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Creating VCE Success

MATHEMATICAL METHODS CAS
Teach Yourself Series
Topic 3: Functions and Relations – Inverse Functions, Hybrid Functions, Modulus Functions, Composite Functions and Functional Equations

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Functions and Relations

Implied Domain

When a function is stated and the domain is not specified, then the functions is assumed to have its implied domain. This is the largest set of x values that belongs to R that the functions can exist over.

Example: If $f(x) = \frac{1}{x}$, the implied domain is $R\backslash\{0\}$

When answering problems and you are working out the implied domain, remember:

- You can not have numbers in in the dominator of a fraction that evaluate to 0.
- You can not have numbers under a square root function that evaluate to a negative number.
- You can not have numbers that result in a number that is equal to or less than 0 inside a log function.

Absolute value

As it appears in Unit 1

To undo the absolute value sign when equation solving you use ±.

Example:

$|-4| = 4$

$|4| = 4$

Example:

Solve for x: $|x + 3| = 5$

$-(x + 3) = 5$

$-x - 3 = 5$

$x = -8$

Or

$x + 3 = 5$

$x = 2$

Inverse functions

As it appears in Unit 1

Finding rules of inverse functions

When you have to find the rule of the inverse function the following process is used:

1. Rewrite $f(x) = g(x)$ into $y = g(x)$.
2. Interchange x and y. ie $x = g(y)$
3. Now rearrange the equation so y is the subject (by itself).
4. When you have found the equation rewrite it as $f^{-1}(x) = h(x)$

Note the following:

1. $domf = ranf^{-1}$.
2. $ranf = domf^{-1}$.
3. x intercept of $f = y$ intercept of f^{-1}.
4. y intercept of $f = x$ intercept of f^{-1}.
5. Asymptotes with the equation $y = k$ on f = asymptotes with the equation $x = k$ on f^{-1}.
6. Asymptotes with the equation $x = h$ on f = asymptotes with the equation $y = h$ on f^{-1}.
7. If $f(x)$ and $f^{-1(x)}$ intersect, they intersect on the line $y = x$.

$$f(x) = x$$
$$f^{-1}(x) = x$$

Calculator Skills

Define functions.
Solve $(f(y) = x, y)$.

Hybrid Functions

As it appears in Unit 1

These are functions that have more than one rule that defines them.
eg

$$f(x)=\begin{cases} x, x \le 0 \\ x^2+3, x>0 \end{cases}$$

When working out the domain and range we need to be careful. The domain for the above example is R, however the range is not. The range is $(-\infty,0]\cup(3,\infty]$.

When sketching hybrid functions the following process should be followed:

1. Draw separate graphs of each separate function.
2. Mark in start and end points that defines each of the separate functions. This is given by the values of x that define the function.
3. Draw a separate graph that now shows the rules defined over the appropriate domain.
4. Be sure that you use "**hollow dots**" if the end point is **not included** or "**filled dots**" if the end point **is included**.

When looking at domain and range check to see if there are gaps in either or both.

Reciprocal Functions

As it appears in Unit 3

These are graphs that are written in the form of $y = \frac{1}{f(x)}$. To sketch these use the basic process of:

1. Sketch $y = f(x)$ first.
2. Mark in vertical asymptotes. These are where $f(x)=0$.
3. Now sketch $y = \frac{1}{f(x)}$ on all side of asymptotes. You need to take note of the reciprocal behaviour of the function.

Modulus graphs

As it appears in Unit 3

Type 1

These are in the form of $y = a \mid f(x) \mid +k$. To sketch them apply the following basic process:

Sketch $f(x)$. Make sure the x intercepts are found.
Draw new sketch where the negative side is flipped to the positive side.
Then apply a
Then apply k.

Type 2

These are in the form $y = f(|x|)$.

Sketch $f(x)$
The y values over the R^+ are then reflected across the y axis.
Logs are reflected across the asymptote

Calculator Skills

Graphing functions

Addition, Subtraction and Multiplication of ordinates

The resulting function has a domain that is the intersecting domain of the two original functions.

Two basic types of questions:

Given the rules of two separate functions. Write the rule state the domain and range.

Given the graphs find the new one.

Note: $f - g = f +^{-} g$.

Calculator skills

Define functions
Evaluate functions

Composite functions

$f \circ g - f(g(x))$ To exist, the range of g must be included in the domain of f.

$g \circ f - g(f(x))$ To exist, the range of g must be included in the domain of f.

In other words the range of the inside function has to be the domain of the outside function

If this is not the case, the domain of the inside function can be restricted to the range of the outside function.

To do this:

1. Sketch the inside function
2. Work out the y values you can't have
3. Work out the x values that correspond to the forbidden y values. These are the x values that are excluded in the domain of the original function.

Calculator skills

Define functions
Evaluate functions

Functional Equations

These are where you have to prove an expression for a given function. For example does $f(x+y) = f(x) + f(y)$ if $f(x) = \frac{1}{x}$

Set the problems out by simplifying the Left Hand Side of the expression, then simplify the Right Hand Side.

Calculator skills

Define functions
Evaluate functions

Basic Matrix Knowledge

Order of a matrix is the number of rows x number of columns.

Example: $\begin{bmatrix} 3 & 1 \\ 3 & 6 \\ -2 & 1 \end{bmatrix}$ This is a 3 x 2 matrix.

Addition and Subtraction

You can only add or subtract matrices that are the same order.

Example

$$\begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 1 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 12 & 3 \end{bmatrix}$$

Each corresponding element is added or subtracted.

Multiplying a matrix by a scalar

This is where a number is multiplied to each element in the matrix.

Example

$$2\begin{bmatrix} 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 4 \end{bmatrix}$$

Multiplying two matrices together

To multiply a pair of matrices - The number of columns in the first matrix = number of rows in the second matrix.

That is: m x n n x o
1st matrix 2nd matrix

This will result in a m x o matrix.

You multiply each element in the rows of the first matrix to elements in each column of the second matrix. If you multiply the first row by the first column then answer is located the position of first row first column. If you multiply the first row by the second column then answer is located the position of first row second column.

Example: $$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix} x \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 1 & 8 \end{bmatrix}$$

9 = 1x1 + 2x2 + 1x4
8 = 3x2 + 1x1 + -1x-1

Note: Multiplying a 2 x 3 matrix to a 3x 2 matrix results in a 2x2 matrix.

Determinant of a matrix

If $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant, $\det_A$, is calculate by:

$$\boldsymbol{det_A = ad - bc}$$

A matrix that has a determinant = 0 is called a singular matrix.

The inverse of a matrix

If $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the inverse, A^{-1}, is calculated by:

$$\boldsymbol{A^{-1} = \frac{1}{det_A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

When you multiply a matrix by its inverse, the result gives the identity matrix.

Calculator Skills

Storing matrices
Calculating determinant
Finding inverse matrices

Review Questions

1. Solve for x:

a. $|x + 5| = 7$

b. $-2|x - 4| = 24$

2. A function has rule, $f(x) = 5x^{1/3} - 1, x \in R$. The rule for the inverse function is:

a. Find the inverse, f^{-1}, of the function $f: R\backslash\{-\frac{3}{5}\} \to R, f(x) = \frac{2x + 1}{5x + 3}$. State the domain and range of the inverse function.

b. Sketch the graph of f: $[3, \infty) \to R$, $f(x) = \sqrt{x-3}$ and find the inverse function f^{-1}. Sketch the graph of f^{-1} on the same set of axes.

3. For the function $f(x) = 2e^{3x} - 4$ find its inverse function, $f^{-1}(x)$. Sketch $f(x)$ and $f^{-1(x)}$ stating the domain and range of each.

4. If $f(x) = \begin{cases} -2x + 6 \text{ if } x \geq -2 \\ x + 2 \text{ if } x < -2 \end{cases}$:

a. Find $f(4)$

b. Sketch $f(x)$. State the range of $f(x)$.

5. If $f(x) = x^2 + 4$, Sketch $\frac{1}{f(x)}$

a. If $f(x) = x^2 + 6x + 8$, Sketch $\frac{1}{f(x)}$

6. Sketch the graph of $f(x) = 2|x + 4| + 1$, State the Domain and Range.

7. Sketch the graph of $f(x) = |x^2 - 6| + 1$, State the Domain and Range.

8. Sketch the graph of $f(x) = e^{|x|}$, State the Domain and Range.

9. Sketch the graph of $f(x) = x^2 - 5x$. Hence sketch $f(|x|)$ and state the domain and range of $f(|x|)$.

10. For $f(x) = x^2$ and $g(x) = 1 - 4x$ find:

a. $f(g(x))$.

b. $g(f(x))$.

c. $f(g(-1))$.

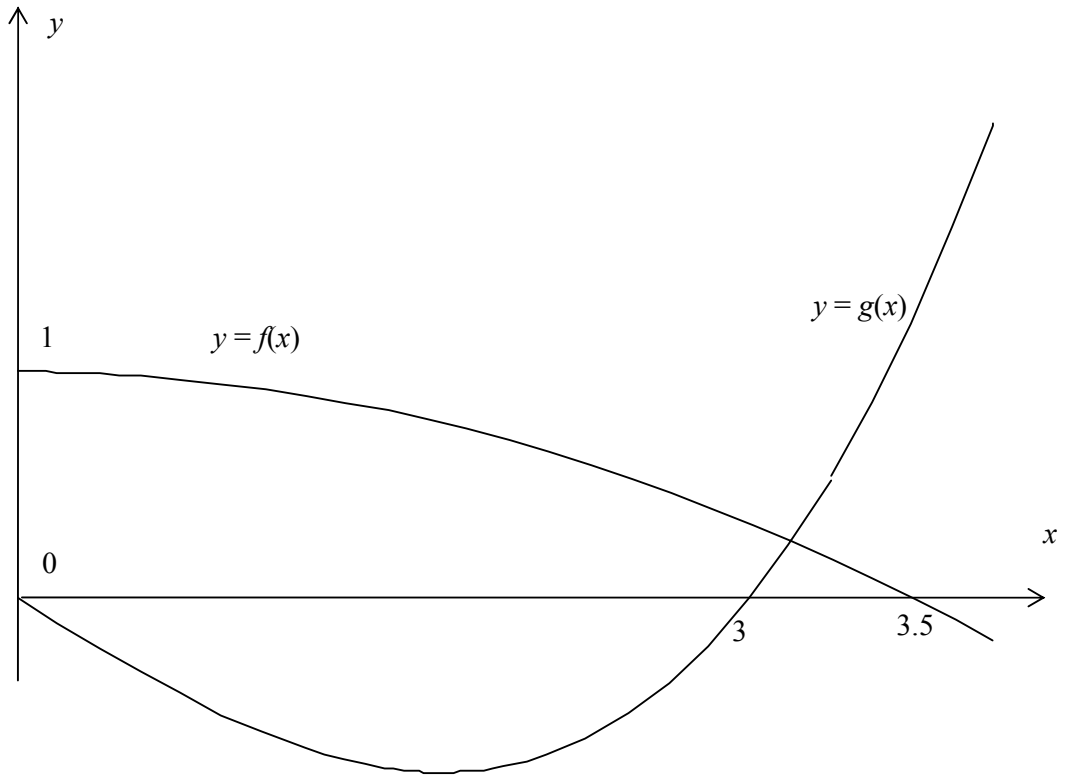
d. $g(f(-1))$.

11. For $f(x) = e^{-x}$ and $g(x) = -x$. Find the rules for:

a. $(f+g)$.

b. (fg) .

12. The graphs of $y = g(x)$ and $y = f(x)$ are shown on the axes below. On the same axes sketch the graphs of $y = f(x) + g(x)$.



13. Prove the following functional equations:

a. $f(x+y) = f(x)f(y)$ for $f(x = e^x)$.

_____

_____

_____

_____

b. $f(xy) = f(x) + f(y)$ for $f(x) = \log_2(x)$.

_____

_____

_____

_____

14. If $A = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ find the following:

a. 3A

b. 2B – C

c. $\det_C$

d. B^{-1}

15. State the implied domain for the following functions:

a. $f(x) = log_e(x-3)$

b. $f(x) = \frac{1}{2x-1}$

c. $f(x) = \sqrt{3x-4}$

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Solutions to Review Questions

1.

a.

$x = 2$ *or* -12

Set up 2 equations

$x + 5 = 7$ and $-(x + 5) = 7$

b.

$-2|x - 4| = 24$

$x = -8$ *or* 16

Set up 2 equations

$x - 4 = -12$ and $-(x - 4) = -12$

c.

$f^{-1}(x) = \left((x+1)/5\right)^3$

set up equation from first step of process $x = 5y^{\frac{1}{3}} - 1$ and solve for y.

2.

$f^{1}(x) = \frac{3}{5} + \frac{1}{5(2 - 5x)}$

Domain: $R \setminus \left\{\frac{2}{5}\right\}$ Range: $R \setminus \left\{\frac{3}{5}\right\}$

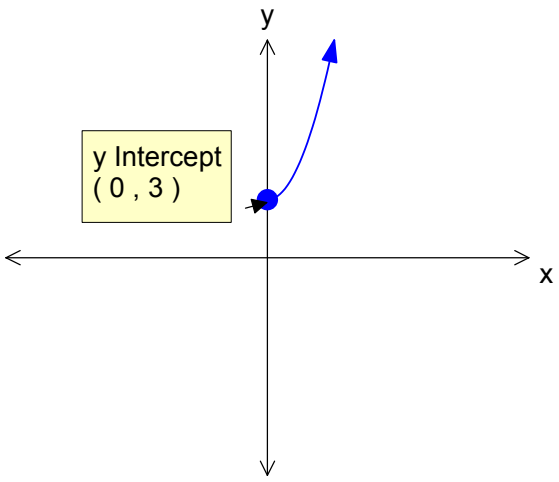
Use long division to change form of function.

set up equation from first step of process and solve for y.

3.

Domain: $R^+ \cup \{0\}$ Range: $[3, \infty)$

$f^{-1}(x) = x^2 + 3$

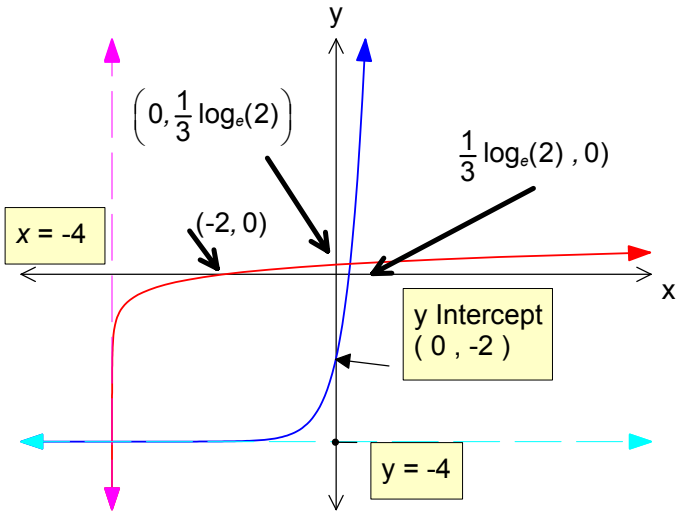


set up equation from first step of process and solve for y

3.

Domain: $(4, \infty)$ Range: R

$f^{-1}(x) = \frac{1}{3}\log_e\left(\frac{x+4}{2}\right)$



Swap x and y around as per process and solve for y to get the inverse function

4.

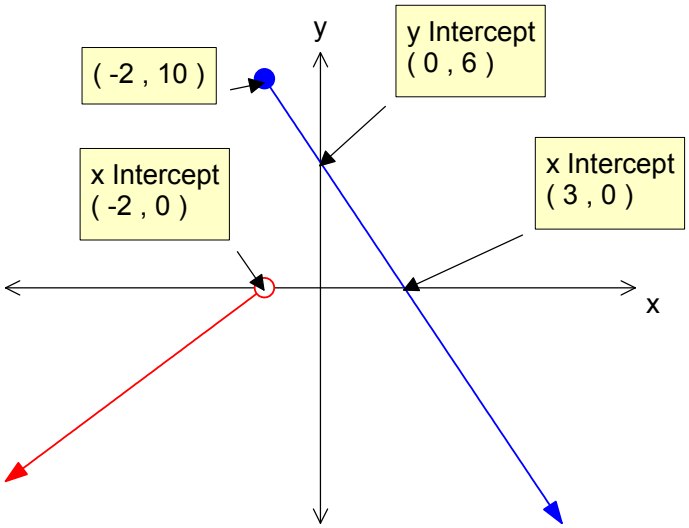
a.

$f(4) = -2$

Use first function as $x = 4$ is in that domain.

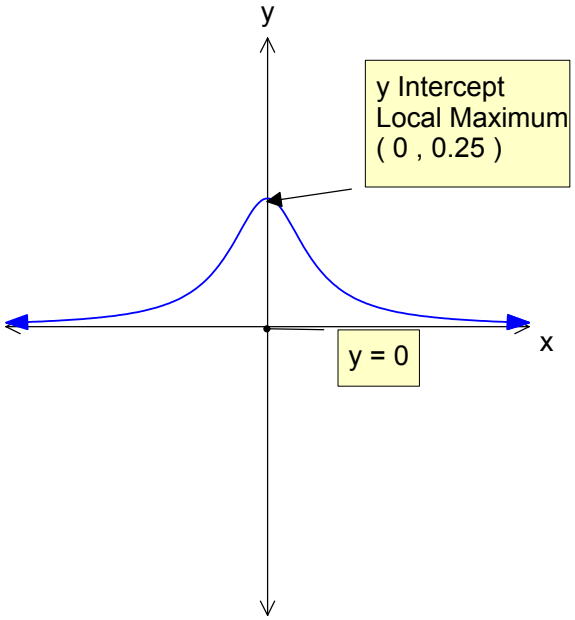
b.

Range: R



Only sketch the section of each rule of the function over the appropriate domain.

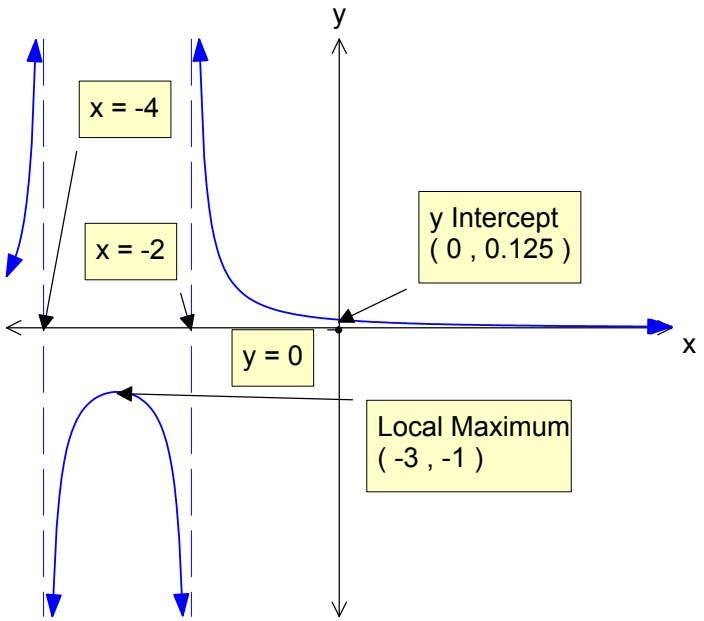
5.



Sketch $y = x^2 + 4$ first.

Then reciprocate the y values.

6.

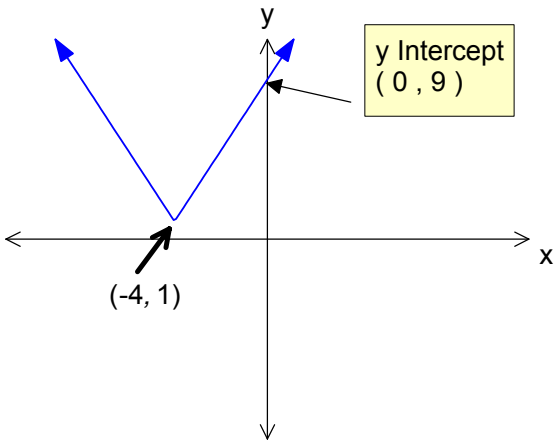


Sketch $x^2 + 6x + 8$ first, then reciprocate y values.

X intercepts on first graph become vertical asymptotes.

7.

Domain is R and Range is $[1 , \infty)$

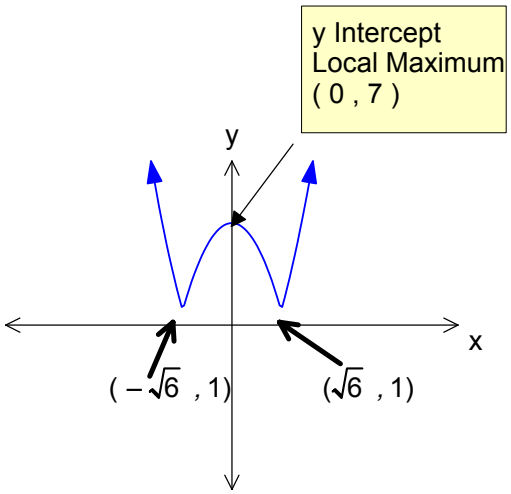


Sketch $y = x + 4$.

Reflect the line below the x axis to above the x axis.

Then multiply y coordinates by the 2 then add the 1 to them.

8. Domain: R Range: $[1, \infty)$



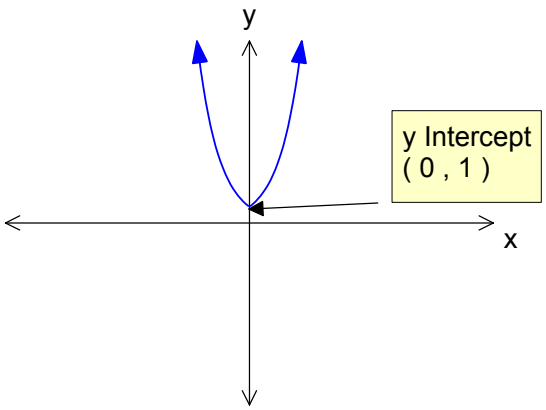
Sketch $y = x^2 - 6$.

Reflect the line below the x axis to above the x axis.

Then add 1 to the y coordinates.

9.

Domain: R Range: $[1, \infty)$

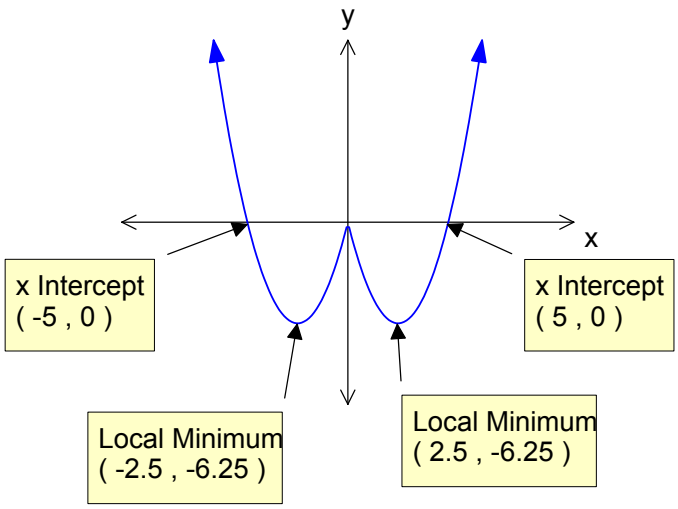


Sketch $y = e^x$.

Reflect the section of the graph that is in the positive domain across the y axis.

10.

Domain: R Range: $[-6.25, \infty)$



Sketch $y = x^2 - 5x$.

Reflect the section of the graph that is in the positive domain across the y axis.

11.

a.

$(4x - 1)^2$

Write out $f(x)$, substituting x for $(1 - 4x)$

b.

$1 - 4x^2$

Write out $g(x)$, substituting x for x^2 .

c.

25

Substitute $x = -1$ into answer for part a.

d.

-3

Substitute x=-1 into answer for part b.

12.

a.

$e^{-x} - x$

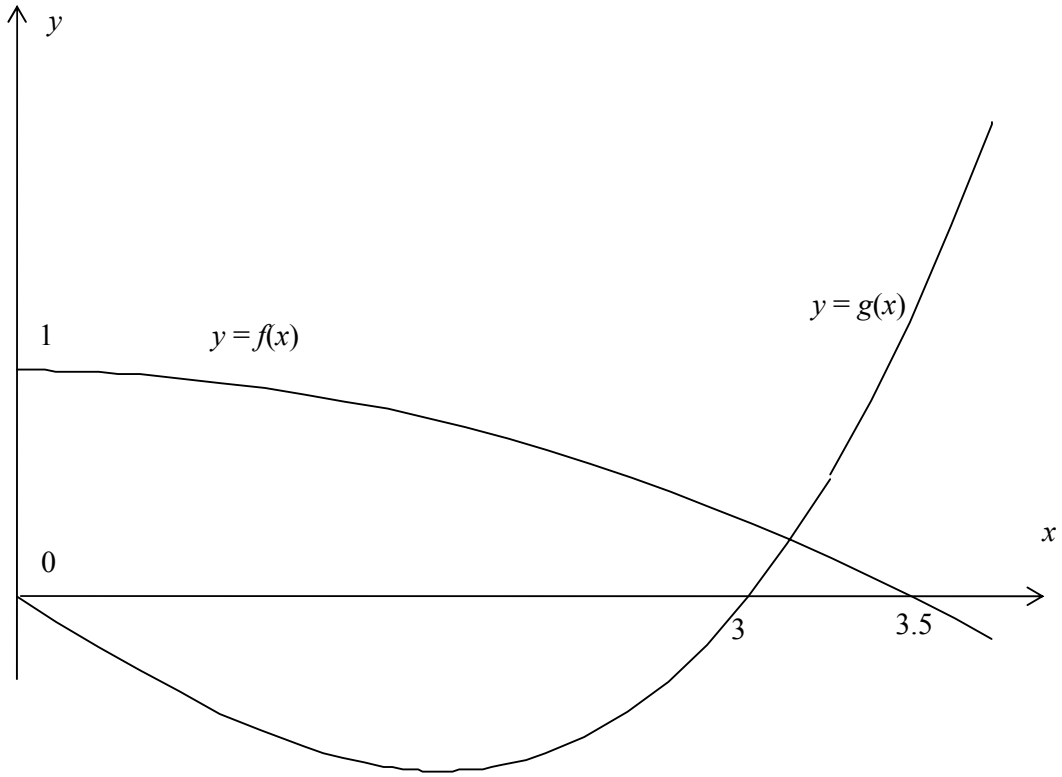
Add the two functions together.

b.

$-xe^{-x}$

Multiply the two functions together.

13.



14. Prove the following functional equations:

a.

$LHS = e^{x+y}$

$RHS = e^{x} \times e^{y}$

$= e^{x+y}$

LHS = RHS as required.

b.

$LHS = \log_2(xy)$

$RHS = \log_2(x) + \log_2(y)$

$= \log_2(xy)$

LHS = RHS as required.

15.

a. 3A

$$\begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Multiply 2 to each element.

b. 2B – C

$$\begin{bmatrix} 0 & 1 \\ -5 & 5 \end{bmatrix}$$

Multiply each element in B by 2. Then subtract each element from C.

c. $\det_C$

2 x 5 – 1 x 3 = 7

d. B^{-1}

det = 9

$$B^{-1} = \frac{1}{9}\begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{5}{9} & \frac{-2}{9} \\ \frac{2}{9} & \frac{1}{9} \end{bmatrix}$$

16.

a.

$x - 3 > 0$

$x > 3$

$(3, \infty)$

b.

$$2x - 1 \neq 0$$
$$x \neq \frac{1}{2}$$
$$R \setminus \left\{\frac{1}{2}\right\}$$

c.

$$3x - 4 \geq 0$$
$$x \geq \frac{4}{3}$$
$$\left[\frac{4}{3}, \infty\right)$$