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MATHEMATICAL METHODS CAS
Teach Yourself Series
Topic 6: Circular Functions

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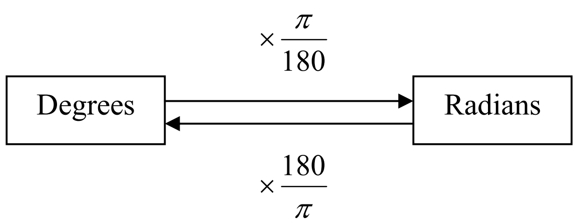
Circular Functions

In this Topic you will learn the basic skills required in circular functions.

Radians and Degrees

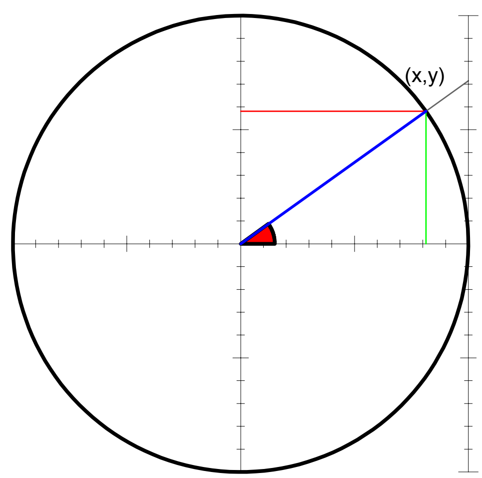
Degrees (°) and radians (^{c}) are two different methods used to measure the magnitude of an angle. In circular functions, and mathematics in general, radian is the preferred unit of measure. For any calculus applications, radians must be used, as in any other measure of angle, the derivative of (say) sine is not equal to cosine!
The method of converting from one form to another is illustrated in the diagram below:



Symmetrical Properties of Circular Functions

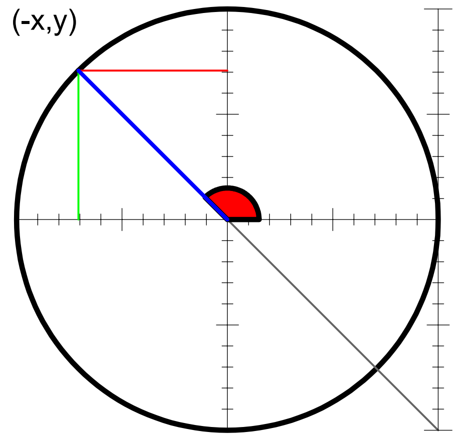
As it appears Unit 2

Quadrant 1



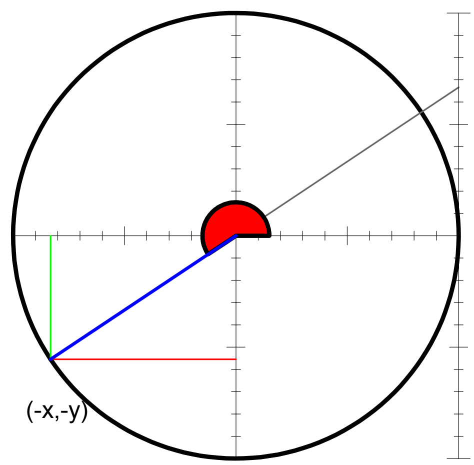
$$x = \cos\theta$$
$$y = \sin\theta$$
$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

Quadrant 2



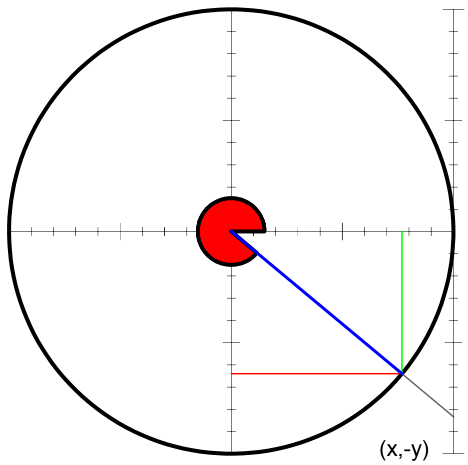
$$\sin(\pi - \theta) = \sin\theta$$
$$\cos(\pi - \theta) = -\cos\theta$$
$$\tan(\pi - \theta) = -\tan\theta$$

Quadrant 3



$$\sin(\pi + \theta) = -\sin\theta$$
$$\cos(\pi + \theta) = -\cos\theta$$
$$\tan(\pi + \theta) = \tan\theta$$

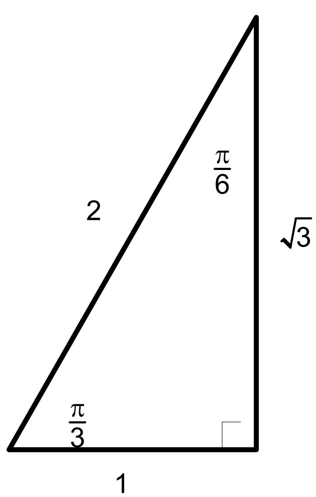
Quadrant 4

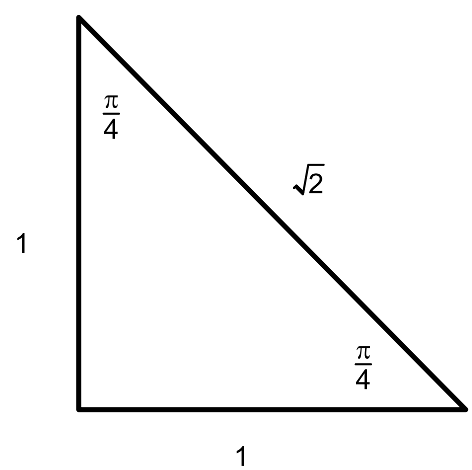


$$\sin(2\pi - \theta) = -\sin\theta$$
$$\cos(2\pi - \theta) = \cos\theta$$
$$\tan(2\pi - \theta) = -\tan\theta$$

Exact Values of Common Angles

As it appears in Unit 2





From these triangles you can evaluate the sin, cos and tan of these angles.

Example: $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Note the the denominator has been rationalised for the $\cos\left(\frac{\pi}{4}\right)$.

Equation Solving

As it appears in Unit 2

The basic process is:

1. Transpose the equations so the trig function is on its own. Be careful here, if you have more than one trig function in the equation you may have to transpose it quite a bit. Hopefully it is in terms of the same angle. If the angles are different you will need to use the calculator.
2. Find the Base Angle.
3. Use the unit circle to work out which quadrants the solutions lie. If n>1 you may have to go around the circle more than once (n times if solutions are between $[0,2\pi]$.
4. Solve the equation.

General Solutions

As it appears in Unit 3

$x = 2\pi n \pm \cos^{-1}(a)$

$x = (2n + 1)\pi - \sin^{-1}(a)$ *or* $x = 2\pi n + \sin^{-1}(a)$

$x = \pi + \tan^{-1}(a)$

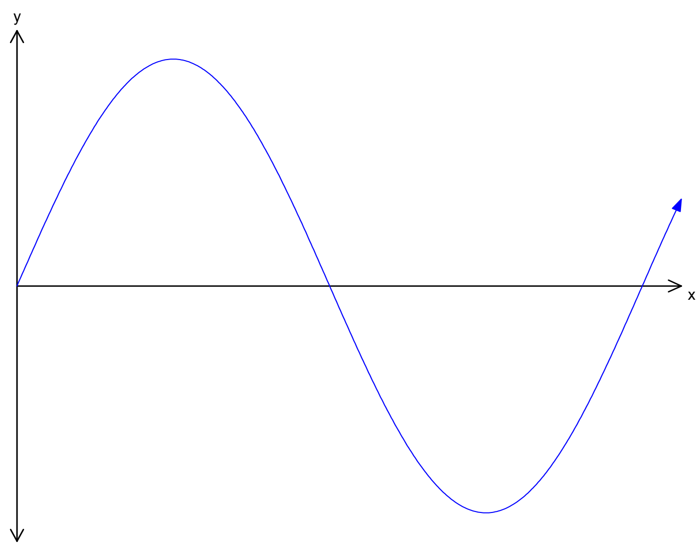
Calculator Skills

Solve equations getting general solutions as well as within a domain.

Graphs of circular functions

As it appears in Unit 2

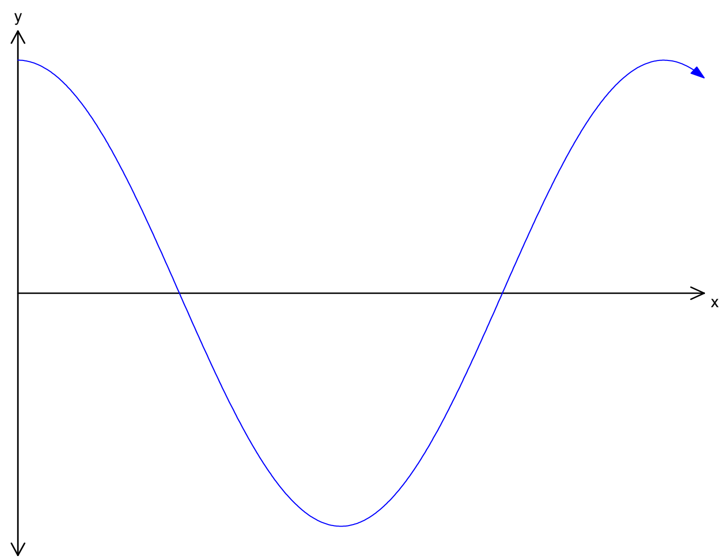
$y = \sin(x)$



$p = \dfrac{2\pi}{n}$

Amp = dist from average position to min or max

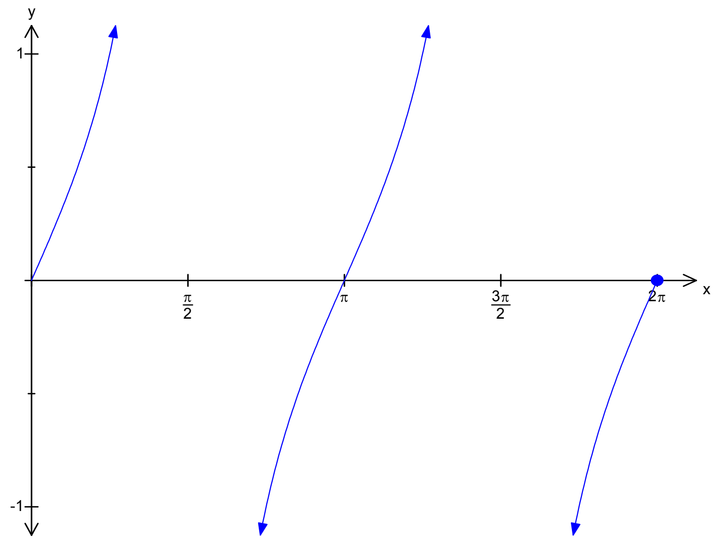
$y = \cos x$



$p = \dfrac{2\pi}{n}$

Amp = dist from average position to min or max.

$y = \tan x$



$p = \dfrac{\pi}{n}$

Asymptotes: at $\dfrac{p}{2}$, then add p to get the next one and so on.

Calculator skills

Graphing functions in appropriate window
Solving equations graphically.

Review Questions

1. Solve the equation $2 \sin 3x = 1$ for $x \in [0, 2\pi]$.

2. $\sin\left(\frac{3\pi}{4}\right) =$

3. $\cos\left(\frac{11\pi}{6}\right) =$

4. $\tan\left(\frac{5\pi}{4}\right) =$

5. The general solution of the equation tan $2x = \sqrt{3}$ is:

____

____

____

____

6. The equation $2\sin x + 1 = b$, where b is a positive real number, has one solution in the interval $(0, 2\pi)$. The value of b is:

A. 1
B. 1.5
C. 2
D. 3
E. 4

7. If $\sin x = -1$ and $x \in [0, 2\pi]$, the value of x is:

A. 0
B. $\frac{3\pi}{2}$
C. $\frac{\pi}{4}$
D. π
E. $\frac{\pi}{2}$

8. The smallest positive value of x for which the graph of $f(x) = 2\cos 2x + 2$ touches the x-axis is:

A. $x = \frac{\pi}{4}$
B. $x = \frac{\pi}{2}$
C. $x = \frac{3\pi}{4}$
D. $x = \frac{3\pi}{2}$
E. $x = \pi$

9. State the range of the function f: $[0, 2\pi] \to R$, $f(x) = 4 \sin\left(2x + \frac{\pi}{4}\right)$

10. State the range, amplitude and period of the function f: $[0, 2\pi] \to R$, $f(x) = -4 \sin\left(3x + \frac{\pi}{4}\right) + 2$

11. For the function with rule $g(x) = -4 \sin\left(\frac{\pi x}{6}\right) + 10$:

a. state the period and range

b. solve the equation $g(x) = 8$ for $x \in [0, 12]$

c. sketch the graph of $y = g(x)$ for $x \in [0, 12]$

12. A function with rule $f(t) = a\cos(nt + \varepsilon)$ with $\varepsilon \in \left[0, \frac{\pi}{2}\right]$ has the following properties:

- range = [–5, 5]
- period = 4
- $f(0) = 2.5$

Find the values of a, n and ε.

13. Find the general solution of each of the following equations:

a. $\cos 5(x - \frac{\pi}{6}) = 1$

b. $\sin 2(x - \frac{\pi}{3}) = \frac{1}{2}$

c. $\tan(2x - \frac{\pi}{4}) = -1$

14. Sketch the graph of $y = 2\tan(3\pi x)$ for $x \in [0, 1]$.

15. Sketch the graph of $y = 2\sin(2\pi x) + 2$ for $x \in [0, 1]$.

16. The height, $H(t)$ metres, of the tide above mean sea level at t hours after midnight on a particular day is given by $H(t) = 4\cos\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$

a. What is the height of the tide when:

$t = 0$?
$t = 24$?

b. What is the period of the function?

c. Sketch the graph of $H(t)$ against t for $0 \le t \le 24$

d. At what time is high tide?

e. A pier can only be accessed when the tide is less than 2 metres above mean sea level. Find, to the nearest minute, the times when the pier can be accessed on this day.

Solutions to Review Questions

1.

$\sin(3x) = \frac{1}{2}$

Basic angle = $\frac{\pi}{6}$

Need to complete three laps of unit circle to get all solutions.

$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

2. $\sin\left(\frac{3\pi}{4}\right) =$

$\frac{\sqrt{2}}{2}$

Symmetrical Property

3. $\cos\left(\frac{11\pi}{6}\right) =$

$\frac{\sqrt{3}}{2}$

Symmetrical Property

4. $\tan\left(\frac{5\pi}{4}\right) =$

1

Symmetrical Property

5.

Use $x = \pi n + \tan^{-1}(x)$

$2x = \pi n + \tan^{-1}(\sqrt{3})$

Basic angle = $\frac{\pi}{3}$

$x = \frac{\pi(3n + 1)}{6}$

6.

D.

Sketch graph to help with answer.

7.

B.

Sketch graph to help with answer.

8.

B.

Sketch graph to help with answer.

9.

[-4,4]

Use amplitude or sketch graph.

10.

Explanation

Amplitude in number multiplied to sin. Amplitude = 4

Period = $\frac{2\pi}{n}$, where n = 3. Period = $\frac{2\pi}{3}$

Range is -4 +2 and 4 + 2. Range = [-2,6]

11.

a.

Period = $\frac{2\pi}{n}$, where n = $\frac{\pi}{6}$. Period = 12

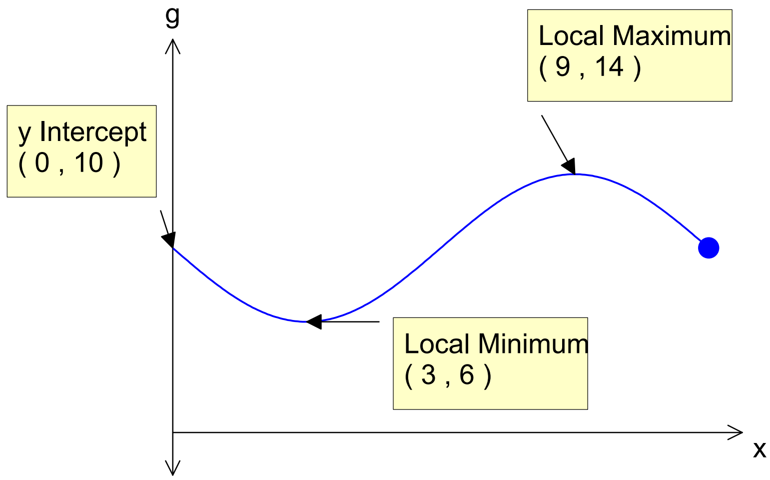
Range is -4 +10 and 4 + 10. Range :[6,14]

b.

Rearrange to get sin term as subject.

$x = 1 \text{ or } 5$

c.



12.

a = amplitude

a = 5

use $p = \frac{2\pi}{n}$ where period = 4 to find *n*.

$n = \frac{\pi}{2}$

Substitute first two answers into expression to evaluate ε

$\varepsilon = \frac{\pi}{6}$

13.

a.

Use $x = 2\pi n \pm \cos^{-1}(a)$

$$5\left(x - \frac{\pi}{6}\right) = 2\pi n \pm 0$$

$$x = \frac{\pi(12n+5)}{6}$$

b.

To get both solution use $x = 2\pi n + \sin^{-1}(a)$ and $x = (2n + 1)\pi - \sin^{-1}(a)$

Basic angle = $\frac{\pi}{6}$.

$x = \frac{\pi(12n + 5)}{12}$ *or* $x = \frac{\pi(4n + 1)}{4}$

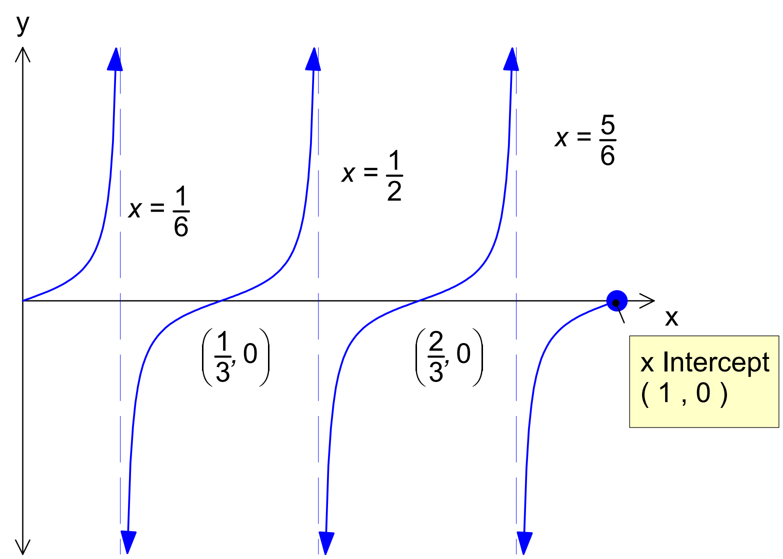
c.

Use $x = \pi n + \tan^{-1}(a)$

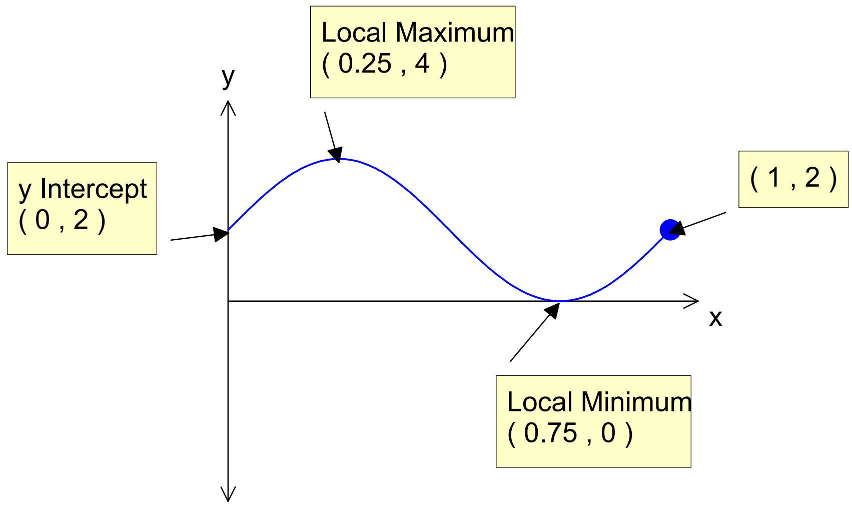
$$2x - \frac{\pi}{4} = \pi n - \frac{\pi}{4}$$

$$x = \frac{\pi n}{2}$$

14.



15.



16.

a.

Substitute t=0 and t=24 into expression.

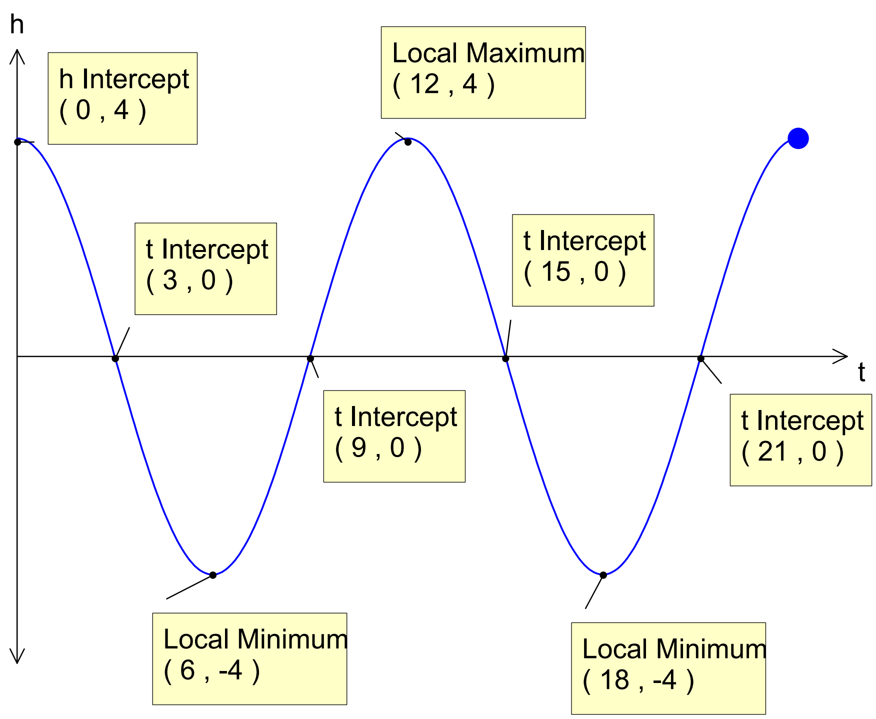
$H(0)=H(24)=4$

b.

Use $p = \frac{2\pi}{n}$

12 hours

c.



d.

Read from graph
Midnight and Midday

e.

Solve the equation $H(t)=2$.
Has to be above this height.
You could graph the problem using a second function of $y=2$ to see what is happening.

Midnight to 2 am

10 am to 2 pm

10 pm to midnight

