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MATHEMATICAL METHODS CAS

Teach Yourself Series

Topic 11: Discrete Random Variables

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Discrete Random Variables

As it appears in Unit 3

A random variable is a variable that can't be predicted. The outcomes are determined experimentally.

A discrete random variable is one that takes on countable outcomes.

eg The sum of the faces on two dice that have been thrown.
The number of heads that appear when three coins have been tossed.

Discrete Probability Distributions

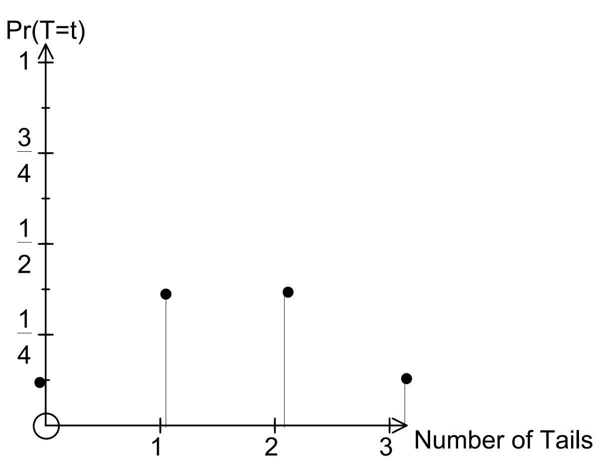
These are tables that list the probability of all outcomes. In generating all possible outcomes you may need to consider using Tree Diagrams or Lattice Diagrams.

eg Let T be the number of tails that will appear when three coins have been thrown. Find the probability distribution of the number of tails.

1. Use a tree diagram to list all out comes and their associated probabilities.
2. Draw up a table headed appropriately
3. Enter the information into the table.

t	0	1	2	3
Pr(T=t)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Note that it is possible to graph probability distributions.



These could also be column or dot graphs.

Properties of DRV's

There are some important facts that are linked with DRV's.

1. $0 \le \Pr(X = x) \le 1$ ie each outcome has a probability within this range.
2. $\sum \Pr(X = x) = 1$ ie the sum of the separate probabilities must equal 1.

These two conditions must be satisfied for a correct probability distribution to exist.

Expected Value of DRV's

This is a value that describes the average outcome of an experiment. It will not necessarily be a discrete value.
The symbol for the expected value is $E(X)$ or μ.
The formula is: $\mathbf{E(x) = \mu = \Sigma x \cdot Pr(X = x)}$

Expectation theorems

$\mathbf{E(aX) = aE(X)}$ where a is a constant.
$\mathbf{E(aX + b) = aE(X) + b}$ where and b are constants.
$\mathbf{E(b) = b}$ where b is a constant.
$\mathbf{E(X + Y) = E(X) + E(Y)}$ where X and Y are DRV's.

Variance

The variance is a calculation that describes the variation from the mean. If the variation of the outcomes is spread a long way from the mean, the variance is a large number.

The symbol is $\mathbf{VAR(X)}$ or σ^2.
The formula is: $\mathbf{VAR(X) = \sigma^2 = E(X - \mu)^2 = E(X^2) - \mu^2}$

Note that there is also a theorem for the variance of DRV's:
$\mathbf{VAR(aX + b) = a^2VAR(X)}$

Standard Deviation

This is another measure of spread. It is the positive square root of the variance.

The symbol is $\mathbf{SD}(\mathbf{X}) = \sigma$.

The formula is: $\mathbf{SD}(\mathbf{X}) = \sigma = \sqrt{\mathbf{VAR}(\mathbf{X})}$

The Standard Deviation is used to calculate some special characteristics of DRV's.

The 95 % Confidence Interval: $\mathbf{Pr}(\mu - 2\sigma \leq \mathbf{X} \leq \mu + 2\sigma) = 0{\cdot}95$

This is the most common characteristic but there are two others:

The 68 % Confidence Interval: $\mathbf{Pr}(\mu - \sigma \leq \mathbf{X} \leq \mu + \sigma) = 0{\cdot}68$

The 99.7 % Confidence Interval: $\mathbf{Pr}(\mu - 3\sigma \leq \mathbf{X} \leq \mu + 3\sigma) = 0{\cdot}997$

When you sum the probabilities you may not get the exact answers for the confidence intervals.

The basic process to follow when answering questions is:

1. Calculate mean.
2. Calculate variance.
3. Calculate Standard deviation.
4. Then find the limits if you are told what CI you are working with OR you may have to add the probabilities together once you have calculated the limits.

Calculator skills

Use lists or spreadsheets on calculator to calculate $E(x)$, $E(X^2)$ etc.

Markov Chains

Key things to look for in a problem: Repeated events occurring.
There is the probability of two things occurring.
Given initial conditions.

Tree diagram approach

1. Start the tree diagram with the initial conditions.
2. Then the second branch is generated from the probabilities of the two separate events happening.
3. Then multiply the branches together to give the probability of each event occurring for the first repetition. These give the initial conditions for the next repetition.

Matrix approach

Transition generated by the probabilities of the two events occurring.

Initial state matrix given by the initial conditions. This could be probabilities or actual whole numbers

The steady state matrix is calculated by using the calculator or algebraically. CALC – T $^{\text{big number}}$ S_o

ALGEBRAICALLY - $T\begin{bmatrix} p \\ 1-p \end{bmatrix} = \begin{bmatrix} p \\ 1-p \end{bmatrix}$

Calculator skills

Store probabilities in matrix.
Do problems using matrices.

Review Questions

1. Calculate the value of k for the following discrete random variable:

X	0	1	2	3	4
$\Pr(X = x)$	$k/2$	$k/4$	k	$k/4$	$k/2$

2. There are five traffic lights on the highway in a city. The city council has determined that number of traffic lights at which a driver will have to stop along the highway has the following distribution.

x	1	2	3	4	5
$\Pr(X = x)$	k	$2k$	k	$k/2$	$k/2$

a. Find the value of k.

b. Calculate the probability that the number of traffic lights at which a driver will have to stop will be more than two.

c. Calculate the expected number of traffic lights to two decimal places that a driver would stop.

3. Suppose that the number of matches that a player wins in a table tennis tournament has the following distribution:

x	0	1	2	3	4	5
$\Pr(X = x)$	0.50	0.25	0.13	0.06	0.03	0.03

Find the mode, median and mean of the number of matches won.

4. When a biased die is rolled, the probability of obtaining a particular outcome (X) is shown in the following table:

x	1	2	3	4	5	6
$\Pr(X = x)$	0.1	0.2	0.3	0.2	0.1	p

Find:

a. the value of p

b. the expected value of X, μ

c. the standard deviation of X, correct to four decimal places

d. $\Pr(X > \mu)$

5. In a particular country it has been found that the probability of snow is dependent on whether or not it snowed the day before. If it snows one day then the probability of snow the next day is 0.82. If it doesn't snow one day then the probability of snow on the next day is only 0.05.

a. Write down the Transition Matrix for this information.

b. If it snows on Monday, calculate the Probability that it will snow on Thursday.

6. Suppose that the probability of squash player winning a game is 0.6 if she has won the preceding game and 0.5 if she has lost the preceding game.

a. Write down a transition matrix T which can be used to represent this information.

b. Write down a matrix equation which can be used to determine the probability that the player wins game 6 if she loses game 1.

7. For the Markov chain defined by the transition matrix:

$$T = \begin{bmatrix} \Pr(W_{i+1}|W_i) & \Pr(W_{i+1}|D_i) \\ \Pr(D_{i+1}|W_i) & \Pr(D_{i+1}|D_i) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix}$$

where W_i is the event day i is wet, and D_i is the event day i is dry, what is the probability that, given that the first day is wet, there will be a run of 3 *more* wet days in a row before the next dry day.

Solutions to Review Questions

1.

Add the probabilities together and make answer equal 1. Then solve.

$k = \frac{2}{5}$

2.

a. Add the probabilities together and make answer equal 1. Then solve.

$k = \frac{1}{5}$

b. Add the probabilities for 3, 4 and 5 lights.

$k = \frac{2}{5}$

c.

Calculate

$E(x) = \sum x.\,Pr(X = x)$

2.5 traffic lights

3.

Median: The value below which there is a probability of 0.5.

Mode: Highest Probability

Mean: Calculate $E(x) = \sum x.\,Pr(X = x)$

Median = 0

Mode = 0

Mean = 0.96

4.

a. Add the probabilities together and make answer equal 1. Then solve.

0.1

b. Calculate $E(x) = \sum x.Pr(X = x)$

3.3

c. Calculate Variance: $E(X^2) = 12.9$ Var(x)=2.01

Squareroot variance to get standard deviation.

1.4177

d. Add the probabilities from 4 and above.

0.4

5.

a.

Probabilities can be summarised into the following table which leads to transistion matrix.

Snow_{now}	Not Snow_{now}	
0.82	0.05	Snow_{next}
0.18	0.95	Not Snow_{next}

$$\begin{bmatrix} 0.82 & 0.05 \\ 0.18 & 0.95 \end{bmatrix}$$

b. $S_3 = T^3 S_0$ Where $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

0.5747

6.

a. Probabilities can be summarised into the following table which leads to transition matrix.

Win_{now}	Loss_{now}	
0.6	0.5	Win_{next}
0.4	0.5	Loss_{next}

$$\begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$$

b.

$s_5 = T^5 S_0$

$$S_5 = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}^5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

c. Use matrix to get probabilities. Run of three wet days followed by a dry day means WWWD. Need to use top left hand side probability matrix.

0.0625