\left(\theta-\frac{\pi}{2}\right)=\cos(\theta)$

D. $\sin\left(\theta-\frac{3\pi}{2}\right)=\cos(\theta)$

E. $\tan\left(\theta-\frac{\pi}{2}\right)=\frac{-1}{\tan(\theta)}$

Question 18 (1 mark)

If $\tan(\theta)=0.5$, find the value of $\tan\left(\frac{\pi}{2}-\theta\right)$.

Answers and marking guide

2.2 Trigonometric symmetry properties

Question 1

$\sin\left(-\frac{4\pi}{3}\right) = -\sin\left(\frac{4\pi}{3}\right) = -\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$

The correct answer is **A**.

Question 2

Option B is false since $\cos^3(\pi) + \sin^3(\pi) = (-1)^3 + 0^3 = -1$. Therefore, A, C, D and E are true.
The correct answer is **B**.

Question 3

It can be seen than A, B, C and D are true. E is false since $\sin(\pi + \theta) + \sin(2\pi - \theta) = -2\sin(\theta)$.
The correct answer is **E**.

Question 4

$\cos(x) = \frac{3}{5} \quad x \in \left[\frac{3\pi}{2}, 2\pi\right]$

x is in the fourth quadrant $\sin(x) = -\frac{4}{5}$

$\sin^2(y) = \frac{25}{169} \quad x \in \left[\frac{3\pi}{2}, 2\pi\right]$

y is in the fourth quadrant $\sin(y) = -\frac{5}{13}$, $\cos(y) = \frac{12}{13}$

$\sin(x) + \cos(y) = -\frac{4}{5} + \frac{12}{13} = \frac{8}{65}$

Question 5

Options A, B, D, and E are true. C is false since $\tan\left(\frac{\pi}{2}\right) - \tan\left(\frac{3\pi}{2}\right)$ is undefined.

Question 6

Options A, B, C, and D are true, therefore E is false.

Question 7

Options A, B, D and E are true, therefore C is false.

Question 8

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\sin^2(\theta) = 1 - \left(\frac{-5}{13}\right)^2$ **[1 mark]**

$\sin^2(\theta) = \frac{144}{169}$

$\sin(\theta) = \frac{12}{13} \quad \text{since } \frac{\pi}{2} < \theta < \pi$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$= \frac{\frac{12}{13}}{-\frac{5}{13}}$$
$$= \frac{-12}{5} \quad \textbf{[1 mark]}$$

Question 9

$$\sin\left(-\frac{4\pi}{3}\right) = -\sin\left(\frac{4\pi}{3}\right) = -\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

Question 10

$$\tan\left(\frac{5\pi}{3}\right) - \sin\left(\frac{3\pi}{4}\right) = -\sqrt{3} - \frac{\sqrt{2}}{2}$$
$$= \frac{-2\sqrt{3} - \sqrt{3}}{2}$$

Question 11

Options A, B, C and E are true, therefore D is false.

$$\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = -\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{4\pi}{3}\right) = \frac{1}{2}$$

but $-\cos\left(\frac{5\pi}{3}\right) = -\frac{1}{2}$

Question 12

Options A, B, C, and E are true, D is false

$$\tan\left(-\frac{3\pi}{4}\right) = -\tan\left(\frac{3\pi}{4}\right) = 1 = \tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{5\pi}{4}\right) \quad \text{but} \ -\frac{\sin\left(-\frac{3\pi}{4}\right)}{\cos\left(-\frac{3\pi}{4}\right)} = -1$$

Question 13

a. $\sin(\pi + \theta) = \sin(\theta) = -0.25$

b. $\cos(-\theta) = \cos(\theta) = 0.67$

c. $\tan(2\pi - \theta) = -\tan(\theta) = -0.54$

Question 14

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right)$$
$$= \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

Question 15

Options A, B, D, and E are true, therefore C is false since $\cos\left(\frac{3\pi}{2}\right) = 0$.

2.3 Trigonometric equations

Question 1

$\tan(\alpha) = d, \quad \alpha = \tan^{-1}(d), \quad 0 < x < \frac{5\pi}{4}$

$\tan(2x) = d, \quad 0 < 2x < \frac{5\pi}{2}$

$2x = \alpha, \quad \pi + \alpha, \quad 2\pi + \alpha$

$$\sum_{i-1}^{3}(2x_i) = 3\pi + 3\alpha = 3(\pi + \alpha)$$

$$\sum_{i-1}^{3}(x_i) = \frac{3(\pi + \alpha)}{2}$$

The correct answer is **E**.

Question 2

$$\sin(2x) = \frac{\sqrt{3}}{2}$$

$$x = -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6} \cdots$$

Add solutions together until the sum is equal to $-\pi$:

$$-\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6} + \frac{\pi}{3} = -\pi$$

The solution $x = \frac{\pi}{3}$ is included in the sum but the solution $x = \frac{7\pi}{6}$ is not included in the sum; hence, $\frac{\pi}{3} \leq d < \frac{7\pi}{6}$.

The correct answer is **C**.

Question 3

$2\cos(2x) = -\sqrt{3}$ for $0 \leq x \leq \pi$

$$\begin{aligned}\cos(2x) &= -\frac{\sqrt{3}}{2}\\ 2x &= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\\ 2x &= \frac{5\pi}{6}, \frac{7\pi}{6}\\ x &= \frac{5\pi}{12}, \frac{7\pi}{12}\end{aligned}$$

Award 1 mark for solving the trigonometric equation.
Award 1 mark for both of the correct values.

Question 4

$\cos(\theta) = \pm 1$

$$\therefore \theta = 0, \text{ or } \theta = \pi, \text{ or } \theta = 2\pi$$

Question 5

LHS:

$$\begin{aligned}\sin(2\theta) &= \sin\left(2 \times \frac{5\pi}{6}\right)\\ &= \sin\left(\frac{5\pi}{3}\right)\\ &= \frac{-\sqrt{3}}{2}\end{aligned}$$

[1 mark]

RHS:

$$2\sin(\theta)\cos(\theta) = 2\sin\left(\frac{5\pi}{6}\right)\cos\left(\frac{5\pi}{6}\right)$$
$$= 2 \times \frac{1}{2} \times \frac{-\sqrt{3}}{2}$$
$$= \frac{-\sqrt{3}}{2}$$

[1 mark]

∴ LHS = RHS

∴ shown.

Question 6

Use a CAS calculator to determine the number of solutions, or another method.

There are 8 values of x that satisfy $3\sin(4x) = 1, x \in [0, 2\pi]$.

Question 7

Use a CAS calculator to determine the number of solutions, or another method.

There are 3 values of x that satisfy $\cos(3x) = -1,\ x \in [0,\ 2\pi]$.

Question 8

$$4x = \sin^{-1}(0.5)$$
$$x = \frac{\pi}{24}, \quad \frac{5\pi}{24}$$
$$\text{Sum} = \frac{\pi}{24} + \frac{5\pi}{24}$$
$$= \frac{\pi}{4}$$

Question 9

$$\frac{3x}{2} = \sin^{-1}\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{6}$$
$$x = \frac{\pi}{9}$$

Question 10

$$2x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{\pi}{6}, \quad \frac{7\pi}{6}$$
$$x = \frac{\pi}{12}, \quad \frac{7\pi}{12}$$

Question 11

$\sqrt{3} = \tan(2x)$ **[1 mark]**

$$2x = \tan^{-1}\left(\sqrt{3}\right)$$
$$= \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$$
$$x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$$

[1 mark]

Question 12

$\sin(x) = a\cos(x)$

$$a = \tan(x) \qquad \textbf{[1 mark]}$$
$$= \tan\left(\frac{\pi}{3}\right)$$
$$= \sqrt{3} \qquad \textbf{[1 mark]}$$

Question 13

$$\sqrt{3}\sin(2x) + \cos(2x) = 0$$
$$\sqrt{3}\sin(2x) = -\cos(2x)$$
$$\tan(2x) = -\frac{1}{\sqrt{3}}$$

If $x \in [0,\ 2\pi]$, then $2x \in [0,\ 4\pi]$.

$$2x = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

Tan is negative, so the angles are in the 2nd and 4th quadrants.
Go around the unit circle twice by adding 2π.

$$2x = \pi - \frac{\pi}{6},\ 2\pi - \frac{\pi}{6},\ 3\pi - \frac{\pi}{6},\ 4\pi - \frac{\pi}{6},\ 5\pi - \frac{\pi}{6}$$
$$= \frac{5\pi}{6},\ \frac{11\pi}{6},\ \frac{17\pi}{6},\ \frac{23\pi}{6}$$
$$x = \frac{5\pi}{12},\ \frac{11\pi}{12},\ \frac{17\pi}{12},\ \frac{23\pi}{12} \Rightarrow 4 \text{ solutions}$$
$$\text{Sum} = \frac{56\pi}{12} = \frac{14\pi}{3}$$

2.4 General solutions of trigonometric equations

Question 1

$$2\cos\left(2x - \frac{\pi}{3}\right) + 1 = 0$$
$$2\cos\left(2x - \frac{\pi}{3}\right) = -1$$
$$\cos\left(2x - \frac{\pi}{3}\right) = -\frac{1}{2}$$
$$2x - \frac{\pi}{3} = 2k\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$$
$$2x - \frac{\pi}{3} = 2k\pi \pm \frac{2\pi}{3}$$
$$2x = 2k\pi + \pi,\ 2k\pi - \frac{\pi}{3}$$
$$2x = \pi(2k+1),\ \frac{\pi}{3}(6k-1)$$
$$x = \frac{\pi}{2}(2k+1),\ \frac{\pi}{6}(6k-1)$$
$$x = \frac{\pi}{6}(6k+3),\ \frac{\pi}{6}(6k-1), k \in Z$$

The correct answer is **D**.

Question 2

$x - \frac{\pi}{3} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2\pi n$ and $x - \frac{\pi}{3} = (2n+1)\pi - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, where $n \in Z$ **[1 mark]**

$x - \frac{\pi}{3} = \frac{\pi}{3} + 2\pi n$ and $x - \frac{\pi}{3} = (2n+1)\pi - \frac{\pi}{3}$, where $n \in Z$

$x = \frac{2\pi}{3} + 2\pi n$ and $x = (2n+1)\pi$ where $n \in Z$ **[1 mark]**

Question 3

$\sin(2x) = -1$

$$2x = \sin^{-1}(-1)$$
$$2x = -\frac{\pi}{2} + 2n\pi, n \in Z$$
$$2x = \frac{-\pi + 4n\pi}{2}, n \in Z$$
$$x = \frac{-\pi + 4n\pi}{4}, n \in Z$$
$$x = n\pi - \frac{\pi}{4}, n \in Z$$

The correct answer is **C**.

Question 4

$$\sin(3x) = \cos(3x)$$
$$\frac{\sin(3x)}{\cos(3x)} = \frac{\cos(3x)}{\cos(3x)}$$
$$\tan(3x) = 1$$

$3x = \tan^{-1}(1) + \pi n$, where $n \in Z$ **[1 mark]**

$$= \frac{\pi}{4} + \pi n$$

$x = \frac{(4n+1)\pi}{12}, n \in Z$ **[1 mark]**

Question 5

$$x = 2n\pi \pm \cos^{-1}\left(\frac{1}{\sqrt{2}}\right), n \in Z$$
$$x = \frac{\pi}{4} + 2n\pi \quad \text{or} \quad x = \frac{7\pi}{4} + 2n\pi, n \in Z$$

Question 6

$$2x - \frac{\pi}{3} = \tan^{-1}(1) + \pi n, \quad \text{where } n \in Z$$
$$2x = \frac{\pi}{4} + \pi n + \frac{\pi}{3}$$
$$= \frac{\pi(12n+7)}{12}$$
$$x = \left(\frac{12n+7}{24}\right)\pi, n \in Z$$

2.5 The sine and cosine functions

Question 1

$f: R \rightarrow R, f(x) = 3\sin\left(\frac{2x}{5}\right) - 2$

Period $T = \frac{2\pi}{\frac{2}{5}} = 5\pi$

Range $[-3-2, 3-2] = [-5, 1]$

The correct answer is **B**.

Question 2

a $2\cos(x) + 1 = 0$

$$2\cos(x) = -1$$

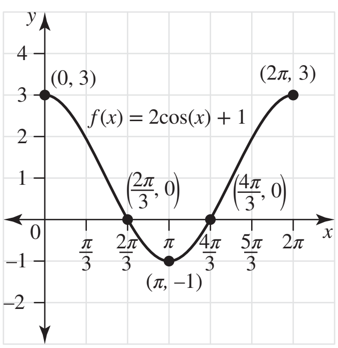
$$\cos(x) = -\frac{1}{2}$$

$$x = \pi + \frac{\pi}{3}, \pi - \frac{\pi}{3}$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Award 1 mark for finding one correct trigonometric value.
Award 1 mark for both correct answers.

b



Award 1 mark for correct minimum and shape.
Award 1 mark for correct axial intercepts.
Award 1 mark for correct end points.

Question 3

$f: R \rightarrow R, f(x) = 1 - 2\cos\left(\frac{\pi x}{2}\right)$

$T = \frac{2\pi}{\frac{\pi}{2}} = 4$, range $[1-2, 1+2] = [-1, 3]$

The correct answer is **B**.

Question 4

$$\sin(2x) = \frac{1}{2}$$
$$2x = \sin^{-1}\left(\frac{1}{2}\right)$$
$$2x = \frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6} \qquad \textbf{[1 mark]}$$
$$x = \frac{-11\pi}{12}, \frac{-7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12} \qquad \textbf{[1 mark]}$$
$$\text{Sum of solutions} = \frac{-11\pi}{12} + \frac{-7\pi}{12} + \frac{\pi}{12} + \frac{5\pi}{12}$$
$$= \frac{-12\pi}{12}$$
$$= -\pi \qquad \textbf{[1 mark]}$$

Question 5

The amplitude is 4 and the period is

$$T = \frac{2\pi}{\frac{\pi}{4}} = 8$$

Question 6

The period is $T = \frac{2\pi}{10} = \frac{\pi}{5}$. Therefore, 'The period is $\frac{\pi}{10}$ ' is false. All other options are true.

Question 7

The amplitude is 5 and the period is

$$T = \frac{2\pi}{\frac{\pi}{5}} = 10$$

Question 8

Options A, B, C, and D are true.

The graph crosses the x-axis at

$$\cos\left(\frac{x}{5}\right) = 0 \quad \Rightarrow \frac{x}{5} = (2k+1)\frac{\pi}{2}$$

when $x = \frac{5(2k+1)\pi}{2}$

when $k \in Z$

Question 9

The graph is obtained from the graph of $y = \cos(x)$ by a reflection in the x -axis, a dilation by a scale factor of 2 units away from the x-axis, followed by a dilation by a scale factor of $\frac{1}{2}$ units away from the y -axis, and then a vertical translation of 1 unit up and away from the x-axis. The translated graph is $y = -2\cos(2x) + 1$, so $a = -2$ $n = 2$ and $c = 1$.

Question 10

$$f(x) = \frac{b}{2} + \frac{b}{2}\sin\left(\frac{2\pi x}{d} - \frac{\pi}{2}\right)$$
$$g(x) = \frac{1}{20} - \frac{1}{20}\cos(4\pi x)$$
$$\frac{1}{20} = \frac{b}{2}$$
$$b = \frac{1}{10}$$

$\sin\left(\frac{2\pi x}{d} - \frac{\pi}{2}\right) = -\cos(4\pi x)$

$-\cos(x)$ is $\sin(x)$ translated $\frac{\pi}{2}$ units right

$\therefore \cos\left(\frac{2\pi x}{d}\right) = \cos(4\pi x)$

$\frac{2\pi}{d} = 4\pi$

$d = \frac{1}{2}$

Question 11

$f(x) = 2\sin(10\pi x)$

Question 12

The transformations used are:
- dilation of factor 2 from the y-axis
- dilation of factor 2 from the x-axis
- translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis.
- translation of 2 units in the positive direction of the y-axis.

Award 1 mark for all correct transformations.

Question 13

The transformations used are:
- dilation of factor 3 from the x-axis
- dilation of factor $\frac{1}{2}$ from the y-axis
- horizontal translation of $\frac{\pi}{2}$ units in the negative direction.

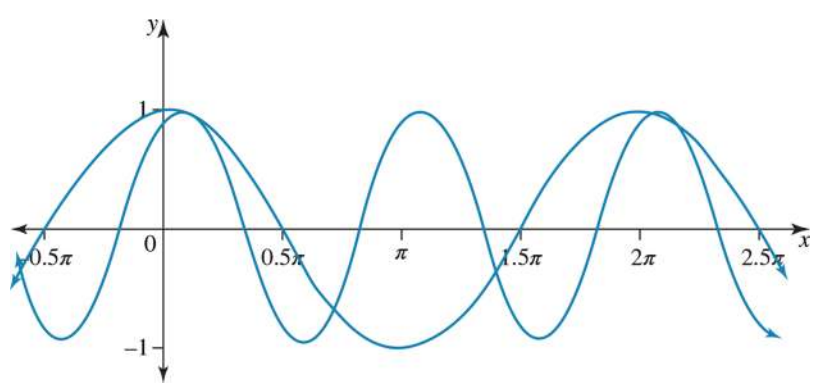
Award 1 mark for all correct transformations.

Question 14

The graph is $y = 2\cos(2x)$, which has an amplitude of 2 and a period of $T = \frac{2\pi}{2} = \pi$.

Question 15

$y = \cos\left(2x - \frac{\pi}{3}\right) = \cos\left(2\left(x - \frac{\pi}{6}\right)\right)$



2.6 The tangent function

Question 1

Period $= \frac{\pi}{\frac{\pi}{2}} = 2$

The correct answer is **B**.

Question 2

$y = \tan(ax + b)$

$P\left(1, \sqrt{3}\right)$

$\sqrt{3} = \tan(a + b)$

$a + b = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} \quad [1]$

$Q(-1, -1)$

$-1 = \tan(b - a)$

$b - a = \tan^{-1}(-1) = -\frac{\pi}{4} \quad [2]$

[1] + [2]:

$2b = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

$b = \frac{\pi}{24}, a = \frac{\pi}{3} - \frac{\pi}{24} = \frac{7\pi}{24}$

Award 1 mark for two correct equations.
Award 1 mark for the correct value of a.
Award 1 mark for the correct value of b.

Question 3

$y = \tan(ax)$

The period of $y = \tan(ax)$ is $\frac{\pi}{a}$.

x-intercepts occur at multiples of the period, $\frac{n\pi}{a}, n \in Z$.

When $a = \frac{1}{2}$ the period is 2π, meaning that the x-intercepts occur at $0, 2\pi, 4\pi$.

$\therefore a = \frac{1}{2}$

The correct answer is **C**.

Question 4

$f(x) = -3\tan(2\pi x)$

Period: $T = \frac{\pi}{n}$

$= \frac{\pi}{2\pi}$

$= \frac{1}{2}$

Question 5

$y = \tan\left(\frac{bx}{3}\right)$ has vertical asymptotes when $\frac{bx}{3} = \frac{\pi}{3}$ at $x = \frac{3\pi}{2b}$.

Question 6

Vertical asymptotes at:

$x = \frac{\pi}{2} + \frac{\pi}{3}$ and $x = \frac{-\pi}{2} + \frac{\pi}{3}$

$x = \frac{5\pi}{6}$ **[1 mark]**

$x = \frac{-\pi}{6}$ **[1 mark]**

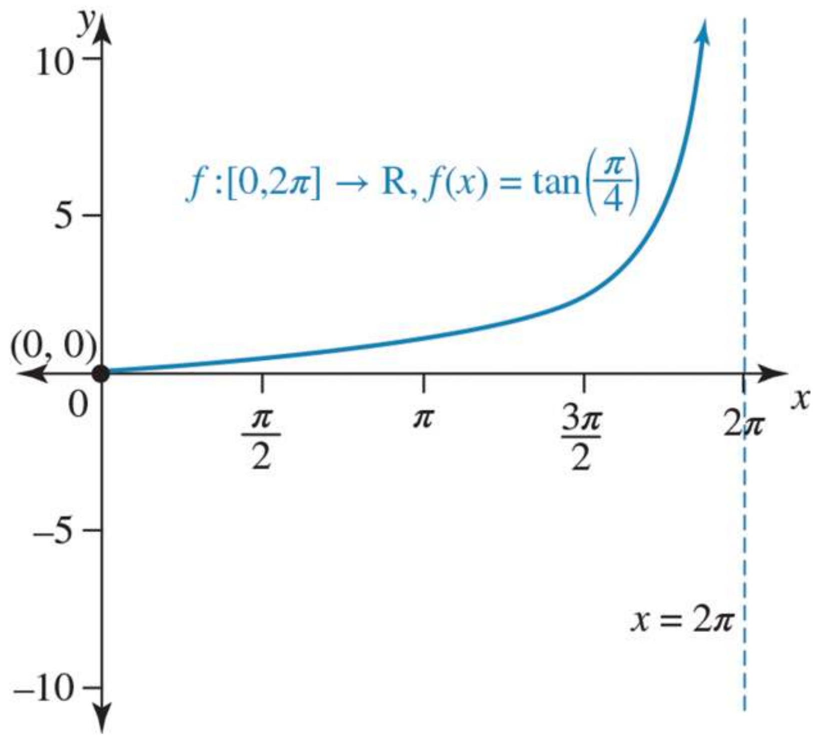
Question 7

a. $T = \frac{\pi}{n}$

$= \frac{\pi}{\frac{1}{4}}$

$= 4\pi$ **[1 mark]**

b. Since the period of the function is 4π, the asymptote is at 2π. The graph is:



Award 1 mark for correct graph and labelled asymptote.

Question 8

Answer E

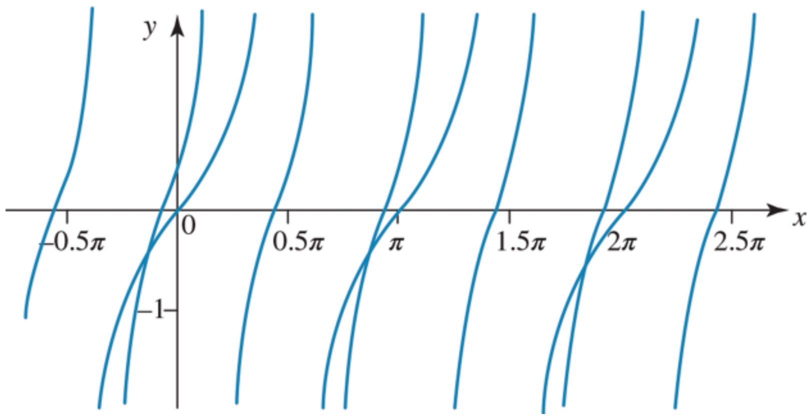
Options A, B, C and D are false.

The period is $T = \frac{\pi}{\frac{1}{5}} = 5\pi$ and the range is R.

Question 9

$y = \tan\left(2x + \frac{\pi}{4}\right) = \tan\left(2\left(x + \frac{\pi}{8}\right)\right)$

This is a dilation by a scale factor of $\frac{1}{2}$ units parallel to the x-axis, followed by a horizontal translation of $\frac{\pi}{8}$ units to the left parallel to the x-axis.



Question 10

The graph of $y = \tan(x)$ has had a transformation of $y = a\tan(n(x - h)) + k$.
The graph of $y = \tan(x)$ is transformed to the graph of $y = 4 - 4\tan(4x)$ by a reflection in the x-axis, a dilation by a scale factor of $\frac{1}{4}$ units from the y-axis, followed by a dilation by a scale factor of 4 units up and away from the x-axis.

Question 11

$$\begin{aligned}\text{Period} &= \frac{7\pi}{4} - \frac{3\pi}{4}\\ &= \frac{4\pi}{4}\\ &= \pi\end{aligned}$$

2.7 Modelling and applications

Question 1

When $t = 0, h = 25$, and when $t = 30, h = 5$; amplitude is 10.

$t = 60 = \frac{2\pi}{n}, n = \frac{\pi}{30}$

$h(t) = 15 + 10\cos\left(\frac{\pi t}{30}\right)$

The correct answer is **E**.

Question 2

$T = 14 = \frac{2\pi}{n} \Rightarrow n = \frac{\pi}{7}$

$f(0) = 0, f(14) = 0$

Range $[0, 10]$

This is only satisfied by $y = f(x) = 5 - 5\cos\left(\frac{\pi t}{7}\right)$.

The correct answer is **B**.

Question 3

a $h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right),\ 0 \le t \le 24$

$$\begin{aligned} h_{\text{min}} &= 14 - 8 \\ &= 6 \text{ m} \end{aligned}$$

b $h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right) = 10$

$$8\sin\left(\frac{\pi t}{12}\right) = 10 - 14 = -4$$

$$\sin\left(\frac{\pi t}{12}\right) = -\frac{1}{2}$$

$$\begin{aligned} \frac{\pi t}{12} &= \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6} \\ &= \frac{7\pi}{6}, \frac{11\pi}{6} \\ t &= 14, 22 \end{aligned}$$

Award 1 mark for solving the trigonometric equation.
Award 1 mark for both correct values of t.

Question 4

a. $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$

The period is $\frac{2\pi}{\frac{\pi}{3}} = 6$.

The amplitude is 400.

Award 1 mark for the correct period.

Award 1 mark for the correct amplitude.

***VCAA Assessment Report* note:**

This question was answered well. However, some students did not answer both parts of the question. Most had the correct period but some expressed the amplitude as. [800, 1600].

b. The maximum is $1200 + 400 = 1600$.

The minimum is $1200 - 400 = 800$.

Award 1 mark for correct maximum.

Award 1 mark for correct minimum.

***VCAA Assessment Report* note:**

This question was answered well. Some students used calculus, which was not necessary. Some gave their answers as coordinate pairs, such as (0, 1600) and (3, 800), but this was incorrect.

c. $$\begin{aligned} n(10) &= 1200 + 400\cos\left(\frac{10\pi}{3}\right) \\ &= 1000 \end{aligned}$$

Award 1 mark for the correct value.

***VCAA Assessment Report* note:**

Some students used their technology in degrees instead of radians and gave the answer 1593 wombats.

d. Solving $n(t) = 1000$ over $0 < t < 12$:

$t = 2, 4, 8, 10$

From 2 to 4 and 10 to 12, there is a total of 4 hours out of 12, so the fraction is $\frac{1}{3}$.

Award 1 mark for solving correctly.
Award 1 mark for the correct fraction.
***VCAA Assessment Report* note:**
Many students obtained 4 months but did not find the fraction of time. Others did not find all the t values. Some wrote their answers in terms of dates.

Question 5
Answer C is false; all the other options are **true**.

$$\begin{aligned} n(0) &= 100 \sin\left(\frac{12\pi}{12}\right) + 300 \\ &= 100 \sin(\pi) + 300 \\ &= 300 \end{aligned}$$

2.8 Review

Question 1

a $[-2, 2]$ **[1 mark]**

b Period $= \frac{2\pi}{2} = \pi$ **[1 mark]**

c $\sin(2x) = \frac{\sqrt{3}}{2}$

$2x = \frac{\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi, k \in Z$ **[2 marks]**

$x = \frac{\pi}{6} + k\pi, \frac{\pi}{3} + k\pi, k \in Z$ **[1 mark]**

Question 2

a $1 - \cos\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)$ for $x \in [-2\pi, \pi]$

$2\cos\left(\frac{x}{2}\right) = 1$

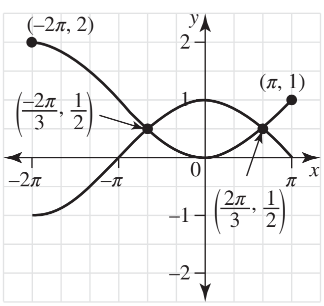
$\cos\left(\frac{x}{2}\right) = \frac{1}{2}$

$\frac{x}{2} = \frac{\pi}{3}, -\frac{\pi}{3}$

$x = \frac{2\pi}{3}, -\frac{2\pi}{3}$

Award 1 mark for solving.
Award 1 mark for the correct answers.

b



Award 1 mark for correct intersection points.
Award 1 mark for the correct graph.

Question 3

Period $T = \frac{2\pi}{\left(\frac{2\pi}{3}\right)} = 3$

The correct answer is **C**.

Question 4

a $(\tan(\theta) - 1)\left(\sin(\theta) - \sqrt{3}\cos(\theta)\right)$

$\left(\sin(\theta) + \sqrt{3}\cos(\theta)\right) = 0$

$(\tan(\theta) - 1)\left(\tan(\theta) - \sqrt{3}\right)\left(\tan(\theta) + \sqrt{3}\right) = 0$

$\tan(\theta) = 1, \pm\sqrt{3}$

Award 1 mark for all three correct values.

b $(\tan(\theta) - 1)\left(\sin(\theta) - \sqrt{3}\cos(\theta)\right)$

$\left(\sin(\theta) + \sqrt{3}\cos(\theta)\right) = 0$

$\Rightarrow (\tan(\theta) - 1)\left(\sin^2(\theta) - 3\cos^2(\theta)\right) = 0$

$\Rightarrow \tan(\theta) = 1, \tan(\theta) = -\sqrt{3}, \tan(\theta) - \sqrt{3}, 0 \le \theta \le \pi$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}$

Award 1 mark for the correct method.

Award 1 mark for all three correct values.

Question 5

$f: R \rightarrow R,\ f(x) = 2\sin(3x) - 3$

Period: $T = \frac{2\pi}{3}$

Range: $[-2, 2] - 3 = [-5, -1]$

The correct answer is **A**.

Question 6

$f: R \rightarrow R, f(x) = 3\sin\left(\frac{2x}{5}\right) - 2$

Period $T = \frac{2\pi}{\frac{2}{5}} = 5\pi$

Range $[-3 - 2, 3 - 2] = [-5, 1]$

Question 7

$$\begin{aligned}\sin(2x) &= 2\sin(x)\cos(x)\\ &= 2 \times \frac{3}{5} \times \frac{-4}{5}\\ &= -\frac{24}{25}\end{aligned}$$

Question 8

$$\begin{aligned}\cos(A)\cos(B) + \sin(A)\sin(B) &= \cos(A - B)\\ &= \cos(3x - 2x)\\ &= \cos(x)\end{aligned}$$

Question 9

$\sin^2(\theta) + \cos^2(\theta) = 1$

$$\sin^2(\theta) = 1 - \left(\frac{5}{13}\right)^2 \qquad \textbf{[1 mark]}$$

$$\sin^2(\theta) = \frac{144}{169}$$

$$\sin(\theta) = \pm\frac{12}{13}$$

$$\sin(\theta) = -\frac{12}{13} \quad \text{since} \quad \frac{3\pi}{2} < \theta < 2\pi \qquad \textbf{[1 mark]}$$

Question 10

For statements A, B, C the right hand side is equal to 4, and E is false; therefore, D is true.

Question 11

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$= 2$$

$$\cos(\theta) = \frac{1}{2} \qquad \textbf{[1 mark]}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$
$$= \sqrt{3} \qquad \textbf{[1 mark]}$$

Question 12

$$\frac{\sin(A)}{1+\cos(A)} + \frac{1+\cos(A)}{\sin(A)}$$
$$= \frac{\sin^2(A) + (1+\cos(A))^2}{(1+\cos(A))\sin(A)}$$
$$= \frac{\sin^2(A) + 1 + 2\cos(A) + \cos^2(A)}{\sin(A)(1+\cos(A))}$$
$$= \frac{2 + 2\cos(A)}{\sin(A)(1+\cos(A))}$$
$$= \frac{2(1+\cos(A))}{\sin(A)(1+\cos(A))}$$
$$= \frac{2}{\sin(A)}$$

Question 13

$$\frac{1-\tan^2(A)}{1+\tan^2(A)}$$
$$= \frac{1 - \frac{\sin^2(A)}{\cos^2(A)}}{1 + \frac{\sin^2(A)}{\cos^2(A)}}$$

$$= \frac{\frac{\cos^2(A)-\sin^2(A)}{\cos^2(A)}}{\frac{\cos^2(A)+\sin^2(A)}{\cos^2(A)}}$$

$$= \frac{\frac{\cos^2(A)-\sin^2(A)}{\cos^2(A)}}{\frac{1}{\cos^2(A)}}$$

$$= \cos^2(A) - \sin^2(A)$$

Question 14

$$\cos\left(\frac{3\pi}{2} - \theta\right) + \sin\left(\frac{3\pi}{2} - \theta\right) = -\sin(\theta) - \cos(\theta)$$

Question 15

$$\begin{aligned}\sin\left(\frac{3\pi}{2} + \alpha\right) &= -\cos(\alpha)\\ &= -0.6\end{aligned}$$

Question 16

Options A, B, C and D are true. E is false since $\tan\left(\frac{\pi}{2} + \theta\right) + \tan\left(\frac{3\pi}{2} + \theta\right) = \frac{-2}{\tan(\theta)}$.

Question 17

Statements A, B, D, and E are true. C is false since $\sin\left(\theta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -\cos(\theta)$.

Question 18

$$\begin{aligned}\tan\left(\frac{\pi}{2} - \theta\right) &= \frac{1}{\tan(\theta)}\\ &= \frac{1}{0.5}\\ &= 2\end{aligned}$$

3 Composite functions, transformations and inverses

Topic	**3**	**Composite functions, transformations and inverses**
Subtopic	**3.2**	**Composite functions**

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Source: *VCE 2020 Mathematical Methods Exam 2, Section A, Q1; © VCAA*

Question 1 (1 mark)

Let f and g be functions such that $f(-1) = 4$, $f(2) = 5$, $g(-1) = 2$, $g(2) = 7$ and $g(4) = 6$.
The value of $g(f(-1))$ is

A. 2
B. 4
C. 5
D. 6
E. 7

Source: *VCE 2018, Mathematical Methods Exam 2, Section A, Q6; © VCAA*

Question 2 (1 mark)

Let f and g be two functions such that $f(x) = 2x$ and $g(x + 2) = 3x + 1$.
The function $f(g(x))$ is

A. $6x - 5$
B. $6x + 1$
C. $6x^2 + 1$
D. $6x - 10$
E. $6x + 2$

Question 3 (1 mark)
If $f(x) = 4x^2$ and $g(x) = 3x + 1$, then $f(g(a))$ is equal to
A. $4(3a + 1)^2$
B. $36a^2 + 4$
C. $(3a + 1)^2$
D. $3a + 1$
E. $12a^2 + 1$

__Source:__ VCE 2021, Mathematical Methods 2, Section A, Q9; © VCAA

Question 4 (1 mark)
Let $g(x) = x + 2$ and $f(x) = x^2 - 4$.
If h is the composite function given by $h: [-5, -1] \to R$, $h(x) = f(g(x))$, then the range of h is
A. $(-3, 5]$
B. $[-3, 5)$
C. $(-3, 5)$
D. $(-4, 5]$
E. $[-4, 5]$

__Source:__ VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q13; © VCAA

Question 5 (1 mark)
The domain of the function h, where $h(x) = \cos(\log_a(x))$ and a is a real number greater than 1 is chosen so that h is a one-to-one function.
Which one of the following could be the domain?
A. $\left(a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$
B. $(0, \pi)$
C. $\left[1, a^{\frac{\pi}{2}}\right]$
D. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$
E. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right]$

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q10; © VCAA*

Question 6 (1 mark)

Which one of the following functions satisfies the functional equation $f(f(x)) = x$ for every real number x?

A. $f(x) = 2x$
B. $f(x) = x^2$
C. $f(x) = 2\sqrt{x}$
D. $f(x) = x - 2$
E. $f(x) = 2 - x$

Question 7 (1 mark)

The function f has the rule $f(x) = \sqrt{x^2 - 9}$ and the function g has the rule $g(x) = x + 2$. The integers c and d, such that $f(g(x)) = \sqrt{(x + c)(x + d)}$, are

A. $c = -5,\ d = 1$
B. $c = 5,\ d = 1$
C. $c = 5,\ d = -1$
D. $c = -2,\ d = -1$
E. $c = 2,\ d = 1$

Topic	3	Composite functions, transformations and inverses
Subtopic	3.3	Transformations

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***Source:** VCE 2019, Mathematical Methods Exam 2, Section A, Q13; © VCAA*

Question 1 (1 mark)

The graph of the function f passes through the point $(-2, 7)$.

If $h(x) = f\left(\frac{x}{2}\right) + 5$, then the graph of the function h must pass through the point

A. $(-1, -12)$
B. $(-1, 19)$
C. $(-4, 12)$
D. $(-4, -14)$
E. $(3, 3.5)$

***Source:** VCE 2016, Mathematical Methods Exam 2, Section A, Q12; © VCAA*

Question 2 (1 mark)

The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x - 5}$ by a reflection in the x-axis followed by a dilation from the y-axis by a factor of $\frac{1}{2}$.

Which one of the following is the rule for the function f ?

A. $f(x) = \sqrt{5 - 4x}$
B. $f(x) = \sqrt{x - 5}$
C. $f(x) = \sqrt{x + 5}$
D. $f(x) = -\sqrt{4x - 5}$
E. $f(x) = -\sqrt{4x - 10}$

Source: *VCE 2015, Mathematical Methods (CAS) Exam 2, Section 1, Q11; © VCAA*

Question 3 (1 mark)

The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A. dilation by a factor of 2 from the y-axis.

B. dilation by a factor of 2 from the x-axis.

C. dilation by a factor of $\frac{1}{2}$ from the x-axis.

D. dilation by a factor of 8 from the y-axis.

E. dilation by a factor of $\frac{1}{2}$ from the y-axis.

Source: *VCE 2021, Mathematical Methods 1, Q5; © VCAA*

Question 4 (4 marks)

Let $f: R \to R, f(x) = x^2 - 4$ and $g: R \to R, g(x) = 4(x-1)^2 - 4$.

a. The graphs of f and g have a common horizontal axis intercept at $(2, 0)$.

Find the coordinates of the other horizontal axis intercept of the graph of g. **(2 marks)**

b. Let the graph of h be a transformation of the graph of f where the transformations have been applied in the following order:

- dilation by a factor of $\frac{1}{2}$ from the vertical axis (parallel to the horizontal axis)
- translation by two units to the right (in the direction of the positive horizontal axis)

State the rule of h and the coordinates of the horizontal axis intercepts of the graph of h. **(2 marks)**

***Source:** VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q1; © VCAA*

Question 5 (1 mark)

The point $P(4, -3)$ lies on the graph of a function f. The graph of f is translated four units vertically up and then reflected in the y-axis.

The coordinates of the final image of P are

A. $(-4, 1)$
B. $(-4, 3)$
C. $(0, -3)$
D. $(4, -6)$
E. $(-4, -1)$

Question 6 (1 mark)

State the equation of the graph if the parabola of $y = x^2$ is dilated by a factor of 2 parallel to the y-axis, translated 1 unit in the negative direction parallel to the x-axis and reflected about the x-axis.

Question 7 (2 marks)

State the transformations of $y = x^3$ to $y = (2x)^3 + 2$.

Question 8 (1 mark)

The graph of $y = \sin(x)$ is transformed to the graph of $y = -4\sin\left(\frac{x}{3}\right)$ by

A. a reflection y-axis, a dilation by a scale factor of 4 units away from the y-axis, followed by a dilation by a scale factor of $\frac{1}{3}$ units away from the x-axis.
B. a reflection y-axis, a dilation by a scale factor of 4 units away from the y-axis, followed by a dilation by a scale factor of 3 units away from the x-axis.
C. a reflection x-axis, a dilation by a scale factor of 4 units away from the x-axis, followed by a dilation by a scale factor of $\frac{1}{3}$ units away from the y-axis.
D. a reflection x-axis, a dilation by a scale factor of 4 units away from the y-axis, followed by a dilation by a scale factor of 3 units away from the x-axis.
E. a reflection x-axis, a dilation by a scale factor of 4 units away from the x-axis, followed by a dilation by a scale factor of 3 units away from the y-axis.

Question 9 (1 mark)

The graph $y = x^2$ is transformed to the graph of $y = 4x^2$ by

A. a dilation parallel to the x-axis by a scale factor of 4.
B. a dilation parallel to the y-axis by a scale factor of 4.
C. a dilation parallel to the x-axis by a scale factor of 2.
D. a dilation parallel to the y-axis by a scale factor of 2.
E. a dilation parallel to the y-axis by a scale factor of $\frac{1}{2}$.

Question 10 (1 mark)

The graph $y = x^2$ is transformed to the graph of $y = \frac{x^2}{4}$ by

A. a dilation from the x-axis by a scale factor of 4.
B. a dilation from the y-axis by a scale factor of 4.
C. a dilation from the x-axis by a scale factor of 2.
D. a dilation from the y-axis by a scale factor of 2.
E. a dilation from the y-axis by a scale factor of $\frac{1}{2}$.

Question 11 (1 mark)

The graph $y = x^3$ is transformed to the graph of $y = 8x^3$ by

A. a dilation parallel to the x-axis by a scale factor of 8.
B. a dilation parallel to the y-axis by a scale factor of 8.
C. a dilation parallel to the x-axis by a scale factor of 2.
D. a dilation parallel to the y-axis by a scale factor of 2.
E. a dilation parallel to the y-axis by a scale factor of $\frac{1}{2}$.

Question 12 (1 mark)

If the graph of $y = (x - 2)^2 + 2$ is reflected in the x-axis and then in the y-axis, it becomes the graph of

A. $y = -(x - 2)^2 - 2$

B. $y = -(x - 2)^2 + 2$

C. $y = (x - 2)^2 - 2$

D. $y = -(x + 2)^2 - 2$

E. $y = (x + 2)^2 - 2$

Question 13 (1 mark)

The graph of $y = f(x)$ is transformed to the graph of $y = -f(x) + c$ where $c \geq 0$ by:

A. a reflection in the x-axis, followed by a vertical translation of c units in the positive direction of the y-axis.

B. a reflection in the x-axis, followed by a vertical translation of c units in the negative direction of the y-axis.

C. a reflection in the y-axis, followed by a vertical translation of c units in the positive direction of the y-axis.

D. a reflection in the y-axis, followed by a vertical translation of c units in the negative direction of the y-axis.

E. a reflection in the x-axis, followed by a horizontal translation of c units in the positive direction of the y-axis.

Question 14 (1 mark)

The graph of $y = f(x)$ is transformed to the graph of $y = Af(x - b)$ where $b \geq 0$ by:

A. a dilation of factor A from the x-axis followed by a horizontal translation of b units in the positive direction of the x-axis.

B. a dilation of factor A from the x-axis followed by a vertical translation of b units in the negative direction of the y-axis.

C. a dilation of factor A from the y-axis followed by a horizontal translation of b units in the positive direction of the x-axis.

D. a dilation of factor A from the y-axis followed by a horizontal translation of b units in the negative direction of the x-axis.

E. a dilation of factor A from the x-axis followed by a horizontal translation of b units in the positive direction of the x-axis.

Question 15 (1 mark)

The graph of $y = f(x)$ is transformed to the graph of $y = f(-x) + c$ by:

A. a reflection in the x-axis, followed by a vertical translation of c units in the positive direction of the y-axis.

B. a reflection in the x-axis, followed by a vertical translation of c units in the negative direction of the y-axis.

C. a reflection in the y-axis, followed by a vertical translation of c units in the positive direction of the y-axis.

D. a reflection in the y-axis, followed by a vertical translation of c units in the negative direction of the y-axis.

E. a reflection in the y-axis, followed by a horizontal translation of c units in the positive direction of the y-axis.

Question 16 (1 mark)

The graph of $y = x^2$ is translated 2 units to the left parallel to the x-axis and then translated 3 units up, parallel to the y-axis, it becomes the graph of

A. $y = (x + 2)^2 + 3$

B. $y = (x - 2)^2 + 3$

C. $y = (x + 2)^2 - 3$

D. $y = (x - 3)^2 + 2$

E. $y = (x + 3)^2 - 2$

Question 17 (1 mark)

The graph of $y = x^2$ is transformed to the graph of $y = 4 - (x - 2)^2$ by

A. a reflection in the y-axis and a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

B. a reflection in the x-axis and a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

C. a reflection in the y-axis and a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

D. a reflection in the x-axis and a translation of 2 units in the negative direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

E. a reflection in the y-axis and a translation of 2 units in the positive direction of the x-axis and a translation of 4 units in the positive direction of the y-axis.

Topic 3 Subtopic 3.3 Transformations

Question 18 (1 mark)

The graph of $y = x^3$ is translated 3 units in the positive direction of the x-axis and then translated 3 units down in the negative direction of the y-axis, it becomes the graph of

A. $y = 3x^3 - 3$
B. $y = (x - 3)^3 + 3$
C. $y = (x - 3)^3 - 3$
D. $y = 3 - (x - 3)^3$
E. $y = 3 - (x + 3)^3$

Question 19 (2 marks)

Let $f: [0, 5] \rightarrow R, f(x) = -(x - 3)^2 + 2$ Find the equation of the image of the graph of f after a reflection in the x-axis followed by a translation of 3 units in the positive direction of the x-axis.

Question 20 (1 mark)

Let $f: \left[0, \frac{\pi}{2}\right] \rightarrow R, f(x) = \sin(2x) - 1$. The graph of f is transformed by a reflection in the x-axis followed by a dilation of factor 2 from in the y-axis. The resulting function is:

A. $g: [0, \pi] \rightarrow R, g(x) = \sin(4x) - 1$
B. $g: [0, 2\pi] \rightarrow R, g(x) = -\sin(4x) + 1$
C. $g: [0, \pi] \rightarrow R, g(x) = 2\sin(2x) + 1$
D. $g: [0, 2\pi] \rightarrow R, g(x) = -2\sin(2x) + 1$
E. $g: [0, \pi] \rightarrow R, g(x) = -\sin(x) + 1$

Topic	3	Composite functions, transformations and inverses
Subtopic	3.4	Inverse graphs

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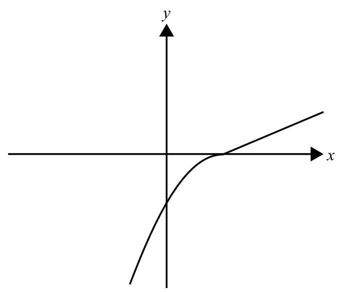
Source: *VCE 2017 Mathematical Methods Exam 2, Section A, Q6; © VCAA*

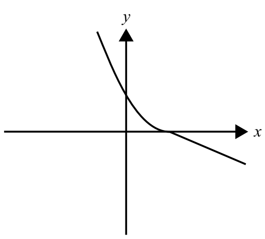
Question 1 (1 mark)

Part of the graph of the function f is shown below. The same scale has been used on both axes.

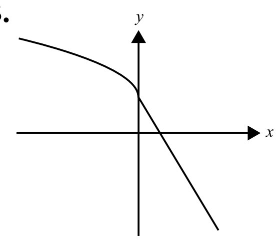


The corresponding part of the graph of the inverse function f^{-1} is best represented by

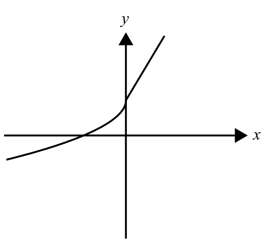
A.



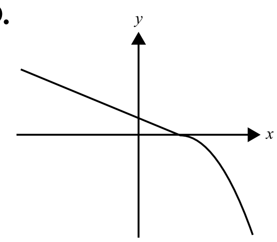
B.



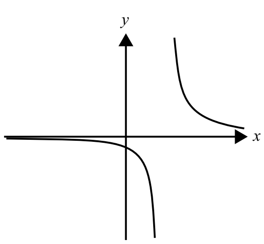
C.



D.



E.



Source: *VCE 2013, Mathematical Methods (CAS) Exam 2, Section 1, Q7; © VCAA*

Question 2 (1 mark)

The function $g : [-a, a] \to R, g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.

The maximum possible value of a is

A. $\frac{\pi}{12}$

B. 1

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

E. $\frac{\pi}{2}$

Question 3 (1 mark)

Select the rule that does not describe a function.

A. $y = x^2 - 5$

B. $y = -5$

C. $x = -5$

D. $y = \sqrt{5 - x^2}$

E. $y = \frac{1}{x - 5}$

Topic	3	Composite functions, transformations and inverses
Subtopic	3.5	Inverse functions

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***Source:** VCE 2018, Mathematical Methods Exam 1, Q5; © VCAA*

Question 1 (3 marks)

Let $f: (2, \infty) \rightarrow R$, where $f(x) = \dfrac{1}{(x-2)^2}$.

State the rule and domain of f^{-1}.

***Source:** VCE 2016, Mathematical Methods Exam 2, Section A, Q5; © VCAA*

Question 2 (1 mark)

Which one of the following is the inverse function of $[3, \infty) \rightarrow R, g(x) = \sqrt{2x-6}$?

A. $g^{-1} : [3, \infty] \rightarrow R, g^{-1}(x) = \dfrac{x^2+6}{2}$

B. $g^{-1} : [0, \infty) \rightarrow R, g^{-1}(x) = (2x-6)^2$

C. $g^{-1} : [0, \infty) \rightarrow R, g^{-1}(x) = \sqrt{\dfrac{x}{2}+6}$

D. $g^{-1} : [0, \infty) \rightarrow R, g^{-1}(x) = \dfrac{x^2+6}{2}$

E. $g^{-1} : R \rightarrow R, g^{-1}(x) = \dfrac{x^2+6}{2}$

Topic 3 Subtopic 3.5 Inverse functions

***Source:** VCE 2015, Mathematical Methods (CAS) Exam 2, Section 1, Q2; © VCAA*

Question 3 (1 mark)

The inverse function of $f:(-2,\infty)\to R,\ f(x)=\dfrac{1}{\sqrt{x+2}}$ is

A. $f^{-1}:R^{+}\to R \quad f^{-1}(x)=\dfrac{1}{x^2}-2$

B. $f^{-1}:R\backslash\{0\}\to R \quad f^{-1}(x)=\dfrac{1}{x^2}-2$

C. $f^{-1}:R^{+}\to R \quad f^{-1}(x)=\dfrac{1}{x^2}+2$

D. $f^{-1}:(-2,\infty)\to R \quad f^{-1}(x)=x^2+2$

E. $f^{-1}:(2,\infty)\to R \quad f^{-1}(x)=\dfrac{1}{x^2-2}$

***Source:** VCE 2019, Mathematical Methods 2, Section A, Q15; © VCAA*

Question 4 (1 mark)

Let $f:[2,\infty)\to R, f(x)=x^2-4x+2$ and $f(5)=7$. The function g is the inverse function of f.
$g'(7)$ is equal to

A. $\dfrac{1}{6}$

B. 5

C. $\dfrac{\sqrt{7}}{14}$

D. 6

E. $\dfrac{1}{7}$

Topic 3 | Subtopic 3.5 Inverse functions

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q6; © VCAA*

Question 5 (1 mark)

The function $f: D \to R$ with rule $f(x) = 2x^3 - 9x^2 - 168x$ will have an inverse function for

A. $D = R$

B. $D = (7, \infty)$

C. $D = (-4, 8)$

D. $D = (-\infty, 0)$

E. $D = \left[-\frac{1}{2}, \infty\right)$

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q9; © VCAA*

Question 6 (1 mark)

The inverse of the function $f: R^+ \to R, f(x) = \dfrac{1}{\sqrt{x}} + 4$ is

A. $f^{-1}: (4, \infty) \to R \quad f^{-1}(x) = \dfrac{1}{(x-4)^2}$

B. $f^{-1}: R^+ \to R \quad f^{-1}(x) = \dfrac{1}{x^2} + 4$

C. $f^{-1}: R^+ \to R \quad f^{-1}(x) = (x+4)^2$

D. $f^{-1}: (-4, \infty) \to R \quad f^{-1}(x) = \dfrac{1}{(x+4)^2}$

E. $f^{-1}: (-\infty, 4) \to R \quad f^{-1}(x) = \dfrac{1}{(x-4)^2}$

Topic 3 Subtopic 3.5 Inverse functions

Source: *VCE 2013, Mathematical Methods (CAS) 2, Section 1, Q7; © VCAA*

Question 7 (1 mark)

The function $g : [-a, a] \to R, g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.

The maximum possible value of a is

A. $\frac{\pi}{12}$

B. 1

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

E. $\frac{\pi}{2}$

Topic	3	Composite functions, transformations and inverses
Subtopic	3.6	Literal equations

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Question 1 (1 mark)

The solution to the equation $\frac{1}{x+a}=\frac{b}{x}$ in terms of x is

A. $1+\frac{b}{a}$

B. $\frac{ab}{1-b}$

C. $\frac{ab}{1+b}$

D. $\frac{a}{1-b}$

E. $\frac{1}{a+b}$

Question 2 (1 mark)

$mx+n=nx+m$ solved for x is

A. $\frac{m+n}{m-n}$

B. $\frac{m-n}{m+n}$

C. $\frac{m}{n}$

D. 0

E. 1

Question 3 (1 mark)

The solutions for the pair of simultaneous equations $ax + by = r$ and $ax - by = s$ in term of x and y are

A. $x = \frac{r+s}{2a}, y = \frac{-s+r}{2b}$

B. $x = \frac{r-by}{a}, y = \frac{r-ax}{b}$

C. $x = \frac{s+by}{a}, y = \frac{s-ax}{-b}$

D. $x = \frac{r+s}{2a}, y = \frac{s+r}{2b}$

E. $x = \frac{r-s}{2a}, y = \frac{-s+r}{2b}$

Question 4 (1 mark)

If $x = 2$ is a solution to the equation $kx + 3 = \frac{x}{2} + 1$, find the value of k.

Question 5 (1 mark)

The solution to the equation $\frac{2+x}{b} = \frac{x-10}{a}$ in term of x is

A. $\frac{2(a+5b)}{a-b}$

B. $-\frac{2(a-5b)}{a-b}$

C. $-\frac{2(a+5b)}{a+b}$

D. $-\frac{2(a+5b)}{a-b}$

E. $\frac{2(a+5b)}{a+b}$

Topic	3	Composite functions, transformations and inverses
Subtopic	3.7	Review

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Source: *VCE 2017, Mathematical Methods Exam 2, Section A, Q4; © VCAA*

Question 1 (1 mark)

Let f and g be functions such that $f(2) = 5$, $f(3) = 4$, $g(2) = 5$, $g(3) = 2$ and $g(4) = 1$.

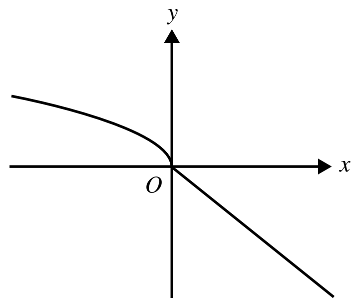
The value of $f(g(3))$ is

A. 1
B. 2
C. 3
D. 4
E. 5

Source: *VCE 2015, Mathematical Methods (CAS) Exam 2, Section 1, Q5; © VCAA*

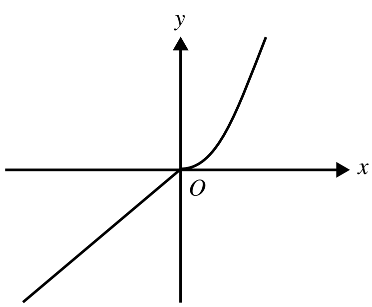
Question 2 (1 mark)

Part of the graph of $y = f(x)$ is shown below.

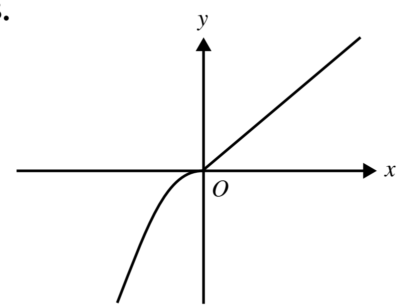


The corresponding part of the graph of the inverse function $y = f^{-1}(x)$ is best represented by

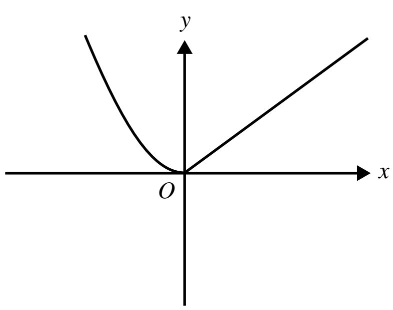
A.



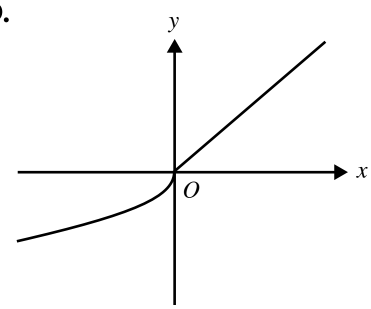
B.



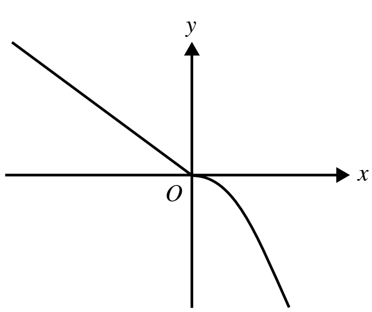
C.



D.



E.



Source: *VCE 2018, Mathematical Methods Exam 2, Section A, Q4; © VCAA*

Question 3 (1 mark)

The point $A(3, 2)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x) = \frac{1}{2}f(x - 1)$. The same transformation maps the point A to the point P.

The coordinates of the point P are

A. (2, 1)
B. (2, 4)
C. (4, 1)
D. (4, 2)
E. (4, 4)

Source: *VCE 2019, Mathematical Methods Exam 1, Q2; © VCAA*

Question 4 (3 marks)

a. Let $f: R\backslash\left\{\frac{1}{3}\right\} \to R,\ f(x) = \frac{1}{3x - 1}$.

Find the rule of f^{-1}. **(2 marks)**

b. State the domain of f^{-1}. **(1 mark)**

Source: VCE 2017, Mathematical Methods Exam 1, Q7; © VCAA

Question 5 (5 marks)

Let $f: [0, \infty] \rightarrow R, \quad f(x) = \sqrt{x + 1}$

a. State the range of f. **(1 mark)**

b. Let $g: (-\infty, c) \rightarrow R, g(x) = x^2 + 4x + 3$, where $c < 0$.

i. Find the largest possible value of c such that the range of g is a subset of the domain of f. **(2 marks)**

ii. For the value of c found in part **b.i.**, state the range of $f(g(x))$. **(1 mark)**

c. Let $h: R \rightarrow R, h(x) = x^2 + 3$.

State the range of $f(h(x))$. **(1 mark)**

Source: *VCE 2021, Mathematical Methods 2, Section A, Q18; © VCAA*

Question 6 (1 mark)

Let $f: R \to R,\ f(x) = (2x-1)(2x+1)(3x-1)$ and $g: (-\infty,\ 0) \to R,\ g(x) = x\log_e(-x)$.

The maximum number of solutions for the equation $f(x-k) = g(x)$ where $k \in R$, is

A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2021, Mathematical Methods 2, Section A, Q4; © VCAA*

Question 7 (1 mark)

The maximum value of the function $h: [0, 2] \to R,\ h(x) = (x-2)e^x$ is

A. $-e$
B. 0
C. 1
D. 2
E. e

Source: *VCE 2020, Mathematical Methods 2, Section 1, Q20; © VCAA*

Question 8 (1 mark)

Let $f: R \to R,\quad f(x) = \cos(ax)$ where $a \in R$ be a function with the property $f(x) = f(x+h)$ for all $h \in Z$.

Let $g: D \to R,\quad g(x) = \log_2(f(x))$ be a function where the range of g is $[-1, 0]$.

A possible interval for D is

A. $\left[\frac{1}{4}, \frac{5}{12}\right]$

B. $\left[1, \frac{7}{6}\right]$

C. $\left[\frac{5}{3}, 2\right]$

D. $\left[-\frac{1}{3}, 0\right]$

E. $\left[-\frac{1}{12}, \frac{1}{4}\right]$

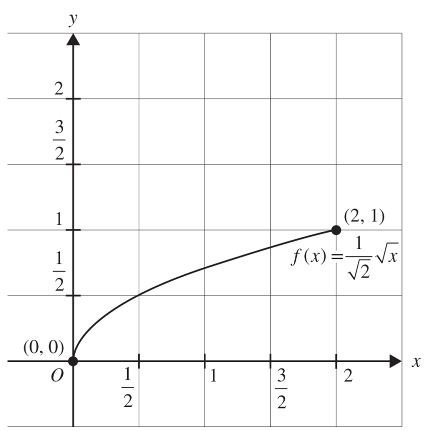
Source: *VCE 2020, Mathematical Methods 1, Q6; © VCAA*

Question 9 (4 marks)

let $f: [0, 2] \rightarrow R$, where $f(x) = \frac{1}{\sqrt{2}}\sqrt{x}$.

a. Find the domain and the rule for f^{-1}, the inverse function of f. **(2 marks)**

b. The graph of $y = f(x)$, where $x \in [0, 2]$, is shown on the axes below.
On the axes, sketch the graph of f^{-1} over its domain. Label the endpoints and point(s) of intersection with the function f, giving their coordinates. **(2 marks)**



Source: *VCE 2018, Mathematical Methods 2, Section A, Q18; © VCAA*

Question 10 (1 mark)

Consider the functions $f: R^+ \to R\ f(x) = x^{\frac{p}{q}}$ and $g: R^+ \to R\ g(x) = x^{\frac{m}{n}}$ where p, q, m and n are positive integers, and $\frac{p}{q}$ and $\frac{m}{n}$ are fractions in simplest form.

If $\{x : f(x) > g(x)\} = (0, 1)$ and $\{x : g(x) > f(x)\} = (1, \infty)$, which of the following must be **false**?

A. $q > n$ and $p = m$

B. $m > p$ and $q = n$

C. $pn < qm$

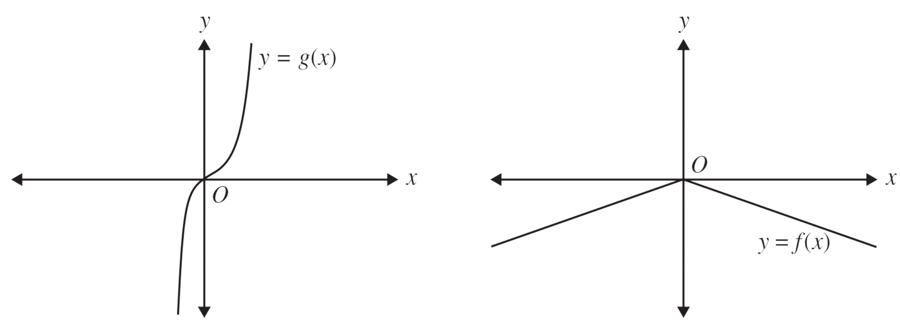
D. $f'(c) = g'(c)$ for some $c \in (0, 1)$

E. $f'(d) = g'(d)$ for some $d \in (1, \infty)$

***Source**: VCE 2015, Mathematical Methods (CAS) 2, Section 1, Q22; © VCAA*

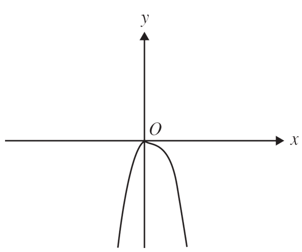
Question 11 (1 mark)

The graphs of the functions with rules $f(x)$ and $g(x)$ are shown below.

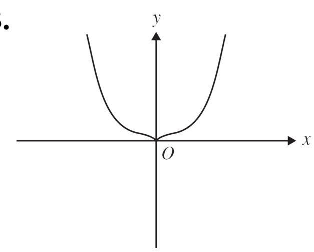


Which one of the following best represents the graph of the function with rule $g(-f(x))$?

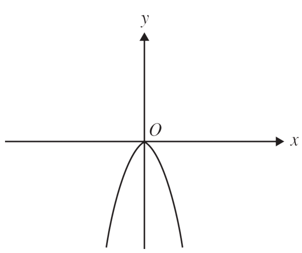
A.



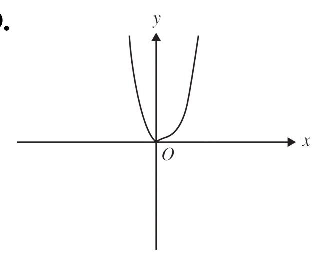
B.



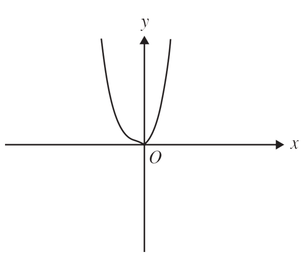
C.



D.



E.



Source: *VCE 2015, Mathematical Methods (CAS) 2, Section 1, Q20; © VCAA*

Question 12 (1 mark)

If $f(x-1)=x^2-2x+3$, then $f(x)$ is equal to

A. x^2-2
B. x^2+2
C. x^2-2x+2
D. x^2-2x+4
E. x^2-4x+6

Question 13 (1 mark)

If $f(x)=-4\cos^2(2x)$ and $g(x)=\sqrt{x}$, then

A. $f(g(x))$ does not exist, and $g(f(x))$ does not exist.
B. $f(g(x))=-4\cos^2\left(2\sqrt{x}\right)$, has domain $[0,\infty)$ and $g(f(x))$ does not exist.
C. $f(g(x))=2\cos(2x)$, has domain $[0,\infty)$ and $f(g(x))$ does not exist.
D. $g(f(x))=4\cos^2\left(\sqrt{x}\right)$, has domain $[0,\infty)$ and $f(g(x))$ does not exist.
E. $g(f(x))=2\cos(2x)$, has domain R and $f(g(x))$does not exist.

Question 14 (1 mark)

If $f(x)=x^2$, $g(x)=\cos(2x)$ and $h(x)=\dfrac{1}{x}$ then $\dfrac{1}{\cos^2(2x)}$ is equal to

A. $g(f(h(x)))$
B. $g(h(f(x)))$
C. $h(f(g(x)))$
D. $h(g(f(x)))$
E. $f(h(g(x)))$

Question 15 (1 mark)

Consider the graph of $y = x^3 - x^2$. Which of the following statements is **true**?

A. When reflected in the x-axis, the graph becomes $y = x^3 + x^2$.
B. When reflected in the x-axis, or the y-axis, the graph becomes $y = -x^3 + x^2$.
C. When reflected in the line, the graph becomes $y = x^3 - x^2$.
D. When reflected in the line, the graph becomes $y = -x^3 + x^2$.
E. When reflected in both the x-axis and y-axis, the graph becomes $y = x^3 + x^2$.

Answers and marking guide

3.2 Composite functions

Question 1

$g(f(-1)) = g(4) = 6$

The correct answer is **D**.

Question 2

$f(x) = 2x, \; g(x+2) = 3x+1$

$$\begin{aligned} g(x) &= 3(x-2)+1 \\ &= 3x-6+1 \\ &= 3x-5 \\ f(g(x)) &= 2(3x-5) \\ &= 6x-10 \end{aligned}$$

The correct answer is **D**.

Question 3

$$\begin{aligned} f(x) &= 4x^2 \\ f(g(x)) &= f(3x+1) \\ &= 4(3x+1)^2 \\ f(g(a)) &= 4(3a+1)^2 \end{aligned}$$

The correct answer is **A**.

Question 4

$g(x) = x+2, \quad f(x) = x^2 - 4$

$h(x) = f(g(x)) = f(x+2) = (x+2)^2 - 4$

$h: [-5, -1] \rightarrow R, \; h(x) = x^2 + 4x$

$h(-5) = 5$

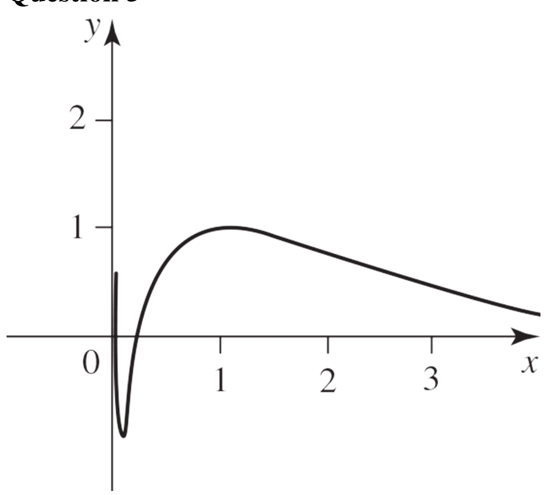
$h(-1) = -3$

$h(-2) = -4$

range $h = [-4, 5]$

The correct answer is **E**.

Question 5



For h to be a one-to-one function, the domain is $\left[1, a^{\frac{\pi}{2}}\right]$.

The correct answer is **C**.

Question 6
$f(x) = 2 - x$
$f(f(x)) = f(2 - x)$
$= 2 - (2 - x)$
$= x$
The correct answer is **E**.

Question 7
$$\begin{aligned} f(g(x)) &= \sqrt{(x+2)^2 - 9} \\ &= \sqrt{x^2 + 4x - 5} \\ &= \sqrt{(x+5)(x-1)} \end{aligned}$$
The correct answer is **C**.

3.3 Transformations

Question 1
Since f passes through $(-2, 7)$, $f(-2) = 7$.
$h(x) = f\left(\frac{x}{2}\right) + 5,\ h(-4) = f(-2) + 5 = 7 + 5 = 12$
Alternatively, double the x- value and add 5 to the y-value:
$f: (-2, 7) \rightarrow h: (-4, 12)$
The correct answer is **C**.

Question 2
$y = \sqrt{2x - 5}$ reflected in the x-axis becomes $y = -\sqrt{2x - 5}$.
Dilation by a factor of $\frac{1}{2}$ from the y-axis, that is parallel to the y-axis, replaces $x \rightarrow 2x$.
Therefore, $f(x) = -\sqrt{4x - 5}$
The correct answer is **D**.

Question 3
To map $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$, replace x with $\frac{x}{2}$, which represents dilation by a factor of 2 parallel to the x-axis or away from the y-axis.
The correct answer is **A**.

Question 4
a. $g(x) = 0\ ,\ \ 4(x-1)^2 - 4 = 0, \quad (x-1)^2 = 1$
$x - 1 = \pm 1, \quad x = 0, 2 \quad (0, 0),\ (2, 0)$
So, the other point is $(0, 0)$
Award 1 mark for locating the two points.
Award 1 mark for identifying the *other* point at the origin

b. $f: y = x^2 - 4$
Dilate $\frac{1}{2}$ from y-axis, replace x with $2x$, $y = 4x^2 - 4$
Translate 2 units to the right, replace x with $(x - 2)$, $\quad h(x) = 4(x-2)^2 - 4\ \ h(x) = 0\ ,\ \ (1, 0),\ \ (3, 0)$
Award 1 mark for the correct function h.
Award 1 mark for the correct axial points.

Question 5
The point $P(4, -3)$ translated up 4 units gives $(4, 1)$ and reflected in the y-axis gives $(-4, 1)$.
The correct answer is **A**.

Question 6

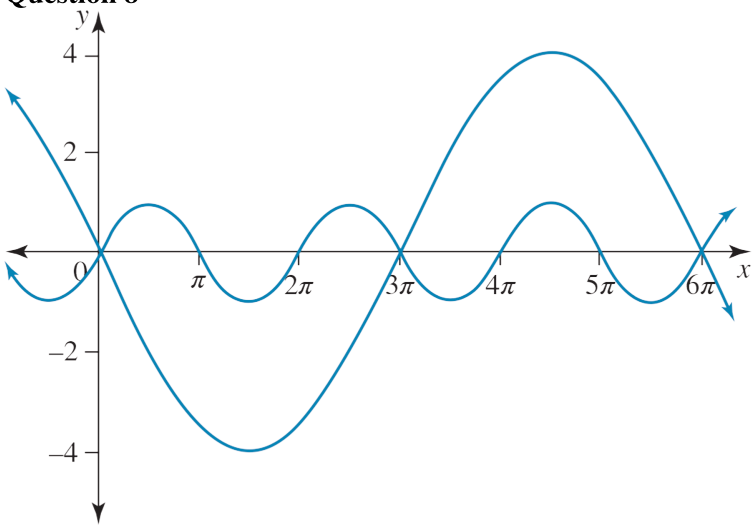
$y = x^2 \rightarrow y = 2x^2 \rightarrow y = 2(x+1)^2 \rightarrow y = -2(x+1)^2$

Question 7

Dilation factor of 8 from the x-axis **[1 mark]**

Translation of 2 units in the positive direction of the -axis **[1 mark]**

Question 8



The correct answer is **E**.

Question 9

a dilation parallel to the y-axis by a scale factor of 4.

The correct answer is **B**.

Question 10

$y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$

The correct answer is **D**.

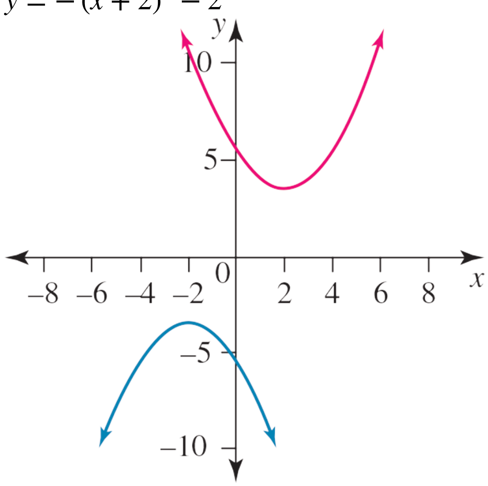
Question 11

a dilation parallel to the y-axis by a scale factor of 8.

The correct answer is **B**.

Question 12

$y = -(x+2)^2 - 2$



The correct answer is **D**.

Question 13

a reflection in the x-axis followed by a vertical translation of c units in the positive direction of the y-axis
The correct answer is **E**.

Question 14

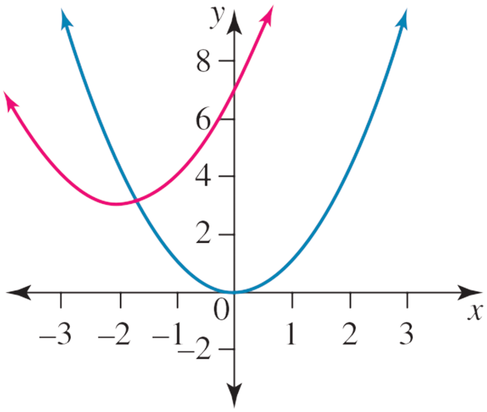
a dilation of factor A from x-axis followed by a horizontal translation of b units in the positive direction of the x-axis.
The correct answer is **E**.

Question 15

a reflection in the y-axis followed by a vertical translation of c units in the positive direction of the y-axis.
The correct answer is **C**.

Question 16

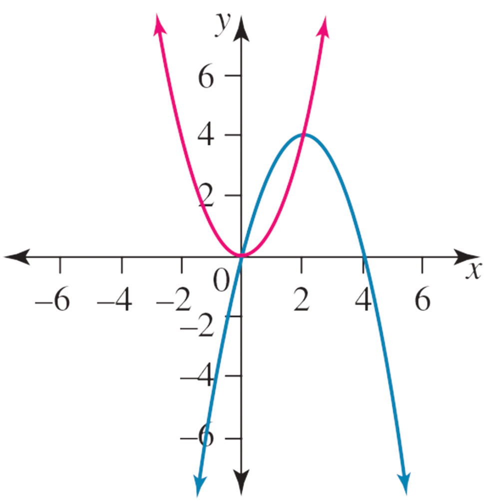
$y = (x + 2)^2 + 3$



The correct answer is **A**.

Question 17

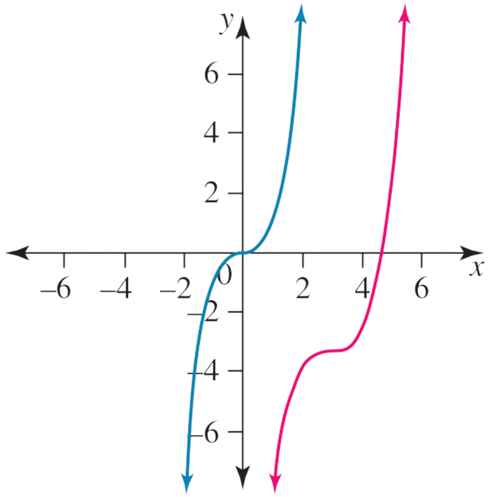
a reflection in the x-axis, a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.



The correct answer is **B**.

Question 18

$y = (x - 3)^3 - 3$



The correct answer is **C**.

Question 19

$$x' = x + 3 \Leftrightarrow x = x' - 3$$
$$y' = -y$$
$$-y' = -(x' - 3 - 3)^2 + 2 \quad \textbf{[1 mark]}$$
$$y' = (x' - 6)^2 - 2$$
$$y = (x - 6)^2 - 2 \quad \textbf{[1 mark]}$$

Question 20

$g : [0, \pi] \to R,\ g(x) = -\sin(x) + 1$

The correct answer is **E**.

3.4 Inverse graphs

Question 1

Reflect the original graph in the line $y = x$, so option C is the correct graph.

The correct answer is **C**.

Question 2

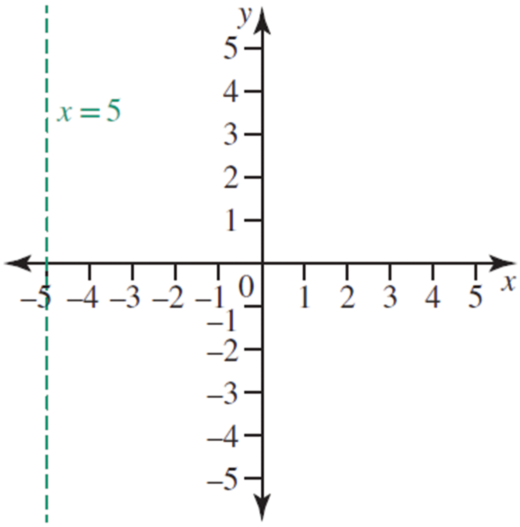
$g : [-a, a] \to R,\ g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$

For the inverse to be a function, the original function must be a one-to-one function.

Only $a = \frac{\pi}{12}$ makes g a one-to-one function. Check with CAS.

The correct answer is **A**.

Question 3
A line parallel to the y-axis crosses the graph at infinitely many points, so the graph of $x = -5$ is not a function.



The correct answer is **C**.

3.5 Inverse functions

Question 1

$f: (2, \infty) \to R$, where $f(x) = \dfrac{1}{(x-2)^2}$

$\text{dom} f = (2, \infty) = \text{range} f^{-1}$ and $\text{dom} f^{-1} = (0, \infty) = \text{range} f$

$f: y = \dfrac{1}{(x-2)^2}$

$f^{-1}: x = \dfrac{1}{(y-2)^2} \Rightarrow (y-2)^2 = \dfrac{1}{x} \Rightarrow y = 2 \pm \dfrac{1}{\sqrt{x}}$

Take the positive value, since range $f^{-1} = (2, \infty)$.

$f^{-1}: (0, \infty) \to R, f^{-1}(x) = 2 + \dfrac{1}{\sqrt{x}}$

Award 1 mark for transposing and rearranging the function.
Award 1 mark for the correct rule.
Award 1 mark for the correct domain of the inverse function.

***VCAA Examination Report* note:**
Students appeared to manage this question confidently. However, some students did not handle the algebraic manipulation correctly and others used incorrect notation, stating their final answer or in stating the domain.

Question 2

$g: [3, \infty) \to R, g(x) = \sqrt{2x-6}$
$\text{dom}\, g = \text{ran}\, g^{-1} = [3, \infty]$
$\text{dom}\, g^{-1} = \text{ran}\, g = [0, \infty]$
$g: y = \sqrt{2x-6}$

$g^{-1}: x = \sqrt{2y-6} \Rightarrow x^2 = 2y - 6, y = g^{-1}(x) = \dfrac{x^2+6}{2}$

$g^{-1}: [0, \infty) \to R, g^{-1}(x) = \dfrac{x^2+6}{2}$

The correct answer is **D**.

Question 3

$f: (-2, \infty) \to R, f(x) = \dfrac{1}{\sqrt{x+2}}$

$f: y = \dfrac{1}{\sqrt{x+2}}$

$f^{-1}: x = \dfrac{1}{\sqrt{y+2}}$

$\Rightarrow x^2 = \dfrac{1}{y+2} \Rightarrow y + 2 = \dfrac{1}{x^2} \Rightarrow y = \dfrac{1}{x^2} - 2$

range $f = R^+ =$ domain f^{-1}

$f^{-1}: R^+ \to R, f^{-1}(x) = \dfrac{1}{x^2} - 2$

The correct answer is **A**.

Question 4

Since $g(x) = f^{-1}(x), f(5) = 7 \Leftrightarrow g(7) = 5$

$f'(x) = 2x - 4 \quad f'(5) = 6 \quad g'(7) = \dfrac{1}{f'(5)} = \dfrac{1}{6}$

Alternatively

$f: y = x^2 - 4x + 2 = \left(x^2 - 4x + 4\right) + 2 - 4$

$f: y = (x-2)^2 - 2$

$g = f^{-1}: x = (y-2)^2 - 2$

$y - 2 = \pm\sqrt{2+x}$

dom $f =$ ran $f^{-1} = [2, \infty)$ dom $f^{-1} =$ ran $f = [-2, \infty)$ take positive

$f^{-1}(x) = g(x) = 2 + \sqrt{2+x}$

$\dfrac{d}{dx}\left(f^{-1}(x)\right) = g'(x) = \dfrac{1}{2\sqrt{x+2}}$

$g'(7) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$

The correct answer is **A**.

Question 5

$f(x) = 2x^3 - 9x^2 - 168x$

$f'(x) = 6x^2 - 18x - 168$

$= 6(x-7)(x+4)$

The turning points are at $x = -4$ and $x = 7$.

For $f(x)$ to be a one-to-one function, $D = (7, \infty)$.

The correct answer is **B**.

Question 6

$f: y = \dfrac{1}{\sqrt{x}} + 4$

dom $f =$ ran $f^{-1} = R^+$

ran $f =$ dom $f^{-1} = (4, \infty)$

$f^{-1}: x = \dfrac{1}{\sqrt{y}} + 4$

$$\frac{1}{\sqrt{y}} = x - 4$$
$$\sqrt{y} = \frac{1}{x-4}$$
$$y = f^{-1}(x)$$
$$y = \frac{1}{(x-4)^2}$$
The correct answer is **A**.

Question 7

$g : [-a, a] \to R,\ g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$

For the inverse to be a function, the original function must be a one-to-one function.
Only $a = \frac{\pi}{12}$ makes g a one-to-one function. Check with CAS.
The correct answer is **A**.

3.6 Literal equations

Question 1

$$\frac{1}{x+a} = \frac{b}{x}$$
$$x = b(x+a)$$
$$x = bx + ab$$
$$x - bx = ab$$
$$x(1-b) = ab$$
$$x = \frac{ab}{1-b}$$
The correct answer is **B**.

Question 2

$$mx + n = nx + m$$
$$mx - nx = m - n$$
$$x(m-n) = m - n$$
$$x = 1$$
The correct answer is **E**.

Question 3

$ax + by = r$ [1]
$ax - by = s$ [2]
[1] + [2]
$2ax = r + s$
$$x = \frac{r+s}{2a}$$
[1] + [2]:
$2by = r - s$
$$y = \frac{r-s}{2b}$$
$$x = \frac{r+s}{2a}$$
$$y = \frac{-s+r}{2b}$$
The correct answer is **A**.

Question 4

$2x + 3 = \frac{2}{2} + 1$

$2k + 3 = 2$

$k = \frac{-1}{2}$

Question 5

$\frac{2+x}{b} = \frac{x-10}{a}$

$a(2+x) = b(x-10)$

$2a + 2x = bx - 10b$

$ax - bx = -2a - 10b$

$x(a-b) = -2(a+5b)$

$x = \frac{-2(a+5b)}{a-b}$

The correct answer is **D**.

3.7 Review

Question 1

$f(g(3)) = f(2) = 5$

The correct answer is **E**.

Question 2

The inverse function is the graph reflected in the line $y = x$.

The correct answer is **E**.

Question 3

A is on $f(x)$; P is on $g(x) = \frac{1}{2}f(x-1)$.

Dilate by a factor of $\frac{1}{2}$ from the x-axis:

$(x, y) \rightarrow \left(x, \frac{y}{2}\right)$, $A(3, 2) \rightarrow (3, 1)$

Translation of one unit in the positive x-direction:

$\left(x, \frac{y}{2}\right) \rightarrow \left(x+1, \frac{y}{2}\right)$, $(3, 1) \rightarrow P(4, 1)$

The correct answer is **C**.

Question 4

a. $f: R\setminus\left\{\frac{1}{3}\right\} \rightarrow R, f(x) = \frac{1}{3x-1}$

$f: y = \frac{1}{3x-1}$

Swap x and y.

$f^{-1}: x = \frac{1}{3y-1}$

$3y - 1 = \frac{1}{x}$

$y = f^{-1}(x) = \frac{1}{3x} + \frac{1}{3} = \frac{x+1}{3x}$

Award 1 mark for swapping x and y and making y the subject.

Award 1 mark for the correct result.

b. Domain of f^{-1} = range of $f = R\setminus\{0\}$ **[1 mark]**

Question 5

a. $f:[0,\infty)\to R,\ f(x)=\sqrt{x+1}$

$\text{dom} f=[0,\infty),\ \text{ran} f=[1,\infty)$ **[1 mark]**

b. i. $g:(-\infty,c]\to R,\ g(x)=x^2+4x+3=(x+2)^2-1$

$x^2+4x+3=(x+3)(x+1)\geq 0\Rightarrow x=-3,-1$

So $c=-3$, so that the range of $g\subseteq[0,\infty)$

Award 1 mark for finding the x-intercepts or a graph with the x-intercepts labelled.

Award 1 mark for the correct value of c.

ii. Since $(-\infty,-3]$ is the domain of g, the range of g becomes $[0,\infty)$, which is the same as the domain of f

Therefore, the range of f is the same as the range of.

$f(g(x))$

$=f\left(x^2+4x+3\right)$

$=\sqrt{x^2+4x+4}$

$=\sqrt{(x+2)^2}$

$=|x+2|$

$=-x-2$ since $x\leq -3$

$\text{dom} f(g(x))=\text{dom}\ g(x)=(-\infty,-3]$

So $\text{ran} f(g(x))=[1,\infty)$. **[1 mark]**

c. $h:R\to R,\ h(x)=x^2+3$

$f(h(x))$

$=f\left(x^2+3\right)$

$=\sqrt{x^2+3+1}$

$=\sqrt{x^2+4}$

$\text{dom}\ (h(x))=R$

$\text{ran}\ (h(x))=[3,\infty)$

$\therefore \text{dom}\ (g(x))=[3,\infty)$

$\text{ran} f(h(x))=[2,\infty)$ **[1 mark]**

Question 6

Translate f to the left; there are 3 points of intersection.

The correct answer is **D**.

Question 7

$h:[0,2]\to R,\quad h(x)=(x-2)e^x$

The maximum value at the end point when $x=2$ is $y=0$.

The correct answer is **B**.

Question 8

$f(x) = \cos(ax) = \cos(a(x+h))$

$f(x) = f(x+h),\ h \in Z,\ a = 2\pi$

$f(x) = \cos(2\pi x)$

$g(x) = \log_2(f(x))$ has range $[-1, 0]$

$-1 \le \log_2(\cos(2\pi x)) \le 0$

$\frac{1}{2} \le \cos(2\pi x) \le 1$

$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2},\ \cos(2\pi) = 1,\ \cos\left(\frac{7\pi}{3}\right) = \frac{1}{2}$

$\frac{5}{6} \le x \le \frac{7}{6}$

Only satisfied by $1 \le x \le \frac{7}{6}$, $\left[1, \frac{7}{6}\right]$

The correct answer is **B**.

Question 9

a. $f: [0, 2] \to R, f(x) = \frac{1}{\sqrt{2}}\sqrt{x}$

$f: y = \sqrt{\frac{x}{2}}$

$\text{dom} f = [0, 2] = \text{ran} f^{-1},\ \text{dom} f^{-1} = [0, 1] = \text{ran} f$

$f^{-1}: x = \sqrt{\frac{y}{2}}$

$\frac{y}{2} = x^2$

$f^{-1}: [0, 1] \to R, f^{-1}(x) = 2x^2$

Award 1 mark for the correct inverse function rule.

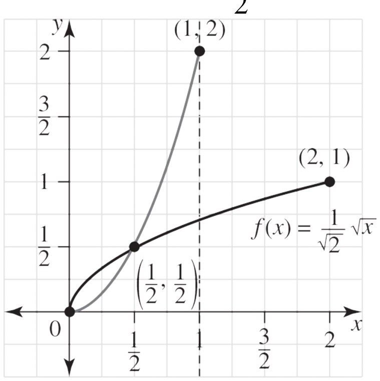
Award 1 mark for the domain of inverse function.

b. $x = f(x) = f^{-1}(x) \Rightarrow 2x^2 = x$

$2x^2 - x = 0$

$x(2x - 1) = 0$

$x = 0, x = \frac{1}{2}$



Award 1 mark for the correct graph shape.

Award 1 mark for the correct endpoints and intersection point.

Question 10

$f: R^+ \to R, f(x) = x^{\frac{p}{q}}$

$g: R^+ \to R,\ g(x) = x^{\frac{m}{n}},\ p, q, m, n \in Z^+$

$f(x) > g(x),\ x \in (0,\ 1)$

$x^{\frac{p}{q}} > x^{\frac{m}{n}}$

$\frac{p}{q} < \frac{m}{n}$

$pn < qm$

$g(x) > f(x),\ x \in (1,\ \infty)$

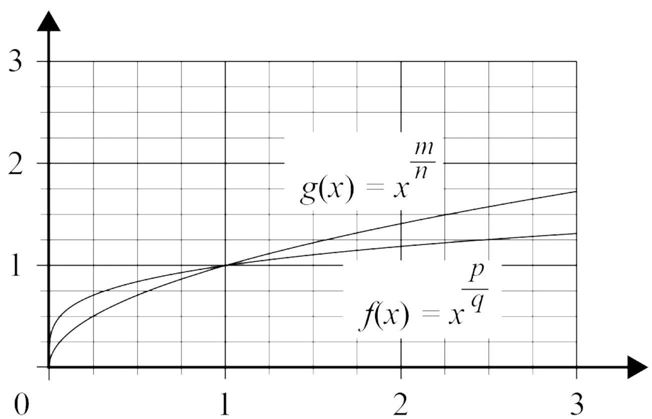
$x^{\frac{m}{n}} > x^{\frac{p}{q}}$

$\frac{p}{q} < \frac{m}{n}$

$pn < qm$

If $p = m,\ q > n$, and if $q = n,\ m > p$

The graphs of $f(x)$ and $g(x)$ intersect at $x = 0$ and $x = 1$.



For $x \in (0,\ 1)$, $f(x) > g(x)$, but since $\lim_{x\to\infty} (f(x)) = \lim_{x\to\infty} (g(x))$ and $f(1) = g(1)$, $f(x)$ is parallel to $g(x)$ for some $x \in (0,\ 1)$. That is, $f'(c) = g'(c)$ for some $c \in (0,\ 1)$. Option D is true.

For $x > 1$ we can see that $g'(x) > f'(x)$. Thus, $f'(d) \neq g'(d),\ d \in (1,\ \infty)$; therefore, option E is false.

The correct answer is **E**.

Question 11

Let $f(x) = -\frac{1}{2}|x|$ and $g(x) = x^3$.

The graph of $g(-f(x)) = g\left(\frac{1}{2}|x|\right) = \frac{1}{8}|x^3|$

$g(-f(x)) \geq 0$ and symmetrical about the y-axis.

The correct answer is **B**.

Question 12

$f(x-1) = x^2 - 2x + 3$

$f(u-1) = u^2 - 2u + 3$

$f(u+1) = (u+1)^2 - 2(u+1) + 3$

$= u^2 + 2u + 1 - 2u - 2 + 3$

$= u^2 + 2$

$f(x) = x^2 + 2$

The correct answer is **B**.

Question 13

$f(x) = -4\cos^2(2x) \quad \text{dom} f = R$ and $\text{ran} f = [-4, 0]$

$g(x) = \sqrt{x} \quad \text{dom } g = [0, \infty)$ and $\text{ran } g = [0, \infty)$

$f(g(x))$ exists if ran $g \subseteq$ dom f, yes $f(g(x))$ exists.

$f(g(x)) = f\left(\sqrt{x}\right) = -4\cos^2\left(2\sqrt{x}\right)$

$\text{dom} f(g(x)) = \text{dom } g = [0, \infty)$

$g(f(x))$ exists if ran $f \subseteq$ dom g, not true

so $g(f(x))$ does not exist

The correct answer is **B**.

Question 14

$f(x) = x^2, \; g(x) = \cos(2x)$ and $h(x) = \dfrac{1}{x}$

$f(g(x)) = f(\cos(2x)) = \cos^2(2x)$

$h(f(g(x))) = h\left(\cos^2(2x)\right) = \dfrac{1}{\cos^2(2x)}$

The correct answer is **C**.

Question 15

If $y = x^3 - x^2$ is reflected in the x-axis, the graph becomes $y = -x^3 + x^2$, and if $y = x^3 - x^2$ is reflected in the y-axis, the graph becomes $y = -x^3 + x^2$.

The correct answer is **B**.

4 Exponential and logarithmic functions

Topic	4	Exponential and logarithmic functions
Subtopic	4.2	Logarithm laws and equations

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Source: *VCE 2020, Mathematical Methods Exam 1, Q4; © VCAA*

Question 1 (3 marks)

Solve the equation $2\log_2(x+5) - \log_2(x+9) = 1$.

Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q10; © VCAA*

Question 2 (1 mark)

Given that $\log_2(n+1) = x$, the values of n for which x is a positive integer are

A. $n = 2^k, k \in Z^+$
B. $n = 2^k - 1, k \in Z^+$
C. $n = 2^{k-1}, k \in Z^+$
D. $n = 2k - 1, k \in Z^+$
E. $n = 2k, k \in Z^+$

Topic 4 | Subtopic 4.2 Logarithm laws and equations

***Source:** VCE 2019, Mathematical Methods Exam 2, Section A, Q20; © VCAA*

Question 3 (1 mark)

The expression $\log_x(y) + \log_y(z)$, where x, y and z are all real numbers greater than 1, is equal to

A. $-\dfrac{1}{\log_y(x)} - \dfrac{1}{\log_z(y)}$

B. $\dfrac{1}{\log_x(y)} + \dfrac{1}{\log_y(z)}$

C. $-\dfrac{1}{\log_x(y)} - \dfrac{1}{\log_y(z)}$

D. $\dfrac{1}{\log_y(x)} - \dfrac{1}{\log_z(y)}$

E. $\log_y(x) + \log_z(y)$

***Source:** VCE 2014, Mathematical Methods (CAS) Exam 1, Q6; © VCAA*

Question 4 (2 marks)

Solve $\log_e(x) - 3 = \log_e\left(\sqrt{x}\right)$ for x, where $x > 0$.

***Source:** VCE 2013, Mathematical Methods (CAS) Exam 2, Section 1, Q18; © VCAA*

Question 5 (1 mark)

let $g(x) = \log_2(x),\ x > 0$.

Which one of the following equations is true for all positive real values of x?

A. $2g(8x) = g(x)^2 + 8$

B. $2g(8x) = g(x)^2 + 6$

C. $2g(8x) = (g(x) + 8)^2$

D. $2g(8x) = g(2x) + 6$

E. $2g(8x) = g(2x) + 64$

Question 6 (1 mark)

$729^{-\frac{2}{3}}$ is equal to

A. $\frac{1}{81}$

B. $-\frac{1}{81}$

C. 486

D. $\frac{1}{19\,683}$

E. -81

Question 7 (1 mark)

$\left(\frac{9y^2}{4x}\right)^{-\frac{1}{2}}$ is equal to

A. $\frac{2\sqrt{x}}{3y}$

B. $\frac{3y}{\sqrt{x}}$

C. $\frac{\sqrt{x}}{3y}$

D. $-\frac{9y^2}{8x}$

E. $-\frac{\sqrt{x}}{3y}$

Question 8 (2 marks)

Simplify $\frac{12^{-5} \times 2^3 \times 9^{-8}}{6^2}$.

Topic 4 | Subtopic 4.2 Logarithm laws and equations

Question 9 (2 marks)

Simplify $\frac{2^{x+3}}{16^{3x}} \times \frac{8}{2^x} \div \frac{4^{3x+1}}{2^4}$.

Question 10 (2 marks)

Solve $\log_2(6 - x) - \log_2(4 - x) = 2$ for x, where $x < 4$.

Question 11 (1 mark)

The solution of the equation $\log_9(x + 1) + \log_9(2x - 7) = 2$ over R is

A. $x = 8$ only

B. $x = \frac{87}{3}$ only

C. $x = 8$ or $x = -\frac{11}{2}$

D. $\log_9 8$ only

E. $x = \frac{11}{2}$ only

Question 12 (1 mark)

If $\log_e(y) = n\log_e(x) + c$ then

A. $y = nx + c$

B. $y = cx^n$

C. $y = nx + e^c$

D. $y = x^n + e^c$

E. $y = e^c x^n$

Question 13 (2 marks)

Solve the equation $2\log_e(x) - \log_e(x + 1) = \log_e\left(\frac{1}{2}\right)$ for x.

Topic	4	Exponential and logarithmic functions
Subtopic	4.3	Logarithmic scales

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Question 1 (2 marks)

Octaves in music can be measured in cents, n. The frequencies of two notes, f_1 and f_2, are related by the equation

$$n = 1200\ \log_{10}\left(\frac{f_2}{f_1}\right).$$

Middle C on the piano has a frequency of 256 hertz; the C an octave higher has a frequency of 512 hertz. Calculate the number of cents between these two Cs.

Question 2 (3 marks)

Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss. A gunshot from a .22 rifle has an intensity of about $(2.5 \times 10^{13})I_0$.

Calculate the loudness, in decibels, of the gunshot sound and state if ear protection should be worn when a person goes to a rifle range for practice shooting. Use the formula $L = 10\ \log_{10}\left(\frac{I}{I_0}\right)$, where I_0 is equal to 10^{-12} W/m^2, and give your answer correct to 2 decimal places.

Question 3 (4 marks)

Early in the 20th century, San Francisco had an earthquake that measured 8.3 on the magnitude scale. In the same year, another earthquake was recorded in South America that was four times stronger than the one in San Francisco. Using the equation $M = 0.67\ \log_{10}\left(\frac{E}{K}\right)$, where M is the magnitude of the earthquake and $\frac{E}{K}$ is the ratio between the largest and smallest waves, calculate the magnitude of the earthquake in South America, correct to 1 decimal place.

Topic	4	Exponential and logarithmic functions
Subtopic	4.4	Indicial equations

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Source: *VCE 2017, Mathematical Methods Exam 2, Section A, Q8; © VCAA*

Question 1 (1 mark)

If $y = a^{b-4x} + 2$, where $a > 0$, then x is equal to

A. $\frac{1}{4}\left(b - \log_a(y-2)\right)$

B. $\frac{1}{4}\left(b - \log_a(y+2)\right)$

C. $b - \log_a\left(\frac{1}{4}(y+2)\right)$

D. $\frac{b}{4} - \log_a(y-2)$

E. $\frac{1}{4}\left(b + 2 - \log_a(y)\right)$

Source: *VCE 2015, Mathematical Methods (CAS) Exam 1, Q7b; © VCAA*

Question 2 (3 marks)

Solve $3e^t = 5 + 8e^{-t}$ for t.

Source: *VCE 2014, Mathematical Methods (CAS) Exam 1, Q4; © VCAA*

Question 3 (2 marks)

Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

Question 4 (1 mark)
If $e^{x+4} = e^{2x-1}$, then x is equal to:

A. $-\frac{5}{3}$

B. 5

C. e^5

D. -5

E. $e^{-\frac{5}{3}}$

Question 5 (1 mark)
If $16\left(4^x\right) - 17 + 4^{-x} = 0$ then

A. $x = 0$ only

B. $x = -2$ only

C. $x = -1$ only

D. $x = -2$ or $x = 0$

E. $x = -1$ or $x = 0$

Question 6 (2 marks)
Find all values of x for the equation $e^{2x} - 3e^x + 2 = 0$.

Question 7 (1 mark)
If $32^{x+1} = \frac{1}{8}$, then x is equal to

A. $\frac{2}{5}$

B. $-\frac{2}{5}$

C. $-\frac{4}{5}$

D. $-\frac{8}{5}$

E. $-\frac{3}{5}$

Topic	4	Exponential and logarithmic functions
Subtopic	4.5	Logarithmic graphs

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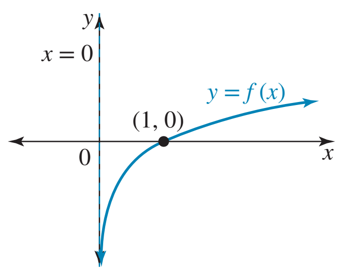
Question 1 (3 marks)

The graph of the equation $y = f(x)$ is shown.



Sketch the graph of:

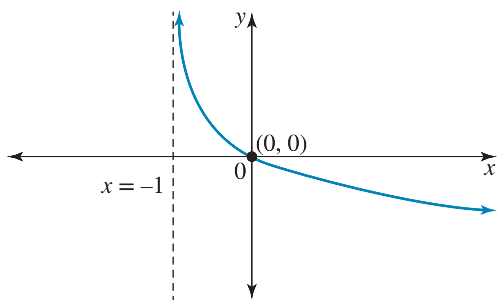
a. $y = f(-x)$ **(1 mark)**

b. $y = f(x - 1)$ **(1 mark)**

c. $y = f\left(\frac{x}{2}\right)$ **(1 mark)**

Question 2 (1 mark)

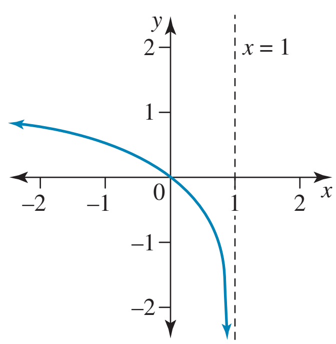
The graph of the function f is shown. The rule for f is most likely to be:



A. $f(x) = 1 + \log_e(x)$
B. $f(x) = \log_e(x) - 1$
C. $f(x) = 1 - \log_e(x)$
D. $f(x) = \log_e(x + 1)$
E. $f(x) = -\log_e(x + 1)$

Question 3 (1 mark)

If the equation of the graph shown is $y = \log_e(ax + b)$, then



A. $a = 1$ and $b = -1$
B. $a = 1$ and $b = 1$
C. $a = -1$ and $b = -1$
D. $a = -1$ and $b = 1$
E. $a = -1$ and $b = 0$

Question 4 (1 mark)

The graph of $y = \log_e(x)$ is transformed to the graph of $y = 3\log_e(3 - x)$ by

A. a reflection in the y-axis, a dilation by a scale factor of 3 units away from the y-axis, followed by a translation of 3 units to the left away from the x-axis.

B. a reflection in the x-axis, a dilation by a scale factor of 3 units away from the y-axis, followed by a translation of 3 units to the left away from the x-axis.

C. a reflection in the y-axis, a dilation by a scale factor of 3 units away from the x-axis, followed by a translation of 3 units to the right away from the y-axis.

D. a reflection in the x-axis, a dilation by a scale factor of 3 units away from the x-axis, followed by a translation of 3 units to the right away from the y-axis.

E. a reflection in the x-axis, a dilation by a scale factor of 3 units away from the x-axis, followed by a translation of $\frac{1}{3}$ units to the left away from the y-axis.

Question 5 (1 mark)

If the graph of $y = \log_e(x)$ is transformed to the graph of $y = 2\ \log_e(2x)$, which of the following options is **not** a correct sequence of transformations?

A. A dilation by a scale factor of 2 units parallel to the y-axis, followed by a dilation by a scale factor of $\frac{1}{2}$ units parallel to the x-axis.

B. A dilation by a scale factor of 2 units away from the x-axis, followed by a dilation by a scale factor of $\frac{1}{2}$ units away from the y-axis.

C. A dilation by a scale factor of 2 units parallel to the y-axis, followed by a dilation by a scale factor of 2 units away from the y-axis.

D. A dilation by a scale factor of 2 units parallel to the y-axis, followed by a vertical translation of $\log_e 4$ up and parallel to the y-axis.

E. A dilation by a scale factor of 2 units away from the x-axis, followed by a vertical translation of $\log_e 4$ up and away from the x-axis.

Question 6 (2 marks)

Let $f: R^+ \to R, f(x) = -\log_e(x)$. Find:

a. $f\left(\frac{x}{2}\right)$ **(1 mark)**

b. $4f(x) + 2$ **(1 mark)**

Topic	4	Exponential and logarithmic functions
Subtopic	4.6	Exponential graphs

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Source: *VCE 2018, Mathematical Methods Exam 2, Section A, Q7; © VCAA*

Question 1 (1 mark)

Let $f: R^+ \to R, f(x) = k\log_2(x), k \in R$.

Given that $f^{-1}(1) = 8$, the value of k is

A. 0

B. $\frac{1}{3}$

C. 3

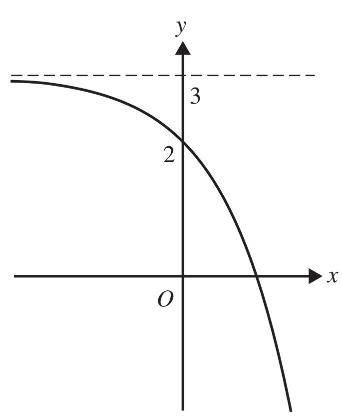
D. 8

E. 12

Source: *VCE 2013, Mathematical Methods (CAS) Exam 2, Section 1, Q4; © VCAA*

Question 2 (1 mark)

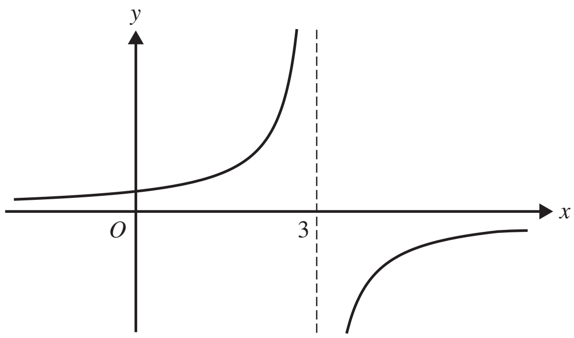
Part of the graph of $y = f(x)$, where $f: R \to R, f(x) = 3 - e^x$, is shown below.



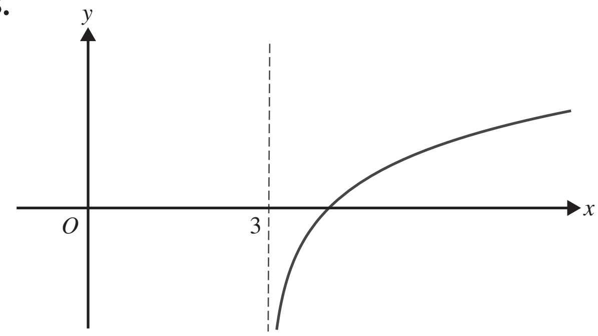
Topic 4 Subtopic 4.6 Exponential graphs

Which one of the following could be the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f?

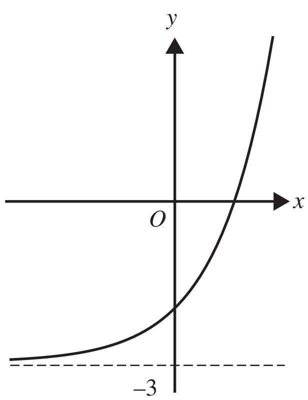
A.



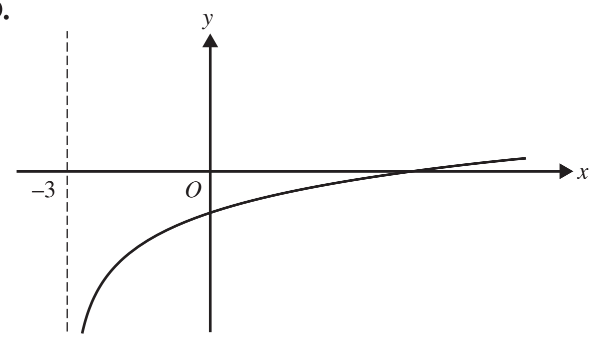
B.



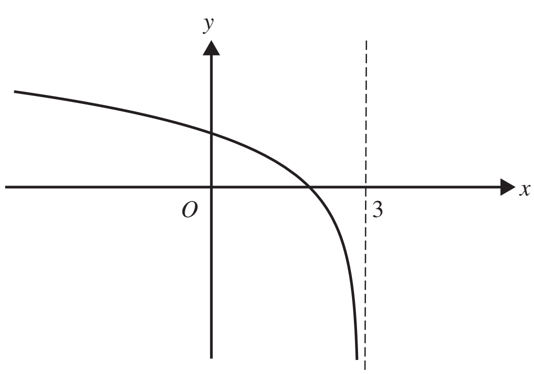
C.



D.



E.



Question 3 (1 mark)

Consider the function $f: R \rightarrow R$ where $f(x) = 2 - e^{-x}$.

Select the false statement from the following.

A. The range is $(-\infty, 2)$.

B. The domain is R.

C. The line $y = 2$ is a horizontal asymptote.

D. The graph crosses the x-axis at $\left(\log_e(2), 0\right)$.

E. The graph crosses the y-axis at $(0, 1)$

Question 4 (3 marks)

For the graphs of $y = x^n$ and $y = x^m$, where n and m are even and $n > m$, show that $x^n = x^m$ when $x = 0$, $x = 1$ and $x = -1$.

Question 5 (1 mark)

Consider the function $f: R \rightarrow R$, where $f(x) = e^{x+1} - 1$

Which of the following statements is **false**?

A. The domain is R.

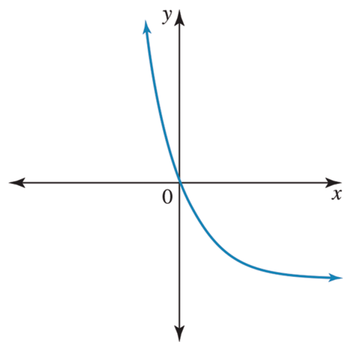
B. The line $y = -1$ is a horizontal asymptote.

C. The line $x = -1$ is a vertical asymptote.

D. The graph crosses the x-axis at $(-1, 0)$.

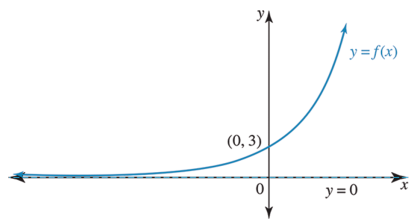
E. The graph crosses the y-axis at $(0, e - 1)$.

Question 6 (1 mark)
If the graph shown below has the equation $y = a\left(1 - e^{-kx}\right)$, then



A. $a < 0$ and $k > 0$
B. $a > 0$ and $k < 0$
C. $a > 0$ and $k > 0$
D. $a < 0$ and $k < 0$
E. $a > 0$ and $k = 0$

Question 7 (3 marks)
The graph of the equation $y = f(x)$ is shown.



Sketch the graph of:

a. $y = -f(x)$ **(1 mark)**

b. $y = f(2x)$ **(1 mark)**

c. $y = f(x) - 2$ **(1 mark)**

Question 8 (1 mark)
The graph of $y = e^x$ is transformed to the graph of $y = 2e^x + 3$ by:

A. a dilation of factor 2 from the y-axis followed by a horizontal translation of 3 units in the positive direction of the x-axis.
B. a dilation of factor 2 from the x-axis followed by a vertical translation of 3 units in the negative direction of the y-axis.
C. a dilation of factor 2 from the x-axis followed by a vertical translation of 3 units in the positive direction of the y-axis.
D. a dilation of factor 2 from in the y-axis followed by a horizontal translation of 3 units in the negative direction of the x-axis.
E. a dilation of factor 2 from the x-axis followed by a horizontal translation of 3 units in the positive direction of the x-axis.

Question 9 (1 mark)

If the graph of $y = e^x$ is dilated by a scale factor of 2 units parallel to the x-axis, reflected in the y-axis, and then translated 2 units up away from the x-axis, it becomes the graph of

A. $y = 2 + e^{-\frac{x}{2}}$
B. $y = 2 - e^{\frac{x}{2}}$
C. $y = 2 + e^{-2x}$
D. $y = 2 - e^{-2x}$
E. $y = -2 + e^{\frac{x}{2}}$

Question 10 (1 mark)

The graph of $y = e^x$ is transformed to the graph of $y = 3e^{-3x}$ by

A. a reflection in the y-axis, a dilation by a scale factor of 3 units away from the y-axis, followed by a dilation by a scale factor of 3 units away from the x-axis.
B. a reflection in the x-axis, a dilation by a scale factor of 3 units away from the y-axis, followed by a dilation by a scale factor of 3 units away from the x-axis.
C. a reflection in the y-axis, a dilation by a scale factor of 3 units away from the x-axis, followed by a dilation by a scale factor of 3 units away from the y-axis.
D. a reflection in the x-axis, a dilation by a scale factor of 3 units away from the x-axis, followed by a dilation by a scale factor of $\frac{1}{3}$ units away from the y-axis.
E. a reflection in the y-axis, a dilation by a scale factor of 3 units away from the x-axis, followed by a dilation by a scale factor of $\frac{1}{3}$ units away from the y-axis.

Question 11 (2 marks)

Let $f: R \to R, f(x) = e^{2x} - 1$. Find:

a. $f(2x)$ **(1 mark)**

b. $3f(x+1)$ **(1 mark)**

Topic	4	Exponential and logarithmic functions
Subtopic	4.7	Applicationss

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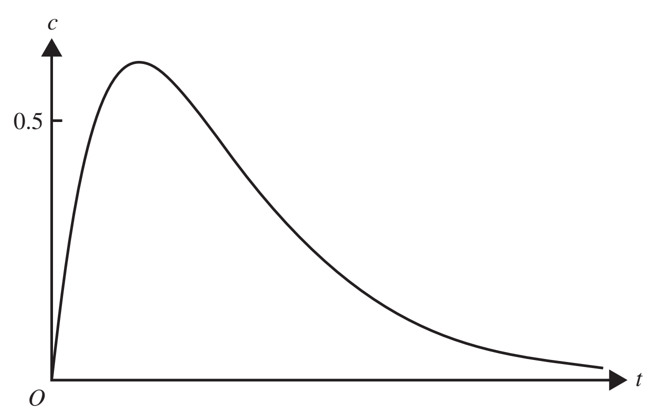
Source: *VCE 2014, Mathematical Methods (CAS) Exam 2, Section 2, Q3; © VCAA*

Question 1 (4 marks)

In a controlled experiment, Juan took some medicine at 8 pm. The concentration of medicine in his blood was then measured at regular intervals. The concentration of medicine in Juan's blood is modelled by the function $c(t) = \frac{5}{2}te^{-\frac{3t}{2}}, t \geq 0$, where c is the concentration of medicine in his blood, in milligrams per litre, t hours after 8 pm. Part of the graph of the function c is shown below.



a. What was the maximum value of the concentration of medicine in Juan's blood, in milligrams per litre, correct to two decimal places? **(1 mark)**

b. Find the value of t, in hours, correct to two decimal places, when the concentration of medicine in Juan's blood first reached 0.5 milligrams per litre. **(1 mark)**

c. Find the length of time that the concentration of medicine in Juan's blood was above 0.5 milligrams per litre. Express the answer in hours, correct to two decimal places. **(2 marks)**

Question 2 (4 marks)
In her chemistry class, Hei is preparing a special solution for an experiment that she has to complete. The concentration of the solution can be modelled by the rule

$$C = A \log_e(kt)$$

where C is the concentration in moles per litre (M) and t represents the time of mixing in seconds. The concentration of the solution after 30 seconds of mixing is 4 M, and the concentration of the solution after 2 seconds of mixing was 0.1 M.

a. Calculate the values of the constants A and k, giving your answers correct to 3 decimal places. **(2 marks)**

b. Determine the concentration of the solution after 15 seconds of mixing. **(1 mark)**

c. Determine how long it will take, in minutes and seconds, for the concentration of the solution to reach 10 M. **(1 mark)**

Question 3 (7 marks)

Manoj pours himself a mug of coffee but gets distracted by a phone call before he can drink the coffee. The temperature of the cooling mug of coffee is given by $T = 20 + 75e^{-0.062t}$, where T is the temperature of the coffee t minutes after it was initially poured into the mug.

a. Calculate the initial temperature of the coffee when it was first poured. **(1 mark)**

b. Determine the temperature to which the coffee will cool if left unattended. **(1 mark)**

c. Determine how long it will take for the coffee to reach a temperature of 65 °C. Give your answer correct to 2 decimal places. **(1 mark)**

d. Manoj returns to the coffee when it has reached 65 °C and decides to reheat the coffee in a microwave. The temperature of the coffee in this warming stage is $T = A + Be^{-0.05t}$.
Given that the temperature of the reheated coffee cannot exceed 85 °C, calculate the values of A and B. **(2 marks)**

e. Sketch a graph showing the temperature of the coffee during its cooling and warming stages. **(2 marks)**

Topic	4	Exponential and logarithmic functions
Subtopic	4.8	Review

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Source: *VCE 2021, Mathematical Methods Exam 2, Q2; © VCAA*

Question 1 (1 mark)

The graph of $y = \log_e(x) + \log_e(2x)$, where $x > 0$, is identical, over the same domain, to the graph of

A. $y = 2\log_e\left(\frac{1}{2}x\right)$

B. $y = 2\log_e(2x)$

C. $y = \log_e\left(2x^2\right)$

D. $y = \log_e(3x)$

E. $y = \log_e(4x)$

Question 2 (7 marks)

A kettle was filled with water and the water heated to 98 °C. The kettle was then switched off at 1 pm and the water began to cool. By 3 pm, the temperature of the water was 58 °C. The temperature, T °C, of the water t hours after 1 pm is modelled by the rule $T = Ae^{-kt} + 18$.

a. Calculate the value of A and k. **(2 marks)**

b. Calculate the temperature of the water at 11 pm. Give your answer to 1 decimal place. **(1 mark)**

c. Sketch the graph of T versus t. **(2 marks)**

d. Find the time after which the temperature of the water will be less than 22 °C. **(1 mark)**

e. Explain what happens to the temperature in the long term. **(1 mark)**

Question 3 (5 marks)

A biologist conducts an experiment to determine conditions that affect the growth of bacteria. Her initial experiment finds the growth of the population of bacteria is modelled by the rule $N = 22 \times 2^t$, where N is the number of bacteria present after t days.

a. Calculate how long it will take for the number of bacteria to reach 2816. **(1 mark)**

b. Explain what will happen to the number of bacteria in the long term according to this model. **(1 mark)**

c. The biologist changes the conditions of her experiment and starts with a new batch of bacteria. She finds that under the changed conditions the growth of the population of bacteria is modelled by the rule $N = \dfrac{66}{1 + 2e^{-0.2t}}$.

i. Show that in both of her experiments the biologist used the same initial number of bacteria. **(2 marks)**

ii. Explain what will happen to the number of bacteria in the long term according to her second model. **(1 mark)**

Question 4 (5 marks)

In some parts of the world there have been measles (rubella) epidemics. For one such epidemic in Wales, the number of people in the population infected was modelled by the rule

$$P(t) = Ae^{kt}$$

where t was the number of days after the epidemic began. At the beginning of the epidemic, 200 cases were reported to authorities, but 30 days later there were 1000 cases.

a. Calculate the values of the constants A and k. Give k correct to 4 decimal places. **(2 marks)**

b. Calculate the expected number of cases after 60 days. Give your answer correct to the nearest integer. **(1 mark)**

c. Calculate how long it would have taken for the number of cases to reach 6000. Give your answer correct to 1 decimal place. **(1 mark)**

d. Thirty-eight thousand young people in Wales were at risk of contracting measles because they had not been immunised against rubella. If the epidemic went unchecked, find how long would it have taken for all these young people to be infected. Give your answer correct to 1 decimal place. **(1 mark)**

Question 5 (2 marks)

Find all values of x for the equation $e^{2x} - 3e^{x} + 2 = 0$.

***Source:** VCE 2018, Mathematical Methods Exam 2, Section A, Q7; © VCAA*

Question 6 (1 mark)

The table below shows the mass m, in grams, of a substance at a time t hours.

t	1	2	3	4	5	6
m	7.4	5.3	3.9	2.8	2.1	1.5

The mass and time would be best modelled using

A. a linear function.
B. an exponential function.
C. a power function.
D. a circular function.
E. a logarithmic function.

Question 7 (1 mark)

Find the domain of the inverse function of f: $[3, \infty) \to R$, $f(x) = \log_e(x - 2)$.

Source: *VCE 2016, Mathematical Methods Exam 1, Q5; © VCAA*

Question 8 (11 marks)

Let $f:(0, \infty) \to R$, where $f(x) = \log_e(x)$ and $g: R \to R$, where $g(x) = x^2 + 1$.

a. **i.** Find the rule for h, where $h(x) = f(g(x))$. **(1 mark)**

ii. State the domain and range of h. **(2 marks)**

iii. Show that $h(x) + h(-x) = f\left((g(x))^2\right)$. **(2 marks)**

iv. Find the coordinates of the stationary point of h and state its nature. **(2 marks)**

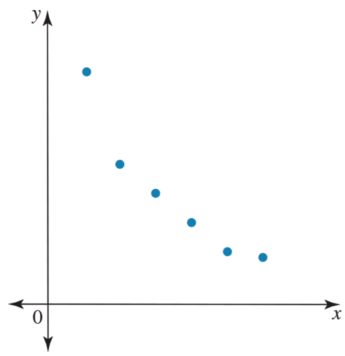
b. Let $k: (-\infty, 0) \to R$, where $k(x) = \log_e\left(x^2 + 1\right)$.

i. Find the rule for k^{-1}. **(2 marks)**

ii. State the domain and range of k^{-1}. **(2 marks)**

Question 9 (1 mark)

The graph below shows the relationship between two variables, x and y.
If a is a positive constant, the equation relating x and y is most likely



A. $y = a\sqrt{x}$
B. $y = ax^2$
C. $y = \frac{a}{x}$
D. $y = ae^x$
E. $y = a\log_e(x)$

Question 10 (1 mark)

Let $g(x) = 2^x$, $x \in R$. Which one of the following equations is true for all positive real values of x?
A. $g(x) + g(y) = 2^{x+y}$
B. $g(2x) = 2g(x)$
C. $2g(2x) = 4g(x)$
D. $g(x)g(y) = 2^{x+y}$
E. $g(x) - g(y) = 2^{2x+y}$

Question 11 (2 marks)

Solve the following for x.

a. $ae^{bx} - c = 0$ **(1 mark)**

b. $b\log_e(x - a) = c$ **(1 mark)**

Answers and marking guide

4.2 Logarithm laws and equations

Question 1

$$2\log_2(x+5) - \log_2(x+9) = 1$$
$$\log_2(x+5)^2 - \log_2(x+9) = 1$$
$$\log_2\left(\frac{(x+5)^2}{x+9}\right) = 1$$
$$\frac{(x+5)^2}{x+9} = 2^1 = 2$$
$$(x+5)^2 = 2(x+9)$$
$$x^2 + 10x + 25 = 2x + 18$$
$$x^2 + 8x + 7 = 0$$
$$(x+7)(x+1) = 0$$
$$x = -7, x = -1 \text{ but } x > -5$$
$$x = -1 \text{ only}$$

Award 1 mark for using the correct log laws.
Award 1 mark for solving.
Award 1 mark for only one correct answer.

***VCAA Examination Report* note:**

Students confidently attempted this question; however, many incorrect uses of the logarithmic laws were observed. Those who did end up with the appropriate quadratic equation and solved it correctly did not always check the validity of their answers; these students failed to reject the solution $x = -7$.

Question 2

$\log_2(n+1) = x, x \in Z^+$
$\log_2(n+1) = k, k \in Z^+$
$n + 1 = 2^k, n = 2^k - 1$
The correct answer is **B**.

Question 3

By the change of base rule:

$$\log_a(b) = \frac{1}{\log_b(a)}$$
$$\log_x(y) + \log_y(z) = \frac{1}{\log_y(x)} + \frac{1}{\log_z(y)}$$

The correct answer is **D**.

Question 4

$$\log_e(x) - 3 = \log_e\left(\sqrt{x}\right) \quad \text{for } x > 0$$
$$\log_e(x) - \log_e\left(\sqrt{x}\right) = 3$$
$$\log_e\left(\frac{x}{\sqrt{x}}\right) = 3$$
$$\log_e\left(\sqrt{x}\right) = 3$$
$$\sqrt{x} = e^3$$
$$x = e^6$$

Award 1 mark for correct log laws.
Award 1 mark for the correct final value of x.

***VCAA Assessment Report* note:**
Poor performance on this question was mainly attributed to the incorrect application of logarithm or index laws. Students should have good facility with logarithm and exponent laws.

Question 5

$$\begin{aligned} g(x) &= \log_2(x) \\ 2g(8x) &= 2\log_2(8x) \\ &= 2\log_2(8) + 2\log_2(x) \\ &= 2\log_2(2^3) + \log_2(x^2) \\ &= 6\log_2(2) + \log_2(x^2) \\ &= 6 + g(x^2) \\ &= g(x^2) + 6 \end{aligned}$$

Question 6

$$729^{-\frac{2}{3}} = \frac{1}{\left(\sqrt[3]{729}\right)^2} = \frac{1}{9^2} = \frac{1}{81}$$

Question 7

$$\left(\frac{9y^2}{4x}\right)^{-\frac{1}{2}} = \sqrt{\frac{4x}{9y^2}} = \frac{2\sqrt{x}}{3y}$$

Question 8

$$\begin{aligned} \frac{12^{-5} \times 2^3 \times 9^{-8}}{6^2} &= \frac{\left(2^2 \times 3\right)^{-5} \times 2^3 \times \left(3^2\right)^{-8}}{(3 \times 2)^2} \\ &= \frac{2^{-10} \times 3^{-5} \times 2^3 \times 3^{-16}}{3^2 \times 2^2} \qquad \textbf{[1 mark]} \\ &= \frac{1}{2^9 \times 3^{23}} \qquad \textbf{[1 mark]} \end{aligned}$$

Question 9

$$\begin{aligned} \frac{2^{x+3}}{16^{3x}} \times \frac{8}{2^x} \div \frac{4^{3x+1}}{2^4} &= \frac{2^x \times 2^3 \times 2^3 \times 2^4}{2^{12x} \times 2^x \times 2^{6x} \times 2^2} \qquad \textbf{[1 mark]} \\ &= \frac{2^x \times 2^{10}}{2^{19x} \times 2^2} \\ &= \frac{2^8}{2^{18x}} \qquad \textbf{[1 mark]} \end{aligned}$$

Question 10

$\log_2(6 - x) - \log_2(4 - x) = 2$ for $x < 4$

$$\begin{aligned} \log\left(\frac{6-x}{4-x}\right) &= 2 \\ \frac{6-x}{4-x} &= 2^2 \\ \frac{6-x}{4-x} &= 4 \\ 6 - x &= 4(4 - x) \end{aligned}$$

$$6 - x = 16 - 4x$$
$$3x = 10$$
$$x = \frac{10}{3}$$

Award 1 mark for correct log laws.
Award 1 mark for the correct final value of x.

Question 11

$\log_9(x+1) + \log_9(2x-7) = 2$
$\log_9(x+1)(2x-7) = 2$
$(x+1)(2x-7) = 9^2$
$2x^2 - 5x - 7 = 81$
$2x^2 - 5x - 88 = 0$
$(2x+11)(x-8) = 0$
$x = -\frac{11}{2}$ or $x = 8$ but $x > -1$
The only answer is $x = 8$.

Question 12

$$\log_e(y) = n\log_e(x) + c$$
$$\log_e(y) = \log_e(x^n) + c$$
$$\log_e(y) - \log_e(x^n) = c$$
$$\log_e\left(\frac{y}{x^n}\right) = c$$
$$\frac{y}{x^n} = e^c$$
$$y = e^c x^n$$

Question 13

$$\log_e(x^2) - \log_e(x+1) = \log_e\left(\frac{1}{2}\right)$$
$$\frac{x^2}{x+1} = \frac{1}{2}$$
$2x^2 - x - 1 = 0$ [**1 mark**]
$$(2x+1)(x-1) = 0$$
$x = 1$ or $x = \frac{-1}{2}$ [**1 mark**]

4.3 Logarithmic scales

Question 1

$n = 1200\log_{10}\left(\frac{f_2}{f_1}\right)$ [**1 mark**]
$f_1 = 256, \quad f_2 = 512$
$n = 1200\log_{10}\left(\frac{512}{256}\right)$
$n = 361$ cents [**1 mark**]

Question 2

$$L = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$$

.22 rifle:

$$I = \left(2.5 \times 10^{13}\right) I_0$$
$$= 2.5 \times 10^{13} \times 10^{-12}$$
$$= 2.5 \times 10 \qquad \textbf{[1 mark]}$$
$$L = 10 \log_{10}\left(\frac{2.5 \times 10}{10^{-12}}\right)$$
$$L = 10\left(\log_{10}(2.5 \times 10) - \log_{10}(10)^{-12}\right)$$
$$L = 10\left(\log_{10}(2.5) + \log_{10}(10) + 12\log_{10}(10)\right)$$
$$L = 10\left(\log_{10}(2.5) + 13\right)$$
$$L = 133.98 \qquad \textbf{[1 mark]}$$

The loudness of the gunshot is about 133.98 dB, so ear protection should be worn. **[1 mark]**

Question 3

$$M = 0.67 \log_{10}\left(\frac{E}{K}\right)$$

San Francisco: $M_{\text{SF}} = 8.3$

$$8.3 = 0.67 \log_{10}\left(\frac{E_{SF}}{K}\right) \qquad \textbf{[1 mark]}$$
$$12.3881 = \log_{10}\left(\frac{E_{SF}}{K}\right)$$
$$10^{12.3881} = \frac{E_{SF}}{K} \qquad \textbf{[1 mark]}$$

South America: $M_{\text{SA}} = 4E_{\text{SF}}$

$$M_{\text{SA}} = 0.67 \log_{10}\left(\frac{4E_{\text{SF}}}{K}\right) \qquad \textbf{[1 mark]}$$

Substitute $10^{12.3881} = \dfrac{E_{\text{SF}}}{K}$.

$$M_{\text{SA}} = 0.67 \log_{10}\left(4 \times 10^{12.3881}\right)$$
$$= 8.7 \qquad \textbf{[1 mark]}$$

The magnitude of the South American earthquake was 8.7.

4.4 Indicial equations

Question 1

$$y = a^{b-4x} + 2$$
$$y - 2 = a^{b-4x}$$
$$\log_a(y-2) = b - 4x$$
$$4x = b - \log_a(y-2)$$
$$x = \frac{1}{4}\left(b - \log_a(y-2)\right)$$

The correct answer is **A**.

Question 2

$$3e^t = 5 + 8e^{-t}$$

Let $u = e^t$: $e^{-t} = \dfrac{1}{e^t} = \dfrac{1}{u}$

$$3u = 5 + \frac{8}{u}$$
$$3u^2 = 5u + 8$$
$$3u^2 - 5u - 8 = 0$$
$$(3u - 8)(u + 1) = 0$$

$u = e^t = \frac{8}{3}$ or $u = e^t = -1$ (no solution)

$\text{t} = \log_e\left(\frac{8}{3}\right)$

Award 1 mark for solving for u.
Award 1 mark for rejecting one of the solutions.
Award 1 mark for the correct final value.

***VCAA Assessment Report* note:**
This question was not answered well. Many students were unable to create the quadratic equation evolved from manipulating e^{-t}. Many students solved via the quadratic formula rather than using simpler factorising techniques. The feasibility of only one answer was generally well handled.

Question 3

$$2^{3x-3} = 8^{2-x}$$
$$2^{3x-3} = \left(2^3\right)^{2-x}$$
$$= 2^{6-3x}$$
$$3x - 3 = 6 - 3x$$
$$6x = 9$$
$$x = \frac{3}{2}$$

Award 1 mark for correct manipulation of indices.
Award 1 mark for the correct answer.

***VCAA Assessment Report* note:**
Some students chose to work with a common base of 8.
Students are reminded to simplify their final answer, especially for fraction answers.

Question 4

$$e^{x+4} = e^{2x-1}$$
$$\Rightarrow x + 4 = 2x - 1$$
$$\Rightarrow x = 5$$

Question 5

$16\left(4^x\right) - 17 + 4^{-x} = 0$ let $u = 4^x$

$$16u - 17 + \frac{1}{u} = 0$$
$$16u^2 - 17u + 1 = 0$$
$$(16u - 1)(u - 1) = 0$$

$u = 4^x = \frac{1}{16}$ and $u = 4^x = 1$

$x = -2$ or $x = 0$

Question 6

Let $w = e^x$.

$$w^2 - 3w + 2 = 0$$

$(w - 2)(w - 1) = 0$ **[1 mark]**

$w = 2$ or $w = 1$

$e^x = 2$ or $e^x = 1$

$x = \log_e(2)$ or $x = 0$ **[1 mark]**

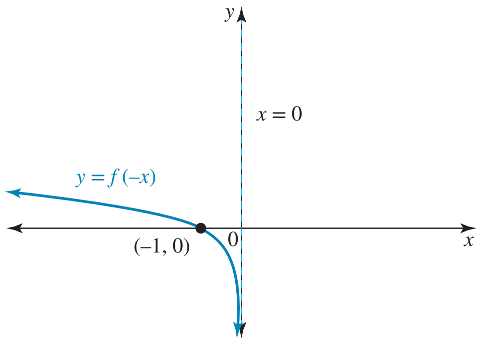
Question 7

$$2^{5x+5} = 2^{-3}$$
$$5x + 5 = -3$$
$$x = \frac{-8}{5}$$

4.5 Logarithmic graphs

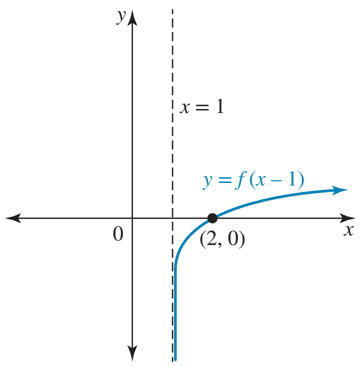
Question 1

a.



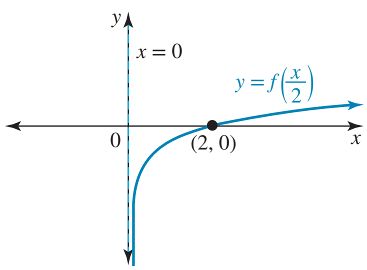
Award 1 mark for correct curve and labelled asymptote.

b.



Award 1 mark for correct curve and labelled asymptote.

c.



Award 1 mark for correct curve and labelled asymptote.

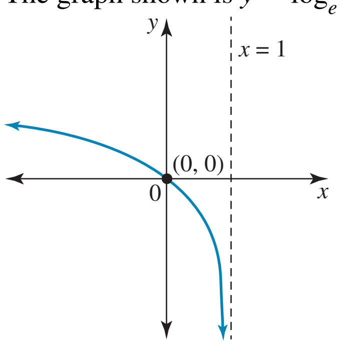
Question 2

$f(x) = -\log_e(x+1)$

The correct answer is **E**.

Question 3

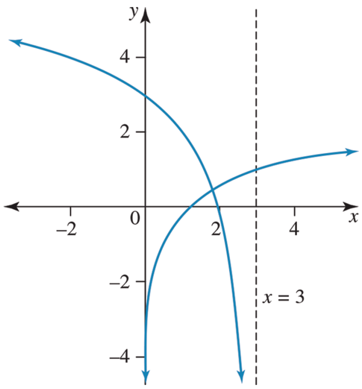
The graph shown is $y = \log_e(1-x)$. It has domain $(-\infty, 1)$ and range R.



Hence, $a = -1$ and $b = 1$.

The correct answer is **D**.

Question 4



Question 5

Since $y = 2\log_e(2x) = 2\left[\log_e(x) + \log_e(2)\right] = 2\log_e(x) + \log_e(4)$, options A, B, D and E are true, and option C is false.

Question 6

a. $f\left(\frac{x}{2}\right) = -\log_e\left(\frac{x}{2}\right)$

b. $4f(x) + 2 = -4\log_e(x) + 2$

4.6 Exponential graphs

Question 1

$f: R^+ \to R, f(x) = k\log_2(x), k \in R$

$f^{-1}(1) = 8 \Leftrightarrow f(8) = 1$

$f(8) = k\log_2(8) = 1$

$\log_2(8) = \frac{1}{k}$

$2^{\frac{1}{k}} = 8 = 2^3$

$\frac{1}{k} = 3$

$k = \frac{1}{3}$

The correct answer is **B**.

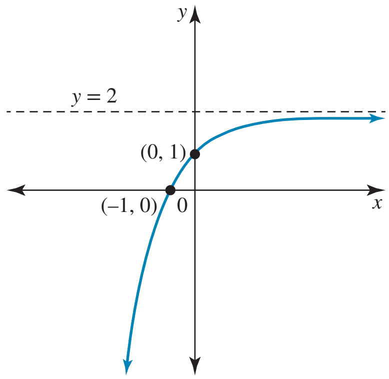
Question 2

For the function f, $y = 3$ is a horizontal asymptote that crosses the y-axis at $(0, 2)$.
The inverse function f^{-1} has $x = 3$ as a vertical asymptote that crosses the x-axis at $(2, 0)$.
The correct answer is **E**.

Question 3

All of A, B, C, and E are true, and D is false when $y = 0$.

$2 - e^{-x} = 0 \Rightarrow e^{-x} = 2$ or $e^{-x} = \frac{1}{2}$, so $x = \log_e\left(\frac{1}{2}\right)$.



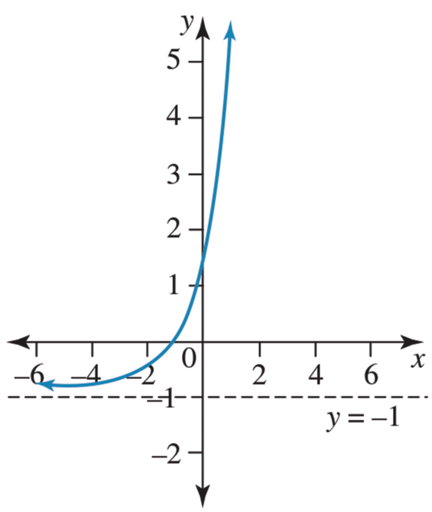
The correct answer is **D**.

Question 4

$$\text{Let} \quad n = 2m$$
$$x^n = x^{2m}$$
$$= \left(x^2\right)^m$$
$$\text{Let} \quad n = 4,\ m = 2,\ x = 0$$
$$(0)^4 = (0)^2$$
$$\text{When } x = 0$$
$$0^n = \left(0^2\right)^m$$
$$= 0 \quad \textbf{[1 mark]}$$
$$\text{Let} \quad n = 4,\ m = 2,\ x = 1$$
$$(1)^4 = (1)^2$$
$$\text{When} \quad x = 1$$
$$1^n = \left(1^2\right)^m$$
$$= 1 \quad \textbf{[1 mark]}$$
$$\text{Let} \quad n = 4,\ m = 2,\ x = -1$$
$$(-1)^4 = (-1)^2$$
$$\text{When } x = -1$$
$$(-1)^n = (-1)^{2m}$$
$$= 1 \quad \textbf{[1 mark]}$$

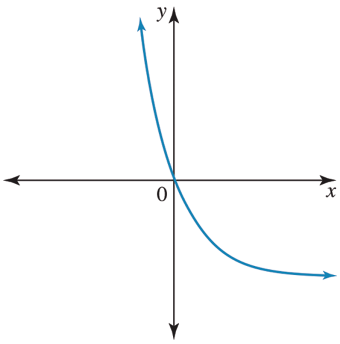
Question 5

A, B, D and E are all true. C is false, as there are no vertical asymptotes, and the domain is R.
When $f(-1) = e^0 - 1 = 0\,(-1, 0)$ and $f(0) = e - 1, (0, e - 1)$



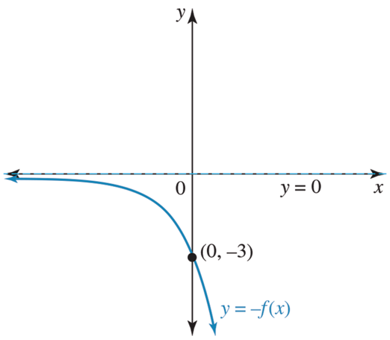
Question 6

A typical graph is $y = -\left(1 - e^{-2x}\right)$ so that $ak > 0$.



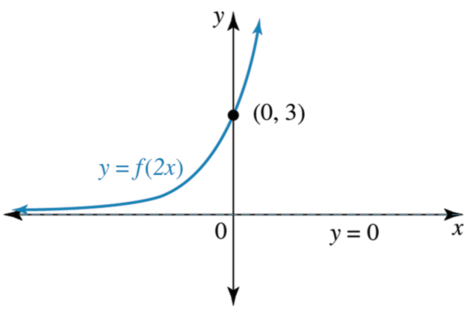
Question 7

a.



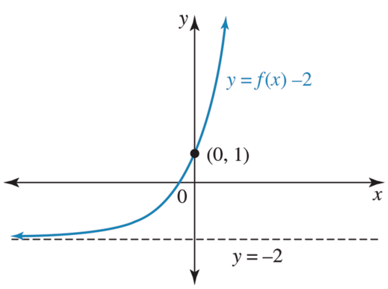
Award 1 mark for correct curve and labelled asymptote.

b.



Award 1 mark for correct curve and labelled asymptote.

c.

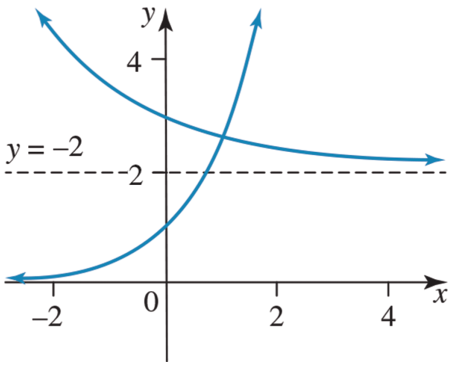


Award 1 mark for correct curve and labelled asymptote.

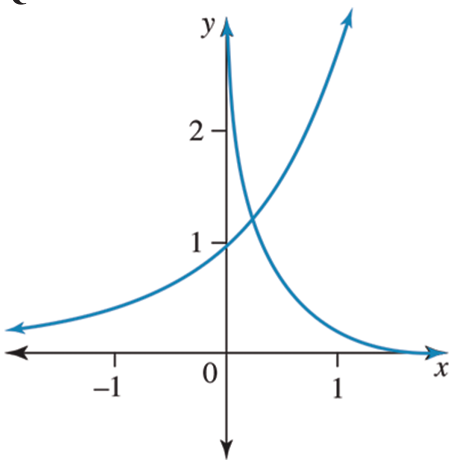
Question 8

a dilation of factor 2 from the x-axis followed by a vertical translation of 3 units in the positive direction of the x-axis.

Question 9



Question 10



Question 11

a. $f(2x) = e^{4x} - 1$ **[1 mark]**

b. $3f(x+1) = 3\left(e^{2(x+1)} - 1\right)$

$= 3e^{2x+2} - 3$ **[1 mark]**

4.7 Applications

Question 1

a. $c(t) = \frac{5}{2}te^{-\frac{3t}{2}}$

$c'(t) = \frac{5}{4}e^{-\frac{3t}{2}}(2 - 3t)$

For maximum or minimum:

$c'(t) = 0$

$\Rightarrow 2 - 3t = 0$

$t = \frac{2}{3}$

$c\left(\frac{2}{3}\right) = \frac{5}{3}e^{-1} = 0.61$ mg/L

Award 1 mark for the correct maximum value.

VCAA Assessment Report **note:**

This question was answered well. Some students had incorrect units, such as mm for milligrams. Some left their answers in exact form. Some found t correct to two decimal places and left their answer as 0.67.

b. Solving using CAS:

$c(t) = 0.5 \Rightarrow t_1 = 0.33$

Award 1 mark for the correct time.

***VCAA Assessment Report* note:**

This question was answered well. Some students gave two answers, 0.33 and 1.19, instead of only the first one, as specified in the question. Some students rounded incorrectly and gave 0.32 as their answer.

c. $c(t) = 0.5 \Rightarrow t_1 = 0.3263\ t_2 = 1.1876$

$$\begin{aligned} t_2 - t_1 &= 1.1876 - 0.3263 \\ &= 0.86 \text{ hours} \end{aligned}$$

Award 1 mark for finding the other time.

Award 1 mark for subtracting times, correct to 2 decimal places

***VCAA Assessment Report* note:**

Students should always work to suitable accuracy in intermediate calculations to support rounding the answer to the required accuracy. Some students wrote down the two values but did not find the difference for the length of time. Some added the two values. Some students incorrectly converted the time to minutes.

Question 2

a. $C = A\log_e(kt)$

When $t = 2,\ C = 0.1,$

$0.1 = A\log_e(2k)$ [1]

When $t = 30,\ C = 4,$

$4 = A\log_e(30k)$ [2]

[2] ÷ [1]:

$$\begin{aligned} \frac{A\log_e(30k)}{A\log_e(2k)} &= \frac{4}{0.1} \\ \log_e(30k) &= 40\log_e(2k) \\ \log_e(30) + \log_e(k) &= 40\left(\log_e(2) + \log_e(k)\right) \\ \log_e(30) + \log_e(k) &= 40\log_e(2) + 40\log_e(k) \\ \log_e(30) - 40\log_e(2) &= 40\log_e(k) - \log_e(k) \\ \log_e(30) - 40\log_e(2) &= 39\log_e(k) \\ -24.3247 &= 39\log_e(k) \\ \frac{-24.3247}{39} &= \log_e(k) \\ -0.6237 &= \log_e(k) \\ e^{-0.6237} &= k \\ k &= 0.536 \quad [\textbf{1 mark}] \end{aligned}$$

Substitute $k = 0.536$ into [1]:

$0.1 = A\ \log_e(2 \times 0.536)$

$0.1 = 0.0695A$

$A = 1.439$

$C = 1.439\ \log_e(0.536t)$ [**1 mark**]

b. When $t = 15$,

$C = 1.439\ \log_e(0.536 \times 15) = 2.999\,\text{M}$

The concentration after 15 minutes is 2.999 M [**1 mark**]

c. When $C = 10$ M,

$$\begin{aligned} 10 &= 1.439\ \log_e(0.536t) \\ 6.9493 &= \log_e(0.536t) \\ e^{6.9493} &= 0.536t \end{aligned}$$

$1042.4198 = 0.536t$

$t = 1945$ (to the nearest second)

After 1945 seconds, or 32 minutes and 25 seconds, the concentration is 10 M. **[1 mark]**

Question 3

$T = 20 + 75_e^{-0.062t}$

a. When $t = 0, T = 20 + 75 = 95$, so the initial temperature was 95 °C. **[1 mark]**

b. The exponential function has a horizontal asymptote at $T = 20$, so as $t \to \infty,\ T = 20$.
The temperature approaches 20 °C. [**1 mark**]

c. Let $T = 65$.

$$\therefore 65 = 20 + 75e^{-0.062t}$$
$$\therefore e^{-0.062t} = \frac{45}{75}$$
$$\therefore e^{-0.062t} = \frac{3}{5}$$
$$\therefore -0.062t = \log_e\left(\frac{3}{5}\right)$$
$$\therefore t = -\frac{1}{0.062}\log_e\left(\frac{3}{5}\right)$$
$$\therefore t \simeq 8.24$$ [**1 mark**]

It takes approximately 8.24 minutes to cool to 65 °C.

d. $T = A + Be^{-0.062t}$

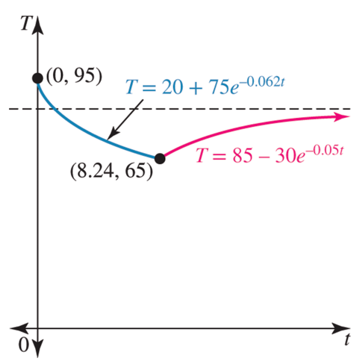
As the temperature cannot exceed 85 °C, $A = 85$. **[1 mark]**

$T = 85 + Be^{-0.062t}$

Substitute (8.24, 65):

$$\therefore 65 = 85 + Be^{-0.05\times 8.24}$$
$$\therefore e^{-0.412} = -20$$
$$\therefore B = -20e^{0.412}$$
$$\therefore B \simeq -30$$ [**1 mark**]

e. $T = 20 + 75e^{-0.062t}$



Award 1 mark for correct minimum and starting point.
Award 1 mark for correct graph shape.

4.8 Review

Question 1

$y = \log_e(x) + \log_e(2x)$
$\quad = \log_e\left(2x^2\right)$

The correct answer is **C**.

Question 2

$T = Ae^{-kt} + 18$

a. When $t = 0,\ T = 98.$

$\therefore 98 = A + 18$

$\therefore A = 80$ **[1 mark]**

The rule becomes $T = 80e^{-kt} + 18.$

At 3 pm, $t = 2$ and $T = 58.$

$$\therefore 58 = 80e^{-2k} + 18$$
$$\therefore e^{-2k} = \frac{1}{2}$$
$$\therefore -2k = \log_e\left(\frac{1}{2}\right)$$
$$\therefore k = -\frac{1}{2}\log_e\left(\frac{1}{2}\right)$$
$$\therefore k \approx 0.3466$$ **[1 mark]**

b. $T = 80e^{-0.3466t} + 18$

At 11 pm, $t = 10.$

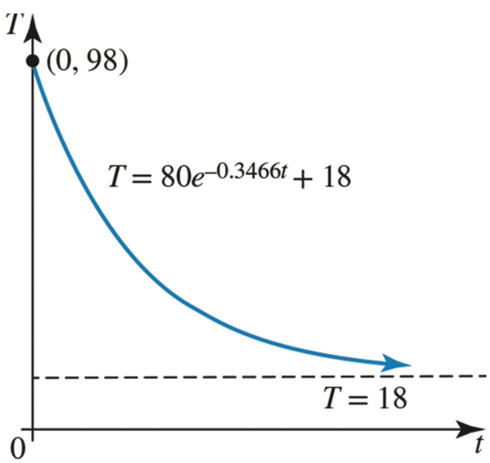
$\therefore T = 80e^{-3.466} + 18$

$\therefore T \approx 20.5$ **[1 mark]**

The temperature is 20.5 °C at 11 pm.

c. $T = 80e^{-0.3466t} + 18,\ t \geq 0$

Horizontal asymptote at $T = 18$, contains the points $(0, 98)$, $(2, 58)$ and $(10, 20.5)$.



Award 1 mark for correct shape and starting point.

Award 1 mark for correct asymptote.

d. $T = 22$; either use a CAS graphing screen or solve as follows.

$$22 = 80e^{-0.3466t} + 18$$
$$\therefore e^{-0.3466t} = \frac{1}{20}$$
$$\therefore -0.3466t = \log_e\left(\frac{1}{20}\right)$$
$$\therefore t = -\frac{1}{0.3466}\log_e\left(\frac{1}{20}\right)$$
$$\therefore t \approx 8.64$$

As the graph is decreasing, the temperature will be less than 22 °C when $t > 8.64$.

The temperature is less than 22 °C from approximately 9:38 pm onwards. **[1 mark]**

e. As $t \to \infty$, $T \to 18$, the horizontal asymptote value. In the long run, the temperature cools to 18 °C. **[1 mark]**

Question 3

a. $N = 22 \times 2^t$

Let $N = 2816$.

$\therefore 2816 = 22 \times 2^t$

$\therefore 2^t = 128$

$\therefore 2^t = 2^7$

$\therefore t = 7$

In 7 days the number of bacteria reaches 2816. **[1 mark]**

b. As $t \to \infty$, $N \to \infty$, so the number of bacteria will increase without limit. **[1 mark]**

c. i. $N = \dfrac{66}{1 + 2e^{-0.2t}}$

Let $t = 0$.

$\therefore N = \dfrac{66}{1 + 2e^0}$

$\therefore N = \dfrac{66}{3}$

$\therefore N = 22$ [**1 mark**]

Reconsider the first model, $N = 22 \times 2^t$.

If $t = 0,\ N = 22$. **[1 mark]**

Both models have an initial number of 22 bacteria.

ii. As $t \to \infty$, $e^{-0.2t} \to 0$.

Therefore $N \to \dfrac{66}{1 + 0} = 66$.

The number of bacteria will never exceed 66. **[1 mark]**

Question 4

a. $P(t) = Ae^{kt}$

When $t = 0$, $P(0) = 200$,

$200 = Ae^0$

$200 = A \times 1$

$A = 200$ [**1 mark**]

$\therefore P(t) = 200e^{kt}$

When $t = 30$, $P(30) = 1000$,

$1000 = 200e^{30k}$

$5 = e^{30k}$

$30k = \log_e(5)$

$k = \dfrac{\log_e(5)}{30}$

$k = 0.0536$ [**1 mark**]

$\therefore P(t) = 200e^{0.0536t}$

b. When $t = 60$,

$\Pr(60) = 200e^{0.0536(60)}$

$\simeq 4986$

Therefore, there were 4986 expected cases after 60 days. **[1 mark]**

c. When $P(t) = 6000$,

$P(t) = 200e^{0.0536t}$

$6000 = 200e^{0.0536t}$

$30 = e^{0.0536t}$

$0.0536t = \log_e(30)$

$$t = \frac{\log_e(30)}{0.0536}$$
$$t = 63.5$$

It took 63.5 days for the number of cases to reach 6000. **[1 mark]**

d. When $P(t) = 38\,000$,

$$P(t) = 200e^{0.0536t}$$
$$38\,000 = 200e^{0.0536t}$$
$$190 = e^{0.0536t}$$
$$0.0536t = \log_e(190)$$
$$t = \frac{\log_e(190)}{0.0536}$$
$$t = 97.9$$

It took 97.9 days for 38 000 young people to be infected. **[1 mark]**

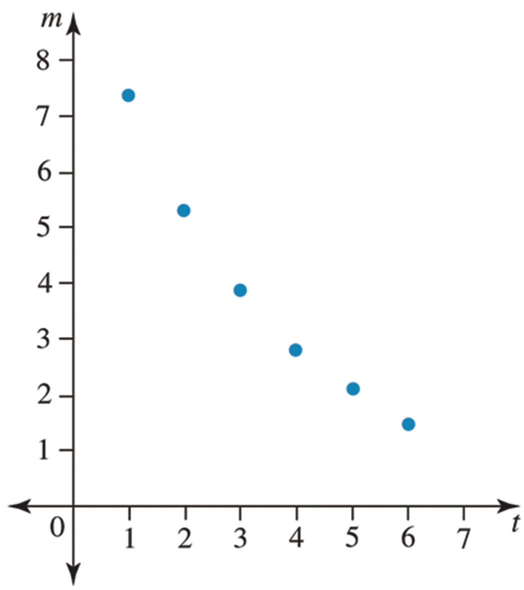
Question 5

Let $w = e^x$.

$$w^2 - 3w + 2 = 0$$
$$(w-2)(w-1) = 0 \qquad [\textbf{1 mark}]$$
$$w = 2 \text{ or } w = 1$$
$$e^x = 2 \text{ or } e^x = 1$$
$$x = \log_e(2) \text{ or } x = 0 \qquad [\textbf{1 mark}]$$

Question 6

An exponential function, decaying.



Question 7

Let $y = \log_e(x-2)$

Swap x and y.

$\Rightarrow x = \log_e(y-2)$

$\Rightarrow e^x = y - 2$

$\Rightarrow e^x + 2 = y$

$f^{-1}(x) = e^x + 2$

Given f: $[3, \infty]$

Therefore, the domain of $f^{-1}(x)$ is $[0, \infty)$.

Question 8

a. i. $h(x) = f(g(x))$
$= f\left(x^2 + 1\right)$
$= \log_e\left(x^2 + 1\right)$

Award 1 mark for the correct rule.

***VCAA Assessment Report* note:**

This question was very well answered. The few errors tended to be the result of poor notation, for example $\log_e x^2 + 1$, rather than a lack of understanding in determining the rule of this composite function.

ii. $\text{dom } h = \text{ dom } g = R, \text{ ran } h = [0, \infty)$

Award 1 mark for the correct domain.
Award 1 mark for the correct range.

***VCAA Assessment Report* note:**

A small proportion of students gained full marks for this question. While poor notation was a contributing factor, students appeared to experience difficulty in determining the range of the composite function, h. A quick sketch over the given domain would have been helpful. Students are reminded that the domain and range of a function are key aspects of a function.

iii. $h(-x) = \log_e\left((-x)^2 + 1\right) = \log_e\left(x^2 + 1\right)$

$\text{LHS} = h(x) + h(-x)$
$= 2\log_e\left(x^2 + 1\right)$
$= \log_e\left(x^2 + 1\right)^2$

$\text{RHS} = f\left((g(x))^2\right)$
$= f\left(\left(x^2 + 1\right)^2\right)$
$= \log_e\left(x^2 + 1\right)^2 = \text{LHS}$

Award 1 mark for the left-hand side.
Award 1 mark for the correct proof and right-hand side.

***VCAA Assessment Report* note:**

Many students were unsure of how to present their working. In the sample working above, both sides were operated on separately to arrive at the same expression and the conclusion that one side was in fact equivalent to the other. Poor notation was again evident, in particular
$\log_e(-x^2 + 1) \neq \log_e((-x)^2 + 1)$.

iv. $h'(x) = \dfrac{2x}{x^2 + 1} = 0 \Rightarrow x = 0$

$h(0) = \log_e(1) = 0$

The point (0, 0) is an absolute minimum turning point.

Award 1 mark for equating the correct derivative to zero.
Award 1 mark for stating the correct coordinates and nature of the point.

***VCAA Assessment Report* note:**

Most students could equate the correct derivative to 00. However, many then were unable to solve the equation, forgetting that a fraction is zero when its numerator is zero. Of those who managed a solution, some overlooked the second part of the question or assumed that the stationary point was a point of inflection.

b. i. $k(x) : y = \log_e\left(x^2 + 1\right)$

$k^{-1}(x) : x = \log_e\left(y^2 + 1\right)$
$y^2 + 1 = e^x$
$y^2 = e^x - 1$
$y = \pm\sqrt{e^x - 1}$

Take the negative result due to the range:

$k^{-1}(x) = -\sqrt{e^x - 1}$

Award 1 mark for swapping x and y.

Award 1 mark for the correct rule.

***VCAA Assessment Report* note:**

Students appeared quite adept at the mechanics of determining the rule for the inverse: swap x and y then rearrange. However, few students took care to determine the range of the inverse function and select for the negative root of their expression.

ii. dom $k^{-1} = [0, \infty)$

ran k^{-1} = dom $k = (-\infty, 0]$

Award 1 mark for the correct domain.

Award 1 mark for the correct range.

***VCAA Assessment Report* note:**

This question was not answered well. Most students utilised the fact that $Range_{k^{-1}} = Domain_k$ but found stating the domain of the inverse function more difficult. Again, poor notation was evident.

Question 9

The shape of the graph is a hyperbola, $y = \frac{a}{x}$.

Question 10

$$g(x) = 2^x$$
$$g(y) = 2^y$$
$$\therefore g(x)\,g(y) = 2^x \times 2^y = 2^{x+y}$$

Question 11

a.
$$ae^{bx} - c = 0$$
$$e^{bx} = \frac{c}{a}$$
$$bx = \log_e\left(\frac{c}{a}\right)$$
$$x = \frac{1}{b}\log_e\left(\frac{c}{a}\right)$$
[**1 mark**]

b.
$$b\log_e(x - a) = c$$
$$\log_e(x - a) = \frac{c}{b}$$
$$x - a = e^{\frac{c}{b}}$$
$$x = e^{\frac{c}{b}} + a$$
[**1 mark**]

5 Differentiation

Topic	5	Differentiation
Subtopic	5.2	Review of differentiation

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Source: *VCE 2021, Mathematical Methods Exam 2, Q7; © VCAA*

Question 1 (1 mark)

The tangent to the graph of $y = x^3 - ax^2 + 1$ at $x = 1$ passes through the origin.

The value of a is

A. $\frac{1}{2}$

B. 1

C. $\frac{3}{2}$

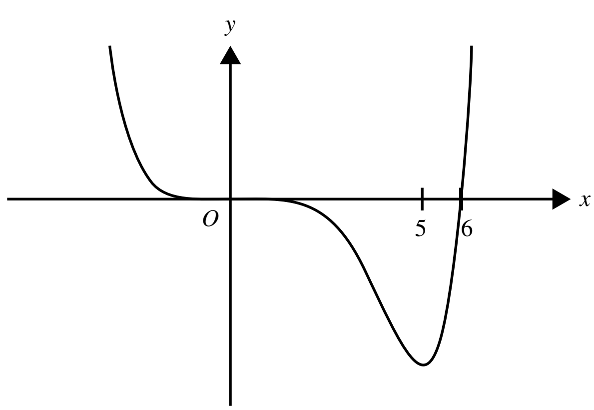
D. 2

E. $\frac{5}{2}$

Source: *VCE 2019, Mathematical Methods Exam 2, section A, Q16; © VCAA*

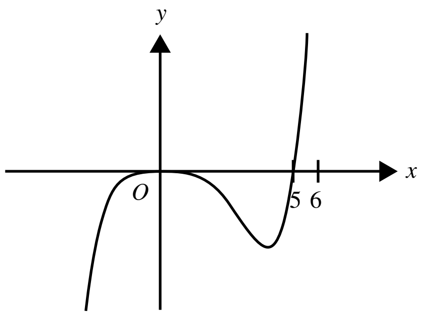
Question 2 (1 mark)

Part of the graph of $y = f(x)$ is shown below.

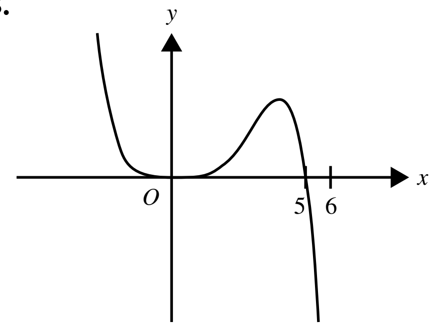


The corresponding part of the graph of $y = f'(x)$ is best represented by

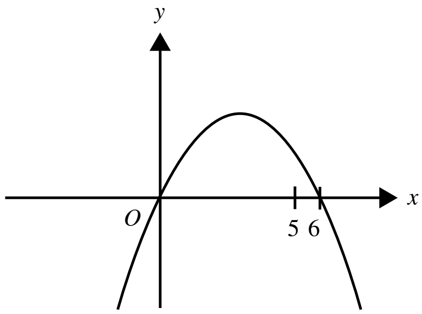
A.



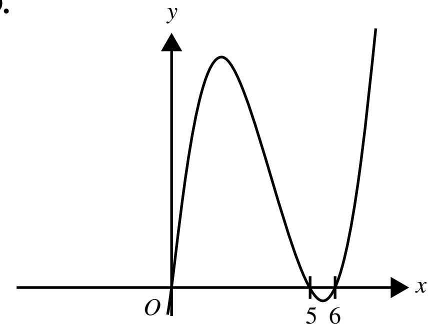
B.



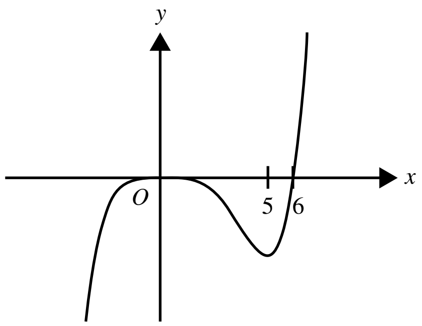
C.



D.



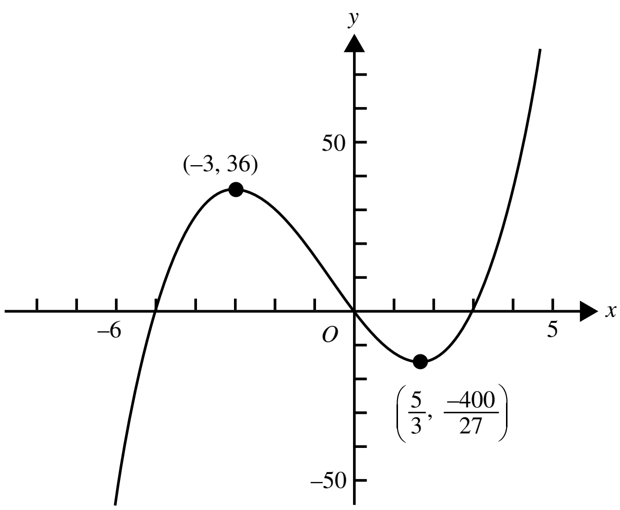
E.



Source: *VCE 2017, Mathematical Methods Exam 2, section A, Q2; © VCAA*

Question 3 (1 mark)

Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.



$f'(x) < 0$ for the interval

A. $(0, 3)$

B. $(-\infty, -5) \cup (0, 3)$

C. $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D. $\left(-3, \frac{5}{3}\right)$

E. $\left(\frac{-400}{27}, 36\right)$

Question 4 (1 mark)

$\frac{d}{dx}\left(x^8\right)$ is equal to

A. $8x^9$

B. $\frac{x^9}{9}$

C. $8x^8$

D. $8x^7$

E. $9x^7$

Question 5 (1 mark)

The derivative of $\frac{1}{x^7}$ is equal to

A. $\frac{1}{7x^6}$

B. $-\frac{7}{x^8}$

C. $\frac{8}{x^8}$

D. $-\frac{1}{6x^6}$

E. $\frac{7}{x^6}$

Question 6 (1 mark)

The value of $f'(9)$ if $f(x) = x^2 + x^{\frac{3}{2}} - 10x$ is

A. $\frac{1}{2}$

B. 18

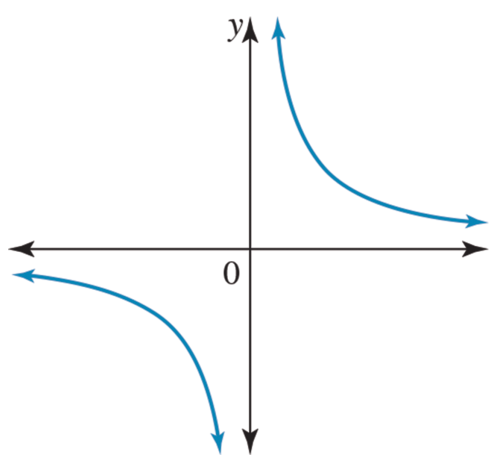
C. $12\frac{1}{2}$

D. 8

E. 0

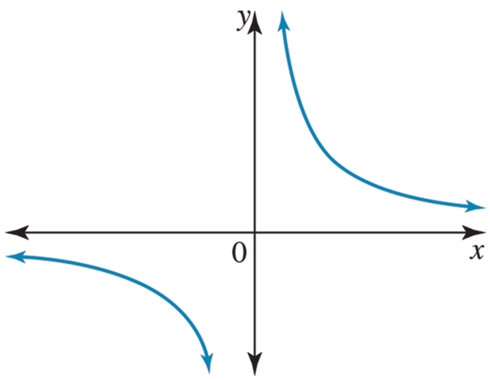
Question 7 (1 mark)

The graph of the function $y = f(x)$ is shown below.

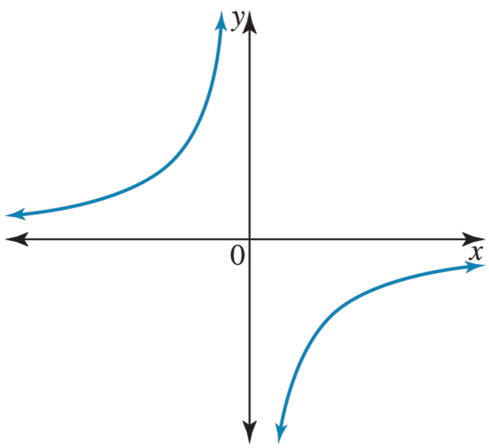


Which one of the following is the graph of the derivative function $y = f'(x)$?

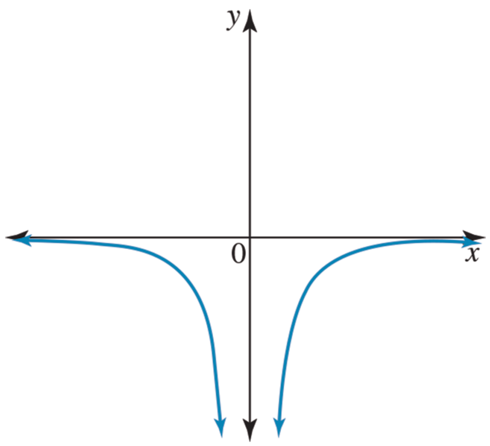
A.



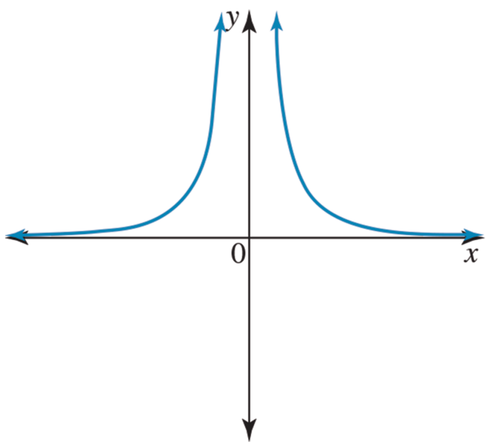
B.



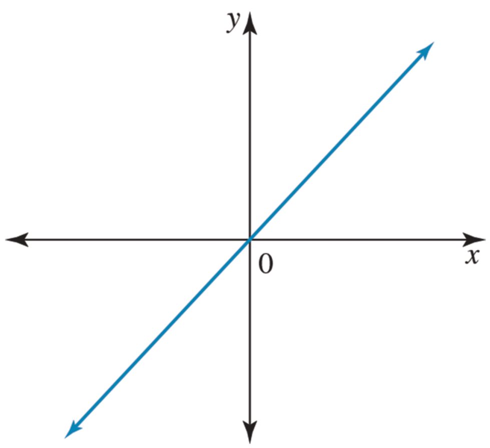
C.



D.

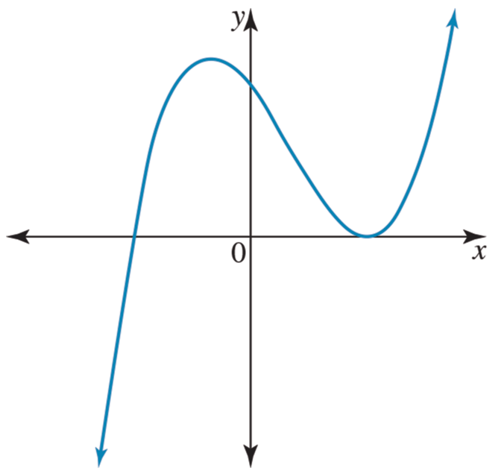


E.



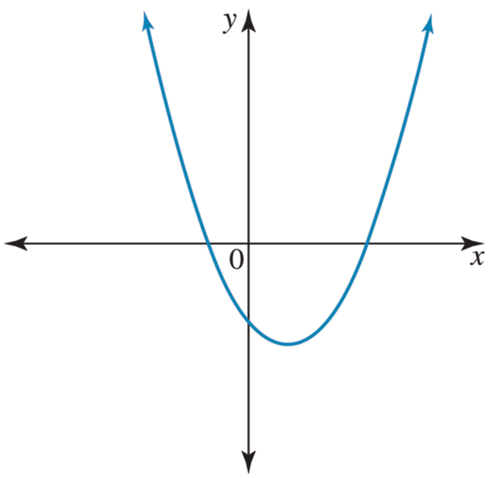
Question 8 (1 mark)

The graph of the function is shown below.

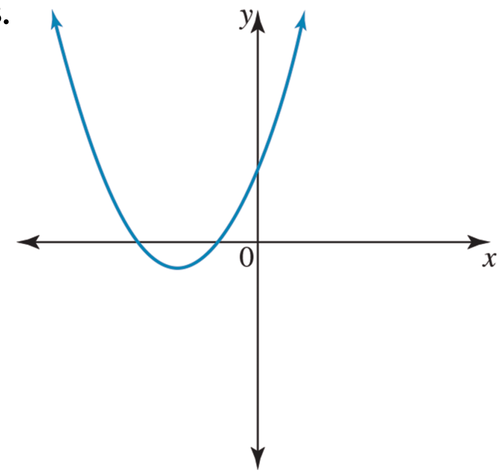


Which one of the following figures is the graph of the **gradient** function $y = f'(x)$?

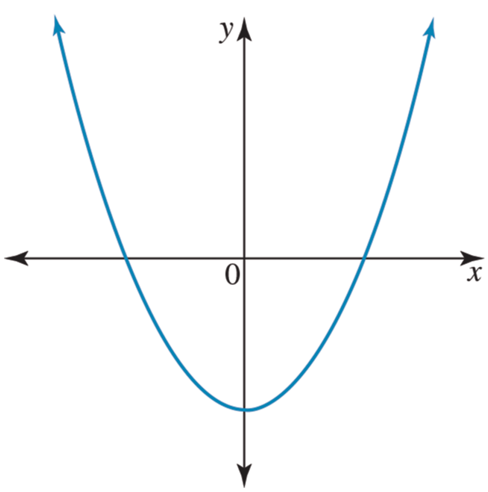
A.



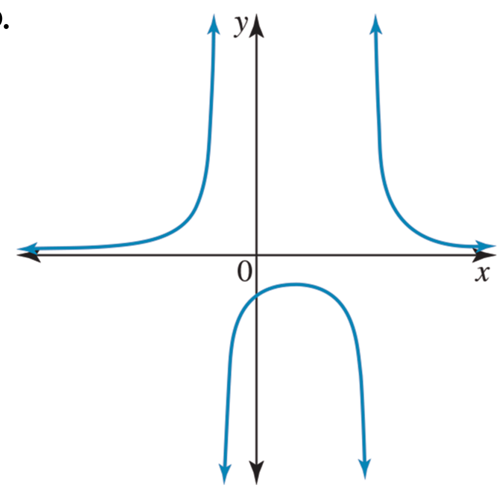
B.



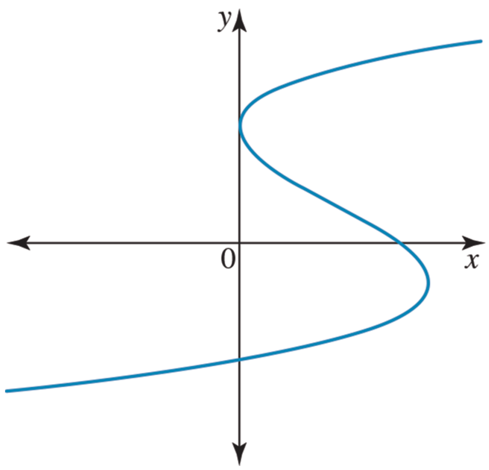
C.



D.



E.



Question 9 (1 mark)

The equation of the line perpendicular to the curve $y = \sqrt{4x + 9}$ at the point where $x = 4$ is given by

A. $2y + 5x = 30$

B. $\dfrac{1}{5y - 2x} = 17$

C. $2y - 5x + 10 = 0$

D. $y + 5x = 25$

E. $4y + 5x = 45$

Question 10 (1 mark)

The point at which the equation of the tangent to the graph of f, where $f: R \to R, f(x) = -\dfrac{x^2}{4} + 4$, has the equation $y = -x + 5$ is

A. $(2, 4)$

B. $(-2, 4)$

C. $\left(2, 3\dfrac{1}{2}\right)$

D. $(2, 3)$

E. $(-2, 3)$

Topic	5	Differentiation
Subtopic	5.3	Differentiation of exponential functions

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Source: *VCE 2021, Mathematical Methods Exam 1, Q1a; © VCAA*

Question 1 (1 mark)

Differentiate $y = 2e^{-3x}$ with respect to x.

Source: *VCE 2013, Mathematical Methods (CAS) Exam 2, Section 1, Q11; © VCAA*

Question 2 (1 mark)

If the tangent to the graph of $y = e^{ax}, a \neq 0$, at $x = c$ passes through the origin, then c is equal to

A. 0

B. $\frac{1}{a}$

C. 1

D. a

E. $-\frac{1}{a}$

Question 3 (2 marks)

Determine the derivative of $y = \dfrac{e^{2x} + e^{-2x}}{e^x}$.

Topic	5	Differentiation
Subtopic	5.4	Application of exponential functions

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Question 1 (2 marks)

A body that is at a higher temperature than its surroundings cools according to Newton's Law of Cooling, which states that

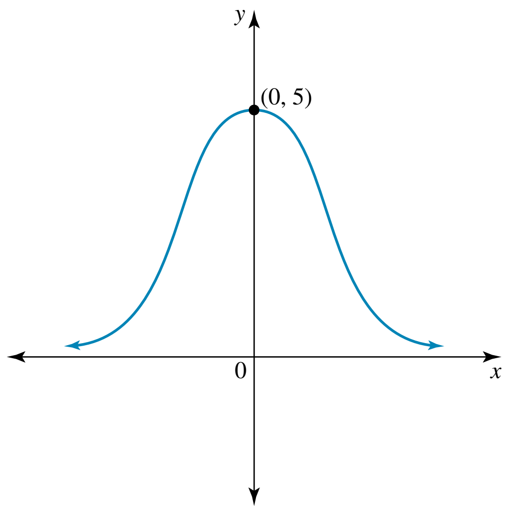
$$T = T_0 e^{-zt}$$

where T_0 is the original *excess* of temperature, T is the excess of temperature in degrees centigrade after t minutes, and z is a constant.

a. The original temperature of the body was 95 °C and the temperature of the surroundings was 20 °C. Determine the value of T_0. **(1 mark)**

b. Determine the rate at which the temperature is decreasing after a quarter of an hour if it is known that $z = 0.034$. Give your answer correct to 3 decimal places. **(1 mark)**

Question 2 (4 marks)
The graph of $y = Ae^{-x^2}$, where A is a constant, is shown. Answer the following questions correct to 2 decimal places where appropriate.



a. If the graph goes through $(0, 5)$, determine the value of A. **(1 mark)**

b. Determine $\frac{dy}{dx}$. **(1 mark)**

c. Determine the gradient of the tangent to the curve at the point where:

i. $x = -0.5$ **(1 mark)**

ii. $x = 1$ **(1 mark)**

Question 3 (3 marks)

The pressure of the atmosphere, P cm of mercury, decreases with the height, h km above sea level, according to the law

$$P = P_0 e^{-kh}$$

where P_0 is the pressure of the atmosphere at sea level and k is a constant. At 500 m above sea level, the pressure is 66.7 cm of mercury, and at 1500 m above sea level, the pressure is 52.3 cm of mercury.

a. Determine the values of P_0 and k, correct to 2 decimal places. **(2 marks)**

b. Determine the rate at which the pressure is falling when the height above sea level is 5 km. Give your answer correct to 2 decimal places. **(1 mark)**

Topic	5	Differentiation
Subtopic	5.5	Differentiation of trigonometric functions

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Question 1 (1 mark)

Consider the following function.

$$f(x) = \begin{cases} \sin(x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Select the true statement from the following.

A. The function is continuous at $x = 0$ and differentiable at $x = 0$.
B. The function is continuous at $x = 0$ and not differentiable at $x = 0$.
C. The function is not continuous at $x = 0$ and differentiable at $x = 0$.
D. The function is not continuous at $x = 0$ and not differentiable at $x = 0$.
E. The $\lim_{x \to 0} f(x)$ exists and $f(0) = 0$. Therefore the function is differentiable at $x = 0$.

Question 2 (1 mark)

The derivative of $h(\cos(2x))$ is

A. $2h'(\cos(2x))\cos(2x)$
B. $h'(\cos(2x))\cos(2x)$
C. $-2h'(\cos(2x))\,(\cos(2x))$
D. $-2h'(\cos(2x))\,(\sin(2x))$
E. $2h'(\cos(2x))\,(\sin(2x))$

Question 3 (1 mark)

If $f(x) = e^{\sin(2x)}$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

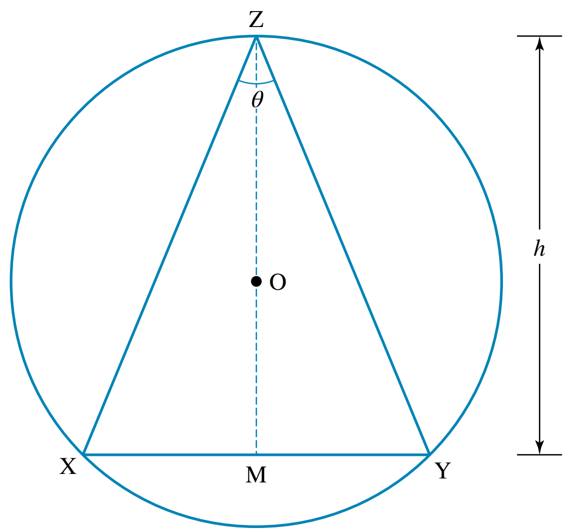
A. $-\frac{1}{e}$
B. 0
C. -1
D. -2
E. 2

Topic	5	Differentiation
Subtopic	5.6	Applications of trigonometric functions

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Question 1 (4 marks)

The triangle XYZ is inscribed by a circle with radius, r cm. The actual placement of the triangle is dependent on the size of the angle XZY, θ radians, and the length of ZM, where M is the midpoint of XY.



a. Show that $\angle XOM = \theta$. **(1 mark)**

b. Show that the relationship between θ, r and h, where $h = d(\overline{ZM})$, is given by $\frac{h}{r} = \cos(\theta) + 1$. **(1 mark)**

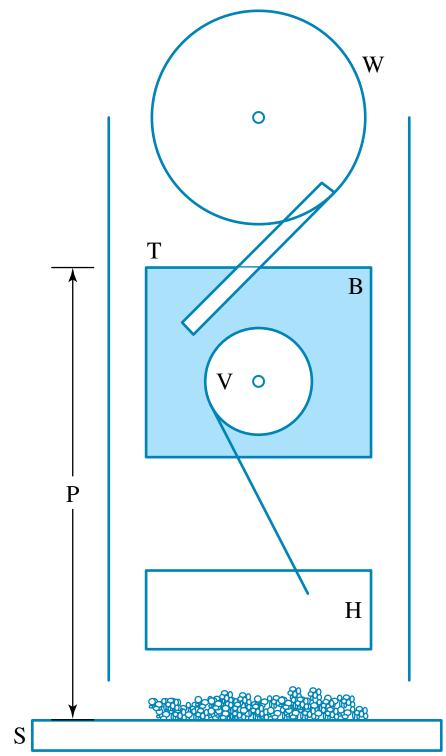
c. If the radius of the circle is 3 cm, determine $\frac{dh}{d\theta}$. **(1 mark)**

d. Determine the exact value of $\frac{dh}{d\theta}$ when $\theta = \frac{\pi}{6}$. **(1 mark)**

Question 2 (4 marks)

A mechanism for crushing rock is shown. Rocks are placed on a steel platform, *S*, and a device raises and lowers a heavy mallet, H. The wheel, W, rotates, causing the upper block, B, to move up and down. The other wheel, V, is attached to the block, B, and rotates independently, causing the mallet to move up and down. T is the top of the block B.

The distance, P metres, between T and the steel platform is modelled by the equation $P = -2\cos(mt) + n$, where t is the time in minutes and m and n are constants. When $t = 0$, T is at its lowest point, 4 metres above the steel platform. The wheel, W, rotates at a rate of 1 revolution per 1.5 minutes.



a. Show that $n = 6$ and $m = \frac{4\pi}{3}$. **(2 marks)**

b. Determine $\frac{dP}{dt}$. **(1 mark)**

c. Calculate the exact rate of change of distance when $t = 0.375$ minutes. **(1 mark)**

Question 3 (7 marks)

An industrial process is known to cause the production of two separate gases that are released into the atmosphere. At a factory where this industrial process occurs, the technicians work a 12-hour day from 6.00 am until 6.00 pm.

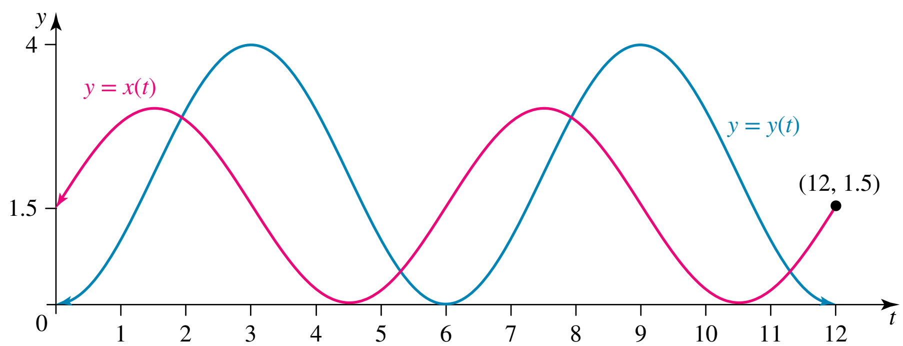
The emission of the gas X can be modelled by the rule

$$x(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 1.5, 0 \leq t \leq 12$$

and the emission of the gas Y can be modelled by the rule

$$y(t) = 2.0 - 2.0\cos\left(\frac{\pi t}{3}\right), 0 \leq t \leq 12.$$

The graphs of these two functions are shown.



a. Determine the time of the day at which the emissions are the same for the first time and how many units of each gas are emitted at that time. Give your answer correct to 2 decimal places, and remember to note whether the time is am or pm. **(2 marks)**

b. The Environment Protection Authority (EPA) has strict rules about the emissions of gases. The total emission of gases for this particular industrial process is given by

$$T(t) = x(t) + y(t)$$

i. Sketch the graph of the function $T(t)$ **(2 marks)**

ii. Determine the maximum and minimum emissions in a 12-hour working day and the times at which these occur. **(2 marks)**

c. If the EPA rules state that all emissions from any one company must lie within the range 0 to 7 units at any one time, indicate whether this company works within the guidelines. **(1 mark)**

Topic	5	Differentiation
Subtopic	5.7	Differentiation and Application of logarithmic functions

online only

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***Source:** VCE 2020, Mathematical Methods Exam 2, section A, Q17; © VCAA*

Question 1 (1 mark)

Let $f(x) = -\log_e(x+2)$.

A tangent to the graph of f has a vertical axis intercept at $(0, c)$.

The maximum value of c is

A. -1

B. $-1 + \log_e(2)$

C. $-\log_e(2)$

D. $-1 - \log_e(2)$

E. $\log_e(2)$

***Source:** VCE 2018, Mathematical Methods 2, section A, Q9; © VCAA*

Question 2 (1 mark)

A tangent to the graph of $y = \log_e(2x)$ has a gradient of 2.

A. 0

B. -0.5

C. -1

D. $-1 - \log_e(2)$

E. $-2\log_e(2)$

Question 3 (2 marks)

If a function $f(x) = \log_e(2x)$, determine $f'(1)$.

Topic	5	Differentiation
Subtopic	5.8	Review

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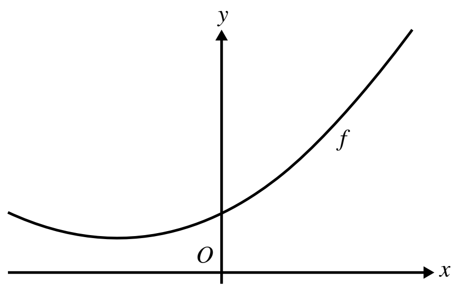
Source: *VCE 2020, Mathematical Methods Exam 1, Q7; © VCAA*

Question 1 (8 marks)

Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.



a. Show that point p is not on the graph of $y = f(x)$. **(1 mark)**

b. Consider a point $Q(a, f(a))$ to be a point on the graph of f.

i. Find the slope of the line connecting points p and Q in terms of a. **(1 mark)**

ii. Find the slope of the tangent to the graph of f at point Q in terms of a. **(1 mark)**

iii. Let the tangent to the graph of f at $x = a$ pass through point p. Find the values of a. **(2 marks)**

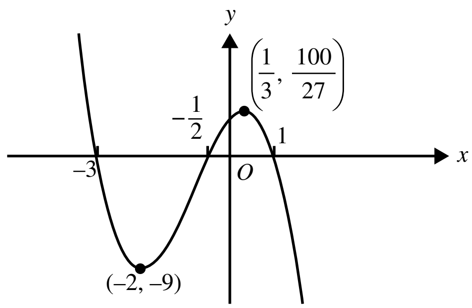
iv. Give the equation of one of the lines passing through point p that is tangent to the graph of f. **(1 mark)**

c. Find the value, k, that gives the shortest possible distance between the graph of the function of $y = f(x - k)$ and point p. **(2 marks)**

Source: *VCE 2016, Mathematical Methods Exam 2, section A, Q3; © VCAA*

Question 2 (1 mark)

Part of the graph $y = f(x)$ of the polynomial function f is shown below.



$f'(x) < 0$ for

A. $x \in (-2, 0) \cup \left(\frac{1}{3}, \infty\right)$

B. $x \in \left(-9, \frac{100}{27}\right)$

C. $x \in (-\infty, -2) \cup \left(\frac{1}{3}, \infty\right)$

D. $x \in \left(-2, \frac{1}{3}\right)$

E. $x \in (-\infty, -2] \cup (1, \infty)$

Source: *VCE 2016, Mathematical Methods Exam 2, section A, Q10; © VCAA*

Question 3 (1 mark)

For the curve $y = x^2 - 5$, the tangent to the curve will be parallel to the line connecting the positive x-intercept and the y-intercept when x is equal to

A. $\sqrt{5}$

B. 5

C. -5

D. $\frac{\sqrt{5}}{2}$

E. $\frac{1}{\sqrt{5}}$

Source: *Adapted from VCE 2016, Mathematical Methods Exam 2, Section B, Q1; © VCAA*

Question 4 (8 marks)

Let $f: [0, 8\pi] \to R, f(x) = 2\cos\left(\frac{x}{2}\right) + \pi$

a. Find the period and range of f. **(2 marks)**

b. State the rule for the derivative function f'. **(1 mark)**

c. Find the equation of the tangent to the graph of f at $x = \pi$. **(1 mark)**

d. Find the equations of the tangents to the graph of $f: [0, 8\pi] \to R, f(x) = 2\cos\left(\frac{x}{2}\right) + \pi$ that have a gradient of 1. **(2 marks)**

e. Find the values of x, $0 \le x \le 8\pi$ such that $f(x) = 2f'(x) + \pi$. **(2 marks)**

Source: *VCE 2015, Mathematical Methods (CAS) Exam 2, section 1, Q4; © VCAA*

Question 5 (1 mark)

Consider the tangent to the graph of $y = x^2$ at the point $(2, 4)$.

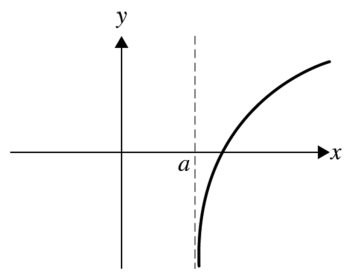
Determine which of the following points lies on this tangent.

A. $(1, -4)$
B. $(3, 8)$
C. $(-2, 6)$
D. $(1, 8)$
E. $(4, -4)$

Source: *VCE 2021, Mathematical Methods 2, section A, Q8; © VCAA*

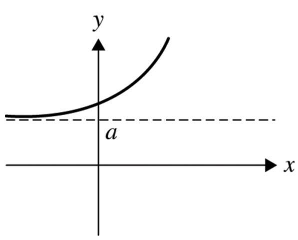
Question 6 (1 mark)

The graph of the function f is shown below.

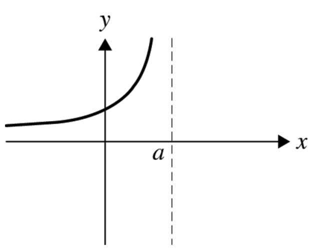


The graph corresponding to f' is

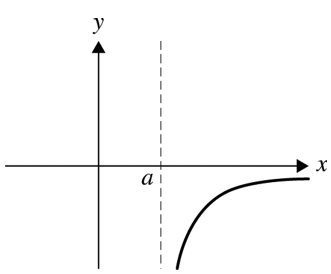
A.



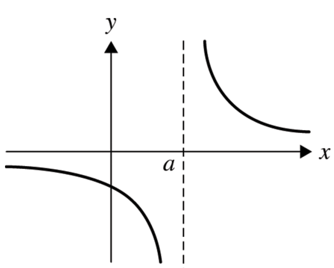
B.



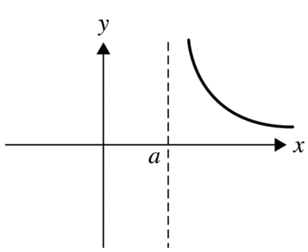
C.



D.



E.



Source: *VCE 2019, Mathematical Methods 1, Q9; © VCAA*

Question 7 (9 marks)

Consider the functions $f : R \to R, f(x) = 3 + 2x - x^2$ and $g : R \to R, g(x) = e^x$.

a. State the rule of $g(f(x))$. **(1 mark)**

b. Find the values of x for which the derivative of $g(f(x))$ is negative. **(2 marks)**

c. State the rule of $f(g(x))$ **(1 mark)**

d. Solve $f(g(x)) = 0$. **(2 marks)**

e. Find the coordinates of the stationary point on the graph of $f(g(x))$. **(2 marks)**

f. State the number of solutions to $g(f(x)) + f(g(x)) = 0$. **(1 mark)**

Question 8 (1 mark)

Assuming that the functions below are defined on their maximal domain, which of the following functions is differentiable at $x = 2$?

A. $f(x) = |x - 2|$

B. $f(x) = \sqrt{x - 2}$

C. $f(x) = \log_e(x - 2)$

D. $f(x) = \dfrac{1}{x - 2}$

E. $f(x) = (x - 2)^3$

Answers and marking guide

5.2 Review of differentiation

Question 1

$y = x^3 - ax^2 + 1$

$x = 1 \quad y(1) = 1 - a + 1 = 2 - a$

$\frac{dy}{dx} = 3x^2 - 2ax, \quad x = 1 \quad \frac{dy}{dx} = 3 - 2a$

$T: \; y - (2 - a) = (3 - 2a)(x - 1)$

at $(0, 0) \quad -(2 - a) = -(3 - 2a)$

$a = 1$

The correct answer is **B**.

Question 2

The gradient is negative for $x < 5$, The gradient is positive for $x > 5$

The gradient is zero at $x = 0$ and $x = 5$

The correct answer is **A**.

Question 3

The gradient is negative for $x \in \left(-3, \frac{5}{3}\right)$

The correct answer is **D**.

Question 4

$y = x^8$

$\frac{dy}{dx} = 8x^7$

$\Rightarrow \frac{d}{dx}\left(x^8\right) = 8x^7$

The correct answer is **D**.

Question 5

$y = \frac{1}{x^7} = x^{-7}$

$\frac{dy}{dx} = -7x^{-8}$

$= \frac{-7}{x^8}$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^7}\right) = \frac{-7}{x^8}$

The correct answer is **B**.

Question 6

$f(x) = x^2 + x^{\frac{3}{2}} - 10x$

$\Rightarrow f'(x) = 2x + \frac{3x^{\frac{1}{2}}}{2} - 10$

$\Rightarrow f'(9) = 2(9) + \frac{3(9)^{\frac{1}{2}}}{2} - 10$

$= 18 + \frac{9}{2} - 10$

$= 12\frac{1}{2}$

The correct answer is **C**.

Question 7
The gradient is always negative, option C is the only correct possible graph.
The correct answer is **C**.

Question 8
Since the graph shown is a positive cubic then its derivative is a positive quadratic.
The correct answer is **A**.

Question 9

$$f(x) = \sqrt{4x+9} = (4x+9)^{\frac{1}{2}} \text{ at } x = 4,\ f(4) = \sqrt{25} = 5 \text{ and } p\,(4,5)$$

$$f'(x) = \frac{4}{2\sqrt{4x+9}}$$

$$f'(x) = \frac{2}{\sqrt{25}} = \frac{2}{5} = m_T \qquad m_p = -\frac{5}{2}$$

$$y - 5 = -\frac{5}{2}(x-4)$$

$2y - 10 = -5x + 20$
$2y + 5x = 30$
The correct answer is **A**.

Question 10

$$y = -x + 5$$

$$m = -1$$

$$\frac{dy}{dx} = \frac{-2x}{4}$$

$$= -\frac{1}{2}x$$

$$-\frac{1}{2}x = -1$$

$$x = 2$$

$$f(2) = -1 + 4$$

$$= 3$$

$\therefore$ the point is $(2, 3)$
The correct answer is **D**.

5.3 Differentiation of exponential functions

Question 1
$y = 2e^{-3x}$

$\frac{dy}{dx} = -6e^{-3x}$ **[1 mark]**

Question 2
$y = e^{ax}, a \neq 0$
AT $x = c$, $y = e^{ax}$ and $P\,(c, e^{ac})$
$\frac{dy}{dx} = ae^{ax}$ at $x = c$, $m_{\text{T}} = ae^{ac}$
Tangent: $y - e^{ac} = ae^{ac}(x - c)$ passes through the origin, $(0, 0)$.

$$y = axe^{ac} - cae^{ac} + e^{ac}$$

$\therefore 0 = -cae^{ac} + e^{ac}$

$e^{ac} = cae^{ac}$

$\Rightarrow ca = 1$

$\Rightarrow c = \frac{1}{a}$

The correct answer is **B**.

Question 3

Simplify:

$y = \frac{e^{2x}}{e^x} + \frac{e^{-2x}}{e^x}$

$= e^x + e^{-3x}$ [**1 mark**]

$\frac{dy}{dx} = e^x - 3e^{-3x}$ [**1 mark**]

5.4 Application of exponential functions

Question 1

a. When $t = 0$, $T = 95 - 20 = 75$.

$75 = T_0 e^{-z(0)}$

$75 = T_0$

So $T = 75e^{-zt}$. [**1 mark**]

b. $T = 75e^{-0.034t}$

$\frac{dT}{dt} = -0.034 \times 75e^{-0.034t}$

$\frac{dT}{dt} = -2.55e^{-0.034t}$

When $t = 15$,

$\frac{dT}{dt} = -2.55e^{-0.034(15)} = -1.531$°C/min **[1 mark]**

Therefore, the temperature is decreasing at a rate of 1.531°C/min.

Question 2

a. $y = Ae^{-x^2}$

When $x = 0,\ y = 5$

$5 = Ae^0$

$A = 5$

Thus, $y = 5e^{-x^2}$. **[1 mark]**

b. $\frac{dy}{dx} = -2x \times 5e^{-x^2}$

$\frac{dy}{dx} = -10xe^{-x^2}$ [**1 mark**]

c. i. when $x = -0.5$, $\frac{dy}{dx} = -10(-0.5)e^{-(-0.5)^2} = 3.89$ **[1 mark]**

ii. When $x = 1$, $\frac{dy}{dx} = -10(1)e^{-(1)^2} = -3.68$ **[1 mark]**

Question 3

a. $P = P_0 e^{-kh}$

When $h = 0.5,\ P = 66.7 \rightarrow 66.7 = P_0 e^{-0.5k}$.

When $h = 1.5,\ P = 52.3 \rightarrow 52.3 = P_0 e^{-1.5k}$. [**1 mark**]

Solve using CAS: $P_0 = 75.32$ cm of mercury, $k = 0.24$ [**1 mark**]

So $P = 75.32e^{-0.24h}$.

b. $P = 75.32e^{-0.24h}$

$\frac{dP}{dh} = -0.24 \times 75.32e^{-0.24h}$

$\frac{dP}{dh} = -18.0768e^{-0.24h}$

When $h = 5$,

$dPdh = -18.0768e^{-0.24(5)} = -5.44$ cm of mercury/km **[1 mark]**

The rate is falling at 5.44 cm of mercury/km.

5.5 Differentiation of trigonometric function

Question 1

Although the function is continuous at $x = 0$, so that $\sin(0) = 0$, it is not differentiable at $x = 0$. **[1 mark]**
This is because there is a sharp point at $x = 0$.
The correct answer is **B**.

Question 2

Let $u = \cos(2x)$.

$\therefore y = h(u)$

$\frac{du}{dx} = -2\sin(2x)$

$\frac{dy}{du} = h'(u)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= h'(u) \times -2\sin(2x)$

$= -2h'(\cos(2x))(\sin(2x))$

The correct answer is **D**.

Question 3

$f(x) = e^{\sin(2x)}$

$f'(x) = 2\cos(2x)\,e^{\sin(2x)}$

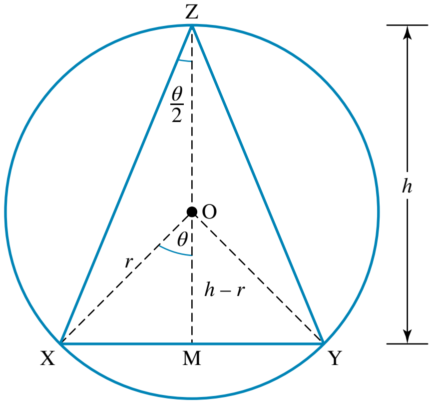
$f'\left(\frac{\pi}{2}\right) = 2\cos(\pi)\,e^{\sin(\pi)} = -2$

The correct answer is **D**.

5.6 Applications of trigonometric functions

Question 1

a.



$\angle XOY = 2\theta$ because the angle at the centre of the circle is twice the angle at the circumference.

$\angle XOM = \angle YOM = \frac{1}{2} \times 2\theta$

$\angle XOM = \theta$ as required **[1 mark]**

b. $XM = r\sin(\theta)$

$$\frac{XM}{h-r} = \tan(\theta)$$

$$\frac{r\sin(\theta)}{h-r} = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\frac{r}{h-r} = \frac{1}{\cos(\theta)}$$

$$\frac{h-r}{r} = \cos(\theta)$$

$$\frac{h}{r} - 1 = \cos(\theta)$$

$$\frac{h}{r} = \cos(\theta) + 1$$ **[1 mark]**

c. $r = 3\,\text{cm}$, $\frac{h}{3} = \cos(\theta) + 1$

$$h = 3\cos(\theta) + 3$$

$$\frac{dh}{d\theta} = -3\sin(\theta)$$ **[1 mark]**

d. When $\theta = \frac{\pi}{6}$, $\frac{dh}{d\theta} = -3\sin\left(\frac{\pi}{6}\right) = -\frac{3}{2}$. **[1 mark]**

Question 2

a. $P = -2\cos(mt) + n$

When $t = 0$, $P = 4$

$$4 = -2\cos(0) + n$$

$$4 + 2 = n$$

$$n = 6$$ **[1 mark]**

The period, m:

$$\frac{3}{2} = \frac{2\pi}{m}$$

$$3m = 4\pi$$

$$m = \frac{4\pi}{3}$$ **[1 mark]**

b. $P = -2\cos\left(\frac{4\pi t}{3}\right) + 6$

$$\frac{dP}{dt} = \frac{8\pi}{3}\sin\left(\frac{4\pi t}{3}\right)$$ **[1 mark]**

c. When $t = 0.375$,

$$\frac{dP}{dt} = \frac{8\pi}{3}\sin\left(\frac{4\pi \times 0.375}{3}\right)$$

$$= \frac{8\pi}{3}\sin\left(\frac{\pi}{2}\right) = \frac{8\pi}{3}\text{m/min}$$ **[1 mark]**

Question 3

a. $x(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 1.5 \quad 0 \le t \le 12$

$y(t) = 2.0 - 2.0\cos\left(\frac{\pi t}{3}\right) \quad 0 \le t \le 12$ [**1 mark**]

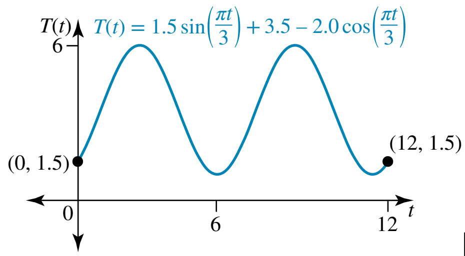
Solve $x(t) = y(t)$ on CAS.

The first time the emissions are equal is at $t = 1.9222$ or 1 hour and 55 minutes after 6 am. So at 7.55 am the emissions are both 2.86 units. **[1 mark]**

b. i. $T(t) = x(t) + y(t)$

$T(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 1.5 + 2 - 2\cos\left(\frac{\pi t}{3}\right)$

$T(t) = 3.5 + 1.5\sin\left(\frac{\pi t}{3}\right) - 2\cos\left(\frac{\pi t}{3}\right)$ [**1 mark**]



[1 mark]

ii. Maximum emission of 6 units at $t = 2.3855$ or 2 hours and 23 minutes after 6 am, which is 8.23 am, and at $t = 8.3855$ or 8 hours and 23 minutes after 6 am, which is 2.23 pm. **[1 mark]**

Minimum emission of 1 unit at $t = 5.3855$ or 5 hours and 23 minutes after 6 am, which is 11.23 am, and again at $t = 11.3855$ or 11 hours and 23 minutes after 6 am, which is 5.23 pm. **[1 mark]**

c. As the emissions range is 1–6 units, they lie within the required range. **[1 mark]**

5.7 Differentiation and Application of logarithmic functions

Question 1

$f(x) = -\log_e(x+2)$

$f(0) = -\log_e(2)$

$c = -\log_e(2)$

T : when $x = 0$ has $c = -\log_e(2)$

The correct answer is **C**.

Question 2

$y = \log_e(2x)$

$\frac{dy}{dx} = \frac{1}{x} = 2$

$x = \frac{1}{2}$

$y = \log_e(1) = 0$

$P\left(\frac{1}{2}, 0\right)$

Tangent:

$y - 0 = 2\left(x - \frac{1}{2}\right)$

$= 2x - 1$

The tangent crosses the y-axis at $y = -1$.

The correct answer is **C**.

Question 3

$f(x) = \log_e(x) + \log_e(2)$

$f'(x) = \dfrac{1}{x}$ **[1 mark]**

$f'(1) = 1$ **[1 mark]**

5.8 Review

Question 1

a. $f(x) = x^2 + 3x + 5$

$f(1) = 1 + 3 + 5 = 9 \neq 0$, so the point $P(1, 0)$ is not on the graph of $y = f(x)$

Award 1 mark for the correct explanation.

b. i. $Q(a, f(a))$

$m(PQ) = \dfrac{f(a) - 0}{a - 1} = \dfrac{a^2 + 3a + 5}{a - 1}$

Award 1 mark for the correct answer.

ii. $f'(x) = 2x + 3,\ f'(a) = 2a + 3$

Award 1 mark for equating and solving.

iii. $f'(a) = m$

$-\left(a^2 + 3a + 5\right) = (2a + 3)(1 - a)$

$-a^2 - 3a - 5 = -2a^2 - a + 3$

$a^2 - 2a - 8 = 0$

$(a - 4)(a + 2) = 0$

$a = -2, 4$

Award 1 mark for equating and solving.

Award 1 mark for both correct values of a.

iv. $a = 4 f'(4) = 11 \quad y = 11(x - 1) = 11x - 11$

or

$a = -2 f'(-2) = -1 \quad y = -1(x - 1) = 1 - x$

Award 1 mark for the correct tangent.

c. The turning point on the graph of f at $x = -\dfrac{3}{2}$ needs to be above the point P,

so translate f, $\dfrac{3}{2} + 1 = \dfrac{5}{2}$ so that $k = \dfrac{5}{2}$.

Award 1 mark for the correct method.

Award 1 mark for the correct value of k.

Question 2

The gradient is negative for $x \in (-\infty, -2) \cup \left(\dfrac{1}{3}, \infty\right)$

The correct answer is **C**.

Question 3

$y = x^2 - 5$

Positive x-intercept: A $\left(\sqrt{5}, 0\right)$, y-intercept B $(0, -5)$

Gradient $m_{AB} = \dfrac{-5 - 0}{0 - \sqrt{5}} = \dfrac{5}{\sqrt{5}} = \sqrt{5}$

$\dfrac{dy}{dx} = 2x = \sqrt{5} \Rightarrow x = \dfrac{\sqrt{5}}{2}$

The correct answer is **D**.

Question 4

a. $f : [0, 8\pi] \to R, f(x) = 2\cos\left(\frac{x}{2}\right) + \pi$

Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$, range $[-2+\pi, 2+\pi]$

Award 1 mark for the correct period.

Award 1 mark for the correct range.

***VCAA Assessment Report* note:**

This question was answered well. However, some students included round brackets instead of square brackets for the range. Range $= [2+\pi, -2+\pi]$ was occasionally seen. Some students gave approximate answers instead of exact answers.

b. $f'(x) = -\sin\left(\frac{x}{2}\right)$

Award 1 mark for the correct derivative.

***VCAA Assessment Report* note:**

This question was answered well. Some students did not write an equation, leaving their answer as $-\sin\left(\frac{x}{2}\right)$. Others made errors when using the chain rule. Some had their technology in degree mode rather than radian mode.

c. $f(\pi) = 2\cos\left(\frac{\pi}{2}\right) + \pi = \pi.$

$f'(\pi) = -\sin\left(\frac{\pi}{2}\right) = -1, P(\pi, \pi)$

$T : y - \pi = -1(x - \pi)$

$y = -x + 2\pi$

Award 1 mark for the correct tangent equation.

***VCAA Assessment Report* note:**

This question was answered well. Students were not required to show any working. The answer could be obtained directly using technology. Some left their answer as $-x + 2\pi$.

d. $f'(x) = -\sin\left(\frac{x}{2}\right) = 1$ and $0 \le x \le 8\pi$

$x = 3\pi, 7\pi,$

$f(3\pi) = \pi, f(7\pi) = \pi$

$T_1: \quad y - \pi = 1(x - 3\pi)$

$y = x - 2\pi$

$T_2: \quad y - \pi = 1(x - 7\pi)$

$y = x - 6\pi$

Award 1 mark for each correct tangent equation. 2 marks total.

***VCAA Assessment Report* note:**

Once students found $x = 3\pi$ or $x = 7\pi$ the rest of the question could be completed using technology. Some students gave only one of the equations of the tangents.

e. $f(x) = 2f'(x) + \pi$

$$2\cos\left(\frac{\pi}{2}\right) + \pi = -2\sin\left(\frac{\pi}{2}\right) + \pi$$

$$2\cos\left(\frac{\pi}{2}\right) = -2\sin\left(\frac{x}{2}\right)$$

$$\tan\left(\frac{x}{2}\right) = -1, \quad 0 \le x \le 8\pi$$

$$\frac{x}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

$$x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}$$

Award 1 mark for solving the tangent equation.
Award 1 mark for all four correct solutions.
***VCAA Assessment Report* note:**
Some students gave $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$ as the answer. Others tried solving $2f'(x) + \pi = 0$ instead of $2f'(x) + \pi = f(x)$.

Question 5

$$y = x^2 \Rightarrow \frac{dy}{dx} = 2x, \left.\frac{dy}{dx}\right|_{x=2} = 4 \; P(2, 4)$$

$$T : y - 4 = 4(x - 2)$$

$$= 4x - 8$$

$$y = 4x - 4$$

When $x = 3$, $y = 12 - 4 = 8$
$(3, 8)$ lies on the tangent.
The correct answer is **B**.

Question 6

The gradient is positive and only defined for $x > a$ as x increases gradient and approaches zero, so the only option is **E**.
The correct answer is **E**.

Question 7

a. $f : R \to R, f(x) = 3 + 2x - x^2$ and $f : R \to R,\ g(x) = e^x$.
$g(f(x)) = e^{3+2x-x^2}$ **[1 mark]**

b. $g'(f(x)) = (2 - 2x)e^{3+2x-x^2} < 0$
Since $e^{3+2x-x^2} > 0 \quad 2 - 2x < 0$
$x > 1$
[2 marks]
***VCAA Examination Report* note:**
Students generally applied the chain rule to find the derivative; however, poor expression resulted in incorrect answers. The expression $(2 - 2x)e^{3+2x-x^2}$ is **not** equivalent to $2 - 2xe^{3+2x-x^2}$. Some students did find the correct answer; however, it was not supported by correct reasoning.

c. $f(g(x)) = 3 + 2e^x - e^{2x}$ **[1 mark]**
***VCAA Examination Report* note:**
This question was done well. Some students incorrectly stated $f(g(x)) = 3 + 2e^x - e^{x^2}$.

d. $f(g(x)) = 3 + 2e^x - e^{2x} = 0$

Let $u = e^x \Leftrightarrow 3 + 2u - u^2 = 0$.

$u^2 - 2u - 3 = 0$

$(u - 3)(u + 1) = 0$

$u = e^x = 3, \quad u = e^x = -1$ no solution

$x = \log\ e^3$

Award 1 mark for correctly reducing to a quadratic.

Award 1 mark for the correct answer.

***VCAA Examination Report* note:**

Most students were able to form a quadratic equation. Some students faltered with the correct factorisation. The inclusion of $x = \log_e(-1)$ was a common error.

e. $f'(g(x)) = 2e^x - 2e^{2x} = 0$ for stationary points.

Let $u = e^x \Leftrightarrow u = u^2$.

$u^2 - u = 0$

$u(u - 1) = 0$

$u = e^x = 1, \qquad u = e^x = 0$ no solution

$x = 0,\ f(g(0)) = 3 + 2 - 1 = 4$

The stationary point is (0, 4).

Award 1 mark for the correct gradient.

Award 1 mark for the correct stationary point.

***VCAA Examination Report* note:**

This question was well attempted but not so well done. Common errors included an incorrect derivative and omitting the y-coordinate of the stationary point.

f. $g(f(x)) + f(g(x)) = 0$

$-f(g(x)) = -(3 + 2e^x - e^{2x})$ from part **c**. As $x \to -\infty,\ f(g(x)) \to -3,\ x \to \infty,\ f(g(x)) \to \infty$

From part **e**, $-f(g(x))$ has a turning point at $(0, -4)$.

$g(f(x)) = 2^{3+2x-x^2}$: from part **b**, gradient is negative for $x > 1$.

As $x \to \infty,\ g(f(x)) \to 0^+,\ x \to -\infty,\ g(f(x)) \to 0^+$.

$g(f(x)) = -f(g(x))$ has only one solution,

So $g(f(x)) + f(g(x)) = 0$ also has only one solution. **[1 mark]**

***VCAA Examination Report* note:**

This question was not well done. Few students attempted to draw a rough sketch of each equation and use addition of ordinates.

Question 8

A has a cusp at $x = 2$ and is not differentiable at $x = 2$.

B has an end point at $x = 2$ and is not differentiable at $x = 2$.

C and D have vertical asymptotes at $x = 2$ and are certainly not differentiable at $x = 2$.

E is differentiable at $x = 2$.

The correct answer is **E**.

6 Further differentiation and applications

Topic	6	Further differentiation and applications
Subtopic	6.2	The chain rule

online only

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***Source:** VCE 2020, Mathematical Methods Exam 2, Section A, Q7; © VCAA*

Question 1 (1 mark)

If $f(x) = e^{g(x^2)}$, where g is a differentiable function, then $f'(x)$ is equal to

A. $2xe^{g(x^2)}$
B. $2xg\left(x^2\right)e^{g(x^2)}$
C. $2xg'(x^2)e^{g(x^2)}$
D. $2xg'(2x)\,e^{g(x^2)}$
E. $2xg'\left(x^2\right)e^{g(2x)}$

***Source:** VCE 2015, Mathematical Methods (CAS) Exam 1, Q1a; © VCAA*

Question 2 (1 mark)

Let $y = (5x + 1)^7$.

Find $\frac{dy}{dx}$.

***Source:** VCE 2014, Mathematical Methods (CAS) Exam 1, Q1b; © VCAA*

Question 3 (3 marks)

If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$.

Topic 6 Subtopic 6.2 The chain rule

Source: *VCE 2017, Mathematical Methods 1, Q1b; © VCAA*

Question 4 (2 marks)

Let $g(x) = \left(2 - x^3\right)^3$.

Evaluate $g'(1)$

Source: *VCE 2013, Mathematical Methods (CAS) 2, Section 1, Q12; © VCAA*

Question 5 (1 mark)

Let $y = 4\cos(x)$ and x be a function of t such $\frac{dx}{dt} = 3e^{2t}$ that and $x = \frac{3}{2}$ when $t = 0$.

The value of $\frac{dy}{dt}$ when $x = \frac{\pi}{2}$ is

A. 0

B. $3\pi \log_e\left(\frac{\pi}{2}\right)$

C. -4π

D. -2π

E. $-12e$

Question 6 (1 mark)

For $y = \sqrt{1 + f(x)}$, $\frac{dy}{dx}$ is equal to

A. $\frac{1}{2}f'(x)\sqrt{1 + f(x)}$

B. $\frac{f'(x)}{2\sqrt{1 + f(x)}}$

C. $f'(x)\sqrt{1 + f(x)}$

D. $\frac{f'(x)}{\sqrt{1 + f(x)}}$

E. $\sqrt{1 + f(x)}$

Question 7 (1 mark)

If $f(x) = e^{\sin(2x)}$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

A. $-\frac{1}{e}$
B. 0
C. -1
D. -2
E. 2

Question 8 (1 mark)

The derivative of $\left(4x^4 + x\right)^5$ with respect to x, is equal to

A. $5\left(4x^4 + x\right)^4$
B. $80x^3\left(4x^4 + x\right)^4$
C. $5\left(16x^3 + 1\right)\left(4x^4 + x\right)^4$
D. $5\left(16x^3\right)^4$
E. $5\left(16x^3 + 1\right)^4$

Topic	6	Further differentiation and applications
Subtopic	6.3	The product rule

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Source: *VCE 2020, Mathematical Methods Exam 1, Q1a; © VCAA*

Question 1 (1 mark)

Let $y = x^2 \sin(x)$.

Find $\frac{dy}{dx}$.

Source: *VCE 2016, Mathematical Methods Exam 1, Q1b; © VCAA*

Question 2 (2 marks)

Let $f(x) = x^2 e^{5x}$.

Evaluate $f'(1)$.

Source: *VCE 2013, Mathematical Methods (CAS) Exam 1, Q1a; © VCAA*

Question 3 (2 marks)

If $y = x^2 \log_e(x)$, find $\frac{dy}{dx}$

Question 4 (1 mark)

If $f(x) = g(x)\cos(3x)$ and $f'(x) = -3e^{-3x}(\cos(3x) + \sin(3x))$ then $g(x)$ is equal to

A. e^{-3x}

B. $-e^{-3x}$

C. $3e^{-3x}$

D. $-3e^{-3x}$

E. $-\frac{1}{3}e^{-3x}$

Question 5 (1 mark)

If $f(x) = g(x)e^{2x}$, $g(1) = 2$ and $g'(1) = 1$ then $f'(1)$ is equal to

A. e^2

B. $2e^2$

C. $3e^2$

D. $4e^2$

E. $5e^2$

Topic	6	Further differentiation and applications
Subtopic	6.4	The quotient rule

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Source: *VCE 2018, Mathematical Methods Exam 1, Q1b; © VCAA*

Question 1 (2 marks)

Let $f(x) = \dfrac{e^x}{\cos(x)}$.

Evaluate $f'(\pi)$.

Source: *VCE 2017, Mathematical Methods Exam 1, Q1a; © VCAA*

Question 2 (2 marks)

Let $f: (-2, \infty) \to R, f(x) = \dfrac{x}{x+2}$.

Differentiate f with respect to x.

Source: *VCE 2016, Mathematical Methods Exam 1, Q1a; © VCAA*

Question 3 (2 marks)

Let $y = \dfrac{\cos(x)}{x^2+2}$.

Find $\dfrac{dy}{dx}$.

Source: *VCE 2015, Mathematical Methods (CAS) 1, Q1b; © VCAA*

Question 4 (3 marks)

Let $f(x) = \dfrac{\log_e(x)}{x^2}$.

a. Find $f'(x)$. **(2 marks)**

b. Evaluate $f'(1)$. **(1 mark)**

Question 5 (1 mark)

If $y = \dfrac{4x-3}{3x+4}$ then $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{4}{3}$

B. $\dfrac{25}{(3x+4)^2}$

C. $\dfrac{-7}{(3x+4)^2}$

D. $\dfrac{7}{(3x+4)^2}$

E. $\dfrac{12x+9}{(3x+4)^2}$

Question 6 (1 mark)

If $y = \dfrac{\log_e(3x)}{3x}$ then $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{1}{x^2} - \dfrac{\log_e(3x)}{3x^2}$

B. $\dfrac{1}{3x^2}\left(1 - \log_e(3x)\right)$

C. $\dfrac{1}{3x^2}\left(\log_e(3x) - 3\right)$

D. $\dfrac{7}{(3x+4)^2}$

E. $\dfrac{12x+9}{(3x+4)^2}$

Question 7 (1 mark)

If $f(x) = \dfrac{e^{2x}}{g(x)}$, $g(1) = 2$ and $g'(1) = 1$ then $f'(1)$ is equal to

A. e^2

B. $2e^2$

C. $\dfrac{3e^2}{4}$

D. $\dfrac{5e^2}{4}$

E. $\dfrac{e^2}{4}$

Topic	6	Further differentiation and applications
Subtopic	6.5	Curve sketching

online only

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Source: *VCE 2017, Mathematical Methods Exam 2, Section A, Q11; © VCAA*

Question 1 (1 mark)

The function $f: R \to R$, $f(x) = x^3 + ax^2 + bx$ has a local maximum at $x = -1$ and a local minimum at $x = 3$. The values of a and b are respectively

A. -2 and -3

B. 2 and 1

C. 3 and -9

D. -3 and -9

E. -6 and -15

Source: *VCE 2013, Mathematical Methods (CAS) Exam 2, Section 1, Q21; © VCAA*

Question 2 (1 mark)

The cubic function $R \to R$, $f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A. $c > \frac{b^2}{4a}$

B. $c < \frac{b^2}{4a}$

C. $c < 4b^2a$

D. $c > \frac{b^2}{3a}$

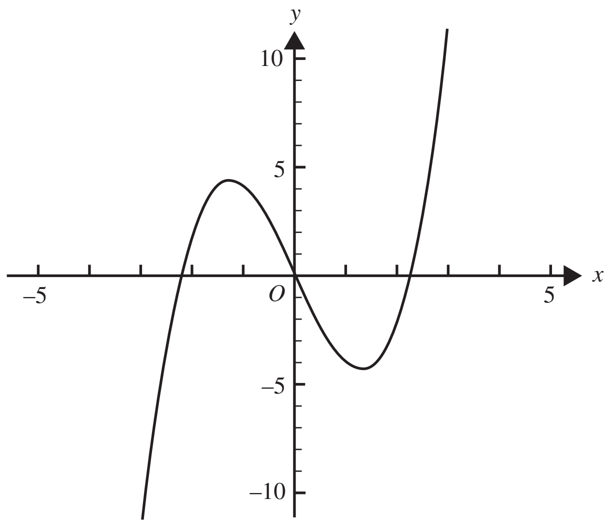
E. $c < \frac{b^2}{3a}$

Topic 6 | Subtopic 6.5 Curve sketching

Source: *VCE 2017, Mathematical Methods Exam 2, Section B, Q1a, b; © VCAA*

Question 3 (5 marks)

Let $f: R \to R,\ f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a. Find the coordinates of the turning points. **(2 marks)**

b. $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

i. Find the equation of the straight line through A and B. **(2 marks)**

ii. Find the distance AB. **(1 mark)**

Source: *VCE 2018, Mathematical Methods 2, Section A, Q5; © VCAA*

Question 4 (1 mark)

Consider $f(x) = x^2 + \frac{p}{x}, x \neq 0, p \in R$

There is a stationary point on the graph of f when $x = -2$.

The value of p is

A. -16

B. -8

C. 2

D. 8

E. 16

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q4*; *© VCAA*

Question 5 (1 mark)

Let f be a function with domain R such that $f'(5) = 0$ and $f'(x) < 0$ when $x \neq 5$.

At $x = 5$, the graph of f has a

A. local minimum.

B. local maximum.

C. gradient of 5.

D. gradient of -5.

E. stationary point of inflection.

Question 6 (1 mark)

Consider the function $f{:}R \to R, f(x) = \frac{1}{27}(ax-1)^3(b-3x)+1$ where a and b real constants. The function will have stationary points at

A. $x = \frac{1}{a}$ and $x = \frac{1+ab}{4a}$

B. $x = \frac{1}{a}$ and $x = \frac{1-ab}{4a}$

C. $x = \frac{1}{9a}$ and $x = \frac{1+ab}{4a}$

D. $x = \frac{1}{a}$ and $x = \frac{b}{3}$

E. $x = \frac{1}{a}$ and $x = \frac{ab-1}{4a}$

Question 7 (1 mark)

Let $f{:}R \to R <, f(x) = (x-2)(x-5)\left(3x^2+ax+6\right)$ where a is a real number. If f has p stationary points, the possible values of p are

A. $p \in \{1, 2, 3, 4\}$

B. $p \in \{1, 2, 3\}$

C. $p \in \{1, 2\}$

D. $p \in \{1\}$

E. $p \in \{0\}$

Topic 6 | Subtopic 6.5 Curve sketching

Question 8 (1 mark)

For the graph of $y = x^4 - 4x^2 + 4$ which of the following statements is most correct?

A. There are local maximums at $\left(\pm\sqrt{2}, 0\right)$

B. There are local minimums at $\left(\pm\sqrt{2}, 0\right)$

C. There are local minimums at $\left(\pm\sqrt{2}, 0\right)$ and a local maximum at $(0, 4)$

D. There are local maximums at $\left(\pm\sqrt{2}, 0\right)$ and a local minimum at $(0, 4)$

E. There is a local maximum at $(0, 4)$

Question 9 (2 marks)

Sketch the graph of the derivative of the function with rule $y = (x + 2)^2 \left(x^2 - 4x + 6\right)$ in your workbook and determine the stationary points of the original curve and their nature.

Question 10 (4 marks)

If $f(x) = ax^3 + bx^2 + cx + d, f'(-1) = 3$ and $f(1) = 1$ and there is a point of inflection at $(2, 4)$, find the values of a, b, c and d.

Question 11 (1 mark)

If $f{:}R \to R$ is a differentiable function such that:

- $f'(x) = 0$ at $x = 2$ and $x = 3$
- $f'(x) > 0$ at $x > 3$ and $2 < x < 3$
- $f'(x) < 0$ at $x < 2$

Which of the following statements is **correct**?

A. The graph has a stationary point of inflection at $x = 3$ and a maximum at $x = 2$.

B. The graph has a stationary point of inflection at $x = 3$ and a minimum at $x = 2$.

C. The graph has a stationary point of inflection at $x = 2$ and a minimum at $x = 3$.

D. The graph has a maximum at $x = 2$ and a minimum at $x = 3$.

E. The graph has a minimum at $x = 2$ and a maximum at $x = 3$.

Question 12 (1 mark)

For the curve $y = (x - 5)^5$, the point $(5,\ 0)$ is best classified as

A. A local minimum turning point.
B. A local maximum turning point.
C. A stationary point of inflection.
D. A point where the function is continuous but not differentiable.
E. An axial intercept.

Question 13 (1 mark)

If $f'(x) < 0$ where $x > 3$ and $f'(x) > 0$ where $x < 3$, then at the point $x = 3$, $f(x)$ has a

A. local minimum.
B. local maximum.
C. point of inflection.
D. discontinuous point.
E. gradient of 3.

Question 14 (1 mark)

For the function $f{:}R^+ \to R, f(x) = \dfrac{4}{\sqrt{x}} + \sqrt{x-5}$, state the interval for which the graph of f is strictly increasing.

Question 15 (1 mark)

Given $h : R \to R, \quad h(x) = x - x^3$, the function is strictly increasing over the interval

A. $(0, 1)$
B. $\left[-\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right]$
C. $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right)$
D. $(-1, 0)$
E. $(-1, 1]$

Question 16 (1 mark)

Find the interval for which the function $f: R \to R,\ f(x) = \dfrac{1}{e^{x+1}}$ is strictly decreasing.

Question 17 (1 mark)

For the function $g: [0, 2\pi] \to R, g(x) = \cos(x), g$, is strictly decreasing over the interval

A. $\left[0, \dfrac{3\pi}{2}\right]$
B. $[-\pi, 0]$
C. $(0, \pi)$
D. $(-\pi, \pi)$
E. $[0, 2\pi]$

Topic	6	Further differentiation and applications
Subtopic	6.6	Maximum and minimum problems

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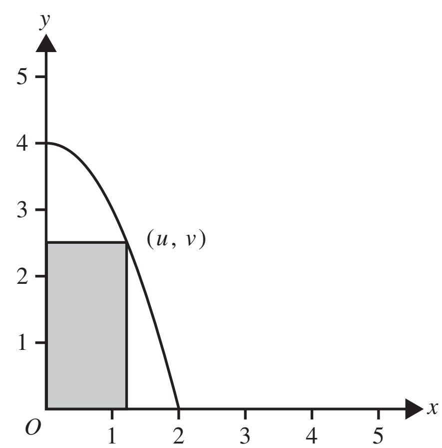
Source: *VCE 2016, Mathematical Methods Exam 2, Section A, Q14; © VCAA*

Question 1 (1 mark)

A rectangle is formed by using part of the coordinate axes and a point (u, v), where $u > 0$ on the parabola $y = 4 - x^2$.



Which one of the following is the maximum area of the rectangle?

A. 4

B. $\dfrac{2\sqrt{3}}{3}$

C. $\dfrac{8\sqrt{3}-4}{3}$

D. $\dfrac{8}{3}$

E. $\dfrac{16\sqrt{3}}{9}$

***Source:** VCE 2018, Mathematical Methods Exam 1, Q7; © VCAA*

Question 2 (5 marks)

Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

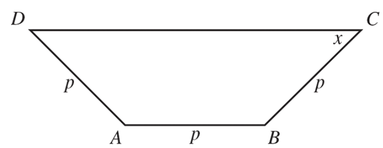
a. Find the coordinates of P. **(3 marks)**

b. Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers. **(2 marks)**

***Source:** VCE 2014, Mathematical Methods (CAS) Exam 2, Section 1, Q21; © VCAA*

Question 3 (1 mark)

The trapezium $ABCD$ is shown below. The sides AB, BC and DA are of equal length, p. The size of the acute angle BCD is x radians.



The area of the trapezium is a maximum when the value of x is

A. $\frac{\pi}{12}$

B. $\frac{\pi}{6}$

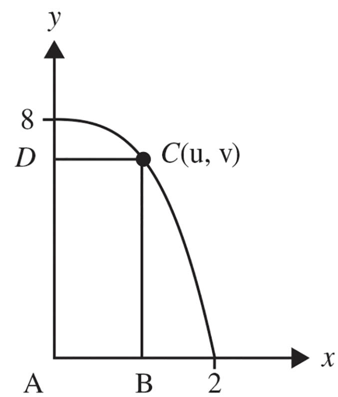
C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

E. $\frac{5\pi}{12}$

Source: *VCE 2017, Mathematical Methods 2, Section A, Q15; © VCAA*

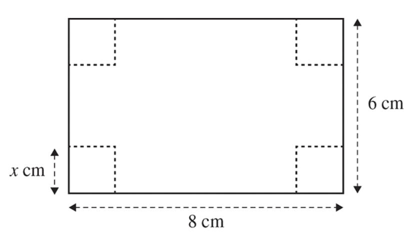
Question 4 (1 mark)
A rectangle $ABCD$ has vertices $A(0, 0)$, $B(u, 0)$, $C(u, v)$ and $D(0, v)$ where (u, v) lies on the graph of $y = -x^3 + 8$, as shown below.



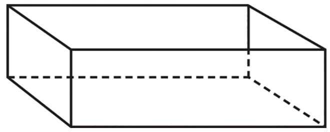
The maximum area of the rectangle is
A. $\sqrt[3]{2}$
B. $6\sqrt[3]{2}$
C. 16
D. 8
E. $3\sqrt[3]{2}$

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q15; © VCAA*

Question 5 (1 mark)
Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.



Zoe turns up the sides to form an open box.



The value of x for which the volume of the box is a maximum is closest to

A. 0.8
B. 1.1
C. 1.6
D. 2.0
E. 3.6

Question 6 (1 mark)

A piece of wire of length 20 cm is bent into the shape of a rectangle. If one side of the rectangle has a length of x cm, then the area A of the rectangle is equal to

A. $A(x) = x(20 - x)$
B. $A(x) = x(10 - x)$
C. $A(x) = 20x$
D. $A(x) = 10x$
E. $A(x) = 5x$

Question 7 (1 mark)

The function $f: [0, 100] \rightarrow R, f(x) = \dfrac{x^2}{16} + \dfrac{(100 - x)^2}{4\pi}$ has a maximum value at

A. 400
B. 625
C. 700
D. 795
E. 800

Question 8 (1 mark)

Find the maximum value for the function $f: [-2, 1] \rightarrow R, f(x) = x^3 + 2$.

Question 9 (1 mark)

Find the minimum value for the function $f: [-1, 3] \rightarrow R, f(x) = x^3 + 2$.

Question 10 (1 mark)

Find the maximum value for the function $f: [0, 2\pi] \rightarrow R, f(x) = \sin(x)$.

Topic	6	Further differentiation and applications
Subtopic	6.7	Rates of change

online only

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***Source:** VCE 2019, Mathematical Methods Exam 2, Section A, Q3; © VCAA*

Question 1 (1 mark)

Let $f : R\backslash\{4\} \to R,\ f(x) = \dfrac{a}{x-4}$, where $a > 0$.

The average rate of change of f from $x = 6$ to $x = 8$ is

A. $a \log_e(2)$

B. $\dfrac{a}{2}\log_e(2)$

C. $2a$

D. $-\dfrac{a}{4}$

E. $-\dfrac{a}{8}$

***Source:** VCE 2017, Mathematical Methods Exam 2, Section A, Q9; © VCAA*

Question 2 (1 mark)

The average rate of change of the function with the rule $f(x) = x^2 - 2x$ over the interval $[1, a]$, where $a > 1$, is 8.

The value of a is

A. 9

B. 8

C. 7

D. 4

E. $1 + \sqrt{2}$

Question 3 (1 mark)

For $y = 2e^{-x+1}\sin(x-1)$, the rate of change of y with respect to x when $x = 1$ is

A. -1
B. 1
C. 4
D. 2
E. -2

Source: *VCE 2021, Mathematical Methods 2, Section A, Q13; © VCAA*

Question 4 (1 mark)

The value of an investment, in dollars, after n months can be modelled by the function

$f(n) = 2500 \times (1.004)^n$

where $n \in \{0, 1, 2, ...\}$

The average rate of change of the value of the investment over the first 12 months is closest to

A. \$10.00 per month.
B. \$10.20 per month.
C. \$10.50 per month.
D. \$125.00 per month.
E. \$127.00 per month.

Source: *VCE 2016, Mathematical Methods 2, Section A, Q4; © VCAA*

Question 5 (1 mark)

The average rate of change of the function f with rule $f(x) = 3x^2 - 2\sqrt{x+1}$, between $x = 0$ and $x = 3$, is

A. 8
B. 25
C. $\frac{53}{9}$
D. $\frac{25}{3}$
E. $\frac{13}{9}$

Source: *VCE 2013, Mathematical Methods (CAS) 2, Section 1, Q6*; © VCAA

Question 6 (1 mark)

For the function $f(x) = \sin(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\frac{1}{4}, 5\right]$ is

A. 0

B. $\frac{34}{19}$

C. $\frac{7}{2}$

D. $\frac{2\pi + 10}{4}$

E. $\frac{23}{4}$

Question 7 (1 mark)

For the function $f(x) = \cos(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\frac{1}{4}, 4\right]$ is

A. $\frac{17}{15}$ **B.** $\frac{32}{15}$ **C.** $\frac{34}{15}$

D. 2 **E.** $\frac{4}{15}$

Question 8 (1 mark)

The average rate of change of the function with the rule $f(x) = x^3 + e^{2x}$ between $x = 0$ and $x = 2$ is equal to

A. $\frac{8 + e^4}{2}$ **B.** $\frac{7 + e^4}{2}$ **C.** $12 + 2e^4$

D. $5 + e^4$ **E.** $7 + e^4$

Topic	6	Further differentiation and applications
Subtopic	6.8	Newton's method

online only

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Question 1 (2 marks)

Determine the root of $2 - x^2 = \sin(x)$ in the interval $[1, 2]$ to an accuracy of 3 decimal places.

Question 2 (3 marks)

Use a suitable equation to calculate $\sqrt[4]{12}$, accurate to 2 decimal places.

Question 3 (3 marks)

The equation $0 = x^4 - 5x - 8$ has only one negative solution. Determine between which two integer values the solution lies, then calculate the solution to an accuracy of 3 decimal places.

Topic	6	Further differentiation and applications
Subtopic	6.9	Review

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Source: *VCE 2018, Mathematical Methods Exam 1, Q1a; © VCAA*

Question 1 (1 mark)

If $y = \left(-3x^3 + x^2 - 64\right)^3$, find $\dfrac{dy}{dx}$.

Source: *VCE 2014, Mathematical Methods (CAS) Exam 1, Q1a; © VCAA*

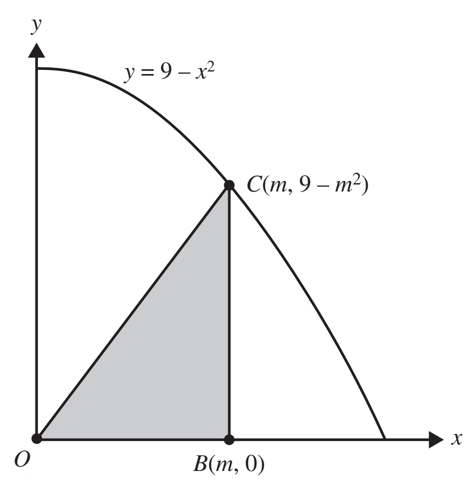
Question 2 (2 marks)

If $y = x^2 \sin(x)$, find $\dfrac{dy}{dx}$.

Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q16; © VCAA*

Question 3 (1 mark)

A right-angled triangle, OBC, is formed using the horizontal axis and the point $C(m, 9 - m^2)$ $m \in (0, 3)$, where, on the parabola $y = 9 - x^2$, as shown below.



The maximum area of the triangle OBC is

A. $\frac{\sqrt{3}}{3}$

B. $\frac{2\sqrt{3}}{3}$

C. $\sqrt{3}$

D. $3\sqrt{3}$

E. $9\sqrt{3}$

Source: *VCE 2015, Mathematical Methods (CAS) Exam 1, Q1a; © VCAA*

Question 4 (1 mark)

The average rate of change of the function f with rule $f(x) = 3x^2 - 2\sqrt{x+1}$, between $x = 0$ and $x = 3$, is

A. 8

B. 25

C. $\frac{53}{9}$

D. $\frac{25}{3}$

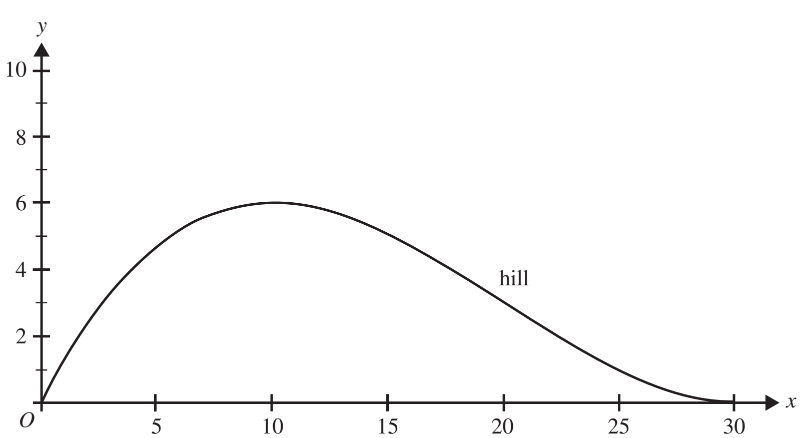
E. $\frac{13}{9}$

Source: *VCE 2019, Mathematical Methods Exam 2, Section B, Q2; © VCAA*

Question 5 (11 marks)

An amusement park is planning to build a zip-line above a hill on its property.

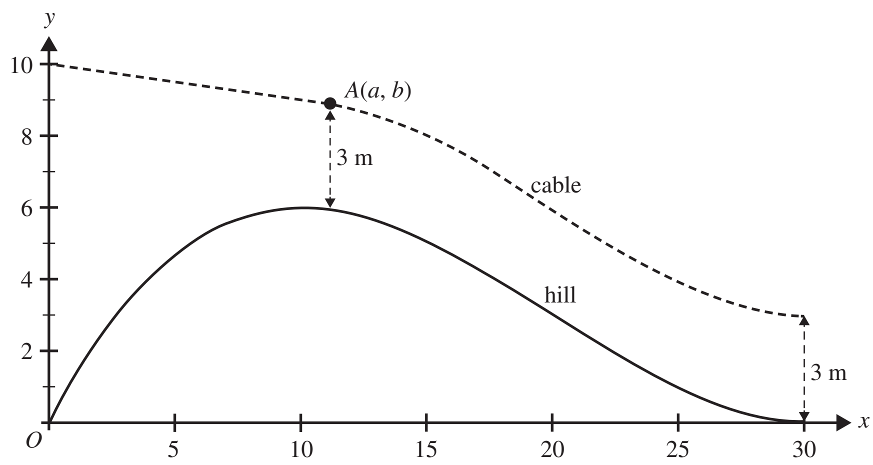
The hill is modelled by $y = \frac{3x(x-30)^2}{2000}$, $x \in [0, 30]$, where x is the horizontal distance, in metres, from an origin and y is the height, in metres, above this origin, as shown in the graph below.



a. Find $\frac{dy}{dx}$. **(1 mark)**

b. State the set of values for which the gradient of the hill is strictly decreasing. **(1 mark)**

The cable for the zip-line is connected to a pole at the origin at a height of 10 m and is straight for $0 \leq x \leq a$, where $10 \leq a \leq 20$. The straight section joins the curved section at $A(a, b)$. The cable is then exactly 3 m vertically above the hill from $a \leq x \leq 30$, as shown in the graph below.



c. State the rule, in terms of x, for the height of the cable above the horizontal axis for x $[a, 30]$. **(1 mark)**

d. Find the values of x for which the gradient of the cable is equal to the average gradient of the hill for $x \in [10, 30]$ **(3 marks)**

The gradients of the straight and curved sections of the cable approach the same value at $x = a$, so there is a continuous and smooth join at A.

e. **i.** State the gradient of the cable at A, in terms of a. **(1 mark)**

ii. Find the coordinates of A, with each value correct to two decimal places. **(3 marks)**

iii. Find the value of the gradient at A, correct to one decimal place. **(1 mark)**

Source: *VCE 2021, Mathematical Methods 2, Section A, Q19; © VCAA*

Question 6 (1 mark)

Which one of the following functions is differentiable for all real values of x?

A. $f(x) = \begin{cases} x & x < 0 \\ -x & x \geq 0 \end{cases}$

B. $f(x) = \begin{cases} x & x < 0 \\ -x & x > 0 \end{cases}$

C. $f(x) = \begin{cases} 8x + 4 & x < 0 \\ (2x + 1)^2 & x \geq 0 \end{cases}$

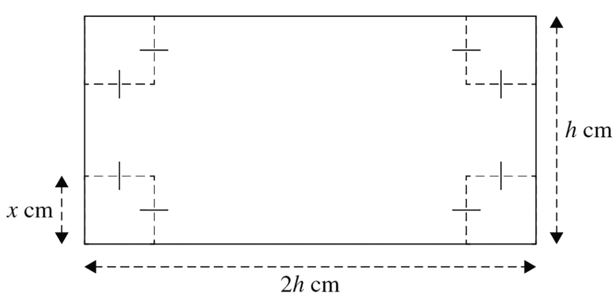
D. $f(x) = \begin{cases} 2x + 1 & x < 0 \\ (2x + 1)^2 & x \geq 0 \end{cases}$

E. $f(x) = \begin{cases} 4x + 1 & x < 0 \\ (2x + 1)^2 & x \geq 0 \end{cases}$

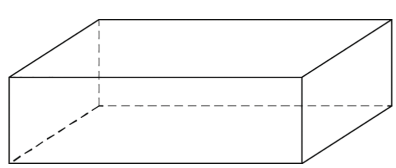
Source: *VCE 2021, Mathematical Methods 2, Section B, Q1; © VCAA*

Question 7 (14 marks)

A rectangular sheet of cardboard has a width of h centimetres. Its length is twice its width.
Squares of side length x centimetres, where $x > 0$, are cut from each of the corners, as shown in the diagram below.



The sides of this sheet of cardboard are then folded up to make a rectangular box with an open top, as shown in the diagram below.
Assume that the thickness of the cardboard is negligible and that $V_{box} > 0$.



A box is to be made from a sheet of cardboard with $h = 25$ cm.

a. Show that the volume, V_{box}, in cubic centimetres, is given by $V_{box}(x) = 2x(25 - 2x)(25 - x)$. **(1 mark)**

b. State the domain of V_{box}. **(1 mark)**

c. Find the derivative of V_{box} with respect to x. **(1 mark)**

d. Calculate the maximum possible volume of the box and for which value of x this occurs. **(3 marks)**

e. Waste minimisation is a goal when making cardboard boxes.
Percentage wasted is based on the area of the sheet of cardboard that is cut out before the box is made.
Find the percentage of the sheet of cardboard that is wasted when $x = 5$. **(2 marks)**

f. Now consider a box made from a rectangular sheet of cardboard where $h > 0$ and the box's length is still twice its width.

i. Let V_{box} be the function that gives the volume of the box.
State the domain of V_{box} in terms of h. **(1 mark)**

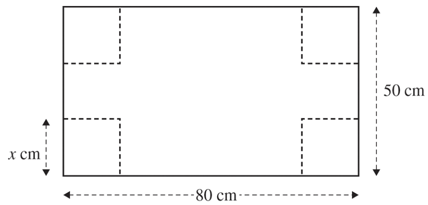
ii. Find the maximum volume for any such rectangular box, V_{box}, in terms of h. **(3 marks)**

g. Now consider making a box from a square sheet of cardboard with side lengths of h centimetres.
Show that the maximum volume of the box occurs when $x = \frac{h}{6}$. **(2 marks)**

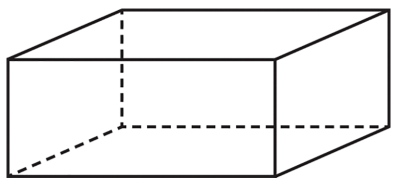
Source: *VCE 2019, Mathematical Methods 2, Section A, Q6*; © VCAA

Question 8 (1 mark)

A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the corners, as shown in the diagram below.



A rectangular box with an open top is then constructed, as shown in the diagram below.



The volume of the box is a maximum when x is equal to

A. 10

B. 20

C. 25

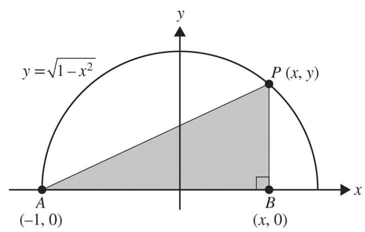
D. $\frac{100}{3}$

E. $\frac{200}{3}$

Source: *VCE 2019, Mathematical Methods 1, Q7*; © VCAA

Question 9 (4 marks)

The graph of the relation $y = \sqrt{1 - x^2}$ is shown on the axes below. P is a point on the graph of this relation, A is the point $(-1, 0)$ and B is the point $(x, 0)$.



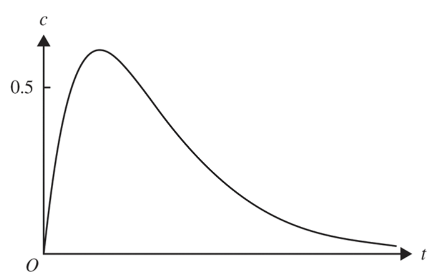
a. Find an expression for the distance PB in terms of x only. **(1 mark)**

b. Find the maximum area of the triangle ABP. **(3 marks)**

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 2, Q3*; © VCAA

Question 10 (11 marks)

In a controlled experiment, Juan took some medicine at 8 pm. The concentration of medicine in his blood was then measured at regular intervals. The concentration of medicine in Juan's blood is modelled by the function $c(t) = \frac{5}{2}te^{\frac{-3t}{2}}, t \geq 0$, where c is the concentration of medicine in his blood, in milligrams per litre, t hours after 8 pm. Part of the graph of the function c is shown below.



a. What was the maximum value of the concentration of medicine in Juan's blood, in milligrams per litre, correct to two decimal places? **(1 mark)**

b. Answer the following.

i. Find the value of t, in hours, correct to two decimal places, when the concentration of medicine in Juan's blood first reached 0.5 milligrams per litre. **(1 mark)**

ii. Find the length of time that the concentration of medicine in Juan's blood was above 0.5 milligrams per litre. Express the answer in hours, correct to two decimal places. **(2 marks)**

c. Answer the following.

i. What was the value of the average rate of change of the concentration of medicine in Juan's blood over the interval $\left[\frac{2}{3}, 3\right]$?

Express the answer in milligrams per litre per hour, correct to two decimal places. **(2 marks)**

ii. At times t_1 and t_2, the instantaneous rate of change of the concentration of medicine in Juan's blood was equal to the average rate of change over the interval $\left[\frac{2}{3}, 3\right]$.

Find the values of t_1 and t_2, in hours, correct to two decimal places. **(2 marks)**

d. Alicia took part in a similar controlled experiment. However, she used a different medicine. The concentration of this different medicine was modelled by the function $n(t) = Ate^{-kt}, t \geq 0$, where A and $k \in R^+$.

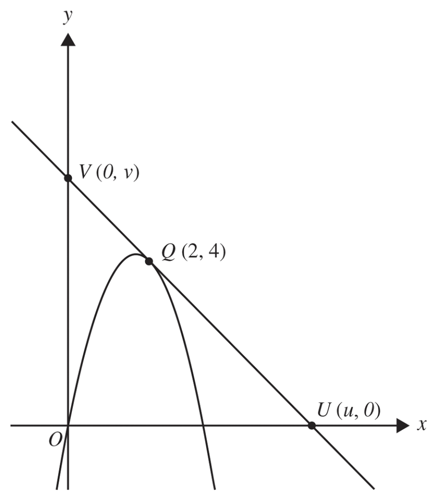
If the **maximum** concentration of medicine in Alicia's blood was 0.74 milligrams per litre at $t = 0.5$ hours, find the value of A, correct to the nearest integer. **(3 marks)**

***Source:** VCE 2014, Mathematical Methods (CAS) 1, Q10*; © VCAA

Question 11 (7 marks)

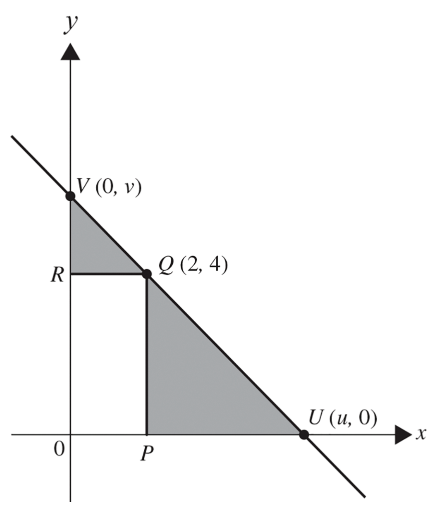
A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \le u \le 6$.

a. When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates $(2, 4)$, as shown. **(3 marks)**



If a and b are non-zero real numbers, find the values of a and b.

b. The rectangle $OPQR$ has a vertex at Q on the line. The coordinates of Q are $(2, 4)$, as shown.



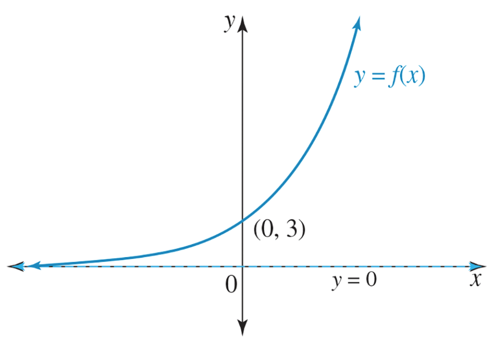
i. Find an expression for v in terms of u. **(1 mark)**

ii. Find the **minimum** total shaded area and the value of u for which the area is a minimum. **(2 marks)**

iii. Find the **maximum** total shaded area and the value of u for which the area is a maximum. **(1 mark)**

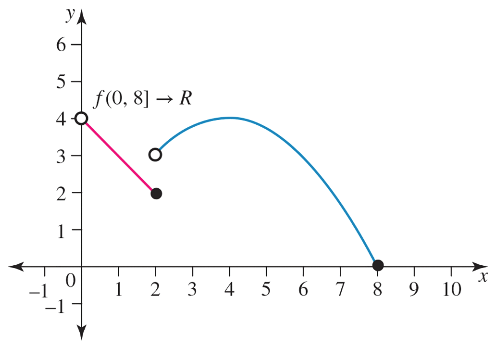
Question 12 (1 mark)

The graph of the function f, with rule $y = f(x)$, is shown.
Sketch the graph of the derivative function $y = f'(x)$.



Question 13 (2 marks)

The graph of the function $f:(0,\ 8] \rightarrow R$ is shown. Sketch the graph of the derivative function.



Answers and marking guide

6.2 The chain rule

Question 1

$f(x) = e^{g\left(x^2\right)}$

$f'(x) = \dfrac{d}{dx}\left(g\left(x^2\right)\right) e^{g\left(x^2\right)}$

$f'(x) = 2xg'\left(x^2\right) e^{g\left(x^2\right)}$

The correct answer is **C**.

Question 2

$y = (5x + 1)^7$

Apply the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= 7 \times 5(5x + 1)^6\\ &= 35(5x + 1)^6\end{aligned}$$ **[1 mark]**

Question 3

$y = f(x) = \sqrt{x^2 + 3}$

$y = \sqrt{u} = u^{\frac{1}{2}}$ where $u = x^2 + 3$ (chain rule)

$\dfrac{dy}{du} = \dfrac{1}{2}u^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{u}} \qquad \dfrac{du}{dx} = 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$f'(x) = \dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 + 3}}$

$f'(1) = \dfrac{1}{\sqrt{1 + 3}} = \dfrac{1}{2}$

Award 1 mark for using the chain rule.
Award 1 mark for the correct derivative.
Award 1 mark for the correct final answer.

***VCAA Examination Report* note:**

Many students only gave the expression for $f'(x)$, not the specific value of $f'(1)$. Students should also note that $\dfrac{1}{\sqrt{4}} \neq \pm\dfrac{1}{2}$, $\dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$.

Question 4

$g(x) = \left(2 - x^3\right)^3$ using the chain rule

$g = u^3 \quad u = \left(2 - x^3\right)$

$g'(u) = 3u^2 \quad u'(x) = -3x^2$

$g'(x) = -9x^2\left(2 - x^3\right)^2$

$g'(1) = -9$

Award 1 mark for using the chain rule.
Award 1 mark for the correct result.

***VCAA Examination Report* note:**

Students competently applied the chain rule; however, some erred with the derivative of $(2 - x^3)^3$, especially with negatives. Some students opted unnecessarily to take the longer route by (often incorrectly) expanding the rule given by g. Others forgot to evaluate $g'(1)$

Question 5

$y = 4\cos(x) \quad \frac{dx}{dt} = 3e^{2t} \quad \text{and } x = \frac{3}{2} \text{ when } t = 0$

$\frac{dy}{dx} = -4\sin(x)$

$x = \int 3e^{2t}dt \quad \Rightarrow \quad x = \frac{3}{2}e^{2t} + c \quad \text{when } x = \frac{3}{2}, t = 0 \Rightarrow c = 0$

$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -4\sin(x) \times 3e^{2t}$

When $x = \frac{\pi}{2}, \quad \frac{\pi}{2} = \frac{3}{2}e^{2t} \quad \Rightarrow 3e^{2t} = \pi$

$\frac{dy}{dt} = -4\sin\left(\frac{\pi}{2}\right) \times \pi = -4\pi$

The correct answer is **C**.

Question 6

Let $u = 1 + f(x)$.

$\therefore y = u^{\frac{1}{2}}$

$\frac{du}{dx} = f'(x)$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times f'(x)$

$\frac{dy}{dx} = f'(x) \times \frac{1}{2\sqrt{1 + f(x)}}$

The correct answer is **B**.

Question 7

$f(x) = e^{\sin(2x)}$

$f'(x) = 2\cos(2x)e^{\sin(2x)}$

$f'\left(\frac{\pi}{2}\right) = 2\cos(\pi)\, e^{\sin(\pi)} = -2$

The correct answer is **D**.

Question 8

$y = \left(4x^4 + x\right)^5$

$y = u^5 \quad \text{where} \quad u = 4x^4 + x$

$\frac{dy}{du} = 5u^4 \frac{du}{dx} = 16x^3 + 1$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5\left(16x^3 + 1\right)u^4$

$= 5\left(16x^3 + 1\right)\left(4x^4 + x\right)^4$

6.3 The product rule

Question 1

$y = x^2 \sin(x)$ using the product rule.

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin(x)) + \sin(x)\frac{d}{dx}\left(x^2\right)$$
$$\frac{dy}{dx} = x^2 \cos(x) + 2x\sin(x)$$
$$= x\,(x\cos(x) + 2\sin(x))$$ **[1 mark]**

Question 2

$f(x) = x^2 e^{5x}$

Using the product rule:

$$u = x^2 \quad v = e^{5x}$$
$$\frac{du}{dx} = 2x \quad \frac{dv}{dx} = 5e^{5v}$$
$$f'(x) = 2xe^{5x} + 5x^2e^{5x}$$
$$f'(1) = 2e^5 + 5e^5$$
$$= 7e^5$$

Award 1 mark for using the product rule.

Award 1 mark for the correct result.

VCAA Assessment Report **note:**

This question was well answered. Most students correctly identified the product rule but did not evaluate (as instructed) or their answers were incomplete. An incorrect combination of the product and chain rule resulted in an answer of $10xe^{5x}$ as a common error.

Question 3

$y = x^2 \log_e(x)$

$$u = x^2 \qquad v = \log_e(x)$$
$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \frac{1}{x}$$
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \text{ (product rule)}$$
$$\frac{dy}{dx} = x^2 \times \frac{1}{x} + 2x\log_e(x)$$
$$\frac{dy}{dx} = x + 2x\log_e(x) = x\left(1 + 2\log_e(x)\right)$$

Award 1 mark for using the product rule.

Award 1 mark for the correct result.

VCAA Assessment Report **note:**

Some students did not simplify the expression or incorrectly combined the terms to obtain $3x\log_e(x)$.

Question 4

$$f(x) = g(x)\cos(3x)$$
$$f'(x) = g'(x)\cos(3x) - 3g(x)\sin(3x)$$
$$= -3e^{-3x}(\cos(3x) + \sin(3x))$$
$$g'(x) = -3e^{-3x} \text{ and } g(x) = e^{-3x}$$

The correct answer is **A**.

Question 5

$f(x) = g(x)e^{2x}$

$f'(x) = g'(x)e^{2x} + 2g(x)e^{2x}$

$= g'(1)e^2 + 2g(1)e^2$ from question $g(1) = 2$ and $g'(1) = 1$

$= e^2 + 4e^2 = 5e^2$

The correct answer is **E**.

6.4 The quotient rule

Question 1

$$f(x) = \frac{e^x}{\cos(x)}$$

Using the quotient rule:

$$f'(x) = \frac{e^x\cos(x) - e^x(-\sin(x))}{\cos^2(x)}$$

$$f'(x) = \frac{e^x(\cos(x) + \sin(x))}{\cos^2(x)}$$

$$f'(\pi) = \frac{e^\pi(\cos(\pi) + \sin(\pi))}{\cos^2(\pi)}$$

$$f'(\pi) = \frac{e^\pi(-1+0)}{(-1)^2}$$

$f'(\pi) = -e^\pi$

Award 1 mark for using the quotient rule.

Award 1 mark for the correct substitution and final answer.

VCAA Examination Report **note:**

Students competently applied the quotient rule; however, many were unable to carry out the required evaluation, often omitting it completely. Students who opted to use the product and chain rules tended to make little progress due to confusion with negative signs or negative exponents. Students should take care with legibility, for example, to distinguishing clearly the variable x and the constant π.

Question 2

$f: (-2, \infty) \to R, f(x) = \dfrac{x}{x+2}$ using the quotient rule.

Let $u = x$, $v = x + 2$.

$u'(x) = 1 \quad v'(x) = 1$

$$f'(x) = \frac{1(x+2) - x}{(x+2)^2}$$

$$f'(x) = \frac{2}{(x+2)}, \text{ for } x > -2$$

Award 1 mark for using the quotient rule.

Award 1 mark for the correct result.

VCAA Examination Report **note:**

This question was well handled. Students choosing to use the quotient rule tended to progress better than those using the product rule.

Some very poor algebraic slips were made. The most common was 'cancelling' $x + 2$ in the numerator with $x + 2$ in the denominator. Others unnecessarily expanded $(x + 2)^2$ and did so incorrectly.

Question 3

$$y = \frac{\cos(x)}{x^2 + 2}$$

Using the quotient rule:

$u = \cos(x) \qquad v = x^2 + 2$

$\frac{du}{dx} = -\sin(x) \qquad \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{-\left(x^2 + 2\right)\sin(x) - 2x\cos(x)}{\left(x^2 + 2\right)^2}$$

Award 1 mark for using the quotient rule.
Award 1 mark for the correct result.

***VCAA Assessment Report* note:**

Most students were able to confidently apply the quotient rule. However, many students did not obtain full marks due to errors caused by, for example, a denominator of $x^4 + 4$ as the supposed expansion of $(x^2 + 2)^2$. Students should very carefully consider the placement and usage of brackets. For example, the expression $x^2 + 2 \times -\sin(x^2)$ is not equivalent to $(x^2 + 2) \times -\sin(x)$.

Question 4

a. $y = \frac{\log_e(x)}{x^2}$

Apply the quotient rule.

$u = \log_e(x) \quad v = x^2$

$\frac{du}{dx} = \frac{1}{x} \quad \frac{dv}{dx} = 2x$

$$f'(x) = \frac{\frac{1}{x} \times x^2 - 2x\log_e(x)}{x^4}$$

$$= \frac{1 - 2\log_e(x)}{x^3}$$

Award 1 mark for using the quotient rule.
Award 1 mark for the final correct derivative.

b. $f'(1) = \frac{1 - 2\log_e(1)}{1}$

$= 1$ [**1 mark**]

Question 5

$$y = \frac{4x - 3}{3x + 4}$$

$$\frac{dy}{dx} = \frac{4(3x + 4) - 3(4x - 3)}{(3x + 4)^2} = \frac{(12x + 16) - (12x - 9)}{(3x + 4)^2}$$

$$= \frac{25}{(3x + 4)^2}$$

The correct answer is **B**.

Question 6

$$y = \frac{\log_e(3x)}{3x}$$

let $u = \log_e(3x)$ and $v = 3x$

$\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 3$

$$\frac{dy}{dx} = \frac{\frac{1}{x} \times 3x - 3\log_e(3x)}{\left(3x^2\right)} = \frac{3\left(1 - \log_e(3x)\right)}{9x^2}$$

$$= \frac{1}{3x^2}\left(1 - \log_e(3x)\right)$$

The correct answer is **B**.

Question 7

$$f(x) = \frac{e^{2x}}{g(x)}$$

$$f'(x) = \frac{2e^{2x}g(x) - g'(x)e^{2x}}{\left[g(x)\right]^2}$$

$$= \frac{2e^2 g(1) - g'(1)e^2}{\left[g(1)\right]^2} \text{ from question } g(1) = 2 \quad \text{and } g'(1) = 1$$

$$= \frac{4e^2 - e^2}{4} = \frac{3e^2}{4}$$

The correct answer is **C**.

6.5 Curve sketching

Question 1

$$f(x) = x^3 + ax^2 + bx$$
$$f'(x) = 3x^2 + 2ax + b$$
$$f'(-1) = 0 \quad \Rightarrow 3 - 2a + b = 0 \quad [1]$$
$$f'(3) = 0 \quad \Rightarrow 27 + 6a + b = 0 \quad [2]$$

$[2] - [1]: 24 + 8a = 0$

$$\Rightarrow a = -3$$

$b = 2a - 3 = -9$

The correct answer is **D**.

Question 2

$f: R \to R,\ f(x) = ax^3 - bx^2 + cx,\ a,\ b,\ c \in R^+$

$f'(x) = 3ax^2 - 2bx + c \neq 0$

There will be no stationary points when $\Delta < 0$.

$\Rightarrow \Delta = (-2b)^2 - 4 \times 3a \times c \ < 0$

$4b^2 - 12ac < 0$ or $c > \dfrac{b^2}{3a}$

The correct answer is **D**.

Question 3

a. $f: R \to R,\ f(x) = x^3 - 5x$

$f'(x) = 3x^2 - 5 = 0$

$$\Rightarrow \quad x = \pm\frac{\sqrt{15}}{3}, f\left(\frac{\sqrt{15}}{3}\right) = -\frac{10\sqrt{15}}{9},$$

$$f\left(-\frac{\sqrt{15}}{3}\right) = \frac{10\sqrt{15}}{9}$$

$$\text{TP}\left(\frac{\sqrt{15}}{3}, -\frac{10\sqrt{15}}{9}\right), \left(-\frac{\sqrt{15}}{3}, \frac{10\sqrt{15}}{9}\right)$$

Award 1 mark for setting the derivative equal to zero.
Award 1 mark for both correct turning points.

b. i. $A(-1, f(-1)) = (-1, 4), B(1, f(1)) = (1, -4)$

$$m_{AB} = \frac{4+4}{-1-1} = -4$$
$$y - 4 = -4(x+1)$$
$$y - 4 = -4x - 4$$
$$y = -4x$$

Award 1 mark for calculating the correct gradient.
Award 1 mark for the correct equation.

ii. $d_{AB} = \sqrt{(4 - -4)^2 + (-1-1)^2}$
$$= \sqrt{68}$$
$$= 2\sqrt{17}$$

Award 1 mark for the correct distance.

Question 4

$$f(x) = x^2 + \frac{p}{x} = x^2 + px^{-1}$$
$$f'(x) = 2x - px^{-2} = 2x - \frac{p}{x^2}$$
$$f'(-2) = -4 - \frac{p}{4} = 0.$$
$$\frac{p}{4} = -4$$
$$p = -16$$

The correct answer is **A**.

Question 5

Since the gradient of f is negative ($f'(x) < 0, x \neq 5$) on either side of the stationary point ($f'(5) = 0$), at $x = 5$ the graph of f has a stationary point of inflection.
The correct answer is **E**.

Question 6

$$f'(x) = \frac{1}{27} \times (ax-1)^3 \times -3 + (b-3x) \times \frac{3}{27}(ax-1)^2 \times a$$
$$= -\frac{1}{9} \times (ax-1)^3 + \frac{a}{9}(ax-1)^2(b-3x)$$
$$= \frac{1}{9} \times (ax-1)^2(-(ax-1) + a(b-3x))$$
$$= \frac{1}{9} \times (ax-1)^2(-ax+1+ab-3ax)$$
$$= \frac{1}{9}(ax-1)^2(1+ab-4ax) = 0$$
$$ax - 1 = 0 \quad \text{or} \quad 1 + ab - 4ax = 0$$
$$x = \frac{1}{a} \quad \text{or} \quad x = \frac{1+ab}{4a}$$

The correct answer is **A**.

Question 7

Sketch $f: R \to R, f(x) = (x-2)(x-5)\left(3x^2 + ax + 6\right)$ using CAS; $p \in \{1, 2, 3\}$.
$\therefore$ The graph of $y = f(x)$ could have 1, 2 or 3 stationary points.
The correct answer is **B**.

Question 8

$\frac{dy}{dx} = 4x^3 - 8x = 4x\left(x^2 - 2\right) = 0$

when $x = 0$ and $y = 4 \Rightarrow (0, 4)$ is a local maximum

$\frac{dy}{dx} = 0 \Rightarrow x = 0$ and $x = \pm\sqrt{2}$

when $x = \pm\sqrt{2}$ and $y = 0 \Rightarrow$ both $\left(\pm\sqrt{2}, 0\right)$ are local minimums

The correct answer is **C**.

Question 9

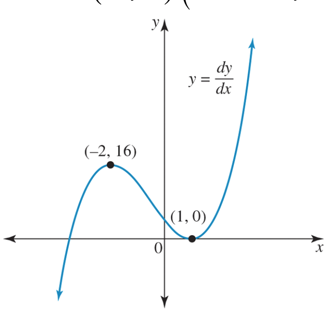
The derivative of $y = (x+2)^2\left(x^2 - 4x + 6\right)$ is found by applying the Product rule.

Let $y = u \times v$

Where $u = (x+2)^2$ and $v = (x^2 - 4x + 6)$

$\frac{du}{dx} = 2(x+2)$ $\qquad \frac{dv}{dx} = 2x - 4$

$$\begin{aligned}\frac{dy}{dx} &= u \times \frac{dv}{dx} + v \times \frac{du}{dx}\\ &= (x+2)^2 \times (2x-4) + \left(x^2 - 4x + 6\right) \times 2(x+2)\\ &= (x+2)^2 \times 2(x-2) + \left(x^2 - 4x + 6\right) \times 2(x+2)\\ &= 2(x+2)\left[(x+2)(x-2) + \left(x^2 - 4x + 6\right)\right]\\ &= 2(x+2)\left[x^2 - 4 + x^2 - 4x + 6\right]\\ &= 2(x+2)\left[2x^2 - 4x + 2\right]\\ &= 4(x+2)\left(x^2 - 2x + 1\right)\end{aligned}$$



Award 1 mark for the correct derivative curve.

The graph of the original function has a minimum stationary point at $x = -2$ and point of inflection at $x = 1$.

Award 1 mark for the correct stationary points.

Question 10

$f'(x) = 3ax^2 + 2bx + c$

$f(-1) = -a + b - c + d$ (equation 1)

$= 1$

$f(2) = 8a + 4b + 2c + d$ (equation 2)

$= 4$ **[1 mark]**

Award 1 mark for the correct substitution and use of $f(x) = ax^3 + bx^2 + cx + d$

$f'(-1) = 3a - 2b + c$ (equation 3)

$= 3$

$f'(2) = 12a + 4b + c$ (equation 4)

$= 0$ **[1 mark]**

Award 1 mark for the correct substitution and use of $f'(x) = 3ax^2 + 2bx + c$
Solve the equations simultaneously.

$a = \frac{1}{9} \quad b = -\frac{2}{3} \quad c = \frac{4}{3} \quad d = \frac{28}{9}$ **[2 marks]**

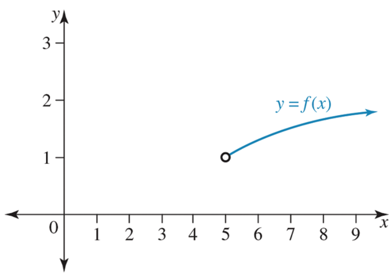
Award $\frac{1}{2}$ mark for each correct value of $a,\ b,\ c$ and d

Question 11
The graph has a stationary point of inflection at $x = 3$ and a minimum at $x = 2$.
The correct answer is **B**.

Question 12
A stationary point of inflection.
The correct answer is **C**.

Question 13
$f'(x) < 0$ implies that the graph has negative gradients for $x > 3$, therefore the graph of $f(x)$ is decreasing when $x > 3$.
$f'(x) > 0$ implies that the graph has positive gradients for $x < 3$, therefore the graph of $f(x)$ is increasing when $x < 3$.
Taking these two facts together implies that there is a local maximum at $x = 3$.
The correct answer is **B**.

Question 14
The graph of $f(x)$ looks like this:



This implies that f is strictly increasing when $x \in [5, \infty)$. **[1 mark]**

Question 15
The graph of $h(x)$ is a negative cubic with stationary points at

$h'(x) = 1 - 3x^2 = 0$

$1 = 3x^2$

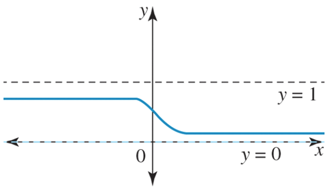
$x^2 = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$

$h(x)$ is increasing when $x \in \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

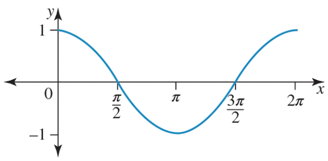
The correct answer is **C**.

Question 16
The graph of $f(x)$ looks like:



The graph of $f(x)$ is strictly decreasing for all values of x. That is $x \in R$. **[1 mark]**.

Question 17
The graph of $g(x)$ looks like:



The graph of $g(x)$ is strictly decreasing when $x \in (0, \pi)$.
The correct answer is **C**.

6.6 Maximum and minimum problems

Question 1

$$\begin{aligned} A &= l \times w \\ &= x \times y \\ A(x) &= x\left(4 - x^2\right) \\ &= 4x - x^3 \\ \frac{dA}{dx} &= 4 - 3x^2 = 0 \text{ for max/min} \end{aligned}$$

$$\Rightarrow x^2 = \frac{4}{3} \Rightarrow u = \frac{2}{\sqrt{3}} \text{ since } u > 0$$

$$A_{\max} = A\left(\frac{2}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\left(4 - \frac{4}{3}\right) = \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}$$

The correct answer is **E**.

Question 2

a. $s(x) = d_{\text{OP}}$

$$\begin{aligned} &= \sqrt{x^2 + y^2} \\ &= \sqrt{x^2 + (2x - 4)^2} \\ &= \sqrt{5x^2 - 16x + 16} \end{aligned}$$

$$s = \left(5x^2 - 16x + 16\right)^{\frac{1}{2}}$$

$$\begin{aligned} \frac{ds}{dx} &= \frac{1}{2}\left(5x^2 - 16x + 16\right)^{-\frac{1}{2}} \times (10x - 16) \\ &= \frac{10x - 16}{2\sqrt{5x^2 - 16x + 16}} \end{aligned}$$

Minimum value occurs where $\dfrac{ds}{dx} = 0$

$$0 = \frac{10x - 16}{2\sqrt{5x^2 - 16x + 16}}$$
$$= 10x - 16$$
$$16 = 10x$$
$$x = \frac{8}{5}$$
$$y = 2 \times \frac{8}{5} - 4$$
$$= -\frac{4}{5}$$
$$P\left(\frac{8}{5}, -\frac{4}{5}\right)$$

Award 1 mark for correctly deriving $\frac{ds}{dx}$.
Award 1 mark for the correct value of x.
Award 1 mark for the correct value of y.

***VCAA Examination Report* note:**
Students who tackled this question by finding an expression for OP in terms of x, then setting the derivative to zero, often had difficulty finding the derivative correctly, which ended up with an incorrect x value. However, most students calculated a y coordinate. Some students opted for a solution by working with similar triangles. A common incorrect response for P was $(2, 0)$, which is the point of intersection of the line $y = 2x - 4$ and the x-axis, while others incorrectly assumed the point P to be midway between the line segment formed by $y = 2x - 4$ and its axial intercepts.

b. $$s\left(\frac{8}{5}\right) = \sqrt{\left(\frac{8}{5}\right)^2 + \left(2 \times \frac{8}{5} - 4\right)^2}$$
$$= \sqrt{\left(\frac{8}{5}\right)^2 + \left(-\frac{4}{5}\right)^2}$$
$$= \sqrt{\frac{64 + 16}{25}}$$
$$= \sqrt{\frac{80}{25}}$$
$$= \frac{\sqrt{16 \times 5}}{5}$$
$$d_{\text{OP}} = \frac{4\sqrt{5}}{5}$$

Award 1 mark for correctly using the distance formula.
Award 1 mark for the correct answer.

***VCAA Examination Report* note:**
This question was attempted well. Some students misquoted the distance formula or made arithmetic errors in their calculations.

Question 3

$$A = 2 \times A_{\text{triangle}} + A_{\text{rectangle}}$$
$$= 2 \times \frac{1}{2}bh + lw$$
$$b = p\cos(x), \quad h = p\sin(x)$$
$$l = p, \quad w = h = p\sin(x)$$
$$A = p\cos(x) \times p\sin(x) + p \times p\sin(x)$$
$$= p^2\left(\cos(x)\sin(x) + \sin(x)\right)$$

$\frac{dA}{dx} = p^2 \left(2\cos^2(x) + \cos(x) - 1\right) = 0$

since $0 < x < \frac{\pi}{2}$, $x = \frac{\pi}{3}$.

The correct answer is **D**.

Question 4

$$\begin{aligned} A(u) &= u.v \\ &= u\left(8 - u^3\right) \\ &= 8u - u^4 \end{aligned}$$

$\frac{dA}{du} = 8 - 4u^3 = 0$ for max/min

$\Rightarrow u^3 = 2$

$\Rightarrow u = \sqrt[3]{2}$ since $u > 0$

$$\begin{aligned} A_{\max} &= A\left(\sqrt[3]{2}\right) \\ &= \sqrt[3]{2}\,(8 - 2) \\ &= 6\sqrt[3]{2} \end{aligned}$$

Question 5

$V(x) = x(6 - 2x)(8 - 2x)$

$\frac{dV}{dx} = 12x^2 - 56x + 48 = 0$

Since $0 < x < 3$, solving $\Rightarrow x \approx 1.13$

Question 6

Let the other side be y cm

$2x + 2y = 20 \Rightarrow y = 10 - x$

$A = xy \Rightarrow A(x) = x(10 - x)$

Question 7

Using CAS technology to graph the function the maximum value for the domain given is when $x = 0$, therefore the maximum value is 795.

Question 8

$f(x)$ is an upright cubic function with a point of inflection so its maximum value occurs at the end point $f(1) = (1)^3 + 2 = 3$.

Maximum is at $y = 3$ when $x = 1$. **[1 mark]**

Question 9

$f(x)$ is a positive quadratic function, so its minimum value occurs at the local minimum.

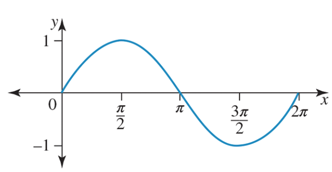
$$\begin{aligned} \text{Minimum} \Rightarrow f'(x) &= 2x = 0 \\ &= x = 0 \end{aligned}$$

$f(0) = 0^2 + 2 = 2$

$\Rightarrow$ Minimum is at $y = 2$ when $x = 0$. **[1 mark]**

Question 10

The graph of $f(x)$ looks like this:



The maximum value is $f(x) = 1$ when $x = \frac{\pi}{2}$. **[1 mark]**

6.7 Rates of change

Question 1

$f: R\backslash(4) \to R,\ f(x) = \dfrac{a}{x-4}$

$f(8) = \dfrac{a}{4}, f(6) = \dfrac{a}{2}$

$$\begin{aligned}\frac{f(8)-f(6)}{8-6} &= \frac{\frac{a}{4}-\frac{a}{2}}{2} \\ &= \frac{1}{2}\left(\frac{a-2a}{4}\right) \\ &= -\frac{a}{8}\end{aligned}$$

The correct answer is **E**.

Question 2

$$f(x) = x^2 - 2x$$

$$f(a) = a^2 - 2a, \quad f(1) = 1 - 2 = -1$$

$$\begin{aligned}\frac{f(a)-f(1)}{a-1} &= \frac{a^2-2a+1}{a-1} \\ &= \frac{(a-1)^2}{a-1} \\ &= a-1\end{aligned}$$

Since $a > 1$,

$$a - 1 = 8 \Rightarrow a = 9$$

The correct answer is **A**.

Question 3

$y = 2e^{-x+1}\sin(x-1)$

Using the product rule:

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= 2e^{-x+1}\cos(x-1) - 2e^{-x+1}\sin(x-1) \\ &= 2e^{-x+1}(\cos(x-1) - \sin(x-1))\end{aligned}$$

When $x = 1$,

$$\begin{aligned}\frac{dy}{dx} &= 2e^0(\cos(0) - \sin(0)) \\ &= 2(1-0) \\ &= 2\end{aligned}$$

The correct answer is **D**.

Question 4

$f(n) = 2500 \times (1.004)^n$

$$\frac{f(12)-f(0)}{12-0} = 10.22$$

The correct answer is **B**.

Question 5

$f(x) = 3x^2 - 2\sqrt{x+1}$

$f(3) = 27 - 2\sqrt{4} = 23 \quad f(0) = -2$

$$\frac{f(3)-f(0)}{3-0} = \frac{23+2}{3-0} = \frac{25}{3}$$

The correct answer is **D**.

Question 6

$f(x) = \sin(2\pi x) + 2x$

$f(5) = \sin(10\pi) + 10 = 10$

$f\left(\frac{1}{4}\right) = \sin\left(\frac{\pi}{2}\right) + \frac{1}{2} = \frac{3}{2}$

The average rate of change over $\left[\frac{1}{4}, 5\right]$ is

$\frac{f(5) - f\left(\frac{1}{4}\right)}{5 - \frac{1}{4}} = \frac{10 - \frac{3}{2}}{5 - \frac{1}{4}} = \frac{34}{19}.$

The correct answer is **B**.

Question 7

$f(x) = \cos(2\pi x) + 2x$

$f(4) = \cos(8\pi) + 8$

$f\left(\frac{1}{4}\right) = \cos\left(\frac{\pi}{2}\right) + \frac{1}{2}$

$$\begin{aligned}\text{Average rate of change} &= \frac{f(4) - f\left(\frac{1}{4}\right)}{4 - \frac{1}{4}} \\ &= \frac{(\cos(8\pi) + 8) - \left(\cos\left(\frac{\pi}{2}\right) + \frac{1}{2}\right)}{\frac{15}{4}} \\ &= \frac{(1 + 8) - \left(\frac{1}{2}\right)}{\frac{15}{4}} \\ &= \frac{17}{2} \times \frac{4}{15} \\ &= \frac{34}{15}\end{aligned}$$

The correct answer is **C**.

Question 8

$f(x) = x^3 + e^{2x}$

$f(x) = x^3 + e^{2x}$

$f(0) = 1$ and $f(2) = 8 + e^4$

$\text{Average rate} = \frac{f(2) - f(0)}{2 - 0} = \frac{7 + e^4}{2}$

6.8 Newton's method

Question 1

$f(x) = 2 - x^2 - \sin(x)$

$f'(x) = -2x - \cos(x)$ **[1 mark]**

Let $x_0 = 1$.

$x_1 = 1 - \frac{f(1)}{f'(1)}$

$\therefore x_1 = 1 - \frac{(1 - \sin(1))}{-2 - \cos(1)}$

$= 1.062\,406$

$x_2 = \text{Ans} - \frac{f(\text{Ans})}{f'(\text{Ans})}$

$= 1.061\,549\,9$

Therefore, $x = 1.062$ (3 d.p.). **[1 mark]**

Question 2

If $x^4 = 12$, then $x = \sqrt[4]{12}$.

$\therefore f(x) = x^4 - 12$

$f'(x) = 4x^3$ **[1 mark]**

As $1^4 = 1$ and $2^4 = 16$, the solution to $x = \sqrt[4]{12}$ lies between 1 and 2, and closer to 2. **[1 mark]**

Let $x_0 = 2$.

$$x_1 = 2 - \frac{f(2)}{f'(2)}$$

$$\therefore x_1 = 2 - \frac{4}{32}$$
$$= 1.875$$

$$x_2 = \text{Ans} - \frac{f(\text{Ans})}{f'(\text{Ans})}$$
$$= 1.861\,361\,1$$

$$x_3 = \text{Ans} - \frac{f(\text{Ans})}{f'(\text{Ans})}$$
$$= 1.861\,209\,74$$

Therefore, $\sqrt[4]{12} = 1.86$ (2 d.p.). **[1 mark]**

Question 3

Let $f(x) = x^4 - 5x - 8$.

$$f(0) = 0^4 - 5(0) - 8$$
$$= -8$$
$$< 0$$

$$f(-1) = (-1)^4 - 5(-1) - 8$$
$$= -2$$
$$< 0$$

$$f(-2) = (-2)^4 - 5(-2) - 8$$
$$= 18$$
$$> 0$$

As there is a sign change between $x = -1$ and $x = -2$, the solution to the equation must lie between these values. Most likely it is closer to -1, as $f(-1)$ is closer to 0 than $f(-2)$. **[1 mark]**

Let $x_0 = -1$.

$f(x) = x^4 - 5x - 8$

$f''(x) = 4x^3 - 5$ **[1 mark]**

$$x_1 = -1 - \frac{f(-1)}{f'(-1)}$$

$$\therefore x_1 = -1 - \frac{-3}{21}$$
$$= -1.222\,222$$

$$x_2 = \text{Ans} - \frac{f(\text{Ans})}{f'(\text{Ans})}$$
$$= -1.194\,373\,21$$

$$x_3 = \text{Ans} - \frac{f(\text{Ans})}{f'(\text{Ans})}$$
$$= -1.193\,793\,76$$

Therefore, $x = -1.194$ (3 d.p.). **[1 mark]**

6.9 Review

Question 1

$$y = \left(-3x^3 + x^2 - 64\right)^3$$
$$\frac{dy}{dx} = 3\left(-9x^2 + 2x\right)\left(-3x^3 + x^2 - 64\right)^2$$
$$= -3\left(9x^2 - 2x\right)\left(3x^3 - x^2 + 64\right)^2 \quad \textbf{[1 mark]}$$

VCAA Examination Report **note:**

Students generally recognised the need to deploy the chain rule; however, a significant number of students could not be awarded the mark. Poor use of brackets (or lack of brackets) resulted in an incorrect expression. For example, the expression $3\left(-3x^3 + x^2 - 64\right)^2 \left(-9x^2 + 2x\right)$ is not equivalent to $3\left(-3x^3 + x^2 - 64\right)^2 - 9x^2 + 2x$. Transcription errors (especially with exponents) and arithmetic errors with unnecessary expansions were also observed.

Question 2

$$y = x^2 \sin(x)$$
$$u = x^2 \quad v = \sin(x) \text{ (product rule)}$$
$$\frac{du}{dx} = 2x \quad \frac{dv}{dx} = \cos(x)$$
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{dy}{dx} = x^2\cos(x) + 2x\sin(x)$$
$$\frac{dy}{dx} = x\left(x\cos(x) + 2\sin(x)\right)$$

Award 1 mark for using the product rule.
Award 1 mark for the correct answer.

VCAA Assessment Report **note:**

Although this question was generally very well handled, some students made errors in an attempt to factorise, which was not necessary.

Question 3

$$A(m) = \frac{1}{2}m\left(9 - m^2\right)$$
$$= \frac{1}{2}\left(9m - m^3\right)$$
$$\frac{dA}{dm} = \frac{1}{2}\left(9 - 3m^2\right) = 0$$
$$m = \sqrt{3}, m > 0$$

$$A\left(\sqrt{3}\right) = 3\sqrt{3}$$

The correct answer is **D**.

Question 4

$$f(x) = 3x^2 - 2\sqrt{x+1}$$
$$f(3) = 27 - 2\sqrt{4} = 23 \quad f(0) = -2$$
$$\frac{f(3) - f(0)}{3 - 0} = \frac{23 + 2}{3 - 0} = \frac{25}{3}$$

The correct answer is **D**.

Question 5

a. $y = f(x) = \dfrac{3x(x-30)^2}{2000}, x \in [0, 30]$

$\dfrac{dy}{dx} = \dfrac{9(x-10)(x-30)}{2000}, x \in (0, 30)$ **[1 mark]**

***VCAA Examination Report* note:**

Other equivalent forms were acceptable.

This question was answered well. Common incorrect answers were

$\dfrac{9x(x-30)(x-15)}{500}$ and $\dfrac{9\left(x^2-40x+30\right)}{2000}$.

b. $m = \dfrac{dy}{dx}$

Solving $\dfrac{dm}{dx} \le 0 \Rightarrow x \in (0,\ 20]$ **[1 mark]**

***VCAA Examination Report* note:**

This question was not done well. Most students interpreted the question as asking where the function modelling the hill was strictly decreasing, rather than the gradient of the hill and so the most common incorrect response was [10, 30] or a combination of round and square brackets with those two values.

c. $g(x) = f(x) + 3 = \dfrac{3x(x-30)^2}{2000} + 3$ **[1 mark]**

***VCAA Examination Report* note:**

This question was generally well done. Some students did not give an equation. Others added 10, instead of 3, to f.

d. Average gradient $\dfrac{f(30)-f(10)}{30-10} = \dfrac{0-6}{20} = \dfrac{-3}{10}$

Solving $\dfrac{dy}{dx} = \dfrac{9(x-10)(x-30)}{2000} = \dfrac{-3}{10}$

gives $x = \dfrac{60 \pm 10\sqrt{3}}{3}$.

Award 1 mark for calculation of the average gradient.

Award 1 mark for solving the derivative equal to the average gradient.

Award 1 mark for the correct answer.

***VCAA Examination Report* note:**

A common incorrect answer for the average gradient

was $\dfrac{3}{10}$.

Some students used $\dfrac{1}{30-10}\displaystyle\int_{10}^{30} h(x)dx$.

instead of $\dfrac{1}{30-10}\displaystyle\int_{10}^{30} h'(x)\,dx$.

Some students gave approximate answers for the x values, 14.23 and 25.77.

Other students did not use brackets correctly, giving $x = \dfrac{\pm 10\left(\sqrt{3}+6\right)}{3}$ as their answer. Another common incorrect answer was $\dfrac{6 \pm 10\sqrt{3}}{3}$.

e. i. $g'(a) = \dfrac{9(a-10)(a-30)}{2000}$ **[1 mark]**

***VCAA Examination Report* note:**
Common incorrect answers were
$\frac{9a^2}{2000} - \frac{9a}{50} + \frac{27}{50}, \frac{3}{2000}a^2 - \frac{9}{100}a - \frac{10}{a} + \frac{27}{20}$ and
$\frac{3a^2 - 180a^2 + 2700a - 20\,000}{2000a}$.
Some students used $\frac{f(a) - 10}{a}$ instead of $\frac{h(a) - 10}{a}$.
Other students wrote $\frac{b - 10}{a}$.
The answer had to be given in terms of a.

ii. At the point (a, b), the gradient of the straight section and the gradient of the curve are equal (smooth join).
Also, the y-values are equal.
$g'(a) = \frac{9(a - 10)(a - 30)}{2000} = \frac{10 - b}{0 - a}$ and $b = g(a)$
Solving gives $a = 11.12$, $b = 8.95 \Rightarrow A(11.12,\ 8.95)$.
Award 1 mark for each equation (up to 2 marks).
Award 1 mark for the correct answer.
***VCAA Examination Report* note:**
Many students did not equate the correct expressions. Some students found the value of a but not the value of b. Other students rounded their answers incorrectly, giving (11.11, 8.94).

iii. $g'(11.12) = -0.094\,85$ **[1 mark]**
$= -0.1$
***VCAA Examination Report* note:**
Students who obtained the correct value for a in Question **2eii** were generally successful with this question.

Question 6
Only **E** is a smooth, continuous graph and differentiable at the origin.
$$f(x) = \begin{cases} 4x + 1 & x < 0 \\ (2x + 1)^2 & x \geq 0 \end{cases}$$
$f(0) = 1$
$f'(x) = 4$
The correct answer is **E**.

Question 7
a. $V_{box} = L \times W \times H$
$L = 25 - 2x,\ H = 50 - 2x = 2(25 - x),\ W = x$
$V = 2x(25 - 2x)(25 - x)$
[1 mark]

b. $L = 25 - 2x > 0,\ H = 2(25 - x) > 0,\ W = x > 0$
$0 < x < \frac{25}{2}$ **[1 mark]**

c. $V_{box} = 4x^3 - 150x^2 + 1250x$
$\frac{dV_{box}}{dx} = 2\left(6x^2 - 150x + 625\right)$ **[1 mark]**

d. Solving $\frac{dV_{box}}{dx} = 2\left(6x^2 - 150x + 625\right) = 0$

gives $x = \frac{25}{6}\left(3 - \sqrt{3}\right)$

$V_{\max} = V_{box}\left(\frac{25}{6}\left(3 - \sqrt{3}\right)\right) = \frac{15625\sqrt{3}}{9}$ cm^3

Award 1 mark for solving.

Award 1 mark for the correct value of x.

Award 1 mark for the correct maximum volume.

e. When $x = 5$, area of the 4 corners is $4 \times 5 \times 5 = 100$, The area of cardboard: $h \times 2h = 25 \times 50 = 1250$

% wasted $\frac{100}{1250} \times 100\% = 8\%$

Award 1 mark for correct calculations.

Award 1 mark for the correct % wastage.

f. i. $V_{box} = 2x(h - 2x)(h - x),\ 0 < x < \frac{h}{2}$ **[1 mark]**

ii. $V_{box} = 4x^3 - 6x^2h + 2h^2x$

Solving $\frac{dV_{box}}{dx} = 12x^2 - 12xh + 2h^2 = 0$

gives $x = \frac{h}{6}\left(3 - \sqrt{3}\right)$

$V_{\max} = V_{box}\left(\frac{h}{6}\left(3 - \sqrt{3}\right)\right) = \frac{\sqrt{3}h^3}{9}$ cm^3

Award 1 mark for solving.

Award 1 mark for the correct value of x.

Award 1 mark for the correct maximum volume.

g. $V_{box} = x(h - 2x)^2 = 4x^3 - 4hx^2 + h^2x$ domain $0 < x < \frac{h}{2}$

$\frac{dV_{box}}{dx} = 12x^2 - 8xh + h^2$ $\frac{dV_{box}}{dx} = (2x - h)(6x - h) = 0$ for max $x = \frac{h}{6}$

$x > \frac{h}{6}$ $\frac{dV}{dx} < 0,$

$x < \frac{h}{6}$ $\frac{dV}{dx} > 0$

So, it is a maximum by the sign test.

Award 1 mark for the correct value of x

Award 1 mark for sign test.

Question 8

$L = 80 - 2x$

$W = 50 - 2x$ $W > 0 \Rightarrow 0 < x < 25$

$V = LWH$

$V = x(80 - 2x)(50 - 2x)$

$V = x\left(4000 - 260x + 4x^2\right)$

$V = 4000x - 260x^2 + 4x^3$

$\frac{dV}{dx} = 4000 - 520x + 12x^2 = 0$

$\frac{dV}{dx} = 4\left(3x^2 - 130x + 1000\right) = 0$

$\frac{dV}{dx} = 4(3x - 100)(x - 10) = 0$

$x = 10$ since $0 < x < 25$

The correct answer is **A**.

Question 9

a. $d(PB) = \sqrt{1 - x^2}$ [**1 mark**]

***VCAA Assessment Report* note:**

Though mostly well done, some students left their answer as an incorrect and unsimplified form of the distance formula.

b. $A(x) = \frac{1}{2}(x + 1)\sqrt{1 - x^2}$

Applying the product rule:

$$\frac{dA}{dx} = \frac{1}{2}\left[1 \times \sqrt{1 - x^2} + (x + 1) \times {}^{-}2x \times \frac{1}{2}\left(1 - x^2\right)^{-\frac{1}{2}}\right]$$

$$\frac{dA}{dx} = \frac{1}{2}\left[\sqrt{1 - x^2} - \frac{x(x + 1)}{\sqrt{1 - x^2}}\right]$$

$$\frac{dA}{dx} = \frac{1}{2}\left[\frac{1 - x^2 - x(x + 1)}{\sqrt{1 - x^2}}\right] = 0 \text{ for a maximum or minimum.}$$

$1 - x^2 - x(x + 1) = 0$

$1 - x^2 - x^2 - x = 0$

$2x^2 + x - 1 = 0$

$(2x - 1)(x + 1) = 0$ but $x > 0$

$x = \frac{1}{2}$

$$A_{\max} = A\left(\frac{1}{2}\right) = \frac{1}{2} \times \frac{3}{2} \times \sqrt{\frac{3}{4}} = \frac{3\sqrt{3}}{8}$$

Award 1 mark for the method.

Award 1 mark for correctly applying the product rule.

Award 1 mark for the correct values of x and maximum area.

***VCAA Assessment Report* note:**

The majority of students used calculus to solve this problem. Some students used geometry and trigonometry to obtain a correct solution. Most students managed to find an expression for the area in terms of x. Many of those who used calculus found the differentiation of the expression difficult, generally as a result of poor setting out, particularly with lack of brackets, or dealing with negative terms. Students are encouraged to practice differentiations involving combinations of product and chain rules.

Question 10

a. $c(t) = \frac{5}{2}te^{-\frac{3t}{2}}$

$c'(t) = \frac{5}{4}e^{-\frac{3t}{2}}(2 - 3t)$

For maximum or minimum:

$c'(t) = 0$

$\Rightarrow 2 - 3t = 0$

$t = \frac{2}{3}$

$c\left(\frac{2}{3}\right) = \frac{5}{3}e^{-1} = 0.61 \text{ mg/L}$

Award 1 mark for the correct maximum value

***VCAA Assessment Report* note:**

This question was answered well. Some students had incorrect units, such as mm for milligrams. Some left their answers in exact form. Some found t correct to two decimal places and left their answer as 0.67.

b. i. Solving using CAS:

$c(t) = 0.5 \Rightarrow t_1 = 0.33$

Award 1 mark for the correct time.

***VCAA Assessment Report* note:**

This question was answered well. Some students gave two answers, 0.33 and 1.19, instead of only the first one, as specified in the question. Some students rounded incorrectly and gave 0.32 as their answer.

ii. $c(t) = 0.5 \Rightarrow t_1 = 0.3263\ t_2 = 1.1876$

$t_2 - t_1 = 1.1876 - 0.3263$

$= 0.86$ hours

Award 1 mark for finding the other time.

Award 1 mark for subtracting times, correct to 2 decimal places

***VCAA Assessment Report* note:**

Students should always work to suitable accuracy in intermediate calculations to support rounding the answer to the required accuracy. Some students wrote down the two values but did not find the difference for the length of time. Some added the two values. Some students incorrectly converted the time to minutes.

c. i. $\dfrac{c(3) - c\left(\frac{2}{3}\right)}{3 - \frac{2}{3}} = -0.227\,064$

$= -0.23$

Award 1 mark for finding the average rate.

Award 1 mark for the correct value of average rate, correct to 2 decimal places.

***VCAA Assessment Report* note:**

Average rate of change $\dfrac{c(3) - c\left(\frac{2}{3}\right)}{3 - \frac{2}{3}} = -0.23$ mg/L/h, correct to two decimal places. Some students worked out the average value of the function. Some used $\dfrac{3 - \frac{2}{3}}{c(3) - c\left(\frac{2}{3}\right)}$. Others had incorrect units. Some changed their answer to 0.23 mg/L/h.

ii. $c'(t) = \dfrac{5}{4}e^{-\frac{3t}{2}}(2 - 3t)$

$= 0.227\,062$

$\Rightarrow t_1 = 0.90,\ t_2 = 2.12$

Award 1 mark for solving the gradient equal to average rate.

Award 1 mark from CAS for both correct times.

VCAA Assessment Report note:

Some rounded their answers incorrectly. Others did not work to the required number of decimal places.

d. $n(t) = Ate^{-kt}$

$n(0.5) = 0.74 = \dfrac{1}{2}Ae^{-1}$

$\Rightarrow A = \dfrac{2 \times 0.74}{e^{-1}} = 4$

$n(0.5) = 0.74 = \dfrac{1}{2}Ae^{-1}$

$\Rightarrow A = \dfrac{2 \times 0.74}{e^{-1}} = 4$

Award 1 mark for differentiating and solving for time.

Award 1 mark for the correct value of k.

Award 1 mark for the correct value of A.

***VCAA Assessment Report* note:**
Many students were able to set up at least one of the equations. $n'(0.5) = 0.74$ was often used. Some students differentiated by hand incorrectly. Some students gave the value of k, not A. Others gave an exact answer.

Question 11

a. $y = ax^2 + bx$

$Q(2, 4) \Rightarrow 4 = 4a + 2b \quad (1)$

$\frac{dy}{dx} = 2ax + b$

$m_T = \left.\frac{dy}{dx}\right|_{x=2} = 4a + b$

$T: y - 4 = (4a + b)(x - 2)$

Crosses at $(6, 0)$:

$-4 = 4(4a + b)$

$4a + b = -1 \quad (2)$

$(1) - (2): b = 5$

So $4a = -1 - b$

$4a = -6$

$a = -\frac{3}{2}$

Award 1 mark for the first equation.
Award 1 mark for the equation involving tangents.
Award 1 mark for both correct values of a and b.

***VCAA Assessment Report* note:**
Most students knew to set up two equations to solve simultaneously. The most common error was to substitute $x = 6$ and $y = 0$ into $y = ax^2 + bx$, stating incorrectly that the point $(6, 0)$ lay on the parabola.

b. i. By similar triangles:

$\frac{v}{u} = \frac{4}{u - 2}$

$v = \frac{4u}{u - 2}$

Award 1 mark for the correct expression.

***VCAA Assessment Report* note:**
Some students were able to set up a suitable equation; however, errors in algebraic manipulation (or poor setting out of working) often resulted in incorrect final answers.

ii. $A = \frac{1}{2}uv - 8$

$A(u) = \frac{2u^2}{u - 2} - 8$ Quotient rule

$\frac{dA}{du} = \frac{4u(u - 2) - 2u^2}{(u - 2)^2} = 0$

$2u^2 - 8u = 0$

$2u(u - 4) = 0$

$u = 0$ or $u = 4$ but $\frac{5}{2} \leq u \leq 6$

$A(4) = \frac{32}{2} - 8$

$A(4) = 8$

The minimum area is 8 and occurs when

Award 1 mark for using the quotient rule and setting the derivative to zero.
Award 1 mark for the correct value of u and the correct minimum area.

***VCAA Assessment Report* note:**
This question was answered poorly. A complete solution to this question required setting up an equation for the area in one variable and then testing turning points as well as endpoints to determine the minimum value. Many students set up overly complex equations for the area or had difficulty in differentiating their equation correctly. Other students did the reverse by simply assuming $u = 6$.

iii. Examine the endpoints.

$$A(6) = \frac{2 \times 36}{4} - 8 = 10$$

$$A\left(\frac{5}{2}\right) = \frac{2 \times \frac{25}{4}}{\frac{1}{2}} - 8$$

$$A\left(\frac{5}{2}\right) = 25 - 8$$

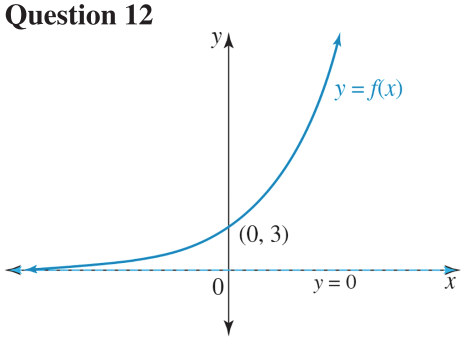
$$A\left(\frac{5}{2}\right) = 17$$

The maximum area is 17 and occurs when $u = \frac{5}{2}$.

Award 1 mark for the correct maximum area and value of u.

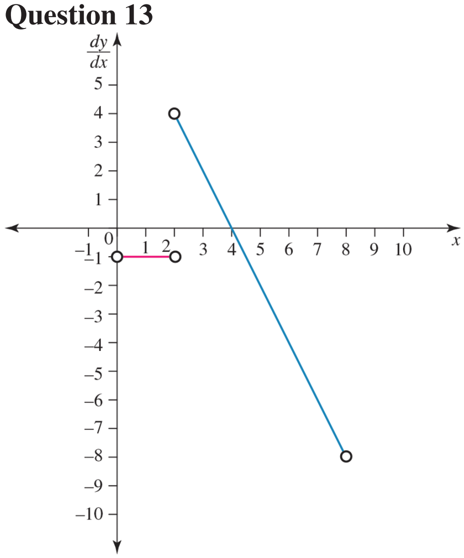
***VCAA Assessment Report* note:**
This question was also answered poorly. Many students who attempted this question incorrectly assumed that the value of the maximum area occurred at a local stationary point. Some students only gave a partial answer to the question and not the values for both u and area.

Question 12



Award 1 mark for the correct curve shape and the correct asymptote.

Question 13



Award 1 mark for the correct placement of the two derivative sections.
Award 1 mark for the use of correct end points on each section.

7 Anti-differentiation

Topic	7	Anti-differentiation
Subtopic	7.2	Anti-differentiation

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***Source:** VCE 2013, Mathematical Methods (CAS) Exam 1, Q2; © VCAA*

Question 1 (2 marks)

Find an anti-derivative of $(4-2x)^{-5}$ with respect to x.

***Source:** VCE 2015, Mathematical Methods (CAS) Exam 2, Section 1, Q16; © VCAA*

Question 2 (1 mark)

Let $f(x)=ax^m$ and $g(x)=bx^n$, where a, b, m and n are positive integers. The domain of f = domain of $g=R$.

If $f'(x)$ is an anti-derivative of $g(x)$, then which one of the following must be true?

A. $\frac{m}{n}$ is an integer

B. $\frac{n}{m}$ is an integer

C. $\frac{a}{b}$ is an integer

D. $\frac{b}{a}$ is an integer

E. $n-m=2$

Question 3 (1 mark)

Find $\int\left(3x^4-\frac{2}{x^2}\right)dx$.

Source: VCE 2015, Mathematical Methods (CAS) 2, Section 1, Q19; © VCAA

Question 4 (1 mark)

If $f(x) = \int_0^x \left(\sqrt{t^2 + 4}\right) dt$, then $f'(-2)$ is equal to

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. $2\sqrt{2}$

D. $-2\sqrt{2}$

E. $4\sqrt{2}$

Question 5 (1 mark)

$\int \frac{6}{x+5} dx$ is equal to:

A. $6\int \frac{1}{x+5} dx$

B. $\int 6dx \int \frac{1}{x+5} dx$

C. $\int 6dx + \int \frac{1}{x+5} dx$

D. $\dfrac{\int 6dx}{\int (x+5)dx}$

E. $\dfrac{6}{\int (x+5)dx}$

Question 6 (1 mark)

An anti-derivative of $\frac{1}{x^8}$ is equal to

A. $\frac{1}{8x^9} + c$

B. $-\frac{8}{x^9} + c$

C. $-\frac{1}{7x^7} + c$

D. $-\frac{9}{x^9} + c$

E. $\frac{1}{9x^9} + c$

Topic	7	Anti-differentiation
Subtopic	7.3	Anti-derivatives of exponential and trigonometric functions

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***Source:** VCE 2016, Mathematical Methods Exam 2, Section A, Q9; © VCAA*

Question 1 (1 mark)

Given that $\frac{d\left(xe^{kx}\right)}{dx} = (kx+1)\,e^{kx}$, then $\int xe^{kx}dx$ is equal to

A. $\frac{xe^{kx}}{kx+1} + c$

B. $\left(\frac{kx+1}{k}\right)e^{kx} + c$

C. $\frac{1}{k}\int e^{kx}dx$

D. $\frac{1}{k}\left(xe^{kx} - \int e^{kx}dx\right) + c$

E. $\frac{1}{k^2}\left(xe^{kx} - e^{kx}\right) + c$

Question 2 (1 mark)

The gradient of a curve is given by $2\sin(2x) - 4e^{-2x}$. The curve passes through the origin. The equation of the curve is given by

A. $2e^{-2x} - 1 - \cos(2x)$

B. $1 + \cos(2x) - 2e^{-2x}$

C. $\cos(2x) + 2e^{-2x} - 3$

D. $4\cos(2x) - 8e^{-2x} + 4$

E. $-4\cos(2x) + 8e^{-2x} - 4$

Question 3 (1 mark)

If $\int ae^{bx}\,dx = -2e^{2x} + c$, then

A. $a = 4$ and $b = -2$
B. $a = -2$ and $b = 2$
C. $a = -1$ and $b = 2$
D. $a = -4$ and $b = 2$
E. $a = -4$ and $b = -2$

Source: *VCE 2013, Mathematical Methods (CAS) 1, Q3; © VCAA*

Question 4 (2 marks)

The function with rule $g(x)$ has derivative $g'(x) = \sin(2\pi x)$.

Given that find $g(1) = \dfrac{1}{\pi}$, find $g(x)$.

Question 5 (3 marks)

Find the derivative of $x\sin(x)$ and hence find an antiderivative of $x\cos(x)$.

Question 6 (1 mark)

Which one of the following options is an anti-derivative of $\frac{1}{x^2} - \frac{1}{\cos^2\left(\frac{x}{2}\right)}$?

A. $-\frac{1}{x} - 2\tan\left(\frac{x}{2}\right)$

B. $-\frac{2}{x^3} - \frac{(2)}{\cos^3\left(\frac{x}{2}\right)}$

C. $\frac{1}{x} - \frac{1}{2}\tan\left(\frac{x}{2}\right)$

D. $\log_e\left(x^2\right) - \tan\left(\frac{x}{2}\right)$

E. $\frac{1}{2}\log_e\left(x^2\right) - 2\tan\left(\frac{x}{2}\right)$

Question 7 (1 mark)

The gradient of a curve is given by $2\cos\left(\frac{x}{2}\right)$. If the x intercept is $x = \frac{5\pi}{3}$ then the y intercept will be at $y =$

A. $-\frac{1}{2}$

B. $\frac{1}{2}$

C. -2

D. $\frac{\sqrt{3}}{2}$

E. 2

Topic	7	Anti-differentiation
Subtopic	7.4	The anti-derivative of $f(x) = \frac{1}{x}$

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Question 1 (1 mark)

Determine $\int \frac{x^2 + 2x - 3}{x^2} dx$.

Question 2 (1 mark)

$\int \frac{6}{x+5} dx$ is equal to

A. $6 \int \frac{1}{x+5} dx$

B. $\int 6dx \int \frac{1}{x+5} dx$

C. $\int 6dx + \int \frac{1}{x+5} dx$

D. $\frac{\int 6dx}{\int (x+5)dx}$

E. $\frac{6}{\int (x+5)dx}$

Question 3 (3 marks)

If $f(x) = x \log_e(x)$, find $f'(x)$ and hence $\int \log_e(x) dx$.

Source: *VCE 2018, Mathematical Methods 1, Q2; © VCAA*

Question 4 (3 marks)

The derivative with respect to x of the function $f:(1, \infty) \to R$ has the rule $f'(x) = \frac{1}{2} - \frac{1}{(2x-2)}$.

Given that $f(2) = 0$, find $f(x)$ in terms of x.

Topic	7	Anti-differentiation
Subtopic	7.5	Families of curves

online only

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Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q3; © VCAA*

Question 1 (1 mark)

Let $f'(x) = \dfrac{2}{\sqrt{2x-3}}$.

If $f(6) = 4$, then

A. $f(x) = 2\sqrt{2x-3}$

B. $f(x) = \sqrt{2x-3} - 2$

C. $f(x) = 2\sqrt{2x-3} - 2$

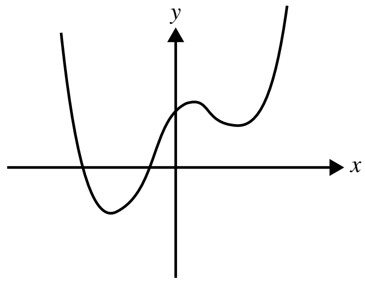
D. $f(x) = \sqrt{2x-3} + 2$

E. $f(x) = \sqrt{2x-3}$

Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q6; © VCAA*

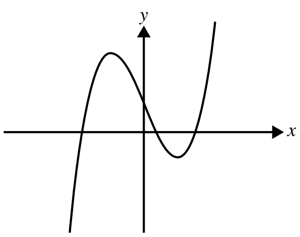
Question 2 (1 mark)

Part of the graph of $y = f'(x)$ is shown below.

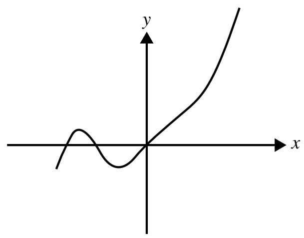


The corresponding part of the graph of $y = f(x)$ is best represented by

A.



B.



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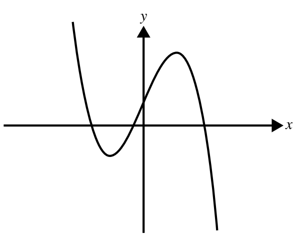

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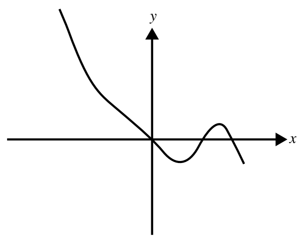
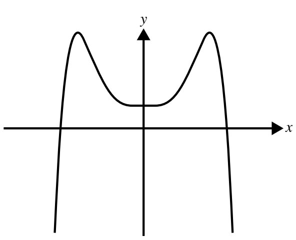


E.



Source: *VCE 2014, Mathematical Methods (CAS) Exam 1, Q7; © VCAA*

Question 3 (3 marks)

If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

Source: *VCE 2021, Mathematical Methods 1, Q2; © VCAA*

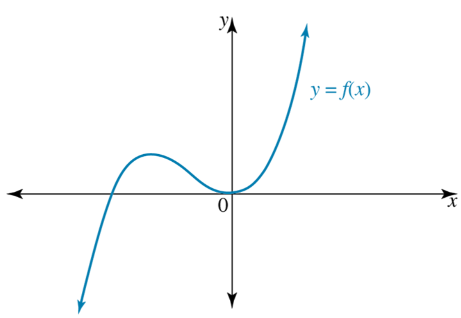
Question 4 (2 marks)

Let $f'(x) = x^3 + x$.

Find $f(x)$ given that $f(1) = 2$.

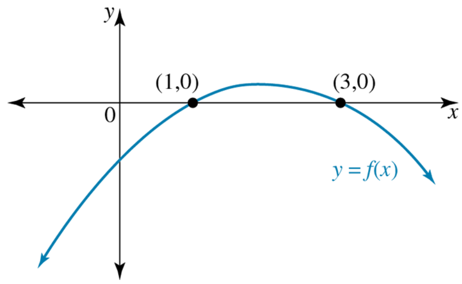
Question 5 (1 mark)

Sketch the graph of the antiderivative of the function f with rule $y = f(x)$.

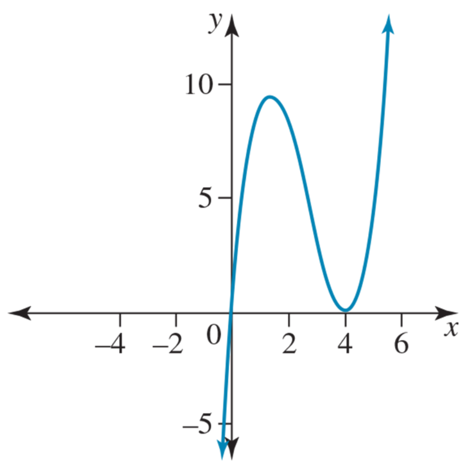


Question 6 (1 mark)

The graph of the function f with rule $y = f(x)$ is shown. Sketch a graph of an antiderivative function of f.



Question 7 (1 mark)



The graph of $y = f'(x)$ is shown above.

Which of the following statements is **true** for the graph of $y = f(x)$?

The graph has a local maximum at $x = 0$ and a stationary point of inflection at $x = 4$.

The graph has a local minimum at $x = 0$ and a stationary point of inflection at $x = 4$.

The graph has a local maximum at $x = 4$ and a stationary point of inflection at $x = 0$.

The graph has a local minimum at $x = 4$ and a stationary point of inflection at $x = 0$.

The graph has a local minimum at $x = 4$, a local maximum at $x = 1$, and a stationary point of inflection at $x = 0$.

Topic	7	Anti-differentiation
Subtopic	7.6	Applications

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Question 1 (3 marks)

A particle moves in a straight line so that its velocity, in metres per second, can be defined by the rule $v = 2t^3 - t + 1,\ t \geq 0$

a. Determine the rule for the position of the particle, if it is known the particle started 1 m to the left of the origin. **(1 mark)**

b. Determine the velocity and position of the particle after 3 seconds. **(2 marks)**

Question 2 (2 marks)

A particle moves in a straight line so that its acceleration, after t seconds, is given by $a = 4 - 2t,\ t \geq 0$. If $v = 0$ and $x = 3$ when $t = 0$, calculate the position and velocity of the particle when $t = 2$.

Question 3 (2 marks)

The rate of growth of bacteria in a petri dish in a laboratory can be modelled by $\frac{db}{dt} = 100t^{\frac{3}{2}}$, where b is the number of bacteria after t hours. Initially there were 80 bacteria cells.

a. Determine an equation for b in terms of t. **(1 mark)**

b. Determine how long, to the nearest minute, it would take the number of bacteria to reach 500. **(1 mark)**

Topic	7	Anti-differentiation
Subtopic	7.7	Review

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Source: *VCE 2019, Mathematical Methods Exam 2, Section A, Q5; © VCAA*

Question 1 (1 mark)

Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$.

The rule of f is

A. $f(x) = x^3 - x^2$

B. $f(x) = x^3 - x^2 + 48$

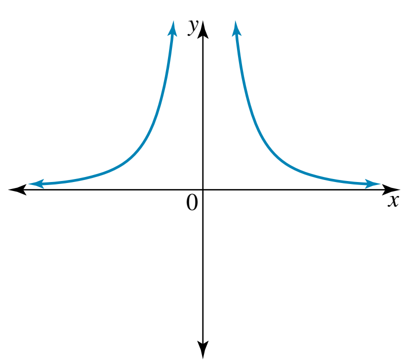
C. $f(x) = x^3 - x^2 - 48$

D. $f(x) = 6x - 2$

E. $f(x) = 6x - 24$

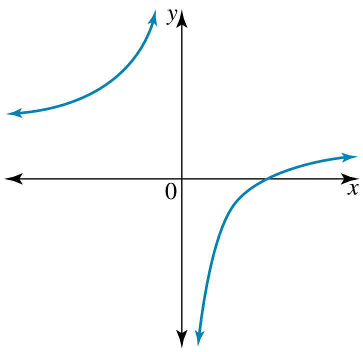
Question 2 (1 mark)

The graph of the function $y = f(x)$ is shown.

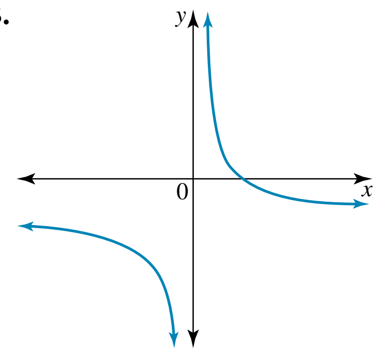


Select the graph from the following that could be the graph of the anti-derivative.

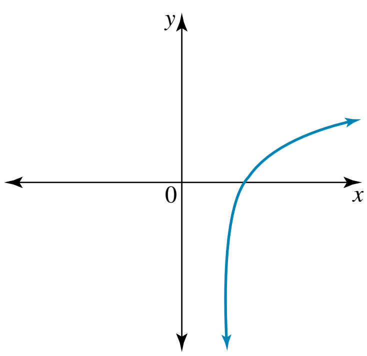
A.



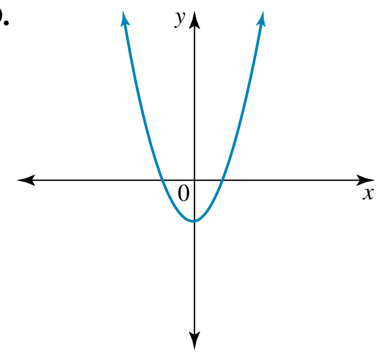
B.



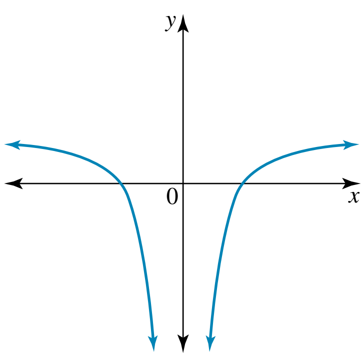
C.



D.



E.



Question 3 (1 mark)

An anti-derivative of $\int \frac{1}{(3x-4)^{\frac{5}{2}}}dx$ is

A. $\frac{1}{3(x-4)^{\frac{3}{2}}}$

B. $\frac{-3}{2(3x-4)^{\frac{3}{2}}}$

C. $\frac{-9}{2(3x-4)^{\frac{3}{2}}}$

D. $\frac{3}{(3x-4)^{\frac{3}{2}}}$

E. $\frac{-2}{9(3x-4)^{\frac{3}{2}}}$

Question 4 (1 mark)

The gradient of a curve is given by $6x - 1$. The curve passes through the point (2, 1). The equation of the curve is given by

A. $y = 3x^2 - x$

B. $y = 3x^2 - x - 9$

C. $y = 3x^2 - 11$

D. $y = 6$

E. $y = 2x^2 - x - 3$

Question 5 (3 marks)

Find the derivative of $x\sin(x)$ and hence find an anti-derivative of $x\cos(x)$.

Source: *VCE 2019, Mathematical Methods 1, Q1; © VCAA*

Question 6 (4 marks)

Let $f: \left(\frac{1}{3}, \infty\right) \to R,\ f(x) = \frac{1}{3x-1}$.

a. i. Find $f'(x)$. **(1 mark)**

ii. Find an antiderivative of $f(x)$. **(1 mark)**

b. Let $g: R\backslash\{-1\} \to R,\ g(x) = \frac{\sin(\pi x)}{x+1}$.

Evaluate $g'(1)$. **(2 marks)**

Source: *VCE 2015, Mathematical Methods (CAS) 1, Q2; © VCAA*

Question 7 (3 marks)

Let $f'(x) = 1 - \frac{3}{x}$ where $x \neq 0$

Given that $f(e) = -2$, find $f(x)$.

Question 8 (3 marks)

Let $f(x) = g'(x)$ and $h(x) = k'(x)$, where $g(x) = \left(x^2 - 2\right)^3$ and $k(x) = \cos^2(x)$.

Find

a. $\int (f(x) + h(x))\,dx$ **(1 mark)**

b. $\int (f(x) - 4)\,dx$ **(1 mark)**

c. $\int 3h(x)dx$ **(1 mark)**

Question 9 (3 marks)

If $y = \dfrac{\tan(x)}{4}$, find $\dfrac{dy}{dt}$, given $\dfrac{dx}{dt} = \dfrac{2}{\sqrt{t}}$ and $x = 4$ when $t = 1$.

Answers and marking guide

7.2 Anti-differentiation

Question 1

$$\int (4-2x)^{-5}\,dx$$
$$= \frac{(4-2x)^{-4}}{-2\times -4}$$
$$= \frac{1}{8(4-2x)^4}$$

Award 1 mark for the correct power.
Award 1 mark for the correct factor out the front.

Question 2

a, b, m and $n \in Z^+$

$$\frac{d}{dx}(ax^m) = max^{m-1} = \int bx^n dx = \frac{b}{n+1}x^{n+1}$$
$$\Rightarrow \frac{b}{a} = m(n+1)x^{m-n-2} \Rightarrow m-n-2=0$$

Since $\frac{b}{a} = m\,(n+1)$ and m, n are both integers, it is clear that $\frac{b}{a}$ is an integer.

$n=1,\ m=3$: $\frac{b}{a}=6,\ \frac{m}{n}=3$

$n=2,\ m=4$: $\frac{b}{a}=12,\ \frac{m}{n}=2$

$n=3,\ m=5$: $\frac{b}{a}=20,\ \frac{m}{n}=\frac{5}{3}$

The correct answer is **D**.

Question 3

$$\int\left(3x^4-\frac{2}{x^2}\right)dx = \int 3x^4\,dx - \int \frac{2}{x^2}\,dx$$
$$= \int 3x^4\,dx - \int 2x^{-2}\,dx$$
$$= \frac{3x^5}{5}+\frac{2}{x}+c \quad \textbf{[1 mark]}$$

Question 4

$$f(x) = \int_0^x \sqrt{t^2+4}\,dt$$
$$f'(x) = \sqrt{x^2+4}$$
$$f'(-2) = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

The correct answer is **C**.

Question 5

From the rule

$$\int k f(x)dx = k\int f(x)dx$$

The correct answer is **A**.

Question 6

$$\int \frac{1}{x^8}dx$$

$$= \int x^{-8}dx$$

$$= -\frac{1}{7}x^{-7} + c$$

$$= -\frac{1}{7x^7} + c$$

The correct answer is **C**.

7.3 Anti-derivatives of exponential and trigonometric functions

Question 1

$$\frac{d}{dx}\left(xe^{kx}\right) = (kx+1)e^{kx}$$

$$\int (kx+1)e^{kx}dx = xe^{kx}$$

$$k\int xe^{kx}dx + \int e^{kx}dx = xe^{kx}$$

$$k\int xe^{kx}dx = xe^{kx} - \int e^{kx}dx$$

$$\int xe^{kx}dx = \frac{1}{k}\left(xe^{kx} - \int e^{kx}dx\right) + c$$

The correct answer is **D**.

Question 2

$$\frac{dy}{dx} = 2\sin(2x) - 4e^{-2x}$$

$$y = \int \left(2\sin(2x) - 4e^{-2x}\right)dx$$

$$= -\cos(2x) + 2e^{-2x} + c$$

When $x = 0$ and $y = 0$,

$0 = -1 + 2 + C \Rightarrow C = -1$

$y = -\cos(2x) + 2e^{-2x} - 1$

The correct answer is **A**.

Question 3

$$a\int e^{bx}dx = \frac{a}{b}e^{bx} + c$$
$$-2e^{2x} = \frac{a}{b}e^{bx}$$

Equating coefficients:

$$e^{2x} = e^{bx}$$
$$b = 2$$
$$-2 = \frac{a}{b}$$
$$-2 = \frac{a}{2}$$
$$a = -4$$

The correct answer is **D**.

Question 4

$g'(x) = \sin(2\pi x)$

$$g(x) = \int \sin(2\pi x)dx$$
$$= -\frac{1}{2\pi}\cos(2\pi x) + c$$
$$g(1) = \frac{1}{\pi} = -\frac{1}{2\pi}\cos(2\pi) + c$$
$$= \frac{1}{\pi} = -\frac{1}{2\pi} + c \Rightarrow c = \frac{3}{2\pi}$$
$$g(x) = \frac{3}{2\pi} - \frac{1}{2\pi}\cos(2\pi x)$$

Award 1 mark for the correct antiderivative.
Award 1 mark for the correct constant of integration.
***VCAA Assessment Report* note:**

Most students could anti-differentiate $\sin(2\pi x)$ but many neglected '$+c$', which was essential in order to move to the next step. Mistakes in the attempt to combine the two fractions for c were common.

Question 5

$$\frac{d}{dx}(x\sin(x)) = \sin(x) + x\cos(x) \quad \textbf{[1 mark]}$$
$$\int (x\cos(x))\,dx = x\sin(x) - \int \sin(x)\,dx \quad \textbf{[1 mark]}$$
$$= x\sin(x) + \cos(x) \quad \textbf{[1 mark]}$$

Question 6

$$\int\left(\frac{1}{x^2} - \frac{1}{\cos^2\left(\frac{x}{2}\right)}\right)dx = -\frac{1}{x} - 2\tan\left(\frac{x}{2}\right) + c$$

The correct answer is **A**.

Question 7

$$\frac{dy}{dx} = 2\cos\left(\frac{x}{2}\right)$$

$$y = \int \left(2\cos\left(\frac{x}{2}\right)\right) dx$$

$$= 4\sin\left(\frac{x}{2}\right) + C$$

when $x = \dfrac{5\pi}{3}$ and $y = 0$

$$0 = 4\sin\left(\frac{5\pi}{6}\right) + C = 2 + C \qquad \Rightarrow C = -2$$

$$y = 4\sin\left(\frac{x}{2}\right) - 2$$

now when $x = 0$, $y = -2$

The correct answer is **C**.

7.4 The anti-derivative of $f(x) = \frac{1}{x}$

Question 1

$$\int \frac{x^2 + 2x - 3}{x^2}\, dx = \int \left(1 + 2x^{-1} - 3x^{-2}\right) dx$$

$$= x + 2\log_e(x) + 3x^{-1} + c$$

$$= x + 2\log_e(x) + \frac{3}{x} + c,\ x > 0 \quad \textbf{[1 mark]}$$

Question 2

From the rule,

$$\int kf(x)\, dx = k\int f(x)\, dx.$$

The correct answer is **A**.

Question 3

Let $u = x, \quad v = \log_e(x)$.

$$\frac{du}{dx} = 1, \quad \frac{dv}{dx} = \frac{1}{x}$$

$$f'(x) = \frac{1}{x} \times x + 1 \times \log_e(x)$$

$$= 1 + \log_e(x) \qquad \textbf{[1 mark]}$$

$$x\log_e(x) = \int (\log_e(x) + 1) dx$$

$$x\log_e(x) = \int \log_e(x) dx + \int 1 dx \qquad \textbf{[1 mark]}$$

$$\int \log_e(x) dx = x\log_e(x) - x \qquad \textbf{[1 mark]}$$

Question 4

$$f'(x) = \frac{1}{2} - \frac{1}{2x-2}$$

$$f(x) = \int \left(\frac{1}{2} - \frac{1}{2x-2}\right) dx$$

$$f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x-2) + c$$ **[1 mark]**

$$f(2) = 0 \Rightarrow 0 = 1 - \frac{1}{2}\log_e(2) + c; \qquad c = \frac{1}{2}\log_e(2) - 1$$ **[1 mark]**

$$f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x-2) + \frac{1}{2}\log_e(2) - 1$$

$$f(x) = \frac{x}{2} - 1 - \frac{1}{2}\log_e\left(\frac{2x-2}{2}\right)$$

$$f(x) = \frac{x}{2} - 1 - \frac{1}{2}\log_e(x-1) \qquad x > 1$$ **[1 mark]**

***VCAA Examination Report* note:**

This question was attempted well. A common misconception was that $\int \left(\frac{1}{2x-2}\right) dx = \log_e(2x-2) + c$, which was incorrect. Some students found a value of c but did not substitute it back into the final answer to state $f(x)$. Some poor notation was observed, for example, $\frac{1}{2x}$ is not the same as $\frac{x}{2}$, and notation for the natural logarithm is $\log_e$ not loge.

7.5 Families of curves

Question 1

$$f'(x) = \frac{2}{\sqrt{2x-3}} = 2(2x-3)^{-\frac{1}{2}}$$

$$f(x) = \frac{2}{2 \times \frac{1}{2}}(2x-3)^{\frac{1}{2}} + c$$

$$f(x) = 2\sqrt{2x-3} + c$$

$$f(6) = 4 \Rightarrow 4 = 2\sqrt{9} + c, \ c = -2$$

$$f(x) = 2\sqrt{2x-3} - 2$$

The correct answer is **C**.

Question 2

The gradient changes from positive to zero to negative to zero, then stays positive. The graph also has to be a quintic graph. Graph B is the only option.
The correct answer is **B**.

Question 3

$$f'(x) = 2\cos(x) - \sin(2x)$$

$$f(x) = \int (2\cos(x) - \sin(2x))\, dx$$

$$f(x) = 2\sin(x) + \frac{1}{2}\cos(2x) + \text{c}$$

$f\left(\frac{\pi}{2}\right) = \frac{1}{2}$

$\frac{1}{2} = 2\ \sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\ \cos(\pi) + c$

$\frac{1}{2} = 2 - \frac{1}{2} + c \quad \Rightarrow c = -1$

$f(x) = 2\ \sin(x) + \frac{1}{2}\cos(2x) - 1$

Award 1 mark for the correct antiderivative.
Award 1 mark for attempting to find c.
Award 1 mark for the correct value of c.

Question 4

$f'(x) = x^3 + x, \quad f(1) = 2$

$f(x) = \int \left(x^3 + x\right)\, dx$

$f(x) = \frac{x^4}{4} + \frac{x^2}{2} + c$

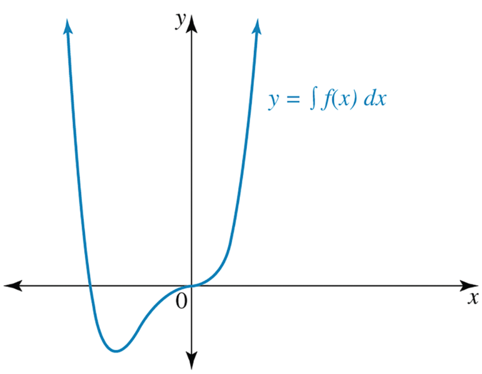
$2 = \frac{1}{4} + \frac{1}{2} + c, \quad c = 2 - \frac{1}{4} - \frac{1}{2} = \frac{5}{4}$

$f(x) = \frac{x^4}{4} + \frac{x^2}{2} + \frac{5}{4}$

Award 1 mark for the correct antiderivative.
Award 1 mark for the correct answer with correct c value.

Question 5

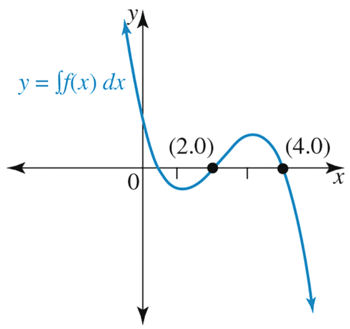
Given that $f(x)$ has a cubic shape, the antiderivative function has to be a polynomial of degree 4.
The function given has two x-intercepts, one of which is also a stationary point. This implies that the quartic has a positive shape, with a point of inflection at $x = 0$ and a stationary point.



Award 1 mark for the correct antiderivative graph showing a quartic curve.

Question 6

The graph $y = f(x)$ of is an inverted quadratic, which implies that the antiderivative function is an inverted cubic function.
The graph $y = f(x)$ of has two x-intercepts, at $x = 1$ and $x = 3$.
This implies that the stationary points of the anti-derivative function will occur when $x = 1$ and $x = 3$.



Award 1 mark for the correct antiderivative graph of a negative cubic curve.

Question 7

At $x = 0$ the gradient goes from negative to positive, and is zero at $x = 0$, this is a local minimum. At $x = 4$ the gradient is zero, but its rate of change goes from negative to positive, this is a stationary point of inflection.

The correct answer is **B**.

7.6 Applications

Question 1

a. $v = \frac{dx}{dt} = 2t^3 - t + 1$

$x = \int 2t^3 - t + 1dt$

$= \frac{1}{2}t^4 - \frac{1}{2}t^2 + t + c$

When $t = 0$, $x = -1$.

$-1 = \frac{1}{2}(0) - \frac{1}{2}(0) + 0 + c$

$c = -1$

Therefore, $x = \frac{1}{2}t^4 - \frac{1}{2}t^2 + t - 1$. **[1 mark]**

b. $t = 3$

$x = \frac{1}{2}(3)^4 - \frac{1}{2}(3)^2 + 3 - 1$

$= \frac{81}{2} - \frac{9}{2} + 2$

$= 36 + 2$

$= 38$ **[1 mark]**

$v = 2t^3 - t + 1$

$= 2(3)^3 - 3 + 1$

$= 54 - 3 + 1$

$= 52$ **[1 mark]**

After 3 seconds, the particle is 38 m to the right of the origin, with a velocity of 52 m/s.

Question 2

$a = \frac{dv}{dt} = 4 - 2t$

$v = \int 4 - 2t dt$

$= 4t - t^2 + c$

When $t = 0$, $v = 0$.

$0 = 4(0) - (0)^2 + c$

$c = 0$

Therefore, $v = 4t - t^2$.

$v = \frac{dx}{dt} = 4t - t^2$

$= \int 4t - t^2 dt$

$= 2t^2 - \frac{1}{3}t^3 + c$

When $t = 0$, $x = 3$.

$3 = 2(0)^2 - \frac{1}{3}(0)^3 + c$

$c = 3$

Therefore, $x = 2t^2 - \frac{1}{3}t^3 + 3$. **[1 mark]**

$t = 2$

$v = 4(2) - (2)^2$

$= 8 - 4$

$= 4$

$x = 2(2)^2 - \frac{1}{3}(2)^3 + 3$

$= 8 - \frac{8}{3} + 3$

$= \frac{25}{3}$

After 3 seconds, the particle is $\frac{25}{3}$ m to the right of the origin, with a velocity of 4 m/s. **[1 mark]**

Question 3

a. $\frac{db}{dt} = 100t^{\frac{3}{2}}$

$b = \int 100t^{\frac{3}{2}} dt$

$= \frac{100t^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= 40t^{\frac{5}{2}} + c$

Initially, $b = 80$.

$80 = 40(0)^{\frac{5}{2}} + c$

$80 = 0 + c$

$c = 80$

Therefore, $b = 40t^{\frac{5}{2}} + 80$. **[1 mark]**

b. $500 = 40t^{\frac{5}{2}} + 80$

$t = 2.56$

$0.56139 \times 60 = 33.68$ **[1 mark]**

Therefore, the bacteria will number 500 after 2 hours, 34 minutes.

7.7 Review

Question 1

$f'(x) = 3x^2 - 2x$

$f(x) = \int (3x^2 - 2x)dx = x^3 - x^2 + c$

$f(4) = 0 \quad \Rightarrow \quad 0 = 46 - 16 + c, \quad c = -48$

$f(x) = x^3 - x^2 - 48$

The correct answer is **C**.

Question 2

$\frac{dy}{dx} = \frac{1}{x^2}$

$y = \int \frac{1}{x^2} dx$

$= -\frac{1}{x} + c$

The correct answer is **A**.

Question 3

$$\int \frac{1}{(3x-4)^{\frac{5}{2}}} dx = \int (3x-4)^{-\frac{5}{2}} dx$$

$$= \frac{(3x-4)^{-\frac{3}{2}}}{3 \times -\frac{3}{2}}$$

$$= \frac{(3x-4)^{-\frac{3}{2}}}{-\frac{9}{2}}$$

$$= -\frac{2(3x-4)^{-\frac{3}{2}}}{9}$$

The correct answer is **E**.

Question 4

$\frac{dy}{dx} = 6x - 1$

$y = \int (6x - 1)\, dx$

$= 3x^2 - x + C$

when $x = 2$ and $y = 1$

$1 = 12 - 2 + C \Rightarrow C = -9$

$y = 3x^2 - x - 9$

The correct answer is **B**.

Question 5

$$\frac{d}{dx}(x\sin(x)) = \sin(x) + x\cos(x) \quad \textbf{[1 mark]}$$

$$\int (x\cos(x))dx = x\sin(x) - \int \sin(x)dx \quad \textbf{[1 mark]}$$

$$= x\sin(x) + \cos(x) \quad \textbf{[1 mark]}$$

Question 6

a. i. $f'(x) = -3(3x-1)^{-2} = \dfrac{-3}{(3x-1)^2}$ using the chain rule. **[1 mark]**

***VCAA Examination Report* note:**

The majority of students correctly applied the chain rule. Errors were generally arithmetic in nature or with the negative exponent.

ii. $\displaystyle\int \frac{1}{(3x-1)}dx = \frac{1}{3}\log_e(3x-1) + c$ since $x > \dfrac{1}{3}$ **[1 mark]**

***VCAA Examination Report* note:**

Students should note that they could easily verify their answer by using the chain rule to differentiate their answer, and checking whether or not this derivative was in fact the rule for f.

b. $g: R\backslash\{-1\} \to R, \quad g(x) = \dfrac{\sin(\pi x)}{x+1}$

Using the quotient rule:

$$g'(x) = \frac{\pi\cos(\pi x)(x+1) - 1\times\sin(\pi x)}{(x+1)^2}$$

$$g'(1) = \frac{2\pi\cos(\pi) - \sin(\pi)}{4}$$

$$g'(1) = -\frac{\pi}{2}$$

Award 1 mark for using quotient rule.

Award 1 mark for correct result.

***VCAA Examination Report* note:**

Though generally well handled, poor placement of, or lack of, brackets when using quotient rule (or the combination of product and chain rules) led to errors in evaluation. Other errors included the misconception that $\cos(\pi) = 1$ or misquoting the relevant differentiation rule (which is listed on the formula sheet).

Some students did not answer the question in its entirety (i.e. completely forgetting to evaluate $g'(1)$).

Question 7

$$f'(x) = 1 - \frac{3}{x}$$

$$f(x) = \int\left(1 - \frac{3}{x}\right)dx$$

$$= x - 3\log_e(|x|) + c$$

To find c, $f(e) = -2$

$$-2 = e - 3\log_e(e) + c \qquad \Rightarrow c = 1 - e$$

$$f(x) = x - 3\log_e(|x|) + 1 - e$$

Award 1 mark for the correct integration.

Award 1 mark for finding the constant.

Award 1 mark for the correct function.

Question 8

a. $\left(x^2-2\right)^3+\cos^2(x)+c$ **[1 mark]**

b. $\left(x^2-2\right)^3-4x+c$ **[1 mark]**

c. $3\cos^2(x)+c$ **[1 mark]**

Question 9

$$x=\int \frac{dx}{dt}dt$$

$$=\int \frac{2}{\sqrt{t}}dt$$

$$=4\sqrt{t}+c$$ **[1 mark]**

$$4=4\sqrt{1}+c$$

$$c=0$$

$$\therefore x=4\sqrt{t}$$ **[1 mark]**

$$y=\frac{\tan\left(4\sqrt{t}\right)}{t}$$

$$\frac{dy}{dt}=\frac{1}{\sqrt{t}\times 2\cos^2\left(4\sqrt{t}\right)}$$

$$=\frac{1}{2\sqrt{t}\cos^2\left(4\sqrt{t}\right)}$$ **[1 mark]**

8 Integral calculus

Topic	8	Integral calculus
Subtopic	8.2	The fundamental theorem of integral calculus

online only

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Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q9; © VCAA*

Question 1 (1 mark)

If $\int_4^8 f(x)dx = 5$, then $\int_0^2 f(2(x+2))dx$ is equal to

A. 12

B. 10

C. 8

D. $\frac{1}{2}$

E. $\frac{5}{2}$

Source: *VCE 2018, Mathematical Methods Exam 2, Section A, Q8; © VCAA*

Question 2 (1 mark)

If $\int_1^{12} g(x)\,dx = 5$ and $\int_{12}^{5} g(x)\,dx = -6$, then $\int_1^5 g(x)\,dx$ is equal to

A. -11

B. -1

C. 1

D. 3

E. 11

Topic 8 | Subtopic 8.2 The fundamental theorem of integral calculus

Source: *VCE 2015, Mathematical Methods (CAS) Exam 1, Q3; © VCAA*

Question 3 (2 marks)

Evaluate $\int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx$.

Source: *VCE 2019, Mathematical Methods 2, Section A, Q4; © VCAA*

Question 4 (1 mark)

$\int_0^{\frac{\pi}{6}} (a\sin(x) + b\cos(x))\, dx$ is equal to

A. $\frac{\left(2-\sqrt{3}\right)a - b}{2}$

B. $\frac{b - \left(2-\sqrt{3}\right)a}{2}$

C. $\frac{\left(2-\sqrt{3}\right)a + b}{2}$

D. $\frac{\left(2-\sqrt{3}\right)b - a}{2}$

E. $\frac{\left(2-\sqrt{3}\right)b + a}{2}$

Topic 8 | Subtopic 8.2 The fundamental theorem of integral calculus

Source: *VCE 2015, Mathematical Methods (CAS) 2, Section 1, Q15; © VCAA*

Question 5 (1 mark)

If $\int_0^5 g(x)dx = 20$ and $\int_0^5 (2g(x) + ax)\,dx = 90$, then the value of a is

A. 0
B. 4
C. 2
D. −3
E. 1

Source: *VCE 2014, Mathematical Methods (CAS) 1, Q2; © VCAA*

Question 6 (2 marks)

Let $\int_4^5 \frac{2}{2x-1}dx = \log_e(b)$.

Find the value of b.

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q8; © VCAAA*

Question 7 (1 mark)

If $\int_1^4 f(x)dx = 6$, then $\int_1^4 (5 - 2f(x))\,dx$ is equal to

A. 3
B. 4
C. 5
D. 6
E. 16

Question 8 (2 marks)

Given that $\int_0^a x^4 dx = \frac{a^5}{5}$ find $\int_{-a}^0 x^4 dx$.

Question 9 (1 mark)

$\int_1^4 (2f(x) + 6)\, dx$ can be written as

A. $2\int_1^4 f(x)dx + 6$

B. $2\int_1^4 f(x)dx + \int_1^4 3dx$

C. $2\int_1^4 (f(x) + 6)\, dx$

D. $2\int_1^4 f(x)dx + 18$

E. $2\int_1 f(x)dx + 6x$

Question 10 (2 marks)

Let f be a differentiable function defined for all real x, where $f(x) \geq 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)dx = a$, find $2\int_0^{5a} \left(f\left(\frac{x}{5}\right) + 2\right) dx$.

Question 11 (1 mark)

Find $4\int_0^{3k}\left(g\left(\frac{x}{3}\right)-1\right)dx$, given $\int_0^k (g(x))\,dx = 2k$, where function g is continuous for $x \in R$ and given $g(x) \geq 0$ for $x \in [0, k]$.

Question 12 (1 mark)

If $\int_1^3 (f(x))\,dx = 4$ then $\int_1^3 (2f(x)-3)\,dx$ is equal to

A. 0
B. 2
C. 3
D. 5
E. 8

Question 13 (1 mark)

Given that $\int_1^5 (h(x))\,dx = 4$, $\int_5^1 (h(x)-2)\,dx$ is equal to

A. 0
B. 1
C. 4
D. 7
E. 8

Question 14 (2 marks)

Given that $\int_2^6 (f(x))\,dx = 14$, find $\int_2^3 (f(x)+2)\,dx + \int_3^6 (f(x))\,dx$.

Topic	8	Integral calculus
Subtopic	8.3	Areas under curves

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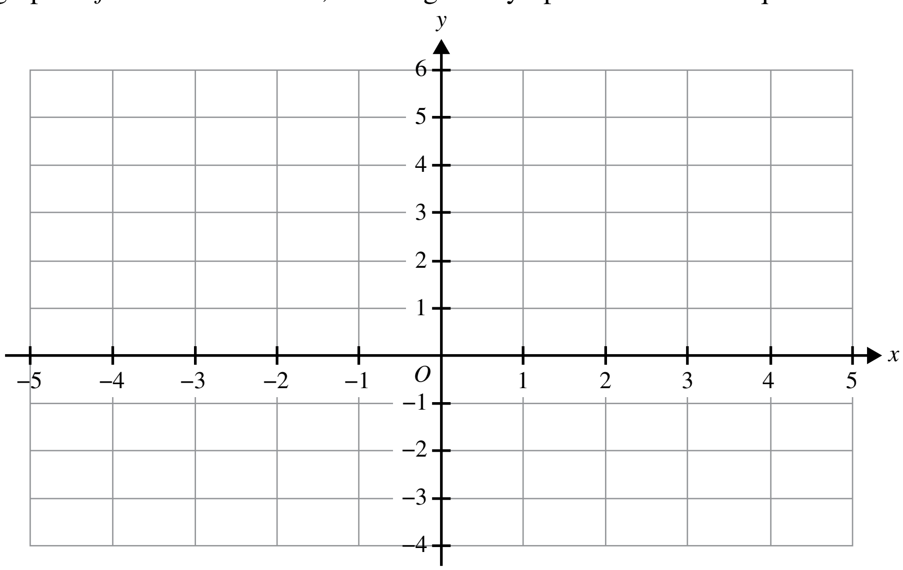
***Source:** VCE 2019, Mathematical Methods Exam 1, Q5; © VCAA*

Question 1 (5 marks)

Let $f: R\backslash\{1\} \to R$, $f(x) = \dfrac{2}{(x-1)^2} + 1$.

a. i. Evaluate $f(-1)$. **(1 mark)**

ii. Sketch the graph of f on the axes below, labelling all asymptotes with their equations. **(2 marks)**

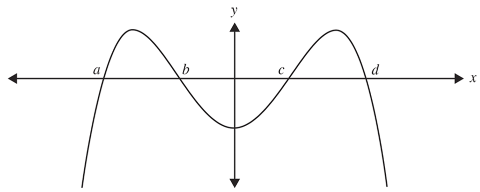


b. Find the area bounded by the graph of f, the x-axis, and the line $x = -1$ and the line $x = 0$. **(2 marks)**

Source: *VCE 2017, Mathematical Methods Exam 2, Section A, Q17; © VCAA*

Question 2 (1 mark)

The graph of a function f, where $f(-x) = f(x)$, is shown below.



The graph has x-intercepts at $(a, 0), (b, 0), (c, 0)$ and $(d, 0)$ only.

The area bound by the curve and the x-axis on the interval $[a, d]$ is

A. $\int_a^d f(x)dx$

B. $\int_a^b f(x)dx - \int_c^d f(x)dx + \int_c^d f(x)dx$

C. $2\int_a^b f(x)dx + \int_b^c f(x)dx$

D. $2\int_a^b f(x)dx - 2\int_b^{b+c} f(x)dx$

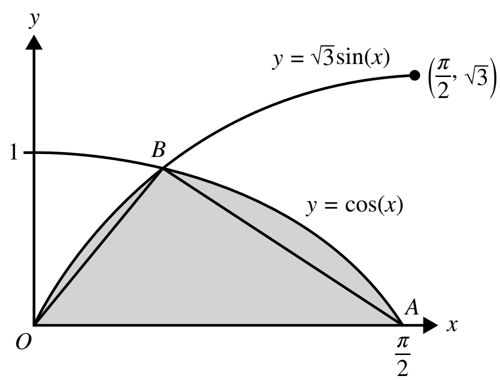
E. $\int_a^b f(x)dx + \int_c^b f(x)dx + \int_d^c f(x)dx$

Source: *VCE 2017, Mathematical Methods Exam 2, Section A, Q20; © VCAA*

Question 3 (1 mark)

The graphs of $f: \left[0, \frac{\pi}{2}\right] \to R, f(x) = \cos(x)$ and $g: \left[0, \frac{\pi}{2}\right] \to R, g(x) = \sqrt{3}\sin(x)$ are shown below.

The graphs intersect at B.



The ratio of the area of the shaded region to the area of triangle OAB is

A. $9:8$

B. $\sqrt{3} - 1 : \frac{\sqrt{3}\pi}{8}$

C. $8\sqrt{3} - 3 : 3\pi$

D. $\sqrt{3} - 1 : \frac{\sqrt{3}\pi}{4}$

E. $1 : \frac{\sqrt{3}\pi}{8}$

***Source:** VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q3; © VCAA*

Question 4 (1 mark)

The area of the region enclosed by the graph of $y = x(x + 2)(x - 4)$ and the x-axis is

A. $\frac{128}{3}$

B. $\frac{20}{3}$

C. $\frac{236}{3}$

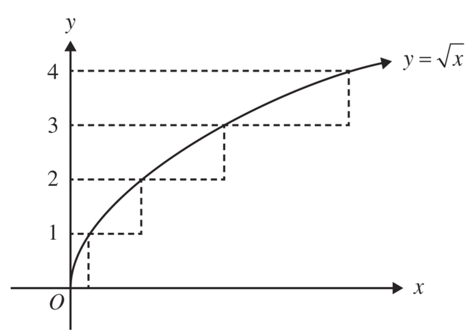
D. $\frac{148}{3}$

E. 36

***Source:** VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q19; © VCAA*

Question 5 (1 mark)

Jake and Anita are calculating the area between the graph of $y = \sqrt{x}$ and the y-axis between $y = 0$ and $y = 4$. Jake uses a partitioning, shown in the diagram below, while Anita uses a definite integral to find the exact area.



The difference between the results obtained by Jake and Anita is

A. 0

B. $\frac{22}{3}$

C. $\frac{26}{3}$

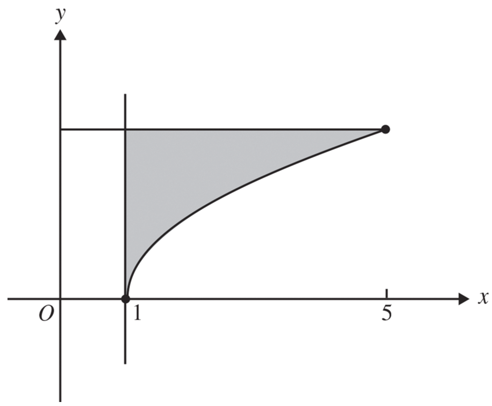
D. 14

E. 35

Source: VCE 2013, Mathematical Methods (CAS) 2, Section 1, Q16; © VCAA

Question 6 (1 mark)

The graph of $f: [1, 5] \rightarrow R, f(x) = \sqrt{x-1}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_1^5 \left(\sqrt{x-1}\right) dx$

B. $\int_0^2 \left(\sqrt{x-1}\right) dx$

C. $\int_0^5 \left(2 - \sqrt{x-1}\right) dx$

D. $\int_0^2 \left(x^2 + 1\right) dx$

E. $\int_0^2 \left(x^2\right) dx$

Question 7 (1 mark)

The area bounded by the graph of $y = -x^2 + 8x + 12$, the x-axis and the lines $x = 1$ and $x = 4$, is equal to

A. 3

B. $2\frac{1}{3}$

C. $5\frac{1}{3}$

D. $7\frac{2}{3}$

E. 7.66

Topic	8	Integral calculus
Subtopic	8.4	Areas between curves and average values

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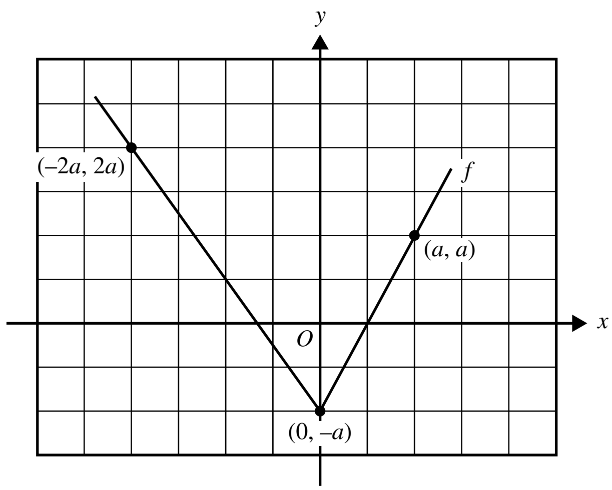
***Source:** VCE 2020, Mathematical Methods Exam 2, Section A, Q15; © VCAA*

Question 1 (1 mark)

Part of the graph of a function f, where $a > 0$, is shown below.



The average value of the function f over the interval $[-2a,\ a]$ is

A. 0

B. $\frac{a}{3}$

C. $\frac{a}{2}$

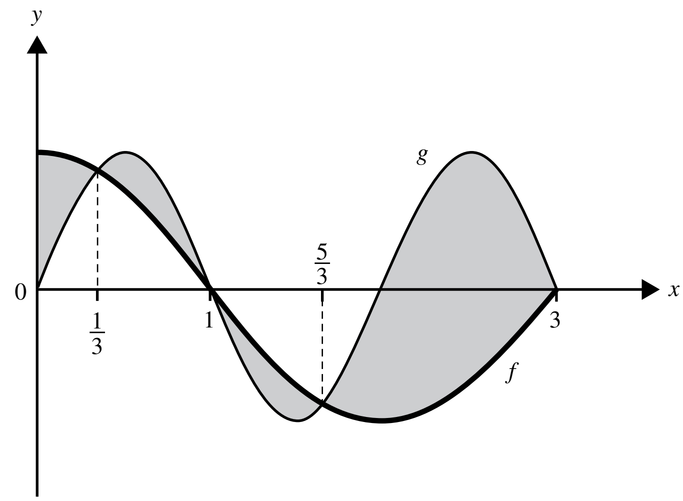
D. $\frac{3a}{4}$

E. a

Source: *VCE 2018, Mathematical Methods Exam 2, Section A, Q19; © VCAA*

Question 2 (1 mark)

The graphs $f: R \to R, f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g: R \to R, g(x) = \sin(\pi x)$ are shown in the diagram below.



An integral expression that gives the total area of the shaded regions is

A. $\int_0^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

B. $2\int_{\frac{5}{3}}^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

C. $\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{1}{3}}^1 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

D. $2\int_1^{\frac{5}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

E. $\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx + 2\int_{\frac{1}{3}}^1 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx + \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

Source: *VCE 2018, Mathematical Methods Exam 1, Q8; © VCAA*

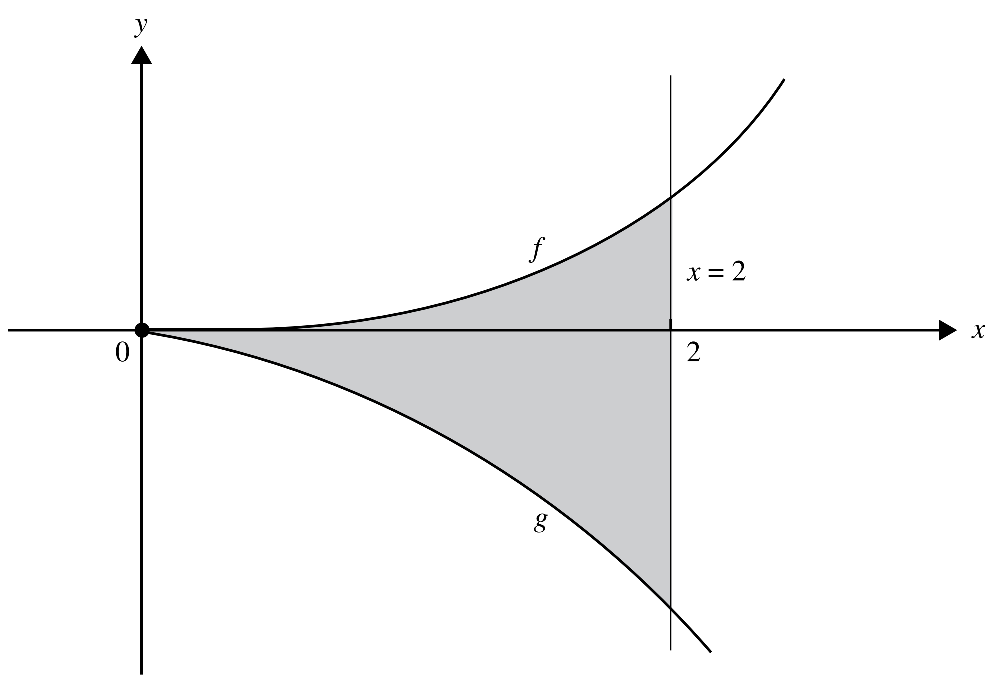
Question 3 (7 marks)

Let $f: R \to R$, $f(x) = x^2 e^{kx}$, where k is a positive real constant.

a. Show that $f'(x) = xe^{kx}(kx + 2)$. **(1 mark)**

b. Find the value of k for which the graphs of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection. **(2 marks)**

Let $g(x) = -\dfrac{2xe^{kx}}{k}$. The diagram below shows sections of the graphs of f and g for $x \geq 0$.



Let A be the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = 2$.

c. Write down a definite integral that gives the value of A. **(1 mark)**

d. Using your result from part **a**, or otherwise, find the value of k such that $A = \dfrac{16}{k}$. **(3 marks)**

Topic 8 Subtopic 8.4 Areas between curves and average values

Source: *VCE 2021, Mathematical Methods 2, Section A, Q14; © VCAA*

Question 4 (1 mark)

A value of k for which the average value of $y = \cos\left(kx - \frac{\pi}{2}\right)$ over the interval $[0, \pi]$ is equal to the average value of $y = \sin(x)$ over the same interval is

A. $\frac{1}{6}$

B. $\frac{1}{5}$

C. $\frac{1}{4}$

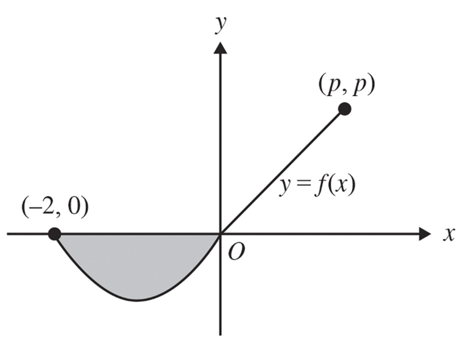
D. $\frac{1}{3}$

E. $\frac{1}{2}$

Source: *VCE 2015, Mathematical Methods (CAS) 2, Section 1, Q8; © VCAA*

Question 5 (1 mark)

The graph of a function $f: [-2, p] \to R$ is shown below.



The average value of f over the interval $[-2, p]$ is zero.

The area of the shaded region is $\frac{25}{8}$.

If the graph is a straight line, for $0 \le x \le p$, then the value of p is

A. 2

B. 5

C. $\frac{5}{4}$

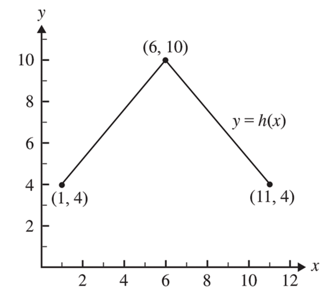
D. $\frac{5}{2}$

E. $\frac{25}{4}$

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q20; © VCAA*

Question 6 (1 mark)

The graph of a function, h, is shown below.



The average value of h is

A. 4
B. 5
C. 6
D. 7
E. 10

Source: *VCE 2013, Mathematical Methods (CAS) 1, Q6; © VCAA*

Question 7 (3 marks)

Let $g : R \to R,\ g(x) = (a - x)^2$, where a is a real constant.

The average value of g on the interval $[-1,\ 1]$ is $\frac{31}{12}$.

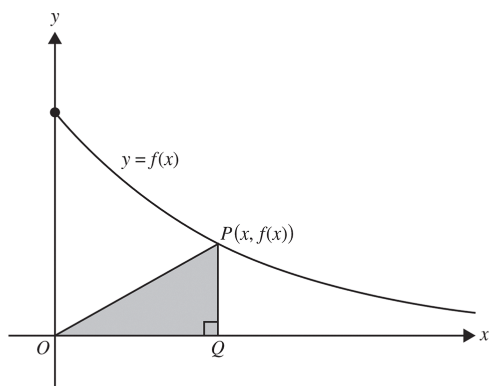
Find all possible values of a.

Source: *VCE 2013, Mathematical Methods (CAS) 1, Q10c; © VCAA*

Question 8 (3 marks)

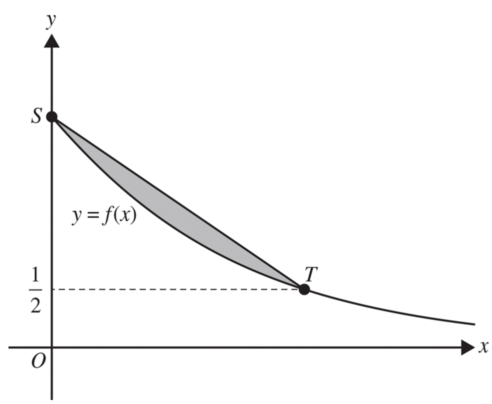
Let $f: [0, \infty] \to R,\ f(x) = 2e^{\frac{-x}{5}}$.

A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.



Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$.

Find the area of the region bounded by the graph of f and the line segment ST.



Question 9 (1 mark)

The average value of the function $y = 3\sin^2(2x)$ over $0 \leq x \leq \frac{\pi}{2}$ is

A. 0.00472

B. 0.0019

C. 1.5

D. 4.5

E. $\frac{3\pi}{4}$

Question 10 (1 mark)

The average value of the function $y = \sqrt{4x + 9}$ over $0 \leq x \leq 4$ is

A. $\frac{1}{2}$

B. $4\frac{1}{12}$

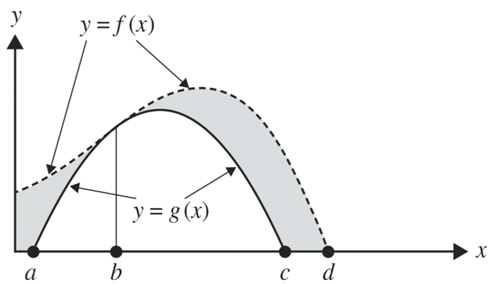
C. $8\frac{1}{6}$

D. $16\frac{1}{3}$

E. $65\frac{1}{3}$

Question 11 (1 mark)

Consider the graphs of the functions f and g shown below.



The area of the shaded region could be represented by

A. $\int_a^d (f(x) - g(x))\, dx$

B. $\int_0^d (f(x) - g(x))\, dx$

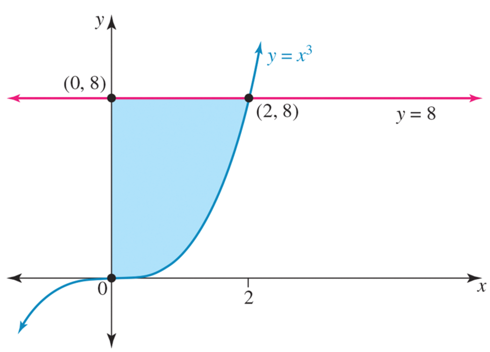
C. $\int_0^b (f(x) - g(x))\, dx + \int_b^c (f(x) - g(x))\, dx$

D. $\int_0^a f(x)dx + \int_a^c (f(x) - g(x))\, dx + \int_b^d f(x)dx$

E. $\int_0^d f(x)dx - \int_a^c g(x)dx$

Question 12 (3 marks)

Find the area of the shaded region between the equations $y = x^3$ and $y = 8$, as shown.



Topic 8 | Subtopic 8.4 Areas between curves and average values

Question 13 (1 mark)

Given $f: R \to R, f(x) = e^{2x} - 1$ and, $g: R \to R, g(x) = -e^{2x}$, the area enclosed between the graphs $y = f(x)$ and $y = g(x)$ and the lines $x = -1$ and $x = 0$ is given by

A. $\int_{-1}^{-\frac{1}{2}\log_e(2)} \left(2e^{2x} - 1\right) dx + \int_{-\frac{1}{2}\log_e(2)}^{0} \left(-2e^{2x} + 1\right) dx$

B. $\int_{-1}^{-\frac{1}{2}\log_e(2)} \left(-2e^{2x} + 1\right) dx + \int_{-\frac{1}{2}\log_e(2)}^{0} \left(2e^{2x} - 1\right) dx$

C. $\int_{-1}^{0} \left(2e^{2x} - 1\right) dx$

D. $\int_{-1}^{0} \left(2e^{2x} + 1\right) dx$

E. $\int_{\frac{1}{2}\log(2)}^{-1} \left(2e^{2x} - 1\right) dx + \int_{0}^{\frac{1}{2}\log(2)} \left(-2e^{2x} + 1\right) dx$

Question 14 (3 marks)

Find the area enclosed by the curves with equations $f(x) = \cos\left(\frac{x}{2}\right)$ and $g(x) = \sin(x)$ over the interval $[0, 2\pi]$.

Topic	8	Integral calculus
Subtopic	8.5	Applications

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Source: *Adapted from VCE 2020, Mathematical Methods Exam 2, Section B, Q2; © VCAA*

Question 1 (11 marks)

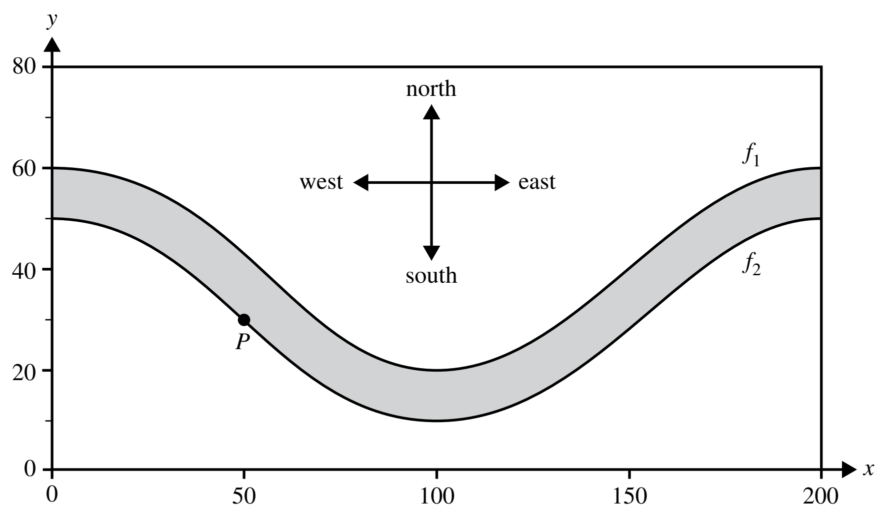
An area of parkland has a river running through it, as shown below. The river is shown shaded.

The north bank of the river is modelled by the function $f_1 : [0, 200] \to R, f_1(x) = 20\cos\left(\frac{\pi x}{100}\right) + 40$

The south bank of the river is modelled by the function $f_2 : [0, 200] \to R, f_2(x) = 20\cos\left(\frac{\pi x}{100}\right) + 30$

The horizontal axis points east and the vertical axis points north.

All distances are measured in metres.



A swimmer always starts at point P, which has coordinates $(50, 30)$.

Assume that no movement of water in the river affects the motion or path of the swimmer, which is always a straight line.

a. The swimmer swims north from point P.
Find the distance, in metres, that the swimmer needs to swim to get to the north bank of the river. **(1 mark)**

b. The swimmer swims east from point P.
Find the distance, in metres, that the swimmer needs to swim to get to the north bank of the river. **(2 marks)**

c. On another occasion, the swimmer swims the minimum distance from point P to the north bank of the river.
Find this minimum distance. Give your answer in metres, correct to one decimal place. **(2 marks)**

d. Calculate the surface area of the section of the river shown on the graph above in square metres. **(1 mark)**

e. A horizontal line is drawn through point P. The section of the river that is south of the line is declared a 'no swimming' zone.
Find the area of the 'no swimming' zone, correct to the nearest square metre. **(3 marks)**

f. Scientists observe that the north bank of the river is changing over time. It is moving further north from its current position. They model its predicted new location using the function with rule $y = kf_1(x)$, where $k \geq 1$.
Find the values of k for which the distance **north** across the river, for all parts of the river, is strictly less than 20 m. **(2 marks)**

Source: *Adapted from VCE 2019, Mathematical Methods Exam 2, Section B, Q3; © VCAA*

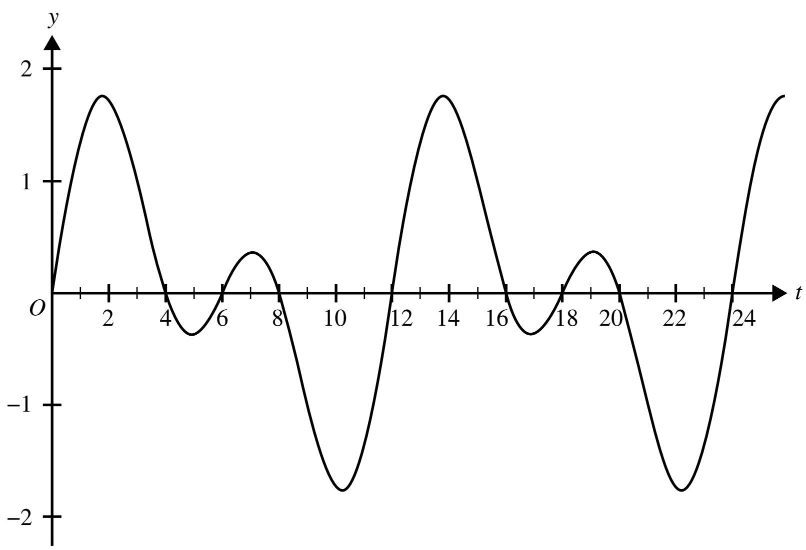
Question 2 (7 marks)

During a telephone call, a phone uses a dual-tone frequency electrical signal to communicate with the telephone exchange.

The strength, f, of a simple dual-tone frequency signal is given by the function

$f(t) = \sin\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{6}\right)$ where t is a measure of time and $t \geq 0$.

Part of the graph of $y = f(t)$ is shown below.



a. State the period of the function. **(1 mark)**

b. Find the values of t where $f(t) = 0$ for the interval $t \in [0, 6]$. **(1 mark)**

c. Find the maximum strength of the dual-tone frequency signal, correct to two decimal places. **(1 mark)**

d. Find the area between the graph of f and the horizontal axis for $t \in [0, 6]$. **(2 marks)**

e. The rectangle bounded by the line $y = k, k \in R^+$, the horizontal axis, and the lines $x = 0$ and $x = 12$ has the same area as the area between the graph of f and the horizontal axis for one period of the dual-tone frequency signal.

Find the value of k. **(2 marks)**

Source: *VCE 2015, Mathematical Methods (CAS) Exam 2, Section 2, Q2; © VCAA*

Question 3 (14 marks)

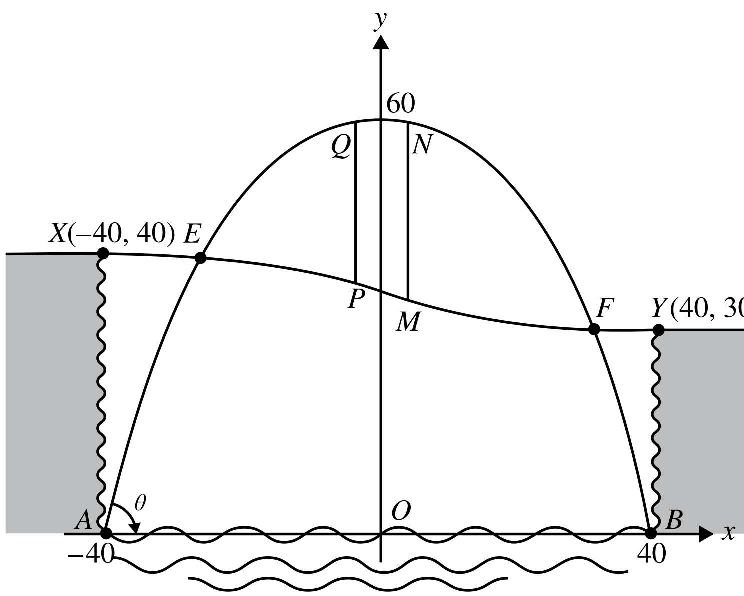
A city is located on a river that runs through a gorge.

The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.

A bridge is to be built that crosses the river and the gorge.

A diagram for the design of the bridge is shown below.

The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by $y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at $B(40, 0)$ and $A(-40, 0)$.



The road across the gorge is modelled by a cubic polynomial function.

a. Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree. **(2 marks)**

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \frac{x^3}{25\,600} - \frac{3x}{16} + 35$

b. Find the maximum downwards slope of the road. Give your answer in the form $-\frac{m}{n}$ where m and n are positive integers. **(2 marks)**

Topic 8 | Subtopic 8.5 Applications

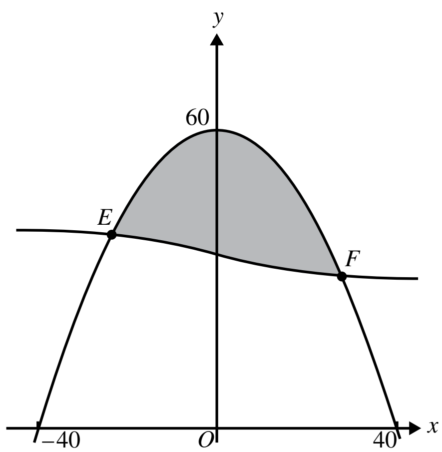
Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame.
The supporting column, MN is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c. Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places. **(3 marks)**

The second supporting column, PQ, has its lowest point at $P(-u,\ w)$.

d. Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ. **(3 marks)**

For the opening of the bridge, a banner is erected on the bridge, as shown by the shaded region in the diagram below.



e. Find the x-coordinates, correct to two decimal places, of E and F, the points at which the road meets the parabolic frame of the bridge. **(3 marks)**

f. Find the area of the banner (shaded region), giving your answer to the nearest square metre. **(1 mark)**

Topic	8	Integral calculus
Subtopic	8.6	Review

online only

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***Source:** VCE 2021, Mathematical Methods Exam 2, Section A, Q11; © VCAA*

Question 1 (1 mark)

If $\int_0^a f(x)dx = k$, then $\int_0^a (3f(x) + 2)dx$ is

A. $3k + 2a$

B. $3k$

C. $k + 2a$

D. $k + 2$

E. $3k + 2$

***Source:** VCE 2017, Mathematical Methods Exam 1, Q2; © VCAA*

Question 2 (4 marks)

Let $y = x\log_e(3x)$.

a. Find $\frac{dy}{dx}$. **(2 marks)**

b. Hence, calculate $\int_1^2 \left(\log_e(3x) + 1\right) dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer. **(2 marks)**

Source: VCE 2016, Mathematical Methods Exam 1, Q3; © VCAA

Question 3 (5 marks)

Let $f: R\backslash\{1\} \to R$, where $f(x) = 2 + \dfrac{3}{x-1}$.

a. Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation. **(3 marks)**

b. Find the area enclosed by the graph of f, the lines $x = 2$ and $x = 4$, and the x-axis. **(2 marks)**

Source: VCE 2019, Mathematical Methods Exam 2, Section A, Q12; © VCAA

Question 4 (1 mark)

If $\int_1^4 f(x)dx = 4$ and $\int_2^4 f(x)dx = -2$, then $\int_1^2 (f(x) + x)$ is equal to

A. 2

B. 6

C. 8

D. $\dfrac{7}{2}$

E. $\dfrac{15}{2}$

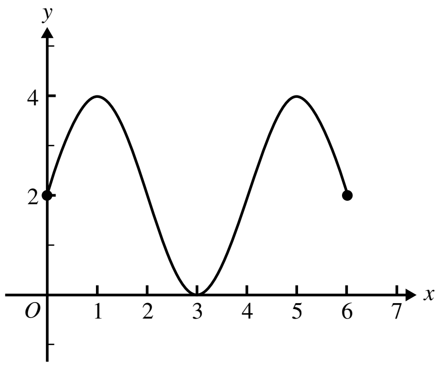
***Source:** VCE 2013, Mathematical Methods (CAS) Exam 2, Section 1, Q15; © VCAA*

Question 5 (1 mark)

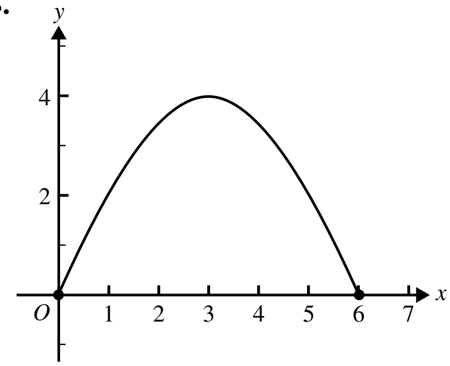
Let h be a function with an average value of 2 over the interval $[0, 6]$.

The graph of h over this interval could be

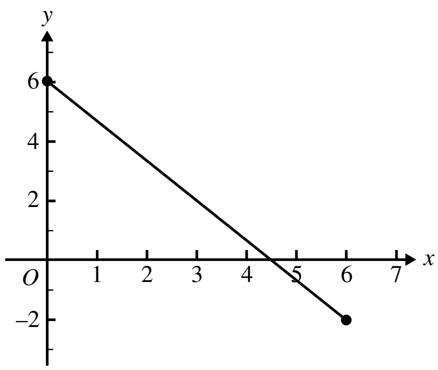
A.



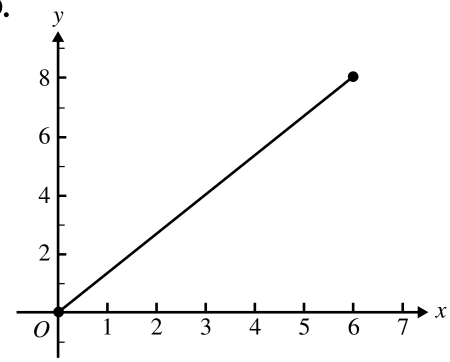
B.



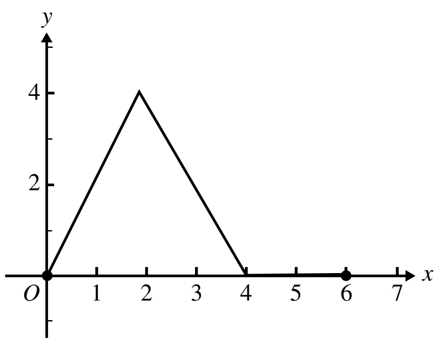
C.



D.



E.



Source: *VCE 2016, Mathematical Methods 1, Q6; © VCAA*

Question 6 (5 marks)

Let $f: [-\pi, \pi] \to R$, where $f(x) = 2\sin(2x) - 1$.

a. Calculate the average rate of change of f between $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{6}$. **(2 marks)**

b. Calculate the average value of f over the interval $-\frac{\pi}{3} \le x \le \frac{\pi}{6}$. **(3 marks)**

Source: *VCE 2014, Mathematical Methods (CAS) 1, Q5; © VCAA*

Question 7 (7 marks)

Consider the function $f: [-1, 3] \to R, f(x) = 3x^2 - x^3$.

a. Find the coordinates of the stationary points of the function. **(2 marks)**

b. On the axes below, sketch the graph of f.
Label any end points with their coordinates. **(2 marks)**

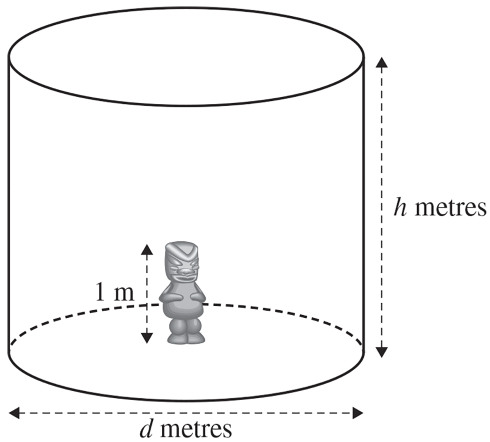
c. Find the area enclosed by the graph of the function and the horizontal line given by $y = 4$. **(3 marks)**

***Source:** VCE 2014, Mathematical Methods (CAS) 2, Section 2, Q2; © VCAA*

Question 8 (13 marks)

On 1 January 2010, Tasmania Jones was walking through an ice-covered region of Greenland when he found a large ice cylinder that was made a thousand years ago by the Vikings.

A statue was inside the ice cylinder. The statue was 1 m tall and its base was at the centre of the base of the cylinder.



The cylinder had a height of h metres and a diameter of d metres. Tasmania Jones found that the volume of the cylinder was $216\,m^3$. At that time, 1 January 2010, the cylinder had not changed in a thousand years. It was exactly as it was when the Vikings made it.

a. Write an expression for h in terms of d. **(2 marks)**

b. Show that the surface area of the cylinder excluding the base, S square metres, is given by the rule $S = \frac{\pi d^2}{4} + \frac{864}{d}$. **(1 mark)**

c. Tasmania found that the Vikings made the cylinder so that S is a minimum.
Find the value of d for which S is a minimum and find this minimum value of S. **(2 marks)**

d. Find the value of h when S is a minimum. **(1 mark)**

e. On 1 January 2010, Tasmania believed that due to recent temperature changes in Greenland, the ice of the cylinder had just started melting. Therefore, he decided to return on 1 January each year to measure the ice cylinder. He observes that the volume of the ice cylinder decreases by a constant rate of 10 m^3 per year. Assume that the cylindrical shape is retained and $d = 2h$ at the beginning and as the cylinder melts. **(1 mark)**
Write down an expression for V in terms of h.

f. Find $\frac{dh}{dt}$ in terms of h. **(3 marks)**

g. Find the rate at which the height of the cylinder will be decreasing when the top of the statue is just exposed. **(1 mark)**

h. Find the year in which the top of the statue will just be exposed. (Assume that the melting started on 1 January 2010.) **(2 marks)**

Source: *VCE 2021, Mathematical Methods 1, Q8; © VCAA*

Question 9 (5 marks)

The gradient of a function is given by $\frac{dy}{dx} = \sqrt{x+6} - \frac{x}{2} - \frac{x}{3}$.

The graph of the function has a single stationary point at $\left(3, \frac{29}{4}\right)$.

a. Find the rule of the function. **(3 marks)**

b. Determine the nature of the stationary point. **(2 marks)**

Source: *VCE 2017, Mathematical Methods 2, Section B, Q1; © VCAA*

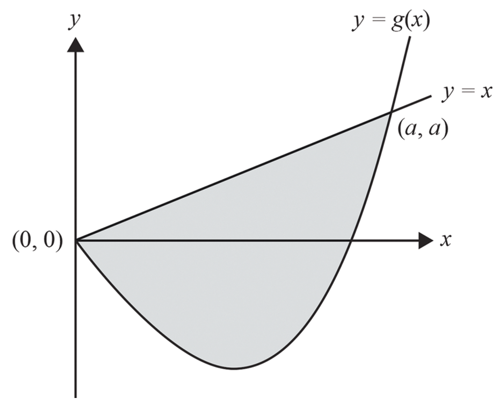
Question 10 (5 marks)

Let $g : R \to R, g(x) = x^3 - kx, k \in R^+$.

a. Let $C(-1,\ g(-1))$ and $D(1,\ g(1))$ be two points on the graph of g.

i. Find the distance CD in terms of k. **(2 marks)**

b. The diagram below shows part of the graphs of g and $y = x$. These graphs intersect at the points with the coordinates $(0, 0)$ and (a, a).



i. Find the values of k such that the distance CD is equal to $k + 1$. **(1 mark)**

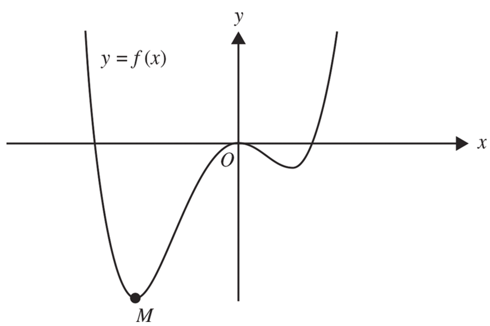
ii. Find the value of a in terms of k. **(1 mark)**

iii. Find the area of the shaded region in terms of k. **(2 marks)**

Source: *VCE 2018, Mathematical Methods 2, Section B, Q1; © VCAA*

Question 11 (12 marks)

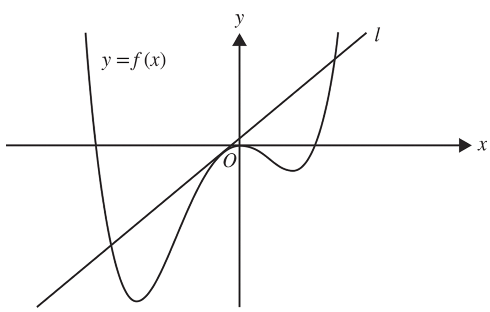
Consider the quartic $f: R \to R,\ f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below.



a. Find the coordinates of the point M, at which the minimum value of the function f occurs. **(1 mark)**

b. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts. **(1 mark)**

c. Part of the tangent, l, to $y = f(x)$ at $x = -\frac{1}{3}$ is shown below. **(1 mark)**



Find the equation of the tangent l.

d. The tangent l intersects $y = f(x)$ at $x = -\frac{1}{3}$ and at two other points.

State the x-values of the two other points of intersection. Express your answers in the form $\frac{a \pm \sqrt{b}}{c}$, where $a,\ b$ and c are integers. **(2 marks)**

e. Find the total area of the regions bounded by the tangent l and $y = f(x)$. Express your answer in the form $\dfrac{a\sqrt{b}}{c}$, where a, b and c are positive integers. **(2 marks)**

f. Let $p: R \to R, p(x) = 3x^4 + 4x^3 + 6(a-2)x^2 - 12ax + a^2, a \in R$.
State the value of a for which $f(x) = p(x)$ for all x. **(1 mark)**

g. Find all solutions to $p'(x) = 0$, in terms of a where appropriate. **(1 mark)**

h. **i.** Find the values of a for which p has only one stationary point. **(1 mark)**

ii. Find the minimum value of p when $a = 2$. **(1 mark)**

iii. If p has only one stationary point, find the values of a for which $p(x) = 0$ has no solutions. **(1 mark)**

***Source:** VCE 2017, Mathematical Methods 1, Q2; © VCAA*

Question 12 (4 marks)

Let $y = x\log_e(3x)$.

a. Find $\dfrac{dy}{dx}$. **(2 marks)**

b. Hence, calculate $\displaystyle\int_1^2 \left(\log_e(3x) + 1\right) dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer. **(2 marks)**

Source: *VCE 2015, Mathematical Methods (CAS) 2, Section 2, Q5; © VCAA*

Question 13 (15 marks)

Answer the following.

a. Let $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, where $0 \le t \le 5$.

i. Find $S(0)$ and $S(5)$. **(1 mark)**

ii. The minimum value of S occurs when $t = \log_e(c)$.
State the value of c and the minimum value of S. **(2 marks)**

iii. On the axes below, sketch the graph of S against t for $0 \le t \le 5$. Label the end points and the minimum point with their coordinates. **(2 marks)**

iv. Find the value of the average rate of change of the function S over the interval $[0, \log_e(c)]$. **(2 marks)**

b. Let $V: [0.5] \to R,\ V(t) = de^{\frac{t}{3}} + (10 - d)\,e^{-\frac{2t}{3}}$, where d is a real number and $d \in (0,\ 10)$.
If the minimum value of the function occurs when $t = \log_e(9)$, find the value of d. **(2 marks)**

c. i. Find the set of possible values of d such that the minimum value of the function occurs when $t = 0$. **(2 marks)**

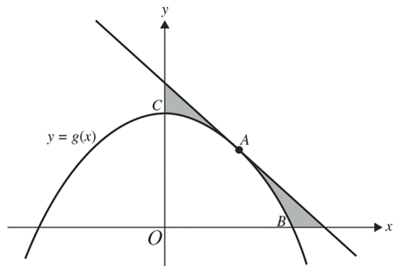
ii. Find the set of possible values of d such that the minimum value of the function occurs when $t = 5$. **(2 marks)**

d. If the function V has a local minimum $(a,\ m)$, where $0 \le a \le 5$, it can be shown that $m = \frac{k}{2}d^{\frac{2}{3}}(10 - d)^{\frac{1}{3}}$
Find the value of k. **(2 marks)**

Source: *VCE 2013, Mathematical Methods (CAS) 2, Section 2, Q4; © VCAA*

Question 14 (16 marks)

Part of the graph of a function $R \to R, g(x) = \dfrac{16 - x^2}{4}$ is shown below.



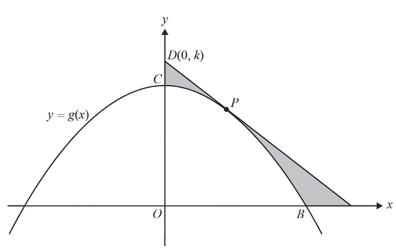
a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram above. The tangent to the graph of g at the point A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A. **(2 marks)**

ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the point A, and the x-axis and y-axis.
Evaluate the area of this shaded region. **(3 marks)**

b. Let Q be a point on the graph of $y = g(x)$. Find the positive value of the x-coordinate of Q, for which the distance OQ is a minimum and find the minimum distance. **(3 marks)**

c. The tangent to the graph of g at a point p has a **negative** gradient and intersects the y-axis at point $D(0, k)$, where $5 \le k \le 8$.



Find the gradient of the tangent in terms of k. **(2 marks)**

d. i. Find the rule $A(k)$ for the function of k that gives the area of the shaded region. **(2 marks)**

ii. Find the maximum area of the shaded region and the value of k for which this occurs. **(2 marks)**

iii. Find the minimum area of the shaded region and the value of k for which this occurs. **(2 marks)**

Question 15 (1 mark)

The temperature $T°$ over a time period of a day is given by the function $T(t) = 17 - 6\sin\left(\frac{\pi t}{12}\right)$, where t is the time in hours. Using the given function the average temperature over the first 12 hours is equal to

A. 17

B. $204 - \frac{12}{\pi}$

C. $17 + \frac{12}{\pi}$

D. $17 - \frac{12}{\pi}$

E. $\frac{12}{\pi}$

Answers for marking guide

8.2 The fundamental theorem of integral calculus

Question 1

$$\int_4^8 f(x)\,dx$$

Let $u = 2(x+2)$, $\frac{du}{dx} = 2$.

Terminals: $x = 0,\ u = 4,\ x = 2,\ u = 8$

$$\begin{aligned}\int_0^2 f(2(x+2))\,dx &= \int_4^8 f(u)\,\frac{1}{2}du \\ &= \frac{1}{2}\int_4^8 f(x)\,dx \\ &= \frac{5}{2}\end{aligned}$$

Alternatively:

$y = f(x)$

$y = f(2(x'+2))$

$x' = \frac{x}{2} - 2$

$x = 4, x' = 0$

$x = 8, x' = 2$

A translation of 2 units to the left parallel to the x-axis does not affect the area.

A dilation by a factor of $\frac{1}{2}$ from the y-axis means the area is halved.

The correct answer is **E**.

Question 2

$$\int_1^{12} g(x)dx = 5,\ \int_{12}^{5} g(x)dx = -6$$

$$\int_1^5 g(x)dx + \int_5^{12} g(x)dx = \int_1^{12} g(x)dx$$

$$\int_1^5 g(x)dx + (-6) = -5$$

$$\int_1^5 g(x)dx = -1$$

The correct answer is **B**.

Question 3

$$\int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx = \int_1^4 x^{-\frac{1}{2}} dx$$
$$= 2\left[x^{\frac{1}{2}}\right]_1^4$$
$$= 2\left[\sqrt{4} - \sqrt{1}\right]$$
$$= 2(2-1)$$
$$= 2$$

Award 1 mark for the correct antiderivative.
Award 1 mark for the correct final answer.

***VCAA Assessment Report* note:**
This question was not answered well. A range of incorrect anti-derivatives were given, the majority of which involved the logarithm function.

Question 4

$$\int_0^{\frac{\pi}{6}} (a\sin(x) + b\cos(x))\,dx$$

$$= \left[-a\cos(x) + b\sin(x)\right]_0^{\frac{\pi}{6}}$$

$$= \left[-a\cos\left(\frac{\pi}{6}\right) + b\sin\left(\frac{\pi}{6}\right)\right]$$
$$-\left[-a\cos(0) + b\sin(0)\right]$$

$$= \left(-\frac{a\sqrt{3}}{2} + \frac{b}{2}\right) - (-a) = \frac{\left(2-\sqrt{3}\right)a + b}{2}$$

The correct answer is **C**.

Question 5

Given $\int_0^5 g(x)dx = 20$

$$\int_0^5 (2g(x) + ax)\,dx = 90$$
$$2\int_0^5 g(x)dx + \int_0^5 ax\,dx = 90$$
$$2\times 20 + \left[\frac{1}{2}ax^2\right]_0^5 = 90$$
$$\frac{a}{2}(25-0) = 50$$
$$\Rightarrow a = 4$$

The correct answer is **B**.

Question 6

$$\int_4^5 \frac{2}{2x-1}dx = \left[\log_e(|2x-1|)\right]_4^5$$
$$= \left(\log_e(9) - \log_e(7)\right)$$
$$= \log_e\left(\frac{9}{7}\right)$$

So $b = \frac{9}{7}$.

Award 1 mark for the correct antiderivative.

Award 1 mark for the correct value of b.

VCAA Assessment Report note:

This was a routine application of $\int \left(\frac{a}{ax+b}\right)dx = \log_e(|ax+b|) + c$.

Question 7

$$\int_1^4 f(x)dx = 6$$

$$\int_1^4 (5 - 2f(x))\,dx = [5x]_1^4 - 2\int_1^4 f(x)dx$$
$$= 20 - 5 - 2 \times 6$$
$$= 3$$

The correct answer is **A**.

Question 8

$$\int_{-a}^0 x^4 dx = -\int_0^0 x^4 dx \qquad \textbf{[1 mark]}$$
$$= -\frac{a^5}{5} \qquad \textbf{[1 mark]}$$

Question 9

$$2\int_1^4 f(x)dx + \int_1^4 6dx = 2\int_1^4 f(x)dx + [6x]_1^4$$
$$= 2\int_1^4 f(x)dx + 18$$

The correct answer is **D**.

Question 10

$$2\int_0^{5a} f\left(\frac{x}{5}\right)dx + 2\int_0^{5a} 2dx = 2 \times 5a + 2\,[2x]_0^{5a} \quad \textbf{[1 mark]}$$
$$= 10a + 20a$$
$$= 30a \qquad \textbf{[1 mark]}$$

Question 11

$$4\int_0^{3k} g\left(\frac{x}{3}\right)dx - 4\int_0^{3k} 1dx = 4\left[3G\left(\frac{x}{3}\right)\right]_0^{3k} - 4\,[x]_0^{3k}$$

$$= 4\,[3\,(G\,(k) - G\,(0))] - 4\,[3k] \quad \textbf{[1 mark]}$$
$$= 4 \times 3 \times 2k - 12k$$
$$= 12k \quad \textbf{[1 mark]}$$

Question 12

$$2\int_1^3 f(x)dx - \int_1^3 3\,dx = 2 \times 4 + [3x]_1^3$$
$$= 8 - (9 - 3)$$
$$= 2$$

The correct answer is **B**.

Question 13

$$\int_5^1 h(x)dx - \int_5^1 2\,dx = -\int_1^5 h(x)dx - [2x]_5^1$$
$$= -4 - (2 - 10)$$
$$= 4$$

The correct answer is **C**.

Question 14

$$\int_3^6 f(x)dx + \int_2^3 f(x)dx + \int_2^3 2\,dx = \int_2^6 f(x)dx + \int_2^3 2\,dx \quad \textbf{[1 mark]}$$
$$= \int_2^6 f(x)dx + [2x]_2^3$$
$$= 14 + (6 - 4)$$
$$= 16 \quad \textbf{[1 mark]}$$

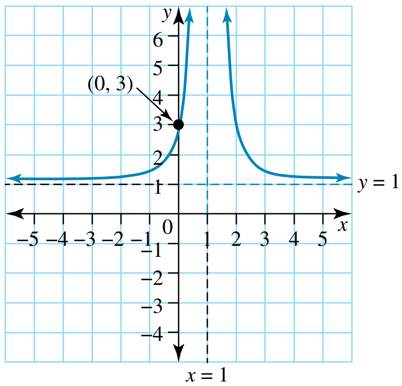
8.3 Areas under curves

Question 1

a. i. $f: R \backslash (1) \to R,\ f(x) = \dfrac{2}{(x-1)^2} + 1$

$$f(-1) = \frac{2}{(-2)^2} + 1 = \frac{3}{2} \quad \textbf{[1 mark]}$$

ii. Note that the graph must pass through the points $\left(-1, \frac{3}{2}\right)$, $(0, 3)$, $(2, 3)$, $\left(3, \frac{3}{2}\right)$.



Award 1 mark for the correct asymptotes.
Award 1 mark for the correct graph.
***VCAA Examination Report* note:**
Most students correctly recognised that a truncus shape was required and most of these students located it correctly. Again, curvature was an issue for some, with graphs 'turning away' from the asymptotes or crossing them. Students who made use of their answer to part **a**, and found a y-intercept were most successful in producing a correct graph. Those who did not recognise the truncus tended to draw a rectangular hyperbola with correct asymptotes.

b. $A = \int_{-1}^{0} \left(\frac{2}{(x-1)^2} + 1\right) dx$

$A = \left[\frac{-2}{(x-1)} + x\right]_{-1}^{0} = (2+0) - (1-1)$

$A = 2 \text{ units}^2$

Award 1 mark for the correct definite integral and anti-derivative.
Award 1 mark for the correct answer.
***VCAA Examination Report* note:**
Students were generally able to set up the correct definite integral, but often did not find the correct anti-derivative or evaluated it incorrectly. The most common errors were an anti-derivative that involved a log component or overlooking the +1 constant when anti-differentiating.

Question 2
Given that $f(-x) = f(x)$, the graph is symmetrical about the y-axis.

$\int_{a}^{b} f(x)dx = \int_{c}^{d} f(x)dx$, and $-\int_{b}^{0} f(x)dx = -\int_{0}^{c} f(x)dx$

$$A = 2\int_{a}^{b} f(x)dx - 2\int_{b}^{0} f(x)dx$$

$$= 2\int_{a}^{b} f(x)dx - 2\int_{b}^{b+c} f(x)dx$$

Since $b + c = 0$

The correct answer is **D**.

Question 3

$f: \left[0, \frac{\pi}{2}\right] \to R,\ f(x) = \cos(x),\ g: \left[0, \frac{\pi}{2}\right] \to R,$

$g(x) = \sqrt{3}\sin(x)$

To find the point of intersection of $f(x)$ and $g(x)$:

$f(x) = g(x)$

$\Rightarrow \cos(x) = \sqrt{3}\ \sin(x)$

$\Rightarrow \tan(x) = \frac{1}{\sqrt{3}}$

$\Rightarrow x = \frac{\pi}{6} \quad B\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

$$\text{Shaded area} = \int_0^{\frac{\pi}{6}} \sqrt{3}\ \sin(x)dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos(x)dx$$

$$= \sqrt{3} - \frac{3}{2} + \frac{1}{2}$$

$$= \sqrt{3} - 1$$

Area of the triangle $OAB = \frac{1}{2} \times \frac{\pi}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}\pi}{8}$

The ratio of areas is $\sqrt{3} - 1 : \frac{\sqrt{3}\pi}{8}$.

The correct answer is **B**.

Question 4

$y = x(x+2)(x-4)$

$A_1 = \int_{-2}^{0} y\,dx = \frac{20}{3}$ using CAS

$A_2 = \left|\int_0^4 y\,dx\right| = \left|-\frac{128}{3}\right|$

$A = \frac{20}{3} + \frac{128}{3} = \frac{148}{3}$

The correct answer is **D**.

Question 5

$y = \sqrt{x} \Rightarrow x = y^2$

Area with y-axis:

$$A = \int_a^b x dy$$

$$= \int_0^4 y^2 dy$$

$$= \frac{1}{3}\left[y^3\right]_0^4$$

$$= \frac{64}{3}$$

y	4	3	2	1
$\sqrt{x}$	16	9	4	1

$J = 1 \times (16 + 9 + 4 + 1) = 30$

$J - A = 30 - \frac{64}{3} = \frac{26}{3}$

The correct answer is **C**.

Question 6

Area with the y-axis, $A = \int_a^b (x_2 - x_1)\, dy$

$f(x) = \sqrt{x-1} \quad f(5) = 2$

$y = \sqrt{x-1} \quad \Rightarrow \quad y^2 = x - 1 \quad \Rightarrow \quad x_2 = y^2 + 1$

$x_1 = 1$

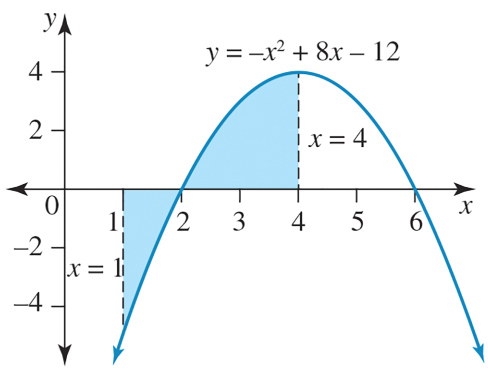
$$A = \int_0^2 \left(\left(y^2 + 1\right) - 1\right) dy = \int_0^2 y^2 dy = \int_0^2 x^2 dx$$

(by dummy variable property)

The correct answer is **E**.

Question 7

$y = -x^2 + 8x + 12 = -(x - 6)(x - 2)$



$$A_1 = \int_2^4 \left(-x^2 + 8x - 12\right) dx$$

$$= \left[-\frac{1}{3}x^3 + 4x^2 - 12x\right]_2^4$$

$$= \left(-\frac{64}{3} + 64 - 48\right) - \left(-\frac{8}{3} + 16 - 24\right)$$

$$= 5\frac{1}{3}$$

$$A_2 = \int_1^2 \left(-x^2 + 8x - 12\right) dx$$
$$= \left[-\frac{1}{3}x^3 + 4x^2 - 12x\right]_1^2$$
$$= \left(-\frac{8}{3} + 16 - 24\right) - \left(-\frac{1}{3} + 4 - 12\right)$$
$$= -2\frac{1}{3}$$

Total area $A = A_1 + |A_2| = 5\frac{1}{3} + 2\frac{1}{3} = 7\frac{2}{3}$

The correct answer is **D**.

8.4 Areas between curves and average values

Question 1

Equation of left-hand graph:

$m = \frac{-3a}{2a} = -\frac{3}{2},\ c = -a$

$y = -\frac{3}{2}x - a$

Equation of right-hand graph:

$m = \frac{2a}{a} = 2,\ c = -a$

$y = 2x - a$

$$\text{Average value} = \frac{1}{a - (-2a)} \int_{-2a}^{a} f(x)dx$$
$$= \frac{1}{3a}\left(\int_{-2a}^{0} \left(-\frac{3}{2}x - a\right) dx + \int_{0}^{a} (2x - a)\, dx\right)$$
$$= \frac{a}{3}$$

The correct answer is **B**.

Question 2

$f: R \to R,\ f(x) = \cos\left(\frac{\pi x}{2}\right)$

$g: R \to R,\ f(x) = \sin(\pi x)$

$$A_1 = \int_0^{\frac{1}{3}} (f(x) - g(x))\ dx$$

$$A_2 = \int_{\frac{1}{3}}^{1} (g(x) - f(x))\ dx$$

$$A_3 = \int_1^{\frac{5}{3}} (f(x) - g(x))\ dx$$

$$A_4 = \int_{\frac{5}{3}}^{3} (g(x) - f(x))\ dx$$

$A = A_1 + A_2 + A_3 + A_4$ but $A_2 = A_3$, so $A = A_1 + 2A_2 + A_4$.

$$= \int_0^{\frac{1}{3}} (f(x) - g(x))\ dx + 2\int_{\frac{1}{3}}^{1} (g(x) - f(x))\ dx + \int_{\frac{5}{3}}^{3} (g(x) - f(x))\ dx$$

$$A = \int_0^{\frac{1}{3}} (f(x) - g(x))\ dx - 2\int_{\frac{1}{3}}^{1} (f(x) - g(x))\ dx - \int_{\frac{5}{3}}^{3} (f(x) - g(x))\ dx$$

The correct answer is **C**.

Question 3

a. $f(x) = x^2e^{kx}$

Differentiating using the product rule:

$f'(x) = 2xe^{kx} + x^2ke^{kx} = e^{kx}\left(kx^2 + 2x\right)$

$f'(x) = xe^{kx}(kx + 2)$

Award 1 mark for correct proof.

b. $f(x) = f'(x)$

$x^2e^{kx} = e^{kx}\left(kx^2 + 2x\right)$

Since $e^{kx} \neq 0$, $x^2 = kx^2 + 2x$ **[1 mark]**

$x^2(k - 1) + 2x = 0$

$x(x(k - 1) + 2) = 0$

When $k = 1$, there is only one solution, $x = 0$. **[1 mark]**

***VCAA Examination Report* note:**

Many students found this question challenging. Most students found the correct quadratic equation to solve but solved for k , rather than the x value that satisfied the quadratic equation. Few students realised that $x = 0$ was the unique solution. Incorrect use of the null factor law and/or the incorrect discriminant of the quadratic were the main sources of error.

c. $g(x) = -\dfrac{2xe^{kx}}{k}$

$$A = \int_0^2 (f(x) - g(x))\ dx$$

$$A = \int_0^2 \left(x^2e^{kx} + \frac{2xe^{kx}}{k}\right)\ dx = \frac{1}{k}\int_0^2 \left(kx^2 + 2x\right)e^{kx}dx$$ **[1 mark]**

***VCAA Examination Report* note:**

This question was attempted well, although students commonly left out the dx, or found the sum of the integral of $f(x)$ and $g(x)$.

d. From part **a**, $\frac{d}{dx}\left[x^2e^{kx}\right] = e^{kx}\left(kx^2 + 2x\right)$ so that $\int \left(kx^2 + 2x\right) e^{kx} dx = x^2e^{kx}$.

$$A = \frac{1}{k}\left[x^2e^{2k}\right]_0^2 = \frac{1}{k}\left(4e^{2k} - 0\right) \quad \textbf{[1 mark]}$$

$$= \frac{4e^{2k}}{k}$$

$$\frac{4e^{2k}}{k} = \frac{16}{k}$$

$$e^{2k} = 4$$

$$2k = \log_e(4) \qquad \textbf{[1 mark]}$$

$$k = \frac{1}{2}\log_e(4)$$

$$k = \log_e(2) \qquad \textbf{[1 mark]}$$

***VCAA Examination Report* note:**

While students could equate their answer to part **c**. to $\frac{16}{k}$, many students did not use their result from part **a**. Incorrect algebraic manipulation made progress difficult for some students.

Question 4

$$y = \cos\left(kx - \frac{\pi}{2}\right) = \sin(kx)$$

$$\bar{y} = \frac{1}{\pi - 0}\int_0^{\pi} \sin(x)\,dx = \frac{2}{\pi}$$

$$\bar{y} = \frac{1}{\pi - 0}\int_0^{\pi} \sin(kx)\,dx = \frac{2}{\pi}$$

$$k = \frac{1}{2}$$

The correct answer is **E**.

Question 5

Since the average value is zero, the areas are equal.

$$\frac{1}{2}p^2 = \frac{25}{8}$$

$$p^2 = \frac{25}{4}$$

$$\Rightarrow p = \frac{5}{2}, \text{ since } p > 0$$

The correct answer is **D**.

Question 6

$$A = 2 \times \left(\frac{1}{2} \times 5 \times 6\right) + 4 \times 10$$

$$= 70$$

$$= 10\bar{y}$$

$$\Rightarrow \bar{y} = 7$$

The correct answer is **D**.

Question 7

$g: R \to R,\ g(x) = (a - x)^2$

$$\frac{1}{1-(-1)}\int_{-1}^{1}(a-x)^2 dx = \frac{31}{12}$$

$$\frac{1}{2}\int_{-1}^{1}\left(a^2 - 2ax + x^2\right)dx = \frac{31}{12}$$

$$\left[a^2x - ax^2 + \frac{x^3}{3}\right]_{-1}^{1} = \frac{31}{6}$$

$$\left(a^2 - a + \frac{1}{3}\right) - \left(-a^2 - a - \frac{1}{3}\right) = \frac{31}{6}$$

$$2a^2 + \frac{2}{3} = \frac{31}{6}$$

$$2a^2 = \frac{31}{6} - \frac{2}{3} = \frac{27}{6}$$

$$a^2 = \frac{27}{12} = \frac{9\times 3}{4\times 3}$$

$$a = \pm\frac{3}{2}$$

Award 1 mark for the correct definite integral.
Award 1 mark for integration.
Award 1 mark for both final answers.

***VCAA Assessment Report* note:**

Most students attempted this question but far too many misunderstood 'average value' to be either 'average rate' or 'average of'. The correct anti-derivative of the bracketed form of $g(x)$ was less evident than for the expanded form, due to the necessary division by the coefficient of x.

The formula for the 'average value' was not on the formula sheet. Students should ensure that they have a good understanding of the concept that the average value of a function f over the interval $[a, b]$ is

$$\frac{1}{b-a}\int_{a}^{b} f(x)dx$$

Question 8

$$f(x) = \frac{1}{2} \Rightarrow 2e^{-\frac{x}{5}} = \frac{1}{2}$$

$$e^{-\frac{x}{5}} = \frac{1}{4}$$

$$x_T = -5\log_e\left(\frac{1}{4}\right) = 5\log_e(4)$$

$$S(0,2)\ T\left(5\log_e(4), \frac{1}{2}\right)\quad m(ST) = \frac{2-\frac{1}{2}}{0-5\log_e(4)}$$

$$= -\frac{3}{10\log_e(4)}$$

The line ST is $y = g(x)$

$$= 2 - \frac{3x}{10\log_e(4)}$$

$$\text{Area} = \int_a^b (g(x) - f(x))\,dx$$

$$= \int_0^{5\log_e(4)} \left(2 - \frac{3x}{10\log_e(4)} - 2e^{-\frac{x}{5}}\right) dx$$

$$= \left[2x - \frac{3x^2}{20\log_e(4)} + 10e^{-\frac{x}{5}}\right]_0^{5\log_e(4)}$$

$$= \left(10\log_e(4) - \frac{3 \times 25\log_e(4)}{20} + 10 \times \frac{1}{4}\right) - (0 - 0 + 10)$$

$$= \frac{25}{4}\log_e(2) - \frac{15}{4}$$

$$= \frac{25}{2}\log_e(2) - \frac{15}{2}$$

Award 1 mark for the correct equation of the line ST (or equivalent methods) and setting up a definite integral involving the area.
Award 1 mark for the correct integration.
Award 1 mark for the correct area.

***VCAA Assessment Report* note:**

Many students made a start with this question, yet most had difficulty with the substitutions required to continue. Students who obtained two marks neglected to take into account the rectangle below the $y = 0.5$ line.

Question 9

$$\bar{y} = \frac{1}{b-a}\int_a^b y(x)dx$$

$$= \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} 3\sin^2(2x)dx$$

$$= \frac{6}{\pi}\int_0^{\frac{\pi}{2}} \sin^2(2x)dx$$

$$= \frac{6}{\pi} \times \frac{\pi}{4} = 1.5$$

The correct answer is **C**.

Question 10

$$\bar{y} = \frac{1}{b-a}\int_a^b y(x)\,dx$$
$$= \frac{1}{4-0}\int_0^4 \left(\sqrt{4x+9}\right) dx$$
$$= \frac{1}{4}\left[\frac{1}{4}\times\frac{2}{3}(4x+9)^{\frac{3}{2}}\right]_0^4 = \frac{1}{24}\left(25^{\frac{3}{2}} - 9^{\frac{3}{2}}\right)$$
$$= \frac{1}{24}(125-27)$$
$$= 4\frac{1}{12}$$

The correct answer is **B**.

Question 11

Top area minus bottom area: $A = \int_0^d f(x)dx - \int_a^c g(x)dx$

The correct answer is **E**.

Question 12

when $x = 2$, $y = 8$

Point of intersection of curves is $(2, 8)$. **[1 mark]**

$$\text{Area} = (2\times 8) - \int_0^2 x^3\,dx$$ [**1 mark**]

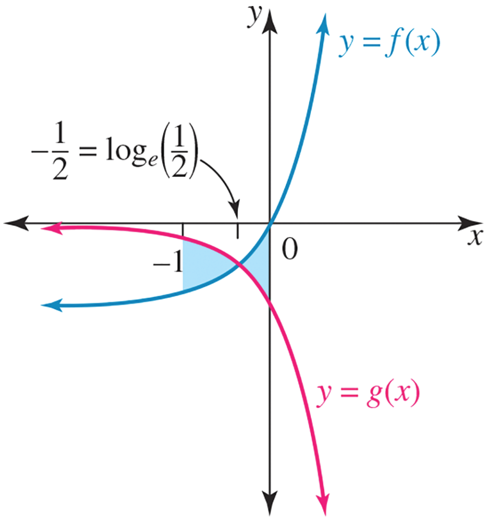
$$= 16 - \left[\frac{x^4}{4}\right]_0^2$$
$$= 16 - 4$$
$$= 12 \text{ units}^2$$ [**1 mark**]

Question 13

POI:

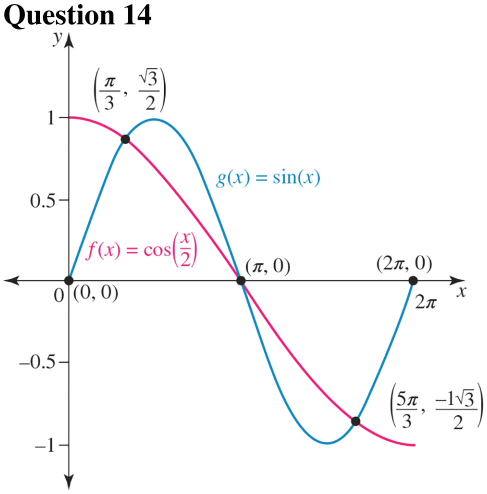
$$f(x) = g(x)$$
$$e^{2x} - 1 = -e^{2x}$$
$$2e^{2x} = 1$$
$$e^{2x} = \frac{1}{2}$$
$$2x = \log_e\left(\frac{1}{2}\right)$$
$$x = \frac{1}{2}\log_e\left(\frac{1}{2}\right)$$
$$= -\frac{1}{2}\log_e(2)$$

$$A = \int_{-1}^{-\frac{1}{2}\log_e(2)} \left(-2e^{2x} + 1\right) dx + \int_{-\frac{1}{2}\log_e(2)}^{0} \left(2e^{2x} - 1\right) dx$$



The correct answer is **B**.

Question 14



From the graph:

$y = \cos\left(\dfrac{x}{2}\right) = \sin(x)$ at $x = \dfrac{\pi}{3}$ and $x = \dfrac{5\pi}{3}$

Award 1 mark for finding correct x-values of points of intersection.

$$\text{Area} = \int_{\frac{\pi}{3}}^{\pi} \left(\sin(x) - \cos\left(\frac{x}{2}\right)\right) dx + \int_{\pi}^{\frac{5\pi}{3}} \left(\cos\left(\frac{x}{2}\right) - \sin(x)\right) dx \qquad \textbf{[1 mark]}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1 \text{ units}^2 \qquad \textbf{[1 mark]}$$

8.5 Applications

Question 1

a. $f_1(x): [0, 200] \to R, f_1(x) = 20\cos\left(\frac{\pi x}{100}\right) + 40$

$f_2(x): [0, 200] \to R, f_2(x) = 20\cos\left(\frac{\pi x}{100}\right) + 30$

$P(50, 30), f_1(50) = 40$

The swimmer needs to swim north 10 metres.

Award 1 mark for the correct answer.

b. Solving $f_1(x) = 30 \Rightarrow x = \frac{200}{3}, \frac{400}{3}$

So the distance is $\frac{200}{3} - 50 = \frac{50}{3}$ metres.

Award 1 mark for solving for x values.

Award 1 mark for the correct distance.

c. Distance from P to a point on $f_1(x)$:

$s(x) = \sqrt{(x-50)^2 + (f_1(x) - 30)^2}$

Solving $\frac{ds}{dx} = 0 \Rightarrow x = 54.477, f_1(54.477) = 8.47$

The minimum distance is 8.5 metres (to 1 d.p.).

Award 1 mark for solving the distance derivative to zero.

Award 1 mark for the correct minimum distance.

d. $A = \int_0^{200} (f_1(x) - f_2(x))\, dx = 2000\,\text{m}^2$

Award 1 mark for the correct area.

e. The required area is

$$N = 30 \times (150 - 50) - \int_{\frac{200}{3}}^{\frac{400}{3}} (30 - f_1(x))\, dx - \int_{50}^{150} f_2(x)\, dx$$

$= 837\,\text{m}^2$

Award 1 mark for each correct definite integral.

Award 1 mark for the correct area.

f. $s(x) = kf_1(x) - f_2(x) < 20$

$s(x) = 20(k-1)\cos\left(\frac{\pi x}{100}\right) + 40k - 30$

The maximum separation occurs when $\cos\left(\frac{\pi x}{100}\right) = 1$ or $x = 0, 200$.

This gives $60k = 70$.

Given that $k \geq 1$, $k \in \left[1, \frac{7}{6}\right)$.

Award 1 mark for solving.

Award 1 mark for the correct interval.

Question 2

a. $f(t) = \sin\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{6}\right), \quad t \geq 0$

The period is 12, since $f(t + 12) = f(t)$ **[1 mark]**

VCAA Examination Report **note:**
This question was answered well. Common incorrect answers were 6, 18 and $12t$.

b. $f(t) = 0 \ t \in [0.6] \Rightarrow \ t = 0, 4, 6$ **[1 mark]**
VCAA Examination Report **note:**
This question was answered well. Some students only gave two values, either 0, 4 or 4, 6.

c. Solving $f'(t) = \frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right) + \frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right) = 0, \ t \in (0, 4)$
$t = 1.8875, \ f(1.8875) = 1.76$ **[1 mark]**
VCAA Examination Report **note:**
This question was answered well. Common incorrect answers were 1.73 and 1.79.

d. $A = \int_0^4 f(t)\,dt - \int_6^4 f(t)\,dt = \frac{15}{\pi}$
Award 1 mark for correct integrals.
Award 1 mark for the correct equation.
VCAA Examination Report **note:**
The most common incorrect answer was $\int_0^4 f(t)dt + \int_4^6 f(t)dt = \frac{12}{\pi}$ or $\int_0^6 f(t)dt = \frac{12}{\pi}$.

e. $12k = 2 \times \frac{15}{\pi}$
$12k = \frac{30}{\pi}, \ k = \frac{5}{2\pi}$
Award 1 mark for correct working.
Award 1 mark for the correct value of k.
VCAA Examination Report **note:**
Many students did not double their answer from Question 3d., giving $12k = \frac{15}{\pi}, k = \frac{5}{4\pi}$.
Other students had the correct method but wrote their final answer as $k = \frac{5\pi}{2}$.

Question 3

a. $y = f(x) = 60 - \frac{3}{80}x^2$

$$\frac{dy}{dx} = \frac{6}{80}x$$

$$\left.\frac{dy}{dx}\right|_{x=-40} = -\frac{6}{80} \times -40$$
$$= 3$$
$$= \tan(\theta)$$
$$\theta = \tan^{-1}(3)$$
$$= 71.56°$$
$$= 72°$$

Award 1 mark for the gradient at the point.
Award 1 mark for the correct angle.
VCAA Assessment Report **note:**
This question was not answered well. A common incorrect response was 71°. Some students did not convert their answer to degrees. Others gave the answer as 56°, using $\tan(\theta) = \frac{ON}{OA} = \frac{60}{40}$. Some found $m = 3$ but were unable to find the angle.

b. $y = g(x) = \frac{x^3}{25\,600} - \frac{3x}{16} + 35$

$p(x) = \frac{dy}{dx} = \frac{3x^2}{25\,600} - \frac{3}{16}$

$\frac{dp}{dx} = \frac{6x}{25\,600} = 0 \Rightarrow x = 0$

$p(0) = -\frac{3}{16}$

The maximum slope is $-\frac{3}{16}$. That is, $\frac{3}{16}$ downwards.

Award 1 mark for the correct derivative.

Award 1 mark for the correct maximum slope.

VCAA Assessment Report **note:**

Some students did not interpret the question correctly and found the gradient of the straight line passing through X and Y. Some solved $h'(x) = 0$ for x.

c. $s(x) = f(x) - g(x)$

$$= \left(60 - \frac{3}{80}x^2\right) - \left(\frac{x^3}{25\,600} - \frac{3x}{16} + 35\right)$$

$\frac{ds}{dx} = 0 \quad \Rightarrow x = u = 2.490\,31$

$g(u) = 34.5337$

$M\,(2.49,\ 34.53)$

Award 1 mark for setting up the difference of functions.

Award 1 mark for solving the derivative equal to zero and obtaining the correct value of u.

Award 1 mark for the correct coordinates.

VCAA Assessment Report **note:**

Some students did not give their answers correct to two decimal places. Some worked to one decimal place and others rounded their answers incorrectly. Others did not set up the distance formula correctly or did not use brackets correctly in the distance formula. Some substituted u into g instead of h. A common incorrect response was $v = 24.53$.

d. $w = g(-u) = 35.4663 = 35.47$ [**1 mark**]

$d(MN) = f(u) - g(u)$

$= 25.23$ [**1 mark**]

$d\,(PQ) = f(-u) - g(-u)$

$= 24.30$ [**1 mark**]

VCAA Assessment Report **note:**

Some students did not work to the required number of decimal places or rounded incorrectly. Others had $PQ = 25.23$ and $MN = 24.30$.

e. $f(x) = 60 - \frac{3}{80}x^2$

$g(x) = \frac{x^3}{25\,600} - \frac{3x}{16} + 35$

Solving $f(x) = g(x)$

$60 - \frac{3}{80}x^2 = \frac{x^3}{25\,600} - \frac{3x}{16} + 35$ [**1 mark**]

$x_e = -23.71$ [**1 mark**]

$x_f = 28.00$ [**1 mark**]

***VCAA Assessment Report* note:**
Some students did not work to the required number of decimal places. Others rounded to 27.00 instead of 28.00. Some students gave answers without showing any working.

f. $A = \int_{xe}^{xf} (f(x) - g(x))\, dx$

$= \int_{-23.7}^{28} \left(60 - \frac{3}{80}x^2\right) - \left(\frac{x^3}{25600} - \frac{3x}{16} + 35\right) dx$

$= 869.62$

$= 870\,\text{m}^2$ **[1 mark]**

***VCAA Assessment Report* note:**
Some students rounded incorrectly to 869 or did not work to the required number of decimal places. Some gave their answers in exact form.

8.6 Review

Question 1

$3\int_0^a f(x)dx + \int_0^a 2dx = 3k + 2a$

The correct answer is **A**.

Question 2

a. $y = x\log_e(3x)$ using product rule

$\frac{dy}{dx} = \log_e(3x)\frac{d}{dx}(x) + x\frac{d}{dx}\left(\log_e(3x)\right)$

$\frac{dy}{dx} = \log_e(3x) + x \times \frac{1}{x}$

$\frac{dy}{dx} = \log_e(3x) + 1$

Award 1 mark for using the product rule.
Award 1 mark for the correct result.

***VCAA Examination Report* note:**
Most students used the product rule; however, many erred with the derivative of $\log_e(3x)$
Common incorrect answers were $\log_e(3x) + 3$ and $\log_e(3x) + \frac{1}{3}$.

b. $\int_1^{-2} \left(\log_e(3x) + 1\right) dx$

$= \left[x\log_e(3x)\right]_1^{\,2}$

$= \left(2\log_e(6) - \log_e(3)\right)$

$= \log_e(36) - \log_e(3)$

$= \log_e(12)$

Award 1 mark for the correct integration by recognition.
Award 1 mark for the correct result.

***VCAA Examination Report* note:**
Students generally were not able to form an integral from their previous answer, ignoring the 'hence' instruction. Some students attempted to integrate the given expression. Some poor application of log laws and/or log notation was observed.

Question 3

a. $f: R\backslash\{1\} \to R,\ f(x) = 2 + \dfrac{3}{x-1}$

Crosses the x-axis at $y = 0 \Rightarrow f(0) = 2 - 3 = -1 \quad (0, -1)$

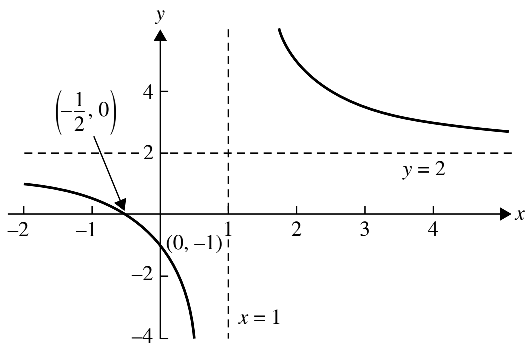
Crosses the y-axis at $y = 0 \Rightarrow x - 1 = -\dfrac{3}{2} \Rightarrow x = -\dfrac{1}{2} \quad \left(-\dfrac{1}{2}, 0\right)$

$x = 1$ is the vertical asymptote; $y = 2$ is a horizontal asymptote.

Award 1 mark for the correct shape.

Award 1 mark for the correct axial intercepts.

Award 1 mark for both correct asymptotes.



***VCAA Assessment Report* note:**

Some well-constructed graphs were presented by students. The highest-scoring graphs were clearly labelled with the correct points/equations, as specified by the question, and with care taken in showing the asymptotic behaviour nature of the curve as it approached an asymptote. Using a dashed line to represent an asymptote indicated that the curve was distinct from its asymptote.

b. $A = \displaystyle\int_2^4 \left(2 + \frac{3}{x-1}\right)$

$A = \left[2x + 3\log_e(x-1)\right]_2^4$

$= \left(8 + 3\log_e(3)\right) - (4 + \log_e(1))$

$= 4 + 3\log_e(3) = 4 + \log_e(27)$

Award 1 mark for correct integration.

Award 1 mark for the correct area value.

***VCAA Assessment Report* note:**

Students were able to identify the required integral; however, many then erred in the evaluation of the terminals. A common error occurring as a result of incorrectly applying logarithmic laws was a final answer of $4 + \log_e(9)$. Many students were unable to recognise that the antiderivative of the reciprocal of $(x - 1)$ involved a logarithmic function. The 'dx' was often omitted.

Question 4

$$\int_1^2 (f(x) + x)\ dx = \int_1^2 f(x)\,dx + \left[\frac{1}{2}x^2\right]_1^2$$

$$= \int_1^4 f(x)\,dx - \int_2^4 f(x)\,dx + \left(2 - \frac{1}{2}\right)$$

$$= \int_1^4 f(x)\,dx - \int_2^4 f(x)\,dx + \frac{3}{2}$$

$$= 4 + 2 + \frac{3}{2} = \frac{15}{2}$$

The correct answer is **E**.

Question 5

Graph C is the only graph that has an average value of 2 over $[0, 6]$; that is, if a line $y = 2$ is drawn, the area bound by the graph and the line is equal above and below the line.

That is, $\frac{1}{6-0}\int_0^6 h(x)\,dx = 2$.

The correct answer is **C**.

Question 6

a. The average rate of change of $f(x)$ over $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$.

$$\frac{f\left(\frac{\pi}{6}\right) - f\left(-\frac{\pi}{3}\right)}{\frac{\pi}{6} - \left(-\frac{\pi}{3}\right)} = \frac{\left(2\sin\left(\frac{\pi}{3}\right) - 1\right) - \left(2\sin\left(\frac{-2\pi}{3}\right) - 1\right)}{\frac{\pi}{6} + \frac{\pi}{3}}$$

$$= \frac{2\sin\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{-2\pi}{3}\right)}{\frac{3\pi}{6}} = \frac{2 \times \frac{\sqrt{3}}{2} - 2 \times \frac{-\sqrt{3}}{2}}{\frac{\pi}{2}}$$

$$= \frac{4\sqrt{3}}{\pi}$$

Award 1 mark for using the correct rate of change.
Award 1 mark for the correct final answer.

***VCAA Assessment Report* note:**

Most students used the correct gradient rule but erred when evaluating, particularly $f\left(-\frac{\pi}{3}\right)$ or in dealing with fractions in the denominator. A few students confused average rate of change with average value, and some incorrectly found the average of derivatives.

b. The average value of a function $f(x)$ over $[a, b]$ is $\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$

$$\bar{f} = \frac{1}{\frac{\pi}{6} - \left(-\frac{\pi}{3}\right)} \int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} (2\sin(2x) + 1)\,dx$$

$$= \frac{1}{\frac{\pi}{2}}\left[-\cos(2x) - x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{6}} = \frac{2}{\pi}\left[\left(-\cos\left(\frac{\pi}{3}\right) - \frac{\pi}{6}\right) - \left(-\cos\left(-\frac{2\pi}{3}\right) + \frac{\pi}{3}\right)\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{2} - \frac{\pi}{6} - \frac{1}{2} - \frac{\pi}{3}\right] = \frac{2}{\pi}\left(-1 - \frac{\pi}{2}\right)$$

$$= \frac{-(\pi + 2)}{\pi}$$

Award 1 mark for using the correct average value.
Award 1 mark for correct integration and evaluation.
Award 1 mark for the correct final answer.

***VCAA Assessment Report* note:**

Most students used the correct expression for average volume. As with the previous part of the question, arithmetic errors, especially when substituting the terminals, caused difficulties. Some students omitted the -1 in the integrand, while others misplaced negative signs or the constant of 2.

Question 7

a. $f: [-1, 3] \rightarrow R, f(x) = 3x^2 - x^3$

For stationary points: $f'(x) = 6x - 3x^2 = 0$

$3x(2 - x) = 0$

$x = 0, x = 2$

$f(0) = 0, f(2) = 12 - 8 = 4$

The stationary points are $(0, 0)$ and $(2, 4)$.

Award 1 mark for solving the derivative equal to zero.

Award 1 mark for both correct coordinates.

***VCAA Assessment Report* note:**

Students must be vigilant in ensuring they answer the specific question. In this question, coordinates were required and not simply x values. Some students omitted the turning point $(0, 0)$ and others incorrectly found x-intercepts.

b. The curve crosses the x-axis when

$3x^2 - x^3 = 0$

$x^2(3 - x) = 0$

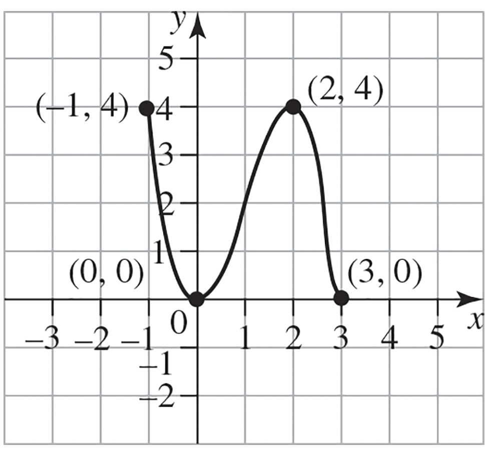
The x-intercepts are at $(0, 0)$ and $(3, 0)$.

Endpoints:

$f(-1) = 3 + 1$

$= 4$

The endpoints are at $(-1, 4)$ and $(3, 0)$.



Award 1 mark for the correct shape and turning points.

Award 1 mark for the correct endpoints and graph on the restricted domain.

***VCAA Assessment Report* note:**

The correct response to this question required a smooth curve (not V shapes at turning points) and endpoints clearly labelled with their coordinates.

c. $A = \int_a^b (y_2 - y_1)\,dx$

$y_2 = 4, y_1 = 3x^2 - x^3, a = -1, b = 2$

$A = \int_{-1}^{2} \left(4 - \left(3x^2 - x^3\right)\right) dx$

$= \int_{-1}^{2} \left(4 - 3x^2 + x^3\right) dx$

$= \left[4x - x^3 + \frac{1}{4}x^4\right]_{-1}^{2}$

$= (8 - 8 + 4) - \left(-4 + 1 + \frac{1}{4}\right)$

$= \frac{27}{4} = 6\frac{3}{4} = 6.75$

Award 1 mark for the correct definite integral.
Award 1 mark for the correct antiderivative and evaluation.
Award 1 mark for the correct final value of the area.

VCAA Assessment Report **note:**

Most students knew to seek a difference of two areas and were adept with basic integration; however, quite often arithmetic errors in evaluations or the incorrect use of negative signs marred their progress towards acquiring full marks. Some students unnecessarily 'overworked' the problem by creating three or four integrations, increasing the likelihood of an error. A few students took a more direct route that involved symmetry of the curve.

Question 8

a. $V = \pi r^2 h, \qquad r = \frac{d}{2}$

$216 = \pi\left(\frac{d}{2}\right)^2 h$

$h = \frac{4 \times 216}{\pi d^2}$

$h = \frac{864}{\pi d^2}$

Award 1 mark for the correct volume.
Award 1 mark for expressing h in terms of d.

VCAA Assessment Report **note:**

This question was quite well answered. Some students used the formula for the volume of a cone instead of a cylinder. Some used poor notation, omitting brackets and writing $\frac{d^2}{2} = \frac{d^2}{4}$. Many left their answer in the form $h = \frac{216}{\pi\left(\frac{d}{2}\right)^2}$, which was accepted; however, it is preferable to write in simplified form.

b. $S = \frac{\pi d^2}{4} + \pi h d$

$S = \frac{\pi d^2}{4} + \pi d\left(\frac{864}{\pi d^2}\right)$

$S = \frac{\pi d^2}{4} + \frac{864}{d}$

Award 1 mark for correctly showing the surface area.

VCAA Assessment Report **note:**

This was a 'show that' question and some students showed sufficient working. Some included the area of the base of the cylinder. Others did not include the area of the top of the cylinder and only considered the curved surface area.

c. $\frac{dS}{dd} = \frac{\pi d}{2} - \frac{864}{d^2}$

$\frac{dS}{dd} = 0$ for maximum or minimum

$\frac{\pi d}{2} - \frac{864}{d^2} = 0$

$d^3 = \frac{2 \times 864}{\pi}$ since $d > 0$

$d = \sqrt[3]{\frac{2 \times 864}{\pi}}$

$d = \frac{12}{\sqrt[3]{\pi}}$

$S_{\min} = s\left(\frac{12}{\sqrt[3]{\pi}}\right)$

$= 108\sqrt[3]{\pi}$

Award 1 mark for finding the correct value of d.

Award 1 mark for the correct minimum value of S.

***VCAA Assessment Report* note:**

Some students answered only part of the question, finding the correct value for d but not attempting to find S. Exact answers were required. Answers such as $d = 8.19 \ldots$ and $S = 158.17 \ldots$ were often given.

d. $h = \frac{864}{\pi d^2}$

$= \frac{864}{\pi}\left(\frac{\sqrt[3]{\pi}}{12}\right)^2$

$= \frac{6}{\sqrt[3]{\pi}}$

Award 1 mark for the correct value of h.

***VCAA Assessment Report* note:**

Some students did not square $\frac{12}{\sqrt[3]{\pi}}$, using $h = \frac{864}{\pi\left(\frac{12}{\sqrt[3]{\pi}}\right)}$ to get $\frac{72}{\pi^{\frac{2}{3}}}$. An exact answer was required, not a decimal expression such as 4.09, as was given by some students. Some substituted $d = \frac{12}{\sqrt[3]{\pi}}$ into $S = \frac{\pi d^2}{4} + \frac{864}{d}$.

e. $V = \pi r^2 h$

$V = \pi\left(\frac{d}{2}\right)^2 h$

$V = \pi\left(\frac{2h}{2}\right)^2 h$ since $d = 2h$

$V = \pi h^3$

Award 1 mark for the correct expression.

***VCAA Assessment Report* note:**
Some students used an incorrect formula, such as
$V = \pi(2h)^2 h = 4\pi h^3$ or $V = 2\pi rh = 2\pi h^2$.

f. Given $\frac{dV}{dt} = -10, \quad \frac{dV}{dh} = 3\pi h^2$

$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$

$= -\frac{10}{3\pi h^2}$

Award 1 mark for the correct given rate and differentiation.
Award 1 mark for correctly using the chain rule.
Award 1 mark for the final correct rate.

***VCAA Assessment Report* note:**
Many students were able to set up the related rates equation and find $\frac{dV}{dh}$. Some students did not find the reciprocal before substituting into $\frac{dh}{dV}$. Many used $\frac{dV}{dt} = 10\,\text{m}^3/\text{year}$.

g. $\left.\frac{dh}{dt}\right|_{h-1} = -\frac{10}{3\pi}$

The height will be decreasing at a rate of $\frac{10}{3\pi}$ m/year.
Award 1 mark for evaluating the rate when $h = 1$.

***VCAA Assessment Report* note:**
Students who answered Question 2f correctly tended to also answer this question correctly. An exact answer was required. Some students gave incorrect units.

h. $\frac{dh}{dt} = -\frac{10}{3\pi h^2}$

$\int 3\pi h^2 dh = -10t + c$

$\pi h^3 = -10t + c$

when $t = 0, \frac{6}{\sqrt[3]{\pi}}$

$\Rightarrow V = \pi h^3 = 216 \Rightarrow c = 216$

$\pi h^3 = -10t + 216$

when $h = 1$

$t = \frac{216 - \pi}{10}$

$= 21.29$

The top of the statue will just be exposed in the year 2031.
Award 1 mark for finding the value of t.
Award 1 mark for the final correct year.

***VCAA Assessment Report* note:**
This question was not answered well. A number of different approaches could have been used. Some students gave 2032 as their final answer. Some did not subtract π from 216 and used $t = \frac{216}{10.}$. Incorrect terminals were often used or if evaluating $t = \int \left(-\frac{3\pi h^2}{10}\right) dh$, a constant of integration was often missing. Some used $t = \int \left(-\frac{10}{3\pi h^2}\right) dh$.

Question 9

a. $\frac{dy}{dx} = \sqrt{x+6} - \frac{x}{2} - \frac{3}{2} \quad \left(3, \frac{29}{4}\right)$

$y = \int \left(\sqrt{x+6} - \frac{x}{2} - \frac{3}{2}\right) dx = \frac{2}{3}(x+6)^{\frac{3}{2}} - \frac{x^2}{4} - \frac{3x}{2} + c$

$\frac{29}{4} = \frac{2}{3} \times 9^{\frac{3}{2}} - \frac{9}{4} - \frac{3}{2} + c, \quad c = \frac{29}{4} - 18 + \frac{9}{4} + \frac{9}{2} = -4$

$y = \frac{2}{3}(x+6)^{\frac{3}{2}} - \frac{x^2}{4} - \frac{3x}{2} - 4$

Award 1 mark for the correct integration.
Award 1 mark for the correct integrals.
Award 1 mark for the correct c value.

b. when $x = 4, \quad \frac{dy}{dx} = \sqrt{10} - 2 - \frac{3}{2} < 0$ and when

$x = 2, \quad \frac{dy}{dx} = \sqrt{8} - 1 - \frac{3}{2} > 0$

Since the graph has a single stationary point when $\left(3, \frac{29}{4}\right)$

This point is an absolute maximum turning point.
Award 1 mark for using the sign test.
Award 1 mark for identifying the correct nature of the stationary point.

Question 10

a. $g: R \to R, g(x) = x^3 - kx$

$g(-1) = -1 + k, \ g(1) = 1 - k, \ C(-1, -1 + k), \ D(1, 1 - k)$

$d(CD) = \sqrt{(1 - {}^{-}1)^2 + ((1-k) - (-1+k))^2}$

$= \sqrt{4 + 4 - 8k + 4k^2}$

$= 2\sqrt{k^2 - 2k + 2}$

Award 1 mark for the correct coordinates.
Award 1 mark for the correct distance.

***VCAA Examination Report* note:**

As in Question 1bii. some students did their solutions by hand and made arithmetic errors, especially sign errors. This would have been time consuming. $\sqrt{2^2 + (2-2k)^2} = 2 + 2 - 2k$ was sometimes given. Some incorrect answers contained x. When defining $g(x) = x^3 - kx$ on the technology, a multiplication sign must be inserted between k and x.

b. **i.** Solving $2\sqrt{k^2 - 2k + 2} = k + 1$, gives $k = 1$ or $k = \frac{7}{3}$

Award 1 mark for both correct values of k.

***VCAA Examination Report* note:**

Students who answered Question c.i. correctly were generally able to answer this question. Some students gave only one value for k.

ii. Solving $g(x) = x^3 - kx = x$ gives $x = a$

$g(a) = a^3 - ka = a$

$a^3 - ka - a = 0$

$a^3 - (k+1)a = 0$

$a\left[a^2 - (k+1)\right] = 0$

$a = \sqrt{k+1}$ since $a > 0$ **[1 mark]**

***VCAA Examination Report* note:**

A common incorrect answer was $a = \pm\sqrt{k+1}$. By inspection of the graph, the answer was positive. Some students found k in terms of a, instead of a in terms of k.

iii. $A = \int_0^{\sqrt{k+1}} (x - g(x))\,dx$

$= \int_0^{\sqrt{k+1}} \left(x - x^3 + kx\right) dx$ **[1 mark]**

To solve using CAS, complete the entry line as:

$\int_0^{\sqrt{k+1}} \left(x - x^3 + k \times x\right) dx$

$A = \frac{(k+1)^2}{4}$ **[1 mark]**

***VCAA Examination Report* note:**

$\int_0^{\sqrt{k+1}} \left(x - x^3 + kx\right) dx$ was a common error, leaving out the brackets in $\int_0^{\sqrt{k+1}} \left(x - x^3 + kx\right) dx$. To avoid these errors it would have been better to use the expression $\int_0^{\sqrt{k+1}} (x - g(x))\,dx$. Some students overcomplicated the question by breaking up the areas into different sections. The easiest approach was to use 'upper function subtract lower function'. There was evidence that students substituted $k + 1$ instead of $\sqrt{k+1}$ and this resulted in the answer of $\frac{-(k-3)(k+1)}{4}$.

Question 11

a. $f: R \to R, f(x) = 3x^4 + 4x^3 - 12x^2$

$f'(x) = 12x^3 + 12x^2 - 24x$

$f'(x) = 12x(x+2)(x-1)$

$f'(x) = 0 \Rightarrow x = -2$

$f(-2) = -32$

Minimum $M(-2, -32)$ **[1 mark]**

***VCAA Examination Report* note:**

This question was answered well. Some students only gave the x value when coordinates were required.

b. $b > 32$ **[1 mark]**

***VCAA Examination Report* note:**

This question was answered well. Common incorrect answers were $(-\infty, 32)$, $b = 32, b \geq 32, [33, \infty]$ and $b = 33$. Others used the x-coordinate and gave $x > 2$ as their answer.

c. $P\left(-\frac{1}{3}, -\frac{13}{9}\right)$

$m_T = f'\left(-\frac{1}{3}\right) = \frac{80}{9}$

Tangent:

$y + \frac{13}{9} = \frac{80}{9}\left(x + \frac{1}{3}\right)$

$y = l(x) = \frac{80x}{9} + \frac{41}{27}$ **[1 mark]**

***VCAA Examination Report* note:**

An equation and exact values were required.

d. Solving $l(x) = f(x)$:

$\frac{80x}{9} + \frac{41}{27} = 3x^4 + 4x^3 - 12x^2$ **[1 mark]**

$x = \frac{-1 \pm \sqrt{42}}{3}, x = -\frac{1}{3}$ **[1 mark]**

***VCAA Examination Report* note:**

Exact values were required. There were many sign errors, for example $x = \frac{1 \pm \sqrt{42}}{3}$. Some students found the values of x where the gradient of l was equal to the gradient of f.

e. $x_1 = \frac{-1 - \sqrt{42}}{3}, x_2 = \frac{-1 + \sqrt{42}}{3}$

$A = \int_{x_1}^{-\frac{1}{3}} (l(x) - f(x))\, dx + \int_{-\frac{1}{3}}^{x_2} (l(x) - f(x))\, dx$ **[1 mark]**

$A = \int_{x_1}^{x_2} \left(\frac{80x}{9} + \frac{41}{27} - \left(3x^4 + 4x^3 - 12x^2\right) \right) dx$

$A = \frac{784\sqrt{42}}{135}$ **[1 mark]**

***VCAA Examination Report* note:**

Students who answered Question 1d. correctly were generally able to answer this question correctly. Some students split the integral, which was unnecessary. Others put a negative sign in front of the integral for the bounded area below the x-axis. Some had their terminals or expressions the reverse of what was required.

f. For $f(x) = p(x)$ we require the following equations to hold:

$6(a - 2)x^2 = -12x^2$

$-12ax = 0$

$a^2 = 0$

The unique solution for this system of equations is $a = 0$.

$f(x) = p(x)$ when $a = 0$. [1 mark]

***VCAA Examination Report* note:**

Some students gave an additional expression $a = -6x(x - 2)$, which was obtained if technology was used rather than equating coefficients.

g. $p'(x) = 12x^3 + 12x^2 + 12(a - 2)x - 12a$

$p'(x) = 12(x - 1)\left(x^2 + 2x + a\right)$

$p'(x) = 0 \Rightarrow x = 1, x = -1 \pm \sqrt{1 - a}$ **[1 mark]**

***VCAA Examination Report* note:**

A common error was $x = 1 \pm \sqrt{1 - a}$. $x = \frac{-1 \pm \sqrt{9 - 4a}}{2}, x = 0$ was often given. This comes from forgetting to differentiate $-12ax$ when differentiating $p(x)$.

h. i. There is only one stationary point at $x = 1$ when $a > 1$. **[1 mark]**

***VCAA Examination Report* note:**

This question was not answered well. Common incorrect answers were $a = 1$, $a = 0$ or $a > 0$

ii. when $a = 2$, $p(1) = -13$.**[1 mark]**

***VCAA Examination Report* note:**

This question was answered well. The minimum value needed to be stated, not just the coordinates of the turning point.

iii. There is only one stationary point at $x = 1$ when $a > 1$,

so $p(1) = a^2 - 6a - 5 > 0$. **[1 mark]**

For no solutions, solving $a^2 - 6a - 5 > 0$ with $a > 1$
gives $a > 3 + \sqrt{14}$. **[1 mark]**
***VCAA Examination Report* note:**
This question was not answered well. Many students did not attempt this question. Some students solved $p(x) = 0$ or $p'(x) = 0$ for x. Others tried to apply the discriminant to a cubic equation. Others, who used a correct method, sometimes gave an incorrect inequality, for example $a < \sqrt{14} + 3$.

Question 12

a. $y = x\log_e(3x)$ using product rule

$$\frac{dy}{dx} = \log_e(3x)\frac{d}{dx}(x) + x\frac{d}{dx}\left(\log_e(3x)\right)$$

$$\frac{dy}{dx} = \log_e(3x) + x \times \frac{1}{x}$$

$$\frac{dy}{dx} = \log_e(3x) + 1$$

Award 1 mark for using the product rule.
Award 1 mark for the correct result.
***VCAA Examination Report* note:**
Most students used the product rule; however, many erred with the derivative of $\log_e(3x)$.
Common incorrect answers were $\log_e(3x) + 3$ and $\log_e(3x) + \frac{1}{3}$.

b. $\int_1^2 \left(\log_e(3x) + 1\right) dx$

$= \left[x\log_e(3x)\right]_1^2$
$= \left(2\log_e(6) - \log_e(3)\right)$
$= \log_e(36) - \log_e(3)$
$= \log_e(12)$

Award 1 mark for the correct integration by recognition.
Award 1 mark for the correct result.
VCAA Examination Report note:
Students generally were not able to form an integral from their previous answer, ignoring the 'hence' instruction. Some students attempted to integrate the given expression. Some poor application of log laws and/or log notation was observed.

Question 13

a. i. $S(t) = 2e^{\frac{t}{3}} + 8e^{-\frac{2t}{3}},\ 0 \le t \le 5$

$S(0) = 2 + 8 = 10$

$S(5) = 2e^{\frac{5}{3}} + 8e^{-\frac{10}{3}}$ **[1 mark]**

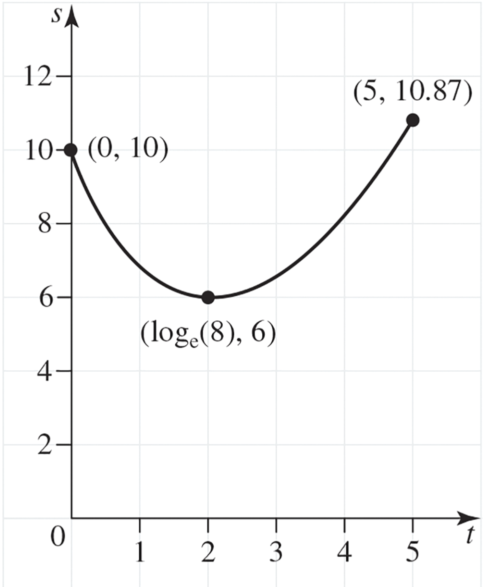
***VCAA Assessment Report* note:**
Some students gave an approximate answer for $S(5)$ when an exact answer was required.

ii. $\frac{ds}{dt} = \frac{2}{3}e^{\frac{t}{3}} - \frac{16}{3}e^{-\frac{2t}{3}} = 0$

Solving using CAS gives $t = 3\log_e(2) = \log_e(8)$
$\Rightarrow c = 8$ **[1 mark]**
$S_{\min} = S\left(\log_e(8)\right) = 6$ **[1 mark]**
***VCAA Assessment Report* note:**
Some students did not find the minimum value.

iii. Endpoints (0, 10), $\left(5,\ 2e^{\frac{5}{3}}+8e^{-\frac{10}{3}}\right)\approx(5,\ 10.87)$

Minimum $\left(\log_e(8),\ 6\right)\approx(2.08,\ 6)$



Award 1 mark for correct minimum and endpoints.
Award 1 mark for a correctly drawn graph.

***VCAA Assessment Report* note:**

Many students drew accurate graphs. Some had the coordinates the wrong way around and others did not put the coordinates on the graph. Some did not sketch the graph over the required domain.

iv. Average value: $\dfrac{f(b)-f(a)}{b-a}$ **[1 mark]**

$$\frac{s\left(\log_e(c)\right)-s(0)}{\log_e(c)-0}=\frac{6-10}{\log_e(8)}$$
$$=\frac{-4}{\log_e(8)}\quad \textbf{[1 mark]}$$

***VCAA Assessment Report* note:**

Some students worked out the average value of the function and not the average rate of change. Others left the negative sign off their answer, writing $\dfrac{4}{\ln 8}$. Some students gave $\dfrac{\log_e(8)-0}{6-10}$.

b. $V:[0.5]\to R,\ V(t)=de^{\frac{t}{3}}+(10-d)\,e^{-\frac{2t}{3}}$

Now $d\in(0,\ 10)$ $\dfrac{dV}{dt}=\dfrac{d}{3}e^{\frac{t}{3}}-\dfrac{2}{3}(10-d)\,e^{-\frac{2t}{3}}$

Now $\dfrac{dV}{dt}=0$ when $t=\log_e(9)$. **[1 mark]**

Solving for d, gives $d=\dfrac{20}{11}$. **[1 mark]**

***VCAA Assessment Report* note:**

This question was answered reasonably well. Some students tried to solve the equation by hand but were unsuccessful.

c. i. If the minimum value occurs at $t=0$ then $d=\dfrac{20}{3}$.

This could be the endpoint of the function or the turning point occurs outside the domain [0, 5]. **[1 mark]**

When $t < 0$:

Using CAS, when $t = -1$, $\frac{dv}{dt} = 0 \Rightarrow d = 8.4 > \frac{20}{3}$

so $\frac{20}{3} \leq d < 10$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students found $d = \frac{20}{3}$ but did not consider the set of values for d. Some had the inequality as $\frac{20}{3} \leq d \leq 10$ or $\frac{20}{3} < d < 10$.

ii. If the minimum value occurs at $t = 5$

This could be the endpoint of the function or the turning point occurs outside the domain $[0, 5]$.

[1 mark]

When $t > 5$:

Using CAS, when $t = 6$ $\frac{dv}{dt} = 0 \Rightarrow d = 0.05 < \frac{20}{2 + e^5}$

so $0 < d \leq \frac{20}{2 + e^5}$

[1 mark]

***VCAA Assessment Report* note:**

Many students found $d = \frac{20}{2 + e^5}$, but did not consider the set of values for d. Some had the inequality written incorrectly.

d. $V(t) = de^{\frac{t}{3}} + (10 - d)e^{-\frac{2t}{3}}$ has local minimum at (a, m), $0 \leq a \leq 5$

(1) $V(a) = m \Rightarrow m = de^{\frac{a}{3}} + (10 - d)e^{-\frac{2a}{3}}$

(2) $V'(a) = 0 \Rightarrow 0 = \frac{d}{3}e^{\frac{a}{3}} - \frac{2}{3}(10 - d)e^{-\frac{2a}{3}}$

$(1) - 3 \times (2) \quad \Rightarrow m = 3(10 - d)e$

$2 \times (1) + 3 \times (2) \Rightarrow 2m = 2de^{\frac{a}{3}}$

$e^{\frac{a}{3}} = \frac{2m}{3d} \Rightarrow e^{-\frac{a}{3}} = \frac{3d}{2m}$

$$\Rightarrow e^{-\frac{2a}{3}} = \left(\frac{3d}{2m}\right)^2 = \frac{9d^2}{4m^2} = \frac{m}{3(10 - d)}$$

$$m^3 = \frac{27d^2(10 - d)}{4}$$

$$m = \frac{3d^{\frac{2}{3}}(10 - d)^{\frac{1}{3}}}{4^{\frac{1}{3}}}$$

$$= \frac{k}{2}d^{\frac{2}{3}}(10 - d)^{\frac{1}{3}}$$

$$k = \frac{2 \times 3}{4^{\frac{1}{3}}} = 3 \times 2^{1-\frac{2}{3}}$$

$$k = 3\sqrt[3]{2}$$

Award 1 mark for eliminating a.

Award 1 mark for the correct value of k.

***VCAA Assessment Report* note:**

This question was not answered well. Some students were able to find $a = \log_e\left(\frac{20}{d} - 2\right)$ but did not substitute it into V. A common incorrect response was $k = -3 \times 2^{\frac{1}{3}}$

Question 14

a. i. $g: R \rightarrow R, g(x) = \dfrac{16 - x^2}{4}$

$B(4,0), C(0,4) \Rightarrow m_{BC} = -1$

$g'(x) = -\dfrac{x}{2}$

$g'(x_A) = -\dfrac{x_A}{2} = -1$

$\Rightarrow x_A = 2$

$A(2,3)$

Tangent at A:

$y - 3 = -1(x - 2)$

$y = -x + 5$

Award 1 mark for equating gradients.

Award 1 mark for the correct equation for the tangent.

ii. Area of triangle minus area under curve:

$$1 \times 5 \times 5 - \int_0^4 \frac{16 - x^2}{4} dx$$

$$= \frac{25}{2} - \frac{32}{3}$$

$$= \frac{11}{6}$$

Award 1 mark for the correct definite integral representing the area.

Award 1 mark for evaluation.

Award 1 mark for the final correct area.

***VCAA Assessment Report* note:**

There were many different approaches to this question. A common incorrect response was

$$\int_0^5 ((-x + 5) - g(x))\, dx = \frac{35}{12}.$$

An exact answer was required.

b. $Q(x, g(x)) \quad OQ = s(x) = \sqrt{x^2 + (g(x))^2}$

$s(x) = \dfrac{\sqrt{x^4 - 16x^2 + 256}}{4}$ for $0 < x < 4$

$\dfrac{ds}{dx} = 0 \Rightarrow 4x^3 - 32x = 0 \Rightarrow x = \sqrt{8} = 2\sqrt{2}$

$s_{\min} = s\left(2\sqrt{2}\right) = 2\sqrt{3}$

Award 1 mark for the correct distance.

Award 1 mark for setting the derivative equal to zero.

Award 1 mark for the correct x-value and minimum distance using CAS.

***VCAA Assessment Report* note:**

Some students found $x = 2\sqrt{2}$ but did not find the minimum distance. Exact answers were required.

c. $P(x_P, g(x_P))$

$\dfrac{dy}{dx} = g'(x) = -\dfrac{x}{2} \Rightarrow m_{x_P} = -\dfrac{x_P}{2}$

Tangent at P: $y - g(x_P) = -\dfrac{x_P}{2}(x - x_P)$ crosses the y-axis at $x = 0,\ y = k$

$k - \dfrac{(16 - x_P)^2}{4} = -\dfrac{x_P}{2}(0 - x_P) \Rightarrow -\dfrac{x_P}{2}(0 - x_P) \Rightarrow x_P = 2\sqrt{k - 4}$

So the gradient at P is $m_{x_P} = -\sqrt{k-4}$
Award 1 mark for attempting to solve an appropriate equation.
Award 1 mark for the correct gradient.

d. **i.** The tangent at P is $y = -x\sqrt{k-4} + k$.

It crosses the y-axis is at $(0,\ k)$ and crosses the x-axis at $\left(\dfrac{k}{\sqrt{k-4}}, 0\right)$.

The area of the triangle minus the area under the curve is:

$$A(k) = \frac{1}{2} \times k \times \frac{k}{\sqrt{k-4}} - \int_0^4 \frac{16 - x^2}{4} dx$$

$$A(k) = \frac{k^2}{2\sqrt{k-4}} - \frac{32}{3} \text{ for } 5 \le k \le 8$$

Award 1 mark for setting up the triangle area.
Award 1 mark for the correct function.

***VCAA Assessment Report* note:**
Some students used the equation of their tangent line to attempt to find $A(k)$.

ii. Examine the end points: $A(5) = \dfrac{11}{6} \approx 1.833 \qquad A(8) = \dfrac{16}{3} \approx 5.333$

The maximum area is $5\frac{1}{3}$. It occurs at the right-hand end point when $k = 8$.
Award 1 mark for investigating the end points of the restricted domain function.
Award 1 mark for the correct maximum area and the value of k for which it occurs.

***VCAA Assessment Report* note:**
Many students did not attempt this question. Most of the students who were successful with **part di.** were successful with this question.

iii. $$\frac{dA}{dk} = \frac{k(3k-16)}{4\left(\sqrt{k-4}\right)^3} = 0$$

$$\Rightarrow k = \frac{16}{3}$$

$$A\left(\frac{16}{3}\right) = \frac{64\sqrt{3}}{9} - \frac{32}{3}$$
$$\approx 1.65$$

The minimum area is $\dfrac{64\sqrt{3}}{9} - \dfrac{32}{3}$, occurring when $k = 5\frac{1}{3}$.
Award 1 mark for finding where the derivative is zero.
Award 1 mark for the correct minimum area and the value of k for which it occurs.

Question 15

$$\overline{T} = \frac{1}{12-0}\int_0^{12} \left(17 - 6\sin\left(\frac{\pi t}{12}\right)\right) dt$$

$$= \frac{1}{12}\left[17t + \frac{72}{\pi}\cos\left(\frac{\pi t}{12}\right)\right]_0^{12}$$

$$= \frac{1}{12}\left(\left(17 \times 12 + \frac{72}{\pi}\cos(\pi)\right) - \left(0 + \frac{72}{\pi}\cos(0)\right)\right)$$

$$= 17 - \frac{12}{\pi}$$

The correct answer is **D**.

9 Discrete random variables

Topic	9	Discrete random variables
Subtopic	9.2	Probability review

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Source: *VCAA 2020, Mathematical Methods Exam 1, Q2*

Question 1

A car manufacturer is reviewing the performance of its car model X. It is known that at any given six-month service, the probability of model X requiring an oil change is $\frac{17}{20}$, the probability of model X requiring an air filter change is $\frac{3}{20}$ and the probability of model X requiring both is $\frac{1}{20}$.

a. State the probability that at any given six-month service model X will require an air filter change without an oil change. **(1 mark)**

b. The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where $m, n \in Z^{+}$.

Determine m in terms of n if the probability of model Y requiring an air filter change without an oil change at any given six-month service is 0.05. **(2 marks)**

Source: *VCAA 2019, Mathematical Methods Exam 2, Section A, Q17*

Question 2 (1 mark)

A box contains n marbles that are identical in every way except colour, of which k marbles are coloured red and the remainder of the marbles are coloured green. Two marbles are drawn randomly from the box.

If the first marble is **not** replaced into the box before the second marble is drawn, then the probability that the two marbles drawn are the same colour is

A. $\dfrac{k^2 + (n-k)^2}{n^2}$

B. $\dfrac{k^2 + (n-k-1)^2}{n^2}$

C. $\dfrac{2k(n-k-1)}{n(n-1)}$

D. $\dfrac{k(n-1) + (n-k)(n-k-1)}{n(n-1)}$

E. ${}^nC_2 \left(\dfrac{k}{n}\right)^2 \left(1 - \dfrac{k}{n}\right)^{n-2}$

Source: *VCAA 2018, Mathematical Methods Exam 2, Section A, Q14*

Question 3 (1 mark)

Two events, A and B, are independent, where $\Pr(B) = 2\Pr(A)$ and $\Pr(A \cup B) = 0.52$. $\Pr(A)$ is equal to

A. 0.1
B. 0.2
C. 0.3
D. 0.4
E. 0.5

***Source:** VCE 2018, Mathematical Methods 2, Section A, Q13; © VCAA*

Question 4 (1 mark)

In a particular scoring game, there are two boxes of marbles and a player must randomly select one marble from each box. The first box contains four white marbles and two red marbles. The second box contains two white marbles and three red marbles. Each white marble scores -2 points and each red marble scores $+3$ points. The points obtained from the two marbles randomly selected by a player are added together to obtain a final score.

What is the probability that the final score will equal $+1$?

A. $\frac{2}{3}$

B. $\frac{1}{5}$

C. $\frac{2}{5}$

D. $\frac{2}{15}$

E. $\frac{8}{15}$

***Source:** VCE 2017, Mathematical Methods 2, Section A, Q3; © VCAA*

Question 5 (1 mark)

A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement.

The probability that the marbles are of **different** colours is

A. $\frac{5}{8}$

B. $\frac{3}{5}$

C. $\frac{15}{28}$

D. $\frac{15}{56}$

E. $\frac{30}{28}$

Source: VCE 2017, Mathematical Methods 1, Q5; © VCAA

Question 6 (4 marks)

For Jac to log on to a computer successfully, Jac must type the correct password. Unfortunately, Jac has forgotten the password. If Jac types the wrong password, Jac can make another attempt. The probability of success on any attempt is $\frac{2}{5}$. Assume that the result of each attempt is independent of the result of any other attempt. A maximum of three attempts can be made.

a. What is the probability that Jac does not log on to the computer successfully? **(1 mark)**

b. Calculate the probability that Jac logs on to the computer successfully. Express your answer in the form $\frac{a}{b}$, where a and b are positive integers. **(1 mark)**

c. Calculate the probability that Jac logs on to the computer successfully on the second or on the third attempt. Express your answer in the form $\frac{c}{d}$, where c and d are positive integers. **(2 marks)**

Topic 9 | Subtopic 9.2 Probability review

Source: *VCE 2016, Mathematical Methods 2, Section A, Q15; © VCAA*

Question 7 (1 mark)

A box contains six red marbles and four blue marbles. Two marbles are drawn from the box, without replacement.

The probability that they are the same colour is

A. $\frac{1}{2}$

B. $\frac{28}{45}$

C. $\frac{7}{15}$

D. $\frac{3}{5}$

E. $\frac{1}{3}$

Source: *VCE 2015, Mathematical Methods (CAS) 1, Q8; © VCAA*

Question 8 (3 marks)

For events A and B from a sample space, $\Pr(A|B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

a. Calculate $\Pr(A \cap B)$ **(1 mark)**

b. Calculate $\Pr(A' \cap B)$, where A' denotes the complement of A. **(1 mark)**

c. If events A and B are independent, calculate $\Pr(A \cup B)$. **(1 mark)**

Source: VCE 2015, Mathematical Methods (CAS) 2, Section 1, Q12; © VCAA

Question 9 (1 mark)

A box contains five red balls and three blue balls. John selects three balls from the box, without replacing them.

The probability that at least one of the balls that John selected is red is

A. $\frac{5}{7}$

B. $\frac{5}{14}$

C. $\frac{7}{28}$

D. $\frac{15}{56}$

E. $\frac{55}{56}$

Source: VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q11; © VCAA

Question 10 (1 mark)

A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are different colours is

A. $\frac{20}{81}$

B. $\frac{5}{18}$

C. $\frac{4}{9}$

D. $\frac{40}{81}$

E. $\frac{5}{9}$

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q22; © VCAA*

Question 11 (1 mark)

John and Rebecca are playing darts. The result of each of their throws is independent of the result of any other throw. The probability that John hits the bullseye with a single throw is $\frac{1}{4}$. The probability that Rebecca hits the bullseye with a single throw is $\frac{1}{2}$. John has four throws and Rebecca has two throws. The ratio of the probability of Rebecca hitting the bullseye at least once to the probability of John hitting the bullseye at least once is

A. $1:1$
B. $32:27$
C. $64:85$
D. $2:1$
E. $192:175$

Source: *VCE 2013, Mathematical Methods (CAS) 2, Section 1, Q17; © VCAA*

Question 12 (1 mark)

A and B are events of a sample space.

Given that $\Pr(A|B) = p, \Pr(B) = p^2$ and $\Pr(A) = p^{\frac{1}{3}}, \Pr(B|A)$, is equal to

A. p
B. $p^{\frac{4}{3}}$
C. $p^{\frac{7}{3}}$
D. $p^{\frac{8}{3}}$
E. p^3

***Source:** VCE 2013, Mathematical Methods (CAS) 2, Section 1, Q10; © VCAA*

Question 13 (1 mark)

For events A and B, $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is

A. 0

B. $\frac{1}{4}$

C. $\frac{3}{8}$

D. $\frac{1}{2}$

E. $\frac{3}{5}$

Question 14 (1 mark)

Consider two events A and B, $\Pr(A) = \frac{1}{4}$ and $\Pr(B) = \frac{1}{5}$.

Which of the following statements is **true**?

A. If A and B are independent events, then $\Pr(A \cup B) = \frac{2}{5}$.

B. If A and B are mutually exclusive events, then $\Pr(A \cup B) = \frac{2}{9}$.

C. If A and B are independent events, then $\Pr(A \cap B) = \frac{9}{20}$.

D. If A and B are mutually exclusive events, then $\Pr(A \cup B) = \frac{1}{20}$.

E. If A and B are mutually exclusive events, then $\Pr(A \cap B) = \frac{1}{20}$.

Topic	9	Discrete random variables
Subtopic	9.3	Discrete random variables

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Source: *VCE 2016 Mathematical Methods Exam 2, Section A, Q7; © VCAA.*

Question 1 (1 mark)

The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$\Pr(X = x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

A. 0.3
B. 0.305
C. 0.355
D. 0.405
E. 0.8

Question 2 (1 mark)

For the table below to represent a probability function, the value(s) of p must be

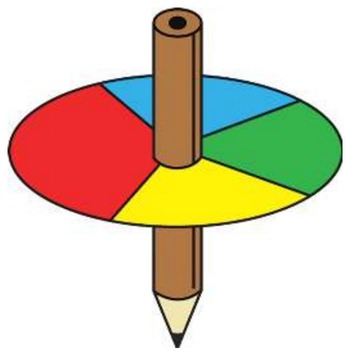
x	0	1	2	3	4
$\Pr(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

A. $p = 0$
B. $p = \frac{4}{5}$
C. $p = \frac{2}{3}$
D. $p = \frac{2}{3}$ or $p = 1$
E. $p = 1$

Question 3 (5 marks)

A game is played using a spinner that has been loaded so that it is more likely to land on the red side. In fact, $\Pr(\text{red}) = \frac{2}{5}$, and $\Pr(\text{blue}) = \Pr(\text{green}) = \Pr(\text{yellow}) = \frac{1}{5}$.

Each player pays $2 to play. The player spins the spinner a total of 3 times; however, once the spinner lands on the red side the game is over. If a player has a combination of any 3 colours, they win $1, but if the player has a combination of 3 colours that are all the same, they win $10. There are a total of 40 different outcomes for the game.



a. List the possible ways in which the game could end. **(1 mark)**

b. List the possible ways in which the player could win $10. **(1 mark)**

c. Suppose X equals the amount of money won by playing the game, excluding the amount the person pays to play, so $X = \{0, 1, 10\}$. Construct the probability distribution. Give your answers correct to 4 decimal places. **(3 marks)**

Question 4 (1 mark)
Given the probability distribution

X	0	1	2
$\Pr(X = x)$	$\frac{1}{2k}$	$\frac{k}{5}$	$\frac{7}{10k}$

Then k is equal to

A. $\frac{5}{7}$
B. 2 only
C. 3 only
D. 2 or 3
E. $\frac{10}{9}$

Question 5 (3 marks)
The discrete random variable X has the probability function

x	1	2	3	4	5
$\Pr(X = x)$	$2p$	$3p$	$4p$	$5p$	$6p$

a. Find the value of p. **(2 marks)**

b. Find $\Pr(3 \leq X \leq 5)$ **(1 mark)**

Question 6 (4 marks)

The number of "no-shows" on a scheduled airline flight has the probability distribution

x	0	1	2	3	4	5	6	7
$\Pr(X = x)$	0.09	0.22	0.26	0.21	0.13	0.06	0.02	p

a. Find p. **(2 marks)**

b. Find the probability that more than five people do not show up for the flight. **(2 marks)**

Question 7 (1 mark)

Given the probability distribution defined by $\Pr(X = x) = cx^2$ for $x = 0, 1, 2, 3$.

Then

A. $c = \frac{1}{13}$

B. $c = \frac{1}{6}$

C. $c = \frac{1}{14}$

D. $c = \frac{1}{15}$

E. $c = \frac{1}{36}$

Question 8 (1 mark)

A probability distribution for the random variable X is shown below.

X	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$

Using the above information $\Pr(X \geq 3/X > 1)$ is equal to

A. $\frac{7}{24}$

B. $\frac{7}{13}$

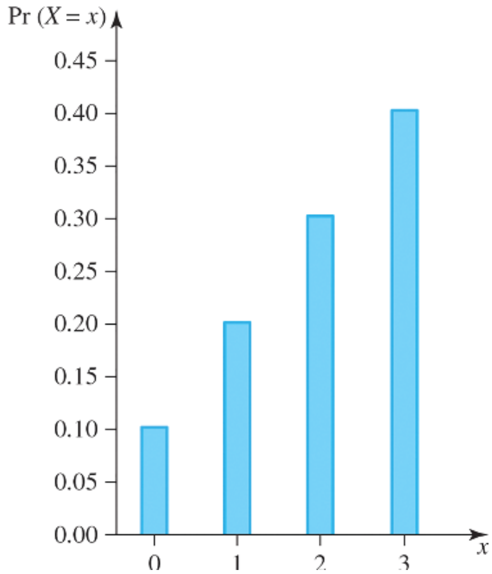
C. $\frac{7}{16}$

D. $\frac{1}{2}$

E. $\frac{3}{13}$

Question 9 (1 mark)

The graph of the probability distribution for the random variable X is shown below.



Which of the following options is **true**?

A. $\Pr(X = x) = \dfrac{x}{10}$ for $x = 0, 1, 2, 3$

B. $\Pr(X = x) = \dfrac{x}{10} + 10$ for $x = 0, 1, 2, 3$

C. $\Pr(X = x) = \dfrac{x + 1}{10}$ for $x = 0, 1, 2, 3$

D. $\Pr(X = x) = \dfrac{x - 1}{10}$ for $x = 1, 2, 3, 4$

E. $\Pr(X = x) = \dfrac{1}{100}(10x + 1)$ for $x = 0, 1, 2, 3$

Topic	9	Discrete random variables
Subtopic	9.4	Measures of centre and spread

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***Source:** VCE 2018, Mathematical Methods Exam 2, Section A, Q12; © VCAA*

Question 1 (1 mark)

The discrete random variable X has the following probability distribution.

x	0	1	2	3	6
$Pr(X=x)$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

Let μ be the mean of X.

$\Pr(X < \mu)$ is

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{17}{20}$

D. $\frac{4}{5}$

E. $\frac{7}{10}$

***Source:** VCE 2017, Mathematical Methods Exam 2, Section A, Q14; © VCAA*

Question 2 (1 mark)

The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

x	-1	0	1
$Pr(X=x)$	p	$2p$	$1-3p$

The variance of X is

A. $2p(1-3p)$

B. $1-4p$

C. $(1-3p)^2$

D. $6p-16p^2$

E. $p(5-9p)$

Question 3 (10 marks)
A dart competition at a local sports centre allows each player to throw one dart at the board, which has a radius of 20 centimetres. The board consists of five concentric circles, each with the same width.

The inner circle has a radius of 4 cm. The probability of landing on each band is determined by the area of that band available on the board.
The outer red band is called band E, the next white band is called band D and so on until you get to the inner red circle, which is band A.

a. Calculate the probability of landing on each of the bands. **(5 marks)**

The competition costs \$1 to enter and the prizes are as follows:
If a dart hits band E, the player receives nothing.
If a dart hits band D, the player receives \$1.
If a dart hits band C, the player receives \$2.
If a dart hits band B, the player receives \$5.
If a dart hits band A, the player receives \$10.

b. If X is a discrete random variable that represents the profit in dollars for the player, construct a probability distribution table for this game. **(1 mark)**

c. Calculate:

i. the expected profit a player could make in dollars **(2 marks)**

ii. the standard deviation. **(2 marks)**

Source: *VCE 2019, Mathematical Methods 2, Section A, Q7; © VCAA*

Question 4 (1 mark)

The discrete random variable X has the following probability distribution.

X	0	1	2	3
$\Pr(X = x)$	a	$3a$	$5a$	$7a$

The mean of X is

A. $\frac{1}{16}$

B. 1

C. $\frac{35}{16}$

D. $\frac{17}{8}$

E. 2

Source: *VCE 2013, Mathematical Methods (CAS) 1, Q7b © VCAA*

Question 5 (3 marks)

The probability distribution of a discrete random variable, X, is given by the table below.

x	0	1	2	3	4
$\Pr(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

Let $p = \frac{2}{3}$.

i. Calculate $E(X)$. **(2 marks)**

ii. Find $\Pr(X \geq E(X))$. **(1 mark)**

Question 6 (1 mark)

Given the discrete random variable with the probability distribution shown below.

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$

Then the standard deviation of X is closest to

A. 4.833
B. 2.1985
C. 1.732
D. 1.4722
E. 1.2134

Question 7 (1 mark)
A discrete random variable has a mean of 5 and a standard deviation of 3. $E(X^2)$ is then equal to
A. 8
B. 22
C. 25
D. 28
E. 34

Question 8 (3 marks)
A random variable X has this probability distribution.

x	1	2	3	4	5
$\Pr(X = x)$	c	0.3	0.1	0.2	0.05

Given that $\text{E}(X) = 2.3$, find the $\text{Var}(X)$.

Question 9 (1 mark)
Given the probability distribution shown below where $0 < k < 1$.

X	1	2
Probability	k	$1-k$

The standard deviation of X is

A. $2-k$
B. $4-3k$
C. $\sqrt{k(1-k)}$
D. $\sqrt{k}-k$
E. $\sqrt{4-3k}$

Question 10 (1 mark)
The random variable X has the following probability distribution.

x	0	1	2	3	4
$\Pr(X=x)$	p	0.2	q	0.2	p

If $E(X) = 2$ and $\text{Var}(X) = 1.2$, the values of p and q are

A. $p = 0.1, q = 0.4$
B. $p = 0.4, q = 0.1$
C. $p = 0.05, q = 0.5$
D. $p = 0.15, q = 0.3$
E. $p = 0.2, q = 0.2$

Topic	9	Discrete random variables
Subtopic	9.5	Applications

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Question 1 (2 marks)

X is a discrete random variable representing the number of touchscreen tablets sold on a particular day. Given $\mathrm{E}(X) = 15$ and $\mathrm{E}(X^2) = 250$, determine the values of x_1 and x_2 in the interval $[x_1, x_2]$. The interval represents a probability of 95% that the number of touchscreen tablets sold on a particular day falls within this range.

Question 2 (1 mark)

Over a twenty-day period, Pete recorded the number of minutes that he spent waiting for the lift to go from the ground floor to a classroom on the sixth floor.

Number of minutes spent waiting x	0	1	2	3	4
Proportion of days on which x minutes were spent waiting	$\frac{5}{20}$	$\frac{3}{20}$	$\frac{4}{20}$	$\frac{6}{20}$	$\frac{2}{20}$

During this twenty-day period, the mean number of minutes that Pete spent waiting for the lift was

A. 1.85
B. 0.75
C. 18.75
D. 37
E. 2.1

Question 3 (8 marks)

In a certain random experiment the events V and W are independent events.

a. If $\Pr(V \cup W) = 0.7725$ and $\Pr(V \cap W) = 0.2275$, calculate $\Pr(V)$ and $\Pr(W)$, given $\Pr(V) < \Pr(W)$. **(3 marks)**

b. Determine the probability that neither V nor W occur. **(1 mark)**

Let X be the discrete random variable that defines the number of times events V and W occur.

$X = 0$ if neither V nor W occurs.

$X = 1$ if only one of V and W occurs.

$X = 2$ if both V and W occur.

c. Specify the probability distribution for X. **(1 mark)**

d. Determine, correct to 4 decimal places where appropriate

i. $\mathrm{E}(X)$ **(1 mark)**

ii. $\mathrm{Var}(X)$ **(1 mark)**

iii. $\mathrm{SD}(X)$. **(1 mark)**

Topic	9	Discrete random variables
Subtopic	9.6	Review

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***Source:** VCE 2021 Mathematical Methods Exam 2, Q20; © VCAA*

Question 1 (1 mark)

Let A and B be two independent events from a sample space.

If $\Pr(A) = p$, $\Pr(B) = p^2$ and $\Pr(A) + \Pr(B) = 1$, then $\Pr(A' \cup B)$ is equal to

A. $1 - p - p^2$
B. $p^2 - p^3$
C. $p - p^3$
D. $1 - p + p^3$
E. $1 - p - p^2 + p^3$

***Source:** VCE 2019 Mathematical Methods Exam 1, Q3; © VCAA*

Question 2 (3 marks)

The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail. Jo has three coins in her pocket; two are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$.

Jo randomly selects a coin from her pocket and tosses it.

a. Find the probability that she tosses a head. **(2 marks)**

b. Find the probability that she selected an unbiased coin, given that she tossed a head. **(1 mark)**

Source: *VCE 2016 Mathematical Methods Exam 2, Section A, Q19; © VCAA*

Question 3 (1 mark)

Consider the discrete probability distribution with random variable X shown in the table below.

x	–1	0	b	$2b$	4
$Pr(X=x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of E(X) are respectively

A. –0.8 and 1

B. –0.8 and 1.6

C. 0 and 2.4

D. 0.2125 and 1

E. 0 and 1

Source: *VCE 2015 Mathematical Methods (CAS) Exam 2, Section 1, Q14; © VCAA*

Question 4 (1 mark)

Consider the following discrete probability distribution for the random variable X.

x	1	2	3	4	5
$Pr(X=x)$	p	$2p$	$3p$	$4p$	$5p$

The mean of this distribution is

A. 2

B. 3

C. $\frac{7}{2}$

D. $\frac{11}{3}$

E. 4

Source: *VCE 2014 Mathematical Methods (CAS) Exam 2, Section 1, Q14; © VCAA*

Question 5 (1 mark)

If X is a random variable such that $\Pr(X > 5) = a$ and $\Pr(X > 8) = b$, then $\Pr(X < 5 \mid X < 8)$ is

A. $\dfrac{a}{b}$

B. $\dfrac{a-b}{1-b}$

C. $\dfrac{1-b}{1-a}$

D. $\dfrac{ab}{1-b}$

E. $\dfrac{a-1}{b-1}$

Source: *VCE 2019, Mathematical Methods 2, Section A, Q11 © VCAA*

Question 6 (1 mark)

$A$and B are events from a sample space such that $\Pr(A) = p$, where $p > 0, \Pr(B|A) = m$ and $\Pr(B \mid A') = n$.
A and B are independent events when

A. $m = n$

B. $m = 1 - p$

C. $m + n = 1$

D. $m = p$

E. $m + n = 1 - p$

Source: *VCE 2017, Mathematical Methods 1, Q8; © VCAA*

Question 7 (5 marks)

For events A and B from a sample space, $\Pr(A|B) = \dfrac{1}{5}$ and $\Pr(B|A) = \dfrac{1}{4}$ Let $\Pr(A \cap B) = p$.

a. Find $\Pr(A)$ in terms of p. **(1 mark)**

b. Find $\Pr(A' \cap B')$ in terms of p. **(2 marks)**

c. Given that $\Pr(A \cup B) \leq \frac{1}{5}$, state the largest possible interval for p. **(2 marks)**

Source: *VCE 2018, Mathematical Methods 1, Q6; © VCAA*

Question 8 (4 marks)

Two boxes each contain four stones that differ only in colour.
Box 1 contains four black stones.
Box 2 contains two black stones and two white stones.
A box is chosen randomly and one stone is drawn randomly from it.
Each box is equally likely to be chosen, as is each stone.

a. What is the probability that the randomly drawn stone is black? **(2 marks)**

b. It is not known from which box the stone has been drawn.
Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1? **(2 marks)**

Source: *VCE 2016, Mathematical Methods 1, Q7; © VCAA*

Question 9 (3 marks)

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B. At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a. What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer. **(2 marks)**

b. The selected motor is found to be faulty.

What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$ where c is a positive integer. **(1 mark)**

Source: *VCE 2015, Mathematical Methods (CAS) 1, Q9; © VCAA*

Question 10 (4 marks)

An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse.

Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

a. An egg is selected at random from the set of all eggs at the warehouse.

Find, in terms of p, the probability that the egg has a white eggshell. **(1 mark)**

b. Another egg is selected at random from the set of all eggs at the warehouse.

i. Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B. **(2 marks)**

ii. If the probability that this egg came from farm B is 0.3, find the value of p. **(1 mark)**

Source: *VCE 2014, Mathematical Methods (CAS) 1, Q9; © VCAA*

Question 11(6 marks)

Sally aims to walk her dog, Mack, most mornings. If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$, and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning.

a. In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning. Find the probability that Sally walked Mack on at least one of these two mornings. **(2 marks)**

b. In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$.

i. Find the probability that on a particular morning in April, Sally walked Mack. **(2 marks)**

ii. Using your answer from **part b.i.**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning. **(2 marks)**

Question 12 (1 mark)

Given that $\Pr(A) = \frac{5}{8}$, $\Pr(B) = \frac{1}{5}$ and $\Pr(A|B) = \frac{1}{8}$, which of the following options is **correct**?

A. $\Pr(A \cap B) = \frac{1}{40}$ and $\Pr(A' \cap B') = \frac{1}{5}$

B. $\Pr(A \cap B) = \frac{5}{8}$ and $\Pr(A' \cap B') = \frac{3}{5}$

C. $\Pr(A \cap B) = \frac{13}{40}$ and $\Pr(A' \cap B') = \frac{39}{40}$

D. $\Pr(A \cap B) = \frac{5}{8}$ and $\Pr(A' \cap B') = \frac{4}{5}$

E. $\Pr(A \cap B) = \frac{1}{40}$ and $\Pr(A' \cap B') = \frac{3}{10}$

Question 13 (1 mark)

A college has 500 students. During a census of the student body it is found that 20% of the students are left-handed and 40% wear glasses. It is also known that 52% of the right handed students do not wear glasses.

One student is selected at random. If the selected student does not wear glasses, the probability that they are left handed is closest to

1. 0.08
2. 0.12
3. 0.31
4. 0.32
5. 0.98

Answers and marking guide

9.2 Probability review

Question 1

a. Model X, O = oil change, F = filter change

$\Pr(O) = \frac{17}{20}, \Pr(F) = \frac{3}{20}$ and $\Pr(F \cap O) = \frac{1}{20}$

	O	O'	
F	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$
F'	$\frac{16}{20}$	$\frac{1}{20}$	$\frac{17}{20}$
	$\frac{17}{20}$	$\frac{3}{20}$	

$\Pr(F \cap O') = \frac{1}{10}$ **[1 mark]**

b. Model Y, $\Pr(O) = \frac{m}{m+n}$, $\Pr(F) = \frac{n}{m+n}$ and

$\Pr(F \cap O) = \frac{1}{m+n}$

	O	O'	
F	$\frac{1}{m+n}$	$\frac{n-1}{m+n}$	$\frac{n}{m+n}$
F'	$\frac{m-1}{m+n}$		
	$\frac{m}{m+n}$		

$\Pr(F \cap O') = \frac{n-1}{m+n} = 0.05 = \frac{1}{20}$

$20(n-1) = 20n - 20 = m + n$

$m = 19n - 20$

Award 1 mark for the correct Karnaugh map (table).

Award 1 mark for the correct answer.

Question 2

There are k red marbles, $n-k$ green marbles, a total of n

$$\Pr(RR) + \Pr(GG) = \frac{k}{n} \times \frac{k-1}{n-1} + \frac{n-k}{n} \times \frac{n-k-1}{n-1}$$

$$= \frac{k(k-1) + (n-k)(n-k-1)}{n(n-1)}$$

The correct answer is **D**.

Question 3

$\Pr(A) = a = ?$ and $\Pr(B) = 2\Pr(A); \Pr(A \cup B) = 0.52$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

A and B are independent, so $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B)$$
$$0.52 = \Pr(A) + 2\Pr(A) - 2(\Pr(A))^2$$
$$0.52 = 3a - 2a^2$$
$$2a^2 - 3a + 0.52 = 0$$
$$a = \Pr(A) = 0.2 \quad 0 < a < 1$$

The correct answer is **B**.

Question 4

Box 1: 4W, 2R; total 6

Box 2: 2W, 3R; total 5

$$\Pr(+1) = \Pr(RW) + \Pr(WR) = \frac{2}{6} \times \frac{2}{5} + \frac{4}{6} \times \frac{3}{5} = \frac{16}{30}$$
$$= \frac{8}{15}$$

The correct answer is **E**.

Question 5

5 red marbles and 3 yellow marbles means 8 marbles in total.

$$\Pr(\text{different}) = \Pr(RY) + \Pr(YR)$$
$$= \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7}$$
$$= \frac{30}{56}$$
$$= \frac{15}{28}$$

The correct answer is **C**.

Question 6

a. Pr(does not log on)

$$= \Pr(FFF)$$
$$= \left(\frac{3}{5}\right)^3$$
$$= \frac{27}{125}$$ **[1 mark]**

***VCAA Examination Report* note:**

Students clearly identified what was required but some students erred with the arithmetic evaluation.

b. $\Pr(\text{log on}) = \Pr(S) + \Pr(FS) + \Pr(FFS)$

$$= \frac{2}{5} + \frac{3}{5} \times \frac{2}{5} + \left(\frac{3}{5}\right)^2 \times \frac{2}{5}$$
$$= \frac{98}{125}$$ **[1 mark]**

alternatively

$$\Pr(\text{log on}) = 1 - \Pr(\text{does not log on})$$
$$= 1 - \frac{27}{125}$$
$$= \frac{98}{125}$$

***VCAA Examination Report* note:**
Students generally recognised that the solution was the complement of their answer to part a. Others used a tree diagram to identify all possibilities for Jac to log on successfully.

c. Pr(logs on second or third attempt) = $\Pr(FS) + \Pr(FFS)$

$$= \frac{3}{5} \times \frac{2}{5} + \left(\frac{3}{5}\right)^2 \times \frac{2}{5}$$

$$= \frac{48}{125}$$

Award 1 mark for the correct method.
Award 1 mark for the correct probability.

***VCAA Examination Report* note:**
Many students who struggled with previous parts of the question generally made use of a tree diagram to find the two required cases. Common errors included use of conditional probability, use of binomial theorem or not realising that once Jac logged in, there was no need to keep attempting (three cases).
A small number of students recognised that

Pr(success on second or third attempt) = Pr(success) – Pr(success on the first attempt) = $\frac{98}{125} - \frac{2}{5} = \frac{48}{125}$

Question 7

$$\begin{aligned}\Pr(\text{same}) &= \Pr(\text{RR}) + \Pr(\text{BB})\\ &= \frac{6}{10} \times \frac{5}{9} + \frac{4}{10} \times \frac{3}{9}\\ &= \frac{42}{90}\\ &= \frac{7}{15}\end{aligned}$$

The correct answer is **C**.

Question 8

a. $\Pr(A|B) = \frac{3}{4}, \Pr(B) = \frac{1}{3}$

$$\begin{aligned}\Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\\ &= \frac{\Pr(A \cap B)}{\frac{1}{3}}\end{aligned}$$

$$\frac{3}{4} \times \frac{1}{3} = \Pr(A \cap B)$$

$\Pr(A \cap B) = \frac{1}{4}$ **[1 mark]**

b. $\Pr(A' \cap B) = \Pr(B) - \Pr(A \cap B)$

$$\begin{aligned}&= \frac{1}{3} - \frac{1}{4}\\ &= \frac{4-3}{12}\\ &= \frac{1}{12}\end{aligned}$$

[1 mark]

***VCAA Assessment Report* note:**
A Karnaugh map was most useful for formulating a solution. Some poor manipulation of fractions was evident in responses.

c. since A and B are independent.

$\Pr(A \cap B) = \Pr(A)\Pr(B)$

$\frac{1}{4} = \Pr(A) \times \frac{1}{3} \Rightarrow \Pr(A) = \frac{3}{4}$

$$\begin{aligned}\Pr(A \cup B) &= \frac{3}{4} + \frac{1}{3} - \frac{1}{4} \\ &= \frac{9+4-3}{12} \\ &= \frac{10}{12} \\ &= \frac{5}{6}\end{aligned}$$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students made little headway into solving this problem due to their lack of understanding of independent events. The addition rule was then applied using an incorrect value for $\Pr(A)$, resulting in final answers well outside the interval [0, 1]. Students must note that a probability must lie within [0, 1] and is never a negative number.

Question 9

Five red, three blue for a total of eight balls.

$$\begin{aligned}\Pr(\text{at least one red}) &= 1 - \Pr(\text{no red}) \\ &= 1 - \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} \\ &= \frac{55}{56}\end{aligned}$$

The correct answer is **E**.

Question 10

Five red (5R) and four blue (4B) marbles = 9 marbles in total.

$$\begin{aligned}\Pr(RB + BR) &= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8} \\ &= \frac{5}{9}\end{aligned}$$

The correct answer is **E**.

Question 11

$\Pr(\text{J bullseye}) = \frac{1}{4}$ $\Pr(\text{R bullseye}) = \frac{1}{2}$

$J \times 4$ $\Pr(\text{at least one}) = 1 - \Pr(\text{none}) = 1 - \left(\frac{3}{4}\right)^4 = \frac{175}{256}$

$R \times 2$ $\Pr(\text{at least one}) = 1 - \Pr(\text{none}) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

Ratio: $\frac{3}{4} \times \frac{256}{175} = 192 : 175$

The correct answer is **E**.

Question 12

$\Pr(A|B) = p,\ \Pr(B) = p^2, \Pr(A) = p^{\frac{1}{3}}$

$\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} \Rightarrow p = \dfrac{\Pr(A \cap B)}{p^2}$

$\Pr(A \cap B) = p^3$

$$\begin{aligned}\Pr(B|A) &= \frac{\Pr(A \cap B)}{\Pr(A)} \\ &= \frac{p^3}{p^{\frac{1}{3}}} \\ &= P^{3-\frac{1}{3}} \\ &= p^{\frac{8}{3}}\end{aligned}$$

The correct answer is **D**.

Question 13

$\Pr(A \cap B) = p,\ \Pr(A' \cap B) = p - \dfrac{1}{8},\ \Pr(A \cap B') = \dfrac{3p}{5}$

	A	A'	
B	p	$p - \frac{1}{8}$	$2p - \frac{1}{8}$
B'	$\frac{3p}{5}$		
	$\frac{8p}{5}$		

Since the events are independent, $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

$p = \left(\dfrac{8p}{5}\right)\left(2p - \dfrac{1}{8}\right)$ solving where $0 < p < 1$

$2p - \dfrac{1}{8} = \dfrac{5}{8}$

$2p = \dfrac{6}{8}$

$\Rightarrow p = \dfrac{3}{8}$

The correct answer is **C**.

Question 14

If A and B are independent events, then

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A) \cdot \Pr(B)$

$\Pr(A \cup B) = \dfrac{1}{4} + \dfrac{1}{5} - \dfrac{1}{4} \times \dfrac{1}{5} = \dfrac{2}{5}$

The correct answer is **A**.

9.3 Discrete random variables

Question 1

$$\begin{aligned}\Pr(\text{same}) &= [\Pr(X=0)]^2 + [\Pr(X=1)]^2 + [\Pr(X=2)]^2 + [\Pr(X=3)]^2 \\ &= 0.5^2 + 0.25^2 + 0.2^2 + 0.05^2 \\ &= 0.355\end{aligned}$$

The correct answer is **C**.

Question 2

$$\sum \Pr(X=x) = 1$$

$$1 = 0.2 + 0.6p^2 + 0.1 + 1 - p + 0.1$$

$$0.6p^2 - p + 0.4 = 0$$

$$6p^2 - 10p + 4 = 0$$

$$3p^2 - 5p + 2 = 0$$

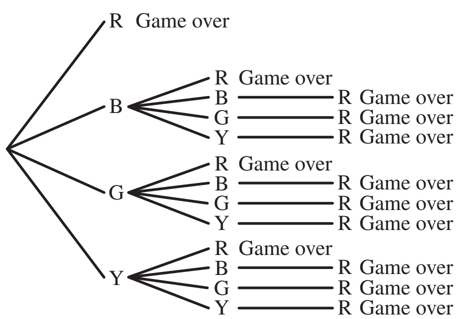
$$(3p-2)(p-1) = 0$$

$$p = 1 \quad \text{or} \quad p = \frac{2}{3}$$

The correct answer is **D**.

Question 3

a. **[1 mark]**



b. Wins \$10 with BBB, GGG or YYY **[1 mark]**

c. $X = \{0,\ 1,\ 10\}$

$$\begin{aligned}\Pr(X=0) &= \frac{2}{5} + 3\left(\frac{1}{5}\times\frac{2}{5}\right) + 9\left(\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}\right) \\ &= \frac{2}{5} + \frac{6}{25} + \frac{9}{125} \\ &= \frac{50}{125} + \frac{30}{125} + \frac{9}{125} = \frac{98}{125}\end{aligned}$$ **[1 mark]**

$$\Pr(X=10) = 3\left(\frac{1}{5}\right)^3 = \frac{3}{125}$$ **[1 mark]**

$$\begin{aligned}\Pr(X=1) &= 1 - (\Pr(X=0) + \Pr(x=10)) \\ &= \frac{125}{125} - \left(\frac{98}{125} + \frac{3}{125}\right) = \frac{24}{125}\end{aligned}$$

x	\$0	\$1	\$10
$\Pr(X=x)$	$\frac{98}{125}$	$\frac{24}{125}$	$\frac{3}{125}$

[1 mark]

Question 4

The sum of the probabilities is one.

$$\frac{1}{2k}+\frac{k}{5}+\frac{7}{10k}=1$$

$$\frac{5+2k^2+7}{10k}=1$$

$$2k^2+12=10k$$

$$k^2-5k+6=0$$

$$(k-3)(k-2)=0$$

Both $k=2$ and $k=3$ are valid answers.

The correct answer is **D**.

Question 5

a. $2p+3p+4p+5p+6p=1$ **[1 mark]**

$p=\frac{1}{20}$ **[1 mark]**

b. $\Pr(3 \le X \le 5)=\frac{4}{20}+\frac{5}{20}+\frac{6}{20}$

$=\frac{15}{20}$

$=0.75$ **[1 mark]**

Question 6

a. $0.09+0.22+0.26+0.21+0.13+0.06+0.02+p=1$ **[1 mark]**

$0.99+p=1$

$p=0.01$ **[1 mark]**

b. $\Pr(X>5)=\Pr(X=6)+\Pr(X=7)$

$=0.02+0.01$ **[1 mark]**

$=0.03$ **[1 mark]**

Question 7

X	0	1	2	3
Probability	0	*c*	4*c*	9*c*

The sum of the probabilities is one.

$c+4c+9c=14c=1$

$c=\frac{1}{14}$

The correct answer is **C**.

Question 8

$$\Pr(X \ge 3/X>1)=\frac{\Pr(X \ge 3)}{\Pr(X>1)}$$

$$=\frac{\frac{1}{6}+\frac{1}{8}}{1-\left(\frac{1}{8}+\frac{1}{3}\right)}$$

$$=\frac{7}{13}$$

The correct answer is **B**.

Question 9
The graph is equivalent to the table

X	0	1	2	3
Probability	0.1	0.2	0.3	0.4

Which is $\Pr(X = x) = \dfrac{x+1}{10}$ for $x = 0, 1, 2, 3$
The correct answer is **C**.

9.4 Measures of centre and spread

Question 1

$$\mu = \mathrm{E}(X) = \sum x \Pr(X = x)$$
$$\mu = 0 \times \frac{1}{4} + 1 \times \frac{9}{20} + 2 \times \frac{1}{10} + 3 \times \frac{1}{20} + 6 \times \frac{3}{20} = \frac{17}{10}$$
$$\begin{aligned}\Pr(X < \mu) &= \Pr(X < 1.7)\\ &= \Pr(X = 0) + \Pr(X = 1)\\ &= \frac{1}{4} + \frac{9}{20}\\ &= \frac{7}{10}\end{aligned}$$

The correct answer is **E**.

Question 2

$$\begin{aligned}\mathrm{E}(X) &= \sum x \Pr(X = x) = -p + 1 - 3p = 1 - 4p\\ \mathrm{E}(X^2) &= \sum x^2 \Pr(X = x) = p + 1 - 3p = 1 - 2p\\ \mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2\\ &= (1 - 2p) - (1 - 4p)^2\\ &= 1 - 2p - \left(1 - 8p + 16p^2\right)\\ &= 6p - 16p^2\end{aligned}$$

The correct answer is **D**.

Question 3

a. Area of the whole board is $\pi(4 \times 5)^2 = 400\pi$.
Band A $= \pi(4)^2 = 16\pi$ and
$\Pr(\text{Band A}) = \dfrac{16\pi}{400\pi} = \dfrac{1}{25}$ **[1 mark]**

Band B $= \pi(8)^2 - 16\pi = 64\pi - 16\pi = 48\pi$ and $\Pr(\text{Band B}) = \dfrac{48\pi}{400\pi} = \dfrac{3}{25}$ **[1 mark]**

Band C $= \pi(12)^2 - 64\pi = 144\pi - 64\pi = 80\pi$ and $\Pr(\text{Band C}) = \dfrac{80\pi}{400\pi} = \dfrac{5}{25}$ **[1 mark]**

Band D $= \pi(16)^2 - 144\pi = 256\pi - 144\pi = 112\pi$ and $\Pr(\text{Band D}) = \dfrac{112\pi}{400\pi} = \dfrac{7}{25}$ **[1 mark]**

Band E $= \pi(20)^2 - 256\pi = 400\pi - 256\pi = 144\pi$ and $\Pr(\text{Band E}) = \dfrac{144\pi}{400\pi} = \dfrac{9}{25}$ **[1 mark]**

b. X is the gain in dollars.

Pr(E) = −\$1, Pr(D) = \$0, Pr(C) = \$1, Pr(B) = \$4, Pr(A) = \$9

x	−\$1	\$0	\$1	\$4	\$9
$\Pr(X = x)$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{5}{25} = \frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{25}$

[1 mark]

c. i. $E(X) = -1\left(\frac{9}{25}\right) + 0\left(\frac{7}{25}\right) + 1\left(\frac{5}{25}\right) + 4\left(\frac{3}{25}\right) + 9\left(\frac{1}{25}\right)$ **[1 mark]**

$E(X) = -\frac{9}{25} + 0 + \frac{5}{25} + \frac{12}{25} + \frac{9}{25}$

$E(X) = \frac{17}{25} = 0.68$ cents **[1 mark]**

ii. $E(X^2) = (-1)^2\left(\frac{9}{25}\right) + 0^2\left(\frac{7}{25}\right) + 1^2\left(\frac{5}{25}\right) + 4^2\left(\frac{3}{25}\right) + 9^2\left(\frac{1}{25}\right)$

$E(X^2) = \frac{9}{25} + 0 + \frac{5}{25} + \frac{48}{25} + \frac{81}{25}$

$E(X^2) = \frac{143}{25}$ **[1 mark]**

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$\text{Var}(X) = \frac{143}{25} - \left(\frac{17}{25}\right)^2$

$\text{Var}(X) - \frac{3575}{625} - \frac{289}{625} = \frac{3286}{625} = \$5.26$

$SD(X) = \sqrt{\frac{3286}{625}} = \$2.29$ [1 mark]

Question 4

$\sum \Pr(X = x) = a + 3a + 5a + 7a = 16a = 1$

$a = \frac{1}{16}$

$E(X) = \sum x \Pr(X = x) = 3a + 10a + 21a = 34a$

$E(X) = 34 \times \frac{1}{16} = \frac{17}{8}$

The correct answer is **D**.

Question 5

i. $E(X) = \sum x \Pr(X = x)$

$E(X) = 0 \times 0.2 + 1 \times 0.6p^2 + 2 \times 0.1 + 3 \times (1 - p) + 4 \times 0.1$

$= \frac{3}{5}p^2 - 3p + 3.6 \quad \text{but } p = \frac{2}{3}$

$= \frac{3}{5} \times \frac{4}{9} - 2 + 3.6 = \frac{4}{15} + \frac{8}{5} = \frac{4 + 24}{15}$

$= \frac{28}{15}$

Award 1 mark for finding the expected value and substituting.

Award 1 mark for the final correct expected value.

***VCAA Assessment Report* note:**

i. Many students had difficulty adding decimals and fractions to give an answer.

ii. Many students gave 0.5 as the answer.

ii. $\Pr(X > E(X)) = \Pr(X \geq 2)$

$$\begin{aligned}\Pr(X > E(X)) &= 1 - (\Pr(X = 0) + \Pr(X = 1)) \\ &= 1 - \left(\frac{1}{5} + \frac{4}{15}\right) \\ &= 1 - \frac{7}{15} \\ &= \frac{8}{15}\end{aligned}$$

Award 1 mark for the correct probability.

***VCAA Assessment Report* note:**

i. Many students had difficulty adding decimals and fractions to give an answer.

ii. Many students gave 0.5 as the answer.

Question 6

$$\begin{aligned}E(X) &= 1 \times \frac{1}{3} + 2 \times \frac{1}{4} + 3 \times \frac{1}{6} + 4 \times \frac{1}{8} \\ &= \frac{11}{6} \\ &= 1.8333\end{aligned}$$

$$\begin{aligned}E(X)^2 &= 1 \times \frac{1}{3} + 4 \times \frac{1}{4} + 9 \times \frac{1}{6} + 16 \times \frac{1}{8} \\ &= \frac{29}{6} \\ &= 4.8333\end{aligned}$$

$$\begin{aligned}\text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= \frac{29}{6} - \left(\frac{11}{6}\right)^2 \\ &= \frac{53}{36} \\ &= 1.4722\end{aligned}$$

$$\begin{aligned}\text{Sd}(X) &= \sqrt{1.4722} \\ &= 1.2134\end{aligned}$$

The correct answer is **E**.

Question 7

Given that $\text{Sd}(X) = 3$ and $E(X) = 5$

$$\begin{aligned}\text{Var}(X) &= E(X^2) - (E(X))^2 \\ 9 &= E(X^2) - 25 \\ E(X^2) &= 34\end{aligned}$$

The correct answer is **E**.

Question 8

$c + 0.3 + 0.1 + 0.2 + .05 = 1$

$c = 0.35$ **[1 mark]**

$$\begin{aligned}\text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= (1 \times 0.35 + 4 \times 0.3 + 9 \times 0.1 + 16 \times 0.2 + 25 \times 0.05) - [2.3]^2 \quad \textbf{[1 mark]} \\ &= 1.61 \quad \textbf{[1 mark]}\end{aligned}$$

Question 9

$$\begin{aligned}E(X) &= 1 \times k + 2 \times (1 - k) \\ &= 2 - k \\ E(X^2) &= 1 \times k + 4 \times (1 - k) \\ &= 4 - 3k \\ \text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= (4 - 3k) - (2 - k)^2 \\ &= 4 - 3k - \left(4 - 4k + k^2\right) \\ &= k - k^2 \\ &= k(1 - k) \\ \text{Sd}(X) &= \sqrt{k(1 - k)}\end{aligned}$$

The correct answer is **C**.

Question 10

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - [\text{E}(X)]^2 \\ 1.2 &= \text{E}(X^2) - [2]^2 \\ \text{E}(X^2) &= 5.2 \\ 5.2 &= 0.2 + 4q + 1.8 + 16p \\ 4q + 16p &= 3.2 \\ q + 4p &= 0.8 \quad (1) \\ \sum \Pr(X = x) &= 1 \\ p + 0.2 + q + 0.2 + p &= 1 \\ 2p + q &= 0.6 \quad (2)\end{aligned}$$

$(1) - (2)$:

$2p = 0.2$

$p = 0.1$

Substitute into (1):

$q + 4 \times 0.1 = 0.8$

$\therefore \; q = 0.4$

The correct answer is **A**.

9.5 Applications

Question 1

$$\begin{aligned}\text{Var}(X) &= \text{E}(X)^2 - [\text{E}(X)]^2 \\ &= 250 - 15^2 \\ &= 25 \\ \text{SD}(X) &= \sqrt{\text{Var}(X)} \\ &= \sqrt{25} \\ &= 5 \quad \textbf{[1 mark]}\end{aligned}$$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$

$\mu - 2\sigma = 15 - 2 \times 5$
$= 5$
$\mu + 2\sigma = 15 + 2 \times 5$
$= 25$ **[1 mark]**
$\therefore x_1 = 5,\ x_2 = 25$

Question 2

$$\text{E}(X) = 1 \times \frac{3}{20} + 2 \times \frac{4}{20} + 3 \times \frac{6}{20} + 4 \times \frac{2}{20}$$
$$= \frac{37}{20}$$
$$= 1.85$$

The correct answer is **A**.

Question 3

a. $\Pr(V \cup W) = 0.7725$ and $\Pr(V \cap W) = 0.2275$.

$\Pr(V \cup W) = \Pr(W) + \Pr(V) - \Pr(W \cap V)$
$0.7725 = \Pr(W) + \Pr(V) - 0.2275$
$1.0000 = \Pr(W) + \Pr(V)$ [1] **[1 mark]**

V and W are independent events.

$\Pr(W \cap V) = \Pr(W)\Pr(V)$
$0.2275 = \Pr(W)\Pr(V)$
$\frac{0.2275}{\Pr(W)} = \Pr(V)$ [2] **[1 mark]**

Substitute [2] into [1]:

$$1 = \frac{0.2275}{\Pr(W)} + \Pr(W)$$

$\Pr(W) = 0.2275 + [\Pr(W)]^2$
$0 = [\Pr(W)]^2 - \Pr(W) + 0.2275$
$\Pr(W) = 0.65$ or 0.35

But $\Pr(V) < \Pr(W)$, so $\Pr(W) = 0.65$ and $\Pr(V) = 0.35$. **[1 mark]**

b.

	W	M'	
V	0.2275	0.1225	0.35
V'	0.4225	0.2275	0.65
	0.35	0.65	1.000

[1 mark]

Note: $\Pr(V' \cap W') = 0.2275$

c.

x	0	1	2
$\Pr(X = x)$	0.2275	0.5450	0.2275

[1 mark]

d. i. $\text{E}(X) = 0(0.2275) + 1(0.545) + 2(0.2275) = 1$ **[1 mark]**

ii. $\text{E}(X^2) = 0^2(0.2275) + 1^2(0.545) + 2^2(0.2275)$
$= 0 + 0.545 + 0.91 = 1.455$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$
$\text{Var}(X) = 1.455 - 1^2$
$\text{Var}(X) = 0.455$ **[1 mark]**

iii. $\text{SD}(X) = \sqrt{0.455} = 0.6745$ **[1 mark]**

9.6 Review

Question 1

	A	A'	
B	p^3	$p^2 - p^3$	p^2
B'			$1 - p^2$
	p	$1 - p$	

$\Pr(A) = p,\ \Pr(B) = p^2$

$\Pr(A \cap B) = \Pr(A)\Pr(B) = p^3$

$$\begin{aligned}\Pr(A' \cup B) &= \Pr(A') + \Pr(B) - \Pr(A' \cap B)\\ &= (1 - p) + p^2 - \left(p^2 - p^3\right)\\ &= 1 - p + p^3\end{aligned}$$

The correct answer is **D**.

Question 2

a. $\Pr(\text{H}) = \text{H} = \text{Head},\ \text{B} = \text{Biased coin}$

$$= \Pr(\text{H} \cap \text{B}') + \Pr(\text{H} \cap \text{B})$$

$$\Pr(\text{H}) = \frac{1}{2} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{3}$$

$$= \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$$

Award 1 mark for the setup of probabilities.

Award 1 mark for the correct answers.

***VCAA Examination Report* note:**

As this question was worth two marks, appropriate working was required to be shown. This could include computations or a probability tree diagram with relevant branches clearly identified. In some instances, it was not clear which fractions were being manipulated or how they were manipulated.

b. $\Pr(\text{unbiased}|\text{H}) = \dfrac{\Pr(\text{unbiased} \cap \text{H})}{\Pr(\text{H})} = \dfrac{\frac{1}{2} \times \frac{2}{3}}{\frac{4}{9}} = \dfrac{\frac{1}{3}}{\frac{4}{9}} = \dfrac{3}{4}$

Award 1 mark for the correct probability.

***VCAA Examination Report* note:**

Most students correctly identified the conditional nature of this probability problem. It was noted that many students who did not simplify their answer to part **a** did not carry out the subsequent calculation successfully.

Question 3

$\sum \Pr(X = x) = a + 4b + 0.2 = 1 \qquad [1]$

$a + 4b = 0.8 \Rightarrow a = 0.8 - 4b$

$$\begin{aligned}\text{E}(X) &= \sum x \Pr(X = x)\\ &= -a + 5b^2 + 0.8\\ &= -(0.8 - 4b) + 5b^2 + 0.8\end{aligned}$$

$\text{E}(X) = 5b^2 + 4b$

$$\frac{d}{db}(E(X)) = 10b + 4 = 0 \Rightarrow b = -\frac{2}{5}$$

but $0 \le b \le 0.2, 0 \le a \le 0.8$ from [1].

Examine end points:
$b = 0,\ a = 0.8,\ \mathrm{E}(X) = 0,$ smallest
$b = 0.2,\ a = 0,\ \mathrm{E}(X) = 1,$ largest
The correct answer is **E**.

Question 4

$$\sum \Pr(X = x) = 1$$

$$\begin{aligned} p + 2p + 3p + 4p + 5p &= 15p \\ &= 1 \\ \Rightarrow p &= \frac{1}{15} \end{aligned}$$

$$\begin{aligned} \mathrm{E}(X) &= \sum x \Pr(X = x) \\ &= p + 4p + 9p + 16p + 25p \\ &= 55p \\ &= \frac{55}{15} \\ &= \frac{11}{3} \end{aligned}$$

The correct answer is **D**.

Question 5

$$\begin{aligned} \Pr(X < 5 | X < 8) &= \frac{\Pr(X < 5)}{\Pr(X < 8)} \\ &= \frac{1 - \Pr(X > 5)}{1 - \Pr(X > 8)} \\ &= \frac{1 - a}{1 - b} \\ &= \frac{a - 1}{b - 1} \end{aligned}$$

The correct answer is **E**.

Question 6

$\Pr(A) = p$

$$\Pr(B|A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{\Pr(B \cap A)}{p} = m, \quad \Pr(B \cap A) = mp$$

$$\Pr(B|\overline{A}) = \frac{\Pr(B \cap \overline{A})}{\Pr(\overline{A})} = \frac{\Pr(B \cap \overline{A})}{1 - p} = n, \Pr(B \cap \overline{A}) = n(1 - p)$$

$\Pr(B) = \Pr(B \cap A) + \Pr(B \cap \overline{A}) = mp + n(1 - p)$
A and B are independent so
$\Pr(A \cap B) = \Pr(A)\Pr(B)$
$mp = p \times (mp + n(1 - p))$
$m = mp + n(1 - p)$
$m(1 - p) = n(1 - p)$
$m = n$
The correct answer is **A**.

Question 7

a. $\Pr(B|A) = \frac{1}{4}$

$\frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1}{4}$

$\frac{p}{\Pr(A)} = \frac{1}{4}$

$\Pr(A) = 4p$ **[1 mark]**

***VCAA Examination Report* note:**

This question was generally answered well. The most common errors included solving for $\Pr(B)$, and incorrectly transposing $\frac{p}{\Pr(A)}$ to yield $\frac{1}{4}$.

b. $\Pr(A|B) = \frac{1}{5}$

$\frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1}{5}$

$\frac{p}{\Pr(B)} = \frac{1}{5}$

$\Pr(B) = 5p$

	A		
B	p	$4p$	$5p$
B'	$3p$	$1 - 8p$	$1 - 5p$
	$4p$	$1 - 4p$	

$\Pr(A' \cap B') = 1 - 8p$

Award 1 mark for the Karnaugh map or Venn diagram or alternative correct method.

Award 1 mark for the correct probability.

***VCAA Examination Report* note:**

Students who scored highly usually used a table or a Venn diagram to arrive at their answer. There were various misconceptions of the connection between conditional probabilities and $\Pr(A' \cap B')$. Many students assumed that events A and B were independent, hence incorrectly used $\Pr(A' \cap B') = \Pr(A') \times \Pr(B')$.

c. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$= 4p + 5p - p$

$= 8p$

$\Pr(A \cup B) \leq \frac{1}{5}$

$8p \leq \frac{1}{5}$

$0 < p \leq \frac{1}{40}$

Award 1 mark for the correct inequality.

Award 1 mark for the correct values.

***VCAA Examination Report* note:**

Most students identified that $\Pr(A \cup B) = 8p$. Only a few students identified the correct interval because students did not consider that in this case $p \neq 0$. Common incorrect answers included $p = \frac{1}{40}$ or $p \leq \frac{1}{40}$ (allowing negative probabilities) and $0 \leq p \leq \frac{1}{40}$.

Question 8

a. $\Pr(B) = \Pr(B|1) + \Pr(B|2)$ **[1 mark]**

$= \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{2}{4} = \frac{1}{2} + \frac{1}{4}$

$= \frac{3}{4}$ **[1 mark]**

***VCAA Examination Report* note:**

This question was generally well answered. Many students showed their reasoning via a tree diagram or some written explanation. Some students overworked the problem by trying to use the binomial distribution.

b. $\Pr(1|B) = \frac{\Pr(1 \cap B)}{\Pr(B)}$ **[1 mark]**

$= \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$ **[1 mark]**

***VCAA Examination Report* note:**

Students generally recognised the conditional probability (reduced sample space). Some students incorrectly worked Pr(Black|Box1), resulting in a probability greater than 1, which is not feasible.

Question 9

a. $\Pr(F|A) = \frac{5}{100}, \ \Pr(F|B) = \frac{8}{100}, \ \Pr(A) = \frac{40}{90}, \ \Pr(B) = \frac{50}{90}$

$\Pr(F) = \Pr(F|A)\Pr(A) + \Pr(F|B)\Pr(B)$

$= \frac{5}{100} \times \frac{40}{90} + \frac{8}{100} \times \frac{50}{90} = \frac{1}{9 \times 5} + \frac{2}{9 \times 5}$

$= \frac{1}{15} \quad (b = 15)$

Award 1 mark for using conditional probability.

Award 1 mark for the correct probability.

***VCAA Assessment Report* note:**

While a tree was not a required to answer the question, it may have assisted some students to determine the two required cases. Many students stated probabilities greater than 1.

b. $\Pr(A|F) = \frac{\Pr(A \cap F)}{\Pr(F)}$

$= \frac{\frac{40}{90} \times \frac{5}{100}}{\frac{1}{15}}$

$= \frac{\frac{1}{45}}{\frac{1}{15}}$

$= \frac{1}{3} \quad (c = 3)$

Award 1 mark for the correct probability

***VCAA Assessment Report* note:**

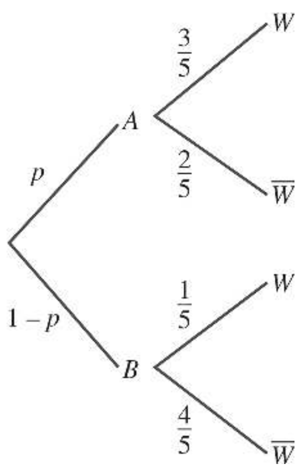
In general the conditional probability was recognised but not the reduced sample space. Often the instruction regarding the form of the final answer was overlooked.

Question 10

a. $\Pr(W) = p \times \frac{3}{5} + (1-p) \times \frac{1}{5}$

$= \frac{3p}{5} + \frac{1-p}{5}$

$\Pr(W) = \frac{2p+1}{5}$



[1 mark]

***VCAA Assessment Report* note:**

Many students made good use of a tree diagram in their formulation of a solution. Some students left their answer unsimplified as a sum of two products. A significant number of students offered a final expression not in terms of *p*.

b. i. $\Pr(B|W) = \frac{\Pr(B \cap W)}{\Pr(W)}$

$= \frac{\frac{1-p}{5}}{\frac{2p+1}{5}}$

$\Pr(B|W) = \frac{1-p}{2p+1}$

Award 1 mark for applying rules for conditional probability.

Award 1 mark for the correct answer.

***VCAA Assessment Report* note:**

While most students recognised that this question involved conditional probability, many could not apply it within the context of the specific question. Algebraic fractions were not handled well.

ii. $\frac{1-p}{2p+1} = 0.3$

$\frac{1-p}{2p+1} = \frac{3}{10}$

$10(1-p) = 3\,(2p+1)$

$10 - 10p = 6p + 3$

$16p = 7$

$p = \frac{7}{16}$ **[1 mark]**

Question 11

a. Pr (at least one) = 1 − Pr (no walks)

$$= 1 - \frac{1}{4} \times \frac{2}{3}$$
$$= 1 - \frac{1}{6}$$
$$= \frac{5}{6}$$

Award 1 mark for considering cases.
Award 1 mark for the correct probability.

VCAA Assessment Report **note:**

Many students attempted to use matrices but did not recognise the basic nature of the problem. A tree diagram or listing the sample space were the best options.

b. i. $\Pr(\text{walk}) = \frac{5}{8} \times \frac{3}{4} + \frac{3}{8} \times \frac{1}{3}$

$$= \frac{15}{32} + \frac{1}{8}$$
$$= \frac{19}{32}$$

Award 1 mark for considering the tree diagram of both cases.
Award 1 mark for the correct final probability.

VCAA Assessment Report **note:**

Many students made a good attempt at this question. Most students correctly identified the required sum of two products; however, made errors in the evaluation of the final fraction.

ii. $\Pr(\text{pleasant}|\text{walk}) = \frac{\Pr(P \cap W)}{\Pr(W)}$

$$= \frac{\frac{5}{8} \times \frac{3}{4}}{\frac{19}{32}}$$
$$= \frac{15}{19}$$

Award 1 mark for the conditional.
Award 1 mark for the correct final probability.

VCAA Assessment Report **note:**

Many students were able to identify the conditional probability and use their answer to part bi. in the denominator; however, used an incorrect numerator.

Question 12

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1}{8} = \frac{\Pr(A \cap B)}{\frac{1}{5}}$$

$$\Pr(A \cap B) = \frac{1}{40}$$

$$\Pr(A' \cap B') = \frac{3}{8} - \frac{7}{40} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

	A	A'	
B	$\frac{1}{40}$	$\frac{7}{40}$	$\frac{1}{5}$
B'	$\frac{3}{5}$	?	$\frac{4}{5}$
	$\frac{5}{8}$	$\frac{3}{8}$	

The correct answer is **A**.

Question 13

Let L be left handed and G wears glasses, given $\Pr(L) = 0.2,\ \Pr(G) = 0.4$

and $\Pr(G'|L') = \dfrac{\Pr(G' \cap L')}{\Pr(L')} = \dfrac{\Pr(G' \cap L')}{0.8} = 0.52$

$\Rightarrow \Pr(G' \cap L') = 0.416$

$\Pr(L|G') = \dfrac{\Pr(L \cap G')}{\Pr(G')} = \dfrac{0.184}{0.6} = 0.307$

	L	L'	
G	0.016	0.384	0.4
G'	0.184	0.416	0.6
	0.2	0.8	

The correct answer is **C**.

10 The binomial distribution

Topic	**10**	**The binomial distribution**
Subtopic	**10.2**	**Bernoulli trials**

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Question 1 (4 marks)

It has been found that when a breast ultrasound is combined with a common mammogram, the rate in which breast cancer is detected in a group of women is 7.2 per 1000. Noa is due for her two-yearly mammography testing, which will involve an ultrasound combined with a mammogram. Let Z be the discrete random variable that breast cancer is detected.

a. Calculate the probability that Noa has breast cancer detected at this next test. **(1 mark)**

b. Construct a probability distribution table for Z. **(1 mark)**

c. Calculate $\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma)$. **(2 marks)**

Question 2 (3 marks)

Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation for this distribution is 0.4936.

a. Calculate the variance of Y correct to 4 decimal places. **(1 mark)**

b. Calculate the probability of success, p, if $p > 1 - p$. **(1 mark)**

c. Determine $\text{E}(Y)$. **(1 mark)**

Question 3 (3 marks)

Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation of Y is 0.3316.

a. Calculate the variance correct to 2 decimal places. **(1 mark)**

b. Calculate the probability of success correct to 4 decimal places if $\Pr(\text{success}) > \Pr(\text{failure})$. **(2 marks)**

Topic	10	The binomial distribution
Subtopic	10.3	The binomial distribution

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***Source:** VCE 2021, Mathematical Methods Exam 2, Section A, Q17; © VCAA*

Question 1 (1 mark)

A discrete random variable X has a binomial distribution with a probability of success of $p = 0.1$ for n trials, where $n > 2$.

If the probability of obtaining at least two successes after n trials is at least 0.5, then the smallest possible value of n is

A. 15
B. 16
C. 17
D. 18
E. 19

***Source:** VCE 2017, Mathematical Methods Exam 2, Section A, Q18; © VCAA*

Question 2 (1 mark)

Let X be a discrete random variable with binomial distribution $X \sim \text{Bi}(n, p)$. The mean and the standard deviation of this distribution are equal.

Given that $0 < p < 1$, the smallest number of trials, n, such that $p \leq 0.01$ is

A. 37
B. 49
C. 98
D. 99
E. 101

Topic 10 | Subtopic 10.3 The binomial distribution

Source: *VCE 2015, Mathematical Methods (CAS) Exam 2, Section 1, Q10; © VCAA*

Question 3 (1 mark)

The binomial random variable, X, has $\mathrm{E}(X) = 2$ and $\mathrm{Var}(X) = \frac{4}{3}$.

$\Pr(X = 1)$ is equal to

A. $\left(\frac{1}{3}\right)^6$

B. $\left(\frac{2}{3}\right)^6$

C. $\frac{1}{3} \times \left(\frac{2}{3}\right)^2$

D. $6 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^5$

E. $6 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^5$

Question 4 (1 mark)

In a shipment of electrical conductors, it is found that 8% are defective.

A sample of 30 conductors is selected at random.

The standard deviation of the number of defective conductors selected is closest to

A. 4.80

B. 2.4

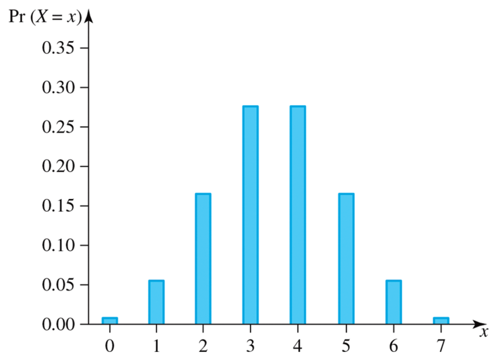
C. 2.19

D. 2.21

E. 1.49

Question 5 (1 mark)

The probability distribution of a binomial random variable X, with, is shown graphically below. The most likely value for p is equal to



A. 0.2
B. 0.3
C. 0.4
D. 0.5
E. 0.6

Question 6 (1 mark)

Describe the probability distribution function $\Pr(X = x) = \binom{n}{x} p^x(1-p)^{n-x}, x = 0, 1....n$, for $n = 5$ and $p = 0.2$.

Topic	10	The binomial distribution
Subtopic	10.4	Applications

online only

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***Source:** VCE 2021, Mathematical Methods Exam 2, Section A, Q6; © VCAA*

Question 1 (1 mark)

The probability of winning a game is 0.25.

The probability of winning a game is independent of winning any other game.

If Ben plays 10 games, the probability that he will win exactly four times is closest to

A. 0.1460
B. 0.2241
C. 0.9219
D. 0.0781
E. 0.7759

***Source:** VCE 2019, Mathematical Methods Exam 2, Section A, Q8; © VCAA*

Question 2 (1 mark)

An archer can successfully hit a target with a probability of 0.9. The archer attempts to hit the target 80 times. The outcome of each attempt is independent of any other attempt.

Given that the archer successfully hits the target at least 70 times, the probability that the archer successfully hits the target exactly 74 times, correct to four decimal places, is

A. 0.3635
B. 0.8266
C. 0.1494
D. 0.3005
E. 0.1701

***Source:** VCE 2013, Mathematical Methods (CAS) Exam 2, Section 1, Q9; © VCAA*

Question 3 (1 mark)

Harry is a soccer player who practises penalty kicks many times each day.

Each time Harry takes a penalty kick, the probability that he scores a goal is 0.7, independent of any other penalty kick. One day Harry took 20 penalty kicks.

Given that he scored at least 12 goals, the probability that Harry scored exactly 15 goals is closest to

A. 0.1789
B. 0.8867
C. 0.8
D. 0.6396
E. 0.2017

Topic	10	The binomial distribution
Subtopic	10.5	Review

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Source: *VCE 2016, Mathematical Methods Exam 1, Q4; © VCAA*

Question 1 (3 marks)

A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then returned to the paddock.

a. What is the probability that the number of tagged sheep selected on a given day is zero? **(1 mark)**

b. What is the probability that at least one tagged sheep is selected on a given day? **(1 mark)**

c. What is the probability that no tagged sheep are selected on each of six consecutive days? Express your answer in the form $\left(\frac{a}{b}\right)^c$ where a, b and c are positive integers. **(1 mark)**

Source: *VCE 2020, Mathematical Methods Exam 1, Q5; © VCAA*

Question 2 (4 marks)

For a certain population the probability of a person being born with the specific gene SPGE1 is $\frac{3}{5}$.
The probability of a person having this gene is independent of any other person in the population having this gene.

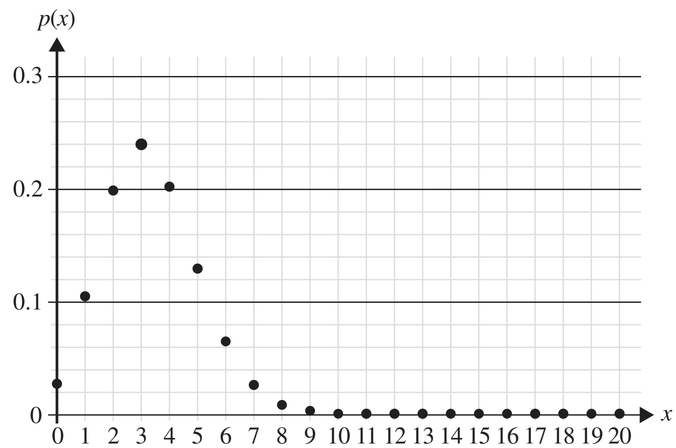
a. In a randomly selected group of four people, what is the probability that three or more people have the SPGE1 gene? **(2 marks)**

b. In a randomly selected group of four people, what is the probability that exactly two people have the SPGE1 gene, given that at least one of those people has the SPGE1 gene? Express your answer in the form $\frac{a^3}{b^4 - c^4}$, where $a, b, c \in Z^+$. **(2 marks)**

Source: *VCE 2020, Mathematical Methods Exam 2, Section A, Q19; © VCAA*

Question 3 (1 mark)

Shown below is the graph of p, which is the probability function for the number of times, x, that a '6' is rolled on a fair six-sided die in 20 trials.



Let q be the probability function for the number of times, w, that *a* '6' is **not** rolled on a fair six-sided die in 20 trials. $q(w)$ is given by

A. $p(20 - w)$

B. $p\left(1 - \frac{w}{20}\right)$

C. $p\left(\frac{w}{20}\right)$

D. $p(w - 20)$

E. $1 - p(w)$

***Source:** VCE 2020, Mathematical Methods Exam 2, Section A, Q8; © VCAA*

Question 4 (1 mark)

Items are packed in boxes of 25 and the mean number of defective items per box is 1.4.

Assuming that the probability of an item being defective is binomially distributed, the probability that a box contains more than three defective items, correct to three decimal places, is

A. 0.037
B. 0.048
C. 0.056
D. 0.114
E. 0.162

Question 5 (1 mark)

In a shipment of electrical conductors, it is found that 8% are defective.

A sample of 30 conductors is selected at random.

The standard deviation of the number of defective conductors selected is closest to

A. 4.80
B. 2.4
C. 2.19
D. 2.21
E. 1.49

***Source:** VCE 2021, Mathematical Methods 2, Section A, Q15; © VCAA*

Question 6 (1 mark)

Four fair coins are tossed at the same time.

The outcome for each coin is independent of the outcome for any other coin.

The probability that there is an equal number of heads and tails, given that there is at least one head, is

A. $\frac{1}{2}$
B. $\frac{1}{3}$
C. $\frac{3}{4}$
D. $\frac{2}{5}$
E. $\frac{4}{7}$

Source: VCE 2021, Mathematical Methods 1, Q6; © VCAA

Question 7 (6 marks)

An online shopping site sells boxes of doughnuts.

A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are:

- glazed, with custard
- glazed, with no custard
- not glazed, with custard
- not glazed, with no custard.

It is known that, in the box:

- $\frac{1}{2}$ of the doughnuts are with custard
- $\frac{7}{10}$ of the doughnuts are not glazed
- $\frac{1}{10}$ of the doughnuts are glazed, with custard.

a. A doughnut is chosen at random from the box.
Find the probability that it is not glazed, with custard. **(1 mark)**

b. The 20 doughnuts in the box are randomly allocated to two new boxes, Box A and Box B.
Each new box contains 10 doughnuts.
One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random.
Let g be the number of glazed doughnuts in Box A.
Find the probability, in terms of g, that the doughnut comes from Box B given that it is glazed. **(2 marks)**

c. The online shopping site has over one million visitors per day.
It is known that half of these visitors are less than 25 years old.
Let $\hat{p}$ be the random variable representing the proportion of visitors who are less than 25 years old in a random sample of five visitors.
Find $\Pr(\hat{p} \geq 0.8)$. Do not use a normal approximation. **(3 marks)**

Question 8 (1 mark)

It is found that 60% of all trains do not run on schedule. In any one day there are 50 trains running, the mean and standard deviation of the number of trains that are not on schedule is given by

A. 30, 12
B. 30, 3.46
C. 3, 2
D. 3, 2.82
E. 3, 1.68

Question 9 (1 mark)

An examination consists of 40 multiple-choice questions, each having 5 possible answers. If a student guesses the answer to every question, determine the expected number of questions that the student expects to get right.

Question 10 (3 marks)

Let X be a discrete random variable with a binomial distribution. The mean of X is 3 and the variance of X is 2.1. Find the values of n and p.

Answers and marking guide

10.2 Bernoulli trials

Question 1

a. Pr(breast cancer) = 0.0072 **[1 mark]**

b.

z	0	1
$\Pr(Z = z)$	0.9928	0.0072

[1 mark]

c. $\mu = \text{E}(Z) = 0.0072$

$\text{Var}(Z) = pq = 0.0072 \times 0.9928 = 0.0071$

$\sigma = \text{SD}(Z) = \sqrt{0.0071} = 0.0845$ **[1 mark]**

$\mu - 2\sigma = 0.0072 - 2(0.0845) = -0.1618$

$\mu + 2\sigma = 0.0072 + 2(0.0845) = 0.1762$

$$\begin{aligned}\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) &= \Pr(-0.1618 \le Z \le 0.1762)\\ &= \Pr(Z = 0)\\ &= 0.9928\end{aligned}$$ **[1 mark]**

Question 2

a. $\text{SD}(Y) = 0.4936$

$\text{Var}(Y) = 0.4936^2 = 0.2436$ **[1 mark]**

b. $\text{Var}(Y) = p(1-p) = 0.2436$

$p - p^2 = 0.2436$

$0 = p^2 - p + 0.2436$

Therefore, $p = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(0.21)436}}{2(1)}$

$p = \dfrac{1 \pm \sqrt{1 - 0.9744}}{2}$

$p = \dfrac{1 \pm 0.16}{2}$

$p = \dfrac{0.84}{2}$ or $\dfrac{1.16}{2}$

$p = 0.42$ or 0.58

But $p > 1 - p$ so $p = 0.58$ **[1 mark]**

c. $\text{E}(Y) = p = 0.58$ **[1 mark]**

Question 3

a. $\text{SD}(Y) = 0.3316$

$\text{Var}(Y) = 0.3316^2 = 0.11$ **[1 mark]**

b. $\text{Var}(Y) = p(1-p) = 0.11$

$p - p^2 = 0.11$

$0 = p^2 - p + 0.11$

$p = 0.1258$ or 0.8742 **[1 mark]**

Since $p > 1 - p$, $p = 0.8742$. **[1 mark]**

10.3 The binomial distribution

Question 1

$X \overset{d}{=} \text{Bi}(n = ?, p = 0.1)$
$\Pr(X \geq 2) \geq 0.5$
$1 - [\Pr(X = 0) + \Pr(X = 1)] \geq 0.5$
$0.9^n + n \times 0.1 \times 0.9^{n-1} = 0.5,\ n = 16.44$
$n = 17$
The correct answer is **C**.

Question 2

$X \sim \text{Bi}\,(n, p)$
$\text{E}(X) = np,\ \ \text{SD}(X) = \sqrt{np(1-p)}$
$\text{E}(X) = \text{SD}(X)$
$np = \sqrt{np(1-p)}$
$n^2p^2 = np(1-p)$
$n^2p^2 - np(1-p) = 0$
$np\,(np - (1-p)) = 0$ since $0 < p < 1, n > 0$
$n = \dfrac{1-p}{p}, p = 0.01 \Rightarrow n = 99$
The correct answer is **D**.

Question 3

$\text{Var}(X) = npq = \dfrac{4}{3},\ \text{E}(X) = np = 2$

$$\begin{aligned}\frac{\text{Var}(X)}{\text{E}(X)} &= \frac{npq}{np} \\ &= q \\ \therefore\ q &= \frac{\frac{4}{3}}{2} \\ &= \frac{2}{3} \\ \Rightarrow p &= \frac{1}{3} \text{ and } n = 6\end{aligned}$$

$\Pr\,(X = 1) = \dbinom{6}{1} \times \dfrac{1}{3} \times \left(\dfrac{2}{3}\right)^5$
The correct answer is **D**.

Question 4

$X \overset{d}{=} Bi\,(n = 30, p = 0.08)$

$$\begin{aligned}\text{Var}\,(X) &= npq \\ &= 30 \times 0.08 \times 0.92 \\ &= 2.21 \\ \text{Sd}\,(X) &= \sqrt{2.21} \\ &= 1.486\end{aligned}$$

The correct answer is **E**.

Question 5
The graph is symmetrical, so $p = 0.5$
The correct answer is **D**.

Question 6
Use a table to analyse the probability distribution function
$\Pr(X = x) = \binom{n}{x} p^x(1-p)^{n-x}, x = 0, 1....n$, for $n = 5$ and $p = 0.2$.

x	0	1	2	3	4	5
$n = 5$	0.328	0.41	0.205	0.051	0.007	0.0003
$p = 0.2$						

The distribution is positively skewed. **[1 mark]**

10.4 Applications

Question 1
$X \overset{d}{=} \text{Bi}(n = 10, p = 0.25)$
$\Pr(X = 4) = 0.1460$
The correct answer is **A**.

Question 2
$A \sim \text{Bi}(80, 0.9)$

$$\begin{aligned}\Pr(A = 74 \mid A \geq 70) &= \frac{\Pr(A = 74)}{\Pr(A \geq 70)} \\ &= \frac{0.1235}{0.8266} = 0.1494\end{aligned}$$

The correct answer is **C**.

Question 3
$X \sim \text{Bi}(20, 0.7)$
$\Pr(X = 15 \mid X \geq 12)$

$$= \frac{\Pr(X = 15)}{\Pr(X \geq 12)} = \frac{0.178\,863}{0.886\,69}$$
$$= 0.2017$$

The correct answer is **E**.

10.5 Review

Question 1

a. $T \sim \text{Bi}\left(4, \frac{1}{3}\right)$

$$\begin{aligned}\Pr(T = 0) &= \left(\frac{2}{3}\right)^4 \\ &= \frac{16}{81}\end{aligned}$$

Award 1 mark for the correct probability.
***VCAA Assessment Report* note:**
Incorrect responses overlooked the stipulation ‘four times each day’, thus not identifying a binomial distribution. Some students considered untagged sheep rather than tagged sheep.

b. $\Pr(T \geq 1) = 1 - \Pr(T = 0)$

$$= 1 - \frac{16}{81}$$
$$= \frac{65}{81}$$

Award 1 mark for the correct probability.

***VCAA Assessment Report* note:**

Students identified that the answer to this part of the question was simply the complement of their previous answer. However, some students wasted time in finding the sum of four probabilities, and others made arithmetic errors.

c. $\Pr(\text{no tagged sheep}) = \left(\frac{16}{81}\right)^6$

Award 1 mark for the correct probability.

***VCAA Assessment Report* note:**

The majority of students made the correct connection to part **a** and the exponent 6. The most common incorrect answer was $\left(\frac{2}{3}\right)^6$.

Question 2

a. $G \overset{d}{=} \text{Bi}\left(n = 4, p = \frac{3}{5}\right)$

$$\Pr(G \geq 3) = \Pr(G = 3) + \Pr(G = 4)$$
$$= \binom{4}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^1 + \binom{4}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^0$$
$$= 4 \times \left(\frac{3}{5}\right)^3 \times \frac{2}{5} + \left(\frac{3}{5}\right)^4$$
$$= \left(\frac{3}{5}\right)^3\left(\frac{8}{5} + \frac{3}{5}\right)$$
$$= \frac{11 \times 3^3}{5 \times 5^3}$$
$$= \frac{297}{625}$$

Award 1 mark for using binomial probabilities.
Award 1 mark for the final correct answer.

b. $\Pr(G = 2 \mid G \geq 1) = \frac{\Pr(G = 2)}{\Pr(G \geq 1)}$

$$= \frac{\Pr(G = 2)}{1 - \Pr(G = 0)}$$
$$= \frac{\binom{4}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2}{1 - \left(\frac{2}{5}\right)^4}$$

$$= \frac{\frac{6\times9\times4}{5^4}}{1-\frac{2^4}{5^4}}$$

$$= \frac{\frac{216}{5^4}}{\frac{5^4-2^4}{5^4}}$$

$$= \frac{6^3}{5^4-2^4}, a = 6, b = 5, c = 2$$

Award 1 mark for using binomial conditional probabilites.
Award 1 mark for the final correct answer.

Question 3

$$p \overset{d}{=} \text{Bi}\left(n = 20, p = \frac{1}{6}\right)$$

$$p(x) = \binom{20}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{20-x}$$

$$q \overset{d}{=} \text{Bi}\left(n = 20, p = \frac{5}{6}\right)$$

$$q(w) = \binom{20}{w}\left(\frac{5}{6}\right)^w\left(\frac{1}{6}\right)^{20-w}$$

$$\begin{aligned} p(20-w) &= \binom{20}{20-w}\left(\frac{1}{6}\right)^{20-w}\left(\frac{5}{6}\right)^{20-(20-w)} \\ &= \binom{20}{w}\left(\frac{5}{6}\right)^w\left(\frac{1}{6}\right)^{20-w} \\ &= q(w) \end{aligned}$$

The correct answer is **A**.

Question 4

$$np = 1.4$$
$$25p = 1.4$$
$$\therefore p = \frac{7}{125}$$
$$\therefore B \overset{d}{=} \text{Bi}\left(n = 25, p = \frac{7}{125}\right)$$

$\Pr(B > 3) = \Pr(B \geq 4) = 0.048$

The correct answer is **B**.

Question 5

$$X \overset{d}{=} \text{Bi}(n = 30, p = 0.08)$$

$$\begin{aligned} \text{Var}(X) &= npq \\ &= 30 \times 0.08 \times 0.92 \\ &= 2.21 \\ \text{SD}(X) &= \sqrt{2.21} \\ &= 1.486 \end{aligned}$$

The correct answer is **E**.

Question 6

$C = Bi\left(n = 4\,,\, p = \frac{1}{2}\right)$

$\Pr\left(C = 2 | C \geq 1\right) = \frac{\Pr(C = 2)}{\Pr(C \geq 1)}$

$= \frac{\Pr(C = 2)}{1 - \Pr(C = 0)} = \frac{\binom{4}{2}\left(\frac{1}{2}\right)^4}{1 - \left(\frac{1}{2}\right)^4} = \frac{\frac{6}{16}}{\frac{15}{16}} = \frac{6}{15}$

$= \frac{2}{5} = 0.4$

The correct answer is **D**.

Question 7

a. Let G represent glazed, and C represent custard doughnuts.

	G	G'	
C	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{1}{2}$
C'	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{2}$
	$\frac{3}{10}$	$\frac{7}{10}$	

$\Pr(G' \cap C) = \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$ **[1 mark]**

b. Total number of glazed doughnuts is 6, g in box A, so $6 - g$ in box B.

$\Pr(B|G) = \frac{\Pr(B \cap G)}{\Pr(G)} = \frac{\frac{1}{2} \times \left(\frac{6-g}{10}\right)}{\frac{1}{2} \times \left(\frac{6-g}{10}\right) + \frac{1}{2} \times \frac{g}{10}}$

$\Pr\left(B|G\right) = \frac{6-g}{6}$

Award 1 mark for using conditional probability.
Award 1 mark for correct probability.

c. $X \overset{d}{=} \text{Bi}\left(n = 5, p = \frac{1}{2}\right),\ \hat{P} = \frac{X}{5}$

$\Pr\left(\hat{P} \geq 0.8\right) = \Pr(X \geq 4) = \Pr(X = 4) + \Pr(X = 5)$

$\Pr\left(\hat{P} \geq 0.8\right) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5 = \frac{5}{32} + \frac{1}{32} = \frac{6}{32}$

$\Pr\left(\hat{P} \geq 0.8\right) = \frac{3}{16}$

Award 1 mark for the correct proportions to binomial.
Award 1 mark for using binomial.
Award 1 mark for the correct answer.

Question 8

$$X \overset{d}{=} Bi(n = 50, p = 0.6)$$

$$\begin{aligned} \mathrm{E}(X) &= np \\ &= 50 \times 0.6 \\ &= 30 \\ \mathrm{Var}(X) &= npq \\ &= 50 \times 0.6 \times 0.4 \\ &= 12 \\ \mathrm{Sd}(X) &= \sqrt{12} \\ &= 3.46 \end{aligned}$$

The correct answer is **B**.

Question 9

$$\begin{aligned} n = 40,\ \ p &= \frac{1}{5} \\ E(X) &= 40 \times \frac{1}{5} \\ &= 8 \end{aligned}$$ [**1 mark**]

Question 10

$$\begin{aligned} E(X) &= np \\ 3 &= np \qquad \text{(equation 1)} \\ \mathrm{Var}(X) &= np(1-p) \\ 2.1 &= np(1-p) \quad \text{(equation 2)} \end{aligned}$$

Award 1 mark for use of simultaneous equations of 1 and 2.

$$\begin{aligned} 3(1-p) &= 2.1 \\ 3 - 3p &= 2.1 \\ 3p &= 0.9 \\ p &= 0.3 \end{aligned}$$ [**1 mark**]

$$\begin{aligned} n \times 0.3 &= 3 \\ n &= 10 \end{aligned}$$ [**1 mark**]

11 Continuous probability distributions

Topic	11	Continuous probability distributions
Subtopic	11.2	Continuous random variables and probability functions

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Source: *VCE 2017 Mathematical Methods Exam 2, Section A, Q19; © VCAA*

Question 1 (1 mark)

A probability density function f is given by

$$f(x) = \begin{cases} \cos(x) + 1 & k < x < (k+1) \\ 0 & \text{elsewhere} \end{cases}$$

where $0 < k < 2$.

The value of K is

A. 1

B. $\dfrac{3\pi - 1}{2}$

C. $\pi - 1$

D. $\dfrac{\pi - 1}{2}$

E. $\dfrac{\pi}{2}$

Source: *VCE 2015 Mathematical Methods (CAS) Exam 2, Section 1, Q13; © VCAA*

Question 2 (1 mark)

The function f is a probability density function with rule

$$f(x) = \begin{cases} ae^x & 0 \le x \le 1 \\ ae & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The value of a is

A. 1

B. e

C. $\frac{1}{e}$

D. $\frac{1}{2e}$

E. $\frac{1}{2e-1}$

Question 3 (1 mark)

A continuous random variable, X, has a probability density function defined by

$$f(x) = \begin{cases} cx^2 & \text{for } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Determine which of the following options is **correct**.

A. $c = \frac{3}{26}$

B. $c = \frac{26}{3}$

C. $c = \frac{1}{14}$

D. $c = \frac{1}{9}$

E. $c = 9$

Question 4 (1 mark)

Which of the following options could not be a valid probability density function?

A. $f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$

B. $f(x) = \begin{cases} \frac{6}{9} & \text{for } 0 \leq x \leq 3 \\ \frac{6-x}{9} & \text{for } 3 \leq x \leq 6 \\ 0 & \text{elsewhere} \end{cases}$

C. $f(x) = \begin{cases} \frac{x+3}{9} & \text{for } -3 \leq x \leq 0 \\ \frac{3-x}{9} & \text{for } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$

D. $f(x) = \begin{cases} \frac{1}{9}\left|x-3\right| & \text{for } 0 \leq x \leq 6 \\ 0 & \text{elsewhere} \end{cases}$

E. $f(x) = \begin{cases} \frac{1}{3}\left(\left|x\right|-3\right) & \text{for } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$

Topic	11	Continuous probability distributions
Subtopic	11.3	The continuous probability density function

online only

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***Source:** VCE 2016 Mathematical Methods Exam 2, Section A, Q18; © VCAA*

Question 1 (1 mark)

The continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi \le x \le 5\pi \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X < a) = \frac{\sqrt{3}+2}{4}$ is

A. $\frac{19\pi}{6}$

B. $\frac{14\pi}{3}$

C. $\frac{10\pi}{3}$

D. $\frac{29\pi}{6}$

E. $\frac{17\pi}{3}$

Topic 11 | Subtopic 11.3 The continuous probability density function

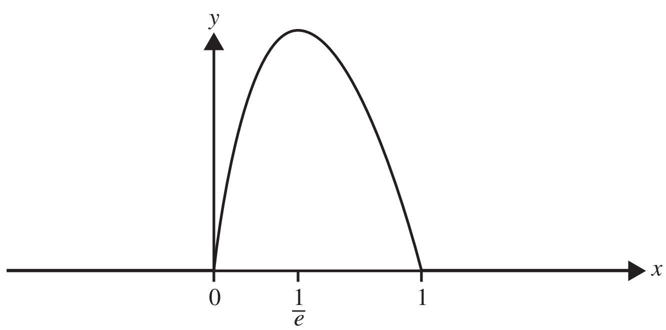
Source: *Adapted from VCE 2016 Mathematical Methods Exam 1, Q8; © VCAA*

Question 2 (4 marks)

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} -4x\log_e(x) & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Part of the graph of f is shown below. The graph has a turning point at $x = \frac{1}{e}$.



a. Show by differentiation that

$$\frac{x^k}{k^2}\left(k\log_e(x) - 1\right)$$

is an antiderivative of $x^{k-1}\log_e(x)$, where k is a positive real number. **(2 marks)**

b. Calculate $\Pr\left(X > \frac{1}{e}\right)$. **(2 marks)**

Question 3 (7 marks)

The continuous random variable Z has a probability density function defined by

$$f(z) = \begin{cases} e^{-\frac{z}{3}}, & 0 \leq z \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant. Determine:

a. the value of the constant a such that $\int_0^a f(z)dz = 1$ **(2 marks)**

b. $\Pr(0 < Z < 0.7)$, correct to 4 decimal places **(1 mark)**

c. $\Pr(Z < 0.7 | Z > 0.2)$, correct to 4 decimal places **(2 marks)**

d. the value of b, correct to 2 decimal places, such that $\Pr(Z \leq b) = 0.54$. **(2 marks)**

Topic	11	Continuous probability distributions
Subtopic	11.4	Measures of centre and spread

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Source: *VCE 2021 Mathematical Methods Exam 1, Q7; © VCAA*

Question 1 (3 marks)

A random variable X had the probability density function f given by

$$f(x) = \begin{cases} \dfrac{k}{x^2}, & 1 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

where k is a positive real number.

a. Show that $k = 2$. **(1 mark)**

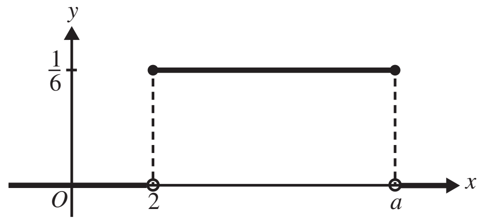
b. Find $\mathrm{E}(X)$. **(2 marks)**

Topic 11 | Subtopic 11.4 Measures of centre and spread

***Source:** VCE 2015 Mathematical Methods (CAS) Exam 2, Section 1, Q9; © VCAA*

Question 2 (1 mark)

The graph of the probability density function of a continuous random variable, X, is shown below.



If $a > 2$, then $\mathrm{E}(X)$ is equal to

A. 8

B. 5

C. 4

D. 3

E. 2

***Source:** VCE 2013 Mathematical Methods (CAS) Exam 1, Q8; © VCAA*

Question 3 (2 marks)

A continuous random variable, X, has a probability density function

$$f(x) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi x}{4}\right) & \text{if } x \in [0,\ 2] \\ 0 & \text{otherwise} \end{cases}$$

Given that $\dfrac{d}{dx}\left(x\sin\left(\dfrac{\pi x}{4}\right)\right) = \dfrac{\pi x}{4}\cos\left(\dfrac{\pi x}{4}\right) + \sin\left(\dfrac{\pi x}{4}\right)$, find $\mathrm{E}(X)$.

Question 4 (1 mark)
A continuous random variable X, has a probability density function defined by

$$f(x)=\begin{cases}12(x-1)(x-2)^2 & \text{for } 1\le x\le 2\\ 0 & \text{elsewhere}\end{cases}$$

The mean is

A. $x = 1$

B. $x = \frac{7}{5}$

C. $x = \frac{4}{3}$

D. $x = \frac{8}{5}$

E. $x = 2$

Question 5 (2 marks)
A continuous random variable X has the probability density function defined by

$$f(x)=\begin{cases}0 & x<0\\ \frac{1}{8} & 0\le x\le 2\\ 0.6e^{-0.8(x-2)} & x>2\end{cases}$$

Find the mean.

Question 6 (3 marks)
A probability density function is defined by

$$f(x)=\begin{cases}k\left(a^2-x^2\right) & |x|<a\\ 0 & \text{otherwise}\end{cases}$$

Find the exact value of a which gives a standard deviation of 2.

Question 7 (1 mark)
A continuous random variable X, has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{x^2}{21} & \text{for } 1 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

Then the standard deviation is closest to
A. 0.527
B. 0.726
C. 3.036
D. 3.123
E. 9.743

Question 8 (1 mark)
A continuous random variable X, has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{\pi}{12} \sin\left(\dfrac{\pi x}{6}\right) & \text{for } 0 \leq x \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

Then the variance is closest to
A. 1.31
B. 1.705
C. 1.73
D. 3.27
E. 10.71

Topic	11	Continuous probability distributions
Subtopic	11.5	Linear transformations

online only

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Question 1 (5 marks)

The mean of the continuous random variable Y is known to be 3.5, and its standard deviation is 1.2.

Determine:

a. $\mathrm{E}(2 - Y)$ **(1 mark)**

b. $\mathrm{E}\left(\frac{Y}{2}\right)$ **(1 mark)**

c. $\mathrm{Var}(Y)$ **(1 mark)**

d. $\mathrm{Var}(2 - Y)$ **(1 mark)**

e. $\mathrm{Var}\left(\frac{Y}{2}\right)$. **(1 mark)**

Question 2 (7 marks)

The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} -\cos(x), & \frac{\pi}{2} \leq x \leq \pi \\ 0, & \text{elsewhere} \end{cases}$$

a. Sketch the graph of f and verify that it is a probability density function. **(2 marks)**

b. Calculate E(X) and Var(X), correct to 4 decimal places. **(2 marks)**

c. Calculate E($3X + 1$) and Var($3X + 1$), correct to 4 decimal places. **(2 marks)**

d. Calculate E(($2X - 1$)($3X - 2$)), correct to 4 decimal places. **(1 mark)**

Question 3 (5 marks)

The daily rainfall, X mm, in a particular Australian town has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{x}{k\pi} \sin\left(\dfrac{x}{3}\right), & 0 \le x \le 3\pi \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant.

a. Determine the value of the constant k. **(2 marks)**

b. Calculate the expected daily rainfall, correct to 2 decimal places. **(2 marks)**

c. During the winter the daily rainfall is better approximated by $W = 2X - 1$. Calculate the expected daily rainfall during winter, correct to 2 decimal places. **(1 mark)**

Topic	11	Continuous probability distributions
Subtopic	11.6	Review

online only

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Source: *VCE 2017, Mathematical Methods Exam 2, Section B, Q3; © VCAA*

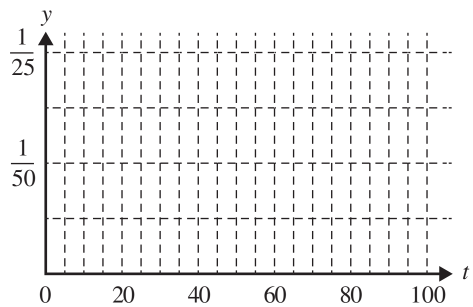
Question 1 (19 marks)

The time Jennifer spends on her homework each day varies, but she does some homework every day.

The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f, where

$$f(t) = \begin{cases} \frac{1}{625}(t-20) & 20 \le t < 45 \\ \frac{1}{625}(70-t) & 45 \le t < 70 \\ 0 & \text{elsewhere} \end{cases}$$

a. Sketch the graph of f on the axes provided below. **(3 marks)**



b. Find $\Pr(25 \le T \le 55)$. **(2 marks)**

c. Find $\Pr(T \le 25 \mid T \le 55)$. **(2 marks)**

d. Find a such that $\Pr(T \geq a) = 0.7$, correct to four decimal places. **(2 marks)**

e. The probability that Jennifer spends more than 50 minutes on her homework on any given day is $\frac{8}{25}$.

Assume that the amount of time spent on her homework on any day is independent of the time spent on her homework on any other day.

i. Find the probability that Jennifer spends more than 50 minutes on her homework on more than three of seven randomly chosen days, correct to four decimal places. **(2 marks)**

ii. Find the probability that Jennifer spends more than 50 minutes on her homework on at least two of seven randomly chosen days, given that she spends more than 50 minutes on her homework on at least one of those days, correct to four decimal places. **(2 marks)**

f. Let p be the probability that on any given day Jennifer spends more than d minutes on her homework. Let q be the probability that on two or three days out of seven randomly chosen days she spends more than d minutes on her homework.
Express q as a polynomial in terms of p. **(2 marks)**

g. i. Find the maximum value of q, correct to four decimal places, and the value of p for which this maximum occurs, correct to four decimal places. **(2 marks)**

ii. Find the value of d for which the maximum found in **part g.i.** occurs, correct to the nearest minute. **(2 marks)**

Question 2 (1 mark)

A continuous random variable X, has a probability density function defined by

$$f(x) = \begin{cases} k\sin\left(\dfrac{\pi x}{8}\right) & \text{for } 0 \le x \le 8. \\ 0 & \text{elsewhere} \end{cases}$$

Determine which of the following options is **correct**.

A. $k = \dfrac{16}{\pi}$

B. $k = \dfrac{\pi}{16}$

C. $k = \dfrac{8}{\pi}$

D. $k = \dfrac{\pi}{8}$

E. $k = 8$

Question 3 (1 mark)

A continuous random variable X, has a probability density function defined by

$$f(x) = \begin{cases} 12(x-1)(x-2)^2 & \text{for } 1 \le x \le 2. \\ 0 & \text{elsewhere} \end{cases}$$

The mean is

A. $x = 1$

B. $x = \dfrac{7}{5}$

C. $x = \dfrac{4}{3}$

D. $x = \dfrac{8}{5}$

E. $x = 2$

Question 4 (3 marks)
A probability density function is defined by

$$f(x) = \begin{cases} k\left(a^2 - x^2\right) & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Determine the exact value of a that gives a standard deviation of 2.

Question 5 (1 mark)
The time T spent waiting for a tram to arrive is a continuous random variable, and has a probability density function defined by

$$f(x) = \begin{cases} 2e^{-2t} & \text{for } t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

The variance is equal to

A. $\frac{1}{2}$

B. $\frac{1}{\sqrt{2}}$

C. $\frac{1}{4}$

D. 0.26

E. 2

Source: *VCE 2014, Mathematical Methods (CAS) 2, Section 1, Q16; © VCAA*

Question 6 (1 mark)
The continuous random variable X, with probability density function $p(x)$, has mean 2 and variance 5.

The value of $\int_{-\infty}^{\infty} x^2 p(x)\, dx$

A. 1

B. 7

C. 9

D. 21

E. 29

Answers and marking guide

11.2 Continuous random variables and probability functions

Question 1

$$f(x) = \begin{cases} \cos(x) + 1, & k < x < k + 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$\int_k^{k+1} f(x)dx = 1$$

To solve this using CAS, complete the entry line as:

$$\left(\int_k^{k+1} (\cos(x) + 1)\ dx = 1, k\right) | 0 < k < 2$$

$$\Rightarrow k = \frac{\pi - 1}{2}$$

The correct answer is **D**.

Question 2

$$f(x) = \begin{cases} ae^x, & 0 \leq x \leq 1 \\ ae, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Since the total area under the curve is 1:

$$\int_0^1 ae^x dx + \int_1^2 ae\ dx = 1$$

$$\left[ae^x\right]_0^1 + \left[aex\right]_1^2 = 1$$

$$ae - a + 2ae - ae = 1$$

$$2ae - a = 1$$

$$a(2e - 1) = 1$$

$$a = \frac{1}{2e - 1}$$

The correct answer is **E**.

Question 3

$$\int_0^3 cx^2 dx = 1$$

$$1 = c\left[\frac{x^3}{3}\right]_0^3$$

$$1 = c\left(\frac{27}{3} - 0\right)$$

$$1 = 9c$$

$$c = \frac{1}{9}$$

The correct answer is **D**.

Question 4

All of A, B, C and D have total areas of 1, (area of triangle $\frac{1}{2}bh$) and have $f(x) \geq 0$.

However option E is false, when $x = 0$ $f(x) = -1 < 0$.

The correct answer is **E**.

11.3 The continuous probability density function

Question 1

$$f(x) = \begin{cases} \frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi \le x \le 5\pi, \\ 0 & \text{elsewhere} \end{cases}$$

$$\Pr(Z < a) = \frac{\sqrt{3}+2}{4}$$

$$\int_{3\pi}^{a} \frac{1}{4}\cos\left(\frac{x}{2}\right) dx = \frac{\sqrt{3}+2}{4}$$

$$\text{LHS} = \left[\frac{1}{2}\sin\left(\frac{x}{2}\right)\right]_{3\pi}^{a} = \frac{1}{2}\sin\left(\frac{a}{2}\right) - \frac{1}{2}\sin\left(\frac{3\pi}{2}\right)$$

$$\frac{1}{2}\sin\left(\frac{a}{2}\right) + \frac{1}{2} = \frac{\sqrt{3}}{4} + \frac{1}{2}$$

$$\sin\left(\frac{a}{2}\right) = \frac{\sqrt{3}}{2}$$

$$a = \frac{14\pi}{3} \text{ since } 3\pi < a < 5\pi$$

The correct answer is **B**.

Question 2

a. Show that $\int x^{k-1}\log_e(x)\,dx = \frac{x^k}{k^2}\left(k\log_e(x) - 1\right)$.

Using the product rule in the first term,

$$\frac{d}{dx}\left[\frac{x^k}{k^2}\left(k\log_e(x) - 1\right)\right]$$

$$= \frac{d}{dx}\left[\frac{1}{k}x^k\log_e(x) - \frac{x^k}{k^2}\right]$$

$$= \frac{1}{k} \times kx^{k-1}\log_e(x) + \frac{1}{k} \times x^k \times \frac{1}{x} - \frac{k \times x^{k-1}}{k^2}$$

$$= x^{k-1}\log_e(x) + \frac{x^{k-1}}{k} - \frac{x^{k-1}}{k}$$

$$= x^{k-1}\log_e(x)$$

It follows that $\int x^{k-1}\log_e(x)\,dx = \frac{x^k}{k^2}\left(k\log_e(x) - 1\right)$.

Award 1 mark for using the product rule.

Award 1 mark for correct integration by recognition.

VCAA Assessment Report **note:**

This question was attempted well. Most students applied the product rule but struggled with the algebraic manipulation, often confusing k (a constant) with the variable x, which gave them an incorrect result. Students who expanded the expression before differentiating or those who made fewer manipulations tended to scored more highly. Some students differentiated the wrong expression.

b. $\Pr\left(X > \frac{1}{e}\right) = \int_{\frac{1}{e}}^{1} -4x\log_e(x)dx$

$$= \left[-x^2\left(2\log_e(x) - 1\right)\right]_{\frac{1}{e}}^{1}$$

$$= \left(-\left(2\log_e(1) - 1\right)\right) + \left(\frac{1}{e^2}\left(2\log_e\left(\frac{1}{e}\right) - 1\right)\right)$$

$$= 1 + \frac{1}{e^2}\left(-2\log_e(e) - 1\right)$$

$$= 1 - \frac{3}{e^2}$$

Award 1 mark for correctly applying the result from part **a**.

Award 1 mark for the correct final probability.

***VCAA Assessment Report* note:**

Students made the connection to part a. and determined $k = 2$. However, few managed to find the correct antiderivative.

Evaluation after substituting terminals was problematic. Some used incorrect terminals.

Question 3

a. $\int_0^a f(z)dz = 1$

$$\int_0^a e^{-\frac{z}{3}}dz = 1$$

$$\left[-3e^{-\frac{z}{3}}\right]_0^a = 1$$

$$-3e^{-\frac{a}{3}} + 3e^0 = 1 \qquad \textbf{[1 mark]}$$

$$-3e^{-\frac{a}{3}} + 3 = 1$$

$$-3e^{-\frac{a}{3}} = -2$$

$$e^{-\frac{a}{3}} = \frac{2}{3}$$

$$\log_e\left(\frac{2}{3}\right) = -\frac{a}{3}$$

$$-3\log_e\left(\frac{2}{3}\right) = a$$

$$-\log_e\left(\frac{3}{2}\right)^{-1} = a$$

$$a = 3\log_e\left(\frac{3}{2}\right) \qquad \textbf{[1 mark]}$$

b. $\Pr(0 < Z < 0.7) = \int_0^{0.7} e^{-\frac{z}{3}} dz$

$\Pr(0 < Z < 0.7) = \left[-3e^{-\frac{z}{3}}\right]_0^{0.7}$

$\Pr(0 < Z < 0.7) = 0.6243$ **[1 mark]**

c. $\Pr(Z < 0.7 | Z > 0.2) = \dfrac{\Pr(0.2 < Z < 0.7)}{\Pr(Z > 0.2)}$

$$\Pr(0.2 < Z < 0.7) = \int_{0.2}^{0.7} e^{-\frac{z}{3}} dz$$
$$= \left[-3e^{-\frac{z}{3}}\right]_{0.2}^{0.7}$$
$$= 0.4308$$
[1 mark]

$$\Pr(Z > 0.2) = \int_{0.2}^{3\log_e\left(\frac{3}{2}\right)} e^{-\frac{z}{3}} dz$$
$$= \left[-3e^{-\frac{z}{3}}\right]_{0.2}^{3\log_e\left(\frac{3}{2}\right)}$$
$$= 0.8065$$

$\dfrac{\Pr(0.2 < Z < 0.7)}{\Pr(Z > 2)} = \dfrac{0.43085}{0.8065} = 0.5342$ **[1 mark]**

d. $\Pr(Z \le b) = 0.54$

$\int_0^b e^{-\frac{z}{3}} dz = 0.54$

$\left[-3e^{-\frac{z}{3}}\right]_0^b = 0.54$ **[1 mark]**

Using CAS,

$b = 0.60$ **[1 mark]**

11.4 Measures of centre and spread

Question 1

a. $$f(x) = \begin{cases} \dfrac{k}{x^2}, & 1 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$\int_1^2 \frac{k}{x^2} dx = 1$$

$$k\int_1^2 x^{-2} dx = k\left[-\frac{1}{x}\right]_1^2 = k\left(-\frac{1}{2} + 1\right) = \frac{k}{2} = 1$$

$$k = 2$$

Award 1 mark for the correct proof.

b. $E(X) = \int_a^b xf(x)\, dx = \int_1^2 \frac{2}{x} dx$

$E(X) = \left[2\log_e(x)\right]_1^2 = 2\left(\log_e(2) - \log_e(1)\right)$

$E(X) = 2\log_e(2) = \log_e(4)$

Award 1 mark for solving for expectation.

Award 1 mark for the correct expectation.

Question 2

Since the total area is 1, $\frac{1}{6}(a-2)=1 \Rightarrow a=8$.

$$f(x)=\begin{cases}\frac{1}{6}, & 2 \le x \le 8\\ 0, & \text{otherwise}\end{cases}$$

$$\begin{aligned}\mathrm{E}(X) &= \int_2^8 \frac{x}{6}dx\\ &= \left[\frac{1}{12}x^2\right]_2^8\\ &= \frac{1}{12}\left(8^2-2^2\right)\\ &= 5\end{aligned}$$

The correct answer is **B**.

Question 3

$$\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)=\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)+\sin\left(\frac{\pi x}{4}\right)$$

So, $\int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)+\sin\left(\frac{\pi x}{4}\right)\,dx = x\sin\left(\frac{\pi x}{4}\right)$

$$\int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx = x\sin\left(\frac{\pi x}{4}\right)-\int \sin\left(\frac{\pi x}{4}\right)dx$$

$$\int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx = x\sin\left(\frac{\pi x}{4}\right)+\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)$$

$$\mathrm{E}(X)=\int_0^2 x\times\frac{\pi}{4}\cos\left(\frac{\pi}{x}4\right)dx$$

$$=\left[x\sin\left(\frac{\pi x}{4}\right)+\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_0^2$$

$$=2\sin\left(\frac{2}{\pi}4\right)+\frac{4}{4}4\pi\cos\left(\frac{2\pi}{4}\right)-\left(0+\frac{4}{\pi}\cos(0)\right)$$

$$=2-\frac{4}{\pi}$$

Award 1 mark for the setup of the integral and using integration by recognition.

Award 1 mark for the evaluation of E(X).

Question 4

$$\begin{aligned}E(x) &= \int_1^2 12x(x-1)(x-2)^2dx\\ &= \frac{7}{5}\end{aligned}$$

The correct answer is **B**.

Question 5

$$\begin{aligned}\mathrm{E}(X) &= \int_0^2 \frac{x}{8}dx+\int_2^\infty 0.6xe^{-0.8(x-2)}dx \quad \textbf{[1 mark]}\\ &= 0.25+2.4375\\ &= 2.6875 \quad \textbf{[1 mark]}\end{aligned}$$

Question 6

$$\int_{-a}^{a} \left(k\left(a^2 - x^2\right)\right) dx = 1$$

$$\frac{4a^3k}{3} = 1$$

$$a^3k = 0.75 \text{ (equation 1)}$$ [**1 mark**]

$$\text{Var}(X) = \text{E}\left(X^2\right) - [\text{E}(X)]^2$$

$$= \int_{-a}^{a} \left(x^2 f(x)\right) dx - \left[\int_{-a}^{a} (xf(x))\, dx\right]^2$$

$$4 = \int_{-a}^{a} \left(x^2 f(x)\right) dx - \left[\int_{-a}^{a} (xf(x))\, dx\right]^2$$

$$4 = \frac{4a^5k}{15}$$

$$15 = a^5k \quad \text{(equation 2)}$$ [**1 mark**]

$$15 = a^2 \times 0.75$$

$$a^2 = 20$$

$$a = 2\sqrt{5}$$ [**1 mark**]

Question 7

$$E(X) = \int_a^b xf(x)dx$$

$$= \int_1^4 \frac{x^3}{21} dx$$

$$= \left[\frac{x^4}{84}\right]_1^4$$

$$= \frac{1}{84}\left(4^4 - 1\right)$$

$$= 3.0357$$

$$E\left(X^2\right) = \int_a^b x^2 f(x)dx$$

$$= \int_1^4 \frac{x^4}{21} dx$$

$$= \left[\frac{x^5}{105}\right]_1^4$$

$$= \frac{1}{105}\left(4^5 - 1\right)$$

$$= 9.7429$$

$$\begin{aligned}\text{Var}(X) &= E\left(X^2\right) - (E(X))^2 \\ &= 9.7429 - 3.0357^2 \\ &= 0.5274 \\ \text{Sd}(X) &= \sqrt{\text{Var}(X)} \\ &= \sqrt{0.5274} \\ &= 0.726\end{aligned}$$

The correct answer is **B**.

Question 8

$$\begin{aligned}E(X) &= \int_a^b xf(x)dx \\ &= \int_0^6 \frac{\pi x}{12}\sin\left(\frac{\pi x}{6}\right)dx \\ &= 3 \quad \text{(using CAS)}\end{aligned}$$

$$\begin{aligned}E\left(X^2\right) &= \int_a^b x^2 f(x)\,dx \\ &= \int_0^6 \frac{\pi x^2}{12}\sin\left(\frac{\pi x}{6}\right)dx \\ &= 10.705 \\ \text{Var}(X) &= E\left(X^2\right) - (E(X))^2 \\ &= 10.705 - 3^2 \\ &= 1.705\end{aligned}$$

The correct answer is **B**.

11.5 Linear transformations

Question 1

$\text{E}(Y) = 3.5$ and $\text{SD}(Y) = 1.2$

a. $\text{E}(2 - Y) = 2 - \text{E}(Y) = 2 - 3.5 = -1.5$ **[1 mark]**

b. $\text{E}\left(\frac{Y}{2}\right) = \frac{1}{2}\text{E}(Y) = \frac{1}{2} \times 3.5 = 1.75$ **[1 mark]**

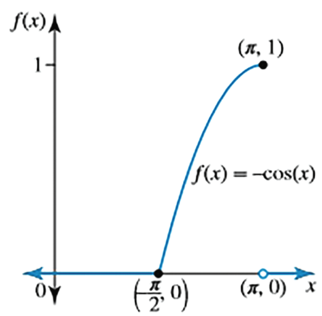
c. $\text{Var}(Y) = [\text{SD}(Y)]^2 = (1.2)^2 = 1.44$ **[1 mark]**

d. $\text{Var}(2 - Y) = (-1)^2\text{Var}(Y) = 1.44$ **[1 mark]**

e. $\text{Var}\left(\frac{Y}{2}\right) = \left(\frac{1}{2}\right)^2\text{Var}(Y) = \frac{1}{4} \times 1.44 = 0.36$ **[1 mark]**

Question 2

a. **[1 mark]**



$$\int_{\frac{\pi}{2}}^{\pi} (-\cos(x))\, dx$$
$$= [-\sin(x)]_{\frac{\pi}{2}}^{\pi}$$
$$= -\sin(\pi) + \sin\left(\frac{\pi}{2}\right)$$
$$= 0 + 1$$
$$= 1$$

This is a probability density function. **[1 mark]**

b. $\mathrm{E}(X) = \int_{\frac{\pi}{2}}^{\pi} xf(x)dx$

$\mathrm{E}(X) = \int_{\frac{\pi}{2}}^{\pi} -x\cos(x)dx$

$\mathrm{E}(X) = 2.5708$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ **[1 mark]**

$\mathrm{E}(X^2) = \int_{\frac{\pi}{2}}^{\pi} x^2 f(x)dx$

$\mathrm{E}(X^2) = \int_{\frac{\pi}{2}}^{\pi} -x^2\cos(x)dx$

$\mathrm{E}(X^2) = 6.7506$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ **[1 mark]**

$\mathrm{Var}(X) = 6.7506 - (2.5708)^2$

$\mathrm{Var}(X) = 0.1416$

c. $\mathrm{E}(3X+1) = 3\mathrm{E}(X) + 1$ **[1 mark]**

$\mathrm{E}(3X+1) = 3(2.5708) + 1$

$\mathrm{E}(3X+1) = 8.7124$

$\mathrm{Var}(3X+1) = 3^2\mathrm{Var}(X)$ **[1 mark]**

$\mathrm{Var}(3X+1) = 9(0.1416)$

$\mathrm{Var}(3X+1) = 1.2743$

d. $\mathrm{E}((2X-1)(3X-2)) = \mathrm{E}\left(6X^2 - 7X + 2\right)$

$\mathrm{E}((2X-1)(3X-2)) = 6\mathrm{E}(X^2) - 7\mathrm{E}(X) + 2$

$\mathrm{E}((2X-1)(3X-2)) = 6(6.7606) - 7(2.5708) + 2$

$\mathrm{E}((2X-1)(3X-2)) = 24.5079$ [**1 mark**]

Question 3

a.
$$\int_0^{3\pi} \frac{x}{k\pi}\sin\left(\frac{x}{3}\right)dx = 1$$
$$\frac{1}{k\pi}\int_0^{3\pi} x\sin\left(\frac{x}{3}\right)dx = 1$$
$$\frac{1}{k\pi}\left[-3x\cos\left(\frac{x}{3}\right)+9\sin\left(\frac{x}{3}\right)\right]_0^{3\pi} = 1 \qquad \textbf{[1 mark]}$$
$$(-3(3\pi)\cos(\pi)+9\sin(\pi))-(-3(0)\cos(0)+9\sin(0)) = k\pi$$
$$9\pi = k\pi$$
$$k = 9 \qquad \textbf{[1 mark]}$$

b. $\text{E}(X) = \int_0^{3\pi} xf(x)dx$ **[1 mark]**

$\text{E}(X) = \int_0^{3\pi} \frac{x^2}{9\pi}\sin\left(\frac{x}{3}\right)dx$ **[1 mark]**

$\text{E}(X) = 5.61$ mm

c. $W = 2X - 1$

$\text{E}(W) = \text{E}(2X-1)$

$\text{E}(W) = 2\text{E}(X) - 1$

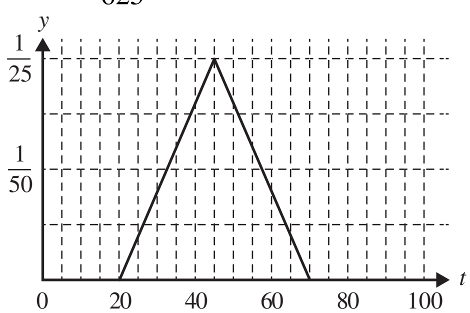
$\text{E}(W) = 2(5.6051) - 1$

$\text{E}(W) = 10.21$ mm [**1 mark**]

11.6 Review

Question 1

a. $f(t) = \begin{cases} \frac{1}{625}(t-20), & 20 \le t < 45 \\ \frac{1}{625}(70-t), & 45 \le t \le 70 \end{cases}$ $\quad f(45) = \frac{1}{25}$



Award 1 mark for the correct shape.

Award 1 mark for the correct coordinate $\left(45, \frac{1}{25}\right)$.

Award 1 mark for clearly showing zero elsewhere.

***VCAA Examination Report* note:**

Many students did not draw their graphs along the t-axis, ignoring $f(t) = 0$. Some had an open circle at $(45, 0.04)$. Others had an open circle over a closed circle at $(45, 0.04)$. Many students did not use rulers to draw the line segments. Some graphs looked like parabolas.

b. $\Pr(25 \le T \le 55)$

$$= \frac{1}{625}\int_{25}^{45}(t-20)dt + \frac{1}{625}\int_{45}^{55}(70-t)dt$$

$$= \frac{12}{25} + \frac{8}{25}$$

$$= \frac{4}{5}$$

Alternatively, solve using areas of triangles.

Award 1 mark for setting up the correct definite integrals.

Award 1 mark for the correct probability.

***VCAA Examination Report* note:**

This question was answered well. Some students had the incorrect terminals. 44 instead of 45 was occasionally given, for example, $\int_{25}^{44}(f(t))\,dt + \int_{44}^{55}(f(t))\,dt$. Others used 20 as the lower limit instead of 25.

c. $\Pr(T \le 25 | T \le 55) = \dfrac{\Pr(T \le 25)}{\Pr(T \le 55)}$

$$= \frac{\frac{1}{625}\int_{20}^{25}(t-20)dt}{\frac{1}{625}\int_{20}^{45}(t-20)dt + \frac{1}{625}\int_{45}^{55}(70-t)dt}$$

$$= \frac{\frac{1}{50}}{\frac{1}{2}+\frac{8}{25}}$$

$$= \frac{1}{41}$$

Alternatively, solve using areas of triangles.

Award 1 mark for setting up the conditional probability.

Award 1 mark for the correct probability.

***VCAA Examination Report* note:**

Many students were able to use the conditional probability formula. A common incorrect answer was $\frac{1}{40}$.

d. $\Pr(T \ge a) = 0.7 \Rightarrow \Pr(T \ge a) = 0.3$

$$\frac{1}{625}\int_{20}^{a}(t-20)dt = 0.3$$

$$\frac{a^2 - 40a + 400}{1250} = 0.3$$

then $a = 39.3649$

Award 1 mark for setting up the equation.

Award 1 mark for the correct value of a.

***VCAA Examination Report* note:**

A number of correct approaches were used. $\int_{20}^{a} f(t)dt = 0.7$, $a = 50.6351$ was a common incorrect answer.

$\int_{a}^{75} \frac{1}{625}(70-t)dt = 0.7$ was occasionally given. Some students attempted to use the inverse normal as a method.

e. i. $\Pr(T \geq 50) = \dfrac{1}{625}\displaystyle\int_{50}^{70}(70-t)dt = \dfrac{8}{25}$

$J \sim \text{Bi}\left(7, \dfrac{8}{25}\right)$

To solve for $\Pr(J > 3) = \Pr(J \geq 4)$ using CAS, complete the entry line as:

$\text{binomCdf}\left(7, \dfrac{8}{25}, 4, 7\right)$

$\Pr(J > 3) = \Pr(J \geq 4) = 0.1534$

Award 1 mark for using binomial probability.

Award 1 mark for the correct probability.

VCAA Examination Report **note:**

Many students recognised that the distribution was binomial and gave the correct n and p values. Some used $\Pr(X \geq 3)$.

ii. $\Pr(J \geq 2 \mid J \geq 1) = \dfrac{\Pr(J \geq 2)}{\Pr(J \geq 1)}$

$= \dfrac{0.711307}{0.93277}$

$= 0.7626$

Award 1 mark for setting up the conditional binomial probability.

Award 1 mark for the correct probability.

VCAA Examination Report **note:**

Many students were able to set up the conditional probability. Some wrote

$\Pr(X \geq 2|X \geq 1) = \dfrac{\Pr(X \geq 2)}{\Pr(X \geq 1)}$. Others rounded incorrectly, giving 0.7625 as the answer.

f. $J \sim \text{Bi}(n = 7, p), q = \Pr(J = 2) + \Pr(J = 3)$

$q = \binom{7}{2} p^2(1-p)^5 + \binom{7}{3} p^3(1-p)^4$

$q = 21p^2(p-1)^4 + 35p^3(1-p)^4$

$q = 7p^2(p-1)^4(2p+3)$

$q(P) = 14p^7 - 35p^6 + 70p^4 - 70p^3 + 21p^2$

Award 1 mark for setting up the sum of binomial terms in terms of p.

Award 1 mark for the correct polynomial (does not need to be expanded).

VCAA Examination Report **note:**

Of those who attempted this question, some students did not realise that the binomial distribution was required.

g. i. Solving $\dfrac{dq}{dp} = 0$ for p, since $0 < p < 1$:

$\dfrac{dq}{dp} = 98p^6 - 210p^5 + 280p^3 - 210p^2 + 42p = 0$ [**1 mark**]

gives $p = 0.3539$

and $q_{max} = q(0.3539) = 0.5665.$ [**1 mark**]

VCAA Examination Report **note:**

Some students knew to solve $q'(p) = 0$ if they had an equation in Question 3f. Others found only p. Some gave exact values for their answers.

ii. $p = \Pr(T > d) = \dfrac{1}{625}\displaystyle\int_{d}^{70}(70-t)dt = 0.3539$ [**1 mark**]

$d = 49$ minutes [**1 mark**]

***VCAA Examination Report* note:**

Some students used q instead of p in their equation, solving $\int_d^{70} f(t)dt = 0.56646...$ for d. Others solved $\int_{20}^{d} f(t)dt = 0.35388...$ obtaining $d = 41$ minutes.

Question 2

$$\int_0^8 k\sin\left(\frac{\pi x}{8}\right)dx = 1$$

$$k\left[-\frac{8}{\pi}\cos\left(\frac{\pi x}{8}\right)\right]_0^8 = 1$$

$$k\left(-\frac{8}{\pi}(\cos(\pi) - \cos(0))\right) = 1$$

$$\frac{16k}{\pi} = 1$$

$$k = \frac{\pi}{16}$$

The correct answer is **B**.

Question 3

$$\mathrm{E}(x) = \int_1^2 12x(x-1)(x-2)^2 dx$$

$$= \frac{7}{5}$$

The correct answer is **B**.

Question 4

$$\int_{-a}^{a} \left(k\left(a^2 - x^2\right)\right) dx = 1$$

$$\frac{4a^3k}{3} = 1$$

$$a^3k = 0.75 \quad \text{(equation 1)}$$ **[1 mark]**

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

$$2^2 = \int_{-a}^{a} \left(x^2 f(x)\right) dx - \left[\int_{-a}^{a} (xf(x))\, dx\right]^2$$

$$4 = \frac{4a^5k}{15}$$

$$15 = a^5k \quad \text{(equation 2)}$$ **[1 mark]**

Substitute equation 1 into equation 2:
$15 = a^2 \times 0.75$
$a^2 = 20$
$a = 2\sqrt{5}$ **[1 mark]**

Question 5

$$\begin{aligned}
\mathrm{E}(T) &= \int_a^b tf(t)dt \\
&= \int_0^\infty 2te^{-2t}dt \\
&= \frac{1}{2} \text{ (using CAS)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left(T^2\right) &= \int_a^b t^2 f(t)dt \\
&= \int_0^\infty 2t^2 e^{-2t}dt \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(T) &= \mathrm{E}\left(T^2\right) - (\mathrm{E}(T))^2 \\
&= \frac{1}{2} - \left(\frac{1}{2}\right)^2 \\
&= \frac{1}{4}
\end{aligned}$$

The correct answer is **C**.

Question 6

$E(X) = 2$
$\mathrm{Var}(X) = 5$
$\int_{-\infty}^{\infty} x^2 p(x) = E(X^2)$
$\mathrm{Var}(X) = E(X^2) - (E(X))^2$
$5 = E(X^2) - 2^2$
$E(X^2) = 5 + 4 = 9$
The correct answer is **C**.

12 The normal distribution

Topic	12	The normal distribution
Subtopic	12.2	The normal distribution

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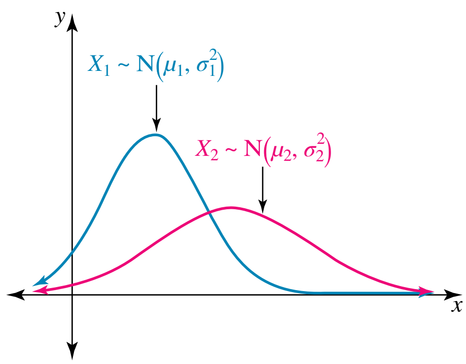
Question 1 (2 marks)
Sketch the probability density curve for the random variable X, which is normally distributed with mean 12 and standard deviation of 1.5.

Question 2 (1 mark)
The diagram shows the graphs of two normal distributions curves, with means μ_1 and μ_2 and standard deviations σ_1 and σ_2 respectively.



Select the **true** statement from the following.
A. $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$
B. $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$
C. $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$
D. $\mu_1 < \mu_2$ and $\sigma_1 > \sigma_2$
E. $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$

Topic 12 Subtopic 12.2 The normal distribution

Question 3 (2 marks)

The study score of a subject is normally distributed with a mean $\mu = 30$ and a standard deviation $\sigma = 7$.

a. Determine the approximate percentage of student scores that are between 23 and 37. **(1 mark)**

b. Determine the approximate percentage of student scores that are above 44. **(1 mark)**

Question 4 (2 marks)

The probability density function for a normal random variable X is random variable $f(x) = \frac{1}{\sqrt{18\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{3}\right)^2}$

Find the mean and standard deviation of X.

Question 5 (1 mark)

A normal distribution curve is defined by $f(x) = \frac{1}{\sqrt{6\pi}} e^{-\frac{(x-4)^2}{6}}$ for $x \in R$.

Which of the following statements is **correct**?

A. The mean is 4 and the standard deviation is 6.

B. The mean is 4 and the standard deviation is $\sqrt{6}$.

C. The mean is 4 and the standard deviation is $\sqrt{3}$.

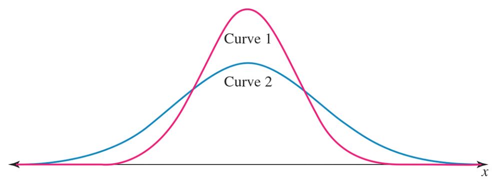
D. The mean is 4 and the standard deviation is 3.

E. The mean is 3 and the standard deviation is 3.

Question 6 (2 marks)

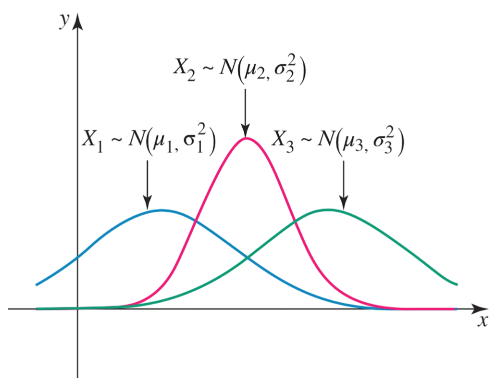
The diagram shows two normal distribution curves with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively.

Compare the means and standard deviations.



Question 7 (1 mark)

The diagram below shows the graphs of three normal distributions curves, with means μ_1, μ_2 and μ_3 standard deviations σ_1, σ_2 and σ_3 respectively.



Which of the following statements is **true**?

A. $\mu_1 < \mu_2 < \mu_3$ and $\sigma_1 = \sigma_2 = \sigma_3$

B. $\mu_1 = \mu_2 = \mu_3$ and $\sigma_1 < \sigma_2 < \sigma_3$

C. $\mu_1 < \mu_2 < \mu_3$ and $\sigma_1 = \sigma_3 > \sigma_2$

D. $\mu_1 < \mu_2 < \mu_3$ and $\sigma_1 = \sigma_2 < \sigma_3$

E. $\mu_1 > \mu_2 > \mu_3$ and $\sigma_1 = \sigma_2 < \sigma_3$

Topic	12	The normal distribution
Subtopic	12.3	Calculating probabilities and the standard normal distribution

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***Source:** VCE 2018 Mathematical Methods Exam 1, Q4; © VCAA*

Question 1 (2 marks)

Let X be a normally distributed random variable with a mean of 6 and a variance of 4. Let Z be a random variable with the standard normal distribution

a. Find $\Pr(X > 6)$. **(1 mark)**

b. Find b such that $\Pr(X > 7) = \Pr(Z < b)$. **(1 mark)**

***Source:** VCE 2016 Mathematical Methods Exam 2, Section A, Q16; © VCAA*

Question 2 (1 mark)

The random variable, X, has a normal distribution with mean 12 and standard deviation 0.25.

If the random variable, Z, has the standard normal distribution, then the probability that X is greater than 12.5 is equal to

A. $\Pr(Z < -4)$

B. $\Pr(Z < -1.5)$

C. $\Pr(Z < 1)$

D. $\Pr(Z \geq 1.5)$

E. $\Pr(Z > 2)$

Question 3 (1 mark)

If Z has the standard normal distribution, and a and b are positive real numbers, state which of the following is **false**.

A. $\Pr(Z \geq -a) = 1 - \Pr(Z < -a)$

B. $\Pr(Z \leq a) = 0.5 + \Pr(0 < Z < a)$

C. $\Pr(-a \leq Z \leq b) = \Pr(-a < Z < 0) + \Pr(0 < Z < b)$

D. $\Pr(a \leq Z \leq b) = \Pr(Z < b) + \Pr(Z > a) - 0.5$

E. $\Pr(Z = a) = 0$

Topic	12	The normal distribution
Subtopic	12.4	The inverse normal distribution

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Source: *VCE 2020 Mathematical Methods Exam 2, Section A, Q11; © VCAA*

Question 1 (1 mark)
The lengths of plastic pipes that are cut by a particular machine are a normally distributed random variable, X, with a mean of 250 mm.
Z is the standard normal random variable.
If $\Pr(X < 259) = 1 - \Pr(Z > 1.5)$, then the standard deviation of the lengths of plastic pipes, in millimetres, is
A. 1.5
B. 3
C. 6
D. 9
E. 12

Source: *VCE 2020 Mathematical Methods Exam 2, Section A, Q14; © VCAA*

Question 2 (1 mark)
The random variable X is normally distributed.
The mean of X is twice the standard deviation of X.
If $\Pr(X > 5.2) = 0.9$, then the standard deviation of X is closest to
A. 7.238
B. 14.476
C. 3.327
D. 1.585
E. 3.169

Topic 12 | Subtopic 12.4 The inverse normal distribution

Source: *VCE 2019 Mathematical Methods Exam 2, Section A, Q14; © VCAA*

Question 3 (1 mark)

The weights of packets of lollies are normally distributed with a mean of 200 g.

If 97% of these packets of lollies have a weight of more than 190 g, then the standard deviation of the distribution, correct to one decimal place, is

A. 3.3 g
B. 5.3 g
C. 6.1 g
D. 9.4 g
E. 12.1 g

Question 4 (1 mark)

Amongst a group of students, the average weight is known to be 80 kg. Approximately 16% of these students have a weight in excess of 88 kg. Assuming a normal distribution, the standard deviation of the weights in kg is closest to

A. 4
B. 5
C. 8
D. 9
E. 13

Question 5 (2 marks)

X is a continuous random variable which is normally distributed, with a mean of 14.2 and a standard deviation of 2.1. The values a_1 and a_2 are evenly distributed either side of the mean such that $\Pr(a_1 < X < a_2) = 0.8$. Find the values of a_1 and a_2.

Topic	12	The normal distribution
Subtopic	12.5	Mixed probability applications

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Source: *VCE 2013 Mathematical Methods (CAS) Exam 2, Section 1, Q22; © VCAA*

Question 1 (1 mark)

Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

A. 7 days
B. 13 days
C. 17 days
D. 21 days
E. 37 days

Source: *VCE 2014 Mathematical Methods (CAS) Exam 2, Section 2, Q4a–e; © VCAA*

Question 2 (8 marks)

Patricia is a gardener and she owns a garden nursery. She grows and sells basil plants and coriander plants. The heights, in centimetres, of the basil plants that Patricia is selling are distributed normally with a mean of 14 cm and a standard deviation of 4 cm. There are 2000 basil plants in the nursery.

a. Patricia classifies the tallest 10 per cent of her basil plants as **super**.
What is the minimum height of a super basil plant, correct to the nearest millimetre? **(1 mark)**

b. Patricia decides that some of her basil plants are not growing quickly enough, so she plans to move them to a special greenhouse. She will move the basil plants that are less than 9 cm in height.
How many basil plants will Patricia move to the greenhouse, correct to the nearest whole number? **(2 marks)**

c. The heights of the coriander plants, x centimetres, follow the probability density function $h(x)$,

$$h(x) = \begin{cases} \frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right) & 0 < x < 50 \\ 0 & \text{otherwise} \end{cases}$$

State the mean height of the coriander plants. **(1 mark)**

d. Patricia thinks that the smallest 15 per cent of her coriander plants should be given a new type of plant food.
Find the maximum height, correct to the nearest millimetre, of a coriander plant if it is to be given the new type of plant food. **(2 marks)**

e. Patricia also grows and sells tomato plants that she classifies as either **tall** or **regular**. She finds that 20 per cent of her tomato plants are tall.
A customer, Jack, selects n tomato plants at random.
Let q be the probability that at least one of Jack's n tomato plants is tall.
Find the minimum value of n so that q is greater than 0.95. **(2 marks)**

Source: *VCE 2015 Mathematical Methods (CAS) Exam 2, Section 2, Q3; © VCAA*

Question 3 (11 marks)

Mani is a fruit grower. After his oranges have been picked, they are sorted by a machine, according to size. Oranges classified as **medium** are sold to fruit shops and the remainder are made into orange juice.

The distribution of the diameter, in centimetres, of medium oranges is modelled by a continuous random variable, X, with probability density function

$$f(x) = \begin{cases} \frac{3}{4}(x-6)^2(8-x) & 6 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

a. i. Find the probability that a randomly selected medium orange has a diameter greater than 7 cm. **(2 marks)**

ii. Mani randomly selects three medium oranges.

Find the probability that exactly one of the oranges has a diameter greater than 7 cm. Express the answer in the form $\frac{a}{b}$, where a and b are positive integers. **(2 marks)**

b. Find the mean diameter of medium oranges, in centimetres. **(1 mark)**

c. For oranges classified as large, the quantity of juice obtained from each orange is a normally distributed random variable with a mean of 74 mL and a standard deviation of 9 mL.
What is the probability, correct to three decimal places, that a randomly selected large orange produces less than 85 mL of juice, given that it produces more than 74 mL of juice? **(2 marks)**

d. Mani also grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.
It is known that 3% of Mani's lemons are underweight.

i. Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places. **(2 marks)**

ii. Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load.
Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5. **(2 marks)**

Topic	12	The normal distribution
Subtopic	12.6	Review

online only

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Source: *VCE 2015 Mathematical Methods (CAS) Exam 1, Q6; © VCAA*

Question 1 (3 marks)

Let the random variable X be normally distributed with mean 2.5 and standard deviation 0.3.

Let Z be the standard normal random variable, such that $Z \sim \mathrm{N}(0, 1)$.

a. Find b such that $\Pr(X > 3.1) = \Pr(Z < b)$. **(1 mark)**

b. Using the fact that, correct to two decimal places, $\Pr(Z < -1) = 0.16$, find $\Pr(X < 2.8 | X < 2.5)$. Write the answer correct to two decimal places. **(2 marks)**

Source: *VCE 2014 Mathematical Methods (CAS) Exam 2, Section 1, Q5; © VCAA*

Question 2 (1 mark)

The random variable X has a normal distribution with mean 12 and standard deviation 0.5. If Z has the standard normal distribution, then the probability that X is less than 11.5 is equal to

A. $\Pr(Z > -1)$

B. $\Pr(Z < -0.5)$

C. $\Pr(Z > 1)$

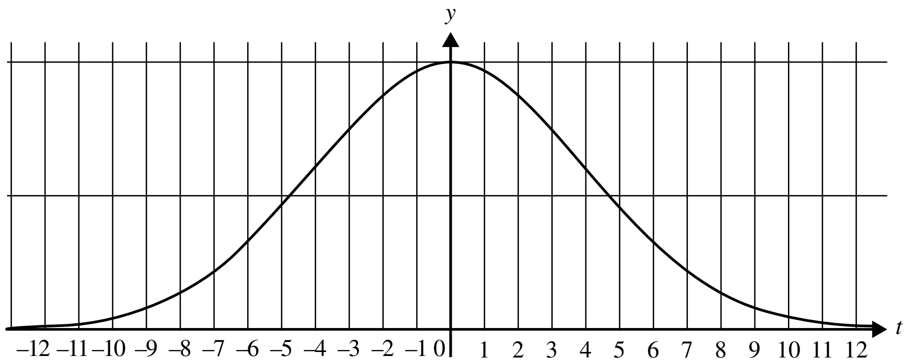
D. $\Pr(Z \geq 0.5)$

E. $\Pr(Z < 1)$

***Source:** Adapted from VCE 2020 Mathematical Methods Exam 2, Section B, Q3; © VCAA*

Question 3 (12 marks)

A transport company has detailed records of all its deliveries. The number of minutes a delivery is made before or after its scheduled delivery time can be modelled as a normally distributed random variable, T, with a mean of zero and a standard deviation of four minutes. A graph of the probability distribution of T is shown below.



a. If $\Pr(T \leq a) = 0.6$, find a to the nearest minute. **(1 mark)**

b. Find the probability, correct to three decimal places, of a delivery being no later than three minutes after its scheduled delivery time, given that it arrives after its scheduled delivery time. **(2 marks)**

c. Using the model described above, the transport company can make 46.48% of its deliveries over the interval $-3 \leq t \leq 2$.

It has an improved delivery model with a mean of k and a standard deviation of four minutes.

Find the values of k, correct to one decimal place, so that 46.48% of the transport company's deliveries can be made over the interval $-4.5 \leq t \leq 0.5$. **(3 marks)**

d. A rival transport company claims that there is a 0.85 probability that each delivery it makes will arrive on time or earlier.
Assume that whether each delivery is on time or earlier is independent of other deliveries.
Assuming that the rival company's claim is true, find the probability that on a day in which the rival company makes eight deliveries, fewer than half of them arrive on time or earlier. Give your answer correct to three decimal places. **(2 marks)**

e. Assuming that the rival company's claim is true, consider a day in which it makes n deliveries.

i. Express, in terms of n, the probability that one or more deliveries will **not** arrive on time or earlier. **(1 mark)**

ii. Hence, or otherwise, find the minimum value of n such that there is at least a 0.95 probability that one or more deliveries will **not** arrive on time or earlier. **(1 mark)**

f. An analyst from a government department believes the rival transport company's claim is only true for deliveries made before 4 pm. For deliveries made after 4 pm, the analyst believes the probability of a delivery arriving on time or earlier is x, where $0.3 \leq x \leq 0.7$.
After observing a large number of the rival transport company's deliveries, the analyst believes that the overall probability that a delivery arrives on time or earlier is actually 0.75.
Let the probability that a delivery is made after 4 pm be y.
Assuming that the analyst's beliefs are true, find the minimum and maximum values of y. **(2 marks)**

Question 4 (1 mark)

For the standard normal distribution, state which one of the following is true.

A. $\Pr(Z > a) = \Pr(Z \leq a)$

B. $\Pr(Z < -a) = \Pr(Z > -a)$

C. $\Pr(Z > a) = 1 - \Pr(Z \geq a)$

D. $\Pr(-a < Z < a) = 1 - 2\Pr(Z \leq -a)$

E. $\Pr(-a < Z < a) = 1 - 2\Pr(Z \leq a)$

Question 5 (2 marks)

X is a continuous random variable that is normally distributed, with a mean of 14.2 and a standard deviation of 2.1. The values a_1 and a_2 are evenly distributed either side of the mean such that $\Pr(a_1 < X < a_2) = 0.8$. Determine the values of a_1 and a_2.

Question 6 (1 mark)

If Z has the standard normal distribution and $\Pr(|Z| < c) = a$, where $0 < c < 3$ and $0 < a < 1$ then $\Pr(Z \geq c)$ is equal to

A. $0.5 - \frac{a}{2}$

B. $\frac{a}{2} - 0.5$

C. $2a - 1$

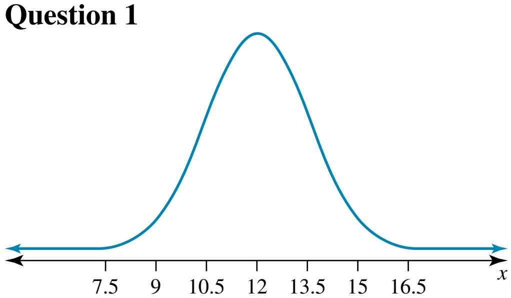
D. $a - 0.5$

E. $0.5 - a$

Answers and marking guide

12.2 The normal distribution

Question 1



Award 1 mark for the correct values on the x-axis.
Award 1 mark for the shape.

Question 2

The mean of X_2 is greater than the mean of X_1. The spread of graph 2 is greater than the spread of graph 1, so $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$
The correct answer is **E**.

Question 3

a. $\Pr(23 < X < 37) = \Pr(-1 < Z < 1)$
$= 0.68$
$\therefore 68\%$ **[1 mark]**

b. $\Pr(X > 44) = \Pr\left(Z > \dfrac{44 - 30}{7}\right)$
$= \Pr(Z > 2)$
$= \dfrac{1 - 0.95}{2}$
$= 0.025$
$= 2.5\%$ **[1 mark]**

Question 4

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$
$$= \frac{1}{\sqrt{18\pi}}e^{-\frac{1}{2}\left(\frac{x-1}{3}\right)^2}$$

Equating coefficents gives:

$x - 1 = x - \mu$
$\mu = 1$ **[1 mark]**

Equating coefficents gives:

$$\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{\sqrt{18\pi}}$$
$$= \frac{1}{3\sqrt{2\pi}}$$
$\sigma = 3$ **[1 mark]**

Question 5

The standard normal curve has the equation

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{6\pi}} e^{-\frac{(x-4)^2}{6}} \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \\ \mu &= 4 \end{aligned}$$

$$\begin{aligned} \sqrt{6\pi} &= \sigma\sqrt{2\pi} \\ 6\pi &= 2\pi\sigma^2 \\ \sigma^2 &= 3 \\ \sigma &= \sqrt{3} \end{aligned}$$

The correct answer is **C**.

Question 6

$\mu_1 = \mu_2$ **[1 mark]**

$\sigma_1{}^2 < \sigma_2{}^2$ **[1 mark]**

Question 7

For the means $\mu_1 < \mu_2 < \mu_3$

However the spreads of X_1 and X_3 are greater than the spread of X_2, therefore $\sigma_1 = \sigma_3 > \sigma_2$.

The correct answer is **C**.

12.3 Calculating probabilities and the standard normal distribution

Question 1

a. $X = \text{N}(6, 4),\ Z = \text{N}(0, 1)$

$\Pr(X > 6) = \Pr(Z > 0) = 0.5$ **[1 mark]**

***VCAA Examination Report* note:**

This question was well answered, with students recognising and applying symmetry of the normal distribution about the mean.

b.
$$\begin{aligned} \Pr(X > 7) &= \Pr\left(Z > \frac{7-6}{2}\right) \\ &= \Pr\left(z > \frac{1}{2}\right) \\ &= \Pr\left(Z < -\frac{1}{2}\right) \\ &= \Pr(Z < b) \\ b &= -\frac{1}{2} \end{aligned}$$
[1 mark]

***VCAA Examination Report* note:**

Most students understood what was required as evident by the sketch graphs of the normal distribution and relevant areas. Some students did not standardise and left their answer as 5 or mistook the variance to be the standard deviation, resulting in an answer of $-\frac{1}{4}$.

Question 2

$X = \text{N}(12, 0.25^2),\ Z = \text{N}(0, 1)$

$$\Pr(X > 12.5) = \Pr\left(Z > \frac{12.5 - 12}{0.25}\right)$$
$$= \Pr(Z > 2)$$

The correct answer is **E**.

Question 3

By symmetry all options A, B, C and E are true; therefore, D is false.

The correct answer is **D**.

12.4 The inverse normal distribution

Question 1

$X \overset{d}{=} \text{N}(\mu = 250, \sigma^2 = ?)$

$\Pr(X < 259) = 1 - \Pr(X > 259) = 1 - \Pr(Z > 1.5)$

$$1.5 = \frac{259 - 250}{\sigma} = \frac{9}{\sigma}$$

$$\sigma = \frac{9}{1.5} = 6$$

The correct answer is **C**.

Question 2

$\text{E}(X) = 2\text{SD}(X),\ X \overset{d}{=} \text{N}(2\sigma, \sigma^2)$

$\Pr(X > 5.2) = 0.9$

$\Pr(Z < z) = 0.9$

$$z = -1.28$$

$$\frac{5.2 - 2\sigma}{\sigma} = -1.28$$

$$\sigma = 7.238$$

The correct answer is **A**.

Question 3

$W \overset{d}{=} \text{N}(200, \sigma^2 = ?)$

$\Pr(W > 190) = 0.97$

$\Pr(W < 190) = 0.03$

$\Pr(Z < z) = 0.03$

$$z = -1.88$$

$$-1.88 = \frac{190 - 200}{\sigma} = -\frac{10}{\sigma}$$

$$\sigma = 5.3$$

The correct answer is **B**.

Question 4

$X \overset{d}{=} N(\mu = 80, \sigma^2 = ?)$

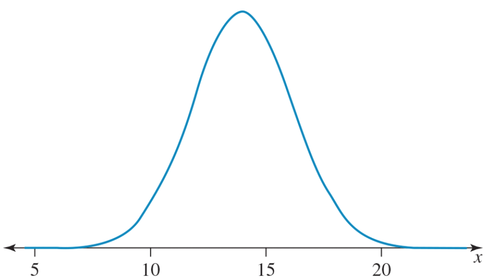
$\Pr(X > 88) = 0.16 = \Pr(Z > 0.9945)$

$$Z = \frac{88 - 80}{\sigma}$$
$$= \frac{8}{\sigma} = 0.9945$$
$$\sigma = \frac{8}{0.9945} \approx 8$$

The correct answer is **C**.

Question 5

Sketch the normal curve to assist in visualising the problem.



$\Pr(X > a_1) = 0.9$
$\quad = 1 - \Pr(X < a_1)$
$\Pr(X < a_1) = 0.1$

$$z = \frac{a_1 - 14.2}{2.1}$$
$$= -1.28155 \text{ as } \Pr(z < a_1) = 0.1$$

$a_1 = 11.5087$ **[1 mark]**

$a_2 = 14.2 + 2.691$

$a_2 = 16.8913$ **[1 mark]**

12.5 Mixed probability applications

Question 1

$T = \mathrm{N}(\mu = 120, \sigma^2 = ?^2)$

$\Pr(T < 90) = \dfrac{150}{2000} = 0.075$

$\Pr(Z < z) = 0.075$

$$z = -1.44$$
$$1.44 = \frac{90 - 120}{\sigma}$$
$$\sigma \approx 20.8$$
$$\sigma = 21$$

The correct answer is **D**.

Question 2

a. $S \sim \mathrm{N}(\mu = 14, \sigma^2 = 4^2)$

$\Pr(S < T) = 0.9$

$T = 19.1$ cm

Award 1 mark for the correct height.

***VCAA Assessment Report* note:**

Many students thought 100 mm = 1 cm, giving their final answer as 1913 mm. Others had incorrect units, such as 19.1 mm. Some entered the incorrect probability into their technology.

b. $\Pr(S < 9) = 0.1057$

$$\begin{aligned} E(S) &= 0.1057 \times 2000 \\ &= 211 \end{aligned}$$

Award 1 mark for the correct probability.

Award 1 mark for the correct number.

***VCAA Assessment Report* note:**

Some students had incorrect working, such as $\Pr(X < 9) = 0.10565... = 0.10565... \times 2000 = 211$ basil plants. Some students used $\Pr(X < 8.9)$ or $\Pr(X < 8)$. Some rounded incorrectly. Some used technology syntax in their working. Correct mathematical notation was required. Other students complicated the question by using z values. Many of these attempts were unsuccessful.

c. $$h(x) = \begin{cases} \dfrac{\pi}{100}\sin\left(\dfrac{\pi x}{50}\right) & 0 < x < 50 \\ 0 & \text{otherwise} \end{cases}$$

$$E(H) = \int_0^{50} x h(x)dx = 25$$

Award 1 mark for the correct mean height.

***VCAA Assessment Report* note:**

This question was answered well. Some students used the median formula. Others used their technology in degrees rather than radians.

d. $$\int_0^d h(x)dx = 0.15 \Rightarrow d = 12.7$$

$$d = 12.7\text{ cm}$$

Award 1 mark for writing a definite integral for the height.

Award 1 mark for solving using CAS for the correct height.

***VCAA Assessment Report* note:**

Some students attempted to use the normal distribution to answer this question. Some had incorrect units or conversions.

e. $J \sim \text{Bi}(n = ?, p = 0.2)$

$$\begin{aligned} &\Pr(J \geq 1) \\ &= 1 - \Pr(J = 0) \\ &= 1 - 0.8^n > 0.95 \end{aligned}$$

$1 - 0.8^n > 0.95$

$\Rightarrow 0.8^n < 0.05$

$\Rightarrow n > 13.85$

so $n = 14$

Award 1 mark for using at least one binomial expression.

Award 1 mark for the correct value of n.

***VCAA Assessment Report* note:**

Many students did not know to use the binomial distribution and others used the inequality sign incorrectly. Many different approaches could have been used. Many different approaches were used, including trial and error.

Question 3

a. i. $f(x)=\begin{cases}\frac{3}{4}(x-6)^2(8-x) & 6\le x\le 8\\ 0 & \text{otherwise}\end{cases}$

$\Pr(X>7)=\int_7^8 f(x)dx$ **[1 mark]**

$=\int_7^8 \frac{3}{4}(x-6)^2(8-x)dx$

$=\frac{11}{16}$ **[1 mark]**

***VCAA Assessment Report* note:**

Some students omitted the dx. Some had incorrect terminals such as

$\int_6^7 f(x)dx$, $\int_{7.0001}^8 f(x)dx$ or $\int_{6.9999}^8 f(x)dx$. Others gave the answer without showing any working.

ii. $Y\sim \text{Bi}\left(n=3, p=\frac{11}{16}\right)$ **[1 mark]**

$\Pr(Y=1)=\binom{3}{1}\times\left(\frac{11}{16}\right)\times\left(\frac{5}{16}\right)^2$

$=\frac{825}{4096}$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students were able to identify the binomial distribution with the correct n and p values.

A common incorrect answer was $\frac{11}{16}\times\left(\frac{5}{16}\right)^2=\frac{275}{4096}$.

b. $E(X)=\int_6^8 xf(x)dx$

$=\int_6^8 \frac{3x}{4}(x-6)^2(8-x)dx$

$=\left[-\frac{3x^5}{20}+\frac{15x^4}{4}-33x^3+108x^2\right]_6^8$

$=-\frac{3(8^5-6^5)}{20}+\frac{15(8^4-6^4)}{4}-33(8^3-6^3)+108\left(8^2-6^2\right)$

$=\frac{36}{5}$

$=7.2$ **[1 mark]**

***VCAA Assessment Report* note:**

Some students worked out the median, solving $\int_6^8 f(x)\,dx=0.5$ for x, instead of the mean. Others

evaluated $\int_6^8 (f(x))dx$, leaving out x.

c. $X \sim N\left(\mu = 74, \sigma^2 = 9^2\right)$

$$\Pr(X < 85 | X > 74) = \frac{\Pr(74 < X < 85)}{\Pr(X > 74)} \quad \textbf{[1 mark]}$$
$$= \frac{0.3892}{0.5}$$
$$= 0.778 \quad \textbf{[1 mark]}$$

***VCAA Assessment Report* note:**

Many students were able to recognise that the problem involved conditional probability. Some students evaluated $\frac{0.389\,18...}{0.499\,99}$ or $\frac{0.889\,188...}{0.5}$.

d. i. $L \sim Bi\left(n = 4, p = \frac{3}{100}\right)$

$$\Pr(L \geq 1) = 1 - \Pr(L = 0) \quad \textbf{[1 mark]}$$
$$= 1 - \left(\frac{97}{100}\right)^4$$
$$= 0.1147 \quad \textbf{[1 mark]}$$

***VCAA Assessment Report* note:**

Some students wrote 3% as 0.3. Others had the incorrect value for n, using $Lemons \sim \text{Bi}(3,\ 0.03)$. Some gave the answer without showing any working, while others attempted to use the normal distribution.

ii. $L \sim \text{Bi}\left(n = ?, p = \frac{3}{100}\right)$

$$\Pr(L \geq 1) = 1 - \Pr(L = 0) > 0.5 \quad \textbf{[1 mark]}$$
$$\Pr(L = 0) = \left(\frac{97}{100}\right)^n < 0.5$$
$$n = 23 \quad \textbf{[1 mark]}$$

***VCAA Assessment Report* note:**

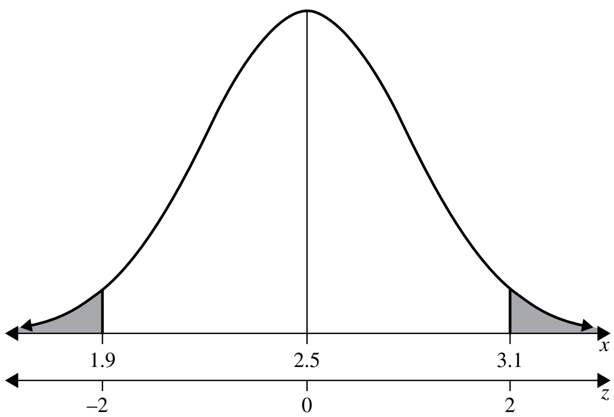
Some students rounded their answer to 22. Others did not state the minimum value, leaving their answer as $n > 22.7566$. Some students used the trial and error methods and this was acceptable. Some students did not show any working.

12.6 Review

Question 1

a. $X \sim \text{N}\left(\mu = 2.5,\ \sigma^2 = 0.3^2\right) \quad Z \sim \text{N}(0, 1) \quad Z = \frac{x - \mu}{\sigma}$

$$\Pr(X > 3.1) = \Pr\left(Z > \frac{3.1 - 2.5}{0.3}\right)$$
$$= \Pr(Z > 2)$$
$$= \Pr(Z < -2)$$
$$b = -2 \quad \textbf{[1 mark]}$$



***VCAA Assessment Report* note:**

Those students who drew a diagram of a 'normal' curve with relevant areas shaded found this helpful. An answer of +2 was common. The answer of 1.9 was also common, and was two standard deviations below the mean of X. This question required a conversion to the standard normal curve.

b. $\Pr(X < 2.8 \mid X > 2.5)$

$$\frac{\Pr(2.5 < X < 2.8)}{\Pr(X < 2.5)} = \frac{\Pr(0 < Z < 1)}{\Pr(Z < 0)}$$
$$= \frac{0.5 - \Pr(Z > 1)}{\Pr(Z < 0)}$$
$$= \frac{0.5 - \Pr(Z < -1)}{\Pr(Z < 0)}$$
$$= \frac{0.5 - 0.16}{0.5}$$
$$= \frac{0.34}{0.5}$$
$$= 0.68$$

Award 1 mark for the correct use of conditional probability.

Award 1 mark for the final correct probability.

***VCAA Assessment Report* note:**

Most students could state the relevant rule and obtained the correct denominator of $\frac{1}{2}$ but then failed to recognise that $\Pr(X < 2.8 \mid X > 2.5) = \frac{\Pr(2.5 < X < 2.8)}{\Pr(X > 2.5)}$. Probabilities greater than 1 or errors in handling decimals and/or fraction simplifications were common.

Question 2

$X \sim \text{N}\left(\mu = 12, \sigma^2 = 4^2\right)$

$$z = \frac{11.5 - 12}{0.5}$$
$$= -1$$

$\Pr(X < 11.5) = \Pr(Z < -1)$

$= \Pr(Z > 1)$ by symmetry

The correct answer is **C**.

Question 3

a. $T \overset{d}{=} \text{N}\left(\mu = 0, \sigma^2 = 16\right)$

$\Pr(T \leq a) = 0.6$

$\therefore$ a = 1 minute

Award 1 mark for the correct answer.

b. $\Pr(T \leq 3 \,|T > 0)$

$= \dfrac{\Pr(0 < T \leq 3)}{\Pr(\text{T} > 0)}$

$= \dfrac{0.273\,37}{0.5}$

$= 0.547$

Award 1 mark for considering conditional probabilities.

Award 1 mark for the correct probability.

c. $\Pr(-3 \leq \text{T} \leq 2) = 0.4648$

$D \overset{d}{=} \text{N}\left(\mu = k, \sigma^2 = 16\right)$

$\Pr(-4.5 \leq D \leq 0.5) = 0.4648,\ k = -1.5$

Using symmetry,

$\Pr(-0.5 \leq D \leq 4.5) = 0.4648,\ k = -2.5$

Award 1 mark for considering symmetry.

Award 1 mark each for the two values of k.

d. $R \overset{d}{=} \text{Bi}(n = 8, p = 0.85)$

$\Pr(R < 4) = \Pr(R \leq 3) = 0.003$

Award 1 mark for considering binomial probabilities.

Award 1 mark for the correct probability.

e. i. $R \overset{d}{=} \text{Bi}\,(n = ?, p = 0.85)$

$\Pr(\text{R} \geq 1) = 1 - \Pr(\text{R} = 0) = 1 - 0.85^n$

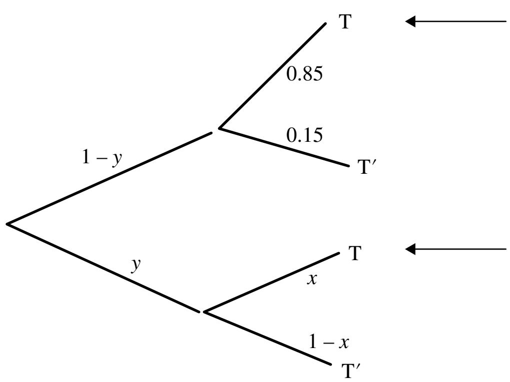
Award 1 mark for the correct expression.

ii. Solving $1 - 0.85^n > 0.95$

$n = 19$

Award 1 mark for the correct value of n.

f.



$\Pr(\text{on time}) = (1 - y) \times 0.85 + xy = 0.75,\ x \in [0.3, 0.7]$

When $x = 0.3$, solving $(1 - y) \times 0.85 + 0.3y = 0.75$ gives the minimum y-value: $y = \dfrac{2}{11}$.

When $x = 0.7$, solving $(1 - y) \times 0.85 + 0.7y = 0.75$ gives the maximum y-value: $y = \dfrac{2}{3.}$

Award 1 mark for the correct maximum y-value and 1 mark for the correct minimum y-value.

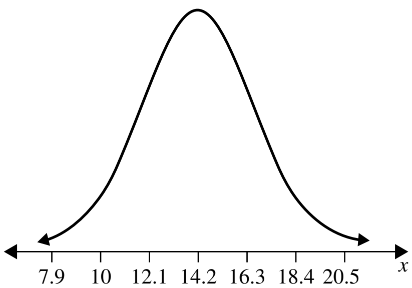
Question 4

$\Pr(-a < Z < a) = 1 - 2\Pr(Z \leq -a)$

The correct answer is **D**.

Question 5

Sketch the normal curve to assist in visualising the problem.



$$\begin{aligned}\Pr(X > a_1) &= 0.9\\ &= 1 - \Pr(X < a_1)\\ \Pr(X < a_1) &= 0.1\\ \Pr(Z < z) &= 0.1\\ z &= -1.282\\ -1.282 &= \frac{a_1 - 14.2}{2.1}\\ a_1 &= 11.5087\end{aligned}$$

[1 mark]

$$14.2 - 11.5087 = 2.691$$

$$\begin{aligned}a_2 &= 14.2 + 2.691\\ a_2 &= 16.8913\end{aligned}$$

[1 mark]

Question 6

$$\begin{aligned}\Pr(|Z| < c) = \Pr(-c < Z < c) &= a\\ \Rightarrow \Pr(0 < Z < c) &= \frac{a}{2}\\ \Pr(Z \geq c) &= 0.5 - \frac{a}{2}\end{aligned}$$

The correct answer is **A**.

13 Statistical inference

Topic	13	Statistical inference
Subtopic	13.2	Population parameters and sample statistics

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Question 1 (1 mark)
In a school population census, it is found that 12% of students attending have no access to the internet at home. The '12%' represents the value of a

A. sample
B. sample statistic
C. sample parameter
D. population
E. population parameter

Question 2 (1 mark)
To estimate the ratio of females to males at a tennis club, the coach determines the number of females and males in a particular tennis lesson. The ratio that she then calculates is called a

A. sample
B. sample statistic
C. sample parameter
D. population
E. population parameter

Question 3 (1 mark)
Select the true statement from the following.

A. Sample parameters are used to estimate population statistics.
B. Sample statistics are used to estimate population parameters.
C. Population parameters are used to estimate sample statistics.
D. Population statistics are used to estimate sample parameters.
E. Population proportion is an estimate called a point estimate.

Question 4 (1 mark)
Every fourth person who enters a particular store is selected. This sampling method is known as
A. simple random.
B. interval.
C. cluster.
D. batch.
E. stratified.

Question 5 (1 mark)
Which is not a random sampling method?
A. Simple random
B. Interval
C. Cluster
D. Quota
E. Stratified

Question 6 (1 mark)
Which of the following statements is *not* true?
A. A sampling distribution describes how a statistics value will change from sample to sample.
B. A simple random sample has a sample of size n.
C. The population proportion p is a population parameter; its value is constant.
D. The sample proportion is a sample statistic; its value is not constant but varies from sample to sample.
E. When the sample size is small, the sample proportion has an approximately normal distribution.

Question 7 (1 mark)
The mean of a population is represented by the symbol
A. $\bar{x}$
B. s
C. σ
D. μ
E. n

Topic	13	Statistical inference
Subtopic	13.3	The distribution of the sample proportion

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Source: *VCE 2017, Mathematical Methods Exam 1, Q4; © VCAA*

Question 1 (2 marks)

In a large population of fish, the proportion of angel fish is $\frac{1}{4}$.

Let $\hat{P}$ be the random variable that represents the sample proportion of angel fish for samples of size n drawn from the population.

Find the smallest integer value of n such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

Source: *VCE 2016, Mathematical Methods Exam 2, Section A, Q17; © VCAA*

Question 2 (1 mark)

Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. (Do not use a normal approximation.)

$\Pr\left(\hat{P} \geq \frac{3}{16}\right)$ is closest to

A. 0.6482
B. 0.8593
C. 0.7543
D. 0.6542
E. 0.3211

Topic 13 Subtopic 13.3 The distribution of the sample proportion

Source: *VCE 2017, Mathematical Methods Exam 2, Section A, Q16; © VCAA*

Question 3 (1 mark)

For random samples of five Australians, $\hat{P}$ is the random variable that represents the proportion who live in a capital city. Given that $\Pr\left(\hat{P} = 0\right) = \frac{1}{243}$, then $\Pr\left(\hat{P} > 0.6\right)$, correct to four decimal places, is

A. 0.0453
B. 0.3209
C. 0.4609
D. 0.5390
E. 0.7901

Question 4 (1 mark)

A random sample of 50 chocolate bars was selected, and 15% were defective. The sample proportion of a defective chocolate, $\hat{p}$, is

A. 0.15
B. 0.75
C. 0.5
D. 0.3
E. 0.45

Question 5 (1 mark)

Five CAS calculators selected from a random sample of 200 CAS calculators were found to be defective. The proportion of defectives in the CAS calculator population is

A. $\hat{p} = \frac{1}{20}$

B. $\hat{p} = \frac{1}{40}$

C. $p = \frac{1}{20}$

D. $p = \frac{1}{40}$

E. $p = \frac{1}{200}$

Question 6 (1 mark)

A random sample of 1000 people is selected from the population of a country. Of this sample, 60 people believed the Prime Minister was doing a bad job. The sample proportion who believed she was doing a good job is

A. 0.6
B. 0.06
C. 0.4
D. 0.04
E. 0.94

Question 7 (1 mark)

175 people out of 225 employees of a company were surveyed to see if they had job satisfaction. Calculate $\hat{p}$, an estimate of the population proportion who are satisfied with their job.

Question 8 (1 mark)

Of a random sample of 150 batteries of a certain brand, 3 were found to be defective. Find $\hat{p}$, an estimate of the population proportion of defective batteries.

Question 9 (2 marks)

Find the mean and standard deviation of the sample proportion if $p = 0.6$ and $n = 10$.

Question 10 (2 marks)
Find the mean and standard deviation of the sample proportion if $p = 0.2$ and $n = 50$.

Question 11 (2 marks)
If $p = 0.4$, find how large a sample is required so that $SD(\hat{p}) \leq 0.01$.

Question 12 (1 mark)
If $p = 0.3$, the sample size required so that $SD(\hat{p}) \leq 0.02$ is

A. 22
B. 23
C. 525
D. 526
E. 600

Question 13 (1 mark)
If $p = 0.4$ and $n = 25$, the mean and standard deviation for the sample proportion is

A. $\mathrm{E}(\hat{p}) = 0.4$ and $\mathrm{SD}(\hat{p}) = 0.0096$
B. $\mathrm{E}(\tilde{p}) = 0.4$ and $\mathrm{SD}(\hat{p}) = 0.098$
C. $\mathrm{E}(\tilde{p}) = 0.4$ and $\mathrm{SD}(\hat{p}) = 0.097$
D. $\mathrm{E}(\tilde{p}) = 0.6$ and $\mathrm{SD}(\hat{p}) = 0.0096$
E. $\mathrm{E}(\tilde{p}) = 0.6$ and $\mathrm{SD}(\hat{p}) = 0.098$

Topic	13	Statistical inference
Subtopic	13.4	Confidence intervals

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Source: *VCE 2021, Mathematical Methods Exam 2, Section A, Q3; © VCAA*

Question 1 (1 mark)

A box contains many coloured glass beads.

A random sample of 48 beads is selected and it is found that the proportion of blue-coloured beads in this sample is 0.125.

Based on this sample, a 95% confidence interval for the proportion of blue-coloured glass beads is

A. $(0.0314, 0.2186)$

B. $(0.0465, 0.2035)$

C. $(0.0018, 0.2482)$

D. $(0.0896, 0.1604)$

E. $(0.0264, 0.2136)$

Source: *VCE 2017, Mathematical Methods Exam 2, Section A, Q5; © VCAA*

Question 2 (1 mark)

The 95% confidence interval for the proportion of ferry tickets that are cancelled on the intended departure day is calculated from a large sample to be $(0.039, 0.121)$.

The sample proportion from which this interval was constructed is

A. 0.080

B. 0.041

C. 0.100

D. 0.062

E. 0.059

Question 3 (1 mark)
From a survey, 747 out of 1168 Year 12 students said they had obtained their learner's permit. The 99% confidence interval for Year 12 students who have obtained their learner's permit is

A. $(0.598, 0.682)$
B. $(0.612, 0.668)$
C. $(0.626, 0.654)$
D. $(0.630, 0.650)$
E. $(0.604, 0.676)$

Question 4 (3 marks)
Find an approximate 95% confidence interval for successful aces in first serves if 15 aces are scored out of 20 first serves.

Question 5 (1 mark)
Of a random sample of 150 batteries, three were found to be defective. An approximate 95% confidence interval for p, the proportion of the total population of batteries which are defective, is

A. $(0, 0.002)$
B. $(0, 0.019)$
C. $(0, 0.031)$
D. $(0, 0.043)$
E. $(0, 0.054)$

Question 6 (1 mark)
Based on sales data, 60% of purchases at a supermarket are done by tap-and-go. If random samples of 100 sales are investigated, the proportion of samples likely to include between 50% and 70% tap-and-go sales is

A. 0.30
B. 0.42
C. 0.68
D. 0.95
E. 0.96

Question 7 (1 mark)
A particular football team is supported by 15% of the population. If a random sample of 300 people is selected, the proportion of the sample which will have between 6% and 20% support for the team is

A. 0.15
B. 0.30
C. 0.90
D. 0.99
E. 1.00

Topic	13	Statistical inference
Subtopic	13.5	Review

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

***Source:** VCE 2019, Mathematical Methods Exam 1, Q6; © VCAA*

Question 1 (3 marks)

Fred owns a company that produces thousands of pegs each day. He randomly selects 41 pegs that are produced on one day and finds eight faulty pegs.

a. What is the proportion of faulty pegs in this sample? **(1 mark)**

b. Pegs are packed each day in boxes. Each box holds 12 pegs. Let $\hat{P}$ be the random variable that represents the proportion of faulty pegs in a box.

The actual proportion of faulty pegs produced by the company each day is $\frac{1}{6}$.

Find $\Pr\left(\hat{P} < \frac{1}{6}\right)$. Express your answer in the form $a(b)^n$, where a and b are positive rational numbers and n is a positive integer. **(2 marks)**

***Source:** VCE 2021, Mathematical Methods Exam 2, Section A, Q12; © VCAA*

Question 2 (1 mark)

For a certain species of bird, the proportion of birds with a crest is known to be $\frac{3}{5}$.

Let $\hat{P}$ be the random variable representing the proportion of birds with a crest in samples of size n for this specific bird.

The smallest sample size for which the standard deviation of $\hat{P}$ is less than 0.08 is

A. 7
B. 27
C. 37
D. 38
E. 43

Source: *Adapted from VCE 2016, Mathematical Methods Exam 2, Section B, Q3; © VCAA*

Question 3 (14 marks)

A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a. Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places. **(2 marks)**

b. A teacher observes that at least one of the returned laptops is **not** correctly plugged into the trolley. Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places. **(2 marks)**

c. The time for which a laptop will work without recharging (the battery life) is normally distributed, with a mean of three hours and 10 minutes and standard deviation of six minutes. Suppose that the laptops remain out of the recharging trolley for three hours.

For any one laptop, find the probability that it will stop working by the end of these three hours. Give your answer correct to four decimal places. **(2 marks)**

d. A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population of laptops with a mean battery life of three hours and 10 minutes and standard deviation of six minutes, $\hat{P}$ is the random variable of the distribution of sample proportions of laptops with a battery life of less than three hours.

Find the probability that $\Pr(\hat{P} \geq 0.06 | \hat{P} \geq 0.05)$. Give your answer correct to three decimal places. Do not use a normal approximation. **(3 marks)**

e. It is known that when laptops have been used regularly in a school for six months, their battery life is still normally distributed but the mean battery life drops to three hours. It is also known that only 12% of such laptops work for more than three hours and 10 minutes.
Find the standard deviation for the normal distribution that applies to the battery life of laptops that have been used regularly in a school for six months, correct to four decimal places. **(2 marks)**

f. The laptop supplier collects a sample of 100 laptops that have been used for six months from a number of different schools and tests their battery life. The laptop supplier wishes to estimate the proportion of such laptops with a battery life of less than three hours.
Suppose the supplier tests the battery life of the laptops one at a time.
Find the probability that the first laptop found to have a battery life of less than three hours is the third one. **(1 mark)**

g. The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.
Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give values correct to two decimal places. **(1 mark)**

h. The supplier also provides laptops to businesses. The probability density function for battery life, x (in minutes), of a laptop after six months of use in a business is

$$f(x) = \begin{cases} \dfrac{(210 - x)e^{\frac{x-210}{20}}}{400} & 0 \le x \le 210 \\ 0 & \text{elsewhere} \end{cases}$$

Find the **mean** battery life, in minutes, of a laptop with six months of business use, correct to two decimal places. **(1 mark)**

Source: *VCE 2019, Mathematical Methods Exam 2, Section B, Q4; © VCAA*

Question 4 (17 marks)

The Lorenz birdwing is the largest butterfly in Town A.

The probability density function that describes its life span, X, in weeks, is given by

$$f(x) = \begin{cases} \frac{4}{625}\left(5x^3 - x^4\right) & 0 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the mean life span of the Lorenz birdwing butterfly. **(2 marks)**

b. In a sample of 80 Lorenz birdwing butterflies, how many butterflies are expected to live longer than two weeks, correct to the nearest integer? **(2 marks)**

c. What is the probability that a Lorenz birdwing butterfly lives for at least four weeks, given that it lives for at least two weeks, correct to four decimal places? **(2 marks)**

d. The wingspans of Lorenz birdwing butterflies in Town A are normally distributed with a mean of 14.1 cm and a standard deviation of 2.1 cm.

Find the probability that a randomly selected Lorenz birdwing butterfly in Town A has a wingspan between 16 cm and 18 cm, correct to four decimal places. **(1 mark)**

e. A Lorenz birdwing butterfly is considered to be **very small** if its wingspan is in the smallest 5% of all the Lorenz birdwing butterflies in Town A.
Find the greatest possible wingspan, in centimetres, for a **very small** Lorenz birdwing butterfly in Town A, correct to one decimal place. **(1 mark)**

f. Each year, a detailed study is conducted on a random sample of 36 Lorenz birdwing butterflies in Town A. A Lorenz birdwing butterfly is considered to be **very large** if its wingspan is greater than 17.5 cm.
The probability that the wingspan of any Lorenz birdwing butterfly in Town A is greater than 17.5 cm is 0.0527, correct to four decimal places.

i. Find the probability that three or more of the butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large**, correct to four decimal places. **(1 mark)**

ii. The probability that n or more butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large** is less than 1%.
Find the smallest value of n, where n is an integer. **(2 marks)**

iii. For random samples of 36 Lorenz birdwing butterflies in Town A, $\hat{P}$ is the random variable that represents the proportion of butterflies that are **very large**.
Find the expected value and the standard deviation of $\hat{P}$, correct to four decimal places. **(2 marks)**

iv. What is the probability that a sample proportion of butterflies that are **very large** lies within one standard deviation of 0.0527, correct to four decimal places? Do not use a normal approximation. **(2 marks)**

g. The Lorenz birdwing butterfly also lives in Town B.
In a particular sample of Lorenz birdwing butterflies from Town B, an approximate 95% confidence interval for the proportion of butterflies that are **very large** was calculated to be (0.0234, 0.0866), correct to four decimal places.
Determine the sample size used in the calculation of this confidence interval. **(2 marks)**

***Source:** VCE 2018, Mathematical Methods Exam 2, Section B, Q4; © VCAA*

Question 5 (16 marks)

Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm).

The resting heart rate of adults in Mathsland is known to be normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm.

a. Find the probability that a randomly selected Mathsland adult has a resting heart rate between 60 bpm and 90 bpm. Give your answer correct to three decimal places. **(1 mark)**

b. The doctors consider a person to have a slow heart rate if the person's resting heart rate is less than 60 bpm. The probability that a randomly chosen Mathsland adult has a slow heart rate is 0.1587.
It is known that 29% of Mathsland adults play sport regularly.
It is also known that 9% of Mathsland adults play sport regularly and have a slow heart rate.
Let S be the event that a randomly selected Mathsland adult plays sport regularly and let H be the event that a randomly selected Mathsland adult has a slow heart rate.

i. Find $\Pr(H|S)$, correct to three decimal places. **(1 mark)**

ii. Are the events H and S independent? Justify your answer. **(1 mark)**

c. i. Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places. **(2 marks)**

ii. For random samples of 16 Mathsland adults, $\hat{P}$ is the random variable that represents the proportion of people who have a slow heart rate.
Find the probability that $\hat{P}$ is greater than 10%, correct to three decimal places. **(2 marks)**

iii. For random samples of n Mathsland adults, $\hat{P}_n$ is the random variable that represents the proportion of people who have a slow heart rate.
Find the least value of n for which $\Pr\left(\hat{P}_n > \frac{1}{n}\right) > 0.99$. **(2 marks)**

d. The doctors took a large random sample of adults from the population of Statsville and calculated an approximate 95% confidence interval for the proportion of Statsville adults who have a slow heart rate. The confidence interval they obtained was (0.102, 0.145).

i. Determine the sample proportion used in the calculation of this confidence interval. **(1 mark)**

ii. Explain why this confidence interval suggests that the proportion of adults with a slow heart rate in Statsville could be different from the proportion in Mathsland. **(1 mark)**

e. Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school. The time taken by a randomly selected student to reach the top of the hill has the probability density function M with the rule

$$M(t) = \begin{cases} \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where t is given in minutes.

Find the expected time, in minutes, for a randomly selected student from Mathsland Secondary College to reach the top of the hill. Give your answer correct to one decimal place. **(2 marks)**

f. Students who take less than 15 minutes to get to the top of the hill are categorised as 'elite'.

Find the probability that a randomly selected student from Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places. **(1 mark)**

g. The Year 12 students at Mathsland Secondary College make up $\frac{1}{7}$ of the total number of students at the school. Of the Year 12 students at Mathsland Secondary College, 5% are categorised as elite.

Find the probability that a randomly selected non-Year-12 student at Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places. **(2 marks)**

Source: *VCE 2015, Mathematical Methods (CAS) 2, Section 2, Q3; © VCAA*

Question 6 (11 marks)

Mani is a fruit grower. After his oranges have been picked, they are sorted by a machine, according to size. Oranges classified as medium are sold to fruit shops and the remainder are made into orange juice.

The distribution of the diameter, in centimetres, of medium oranges is modelled by a continuous random variable, X, with probability density function

$$f(x) = \begin{cases} \frac{3}{4}(x-6)^2(8-x) & 6 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

a. i. Find the probability that a randomly selected medium orange has a diameter greater than 7 cm. **(2 marks)**

ii. Mani randomly selects three medium oranges.

Find the probability that exactly one of the oranges has a diameter greater than 7 cm.

Express the answer in the form $\frac{a}{b}$, where a and b are positive integers. **(2 marks)**

b. Find the mean diameter of medium oranges, in centimetres. **(1 mark)**

c. For oranges classified as large, the quantity of juice obtained from each orange is a normally distributed random variable with a mean of 74 mL and a standard deviation of 9 mL.
What is the probability, correct to three decimal places, that a randomly selected large orange produces less than 85 mL of juice, given that it produces more than 74 mL of juice? **(2 marks)**

d. Mani also grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.
It is known that 3% of Mani's lemons are underweight.

i. Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places. **(2 marks)**

ii. Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load.
Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5 **(2 marks)**

Answers and marking guide

13.2 Population parameters and sample statistics

Question 1

The 12% is a statistic from a population; therefore, it represents a population parameter.
The correct answer is **E**.

Question 2

The sample statistic of the tennis group lesson is used to predict the population.
The correct answer is **B**.

Question 3

Sample statistics are used to estimate population parameters.
The correct answer is **B**.

Question 4

Interval sampling is a form of systematic sampling.
The correct answer is **B**.

Question 5

Quota sampling involves selecting a predetermined number of items/individuals from different population sectors for specified criteria.
The correct answer is **D**.

Question 6

When the sample size is large, the sample proportion has an approximately normal distribution.
The correct answer is **E**.

Question 7

The population parameter of the mean is μ.
The correct answer is **D**.

13.3 The distribution of the sample proportion

Question 1

$p = \frac{1}{4} \quad \text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$

$$\sqrt{\frac{\frac{1}{4} \times \frac{3}{4}}{n}} \leq \frac{1}{100}$$

$$\frac{\sqrt{3}}{4\sqrt{n}} \leq \frac{1}{100}$$

$$\frac{100\sqrt{3}}{4} \leq \sqrt{n}$$

$$\sqrt{n} \geq 25\sqrt{3}$$

$$n \geq 625 \times 3$$

$$n \geq 1875$$

Hence, the smallest value of n is 1875.
Award 1 mark for the correct method.
Award 1 mark for the correct value of n.

***VCAA Examination Report* note:**
Most students identified the correct formula; however, many were unable to correctly transpose the inequality to solve for n or to correctly manipulate the arithmetic involving rational numbers. Some students had poor use of notation work, in that they did not extend the square root sign to include n.

Question 2

$n = 16, p = 0.2, X = \text{Bi}\,(n = 16, p = 0.2)$

$\hat{p} = \dfrac{X}{n}$

$\Pr\left(\hat{p} \geq \dfrac{3}{16}\right) = \Pr(X \geq 3) = 0.6482$

The correct answer is **A**.

Question 3

$X = \text{Bi}(5, p),\ \hat{p} = \dfrac{X}{5}$

$$\begin{aligned}\Pr(\hat{p} = 0) &= \Pr(X = 0)\\ &= q^5\\ &= \frac{1}{243}\\ &\Rightarrow q = \frac{1}{3},\ p = \frac{2}{3},\ X = \text{Bi}\left(5, \frac{2}{3}\right)\end{aligned}$$

$$\begin{aligned}\Pr(\hat{p} > 0.6) &= \Pr(X > 3)\\ &= \Pr(X \geq 4)\\ &= 0.4609\end{aligned}$$

The correct answer is **C**.

Question 4

$$\begin{aligned}\hat{p} &= \frac{15}{100}\\ &= 0.15\end{aligned}$$

The correct answer is **A**.

Question 5

$$\begin{aligned}\hat{p} &= \frac{5}{200}\\ &= \frac{1}{40}\end{aligned}$$

The correct answer is **B**.

Question 6

$$\begin{aligned}\hat{p} &= \frac{940}{1000}\\ &= 0.94\end{aligned}$$

The correct answer is **E**.

Question 7

$$\begin{aligned}\hat{p} &= \frac{175}{225}\\ &= 0.78\end{aligned}$$

[**1 mark**]

Question 8

$$\begin{aligned}\hat{p} &= \frac{3}{150}\\ &= 0.02\end{aligned}$$

[**1 mark**]

Question 9

$$\begin{aligned} E(\hat{p}) &= p \\ &= 0.6 \quad \textbf{[1 mark]} \\ sd(\hat{p}) &= \sqrt{\frac{0.6 \times 0.4}{10}} \\ &= 0.155 \quad \textbf{[1 mark]} \end{aligned}$$

Question 10

$$\begin{aligned} E(\hat{p}) &= p \\ &= 0.2 \quad \textbf{[1 mark]} \\ SD(\hat{p}) &= \sqrt{\frac{0.2 \times 0.8}{50}} \\ &= 0.0566 \quad \textbf{[1 mark]} \end{aligned}$$

Question 11

$$\sqrt{\frac{0.4 \times 0.6}{n}} \le 0.01 \quad \textbf{[1 mark]}$$

$$n = 2400 \quad \textbf{[1 mark]}$$

Question 12

$$\sqrt{\frac{0.3 \times 0.7}{n}} \le 0.02$$

$n = 525$

The correct answer is **C**.

Question 13

$$\begin{aligned} E(\hat{p}) &= 0.4 \\ SD(\hat{p}) &= \sqrt{\frac{0.4 \times 0.6}{25}} \\ &= 0.098 \end{aligned}$$

The correct answer is **B**.

13.4 Confidence intervals

Question 1

$n = 48,\ \hat{p} = 0.125,\ 95\% \quad z = 1.96$

$$SD(\hat{P}) = \sqrt{\frac{0.125 \times (1 - 0.125)}{48}} = 0.0477$$

$\hat{p} \pm z\,SD(\hat{P}) = 0.125 \pm 1.96 \times 0.0477$

$\therefore CI = (0.0314, 0.2186)$

The correct answer is **A**.

Question 2

$$p = \frac{0.039 + 0.121}{2}$$

$p = 0.08$

The correct answer is **A**.

Question 3

A 99% confidence interval implies:

$0.64 \pm 2.58 \times 0.014$

$\Rightarrow (0.603, 0.676)$

Alternatively, use the inbuilt functions in CAS.

The correct answer is **E**.

Question 4

$$\hat{p} = \frac{15}{20}$$
$$= 0.75 \quad [\mathbf{1\ mark}]$$

$$\text{SD}(\hat{p}) = \sqrt{\frac{0.75 \times 0.25}{20}}$$
$$= 0.0968 \quad [\mathbf{1\ mark}]$$

$$95\%\ \text{confidence interval} = (0.75 - 2 \times 0.096\,825, 0.75 + 2 \times 0.096\,825)$$
$$= (0.556, 0.944) \quad [\mathbf{1\ mark}]$$

Question 5

$$\hat{p} = \frac{3}{150}$$
$$= 0.02$$

$$\text{SD}(\hat{p}) = \sqrt{\frac{0.02 \times 0.98}{150}}$$
$$= 0.0114 \quad [\mathbf{1\ mark}]$$

$$95\%\ \text{confidence interval} = (0.02 - 2 \times 0.011431, 0.02 + 2 \times 0.011431)$$
$$= (0.0029, 0.043) \quad [\mathbf{1\ mark}]$$
$$\approx (0, 0.043)$$

The correct answer is **D**.

Question 6

$$p = 0.6$$

$$\sigma = \sqrt{\frac{0.6 \times 0.4}{100}}$$
$$= 0.04899$$

$$\hat{p} = 0.5 \quad z = \frac{0.5 - 0.6}{0.04899}$$
$$= -2.04123$$

$$\hat{p} = 0.7 \quad z = \frac{0.7 - 0.6}{0.04899}$$
$$= 2.04123$$

$$\Pr(0.5 < \hat{p} < 0.7) = \Pr(-2.04123 < Z < 2.04123)$$
$$= 0.9587$$

The correct answer is **E**.

Question 7

$$\hat{p} = 0.15$$

$$\sigma = \sqrt{\frac{0.15 \times 0.85}{300}}$$
$$= 0.0206$$

$$\hat{p} = 0.06 \ z = \frac{0.06 - 0.15}{0.0206}$$
$$= -4.368$$

$$\hat{p} = 0.2 \ z = \frac{0.20 - 0.15}{0.0206}$$
$$= 2.42718$$

$$\Pr(0.06 < \hat{p} < 0.2) = \Pr(-4.368 < Z < 2.427)$$
$$= 0.992$$

The correct answer is **D**.

13.5 Review

Question 1

a. The proportion of faulty pegs $= \dfrac{8}{41}$ **[1 mark]**

b. $\hat{p} = \dfrac{X}{12} \quad X \stackrel{d}{=} \text{Bi}\left(n = 12, p = \dfrac{1}{6}\right)$

$$\Pr\left(\hat{p} < \frac{1}{6}\right) = \Pr(X < 2)$$
$$= \Pr(X = 0) + \Pr(X = 1)$$
$$\Pr\left(\hat{p} < \frac{1}{6}\right) = \left(\frac{5}{6}\right)^{12} + 12 \times \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11}$$
$$= \left(\frac{5}{6}\right)^{11}\left(\frac{5}{6} + 2\right)$$
$$= \frac{17}{6}\left(\frac{5}{6}\right)^{11}$$

Award 1 mark for the correct method.

Award 1 mark for the correct answer.

***VCAA Examination Report* note:**

Most students recognised this as a binomial distribution; however, few managed to correctly find the two component expressions. Even fewer successfully managed to manipulate these expressions to the format specified by the question. Another common error was to apply the standard deviation formula.

Question 2

$n = ?, \quad \hat{p} = \dfrac{3}{5}$

$$\text{SD}(\hat{P}) = \sqrt{\frac{0.6 \times (1 - 0.6)}{n}} < 0.08$$
$$\sqrt{n} > \frac{\sqrt{0.24}}{0.08}$$
$$n > \frac{0.24}{0.08^2} = 37.5$$

$n = 38$

The correct answer is **D**.

Question 3

a. $L = \text{Bi}\,(n = 22, p = 0.1)$

$$\Pr(L \geq 1) = 1 - \Pr(L = 0)$$
$$= 1 - 0.9^{22}$$
$$= 0.9015$$

Award 1 mark for using the binomial.

Award 1 mark for the correct probability.

***VCAA Assessment Report* note:**

This question was answered well. Most students recognised that it was binomial and gave the correct n and p values. Some used $\Pr(X > 1)$ instead of $\Pr(X \geq 1)$.

b. $\Pr(L < 5 \mid L \geq 1)$

$$= \frac{\Pr(1 \leq L < 5)}{\Pr(L \geq 1)}$$
$$= \frac{0.839389}{0.9015}$$
$$= 0.9311$$

Award 1 mark for using conditional probability and the binomial.
Award 1 mark for the correct probability.
VCAA Assessment Report **note:**
Many students recognised that this was a conditional probability question but had the incorrect numerator or denominator. Some included 5 in their calculations, getting 0.9798. Others rounded too soon and gave 0.9312 as the answer.

c. $X = \text{N}\left(\mu = 190,\ \sigma^2 = 36\right)$
$\Pr(X < 180) = 0.0478$
Award 1 mark for using normal cdf.
Award 1 mark for the correct probability.
VCAA Assessment Report **note:**
Some students thought 3 hours and 10 minutes was 3.1 hours and 6 minutes was 0.6 hours. Others had the standard deviation as 10 minutes. Some gave the answer without showing any working. Students are reminded that some working must be shown for questions worth more than one mark.
$\Pr(Y > 180) = 0.9522$ was a common incorrect response.

d. $S = \text{Bi}(n = 100, p = 0.0478)$
$\Pr(\hat{P} \geq 0.06 \mid \hat{P} \geq 0.05)$
$= \dfrac{\Pr(\hat{p} \geq 0.06)}{\Pr(\hat{p} \geq 0.05)}$
$= \dfrac{\Pr(S \geq 6)}{\Pr(S \geq 5)}$
$= \dfrac{0.344511}{0.523588}$
$= 0.658$
Award 1 mark for using conditional probability.
Award 1 mark for converting to the binomial.
Award 1 mark for the correct probability.
VCAA Assessment Report **note:**
Most students used the conditional probability formula but tried to use the normal distribution rather than the binomial distribution.

e. $Y = \text{N}\left(\mu = 180, \sigma^2 = ?\right)$
$\Pr(Y > 190) = 0.12$
$\Rightarrow \Pr(Y < 190) = 0.88$
$\Pr(\text{Z} < z) = 0.88$
$z = 1.17499$
$1.17499 = \dfrac{190 - 180}{\sigma} = \dfrac{10}{\sigma}$
$\sigma = 8.5107$
Award 1 mark for using the inverse normal distribution.
Award 1 mark for the correct standard deviation.

f. Since n is large, use the binomial approximation.
$\Pr(\text{third}) = \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{8}$
Award 1 mark for the correct probability.

g. $\hat{p} = \frac{6}{100} = 0.06, n = 100, 95\% \Rightarrow z = 1.96$

Confidence interval for p:

$$\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= 0.06 \pm 1.96\sqrt{\frac{0.06(1-0.06)}{100}}$$

$$= (0.01, 0.11)$$

Alternatively, use inbuilt CAS functions.

Award 1 mark for the correct confidence interval.

***VCAA Assessment Report* note:**

Students were not expected to write out the formula; the relevant computation could be done directly using technology. There were some rounding errors. A common incorrect interval was (0.01, 0.12).

h. $\text{E}(X) = \int_0^{210} \frac{x(210-x)e^{\frac{x-210}{20}}}{400}dx$

$= 170.01$ (solved using CAS)

Award 1 mark for the correct expectation.

Question 4

a. $f(x) = \begin{cases} \frac{4}{625}\left(5x^3 - x^4\right) & 0 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$

$$\text{E}(X) = \int_0^5 xf(x)dx = \int_0^5 \frac{4}{625}\left(5x^4 - x^5\right)dx = \frac{10}{3}$$

The mean life span $= 3\frac{1}{3}$ weeks

Award 1 mark for the correct method.

Award 1 mark for the correct answer.

***VCAA Examination Report* note:**

This question was done well. Some students worked out the median instead of the mean or evaluated $\int_0^5 \left(\frac{4}{625}\left(5x^3 - x^4\right)\right)dx$. Other students gave an approximate answer. Some students tried to treat f as a discrete random variable.

b. $\Pr(X > 2) = \int_2^5 f(x)dx$

$$\Pr(X > 2) = \int_2^5 \frac{4}{625}\left(5x^3 - x^4\right)dx = \frac{2853}{3125}$$

$80\Pr(X > 2) = 73$

Award 1 mark for the correct method.

Award 1 mark for the correct answer.

***VCAA Examination Report* note:**
Some students found the probability but did not multiply by 80. Other students used a discrete random variable or the normal distribution. Some students evaluated $80\int_0^2 \left(\frac{4}{625}\left(5x^3 - x^4\right)\right) dx$ or $80\int_3^5 \left(\frac{4}{625}\left(5x^3 - x^4\right)\right) dx$. Other students rounded to 74.

c. $\Pr(X \geq 4 | X \geq 2) = \dfrac{\Pr(X \geq 4)}{\Pr(X \geq 2)}$

$$\Pr(X \geq 4 | X \geq 2) = \frac{\int_4^5 f(x)dx}{\int_2^5 f(x)dx} = 0.2878$$

Award 1 mark for the correct method.
Award 1 mark for the correct answer.

***VCAA Examination Report* note:**
Many students used conditional probability. Some students used $\Pr(X \leq 4 | X \leq 2)$. Other students rounded answers too early.

d. $A \stackrel{d}{=} \mathrm{N}\left(14.1, 2.1^2\right)$
$\Pr(16 \leq A \leq 18) = 0.1512$
Award 1 mark for the correct probability.

***VCAA Examination Report* note:**
This question was answered well. Some students rounded incorrectly, giving their answer as 0.1511 or 0.1516.

e. $\Pr(A < v) = 0.05$
$v = 10.6$
Award 1 mark for the correct answer.

***VCAA Examination Report* note:**
This question was answered reasonably well. A common incorrect answer was 9.9.

f. i. $V \stackrel{d}{=} \mathrm{Bi}\,(n = 36, p = 0.0527)$
$\Pr(V \geq 3) = 0.2947$
Award 1 mark for the correct probability.

ii. $\Pr(V \geq n) \leq 0.01$
$\Pr(V \geq 6) = 0.0107 > 0.010$
$\Pr(V \geq 7) = 0.0024 < 0.01, n = 7$
Award 1 mark for the correct method.
Award 1 mark for the correct answer.

***VCAA Examination Report* note:**
A common incorrect answer was $n = 6$. Some students gave an answer without any working. Trial and error is an acceptable method.

iii. $\mathrm{E}(\hat{P}) = 0.0527$

$$\mathrm{SD}(\hat{P}) = \sqrt{\frac{p\,(1-p)}{n}} = \sqrt{\frac{0.0527\,(1-0.0527)}{36}} = 0.0372$$

Award 1 mark for the correct method.
Award 1 mark for the correct answer.

***VCAA Examination Report* note:**
Some students found $E(X) = 36 \times 0.0527 = 1.8972$. Many students were able to find the standard deviation.

iv.
$$\begin{aligned}\Pr(p - SD < \hat{P} < p + SD) &= \Pr(0.0527 - 0.0372 < \hat{P} < 0.0527 + 0.0372)\\ &= \Pr(0.0155 < \hat{P} < 0.0899) \times 36\\ &= \Pr(0.55 < V < 3.2)\\ &= \Pr(1 \le V \le 3)\\ &= 0.7380\end{aligned}$$

Award 1 mark for the correct method.
Award 1 mark for the correct answer.

***VCAA Examination Report* note:**
Many students were able to find the first interval. Some students used the normal distribution. Others rounded their answer to 0.738.

g. $$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right), \left(\hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = (0.0234, 0.0866)$$

$\hat{p} = \dfrac{0.0234 + 0.0866}{2} = 0.055$, 95% $\quad z = 1.96$

$$1.96\sqrt{\frac{0.055(1-0.055)}{n}} = \frac{0.0866 - 0.0234}{2} = 0.0316$$

$n = 199.95$
Sample size $n = 200$
Award 1 mark for the correct method.
Award 1 mark for the correct answer.

***VCAA Examination Report* note:**
Many students had the proportion as 0.0527 or 0.55 instead of 0.055. Others did not include the 1.96.

Question 5

a. Mathsland $(M) \sim N(68, 64)$
$\Pr(60 < M < 90) = 0.838$ **[1 mark]**

b. i. $\Pr(H|S) = \dfrac{\Pr(H \cap S)}{\Pr(S)} = \dfrac{0.09}{0.29} = 0.310$ **[1 mark]**

***VCAA Examination Report* note:**
This question was answered reasonably well. Some students gave their answer as 0.31. A common mistake was $\dfrac{\Pr(H)}{\Pr(S)} = \dfrac{0.1587}{0.29} = 0.547$ or $\dfrac{\Pr(H \cap S)}{\Pr(S)} = \dfrac{0.9}{0.1857}$, giving an answer greater than 1.

ii. $\Pr(H) = 0.1587, \Pr(S) = 0.29$
$\Pr(H \cap S) = 0.09$
$$\begin{aligned}\Pr(H) \times \Pr(S) &= 0.1587 \times 0.29\\ &= 0.0460\\ &\ne \Pr(H \cap S)\\ &= 0.09\end{aligned}$$ **[1 mark]**

Events H and S are not independent.

***VCAA Examination Report* note:**
A mathematical explanation was required. Some students confused mutually exclusive events with independent events. A common mistake was $\Pr(H|S) = \Pr(S)$.

c. i. $X \sim Bi(n = 16, p = 0.1587)$
$\Pr(X = 1) = \binom{16}{1} 0.1587 \times (1 - 0.1587)^{15}$ **[1 mark]**
$= 0.190$ **[1 mark]**

VCAA Examination Report **note:**
This question was reasonably well done. A method was required to get full marks. Stating the correct n and p value was sufficient. Some students gave their answer as 0.19.

ii. $n = 16, \hat{P} = \dfrac{X}{16}$

$\Pr(\hat{P} > 0.1) = \Pr(X > 1.6)$ **[1 mark]**

$= \Pr(X \geq 2) = 0.747$ **[1 mark]**

VCAA Examination Report **note:**
Some students used the normal approximation to the binomial distribution. There was poor use of variables, for example, $\Pr(\hat{P} > 0.1) = \Pr(\hat{P} > 1.6) = \Pr(\hat{P} \geq 2)$.

iii. $X \sim \text{Bi}(n = ?, p = 0.1587), \hat{P} = \dfrac{X}{n}$

$$\Pr\left(\hat{P} > \frac{1}{n}\right) = \Pr\left(\frac{X}{n} > \frac{1}{n}\right)$$

$= \Pr(X > 1) = \Pr(X \geq 2)$

$\Pr(X \geq 2) > 0.99$ **[1 mark]**

$\Pr(X \leq 1) = \Pr(X = 0) + \Pr(X = 1) \leq 0.01$

Solving $0.8413^n + n \times 0.8413^{n-1} \times 0.1587 = 0.01$

gives $n = 38.93$

so n must be at least 39. **[1 mark]**

VCAA Examination Report **note:**
This question was not answered well. Many students appeared to be confused by the terminology $\Pr\left(\hat{p}_n > \dfrac{1}{n}\right)$. $1 - \Pr(X = 0) < 0.01$ was often evaluated, giving $n = 27$. Trial and error was an acceptable method.

d. i. $\hat{p} = \dfrac{0.145 + 0.102}{2} = 0.1235$ **[1 mark]**

VCAA Examination Report **note:**
Many students tried to find the sample size rather than the proportion. $n = 900$ was often given.

ii. p (Mathsland) $= 0.1587$ is not contained within the confidence interval for Statsville, which suggests that the proportions between the two towns differ. **[1 mark]**

VCAA Examination Report **note:**
The confidence interval needed to be referred to in the answer.

e. $$M(t) = \begin{cases} \dfrac{3}{50}\left(\dfrac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} & t \geq 0 \\ 0 & t \geq 0 \end{cases}$$

$$\text{E}(T) = \int_0^{\infty} \frac{3t}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} dt$$ **[1 mark]**

$= 44.6$ min **[1 mark]**

VCAA Examination Report **note:**
Some students found the median or the mode. Others found the area under the curve. Some had one of the terminals incorrect, for example, $\int_0^{437} (t \times M(t))\, dt$. There were rounding errors; 44.7 was occasionally given.

f. $$\Pr(T < 15) = \int_0^{15} \left(\frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3}\right) dt$$

$= 0.0266$ **[1 mark]**

VCAA Examination Report **note:**

This question was answered reasonably well. $\int_0^{15} (t \times M(t))\,dt = 0.2991$ was a common incorrect answer.

g. Let x be the probability that a non-Year-12 student is elite.

$0.05 \times \frac{1}{7} + x \times \frac{6}{7} = 0.0266$ [**1 mark**]

$x = 0.0227$ [**1 mark**]

VCAA Examination Report **note:**

This question was not answered well. There were a number of other approaches to this question, for example, Karnaugh maps, tree diagrams or a conditional probability statement. A common incorrect answer was $\frac{6}{7} \times 0.0266 = 0.0228$.

Question 6

a. i. $f(x) = \begin{cases} \frac{3}{4}(x-6)^2(8-x) & 6 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$ **[1 mark]**

$$\Pr(X > 7) = \int_7^8 f(x)dx$$

$$= \int_7^8 \frac{3}{4}(x-6)^2(8-x)dx$$

$$= \frac{11}{6}$$ [**1 mark**]

VCAA Assessment Report **note:**

Some students omitted the dx. Some had incorrect terminals such as $\int_6^7 f(x)dx$, $\int_{7.0001}^8 f(x)dx$ or $\int_{6.9999}^8 f(x)dx$. Others gave the answer without showing any working.

ii. $Y \sim Bi\left(n = 3, p = \frac{11}{16}\right)$ **[1 mark]**

$$\Pr(Y = 1) = \binom{3}{1} \times \left(\frac{11}{16}\right) \times \left(\frac{5}{16}\right)^2$$

$$= \frac{825}{4096}$$ [**1 mark**]

VCAA Assessment Report **note:**

Many students were able to identify the binomial distribution with the correct n and p values.

A common incorrect answer was $\frac{11}{16} \times \left(\frac{5}{16}\right)^2 = \frac{275}{4096}$.

b. $E(X) = \int_6^8 xf(x)dx$

$$= \int_6^8 \frac{3x}{4}(x-6)^2(8-x)dx$$

$$= \frac{36}{5}$$

$= 7.2$ [**1 mark**]

***VCAA Assessment Report* note:**

Some students worked out the median, solving $\int_6^8 f(x)dx = 0.5$ for x, instead of the mean. Others evaluated $\int_6^8 (f(x))dx$, leaving out x.

c. $X \sim N\left(\mu = 74, \sigma^2 = 9^2\right)$

$$\Pr(X < 85 | X > 74) = \frac{\Pr(74 < X < 85)}{\Pr(X > 74)} \quad \textbf{[1 mark]}$$

$$= \frac{0.3892}{0.5}$$

$$= 0.778 \quad \textbf{[1 mark]}$$

***VCAA Assessment Report* note:**

Many students were able to recognise that the problem involved conditional probability. Some students evaluated $\frac{0.38918...}{0.49999}$ or $\frac{0.889188...}{0.5}$.

d. i. $M \sim Bi\left(n = 4, p = \frac{3}{100}\right)$

$$\Pr(M \geq 1) = 1 - \Pr(M = 0) \quad \textbf{[1 mark]}$$

$$= 1 - \left(\frac{97}{100}\right)^4$$

$$= 0.1147 \quad \textbf{[1 mark]}$$

***VCAA Assessment Report* note:**

Some students wrote 3% as 0.3. Others had the incorrect value for n, using *Lemons* ~ Bi (3, 0.03). Some gave the answer without showing any working, while others attempted to use the normal distribution.

ii. $L \sim Bi\left(n = ?, p = \frac{3}{100}\right)$ **[1 mark]**

$$\Pr(L \geq 1) = 1 - \Pr(L = 0) \geq 0.5$$

$$\Pr(L = 0) = \left(\frac{97}{100}\right)^n \leq 0.5$$

$$n = 23 \quad \textbf{[1 mark]}$$

***VCAA Assessment Report* note:**

Some students rounded their answer to 22. Others did not state the minimum value, leaving their answer as $n > 22.7566$. Some students used the trial and error methods and this was acceptable. Some students did not show any working.