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MATHEMATICAL METHODS CAS
Teach Yourself Series
Topic 8: Differential Calculus

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Contents

Differential Calculus

Limits

As it appears in Unit 2

A limit exist if the limit from the right is the same as the limit from the left.

To evaluate limits you must remember:

If you have $\frac{\infty}{\infty}$, $\frac{0}{0}$ or $\frac{x}{0}$ then are undefined results.

Before you say that the limit is undefined, you need to try and simplify the problem.

To simplify, use factorising skills.

If you can't do this, the answer is undefined.

Calculator skills

Limits on calculator
Factor

Average rate of change

As it appears in Unit 2

This can be approximated by: $m_{tangent} = \frac{f(x+h)-f(x)}{h}$ OR $m_{tangent} = \frac{f(x)-f(x-h)}{h}$

This is the **averag**e rate of change.

Simply it is the **gradient between two points**.

Calculator skills

Define functions.
Evaluate functions.

Differentiation by first principles

As it appears in Unit 2

This is a rule that relates the gradient of the tangent to any value of x. This is calculated by:

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternative notations for the derivative function are: $\frac{dy}{dx}, y', f'$.

Calculator skills

Differentiate functions.
Define and evaluate functions

Conditions of Differentiability

As it appears in Unit 2

A function is differentiable at a point *x* if:

- There are no holes or breaks at *x*
- The derivative of the function approaches the same value from both sides of *x*. (No Sharp points)

Note that on a restricted domain the function is not differentiable at the end points.

Rules to differentiate Different Functions

As it appears in Units 2 and 3

$y = x^n$	$y = e^{f(x)}$	$y = \sin(f(x))$
$y' = nx^{n-1}$	$y' = f'(x)e^{f(x)}$	$y' = f'(x)\cos(f(x))$
$y = \cos(f(x))$	$y = \tan(f(x))$	$y = a\ln(f(x))$
$y' = -f'(x)\sin(f(x))$	$y' = f'(x)\sec^2(f(x))$	$y' = a\frac{f'(x)}{f(x)}$

Chain Rule
As it appears in Units 2 and 3

$$\frac{dy}{dx} = \frac{du}{dx} \bullet \frac{dy}{du}$$

Product Rule
As it appears in Units 2 and 3

$$y' = vu' + uv'$$

Quotient Rule
As it appears in units 2 and 3

$$y' = \frac{vu' - uv'}{v^2}$$

Calculator Skills

Differentiate functions
Define and evaluate functions
Proper fraction

Must be able to recognise when to use which technique.

There are some important tips that you can look for when simplifying these expressions:

- Look for common factors.
- Expanding
- Index laws
- Find **lowest common denominator** to combine expression into a singular fraction.

Sketching $f'(x)$ given $f(x)$

As it appears in Units 2 and 3

When you find the gradient function, it can then be graphed. We must understand what information a gradient v x graph can give us:

1. Where the function crosses the x axis – a stationary point
2. Where the function is below the x axis – the gradient function is negative (the tangents that would be drawn on the original function would have a negative gradient)
3. Where the function is above the x axis – the gradient function is positive (the tangents that would be drawn on the original function would have a positive gradient)

Sketching gradient function given graphs

The following process is used:

1. Locate stationary points.
2. Then look at the rate of change of the graph (tangents) on both the LHS and the RHS of the stationary points.
3. If the tangents are negative, the graph of the gradient function will be negative – draw the line.
4. If the tangents are positive, the graph of the gradient function will be positive – draw the line.

Calculator skills

Graphing derivative functions

Applications

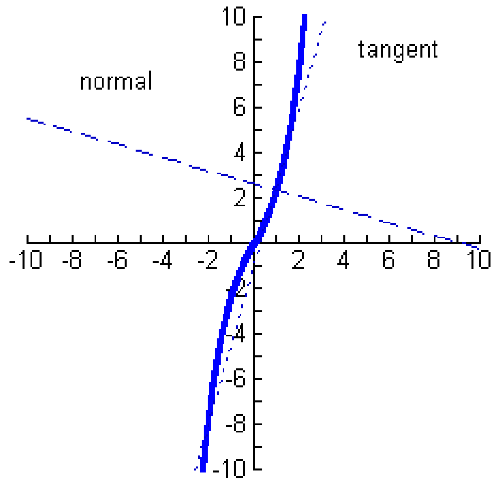
As it appears in Units 2 and 3

Approximation

$$f(x+h) \approx f(x) + hf'(x)$$

Tangent /Normal Problems

Tangents and Normals are at right angles to one another.



The gradients are related by the following rule: $m_t m_n = -1$ or $m_n = -\frac{1}{m_t}$

You will need to use $y - y_1 = m(x - x_1)$ to find the equation.

Questions end to be in two basic types

1. Given info about points – need to find gradient
2. Given info about gradient – need to find points

Stationary points

At Stationary points $\frac{dy}{dx} = 0$. This is very important to remember.

The process is as follows:

1. Differentiate the function.
2. Let the derivative function equal zero.
3. Solve for *x*.

You often have to justify the type of stationary point. To do this you need to fill in a gradient table:

1. Find the gradient function.
2. Choose a point that is "left" of the stationary point. Make sure you do not go beyond another stationary point when you make this choice. Take note of the sign.
3. Choose a point that is to the "right" of the stationary point. Make sure you do not go beyond another stationary point when you make this choice. Take note of the sign.
4. Fill in the following table:

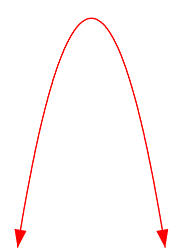
x	$x - \Delta x$ (left)	x=point at S.P.	$x + \Delta x$ (right)
$\frac{dy}{dx}$		**0**	

5. Then decide on the type of stationary point that you have.

The type of stationary point and their table follows:

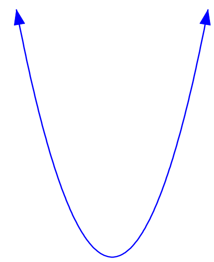
Maximum

x	$x - \Delta x$	*x*=point at S.P.	$x + \Delta x$
$\frac{dy}{dx}$	+	**0**	**-**



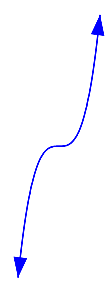
Minimum

x	$x - \Delta x$	x=point at S.P.	$x + \Delta x$
$\frac{dy}{dx}$	-	**0**	+



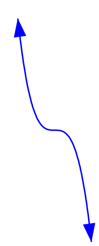
Increasing point of inflection

x	$x - \Delta x$	x=point at S.P.	$x + \Delta x$
$\frac{dy}{dx}$	+	**0**	+



Decreasing point of inflection

x	$x - \Delta x$	x=point at S.P.	$x + \Delta x$
$\frac{dy}{dx}$	-	**0**	-



Maximum and Minimum Problems

These are problem where you often have to find the maximum of minimum **value** of a function. You also have to often find **where** it occurs ie what value of '*x*'.

The following process is used in solving maximum and minimum problems:

1. Identify the rule that has to be maximised or minimised. This is the function that has to be differentiated. You may have to derive the rule. (Area of shape, Volume of solid, etc)
2. If the rule that is to be minimised is in more than one variable, use the other information in the question so that you can make relevant substitutions. When this is done you should have just the function in terms on one variable only.
3. Differentiate the function and make it equal to zero.
4. Solve the function.
5. Answer the question.

You need to be very careful when drawing graphs of these functions. These are often drawn over restricted domains that suit the context of the problem. Not all solutions that have been reached may be sensible. Make sure you check this. Sometimes the maximum or minimum may not be in the domain that is being considered. It is a good idea to calculate end points and graph the function over the domain to check.

Rates of change

Some of these are problems that involve using calculus but time is plotted on the x axis (independent variable). Eg $\frac{ds}{dt}$

Motion in a straight line

Some special rates of change involve the study of motion. In studying motion we have to be aware of a starting point.

Displacement is the distance an object is from the starting point.

Velocity is the rate at which displacement changes.

Acceleration is the rate at which the velocity changes.

From the above definitions:

- $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ where:

  x is displacement.

  v is velocity.
  a is acceleration.

Related Rates

When one variable changes with respect to another. Often use the chain rule to find it.

Note: $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ this is often useful to use.

Easy questions

Given a rate in terms of a number
Given a rule to differentiate to get the other rate
Sub info into the chain rule to find rate asked for.

Hard questions

Rule that has to be differentiated has more than one variable in it. Need to use information in question to help get rid of one.

Often have to use similar triangles, trig ratios, Pythagoras' theorem.

Review Questions

1. Evaluate:

a. $\lim_{x \to 2} \frac{x^2 - 4}{x + 2}$

b. $\lim_{x \to \infty} \frac{1}{e^x}$

2. Given $f(x) = x^3 - 2x$, calculate the average rate of change of $f(x)$ with respect to x for the interval $[-2, 0]$.

3. Find the gradient of the chord of the curve $y = 2x^2$ between the points where $x = 1$ and $x = 1 + h$.

4. Find the derivative of $f(x) = x^2 - 4x$ by first principles.

5. Find the derivative of $(3x^2 - 2x)^3$ with respect to x.

6. Find the gradient of the curve with equation $y = \dfrac{2x^2 - 1}{x^2 + 2}$ at the point where $x = 2$.

7. Find the coordinates of the point on the curve with equation $y = 4x^2 - 2x + 3$ for which the gradient is – 10.

8. For $f(x) = x^3 + x$, find $f'(x)$ by first principles.

9. The gradient of the curve with equation $y = \frac{a}{x} + bx^2$ at the point with coordinates (3, 6) is 7. Calculate the values of a and b.

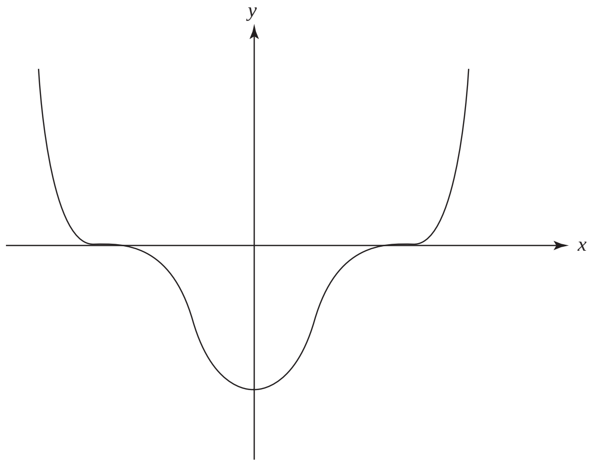
10. Find the derivative of each of the following with respect to x:

a $(x^2 - 2x + 3)^{-2}$

b $\dfrac{2x^2 - 3}{x^2}$

c $\sqrt{x^2 - 3x}$

11. The graph of $y = f(x)$ is as shown:



Copy the graph and on the same set of axes sketch the graph of $y = f'(x)$.

12. Find the equation of the tangent to the curve with equation $y = (4x^2 - 1)^3$ at the point where:

a. $x = \dfrac{1}{2}$

b. $x = 1$

13. For the function with rule $f: R^+ \rightarrow R, f(x) = 3x^{1/3}$ find the equation of the normal to the curve at the point with coordinates (1, 3)

14. Calculate the equation of the tangent to the curve with equation $y = e^{-x}$, at the point $(-1, e)$.

15. Find the value of x where the minimum value of $e^{-x} + 2e^x$ occurs.

16. Find the derivative of each of the following with respect to x:

a. $2 \sin (x^2)$

b. $x^3 \sin x$

c. $\dfrac{\sin x}{e^x}$

d. $\dfrac{\log_e x}{3x}$

e. $5e^{3x} \cos (2x)$

17. Find the equation of the tangent to the curve with equation $y = \sin (2x)$ at the point where $x = \frac{\pi}{8}$

18. *Let* $f: R \rightarrow R, f(x) = 2e^{x} + 3$

a. Sketch $f(x)$.

b. Find the derivave of $f(x)$.

c. Find the equation of the normal at $x = 1$.

19. Using the approximation $f(x + h) \approx f(x) + h f'(x)$ where $f(x) = e^x$, with $x = 0$, the approximate value of $e^{0.025}$ is found to be:

A. 0.025
B. 0.975
C. 1.025
D. 1.0253

20. The volume V (litres) of water in a tank at time t (hours) is given by

$$V(t) = 3\sin\left(\frac{\pi t}{12}\right) + \frac{1}{2}$$

a. Find the volume of water in the tank at time $t = 24$

b. Find the rate of change of volume of the water in the tank when $t = 24$

21. Find the stationary point and the nature of the stationary point for $y = (1 + x^2)\, e^x$

22. The cost C (\$ / h) of running a ferry at a constant speed of v (km / h) is $C = \frac{5625}{v} + \frac{4v^2}{75}$.
At what speed will the cost be a minimum?

23. A farmer plans to use a river as one boundary of a rectangular paddock. If the farmer has 960 metres of fencing to be used to fence the other 3 sides, what dimensions should the paddock be to ensure maximum area?

24. The sum of two numbers is 80. What are the two numbers if the sum of their squares is a minimum?

25. A closed box is to be constructed that has the following conditions:

i. a square base

ii. a volume of $27\ 000\ \text{cm}^3$

iii. minimum surface area

What are the dimensions of the box?

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Solutions to Review Questions

1.

a.

Factorise Numerator. $x + 2$ cancels out.

0

b.

As x increases the denominator gets bigger, the fraction gets smaller.

0

2.

Average rate of change $= \dfrac{f(0) - f(-2)}{0 - -2}$

2

3.

Average rate of change $= \dfrac{f(1 + h) - f(1)}{h}$

2h + 4

4.

$f'(x) = \lim\limits_{h \to 0} \dfrac{(x + h)^2 - 4(x + h) - (x^2 - 4x)}{h}$ (from first principles definition)

Expand top line. Take out h as a common factor from the top line and cancel it. Then evaluate the limit.

$2x - 4$

5. Expand brackets before differentiating.

$162x^5 - 270x^4 + 144x^3 - 24x^2$

6.

$\dfrac{2x^2 - 1}{x^2 + 2} = 2 - \dfrac{5}{x^2 + 2}$ - long divide

$y' = \dfrac{10x}{(x^2 + 2)^2}$ - chain rule

$\dfrac{5}{9}$

7. $y' = 8x - 2$ Differentiate, make expression =10 and solve.

$-10 = 8x - 2$

$x = -1$

$f(-1) = 9$ Evaluate.

(-1, 9)

8.

$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 + (x+h) - (x^3 + x)}{h}$ - From definition.

Expand top line. Take out h as a common fact from the top line and cancel it. Then evaluate the limit.
$3x^2 + 1$

9. $6 = \frac{a}{3} + 9b$ - Substitute point into function.

$7 = -\frac{a}{9} + 6b$ - Substitute the gradient and x into derivative.

Solve equations simultaneously.
$a = -9$ $b = 1$

10.

a.

Use chain rule - $u = x^2 - 2x + 3$ $\frac{du}{dx} = 2x - 2$

$y = u^{-2}$ $\frac{dy}{du} = -2u^{-3}$

$\frac{-4(x-1)}{(x^2 - 2x + 3)^3}$

b. Divide denominator into each term of the numerator. The differentiate.

$\frac{6}{x^3}$

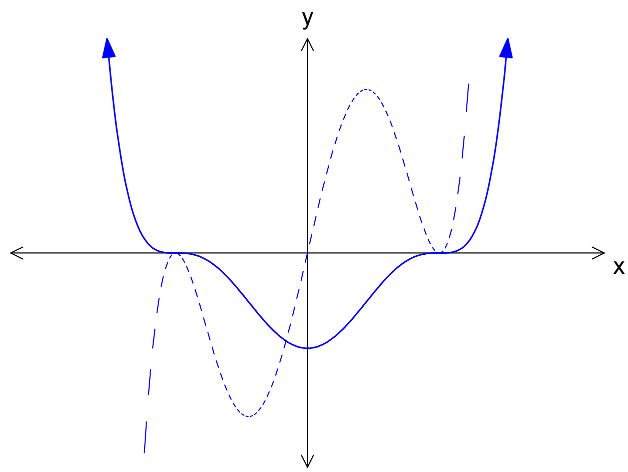
c.

Chain rule - $u = x^2 - 3x$ $\frac{du}{dx} = 2x - 3$

$y = u^{\frac{1}{2}}$ $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

$\frac{2x-3}{2\left(\sqrt{x^2 - 3x}\right)}$

11.



12.

a. $y' = 24x(4x^2 - 1)^2$

$y'\left(\frac{1}{2}\right) = 0$

$y(0) = 0$

b. $y'(1) = 216$

$y(1) = 27$

$y - 27 = 216(x - 1)$

$y = 216x - 189$

13.

$$f'(x) = \frac{1}{x^{\frac{2}{3}}}$$

$$f'(1) = 1$$

$$m_{normal} = -1$$

$$y - 3 = -1(x - 1)$$
$$y = -x + 4$$

14.

$$y' = -e^{-x}$$

$$y'(-1) = -e$$

$$y - e = -e(x + 1)$$
$$y = -ex$$

15.

$$f'(x) = 0$$

$$2e^{x} - e^{-x} = 0$$

$$x = -\frac{\log_e(2)}{2}$$

$$f\left(-\frac{\log_e(2)}{2}\right) = 2\sqrt{2}$$

$$x = \frac{-\log_e(2)}{2}$$

16.

a.

Chain rule - $u = x^2$ $\frac{du}{dx} = 2x$

$y = 2\sin u$ $\frac{dy}{du} = 2\cos u$

$4x\cos(x^2)$

b.

Product Rule - $u = x^3$ $\frac{du}{dx} = 3x^2$

$v = \sin x$ $\frac{dv}{dx} = \cos x$

Take out common factor after chain rule has been applied.

$x^2(xcox(x) + 3\sin(x))$

c.

Use quotient rule - $u = \sin(x)$ $\frac{du}{dx} = \cos(x)$

$v = e^x$ $\frac{dv}{dx} = e^x$

Take out common factor when the quotient rule has been applied, then cancel down.

$\frac{(\cos(x) - \sin(x))}{e^x}$

d.

Use quotient rule - $u = \log_e(x)$ $\frac{du}{dx} = \frac{1}{x}$

$v = 3x$ $\frac{dv}{dx} = 3$

$\frac{1 - \log_e(x)}{3x^2}$

e.

Product Rule - $u = 5e^{3x}$ $\frac{du}{dx} = 15e^{3x}$

$v = \cos(2x)$ $\frac{dv}{dx} = 2\sin(2x)$

Take out common factor when the product rule has been applied.

$5e^{3x}(-2\sin(2x) + 3\cos(3x))$

17.

$$y' = 2\sin(2x)$$

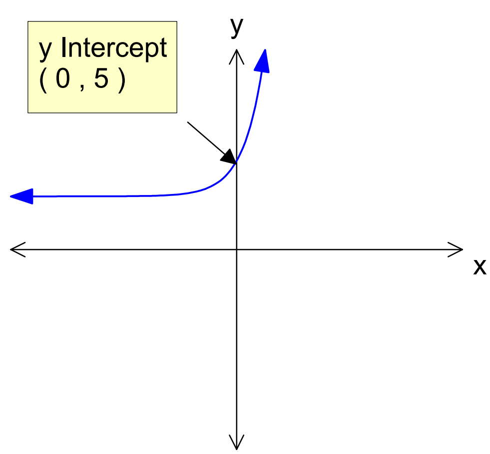
$$y'\left(\frac{\pi}{8}\right) = \sqrt{2}$$

$$y\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2}$$

$$y - \frac{\sqrt{2}}{2} = \sqrt{2}\left(x - \frac{\pi}{8}\right)$$
$$y = \sqrt{2}\,x - \frac{\sqrt{2}\,\pi}{8} + \frac{\sqrt{2}}{2}$$

18.

a.



b. Use rule to differentiate.

$2e^{x}$

c.

$f'(1) = 2e$

$m_{normal} = \frac{1}{2e}$

$y = 2e + 3$

$y - (2e + 3) = \frac{1}{2e}(x - 1)$

$y = \frac{e^{-1}x}{2} - 2e^{-1} + 2e + 3$

19.

C.

$h = 0.025$ $\quad f(x) = e^x$

$x = 0$

$f'(x) = e^x$

$f(0.025) = f(0) + 0.025f'(0)$

20.

a.

Calculate $v(24)$

0.25 litre

b.

$V'(t) = \frac{\pi}{4}\cos\left(\frac{\pi t}{12}\right)$

Evaluate V'(24)

$\frac{\pi}{4}$ litres/hour

21.

$y' = (x^2 + 2x + 1)e^x$

$y' = 0$

Solve for x – $x = -1$

$f(-1) = 2e^{-1}$

$y'(-2)$ is positive.

$y'(0)$ is positive.

Point of inflection at $(-1, 2e^{-1})$

22.

$c'(v) = \frac{8v}{75} - \frac{5625}{v^2}$ Solve $c'(v) = 0$

$\frac{75}{2}$ km/h

23.

$960 = 2x + y$

$y = 960 - 2x = 480 - x$

$A = x(480 - x)$

$= 480x - x^2$

$A' = 480 - 2x$

$0 = 480 - 2x$

$x = 240$

$y = 480$

Length = 480m Width = 240m

24.

$x + y = 80$

$y = 80 - x$

$s = x^2 + y^2$

$= x^2 + (80 - x)^2$

$= 2x^2 - 160x + 6400$

$s'(x) = 4x - 160$

$0 = 4x - 160$

$x = 40$

$y = 40$

$x = y = 40$

25.

Draw a diagram with a square base of x cm by x cm and a height of h cm.

$$V = x^2h$$

$$27000 = x^2h$$

$$h = \frac{27000}{x^2}$$

$$SA = 2x^2 + 4xh$$

$$= 2x^2 + 4x\left(\frac{27000}{x^2}\right)$$

$$= 2x^2 + \frac{108000}{x}$$

Solve SA' = 0

Length = width = 30cm Height = 30 cm