TSSM™

Creating VCE Success

MATHEMATICAL METHODS (CAS)

Teach Yourself Series

Topic 4: Exponential and Log Functions

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Exponential and Log Functions

In this topic you will learn the basic concepts around exponential and logarithmic functions.

Index laws

As it appears in Unit 2 & 3

$$a^x \times a^y = a^{x+y}$$
$$\frac{a^x}{a^y} = a^{x-y}$$
$$\left(a^x\right)^y = a^{xy}$$
$$a^{-x} = \frac{1}{a^x}$$
$$a^0 = 1$$
$$a^{\frac{x}{y}} = \sqrt[y]{a^x}$$

Some basic tips to remember:

1. Express surds as fractional powers
2. Remove brackets (3^{rd} law)
3. Apply 1^{st} and 2^{nd} laws
4. Use the Reciprocal Law last.
5. Numbers raised to powers expressed in lowest base

Log laws

As it appears in Unit 2 & 3

$$\log_x a + \log_x b = \log(ab)$$
$$\log_x a - \log_x b = \log_x \frac{a}{b}$$
$$\log_x a^n = n\log_x a$$
$$\log_x a = 1$$
$$\log_a x = \frac{\log_{10} x}{\log_{10} a} = \frac{\log_e x}{\log_e a}$$

Some basic rules to remember when using the log laws in simplifying expressions:

- Attempt to reduce the number of logs in the expression to just the one.
- Use 3^{rd} log law before you use the 1^{st} and 2^{nd}.
- Index laws can sometimes help to simplify expressions.
- Look for log statements that can be evaluated.

Equation Solving

Exponential

As it appears in Unit 2 & 3

Type 1: Like base method
Type 2: Use logs – looks like type one
Type 3: Terms added together – substitution method
Type 4: unknown in base – use opposite operations – watch for ± answers
May have to rearrange equation to make it look like one of the above types - particularly if the equation is written as a fraction.

Log

As it appears in Unit 2 & 3

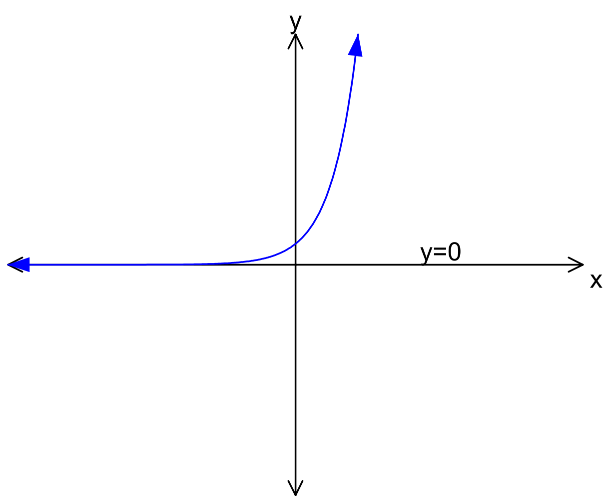
Type 1: Unknown inside log – use log laws to get multiple logs into singular logs. Opposite to $\log_a$ is to the power of a.
Type 2: Unknown is the base – rewrite in index form and solve.

Exponential Graphs

As it appears in Unit 2 & 3

Exponential functions are written in the form $y = a^x$ where a can be any number.
When sketched these have **horizontal** asymptotes. All transformations apply to these graphs.

The basic shape for exponential function:

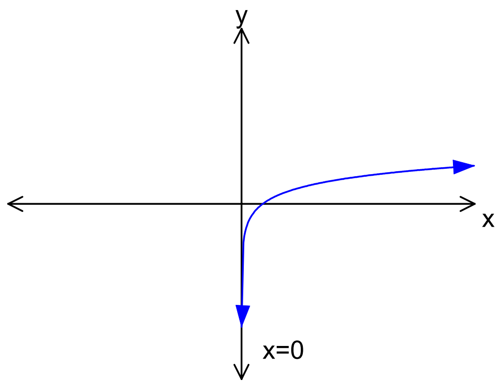


For functions of the form $f(x) = n.a^{x-h} + k$, $n, h, k \in R$ and $a \in R \setminus \{1\}$:
Maximal Dom is R.
Range is (k,∞).
Asymptote: $y = k$.

Log Graphs
As it appears in Unit 2 & 3

Log functions are written in the form of $y = \log_a x$. a is the base in which you have to draw the graph in. the calculator will only graph $\log_{10}$ and $\log_e$ or $\ln$. Other bases need to worked out by you.
When these are sketched, they have **vertical** asymptotes. All transformations apply to these graphs.

The basic shape for log functions are:



For graphs of the form $f(x) = n\log_a(x-h)+k$, where $n,h,k \in R$ and $a \in R^+ \setminus \{1\}$:
Maximal Domain (h,∞)
Range R
Vertical Asymptote: $x = h$. This can be found by equating the bracket to zero and solving.

Be careful of reflections across the y axis.

Calculator Skills

Define functions and evaluate them
Solve functions
Graph functions

Review Questions

1. Simplify $\frac{4\left(x\,y^{2}\right)^{3}\times\left(2x^{3}y\right)^{2}}{x\,y^{2}\left(2x\right)^{4}}$.

2. Simplify $\left(\frac{64}{27}\right)^{-\frac{2}{3}}\div\left(\frac{9}{16}\right)^{\frac{3}{2}}$.

3. Simplify $\frac{x}{1+x^{-1}}+\frac{x}{x^{-1}-1}$.

a. Solve for x where: $2\log_{5}\left(x-3\right)=4$.

b. Simplify $\frac{\log_4 x^5}{\log_4 \sqrt[3]{x}}$.

4. Solve for x where $2^{2x} + 4 \times 2^x - 32 = 0$.

5. A solution for x in $4^{-x} = 9$ exact to three decimal places is.

6. Solve for x where: $\log_2 x + 2\log_2 10 - 3\log_2 5 = 3$.

7. Solve for x where $\log_x 96 - \log_x 12 = 6$.

8. Solve for x where: $(\log_3 x)^2 - 4\log_3 x + 3 = 0$.

9. Simplify $5\log_3 243 - 6\log_2\left(\frac{1}{32}\right)$.

10. If $R = 5\log_3\left(\frac{a}{T}\right) + B$, calculate a value for R, when $a = 72, T = 8$ and $B = -4$.

11. Solve for x where: $256^{-x} = \frac{4^x}{16^{3-x}}$.

12. Solve for x where $e^{5-x} = \frac{2}{3e^{4x}}$.

13. Solve for x where: $\log_e(1-x)=3$.

14. Solve for x where $\log_e(x-3)+\log_e(x-2)=\log_e 12$.

15. Solve for y where: $4-\log_e x = 2\log_e y$.

16. The population of bacteria is given by: $P(t)=500e^{0.3t}$, where t is the time measured in weeks. The number of bacteria in the pond after 10 weeks would be closest to:

17. The height of a certain species of tree can be modelled by the function: $H = \frac{12}{8 + 100e^{-0.3t}}$, where H is the height of the tree in metres and t is the age of the tree in years since it was planted. The height of the tree when planted is approximately:

18. Sketch the graphs of the following stating domain and range:

a. $f(x) = 2e^{3x} - 6$.

b. $f(x) = -3e^{x} + 4$.

19. Sketch the following graphs stating domain and range:

a. $f(x) = \log_e(2 - x)$.

b. $f(x) = -3\log_e(2x + 3) + 4$.

20. The size of a population of rabbits is determined by the rule $P = 6400 \times 3^{0.2t} - 400$ where P is the size of the population t years after January 2006.

a. Find the size of the population when $t = 0$ and $t = 15$.

b. Find the value of t when the population exceeds 1,000,000.

c. Sketch the graph of P vs. t.

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Solutions to Review Questions

1.

x^4y^6

Remove brackets.
Then use the addition and subtraction laws.

2.

$\frac{4}{3}$

Express numbers in bases in lowest base (2 and 3).
Remove brackets and simplify.

3.

$\frac{x^2}{(x+1)(x-1)}$

Express denominators over lowest common denominator.
Then add fractions.

4.

$x = 28$

Divide through by 2.
Raise both sides to the power of 5.

5.

15

Express the cube root as a power of $\frac{1}{3}$.
Bring both powers to the front using index law.
Cancel logs

6.

$x = 2$

Let $2^x = A$ to set up quadratic equation $A^2 - 4A - 32 = 0$
Solve in terms of A
Then substitute 2^x back instead of having A and solve.

7.

$x = -\frac{\log 9}{\log 4} = -\frac{\log 3}{\log 2} = -1.585$

Use index law to bring the power of x out of the log. Then solve.

8.

$x = 10$

Put the 2 and the 3 inside the log using log law.
Simplify the multiple logs into single log expression

9.

$x = \sqrt{2}$

Use division log law. Rewrite into index form then solve.

10.

$x = 27$ *or* $x = 3$

Let $\log_3 x = A$ to set up quadratic equation $A^2 - 4A + 3 = 0$
Solve in terms of A
Then substitute $\log_3 x$ back instead of having A and solve.

11.

55

Evaluate logs.

12.

$R = 6$

Substitute values into expression and simplify.

13.

$x = \frac{6}{7}$

Express each term as a base of 2.
Express right hand side in terms of one base.
Equate the powers.

14.

$x = \frac{1}{3}\log_e\left(\frac{2}{3e^5}\right)$

Multiply both sides by e^{4x}.
Add the powers of *e*.
Divide through by e^5
Take $\log_e$ of both sides of the equation.

15.
$x = 1 - e^3$

Raise both sides as a power of *e*.

16.
$x = 6$

Use addition log law.
Equate expression inside log.
Solve quadratic equation by factorising.
Can't have $x = -1$ as you can't log a negative number when you test the solution.

17.
$y = e^2 / \sqrt{x}$

Divide through by 2.
Raise both sides as a power of *e.*

18.

10043
Evaluate P(10).

19.

11.1 cm

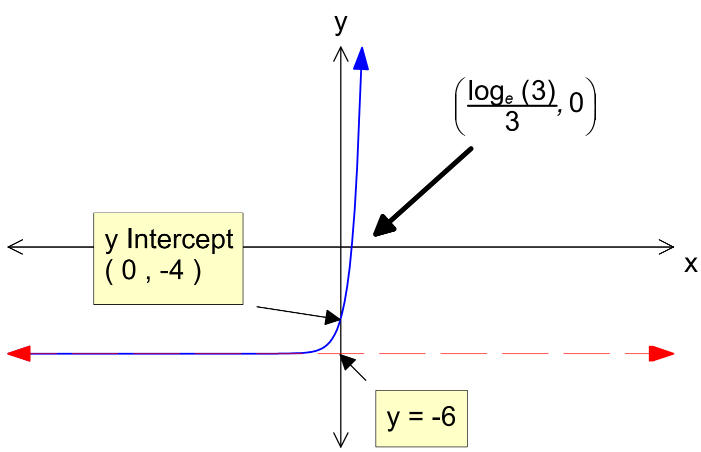
Evaluate H(0)

20.

a.

Dom: R

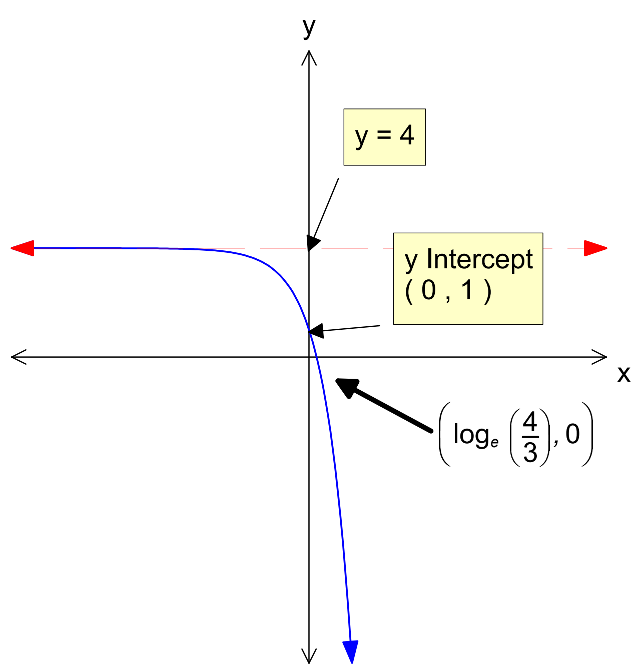
Ran: (-6 , ∞)



b.

Dom: R

Ran: $(-\infty, 4)$

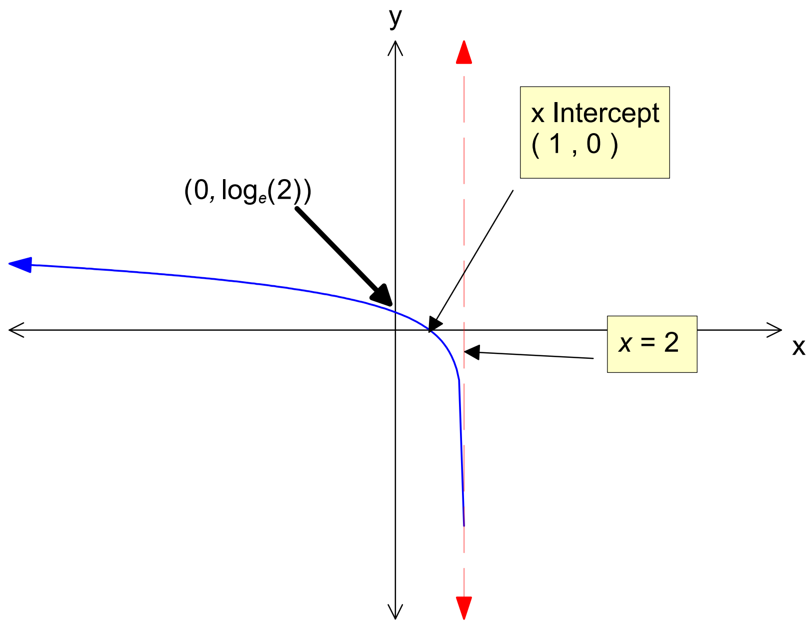


21.

a.

Dom: $(-\infty, 2)$

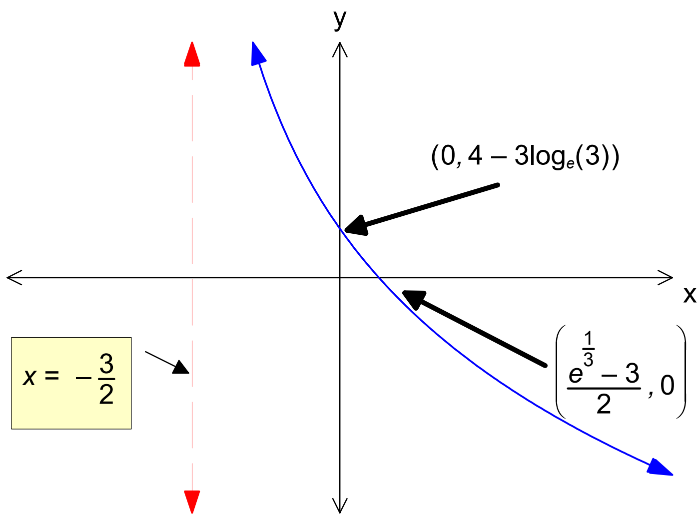
Ran: R



b.

Dom: $\left(-\frac{3}{2}, \infty\right)$

Ran: R



22.

a.

P(0)=6000

P(15)=172400

b.

$t \approx 23$ *years*

Solve 1000000=p(t)

Could use calculator.

c.