TSSM™
Creating VCE Success

Mathematical Methods (CAS)
Teach Yourself Series
Topic 1: Curve Sketching 1 – Linear, Quadratic, Cubic & Quartic Functions

A: Level 14, 474 Flinders Street Melbourne VIC 3000
T: 1300 134 518 **W:** tssm.com.au **E:** info@tssm.com.au

Contents

Curve Sketching 1

In this topic you will learn how to sketch different types of polynomial functions.

Remainder and Factor theorems

As it appears in Unit 1 and 3

Remainder theorem:

If $p(x)$ is divided by $(ax+b)$ then the remainder is given by evaluating $p\left(\frac{-b}{a}\right)$

Factor theorem:

If $p(x)$ is divided by $(ax+b)$ and the remainder by evaluating $p\left(\frac{-b}{a}\right)$ equals 0, $(ax+b)$ is a factor.

Types of questions: Evaluating to see if $(ax+b)$ is a factor of $p(x)$ find the remainder.
Finding coefficients of $p(x)$ when given the factor or the remainder.

Equation solving:

Quadratic

$f(x)=0$
Factorise and solve
Complete the square and solve
Quadratic formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

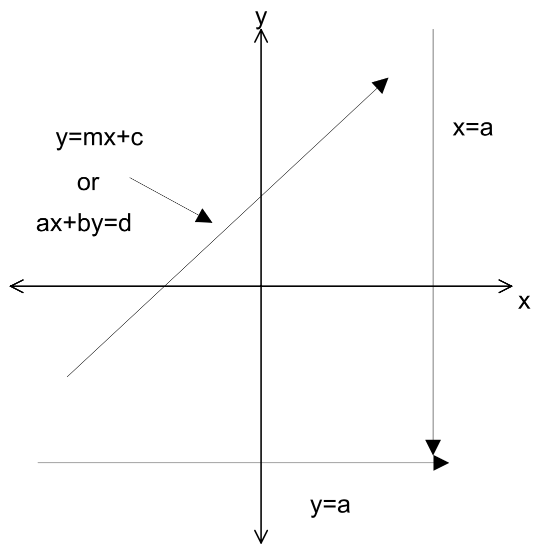
Cubic

$f(x)=0$
Factorise and solve
May have to use quadratic formula or some other method to get solutions

Quartic

$f(x)=0$
Factorise and solve
May have to use quadratic formula or some other method to get solutions

Linear Graphs



The following should be familiar:

Linear function can be written as $y = mx + c$ or $ax + by = d$.

The gradient can be calculated by: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Linear functions can be sketched by: gradient intercept method or intercept/intercept method.

If you know two points that lie on the line or the gradient and one point you can find the equation using: $y - y_1 = m(x - x_1)$. Remember $m = \frac{y_2 - y_1}{x_2 - x_1}$.

The angle that the line makes with the positive direction with the x-axis can be calculated using: $\tan\theta = m, \theta = \tan^{-1}(m)$

The distance between two points = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The angle β, which is the angle between two lines, is: $\beta = \theta_2 - \theta_1$, where θ_2, θ_1 are the angles that the two separate lines make with the positive direction with the x-axis.

The midpoint of a straight line that joins (x_1, y_1) and (x_2, y_2) is given by: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Simultaneous equations

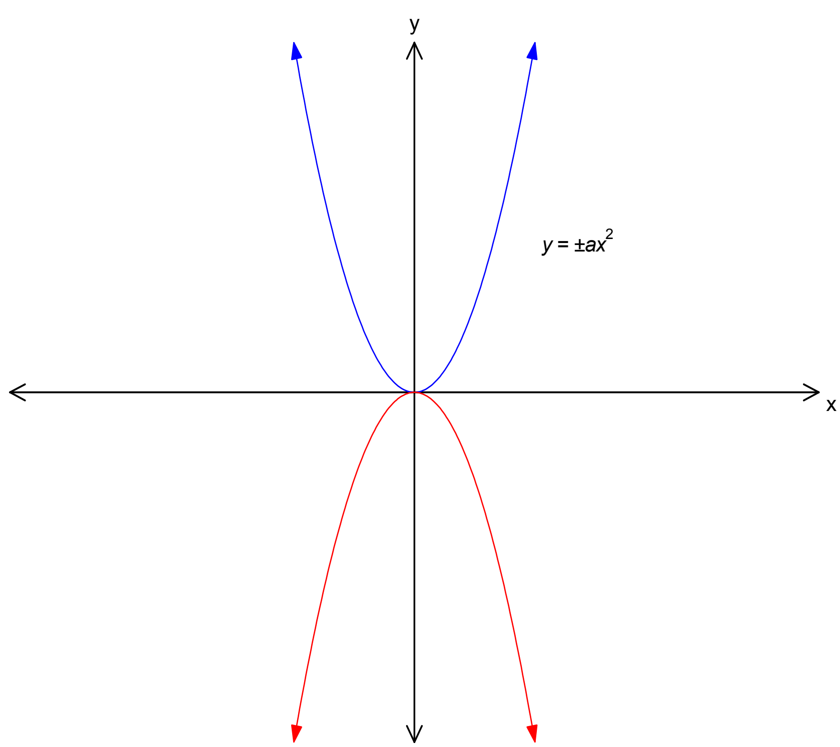
Unique solutions:
Where two equation cross – solve them

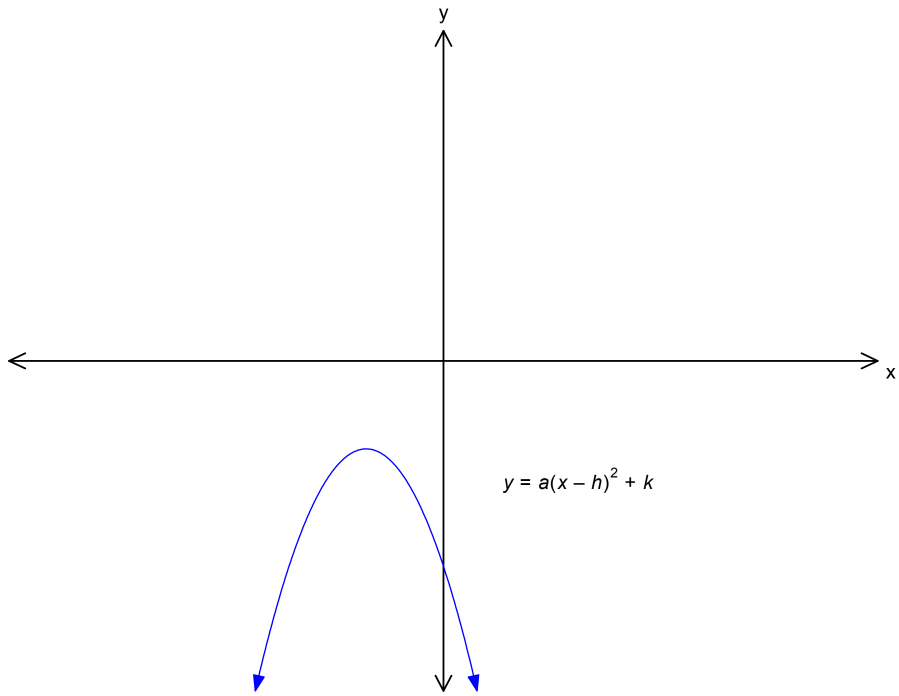
Infinite solutions:
Where one line lies on top of another – one equation is a multiple of the other. Set up an equation where the ratios of the coefficients are equal.

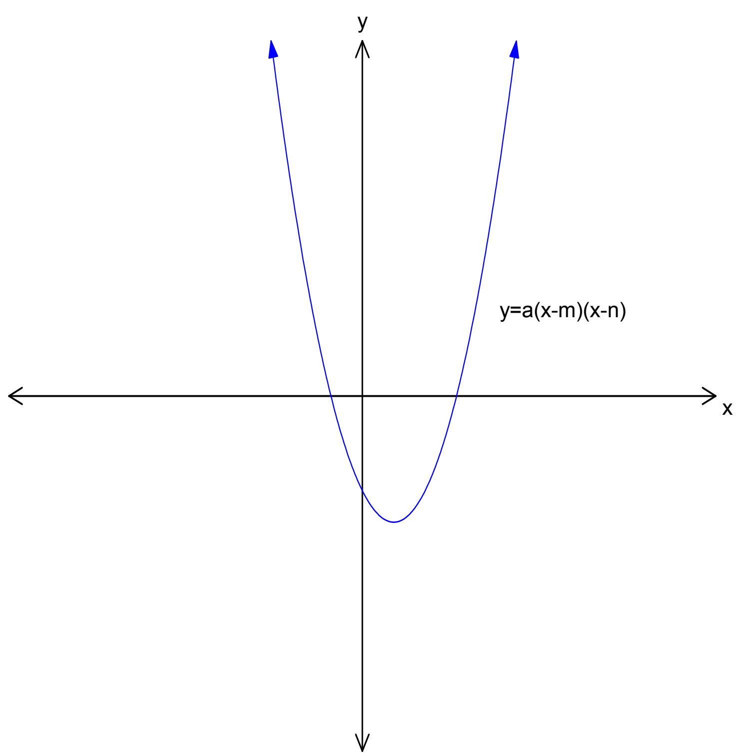
No Solutions questions:
Where lines have the same gradient. You can equate the gradients or you can set up a matrix equation and make the determinant equal to zero.

Calculator skills
Solve equations using solve
Solve equations using Matrices

Quadratic Graphs:




y
x
y = a(x – h)^2 + k


y
x
y=a(x-m)(x-n)

To find equations to the graphs:

Substitute relevant information into the relevant rule.

Set up simultaneous equations and solve.

The format of the equations can be:

$y = ax^2 + bx + c$

$y = a(x - h)^2 + k$

$y = a(x + m)(x + n)$

x intercepts can be calculated using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Also remember what the discriminant tells us about the solutions to an equation:

$\Delta > 0$, two solutions (intercepts)

$\Delta < 0$, no solutions

$\Delta = 0$, 1 solution (turning point on the x axis)

The turning point can be found by:

Completing the square to get equation in turning point form.

Use $x = \frac{-b}{2a}$ and then substitute the x coordinate into $f(x)$

Calculus $\frac{dy}{dx} = 0$ and solve for x. Then find $f(x)$

Cubic Graphs:

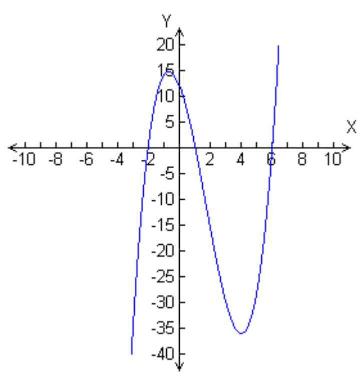
Cubic functions can be expressed in the form of:

Expanded form:

$f(x) = ax^3 + bx^2 + cx + d$

e.g.

$f(x) = (x-1)(x+2)(x-6)$

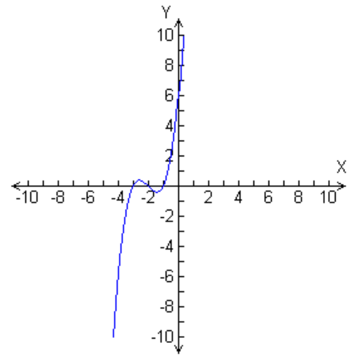


(Intercept form for three intercepts):

$f(x) = a(x-m)(x-n)(x-o)$

e.g.

$f(x) = (x+1)(x+2)(x+3)$



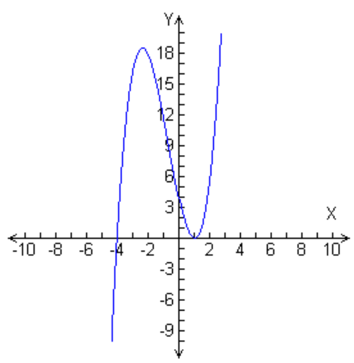
Intercept form with repeated factor:

A repeated factor indicates that there is a turning point on the x-axis at x=n.

$f(x) = a(x-m)(x-n)^2$

e.g.

$f(x) = (x+4)(x-1)^2$



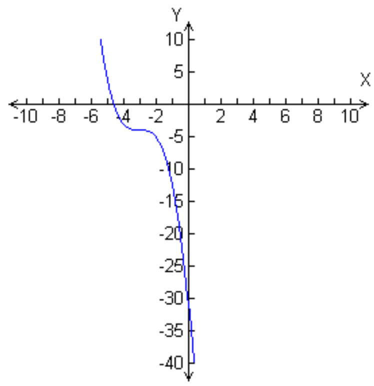
Inflection point form:

This tells us how the original form of the graph has been dilated, reflected and translated.

$f(x) = \pm a(x-h)^3 + k$.

e.g.

$f(x) = -(x+3)^3 - 4$



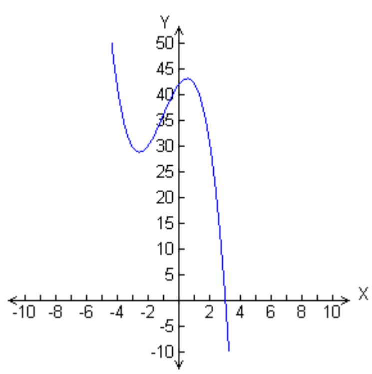
This is the form where the cubic can be factorised into one linear factor and into a quadratic factor:

In this case there in only one x intercept at $x = m$. The quadratic factor does not solve.

$f(x) = (x-m)(ax^2 + bx + c)$

e.g.

$f(x) = (x-3)(-x^2 - 6x + 14)$



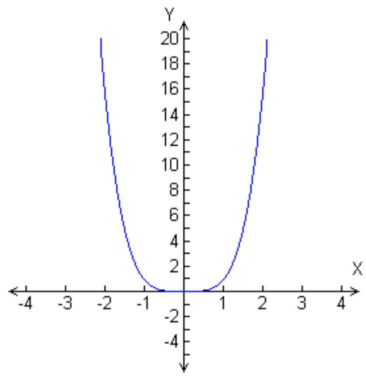
As it appears in Unit 3

Quartic Graphs:

These are graphs that are a polynomial of degree 4. The basic quartic function is $f(x)=x^4$.

e.g.

$f(x)=x^4$



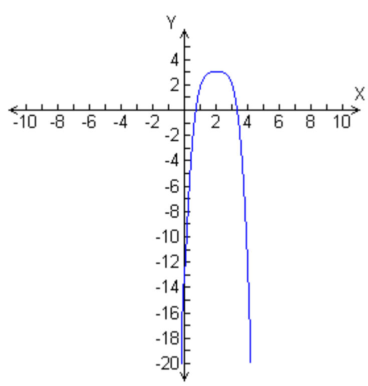
Quartics can be written in the following ways:

This shows how the basic shape of a quartic can be transformed.
The transformations $af(x)$, $f(ax), -f(x), f(-x), f(x-h)$ and $f(x)+h$ can be seen easily using this form.

$f(x)=\pm a(x-h)^4+k$

e.g.

$f(x)=-(x-2)^4+3$

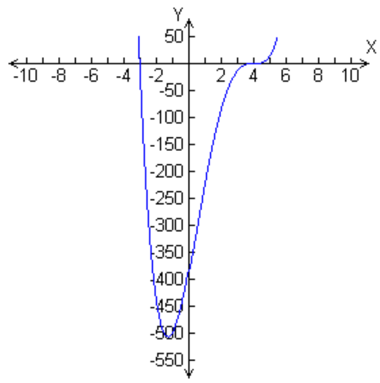


This is the form of a quartic that shows two intercepts. At $x = n$ you have a point of inflection on the x-axis.

$f(x) = a(x-m)(x-n)^3$

e.g.

$f(x) = 2(x+3)(x-4)^3$



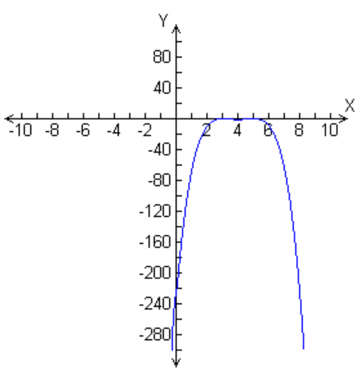
This in the form of a quartic that again shows two intercepts but they are both turning points. Note the repeated factors at $x = m$ and $x = n$

$f(x) = a(x-m)^2(x-n)^2$

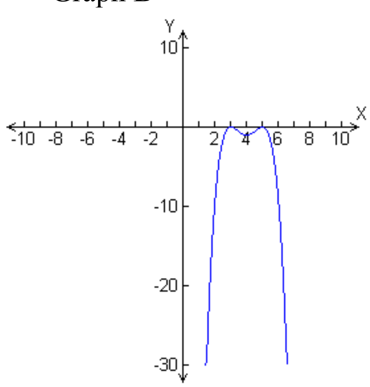
e.g.

$f(x) = -(x-3)^2(x+5)^2$

Graph A



Graph B



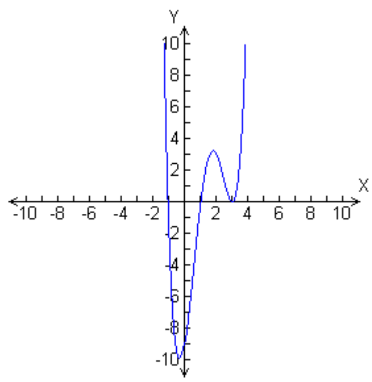
We need to be careful when we are using technology that we carefully investigate the graphs. All sketches should include important features of a graph. On **Graph A** you cannot see the turning points. This is why you need to understand graphing concepts. This advice holds for any graphing that you do.

This is the form of a quartic that shows three x intercepts.
Note that one of the intercepts is a turning point, again represented by the repeated factor.

$$f(x) = a(x-m)(x-n)(x-o)^2$$

e.g.

$$f(x) = (x-1)(x+1)(x-3)^2$$

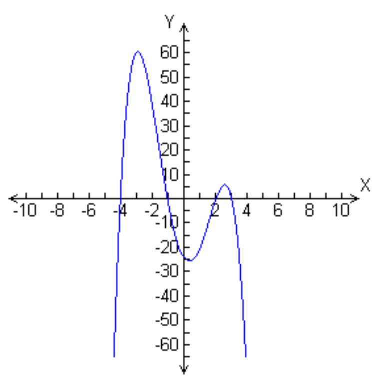


This is the form of a quartic that shows four x intercepts.

$$f(x) = a(x-m)(x-n)(x-o)(x-p)$$

e.g.

$$f(x) = (2-x)(x+4)(x-3)(x+1)$$



Note:

We need to be careful about deciding if we have a positive or negative quartic. You can work this out by expanding the x terms only. You need to be very careful when you have factors like $(m-x)$ in the equation. This advice applies to all polynomial graphs.
The basic advice for sketching quartic functions is the same for cubics. When the quartic is in factorised form you should be familiar with the basic shape according to the number of intercepts.

Review Questions

1. Without using division, show that the first polynomial is divisible by the second

a. $x^3 - x^2 + x - 1$, $x - 1$

b. $2x^3 - 13x^2 + 27x - 18$, $2x - 3$

2. Find the value of k if the first polynomial is exactly divisible by the second:

a. $x^3 - 4x^2 + x + k$, $x - 3$

b. $x^3 - (k + 1)x^2 - x + 30$, $x + 3$

3. Factorise the following cubic functions

a. $x^3 - 4x^2 + x + 6$

b. $2x^3 - 3x^2 - 11x + 6$

4. Sketch the graphs of the following linear functions, stating domain and range:

a. $3x + 4y = 12$

b. $3x - 5y + 15 = 0$

5. Find the equation of the line in the form of $ax + by = d$ in each of the following:

a. A line that has a gradient of ½ and passes through $(6,2)$

b. A line that passes through $(a,2)$ and $(4,-3)$.

6. A line joins the points $(-4,3)$ and $(-2,-1)$.

a. Find the exact distance between the two points.

b. Find the angle the line makes with the positive direction of the x axis

7. Find the angle between the lines $y = 2x - 5$ and $y = -3x + 5$ to 1 decimal place.

8. Find exact solutions to the following equations:

a. $x^2 + 2x - 15 = 0$

b. $8m^2 = 800$

c. $x^2 + 10 = 7x$

d. $7x^2 = 2(17x - 12)$

9. Sketch the graphs of the following stating domain and range:

a. $y = x^2 + 6x + 8$

b. $y = (x - 1)^2 + 4$

c. $f(x) = -x^2 - x + 6$

d. $f(x) = x^2 - 3x - 2,\ -5 \le x \le 3$

10. Let f: R→R, $f(x) = x^2 + 5x - 8$

a. Write the function in turning point form. Hence state the coordinates of the turning point.

b. Use the discriminant to determine the number of x-intercepts

c. Sketch the graph of $f(x)$ stating the domain and range.

d. Solve the equation $2x^3 - 7x^2 + 2x + 3 = 0$

11. Sketch the following graphs showing all relevant information:

a. $f(x) = (x - 2)^3 - 2$

b. $f(x) = (x + 3)(1 - x)(x - 4)$

c. $f(x) = (x + 2)^2(x - 3)$

d. $f(x) = x^3 - 5x^2 + x + 10$

12. Sketch the following graphs showing all relevant information:

a. $f(x) = (x + 2)(x - 2)(x + 4)(x - 5)$

b. $f(x) = x^2(x + 4)(5 - x)$

c. $f(x) = (x + 2)^2(x - 6)^2$

d. $f(x) = (x + 2)^3(x - 6)$

13. For this set of simultaneous equations: $(m-3)x+8y=10$ and $5x+(m+3)y=11$, find the values of m for which the equations have:

No solutions
Unique solutions
Infinite Solutions

Solutions to Review Questions

1.

a.

$p(1) = 0$

Use factor theorem

b.

$p\left(\frac{3}{2}\right) = 0$

Use factor theorem

2.

a.

$k = 6$

Use Factor theorem

$p(3) = 0$

$k - 6 = 0$

b.

$k = -\frac{1}{3}$

Use Factor theorem

$p(-3) = 0$

$-9k - 3 = 0$

3.

a.

$(x-3)(x-2)(x+1)$

$p(3)=0 \Rightarrow x-3$ is a factor

$$\frac{x^3-4x^2+x+6}{x-3}=x^2-x-2=(x-2)(x+1)$$

b.

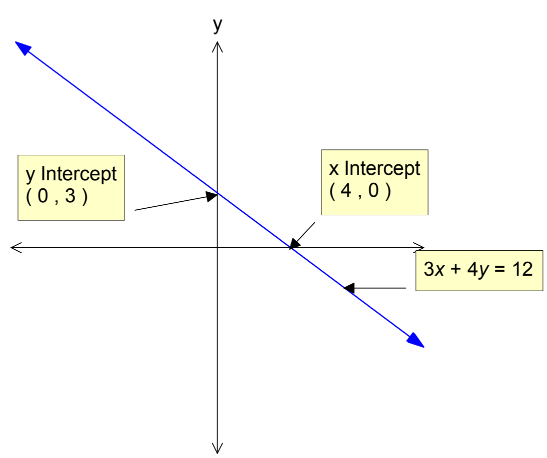
$(x+2)(2x-1)(x-3)$

$p(-2)=0 \Rightarrow x+2$ is a factor

$$\frac{2x^3-3x^2-11x+6}{x+2}=2x^2-7x+3=(2x-1)(x-3)$$

4.

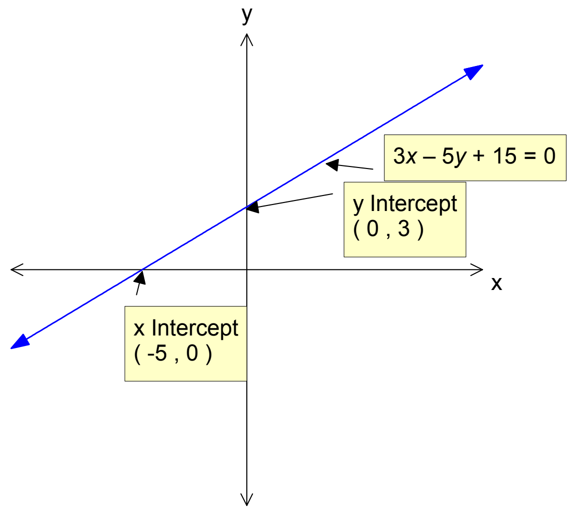
a.



Domain is R
Range is R
Explanation

X intercept, let $y=0$, solve
Y intercept, let $x=0$, solve

b.



Domain is R
Range is R
Explanation

X intercept, let $y = 0$, solve
Y intercept, let $x = 0$, solve

5.

a.
$2y - x = -2$

$x - 2y = 2$ $x - 2y$ (this form is asked in the question)

$$y - 2 = \frac{1}{2}(x - 6)$$

$$y - 2 = \frac{x}{2} - 3$$

b.

$5x-(a-4)y=8+3a$

$m = \frac{5}{a-4}$

$y + 3 = \frac{5}{a-4}(x-4)$

$= \frac{5x-4}{a-4}$

$(y + 3)(a - 4) = 5x - 4$

6.

a.

$2\sqrt{5}$ *units*

Distance between two points formula

b.

$m = -2$

$\tan\theta = m$

$\theta = \tan^{-1}(-2)$

$= -63.34^{\circ}$

$\text{angle} = 180 - 63.43 = 116.57^{\circ}$

7.

45°

$\theta_1 = \tan^{-1}(2)$ $\qquad$ $\theta_2 = 180 + \tan^{-1}(-3)$

$= 63.43^{\circ}$ $\qquad$ $= 108.43^{\circ}$

$\text{angle} = \theta_2 - \theta_1$

8.

a.

$x = -5 \text{ or } 3$

Factorise, solve
Could use calculator.

b.

$m = \pm 10$

Solve
Could use calculator

c.

$x = 5 \text{ or } 2$

Factorise, solve
Could use calculator

d.

$x = \frac{6}{7} \text{ or } 4$

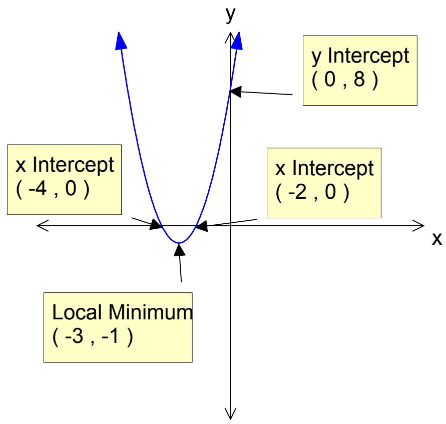
Expand then make equation =0
Factorise, then solve
Could use calculator

9.

a.

Dom:R

Ran:[-1, ∞)



x intercept, let $y = 0$

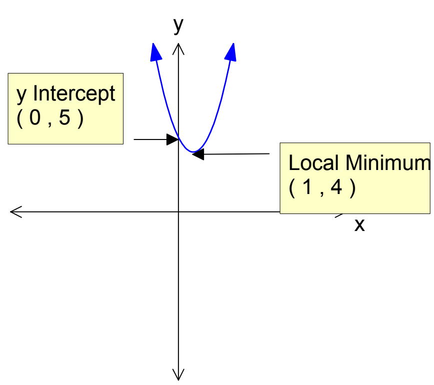
y intercept, let $x = 0$

Complete square for turning points or one of the other method

b.

Dom:R

Ran:[4, ∞)



x intercept, let $y = 0$

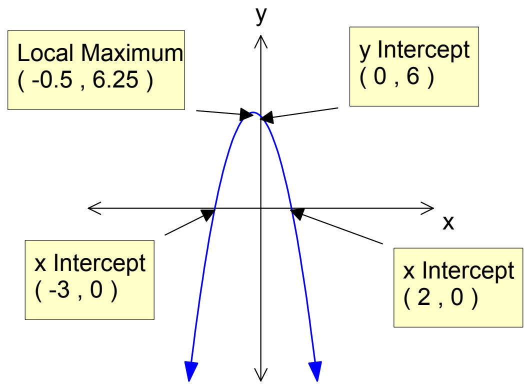
y intercept, let $x = 0$

Read Turning Point from graph.

c.

Dom:R

Ran: $\left(-\infty, 6\frac{1}{4}\right]$



x intercept, let $y = 0$

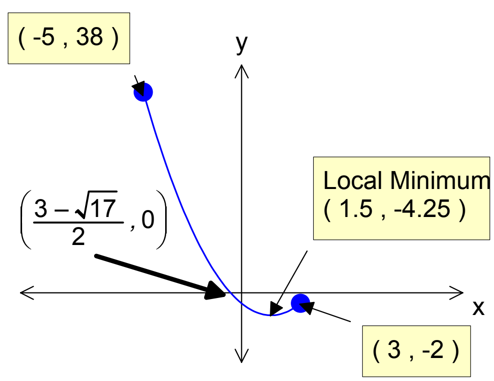
y intercept, let $x = 0$

Complete square or other method for turning point.

d.

Dom:[-5,3]

Ran: $\left[-4\frac{1}{4}, 38\right]$



x intercept, let $y = 0$

y intercept, let $x = 0$

Complete square or other method for turning point.

10.

a.

$$y = \left(x + \frac{5}{2}\right)^2 - \frac{57}{4}$$

Turning Point: $\left(-\frac{5}{2}, -\frac{57}{4}\right)$

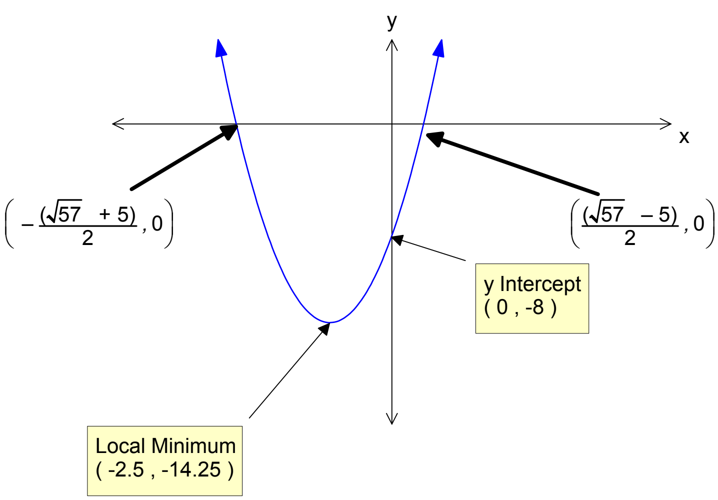
Complete the Square to get into turning point form.

b.

2 intercepts
$b^2 - 4ac = 57$

c.

Dom:R
Ran: $\left[-\frac{57}{4}, \infty\right)$



y intercept, let $x = 0$
x intercept, let $y = 0$, then use quadratic formula.
Any method for turning point.

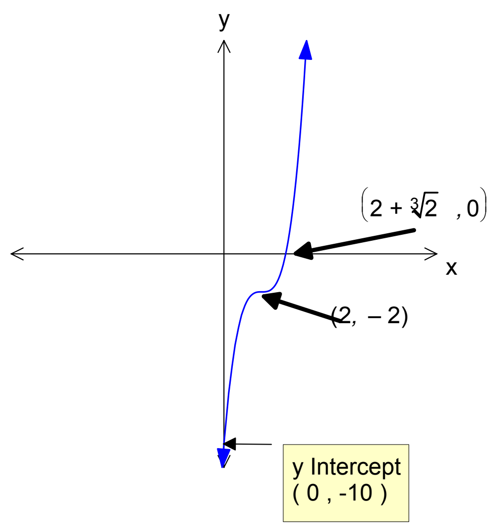
11.

$x = -\frac{1}{2}\ ,\ 3\ ,\ 1$

Use factor theorem then solve.
Could use calculator.

12.

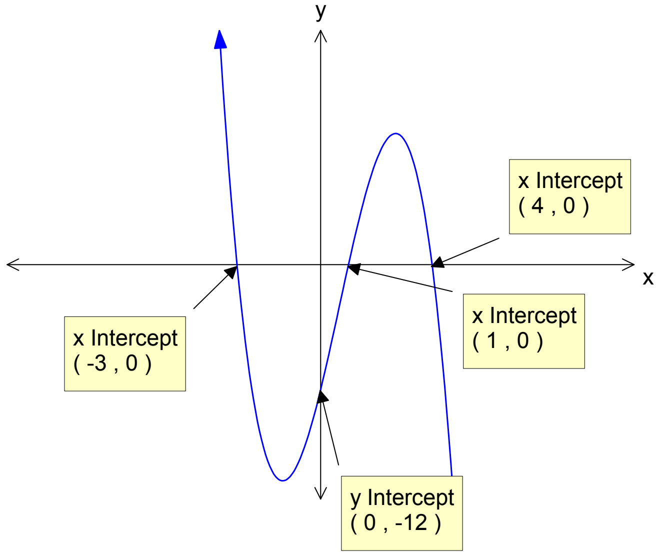
a.
Dom:R
Ran:R



Let $y = 0$ then solve for x to get x intercepts.
$x = 0$ for y intercepts.

b.

Dom: R
Ran:R

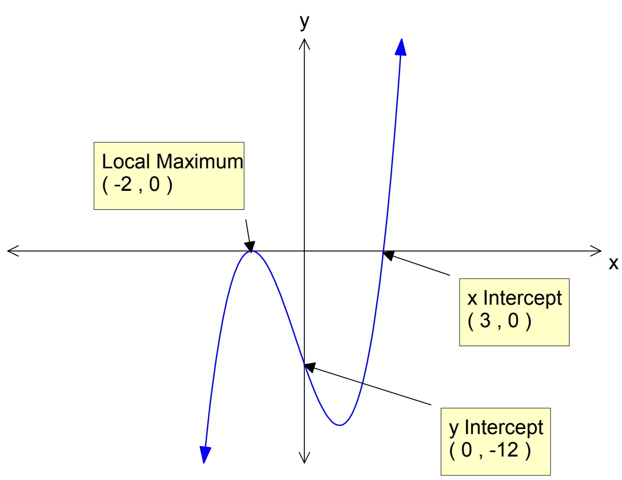


y intercept let $x = 0$

x intercept let $y = 0$

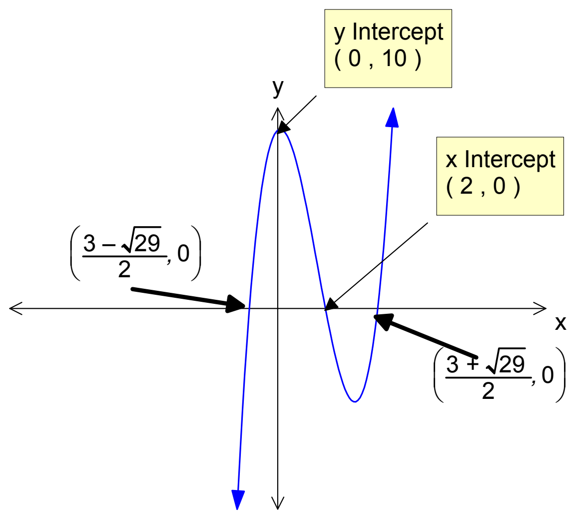
c.

Dom: R
Ran:R



Repeated factor leads to turning point on corresponding intercept.

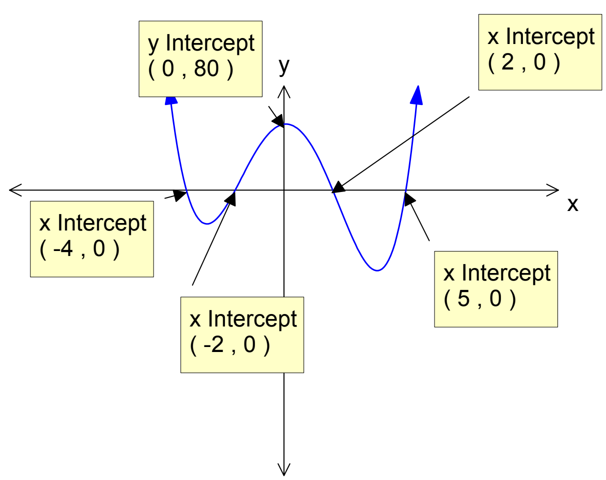
d.



Use Factor theorem to help find x intercepts.
Will need to use quadratic formula to solve the quadratic factor.

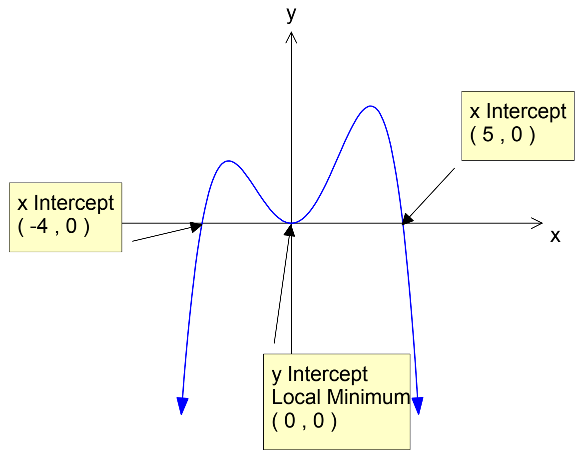
13.

a.



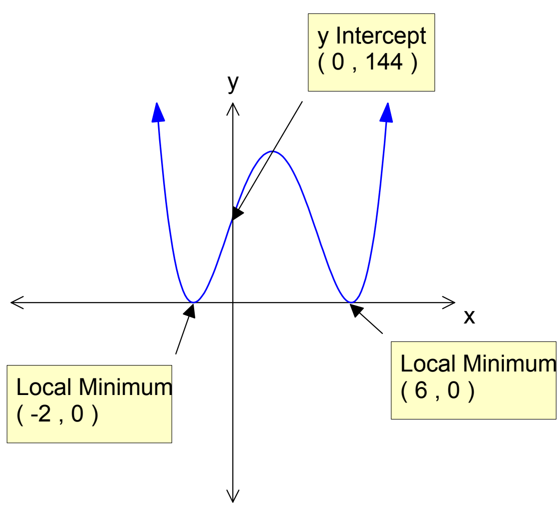
Quartic with 4 x intercepts

b.



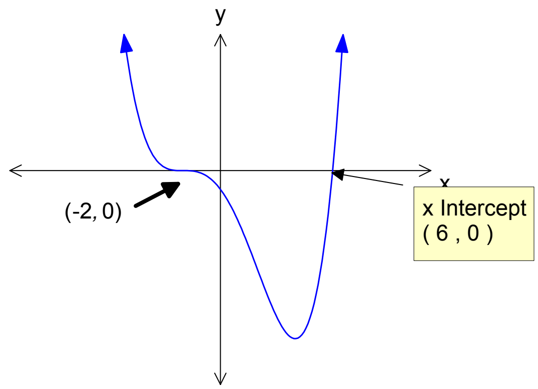
Quartic with x intercepts. Repeated factor give s a turning point.

c.



Quartic with 2 x intercepts. Two repeated factors leading to the 2 turning points.

d.



Quartic with 2 x intercepts. Cubed factor leading to the inflexion point.

14.

No solutions $- m = 7$
Unique solutions $- R \setminus \{-7, 7\}$
Infinite Solutions $- m = -7$

Set up matrix equation to solve the simultaneous equations.
Make the determinant $= 0$ and solve for m. $m = \pm 7$. All values other than these give unique solutions i.e. lines that cross. Use calculator to test the effect of 7 and –7.