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MATHEMATICAL METHODS CAS
Teach Yourself Series
Topic 9: Integral Calculus

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Contents

Rules for Antidifferentiation of functions

As it appears in Units 2 and 4

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

$$\int (ax+b)^r dx = \frac{(ax+b)^{r+1}}{a(r+1)} + c, r \neq -1$$

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + c$$

$$\int \sin(kx+a) dx = -\frac{1}{k}\cos(kx+a) + c$$

$$\int \cos(kx+a) dx = \frac{1}{k}\sin(kx+a) + c$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln(ax+b) + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$$

$$\int \sec^2(ax+b) dx = \frac{1}{a}\tan(ax+b) + c$$

Other general rules

- $\int f'(x)dx = f(x) + c$
- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
- $\int af(x)dx = a\int f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Calculator skills

Define functions
Annti differentiate functions

Finding $f(x)$ given $f'(x)$

Let's consider the functions: $f(x) = x^2 + 2x$

$$f(x) = x^2 + 2x + 1$$

$$f(x) = x^2 + 2x - 3$$

$$f(x) = x^2 + 2x + \frac{61}{2}$$

When all of these different functions are differentiated, you arrive at the same gradient function: $f'(x) = 2x + 2$

Why is this the case?

If you start off with the gradient function and have to find $f(x)$ we write 'c' at the end of the function.

<u>If you are given information about $f(x)$ (a point) then you can use this information to find out the specific function.</u>

Calculator skills

Antidifferentiate functions

Area under curves – Definite Integral

As it appears in Units 2 and 4

This is used to calculate area under curves between the **curve and the x axis**.

$\int_0^3 x^2 dx$. This means that you wish to calculate the area between the curve and the x-axis between x=0 and x=3.

Calculating Area Algebraically

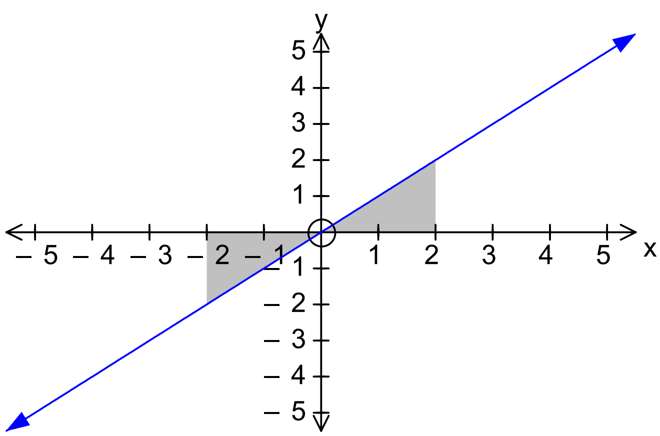
This uses the Fundamental Theorem of Integral Calculus:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where $F(x)$ is the antiderivative of $f(x)$.

Consider the following example:

Find the area between the curve and the x axis between x = -2 and x = 2 for $y = x$.



$$\int_{-2}^{2} x\,dx = 0 \quad (Area = 4)$$

When this is calculated you notice that the area= 0. This is because the area below the x axis is negative. This is the signed area.

If the true area is to be calculated the above example needs to be worked out in two calculations.

area= $\int_{-2}^{0} x\,dx + \int_{0}^{2} x\,dx$

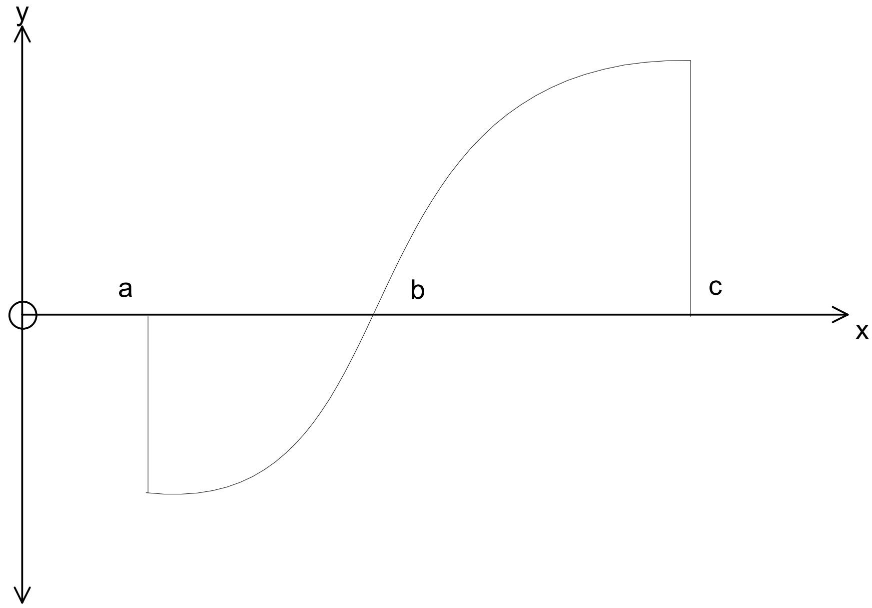
But $\int_{-2}^{0} x\,dx$ =-2

However $\int_{0}^{-2} x\,dx$ =2

$\int_{-2}^{0} x\,dx + \int_{0}^{2} x\,dx = 2 + 2$

=4 sq units.

In summary:



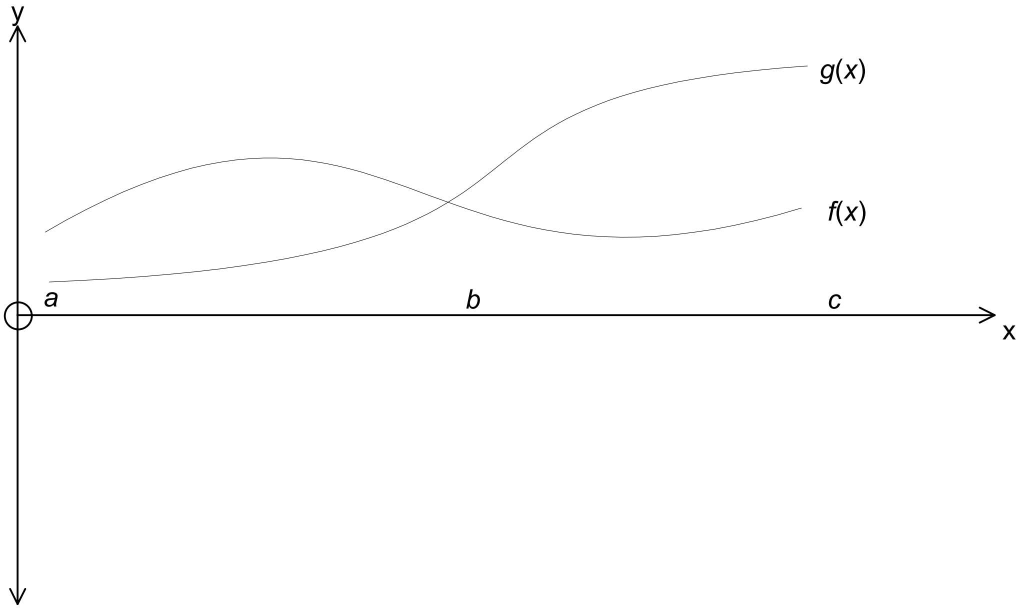
Signed Area= $\int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx$

True Area= $\int_{b}^{a} f(x)dx + \int_{b}^{c} f(x)dx$

Area between curves

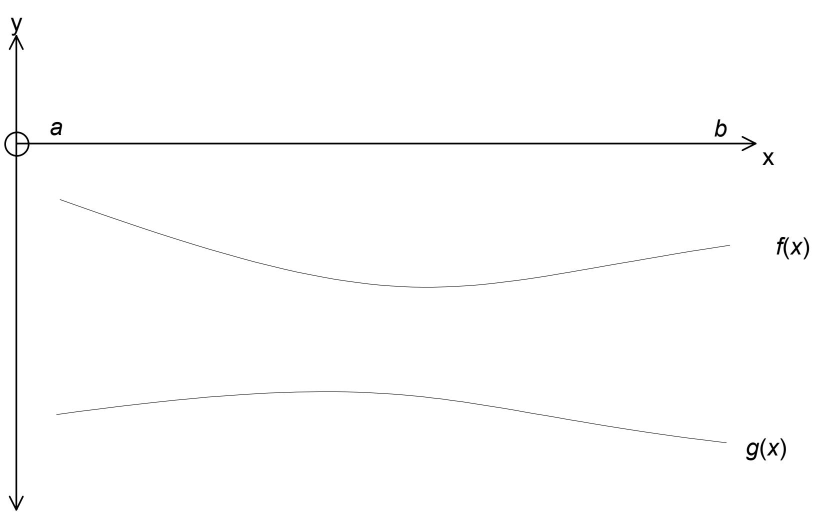
As it appears in Units 2 and 4

Type 1



$$\text{Area} = \int_a^b [f(x) - g(x)]dx + \int_b^c [g(x) - f(x)]dx$$

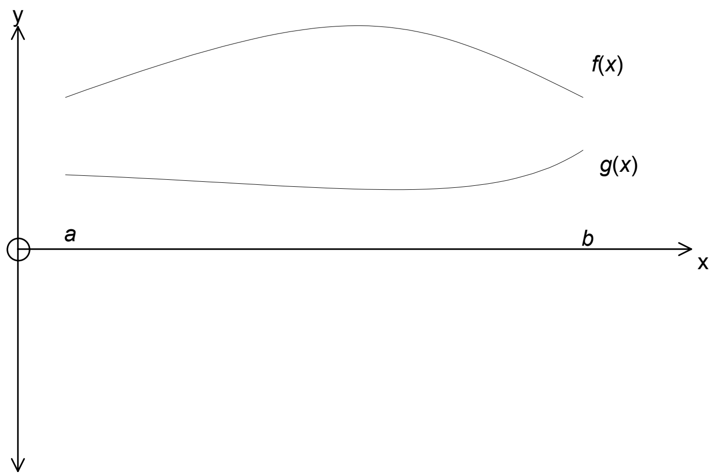
Type 2



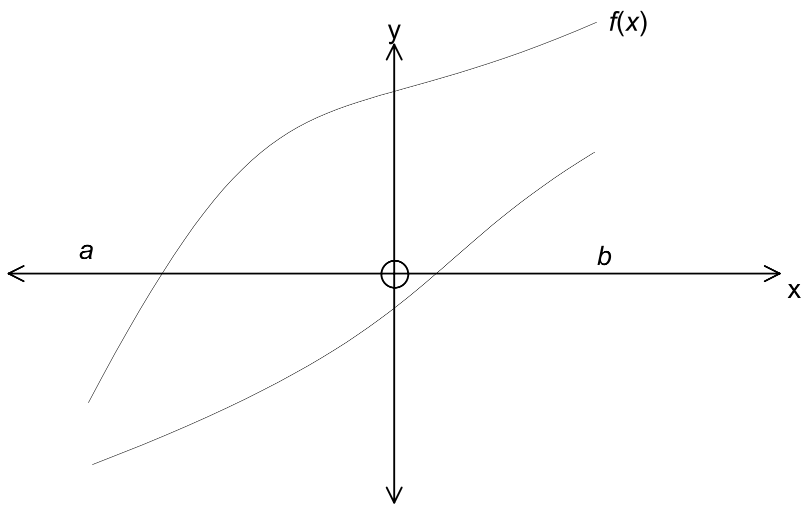
$$\text{Area} = \int_a^b [f(x) - g(x)]dx$$

Type 3

$$\textbf{Area} = \int_a^b [f(x) - g(x)]\,dx$$



Type 4



$$\textbf{Area} = \int_a^b [f(x) - g(x)]\,dx$$

Calculator skills

Define Functions
Graph functions
Antidifferentiate functions and evaluate the antiderivative

Rectangle approximations

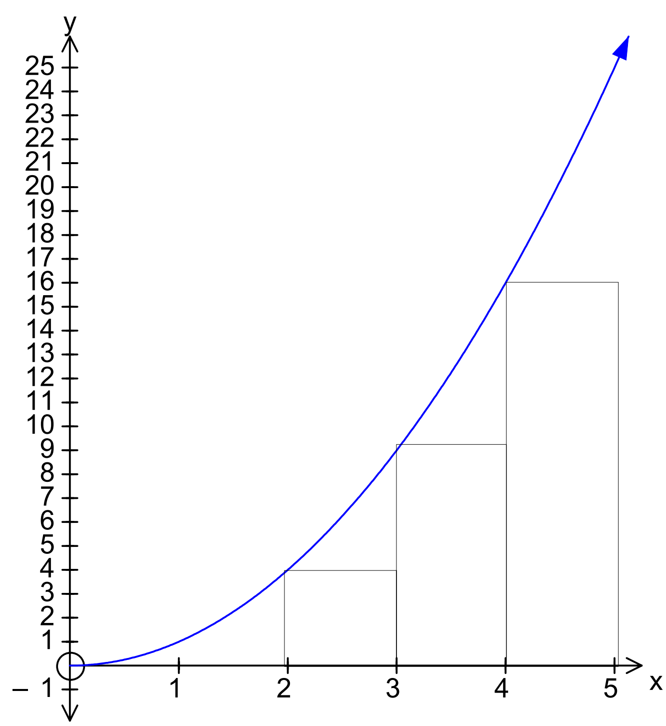
As it appears in Unit 4

Using rectangles, which are drawn under the graph, you can approximate area under curves. The area of each rectangle is calculated and the answer is obtained when you add the area of all rectangles together.

eg

Using rectangles to approximate the area, Find the area under the curve of $y = x^2$ between x=2 and x=5 and using three intervals.

Three intervals mean that this has to be done where each rectangle is one unit wide (rectangles have to be uniform in length).



Note that for this example the height of the rectangles is calculated by the left edge of the rectangles.

$1 \times 4 = 4$

$1 \times 9 = 9$

$1 \times 16 = 16$

Area $= 4 + 9 + 16$

$= 29$ **sq units**

In this case the area is an **under** approximation to the "real" area. The more rectangles that are used the better the approximation will be.

Left rectangle approximation

These are approximations that use the left side of the rectangles to approximate the area.

It can be cumbersome to construct the rectangles by hand, so a formula can be used.

$$\text{leftarea} = \frac{b - a}{n}(f(x_0) + f(x_1) + .. + f(x_{(n-1)}))$$

a: lower bound

b: upper bound

n: number of intervals

$f(x_0), f(x_1)$ etc: height of rectangles.

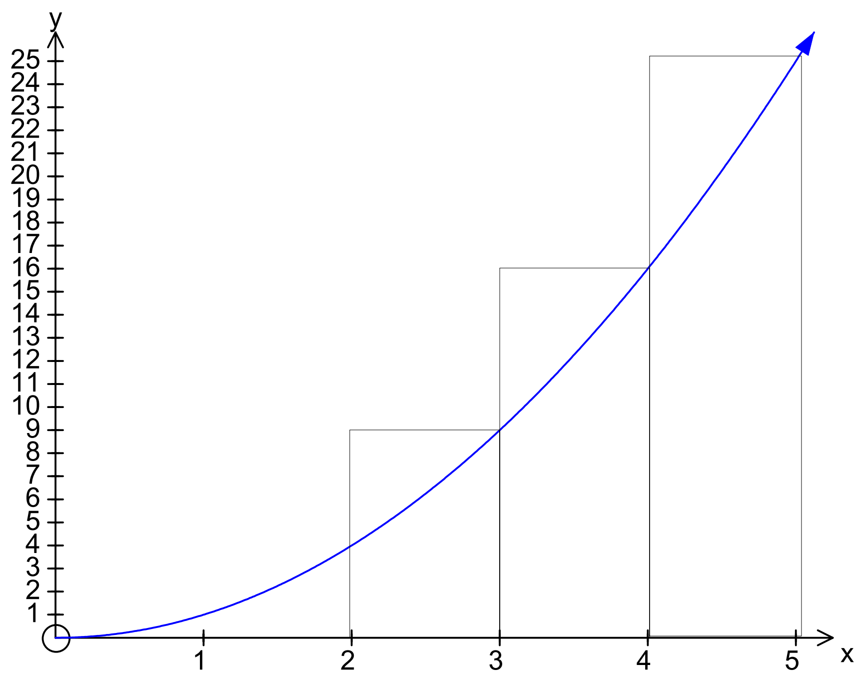
$\frac{b-a}{n}$: width of rectangles.

When the function is increasing, these tend to be an under estimation of the true area.
When the function is decreasing, these tend to be an over estimation of the true area.

Right rectangle approximation

These are approximations that use the right side of the rectangles to approximate the area.

Using right rectangles to approximate the area, Find the area under the curve of $y = x^2$ between x=2 and x=5 and using three intervals.



$$\textbf{rightarea} = 1 \times 9 + 1 \times 16 + 1 \times 25$$

$$= 50 \textbf{ sq units}$$

The formula is:

$$\textbf{rightarea} = \frac{b - a}{n}(f(x_1) + f(x_2) + .. + f(x_n))$$

a: lower bound

b: upper bound

n: number of intervals

$f(x_1), f(x_2)$ etc: height of rectangles.

$\frac{b-a}{n}$: width of rectangles.

Calculator Skills

Define functions
Evaluate functions

Integration by recognition

As it appears in Unit 4

The important thing to remember here is:
$\int f'(x)dx = f(x) + c$

The basic process is:

1. Differentiate the function.
2. Integrate both sides.
3. Rearrange to find integral and solve.

Sketching $f(x)$ given $f'(x)$

This is where you have been given a graph of $f'(x)$ vs x and you have to sketch $f(x)$.

Some points to remember:

When the graph is below the x axis, the gradient is negative. (negative tangents on $f(x)$))

When the graph is above the x axis, the gradient is positive. (positive tangents on $f(x)$))

The process in sketching these functions is as follows:

1. Look at the gradient function. Locate x intercepts as these indicate stationary points. Mark these on a graph.
2. Focus on one stationary point at a time. Look at the gradient function either side of the x intercept.
3. Use the gradient function to decide on the type of stationary point that you have.
4. When stationary points have been identified mark in any additional information. ie x intercepts, y intercepts, etc.
5. Draw the graph of $f(x)$

Review Questions

1. Find the following indefinite integrals:

a. $\int (x-2)^2 (x+3)\ dx$

b. $\int (4-3x)^{-3}\ dx$

2. Antidifferentiate the following:

a. $\int \frac{4}{3x+2}\,dx$

b. $\int \frac{5+2x-x^3}{\sqrt{x}}\,dx$

3. A curve has the gradient, $g'(x)=k\sqrt{x}-3x^3$ where k is a constant and a stationary point (1, −2). Find:

a. the value of k

b. $g(x)$

4. Find an antiderivative of $\left(2-e^{-x}\right)^2$.

5. Find $f(x)$, if $f'(x) = 5\sin 2x$ and $f(\pi) = -1$.

6. Show that $\dfrac{3-2x}{1-x} = 2 + \dfrac{1}{1-x}$ and hence find the antiderivative of $\dfrac{3-2x}{1-x}$.

7. Evaluate $\int_1^3 4 - 3x + 2x^2 \, dx$

8. Evaluate $\int_{0}^{\frac{\pi}{4}} 3\cos 2x \, dx$

9. Evaluate $\int_{-4}^{-3} \frac{3}{4-2m} \, dm$

10. Evaluate $\int_{-3}^{0} e^{-2x} \, dx$

11. Sketch the graph of $y = 1 + 2\sin 2x$, showing all the intercepts from 0 to π. Find the area between the curve and the $x-$axis from:

a. $x = 0$ to $x = \frac{7\pi}{12}$.

b. $x = \frac{7\pi}{12}$ to $x = \frac{11\pi}{12}$.

c. $x = 0$ to $x = \frac{11\pi}{12}$.

12. Find the exact area between the curve $y = 4e^{2x}$, the x-axis and the lines $x = 0$ and $x=1$

13. Find an approximation for the area between the curve $y = \frac{x^2}{4}$ and the x-axis over the interval [0,4] using:

a. the upper rectangles.

b. the lower rectangles.

c. the average of the upper and lower rectangles.

14. Find the exact area between $y = \cos x$ and $y = \sin x$ between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

15. A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity (v cm/s) is given by $v = 4t - 6$. Find:

a. its position x in terms of t

b. its position after 3 seconds

c. the distance travelled in the first 3 seconds

d. its average velocity in the first 3 seconds

e. its average speed in the first 3 seconds.

16. Find the average value of $f(x) = \frac{1}{x}$ between the interval of $[1, e^2]$.

17.

a. Find the derivative of $(2x^2 - 5x)e^x$.

b. Hence find $\int_0^5 (2x^2 - 5x)e^x \, dx$.

Solutions to Review Questions

1.

a.

Add one to each power, divide by the new power, simplify.

$\frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + c$

b.

Use $\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{(n + 1)a}$

$\frac{1}{6(4 - 3x)^2} + c$

2.

a.

Use $\int \frac{1}{ax + b} = \frac{1}{a} \log_e(|ax + b|) + c$

$\frac{4}{3}\log_e(|3x + 2|) + c$

b.

Change surds into powers, divide denominator into each term of the numerator, add one to each power and divide by the new power.

$10\sqrt{x} + \frac{4}{3}x^{\frac{3}{2}} - \frac{2x^{\frac{7}{2}}}{7} + c$

3.

a.

$0 = k(1) - 3$

$k = 3$

b. Change surds into powers, add one to each power and divide b the new power, substitute point to find c.

$g(x) = 2x^{\frac{3}{2}} - \frac{3}{4}x^4 - \frac{13}{4}$

4.

Expand brackets, use rule for e^{kx}, the wording "an: means c = 0.

$4e^{-x} - \frac{e^{-2x}}{2} + 4x$

5.

Use $\int \sin kx \, dx = \frac{1}{k}\cos(kx)$, substitute point to find c.

$f(x) = -\frac{5}{2}\cos(2x) + \frac{3}{2}$

6.

Use long division for first part of question. Then apply rules.

$2x - \log_e(|1 - x|) + c$

7.

Antiderivative = $\left[\frac{2}{3}x^3 - \frac{3}{2}x^2 + 4x\right]_1^3$, then evaluate.

$\frac{40}{3}$

8.

Antiderivative = $\left[\frac{3}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}$, then evaluate.

$\frac{3}{2}$

9.

Antiderivative = $\left[-\frac{3}{2}\log_e(|4 - 2m|)\right]_{-4}^{-3}$, then evaluate.

$-\frac{3}{2}\log_e\left(\frac{5}{6}\right)$

10.

Antiderivative = $\left[\frac{e^{-2x}}{-2}\right]_{-3}^{0}$, then evaluate.

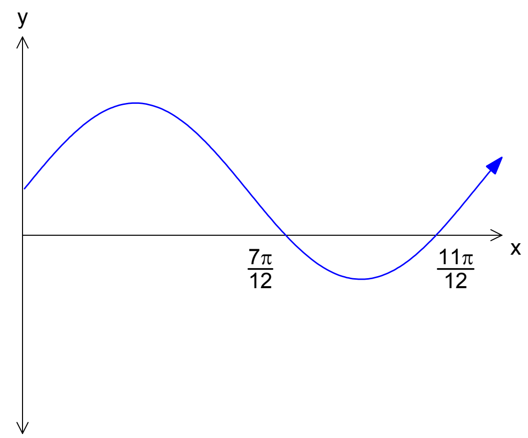
$\frac{e^6 - 1}{2}$

11.

a. $\frac{7\pi}{12} + \frac{\sqrt{3}}{2} + 1$ *sq units*

b. $\sqrt{3} - \frac{\pi}{3}$ *sq units*

c. $\frac{\pi}{4} + \frac{3\sqrt{3}}{2} + 1$ *sq units*



Add answers to parts a and b to get the answers to part c.

12.

Draw a graph to help, Antiderivative = $\left[2e^{2x}\right]_0^1$ then evaluate.

$2(e^2 - 1)$ *sq units*

13.

a. Area = $1(f(1) + f(2) + f(3) + f(4))$

$\frac{15}{2}$ *sq units*

b. Area = $1(f(0) + f(1) + f(2) + f(3))$

$\frac{7}{2}$ *sq units*

c. Add answers to parts a and b together and divide result by 2.

$\frac{11}{2}$ *sq units*

14.

Area = $\left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$2\sqrt{2}$ *sq units*

15.

a. Antidifferntiate, then susbstitute t=0 to find c.

$x(t) = 2t^2 - 6t$

b. Evaluate x(3)

0 cm

c. Find $2 \times \int_0^{1.5} 4t - 6 \, dt$

9 cm.

d. Answer of part b divides by 3.

0 cm/s

e. Answer to part c divide by 3.

3 cm/s

16.

Use $\frac{1}{b-a}\int_a^b \frac{1}{x} \, dx = \frac{1}{e^2-1}\left[\log_e(x)\right]_1^{e^2}$

$\frac{2}{e^2-1}$

17.

a.

Use product rule and take out common factor of e^x .

$e^x(2x^2 - x - 5)$

b. Antidifferentiate the answer from part a. remember the antiderivative of y' = y. Then evaluate.

$25\,e^5$