ATARNotes

Specialist VCE Units 3/4

ATARNotes Exam Cram

Presented by:
Nguyen

About Me

Hi, I’m Nguyen!

- Graduated in 2023
- Received a study score of 41 in Specialist Maths
- Received 40+ study scores in English Language, Math Methods, Chemistry, Physics, Software Development
- Currently studying a Bachelor of Electrical and Electronics Engineering at Deakin University

TODAY'S PLAN

Topics to be covered

- Statistics – Hypothesis testing + probability
- Complex Numbers
- Calculus
- Circular Functions
- Vectors Equations
- Kinematics and Vector Calculus
- Proofs

Announcements

- Please ask any questions!
- We will be covering stats in detail then revise over other topics

Overview of statistics

- Linear combinations of random variables
    - Similar to methods probability/sampling
        - Discrete, continuous, normal, normal approx of binomial, central limit theorem, confidence intervals, sampling
    - Difference between aX vs $\underbrace{X + X + \cdots + X}_{a\ times}$
- Hypothesis testing
    - Hard to understand, but honestly EASIEST marks on the exam!
    - Drawing is important: normal distribution
    - “If company claims mean is x, how true is this if we take a sample?”
        - Null/alternative hypothesis, p-value, level of significance

Statistics

SUMMARY PROBABILITY FORMULAS

$$E(aX \pm b) = aE(x) \pm b$$

$$E(aX \pm bY) = aE(X) \pm bE(Y)$$

$$Var(aX \pm b) = a^2Var(X)$$

$$Var(aX \pm bY) = a^2Var(X) + b^2Var(Y)$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

Recommend you have a page in ur book listing all the formulas for each distribution type

ALWAYS ADD VARIANCE!!! SD ONLY THRU $\sqrt{Var(X)}$

Discrete Mean

Continuous Mean

Normal

Inverse Norm

Central Limit Theorem

All of this is methods ¾ knowledge

LINEAR COMBINATIONS

- aX vs $\underbrace{X + X + \cdots + X}_{a \text{ times}}$

- Eg. The Devil of Gambling offers you two options to win money:

- 1. You roll a die <u>once</u>. You'll earn 2 times the number you roll
- 2. You roll a die <u>twice</u>. You'll earn the sum of the numbers you roll

- Which game yields more consistent income (i.e. less variability)?

LINEAR COMBINATIONS

- If you do X **once**, then scale it by factor a afterwards, then $Y = aX$, but if you do X a times, without scaling by any factor, then $Y = \underbrace{X + X + \cdots + X}_{a\ times}$

Generally,

- If $Y = aX$
- $E(Y) = aE(X)$
- $Var(Y) \boxed{= a^2Var(X)}$
- If$Y = \underbrace{X + X + \cdots + X}_{a\ times}$
- $E(Y) \boxed{= aE(X)}$

Notice the difference in the variance and no difference in the expected value!

Statistics

VCAA 2016

Oranges grown on a citrus farm have a mean mass of 204 grams with a standard deviation of 9 grams. Lemons grown on the same farm have a mean mass of 76 grams with a standard deviation of 3 grams.

The masses of the lemons are independent of the masses of the oranges.

The mean mass and standard deviation, in grams, respectively of a set of three of these oranges and two of these lemons are

[✓ **A.**] 764, $3\sqrt{29}$
B. 636, 12
C. 764, $\sqrt{33}$
D. 636, $3\sqrt{10}$
E. 636, 33

$O \sim N(204, 9^2)$ and $L \sim N(76, 3^2)$

$$X = O + O + O + L + L$$

$$E(X) = 3E(O) + 2E(L) = 3 \cdot 204 + 2 \cdot 76 = 764$$

$$Var(X) = 3Var(O) + 2Var(L) = (3 \times 9^2) + (2 \times 3^2)$$
$$Var(X) = 261$$

$$\therefore sd(X) = \sqrt{261} = 3\sqrt{29}$$

Statistics

VCAA 2018 NHT

A farm grows oranges and lemons. The oranges have a mean mass of 200 grams with a standard deviation of 5 grams and the lemons have a mean mass of 70 grams with a standard deviation of 3 grams.

Assuming masses for each type of fruit are normally distributed, what is the probability, correct to four decimal places, that a randomly selected orange will have at least three times the mass of a randomly selected lemon?

A. 0.0062
B. 0.0828
[✓ **C.**] 0.1657
D. 0.8343
E. 0.9172

$$O \sim N(200, 5^2) \quad \& \quad L \sim N(70, 3^2)$$

$$\Pr(O \geq 3L) = \Pr(O - 3L \geq 0)$$

Set $X = O - 3L$

$$E(X) = E(O) - 3E(L) = -10$$
$$Var(X) = Var(O) + 9Var(L) = 106$$
$$\therefore X \sim N\left(-10, \sqrt{106}^2\right)$$

Now we can use this to calculate $\Pr(O - 3L \geq 0) = \Pr(Z \geq 0)$
$= 0.1657$

SAMPLE DISTRIBUTION

- Generally, looking at ONE sample:
- $X \sim N(\mu, \sigma^2)$

As sample size gets bigger = the more accurate the sample mean is to the population mean

- **Sample distribution**:
- If we take a sample of n and calculate the mean
- Then repeat for more samples of n
- This gives us: $\bar{X}$ (Sample Mean Distribution)

$$\boxed{\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)}$$

Statistics

VCAA 2017

The petrol consumption of a particular model of car is normally distributed with a mean of 12 L/100 km and a standard deviation of 2 L/100 km.

$X \sim N(12, 2^2)$

The probability that the average petrol consumption of 16 such cars exceeds 13 L/100 km is closest to

A. 0.0104

B. 0.0193

[✓ **C.**] 0.0228

D. 0.3085

E. 0.3648

$n = 16$

$\Pr(\bar{X} > 13)$

$$\therefore E(\bar{X}) = E(X) = 12$$

$$sd(\bar{X}) = \frac{sd(X)}{\sqrt{16}} = \frac{1}{2}$$

We therefore have the distribution $\bar{X} \sim N(12, 0.5^2)$

$$\Pr(\bar{X} > 13) \approx 0.0228$$

Statistics

VCAA 2019 NHT

Question 18

Consider a random variable X with probability density function

$$f(x)=\begin{cases}2x, & 0\le x\le 1\\ 0, & x<0 \text{ or } x>1\end{cases}$$

$n = 100$

If a large number of samples, each of size 100 is taken from this distribution, then the distribution of the sample means, $\bar{X}$, will be approximately normal with mean $\mathrm{E}\left(\bar{X}\right)=\frac{2}{3}$ and standard deviation $\mathrm{sd}\left(\bar{X}\right)$ equal to

[✓ **A.**] $\frac{\sqrt{2}}{60}$

B. $\frac{\sqrt{2}}{6}$

C. $\frac{1}{180}$

D. $\frac{1}{18}$

E. $\frac{\sqrt{2}}{30}$

$$sd(\bar{X})=\frac{sd(X)}{\sqrt{n}}=\frac{sd(X)}{\sqrt{100}}$$

$$Var(X)=\int_0^1 2x^3\,dx-\left(\frac{2}{3}\right)^2=\frac{1}{2}-\frac{4}{9}$$

$$Var(X)=\frac{1}{18}$$

$$\therefore sd(X)=\frac{\sqrt{2}}{6}$$

Hence, $sd(\bar{X})=\frac{\sqrt{2}}{60}$

CONFIDENCE INTERVAL

- What does a confidence interval mean?
- We can say with ___% certainty that the population proportion falls within ___ and ___

$$\left(\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

- If you have 100 samples and do a 95% confidence interval for each, 95 of the intervals that you find will **actually** contain the population proportion
- The more ‘certain’ you are that it will fall in your interval, the less useful that information becomes, as there are more possible values for population proportion

CONFIDENCE INTERVALS

- Each sample will have its associated confidence interval
- $CI = \left(\bar{x} - k\frac{s}{\sqrt{n}}, \bar{x} + k\frac{s}{\sqrt{n}}\right) = (\bar{x} - M, \bar{x} + M)$
- $\bar{x}$ Sample Mean
- n Sample Size
- s Sample SD (Note that $s \approx \sigma_X$)
- M Margin of Error

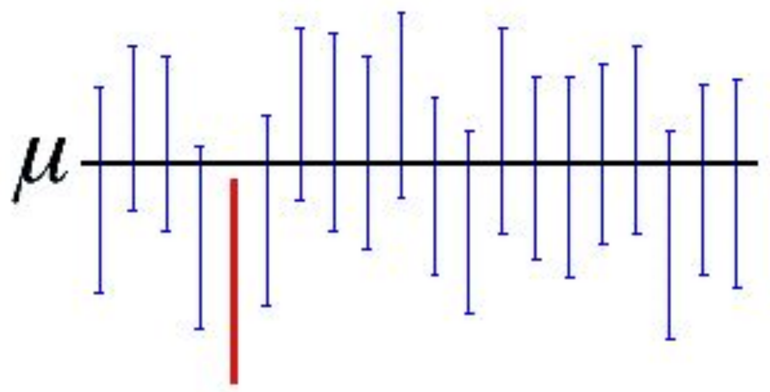
CONFIDENCE INTERVAL

Degree of Confidence	
90%	1.645
95%	
99%	

It is a good idea to memorise these numbers as they can ask you to calculate the CI in Exam 1!

If exam says:
“Integer multiple of SD”

- 95% = 2



A 95% confidence interval indicates that 19 out of 20 samples (95%) from the same population will produce confidence intervals that contain the population parameter.

Statistics

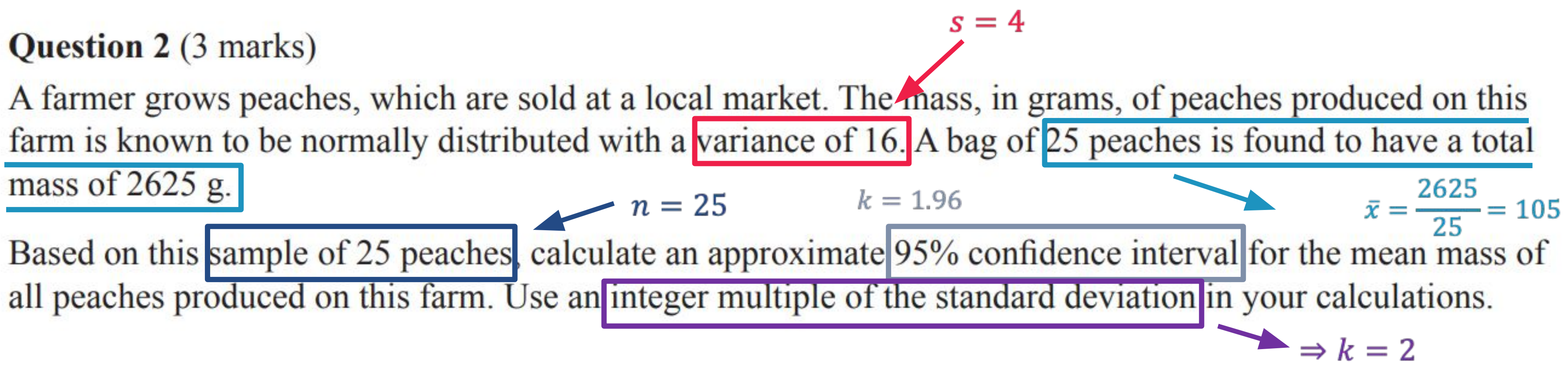
k VALUES

- You can determine specific k values for each C percentage of confidence using the following:

- Where $\Pr(-k < Z < k) = \frac{C}{100}$
- $2\Pr(Z < -k) = 1 - \frac{C}{100}$
- $\Pr(Z < -k) = \frac{1}{2}\left(1 - \frac{C}{100}\right)$
- $k = -\text{invNorm}\left(\frac{1}{2}\left(1 - \frac{C}{100}\right), 0{,}1\right)$

Statistics

VCAA 2016 EXAM 1



Question 2 (3 marks)

A farmer grows peaches, which are sold at a local market. The mass, in grams, of peaches produced on this farm is known to be normally distributed with a variance of 16. A bag of 25 peaches is found to have a total mass of 2625 g.

Based on this sample of 25 peaches, calculate an approximate 95% confidence interval for the mean mass of all peaches produced on this farm. Use an integer multiple of the standard deviation in your calculations.

$s = 4$

$n = 25$

$k = 1.96$

$\bar{x} = \frac{2625}{25} = 105$

$\Rightarrow k = 2$

$$CI = \left(\bar{x} - k\frac{s}{\sqrt{n}}, \bar{x} + k\frac{s}{\sqrt{n}}\right)$$

$$CI = \left(105 - 2 \cdot \frac{4}{\sqrt{25}}, 105 + 2 \cdot \frac{4}{\sqrt{25}}\right)$$

$$CI = (103.4, 106.6)$$

Statistics

IMPORTANT!

- KNOW THE DIFFERENCE BETWEEN:
  - POPULATION STANDARD DEVIATION
    - SD of the POPULATION!! NO SAMPLE SIZES HERE

$$\sigma$$

DON'T MIX THE TWO UP ITS VERY EASY

  - SAMPLE STANDARD DEVIATION
    - SD of the SAMPLE!! THERE IS A SAMPLE HERE!

$$sd(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

HYPOTHESIS TESTING

1. Set up statistical hypotheses

- In HT, we assume the rule of “not guilty until proven”.
- Null hypothesis is when there is NO EFFECT - the treatment is ineffective → H_0.
- $H_0\colon \mu = \mu_{\text{before}}$
- Alternative hypothesis is a DESIRED/FEARED effect – the treatment is effective.
- $H_1\colon \mu < \mu_{\text{before}}$

There are three possibilities for H_1:

$\mu > \mu_{\text{before}}$

$\mu < \mu_{\text{before}}$

$\mu \neq \mu_{\text{before}}$

One-tailed test: “Difference/change”

Two-tailed test: “Above/Below”

2. Set up a level of significance (p value/α significance)

- To reject the Null hypothesis, we need to prove that the probability of getting a certain sample mean is EXTREMELY SMALL, assuming the null hypothesis is true
  - Eg. If we flip a coin 20 times and we get 2 heads, can we assume the coin is fair?
- P value is the probability of getting an extreme value when assuming null is correct.
- α significance level (default is 0.05). If p-value is less than 0.05, we can state that there is a less

HYPOTHESIS TESTING

-
- **3. Calculate p-value (one tailed)**
- Predict the directionality
  - the size of virus will **decrease** after receiving treatment X
  - then $H_1 : \mu < \mu_{\text{before}}$

- The P-value is given by:
  - $\boxed{\text{P-value} = \Pr(\bar{X} < \bar{x} | \mu = \mu_0)}$

Obvs replace the sign

P VALUE

Method 1 (Convert into Standard Normal Distribution)

STEPS:

1. Define Null/Alternative
2. $z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$ for standard normal
3. NormCdf on calc Z~N(0,1)

- P-value $= \Pr(\bar{X} < \bar{x} | \mu = \mu_0)$

$$\Pr\left(Z \leq \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

Method 2 (NormCdf)

STEPS:

1. Define Null/Alternative
2. NormCdf on calc
3. DON'T FORGET TO CONVERT TO SAMPLE SD

Method 3 (Z-test)

STEPS:

1. Z-test (Menu>6>7>1)
2. Choose 'Stats' not 'Data'
3. Fill in calc, using POPULATION SD
4. Change the 'Alt Hypo' to relevant
5. Ctrl+Enter and look at the 'PVal'

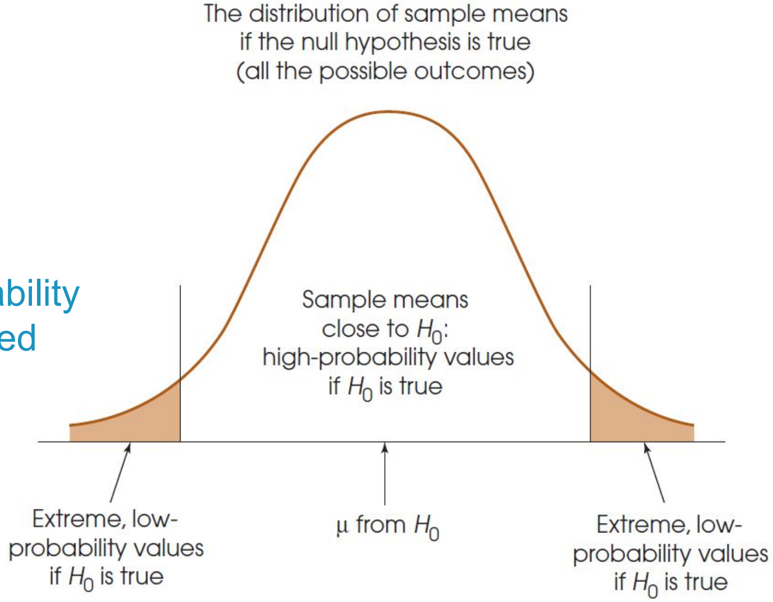
HYPOTHESIS TESTING

- **3. Calculate p-value (two tailed)**
- If we have no direction ”H_1 is different than expected”
- $H_1: \mu \neq \mu_0$
- Basically just half of the α for one side

$$p_{2\text{ tailed}} = 2p_{1\text{ tailed}}$$
$$= 2\,\Pr\left(Z \geq \left|\frac{\mu_0 - \mu}{\frac{\delta}{\sqrt{n}}}\right|\right)$$

Otherwise use z-test and alt hypo as $\text{u} \neq u_0$

It’s the probability of both shaded regions!



4. Making inferences

If the p value is:

- Above α = ”insufficient evidence to reject H_0”

- Below α =
    - “good evidence to reject H_0” >0.05
    - “Strong evidence to reject H_0” >0.01
    - “very strong evidence to reject H_0” >0.001

Statistics

HYPOTHESIS TESTING

- There’s ALWAYS a tiny chance the p-value is erroneous/wrong due to randomness. There are 2 types of errors in HT:

Type I (False positive)	Correct conclusion
Correct conclusion	Type II (False negative)

- Type I: Rejecting H_0 when it is true
- Type II: Not rejecting H_0 when it is false

ERRORS

You're probs not gonna need to do this for the exam but good to understand

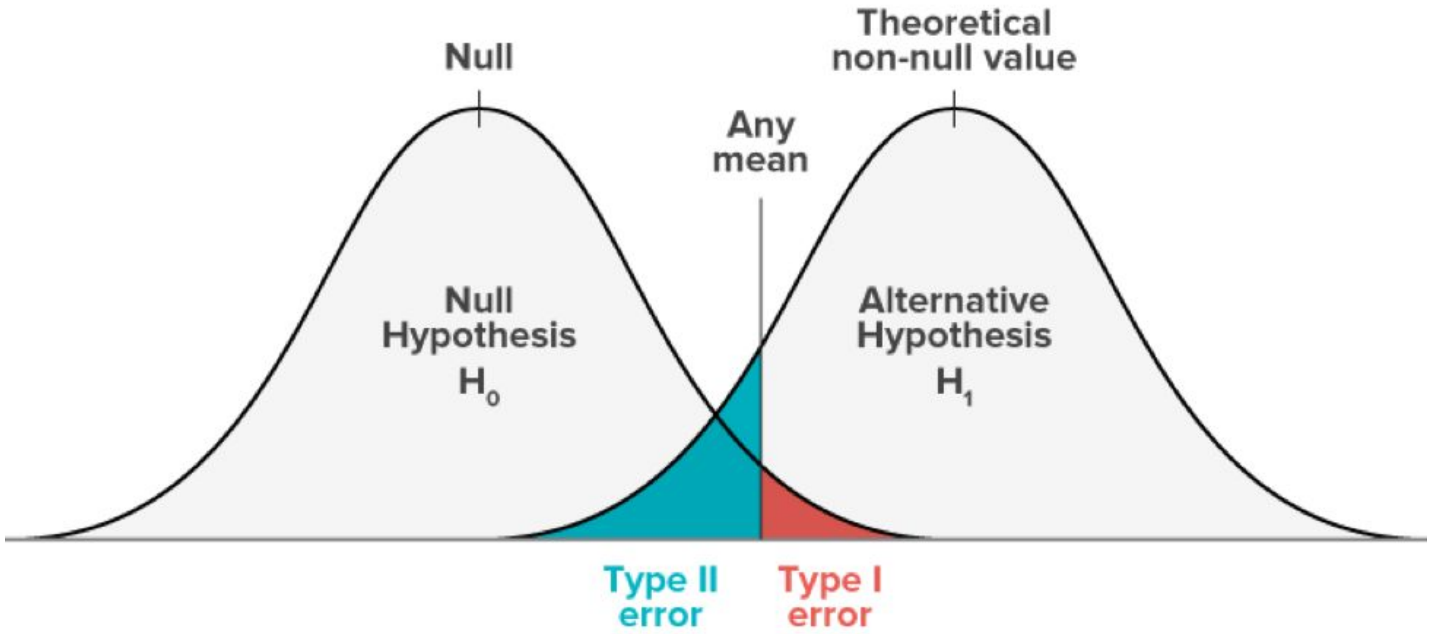
- We want to reduce the chance of both errors.
- Type I: related to significance level.
  - You reject H_0 below $\boldsymbol{\alpha}$ = 0.05, but you shouldn't have
  - Therefore, you can reduce type I error by decreasing $\boldsymbol{\alpha}$ level, but this would increase chance of Type II error

- Steps:

1. Let p-value = 0.05
2. Use $z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$ and solve for x

ERRORS

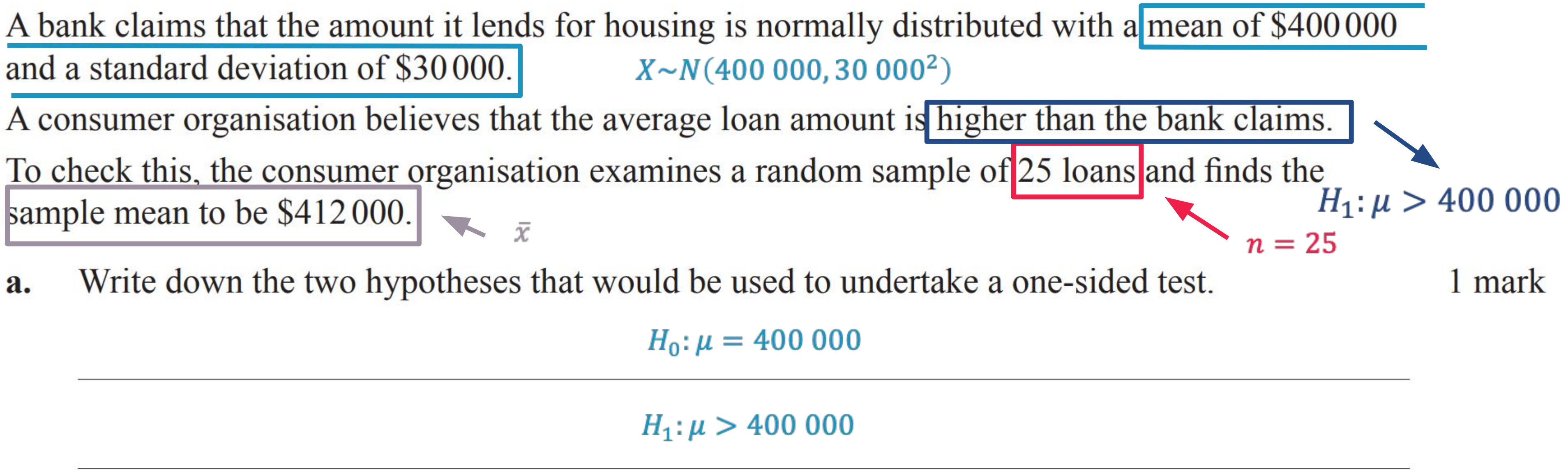
- Type II: (Decreases with sample size)



- Steps:
  - Use InvNorm to find the value where type I meets type II
  - Pr(Accept H_0|H_0 false)

Statistics

VCAA 2017 NHT



A bank claims that the amount it lends for housing is normally distributed with a mean of $400 000 and a standard deviation of $30 000.

$X \sim N(400\,000, 30\,000^2)$

A consumer organisation believes that the average loan amount is higher than the bank claims.

$H_1: \mu > 400\,000$

To check this, the consumer organisation examines a random sample of 25 loans and finds the sample mean to be $412 000.

$\bar{x}$

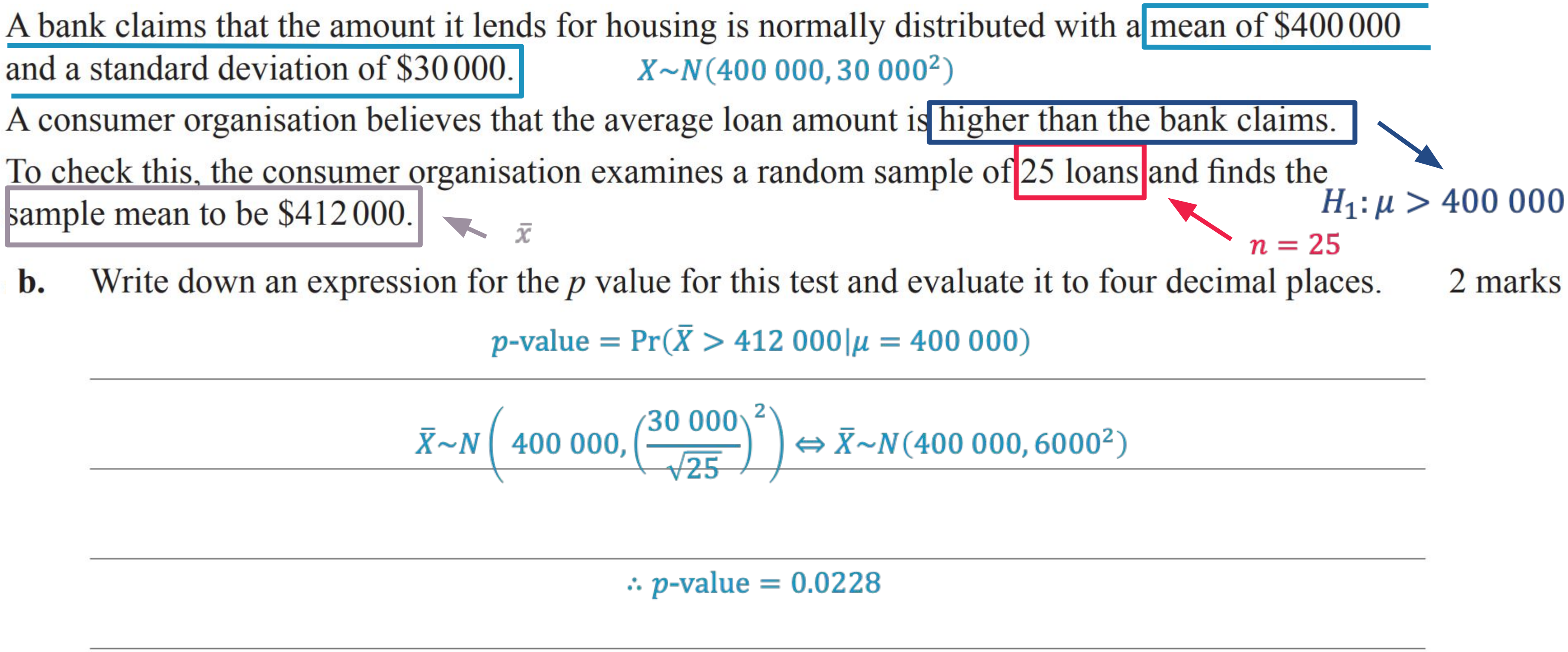
$n = 25$

a. Write down the two hypotheses that would be used to undertake a one-sided test. 1 mark

$H_0: \mu = 400\,000$

$H_1: \mu > 400\,000$

Statistics



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$X \sim N(400\,000, 30\,000^2)$

A consumer organisation believes that the average loan amount is higher than the bank claims.

$H_1: \mu > 400\,000$

To check this, the consumer organisation examines a random sample of 25 loans and finds the sample mean to be \$412 000.

$\bar{x}$

$n = 25$

b. Write down an expression for the p value for this test and evaluate it to four decimal places. 2 marks

$$p\text{-value} = \Pr(\bar{X} > 412\,000 | \mu = 400\,000)$$

$$\bar{X} \sim N\left(400\,000, \left(\frac{30\,000}{\sqrt{25}}\right)^2\right) \Leftrightarrow \bar{X} \sim N(400\,000, 6000^2)$$

$$\therefore p\text{-value} = 0.0228$$

Statistics

VCAA 2017 NHT CONT.

c. State with a reason whether the bank's claim should be rejected at the 5% level of significance. 1 mark

Since p-value $= 0.0288 < 0.05$, this means that we reject the null hypothesis

which is the bank's claim.

d. What is the largest value of the sample mean that could be observed before the bank's claim was rejected at the 5% level of significance? Give your answer correct to the nearest 10 dollars. 1 mark

We want p-value $= 0.05$

$$\Pr(\bar{X} > \bar{x} | \mu = 400\,000) = 0.05$$

$$\Pr\left(Z > \frac{\bar{x} - 400\,000}{6000}\right) = 0.05 = \Pr(Z > 1.644 \ldots)$$

$$\therefore \bar{x} \approx 409870$$

Complex Numbers

Polar Form

- Complex's fancy way of using its fancy cartesian plane.

Use when multiplying/division/finding solutions with De Moivre

To convert between Cartesian and Polar form:

$$z = x + yi \quad \rightarrow \quad z = rcis(\theta) \quad or \quad z = rcos(\theta) + irsin(\theta)$$

Where:

- r is the distance from the point to the origin.

$$r = \sqrt{x^2 + y^2}$$

- θ : The angle between the positive x axis and point

$$\tan(\theta) = \frac{y}{x}$$

Complex Numbers

Polar Form

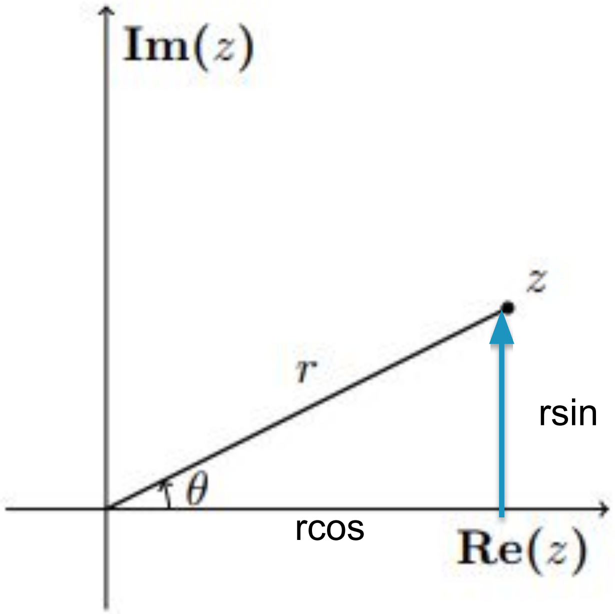
-

Thus, we can represent *z* as

$$r(\cos\theta \ + \ i\sin\theta)$$

Or, for short:

$$r\ cis\ \theta$$



SUPER important: DOMAINS

If you are given Arg (not arg) the domain is $\mathrm{Arg}(\theta)\ \epsilon\ (-\pi,\pi]$

You have to draw the CAST quadrants

Complex Numbers

Multiplication and Division

- Rule: multiply/divide to r or add/subtract θ

Consider two complex numbers: (USE POLAR FOR THIS)

$$z_1 = r_1 cis(\theta_1) \quad z_2 = r_2 cis(\theta_2)$$

Multiplication:

$$\boxed{Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)}$$

$$z_1 \cdot z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

Conjugate:

$$\bar{z} = rcis(-\theta)$$

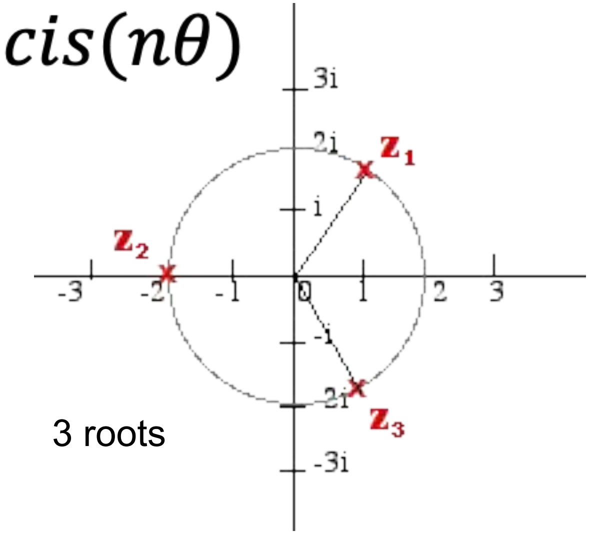
Reciprocal:

$$z^{-1} = \frac{1}{r} cis(-\theta)$$

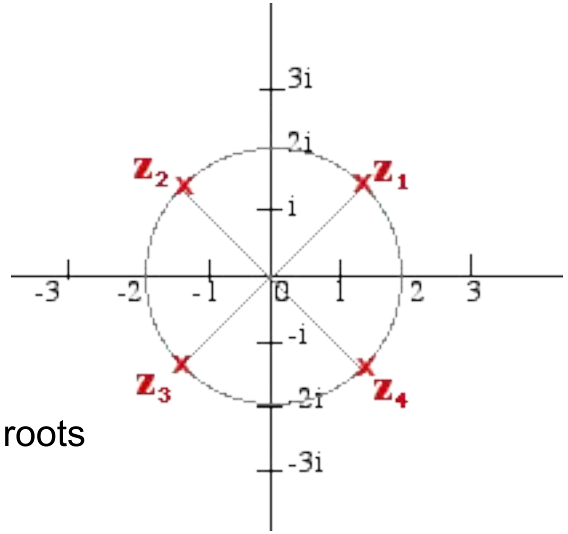
Complex Numbers

De Moivre's Theorem

- We use this for finding nth roots of a complex number
- (eg. Numbers have can 2 square roots, 3 cube roots, etc……)
- Therefore: There are n solutions for nth roots around a circle
    - MUST BE IN POLAR FORM AND IN $Arg(\theta)\epsilon(-\pi,\pi]$
- $z^n = r^n cis(n\theta)$



3 roots



4 roots

Solving De Moivres

- Super simple! Just need to equate r variables and θ variables

- Steps:

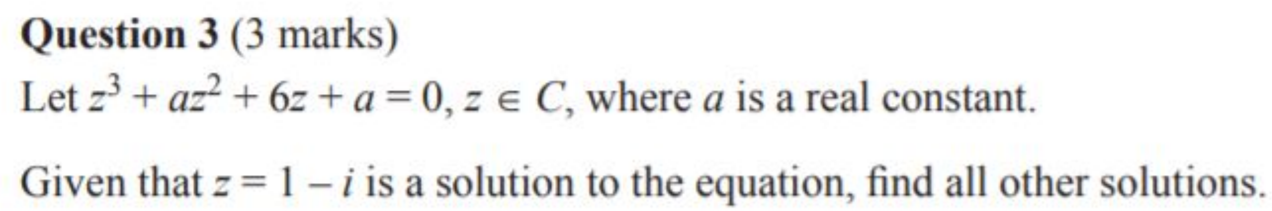
1. Write the formula: $z^n = r^n cis(n\theta)$
2. Convert the complex number into Polar form
3. Let r^n = magnitude and $n\theta$ = arg
   - For angles if you find one angle, just $\pm$ angles equally apart
4. Make sure $Arg(\theta)\epsilon(-\pi,\pi]$
5. Convert into cartesian if needed

Polynomials

- **Conjugate root theorem:**
- If all coefficients are REAL numbers in eg
- $p = az^3 + bz^2 + cz + d$
- The solution of z=x+yi will also have $\bar{z}$=x-yi as a solution

- **Simplifying:** Let’s say I have
  - $p = z^3 + 8z^2 + 25z + 26$ with a solution of (z-2)

Complex Numbers



Question 3 (3 marks)

Let $z^3 + az^2 + 6z + a = 0$, $z \in C$, where a is a real constant.

Given that $z = 1 - i$ is a solution to the equation, find all other solutions.

$$z = 1 + i$$

$$(z - 1 + i)(z - 1 - i) = z^2 - 2z + 2$$

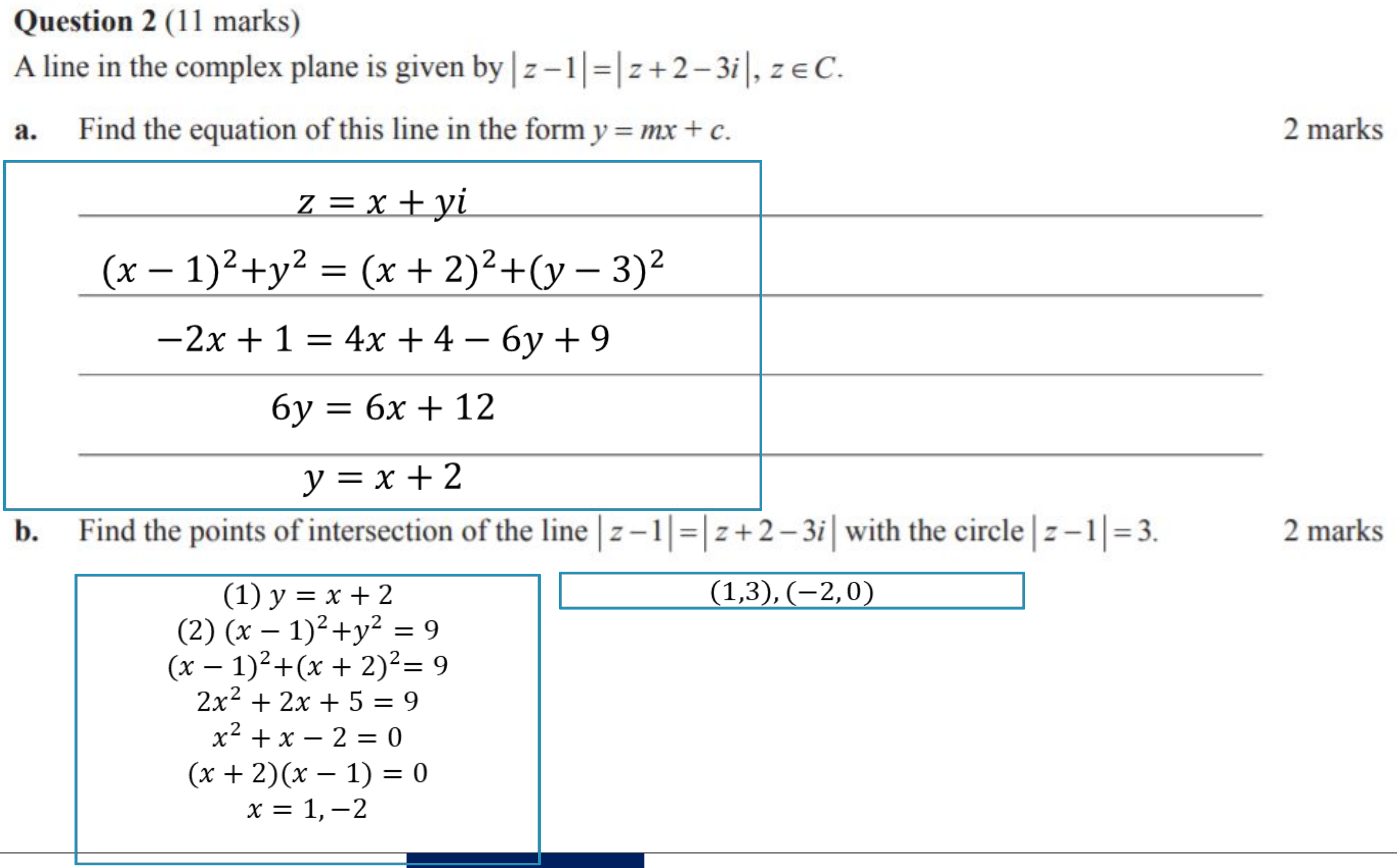
$$(z^2 - 2z + 2)(z - n) = z^3 + (-n - 2)z^2 + (2 + 2n)z - 2n = z^3 + az^2 + 6z + a$$

$$6 = 2 + 2n \Rightarrow n = 2$$

$$\boldsymbol{z = 1 + i, 1 - i, 2}$$

Complex Numbers

Questions



Question 2 (11 marks)

A line in the complex plane is given by $|z-1| = |z+2-3i|$, $z \in C$.

a. Find the equation of this line in the form $y = mx + c$. 2 marks

$$z = x + yi$$

$$(x-1)^2 + y^2 = (x+2)^2 + (y-3)^2$$

$$-2x + 1 = 4x + 4 - 6y + 9$$

$$6y = 6x + 12$$

$$y = x + 2$$

b. Find the points of intersection of the line $|z-1| = |z+2-3i|$ with the circle $|z-1| = 3$. 2 marks

$$(1)\ y = x + 2$$

$$(2)\ (x-1)^2 + y^2 = 9$$

$$(x-1)^2 + (x+2)^2 = 9$$

$$2x^2 + 2x + 5 = 9$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = 1, -2$$

$$(1,3), (-2,0)$$

Complex Numbers

Questions

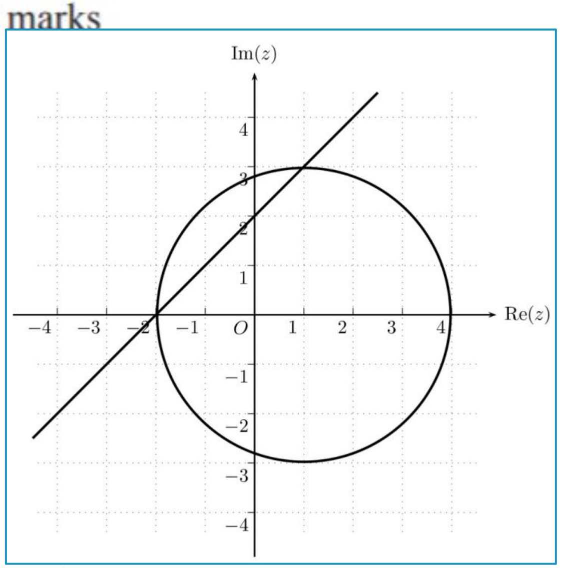
d. The line $|z-1| = |z+2-3i|$ cuts the circle $|z-1| = 3$ into two segments.

Find the area of the major segment.

2 marks

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$

$$Area = \frac{9}{2}\left(\frac{3\pi}{2} - \sin\frac{3\pi}{2}\right)$$
$$= \frac{9}{2}(\frac{3\pi}{2} + 1)$$
$$= \frac{27\pi}{4} + \frac{9}{2}$$



f. Write down the range of values of α, $\alpha \in R$, for which a ray with equation $\text{Arg}(z) = \alpha\pi$ intersects the line $|z-1| = |z+2-3i|$.

2 marks

$$\alpha \in \left(-1, \frac{-3}{4}\right) \cup \left(\frac{1}{4}, 1\right]$$

To intersect, the argument must produce a ray that has a gradient towards the line

Circular Functions

Compound Formulas

Double Angle Formulas:

- On formula sheet, but REALLY good to memorise
- Used for trig with WEIRD angles

Note: sometimes you will need to use **Half Angle Formulas** to proof. Just replace with half the x

Circular Functions

Converting Inverse Domain and Range

- If we had $y = a\sin^{-1}(bx - c) + d$:

DOMAINS: UNDO

1. Take the bracket
2. Apply to the default domain of the trig
3. 'Undo' the transformations until you only have x

$$bx - c \epsilon [-1,1]$$

$$bx \epsilon [c - 1, c + 1]$$

$$x \epsilon [\frac{c-1}{b}, \frac{c+1}{b}]$$

RANGES: BUILD UP

1. Rearrange equation to 'Build Up' y side
2. Let the y side equal the default range
3. 'Undo' the transformations until you only have y

$$\frac{y-d}{a} \epsilon [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$y - d \epsilon [\frac{-\pi a}{2}, \frac{\pi a}{2}]$$

$$y \epsilon [d - \frac{\pi a}{2}, d + \frac{\pi a}{2}]$$

Circular Functions

Questions

a. Solve $\sin(2x) = \sin(x)$, $x \in [0, 2\pi]$. 3 marks

$$2\sin(x)\cos(x) = \sin(x)$$

$$2\sin(x)\cos(x) - \sin(x) = 0$$

$$(2\cos(x) - 1)\sin(x) = 0$$

$$\cos(x) = \frac{1}{2}, \sin(x) = 0$$

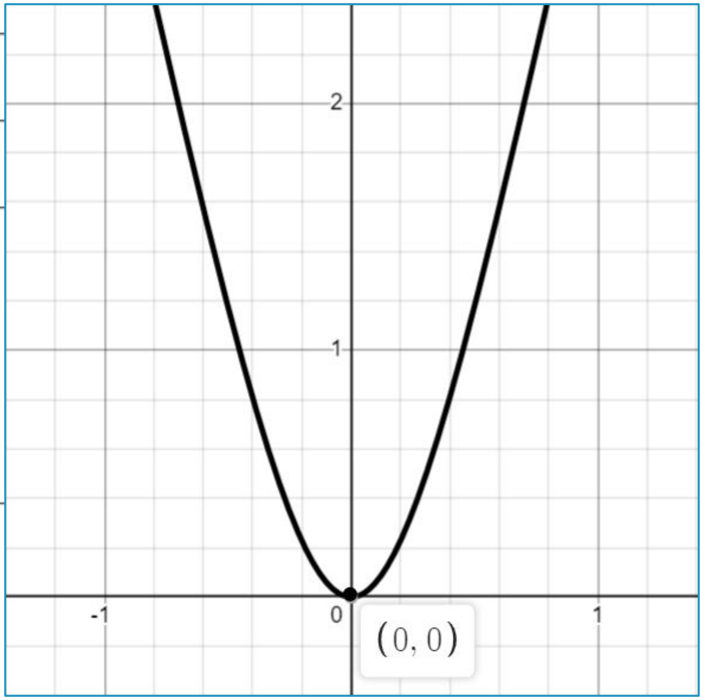
$$x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$$

Question 7 (5 marks)

Consider $f(x) = 3x\arctan(2x)$.

a. Write down the range of f.

$ran \in [0, \infty)$



Calculus

Compound Differentiation

- Product Rule (use this unless stated by inversing the denominator)

$$\frac{d}{dx}(uv) = u'v + uv'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(u'v - uv')}{v^2}$$

Tips:

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Add examples of differentiation that are tricky!

Calculus

Differentiation Rules

These are the most useful equations to memorize outside of formula sheet:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$

$$\frac{d}{dx}(\log_e f(x)) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}(a^x) = a^x \log_e a$$

$$\frac{d}{dx}(f(g(x)) = g'(x)f'(g(x)$$

This one in particular! Chain rule (work in layers)

CAREFUL OF DOMAINS

$$\frac{d}{dx}(\sin^{-1} f(x)) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\frac{d}{dx}(\cos^{-1} f(x)) = \frac{-f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\frac{d}{dx}(\tan^{-1} f(x)) = \frac{f'(x)}{1+(f(x))^2}$$

Implicit Differentiation + Concavity

Implicit Differentiation:

- If you see ‘y’ differentiate as per normal but put $\frac{dy}{dx}$ after every time

Points of Inflection/Concavity

- Used to verify TP/direction of gradient
- $f''(x) = +ve \rightarrow local\ min$
- $f''(x) = -ve \rightarrow local\ max$
- $POI: f''(x) = 0\ but\ f'(x) \neq 0$
- $SPOI: f''(x) = 0\ but\ f'(x) = 0$

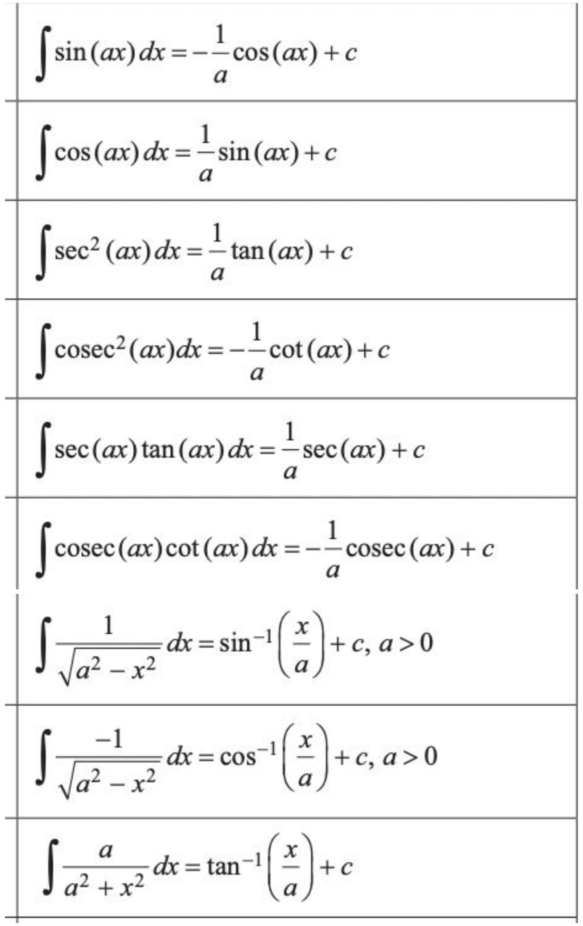
Related Rates

- Related Rates
- Look at the units in question – gives big clue to which variable is over which variable
- Is based upon Chain Rule
- Parametric is the same – derivative of both equations combined together
- Often used in applications involving time and shapes

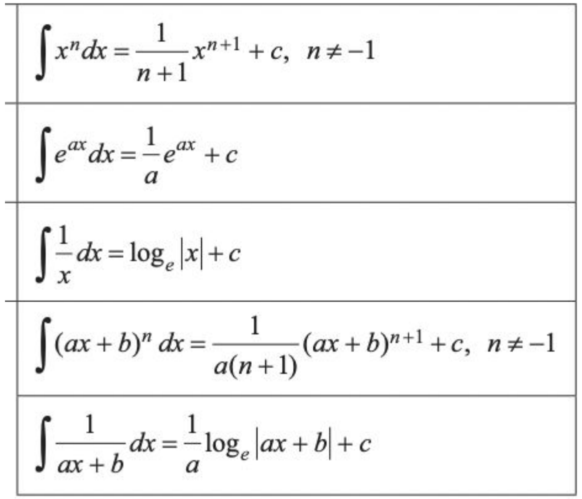
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Summary: Integration Rules

Don’t forget Modulus – must reject one side



$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\int \text{cosec}^2(ax)dx = -\frac{1}{a}\cot(ax) + c$
$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + c$
$\int \text{cosec}(ax)\cot(ax)\,dx = -\frac{1}{a}\text{cosec}(ax) + c$
$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\int \frac{-1}{\sqrt{a^2 - x^2}}dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\int \frac{a}{a^2 + x^2}dx = \tan^{-1}\left(\frac{x}{a}\right) + c$



$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ \ n \neq -1$
$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\int \frac{1}{x}dx = \log_e \lvert x \rvert + c$
$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ \ n \neq -1$
$\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$

VCAA Specialist Mathematics Formula Sheet Accessed: https://vcaa.vic.edu.au/Documents/exams/mathematics/specmaths-formula-w.pdf

INTEGRATION BY SUBSTITUTION "LET U"

- AKA Let u when there is a STEP DOWN in power eg. ($g'(x)f(g(x))$)
- You will make mistakes. Its ok!
- If you do the wrong sub, try the other
- Almost always more complex/denominator is the Let u

Steps:

1. Let a component = u
2. Find $\frac{du}{dx}$ and rearrange into smth that looks like other component
3. Sub u and $\frac{du}{dx}$ into back into original (if you have terminals, convert to u too)
4. Solve for u
5. Sub u original back

INTEGRATION WITH TRIG

- 2 cases: ANY odd powers or ALL even powers

ANY ODD POWERS $(\text{eg.} \sin^5 x \cos^2 x)$

1. Let u=opposite trig of odd power

$$Let\ u = \cos x , \qquad -\frac{du}{dx} = \sin x$$

2. Cancel the trig with odd power to have it be an even power

$$-\sin^4 x\, u^2$$

3. Remember $\sin^2 x + \cos^2 x = 1$ and convert the other even powers to u

$$(1 - u^2)^2 u^2$$

4. Solve like normal

5. (If tanx is odd) Convert into $\frac{\sin x}{\cos x}$

INTEGRATION WITH TRIG

Super hard!

- 2 cases: ANY odd powers or ALL even powers
- **ALL EVEN POWERS** MUST MEMORISE THESE FORMULAS!
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\tan^2 kx = \sec^2 kx - 1$$

Calculus

INTEGRATION TECHNIQUES

-
- For a rational function in the form $y = \frac{f(x)}{g(x)}$ where f and g are polynomials,
- If degree of $f(x) <$ degree of $g(x)$, use partial fractions
- If degree of $f(x) =$ degree of $g(x)$, break it up and then use partial fractions
- If degree of $f(x) >$ degree of $g(x)$, break it up using long division of polynomials and then use

Integrating partial fractions

Integrating partial fractons

- A fraction of polynomials can be split into respective partial fractions
  - Integrating via log form
  - Integrating via tan inverse form

Calculus

INTEGRATION BY PARTS/RECOGNITION

We use this technique when the integrand is in the form of

$$fg'$$

Where

- f is a function that doesn't have a standard integral, but can be differentiated
- g' is a function that can easily be integrated
- Or use ILATE/LIATE

Examples:

$$x^2 \log_e(x) \qquad 3x^2 \sin^{-1}(2x)$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

SPECIAL CASE: NEWTON’S LAW OF COOLING

- Rate at which a body cools(dT/dt) is PROPORTIONAL to the difference between its temp and ambient temp(T - T_s)

First Principles

Just memorise this unless

$$\frac{dT}{dt} = -k(T - T_s) \longrightarrow \boxed{T = Ae^{kt} + Ts, \qquad k > 0}$$

- Where:
- T_s is the temperature of the surroundings
- k is the proportional constant

SPECIAL CASE: LOGISTIC DIFFERENTIAL EQUATIONS

- Rate at which a population (dP/dt) is PROPORTIONAL to the Population and the limiting population

First Principles

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) \longrightarrow$$

Just memorise this unless

$$P = \frac{L}{e^{c-kt} + 1}, \qquad k > 0$$

- Where:
- P = population
- L is the limiting population ($P \rightarrow L$ when $t \rightarrow \infty$)
- k = constant which changes based on the question

CONCENTRATION/MIXING PROBLEMS

- When you have a substance pouring into a tank, mixing with something else and then a mixture flows out.

$$\boxed{\frac{dQ}{dt} = Inflow\ - Outflow}$$

both usually in g/min

$$Inflow = \frac{Amount\ (g)}{\underline{(L)}} \times \frac{flow\ in\ \underline{(L)}}{(min)} = \frac{Amount(g)}{(\text{min})} \times flow\ in$$

$$Outflow = \frac{Q\ (g)}{volume +\ changes\ \underline{(L)}} \times \frac{flow\ out\ \underline{(L)}}{(\text{min})} = \frac{Q \times flow\ out}{volume +\ changes}$$

Where:
- Q is outflow substance (in grams)
- If Q is conc, remember $C = \frac{mass}{Volume}$

If they give you initial conditions in qn:
- *THEY DON'T COUNT FOR THIS FORMULA*
- *USE LATER FOR +c*

CONCENTRATION/MIXING PROBLEMS

- Two situations: This will change your volume in outflow.
- Inflow = outflow
- Inflow ≠ outflow

Note! If one of your substances is water, you are adding no grams/diluting another solution. Inflow will be 0.

Inflow L/min

capacity

Outflow L/min

Steps:

1. Draw the situation: inflow, outflow, capacity of tank
2. Define your outflow/inflow formulas
3. $\frac{dQ}{dt} = Inflow - Outflow$
4. Solve so Q is subject

SLOPE FIELDS

Steps:

1. Look at x-axis/y-axis for patterns
    a) +ve/-ve gradients, 0 gradients, undefined gradients
2. Look at the question
    a) If want $\frac{dy}{dx}$: look for TP or asymptote lines.
        a) Vertical lines: Undefined
        b) Horizontal: 0 gradient
        c) Pos/neg gradients
    b) If want y: draw curve on slope field and look at shape.
    THIS IS THE IMPLIED CURVE NOT $\frac{dy}{dx}$!

EULER'S METHOD

-
- $$y_{n+1} = y_n + hf'(x_n, y_n)$$

h is step size

- **MUST MEMORISE FORMULA**
- DON'T GET CONFUSED W/ PREVIOUS x, y TERMS
- BIGGEST ERROR IS NOT ORGANISING YOUR WORKING!!!!!

Steps:

1. Write $y_{n+1} = y_n + h\,\frac{dy}{dx}$
2. COUNT HOW MANY STEPS BEFORE SUB INTO FORMULA
3. Sub and DON'T SKIP STEPS! ORGANISE

SUMMARY BOOK - DIFFERENTIAL EQUATIONS

- Fundamental Theorem of Calculus

$$y_{final} = \int_{x_{initial}}^{x_{final}} f'(x)\, dx + y_{initial}$$

- Separation of Variables
  - Like terms: put dx with x terms, dy with y terms
  - Integrate both sides
  - +c on x side, arrange so y is subject
- Growth/Decay
  - Key words: ‘Proportional’
  - A = initial conditions, k = growth/decay, t = time

SUMMARY BOOK - DIFFERENTIAL EQUATIONS

- **Newton’s Law of Cooling**
  - k = cooling/heating, T = temp, T_0 = environment temp, t = time

$$\frac{dT}{dt} = k(T - T_0) \rightarrow T = Ae^{kt} + T_0$$

- **Concentration/mixing problems**

$$\frac{dQ}{dt} = inflow - outflow$$

$$Inflow = \frac{amount(g)}{L} \times \frac{flow\,rate(L)}{min}$$

$$Inflow = \frac{Q(g)}{capacity \pm \Delta volume} \times \frac{flow\,rate(L)}{min}$$

$$\frac{dQ}{dt} = \frac{dQ}{dV}_{in} \times \frac{dV}{dt}_{in} - \frac{dQ}{dV}_{out} \times \frac{dV}{dt}_{out}$$

SUMMARY BOOK - DIFFERENTIAL EQUATIONS

Slope fields

1. Look at **x&y axis** for patterns (0, undefined, +/- gradients)
2. Look at **question:**
    a. If f’(x) - look for patterns(quadrants, 0, undefined, +/- grad)
    b. If y - trace the shape of the slope
3. Special:
    a. sin/cos - look at shape and period
    b. circle - look for centre

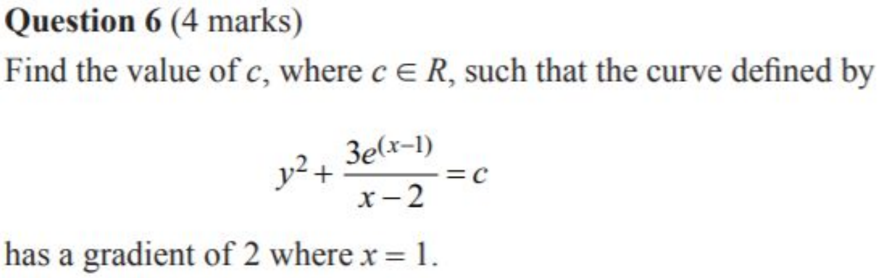
Euler’s method

- ORGANISE your working!

$$y_{n+1} = y_n + hf'(x_n, y_n)$$

- CAS: ‘euler(dy/dx, x,y,{x0,xn},y0,h)

IMPLICIT DIFFERENTIATION

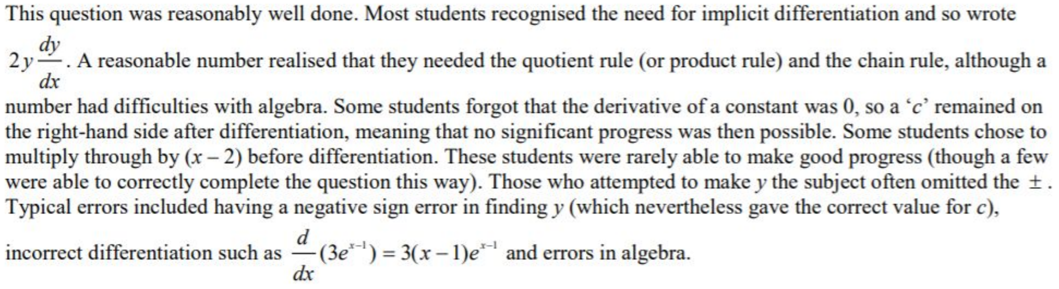


Question 6 (4 marks)

Find the value of c, where $c \in R$, such that the curve defined by

$$y^2 + \frac{3e^{(x-1)}}{x-2} = c$$

has a gradient of 2 where $x = 1$.

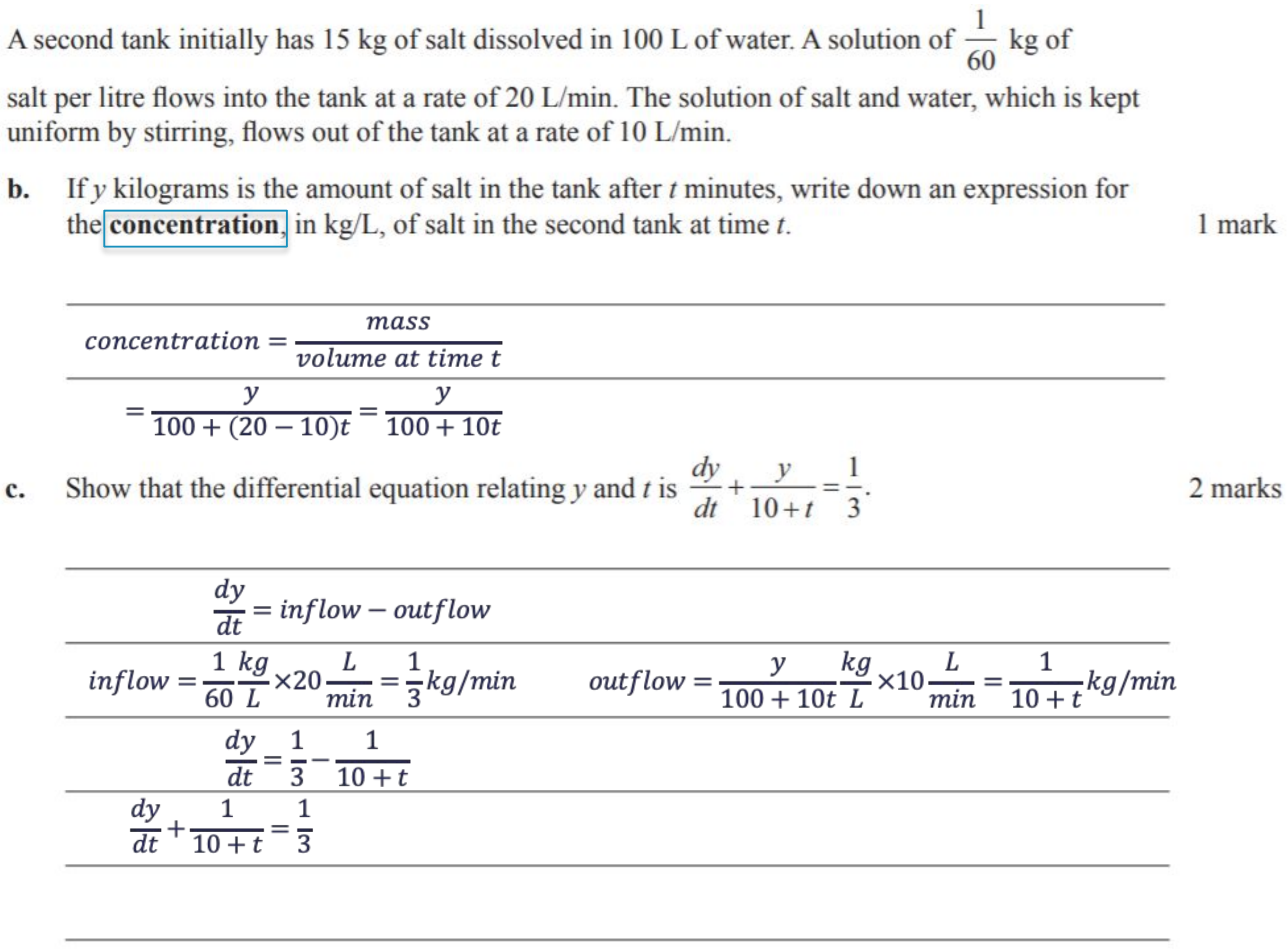


This question was reasonably well done. Most students recognised the need for implicit differentiation and so wrote $2y\frac{dy}{dx}$. A reasonable number realised that they needed the quotient rule (or product rule) and the chain rule, although a number had difficulties with algebra. Some students forgot that the derivative of a constant was 0, so a 'c' remained on the right-hand side after differentiation, meaning that no significant progress was then possible. Some students chose to multiply through by $(x-2)$ before differentiation. These students were rarely able to make good progress (though a few were able to correctly complete the question this way). Those who attempted to make y the subject often omitted the $\pm$. Typical errors included having a negative sign error in finding y (which nevertheless gave the correct value for c), incorrect differentiation such as $\frac{d}{dx}(3e^{x-1}) = 3(x-1)e^{x-1}$ and errors in algebra.

$(f(x))^2 + \frac{3e^{(x-1)}}{x-2} = c$	$(y)^2 + \frac{3e^{(x-1)}}{x-2} = c$
$2 \times f'(x) \times f(x) + \frac{(x-2) \times 3 \times e^{(x-1)} - (1) \times 3e^{(x-1)}}{(x-2)^2} = 0$	$x = 1 \qquad y = \frac{3}{2}$
$2 \times \frac{dy}{dx} \times y + \frac{3(x-2)e^{(x-1)} - 3e^{(x-1)}}{(x-2)^2} = 0$	$\left(\frac{3}{2}\right)^2 + \frac{3e^{(1-1)}}{1-2} = c$
Let $x = 1$, $\frac{dy}{dx} = 2$	$c = \frac{9}{4} + \frac{3 \times 1}{-1}$
$2 \times 2 \times y + \frac{3(1-2)e^{(1-1)} - 3e^{(1-1)}}{(1-2)^2} = 0$	$c = \frac{9}{4} + \frac{-12}{4} = -\frac{3}{4}$
$4y + \frac{3(-1)e^0 - 3e^0}{(-1)^2} = 0$	
$\frac{3 \times -1 \times 1 - 3 \times 1}{1} = -4y$	
$-3 - 3 = -4y$	
$y = \frac{3}{2}$	

Calculus

RATE IN – RATE OUT



A second tank initially has 15 kg of salt dissolved in 100 L of water. A solution of $\frac{1}{60}$ kg of salt per litre flows into the tank at a rate of 20 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 10 L/min.

b. If y kilograms is the amount of salt in the tank after t minutes, write down an expression for the **concentration**, in kg/L, of salt in the second tank at time t. 1 mark

$$concentration = \frac{mass}{volume\ at\ time\ t}$$

$$= \frac{y}{100 + (20 - 10)t} = \frac{y}{100 + 10t}$$

c. Show that the differential equation relating y and t is $\frac{dy}{dt} + \frac{y}{10 + t} = \frac{1}{3}$. 2 marks

$$\frac{dy}{dt} = inflow - outflow$$

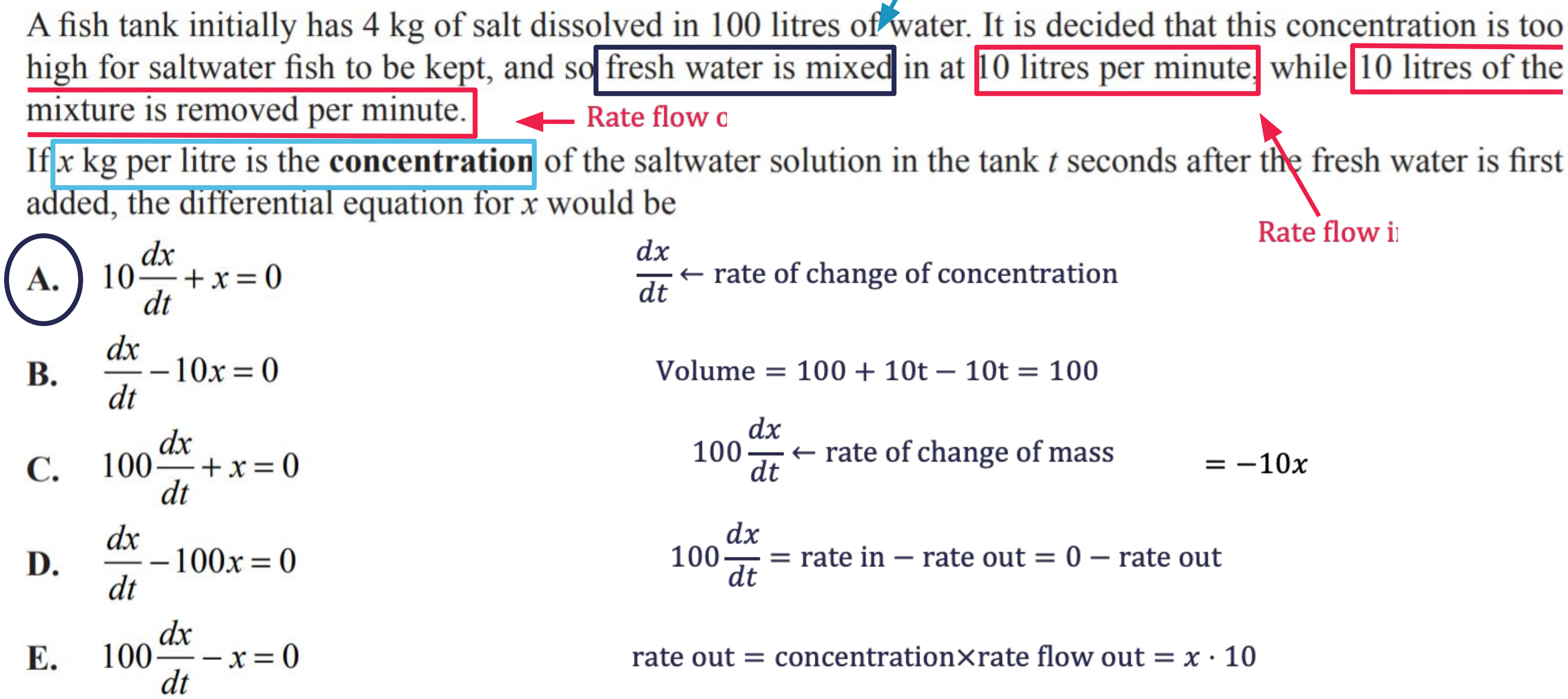
$$inflow = \frac{1}{60}\frac{kg}{L} \times 20\frac{L}{min} = \frac{1}{3}kg/min \qquad outflow = \frac{y}{100 + 10t}\frac{kg}{L} \times 10\frac{L}{min} = \frac{1}{10 + t}kg/min$$

$$\frac{dy}{dt} = \frac{1}{3} - \frac{1}{10 + t}$$

$$\frac{dy}{dt} + \frac{1}{10 + t} = \frac{1}{3}$$

Calculus

VCAA 2009



rate in $= 0$

A fish tank initially has 4 kg of salt dissolved in 100 litres of water. It is decided that this concentration is too high for saltwater fish to be kept, and so fresh water is mixed in at 10 litres per minute, while 10 litres of the mixture is removed per minute.

Rate flow o

If x kg per litre is the **concentration** of the saltwater solution in the tank t seconds after the fresh water is first added, the differential equation for x would be

Rate flow i

[✓ **A.**] $10\frac{dx}{dt}+x=0$

B. $\frac{dx}{dt}-10x=0$

C. $100\frac{dx}{dt}+x=0$

D. $\frac{dx}{dt}-100x=0$

E. $100\frac{dx}{dt}-x=0$

$\frac{dx}{dt} \leftarrow$ rate of change of concentration

Volume $= 100 + 10t - 10t = 100$

$100\frac{dx}{dt} \leftarrow$ rate of change of mass $\qquad = -10x$

$$100\frac{dx}{dt} = \text{rate in} - \text{rate out} = 0 - \text{rate out}$$

$$\text{rate out} = \text{concentration} \times \text{rate flow out} = x \cdot 10$$

$$\therefore 100\frac{dx}{dt} = -10x \Rightarrow 10\frac{dx}{dt} + x = 0$$

Calculus

VCAA 2013

The number of mobile phones, N, owned in a certain community after t years, may be modelled by $\log_e(N) = 6 - 3e^{-0.4t}$, $t \geq 0$.

$\log_e(N) = 6 - 3e^{-0.4t}$ satisfies the differential equation

$$\frac{1}{N}\frac{dN}{dt} + 0.4\log_e(N) - 2.4 = 0\,.$$

c. Using this mathematical model, find the limiting number of mobile phones that would eventually be owned in the community.

Give your answer correct to the nearest integer. 2 marks

Two ways:

1. As $t \to \infty, \log_e(N) \to 6$

OR

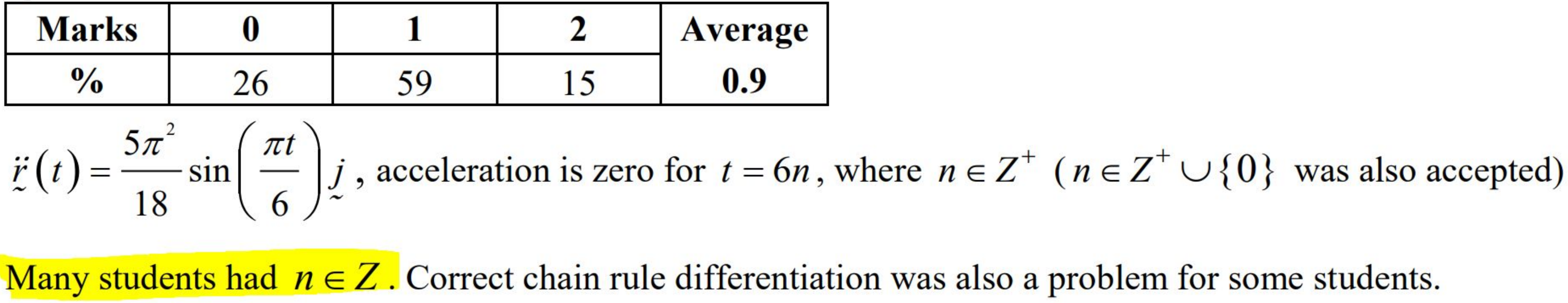
2. $\frac{dN}{dt} = 0 \Rightarrow 0.4\log_e(N) - 2.4 = 0 \Rightarrow \log_e(N) = 6 \backslash$

$$\therefore N = e^6 \approx 403$$

Calculus

TIME IS POSITIVE

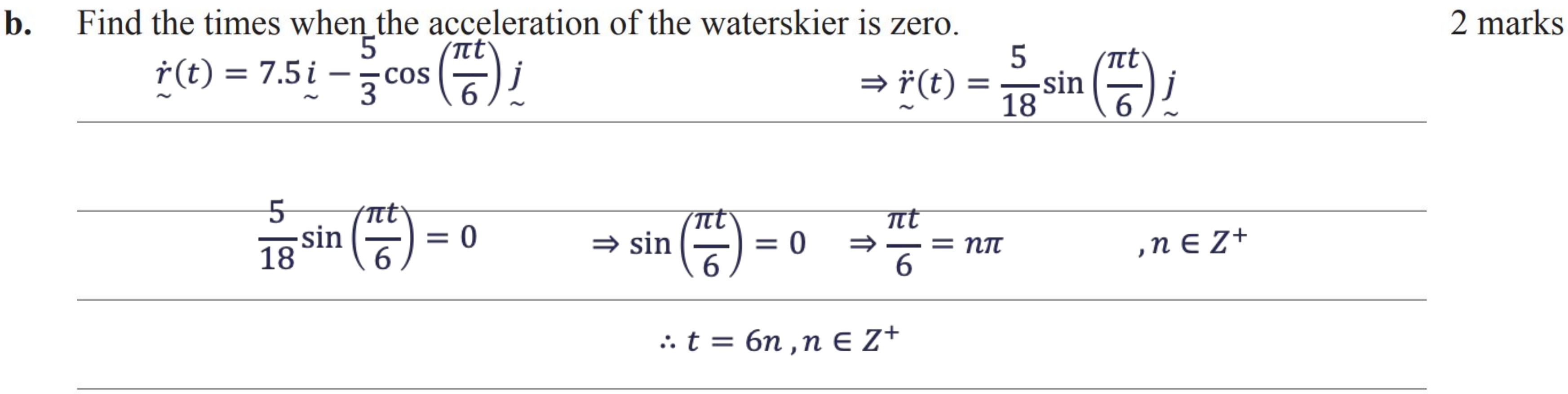
- Sometimes, you may be asked to find the values of *t* so please ensure that the values of *t* you give are positive!



Marks	0	1	2	Average
%	26	59	15	0.9

$\ddot{\underset{\sim}{r}}(t) = \frac{5\pi^2}{18}\sin\left(\frac{\pi t}{6}\right)\underset{\sim}{j}$, acceleration is zero for $t = 6n$, where $n \in Z^+$ ($n \in Z^+ \cup \{0\}$ was also accepted)

Many students had $n \in Z$. Correct chain rule differentiation was also a problem for some students.



b. Find the times when the acceleration of the waterskier is zero. 2 marks

$\dot{\underset{\sim}{r}}(t) = 7.5\underset{\sim}{i} - \frac{5}{3}\cos\left(\frac{\pi t}{6}\right)\underset{\sim}{j}$ $\Rightarrow \ddot{\underset{\sim}{r}}(t) = \frac{5}{18}\sin\left(\frac{\pi t}{6}\right)\underset{\sim}{j}$

$\frac{5}{18}\sin\left(\frac{\pi t}{6}\right) = 0$ $\Rightarrow \sin\left(\frac{\pi t}{6}\right) = 0$ $\Rightarrow \frac{\pi t}{6} = n\pi$ $, n \in Z^+$

$\therefore t = 6n, n \in Z^+$

Calculus

EULER’S METHOD

Question 9

Euler’s formula is used to find y_2, where $\dfrac{dy}{dx} = \cos(x)$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

The value of y_2 correct to four decimal places is

A. 1.1000 and this is an underestimate of $y(0.2)$

[✓ **B.**] 1.1995 and this is an overestimate of $y(0.2)$

C. 1.1995 and this is an underestimate of $y(0.2)$

D. 1.2975 and this is an underestimate of $y(0.2)$

E. 1.2975 and this is an overestimate of $y(0.2)$

$$\frac{dy}{dx} = \cos(x) \Rightarrow y = \sin(x) + c$$

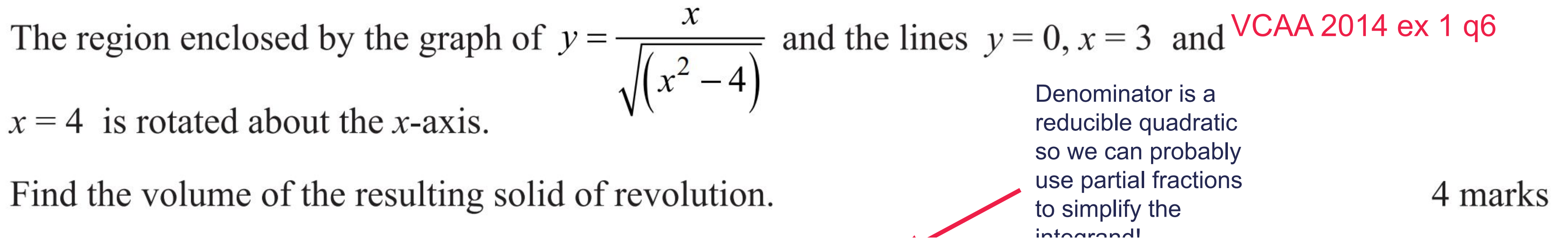
$$x = 0, y = 1$$

$$1 = \sin(0) + c \Rightarrow c = 1$$

$$y = \sin(x) + 1$$

$$y = \sin(0.2) + 1 = 1.19867$$

Calcu



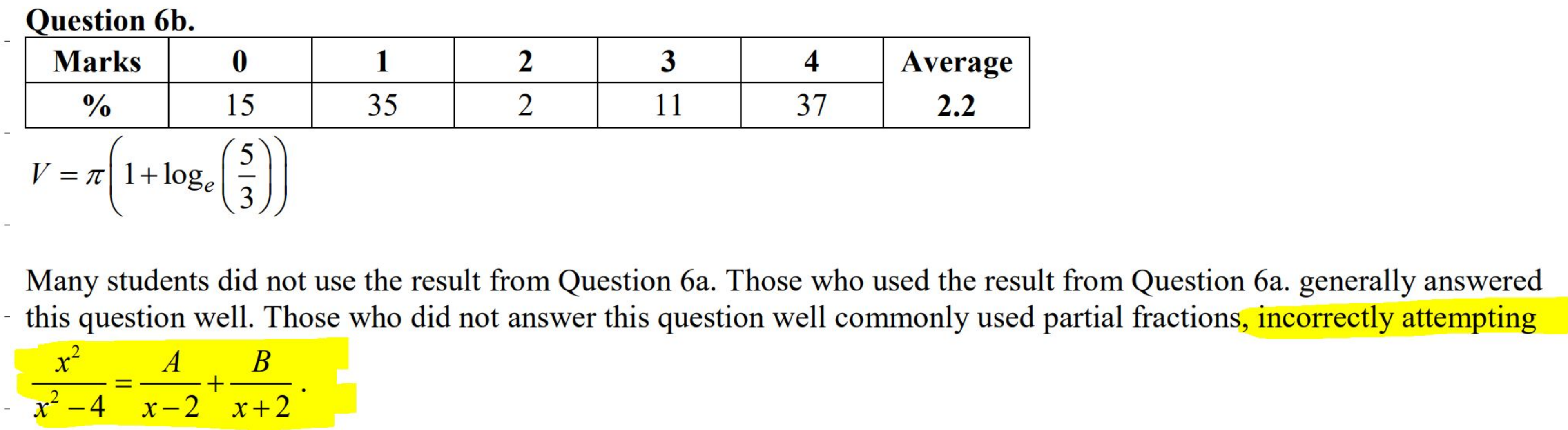
The region enclosed by the graph of $y=\frac{x}{\sqrt{(x^2-4)}}$ and the lines $y=0,\ x=3$ and $x=4$ is rotated about the x-axis.

Find the volume of the resulting solid of revolution.

4 marks

VCAA 2014 ex 1 q6

Denominator is a reducible quadratic so we can probably use partial fractions to simplify the integrand!



Question 6b.

Marks	**0**	**1**	**2**	**3**	**4**	**Average**
%	15	35	2	11	37	**2.2**

$$V=\pi\left(1+\log_e\left(\frac{5}{3}\right)\right)$$

Many students did not use the result from Question 6a. Those who used the result from Question 6a. generally answered this question well. Those who did not answer this question well commonly used partial fractions, incorrectly attempting $\frac{x^2}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$.

$$1+\frac{}{x^2-4}=1+\frac{}{x-2}-\frac{}{x+2}$$

$$\therefore V=\pi\int_3^4\left(1+\frac{1}{x-2}-\frac{1}{x+2}\right)dx=\pi\left[x+\log_e\left|\frac{x-2}{x+2}\right|\right]_3^4=\pi\left(1+\log_e\left(\frac{5}{3}\right)\right)$$

Vectors & Vector Calc

Properties of Vectors

- <u>Length/Magnitude of Vector:</u>
- If a vector is r = xi + yj + zk:
- $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$
- <u>Parallel Vectors:</u>
- Two vectors, $\vec{u}$ and $\vec{v}$, are **parallel** if $\vec{u} = k\vec{v}$ where k is a scalar (this stretches/squishes the magnitude).
- <u>Unit Vectors:</u>
- Special Vectors that have a magnitude of 1 to SPECIFY direction. We just divide the vector by its magnitude.

Scalar (Dot) Product

- For multiplying vectors together = gives you a real number!

Super simple: Remember to multiply like and like together and add. Eg:

$$\vec{a} = a_1\vec{\imath} + b_1\vec{\jmath} \qquad \vec{b} = a_2\vec{\imath} + b_2\vec{\jmath}$$

Their dot product is:

$$\vec{a} \cdot \vec{b} = a_1a_2 + b_1b_2$$

Vector Angles

To find angles between 2 vectors, THEY MUST BE TAIL TO TAIL

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

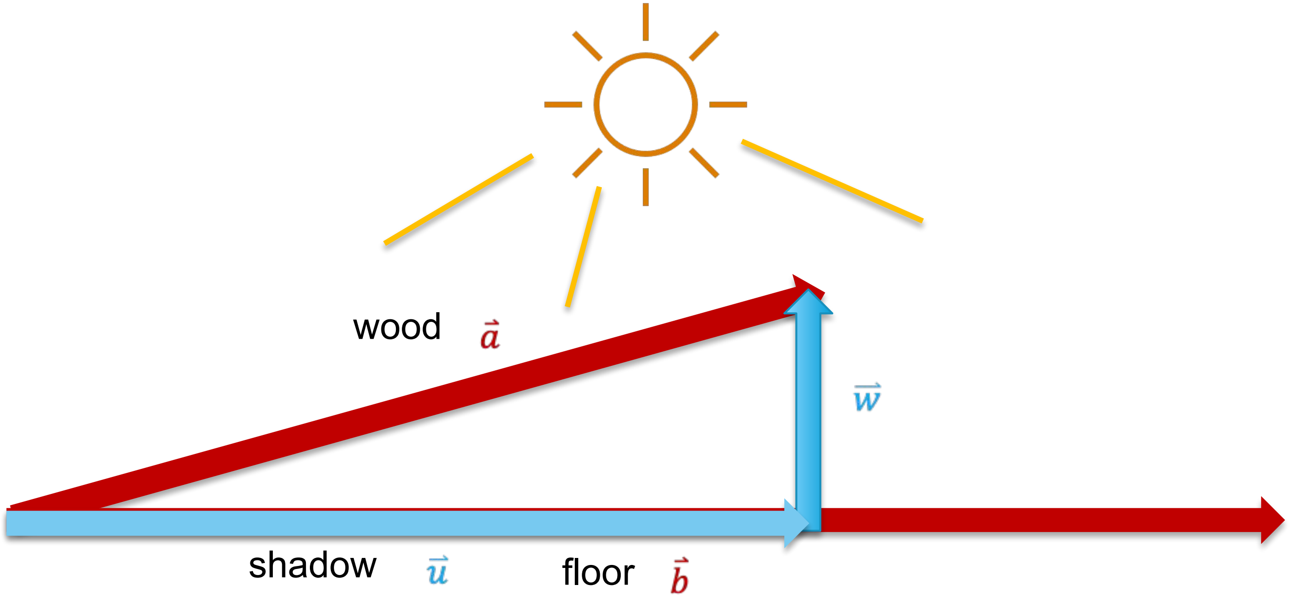
SOME IMPORTANT PROPERTIES

- $\vec{a}\cdot\vec{a} = |a|^2$
- $\vec{a}\cdot\vec{b} = 0$ if $\vec{a}$ and $\vec{b}$ are perpendicular

Vectors & Vector Calc

Vector Projections/Resolutes

-



Vector Resolutes ask “What is the shadow $(\vec{u})$ cast by the FIRST vector $(\vec{a})$ on the SECOND vector $(\vec{b})$”

Basically, we are trying to find the components of vector a.

Main gimmick: Shortest distance aka perpendicular distance

Vectors & Vector Calc

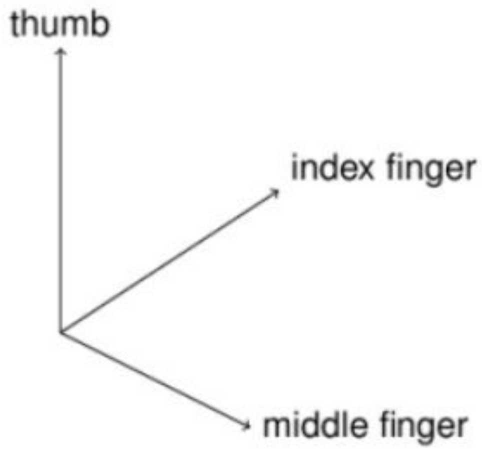
Linear Dependency

- $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent
- $\begin{cases} ma_x + na_y = a_z \\ mb_x + nb_y = b_z \\ mc_x + nc_y = c_z \end{cases}$ has a unique solution (m, n)

- Find k so that $\vec{a}, \vec{b}, \vec{c}$ are linearly independent
- Solve det $\left(\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix}\right) = 0$ for k

Cross Product

- The vector cross product finds the vector perpendicular to the plane which houses two vectors and is given by $\vec{a} \times \vec{b}$
- The direction of vector is given by the right hand rule
- Two methods to find it using the cross method or matrices method



$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

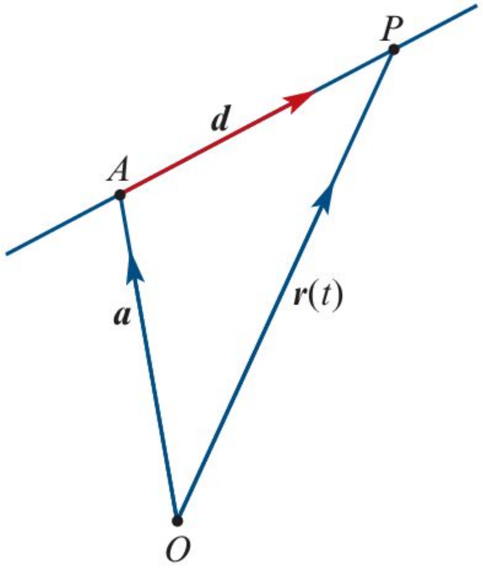
$$= (a_2b_3 - a_3b_2)\vec{\imath} - (a_1b_3 - a_3b_1)\vec{\jmath}$$

$$+ (a_1b_2 - a_2b_1)\vec{k}$$

Equation of straight line

- Vector equation lines can be thought of as an initial point and the relative points that stem from it based on the direction vector
- Coincident, parallel, intersecting, skew

$$\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{d}, \quad t \in \mathbb{R}$$



2D vector equations:

- $\vec{r}(t) = \begin{pmatrix} a_1 + kd_1 \\ a_2 + kd_2 \end{pmatrix}$

3D vector equations:

$$\vec{r}(t) = \begin{pmatrix} a_1 + kd_1 \\ a_2 + kd_2 \\ a_3 + kd_3 \end{pmatrix}$$

2D parametric conversion:

- $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2}$
- $y = \frac{d_2(x-a_1)}{d_1} + a_2$

3D parametric conversion:

$$k = \frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

$$\vec{r}(t) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} + k(d_1\vec{i} + d_2\vec{j} + d_3\vec{k})$$

Planes of vectors

- Normal vector perpendicular to all vectors that exist on the plane
- Defined by 2 vectors acting similarly to axes (same idea as linear dependency)

Vector Equation

$$(\vec{r}-\vec{a})\cdot\vec{n}=0$$

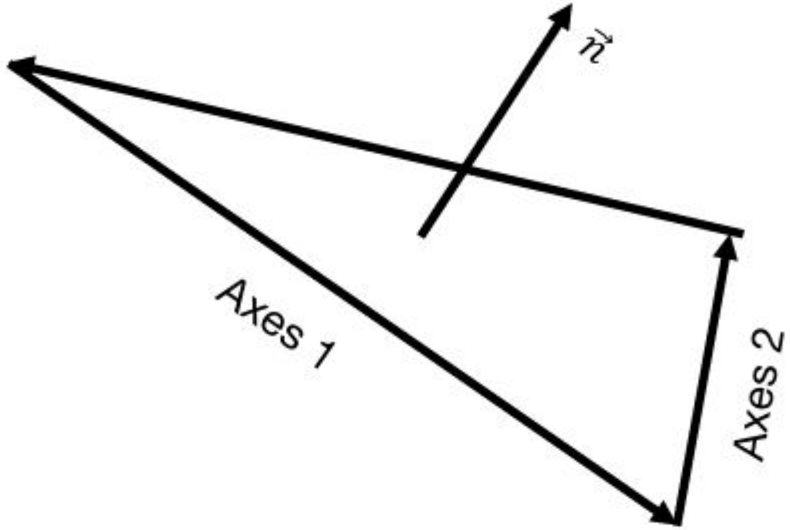
$$\vec{r}\cdot\vec{n}-\vec{a}\cdot\vec{n}=0$$

$$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

Cartesian Equation

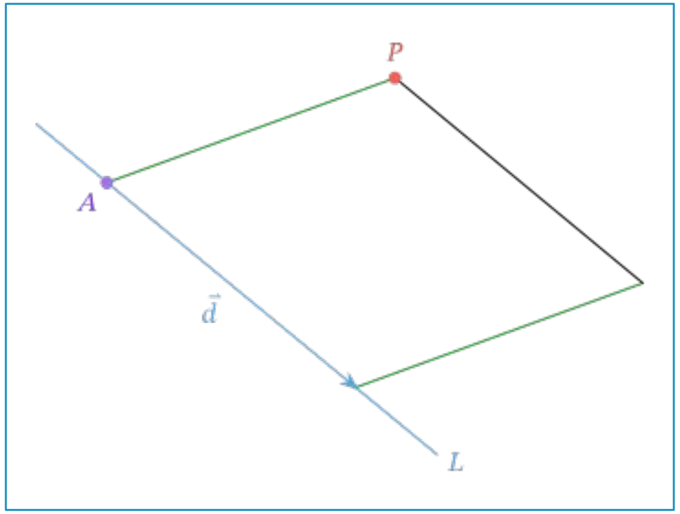
$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$$

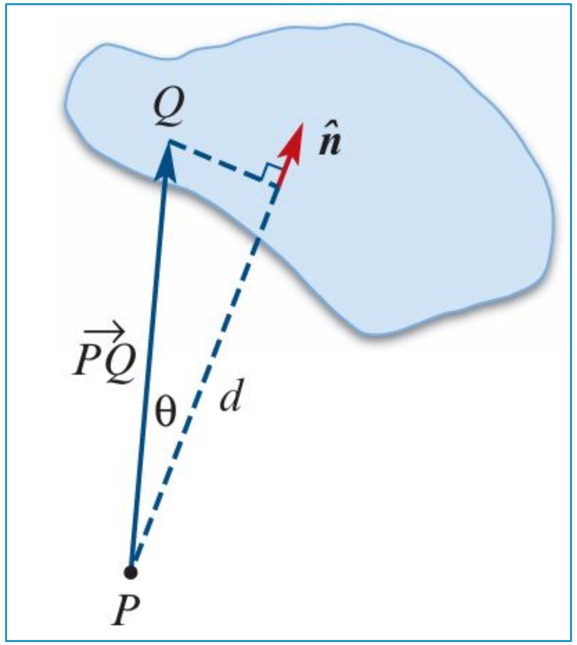
$$ax+by+cz=d$$



Distances & Intersection & Angles

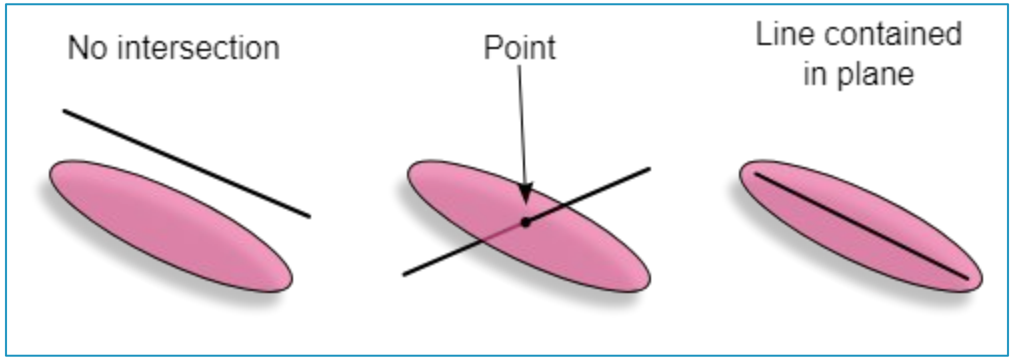
- **Distances**
  - **Line & Line**:
    - Skew: $d = \left|\overrightarrow{PQ} \cdot \hat{n}\right| \rightarrow$ *whereby*: $(\vec{n} = \vec{d}_1 \times \vec{d}_2)$
    - Parallel: $d = |\overrightarrow{AP} \times \hat{d}|$
  - **Point & Line**: $d = |\overrightarrow{AP} \times \hat{d}|$ OR finding point on the line where it is perpendicular to the line itself
  - **Point & Plane**: $d = |\overrightarrow{PQ} \cdot \hat{n}|$
  - **Line & Plane**: $d = |\overrightarrow{PQ} \cdot \hat{n}|$
  - **Plane & Plane**: $d = |\overrightarrow{PQ} \cdot \hat{n}|$

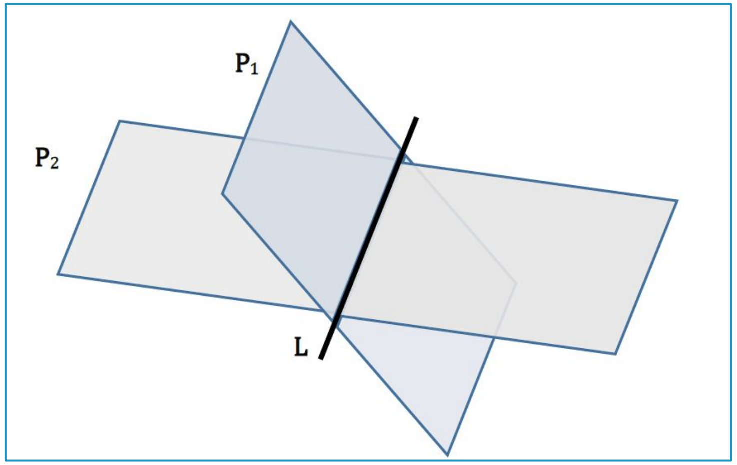




Distances & Intersection & Angles

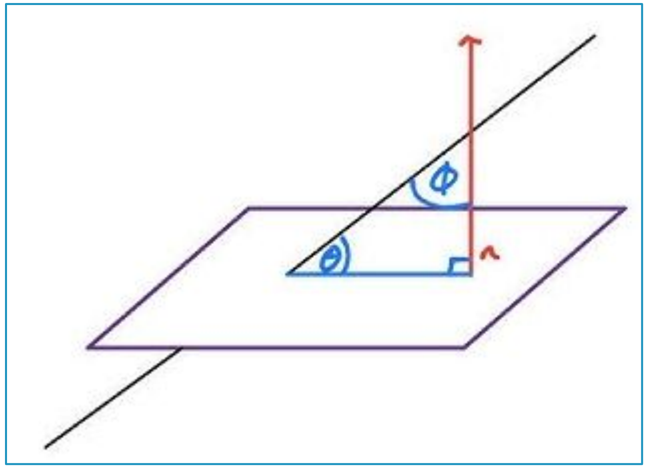
- **Intersections**
  - **Line & Line**: Equate 2 lines with different indep. variables. Solve for each variables: Analyse if it is skew or intersect based on if solutions are possible.
  - **Line & Plane**: Meets at point – sub equation of line into vector equation of the plane to find the point.
  - **Plane & Plane**: Forms a line – direction vector perpendicular to normal, a point is found that exists on the line by subbing in a random coordinate

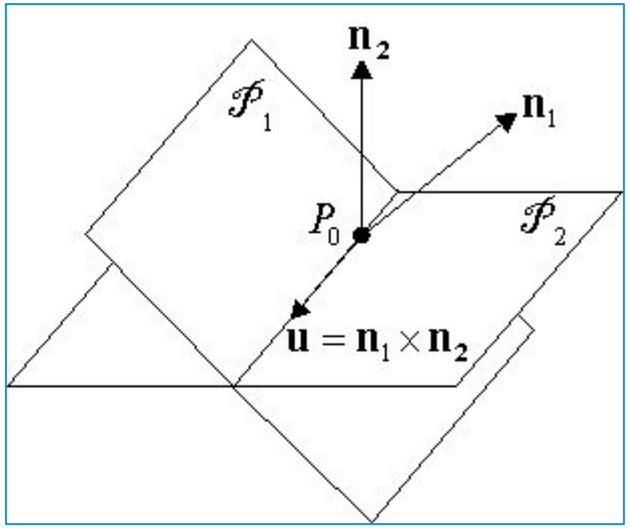




Distances & Intersection & Angles

- **Angles**
  - **Line & Line**: Use direction vectors
  - **Line & Plane**: Use direction vector of line and normal vector of plane, the answer is the 90 degrees – the resultant angle.
  - **Plane & Plane**: Use normal vectors





- Find the equation of the line which passes through point A, given by position vector = $\overrightarrow{OA} = 3\vec{\imath} - 3\vec{\jmath} + 4\vec{k}$ and is parallel to vector $\overrightarrow{OB} = -\vec{\imath} + 2\vec{\jmath} + 5\vec{k}$

$$\vec{d} = \overrightarrow{OA} - \overrightarrow{OB} = 4i - 5j - k$$

$$\vec{r} = (3i - 3j + 4k) + t(4i - 5j - k)$$

- Find the cartesian equation of the plane which passes through the points of A (-1,3,2), B (-2,-5,7) and C = (9,1, -5)

$$\vec{n} = \overrightarrow{AC} \times \overrightarrow{AB} = \begin{matrix} 10 \\ -2 \\ -7 \end{matrix} \times \begin{matrix} -1 \\ -8 \\ 5 \end{matrix} = \begin{matrix} -66 \\ -43 \\ -82 \end{matrix}$$

$$\vec{r} \cdot \vec{n} = (-66i - 43j - 82k) \cdot (-i + 3j + 2k)$$

$$\vec{r} \cdot (66i + 43j + 82k) = 227$$

- Find the intersection of the line and the plane found in two previous slides:

$$\vec{r} = i(3+4t) + j(-3-5t) + k(4-t)$$

$$\left(i(3+4t) + j(-3-5t) + k(4-t)\right) \cdot (66i + 43j + 82k) = 227$$

$$33t - 397 = 227$$

$$t = \frac{208}{11}$$

$$\vec{r} = \frac{865}{11}i - \frac{1073}{11} - \frac{164}{11}k$$

KINEMATICS

Signed area is displacement

- Motion Variables

Vector	Scalar	Representation on velocity – time graph
Displacement	Distance	Area under graph
Velocity	Speed	Coordinate of point
Acceleration	Acceleration	Gradient at a point

- $x \rightarrow$ position
- $v \rightarrow$ velocity
- $a \rightarrow$ acceleration

MEASUREMENTS AND FORMULAS

- **Velocity/Speed:**

Average rate of change/ Average velocity	
Instantaneous velocity	
Average speed	

- **Units:** Always ms^{-1} unless specified (divide km/hr by 3.6)

MEASUREMENTS AND FORMULAS

- **Acceleration**

Average acceleration	
Instantaneous acc	

- **IMPORTANT:** (physics time)
- Motion direction ≠ Acceleration: eg. Moving to the right but slowing down

CONSTANT ACCELERATION

Not on your formula sheet

- When you have **unchanging acceleration**, you can use 5 equations called SUVAT to work out displacement, initial/final velocity, acceleration and time.
- Provided in formula sheet
- *tip: Each equation is missing a variable

Most important ones in yellow

S= displacement
U= initial velocity
V= final velocity
A= acceleration
T= time

*Don't use previous variables you've found

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$s = vt - \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{1}{2}(u + v)t$$

SPECIAL CASE: PROJECTILE MOTION

- Almost always SUVAT related. When you drop/shoot something in the air. It will have vertical and/or horizontal movement

Horizontal:

- NO ACCELERATION UNLESS STATED (like engine propelling forward/constant air resistance)
- Speed is ALWAYS constant unless air resistance. Newton's First Law. No force acting on it

Vertical:

- GRAVITY! Direction is important

Just remember:
- Displacement is FINAL position – INITIAL position
- Define your positive direction of motion

OTHER ACCELERATION FORMULAS

- Pretty common question. Acceleration can be written in many ways. Just look at what info you are given in the question to decide which formula you need to use.
- $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

This one is specific to a question type

Vectors & Vector Calc

KINEMATICS

- Example) The acceleration $a\ ms^{-1}$ of a body moving in a straight line in terms of the velocity $v\ ms^{-1}$ given by $a = 4v^2$.
- Given that when $v = e$ when $x = 1$, where x is displacement of the body in metres, find the velocity of the body when $x = 2$ (VCAA – exam 1 – 2015)

$$a = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = 4v^2 \Rightarrow \frac{dv}{dx} = 4v$$

$$\frac{dx}{dv} = \frac{1}{4v}$$

$$x = \frac{1}{4}log_e|v| + c$$

Let $v = e$ when $x = 1$

$$1= \frac{1}{4}log_e|e| + c \Rightarrow c = 1 - \frac{1}{4} = \frac{3}{4}$$

$$x = \frac{1}{4}log_e|v| + \frac{3}{4}$$

$$x - \frac{3}{4} = \frac{1}{4}log_e|v|$$

$$4x - 3 = log_e|v|$$

$$v = e^{4x-3}$$

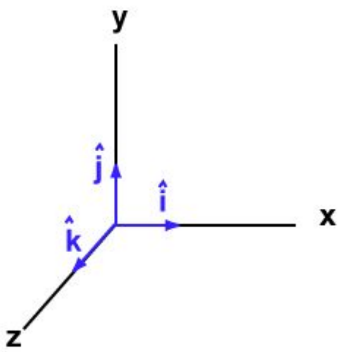
Let $x = 2$

$$v = e^{4\times 2-3} = e^5$$

Vectors & Vector Calc

Awful. Horrible. Terrible.

- Mixing vectors and calculus amazing.



- Either 2D or 3D brain is needed! (z axis only in 3D)
- Vectors with respect to time given by

$$\begin{tikzcd}
\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k} \arrow[d, bend right, "\frac{d}{dt}"'] \\
\vec{v}(t) = x'(t)\vec{\imath} + y'(t)\vec{\jmath} + z'(t)\vec{k} \arrow[u, bend right, "\int v\,dt"'] \arrow[d, bend right, "\frac{d}{dt}"'] \\
\vec{a}(t) = x''(t)\vec{\imath} + y''(t)\vec{\jmath} + z''(t)\vec{k} \arrow[u, bend right, "\int a\,dt"']
\end{tikzcd}$$

Just diff/integrate normally with components separate

- **Total distance travel from t_0 to t_1**
- Arclength/total area under a velocity-time graph of the curve $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$
- Distance $= \int_{t_0}^{t_1} \left|\dot{\underset{\sim}{r}}(t)\right| dt = \int_{t_0}^{t_1} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, dt$
- **Distance between two objects**
- Distance $= |\overrightarrow{r_1}(t) - \overrightarrow{r_2}(t)|$
- **Direction of motion**
- Given by $\dot{r}(t)$ or $\underset{\sim}{v}(t)$. IT IS THE VELOCITY!!!

Vectors & Vector Calc

- **Two objects crossing paths:**
- Share x, y points but not necessarily at the same t

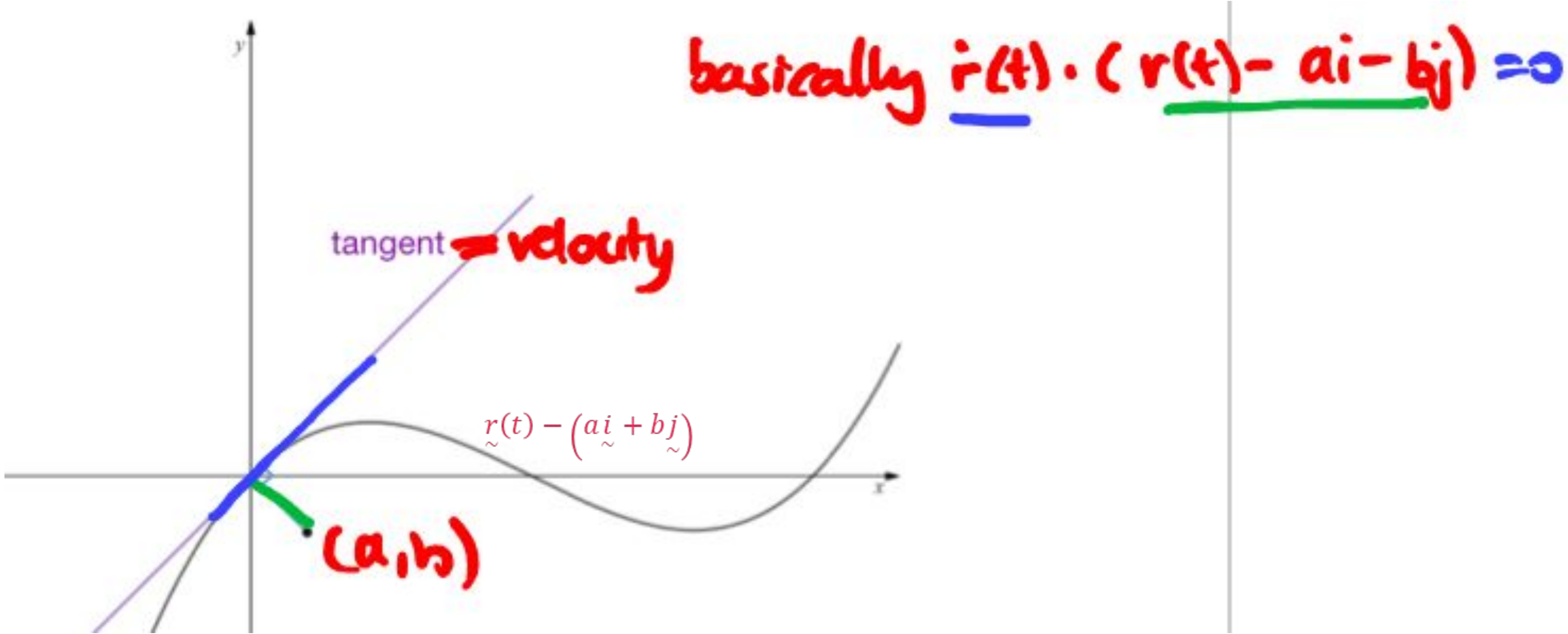
- **Two objects collide:**
- If and only if the simultaneous system have the SAME t
- $x_1(t) = x_2(t)$

$$y_1(t) = y_2(t)$$

Vectors & Vector Calc

SHORTEST DISTANCE

- To find the shortest distance between a point and curve, we want to be PERPENDICULAR to the curve to go STRAIGHT THERE.
- Remember vector property:
    - dot product of 2 vectors = 0 is perpendicular



basically $\underline{\dot{r}(t)} \cdot (\underline{r(t) - ai - bj}) = 0$

SHORTEST DISTANCE

-

- Steps:

1. Convert the point into a vector
2. Subtract from curve vector
3. Multiply against velocity curve vector
4. Let dot product = 0 and solve for t
5. Distance formula $|\overrightarrow{r_1}(t) - \overrightarrow{r_2}(t)|$

Vectors & Vector Calc

- Example) The velocity of a particle at time t seconds is given by
- $\dot{\underset{\sim}{r}} = (4t - 3)\underset{\sim}{i} + 2tj - 5\underset{\sim}{k}$ when components are measures in metres per second.
- Find the distance of the particle from the origin in metres when t=2 given that $\underset{\sim}{r}(0)=\underset{\sim}{i}-2\underset{\sim}{k}$ (VCAA - exam 1 – 2015)

$$\underset{\sim}{r} = (2t^2 - 3t)\underset{\sim}{i} + t^2 j - 5t\underset{\sim}{k} + c$$

$$\text{Let } t = 0,\ \underset{\sim}{r}(0) = \underset{\sim}{i} - 2\underset{\sim}{k}$$

$$\underset{\sim}{i} - 2\underset{\sim}{k} = (2(0^2) - 3 \times 0)\underset{\sim}{i} + 0^2 j - 5 \times 0 \times \underset{\sim}{k} +$$

$$\underset{\sim}{i} - 2\underset{\sim}{k} = (0)\underset{\sim}{i} + c \Rightarrow c = \underset{\sim}{i} - 2\underset{\sim}{k}$$

$$\underset{\sim}{r}(t) = (2t^2 - 3t)\underset{\sim}{i} + t^2 j - 5t\underset{\sim}{k} + \underset{\sim}{i} - 2\underset{\sim}{k}$$

$$\underset{\sim}{r}(t) = (2t^2 - 3t + 1)\underset{\sim}{i} + t^2 j + (-5t - 2)\underset{\sim}{k}$$

$$\underset{\sim}{r}(2) = (3)\underset{\sim}{i} + 4j + (-12)\underset{\sim}{k}$$

$$distance = \sqrt{(3)^2 + (4)^2 + (-12)^2} = \sqrt{169} = 13\ units$$

Logic & Proofs

Logic & Proof Language

Name	Definition	Symbol or denoted by:
Negation	Corresponds to the idea of complement in probability or “Not-P”	
Conjunction	The idea of intersection between two statements, we need both to be true	
Disjunction	The idea of union between statements, we need either to be true	
Premise	The statement assumed to be true	Premise Conclusion (Implication)
Conclusion	The statement concluded from the premise	
For all		All elements in set need to fulfill the condition
There exists		At least one element in the set needs to fulfill the condition
Example	Used to prove something true	
Counterexample	Used to disprove something	

Logic & Proof Language

- Negation, converse and contrapositive of implications
- Negation of implication: $\sim(P \Rightarrow\ Q) \Leftrightarrow\ P \wedge \sim Q$ (
- Contrapositive of implication: $P \Rightarrow\ Q \Leftrightarrow \sim Q\ \Rightarrow \sim P$
- Converse of implication of $P \Rightarrow\ Q$ becomes $Q \Rightarrow\ P$ (These two statements have no relation to one another, if we can prove the implication and the converse of the implication $P \Leftrightarrow Q$, which as otherwise known as equivalence (P if and only if Q, P is necessary and sufficient for Q)

- Negation of quantifiers: swap the symbol of $\forall$ *and* $\exists$ and negate the expression after the symbol

Proofing Techniques

- Direct Proof:
- Use all the fancy skills and formulas you have learnt in other topics like vectors and circular functions

Proof by cases:
Break into bite-size pieces, common examples:

- Prove NOT divisible
- Modulus functions

Proof by contradiction:

- Assume that the statement is false or negate the statement.
- Prove until a contradiction is reach at which we can conclude that the original assumption is false
- Now we can say the original statement is true

Proofing Techniques

- Proof by contrapositive
- Change the statement into its contrapositive statement (which is equivalent to the original statement)

Proof by mathematical induction:

- Prove the base case
- Assume the general case is true (n=k is true)
- Induction step = prove the statement works for n = k + m, where m is the step size between subsequent n values

Key Proofs Concepts

- Prove/disprove a number is irrational via contraction
- Be comfortable with using differentiation, integration, algebraic manipulation in induction questions
- Inequality proofs
- Factorials
- Trig proofs

Summary/Exam Tips

Exam 1:

- **Monday 11 November:** 9:00am □ 10:15am (40 marks)
- D-18

Exam 2:

- **Wednesday 13 November**: 11:45am □ 2:00pm (80 marks)
- D-20

Summary/Exam Tips

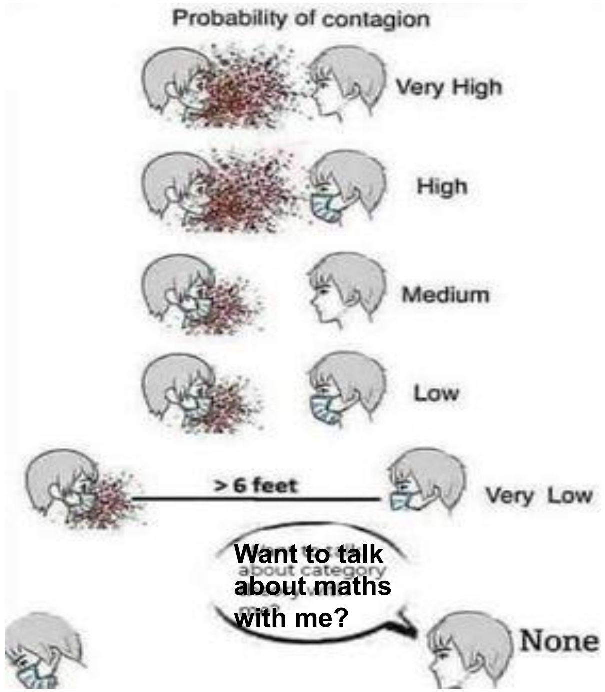
- Aim to finish content in your bound reference first
- Add to your bound reference as you go with practice exams and questions
- You bound reference needs to work for YOU not for anyone else
- Put in formula, calculator commands or mathematica code and content examples
- Prioritise information that is key to your success and motivating (reminders etc.)
- Do not have a huge bound reference, you will waste time trying to find things

Summary/Exam Tips

- PRACTISE EXAMS!!!!
  - My recommendation is MAV, Heffernan, NEAP, VCAA, VCAA NHT (2016-2024)
  - Do an intermixing between VCAA & company exams, ensure you finish all the VCAA available
- Try to get ur teacher to get SOME papers bc remember VCE is school vs school not peers vs peers. Help each other out! https://vcaa.vic.edu.au/assessment/vce-assessment/past-examinations/Pages/Index.aspx
- Correct it yourself HARSHLY and analyse your performance (what were the errors you made, how can you avoid doing that next time)

EXAM ADVICE - PREPARATION

- Go to sleep early every night – routine is important
- Make sure you are comfortable using your calculator. Have a play around with your calculator at some point and explore the different functionality. You may discover more functions on your calculator that come in use
    - Euler’s method
    - Solving DE
    - 3D model of vectors
- Try teaching someone else. Explaining a concept to someone in your own words can help to cement your understanding
- Don’t stress! A good mindset comes with good study scores!



EXAM ADVICE – EXAM 1

- Exam 1: (1 hour writing time + 15 mins reading time)
- Use reading time wisely!
- During reading time, go through each question and <u>form a plan of attack</u>. (vector calc and closest distance □ dot product being zero). Do this for as many questions as possible.
- You only have one hour to complete the exam so manage your time wisely!
- If you can’t figure out what to do in 10secs, move on. Come back to it later. ~30-45min you should have attempted every page

Summary/Exam Tips

EXAM ADVICE – EXAM 2

- Exam 2: (2 hours writing time + 15 mins reading time)
- Reading time
- Reading time!! USE IT TO READ EXTENDED RESPONSE
- DO extended response questions FIRST!!!! You can always guess MC if you run out of time. If worse comes to worse, damage control and get as many working/formula marks.
- USE YOUR CALC! Set up a new problem tab for EVERY ER and 1 page for MC for ease
- Each mark is ~90 sec so 30min for MC is good

Summary/Exam Tips

- Ultimately, you should do the exam in your own preferred way.
- This is why doing practice exams is valuable
- In doing many practice exams, you should develop a strategy that works for you.

EXAM ADVICE – IN THE EXAM

- UNDERLINE, DRAW, CHECK
- Underline IMPORTANT WORDS so you actively retain info. Look for clues (no. of marks, rounding, ‘hence’ etc.)
- Draw the qns visually if confused, could help
- Check to make sure you answered the qn, have units, correct form

- Save these tips in your BOUND REFERENCE
- Unless stated, give answers in exact form.
- Ensure your graphs and sketches are clear and no fraying

EXAM ADVICE – DURING EXAM SEASON

- In between exams, don’t be afraid to pursue a lot of leisure tasks.
- Spend time with family, read a book, exercise, talk with friends. Destress!!!
- Always make sure each night you are going over revision whether that be in the form of an exam analysis doc, ‘what to remember’ list etc. I personally had a mistake log that I print out once a week to redo.

Key skills

- Statistics – Hypothesis testing + probability
- Complex Numbers
- Circular Functions
- Vectors
- Kinematics and Vector Calculus

Reminders

- Try to study smarter – study in blocks
- Good Luck!

Questions?

Please leave us a review

SCAN

Thanks so much!

ATARNotes