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MATHEMATICAL METHODS CAS
Teach Yourself Series
Topic 5: Transformations of Functions

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Transformations of Functions

Type 1 – Functions

As it appears in Unit 1

All graphs can be transformed by the following:

Reflections: $-f(x)$

Reflection across x axis. y coordinates on the original functions are multiplied by -1.

$f(-x)$

Reflection across y axis. x coordinates on the original functions are multiplied by -1.

Dilations $af(x)$

Dilation of a from x axis (parallel to y axis) – y coordinates on the original functions are multiplied by "a".

$f(ax)$

Dilation of $\frac{1}{a}$ from y axis (parallel to x axis) - x coordinates on the original functions are multiplied by "$\frac{1}{a}$".

Translations $f(x)+k$

Translation of k along y axis (from x axis) – y coordinates on the original functions have "k" added to them.

$f(x-h)$

Translation of h along x axis (from y axis) - x coordinates on the original functions have "-h" added to them.

Watch for the tricky questions – The dilation from y

The transformations given out of order.

Example

Describe the transformations that have been applied to $\boldsymbol{f(x) = \sqrt{x}}$ to result in the function $\boldsymbol{f(x) = -2\sqrt{3(x-1)} + 4}$.

Answer:

Reflection across the *x* axis.
Dilation of 2 units from the *x* axis and $\frac{1}{3}$ of a unit from the *y* axis.

Translation of 1 units along *x* axis and 4 units along *y* axis.

Calculator skills

Define functions
Type transformation in function form once function is defined.

Type 2 – Matrices

As it appears in Unit 3

Reflection in x axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflection in y axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection across $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Dilation from x axis $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

Dilation from y axis $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

Translation along y axis $\begin{bmatrix} 0 \\ \mathbf{k} \end{bmatrix}$

Translation along x axis $\begin{bmatrix} h \\ 0 \end{bmatrix}$

Note: You put the transformations in to the matrices as you read them.

- Have to solve the matrix equation: $X = \mathrm{TX'} + B$

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

$T = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ ie dilation and reflection matrix

$X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$B = \begin{bmatrix} h \\ k \end{bmatrix}$ ie translation matrix

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- Solve for X"
- Swap *x* for *x'* and *y* for *y'* in the rule that is being transformed

Example

Use matrix methods to apply the following transformations to $f(x) = e^x$. State the new function.

Reflection across the *x* axis.
Dilation of 2 units from the *x* axis and $\frac{1}{3}$ of a unit from the *y* axis.
Translation of 1 units along *x* axis and 4 units along *y* axis.

Answer:

The matrix equation for this question is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

This results in the equation being:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{x}{3} + 1 \\ -2y + 4 \end{bmatrix}$$

Therefore:

$$x' = \frac{x}{3} + 1 \; so \; 3(x' - 1) = x$$

and

$$y' = -2y + 4 \; so \; \frac{y' - 4}{-2} = y$$

from the original function:

$$y = e^x$$

substitute the above results:

$$\frac{y' - 4}{-2} = e^{3(x'-1)}$$

This transposes to:

$$y' = -2e^{3(x'-1)} + 4$$

So the new function is

$$f(x) = -2e^{3(x-1)} + 4$$

Review Questions

1. If $f(x) = \frac{1}{x}$ perform the following transformations using matrices and state the new function:

Reflection across *y* axis, Dilation of 3 units from *x* axis, Translation 2 units along *x* axis.

2. If $f(x) = \sqrt{x}$ perform the following transformations using matrices and state the new function:

Reflection across *y* axis and *x* axis, Dilation of 2 units from *x* axis and 3 units from the *y* axis, Translation 2 units along *y* axis and 7 units along the *x* axis.

3. The graph of $y = \frac{1}{x^2}$ is transformed by the following sequence of transformations

- dilation of factor 2 from the x-axis
- dilation of factor 3 from the y-axis
- reflection in the x-axis
- translation of 3 units in the positive direction of the y-axis.

Find the equation of the image of $y = \frac{1}{x^2}$ under this sequence of transformations using a functions approach.

4. State a sequence of transformations which takes the graph of $y = \frac{1}{x}$ to the graph of

$y = \frac{2}{2 - x} + 4$

5. The graph of $y = e^x$ is transformed by the following sequence of transformations

- dilation of factor 2 from the x-axis
- dilation of factor 3 from the y-axis
- reflection in the x-axis
- translation of 3 units in the positive direction of the y-axis.

Find the equation of the image of $y = e^x$ under this sequence of transformations using a functions approach.

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6. The graph of $y = x^3$ is transformed by the following sequence of transformations

- dilation of factor $\frac{1}{2}$ from the x-axis
- dilation of factor 4 from the y-axis
- reflection in the x-axis
- translation of 3 units in the positive direction of the y-axis.

Find the equation of the image of $y = x^3$ under this sequence of transformations using a matrix approach.

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7. The graph of $y = \log_e(x)$ is transformed by the following sequence of transformations

- dilation of factor 5 from the x-axis
- dilation of factor 1/2 from the y-axis
- reflection in the x-axis
- translation of 3 units in the positive direction of the y-axis.

Find the equation of the image of $y = \log_e(x)$ under this sequence of transformations using a functions approach.

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8. The graph of $y = \frac{1}{x}$ is transformed by the following sequence of transformations

- dilation of factor 2 from the x-axis
- dilation of factor 3 from the y-axis
- reflection in the x-axis
- translation of 3 units in the direction of the y-axis and -2 units along the x axis.

Find the equation of the image of $y = \frac{1}{x}$ under this sequence of transformations using a matrix approach.

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9. The graph of $y = \cos(x)$ is transformed by the following sequence of transformations

- dilation of factor 2 from the x-axis
- reflection in the x-axis
- translation of 1 units in the positive direction of the y-axis and $\frac{\pi}{4}$ along the x axis.

Find the equation of the image of $y = \cos(x)$ under this sequence of transformations using a functions approach.

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Solutions to Review Questions

1.

$$3f(-(x-2)) = \frac{3}{-(x-2)}$$

$$\frac{3}{2-x}$$

2.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -3x + 2 \\ -2y + 7 \end{bmatrix}$$

$x = \frac{x' - 2}{-3}$ and $y = \frac{y' - 7}{-2}$ Substitute these into the function to be transformed and transpose to the answer.

$$f(x) = 2\sqrt{-\frac{1}{3}(x-2)} + 7$$

3.

$$-2f\left(\frac{x}{3}\right) + 3 = -\frac{2}{\left(\frac{x}{3}\right)^2} + 3$$

$$f(x) = -\frac{18}{x^2} + 3$$

4.

Transpose function to $\dfrac{2}{-(x-2)} + 4$ before describing transformations.

Reflection in *y* axis

Dilation of 2 units from *x* axis

Translation of 2 units along *x* and 4 units along *y*.

5.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3x \\ 2y+3 \end{bmatrix}$$

$x = \frac{x'}{3}$ and $y = \frac{y'-3}{2}$ Substitute these into the function to be transformed and transpose to the answer.

$f(x) = -2e^{\frac{x}{3}} + 3$

6.

Set up the following matrix equation: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$x' = 4x \qquad y' = -\frac{y}{2} + 3$

$x = \frac{x'}{4} \qquad y = -2(y' - 3)$

Make these substitutions and transpose to make *y'* the subject

$y = -\frac{1}{128}x^3 + 3$

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7.

Apply the transformations $-5f(2x)+3$

$y = -5\log_e(2x) + 3$

8.

Set up matrix equation: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

This leads to

$x' = 3x - 2$

$x = \dfrac{x' + 2}{3}$

and

$y' = -2y + 3$

$y = \dfrac{y' - 3}{-2}$

Substitute these into $y = \dfrac{1}{x}$ and transpose to make y' the subject of the formula.

$y = -\dfrac{6}{x+2} + 3$

9.

Apply the transformations of $-2f\left(x - \frac{\pi}{4}\right) + 1$.

$y = -2\cos\left(x - \frac{\pi}{4}\right) + 1$