Chapter 1 – Algebra I

Solutions to Exercise 1A

1 **a** Add indices:
$x^3 \times x^4 \times x^{3+4} = x^7$

b Add indices:
$a^5 \times a^{-3} = a^{5+-3} = a^2$

c Add indices:
$x^2 \times x^{-1} \times x^2 = x^{2+-1+2} = x^3$

d Subtract indices:
$\frac{y^3}{y^7} = y^{3-7} = y^{-4}$

e Subtract indices:
$\frac{x^8}{x^{-4}} = x^{8-(-4)} = x^{12}$

f Subtract indices:
$\frac{p^{-5}}{p^2} = p^{-5-2} = p^{-7}$

g Subtract indices:
$a^{\frac{1}{2}} \div a^{\frac{2}{3}} = a^{\frac{3}{6}-\frac{4}{6}} = a^{-\frac{1}{6}}$

h Multiply indices:
$(a^{-2})^4 = a^{-2\times 4} = a^{-8}$

i Multiply indices:
$(y^{-2})^{-7} = y^{-2\times(-7)} = y^{14}$

j Multiply indices:
$(x^5)^3 = x^{5\times 3} = x^{15}$

k Multiply indices:
$(a^{-20})^{\frac{3}{5}} = a^{-20\times\frac{3}{5}} = a^{-12}$

l Multiply indices:
$\left(x^{-\frac{1}{2}}\right)^{-4} = x^{-\frac{1}{2}\times -4} = x^2$

m Multiply indices:
$(n^{10})^{\frac{1}{5}} = n^{10\times\frac{1}{5}} = n^2$

n Multiply the coefficients and add the indices:
$2x^{\frac{1}{2}} \times 4x^3 = (2 \times 4)x^{\frac{1}{2}+3} = 8x^{\frac{7}{2}}$

o Multiply the first two indices and add the third:

$$\begin{aligned}(a^2)^{\frac{5}{2}} \times a^{-4} &= a^{2\times\frac{5}{2}} \times a^{-4}\\ &= a^{5+(-4)}\\ &= a^1 = a\end{aligned}$$

p $\frac{1}{x^{-4}} = x^{1\div\frac{1}{4}} = x^4$

q

$$\begin{aligned}\left(2n^{-\frac{2}{5}}\right)^5 \div (4^3n^4) &= 2^5n^{-\frac{2}{5}\times 5} \div ((2^2)^3n^4)\\ &= 2^5n^{-2} \div (2^6n^4)\\ &= 2^{5-6}n^{-2-4}\\ &= 2^{-1}n^{-6} = \frac{1}{2n^6}\end{aligned}$$

r Multiply the coefficients and add the indices.

$$\begin{aligned}&x^3 \times 2x^{\frac{1}{2}} \times -4x^{-\frac{3}{2}}\\ &= (1 \times 2 \times -4)x^{3+\frac{1}{2}+\left(-\frac{3}{2}\right)}\\ &= -8x^2\end{aligned}$$

s

$$\begin{aligned}&(ab^3)^2 \times a^{-2}b^{-4} \times \frac{1}{a^2b^{-3}}\\ &= a^2b^6 \times a^{-2}b^{-4} \times a^{-2}b^3\\ &= a^{2+-2+-2}b^{6+-4+3}\\ &= a^{-2}b^5\end{aligned}$$

t $(2^2p^{-3} \times 4^3p^5 \div ((6p^{-3}))^0 = 1$
Anything to the power zero is 1.

2 **a** $25^{\frac{1}{2}} = \sqrt{25} = 5$

b $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

c $\left(\frac{16}{9}\right)^{\frac{1}{2}} = \frac{16^{\frac{1}{2}}}{9^{\frac{1}{2}}}$

$= \frac{\sqrt{16}}{\sqrt{9}} = \frac{4}{3}$

d $16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}}$

$= \frac{1}{\sqrt{16}} = \frac{1}{4}$

e $\left(\frac{49}{36}\right)^{-\frac{1}{2}} = \frac{1}{\left(\frac{49}{36}\right)^{\frac{1}{2}}}$

$= \frac{1}{\frac{\sqrt{49}}{\sqrt{36}}}$

$= \frac{\sqrt{36}}{\sqrt{49}} = \frac{6}{7}$

f $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

g $144^{\frac{1}{2}} = \sqrt{144} = 12$

h $64^{\frac{2}{3}} = \left(64^{\frac{1}{3}}\right)^2 = 4^2 = 16$

i $9^{\frac{3}{2}} = \left(9^{\frac{1}{2}}\right)^3$

$= 3^3 = 27$

j $\left(\frac{81}{16}\right)^{\frac{1}{4}} = \frac{81^{\frac{1}{4}}}{16^{\frac{1}{4}}}$

$= \frac{3}{2}$

k $\left(\frac{23}{5}\right)^0 = 1$

l $128^{\frac{3}{7}} = \left(128^{\frac{1}{7}}\right)^3$

$= 2^3 = 8$

3 **a** $4.35^2 = 18.9225 \approx 18.92$

b $2.4^5 = 79.62624 \approx 79.63$

c $\sqrt{34.6921} = 5.89$

d $0.02^{-3} = 125\,000$

e $\sqrt[3]{0.729} = 0.9$

f $\sqrt[4]{2.3045} = 1.23209\ldots \approx 1.23$

g $(345.64)^{-\frac{1}{3}} = 0.14249\ldots \approx 0.14$

h $(4.558)^{\frac{2}{5}} = 1.83607\ldots \approx 1.84$

i $\frac{1}{(0.064)^{-\frac{1}{3}}} = (0.064)^{\frac{1}{3}} = 0.4$

4 **a** $\frac{a^2b^3}{a^{-2}b^{-4}} = a^{2--2}b^{3--4}$

$= a^4b^7$

b $\frac{2a^2(2b)^3}{(2a)^{-2}b^{-4}} = \frac{2a^2 \times 2^3b^3}{2^{-2}a^{-2}b^{-4}}$

$= \frac{2^4a^2b^3}{2^{-2}a^{-2}b^{-4}}$

$= 2^{4--2}\,a^{2--2}b^{3--4}$

$= 2^6a^4b^7 = 64a^4b^7$

c $\frac{a^{-2}b^{-3}}{a^{-2}b^{-4}} = a^{-2--2}b^{-3--4}$

$= a^0b^1 = b$

d $\dfrac{a^2b^3}{a^{-2}b^{-4}} \times \dfrac{ab}{a^{-1}b^{-1}}$

$= \dfrac{a^{2+1}b^{3+1}}{a^{-2+-1}b^{-4+-1}}$

$= \dfrac{a^3b^4}{a^{-3}b^{-5}}$

$= a^{3--3}b^{4--5} = a^6b^9$

e $\dfrac{(2a)^2 \times 8b^3}{16a^{-2}b^{-4}} = \dfrac{4a^2 \times 8b^3}{16a^{-2}b^{-4}}$

$= \dfrac{32a^2b^3}{16a^{-2}b^{-4}}$

$= \dfrac{32}{16}a^{2--2}b^{3--4}$

$= 2a^4b^7$

f $\dfrac{2a^2b^3}{8a^{-2}b^{-4}} \div \dfrac{16ab}{(2a)^{-1}b^{-1}}$

$= \dfrac{2a^2b^3}{8a^{-2}b^{-4}} \times \dfrac{(2a)^{-1}b^{-1}}{16ab}$

$= \dfrac{2a^2b^3}{8a^{-2}b^{-4}} \times \dfrac{2^{-1}a^{-1}b^{-1}}{16ab}$

$= \dfrac{2^{1+-1}a^{2+-1}b^{3+-1}}{8 \times 16 \times a^{-2+1}b^{-4+1}}$

$= \dfrac{2^0a^1b^2}{128a^{-1}b^{-3}}$

$= \dfrac{1}{128}a^{1--1}b^{2--3} = \dfrac{a^2b^5}{128}$

5 $\dfrac{2^n \times 8^n}{2^{2n} \times 16} = \dfrac{2^n \times (2^3)^n}{2^{2n} \times 2^4}$

$= \dfrac{2^n \times 2^{3n}}{2^{2n} \times 2^4}$

$= \dfrac{2^{n+3n-2n}}{2^4}$

$= 2^{2n} \times 2^{-4}$

$= 2^{2n-4}$

6 $2^{-x} \times 3^{-x} \times 6^{2x} \times 3^{2x} \times 2^{2x}$

$= (2 \times 3)^{-x} \times 6^{2x} \times (2 \times 3)^{2x}$

$= 6^{-x} \times 6^{2x} \times 6^{2x}$

$= 6^{-x+2x+2x}$

$= 6^{3x}$

7 In each case, add the fractional indices.

a $2^{\frac{1}{3}} \times 2^{\frac{1}{6}} \times 2^{-\frac{2}{3}} = 2^{\frac{2}{6}+\frac{1}{6}+-\frac{4}{6}}$

$= 2^{-\frac{1}{6}} = \left(\dfrac{1}{2}\right)^{\frac{1}{6}}$

b $a^{\frac{1}{4}} \times a^{\frac{2}{5}} \times a^{-\frac{1}{10}} = a^{\frac{5}{20}+\frac{8}{20}+-\frac{2}{20}}$

$= a^{\frac{11}{20}}$

c $2^{\frac{2}{3}} \times 2^{\frac{5}{6}} \times 2^{-\frac{2}{3}} = 2^{\frac{4}{6}+\frac{5}{6}+-\frac{4}{6}}$

$= 2^{\frac{5}{6}}$

d $\left(2^{\frac{1}{3}}\right)^2 \times \left(2^{\frac{1}{2}}\right)^5 = 2^{\frac{2}{3}} \times 2^{\frac{5}{2}}$

$= 2^{\frac{4}{6}+\frac{15}{6}} = 2^{\frac{19}{6}}$

e $\left(2^{\frac{1}{3}}\right)^2 \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}} = 2^{\frac{2}{3}} \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}}$

$= 2^{\frac{2}{3}+\frac{1}{3}+-\frac{2}{5}} = 2^{\frac{3}{5}}$

8 **a**

$\sqrt[3]{a^3b^2} \div \sqrt[3]{a^2b^{-1}} = (a^3b^2)^{\frac{1}{3}} \div (a^2b^{-1})^{\frac{1}{3}}$

$= a^1b^{\frac{2}{3}} \div a^{\frac{2}{3}}b^{-\frac{1}{3}}$

$= a^{1-\frac{2}{3}}b^{\frac{2}{3}--\frac{1}{3}} = a^{\frac{1}{3}}b$

b $\sqrt{a^3b^2} \times \sqrt{a^2b^{-1}}$

$$= (a^3b^2)^{\frac{1}{2}} \times (a^2b^{-1})^{\frac{1}{2}}$$

$$= a^{\frac{3}{2}}b^1 \times a^1b^{-\frac{1}{2}}$$

$$= a^{\frac{3}{2}+1}b^{1+-\frac{1}{2}} = a^{\frac{5}{2}}b^{\frac{1}{2}}$$

c

$$\sqrt[5]{a^3b^2} \times \sqrt[5]{a^2b^{-1}} = (a^3b^2)^{\frac{1}{5}} \times (a^2b^{-1})^{\frac{1}{5}}$$

$$= a^{\frac{3}{5}}b^{\frac{2}{5}} \times a^{\frac{2}{5}}b^{-\frac{1}{5}}$$

$$= a^{\frac{3}{5}+\frac{2}{5}}b^{\frac{2}{5}+-\frac{1}{5}} = ab^{\frac{1}{5}}$$

d $\sqrt{a^{-4}b^2} \times \sqrt{a^3b^{-1}}$

$$= (a^{-4}b^2)^{\frac{1}{2}} \times (a^3b^{-1})^{\frac{1}{2}}$$

$$= a^{-2}b^1 \times a^{\frac{3}{2}}b^{-\frac{1}{2}}$$

$$= a^{-2+\frac{3}{2}}b^{1+-\frac{1}{2}}$$

$$= a^{-\frac{1}{2}}b^{\frac{1}{2}}$$

$$= \frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}} = \left(\frac{b}{a}\right)^{\frac{1}{2}}$$

e $\sqrt{a^3b^2c^{-3}} \times \sqrt{a^2b^{-1}c^{-5}}$

$$= (a^3b^2c^{-3})^{\frac{1}{2}} \times (a^2b^{-1}c^{-5})^{\frac{1}{2}}$$

$$= a^{\frac{3}{2}}b^1c^{-\frac{3}{2}} \times a^1b^{-\frac{1}{2}}\,c^{-\frac{5}{2}}$$

$$= a^{\frac{3}{2}+1}b^{1+-\frac{1}{2}}c^{-\frac{3}{2}+-\frac{5}{2}}$$

$$= a^{\frac{5}{2}}b^{\frac{1}{2}}c^{-4}$$

f $\sqrt[5]{a^3b^2} \div \sqrt[5]{a^2b^{-1}}$

$$= (a^3b^2)^{\frac{1}{5}} \div (a^2b^{-1})^{\frac{1}{5}}$$

$$= a^{\frac{3}{5}}b^{\frac{2}{5}} \div a^{\frac{2}{5}}b^{-\frac{1}{5}}$$

$$= a^{\frac{3}{5}-\frac{2}{5}}b^{\frac{2}{5}--\frac{1}{5}} = a^{\frac{1}{5}}b^{\frac{3}{5}}$$

g

$$\frac{\sqrt{a^3b^2}}{a^2b^{-1}c^{-5}} \times \frac{\sqrt{a^{-4}b^2}}{a^3b^{-1}} \times \sqrt{a^3b^{-1}}$$

$$= \frac{(a^3b^2)^{\frac{1}{2}}}{a^2b^{-1}c^{-5}} \times \frac{(a^{-4}b^2)^{\frac{1}{2}}}{a^3b^{-1}} \times (a^3b^{-1})^{\frac{1}{2}}$$

$$= \frac{a^{\frac{3}{2}}b^1}{a^2b^{-1}c^{-5}} \times \frac{a^{-2}b^1}{a^3b^{-1}} \times a^{\frac{3}{2}}b^{-\frac{1}{2}}$$

$$= a^{\frac{3}{2}-2}b^{1--1}c^{0--5} \times a^{-2-3}b^{1--1}$$

$$\times\, a^{\frac{3}{2}}b^{-\frac{1}{2}}$$

$$= a^{-\frac{1}{2}}b^2c^5 \times a^{-5}b^2 \times a^{\frac{3}{2}}b^{-\frac{1}{2}}$$

$$= a^{-\frac{1}{2}+-5+\frac{3}{2}}b^{2+2+-\frac{1}{2}}c^5$$

$$= a^{-4}b^{\frac{7}{2}}c^5$$

Solutions to Exercise 1B

1 a $47.8 = 4.78 \times 10^1 = 4.78 \times 10$

b $6728 = 6.728 \times 10^3$

c $79.23 = 7.923 \times 10^1 = 7.923 \times 10$

d $43\,580 = 4.358 \times 10^4$

e $0.0023 = 2.3 \times 10^{-3}$

f $0.000\,000\,56 = 5.6 \times 10^{-7}$

g $12.000\,34 = 1.2000\,34 \times 10^1$

$= 1.2000\,34 \times 10$

h Fifty million $= 50\,000\,000$

$= 5.0 \times 10^7$

i $23\,000\,000\,000 = 2.3 \times 10^{10}$

j $0.000\,000\,0013 = 1.3 \times 10^{-9}$

k 165 thousand $= 165\,000$

$= 1.65 \times 10^5$

l $0.000\,014\,567 = 1.4567 \times 10^{-5}$

2 a The decimal point moves 8 places to the right $= 1.0 \times 10^{-8}$

b The decimal point moves 24 places to the right $= 1.67 \times 10^{-23}$

c The decimal point moves 5 places to the right $= 5.0 \times 10^{-5}$

d The decimal point moves 3 places to the left $= 1.853\,18 \times 10^3$

e The decimal point moves 12 places to the left $= 9.461 \times 10^{12}$

f The decimal point moves 10 places to the right $= 2.998 \times 10^{10}$

3 a The decimal point move 13 places to the right $= 81\,280\,000\,000\,000$

b The decimal point move 8 places to the right $= 270\,000\,000$

c The decimal point move 13 places to the left $= 0.000\,000\,000\,000\,28$

4 a $456.89 \approx 4.569 \times 10^2$
(4 significant figures)

b $34567.23 \approx 3.5 \times 10^4$
(2 significant figures)

c $5679.087 \approx 5.6791 \times 10^3$
(5 significant figures)

d $0.04536 \approx 4.5 \times 10^{-2}$
(2 significant figures)

e $0.09045 \approx 9.0 \times 10^{-2}$
(2 significant figures)

f $4568.234 \approx 4.5682 \times 10^3$
(5 significant figures)

5 a $\dfrac{324\,000 \times 0.000\,000\,7}{4000}$

$$= \frac{3.24 \times 10^5 \times 7 \times 10^{-7}}{4 \times 10^3}$$

$$= \frac{3.24 \times 7}{4} \times 10^{5+-7-3}$$

$$= 5.67 \times 10^{-5}$$

$$= 0.0000567$$

b $\dfrac{5\,240\,000 \times 0.8}{42\,000\,000}$

$$= \frac{5.24 \times 10^6 \times 8 \times 10^{-1}}{4.2 \times 10^7}$$

$$= \frac{41.92 \times 10^5}{4.2 \times 10^7}$$

$$= \frac{4192 \times 10^3}{42\,000 \times 10^3}$$

$$= \frac{4192}{42\,000} = \frac{262}{2625}$$

6 **a** $\dfrac{\sqrt[3]{a}}{b^4} = \dfrac{\sqrt[3]{2 \times 10^9}}{3.215^4}$

$$= \frac{\sqrt[3]{2} \times \sqrt[3]{10^9}}{106.8375\ldots}$$

$$= \frac{1.2599\ldots \times 10^3}{106.8375\ldots}$$

$$= 0.011\,792\ \ldots \times 10^3 \approx 11.8$$

b

$$\frac{\sqrt[4]{a}}{4b^4} = \frac{\sqrt[4]{2 \times 10^{12}}}{4 \times 0.05^4}$$

$$= \frac{\sqrt[4]{2} \times \sqrt[4]{10^{12}}}{4 \times 0.000\,006\,25}$$

$$= \frac{1.189\,2\ldots \times 10^3}{4 \times 6.25 \times 10^{-6}}$$

$$= 0.047\,568\ldots \times 10^9 \approx 4.76 \times 10^7$$

Solutions to Exercise 1C

1 a $3x + 7 = 15$

$$3x = 15 - 7$$
$$= 8$$
$$x = \frac{8}{3}$$

b $8 - \frac{x}{2} = -16$

$$-\frac{x}{2} = -16 - 8$$
$$= -24$$
$$-\frac{x}{2} \times -2 = -24 \times -2$$
$$x = 48$$

c $42 + 3x = 22$

$$3x = 22 - 42$$
$$= -20$$
$$x = -\frac{20}{3}$$

d $\frac{2x}{3} - 15 = 27$

$$\frac{2x}{3} = 27 + 15$$
$$= 42$$
$$\frac{2x}{3} \times \frac{3}{2} = 42 \times \frac{3}{2}$$
$$x = 63$$

e $5(2x + 4) = 13$

$$10x + 20 = 13$$
$$10x = 13 - 20$$
$$= -7$$
$$x = -\frac{7}{10} = -0.7$$

f $-3(4 - 5x) = 24$

$$-12 + 15x = 24$$
$$15x = 24 + 12$$
$$= 36$$
$$x = \frac{36}{15}$$
$$= \frac{12}{5} = 2.4$$

g $3x + 5 = 8 - 7x$

$$3x + 7x = 8 - 5$$
$$10x = 3$$
$$x = \frac{3}{10} = 0.3$$

h $2 + 3(x - 4) = 4(2x + 5)$

$$2 + 3x - 12 = 8x + 20$$
$$3x - 10 = 8x + 20$$
$$3x - 8x = 20 + 10$$
$$-5x = 30$$
$$x = \frac{30}{-5} = -6$$

i $\frac{2x}{5} - \frac{3}{4} = 5x$

$$\frac{2x}{5} \times 20 - \frac{3}{4} \times 20 = 5x \times 20$$
$$8x - 15 = 100x$$
$$8x - 100x = 15$$
$$-92x = 15$$
$$x = -\frac{15}{92}$$

j

$$6x + 4 = \frac{x}{3} - 3$$
$$6x \times 3 + 4 \times 3 = \frac{x}{3} \times 3 - 3 \times 3$$
$$18\ x + 12 = x - 9$$
$$18\ x - x = -9 - 12$$
$$17x = -21$$
$$x = -\frac{21}{17}$$

2 a

$$\frac{x}{2} + \frac{2x}{5} = 16$$
$$\frac{x}{2} \times 10 + \frac{2x}{5} \times 10 = 16 \times 10$$
$$5x + 4x = 160$$
$$9x = 160$$
$$x = \frac{160}{9}$$

b

$$\frac{3x}{4} - \frac{x}{3} = 8$$
$$\frac{3x}{4} \times 12 - \frac{x}{3} \times 12 = 8 \times 12$$
$$9x - 4x = 96$$
$$5x = 96$$
$$x = \frac{96}{5} = 19.2$$

c

$$\frac{3x - 2}{2} + \frac{x}{4} = -8$$
$$\frac{3x - 2}{2} \times 4 + \frac{x}{4} \times 4 = -8 \times 4$$
$$2(3x - 2) + x = -32$$
$$6x - 4 + x = -32$$
$$7x = -32 + 4$$
$$= -28$$
$$x = -4$$

d

$$\frac{5x}{4} - \frac{4}{3} = \frac{2x}{5}$$
$$\frac{5x}{4} \times 60 - \frac{4}{3} \times 60 = \frac{2x}{5} \times 60$$
$$75x - 80 = 24x$$
$$75x - 24x = 80$$
$$51x = 80$$
$$x = \frac{80}{51}$$

e

$$\frac{x - 4}{2} + \frac{2x + 5}{4} = 6$$
$$\frac{x - 4}{2} \times 4 + \frac{2x + 5}{4} \times 4 = 6 \times 4$$
$$2(x - 4) + (2x + 5) = 24$$
$$2x - 8 + 2x + 5 = 24$$
$$4x = 24 + 8 - 5$$
$$= 27$$
$$x = \frac{27}{4} = 6.75$$

f

$$\frac{3 - 3x}{10} - \frac{2(x + 5)}{6} = \frac{1}{20}$$
$$\frac{3 - 3x}{10} \times 60 - \frac{2(x + 5)}{6} \times 60 = \frac{1}{20} \times 60$$
$$6(3 - 3x) - 20(x + 5) = 3$$
$$18 - 18x - 20x - 100 = 3$$
$$-38x = 3 - 18 + 100$$
$$= 85$$
$$x = -\frac{85}{38}$$

g
$$\frac{3-x}{4} - \frac{2(x+1)}{5} = -24$$
$$\frac{3-x}{4} \times 20 - \frac{2(x+1)}{5} \times 20 = -24 \times 20$$
$$5(3-x) - 8(x+1) = -480$$
$$15 - 5x - 8x - 8 = -480$$
$$-13x = -480 - 15 + 8$$
$$= -487$$
$$x = \frac{487}{13}$$

h
$$\frac{-2(5-x)}{8} + \frac{6}{7} = \frac{4(x-2)}{3}$$
$$\frac{-2(5-x)}{8} \times 168 + \frac{6}{7} \times 168 = \frac{4(x-2)}{3} \times 168$$
$$-42(5-x) + 144 = 224(x-2)$$
$$-210 + 42x + 144 = 224x - 448$$
$$42x - 224x = -448 + 210 - 144$$
$$-182x = -382$$
$$x = \frac{382}{182} = \frac{191}{91}$$

3 **a** $3x + 2y = 2;\ 2x - 3y = 6$

Use elimination. Multiply the first equation by 3 and the second equation by 2.

$9x + 6y = 6$ ①

$4x - 6y = 12$ ②

① + ②:
$$13x = 18$$
$$x = \frac{18}{13}$$

Substitute into the first equation:
$$3 \times \frac{18}{13} + 2y = 2$$
$$\frac{54}{13} + 2y = 2$$
$$2y = 2 - \frac{54}{13}$$
$$= -\frac{28}{13}$$
$$y = -\frac{14}{13}$$

b $5x + 2y = 4;\ 3x - y = 6$

Use elimination. Multiply the second equation by 2.

$5x + 2y = 4$ ①

$6x - 2y = 12$ ②

① + ②:
$$11x = 16$$
$$x = \frac{16}{11}$$

Substitute into the second, simpler equation:
$$3 \times \frac{16}{11} - y = 6$$
$$\frac{48}{11} - y = 6$$
$$-y = 6 - \frac{48}{11}$$
$$y = \frac{18}{11}$$

c $2x - y = 7;\ 3x - 2y = 2$

Use substitution. Make y the subject of the first equation.

$y = 2x - 7$

Substitute into the second equation:
$$3x - 2(2x - 7) = 2$$
$$3x - 4x + 14 = 2$$
$$-x = 2 - 14$$
$$x = 12$$

Substitute into the equation in which y is the subject:
$$y = 2 \times 12 - 7$$
$$= 17$$

d $x + 2y = 12;\ x - 3y = 2$

Use substitution. Make x the subject of the first equation.

$x = 12 - 2y$

Substitute into the second equation:

$$12 - 2y - 3y = 2$$
$$-5y = 2 - 12$$
$$= -10$$
$$y = 2$$

Substitute into the first equation:

$$x + 2 \times 2 = 12$$
$$x + 4 = 12$$
$$x = 8$$

e $7x - 3y = -6; x + 5y = 10$

Use substitution. Make x the subject of the second equation.

$x = 10 - 5y$

Substitute into the first equation:

$$7(10 - 5y) - 3y = -6$$
$$70 - 35y - 3y = -6$$
$$-38y = -6 - 70$$
$$= -76$$
$$y = \frac{-76}{-38} = 2$$

Substitute into the second equation:

$$x + 5 \times 2 = 10$$
$$x + 10 = 10$$
$$x = 0$$

f $15x + 2y = 27; 3x + 7y = 45$

Use elimination. Multiply the second equation by 5.

$$15x + 2y = 27 \quad ①$$
$$15x + 35y = 225 \quad ②$$

① − ②:

$$-33y = -198$$
$$y = \frac{-198}{-33} = 6$$

Substitute into the second equation:

$$3x + 7 \times 6 = 45$$
$$3x + 42 = 45$$
$$3x = 45 - 42$$
$$= 3$$
$$x = 1$$

Solutions to Exercise 1D

1 **a** $4(x-2) = 60$

$4x - 8 = 60$

$4x = 60 + 8$

$= 68$

$x = 17$

b The length of the square is $\frac{2x+7}{4}$.

$\left(\frac{2x+7}{4}\right)^2 = 49$

$\frac{2x+7}{4} = 7$

$2x + 7 = 7 \times 4 = 28$

$2x = 28 - 7 = 21$

$x = 10.5$

c The equation is length = twice width.

$x - 5 = 2(12 - x)$

$x - 5 = 24 - 2x$

$x + 2x = 24 + 5$

$3x = 29$

$x = \frac{29}{3}$

d $y = 2((2x+1) + (x-3))$

$= 2(2x + 1 + x - 3)$

$= 2(3x - 2)$

$= 6x - 4$

e $Q = np$

f If a 10% service charge is added, the total price will be multiplied by 110%, or 1.1.

$R = 1.1pS$

g Using the fact that there are 12 lots of 5 min in an hour ($60 \div 12 = 5$),

$\frac{60n}{5} = 2400$

h $a = \text{circumference} \times \frac{60}{360}$

$= 2\pi(x+3) \times \frac{60}{360}$

$= 2\pi(x+3) \times \frac{1}{6}$

$= \frac{\pi}{3}(x+3)$

2 Let the value of Bronwyn's sales in the first week be $\$s$.

$s + (s + 500) + (s + 1000)$

$+ (s + 1500) + (s + 2000)$

$= 17500$

$5s + 5000 = 17500$

$5s = 12500$

$s = 2500$

The value of her first week's sales is $2500.

3 Let d be the number of dresses bought and h the number of handbags bought.

$65d + 26h = 598$

$d + h = 11$

Multiply the second equation by 26 (the smaller number).

$65d + 26h = 598$ ①

$26d + 26h = 286$ ②

① − ②:

$$39d = 312$$

$$d = \frac{312}{39} = 8$$

$$h + 8 = 11$$

$$h = 3$$

Eight dresses and three handbags.

4 Let the courtyard's width be w metres.

$$3w + w + 3w + w = 67$$

$$8w = 67$$

$$w = 8.375$$

The width is 8.375 m.

The length = 3 × 8.375 = 25.125 m.

5 Let p be the full price of a case of wine. The merchant will pay 60% (0.6) on the 25 discounted cases.

$$25p + 25 \times 0.6p = 2260$$

$$25p + 15p = 2260$$

$$40p = 2260$$

$$p = 56.5$$

The full price of a case is $56.50.

6 Let x be the number of houses with an $11 500 commission and y be the number of houses with a $13 000 commission.

We only need to find x.

$x + y = 22$

$11\,500x + 13\,000y = 272\,500$

To simplify the second equation, divide both sides by 500.

$23x + 26y = 545$

Using the substitution method:

$$23x + 26y = 545$$

$$y = 22 - x$$

$$23x + 26(22 - x) = 545$$

$$23x + 572 - 26x = 545$$

$$-3x = 545 - 572$$

$$= -27$$

$$x = 9$$

He sells nine houses with an $11 500 commission.

7 It is easiest to let the third boy have m marbles, in which case the second boy will have $2m$ marbles and the first boy will have $2m - 14$.

$$(2m - 14) + 2m + m = 71$$

$$5m - 14 = 71$$

$$5m = 85$$

$$m = 17$$

The first boy has 20 marbles, the second boy has 34 and the third boy has 17 marbles, for a total of 71.

8 Let Belinda's score be b.

Annie's score will be 110% of Belinda's or $1.1b$.

Cassie's will be 60% of their combined scores:

$$0.6(1.1b + b) = 0.6 \times 2.1b$$

$$= 1.26b$$

$$1.1b + b + 1.26b = 504$$

$$3.36b = 504$$

$$b = \frac{5.04}{3.36}$$

$$= 150$$

Belinda scores 150

Annie scores $1.1 \times 150 = 165$

Cassie scores $0.6 \times (150 + 165) = 189$

9 Let r km/h be the speed Kim can run. Her cycling speed will be $(r + 30)$ km/h. Her time cycling will be $48 + 48 \div 3 = 64$ min. Converting the times to hours ($\div 60$) and using distance = speed × time gives the following equation:

$$r \times \frac{48}{60} + (r + 30) \times \frac{64}{60} = 60$$

$$48r + 64(r + 30) = 60 \times 60$$

$$48r + 64r + 1920 = 3600$$

$$112r + 1920 = 3600$$

$$112r = 1680$$

$$r = \frac{1680}{112} = 15$$

She can run at 15 km/h

10 Let c g be the mass of a carbon atom and x g be the mass of an oxygen atom. (o is too confusing a symbol to use)

$$2c + 6x = 2.45 \times 10^{-22}$$

$$x = \frac{c}{3}$$

Use substitution.

$$2c + 6 \times \frac{c}{3} = 2.45 \times 10^{-22}$$

$$2c + 2c = 2.45 \times 10^{-22}$$

$$4c = 2.45 \times 10^{-22}$$

$$c = \frac{2.45 \times 10^{-22}}{4}$$

$$= 6.125 \times 10^{-23}$$

$$x = \frac{c}{3}$$

$$= \frac{6.125 \times 10^{-23}}{3}$$

$$\approx 2.04 \times 10^{-23}$$

The mass of an oxygen atom is 2.04×10^{-23} g.

11 Let x be the number of pearls.

$$\frac{x}{6} + \frac{x}{3} + \frac{x}{5} + 9 = x$$

$$\frac{5x + 10x + 6x}{30} + 9 = x$$

$$21x + 270 = 30x$$

$$7x + 90 = 10x$$

$$3x = 90$$

$$x = 30$$

There are 30 pearls.

12 Let the oldest receive \$$x$.

The middle child receives \$$(x - 12)$.

The youngest child receives \$$\left(\frac{x - 12}{3}\right)$

$$x + x - 12 + \frac{x - 12}{3} = 96$$

$$2x - 12 + \frac{x - 12}{3} = 96$$

$$2x - 12 + \frac{x}{3} = 100$$

$$6x - 36 + x = 300$$

$$7x = 336$$

$$x = 48$$

Oldest \$48, Middle \$35, Youngest \$12

13 Let S be the sum of her marks on the first four tests.

Then $\frac{S}{4} = 88$

$\therefore S = 352$

Let x be her mark on the fifth test.

$\frac{S + x}{5} = 90$

$352 + x = 450$

$x = 98$

Her mark on the fifth test has to be 98%

14 Let N be the number of students in the class.

$0.72N$ students have black hair

After 5 leave the class there are $0.72N - 5$ students with black hair.

There are now $N - 5$ students in the class.

Hence $\frac{0.72N - 5}{N - 5} = 0.65$

$$\therefore 0.72N - 5 = 0.65(N - 5)$$

$$\therefore 0.72N = 0.65N + 1.75$$

$$\therefore 0.07N = 1.75$$

$$7N = 175$$

$$N = 25$$

There were originally 25 students

15 Amount of water in tank A at time t minutes $= 100 - 2t$

Amount of water in tank B at time t minutes $= 120 - 3t$

$$100 - 2t = 120 - 3t$$

$$t = 20$$

After 20 minutes the amount of water in the tanks will be the same.

16 Height candle A at t minutes $= 10 - 5t$

Height of candle B at t minutes $= 8 - 2t$

a $10 - 5t = 8 - 2t$

$$3t = 2$$

$$t = \frac{2}{3}$$

$\therefore$ equal after 40 minutes.

b $10 - 5t = \frac{1}{2}(8 - 2t)$

$$10 - 5t = 4 - t$$

$$4t = 6$$

$$t = \frac{3}{2}$$

$\therefore$ half the height after 90 minutes.

c $10 - 5t = 8 - 2t + 1$

$$10 - 5t = 9 - 2t$$

$$3t = 1$$

$$t = \frac{1}{3}$$

$\therefore$ one centimetre more after 20 minutes.

17 Let t be the time the motorist drove at 100 km/h

$$100t + 90\left(\frac{10}{3} - t\right) = 320$$

$$100t + 300 - 90t = 320$$

$$10t = 20$$

$$t = 2$$

Therefore the motorist travelled 200 km at 100 km/h

18 Let v km/h be Jarmila's usual speed

Therefore distance travelled= $\frac{14v}{3}$ km

$v + 3$ is the new speed and it takes $\frac{13}{3}$ hours.

$$\therefore \frac{13}{3}(v + 3) = \frac{14v}{3}$$

$$13(v + 3) = 14v$$

$$v = 39$$

Her usual speed is 39 km/h

Solutions to Exercise 1E

1 Let k be the number of kilometres travelled in a day. The unlimited kilometre alternative will become more attractive when $0.32k + 63 > 108$.
Solve for $0.32k + 63 = 108$:

$$\begin{aligned} 0.32k &= 108 - 63 \\ &= 45 \\ k &= \frac{45}{0.32} = 140.625 \end{aligned}$$

The unlimited kilometre alternative will become more attractive when you travel more than 140.625 km.

2 Let g be the number of guests. Solve for the equality.

$$\begin{aligned} 300 + 43g &= 450 + 40g \\ 43g - 40g &= 450 - 300 \\ 3g &= 150 \\ g &= 50 \end{aligned}$$

Company A is cheaper when there are more than 50 guests.

3 Let a be the number of adults and c the number of children.

$$\begin{aligned} 45a + 15c &= 525\,000 \\ a + c &= 15\,000 \end{aligned}$$

Multiply the second equation by 15.

$$\begin{aligned} 45a + 15c &= 525\,000 \quad (1) \\ 15a + 15c &= 225\,000 \quad (2) \end{aligned}$$

(1) − (2):

$$\begin{aligned} 30a &= 300\,000 \\ a &= 10\,000 \end{aligned}$$

10 000 adults and 5000 children bought tickets.

4 Let \$$m$ be the amount the contractor paid a man and \$$b$ the amount he paid a boy.

$$\begin{aligned} 8m + 3b &= 2240 \\ 6m + 18b &= 4200 \end{aligned}$$

Multiply the first equation by 6.

$$\begin{aligned} 48m + 18b &= 13\,440 \quad (1) \\ 6m + 18b &= 4200 \quad (2) \end{aligned}$$

(1) − (2):

$$\begin{aligned} 42m &= 9240 \\ m &= 220 \end{aligned}$$

Substitute into the first equation:

$$\begin{aligned} 8 \times 220 + 3b &= 2240 \\ 1760 + 3b &= 2240 \\ 3b &= 2240 - 1760 \\ &= 480 \\ b &= 160 \end{aligned}$$

He paid the men \$220 each and the boys \$160.

5 Let the numbers be x and y.

$$\begin{aligned} x + y &= 212 \quad (1) \\ x - y &= 42 \quad (2) \end{aligned}$$

(1) + (2):

$$\begin{aligned} 2x &= 254 \\ x &= 127 \\ 127 + y &= 212 \\ y &= 85 \end{aligned}$$

The numbers are 127 and 85.

6 Let x L be the amount of 40% solution and y L be the amount of 15% solution. Equate the actual substance.

$$0.4x + 0.15y = 0.24 \times 700$$
$$= 168$$
$$x + y = 700$$

Multiply the second equation by 0.15.

$$0.4x + 0.15y = 168 \quad ①$$
$$0.15x + 0.15y = 105 \quad ②$$

① − ②:

$$0.25x = 63$$
$$x = 63 \times 4$$
$$= 252$$
$$252 + y = 700$$
$$y = 448$$

Use 252 L of 40% solution and 448 L of 15% solution.

7 Form two simultaneous equations.

$$x + y = 220 \quad ①$$
$$x - \frac{x}{2} = y - 40$$
$$\frac{x}{2} - y = -40 \quad ②$$

① + ②:

$$\frac{3x}{2} = 180$$
$$x = 120$$
$$120 + y = 220$$
$$y = 100$$

They started with 120 and 100 marbles and ended with 60 each.

8 Let \$$x$ be the amount initially invested at 10% and \$$y$ the amount initially invested at 7%. This earns \$31 000.

$0.1x + 0.07y = 31\,000$

When the amounts are interchanged, she earns \$1000 more, i.e. \$32 000.

$0.07x + 0.1y = 32\,000$

Multiply the first equation by 100 and the second equation by 70.

$$10x + 7y = 3\,100\,000 \quad ①$$
$$4.9x + 7y = 2\,240\,000 \quad ②$$

① − ②:

$$5.1x = 860\,000$$
$$x = \frac{860\,000}{5.1} \approx 168\,627.451$$
$$10 \times 168\,627.451 + 7y = 3\,100\,000$$
$$1\,686\,274.51 + 7y = 3\,100\,000$$
$$7y = 1\,413\,725.49$$
$$y = 201\,960.78$$

The total amount invested is

$$x + y = 168\,627.45 + 201\,960.78$$
$$= \$370\,588.23$$
$$= \$370\,588$$

correct to the nearest dollar.

9 Let a be the number of adults and s the number of students who attended.

$$30a + 20s = 37\,000 \quad ①$$
$$a + s = 1600$$
$$20a + 20s = 1600 \times 20$$
$$= 32\,000 \quad ②$$

① − ②:

$$10a = 5000$$
$$a = 500$$
$$500 + s = 1600$$
$$s = 1100$$

500 adults and 1100 students attended the concert.

Solutions to Exercise 1F

1 a $v = u + at$

$= 15 + 2 \times 5$

$= 25$

b $I = \dfrac{PrT}{100}$

$= \dfrac{600 \times 5.5 \times 10}{100}$

$= 330$

c $V = \pi r^2 h$

$= \pi \times 4.25^2 \times 6$

≈ 340.47

d $S = 2\pi r(r + h)$

$= 2\pi \times 10.2 \times (10.2 + 15.6)$

≈ 1653.48

e $V = \dfrac{4}{3}\pi r^2 h$

$= \dfrac{4\pi \times 3.58^2 \times 11.4}{3}$

≈ 612.01

f $s = ut + \dfrac{1}{2}at^2$

$= 25.6 \times 3.3 + \dfrac{1}{2} \times -1.2 \times 3.3^2$

≈ 77.95

g $T = 2\pi\sqrt{\dfrac{l}{g}}$

$= 2\pi \times \sqrt{\dfrac{1.45}{9.8}}$

$= 2\pi \times 0.3846\ldots$

≈ 2.42

h $\dfrac{1}{f} = \dfrac{1}{v} + \dfrac{1}{u}$

$= \dfrac{1}{3} + \dfrac{1}{7} = \dfrac{10}{21}$

$f = \dfrac{21}{10}$

$= 2.1$

i $c^2 = a^2 + b^2$

$= 8.8^2 + 3.4^2$

$= 89$

$c = \sqrt{89}$

≈ 9.43

j $v^2 = u^2 + 2as$

$= 4.8^2 + 2 \times 2.25 \times 13.6$

$= 91.04$

$v = \sqrt{91.04}$

≈ 9.54

2 a $v = u + at$

$v - u = at$

$\therefore \quad a = \dfrac{v - u}{t}$

b $S = \dfrac{n}{2}(a + l)$

$2S = n(a + l)$

$a + l = \dfrac{2S}{n}$

$\therefore \quad l = \dfrac{2S}{n} - a$

c $A = \frac{1}{2}bh$

$2A = bh$

$\therefore b = \frac{2A}{h}$

d $P = I^2R$

$\frac{P}{R} = I^2$

$\therefore I = \pm\sqrt{\frac{P}{R}}$

e $s = ut + \frac{1}{2}at^2$

$s - ut = \frac{1}{2}at^2$

$2(s - ut) = at^2$

$\therefore \quad a = \frac{2(s - ut)}{t^2}$

f $E = \frac{1}{2}mv^2$

$2E = mv^2$

$v^2 = \frac{2E}{m}$

$\therefore v = \pm\sqrt{\frac{2E}{m}}$

g $Q = \sqrt{2gh}$

$Q^2 = 2gh$

$\therefore h = \frac{Q^2}{2g}$

h $-xy - z = xy + z$

$-xy - xy = z + z$

$-2xy = 2z$

$\therefore \quad x = \frac{2z}{-2y}$

$= -\frac{z}{y}$

i $\frac{ax + by}{c} = x - b$

$ax + by = c(x - b)$

$ax + by = cx - bc$

$ax - cx = -bc - by$

$x(a - c) = -b(c + y)$

$\therefore \quad x = \frac{-b(c + y)}{a - c}$

$= \frac{b(c + y)}{c - a}$

j $\frac{mx + b}{x - b} = c$

$mx + b = c(x - b)$

$mx + b = cx - bc$

$mx - cx = -bc - b$

$x(m - c) = -b(c + 1)$

$\therefore \quad x = \frac{-b(c + 1)}{m - c}$

3 **a** $F = \frac{9C}{5} + 32$

$= \frac{9 \times 28}{5} + 32$

$= 82.4°$

b $F = \frac{9C}{5} + 32$

$F - 32 = \frac{9C}{5}$

$9C = 5(F - 32)$

$\therefore \quad C = \frac{5(F - 32)}{9}$

Substitute $F = 135$.

$C = \frac{5(135 - 32)}{9}$

$= \frac{515}{9}$

$\approx 57.22°$

4 a $S = 180(n-2)$

$= 180(8-2)$

$= 1080°$

b $S = 180(n-2)$

$\frac{S}{180} = n-2$

$\therefore \quad n = \frac{S}{180} + 2$

$= \frac{1260}{180} + 2$

$= 7 + 2 = 9$

Polygon has 9 sides (a nonagon).

5 a $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 3.5^2 \times 9$

$\approx 115.45 \text{ cm}^3$

b $V = \frac{1}{3}\pi^2 h$

$3V = \pi r^2 h$

$\therefore h = \frac{3V}{\pi r^2}$

$= \frac{3 \times 210}{\pi 4^2}$

≈ 12.53 cm

c $V = \frac{1}{3}\pi r^2 h$

$3V = \pi r^2 h$

$r^2 = \frac{3V}{\pi h}$

$\therefore \quad r = \sqrt{\frac{3V}{\pi h}}$

$= \sqrt{\frac{3 \times 262}{\pi \times 10}}$

≈ 5.00 cm

6 a $S = \frac{n}{2}(a + l)$

$= \frac{7}{2}(-3 + 22)$

$= 66.5$

b $S = \frac{n}{2}(a + l)$

$2S = n(a + l)$

$\frac{2S}{n} = a + l$

$\therefore a = \frac{2S}{n} - l$

$= \frac{2 \times 1040}{13} - 156$

$= 4$

c $S = \frac{n}{2}(a + l)$

$2S = n(a + l)$

$\therefore n = \frac{2S}{a + l}$

$= \frac{2 \times 110}{25 + -5}$

$= 11$

There are 11 terms.

Solutions to Exercise 1G

1 a $\frac{2x}{3} + \frac{3x}{2} = \frac{4x + 9x}{6}$

$= \frac{13x}{6}$

b $\frac{3a}{2} - \frac{a}{4} = \frac{6a - a}{4}$

$= \frac{5a}{4}$

c $\frac{3h}{4} + \frac{5h}{8} - \frac{3h}{2} = \frac{6h + 5h - 12h}{8}$

$= -\frac{h}{8}$

d $\frac{3x}{4} - \frac{y}{6} - \frac{x}{3} = \frac{9x - 2y - 4x}{12}$

$= \frac{5x - 2y}{12}$

e $\frac{3}{x} + \frac{2}{y} = \frac{3y + 2x}{xy}$

f $\frac{5}{x - 1} + \frac{2}{x} = \frac{5x + 2(x - 1)}{x(x - 1)}$

$= \frac{5x + 2x - 2}{x(x - 1)}$

$= \frac{7x - 2}{x(x - 1)}$

g $\frac{3}{x - 2} + \frac{2}{x + 1} = \frac{3(x + 1) + 2(x - 2)}{(x - 2)(x + 1)}$

$= \frac{3x + 3 + 2x - 4}{(x - 2)(x + 1)}$

$= \frac{5x - 1}{(x - 2)(x + 1)}$

h $\frac{2x}{x + 3} - \frac{4x}{x - 3} - \frac{3}{2}$

$= \frac{4x(x - 3) - 8x(x + 3) - 3(x + 3)(x - 3)}{2(x + 3)(x - 3)}$

$= \frac{4x^2 - 12x - 8x^2 - 24x - 3(x^2 - 9)}{2(x + 3)(x - 3)}$

$= \frac{4x^2 - 12x - 8x^2 - 24x - 3x^2 + 27}{2(x + 3)(x - 3)}$

$= \frac{-7x^2 - 36x + 27}{2(x + 3)(x - 3)}$

i $\frac{4}{x + 1} + \frac{3}{(x + 1)^2} = \frac{4(x + 1) + 3}{(x + 1)^2}$

$= \frac{4x + 4 + 3}{(x + 1)^2}$

$= \frac{4x + 7}{(x + 1)^2}$

j $\frac{a - 2}{a} + \frac{a}{4} + \frac{3a}{8}$

$= \frac{8(a - 2) + 2a^2 + 3a^2}{8a}$

$= \frac{5a^2 + 8a - 16}{8a}$

k $2x - \frac{6x^2 - 4}{5x} = \frac{10x^2 - (6x^2 - 4)}{5x}$

$= \frac{10x^2 - 6x^2 + 4}{5x}$

$= \frac{4x^2 + 4}{5x}$

$= \frac{4(x^2 + 1)}{5x}$

l $\dfrac{2}{x+4} - \dfrac{3}{x^2+8x+16}$

$= \dfrac{2}{x+4} - \dfrac{3}{(x+4)^2}$

$= \dfrac{2(x+4)-3}{(x+4)^2}$

$= \dfrac{2x+8-3}{(x+4)^2}$

$= \dfrac{2x+5}{(x+4)^2}$

m $\dfrac{3}{x-1} + \dfrac{2}{(x-1)(x+4)}$

$= \dfrac{3(x+4)+2}{(x-1)(x+4)}$

$= \dfrac{3x+12+2}{(x-1)(x+4)}$

$= \dfrac{3x+14}{(x-1)(x+4)}$

n $\dfrac{3}{x-2} - \dfrac{2}{x+2} + \dfrac{4}{x^2-4}$

$= \dfrac{3}{x-2} - \dfrac{2}{x+2} + \dfrac{4}{(x-2)(x+2)}$

$= \dfrac{3(x+2)-2(x-2)+4}{(x-2)(x+2)}$

$= \dfrac{3x+6-2x+4+4}{(x-2)(x+2)}$

$= \dfrac{x+14}{(x-2)(x+2)}$

o

$\dfrac{5}{x-2} - \dfrac{3}{x^2+5x+6} + \dfrac{2}{x+3}$

$= \dfrac{5}{x-2} - \dfrac{3}{(x+2)(x+3)} + \dfrac{2}{x+3}$

$= \dfrac{5(x+3)(x+2)-3(x-2)+2(x-2)(x+2)}{(x-2)(x+2)(x+3)}$

$= \dfrac{5(x^2+5x+6)-3x+6+2(x^2-4)}{(x-2)(x+2)(x+3)}$

$= \dfrac{5x^2+25x+30-3x+6+2x^2-8}{(x-2)(x+2)(x+3)}$

$= \dfrac{7x^2+22x+28}{(x-2)(x+2)(x+3)}$

p $x - y - \dfrac{1}{x-y} = \dfrac{(x-y)(x-y)-1}{x-y}$

$= \dfrac{(x-y)^2-1}{x-y}$

q $\dfrac{3}{x-1} - \dfrac{4x}{1-x} = \dfrac{3}{x-1} + \dfrac{4x}{x-1}$

$= \dfrac{4x+3}{x-1}$

r $\dfrac{3}{x-2} + \dfrac{2}{2-x} = \dfrac{3}{x-2} - \dfrac{2x}{x-2}$

$= \dfrac{3-2x}{x-2}$

2 a $\dfrac{x^2}{2y} \times \dfrac{4y^3}{x} = \dfrac{4y^3x^2}{2yx}$

$= 2xy^2$

b $\dfrac{3x^2}{4y} \times \dfrac{y^2}{6x} = \dfrac{3x^2y^2}{24yx}$

$= \dfrac{xy}{8}$

c $\dfrac{4x^3}{3} \times \dfrac{12}{8x^4} = \dfrac{48x^3}{24x^4}$

$= \dfrac{2}{x}$

d $\dfrac{x^2}{2y} \div \dfrac{3xy}{6} = \dfrac{x^2}{2y} \times \dfrac{6}{3xy}$

$= \dfrac{6x^2}{6xy^2}$

$= \dfrac{x}{y^2}$

e $\dfrac{4-x}{3a} \times \dfrac{a^2}{4-x} = \dfrac{a^2(4-x)}{3a(4-x)}$

$= \dfrac{a}{3}$

f $\dfrac{2x+5}{4x^2+10x} = \dfrac{2x+5}{2x(2x+5)}$

$= \dfrac{1}{2x}$

g $\dfrac{(x-1)^2}{x^2+3x-4} = \dfrac{(x-1)^2}{(x-1)(x+4)}$

$= \dfrac{x-1}{x+4}$

h $\dfrac{x^2-x-6}{x-3} = \dfrac{(x-3)(x+2)}{x-3}$

$= x+2$

i $\dfrac{x^2-5x+4}{x^2-4x} = \dfrac{(x-1)(x-4)}{x(x-4)}$

$= \dfrac{x-1}{x}$

j $\dfrac{5a^2}{12b^2} \div \dfrac{10a}{6b} = \dfrac{5a^2}{12b^2} \times \dfrac{6b}{10a}$

$= \dfrac{30a^2b}{120ab^2}$

$= \dfrac{a}{4b}$

k $\dfrac{x-2}{x} \div \dfrac{x^2-4}{2x^2}$

$= \dfrac{x-2}{x} \times \dfrac{2x^2}{x^2-4}$

$= \dfrac{x-2}{x} \times \dfrac{2x^2}{(x-2)(x+2)}$

$= \dfrac{2x^2}{x(x+2)}$

$= \dfrac{2x}{x+2}$

l $\dfrac{x+2}{x(x-3)} \div \dfrac{4x+8}{x^2-4x+3}$

$= \dfrac{x+2}{x(x-3)} \div \dfrac{4(x+2)}{(x-1)(x-3)}$

$= \dfrac{x+2}{x(x-3)} \times \dfrac{(x-1)(x-3)}{4(x+2)}$

$= \dfrac{1}{x} \times \dfrac{x-1}{4}$

$= \dfrac{x-1}{4x}$

m

$\dfrac{2x}{x-1} \div \dfrac{4x^2}{x^2-1} = \dfrac{2x}{x-1} \times \dfrac{x^2-1}{4x^2}$

$= \dfrac{2x}{x-1} \times \dfrac{(x-1)(x+1)}{4x^2}$

$= \dfrac{2x(x+1)}{4x^2}$

$= \dfrac{x+1}{2x}$

n $\dfrac{x^2-9}{x+2} \times \dfrac{3x+6}{x-3} \div \dfrac{9}{x}$

$= \dfrac{(x-3)(x+3)}{x+2} \times \dfrac{3(x+2)}{x-3} \times \dfrac{x}{9}$

$= \dfrac{3x(x-3)(x+3)(x+2)}{9(x+2)(x-3)}$

$= \dfrac{x(x+3)}{3}$

o $\dfrac{3x}{9x-6} \div \dfrac{6x^2}{x-2} \times \dfrac{2}{x+5}$

$= \dfrac{3x}{3(3x-2)} \times \dfrac{x-2}{6x^2} \times \dfrac{2}{x+5}$

$= \dfrac{2x(x-2)}{6x^2(3x-2)(x+5)}$

$= \dfrac{x-2}{3x(3x-2)(x+5)}$

3 **a** $\dfrac{1}{x-3} + \dfrac{2}{x-3} = \dfrac{3}{x-3}$

b $$\frac{2}{x-4}+\frac{2}{x-3}=\frac{2(x-3)+2(x-4)}{(x-4)(x-3)}$$
$$=\frac{2x-6+2x-8}{x^2-7x+12}$$
$$=\frac{4x-14}{x^2-7x+12}$$

c $$\frac{3}{x+4}+\frac{2}{x-3}=\frac{3(x-3)+2(x+4)}{(x+4)(x-3)}$$
$$=\frac{3x-9+2x+8}{x^2+x-12}$$
$$=\frac{5x-1}{x^2+x-12}$$

d
$$\frac{2x}{x-3}+\frac{2}{x+4}=\frac{2x(x+4)+2(x-3)}{(x-3)(x+4)}$$
$$=\frac{2x^2+8x+2x-6}{x^2+x-12}$$
$$=\frac{2x^2+10x-6}{x^2+x-12}$$

e $$\frac{1}{(x-5)^2}+\frac{2}{x-5}=\frac{1+2(x-5)}{(x-5)^2}$$
$$=\frac{1+2x-10}{x^2-10x+25}$$
$$=\frac{2x-9}{x^2-10x+25}$$

f $$\frac{3x}{(x-4)^2}+\frac{2}{x-4}=\frac{3x+2(x-4)}{(x-4)^2}$$
$$=\frac{3x+2x-8}{x^2-8x+16}$$
$$=\frac{5x-8}{x^2-8x+16}$$

g $$\frac{1}{x-3}-\frac{2}{x-3}=\frac{-1}{x-3}$$
$$=\frac{1}{3-x}$$

h $$\frac{2}{x-3}-\frac{5}{x+4}=\frac{2(x+4)-5(x-3)}{(x-3)(x+4)}$$
$$=\frac{2x+8-5x+15}{x^2+x-12}$$
$$=\frac{23-3x}{x^2+x-12}$$

i
$$\frac{2x}{x-3}+\frac{3x}{x+3}=\frac{2x(x+3)+3x(x-3)}{(x-3)(x+3)}$$
$$=\frac{2x^2+6x+3x^2-9x}{x^2-9}$$
$$=\frac{5x^2-3x}{x^2-9}$$

j $$\frac{1}{(x-5)^2}-\frac{2}{x-5}=\frac{1-2(x-5)}{(x-5)^2}$$
$$=\frac{1-2x+10}{x^2-10x+25}$$
$$=\frac{11-2x}{x^2-10x+25}$$

k $$\frac{2x}{(x-6)^3}-\frac{2}{(x-6)^2}=\frac{2x-2(x-6)}{(x-6)^3}$$
$$=\frac{2x-2x+12}{(x-6)^3}$$
$$=\frac{12}{(x-6)^3}$$

l
$$\frac{2x+3}{x-4}-\frac{2x-4}{x-3}$$
$$=\frac{(2x+3)(x-3)-(2x-4)(x-4)}{(x-4)(x-3)}$$
$$=\frac{(2x^2-3x-9)-(2x^2-12x+16)}{x^2-7x+12}$$
$$=\frac{2x^2-3x-9-2x^2+12x-16}{x^2-7x+12}$$
$$=\frac{9x-25}{x^2-7x+12}$$

4 a $\sqrt{1-x}+\dfrac{2}{\sqrt{1-x}}$

$=\dfrac{\sqrt{1-x}\sqrt{1-x}+2}{\sqrt{1-x}}$

$=\dfrac{1-x+2}{\sqrt{1-x}}$

$=\dfrac{3-x}{\sqrt{1-x}}$

b $\dfrac{2}{\sqrt{x-4}}+\dfrac{2}{3}=\dfrac{2\sqrt{x-4}+6}{3\sqrt{x-4}}$

c $\dfrac{3}{\sqrt{x+4}}+\dfrac{2}{\sqrt{x+4}}=\dfrac{5}{\sqrt{x+4}}$

d $\dfrac{3}{\sqrt{x+4}}+\sqrt{x+4}$

$=\dfrac{3+\sqrt{x+4}\sqrt{x+4}}{\sqrt{x+4}}$

$=\dfrac{3+x+4}{\sqrt{x+4}}$

$=\dfrac{x+7}{\sqrt{x+4}}$

e $\dfrac{3x^3}{\sqrt{x+4}}-3x^2\sqrt{x+4}$

$=\dfrac{3x^3-3x^2\sqrt{x+4}\sqrt{x+4}}{\sqrt{x+4}}$

$=\dfrac{3x^3-3x^2(x+4)}{\sqrt{x+4}}$

$=\dfrac{3x^3-3x^3-12x^2}{\sqrt{x+4}}$

$=-\dfrac{12x^2}{\sqrt{x+4}}$

f $\dfrac{3x^3}{2\sqrt{x+3}}+3x^2\sqrt{x+3}$

$=\dfrac{3x^3+6x^2\sqrt{x+3}\sqrt{x+3}}{\sqrt{x+3}}$

$=\dfrac{3x^3+6x^2(x+3)}{\sqrt{x+3}}$

$=\dfrac{3x^3+6x^3+18x^2}{\sqrt{x+3}}$

$=\dfrac{9x^3+18x^2}{\sqrt{x+3}}$

$=\dfrac{9x^2(x+2)}{\sqrt{x+3}}$

5 a $(6x-3)^{\frac{1}{3}}-(6x-3)^{-\frac{2}{3}}$

$=(6x-3)^{\frac{1}{3}}-\dfrac{1}{(6x-3)^{\frac{2}{3}}}$

$=\dfrac{(6x-3)^{\frac{1}{3}}(6x-3)^{\frac{2}{3}}-1}{(6x-3)^{\frac{2}{3}}}$

$=\dfrac{6x-3-1}{(6x-3)^{\frac{2}{3}}}$

$=\dfrac{6x-4}{(6x-3)^{\frac{2}{3}}}$

b $(2x+3)^{\frac{1}{3}}-2x(2x+3)^{-\frac{2}{3}}$

$=(2x+3)^{\frac{1}{3}}-\dfrac{2x}{(2x+3)^{\frac{2}{3}}}$

$=\dfrac{(2x+3)^{\frac{1}{3}}(2x+3)^{\frac{2}{3}}-2x}{(2x+3)^{\frac{2}{3}}}$

$=\dfrac{2x+3-2x}{(2x+3)^{\frac{2}{3}}}$

$=\dfrac{3}{(2x+3)^{\frac{2}{3}}}$

c $(3-x)^{\frac{1}{3}} - 2x(3-x)^{-\frac{2}{3}}$

$$= (3-x)^{\frac{1}{3}} - \frac{2x}{(3-x)^{\frac{2}{3}}}$$

$$= \frac{(3-x)^{\frac{1}{3}}(3-x)^{\frac{2}{3}} - 2x}{(3-x)^{\frac{2}{3}}}$$

$$= \frac{3-x-2x}{(3-x)^{\frac{2}{3}}}$$

$$= \frac{3-3x}{(3-x)^{\frac{2}{3}}}$$

Since $(3-x)^2 = (x-3)^2$, the answer is equivalent to $\frac{3-3x}{(x-3)^{\frac{2}{3}}}$.

Solutions to Exercise 1H

1 **a** $ax + n = m$

$$ax = m - n$$

$$x = \frac{m - n}{a}$$

b $ax + b = bx$

$$ax - bx = -b$$

$$x(a - b) = -b$$

$$x = \frac{-b}{a - b}$$

This answer is correct, but to avoid a negative sign, multiply numerator and denominator by -1.

$$x = \frac{-b}{a - b} \times \frac{-1}{-1}$$

$$= \frac{b}{b - a}$$

c $\frac{ax}{b} + c = 0$

$$\frac{ax}{b} = -c$$

$$ax = -bc$$

$$x = -\frac{bc}{a}$$

d $px = qx + 5$

$$px - qx = 5$$

$$x(p - q) = 5$$

$$x = \frac{5}{p - q}$$

e $mx + n = nx - m$

$$mx - nx = -m - n$$

$$x(m - n) = -m - n$$

$$x = \frac{-m - n}{m - n}$$

$$= \frac{m + n}{n - m}$$

f $\frac{1}{x + a} = \frac{b}{x}$

Take reciprocals of both sides:

$$x + a = \frac{x}{b}$$

$$x - \frac{x}{b} = -a$$

$$\frac{x}{b} - x = a$$

$$\frac{x - xb}{b} = a$$

$$\frac{x - xb}{b} \times b = ab$$

$$x - xb = ab$$

$$x(1 - b) = ab$$

$$x = \frac{ab}{1 - b}$$

g $\frac{b}{x - a} = \frac{2b}{x + a}$

Take reciprocals of both sides:

$$\frac{x - a}{b} = \frac{x + a}{2b}$$

$$\frac{x - a}{b} \times 2b = \frac{x + a}{2b} \times 2b$$

$$2(x - a) = x + a$$

$$2x - 2a = x + a$$

$$2x - x = a + 2a$$

$$x = 3a$$

h

$$\frac{x}{m} + n = \frac{x}{n} + m$$

$$\frac{x}{m} \times mn + n \times mn = \frac{x}{n} \times mn + m \times mn$$

$$nx + mn^2 = mx + m^2n$$

$$nx - mx = m^2n - mn^2$$

$$x(n - m) = mn(m - n)$$

$$x = \frac{mn(m - n)}{n - m}$$

Note that $n - m = -m + n$

$$= -1(m - n)$$

$$\therefore \quad x = \frac{-mn(n - m)}{n - m}$$

$$= -mn$$

i $-b(ax + b) = a(bx - a)$

$$-abx - b^2 = abx - a^2$$

$$-abx - abx = -a^2 + b^2$$

$$-2abx = -a^2 + b^2$$

$$x = -\frac{(-a^2 + b^2)}{2ab}$$

$$= \frac{a^2 - b^2}{2ab}$$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

$$p^2 - p^2x - 2pqx = q^2 + q^2x$$

$$-p^2x - 2pqx - q^2x = q^2 - p^2$$

$$-x(p^2 + 2pq + q^2) = q^2 - p^2$$

$$x = \frac{-(q^2 - p^2)}{p^2 + 2pq + q^2}$$

$$= \frac{p^2 - q^2}{(p + q)^2}$$

$$= \frac{(p - q)(p + q)}{(p + q)^2}$$

$$= \frac{p - q}{p + q}$$

k $\frac{x}{a} - 1 = \frac{x}{b} + 2$

$$\frac{x}{a} \times ab - ab = \frac{x}{b} \times ab + 2ab$$

$$bx - ab = ax + 2ab$$

$$bx - ax = 2ab + ab$$

$$x(b - a) = 3ab$$

$$x = \frac{3ab}{b - a}$$

l

$$\frac{x}{a - b} + \frac{2x}{a + b} = \frac{1}{a^2 - b^2}$$

$$\frac{x(a - b)(a + b)}{a - b} + \frac{2x(a + b)(a - b)}{a + b} = \frac{(a + b)(a - b)}{a^2 - b^2}$$

$$x(a + b) + 2x(a - b) = 1$$

$$ax + bx + 2ax - 2bx = 1$$

$$3ax - bx = 1$$

$$x(3a - b) = 1$$

$$x = \frac{1}{3a - b}$$

m

$$\frac{p - qx}{t} + p = \frac{qx - t}{p}$$

$$\frac{pt(p - qx)}{t} + p \times pt = \frac{pt(qx - t)}{p}$$

$$p(p - qx) + p^2t = t(qx - t)$$

$$p^2 - pqx + p^2t = qtx - t^2$$

$$-pqx - qtx = -t^2 - p^2 - p^2t$$

$$-qx(p + t) = -(t^2 + p^2 + p^2t)$$

$$x = \frac{t^2 + p^2 + p^2t}{q(p + t)} \text{ or } \frac{p^2 + p^2t + t^2}{q(p + t)}$$

n $\frac{1}{x + a} + \frac{1}{x + 2a} = \frac{2}{x + 3a}$

Multiply each term by $(x + a)(x + 2a)(x + 3a)$.

$$(x + 2a)(x + 3a) + (x + a)(x + 3a) = 2(x + a)(x + 2a)$$

$$x^2 + 5ax + 6a^2 + x^2 + 4ax + 3a^2 = 2x^2 + 6ax + 4a^2$$

$$2x^2 + 9ax + 9a^2 = 2x^2 + 6ax + 4a^2$$

$$2x^2 - 9ax - 2x^2 - 6ax = 4a^2 - 9a^2$$

$$3ax = -5a^2$$

$$x = \frac{-5a^2}{3a}$$

$$= -\frac{5a}{3}$$

2 $ax + by = p; bx - ay = q$

Multiply the first equation by a and the second equation by b.

$$a^2x + aby = ap \quad ①$$

$$b^2x - aby = bp \quad ②$$

①s + ②:

$$x(a^2 + b^2) = ap + bq$$

$$x = \frac{ap + bq}{a^2 + b^2}$$

Substitute into $ax + by = p$:

$$a \times \frac{ap + bq}{a^2 + b^2} + by = p$$

$$a(ap + bq) + by(a^2 + b^2) = p(a^2 + b^2)$$

$$a^2p + abq + by(a^2 + b^2) = a^2p + b^2p$$

$$by(a^2 + b^2) = a^2p + b^2p - a^2p - abq$$

$$by(a^2 + b^2) = b^2p - abq$$

$$y = \frac{b(bp - aq)}{b(a^2 + b^2)}$$

$$= \frac{bp - aq}{a^2 + b^2}$$

3 $\frac{x}{a} + \frac{y}{b} = 1; \frac{x}{b} + \frac{y}{a} = 1$

First, multiply both equations by ab, giving the following:

$$bx + ay = ab$$

$$ax + by = ab$$

Multiply the first equation by b and the second equation by a:

$$b^2x + aby = ab^2 \quad ①$$

$$a^2x + aby = a^2b \quad ②$$

① − ②:

$$x(b^2 - a^2) = ab^2 - a^2b$$

$$x = \frac{ab^2 - a^2b}{b^2 - a^2}$$

$$= \frac{ab(b - a)}{(b - a)(b + a)}$$

$$= \frac{ab}{a + b}$$

Substitute into $bx + ay = ab$:

$$b \times \frac{ab}{a + b} + ay = ab$$

$$\frac{ab^2(a + b)}{a + b} + ay(a + b) = ab(a + b)$$

$$ab^2 + ay(a + b) = a^2b + ab^2$$

$$ay(a + b) = a^2b + ab^2 - ab^2$$

$$ay(a + b) = a^2b$$

$$y = \frac{a^2b}{a(a + b)}$$

$$= \frac{ab}{a + b}$$

4 **a** Multiply the first equation by b.

$$abx + by = bc \quad ①$$

$$x + by = d \quad ②$$

① − ②:

$$x(ab - 1) = bc - d$$

$$x = \frac{bc - d}{ab - 1}$$

$$= \frac{d - bc}{1 - ab}$$

It is easier to substitute in the first equation for x:

$$a \times \frac{bc - d}{ab - 1} + y = c$$

$$\frac{a(bc - d)(ab - 1)}{ab - 1} + y(ab - 1) = c(ab - 1)$$

$$abc - ad + y(ab - 1) = abc - c$$

$$y(ab - 1) = abc - c$$

$$- abc + ad$$

$$y(ab - 1) = -c + ad$$

$$y = \frac{ad - c}{ab - 1}$$

$$= \frac{c - ad}{1 - ab}$$

b Multiply the first equation by a and the second equation by b.

$$a^2x - aby = a^3 \quad ①$$

$$b^2x - aby = b^3 \quad ②$$

① − ②:

$$x(a^2 - b^2) = a^3 - b^3$$

$$x = \frac{a^3 - b^3}{a^2 - b^2}$$

$$= \frac{(a - b)(a^2 + ab + b^2)}{(a - b)(a + b)}$$

$$= \frac{a^2 + ab + b^2}{a + b}$$

In this case it is easier to start again, but eliminate x.

Multiply the first equation by b and the second equation by a.

$$abx - b^2y = a^2b \quad ③$$

$$abx - a^2y = ab^2 \quad ④$$

③ − ④:

$$y(-b^2 + a^2) = a^2b - ab^2$$

$$y(a^2 - b^2) = ab(a - b)$$

$$y = \frac{ab(a - b)}{a^2 - b^2}$$

$$= \frac{ab(a - b)}{(a - b)(a + b)}$$

$$= \frac{ab}{a + b}$$

c Add the starting equations:

$$ax + by + ax - by = t + s$$

$$2ax = t + s$$

$$x = \frac{t + s}{2a}$$

Subtract the starting equations:

$$ax + by - (ax - by) = t - s$$

$$2by = t - s$$

$$y = \frac{t - s}{2b}$$

d Multiply the first equation by a and the second equation by b.

$$a^2x + aby = a^3 + 2a^2b - ab^2 \quad ①$$

$$b^2x + aby = a^2b + b^3 \quad ②$$

① − ②:

$$x(a^2 - b^2) = a^3 + a^2b - ab^2 - b^3$$

$$x = \frac{a^3 + a^2b - ab^2 - b^3}{a^2 - b^2}$$

$$= \frac{a^2(a + b) - b^2(a + b)}{a^2 - b^2}$$

$$= \frac{(a^2 - b^2)(a + b)}{a^2 - b^2}$$

$$= a + b$$

Substitute into the second, simpler equation.

$$b(a + b) + ay = a^2 + b^2$$

$$ab + b^2 + ay = a^2 + b^2$$

$$ay = a^2 + b^2 - ab - b^2$$

$$ay = a^2 - ab$$

$$y = \frac{a^2 - ab}{a}$$

$$= a - b$$

e Rewrite the second equation, then multiply the first equation by $b + c$

and the second equation by c.

$(a + b)(b + c)x + c(c + c)y$

$= bc(b + c)$ ①

$acx + c(b + c)y$

$= -abc$ ②

① − ②:

$x((a + b)(b + c) - ac)$

$= bc(b + c) + abc$

$x(ab + ac + b^2 + bc - ac)$

$= bc(b + c + a)$

$x(ab + b^2 + bc) = bc(a + b + c)$

$xb(a + b + c) = bc(a + b + c)$

$$x = \frac{bc(a + b + c)}{b(a + b + c)}$$

$= c$

Substitute into the first equation. (It has the simpler y term.)

$c(a + b) + cy = bc$

$ac + bc + cy = bc$

$cy = bc - ac - bc$

$cy = -ac$

$$y = \frac{-ac}{c}$$

$= -a$

f First simplify the equations.

$3x - 3a - 2y - 2a = 5 - 4a$

$3x - 2y = 5 - 4a$

$+ 3a + 2a$

$3x - 2y = a + 5$ ①

$2x + 2a + 3y - 3a = 4a - 1$

$2x + 3y = 4a - 1$

$- 2a + 3a$

$2x + 3y = 5a - 1$ ②

Multiply① by 3 and ② by 2.

$9x - 6y = 3a + 15$ ③

$4x + 6y = 10a - 2$ ④

③ + ④:

$13x = 13a + 13$

$x = a + 1$

Substitute into ②:

$2(a + 1) + 3y = 5a - 1$

$2a + 2 + 3y = 5a - 1$

$3y = 5a - 1 - 2a - 2$

$3y = 3a - 3$

$y = a - 1$

5 a $s = ah$

$= a(2a + 1)$

b Make h the subject of the second equation.

$h = a(2 + h)$

$= 2a + ah$

$h - ah = 2a$

$h(1 - a) = 2a$

$$h = \frac{2a}{1 - a}$$

Substitute into the first equation.

$s = ah$

$$= a \times \frac{2a}{1 - a}$$

$$= \frac{2a^2}{1 - a}$$

c $h + ah = 1$

$h(1 + a) = 1$

$$h = \frac{1}{(1+a)} = \frac{1}{a+1}$$

$$as = a + h$$

$$= a + \frac{1}{a+1}$$

$$= \frac{a(a+1)+1}{a+1}$$

$$= \frac{a^2+a+1}{a+1}$$

$$s = \frac{a^2+a+1}{a(a+1)}$$

d Make h the subject of the second equation.

$$ah = a + h$$

$$ah - h = a$$

$$h(a-1) = a$$

$$h = \frac{1}{a-1}$$

Substitute into the first equation.

$$as = s + h$$

$$as = s + \frac{a}{a-1}$$

$$as - s = \frac{a}{a-1}$$

$$s(a-1) = \frac{a}{a-1}$$

$$s(a-1)(a-1) = \frac{a(a-1)}{a-1}$$

$$s(a-1)^2 = a$$

$$s = \frac{a}{(a-1)^2}$$

e $s = h^2 + ah$

$$= (3a^2)^2 + a(3a^2)$$

$$= 9a^4 + 3a^3$$

$$= 3a^3(3a+1)$$

f $as = a + 2h$

$$= a + 2(a - s)$$

$$= a + 2a - 2s$$

$$as + 2s = 3a$$

$$s(a+2) = 3a$$

$$s = \frac{3a}{a+2}$$

g $s = 2 + ah + h^2$

$$= 2 + a\left(a - \frac{1}{a}\right) + \left(a - \frac{1}{a}\right)^2$$

$$= 2 + a^2 - 1 + a^2 - 2 + \frac{1}{a^2}$$

$$= 2a^2 - 1 + \frac{1}{a^2}$$

h Make h the subject of the second equation.

$$as + 2h = 3a$$

$$2h = 3a - as$$

$$h = \frac{3a - as}{2}$$

Substitute into the first equation.

$$3s - ah = a^2$$

$$3s - \frac{a(3a-as)}{2} = a^2$$

$$6s - a(3a - as) = 2a^2$$

$$6s - 3a^2 + a^2s = 2a^2$$

$$a^2s + 6s = 2a^2 + 3a^2$$

$$s(a^2+6) = 5a^2$$

$$s = \frac{5a^2}{a^2+6}$$

Solutions to Exercise 1I

Use your CAS calculator to find the solutions to these problems. The exact method will vary depending on the calculator used.

1 a $x = a - b$

b $x = 7$

c $x = -\frac{a \pm \sqrt{a^2 + 4ab - 4b^2}}{2}$

d $x = \frac{a + c}{2}$

2 a $(x - 1)(x + 1)(y - 1)(y + 1)$

b $(x - 1)(x + 1)(x + 2)$

c $(a^2 - 12b)(a^2 + 4b)$

d $(a - c)(a - 2b + c)$

3 a $axy + b = (a + c)y$

$bxy + a = (b + c)y$

Dividing by y yields:

$ax + \frac{b}{y} = a + c$

$bx + \frac{a}{y} = b + c$

let $n = \frac{1}{y}$ and the equations become:

$ax + bn = a + c$

$bx + an = b + c$

$\therefore \quad x = \frac{a + b + c}{a + b}$

$y = \frac{a + b}{c}$

b

$$x(b - c) + by - c = 0$$

$$y(c - a) - ax + c = 0$$

$$(b - c)x + by = c$$

$$-ax + (c - a)y = -c$$

$\therefore \quad x = \frac{-(a - b - c)}{a + b - c}$

$y = \frac{a - b + c}{a + b - c}$

Solutions to technology-free questions

1 a $(x^3)^4 = x^{3\times4}$

$= x^{12}$

b $(y^{-12})^{\frac{3}{4}} = y^{-12\times\frac{3}{4}}$

$= y^{-9}$

c $3x^{\frac{3}{2}} \times -5x^4 = (3\times -5)x^{\frac{3}{2}+4}$

$= -15x^{\frac{11}{2}}$

d $(x^3)^{\frac{4}{3}} \times x^{-5} = x^{3\times\frac{4}{3}} \times x^{-5}$

$= x^{4-5}$

$= x^{-1}$

2 $32\times10^{11}\times12\times10^{-5}$

$= (32\times12)\times10^{11-5}$

$= 384\times10^6$

$= 3.84\times10^8$

3 a $\frac{3x}{5}+\frac{y}{10}-\frac{2x}{5} = \frac{6x+y-4x}{10}$

$= \frac{2x+y}{10}$

b $\frac{4}{x}-\frac{7}{y} = \frac{4y-7x}{xy}$

c $\frac{5}{x+2}+\frac{2}{x-1} = \frac{5(x-1)+2(x+2)}{(x+2)(x-1)}$

$= \frac{5x-5+2x+4}{(x+2)(x-1)}$

$= \frac{7x-1}{(x+2)(x-1)}$

d $\frac{3}{x+2}+\frac{4}{x+4} = \frac{3(x+4)+4(x+2)}{(x+2)(x+4)}$

$= \frac{3x+12+4x+8}{(x+2)(x+4)}$

$= \frac{7x+20}{(x+2)(x+4)}$

e $\frac{5x}{x+4}+\frac{4x}{x-2}-\frac{5}{2}$

$= \frac{10x(x-2)+8x(x+4)-5(x+4)(x-2)}{2(x+4)(x-2)}$

$= \frac{10x^2-20x+8x^2+32x-5(x^2+2x-8)}{2(x+4)(x-2)}$

$= \frac{10x^2-20x+8x^2+32x-5x^2-10x+40}{2(x+4)(x-2)}$

$= \frac{37x^2+2x+40}{2(x+4)(x-2)}$

f $\frac{3}{x-2}-\frac{6}{(x-2)^2} = \frac{3(x-2)-6}{(x-2)^2}$

$= \frac{3x-6-6}{x-2}$

$= \frac{3x-12}{x-2}$

$= \frac{3(x-4)}{x-2}$

4 a $\frac{x+5}{2x-6} \div \frac{x^2+5x}{4x-12}$

$= \frac{x+5}{2x-6} \times \frac{4x-12}{x^2+5x}$

$= \frac{x+5}{2(x-3)} \times \frac{4(x-3)}{x(x+5)}$

$= \frac{4}{2x} = \frac{2}{x}$

b $\frac{3x}{x+4} \div \frac{12x^2}{x^2-16}$

$= \frac{3x}{x+4} \times \frac{x^2-16}{12x^2}$

$= \frac{3x}{x+4} \times \frac{(x-4)(x+4)}{12x^2}$

$= \frac{3x(x-4)}{12x^2}$

$= \frac{x-4}{4x}$

c $\frac{x^2-4}{x-3} \times \frac{3x-9}{x+2} \div \frac{9}{x+2}$

$= \frac{x^2-4}{x-3} \times \frac{3x-9}{x+2} \times \frac{x+2}{9}$

$= \frac{(x-2)(x+2)}{x-3} \times \frac{3(x-3)}{x+2}$

$\times \frac{x+2}{9}$

$= \frac{(x+2)(x-2)}{3} = \frac{x^2-4}{3}$

d $\frac{4x+20}{9x-6} \times \frac{6x^2}{x+5} \div \frac{2}{3x-2}$

$= \frac{4(x+5)}{3(3x-2)} \times \frac{6x^2}{x+5} \times \frac{3x-2}{2}$

$= \frac{4 \times 6x^2}{3 \times 2} = 4x^2$

5 Let t seconds be the required time. The number of red blood cells to be replaced is $\frac{1}{2} \times 5 \times 10^{12} = 2.5 \times 10^{12}$

$2.5 \times 10^6 \times t = 2.5 \times 10^{12}$

$t = \frac{2.5 \times 10^{12}}{2.5 \times 10^6}$

$= 10^6$

Time $= 10^6$ seconds

$= 10^6 \div 3600 \div 24$ days

≈ 11.57 or $11\frac{31}{54}$ days

6 $\frac{1.5 \times 10^8}{3 \times 10^6} = 0.5 \times 10^2$

$= 50$ times further

7 Let g be the number of games the team lost. They won $2g$ games and drew one third of 54 games, i.e. 18 games.

$g + 2g + 18 = 54$

$3g = 54 - 18$

$= 36$

$g = 12$

They have lost 12 games.

8 Let b be the number of blues CDs sold. The store sold l.lb classical and $1.5(b + 1.1b)$ heavy metal CDs, totalling 420 CDs.

$b + 1.1\,b + 1.5 \times 2.1b = 420$

$5.25b = 420$

$b = \frac{420}{5.25}$

$= 80$

$1.1b = 1.1 \times 80 = 88$

$1.5 \times 2.1b = 1.5 \times 2.1 \times 80$

$= 252$

80 blues, 88 classical and 252 heavy metal (totalling 420)

9 a $V = \pi r^2 h$

$= \pi \times 5^2 \times 12$

$= 300\pi \approx 942\text{ cm}^3$

b $h = \dfrac{V}{\pi r^2}$

$= \dfrac{585}{\pi \times 5^2}$

$= \dfrac{117}{5\pi} \approx 7.4$ cm

c $r^2 = \dfrac{V}{\pi h}$

$r = \sqrt{\dfrac{V}{\pi h}}$ (use positive root)

$= \sqrt{\dfrac{786}{\pi \times 6}}$

$= \sqrt{\dfrac{128}{\pi}} \approx 40.7$ cm

d

$$\frac{a - dx}{d} + b = \frac{ax + d}{b}$$

$$\frac{bd(a - dx)}{d} + bd \times b = \frac{bd(ax + d)}{b}$$

$$b(a - dx) + b^2d = d(ax + d)$$

$$ab - bdx + b^2d = adx + d^2$$

$$-bdx - adx = d^2 - ab - b^2d$$

$$-x(bd + ad) = -(ab + b^2d - d^2)$$

$$x = \frac{-(ab + b^2d - d^2)}{-(bd + ad)}$$

$$= \frac{ab + b^2d - d^2}{bd + ad}$$

10 a $xy + ax = b$

$x(y + a) = b$

$x = \dfrac{b}{a + y}$

b $\dfrac{a}{x} + \dfrac{b}{x} = c$

$\dfrac{ax}{x} + \dfrac{bx}{x} = cx$

$a + b = cx$

$x = \dfrac{a + b}{c}$

c $\dfrac{x}{a} = \dfrac{x}{b} + 2$

$\dfrac{xab}{a} = \dfrac{xab}{b} + 2ab$

$bx = ax + 2ab$

$bx - ax = 2ab$

$x(b - a) = 2ab$

$x = \dfrac{2ab}{b - a}$

11 a $\dfrac{p}{p + q} + \dfrac{q}{p - q} = \dfrac{p(p - q) + q(p + q)}{(p + q)(p - q)}$

$= \dfrac{p^2 - qp + qp + q^2}{p^2 - pq + pq - q^2}$

$= \dfrac{p^2 + q^2}{p^2 - q^2}$

b $\dfrac{1}{x} - \dfrac{2y}{xy - y^2} = \dfrac{(xy - y^2) - 2xy}{x(xy - y^2)}$

$= \dfrac{-xy - y^2}{x^2y - xy^2}$

$= \dfrac{y(-x - y)}{xy(x - y)}$

$= \dfrac{-x - y}{x(x - y)}$

$= \dfrac{x + y}{x(y - x)}$

c $\dfrac{x^2 + x - 6}{x + 1} \times \dfrac{2x^2 + x - 1}{x + 3}$

$= \dfrac{(x - 2)(x + 3)}{x + 1} \times \dfrac{(x + 1)(2x - 1)}{x + 3}$

$= (x - 2)(2x - 1)$

d $\dfrac{2a}{2a+b} \times \dfrac{2ab+b^2}{ba^2}$

$= \dfrac{2a}{2a+b} \times \dfrac{b(2a+b)}{ba^2}$

$= \dfrac{2ab}{ba^2}$

$= \dfrac{2}{a}$

12 Let A's age be a, B's age be b and C's age be c.

$a = 3b$

$b + 3 = 3(c + 3)$

$a + 15 = 3(c + 15)$

Substitute for a and simplify:

$b + 3 = 3(c + 3)$

$b + 3 = 3c + 9$

$b = 3c + 6$ ①

$3b + 15 = 3(c + 15)$

$3b + 15 = 3c + 45$

$3b = 3c + 30$

$b = c + 10$ ②

① = ②:

$3c + 6 = c + 10$

$3c - c = 10 - 6$

$2c = 4$

$c = 2$

$b = 3 \times 2 + 6$

$= 12$

$a = 3 \times 12$

$= 36$

A, B and C are 36, 12 and 2 years old respectively.

13 a Simplify the first equation:

$a - 5 = \frac{1}{7}(b + 3)$

$7(a - 5) = b + 3$

$7a - 35 = b + 3$

$7a - b = 38$

Simplify the second equation:

$b - 12 = \frac{1}{5}(4a - 2)$

$5(b - 12) = 4a - 2$

$5b - 60 = 4a - 2$

$-4a + 5b = 58$

Multiply the first equation by 5, and add the second equation.

$35a - 5b = 190$ ①

$-4a + 5b = 58$ ②

① + ②:

$31a = 248$

$a = 8$

Substitute into the first equation:

$7 \times 8 - b = 38$

$56 - b = 38$

$b = 56 - 38 = 18$

b Multiply the first equation by p.

$(p - q)x + (p + q)y = (p + q^2)$

$p(p - q)x + p(p + q)y = p(p + q^2)$ ①

Multiply the second by $(p + q)$.

$qx - py = q^2 - pq$

$q(p + q)x - p(p + q)y$

$= (p + q)(q^2 - pq)$ ②

① + ②:

$(p(p - q) + q(p + q))x$

$= p(p + q)^2 + (p + q)(q^2 - pq)$

$(p^2 - pq + pq + q^2)x$

$= p(p^2 + 2pq + q^2)$

$+ pq^2 - p^2q + q^3 - pq^2$

$(p^2 + q^2)x$

$= p^3 + 2p^2q + pq^2 - p^2q + q^3$

$= p^3 + p^2q + pq^2 + q^3$

$= p^2(p + q) + q^2(p + q)$

$= (p + q)(p^2 + q^2)$

$x = p + q$

Substitute into the second equation, factorising the right side.

$q(p + q) - py = q^2 - pq$

$pq + q^2 - py = q^2 - pq$

$-py = q^2 - pq - pq - q^2$

$-py = -2pq$

$y = \dfrac{-2pq}{-p}$

$= 2q$

14 Time $= \dfrac{\text{distance}}{\text{speed}}$

Remainder = 50 − 7 − 7 = 36 km

$\dfrac{7}{x} + \dfrac{7}{4x} + \dfrac{36}{6x + 3} = 4$

$\dfrac{7}{x} + \dfrac{7}{4x} + \dfrac{12}{2x + 1} = 4$

$(4x(2x + 1)) \times \left(\dfrac{7}{x} + \dfrac{7}{4x} + \dfrac{12}{2x + 1}\right)$

$= 4 \times 4x(2x + 1)$

$28(2x + 1) + 7(2x + 1) + 48x$

$= 16x(2x + 1)$

$56x + 28 + 14x + 7 + 48x$

$= 32x^2 + 16x$

$56x + 28 + 14x + 7 + 48x$

$- 32x^2 - 16x = 0$

$-32x^2 + 102x + 35 = 0$

$32x^2 - 102x - 35 = 0$

$(2x - 7)(16x + 5) = 0$

$2x - 7 = 0$ or $16x + 5 = 0$

$x > 0$, so $2x - 7 = 0$

$x = 3.5$

15 **a** $2n^2 \times 6nk^2 \div 3n = \dfrac{2n^2 \times 6nk^2}{3n}$

$= \dfrac{12n^3k^2}{3n}$

$= 4n^2k^2$

b $\dfrac{8c^2x^3y}{6a^2b^3c^3} \div \dfrac{\frac{1}{2}xy}{15abc^2}$

$= \dfrac{8c^2x^3y}{6a^2b^3c^3} \div \dfrac{xy}{30abc^2}$

$= \dfrac{8c^2x^3y}{6a^2b^3c^3} \times \dfrac{30abc^2}{xy}$

$= \dfrac{240abc^4x^3y}{6a^2b^3c^3xy}$

$= \dfrac{40cx^2}{ab^2}$

16 $\dfrac{x + 5}{15} - \dfrac{x - 5}{10} = 1 + \dfrac{2x}{15}$

$\dfrac{30(x + 5)}{15} - \dfrac{30(x - 5)}{10} = 30 \times \left(1 + \dfrac{2x}{15}\right)$

$2(x + 5) - 3(x - 5) = 30 + 4x$

$2x + 10 - 3x + 15 = 30 + 4x$

$2x - 3x - 4x = 30 - 10 - 15$

$-5x = 5$

$x = -1$

Solutions to multiple-choice questions

1 A

$$5x + 2y = 0$$
$$2y = -5x$$
$$\frac{y}{x} = -\frac{5}{2}$$

2 A Multiply both sides of the second equation by 2.

$$3x + 2y = 36 \quad ①$$
$$6x - 2y = 24 \quad ②$$

① + ②:

$$9x = 60$$
$$x = \frac{20}{3}$$
$$3 \times \frac{20}{3} - y = 12$$
$$20 - y = 12$$
$$y = 8$$

3 C

$$t - 9 = 3t - 17$$
$$t - 3t = 9 - 17$$
$$-2t = -8$$
$$t = 4$$

4 A

$$m = \frac{n - p}{n + p}$$
$$m(n + p) = n - p$$
$$mn + mp = n - p$$
$$mp + p = n - mn$$
$$p(m + 1) = n(1 - m)$$
$$p = \frac{n(1 - m)}{1 + m}$$

5 B

$$\frac{3}{x - 3} - \frac{2}{x + 3} = \frac{3(x + 3) - 2(x - 3)}{(x - 3)(x + 3)}$$
$$= \frac{3x + 9 - 2x + 6}{x^2 - 9}$$
$$= \frac{x + 15}{x^2 - 9}$$

6 E

$$9x^2y^3 \div 15(xy)^3 = \frac{9x^2y^3}{15(xy)^3}$$
$$= \frac{9x^2y^3}{15x^3y^3}$$
$$= \frac{9}{15x}$$
$$= \frac{3}{5x}$$

7 B

$$V = \frac{1}{3}h(l + w)$$
$$3V = h(l + w)$$
$$3V = hl + hw$$
$$hl = 3V - hw$$
$$l = \frac{3V - hw}{h}$$
$$= \frac{3V}{h} - w$$

8 B

$$\frac{(3x^2y^3)^2}{2x^2y} = \frac{9x^4y^6}{2x^2y}$$
$$= \frac{9x^2y^5}{2}$$
$$= \frac{9}{2}x^2y^5$$

9 B $Y = 80\% \times Z = \frac{4}{5}Z$

$X = 150\% \times Y = \frac{3}{2}Y$

$= \frac{3}{2} \times \frac{4Z}{5}$

$= \frac{12Z}{10}$

$= 1.2Z$

= 20% greater than Z

10 B Let the other number be n.

$\frac{x + n}{2} = 5x + 4$

$x + n = 2(5x + 4)$

$= 10x + 8$

$n = 10x + 8 - x$

$= 9x + 8$

Solutions to extended-response questions

1 Jack cycles $10x$ km.
Benny drives $40x$ km.

a
$$\text{Distance} = \text{speed} \times \text{time}$$
$$\therefore \quad \text{time} = \frac{\text{distance}}{\text{speed}}$$
$$\therefore \quad \text{time taken by Jack} = \frac{10x}{8}$$
$$= \frac{5x}{4} \text{ hours}$$

b
$$\text{Time taken by Benny} = \frac{40x}{70}$$
$$= \frac{4x}{7} \text{ hours}$$

c
$$\text{Jack's time–Benny's time} = \frac{5x}{4} - \frac{4x}{7}$$
$$= \frac{(35 - 16)x}{7}$$
$$= \frac{19x}{28} \text{ hours}$$

d **i** If the difference is 30 mins $= \frac{1}{2}$ hour
$$\text{then } \frac{19x}{28} = \frac{1}{2}$$
$$\therefore \quad x = \frac{14}{19}$$
$$= 0.737 \text{ (correct to three decimal places)}$$

ii
$$\text{Distance for Jack} = 10 \times \frac{14}{19}$$
$$= \frac{140}{19}$$
$$= 7 \text{ km (correct to the nearest km)}$$
$$\text{Distance for Benny} = 40 \times \frac{14}{19}$$
$$= \frac{560}{19}$$
$$= 29 \text{ km (correct to the nearest km)}$$

2 a Dinghy is filling with water at a rate of
$27\,000 - 9\,000 = 18\,000$ cm^3 per minute.

b After t minutes there are $18\,000t$ cm^3 water in the dinghy,
i.e. $V = 18\,000t$

c $V = \pi r^2 h$ is the formula for the volume of a cylinder

$$\therefore \quad h = \frac{V}{\pi r^2}$$
$$= \frac{18\,000t}{\pi r^2}$$

The radius of this cylinder is 40 cm

$$\therefore \quad h = \frac{18\,000t}{1600\pi} = \frac{45t}{4\pi}$$

i.e. the height h cm water at time t is given by $h = \dfrac{45t}{4\pi}$

d When $t = 9$, $h = \dfrac{45 \times 9}{4\pi}$
$\approx 32.228\ldots$
The dinghy has filled with water, before $t = 9$, i.e. Sam is rescued after the dinghy completely filled with water.

3 a Let Thomas have a cards. Therefore Henry has $\dfrac{5a}{6}$ cards, George has $\dfrac{3a}{2}$ cards, Sally has $(a - 18)$ cards and Zeb has $\dfrac{a}{3}$ cards.

b $\dfrac{3a}{2} + a - 18 + \dfrac{a}{3} = a + \dfrac{5a}{6} + 6$

c $\therefore \quad 9a + 6a - 108 + 2a = 6a + 5a + 36$

$$\therefore \quad 6a = 144$$
$$\therefore \quad a = 24$$

Thomas has 24 cards, Henry has 20 cards, George has 36 cards, Sally has 6 cards and Zeb has 8 cards.

4 a $F = \dfrac{6.67 \times 10^{-11} \times 200 \times 200}{12^2}$

$= 1.852\ldots \times 10^{-8}$

$= 1.9 \times 10^{-8}$ (correct to two significant figures)

b $m_1 = \dfrac{Fr^2}{m_2 \times 6.67 \times 10^{-11}}$

$= \dfrac{Fr^2 \times 10^{11}}{6.67m_2}$

c If $F = 2.4 \times 10^4$

$r = 6.4 \times 10^6$

and $m_2 = 1500$

$m_1 = \dfrac{2.4 \times 10^4 \times (6.4 \times 10^6)^2 \times 10^{11}}{6.67 \times 1500}$

$= 9.8254\ldots \times 10^{24}$

The mass of the earth is 9.8×10^{24} kg (correct to two significant figures).

5 **a** $V = 3 \times 10^3 \times 6 \times 10^3 \times d$

$= 18 \times 10^6 d$

b When $d = 30$, $V = 18 \times 10^6 \times 30$

$= 540\,000\,000$

$= 5.4 \times 10^8$

The volume of the reservoir is 5.4×10^8 m^3.

c $E = kVh$

$1.06 \times 10^{15} = k \times 200 \times 5.4 \times 10^8$

$k = \dfrac{1.06 \times 10^{15}}{200 \times 5.4 \times 10^8}$

$= 9.81\ldots \times 10^3$

$k = 9.81 \times 10^3$ correct to three significant figures.

d $E = (9.81 \times 10^3) \times 5.4 \times 10^8 \times 250$

$= 1.325 \times 10^{15}$ correct to four significant figures.

The amount of energy produced is 1.325×10^{15} J.

e Let t be the time in seconds.

$5.2 \times t = 5.4 \times 10^8$

$t = 103.846\,153\,8$

$\therefore$ number of days $= 103.846\,153\,8 \div (24 \times 60 \times 60)$

$= 1201.92\ldots$

The station could operate for approximately 1202 days.

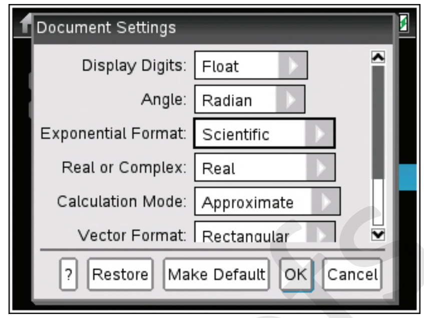
CAS calculator techniques for Question 5

5 b Calculations involving scientific notation and significant figures can be accomplished with the aid of a graphics calculator.

When $d = 30$, $V = 18 \times 10^6 \times 30$

$$= 540\,000\,000$$

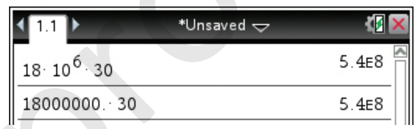
This calculation can be completed as shown here.



T1: Press c → **5**: **Settings** → **2**: **Document Settings** and change the Exponential Format to Scientific. Click on Make Default.

Return to the Calculator application.

Type **18** × **10^6** × **30** or **18i6** × **30**



CP: In the Main application tap ◯ → **Basic Format**

Change the Number Format to Sci2

Type **18** × **10^6** × **30**

c **T1**: Press c→ **5**: **Settings** → **2**: **Document Settings** and change the Display Digits to Float 3. Click on Make Default.

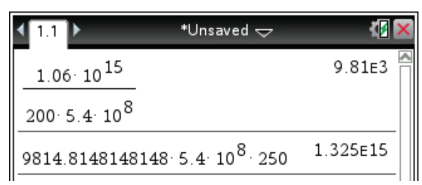
Return to the home screen and press and complete as shown.

CP: tap ◯ → **Basic Format**

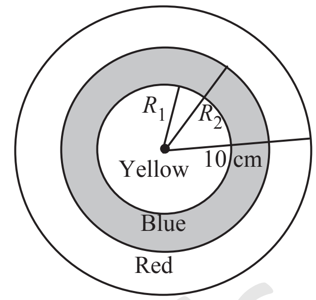
Change the Number Format to Sci3 Complete calculation as shown

d The calculation is as shown. **T1**: Display Digits is Float 4 **CP**: Number Format is Sci4

Simply type ×**5.4** × **10^8** × **25**



6 Let R_1 cm and R_2 cm be the radii of the inner circles.



$$\therefore \quad \text{Yellow area} = \pi R_1^2$$

$$\text{Blue area} = \pi R_2^2 - \pi R_1^2$$

$$\text{Red area} = 100\pi - \pi R_2^2$$

$$\therefore \quad 100\pi - \pi R_2^2 = \pi R_2^2 - \pi R_1^2 = \pi R_1^2$$

Firstly, $\quad \pi R_2^2 - \pi R_1^2 = \pi R_1^2$

implies $\quad R_2^2 = 2R_1^2 \qquad ①$

and $\quad 100\pi - \pi R_2^2 = \pi R_2^2 - \pi R_1^2$

implies $\quad 100 = 2R_2^2 - R_1^2 \qquad ②$

Substitute from ①in ②

$$\therefore \quad 100 = 4R_1^2 - R_1^2$$

$$100 = 3R_1^2$$

and $\quad R_1 = \dfrac{10}{\sqrt{3}}$

$$= \frac{10\sqrt{3}}{3} \qquad \left(\text{Note}: \ R_2^2 = \frac{200}{3}\right)$$

The radius of the innermost circle is $\dfrac{10\sqrt{3}}{3}$ cm.

7 If $C = F$,

$$F = \frac{5}{9}(F - 32)$$

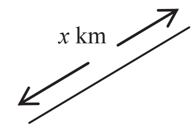
$$9F = 5F - 160$$

$$\therefore 4F = -160$$

$$\therefore F = -40$$

Therefore − 40°F = −40°C.

8 Let x km be the length of the slope.



$$\therefore \quad \text{time to go up} = \frac{x}{15}$$

$$\therefore \quad \text{time to go down} = \frac{x}{40}$$

$$\begin{aligned}\therefore \quad \text{total time} &= \frac{x}{15} + \frac{x}{40}\\ &= \frac{11x}{120}\end{aligned}$$

$$\begin{aligned}\therefore \quad \text{average speed} &= \frac{\text{total distance}}{\text{total time}}\\ &= 2x \div \frac{11x}{120}\\ &= 2x \times \frac{120}{11x}\\ &= \frac{240}{11}\\ &\approx 21.82 \text{ km/h}\end{aligned}$$

9 1 litre = 1000 cm^3

a

$$\begin{aligned}\text{Volume} &= \text{Volume of cylinder} + \text{Volume of hemisphere}\\ &= \pi r^2 h + \frac{2}{3}\pi r^3\end{aligned}$$

It is known that $r + h = 20$

$$\therefore \quad h = 20 - r$$

b **i**

$$\begin{aligned}\text{Volume} &= \pi r^2(20 - r) + \frac{2}{3}\pi r^3\\ &= 20\pi r^2 - \pi r^3 + \frac{2}{3}\pi r^3\\ &= 20\pi r^2 - \frac{\pi}{3} r^3\end{aligned}$$

ii If Volume = 2000 cm^3
then $20\pi r^2 - \frac{\pi}{3}r^3 = 2000$
Use a CAS calculator to solve this equation for r, given that $0 < r < 20$. This gives $r = 5.943999\cdots$
Therefore

$$\begin{aligned}h &= 20 - r\\ &= 20 - 5.943\,99\ldots\\ &= 14.056\,001\ldots\end{aligned}$$

The volume is two litres when $r = 5.94$ and $h = 14.06$, correct to two decimal places.

10 a Let x and y be the amount of liquid (in cm^3) taken from bottles A and B respectively. Since the third bottle has a capacity of 1000 cm^3,

$$x + y = 1000 \qquad ①$$

Now $$x = \frac{2}{3}x \text{ wine} + \frac{1}{3}x \text{ water}$$

and $$y = \frac{1}{6}y \text{ wine} + \frac{5}{6}y \text{ water}$$

$$\therefore \quad \frac{2}{3}x + \frac{1}{6}y = \frac{1}{3}x + \frac{5}{6}y \text{ since the proportion of wine and water must be the same.}$$

$$\therefore \quad 4x + y = 2x + 5y$$

$$\therefore \quad 2x = 4y$$

$$\therefore \quad x = 2y$$

From ② $$2y + y = 1000$$

$$\therefore \quad y = \frac{1000}{3} \text{ and } x = \frac{2000}{3}$$

Therefore, $\frac{2000}{3}$ cm^3 and $\frac{1000}{3}$ cm^3 must be taken from bottles A and B respectively so that the third bottle will have equal amounts of wine and water, i.e. $\frac{2}{3}L$ from A and $\frac{1}{3}L$ from B

b

$$x + y = 1000 \qquad ①$$

$$\frac{1}{3}x + \frac{3}{4}y = \frac{2}{3}x + \frac{1}{4}y$$

$$\therefore \quad 4x + 9y = 8x + 3y$$

$$\therefore \quad 6y = 4x$$

$$\therefore \quad x = \frac{3}{2}y \qquad ②$$

From ① $$\frac{3}{2}y + y = 1000$$

$$\therefore \quad y = \frac{2}{5} \times 1000$$

$$= 400$$

$$\therefore \quad x = 600$$

Therefore, 600 cm^3 and 400 cm^3 must be taken from bottles *A* and *B* respectively so that the third bottle will have equal amounts of wine and water, i.e. 600 mL from A and 400 mL from B

c

$$x + y = 1000 \qquad ①$$

$$\frac{m}{m+n}x + \frac{p}{p+q}y = \frac{n}{m+n}x + \frac{q}{p+q}y$$

$$\therefore \quad m(p+q)x + p(m+n)y = n(p+q)x + q(m+n)y$$

$$\therefore \quad (m(p+q) - n(p+q))x = (q(m+n) - p(m+n))y$$

$$\therefore \quad (m-n)(p+q)x = (q-p)(m+n)y$$

$$\therefore \quad x = \frac{(m+n)(q-p)}{(m-n)(p+q)}y,\ m \neq n, p \neq q \qquad ②$$

From ①
$$\frac{(m+n)(q-p)}{(m-n)(p+q)}y + y = 1000$$

$$\therefore \quad \frac{(m+n)(q-p) + (m-n)(p+q)}{(m-n)(p+q)}y = 1000$$

$$\therefore \quad \frac{mq - mp + nq - np + mp + mq - np - nq}{(m-n)(p+q)}y = 1000$$

$$\therefore \quad \frac{2(mq-np)}{(m-n)(p+q)}y = 1000$$

$$\therefore \quad y = \frac{500(m-n)(p+q)}{mq-np},$$

$$mq \neq np$$

From ① $x = \dfrac{(m+n)(q-p)}{(m-n)(p+q)} \times \dfrac{500(m-n)(p+q)}{mq-np}$

$$= \frac{500(m+n)(q-p)}{mq-np},\ \frac{n}{q} \neq \frac{q}{p}$$

Therefore, $\dfrac{500(m+n)(q-p)}{mq-np}$ cm^3 and $\dfrac{500(m-n)(p+q)}{mq-np}$ cm^3 must be taken from bottles *A* and *B* respectively so that the third bottle will have equal amounts of wine and water. In litres this is $\dfrac{(m+n)(q-p)}{2(mq-np)}$ litres from A and $\dfrac{(m-n)(p+q)}{2(mq-np)}$ litres from B. Also note that $\dfrac{n}{m} \geq 1$ and $\dfrac{q}{p} \leq 1$ or $\dfrac{n}{m} \leq 1$ and $\dfrac{q}{p} \geq 1$.

11 a $\dfrac{20-h}{20} = \dfrac{r}{10}$

$\therefore \quad 10(20-h) = 20r$

$\therefore \quad 200 - 10h = 20r$

$\therefore \quad 20 - h = 2r$

$\therefore \quad h = 20 - 2r$

$= 2(10 - r)$

b $V = \pi r^2 h$

$= 2\pi r^2(10 - r)$

c Use CAS calculator to solve the equation $2\pi x^2(10 - r) = 500$, given that $0 < r < 10$.
This gives $r = 3.49857\ldots$ or $r = 9.02244\ldots$
When $r = 3.498\,57\ldots$, $h = 2(10 - 3.498\,57\ldots)$

$= 13.002\,85\ldots$

When $r = 9.022\,44\ldots$, $h = 2(10 - 9.022\,44\ldots)$

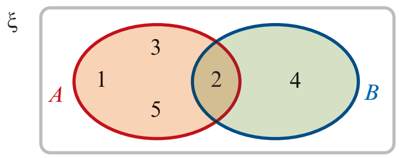
$= 1.955\,11\ldots$

Therefore the volume of the cylinder is 500 cm^3 when $r = 3.50$ and $h = 13.00$ or when $r = 9.02$ and $h = 1.96$, correct to two decimal places.

Chapter 2 – Number systems and sets

Solutions to Exercise 2A

1



a $A' = \{4\}$

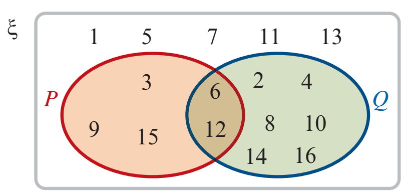
b $B' = \{1, 3, 5\}$

c $A \cup B = \{1, 2, 3, 4, 5\}$, or ξ

d $(A \cup B)' = \varnothing$

e $A' \cap B' = \varnothing$

2



a $P' = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16\}$

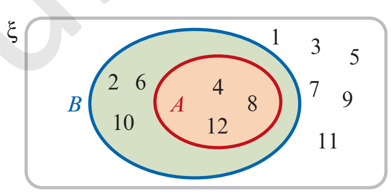
b $Q' = \{1, 3, 5, 7, 9, 11, 13, 15\}$

c $P \cup Q =$
$\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}$

d $(P \cup Q)' = \{1, 5, 7, 11, 13\}$

e $P' \cap Q' = \{1, 5, 7, 11, 13\}$

3



a $A' = \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$

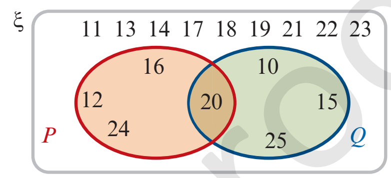
b $B' = \{1, 3, 5, 7, 9, 11\}$

c $A \cup B = \{2, 4, 6, 8, 10, 12\}$

d $(A \cup B)' = \{1, 3, 5, 7, 9, 11\}$

e $A' \cap B' = \{1, 3, 5, 7, 9, 11\}$

4



a $P' =$
$\{10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25\}$

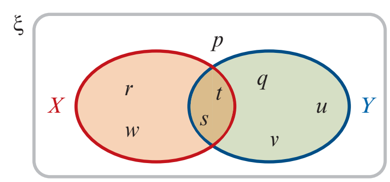
b $Q' =$
$\{11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24\}$

c $P \cup Q = \{10, 12, 15, 16, 20, 24, 25\}$

d $(P \cup Q)' =$
$\{11, 13, 14, 17, 18, 19, 21, 22, 23\}$

e $P' \cap Q' =$
$\{11, 13, 14, 17, 18, 19, 21, 22, 23\}$

5



a $X' = \{p,\ q,\ u,\ v\}$

b $Y' = \{p,\ r,\ w\}$

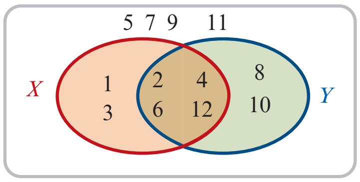
c $X' \cap Y' = \{p\}$

d $X' \cup Y' = \{p,\ q,\ r,\ u,\ v,\ w\}$

e $X \cup Y = \{q,\ r,\ s,\ t,\ u,\ v,\ w\}$

f $(X \cup Y)' = \{p\}$ **c** and **f** are equal.

6 ξ



a $X' = \{5, 7, 8, 9, 10, 11\}$

b $Y' = \{1, 3, 5, 7, 9, 11\}$

c $X' \cup Y' = \{1, 3, 5, 7, 8, 9, 10, 11\}$

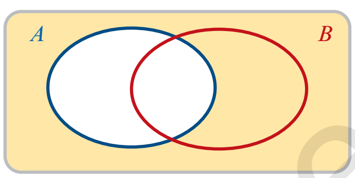
d $X' \cap Y' = \{5, 7, 9, 11\}$

e $X \cup Y = \{1, 2, 3, 4, 6, 8, 10, 12\}$

f $(X \cup Y)' = \{5, 7, 9, 11\}$ **d** and **f** are equal.

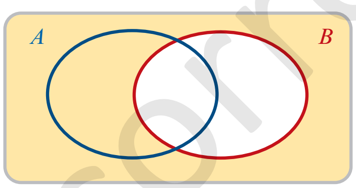
7 a ξ

A'



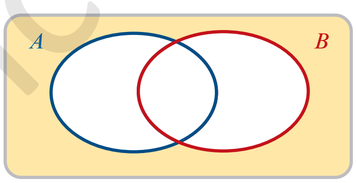
b ξ

B'



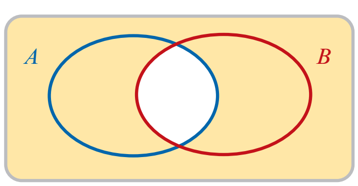
c ξ

$A' \cap B'$



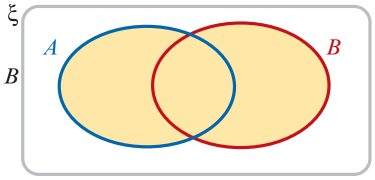
d ξ

$A' \cup B'$



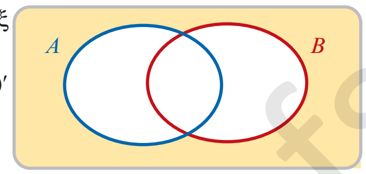
e ξ

$A \cup B$

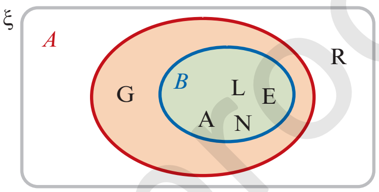


f ξ

$(A \cup B)'$



8 ξ



a $A' = \{\text{R}\}$

b $B' = \{\text{G, R}\}$

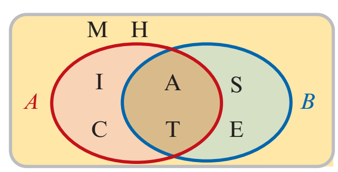
c $A \cap B = \{\text{L, E, A, N}\}$

d $A \cup B = \{\text{A, N, G, E, L}\}$

e $(A \cup B)' = \{\text{R}\}$

f $A' \cup B' = \{\text{G, R}\}$

9 ξ



a $A' = \{\text{E, H, M, S}\}$

b $B' = \{\text{C, H, I, M}\}$

c $A \cap B = \{\text{A, T}\}$

d $(A \cup B)' = \{\text{H, M}\}$

e $A' \cup B' = \{\text{C, E, H, I, M, S}\}$

f $A' \cap B' = \{\text{H, M}\}$

Solutions to Exercise 2B

1 **a** Ye

b Yes

c Yes

2 **a** The sum may be rational or irrational, for instance, $\sqrt{2} + \sqrt{3}$ is irrational; $\sqrt{2} + (3 - \sqrt{2}) = 3$ is rational.

b The product may be rational or irrational. For instance, $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ is irrational; $\sqrt{2} \times 3\sqrt{2} = 6$ is rational.

c The quotient may be rational or irrational. For instance $\dfrac{\sqrt{2}}{\sqrt{3}}$ is irrational; $\dfrac{3\sqrt{2}}{\sqrt{2}} = 3$ is rational.

3 **a** $0.45 = \dfrac{45}{100} = \dfrac{9}{20}$

b
$$\begin{aligned} 0.\dot{2}\dot{7} &= 0.272727\ldots \\ 0.\dot{2}\dot{7} \times 100 &= 27.272727\ldots \\ 0.\dot{2}\dot{7} \times 99 &= 27 \\ \therefore 0.\dot{2}\dot{7} &= \frac{27}{99} = \frac{3}{11} \end{aligned}$$

c $0.12 = \dfrac{12}{100} = \dfrac{3}{25}$

d
$$\begin{aligned} 0.\dot{2}8571\dot{4} &= 0.285714285714\ldots \\ 0.\dot{2}8571\dot{4} \times 10^6 &= 285714.285714\ldots \\ 0.\dot{2}8571\dot{4} \times (10^6 - 1) &= 285714 \\ \therefore \quad 0.\dot{2}8571\dot{4} &= \frac{285714}{999999} = \frac{2}{7} \end{aligned}$$

e
$$\begin{aligned} 0.\dot{3}\dot{6} &= 0.363636\ldots \\ 0.\dot{3}\dot{6} \times 100 &= 36.3636\ldots \\ 0.\dot{3}\dot{6} \times 99 &= 36 \\ \therefore \quad 0.\dot{3}\dot{6} &= \frac{36}{99} = \frac{4}{11} \end{aligned}$$

f
$$\begin{aligned} 0.\dot{2} &= 0.22222\ldots \\ 0.\dot{2} \times 10 &= 2.2222\ldots \\ 0.\dot{2} \times 9 &= 2 \\ \therefore 0.\dot{2} &= \frac{2}{9} \end{aligned}$$

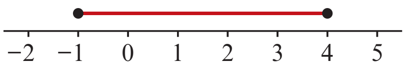
4 **a**
$$\begin{aligned} \frac{2}{7} &= 7\overline{)2.000000}\ldots \\ &= 0.2857142857\ldots \\ &= 0.\dot{2}8571\dot{4} \end{aligned}$$

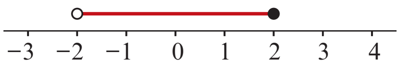
b
$$\begin{aligned} \frac{5}{11} &= 11\overline{)5.000000}\ldots \\ &= 0.454545\ldots \\ &= 0.\dot{4}\dot{5} \end{aligned}$$

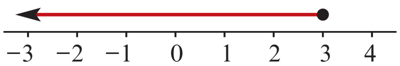
c
$$\begin{aligned} \frac{7}{20} &= 20\overline{)7.00} \\ &= 0.35 \end{aligned}$$

d
$$\begin{aligned} \frac{4}{13} &= 13\overline{)4.000000}\ldots \\ &= 0.30769230\ldots \\ &= 0.\dot{3}0769\dot{2} \end{aligned}$$

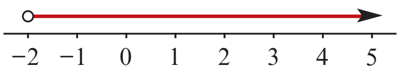
e
$$\begin{aligned} \frac{1}{17} &= 17\overline{)1.0000000000000000}\ldots \\ &= 0.0588235294117647058\ldots \\ &= 0.\dot{0}58823529411764\dot{7} \end{aligned}$$

5 **a** 

b 

c 

d 

e 

6 **a** $(-\infty, 3)$

b $[-3, \infty)$

c $(-\infty, -3]$

d $(5, \infty)$

e $[-2, 3)$

f $[-2, 3]$

g $(-2, 3]$

h $(-5, 3)$

Solutions to Exercise 2C

1 a 8

b 8

c 2

d −2

e −2

f 4

2 a $|x-1| = 2$

Case 1: If $x \geq 1$

$$x - 1 = 2$$
$$x = 3$$

Case 2: If $x < 1$

$$1 - x = 2$$
$$x = -1$$

b $|2x-3| = 4$

Case 1: If $x \geq \frac{3}{2}$

$$2x - 3 = 4$$
$$x = \frac{7}{2}$$

Case 2: If $x < \frac{3}{2}$

$$3 - 2x = 4$$
$$x = -\frac{1}{2}$$

c $|5x-3| = 9$

Case 1: If $x \geq \frac{3}{5}$

$$5x - 3 = 9$$
$$x = \frac{12}{5}$$

Case 2: If $x < \frac{3}{5}$

$$3 - 5x = 9$$
$$x = -\frac{6}{5}$$

d $|x-3| = 9$

Case 1: If $x \geq 3$

$$x - 3 = 9$$
$$x = 12$$

Case 2: If $x < 3$

$$3 - x = 9$$
$$x = -6$$

e $|x-3| = 4$

Case 1: If $x \geq 3$

$$x - 3 = 4$$
$$x = 7$$

Case 2: If $x < 3$

$$3 - x = 4$$
$$x = -1$$

f $|3x + 4| = 8$

Case 1: If $x \geq -\dfrac{4}{3}$

$$3x + 4 = 8$$

$$x = \frac{4}{3}$$

Case 2: If $x < -\dfrac{4}{3}$

$$-3x - 4 = 8$$

$$x = -4$$

g $|5x + 11| = 9$

Case 1: If $x \geq -\dfrac{11}{5}$

$$5x + 11 = 9$$

$$x = -\frac{2}{5}$$

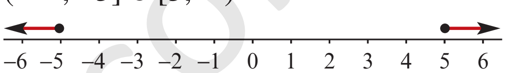
Case 2: If $x < -\dfrac{11}{5}$

$$-5x - 11 = 9$$

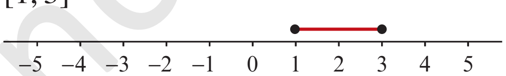
$$x = -4$$

3 a $(-3, 3)$

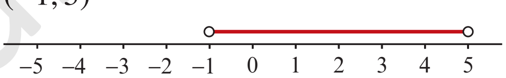
b $(-\infty, -5] \cup [5, \infty)$



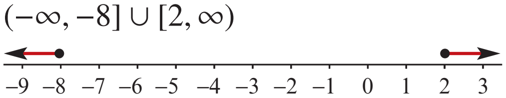
c $[1, 3]$



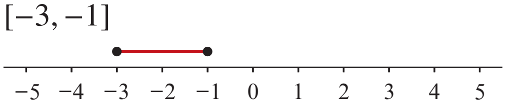
d $(-1, 5)$



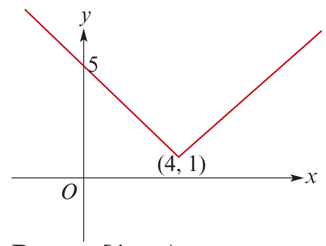
e $(-\infty, -8] \cup [2, \infty)$



f $[-3, -1]$

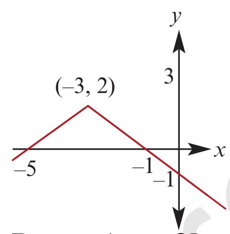


4 a



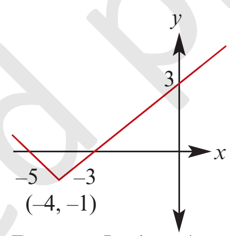
Range $[1, \infty)$

b



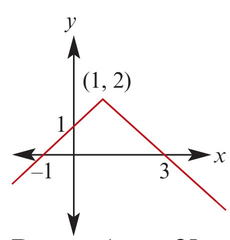
Range $(-\infty, 2]$

c



Range $[-1, \infty)$

d



Range $(-\infty, 2]$

5 a $\{x : -5 \leq x \leq 5\}$

b $\{x : x \leq -2\} \cup \{x : x \geq 2\}$

c $\{x : 1 \leq x \leq 2\}$

d $\{x : -\frac{1}{5} < x < 1\}$

e $\{x : x \leq -4\} \cup \{x : x \geq 10\}$

f $\{x : 1 \leq x \leq 3\}$

6 **a** $|x - 4| - |x + 2| = 6$

Case 1: If $x \geq 4$

$x - 4 - x - 2 = 6$ (no solution)

Case 2: If $x \leq -2$

$4 - x - (-x - 2) = 6$ Always true:

Case 3: If $-2 < x < 4$

$4 - x - (x + 2) = 6$

$4 - 2x - 2 = 6$

$-2x = 8$

$x = -4$

Soln not acceptable.

Therefore $x \leq -2$ is the solution

b $x = -9$ or $x = 11$

c $x = -\frac{5}{4}$ or $x = \frac{15}{4}$

7 $a = 1, \ b = 1$

8

$$x^2 + y^2 + 2|x||y| \geq x^2 + y^2 + 2xy$$
$$(|x| + |y|)^2 \geq |x + y|^2$$
$$\therefore |x| + |y| \geq |x + y|$$

Hence

$|x - y| = |x + (-y)| \geq |x| + | - y| = |x| + |y|$

9

$$x^2 + y^2 - 2|x||y| \leq x^2 + y^2 - 2xy$$
$$(|x| - |y|)^2 \leq |x - y|^2$$
$$\therefore |x| - |y| \leq |x - y|$$

We can assume $|x| \geq |y|$ without loss of generality.

10 $|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$

Solutions to Exercise 2D

1 **a** $\sqrt{8} = \sqrt{4} \times \sqrt{2}$

$= 2\sqrt{2}$

b $\sqrt{12} = \sqrt{4} \times \sqrt{3}$

$= 2\sqrt{3}$

c $\sqrt{27} = \sqrt{9} \times \sqrt{3}$

$= 3\sqrt{3}$

d $\sqrt{50} = \sqrt{25} \times \sqrt{2}$

$= 5\sqrt{2}$

e $\sqrt{45} = \sqrt{9} \times \sqrt{5}$

$= 3\sqrt{5}$

f $\sqrt{1210} = \sqrt{121} \times \sqrt{10}$

$= 11\sqrt{10}$

g $\sqrt{98} = \sqrt{49} \times \sqrt{2}$

$= 7\sqrt{2}$

h $\sqrt{108} = \sqrt{36} \times \sqrt{3}$

$= 6\sqrt{3}$

i $\sqrt{25} = 5$

j $\sqrt{75} = \sqrt{25} \times \sqrt{3}$

$= 5\sqrt{3}$

k $\sqrt{512} = \sqrt{256} \times \sqrt{2}$

$= 16\sqrt{2}$

2 **a** $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$

$= \sqrt{4 \times 2} + \sqrt{9 \times 2} - 2\sqrt{2}$

$= 2\sqrt{2} + 3\sqrt{2} - 2\sqrt{2}$

$= 3\sqrt{2}$

b

$\sqrt{75} + 2\sqrt{12} - \sqrt{27}$

$= \sqrt{25 \times 3} + 2\sqrt{4 \times 3} - \sqrt{9 \times 3}$

$= 5\sqrt{3} + 4\sqrt{3} - 3\sqrt{3}$

$= 6\sqrt{3}$

c

$\sqrt{28} + \sqrt{175} - \sqrt{63}$

$= \sqrt{4 \times 7} + \sqrt{25 \times 7} - \sqrt{9 \times 7}$

$= 2\sqrt{7} + 5\sqrt{7} - 3\sqrt{7}$

$= 4\sqrt{7}$

d

$\sqrt{1000} - \sqrt{40} - \sqrt{90}$

$= \sqrt{100 \times 10} - \sqrt{4 \times 10} - \sqrt{9 \times 10}$

$= 10\sqrt{10} - 2\sqrt{10} - 3\sqrt{10}$

$= 5\sqrt{10}$

e

$\sqrt{512} + \sqrt{128} + \sqrt{32}$

$= \sqrt{256 \times 2} + \sqrt{64 \times 2} + \sqrt{16 \times 2}$

$= 16\sqrt{2} + 8\sqrt{2} + 4\sqrt{2}$

$= 28\sqrt{2}$

f

$\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

$= \sqrt{4 \times 6} - 3\sqrt{6} - \sqrt{36 \times 6} + \sqrt{49 \times 6}$

$= 2\sqrt{6} - 3\sqrt{6} - 6\sqrt{6} + 7\sqrt{6}$

$= 0$

3 a $\sqrt{75}+\sqrt{108}+\sqrt{14}$

$= \sqrt{25\times 3}+\sqrt{36\times 3}+\sqrt{14}$

$= 5\sqrt{3}+6\sqrt{3}+\sqrt{14}$

$= 11\sqrt{3}+\sqrt{14}$

b $\sqrt{847}-\sqrt{567}+\sqrt{63}$

$= \sqrt{121\times 7}-\sqrt{81\times 7}$

$+\sqrt{9\times 7}$

$= 11\sqrt{7}-9\sqrt{7}+3\sqrt{7}$

$= 5\sqrt{7}$

c $\sqrt{720}-\sqrt{245}-\sqrt{125}$

$= \sqrt{144\times 5}-\sqrt{49\times 5}$

$-\sqrt{25\times 5}$

$= 12\sqrt{5}-7\sqrt{5}-5\sqrt{5}$

$= 0$

d $\sqrt{338}-\sqrt{288}+\sqrt{363}-\sqrt{300}$

$= \sqrt{169\times 2}-\sqrt{144\times 2}$

$+\sqrt{121\times 3}-\sqrt{100\times 3}$

$= 13\sqrt{2}-12\sqrt{2}+11\sqrt{3}$

$-10\sqrt{3}$

$= \sqrt{2}+\sqrt{3}$

e $\sqrt{12}+\sqrt{8}+\sqrt{18}+\sqrt{27}+\sqrt{300}$

$= \sqrt{4\times 3}+\sqrt{4\times 2}+\sqrt{9\times 2}$

$+\sqrt{9\times 3}+\sqrt{100\times 3}$

$= 2\sqrt{3}+2\sqrt{2}+3\sqrt{2}$

$+3\sqrt{3}+10\sqrt{3}$

$= 5\sqrt{2}+15\sqrt{3}$

f

$2\sqrt{18}+3\sqrt{5}-\sqrt{50}+\sqrt{20}-\sqrt{80}$

$= 2\sqrt{9\times 2}+3\sqrt{5}-\sqrt{25\times 2}$

$+\sqrt{4\times 5}-\sqrt{16\times 5}$

$= 6\sqrt{2}+3\sqrt{5}-5\sqrt{2}+2\sqrt{5}-4\sqrt{5}$

$= \sqrt{2}+\sqrt{5}$

4 a $\frac{1}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{5}$

b $\frac{1}{\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}=\frac{\sqrt{7}}{7}$

c $-\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$

d $\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

e $\frac{3}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}=\frac{3\sqrt{6}}{6}=\frac{\sqrt{6}}{2}$

f $\frac{1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{4}$

g $\frac{1}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}=\frac{\sqrt{2}-1}{2-1}$

$=\frac{\sqrt{2}-1}{1}$

$=\sqrt{2}-1$

h $\frac{1}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\frac{2+\sqrt{3}}{4-3}$

$=2+\sqrt{3}$

i $\frac{1}{4-\sqrt{10}}\times\frac{4+\sqrt{10}}{4+\sqrt{10}}=\frac{4+\sqrt{10}}{16-10}$

$=\frac{4+\sqrt{10}}{6}$

j $\dfrac{2}{\sqrt{6}+2} \times \dfrac{\sqrt{6}-2}{\sqrt{6}-2} = \dfrac{2\sqrt{6}-4}{6-4}$

$= \dfrac{2\sqrt{6}-4}{2}$

$= \sqrt{6}-2$

k $\dfrac{1}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \dfrac{\sqrt{5}+\sqrt{3}}{5-3}$

$= \dfrac{\sqrt{5}+\sqrt{3}}{2}$

l

$\dfrac{3}{\sqrt{6}-\sqrt{5}} \times \dfrac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}} = \dfrac{3(\sqrt{6}+\sqrt{5})}{6-5}$

$= 3(\sqrt{6}+\sqrt{5})$

m $\dfrac{1}{3-2\sqrt{2}} \times \dfrac{3+2\sqrt{2}}{3+2\sqrt{2}} = \dfrac{3+2\sqrt{2}}{9-8}$

$= 3+2\sqrt{2}$

5 a $\dfrac{2}{3-2\sqrt{2}} \times \dfrac{3+2\sqrt{2}}{3+2\sqrt{2}} = \dfrac{6+4\sqrt{2}}{9-8}$

$= 6+4\sqrt{2}$

b $(\sqrt{5}+2)^2 = (\sqrt{5})^2 + 4\sqrt{5} + 4$

$= 5 + 4\sqrt{5} + 4$

$= 9 + 4\sqrt{5}$

c $(1+\sqrt{2})(3-2\sqrt{2})$

$= 3 - 2\sqrt{2} + 3\sqrt{2} - 4$

$= -1 + \sqrt{2}$

d $(\sqrt{3}-1)^2 = 3 - 2\sqrt{3} + 1$

$= 4 - 2\sqrt{3}$

e

$\dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{27}} = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{27}}{\sqrt{27}} - \dfrac{1}{\sqrt{27}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{3\sqrt{3}-\sqrt{3}}{9}$

$= \dfrac{2\sqrt{3}}{9}$

f $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1} = \dfrac{\sqrt{3}+2}{2\sqrt{3}-1} \times \dfrac{2\sqrt{3}+1}{2\sqrt{3}+1}$

$= \dfrac{6+\sqrt{3}+4\sqrt{3}+2}{12-1}$

$= \dfrac{8+5\sqrt{3}}{11}$

g $\dfrac{\sqrt{5}+1}{\sqrt{5}-1} = \dfrac{\sqrt{5}+1}{\sqrt{5}-1} \times \dfrac{\sqrt{5}+1}{\sqrt{5}+1}$

$= \dfrac{5+2\sqrt{5}+1}{5-1}$

$= \dfrac{6+2\sqrt{5}}{4}$

$= \dfrac{3+\sqrt{5}}{2}$

h $\dfrac{\sqrt{8}+3}{\sqrt{18}+2} = \dfrac{2\sqrt{2}+3}{3\sqrt{2}+2}$

$= \dfrac{2\sqrt{2}+3}{3\sqrt{2}+2} \times \dfrac{3\sqrt{2}-2}{3\sqrt{2}-2}$

$= \dfrac{12-4\sqrt{2}+9\sqrt{2}-6}{18-4}$

$= \dfrac{6+5\sqrt{2}}{14}$

6 a $(2\sqrt{a}-1)^2 = (2\sqrt{a}-1)(2\sqrt{a}-1)$

$= 4a - 2\sqrt{a} - 2\sqrt{a} + 1$

$= 4a - 4\sqrt{a} + 1$

b $(\sqrt{x+1} + \sqrt{x+2})^2$

$= (\sqrt{x+1} + \sqrt{x+2})$

$\times (\sqrt{x+1} + \sqrt{x+2})$

$= x + 1 + 2\sqrt{(x+1)(x+2)}$

$+ x + 2$

$= 2x + 3 + 2\sqrt{(x+1)(x+2)}$

7 a $(5 - 3\sqrt{2}) - (6\sqrt{2} - 8)$

$= 5 - 3\sqrt{2} - 6\sqrt{2} + 8$

$= 13 - 9\sqrt{2}$

$= \sqrt{169} - \sqrt{162}$

> 0

$5 - 3\sqrt{2}$ is larger.

b $(2\sqrt{6} - 3) - (7 - 2\sqrt{6})$

$= 2\sqrt{6} - 3 - 7 + 2\sqrt{6}$

$= 4\sqrt{6} - 10$

$= \sqrt{96} - \sqrt{100}$

< 0

$7 - 2\sqrt{6}$ is larger.

8 a $\frac{4}{3} < \frac{9}{2} \Rightarrow \frac{2}{\sqrt{3}} < \frac{3}{\sqrt{2}}$

b $\frac{7}{9} < \frac{5}{4} \Rightarrow \frac{\sqrt{7}}{3} < \frac{\sqrt{5}}{2}$

c $\frac{3}{49} < \frac{1}{5} \Rightarrow \frac{\sqrt{3}}{7} < \frac{\sqrt{5}}{5}$

d $\frac{10}{4} < \frac{64}{3} \Rightarrow \frac{\sqrt{10}}{2} < \frac{8}{\sqrt{3}}$

9 a $(x - \sqrt{3})(x + \sqrt{3}) = x^2 - 3$

Therefore $b = 0$ and $c = -3$

b $(x - 2\sqrt{3})(x + 2\sqrt{3}) = x^2 - 12$

Therefore $b = 0$ and $c = -12$

c $(x - (1 - \sqrt{2})(x - (1 + \sqrt{2}) =$
$x^2 - 2x - 1$

Therefore $b = -2$ and $c = -1$

d $(x - (2 - \sqrt{3})(x - (2 + \sqrt{1}) =$
$x^2 - 4x + 1$

Therefore $b = -4$ and $c = 1$

e $(x - (3 - 2\sqrt{2})(x - (3 + 2\sqrt{2}) =$
$x^2 - 6x + 1$

Therefore $b = -6$ and $c = 1$

f $(x - (4 - 7\sqrt{5})(x - (3 + 2\sqrt{5}) =$
$x^2 - (= -7 + 5\sqrt{5})x - 58 - 13\sqrt{5}$

Therefore $b = -7 + 5\sqrt{5}$ and
$c = -58 - 13\sqrt{5}$

10 $\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}} \times \frac{\sqrt{2} + \sqrt{3} - \sqrt{5}}{\sqrt{2} + \sqrt{3} - \sqrt{5}}$

$= \frac{\sqrt{2} + \sqrt{3} - \sqrt{5}}{(\sqrt{2} + \sqrt{3})^2 - 5}$

$= \frac{\sqrt{2} + \sqrt{3} - \sqrt{5}}{(2 + 3 + 2\sqrt{6} - 5}$

$= \frac{\sqrt{2} + \sqrt{3} - \sqrt{5}}{2\sqrt{6}}$

$= \frac{\sqrt{2} + \sqrt{3} - \sqrt{5}}{2\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$

$= \frac{\sqrt{12} + \sqrt{18} - \sqrt{30}}{12}$

$= \frac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}$

11 a Note $a - b = \left(a^{\frac{1}{3}}\right)^3 - \left(b^{\frac{1}{3}}\right)^3$

b $\dfrac{1}{1-2^{\frac{1}{3}}} \times \dfrac{1+2^{\frac{1}{3}}+2^{\frac{2}{3}}}{1+2^{\frac{1}{3}}+2^{\frac{2}{3}}}$

$= -(1+2^{\frac{1}{3}}+2^{\frac{2}{3}})$

Solutions to Exercise 2E

1 **a** $2^2 \times 3 \times 5$

b $2^2 \times 13^2$

c $2^2 \times 57$

d $2^2 \times 3^2 \times 5^2$

e $2^2 \times 3^2 \times 7$

f

$$\begin{array}{r|r} 2 & 68\,640 \\ 2 & 34\,320 \\ 2 & 17\,160 \\ 2 & 8580 \\ 2 & 4290 \\ 3 & 2145 \\ 5 & 715 \\ 11 & 143 \\ 13 & 13 \\ \hline & 1 \end{array}$$

Prime decomposition
$= 2^5 \times 3 \times 5 \times 11 \times 13$

g

$$\begin{array}{r|r} 2 & 96\,096 \\ 2 & 48\,048 \\ 2 & 24\,024 \\ 2 & 12\,012 \\ 2 & 6006 \\ 3 & 3003 \\ 7 & 1001 \\ 11 & 143 \\ 13 & 13 \\ \hline & 1 \end{array}$$

Prime decomposition
$= 2^5 \times 3 \times 7 \times 11 \times 13$

h

$$\begin{array}{r|r} 2 & 32\,032 \\ 2 & 16\,016 \\ 2 & 8008 \\ 2 & 4004 \\ 2 & 2002 \\ 7 & 1001 \\ 11 & 143 \\ 13 & 13 \\ \hline & 1 \end{array}$$

Prime decomposition
$= 2^5 \times 7 \times 11 \times 13$

i

$$\begin{array}{r|r}
2 & 544\,544 \\
2 & 272\,272 \\
2 & 136\,136 \\
2 & 68\,068 \\
2 & 34\,034 \\
7 & 17\,017 \\
11 & 2431 \\
13 & 221 \\
17 & 17 \\
\hline
 & 1
\end{array}$$

Prime decomposition

$= 2^5 \times 7 \times 11 \times 13 \times 17$

2 For each part, first find the prime decomposition of each number.

a $4361 = 7^2 \times 89$
Neither 7 nor 89 are factors of 9281.
HCF = 1

b $999 = 3^3 \times 37$
$2160 = 2^4 \times 3^3 \times 5$
$\text{HCF} = 3^3 = 27$

c $5255 = 5 \times 1051$
716 845 is divisible by 5 but not 1051.
HCF = 5

d $1271 = 31 \times 41$
$3875 = 5^3 \times 31$
HCF = 31

e $804 = 2^2 \times 3 \times 67$
$2358 = 2 \times 3^2 \times 131$
$\text{HCF} = 2 \times 3 = 6$

3 **a** $18 = 3^2 \times 2$
Factors: 1, 2, 3,6, 9,18.
$36 = 3^2 \times 2^2$
Factors: 1, 2, 4, 3, 6, 12, 9, 18, 36

b 36 is a perfect square

c $121 = 11^2$. It has to be a perfect square to have an odd number of factors. To have 3 it must be the perfect square of a prime.

4 $1050 = 7 \times 5^2 \times 3 \times 2$
Children are teenagers: Ages:
$7 \times 2 = 14$
$5 \times 3 = 15$
5

5 $22^2 \times 55^2 = 10^2 \times n^2$
$(11 \times 2)^2 \times (11 \times 5^2) = 10^2 \times n^2$
$\therefore 11^2 \times 11^2 \times (5 \times 2)^2 = 10^2 \times n^2$
$\therefore n = 121$

6 $5 \times 3 \times 7 \times 3 = 7 \times 5 \times 3^2$.
This has 12 factors Therefore the starting number is $7 \times 5 \times 3 = 105$. It has 8 factors.

7 $720 = 5 \times 3^2 \times 2^4$
$720 = 2^3 \times 2 \times 3^2 \times 5$
$720 = 8 \times 9 \times 10$. $n = 8$

8 $30 = 2 \times 3 \times 5$
Factors are:1, 3, 5, 2, 2×3, 2×5, 3×5, $2 \times 3 \times 5$
$\text{Product} = 2^4 \times 3^4 \times 5^4 = 30^4$

9 LCM is 252 which is 4 hours and 12 minutes. That is 1:12 pm.

10 600 and 108 000
2400 and 27 000
3000 and 21 600
5400 and 12 000

Solutions to Exercise 2F

1 a If a solution is not readily seen, use trial and error on the variable with the largest coefficient, as you will expect fewer trials until you find a multiple of the other variable.
Try $x = 0 : 3y = 1$ has no integral solutions.
Try $x = 1 : 11 + 3y = 1$ has no integral solutions.
Try $x = 2 : 22 + 3y = 1$ has the solution $y = -7$.
The HCF of 11 and 3 is 1.
The general solution will be
$x = 2 + 3t, y = -7 - 11t, t \in \mathbb{Z}$

b An obvious solution is $x = 1, y = 0$.
The HCF of 2 and 7 is 1.
The general solution will be
$x = 1 + 7t, y = -2t, t \in \mathbb{Z}$
Alternatively, if you spot the solution $x = 8, y = -2$, then the general solution will be:
$x = 8 + 7t, y = -2 - 2t, t \in \mathbb{Z}$

c This equation is equivalent to $8x + 21y = 33$, and then the HCF of 8 and 21 is 1.
It is also obvious that y must be odd, and x must be a multiple of 3.
$y = 5$ gives the solution $x = -9$.
$x = -9 + 21t, y = 5 - 8t, t \in \mathbb{Z}$
Alternatively, use a CAS calculator's table feature with $y = -\frac{8}{21}x + \frac{33}{21}$ and find an integer solution. Once such is $x = 264, y = -99$, so the general solution can also be written as:
$x = 264 + 21t, y = -98 - 8t, t \in \mathbb{Z}$

d Dividing through by 2 shows that this is the same equation as in part **a**, hence the solution will be the same.
$x = 2 + 3t, y = -7 - 11t, t \in \mathbb{Z}$

e Any even value of y will give a solution.
If $y = 2, x = 4$.
The HCF of this solution is 2.
The general solution will be
$x = 4 + 7t, y = 2 - 2t, t \in \mathbb{Z}$
To get integer solutions, replace t by $2\,t$.
$x = 4 + 7t, y = 2 - 2t, t \in \mathbb{Z}$

f Dividing through by 5 shows that this is the same equation as in part **e**, hence the solution will be the same.
$x = 4 + 7t, y = 2 - 2t, t \in \mathbb{Z}$

2 From the general solution, when $t = 0$, $x = 4$ and $y = 2$. If $t \geq 1, y \leq 0$
If $t \leq -1, x < 0$ so only $t = 0$ works.
There is one solution: $x = 4, y = 2$.

3 Let h be the highest common factor of a and b.
a, b and c can be written as $a = hp$, $b = hq, c = hr + k$, where $0 < k < h$.
The equation becomes
$hpx + hqy = hr + k$.
For all integer values of x and y, the left side of the equation will be a multiple of h, while the right side will not be.
Therefore the equation can have no integral solutions.

4 a Let s be the number of spiders and b the number of beetles.

Equating the numbers of legs gives $8s + 6b = 54$.

b This equation simplifies to $4s + 3b = 27$.

$$4s = 27 - 3y$$
$$= 3(9 - b)$$
$$s = \frac{3(9 - b)}{4}$$

Solutions will only exist when $9 - b$ is a multiple of 4, and $b > 0, 9 - b > 0$. This occurs when $b = 1, s = 6$ and when $b = 5, s = 3$.

The answer could be written '3 spiders and 5 beetles, or 6 spiders and 1 beetle'.

5 Equating the value of the coins,

$$20x + 50y = 500$$
$$2x + 5y = 50$$
$$5y = 50 - 2x$$
$$= 2(25 - x)$$
$$y = \frac{2(25 - x)}{5}$$
$$= 2\left(5 - \frac{x}{5}\right)$$

This gives the results as in the table below.

50c coins	0	2	4	6	8	10
20c coins	25	20	15	10	5	0

6 All solutions are given by $x = 100 + 83t, y = 1 - 19t$

$$100 + 83t > 0$$
$$83t > -100$$
$$t > -\frac{100}{83}$$
$$1 - 19t > 0$$
$$-19t > -1$$
$$t < \frac{1}{19}$$

Since t is an integer, $-1 \le t \le 0$.

The second solution occurs when $t = -1$.

$$x = 100 - 83$$
$$= 17$$
$$y = 1 + 19$$
$$= 20$$

For $19x + 98y = 1998$, one obvious solution is $x = 100, y = 1$.

$x = 100 + 98t, y = 1 - 19t$

$100 + 98t > 0$

$98t > -100$

$t > -\frac{100}{98}$

$1 - 19t > 0$

$-19t > -1$

$t < \frac{1}{19}$

Since t is an integer, $-1 \le t \le 0$.

The second solution occurs when $t = -1$.

$x = 100 - 98$

$= 2$

$y = 1 + 19$

$= 20$

7 Equating the value of the notes,

$10x + 50y = 500$

$x + 5y = 50$

$x = 50 - 5y$

$= 5(10 - y)$

This gives the results as in the table below.

\$50 notes	0	1	2	3	4	5
\$10 notes	50	45	40	35	30	25
	6	7	8	9	10	
	20	15	10	5	0	

8 Total number of pieces of fruit = $63x + 7$.

$y = \frac{63x + 7}{23}$

$= \frac{7(9x + 1)}{23}$

$y = \frac{63x + 7}{23} = \frac{7(9x + 1)}{23}$

$9x + 1$ must be a multiple of 23.

$9x + 1 = 23n$

$9x = 23n - 1$

$x = \frac{23n - 1}{9}$

If $n = 2$, $x = 5$ and $y = 14$.

If $n = 9t + 2$,

$x = \frac{23n - 1}{9}$

$= \frac{23(9t + 2) - 1}{9}$

$= 23t + \frac{23 \times 2 - 1}{9}$

$= 23t + 5$

$y = \frac{7(9x + 1)}{23}$

$= \frac{7((23n - 1) + 1)}{23}$

$= 7n$

$= 7(9t + 2)$

The next solution will be $x = 28$, $y = 112$.

The general solution is

$x = 5 + 23t$,

$y = 14 + 63t$; $t \ge 0$ and $t \in \mathbb{Z}$.

9 Consider the value of the two types of cattle.

$410x + 530y = 10\,000$

$41x + 53y = 1000$

Using a CAS calculator, a spreadsheet, or trial and error,

$x = 5$, $y = 15$.

5 of the \$410 cattle and 15 of the \$530 cattle.

10 Let the required number be x.

If it leaves a remainder of 6 when divided by 7, then $x = 7n + 6$.

If it leaves a remainder of 9 when

divided by 11, then $x = 11m + 9$.

$$7n + 6 = 11m + 9$$

$$7n - 11m = 3$$

One solution is $n = 2, m = 1$.

The general solution is $n = 2 + (-11)t$, $m = 1 - 7t$.

Replacing t with $-t$ gives $n = 2 + 11t$, $m = 1 + 7t$.

$t = 0$ gives $n = 2, m = 1, x = 7 \times 2 + 6 = 20$.

The smallest positive number is 20.

The general form is

$$\begin{aligned} x &= 7n + 6 \\ &= 7(2 + 11t) + 6 \\ &= 77t + 20 \text{ for } t \in \mathbb{N} \cup \{0\} \end{aligned}$$

11 Let x be the number of 5-litre jugs used and y the number of 3-litres jugs used.

$$\begin{aligned} 5x + 3y &= 7 \\ 5x &= 7 - 3y \\ x &= \frac{7 - 3y}{5} \end{aligned}$$

Solutions will only exist when $7 - 3y$ is a multiple of 3.

This occurs when $y = -1$:

$$x = \frac{7 + 3}{5} = 2$$

To measure exactly 7 litres, you would pour two full 5-litre jugs into a container and then remove one 3-litre jugful.

12 Obviously the post office can't sell 1c or 2c worth of postage. Nor can it sell 4c or 7c worth, because there's no way to arrange 3c and 5c to get those values.

It can sell 6c worth $(3 + 3 = 6)$ and 8c worth $(3 + 5 = 8)$.

So the problem can be rephrased as $3x + 5y = n, n \geq 8$ where x is the number of 3c stamps and y the number of 5c stamps.

If $n = 8, 3x + 5y = 8$; the obvious solution is $x = 1, y = 1$.

If $n = 9, 3x + 5y = 9$; the obvious solution is $x = 3, y = 0$.

If $n = 10, 3x + 5y = 10$; the obvious solution is $x = 0, y = 2$.

Since this set of three can be made using $3x + 5y$, the next set of three amounts (11, 12, 13) can be made as $3x + 5y + 3$, or by adding another 3c stamp.

Similarly, every set of three consecutive amounts can be made by adding an additional 3c stamp.

Therefore it's possible to create all amounts in excess of 3c, except for 4c and 7c.

13 Consider total cost.

$$1.7a + b = 29.6$$

$$17a + 10b = 296$$

Using a CAS calculator, a spreadsheet, or trial and error,

$a = 8, b = 16$.

8 of type A and 16 of type B.

14 $6x - 9y = 10$ has no integer sollutions since the left-hand-side is divisible by 3 while the right side is not.

15 $13k - 18m = 5$
$k = -1$ and $m = -1$ is a solution.
Next solution is $k = -1 + 18$ and $m = -1 + 13$ Therefore the multiple of 13 is 221

16 $10a + b - 5(a + b) = 17$
$5a - 4b = 17$
$a = 5$ and $b = 2$ is a solution.
The general solution is $a = 5 + 4n$ and $b = 2 + 5n, n \in \mathbb{Z}$.
A second solution is given by $n = 1$.
That is $a = 9$ and $b = 7$.

Solutions to Exercise 2G

1 a $b = aq + r,\ 0 \le r < a$

$43 = 5q + r$

$= 5 \times 8 + 3$

$(5, 43) = (5, 3)$ by theorem 2

$5 = 3 \times 1 + 2$

$(3, 5) = (3, 2)$ by theorem 2

$3 = 2 \times 1 + 1$

$(2, 3) = (2, 1)$ by theorem 2

$2 = 2 \times 1 + 0$

$\therefore\ (43, 5) = (5, 3) = 1$

b $b = aq + r,\ 0 \le r < a$

$39 = 13q + r$

$= 13 \times 3 + 0$

$\therefore\ (39, 13) = (13, 0) = 13$

c $b = aq + r,\ 0 \le r < a$

$37 = 17q + r$

$= 17 \times 2 + 3$

$(17, 37) = (17, 3)$ by theorem 2

$17 = 3 \times 5 + 2$

$(3, 17) = (3, 2)$ by theorem 2

$3 = 2 \times 1 + 1$

$(2, 3) = (2, 1)$ by theorem 2

$2 = 2 \times 1 + 0$

$\therefore\ (37, 17) = (17, 3) = 1$

d $b = aq + r,\ 0 \le r < a$

$128 = 16q + r$

$= 16 \times 8 + 0$

$\therefore\ (128, 16) = (16, 0) = 16$

2 If d is a common factor of a and b, then $a = nd$ and $b = md$.

$a + b = nd + md$

$= d(n + m)$

$\therefore$ d is a divisor of $a + b$.

If d is a common factor of a and b, then $a = nd$ and $b = md$.

$a - b = nd - md$

$= d(n - m)$

$\therefore$ d is a divisor of $a - b$.

3 a $9284 = 4361 \times 2 + 562$

$(4361, 9284) = (4361, 562)$

$4361 = 562 \times 7 + 427$

$(4361, 562) = (427, 562)$

$562 = 427 \times 1 + 135$

$(427, 562) = (135, 427)$

$427 = 135 \times 3 + 22$

$(135, 427) = (22, 135)$

This process could continue, but at this point it is quicker and easy to notice that the two numbers have no common factor other than 1, so $(4361, 9284) = 1$.

b $2160 = 999 \times 2 + 162$

$(999, 2160) = (162, 999)$

$999 = 162 \times 6 + 27$

$(162, 999) = (27, 162)$

$162 = 27 \times 6 + 0$

$(999, 2160) = 27$

c $(-372, 762) = (372, 762)$

$762 = 372 \times 2 + 18$

$(372, 762) = (372, 18)$

$372 = 18 \times 20 + 12$

$(372, 18) = (12, 18)$

$18 = 12 \times 1 + 6$

$(12, 18) = (6, 12)$

$12 = 6 \times 2 + 0$

$(-372, 762) = 6$

d $716\,485 = 5255 \times 136 + 1805$

$(716\,485, 5255) = (1805, 5255)$

$5255 = 1805 \times 2 + 1645$

$(1805, 5255) = (1805, 1645)$

$1805 = 1645 \times 1 + 160$

$(1805, 1645) = (160, 1645)$

$1645 = 160 \times 10 + 45$

$(160, 1645) = (45, 160)$

This process could continue, but at this point it is quicker and easy to notice that the two numbers have a highest common factor of 5, so $(716\,485, 5255) = 5$.

4 a Apply the division algorithm to 804 and 2358.

$2358 = 804 \times 2 + 750$

$804 = 750 \times 1 + 54$

$750 = 54 \times 13 + 48$

$54 = 48 \times 1 + 6$

$48 = 6 \times 8$

Working backwards with these results,

$6 = 54 - 48 \times 1$

$6 = 54 - (750 - 54 \times 13)$

$6 = 54 - 750 + 54 \times 13$

$6 = 54 \times 14 - 750$

$6 = (804 - 750 \times 1) \times 14 - 750$

$6 = 804 \times 14 - 750 \times 14 - 750$

$6 = 804 \times 14 - 750 \times 15$

$6 = 804 \times 14 - (2358 - 804 \times 2) \times 15$

$6 = 804 \times 14 - 2358 \times 15 + 804 \times 30$

$6 = 804 \times 44 - 2358 \times 15$

A solution is $x = 44, y = -15$.

The general solution is

$x = 44 + \frac{2358}{6}t$

$= 44 + 393t$

$y = -15 - \frac{804}{6}t$

$= -15 - 134t,\ t \in \mathbb{Z}$

b This is equivalent to $3x + 4y = 1$.

The algorithm is still useful.

$4 = 3 \times 1 + 1$

$1 = 3 \times -1 + 4$

A solution is $x = -1, y = 1$.

The general solution is

$x = -1 + 4t$

$y = 1 - 3t,\ t \in \mathbb{Z}$

c $478 = 3 \times -478 + 4 \times 478$

A solution is $x = -478, y = 478$.

The general solution is

$x = -478 + 4t$

$y = 478 - 3t,\ t \in \mathbb{Z}$

(If you use the algorithm, you will find the solution $x = 1434, y = -956$.

Then the general solution can be expressed as
$x = 1434 + 4t$
as
$y = -956 - 3t,\ t \in \mathbb{Z}$.)

d The algorithm is still useful.

$$-5 = 3 \times -2 + 1$$
$$1 = 3 \times 2 + -5$$
$$1 = 3 \times 2 - 5 \times 1$$
$$38 = 3 \times 76 - 5 \times 38$$

A solution is $x = 76, y = 38$
The general solution is

$$x = 76 + 5t$$
$$y = 38 + 3t$$

This can be simplified. If $t - 15$ is used instead of t, then

$$x = 76 + 5(t - 15)$$
$$= 1 + 5t$$
$$y = 38 + 3(t - 15)$$
$$= -7 + 3t,\ t \in R$$

e Apply the division algorithm to 804 and 2688.

$$2688 = 804 \times 3 + 276$$
$$804 = 276 \times 2 + 252$$
$$276 = 252 \times 1 + 24$$
$$252 = 24 \times 10 + 12$$
$$24 = 12 \times 2$$

Working backwards with these results,

$$12 = 252 - 24 \times 10$$
$$12 = 252 - (276 - 252 \times 1) \times 10$$
$$12 = 252 - 276 \times 10 + 252 \times 10$$
$$12 = 252 \times 11 - 276 \times 10$$
$$12 = (804 - 276 \times 2) \times 11 - 276 \times 10$$
$$12 = 804 \times 11 - 276 \times 22 - 276 \times 10$$
$$12 = 804 \times 11 - 276 \times 32$$
$$12 = 804 \times 11 - (2688 - 804 \times 3) \times 32$$
$$12 = 804 \times 11 - 2688 \times 32 + 804 \times 96$$
$$12 = 804 \times 107 - 2688 \times 32$$

A solution is $x = 107, y = -32$.
The general solution is

$$x = 107 + \frac{2688}{12}t$$
$$= 107 + 224t$$
$$y = -32 - \frac{804}{12}t$$
$$= -32 - 67t,\ t \in R$$

f Apply the division algorithm to 1816 and 2688.

$$2688 = 1816 \times 1 + 872$$
$$1816 = 872 \times 2 + 72$$
$$872 = 72 \times 12 + 8$$
$$72 = 8 \times 9$$

Working backwards with these results,

$8 = 872 - 72 \times 12$

$8 = 872 - (1816 - 872 \times 2) \times 12$

$8 = 872 - 1816 \times 12 + 872 \times 24$

$8 = 872 \times 25 - 1816 \times 12$

$8 = (2688 - 1816 \times 1) \times 25$
$\quad - 1816 \times 12$

$8 = 2688 \times 25 - 1816 \times 25$
$\quad - 1816 \times 12$

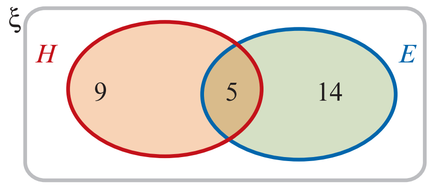
$8 = 2688 \times 25 - 1816 \times 37$

A solution is $x = -37, y = 25$.
The general solution is

$$x = -37 + \frac{2688}{8}t$$
$$= -37 + 336t$$
$$y = 25 - \frac{1816}{8}t$$
$$= 25 - 227t, t \in \mathbb{Z}$$

Solutions to Exercise 2H

1 a



Since all students do at least one of these subjects, $9 + 5 + x = 28$

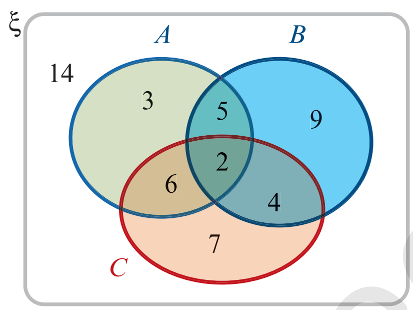
$x = 14$

b **i** $5 + 14 = 19$

ii 9

iii $9 + 14 = 23$ or $28 - 5 = 23$

2 a



b **i** $n(A' \cap C') = 9 + 14 = 23$

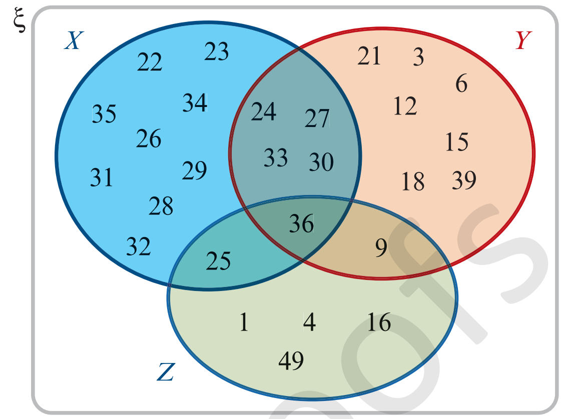
ii

$n(A \cup B') = 3 + 6 + 5 + 2 + 7 + 14$

$= 37$

iii $n(A' \cap B \cap C') = 9$

3



Since 40% don't speak Greek,

$y + 20\% = 40\%$

$y = 20\%$

Since 40% speak Greek,

$x + 20\% = 40\%$

$x = 20\%$

20% speak both languages.

4 Since $40 - 25 = 15$ don't own a cat,

$y + 6 = 15$

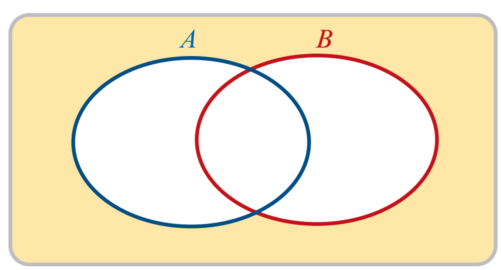
$y = 9$

Since 16 own a dog, $x + 9 = 16$

$x = 7$

Seven students own both.

5 a



$(A \cup B)' = A' \cap B'$ is shaded

We must assume every delegate spoke at least one of these languages. If 70 spoke English, and 25 spoke English and French, 45 spoke English but not French.

$\therefore 45 + 50 = 95$ spoke either

English or French or both.

$\therefore$ 105 − 95 = 10 spoke only Japanese.

If 50 spoke French, and 15 spoke French and Japanese, 35 spoke French but not Japanese.

$\therefore$ 35 + 50 = 85 spoke either French or Japanese or both.

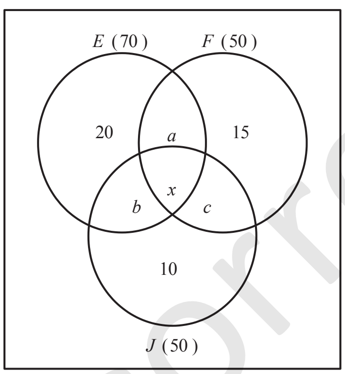
$\therefore$ 105 − 85 = 20 spoke only English.

If 50 spoke Japanese, and 30 spoke Japanese and English, 20 spoke Japanese but not English.

$\therefore$ 20 + 70 = 90 spoke either Japanese or English or both.

$\therefore$ 105 − 90 = 15 spoke only French.

We can now fill in more of the Venn diagram.



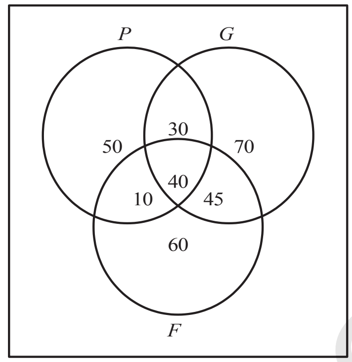
c is the number who don't speak English.

$$105 - 70 = 10 + c + 15$$
$$c + 25 = 35$$
$$c = 10$$
$$x + c = 15$$
$$x = 5$$

5 delegates speak all five languages.

b We have already found that 10 spoke only Japanese.

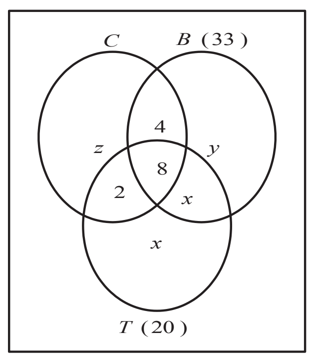
6 Enter the information into a Venn diagram.



Number having no dessert

$$= 350 - 50 - 30 - 70 - 10$$
$$- 40 - 45 - 60$$
$$= 45$$

7 Insert the given information on a Venn diagram. Place y as the number taking a bus only, and z as the number taking a car only.



a Using $n(T) = 20$, $2x + 10 = 20$

$$x = 5$$

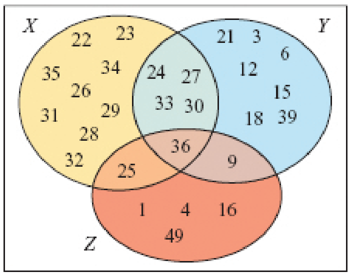
b Using $n(B) = 33$ and $x = 5$,

$$12 + 5 + y = 33$$
$$y = 16$$

c Assume they all used at least one of these forms of transport.

$$z + 4 + 8 + 16 + 2 + 5 + 5 = 40$$
$$z = 0$$

8 **a**



b **i**

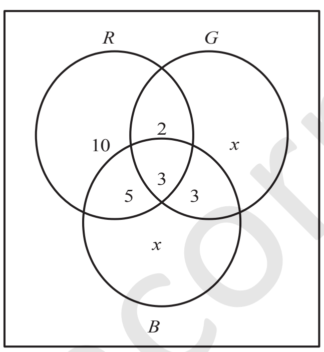
$(X \cap Y \cap \mathbb{Z})$ = intersection of all sets

= 36 (from diagram)

ii $n(X \cap Y)$ = number of elements in both X and Y

= 5 (from diagram)

9 The following information can be placed on a Venn diagram.



The additional information gives $5 > x$ and $x > 3$.

$\therefore \quad x = 4$

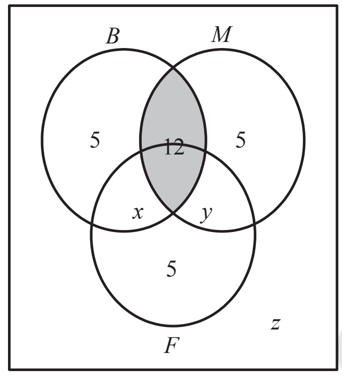
Number of students

$= 10 + 2 + 4 + 5 + 3 + 3 + 4$

$= 31$

20 bought red pens, 12 bought green pens and 15 bought black pens.

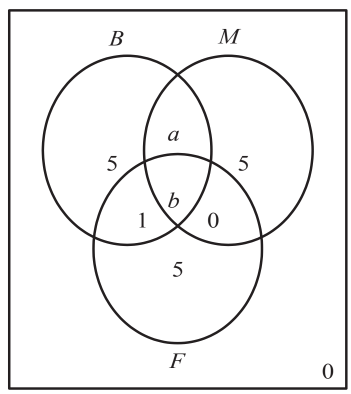
10 Enter the given information as below.
$B \cap M$ is shaded.



$$5 + 12 + 5 + 5 + x + y + z = 28$$
$$27 + x + y + z = 28$$
$$x + y + z = 1$$

This means that exactly one of x, y and z must equal 1, and the other two will equal zero.
Since $n(F \cap B) > n(M \cap F)$, the Venn diagram shows that this means $x > y$.

$\therefore \quad x = 1, y = z = 0$



$a + b = 12$

$n(M \cap F \cap B) = n(F')$

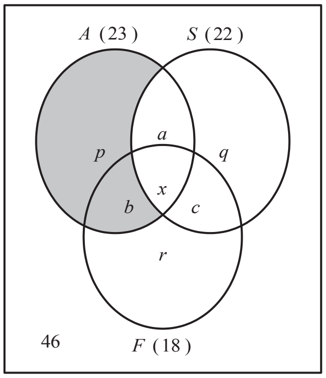
$\therefore \quad b = a + 10$

Substitute in $a + b = 12$:

$$a + (a + 10) = 12$$
$$2a = 2$$
$$a = 1$$
$$b = a + 10 = 11$$

$n(M \cap F) = b + 0 = 11$

11 Enter the given information as below.



$a + x = n(A \cap S) = 10$

The number of elements in the shaded region is given by

$$n(A \cap S') = n(A) - (a + x)$$
$$= 23 - 10$$
$$= 13$$
$$n(A \cup S) = 10 + 22$$
$$= 32$$
$$\therefore r + 46 = 80 - 32 = 48$$
$$r = 2$$

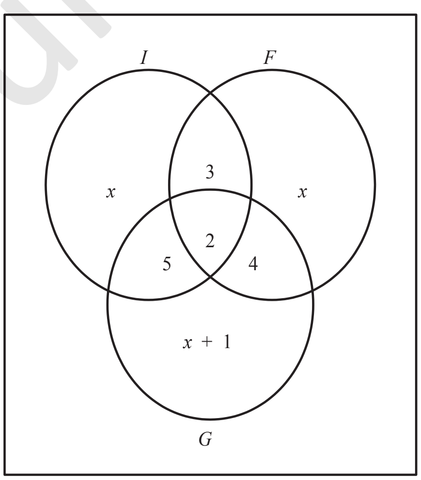
Use similar reasoning to show

$$c + r = 18 - (b + x)$$
$$= 18 - 11 = 7$$

Since $r = 2$, $c = 5$

Since $x + c = n(S \cup F) = 6$ and $c = 5$, $x = 1$ One person plays all three sports.

12 Enter the information into a Venn diagram.



Since they are all proficient in at least one language,

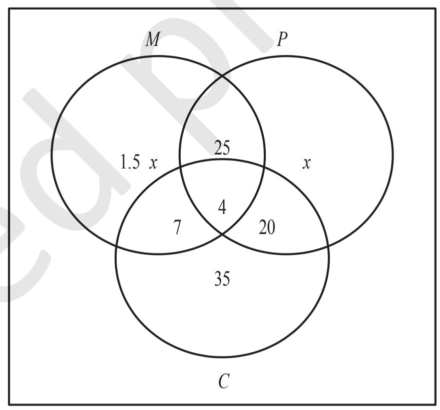
$$x + 3 + x + 5 + 2 + 4 + x + 1 = 33$$
$$3x + 15 = 33$$
$$3x = 18$$
$$x = 6$$

The number proficient in Italian

$= 6 + 3 + 2 + 5$

$= 16$

13 Enter the given information into a Venn diagram.



$$1.5x + 25 + x + 7 + 4 + 20 + 35 = 201$$
$$2.5x + 91 = 201$$
$$2.5x = 110$$
$$x = \frac{110}{2.5}$$
$$= 44$$

The number studying Mathematics

$= 1.5x + 25 + 7 + 4$

$= 66 + 25 + 7 + 4$

$= 102$

Solutions to Technology-free questions

1 a

$$0.0\dot{7} = 0.07777\ldots$$
$$0.0\dot{7} \times 10 = 0.7777\ldots$$
$$0.0\dot{7} \times 9 = 0.7 = \frac{7}{10}$$
$$0.0\dot{7} = \frac{7}{90}$$

b

$$0.\dot{4}\dot{5} = 0.454545\ldots$$
$$0.\dot{4}\dot{5} \times 100 = 45.4545\ldots$$
$$0.\dot{4}\dot{5} \times 99 = 45$$
$$0.\dot{4}\dot{5} = \frac{45}{99} = \frac{5}{11}$$

c $0.005 = \dfrac{5}{1000} = \dfrac{1}{200}$

d $0.405 = \dfrac{405}{1000} = \dfrac{81}{200}$

e

$$0.2\dot{6} = 0.26666\ldots$$
$$0.2\dot{6} \times 10 = 2.6666\ldots$$
$$0.2\dot{6} \times 9 = 2.4 = \frac{24}{10}$$
$$0.2\dot{6} = \frac{24}{90} = \frac{4}{15}$$

f $0.1\dot{7}1428\dot{5}$

$$= 0.1714825714\ldots$$
$$0.1\dot{7}1428\dot{5} \times 10^6$$
$$= 171\,428.5714285\ldots$$
$$0.1\dot{7}1428\dot{5} \times (10^6 - 1)$$
$$= 171\,428.4$$
$$= \frac{1\,714\,284}{10}$$
$$0.1\dot{7}1428\dot{5}$$
$$= \frac{1\,714\,284}{9\,999\,990}$$
$$= \frac{6}{35}$$

2

$$2\,\overline{)504}$$
$$2\,\overline{)252}$$
$$2\,\overline{)126}$$
$$3\,\overline{)\ 63}$$
$$3\,\overline{)\ 21}$$
$$7\,\overline{)\ \ 7}$$
$$1$$

$504 = 2^3 \times 3^2 \times 7$

3 a $|n^2 - 9|$ is prime.

$|n^2 - 9| = |n - 3||n + 3|$

For it to be prime either $|n - 3| = 1$ or $|n + 3| = 1$

If $|n - 3| = 1$, then $n = 4$ or $n = 2$

If $|n + 3| = 1$, then $n = -4$ or $n = -2$

b i $x^2 + 5|x| - 6 = 0$

Consider two cases:

$x \geq 0: \quad x^2 + 5x - 6 = 0$

$(x + 6)(x - 1) = 0$

$\therefore x = 1$

$x < 0 : \quad x^2 - 5x - 6 = 0$

$(x - 6)(x + 1) = 0$

$\therefore x = -1$

ii $x + |x| = 0$

Consider two cases:

$x \geq 0 : 2x = 0 \Rightarrow x = 0$

$x < 0$:Always true

Therefore the solution is $x \leq 0$

c $5 - |x| < 4$

$|x| > 1$

$\therefore x > 1$ or $x < -1$.

4 a $\dfrac{2\sqrt{3} - 1}{\sqrt{2}} = \dfrac{2\sqrt{3} - 1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{2\sqrt{6} - \sqrt{2}}{2}$

b $\dfrac{\sqrt{5} + 2}{\sqrt{5} - 2} = \dfrac{\sqrt{5} + 2}{\sqrt{5} - 2} \times \dfrac{\sqrt{5} + 2}{\sqrt{5} + 2}$

$= \dfrac{5 + 4\sqrt{5} + 4}{5 - 4}$

$= 4\sqrt{5} + 9$

c $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} \times \dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}}$

$= \dfrac{3 + 2\sqrt{6} + 2}{3 - 2}$

$= 2\sqrt{6} + 5$

5 $\dfrac{3 + 2\sqrt{75}}{3 - \sqrt{12}} = \dfrac{3 + 2\sqrt{25 \times 3}}{3 - \sqrt{4 \times 3}}$

$= \dfrac{3 + 2 \times 5\sqrt{3}}{3 - 2\sqrt{3}}$

$= \dfrac{3 + 10\sqrt{3}}{3 - 2\sqrt{3}} \times \dfrac{3 + 2\sqrt{3}}{3 + 2\sqrt{3}}$

$= \dfrac{9 + 6\sqrt{3} + 30\sqrt{3} + 60}{9 - 12}$

$= \dfrac{69 + 36\sqrt{3}}{-3}$

$= -23 - 12\sqrt{3}$

6 a $\dfrac{6\sqrt{2}}{3\sqrt{2} - 2\sqrt{3}} = \dfrac{6\sqrt{2}}{3\sqrt{2} - 2\sqrt{3}}$

$\times \dfrac{3\sqrt{2} + 2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}}$

$= \dfrac{36 + 12\sqrt{6}}{18 - 12}$

$= \dfrac{36 + 12\sqrt{6}}{6}$

$= 6 + 2\sqrt{6}$

b $\dfrac{\sqrt{a + b} - \sqrt{a - b}}{\sqrt{a + b} + \sqrt{a - b}}$

$= \dfrac{\sqrt{a + b} - \sqrt{a - b}}{\sqrt{a + b} + \sqrt{a - b}}$

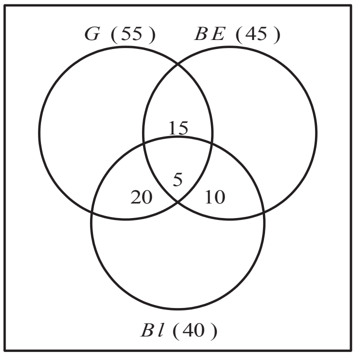
$\times \dfrac{\sqrt{a + b} - \sqrt{a - b}}{\sqrt{a + b} - \sqrt{a - b}}$

$= \dfrac{a + b - 2\sqrt{(a + b)(a - b)} + a - b}{(a + b) - (a - b)}$

$= \dfrac{2a - 2\sqrt{a^2 - b^2}}{2b}$

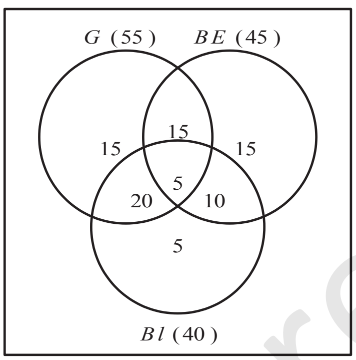
$= \dfrac{a - \sqrt{a^2 - b^2}}{b}$

7 First enter the information on a Venn diagram.



a It is obvious to make up the 40 blonds that 5 must be blond only, so the number of boys (not girls) who are blond is 5 + 10 = 15.

b The rest of the Venn diagram can be filled in the same way:

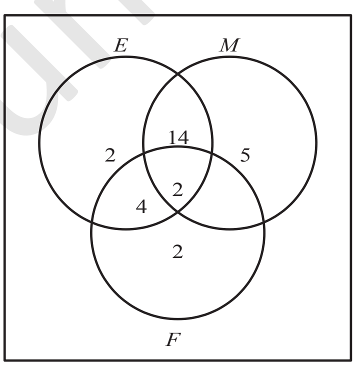


Boys not blond or blue-eyed

= 100 – 15 – 15 – 15 – 20 – 5 – 10 – 5

= 15

8 Fill in a Venn diagram.

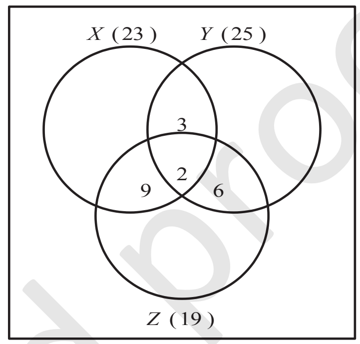


a 30 – 2 – 14 – 5 – 4 – 2 – 2 = 1 (since all received at least one prize.)

b 14 + 5 + 2 + 1 = 22

c 2 + 14 + 4 + 2 = 22

9 Enter the given information on a Venn diagram as below.



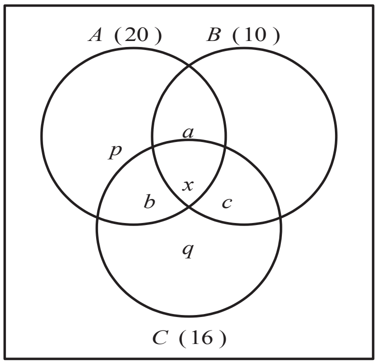
The numbers liking *X* only, *Y* only and *Z* only are 9, 14 and 2 respectively.

The number who like none of them

= 50 – 9 – 3 – 14 – 9 – 2 – 6 – 2

= 5

10 The rectangles can be represented by circles for clarity. Enter the data:



Note: $a + x = 3$, $b + x = 6$ and $c + x = 4$

$$p + b + a + x = 20$$
$$p + b + 3 = 20$$
$$p + b = 17$$
$$q + (p + b) + n(B) = 35$$
$$q + 17 + 10 = 35$$
$$\therefore q = 8$$
$$q + (b + x) + c = n(C) = 16$$
$$8 + 6 + c = 16$$
$$\therefore c = 2$$
$$c + x = 4$$
$$\therefore x = 2$$

There is 2 cm^2 in common.

11 $\sqrt{112} - \sqrt{63} - \dfrac{224}{\sqrt{28}}$

$$= \sqrt{16 \times 7} - \sqrt{9 \times 7} - \frac{224}{\sqrt{4 \times 7}}$$
$$= 4\sqrt{7} - 3\sqrt{7} - \frac{224}{2\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$
$$= 4\sqrt{7} - 3\sqrt{7} - \frac{224\sqrt{7}}{14}$$
$$= 4\sqrt{7} - 3\sqrt{7} - 16\sqrt{7}$$
$$= -15\sqrt{7}$$

12 Cross multiply:

$$(\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3}) = x^2$$
$$7 - 3 = x^2$$
$$4 = x^2$$
$$x = \pm 2$$

13 $\dfrac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} + \dfrac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}}$

$$= \frac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} \times \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}} + \frac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}}$$
$$= \frac{\sqrt{5} - \sqrt{5} + \sqrt{10} - \sqrt{6}}{5 - 3} + \frac{\sqrt{5} + \sqrt{5} - \sqrt{10} - \sqrt{6}}{5 - 3}$$
$$= \frac{2\sqrt{5} - 2\sqrt{6}}{2}$$
$$= \sqrt{5} - \sqrt{6}$$

14 $\sqrt{27} - \sqrt{12} + 2\sqrt{75} - \sqrt{\dfrac{48}{25}}$

$$= \sqrt{9 \times 3} - \sqrt{4 \times 3} + 2\sqrt{25 \times 3} - \frac{\sqrt{16 \times 3}}{\sqrt{25}}$$
$$= 3\sqrt{3} - 2\sqrt{3} + 10\sqrt{3} - \frac{4\sqrt{3}}{5}$$
$$= \frac{15\sqrt{3} - 10\sqrt{3} + 50\sqrt{3} - 4\sqrt{3}}{5}$$
$$= \frac{51\sqrt{3}}{5}$$

15 **a** $|A \cup B| = 32 + 7 + 15 + 3 = 57$

b $C = 3$

c $B' \cap A = 32$

16 $17 + 6\sqrt{8} = 17 + 2 \times \sqrt{9} \times \sqrt{8}$

$= 17 + 2\sqrt{72}$

$a + b = 17$; $ab = 72$

$a = 8$, $b = 9$ (or $a = 9$, $b = 8$, giving the same answer.)

$(\sqrt{8} + \sqrt{9})^2 = 17 + 6\sqrt{8}$

So the square root of

$17 + 6\sqrt{8} = \sqrt{8} + \sqrt{9}$

$= 2\sqrt{2} + 3$

17 $1885 = 365 \times 5 + 60$

$(1885, 365) = (60, 365)$

$365 = 60 \times 6 + 5$

$(60, 365) = (60, 5)$

$60 = 5 \times 12 + 0$

$(1885, 365) = 5$

18 a Apply the division algorithm to 43 and 9.

$43 = 9 \times 4 + 7$

$9 = 7 \times 1 + 2$

$7 = 2 \times 3 + 1$

$2 = 2 \times 1$

Working backwards with these results,

$1 = 7 - 2 \times 3$

$1 = 7 - (9 - 7 \times 1) \times 3$

$1 = 7 - 9 \times 3 + 7 \times 3$

$1 = 7 \times 4 - 9 \times 3$

$1 = (43 - 9 \times 4) \times 4 - 9 \times 3$

$1 = 43 \times 4 - 9 \times 16 - 9 \times 3$

$1 = 43 \times 4 - 9 \times 19$

A solution to $9x + 43y = 1$ is $x = -19, y = 4$.

A solution to $9x + 43y = 7$ is $x = -19 \times 7 = -133, y = 4 \times 7 = 28$.

The general solution is

$x = -133 + 43t$

$y = 28 - 9t, t \in R$

Other solutions are possible.

$t = 4$ gives a specific solution of $x = 39$, $y = -8$, leading to a general solution of

$x = 39 + 43t$

$y = -8 - 9t, \ t \in R$

b If $x > 0, 39 + 43t > 0$

$t > -\frac{39}{43}$

If $y > 0, -8 - 9t > 0$

$t < -\frac{8}{9}$

These two inequations cannot both be true if x is an integer.

There is no solution for $x \in \mathbb{Z}^+, \ y \in \mathbb{Z}^+$.

19 If a and b are odd, they may be written as $2n + 1$ and $2m + 1$ respectively, where n and m are integers.

$ab = (2n + 1)(2m + 1)$

$= 4mn + 2n + 2m + 1$

$= 2(2mn + n + m) + 1$

This will be an odd number since $2mn + n + m$ is an integer.

20

$$12\,121 = 10\,659 \times 1 + 1462$$
$$(12\,121, 10\,659) = (1462, 10\,659)$$
$$10\,659 = 1462 \times 7 + 425$$
$$(1462, 10\,659) = (1462, 425)$$
$$1462 = 425 \times 3 + 187$$
$$(1462, 425) = (187, 425)$$
$$425 = 187 \times 2 + 51$$
$$(187, 425) = (187, 51)$$
$$187 = 51 \times 3 + 34$$
$$(187, 51) = (51, 34)$$
$$51 = 34 \times 1 + 17$$
$$(51, 34) = (34, 17)$$
$$34 = 17 \times 2 + 0$$
$$(12\,121, 10\,659) = 17$$

21 **a** The algorithm is still useful.

$$7 = 5 \times 1 + 2$$
$$5 = 2 \times 2 + 1$$
$$1 = 5 - 2 \times 2$$
$$1 = 5 - (7 - 5 \times 1) \times 2$$
$$1 = 5 - 7 \times 2 + 5 \times 2$$
$$1 = 5 \times 3 - 7 \times 2$$

A solution is $x = 3, y = -2$.
The general solution is

$$x = 3 + 7t$$
$$y = -2 - 5t,\ t \in R$$

b If $1 = 5 \times 3 - 7 \times 2$, then
$100 = 5 \times 300 - 7 \times 200$.
A solution is $x = 300, y = -200$.
The general solution is

$$x = 300 + 7t$$
$$y = -200 - 5t,\ t \in \mathbb{R}R$$
$$x = 300 + 7t, y = -200 - 5t$$

c If $y \geq x$,

$$-2 - 5t \geq 3 + 7t$$
$$-12t \geq 5$$
$$t \leq -\frac{5}{12}$$

Since t is an integer, $t \leq -1$.
The solution is

$$x = 3 + 7t$$
$$y = -2 - 5t, t \leq -1, t \in \mathbb{R}$$

22 First, let Tom's age be t and Fred's age be f.
Since it appears Tom is older than Fred, and we must look at the time when Tom was Fred's age, we will define d as the difference in ages, specifically how many years Tom is older than Fred.

$$t = f + d$$
$$t + f = 63$$
$$\therefore\ (f + d) + f = 63$$
$$2f + d = 63$$

When Tom was Fred's age, d years ago, Fred was aged $f - d$.
Tom is now twice that age, $2(f - d)$.

$$\therefore\ t = 2(f - d)$$
$$\therefore\ t = 2(f - d)$$

Since $t = f + d$,

$$f + d = 2(f - d)$$
$$= 2f - 2d$$
$$3d = f$$

Substitute $f = 3d$ into $2f + d = 63$.

$$\begin{aligned}
6d + d &= 63 \\
7d &= 63 \\
d &= 9 \\
f &= 3d \\
&= 27 \\
t + f &= 63 \\
t &= 36
\end{aligned}$$

Tom is 36 and Fred is 27.

Solutions to multiple-choice questions

1 A
$$\frac{4}{3+2\sqrt{2}} = \frac{4}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$
$$= \frac{12-8\sqrt{2}}{9-8}$$
$$= 12-8\sqrt{2}$$

2 D
$2\overline{)86\,400}$
$2\overline{)43\,200}$
$2\overline{)21\,600}$
$2\overline{)10\,800}$
$2\overline{)5400}$
$2\overline{)2700}$
$2\overline{)1350}$
$3\overline{)675}$
$3\overline{)225}$
$3\overline{)75}$
$5\overline{)25}$
$5\overline{)5}$
$\quad 1$

Prime decomposition
$= 2^7 \times 3^3 \times 5^2$

3 D $(\sqrt{6}+3)(\sqrt{6}-3)$
$= (\sqrt{6})^2 + 3\sqrt{6} - 3\sqrt{6} - 9$
$= 6 - 9$
$= -3$

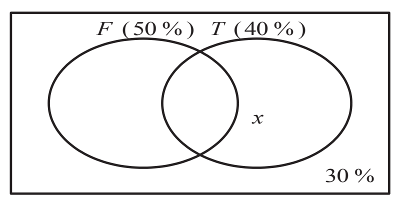
4 D $B' \cap A$ = numbers in set A that are not also in set B
$= \{1, 2, 4, 5, 7, 8\}$

5 C $(3, \infty) \cap (-\infty, 5]$
$= \{x \in R : x > 3\} \cap \{x \in R : x \le 5\}$
$= \{x \in R : 3 < x \le 5\}$
$= (3, 5]$

6 D The next time will be both a multiple of 6 and a multiple of 14.
$$\text{LCM} = \frac{6 \times 14}{3}$$
$$= 42$$
The next time is in 42 minutes.

7 B
$X \cap Y \cap \mathbb{Z}$ = set of numbers that are multiples of 2, 5 and 7
$\text{LCM} = 2 \times 5 \times 7$
$= 35$

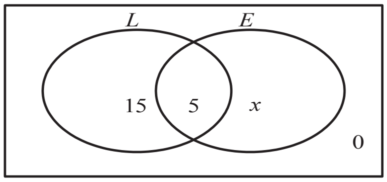
8 B Draw a Venn diagram.



Since 50% don't play football,
$x + 30\% = 50\%$
$x = 20\%$
Since 40% play tennis, it can be seen that 20% play both sports.

9 C
$$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} = \frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} \times \frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}-\sqrt{6}}$$
$$= \frac{7-2\sqrt{42}+6}{7-6}$$
$$= 13-2\sqrt{42}$$

10 A Draw a Venn diagram.



$15 + 5 + x = 40$

$x = 20$

20 students take only Economics.

11 D

12 D You can choose any number of 2s from 0 to p in $(p + 1)$ ways. For each of these, you can choose any number of 3s from 0 to q in $(q + 1)$ ways, and for each of these combinations you can choose any number of 5s from 0 to r in $(r + 1)$ ways.

The total number of ways = $(p + 1)(q + 1)(r + 1)$

13 B $m + n = mn$

$$n = mn - m$$

$$= m(n - 1)$$

$$m = \frac{n}{n-1}$$

This will only be an integer if $n = 2, m = 2$ or $n = 0, m = 0$.

There are two solutions.

Solutions to extended-response questions

1 **a** $(\sqrt{x} + \sqrt{y})^2 = (\sqrt{x} + \sqrt{y})(\sqrt{x} + \sqrt{y})$

$= \sqrt{x}(\sqrt{x} + \sqrt{y}) + \sqrt{y}(\sqrt{x} + \sqrt{y})$

$= x + \sqrt{x}\sqrt{y} + \sqrt{y}\sqrt{x} + y$

$= x + y + 2\sqrt{x}\sqrt{y}$

$= x + y + 2\sqrt{xy}$

b From **a**, $(\sqrt{3} + \sqrt{5})^2 = 3 + 5 + 2\sqrt{3}\sqrt{5}$

$= 8 + 2\sqrt{15}$

$\therefore \quad \sqrt{3} + \sqrt{5} = \sqrt{8 + 2\sqrt{15}}$

c **i** $(\sqrt{11} + \sqrt{3})^2 = 11 + 3 + 2\sqrt{11}\sqrt{3}$

$= 14 + 2\sqrt{33}$

$\therefore \sqrt{14 + 2\sqrt{33}} = \sqrt{11} + \sqrt{3}$

ii $(\sqrt{8} - \sqrt{7})^2 = 8 + 7 - 2\sqrt{8}\sqrt{7}$ (also consider $-\sqrt{8} + \sqrt{7}$)

$= 15 - 2\sqrt{56}$

$\therefore \sqrt{15 - 2\sqrt{56}} = \sqrt{8} - \sqrt{7}$

$= 2\sqrt{2} - \sqrt{7}$

iii $(\sqrt{27} - \sqrt{24})^2 = 27 + 24 - 2\sqrt{27}\sqrt{24}$

$= 51 - 2 \times 3\sqrt{3} \times 2\sqrt{3}\sqrt{2}$

$= 51 - 36\sqrt{2}$

$\therefore \sqrt{51 - 36\sqrt{2}} = \sqrt{27} - \sqrt{24}$

$= 3\sqrt{3} - 2\sqrt{6}$

2 **a** $(2 + 3\sqrt{3}) + (4 + 2\sqrt{3}) = 2 + 4 + 3\sqrt{3} + 2\sqrt{3}$

$= 6 + 5\sqrt{3}$

Hence $a = 6$ and $b = 5$.

b $(2 + 3\sqrt{3})(4 + 2\sqrt{3}) = 2(4 + 2\sqrt{3}) + 3\sqrt{3}(4 + 2\sqrt{3})$

$$= 8 + 4\sqrt{3} + 12\sqrt{3} + 18$$

$$= 26 + 16\sqrt{3}$$

Hence $p = 26$ and $q = 16$.

c $\dfrac{1}{3 + 2\sqrt{3}} = \dfrac{1}{3 + 2\sqrt{3}} \times \dfrac{3 - 2\sqrt{3}}{3 - 2\sqrt{3}}$

$$= \frac{3 - 2\sqrt{3}}{9 - 12}$$

$$= \frac{3 - 2\sqrt{3}}{-3}$$

$$= -1 + \frac{2}{3}\sqrt{3}$$

Hence $a = -1$ and $b = \dfrac{2}{3}$.

d **i** $(2 + 5\sqrt{3})x = 2 - \sqrt{3}$

$$\therefore x = \frac{2 - \sqrt{3}}{2 + 5\sqrt{3}}$$

$$= \frac{2 - \sqrt{3}}{2 + 5\sqrt{3}} \times \frac{2 - 5\sqrt{3}}{2 - 5\sqrt{3}}$$

$$= \frac{(2 - \sqrt{3})(2 - 5\sqrt{3})}{4 - 75}$$

$$= \frac{2(2 - 5\sqrt{3}) - \sqrt{3}(2 - 5\sqrt{3})}{-71}$$

$$= \frac{4 - 10\sqrt{3} - 2\sqrt{3} + 15}{-71}$$

$$= \frac{19 - 12\sqrt{3}}{-71}$$

$$= \frac{12\sqrt{3} - 19}{71}$$

ii $(x - 3)^2 - 3 = 0$

$\therefore \quad (x - 3)^2 = 3$

$\therefore \quad x - 3 = \pm\sqrt{3}$

$\therefore \quad x = 3 \pm \sqrt{3}$

iii $(2x-1)^2 - 3 = 0$

$\therefore\quad (2x-1)^2 = 3$

$\therefore\quad 2x - 1 = \pm\sqrt{3}$

$\therefore\quad 2x = 1 \pm \sqrt{3}$

$\therefore\quad x = \dfrac{1 \pm \sqrt{3}}{2}$

e If $b = 0$, $a + b\sqrt{3} = a$. Hence every rational number, a, is a member of $\{a + b\sqrt{3} : a, b \in Q\}$.

3 a
$$\frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$$
$$= \frac{2-\sqrt{3}}{4-3}$$
$$= 2-\sqrt{3}$$

b
$$\left(\sqrt{2+\sqrt{3}}\right)^x + \left(\sqrt{2-\sqrt{3}}\right)^x = 4$$
$$\therefore \left(\sqrt{2+\sqrt{3}}\right)^x + \left(\sqrt{\frac{1}{2+\sqrt{3}}}\right)^x = 4 \left(\text{as } \frac{1}{2+\sqrt{3}} = 2-\sqrt{3} \text{ from } \mathbf{a}\right)$$
$$\therefore \left(\sqrt{2+\sqrt{3}}\right)^x + \frac{1}{\left(\sqrt{2+\sqrt{3}}\right)^x} = 4$$
$$\therefore t + \frac{1}{t} = 4 \quad \text{where } t = \left(\sqrt{2+\sqrt{3}}\right)^x$$

c
$$t + \frac{1}{t} = 4$$
$$\therefore\quad t^2 + 1 = 4t$$
$$\therefore\quad t^2 - 4t + 1 = 0$$

Using the general quadratic formula $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1$, $b = -4$ and $c = 1$ gives

$$t = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 1}}{2 \times 1}$$
$$= \frac{4 \pm \sqrt{16 - 4}}{2}$$
$$= \frac{4 \pm \sqrt{12}}{2}$$
$$= \frac{4 \pm 2\sqrt{3}}{2}$$
$$= 2 \pm \sqrt{3}$$

d From **c**, $t = 2 + \sqrt{3}$ or $t = 2 - \sqrt{3}$

but $t = (\sqrt{2 + \sqrt{3}})^x$,

$$\therefore \quad (\sqrt{2 + \sqrt{3}})^x = 2 + \sqrt{3} \qquad ①$$
$$\text{or } (\sqrt{2 + \sqrt{3}})^x = 2 - \sqrt{3} \qquad ②$$

From ① $(2 + \sqrt{3})^{\frac{x}{2}} = 2 + \sqrt{3}$
$$\therefore \frac{x}{2} = 1$$
$$\therefore x = 2$$

and from ② $(2 + \sqrt{3})^{\frac{x}{2}} = \frac{1}{2 + \sqrt{3}} \quad \left(\text{as } \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3} \text{ from } \mathbf{a}\right)$
$$= (2 + \sqrt{3})^{-1}$$
$$\therefore \frac{x}{2} = -1$$
$$\therefore x = -2$$

Hence the solutions of the equation $(\sqrt{2 + \sqrt{3}})^x + (\sqrt{2 - \sqrt{3}})^x = 4$ are $x = \pm 2$.

Graphics calculator techniques for Question 3

CAS calculator techniques for Question 3

3 d A CAS calculator can be used to help understand the structure of this question.

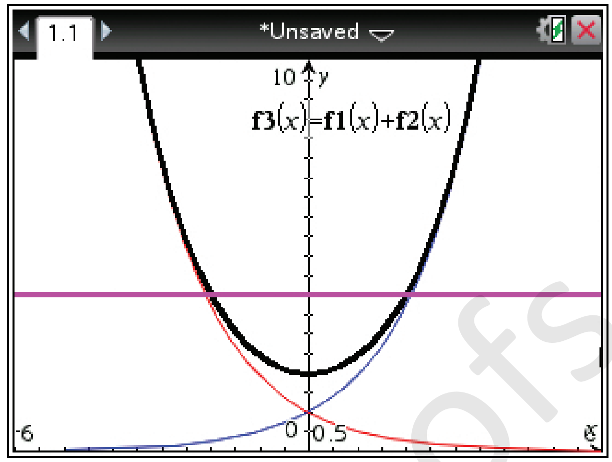
TI: Sketch the graphs of

$\boldsymbol{f1 = \left(\sqrt{2 + \sqrt{3}}\right)^x}$,

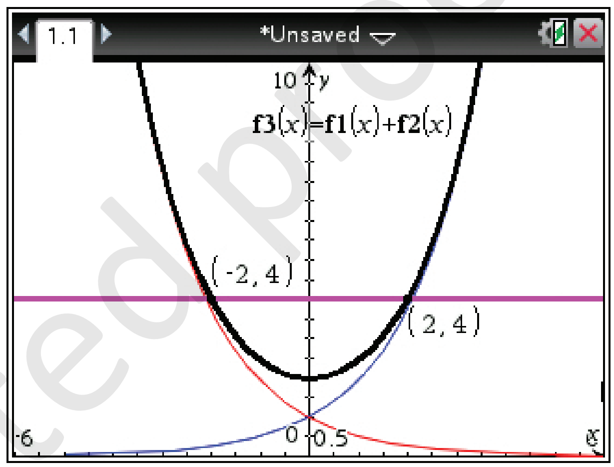
$\boldsymbol{f2 = \left(\sqrt{2 - \sqrt{3}}\right)^x}$, $\boldsymbol{f3 = f1(x) + f2(x)}$ and

$\boldsymbol{f4 = 4}$

Press **Menu** → **6**:**Analyze Graph** → **4**:**Intersection**
Repeat this process to find the other intersection point



Altematively, with the graphs still active, type **solve**(**f3**(**x**) = **4**, **x**) in the Calculator application



CP: Sketch the graphs of

$\boldsymbol{y1 = \left(\sqrt{2+\sqrt{3}}\right)^x}$,

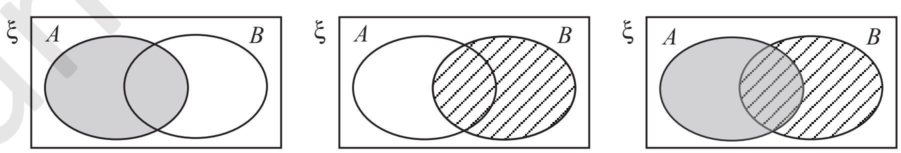
$\boldsymbol{y2 = \left(\sqrt{2-\sqrt{3}}\right)^x}$,

$\boldsymbol{y3 = y1(x) + y2(x)}$ and

$\boldsymbol{y4 = 4}$

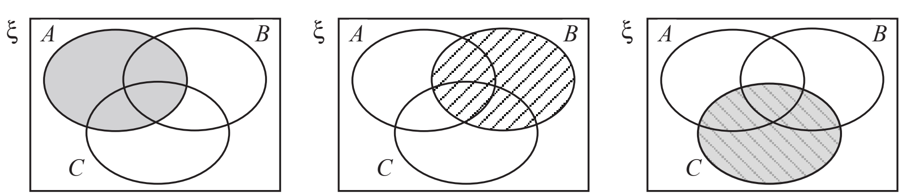
Tap **Analysis** → **G-Solve** → **Intersect**
Use the Up and Down arrows on the Keypad to select the graph of y3 and y4
To display the other point of intersection use the Left and Right arrows

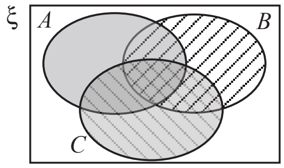
4 **a**



$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

b





$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

5 a If $x^2 + bx + c = 0$ and $x = 2 - \sqrt{3}$

then $\quad (2 - \sqrt{3})^2 + b(2 - \sqrt{3}) + c = 0$

$\therefore \quad 4 - 4\sqrt{3} + 3 + 2b - \sqrt{3}b + c = 0$

$\therefore \quad (7 + 2b + c) + (-4 - b)\sqrt{3} = 0$

$\therefore \quad 7 + 2b + c = 0 \quad \text{and} \quad -4 - b = 0$

$\therefore \quad 7 + 2(-4) + c = 0 \quad b = -4$

$\therefore \quad 7 - 8 + c = 0$

$\therefore \quad -1 + c = 0$

$\therefore \quad c = 1$

b $x^2 - 4x + 1 = 0$

Using the same procedure as in **3 c**, $x = 2 \pm \sqrt{3}$.

Hence $2 + \sqrt{3}$ is the other solution.

c i If $x^2 + bx + c = 0$ and $x = m - n\sqrt{q}$

then $\quad (m - n\sqrt{q})^2 + b(m - n\sqrt{q}) + c = 0$

$\therefore \quad m^2 - 2mn\sqrt{q} + n^2q + bm - bn\sqrt{q} + c = 0$

$\therefore \quad (m^2 + n^2q + bm + c) + (-2mn - bn)\sqrt{q} = 0$

$\therefore \quad m^2 + n^2q + bm + c = 0 \text{ and } -2mn - bn = 0$

$-2mn = bn$

$-2m = b$

ii $\quad m^2 + n^2q + (-2m)m + c = 0$

$\therefore \quad m^2 + n^2q + -2m^2 + c = 0$

$\therefore \quad n^2q - m^2 + c = 0$

$\therefore \quad c = m^2 - n^2q$

iii If $x^2 + bx + c = 0$, the general quadratic formula gives

$x = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$ (as $a = 1$)

Given $b = -2m$ and $c = m^2 - n^2q$

$$x = \frac{2m \pm \sqrt{4m^2 - 4(m^2 - n^2q)}}{2}$$

$$= \frac{2m \pm \sqrt{4m^2 - 4m^2 + 4n^2q}}{2}$$

$$= \frac{2m \pm 2n\sqrt{q}}{2}$$

$$= m \pm n\sqrt{q}$$

$$\therefore \quad x^2 + bx + c = (x - (m - n\sqrt{q}))(x - (m + n\sqrt{q}))$$

or, by completing the square,

$$x^2 - 2mx + m^2 - n^2q = x^2 - 2mx + m^2 + m^2 - n^2q - m^2$$

$$= (x - m)^2 - (n\sqrt{q})^2$$

$$= (x - m - n\sqrt{q})(x - m + n\sqrt{q})$$

6 a $x = 2mn$

$= 2 \times 5 \times 2$

$= 20$

$y = m^2 - n^2$

$= 5^2 - 2^2$

$= 25 - 4$

$= 21$

$z = m^2 + n^2$

$= 5^2 + 2^2$

$= 25 + 4$

$= 29$

b $x^2 + y^2 = (2mn)^2 + (m^2 - n^2)^2$

$= 4m^2n^2 + m^4 - 2m^2n^2 + n^4$

$= 2m^2n^2 + m^4 + n^4$

$z^2 = (m^2 + n^2)^2$

$= m^4 + 2m^2n^2 + n^4$

$\therefore x^2 + y^2 = z^2$

7 a i $2^3 = 8$. Factors of 8 are 1, 2, 4 and 8. Hence 2^3 has four factors.

ii $3^7 = 2187$. Factors of 2187 are 1, 3, 9, 27, 81, 243, 729 and 2187. Hence 3^7 has eight factors.

b

$2^1 = 2$ Factors are 1, 2. Hence 2^1 has two factors.
$2^2 = 4$ Factors are 1, 2, 4. Hence 2^2 has three factors.
$2^3 = 8$ Factors are 1, 2, 4, 8. Hence 2^3 has four factors.
$2^4 = 16$ Factors are 1, 2, 4, 8, 16. Hence 2^4 has five factors.
2^n has $n + 1$ factors.

c i $2^1.3^1 = 6$. Factors are 1, 2, 3, 6. There are four factors.
$2^1.3^2 = 18$. Factors are 1, 2, 3, 6, 9, 18. There are six factors.
$2^2.3^2 = 36$. Factors are 1, 2, 3, 4, 6, 9, 12, 18, 36. There are nine factors.
$2^2.3^3 = 108$. Factors are 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108. There are twelve factors.
$2^3.3^7$ has $(3 + 1)(7 + 1) = 32$ factors.

ii $2^n.3^m$ has $(n + 1)(m + 1)$ factors.

d The following table investigates the relationship between the number of factors of x and its prime factorisation.

x	Factors	Number of factors	Prime factorisation	Number of factors
1	1	1		0 + 1
2	1, 2	2	2^1	1 + 1
3	1, 3	2	3^1	1 + 1
4	1, 2, 4	3	2^2	2 + 1
5	1, 5	2	5^1	1 + 1
6	1, 2, 3, 6	4	$2^1.3^1$	(1 + 1)(1 + 1)

e For any number x, there are $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_n + 1)$ factors.

$$\begin{aligned} 8 &= 4 \times 2 \\ &= (3 + 1)(1 + 1) \end{aligned}$$

Now $2^3.3^1 = 24$
The smallest number which has eight factors is 24.

8 a $1080 = 2^3 \times 3^3 \times 5$ $25\,200 = 2^4 \times 3^2 \times 5^2 \times 7$

b Least common multiple of 1080 and 25 200 is $2^4 \times 3^3 \times 5^2 \times 7 = 75\,600$

c HCF of m and $n = p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \cdots p_n^{\min(\alpha_n,\beta_n)}$

$\therefore$ the product of the HCF and LCM

$$= p_1^{\min(\alpha_1,\beta_1)+\max(\alpha_1,\beta_1)}\ p_2^{\min(\alpha_2,\beta_2)+\max(\alpha_2,\beta_2)} \cdots\ Pn^{\min(\alpha_n,\beta_n)+\max(\alpha_n,\beta_n)}$$

$$= p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2}\ \ p_n^{\alpha_n+\beta_n}$$

$$= mn$$

d **i** The lowest common multiple of 5, 7, 9 and 11 is 3465.
Now 3465 + 11 is divisible by 11, 3465 + 9 is divisible by 9, 3465 + 7 is divisible by 7, 3465 + 5 is divisible by 5.
Therefore choose numbers 3476, 3474, 3472 and 3470.

ii Divide by 2 to obtain 4 consecutive natural numbers, i.e. 1738, 1737, 1736, 1735.

9 **a** **i** Region 8, $B' \cap F' \cap R'$

ii Region 1, $B \cap F' \cap R$ represents red haired, blue eyed males.

iii Region 2, $B \cap F' \cap R'$ represents blue eyed males who do not have red hair.

b Let ξ be the set of all students at Argos Secondary College studying French, Greek or Japanese.

$n(\xi) = n(F \cup G \cup J) = 250$

$n(F' \cap G' \cap J') = 0$

$n((G \cup J) \cap F') = 41$

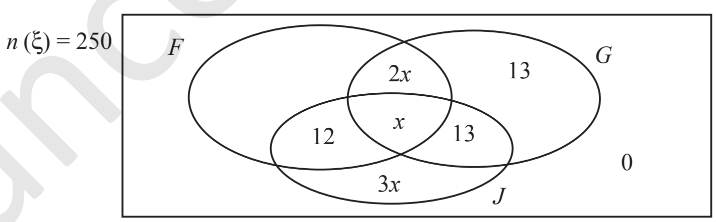
$n(F \cap J \cap G') = 12$

$n(J \cap G \cap F') = 13$

$n(G \cap J' \cap F') = 13$

$n(F \cap G \cap J') = 2 \times n(F \cap G \cap J)$

$n(J \cap G' \cap F') = n(F \cap G)$



Now $n((G \cup J) \cap F') = 13 + 13 + 3x$

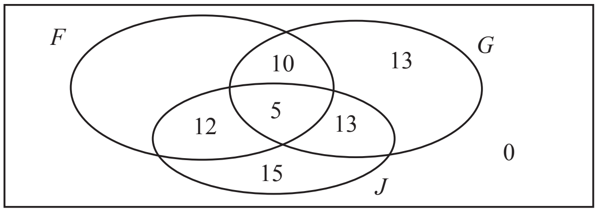
$= 26 + 3x$

$\therefore \quad 26 + 3x = 41$

$\therefore \quad 3x = 15$

$\therefore \quad x = 5$

$n(\xi) = 250$

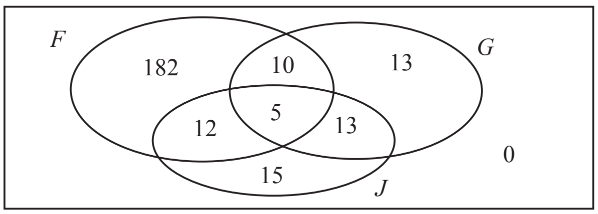


$n(F \cap G' \cap J') = 250 - (10 + 12 + 5 + 13 + 13 + 15 + 0)$

$= 250 - 68$

$= 182$

$n(\xi) = 250$



i $n(F \cap G \cap J) = 5$, the number studying all three languages.

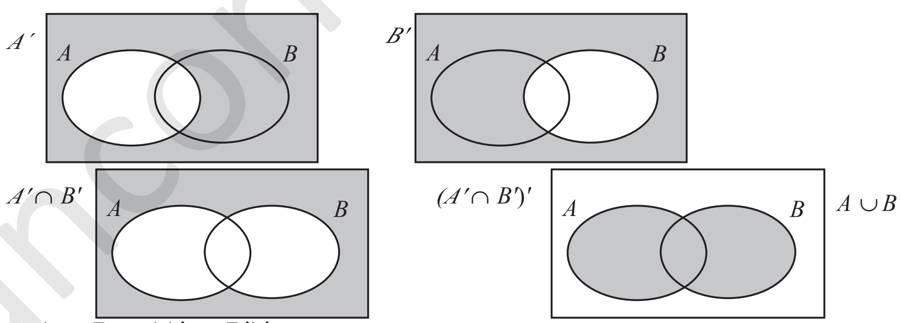
ii $n(F \cap G' \cap J') = 182$, the number studying only French.

10 a i B' denotes the set of students at Sounion Secondary College 180 cm or shorter.

ii $A \cup B$ denotes the set of students at Sounion Secondary College either female or taller than 180 cm or both.

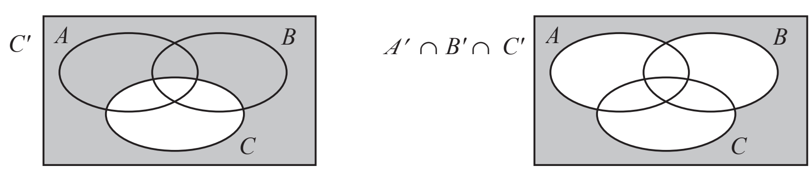
iii $A' \cap B'$ denotes the set of students at Sounion Secondary College who are males 180 cm or shorter.

b

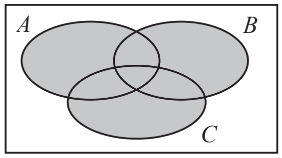


$\therefore A \cup B = (A' \cap B')'$

c



$(A' \cap B' \cap C')'$



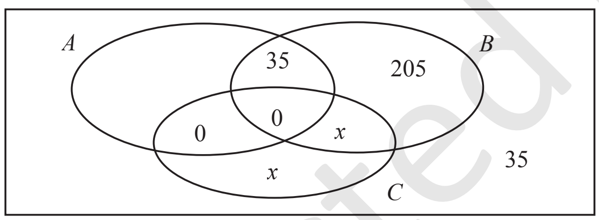
$A \cup B \cup C$

$\therefore A \cup B \cup C = (A' \cap B' \cap C')'$

11

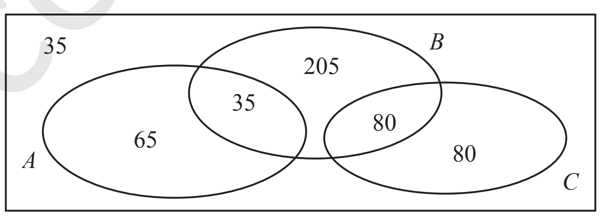
$$n(\xi) = 500$$
$$n(A \cap C) = 0$$
$$n(A) = 100$$
$$n(B \cap A' \cap C') = 205$$
$$n(C) = 2 \times n(B \cap C)$$
$$n(A \cap B \cap C') = 35$$
$$n(A' \cap B' \cap C') = 35$$

a $n(\xi) = 500$



$$n(A \cap B' \cap C') = 100 - 35$$
$$= 65$$
$$2x + 35 + 65 + 205 + 35 = 500$$
$$\therefore \quad 2x + 340 = 500$$
$$\therefore \quad 2x = 160$$
$$\therefore \quad x = 80$$

$n(\xi) = 500$



b $n(C) = 160$, regular readers of C.

c $n(A \cap B' \cap C') = 65$, regular readers of A only.

d $n(A \cap B \cap C) = 0$, regular readers of A, B and C.

12 **a** All possible ways to combine 5c and 8c stamps for a total value of 38c

Use linear Diophantine equations

let x = number of 5c stamps, y = number of 8c stamps

then 5x+8y=38

By inspection a solution for this equation is $x = 6$, $y = 1$

Let

$x = 6 + 8t$

$y = 1 - 5t, \quad t \in \mathbb{Z}$

Only want when $x, y \geq 0$

So $6 + 8t \geq 0$ and $-5t \geq -1$

$$t \geq -\frac{6}{8} \qquad 5t \leq 1$$

$$\geq -\frac{3}{4} \qquad t \leq \frac{1}{5}$$

So $t = \{t \in \mathbb{Z} : -\frac{3}{4} \leq t \leq \frac{1}{5}\}$

$= 0$

So only $6 \times 5\text{c} + 1 \times 8\text{c} = 38\text{c}$

b As above let x = number of 5c stamps, y = number of 8c stamps

now 5x+8y=120

By setting $x = 0$ we see that $x = 0,\ y = 15$ is a solution

Let

$x = 0 + 8t$

$y = 15 - 5t, \quad t \in \mathbb{Z}$

Only want when $x, y \geq 0$

So $0 + 8t \geq 0$ and $15 - 5t \geq 0$

$$t \geq 0 \qquad -5t \geq -15$$

$$t \leq \frac{15}{5}$$

$$t \leq 3$$

That is $t = \{t \in \mathbb{Z} : 0 \leq t \leq 3\}$

$= \{0, 1, 2, 3, 4\}$

(1) $t = 0$,

$x = 0 + 8 \times 0$ and $y = 15 - 5 \times 0$

$= 0$ $= 15$

then $5x + 8y = 5 \times 0 + 8 \times 15$

$= 120$

(2) $t = 1$,

$x = 0 + 8 \times 1$ and $y = 15 - 5 \times 1$

$= 8$ $= 10$

then $5x + 8y = 5 \times 8 + 8 \times 10$

$= 120$

(3) $t = 2$,

$x = 0 + 8 \times 2$ and $y = 15 - 5 \times 2$

$= 16$ $= 5$

then $5x + 8y = 5 \times 16 + 8 \times 5$

$= 120$

(4) $t = 3$,

$x = 0 + 8 \times 3$ and $y = 15 - 5 \times 3$

$= 24$ $= 0$

then $5x + 8y = 5 \times 24 + 8 \times 0$

$= 120$

So solutions are (0, 15), (8, 10), (16, 5), (24, 0)

Chapter 4 – Sequences and series

Solutions to Exercise 4A

1 a $t_1 = 3$

$t_2 = 3 + 4 = 7$

$t_3 = 7 + 4 = 11$

$t_4 = 11 + 4 = 15$

$t_5 = 15 + 4 = 19$

b $t_1 = 5$

$t_2 = 3 \times 5 + 4 = 19$

$t_3 = 3 \times 19 + 4 = 61$

$t_4 = 3 \times 61 + 4 = 187$

$t_5 = 3 \times 187 + 4 = 565$

c $t_1 = 1$

$t_2 = 5 \times 1 = 5$

$t_3 = 5 \times 5 = 25$

$t_4 = 5 \times 25 = 125$

$t_5 = 5 \times 125 = 625$

d $t_1 = -1$

$t_2 = -1 + 2 = 1$

$t_3 = 1 + 2 = 3$

$t_4 = 3 + 2 = 5$

$t_5 = 5 + 2 = 7$

e $t_1 = 1$

$t_2 = 3$

$t_3 = 2 \times 3 + 1 = 7$

$t_4 = 2 \times 7 + 3 = 17$

$t_5 = 2 \times 17 + 7 = 41$

2 a $t_2 = t_1 + 3$

$t_3 = t_2 + 3$

$\therefore t_n = t_{n-1} + 3, \ t_1 = 3$

b $t_2 = 2t_1$

$t_3 = 2t_2$

$\therefore t_n = 2t_{n-1}, \ t_1 = 1$

c $t_2 = -2 \times t_1$

$t_3 = -2 \times t_2$

$\therefore t_n = -2t_{n-1}, \ t_1 = 3$

d $t_2 = t_1 + 3$

$t_3 = t_2 + 3$

$\therefore t_n = t_{n-1} + 3, \ t_1 = 4$

e $t_2 = t_1 + 5$

$t_3 = t_2 + 5$

$\therefore t_n = t_{n-1} + 5, \ t_1 = 4$

3 a $t_n = \frac{1}{n}$

$t_1 = \frac{1}{1} = 1$

$t_2 = \frac{1}{2}$

$t_3 = \frac{1}{3}$

$t_4 = \frac{1}{4}$

b $t_n = n^2 + 1$

$t_1 = 1^2 + 1 = 2$

$t_2 = 2^2 + 1 = 5$

$t_3 = 3^2 + 1 = 10$

$t_4 = 4^2 + 1 = 17$

c $t_n = 2n$

$t_1 = 2 \times 1 = 2$

$t_2 = 2 \times 2 = 4$

$t_3 = 2 \times 3 = 6$

$t_4 = 2 \times 4 = 8$

d $t_n = 2^n$

$t_1 = 2^1 = 2$

$t_2 = 2^2 = 4$

$t_3 = 2^3 = 8$

$t_4 = 2^4 = 16$

e $t_n = 3n + 2$

$t_1 = 3 \times 1 + 2 = 5$

$t_2 = 3 \times 2 + 2 = 8$

$t_3 = 3 \times 3 + 2 = 11$

$t_4 = 3 \times 4 + 2 = 14$

f $t_n = (-1)^n n^3$

$t_1 = (-1)^1 \times 1^3 = -1$

$t_2 = (-1)^2 \times 2^3 = 8$

$t_3 = (-1)^3 \times 3^3 = -27$

$t_4 = (-1)^4 \times 4^3 = 64$

g $t_n = 2n + 1$

$t_1 = 2 \times 1 + 1 = 3$

$t_2 = 2 \times 2 + 1 = 5$

$t_3 = 2 \times 3 + 1 = 7$

$t_4 = 2 \times 4 + 1 = 9$

h $t_n = 2 \times 3^{n-1}$

$t_1 = 2 \times 3^0 = 2$

$t_2 = 2 \times 3^1 = 6$

$t_3 = 2 \times 3^2 = 18$

$t_4 = 2 \times 3^3 = 54$

4 **a** $t_n = 3n$

b $t_n = 2^{n-1}$

c $t_n = \dfrac{1}{n^2}$

d $t_n = 3(-2)^{n-1}$

e $t_n = 3n + 1$

f $t_n = 5n - 1$

5 $t_n = 3n + 1$

$t_{n+1} = 3(n + 1) + 1$

$= 3n + 4$

$t_{2n} = 3(2n) + 1$

$= 6n + 1$

6 **a** $t_n = t_{n-1} + 3, \ t_1 = 15$

b $t_1 = 15$

$t_2 = 15 + 3$

$t_3 = (15 + 3) + 3$

$= 15 + 2 \times 3$

$\therefore t_n = 15 + (n - 1) \times 3$

$= 3n + 12$

c $t_{13} = 3 \times 13 + 12$

$= 51$

7 a 4% reduction is equivalent to 96% of the original.
$t_n = 0.96t_{n-1}$
$t_1 = 3$

b $t_1 = 94.3$

$t_2 = 0.96 \times 94.3$

$t_3 = 0.96 \times (0.96 \times 94.3)$

$= 0.96^2 \times 94.3$

$\therefore t_n = 94.3 \times 0.96^{n-1}$

c $t_9 = 94.3 \times 0.96^8$

≈ 68.03 seconds

8 a $t_n = 1.8t_{n-1} + 20$
$t_0 = 100$

b $t_1 = 1.8 \times 100 + 20 = 200$

$t_2 = 1.8 \times 200 + 20 = 380$

$t_3 = 1.8 \times 380 + 20 = 704$

$t_4 = 1.8 \times 704 + 20 = 1287$

$t_5 = 1.8 \times 1287 + 20 = 2336$

9 a $t_1 = 2000 \times 1.06$

$= \$2120$

$t_2 = (2120 + 400) \times 1.06$

$= \$2671.20$

$t_3 = (2671.2 + 400) \times 1.06$

$= \$3255.47$

b $t_n = (t_{n-1} + 400) \times 1.06$

$= 1.06(t_{n-1} + 400),\ t_1 = 2120$

c Method will depend on the calculator or spreadsheet used.
$t_{10} = \$8454.02$

10 a 1, 4, 7, 10, 13, 16

b 3, 1, −1, −3, −5, −7

c $\frac{1}{2}$, 1, 2, 4, 8, 16

d 32, 16, 8, 4, 2, 1

11 a 1.1, 1.21, 1.4641, 2.144, 4.595, 21.114

b 27, 18, 12, 8, $\frac{16}{3}$, $\frac{32}{9}$

c −1, 3, 11, 27, 59, 123

d −3, 7, −3, 7, −3, 7

12 a $t_n = 2^{n-1}$

$t_1 = 2^0 = 1$

$t_2 = 2^1 = 2$

$t_3 = 2^2 = 4$

b $u_n = \frac{1}{2}(n^2 - n) + 1$

$u_1 = \frac{1}{2}(1^2 - 1) + 1 = 1$

$u_2 = \frac{1}{2}(2^2 - 2) + 1 = 2$

$u_3 = \frac{1}{2}(3^2 - 3) + 1 = 4$

c The sequences are the same for the first three terms.

$t_1 = u_1$

$t_2 = u_2$

$t_3 = u_3$

d $t_4 = 2^3 = 8$

$u_4 = \frac{1}{2}(4^2 - 4) + 1 = 7$

The sequences are not the same after the first three terms.

13 $S_1 = a \times 1^2 + b \times 1 = a + b$

$S_2 = a \times 2^2 + b \times 2 = 4a + 2b$

$S_3 = a \times 3^2 + b \times 3 = 9a + 3b$

$S_{n+1} - S_n$

$= a(n + 1)^2 + b(n + 1) - an^2 - bn$

$= a(n^2 + 2n + 1) + bn + b - an^2 - bn$

$= an^2 + 2an + a + b - an^2$

$= 2an + a + b$

14 $t_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$

$t_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \approx 1.4168$

$t_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{577}{408} \approx 1.4142$

Comparing the terms to real numbers between 1 and 1.5, it can be seen that the sequence gives an approximation of $\sqrt{2} = 1.4142$

15

$$\begin{aligned} t_3 &= t_2 + t_1 \\ &= 1 + 1 = 2 \\ t_4 &= t_3 + t_2 \\ &= 2 + 1 = 3 \\ t_5 &= t_4 + t_3 \\ &= 3 + 2 = 5 \\ t_{n+2} &= t_{n+1} + t_n \\ \therefore t_{n+1} &= t_n + t_{n-1} \\ \therefore t_{n+2} &= (t_n + t_{n-1}) + t_n \\ &= 2t_n + t_{n-1} \end{aligned}$$

Solutions to Exercise 4B

1 $t_n = a + (n-1)d$

a $t_1 = 0 + (1-1) \times 2 = 0$

$t_2 = 0 + (2-1) \times 2 = 2$

$t_3 = 0 + (3-1) \times 2 = 4$

$t_4 = 0 + (4-1) \times 2 = 6$

b $t_1 = -3 + (1-1) \times 5 = -3$

$t_2 = -3 + (2-1) \times 5 = 2$

$t_3 = -3 + (3-1) \times 5 = 7$

$t_4 = -3 + (4-1) \times 5 = 12$

c $t_1 = -\sqrt{5} + (1-1) \times -\sqrt{5} = -\sqrt{5}$

$t_2 = -\sqrt{5} + (2-1) \times -\sqrt{5} = -2\sqrt{5}$

$t_3 = -\sqrt{5} + (3-1) \times -\sqrt{5} = -3\sqrt{5}$

$t_4 = -\sqrt{5} + (4-1) \times -\sqrt{5} = -4\sqrt{5}$

d $t_1 = 11 + (1-1) \times -2 = 11$

$t_2 = 11 + (2-1) \times -2 = 9$

$t_3 = 11 + (3-1) \times -2 = 7$

$t_4 = 11 + (4-1) \times -2 = 5$

2 **a** $t_{13} = a + 12d$

$= 5 + 12 \times -3 = -31$

b $t_{10} = a + 9d$

$= -12 + 9 \times 4 = 24$

c $t_9 = a + 8d$

$= 25 + 8 \times -2.5 = 5$

d $t_5 = a + 4d$

$= 2\sqrt{3} + 4 \times \sqrt{3}$

$= 6\sqrt{3}$

3 **a** $a + (1-1)d = 3$

$a = 3$

$3 + (2-1)d = 7$

$d = 7 - 3 = 4$

$\therefore t_n = 3 + 4(n-1)$

$= 4n - 1$

b $a + (1-1)d = 3$

$a = 3$

$3 + (2-1)d = -1$

$d = -1 - 3 = -4$

$\therefore t_n = 3 + -4(n-1)$

$= 7 - 4n$

c $a + (1-1)d = -\frac{1}{2}$

$a = -\frac{1}{2}$

$-\frac{1}{2} + (2-1)d = \frac{3}{2}$

$d = \frac{3}{2} - \frac{1}{2} = 2$

$t_n = -\frac{1}{2} + 2(n-1)$

$= 2n - \frac{5}{2}$

d $a + (1 - 1)d = 5 - \sqrt{5}$

$a = 5 - \sqrt{5}$

$(5 - \sqrt{5}) + (2 - 1)d = 5$

$d = 5 - (5 - \sqrt{5})$

$= \sqrt{5}$

$t_n = (5 - \sqrt{5}) + \sqrt{5}(n - 1)$

$= n\sqrt{5} + 5 - 2\sqrt{5}$

4 **a** $a = 6$ and $d = 4$

$6 + 4(n - 1) = 54$

$4(n - 1) = 48$

$n - 1 = 12$

$n = 13$

b $a = 5$ and $d = -3$

$5 - 3(n - 1) = -16$

$-3(n - 1) = -21$

$n - 1 = 7$

$n = 8$

c $a = 16$ and $d = 16 - 13 = 3$

$16 + 3(n - 1) = -41$

$-3(n - 1) = -57$

$n - 1 = 19$

$n = 20$

d $a = 7$ and $d = 11 - 7 = 4$

$7 + 4(n - 1) = 227$

$4(n - 1) = 220$

$n - 1 = 55$

$n = 56$

5 $t_4 = 7$

$t_{30} = 85$

$a + 3d = 7 \ldots (1)$

$a + 29d = 85 \ldots (2)$

Equation (2) − Equation (1)

$26d = 78$

$d = 3$

$\therefore a = -2$

$t_7 = -2 + 6 \times 3$

$= 16$

6

$a + 2d = 18 \quad \ldots (1)$

$a + 5d = 486 \quad \ldots (2)$

Equation (2) − Equation (1)

$3d = 468$

$d = 156$

$a + 2 \times 156 = 18$

$a + 312 = 18$

$a = -294$

$\therefore t_n = -294 + 156(n - 1)$

$= 156n - 450$

7

$$a + 6d = 0.6 \ldots (1)$$
$$a + 11d = -0.4 \ldots (1)$$

Equation (2) − Equation (1)

$$5d = -1.0$$
$$d = -0.2$$
$$a + 6 \times -0.2 = 0.6$$
$$a - 1.2 = 0.6$$
$$a = 1.8$$
$$\therefore t_{20} = 1.8 + 19 \times -0.2$$
$$= -2$$

8

$$a + 4d = 24 \ldots (1)$$
$$a + 9d = 39 \ldots (2)$$

Equation (2) − Equation (1)

$$5d = 15$$
$$d = 3$$
$$a + 4 \times 3 = 24$$
$$a + 12 = 24$$
$$a = 12$$
$$\therefore t_{15} = 12 + 14 \times 3$$
$$= 54$$

9

$$a + 14d = 3 + 9\sqrt{3} \ldots (1)$$
$$a + 19d = 38 - \sqrt{3} \ldots (2)$$

Equation (2) − Equation (1)

$$5d = 35 - 10\sqrt{3}$$
$$d = 7 - 2\sqrt{3}$$
$$a + 14 \times (7 - 2\sqrt{3}) = 3 + 9\sqrt{3}$$
$$a + 98 - 28\sqrt{3} = 3 + 9\sqrt{3}$$
$$a = 37\sqrt{3} - 95$$
$$t_6 = 37\sqrt{3} - 95 + 5 \times (7 - 2\sqrt{3})$$
$$= 37\sqrt{3} - 95 + 35 - 10\sqrt{3}$$
$$= 27\sqrt{3} - 60$$

10 **a** 672

b 91st week

11 **a** P is the 16th row. $a = 25,\ d = 3$

$$t_{16} = a + 15d$$
$$= 25 + 15 \times 3$$
$$= 70 \text{ seats}$$

b X is the 24th row. $a = 25,\ d = 3$

$$t_{24} = a + 23d$$
$$= 25 + 23 \times 3$$
$$= 94 \text{ seats}$$

c

$$t_n = 25 + 3(n - 1) = 40$$
$$3(n - 1) = 15$$
$$n - 1 = 5$$
$$n = 6$$

Row F

12 $t_6 = 3 + 5d = 98$

$5d = 95$

$d = 19$

$t_7 = t_6 + 19$

$= 117$

13 $4 + 9d = 30$

$9d = 26$

$d = \frac{29}{9}$

$t_2 = 4 + 1 \times \frac{26}{9} = \frac{62}{9}$

$t_3 = 4 + 2 \times \frac{26}{9} = \frac{88}{9}$

$t_4 = 4 + 3 \times \frac{26}{9} = \frac{38}{3}$

$t_5 = 4 + 4 \times \frac{26}{9} = \frac{140}{9}$

$t_6 = 4 + 5 \times \frac{26}{9} = \frac{166}{9}$

$t_7 = 4 + 6 \times \frac{26}{9} = \frac{64}{3}$

$t_8 = 4 + 7 \times \frac{26}{9} = \frac{218}{9}$

$t_9 = 4 + 8 \times \frac{26}{9} = \frac{244}{9}$

14 $5 + 5d = 15$

$5d = 10$

$d = 2$

$t_2 = 5 + 1 \times 2 = 7$

$t_3 = 5 + 2 \times 2 = 9$

$t_4 = 5 + 3 \times 2 = 11$

$t_5 = 5 + 4 \times 2 = 13$

15 $a + (m - 1)d = 0$

$(m - 1)d = -a$

$d = -\frac{a}{m - 1}$

$t_n = a - \frac{a(n - 1)}{m - 1}$

This could be simplified as follows:

$t_n = \frac{a(m - 1) - a(n - 1)}{m - 1}$

$= \frac{a(m - 1 + n + 1)}{m - 1}$

$= \frac{a(m - n)}{m - 1}$

16 **a** $c = \frac{a + b}{2}$

$= \frac{8 + 15}{2} = 11.5$

b $c = \frac{a + b}{2}$

$= \frac{1}{2}\left(\frac{1}{2\sqrt{2} - 1} + \frac{1}{2\sqrt{2} + 1}\right)$

$= \frac{2\sqrt{2} + 1 + 2\sqrt{2} - 1}{2(2\sqrt{2} - 1)(2\sqrt{2} + 1)}$

$= \frac{4\sqrt{2}}{2 \times (8 - 1)}$

$= \frac{2\sqrt{2}}{7}$

17 $3x - 2 = \frac{5x + 1 + 11}{2}$

$6x - 4 = 5x + 12$

$x = 16$

18 Use the fact that the difference is constant.

$$(8a - 13) - (4a - 4) = (4a - 4) - a$$
$$8a - 13 - 4a + 4 = 4a - 4 - a$$
$$4a - 9 = 3a - 4$$
$$a = 5$$

19 $t_m = a + (m - a)d = n$

$t_n = a + (n - a)d = m$

Subtract:

$(m - n)d = n - m$

$= -1(m - n)$

$d = \dfrac{-1(m - n)}{m - n}$

$= -1$

Substitute:

$a + (m - a) \times -1 = n$

$a = m + n - 1$

$t_{m+n} = a + (m + n - 1)d$

$= n + m - 1 + (m + n - 1) \times -1$

$= n + m - 1 - m - n + 1$

$= 0$

20 Use the fact that the difference is constant.

$a^2 - 2a = 2a - a$

$a^2 - 3a = 0$

$a(a - 3) = 0$

$a = 3$ (since $a \neq 0$)

21 If a is a prime number, then the nth term is $a + (n - 1)d$ Since a is a natural number there is an n such that $n - 1 = a$. The term $t_{a+1} = a + ad$ which is divisible by a ($=a(d + 1)$ is composite since $d + 1 \geq 2$ and $a \geq 2$). Hence no infinite arithmetic sequence of primes exists.

Solutions to Exercise 4C

1 a $a = 8,\ d = 5,\ n = 12$

$t_{12} = 8 + 11 \times 5 = 63$

$S_{12} = \frac{12}{2}(8 + 63)$

$= 6 \times 71$

$= 426$

b $a = -3.5,\ d = 2,\ n = 10$

$t_{10} = -3.5 + 9 \times 2 = 14.5$

$S_{10} = \frac{10}{2}(-3.5 + 14.5)$

$= 5 \times 11$

$= 55$

c $a = \frac{1}{\sqrt{2}},\ d = \frac{1}{\sqrt{2}},\ n = 15$

$t_{15} = \frac{1}{\sqrt{2}} + 14 \times \frac{1}{\sqrt{2}}$

$= \frac{15}{\sqrt{2}}$

$S_{15} = \frac{15}{2}\left(\frac{1}{\sqrt{2}} + \frac{15}{\sqrt{2}}\right)$

$= 60\sqrt{2}$

d $a = -4,\ d = 5,\ n = 8$

$t_8 = -4 + 7 \times 5 = 31$

$S_8 = \frac{8}{2}(-4 + 31)$

$= 108$

2 $a = 7,\ d = 3,\ n = 7$

$S_7 = \frac{7}{2}(14 + 6 \times 3)$

$= 112$

3 $a = 5,\ d = 5,\ n = 16$

$S_{16} = \frac{16}{2}(10 + 15 \times 5)$

$= 680$

4 There will be half of 98 = 49 numbers:

$a = 2,\ d = 2,\ n = 49$

$S_{49} = \frac{49}{2}(4 + 48 \times 2)$

$= 2450$

5 a 14

b 322

6 a 20

b −280

7 a 12

b 105

8 a 180

b $S_n = \frac{n}{2}(8 + (n - 1) \times 4)$

$= 180$

$n(8 + 4n - 4) = 360$

$4n^2 + 4n - 360 = 0$

$n^2 + n - 90 = 0$

$(n - 9)(n + 10) = 0$

$n = 9$ as $n > 0$

So $\{n : S_n = 180\} = \{n : n = 9\}$

9 $S_n = \frac{n}{2}(30 + (n-1) \times -1) = 110$

$n(30 - n + 1) = 220$

$-n^2 + 31n - 220 = 0$

$n^2 - 31n + 220 = 0$

$(n - 11)(n - 20) = 0$

$n = 11$ or $n = 20$

Reject any value of $n > 15$, as this would involve a negative number of logs in a row. There will be 11 layers.

10 $a = -5, \ d = 4$

$S_m = \frac{m}{2}(-10 + (m-1) \times 4)$

$= 660$

$m(-10 + 4m - 4) = 1320$

$4m^2 - 14m - 1320 = 0$

$(m - 20)(4m + 66) = 0$

$m = 20$ as $m > 0$

11 $S_n = \frac{n}{2}(a + \ell) \therefore S_n = 0$

12 a $a = 6$

$t_{15} = 6 + 14d = 27$

$14d = 21$

$d = 1.5$

$t_8 = 6 + 7 \times 1.5$

$= 16.5$ km

b $S_5 = \frac{5}{2}(12 + 4 \times 1.5)$

$= 45$ km

c 7 walks

d Total distance:

$S_{15} = \frac{15}{2}(12 + 14 \times 1.5)$

$= 247.5$

Distance missed $= 18 + 19.5 + 21$

$= 58.5$ km

(8th day = 16.5 km)

Distance Dora walks $= 247.5 - 58.5$

$= 189$ km

13 a $a = 30, \ d = 5$

$S_n = \frac{n}{2}(60 + (n-1) \times 5)$

$= 500$

$n(60 + 5n - 10) = 1000$

$5n^2 + 50n - 1000 = 0$

$n^2 + 10n - 200 = 0$

$(n - 10)(n + 20) = 0$

$n = 10, \ $ as $n > 0$

10 days

b $a = 50, \ n = 5$

$S_5 = \frac{5}{2}(100 + 4d)$

$= 500$

$100 + 4d = 200$

$d = \frac{200 - 100}{4}$

$= 25$ pages per day

14 a Row J $= t_{10}$

$= 50 + 9 \times 4 = 86$

b $S_{26} = \frac{26}{2}(100 + 25 \times 4)$

$= 2600$

c $50 + 54 + 58 + 62 = 224$

d $2600 - 224 = 2376$

e
$$\begin{aligned} S_n &= \frac{n}{2}(100 + (n-1) \times 4) \\ &= 3410 \end{aligned}$$

$$\begin{aligned} n(100 + 4n - 4) &= 6820 \\ 4n^2 + 96n - 6820 &= 0 \\ n^2 + 24n - 1705 &= 0 \\ (n - 31)(n + 55) &= 0 \\ n &= 31 \text{ as } n > 0 \end{aligned}$$

There are 5 extra rows (from 26 to 31).

15 Total members

$$\begin{aligned} S_{12} &= \frac{12}{2}(80 + 11 \times 15) \\ &= 1470 \\ \text{Total fees} &= 1470 \times \$120 \\ &= \$176\,400 \end{aligned}$$

16

$$\begin{aligned} a + d &= -12 \\ 6(2a + 11d) &= 18 \\ 2a + 11d &= 3 \end{aligned}$$

Substitute $a = -12 - d$:

$$\begin{aligned} -24 - 2d + 11d &= 3 \\ 9d - 24 &= 3 \\ d &= 3 \\ a + 3 &= -12 \\ a &= -15 \\ t_6 &= -15 + 5 \times 3 \\ &= 0 \\ S_6 &= \frac{6}{2}(-30 + 5 \times 3) \\ &= -45 \end{aligned}$$

17

$$\begin{aligned} 5(2a + 9d) &= 120 \\ 2a + 9d &= 24 \ldots (1) \\ 10(2a + 19d) &= 840 \\ 2a + 19d &= 84 \ldots (2) \end{aligned}$$

Equation (2) – Equation (1)

$$\begin{aligned} 10d &= 60 \\ d &= 6 \\ 2a + 9 \times 6 &= 24 \\ a &= -15 \\ S_{30} &= \frac{30}{2}(-30 + 29 \times 6) \\ &= 2160 \end{aligned}$$

18

$$\begin{aligned} a + 5d &= 16 \ldots (1) \\ a + 11d &= 28 \ldots (2) \end{aligned}$$

Equation (2) – Equation (1)

$$\begin{aligned} 6d &= 12 \\ d &= 2 \\ a + 10 &= 16 \\ a &= 6 \\ S_{14} &= \frac{14}{2}(12 + 13 \times 2) \\ &= 266 \end{aligned}$$

19 a

$$a + 2d = 6.5 \ldots (1)$$
$$4(2a + 7d) = 67$$
$$a + 3.5d = \frac{67}{8} = 8.375 \ldots (2)$$

Equation (2) − Equation (1)

$$1.5d = 1.875$$
$$d = 1.25$$
$$a + 1.25 \times 2 = 6.5$$
$$a = 4$$
$$t_n = 4 + 1.25(n - 1)$$
$$= 2.75 + 1.25n$$
$$= \frac{5}{4}n + \frac{11}{4}$$

b

$$a + 3d = \frac{6}{\sqrt{5}}$$
$$= \frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{6\sqrt{5}}{5} \ldots (1)$$
$$\frac{5}{2}(2a + 4d) = 16\sqrt{5}$$
$$5(a + 2d) = 16\sqrt{5}$$
$$a + 2d = \frac{16\sqrt{5}}{5} \ldots (2)$$

Equation (2) − Equation (1)

$$d = \frac{6\sqrt{5}}{5} - \frac{16\sqrt{5}}{5}$$
$$= -\frac{10\sqrt{5}}{5}$$
$$a + 2 \times \frac{-10\sqrt{5}}{5} = \frac{16\sqrt{5}}{5}$$
$$a = \frac{16\sqrt{5}}{5} + \frac{20\sqrt{5}}{5}$$
$$= 36\sqrt{5}$$
$$t_n = \frac{36\sqrt{5}}{5} - \frac{10\sqrt{5}}{5}$$
$$(n - 1)$$
$$= \frac{46\sqrt{5}}{5} - \frac{10\sqrt{5}}{5}n$$
$$= \frac{46\sqrt{5}}{5} - 2\sqrt{5}n$$

20 a

$$t_{n+1} - t_n = b(n + 1) - bn$$
$$= b$$

b

$$S_n = \frac{n}{2}(2b + (n - 1)b)$$
$$= \frac{n}{2}(2b + nb - b)$$
$$= \frac{n}{2}(nb + b)$$

This can be factorised to $\dfrac{nb(n+1)}{2}$.

21 $a = 10,\ d = -5$

$$\begin{aligned} t_5 &= 10 + 4 \times -5 \\ &= -10 \\ S_{25} &= \frac{25}{2}(20 + 24 \times -5) \\ &= -1250 \end{aligned}$$

22

$$\begin{aligned} S_{20} &= 10(2a + 19d) \\ &= 25a \\ 20a + 190d &= 25a \\ 190d &= 5a \\ a &= 38d \\ S_{30} &= 15(76d + 29d) \\ &= 1575d \end{aligned}$$

23 **a**

$$\begin{aligned} S_{n-1} &= 17(n-1) - 3(n-1)^2 \\ &= 17n - 17 - 3(n^2 - 2n + 1) \\ &= 17n - 17 - 3n^2 + 6n - 3 \\ &= 23n - 3n^2 - 20 \end{aligned}$$

b

$$\begin{aligned} t_n &= S_n - S_{n-1} \\ &= 17n - 3n^2 - 23n + 3n^2 + 20 \\ &= 20 - 6n \end{aligned}$$

c

$$\begin{aligned} t_{n+1} - t_n &= 20 - 6(n+1) - (20 - 6n) \\ &= 20 - 6n - 6 - 20 + 6n \\ &= -6 \end{aligned}$$

The sequence has a constant difference of −6 and so is arithmetic.

$$\begin{aligned} a &= t_1 \\ &= 20 - 6 \times 1 = 14 \\ d &= 14 \end{aligned}$$

24 Let the terms be $a,\ a + d,\ a + 2d$.

$$\begin{aligned} \text{Sum } &= 3a + 3d = 36 \\ a + d &= 12 \\ \text{Product} &= a(a+d)(a+2d) \\ &= 1428 \end{aligned}$$

Substitute $d = 12 - a$.

$$\begin{aligned} a(a + 12 - a)(a + 24 - 2a) &= 1428 \\ 12a(24 - a) &= 1428 \\ a(24 - a) &= 119 \\ 24a - a^2 &= 119 \\ a^2 - 24a + 119 &= 0 \\ (a-7)(a-17) &= 0 \\ a = 7 \text{ or } a &= 17 \\ \therefore d = 12 - 7 &= 5 \\ \text{or } d = 12 - 17 &= -5 \end{aligned}$$

The three terms are either 7, 12, 17 or 17, 12, 7.
Note: in cases like this, it is sometimes easier to call the terms $a - d,\ a,\ a + d$.

25 The middle terms will be t_n and t_{n+1}.

$$\begin{aligned} t_n &= a + (n-1)d \\ t_{n+1} &= a + nd \\ t_n + t_{n+1} &= 2a + (2n-1)d \\ n(t_n + t_{n+1}) &= n(2a + (2n-1)d) \\ S_{2n} &= \frac{2n}{2}(2a + (2n-1)d) \\ &= n(2a + (2n-1)d) \\ &= n(t_n + t_{n+1}) \end{aligned}$$

26 There are 60 numbers divisible by 2.
$S_{60} = 30(4 + 59 \times 2) = 3660$
There are 40 numbers divisible by 3.
$S_{40} = 20(6 + 39 \times 3) = 2460$
There are 20 numbers divisible by 6
$S_{60} = 10(12 + 19 \times 6) = 1260$
The sum of the numbers divisible by 2 or 3 = 3660 + 2460 − 1260 = 4860

27 Let the numbers be $a - d, a, a + d, a + 2d$.
The sum is $4a + 2d = 100$ which simplifies to $2a + d = 50$.
One solution is $a = 25$ and $d = 0$.
The others are $(24, 2), (23, 4), \ldots (1, 24)$
The sequence for the first solution is 25,25,25,25.
The sequence for the second solution is 22,24,26,28

28 Let the angles be $a - d$, a and $a + d$.
Then $3a = 180$. Hence $a = 60$. The angles are, $60 - d$, 60 and $60 + d$.
There are 60 such triangles: Listing:
$(1, 60, 119), (2, 60, 118), \ldots, (60, 60, 60)$

Solutions to Exercise 4D

1 $t_n = ar^{n-1}$

a $t_1 = 3 \times 2^{1-1} = 3$

$t_2 = 3 \times 2^{2-1} = 6$

$t_3 = 3 \times 2^{3-1} = 12$

$t_4 = 3 \times 2^{4-1} = 24$

b $t_1 = 3 \times -2^{1-1} = 3$

$t_2 = 3 \times -2^{2-1} = -6$

$t_3 = 3 \times -2^{3-1} = 12$

$t_4 = 3 \times -2^{4-1} = -24$

c $t_1 = 10\,000 \times 0.1^{1-1} = 10\,000$

$t_2 = 10\,000 \times 0.1^{2-1} = 1000$

$t_3 = 10\,000 \times 0.1^{3-1} = 100$

$t_4 = 10\,000 \times 0.1^{4-1} = 10$

d $t_1 = 3 \times 3^{1-1} = 3$

$t_2 = 3 \times 3^{2-1} = 9$

$t_3 = 3 \times 3^{3-1} = 27$

$t_4 = 3 \times 3^{4-1} = 81$

2 a $a = \frac{15}{7}$

$r = \frac{1}{3}$

$t_6 = \frac{15}{7} \times \left(\frac{1}{3}\right)^5 = \frac{5}{567}$

b $a = 1$

$r = -\frac{1}{4}$

$t_5 = 1 \times \left(-\frac{1}{4}\right)^4 = \frac{1}{256}$

c $a = \sqrt{2}$

$r = \sqrt{2}$

$t_{10} = \sqrt{2} \times (\sqrt{2})^9 = 32$

d $a = a^x$

$r = a$

$t_6 = a^x \times a^5 = a^{x+5}$

3 a $a = 3$

$r = \frac{2}{3}$

$t_n = 3 \times \left(\frac{2}{3}\right)^{n-1}$

b $a = 2$

$r = \frac{-4}{2} = -2$

$t_n = 2 \times (-2)^{n-1}$

c $a = 2$

$r = \frac{2\sqrt{5}}{2} = \sqrt{5}$

$t_n = 2 \times (\sqrt{5})^{n-1}$

4 a $a = 2$ and $t_6 = 486$

Let r be the common ratio

$\therefore 2 \times r^5 = 486$

$\therefore r^5 = 243$

$\therefore r = 3$

b $a = 25$ and $t_5 = \frac{16}{25}$

Let r be the common ratio

$\therefore 25 \times r^4 = \frac{16}{25}$

$\therefore r^4 = \frac{16}{625}$

$$\therefore r = \pm\frac{2}{5}$$

5 $$\frac{1}{4}2^{n-1} = 64$$
$$2^{n-1} = 64 \times 4$$
$$= 2^8$$
$$n = 9$$

Thus t_9, the ninth term.

6 **a** $$a = 2,\ r = 3$$
$$2 \times 3^{n-1} = 486$$
$$3^{n-1} = 243$$
$$= 3^5$$
$$n = 6$$

b $$a = 5,\ r = 2$$
$$5 \times 2^{n-1} = 1280$$
$$2^{n-1} = 256$$
$$= 2^8$$
$$n = 9$$

c $$a = 768,\ r = \frac{1}{2}$$
$$768 \times \left(\frac{1}{2}\right)^{n-1} = 3$$
$$\frac{1}{2^{n-1}} = \frac{3}{768}$$
$$= \frac{1}{256} = \frac{1}{2^8}$$
$$n = 9$$

d $$a = \frac{8}{9},\ r = \frac{3}{2}$$
$$\frac{8}{9} \times \frac{3^{n-1}}{2^{n-1}} = \frac{27}{4}$$
$$\frac{3^{n-1}}{2^{n-1}} = \frac{27}{4} \times \frac{9}{8}$$
$$= \frac{3^5}{2^5}$$
$$n = 6$$

e $$a = -\frac{4}{3},\ r = -\frac{1}{2}$$
$$-\frac{4}{3} \times \left(-\frac{1}{2}\right)^{n-1} = \frac{1}{96}$$
$$\left(-\frac{1}{2}\right)^{n-1} = \frac{1}{96} \times -\frac{3}{4}$$
$$= -\frac{1}{32 \times 4}$$
$$= -\frac{1}{2^7} = \left(-\frac{1}{2}\right)^7$$
$$n = 8$$

7 $$ar^{14} = 54$$
$$ar^{11} = 2$$
$$r^3 = \frac{54}{2} = 27$$
$$r = 3$$
$$a \times 3^{11} = 2$$
$$a = \frac{2}{3^{11}}$$
$$t_7 = \frac{2}{3^{11}} \times 3^6$$
$$= \frac{2}{3^5}$$

8 $$ar^1 = \frac{1}{2\sqrt{2}}$$
$$ar^3 = \sqrt{2}$$
$$r^2 = \sqrt{2} \div \frac{1}{2\sqrt{2}}$$
$$= 4$$
$$r = 2$$
$$a \times 2 = \frac{1}{2\sqrt{2}}$$
$$a = \frac{1}{4\sqrt{2}}$$
$$t_8 = \frac{1}{4\sqrt{2}} \times 2^7$$
$$= \frac{32}{\sqrt{2}}$$
Rationalise the denominator:
$$t_8 = \frac{32}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{32\sqrt{2}}{2} = 16\sqrt{2}$$

9 **a** $$ar^5 = 768$$
$$ar^2 = 96$$
$$r^3 = \frac{768}{96} = 8$$
$$r = 2$$
$$a \times 2^2 = 96$$
$$a = 24 \text{ fish}$$

b $24 \times 2^9 = 12\,888$ fish

10 **a** At the end of 7 days, it will have increased 7 times.
$10 \times 3^7 = 21870 \text{ m}^2$

b $$10 \times 3^n \geq 200\,000$$
$$3^n \geq 20\,000$$
$$n \log_{10} 3 \geq \log_{10} 20\,000$$
$$n \geq 9.014\ldots$$
It will cover the lake early in the tenth day.

11 $r = \frac{3}{4}$.
First bounce: $\frac{3}{2}$ m
Second bounce: $\frac{9}{8}$ m
Third bounce: $\frac{27}{32}$ m
Fourth bounce: $\frac{81}{128}$ m
Fifth bounce: $\frac{243}{512}$ m

12 **a** At the end of 10 years, it will have increased 10 times.
$2500 \times 1.08^9 = \$5397.31$

b $$2500 \times 1.08^n \geq 100\,000$$
$$1.08^n \geq \frac{100\,000}{2500} = 40$$
$$n \log_{10} 1.08 \geq \log_{10} 40$$
$$n \geq 47.93\ldots$$
It will take 48 years until the value exceeds \$100 000. Alternatively, use the solve command of a CAS calculator to solve $2500 \times 1.08^n \geq 100\,000$.

This gives $n > 47.93\ldots$ directly.

13 **a** $120 \times 0.9^7 \approx 57.4$ km

b
$$120 \times 0.9^{n-1} = 30.5$$
$$0.9^{n-1} = \frac{30.5}{120}$$
$$= 0.251\ldots$$
$$(n-1)\log_{10} 0.9 = \log_{10} 0.251\ldots$$
$$n - 1 = 13.0007\ldots$$
$$n = 14$$
The 14th day.

14 $a = 1$ and $r = 2$
$t_{30} = 2^{29} = 5\,368\,709.12$
She would receive \$ 5 368 709.12

15 **a** At the end of 10 years:
$$\text{value} = 5000 \times 1.06^6$$
$$= \$7092.60$$

b
$$1.06^n \geq 2$$
$$n \log_{10} 1.06 \geq \log_{10} 2$$
$$n \geq 11.89\ldots$$
In the 12th year.

16
$$A \times 1.085^{12} = 8000$$
$$A \times 2.6616\ldots = 8000$$
$$A = \$3005.61$$

17 Let the rate be r.
$$r^{10} = 3$$
$$r = 3^{0.1} = 0.11612\ldots$$
Approximately 11.6% per annum.

18
$$a = 4,\ r = 2$$
$$4 \times 2^{n-1} > 2000$$
$$2^{n-1} > 500$$
$$2^9 = 512$$
The tenth term, which is
$t_{10} = 4 \times 2^9 = 2048$

19
$$a = 3,\ r = 3$$
$$3 \times 3^{n-1} > 500$$
$$3^n > 500$$
$$3^5 = 243 \text{ and } 3^6 = 729$$
The sixth term, which is $t_6 = 729$

20 Solve for n:
$$40\,960 \times \left(\frac{1}{2}\right)^{n-1} = 40 \times 2^{n-1}$$
$$\frac{40\,960}{40} = 2^{n-1} \times 2^{n-1}$$
$$1024 = 2^{2n-2} = 2^{10}$$
$$2n - 2 = 10$$
$$n = 6$$
But $n = 1$ corresponds to the initial numbers present, so they are equal after 5 weeks.

21 **a** $\sqrt{5 \times 720} = \sqrt{3600} = 60$

b $\sqrt{1 \times 6.25} = \sqrt{6.25} = 2.5$

c $\sqrt{\frac{1}{\sqrt{3}} \times \sqrt{3}} = \sqrt{1} = 1$

d
$$\sqrt{x^2y^3 \times x^6y^{11}} = \sqrt{x^8y^{14}}$$
$$= x^4y^7$$

22

$$r = \frac{t_7}{t_4} = \frac{t_{16}}{t_7}$$

$$\frac{a + 6d}{a + 3d} = \frac{a + 15d}{a + 6d}$$

$$(a + 6d)^2 = (a + 15d)(a + 3d)$$

$$a^2 + 12ad + 36d^2 = a^2 + 18ad + 45d^2$$

$$9d^2 + 6ad = 0$$

$$3d(3d + 2a) = 0$$

$$3d + 2a = 0 \text{ (see below)}$$

$$d = -\frac{2}{3}a$$

$$r = \frac{a + 6d}{a + 3d} = \frac{a - 4a}{a - 2a} = \frac{-3a}{-a} = 3$$

Note: $d = 0$ gives the trivial case $r = \frac{a}{a} = 1$.
(All the terms are the same.)

23 $a^{n-1} + a^n = a^{n+1}$

$$\therefore a^{n-1}(1 + a - a^2) = 0$$

$$\therefore a = \frac{1 \pm \sqrt{5}}{2} \text{ or } a = 0$$

24 **a** When the first 300 ml is withdrawn there is 700 mL of ethanol left.
When the second 300 ml withdrawn there is $0.7^2 \times 1000$ mL of ethanol left
After 5 such witthdrawals there is $0.7^5 \times 1000 \approx 168.07mL$ left.

b Solve the inequality $1000 \times 0.7^n < 1$ for n an integer t find n =20.

25 **a** The perimeter of the rectangle is $2a + 2b$. Each side of the corresponding square will be $\frac{a+b}{2}$, the arithmetic mean of a and b.

b The area of the rectangle is ab. The side length of the corresponding square is $\sqrt{ab}$, the geometric mean of a and b.

Solutions to Exercise 4E

1 $S_n = \dfrac{a(r^n - 1)}{r - 1}$

a $a = 5$

$r = \dfrac{10}{5} = 2$

$S_{10} = \dfrac{5(2^{10} - 1)}{2 - 1}$

$= 5115$

b $a = 1$

$r = \dfrac{-3}{1} = -3$

$S_6 = \dfrac{1(-3^6 - 1)}{-3 - 1}$

$= -182$

c $a = -\dfrac{4}{3}$

$r = \dfrac{2}{3} \div -\dfrac{4}{3} = -\dfrac{1}{2}$

$S_9 = \dfrac{-\frac{4}{3}\left(\left(-\frac{1}{2}\right)^9 - 1\right)}{-\frac{1}{2} - 1}$

$= -\dfrac{57}{64}$

2 a $a = 2$

$r = \dfrac{-6}{2} = -3$

$t_n = 1458 = 2 \times -3^{n-1}$

$-3^{n-1} = 729$

$n = 7$

$S_7 = \dfrac{2 \times (-3^7 - 1)}{-3 - 1}$

$= 1094$

b $a = -4$

$r = \dfrac{8}{-4} = -2$

$t_n = -1024 = -4 \times -2^{n-1}$

$-2^{n-1} = 256$

$n = 9$

$S_9 = \dfrac{-4 \times (-2^9 - 1)}{-2 - 1}$

$= -684$

c $a = 6250$

$r = \dfrac{1250}{6250} = 0.2$

$t_n = 2 = 6250 \times (0.2)^{n-1}$

$(0.2)^{n-1} = \dfrac{2}{6250} = \dfrac{1}{3125}$

$n = 6$

$S_6 = \dfrac{6250 \times ((0.2)^6 - 1)}{0.2 - 1}$

$= 7812$

3 $a = 3$ and $r = 2$

$\therefore S_n = \dfrac{3(2^n - 1)}{2 - 1}$

If $S_n = 3069$ then

$3(2^n - 1) = 3069$

$2^n - 1 = 1023$

$2^n = 1024$

$n = 10$

4 $a = 24$ and $r = -\dfrac{1}{2}$

$$\therefore S_n = \frac{24(1 + (\frac{1}{2})^n)}{1 + \frac{1}{2}}$$

$$S_n = 16(1 - (\tfrac{1}{2})^n)$$

If $S_n = \dfrac{129}{8}$ then

$$16(1 + (\tfrac{1}{2})^n) = \frac{129}{8}$$

$$1 + (\tfrac{1}{2})^n = \frac{129}{128}$$

$$(\tfrac{1}{2})^n = \frac{1}{128}$$

$$n = 7$$

5 $a = 600,\ r = 1.1$

a $t_7 = 600 \times 1.1^6$

$= 1062.9366$

About 1062.9 mL

b $S_7 = \dfrac{600 \times (1.1^7 - 1)}{1.1 - 1}$

$= 5692.3026$

About 5692.3 mL

c 11 days

6 $a = 20,\ r = \dfrac{25}{20} = 2.5$

a $t_5 = 20 \times 1.25^4$

$= 48.828125$

49 minutes (to the nearest minute)

b $S_5 = \dfrac{20 \times (1.25^5 - 1)}{1.25 - 1}$

$= 164.140625$

164 minutes, or 2 hours and 44 minutes

c

$$S_n > 15 \times 60 = 900$$

$$\frac{20 \times (1.25^n - 1)}{0.25} > 900$$

$$1.25^n - 1 > 900 \times \frac{0.25}{20}$$

$$= 11.25$$

$$1.25^n > 12.25$$

$$n \log_{10} 1.25 > \log_{10} 12.25$$

$$n > 11.228$$

$12 - 7 = 5$, so Friday.

7 $a = 15,\ r = \dfrac{2}{3}$

$$S_{10} = \frac{15 \times \left(1 - \left(\frac{2}{3}\right)^{10}\right)}{1 - \frac{2}{3}}$$

$$= 3 \times 15 \times \frac{3^{10} - 2^{10}}{3^{10}}$$

$$= 5 \times \frac{3^{10} - 2^{10}}{3^8}$$

$$= \frac{5 \times 58\,025}{6561}$$

$$= \frac{290\,125}{6561}$$

The bounces will all be doubled (up and down) except for the first (down only).

$$\text{Distance} = 2 \times \frac{290\,125}{6561} - 15$$

$$= \frac{481\,835}{6561}$$

$$= 73\frac{2882}{6561} \text{ m}$$

8 $a = \$15\,000,\ r = 1.05$

a $t_5 = 15\,000 \times 1.05^4$

$= 18\,232.593\ldots$

\$18 232.59

b $S_5 = \dfrac{15\,000 \times (1.05^5 - 1)}{1.05 - 1}$

$= 82\,844.4686$

\$82 884.47

9 Andrew: Interest = $1000 \times 0.20 \times 10$

= \$2000

His investment is worth

\$1000 + \$2000 = \$3000.

Bianca's investment is worth

$1000 \times 1.125^{10} = \$3247.32$

Bianca's investment is worth more.

10 a

$$ar^2 = 20$$
$$ar^5 = 160$$
$$r^3 = \frac{160}{20} = 8$$
$$r = 2$$
$$a \times 2^2 = 20$$
$$a = 5$$
$$S_5 = \frac{5 \times (2^5 - 1)}{2 - 1}$$
$$= 155$$

b

$$ar^2 = \sqrt{2}$$
$$ar^7 = 8$$
$$r^5 = \frac{8}{\sqrt{2}}$$
$$= \frac{\sqrt{64}}{\sqrt{2}}$$
$$= \sqrt{32} = (\sqrt{2})^5$$
$$r = \sqrt{2}$$
$$a \times (\sqrt{2})^2 = \sqrt{2}$$
$$a = \frac{1}{\sqrt{2}}$$
$$S_8 = \frac{\frac{1}{\sqrt{2}} \times ((\sqrt{2})^8 - 1)}{\sqrt{2} - 1}$$
$$= \frac{\frac{1}{\sqrt{2}} \times 15}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$$
$$= \frac{\frac{15}{\sqrt{2}} \times (\sqrt{2} + 1)}{2 - 1}$$
$$= 15 + \frac{15\sqrt{2}}{2}$$

11 $a = 1,\ r = 2$

a

$$S_n = 255$$
$$\frac{1 \times (2^n - 1)}{2 - 1} = 255$$
$$2^n - 1 = 255$$
$$2^n = 256$$
$$n = 8$$

b $$S_n > 1\,000\,000$$

$$\frac{1 \times (2^n - 1)}{2 - 1} > 1\,000\,000$$

$$2^n - 1 > 1\,000\,000$$

$$2^n > 1\,000\,001$$

$$n \log_{10} 2 > \log_{10} 1\,000\,001$$

$$n > 19.931\ldots$$

$\{n : n > 19\}$ or $\{n : n \geq 20\}$, since n is a positive integer.

12 $a = 1,\ r = -x^2$

Note that there are $(m + 1)$ terms.

$$S_{m+1} = \frac{1 \times (-x^2)^{m+1} - 1}{-x^2 - 1}$$

$$= \frac{-x^{2(m+1)} - 1}{-x^2 - 1}$$

$$= \frac{x^{2m+2} + 1}{x^2 + 1}$$

13 **a** The thickness of each piece is 0.05 mm.

There are $1 + 2 + 4 + \cdots 2^{40}$ pieces of paper of this thickness.

That is, $1 + 2 + 4 + \cdots 2^{40} = \dfrac{2^{40} - 1}{2 - 1}$.

The thickness is $\dfrac{2^{40} - 1}{2 - 1} \times 0.05 \approx$ 54976 km

b Solve the inequality $0.05 \times 2^n \geq 384400 \times 10^6$ for n an integer to find $n = 43$.

14 Option 1: \$52 million;

Option 2: \$45 040 000 million

Solutions to Exercise 4F

1 $S_\infty = \dfrac{a}{1-r}$

a $a = 1$

$r = \dfrac{1}{5} \div 1 = \dfrac{1}{5}$

$S_\infty = \dfrac{1}{1-\frac{1}{5}}$

$= \dfrac{5}{4}$

b $a = 1$

$r = -\dfrac{2}{3} \div 1 = -\dfrac{2}{3}$

$S_\infty = \dfrac{1}{1 - -\frac{2}{3}}$

$= \dfrac{3}{5}$

2 Each side, and hence each perimeter, will be half the larger side.

$r = \dfrac{1}{2},\ a = p$

Perimeter of nth triangle $= p \times \left(\dfrac{1}{2}\right)^{n-1}$

$= \dfrac{p}{2^{n-1}}$

$S_\infty = \dfrac{p}{1-\frac{1}{2}}$

$= 2p$

Area $= \dfrac{p^2\sqrt{3}}{9 \times 4^n}$

Sum of the areas $= \dfrac{p^2\sqrt{3}}{27}$

3 $a = 200,\ r = 0.94$

$S_\infty = \dfrac{200}{1-0.94}$

$= 3333\frac{1}{3}$ m

4 $a = 450,\ r = 0.65$

$S_\infty = \dfrac{450}{1-0.65}$

≈ 1285.7

Yes, it will kill him.

5 $a = 3,\ r = 0.5$

$S_\infty = \dfrac{3}{1-0.5} = 6$

He can only make the journey if he walks for an infinite time (which isn't very likely).

6 $a = 2,\ r = \dfrac{3}{4}$

$S_\infty = \dfrac{2}{1-0.75} = 8$

The frog will approach a limit of 8 m.

7 $a = \dfrac{1}{3},\ r = \dfrac{1}{3}$

$S_\infty = \dfrac{\frac{1}{3}}{1-\frac{1}{3}}$

$= \dfrac{1}{2}$ or 50%

8 $r = 70\% = 0.7$

$S_\infty = \dfrac{a}{1-0.7} = 40$

$a = 0.3 \times 40$

$= 12$ m

9 Note: all distances will be double (up and down) except the first (down only).

Use $a = 30,\ r = \dfrac{2}{3}$ and subtract 15 m from the answer.

$$S_\infty = \frac{30}{1-\frac{2}{3}} = 90$$

Distance = 90 − 15 = 75 m

10 **a** $a = 0.4,\ r = 0.1$

$$S_\infty = \frac{0.4}{1-0.1} = \frac{4}{9}$$

b $a = 0.03,\ r = 0.1$

$$S_\infty = \frac{0.03}{1-0.1} = \frac{3}{90} = \frac{1}{30}$$

c $a = 0.3,\ r = 0.1$

$$S_\infty = \frac{0.3}{1-0.1} = \frac{3}{9} = \frac{1}{3}$$

Decimal $= 10\frac{1}{3} = \frac{31}{3}$

d $a = 0.035,\ r = 0.01$

$$S_\infty = \frac{0.035}{1-0.01} = \frac{35}{990} = \frac{7}{198}$$

e $a = 0.9,\ r = 0.1$

$$S_\infty = \frac{0.9}{1-0.1} = \frac{9}{9} = 1$$

f $a = 0.1,\ r = 0.1$

$$S_\infty = \frac{0.1}{1-0.1} = \frac{1}{9}$$

Decimal $= 4\frac{1}{9} = \frac{37}{9}$

11 $S_4 = \dfrac{a(1-r^4)}{1-r} = 30$

$S_\infty = \dfrac{a}{1-r} = 32$

$a = 32(1-r)$

Substitute for a:

$$\frac{32(1-r)(1-r^4)}{1-r} = 30$$

$$32(1-r^4) = 30$$

$$1 - r^4 = \frac{30}{32}$$

$$r^4 = 1 - \frac{30}{32} = \frac{2}{32} = \frac{1}{16}$$

$$r = \frac{1}{2} \text{ or } r = -\frac{1}{2}$$

If $r = \frac{1}{2}$: $a = 32\left(1 - \frac{1}{2}\right) = 16$

If $r = -\frac{1}{2}$: $a = 32\left(1 - \left(-\frac{1}{2}\right)\right) = 48$

The first two terms are 16 and 8, or 48 and −24

12 $S_\infty = \dfrac{a}{1+\frac{1}{4}}$

$= \dfrac{4a}{5} = 8$

$a = 10$

$t_3 = 10 \times \left(-\dfrac{1}{4}\right)^2$

$= \dfrac{5}{8}$

13 $\dfrac{5}{1-r} = 15$

$5 = 15(1-r)$

$1 - r = \dfrac{1}{3}$

$r = 1 - \dfrac{1}{3}$

$= \dfrac{2}{3}$

14 $\dfrac{2}{1-r} = x$

Solve for r

$\dfrac{2}{x} = 1 - r$

$r = 1 - \dfrac{2}{x}$

Since $x > 2$, $\dfrac{2}{x} < 1$ and so $r = 1 - \dfrac{2}{x} < 1$

Solutions to short-answer questions (technology-free)

1 a $t_1 = 3$

$t_2 = 3 - 4 = -1$

$t_3 = -1 - 4 = -5$

$t_4 = -5 - 4 = -9$

$t_5 = -9 - 4 = -13$

$t_6 = -13 - 4 = -17$

b $t_1 = 5$

$t_2 = 2 \times 5 + 2 = 12$

$t_3 = 2 \times 12 + 2 = 26$

$t_4 = 2 \times 26 + 2 = 54$

$t_5 = 2 \times 54 + 2 = 110$

$t_6 = 2 \times 110 + 2 = 222$

2 a $t_1 = 2 \times 1 = 2$

$t_2 = 2 \times 2 = 4$

$t_3 = 2 \times 3 = 6$

$t_4 = 2 \times 4 = 8$

$t_5 = 2 \times 5 = 10$

$t_6 = 2 \times 6 = 12$

b $t_1 = -3 \times 1 + 2 = -1$

$t_2 = -3 \times 2 + 2 = -4$

$t_3 = -3 \times 3 + 2 = -7$

$t_4 = -3 \times 4 + 2 = -10$

$t_5 = -3 \times 5 + 2 = -13$

$t_6 = -3 \times 6 + 2 = -16$

3 a End of first year:
$\$5000 \times 1.05 = \$5250$
Start of second year:
$\$5250 + \$500 = \$5750$
End of second year:
$\$5750 \times 1.05 = \$6037.50$

b $t_n = 1.05(t_{n-1} + 500),\ t_1 = 5250$

4

$a + 3d = 19 \ldots (1)$

$a + 6d = 43 \ldots (2)$

Equation (2) − Equation (1)

$3d = 24$

$d = 8$

$a + 3 \times 8 = 19$

$a = -5$

$t_{20} = -5 + 19 \times 8$

$= 147$

5

$a + 4d = 0.35 \ldots (1)$

$a + 8d = 0.15 \ldots (2)$

Equation (2) − Equation (1)

$4d = -0.2$

$d = -0.05$

$a + 4 \times -0.05 = 0.35$

$a = 0.35 + 0.2$

$= 0.55$

$t_{14} = 0.55 + 13 \times -0.55$

$= -0.1$

6
$$a + 5d = -24 \ldots (1)$$
$$a + 13d = 6 \ldots (2)$$
Equation (2) − Equation (1)
$$8d = 30$$
$$d = 3.75$$
$$a + 5 \times 3.75 = -24$$
$$a = -24 - 18.75$$
$$= -42.75$$
$$S_{10} = 5 \times (-85.5 + 9 \times 3.75)$$
$$= -258.75$$

7 $a = -5,\ d = 7$
$$S_n = \frac{n}{2}(-10 + 7(n-1))$$
$$= 402$$
$$n(-10 + 7(n-1)) = 804$$
$$7n^2 - 10n - 7n = 804$$
$$7n^2 - 17n - 804 = 0$$
$$(7n + 67)(n - 12) = 0$$
$$n = 12 \text{ (since } n > 0)$$
$$\{n : S_n = 402\} = \{n : n = 12\}$$

8 $ar^5 = 9$
$$ar^9 = 729$$
$$r^4 = 81$$
$$r = 3 \text{ or } r = -3$$
$$r = 3 : a \times 3^5 = 9$$
$$a = \frac{9}{243} = \frac{1}{27}$$
$$t_4 = \frac{1}{27} \times 3^3 = 1$$
$$r = -3 : a \times (-3)^5 = 9$$
$$a = -\frac{1}{27}$$
$$t_4 = -\frac{1}{27} \times (-3)^3 = 1$$
So for either case, $t_4 = 1$

9 $a = 1000$
$$r = 1.035$$
$$t_n = ar^n$$
$$= 1000 \times 1.035^n$$

10 $9r^2 = 4$
$$r^2 = \frac{4}{9}$$
$$r = \pm\frac{2}{3}$$
$$t_2 = ar = \pm 6$$
$$t_4 = ar^3 = \pm\frac{8}{3}$$
Terms $= 6, \frac{8}{3}$ or $-6, -\frac{8}{3}$

11
$$a + ar + ar^2 = 24$$
$$ar^3 + ar^4 + ar^5 = 24$$
$$r^3(a + ar + ar^2) = 24$$
$$r^3 = 1$$
$$r = 1$$
All terms will be the same: $t_n = \frac{24}{3} = 8$
$$S_{12} = 12 \times 8 = 96$$

12 $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$S_8 = \dfrac{6 \times (-3^8 - 1)}{-3 - 1}$

$= -9840$

13 $a = 1,\ r = -\dfrac{1}{3}$

$S_\infty = \dfrac{1}{1 - -\dfrac{1}{3}}$

$= \dfrac{3}{4}$

14

$$\frac{x+4}{x} = \frac{2x+2}{x+4}$$

$$(x+4)^2 = x(2x+2)$$

$$x^2 + 8x + 16 = 2x^2 + 2x$$

$$2x^2 + 2x - x^2 - 8x - 16 = 0$$

$$x^2 - 6x - 16 = 0$$

$$(x-8)(x+2) = 0$$

$$x = 8 \text{ or } x = -2$$

Solutions to multiple-choice questions

1 D
$$t_1 = 3 \times 1 + 2 = 5$$
$$t_2 = 3 \times 2 + 2 = 8$$
$$t_3 = 3 \times 3 + 2 = 11$$

2 B
$$t_2 = 3 + 3 = 6$$
$$t_3 = 6 + 3 = 9$$
$$t_4 = 9 + 3 = 12$$

3 A
$$a = 10$$
$$d = 8 - 10 = -2$$
$$t_{10} = 10 + (9 \times -2)$$
$$= -8$$

4 A
$$a = 10,\ d = -2$$
$$S_{10} = \frac{10}{2}(10 + -8)$$
$$= 10$$

5 B
$$a = 8$$
$$d = 13 - 8 = 5$$
$$t_n = 8 + 5(n - 1) = 58$$
$$5(n - 1) = 50$$
$$n - 1 = 10$$
$$n = 11$$

6 D
$$a = 12$$
$$r = \frac{8}{12} = \frac{2}{3}$$
$$t_6 = 12 \times \left(\frac{2}{3}\right)^5$$
$$= \frac{128}{81}$$

7 E
$$a = 8$$
$$r = \frac{4}{8} = \frac{1}{2}$$
$$S_6 = \frac{8 \times \left(1 - \left(\frac{1}{2}\right)^6\right)}{1 - \frac{1}{2}}$$
$$= 15\frac{3}{4}$$

8 C
$$a = 8$$
$$r = \frac{4}{8} = \frac{1}{2}$$
$$S_\infty = \frac{8}{1 - \frac{1}{2}}$$
$$= 16$$

9 E
$$\text{Value} = 2000 \times 1.055^6$$
$$= \$2757.69$$

10 D
$$\frac{a}{1 - \frac{1}{3}} = 37.5$$
$$a = 37.5 \times \frac{2}{3}$$
$$= 25$$

Solutions to extended-response questions

1 a $0.8, 1.5, 2.2, \ldots$

b $d = 0.7$ and so the sequence is conjectured to be arithmetic.

c
$$t_n = 0.8 + (n-1) \times 0.7$$
$$\therefore t_{12} = 0.8 + (12-1) \times 0.7$$
$$= 8.5$$

The length of moulding in the kit size 12 is 8.5 metres.

2 a $d = 25$ and so the sequence is arithmetic.

b
$$t_n = a + (n-1)d$$
$$= 50 + (n-1) \times 25$$
$$= 50 + 25n - 25$$
$$= 25n + 25$$

c
$$t_{25} = 25 \times 25 + 25$$
$$= 650$$

There are 650 seeds in the 25th size packet.

3 The distances $5, 5-d,\ 5-2d, \ldots, 5-6d$ form an arithmetic sequence of seven terms with common difference $-d$.

$$\text{Now } S_n = \frac{n}{2}(a + \ell)$$
$$\therefore S_7 = \frac{7}{2}(5 + 5 - 6d)$$
$$\text{Since } S_7 = 32 - 3 = 29,\ 29 = \frac{7}{2}(10 - 6d)$$
$$\therefore \frac{58}{7} = 10 - 6d$$
$$\therefore 6d = \frac{12}{7}$$
$$\therefore d = \frac{2}{7}$$

The distance of the fifth pole from town A is given by S_5.

$$S_5 = \frac{5}{2}\left(5 + 5 - 4 \times \frac{2}{7}\right)$$
$$= \frac{155}{7}$$
$$= 22\frac{1}{7} \text{ and } 32 - 22\frac{1}{7} = 9\frac{6}{7}$$

The fifth pole is $22\frac{1}{7}$ km from town A and $9\frac{6}{7}$ km from town B.

4 **a** 20, 36, 52, 68, 84, 100, 116, 132, . . .

b $T_n = a + (n - 1)d$

$= 20 + (n - 1) \times 16$

$= 20 + 16n - 16$

$= 16n + 4$

c Let $T_n = 196$

$\therefore 16n + 4 = 196$

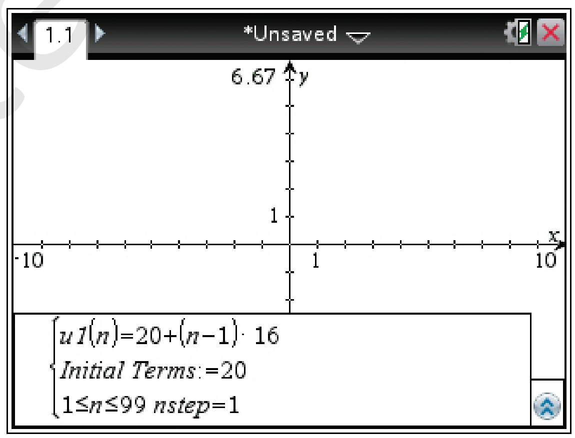
$\therefore 16n = 192$

$\therefore n = 12$

Yes, size 12 will handle 196 lines.

CAS calculator techniques for Question 4

TI: Open a Graphs page. Press **Menu** → **3** : **Graph Entry/Edit** → **6** : **Sequence** → **1** : **Sequence** and input the equation and initial term as shown. Press ENTER then press /T to view the sequence.

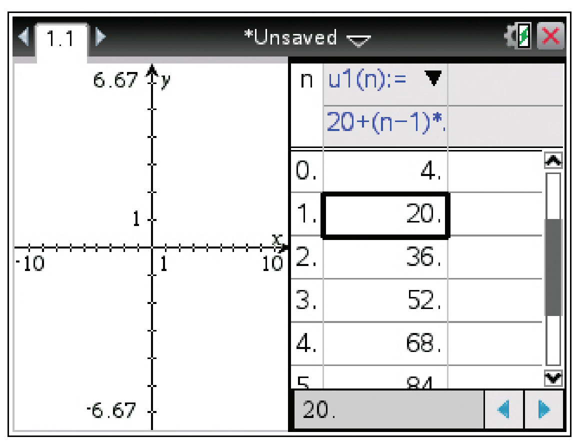


CP: Open the Sequence application. Input the following:

$a_{n+1} = 20 + (n - 1) \times 16$

$a_0 = 20$

Tap # to view the sequence.



5 a $D_n = a + (n - 1)d$

$= 2 + (n - 1) \times 7$

$= 2 + 7n - 7$

$= 7n - 5$

b $D_{n+1} = 191$

$\therefore 7(n + 1) - 5 = 191$

$\therefore 7(n + 1) = 196$

$\therefore n + 1 = 28$

$\therefore n = 27$

The firm made 27 different thicknesses.

6 $t_1 = 4,\ t_2 = 16,\ t_3 = 28 \qquad \therefore d = 12$

$t_{40} = a + (40 - 1)d$

$= 4 + 39 \times 12$

$= 472$

The house will slip 472 mm in the 40th year.

7 $t_1 = 16,\ t_2 = 24,\ t_3 = 32 \qquad \therefore d = 8$

$S_{10} = \frac{10}{2}(2 \times 16 + (10 - 1) \times 8)$

$= 5\ (32 + 72)$

$= 520$

She will have sent 520 cards altogether in 10 years.

8 a $a = 90,\ r = \frac{1}{10},$

$$\therefore S_6 = \frac{90\left(1 - \left(\frac{1}{10}\right)^6\right)}{1 - \frac{1}{10}}$$

$= 99.9999$

After six rinses, Joan will have washed out 99.9999 mg of shampoo.

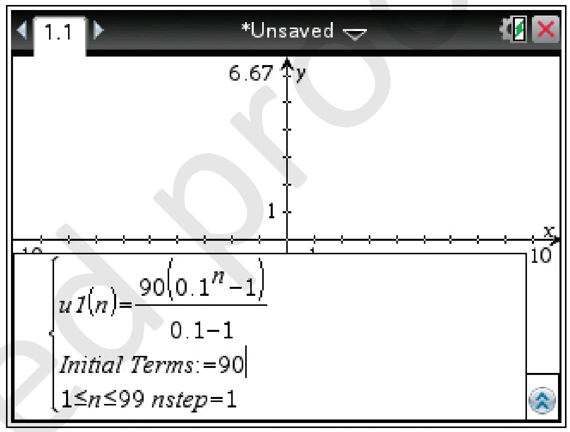
b

$$S_\infty = \frac{a}{1-r} = \frac{90}{1-\frac{1}{10}} = 100$$

There were 100 mg present at the beginning.

CAS calculator techniques for Question 8

TI: Open a Graphs page. Press **Menu** → **3** : **Graph Entry/Edit** → **6** : **Sequence** → **1** : **Sequence** and input the equation and initial term as shown. Press ENTER then press /T to view the sequence.

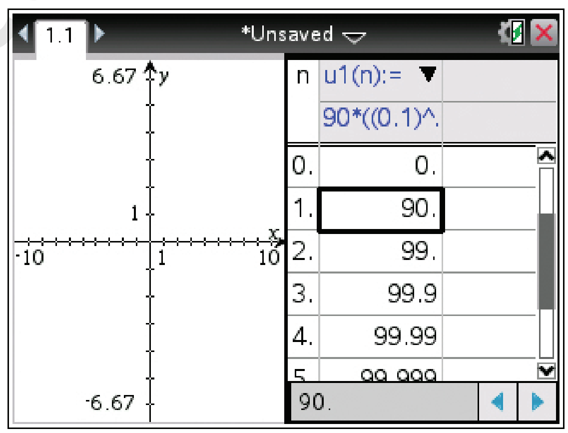


CP: Open the Sequence application.
Input the following:

$$a_{n+1} = \frac{90(0.1^n - 1)}{0.1 - 1}$$

$$a_0 = 90$$

Tap # to view the sequence.



9 **a** $t_1 = \frac{1}{3},\ t_2 = \left(\frac{1}{3}\right)^2,\ t_6 = \left(\frac{1}{3}\right)^6 = \frac{1}{729}$

The water level will rise by $\frac{1}{729}$ metres at the end of the sixth hour.

b $S_6 = \dfrac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^6\right)}{1 - \frac{1}{3}}$

$= \dfrac{364}{729}$

$= 0.499314\ldots$

The total height of the water level after six hours will be 1.499 m, correct to three decimal places.

$S_\infty = \dfrac{\frac{1}{3}}{1 - \frac{1}{3}}$

$= 0.5$

The maximum height the water will reach is 1.5 metres. If the prisoner is able to keep his head above this level, he will not drown.

10 a $\dfrac{400}{500} = \dfrac{320}{400} = 0.8$

$a = 500,\ r = 0.8,$

$\therefore t_n = 500(0.8)^{n-1}$

$t_{14} = 500(0.8)^{14-1}$

$= 27.487\,790\ldots$

On the 14th day they were subjected to 27.49 curie hours, correct to two decimal places.

b $S_n = \dfrac{a(1 - r^n)}{1 - r}$

$S_5 = \dfrac{500(1 - 0.8^5)}{1 - 0.8}$

$= 1680.8$

During the first five days, they were subjected to 1680.8 curie hours.

11 a $t_1 = \dfrac{2}{3} \times 81$

$t_2 = \left(\dfrac{2}{3}\right)^2 \times 81$

$t_6 = \left(\dfrac{2}{3}\right)^6 \times 81$

$= 7\dfrac{1}{9}$

After the sixth bounce, the ball reaches a height of $7\frac{1}{9}$ metres.

b Total distance $= 81 + \frac{2}{3} \times 81 + \frac{2}{3} \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \ldots$

$$= 81 + 2\left(\frac{2}{3} \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \ldots\right)$$

$$= 81 + 2 \times \frac{\frac{2}{3} \times 81}{1 - \frac{2}{3}}$$

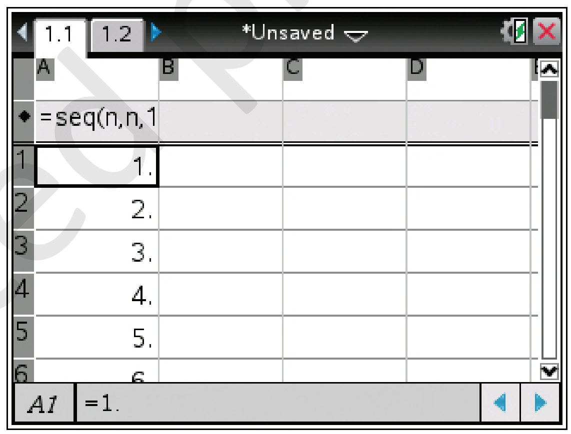
$$= 81 + 324$$

$$= 405$$

The total distance travelled by the ball is 405 metres.

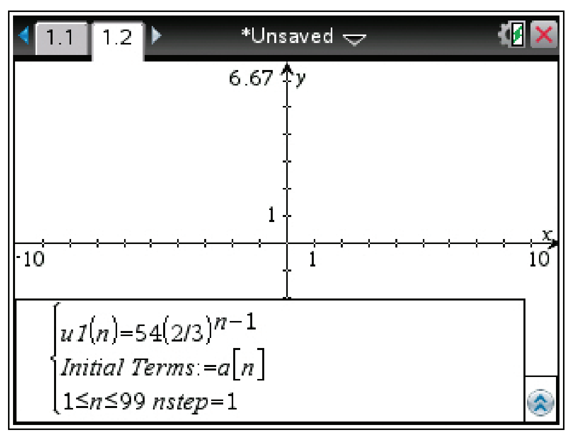
CAS calculator techniques for Question 11

TI: Open a Lists & Spreadsheet application. Type **seq**(**n**, **n**, **1**, **30**, **1**) into the formula cell for column A. This will place the number 1–30 into column A.



Open a Graphs application and input the following sequence.

Navigate back to the Lists & Spread-sheet page. Type **seq**(**ul**(**n**), **n**, **1**, **30**, **1**) into the formula cell for columns B.



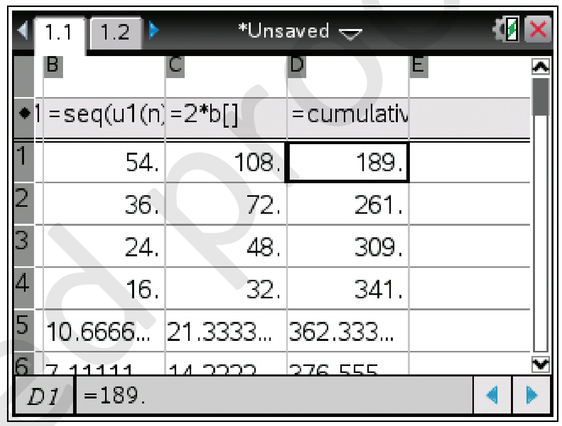
Type **2** × **b**[] into the formula cell for column C.

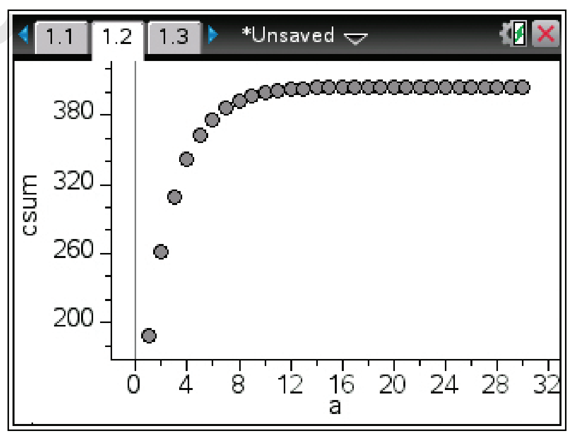
Type **cumulativeSum**(**c**[]) + **81** into the formula cell for column D.

Give column D the name **csum** and column A the name **a**

The graphs of these relations can now be considered. In a Data & Statistics application sketch the graph of **csum** against **a** as shown. This is the total distance travelled against the number of bounces.

The limiting behaviour is demonstrated by this graph.





12 $t_1 = 1 = 2^0$

$t_2 = 2 = 2^1$

$t_3 = 4 = 2^2$

$\therefore t_n = 2^{n-1}$

$S_n = \dfrac{a(1 - r^n)}{1 - r}$ where $a = 1,\ r = 2$

$\therefore S_{64} = \dfrac{1(1 - 2^{64})}{1 - 2}$

$= 2^{64} - 1$

The king had to pay $2^{64} - 1 = 1.845 \times 10^{19}$ grains of rice.

13 a i The amount of cement produced is an arithmetic sequence.

Let C_n be the amount of cement produced (in tonnes) in the nth month.

$C_n = a + (n - 1)d$ where $a = 4000,\ d = 250$

$= 4000 + (n - 1) \times 250$

$= 4000 + 250n - 250$

$\therefore C_n = 250n + 3750$

ii Let S_n be the amount of cement (in tonnes) produced in the first n months.

$S_n = \dfrac{n}{2}(a + 1)$ where $a = 4000,\ l = 250n + 3750$

$= \dfrac{n}{2}(4000 + 250n + 3750)$

$= \dfrac{n}{2}(250n + 7750)$

$= n(125n + 3875)$

$\therefore S_n = 125n(n + 31)$

$= 3875n + 125n^2$

iii When $C_n = 9250$,

$250n + 3750 = 9250$

$\therefore 250n = 5500$

$\therefore n = 22$

The amount of cement produced is 9250 tonnes in the 22nd month.

iv $C_n = 250n + 3750$

$\therefore T = 250m + 3750$

$\therefore m = \dfrac{1}{250}T - 15$

v $S_p = 522\,750$ and $S_p = \dfrac{p}{2}(a + l)$

$$\therefore 522\,750 = \frac{p}{2}(4000 + 250p + 3750)$$

$$\therefore 1045\,500 = p(250p + 7750)$$

$$\therefore 4182 = p(p + 31)$$

$\therefore p^2 + 31p - 4182 = 0$

Using the general quadratic formula,

$$p = \frac{-31 \pm \sqrt{31^2 - 4 \times 1 \times (-4182)}}{2}$$

$$= \frac{-31 \pm 131}{2}$$

$$= -82 \text{ or } 51$$

$$= 51 \text{ as } p > 0$$

b **i** The total amount of cement produced is a geometric series. Total amount of cement produced after n months is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ where } a = 3000,\ r = 1.08$$

$$= \frac{3000(1.08^n - 1)}{0.08}$$

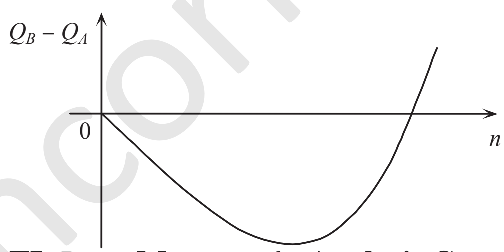
$$\therefore S_n = 37500(1.08^n - 1)$$

ii $Q_A = 125n(n + 31)$ and $Q_B = 37\,500(1.08^n - 1)$

$\therefore Q_B - Q_A = 37\,500(1.08^n - 1) - 125n(n + 31)$

Using a CAS calculator,

sketch ***f*1** = **37 500(1.08**^{x} − 1) − **125*x*(*x* + 31**)



TI: Press **Menu** → **6** : **Analysis Graph** → **1** : **Zero**

CP: Tap **Analysis** → **G - Solve** → **Root** to yield a horizontal axis intercept at (17.28, 0), correct to two decimal places. Hence, the smallest value of n for which $Q_B - Q_A \geq 0$ is 18.

14 **a** Geometric sequence with $a = 1$ and $r = 3$:

Number of white triangles after step n is 3^{n-1}

b Geometric sequence with $a = 1$ and $r = \dfrac{1}{2}$

Side length of white triangle in diagram n is $\left(\frac{1}{2}\right)^{n-1}$

c Geometric sequence with $a = 1$ and $r = \frac{3}{4}$:

Fraction that is white $= \left(\frac{3}{4}\right)^{n-1}$

d As $n \to \infty$ the fraction that is white approaches 0.

15 **a** Geometric sequence with $a = 1$ and $r = 8$:
Number of white squares after step n is 8^{n-1}

b Geometric sequence with $a = 1$ and $r = \frac{1}{3}$:

Side length of white square in diagram n is $\left(\frac{1}{3}\right)^{n-1}$

c Geometric sequence with $a = 1$ and $r = \frac{8}{9}$:

Fraction that is white $= \left(\frac{8}{9}\right)^{n-1}$

d As $n \to \infty$ the fraction that is white approaches 0.

Chapter 5 – Algebra II

Solutions to Exercise 5A

1 $ax^2 + bx + c = 10x^2 - 7$

$= 10x^2 + 0x - 7$

$a = 10,\ b = 0,\ c = -7$

2

$2a - b = 4$ ①

$a + 2b = -3$ ②

$4a - 2b = 8$ ③

② + ③:

$5a = 5$

$a = 1$

$a \times 1 - b = 4$

$b = -2$

3

$2a - 3b = 7$ ①

$3a + b = 5$ ②

① + 3 × ②:

$11a = 22$

$a = 2$

$3 \times 2 + b = 5$

$b = -1$

$c = 7$

4 $a(x + b)^2 + c = ax^2 + 2abx + ab^2 + c$

$a = 2$

$2ab = 4$

$b = 1$

$ab^2 + c = 5$

$2 + c = 5$

$c = 3$

5 $c(x + 2)^2 + a(x + 2) + 2$

$= cx^2 + 4cx + 4c + ax + 2a + d$

$c = 1$

$4c + a = 0$

$a = -4$

$4c + 2a + d = 0$

$4 - 8 + d = 0$

$d = 4$

$\therefore\ x^2 = (x + 2)^2 - 4(x + 2) + 4$

6 $(x + 1)^3 + a(x + 1)^2 + b(x + 1) + c$

$= x^3 + 3x^2 + 3x + 1 + ax$

$+ a + bx + b + c$

$3 + a = 0$

$a = -3$

$3 + 2a + b = 0$

$3 - 6 + b = 0$

$b = 3$

$1 + a + b + c = 0$

$c = -1$

$\therefore\ x^3 = (x + 1)^3 - 3(x + 1)^2$

$+ 3(x + 1) - 1$

7 $ax^2 + 2ax + a + bx + c = x^2$

$$a = 1$$

$$2a + b = 0$$

$$b = -2$$

$$a + c = 0$$

$$c = -1$$

8 a $a(x+b)^3 + c = ax^3 + 3abx^2 + 3ab^2x + ab^3 + c$

$$= 3x^3 - 9x^2 + 8x + 12$$

$$a = 3$$

$$3ab = -9$$

$$3 \times 3 \times b = -9$$

$$b = -1$$

Equating x terms:

$$3ab^2 = 8$$

$$3ab^2 = 3 \times 3 \times (-1)^2 = 9$$

The equality is impossible.

b Clearly this expression can be expressed in this form, if $a = 3,\ b = -1$ and

$$ab^3 + c = 2$$

$$-3 + c = 2$$

$$c = 5$$

9 Expanding gives the following:

$$n^3 = an^3 + 6an^2 + 11an + 6a + bn^2 + 3bn + 2b + cn + c + d$$

$$a = 1$$

$$6a + b = 0$$

$$b = -6$$

$$11a + 3b + c = 0$$

$$11 - 18 + c = 0$$

$$c = 7$$

$$6a + 2b + c + d = 0$$

$$6 - 12 + 7 + d = 0$$

$$d = -1$$

10 a Expanding gives the following:

$$n^2 = an^2 + 3an + 2a + bn^2 + 5bn + 6b$$

$$a + b = 1 \quad ①$$

$$3a + 5b = 0 \quad ②$$

$$2a + 6b = 0$$

$$a + 3b = 0 \quad ③$$

③ − ①:

$$2b = -1$$

$$b = -\frac{1}{2}$$

$$a + -\frac{1}{2} = 1$$

$$a = 1\frac{1}{2}$$

These do not satisfy the second equation, as $3 \times 1\frac{1}{2} + 5 \times -\frac{1}{2} = 2$.

b $$n^2 = an^2 + 3an + 2a + bn + b + c$$

$$a = 1$$

$$3a + b = 0$$

$$b = -3$$

$$2a + b + c = 0$$

$$2 - 3 + c = 0$$

$$c = 1$$

$$\therefore \quad n^2 = (n + 1)(n + 2) - 3(n + 1) + 1$$

11 a $a(x^2 + 2bx + b^2) + c =$
$ax^2 + 2abx + ab^2 + c$

b $$ax^2 + bx + c = A(x + B)^2 + C$$

$$= Ax^2 + 2ABx + AB^2 + C$$

$$A = a$$

$$2AB = b$$

$$B = \frac{b}{2a}$$

$$AB^2 + C = c$$

$$a \times \frac{b^2}{4a^2} + C = c$$

$$C = c - \frac{b^2}{4a}$$

$$\therefore \quad ax^2 + bx + x = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

12 $$(x - 1)^2(px + q) = (x^2 - 2x + 1)(px + q)$$

$$= px^3 + (q - 2p)x^2 + (p - 2q)x + q$$

Equating x^3 and x^2 terms:

$$p = a$$

$$q - 2p = b$$

$$q - 2a = b$$

$$q = 2a + b$$

Equating x and constant terms:

$$q = d$$

$$p - 2q = c$$

$$p = c + 2d$$

Equating the two different expressions for p and q gives:

$$d = 2a + b \quad (q)$$

$$\therefore \quad b = d - 2a$$

$$a = c + 2d \quad (p)$$

$$\therefore \quad c = a - 2d$$

13 $$c(x - a)(x - b) = cx^2 - acx - bcx + abc$$

$$= 3$$

$$c = 3$$

$$-ac - bc = 10$$

$$-3a - 3b = 10$$

$$abc = 3$$

$$3ab = 3$$

$$ab = 1$$

$$b = \frac{1}{a}$$

$$-3a - \frac{3}{a} = 10$$

$$3a^2 + 3 = -10a$$
$$3a^2 + 10a + 3 = 0$$
$$(3a + 1)(a + 3) = 0$$
$$a = -\frac{1}{3},\ b = -3,\ c = 3$$
$$\text{or } a = -3,\ b = -\frac{1}{3},\ c = 3$$

14 $n^2 = a(n-1)^2 + b(n-2)^2 + c(n-3)^2$

$= an^2 - 2an + a + bn^2 - 4bn + 4b + cn^2 + 9c$

$a + b + c = 1$ ①

$-2a - 4b - 6c = 0$

$a + 2b + 3c = 0$ ②

$a + 4b + 9c = 0$ ③

② − ①:

$b + 2c = -1$ ④

③ − ②:

$2b + 6c = 0$

$b + 3c = 0$ ⑤

⑤ − ④:

$c = 1$

$b + 3 \times 1 = 0$

$b = -3$

$a + b + c = 1$

$a - 3 + 1 = 1$

$a = 3$

$\therefore \quad n^2 = 3(n-1)^2 - 3(n-2)^2 + (n-3)^2$

15 $(x-a)^2(x-b) = (x^2 - 2ax + a^2)(x-b)$

$= x^3 - 2ax^2 - bx^2 + a^2x + 2abx - a^2b$

$-2a - b = 3$

$a^2 + 2ab = -9$

Substitute $b = -2a - 3$:

$a^2 + 2a(-2a - 3) = -9$

$a^2 - 4a^2 - 6a = -9$

$-3a^2 - 6a + 9 = 0$

$a^2 + 2a - 3 = 0$

$(a + 3)(a - 1) = 0$

$a = -3$ or $a = 1$

$b = -2a - 3$

$b = 3$ or $b = -5$

Comparing the constant terms:

$c = -a^2b$

$c = (-3)^2 \times 3 = -27$

or $c = (-1)^2 \times -5 = 5$

So $a = 1,\ b = -5, c = 5$

or $a = -3,\ b = 3, c = -27$

16 **a**

WTS $\quad P(-x) = P(x) \rightarrow b = d = 0$

$P(-x) = P(x)$

$ax^4 - bx^3 + cx^2 - dx + e = ax^4 + bx^3 + cx^2 + dx + e$

$-bx^3 - dx = bx^3 + dx$

$-2bx^3 - 2dx = 0$

$-2x(bx^2 + d) = 0$

$bx^2 + d = 0$

$\therefore b = d = 0$

b next page

WTS $P(-x) = -P(x) \rightarrow b = d = f = 0$

$$P(-x) = P(x)$$

$$a(-x)^5 + b(-x)^4 + c(-x)^3 + d(-x)^2 + e(-x) + f = -(a(x)^5 + b(x)^4 + c(x)^3 + d(x)^2 + ex + f)$$

$$-a(x)^5 + b(x)^4 - c(x)^3 + d(x)^2 - ex + f = -a(x)^5 - b(x)^4 - c(x)^3 - d(x)^2 - ex - f$$

$$b(x)^4 + d(x)^2 + f = -b(x)^4 - d(x)^2 - f$$

$$2b(x)^4 + 2d(x)^2 + 2f = 0$$

$$b(x)^4 + d(x)^2 + f = 0$$

$$\therefore b = d = f = 0$$

Solutions to Exercise 5B

1 a
$$x^2 - 2x = -1$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

b
$$x^2 - 6x + 9 = 0$$
$$(x-3)^2 = 0$$
$$x = 3$$

c Divide both sides by 5:
$$x^2 - 2x = \frac{1}{5}$$
$$x^2 - 2x + 1 = \frac{6}{5}$$
$$(x-1)^2 = \frac{6}{5} = \frac{30}{25}$$
$$x - 1 = \pm\frac{\sqrt{30}}{5}$$
$$x = 1 \pm \frac{\sqrt{30}}{5}$$

d Divide both sides by −2:
$$x^2 - 2x = -\frac{1}{2}$$
$$x^2 - 2x + 1 = \frac{1}{2}$$
$$(x-1)^2 = \frac{1}{2} = \frac{2}{4}$$
$$x - 1 = \pm\frac{\sqrt{2}}{2}$$
$$x = 1 \pm \frac{\sqrt{2}}{2}$$

e Divide both sides by 2:
$$x^2 + 2x = \frac{7}{2}$$
$$x^2 + 2x + 1 = \frac{9}{2}$$
$$(x+1)^2 = \frac{9}{2} = \frac{9 \times 2}{4}$$
$$x + 1 = \pm\frac{3\sqrt{2}}{2}$$
$$x = -1 \pm \frac{3\sqrt{2}}{2}$$

f $6x^2 + 13x + 1$
$$= 0$$
x
$$= \frac{-13 \pm \sqrt{169 - 4 \times 6 \times 1}}{12}$$
$$= \frac{-13 \pm \sqrt{145}}{12}$$

2 a
$$\Delta = 9 - 4m$$
No solutions: $\Delta < 0$
$$9 - 4m < 0$$
$$m > \frac{9}{4}$$

b
$$\Delta = 25 - 4m$$
Two solutions: $\Delta > 0$
$$25 - 4m > 0$$
$$m < \frac{25}{4}$$

c
$$\Delta = 25 + 32m$$
One solution: $\Delta = 0$
$$25 + 32m = 0$$
$$m = -\frac{25}{32}$$

d $\Delta = m^2 - 36$

Two solutions: $\Delta > 0$

$m^2 - 36 > 0$

$m > 6$ or $m < -6$

e $\Delta = m^2 - 16$

Nosolutions: $\Delta < 0$

$m^2 - 16 < 0$

$-4 < m < 4$

f $\Delta = m^2 + 16m$

One solution: $\Delta = 0$

$m^2 + 16m = 0$

$m = -16$ or $m = 0$

3 a $2x^2 - x - 4t = 0$

$$x = \frac{1 \pm \sqrt{1 - 4 \times 2 \times -4t}}{4}$$

$$= \frac{1 \pm \sqrt{32t + 1}}{4}$$

$32t + 1 \geq 0$

$32t \geq -1$

$t \geq -\frac{1}{32}$

b $4x^2 + 4x - t - 2 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 4 \times -(t+2)}}{8}$$
$$= \frac{-4 \pm \sqrt{16 + 32 + 16t}}{8}$$
$$= \frac{-4 \pm \sqrt{16t + 48}}{8}$$
$$= \frac{-4 \pm 4\sqrt{t+3}}{8}$$
$$= \frac{-1 \pm \sqrt{t+3}}{2}$$

$t + 3 \geq 0$

$t \geq -3$

c $5x^2 + 4x - t + 10 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 5 \times (-t + 10)}}{10}$$
$$= \frac{-4 \pm \sqrt{16 + 20t - 200}}{10}$$
$$= \frac{-4 \pm \sqrt{20t - 184}}{10}$$
$$= \frac{-4 \pm \sqrt{4(5t - 46)}}{10}$$
$$= \frac{-4 \pm 2\sqrt{5t - 46}}{10}$$
$$= \frac{-2 \pm \sqrt{5t - 46}}{5}$$

$5t - 46 \geq 0$

$5t \geq 46$

$t \geq \frac{46}{5}$

d $tx^2 + 4tx - t + 10 = 0$

$$x = \frac{-4t \pm \sqrt{16t^2 - 4 \times t \times (-t + 10)}}{2t}$$
$$= \frac{-4t \pm \sqrt{16t^2 + 4t^2 - 40t}}{2t}$$
$$= \frac{-4t \pm \sqrt{20t^2 - 40t}}{2t}$$
$$= \frac{-4t \pm 2\sqrt{5t^2 - 10t}}{2t}$$
$$= \frac{-2t \pm \sqrt{5t(t-2)}}{t}$$

$5t(t - 2) \geq 0$

This is a quadratic with a minimum and solutions $t = 0,\ t = 5$.

$\therefore\ t < 0,\ t \geq 5$

Note: $t = 0$ gives denominator zero, so it must be checked by substituting $t = 0$ in the original equation. In this case it gives $10 = 0$, and so is not a solution, but it should be checked.

(e.g. $tx^2 + 5x + 4 = t$ gives a solution with t on the denominator, but substituting $t = 0$ gives $5x + 4 = 0$, which has a solution.)

4 a $x = \frac{-p \pm \sqrt{p^2 - 4 \times 1(-16)}}{2}$

$$= \frac{-p \pm \sqrt{p^2 + 64}}{2}$$

b $p = 0$ gives $x = \frac{0 + \sqrt{64}}{2} = 4$

$p = 6$ gives $x = \frac{-6 + \sqrt{100}}{2} = 2$

5 a $2x^2 - 3px + (3p - 2) = 0$

$$\Delta = 9p^2 - 8(3p - 2)$$
$$= 9p^2 - 24p + 16$$
$$= (3p - 4)^2$$

$\Delta$is a perfect square

b $\Delta = 0 \Rightarrow p = \frac{4}{3}$

c Solution is $x = \frac{3p \pm (3p - 4)}{4}$

i When $p = 1, x = \dfrac{3 \pm 1}{4}$

$\therefore x = 1$ or $x = \dfrac{1}{2}$

ii When $p = 2, x = \dfrac{6 \pm 2}{4}$

$\therefore x = 2$ or $x = -\dfrac{1}{2}$

iii When $p = -1, x = \dfrac{-3 \pm 7}{4}$

$\therefore x = 1$ or $x = -\dfrac{5}{2}$

6 a $4(4p - 3)x^2 - 8px + 3 = 0$

$$\begin{aligned}\Delta &= 64p^2 - 48(4p - 3)\\ &= 64p^2 - 192p + 144\\ &= 16(4p^2 - 12p + 9)\\ &= 4(2p - 3)^2\end{aligned}$$

$\Delta$is a perfect square

b $\Delta = 0 \Rightarrow p = \dfrac{3}{2}$

c Solution is $x = \dfrac{8p \pm 4(2p - 3)}{8(4p - 3)}$

i When $p = 1, x = \dfrac{8 \pm 4}{8}$

$\therefore x = \dfrac{1}{2}$ or $x = \dfrac{3}{2}$

ii When $p = 2, x = \dfrac{16 \pm 4}{40}$

$\therefore x = \dfrac{1}{2}$ or $x = \dfrac{3}{10}$

iii When $p = -1, x = \dfrac{-8 \pm 20}{-56}$

$\therefore x = \dfrac{1}{2}$ or $x = -\dfrac{3}{14}$

7 Use Pythagoras' Theorem:

$$\begin{aligned}(8 - x)^2 + (6 + x)^2 &= 100\\ 64 - 16x + x^2 + 36 + 12x + x^2 &= 100\\ 2x^2 - 4x &= 0\\ 2x(x - 4) &= 0\\ x &= 2 \text{ since}\\ x &\neq 0\end{aligned}$$

8 Let x be the length of one part.
The other part has length $100 - x$
Let the second one be the larger.

$$\begin{aligned}\left(\frac{200 - x}{4}\right)^2 &= 9\frac{x^2}{16}\\ (200 - x)^2 &= 9x^2\\ 200 - x &= 3x\\ x &= 50\end{aligned}$$

$\therefore 200 - x = 150$

The length of the sides of the larger square is 37.5 cm

9 a Let $a = \sqrt{x}$

$$\begin{aligned}a^2 - 8a + 12 &= 0\\ (a - 6)(a - 2) &= 0\\ a = 6 \text{ or } a &= 2\\ \therefore x = 36 \text{ or } x &= 4\end{aligned}$$

b Let $a = \sqrt{x}$

$$\begin{aligned}a^2 - 2a - 8 &= 0\\ (a - 4)(a + 2) &= 0\\ a = 4 \text{ or } a &= -2\\ \therefore x &= 16\end{aligned}$$

c Let $a = \sqrt{x}$

$$a^2 - 5a - 14 = 0$$
$$(a - 7)(a + 2) = 0$$
$$a = 7 \text{ or } a = -2$$
$$\therefore x = 49$$

d Let $a = \sqrt[3]{x}$
$$a^2 - 9a + 8 = 0$$
$$(a - 8)(a - 1) = 0$$
$$a = 8 \text{ or } a = 1$$
$$\therefore x = 512 \text{ or } x = 1$$

e Let $a = \sqrt[3]{x}$
$$a^2 - a - 6 = 0$$
$$(a - 3)(a + 2) = 0$$
$$a = 3 \text{ or } a = -2$$
$$\therefore x = 27 \text{ or } x = -8$$

f Let $a = \sqrt{x}$
$$a^2 - 29a + 100 = 0$$
$$(a - 25)(a - 4) = 0$$
$$a = 25 \text{ or } a = 4$$
$$\therefore x = 625 \text{ or } x = 16$$

10 $3x^2 - 5x + 1 = a(x^2 + 2bx + b^2) + c$
Equating coefficients:
x^2: $3 = a$
x: $-5 = 2ba \Rightarrow b = -\frac{5}{6}$
constant: $1 = b^2a + c \Rightarrow c = -\frac{13}{12}$
Minimum value is $-\frac{13}{12}$

11 $$2 - 4x - x^2 = 24 + 8x + x^2$$
$$2x^2 + 12x + 22 = 0$$
$$x^2 + 6x + 11 = 0$$
$$\Delta = 36 - 4 \times 11 < 0$$
Therefore no intersection

12 $(b - c)x^2 + (c - a)x + (a - b) = 0$
$$((b - c)x - (a - b))(x - 1)) = 0$$
$$x = \frac{a - b}{b - c} \text{ or } x = 1$$

13 $2x^2 - 6x - m = 0$
$$x = \frac{6 \pm \sqrt{36 + 8m}}{4}$$
The difference of the two solutions
$$= \frac{\sqrt{36 + 8m}}{2}$$
$$\frac{\sqrt{36 + 8m}}{2} = 5$$
$$36 + 8m = 100$$
$$8m = 64$$
$$m = 8$$

14 a $(b^2 - 2ac)x^2 + 4(a + c)x - 8 = 0$
$$\Delta = 16(a + c)^2 + 32(b^2 - 2ac)$$
$$= 16(a^2 + 2ac + c^2) + 32b^2 - 64ac$$
$$= 16a^2 - 32ac + 16c^2 + 32b^2$$
$$= 16(a^2 - 2ac + c^2 + 2b^2)$$
$$= 16((a - c)^2 + 2b^2) > 0$$

b One solution if $a = c$ and $b = 0$

15 $$\frac{1}{2} + \frac{1}{x+k} = \frac{1}{x}$$

$$x(x+k) + 2x = 2(x+k)$$

$$x^2 + xk + 2x = 2x + 2k$$

$$x^2 + kx - 2k = 0$$

$$\Delta = k^2 + 8k$$

$$\Delta < 0 \Rightarrow k^2 + 8k < 0$$

$$k^2 + 8k < 0$$

$$k(k+8) < 0$$

$$-8 < k < 0$$

16 $3x^2 + px + 7 = 0$ $\Delta = p^2 - 84$

$p^2 > 84$

The smallest such integer is 10.

Solutions to Exercise 5C

1 a $\dfrac{6(x+3)-6x}{x(x+3)} = \dfrac{18}{x(x+3)}$

b $\dfrac{18}{x(x+3)} = 1$

$\dfrac{18 - x(x+3)}{x(x+3)} = 0$

$18 - x(x+3) = 0$

$18 - x - 3x = 0$

Re-arrange and divide by -1:

$x^2 + 3x - 18 = 0$

$(x-3)(x+6) = 0$

$x = 3$ or $x = -6$

2

$\dfrac{300}{x+5} = \dfrac{300}{x} - 2$

$300x = 300(x+5) - 2x(x+5)$

$300x = 300x + 1500 - 2x^2 + 10x$

$2x^2 - 10x - 1500 = 0$

$x^2 - 5x - 750 = 0$

$(x+25)(x-30) = 0$

$x = 25$ or $x = -30$

3 Let the numbers be n and $n + 2$.

$\dfrac{1}{n} + \dfrac{1}{n+2} = \dfrac{36}{323}$

$\dfrac{1}{n} + \dfrac{1}{n+2} - \dfrac{36}{323} = 0$

$\dfrac{323(n+2) + 323n - 36n(n+2)}{323n(n+2)} = 0$

$323n + 646 + 323n - 36n^2 - 72n = 0$

Re-arrange and divide by -1:

$36n^2 - 574n - 646 = 0$

$18n^2 - 287 - 323 = 0$

$(n-17)(18n+19) = 0$

$n = 17$

The numbers are 17 and 19.

4 a $\dfrac{40}{x}$

b $\dfrac{40}{x-2}$

$\dfrac{40}{x-2} - \dfrac{40}{x} = 1$

$40x - 40(x-2) = x(x-2)$

$80 = x^2 - 2x$

c

$x^2 - 2x - 80 = 0$

$(x-10)(x+8) = 0$

$\therefore x = 10$

5 a Car $= \dfrac{600}{x}$ km/h; Plane $= \dfrac{600}{x} + 220$ km/h

b Since the plane takes $x - 5.5$ hours to cover 600 km its average speed is also given by $\dfrac{600}{x-5.5}$. Hence:

$$\frac{600}{x} + 220 = \frac{600}{x - 5.5}$$

$$600(x - 5.5) + 220x(x - 5.5) = 600x$$

$$600x - 3300 + 220x^2 - 1210x = 600x$$

$$220x^2 - 1210x - 3300 = 0$$

$$2x^2 - 11x - 30 = 0$$

$$(2x - 15)(x + 2) = 0$$

$$x = 7.5$$

$$(x > 0)$$

Average speed of car = $\frac{600}{7.5}$ = 80 km/h

Average speed of plane = 80 + 220

= 300 km/h

6 Time taken by car = $\frac{200}{x}$ h

Time taken by train = $\frac{200}{x+5}$ h = $\frac{200}{x} - 2$ h

$$\frac{200}{x+5} = \frac{200}{x} - 2$$

$$\frac{200}{x+5} \times x(x+5) = \frac{200}{x} \times x(x+5) - 2 \times x(x+5)$$

$$200x = 200(x + 5) - 2x(x + 5)$$

$$= 200x + 1000 - 2x^2 - 10x$$

$$2x^2 + 10x - 1000 = 0$$

$$x^2 + 5x - 500 = 0$$

$$(x - 20)(x + 25) = 0$$

$$x = 20 \text{ since } x > 0$$

7 Let his average speed be x km/h.

His time for the journey is $\frac{108}{x}$ h.

$$\frac{108}{x} - 4\frac{1}{2} = \frac{108}{x+2}$$

$$108 \times 2(x + 2) - 4\frac{1}{2} \times 2x(x + 2) = 108 \times 2x$$

$$216x + 432 - 9x^2 - 18x = 216x$$

$$-9x^2 - 18x + 432 = 0$$

$$x^2 + 2x - 48 = 0$$

$$(x - 6)(x + 8) = 0$$

$$x = 6$$

since $x > 0$

His average speed is 6 km/h.

8 **a** Usual time = $\frac{75}{x}$ h.

$$\frac{75}{x} - \frac{18}{60} = \frac{75}{x + 12.5}$$

$$\frac{75}{x} - \frac{3}{10} = \frac{75}{x + 12.5}$$

$$75(x + 12.5) - 0.3x(x + 12.5) = 75x$$

$$75x + 937.5 - 0.3x^2 - 3.75x = 75x$$

$$-0.3x^2 - 3.75x + 937.5 = 0$$

Divide by 0.15:

$$2x^2 + 25x - 6250 = 0$$

$$(x - 50)(2x + 125) = 0$$

$$x = 50$$

b Average speed = $x + 12.5 = 62.5$

Time = $\frac{75}{62.5}$ = 1.2 h,
or 1 hour 12 minutes, or 72 minutes.

9 Let the speed of the slow train be x km/h.

The slow train takes $3\frac{1}{2} - \frac{10}{60} = \frac{7}{2} - \frac{1}{6}$

$$= \frac{20}{6}$$

$$= \frac{10}{3}$$

hours longer.

Compare the times:

$$\frac{250}{x+20} + \frac{10}{3} = \frac{250}{x}$$

$$750x + 10x(x+20) = 750(x+20)$$

$$750x + 10x^2 + 200x = 750x + 15\,000$$

$$10x^2 + 200x - 15\,000 = 0$$

$$x^2 + 20x - 1500 = 0$$

$$(x-30)(x+50) = 0$$

$$x = 30$$

Slow train: 30 km/h

Fast train: 50 km/h

10 Let the original speed of the car be x km/h. Compare the times:

$$\frac{105}{x+10} = \frac{105}{x} - \frac{1}{4}$$

$$420x = 420(x+10) - x(x+10)$$

$$420x = 420x + 4200 - x - 10x$$

$$x^2 + 10x - 4200 = 0$$

$$(x-60)(x+70) = 0$$

$$x = 60 \text{ km/h}$$

11 Let x min be the time the larger pipe takes, and C the capacity of the tank. Form an equation using the rates:

$$\frac{C}{x} + \frac{C}{x+5} = \frac{C}{11\frac{1}{9}}$$

$$\frac{C}{x} + \frac{C}{x+5} = \frac{9C}{100}$$

$$\frac{1}{x} + \frac{1}{x+5} = \frac{9}{100}$$

$$100(x+5) + 100x = 9x(x+5)$$

$$100x + 500 + 100x = 9x^2 + 45x$$

$$200x + 500 = 9x^2 + 45x$$

$$9x^2 - 155x - 500 = 0$$

$$(x-20)(9x+25) = 0$$

$$x = 20 \text{ since } x > 0$$

The larger pipe takes 20 min and the smaller pipe takes 25 min.

12 Let x min be the original time the first pipe takes, and y min be the original time the second pipe takes.

Let C be the capacity of the tank.

The original rates are $\frac{C}{x}$ and $\frac{C}{y}$.

The combined rate is $\frac{C}{x} + \frac{C}{y}$.

Total time taken = capacity ÷ rate

$$C \div \left(\frac{C}{x} + \frac{C}{y}\right) = C \div \frac{Cy + Cx}{xy}$$

$$= C \times \frac{xy}{Cx + Cy}$$

$$= \frac{xy}{x+y} = \frac{20}{3}$$

New rates are $\frac{C}{x-1}$ and $\frac{C}{y+2}$.

The combined rate is $\frac{C}{x-1} + \frac{C}{y+2}$.

$$C \div \left(\frac{C}{x-1} + \frac{C}{y+2}\right)$$

$$= C \div \frac{C(y+2) + C(x-1)}{(x-1)(y+2)}$$

$$= C \times \frac{(x-1)(y+2)}{Cx + Cy + C}$$

$$= \frac{(x-1)(y+2)}{x+y+1} = 7$$

Solve the simultaneous equations:

$$\frac{xy}{x+y} = \frac{20}{3}$$

$$\frac{(x-1)(y+2)}{x+y+1} = 7$$

Multiply both sides of the first equation by $3(x + y)$:

$$3xy = 20x + 20y$$

$$3xy - 20y = 20x$$

$$y(3x - 20) = 20x$$

$$y = \frac{20x}{3x-20}$$

Substitute into the second equation, after multiplying both sides by $x + y + 1$:

$$(x-1)(y+2)$$

$$= 7x + 7y + 7$$

$$(x-1)\left(\frac{20x}{3x-20} + 2\right)$$

$$= 7x + \frac{140x}{3x-20} + 7$$

$$(x-1)\frac{20x + 2(3x-20)}{3x-20}$$

$$= 7x + \frac{140x}{3x-20} + 7$$

$$(x-1)\frac{26x-40}{3x-20}$$

$$= 7x + \frac{140x}{3x-20} + 7$$

$$(x-1)(26x-40)$$

$$= 7x(3x-20)$$

$$+ 140x + 7(3x-20)$$

$$26x^2 - 66x + 40$$

$$= 21x^2 - 140x$$

$$+ 140x + 21x - 140$$

$$5x^2 - 87x + 180$$

$$= 0$$

$$(5x-12)(x-15)$$

$$= 0$$

$x = 2.4$ or $x = 15$

$$y = \frac{20x}{3x-20} < 0 \text{ if } x = 2.4$$

$$\therefore \quad x = 15$$

$$y = \frac{20 \times 15}{3 \times 15 - 20} = 12$$

The first pipe now takes one minute less, i.e. $15 - 1 = 14$ minutes.
The second pipe now takes two minutes more, i.e. $12 + 2 = 14$ minutes.

13 Let the average speed for rail and sea be $x + 25$ km/h and x km/h respectively.

Time for first route = $\frac{233}{x+25} + \frac{126}{x}$ hours.

Time for second route = $\frac{405}{x+25} + \frac{39}{x}$ hours.

$$\frac{233}{x+25} + \frac{126}{x}$$

$$= \frac{405}{x+25} + \frac{39}{x} + \frac{5}{6}$$

$$233 \times 6x + 126 \times 6(x+25)$$

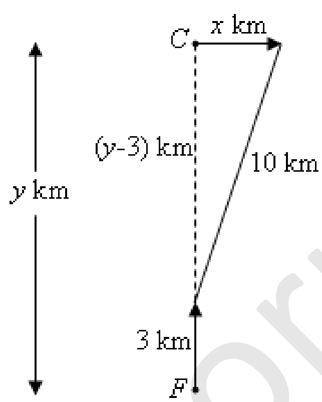
$$= 405 \times 6x + 39 \times 6(x+25) + 5x(x+25)$$

$$1398x + 756x + 18\,900$$

$$= 2430x + 234x + 5850 + 5x^2 + 125x$$

$$-5x^2 - 635x + 13\,050 = 0$$
$$x^2 + 127x - 2625 = 0$$
$$x = \frac{-127 + \sqrt{127^2 - 4 \times 1 \times 2625}}{2}$$
$$\approx 18.09$$
(Ignore negative square root as $x > 0$.)
Speed by rail is 18 + 25 = 43 km/h and by sea is 18 km/h.

14 After 15 min, the freighter has travelled 3 km, bringing it to 12 km from where the cruiser was.
Let x km be the distance the cruiser has travelled in 15 minutes and y km the original distance apart of the ships.
The distance the cruiser has travelled can be calculated using Pythagoras' theorem.



$$x^2 + (y - 3)^2 = 10^2 = 100$$
After a further 15 minutes, the distances will be $2x$ km and $(y - 6)$ km.
$$(2x)^2 + (y - 6)^2 = 13^2$$
$$4x^2 + (y - 6)^2 = 169$$
Multiply the first equation by 4 and subtract:
$$4(y - 3)^2 - (y - 6)^2 = 400 - 169$$
$$4y^2 - 24y + 36 - y^2 + 12y - 36 = 231$$
$$3y^2 - 12y - 231 = 0$$
$$y^2 - 4y - 77 = 0$$
$$(y - 11)(y + 7) = 0$$
$$y = 11$$
$$x^2 + 8^2 = 10^2$$
$$x = 6$$
The speed of the cruiser is $6 \div 0.25 = 24$ km/h. The cruiser will be due east of the freighter when the freighter has travelled 11 km.
This will take $\frac{11}{12}$ hours. During that time the cruiser will have travelled $24 \times \frac{11}{12} = 22$ km.
They will be 22 km apart.

15 Let x be the amount of wine first taken out of cask A.
After water is added, the concentration of wine in cask B is $\frac{x}{20}$.
If cask A is filled, it will receive x litres at concentration $\frac{x}{20}$.
The amount of wine in cask A will be $(20 - x) + x \times \frac{x}{20} = 20 - x + \frac{x^2}{20}$.

The concentration of wine in cask A will be $\dfrac{20 - x + \frac{x^2}{20}}{20} = 1 - \dfrac{x}{20} + \dfrac{x^2}{400}$.

The amount of wine in cask B will be

$(20 - x) \times \dfrac{x}{20} = x - \dfrac{x^2}{20}$.

Mixture is transferred again.

The amount of wine transferred is

$\left(1 - \dfrac{x}{20} + \dfrac{x^2}{400}\right) \times \dfrac{20}{3} = \dfrac{20}{3} - \dfrac{x}{3} + \dfrac{x^2}{60}$.

Amount of wine in A =
$\left(20 - x + \dfrac{x^2}{20}\right) - \left(\dfrac{20}{3} - \dfrac{x}{3} + \dfrac{x^2}{60}\right)$.

Amount of wine in B =
$\left(x - \dfrac{x^2}{20}\right) + \left(\dfrac{20}{3} - \dfrac{x}{3} + \dfrac{x^2}{60}\right)$

$$\left(20 - x + \frac{x^2}{20}\right) - \left(\frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}\right)$$

$$= \left(x - \frac{x^2}{20}\right) + \left(\frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}\right)$$

$$20 - x + \frac{x^2}{20} - \frac{20}{3} + \frac{x}{3} - \frac{x^2}{60}$$

$$= x - \frac{x}{20} + \frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}$$

$$-\frac{4x^2}{60} - \frac{4x}{3} + \frac{20}{3} = 0$$

$$\frac{x^2}{15} + \frac{4x}{3} - \frac{20}{3} = 0$$

$$x^2 + 20x - 100 = 0$$

$$(x - 10)^2 = 0$$

10 litres was first taken out of cask A.

16 Let v km/h be the speed of train B

The speed of train A is $v + 5$ km/h.

Time for train A = $\dfrac{80}{v + 5}$

Time for train B = $\dfrac{80}{v}$

$$\frac{80}{v} - \frac{80}{v + 5} = \frac{1}{3}$$

$$80(v + 5) - 80v = \frac{1}{3}(v(v + 5)$$

$$80v + 400 - 80v = \frac{1}{3}(v(v + 5))$$

$$1200 = v^2 + 5v$$

$$v^2 + 5v - 1200 = 0$$

$$v = \frac{5(\sqrt{193} - 1}{2}$$

or $v = -\dfrac{5(\sqrt{193} + 1)}{2}$.

The speed of train B is ≈ 37.23 km/h and the speed of train A is ≈ 42.23km/h

17 **a** $a + \sqrt{a^2 - 24a}$ minutes,
$a - 24 + \sqrt{a^2 - 24a}$ minutes

b **i** 84 minutes, 60 minutes

ii 48 minutes, 24 minutes

iii 36 minutes, 12 minutes

iv 30 minutes, 6 minutes

18 **a** 120 km

b 20 km/h, 30 km/h

Solutions to Exercise 5D

1 **a**

$$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$
$$= \frac{Ax+Bx+2A-B}{(x-1)(x+2)}$$

$A + B = 5$ ①

$2A - B = 1$ ②

① + ②:

$3A = 6$

$A = 2$

$2 + B = 5$

$B = 3$

$$\therefore \frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$$

b

$$\frac{-1}{(x+1)(2x+1)}$$
$$= \frac{A}{x+1} + \frac{B}{2x+1}$$
$$= \frac{A(2x+1)+B(x+1)}{(x+1)(2x+1)}$$
$$= \frac{2Ax+Bx+A+B}{(x+1)(2x+1)}$$

$2A + B = 0$ ①

$A + B = -1$ ②

① − ②:

$A = 1$

$1 + B = -1$

$B = -2$

$$\therefore \frac{-1}{(x+1)(2x+1)} = \frac{1}{x+1} - \frac{2}{2x+1}$$

c

$$\frac{3x-2}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$
$$= \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$
$$= \frac{Ax+Bx-2A+2B}{(x+2)(x-2)}$$

$A + B = 3$

$2A + 2B = 6$ ①

$-2A + 2B = -2$ ②

① + ②:

$4B = 4$

$B = 1$

$A + 1 = 3$

$A = 2$

$$\therefore \frac{3x-2}{(x+2)(x-2)} = \frac{2}{x+2} + \frac{1}{x-2}$$

d

$$\frac{4x+7}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$
$$= \frac{A(x-2)+B(x+3)}{(x+3)(x-2)}$$
$$= \frac{Ax+Bx-2A+3B}{(x+3)(x-2)}$$

$A + B = 4$

$2A + 2B = 8$ ①

$-2A + 3B = 7$ ②

① + ②:

$5B = 15$

$B = 3$

$A + 3 = 4$

$A = 1$

$$\therefore \frac{4x+7}{(x+3)(x-2)} = \frac{1}{x+3} + \frac{3}{x-2}$$

e

$$\frac{7-x}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$$
$$= \frac{Ax+Bx+A-4B}{(x-4)(x+1)}$$

$A + B = -1$ ①

$A - 4B = 7$ ②

① − ②:

$5B = -8$

$B = -\frac{8}{5}$

$A - \frac{8}{5} = -1$

$A = \frac{3}{5}$

$$\therefore \quad \frac{7-x}{(x-4)(x+1)} = \frac{3}{5(x-4)} - \frac{8}{5(x+1)}$$

2 a

$$\frac{2x+3}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$
$$= \frac{A(x-3)+B}{(x-3)^2}$$
$$= \frac{Ax-3A+B}{(x-3)^2}$$

$A = 2$

$-3A + B = 3$

$-6 + B = 3$

$B = 9$

$$\therefore \quad \frac{2x+3}{(x-3)^2} = \frac{2}{x-3} + \frac{9}{(x-3)^2}$$

b

$$\frac{9}{(1+2x)(1-x)^2}$$
$$= \frac{A}{1+2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$
$$= \frac{A(1-x)^2 + B(1+2x)(1-x) + C(1+2x)}{(1+2x)(1-x)^2}$$
$$= \frac{A - 2Ax + Ax^2 + B + Bx - 2Bx^2 + C + 2Cx}{(1+2x)(1-x)^2}$$

$A - 2B = 0$ ①

$-2A + B + 2C = 0$ ②

$A + B + C = 9$ ③

$2A + 2B + 2C = 18$ ④

④ − ②:

$4A + B = 18$

① × ④: $4A - 8B = 0$

$9B = 18$

$B = 2$

$4A + 2 = 18$

$A = 4$

$4 + 2 + C = 9$

$C = 3$

$$\therefore \quad \frac{9}{(1+2x)(1-x)^2} = \frac{4}{1+2x} + \frac{2}{1-x} + \frac{3}{(1-x)^2}$$

c

$$\frac{2x-2}{(x+1)(x-2)^2}$$
$$= \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$
$$= \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2}$$
$$= \frac{Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C}{(x+1)(x-2)^2}$$

$$A + B = 0 \quad ①$$
$$-4A - B + C = 2 \quad ②$$
$$4A - 2B + C = -2 \quad ③$$
$$③ - ②: 8A - B = -4 \quad ④$$
$$④ + ①: 9A = -4$$
$$A = -\frac{4}{9}$$
$$A + B = 0$$
$$B = \frac{4}{9}$$
$$4A - 2B + C = -2$$
$$-\frac{16}{9} - \frac{8}{9} + C = -2$$
$$C = -2 + \frac{24}{9} = \frac{2}{3}$$
$$\therefore \quad \frac{2x-2}{(x+1)(x-2)^2} = -\frac{4}{9(x+1)} + \frac{4}{9(x-2)} + \frac{2}{3(x-2)^2}$$

3 a $$\frac{3x+1}{(x+1)(x^2+x+1)}$$
$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$
$$= \frac{A(x^2+x+1) + (Bx+C)(x+1)}{(x+1)(x^2+x+1)}$$
$$= \frac{Ax^2 + Ax + A + Bx^2 + Bx + Cx + C}{(x+1)(x^2+x+1)}$$

$$A + B = 0 \quad ①$$
$$A + B + C = 3 \quad ②$$
$$A + C = 1 \quad ③$$
$$② - ①: C = 3$$
$$A + 3 = 1$$
$$A = -2$$
$$A + B + C = 3$$
$$-2 + B + 3 = 3$$
$$B = 2$$
$$\therefore \quad \frac{3x+1}{(x+1)(x^2+x+1)} = -\frac{2}{x+1} + \frac{2x+3}{x^2+x+1}$$

b $$\frac{3x^2+2x+5}{(x^2+2)(x+1)}$$
$$= \frac{Ax+B}{x^2+2} + \frac{C}{x+1}$$
$$= \frac{(Ax+B)(x+1) + C(x^2+2)}{(x^2+2)(x+1)}$$
$$= \frac{Ax^2 + Ax + Bx + B + Cx^2 + 2C}{(x^2+2)(x+1)}$$
$$A + C = 3 \quad ①$$
$$A + B = 2 \quad ②$$
$$B + 2C = 5 \quad ③$$
$① - ②$:
$$C - B = 1 \quad ④$$
$③ + ④$:
$$3C = 6$$
$$C = 2$$
$$A + 2 = 3$$

$$A = 1$$
$$1 + B = 2$$
$$B = 1$$
$$\therefore \quad \frac{3x^2 + 2x + 5}{(x^2 + 2)(x + 1)} = \frac{x + 1}{x^2 + 2} + \frac{2}{x + 1}$$

c Factorise the denominator:

$$2x^3 + 6x^2 + 2x + 6$$
$$= 2x^2(x + 3) + 2(x + 3)$$
$$= 2(x^2 + 1)(x + 3)$$

The 2 factor can be put with either fraction.

$$\frac{x^2 + 2x - 13}{2(x^2 + 1)(x + 3)}$$
$$= \frac{Ax + B}{x^2 + 1} + \frac{C}{2(x + 3)}$$
$$= \frac{2(Ax + B)(x + 3) + C(x^2 + 1)}{2(x^2 + 1)(x + 3)}$$
$$= \frac{2Ax^2 + 6Ax + 2Bx + 6B + Cx^2 + C}{2(x^2 + 1)(x + 3)}$$

$2A + C = 1$ ①

$6A + 2B = 2$

$9A + 3B = 3$ ②

$6B + C = -13$ ③

① − ③:

$2A - 6B = 14$

$A - 3B = 7$ ④

② + ④:

$10A = 10$

$A = 1$

$2 + C = 1$

$C = -1$

$3A + B = 1$

$A + B = 1$

$B = -2$

$$\therefore \quad \frac{x^2 + 2x - 13}{2(x^2 + 1)(x + 3)} = \frac{x - 2}{x^2 + 1} - \frac{1}{2(x + 3)}$$

4 $(x - 1)(x - 2) = x^2 - 3x + 2$

First divide:

$$3x^2 - 4x - 2 = 3(x^2 - 3x + 2) + 5x - 8$$
$$\frac{3x^2 - 4x - 2}{(x - 1)(x - 2)} = \frac{5x - 8}{(x - 1)(x - 2)}$$
$$\frac{5x - 8}{(x - 1)(x - 2)} = \frac{A}{x - 1} + \frac{B}{x - 2}$$
$$= \frac{A(x - 2) + B(x - 1)}{(x - 1)(x - 2)}$$
$$= \frac{Ax + Bx - 2A - B}{(x - 1)(x - 2)}$$

$A + B = 5$ ①

$-2A - B = -8$ ②

① + ②:

$-A = -3$

$A = 3$

$3 + B = 5$

$B = 2$

$$\therefore \quad \frac{5x - 8}{(x - 1)(x - 2)} = \frac{3}{x - 1} + \frac{2}{x - 2}$$

Use the previous working:

$$\frac{3x^2 - 4x - 2}{(x - 1)(x - 2)} = 3 + \frac{3}{x - 1} + \frac{2}{x - 2}$$

5 $\dfrac{2x+10}{(x+1)(x-1)^2}$

$= \dfrac{A}{x+1} + \dfrac{C}{(x-1)^2}$

$= \dfrac{A(x-1)^2 + C(x+1)}{(x+1)(x-1)^2}$

$= \dfrac{Ax^2 - 2Ax + A + Cx + C}{(x+1)(x-1)^2}$

$A = 0$

$-2A + C = 2$

$C = 2$

$A + C = 10$

$0 + 2 \neq 10$

It is impossible to find A and C to satisfy this equation.

6 a

$\dfrac{1}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

$= \dfrac{A(x+1) + B(x-1)}{(x-1)(x+1)}$

$= \dfrac{Ax + Bx + A - B}{(x-1)(x+1)}$

$A + B = 0$ ①

$A - B = 1$ ②

① + ②:

$2A = 1$

$A = \dfrac{1}{2}$

$\dfrac{1}{2} + B = 0$

$B = -\dfrac{1}{2}$

$\therefore \quad \dfrac{1}{(x-1)(x+1)} = \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

b

$\dfrac{x}{(x-2)(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{x+3}$

$= \dfrac{A(x+3) + B(x-2)}{(x-2)(x+3)}$

$= \dfrac{Ax + Bx + 3A - 2B}{(x-2)(x+3)}$

$A + B = 1$

$2A + 2B = 2$ ①

$3A - 2B = 0$ ②

① + ②:

$5A = 2$

$A = \dfrac{2}{5}$

$\dfrac{2}{5} + B = 1$

$B = \dfrac{3}{5}$

$\therefore \quad \dfrac{x}{(x-2)(x+3)} = \dfrac{2}{5(x-2)} + \dfrac{3}{5(x+3)}$

c

$\dfrac{3x+1}{(x-2)(x+5)} = \dfrac{A}{x-2} + \dfrac{B}{x+5}$

$= \dfrac{A(x+5) + B(x-2)}{(x-2)(x+5)}$

$= \dfrac{Ax + Bx + 5A - 2B}{(x-2)(x+5)}$

$A + B = 3$

$2A + 2B = 6$ ①

$5A - 2B = 1$ ②

① + ②:

$7A = 7$

$A = 1$

$1 + B = 3$

$B = 2$

$\therefore \quad \dfrac{3x+1}{(x-2)(x+5)} = \dfrac{1}{x-2} + \dfrac{2}{x+5}$

d $\dfrac{1}{(2x-1)(x+2)}$

$$= \frac{A}{2x-1} + \frac{B}{x+2}$$

$$= \frac{A(x+2) + B(2x-1)}{(2x-1)(x+2)}$$

$$= \frac{Ax + 2Bx + 2A - B}{(2x-1)(x+2)}$$

$$A + 2B = 0$$

$$2A + 4B = 0 \quad ①$$

$$2A - B = 1 \quad ②$$

① + ②:

$$5B = -1$$

$$B = -\frac{1}{5}$$

$$A + 2B = 0$$

$$A = \frac{2}{5}$$

$$\therefore \quad \frac{1}{(2x-1)(x+2)} = \frac{2}{5(2x-1)} - \frac{1}{5(x+2)}$$

e

$$\frac{3x+5}{(3x-2)(2x+1)}$$

$$= \frac{A}{3x-2} + \frac{B}{2x+1}$$

$$= \frac{A(2x+1) + B(3x-2)}{(3x-2)(2x+1)}$$

$$= \frac{2Ax + 3Bx + A - 2B}{(3x-2)(2x+1)}$$

$$2A + 3B = 3 \quad ①$$

$$A - 2B = 5$$

$$2A - 4B = 10 \quad ②$$

① − ②:

$$7B = -7$$

$$B = -1$$

$$A - 2 \times -1 = 5$$

$$A = 3$$

$$\therefore \quad \frac{3x+5}{(3x-2)(2x+1)} = \frac{3}{3x-2} - \frac{1}{2x+1}$$

f $\dfrac{2}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$$= \frac{A(x-1) + Bx}{x(x-1)}$$

$$= \frac{Ax + Bx - A}{x(x-1)}$$

$$A + B = 0$$

$$-A = 2$$

$$A = -2$$

$$-2 + B = 0$$

$$B = 2$$

$$\therefore \quad \frac{2}{x(x-1)} = \frac{2}{x-1} - \frac{2}{x}$$

g $\dfrac{3x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2+1}$

$$= \frac{A(x^2+1) + x(Bx + C)}{x(x^2+1)}$$

$$= \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2+1)}$$

$$A + B = 0$$

$$C = 3$$

$$A = 1$$

$$1 + B = 0$$

$$B = -1$$

$$\therefore \quad \frac{3x+1}{x(x^2+1)} = \frac{1}{x} + \frac{3-x}{x^2+1}$$

h $\dfrac{3x^2 + 8}{x(x^2 + 4)}$

$$= \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$= \frac{A(x^2 + 4) + x(Bx + C)}{x(x^2 + 4)}$$

$$= \frac{Ax^2 + 4A + Bx^2 + Cx}{x(x^2 + 4)}$$

$$A + B = 3$$

$$C = 0$$

$$4A = 8$$

$$A = 2$$

$$2 + B = 3$$

$$B = 1$$

$$\therefore \quad \frac{3x^2 + 8}{x(x^2 + 4)} = \frac{2}{x} + \frac{x}{x^2 + 4}$$

i $\dfrac{1}{x(x - 4)} = \dfrac{A}{x} + \dfrac{B}{x - 4}$

$$= \frac{A(x - 4) + Bx}{x(x - 4)}$$

$$= \frac{Ax + Bx - 4A}{x(x - 4)}$$

$$A + B = 0$$

$$-4A = 1$$

$$A = -\frac{1}{4}$$

$$-\frac{1}{4} + B = 0$$

$$B = \frac{1}{4}$$

$$\therefore \quad \frac{1}{x(x - 4)} = \frac{1}{4(x - 4)} - \frac{1}{4x}$$

j $\dfrac{x + 3}{x(x - 4)} = \dfrac{A}{x} + \dfrac{B}{x - 4}$

$$= \frac{A(x - 4) + Bx}{x(x - 4)}$$

$$= \frac{Ax + Bx - 4A}{x(x - 4)}$$

$$A + B = 1$$

$$-4A = 3$$

$$A = -\frac{3}{4}$$

$$-\frac{3}{4} + B = 1$$

$$B = \frac{7}{4}$$

$$\therefore \quad \frac{x + 3}{x(x - 4)} = \frac{7}{4(x - 4)} - \frac{3}{4x}$$

k First divide $x^2 - x^2 - 1$ by $x^2 - x$. You might observe a pattern in the question.

$$\frac{x^3 - x^2 - 1}{x^2 - x} = \frac{x(x^2 - x) - 1}{x^2 - x} =$$

$$x - \frac{1}{x^2 - x}$$

Express $-\dfrac{1}{x^2 - x}$ in partial fractions.

$$-\frac{1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}$$

$$= \frac{A(x - 1) + Bx}{x(x - 1)}$$

$$= \frac{Ax + Bx - A}{x(x - 1)}$$

$$A + B = 0$$

$$-A = -1$$

$$A = 1$$

$$1 + B = 0$$
$$B = -1$$
$$\therefore \quad \frac{-1}{x(x-1)} = \frac{1}{x} - \frac{1}{x-1}$$
$$\frac{x^3 - x^2 - 1}{x^2 - x} = x + \frac{1}{x} - \frac{1}{x-1}$$

l First divide $(x^2 - x^2 - 6)$ by $(-x^2 + 2x)$.

$$\begin{array}{r} -x - 1 \\ -x^2 + 2x \overline{) x^3 - x^2 - 6} \\ \underline{x^3 - 2x^2} \\ x^2 - 6 \\ \underline{x^2 - 2x} \\ 2x - 6 \end{array}$$

$\therefore$ $(x^3 - x^2 - 6) \div (-x^2 + 2x) = -x - 1 + \dfrac{2x-6}{x(2-x)}$

Separate $\dfrac{2x-6}{x(2-x)}$ into partial fractions.

$$\frac{2x-6}{x(2-x)} = \frac{A}{x} + \frac{B}{2-x}$$
$$= \frac{A(2-x) + Bx}{x(2-x)}$$
$$= \frac{-Ax + Bx + 2A}{x(2-x)}$$

$$-A + B = 2$$
$$2A = -6$$
$$A = -3$$
$$3 + B = 2$$
$$B = -1$$
$$\therefore \quad \frac{2x-6}{x(2-x)} = -\frac{3}{x} - \frac{1}{2-x}$$
$$\frac{x^3 - x^2 - 6}{2x - x^2} = -x - 1 - \frac{3}{x} - \frac{1}{2-x}$$

m $\dfrac{x^2 - x}{(x+1)(x^2+2)}$

$$= \frac{A}{x+1} + \frac{Bx + C}{x^2 + 2}$$
$$= \frac{A(x^2+2) + (Bx + C)(x+1)}{(x+1)(x^2+2)}$$
$$= \frac{Ax^2 + 2A + Bx^2 + Bx + Cx + C}{(x+1)(x^2+2)}$$

$$A + B = 1 \quad ①$$
$$B + C = -1 \quad ②$$
$$2A + C = 0 \quad ③$$
$$① - ②: A - C = 2 \quad ④$$
$$③ + ④: 3A = 2$$
$$A = \frac{2}{3}$$
$$\frac{2}{3} + B = 1$$
$$B = \frac{1}{3}$$
$$\frac{1}{3} + C = -1$$
$$C = -\frac{4}{3}$$
$$\therefore \quad \frac{x^2 - x}{(x+1)(x^2+2)} = \frac{2}{3(x+1)} + \frac{x-4}{3(x^2+2)}$$

n $x^3 - 3x - 2$ can be factorised into $(x-2)(x+1)^2$.

$$\frac{x^2 + 2}{(x-2)(x+1)^2}$$
$$= \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$= \frac{A(x+1)^2 + B(x+1)(x-2) + C(x-2)}{(x-2)(x+1)^2}$$
$$= \frac{Ax^2 + 2Ax + A + Bx^2 - Bx - 2B + Cx - 2C}{(x-2)(x+1)^2}$$

$$A + B = 1 \quad ①$$

$$2A - B + C = 0$$

$$4A - 2B + 2C = 0 \quad ②$$

$$A - 2B - 2C = 2 \quad ③$$

② + ③:

$$5A - 4B = 2 \quad ④$$

④ − 4 × ①:

$$9A = 6$$

$$A = \frac{2}{3}$$

$$A + B = 1$$

$$B = \frac{1}{3}$$

$$\frac{4}{3} - \frac{1}{3} + C = 0$$

$$C = -1$$

$$\therefore \quad \frac{x^2 + 2}{(x - 2)(x + 1)^2} = \frac{2}{3(x - 2)} + \frac{1}{3(x + 1)} - \frac{1}{(x + 1)^2}$$

o $\dfrac{2x^2 + x + 8}{x(x^2 + 4)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 4}$

$$= \frac{A(x^2 + 4) + x(Bx + C)}{x(x^2 + 4)}$$

$$= \frac{Ax^2 + 4A + Bx^2 + Cx}{x(x^2 + 4)}$$

$$A + B = 2$$

$$C = 1$$

$$4A = 8$$

$$A = 2$$

$$2 + B = 2$$

$$B = 0$$

$$\therefore \quad \frac{2x^2 + x + 8}{x(x^2 + 4)} = \frac{2}{x} + \frac{1}{x^2 + 4}$$

p $2x^2 + 7x + 6 = (2x + 3)(x + 2)$

$$\frac{1 - 2x}{(2x + 3)(x + 2)} = \frac{A}{2x + 3} + \frac{B}{x + 2}$$

$$= \frac{A(x + 2) + B(2x + 3)}{(2x + 3)(x + 2)}$$

$$= \frac{Ax + 2Bx + 2A + 3B}{(2x + 3)(x + 2)}$$

$$A + 2B = -2$$

$$2A + 4B = -4 \quad ①$$

$$2A + 3B = 1 \quad ②$$

① − ②:

$$B = -5$$

$$A + 2 \times -5 = -2$$

$$A = 8$$

$$\therefore \quad \frac{1 - 2x}{(2x + 3)(x + 2)} = \frac{8}{2x + 3} - \frac{5}{x + 2}$$

q $\dfrac{3x^2 - 6x + 2}{(x - 1)^2(x + 2)}$

$$= \frac{A}{x + 2} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}$$

$$= \frac{A(x - 1)^2 + B(x + 2)(x - 1) + C(x + 2)}{(x - 1)^2(x + 2)}$$

$$= \frac{Ax^2 - 2Ax + A + Bx^2 + Bx - 2B + Cx + 2C}{(x - 1)^2(x + 2)}$$

$$A + B = 3$$
$$4A + 4B = 12 \quad ①$$
$$-2A + B + C = -6 \quad ②$$
$$A - 2B + 2C = 2 \quad ③$$
③ − ②:
$$5A - 4B = 14 \quad ④$$
①+④:
$$9A = 26$$
$$A = \frac{26}{9}$$
$$\frac{26}{9} + B = 3$$
$$B = \frac{1}{9}$$
$$-\frac{52}{9} + \frac{1}{9} + C = -6$$
$$C = -\frac{1}{3}$$
$$\therefore \frac{3x^2 - 6x + 2}{(x-1)^2(x+2)} = \frac{26}{9(x+2)} + \frac{1}{9(x-1)} - \frac{1}{3(x-1)^2}$$

r $\frac{4}{(x-1)^2(2x+1)}$
$$= \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2 + B(2x+1)(x-1) + C(2x+1)}{(x-1)^2(2x+1)}$$
$$= \frac{Ax^2 - 2Ax + A + 2Bx^2 - Bx - B + 2Cx + C}{(x-1)^2(2x+1)}$$
$$A + 2B = 0 \quad ①$$
$$-2A - B + 2C = 0 \quad ②$$
$$A - B + C = 4$$
$$2A - 2B + 2C = 8 \quad ③$$
③ − ②:
$$4A - B = 8$$
$$8A - 2B = 16 \quad ④$$
① + ④:
$$9A = 16$$
$$A = \frac{16}{9}$$
$$\frac{16}{9} + 2B = 0$$
$$B = -\frac{8}{9}$$
$$\frac{16}{9} + \frac{8}{9} + C = 4$$
$$C = \frac{4}{3}$$
$$\therefore \frac{4}{(x-1)^2(2x+1)} = \frac{16}{9(2x+1)} - \frac{8}{9(x-1)} + \frac{4}{3(x-1)^2}$$

s Divide:
$$\begin{array}{r|l} & x - 2 \\ \hline x^2 - 4 & x^3 - 2x^2 - 3x + 9 \\ & \underline{x^3 - 0x^2 - 4x} \\ & \quad -2x^2 + x \\ & \quad \underline{-2x^2 + 8} \\ & \qquad\quad x + 1 \end{array}$$

$$\frac{x^3 - 2x^2 - 3x + 9}{x^2 - 4}$$

$$= x - 2 + \frac{x + 1}{x^2 - 4}$$

$$\frac{x + 1}{(x + 2)(x - 2)}$$

$$= \frac{A}{x + 2} + \frac{B}{x - 2}$$

$$= \frac{A(x - 2) + B(x + 2)}{(x + 2)(x - 2)}$$

$$= \frac{Ax + Bx - 2A + 2B}{(x + 2)(x - 2)}$$

$$A + B = 1$$

$$2A + 2B = 2 \quad ①$$

$$-2A + 2B = 1 \quad ②$$

① + ②:

$$4B = 3$$

$$B = \frac{3}{4}$$

$$A + \frac{3}{4} = 1$$

$$A = \frac{1}{4}$$

$$\therefore \quad \frac{x + 1}{(x + 2)(x - 2)} = \frac{1}{4(x + 2)} + \frac{3}{4(x - 2)}$$

$$\frac{x^3 - 2x^2 - 3x + 9}{x^2 - 4} = x - 2 + \frac{1}{4(x + 2)} + \frac{3}{4(x - 2)}$$

t Divide:

$$\begin{array}{r} x \\ x^2 - 1 \overline{\smash{\big)} x^3 + 3} \\ \underline{x^3 - x} \\ x + 3 \end{array}$$

$$\frac{x^3 + 3}{(x + 1)(x - 1)} = x + \frac{x + 3}{(x + 1)(x - 1)}$$

$$\frac{x + 3}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

$$= \frac{A(x - 1) + B(x + 1)}{(x + 1)(x - 1)}$$

$$= \frac{Ax + Bx - A + B}{(x + 1)(x - 1)}$$

$$A + B = 1 \quad ①$$

$$-A + B = 3 \quad ②$$

① + ②:

$$2B = 4$$

$$B = 2$$

$$A + 2 = 1$$

$$A = -1$$

$$\therefore \quad \frac{x + 3}{(x + 1)(x - 1)} = -\frac{1}{x + 1} + \frac{2}{x - 1}$$

$$\frac{x^3 + 3}{(x + 1)(x - 1)} = x - \frac{1}{x + 1} + \frac{2}{x - 1}$$

u $\dfrac{2x - 1}{(x + 1)(3x + 2)}$

$$= \frac{A}{x + 1} + \frac{B}{3x + 2}$$

$$= \frac{A(3x + 2) + B(x + 1)}{(x + 1)(3x + 2)}$$

$$= \frac{3Ax + Bx + 2A + B}{(x + 1)(3x + 2)}$$

$$3A + B = 2 \quad ①$$

$$2A + B = -1 \quad ②$$

① − ②: $A = 3$

$$9 + B = 2$$

$$B = -7$$

$$\therefore \quad \frac{2x - 1}{(x + 1)(3x + 2)} = \frac{3}{x + 1} - \frac{7}{3x + 2}$$

Solutions to Exercise 5E

1 **a** A simple start is often to subtract the equations.

$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } x = 1$$
$$\text{If } x = 0,\ y = 0$$
$$\text{If } x = 1,\ y = 1$$

The points of intersection are $(0, 0)$ and $(1, 1)$.

b Subtract the equations:

$$2x^2 - x = 0$$
$$x(2x-1) = 0$$
$$x = 0 \text{ or } x = \frac{1}{2}$$

If $x = 0,\ y = 0$

If $x = \frac{1}{2},\ y = \frac{1}{2}$

The points of intersection are $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$.

c Subtract the equations:

$$x^2 - 3x - 1 = 0$$
$$x = \frac{3 \pm \sqrt{9 - 4 \times 1 \times -1}}{2}$$
$$= \frac{3 \pm \sqrt{13}}{2}$$
$$= \frac{3 + \sqrt{13}}{2} \text{ or } \frac{3 - \sqrt{13}}{2}$$

$$\text{If } x = \frac{3 + \sqrt{13}}{2},\ y = 2 \times \frac{3 + \sqrt{13}}{2} + 1$$
$$= 4 + \sqrt{13}$$

$$\text{If } x = \frac{3 - \sqrt{13}}{2},\ y = 2 \times \frac{3 - \sqrt{13}}{2} + 1$$
$$= 4 - \sqrt{13}$$

The points of intersection are

$$\left(\frac{3 + \sqrt{13}}{2}, 4 + \sqrt{13}\right) \text{ and}$$
$$\left(\frac{3 - \sqrt{13}}{2}, 4 - \sqrt{13}\right).$$

2 **a** Substitute $y = 16 - x$ into $x^2 + y^2 = 178$

$$x^2 + (16 - x)^2 = 178$$
$$x^2 + 256 - 32x + x^2 = 178$$
$$2x^2 - 32x + 78 = 0$$
$$x^2 - 16x + 39 = 0$$
$$(x - 3)(x - 13) = 0$$
$$x = 3 \text{ or } x = 13$$

If $x = 3, y = 16 - x = 13$

If $x = 13, y = 16 - x = 3$

The points of intersection are $(3, 13)$ and $(13, 3)$.

b Substitute $y = 15 - x$ into $x^2 + y^2 = 125$.

$$x^2 + (15 - x)^2 = 125$$
$$x^2 + 225 - 30x + x^2 = 125$$
$$2x^2 - 30x + 100 = 0$$
$$x^2 - 15x + 50 = 0$$
$$(x - 5)(x - 10) = 0$$
$$x = 5 \text{ or } x = 10$$

If $x = 5,\ y = 15 - x = 10$

If $x = 10,\ y = 15 - x = 5$

The points of intersection are $(5, 10)$ and $(10, 5)$.

c Substitute $y = x - 3$ into $x^2 + y^2 = 185$.

$$x^2 + (x - 3)^2 = 185$$
$$x^2 + x^2 - 6x + 9 = 185$$
$$2x^2 - 6x - 176 = 0$$
$$x^2 - 3x - 88 = 0$$
$$(x - 11)(x + 8) = x = 0$$
$$x = 11 \text{ or } x = -8$$

If $x = 11, y = x - 3 = 8$

If $x = -8, y = x - 3 = -11$

The points of intersection are $(11, 8)$ and $(-8, -11)$.

d Substitute $y = 13 - x$ into $x^2 + y^2 = 97$.

$$x^2 + (13 - x)^2 = 97$$
$$x^2 + 169 - 26x + x^2 = 97$$
$$2x^2 - 26x + 72 = 0$$
$$x^2 - 13x + 36 = 0$$
$$(x - 4)(x - 9) = 0$$
$$x = 4 \text{ or } x = 9$$

If $x = 4,\ y = 13 - x = 9$

If $x = 9,\ y = 13 - x = 4$

The points of intersection are $(4, 9)$ and $(9, 4)$.

e Substitute $y = x - 4$ into $x^2 + y^2 = 106$.

$$x^2 + (x - 4)^2 = 106$$
$$x^2 + x^2 - 8x + 16 = 106$$
$$2x^2 - 8x - 90 = 0$$
$$x^2 - 4x - 45 = 0$$
$$(x - 9)(x + 5) = 0$$
$$x = 9 \text{ or } x = -5$$

If $x = 9,\ y = x - 4 = 5$

If $x = -5,\ y = x - 4 = -9$

The points of intersection are $(9, 5)$ and $(-5, -9)$.

3 a Substitute $y = 28 - x$ into $xy = 187$.

$$x(28 - x) = 187$$
$$28x - x^2 = 187$$
$$x^2 - 28x + 187 = 0$$
$$(x - 11)(x - 17) = 0$$
$$x = 11 \text{ or } x = 17$$

If $x = 11,\ y = 28 - x = 17$

If $x = 17,\ y = 28 - x = 11$

The points of intersection are $(11, 17)$ and $(17, 11)$.

b Substitute $y = 51 - x$ into $xy = 518$.

$$x(51 - x) = 518$$
$$51x - x^2 = 518$$
$$x^2 - 51x + 518 = 0$$
$$(x - 14)(x - 37) = 0$$
$$x = 14 \text{ or } x = 37$$

If $x = 14,\ y = 51 - x = 37$

If $x = 37,\ y = 51 - x = 14$

The points of intersection are $(14, 37)$ and $(37, 14)$.

c Substitute $y = x - 5$ into $xy = 126$.

$$x(x - 5) = 126$$
$$x^2 - 5x = 126$$
$$x^2 - 5x - 126 = 0$$
$$(x - 14)(x + 9) = 0$$
$$x = 14 \text{ or } x = -9$$

If $x = 14,\ y = x - 5 = 9$

If $x = -9,\ y = x - 5 = -14$

The points of intersection are $(14, 9)$ and $(-9, -14)$.

4 Substitute $y = 2x$ into the equation of the circle.

$$(x-5)^2 + (2x)^2 = 25$$
$$x^2 - 10x + 25 + 4x^2 = 25$$
$$5x^2 - 10x = 0$$
$$5x(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

If $x = 0, y = 2x = 0$
If $x = 2, y = 2x = 4$

The points of intersection are (0, 0) and (2, 4).

5 Substitute $y = x$ into the equation of the second curve.

$$x = \frac{1}{x-2} + 3$$
$$x(x-2) = 1 + 3(x-2)$$
$$x^2 - 2x = 1 + 3x - 6$$
$$x^2 - 5x + 5 = 0$$
$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 5}}{2}$$
$$= \frac{5 \pm \sqrt{5}}{2}$$
$$= \frac{5 + \sqrt{5}}{2} \text{ or } \frac{5 - \sqrt{5}}{2}$$

Since $y = x$, the points of intersection are

$\left(\frac{5+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$ and

$\left(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$.

6 Substitute $x = 3y$ into the equation of the circle.

$$9y^2 + y^2 - 30y - 5y + 25 = 0$$
$$10y^2 - 35y + 25 = 0$$
$$2y^2 - 7y + 5 = 0$$
$$(2y-5)(y-1) = 0$$
$$y = \frac{5}{2} \text{ or } y = 1$$

If $y = \frac{5}{2},\ x = 3y = \frac{15}{2}$
If $y = 1,\ x = 3y = 3$

The points of intersection are $\left(\frac{15}{2}, \frac{5}{2}\right)$ and (3, 1).

7 Make y the subject in $\frac{y}{4} - \frac{x}{5} = 1$.

$$\frac{y}{4} = \frac{x}{5} + 1$$
$$y = \frac{4x}{5} + 4$$

Substitute into $x^2 + 4x + y^2 = 12$.

$$x^2 + 4x + \left(\frac{4x}{5} + 4\right)^2 = 12$$
$$x^2 + 4x + \frac{16x^2}{25} + \frac{32x}{5} + 16 = 12$$
$$25x^2 + 100x + 16x^2 + 160x + 400 = 300$$
$$41x^2 + 260x + 100 = 0$$
$$x = \frac{-260 \pm \sqrt{67\,600 - 4 \times 41 \times 100}}{82}$$
$$= \frac{-260 \pm \sqrt{51\,200}}{82}$$
$$= \frac{-260 \pm \sqrt{25\,600 \times 2}}{82}$$
$$= \frac{-260 \pm 160\sqrt{2}}{82}$$
$$= \frac{-130 \pm 80\sqrt{2}}{41}$$

If $x = \dfrac{-130 + 80\sqrt{2}}{41}$,

$$y = \frac{4 \times (-130 + 80\sqrt{2})}{5 \times 41} + 4$$

$$= \frac{4 \times (-26 + 16\sqrt{2})}{41} + \frac{4 \times 41}{41}$$

$$= \frac{-104 + 64\sqrt{2} + 164}{41}$$

$$= \frac{60 + 64\sqrt{2}}{41}$$

Likewise, if $x = \dfrac{-130 - 80\sqrt{2}}{41}$,

$$y = \frac{60 - 64\sqrt{2}}{41}$$

The points of intersection are

$\left(\dfrac{-130 + 80\sqrt{2}}{41}, \dfrac{60 + 64\sqrt{2}}{41}\right)$ and $\left(\dfrac{-130 - 80\sqrt{2}}{41}, \dfrac{60 - 64\sqrt{2}}{41}\right)$.

8 Subtract the second equation from the first.

$$\frac{1}{x+2} - 3 + x = 0$$

$$1 - 3(x + 2) + x(x + 2) = 0$$

$$1 - 3x - 6 + x^2 + 2x = 0$$

$$x^2 - x - 5 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4 \times 1 \times -5}}{2}$$

$$= \frac{1 \pm \sqrt{21}}{2}$$

If $x = \dfrac{1 + \sqrt{21}}{2}$, $y = -x = \dfrac{-1 - \sqrt{21}}{2}$

If $x = \dfrac{1 - \sqrt{21}}{2}$, $y = -x = \dfrac{-1 + \sqrt{21}}{2}$

The points of intersection are

$\left(\dfrac{1 + \sqrt{21}}{2}, \dfrac{-1 - \sqrt{21}}{2}\right)$ and $\left(\dfrac{1 - \sqrt{21}}{2}, \dfrac{-1 + \sqrt{21}}{2}\right)$

9 Substitute $y = \dfrac{9x + 4}{4}$ into the equation of the parabola.

$$\left(\frac{9x + 4}{4}\right)^2 = 9x$$

$$\frac{(9x + 4)^2}{16} = 9x$$

$$(9x + 4)^2 = 9x \times 16$$

$$81x^2 + 72x + 16 = 144x$$

$$81x^2 - 72x + 16 = 0$$

$$(9x - 4)^2 = 0$$

$$x = \frac{4}{9}$$

$$y = \frac{9x + 4}{4}$$

$$= \frac{4 + 4}{4} = 2 \left(\frac{4}{9}, 2\right)$$

Note: Substitute into the linear equation, as substituting into the quadratic introduces a second answer that is not actually a solution.

10 Substitute $y = 2x + 3\sqrt{5}$ into the equation of the circle.

$$x^2 + (2x + 3\sqrt{5})^2 = 9$$

$$x^2 + 4x^2 + 12\sqrt{5}x + 45 = 9$$

$$5x^2 + 12\sqrt{5}x + 36 = 0$$

$$x^2 + \frac{12\sqrt{5}}{5}x + \frac{36}{5} = 0$$

$$x^2 + \frac{2 \times 6\sqrt{5}}{5}x + \frac{(6\sqrt{5})^2}{25} = 0$$

$$\left(x + \frac{6\sqrt{5}}{5}\right)^2 = 0$$

$$x = -\frac{6\sqrt{5}}{5}$$

$$y = 2x + 3\sqrt{5}$$

$$= -\frac{12\sqrt{5}}{5} + \frac{15\sqrt{5}}{5}$$

$$= \frac{3\sqrt{5}}{5}\left(-\frac{6\sqrt{5}}{5},\ \frac{3\sqrt{5}}{5}\right)$$

11 Substitute $y = \frac{1}{4}x + 1$ into $y = -\frac{1}{x}$.

$$\frac{1}{4}x + 1 = -\frac{1}{x}$$

$$\frac{x+4}{4} = -\frac{1}{x}$$

$$x(x+4) = -4$$

$$x^2 + 4x + 4 = 0$$

$$(x+2)^2 = 0$$

$$x = -2$$

$$y = -\frac{1}{x}$$

$$= \frac{1}{2}\left(-2,\ \frac{1}{2}\right)$$

12 Substitute $y = x - 1$ into $y = \frac{2}{x-2}$.

$$x - 1 = \frac{2}{x-2}$$

$$(x-1)(x-2) = 2$$

$$x^2 - 3x + 2 = 2$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0 \text{ or } x = 3$$

If $x = 0$, $y = x - 1 = -1$

If $x = 3$, $y = x - 1 = 2$

The points of intersection are $(0, -1)$ and $(3, 2)$.

13 **a** $2x^2 - 4x + 1 = 2x^2 - x - 1$

$$-3x = -2$$

$$x = \frac{2}{3}$$

$$y = -\frac{7}{9}$$

b

$$-2x^2 + x + 1 = 2x^2 - x - 1$$

$$4x^2 - 2x - 2 = 0$$

$$2x^2 - x - 1 = 0$$

$$(2x+1)(x-1) = 0$$

$$x = -\frac{1}{2} \text{ or } x = 1$$

Solutions: $\left(\frac{-1}{2}, 0\right), (1, 0)$

c $x^2 + x + 1 = x^2 - x - 2$

$$2x + 3 = 0$$

$$x = -\frac{3}{2}$$

$$y = \frac{7}{4}$$

d

$$3x^2 + x + 2 = x^2 - x + 2$$

$$2x^2 + 2x- = 0$$

$$2x(x+1) = 0$$

$x = 0$ or $x = -1$

Solutions: $(-1, 4), (0, 2)$

14 **a** $k = -2, k = 1$

b $-10 < c < 10$

c $p = 5$

Solutions to technology-free questions

1

$$3a + b = 11$$
$$6a + 2b = 22 \quad ①$$
$$a - 2b = -1 \quad ②$$
$① + ②$:
$$7a = 21$$
$$a = 3$$
$$3 \times 3 + b = 11$$
$$b = 2$$
$$2 + 2c = 4$$
$$c = 1$$

2 $x^3 = (x-1)^3$

$+ a(x-1)^2 + b(x-1) + c$

$= x^3 - 3x^2 + 3x - 1 + ax^2$

$- 2ax + a + bx - b + c$

$$-3 + a = 0$$
$$a = 3$$
$$3 - 2 \times 3 + b = 0$$
$$b = 3$$
$$-1 + 3 - 3 + c = 0$$
$$c = 1$$
$$\therefore \quad x^3 = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$$

3 $(x+1)^2(px+q)$

$= (x^2 + 2x + 1)(px + q)$

$= px^3 + (q + 2p)x^2$

$+ (p + 2q)x + q$

$$a = p$$
$$b = q + 2p$$
$$c = p + 2q$$
$$d = q$$
$$2a + d = 2p + q = b$$
$$a + 2d = p + 2q = c$$

4 $(x-2)^2(px+q) = (x^2 - 4x + 4)(px + q)$

$= px^3 + (q - 4p)x^2$

$+ (4p - 4q)x + 4q$

$$a = p$$
$$b = q - 4p$$
$$c = 4p - 4q$$
$$d = 4q$$
$$-4a + \frac{1}{4}d = -4p + q = b$$
$$4a - d = 4p - 4q = c$$

5 a

$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x = -4 \text{ or } x = 3$$

b

$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1 \text{ or } 2$$

c

$$x^2 - 3x - 11 = -1$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$
$$x = 5 \text{ or } x = -2$$

d $2x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4 \times 2 \times 1}}{4}$$

$$= \frac{4 \pm \sqrt{8}}{4}$$

$$= \frac{2 \pm \sqrt{2}}{2}$$

e $3x^2 - 2x + 5 - t = 0$

$$x = \frac{2 \pm \sqrt{4 - 4 \times 3 \times (5 - t)}}{6}$$

$$= \frac{2 \pm \sqrt{4 - 60 + 12t}}{6}$$

$$= \frac{2 \pm \sqrt{12t - 56}}{6}$$

$$= \frac{2 \pm \sqrt{4(3t - 14)}}{6}$$

$$= \frac{2 \pm 2\sqrt{3t - 14}}{6}$$

$$= \frac{1 \pm \sqrt{3t - 14}}{3}$$

f $tx^2 - tx + 4 = 0$

$$x = \frac{t \pm \sqrt{t^2 - 4 \times t \times 4}}{2t}$$

$$= \frac{t \pm \sqrt{t^2 - 16t}}{2t}$$

6 $\dfrac{2(x + 2) - 3(x - 1)}{(x - 1)(x + 2)} = \dfrac{1}{2}$

$$2(2x + 4 - 3x + 3) = (x - 1)(x + 2)$$

$$2(-x + 7) = x^2 + x - 2$$

$$-2x + 14 = x^2 + x - 2$$

$$x^2 + 3x - 16 = 0$$

$a = 1,\ b = 3,\ c = -16$

$$x = \frac{-3 \pm \sqrt{9 - 4 \times 1 \times -16}}{2}$$

$$= \frac{-3 \pm \sqrt{73}}{2}$$

7 **a** $\dfrac{-3x + 4}{(x - 3)(x + 2)} = \dfrac{A}{x - 3} + \dfrac{B}{x + 2}$

$$= \frac{A(x + 2) + B(x - 3)}{(x - 3)(x + 2)}$$

$$= \frac{Ax + Bx + 2A - 3B}{(x - 3)(x + 2)}$$

$A + B = -3$

$3A + 3B = -9$ ①

$2A - 3B = 4$ ②

① + ②:

$5A = -5$

$A = -1$

$-1 + B = -3$

$B = -2$

$$\therefore \frac{-3x + 4}{(x - 3)(x + 2)} = -\frac{1}{x - 3} - \frac{2}{x + 2}$$

b $\dfrac{7x + 2}{(x + 2)(x - 2)} = \dfrac{A}{x + 2} + \dfrac{B}{x - 2}$

$$= \frac{A(x - 2) + B(x + 2)}{(x + 2)(x - 2)}$$

$$= \frac{Ax + Bx - 2A + 2B}{(x + 2)(x - 2)}$$

$A + B = 7$

$2A + 2B = 14$ ①

$-2A + 2B = 2$ ②

① + ②:

$4B = 16$

$B = 4$

$A + 4 = 7$

$A = 3$

$$\therefore \frac{7x + 2}{(x + 2)(x - 2)} = \frac{3}{x + 2} + \frac{4}{x - 2}$$

c

$$\frac{7-x}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$
$$= \frac{A(x+5)+B(x-3)}{(x-3)(x+5)}$$
$$= \frac{Ax+Bx+5A-3B}{(x-3)(x+5)}$$

$$A + B = -1$$
$$3A + 3B = -3 \qquad \textcircled{1}$$
$$5A - 3B = 7 \qquad \textcircled{2}$$

$\textcircled{1} + \textcircled{2}$:

$$8A = 4$$
$$A = \frac{1}{2}$$
$$\frac{1}{2} + B = -1$$
$$B = -\frac{3}{2}$$

$$\therefore \quad \frac{7-x}{(x-3)(x+5)} = \frac{1}{2(x-3)} - \frac{3}{2(x+5)}$$

d

$$\frac{3x-9}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+B(x-5)}{(x-5)(x+1)}$$
$$= \frac{Ax+Bx+A-5B}{(x-5)(x+1)}$$

$$A + B = 3$$
$$5A + 5B = 15 \qquad \textcircled{1}$$
$$A - 5B = -9 \qquad \textcircled{2}$$

$\textcircled{1} + \textcircled{2}$:

$$6A = 6$$
$$A = 1$$
$$1 + B = 3$$
$$B = 2$$

$$\therefore \quad \frac{3x-9}{(x-5)(x+1)} = \frac{1}{x-5} + \frac{2}{x+1}$$

e

$$\frac{3x-4}{(x+3)(x+2)^2}$$
$$= \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$
$$= \frac{A(x+2)^2 + B(x+3)(x+2) + C(x+3)}{(x+3)(x+2)^2}$$
$$= \frac{Ax^2 + 4Ax + 4A + Bx^2 + 5Bx + 6B + Cx + 3C}{(x+3)(x+2)^2}$$

$$A + B = 0$$
$$8A + 8B = 0 \qquad \textcircled{1}$$
$$4A + 5B + C = 3$$
$$12A + 15B + 3C = 9 \qquad \textcircled{2}$$
$$4A + 6B + 3C = -4 \qquad \textcircled{3}$$

$\textcircled{2} - \textcircled{3}$:

$$8A + 9B = 13 \qquad \textcircled{4}$$

$\textcircled{4} - \textcircled{1}$:

$$B = 13$$
$$A + 13 = 0$$
$$A = -13$$
$$4 \times -13 + 5 \times 13 + C = 3$$
$$C = -10$$

$$\therefore \quad \frac{3x-4}{(x+3)(x+2)^2} = -\frac{13}{x+3} + \frac{13}{x+2} - \frac{10}{(x+2)^2}$$

f

$$\frac{6x^2-5x-16}{(x-1)^2(x+4)}$$
$$= \frac{A}{x+4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2 + B(x+4)(x-1) + C(x+4)}{(x-1)^2(x+4)}$$
$$= \frac{Ax^2 - 2Ax + A + Bx^2 + 3Bx - 4B + Cx + 4C}{(x-1)^2(x+4)}$$

$$A + B = 6$$

$$16A + 16B = 96 \quad ①$$

$$-2A + 3B + C = -5$$

$$-8A + 12B + 4C = -20 \quad ②$$

$$A - 4B + 4C = -16 \quad ③$$

③ − ②:

$$9A - 16B = 4 \quad ④$$

① + ④:

$$25A = 100$$

$$A = 4$$

$$4 + B = 6$$

$$B = 2$$

$$-2 \times 4 + 3 \times 2 + C = -5$$

$$C = -3$$

$$\therefore \quad \frac{6x^2 - 5x - 16}{(x-1)^2(x+4)} = \frac{4}{x+4} + \frac{2}{x-1} - \frac{3}{(x-1)^2}$$

g $\dfrac{x^2 - 6x - 4}{(x^2+2)(x+1)}$

$$= \frac{Ax + B}{x^2 + 2} + \frac{C}{x+1}$$

$$= \frac{(Ax+B)(x+1) + C(x^2+2)}{(x^2+2)(x+1)}$$

$$= \frac{Ax^2 + Ax + Bx + B + Cx^2 + 2C}{(x^2+2)(x+1)}$$

$$A + C = 1 \quad ①$$

$$A + B = -6 \quad ②$$

$$B + 2C = -4 \quad ③$$

① − ②:

$$C - B = 7 \quad ④$$

③ + ④:

$$3C = 3$$

$$C = 1$$

$$A + 1 = 1$$

$$A = 0$$

$$0 + B = -6$$

$$B = -6$$

$$\therefore \quad \frac{x^2 - 6x - 4}{(x^2+2)(x+1)} = \frac{1}{x+1} - \frac{6}{x^2+2}$$

h $\dfrac{-x + 4}{(x-1)(x^2+x+1)}$

$$= \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}$$

$$= \frac{A(x^2+x+1) + (Bx+C)(x-1)}{(x-1)(x^2+x+1)}$$

$$= \frac{Ax^2 + Ax + A + Bx^2 - Bx + Cx - C}{(x-1)(x^2+x+1)}$$

$$A + B = 0 \quad ①$$

$$A - B + C = -1 \quad ②$$

$$A - C = 4 \quad ③$$

② + ③:

$$2A - B = 3 \quad ④$$

① + ④:

$$3A = 3$$
$$A = 1$$
$$B = -1$$
$$1 - C = 4$$
$$C = -3$$
$$\therefore \quad \frac{-x+4}{(x-1)(x^2+x+1)} = \frac{1}{x-1} - \frac{x+3}{x^2+x+1}$$

i $$\frac{-4x+5}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$
$$= \frac{A(x-3) + B(x+4)}{(x+4)(x-3)}$$
$$= \frac{Ax + Bx - 3A + 4B}{(x+4)(x-3)}$$
$$A + B = -4$$
$$3A + 3B = -12 \quad ①$$
$$-3A + 4B = 5 \quad ②$$
$$① + ②: 7B = 7$$
$$B = -1$$
$$A - 1 = -4$$
$$A = -3$$
$$\therefore \quad \frac{-4x+5}{(x+4)(x-3)} = -\frac{3}{x+4} - \frac{1}{x-3}$$
$$= \frac{1}{3-x} - \frac{3}{x+4}$$

j $$\frac{-2x+8}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$
$$= \frac{A(x-3) + B(x+4)}{(x+4)(x-3)}$$
$$= \frac{Ax + Bx - 3A + 4B}{(x+4)(x-3)}$$
$$A + B = -2$$
$$3A + 3B = -6 \quad ①$$
$$-3A + 4B = 8 \quad ②$$
$$① + ②: 7B = 2$$
$$B = \frac{2}{7}$$
$$A + \frac{2}{7} = -2$$
$$A = -\frac{16}{7}$$
$$\therefore \quad \frac{-2x+8}{(x+4)(x-3)} = \frac{2}{7(x-3)} - \frac{16}{7(x+4)}$$

8 a $$\frac{14x-28}{(x-3)(x^2+x+2)}$$
$$= \frac{A}{x-3} + \frac{Bx+C}{x^2+x+2}$$
$$= \frac{A(x^2+x+2) + (Bx+C)(x-3)}{(x-3)(x^2+x+2)}$$
$$= \frac{Ax^2 + Ax + 2A + Bx^2 - 3Bx + Cx - 3C}{(x-3)(x^2+x+2)}$$

$$A + B = 0$$
$$9A + 9B = 0 \quad ①$$
$$A - 3B + C = 14$$
$$3A - 9B + 3C = 42 \quad ②$$
$$2A - 3C = -28 \quad ③$$
$$② + ③: 5A - 9B = 14 \quad ④$$
$$① + ④: 14A = 14$$
$$A = 1$$
$$1 + B = 0$$
$$B = -1$$
$$1 - 3 \times -3 + C = 14$$
$$C = 10$$
$$\therefore \frac{14x - 28}{(x-3)(x^2+x+2)} = \frac{1}{x-3} + \frac{-x+10}{x^2+x+2} = \frac{1}{x-3} - \frac{x-10}{x^2+x+2}$$

b $\frac{1}{(x+1)(x^2-x+2)}$

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2-x+2}$$
$$= \frac{A(x^2-x+2) + (Bx+C)(x+1)}{(x+1)(x^2-x+2)}$$
$$= \frac{Ax^2 - Ax + 2A + Bx^2 + Bx + Cx + C}{(x+1)(x^2-x+2)}$$

$$A + B = 0 \quad ①$$
$$-A + B + C = 0 \quad ②$$
$$2A + C = 1 \quad ③$$
$$③ - ②: 3A - B = 1 \quad ④$$
$$① + ④: 4A = 1$$
$$A = \frac{1}{4}$$
$$\frac{1}{4} + B = 0$$
$$B = -\frac{1}{4}$$
$$-\frac{1}{4} - \frac{1}{4} + C = 0$$
$$C = \frac{1}{2}$$
$$\therefore \frac{1}{(x+1)(x^2-x+2)} = \frac{1}{4(x+1)} + \frac{-x+2}{4(x^2-x+2)} = \frac{1}{4(x+1)} - \frac{x-2}{4(x^2-x+2)}$$

c First divide $3x^3$ by $x^2 - 5x + 4$.

$$\begin{array}{r} 3x + 15 \\ x^2 - 5x + 4 \overline{\smash{)}3x^3 } \\ 3x^3 - 15x^2 + 12x \\ \hline 15x^2 - 12x \\ 15x^2 - 75x + 60 \\ \hline 63x - 60 \end{array}$$

$\frac{3x^3}{x^2-5x+4} = 3x + 15 + \frac{63x - 60}{(x-4)(x-1)}$ (factorising the denominator)

$$\frac{63x-60}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$$
$$= \frac{A(x-1) + B(x-4)}{(x-4)(x-1)}$$
$$= \frac{Ax + Bx - A - 4B}{(x-4)(x-1)}$$

$$A + B = 63 \quad ①$$
$$-A - 4B = -60 \quad ②$$
$$① + ②: -3B = 3$$
$$B = -1$$
$$A - 1 = 63$$
$$A = 64$$
$$\therefore \quad \frac{63x - 60}{(x-4)(x-1)} = \frac{64}{x-4} - \frac{1}{x-1}$$
$$\frac{3x^3}{x^2 - 5x + 4} = 3x + 15 + \frac{64}{x-4} - \frac{1}{x-1}$$

9 a
$$x^2 = -x$$
$$x^2 + x = 0$$
$$x(x+1) = 0$$
$$x = 0 \text{ or } x = -1$$
If $x = 0,\ y = 0$
If $x = -1,\ y = 1$
The points of intersections are $(0, 0)$ and $(-1, 1)$.

b Substitute $y = 4 - x$ into $x^2 + y^2 = 16$.
$$x^2 + (4 - x)^2 = 16$$
$$x^2 + 16 - 8x + x^2 = 16$$
$$2x^2 - 8x = 0$$
$$x^2 - 4x = 0$$
$$x(x - 4) = 0$$
$$x = 0 \text{ or } x = 4$$
If $x = 0,\ y = 4$
If $x = 4,\ y = 0$
The points of intersections are $(0, 4)$ and $(4,\ 0)$.

c Substitute $y = 5 - x$ into $xy = 4$.
$$x(5 - x) = 4$$
$$5x - x^2 - 4 = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x - 4)(x - 1) = 0$$
$$x = 4 \text{ or } x = 1$$
If $x = 4,\ y = 1$
If $x = 1,\ y = 4$
The points of intersections are (4, 1) and (1, 4).

10 Substitute $x = 3y - 1$ into the circle.
$$(3y - 1)^2 + 2(3y - 1) + y^2 = 9$$
$$9y^2 - 6y + 1 + 6y - 2 + y^2 = 9$$
$$10y^2 - 10 = 0$$
$$y^2 - 1 = 0$$
$$(y + 1)(y - 1) = 0$$
$$y = 1 \text{ or } y = -1$$
If $y = -1,\ x = -4$
If $y = 1,\ x = 2$
The points of intersections are (2, 1) and $(-4, -1)$.

11 a $t = \dfrac{135}{x}$

b $t = \dfrac{135}{x - 15}$

c $x = 60$

d 60 km/h, 45 km/h

Solutions to multiple-choice questions

1 C $x^2 = (x+1)^2 + b(x+1) + c$

$= x^2 + 2x + 1 + bx + b + c$

$b + 2 = 0$

$b = -2$

$b + c + 1 = 0$

$c = 1$

2 D $x^3 = a(x+2)^3 + b(x+2)^2$

$+ c(x+2) + d$

$= ax^3 + 6ax^2 + 12ax + 8a$

$+ bx^2 + 4bx + 4b$

$+ cx + 2c + d$

$a = 1$

$b + 6a = 0$

$b = -6$

$12a + 4b + c = 0$

$c = 12$

$8a + 4b + 2c + d = 0$

$d = -8$

3 D $a = 3,\ b = -6,\ c = 3$

$$x = \frac{6 \pm \sqrt{36 - 4 \times 3 \times 3}}{2 \times 3}$$

$$= \frac{6 \pm \sqrt{0}}{6}$$

$= 1$

4 C $(x-4)(x+6) = 0$

$x^2 + 2x - 24 = 0$

$x^2 + 2x = 24$

$2x^2 + 4x = 48$

5 E $$\frac{3}{x+4} - \frac{5}{x-2}$$

$$= \frac{3(x-2) - 5(x+4)}{(x+4)(x-2)}$$

$$= \frac{3x - 6 - 5x - 20}{(x+4)(x-2)}$$

$$= \frac{-2x - 26}{(x+4)(x-2)}$$

$$= \frac{-2(x+13)}{(x+4)(x-2)}$$

6 E $$\frac{4}{(x+3)^2} + \frac{2x}{x+1}$$

$$= \frac{4(x+1) + 2x(x+3)^2}{(x+3)^2(x+1)}$$

$$= \frac{4x + 4 + 2x^3 + 12x^2 + 18x}{(x+3)^2(x+1)}$$

$$= \frac{2x^3 + 12x + 22x + 4}{(x+3)^2(x+1)}$$

$$= \frac{2(x^3 + 6x^2 + 11x + 2)}{(x+3)^2(x+1)}$$

7 C $$\frac{7x^2 + 13}{(x-1)(x^2+x+2)}$$

$$= \frac{a}{x-1} + \frac{bx+c}{x^2+x+2}$$

$$= \frac{a(x^2+x+2) + (bx+c)(x-1)}{(x-1)(x^2+x+2)}$$

$$= \frac{ax^2 + ax + 2a + bx^2 - bx + cx - c}{(x-1)(x^2+x+2)}$$

$$a + b = 7 \quad ①$$
$$a - b + c = 0 \quad ②$$
$$2a - c = 13 \quad ③$$
② + ③:
$$3a - b = 13 \quad ④$$
① + ④:
$$4a = 20$$
$$a = 5$$
$$5 + b = 7$$
$$b = 2$$
$$a - b + c = 0$$
$$C = -3$$

8 D $$\frac{4x-3}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$
$$= \frac{A(x-3)+B}{(x-3)^2}$$
$$= \frac{Ax-3A+B}{(x-3)^2}$$
$$A = 4$$
$$-3 \times 4 + B = -3$$
$$B = 9$$
$$\therefore \quad \frac{4x-3}{(x-3)^2} = \frac{4}{x-3} + \frac{9}{(x-3)^2}$$

9 B $2x^2 + 5x + 2 = (2x+1)(x+2)$
$$\frac{8x+7}{(2x+1)(x+2)}$$
$$= \frac{A}{2x+1} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(2x+1)}{(2x+1)(x+2)}$$
$$= \frac{Ax+2Bx+2A+B}{(2x+1)(x+2)}$$
$$A + 2B = 8$$
$$2A + 4B = 16 \quad ①$$
$$2A + B = 7 \quad ②$$
① − ②:
$$3B = 9$$
$$B = 3$$
$$A + 2B = 8$$
$$A = 2$$
$$\therefore \quad \frac{8x+7}{(2x+1)(x+2)} = \frac{2}{2x+1} + \frac{3}{x+2}$$

10 B $$\frac{-3x^2+2x-1}{(x^2+1)(x+1)}$$
$$= \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$
$$= \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}$$
$$= \frac{Ax^2+Ax+Bx+B+Cx^2+C}{(x^2+1)(x+1)}$$
$$A + C = -3 \quad ①$$
$$A + B = 2 \quad ②$$
$$B + C = -1 \quad ③$$
① − ②:
$$C - B = -5$$
$$2C = -6$$
$$C = -3$$
$$A + -3 = -3$$
$$A = 0$$
$$0 + B = 2$$
$$B = 2$$
$$\therefore \quad \frac{-3x+2x+5}{(x^2+1)(x+1)} = \frac{2}{x^2+1} - \frac{3}{x+1}$$

Solutions to extended-response questions

1 a Let V km/h be the initial speed.

$V - 4$ is the new speed.

It takes 2 more hours to travel at the new speed,

$$\therefore \quad \frac{240}{V} + 2 = \frac{240}{V-4} \qquad \ldots \boxed{1}$$

$$\therefore \quad 240(V-4) + 2V(V-4) = 240V$$

$$\therefore \quad 240V - 960 + 2V^2 - 8V = 240V$$

$$\therefore \quad 2V^2 - 8V - 960 = 0$$

$$\therefore \quad V^2 - 4V - 480 = 0$$

$$\therefore \quad (V-24)(V+20) = 0$$

$$\therefore \quad V = 24 \text{ or } V = -20$$

Actual speed is 24 km/h.

b If it travels at $V - a$ km/h and takes 2 more hours, equation $\boxed{1}$ from **a** becomes

$$\frac{240}{V} + 2 = \frac{240}{V-a}$$

$$\therefore \quad 240(V-a) + 2V(V-a) = 240V$$

$$\therefore \quad 240V - 240a + 2V^2 - 2Va = 240V$$

$$\therefore \quad 2V^2 - 2aV - 240a = 0$$

$$\therefore \quad V^2 - aV - 120a = 0$$

Using the general quadratic formula,

$$V = \frac{a + \sqrt{a^2 + 480a}}{2}$$

When $a = 60$, $V = 120$, i.e. the speed is 120 km/h, a fairly fast speed. So if speed is less than this, practical values are $0 < a < 60$ and then $0 < V < 120$.

c If it travels at $V - a$ km/h and takes a more hours, equation $\boxed{1}$ from **a** becomes

$$\frac{240}{V} + a = \frac{240}{V-a}$$

$$\therefore \quad 240(V-a) + aV(V-a) = 240V$$

$$\therefore \quad 240V - 240a + aV^2 - a^2V = 240V$$

$$\therefore \quad aV^2 - a^2V - 240a = 0$$

$$\therefore \quad V^2 - aV - 240 = 0$$

Using the general quadratic formula,

$$V = \frac{a + \sqrt{a^2 + 960}}{2}$$

The only pairs of integers for a and V are found in the table below.

a	1	8	14	22	34	43	56	77	118
V	16	20	24	30	40	48	60	80	120

2 A table is a useful way to display the speed, time taken and distance covered for each train.

	distance (km)	time (h)	speed (km/h)
Faster train	b	$\frac{b}{v}$	v
Slower train	b	$\frac{b}{v} + a$	$b \div \left(\frac{b}{v} + a\right) = \frac{bv}{b + av}$

a In c hours, the faster train travels a distance of cv km.

In c hours, the slower train travels a distance of $\frac{bcv}{b + av}$ km.

Since the slower train travels 1 km less than the faster one in c hours,

$$cv - 1 = \frac{bcv}{b + av}$$

$$\therefore (cv - 1)(b + av) = bcv$$

$$\therefore bcv + acv^2 - b - av = bcv$$

$$\therefore acv^2 - av - b = 0$$

Using the general quadratic formula,

$$v = \frac{a \pm \sqrt{a^2 + 4abc}}{2ac}$$

$$= \frac{a + \sqrt{a^2 + 4abc}}{2ac} \text{ since } v > 0$$

Therefore the speed of the faster train is $\frac{a + \sqrt{a^2 + 4abc}}{2ac}$ km/h.

b If the speed of the faster train is a rational number, then $a^2 + 4abc$ must be a square number.

Set 1

If $a = 1$,

then $a^2 + 4abc = 1 + 4bc$

e.g. $a = 1,\ b = 1,\ c = 2$

in which case $v = \frac{a + \sqrt{a^2 + 4abc}}{2ac}$

becomes $v = \frac{1 + \sqrt{1^2 + 4 \times 1 \times 1 \times 2}}{2 \times 1 \times 2}$

$$= \frac{1 + \sqrt{9}}{4}$$

$$= 1 \text{ km/h}$$

Set 2

If $a = 1$ and $b = 100$,

then $a^2 + 4abc = 1 + 400c$

Choose $c = \frac{11}{10}$

$$\text{then } a^2 + 4ac = 1 + 400 \times \frac{11}{10}$$

$$= 441$$

$$= 21^2$$

When $a = 1,\ b = 100$ and $c = \frac{11}{10}$,

$$v = \frac{a + \sqrt{a^2 + 4abc}}{2ac}$$

$$\text{becomes } v = \frac{1 + 21}{2 \times 1 \times \frac{11}{10}}$$

$$= \frac{22 \times 10}{22}$$

$$= 10 \text{ km/h}$$

Set 3

If $a = \frac{1}{2},\ b = 15,\ c = 1$

$$\text{then } v = \frac{a + \sqrt{a^2 + 4abc}}{2ac}$$

$$\text{becomes } v = \frac{\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + 4 \times \frac{1}{2} \times 15 \times 1}}{2 \times \frac{1}{2} \times 1}$$

$$= \frac{\frac{1}{2} + \sqrt{\frac{121}{4}}}{1}$$

$$= \frac{1}{2} + \frac{11}{2}$$

$$= 6 \text{ km/h}$$

Set 4

If $a = \frac{1}{4}$,

then $a^2 + 4abc = \frac{1}{16} + bc$

e.g. $a = 1,\ b = 5,\ c = 1$

in which case $v = \dfrac{a + \sqrt{a^2 + 4abc}}{2ac}$

becomes $v = \dfrac{\frac{1}{4} + \sqrt{\left(\frac{1}{4}\right)^2 + 4 \times \frac{1}{4} \times 5 \times 1}}{2 \times 1 \times 1}$

$= \dfrac{\frac{1}{4} + \sqrt{\frac{81}{16}}}{2}$

$= \dfrac{5}{4}$ km/h

Set 5

If $a = 1$ and $b = 1$,

then $a^2 + 4abc = 1 + 4c$

Choose $c = 6$

then $a^2 + 4ac = 1 + 4 \times 6$

$= 25$

$= 5^2$

When $a = 1$, $b = 1$ and $c = 6$,

$v = \dfrac{a + \sqrt{a^2 + 4abc}}{2ac}$

becomes $v = \dfrac{1 + \sqrt{1^2 + 4 \times 1 \times 1 \times 6}}{2 \times 1 \times 6}$

$= \dfrac{1 + 5}{12}$

$= \dfrac{1}{2}$ km/h

3 a

	Volume	Time	Rate
Large pipe	1	T_L	r_L
Small pipe	1	T_S	r_S
Both pipes	1	T_B	$r_L + r_S$

T_L is the time for the large pipe to fill the tank

T_S is the time for the small pipe to fill the tank

T_B is the time for both pipes to fill the tank

where it is assumed without loss of generality that the volume of the tank is 1 unit.

Given

$T_S = T_L + a \qquad \ldots \boxed{1}$

$T_S = T_B + b \qquad \ldots \boxed{2}$

Note that $r_B = r_S + r_L$.

$$T_B = \frac{1}{r_B}$$

$$= \frac{1}{r_S + r_L}$$

$$= \frac{1}{\frac{1}{T_S} + \frac{1}{T_L}}$$

$$= \frac{T_S T_L}{T_S + T_L}$$

From $\boxed{1}$ and $\boxed{2}$

$$T_L + a = T_B + b$$

$$= \frac{T_S T_L}{T_S + T_L} + b$$

$$\therefore T_L(T_L + T_S) + a(T_L + T_S) = T_S T_L + b(T_L + T_S)$$

$$\therefore T_L(2T_L + a) + a(2T_L + a) = T_L(T_L + a) + b(2T_L + a)$$

$$\therefore 2T_L^2 + aT_L + 2aT_L + a^2 = T_L^2 + aT_L + 2bT_L + ba$$

$$\therefore T_L^2 + 2(a - b)T_L + a^2 - ba = 0$$

$$\therefore \quad T_L = \frac{2(b - a) + \sqrt{4(a^2 - 2ab + b^2) - 4(a^2 - ba)}}{2} \text{ since } T_L > 0$$

$$= \frac{2(b - a) + \sqrt{4a^2 - 8ab + 4b^2 - 4a^2 + 4ba}}{2}$$

$$= b - a + \sqrt{-ab + b^2}$$

Also from $\boxed{1}$ $T_S = T_L + a$

$$= b - a + \sqrt{b^2 - ab} + a$$

$$= b + \sqrt{b^2 - ab}$$

b If $a = 24$ and $b = 32$,

$$T_S = 32 + \sqrt{32^2 - 32 \times 24}$$

$$= 48$$

$$T_L = T_S - a$$

$$= 48 - 24$$

$$= 24$$

c $b^2 - ab$ is a perfect square, and $T_S = b + \sqrt{b^2 - ab}$.

Let $b = a + 1$. Then $T_S = a + 1 + \sqrt{(a + 1)^2 - a(a + 1)}$

$$= a + 1 + \sqrt{a^2 + 2a + 1 - a^2 - a}$$

$$= a + 1 + \sqrt{a + 1}$$

Note: This means b must be a perfect square.

a	3	8	15	24	35
b	4	9	16	25	36
T_S	8	18	32	50	72
T_L	5	10	17	26	37

Chapter 7 – Principles of Counting

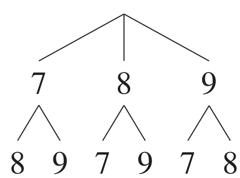
Solutions to Exercise 7A

1 We multiply the number of ways of making each choice. Therefore, there are $5 \times 3 \times 3 = 45$ different outfits.

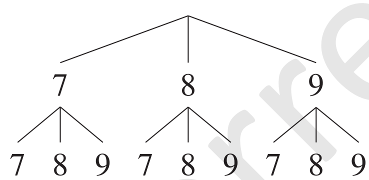
2 As a customer cannot select both chicken and beef, the total number of meals is $5 + 3 = 8$.

3 Each of the ten boys shakes hands with twelve girls so there are $10 \times 12 = 120$ handshakes.

4 a



b



5 a There are 3 choices for each of the 3 digits so $3 \times 3 \times 3 = 27$ three-digit numbers can be formed.

b For the first digit there are 3 choices. For the second there are 2 choices. For the final digit there is only 1 choice. Therefore, $3 \times 2 \times 1 = 6$ three digit numbers can be formed.

6 There are $4 + 2 = 6$ ways to get from Sydney to Adelaide and $2 + 3 = 5$ ways to get from Adelaide to Perth. Therefore, there is a total of $6 \times 5 = 30$ ways to get from Sydney to Perth.

7 a $2 \times 3 = 6$

b $3 \times 2 \times 3 = 18$

c $(1 \times 3 \times 2 \times 2) + (2 \times 2 \times 2 \times 1)$

$= 12 + 8$

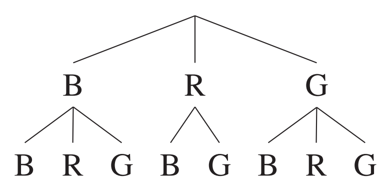
$= 20$

d $3 \times (2 \times 2 + 1)$

$= 3 \times 5$

$= 15$

8

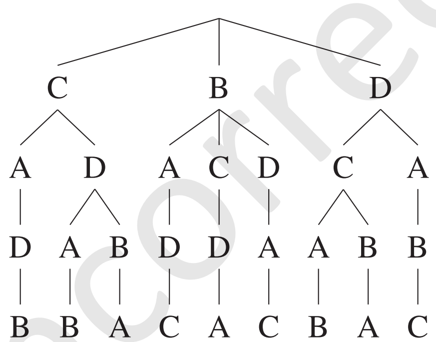


Reading off the tree diagram, the different arrangements are BB,BR,BG,RB,RG,GB,GR,GG.

9 Start by considering coins of the highest value and successively replace these with coins of lower values. This gives 12 ways of making change for 50 cents:

(20, 20, 10)
(20, 20, 5, 5)
(20, 10, 10, 10)
(20, 10, 10, 5, 5)
(20, 10, 5, 5, 5, 5)
(20, 5, 5, 5, 5, 5, 5)
(10, 10, 10, 10, 10)
(10, 10, 10, 10, 5, 5)
(10, 10, 10, 5, 5, 5, 5)
(10, 10, 5, 5, 5, 5, 5, 5)
(10, 5, 5, 5, 5, 5, 5, 5, 5)
(5, 5, 5, 5, 5, 5, 5, 5, 5, 5)

10 Let us denote the desks belonging to the first, second, third and fourth teacher by A, B, C and D respectively. When the teachers select new desks this corresponds to a rearrangement of the letters ABCD. We are interested in the rearrangements in which no letter is in its original position. We illustrate the possibilities on a tree diagram.



This gives 9 possible arrangements.

11 a There are three choices for first position, two for the second and one for the third. This gives a total of $3 \times 2 \times 1 = 6$ arrangements.

b Call the runners A, B, C. Without a tie there are 6 arrangements. If there is two-way tie for first then either $(A, B), (A, C)$ or (B, C) tie. This gives 3 possibilities. If there is a two-way tie for second then there are also 3 possibilities. Finally, if there is a three-way tie for first then there is only one possibility. This gives a total of $6 + 3 + 3 + 1 = 13$ different arrangements.

12 There are more efficient strategies but the easiest method is to count the number of unique entries in the table below.

×	0	2	3	5	7	11
0	0	0	0	0	0	0
2	0	4	6	10	14	22
3	0	6	9	15	21	33
5	0	10	15	25	35	55
7	0	14	21	35	49	77
11	0	22	33	55	77	121

There are 16 unique entries in this table.

Solutions to Exercise 7B

1 1,1, 2, 6, 24,120, 720, 5040, 40320, 362880, 3628800

2 **a** $\dfrac{5!}{4!} = \dfrac{5 \cdot 4!}{4!}$
$= 5$

b $\dfrac{10!}{8!} = \dfrac{10 \cdot 9 \cdot 8!}{8!}$
$= 10 \cdot 9$
$= 90$

c $\dfrac{12!}{10! \cdot 2!} = \dfrac{12 \cdot 11 \cdot 10!}{10! \cdot 2!}$
$= \dfrac{12 \cdot 11}{2}$
$= 66$

d $\dfrac{100!}{97! \cdot 3!} = \dfrac{100 \cdot 99 \cdot 98 \cdot 97!}{97! \cdot 3!}$
$= \dfrac{100 \cdot 99 \cdot 98}{6}$
$= 161700$

3 **a** $\dfrac{(n+1)!}{n!} = \dfrac{(n+1) \cdot n!}{n!}$
$= n + 1$

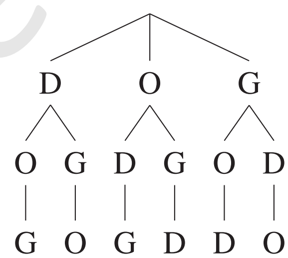
b $\dfrac{(n+2)!}{(n+1)!} = \dfrac{(n+2) \cdot (n+1)!}{(n+1)!}$
$= n + 2$

c $\dfrac{n!}{(n-2)!} = \dfrac{n \cdot (n-1) \cdot (n-2)!}{(n-2)!}$
$= n(n-1)$

d $\dfrac{1}{n!} + \dfrac{1}{(n+1)!} = \dfrac{n+1}{(n+1)n!} + \dfrac{1}{(n+1)!}$
$= \dfrac{n+1}{(n+1)!} + \dfrac{1}{(n+1)!}$
$= \dfrac{n+2}{(n+1)!}$

4 ${}^4P_0 = \dfrac{4!}{(4-0)!} = \dfrac{4!}{4!} = 1$
${}^4P_1 = \dfrac{4!}{(4-1)!} = \dfrac{4!}{3!} = 4$
${}^4P_2 = \dfrac{4!}{(4-2)!} = \dfrac{4!}{2!} = 4 \cdot 3 = 12$
${}^4P_3 = \dfrac{4!}{(4-3)!} = \dfrac{4!}{1!} = 4 \cdot 3 \cdot 2 = 24$
${}^4P_4 = \dfrac{4!}{(4-4)!} = \dfrac{4!}{0!} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

5



Reading the permutations off the tree gives:
DOG, DGO, ODG, OGD, GOD, GDO.

6 Five items can be arranged in $5! = 120$ ways.

7 Nine different letters can be arranged in $9! = 362880$ ways.

8 The first letter can be chosen 4 ways leaving 3 choices for the second letter. Therefore, there are $4 \times 3 = 12$ two letter permutations of the letters in FROG.

9 **a** 6 students can be arranged in $6! = 720$ ways.

b $6 \times 5 \times 4 \times 3 \times 2 = 720$. Note that this is the same as the previous question since if there are six seats then we have no choice but to allocate the final seat to the remaining student.

c $6 \times 5 \times 4 \times 3 = 360$

10 **a** Five different digits can be arranged in $5! = 120$ ways.

b $5 \times 4 \times 3 \times 2 = 120$

c $5 \times 4 \times 3 = 60$

11 There are eight choices of desk for the first student, seven for the second, and so on. This gives $8 \times 7 \times 6 \times 5 \times 4 \times 3 = 20160$ allocations.

12 **a** There are 5 choices for each letter. Therefore, there are $5^3 = 125$ ways that the three letters can be posted.

b There are 5 choices of mailbox for the first letter, 4 for the second and 3 for the third. This gives $5 \times 4 \times 3 = 60$ ways that the three letters can be posted.

13 **a** $6 \times 5 \times 4 = 120$

b $6 \times 5 \times 4 \times 3 = 360$

c $6 \times 5 \times 4 \times 3 \times 2 = 720$

14 Four flags can be arranged in $4! = 24$ ways. Four flags taken three at a time can be arranged in $4 \times 3 \times 2 = 24$ ways. Four flags taken two at a time can be arranged in $4 \times 3 = 12$ ways. Therefore, there are a total of $24 + 24 + 12 = 60$ different signals.

15 **a** $26^3 \times 10^3 = 17576000$

b $(26 \times 25 \times 24) \times (10 \times 9 \times 8) = 11232000$

16 **a** The 3 tiles can be arranged in 3! ways. Each of the 3 tiles can be rotated four different ways so there are $3! \times 4 \times 4 \times 4 = 384$ arrangements.

b The 4 tiles can be arranged in 4! ways. The first 3 tiles can be rotated 4 different ways. The last tile is rotationally symmetric, and can only be rotated 2 different ways. Therefore, there are $4! \times 4 \times 4 \times 4 \times 2 = 3072$ arrangements.

17 Write the equation as $m! = \dfrac{720}{n!}$. We then substitute in values for n beginning with $n = 0$

n	$n!$	$m! = 720/n!$	m
0	1	720	6
1	1	720	6
2	2	360	-
3	6	120	5
4	24	30	-
5	120	6	3
6	720	1	0,1

Since $m > n$ the only solutions are $(m, n) = (6, 0), (6, 1), (5, 3)$.

18 $(n^2 - n) \cdot (n - 2)! = n \cdot (n - 1) \cdot (n - 2)!$
$$= n!$$

19 Without considering rotations, the number of ways to paint the 6 faces with 6 colours is 6!. However we must divide out the number of orientations a cube has to ensure we are not counting the same colouring twice. Choose one colour as the top face. We can then rotate the cube 4 times, keeping the top face still. A cube has 6 possible faces to choose as the top face, so there are $6 \times 4 = 24$ orientations of the cube. Thus the number of colourings is $6! \div 24 = 30$.

Solutions to Exercise 7C

1 **a** Five digits can be arranged in $5! = 120$ ways.

b As the number is odd, there are three possibilities for the last digit (either $1, 3$ or 5). Once this number is selected there are 4 options for the first digit, 3 for the second and so on. This gives a total of $(4 \times 3 \times 2 \times 1) \times 3 = 72$ numbers.

c The first number is 5. There are 4 options for the second digit, 3 for the third and so on. This gives a total of $1 \times 4 \times 3 \times 2 \times 1 = 24$ numbers.

d Without restriction, there are $5! = 120$ numbers. Of these, 24 start with a 5. Therefore, $120 - 24 = 96$ do not begin with a 5.

2 **a** Five children can be arranged in $5! = 120$ ways.

b Label the three girls with the numbers $1, 2, 3$ and the boys with the numbers 4 and 5. If we group the four boys together we must arrange four objects: $1, 2, 3, \{4, 5\}$. These can be arranged in $4!$ ways. Lastly, the two boys can be arranged in in $2!$ ways. We then use the Multiplication Principle so that there are $4! \times 2! = 48$ arrangements.

c There are 120 arrangements without restriction. Of these, 48 have the boys sitting together. Therefore, $120 - 48 = 72$ will have the boys sitting apart.

d If the boys and girls alternate then the first position must be filled by a girl. This can be filled in 3 ways. The next position must be a boy, and this can be filled in 2 ways. Continuing down the row gives a total of $3 \times 2 \times 2 \times 1 \times 1 = 12$ arrangements.

3 **a** There are three vowels. Therefore, there are 3 choices for the first letter. Having chosen this letter, there are 5 choices for the second, 4 for the third and so on. This gives $3 \times 5 \times 4 \times 3 \times 2 \times 1 = 360$ permutations.

b There are three vowels. Therefore, there are 3 choices for the first letter. Having chosen this letter, there are 2 choices for the final letter. This leaves 4 choices for the second, 3 for the third and so on. This gives $3 \times 4 \times 3 \times 2 \times 1 \times 2 = 144$ permutations.

c We group the vowels so that we must now arrange four objects: Q,Z,Y,{U,E,A}. This can be done in $4!$ ways. The 3 vowels can be arranged in $3!$ ways. The Multiplication Principle then gives a total of $4! \times 3! = 144$ permutations.

d We group the vowels and consonants together so that we have to arrange two objects: {Q,Z,Y},{U,E,A}.

This can be done 2! ways. The 3 vowels can be arranged in 3! ways and the 3 consonants in 3! ways. The Multiplication Principle then gives a total of $2! \times 3! \times 3! = 72$ permutations.

4 **a** The boys and girls must sit in alternate positions. The 4 boys can be arranged in 4! ways and the 4 girls in 4! ways. Since the first position is either a boy or a girl we must also multiply by 2. This gives a total of $2 \times 4! \times 4! = 1152$ arrangements.

b We group the boys and girls so that we now have to arrange two groups: {boys}, and {girls}. This can be done in 2! ways. The 4 boys can be arranged in 4! ways and the 4 girls in 4! ways. The Multiplication Principle then gives a total of $2! \times 4! \times 4! = 1152$.

5 **a** The first digits cannot be 0 so there are only 5 choices for the first digit. The remaining 5 digits can be arranged in 5! ways. This gives a total of $5 \times 5! = 600$ arrangements.

b There are two choices for the last digit (0 or 5).
(Case 1) If the last digit is 0 then there are 5 choices for the first digit, 4 for the second and 3 for the third. This gives $5 \times 4 \times 3 \times 1 = 60$ arrangements.
(Case 2) If the last digit is 5 then the first digit cannot be 0. Therefore, there are only 4 choices for the first digit, 4 for the second and 3 for the third. This gives $4 \times 4 \times 3 \times 1 = 48$ arrangements.
This gives a total of $60 + 48 = 108$ arrangements.

c A number less than 6000 will have either 1, 2, 3 or 4 digits. Note that the first digit can never be 0.

digits	arrangements
1	6
2	$5 \times 5 = 25$
3	$5 \times 5 \times 4 = 100$
4	$5 \times 5 \times 4 \times 3 = 300$

This gives a total of $6 + 25 + 100 + 300 = 431$ different arrangements.

d There are two cases to consider.
(Case 1) If the last digit is 0 then there are 5 choices for the first digit and 4 for the second. This gives $5 \times 4 \times 1 = 20$ arrangements.
(Case 2) If the last digit is 2 or 4 then there are 2 choices for the last digit. The first digit cannot be 0. Therefore, there are only 4 choices for the first digit and 4 for the second. This gives $4 \times 4 \times 2 = 32$ arrangements.
We obtain a total of $20 + 32 = 52$ arrangements.

6 **a** There are a total of six people to be arranged. This can be done in $6! = 720$ ways.

b There are 2 ways of filling the first position, and there is 1 way of filling the last position. There are 4 choices for the second seat, 3 for the third and so on. This gives a total of $2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$ arrangements.

c Label the two parents with the letters A and B and the children with the letters C, D, E and F. If the children sit together then we have to arrange three objects: $A, B, \{C, D, E, F\}$. This can be done in 3! ways. The 4 children can then be arranged in 4! ways. This gives a total of $3! \times 4! = 144$ arrangements.

d Label the two parents with the letters A and B and the children with the letters C, D, E and F. If the parents sit together and the children sit together then we first arrange two groups: $\{A, B\}, \{C, D, E, F\}$. This can be done in 2! ways. The 2 adults can be arranged in 2! ways and the 4 children can then be arranged in 4! ways. This gives a total of $2! \times 2! \times 4! = 96$ arrangements.

e Label the two parents with the letters A and B and the children with the letters C, D, E and F, where C is the youngest child. First assume that C sits next to A. We then have to arrange 5 objects: $\{A, C\}, B, D, E, F$. This can be done in 5! ways. However A and C can then be arranged in 2! ways. We also multiply our answer by 2 since C could also sit beside B. This gives a total of $5! \times 2! \times 2 = 480$ arrangements.

7 **a** As the first digit cannot be 0 there are 9 choices for the first digit. The next digit can now be 0 so there are now 10 choices for the second digit. There are also 10 choices for the third digit. There is only one choice for the fourth and fifth digits. This gives a total of $9 \times 10 \times 10 \times 1 \times 1 = 900$ five-digit palindromic numbers.

b As the first digit cannot be 0 there are 9 choices for the first digit. The next digit can now be 0 so there are now 10 choices for the second digit. There are also 10 choices for the third digit. There is only one choice for the fourth, fifth and sixth digits. This gives a total of $9 \times 10 \times 10 \times 1 \times 1 \times 1 = 900$ six-digit palindromic numbers.

8 The total number of arrangements is $5! = 120$. We now consider those arrangements that begin and end in a vowel. There are 3 choices of vowel for the first position, leaving 2 choices for the last position. The second letter can be chosen 3 ways, the third can be chosen 2 ways and the fourth in 1 way. Therefore, there are $3 \times 3 \times 2 \times 1 \times 2 = 36$ arrangements that begin and end in a vowel.
Finally, the number of arrangements that do not begin and end in a vowel must be $120 - 36 = 84$.

9 **(1-digit numbers)** There are only 2 possibilities.
(2-digit numbers) There are $3 \times 2 = 6$ possibilities.
(3-digit numbers) There are $3 \times 2 \times 2 = 12$ possibilities.
(4-digit numbers) There are

$3 \times 2 \times 1 \times 2 = 12$ possibilities. Adding these gives a total of $2 + 6 + 12 + 12 = 32$ possibilities.

10 **a** The total number of arrangements of six girls is $6! = 720$. We now consider those arrangements where the girls sit together. There are five objects to arrange: $\{A, B\}, C, D, E, F$. This can be done is 5! ways. Girls A and B can be arranged in 2! ways. This gives a total of $5! \times 2! = 240$ arrangements. Therefore, there are $720 - 240 = 480$ arrangements where the girls do not sit together.

b Assume that girl F goes between girls A and B. We therefore have only four objects to arrange:$\{A, B\}, C, D, E$. Girl F will simply be slotted in-between A and B. Four objects can be arranged in 4! ways. Girls A and B can be arranged in 2! ways. This gives a total of $4! \times 2! = 48$ arrangements. However, you must multiply this answer by 4 because there are four choices of girl to go between A and B. This gives $48 \times 4 = 192$ arrangements.

11 There are four different patterns possible:

GBGBGB,

BGBGBG,

GBBGBG,

GBGBBG.

For each of these patterns, the boys can be arranged in 3! ways, and the girls can be arranged in 3! ways. This gives a total of $4 \times 3! \times 3! = 144$ arrangements.

Solutions to Exercise 7D

1 There is a total of 7 coins, of which a group of 4 are alike and another group of 3 are alike. These can be arranged in $\frac{7!}{4! \cdot 3!} = 35$ ways.

2 There is a total of 11 letters. Of these letters, 1 is M, 4 are S, 4 are I, and 2 are P. The 11 letters can therefore be arranged in $\frac{11!}{4! \cdot 4! \cdot 2!} = 34650$ ways.

3 There is a total of 11 letters. Of these letters, 2 are A, 2 are R, and 2 are O. The 11 letters can therefore be arranged in $\frac{11!}{2! \cdot 2! \cdot 2!} = 4989600$ ways.

4 There is a total of 8 digits. Of these digits, 5 are nine, and 3 are seven. Therefore, the 8 digits can be arranged in $\frac{8!}{5! \cdot 3!} = 56$ ways.

5 There is a total of 12 letters. Of these letters, 3 are A, 4 are B and 5 are C. Therefore, the 12 letters can be arranged in $\frac{12!}{3! \cdot 4! \cdot 5!} = 27720$ ways.

6 **a** There is a total of 8 flags. Of these, 2 are red, 2 are black, and 4 are blue. These can be arranged in $\frac{8!}{2! \cdot 2! \cdot 4!} = 420$ ways.

b The first flag is red so now there are 7 flags to arrange: 1 red, 2 black and 4 blue. This can be done in $\frac{7!}{2! \cdot 4!} = 105$ ways.

c The first and last flags are blue so so there are 6 flags to arrange: 2 red, 2 black and 2 blue. These can be arranged in $\frac{6!}{2! \cdot 2! \cdot 2!} = 90$ ways.

d Every alternate flag is blue. There are 4 flags to arrange in-between the blue flags: 2 red and 2 black. These can be arranged in $\frac{4!}{2! \cdot 2!} = 6$ ways. However we must multiply this by 2 as the blue flags can be placed in either odd or even positions. This gives a total of $6 \times 2 = 12$ arrangements.

e We group the two red flags together as one item. We then need to arrange this single group as well as 2 black flags and 4 blue flags. This makes 7 items in total. These can be arranged in $\frac{7!}{2! \cdot 4!} = 105$ ways.

7 Each path from A to B can be described by four R's and three D's in some order. As there are seven letters in total, there are $\frac{7!}{3! \cdot 4!} = 35$ paths.

8 **a** Let N be a movement of one unit in the north direction, and E be a movement of one unit in the east direction. Then each path from $(0, 0)$ to $(2, 4)$ can be described by two N's and four E's in some order. As there are six letters in total, there are $\frac{6!}{2! \cdot 4!} = 15$ paths.

b Let E be a movement of one unit

in the east direction, and N be a movement of one unit in the north direction. Then each path from $(0, 0)$ to (m, n) can be described by m letter N's and n letter E's in some order. As there are $m + n$ letters in total, there are $\frac{(m+n)!}{m! \cdot n!}$ paths.

9 **a** The 52 playing cards can be arranged in 52! ways.

b There are now 104 playing cards, however each of the 52 playing cards occurs twice. Therefore, these cards can be arranged in $\frac{104!}{(2!)^{52}}$ ways.

c If there are n decks then there are $52n$ playing cards. Each of the 52 playing cards occurs n times. Therefore, these cards can be arranged in $\frac{(52n)!}{(n!)^{52}}$ ways.

10 Let N,E,S,W denote a movement of one unit in the north, east, south and west directions respectively. Any path of length 8 can be described by a combination of 8 of these letters in some order. Any path that starts at $(0, 0)$ and finishes at $(0, 0)$ has an equal number of N's and S's and an equal number of E's and W's. We list the various ways in which this can happen.

N	E	S	W	arrangements
0	0	4	4	$\frac{8!}{4! \cdot 4!} = 70$
1	1	3	3	$\frac{8!}{3! \cdot 3!} = 1120$
2	2	2	2	$\frac{8!}{2! \cdot 2! \cdot 2! \cdot 2!} = 2520$
3	3	1	1	$\frac{8!}{3! \cdot 3!} = 1120$
4	4	0	0	$\frac{8!}{4! \cdot 4!} = 70$

This gives a total of
$70 + 1120 + 2520 + 1120 + 70 = 4900$.

11 There are various ways to solve this problem. We show one solution using arrangements. Jessica can either take 2 steps or 1 step at a time. Therefore, each of Jessica's paths up the stairs can be described by some sequence of 2's and 1's whose sum is 10. We list the various ways that this can happen.

digits	arrangements
1111111111	$\frac{10!}{10!} = 1$
111111112	$\frac{9!}{8!} = 9$
11111122	$\frac{8!}{6!2!} = 28$
1111222	$\frac{7!}{4!3!} = 35$
112222	$\frac{6!}{2!4!} = 15$
22222	$\frac{5!}{5!} = 1$

This gives a total of 89.

Solutions to Exercise 7E

1

$$\begin{aligned} {}^5C_0 &= \frac{5!}{0! \cdot (5-0)!} \\ &= \frac{5!}{0! \cdot 5!} \\ &= 1 \end{aligned}$$

$$\begin{aligned} {}^5C_1 &= \frac{5!}{1! \cdot (5-1)!} \\ &= \frac{5!}{1! \cdot 4!} \\ &= \frac{5 \cdot 4!}{1! \cdot 4!} \\ &= \frac{5 \cdot 4}{1} \\ &= 5 \end{aligned}$$

$$\begin{aligned} {}^5C_2 &= \frac{5!}{2! \cdot (5-2)!} \\ &= \frac{5!}{2! \cdot 3!} \\ &= \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} \\ &= \frac{5 \cdot 4}{2} \\ &= 10 \end{aligned}$$

$$\begin{aligned} {}^5C_3 &= \frac{5!}{3! \cdot (5-3)!} \\ &= \frac{5!}{3!2!} \\ &= \frac{5 \cdot 4 \cdot 3!}{3! \cdot 2!} \\ &= \frac{5 \cdot 4}{2} \\ &= 10 \end{aligned}$$

$$\begin{aligned} {}^5C_4 &= \frac{5!}{4! \cdot (5-4)!} \\ &= \frac{5!}{4! \cdot 1!} \\ &= \frac{5 \cdot 4!}{4! \cdot 1!} \\ &= \frac{5 \cdot 4}{1} \\ &= 5 \end{aligned}$$

$$\begin{aligned} {}^5C_5 &= \frac{5!}{5! \cdot (5-5)!} \\ &= \frac{5!}{5! \cdot 0!} \\ &= 1 \end{aligned}$$

2 a

$$\begin{aligned} {}^7C_1 &= \frac{7!}{1! \cdot (7-1)!} \\ &= \frac{7!}{1! \cdot 6!} \\ &= \frac{7 \cdot 6!}{1! \cdot 6!} \\ &= \frac{7}{1} \\ &= 7 \end{aligned}$$

b

$$\begin{aligned} {}^6C_5 &= \frac{6!}{5! \cdot (6-5)!} \\ &= \frac{6!}{5! \cdot 1!} \\ &= \frac{6 \cdot 5!}{5! \cdot 1!} \\ &= \frac{6}{1} \\ &= 6 \end{aligned}$$

c

$$\begin{aligned} {}^{12}C_{10} &= \frac{12!}{10! \cdot (12-10)!} \\ &= \frac{12!}{10! \cdot 2!} \\ &= \frac{12 \cdot 11 \cdot 10!}{10! \cdot 2!} \\ &= \frac{12 \cdot 11}{2} \\ &= 66 \end{aligned}$$

d

$$\begin{aligned} {}^8C_5 &= \frac{8!}{5! \cdot (8-5)!} \\ &= \frac{8!}{5! \cdot 3!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3!} \\ &= \frac{8 \cdot 7 \cdot 6}{6} \\ &= 56 \end{aligned}$$

e

$$\begin{aligned} {}^{100}C_{99} &= \frac{100!}{99! \cdot (100-99)!} \\ &= \frac{100!}{99! \cdot 1!} \\ &= \frac{100 \cdot 99!}{99! \cdot 1!} \\ &= \frac{100}{1} \\ &= 100 \end{aligned}$$

f $$^{1000}C_{998} = \frac{1000!}{998! \cdot (1000-998)!} = \frac{1000!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999}{2} = 499500$$

3 a $$^{n}C_1 = \frac{n!}{1! \cdot (n-1)!} = \frac{n!}{(n-1)!} = \frac{n \cdot (n-1)!}{(n-1)!} = n$$

b $$^{n}C_2 = \frac{n!}{2! \cdot (n-2)!} = \frac{n!}{2 \cdot (n-2)!} = \frac{n \cdot (n-1) \cdot (n-2)!}{2 \cdot (n-2)!} = \frac{n(n-1)}{2}$$

c $$^{n}C_{n-1} = \frac{n!}{(n-1)! \cdot (n-(n-1))!} = \frac{n!}{(n-1)! \cdot 1!} = \frac{n \cdot (n-1)!}{(n-1)!} = n$$

d $$^{n+1}C_1 = \frac{(n+1)!}{1! \cdot (n+1-1)!} = \frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n+1$$

e $$^{n+2}C_n = \frac{(n+2)!}{n! \cdot (n+2-n)!} = \frac{(n+2)!}{n! \cdot 2!} = \frac{(n+2) \cdot (n+1) \cdot n!}{n! \cdot 2} = \frac{(n+2)(n+1)}{2}$$

f
$$^{n+1}C_{n-1} = \frac{(n+1)!}{(n-1)! \cdot ((n+1)-(n-1))!} = \frac{(n+1)!}{(n-1)! \cdot 2!} = \frac{(n+1) \cdot n \cdot (n-1)!}{(n-1)! \cdot 2!} = \frac{n(n+1)}{2}$$

4 a There are 10 items and 3 to arrange. This can be done in $10 \times 9 \times 8 = 720$ ways.

b There are 10 items and 3 to select. This can be done in $^{10}C_3 = 120$ ways.

5 5 objects can be selected from 52 in $^{52}C_5 = 2598960$ ways.

6 a $^{10}C_1 = 10$

b $^{10}C_2 = 45$

c $^{10}C_8 = 45$

d $^{10}C_9 = 10$

7 $^{45}C_7 = 45379620$

8 3 vertices are to be selected from 8. This can be done in $^8C_3 = 56$ ways.

9 **a** This is the same as asking how many ways can 2 teams be selected from 10. This can be done in $^{10}C_2 = 45$ ways.

b Let n be the number of teams. Then,

$$\begin{aligned} {}^nC_2 &= 120 \\ \frac{n!}{2!\cdot(n-2)!} &= 120 \\ \frac{n\cdot(n-1)\cdot(n-2)!}{2!\cdot(n-2)!} &= 120 \\ \frac{n\cdot(n-1)}{2} &= 120 \\ n(n-1) &= 240 \\ n^2-n-240 &= 0 \\ (n-16)(n+15) &= 0 \\ \Rightarrow n &= 16 \text{ as } n > 0. \end{aligned}$$

10 Let n be the number of people at the party. Then

$$\begin{aligned} {}^nC_2 &= 105 \\ \frac{n!}{2!\cdot(n-2)!} &= 105 \\ \frac{n\cdot(n-1)\cdot(n-2)!}{2!\cdot(n-2)!} &= 105 \\ \frac{n(n-1)}{2} &= 105 \\ n(n-1) &= 210 \\ n^2-n-210 &= 0 \\ (n-15)(n+14) &= 0 \\ \Rightarrow n &= 15 \text{ as } n > 0 \end{aligned}$$

11

$$\begin{aligned} \text{RHS} &= {}^nC_{n-r} \\ &= \frac{n!}{(n-r)!\cdot(n-(n-r))!} \\ &= \frac{n!}{(n-r)!\cdot r!} \\ &= \frac{n!}{r!\cdot(n-r)!} \\ &= {}^nC_r \\ &= \text{LHS} \end{aligned}$$

12 Each diagonal is obtained by selecting 2 vertices from n. This can be done in $^nC_2$ ways. However, n of these choices will define the side of the polygon, not a diagonal. Therefore, there are $^nC_2 - n$ diagonals.

13 There are $^{10}C_5$ ways of choosing five students to belong to team A. The remaining five students will belong to team B. However, the labelling of the teams doesn't matter, so we must divide by 2.

14 There are ${}^{12}C_6$ ways of choosing six students to belong to team A. The remaining students will belong to team B. However, the labelling of the teams doesn't matter, so we must divide by 2. This gives $\dfrac{{}^{12}C_6}{2} = 462$

15 We begin with the right hand side. This gives,

$$\begin{aligned}
\text{RHS} &= {}^{n-1}C_{r-1} + {}^{n-1}C_r \\
&= \frac{(n-1)!}{(r-1)!(n-1-(r-1))!} + \frac{(n-1)!}{r!(n-1-r)!} \\
&= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!} \\
&= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right) \\
&= \frac{(n-1)!}{(r-1)!(n-r-1)!}\,\frac{n}{r(n-r)} \\
&= \frac{n!}{r!(n-r)!} \\
&= \text{LHS.}
\end{aligned}$$

16 **a** The number of ways of selecting 3 dots from 25 is ${}^{25}C_3 = 2300$.

b There are 12 lines of 5 dots. From each of these lines we select 3 from 5 dots. This can be done in $12 \times {}^{5}C_3 = 120$ ways.
There are 4 lines of 4 dots. From each of these lines we select 3 from 4 dots. This can be done in $4 \times {}^{4}C_3 = 16$ ways.
There are 16 lines of 3 dots. Note: if you've found only 4 lines, then look harder! From each of these lines we select 3 from 3 dots. This can be done in $16 \times {}^{3}C_3 = 16$ ways.
Therefore, there are $120 + 16 + 16 = 152$ ways of selecting 3 dots that lie on a straight line.

c The three dots will form a triangle if they don't form a line. So the number of ways of selecting three dots to form a triangle will be the total number of selections, minus those selections that form a line. This gives $2300 - 15223 = 2148$.

Solutions to Exercise 7F

1 Since Jane and Jenny are to be included we still have to select 2 students from the remaining 18. This can be done in ${}^{18}C_2 = 153$ ways.

2 Since the subset already contains the number 5, there are still 4 numbers to be selected from the remaining 9. This can be done in ${}^{9}C_4 = 126$ ways.

3 Since the hand already contains the jack, queen and king of hearts, there are still 2 cards to be selected from the remaining 49. This can be done in ${}^{49}C_2 = 1176$ ways.

4 There are ${}^{10}C_6$ ways of selecting 6 students from 10. We then subtract the number of combinations that include both Rachel and Nethra. If Rachel and Nethra are on the team then we must select 4 more students from the 8 that remain in ${}^{8}C_4$ ways. Therefore, the required answer is

$${}^{10}C_6 - {}^{8}C_4 = 140.$$

5 **a** 7 students are to be selected from a total of 13. This can be done in ${}^{13}C_7 = 1716$ ways.

b 4 girls are selected from 8 and 3 boys are selected from 5 applying the Multiplication Principle gives ${}^{8}C_4 \cdot {}^{5}C_3 = 700$

c There can be either 4 girls and 3 boys or 3 girls and 4 boys. Applying both the Addition and Multiplication Principles, this can be done in ${}^{8}C_4 \cdot {}^{5}C_3 + {}^{8}C_3 \cdot {}^{5}C_4 = 980$ ways.

d We first consider the number of ways of selecting teams with fewer than 2 boys. There can be either 7 girls and 0 boys or 6 girls and 1 boy. This can be done in ${}^{8}C_7 \cdot {}^{5}C_0 + {}^{8}C_6 \cdot {}^{5}C_1 = 148$ ways. Therefore, the number of teams with at least two boys will be $1716 - 148 = 1568$.

6 **a** 4 students are to be selected from 10 for the first committee and 3 are to be selected from 10 for the second committee. This can be done in ${}^{10}C_4 \cdot {}^{10}C_3 = 25200$ ways.

b First choose 4 students from 10 for the first committee. This can be done in ${}^{10}C_4$ ways. There are now 6 students left, from which we select 3 for the second committee. This can be done in ${}^{6}C_3$ ways. This gives a total of ${}^{10}C_4 \cdot {}^{6}C_3 = 4200$ different selections.

7 **a** 7 students are to be selected from 18 for the basketball team and 8 are to be selected from 18 for the netball team. This can be done in ${}^{18}C_7 \cdot {}^{18}C_8 = 1392554592$ ways.

b First choose 7 students from 18 for the basketball team. There are now 11 students left, from which we select 8 for the netball team. This gives a total of ${}^{18}C_7 \cdot {}^{11}C_8 = 5250960$ different selections.

8 **a** 5 senators are to be selected from a total of 20. This can be done in ${}^{20}C_5 = 15504$ ways.

b There can be either 2 Labor and 3 Liberal senators or 3 Labor and 2 Liberal senators. As there are equal numbers of both types, the number of ways these can be selected is $2 \cdot {}^{10}C_3 \cdot {}^{10}C_2 = 10800$.

c The total number of unrestricted selections is ${}^{20}C_5 = 15504$. We now consider the number of ways of selecting no Labor senator. There are ${}^{10}C_5 = 252$ ways of selecting 5 Liberal senators out of 10. Therefore, there will be $15504 - 252 = 15252$ selections.

9 **a** There are ${}^{7}C_5 = 21$ ways of selecting 5 numbers out of 7.

b As the sets already contain the numbers 2 and 3, there are still 3 numbers to be chosen from the 5 numbers that remain. This can be done in ${}^{5}C_3 = 10$ ways.

c We subtract the number of subsets that contain both numbers from the total number of subsets. This gives $21 - 10 = 11$ subsets.

10 We select 2 of 5 vowels and 2 of 21 consonants. This can be done in ${}^{5}C_2 \cdot {}^{21}C_2 = 2100$ ways.

11 **a** 4 hearts are to be selected from 13 and 3 spades are to be selected from 13. Using the Multiplication Principle, this can be done in ${}^{13}C_4 \cdot {}^{13}C_3 = 204490$ different ways.

b 2 hearts are to be selected from 13 and 3 spades are to be selected from 13. The remaining 2 cards are to be selected by amongst the 26 cards that are neither diamonds nor clubs. Using the Multiplication Principle, this can be done in ${}^{13}C_2 \cdot {}^{13}C_3 \cdot {}^{26}C_2 = 7250100$ ways.

12 **a** 3 doctors are to be selected from 4 and 1 dentist is to be selected from 4. The remaining position is to be filled with 1 of 3 physiotherapists. Using the Multiplication Principle, this can be done in ${}^{4}C_3 \times 4 \times 3 = 48$ ways.

b 2 doctors are to be selected from 4. The remaining 3 positions are to be chosen from among the $4 + 3 = 7$ non-doctors. Using the Multiplication Principle, this can be done in ${}^{4}C_2 \cdot {}^{7}C_3 = 210$ ways.

13 The girls can be selected in ${}^{4}C_2$ ways. The boys can be selected in ${}^{5}C_2 =$ ways. The children can then be arranged in 4! ways. Using the Multiplication Principle gives a total of ${}^{4}C_2 \cdot {}^{5}C_2 \cdot 4! = 6 \cdot 10 \cdot 24 = 1440$ arrangements.

14 The women can be selected in ${}^{6}C_2$ ways. The men can be selected in

$^5C_2$ = ways. The four people can fill the positions in 4! ways. Using the Multiplication Principle gives a total of $^6C_2 \cdot {}^5C_2 \cdot 4! = 3600$ arrangements.

15 There are 4 vowels and 6 consonants. The vowels can be chosen in $^4C_2$ ways and the consonants can be chosen in $^6C_3$ ways. The 5 letters can then be arrangned in 5! ways. Using the Multiplication Principle gives a total of $^4C_2 \cdot {}^6C_3 \cdot 5! = 14400$ arrangements.

16 Each rectangle is defined by a choice of 2 out of 6 vertical lines and 2 out of 5 horizontal lines. This gives a total of $^6C_2 \cdot {}^5C_2 = 150$.

17 There are 13 choices of rank for the first card. 4 cards have this rank, from which we choose 3. There are 12 choices of rank for the second card. 4 cards have this rank, from which 2 will be chosen. This gives a total of $13 \times {}^4C_3 \times 12 \times {}^4C_2 = 3744$ hands.

Solutions to Exercise 7G

1 ${}^7C_2 = 21, {}^6C_2 = 15$ and ${}^6C_1 = 6$.
Clearly, ${}^7C_2 = {}^6C_2 + {}^6C_1$.

2 The $n = 7$ row is:

1 7 21 35 35 21 7 1

${}^7C_2 = 21$ since this is the third entry in the row.
${}^7C_4 = 35$ since this is the fifth entry in the row.

3 The $n = 8$ row is:

1 8 28 56 70 56 28 8 1

${}^8C_4 = 70$ since this is the fifth entry in the row.
${}^8C_6 = 28$ since this is the seventh entry in the row.

4 A set with 6 elements has $2^6 = 64$ subsets. Note that this includes the empty subset, which corresponds to selecting none of the DVDs.

5 A set of 5 elements has $2^5 = 32$ subsets.

6 A set with 10 elements has $2^{10} = 1024$ subsets.

7 A set with 6 elements has $2^6 - 1 = 63$ nonempty subsets.

8 A set with 8 elements has $2^8 - {}^8C_1 - {}^8C_0 = 256 - 8 - 1 = 247$ subsets with at least 2 elements.

9 If the set already contains the numbers 9 and 10, then we need to find the number of subsets of $\{1, 2, \ldots, 8\}$. There are $2^8 = 256$ of these.

10 Each subset of coins creates a different sum of money. We therefore need to find the number of nonempty subsets of a 4 element set. There are $2^4 - 1 = 15$ of these.

11 **a** We consider the selfish subsets of size 1 through to 8. There is 1 selfish set of size 1, namely $\{1\}$.
If a selfish set has size 2, then it is of the form $\{2, a\}$ where a is chosen from the remaining 7 numbers. This can be done in ${}^7C_1$ ways.
If a selfish set has size 3, then it is of the form $\{3, a, b\}$ where the two numbers a and b are chosen from the remaining 7 numbers. This can be done in ${}^7C_2$ ways.
Continuing in this fashion, we find that the number of selfish sets is just the sum of entries in row $n = 7$ of Pascal's Triangle. Therefore, there are $2^7 = 128$ selfish sets.

b We consider the selfish subsets of size 1 through to 8.
There is 1 selfish subset of size 1. Its compliment is also selfish, as it has 7 elements and contains the number 7.
A selfish set of size 2 is of the form $\{2, a\}$, where $a \neq 2$. Since the compliment is also selfish, $a \neq 6$.
Therefore, a can be chosen from the remaining 6 numbers. This can be

done in ${}^6C_1$ ways.
A selfish set of size 3 is of the form $\{3, a, b\}$, where $a, b \neq 3$. Since the compliment is also selfish, $a, b \neq 5$. Therefore, a and b can be chosen from the remaining 6 numbers. This can be done in ${}^6C_2$ ways.
A selfish set of size 4 is of the form $\{4, a, b, c\}$, where $a, b, c \neq 4$. The compliment cannot also be selfish, since the compliment has 4 elements but does not contain the number 4.
A selfish set of size 5 is of the form $\{5, a, b, c, d\}$, where $a, b, c, d \neq 5$. Since the compliment is also selfish, $a, b, c, d \neq 3$. Therefore, a, b, c, d can be chosen from the remaining 6 numbers. This can be done in ${}^6C_4$ ways.
Continuing in this fashion, we find that the number of selfish sets with a selfish compliment is just the sum of entries in row $n = 6$ of Pascal's Triangle, less ${}^6C_3$. Therefore, there are $2^6 - {}^6C_3 = 44$ selfish sets whose compliment is also selfish.

Solutions to Exercise 7H

1 Label three holes with the colours red, blue and green.

R | B | G

Select four socks and place each sock in the hole whose label corresponds to the colour of the sock. As there are four socks and three holes, the Pigeonhole Principle guarantees that some hole contains at least two socks. This is the required pair. Clearly, selecting three socks is not sufficient as you might pick one sock of each colour.

2 Create 26 holes, each labelled with a letter from A to Z.

A | B $\cdots$ Z

Put each word in the hole whose label corresponds to its first letter. There are 27 words to place in 26 holes, so there is some hole that contains at least 2 words. Therefore, there are at least two words that begin with the same letter.

3 Create 4 holes with labels $0, 1, 2$ and 3.

0 | 1 | 2 | 3

Put each natural number in the hole whose label corresponds to the remainder when it is divided by 4. There are 5 numbers to place in 4 holes, so there is some hole that contains at least 2 numbers. Therefore, there is at least 2 numbers that leave the same remainder when divided by 4.

4 **a** As there are only 2 colours, if 3 cards are dealt, at least two will be the same colour.

b As there are only 4 suits, if 5 cards are dealt, there is some suit that occurs at least twice.

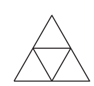
c As there are 13 kinds, if 14 card are dealt, there is at least one kind that occurs at least twice.

5 Divide the number line into ten intervals:

$$[0, 0.1), [0.1, 0.2), \ldots [0.9, 1].$$

Eleven points are located in 10 intervals, so some interval contains at least two numbers. The distance between any two number in this interval is no more than 0.1 units.

6 Divide the equilateral triangle into 4 equilateral triangles of side length 1 unit, as shown:



There are 5 points located in 4 triangles, so some triangle contains at least 2 points. The distance between any two points in this triangle is no more than 1 unit.

7 Divide the rectangle up into square of size 2 metres by 2 metres. There are 13 points located in $3 \times 4 = 12$ squares, so some square contains at least two points. The distance between any two of these points can not exceed length of the square's diagonal, $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

8 **a** For two-digit numbers, there are 18 possible digital sums: $1, 2, \dots, 18$. So if there are 19 numbers then there there is some digital sum that occurs at least twice.

b For three-digit numbers, there are 27 possible digital sums: $1, 2, \dots, 27$. Since $82 = 3 \times 27 + 1$, by the Generalised Pigeonhole Principle, there is some digital sum that occurs at least 4 times.

9 Label four holes with each of four possible remainders when a number is divided by 4, namely 0, 1, 2 and 3. Each of the 13 numbers belongs to one of 4 holes. Since $13 = 3 \times 4 + 1$, there is some hole that contains at least 4 numbers.

10 There are ${}^8C_2 = 28$ ways that 2 teams can be chosen to compete from 8 choices. There are 29 games of football, so there is some pair of teams that play each other at least twice.

11 Label 25 holes as shown below

$$\boxed{1 \text{ or } 49}\ \boxed{2 \text{ or } 48} \cdots \boxed{24 \text{ or } 26}\ \boxed{50}$$

If there are 26 numbers, then there is some hole that contains at least 2 numbers. As these two number are distinct, their sum must be 50. Clearly, this is the smallest number of students required, since 25 students could select the numbers $1, 2, 3, \dots, 25$, no two of which add to 50.

12 Label the chairs with numbers from 1 to 14. There are 14 groups of three consecutive chairs:

$$\{1, 2, 3\}, \{2, 3, 4\}, \dots \{13, 14, 1\}, \{14, 1, 2\}.$$

Each of the 10 people will belong to 3 of these groups, so there are 30 people to be allocated to 14 groups. Since

$$30 > 2 \times 14 + 1,$$

the Generalised Pigeonhole Principle guarantees that some group must contain 3 people.

13 Pick any one of the 4 points and draw a diameter through that point. This splits the circle into two half circles. Considering the remaining 3 points, the pigeonhole principle says that one of the half circles must contain at least 2 of those 3 points. Together with the initial point chosen, that half circle contains at least 3 points.

14 Using two distinct integers chosen from $\{1, 2, \dots, 98, 99\}$, there are 195 different sums possible: $3, 4, \dots, 197$. There are 35 different players and ${}^{35}C_2 = 595$ ways of pair these players up. Since

$$595 = 3 \times 195 + 10 > 3 \times 195 + 1,$$

the Generalised Pigeonhole Principle guarantees that at least four pairs will have the same sum.

15 Label the chairs with numbers from 1 to 12 are then create 6 pairs of opposite seats:

$$\{1,7\}, \{2,8\}, \{3,9\}, \{4,8\}, \{5,9\}, \{6,12\}.$$

Each of the 7 boys belongs to one of 6 pairs, so some pair contains two boys.

16 Create n holes with labels

$$0, 1, 2, \ldots, n-1.$$

Place each of the n guests in the hole corresponding to the number of hands that they shake. Note that either the first or last hole must be empty since if some guest shakes hands with 0 people then no guest shakes hands with all people. Likewise, if some guest shakes hands with all people, then no guest shakes hands with 0 people. This leaves $n-1$ holes in which n guests are located. The Pigeonhole Principle guarantees that some hole has at least two guests.

Solutions to Exercise 7I

1 **a** $\{1, 3, 4\}$

b $\{1, 3, 4, 5, 6\}$

c $\{4\}$

d $\{1, 2, 3, 4, 5, 6\}$

e 3

f $\emptyset, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{4, 5, 6\}$

2 Let T be the set of track athletes and F be the set of field athletes. Then the number of athletes in the team will be given by:

$$\begin{aligned}|T \cup F| &= |T| + |F| - |T \cap F|\\ &= 25 + 23 - 12\\ &= 36.\end{aligned}$$

3 Let A be the set of patients taking medication A and let B be the set of patients taking medication B. Then,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B|\\ 50 &= 25 + 29 - |A \cap B|\\ 50 &= 54 - |P \cap C|\\ \therefore |A \cap B| &= 4.\end{aligned}$$

4 Let A and B be the sets comprising of multiples 7 and 9 respectively. Clearly $A \cap B$ consists of the multiples of 7 and 9, that is, multiples of 63. Therefore, $|A| = 90$, $|B| = 70$ and $|A \cap B| = 10$. We then use the Inclusion Exclusion Principle to find that,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B|\\ &= 90 + 70 - 10\\ &= 150.\end{aligned}$$

5 **a** Let A and B be the sets comprising of multiples 2 and 3 respectively. Clearly $A \cap B$ consists of the multiples of 2 and 3, that is, multiples of 6. Therefore, $|A| = 48$, $|B| = 32$ and $|A \cap B| = 16$. We then use the Inclusion Exclusion Principle to find that,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B|\\ &= 48 + 32 - 16\\ &= 64.\end{aligned}$$

b We have already found that 64 are divisible by 2 or 3. Therefore, $96 - 64 = 32$ will not be divisible by 2 or 3.

6 **a** There are two vowels so there are 2 choices for the first letter, 4 for the second letter, 3 for the third, and so on. This gives $2 \times 4 \times 3 \times 2 \times 1 = 48$ arrangements.

b There are two vowels so there are 2 choices for the last letter, 4 for the first letter, 3 for the second, and so on. This gives $4 \times 3 \times 2 \times 1 \times 2 = 48$ arrangements.

c There are two vowels so there are 2 choices for the first letter and then 1 for the last letter. As two letters have be used, there are then 3 choices for the second letter, 2 for the third, and 1 for the fourth. This gives $2 \times 3 \times 2 \times 1 \times 1 = 12$ arrangements.

d We let A be set of arrangements that begin with a vowel, and B be the set of arrangements that end with a vowel. Then $A \cap B$ is the set arrangements that begin and end with a vowel. We then use the Inclusion-Exclusion Principle to find that,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B|\\ &= 48 + 48 - 12\\ &= 84.\end{aligned}$$

7 a There are ten perfect squares,

$$\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}.$$

There are 4 perfect cubes,

$$\{1, 8, 27, 64\}.$$

Only 1 and 64 are common to both lists, so there are $10 + 4 - 2 = 12$ integers that are perfect squares or perfect cubes.

b Let A be the set of perfect squares. Since $31^2 = 961$ and $32^2 = 1024$, there are exactly 31 perfect squares no greater than 1000,

$$A = \{1^2, 2^2, \ldots, 31^2\}.$$

Let B be the set of perfect cubes. Since $10^3 = 1000$, there are exactly 10 perfect cubes no greater than 1000,

$$B = \{1^3, 2^3, \ldots, 10^3\}.$$

Clearly $A \cap B$ consists integers that are perfect powers of 2 and 3, that is, powers of 6. Since $3^6 = 729$ and $4^6 = 4096$, there are exactly 3 powers of 6 no greater than 1000,

$$A \cap B = \{1^6, 2^6, 3^6\}.$$

We then use the Inclusion-Exclusion Principle to find that,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B|\\ &= 31 + 10 - 3\\ &= 38.\end{aligned}$$

8 Let sets A, B and C consist of those integers that are divisible by of 2, 3 and 5 respectively. We then have

set	multiples of	size
A	2	$\|A\| = 60$
B	3	$\|B\| = 40$
C	5	$\|C\| = 24$
$A \cap B$	6	$\|A \cap B\| = 20$
$A \cap C$	10	$A \cap C\| = 12$
$B \cap C$	15	$\|B \cap C\| = 8$
$A \cap B \cap C$	30	$\|A \cap B \cap C\| = 4$

We then use the Inclusion-Exclusion Principle to give,

$$\begin{aligned}&|A \cup B \cup C|\\ &=|A| + |B| + |C|\\ &-|A \cap B| - |A \cap C| - |B \cap C|\\ &+|A \cap B \cap C|\\ &=60 + 40 + 24 - 20 - 12 - 8 + 4\\ &=88\end{aligned}$$

Therefore, there are 88 integers from 1 to 120 inclusive that are divisible by 2, 3 or 5.

9 Let sets A, B and C consist of those integers that are divisible by of 2, 5 and 11 respectively. We then have

set	multiples of	size of set
A	2	$\lvert A\rvert = 110$
B	5	$\lvert B\rvert = 44$
C	11	$\lvert C\rvert = 20$
$A \cap B$	10	$\lvert A \cap B\rvert = 22$
$A \cap C$	22	$A \cap C\rvert = 10$
$B \cap C$	55	$\lvert B \cap C\rvert = 4$
$A \cap B \cap C$	110	$\lvert A \cap B \cap C\rvert = 2$

We then use the Inclusion-Exclusion Principle to give,

$$\begin{aligned} &|A \cup B \cup C| \\ =&|A| + |B| + |C| \\ -&|A \cap B| - |A \cap C| - |B \cap C| \\ +&|A \cap B \cap C| \\ =&110 + 44 + 20 - 22 - 10 - 4 + 2 \\ =&140 \end{aligned}$$

Therefore, there are 140 integers from 1 to 120 inclusive that are divisible by 2, 3 or 5, and 220 − 140 = 80 integers that are not.

10 Let B be the set of students that study biology, let P be the set of students that study physics, and let C be the set of students that study chemistry. Then,

$$\begin{aligned} &|B \cup P \cup C| \\ &= |B| + |P| + |C| \\ &- |B \cap P| - |B \cap C| - |P \cap C| \\ &+ |B \cap P \cap C| \\ 90 &= 36 + 42 + 40 - 9 - 8 - 7 + |B \cap P \cap C| \\ 90 &= 94 - |B \cap P \cap C| \end{aligned}$$

Therefore, $|B \cap P \cap C| = 4$.

11 **a** There are ${}^4C_2$ ways of choosing 2 Year Ten students from 4, and ${}^9C_4$ ways of choosing 4 more students from those in Years Eleven and Twelve. This gives a total of ${}^4C_2 \times {}^9C_4 = 756$ selections.

b There are ${}^5C_2$ ways of choosing 2 Year Eleven students from 5, and ${}^8C_4$ ways of choosing 4 more students from those in Years Ten and Twelve. This gives a total of ${}^5C_2 \times {}^8C_4 = 700$ selections.

c There are ${}^4C_2$ ways of choosing 2 Year Ten students from 4. There are ${}^5C_2$ ways of choosing 2 from Year Eleven. There are ${}^4C_2$ ways of choosing 2 remaining students from Year Twelve. This gives a total of ${}^4C_2 \times {}^5C_2 \times {}^4C_2 = 360$ selections.

d We let A be the set of selections with exactly 2 Year Ten students and let B be the set of selections with exactly 2 Year Eleven students. Then $A \cap B$ is set selection with exactly 2 Year Ten and Year Eleven students. We then use the Inclusion-Exclusion Principle to find that,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 756 + 700 - 360 \\ &= 1096. \end{aligned}$$

12 We let A be the set of hands with exactly one heart. Then there are ${}^{13}C_1$ ways of choosing the heart and ${}^{39}C_4$ ways of choosing the 4 non-heart cards. Therefore,

$$|A| = {}^{13}C_1 \times {}^{39}C_4 = 1069263$$

We let B be the set of hands with exactly two diamonds. Then there are ${}^{13}C_2$ ways of choosing the two diamonds and ${}^{39}C_3$ ways of choosing the 3 non-diamond cards. Therefore,

$$|B| = {}^{13}C_2 \times {}^{39}C_3 = 712842$$

Then $A \cap B$ is the set hands with exactly one heart and two diamonds. Then there are ${}^{13}C_1$ ways of choosing the one heart, ${}^{13}C_2$ ways of choosing the two diamonds and ${}^{26}C_2$ ways of choosing the 2 remaining cards. Therefore,

$$|A \cap B| = {}^{13}C_1 \times {}^{13}C_2 \times {}^{26}C_2 = 329550.$$

We then use the Inclusion-Exclusion Principle to find that,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 1069263 + 712842 - 329550 \\ &= 1452555. \end{aligned}$$

13 The sum of numbers divisible by 2 will be

$$2 + 4 + \ldots + 100 = 2550.$$

The sum of numbers divisible by 3 will be

$$3 + 6 + \ldots + 99 = 1683.$$

Adding these together does not give the desired answer, since it double counts those numbers divisible by 2 and 3, that is, those divisible by 6. The sum of these numbers is

$$6 + 12 + \ldots + 96 = 816.$$

Therefore, the required answer is $2550 + 1683 - 816 = 3417$.

14 Let F, C and G be the sets of students studying French, Chinese and German, respectively. Then using the Exclusion-Inclusion Principle gives,

$$\begin{aligned} &|F \cup C \cup G| \\ =&|F| + |C| + |G| \\ &- |F \cap C| - |F \cap G| - |C \cap G| \\ &+ |F \cap C \cap G| \\ 80 =&30 + 45 + 30 - |F \cap C| - |F \cap G| - |C \cap G| + 15 \\ 80 =&120 - |F \cap C| - |F \cap G| - |C \cap G| \\ 40 =&|F \cap C| + |F \cap G| + |C \cap G| \end{aligned}$$

Now $|F \cap C| + |F \cap G| + |C \cap G|$ counts the number of students who study at two subjects, but triple counts those who study three subjects. Therefore, the number that study exactly 2 subjects will be $40 - 2 \times 15 = 10$.

Solutions to Technology-free questions

1 **a** $$ {}^{6}C_3 = \frac{6!}{3! \cdot (6-3)!} = \frac{5!}{2! \cdot 3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{6} = 20 $$

b $$ {}^{20}C_2 = \frac{20!}{2! \cdot (20-2)!} = \frac{20!}{2! \cdot 18!} = \frac{20 \cdot 19 \cdot 18!}{2! \cdot 18!} = \frac{20 \cdot 19}{2} = 190 $$

c This can evaluated by using the fact that ${}^{n}C_1 = n$. Otherwise, we simply use the formula for ${}^{n}C_r$ to give,

$$ {}^{300}C_1 = \frac{300!}{1! \cdot (300-1)!} = \frac{300!}{1! \cdot 299!} = \frac{300 \cdot 299!}{299!} = 300. $$

d $$ {}^{100}C_{98} = \frac{100!}{98! \cdot (100-98)!} = \frac{100!}{98! \cdot 2!} = \frac{100 \cdot 99 \cdot 98!}{98! \cdot 2!} = \frac{100 \cdot 99}{2} = 4950 $$

2 We solve the following equation,

$$ {}^{n}C_2 = 55 $$
$$ \frac{n!}{2! \cdot (n-2)!} = 55 $$
$$ \frac{n \cdot (n-1) \cdot (n-2)!}{2! \cdot (n-2)!} = 55 $$
$$ \frac{n \cdot (n-1)}{2} = 55 $$
$$ n(n-1) = 110 $$
$$ n^2 - n - 110 = 0 $$
$$ (n-11)(n+10) = 0 $$
$$ \Rightarrow n = 11 \text{ as } n > 0. $$

3 **a** If the digits can be repeated there are 3 choices for each of the 3 positions. This gives a total of $3^3 = 27$ numbers.

b 3 digits can be arranged (without repetition) in $3! = 6$ ways.

4 There are 6 choices of student for the first position, 5 for the second and 4 for the third. This gives a total of $6 \times 5 \times 4 = 120$ arrangements.

5 There are 5 choices of desk for the first student, 4 for the second and 3 for the third. This gives a total of $5 \times 4 \times 3 = 60$ allocations.

6 There are ${}^4C_2$ ways of selecting 2 Year 11 student out of 4 and ${}^3C_2$ ways of selecting 2 Year 12 student out of 3. Using the Multiplication Principle gives a total of ${}^4C_2 \times {}^3C_2 = 6 \times 3 = 18$ selections.

7 Without restriction, there are ${}^7C_3 = 35$ ways of selecting 3 children out of 7. We then consider those selections that have no boy. There are are ${}^4C_3 = 4$ ways of selecting 3 of 4 girls. Therefore, there are $35 - 4 = 31$ selections that have at least one boy.

8 There are 5 flags, of which a group of 2 are alike, and a group of 3 are alike. These flags can be arranged in $\dfrac{5!}{2! \cdot 3!} = 10$ ways.

9 Label 26 holes with each of the letters from A to Z. Place each word in the hole according to its first letter. Since $53 = 2 \times 26 + 1$ there is some hole that contains at least 3 words.

10 Let N and B be the set of students playing netball and basketball respectively. Then using the Inclusion-Exclusion Principle gives,

$$\begin{aligned} |N \cup B| &= |N| + |B| - |N \cap B| \\ 20 &= |N| + 12 - 4 \\ 20 &= |N| + 8 \\ |N| &= 12. \end{aligned}$$

So 12 student play netball.

11 Let's denote the six people by letters A, B, C, D, E and F. First suppose person C is between person A and B. We then have to arrange four items: $\{A, C, B\}, D, E, F$. This can be done in 4! ways. Person C is fixed but A and B can be arranged in 2 ways. We then multiply by 4 as there are 4 different people who can go between A and B. This gives a total of $4! \times 2 \times 4 = 192$ arrangements.

Solutions to multiple-choice questions

1 C Using the Multiplication Principle there are $4 \times 3 \times 4 = 48$ ways that Sam can select his remaining three subjects.

2 B Using both the Addition and Multiplication Principle gives a total of $3 + 2 \times 4 = 11$ different paths.

3 A 10 people can be arranged in 10! ways.

4 D The first digit can be chosen in 6 ways, the second digit in 5 ways on the third digit in 4 ways. This gives a total of $6 \times 4 \times 3$ arrangements.

5 B There are 4 vowels, so there are 4 choices for the first letter and 3 for the last. After filling these two positions, there are 4 choices for the second letter, 3 for the third, and so on. This gives a total of $4 \times 4 \times 3 \times 2 \times 1 \times 3 = 288$ arrangements.

6 B There are 7 flags in total, of which a group of 4 are alike and another group of 3 are alike. These can be arranged in $\dfrac{7!}{4! \cdot 3!}$ ways.

7 C 3 items can be chosen out of 9 in ${}^9C_3$ ways.

8 D A set with 6 elements has 2^6 subsets. One of these is the empty set, so there are $2^6 - 1$ subsets with at least one element.

9 C There are ${}^9C_2$ ways of selecting 2 girls out of 9 and ${}^8C_2$ ways of selecting 2 boys out of 8. Using the Multiplication Principle gives a total of ${}^9C_2 \times {}^8C_2$ selections.

10 C Labels two holes with the colours blue and red. Select 5 balls. Then since $5 = 2 \times 2 + 1$, there is some hole that contains at least 3 balls. Clearly 5 is the smallest number since selecting 4 balls might give 2 blue and 2 red balls.

11 A Let F, C and G be the sets of students studying French, Chinese and German, respectively. Then using the Exclusion-Inclusion Principle gives,

$$\begin{aligned} |F \cup C \cup G| &= |F| + |C| + |G| \\ &\quad -|F \cap C| - |F \cap G| - |C \cap G| \\ &\quad +|F \cap C \cap G| \\ 30 &= 15 + 15 + 17 \\ &\quad -|F \cap C| - |F \cap G| - |C \cap G| \\ &\quad +|F \cap C \cap G| \end{aligned}$$

Therefore,

$$17 = |F \cap C| + |F \cap G| + |C \cap G| - |F \cap C \cap G|$$

Since 15 student study more than one subject we also know that,

$$15 = |F \cap C| + F \cap G| + |C \cap G| - 2|F \cap C \cap G|.$$

Subtracting the second equation from the first gives,

$$|F \cap C \cap G| = 2.$$

Solutions to extended-response questions

1 **a** The first digit is 5 so there is only 1 choice for the that position. The remaining 5 numbers can be arranged in 5! ways. This gives a total of $1 \times 5! = 120$ arrangements.

b If the first digit is even then there are 3 choices for that position. The remaining 5 numbers can be arranged in 5! ways. This gives a total of $3 \times 5! = 360$ arrangements.

c There are 2 choices for if the first number is even or odd. The 3 odd numbers can then be arranged in 3! ways, and the 3 even numbers can be arranged in 3! ways. This gives a total of $2 \times 3! \times 3! = 72$ arrangements.

d If we group the even numbers, we need only arrange 4 items: $1, 3, 5, \{2, 4, 6\}$ and $\{2, 4, 6\}$. This can be done in 4! ways. The 3 even numbers can then be arranged in 3! ways. This gives a total of $4! \times 3! = 144$ arrangements.

2 **a** If the first letter is E, then there is only 1 choice for the that position. There are 5 remaining choices for the second letter, and 4 for the third. This gives a total of $1 \times 5 \times 4 = 20$ arrangements.

b If the first letter is a vowel then there are 4 choices for that position. There are 5 remaining choices for the second letter and 4 for the third. This gives a total of $4 \times 5 \times 4 = 80$ arrangements.

c If the letter E is used then there are 3 choices of position for that letter. There are 5 and then 4 choices for the remaining two positions. This gives a total of $3 \times 5 \times 4 = 60$ arrangements.

3 **a** 4 students can be chosen from 10 in ${}^{10}C_4 = 210$ ways.

b If the school captain must be chosen then we are still to chose 3 students from the 9 that remain. This can be done in ${}^{9}C_3 = 84$ ways.

c We must select 2 boys from 4 and 2 girls from 6. This can be done in ${}^{4}C_2 \times {}^{6}C_2 = 6 \times 15 = 90$ ways.

d The total number of selections is ${}^{10}C_4 = 210$. The number of selections with no boys is ${}^{6}C_4 = 15$. Therefore, the number of selections with at least one boy will be $210 - 15 = 195$.

4 **a** There are 8 letters in total, 2 of which are N, 2 of which are J and 4 of which are T. These can be arranged in $\dfrac{8!}{2! \cdot 2! \cdot 4!} = 420$ ways.

b The first and last letter are both N. Therefore, we need only arrange 6 letters, 2 of which are J and 4 of which are T. This can be done in $\frac{6!}{2! \cdot 4!} = 15$ ways.

c We group the two J's so that we now must arrange {J,J},N,N,T,T,T,T. There are 7 items to arrange, of which 2 are N and 4 are T. This can be done in $\frac{7!}{2! \cdot 4!} = 105$ ways.

d If no two T's are adjacent then they are either all in the even positions or all in the odd positions.

The 4 remaining letters are: N,N,J,J. These can be arranged in $\frac{4!}{2! \cdot 2!} = 6$ ways. We multiply this by 2 because the J's can be placed in either even or odd positions. This gives a total of $6 \times 2 = 12$ arrangements.

5 **a** **i** 3 toppings can be chosen from 6 in ${}^{6}C_{3} = 20$ ways.

ii 2 additional toppings have to be chosen from among 5 that remain. This can be done in ${}^{5}C_{2} = 10$ ways.

iii A set of size 6 has $2^6 = 64$ subsets.

b We want to find the smallest value of n such that $2^n > 200$. Since $2^7 = 128$ and $2^8 = 256$ they must use at least 8 different toppings.

6 **a** 4 people can be chosen from 10 in ${}^{10}C_{4} = 210$ ways.

b We must choose 2 women from 5 and 2 men from 5. Using the Multiplication Principle, this can be done in ${}^{5}C_{2} \times 5C_{2} = 100$ ways.

c We must choose 2 married couples from 5. This can be done in ${}^{5}C_{2} = 10$ ways.

d We first chose 4 couples from 5. This can be done in ${}^{5}C_{4} = 5$ ways. Then from each of the 4 couples there are 2 choices of either husband or wife. Using the Multiplication Principles gives a total of $5 \times 2 \times 2 \times 2 \times 2 = 80$ selections.

7 **a** There are 26 choices for both the first and second letter so there are $26 \times 26 = 676$ two letter initials.

b We first consider the two letter initials that contain no vowel. There are 21 choices of consonant for each of the two letters, giving a total of $21 \times 21 = 441$. Therefore, $676 - 441 = 235$ two letter initials contain at least one vowel.

c Label 676 holes with each of the different two letter initials. Since $50000/676 > 73$, there must be some hole containing 74 items. Therefore, there are at least 74 people who share the same initials.

8 **a** The lowest common multiple is 24.

b $A \cap B$ consists of numbers that multiples of 6 and 8. That is, it contains multiples of 24. Therefore, $A \cap B = \{24, 48, 72, 96\}$, so there are 4 elements.

c We have,

A	multiples of 6	$\lvert A\rvert = 96 \div 6 = 16$
B	multiples of 8	$\lvert B\rvert = 96 \div 8 = 12$
$A \cap B$	multiples of 24	$\lvert A \cap B\rvert = 96 \div 24 = 4$

Using the Inclusion-Exclusion Principle gives,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 16 + 12 - 4 \\ &= 24. \end{aligned}$$

d There are 24 numbers that are divisible by 6 or 8 and $96 - 24 = 72$ that are not. Therefore, the probability that the integer is not divisible by 6 or 8 is $\frac{72}{96} = \frac{3}{4}$.

9 **a** Let N be a movement of one unit in the north direction, and E be a movement of one unit in the east direction. Then each path fro H to G is described by six N's and six E's in some order. As there are twelve letters in total, there are $\frac{12!}{6! \cdot 6!} = 924$ paths.

b In three years there are at least $365 \times 3 = 1095$ days and there are 924 different paths. Since $1095 > 924$, by the Pigeonhole Principle, there is some path taken at least twice in the course of 3 years.

c **i** Each path from H to C is described by two N's and two E's in some order. As there are four letters in total, there are $\frac{4!}{2! \cdot 2!} = 6$ paths.

ii Each path from C to G is described by four N's and four E's in some order. As there are eight letters in total, there are $\frac{8!}{4! \cdot 4!} = 70$ paths.

iii There are 6 paths from H to C and 70 paths from C to G. Using the Multiplication Principle gives a total of $6 \times 70 = 420$ paths from H to C to G.

d Let X be the set of paths from H to C to G. We have found that $|X| = 420$. Let Y be the set of paths from H to B to G. Clearly, $|Y| = 420$.
Now $X \cap Y$ is the set of paths from H to C to B to G. Since there are 6 paths from each of three steps, $|X \cap Y| = 6 \times 6 \times 6 = 216$. Lastly, applying the Inclusion-

Exclusion Principle gives,

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$
$$= 420 + 420 - 216$$
$$= 624$$

10 The ratio of green to yellow to orange balls is 1 : 3 : 6 = 2 : 6 : 12. Therefore, the ratio of blue to red to green to yellow to orange balls is 1 : 4 : 2 : 6 : 12. Using this, we can find the number of balls of each type. This gives:

blue $\frac{1}{25} \times 400 = 16$

red $\frac{4}{25} \times 400 = 64$

green $\frac{2}{25} \times 400 = 32$

yellow $\frac{6}{25} \times 400 = 96$

orange $\frac{12}{25} \times 400 = 192$

Label five holes with each of the above colours. In the worst case, you might first select 16 blue, 32 green, 49 red, 49 yellow and 49 orange. Selecting one additional ball must give either a red, yellow or orange ball. This gives a total of $16 + 32 + 3 \times 49 + 1 = 196$ balls.

Chapter 8 – Number and Proof

Solutions to Exercise 8A

1 **a** As m and n are even, $m = 2p$ and $n = 2q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m + n &= 2p + 2q \\ &= 2(p + q), \end{aligned}$$

is an even number.

b As m and n are even, $m = 2p$ and $n = 2q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (2p)(2q) \\ &= 4pq \\ &= 2(2pq), \end{aligned}$$

is an even number.

2 As m and n are odd, $m = 2p + 1$ and $n = 2q + 1$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m + n &= (2p + 1) + (2q + 1) \\ &= 2p + 2q + 2 \\ &= 2(p + q + 1), \end{aligned}$$

is an even number.

3 As m is even and n is odd, $m = 2p$ and $n = 2q + 1$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= 2p(2q + 1) \\ &= 2(2pq + p), \end{aligned}$$

is an even number.

4 **a** If m is divisible by 3 and n is divisible by 7, then $m = 3p$ and $n = 7q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (3p)(7q) \\ &= 21pq, \end{aligned}$$

is divisible by 21.

b If m is divisible by 3 and n is divisible by 7, then $m = 3p$ and $n = 7q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m^2n &= (3p)^2(7q) \\ &= 9p^2(7q) \\ &= 63p^2q \end{aligned}$$

is divisible by 63.

5 If m and n are perfect squares then $m = a^2$ and $n = b^2$ for some $a, b \in \mathbb{Z}$. Therefore,

$$mn = (a^2)(b^2) = (ab)^2,$$

is also a perfect square.

6 Expanding both brackets gives,

$$\begin{aligned} &(m + n)^2 - (m - n)^2 \\ &= m^2 + 2mn + n^2 - (m^2 - 2mn + n^2) \\ &= m^2 + 2mn + n^2 - m^2 + 2mn - n^2 \\ &= 4mn, \end{aligned}$$

which is divisible by 4.

7 (Method 1) If n is even then n^2 is even and $6n$ is even. Therefore the expression is of the form

$$\text{even} - \text{even} + \text{odd} = \text{odd}.$$

(Method 2) If n is even then $n = 2k$

where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 - 6n + 5 &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 4k^2 - 12k + 4 + 1 \\ &= 2(2k^2 - 6k + 2) + 1, \quad = \end{aligned}$$

is odd.

8 (Method 1) If n is odd then n^2 is odd and $8n$ is even. Therefore the expression is of the form

odd + even + odd = even.

(Method 2) If n is odd then $n = 2k + 1$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 + 8n + 5 &= (2k + 1)^2 + 8(2k + 1) + 3 \\ &= 4k^2 + 4k + 1 + 16k + 8 + 3 \\ &= 4k^2 + 20k + 12 \\ &= 2(2k^2 + 10k + 6), \end{aligned}$$

is even.

9 First suppose n is even. Then $5n^2$ and $3n$ are both even. Therefore the expression is of the form

even + even + odd = odd.

Now suppose n is odd. Then $5n^2$ and $3n$ are both odd. Therefore the expression is of the form

odd + odd + odd = odd.

10 Firstly, if $x > y$ then $x - y > 0$. Secondly, since x and y are positive, $x + y > 0$.

Therefore,

$$\begin{aligned} &x^4 - y^4 \\ =&(x^2 - y^2)(x^2 + y^2) \\ =&(x - y)(x + y)(x^2 + y^2) \\ =&\underbrace{(x - y)}_{\text{positive}}\underbrace{(x + y)}_{\text{positive}}\underbrace{(x^2 + y^2)}_{\text{positive}} \\ >&0. \end{aligned}$$

Therefore, $x^4 > y^4$.

11 We have,

$$\begin{aligned} &x^2 + y^2 - 2xy \\ =&x^2 - 2xy + y^2 \\ =&(x - y)^2 \\ \geq&2xy. \end{aligned}$$

Therefore, $x^2 + y^2 \geq 2xy$.

12 **a** We prove that Alice is a knave, and Bob is a knight.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice and Bob are both knaves
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice and Bob are not both knaves
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Alice is a knave, and Bob is a knight

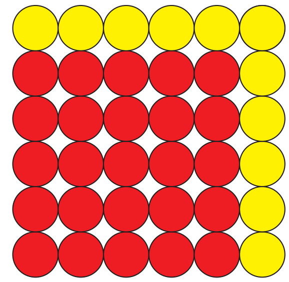
b We prove that Alice is a knave, and Bob is a knight.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ They are both of the same kind
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Bob is lying
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Bob is a knight and a knave.
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice and Bob are of a different kind
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Alice is a knave, and Bob is a knight

c We will prove that Alice is a knight, and Bob is a knave.

Suppose Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Bob is telling the truth
$\Rightarrow$ Neither of them are knaves
$\Rightarrow$ Both of them are knights
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Bob is lying
$\Rightarrow$ At least one of them is a knave
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Alice is a knight, and Bob is a knave.

13 a In the diagram below, there are 11 yellow tiles. We can also count the yellow tiles by subtracting the number of red tiles, 5^2, from the total number of tiles, 6^2. Therefore $11 = 6^2 - 5^2$.



b Every odd number is of the form $2k + 1$ for some $k \in \mathbb{Z}$. Moreover,

$$\begin{aligned}(k+1)^2 - k^2 &= k^2 + 2k + 1 - k^2\\ &= 2k + 1,\end{aligned}$$

so that every odd number can be written as the difference of two squares.

c Since $101 = 2 \times 50 + 1$, we have,

$$51^2 - 50^2 = 101.$$

14 a Since

$$\frac{9}{10} = \frac{99}{110} \text{ and } \frac{10}{11} = \frac{100}{110},$$

it is clear that

$$\frac{10}{11} > \frac{9}{10}.$$

b We have,

$$\begin{aligned}&\frac{n}{n+1} - \frac{n-1}{n}\\ &= \frac{n^2}{n(n+1)} - \frac{n(n-1)}{n(n+1)}\\ &= \frac{n^2 - n(n-1)}{n(n+1)}\\ &= \frac{n^2 - n^2 + n}{n(n+1)}\\ &= \frac{1}{n(n+1)}\\ &> 0\end{aligned}$$

since $n(n+1) > 0$. Therefore,

$$\frac{n}{n+1} > \frac{n-1}{n}.$$

15 **a** We have,

$$\begin{aligned}&\frac{1}{10}-\frac{1}{11}\\=&\frac{11}{110}-\frac{10}{110}\\=&\frac{1}{110}\\<&\frac{1}{100},\end{aligned}$$

since 110 > 100.

b We have,

$$\begin{aligned}\frac{1}{n}-\frac{1}{n+1}&=\frac{n+1}{n(n+1)}-\frac{n}{n(n+1)}\\&=\frac{n+1-n}{n(n+1)}\\&=\frac{1}{n(n+1)},\\&=\frac{1}{n^2+n},\\&<\frac{1}{n^2},\end{aligned}$$

since $n^2 + n > n^2$.

16 We have,

$$\begin{aligned}&\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\\=&\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}\\=&\frac{2a^2+2b^2}{4}-\frac{a^2+2ab+b^2}{4}\\=&\frac{2a^2+2b^2-a^2-2ab-b^2}{4}\\=&\frac{a^2-2ab+b^2}{4}\\=&\frac{(a-b)^2}{4}\\\geq&0.\end{aligned}$$

17 **a** Expanding gives,

$$\begin{aligned}&(x-y)(x^2+xy+y^2)\\=&x^3+x^2y+xy^2-x^2y-xy^2-y^3\\=&x^3-y^3,\end{aligned}$$

which is the difference of two cubes.

b Completing the square by treating y as a constant gives,

$$\begin{aligned}&x^2+yx+y^2\\=&x^2+yx+\frac{y^2}{4}-\frac{y^2}{4}+y^2\\=&\left(x^2+yx+\frac{y^2}{4}\right)+\frac{3y^2}{4}\\=&\left(x+\frac{y}{2}\right)^2+\frac{3y^2}{4}\\\geq&0\end{aligned}$$

c Firstly, if $x \geq y$ then $x - y \geq 0$. Therefore,

$$\begin{aligned}&x^3-y^3\\=&\overbrace{(x-y)}^{\geq 0}\overbrace{(x^2+xy+y^2)}^{\geq 0}\\\geq&0.\end{aligned}$$

Therefore, $x^3 > y^3$.

18 **a** Let D be the distance to and from work. The time taken to get to work is $D/12$ and the time taken to get home from work is $D/24$. The total distance is $2D$ and the total time is

$$\frac{D}{12} + \frac{D}{24}$$
$$= \frac{2D}{24} + \frac{D}{24}$$
$$= \frac{3D}{24}$$
$$= \frac{D}{8}$$

The average speed will then be

$$\text{distance} \div \text{time}$$
$$= 2D \div \frac{D}{8}$$
$$= 2D \times \frac{8}{D}$$
$$= 16 \text{ km/hour.}$$

b Let D be the distance to and from work. The time taken to get to work is D/a and the time taken to get home from work is D/b. The total distance is $2D$ and the total time is

$$\frac{D}{a} + \frac{D}{b}$$
$$= \frac{bD}{ab} + \frac{aD}{ab}$$
$$= \frac{aD + bD}{ab}$$
$$= \frac{(a+b)D}{ab}$$

The average speed will then be

$$\text{distance} \div \text{time}$$
$$= 2D \div \frac{(a+b)D}{ab}$$
$$= 2D \times \frac{ab}{(a+b)D}$$
$$= \frac{2ab}{a+b} \text{ km/hour.}$$

c We first note that $a + b > 0$. Secondly,

$$\frac{a+b}{2} - \frac{2ab}{a+b}$$
$$= \frac{(a+b)^2}{2(a+b)} - \frac{4ab}{2(a+b)}$$
$$= \frac{(a+b)^2 - 4ab}{2(a+b)}$$
$$= \frac{a^2 + 2ab + b^2 - 4ab}{2(a+b)}$$
$$= \frac{a^2 - 2ab + b^2}{2(a+b)}$$
$$= \frac{(a-b)^2}{2(a+b)}$$
$$\geq 0$$

since $(a - b) \geq 0$ and $a + b > 0$. Therefore,

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}.$$

Solutions to Exercise 8B

1 **a** $P : 1 > 0$ (true)
not $P : 1 \le 0$ (false)

b P : 4 is divisible by 8 (false)
not P : 4 is not divisible by 8 (true)

c P : Each pair of primes has an even sum (false)
not P : Some pair of primes does not have an even sum (true)

d P : Some rectangle has 4 sides of equal length (true)
not P : No rectangle has 4 sides of equal length (false)

2 **a** P : 14 is divisible by 7 and 2 (true)
not P : 14 is not divisible by 7 or 14 is not divisible by 2 (false)

b P : 12 is divisible by 3 or 4 (true)
not P : 12 is not divisible by 4 and 12 is not divisible by 3 (false)

c P : 15 is divisible by 3 and 6 (false)
not P : 15 is not divisible by 3 or 15 is not divisible by 6 (true)

d P : 10 is divisible by 2 or 5 (true)
not P : 10 is not divisible by 2 or 10 is not divisible by 5 (false)

3 We will prove that Alice is a knave, and Bob is a knave.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice is a knight OR Bob is a knave
$\Rightarrow$ Bob is a knave, as Alice is not a knight
$\Rightarrow$ Alice and Bob are both knaves.

4 **a** If there are no clouds in the sky, then it is not raining.

b If you are not happy, then you are not smiling.

c If $2x \ne 2$, then $x \ne 1$.

d If $x^5 \le y^5$, then $x \le y$.

e Option 1: If n is not odd, then n^2 is not odd.
Option 2: If n is even, then n^2 is even.

f Option 1: If mn is not odd, then n is not odd or m is not odd.
Option 2: If If mn is even, then n is even or m is even.

g Option 1: If n and n are not both even or both odd, then $m + n$ is not even.
Option 2: If n and n are not both even or both odd, then $m + n$ is odd.

5 **a** Contrapositive: If n is even then $3n + 5$ is odd.
Proof: Suppose n is even. Then

$n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 3n + 5 &= 3(2k) + 5 \\ &= 6k + 5 \\ &= 6k + 4 + 1 \\ &= 2(3k + 2) + 1 \end{aligned}$$

is odd.

b Contrapositive: If n is even, then n^2 is even.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

is even.

c Contrapositive: If n is even, then $n^2 - 8n + 3$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 - 8n + 3 &= (2k)^2 - 8(2k) + 3 \\ &= 4k^2 - 16k + 3 \\ &= 4k^2 - 16k + 2 + 1 \\ &= 2(2k^2 - 8k + 1) + 1 \end{aligned}$$

is odd.

d Contrapositive: If n is divisible by 3, then n^2 is divisible by 3.
Proof: Suppose n is divisible by 3. Then $n = 3k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (3k)^2 \\ &= 9k^2 \\ &= 3(3k^2) \end{aligned}$$

is divisible by 3.

e Contrapositive: If n is even, then $n^3 + 1$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 + 1 &= (2k)^3 + 1 \\ &= 8k^3 + 1 \\ &= 2(4k^3) + 1 \end{aligned}$$

is odd.

f Contrapositive: If m or n are divisible by 3, then mn is divisible by 3.
Proof: If m or n is divisible by 3 then we can assume that m is divisible by 3. Then, $m = 3k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (3k)n \\ &= 3(kn) \end{aligned}$$

is divisible by 3.

g Contrapositive: If $m = n$, then $m + n$ is even.
Proof: Suppose that $m = n$. Then

$$\begin{aligned} m + n &= n + n \\ &= 2n \end{aligned}$$

is even.

6 a Contrapositive: If $x \geq 0$, then $x^2 + 3x \geq 0$.
Proof: Suppose that $x \geq 0$. Then,

$$x^2 + 3x = x(x + 3) \geq 0,$$

since $x \geq 0$ and $x + 3 \geq 0$.

b Contrapositive: If $x \leq -1$, then $x^3 - x \leq 0$.
Proof: Suppose that $x \leq -1$. Then,

$$x^3 - x = x^2(x - 1) \leq 0,$$

since $x^2 \geq 0$ and $x - 1 \geq 0$.

c Contrapositive: If $x < 1$ and $y < 1$, then $x + y < 2$.
Proof: If $x < 1$ and $y < 1$ then,

$$x + y < 1 + 1 = 2,$$

as required.

d Contrapositive: If $x < 3$ and $y < 2$, then $2x + 3y < 12$.
Proof: If $x < 3$ and $y < 2$ then,

$$2x + 3y < 2 \times 3 + 3 \times 2 = 6 + 6 = 12,$$

as required.

7 a Contrapositive: If m is odd or n is odd, then mn is odd or $m + n$ is odd.

b Proof:
(Case 1) Suppose m is odd and n is odd. Then clearly mn is odd.
(Case 2) Suppose m is odd and n is even. Then clearly $m + n$ will be odd. It is likewise, if m is even and n is odd.

8 a We rationalise the right hand side to give,

$$\begin{aligned}&\frac{x-y}{\sqrt{x}+\sqrt{y}}\\ &=\frac{x-y}{\sqrt{x}+\sqrt{y}}\,\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}\\ &=\frac{(x-y)\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)}\\ &=\frac{(x-y)\left(\sqrt{x}-\sqrt{y}\right)}{(x-y)}\\ &=\sqrt{x}-\sqrt{y}.\end{aligned}$$

b If $x > y$ then $x - y > 0$. Then, using the above equality, we see that,

$$\sqrt{x}-\sqrt{y}=\frac{x-y}{\sqrt{x}+\sqrt{y}} > 0,$$

since the numerator and denominator are both positive. Therefore, $\sqrt{x} > \sqrt{y}$.

c Contrapositive: If $\sqrt{x} \le \sqrt{y}$, then $x \le y$.
Proof: If $\sqrt{x} \le \sqrt{y}$ then, since both sides are positive, we can square both sides to give $x \le y$.

Solutions to Exercise 8C

1 If all three angles are less than 60°, then the sum of interior angles of the triangle would be less than 180°. This is a contradiction as the sum of interior angles is exactly 180°.

2 Suppose there is some least positive rational number $\frac{p}{q}$. Then since,

$$\frac{p}{2q} < \frac{p}{q},$$

there is some lesser positive rational number, which is a contradiction. Therefore, there is no least positive rational number.

3 Suppose that $\sqrt{p}$ is an integer. Then

$$\sqrt{p} = n,$$

for some $n \in \mathbb{Z}$. Squaring both sides gives

$$p = n^2.$$

Since $n \neq 1$, this means that p has three factors: $1, n$ and n^2. This is a contradiction since every prime number has exactly two factors.

4 Suppose that x is rational so that $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} & 3^x = 2 \\ \Rightarrow\ & 3^{\frac{p}{q}} = 2 \\ \Rightarrow\ & \left(3^{\frac{p}{q}}\right)^q = 2^q \\ \Rightarrow\ & 3^p = 2^q \end{aligned}$$

The left hand side of this equation is odd, and the right hand side is even. This gives a contradiction, so x is not rational.

5 Suppose that $\log_2 5$ is rational so that $\log_2 5 = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} & 2^{\frac{p}{q}} = 5 \\ \Rightarrow\ & 2^{\frac{p}{q}} = 5 \\ \Rightarrow\ & \left(2^{\frac{p}{q}}\right)^q = 5^q \\ \Rightarrow\ & 2^p = 5^q \end{aligned}$$

The left hand side of this equation is odd, and the right hand side is even. This gives a contradiction, so x is not rational.

6 Suppose the contrary, so that $\sqrt{x}$ is rational. Then

$$\sqrt{x} = \frac{p}{q},$$

where $p, q \in \mathbb{Z}$. Then, squaring both sides of the equation gives,

$$x = \frac{p^2}{q^2},$$

where $p^2, q^2 \in \mathbb{Z}$. Therefore, x is rational, which is a contradiction.

7 Suppose, on the contrary that $a + b$ is rational. Then

$$b = \overbrace{(a+b)}^{\text{rational}} - \overbrace{b}^{\text{rational}}.$$

Therefore, b is the difference of two rational numbers, which is rational. This is a contradiction.

8 Suppose b and c are both natural numbers. Then

$$c^2 - b^2 = 4$$
$$(c - b)(c + b) = 4.$$

The only factors of 4 are 1, 2 and 4. And since $c + b > c - b$,

$$c - b = 1 \text{ and } c + b = 4.$$

Adding these two equations gives $2c = 5$ so that $c = \frac{2}{5}$, which is not a whole number.

9 Suppose that there are two different solutions, x_1 and x_2. Then,

$$ax_1 + b = c \text{ and } ax_2 + b = c.$$

Equating these two equations gives,

$$ax_1 + b = ax_2 + b$$
$$ax_1 = ax_2$$
$$x_1 = x_2, \qquad \text{(since } a \neq 0\text{)}$$

which is a contradiction since the two solutions were assumed to be different.

10 a Every prime $p > 2$ is odd since if it were even then p would be divisible by 2.

b Suppose there are two primes p and p such that $p + q = 1001$. Then since the sum of two odd numbers is even, one of the primes must be 2. Assume $p = 2$ so that $q = 999$. Since 999 is not prime, this gives a contradiction.

11 a Suppose that

$$42a + 7b = 1.$$

Then

$$7(6a + b) = 1.$$

This implies that 1 is divisible by 7, which is a contradiction since the only factor of 1 is 1.

b Suppose that

$$15a + 21b = 2.$$

Then

$$3(5a + 7b) = 2.$$

This implies that 2 is divisible by 3, which is a contradiction since the only factors of 2 are 1 and 2.

12 a Contrapositive: If n is not divisible by 3, then n^2 is not divisible by 3.
Proof: If n is not divisible by 3 then either $n = 3k + 1$ or $n = 3k + 2$.
(Case 1) If $n = 3k + 1$ then,

$$\begin{aligned} n^2 &= (3k + 1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

is not divisible by 3.
(Case 2) If $n = 3k + 2$ then,

$$\begin{aligned} n^2 &= (3k + 2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

is not divisible by 3.

b This will be a proof by contradiction. Suppose $\sqrt{3}$ is rational so that $\sqrt{3} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$p^2 = 3q^2 \qquad (1)$$
$$\Rightarrow \; p^2 \text{ is divisible by } 3$$
$$\Rightarrow \; p \text{ is divisible by } 3$$
$$\Rightarrow \; p = 3k \text{ for some } k \in \mathbb{N}$$
$$\Rightarrow \; (3k)^2 = 3q^2 \text{(substitituting into (1))}$$
$$\Rightarrow \; 3q^2 = 9k^2$$
$$\Rightarrow \; q^2 = 3k^2$$
$$\Rightarrow \; q^2 \text{ is divisible by } 3$$
$$\Rightarrow \; q \text{ is divisible by } 3.$$

So p and q are both divisible by 3, which contradicts the fact that they have no factors in common.

13 a Contrapositive: If n is odd, then n^3 is odd.
Proof: If n is odd then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$n^3 = (2k + 1)^3$$
$$= 8k^3 + 12k^2 + 6k + 1$$
$$= 2(4k^3 + 6k^2 + 3k) + 1$$

is odd. Otherwise, we can simply quote the fact that the product of 3 odd numbers will be odd.

b This will be a proof by contradiction. Suppose $\sqrt[3]{2}$ is rational so that $\sqrt[3]{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$p^3 = 2q^3 \qquad (1)$$
$$\Rightarrow \; p^3 \text{ is divisible by } 2$$
$$\Rightarrow \; p \text{ is divisible by } 2$$
$$\Rightarrow \; p = 2k \text{ for some } k \in \mathbb{N}$$
$$\Rightarrow \; (2k)^3 = 2q^3 \text{(substitituting into (1))}$$
$$\Rightarrow \; 2q^3 = 8k^3$$
$$\Rightarrow \; q^3 = 4k^3$$
$$\Rightarrow \; q^3 \text{ is divisible by } 2$$
$$\Rightarrow \; q \text{ is divisible by } 2.$$

So p and q are both divisible by 2, which contradicts the fact that they have no factors in common.

14 This will be a proof by contradiction, so we suppose there is some $a, b \in \mathbb{Z}$ such that

$$a^2 - 4b - 2 = 0$$
$$\Rightarrow a^2 = 4b + 2$$
$$\Rightarrow a^2 = 2(2b + 1) \qquad (1)$$

which means that a^2 is even. However, this implies that a is even, so that $a = 2k$, for some $k \in \mathbb{Z}$. Substituting this into equation (1) gives,

$$(2k)^2 = 2(2b + 1)$$
$$4k^2 = 2(2b + 1)$$
$$2k^2 = 2b + 1$$
$$2k^2 - 2b = 1$$
$$2(k^2 - b) = 1.$$

This implies that 1 is divisible by 2, which is a contradiction since the only factor of 1 is 1.

15 a Suppose on the contrary, that $a > \sqrt{n}$

and $b > \sqrt{n}$. Then

$$ab > \sqrt{n}\sqrt{n} = n,$$

which is a contradiction since $ab = n$.

b If 97 were not prime then we could write $97 = ab$ where $1 < a < b < n$. By the previous question, we know that

$$a \le \sqrt{97} < \sqrt{100} = 10.$$

Therefore a is one of

$$\{2, 3, 4, 5, 6, 7, 8, 9\}.$$

However 97 is not divisible by any of these numbers, which is a contradiction. Therefore, 97 is a prime number.

16 a Let $m = 4n + r$ where $r = 0, 1, 2, 3$.
($r = 0$) We have,

$$\begin{aligned} m^2 &= (4n)^2 \\ &= 16n^2 \\ &= 4(4n^2) \end{aligned}$$

is divisible by 4.
($r = 1$) We have,

$$\begin{aligned} m^2 &= (4n+1)^2 \\ &= 16n^2 + 8n + 1 \\ &= 4(4n^2 + 2n) + 1 \end{aligned}$$

has a remainder of 1.
($r = 2$) We have,

$$\begin{aligned} m^2 &= (4n+2)^2 \\ &= 16n^2 + 16n + 4 \\ &= 4(4n^2 + 4n + 1) \end{aligned}$$

is divisible by 4.
($r = 3$) We have,

$$\begin{aligned} m^2 &= (4n+3)^2 \\ &= 16n^2 + 24n + 9 \\ &= 16n^2 + 24n + 8 + 1 \\ &= 4(4n^2 + 6n + 2) + 1 \end{aligned}$$

has a remainder of 1.

Therefore, the square of every integer is divisible by 4 or leaves a remainder of 1.

b Suppose the contrary, so that both a and b are odd. Then $a = 2k + 1$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k+1)^2 + (2m+1)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 = 4m + 1 \\ &= 4(k^2 + m^2 + k + m) + 2. \end{aligned}$$

This means that c^2 leaves a remainder of 2 when divided by 4, which is a contradiction.

17 a Suppose by way of contradiction either $a \ne c$ or $b \ne d$. Then clearly both $a \ne c$ and $b \ne d$. Therefore,

$$\begin{aligned} a + b\sqrt{2} &= c + d\sqrt{2} \\ (b - d)\sqrt{2} &= c - a \\ \sqrt{2} &= \frac{c-a}{b-d} \end{aligned}$$

Since $\frac{c-a}{b-d} \in \mathbb{Q}$, this contradicts the irrationality of $\sqrt{2}$.

b Squaring both sides gives,

$$3 + 2\sqrt{2} = \left(c + d\sqrt{2}\right)^2$$
$$3 + 2\sqrt{2} = c^2 + 2cd\sqrt{2} + 2d^2$$
$$3 + 2\sqrt{2} = c^2 + 2d^2 + 2cd\sqrt{2}$$

Therefore

$$c^2 + 2d^2 = 3 \qquad (1)$$
$$cd = 1 \qquad (2)$$

Since c and d are integers, this implies that $c = d = 1$.

18 There are many ways to prove this result. We will take the most elementary approach (but not the most elegant). Suppose that

$$ax^2 + bx + c = 0 \qquad (1)$$

has a rational solution, $x = \frac{p}{q}$. We can assume that p and q have no factors in common (or else we could cancel).

Equation (1) then becomes

$$ax^2 + bx + c = 0$$
$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0$$
$$ap^2 + bpq + cq^2 = 0 \qquad (2)$$

Since p and q cannot both be even, we need only consider three cases.
(Case 1) If p is odd and q is odd then equation (2) is of the form

$$\text{odd} + \text{odd} + \text{odd} = \text{odd} = 0.$$

This is not possible since 0 is even.
(Case 2) If p is odd and q is even then equation (2) is of the form

$$\text{odd} + \text{even} + \text{even} = \text{odd} = 0.$$

This is not possible since 0 is even.
(Case 3) If p is even and q is odd then equation (2) is of the form

$$\text{even} + \text{even} + \text{odd} = \text{odd} = 0.$$

This is not possible since 0 is even.

Solutions to Exercise 8D

1 **a** Converse: If $x = 1$, then $2x + 3 = 5$.
Proof: If $x = 1$ then

$$2x + 3 = 2 \times 1 + 3 = 5.$$

b Converse: If $n - 3$ is even, then n is odd.
Proof: If $n - 3$ is even then $n - 3 = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$n = 2k + 3 = 2k + 2 + 1 = 2(k + 1) + 1$$

is odd.

c Converse: If m is odd, then $m^2 + 2m + 1$ is even.
Proof 1: If m is odd then the expression $m^2 + 2m + 1$ is of the form,

$$\text{odd} + \text{even} + \text{odd} = \text{even}.$$

Proof 2: If m is odd then $m = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} &m^2 + 2m + 1 \\ &= (2k + 1)^2 + 2(k + 1) + 1 \\ &= 4k^2 + 4k + 1 + 2k + 2 + 1 \\ &= 4k^2 + 6k + 3 \\ &= 4k^2 + 6k + 2 + 1 \\ &= 2(2k^2 + 3k + 1) + 1, \end{aligned}$$

is clearly odd.

d Converse: If n is divisible by 5, then n^2 is divisible by 5.
Proof: If n is divisible by 5 then $n = 5k$ for some $k \in \mathbb{Z}$. Therefore,

$$n^2 = (5k)^2 = 25k^2 = 5(5k^2),$$

which is divisible by 5.

2 **a** Converse: If mn is a multiple of 4, then m and n are even.

b This statement is not true. For instance, 4×1 is a multiple of 4, and yet 1 is clearly not even.

3 **a** These statements are not equivalent.
($P \Rightarrow Q$) If Vivian is in China then she is in Asia, since Asia is a country in China.
($Q \not\Rightarrow P$) If Vivian is in Asia, she is not necessarily in China. For example, she could be in Japan.

b These statements are equivalent.
($P \Rightarrow Q$) If $2x = 4$, then dividing both sides by 2 gives $x = 2$.
($Q \Rightarrow P$) If $x = 2$, then multiplying both sides by 2 gives $2x = 4$.

c These statements are not equivalent.
($P \Rightarrow Q$) If $x > 0$ and $y > 0$ then $xy > 0$ since the product of two positive numbers is positive.
($Q \not\Rightarrow P$) If $xy > 0$, then it may not be true that $x > 0$ and $y > 0$. For example, $(-1) \times (-1) > 0$, however $-1 < 0$.

d These statements are equivalent.
($P \Rightarrow Q$) If m or n are even then mn will be even.
($Q \Rightarrow P$) If mn is even then either m or n are even since otherwise the product of two odds numbers would give an odd number.

4 ($\Rightarrow$) If $n + 1$ is odd then, $n + 1 = 2k + 1$, where $k \in \mathbb{Z}$. Therefore,

$$n+2 = 2k+2$$
$$= 2(k+1),$$

so that $n+2$ is even.
($\Leftarrow$) If $n+2$ is even then, $n+2 = 2k$, where $k \in \mathbb{Z}$. Therefore,

$$n+1 = 2k-1$$
$$= 2k-2+1$$
$$= 2(k-1)+1$$

so that $n+1$ is odd.

5 ($\Rightarrow$) Suppose that $n^2 - 4$ is prime. Since

$$n^2 - 4 = (n-2)(n+2)$$

expresses $n^2 - 4$ as the product of two numbers, either $n-2 = 1$ or $n+2 = 1$. Therefore, $n = 3$ or $n = -1$. However, n must be positive, so $n = 3$.
($\Leftarrow$) If $n = 3$ then

$$n^2 - 4 = 3^2 - 4 = 5$$

is prime.

6 ($\Rightarrow$) We prove this statement in the contrapositive. Suppose n is not even. Then $n = 2k+1$ where $k \in \mathbb{Z}$. Therefore,

$$n^3 = (2k+1)^3$$
$$= 8k^3 + 12k^2 + 6k + 1$$
$$= 2(4k^4 + 6k^2 + 3k) + 1$$

is odd.

($\Leftarrow$) If n is even then $n = 2k$. Therefore,

$$n^3 = (2k)^3$$
$$= 8k^3$$
$$= 2(4k^3)$$

is even.

7 ($\Rightarrow$) Suppose that n is odd. Then $n = 2m+1$, for some $m \in \mathbb{Z}$. Now either m is even or m is odd. If m is even, then $m = 2k$ so that

$$n = 2m+1$$
$$= 2(2k)+1$$
$$= 4k+1.$$

as required. If m is odd then $m = 2q+1$ so that

$$n = 2m+1$$
$$= 2(2q+1)+1$$
$$= 4q+3$$
$$= 4q+4-1$$
$$= 4(q+1)-1$$
$$= 4k-1, \text{ where } k = q+1,$$

as required.
($\Leftarrow$) If $n = 4k \pm 1$ then either $n = 4k+1$ or $n = 4k-1$. If $n = 4k+1$, then

$$n = 4k+1$$
$$= 2(2k)+1$$
$$= 2m+1, \text{ where } m = 2k,$$

is odd, as required. Likewise, if

$n = 4k - 1$, then

$$\begin{aligned} n &= 4k - 1 \\ &= 4k - 2 + 1 \\ &= 2(2k - 1) + 1 \\ &= 2m + 1, \text{ where } m = 2k - 1, \end{aligned}$$

is odd, as required.

8 (⇒) Suppose that,

$$\begin{aligned} (x + y)^2 &= x^2 + y^2 \\ x^2 + 2xy + y^2 &= x^2 + y^2 \\ 2xy &= 0 \\ xy &= 0 \end{aligned}$$

Therefore, $x = 0$ or $y = 0$.
(⇐) Suppose that $x = 0$ or $y = 0$. We can assume that $x = 0$. Then

$$\begin{aligned} (x + y)^2 &= (0 + y)^2 \\ &= y^2 \\ &= 0^2 + y^2 \\ &= x^2 + y^2, \end{aligned}$$

as required.

9 **a** Expanding gives

$$\begin{aligned} &(m - n)(m^2 + mn + n^2) \\ &= m^3 + m^2n + mn^2 - m^2n - mn^2 - n^3 \\ &= m^3 - n^3. \end{aligned}$$

b (⇐) We will prove this in the contrapositive. Suppose that $m - n$ were odd. Then either m is odd and n is even or visa versa.
Case 1 - If m is odd and n is even
The expression $m^2 + mn + n^2$ is of the form,

$$\text{odd} + \text{even} + \text{even} = \text{odd}.$$

Case 2 - m is even and n is odd
The expression $m^2 + mn + n^2$ is of the form,

$$\text{even} + \text{even} + \text{odd} = \text{odd}.$$

In both instances, the expression $m^2 + mn + n^2$ is odd. Therefore,

$$m^3 - n^3 = (m - n)(m^2 + mn + n^2)$$

is the product of two odd numbers, and will therefore be odd.

10 We first note that any integer n can be written in the form $n = 100x + y$ where $x, y \in \mathbb{Z}$ and y is the number formed by the last two digits. For example, $1234 = 100 \times 12 + 34$. Then

$$\begin{aligned} &n \text{ is divisible by } 4 \\ \Leftrightarrow\ &n = 100x + y = 4k, \text{ for some } k \in \mathbb{Z} \\ \Leftrightarrow\ &y = 4k - 100x \\ \Leftrightarrow\ &y = 4(k - 25x) \\ \Leftrightarrow\ &y \text{ is divisible by } 4. \end{aligned}$$

Solutions to Exercise 8E

1 a If we let $n = 31$ it is clear that

$$2n^2 - 4n + 31 = 2 \times 31^2 - 4 \times 31 + 31$$

is divisible by 31 and so cannot be prime.

b Let $x = 1$ and $y = -1$ so that

$$(x + y)^2 = (1 + (-1))^2 = 0,$$

while,

$$x^2 + y^2 = 1^2 + (-1)^2 = 1 + 1 = 2,$$

c If $x = \frac{1}{2}$, then,

$$x^2 = \frac{1}{4} < \frac{1}{2} = x.$$

d If $n = 3$ then,

$$n^3 - n = 27 - 3 = 24$$

is even, although 3 is not.

e If $m = n = 1$ then $m + n = 2$ while $mn = 1$.

f Since 6 divides $2 \times 3 = 6$ but 6 does not divide 2 or 3, the statement is false.

2 a Negation: For all $n \in \mathbb{N}$, the number $9n^2 - 1$ is not a prime number.
Proof: Since

$$9n^2 - 1 = (3n - 1)(3n + 1),$$

and since each factor is greater than 1, the number $9n^2 - 1$ is not a prime number.

b Negation: For all $n \in \mathbb{N}$, the number $n^2 + 5n + 6$ is not a prime number.
Since

$$n^2 + 5n + 6 = (n + 2)(n + 3),$$

and since each factor is greater than 1, the number $9n^2 + 5n + 6$ is not a prime number.

c Negation: For all $x \in \mathbb{R}$, we have $2 + x^2 \neq 1 - x^2$
Proof: Suppose that $2 + x^2 = 1 - x^2$.
Rearranging the equation gives,

$$2 + x^2 = 1 - x^2$$
$$2x^2 = -1$$
$$x^2 = -\frac{1}{2},$$

which is impossible since $x^2 \geq 0$.

3 a Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then clearly each of a and b are irrational, although $ab = 2$ is not.

b Let $a = \sqrt{2}$ and $b = -\sqrt{2}$. Then clearly each of a and b are irrational, although $a + b = 0$ is not.

c Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then clearly each of a and b are irrational, although $\frac{a}{b} = 1$ is not.

4 a If a is divisible by 4 then $a = 4k$ for some $k \in \mathbb{Z}$. Therefore,

$$a^2 = (4k)^2 = 16k^2 = 4(4k^2)$$

is divisible by 4.

b Converse: If a^2 is divisible by 4 then a is divisible by 4.
This is clearly not true, since $2^2 = 4$ is divisible by 4, although 2 is not.

5 a If $a - b$ is divisible by 3 then

$a - b = 3k$ for some $k \in \mathbb{Z}$. Therefore,

$$a^2 - b^2 = (a - b)(a + b) = 3k(a + b)$$

is divisible by 3.

b Converse: If $a^2 - b^2$ is divisible by 3 then $a - b$ is divisible by 3.
The converse is not true, since $2^2 - 1^2 = 3$ is divisible by 3, although $2 - 1 = 1$ is not.

6 a This statement is not true since for all $a, b \in \mathbb{R}$,

$$a^2 - 2ab + b^2 = (a - b)^2 \geq 0 > -1.$$

b This statement is not true since for all $x \in \mathbb{R}$, we have,

$$\begin{aligned} & x^2 - 4x + 5 \\ =& x^2 - 4x + 4 - 4 + 5 \\ =& (x - 2)^2 + 1 \\ \geq& 1 \\ >& \frac{3}{4}. \end{aligned}$$

7 a The numbers can be paired as follows:

$$\begin{aligned} 16 + 9 &= 25, & 15 + 10 &= 25 \\ 14 + 11 &= 25, & 13 + 12 &= 25 \\ 1 + 8 &= 9, & 2 + 7 &= 9, \\ 4 + 5 &= 9, & 3 + 6 &= 9. \end{aligned}$$

b We now list each number, in descending order, with each of its potential pairs.

12	4
11	5
10	6
9	7
8	1
7	2, 9
6	3, 10
5	4
4	5
3	1, 6
2	7
1	3, 8

Notice that the numbers 2 and 9 must be paired with 7. Therefore, one cannot pair all numbers in the required fashion.

8 If we let $x = c$, then

$$f(c) = ac^2 + bc + c = c(ac + b + 1)$$

is divisible by $c \geq 2$.

Solutions to Exercise 8F

1 a $\boxed{P(n)}$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1$$

and

$$\text{RHS} = \frac{1(1+1)}{2} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$=1 + 2 + \cdots + k + (k+1)$

$=\dfrac{k(k+1)}{2} + (k+1) \quad$ (by (1))

$=\dfrac{k(k+1)}{2} + \dfrac{2(k+1)}{2}$

$=\dfrac{k(k+1) + 2(k+1)}{2}$

$=\dfrac{(k+1)(k+2)}{2}$

$=\dfrac{(k+1)((k+1)+1)}{2}$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 + x$$

and

$$\text{RHS} = \frac{(1 - x^2)}{1 - x} = \frac{(1-x)(1+x)}{1-x} = 1 + x.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$=1 + x + x^2 + \cdots + x^k + x^{k+1}$

$=\dfrac{1 - x^{k+1}}{1 - x} + x^{k+1} \quad$ (by (1))

$=\dfrac{1 - x^{k+1}}{1 - x} + \dfrac{x^{k+1}(1 - x)}{1 - x}$

$=\dfrac{1 - x^{k+1} + x^{k+1}(1 - x)}{1 - x}$

$=\dfrac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1 - x}$

$=\dfrac{1 - x^{k+2}}{1 - x}$

$=\dfrac{1 - x^{(k+1)+1}}{1 - x}$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^2 - 1$$

and

$$\text{RHS} = \frac{1(1+1)(2+1)}{6} = 1.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \quad (1)$$

$P(k+1)$

$$
\begin{aligned}
&\text{LHS of } P(k+1)\\
&= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2\\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by (1))}\\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}\\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}\\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}\\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}\\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6}\\
&= \frac{(k+1)(k+2)(2k+3)}{6}\\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\\
&= \text{RHS of } P(k+1)
\end{aligned}
$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $P(n)$

$$1 \cdot 2 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

$P(1)$

If $n = 1$ then

$$\text{LHS} = 1 \times 2 = 2$$

and

$$\text{RHS} = \frac{1 \times 2 \times 3}{3} = 2.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$1 \cdot 2 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}. \quad (1)$$

$P(k+1)$

$$
\begin{aligned}
&\text{LHS of } P(k+1)\\
&= 1 \cdot 2 + \cdots + k \cdot (k+1) + (k+1) \cdot (k+2)\\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{(by (1))}\\
&= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}\\
&= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}\\
&= \frac{(k+1)(k+2)(k+3)}{3}\\
&= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}\\
&= \text{RHS of } P(k+1)
\end{aligned}
$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

e $P(n)$

$$\frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

$P(1)$

If $n = 1$ then

$$\text{LHS} = \frac{1}{1 \times 3} = \frac{1}{3}$$

and

$$\text{RHS} = \frac{1}{2 \times 1 + 1} = \frac{1}{3}.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}. \quad (1)$$

$P(k+1)$

LHS of $P(k+1)$

$$=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \quad \text{(by (1))}$$

$$=\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$=\frac{k+1}{(2k+3)}$$

$$=\frac{k+1}{(2(k+1)+1)}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f $P(n)$

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

$P(2)$

If $n=2$ then

$$\text{LHS}=1-\frac{1}{2^2}=\frac{3}{4}$$

and

$$\text{RHS}=\frac{2+1}{2\times 2}=\frac{3}{4}.$$

Therefore $P(2)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

$P(k+1)$

LHS of $P(k+1)$

$$=\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \quad \text{(by (1))}$$

$$=\frac{k+1}{2k}\left(\frac{(k+1)^2}{(k+1)^2}-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\frac{(k+1)(k^2+2k)}{2k(k+1)^2}$$

$$=\frac{k(k+1)(k+2)}{2k(k+1)^2}$$

$$=\frac{(k+2)}{2(k+1)}$$

$$=\frac{(k+1)+1}{2(k+1)}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 a $P(n)$

11^n-1 is divisible by 10

$P(1)$

If $n=1$ then

$$11^1-1=11-1=10$$

is divisible by 10. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$11^k - 1 = 10m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned} 11^{k+1} - 1 &= 11 \times 11^k - 1 \\ &= 11 \times (10m + 1) - 1 \quad \text{(by (1))} \\ &= 110m + 11 - 1 \\ &= 110m + 10 \\ &= 10(11m + 1) \end{aligned}$$

is divisible by 10. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $P(n)$

$3^{2n} + 7$ is divisible by 8

$P(1)$

If $n = 1$ then

$$3^{2\times1} + 7 = 9 + 7 = 16 = 2 \times 8$$

is divisible by 8. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$3^{2k} + 7 = 8m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned} 3^{2(k+1)} + 7 &= 3^{2k+2} + 7 \\ &= 3^{2k} \times 3^2 + 7 \\ &= (8m - 7) \times 9 + 7 \quad \text{(by (1))} \\ &= 72m - 63 + 7 \\ &= 72m - 56 \\ &= 8(9m - 7) \end{aligned}$$

is divisible by 8. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $P(n)$

$7^n - 3^n$ is divisible by 4

$P(1)$

If $n = 1$ then

$$7^1 - 3^1 = 7 - 3 = 4$$

is divisible by 4. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$7^k - 3^k = 4m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned} &7^{k+1} - 3^{k+1} \\ &= 7 \times 7^k - 3^{k+1} \\ &= 7 \times (4m + 3^k) - 3 \times 3^k \quad \text{(by (1))} \\ &= 28m + 7 \times 3^k - 3 \times 3^k \\ &= 28m + 4 \times 3^k \\ &= 4(7m + 3^k) \end{aligned}$$

is divisible by 4. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $P(n)$

$5^n + 6 \times 7^n + 1$ is divisible by 4

$P(1)$

If $n = 1$ then

$$5^1 + 6 \times 7^1 + 1 = 48 = 4 \times 12$$

is divisible by 4. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$5^k + 6 \times 7^k + 1 = 4m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned} &5^{k+1} + 6 \times 7^{k+1} + 1 \\ &= 5 \times 5^k + 6 \times 7 \times 7^k + 1 \\ &= 5 \times (4m - 6 \times 7^k - 1) + 42 \times 7^{k+1} \\ &= 20m - 30 \times 7^k - 5 + 42 \times 7^k + 1 \\ &= 20m + 12 \times 7^k - 4 \\ &= 4(5m + 3 \times 7^k - 1) \end{aligned}$$

is divisible by 4. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

3 a $P(n)$

$4^n > 10 \times 2^n$ where $n \geq 4$

$P(4)$

If $n = 4$ then

LHS $= 4^4 = 256$ and RHS $= 10 \times 2^4 = 160$.

Since LHS > RHS, $P(4)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$4^k > 10 \times 2^k \text{ where } k \geq 4. \qquad (1)$$

$P(k+1)$

We have to show that

$$4^{k+1} > 10 \times 2^{k+1}.$$

$$\begin{aligned} \text{LHS of } P(k+1) &= 4^{k+1} \\ &= 4 \times 4^k \\ &> 4 \times 10 \times 2^k \quad \text{(by (1))} \\ &= 40 \times 2^k \quad \text{(as } 10 > 2) \\ &= 20 \times 2^{k+1} \\ &> 10 \times 2^{k+1} \\ &= \text{RHS of } P(k+1) \end{aligned}$$

Therefore $P(k + 1)$ is true.

Since $P(5)$ is true and $P(k + 1)$ is true whenever $P(k)$ is true, $P(n)$ is true for all integers $n \geq 4$ by the principle of mathematical induction.

b $P(n)$

$3^n > 5 \times 2^n$ where $n \geq 5$

$P(5)$

If $n = 5$ then

LHS $= 3^5 = 243$ and RHS $= 5 \times 2^5 = 160$.

Since LHS > RHS, $P(5)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$3^k > 5 \times 2^k \text{ where } k \geq 5. \qquad (1)$$

$P(k+1)$

We have to show that

$$3^{k+1} > 5 \times 2^{k+1}.$$

$$\begin{aligned} \text{LHS of } P(k+1) &= 3^{k+1} \\ &= 3 \times 3^k \\ &> 3 \times 5 \times 2^k \quad \text{(by (1))} \\ &= 15 \times 2^k \quad \text{(as } 10 > 2) \\ &> 10 \times 2^k \\ &= 5 \times 2^{k+1} \\ &= \text{RHS of } P(k+1) \end{aligned}$$

Therefore $P(k+1)$ is true.

Since $P(5)$ is true and $P(k+1)$ is true whenever $P(k)$ is true, $P(n)$ is true for all integers $n \geq 5$ by the principle of mathematical induction.

c $P(n)$

$2^n > 2n$ where $n \geq 3$

$P(3)$

If $n = 3$ then

$$\text{LHS} = 2^3 = 8 \text{ and RHS} = 2 \times 3 = 6.$$

Since LHS > RHS, $P(3)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$2^k > 2k \text{ where } k \geq 3. \qquad (1)$$

$P(k+1)$

We have to show that

$$2^{k+1} > 2(k+1).$$

$$\begin{aligned}\text{LHS of } P(k+1) &= 2^{k+1}\\ &= 2 \times 2^k\\ &> 2 \times 2k \quad \text{(by (1))}\\ &= 4k\\ &= 2k + 2k\\ &\geq 2k + 2 \quad (\text{as } 2k \geq 2)\\ &= 2(k+1)\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all integers $n \geq 3$ by the principle of mathematical induction.

d $P(n)$

$n! > 2^n$ where $n \geq 4$

$P(4)$

If $n = 4$ then

$$\text{LHS} = 4! = 24 \text{ and RHS} = 2^4 = 16.$$

Since LHS > RHS, $P(4)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$k! > 2^k \text{ where } k \geq 4. \qquad (1)$$

$P(k+1)$

We have to show that

$$(k+1)! > 2^{k+1}.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= (k+1)!\\ &= (k+1)k!\\ &> (k+1) \times 2^k \quad \text{(by (1))}\\ &> 2 \times 2^k \quad (\text{as } k+1 > 2)\\ &= 2^{k+1}\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all integers $n \geq 4$ by the principle of mathematical induction.

4 a $P(n)$

$a_n = 2^n + 1$

$P(1)$

If $n = 1$ then

$$\text{LHS} = a_1 = 3 \text{ and RHS} = 2^1 + 1 = 3.$$

Since LHS = RHS, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$a_k = 2^k + 1. \qquad (1)$$

$P(k+1)$

We have to show that

$$a^{k+1} = 2^{k+1} + 1.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= a_{k+1} \\ &= 2a_k - 1 \quad \text{(by definition)} \\ &= 2(2^k + 1) - 1 \quad \text{(by (1))} \\ &= 2^{k+1} + 2 - 1 \\ &= 2^{k+1} + 1 \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $P(n)$

$a_n = 5^n - 1$

$P(1)$

If $n = 1$ then

LHS $= a_1 = 4$ and RHS $= 5^1 - 1 = 4$.

Since LHS = RHS, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$a_k = 5^k - 4. \qquad (1)$$

$P(k+1)$

We have to show that

$$a^{k+1} = 5^{k+1} - 4.$$

$$\begin{aligned}\text{LHS} &= a_{k+1} \\ &= 5a_k + 4 \quad \text{(by definition)} \\ &= 5(5^k - 1) + 4 \quad \text{(by (1))} \\ &= 5^{k+1} - 5 + 4 \\ &= 5^{k+1} - 1 \\ &= \text{RHS}\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $P(n)$

$a_n = 2^n + n$

$P(1)$

If $n = 1$ then

LHS $= a_1 = 3$ and RHS $= 2^1 + 1 = 3$.

Since LHS = RHS, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$a_k = 2^k + k. \qquad (1)$$

$P(k+1)$

We have to show that

$$a^{k+1} = 2^{k+1} + k + 1.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= a_{k+1} \\ &= 2a_k - k + 1 \quad \text{(by definition)} \\ &= 2(2^k + k) - k + 1 \quad \text{(by (1))} \\ &= 2^{k+1} + 2k - k + 1 \\ &= 2^{k+1} + k + 1 \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

5 $P(n)$

3^n is odd where $n \in \mathbb{N}$

$P(1)$

If $n = 1$ then clearly

$$3^1 = 3$$

is odd. Therefore, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$3^k = 2m + 1 \qquad (1)$$

for some $m \in \mathbb{Z}$.

$P(k+1)$

$$3^{k+1} = 3 \times 3^k$$
$$= 3 \times (2m + 1) \quad \text{(by (1))}$$
$$= 6m + 3$$
$$= 6m + 2 + 1$$
$$= 2(3m + 1) + 1$$

is odd, so that $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

6 a $\boxed{P(n)}$

$n^2 - n$ is even, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$1^2 - \times 1 = 0$$

is even. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^2 - k$ is even. Therefore,

$$k^2 - k = 2m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$(k + 1)^2 - (k + 1)$$
$$= k^2 + 2k + 1 - k - 1$$
$$= k^2 + k$$
$$= (k^2 - k) + 2k$$
$$= 2m + 2k \quad \text{(by (1))}$$
$$= 2(m + k)$$

Since this is even, $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^2 - n = n(n - 1).$$

As this is the product of two consecutive numbers, one of them must be even, so that the product will also be even.

7 a $\boxed{P(n)}$

$n^3 - n$ is divisible by 3, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$1^3 - 1 = 0$$

is divisible by 3. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^3 - k$ is divisible by 3. Therefore,

$$k^3 - k = 3m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

We have to show that $(k+1)^3 - (k+1)$ is divisible by 3.

$$(k + 1)^3 - (k + 1)$$
$$= k^3 + 3k^2 + 3k + 1 - k - 1$$
$$= k^3 - k + 3k^2 + 3k$$
$$= (k^3 - k) + 3k^2 + 3k$$
$$= 3m + 3k^2 + 3k \quad \text{(by (1))}$$
$$= 3(m + k^2 + k)$$

Since this is divisible by 3, $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^3 + n = n(n^2 - 1) = n(n - 1)(n + 1).$$

As this is the product of three consecutive numbers, one of them must be divisible by 3, so that the product will also be divisible by 3.

8 **a**

n	1	2	3	4	5
a_n	9	99	999	9999	99999

b We claim that $a_n = 10^n - 1$.

c $\boxed{P(n)}$

$a_n = 10^n - 1$

$\boxed{P(1)}$

If $n = 1$, then

LHS $= a_1 = 9$ and RHS $= 10^1 - 1 = 9$.

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 10^k - 1. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$a^{k+1} = 10^{k+1} - 1.$$

$$\begin{aligned} \text{LHS} &= a_{k+1} \\ &= 10a_k + 9 \quad \text{(by definition)} \\ &= 10(10^k - 1) + 9 \quad \text{(by (1))} \\ &= 10^{k+1} - 10 + 9 \\ &= 10^{k+1} - 1 \\ &= \text{RHS} \end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

9 **a**

n	1	2	3	4	5	6	7	8	9	10
f_n	1	1	2	3	5	8	13	21	34	55

b $\boxed{P(n)}$

$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = f_1 = 1$$

and

$$\text{RHS} = f_3 - 1 = 2 - 1 = 1.$$

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$f_1 + f_2 + \cdots + f_k = f_{k+2} - 1. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned} \text{LHS of } P(k+1) &= f_1 + f_2 + \cdots + f_k + f_{k+1} \\ &= f_{k+2} - 1 + f_{k+1} \quad \text{(by (1))} \\ &= f_{k+1} + f_{k+2} - 1 \\ &= f_{k+3} - 1 \quad \text{(by definition)} \\ &= f_{(k+1)+2} - 1 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $f_1 = 1$

$f_1 + f_3 = 1 + 2 = 3$

$f_1 + f_3 + f_5 = 3 + 5 = 8$

$f_1 + f_3 + f_5 + f_7 = 8 + 13 = 21$

d From the pattern observed above, we claim that

$$f_1 + f_3 + ... + f_{2n-1} = f_{2n}.$$

e $\boxed{P(n)}$

$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = f_1 = 1$$

and

$$\text{RHS} = f_2 - 1 = 2 - 1 = 1.$$

Since LHS = RHS, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}. \quad (1)$$

$P(k+1)$

$$\begin{aligned} \text{LHS} &= f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} \\ &= f_{2k} + f_{2k+1} \quad \text{(by (1))} \\ &= f_{2k+2} \quad \text{(by definition)} \\ &= f_{2(k+1)} \\ &= \text{RHS} \end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f $P(n)$

The Fibonacci number f_{3n} is even.

$P(1)$

If $n = 1$ then

$$f_3 = 2$$

is even, therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that f_{3k} is even. That is,

$$f_{3k} = 2m \quad (1)$$

for some $m \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned} f_{3(k+1)} &= f_{3k+3} \\ &= f_{3k+2} + f_{3k+1} \quad \text{(by definition)} \\ &= f_{3k+1} + f_{3k} + f_{3k+1} \\ &= 2f_{3k+1} + f_{3k} \\ &= 2f_{3k+1} + 2m \quad \text{(by (1))} \\ &= 2(f_{3k+1} + m) \end{aligned}$$

Since this is even, $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

10 $P(n)$

Since we're only interested in odd numbers our proposition is:
$4^{2n-1} + 5^{2n-1}$ is divisible by 9, where $n \in \mathbb{N}$.

$P(1)$

If $n = 1$ then

$$4^1 + 5^1 = 9$$

is divisible by 9. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$4^{2k-1} + 5^{2k-1} = 9m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$P(k+1)$

The next odd number will be $2k + 1$. Therefore, we have to prove that

$$4^{2k+1} + 5^{2k+1}$$

is divisible by 9.

$$4^{2k+1} + 5^{2k+1}$$
$$= 4^2 \times 4^{2k-1} + 5^2 \times 5^{2k-1}$$
$$= 16 \times (9m - 5^{2k-1}) + 25 \times 5^{2k-1} \quad \text{(by (1))}$$
$$= 144m - 16 \times 5^{2k-1} + 25 \times 5^{2k-1}$$
$$= 144m + 9 \times 5^{2k-1}$$
$$= 9(16 + 5^{2k-1})$$

Since this is divisible by 9, we've shown that $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

11 $P(n)$

A set of numbers S with n numbers has a largest element.

$P(1)$

If $n = 1$, then set S has just one element. This single element is clearly the largest element in the set.

$P(k)$

Assume that $P(k)$ is true. This means that a set of numbers S with k numbers has a largest element.

$P(k + 1)$

Suppose set S has $k + 1$ numbers. Remove one of the elements, say x, so that we now have a set with k numbers. The reduced set has a largest element, y. Put x back in set S, so that its largest element will be the larger of x and y. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

12 $P(n)$

It is possible to walk around a circle whose circumference includes n friends and n enemies (in any order) without going into debt.

$P(1)$

If $n = 1$, there is one friend and one enemy on the circumference of a circle. Start your journey at the friend, receive \$1, then walk around to the enemy and lose \$1. At no point will you be in debt, so $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true. This means that it is possible to walk around a circle with k friends and k enemies (in any order) without going into debt, provided you start at the correct point.

$P(k + 1)$

Suppose there are $k + 1$ friends and $k + 1$ enemies located on the circumference of the circle, in any order. Select a friend whose next neighbour is an enemy (going clockwise), and remove these two people. As there are now k friends and k enemies, it is possible to walk around the circle without going into debt, provided you start at the correct point. Now reintroduce the two people, and start walking from the same point. For every part of the journey you'll have the same amount of money as before except when you meet the added friend, who gives you \$1, which is immediately lost to the added enemy.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

13 $P(n)$

Every integer j such that $2 \leq j \leq n$ is divisible by some prime.

$P(2)$

If $n = 2$, then $j = 2$ is clearly divisible by a prime, namely itself. Therefore $P(2)$ is true.

$P(k)$

Assume that $P(k)$ is true. Therefore, every integer j such that $2 \le j \le k$ is divisible by some prime.

$P(k + 1)$

We need to show that integer j such that $2 \le j \le k + 1$ is divisible by some prime. By the induction assumption, we already know that every j with $2 \le j \le k$ is divisible by some prime. We need only prove that $k + 1$ is divisible by a prime. If $k + 1$ is a prime number, then we are finished. Otherwise we can find integers a and b such that $k + 1 = ab$ and $2 \le a \le k$ and $2 \le b \le k$. By the induction assumption, the number a will be divisible by some prime number. Therefore $k + 1$ is divisible by some prime number.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

14 If such a colouring of the regions is possible we will call it a **satisfactory colouring**.

$P(n)$

If n lines are drawn then the resulting regions have a satisfactory colouring.

$P(1)$

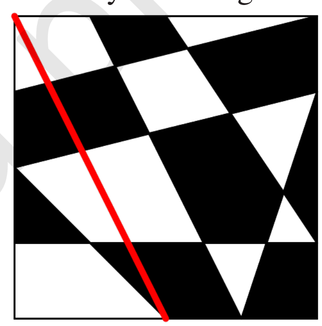
If $n = 1$, then there is just one line. We colour one side black and one side white. This is a satisfactory colouring. Therefore $P(1)$ is true.

$P(k)$

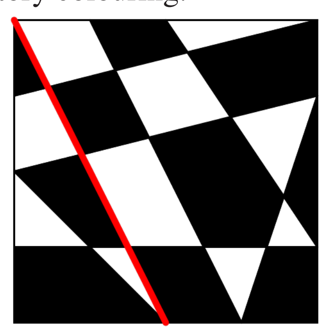
Assume that $P(k)$ is true. This means that we can obtain a satisfactory colouring if there are k lines drawn.

$P(k + 1)$

Now suppose that there are $k + 1$ lines drawn. Select one of the lines, and remove it. There are now k lines, and the resulting regions have a satisfactory colouring since we assumed $P(k)$ is true. Now add the removed line. This will divide some regions into into two new regions with the same colour, so this is not a satisfactory colouring.



However, if we switch each colour on **one** side of the line we obtain a satisfactory colouring.



This is because inverting a satisfactory colouring will always give a satisfactory colouring, and regions separated the new line will not have the same colour.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to Technology-free questions

1 a Let the 3 consecutive integers be $n, n+1$ and $n+2$. Then,

$$\begin{aligned} n + (n+1) + (n+2) &= 3n + 3 \\ &= 3(n+1) \end{aligned}$$

is divisible by 3.

b This statement is not true. For example, $1 + 2 + 3 + 4 = 10$ is not divisible by 4

2 (Method 1) If n is even then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 - 3n + 1 &= (2k)^2 - 2(2k) + 1 \\ &= 4k^2 - 4k + 1 \\ &= 2(2k^2 - 2k) + 1 \end{aligned}$$

is odd.

(Method 2) If n is even then $n^2 - 3n + 1$ is of the form

even − even + odd = odd.

3 a (Contrapositive) If n is not even, then n^3 is not even. (Alternative) If n is odd, then n^3 is odd.

b If n is odd then $n = 2k + 1$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

is odd.

c This will be a proof by contradiction. Suppose $\sqrt[3]{6}$ is rational so that $\sqrt[3]{6} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$\begin{aligned} & p^3 = 6q^3 \qquad (1) \\ \Rightarrow\ & p^3 \text{ is divisible by } 2 \\ \Rightarrow\ & p \text{ is divisible by } 2 \\ \Rightarrow\ & p = 2k \text{ for some } k \in \mathbb{N} \\ \Rightarrow\ & (2k)^3 = 6q^3 \text{(substitituting into (1))} \\ \Rightarrow\ & 8k^3 = 6q^3 \\ \Rightarrow\ & 4k^2 = 3q^2 \\ \Rightarrow\ & q^2 \text{ is divisible by } 2 \\ \Rightarrow\ & q \text{ is divisible by } 2. \end{aligned}$$

So p and q are both divisible by 2, which contradicts the fact that they have no factors in common.

4 a Suppose n is the first of three consecutive numbers. If n is divisible by 3 then there is nothing to prove. Otherwise, it is of the form $n = 3k + 1$ or $n = 3k + 2$. In the first case,

$$\begin{aligned} n &= 3k + 1 \\ n + 1 &= 3k + 2 \\ n + 2 &= 3k + 3 = 3(k+1) \end{aligned}$$

so that the third number is divisible by 3. In the second case,

$$\begin{aligned} n &= 3k + 2 \\ n + 1 &= 3k + 3 = 3(k+1) \\ n + 2 &= 3k + 4 \end{aligned}$$

so that the second number is divisible by 3.

b The expression can be readily factorised so that

$$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2)$$
$$= n(n+1)(n+2)$$

is the product of 3 consecutive integers. As one of these integers must be divisible by 3, the product must also be divisible by 3.

5 a if m and n are divisible by d then $m = pd$ and $n = qd$ for some $p, q \in \mathbb{Z}$. Therefore,

$$m - n = pd - qd$$
$$= d(p - q)$$

is divisible by d.

b Take any two consecutive numbers n and $n + 1$. If d divides n and $n + 1$ then d must divide $(n + 1) - n = 1$. As the only integer that divides 1 is 1, the highest common factor must be 1, as requried.

c We know that any factor of 1002 and 999 must also divide $1002 - 999 = 3$. As the only factors of 3 are 1 and 3, the highest common factor must be 3.

6 a If $x = 9$ and $y = 16$ then the left hand side equals

$$\sqrt{9+16} = \sqrt{25} = 5$$

while the right hand side equals

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

b $(\Rightarrow)$

$$[t]\ \sqrt{x+y} = \sqrt{x} + \sqrt{y}$$
$$\Rightarrow x + y = \left(\sqrt{x} + \sqrt{y}\right)^2$$
$$\Rightarrow x + y = x + \sqrt{xy} + y$$
$$\Rightarrow 0 = sqrtxy$$
$$\Rightarrow xy = 0$$
$$\Rightarrow x = 0 \text{ or } y = 0$$

$(\Leftarrow)$ Suppose that $x = 0$ or $y = 0$. We can assume that $x = 0$. Then

$$\sqrt{x+y} = \sqrt{y+0}$$
$$= \sqrt{y}$$
$$= \sqrt{y} + \sqrt{0}$$
$$= \sqrt{y} + \sqrt{x},$$

as required.

7 (Case 1) If n is even then the expression is of the form

even + even + even = even.

(Case 1) If n is odd then the expression is of the form

odd + odd + even = even.

8 a If $a = b = c = d = 1$ then the left hand side equals

$$\frac{1}{1} + \frac{1}{1} = 2$$

while the right hand side equals

$$\frac{1+1}{1+1} = 1.$$

b first note that if $\frac{c}{d} > \frac{a}{b}$ then $bc > ad$.

Therefore,

$$\frac{a+c}{b+d}-\frac{a}{b}$$
$$=\frac{b(a+c)}{b(b+d)}-\frac{a(b+d)}{b(b+d)}$$
$$=\frac{b(a+c)-a(b+d)}{b(b+d)}$$
$$=\frac{ab+bc-ab-ad}{b(b+d)}$$
$$=\frac{bc-ad}{b(b+d)}$$
$$>0$$

since $bc > ad$. This implies that

$$\frac{a+c}{b+d} > \frac{a}{b}.$$

Similarly, we can show that

$$\frac{a+c}{b+d} < \frac{c}{d}.$$

9 a $P(n)$

$6^n + 4$ is divisible by 10

$P(1)$

If $n = 1$ then

$$6^1 + 4 = 10$$

is divisible by 10. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$6^k + 4 = 10m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned}6^{k+1} + 4 &= 6 \times 6^k + 4\\ &= 6 \times (10m - 4) + 4 \quad \text{(by (1))}\\ &= 60m - 24 + 4\\ &= 60m - 20 \times 3^k\\ &= 10(6m - 2)\end{aligned}$$

is divisible by 10. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $P(n)$

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

$P(1)$

If $n = 1$ then LHS$= 1^2 = 1$and

$$\text{RHS} = \frac{1(2\times 1 - 1)(2 \times 1 + 1)}{3} = 1.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$1^2 + 3^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}. \qquad (1)$$

$P(k+1)$

LHS of $P(k+1)$

$$=1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2$$
$$=\frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \quad \text{(by (1))}$$
$$=\frac{k(2k-1)(2k+1)}{3} + \frac{3(2k+1)^2}{3}$$
$$=\frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$
$$=\frac{(2k+1)(k(2k-1) + 3(2k+1))}{3}$$
$$=\frac{(2k+1)(2k^2 - k + 6k + 3)}{3}$$
$$=\frac{(2k+1)(2k+3)(k+1)}{3}$$
$$=\frac{(k+1)(2k+1)(2k+3)}{3}$$
$$=\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to multiple-choice questions

1 E The expression $m - 3n$ is of the form

$$\text{even} - \text{odd} = \text{odd}.$$

2 E If m is divisible by 6 and n is divisible by 15 then $m = 6p$ and $n = 15q$ for $p, q \in \mathbb{Z}$. Therefore,

$$m \times n = 90pq$$

$$m + n = 6p + 15q = 3(2p + 5q)$$

From these two expressions, it should be clear that A,B,C and D are true, while E might be false. For example, if $m = 6$ and $n = 15$ then $m + n = 21$ is not divisible by 15.

3 C We obtain the contrapositive by switching P and Q and negating both. Therefore, the contrapositive will be

$$\text{not } Q \Rightarrow \text{not } P$$

.

4 B We obtain the converse by switching P and Q. Therefore, the converse will be

$$Q \Rightarrow P$$

.

5 C If $m + n = mn$ then

$$n = mn - m$$

$$n = m(n - 1)$$

This means that n is divisible by $n - 1$, which is only possible if $n = 2$ or $n = 0$. If $n = 0$, then $m = 0$. If $n = 2$, then $m = 2$. Therefore there are only two solutions, $(0, 0)$ and $(2, 2)$.

6 D The only statement that is true for all real numbers a, b and c is D. Counterexamples can be found for each of the other expressions, as shown below.

A $\quad \frac{1}{3} < \frac{1}{2}$

B $\quad \frac{1}{2} > \frac{1}{-1}$

C $\quad 3 \times -1 < 2 \times -1$

E $\quad 1^2 < (-2)^2$

7 D As n is the product of 3 consecutive integers, one of which will be divisible by 3 and one of which will be divisible by 2. The product will be then be divisible by $1, 2, 3$ and 6. On the other hand, it won't necessarily be divisible by 5 since $2 \times 3 \times 4$ is not divisible by 5.

8 C Each of the statements is true except the third. In this instance, $1 + 3$ is even, although 1 and 3 are not even.

Solutions to extended-response questions

1 **a** The number of dots can be calculated two ways, either by addition,

$$(1 + 2 + 3 + 4) + (1 + 2 + 3 + 4)$$

or by multiplication,

$$4 \times 5.$$

Equating these two expressions gives,

$$\begin{aligned}(1 + 2 + 3 + 4) + (1 + 2 + 3 + 4) &= 4 \times 5\\ 2(1 + 2 + 3 + 4) &= 4 \times 5\\ 1 + 2 + 3 + 4 &= \frac{4 \times 5}{2}\end{aligned}$$

The argument obviously generalises to more dots, giving equation (1).

b We have,

$$\begin{aligned}1 + 2 \cdots + 99 &= \frac{99 \times 100}{2}\\ &= 99 \times 50,\end{aligned}$$

which is divisible by 99.

c Suppose that m is the first number, so that the n connective numbers are

$$m, m + 1, \ldots, m + n - 1.$$

Then,

$$\begin{aligned}&m + (m + 1) + (m + 2) + \cdots + (m + n - 1)\\ &= n \times m + (1 + 2 + \cdots (n - 1))\\ &= nm + \frac{(n - 1)n}{2}\\ &= n\left(m + \frac{n - 1}{2}\right)\end{aligned}$$

Since n is odd, $n - 1$ is even. This means that $\frac{n-1}{2}$ is an integer. Therefore, the term in brackets is an integer, which means the expression is divisible by n.

d Since

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2},$$

we need to prove the following statement:

$P(n)$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^3 = 1$$

and

$$\text{RHS} = \frac{1^2(1+1)^2}{4} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
&\text{LHS of } P(k+1) \\
&= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{(by (1))} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 **a** The first number is divisible by 2, the second by 3, the third by 4 and so on. As each number has a factor greater than 1, each is a composite number. Therefore this is a sequence of 9 consecutive composite numbers.

b We consider the this sequence of 10 consecutive numbers,

$$11! + 2, 11! + 3, \ldots, 11! + 11.$$

The first number is divisible by 2, the second by 3 and so on. Therefore as each number has a factor greater than 1, each is a composite number.

3 **a** Since (a, b, c) is a Pythagorean triple, we know that $a^2 + b^2 = c^2$. Then (na, nb, nc) is also a Pythagorean triple since,

$$\begin{aligned}(na)^2 + (nb)^2 &= n^2a^2 + n^2b^2\\ &= n^2(a^2 + b^2)\\ &= n^2(c^2)\\ &= (nc)^2,\end{aligned}$$

as required.

b Suppose that $(n, n+1, n+2)$ is a Pythagorean triple. Then

$$\begin{aligned}n^2 + (n+1)^2 &= (n+2)^2\\ n^2 + n^2 + 2n + 1 &= n^2 + 4n + 4\\ n^2 - 2n - 3 &= 0\\ (n-3)(n+1) &= 0\\ n &= 3, -1.\end{aligned}$$

However, since $n > 0$, we obtain only one solution, $n = 3$, which corresponds to the famous $(3, 4, 5)$ triangle.

c Suppose some triple (a, b, c) contained the number 1. Then clearly, 1 will be the smallest number. Therefore, we can suppose that

$$\begin{aligned}1^2 + b^2 &= c^2\\ c^2 - b^2 &= 1\\ (c-b)(c+b) &= 1\end{aligned}$$

Since the only divisor of 1 is 1, we must have

$$c + b = 1$$
$$c - b = 1$$
$$\Rightarrow b = 0 \text{ and } c = 1.$$

This is a contradiction, since b must be a positive integer. Now suppose some triple (a, b, c) contained the number 2. Then 2 will be smallest number. Therefore, we can suppose that

$$2^2 + b^2 = c^2$$
$$c^2 - b^2 = 4$$
$$(c - b)(c + b) = 4$$

Since the only divisors of 4 are 1, 2 and 4, we must have

$$c + b = 4$$
$$c - b = 1$$
$$\Rightarrow b = \frac{3}{2}, c = \frac{5}{2}$$

or

$$c + b = 2$$
$$c - b = 2$$
$$\Rightarrow b = 0, c = 2$$

In both instances, we have a contradiction since b must be a positive integer.

4 **a** (Case 1) If $a = 3k + 1$ then

$$\begin{aligned} a^2 &= (3k + 1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

leaves a remainder of 1 when divided by 3.
(Case 2) If $a = 3k + 2$ then

$$\begin{aligned} a^2 &= (3k + 2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

also leaves a remainder of 1 when divided by 3.

b Suppose by way of contradiction that neither a nor b are divisible by 3. Then using the previous question, each of a^2 and b^2 leave a remainder of 1 when divided by 3. Therefore $a^2 = 3k + 1$ and $b^2 = 3m + 1$, for some $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 3k + 1 + 3m + 1 \\ &= 3(k + m) + 2. \end{aligned}$$

This means that c^2 leaves a remainder of 2 when divided by 3, which is not possible.

5 a $\boxed{P(n)}$

$n^2 + n$ is divisible by 2, where $n \in \mathbb{Z}$.

$\boxed{P(1)}$

If $n = 1$ then $1^2 + 1 = 2$ is divisible by 2. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k^2 + k = 2m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

Letting $n = k + 1$ we have,

$$\begin{aligned}
&(k+1)^2 + (k+1) \\
=&k^2 + 2k + 1 + k + 1 \\
=&k^2 + 3k + 2 \\
=&(k^2 + k) + (2k + 2) \\
=&2m + 2(k+1) \qquad \text{(by (1))} \\
=&2(m + k + 1)
\end{aligned}$$

is divisible by 2. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Since

$$n^2 + n = n(n+1)$$

is the product of two consecutive integers, one of them must be even. Therefore the product will also be even.

c If n is odd, then $n = 2k + 1$. Therefore

$$\begin{aligned}
n^2 - 1 &= (2k+1)^2 - 1 \\
&= 4k^2 + 4k + 1 - 1 \\
&= 4k^2 + 4k \\
&= 4k(k+1) \\
&= 4 \times 2k \qquad \text{(since the product of consecutive integers is even)} \\
&= 8k
\end{aligned}$$

as required.

6 a If n is divisible by 8, then $n = 8k$ for some $k \in \mathbb{Z}$. Therefore

$$n^2 = (8k)^2 = 64k^2 = 8(8k^2)$$

is divisible by 8.

b (Converse) If n^2 is divisible by 8, then n is divisible by 8.

c The converse is not true. For example, $4^2 = 16$ is divisible by 8 however 4 is not divisible by 8.

7 **a** There are many possibilities. For example $3 + 97 = 100$ and $5 + 97 = 102$.

b Suppose 101 could be written as the sum of two prime numbers. Then one of these primes must be 2, since all other pairs of primes have an even sum. Therefore $101 = 2 + 99$, however 99 is not prime.

c There are many possibilities. For example, $7 + 11 + 83 = 101$.

d Consider any odd integer n greater than 5. Then $n - 3$ will be an even number greater than 2. If the Goldbach Conjecture is true, then $n - 3$ is the sum of two primes, say p and q. Then $n = 3 + p + q$, as required.

8 **a** We have,

$$\begin{aligned}\frac{1}{n-1} - \frac{1}{n} &= \frac{n}{n(n-1)} - \frac{n-1}{n(n-1)} \\ &= \frac{n-(n-1)}{n(n-1)} \\ &= \frac{n-n+1}{n(n-1)} \\ &= \frac{1}{n(n-1)}.\end{aligned}$$

b Using the identity developed in the previous question, we have,

$$\begin{aligned}&\frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{n(n+1)} \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n} \\ &= \frac{1}{1} - \frac{1}{n} \\ &= 1 - \frac{1}{n}\end{aligned}$$

as required.

c

d Since $k^2 > k(k-1)$ for all $k \in \mathbb{N}$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{n^2}$$

$$= \frac{1}{1^2} + \left(\frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{n^2}\right)$$

$$< \frac{1}{1^2} + \left(\frac{1}{2 \times 1} + \frac{1}{3 \times 2} \cdots + \frac{1}{n(n-1)}\right)$$

$$= \frac{1}{1^2} + 1 - \frac{1}{n}$$

$$= 2 - \frac{1}{n}$$

$$< 2,$$

as required.

9 **a** We have,

$$\begin{aligned}\frac{x+y}{2} - \sqrt{xy} &= \frac{a^2+b^2}{2} - \sqrt{a^2b^2}\\ &= \frac{a^2+b^2}{2} - ab\\ &= \frac{a^2+b^2}{2} - \frac{2ab}{2}\\ &= \frac{a^2-2ab+b^2}{2}\\ &= \frac{(a-b)^2}{2}\\ &\geq 0.\end{aligned}$$

It is also worth noting that we get equality if and only if $x = y$.

b **i** Using the above inequality, we obtain,

$$\begin{aligned}a + \frac{1}{a} &\geq 2\sqrt{a \cdot \frac{1}{a}}\\ &= 2\sqrt{1}\\ &= 2.\end{aligned}$$

as required.

ii Using the above inequality three times, we obtain,

$$\begin{aligned}(a+b)(b+c)(c+a) &\geq 2\sqrt{ab} \times 2\sqrt{bc} \times 2\sqrt{ca} \\ &= 8(\sqrt{a})^2(\sqrt{b})^2(\sqrt{c})^2 \\ &= 8abc,\end{aligned}$$

as required.

iii This inequality is a little trickier. We have,

$$\begin{aligned}a^2+b^2+c^2 &= \left(\frac{a^2}{2}+\frac{b^2}{2}\right)+\left(\frac{b^2}{2}+\frac{c^2}{2}\right)+\left(\frac{a^2}{2}+\frac{c^2}{2}\right) \\ &= \frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{a^2+c^2}{2} \\ &\geq \sqrt{a^2b^2}+\sqrt{b^2c^2}+\sqrt{a^2c^2} \\ &= ab+bc+ac,\end{aligned}$$

as required.

c If a rectangle has length x and width y then its perimeter will be $2x + 2y$. A square with the same perimeter will have side length,

$$\frac{2x+2y}{4} = \frac{x+y}{2}.$$

Therefore,

$$A(\text{square}) = \left(\frac{x+y}{2}\right)^2 \geq xy = A(\text{rectangle}).$$

10 We show that it is only possible for Kaye to be the liar.

case 1

Suppose Jaye is lying

⇒ Kaye is not lying

⇒ Elle is lying

⇒ There are two liars

⇒ This is impossible.

case 2

Suppose Kaye is lying

⇒ Jaye is not lying and Elle is not lying

⇒ Kaye is the only liar

case 3

Suppose Elle is lying

⇒ Mina is not lying

⇒ Karl is lying

⇒ There are two liars

⇒ This is impossible.

11 First note that the four sentences can be recast as:

- Exactly three of these statements are true.
- Exactly two of these statements are true.
- Exactly one of these statements are true.
- None of these statements are true.

At most one of these statements can be true, or else we obtain a contradiction. If none of the statements is true, then the last statement is true. This means that at least one of the statements is true. This also gives a contradiction. Therefore, only one of the statements is true, that is, the third statement.

12 **a** There is only one possibility,

$\boxed{1, 2, 4, 8}$ $\boxed{3, 5, 6, 7}$

b We know that we can split the numbers $1, 2, \ldots, 8$,

$\boxed{1, 2, 4, 8}$ $\boxed{3, 5, 6, 7}$

Deleting the largest number, 8, will give a splitting of $1, 2, \ldots, 7$.

$\boxed{1, 2, 4}$ $\boxed{3, 5, 6, 7}$

Continuing this process, deleting the 7, will be a splitting of the numbers $1, 2, \ldots, 6$, and so on.

c We first note that if a set can be split then two numbers can't appear in the same group as their difference. To see this, if x and y and $x - y$ all belong to the same group then $(x - y) + y = x$. Let's now try to split the numbers $1, 2, \ldots, 9$. Call the two groups X and Y. We can assume that $1 \in X$. We now consider four cases for the groups containing elements 2 and 9.

(case 1) Suppose $2 \in X$ and $9 \in X$

Reason	X	Y	Reason
(assumed)	1		
(assumed)	2		
(assumed)	9		
		3	$(1, 2 \in X)$
		7	$(2, 9 \in X)$
$(3, 7 \in Y)$	4		
		5	$(1, 4 \in X)$
		6	$(2, 4 \in X)$
$(5, 6 \in Y)$	8		

This doesn't work, since X is forced to contain the numbers $1, 8$ and 9.

(case 2) Suppose $2 \in X$ and $9 \in Y$

Reason	X	Y	Reason
(assumed)	1		
(assumed)	2		
		9	(assumed)
		3	$(1, 2 \in X)$
$(3, 9 \in Y)$	6		
		4	$(2, 6 \in X)$
		5	$(1, 6 \in X)$

This doesn't work, since Y is forced to contain the numbers $4, 5$ and 9.

(case 3) Suppose $2 \in Y$ and $9 \in X$

Reason	X	Y	Reason
(assumed)	1		
		2	(assumed)
(assumed)	9		
		8	$(1, 9 \in X)$
$(2, 8 \in Y)$	6		
		3	$(6, 8 \in X)$
$(2, 8 \in Y)$	5		$(3, 8 \in X)$

This doesn't work, since X is forced to contain the numbers $1, 5$ and 6.

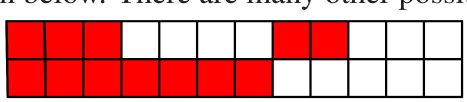
(case 4) Suppose $2 \in Y$ and $9 \in Y$

Reason	X	Y	Reason
(assumed)	1		
		2	(assumed)
		9	(assumed)
$(2, 9 \in Y)$	7		
		6	$(1, 7 \in X)$
$(2, 8 \in Y)$	4		
		3	$(4, 7 \in X)$

This doesn't work, since Y is forced to contain the numbers $3, 6$ and 9.

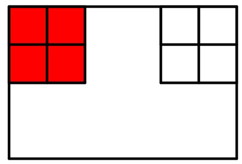
d If the numbers $1, 2, \ldots, n$ could be split, where $n \geq 9$, then we could successively eliminate the largest term to obtain a splitting of the numbers $1, 2, \ldots, 9$. However, we already know that this is impossible.

13 **a** A suitable tiling is shown below. There are many other possibilities.

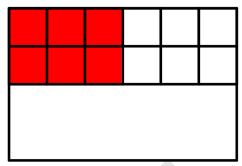


b Tile E must go into a corner. This is because there are only two other tiles (A and

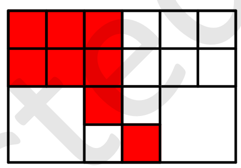
B) that it can go next to. Tile F must also go into a corner. This is because there are only two other tiles (B and C) that it can go next to.
(Case 1) Tile E and tile F are in different rows
Since tile B must go next to both tiles E and F, this is impossible.
(Case 2) Tile E and tile F are in the same row
Assume tile F is in the top left position.
Then tile E goes in the top right position.



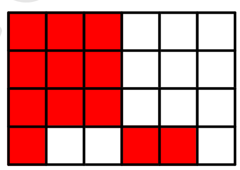
Therefore tile B must go between them.



Tile C must then go beneath tile F and tile A must go beneath tile E. Consequently, tile D must go beneath tile B. Therefore, there is only one valid orientation of tile D.



This fixes the orientation of tiles A and C.



Since tile F could have gone into any one of the four corners, there are only four ways to tile the grid.

Chapter 9 – Geometry in the plane and proof

Solutions to Exercise 9A

1 **a** **i** obtuse

ii straight

iii acute

iv right

b **i** $\angle HFB$

ii $\angle BFE$

iii $\angle HFG$

iv $\angle BFE$

c **i** $\angle CBD, \angle BFE, \angle ABF, \angle HFG$

ii $\angle CBA, \angle BFH, \angle DBF, \angle EFG$

2 **a** $a = 65°$ (supplementary angles),
$b = 65°$ (vertically opposite angles)

b $2x + 10 = 90$ (complementary angles
$x = 40°$,
$y = 130°$(supplementary angles)

c $a = 60°$(vertically opposite angles),
$b = 70°$(vertically opposite angles),
$c = 50°$ (angle sum of triangle),
$d = 60°$ (alternate angles),
$e = 50°$ (supplementary angles),
$f = 130°$ (supplementary angles)

d $\angle FCB = 60°$(supplementary angles,
$\beta = 120°$(co-interior angles)
$\alpha = 60°$(co-interior angles)

e $\alpha = 90°$(alternate angles),
$\angle LMC = 87°$(alternate angles)
$\beta = 93°$(supplementary angles)

f $\theta = 108°$ (vertically opposite angles),
$\alpha = 108°$ (corresponding angles),
$\beta = 90°$(supplementary angles)

3 $\angle ACB = \angle CBD$(alternate angles,
$AC \parallel BD$)
$\angle CAB = \angle DBX$(corresponding angles,
$AC \parallel BD$)
$\angle CBX = \angle CBD + \angle DBX$
$\therefore \angle CBX = \angle ACB + \angle CAB$
We have proved that the sum of two interior angles of a triangle is equal to the opposite exterior angle of the triangle.

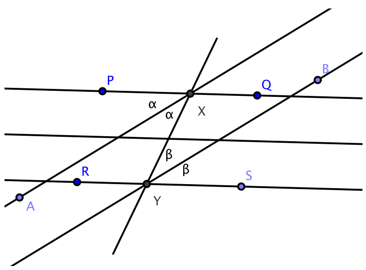
4 $\angle B = 180° - \alpha$ (co-interior angles,
$AD \parallel BC$),
$\angle D = 180° - \alpha$ (co-interior angles,
$AB \parallel DC$),
$\therefore \angle C = \alpha$
We have proved that diagonally opposite angles of a parallelogram are equal.

5 Assume that the opposite angles of a quadrilateral $ABCD$ are equal. Let
$\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$
Then $2\alpha + 2\beta = 360°$(Angle sum of a quadrilateral)
$\therefore \alpha + \beta = 180°$
$\therefore \angle A + \angle B = 180°$
That is, co-interior angles A and B are supplementary
which implies
$AD \parallel BC$

Similarly $AB \parallel CD$
$\therefore ABCD$ is a parallelogram

6 $2\alpha + 2\beta = 180°$ (co-interior angles)
$\therefore \alpha + \beta = 90°$
$\therefore \angle AEB$ is a right angle.

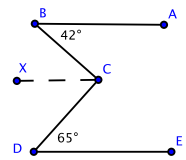
7



Let $\angle PXA = \angle YXA = \alpha$
Let $\angle XYB = \angle BYS = \beta$
$2\alpha = 2\beta$ (alternate angles, $PQ \parallel RS$)
$\therefore \alpha = \beta$
$\therefore AX \parallel BY$ (alternate angles are equal)

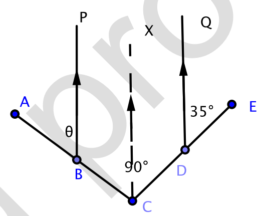
8 $\angle AOX = \angle FAO = \alpha$ (alternate angles, $AF \parallel XO$)
$\angle BOX = \angle CBO = \beta$ (alternate angles, $BG \parallel XO$)
$\therefore \alpha + \beta = \angle AOB = 90°$

9 **a** Draw a line CX through C parallel to both AB and DE.



$\angle BCX = 42°$ (alternate angles, $BA \parallel XC$)
$\angle DCX = 65°$ (alternate angles, $DE \parallel XC$)
$\therefore \theta = \angle BCD = (42 + 65)° = 107°$

b



Draw line XC parallel to both QD and PB
$\angle XCE = 35°$(corresponding angles$XC \parallel QD$
$\therefore \angle XCB = 55°$(complementary angles)
$\therefore \theta = 55°$(corresponding angles $XC \parallel PB$

Solutions to Exercise 9B

1 **a** Yes (satisfies triangle inequality)

b Yes (satisfies triangle inequality)

c Yes (satisfies triangle inequality)

d No (does not satisfy triangle inequality)

2 **a** Scalene

b Isosceles

c Equilateral

3 Must be greater than 10 cm

4 **a** 6, 6.5, 7

b No

5 If $2n - 1 = n + 7$ Then $n = 8$ and the values are 15, 15, 15
If $2n - 1 = 3n - 9$ then $n = 8$ and the sides are 15, 15, 15
If $3n - 9 = n + 7$ then $n = 8$ and the values are 15,15,15

6 **a** $\theta = 46°$, straight angle;
$\beta = 70°$, complementary to $\angle EBC$;
$\gamma = 64°$, alternate angles ($\angle CBD$);
$\alpha = 46°$, corresponding angles ($\angle EBD$)

b $\gamma = 80°$, angle sum of triangle;
$\beta = 80°$, vertically opposite (γ);
$\theta = 100°$, supplementary to β;
$\alpha = 40°$, alternate angles ($\angle BAD$)

c $\alpha = 130°$, supplementary to $\angle ADC$;
$\beta = 65°$, co-interior angles $\angle CDA$;
$\gamma = 65°$, co-interior angles $\angle ACD$

d $\alpha = 60°$, equilateral triangle

e $\alpha = 60°$, straight angle;
$\beta = 60°$, angle sum of triangle

f $a = 55°$, straight angle;
$b = 55°$, corresponding angles (a);
$g = 45°$, vertically opposite;
$c = 80°$, angle sum of triangle;
$e = 100°$, straight angle;
$f = 80°$, corresponding angles (c)

g $m = 68°$, corresponding angles;
$n = 60°$, angle sum of triangle;
$p = 52°$, straight angle;
$q = 60°$, alternate angles (n);
$r = 68°$, alternate angles (m)

7 **a** Sum = 720°; Angles = 120°

b Sum = 1800°; Angles = 150°

c Sum = 3240°; Angles = 162°

8 **a** Together they form 10 straight angles

b 360°

9 The proof is the same as that for **9**
The exterior angles plus the interior angles add to $n \times 180°$
The interior angles sum to $(n - 2)180°$
Therefor the sum of the exterior angles is 360°

10 $(n - 2)180 = 4 \times 360$

$n - 2 = 8$

$n = 10$

11 $(n - 2)180^\circ = k360^\circ$

$\therefore 180n - 360 = 360k$ Solving for n

$n = 2(k + 1)$

Solutions to Exercise 9C

1 **a** A and C (SAS)

b All of them (AAS)

c A and B (SSS)

2 **a** $\triangle ABC \equiv \triangle CDA$ (SSS)

b $\triangle CBA \equiv \triangle CDE$ (SAS)

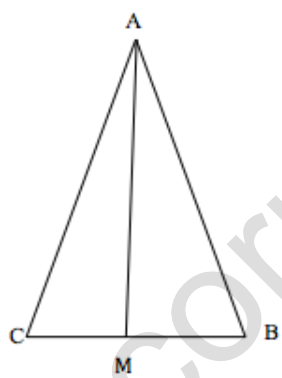
c $\triangle CAD \equiv \triangle CAB$ (SAS)

d $\triangle ADC \equiv \triangle CBA$ (RHS)

e $\triangle DAB \equiv \triangle DCB$ (SSS)

f $\triangle DAB \equiv \triangle DBC$ (SAS)

3



Let AM be the bisector of $\angle CAB$.
Then

$AC = AB$ (Definition of isosceles) That

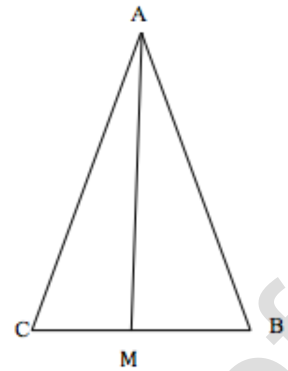
$\angle CAM = \angle BAM$ (Construction)

$AM = AM$ (Common)

$\triangle ACM \equiv \triangle ABM$ (SAS)

$\therefore \angle ACM = \angle ABM$
is $\angle ACB = \angle ABC$

4



Let AM be the bisector of $\angle CAB$.
Then

$\angle ACM = \angle ABM$ (Given)

$\angle CAM = \angle BAM$ (Construction)

$AM = AM$ (Common)

$\triangle ACM \equiv \triangle ABM$ (AAS)

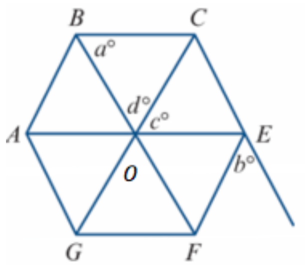
$\therefore AC = AB$

5

$2\alpha + 2\beta = 360°$
(Angle sum of quadrilateral)
$\therefore \alpha + \beta = 180°$
Hence, cointerior angles are supplementary.
Therefore, $AB \parallel DC$

6 **a** $a = b = c = d = 60°$

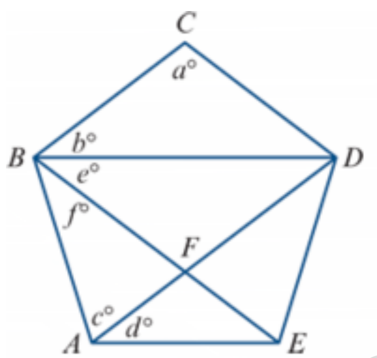
b



$\angle CB0 = \angle BOA = 60° \therefore BC \parallel AE$ (alternate angles equal) Similarly $BE \parallel BA$

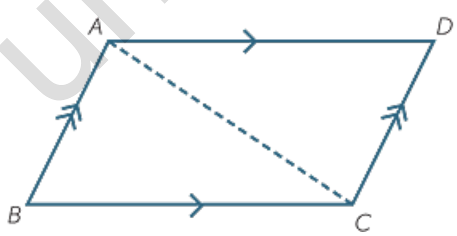
7 **a** $a = 108°,\ b = 36°,\ c = 72°,$
$d = 36°,\ e = 36°,\ f = 36°$

b



$c° + d° = 108°$ and $e° + f° = 72°$
$\therefore BD \parallel AE$ (co-interior angles supplementary)
$b° + e° = 72°$ and $a° = 108°$
$\therefore BE \parallel CD$ (co-interior angles supplementary)

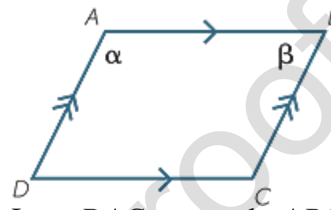
8 **a**



First prove opposite sides are equal.
$ABCD$ is a parallelogram, $AD \parallel BC$ and $AB \parallel DC$

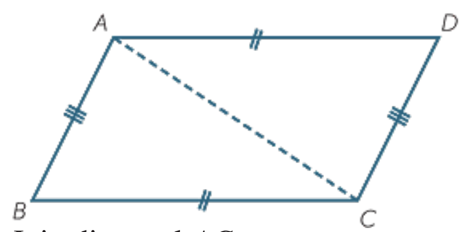
Join diagonal AC
In $\triangle ABC$ and $\triangle CDA$
$\angle BAC = \angle DCA$ (alternate angles, $AB \parallel DC$)
$\angle BCA = \angle DAC$ (alternate angles, $AD \parallel toBC$
$AC = CA$ (common)
$\therefore \triangle ABC \equiv \triangle CDA$ (AAS)
$\therefore AB = CD$ and $AD = BC$

To prove opposite angles are equal.



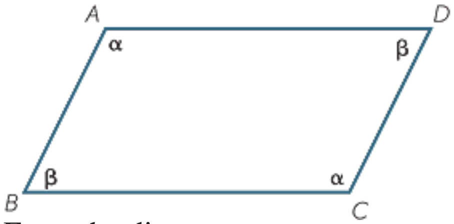
Let $\angle DAC = \alpha$ and $\angle ABC = \beta$
$\alpha + \beta = 180°$ (co-interior angles, $AD \parallel BC$)
$\therefore \angle ADC = \beta$ (co-interior angles, $AB \parallel DC$)
$\therefore \angle BCD = \alpha$ (co-interior angles, $AB \parallel DC$)

b



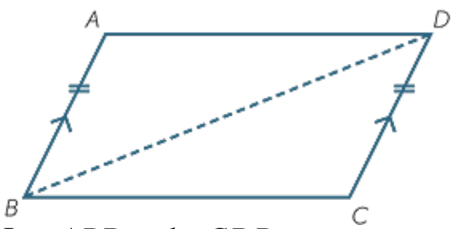
Join diagonal AC
In $\triangle ABC$ and $\triangle CDA$
$AD = CB$ (opposite sides equal)
$AB = CD$ (opposite sides equal)
$AC = CA$ (common)
$\therefore \triangle ABC \equiv \triangle CDA$ (SSS)
$\therefore \angle BAC = \angle DCA$
$\therefore AB \parallel DC$ (alternate angles equal
Furthermore,
$\therefore \angle DAC = \angle BCA$
$\therefore AD \parallel BC$ (alternate angles equal

c



From the diagram ,
$2\alpha + 2\beta = 360°$ (angle sum of quadilateral)
$\therefore \alpha + \beta = 180°$
Co-interior angles are supplementary.
$\therefore AB \parallel DC$ and $AD \parallel BC$

d



In $\triangle ABD$ and $\triangle CDB$

$AB = DC$ (given

$\angle ABD = \angle CDB$ (alternate angles

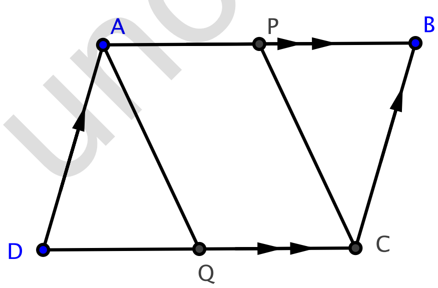
$BD = DB$ (common)

$\therefore \triangle ABD \equiv \triangle CDB$ (SAS)

$\therefore AD = BC$

$ABCD$ is a parallelogram

9



To prove:
$XYZW$ is a parallelogram.
In $\triangle ADQ$ and $\triangle BPC$

$AD = CB$(opposite sides of a parallelogram)

$\angle D = \angle B$ opposite angles of a parallelogram

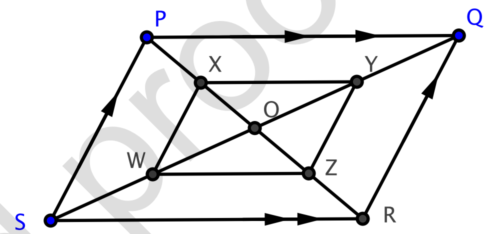
$DQ = BP$ (construction)

$\therefore \triangle ADQ \equiv \triangle CDP$ (SAS)

$\therefore AQ = PC$

$\therefore APCQ$ is a parallelogram (opposite sides are equal in length)

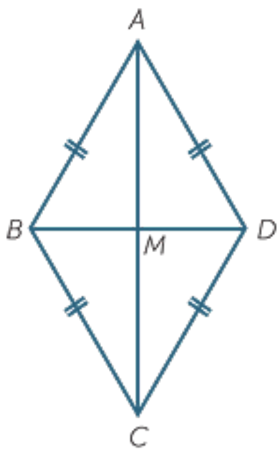
10



To prove:
$APCQ$ is a parallelogram.
The diagonals of a parallelogram bisect each other.
$\therefore XO = OZ$ and $WO = OY$
$\angle XOY = \angle WOZ$ and $\angle XOW = \angle YOZ$
$\therefore \triangle XOY \equiv \triangle WOZ$ and $\triangle XOW \equiv \triangle YOZ$
$\therefore XY = WZ$ and $WX = ZY$
$\therefore XYZW$ is a parallelogram (opposite sides of equal length)

11 A rhombus is defined as a parallelogram with a pair of adjacent sides equal in length. Therefore all the sides are equal in length. You should also prove that if a quadrilateral has all sides of equal length then it is a rhombus.

a



$\triangle ABC \equiv \triangle ADC$ (SSS)

$\therefore \angle BAC = \angle DAM$

$\therefore \triangle ABM \equiv \triangle ADM$ (SAS)

$\therefore \angle BMA = \angle DMA = 90^\circ$ (equal and supplementary)

$\therefore AC \perp BD$

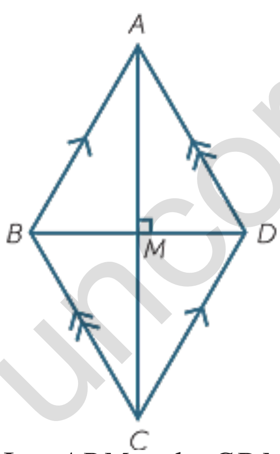
b Refer to the diagram for **a**

$\triangle ABC \equiv \triangle ADC$ (SSS)

$\therefore \angle BAC = \angle DAM$

Similarly for the other vertex angles

c



In $\triangle ABM$ and $\triangle CDM$

$AM = MC$ (diagonals bisect each other)

$MM = DM$ (diagonals bisect each other)

$\angle BMA = \angle CMD = 90^\circ$ (diagonals are perpendicular)

$\therefore \triangle ABM \equiv \triangle CDM$(SAS)

$\therefore AB = CD$ and $\angle MCD$

$\therefore AB \parallel DC$ (alternate angles equal)

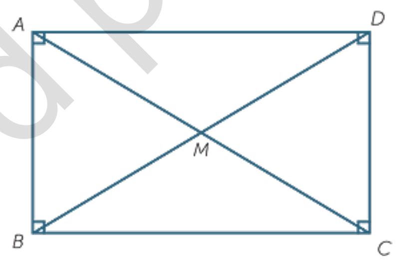
Similarly $BC = AD$ and $BC \parallel AD$

Finally $\triangle ABM \equiv \triangle CDM$(SAS)

Hence $AB = AD$

We note that a shorter proof is available but we have proven several properties of rhombuses on the way through.

12 a



$\triangle ABC \equiv \triangle DCB$ (SAS)

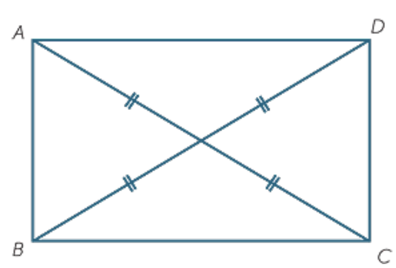
$\therefore AC = BD$.

$ABCD$ is arectangle and therefore a parallelogram

$\therefore$ diagonals bisect each other

b If a parallelogram has one right angle then:
the opposite angle is a right angle (opposite angles equal in a parallelo-gram). the cointerior angles are right angles.

c



Let

M be the pont of intersection of the diagonals.

$$\triangle AMD \equiv \triangle BMC \text{(SAS)}$$

$$\triangle AMB \equiv \triangle DMC \text{(SAS)}$$

All of these triangles are isosceles

$$\therefore \angle BAM = \angle DCM$$

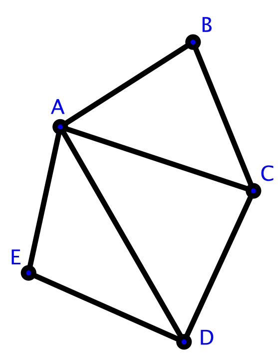
$$\therefore AB \parallel DC$$

Similarly $AD \parallel BC$

$$\angle A = \angle B = \angle C = \angle D$$

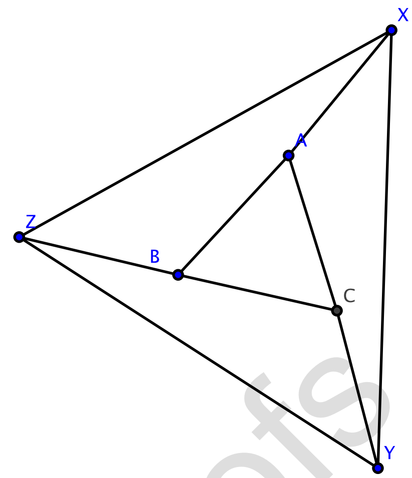
Therefore all right angles. Hence $ABCD$ is a rectangle.

13



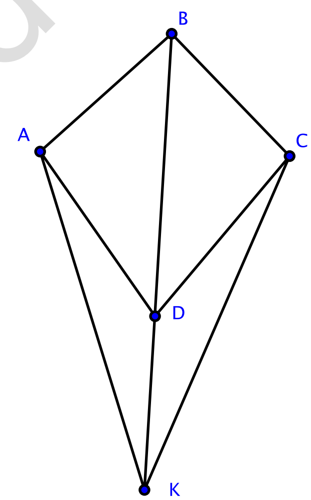
$\triangle ABC \equiv \triangle AED$ (SSS)
$\therefore \angle ABC = \angle AED$

14



$\triangle ZBX \equiv \triangle XAY \equiv \triangle YCZ$ (SAS)
$\therefore ZX = XY = YZ$

15



$$\triangle ABD \equiv \triangle CBD \text{ (SSS)}$$

$$\therefore \angle ABD = \angle CBD$$

$$\therefore \triangle ABK \equiv \triangle CBK \text{ (SAS)}$$

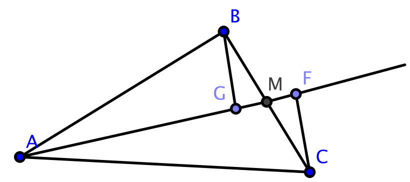
$$\therefore AK = CK$$

16 $\angle C = \angle A + \angle B$ implies that $\angle C = 90°$. $\triangle ABC$is a right-angled triangle. Choose point D to complete the

rectangle $ABCD$.
The rectangle has diagonals AB and CD which are of equal length and bisect each other.
Let M be the midpoint of AB.
Then $AB = 2CM$.

17 Let $\angle MNO = \angle MON = x°$
Then $\angle ANO = (90 - x)°$ and
$\angle NMO = (180 - 2x)°$

18



M is the midpoint of BC.
BG and CF are perpendicular to the median AM extended
$\triangle BMG \equiv \triangle CMF$ (ASA)
$\therefore BG = CF$

Solutions to Exercise 9D

1 Height up the wall $= \sqrt{18^2 - 7^2}$

$= 5\sqrt{11}$ metres

2 Length of diagonal $= \sqrt{40^2 + 9^2}$

$= 41$ metres

3

Distance of the chord from $O = \sqrt{14^2 - 2^2}$

$= \sqrt{192}$

$= 8\sqrt{3}$ cm

4 Lngth of diagonal $= \sqrt{13^2 + 13^2}$

$= 13\sqrt{2}$ cm

5 **a** Let$x$ cm be the length of a side of the square.

$2x^2 = 100$

$x^2 = 50$

$x = \sqrt{50}$

The length of a side is $5\sqrt{2}$ cm. The area = 50 cm^2.

b Let$x$ cm be the length of a side of the square.

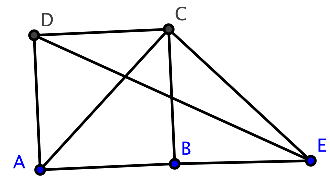
$2x^2 = 64$

$x^2 = 32$

$x = 4\sqrt{2}$

The length of a side is $4\sqrt{2}$ cm Area = 32cm^2

6



$\triangle ACB \equiv ECB$ (RHS)

$\therefore AB = BE$

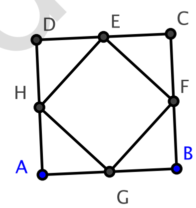
Each side length of the square has length = 2cm

$\therefore DE^2 = 2^2 + (2 \times 2)^2$

$\therefore DE^2 = 20$

$\therefore DE = 2\sqrt{5}$cm

7



E, F, G and H are the midpoints of sides DC, CB, BA, AD repectively.

$DE = EC = CF = FB = BG = GA = AH = HD = 1$cm

We seethat:

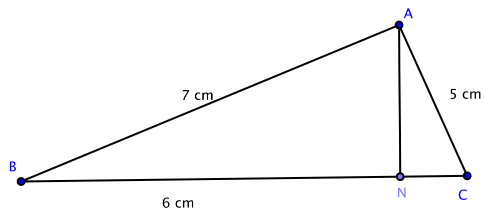
$HE^2 = 1 + 1 = 2$ and therefore $HE = EF = FB = GA = \sqrt{2}$. $\therefore EFGH$ is a rhombus.

$\triangle HDF \equiv \triangle CEF$ (SAS) These triangles are right-angled isosceles triangles and therefore $\angle DEH = \angle CEF = 45°$.

Therefore $\angle HED$ is a right angle and $EFGH$ is a square.

The area of $EFGH$ is 2 cm^2

8



Let $CN = x$ cm

Then in $\triangle ABN$

$(6 - x)^2 + AN^2 = 7^2 \ldots (1)$

In $\triangle ACN$

$x^2 + AN^2 = 25 \ldots (2)$

Subtract (2) from (1).

Then

$$-12x + 36 = 49 - 25$$

$$-12x = -12$$

$$x = 1$$

Substitute in (2)

$$1 + AN^2 = 25$$

$$AN = \sqrt{24}$$

$$\therefore AN = 2\sqrt{6} \text{ cm}$$

9 **a** $7^2 \neq 5^2 + 6^2$ (Not three sides of a right-angled triangle)

b $3.9^2 = 3.6^2 + 1.5^2$ (Three sides of a right-angled triangle)

c $4^2 \neq 2.4^2 + 2.4^2$ (Not three sides of a right-angled triangle)

d $82^2 = 18^2 + 80^2$ (Three sides of a right-angled triangle)

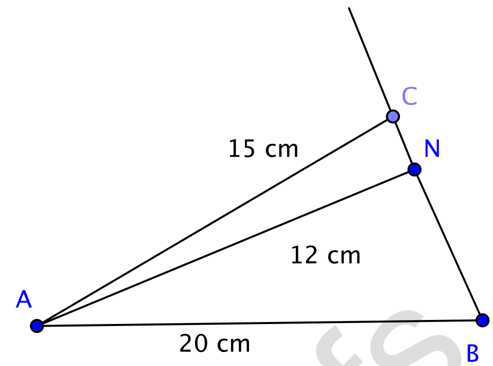
10 $$(x^2 - 1)^2 + 4x^2 = x^4 - 2x^2 + 1 + 4x^2$$

$$= x^4 + 2x^2 + 1$$

$$= (x^2 + 1)^2$$

The converse of Pythagoras' theorem gives that the triangle is right-angled.

11



Let $NC = x$ cm

In $\triangle ACN$

$x^2 = 15^2 - 12^2$ Let $NB = y$ cm In

$$x^2 = 225 - 144$$

$$x^2 = 81$$

$$x = 9$$

$\triangle ABN$

$y^2 = 20^2 - 12^2$ $\therefore BC = x + y = 25$

$$y^2 = 400 - 144$$

$$y^2 = 256$$

$$y = 16$$

The sidea of $\triangle ABC$ are 20,15, and 25.

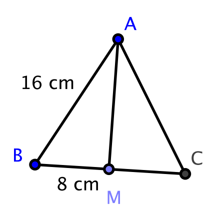
$$20^2 + 15^2 = 400 + 225$$

$$= 625$$

$$= 25^2$$

The converse of Pythagoras' theorem gives that the triangle is right-angled.

12



$AM^2 = 16^2 - 8^2 = 192$

$\therefore AM = \sqrt{192} = 8\sqrt{3}$

13 $$A_3 = \frac{1}{2}\pi\left(\frac{c}{2}\right)^2$$

$$A_2 = \frac{1}{2}\pi\left(\frac{b}{2}\right)^2$$

$$A_1 = \frac{1}{2}\pi\left(\frac{a}{2}\right)^2$$

Adding A_1 and A_2

$$\frac{1}{2}\pi\left(\frac{b}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 = \frac{1}{8}\pi(a^2 + b^2)$$

$$= \frac{1}{8}\pi^2$$

$$= A_3$$

14 $BD^2 = 8^2 + 6^2 = 100$

$\therefore BD = 10$ $\triangle AXB \equiv \triangle CYD$ (ASA)

Let $BX = DY = x$

In $\triangle AXB$

$$AX = \sqrt{36 - x^2}$$

In $\triangle AXD$

$$AX^2 + XD^2 = 64$$

$$36 - x^2 + (10 - x)^2 = 64$$

$$36 - x^2 + 100 - 20x + x^2 = 64$$

$$20x = 72$$

$$x = \frac{18}{5}$$

$$\therefore XY = 10 - 2x$$

$$= 2.8$$

15 From the two right-angled triangles

$$36 = (x + 4)^2 + y^2 \ldots (1)$$

$$9 = x^2 + y^2 \ldots (2)$$

Subtract (2) from (1)

$$27 = x^2 + 8x + 16 - x^2$$

$$27 = 8x + 16$$

$$11 = 8x$$

$$x = \frac{11}{8}$$

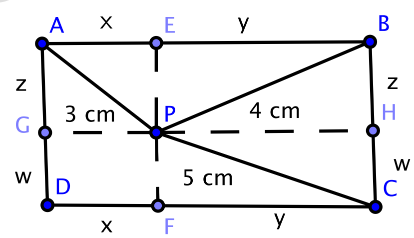
Substitute in (2)

$$9 = \left(\frac{11}{8}\right)^2 + y^2$$

$$9 - \frac{121}{64} = y^2$$

$$y = \frac{\sqrt{455}}{8}$$

16



Let $AE = DF = x$

Let $BE = CF = y$

Let $AG = BF = z$

Let $GD = HC = w$

Using Pythagoras's theorem 3 times

$x^2 + z^2 = 9 \ldots (1)$

$y^2 + z^2 = 16 \ldots (2)$

$w^2 + y^2 = 25 \ldots (3)$

Subtract (1) from (2)

$y^2 - x^2 = 7 \ldots (4)$

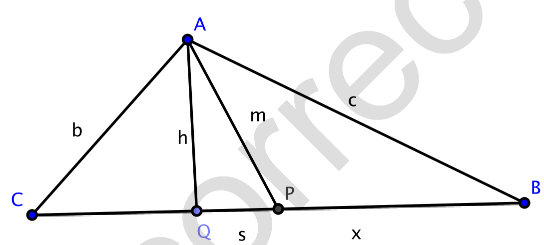
Subtract (4) from (3)

$w^2 + x^2 = 18$

$PD^2 = w^2 + x^2 = 18$

$PD = 3\sqrt{2}$ cm

17



Let $AB = c, AC = b, PB = x,$

$AP = m, AQ = h, CQ = t, QP = s$

Then,

$s + t = x$

$m^2 = h^2 + s^2$

$c^2 = h^2 + (s + x)^2$

$b^2 = h^2 + t^2$

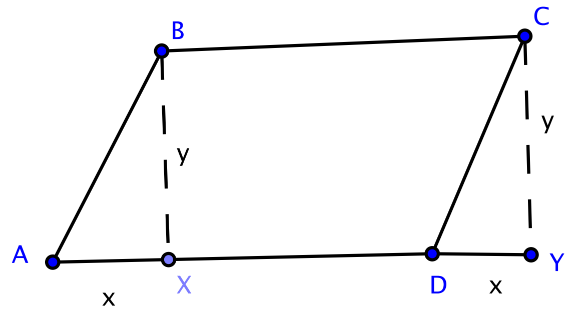
We start with,

$$\begin{aligned}
&AB^2 + AC^2 - 2AP^2\\
&= c^2 + b^2 - 2m^2\\
&= h^2 + (s + x)^2 + h^2 + t^2 - 2h^2 - 2s^2\\
&= h^2 + s^2 + 2sx + x^2 + h^2 + t^2 - 2h^2 - 2s^2\\
&= s^2 + 2xs + x^2 + t^2 - 2s^2\\
&= x^2 + 2xs + t^2 - s^2\\
&= x^2 + 2xs + (t - s)(t + s)\\
&= x^2 + 2sx + (t - s)x\\
&= x^2 + sx + tx\\
&= x^2 + x(s + t)\\
&= 2x^2\\
&= 2PB^2
\end{aligned}$$

$\therefore AB^2 + AC^2 - 2AP^2 = 2PB^2$

$\therefore AB^2 + AC^2 = 2PB^2 + 2AP^2$

18



$\triangle ABX \equiv \triangle CYD \ (RHS)$

Let $AX = BY = x$

Let $BX = CY = y$

$$AC^2 = (AD + x)^2 + y^2 \ldots (1)$$

$$BD^2 = (AD - x)^2 + y^2 \ldots (2)$$

Add (1) and (2)

$$AC^2 + BD^2 = AD^2 + 2xAD + x^2 + AD^2 - 2xAD + y^2$$

$$= 2AD^2 + 2(x^2 + y^2)$$

$$\therefore AC^2 + BD^2 = 2AD^2 + 2AB^2.$$

Solutions to Exercise 9E

1 One part = 9000 ÷ 9 = 1000
Two parts = 1000 × 2 = 2000
Seven parts = 1000 × 7 = 7000

2 One part = 15 000 ÷ 5 = 3000
Two parts = 3000 × 2 = 6000

3 $\frac{x}{6} = \frac{9}{15}$

$x = \frac{9 \times 6}{15} = 3.6$

4 $\frac{144}{p} = \frac{6}{11}$

$\frac{p}{144} = \frac{11}{6}$

$p = \frac{11 \times 144}{6} = 264$

5 $\frac{x}{3} = \frac{15}{2}$

$x = \frac{15 \times 3}{2} = 22.5$

6 6 : 5 : 7 = 180°
One part = 180° ÷ 18 = 10°
Six parts = 10° × 6 = 60°
Five parts = 10° × 5 = 50°
Seven parts = 10° × 7 = 70°

7 Suppose they receive $\$x$, $\$y$ and $\$z$ respectively.

$\frac{x+2}{x} = \frac{3}{2}$

$2(x + 2) = 3x$

$2x + 4 = 3x$

$x = 4$

X receives \$4 and *Y* receives \$6.
Two parts = \$4
One part = \$2
Seven parts = \$14
Z receives \$14.

8 One part = 10 g
Three parts = 10 g × 3 = 30 g (zinc)
Four parts = 10 g × 4 = 40 g (tin)

9 Seven parts = 56
One part = 56 ÷ 7 = 8 green beads
Two parts = 8 × 2 = 16 white beads

10 One part = 45 mm
125 000 parts = 45 mm × 125 000
= 5 625 000 mm
= 5.625 km

11 One part = \$5200 ÷ 13 = \$400
Eight parts = \$400 × 8
= \$3200 (mother)
Five parts = \$400 × 5
= \$2000 (daughter)
Difference = \$1200

12 If BC is one part, AB and CD are each two parts. AD is 5 parts and BD is 3 parts, so $BD = \frac{3}{5}AD$.

13 The ratio will be $\pi : 1$, as for any circle.

14 One part = 30 ÷ 5 = 6
Two parts = 6 × 2 = 12 (boys)
Three parts = 6 × 3 = 18 (girls)
After six boys join the class, there are 18 boys and 18 girls, so the ratio is 1:1.

15 $\frac{b}{a} = \frac{4}{3}$ and $\frac{b+c}{a} = \frac{5}{2}$

$$\frac{b+c}{a} = \frac{5}{2}$$

$$\frac{b}{a} + \frac{c}{a} = \frac{5}{2}$$

$$\frac{4}{3} + \frac{c}{a} = \frac{5}{2}$$

$$\frac{c}{a} = \frac{5}{2} - \frac{4}{3}$$

$$= \frac{15-8}{6} = \frac{7}{6}$$

$$\therefore \quad a : c = 6 : 7$$

16 One part = 3.5 cm

$$250\,000 \text{ parts} = 3.5 \text{ cm} \times 250\,000$$

$$= 875\,000 \text{ cm}$$

$$= 8.75 \text{ km}$$

17 $\frac{a-c}{b-d} = \frac{c}{d}$

$$\Leftrightarrow (a-c)d = (b-d)c$$

$$\Leftrightarrow ad - cd = bc - dc$$

$$\Leftrightarrow ad = bc$$

$$\Leftrightarrow \frac{a}{b} = \frac{c}{d}$$

18 $a = \frac{2}{3}x, b = \frac{2}{3}y, c = \frac{2}{3}z$

$$\therefore \frac{a+b+c}{x+y+z} = \frac{\frac{2}{3}(x+y+z)}{x+y+z} = \frac{2}{3}$$

19

$$\frac{x+y}{x-y} = \frac{m+n}{m-n}$$

$$\Leftrightarrow (x+y)(m-n) = (x-y)(m+n)$$

$$\Leftrightarrow xm - xn + ym - yn = xm + xn - ym - yn$$

$$\Leftrightarrow 2ym = 2xn$$

$$\Leftrightarrow \frac{m}{n} = \frac{x}{y}$$

Solutions to Exercise 9F

1 a AAA

$$\frac{x}{5} = \frac{9}{4}$$

$$x = \frac{9 \times 5}{4} = 11.25 \text{ cm}$$

b AAA

Note that *E* corresponds with *B*, so *x* corresponds with 14 cm.

$$\frac{x}{14} = \frac{10}{12}$$

$$x = \frac{10 \times 14}{12} = 11\frac{2}{3} \text{ cm}$$

c AAA

$$\frac{x}{2} = \frac{6}{4}$$

$$x = \frac{6 \times 2}{4} = 3 \text{ cm}$$

d AAA

Note that *Q* corresponds with *B* and *R* corresponds with *C*, so *x* corresponds with 6 cm.

$$\frac{x}{6} = \frac{10}{8}$$

$$x = \frac{10 \times 6}{8} = 7.5 \text{ cm}$$

2 a AAA

$$\frac{x+12}{12} = \frac{24}{16} = \frac{3}{2}$$

$$2x + 24 = 36$$

$$2x = 12$$

$$x = 6 \text{ cm}$$

b AAA

$$\frac{x+2}{2} = \frac{5}{3}$$

$$3x + 6 = 10$$

$$3x = 4$$

$$x = 1\frac{1}{3} \text{ cm}$$

c AAA

$$\frac{x+8}{x} = \frac{8}{2} = 4$$

$$x + 8 = 4x$$

$$3x = 8$$

$$x = 2\frac{2}{3} \text{ cm}$$

d AAA

$$\frac{x+1.5}{x} = \frac{12}{10} = \frac{6}{5}$$

$$5x + 7.5 = 6x$$

$$x = 7.5 \text{ cm}$$

3

$$\frac{AC}{14} = \frac{15}{12} = \frac{5}{4}$$

$$AC = \frac{5 \times 14}{4} = 17.5 \text{ cm}$$

$$\frac{AE+4}{AE} = \frac{5}{4}$$

$$4AE + 16 = 5AE$$

$$AE = 16 \text{ cm}$$

$$AB = AE + EB$$

$$= 20 \text{ cm}$$

4

$$\frac{\text{tree}}{33} = \frac{30}{224} = \frac{15}{112}$$

$$\text{Tree height} = \frac{15 \times 33}{112} = 4.42 \text{ m}$$

Note: It is valid to leave the measurements of the stick and its shadow in cm, as you are comparing the ratio of measurements with the same units.

5 $\dfrac{h}{15} = \dfrac{20}{40} = \dfrac{1}{2}$

$h = \dfrac{15}{2} = 7.5$ m high

6 $\dfrac{h}{300} = \dfrac{1}{20}$

$h = \dfrac{300}{20} = 15$ m high

7 $\dfrac{CY}{45} = \dfrac{15}{30} = \dfrac{1}{2}$

$CY = \dfrac{45}{2} = 22.5$ m

8 $\dfrac{h}{32} = \dfrac{2}{6.2}$

$h = \dfrac{64}{6.2} = 10\dfrac{10}{31}$ m high

9 $\dfrac{x}{4} = \dfrac{20 - x}{8}$

$\dfrac{2x}{8} = \dfrac{20 - x}{8}$

$2x = 20 - x$

$3x = 20$

$x = \dfrac{20}{3} = 6\dfrac{2}{3}$ cm high

10 Let x be the height of A above the 80 cm leg of the table.

$\dfrac{x}{30} = \dfrac{12}{100}$

$h = \dfrac{12 \times 30}{100} = 3.6$

Height = 80 cm + 3.6 cm

= 83.6 cm

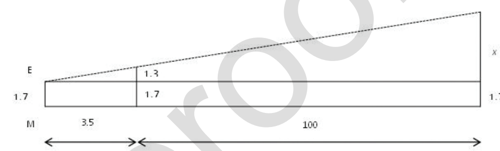
11 $\dfrac{x}{1.3 - x} = \dfrac{1.5}{0.8} = \dfrac{15}{8}$

$8x = 19.5\ x - 15\ x$

$23\ x = 19.5$

$x = \dfrac{19.5}{23} = \dfrac{39}{46}$ m

12



$\dfrac{x}{103.5} = \dfrac{1.3}{3.5}$

$x = \dfrac{1.3 \times 103.5}{3.5}$

$= \dfrac{269.1}{7}$

$= \dfrac{260}{7} \approx 37.1$

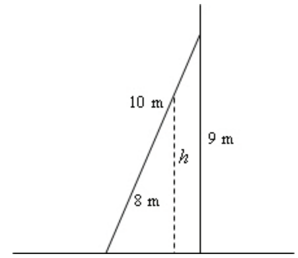
Height $= \dfrac{269.1}{7} + 1.7$

$= \dfrac{269.1 + 11.9}{7}$

$= \dfrac{281}{7}$

$= 40\dfrac{1}{7}$ m

13



$$\frac{h}{8} = \frac{9}{10}$$
$$h = \frac{72}{10} = 7.2 \text{ m high}$$

14 Taking the heights above the spotlight,
$$\frac{h - 0.6}{8} = \frac{0.5}{3} = \frac{1}{6}$$
$$h - \frac{6}{10} = \frac{8}{6} = \frac{4}{3}$$
$$h = \frac{4}{3} + \frac{3}{5}$$
$$= \frac{20 + 9}{15}$$
$$= 1\frac{14}{15} \text{ m high}$$

15 **a** Vertically opposite angles at C are equal:
$\angle B = \angle D = 90°$
The third angles in the triangle must be equal: $\angle A = \angle E$
$\therefore$ $\triangle ABC \sim \triangle EDC$

b $$\frac{x}{4} = \frac{5}{2}$$
$$x = \frac{20}{2} = 10$$

c $$y^2 = 2^2 + 4^2$$
$$= 4 + 16 = 20$$
$$y = \sqrt{20}$$
$$= \sqrt{4 \times 5} = 2\sqrt{5}$$
$$z^2 = 10^2 + 5^2$$
$$= 100 + 25 = 125$$
$$z = \sqrt{125}$$
$$= \sqrt{25 \times 5} = 5\sqrt{5}$$

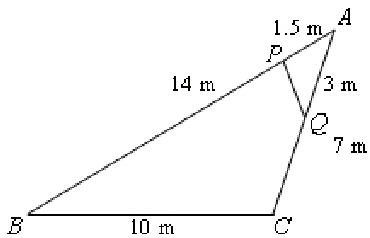
d $$y : z = 2\sqrt{5} : 5\sqrt{5}$$
$$= 2 : 5$$
$$ED : AB = 2 : 5$$
$$\therefore \quad y : z = ED : AB$$

16 $$\frac{a + 12}{12} = \frac{10}{7}$$
$$a + 12 = \frac{120}{7}$$
$$a = \frac{120}{7} - 12$$
$$= \frac{36}{7} = 5\frac{1}{7}$$

17 $$\frac{h}{3} = \frac{1.8}{0.76}$$
$$h = \frac{1.8 \times 3}{0.76} \approx 7.11 \text{ m}$$

18 In $\triangle TRN$, $\angle TRN = 90° - \angle T$
In $\triangle RST$, $\angle S = 90° - \angle T$
$$\angle TRN = \angle S$$
$$\angle SRN = \angle T$$
$$\triangle TRN \sim \triangle TSR$$
$$\frac{NT}{RT} = \frac{RT}{ST}$$
$$= \frac{4}{10} = \frac{2}{5}$$
$$\frac{NT}{4} = \frac{2}{5}$$
$$NT = \frac{2 \times 4}{5} = 1.6 \text{ m}$$

19



In $\triangle APQ$ and $\triangle ACB$,

$$\frac{AQ}{AB} = \frac{3}{14}$$

$$\frac{AP}{AC} = \frac{1.5}{7} = \frac{3}{14}$$

$$\therefore \quad \frac{AQ}{AB} = \frac{AP}{AC}$$

$\angle A$ is common to both triangles.

$\triangle APQ \sim \triangle ACB$

$$\frac{PQ}{BC} = \frac{AQ}{AB}$$

$$\frac{PQ}{10} = \frac{3}{14}$$

$$PQ = \frac{30}{14} = 2\frac{1}{7} \text{ m}$$

20 Note that the three triangles are all similar, as shown in **Q**.18.

$$\frac{x+4}{6} = \frac{6}{4} = \frac{3}{2}$$

$$x + 4 = \frac{3 \times 6}{2} = 9$$

$$\therefore \quad x = 5$$

$$\frac{y}{x} = \frac{4}{y}$$

$$y^2 = 4x = 4 \times 5$$

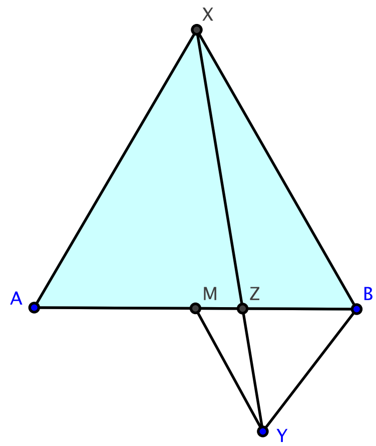
$$\therefore \quad y = 2\sqrt{5}$$

$$\frac{a}{y} = \frac{6}{4} = \frac{3}{2}$$

$$\therefore \quad a = \frac{3y}{2} = 3\sqrt{5}$$

Solutions to Exercise 9G

1



In $\triangle XAZ$ and $\triangle YBZ$

$\angle XZA = \angle YZB$, (vertically opposite)

$\angle XAZ = \angle YBZ = 60^\circ$

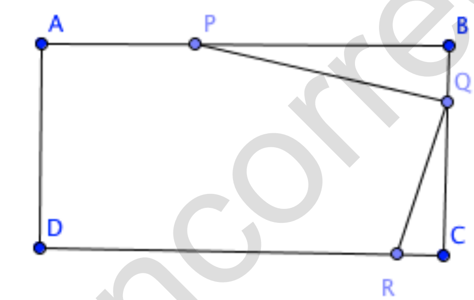
$\triangle XAB$ and $\triangle BMY$ are equilateral - given)

$\therefore \triangle XAZ \sim \triangle YBZ$, (AAA)

$XA = 2YB$ (given)

$\therefore AZ = 2ZB$

2



In $\triangle PBQ$ and $\triangle QCR$

$\angle PBQ = \angle QCR$, (right angles)

$\angle PQR = 90^\circ$, (given)

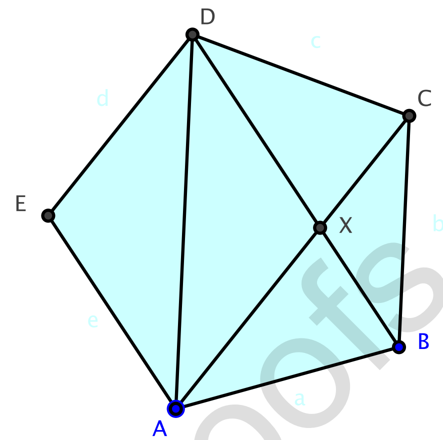
$\therefore \angle RQC = 90^\circ - \angle PQR = \angle BPQ$

$\therefore \triangle PBQ \sim \triangle QCR$, (AAA)

$$\therefore \frac{PB}{QC} = \frac{BQ}{CR}$$

$\therefore PB \times CR = BQ \times QC$

3



a Interior angles of a regular pentagon are each 108°
In $\triangle DEF$
$\angle DEF = 108^\circ$ and
$\angle EDA = \angle EAD = 36^\circ$($\triangle DEF$ is isosceles)
Similarly $\angle BAC = 36^\circ$
$\therefore \angle CAE = 72^\circ$

b AC and BD meet at X. In $\triangle BXA$ and $\triangle BAD$

$\angle XAB = 36^\circ = \angle BDA$

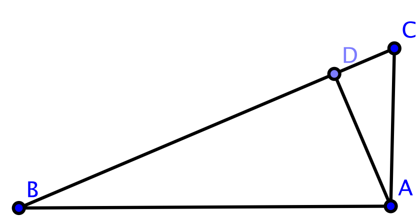
$\angle XBA = 72^\circ = \angle ABD$

$\therefore \triangle BXA \sim \triangle BAD$ (AAA)

$$\therefore \frac{BX}{AB} = \frac{BD}{AB}$$

$\therefore BX \times BD = AB^2$

4



a $\triangle BAD \sim \triangle BCA$ (AAA) $\cdots$ (1)

From(1)

$$\frac{AD}{AC} = \frac{AB}{BC}$$

$$\therefore AD \times BC = AB \times AC$$

b $\triangle BAD \sim \triangle ACD$ (AAA) $\cdots$ (2)

$$\therefore \frac{DA}{DC} = \frac{DB}{DA}$$

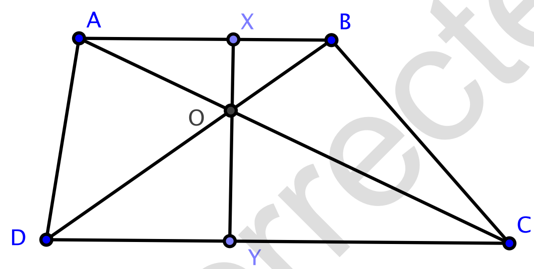
$$\therefore DA^2 = DC \times DB$$

c $\triangle BAD \sim \triangle BCA$ (AAA) $\cdots$ (1)

$$\therefore \frac{BA}{BC} = \frac{DB}{BA}$$

$$\therefore BA^2 = BC \times BD$$

5



$\triangle AXO \sim \triangle CYO$ (AAA) . . . (1)

$\triangle AOB \sim \triangle COD$ (AAA) . . . (2)

From (1)

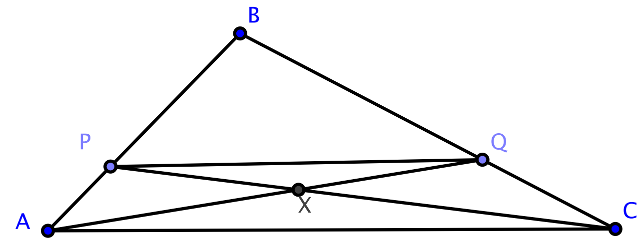
$$\frac{OX}{OY} = \frac{OA}{OC} = \frac{AX}{CY}$$

From (2)

$$\frac{OA}{OC} = \frac{AB}{CD}$$

$$\therefore \frac{OX}{OY} = \frac{OA}{OC} = \frac{AB}{CD}$$

6



$\triangle PBQ \sim \triangle ABC$

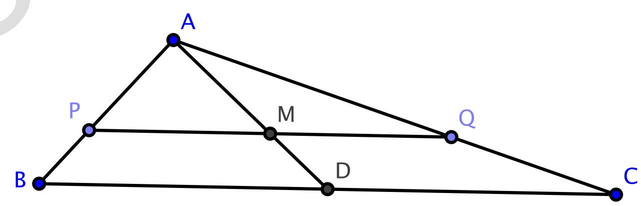
$$\frac{PQ}{AC} = \frac{2}{3}$$

$\triangle AXC \sim \triangle PXQ$

$$\frac{PQ}{AC} = \frac{XQ}{XA} = \frac{2}{3}$$

$$\therefore AX : AQ = 3 : 5$$

7



$\triangle APM \sim \triangle ABD$ (AAA)

$\triangle AMQ \sim \triangle ADC$ (AAA)

$$\frac{PM}{BD} = \frac{AM}{AD}$$

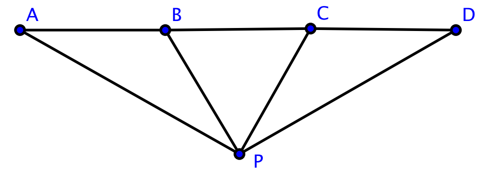
$$\frac{MQ}{DC} = \frac{AM}{AD}$$

$$\therefore \frac{PM}{BD} = \frac{MQ}{DC}$$

Also $BD = DC$

$$\therefore PM = MQ$$

8



Since $\triangle PBC$ is equilateral:

$\angle PBC = \angle PCB = \angle BPC = 60^\circ$

$\angle PCB = \angle PBC = 120^\circ$

$\triangle PCB$ and $\triangle PAB$ are isosceles.

$\therefore \angle CPD = \angle CDP = 30^\circ$

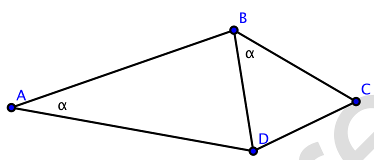
$\angle PAB = \angle BPA = 30^\circ$

$\therefore \triangle APD \sim \triangle ABP \sim \triangle DCP$

$$\therefore \frac{AP}{PB} = \frac{AD}{AP}$$

$\therefore AP^2 = AB \times AD$

9



$\angle BAD = \angle DBC$

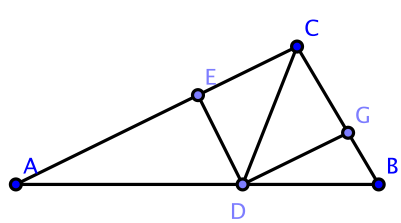
$$\frac{DA}{AB} = \frac{DB}{BC}$$

$\therefore \triangle BAD \sim \triangle CBD$ (SAS)

$\therefore \angle ADB = \angle BDC$

DB bisects $\angle ADC$

10



$\triangle AED \sim \triangle ACB$ AAA

$$\therefore \frac{AE}{AC} = \frac{ED}{CB}$$

$\therefore AE \times CB = ED \times AC$

$\therefore (AC - EC) \times CB = ED \times AC$

$\therefore AC \times CB - EC \times CB = ED \times AC$

$\therefore AC \times CB = EC \times CB + ED \times AC$

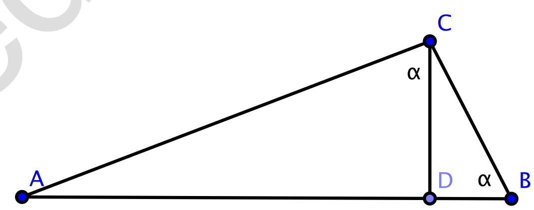
But$EC = ED$

$\therefore AC \times CB = ED(CB + AC)$

$$\therefore \frac{1}{ED} = \frac{CB + AC}{AC \times CB}$$

$$= \frac{1}{AC} + \frac{1}{CB}$$

11



a $\triangle ABC \sim \triangle ACD \sim \triangle CBD$ (AAA)

b We have from $\triangle ABC$ and $\triangle ACD$

$$\frac{AC}{AD} = \frac{AB}{AC}$$ We have

$\therefore AC^2 = AD \times AB \ldots (1)$

from $\triangle ABC$ and $\triangle CBD$

$$\frac{CB}{BA} = \frac{BD}{CB}$$

$\therefore CB^2 = BA \times BD \ldots (2)$

Add (1) and (2)

$AC^2 + CB^2 = AD \times AB + AB \times BD$

$= AB(AD + DB)$

$= AB^2$

Solutions to Exercise 9H

1 **a** $2 : 4 : 6 : 8 = 1 : 2 : 3 : 4$

b $2 : 8 : 18 : 32 = 1 : 4 : 9 : 16$

c The second ratio is the square of the first.

2 **a** $2 : 4 : 6 : 8 = 1 : 2 : 3 : 4$

b $1 : 4 : 9 : 16$

c The second ratio is the square of the first.

3 $\frac{A'B'}{AB} = \frac{5}{3}$

$$\text{Area } A'B'C'D' = 7 \times \left(\frac{5}{3}\right)^2 = \frac{7 \times 25}{9} = 19\frac{4}{9} \text{ cm}^2$$

4 $\frac{20}{2.1^2} = \frac{20}{4.41}$

$= 4.54 \text{ cm}^2$

5 **a** F is the midpoint of AC, so $AF = 1$ cm.

$$BF^2 = BA^2 - AF^2 = 2^2 - 1^2 = 3$$

$$BF = \sqrt{3} \text{ cm}$$

b $\frac{A'C'}{AC} = \frac{B'F'}{BF}$

$$\frac{a}{2} = \frac{2}{\sqrt{3}}$$

$$a = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

c $$\frac{\text{Area } A'B'C'}{\text{Area } ABC} = \left(\frac{B'F'}{BF}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$$

6 Area ratio = 16 : 25

$$\text{Side ratio} = \sqrt{\frac{16}{25}} = \sqrt{\frac{4^2}{5^2}} = 4 : 5$$

7 $30 \times \frac{9}{12} = 22.5$ cm

8 **a** $1 : 2 : 3$

b $1 : 2 : 3$

c $1 : 8 : 27$

d The third ratio is the cube of the first.

9 **a** **i** $8 : 12 = 2 : 3$

ii $4 : 6 = 2 : 3$

iii $3 : 4\frac{1}{2} = 2 : 3$

b $8 \times 4 \times 3 : 12 \times 6 \times 4\frac{1}{2} = 96 : 324$

$= 8 : 27$

c The ratio in **b** is the cube of the ratios in **a**.

10 a 3 : 2 : 5

b Sphere 1: $V = \frac{4}{3} \times \pi \times 3^3 = 36\pi$

Sphere 2: $V = \frac{4}{3} \times \pi \times 2^3 = \frac{32\pi}{3}$

Sphere 3: $V = \frac{4}{3} \times \pi \times 5^3 = \frac{500\pi}{3}$

$36 : \frac{32}{3} : \frac{500}{3} = 108 : 32 : 500$

$= 27 : 8 : 125$

c The second ratio is the cube of the first.

11 $(2 : 1)^3 = 2^3 : 1^3$

$= 8 : 1$

12 $(3 : 4)^3 = 3^3 : 4^3$

$= 27 : 64$

13 $\sqrt[3]{8 : 27} = \sqrt[3]{8} : \sqrt[3]{27}$

$= 2 : 3$

14 Volume ratio = 64 : 27

a Height ratio = $\sqrt[3]{64 : 27}$

$= 4 : 3$

b Radius ratio = $\sqrt[3]{64 : 27}$

$= 4 : 3$

15 Height ratio = 2 : 1

a Area ratio = $(2 : 1)^2$

$= 4 : 1$

b Capacity ratio = $(2 : 1)^3$

$= 8 : 1$

16 a $(1 : 10)^2 = 1 : 100$

b $(1 : 10)^3 = 1 : 1000$

c $(1 : 10)^1 = 1 : 10$

d Both models will have the same number of wheels, so 1 : 1.

17 $\frac{1}{2} \times \left(\frac{12}{8}\right)^3 = \frac{1}{2} \times \left(\frac{3}{2}\right)^3$

$= \frac{27}{16}$ litres

$\frac{1}{2} \times \left(\frac{16}{8}\right)^3 = \frac{1}{2} \times 2^3$

$= 4$ litres

18 $343 \times \left(\frac{7.5}{10.5}\right)^3 = 343 \times \left(\frac{5}{7}\right)^3$

$= 125$ mL

$343 \times \left(\frac{9}{10.5}\right)^3 = 343 \times \left(\frac{6}{7}\right)^3$

$= 216$ mL

19 a Length ratio = $\sqrt{1 : 2500}$

$= 1 : 50$

b Capacity ratio = (area ratio)^3

$= (1 : 50)^3$

$= 1 : 125\,000$

c Width $= 150 \times \frac{1}{50}$

$= 3$ cm

d Area $= 3 \div \frac{1}{2500}$

$= 3 \times 2500 = 7500$ cm^2

20 **a** Height ratio $= \sqrt{144 : 169}$

$= 12 : 13$

b Capacity ratio $= (12 : 13)^3$

$= 1728 : 2197$

21 **a** Ratio of sides = 1 : 2

Ratio of areas $= 1^2 : 2^2 = 1 : 4$

Four times

b Area $\triangle AKM = \frac{15}{4} = 3.75$

22 $\triangle BDE \sim \triangle CAF$

and $AB = AC = 2AD$

$\therefore\ BD^2 = BA^2 - AD^2$

$= (2AD)^2 - AD^2$

$= 3AD^2$

Ratio of areas $= \frac{BD^2}{AC^2}$

$= \frac{3AD^2}{(2AD)^2}$

$= \frac{3AD^2}{4AD^2} = \frac{3}{4}$

So the ratio is 3:4

Note: It is easier to express lengths in terms of AD as fractions are avoided.

23 Area ratio = 144 : 81

$= 12^2 : 9^2$

Length ratio = 12 : 9

Length in second triangle $= \frac{9}{12} \times 6$

$= 4.5$ cm

Solutions to Exercise 9I

1 **a** $\phi - 1 = \dfrac{1+\sqrt{5}}{2} - 1$

$= \dfrac{1+\sqrt{5}-2}{2}$

$= \dfrac{\sqrt{5}-1}{2}$

$\therefore \quad \dfrac{1}{\phi} = \phi - 1$

b $\phi^3 = \dfrac{(1+\sqrt{5})^2(1+\sqrt{5})}{8}$

$= \dfrac{(1+2\sqrt{5}+5)(1+\sqrt{5})}{8}$

$= \dfrac{(6+2\sqrt{5})(1+\sqrt{5})}{8}$

$= \dfrac{6+8\sqrt{5}+10}{8}$

$= \dfrac{16+8\sqrt{5}}{8} = 2+\sqrt{5}$

$2\phi + 1 = 1+\sqrt{5}+1$

$= 2+\sqrt{5}$

$\therefore \quad \phi^3 = 2\phi + 1$

c As shown above, $\phi - 1 = \dfrac{1}{\phi}$.

$\therefore \quad (\phi - 1)^2 = \dfrac{1}{\phi^2}$

$2 - \phi = 2 - \dfrac{1+\sqrt{5}}{2}$

$= \dfrac{4-1-\sqrt{5}}{2}$

$= \dfrac{3-\sqrt{5}}{2}$

$(\phi - 1)^2 = \left(\dfrac{1+\sqrt{5}-2}{2}\right)^2$

$= \dfrac{(\sqrt{5}-1)^2}{4}$

$= \dfrac{5-2\sqrt{5}+1}{4}$

$= \dfrac{3-\sqrt{5}}{2} = 2 - \phi$

$\therefore \quad 2 - \phi = (\phi - 1)^2 = \dfrac{1}{\phi^2}$

2 **a** In $\triangle ACX$, $\angle ACX = 90° - \angle BCX$

In $\triangle CBX$, $\angle B = 90° - \angle BCX$

$\angle ACX = \angle B$

$\angle A = \angle BCX$

$\triangle ACX \sim \triangle CBX$

$\therefore \quad \dfrac{AX}{CX} = \dfrac{CX}{BX}$

b Multiply both sides of the above equation by $CX \times BX$

i $CX^2 = AX \times BX$

$= 2 \times 8 = 16$

$CX = 4$

ii $CX^2 = AX \times BX$

$= 1 \times 10 = 10$

$CX = \sqrt{10}$

3 Join AB and BC. This will produce a right-angled triangle with an altitude. In **Q**.2 we proved that the altitude was the geometric mean of the two segments that divided the base. Therefore, as in **Q**.2:

$$\frac{AD}{BD} = \frac{BD}{CD}$$

$$\frac{EC}{DE} = \frac{DE}{DE + EC}$$

Since $BD = DE$,

$AD = EC$ and $CD = DE + EC$

$$\frac{DE}{EC} = \frac{DE + EC}{DE}$$

$$= 1 + \frac{EC}{DE}$$

$$x = \frac{DE}{EC}$$

$$= 1 + \frac{1}{x}$$

$\therefore \quad x^2 - x - 1 = 0$

Using the quadratic formula:

$$x = \frac{-1 + \sqrt{1 - 4 \times 1 \times -1}}{2}$$

$$= \frac{-1 + \sqrt{5}}{2} = \phi$$

(Rejecting the negative root as $x > 0$)

$$\frac{EC}{DE} = \frac{1}{\phi} = \phi - 1$$

$$\frac{AD}{BD} = \frac{EC}{DE} = \phi - 1$$

$$\therefore \quad \frac{AD}{BD} = \frac{BD}{CD}$$

$$= \phi - 1$$

4 a i $\angle AOB = \frac{360}{10} = 36°$

ii $\angle OAB = \frac{180 - 36}{2}$

$= 72°$

b i $\angle XAB = \frac{72}{2} = 36°$

$\angle ABO = \angle OAB = 72°$

$\angle AXB = 180 - 36 - 72$

$= 72°$

$\angle ABO = \angle AXB$

$\therefore \quad AX = AB$

ii $\angle XAO = \frac{72}{2}$

$= 36° = \angle AOX$

$\therefore \quad AX = OX$

iii Corresponding angles are equal, so the triangles must be similar.

c $\triangle AOB \sim \triangle XAB$

$$\frac{OB}{AB} = \frac{AB}{XB}$$

$$\frac{OX + XB}{AB} = \frac{AB}{XB}$$

$OX = XA = AB$

$$\frac{AB + XB}{AB} = \frac{AB}{XB}$$

$$1 + \frac{XB}{AB} = \frac{AB}{XB}$$

$$x = \frac{XB}{AB}$$

$$= 1 + \frac{1}{x}$$

$\therefore \quad x^2 - x - 1 = 0$

Using the quadratic formula:

$$x = \frac{-1 + \sqrt{1 - 4 \times 1 \times -1}}{2}$$

$$= \frac{-1 + \sqrt{5}}{2} = \phi$$

(Rejecting the negative root as $x > 0$)

$$\frac{XB}{AB} = \frac{1}{\phi}$$

$$= \phi - 1$$

$$= \frac{-1 + \sqrt{5}}{2}$$

(Refer to **Q**.1 part **a**.)

$$\frac{XB}{AB} = \frac{AB}{OB}$$

$$= AB$$

$$= \phi - 1 \text{ since } OB = 1$$

$$AB = \frac{-1 + \sqrt{5}}{2} \approx 0.62$$

d **i** Draw a circle of radius 1 unit. Use the construction in section 9.5 of the textbook to find ϕ, then cut off a length of 1 unit to obtain a length of $\phi - 1$. Mark off this length around the circumference of the circle to divide the circumference into ten equal parts. Join these points to produce a regular decagon.

ii Repeat **i** but join every second point.

5 $\phi^\circ = 1$

$$\phi^1 = \phi = \frac{1 + \sqrt{5}}{2}$$

$$\phi - 1 = \frac{1}{\phi}$$

$$\therefore \quad \phi = \frac{1}{\phi} + 1$$

$$\phi^2 = \phi\left(\frac{1}{\phi} + 1\right)$$

$$= 1 + \phi = \frac{3 + \sqrt{5}}{2}$$

$$\phi^3 = \phi(1 + \phi)$$

$$= \phi^2 + \phi$$

$$= (1 + \phi) + \phi$$

$$= 1 + 2\phi$$

$$= \frac{4 + 2\sqrt{5}}{2} = 2 + \sqrt{5}$$

$$\phi^4 = \phi(1 + 2\phi)$$

$$= \phi + 2\phi^2$$

$$= \phi + 2(1 + \phi)$$

$$= 2 + 3\phi$$

$$= \frac{4 + 3(1 + \sqrt{5})}{2} = \frac{7 + 3\sqrt{5}}{2}$$

$$\phi^{-1} = \frac{1}{\phi}$$

$$= \phi - 1$$

$$= \frac{1 + \sqrt{5} - 2}{2} = \frac{-1 + \sqrt{5}}{2}$$

$$\phi^{-2} = \frac{1}{\phi}(\phi - 1)$$

$$= 1 - (\phi - 1)$$

$$= 2 - \phi$$

$$= \frac{4 - (1 + \sqrt{5})}{2} = \frac{3 - \sqrt{5}}{2}$$

$$\phi^{-3} = \frac{1}{\phi}(2 - \phi)$$

$$= 2\left(\frac{1}{\phi}\right) - 1$$

$$= 2(\phi - 1) - 1$$
$$= 2\phi - 3$$
$$= \frac{2 + 2\sqrt{5} - 6}{2} = \sqrt{5} - 2$$
$$\phi^{-4} = \frac{1}{\phi}(2\phi - 3)$$
$$= 2 - \frac{3}{\phi}$$
$$= 2 - 3(\phi - 1)$$
$$= 5 - 3\phi$$
$$= \frac{10 - 3 - 3\sqrt{5}}{2} = \frac{7 - 3\sqrt{5}}{2}$$

Alternatively, the surd expressions can be multiplied and simplified, for the same answers:

$$\phi - 1 = \frac{1}{\phi}$$
$$\phi = 1 + \frac{1}{\phi}$$
$$\phi^{n+1} = \phi \times \phi^n$$
$$= \left(1 + \frac{1}{\phi}\right) \times \phi^b$$
$$= \phi^n + \phi^{n-1}$$

6 $t_n > t_{n-1}$

$$\frac{t_{n+1}}{t_n} = 1 + \frac{t_{n-1}}{t_n}$$

Since the Fibonacci sequence is increasing, $1 < \frac{t_{n+1}}{t_n} < 2$.

This means the sequence is not diverging to infinity, and has a limit between 1 and 2.

If there is a limit, then when n is large,

$$\frac{t_{n+1}}{t_n} \approx \frac{t_n}{t_{n-1}}$$
$$= 1 + \frac{t_{n-1}}{t_n}$$
$$= 1 + \frac{1}{\frac{t_{n-1}}{t_n}}$$
$$x = \frac{t_{n+1}}{t_n}$$
$$\approx \frac{t_{n-1}}{t_n}$$
$$= 1 + \frac{1}{x}$$
$$\therefore \quad x^2 - x - 1 = 0$$

Using the quadratic formula:

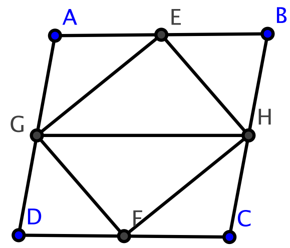
$$x = \frac{-1 + \sqrt{1 - 4 \times 1 \times -1}}{2}$$
$$= \frac{-1 + \sqrt{5}}{2} = \phi$$

(Rejecting the negative root as $x > 0$.)

Thus the sequence will approach ϕ as $n \to \infty$.

Solutions to technology-free questions

1



a

$\triangle GAE \equiv HAF$ (SAS)

$\triangle EBH \equiv FDG$ (SAS)

$\therefore GE = FH$ and $GF = EH$

$\therefore GEHF$ is a parallelogram

$\angle B + \angle A = 180°$ (co-interior angles)

$\angle BEH = (90° - \frac{1}{2}B)$ ($\triangle BEH$ is isosceles)

$\angle AEG = (90° - \frac{1}{2}A)$; ($\triangle AEG$ is isosceles)

$\therefore \angle GAE = 90°$

$\therefore GEHF$ is a rectangle

b 16

2

$$(x^2 - y^2)^2 + (2xy)^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2$$
$$= x^4 + 2x^2y^2 + y^4$$
$$= (x^2 + y^2)^2$$

The converse of Pythagoras' theorem gives the result.

3 The diagonals of a rhombus bisect each other at right angles.
Therefore if x cm is the length of each side length of the rhombus
$x = \sqrt{9 + 25} = \sqrt{34}$

4 **a** $x = 7$ cm, $y = 7$ cm, $\alpha = 45°$, $\beta = 40°$

b $\alpha = 125°$, $\beta = 27.5°$

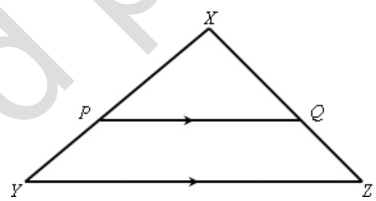
c $\theta = 52°$, $\alpha = 52°$, $\beta = 65°$, $\gamma = 63°$

5 8 m

6 **a** $\triangle PAQ \equiv \triangle QBO$ (RHS)

b Use Pythagoras' theorem: $\triangle PQR \equiv \triangle ORQ$ (SSS)

7 **a**



Both triangles share a common angle X.

$\angle XPQ = \angle XYZ$

$\angle XQP = \angle XYZ$
(alternate angles on parallel lines)

$\therefore$ $\triangle XPQ \sim \triangle XYZ$ (AAA)

b **i** $\frac{XQ}{XZ} = \frac{ZP}{XY}$

$\frac{XQ}{30} = \frac{24}{36} = \frac{2}{3}$

$XQ = 20$ cm

ii $QZ = XZ - XQ$

$QZ = 30 - 20$

$= 10$ cm

c $XP : PY = 24 : 12 = 2 : 1$

$PQ : YZ = 2 : 3$

8 **a** Ratio of areas $ABC : DEF$

$= 12.5 : 4.5$

$= 25 : 9$

$AB : DE = 5 : 3$

$DE = 3$ cm

b $AC : DF = 5 : 3$

c $EF : BC = 3 : 5$

9 $\dfrac{h}{21} = \dfrac{1}{2.3}$

$h = \dfrac{2.1}{23} = \dfrac{210}{23}$ m

10 $BC = 5$ (3–4–5 triangle)

So $YB = 2.5$

$\triangle BAC \sim \triangle BYX$

$\dfrac{XY}{YB} = \dfrac{CA}{AB}$

$\dfrac{XY}{2.5} = \dfrac{3}{4}$

$XY = \dfrac{3}{4} \times 2.5 = \dfrac{15}{8}$

11 The triangles are similar (AAA).

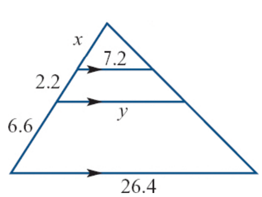
$\dfrac{x-7}{7} = \dfrac{3}{4}$

$4x - 28 = 21$

$4x = 49$

$x = 12.25$

12



If the two sloping lines were extended to form a triangle, then the left side of the top triangle would be given by:

$\dfrac{x}{x+8.8} = \dfrac{7.2}{26.4}$

$= \dfrac{72}{264} = \dfrac{3}{11}$

$11x = 3x + 26.4$

$8x = 26.4$

$x = 3.3$

Now compare the top two triangles:

$\dfrac{y}{7.2} = \dfrac{5.5}{3.3} = \dfrac{5}{3}$

$y = \dfrac{5 \times 7.2}{3}$

$= 12$

13 **a** Volume of block = 64 cm^3

8 parts = 64 cm^3

1 part = 8 cm^3

5 parts = 40 cm^3

3 parts = 24 cm^3

Mass of $X = 40 \times \dfrac{8}{5} = 64$ g

Mass of $Y = 24 \times \dfrac{4}{3} = 32$ g

Total mass = 96 g

b $X : Y = 64 : 32 = 2 : 1$ (by mass)

c Volume (cm^3) : mass (g)

$= 64 : 96$

$= 2 : 3$

$= 1000 : 1500$

Volume of 1500 g block is 1000 cm^3.

d $\sqrt[3]{1000} = 10$ cm $= 100$ mm

14 **a** Consider $\triangle BMA$ and $\triangle PAD$.

$\angle B = \angle P = 90^\circ$

$\angle BAM = \angle PDA$

$= 90^\circ - \angle PAD$

$\angle BMA = \angle PAD$

$= 90^\circ - \angle BAM$

$\triangle BMA \sim \triangle PAD$ (AAA)

b $BM = 30$ cm

$AM = 50$ cm (3–4–5triangle)
Comparing corresponding sides AM and AD:
$AM : AD = 50 : 60 = 5 : 6$
Ratio of areas $= 5^2 : 6^2$

$= 25 : 36$

c $\frac{PD}{BA} = \frac{AD}{MA}$

$\frac{PD}{40} = \frac{60}{50} = \frac{6}{5}$

$PD = \frac{6 \times 40}{5} = 48$ cm

15 a The same units (cm) must be used to compare these quantities.
200 : 30 = 20 : 3

b $\frac{A}{360} = \frac{20^2}{3^2} = \frac{400}{9}$

$A = \frac{400}{9} \times 360$

$= 16\,000 \text{ cm}^2 = 1.6 \text{ m}^2$

c $\frac{V}{1000} = \frac{20^3}{3^3} = \frac{8000}{27}$

$V = \frac{8000}{27} \times 1000$

$= \frac{8\,000\,000}{27} \text{ cm}^3$

$= \frac{8}{27} \text{ m}^3$

16 a Ratio of radii = 101 : 100 = 1.01 : 1

Ratio of areas $= 1.01^2 : 1$

$= 1.0201 : 1$

$= 102.01 : 100$

Percentage increase = 2.01% ≈ 2%

b Ratio of volumes $= 1.01^3 : 1$

$= 1.030301 : 1$

$= 103.0301 : 100$

Percentage increase ≈ 3%

17 a $\frac{XY}{BC} = \frac{AX}{AB}$

$= \frac{3}{9} = \frac{1}{3}$

b $\frac{AY}{AC} = \frac{AX}{AB}$

$= \frac{3}{9} = \frac{1}{3}$

c $\frac{CY}{AC} = \frac{2}{3}$

d $\frac{YZ}{AD} = \frac{CY}{AC}$

$= \frac{2}{3}$

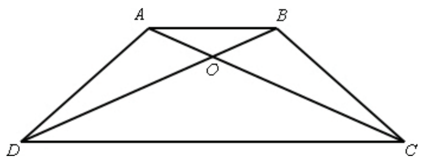
e $\frac{\text{area } AXY}{\text{area } ABC} = \frac{1^2}{3^2}$

$= \frac{1}{9}$

f $\frac{\text{area } CYZ}{\text{area } ACD} = \frac{2^2}{3^2}$

$= \frac{4}{9}$

18



Consider $\triangle AOB$ and $\triangle COD$

$\angle AOB = \angle COD$

(vertically opposite angles)

$\angle ABO = \angle CDO$

(alternate angles on parallel lines)

$\angle OAB = \angle OCD$

(alternate angles on parallel lines)

$\triangle AOB \sim \triangle COD$ (AAA)

$$\frac{CO}{AO} = \frac{CD}{AB}$$

$$= \frac{3}{1} = 3$$

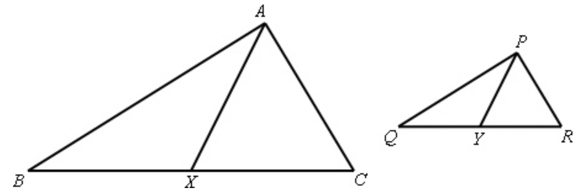
$$CO = 3AO$$

$$CO + AO = 4AO$$

$$AC = \$AO$$

$$AO = \frac{1}{4}AC$$

19 **a**



$$\frac{PQ}{AB} = \frac{YQ}{XB}$$

(corresponding sides of similar triangles)

$\angle B = \angle Q$

(corresponding angles of similar triangles)

$\therefore \triangle ABX \sim \triangle PQY$ (PAP)

b $\frac{AX}{PY} = \frac{AB}{PQ}$

(similar triangles proven above)

$$\frac{AB}{PQ} = \frac{BC}{QR}$$

(ABC and PQR are similar)

$$\therefore \frac{AX}{PY} = \frac{BC}{QR}$$

Solutions to multiple-choice questions

1 C $3x + 66 = 180$

$3x = 114$

$x = 38$

2 B $2x + 270 = 540$

$2x = 270$

$x = 135$

3 B

4 B $BC = 10$ by Pythagoras' theorem

Use similar triangles

$\triangle BAD \sim \triangle BCA$

$$\frac{AD}{AB} = \frac{CA}{BC}$$

$$AD = \frac{24}{5}$$

5 A

6 D $\frac{x}{7} = \frac{3}{5}$

$$x = \frac{3 \times 7}{5} = \frac{21}{5}$$

7 B 100 parts = 400 kg

One part = 4 kg

85 parts = 85×4

= 340 kg (copper)

8 D Cost of one article is $\frac{Q}{P}$.

Cost of R articles $= \frac{Q}{P} \times R = \frac{QR}{P}$

9 C 100 parts = 3.2 m

1 part $= \frac{3.2}{100}$

= 0.032 m = 3.2 cm

10 B 75 parts = 9 seconds

1 part $= \frac{9}{75} = \frac{3}{25}$ seconds

100 parts $= \frac{3}{25} \times 100$

= 12 seconds

11 D 10 parts = 50

One part = 5

Largest part is 6 parts = 30

12 C Ratio of lengths = 10 : 30 = 1 : 3

Ratio of volumes $= 1^3 : 3^3$

= 1 : 27

13 E Ratio of lengths = 4 : 5

Ratio of volumes $= 4^3 : 5^3$

= 64 : 125

14 E $\frac{XY}{3} = \frac{12}{10} = \frac{6}{5}$

$$XY = \frac{6 \times 3}{5}$$

= 3.6 cm

15 E $XY' = \frac{2}{3}XY$

Area of triangle $XY'Z'$

$= \frac{4}{9}$ area of triangle XYZ

$= \frac{4}{9} \times 60 = \frac{80}{3}\ \text{cm}^2$

Solutions to extended-response questions

1 **a** $\triangle DAC$ and $\triangle EBC$ share a common angle $\angle ACE$ and each has a right angle. Hence $\triangle EBC$ is similar to $\triangle DAC$.

b $\dfrac{h}{p} = \dfrac{y}{x+y}$ because corresponding side lengths of similar triangles have the same ratio.

c Using similar triangles $\triangle FAC$ and $\triangle EAB$ (which share a common angle $\angle EAB$ and have a right angle), $\dfrac{h}{q} = \dfrac{y}{x+y}$

d $\dfrac{h}{p} + \dfrac{h}{q} = h\left(\dfrac{1}{q} + \dfrac{1}{q}\right)$ and $\dfrac{h}{p} + \dfrac{h}{q} = \dfrac{y}{x+y} + \dfrac{x}{x+y}$

$$= \frac{x+y}{x+y}$$

$$= 1$$

$$\therefore \quad h\left(\frac{1}{p} + \frac{1}{q}\right) = 1$$

e When $p = 4$ and $q = 5$,

$$h\left(\frac{1}{4} + \frac{1}{5}\right) = 1$$

$$\therefore \quad h\left(\frac{5}{20} + \frac{4}{20}\right) = 1$$

$$\therefore \quad \frac{9}{20}h = 1$$

$$\therefore \quad h = \frac{20}{9}$$

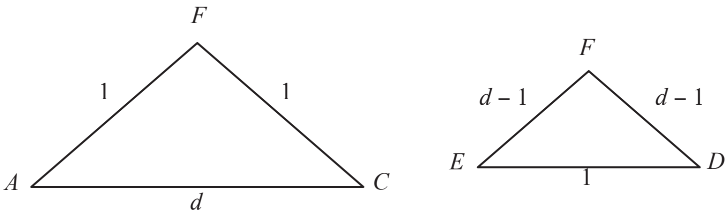
2 **a** AF is parallel to BC and AB is parallel to CF
Hence $ABCF$ is a rhombus and the length of CF is 1 unit.

b $EF = CE - CF$

$= d - 1$, as required.

c $\triangle ACF$ and $\triangle DEF$ have vertically opposite angles which are equal and they are both isosceles.
Hence $\triangle ACF$ and $\triangle DEF$ are similar.

d



$$\frac{d}{1} = \frac{1}{d-1}$$

$\therefore \quad d(d-1) = 1$

$\therefore \quad d^2 - d = 1$

$\therefore \quad d^2 - d - 1 = 0$

e Using the general quadratic formula,

$$d = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$= \frac{1 + \sqrt{5}}{2}, \text{ as } d > 0$$

3 If $DE \parallel AB$ then $\triangle CDE$ is similar to $\triangle ABC$

$$\therefore \quad \frac{CD}{AC} = \frac{CE}{BC}$$

$$\therefore \quad \frac{x-3}{3x - 19 + x - 3} = \frac{4}{x - 4 + 4}$$

$$\therefore \quad \frac{x-3}{4x-22} = \frac{4}{x}$$

$\therefore \quad x(x-3) = 4(4x-22)$

$\therefore \quad x^2 - 3x = 16x - 88$

$\therefore \quad x^2 - 19x + 88 = 0$

$\therefore \quad (x-11)(x-8) = 0$

$\therefore \quad x = 11$ or 8

4 a $\triangle BDR$ and $\triangle CDS$ share a common angle $\angle CDS$ and each has a right angle. Hence $\triangle BDR$ and $\triangle CDS$ are similar.

$\triangle BDT$ and $\triangle BCS$ share a common angle $\angle CBS$ and each has a right angle. Hence $\triangle BDT$ and $\triangle BCS$ are similar.

$\triangle RSB$ and $\triangle DST$ are similar as $\angle RSB = \angle TSD$ (vertically opposite) and $\angle RBS = \angle STD$ (alternate angles).

b $\dfrac{CS}{DT} = \dfrac{BC}{BD}$

$\Rightarrow \dfrac{z}{y} = \dfrac{p}{p+q}$

c $\dfrac{CS}{BR} = \dfrac{CD}{BD}$

$\Rightarrow \dfrac{z}{x} = \dfrac{q}{p+q}$

d $\dfrac{z}{x} + \dfrac{z}{y} = z\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$ and $\dfrac{z}{x} + \dfrac{z}{y} = \dfrac{p}{p+q} + \dfrac{p}{p+q}$

$$= \dfrac{p+q}{p+q}$$

$$= 1$$

$$\therefore \quad z\left(\dfrac{1}{x} + \dfrac{1}{y}\right) = 1$$

$\therefore \quad \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$, as required.

5 a i $\dfrac{QC}{AQ} = \dfrac{PB}{AP}$

$\therefore \quad \dfrac{6}{2} = \dfrac{PB}{3}$

$\therefore \quad 3 \times 3 = PB$

$\therefore \quad PB = 9$ cm

ii $\dfrac{PB}{AP} = \dfrac{BR}{PQ}$

$\therefore \quad \dfrac{9}{3} = \dfrac{BR}{4}$

$\therefore \quad 3 \times 4 = BR$

$BR = 12$ cm

iii $\dfrac{\text{area } \triangle APQ}{\text{area } \triangle ABC} = \dfrac{1^2}{4^2}$

$$= \dfrac{1}{16}$$

iv $\dfrac{\text{area } \triangle BPR}{\text{area } \triangle ABC} = \dfrac{9^2}{12^2}$

$= \dfrac{81}{144}$

$= \dfrac{9}{16}$

b **i** area $\triangle ABC = 9 \times$ area $\triangle APQ$

$= 16a$

Hence area of $\triangle ABC$ is $160a$ cm^2.

ii area $\triangle CPQ = \dfrac{1}{2}$ (area $\triangle ABC$ − area $\triangle APQ$ − area $\triangle BPR$)

$= \dfrac{1}{2}\left(16a - a - \dfrac{9 \times 16a}{16}\right)$

$= \dfrac{1}{2} \times 6a$

$= 3a$

Hence area of $\triangle CPQ$ is $3a$ cm^2.

6 $\dfrac{\text{area } \triangle ADE}{\text{area } \triangle ABC} = \dfrac{1}{9}$

$= \dfrac{1^2}{3^2}$

$\therefore \quad \dfrac{AD}{AB} = \dfrac{AE}{AC}$

$= \dfrac{1}{3}$

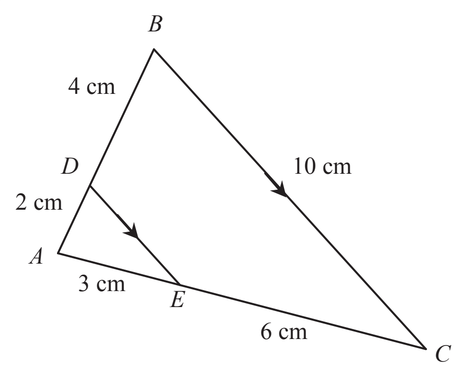
$\therefore \quad AD = \dfrac{1}{3}AB$

$= \dfrac{1}{3} \times 6$

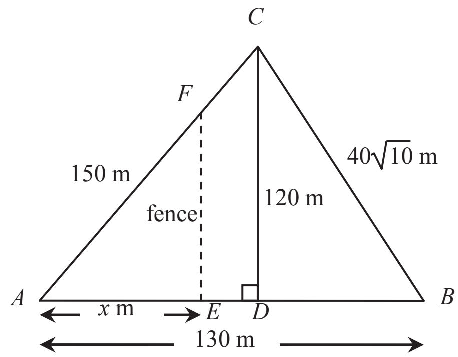
$= 2$

$\therefore \quad AE = \dfrac{1}{3}AC$

$= \dfrac{1}{3} \times 9 = 3$



7 The length of BC should be given as $40\sqrt{10}$ metres.



$$\text{area } \triangle AEF = \frac{1}{2}\text{area } \triangle ABC$$

$$= \frac{1}{2}\text{ (area } \triangle ACD + \text{area } \triangle BCD)$$

$$= \frac{1}{2}\left(\frac{1}{2}\sqrt{150^2 - 120^2}(120) + \frac{1}{2}\sqrt{(40\sqrt{10})^2 - 120^2(120)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}(90)(120) + \frac{1}{2}(40)(120)\right)$$

$$= \frac{1}{2}(5400 + 2400)$$

$$= 3900$$

$$\frac{\text{area } \triangle AEF}{\text{area } \triangle ACD} = \frac{3900}{5400}$$

$$= \frac{13}{18}$$

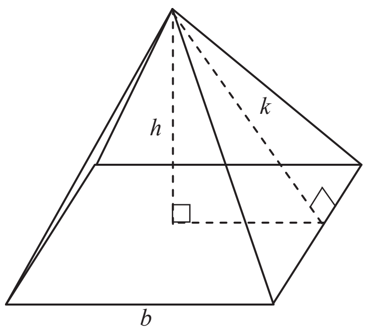
$$= \left(\frac{\sqrt{13}}{\sqrt{18}}\right)^2$$

$$\therefore \quad \frac{x}{AD} = \frac{\sqrt{13}}{\sqrt{18}}$$

$$\therefore \quad x = \frac{\sqrt{13} \times 90}{\sqrt{18}}$$

$$= 15\sqrt{26}\text{ m}$$

8



$$\text{Area of a triangular face} = \frac{1}{2}bk$$

$$h^2 = \frac{1}{2}bk$$

$$h^2 = k^2 - \left(\frac{1}{2}b\right)^2$$

$$= k^2 - \frac{1}{4}b^2$$

$$\therefore \quad k^2 - \frac{1}{4}b^2 = \frac{1}{2}bk$$

$$\therefore \quad 4k^2 - b^2 = 2bk$$

$$\therefore \quad 4k^2 - 2bk - b^2 = 0$$

$$\therefore \quad k = \frac{2b \pm \sqrt{4b^2 + 16b^2}}{8}$$

$$= \frac{2b \pm \sqrt{20b^2}}{8}$$

$$= \frac{b \pm \sqrt{5}b}{4}$$

$$= \frac{b(1 + \sqrt{5})}{4} \qquad \text{since } k > 0$$

$$\therefore \quad k = \frac{b}{2}\phi \qquad \text{since } \phi = \frac{1 + \sqrt{5}}{2}$$

$$\therefore \quad k : \frac{b}{2} = \phi$$

Chapter 10 – Circle geometry

Solutions to Exercise 10A

1 a $50° = \frac{1}{2}x$

$x = 100°$

$y = \frac{1}{2}x$

$= 50°$

b $y = 360° - 108° = 252°$

$x = \frac{1}{2} \times 252 = 126°$

$z = \frac{1}{2} \times 108° = 54°$

c Acute $\angle O = 2 \times 35 = 70°$

$z = 360° - 70° = 290°$

$y = \frac{1}{2} \times 290 = 145°$

d $O = 180°$

$x = 360 - 180 = 180°$

$y = 90°$ (Theorem 3)

e $3x + x = 180°$

$4x = 180°$

$x = 45°$

$z = 2 \times 3x$

$= 2 \times 3 \times 45° = 270°$

$y = 360° - 270°$

$= 90°$

2 The opposite angles of a cyclic quadrilateral are supplementary.

a $x + 112° = 180°$

$x = 68°$

$y + 59° = 180°$

$y = 121°$

b $x + 68° = 180°$

$x = 112°$

$y + 93° = 180°$

$y = 87°$

c $x + 130° = 180°$

$x = 50°$

$y + 70° = 180°$

$y = 110°$

3 Let the equal angles be $x°$.

$2x + 40° = 180°$

$2x = 140°$

$x = 70°$

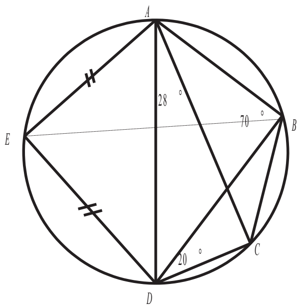
The angles in the minor segments will be the opposite angles of cyclic quadrilaterals.

$180° - 70° = 110°$

$180° - 70° = 110°$

$180° - 40° = 140°$

4

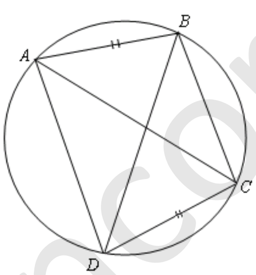


In cyclic quadrilateral *ABDE*,
$\angle DEA = 110°$
On arc *DC*, $\angle DBC = 28°$
$\therefore \angle ABC = 70 + 28 = 98°$
Join *EB*. Equal chords will subtend equal angles at the circumference.
$\therefore \angle ABE = \angle EBD = 35°$

$\angle EAD = 35°$ (also on equal arcs)
On arc *BC*, $\angle BAC = \angle BDC = 20°$
$\therefore \angle EAB = 35° + 28° + 20° = 83°$
In cyclic quadrilateral *ABDE*,
$\angle EDB = 180° - 83° = 97°$

$\therefore \angle EDC = 97° + 20° = 117°$
In cyclic quadrilateral *ABCD*,
$\angle BCD = 180° - (28° + 20°) = 132°$

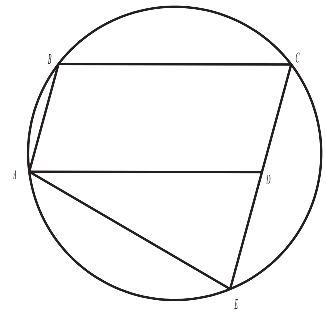
5



$\angle BAC = \angle BDC$ (subtended by the same arc)
$\angle DAC = \angle BDA$ (subtended by equal arcs)
$\therefore \angle BAC + \angle DAC = \angle BDC + \angle BDA$
$\angle BAD = \angle ADC$
$\angle ADC + \angle ABC = 180°$ (opposite angles in a cyclic quadrilateral)
$\therefore \angle BAD + \angle ABC = 180°$

BC and *AD* are thus parallel, as co-interior angles are supplementary

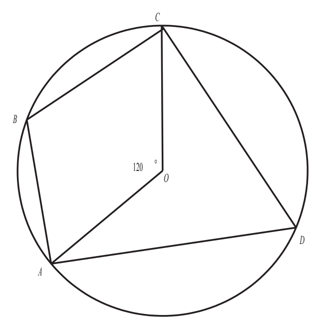
6



$\angle ADE + \angle ADC = 180°$
$\angle ABC = \angle ADC$ (opposite angles in a parallelogram)
$\therefore \angle ADE + \angle ABC = 180°$
$\angle AED + \angle ABC = 180°$ (opposite angles in a cyclic quadrilateral)
$\therefore \angle ADE = \angle AED$

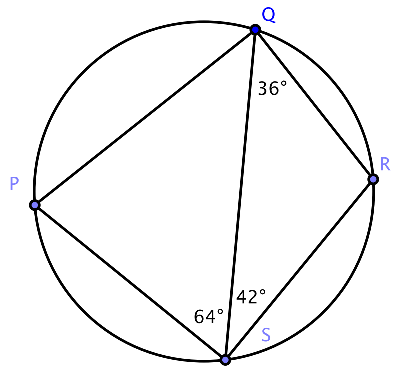
$AE = AD$

7



$\angle ADC = \dfrac{120°}{2} = 60°$
If *B* and *D* are on opposite sides of *AOC*, then $\angle ADC = \dfrac{240°}{2} = 120°$.
(Reflex angle $ADC = 360° - 120°$ will be used.)

8



In $\triangle QRS$, $\angle QRS = 102°$ (angle sum of triangle)

$\angle PSR = 64° + 42° = 106°$

$\angle PQR = 74°$ (opposite angles in a cyclic quadrilateral)

$\angle QPS = 78°$ (opposite angles in a cyclic quadrilateral)

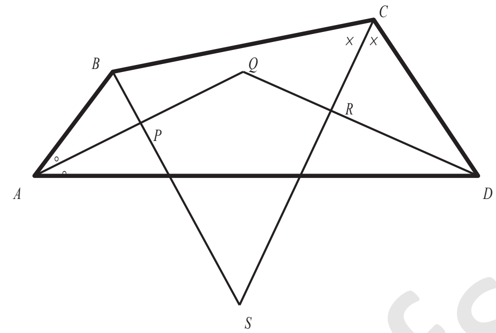
9 The opposite angles in a parallelogram are equal.

In a cyclic parallelogram, the opposite angles will add to 180°.

$\therefore$ the opposite angles equal 90°.

$\therefore$ all angles are 90°, i.e. the parallelogram is a rectangle (subtended by the same arc).

10



In triangle BCS,

$$\angle BSC = 180° - \angle SBC - \angle BCS$$

$$= 180° - \frac{1}{2}\angle ABC - \frac{1}{2}\angle BCD$$

Likewise, in triangle AQD

$$\angle AQD = 180° - \frac{1}{2}\angle BAD - \frac{1}{2}\angle CDA$$

$$\therefore \angle BSC + \angle AQD$$

$$= 180 - \frac{1}{2}\angle ABC$$

$$- \frac{1}{2}\angle BCD + 180° - \frac{1}{2}\angle BAD$$

$$- \frac{1}{2}\angle CDA$$

$$= 360° - \frac{1}{2}(\angle ABC + \angle BCD$$

$$+ \angle BAD + \angle CDA)$$

$$\angle ABC + \angle BCD + \angle BAD + \angle CDA$$

$$= 360° \text{ (angle sum of quadrilateral)}$$

$$\angle BSC + \angle AQD = 360° - 180°$$

$$= 180°$$

$\therefore$ both pairs of opposite angles in $PQRS$ will add to 180°.

$\therefore$ $PQRS$ is a cyclic quadrilateral.

Solutions to Exercise 10B

1 **a** $x = 73°$ (alternate segments)

$y = 81°$ (alternate segments)

b $\angle T = 90°$

$\therefore x = 90° - 33° = 57°$
$q = 57°$ (alternate segment theorem)

c $y = 74°$ (alternate segments)

$$z = \frac{180° - 74°}{2}$$
$$= 53°$$

$x = 53°$ (alternate segments)

d $x = 180° - 80° - 40° = 60°$
Use the alternate segment theorem to find the other angles.

$y = 180° - 60° - 60° = 60°$

$w = 180° - 40° - 40° = 100°$

$z = 180° - 80° - 80° = 20°$

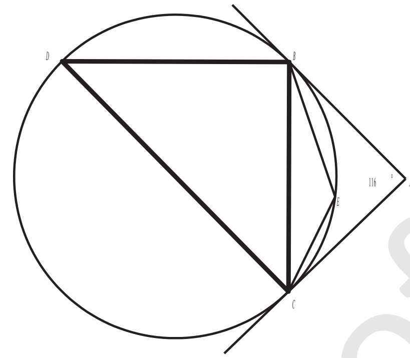
e $w = z = x = 54°$ (alternate segment, alternate angles and isosceles triangle PTS)
$y = 180° - 54° - 54° = 72°$

2 **a** $\angle BCX = 40°$

b $\angle CBD = 40°$

c $\angle ABC = 2 \times 40° = 80°$

3



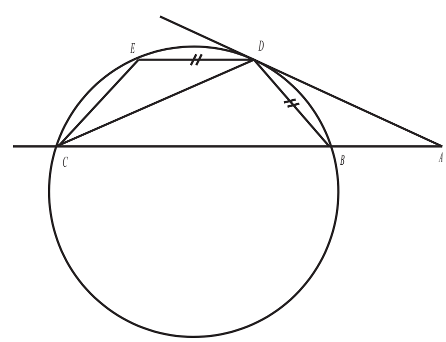
Triangle ABC is isosceles;
$\angle ABC = \angle ACB$

$$= \frac{180° - 116°}{2} = 32°$$

Using the alternate angle theorem,
$\angle BDC = \angle ACB = 32°$.
$\angle BEC + \angle BDC = 180°$
(opposite angles in cyclic quadrilateral $BCED$)
$\therefore BEC = 180° - 32° = 148°$

4 In $\triangle CAT$,
$\angle ACB = 180° - 30° - 110° = 40°$
The alternate segment theorem shows
$\angle BAT = 40°$
$\therefore$ $\angle CAB = 110° - 40° = 70°$
In $\triangle CAB$,
$\angle ABC = 180° - 40° - 70° = 70°$

5



There are multiple ways of proving this result.

$\angle ADB = \angle DCB$ (alternate segment theorem)

$\angle DCB = \angle DCE$ (subtended by equal arcs)

$\therefore \angle ADB = \angle DCE$

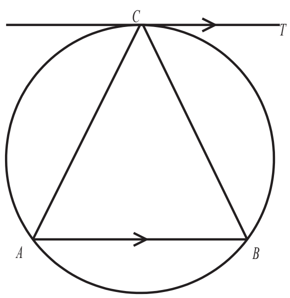
$\angle DBA + \angle DBC = 180^\circ$

$\angle DEC + \angle DBC = 180^\circ$

$\therefore \angle DBA = \angle DEC$

$\therefore$ triangles ABD and CDE are similar, since two pairs of opposite angles are equal.

6



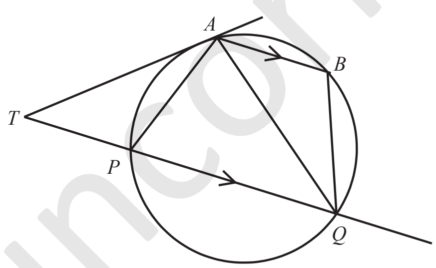
$\angle TCB = \angle CBA$ (alternate angles)

$\angle TCB = \angle CAB$ (alternate segment)

$\therefore \angle CBA = \angle CAB$

ABC is an isosceles triangle with $CA = CB$.

7



$\angle TAP = \angle AQP$ (alternate segment)

$\angle AQP = \angle BAQ$ (alternate angles)

$\therefore \angle TAP = \angle BAQ$

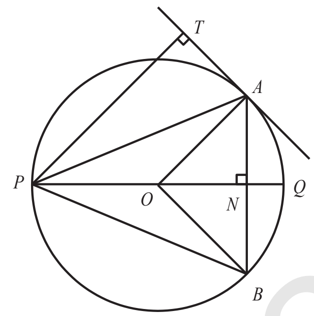
$\angle APT + \angle APQ = 180^\circ$ (adjacent angles)

$\angle AQB + \angle APQ = 180^\circ$ (opposite angles)

$\therefore \quad \angle APT = \angle ABQ$

Triangles PAT and BAQ are similar, since two pairs of opposite angles are equal.

8



Let T be the point where the perpendicular from P meets the tangent at A

Let O be the centre of the circle.

Join PA and PB.

Consider triangles OAN and OBN:

$\angle ANO = \angle BNO = 90^\circ$

$OA = OB$ (radii)

ON is common to both triangles.

$\therefore \angle AON \equiv \angle BON$ (RHS)

$AN = BN$

Now consider triangles PAN and PBN:

$AN = BN$

$\angle PNA = \angle PNB = 90^\circ$

PN is common to both triangles.

$\therefore PAN \equiv PBN$ (SAS)

$\angle PAN = \angle PBN$

Now consider triangles PAT and PAN:

$\angle PBN = \angle PAT$ (alternate segment theorem)

$\therefore \angle PAT = \angle PAN$

$\angle PTA = \angle PNA = 90^\circ$

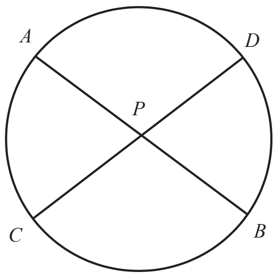
PA is common to both triangles.

$\therefore \angle PAT \equiv \angle PAN$ (AAS)

$PT = PN$

Solutions to Exercise 10C

1



a $AP \cdot PB = CP \cdot PD$

$5 \times 4 = 2PD$

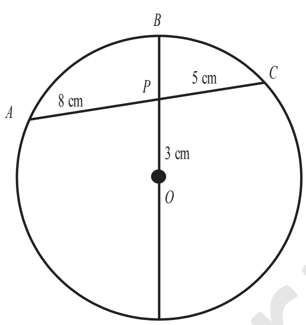
$PD = 10 \text{ cm}$

b $AP \cdot PB = CP \cdot PD$

$4PB = 3 \times 8$

$PB = 6 \text{ cm}$

2



Let the centre of the circle be O and the length of the radius r cm.
Extend OP to meet the circumference of the circle at C and D.
$CP = r - 3$ and $PD = r + 3$

$CP \cdot PD = AP \cdot PB$

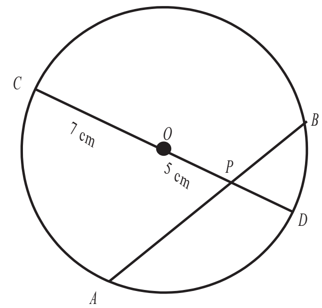
$(r - 3)(r + 3) = 8 \times 5$

$r^2 - 9 = 40$

$r^2 = 49$

$r = 7 \text{ cm}$

3



$PD = 7 \text{ cm} - 5 \text{ cm} = 2 \text{ cm}$
Let $PB = x$ cm
$PA = 4x$ cm
$AP \cdot PB = CP \cdot PD$

$4x \times x = 12 \times 2$

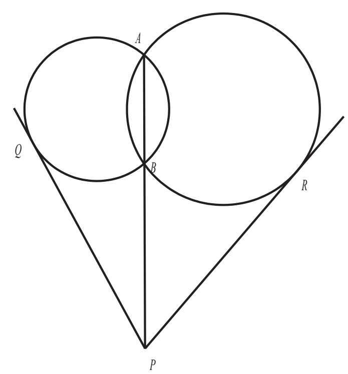
$x^2 = 6$

$x = \sqrt{6}$

$\therefore \quad AB = 4\sqrt{6} + \sqrt{5}$

$= 5\sqrt{6} \text{ cm}$

4



Use theorem 9:

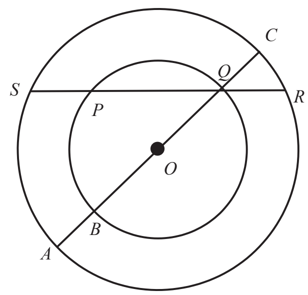
$PQ^2 = PA \cdot PB$

$PR^2 = PA \cdot PB$

$\therefore \quad PQ^2 = PR^2$

$PQ = PR$

5



Let the centre of the circles be O.
Let the radii of the larger and smaller circles be R and r respectively.
Let QP produced meet the larger circle at S.
By symmetry, $SP = RQ$.
Extend OQ to meet the larger circle at A and C, and the smaller circle at B.
Since $SP = RQ$,

$$SP + PQ = RQ + PQ$$

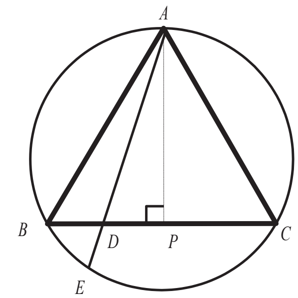
$$\therefore \quad SQ = PR$$

Using the large circle,

$$SQ \cdot RQ = AQ \cdot CQ$$

$$PR \cdot RQ = (R + r)(R - r)\text{, which is constant}$$

6



Let P be a point on BC such that AP is perpendicular to BC.
Because ABC is isosceles, AP will bisect AB. Let $AP = x$ and $PC = PB = y$.

$DP = y - BD$

$CD = 2y - BD$

Using Pythagoras' theorem twice, we get $AB^2 = x^2 + y^2$ in triangle ABP and in triangle ADP.

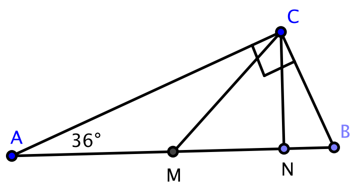
$$\begin{aligned} AD^2 &= x^2 + (y - BD)^2 \\ &= x^2 + y^2 - 2y \times BD + BD^2 \\ &= AB^2 - BD(2y - BD) \\ &= AB^2 - BD \cdot CD \end{aligned}$$

$$BD \cdot CD = DEAD$$

$$\begin{aligned} \therefore \quad AD^2 &= AB^2 - DE \cdot AD \\ AB^2 &= AD^2 + DE \cdot AD \\ &= AD(AD + DE) \\ &= AD \cdot AE \end{aligned}$$

Solutions to technology-free questions

1



AB can be considered as a diameter of the circle centre M passing through C

$MC = MA$ (radii of the circle)

$\angle ACM = 36°$ (isosceles triangle)

$\angle CMN = 72°$ (sum of two interior angles is equal to the opposite exterior angle)

$\angle MCN = (180 - 72 - 90)° = 18°$

2 **a** Theorem 1: $y = \dfrac{140°}{2} = 70°$

Theorem 4: $x + y = 180°$

$$x = 180° - 70° = 110°$$

b Name the quadrilateral $ABCD$, in which y is at A and x is at B.

Let P be the point of intersection of AC and BD.

In triangle XCD,

$$\angle CDX = 180° - 50° - 75°$$
$$= 55°$$

$\angle BCD = 90°$

(angle subtended by a diameter)

In triangle BCD,

$$x = \angle BDC$$
$$= 180° - 90° - 55°$$
$$= 35°$$

$y = x = 35°$(angles subtended by the same arc)

c Angles in the same segment are equal:

$x = 47°$

$y = 53°$

z is the exterior angle angle of either triangle.

Using the left triangle:

$$z = x + 53°$$
$$= 47 + 53°$$
$$= 100°$$

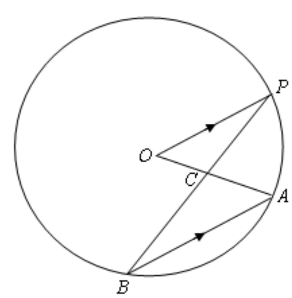
d First note that $y = x$.

Consider the concave quadrilateral containing the 30° angle.

Its angles are 30°, $180° - 70° = 110$, $x + 70°$ and $x + 70°$, using supplementary angles, vertically opposite angles and exterior angles of a triangle.

$$x + 70 + x + 70 + 110 + 30 = 260$$
$$2x + 280 = 260$$
$$x = 40°$$
$$y = 40°$$
$$z = 180 - (x + 70)$$
$$= 70°$$

3



a Using angles on arc AP,

$\angle POA = 2\angle CBA$

Using alternate angles,

$\angle POA = \angle CAB$

$\therefore \quad \angle CAB = 2\angle CBA$

b Using angles on arc AP,

$\angle POA = 2\angle CBA$

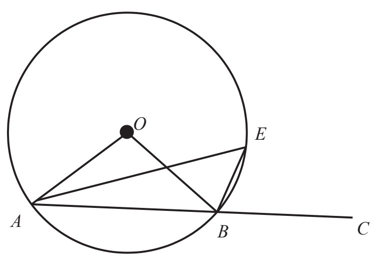
Using alternate angles,

$\angle OPC = \angle CBA$

Using the exterior angle of triangle OCP,

$$\angle PCA = \angle POC + \angle OPC$$
$$= \angle POA + \angle OPC$$
$$\angle PCA = 2\angle CBA + \angle CBA$$
$$= 3\angle CBA$$

4



$\angle OBC = \angle OAB + \angle AOB$ (exterior angle of triangle AOB)

$\angle OBC = \angle OAB + \angle AOB$ (exterior angle of triangle AEB)

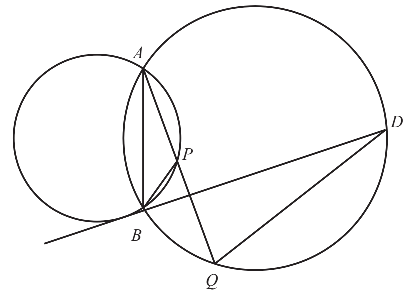
$$\angle BAE = \frac{1}{2}\angle OAB$$

$$\angle BEA = \frac{1}{2}\angle AOB \text{ (angles on arc } AB\text{)}$$

$$\therefore \quad \angle EBC = \frac{1}{2}(\angle OAB + \angle AOB)$$
$$= \frac{1}{2}\angle OBC$$

i.e. EB bisects $\angle OBC$.

5



$\angle PBD = \angle BAP$ (alternate segment)

$\angle BAP = \angle BDQ$ (angles on BQ)

$\therefore \quad PBD = \angle BDQ$

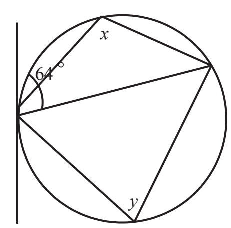
These are alternate angles on BP and QD.

$\therefore \quad BP$ is parallel to QD.

6 **a** The base angle of the isosceles triangle is 57° (alternate segment theorem)

$$x = 180° - 57° - 57°$$
$$= 66°$$

b Make a construction as shown below.

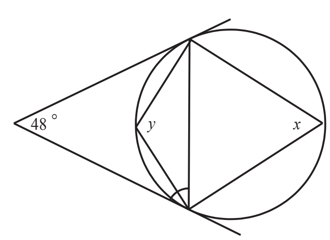


$y = 64°$ (alternate segment theorem)

$x = 180° - 64°$

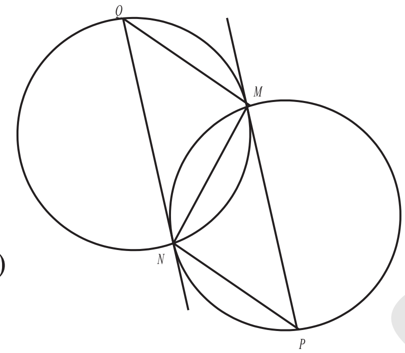
$= 116°$ (cyclic quadrilateral)

c Make a construction as shown below.



$$\text{Marked angle} = \frac{180° - 48°}{2}$$
$$= 66°$$
$$x = 66° \text{ (alternate segment)}$$
$$y = 180° - 66°$$
$$= 114° \text{ (cyclic quadrilateral)}$$

7



Consider triangles MNQ and NPM.

$\angle MQN = \angle NMP$ (alternate segment)

$\angle MNQ = \angle NPM$ (alternate segment)

$\therefore$ the triangles are similar and

$\frac{MN}{Np} = \frac{QM}{MN}$.

Cross multiplying gives

$MN^2 = NP \cdot QM$

8 $AE \cdot EB = CE \cdot ED$

$15 \times 5 = 25ED$

$ED = DE$

$= 3$ cm

Solutions to multiple-choice questions

1 B In isosceles triangle ABD,

$\angle ABD = \angle ADB$

$= \dfrac{180^\circ - 70^\circ}{2} = 55^\circ$

$\angle ACD$ is subtended by the same arc, so $\angle ACD = 55^\circ$.

2 A In quadrilateral $OAPB$,

$\angle OAP = \angle OBP = 90^\circ$

$\angle APB = 360^\circ - 150^\circ - 90^\circ - 90^\circ$

$= 30^\circ$

The angle subtended at the circumference on minor arc AB is $\dfrac{150^\circ}{2} = 75^\circ$.

This angle is opposite Q in a cyclic quadrilateral.

$\therefore$ $\angle AQB = 180^\circ - 75^\circ = 105^\circ$

3 E There are multiple ways to solve this problem.

$\angle OAB = 68^\circ$

$\angle BAT = 90^\circ - 68^\circ = 22^\circ$

$\angle ABT = 180^\circ - 20^\circ - 68^\circ = 92^\circ$

$\angle ATB = 180^\circ - 22^\circ - 92^\circ = 66^\circ$

4 A $\angle BAC = 60^\circ$

Reflex $\angle BOC = 360^\circ - 120^\circ$

$= 240^\circ$

In quadrilateral $ABOC$,

$\angle ABO = 360^\circ - 240^\circ - 42^\circ - 60^\circ$

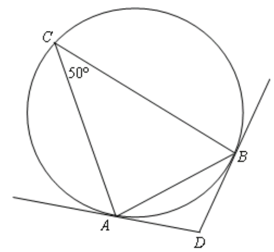
$= 18^\circ$

5 C $\angle DAB = 180^\circ - 65^\circ = 115^\circ$

Corresponding angles on parallel lines

$\therefore$ $\angle CBE = 115^\circ$

6 A



$\angle BAD = 50^\circ$ (alternate segment theorem)

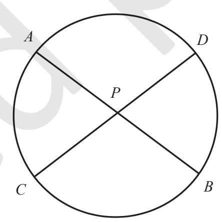
$\angle ABD = 50^\circ$ (alternate segment theorem)

In triangle ABD,

$\angle ADB = 180^\circ - 50^\circ - 50^\circ$

$= 80^\circ$

7 C



$AP \cdot PB = CP \cdot PD$

$12 \times 6 = 2PD$

$PD = 36$ cm

8 B $NB = 13 - 5 = 8$ cm

$NQ = PN$

$AN \cdot NB = PN \cdot NQ$

$= PN^2$

$18 \times 8 = PN^2$

$PN = \sqrt{144} = 12$ cm

$PB^2 = 12^2 + 8^2 = 208$

$PB = \sqrt{208}$

$= \sqrt{16 \times 13}$

$= 4\sqrt{13}$ cm

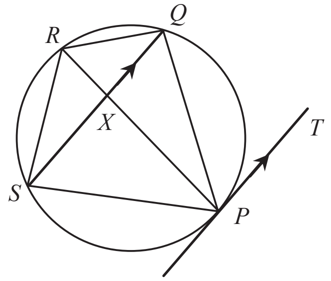
9 A In triangle BAX,

$\angle BAX = 180° - 40° - 105°$
$= 35°$
Angles are subtended by the same arc
$\angle XSC = \angle BAX = 35°$

10 A $\angle CDA = 90°$ (angle subtended by a diameter)
In triangle ACD,
$\angle CAD = 180° - 90° - 25°$
$= 65°$
$\angle CBD = \angle ACD = 65°$
$\angle BCD = 180° - 75° = 105°$
In triangle BCD,
$\angle BDC = 180° - 105° - 65°$
$= 10°$

Solutions to extended-response questions

1 a Let PT be the tangent.



$\angle PSQ = \angle QPT$ (alternate segment)

$\angle PQS = \angle QPT$ (alternate angles)

Since $\angle PSQ = \angle PQS$, triangle PQS is isosceles with $PQ = PS$, as required to prove.

b $\angle PRS = \angle PQS$ (same segment)

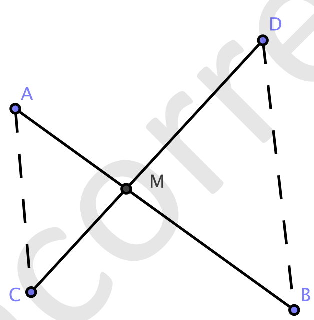
$\angle PRQ = \angle QPT$ (alternate segment)

$= \angle PQS$ (alternate angles)

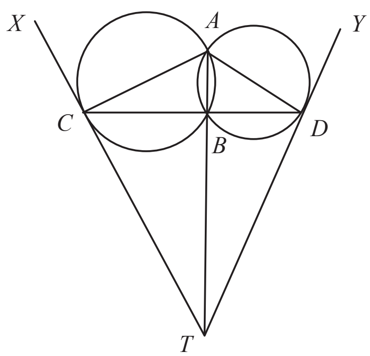
$= \angle PRS$

$\angle QRS = \angle PRQ + \angle PRS = 2\angle PRS$
Therefore PR bisects $\angle QRS$, as required to prove.

2



$AM \times BM = CM \times DM$ implies $\dfrac{AM}{CM} = \dfrac{DM}{BM}$
$\angle AMC = \angle BMD$
Hence $\triangle AMD \sim \triangle DMB$ (SAS). Hence $\angle CAM = \angle BDM$
Hence $ABCD$ is cyclic (converse of Theorem 2)

3 a 

$\angle XCA + \angle ACB + \angle BCT = 180^\circ$ (supplementary)

$\angle XCA = \angle CBA$ (alternate segment)

$\angle TCB = \angle CAB$ (alternate segment)

$\angle BCA + \angle CAB = \angle ABD$ (exterior angle of triangle)

$\angle YDA = \angle ABD$ (alternate segment)

$\angle YDA + \angle ADB + \angle BDT = 180^\circ$

$\therefore \quad \angle ABD + \angle ADB + \angle BDT = 180^\circ$

$\therefore \quad \angle BCA + \angle CAB + \angle ADB + \angle BDT = 180^\circ$

But $\angle CAB = \angle TCB$

$\therefore \quad \angle BCA + \angle TCB + \angle ADB + \angle BDT = 180^\circ$

$\therefore \quad \angle ACT + \angle ADT = 180^\circ$

$\therefore$ $TCAD$ is a cyclic quadrilateral, as required to prove.

b $\angle TAC = \angle BAC$

$= \angle BCT$ (alternate segments in circle ABC)

$= \angle DCT$

$= \angle TAD$ (same segment in circle $ACTD$)

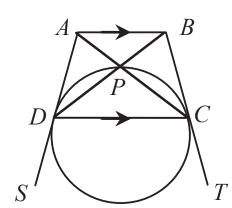
$\therefore \quad \angle TAC = \angle TAD$, as required to prove.

c $TC^2 = TB \cdot TA$ and $TD^2 = TB.\ TA$ (tangent/secant theorem)

$\therefore \quad TC^2 = TD^2$

$\therefore \quad TC = TD$, as required to prove.

4 a AD is a tangent to the circle CDP.



$\angle BAC = \angle ACD$ (alternate angles, $AB \parallel CD$)

$= \angle DCP$

$= \angle ADP$ (alternate segment)

$= \angle ADB$, as required to prove.

b $\angle BAP = \angle BAC$

$= \angle ADB$ (from **a**)

$= \angle ADP$

$\therefore$ AB is a tangent to the circle ADP (alternate segment), with point of contact A, i.e. the circle ADP touches AB at the point A, as required to prove.

c Let $\angle BAC = x^\circ$, $\angle ABD = y^\circ$,

$\therefore$ $\angle ACD = x^\circ$ (alternate angles)

$\angle ADB = x^\circ$ (alternate segment)

$\angle BDC = y^\circ$ (alternate angles)

$\angle BCA = y^\circ$ (alternate segment)

$\therefore$ $\angle ADC = (x + y)^\circ$

and $\angle DCB = (x + y)^\circ$

$\angle ABC + \angle DCB = 180^\circ$ (co-interior angles)

$\therefore$ $\angle ABC + \angle ADC = 180^\circ$

$\therefore$ $ABCD$ is a cyclic quadrilateral, as required to prove.

5 **a** **i** $\angle MSX = \angle MSR$ (supplementary angles)

$= 90^\circ$

Also $MP = MS$ (M midpoint of PS) and $\angle SMX = \angle PMQ$ (vertically opposite)

$\therefore$ triangles MPQ and MSX are congruent.

Therefore $SX = PQ$

$= RS$ (opposite sides of a square)

$\therefore$ S is the midpoint of RX.

Also $\angle PSX = \angle PSR$

$= 90^\circ$ (angle in a square)

Therefore triangle XPS is congruent to triangle RPS.

$\therefore$ $XP = RP$ and triangle XPR is isosceles, as required to prove.

ii $PS = RS$ (sides of a square)

$\therefore$ triangle PRS is isosceles with $\angle RPS = 45^\circ$

$\therefore$ $\angle RPX = 90^\circ$

$\therefore$ $PX \perp OP$ and PX is a tangent to the circle at P, as required to prove.

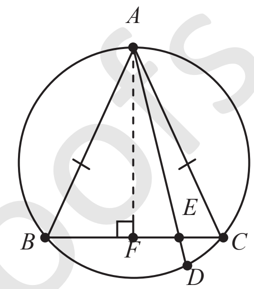
b Area of trapezium = area of square $PQRS$ + area of triangle PSX

$$= 4^2 + \frac{1}{2} \times 4^2$$

$$= 16 + 8$$

$$= 24 \text{ cm}^2$$

6 a Let AF be the perpendicular bisector of BC (since $AB = AC$).



$AB^2 = BF^2 + AF^2$ ① (Pythagoras' theorem)

$AE^2 = AF^2 + FE^2$ ② (Pythagoras' theorem)

① − ② yields

$$AB^2 - AE^2 = BF^2 + AF^2 - AF^2 - FE^2$$

$$= BF^2 - FE^2$$

$$= (BE - FE)^2 - FE^2$$

$$= BE^2 - 2BE \cdot FE + FE^2 - FE^2$$

$$= BE^2 - 2BE \cdot FE$$

$$= BE(BE - 2FE)$$

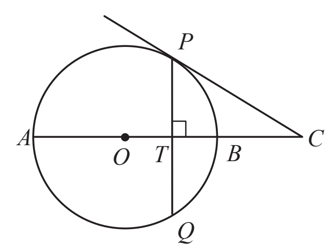
$$= BE(BF + FE - 2FE)$$

$$= BE(BF - FE)$$

$$= BE(CF - FE) \text{ (since } BF = CF\text{)}$$

$$= BE \cdot CE, \text{ as required to prove.}$$

b Extend PT to meet the circle again at Q.



$\therefore$ $PT \cdot QT = AT \cdot BT$ (intersecting chords)

but $QT = PT$ since AB is a diameter

$\therefore$ $PT^2 = AT \cdot BT$

and, by the tangent/secant theorem

$$CP^2 = CA \cdot CB$$

Also $CP^2 = CT^2 + PT^2$ (Pythagoras'theorem)

$$\therefore \quad CA \cdot CB = CT^2 + AT \cdot BT$$

$$\therefore \quad CA \cdot CB - TA \cdot TB = CT^2, \text{ as required to prove.}$$

Chapter 12 – Sampling and sampling distributions

Solutions to Exercise 12A

1 No, the sample is biased towards students who use the internet, because of the email collection method.

2 No, the sample is biased because she is collecting the data at a particular time of day. Some age groups would be more likely to use the restaurant at that time – probably school children and 'young' families.

3 No, the sample is biased towards viewers of that station. Only people with strong opinions will call, and people may call more than once.

4 Answers will vary

5 **a** 0.48

b $\hat{p}$, This is a sample proportion, because it was determined from a sample.

6 **a** The population is all students at this school.

b The population proportion is the proportion of students in the whole school who travel by public transport,0.42.

c The sample proportion is the proportion of students in the sample who travel by public transport,0.37.

7 **a** The population is all Australian adults.

b The population mean is the mean number of hours that Australian adults watch TV per day = 4 hours.

c The sample mean is the mean number of hours that the Australian adults in the sample watch TV per day = 3.5 hours.

Solutions to Exercise 12B

1 a

x	0	1	2
$\Pr(X = x)$	0.16	0.48	0.36

b $\Pr(X \geq 1) = 0.36 + 0.48$

$= 0.84$

2 a $\Pr(X = 3) = 0.35$

b $\Pr(X < 3) = \Pr(X = 1) + \Pr(X = 2)$

$= 0.05 + 0.15$

$= 0.20$

c

$\Pr(X \geq 4) = \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6)$

$= 0.25 + 0.15 + 0.05$

$= 0.45$

d

$\Pr(1 < X < 5) = \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4)$

$= 0.75$

e $\Pr(X \neq 5) = 1 - \Pr(X = 5)$

$= 10.15$

$= 0.85$

f $\Pr(1 < X < 5 \mid X > 1) = \dfrac{0.75}{0.95} = \dfrac{15}{19}$

3 a $\Pr(\hat{P} = 0.2) = 0.0034$

b

$\Pr(\hat{P} < 0.4) = \Pr(\hat{P} = 0) + \Pr(\hat{P} = 0.2)$

$= 0.0001 + 0.0034$

$= 0.0035$

c

$\Pr(\hat{P} \geq 0.8) = \Pr(\hat{P} = 0.8) + \Pr(\hat{P} = 1)$

$= 0.4372 + 0.3060$

$= 0.7342$

d

$\Pr(0.2 < \hat{P} < 0.8) = \Pr(\hat{P} = 0.4) + \Pr(\hat{P} = 0.6)$

$= 0.0422 + 0.2111 + 0.3060$

$= 0.2533$

e

$\Pr(\hat{P} < 0.8 \mid \hat{P} > 0) = \dfrac{\Pr(0 < \hat{P} < 0.8)}{\Pr(\hat{P} > 0)}$

$= \dfrac{0.2567}{0.9999}$

$= 0.2567$

f $\Pr(0.2 < \hat{P} < 0.8 \mid \hat{P} > 0.4)$

$= \dfrac{\Pr(0.2 < \hat{P} < 0.8)}{\Pr(\hat{P} > 0.4)}$

$= \dfrac{0.2533}{0.2111 + 0.4372 + 0.3060}$

$= 0.2654$

4 a $p = \dfrac{8}{16} = 0.5$

b Number of soft centred chocolates could be 0, 1, 2 or 3. Thus, possible values of $\hat{P}$ are $0, \frac{1}{3}, \frac{2}{3}, 1$

c $\Pr(\hat{P} = 0) = \Pr(X = 0)$

$= \dfrac{\binom{8}{0}\binom{8}{3}}{\binom{16}{3}}$

$= \dfrac{56}{560}$

$= 0.1$

$$\Pr(\hat{P} = \tfrac{1}{3}) = \Pr(X = 1)$$
$$= \frac{\binom{8}{1}\binom{8}{2}}{\binom{16}{3}}$$
$$= \frac{224}{560}$$
$$= 0.4$$

$$\Pr(\hat{P} = \tfrac{2}{3}) = \Pr(X = 2)$$
$$= \frac{\binom{8}{2}\binom{8}{1}}{\binom{16}{3}}$$
$$= \frac{224}{560}$$
$$= 0.4$$

$$\Pr(\hat{P} = 1) = \Pr(X = 3)$$
$$= \frac{\binom{8}{3}\binom{8}{0}}{\binom{16}{3}}$$
$$= \frac{56}{560}$$
$$= 0.1$$

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.1	0.4	0.4	0.1

d $\Pr(\hat{P} > 0.25) = 0.9$

5 a $p = \dfrac{12}{20} = 0.6$

b Number of male swimmers could be 0, 1, 2, 3, 4
The values of $\hat{P}$ are
0, 0.2, 0.4, 0.6, 0.8, 1

c $\Pr(\hat{P} = 0) = \Pr(X = 0)$
$$= \frac{\binom{12}{0}\binom{8}{5}}{\binom{20}{5}}$$
$$= 0.0036$$

$$\Pr(\hat{P} = \tfrac{1}{5}) = \Pr(X = 1)$$
$$= \frac{\binom{12}{1}\binom{8}{4}}{\binom{20}{5}}$$
$$= 0.0545$$

$$\Pr(\hat{P} = \tfrac{2}{5}) = \Pr(X = 2)$$
$$= \frac{\binom{12}{2}\binom{8}{3}}{\binom{16}{3}}$$
$$= 0.2384$$

$$\Pr(\hat{P} = \tfrac{3}{5}) = \Pr(X = 3)$$
$$= \frac{\binom{12}{3}\binom{8}{2}}{\binom{20}{5}}$$
$$= 0.3973$$

$$\Pr(\hat{P} = \tfrac{4}{5}) = \Pr(X = 4)$$
$$= \frac{\binom{12}{4}\binom{8}{1}}{\binom{20}{5}}$$
$$= 0.2554$$

$$\Pr(\hat{P} = 1) = \Pr(X = 5)$$
$$= \frac{\binom{12}{5}\binom{8}{0}}{\binom{20}{5}}$$
$$= 0.0511$$

$\hat{p}$	0	0.2	0.4
$\Pr(\hat{P} = \hat{p})$	0.0036	0.0542	0.2384

$\hat{p}$	0.6	0.8	1
$\Pr(\hat{P} = \hat{p})$	0.3973	0.2554	0.0511

d $\Pr(\hat{P} > 0.7) = 0.3065$

e $\Pr(0 < \hat{P} < 0.8) = 0.6899$,
$$\Pr(\hat{P} < 0.8 \mid \hat{P} > 0) = \frac{\Pr(0 < \hat{P} < 0.8)}{\Pr(\hat{P} > 0)}$$
$$= 0.6924$$

6 a $p = \dfrac{10}{50} = 0.2$

b Possible number of defectives could be 0, 1, 2, 3, or 4. Therefore values of $\hat{P}$ are $0, 0.25, 0.5, 0.75, 1$.

c $\Pr(\hat{P} = 0) = \Pr(X = 0)$

$$= \frac{\binom{15}{0}\binom{35}{4}}{\binom{50}{4}}$$

$$= 0.2274$$

$\Pr(\hat{P} = \frac{1}{4}) = \Pr(X = 1)$

$$= \frac{\binom{15}{1}\binom{35}{3}}{\binom{50}{4}}$$

$$= 0.4263$$

$\Pr(\hat{P} = \frac{2}{4}) = \Pr(X = 2)$

$$= \frac{\binom{15}{2}\binom{35}{2}}{\binom{50}{4}}$$

$$= 0.2713$$

$\Pr(\hat{P} = \frac{3}{4}) = \Pr(X = 3)$

$$= \frac{\binom{15}{3}\binom{35}{1}}{\binom{50}{4}}$$

$$= 0.0691$$

$\Pr(\hat{P} = 1) = \Pr(X = 4)$

$$= \frac{\binom{12}{4}\binom{35}{0}}{\binom{50}{4}}$$

$$= 0.0059$$

$\hat{p}$	0	0.2	0.5
$\Pr(\hat{P} = \hat{p})$	0.2274	0.4263	0.2713

$\hat{p}$	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0691	0.0059

d $\Pr(\hat{P} > 0.5) = 0.075$

e $\Pr(0 < \hat{P} < 0.5) = 0.4263$,
$\Pr(\hat{P} < 0.5 \mid \hat{P} > 0) = 0.5518$

7 a $\Pr(\hat{P} > 0.6) = \Pr(\hat{P} = \frac{2}{3}) + \Pr(\hat{P} = 1)$

$$= 0.028$$

b $\Pr(0 < \hat{P} < 0.6) = \Pr(\hat{P} = \frac{1}{3}) = 0.243$,

$$\Pr(\hat{P} < 0.6 \mid \hat{P} > 0) = \frac{\Pr(0 < \hat{P} < 0.6)}{\Pr(\hat{P} > 0)}$$

$$= 0.897$$

8 a $p = 0.5$

b Values of $\hat{P}$are
$0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$

c Binomial with $n = 10, p = 0.5$

$\Pr(\hat{P} = 0) = \Pr(X = 0)$

$$= \binom{10}{0}(0.5)^0(0.5)^{10}$$

$$= 0.00098$$

$\Pr(\hat{P} = 0.1) = \Pr(X = 1)$

$$= \binom{10}{1}(0.5)^1(0.5)^9$$

$$= 0.0098$$

The following values are obtained in the same way

$\hat{p}$	0	0.1	0.2	0.3
$\Pr(\hat{P} = \hat{p})$	0.00098	0.0098	0.0440	0.1172

$\hat{p}$	0.4	0.5	0.6	0.7
$\Pr(\hat{P} = \hat{p})$	0.2051	0.2461	0.2051	0.1172

$\hat{p}$	0.8	0.9	1
$\Pr(\hat{P} = \hat{p})$	0.0440	0.0098	0.00098

d $\Pr(\hat{P} > 0.5) = 0.3771$

9 a The values that $\hat{P}$ can take are $0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$

b Binomial with $n = 6, p = 0.52$

$$\Pr(\hat{P} = 0) = \Pr(X = 0) = \binom{6}{0}(0.52)^0(0.48)^6 = 0.0122$$

Binomial with $n = 6, p = 0.52$

$$\Pr(\hat{P} = \tfrac{1}{6}) = \Pr(X = 1) = \binom{6}{1}(0.52)^1(0.48)^5 = 0.0795$$

The following values are obtained in the same way.

$\hat{p}$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
$\Pr(\hat{P} = \hat{p})$	0.0122	0.0795	0.2153	0.3110

$\hat{p}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$\Pr(\hat{P} = \hat{p})$	0.2527	0.1095	0.0198

c $\Pr(\hat{P} > 0.6) = 0.307$

d $$\Pr(\hat{P} < 0.3 \mid \hat{P} < 0.8) = \frac{\Pr(\hat{P} < 0.3)}{\Pr(\hat{P} < 0.8)} = 0.1053$$

10 a The values that $\hat{P}$ can take are $0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1$

b Binomial with $n = 8, p = 0.8$

$$\Pr(\hat{P} = 0) = \Pr(X = 0) = \binom{8}{0}(0.8)^0(0.2)^6 = 0.000003$$

Binomial with $n = 8, p = 0.8$

$$\Pr(\hat{P} = \tfrac{1}{8}) = \Pr(X = 1) = \binom{8}{1}(0.8)^1(0.2)^5 = 0.00008$$

The following values are obtained in the same way.

$\hat{p}$	0	$\frac{1}{8}$	$\frac{1}{4}$
$\Pr(\hat{P} = \hat{p})$	0.000003	0.00008	0.00115

$\hat{p}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
$\Pr(\hat{P} = \hat{p})$	0.0092	0.0459	0.1468

$\hat{p}$	$\frac{3}{4}$	$\frac{7}{8}$	1
$\Pr(\hat{P} = \hat{p})$	0.2936	0.3355	0.1678

c $\Pr(\hat{P} > 0.6) = 0.1468 + 0.2936 + 0.3355 + 0.1678 = 0.9437$

d

$$\Pr(\hat{P} > 0.6 \mid \hat{P} > 0.25) = \frac{\Pr(\hat{P} > 0.6)}{\Pr(\hat{P} > 0.25)} = 0.9448$$

11 a

$\hat{p}$	0	0.25	0.5	0.75	1
Hyp	0.0587	0.2499	0.3827	0.2499	0.0587
Bin	0.0625	0.25	0.375	0.25	0.0625

b

$\hat{p}$	0	0.1	0.2	0.3
Hyp	0.0006	0.0072	0.0380	0.1131
Bin	0.00098	0.0098	0.0440	0.1172

$\hat{p}$	0.4	0.5	0.6	0.7
Hyp	0.2114	0.2593	0.2114	0.1131
Bin	0.2051	0.2461	0.2051	0.1172

$\hat{p}$	0.8	0.9	1
Hyp	0.0380	0.0072	0.0006
Bin	0.0440	0.0098	0.00098

c Not much

Solutions to Exercise 12C

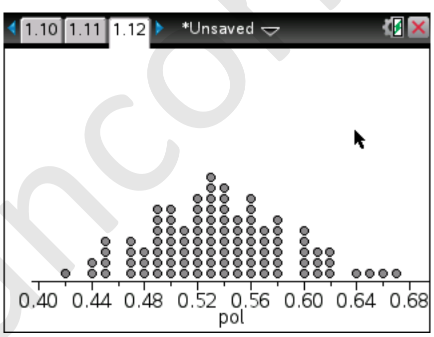
1 **a** a. There are eight dots which represent sample proportions of 0.8 or more from the 100 samples simulated. Thus we can estimate $\Pr(\hat{P} \geq 0.8) = 0.08$.

b b. There is one dot which represent a sample proportions of 0.5 or less from the 100 samples simulated. Thus we can estimate $\Pr(\hat{P} \leq 0.5) = 0.01$.

2 **a** There is one dot which represent sample proportions of 0.7 or more from the 100 samples simulated. Thus we can estimate $\Pr(\hat{P} \geq 0.7) = 0.01$.

b There are seven dots which represent a sample proportion of 0.25 or less from the 100 samples simulated. Thus we can estimate $\Pr(\hat{P} \leq 0.25) = 0.07$.

3 **a**

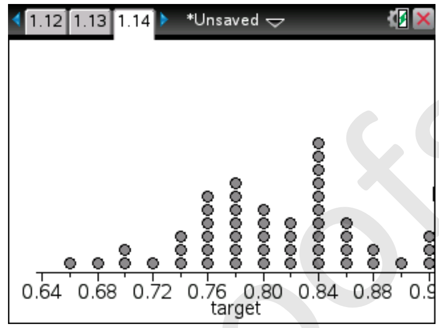


b see **a**

c **i** $\Pr(\hat{P} \geq 0.64) \approx 0.04$.(Answers will differ)

ii $\Pr(\hat{P} \leq 0.44) \approx 0.03$. (Answers will differ)

4 **a**

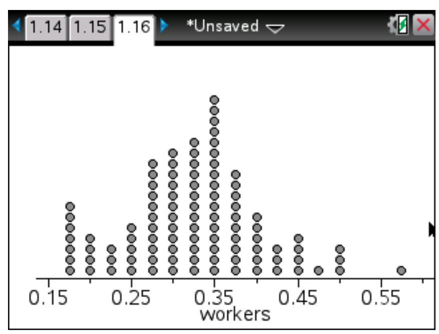


b see **a**

c **i** $\Pr(\hat{P} \geq 0.9) \approx 0.06$.(Answers will differ)

ii $\Pr(\hat{P} \leq 0.7) \approx 0.08$. (Answers will differ)

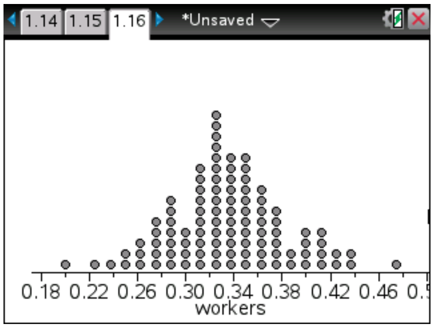
5 **a**



b see **a**

i $\Pr(\hat{P} \geq 0.45) \approx 0.18$.(Answers will differ)

ii $\Pr(\hat{P} \leq 0.25) \approx 0.38$.(Answers will differ)

6 **a** 

b see **a**

i $\Pr(\hat{P} \geq 0.45) \approx 0.01$.(Answers will differ)

ii $\Pr(\hat{P} \leq 0.25) \approx 0.06$(.Answers will differ)

Solutions to Exercise 12D

1 a There are two dots which represent sample means of 25 or more from the 100 samples simulated. Thus we can estimate $\Pr(\bar{X} \geq 25) = 0.02$.

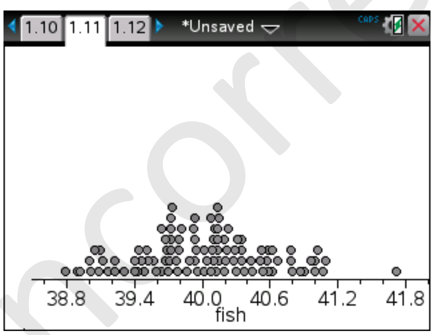
b There is one dot which represent a sample means of 23 or less from the 100 samples simulated. Thus we can estimate $\Pr(\bar{X} \leq 23) = 0.01$.

2 a There are four dots which represent sample means of 163 or more from the 100 samples simulated. Thus we can estimate $\Pr(\bar{X} \geq 163) = 0.04$

b b. There are five dots which represent sample means of 158 or less from the 100 samples simulated. Thus we can estimate $\Pr(\bar{X} \leq 158) = 0.05$

3 a calculator

b

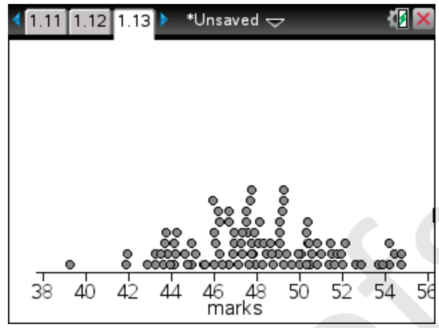


c i $\Pr(\bar{X} \geq 41) \approx 0.04/$

ii $\Pr(\bar{X} \leq 39) \approx 0.04.$

4 a calculator

b

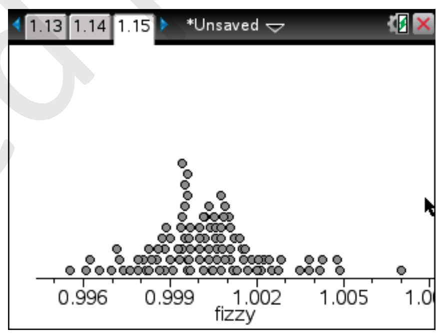


i $\Pr(\bar{X} \geq 55) \approx 0.01.$

ii $\Pr(\bar{X} \leq 40) \approx 0.01.$

5 a calculator

b

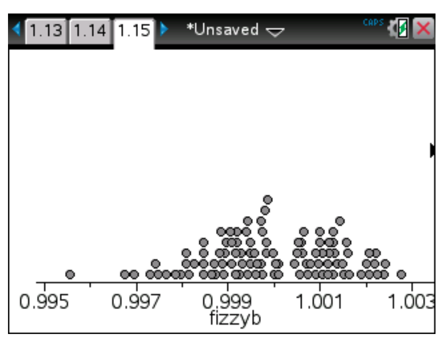


i $\Pr(\bar{X} \geq 1.003) \approx 0.07$

ii $\Pr(\bar{X} \leq 0.995) \approx 0.01$

6 a calculator

b



i $\Pr(\bar{X} \geq 1.003) \approx 0$

ii $\Pr(\bar{X} \leq 0.995) \approx 0$

Solutions to technology-free questions

1 **a** Employees of the company

b $p = \dfrac{\text{number of females in company}}{\text{number of people in company}} = 0.35$

c $\hat{p} = \dfrac{\text{number of females in the sample}}{\text{number of people in the sample}} = 0.4$

2 No, this sample (people already interested in yoga) is not representative of the population

3 No, people who choose to live in houses with gardens may not be representative of the population

4 **a** People with Type II diabetes

b Population is too large and dispersed to use for such an experiment.

c Unknown

d $\bar{x} = 1.5$

5 **a** All of the employees of the company

b p = number of people in the company who are tertiary qualified divided by the number of people in the company = 0.2

c $\hat{p}$ number of people in the sample who are tertiary qualifieddivided by the number of people in the sample=0.22

6 **a** p =number of people in the team who are femaledivided by the number of people in the team $=\dfrac{3}{5}$

b The values of $\hat{P}$ are $\dfrac{1}{3}, \dfrac{2}{3}, 1$

c $\Pr(\hat{P} = \frac{1}{3}) = \Pr(X = 1)$

$$= \frac{\binom{3}{1}\binom{2}{2}}{\binom{5}{3}}$$

$$= \frac{3}{10}$$

$\Pr(\hat{P} = \frac{2}{3}) = \Pr(X = 2)$

$$= \frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}}$$

$$= \frac{6}{10}$$

$\Pr(\hat{P} = 1) = \Pr(X = 3)$

$$= \frac{\binom{3}{3}\binom{2}{0}}{\binom{5}{3}}$$

$$= \frac{1}{10}$$

$\hat{p}$	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{1}{10}$

d $\Pr(\hat{P} > 0.5) = \dfrac{6}{10} + \dfrac{1}{10} = \dfrac{7}{10}$

e $\Pr(0 < \hat{P} < 0.5) = \dfrac{3}{10}$,

$\Pr(\hat{P} < 0.5 \,|\, \hat{P} > 0) = \dfrac{3}{10}$

7 **a** Values of $\hat{P}$ are 0, 0.25, 0.5, 0.75, 1

b Binomial with $n = 4$, $p = 0.5$

$$\Pr(\hat{P} = 0) = \Pr(X = 0)$$
$$= \binom{4}{0}(0.5)^0(0.5)^4$$
$$= 0.0625$$
$$\Pr(\hat{P} = 0.25) = \Pr(X = 1)$$
$$= \binom{4}{1}(0.5)^1(0.5)^3$$
$$= 0.25$$
$$\Pr(\hat{P} = 0.5) = \Pr(X = 2)$$
$$= \binom{4}{2}(0.5)^2(0.5)^2$$
$$= 0.375$$
$$\Pr(\hat{P} = 0.75) = \Pr(X = 3)$$
$$= \binom{4}{3}(0.5)^3(0.5)^1$$
$$= 0.25$$
$$\Pr(\hat{P} = 1) = \Pr(X = 4)$$
$$= \binom{4}{4}(0.5)^4(0.5)^0$$
$$= 0.0625$$

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0625	0.25	0.375	0.25	0.0625

c $\Pr(\hat{P} < 0.5) = 0.3125$

d $\Pr(\hat{P} < 0.5 \mid \hat{P} < 0.8) = \dfrac{\Pr(\hat{P} < 0.5)}{\Pr(\hat{P} < 0.8)}$

$$= \frac{1}{3}$$

8 **a** **i** There are three dots which represent sample proportions of 0.7 or more from the 100 samples simulated. Thus we can estimatePr($\hat{P} \geq 0.7$) = 0.03

ii There are four dots which represent a sample proportion of 0.38 or less from the 100 samples simulated.Thus we can estimate $\Pr(\hat{P} \leq 0.38) = 0.04$

b **i** $\hat{p} = \dfrac{\text{number of people in the sample who will vote for Bill Bloggs}}{\text{number of people in the sample}} = 0.42$

ii From the plot there are 8 samples where the sample proportion is 0.42 or less, from the 100 simulations.Thus we can estimate that $\Pr(\hat{P} \leq 0.42) = 0.08$

Solutions to multiple-choice questions

1 B Since this ratio is determined from a sample it is a sample statistic.

2 C Since this percentage is determined from complete census it is a population parameter

3 A In reality we rarely know the value of a population parameter, whereas we can determine the value of a sample statistic. So generally, we are using the sample statistic to estimate the value of a population parameter.

4 B The sampling distribution describes how the values of the sampling distribution vary from sample to sample.

5 B Since sample statics are estimates of population parameters.

6 E

$$\begin{aligned}\Pr(\hat{P} \geq 0.7 | \hat{P} > 0.2) &= \frac{\Pr(\hat{P} \geq 0.7)}{\Pr(\hat{P} > 0.2)}\\ &= \frac{(0.048 + 0.005}{(0.572}\\ &= \frac{0.053}{0.572}\\ &= 0.092657\ldots\end{aligned}$$

7 A There are 4 vegetarians and 6 non-vegetarians. If the proportion of vegetarians with plastic plates is $\hat{P}$, then:

$\hat{P}$ takes values $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

If $\hat{P} = \frac{3}{4}$ then three of the 4 vegetarians got a plastic plate, and 4 of the 6 non-vegetarians got a plastic plate, that is:

$$\Pr(\hat{P} = 1) = \frac{\binom{4}{4}\binom{6}{0}}{\binom{10}{4}}$$

$$= 0.1143$$

and

$$\Pr(\hat{P} = \tfrac{3}{4}) = \frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}}$$

$$= 0.0048$$

Thus $\Pr(\hat{P} > 0.5) = 0.1191$

8 D There are 12 gold fish and 8 black fish. If the proportion of gold fish in the sample of five is $\hat{p}$, then $\hat{P}$ takes values 0, 0.2, 0.4, 0.6, 0.8, 1.

If $\hat{P} = 0.8$, then 4 of the 5 fish are gold, that is:

$$\Pr(\hat{P} = 0.8) = \frac{\binom{12}{4}\binom{8}{1}}{\binom{20}{5}}$$

$$= 0.2554$$

and

$$\Pr(\hat{P} = 1) = \frac{\binom{12}{5}\binom{8}{0}}{\binom{20}{5}}$$

$$= 0.0511$$

Thus $\Pr(\hat{P} > 0.8) = 0.3065$

9 E X = number of students who study Chinese in the sample of size 20.

X has a binomial distribution,

$n = 20, p = 0.2$

$\hat{P}$ = proportion of studenrs who speak Chinese in the sample of size 20. $\hat{P}$ takes values 0, 0.05, 0.1,0.15, 0.20,0.951

$\Pr(\hat{P} < 0.1) = \Pr(\hat{P} = 0) + \Pr(\hat{P} = 0.05)$

We have

Pr $\hat{P}$ = 0) = $\Pr(X = 0) = 0.0015$

$\Pr(\hat{P} = 0.05) = \Pr(X = 1) = 0.0576$
$\therefore \Pr(\hat{P} < 0.10) = 0.0691$

10 B X = number of heads in the sample of size 10.
X has a binomial distribution,
$n = 10, p = 0.5$
$\hat{P}$ = proportion of heads in the sample of size 10.
$\hat{P}$ takes values $0, 0.1, 0.2, 0.3 \ldots$
$\Pr(\hat{P} < 0.2 \text{ or } \hat{P} > 0.8) = \Pr(\hat{P} < 0.2) + \Pr(\hat{P} > 0.8)$
$\Pr(\hat{P} < 0.2) = \Pr(X < 2) = 0.01074$
$\Pr(\hat{P} > 0.8) = \Pr(X > 8) = 0.01074$
$\Pr(\hat{P} < 0.2 \text{ or } \Pr \hat{P} > 0.8) = 0.02148 \approx 2\%$

11 C The exact sampling distribution is hypergeometric, but when the sample size is small compared to the population size, the binomial distribution gives a good approximation.

12 E Thus increasing the sample size will result in a decrease in the variability of the sample estimates, as we have seen from the sampling dsitributions.

Solutions to extended-response questions

1 a

p	a	b
0.1	0.03	0.17
0.2	0.11	0.29
0.3	0.19	0.41
0.4	0.29	0.51
0.5	0.38	0.62
0.6	0.49	0.71

b **i** $\hat{p} = 0.34$

ii $p = 0.3$ or $p = 0.4$

2 a iii mean ≈ 50, s.d. ≈ 1.12

b iii mean ≈ 50, s.d. ≈ 0.71

c iii mean ≈ 50, s.d. ≈ 0.50

Chapter 13 – Trigonometric ratios and applications

Solutions to Exercise 13A

1 a $\frac{x}{5} = \cos 35°$

$x = 5 \times 0.8191$

$= 4.10$ cm

b $\frac{x}{10} = \sin 45°$

$x = 10 \times 0.0871$

$= 0.87$ cm

c $\frac{x}{8} = \tan 20.16°$

$x = 8 \times 0.3671$

$= 2.94$ cm

d $\frac{x}{7} = \tan 30°15'$

$x = 7 \times 0.9661$

$= 4.08$ cm

e $\tan x° = \frac{10}{15}$

$= 0.666$

$x = 33.69°$

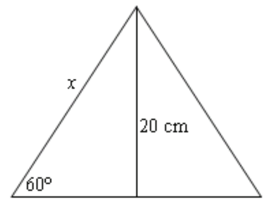
f $\frac{10}{x} = \tan 40°$

$10 = x \times 0.8390$

$x = \frac{10}{0.8390}$

$= 11.92$ cm

2

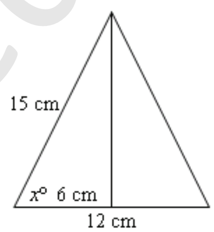


$\frac{20}{x} = \sin 60°$

$20 = x \times \frac{\sqrt{3}}{2}$

$x = \frac{40}{\sqrt{3}} = \frac{40\sqrt{3}}{3}$ cm

3



$\cos x° = \frac{6}{15} = 0.4$

$x° = 66.42°$

The third angle $= 180° - 2 \times 66.42°$

$= 47.16°$

4 $\frac{h}{20} = \tan 49°$

$x = 20 \times 1.1503$

≈ 23 m

5 a $\sin \angle ACB = \frac{1}{6}$

$\angle ACB = 9.59°$

b $BC^2 = 6^2 - 1^2 = 35$

$BC = \sqrt{35}$ m

$= 5.92$ m

6 a $\cos\theta = \frac{10}{20} = 0.5$

$\theta = 60°$

b $\frac{PQ}{20} = \sin 60°$

$PQ = 20 \times 0.866$

$= 17.32$ m

7 a $\frac{3}{L} = \sin 26°$

where L m is the length if the ladder

$3 = L \times 0.4383$

$L = \frac{3}{0.4383}$

$= 6.84$ m

b $\frac{3}{h} = \tan 26°$

where h m is the height above the ground.

$3 = h \times 0.4877$

$h = \frac{3}{0.4877}$

$= 6.15$ m

8 $\sin\theta = \frac{13}{60} = 0.21666\ldots$

$\theta = 12.51°$

9 $\frac{h}{200} = \sin 66°$

$x = 200 \times 0.9135$

$= 182.7$ m

10 $\frac{400}{d} = \sin 16°$

$400 = d \times 0.2756$

$d = \frac{400}{0.2756}$

$= 1451$ m

11 Since the diagonals are equal in length, the rhombus must be a square.

a $AC^2 = BC^2 + BA^2 = 2BC^2$

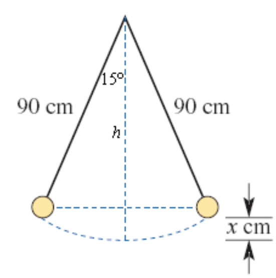
$100 = 2BC^2$

$BC^2 = 50$

$BC = \sqrt{50} = 5\sqrt{2}$ cm

b As the rhombus is a square, $\angle ABC = 90°$.

12 Find the vertical height, h cm.



$\frac{h}{90} = \cos 15°$

$h = 90 \times 0.9659$

$h = 86.93$ cm

$x = 90 - 86.93 = 3.07$ cm

13 $\dfrac{15}{\left(\dfrac{L}{2}\right)} = \sin 52.5°$

$$15 = \frac{L}{2} \times 0.7933$$

$$L = \frac{30}{0.7933}$$

$$= 37.8 \text{ cm}$$

14 $\dfrac{w}{50} = \tan 32°$

$$w = 50 \times 0.6248$$

$$= 31.24 \text{ cm}$$

15 $h^2 + 1.7^2 = 4.7^2$

$$h^2 = 4.7^2 - 1.7^2$$

$$= 19.2$$

$$h = 4.38 \text{ m}$$

16 $\dfrac{50}{d} = \sin 60°$

$$50 = d \times 0.866$$

$$d = \frac{50}{0.866}$$

$$= 57.74 \text{ m}$$

17 Let length of the flagpole be l

$$\sin 60 = \frac{l}{l+2}$$

$$\frac{\sqrt{3}}{2} = \frac{l}{l+2}$$

$$(l+2)\frac{\sqrt{3}}{2} = l$$

$$(\frac{\sqrt{3}}{2} - 1)l = -\sqrt{3}$$

$$l = \frac{\sqrt{3}}{\frac{-\sqrt{3}}{2} - 1}$$

$$l = \frac{2\sqrt{3}}{2 - \sqrt{3}}$$

18 Perimeter $=10 \Rightarrow x + h + opp = 10$

$$\cos 30 = \frac{x}{h}$$

$$h = \frac{x}{\cos 30} = \frac{x}{\frac{\sqrt{3}}{2}} = \frac{2x}{\sqrt{3}}$$

$$\tan 30 = \frac{opp}{x}$$

$$opp = x \tan 30 = \frac{1}{\sqrt{3}}x$$

$$x + \frac{x}{\cos 30} + x \tan 30 = 10$$

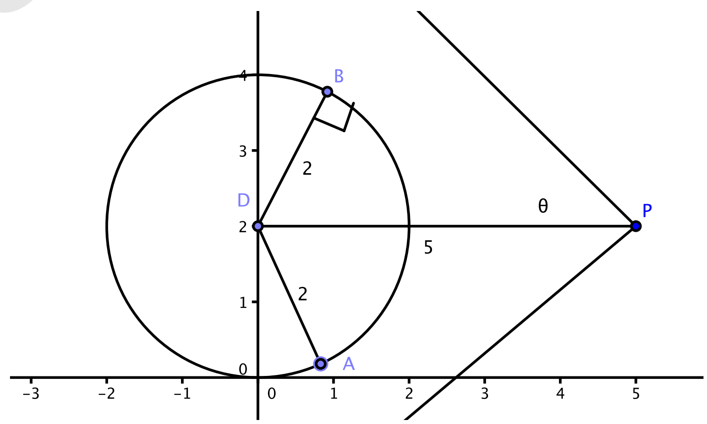
$$x + \frac{2x}{\sqrt{3}} + \frac{1}{\sqrt{3}}x = 10$$

$$(\sqrt{3} + 1)x = 10$$

$$x = \frac{10}{\sqrt{3}+1} = 5(\sqrt{3} - 1)$$

$$x \approx 3.66$$

19



In $\triangle PDB$, Let $\angle PBD = \theta$

Then $\sin\theta = \dfrac{2}{5}$

Hence $\theta = 23.578...°$

$\angle APB = 2\theta \approx 47.16°$

Solutions to Exercise 13B

1 a $\dfrac{x}{\sin 50^\circ} = \dfrac{10}{\sin 70^\circ}$

$x = \dfrac{10 \times \sin 50^\circ}{\sin 70^\circ}$

$= 8.15$ cm

b $\dfrac{y}{\sin 37^\circ} = \dfrac{6}{\sin 65^\circ}$

$x = \dfrac{6 \times \sin 37^\circ}{\sin 65^\circ}$

$= 3.98$ cm

c $\dfrac{x}{\sin 100^\circ} = \dfrac{5.6}{\sin 28^\circ}$

$x = \dfrac{5.6 \times \sin 100^\circ}{\sin 28^\circ}$

$= 11.75$ cm

d $x = 180^\circ - 38^\circ - 90^\circ$

$= 52^\circ$

$\dfrac{x}{\sin 52^\circ} = \dfrac{12}{\sin 90^\circ}$

$x = \dfrac{12 \times \sin 52^\circ}{\sin 90^\circ}$

$= 9.46$ cm

2 a $\dfrac{\sin\theta}{7} = \dfrac{\sin 72^\circ}{8}$

$\sin\theta = \dfrac{7 \times \sin 72^\circ}{8}$

$= 0.8321$

$\theta = 56.32^\circ$

In this case θ cannot be obtuse. Since it is opposite a smaller side.

b $\dfrac{\sin\theta}{8.3} = \dfrac{\sin 42^\circ}{9.4}$

$\sin\theta = \dfrac{8.3 \times \sin 42^\circ}{9.4}$

$= 0.5908$

$\theta = 36.22^\circ$

In this case θ cannot be obtuse. Since it is opposite a smaller side.

c $\dfrac{\sin\theta}{8} = \dfrac{\sin 108^\circ}{10}$

$\sin\theta = \dfrac{8 \times \sin 108^\circ}{10}$

$= 0.7608$

$\theta = 49.54^\circ$

In this case θ cannot be obtuse. Since the given angle is obtuse.

d $\dfrac{\sin\theta}{9} = \dfrac{\sin 38^\circ}{8}$

$\sin\theta = \dfrac{9 \times \sin 38^\circ}{8}$

$= 0.6929$

$\theta = 43.84^\circ$ or $180 - 43.84$

$= 131.16^\circ$

$\theta = 180 - 43.84 - 38 = 98.16^\circ$

or $180 - 136.16 - 38 = 5.84^\circ$

3 a
$$A = 180^\circ - 59^\circ - 73^\circ = 48^\circ$$
$$\frac{b}{\sin 59^\circ} = \frac{12}{\sin 48^\circ}$$
$$b = \frac{12 \times \sin 59^\circ}{\sin 48^\circ} = 13.84 \text{ cm}$$
$$\frac{c}{\sin 73^\circ} = \frac{12}{\sin 48^\circ}$$
$$c = \frac{12 \times \sin 73^\circ}{\sin 48^\circ} = 15.44 \text{ cm}$$

b
$$C = 180^\circ - 75.3^\circ - 48.25^\circ = 56.45^\circ$$
$$\frac{a}{\sin 75.3^\circ} = \frac{5.6}{\sin 48.25^\circ}$$
$$a = \frac{5.6 \times \sin 75.3^\circ}{\sin 48.25^\circ} = 7.26 \text{ cm}$$
$$\frac{c}{\sin 56.45^\circ} = \frac{5.6}{\sin 48.25^\circ}$$
$$c = \frac{5.6 \times \sin 56.45^\circ}{\sin 48.25^\circ} = 6.26 \text{ cm}$$

c
$$B = 180^\circ - 123.2^\circ - 37^\circ = 19.8^\circ$$
$$\frac{b}{\sin 19.8^\circ} = \frac{11.5}{\sin 123.2^\circ}$$
$$b = \frac{11.5 \times \sin 19.8^\circ}{\sin 123.2^\circ} = 4.66 \text{ cm}$$
$$\frac{c}{\sin 37^\circ} = \frac{11.5}{\sin 123.2^\circ}$$
$$c = \frac{11.5 \times \sin 37^\circ}{\sin 123.2^\circ} = 8.27 \text{ cm}$$

d
$$C = 180^\circ - 23^\circ - 40^\circ = 117^\circ$$
$$\frac{b}{\sin 40^\circ} = \frac{15}{\sin 23^\circ}$$
$$b = \frac{15 \times \sin 40^\circ}{\sin 23^\circ} = 24.68 \text{ cm}$$
$$\frac{c}{\sin 117^\circ} = \frac{15}{\sin 23^\circ}$$
$$c = \frac{15 \times \sin 117^\circ}{\sin 23^\circ} = 34.21 \text{ cm}$$

e
$$C = 180^\circ - 10^\circ - 140^\circ = 30^\circ$$
$$\frac{a}{\sin 10^\circ} = \frac{20}{\sin 140^\circ}$$
$$a = \frac{20 \times \sin 10^\circ}{\sin 140^\circ} = 5.40 \text{ cm}$$
$$\frac{c}{\sin 30^\circ} = \frac{20}{\sin 140^\circ}$$
$$c = \frac{20 \times \sin 30^\circ}{\sin 140^\circ} = 15.56 \text{ cm}$$

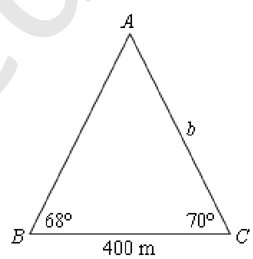
4 a
$$\frac{\sin B}{17.6} = \frac{\sin 48.25^\circ}{15.3}$$
$$\sin B = \frac{17.6 \times \sin 48.25^\circ}{15.3} = 0.8582$$
$$B = 59.12^\circ \text{ or } 180^\circ - 59.12^\circ = 120.88^\circ$$

$$A = 180° - 48.25° - 59.12°$$
$$= 72.63°$$
$$\text{or } 180 - 48.25° - 120.88°$$
$$= 10.87°$$
$$\frac{15.3}{\sin 48.25°} = \frac{a}{\sin 72.63°} \text{ or } \frac{a}{\sin 10.87°}$$
$$a = \frac{15.3 \times \sin 72.63°}{\sin 48.25°}$$
$$\text{or } \frac{15.3 \times \sin 10.87°}{\sin 48.25°}$$
$$= 19.57 \text{ cm or } 3.87 \text{ cm}$$

b
$$\frac{\sin C}{4.56} = \frac{\sin 129°}{7.89}$$
$$\sin C = \frac{4.56 \times \sin 129°}{7.89}$$
$$= 0.4991$$
$$C = 26.69°$$
$$A = 180° - 129° - 26.69°$$
$$= 24.31°$$
$$\frac{a}{\sin 24.31°} = \frac{7.89}{\sin 129°}$$
$$a = \frac{7.89 \times \sin 24.31°}{\sin 129°}$$
$$= 4.18 \text{ cm}$$

c
$$\frac{\sin B}{14.8} = \frac{\sin 28.35°}{8.5}$$
$$\sin B = \frac{14.8 \times \sin 28.35°}{85}$$
$$= 0.8268$$
$$B = 55.77° \text{ or } 180 - 55.77 = 124.23°$$
$$C = 180° - 55.77° - 28.35° = 95.88°$$
$$\text{or } 180° - 124.23° - 28.35°$$
$$= 27.42°$$
$$\frac{8.5}{\sin 28.35°} = \frac{c}{\sin 95.88°} \text{ or } \frac{c}{\sin 27.42°}$$
$$c = \frac{8.5 \times \sin 95.88°}{\sin 28.35°}$$
$$\text{or } \frac{8.5 \times \sin 27.42°}{\sin 28.35°}$$
$$= 17.81 \text{ cm or } 8.24 \text{ cm}$$

5

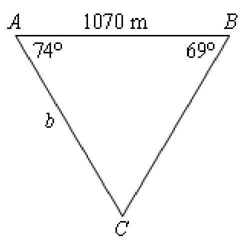


$$A = 180° - 68° - 70°$$
$$= 42°$$
$$\frac{b}{\sin 68°} = \frac{400}{\sin 42°}$$
$$b = \frac{400 \times \sin 68°}{\sin 42°}$$
$$= 554.26 \text{ m}$$

6

$$\angle APB = 46.2° - 27.6°$$
$$= 18.6° \text{ (exterior angle property)}$$
$$\frac{a}{\sin 27.6°} = \frac{34}{\sin 18.6°}$$
$$PB = a = \frac{34 \times \sin 27.6°}{\sin 18.6°}$$
$$= 49.385 \text{ m}$$
$$\frac{h}{PB} = \sin 46.2°$$
$$h = 49.385 \times 0.7217$$
$$= 35.64 \text{ m}$$

7



$$C = 180° - 69° - 74°$$
$$= 37°$$
$$\frac{b}{\sin 69°} = \frac{1070}{\sin 37°}$$
$$b = \frac{1070 \times \sin 69°}{\sin 37°}$$
$$= 1659.86 \text{ m}$$

8 **a**

$$X = 180° - 120° - 20°$$
$$= 40°$$
$$\frac{AX}{\sin 20°} = \frac{50}{\sin 40°}$$
$$= \frac{50 \times \sin 20°}{\sin 40°}$$
$$= 26.60 \text{ m}$$

b

$$Y = 180° - 109° - 32°$$
$$= 39°$$
$$\frac{AY}{\sin 109°} = \frac{50}{\sin 39°}$$
$$AY = \frac{50 \times \sin 109°}{\sin 39°}$$
$$= 75.12 \text{ m}$$

9 **a** From the sine rule:

$$a = \frac{b \sin A}{\sin B}$$
$$b = \frac{a \sin B}{\sin A}$$
$$c = \frac{b \sin C}{\sin B}$$

$$a + b = \frac{b \sin A}{\sin B} + \frac{a \sin B}{\sin A}$$
$$a + b = \frac{b \sin^2 A + a \sin^2 B}{\sin A \sin B}$$
$$\frac{a+b}{c} = \frac{b \sin^2 A + a \sin^2 B}{\sin A \sin B} \times \frac{\sin B}{b \sin C}$$
$$= \frac{b \sin^2 A + a \sin^2 B}{b \sin A \sin C}$$
$$= \frac{b \sin^2 A + b \sin A \sin B}{b \sin A \sin C}$$
$$= \frac{\sin A + \sin B}{\sin C}$$

b Repeat the steps above but for $a - b$

Solutions to Exercise 13C

1 $BC^2 = a^2$

$= b^2 + c^2 - 2bc\cos A$

$= 15^2 + 10^2 - 2 \times 15 \times 10 \times \cos 15°$

$= 325 - 300 \times \cos 15°$

$= 35.222$

$BC = 5.93$ cm

2 $\angle ABC = \angle B$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{5^2 + 8^2 - 10^2}{2 \times 5 \times 8}$$

$= -0.1375$

$\therefore \angle ABC \approx 97.90°$

$\angle ACB = \angle C$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ac}$$

$$= \frac{5^2 + 10^2 - 8^2}{2 \times 5 \times 10}$$

$= 0.61$

$\therefore \angle ACB \approx 52.41°$

3 a $a^2 = b^2 + c^2 - 2bc\cos a$

$= 16^2 + 30^2 - 2 \times 16 \times 30 \times \cos 60°$

$= 1156 - 960 \times \cos 60°$

$= 676$

$a = 26$

b $b^2 = a^2 + c^2 - 2ac\cos B$

$= 14^2 + 12^2 - 2 \times 14 \times 12 \times \cos 53°$

$= 340 - 336 \times \cos 53°$

$= 137.7901$

$a \approx 11.74$

c $\angle ABC = \angle B$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{27^2 + 46^2 - 35^2}{2 \times 27 \times 46}$$

$= 0.6521$

$\therefore \angle ABC \approx 49.29°$

d $b^2 = a^2 + c^2 - 2ac\cos B$

$= 17^2 + 63^2 - 2 \times 17 \times 63 \times \cos 120°$

$= 4258 - 2142 \times \cos 120°$

$= 5329$

$b = 73$

e $c^2 = a^2 + b^2 - 2ab\cos C$

$= 31^2 + 42^2 - 2 \times 31 \times 42 \times \cos 140°$

$= 2642 - 2604 \times \cos 140°$

$= 4719.77$

$c \approx 68.70$

f $\angle BCA = \angle C$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$= \frac{10^2 + 12^2 - 9^2}{2 \times 10 \times 12}$$
$$= 0.6791$$
$$\therefore \angle BCA \approx 47.22^\circ$$

g $c^2 = a^2 + b^2 - 2ab\cos C$

$$= 11^2 + 9^2 - 2 \times 11 \times 9 \times \cos 43.2^\circ$$
$$= 202 - 198 \times \cos 43.2^\circ$$
$$= 57.6642$$
$$c \approx 7.59$$

h $\angle CBA = \angle B$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{8^2 + 15^2 - 10^2}{2 \times 8 \times 15}$$
$$= 0.7875$$
$$\therefore \angle ABC \approx 38.05^\circ$$

4 $c^2 = a^2 + b^2 - 2ab\cos C$

$$= 4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 20^\circ$$
$$= 52 - 48 \times \cos 20^\circ$$
$$= 6.8947$$
$$c \approx 2.626 \text{ km}$$

5 $AB^2 = a^2 + b^2 - 2ab\cos O$

$$= 4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 30^\circ$$
$$= 52 - 48 \times \cos 30^\circ$$
$$= 10.4307$$
$$AB \approx 3.23 \text{ km}$$

6 Label the points suitably: A and B are the hooks, and C is the 70° angle.

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$BD^2 = 42^2 + 54^2 - 2 \times 42 \times 54 \times \cos 70^\circ$$
$$= 4680 - 4536 \times \cos 70^\circ$$
$$= 3128.5966$$
$$BD \approx 55.93 \text{ cm}$$

7 **a** $\angle B = 180^\circ - 48^\circ = 132^\circ$

$$AC^2 = a^2 + c^2 - 2ac\cos B$$
$$= 5^2 + 4^2 - 2 \times 5 \times 4 \times \cos 132^\circ$$
$$= 41 - 40 \times \cos 132^\circ$$
$$= 67.7652$$
$$AC \approx 8.23 \text{ cm}$$

b $BD^2 = b^2 + d^2 - 2bd\cos A$

$$= 5^2 + 4^2 - 2 \times 5 \times 4 \times \cos 48^\circ$$
$$= 41 - 40 \times \cos 48^\circ$$
$$= 14.2347$$
$$BD \approx 3.77 \text{ cm}$$

8 **a** Use $\triangle ABD$.

$$BD^2 = b^2 + d^2 - 2bd\cos A$$
$$= 6^2 + 4^2 - 2 \times 6 \times 4 \times \cos 92^\circ$$
$$= 52 - 48 \times \cos 92^\circ$$
$$= 53.6751$$
$$BD \approx 7.326 \text{ cm}$$

b

$$\angle D = \angle BDC$$

$$\frac{\sin D}{5} = \frac{\sin 88^\circ}{7.3263}$$

$$\sin D = \frac{5 \times \sin 88^\circ}{7.3263}$$

$$= 0.6820$$

$$D = 43.0045^\circ$$

$$B = 180^\circ - 88^\circ - 43.0045^\circ$$

$$= 48.9954^\circ$$

$$\frac{b}{\sin 48.9954^\circ} = \frac{7.3263}{\sin 88^\circ}$$

$$b = \frac{7.3263 \times \sin 48.9956^\circ}{\sin 88^\circ}$$

$$\approx 5.53 \text{ cm}$$

9 a $\cos \angle AO'B = \dfrac{6^2 + 6^2 - 8^2}{2 \times 6 \times 6}$

$$= 0.111$$

$$\angle AO'B \approx 83.62^\circ$$

b $\cos \angle AOB = \dfrac{7.5^2 + 7.5^2 - 8^2}{2 \times 7.5 \times 7.5}$

$$= 0.43111$$

$$\angle AOB \approx 64.46^\circ$$

10 a Treat AB as c.

$$c^2 = a^2 + b^2 - 2ab\cos O$$

$$AB^2 = 70^2 + 90^2 - 2 \times 70 \times 90 \times \cos 65^\circ$$

$$= 13\,000 - 12\,600 \times \cos 65^\circ$$

$$= 7675.0099$$

$$AB \approx 87.61 \text{ m}$$

b $\cos \angle B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$$= \frac{70^2 + 87.6071^2 - 90^2}{2 \times 70 \times 87.6071}$$

$$= 0.3648$$

$$\angle AOB \approx 68.6010^\circ$$

Now use $\triangle OCB$.

Let $CB = a,\ OB = b,\ OC = c.$

$$CB = \frac{AB}{2} = 43.80$$

$$c^2 = a^2 + b^2 - 2ab\cos O$$

$$OC^2 = 43.8035^2 + 70^2 - 2 \times 43.8035 \times 70 \times 0.3648$$

$$= 4581.24$$

$$OC \approx 67.7 \text{ m}$$

Solutions to Exercise 13D

1 a
$$\begin{aligned}\text{Area} &= \frac{1}{2}ab\sin C\\ &= \frac{1}{2}\times 6\times 4\times \sin 70^\circ\\ &= 11.28\text{ cm}^2\end{aligned}$$

b
$$\begin{aligned}\text{Area} &= \frac{1}{2}yz\sin X\\ &= \frac{1}{2}\times 5.1\times 6.2\times \sin 72.8^\circ\\ &= 15.10\text{ cm}^2\end{aligned}$$

c
$$\begin{aligned}\text{Area} &= \frac{1}{2}nl\sin M\\ &= \frac{1}{2}\times 3.5\times 8.2\times \sin 130^\circ\\ &= 10.99\text{ cm}^2\end{aligned}$$

d
$$\angle C = 180 - 25 - 25 = 130^\circ$$
$$\begin{aligned}\text{Area} &= \frac{1}{2}ab\sin C\\ &= \frac{1}{2}\times 5\times 5\times \sin 130^\circ\\ &= 9.58\text{ cm}^2\end{aligned}$$

2 a Use the cosine rule to find $\angle B$. (Any angle will do.)
$$\begin{aligned}\cos\angle B &= \frac{a^2+c^2-b^2}{2ac}\\ &= \frac{3.2^2+4.1^2-5.9^2}{2\times 3.1\times 4.1}\\ &= -0.2957\\ \angle B &= 107.201^\circ\end{aligned}$$
$$\begin{aligned}\text{Area} &= \frac{1}{2}ac\sin B\\ &= \frac{1}{2}\times 3.2\times 4.1\\ &\quad \times \sin 107.201^\circ\\ &\approx 6.267\text{ cm}^2\end{aligned}$$

b Use the sine rule to fmd $\angle C$.
$$\begin{aligned}\frac{\sin C}{7} &= \frac{\sin 100^\circ}{9}\\ \sin C &= \frac{7\times \sin 100^\circ}{9}\\ &= 0.7659\\ C &= 49.992^\circ\\ \angle A &= 180^\circ - 100^\circ - 49.992^\circ\\ &= 30.007^\circ\end{aligned}$$
$$\begin{aligned}\text{Area} &= \frac{1}{2}bc\sin A\\ &= \frac{1}{2}\times 9\times 7\times \sin 30.007^\circ\\ &\approx 15.754\text{ cm}^2\end{aligned}$$

c
$$\begin{aligned}E &= 180^\circ - 65^\circ - 66^\circ\\ &= 60^\circ\end{aligned}$$
$$\begin{aligned}\frac{e}{\sin 60^\circ} &= \frac{6.3}{\sin 55^\circ}\\ e &= \frac{6.3\times \sin 60^\circ}{\sin 55^\circ}\\ &= 6.6604\text{ cm}\end{aligned}$$
$$\begin{aligned}\text{Area} &= \frac{1}{2}ef\sin D\\ &= \frac{1}{2}\times 6.6604\times 6.3\times \sin 65^\circ\\ &\approx 19.015\text{ cm}^2\end{aligned}$$

d Use the cosine rule to find $\angle D$.

$$\cos\angle D = \frac{e^2 + f^2 - d^2}{2ef}$$
$$= \frac{5.1^2 + 5.7^2 - 5.9^2}{2 \times 5.1 \times 5.7}$$
$$= -0.4074$$
$$\angle D = 65.95^\circ$$
$$\text{Area} = \frac{1}{2} ef \sin D$$
$$= \frac{1}{2} \times 5.1 \times 5.7 \times \sin 65.95^\circ$$
$$\approx 13.274 \text{ cm}^2$$

e $$\frac{\sin I}{12} = \frac{\sin 24^\circ}{5}$$
$$\sin I = \frac{12 \times \sin 24^\circ}{5}$$
$$= 0.9671$$
$$I = 77.466^\circ \text{ or } 180^\circ - 74.466^\circ$$
$$= 102.533^\circ$$
$$G = 180^\circ - 24^\circ - 108.533^\circ$$
$$\text{or } 180^\circ - 24^\circ - 77.466^\circ$$
$$= 53.466^\circ \text{ or } 78.534^\circ$$
$$\text{Area} = \frac{1}{2} hi \sin G$$
$$= \frac{1}{2} \times 5 \times 12 \times \sin 53.466^\circ$$
$$\text{or } \frac{1}{2} \times 5 \times 12 \times \sin 78.534^\circ$$
$$\approx 24.105 \text{ cm}^2 \text{ or } 29.401 \text{ cm}^2$$

Note that although the diagram is drawn as if I is obtuse, you should not make this assumption. Diagrams are not neccessarily drawn to scale.

f $$I = 180 - 10 - 19$$
$$= 151^\circ$$
$$\frac{i}{\sin 151^\circ} = \frac{4}{\sin 19^\circ}$$
$$i = \frac{4 \times \sin 151^\circ}{\sin 19^\circ}$$
$$= 5.9564$$
$$\text{Area} = \frac{1}{2} ih \sin G$$
$$= \frac{1}{2} \times 5.9564 \times 4 \times \sin 10^\circ$$
$$\approx 2.069 \text{ cm}^2$$

Solutions to Exercise 13E

1 $l = \frac{105}{360} \times 2\pi r$

$= \frac{105}{360} \times 2 \times \pi \times 25$

≈ 45.81 cm

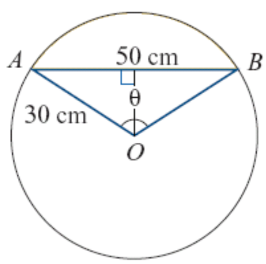
2 **a** $\theta = \frac{50}{30} = \frac{5}{3}$ radians

$= \frac{5}{3} \times \frac{180}{\pi}$ degrees

$= 95.4929°$

$= 95°30'$

b



$\sin\frac{\theta}{2} = \frac{25}{30} = 0.8333$

$\frac{\theta}{2} = 56.4426°$

$\theta = 112.885°$

$= 112°53'$

3 **a** Set your calculator to radian mode.

$\sin\frac{\theta}{2} = \frac{3}{7} = 0.4285$

$\frac{\theta}{2} = 0.4429$

$\theta = 0.8858$

$l = r\theta$

$= 7 \times 0.8858$

$= 6.20$ cm

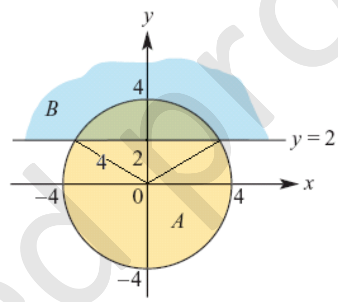
b This represents the minor segment area.

$A = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2} \times 7^2 \times (0.8858 - \sin\ 0.8858)$

$= 2.73\ \text{cm}^2$

4 A represents the interior of a circle of radius 4, centre the origin.



$\cos\frac{\theta}{2} = \frac{2}{4} = \frac{1}{2}$

$\frac{\theta}{2} = \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$

$A \cap B$ is a segment where $r = 4,\ \theta = \frac{2\pi}{3}$

$A = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2} \times 4^2 \times \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= 9.83\ \text{cm}^2$

5 $\cos\theta = \cos 2\left(\frac{\theta}{2}\right)$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$\cos^2\frac{\theta}{2} = 1 - \sin^2\frac{\theta}{2}$$

$$\therefore \quad \cos\theta = 1 - \sin^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= 1 - 2\sin^2\frac{\theta}{2}$$

$$\sqrt{2r^2(1-\cos\theta)}$$

$$= \sqrt{2r^2\left(1-\left(1-2\sin^2\frac{\theta}{2}\right)\right)}$$

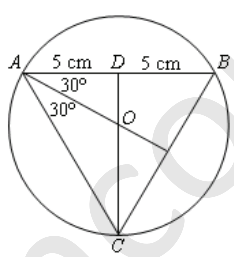
$$= \sqrt{2r^2\left(1-1+2\sin^2\frac{\theta}{2}\right)}$$

$$= \sqrt{2r^2\left(2\sin^2\frac{\theta}{2}\right)}$$

$$= \sqrt{4r^2\sin^2\frac{\theta}{2}}$$

$$= 2r\ \sin\frac{\theta}{2}$$

6



Altitude $CD = 5\tan 60°$

$$= 5\sqrt{3}\text{ cm}$$

$OD = 5\tan 30°$

$$= \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}\text{ cm}$$

$$\text{Radius} = 5\sqrt{3} - \frac{5\sqrt{3}}{3}$$

$$= \frac{15\sqrt{3} - 5\sqrt{3}}{3}$$

$$= \frac{10\sqrt{3}}{3}\text{ cm}$$

$\angle AOD = 60°$

$\therefore\ \angle AOB = 120° = \frac{2\pi}{3}$ radians

Area = 3 × segment area

$$= \frac{3}{2}\times r^2\times(\theta - \sin\theta)$$

$$= \frac{3}{2}\times\frac{300}{9}\times\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$$

$$= 50\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$$

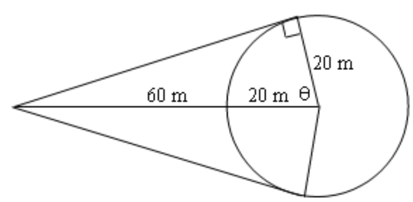
$$= 61.42\text{ cm}^2$$

7 a $C = 2\pi r$

$$= 2\times\pi\times 20$$

$$= 40\pi \approx 125.66\text{ cm}$$

b



$$\cos\theta = \frac{20}{20+60} = 0.25$$

$$\theta = 1.3181\text{ radians}$$

$$2\theta = 2.6362$$

$$\text{Proportion visible} = \frac{2.6362}{2\pi}$$

$$= 0.41956$$

$$\approx 41.96\%$$

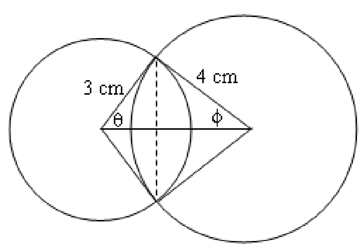
8 a Use fractions of an hour (minutes).

$$l = \frac{25}{60} \times 2\pi r$$
$$= \frac{25}{60} \times 2 \times \pi \times 4$$
$$= \frac{10\pi}{3} \approx 10.47 \text{ m}$$

b Angle $= \frac{25}{60} \times 2\pi = \frac{5\pi}{6}$

$$\text{Area} = -\frac{1}{2}r^2\theta$$
$$= \frac{1}{2} \times 4^2 \times \frac{5\pi}{6}$$
$$= \frac{20\pi}{3} \approx 20.94 \text{ m}^2$$

9



The required area is the sum of two segments.
Let the left area be A_1 and the right area A_2.

$$\tan\theta = \frac{4}{3}$$
$$\theta = 0.9272$$
$$2\theta = 1.8545$$
$$A_1 = \frac{1}{2} \times 3^2 \times (1.8545 - \sin\ 1.8545)$$
$$= 4.0256$$

$$\tan\phi = \frac{3}{4}$$
$$\phi = 0.6435$$
$$2\phi = 1.2870$$
$$A_2 = \frac{1}{2} \times 4^2 \times (1.2870 - \sin\ 1.2870)$$
$$= 2.6160$$

Total area $= 4.0256 + 2.6160$
$= 6.64 \text{ cm}^2$

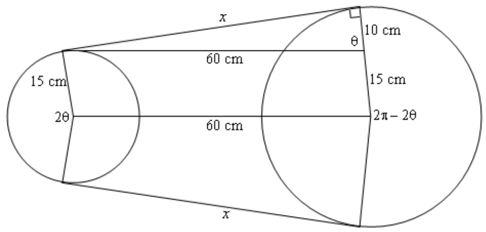
10

$$A = \frac{1}{2}r^2\theta = 63$$
$$r^2\theta = 126$$
$$\theta = \frac{126}{r^2}$$
$$P = r + r + r\theta = 32$$
$$2r + r \times \frac{126}{r^2} = 32$$
$$2r + \frac{126}{r} = 32$$
$$2r^2 + 126 = 32r$$
$$2r^2 - 32r + 126 = 0$$
$$r^2 - 16r + 63 = 0$$
$$(r - 7)(r - 9) = 0$$
$$r = 7 \text{ or } 9 \text{ cm}$$
$$\theta = \frac{126}{r^2}$$

When $r = 7$, $\theta = \frac{126}{7^2} = \left(\frac{18}{7}\right)^c$

When $r = 9$, $\theta = \frac{126}{9^2} = \left(\frac{14}{9}\right)^c$

11 The following diagram can be deduced from the data:



$$x^2 = 60^2 - 10^2 = 3500$$

$$x = 10\sqrt{35}$$

$$\cos\theta = \frac{10}{60} = \frac{1}{6}$$

$$\theta = 1.4033$$

$$2\theta = 2.8066$$

$$2\pi - 2\theta = 3.4764$$

Length of belt on left wheel:

$$l = r\theta$$

$$= 15 \times 2.8066 = 42.1004$$

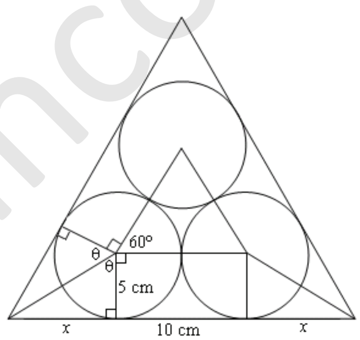
Length of belt on right wheel:

$$l = r\theta$$

$$= 25 \times 3.4764 = 86.9122$$

$$\text{Total} = 12 \times 10\sqrt{25} + 42.1004 + 86.9112 \approx 247.33 \text{ cm}$$

12 a



The balls can be enclosed as in the diagram above.

$$2\theta = 360 - 90 - 60 - 90$$

$$= 120^\circ$$

$$\theta = 60^\circ$$

$$\frac{x}{5} = \tan 60^\circ = \sqrt{3}$$

$$x = 5\sqrt{3}$$

$$\text{Perimeter} = 6 \times 5\sqrt{3} + 3 \times 10 \approx 81.96 \text{ cm}$$

b Height of large triangle

$$= (2x + 10) \times \sin 60^\circ$$

$$= (10\sqrt{3} + 10) \times \frac{\sqrt{3}}{2}$$

$$= 15 + 5\sqrt{3} \text{ cm}$$

Area of large triangle

$$= \frac{1}{2}(10\sqrt{3} + 10)(15 + 5\sqrt{3})$$

$$\approx 173.2050 \text{ cm}^2$$

Area of three discs = 10 cm triangle − half a circle

Height of 10 cm triangle

$$= 10 \times \sin 60^\circ$$

$$= 5\sqrt{3} \text{ cm}$$

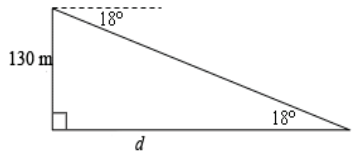
$$\text{Area} = \frac{1}{2} \times 10 \times 5\sqrt{3} - \frac{1}{2} \times \pi \times 5^2$$

$$= 50\sqrt{3} - 12.5\pi$$

$$\approx 4.03 \text{ cm}^2$$

Solutions to Exercise 13F

1

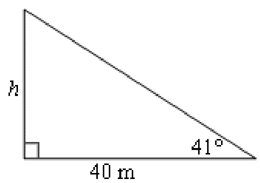


$$\frac{130}{d} = \tan 18°$$

$$d = \frac{130}{\tan 18°}$$

$$= 400.10 \text{ m}$$

2

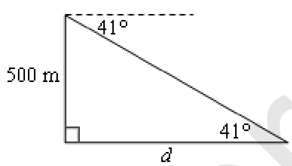


$$\frac{h}{40} = \tan 41°$$

$$h = 40 \times 0.869$$

$$= 34.77 \text{ m}$$

3

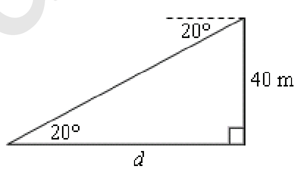


$$\frac{500}{d} = \tan 41°$$

$$d = \frac{500}{\tan 41°}$$

$$= 575.18 \text{ m}$$

4

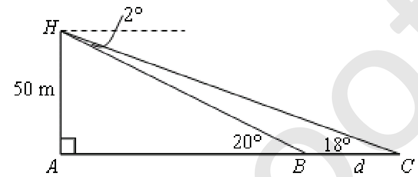


$$\frac{40}{d} = \tan 20°$$

$$d = \frac{40}{\tan 20°}$$

$$= 109.90 \text{ m}$$

5



$$\frac{50}{AB} = \tan 20°$$

$$AB = \frac{50}{\tan 20°}$$

$$= 137.373 \text{ m}$$

$$\frac{50}{AC} = \tan 18°$$

$$AC = \frac{50}{\tan 18°}$$

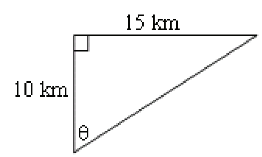
$$= 153.884 \text{ m}$$

$$d = AC - AB$$

$$= 153.884 - 137.373$$

$$\approx 16.51 \text{ m}$$

6

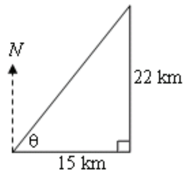


$$\tan\theta = \frac{15}{10} = 1.5$$

$$\theta \approx 56°$$

The bearing is 056°.

7 a



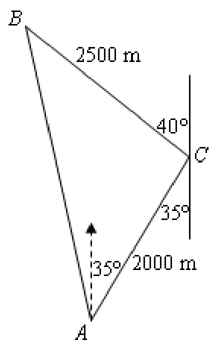
$\tan\theta = \dfrac{22}{15} = 1.466$

$\theta = 55.713°$

The bearing is $90° - \theta \approx 034°$.

b $180° + 34° = 214°$

8



Use the cosine rule, where

$\angle C = 180 - 40 - 35 = 105°$

$AB^2 = c^2$

$= a^2 + b^2 - 2ab\cos C$

$= 2500^2 + 2000^2$

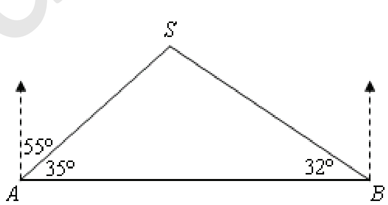
$- 2 \times 2500 \times 2000 \times \cos 105°$

$= 12\,838\,190.4510$

$AB = 3583.04$ m

9 $207° - 180° = 027°$

10

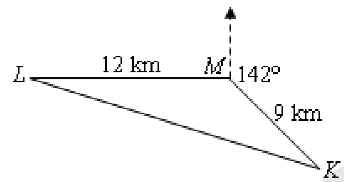


$\angle SAB = 90° - 55° = 35°$

$\angle SBA = 302° - 270° = 32°$

$\angle ASB = 180° - 35° - 32° = 113°$

11



$\angle LMK = 360° - 90° - 142°$

$= 128°$

First, use the cosine rule to find LK.

$LK^2 = m^2$

$= k^2 + l^2 - 2kl\cos M$

$= 12^2 + 9^2 - 2 \times 12 \times 9 \times \cos 128°$

$= 357.9829$

$LK = 18.920$

It is easier to use the sine rule to find $\angle MLK$.

$\dfrac{\sin L}{9} = \dfrac{\sin 128°}{18.920}$

$\sin L = \dfrac{\sin 128° \times 9}{18.920}$

$= 0.3748$

$\angle MLK = \angle L$

$\approx 22.01°$

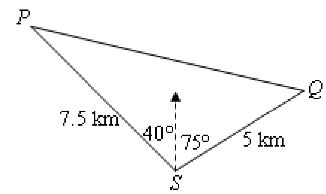
12 a $\angle BAN = 360° - 346° = 14°$

$\angle BAC = 14° + 35° = 49°$

b Use the cosine rule:

$$
\begin{aligned}
BC^2 &= a^2 \\
&= b^2 + c^2 = 2bc\cos A \\
&= 340^2 + 160^2 - 2 \times 340 \\
&\quad \times 160 \times \cos 49^\circ \\
&= 69\,820.7776 \\
BC &= 264.24 \text{ km}
\end{aligned}
$$

13



Use the cosine rule:

$$
\begin{aligned}
\angle PSQ &= 115^\circ \\
PQ^2 &= s^2 \\
&= p^2 + q^2 - 2pq\cos A \\
&= 5^2 + 7.5^2 - 2 \times 5 \\
&\quad \times 7.5 \times \cos 115^\circ \\
&= 112.9464 \\
PQ &= 10.63 \text{ km}
\end{aligned}
$$

Solutions to Exercise 13G

1 a $FH^2 = 12^2 + 5^2$

$= 169$

$FH = 13$ cm

b $BH^2 = 13^2 + 8^2$

$= 233$

$BH = \sqrt{233} \approx 15.26$ cm

c $\tan \angle BHF = \dfrac{8}{13}$

$= 0.615$

$\angle BHF = 31.61°$

d $\angle BGH = 90°$ and $BH = \sqrt{233}$

$\cos \angle BGH = \dfrac{12}{\sqrt{233}}$

$= 0.786$

$\angle BGH = 38.17°$

2 a $AB = 2EF$

$EF = 4$ cm

b $\tan \angle VEF = \dfrac{VE}{EF}$

$= \dfrac{12}{4} = 3$

$\angle VEF = 71.57°$

c $VE^2 = 4^2 + 12^2$

$= 160$

$VE = \sqrt{160}$

$= 4\sqrt{10} \approx 12.65$ cm

d All sloping sides are equal in length. Choose VA.

$VA^2 = VE^2 + EA^2$

$= 160 + 4^2 = 176$

$VA = \sqrt{176}$

$= 4\sqrt{11} \approx 13.27$ cm

e $\angle VAD = \angle VAE$

$\tan \angle VAE = \dfrac{VE}{EA}$

$= \dfrac{4\sqrt{10}}{4}$

$= \sqrt{10} \approx 3.162$

$\angle VAE = 72.45°$

f Area of a triangular face

$= \dfrac{1}{2} \times AD \times VE$

$= \dfrac{1}{2} \times 8 \times 4\sqrt{10}$

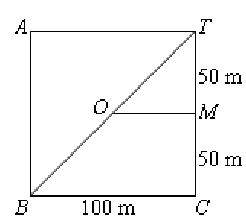
$= 16\sqrt{10}$ cm^2

Area of base $= 8 \times 8 = 64$ cm^2

Surface area $= 4 \times 16\sqrt{10} + 64$

≈ 266.39 cm^2

3 First, sketch the square base, and find the height h of the tree. Mark M as the mid-point of TC and O as the centre of the square.



$OM = TM = 50$ m

$$OT^2 = 50^2 + 50^2 = 5000$$

$$OT = \sqrt{5000}\text{ m}$$

$$\frac{h}{\sqrt{5000}} = \tan 20^\circ$$

$$h = \sqrt{5000} \times \tan 20^\circ$$

$$= 25.7365$$

At A and C,

$$\tan\theta = \frac{25.7365}{100} = 0.2573$$

$$\theta = 14.43^\circ$$

At B, $TB = 2 \times OT = 2\sqrt{5000}$ m

$$\tan\theta = \frac{25.7365}{\sqrt{5000}} = 0.1819$$

$$\theta = 10.31^\circ$$

4 **a** $\angle ABC = 180^\circ - 90^\circ - 45^\circ$

$$= 45^\circ$$

ABC is isosceles, and

$CB = AC = 85$ m.

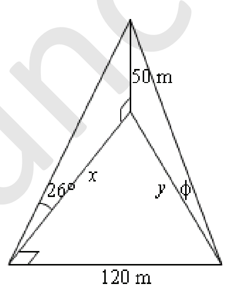
b $\dfrac{XB}{BC} = \sin 32^\circ$

$$\frac{XB}{85} = \sin 32^\circ$$

$$XB = 85 \times \sin 32^\circ$$

$$= 45.04\text{ m}$$

5



$$\frac{50}{x} = \tan 26^\circ$$

$$x = \frac{50}{\tan 26^\circ}$$

$$= 102.515\text{ m}$$

$$y^2 = x^2 + 120^2$$

$$= 24\,909.364$$

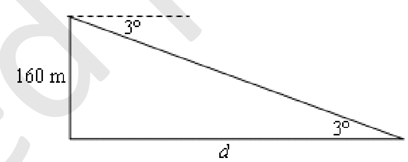
$$y = \sqrt{24\,909.364}$$

$$= 157.827\text{ m}$$

$$\tan\phi = \frac{50}{y} = 0.316$$

$$\phi = 17.58^\circ$$

6 From the top of the cliff:



For the first buoy:

$$\frac{160}{d} = \tan 3^\circ$$

$$d = \frac{160}{\tan 3^\circ}$$

$$= 3052.981\text{ m}$$

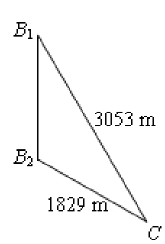
For the second buoy

$$\frac{160}{d} = \tan 5^\circ$$

$$d = \frac{160}{\tan 5^\circ}$$

$$= 1828.808\text{ m}$$

From the cliff:



$\angle C = 337 - 308 = 29^\circ$

Use the cosine rule.

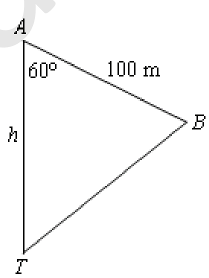
$$c^2 = 3052.981^2 + 1828.808^2 - 2 \times 3052.981 \times 1828.808 \times \cos 29^\circ$$
$$= 2\,898\,675.1436$$
$$c = 1702.55 \text{ m}$$

7 a
$$AC^2 = 12^2 + 5^2 = 169$$
$$AC = 13 \text{ cm}$$
$$\tan \angle ACE = \frac{6}{13} = 0.4615$$
$$\angle ACE = 24.78^\circ$$

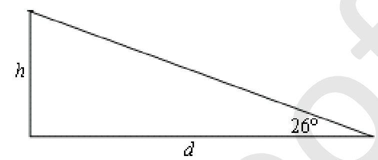
b Triangle *HDF* is identical (congruent) to triangle *AEC*.
$$\therefore \angle HFD = \angle ACE$$
$$\angle HDF = 90^\circ - 24.28^\circ = 65.22^\circ$$

c
$$CH^2 = 12^2 + 6^2 = 180$$
$$CH = \sqrt{180}$$
$$\tan \angle ECH = \frac{EH}{CH} = \frac{5}{\sqrt{180}} = 0.3726$$
$$\angle ECH = 20.44^\circ$$

8 Looking from above, the following diagram applies.

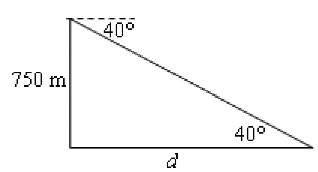


Because the angle of elevation is 45°, *AT* will equal the height of the tower, *h* m. Use the cosine rule.
$$BT^2 = h^2 + 100^2 - 2 \times h \times 100 \times \cos 60^\circ$$
$$= h^2 + 100^2 - 200h \times \frac{1}{2}$$
$$= h^2 - 100h + 100^2$$
From point *B*:



$$\frac{h}{d} = \tan 26^\circ$$
$$d = \frac{h}{\tan 26^\circ} = 2.050h$$
$$\therefore \quad 2.050^2 h^2 = h^2 - 100h + 100^2$$
$$4.2037h^2 = h^2 - 100h + 10\,000$$
$$3.2037h^2 + 100h = 10\,000$$
Using the quadratic formula:
$h \approx 42.40$ m

9 Find the horizontal distance of *A* from the balloon.



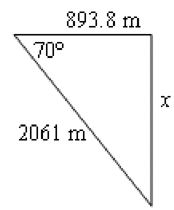
$$\frac{750}{d} = \tan 40^\circ$$
$$d = \frac{750}{\tan 40^\circ} = 893.815 \text{ m}$$
The distance of *B* from the balloon may be calculated in the same way:

$$\frac{750}{d} = \tan 20°$$

$$d = \frac{750}{\tan 20°}$$

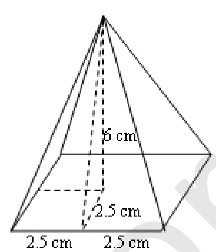
$$= 2060.608 \text{ m}$$

Draw the view from above and use the cosine rule.



$$x^2 = 893.8152 + 2060.6082$$
$$- 2 \times 893.815 \times 2060.608$$
$$\times \cos 70°$$
$$= 3\,785\,143.5836$$
$$x = 1945.54 \text{ m}$$

10 **a** Find the length of an altitude:



$$a^2 = 2.5^2 + 6^2 = 42.45$$
$$a \approx 6.5 \text{ cm}$$

The sloping edges are also the hypotenuse of a right-angled triangle.

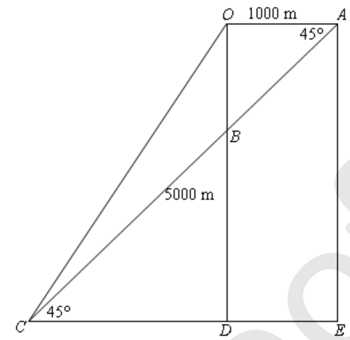
$$s^2 = 2.5^2 + 6.5^2 = 48.5$$
$$s \approx 6.96 \text{ cm}$$

b $\text{Area} = \frac{1}{2} \times 5 \times 6.5$

$$= 16.25 \text{ cm}^2$$

11 **a** $\text{Distance} = 300 \times \frac{1}{60} = 5 \text{ km}$

b Looking from above:



$$AE = 5000 \times \sin 45°$$
$$= \frac{5000}{\sqrt{2}} \approx 3535.433$$
$$CE = 5000 \times \sin 45°$$
$$= \frac{5000}{\sqrt{2}} \approx 3535.433$$
$$CD = CE - DE$$
$$= 3535.533 - 1000$$
$$= 2535.533$$
$$\tan \angle COD = \frac{2535.533}{3535.533}$$
$$= 0.7171$$
$$\angle COD = 35.65°$$
$$\text{Bearing} = 180° + 35.65°$$
$$= 215.65°$$

c Let the angle of elevation be θ.

$$OC^2 = 3535.533^2 + 2535.533^2$$
$$= 18\,928\,932$$
$$OC = 4350.739$$
$$\tan \theta = \frac{500}{4350.739}$$
$$= 0.1149$$
$$\theta = 6.56° = 6°33'$$

Solutions to Exercise 13H

1 a Area $ABFE = AB \times GC$

$= 4a \times a = 4a^2$ units

Area $BCGF = BC \times GC$

$= 3a \times a = 3a^2$ units

Area $ABCD = AB \times BC$

$= 4a \times 3a = 12a^2$ units

b This is equivalent to $\angle FAB$.

$\tan \angle FAB = \dfrac{FB}{AB}$

$= \dfrac{a}{4a} = 0.25$

$\angle FAB = 14.04°$

c $\tan \angle GBC = \dfrac{GC}{BC}$

$= \dfrac{a}{3a} = 0.333$

$\angle GBC = 18.43°$

d $AC = \sqrt{(4a)^2 + (3a)^2}$

$= \sqrt{25a^2} = 5a$

$\tan \angle GAC = \dfrac{GC}{AC}$

$= \dfrac{a}{5a} = 0.2$

$\angle GAC = 11.31°$

2 a Let the altitude of triangle FAB be a.

$s = \sqrt{a^2 + a^2}$

$= \sqrt{2a^2} = a\sqrt{2}$

$\sin \angle VA0 = \dfrac{OV}{VA}$

$= \dfrac{a}{a\sqrt{3}}$

$= \dfrac{1}{\sqrt{3}} = 0.577$

$\angle VA0 = 35.26°$

b This will be the slope of the altitude.

$\sin\theta = \dfrac{a}{a\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\theta = 45°$

3 a $BE = \sqrt{5^2 + 12^2}$

$= \sqrt{169} = 13$

Triangle BEF is isosceles, so

$BE = EF$

$BF = \sqrt{13^2 + 13^2}$

$= \sqrt{338}$

$BD = \sqrt{338 - 5^2}$

$= \sqrt{313}$

Gradient of$BF = \dfrac{DF}{DB}$

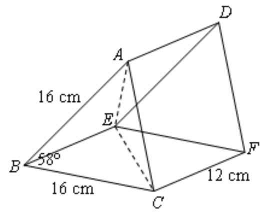
$= \dfrac{5}{\sqrt{313}} \approx 0.28$

b $\tan \angle FBD = \dfrac{5}{\sqrt{313}}$

$= 0.2826$

$\angle FBD = 15.78°$

4



a Use the cosine rule.

$$AC^2 = b^2$$
$$= a^2 + c^2 - 2ac\cos B$$
$$= 16^2 + 16^2 - 2 \times 16 \times 16 \times \cos 58^\circ$$
$$= 240.681$$
$$AC \approx 15.51 \text{ cm}$$

Hint: Keep the exact value in your calculator for part **c**.

b $AE = \sqrt{16^2 + 12^2}$

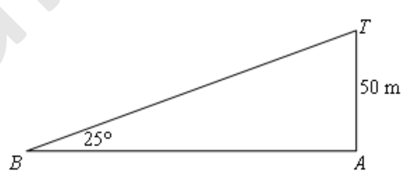
$= \sqrt{400} = 20$ cm

c $AE = CE = 20$ cm

Use the cosine rule in triangle AEC.

$$\cos E = \frac{a^2 + c^2 - e^2}{2ac}$$
$$= \frac{20^2 + 20^2 - 240.681}{2 \times 20 \times 20}$$
$$= 0.699$$
$$\angle AEC = 45.64^\circ$$

5 a First calculate the distances of B and C from the tower.



i $\dfrac{50}{AB} = \tan 25^\circ$

$$AB = \frac{50}{\tan 25^\circ}$$
$$= 107.225 \approx 107 \text{ m}$$

ii Likewise,

$$AC = \frac{50}{\tan 30^\circ}$$
$$= 86.602 \approx 87 \text{ m}$$

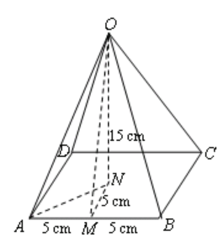
iii Use Pythagoras' theorem:

$$BC = \sqrt{107.225^2 + 86.602^2}$$
$$\approx 138 \text{ m}$$

b $MA = \dfrac{1}{2}AB$

$$= \frac{25}{\tan 25^\circ}$$
$$= 53.612$$
$$\tan \angle TMA = \frac{50}{53.612}$$
$$= 0.9326$$
$$\angle TMA = 43.00^\circ$$

6 Let M be the midpoint of AB.



a $OM = \sqrt{5^2 + 15^2}$

$= \sqrt{250}$

$OA = \sqrt{250 + 5^2}$

$= \sqrt{275} = 5\sqrt{11}$ cm

b $\sin \angle OAM = \dfrac{15}{5\sqrt{11}}$

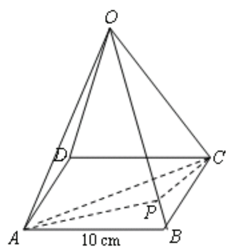
$= \dfrac{3}{\sqrt{11}} = 0.9045$

$\angle OAM = 64.76°$

c $\tan \angle OMN = \dfrac{15}{5} = 3$

$\angle OMN = 71.57°$

d Draw the perpendiculars from *C* and *A* to meet *OB* at the common point *P*.



Find $\angle AOB$ using the cosine rule in triangle *AOB*.

$\cos \angle AOB = \dfrac{(5\sqrt{11})^2 + (5\sqrt{11})^2 - 10^2}{2 \times 5\sqrt{11} \times 5\sqrt{11}}$

$= \dfrac{275 + 275 - 100}{550}$

$= \dfrac{450}{550}$

$= 0.8181$

$\angle AOB = 35.096°$

$\sin \angle AOP = \dfrac{AP}{OA}$

$0.5749 = \dfrac{AP}{5\sqrt{11}}$

$AP = 5\sqrt{11} \times 0.5749$

$= 9.534$

$AC = \sqrt{10^2 + 10^2}$

$= \sqrt{200} = 10\sqrt{2}$

Use the cosine rule in triangle *APC* to find the required angle, $\angle APC$.

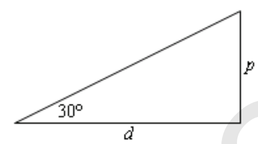
$\cos \angle APC = \cos P$

$= \dfrac{9.534^2 + 9.534^2 - 200}{2 \times 9.534 \times 9.634}$

$= -0.1$

$\angle APC = 95.74°$

7 Let the height of the post be *p*.



The distance away of the first corner is given by:

$\dfrac{p}{d} = \tan 30°$

$d = \dfrac{p}{\tan 30°}$

$= p\sqrt{3}$

Likewise, the distance away of the second corner is given by

$d = \dfrac{p}{\tan 45°}$

$= p$

The distance of the diagonal from the post is

$\sqrt{(p\sqrt{3})^2 + p^2} = \sqrt{3p^2 + p^2}$

$= \sqrt{4p^2}$

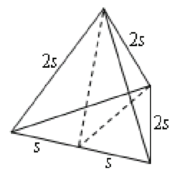
$= 2p$

The elevation from the diagonally opposite corner is

$\tan \theta = \dfrac{p}{2p} = \dfrac{1}{2}$

$\theta = 26.57°$

8 a Let each side of the tetrahedron be $2s$.



$$\text{Height of altitudes} = \sqrt{(2s)^2 - s^2}$$
$$= \sqrt{3s^2}$$
$$= \sqrt{3}s$$

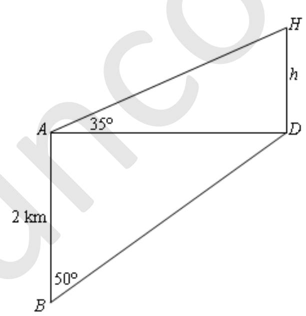
Use the cosine rule:

$$\cos\theta = \frac{(2s)^2 + (\sqrt{3}s)^2 - (\sqrt{3}s)^2}{2 \times 2s \times \sqrt{3}s}$$
$$= \frac{4s^2}{4\sqrt{3}s^2}$$
$$= \frac{1}{\sqrt{3}}$$
$$\theta = 54.74°$$

b Use the cosine rule:

$$\cos\theta = \frac{(\sqrt{3}s)^2 + (\sqrt{3}s)^2 - (2s)^2}{2 \times \sqrt{3}s \times \sqrt{3}s}$$
$$= \frac{2s^2}{6s^2} = \frac{1}{3}$$
$$\theta = 70.53°$$

9



$$\frac{AD}{2} = \tan 50°$$
$$AD = 2\tan 50°$$
$$\frac{h}{AD} = \tan 35°$$
$$h = AD\tan 35°$$
$$= 2\tan 50° \tan 35°$$
$$= 1.6689 \approx 1.67 \text{ km}$$

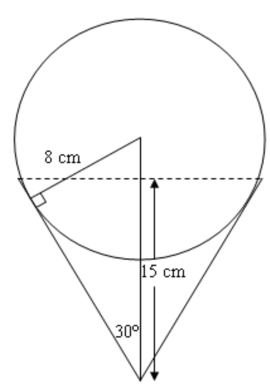
10 a This is the hypotenuse of right-angled triangle ABF.

$$AF = \sqrt{100^2 + 100^2}$$
$$\approx 141.42 \text{ m}$$

b

$$\sin\theta = \frac{AD}{AF}$$
$$\frac{AD}{AE} = \sin 30°$$
$$= \frac{1}{2}$$
$$AD = \frac{1}{2}AE$$
$$= 50 \text{ m}$$
$$\therefore \sin\theta = \frac{50}{141.42}$$
$$= 0.3535$$
$$\theta = 20.70°$$

11



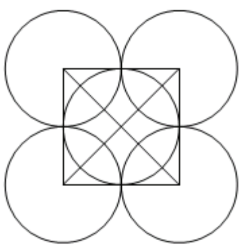
$$\frac{8}{\text{Height}} = \sin 30°$$

$$= \frac{1}{2}$$

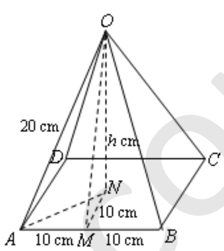
$$\text{Height} = \frac{8}{0.5}$$

$$= 16 \text{ cm}$$

12 Joining the centres of the four balls to each other (excluding diagonal balls), and to the top ball, will form a square pyramid with each side 20 cm. Each line will go through the point where the spheres just touch each other.
The diagram shows the view from above:



Find the height of this square pyramid.



$$OM = \sqrt{20^2 - 10^2}$$

$$= \sqrt{300} \text{ cm}$$

$$ON = \sqrt{OM^2 - 10^2}$$

$$= \sqrt{300 - 100}$$

$$= \sqrt{200} \approx 14.14 \text{ cm}$$

There will be the radius of the top sphere above this, and of the bottom spheres below.

The height of the top will be $14.14 + 10 + 10 = 34.14$ cm.

13 **a** The diagonal of the cube will be the diameter of the sphere.
Applying Pythagoras' rule twice gives

$$d = \sqrt{3a^2}$$

$$= a\sqrt{3} \text{ cm}$$

$$r = \frac{a\sqrt{3}}{2} \text{ cm}$$

b The diameter will be the length of one side of the cube.

$$r = \frac{a}{2} \text{ cm}$$

14 **a** Let the required angle be θ.

$$\tan\theta = \frac{AB}{BD}$$

$$= \frac{20}{40} = 0.5$$

$$\theta = 26.57°$$

b Let the required angle be ϕ.

$$\angle BED = 90^\circ$$

$$\tan \angle BCD = \frac{BD}{BC} = \frac{40}{30} = \frac{4}{3}$$

$$\angle BCD = 53.130^\circ$$

$$\frac{BE}{BC} = \sin \angle BCE = \sin 53.130^\circ = 0.8$$

$$BE = 30 \times 0.8 = 24 \text{ m}$$

$$\tan \phi = \frac{AB}{BE} = \frac{20}{24} = 0.8333$$

$$\phi = 39.81^\circ$$

Note: BE may also be found using similar triangles, and/or noticing that triangles CBD and CBE are 3–4–5 triangles.

c Let the required angle be α.

$$CD = \sqrt{30^2 + 40^2} = 50 \text{ m}$$

$$CE = 25 \text{ m}$$

Use the cosine rule to find BE.

$$\cos c = \frac{CB}{CD} = \frac{30}{50} = 0.6$$

$$BE^2 = CB^2 + CE^2 - 2 \times CB \times CE \times \cos C$$

$$= 30^2 + 25^2 - 2 \times 30 \times 25 \times 0.6$$

$$= 625$$

$$BE = 25 \text{ m}$$

$$\tan \alpha = \frac{AB}{BE} = \frac{20}{25} = 0.8$$

$$\alpha = 38.66^\circ$$

Solutions to technology-free questions

1 a $a^2 = b^2 + c^2 - 2bc\cos A$

$$6^2 = x^2 + 10^2 - 2x \times 10 \times \frac{\sqrt{3}}{2}$$

$$x^2 - 10\sqrt{3}x + 64 = 0$$

$$x = \frac{10\sqrt{3} \pm \sqrt{300 - 4 \times 1 \times 64}}{2}$$

$$= \frac{10\sqrt{3} \pm \sqrt{44}}{2}$$

$$= \frac{10\sqrt{3} \pm 2\sqrt{11}}{2}$$

$$= 5\sqrt{3} \pm \sqrt{11}$$

b $\dfrac{\sin y}{10} = \dfrac{\sin 30^\circ}{6}$

$$\sin y = \frac{10 \times \sin 30^\circ}{6}$$

$$= \frac{10}{12} = \frac{5}{6}$$

$$y = \sin^{-1}\left(\frac{5}{6}\right)$$

$$\text{or } 180^\circ - \sin^{-1}\left(\frac{5}{6}\right)$$

Since both answers to **a** are positive, this must be an ambiguous case.

2 a Triangle is isosceles, so $\angle B = 30^\circ$ and $\angle C = 120^\circ$

$$\text{Area} = \frac{1}{2} \times 40 \times 40 \times \sin 120^\circ$$

$$= 400\sqrt{3}\text{ cm}^2$$

$$= \frac{1}{2}bh$$

$$\therefore \quad 20h = 400\sqrt{3}$$

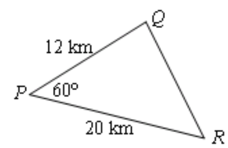
$$h = 20\sqrt{3}\text{ cm}$$

b $\dfrac{CM}{40} = \sin 30^\circ$

$$CM = 40 \times \sin 30^\circ$$

$$= 20\text{ cm}$$

3



$$QR^2 = 12^2 + 20^2 - 2 \times 12 \times 20 \times \cos 60^\circ$$

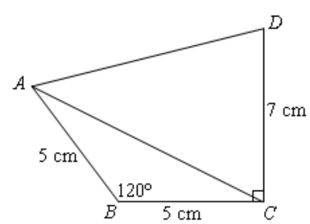
$$= 144 + 400 - 240$$

$$= 304$$

$$QR = \sqrt{304}$$

$$= \sqrt{16 \times 19} = 4\sqrt{19}\text{ km}$$

4



a Use the cosine rule.

$$AC^2 = 5^2 + 5^2 - 2 \times 5 \times 5 \times \cos 120^\circ$$

$$= 25 + 25 + 25$$

$$= 75$$

$$AC = \sqrt{75} = 5\sqrt{3}\text{ cm}$$

b $\text{Area} = \dfrac{1}{2} \times 5 \times 5 \times \sin 120^\circ$

$$= \frac{25\sqrt{3}}{4}\text{ cm}^2$$

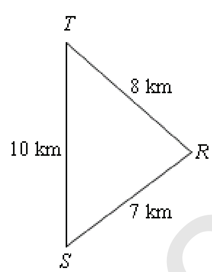
c In isosceles triangle ABC,

$$\angle ACB = \angle BAC$$
$$= \frac{1}{2}(180° - 120°) = 30°$$
$$\angle ACD = 90° - 30° = 60°$$
$$\text{Area of } ADC = \frac{1}{2} \times 7 \times AC \times \sin 60°$$
$$= \frac{1}{2} \times 7 \times 5\sqrt{3} \times \frac{\sqrt{3}}{2}$$
$$= \frac{105}{4} \text{ cm}^2$$

d $$\text{Total area} = \frac{25\sqrt{3}}{4} + \frac{105}{4}$$
$$= \frac{25\sqrt{3} + 105}{4}$$
$$= \frac{5(5\sqrt{3} + 21)}{4} \text{ cm}^2$$

5 $x = 180° - 37° = 143°$

6



$$\cos S = \frac{10^2 + 7^2 - 8^2}{2 \times 10 \times 7}$$
$$= \frac{85}{140} = \frac{17}{28}$$

7 First note that $AB = c = 5$ cm $\angle BAC = A = 60°$ and $AC = b = 6$ cm, so the angle is included. So start by finding $a = BC$ by the cosine rule.

$$a^2 = b^2 + c^2 2bc \cos A$$
$$= 36 + 25 - 60 \cos 60°$$
$$= 36 + 25 - 30$$
$$= 31$$
$$a = \sqrt{31}$$

Now use the sine rule.

$$\frac{\sin B}{6} = \frac{\sin 60°}{\sqrt{31}}$$
$$\sin \angle ABC = \frac{6 \sin 60°}{\sqrt{31}}$$
$$= \frac{3\sqrt{3}}{\sqrt{31}} = \frac{3\sqrt{93}}{31}$$

8 $$A = \frac{1}{2}r^2\theta$$
$$33 = \frac{1}{2} \times 6^2 \times \theta$$
$$= 18\theta$$
$$\theta = \frac{33}{18} = \frac{11}{6} \text{ (radians)}$$

9 **a** **i** $\angle TAB = 90° - 60° = 30°$

ii $\angle ATB = 180° - 30° - (90° + 45°)$
$$= 15°$$

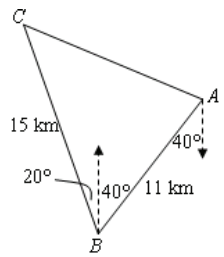
b $$\frac{AT}{\sin 135°} = \frac{300}{\sin 15°}$$
$$AT = \sin 135° \times 300$$
$$\times \frac{4}{\sqrt{6} - \sqrt{2}}$$
$$= \frac{1}{\sqrt{2}} \times \frac{1200}{\sqrt{6} - \sqrt{2}}$$
$$= \frac{1200}{\sqrt{12} - 2}$$
$$= \frac{1200}{2\sqrt{3} - 2} = \frac{600}{\sqrt{3} - 1}$$

$= \dfrac{600}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \dfrac{600(\sqrt{3}+1)}{3-1}$

$= 300(\sqrt{3}+1)$ m

10



Use the cosine rule.

$AC^2 = b^2$

$= a^2 + c^2 - 2ac\cos B$

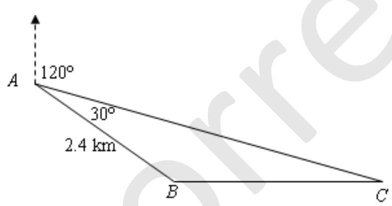
$= 11^2 + 15^2 - 2 \times 11 \times 15\cos 60°$

$= 121 + 225 - 165$

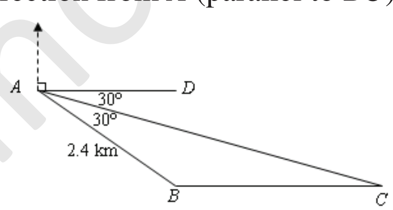
$= 181$

$AC = \sqrt{181}$ km

11 a



Draw a line *AD* in an easterly direction from *A* (parallel to *BC*).



$\angle DAC = 30°$

$\angle ACB = \angle DAC = 30°$

$\angle ABC = 180° - 30 - 30$

$= 120°$

$\therefore \quad BC = 2.4$ km

Use the cosine rule to find *AC*.

$AC^2 = b^2$

$= a^2 + c^2 - 2ac\cos B$

$= 2.4^2 + 2.4^2 - 2 \times 2.4$

$\times 2.4 \times \cos 120°$

$= 5.76 + 5.76 + 5.76 = 17.28$

$AC = \sqrt{1728}$

$= \sqrt{5.76 \times 3}$

$= 2.4\sqrt{3}$ or $\dfrac{12\sqrt{3}}{5}$ km

12 $\quad l = r\theta$

$30 = 12\theta$

$\theta = \dfrac{30}{12} = \left(\dfrac{5}{2}\right)^c$

$A = \dfrac{1}{2} \times 12^2 \times \dfrac{5}{2}$

$= 180 \text{ cm}^2$

13 The reflex angle $= 2\pi - 2$

$\approx 2 \times 3.14 - 2$

≈ 4.28 radians

Arc length $\approx 5 \times 4.28$

$= 21.4$ cm

14 a Draw a perpendicular from O to bisect AB at D

$$\sin \angle AOD = \frac{12}{13}$$

$$\angle AOD = \sin^{-1}\frac{12}{13}$$

$$\angle AOB = 2\sin^{-1}\frac{12}{13}$$

$$\text{arc } AB = r\theta$$

$$= 13 \times 2\sin^{-1}\frac{12}{13}$$

$$= 26\sin^{-1}\frac{12}{13}$$

b Reflex $\angle AOB = 2\pi - 2\sin^{-1}\frac{12}{13}$

$$\text{area} = \frac{1}{2} \times 13^2 \times \left(2\pi - 2\sin^{-1}\frac{12}{13}\right)$$

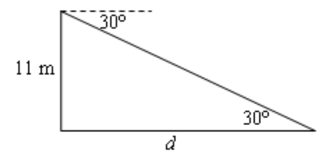
$$= 169\left(\pi - \sin^{-1}\frac{12}{13}\right) \text{ cm}^2$$

Note: the perpendicular distance from O to AB can be calculated to be 5 cm using Pythagoras' theorem, and so $\sin^{-1}\frac{12}{13} = \cos^{-1}\frac{5}{13} = \tan^{-1}\frac{12}{5}$.

Either these three angles may be used interchangeably.

15 First calculate the distance of each boat from the cliff.

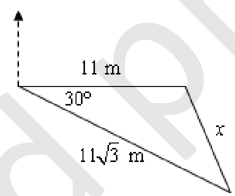
The first boat will form a right-angled isosceles triangle and is 11 m from the cliff.



For the second boat,

$$\frac{11}{d} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$d = 11\sqrt{3} \text{ m}$$



Use the cosine rule.

$$x^2 = 11^2 + (11\sqrt{3})^2 - 2 \times 11 \times 11\sqrt{3} \times \cos 30^\circ$$

$$= 121 + 363 - 363$$

$$= 121$$

$$x = 11 \text{ m}$$

Solutions to multiple-choice questions

1 D Use the sine rule.

$$\frac{\sin Y}{y} = \frac{\sin X}{x}$$

$$\frac{\sin Y}{18} = \frac{\sin 62^\circ}{21}$$

$$\sin Y = 18 \times \frac{\sin 62^\circ}{21}$$

$$= 0.7568$$

$$Y = 49.2^\circ$$

2 C Use the cosine rule.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= 30^2 + 21^2 - 2 \times 30 \times 21 \times \frac{51}{53}$$

$$= 128.547$$

$$c \approx 11$$

3 C Use the cosine rule.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{5.2^2 + 6.8^2 - 7.3^2}{2 \times 5.2 \times 6.8}$$

$$= 0.2826$$

$$C \approx 74^\circ$$

4 B $\text{Area} = \frac{1}{2}bc\sin A$

$$= \frac{1}{2} \times 5 \times 3 \times \sin 30^\circ$$

$$= 3.75 \text{ cm}^2$$

5 A The other angles in the (isosceles) triangle are both

$$\frac{180^\circ - 130^\circ}{2} = 25^\circ.$$

Use the sine rule.

$$\frac{10}{\sin 130^\circ} = \frac{r}{\sin 25^\circ}$$

$$r = \frac{10 \times \sin 25^\circ}{\sin 130^\circ}$$

$$\approx 5.52 \text{ cm}$$

6 A First find the angle at the centre using the cosine rule.

$$\cos C = \frac{6^2 + 6^2 - 5^2}{2 \times 6 \times 6}$$

$$= 0.6527$$

$$C = 49.248^\circ = 0.8595^c$$

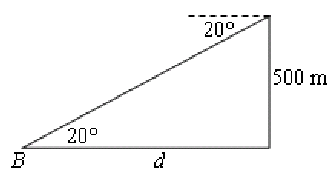
Segment area

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

$$= \frac{1}{2} \times 6^2 \times (0.8595 - \sin 0.8595)$$

$$\approx 1.8 \text{ cm}^2$$

7 D



$$\frac{500}{d} = \tan 20^\circ$$

$$d = \frac{500}{\tan 20^\circ}$$

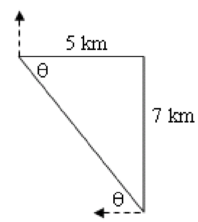
$$\approx 1374 \text{ m}$$

8 B $\tan\theta = \frac{80}{1300}$

$$= 0.0615$$

$$\theta = 3.521^\circ \approx 4^\circ$$

9 C



$$\tan\theta = \frac{7}{5} = 1.4$$

$$\theta = 54°$$

Bearing = 270° + 54° = 324°

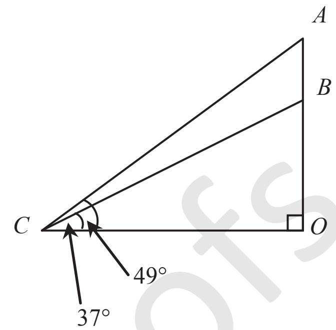
10 A 215° − 180° = 035°

Solutions to extended-response questions

1 **a** $\angle ACB = 12°, \ \angle CBO = 53°, \ \angle CBA = 127°$

b $\angle CAB = 41°$

The sine rule applied to triangle ABC gives



$$\frac{CB}{\sin 41°} = \frac{60}{\sin 12°}$$

$$\therefore \quad CB = \frac{60 \sin 41°}{\sin 12°}$$

$$= 189.33, \text{ correct to two decimal places}$$

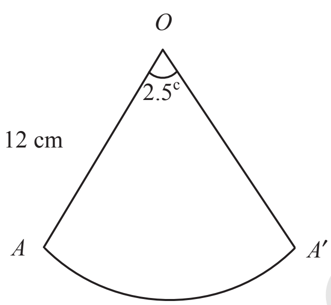
c

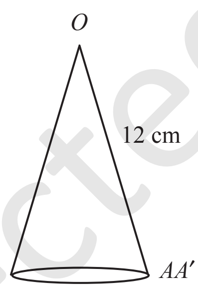
$$\frac{OB}{CB} = \sin 37°$$

$$\therefore \quad OB = CB \sin 37°$$

$$= 113.94 \text{ m}$$

2 **a**





The circumference of the circular base $= 2.5 \times 12$

$= 30$ cm

Therefore $2\pi r = 30$

Solve for r, the radius of the base.

$$r = \frac{30}{2\pi}$$

$$= 4.77 \text{ cm, correct to two decimal places}$$

b Curved surface area of the cone = area of the sector

$$= \frac{1}{2} \times 144 \times 2.5$$

$$= 180 \text{ cm}^2$$

c The diameter length is required.

$$\text{Diameter} = 2r = \frac{30}{\pi} = 9.55 \text{ cm}$$

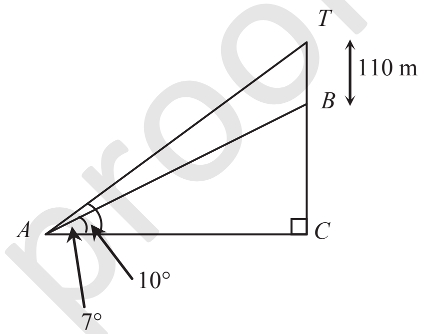
3 a $\angle TAB = 3°$, $\angle ABT = 97°$

$$\angle ATB = (83 - 3)° = 80°$$

b The sine rule applied to triangle ATB gives

$$\frac{AB}{\sin 80°} = \frac{110}{\sin 3°}$$

$$\therefore \quad CB = \frac{110 \sin 80°}{\sin 3°} = 2069.87$$



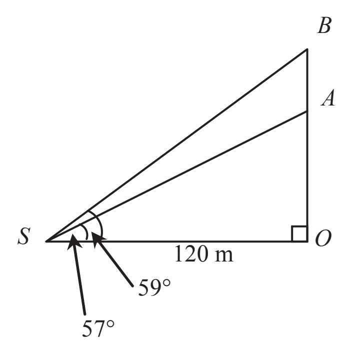
c $CB = AB \sin 7°$

$$= 252.25 \text{ m}$$

4 a In right-angled triangle AOS

$$\frac{OA}{120} = \tan 57°$$

$$\therefore \quad OA = 120 \tan 57° = 184.78 \text{ m, correct to two decimal places}$$



b In right-angled triangle SOB

$$\frac{OB}{120} = \tan 59°$$

$$\therefore \quad OB = 120 \tan 59° = 199.71 \text{ m, correct to two decimal places}$$

c The distance $AB = OB - OA = 14.93$ m, correct to two decimal places.

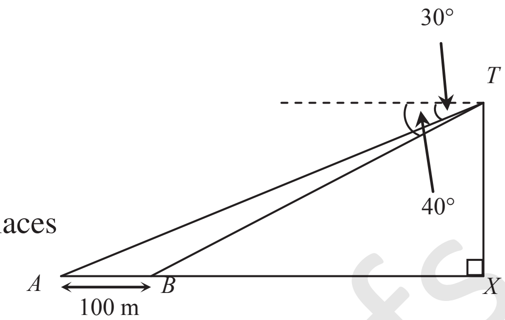
5 a $\angle ATB = 10°$

In triangle ATB the sine rule gives

$$\frac{100}{\sin 10°} = \frac{AT}{\sin 140°}$$

$$\therefore \quad AT = \frac{100 \sin 140°}{\sin 10°}$$

$$= 370.17 \text{ m, correct to two decimal places}$$



b Applying the sine rule again gives

$$\frac{BT}{\sin 30°} = \frac{100}{\sin 10°}$$

$$\therefore \quad BT = 287.94 \text{ m, correct to two decimal places}$$

c In right-angled-triangle TBX

$$\frac{XT}{BT} = \sin 40°$$

$$\therefore \quad XT = BT \sin 40°$$

$$= 185.08 \text{ m, correct to two decimal places}$$

6 a Applying Pythagoras' theorem in triangle VBA

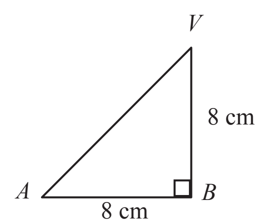
$$VA^2 = 8^2 + 8^2$$

$$= 64 + 64$$

$$= 128$$

$$\therefore \quad VA = 8\sqrt{2}$$

The distance VA is $8\sqrt{2}$ cm.



b Applying Pythagoras' theorem in triangle VBC

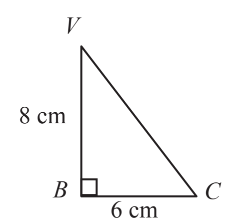
$$VC^2 = 8^2 + 6^2$$

$$= 64 + 36$$

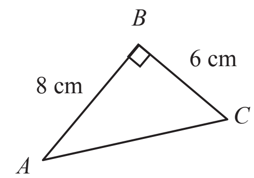
$$= 100$$

$$\therefore \quad VC = 10$$

The distance VC is 10 cm.



c Applying Pythagoras' theorem in triangle ABC

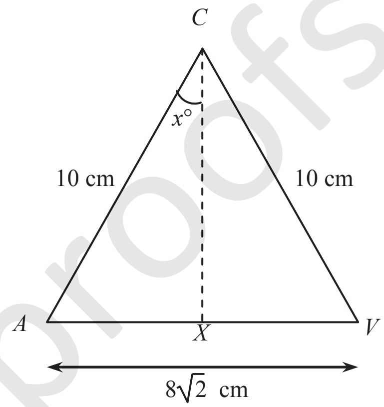


$$AC^2 = 8^2 + 6^2$$
$$= 64 + 36$$
$$= 100$$
$$\therefore \quad AC = 10$$

The distance AC is 10 cm.

d Triangle VCA is isosceles with $VC = AC$

In right-angled triangle CXA



$$\sin x^\circ = \frac{4\sqrt{2}}{10}$$
$$= \frac{2\sqrt{2}}{5}$$

$$\text{Therefore } x^\circ = 34.4490\ldots^\circ$$
$$\text{and } \angle ACV = 68.899\ldots^\circ$$
$$= 68.9^\circ, \text{correct to one decimal place}$$

7 Let L be the perimeter of triangle ABC and α, β and γ the angles at A, B and C respectively.

The sine rule gives: $\dfrac{AB}{\sin\gamma} = \dfrac{AC}{\sin\beta} = \dfrac{BC}{\sin\alpha}$

Let $AB = x$

$$\frac{x}{\sin\gamma} = \frac{AC}{\sin\beta} = \frac{BC}{\sin\alpha}$$

Therefore $AC = \dfrac{x\sin\beta}{\sin\gamma}$ and $BC = \dfrac{x\sin\alpha}{\sin\gamma}$

Next

$$L = AB + AC + BC$$
$$= x + \frac{x\sin\beta}{\sin\gamma} + \frac{x\sin\alpha}{\sin\gamma}$$
$$= x\left(1 + \frac{\sin\beta}{\sin\gamma} + \frac{\sin\alpha}{\sin\gamma}\right)$$
$$= x\left(\frac{\sin\gamma + \sin\beta + \sin\alpha}{\sin\gamma}\right)$$
$$\therefore x = \frac{L\sin\gamma}{\sin\gamma + \sin\beta + \sin\alpha}$$

$$\text{Area} = \frac{1}{2}AC \times AB \times \sin\alpha$$

$$= \frac{1}{2}\frac{L\sin\gamma}{\sin\gamma + \sin\beta + \sin\alpha} \times \frac{L\sin\gamma}{\sin\gamma + \sin\beta + \sin\alpha} \times \frac{\sin\beta}{\sin\gamma} \times \sin\alpha$$

$$= \frac{L^2 \sin\alpha \sin\beta \sin \sin\gamma}{2(\sin\gamma + \sin\beta + \sin\alpha)^2}$$

Chapter 14 – Further trigonometry

Solutions to Exercise 14A

1 **a** $\cos\left(\frac{3\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

b $\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

c $\sin\left(\frac{25\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

d $\sin\left(\frac{15\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

e $\cos\left(\frac{17\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

f $\sin\left(-\frac{15\pi}{4}\right) = \sin\left(\pi - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

g $\sin 27\pi) = 0$

h $\sin\left(-\frac{17\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

i $\cos\left(\frac{75\pi}{6}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

j $\cos\left(-\frac{15\pi}{6}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

k $\sin\left(-\frac{35\pi}{2}\right) = 1$

l $\cos\left(-\frac{45\pi}{6}\right) = 0$

m $\cos\left(-\frac{16\pi}{3}\right) = -\frac{1}{2}$

n $\sin\left(-\frac{105\pi}{2}\right) = -1$

o $\cos(1035\pi) = -1$

2 **a** $\cos(-\alpha) = \cos\alpha$

$= 0.6$

b $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$

$= 0.6$

c $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$= 0.3$

d $\sin(-x) = -\sin x$

$= -0.3$

e $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

$= -0.3$

f $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$

$= 0.6$

g $\sin\left(\frac{3\pi}{2} + \alpha\right) = -\sin\left(\frac{\pi}{2} + \alpha\right)$

$= -\cos\alpha = -0.6$

h $\cos\left(\frac{3\pi}{2} - x\right) = -\cos\left(\frac{\pi}{2} - x\right)$

$= -\sin x = -0.3$

Solutions to Exercise 14B

1 a $\tan x = -1$

$$x = \frac{3\pi}{4} \text{ or } x = \frac{7\pi}{4}$$

b $\tan x = \sqrt{3}$

$$x = \frac{\pi}{3} \text{ or } x = \frac{4\pi}{3}$$

c $\tan x = \frac{1}{\sqrt{3}}$

$$x = \frac{\pi}{6} \text{ or } x = \frac{7\pi}{6}$$

d

$\tan 2x = 1$

$$2x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \text{ or } -\frac{3\pi}{4} \text{ or } -\frac{7\pi}{4}$$

$$x = -\frac{7\pi}{8} \text{ or } x = -\frac{3\pi}{8} \text{ or } x = \frac{\pi}{8} \text{ or } x = \frac{5\pi}{8}$$

e $\tan 2x = \sqrt{3}$

$$2x = \frac{\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{3}$$

$$x = -\frac{5\pi}{6} \text{ or } x = -\frac{\pi}{3} \text{ or } x = \frac{\pi}{6}$$

$$\text{or } x = \frac{2\pi}{3}$$

f

$\tan 2x = -\frac{1}{\sqrt{3}}$

$$2x = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } x = -\frac{\pi}{6} \text{ or } -\frac{7\pi}{6}$$

$$x = -\frac{7\pi}{12} \text{ or } x = -\frac{\pi}{12} \text{ or } \frac{5\pi}{12} \text{ or } x = \frac{11\pi}{12}$$

2 a

$$\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = 1$$

$$2\left(x - \frac{\pi}{4}\right) = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \text{ or } \frac{9\pi}{4} \text{ or } \frac{13\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{\pi}{8} \text{ or } \frac{5\pi}{8} \text{ or } \frac{9\pi}{4} \text{ or } \frac{13\pi}{8}$$

$$x = \frac{3\pi}{8} \text{ or } \frac{7\pi}{8} \text{ or } \frac{11\pi}{8} \text{ or } \frac{15\pi}{8}$$

b

$$\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = -1$$

$$2\left(x - \frac{\pi}{4}\right) = \frac{3\pi}{4} \text{ or } -\frac{\pi}{4} \text{ or } -\frac{5\pi}{4} \text{ or } -\frac{9\pi}{4}$$

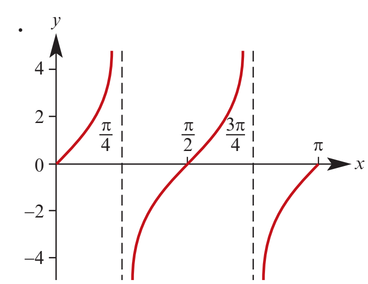
$$x - \frac{\pi}{4} = \frac{3\pi}{8} \text{ or } \frac{-\pi}{8} \text{ or } -\frac{5\pi}{8} \text{ or } -\frac{9\pi}{8}$$

$$x = -\frac{7\pi}{8} \text{ or } -\frac{3\pi}{8} \text{ or } \frac{\pi}{8} \text{ or } \frac{5\pi}{8}$$

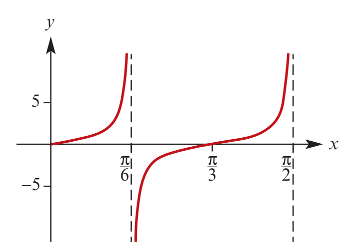
c $-\frac{13\pi}{18}, -\frac{7\pi}{18}, -\frac{\pi}{18}, \frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}$

d $-\frac{\pi}{6}$

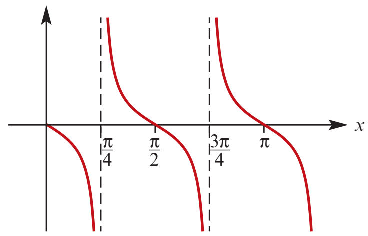
3 a .



b

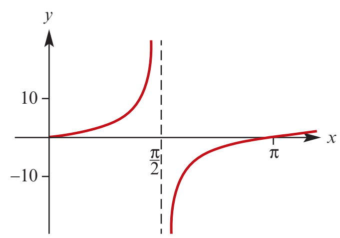


c



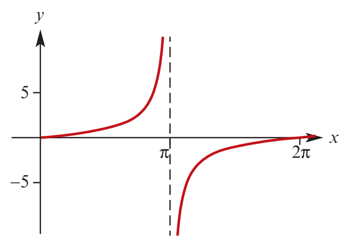
y
x
π/4
π/2
3π/4
π

d



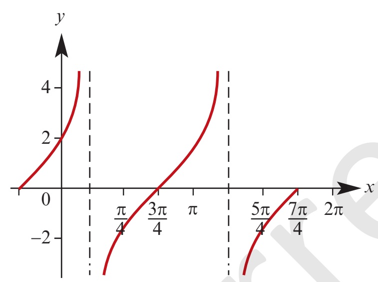
y
10
−10
x
π/2
π

e



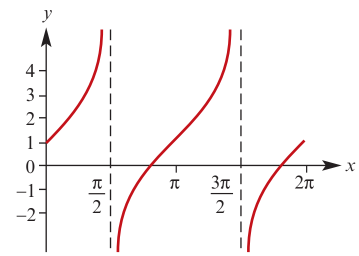
y
5
−5
x
π
2π

f



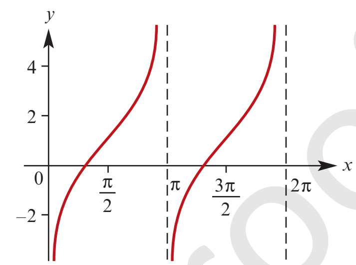
y
4
2
0
−2
x
π/4
3π/4
π
5π/4
7π/4
2π

g



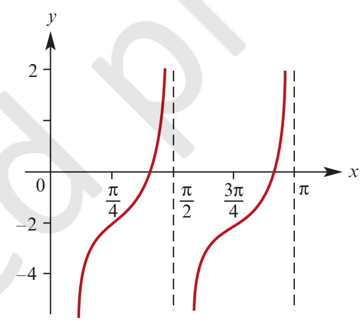
y
4
3
2
1
0
−1
−2
x
π/2
π
3π/2
2π

h



y
4
2
0
−2
x
π/2
π
3π/2
2π

i



y
2
0
−2
−4
x
π/4
π/2
3π/4
π

Solutions to Exercise 14C

1 a $\cot\frac{3\pi}{4} = \dfrac{\cos\frac{3\pi}{4}}{\sin\frac{3\pi}{4}}$

$= -\frac{1}{\sqrt{2}} \div \frac{1}{\sqrt{2}}$

$= -1$

b $\operatorname{cosec}\frac{5\pi}{4} = \dfrac{1}{\sin\frac{5\pi}{4}}$

$= -\dfrac{1}{\frac{1}{\sqrt{2}}}$

$= -\sqrt{2}$

c $\sec\frac{5\pi}{6} = \dfrac{1}{\cos\frac{5\pi}{6}}$

$= \dfrac{1}{\frac{-\sqrt{3}}{2}}$

$= -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

d $\operatorname{cosec}\frac{\pi}{2} = \dfrac{1}{\sin\frac{\pi}{2}}$

$= \frac{1}{1} = 1$

e $\sec\frac{4\pi}{3} = \dfrac{1}{\cos\frac{4\pi}{3}}$

$= \dfrac{1}{-\frac{1}{2}} = -2$

f $\operatorname{cosec}\frac{13\pi}{6} = \dfrac{1}{\sin\frac{13\pi}{6}}$

$= \dfrac{1}{\sin\frac{\pi}{6}}$

$= \dfrac{1}{\frac{1}{2}} = 2$

g $\cot\frac{7\pi}{3} = \dfrac{\cos\frac{7\pi}{3}}{\sin\frac{7\pi}{3}}$

$= \frac{1}{2} \div \frac{\sqrt{3}}{2}$

$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

h $\sec\frac{5\pi}{3} = \dfrac{1}{\cos\frac{5\pi}{3}}$

$= \dfrac{1}{\frac{1}{2}} = 2$

2 a $\cot 135^\circ = \dfrac{\cos 135^\circ}{\sin 135^\circ}$

$= -\frac{1}{\sqrt{2}} \div \frac{1}{\sqrt{2}}$

$= -1$

b $\sec 150^\circ = \dfrac{1}{\cos 150^\circ}$

$= \dfrac{1}{-\frac{\sqrt{3}}{2}}$

$= -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

c $\operatorname{cosec} 90° = \dfrac{1}{\sin 90°}$

$= \dfrac{1}{1} = 1$

d $\cot 240° = \dfrac{\cos 240°}{\sin 240°}$

$= -\dfrac{1}{2} \div -\dfrac{\sqrt{3}}{2}$

$= \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

e $\sec 225° = \dfrac{1}{\cos 225°}$

$= \dfrac{1}{-\dfrac{1}{\sqrt{2}}}$

$= -\sqrt{2}$

f $\sec 330° = \dfrac{1}{\cos 330°}$

$= \dfrac{1}{\dfrac{\sqrt{3}}{2}}$

$= \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

g $\cot 315° = \dfrac{\cos 315°}{\sin 315°}$

$= \dfrac{1}{\sqrt{2}} \div -\dfrac{1}{\sqrt{2}}$

$= -1$

h $\operatorname{cosec} 300° = \dfrac{1}{\sin 300°}$

$= \dfrac{1}{\dfrac{-\sqrt{3}}{2}}$

$= -\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$

i $\cot 420° = \dfrac{\cos 420°}{\sin 420°}$

$= \dfrac{\cos 60°}{\sin 60°}$

$= \dfrac{1}{2} \div \dfrac{\sqrt{3}}{2}$

$= \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

3 a $\operatorname{cosec} x = 2$

$\sin x = \dfrac{1}{2}$

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

b $\cot x = \sqrt{3}$

$\tan x = \dfrac{1}{\sqrt{3}}$

$x = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$

c $\sec x = -\sqrt{2}$

$\cos x = -\dfrac{1}{\sqrt{2}}$

$x = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

d $\operatorname{cosec} x = \sec x$

$\sin x = \cos x$

$\tan x = 1$

$x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

4 a $\cos\theta = \dfrac{1}{\sec\theta}$

$= -\dfrac{8}{17}$

b $\cos^2\theta + \sin^2\theta = 1$

$$\frac{64}{289} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{225}{289}$$

$$\sin\theta = \frac{15}{17} \text{ (Since } \sin\theta > 0\text{)}$$

c $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{15}{17} \div -\frac{8}{17}$$

$$= -\frac{15}{8}$$

5 $1 + \tan^2\theta = \sec^2\theta$

$$\sec^2\theta = 1 + \frac{49}{576} = \frac{625}{576}$$

$$\sec\theta = \frac{25}{24} \text{ (since } \cos\theta > 0\text{)}$$

$$\cos\theta = \frac{24}{25}$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = -\frac{7}{24}$$

$$\sin\theta = -\frac{7}{24} \times \frac{24}{25}$$

$$= -\frac{7}{25}$$

6 $1 + \tan^2\theta = \sec^2\theta$

$\sec^2\theta = 1 + 0.16 = 1.16$

$$\sec\theta = -\sqrt{\frac{116}{100}}$$

(Since θ is in the 3rd quadrant)

$$= -\sqrt{\frac{29}{25}}$$

$$= -\frac{\sqrt{29}}{5}$$

7 $\cot\theta = \dfrac{3}{4}$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\sec^2\theta = 1 + \frac{16}{9} = \frac{25}{9}$$

$$\sec\theta = -\frac{5}{3} (\cos\theta < 0)$$

$$\cos\theta = -\frac{3}{5}$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{4}{3}$$

$$\sin\theta = \frac{4}{3} \times -\frac{3}{5}$$

$$= -\frac{4}{5}$$

$$\frac{\sin\theta - 2\cos\theta}{\cot\theta - \sin\theta} = \frac{-\frac{4}{5} - -\frac{6}{5}}{\frac{3}{4} - -\frac{4}{5}}$$

$$= \frac{2}{5} \div \frac{31}{20}$$

$$= \frac{2}{5} \times \frac{20}{31} = \frac{8}{31}$$

8 $\cos^2\theta + \sin^2\theta = 1$

$$\frac{4}{9} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{5}{9}$$

$$\sin\theta = -\frac{\sqrt{5}}{3}\left(\frac{3\pi}{2} < \theta < 2\pi\right)$$

$$\tan\theta = -\frac{\sqrt{5}}{3} \div \frac{2}{3} = -\frac{\sqrt{5}}{2}$$

$$\cot\theta = -\frac{2}{\sqrt{5}}$$

$$\frac{\tan\theta - 3\sin\theta}{\cos\theta - 2\cot\theta} = \frac{-\frac{\sqrt{5}}{2} - \left(-\sqrt{5}\right)}{\frac{2}{3} - \left(-\frac{4}{\sqrt{5}}\right)}$$

$$= \frac{\sqrt{5}}{2} \div \frac{2\sqrt{5} + 12}{3\sqrt{5}}$$

$$= \frac{\sqrt{5}}{2} \times \frac{3\sqrt{5}}{2\sqrt{5} + 12}$$

$$= \frac{15}{4(\sqrt{5} + 6)} \times \frac{6 - \sqrt{5}}{6 - \sqrt{5}}$$

$$= \frac{15(6 - \sqrt{5})}{4 \times (36 - 5)}$$

$$= \frac{15(6 - \sqrt{5})}{124}$$

9 **a** $(1 - \cos^2\theta)(1 + \cot^2\theta)$

$= \sin^2\theta \times \cot^2\theta$

$= \sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta}$

$= \cos^2\theta$, provided $\sin\theta \neq 0$

If $\sin\theta = 0$, $\cot\theta$ would be undefined.

b $\cos^2\theta \tan^2\theta + \sin^2\theta \cot 2\theta$

$= \cos^2\theta \times \frac{\sin^2\theta}{\cos^2\theta} + \sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta}$

$= \sin^2\theta + \cos^2\theta$

$= 1$, provided $\sin\theta \neq 0$ and $\cos\theta \neq 0$

c In cases like this, it is a good strategy to start with the more complicated expression.

$$\frac{\tan\theta + \cot\phi}{\cot\theta + \tan\phi}$$

$$= \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\phi}{\sin\phi}}{\frac{\cos\theta}{\sin\theta} + \frac{\sin\phi}{\cos\phi}}$$

$$= \frac{\frac{\sin\theta\sin\phi + \cos\phi\cos\theta}{\cos\theta\sin\phi}}{\frac{\cos\phi\cos\theta + \sin\theta\cos\phi}{\cos\phi\sin\theta}}$$

$$= \frac{\sin\theta\sin\phi + \cos\phi\cos\theta}{\cos\theta\sin\phi} \times \frac{\cos\phi\sin\theta}{\cos\phi\cos\theta + \sin\theta\cos\phi}$$

$$\frac{\tan\theta + \cot\phi}{\cot\theta + \tan\phi} = \frac{\cos\phi\sin\theta}{\cos\theta\sin\phi}$$

$$= \frac{\sin\theta}{\cos\theta} \times \frac{\cos\phi}{\sin\phi}$$

$$= \frac{\sin\theta}{\cos\theta} \div \frac{\sin\phi}{\cos\phi}$$

$$= \frac{\tan\theta}{\tan\phi}$$

This is provided $\cot\theta + \tan\phi \neq 0$ and the tangent and cotangent are defined.

d $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$

$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$

$+ \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$

$= 2\sin^2\theta + 2\cos^2\theta$

$= 2$

There are no restrictions on θ.

e $$\frac{1+\cot^2\theta}{\cot\theta\operatorname{cosec}\theta} = \frac{\operatorname{cosec}^2\theta}{\cot\theta\operatorname{cosec}\theta}$$
$$= \frac{\operatorname{cosec}\theta}{\cot\theta}$$
$$= \frac{1}{\sin\theta}\times\frac{\sin\theta}{\cos\theta}$$
$$= \frac{1}{\cos\theta}$$
$$= \sec\theta$$

Conditions: $\sin\theta \neq 0, \cos\theta \neq 0$

f $$\sec\theta+\tan\theta = \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}$$
$$= \frac{1+\sin\theta}{\cos\theta}\times\frac{1-\sin\theta}{1-\sin\theta}$$
$$= \frac{(1+\sin\theta)(1-\sin\theta)}{\cos\theta(1-\sin\theta)}$$
$$= \frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}$$
$$= \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)}$$
$$= \frac{\cos\theta}{1-\sin\theta}$$

Conditions: $\cos\theta \neq 0$ (includes $\sin\theta \neq 1$)

Solutions to Exercise 14D

1 Different angles may be used.

a $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2}$

$= \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$

b $\cos 105^\circ = \cos(45 + 60)^\circ$

$= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$

$= \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2} - \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2}$

$= \dfrac{1-\sqrt{3}}{2\sqrt{2}}$

$= \dfrac{\sqrt{2}-\sqrt{6}}{4}$

2 Different angles may be used.

a $\sin 165^\circ = \sin(180 - 15)^\circ$

$= \sin 15^\circ$

$\sin 15^\circ = \sin(45 - 30)^\circ$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} - \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2}$

$= \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

b $\tan 75^\circ = \tan(45 + 30)^\circ$

$= \dfrac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$

$= \dfrac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{\sqrt{3}+1}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \dfrac{3 + 2\sqrt{3} + 1}{3 - 1} = 2 + \sqrt{3}$

3 Different angles may be used.

a $\cos \dfrac{5\pi}{12} = \cos\left(\dfrac{\pi}{4} + \dfrac{\pi}{6}\right)$

$= \cos \dfrac{\pi}{4} \cos \dfrac{\pi}{6} - \sin \dfrac{\pi}{4} \sin \dfrac{\pi}{6}$

$= \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} - \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2}$

$= \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

b $\sin\dfrac{11\pi}{12} = \sin\left(\pi - \dfrac{\pi}{12}\right)$

$= \sin\dfrac{\pi}{12}$

$\sin\dfrac{\pi}{12} = \sin\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right)$

$= \sin\dfrac{\pi}{3}\cos\dfrac{\pi}{4} - \cos\dfrac{\pi}{3}\sin\dfrac{\pi}{4}$

$= \dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}} - \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

$= \dfrac{\sqrt{6} - \sqrt{2}}{4}$

c $\tan\left(-\dfrac{\pi}{12}\right) = \tan\left(\dfrac{\pi}{4} - \dfrac{\pi}{3}\right)$

$= \dfrac{\tan\dfrac{\pi}{4} - \tan\dfrac{\pi}{3}}{1 + \tan\dfrac{\pi}{4}\tan\dfrac{\pi}{3}}$

$= \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}} \times \dfrac{1 - \sqrt{3}}{1 - \sqrt{3}}$

$= \dfrac{1 - 2\sqrt{3} + 3}{1 - 3}$

$= -2 + \sqrt{3}$

4 $\cos^2 u = 1 - \sin^2 u$

$= 1 - \dfrac{144}{169} = \dfrac{25}{169}$

$\cos u = \pm\dfrac{5}{13}$

$\cos^2 v = 1 - \sin^2 v$

$= 1 - \dfrac{9}{25} = \dfrac{16}{25}$

$\cos v = \pm\dfrac{4}{5}$

$\sin(u + v) = \sin u\cos v + \cos u\sin v$

$= \pm\dfrac{3}{5} \times \dfrac{5}{13} \pm \dfrac{4}{5} \times \dfrac{12}{13}$

$= \dfrac{\pm 15 \pm 48}{65}$

There are four possible answers:

$\dfrac{63}{65}, \dfrac{33}{65}, -\dfrac{33}{65}, -\dfrac{63}{65}$

5 **a** $\sin\left(\theta + \dfrac{\pi}{6}\right) = \sin\theta\cos\dfrac{\pi}{6} + \cos\theta\sin\dfrac{\pi}{6}$

$= \dfrac{\sqrt{3}}{2}\sin\theta + \dfrac{1}{2}\cos\theta$

b $\cos\left(\pi - \dfrac{\pi}{4}\right)$

$= \cos\phi\cos\dfrac{\pi}{4} + \sin\phi\sin\dfrac{\pi}{4}$

$= \dfrac{1}{\sqrt{2}}\cos\phi + \dfrac{1}{\sqrt{2}}\sin\phi$

$= \dfrac{1}{\sqrt{2}}(\cos\phi + \sin\phi)$

c $\tan\left(\theta + \dfrac{\pi}{6}\right) = \dfrac{\tan\theta + \tan\dfrac{\pi}{3}}{1 - \tan\theta\tan\dfrac{\pi}{3}}$

$= \dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$

d $\sin\left(\theta - \dfrac{\pi}{4}\right) = \sin\theta\cos\dfrac{\pi}{4} - \cos\theta\sin\dfrac{\pi}{4}$

$= \dfrac{1}{\sqrt{2}}\sin\theta - \dfrac{1}{\sqrt{2}}\cos\theta$

$= \dfrac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$

6 **a** $\sin(v + (u - v)) = \sin u$

b $\cos((u + v) - v) = \cos u$

7 $\cos^2\theta = 1 - \sin^2\theta$

$= 1 - \frac{9}{25} = \frac{16}{25}$

$\cos\theta = -\frac{4}{5}$

(Since $\cos\theta < 0$)

$\sin^2\phi = 1 - \cos^2\phi$

$= 1 - \frac{25}{169} = \frac{144}{169}$

$\sin\phi = \frac{12}{13}$

(Since $\sin\theta > 0$)

a $\cos 2\phi = \cos^2\phi - \sin^2\phi$

$= \frac{25}{169} - \frac{144}{169}$

$= -\frac{119}{169}$

b $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times -\frac{3}{5} \times -\frac{4}{5}$

$= \frac{24}{25}$

c $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$= \frac{-3}{-4} = \frac{3}{4}$

$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$= \frac{\frac{3}{2}}{1 - \frac{9}{16}}$

$= \frac{3}{2} \times \frac{16}{7}$

$= \frac{24}{7}$

d $\sec 2\phi = \frac{1}{\cos 2\phi}$

$= -\frac{169}{119}$

e $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$= -\frac{3}{5} \times -\frac{5}{13} + -\frac{4}{5} \times \frac{12}{13}$

$= \frac{14 - 48}{65}$

$= -\frac{33}{65}$

f $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$= -\frac{4}{5} \times -\frac{5}{13} + -\frac{3}{5} \times \frac{12}{13}$

$= \frac{20 - 36}{65}$

$= -\frac{16}{65}$

g $\operatorname{cosec}(\theta + \phi) = \frac{1}{\sin(\theta + \phi)}$

$= -\frac{65}{33}$

h $\cot 2\theta = \frac{1}{\tan 2\theta}$

$= \frac{7}{24}$

8 **a** $\tan(u + v)$

$= \frac{\tan u + \tan v}{1 - \tan u \tan v}$

$= \left(\frac{4}{3} + \frac{5}{12}\right) \div \left(1 - \frac{4}{3} \times \frac{5}{12}\right)$

$= \frac{21}{12} \div \frac{4}{9}$

$= \frac{21}{12} \times \frac{9}{4}$

$= \frac{63}{16}$

b $\tan 2u = \dfrac{2\tan u}{1-\tan^2 u}$

$= \dfrac{\frac{8}{3}}{1-\frac{16}{9}}$

$= \dfrac{8}{3} \times \dfrac{9}{-7}$

$= -\dfrac{24}{7}$

c $\sec^2 u = 1 + \tan^2 u$

$= 1 + \dfrac{16}{9} = \dfrac{25}{9}$

$\cos^2 u = \dfrac{9}{25}$

$\cos u = \dfrac{3}{5}$ (since u is acute)

$\sec^2 v = 1 + \tan^2 v$

$= 1 + \dfrac{25}{144} = \dfrac{169}{144}$

$\cos^2 v = \dfrac{144}{169}$

$\cos v = \dfrac{12}{13}$ (since v is acute)

$\cos(u - v) = \cos u \cos v + \sin u \sin v$

$= \dfrac{3}{5} \times \dfrac{12}{13} + \dfrac{4}{5} \times \dfrac{5}{13}$

$= \dfrac{56}{65}$

d $\sin 2u = 2\sin u \cos u$

$= 2 \times \dfrac{4}{5} \times \dfrac{3}{5}$

$= \dfrac{24}{25}$

9 $\cos\alpha = -\dfrac{4}{5}$

$\cos^2\beta = 1 - \sin^2\beta$

$= 1 - \dfrac{576}{625} = \dfrac{29}{625}$

$\cos\beta = -\dfrac{7}{25}$

$\cos^2\alpha = 1 - \sin^2\alpha$

$= 1 - \dfrac{9}{25} = \dfrac{16}{25}$

a $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= \dfrac{16}{25} - \dfrac{9}{25}$

$= \dfrac{7}{25}$

b $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \dfrac{3}{5} \times -\dfrac{7}{25} - -\dfrac{4}{5} \times \dfrac{24}{25}$

$= \dfrac{75}{125} = \dfrac{3}{5}$

c $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

$= -\dfrac{3}{4}$

$\tan\beta = \dfrac{\sin\beta}{\cos\beta}$

$= -\dfrac{24}{7}$

$\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{-\frac{3}{4} + -\frac{24}{7}}{1 - -\frac{3}{4} \times \frac{24}{7}}$

$= -\dfrac{117}{28} \times -\dfrac{7}{11}$

$= \dfrac{117}{44}$

d $\sin 2\beta = 2\sin\beta\cos\beta$

$$= 2 \times \frac{7}{25} \times -\frac{24}{25}$$

$$= -\frac{336}{625}$$

10 a $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times -\frac{\sqrt{3}}{2} \times \frac{1}{2}$$

$$= -\frac{\sqrt{3}}{2}$$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \frac{1}{4} - \frac{3}{4}$$

$$= -\frac{1}{2}$$

11 a $(\sin\theta - \cos\theta)^2$

$$= \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$$

$$= 1 - \sin 2\theta$$

b $\sin^4\theta - \cos^4\theta$

$$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$$

$$= \cos 2\theta \times 1$$

$$= \cos 2\theta$$

12 a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$

$$= \sqrt{2}\left(\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right)$$

$$= \sin\theta - \cos\theta$$

b $\cos\left(\theta - \frac{\pi}{3}\right) = \cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}$

$$= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$$

$$\cos\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$

Add the last two equations:

$$\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$$

c $\tan\left(\theta + \frac{\pi}{4}\right)\tan\left(\theta - \frac{\pi}{4}\right)$

$$= \frac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}} \times \frac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\theta\tan\frac{\pi}{4}}$$

$$= \frac{\tan\theta + 1}{1 - \tan\theta} \times \frac{\tan\theta - 1}{1 + \tan\theta}$$

$$= -1$$

d $\cos\left(\theta + \frac{\pi}{6}\right) = \cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6}$

$$= \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta$$

$$\sin\left(\theta + \frac{\pi}{3}\right) = \sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$$

Add the two equations:

$$\cos\left(\theta + \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{3}\cos\theta$$

e $\tan\left(\theta + \frac{\pi}{4}\right) = \dfrac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}}$

$$= \frac{\tan\theta + 1}{1 - \tan\theta}$$

$$= \frac{1 + \tan\theta}{1 - \tan\theta}$$

f $\dfrac{\sin(u+v)}{\cos u \cos v}$

$= \dfrac{\sin u \cos v + \cos u \sin v}{\cos u \cos v}$

$= \dfrac{\sin u \cos v}{\cos u \cos v} + \dfrac{\cos u \sin v}{\cos u \cos v}$

$= \tan u + \tan v$

g $\dfrac{\sin(u+v)}{\sin(u-v)} = \dfrac{\sin u \cos v + \cos u \sin v}{\sin u \cos v - \cos u \sin v}$

Divide numerator and denominator by $\cos u \cos v$.

$\dfrac{\sin(u+v)}{\sin(u-v)} = \dfrac{\tan u + \tan v}{\tan u - \tan v}$

h $\cos 2\theta + 2\sin^2\theta$

$= \cos^2\theta - \sin\theta + 2\sin^2\theta$

$= \cos^2\theta + \sin^2\theta$

$= 1$

i

$\sin 4\theta = \sin(2 \times 2\theta)$

$= 2\sin 2\theta \cos 2\theta$

$= 2 \times 2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

$= 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

j $\dfrac{1-\sin 2\theta}{\sin\theta - \cos\theta}$

$= \dfrac{1-\sin 2\theta}{\sin\theta - \cos\theta} \times \dfrac{\sin\theta - \cos\theta}{\sin\theta - \cos\theta}$

$= \dfrac{(1-\sin 2\theta)(\sin\theta - \cos\theta)}{\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta}$

$= \dfrac{(1-\sin 2\theta)(\sin\theta - \cos\theta)}{1 - 2\sin\theta\cos\theta}$

$= \dfrac{(1-\sin 2\theta)(\sin\theta - \cos\theta)}{1 - \sin 2\theta}$

$= \sin\theta - \cos\theta$

Solutions to Exercise 14E

1 a Maximum = $\sqrt{4^2 + 3^2} = 5$

Minimum = -5

b Maximum = $\sqrt{3 + 1} = 2$

Minimum = -2

c Maximum = $\sqrt{1 + 1} = \sqrt{2}$

Minimum = $-\sqrt{2}$

d Maximum = $\sqrt{1 + 1} = \sqrt{2}$

Minimum = $-\sqrt{2}$

e Maximum = $\sqrt{9 + 3} = \sqrt{12} = 2\sqrt{3}$

Minimum = $-2\sqrt{3}$

f Maximum = $\sqrt{1 + 3} = 2$

Minimum = -2

g Maximum = $\sqrt{1 + 3} + 2 = 4$

Minimum = $-\sqrt{1 + 3} + 2 = 0$

h Maximum = $5 + \sqrt{3^2 + 2^2}$

$= 5 + \sqrt{13}$

Minimum = $5 - \sqrt{3^2 + 2^2}$

$= 5 - \sqrt{13}$

2 a

$$r = \sqrt{1 + 1} = \sqrt{2}$$

$$\cos\alpha = \frac{1}{\sqrt{2}};\ \sin\alpha = -\frac{1}{\sqrt{2}}$$

$$\alpha = -\frac{\pi}{4}$$

$$\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 1$$

$$\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$x - \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$x = \frac{\pi}{2}, \pi$$

b

$$r = \sqrt{3 + 1} = 2$$

$$\cos\alpha = \frac{\sqrt{3}}{2};\ \sin\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$2\sin\left(x + \frac{\pi}{6}\right) = 1$$

$$\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{5}, \frac{7\pi}{6}$$

$$x = 0, \frac{2\pi}{3}, 2\pi$$

c

$$r = \sqrt{3+1} = 2$$

$$\cos\alpha = \frac{1}{2};\ \sin\alpha = -\frac{\sqrt{3}}{2}$$

$$\alpha = -\frac{\pi}{3}$$

$$2\sin\left(x - \frac{\pi}{3}\right) = -1$$

$$\sin\left(c - \frac{\pi}{3}\right) = -\frac{1}{2}$$

$$x - \frac{\pi}{3} = -\frac{\pi}{6},\ \frac{7\pi}{6}$$

$$x = \frac{\pi}{6},\ \frac{3\pi}{2}$$

d

$$r = \sqrt{9+3} = \sqrt{12}$$

$$= 2\sqrt{3}$$

$$\cos\alpha = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\sin\alpha = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$2\sqrt{3}\cos\left(x + \frac{\pi}{6}\right) = 3$$

$$\cos\left(x + \frac{\pi}{6}\right) = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$x + \frac{\pi}{6} = \frac{\pi}{6},\ \frac{11\pi}{6},\ \frac{13\pi}{6}$$

$$x = 0,\ \frac{5\pi}{3},\ 2\pi$$

e

$$r = \sqrt{4^2 + 3^2}$$

$$= \sqrt{25} = 5$$

$$\cos\alpha = \frac{4}{5};\ \sin\alpha = \frac{3}{5}$$

$$\alpha \approx 36.87^\circ$$

$$5\sin(\theta + 36.87) \approx 5$$

$$\sin(\theta + 36.87) \approx 1$$

$$\theta + 36.87 \approx 90^\circ$$

$$\theta \approx 53.13^\circ$$

f

$$r = \sqrt{8+4} = \sqrt{12} = 2\sqrt{3}$$

$$\cos\alpha = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\sin\alpha = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\alpha \approx -35.26^\circ$$

$$2\sqrt{3}\sin(\theta - 35.26) \approx 3$$

$$\sin(\theta - 35.26) \approx \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\theta - 35.26 \approx 60^\circ, 120^\circ$$

$$\theta \approx 95.26^\circ, 155.26^\circ$$

3

$$r = \sqrt{3+1} = 2$$

$$\cos\alpha = \frac{\sqrt{3}}{2};\ \sin\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$2\cos\left(2x + \frac{\pi}{6}\right)$$

4 $r = \sqrt{1+1} = \sqrt{2}$

$\cos\alpha = \dfrac{1}{\sqrt{2}};\ \sin\alpha = \dfrac{1}{\sqrt{2}}$

$\alpha = \dfrac{\pi}{4}$

$\sqrt{2}\sin\left(3x - \dfrac{\pi}{4}\right)$

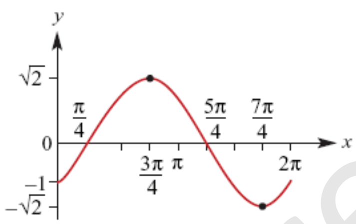
5 a $r = \sqrt{1+1} = \sqrt{2}$

$\cos\alpha = \dfrac{1}{\sqrt{2}};\ \sin\alpha = -\dfrac{1}{\sqrt{2}}$

$\alpha = -\dfrac{\pi}{4}$

$f(x) = \sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right)$

The graph will have amplitude $\sqrt{2}$, period 2π, and be translated $\dfrac{\pi}{4}$ units right.



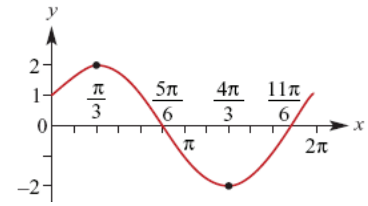
b $r = \sqrt{3+1} = 2$

$\cos\alpha = \dfrac{\sqrt{3}}{2},\ \sin\alpha = \dfrac{1}{2}$

$\alpha = \dfrac{\pi}{6}$

$f(x) = 2\sin\left(x + \dfrac{\pi}{6}\right)$

The graph will have amplitude 2, period 2π, and be translated $\dfrac{\pi}{6}$ units left.



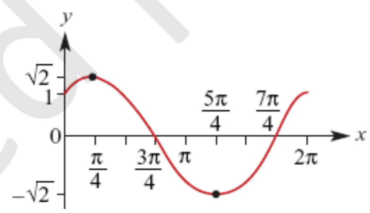
c $r = \sqrt{1+1} = \sqrt{2}$

$\cos\alpha = \dfrac{1}{\sqrt{2}};\ \sin\alpha = \dfrac{1}{\sqrt{2}}$

$\alpha = \dfrac{\pi}{4}$

$f(x) = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$

The graph will have amplitude $\sqrt{2}$, period 2π, and be translated $\dfrac{\pi}{4}$ units left.



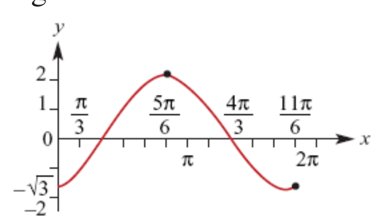
d $r = \sqrt{1+3} = 2$

$\cos\alpha = \dfrac{1}{2};\ \sin\alpha = -\dfrac{\sqrt{3}}{2}$

$\alpha = -\dfrac{\pi}{3}$

$f(x) = 2\sin\left(x - \dfrac{\pi}{3}\right)$

The graph will have amplitude 2, period 2π, and be translated $\dfrac{\pi}{3}$ units right.



Solutions to technology-free questions

1 **a** $\sec\theta + \operatorname{cosec}\theta\cot\theta$

$= \dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta} \times \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{1}{\cos\theta}\left(1 + \dfrac{\cos\theta}{\sin\theta} \times \dfrac{\cos\theta}{\sin\theta}\right)$

$= \dfrac{1}{\cos\theta}(1 + \cot^2\theta)$

$= \sec\theta\operatorname{cosec}^2\theta$

b $\dfrac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta}$

$= \dfrac{\tan^2\theta + 1 - \sin^2\theta}{\sec\theta + \sin\theta}$

$= \dfrac{\sec^2\theta - \sin^2\theta}{\sec\theta + \sin\theta}$

$= \dfrac{(\sec\theta - \sin\theta)(\sec\theta + \sin\theta)}{\sec\theta + \sin\theta}$

$= \sec\theta - \sin\theta$

2 **a** Maximum = 5, minimum = 1

b Maximum = 4, minimum = −2

c Maximum = 4, minimum = −4

d Maximum = 2, minimum = 0

e Maximum = 1 (when $\cos\theta = -1$), minimum = $\dfrac{1}{3}$

3 **a** $\sin^2\theta = \dfrac{1}{4}$

$\sin\theta = \pm\dfrac{1}{2}$

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

b $\sin 2\theta = \dfrac{1}{2}$

$2\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$

$\theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{13\pi}{12}, \dfrac{17\pi}{12}$

c $\cos 3\theta = \dfrac{\sqrt{3}}{2}$

$3\theta = \dfrac{\pi}{6}, \dfrac{11\pi}{6}, \dfrac{13\pi}{6}, \dfrac{23\pi}{6}, \dfrac{25\pi}{6}, \dfrac{35\pi}{6}$

$\theta = \dfrac{\pi}{18}, \dfrac{11\pi}{18}, \dfrac{13\pi}{18}, \dfrac{23\pi}{18}, \dfrac{25\pi}{18}, \dfrac{35\pi}{18}$

d $\sin^2 2\theta = 1$

$\sin 2\theta = \pm 1$

$2\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$

$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

e $\tan^2\theta = \dfrac{1}{3}$

$\tan\theta = \pm\dfrac{1}{\sqrt{3}}$

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

f $\tan 2\theta = -1$

$2\theta = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \dfrac{15\pi}{4}$

$\theta = \dfrac{3\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}, \dfrac{15\pi}{8}$

g $\sin 3\theta = -1$

$$3\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$$

$$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

h $\cos 2\theta = \frac{1}{\sqrt{2}}$

$$2\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$$

$$\theta = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$$

4

$$\tan\theta = 2\sin\theta$$

$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$$

$$\frac{\sin\theta}{\cos\theta} - 2\sin\theta = 0$$

$$\sin\theta\left(\frac{1}{\cos\theta} - 2\right) = 0$$

$$\sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$$

$$\theta = 60°, 300°, 0°,$$
$$180°, 360°$$

5 $\cos^2 A = 1 - \sin^2 A$

$$= 1 - \frac{25}{169} = \frac{144}{169}$$

$\cos A = \frac{12}{13}$ (Since A is acute)

$\cos^2 B = 1 - \sin^2 B$

$$= 1 - \frac{64}{289} = \frac{225}{289}$$

$\cos B = \frac{15}{17}$ (Since B is acute)

a $\cos(A + B)$

$$= \cos A\cos B - \sin A\sin B$$

$$= \frac{12}{13} \times \frac{15}{17} - \frac{5}{13} \times \frac{8}{17}$$

$$= \frac{140}{221}$$

b $\sin(A + B)$

$$= \sin A\cos B - \cos A\sin B$$

$$= - \times \frac{5}{13} \times \frac{15}{17} - \frac{12}{13} \times \frac{8}{17}$$

$$= -\frac{21}{221}$$

c

$$\tan A = \frac{\sin A}{\cos A} = \frac{5}{12}$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{8}{15}$$

$$\tan(A + B) = \frac{\tan a + \tan B}{1 - \tan A\tan B}$$

$$= \left(\frac{5}{12} + \frac{8}{15}\right)$$

$$\div\left(1 - \frac{5}{12} \times \frac{8}{15}\right)$$

$$= \frac{57}{60} \div \frac{7}{9}$$

$$= \frac{19}{20} \times \frac{9}{7}$$

$$= \frac{171}{140}$$

6 **a** Expression $= \cos(80° - 20°)$

$$= \cos 60° = \frac{1}{2}$$

b Expression $= \tan(15° + 30°)$

$$= \tan 45° = 1$$

7 **a** Expression $= \sin(A + B)$

$$= \sin\frac{\pi}{2} = 1$$

b Expression $= \cos(A + B)$

$$= \cos\frac{\pi}{2} = 0$$

8 **a** Maximum = 5, minimum = 1

b Maximum = 9, minimum = −1

9 **a** $\sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$$= \sin^2 A(1 - \sin^2 B)$$
$$- (1 - \sin^2 A)\sin^2 B$$
$$= \sin^2 A - \sin^2 A\sin^2 B$$
$$- \sin^2 B + \sin^2 A\sin^2 B$$
$$= \sin^2 A - \sin^2 B$$

b Left side

$$= \frac{\sin^2\theta + (1 + \cos\theta)^2}{\sin\theta(1 + \cos\theta)}$$
$$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1 + \cos\theta)}$$
$$= \frac{2 + 2\cos\theta}{\sin\theta(1 + \cos\theta)}$$
$$= \frac{2(1 + \cos\theta)}{\sin\theta(1 + \cos\theta)}$$
$$= \frac{2}{\sin\theta}$$

c Left side

$$= \frac{\sin\theta(1 - 2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)}$$
$$= \frac{\sin\theta(1 - \sin^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta + \cos^2\theta - 1)}$$
$$= \frac{\sin\theta(\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - (1 - \cos^2\theta))}$$
$$= \frac{\sin\theta(\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - \sin^2\theta)}$$
$$= \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta$$

10 $\cos^2 A = 1 - \sin^2 A$

$$= 1 - \frac{5}{9} = \frac{4}{9}$$

$\cos A = -\frac{2}{3}$ (Since A is obtuse)

a $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{4}{9} - \frac{5}{9}$$
$$= -\frac{1}{9}$$

b $\sin 2A = 2\sin A\cos A$

$$= 2 \times \frac{\sqrt{5}}{3} \times -\frac{2}{3}$$
$$= -\frac{4\sqrt{5}}{9}$$

c $\sin 4A = 2\sin 2A\cos 2A$

$$= 2 \times -\frac{4\sqrt{5}}{9} \times -\frac{1}{9}$$
$$= \frac{8\sqrt{5}}{81}$$

11 a Left side $= \dfrac{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}{1 + \dfrac{\sin^2\theta}{\cos^2\theta}}$

$= \dfrac{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}{1 + \dfrac{\sin^2\theta}{\cos^2\theta}} \times \dfrac{\cos^2\theta}{\cos^2\theta}$

$= \dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$

$= \dfrac{\cos 2\theta}{1} = \cos 2\theta$

b Left side

$= \dfrac{\sin^2 A + (1 + \cos A)^2}{\sin A(1 + \cos A)}$

$= \dfrac{\sin^2 A + 1 + 2\cos A + \cos^2 A}{\sin A(1 + \cos A)}$

$= \dfrac{2 + 2\cos A}{\sin A(1 + \cos A)}$

$= \dfrac{2(1 + \cos A)}{\sin A(1 + \cos A)}$

$= \dfrac{2}{\sin A}$

12 a $\tan 15° = \tan(60 - 45)°$

$= \dfrac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$

$= \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}}$

$= \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \dfrac{\sqrt{3} - 1}{\sqrt{3} - 1}$

$= \dfrac{3 - 2\sqrt{3} + 1}{3 - 1}$

$= 2 - \sqrt{3}$

b $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

Add the two equations:

$\sin(x + y) + \sin(x - y) = 2\sin x \cos y$

13 a Express in the form $r\sin(x + \alpha) = 1$.

$r = \sqrt{1 + 1} = \sqrt{2}$

$\cos\alpha = \dfrac{1}{\sqrt{2}};\ \sin\alpha = \dfrac{1}{\sqrt{2}}$

$\alpha = \dfrac{\pi}{4}$

$\sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right) = 1$

$\sin\left(x + \dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

$x + \dfrac{\pi}{4} = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}$

$x = 0, \dfrac{\pi}{2}, 2\pi$

b $2\sin\dfrac{x}{2}\cos\dfrac{x}{2} = -\dfrac{1}{2}$

$\sin\left(2 \times \dfrac{x}{2}\right) = -\dfrac{1}{2}$

$\sin x = -\dfrac{1}{2}$

$x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

c $3 \times \dfrac{2\tan x}{1 - \tan^2 x} = 2\tan x$

$2\tan x\left(\dfrac{3}{1 - \tan^2 x} - 1\right) = 0$

$2\tan x\left(\dfrac{3 - (1 - \tan^2 x)}{1 - \tan^2 x}\right) = 0$

$\tan x = 0$ (since$2 + \tan^2 x \neq 0$)

$x = 0, \pi, 2\pi$

d $\sin^2 x - \cos^2 x = 1$

$$\cos 2x = -1$$

$$2x = \pi, 3\pi$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

e $\sin(3x - x) = \dfrac{\sqrt{3}}{2}$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

f $\cos\left(2x - \dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

$$2x - \frac{\pi}{3} = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$$

$$2x = \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{19\pi}{6}, \frac{21\pi}{6}$$

$$x = \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$$

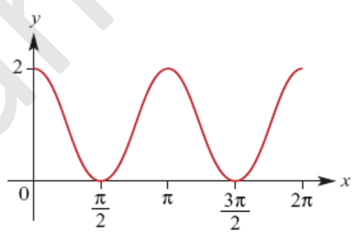
14 a $y = 2\cos^2 x$

$$= \cos^2 x + (1 - \sin^2 x)$$

$$= \cos^2 x - \sin^2 x + 1$$

$$= \cos 2x + 1$$

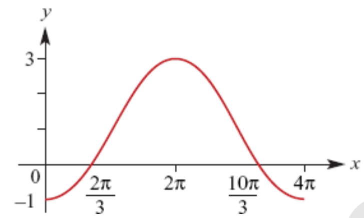
The graph of $y = \cos 2x$ (amplitude 1, period π) raised 1 unit.



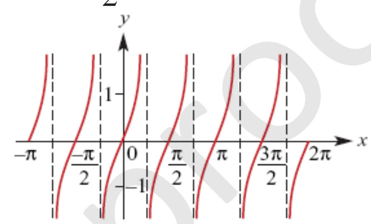
b The graph is

$$y = 1 - 2\sin\left(\frac{\pi}{2} - \frac{x}{2}\right) = 1 - 2\cos\frac{\pi}{2}.$$

It is $y = 2\cos\dfrac{x}{2}$ (period 4π) reflected in the x-axis and raised 1 unit.



c The normal tangent graph, but with period $\dfrac{\pi}{2}$.



15

$$\tan(\theta + A) = 4$$

$$\frac{\tan\theta + \tan A}{1 - \tan\theta\tan A} = 4$$

$$\frac{\tan\theta + 2}{1 - 2\tan\theta} = 4$$

$$\tan\theta + 2 = 4(1 - 2\tan\theta)$$

$$= 4 - 8\tan\theta$$

$$9\tan\theta = 2$$

$$\tan\theta = \frac{2}{9}$$

16 a $r = \sqrt{4 + 81} = \sqrt{85}$

$$\cos\alpha = \frac{2}{\sqrt{85}};\ \sin\alpha = \frac{9}{\sqrt{85}}$$

$\sqrt{85}\cos(\theta - \alpha)$, where

$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$$

b i $\sqrt{85}$

ii $\cos(\theta - \alpha) = 1$

$$\theta - \alpha = 0$$

$$\theta = -\alpha$$

$$\cos\theta = \cos\alpha$$

$$= \frac{2}{\sqrt{85}}$$

iii Solve $\sqrt{85}\cos(\theta + \alpha) = 1$.

$$\cos(\theta - \alpha) = \frac{1}{\sqrt{85}}$$

$$\theta - \alpha = \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$$

$$\theta = \alpha + \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$$

$$+ \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$$

Solutions to multiple-choice questions

1 A

$$\operatorname{cosec}x - \sin x = \frac{1}{\sin x} - \sin x$$
$$= \frac{1 - \sin^2 x}{\sin x}$$
$$= \frac{\cos^2 x}{\sin x}$$
$$= \cos x \times \frac{\cos x}{\sin x}$$
$$= \cos x \cot x$$

2 A

$$\cos x = -\frac{1}{3}$$
$$\cos^2 x + \sin^2 x = 1$$
$$\left(-\frac{1}{3}\right)^2 + \sin^2 x = 1$$
$$\sin^2 x = 1 - \frac{1}{9} = \frac{8}{9}$$
$$\sin x = \pm\sqrt{\frac{8}{9}}$$
$$= -\frac{2\sqrt{2}}{3}, \ \frac{2\sqrt{2}}{3}$$

3 B

$$\sec\theta = \frac{b}{a}$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\tan^2\theta = \frac{b^2}{a^2} = 1$$
$$= \frac{b^2 - a^2}{a^2}$$
$$\tan\theta = \frac{\sqrt{b^2 - a^2}}{a}$$

(Since $\tan\theta > 0$)

4 A $\angle ABC = u$; $\angle XBC = v$

$$\tan u = \frac{x+4}{2}; \ \tan v = \frac{x}{2}$$
$$\tan\theta = \tan(u - v)$$
$$= \frac{\tan u - \tan v}{1 + \tan u \tan v}$$
$$= \frac{\frac{x+4}{2} - \frac{x}{2}}{1 + \frac{x+4}{2} \times \frac{x}{2}}$$
$$= \frac{4}{2} \div \frac{4 + x(x+4)}{4}$$
$$= 2 \times \frac{4}{x^2 + 4x + 4}$$
$$= \frac{8}{(x+2)^2}$$
$$\sin^2 A = 1 - \cos^2 A$$
$$= 1 - t^2$$

5 C

$$\sin A = \sqrt{1 - t^2}$$

(Since $\sin A > 0$)

$$\cos^2 B = 1 - \sin^2 B$$
$$= 1 - t^2$$
$$\cos B = -\sqrt{1 - t^2}$$

Since $\cos B < 0$)

$$\sin(B + A) = \sin B \cos A + \cos B \sin A$$
$$= t \times t + \left(-\sqrt{1 - t^2}\right) \times \sqrt{1 - t^2}$$
$$= t^2 - (1 - t^2)$$
$$= 2t^2 - 1$$

6 E $\dfrac{\sin 2A}{\cos 2A - 1}$

$= \dfrac{2\sin A\cos A}{\cos^2 A - \sin^2 A - 1}$

$= \dfrac{2\sin A\cos A}{-\sin^2 A - (1 - \cos^2 A)}$

$= \dfrac{2\sin A\cos A}{-\sin^2 A - \sin^2 A}$

$= \dfrac{2\sin A\cos A}{-2\sin^2 A}$

$= \dfrac{\cos A}{\sin A}$

$= -\cot A$

7 C Check the symmetry properties:

$\sin\left(\dfrac{\pi}{2} - x\right) = \sin\left(\dfrac{\pi}{2} + x\right)$

$= -\sin\left(\dfrac{3\pi}{2} + x\right)$

$= \cos x$

$= \cos(-x)$

$\cos(-x) = \cos(2\pi - x)$

$\therefore \quad \sin\left(\dfrac{\pi}{2} - x\right) \neq \sin x$

8 E $(1 + \cot x)^2 + (1 - \cot x)^2$

$= 1 + 2\cot x + \cot^2 x + 1$

$- 2\cot x + \cot^2 x$

$= 2 + 2\cot^2 x$

$= 2(1 + \cot^2 x)$

$= 2\text{cosec}^2 x$

9 A $\sin 2A = 2\sin A\cos A$

$m = 2\sin A \times n$

$\sin A = \dfrac{m}{2n}$

$\tan A = \dfrac{\sin A}{\cos A}$

$= \dfrac{m}{2n} \times \dfrac{1}{n}$

$= \dfrac{m}{2n^2}$

10 D $r = \sqrt{1 + 1} = \sqrt{2}$

$\cos\alpha = \dfrac{1}{\sqrt{2}}$; $\sin\alpha = -\dfrac{1}{\sqrt{2}}$

A positive angle must be chosen,

$\therefore \quad \alpha = \dfrac{7\pi}{4}$

$\sqrt{2}\sin\left(x + \dfrac{7\pi}{4}\right)$

Solutions to extended-response questions

1 a $P = AD + DC + CB + BA$

$= 2AO + BA + 2AO + BA$

$= 4AO + 2BA$

$= 4 \times 5\cos\theta + 2 \times 5\sin\theta$

$= 20\cos\theta + 10\sin\theta$, as required.

b $a = 20,\ b = 10$ and $R = \sqrt{a^2 + b^2}$

$= \sqrt{20^2 + 10^2}$

$= \sqrt{500}$

$= 10\sqrt{5}$

Now $\cos\alpha = \dfrac{a}{R}$

$= \dfrac{20}{10\sqrt{5}}$

$= \dfrac{2}{\sqrt{5}}$

$= \dfrac{2\sqrt{5}}{5}$

Also $\sin\alpha = \dfrac{b}{R}$

$= \dfrac{10}{10\sqrt{5}}$

$= \dfrac{1}{\sqrt{5}}$

$= \dfrac{\sqrt{5}}{5}$

Hence, $0 < \alpha < 90$ and $\alpha^\circ = \cos^{-1}\left(\dfrac{2}{\sqrt{5}}\right)^\circ = (26.565\,05\ldots)^\circ$

Hence $P = R\cos(\theta - \alpha)$

$= 10\sqrt{5}\cos(\theta - \alpha)$ where $\alpha = \cos^{-1}\left(\dfrac{2}{\sqrt{5}}\right)$

When $P = 16$,

$10\sqrt{5}\cos(\theta - \alpha) = 16$

$\therefore \quad \cos(\theta - \alpha) = \dfrac{16}{10\sqrt{5}}$

$\therefore \quad (\theta - \alpha)^\circ = \cos^{-1}\left(\dfrac{8}{5\sqrt{5}}\right)^\circ$

$\therefore \quad \theta^\circ = \cos^{-1}\left(\dfrac{8}{5\sqrt{5}}\right)^\circ + \cos^{-1}\left(\dfrac{2}{\sqrt{5}}\right)^\circ$

When $P = 16, \theta = 70.88^\circ$

c Area of rectangle $= AB \times AD$

$= 5\sin\theta \times 2AO$

$= 5\sin\theta \times 2 \times 5\cos\theta$

$= 50\sin\theta\cos\theta$

$= 25 \times 2\sin\theta\cos\theta$

$= 25\sin 2\theta$

$\therefore \quad k\sin 2\theta = 25\sin 2\theta$

$\therefore \quad k = 25$

d Area is a maximum when $\sin 2\theta = 1$

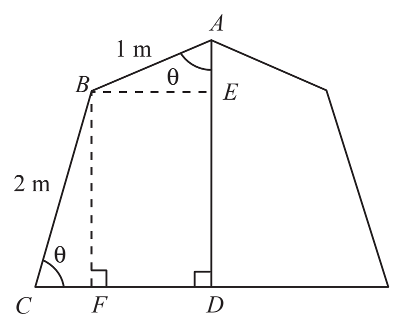
$\therefore \quad 2\theta = 90^\circ$

$\therefore \quad \theta = 45^\circ$

2 a $AD = AE + ED$

$= \cos\theta + BF$

$= \cos\theta + 2\sin\theta$



b $a = 1,\ b = 2$ and $R = \sqrt{a^2 + b^2}$

$$= \sqrt{1^2 + 2^2}$$

$$= \sqrt{5}$$

$$\text{Now } \cos\alpha = \frac{a}{R}$$

$$= \frac{1}{\sqrt{5}}$$

$$= \frac{\sqrt{5}}{5}$$

$$\text{Also } \sin\alpha = \frac{b}{R}$$

$$= \frac{2}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{5}$$

Hence, $0 < \alpha < 90$ and $\alpha^\circ = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)^\circ = (63.434\,94\ldots)^\circ$

Hence $AD = \sqrt{5}\cos(\theta - 63)^\circ$

c The maximum length of AD is $\sqrt{5}$ metres.
When $AD = \sqrt{5}$,

$$\sqrt{5}\cos(\theta - 63)^\circ = \sqrt{5}$$

$$\therefore \quad \cos(\theta - 63)^\circ = 1$$

$$\therefore \quad \theta - 63 = 0$$

$$\therefore \quad \theta = 63$$

d When $AD = 2.15$,

$$\sqrt{5}\cos(\theta - \alpha)^\circ = 2.15$$

$$\therefore \quad \cos(\theta - \alpha)^\circ = \frac{2.15}{\sqrt{5}}$$

$$\therefore \quad (\theta - \alpha)^\circ = \cos^{-1}\left(\frac{2.15}{\sqrt{5}}\right)^\circ$$

$$= (15.948\,46\ldots)^\circ$$

$$\therefore \quad \theta = (15.948 + 63.435)^\circ$$

The value of θ, for which $\theta > \alpha$, is 79.38°.

3 a $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\theta}{\sec^2\theta}$$

$$= \cos^2\theta\,(1-\tan^2\theta)$$

$$= \cos^2\theta - \frac{\cos^2\theta\sin^2\theta}{\cos^2\theta}$$

$$= \cos^2\theta - \sin^2\theta$$

Hence, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$, as required.

b i From **a**, $\cos\left(2\times 67\frac{1}{2}^\circ\right) = \dfrac{1-\tan^2\left(67\frac{1}{2}^\circ\right)}{1+\tan^2\left(67\frac{1}{2}^\circ\right)}$

$$\therefore \quad \cos 135^\circ = \frac{1-x^2}{1+x^2} \text{ where } x = \tan\left(67\frac{1}{2}^\circ\right)$$

$$\therefore \quad -\cos 45^\circ = \frac{1-x^2}{1+x^2}$$

$$\therefore \quad -\frac{1}{\sqrt{2}} = \frac{1-x^2}{1+x^2}$$

$$\therefore \quad -\sqrt{2} = \frac{1+x^2}{1-x^2}$$

$$\therefore \quad 1+x^2 = -\sqrt{2}(1-x^2)$$

$$\therefore \quad 1+x^2 = \sqrt{2}x^2 - \sqrt{2}, \text{ as required.}$$

ii $1+x^2 = \sqrt{2}x^2 - \sqrt{2}$

$$\therefore 1+\sqrt{2} = \sqrt{2}x^2 - x^2$$

$$= x^2(\sqrt{2}-1)$$

$$\therefore \quad x^2 = \frac{1+\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$= \frac{\sqrt{2}+1+2+\sqrt{2}}{2-1}$$

$$= 3+2\sqrt{2} \qquad \ldots \boxed{1}$$

Given $\tan\left(67\frac{1}{2}^\circ\right) = a + b\sqrt{2}$

$$\therefore \quad x = a + b\sqrt{2} \text{ where } x = \tan\left(67\frac{1}{2}^\circ\right)$$

$$\therefore \quad x^2 = (a + b\sqrt{2})^2$$

$$= a^2 + 2\sqrt{2}ab + 2b^2$$

$$= (a^2 + 2b^2) + (2ab)\sqrt{2} \qquad \ldots \boxed{2}$$

Equating $\boxed{1}$ and $\boxed{2}$

$$a^2 + 2b^2 = 3 \qquad \ldots \boxed{3}$$

$$2ab = 2$$

$$ab = 1$$

As a and b are integers, $a = 1, b = 1$ or $a = -1, b = -1$ and $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$

Note: An alternative method is to note

$$x^2 = \frac{\sqrt{2} + 1}{\sqrt{2} - 1}$$

$$= \frac{(\sqrt{2} + 1)^2}{(\sqrt{2} - 1)(\sqrt{2} + 1)}$$

$$= (\sqrt{2} + 1)^2$$

$\therefore x = \pm(\sqrt{2} + 1)$

When $b = -1, a = -1$,

$a + b\sqrt{2} = -1 - \sqrt{2}$

When $b = 1, a = 1$,

$a + b\sqrt{2} = 1 + \sqrt{2}$

But $\tan\left(67\frac{1}{2}^\circ\right) > 0$,

$$\therefore a + b\sqrt{2} = \sqrt{2} + 1$$

$$= 1 + \sqrt{2}$$

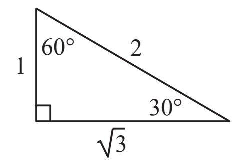
$$\therefore \quad a = 1, \; b = 1$$

c $\tan\left(7\frac{1}{2}^\circ\right) = \tan\left(67\frac{1}{2}^\circ - 60^\circ\right)$

$$= \frac{\tan\left(67\frac{1}{2}^\circ\right) - \tan(60^\circ)}{1 + \tan\left(67\frac{1}{2}^\circ\right)\tan(60^\circ)}$$

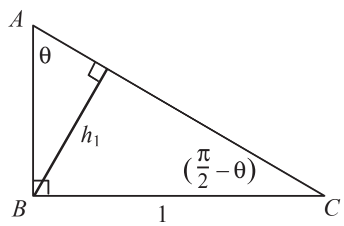
$$= \frac{1 + \sqrt{2} - \sqrt{3}}{1 + (1 + \sqrt{2})\sqrt{3}}$$

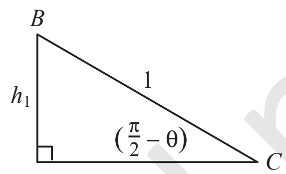
$$= \frac{1 + \sqrt{2} - \sqrt{3}}{1 + \sqrt{3} + \sqrt{6}}$$



$\tan 60^\circ = \sqrt{3}$

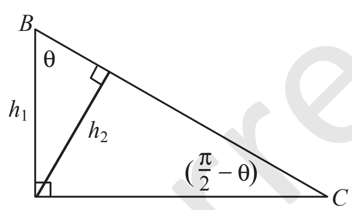
4 a i

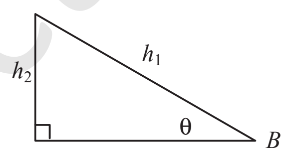




$$h_1 = \sin\left(\frac{\pi}{2} - \theta\right)$$
$$= \cos\theta$$

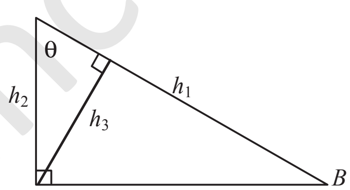
ii

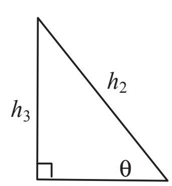




$$h_2 = h_1 \sin\theta$$
$$= \sin\theta\cos\theta$$

iii



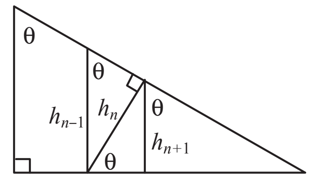


$$h_3 = h_2 \sin\theta$$
$$= \cos\theta\sin^2\theta$$

iv From the diagram, for $n \in N$, $\quad h_n = h_{n-1}\sin\theta$

and $h_{n+1} = h_n \sin\theta$

$\therefore \quad h_n = \sin^{n-1}\theta\cos\theta$



b $h_1 + h_2 + h_3 + \ldots = \cos\theta + \sin\theta\cos\theta + \sin^2\theta\cos\theta + \sin^3\theta\cos\theta + \ldots$

$= \cos\theta(1 + \sin\theta + \sin^2\theta + \ldots)$

Note: $0 < \sin\theta < 1$ for a given triangle.

$$\therefore \quad h_1 + h_2 + h_3 + \ldots = \cos\theta\left(\frac{1}{1-\sin\theta}\right)$$

$$= \frac{\cos\theta}{1-\sin\theta}$$

c If $\quad \dfrac{\cos\theta}{1-\sin\theta} = \sqrt{2}$

then $\quad \cos\theta = \sqrt{2} - \sqrt{2}\sin\theta$

$\therefore \quad \cos\theta + \sqrt{2}\sin\theta = \sqrt{2}$

Let $\quad \cos\theta + \sqrt{2}\sin\theta = r\sin(\theta + \alpha)$

$= r\sin\theta\cos\alpha + r\sin\alpha\cos\theta$

When $\theta = \dfrac{\pi}{2}$, $\quad \sqrt{2} = r\cos\alpha \quad \ldots$ [1]

When $\theta = 0$, $\quad 1 = r\sin\alpha \quad \ldots$ [2]

Squaring [1] and [2] and then adding gives

$2 + 1 = r^2\cos^2\alpha + r^2\sin^2\alpha$

$\therefore \quad 3 = r^2(\cos^2\alpha + \sin^2\alpha)$

$= r^2$

$\therefore \quad r = \sqrt{3}$

Substituting in [1] and [2]

$\sqrt{2} = \sqrt{3}\cos\alpha$

$$\therefore \quad \cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}$$

$$= \frac{\sqrt{6}}{3}$$

and $\quad 1 = \sqrt{3}\sin\alpha 1$

$$\therefore \quad \sin\alpha = \frac{1}{\sqrt{3}}$$

Now $\sqrt{2} = r\sin(\theta + \alpha)$

$\therefore$ $\sqrt{2} = \sqrt{3}\sin(\theta + \alpha)$

$\therefore$ $\sin(\theta + \alpha) = \dfrac{\sqrt{2}}{\sqrt{3}}$

and $\alpha^\circ < (\theta + \alpha)^\circ < (90 + \alpha)^\circ$

$\therefore$ $\theta = \sin^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^\circ - \cos^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^\circ$ or $180 - \sin^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^\circ + \cos^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^\circ$

But $180 - \sin^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^\circ + \cos^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^\circ = 90^\circ$

$\therefore$ $\theta = \sin^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^\circ - \cos^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^\circ$

$\approx 19.47^\circ$

5 a i $\angle CBA = \pi - \dfrac{2\pi}{5} = \dfrac{3\pi}{5}$

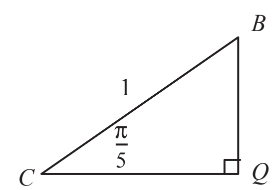
$\angle BCA = \dfrac{1}{2}\left(\pi - \dfrac{3\pi}{5}\right)$ as $\angle BCA = \angle BAC$ ($\triangle ABC$ is isosceles)

$= \dfrac{1}{2} \times \dfrac{2\pi}{5} = \dfrac{\pi}{5}$, as required.

ii $CA = 2CQ$

$= 2\cos\dfrac{\pi}{5}$

The length of CA is $2\cos\dfrac{\pi}{5}$ units.



b i $\angle DCP = \angle BCD - \angle BCA$

$= \angle CBA - \angle BCA$

$= \dfrac{3\pi}{5} - \dfrac{\pi}{5} = \dfrac{2\pi}{5}$, as required.

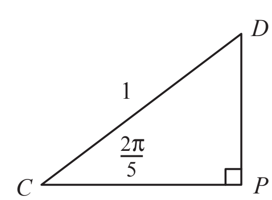
ii $AC = 2CP + PR$

$= 2\cos\dfrac{2\pi}{5} + DE$

$= 2\cos\dfrac{2\pi}{5} + 1$

But $AC = 2\cos\dfrac{\pi}{5}$ (from **a ii**)

$\therefore$ $2\cos\dfrac{\pi}{5} = 2\cos\dfrac{2\pi}{5} + 1$, as required.



iii $$2\cos\frac{\pi}{5} = 2\cos\frac{2\pi}{5} + 1$$

$$\therefore \quad 2\cos\frac{2\pi}{5} = 2\cos\frac{\pi}{5} - 1$$

$$\therefore \quad \cos\frac{2\pi}{5} = \cos\frac{\pi}{5} - \frac{1}{2}$$

$$\therefore \quad 2\cos^2\frac{\pi}{5} - 1 = \cos\frac{\pi}{5} - \frac{1}{2}$$

$$\therefore \quad 2\cos^2\frac{\pi}{5} - \cos\frac{\pi}{5} - \frac{1}{2} = 0 \text{ or equivalently } 4\cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5} - 1 = 0$$

iv $$2\cos^2\frac{\pi}{5} - \cos\frac{\pi}{5} - \frac{1}{2} = 0$$

$$\therefore \quad 2\left(\cos^2\frac{\pi}{5} - \frac{1}{2}\cos\frac{\pi}{5} - \frac{1}{4}\right) = 0$$

$$\therefore \quad 2\left(\cos^2\frac{\pi}{5} - \frac{1}{2}\cos\frac{\pi}{5} + \frac{1}{16} - \frac{5}{16}\right) = 0$$

$$\therefore \quad 2\left(\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 - \frac{5}{16}\right) = 0$$

$$\therefore \quad 2\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 - \frac{5}{8} = 0$$

$$\therefore \quad 2\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 = \frac{5}{8}$$

$$\therefore \quad \left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 = \frac{5}{16}$$

$$\therefore \quad \cos\frac{\pi}{5} - \frac{1}{4} = \pm\frac{\sqrt{5}}{4}$$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1}{4} \pm \frac{\sqrt{5}}{4}$$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1-\sqrt{5}}{4},\ \frac{1+\sqrt{5}}{4}$$

but $\cos\frac{\pi}{5} > 0$, as $0 < \frac{\pi}{5} < \frac{\pi}{2}$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$$

6 a i LHS $= \cos\theta$

$$= \cos\left(2 \times \frac{\theta}{2}\right)$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$\text{RHS} = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

$$= \frac{1 - \tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$= \cos^2\frac{\theta}{2}\left(1 - \tan^2\frac{\theta}{2}\right)$$

$$= \cos^2\frac{\theta}{2} - \frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

Therefore LHS = RHS.

Hence $\cos\theta = \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$, as required.

ii $\text{RHS} = \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

$$= \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$= \cos^2\frac{\theta}{2} \times 2\tan\frac{\theta}{2}$$

$$= \frac{2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= \sin\left(2 \times \frac{\theta}{2}\right)$$

$$= \sin\theta$$

$$= \text{LHS}$$

Hence $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$, as required.

b $$8\cos\theta - \sin\theta = 4$$

$$\therefore\quad 8\left(\frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}\right) - \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = 4$$

$$\therefore\quad 8\left(1 - \tan^2\frac{\theta}{2}\right) - 2\tan\frac{\theta}{2} = 4\left(1 + \tan 2\frac{\theta}{2}\right)$$

$$\therefore\quad 8 - 8\tan^2\frac{\theta}{2} - 2\tan\frac{\theta}{2} = 4 + 4\tan^2\frac{\theta}{2}$$

$$\therefore\quad 12\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - 4 = 0$$

$$\therefore\quad 6\tan^2\frac{\theta}{2} + \tan\frac{\theta}{2} - 2 = 0$$

$$\therefore\quad \left(3\tan\frac{\theta}{2} + 2\right)\left(2\tan\frac{\theta}{2} - 1\right) = 0$$

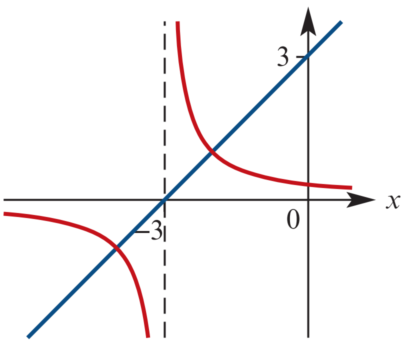
$$\therefore\quad 3\tan\frac{\theta}{2}+2 = 0 \text{ or } 2\tan\frac{\theta}{2} - 1 = 0$$

$$\therefore\quad \tan\frac{\theta}{2} = \frac{-2}{3} \quad \tan\frac{\theta}{2} = \frac{1}{2}$$

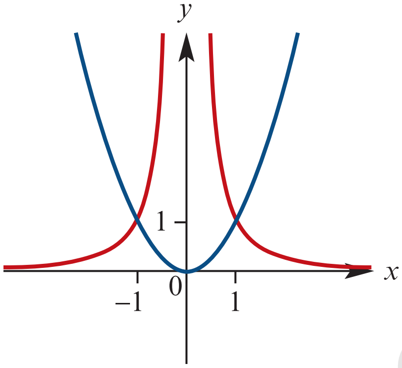
Chapter 15 – Graphing techniques

Solutions to Exercise 15A

1 a



b



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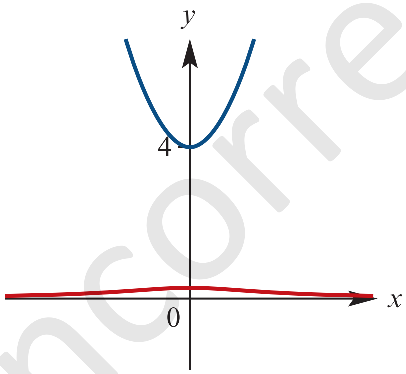
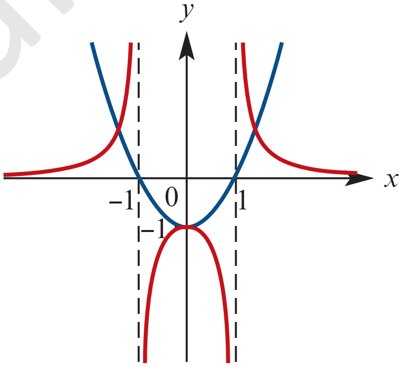
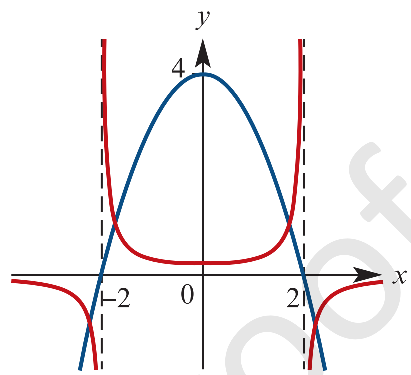


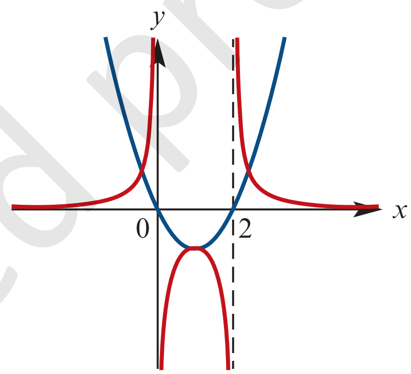
d



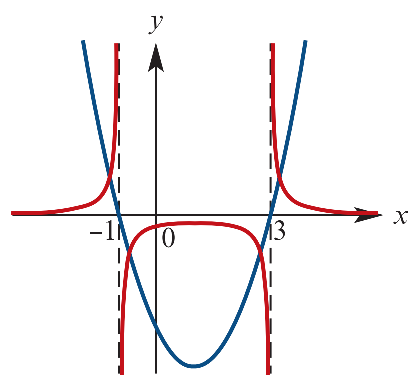
e



f



g



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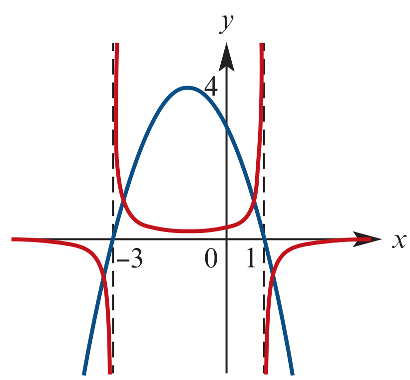
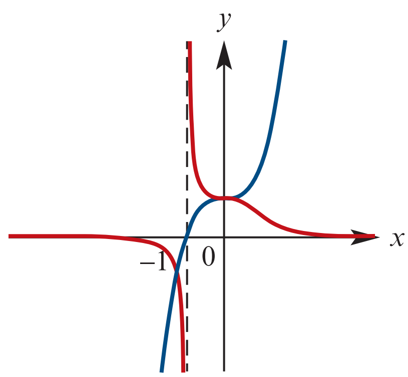
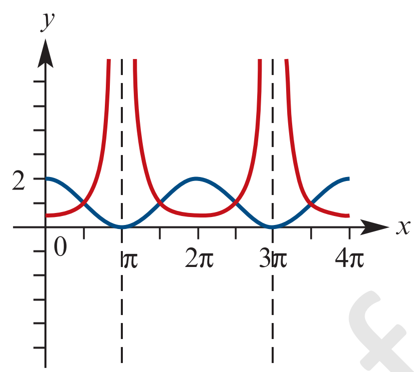


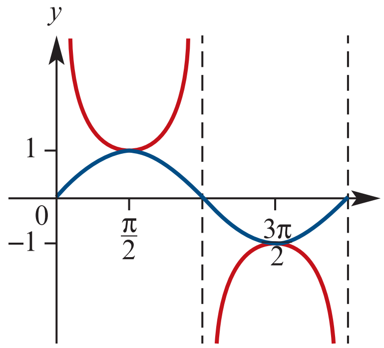
i


y
x
−1
0

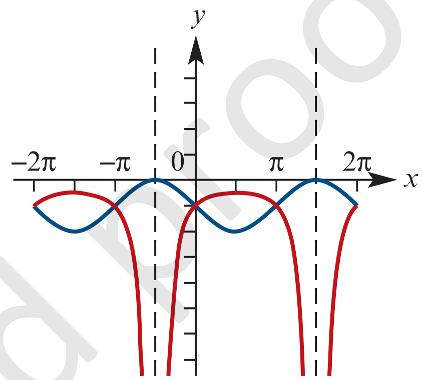
d


y
x
2
0
π
2π
3π
4π

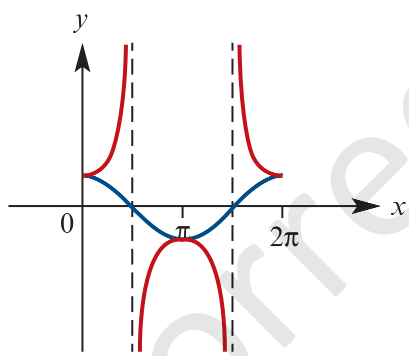
2 a


y
x
1
0
−1
π/2
3π/2

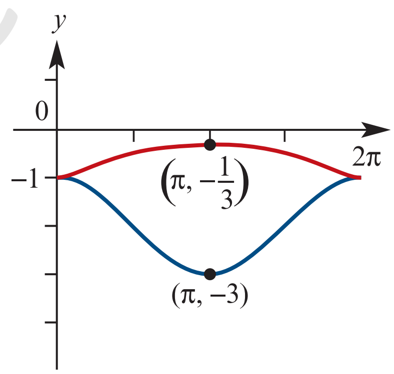
e


y
x
−2π
−π
0
π
2π

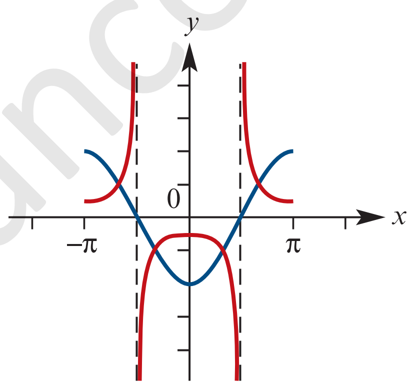
b


y
x
0
π
2π

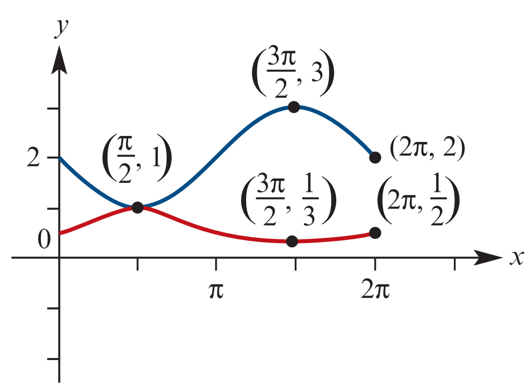
f


y
x
0
−1
2π
(π, −1/3)
(π, −3)

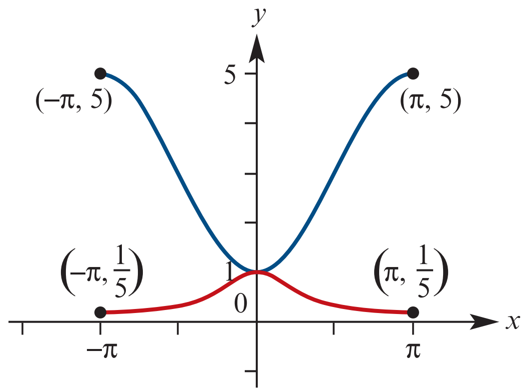
c


y
x
0
−π
π

g


y
x
2
0
π
2π
(3π/2, 3)
(π/2, 1)
(2π, 2)
(3π/2, 1/3)
(2π, 1/2)

h

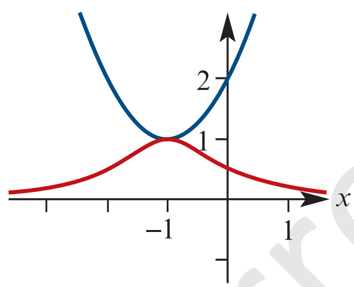


3 **a** We complete the square so that

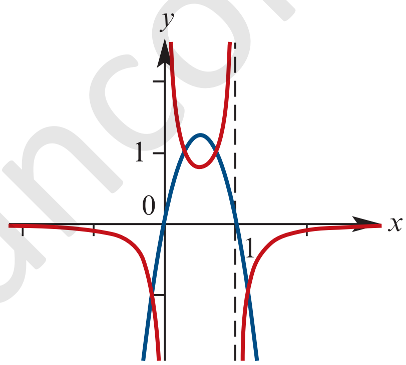
$$f(x) = x^2 + 2x + 2$$
$$= (x^2 + 2x + 1) - 1 + 2$$
$$= (x + 1)^2 + 1.$$

Therefore, a minimum turning point is located at point $(-1, 1)$.

b



4 **a**



b To find points of intersection we solve two equations: $f(x) = 1$ and $f(x) = -1$. If $f(x) = 1$ then

$$5x(1 - x) = 1.$$

Solving this quadratic equation (using the quadratic equation or your calculator) gives

$$x = \frac{5 \pm \sqrt{5}}{10}.$$

Since $f(x) = 1$, the coordinates are

$$\left(\frac{5 \pm \sqrt{5}}{10}, 1\right).$$

If $f(x) = 1$ then

$$5x(1 - x) = -1.$$

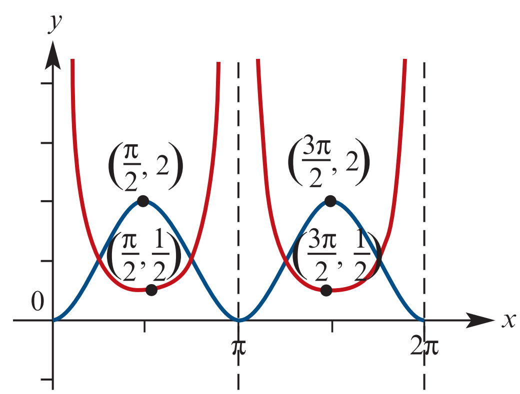
Solving this quadratic equation gives

$$x = \frac{5 \pm 3\sqrt{5}}{10}.$$

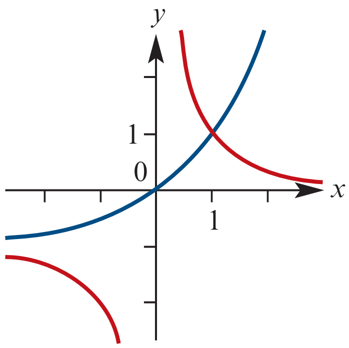
Since $f(x) = -1$, the coordinates are

$$\left(\frac{5 \pm 3\sqrt{5}}{10}, -1\right).$$

5 Notice that $y = 2\sin^2 x$ will have the same x-intercepts as $y = 2\sin x$ but will be non-negative for all values of x.



6



7 a We complete the square so that

$$f(x) = x^2 + 2kx + 1$$
$$= (x^2 + 2x + k^2) - k^2 + 1$$
$$= (x + k)^2 + 1 - k^2.$$

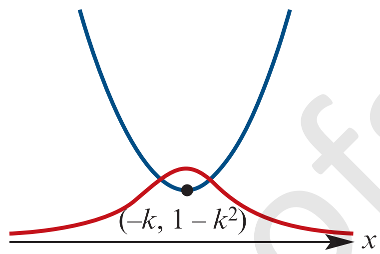
Therefore, a minimum turning point is located at point $(-k, 1 - k^2)$.

b **i** The graph of $y = f(x)$ will have no x-intercept provided $1 - k^2 > 0$. This means that $-1 < k < 1$.

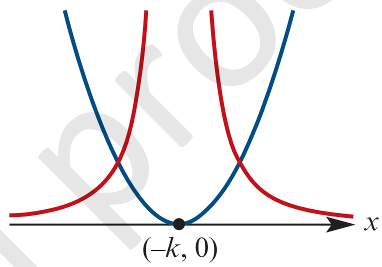
ii The graph of $y = f(x)$ will have one x-intercept provided $1 - k^2 - 0$. This means that $k = \pm 1$.

iii The graph of $y = f(x)$ will have two x-intercepts provided $1 - k^2 < 0$. This means that $k > 1$ or $k < -1$.

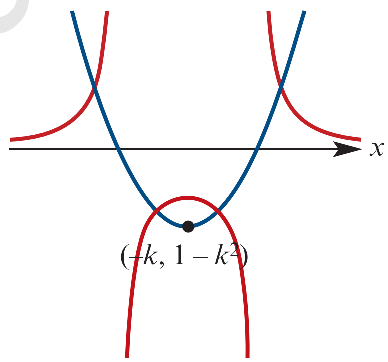
c **i**



ii



iii



Solutions to Exercise 15B

1 We know that the point $P(x, y)$ satisfies,

$$QP = 4$$

$$\sqrt{(x-1)^2 + (y-(-2))^2} = 4$$

$$(x-1)^2 + (y+2)^2 = 4^2.$$

This is a circle with centre $(1, -2)$ and radius 4.

2 We know that the point $P(x, y)$ satisfies,

$$QP = 5$$

$$\sqrt{(x-(-4))^2 + (y-3)^2} = 5$$

$$(x+4)^2 + (y-3)^2 = 5^2.$$

This is a circle with centre $(-4, 3)$ and radius 5.

3 **a** We know that the point $P(x, y)$ satisfies,

$$QP = RP$$

$$\sqrt{(x-(-1))^2 + (y-(-1))^2} = \sqrt{(x-1)^2 + (y-1)^2}$$

$$(x+1)^2 + (y+1)^2 = (x-1)^2 + (y-1)^2$$

$$x^2 + 2x + 1 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2 - 2y + 1$$

$$2x + 2y + = -2x - 2y$$

$$y = -x$$

b The above line has gradient -1. The straight line through $Q(-1, -1)$ and $R(1, 1)$ has equation

$$y = x$$

and thus has gradient 1. Because the product of the two gradients is -1, the two lines are perpendicular. Lastly, the midpoint of points Q and R is $(0, 0)$. This point is on the line $y = -x$ since if $x = 0$ then $y = 0$.

4 **a** We know that the point $P(x, y)$ satisfies,

$$QP = RP$$
$$\sqrt{(x-0)^2 + (y-2)^2} = \sqrt{(x-1)^2 + y^2}$$
$$x^2 + (y-2)^2 = (x-1)^2 + y^2$$
$$x^2 + y^2 - 4y + 4 = x^2 - 2x + 1 + y^2$$
$$-4y + 4 = -2x + 1$$
$$y = \frac{x}{2} + \frac{3}{4}$$

b The above line has gradient 1/2. The straight line through $Q(0, 2)$ and $R(1, 0)$ has gradient

$$m = \frac{0-2}{1-0} = -2$$

and equation

$$y = -2x + 2.$$

Because the product of the two gradients is −1, the two lines are perpendicular. Lastly, the midpoint of points Q and R is (1/2, 1). This point is on the line

$$y = \frac{x}{2} + \frac{3}{4}$$

since if $x = 1/2$ then

$$y = 1/4 + 3/4 = 1.$$

5 Since $P(x, y)$ is equidistant from points $Q(0, 1)$ and $R(2, 3)$ we know that

$$QP = RP$$
$$\sqrt{x^2 + (y-1)^2} = \sqrt{(x-2)^2 + (y-3)^2}$$
$$x^2 + (y-1)^2 = (x-2)^2 + (y-3)^2$$
$$x^2 + y^2 - 2y + 1 = x^2 - 4x + 4 + y^2 - 6y + 9$$
$$-2y + 1 = -4x - 6y + 13$$
$$4y + 4x = 12$$
$$y = -x + 3 \qquad (1)$$

We also know that $P(x, y)$ is 3 units away from $S(3, 3)$. Therefore $P(x, y)$ must lie on the circle whose equation is

$$(x-3)^2 + (y-3)^2 = 3^2. \qquad (2)$$

Substituting equation (1) into equation (2) gives

$$
\begin{aligned}
(x-3)^2 + (-x+3-3)^2 &= 9 \\
(x-3)^2 + x^2 &= 9 \\
x^2 - 6x + 9 + x^2 &= 9 \\
2x^2 - 6x &= 0 \\
2x(x-3) &= 0
\end{aligned}
$$

Therefore, either $x = 0$ or $x = 3$. Substituting $x = 0$ into (1) gives $y = 3$. Substituting $x = 3$ into (1) gives $y = 0$. Therefore, there are two answers: coordinates $(0, 3)$ and $(3, 0)$.

6 Since $P(x, y)$ is equidistant from points $Q(0, 1)$ and $R(2, 0)$ we know that

$$
\begin{aligned}
QP &= RP \\
\sqrt{x^2 + (y-1)^2} &= \sqrt{(x-2)^2 + y^2} \\
x^2 + y^2 - 2y + 1 &= x^2 - 4x + 4 + y^2 \\
-2y + 1 &= -4x + 4 \\
-2y &= -4x + 3 \\
4x - 2y &= 3. \qquad (1)
\end{aligned}
$$

Since $P(x, y)$ is equidistant from points $S(-1, 0)$ and $T(0, 2)$ we know that

$$
\begin{aligned}
SP &= TP \\
\sqrt{(x+1)^2 + y^2} &= \sqrt{x^2 + (y-2)^2} \\
x^2 + 2x + 1 + y^2 &= x^2 + y^2 - 4y + 4 \\
2x + 1 &= -4y + 4 \\
-4y &= 2x - 3 \\
2x + 4y &= 3. \qquad (2)
\end{aligned}
$$

Solving equations (1) and (2) simultaneously gives $x = \frac{9}{10}$ and $y = \frac{3}{10}$.

7 Since the treasure is 10 metres from a tree stump located at coordinates $T(0, 0)$, it lies on the circle whose equation is

$$x^2 + y^2 = 10^2. \qquad (1)$$

Since the treasure is 2 metres from a rock at coordinates $R(6, 10)$, it lies on the circle whose equation is

$$(x-6)^2 + (y-10)^2 = 2^2 \qquad (2)$$

Solving equations (1) and (2) simultaneously or by using your calculator gives two possible coordinates: either $(6, 8)$ or $\left(\frac{72}{17}, \frac{154}{17}\right)$.

8 **a** Since $P(x, y)$ is equidistant from points $R(4, 5)$ and $S(6, 1)$ we know that

$$RP = SP$$

$$\sqrt{(x-4)^2 + (y-5)^2} = \sqrt{(x-6)^2 + (y-1)^2}$$

$$x^2 - 8x + 16 + y^2 - 10y + 25 = x^2 - 12x + 36 + y^2 - 2y + 1$$

$$-8x + 16 - 10y + 25 = -12x + 36 - 2y + 1$$

$$8y - 4x = 4$$

$$2y - x = 1 \qquad (1)$$

b Since $P(x, y)$ is equidistant from points $S(6, 1)$ and $T(1, -4)$ we know that

$$SP = TP$$

$$\sqrt{(x-6)^2 + (y-1)^2} = \sqrt{(x-1)^2 + (y+4)^2}$$

$$x^2 - 12x + 36 + y^2 - 2y + 1 = x^2 - 2x + 1 + y^2 + 8y + 16$$

$$-12x + 36 - 2y + 1 = -2x + 1 + 8y + 16$$

$$x + y = 2 \qquad (1)$$

c Solving equations (1) and (2) simultaneously gives $x = 1$ and $y = 1$ so that the point required is $P(1, 1)$.

d The centre of the circle is $P(1, 1)$ and its radius will be the distance from $P(1, 1)$ to $R(4, 5)$. This is

$$r = PR = \sqrt{(4-1)^2 + (5-1)^2} = 5.$$

Therefore the equation of the circle must be

$$(x-1)^2 + (y-1)^2 = 5^2.$$

9 Let the point be $P(x, y)$. The gradient of AB is

$$\frac{5-1}{2-0} = 2$$

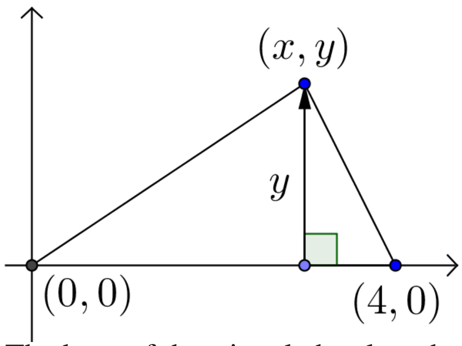
. The gradient of BP is

$$\frac{y-5}{x-2}.$$

Equating the two gradients gives,

$$\frac{y-5}{x-2} = 2$$
$$y - 5 = 2(x-2)$$
$$y = 2x + 1.$$

10 The triangle is shown below.



The base of the triangle has length 4 and its height if y. Therefore,

$$A = \frac{bh}{2}$$
$$12 = \frac{4y}{2}$$
$$y = 6.$$

11 **a** Let the point be $P(x, y)$. Then as the distance from P to the origin is equal to the sum of its x and y coordinates,

$$\sqrt{x^2 + y^2} = x + y$$
$$x^2 + y^2 = (x+y)^2$$
$$x^2 + y^2 = x^2 + 2xy + y^2$$
$$2xy = 0$$

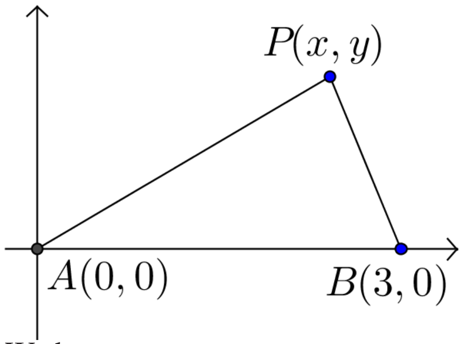
Therefore either $x = 0$ or $y = 0$. This is just both coordinate axes.

b Let the point be $P(x, y)$. Then as the distance from P to the origin is equal to the square of the sum of its x and y coordinates,

$$x^2 + y^2 = x + y$$
$$x^2 - x + y^2 - y = 0$$
$$\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4} = 0$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

This is a circle with centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ with radius $\frac{1}{\sqrt{2}}$.

12 Consider point $P(x, y)$. The triangle is shown below.



We have

$$AP = \sqrt{x^2 + y^2},$$

and

$$BP = \sqrt{(x-3)^2 + y^2}.$$

Since $AP : BP = 2$, we have

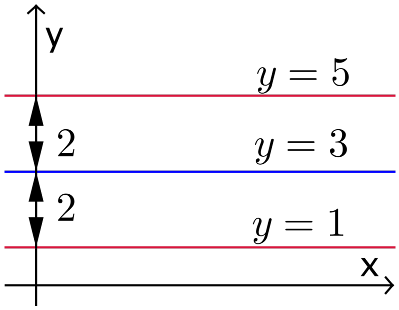
$$\begin{aligned}
\frac{AP}{BP} &= 2 \\
\frac{\sqrt{x^2 + y^2}}{\sqrt{(x-3)^2 + y^2}} &= 2 \\
\frac{x^2 + y^2}{x^2 - 6x + 9 + y^2} &= 4 \\
x^2 + y^2 &= 4(x^2 - 6x + 9 + y^2) \\
x^2 + y^2 &= 4x^2 - 24x + 36 + 4y^2 \\
3x^2 - 24x + 36 + 3y^2 &= 0 \\
x^2 - 8x + y^2 &= -12 \\
(x^2 - 8x + 16) - 16 + y^2 &= -12 \\
(x-4)^2 + y^2 &= 4
\end{aligned}$$

This is a circle of radius 2 and centre $(4, 0)$.

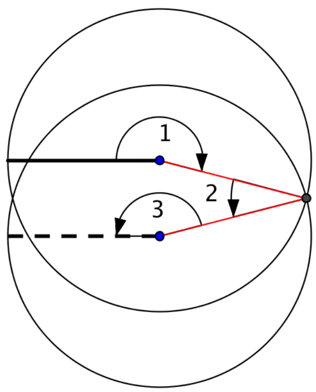
13 The distance from the point $P(x, y)$ to the line $y = 3$ is 2. Therefore,

$$\begin{aligned}
|y - 3| &= 2 \\
y - 3 &= \pm 2 \\
y &= 3 \pm 2 \\
y &= 1 \text{ or } y = 5
\end{aligned}$$

This pair of lines are shown in red on the diagram below.



14 To solve this problem, draw two circles whose radii are equal to the length of the pipe, and whose centres are the endpoints of the pipe. The pipe can then be moved in a minimum of 3 moves. These are indicated on the diagram below.



Solutions to Exercise 15C

1 We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{x^2 + (y - 3))^2} = \sqrt{(y - (-3))^2}$$

$$x^2 + (y - 3)^2 = (y + 3)^2$$

$$x^2 + y^2 - 6y + 9 = y^2 + 6y + 9$$

$$x^2 - 12y = 0$$

$$y = \frac{x^2}{12}.$$

Therefore, the set of points is a parabola whose equation is

$$y = \frac{x^2}{12}.$$

2 We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{x^2 + (y - (-4))^2} = \sqrt{(y - 2)^2}$$

$$x^2 + (y + 4)^2 = (y - 2)^2$$

$$x^2 + y^2 + 8y + 16 = y^2 - 4y + 4$$

$$x^2 + 12y = -12$$

$$y = -\frac{x^2}{12} - 1.$$

Therefore, the set of points is a parabola whose equation is

$$y = -\frac{x^2}{12} - 1.$$

3 We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{(x - 2)^2 + y^2} = \sqrt{(x - (-4))^2}$$

$$(x - 2)^2 + y^2 = (x + 4)^2$$

$$x^2 - 4x + 4 + y^2 = x^2 + 8x + 16$$

$$-12x + y^2 = 12$$

$$x = \frac{y^2}{12} - 1.$$

Therefore, the set of points is a (sideways) parabola whose equation is

$$x = \frac{y^2}{12} - 1.$$

4 **a** We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{(x - c)^2 + y^2} = \sqrt{(x - (-c))^2}$$

$$(x - c))^2 + y^2 = (x + c)^2$$

$$x^2 - 2cx + c^2 + y^2 = x^2 + 2cx + c^2$$

$$y^2 - 2cx = +2cx$$

$$y^2 = 4cx$$

$$x = \frac{y^2}{4c}.$$

Therefore, the set of points is a (sideways) parabola whose equation is

$$x = \frac{y^2}{4c}.$$

b The parabola with equation $x = -\frac{y^2}{4c}$ has focus $F(0, c)$ and directrix $x = -c$. For the parabola $x = 3y^2$, we have $\frac{1}{4c} = 3$ so that $c = \frac{1}{12}$. Therefore, its focus is $(1/12, 0)$ and its directrix is at $x = -1/12$.

5 **a** We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(y-c)^2}$$

$$(x-a)^2 + (y-b)^2 = (y-c)^2$$

$$x^2 - 2ax + a^2 - 2by + b^2 = -2cy + c^2$$

$$x^2 - 2ax + a^2 + b^2 - c^2 = 2by - 2cy$$

$$x^2 - 2ax + a^2 + b^2 - c^2 = (2b - 2c)y$$

Solving for y gives,

$$y = \frac{1}{2b - 2c}(x^2 - 2ax + a^2 + b^2 - c^2).$$

b Let $a = 1, b = 2$ and $c = 3$ in the above equation. This gives,

$$y = \frac{1}{2b - 2c}(x^2 - 2ax + a^2 + b^2 - c^2)$$

$$= -\frac{1}{2}(x^2 - 2x - 4).$$

6 Since the parabola has a vertical line of symmetry, is directrix will be a horizontal line, $y = c$. The point $P(7, 9)$ is on the parabola. Therefore, the distance from $P(7, 9)$ to the focus $F(1, 1)$ is the same as the distance from $P(x, y)$ to the line $y = c$ Therefore,

$$FP = RP$$

$$\sqrt{(7-1)^2 + (9-1)^2} = \sqrt{(9-c)^2}$$

$$6^2 + 8^2 = (9-c)^2$$

$$(9-c)^2 = 100$$

$$9 - c = \pm 10$$

$$c = 9 \pm 10$$

$$= -1, 19$$

Therefore there are two possibilities for the equation of the directrix: $y = -1$ and $y = 19$.

7 As the focus lies on the line of symmetry, we can suppose that the coordinates of the focus are $(2, a)$. The distance from the focus $(2, a)$ to $P(1, 1)$ is the same as the distance from the line $y = 3$ to the point $P(1, 1)$. Therefore,

$$FP = RP \qquad \text{There-}$$

$$\sqrt{(1-2)^2 + (1-a)^2} = 2$$

$$1 + (1-a)^2 = 4$$

$$(1-a)^2 = 3$$

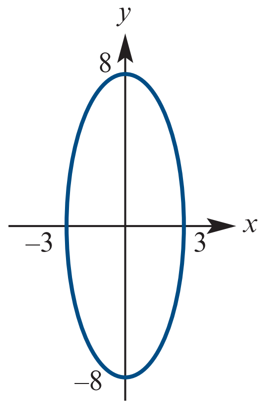
$$1 - a = \pm\sqrt{3}$$

$$a = 1 \pm \sqrt{3}$$

fore the coordinates of the focus are either $(2, 1 + \sqrt{3})$ or $(2, 1 - \sqrt{3})$

Solutions to Exercise 15D

1 a



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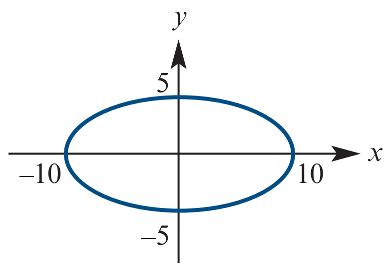

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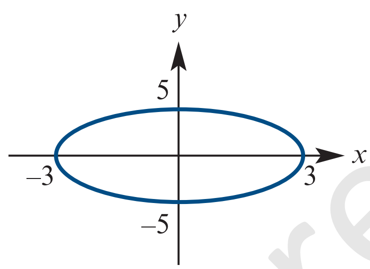
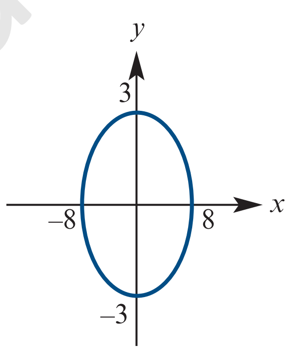
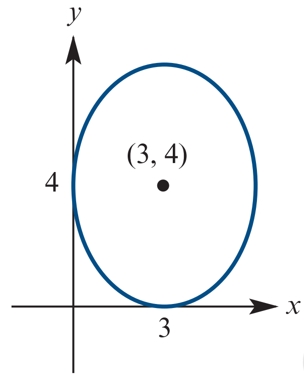


d Dividing both sides of the expression by 225 gives

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$



2 a



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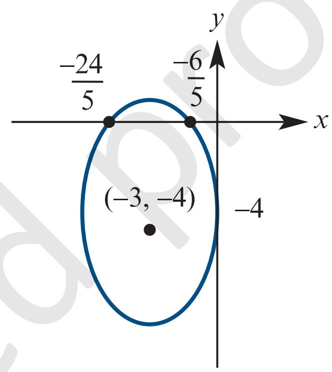

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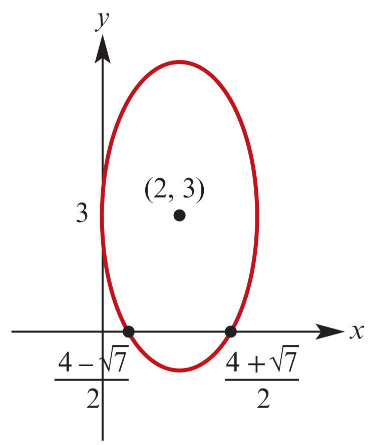

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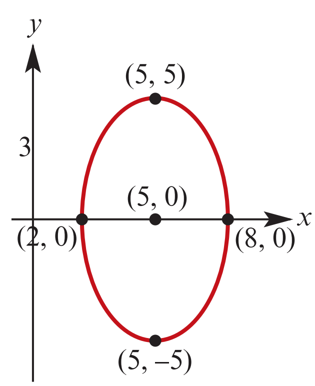


3 **a** The equation can be found by noting that the x-intercepts are $x = \pm 5$ and the y-intercepts are $y = \pm 4$. Therefore the equation must be

$$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$$

or $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

b The centre of the ellipse is $(2, 0)$. Therefore the equation of the ellipse must be

$$\frac{(x-2)^2}{3^2} + \frac{y^2}{2^2} = 1$$

or $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$.

c The centre of the ellipse is $(-1, 1)$. Therefore the equation of the ellipse must be

$$\frac{(x+1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$$

or $\frac{(x+1)^2}{4} + (y-1)^2 = 1$.

4 Let (x, y) be the coordinates of point P. If $AP + BP = 4$ then,

$$\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4,$$

$$\sqrt{(x-1)^2 + y^2} = 4 - \sqrt{(x+1)^2 + y^2}.$$

Squaring both sides gives,

$$(x-1)^2 + y^2 = 16 - 8\sqrt{(x+1)^2 + y^2} + (x+1)^2 + y^2.$$

Now expand and simplify to obtain

$$x^2 - 2x + 1 + y^2 = 16 - 8\sqrt{(x+1)^2 + y^2} + x^2 + 2x + 1 + y^2$$

$$-2x = 16 - 8\sqrt{(x+1)^2 + y^2} + 2x,$$

$$4x + 16 = 8\sqrt{(x+1)^2 + y^2}$$

$$x + 4 = 2\sqrt{(x+1)^2 + y^2}$$

Squaring both sides again gives

$$x^2 + 8x + 16 = 4(x^2 + 2x + 1 + y^2).$$

Simplifying yields

$$12 = 3x^2 + 4y^2 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{3} = 1.$$

This is an ellipse with centre the origin, and intercepts at $x = \pm 2$ and $y = \pm\sqrt{3}$.

5 Let (x, y) be the coordinates of point P. If $AP + BP = 6$ then,

$$\sqrt{x^2 + (y-2)^2} + \sqrt{x^2 + (y+2)^2} = 6,$$

$$\sqrt{x^2 + (y-2)^2} = 6 - \sqrt{x^2 + (y+2)^2}$$

Squaring both sides gives,

$$x^2 + (y-2)^2 = 36 - 12\sqrt{x^2 + (y+2)^2} + x^2 + (y+2)^2.$$

Now expand and simplify to obtain

$$x^2 + y^2 - 4y + 4$$

$$= 36 - 12\sqrt{x^2 + (y+2)^2} + x^2 + y^2 + 4y + 4$$

$$-4y = 36 - 12\sqrt{x^2 + (y+2)^2} + 4y$$

$$8y + 36 = 12\sqrt{x^2 + (y+2)^2}$$

$$2y + 9 = 3\sqrt{x^2 + (y+2)^2}$$

Squaring both sides again gives

$$4y^2 + 36y + 81 = 9(x^2 + y^2 + 4y + 4).$$

Simplifying yields

$$9x^2 + 5y^2 = 45 \quad \text{or} \quad \frac{x^2}{5} + \frac{y^2}{9} = 1.$$

This is an ellipse with centre the origin, and intercepts at $x = \pm\sqrt{5}$ and y-intercepts $y = \pm 3$.

6 Let (x, y) be the coordinates of point P. If $FP = \frac{1}{2}MP$ then

$$\sqrt{(x-2)^2 + y^2} = \frac{1}{2}\sqrt{(x+4)^2}.$$

Squaring both sides gives

$$(x-2)^2 + y^2 = \frac{1}{4}(x+4)^2$$
$$4(x^2 - 4x + 4) + 4y^2 = x^2 + 8x + 16$$
$$4x^2 - 16x + 16 + 4y^2 = x^2 + 8x + 16$$
$$3x^2 - 24x + 4y^2 = 0.$$

Completing the square gives,

$$3(x^2 - 8x) + 4y^2 = 0$$
$$3((x^2 - 8x + 16) - 16) + 4y^2 = 0$$
$$3((x-4)^2 - 16) + 4y^2 = 0$$
$$3(x-4)^2 + 4y^2 = 48$$

Or equivalently,

$$\frac{(x-4)^2}{16} + \frac{y^2}{12} = 1.$$

7 The transformation is defined by the rule $(x, y) \to (5x, 3y)$. Therefore let $x' = 5x$ and $y' = 3y$ where (x', y') is the image of (x, y) under the transformation. Hence $x = \frac{x'}{5}$ and $x = \frac{y'}{3}$. The equation

$$x^2 + y^2 = 1$$

becomes,

$$\frac{(x')^2}{25} + \frac{(y')^2}{9} = 1$$

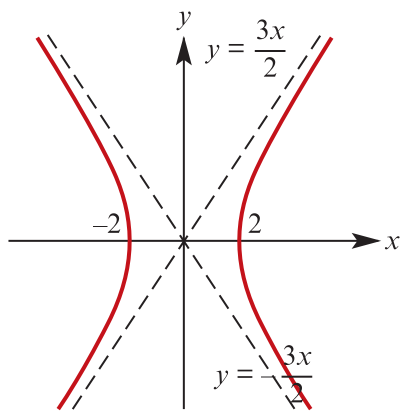
Ignoring the apostrophes gives,

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

This is an ellipse with centre the origin, with intercepts at $(\pm 5, 0)$ and $(0, \pm 3)$.

Solutions to Exercise 15E

1 a



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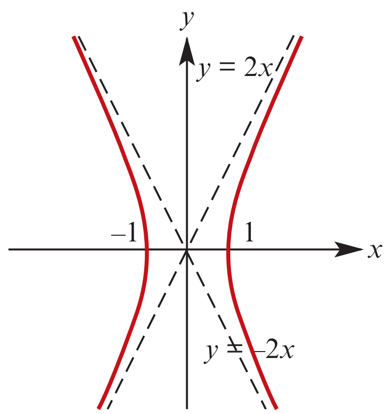

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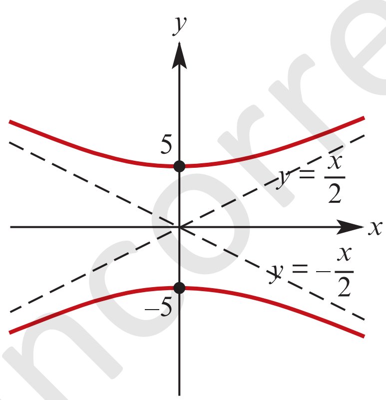

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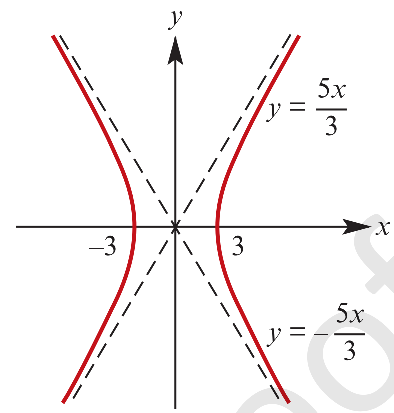

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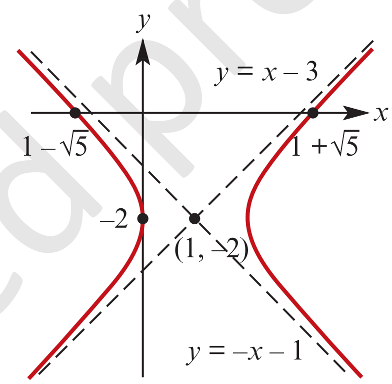

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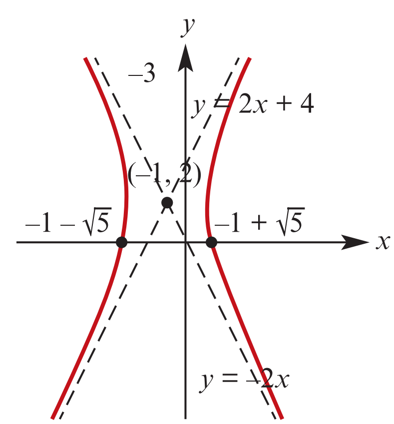

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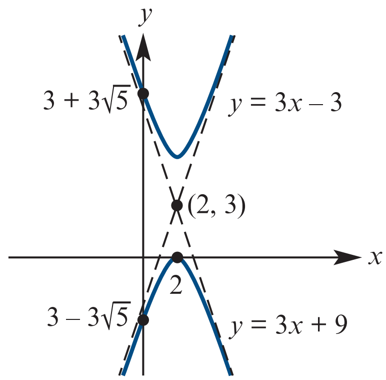

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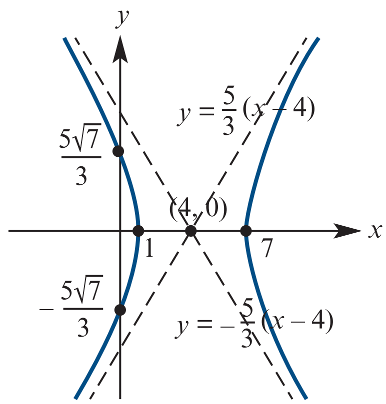


e For this question, we must first complete the square in both x and y variables. This gives,

$$x^2 - 4y^2 - 4x - 8y - 16 = 0$$

$$(x^2 - 4x) - 4(y^2 + 2y) - 16 = 0$$

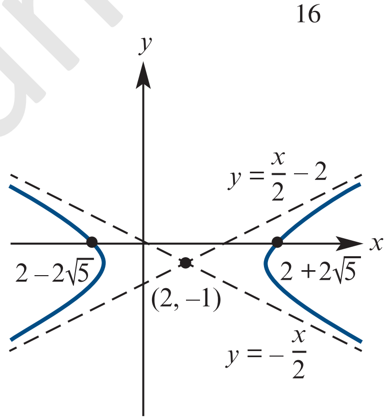
$$(x^2 - 4x + 4 - 4) - 4(y^2 + 2y + 1 - 1) - 16 = 0$$

$$((x - 2)^2 - 4) - 4((y + 1)^2 - 1) - 16 = 0$$

$$(x - 2)^2 - 4 - 4(y + 1)^2 + 4 - 16 = 0$$

$$(x - 2)^2 - 4(y + 1)^2 = 16$$

$$\frac{(x - 2)^2}{16} - \frac{(y + 1)^2}{4} = 1$$



f For this question, we must first complete the square in both x and y variables. This gives,

$$9x^2 - 25y^2 - 90x + 150y = 225$$

$$9(x^2 - 10x) - 25(y^2 - 6y) = 225$$

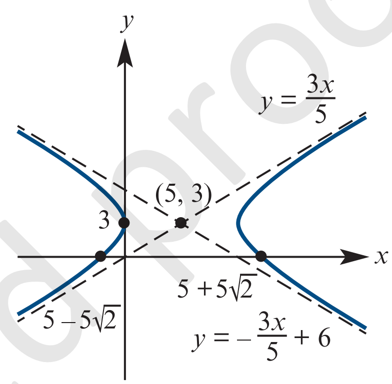
$$9(x^2 - 10x + 25 - 25) - 25(y^2 - 6y + 9 - 9) = 225$$

$$9((x - 5)^2 - 25) - 25((y - 3)^2 - 9) = 225$$

$$9(x - 5)^2 - 225 - 25(y - 3)^2 + 225 = 225$$

$$9(x - 5)^2 - 25(y - 3)^2 = 225$$

$$\frac{(x - 5)^2}{25} - \frac{(y - 3)^2}{9} = 1$$



3 Let (x, y) be the coordinates of point P.
If $AP - BP = 6$, then

$$\sqrt{(x - 4)^2 + y^2} - \sqrt{(x + 4)^2 + y^2} = 3$$

$$\sqrt{(x - 4)^2 + y^2} = 6 + \sqrt{(x + 4)^2 + y^2}.$$

Squaring both sides gives

$$(x-4)^2 + y^2 = 36 + 12\sqrt{(x + 4)^2 + y^2} + (x+4)^2 + y^2$$

Expanding and simplifying

$$x^2 - 8x + 16 + y^2 = 36 + 12\sqrt{(x + 4)^2 + y^2} + x^2 + 8x + 16 + y^2$$

$$-16x - 36 = 12\sqrt{(x + 2)^2 + y^2}$$

$$-4x - 9 = 3\sqrt{(x + 4)^2 + y^2}$$

Note that this only holds if $x \leq -\dfrac{9}{4}$.
Squaring both sides agains gives,

$$16x^2 + 72x + 81 = 9(x^2 + 8x + 16 + y^2)$$

Expanding and simplifying yields

$$16x^2 + 72x + 81 = 9x^2 + 72x + 144 + 9y^2$$
$$7x^2 - 9y^2 = 63$$
$$\frac{x^2}{9} - \frac{y^2}{7} = 1, \quad x \le -\frac{9}{4}.$$

4 Let (x, y) be the coordinates of point P.
If $AP - BP = 4$, then

$$\sqrt{(x+3)^2 + y^2} - \sqrt{(x-3)^2 + y^2} = 4$$
$$\sqrt{(x+3)^2 + y^2} = 4 + \sqrt{(x-3)^2 + y^2}$$

Squaring both sides gives

$$(x+3)^2 + y^2 = 16 + 8\sqrt{(x-3)^2 + y^2} + (x-3)^2 + y^2$$

Expanding and simplifying

$$x^2 + 6x + 9 + y^2 = 16 + 8\sqrt{(x-3)^2 + y^2} + x^2 - 6x + 9 + y^2$$
$$12x - 16 = 8\sqrt{(x-3)^2 + y^2}$$
$$3x - 4 = 2\sqrt{(x-3)^2 + y^2}$$

Note that this only holds if $x \ge \frac{4}{3}$.
Squaring both sides agains gives,

$$9x^2 - 24x + 16 = 4(x^2 - 6x + 9 + y^2)$$

Expanding and simplifying yields

$$9x^2 - 24x + 16 = 4x^2 - 24x + 36 + 4y^2$$
$$5x^2 - 4y^2 = 20$$

5 Let (x, y) be the coordinates of point P.
If $FP = 2MP$

$$\sqrt{(x-5)^2 + y^2} = 2\sqrt{(x+1)^2}$$

Squaring both sides

$$(x-5)^2 + y^2 = 4(x+1)^2$$
$$x^2 - 10x + 25 + y^2 = 4(x^2 + 2x + 1)$$
$$x^2 - 10x + 25 + y^2 = 4x^2 + 8x + 4$$
$$0 = 3x^2 + 18x - y^2 - 21$$

Completing the square gives,

$$0 = 3(x^2 + 6x) - y^2 - 21$$
$$0 = 3(x^2 + 6x + 9 - 9) - y^2 - 21$$
$$0 = 3((x+3)^2 - 9) - y^2 - 21$$
$$0 = 3(x+3)^2 - y^2 - 48$$
$$\frac{(x+3)^2}{16} - \frac{y^2}{48} = 1.$$

This is a hyperbola with centre $(-3, 0)$

6 Let (x, y) be the coordinates of point P.
If $FP = 2MP$

$$\sqrt{x^2 + (y+1)^2} = 2\sqrt{(y+4)^2}$$

Squaring both sides

$$x^2 + (y+1)^2 = 4(y+4)^2$$
$$x^2 + y^2 + 2y + 1 = 4(y^2 + 8y + 16)$$
$$x^2 + y^2 + 2y + 1 = 4y^2 + 32y + 64$$
$$0 = 3y^2 + 30y - x^2 + 63$$

Completing the square gives,

$$0 = 3(y^2 + 10y) - x^2 + 63$$
$$0 = 3(y^2 + 10y + 25 - 25) - x^2 + 63$$
$$0 = 3((y+5)^2 - 25) - x^2 + 63$$
$$0 = 3(y+5)^2 - 75 - x^2 + 63$$
$$0 = 3(y+5)^2 - x^2 - 12$$
$$\frac{(y+5)^2}{4} - \frac{x^2}{12} = 1.$$

This is a hyperbola with centre $(0, -5)$

Solutions to Exercise 15F

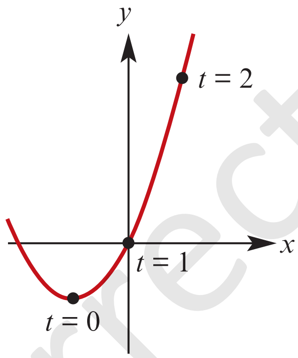
1 **a** From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$y = (x+1)^2 - 1$$
$$= x^2 + 2x.$$

b To sketch the curve it helps to write $y = x(x + 2)$. This is a parabola with intercepts at $x = 0$ and $x = -2$. To label the points corresponding to $t = 0, 1, 2, 3$, we first complete the table shown below.

t	0	1	2
$x = t - 1$	−1	0	1
$y = t^2 - 1$	−1	0	3

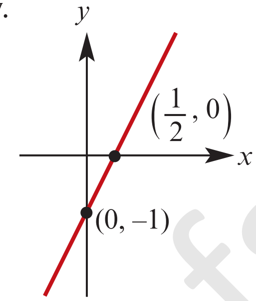
The curve and the required points are shown below.



2 **a** From the first equation we know that $t = x - 1$. Substitute this into the second equation to get

$$y = 2t + 1$$
$$= 2(x - 1) + 1$$
$$= 2x - 2 + 1$$
$$= 2x - 1.$$

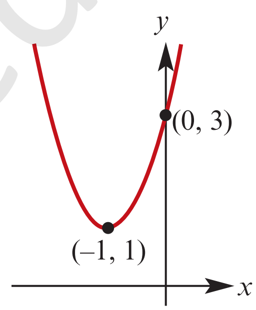
We obtain straight line whose equation is $y = 2x - 1$, and whose graph is shown below.



b From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$y = 2t^2 + 1$$
$$= 2(x + 1)^2 + 1.$$

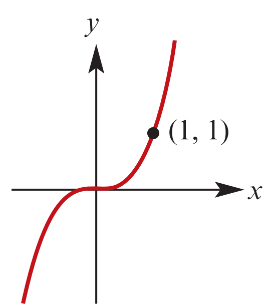
We obtain a parabola whose equation is $y = 2(x + 1)^2 + 1$, and whose graph is shown below.



c For this question, we note that $y = (t^2)^3$. Therefore,

$$y = (t^2)^3 = x^3.$$

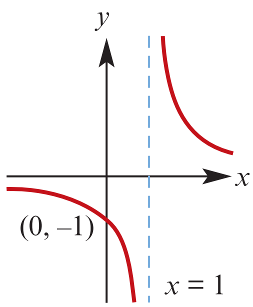
This is clearly a cubic equation whose graph is shown below.



d From the first equation we know that $t = x - 2$. Substitute this into the second equation to get

$$y = \frac{1}{t+1}$$
$$= \frac{1}{x-2+1}$$
$$= \frac{1}{x-1}$$

We obtain straight a hyperbola who=se equation is $y = \dfrac{1}{x-1}$, and whose graph is shown below.



3 a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x}{2} = \cos t \text{ and } \frac{y}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

Multiplying both sides by 2^2 gives the cartesian equation as

$$x^2 + y^2 = 2^2,$$

which is a circle centred at the origin on radius 2.

b We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x+1}{3} = \cos t \text{ and } \frac{y-2}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x+1}{3}\right)^2 + \left(\frac{y-2}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{2^2} = 1,$$

which is an ellipse centred at the point $(-1, 2)$.

c We divide both sides of the equation by 9 so that the equation becomes,

$$\left(\frac{x+3}{3}\right)^2 + \left(\frac{y-2}{2}\right)^2.$$

We then let

$$\cos t = \frac{x+3}{3} \text{ and } \sin t = \frac{y-2}{3}.$$

Therefore, the required equations are

$$x = 3\cos t - 3 \text{ and } y = 3\sin t + 2.$$

d We write this equation as

$$\left(\frac{x+2}{3}\right)^2 + \left(\frac{y-1}{2}\right)^2.$$

We then let

$$\cos t = \frac{x+2}{3} \text{ and } \sin t = \frac{y-1}{2}.$$

so that

$$x = 3\cos t - 2 \text{ and } y = 2\sin t + 1.$$

4 The gradient of the line through points A and B is

$$m = \frac{4-(-2)}{1-(-1)} = \frac{6}{2} = 3.$$

Therefore, the line has equation
$y - 4 = 3(x - 1)$ We can simply let $x = t$

$$y = 3x + 1.$$

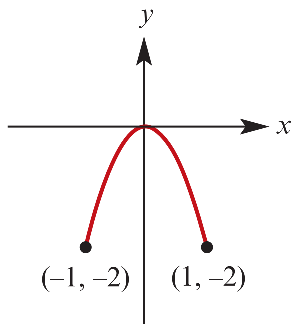
so that $y = 3t + 1$. Note that this is not the only possible answer.

5 a From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$y = -2t^2 + 4t - 2$$
$$= -2(x+1)^2 + 4(x+1) - 2$$
$$= -2(x^2 + 2x + 1) + 4x + 4 - 2$$
$$= -2x^2 - 4x - 2 + 4x + 4 - 2$$
$$= -2x^2$$

Moreover, since $0 \le t \le 2$, we know that $-1 \le x \le 1$.

b We sketch the curve over the domain $-1 \le x \le 1$.



6 The cartesian equation of the circle is

$$x^2 + y^2 = 1. \quad (1)$$

It is a little harder to find the cartesian equation of the straight line. Solving both equations for t gives,

$$t = \frac{x-6}{3} \text{ and } t = \frac{y-8}{4}.$$

Therefore,

$$\frac{x-6}{4} = \frac{y-8}{4}$$
$$4(x-6) = 3(y-8)$$
$$4x - 24 = 3y - 24$$
$$y = \frac{4x}{3} \quad (2)$$

Solving equations (1) and (2) simultaneously gives $x = -\frac{3}{5}$ and $x = \frac{3}{5}$. Substituting these two values into the equation $y = \frac{4x}{3}$ gives $y = -\frac{4}{5}$ and $x = \frac{4}{5}$ respectively. Therefore, the required coordinates are $(-3/5, -4/5)$ and $(3/5, 4/5)$.

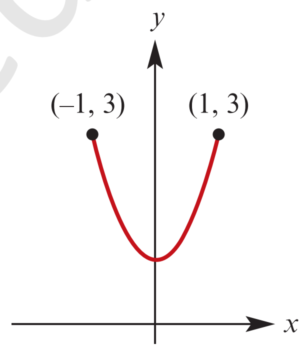
7 **a** We substitute $x = \sin t$ into the second equation to give,

$$y = 2\sin^2 t + 1$$
$$= 2x^2 + 1.$$

b Since the domain is the set of possible x-values and $x = \sin t$ where $0 \le t \le 2\pi$, the domain will be $-1 \le x \le 1$.

c Since the domain is the set of x such that $-1 \le x \le -1$, the range must be the set of y such that $1 \le y \le 3$.

d The curve is sketched below over the interval $-1 \le x \le 1$.



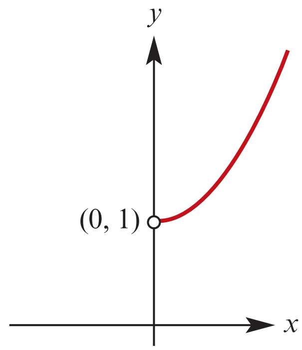
8 **a** We substitute $x = 2^t$ into the second equation to give,

$$y = 2^{2t} + 1$$
$$= (2^t)^2 + 1$$
$$= x^2 + 1.$$

b The domain is the set of possible x-values. Since $x = 2^t$ and $t \in \mathbb{R}$, we know that the domain will be $x > 0$.

c Since the domain is the set of all x such that $x > 0$, the range must be the set of y such that $y > 1$.

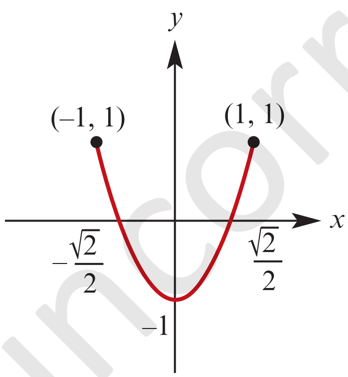
d The curve is sketched below over the interval $x > 0$.



9 Here, we must make use of the identity $\cos^2 t + \sin^2 t = 1$. Since $x = \cos t$ we have,

$$\begin{aligned} y &= 1 - 2\sin^2 t \\ &= 1 - 2(1 - \cos^2 t) \\ &= 1 - 2 + 2\cos^2 t \\ &= -1 + 2x^2 \end{aligned}$$

The domain is the set of possible x-values. Since $x = \cos t$ and $0 \le t \le 2\pi$, we know that the domain will be $-1 \le x \le 1$. We sketch the curve over this interval.

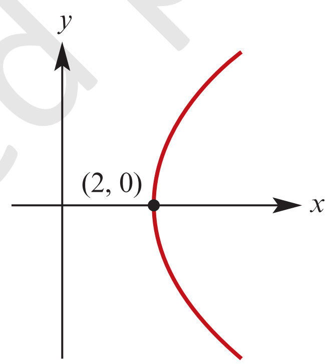


10 **a** We substitute $x = 2^t + 2^{-t}$ and $y = 2^t - 2^{-t}$ into the left hand side of the cartesian equation. This gives,

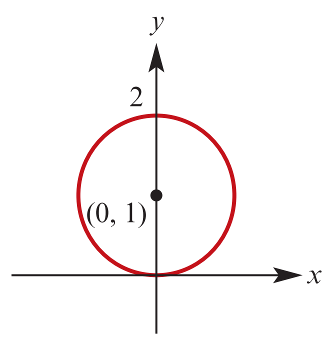
$$\begin{aligned} \text{LHS} &= \frac{x^2}{4} - \frac{y^2}{4} \\ &= \frac{(2^t + 2^{-t})^2}{4} - \frac{(2^t - 2^{-t})^2}{4} \\ &= \frac{2^{2t} + 2 + 2^{-2t}}{4} - \frac{(2^{2t} - 2 + 2^{-2t})}{4} \\ &= \frac{2^{2t} + 2 + 2^{-2t} - 2^{2t} + 2 - 2^{-2t}}{4} \\ &= \frac{4}{4} \\ &= 1 \\ &= \text{RHS}, \end{aligned}$$

as required.

b The curves is one side of a hyperbola centred at the origin.



11 **a** This is the equation of a circle of radius 1 centred at $(0, 1)$. Its graph is shown below.



b Since $x = \cos t$ and $y - 1 = \sin t$, we have

$$x^2 + (y - 1)^2 = \cos^2 t + \sin^2 t = 1.$$

c We will find the points of intersection of the line,

$$y = 2 - tx \quad (1)$$

and the circle,

$$x^2 + (y-1)^2 = 1. \quad (2)$$

Substituting equation (1) into equation (2), we find that,

$$x^2 + (2 - tx - 1)^2 = 1$$
$$x^2 + (1 - tx)^2 = 1$$
$$x^2 + 1 - 2tx + t^2x^2 = 1$$
$$(1 + t^2)x^2 - 2tx = 0$$
$$x((1 + t^2)x - 2t) = 0$$

Since $x \neq 0$, we see that

$$x = \frac{2t}{1+t^2}.$$

We can find y by substituting this into equation (1). This gives,

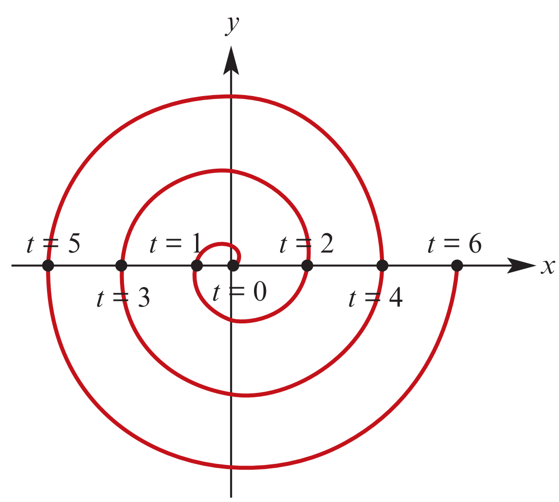
$$\begin{aligned} y &= 2 - tx \\ &= 2 - \frac{2t^2}{1+t^2} \\ &= \frac{2(1+t^2)}{1+t^2} - \frac{2t^2}{1+t^2} \\ &= \frac{2}{1+t^2}. \end{aligned}$$

d To verify that these equations parameterise the same circle we note that

$$\begin{aligned} &x^2 + (y-1)^2 \\ &= \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{2}{1+t^2} - 1\right)^2 \\ &= \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{2}{1+t^2} - \frac{1+t^2}{1+t^2}\right)^2 \\ &= \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2 \\ &= \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2} \\ &= \frac{4t^2}{(1+t^2)^2} + \frac{(1-2t^2+t^4)}{(1+t^2)^2} \\ &= \frac{t^4 + 2t^2 + 1}{(1+t^2)^2} \\ &= \frac{(1+t^2)^2}{(1+t^2)^2} \\ &= 1, \end{aligned}$$

as required.

12 **a**



b The points corresponding to

$$t = 0, 1, 2, 3, 4, 5, 6$$

are all on the x-axis. The values of t correspond to the number of half turns through which the spiral has turned.

Solutions to Exercise 15G

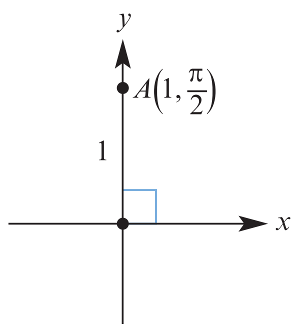
1 a We have

then $x = r\cos\theta \qquad y = r\sin\theta$

$= 1\cos\pi/2 \qquad = 1\sin\pi/2$

$= 0 \qquad = 1$

so that the cartesian coordinates are (0, 1).



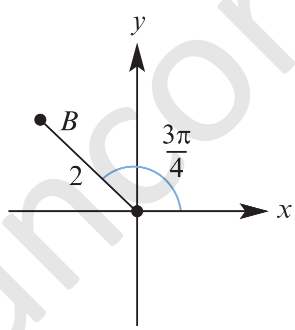
b We have

$x = r\cos\theta \qquad y = r\sin\theta$

$= 2\cos 3\pi/4 \qquad = 2\sin 3\pi/4$

$= -\sqrt{2} \qquad = \sqrt{2}$

so that the cartesian coordinates are $(-\sqrt{2}, \sqrt{2})$.



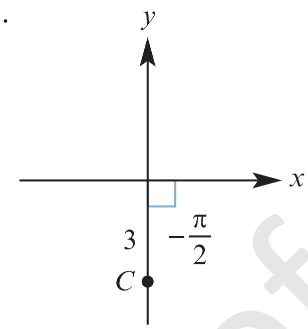
c We have

$x = r\cos\theta \qquad y = r\sin\theta$

$= 3\cos(-\pi/2) \qquad = 3\sin(-\pi/2)$

$= 0 \qquad = -3$

so that the cartesian coordinates are (0, −3).



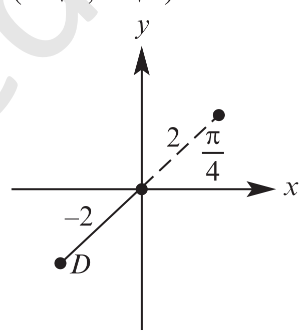
d We have

$x = r\cos\theta \qquad y = r\sin\theta$

$= -2\cos\pi/4 \qquad = -2\sin\pi/4$

$= -\sqrt{2} \qquad = -\sqrt{2}$

so that the cartesian coordinates are $(-\sqrt{2}, -\sqrt{2})$.



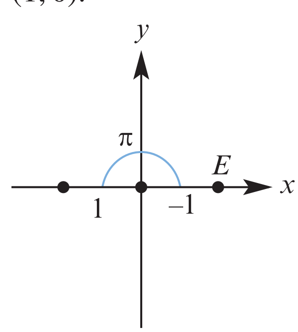
e We have

$x = r\cos\theta \qquad y = r\sin\theta$

$= -1\cos\pi \qquad = -1\sin\pi$

$= 1 \qquad = 0$

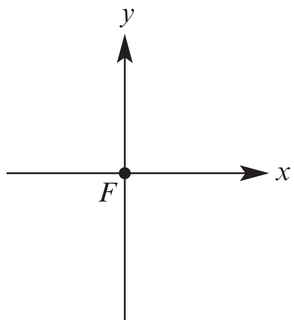
so that the cartesian coordinates are (1, 0).



f We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= 0\cos\pi/4 \qquad = 0\sin\pi/4$$
$$= 0 \qquad = 0$$

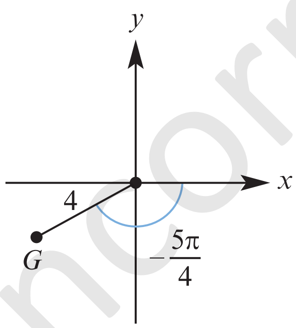
so that the cartesian coordinates are (0, 0).



g We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= 4\cos -5\pi/6 \qquad = 4\sin -5\pi/6$$
$$= -2\sqrt{3} \qquad = -2$$

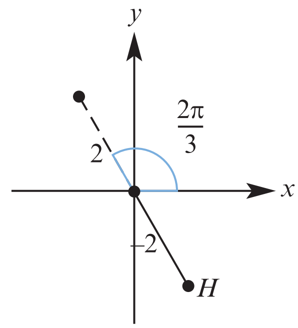
so that the cartesian coordinates are $(-2\sqrt{3}, -2)$.



h We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= -2\cos 2\pi/3 \qquad = -2\sin 2\pi/3$$
$$= 1 \qquad = -\sqrt{3}$$

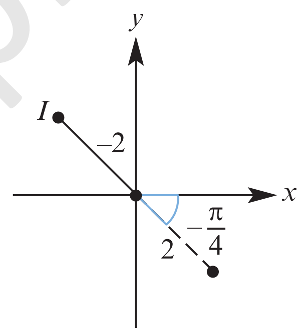
so that the cartesian coordinates are $(1, -\sqrt{3})$.



i We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= -2\cos(-\pi/4) \qquad = -2\sin(-\pi/4)$$
$$= -\sqrt{2} \qquad = \sqrt{2}$$

so that the cartesian coordinates are $(-\sqrt{2}, \sqrt{2})$.



2 **a** $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1} -1 = -\dfrac{\pi}{4}$

The point has polar coordinates $(\sqrt{2}, -\pi/4)$. We could also let $r = -\sqrt{2}$ and add π to the found angle, giving coordinate $(-\sqrt{2}, 3\pi/4)$.

b $r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$

$\theta = \tan^{-1} \sqrt{3} = \dfrac{\pi}{3}$

The point has polar coordinates $(2, \pi/3)$. We could also let $r = -2$ and add π to the found angle, giving coordinate $(-2, 4\pi/3)$.

c $r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$

$\theta = \tan^{-1} -1 = -\dfrac{\pi}{4}$

The point has polar coordinates $(2\sqrt{2}, -\pi/4)$. We could also let$r = -2\sqrt{2}$ and add π to the found angle, giving $(-2\sqrt{2}, 3\pi/4)$.

d $r = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2} = 2$

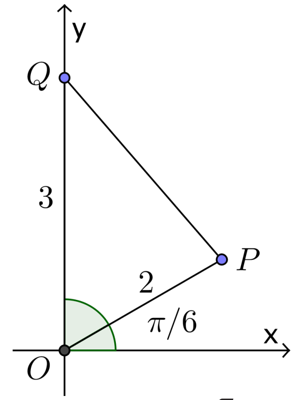
$\theta = -\dfrac{3\pi}{4}$

The point has polar coordinates $(2, -3\pi/4)$. We could also let$r = -2$ and add π to the found angle, giving coordinate$(-2, \pi/4)$.

e Clearly, $r = 3$ and $\theta = 0$ so that the point has polar coordinates $(3, 0)$. We could also let$r = -3$ and add π to the found angle, giving coordinate $(-3, \pi)$.

f Clearly, $r = 2$ and $\theta = -\dfrac{\pi}{2}$ so that the point has polar coordinates $(2, -\dfrac{\pi}{2})$. We could also let$r = -2$ and add π to the found angle, giving coordinate $(-2, \pi/2)$.

3 Points P and Q are shown on the diagram below.

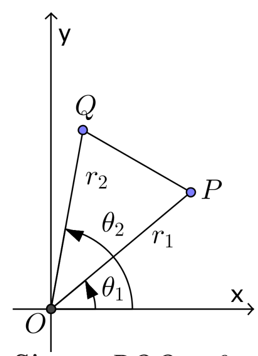


Since $\angle POQ = \dfrac{\pi}{3}$, we can use the cosine rule to find that

$$\begin{aligned} PQ^2 &= OP^2 + OQ^2 - 2 \cdot OP \cdot OQ \cos(\pi/3) \\ &= 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \frac{1}{2} \\ &= 4 + 9 - 6 \\ &= 7. \end{aligned}$$

Therefore, $PQ = \sqrt{7}$.

4 Points P and Q are shown on the diagram below.



Since $\angle POQ = \theta_2 - \theta_1$, we can use the cosine rule to find that

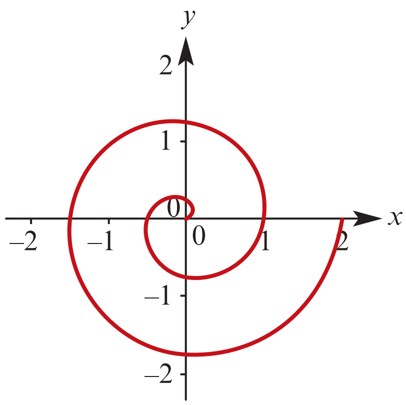
$PQ^2 = {r_1}^2 + {r_2}^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$.

Therefore,

$$PQ = \sqrt{{r_1}^2 + {r_2}^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}.$$

Solutions to Exercise 15H

1



2 **a** Since $x = r\cos\theta$, this equation becomes

$$x = 4$$
$$r\cos\theta = 4$$
$$r = \frac{4}{\cos\theta}$$

b Since $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$xy = 1$$
$$r^2\cos\theta\sin\theta = 1$$
$$r^2 = \frac{1}{\cos\theta\sin\theta}$$

c Since $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$y = x^2$$
$$r\sin\theta = r^2\cos^2\theta$$
$$r\cos^2\theta = \sin\theta$$
$$r = \frac{\sin\theta}{\cos^2\theta}$$
$$= \tan\theta\sec\theta$$

d This is just a circle of radius 3 centred at the origin and so has equation $r = 3$. We can check this by letting $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$x^2 + y^2 = 9$$
$$r^2\cos^2\theta + r^2\sin^2\theta = 9$$
$$r^2(\cos^2\theta + \sin^2\theta) = 9$$
$$r^2 = 9$$

In fact, since we are allowing negative r values we could take either $r = 3$ or $r = -3$ as the equation of this circle.

e Since $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$x^2 - y^2 = 9$$
$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2(\cos^2\theta - \sin^2\theta) = 1$$
$$r^2\cos 2\theta = 1$$
$$r^2 = \frac{1}{\cos 2\theta}$$

f Since $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$2x - 3y = 5$$
$$2r\cos\theta - 3r\sin\theta = 5$$
$$r(2\cos\theta - 3\sin\theta) = 5$$
$$r = \frac{5}{2\cos\theta - 3\sin\theta}$$

3 **a** The trick here is to first multiply both sides of the expression through by $\cos\theta$ to get

$$r\cos\theta = 2$$

Since $r\cos\theta = x$, this equation simply becomes,

$$x = 2.$$

b Since $r = 2$, this is just a circle of radius 2 centred at the origin. Its cartesian equation will then be simply

$$x^2 + y^2 = 2^2.$$

c Here, for all values of r the angle is constant and equal to $\pi/4$. This corresponds to the straight line through the origin, $y = x$. To see this algebraically, note that

$$\frac{y}{x} = \tan(\pi/4) = 1.$$

Therefore, $y = x$.

d Rearranging the equation we find that

$$\frac{4}{3\cos\theta - 2\sin\theta} = r$$

$$r(3\cos\theta - 2\sin\theta) = 4$$

$$3r\cos\theta - 2r\sin\theta = 4 \quad (1)$$

Then since $x = r\cos\theta$ and $y = r\sin\theta$, equation (1) becomes

$$3x - 2y = 4.$$

4 a The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 6r\cos\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, equation (1) becomes,

$$x^2 + y^2 = 6x$$

$$x^2 - 6x + y^2 = 0$$

(completing the square)

$$(x^2 - 6x + 9) - 9 + y^2 = 0$$

$$(x - 3)^2 + y^2 = 9.$$

This is a circles whose centre is $(3, 0)$ and whose radius is 3.

b The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 4r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, equation (1) becomes,

$$x^2 + y^2 = 4y$$

$$x^2 + y^2 - 4y = 0$$

(completing the square)

$$x^2 + (y^2 - 4y + 4) - 4 = 0$$

$$x^2 + (y - 2)^2 = 4.$$

This is a circles whose centre is $(0, 2)$ and whose radius is 2.

c The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 2r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, equation (1) becomes,

$$x^2 + y^2 = -6x$$

$$x^2 + 6x + y^2 = 0$$

(completing the square)

$$(x^2 + 6x + 9) - 9 + y^2 = 0$$

$$(x + 3)^2 + y^2 = 9$$

This is a circles whose centre is $(-3, 0)$ and whose radius is 3.

d The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 2r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, equation (1) becomes,

$$x^2 + y^2 = -8y$$

$$x^2 + y^2 + 8y = 0$$

$$x^2 + (y^2 + 8y + 16) - 16 = 0$$

(completing the square)

$$x^2 + (y + 4)^2 = 16.$$

This is a circles whose centre is $(0, -4)$ and whose radius is 4.

5 The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 2ar\cos\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, equation (1) becomes,

$$x^2 + y^2 = 2ax$$

$$x^2 - 2ax + y^2 = 0$$

$$(x^2 - 2ax + a^2) - a^2 + y^2 = 0$$

(completing the square)

$$(x - a)^2 + y^2 = a^2.$$

This is a circles whose centre is $(a, 0)$ and whose radius is a.

6 **a** The trick here is to first multiply both sides of the expression through by $\cos\theta$ to obtain,

$$r\cos\theta = a$$

$$x = a,$$

which is the equation of a vertical line.

b Let $y = r\sin\theta$ so that

$$r\sin\theta = a$$

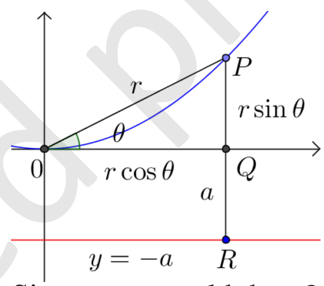
$$r = \frac{a}{\sin\theta}.$$

7 **a** The distance from P to the line is

$$RP = RQ + QP$$

$$= a + r\sin\theta.$$

b Consider the complete diagram shown below.



Since we are told that $OP = RP$, this implies that

$$OP = RP$$

$$r = a + r\sin\theta$$

$$r - r\sin\theta = a$$

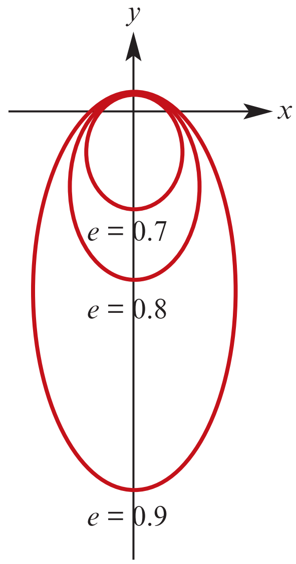
$$r(1 - \sin\theta) = a$$

$$r = \frac{a}{1 - \sin\theta},$$

as required.

Solutions to Exercise 15I

1 a

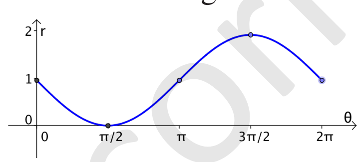


b The ellipse increases in size and become more narrow as e is increased.

2 a To help sketch this curve we first graph the function

$$r = 1 - \sin\theta$$

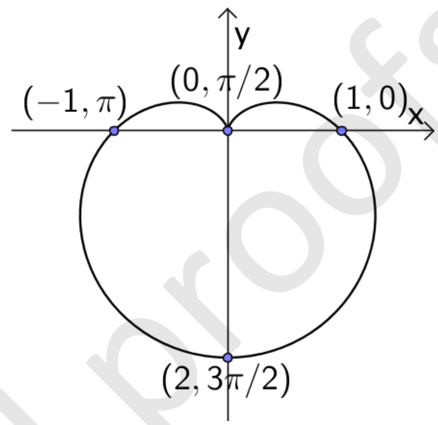
as shown below. This allows us to see how r changes as θ increases.



Note that:

- As angle θ increases from 0 to $\pi/2$, the radius r decreases from 1 to 0.
- As angle θ increases from $\pi/2$ to π, the radius r increases from 0 to 1.
- As angle θ increases from π to $3\pi/2$, the radius r increases from 1 to 2.
- As angle θ increases from $3\pi/2$ to 2π, the radius r decreases from 2 to 1.

This gives the graph shown below. The points are labelled using polar coordinates.



b The trick, once again, is to multiply both sides of the equation through by r. This gives,

$$r^2 = r - r\sin\theta \text{ as}$$

$$x^2 + y^2 = r - y$$

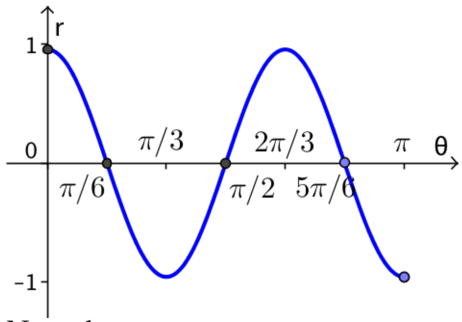
$$x^2 + y^2 + y = r$$

$$x^2 + y^2 + y = \sqrt{x^2 + y^2}$$

$$(x^2 + y^2 + y)^2 = x^2 + y^2,$$

required.

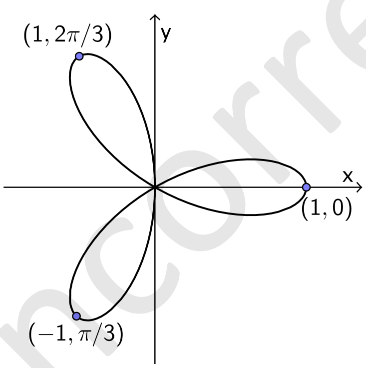
3 a To help sketch this curve we first graph the function $r = \cos 3\theta$ as shown below. This allows us to see how r changes as θ increases.



Note that:

- As angle θ increases from 0 to $\pi/6$, the radius r varies from 1 to 0.
- As angle θ increases from $\pi/6$ to $\pi/3$, the radius r varies from 0 to -1.
- As angle θ increases from $\pi/3$ to $\pi/2$, the radius r varies from from -1 to 0.

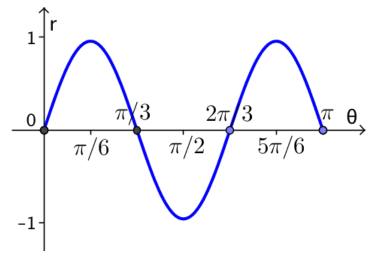
Continuing in this manner, we obtain the following graph shown below. Note that the labelled points are polar coordinates.



b To help sketch this curve we first graph the function

$$r = \cos 3\theta$$

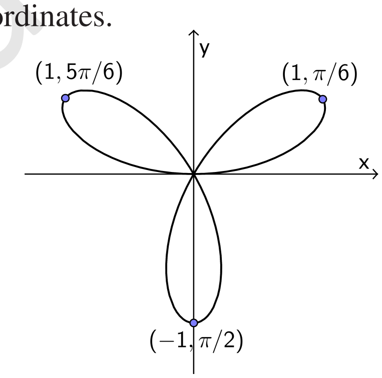
as shown below. This allows us to see how r changes as θ increases.



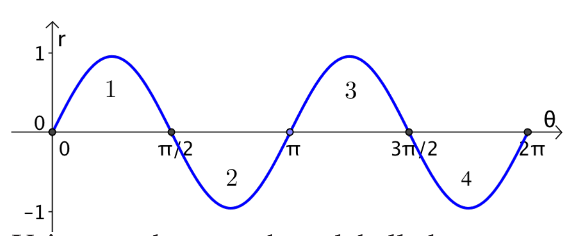
Note that:

- As angle θ increases from 0 to $\pi/6$, the radius r varies from 0 to 1.
- As angle θ increases from $\pi/6$ to $\pi/3$, the radius r varies from 1 to 0.

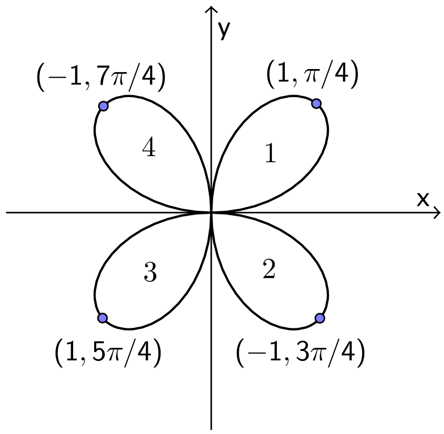
Continuing in this manner, we obtain the following graph shown below. Note that the labelled points are polar coordinates.



4 **a** To help sketch this curve we first graph the function $r = \sin 2\theta$ as shown below. This allows us to see how r changes as θ increases.



Using numbers, we have labelled how each section of this graph corresponds to a each section in the rose below. Note that the labelled points are polar coordinates.

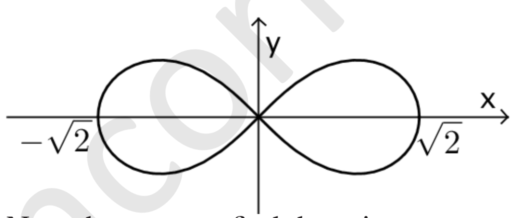


b Since $\sin 2\theta = 2\sin\theta\cos\theta$, we have

$$\begin{aligned} r &= \sin 2\theta \\ r &= 2\sin\theta\cos\theta \\ r^3 &= 2\cdot r\sin\theta\cdot r\cos\theta \\ r^3 &= 2xy \\ \left(x^2+y^2\right)^{\frac{3}{2}} &= 2xy \\ (x^2+y^2)^3 &= 4x^2y^2, \end{aligned}$$

as required.

5 a



Note that we can find the x-intercepts by letting $\theta = 0$ and $\theta = \pi$. This gives $x = \pm\sqrt{2}$.

b Since $\cos 2\theta =$ $\cos^2\theta + \sin^2\theta$, we have

$$\begin{aligned} r^2 &= 2\cos 2\theta \\ r^2 &= 2(\cos^2\theta - \sin^2\theta) \\ r^2 &= 2\cos^2\theta - 2\sin^2\theta \\ r^2 &= 2\left(\frac{x}{r}\right)^2 - 2\left(\frac{y}{r}\right)^2 \\ r^2 &= 2\frac{x^2}{r^2} - 2\frac{y^2}{r^2} \\ r^4 &= 2x^2 - 2y^2 \\ \left(\sqrt{x^2+y^2}\right)^4 &= 2x^2 - 2y^2 \\ (x^2+y^2)^2 &= 2x^2 - 2y^2 \end{aligned}$$

c If $d_1d_2 = 1$ then using the distance formula, we find that

$$\begin{aligned} \sqrt{(x-1)^2+y^2}\,\sqrt{(x+1)^2+y^2} &= 1 \\ [(x-1)^2+y^2][(x+1)^2+y^2] &= 1 \\ (x-1)^2(x+1)^2 + (x-1)^2y^2 + (x+1)^2y^2 + y^4 &= 1 \\ (x^2-1)^2 + [(x-1)^2+(x+1)^2]y^2 + y^4 &= 1 \\ (x^2-1)^2 + (2x^2+2)y^2 + y^4 &= 1 \\ x^4 - 2x^2 + 1 + 2x^2y^2 + 2y^2 + y^4 &= 1 \\ x^4 + 2x^2y^2 + y^4 - 2x^2 + 2y^2 &= 0 \\ (x^2+y^2)^2 - 2x^2 + 2y^2 &= 0. \end{aligned}$$

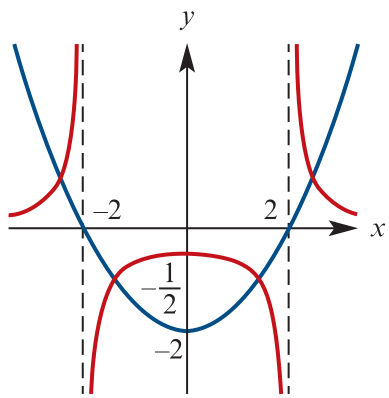
Therefore,

$$(x^2+y^2)^2 = 2x^2 - 2y^2,$$

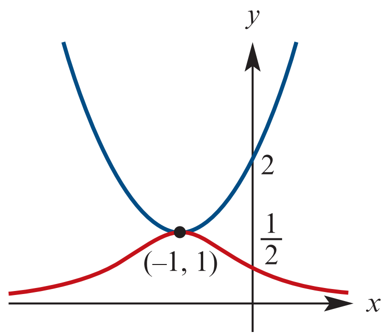
which is the same equation as that found previously.

Solutions to technology-free questions

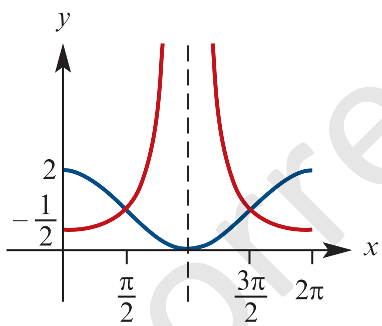
1 a



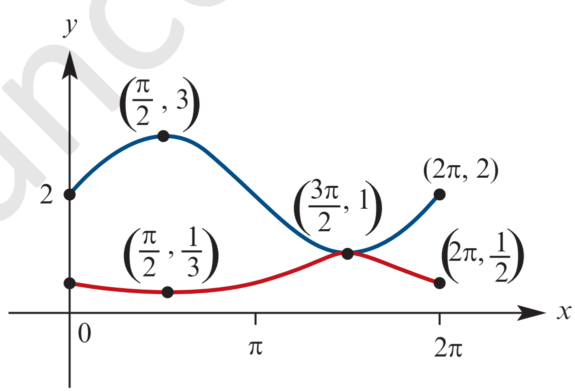
b



c



d



2 We know that the point $P(x, y)$ satisfies,

$$QP = RP$$

$$\sqrt{(x-2)^2 + (y-(-1))^2} = \sqrt{(x-1)^2 + (y-2)^2}$$

$$(x-2)^2 + (y+1)^2 = (x-1)^2 + (y-2)^2$$

$$x^2 - 4x + 4 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2 - 4y + 4$$

$$6y - 2x = 0$$

$$y = \frac{x}{3}.$$

Therefore, point P lies on the straight line with equation $y = \dfrac{x}{3}$.

3 We know that the point $P(x, y)$ satisfies,

$AP = 5$ This is a circle with centre $(3, 2)$ and radius 6 units.

$$\sqrt{(x-3)^2 + (y-2)^2} = 6$$

$$(x-3)^2 + (y-2)^2 = 6^2.$$

4 We complete the square to find that

$$x^2 + 4x + y^2 - 8y = 0$$

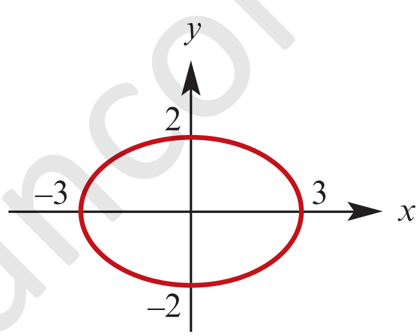
$$[(x^2 + 4x + 4) - 4] + [(y^2 - 8y + 16) - 16] = 0$$

$$(x+2)^2 - 4 + (y-4)^2 - 16 = 0$$

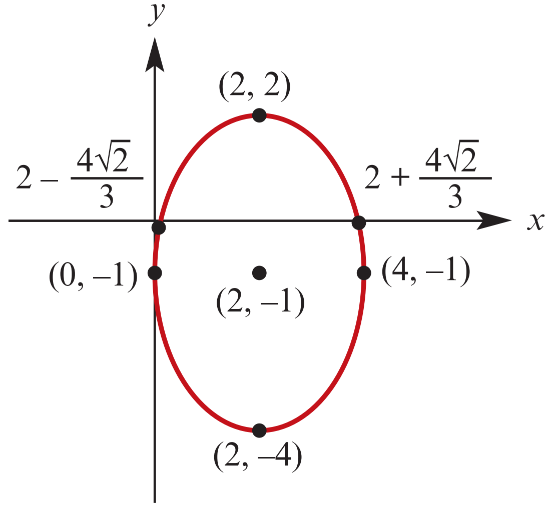
$$(x+2)^2 + (y-4)^2 = 20.$$

This is the equation of a circle with centre $(-2, 4)$ and radius $\sqrt{20}$ units.

5 a



b



6 We complete the square to find that,

$$\begin{aligned} x^2 + 4x + 2y^2 &= 0 \\ (x^2 + 4x + 4) - 4 + 2y^2 &= 0 \\ (x+2)^2 + 2y^2 &= 4 \\ \frac{(x+2)^2}{4} + \frac{y^2}{2} &= 1 \end{aligned}$$

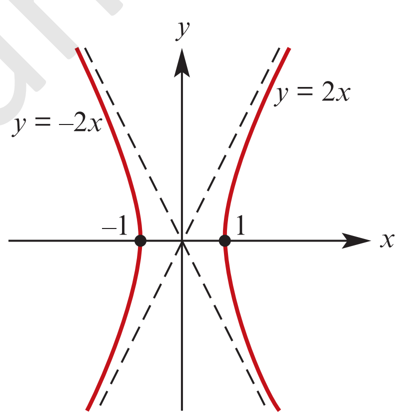
The centre is then $(-2, 0)$. To find the x-intercepts we let $y = 0$. Therefore,

$$\begin{aligned} \frac{(x+2)^2}{4} &= 1 \\ (x+2)^2 &= 4 \\ x + 2 &= \pm 2 \\ x &= -4, 0 \end{aligned}$$

To find the y-intercepts we let $x = 0$ (in the original equation). Therefore,

$$\begin{aligned} 2y^2 &= 0 \\ y &= 0. \end{aligned}$$

7 **a**



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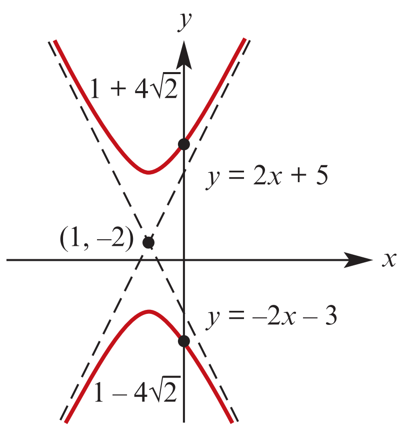


8 We know that the point $P(x, y)$ satisfies,

$$KP = 2MP$$

$$\sqrt{(x-(-2))^2 + (y-5)^2} = 2\sqrt{(x-1)^2}$$

$$(x+2)^2 + (y-5)^2 = 4(x-1)^2$$

$$x^2 + 4x + 4 + y^2 - 10y + 25 = 4(x^2 - 2x + 1)$$

$$x^2 + 4x + 4 + y^2 - 10y + 25 = 4x^2 - 8x + 4$$

$$3x^2 - 12x - y^2 + 10y - 25 = 0$$

Completing the square then gives,

$$3(x^2 - 4x) - (y^2 - 10y) - 25 = 0$$

$$3(x^2 - 4x + 4 - 4) - (y^2 - 10y + 25 - 25) - 25 = 0$$

$$3((x-2)^2 - 4) - ((y-5)^2 + 25) - 25 = 0$$

$$3(x-2)^2 - 12 - (y-5)^2 - 25 - 25 = 0$$

$$3(x-2)^2 - (y-5)^2 = 12$$

$$\frac{(x-2)^2}{4} - \frac{(y-5)^2}{12} = 1$$

Therefore, the set of points is a hyperbola with centre $(2, 5)$.

9 a From the first equation we know that $t = \dfrac{x+1}{2}$. Substitute this into the second equation to get

$$\begin{aligned} y &= 6 - 4t \\ &= 6 - 4\frac{x+1}{2} \\ &= 6 - 2(x+1) \\ &= 6 - 2x - 2 \\ &= 4 - 2x \end{aligned}$$

We obtain straight line whose equation is $y = 4 - 2x$.

b We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x}{2} = \cos t \text{ and } \frac{y}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\begin{aligned} \frac{x^2}{2^2} + \frac{y^2}{2^2} &= 1 \\ x^2 + y^2 &= 2^2 \end{aligned}$$

which is a circle of radius 2 centred at the origin.

c We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x-1}{3} = \cos t \text{ and } \frac{y+1}{5} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{5}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x-1)^2}{3^2} + \frac{(y+1)^2}{5^2} = 1,$$

which is an ellipse centred at the point $(1, -1)$.

d Since $x = \cos t$, we have,

$$\begin{aligned} y &= 3\sin^2 t - 2 \\ &= 3(1 - \cos^2 t) - 2 \\ &= 3 - 3\cos^2 t - 2 \\ &= 1 - 3\cos^2 t \\ &= 1 - 3x^2 \end{aligned}$$

Note that these doesn't give the entire parabola. Since $x = \cos t$, the domain will be $-1 \le x \le 1$. Therefore, the cartesian equation of the curve is

$$y = 1 - 3x^2, \text{ where } -1 \le x \le 1.$$

10 **a** From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$\begin{aligned} y &= 2t^2 - 1 \\ &= 2(x+1)^2 - 1. \end{aligned}$$

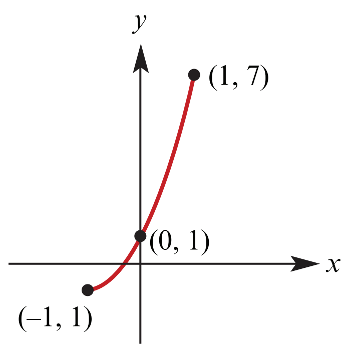
b Since $0 \le t \le 2$ and $x = t - 1$, we know that $-1 \le x \le 1$.

c The parabola has a minimum at $(-1, -1)$. It increases after this point. The maximum value of y is obtained when $x = 1$. Therefore,

$$y = 2(1+1)^2 - 1 = 7.$$

The range is the interval $-1 \le y \le 7$.

d We sketch the curve over the domain $-1 \le x \le 1$.



11 We have

$$\begin{aligned} x &= r\cos\theta & \qquad y &= r\sin\theta \\ &= 2\cos 3\pi/4 & &= 2\sin 3\pi/4 \\ &= -\sqrt{2} & &= \sqrt{2} \end{aligned}$$

so that the cartesian coordinates are $(-\sqrt{2}, \sqrt{2})$.

12 $r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$

$\theta = \tan^{1} \dfrac{-2\sqrt{3}}{2} = \tan^{-1} - \sqrt{3} = -\dfrac{\pi}{3}$

The point has polar coordinates $(4, -\pi/3)$. We could also let$r = -4$ and add π to the found angle, giving coordinate $(-4, 2\pi/3)$.

13 Since $x = r\cos\theta$ and $y = r\sin\theta$ the equation becomes,

$$2x + 3y = 5$$

$$2r\cos\theta + 3r\sin\theta = 5$$

$$r(2\cos\theta + 3\sin\theta) = 5$$

Therefore the polar equation is,

$$r = \frac{5}{2\cos\theta + 3\sin\theta}.$$

14 The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 6r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, equation (1) becomes,

$$x^2 + y^2 = 6y$$

$$x^2 + y^2 - 6y = 0$$

$$x^2 + (y^2 - 6y + 9) - 9 = 0 \qquad \text{(completing the square)}$$

$$x^2 + (y - 3)^2 = 9.$$

This is a circle whose centre is $(0, 3)$ and whose radius is 3, as required.

Solutions to multiple-choice questions

1 B The graph will have two vertical asymptotes provided that the denominator has two x-intercepts. Therefore the discriminant of quadratic must satisfy,

$$\begin{aligned} \Delta &> 0 \\ b^2 - 4ac &> 0 \\ 64 - 4(1)k &> 0 \\ 64 - 4k &> 0 \\ k &< 16. \end{aligned}$$

2 A We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} AP &= BP \\ \sqrt{(x-2)^2 + (y+5)^2} &= \sqrt{(x+4)^2 + (y-1)^2} \\ (x-2)^2 + (y+5)^2 &= (x+4)^2 + (y-1)^2 \\ x^2 - 4x + 4 + y^2 + 10y + 25 &= x^2 + 8x + 16 + y^2 - 2y + 1 \\ y &= x - 1 \end{aligned}$$

Therefore, the set of points is a straight line with equation $y = x - 1$. Alternatively, one could also just find the perpendicular bisector of line AB. This will give the same equation for about the same effort.

3 D One can answer this questions either by reasoning geometrically, or by finding the equation of the parabola. Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} FP &= MP \\ \sqrt{x^2 + (y-2)^2} &= \sqrt{(y-(-2))^2} \\ x^2 + (y-2)^2 &= (y+2)^2 \\ x^2 + y^2 - 4y + 4 &= y^2 + 4y + 4 \\ x^2 &= 8y \\ y &= \frac{x^2}{8}. \end{aligned}$$

Clearly **A**,**B** and **C** are true. The point $(2, 1)$ does not lie on the parabola since when $x = 2$,

$$y = \frac{x^2}{8} = \frac{2^2}{8} \neq 1.$$

The point $(4, 2)$ does lie on the parabola since when $x = 4$,

$$y = \frac{x^2}{8} = \frac{4^2}{8} = 2.$$

4 C Since the x-intercepts are $x = \pm 3$ and the y-intercepts are $y = \pm 2$ the equation must be

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

which clearly corresponds to item C.

5 D We can find the the coordinates of the x-axis intercepts by letting $y = 0$. This gives,

$$\begin{aligned} \frac{x^2}{25} + \frac{0^2}{9} &= 1 \\ \frac{x^2}{25} &= 1 \\ x^2 &= 25 \\ x &= \pm 5, \end{aligned}$$

so that the required coordinates are $(-5, 0)$ and $(5, 0)$.

6 D The hyperbola is centred at the point $(2, 0)$. This means that we can exclude options **A**,**C** and **E**, each of which are centred at the point $(-2, 0)$. The x-intercepts of the hyperbola occur when $x = -7$ and $x = 11$. We let $y = 0$ in option **B** and **D**, and see that only option **D** has the correct intercepts.

7 C The graph of

$$\frac{y^2}{9} - \frac{x^2}{4} = 1$$

is centred at the point $(0, 0)$. If we translate this by 3 units to the left and 2 units up we obtain the given equation. It will now be centred at the point $(-3, 2)$.

8 C We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x-1}{4} = \cos t \text{ and } \frac{y+1}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x-1}{4}\right)^2 + \left(\frac{y+1}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x-1)^2}{4^2} + \frac{(y+1)^2}{2^2} = 1.$$

To find the x-intercepts, we let $y = 0$. Solving for x gives,

$$\begin{aligned}
\frac{(x-1)^2}{4^2} + \frac{(0+1)^2}{2^2} &= 1 \\
\frac{(x-1)^2}{4^2} + \frac{1}{4} &= 1 \\
\frac{(x-1)^2}{4^2} &= \frac{3}{4} \\
(x-1)^2 &= 12 \\
x - 1 &= \pm\sqrt{12} \\
x &= 1 \pm 2\sqrt{3}
\end{aligned}$$

9 **E** **Option A.** These points are in quadrants 1 and 2 respectively and so cannot represent the same point.

Option B. These are located on the y-axis, but on opposite sides.

Option C. These points are in quadrants 1 and 4 respectively so cannot represent the same point.

Option D. These points are in quadrants 1 and 3 respectively so cannot represent the same point.

Option E. These coordinates do represent the same point. Recall that the coordinate $(-1, 7\pi/6)$ means that we locate direction $7\pi/6$, then move 1 unit in the opposite direction. This is the same as moving 1 unit in the direction $\pi/6$.

10 **B** The trick is to multiply both sides of the equation through by r. This gives,

$$\begin{aligned}
r^2 &= r + r\cos\theta \\
x^2 + y^2 &= r + x \\
x^2 + y^2 - x &= r \\
x^2 + y^2 - x &= \sqrt{x^2 + y^2} \\
(x^2 + y^2 - x)^2 &= x^2 + y^2,
\end{aligned}$$

as required.

Solutions to extended-response questions

1 **a** We know that the point $P(x, y)$ satisfies,

$$
\begin{aligned}
AP &= BP \\
\sqrt{x^2 + (y-3)^2} &= \sqrt{(x-6)^2 + y^2} \\
x^2 + (y-3)^2 &= (x-6)^2 + y^2 \\
x^2 + y^2 - 6y + 9 &= x^2 - 12x + 36 + y^2 \\
-6y + 9 &= -12x + 36 \\
y &= 2x - \frac{9}{2}
\end{aligned}
$$

Therefore, the set of points is a straight line with equation $y = 2x - \frac{9}{2}$.

b We know that the point $P(x, y)$ satisfies,

$$
\begin{aligned}
AP &= 2BP \\
\sqrt{x^2 + (y-3)^2} &= 2\sqrt{(x-6)^2 + y^2} \\
x^2 + (y-3)^2 &= 4[(x-6)^2 + y^2] \\
x^2 + y^2 - 6y + 9 &= 4[x^2 - 12x + 36 + y^2]
\end{aligned}
$$

$3x^2 - 48x + 3y^2 + 6y + 135 = 0$

Completing the square then gives,

$$
\begin{aligned}
3x^2 - 48x + 3y^2 + 6y + 135 &= 0 \\
3(x^2 - 16x) + 3(y^2 + 2y) + 135 &= 0 \\
3[(x^2 - 16x + 64) - 64] + 3[(y^2 + 2y + 1) - 1] + 135 &= 0 \\
3[(x-8)^2 - 64] + 3[(y+1)^2 - 1] + 135 &= 0 \\
3(x-8)^2 + 3(y+1)^2 &= 60 \\
(x-8)^2 + (y+1)^2 &= 20
\end{aligned}
$$

This defines a circle with centre $(8, -1)$ and radius $\sqrt{20}$.

2 **a** Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$FP = MP$$

$$\sqrt{x^2 + (y-4)^2} = \sqrt{(y-(-2))^2}$$

$$x^2 + (y-4)^2 = (y+2)^2$$

$$x^2 + y^2 - 8y + 16 = y^2 + 4y + 4$$

$$12y = x^2 + 12$$

$$y = \frac{x^2}{12} + 1.$$

Therefore, the set of points is a parabola.

b Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$FP = \frac{1}{2}MP$$

$$\sqrt{x^2 + (y-4)^2} = \frac{1}{2}\sqrt{(y-(-2))^2}$$

$$x^2 + (y-4)^2 = \frac{1}{4}(y+2)^2$$

$$4[x^2 + (y-4)^2] = (y+2)^2$$

$$4(x^2 + y^2 - 8y + 16) = y^2 + 4y + 4$$

$$4x^2 + 4y^2 - 32y + 64 = y^2 + 4y + 4$$

$$4x^2 + 3y^2 - 36y + 60 = 0$$

Completing the square then gives,

$$4x^2 + 3y^2 - 36y + 60 = 0$$

$$4x^2 + 3[y^2 - 12y + 20] = 0$$

$$4x^2 + 3[(y^2 - 12y + 36) - 36 + 20] = 0$$

$$4x^2 + 3[(y-6)^2 - 16] = 0$$

$$4x^2 + 3(y-6)^2 = 48$$

$$\frac{x^2}{12} + \frac{(y-6)^2}{16} = 1$$

Therefore, the set of points is an ellipse.

c Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$
\begin{aligned}
FP &= 2MP\\
\sqrt{x^2+(y-4)^2} &= 2\sqrt{(y-(-2))^2}\\
x^2+(y-4)^2 &= 4(y+2)^2\\
x^2+(y-4)^2 &= 4(y+2)^2\\
x^2+y^2-8y+16 &= 4(y^2+4y+4)\\
x^2+y^2-8y+16 &= 4y^2+16y+16\\
x^2-3y^2-24y &= 0
\end{aligned}
$$

Completing the square then gives,

$$
\begin{aligned}
x^2-3y^2-24y &= 0 \quad \text{Therefore, the set of points is a hyperbola.}\\
x^2-3[y^2+8y] &= 0\\
x^2-3[y^2+8y+16-16] &= 0\\
x^2-3[(y+4)^2-16] &= 0\\
3(y+4)^2-x^2 &= 48\\
\frac{(y+4)^2}{16}-\frac{x^2}{48} &= 1
\end{aligned}
$$

3 **a** Since $x = 10t$, we know that $t = \frac{x}{10}$. We substitute this into the second equation to give

$$
\begin{aligned}
y &= 20t-5t^2\\
&= 20\left(\frac{x}{10}\right)-5\left(\frac{x}{10}\right)^2\\
&= 2x-5\frac{x^2}{100}\\
&= 2x-\frac{x^2}{20}
\end{aligned}
$$

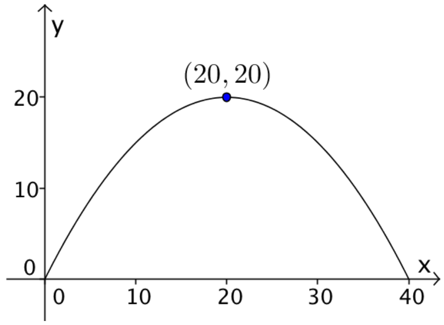
It will also help later if we consider the factorised expression. That is,

$$y = \frac{1}{20}x(40-x).$$

b The equation of the ball's path is

$$y = \frac{1}{20}x(40-x).$$

We note that the x-intercepts are $x = 0, 40$. The turning point will be located half-way between at $x = 20$. When $x = 20$, we find that $y = 20$. The graph of the ball's path is shown below.



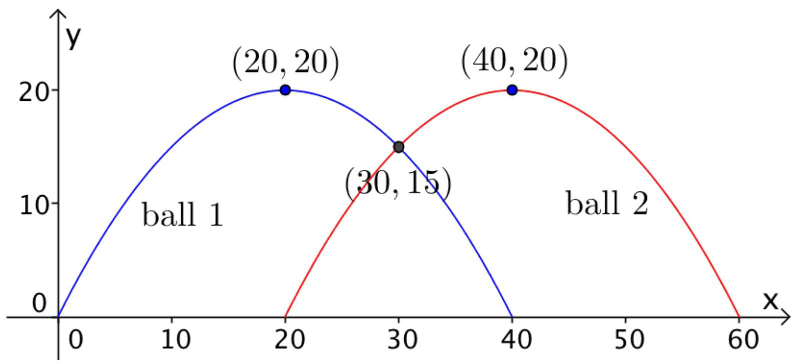
c The maximum height reached by the balls is 20 metres, and occurs when $x = 20$.

d Since $x = 10t$, we know that $t = \dfrac{60 - x}{10}$. We substitute this into the second equation to give

$$
\begin{aligned}
y &= 20t - 5t^2 \\
&= 20\left(\frac{60 - x}{10}\right) - 5\left(\frac{60 - x}{10}\right)^2 \\
&= 120 - 2x - 5\frac{(60 - x)^2}{100} \\
&= 120 - 2x - \frac{(60 - x)^2}{20} \\
&= 120 - 2x - \frac{(3600 - 120x + x^2)^2}{20} \\
&= 120 - 2x - 180 + 6x - \frac{x^2}{20} \\
&= 4x - 60 - \frac{x^2}{20} \\
&= -\frac{1}{20}(x^2 - 80x + 1200) \\
&= -\frac{1}{20}(x - 20)(x - 60)
\end{aligned}
$$

e The second balls path has been included on the diagram below. The point of intersection has been identified in the following question.

f



g To find where the paths meet, we solve the following pair of equations simultaneously (or using your calculator),

$y = -\frac{1}{20}(x - 20)(x - 60) \quad (1)$

$y = \frac{1}{20}x(40 - x) \quad (2)$

This gives a solution of $x = 30$ and $y = 15$.

h Note: just because the paths cross does *not* automatically mean that the balls collide. For this to happen, they must be at the same point at the same *time*. For the first ball, when $x = 30$, we find that

$$t = \frac{x}{10} = \frac{30}{10} = 3.$$

For the second ball, when $x = 30$, we find that

$$t = \frac{60 - x}{10} = \frac{60 - 30}{10} = 3.$$

So the balls are at the same position at the same time. Therefore, they collide.

4 a if $\angle AOB = 90°$ then we can use Pythagoras' theorem on right-triangle AOB. Therefore,

$$\begin{aligned} AB^2 &= OA^2 + OB^2 \\ (m - n)^2 &= (\sqrt{1^2 + m^2})^2 + (\sqrt{1^2 + n^2})^2 \\ (m - n)^2 &= (1 + m^2) + (1 + n^2) \\ m^2 - 2mn + n^2 &= 2 + m^2 + n^2 \\ -2mn &= 2 \\ mn &= -1, \end{aligned}$$

as required.

b Suppose that $mn = -1$. Since we don't yet know that triangle AOB is right-angled, we now use the cosine rule. Suppose $\angle AOB = \theta$. Then,

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\theta$$

$$(m-n)^2 = (\sqrt{1^2+m^2})^2 + (\sqrt{1^2+n^2})^2 - 2\sqrt{1^2+m^2}\sqrt{1^2+n^2}\cos\theta$$

$$(m-n)^2 = (1+m^2) + (1+n^2) - 2\sqrt{(1+m^2)(1+n^2)}\cos\theta$$

$$m^2 - 2mn + n^2 = 2 + m^2 + n^2 - 2\sqrt{(1+m^2)(1+n^2)}\cos\theta$$

$$-2mn = 2 - 2\sqrt{(1+m^2)(1+n^2)}\cos\theta$$

Since $mn = -1$, this simplifies to give,

$$2 = 2 - 2\sqrt{(1+m^2)(1+n^2)}\cos\theta$$

$$0 = 2\sqrt{(1+m^2)(1+n^2)}\cos\theta$$

Since $2\sqrt{(1+m^2)(1+n^2)} \neq 0$, this implies that $\cos\theta = 0$. Therefore,

$$\angle AOB = \theta = 90°,$$

as required.

c Consider the point $P(x, y)$. The gradient of AP is

$$m_1 = \frac{y-4}{x}.$$

The gradient of BP is

$$m_2 = \frac{y-10}{x-8}.$$

Since $AP \perp BP$ we know that

$$m_1 m_2 = -1$$

$$\frac{y-4}{x}\frac{y-10}{x-8} = -1$$

$$\frac{y^2-14y+40}{x^2-8x} = -1$$

$$y^2 - 14y + 40 = -x^2 + 8x$$

$$x^2 - 8x + y^2 - 14y + 40 = 0$$

Completing both squares gives,

$$[(x^2-8x+16)-16] + [(y^2-14y+49)-49] + 40 = 0$$

$$(x-4)^2 - 16 + (y-7)^2 - 49 + 40 = 0$$

$$(x-4)^2 + (y-7)^2 = 25$$

This a circle with centre $(4, 7)$ and radius 5.

5 The ladder is initially vertical with its midpoint located 3 metres up the wall at coordinate $(0, 3)$. The ladder comes to a rest lying horizontally. Its midpoint is located 3 metres to the right of the wall at coordinate $(3, 0)$. So if the midpoint is to move along a circular path then it must be along the circle

$$x^2 + y^2 = 3^2 \quad (1)$$

To check that this is indeed true, we suppose that the ladder is t units from the base of the wall. Then by Pythagoras' theorem, the ladder reaches

$$s = \sqrt{6^2 - t^2} = \sqrt{36 - t^2}$$

units up the wall. The midpoint of the ladder will then be

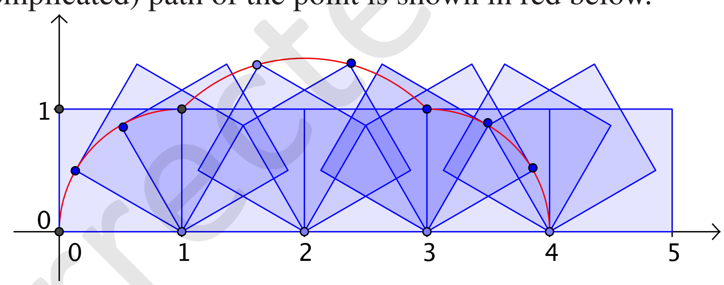
$$P\left(\frac{t}{2}, \frac{\sqrt{36 - t^2}}{2}\right).$$

We just need to check that this point lies on the circle whose equation is (1). Indeed,

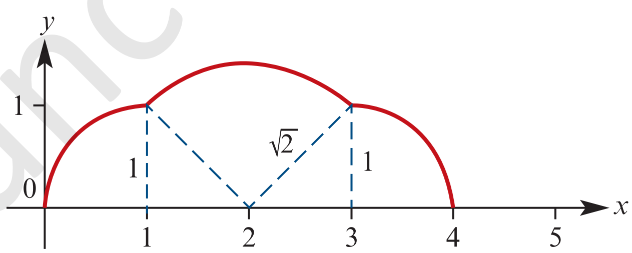
$$\begin{aligned} x^2 + y^2 &= \left(\frac{t}{2}\right)^2 + \left(\frac{\sqrt{36 - t^2}}{2}\right)^2 \\ &= \frac{t^2}{4} + \frac{36 - t^2}{4} \\ &= \frac{36}{4} \\ &= 9 \\ &= 3^2 \end{aligned}$$

Therefore point P lies on the circle whose equation is $x^2 + y^2 = 3^2$.

6 **a** The (rather complicated) path of the point is shown in red below.



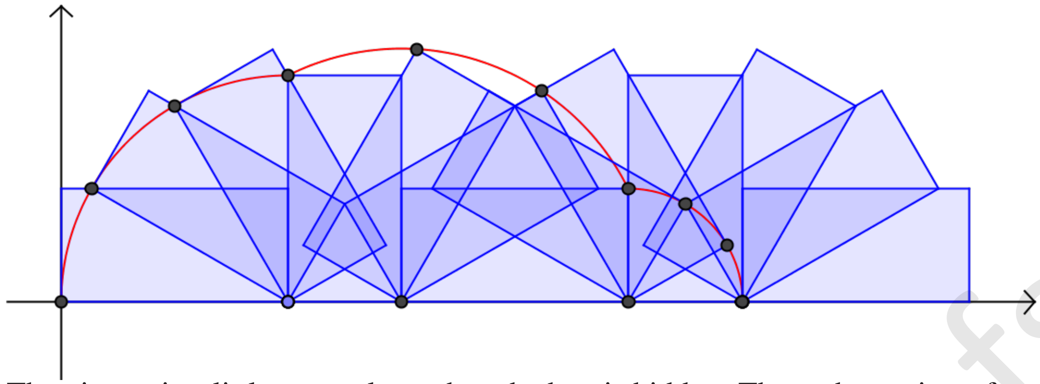
The picture is a little more clear when the box is hidden. The path consists of a quarter circle of of radius 1, another quarter circle of radius $\sqrt{2}$ (the diagonal length) and then another quarter circle of length 1.



b The total distance covered by point P will be

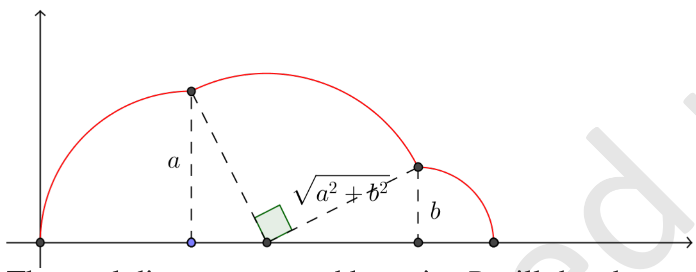
$$D = \frac{1}{4}(2\pi \cdot 1 + 2\pi \cdot \sqrt{2} + 2\pi \cdot 1) = \frac{1}{2}(2\pi + \pi\sqrt{2}).$$

c The (rather complicated) path of the point is shown in red below.



The picture is a little more clear when the box is hidden. The path consists of a quarter circle of of radius a, another quarter circle of radius $\sqrt{a^2+b^2}$ (the diagonal length) and then another quarter circle of length b.

d



The total distance covered by point P will then be

$$D = \frac{1}{4}(2\pi a + 2\pi \cdot \sqrt{a^2+b^2} + 2\pi b) = \frac{\pi}{2}(a + \sqrt{a^2+b^2} + b).$$

The area consists of three quarter circles and two triangles. The total area will then be. $A = \frac{1}{4}(\pi a^2 + \pi(\sqrt{a^2+b^2})^2 + \pi b^2) + 2 \times \frac{1}{2}ab$

$$= \frac{1}{4}(2\pi a^2 + 2\pi b^2) + ab$$

$$= \frac{\pi}{2}(a^2+b^2) + ab.$$

Chapter 16 – Complex numbers

Solutions to Exercise 16A

1 **a** $\text{Re}(z) = a = 2$

$\text{Im}(z) = b = 3$

b $\text{Re}(z) = a = 4$

$\text{Im}(z) = b = 5$

c $\text{Re}(z) = a = \frac{1}{2}$

$\text{Im}(z) = b = -\frac{3}{2}$

d $\text{Re}(z) = a = -4$

$\text{Im}(z) = b = 0$

e $\text{Re}(z) = a = 0$

$\text{Im}(z) = b = 3$

f $\text{Re}(z) = a = \sqrt{2}$

$\text{Im}(z) = b = -2\sqrt{2}$

2 **a** $2a - 3bi = 4 + 6i$

$2a = 4$

$a = 2$

$-3bi = 6i$

$b = -2$

b $a + b = 5$

$b = 5 - a$

$-2ab = -12$

$ab = 6$

$a(5 - a) = 6$

$5a - a^2 = 6$

$a^2 - 5a + 6 = 0$

$(a - 2)(a - 3) = 0$

When $a = 2$

$b = 5 - 2 = 3$

When $a = 3$

$b = 5 - 3 = 2$

c $2a + bi = 10$

$= 10 + 0i$

$2a = 10$

$a = 5$

$b = 0$

d $3a = 2$

$a = \frac{2}{3}$

$a - b = 1$

$\frac{2}{3} - b = 1$

$b = \frac{2}{3} - 1 = -\frac{1}{3}$

3 **a** $(2 - 3i) + (4 - 5i) = 2 + 4 - 3i - 5i$

$= 6 - 8i$

b $(4 + i) + (2 - 2i) = 4 + 2 + i - 2i$

$= 6 - i$

c $(-3 - i) - (3 + i) = -3 - 3 - i - i$

$= -6 - 2i$

d $(2-\sqrt{2}i)+(5-\sqrt{8}i)$

$= 2+5-\sqrt{2}i-\sqrt{8}i$

$= 7-\sqrt{2}i-2\sqrt{2}i$

$= 7-3\sqrt{2}i$

e $(1-i)-(2i+3) = 1-3-i-2i$

$= -2-3i$

f $(2+i)-(-2-i) = 2+2+i+i$

$= 4+2i$

g $4(2-3i)-(2-8i)$

$= 8-2-12i+8i$

$= 6-4i$

h $-(5-4i)+(1+2i)$

$= -5+1+4i+2i$

$= -4+6i$

i $5(i+4)+3(2i-7)$

$= 20-21+5i+6i$

$= -1+11i$

j $\frac{1}{2}(4-3i)-\frac{3}{2}(2-i)$

$= 2-3-\frac{3}{2}i+\frac{3}{2}i$

$= -1$

4 a $\sqrt{-16} = \sqrt{16\times -1}$

$= 4i$

b $2\sqrt{-9} = 2\sqrt{9\times -1}$

$= 6i$

c $\sqrt{-2} = \sqrt{2\times -1}$

$= \sqrt{2}i$

d $i^3 = i^2\times i$

$= -i$

e $i^{14} = i^{4\times 3+2}$

$= -1$

f $i^{20} = i^{4\times 5}$

$= 1$

g $-2i\times i^3 = -2i^4$

$= -2$

h $4i^4\times 3i^2 = 4\times 3\times i^4\times i^2$

$= 12i^6$

$= -12$

i $\sqrt{8}i^5\times\sqrt{-2} = \sqrt{8}i^4\times i\times\sqrt{2}i$

$= \sqrt{16}\times 1\times -1$

$= -4$

5 a $i(2-i) = 2i-i^2$

$= 2i-(-1)$

$= 1+2i$

b $i^2(3-4i) = -1(3-4i)$

$= -3+4i$

c $\sqrt{2}i(i-\sqrt{2}) = \sqrt{2}i^2-2i$

$= -\sqrt{2}-2i$

d

$-\sqrt{3}(\sqrt{-3}+\sqrt{2}) = -\sqrt{3}(\sqrt{3}i+\sqrt{2})$

$= -3i-\sqrt{6}$

$= -\sqrt{6}-3i$

Solutions to Exercise 16B

1 a $(4 + i)^2 = 16 + 8i + i^2$
$= 15 + 8i$

b $(2 - 2i)^2 = 4 - 8i + 4i^2$
$= -8i$

c $(3 + 2i)(2 + 4i) = 6 + 12i + 4i + 8i^2$
$= -2 + 16i$

d $(-1 - i)^2 = 1 + 2i + i^2$
$= 2i$

e $(\sqrt{2} - \sqrt{3}i)(\sqrt{2} + \sqrt{3}i) = 2 - 3i^2$
$= 2 + 3$
$= 5$

f $(5 - 2i)(-2 + 3i)$
$= -10 + 15i + 4i - 6i^2$
$= -4 + 19i$

2 a $z = 2 - 5i$
$\bar{z} = 2 + 5i$

b $z = -1 + 3i$
$\bar{z} = -1 - 3i$

c $z = \sqrt{5} - 2i$
$\bar{z} = \sqrt{5} + 2i$

d $z = 0 - 5i$
$\bar{z} = 0 + 5i = 5i$

3 a $\bar{z}_1 = 2 + i$

b $\bar{z}_2 = -3 - 2i$

c $z_1 z_2 = (2 - i)(-3 + 2i)$
$= -6 + 4i + 3i - 2i^2$
$= -4 + 7i$

d $\overline{z_1 z_2} = -4 - 7i$

e $\bar{z}_1 \bar{z}_2 = (2 + i)(-3 - 2i)$
$= -6 - 4i - 3i - 2i^2$
$= -4 - 7i$

f $z_1 + z_2 = (2 - i) + (-3 + 2i)$
$= -1 + i$

g $\overline{z_1 + z_2} = -1 - i$

h $\bar{z}_1 + \bar{z}_2 = (2 + i) + (-3 - 2i)$
$= -1 - i$

4 $z = 2 - 4i$

a $\bar{z} = 2 + 4i$

b $z\bar{z} = (2 - 4i)(2 + 4i)$
$= 4 - 16i^2$
$= 20$

c $z + \bar{z} = (2 - 4i) + (2 + 4i)$
$= 4$

d $z(z + \bar{z}) = 4z$
$= 8 - 16i$

e $z - \bar{z} = (2 - 4i) - (2 + 4i)$
$= -8i$

f $i(z - \overline{z}) = i \times -8i$

$= -8i^2 = 8$

g $z^{-1} = \dfrac{1}{2 - 4i}$

$= \dfrac{1}{2 - 4i} \times \dfrac{2 + 4i}{2 + 4i}$

$= \dfrac{2 + 4i}{4 - 16i^2}$

$= \dfrac{2 + 4i}{20}$

$= \dfrac{1}{10}(1 + 2i)$

h $\dfrac{z}{i} = \dfrac{z}{i} \times \dfrac{i}{i}$

$= \dfrac{i(2 - 4i)}{-1}$

$= -1 \times (2i - 4i^2)$

$= -4 - 2i$

5 $(a + bi)(2 + 5i) = 2a + 5ai + 2bi - 5b$

$= 3 - i$

$2a - 5b = 3$

$5a + 2b = -1$

Multiply the first equation by 2 and the second equation by 5.

$4a - 10b = 6 \quad ①$

$25a + 10b = -5 \quad ②$

① + ②:

$29a = 1$

$a = \dfrac{1}{29}$

$\dfrac{2}{29} - 5b = 3$

$5b = \dfrac{2}{29} - 3$

$= -\dfrac{85}{29}$

$b = -\dfrac{17}{29}$

6 **a** $\dfrac{2 - i}{4 + 1} = \dfrac{2 - i}{4 + 1} \times \dfrac{4 - i}{4 - i}$

$= \dfrac{8 - 2i - 4i + i^2}{16 - i^2}$

$= \dfrac{7 - 6i}{17}$

$= \dfrac{7}{17} - \dfrac{6}{17}i$

b $\dfrac{3 + 2i}{2 - 3i} = \dfrac{3 + 2i}{2 - 3i} \times \dfrac{2 + 3i}{2 + 3i}$

$= \dfrac{6 + 9i + 4i + 6i^2}{4 - 9i^2}$

$= \dfrac{13i}{13} = i$

c $\dfrac{4 + 3i}{1 + i} = \dfrac{4 + 3i}{1 + i} \times \dfrac{1 - i}{1 - i}$

$= \dfrac{4 - 4i + 3i - 3i^2}{1 - i^2}$

$= \dfrac{7 - i}{2}$

$= \dfrac{7}{2} - \dfrac{1}{2}i$

d $\dfrac{2-2i}{4i} = \dfrac{2-2i}{4i} \times \dfrac{i}{i}$

$= \dfrac{2i-2i^2}{-4}$

$= \dfrac{2+2i}{-4}$

$= \dfrac{-1-i}{2}$

$= -\dfrac{1}{2} - \dfrac{1}{2}i$

e $\dfrac{1}{2-3i} = \dfrac{1}{2-3i} \times \dfrac{2+3i}{2+3i}$

$= \dfrac{2+3i}{4-9i^2}$

$= \dfrac{2+3i}{13}$

$= \dfrac{2}{13} + \dfrac{3}{13}i$

f $\dfrac{i}{2+6i} = \dfrac{i}{2+6i} \times \dfrac{2-6i}{2-6i}$

$= \dfrac{2i+6}{4-36i^2}$

$= \dfrac{2i+6}{40}$

$= \dfrac{3}{20} + \dfrac{1}{20}i$

7 $(3-i)(a+bi) = 3a + 3bi - ai + b$

$= 6 - 7i$

$3a + b = 6$ ①

$-a + 3b = -7$

$-3a + 9b = -21$ ②

① + ②:

$10b = -15$

$b = -\dfrac{3}{2}$

$3a - \dfrac{3}{2} = 6$

$3a = 6 + \dfrac{3}{2} = \dfrac{15}{2}$

$a = \dfrac{5}{2}$

8 a $z = \dfrac{42i}{2-i}$

$= \dfrac{42i}{2+i} \times \dfrac{2+i}{2+i}$

$= \dfrac{84i + 42i^2}{4 - i^2}$

$= \dfrac{-42 + 84i}{5}$

$= -\dfrac{42}{5} + \dfrac{84i}{5}$

b $z = \dfrac{-2-i}{1+3i}$

$= \dfrac{-2-i}{1+3i} \times \dfrac{1-3i}{1-3i}$

$= \dfrac{-2+6i-i+3i^2}{1-9i^2}$

$= \dfrac{-5+5i}{10}$

$= -\dfrac{1}{2}(1-i)$

c $z = \dfrac{1+i}{5+3i}$

$= \dfrac{1+i}{5+3i} \times \dfrac{5-3i}{5-3i}$

$= \dfrac{5-3i+5i-3i^2}{25-9i^2}$

$= \dfrac{8+2i}{34}$

$= \dfrac{1}{17}(4+i)$

d $z = \dfrac{5+2i}{2(4-7i)}$

$= \dfrac{5+2i}{2(4-7i)} \times \dfrac{4+7i}{4+7i}$

$= \dfrac{20+35i+8i+14i^2}{2(16-49i^2)}$

$= \dfrac{6+43i}{130}$

$= \dfrac{1}{130}(6+43i)$

e $z = \dfrac{4}{1+i}$

$= \dfrac{4}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{4-4i}{1-i^2}$

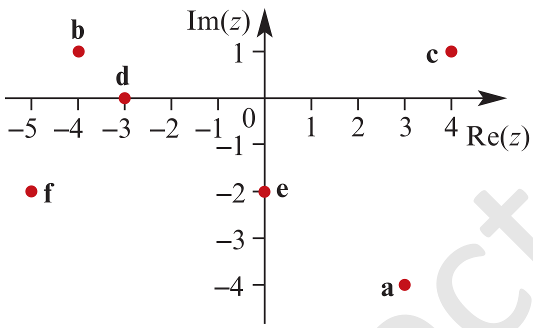
$= \dfrac{4-4i}{2}$

$= 2-2i$

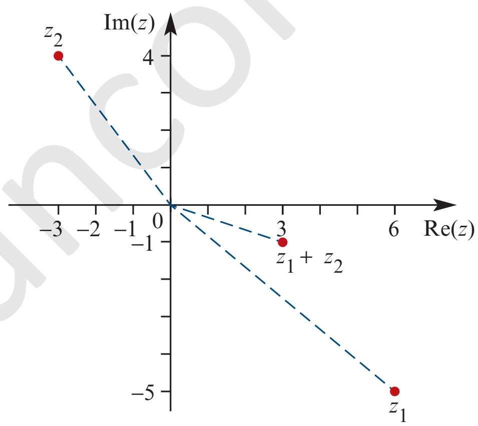
Solutions to Exercise 16C

1 $A = 3 + i$
$B = 2i$
$C = -3 - 4i$
$D = 2 - 2i$
$E = -3$
$F = -1 - i$

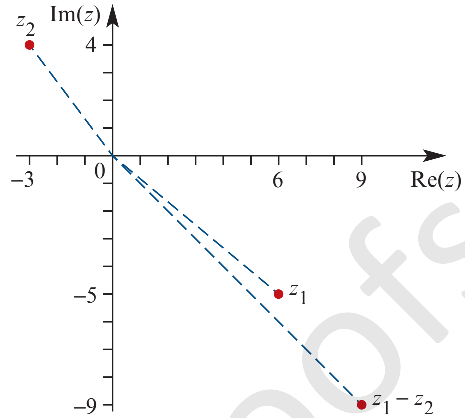
2 $A = 3 - 4i$
$B = -4 + i$
$C = 4 + i$
$D = -3$
$E = -2i$
$F = -5 - 2i$



3 **a** $z_1 + z_2 = (6 - 5i) + (-3 + 4i)$
$= 3 - i$



b $z_1 - z_2 = (6 - 5i) - (-3 + 4i)$
$= 9 - 9i$



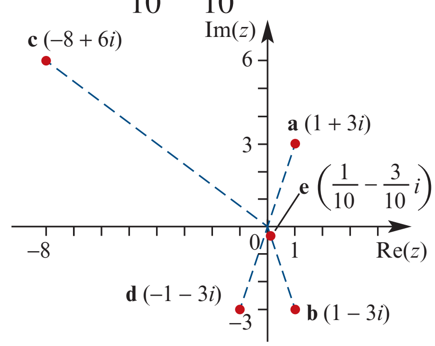
4 **a** $A : z = 1 + 3i$

b $B : \bar{z} = 1 - 3i$

c $C : z^2 = 1 + 6i + 9i^2$
$= -8 + 6i$

d $D : -z = -(1 + 3i)$
$= -1 - 3i$

e $E : \dfrac{1}{z} = \dfrac{1}{1 + 3i}$
$= \dfrac{1}{1 + 3i} \times \dfrac{1 - 3i}{1 - 3i}$
$= \dfrac{1 - 3i}{1 + 9i^2}$
$= \dfrac{1}{10} - \dfrac{3}{10}i$



5 **a** $A : z = 2 - 5i$

b $B : zi = i(2 - 5i)$

$$= 2i - 5i^2$$

$$= 5 + 2i$$

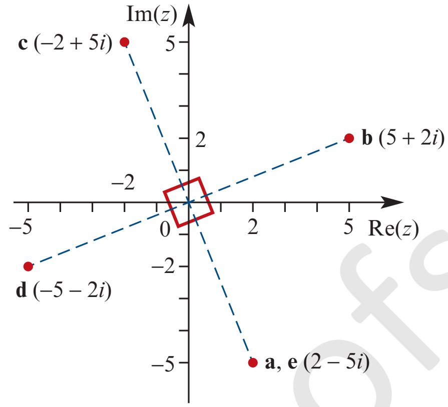
c $C : zi^2 = -z = -2 + 5i$

d $D : zi^3 = -iz$

$$= -i(2 - 5i)$$

$$= -5 - 2i$$

e $E : zi^4 = z = 2 - 5i$



Solutions to Exercise 16D

1 a

$$z^2 + 4 = 0$$
$$z^2 - 4i^2 = 0$$
$$(z - 2i)(z + 2i) = 0$$
$$z = \pm 2i$$

b

$$2x^2 + 18 = 0$$
$$z^2 + 9 = 0$$
$$z^2 - 9i^2 = 0$$
$$(z - 3i)(z + 3i) = 0$$
$$z = \pm 3i$$

c

$$3z^2 + 15 = 0$$
$$z^2 + 5 = 0$$
$$z^2 - 5i^2 = 0$$
$$(z - \sqrt{5}i)(z + \sqrt{5}i) = 0$$
$$z = \pm\sqrt{5}i$$

d $(z - 2)^2 = -16$

$$z - 2 = \pm 4i$$
$$z = 2 \pm 4i$$

e $(z + 1)^2 = -49$

$$z + 1 = \pm 7i$$
$$z = -1 \pm 7i$$

f Complete the square.

$$z^2 - 2z + 1 + 2 = 0$$
$$(z - 1)^2 - 2i^2 = 0$$
$$(z - 1 - \sqrt{2}i)(z - 1 + \sqrt{2}i) = 0$$
$$z = 1 \pm \sqrt{2}i$$

g Use the quadratic formula.

$$z = \frac{-3 \pm \sqrt{9 - 12}}{2}$$
$$= \frac{-3 \pm \sqrt{-3}}{2}$$
$$= \frac{1}{2}(-3 \pm \sqrt{3}i)$$

h Use the quadratic formula.

$$z = \frac{-5 \pm \sqrt{25 - 32}}{4}$$
$$= \frac{-5 \pm \sqrt{-7}}{4}$$
$$= \frac{1}{4}(-5 \pm \sqrt{7}i)$$

i Use the quadratic formula.

$3z^2 - z + 2 = 0$

$$z = \frac{1 \pm \sqrt{1 - 24}}{6}$$
$$= \frac{1 \pm \sqrt{-23}}{6}$$
$$= \frac{1}{6}(1 \pm \sqrt{23}i)$$

j Complete the square.

$$z^2 - 2z + 5 = 0$$
$$z^2 - 2z + 1 + 4 = 0$$
$$(z - 1)^2 - 4i^2 = 0$$
$$(z - 1 - 2i)(z - 1 + 2i) = 0$$
$$z = 1 \pm 2i$$

k Use the quadratic formula.

$$2z^2 - 6z + 10 = 0$$
$$z^2 - 3z + 5 = 0$$
$$z = \frac{3 \pm \sqrt{9 - 20}}{2}$$
$$= \frac{3 \pm \sqrt{-11}}{2}$$
$$= \frac{1}{2}(3 \pm \sqrt{11}i)$$

l Complete the square.

$$z^2 - 6z + 14 = 0$$
$$z^2 - 6z + 9 + 5 = 0$$
$$(z - 3)^2 - 5i^2 = 0$$
$$(z - 3 - \sqrt{5}i)(z - 3 + \sqrt{5}i) = 0$$
$$z = 3 \pm \sqrt{5}i$$

Solutions to Exercise 16E

1 a The point is in the first quadrant.

$$r = \sqrt{1^2 + (\sqrt{3})^2}$$
$$= \sqrt{1 + 3} = 2$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}$$
$$\therefore \quad 1 + \sqrt{3}i = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$$

b The point is in the fourth quadrant.

$$r = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\theta = -\frac{\pi}{4}$$
$$\therefore \quad 1 - i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$$

c The point is in the second quadrant.

$$r = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2}$$
$$= \sqrt{16} = 4$$
$$\cos\theta = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$
$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$
$$\therefore \quad -2\sqrt{3} + 2i = 4\operatorname{cis}\left(\frac{5\pi}{6}\right)$$

d The point is in the third quadrant.

$$r = \sqrt{4^2 + 4^2}$$
$$= \sqrt{32} = 4\sqrt{2}$$
$$\cos\theta = -\frac{4}{4\sqrt{2}} = -\frac{1}{2}$$
$$\theta = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$
$$\therefore \quad -4 - 4i = 4\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

e The point is in the fourth quadrant.

$$r = \sqrt{12^2 + 12^2 \times 3}$$
$$= \sqrt{4 \times 144} = 24$$
$$\cos\theta = -\frac{12}{24}$$
$$= -\frac{1}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\therefore \quad 12 - 12\sqrt{3}\,i = 24\operatorname{cis}\left(-\frac{\pi}{3}\right)$$

f The point is in the second quadrant.

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$
$$= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$
$$\cos\theta = -\frac{1}{2} \div \frac{1}{\sqrt{2}}$$
$$= -\frac{1}{2} \times \sqrt{2} = -\frac{1}{\sqrt{2}}$$
$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$
$$\therefore \quad -\frac{1}{2} + \frac{1}{2}i = \frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{3\pi}{4}\right)$$

2 a $3\operatorname{cis}\frac{\pi}{2} = 3\cos\frac{\pi}{2} + 3i\sin\frac{\pi}{2}$

$$= 3i$$

b $\sqrt{2}\text{ cis }\frac{\pi}{3} = \sqrt{2}\cos\frac{\pi}{3} + \sqrt{2}i\sin\frac{\pi}{3}$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$

$= \frac{\sqrt{2}}{2}(1 + \sqrt{3}\,i)$

c $2\text{ cis }\frac{\pi}{6} = 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6}$

$= \sqrt{3} + i$

d $5\text{ cis }\frac{3\pi}{4} = 5\cos\frac{3\pi}{4} + 5i\sin\frac{3\pi}{4}$

$= -\frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}}i$

$= -\frac{5\sqrt{2}}{2}(1 - i)$

e $12\text{ cis }\frac{5\pi}{6} = 12\cos\frac{5\pi}{6} + 12i\sin\frac{5\pi}{6}$

$= -6\sqrt{3} + 6i$

$= -6(\sqrt{3} - i)$

f $3\sqrt{2}\text{ cis }\left(-\frac{\pi}{4}\right) = 3\sqrt{2}\cos\left(-\frac{\pi}{4}\right)$

$+ 3\sqrt{2}i\sin\left(-\frac{\pi}{4}\right)$

$= 3 - 3i$

$= 3(1 - i)$

g $5\text{ cis }\frac{4\pi}{3} = 5\cos\frac{4\pi}{3} + 5i\sin\frac{4\pi}{3}$

$= -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

$= -\frac{5}{2}(1 + \sqrt{3}i)$

h $5\text{ cis }\left(-\frac{2\pi}{3}\right) = 5\cos\left(-\frac{2\pi}{3}\right)$

$+ 5i\sin\left(-\frac{2\pi}{3}\right)$

$= -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

$= -\frac{5}{2}(1 + \sqrt{3}i)$

3 $z_1 z_2 = r_1 r_2\text{ cis }(\theta_1 + \theta_2)$

a $\left(2\text{ cis }\frac{\pi}{6}\right)\cdot\left(3\text{ cis }\frac{\pi}{12}\right)$

$= 6\text{ cis }\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$

$= 6\text{ cis }\frac{\pi}{4}$

$= 6\cos\frac{\pi}{4} + 6i\sin\frac{\pi}{4}$

$= \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}}i$

$= 3\sqrt{2}(1 + i)$

b $\left(4\text{ cis }\frac{\pi}{12}\right)\cdot\left(3\text{ cis }\frac{\pi}{4}\right)$

$= 12\text{ cis }\left(\frac{\pi}{12} + \frac{\pi}{4}\right)$

$= 12\text{ cis }\frac{\pi}{3}$

$= 12\cos\frac{\pi}{3} + 12i\sin\frac{\pi}{3}$

$= 6 + 6\sqrt{3}i$

$= 6(1 + \sqrt{3}i)$

c $\left(\text{cis}\,\frac{\pi}{4}\right)\cdot\left(5\,\text{cis}\,\frac{5\pi}{12}\right)$

$= 5\,\text{cis}\left(\frac{\pi}{4}+\frac{5\pi}{12}\right)$

$= 5\,\text{cis}\,\frac{2\pi}{3}$

$= 5\cos\frac{2\pi}{3} + 5i\sin\frac{2\pi}{3}$

$= -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$

$= -\frac{5}{2}(1 - \sqrt{3}i)$

d $\left(12\,\text{cis}\left(-\frac{\pi}{3}\right)\right)\cdot\left(3\,\text{cis}\,\frac{2\pi}{3}\right)$

$= 36\,\text{cis}\left(-\frac{\pi}{3}+\frac{2\pi}{3}\right)$

$= 36\,\text{cis}\,\frac{\pi}{3}$

$= 36\cos\frac{\pi}{3} + 36i\sin\frac{\pi}{3}$

$= 18 + 18\sqrt{3}i$

$= 18(1 + \sqrt{3}i)$

e $\left(12\,\text{cis}\,\frac{5\pi}{6}\right)\cdot\left(3\,\text{cis}\,\frac{\pi}{2}\right)$

$= 36\,\text{cis}\left(\frac{5\pi}{6}+\frac{\pi}{2}\right)$

$= 36\,\text{cis}\,\frac{4\pi}{3}$

$= 36\cos\frac{4\pi}{3} + 36i\sin\frac{4\pi}{3}$

$= -18 - 18\sqrt{3}i$

$= -18(1 + \sqrt{3}i)$

f $\left(\sqrt{2}\,\text{cis}\,\pi\right)\cdot\left(\sqrt{3}\,\text{cis}\left(-\frac{3\pi}{4}\right)\right)$

$= \sqrt{6}\,\text{cis}\left(\pi - \frac{3\pi}{4}\right)$

$= \sqrt{6}\,\text{cis}\,\frac{\pi}{4}$

$= \sqrt{6}\cos\frac{\pi}{4} + \sqrt{6}i\sin\frac{\pi}{4}$

$= \sqrt{3} + \sqrt{3}i$

$= \sqrt{3}(1 + i)$

g $\dfrac{10\,\text{cis}\,\frac{\pi}{4}}{5\,\text{cis}\,\frac{\pi}{12}} = \frac{10}{5}\,\text{cis}\left(\frac{\pi}{4}-\frac{\pi}{12}\right)$

$= 2\,\text{cis}\,\frac{\pi}{6}$

$= 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6}$

$= \sqrt{3} + i$

h

$\dfrac{12\,\text{cis}\left(-\frac{\pi}{3}\right)}{3\,\text{cis}\,\frac{2\pi}{3}} = \frac{12}{3}\,\text{cis}\left(-\frac{\pi}{3}-\frac{2\pi}{3}\right)$

$= 4\,\text{cis}\,(-\pi)$

$= 4\cos(-\pi) + 4i\sin(-\pi)$

$= -4 + 0 = -4$

i $\dfrac{12\sqrt{8}\,\text{cis}\,\frac{3\pi}{4}}{3\sqrt{2}\,\text{cis}\,\frac{\pi}{12}} = \frac{12\sqrt{8}}{3\sqrt{2}}\,\text{cis}\left(\frac{3\pi}{4}-\frac{\pi}{12}\right)$

$= 8\,\text{cis}\,\frac{2\pi}{3}$

$= 8\cos\frac{2\pi}{3} + 8i\sin\frac{2\pi}{3}$

$= -4 + 4\sqrt{3}i$

$= -4(1 - \sqrt{3}i)$

j $\dfrac{20 \text{ cis}\left(-\dfrac{\pi}{6}\right)}{8 \text{ cis } \dfrac{5\pi}{6}}$

$$= \frac{20}{8} \text{ cis}\left(-\frac{\pi}{6} - \frac{5\pi}{6}\right)$$

$$= \frac{5}{2} \text{ cis } (-\pi)$$

$$= \frac{5}{2} \cos(-\pi) + \frac{5}{2} i \sin(-\pi)$$

$$= -\frac{5}{2} + 0$$

$$= -\frac{5}{2}$$

Solutions to Technology-free questions

1 a $2z_1 + 3z_2 = 2m + 2ni + 3p + 3qi$

$= (2m + 3p) + (2n + 3q)i$

b $\bar{z}_2 = p - qi$

c $z_1\,\bar{z}_2 = (m + ni)(p - qi)$

$= mp + npi - mqi - nqi^2$

$= (mp + nq) + (np - mq)i$

d $\dfrac{z_1}{z_2} = \dfrac{m + ni}{p + qi}$

$= \dfrac{m + ni}{p + qi} \times \dfrac{p - qi}{p - qi}$

$= \dfrac{mp + npi - mqi - nqi^2}{p^2 + q^2}$

$= \dfrac{(mp + nq) + (np - mq)i}{p^2 + q^2}$

e $z_1 + \bar{z}_1 = (m + ni) + (m - ni)$

$= 2m$

f

$(z_1 + z_2)(z_1 - z_2)$

$= z_1^2 - z_2^2$

$= m^2 + 2mni + n^2i^2 - (p^2 + 2pqi + q^2i^2)$

$= m^2 + 2mni - n^2 - (p^2 + 2pqi - q^2)$

$= (m^2 - n^2 - p^2 + q^2) + (2mn - 2pq)i$

g $\dfrac{1}{z_1} = \dfrac{1}{m + ni}$

$= \dfrac{1}{m + ni} \times \dfrac{m - ni}{m - ni}$

$= \dfrac{m - ni}{m^2 + n^2}$

h $\dfrac{z_2}{z_1} = \dfrac{p + qi}{m + ni}$

$= \dfrac{p + qi}{m + ni} \times \dfrac{m - ni}{m - ni}$

$= \dfrac{mp + nq + (mq - np)i}{m^2 + n^2}$

i $\dfrac{3z_1}{z_2} = \dfrac{3(m + ni)}{p + qi}$

$= \dfrac{3(m + ni)}{p + qi} \times \dfrac{p - qi}{p - qi}$

$= \dfrac{3(mp + npi - mqi - nqi^2)}{p^2 + q^2}$

$= \dfrac{3[(mp + nq) + (np - mq)i]}{p^2 + q^2}$

2 a $A : z = 1 - \sqrt{3}i$

b $B : z^2 = (1 - \sqrt{3}i)^2$

$= 1 - 2\sqrt{3}i + 3i^2$

$= -2 - 2\sqrt{3}i$

c $C : z^3 = z^2 \times z$

$= (-2 - 2\sqrt{3}i)(1 - \sqrt{3}i)$

$= -2 + 2\sqrt{3}i - 2\sqrt{3}i + 6i^2$

$= -8$

d $D : \dfrac{1}{z} = \dfrac{1}{1 - \sqrt{3}i}$

$= \dfrac{1}{1 - \sqrt{3}i} \times \dfrac{1 + \sqrt{3}i}{1 + \sqrt{3}i}$

$= \dfrac{1 + \sqrt{3}i}{4}$

e $E : \bar{z} = 1 + \sqrt{3}i$

f $F : \dfrac{1}{\bar{z}} = \dfrac{1}{1 + \sqrt{3}i}$

$= \dfrac{1}{1 + \sqrt{3}i} \times \dfrac{1 - \sqrt{3}i}{1 - \sqrt{3}i}$

$= \dfrac{1 - \sqrt{3}i}{4}$

Note: use existing diagram from answers

3 **a** The point is in the first quadrant.

$r = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$\cos\theta = \dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{\pi}{4}$

$\therefore \quad 1 + i = \sqrt{2}\text{ cis}\left(\dfrac{\pi}{4}\right)$

b The point is in the fourth quadrant.

$r = \sqrt{1 + 3}$

$= 2$

$\cos\theta = \dfrac{1}{2}$

$\theta = -\dfrac{\pi}{3}$

$\therefore \quad 1 - \sqrt{3}i = 2\text{ cis}\left(-\dfrac{\pi}{3}\right)$

c The point is in the first quadrant.

$r = \sqrt{12 + 1}$

$= \sqrt{13}$

$\tan\theta = \dfrac{1}{2\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{\sqrt{3}}{6}$

$\therefore \quad 2\sqrt{3} + i = \sqrt{13}\text{ cis}\left(\tan^{-1}\dfrac{\sqrt{3}}{6}\right)$

d The point is in the first quadrant.

$r = \sqrt{18 + 18}$

$= \sqrt{36} = 6$

$\cos\theta = \dfrac{3\sqrt{2}}{6} = \dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{\pi}{4}$

$\therefore \quad 3\sqrt{2} + 3\sqrt{2}i = 6\text{ cis}\left(\dfrac{\pi}{4}\right)$

e The point is in the third quadrant.

$r = \sqrt{18 + 18}$

$= \sqrt{36} = 6$

$\cos\theta = -\dfrac{3\sqrt{2}}{6} = -\dfrac{1}{\sqrt{2}}$

$\theta = -\pi + \dfrac{\pi}{4} = -\dfrac{3\pi}{4}$

$\therefore \quad -3\sqrt{2} - 3\sqrt{2}i = 6\text{ cis}\left(-\dfrac{3\pi}{4}\right)$

f The point is in the fourth quadrant.

$$r = \sqrt{3+1}$$
$$= 2$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$
$$\theta = -\frac{\pi}{6}$$
$$\therefore \quad \sqrt{3} - i = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$$

4 a $x = -2\cos\left(\frac{\pi}{3}\right)$

$$= -1$$
$$y = -2\sin\left(\frac{\pi}{3}\right)$$
$$= -\sqrt{3}$$
$$\therefore \quad z = -1 - \sqrt{3}i$$

b $x = 3\cos\left(\frac{\pi}{4}\right)$

$$= \frac{3\sqrt{2}}{2}$$
$$y = 3\sin\left(\frac{\pi}{4}\right)$$
$$= \frac{3\sqrt{2}}{2}$$
$$\therefore \quad z = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

c $x = 3\cos\left(\frac{3\pi}{4}\right)$

$$= -\frac{3\sqrt{2}}{2}$$
$$y = 3\sin\left(\frac{3\pi}{4}\right)$$
$$= \frac{3\sqrt{2}}{2}$$
$$\therefore \quad z = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

d $x = -3\cos\left(-\frac{3\pi}{4}\right)$

$$= \frac{3\sqrt{2}}{2}$$
$$y = -3\sin\left(-\frac{3\pi}{4}\right)$$
$$= \frac{3\sqrt{2}}{2}$$
$$\therefore \quad z = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

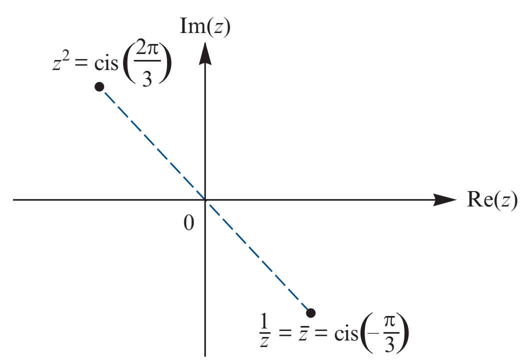
e $x = 3\cos\left(-\frac{5\pi}{6}\right)$

$$= -\frac{3\sqrt{3}}{2}$$
$$y = 3\sin\left(-\frac{5\pi}{6}\right)$$
$$= -\frac{3}{2}$$
$$\therefore \quad z = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

f $x = \sqrt{2}\cos\left(-\frac{\pi}{4}\right)$

$$= 1$$
$$y = \sqrt{2}\sin\left(-\frac{\pi}{4}\right)$$
$$= -1$$
$$\therefore \quad z = 1 - i$$

5



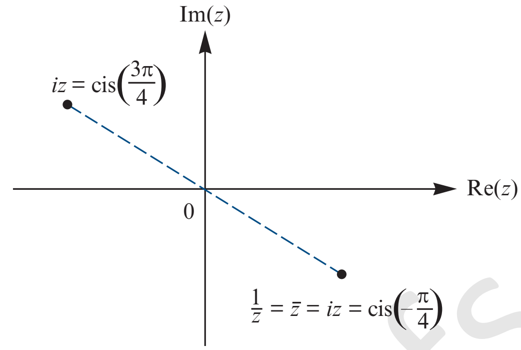
a $z^2 = \text{cis}\left(\frac{2\pi}{3}\right)$

b $\bar{z} = \text{cis}\left(-\frac{\pi}{3}\right)$

c $\frac{1}{z} = \text{cis}\left(-\frac{\pi}{3}\right)$

d $\text{cis}\left(\frac{2\pi}{3}\right)$

6



a $iz = \text{cis}\left(\frac{3\pi}{4}\right)$

b $\bar{z} = \text{cis}\left(-\frac{\pi}{4}\right)$

c $\frac{1}{z} = \text{cis}\left(-\frac{\pi}{4}\right)$

d $-iz = \text{cis}\left(-\frac{\pi}{4}\right)$

Solutions to multiple-choice questions

1 C $$\frac{1}{2-u} = \frac{1}{1-i}$$
$$= \frac{1}{1-i} \times \frac{1+i}{1+i}$$
$$= \frac{1+i}{2}$$
$$= \frac{1}{2} + \frac{1}{2}i$$

2 D $i = \text{ cis } \frac{\pi}{2}$, so the point will be rotated by $\frac{\pi}{2}$.

3 C $$|z| = 5$$
$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$
$$= \frac{1}{5}$$

4 D $$(x+yi)^2 = x^2 + 2xyi + y^2i^2$$
$$= (x^2 - y^2) + 2xyi$$
Therefore
$$x^2 - y^2 = 0 \text{ and } 2xy = -32.$$
Therefore
$$x^2 - y^2 = 0 \Rightarrow y = \pm x$$
If $y = x$ then
$$2xy = -32$$
has no solution. If $y = -x$, then
$$2xy = -32$$
$$-2x^2 = -32$$
$$x^2 = 16$$
$$x = \pm 4$$
Therefore, $x = 4, y = -4$ or $x = 4, y = -4$.

5 D Completing the square gives,
$$z^2 + 6z + 10 = z^2 + 6z + 9 + 1$$
$$= (z+3)^2 + 1$$
$$= (z+3)^2 - i^2$$
$$= (z+3-i)(z+3+i).$$

6 E Completing the square gives,
$$\frac{1}{1-i} = \frac{1}{1-i}\frac{1+i}{1+i}$$
$$= \frac{1+i}{2}$$
$$= \frac{1}{2} + \frac{1}{2}i$$
Therefore,
$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$
and
$$\theta = \frac{\pi}{4}.$$

7 D $$\frac{z-2i}{z-(3-2i)} = 2$$
$$z - 2i = 2(z - (3-2i))$$
$$z - 2i = 2z - 2(3-2i)$$
$$z = 2(3-2i) - 2i$$
$$= 6 - 6i$$

8 D $$z^2(1+i) = 2 - 2i$$
$$z^2 = \frac{2-2i}{1+i}$$
$$= \frac{(2-2i)(1-i)}{2}$$
$$= (1-i)^2$$
$$= (-1+i)^2$$

9 B $\Delta = b^2 - 4ac$

$= (8i)^2 - 4(2 + 2i)(-4(1 - i))$

$= 64i^2 + 16(2 + 2i)(1 - i)$

$= -64 + 32(1 + i)(1 - i)$

$= -64 + 32(1 - i^2)$

$= -64 + 32 \times 2$

$= 0$

10 D $\text{Arg}(1 + ai) = \frac{\pi}{6}$

$\tan^{-1} a = \frac{\pi}{6}$

$a = \tan\left(\frac{\pi}{6}\right)$

$= \frac{1}{\sqrt{3}}$

Solutions to extended-response questions

1 **a** $z^2 - 2\sqrt{3}z + 4 = 0$

Completing the square gives

$$z^2 - 2\sqrt{3}z + 3 + 1 = 0$$

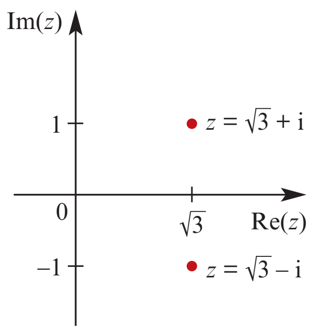
$$\therefore \quad (z - \sqrt{3})^2 + 1 = 0$$

$$\therefore \quad (z - \sqrt{3})^2 - i^2 = 0$$

$$\therefore \quad (z - \sqrt{3} + i)(z - \sqrt{3} - i) = 0$$

$$\therefore \quad z = \sqrt{3} \pm i$$

b **i**



ii $|\sqrt{3} + i| = |\sqrt{3} - i| = 2$

The circle has centre the origin and radius 2.

The cartesian equation is $x^2 + y^2 = 4$.

iii The circle passes through (0, 2) and (0, −2). The corresponding complex numbers are $2i$ and $-2i$. So $a = 2$

2 $|z| = 6$

a
$$\begin{aligned}|(1 + i)z| &= |1 + i||z| \\ &= \sqrt{2} \times 6 \\ &= 6\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}|(1 + i)z - z| &= |z + iz - z| \\ &= |iz| \\ &= |i|\,|z| \\ &= 6\end{aligned}$$

c A is the point corresponding to z, and $|OA| = 6$.

B is the point corresponding to $(1 + i)z$, and $|OB| = 6\sqrt{2}$.

From part **b**, $|AB| = |(1 + i)z - z|$

$= 6$

Therefore ΔOAB is isosceles.

Note also that

$$|OA|^2 + |AB|^2 = 6^2 + 6^2 = 72$$

and $\quad |OB|^2 = (6\sqrt{2})^2$

$= 72$

The converse of Pythagoras' theorem gives the triangle is right-angled at A.

3 $z = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

a $1 + z = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

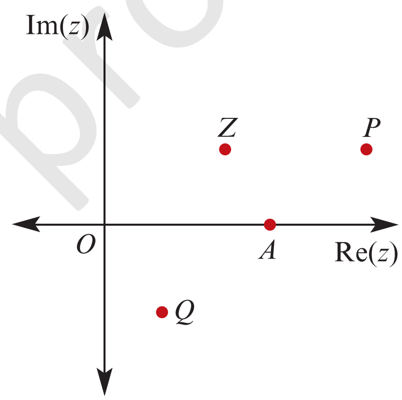
$$= \left(1 + \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}i$$

$$= \frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

and $1 - z = 1 - \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i$

$$= \left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}i$$

$$= \frac{\sqrt{2} - 1}{\sqrt{2}} \frac{1}{\sqrt{2}}i$$



b $|OP|^2 = \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2}(2 + 2\sqrt{2} + 1 + 1)$

$= 2 + \sqrt{2}$

$|OQ|^2 = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2}(2 - 2\sqrt{2} + 1 + 1)$

$= 2 - \sqrt{2}$

$|QP| = |-1 + z + 1 + z|$

$= |2z|$

$= 2|z|$

$= 2$

and $|QP|^2 = 4$

Therefore $|QP|^2 = |OP|^2 + |OQ|^2$

By the converse of Pythagoras' theorem $\angle POQ$ is a right angle, i.e. $\angle POQ = \frac{\pi}{2}$

Now $\frac{|OP|}{|OQ|} = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$

$= \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} \times \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$

$= \frac{2+\sqrt{2}}{\sqrt{2}}$

$= \sqrt{2} + 1$

4 For this question we will use the fact that $|z|^2 = z\bar{z}$. This is easy to prove.

a $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$

$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2$

$= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$

b $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2})$

$$= (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$

$$= z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - \overline{z_1}z_2$$

$$= |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$$

c Since

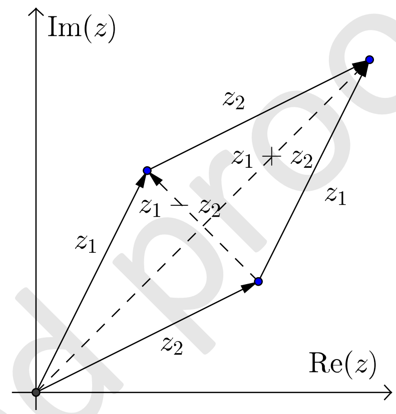
$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$$

and

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$$

we can add these two equations to give,

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2.$$



This result has a geometric interpretation. By interpreting complex numbers z_1 and z_2 as vectors, we obtain a parallelogram with diagonals whose vectors are $z_1 + z_2$ and $z_1 - z_2$. This result then shows that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals

5 a For this question we will use the fact that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$. This is easy to prove if you haven't already seen it done.

i $\overline{\overline{z_1} z_2} = \overline{\overline{z_1}}\ \overline{z_2}$

$$= z_1\overline{z_2}$$

ii First note that $z + \overline{z} = 2\text{Re}\,(z)$. Now using part (i) we have

$$z_1\overline{z_2} + \overline{z_1}z_2 = \overline{\overline{z_1}z_2} + \overline{z_1}z_2$$

$$= 2\text{Re}\,(\overline{z_1}z_2),$$

which is a real number.

iii First note that $z - \overline{z} = 2i\,\text{Im}\,(z)$. Now using part (i) we have

$$z_1\overline{z_2} - \overline{z_1}z_2 = \overline{\overline{z_1}z_2} - \overline{z_1}z_2$$
$$= 2i\ \mathrm{Im}\ (\overline{z_1}z_2),$$

which is an imaginary number.

iv Adding the results of the two previous questions gives

$$\begin{aligned}
(z_1\overline{z_2} + \overline{z_1}z_2)^2 + (z_1\overline{z_2} - \overline{z_1}z_2)^2 &= (2\mathrm{Re}\ (\overline{z_1}z_2))^2 - (2i\ \mathrm{Im}\ (\overline{z_1}z_2))^2 \\
&= 4(\mathrm{Re}\ (\overline{z_1}z_2))^2 + 4(\mathrm{Im}\ (\overline{z_1}z_2))^2 \\
&= 4((\mathrm{Re}\ (\overline{z_1}z_2))^2 + (\mathrm{Im}\ (\overline{z_1}z_2))^2) \\
&= 4|\overline{z_1}z_2|^2 \\
&= 4|\overline{z_1}||z_2|^2 \\
&= 4|z_1||z_2|^2 \\
&= 4|z_1z_2|^2.
\end{aligned}$$

b

$$\begin{aligned}
(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - (z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2) \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - (|z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2) \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - |z_1|^2 - |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2) \\
&= 2|z_1||z_2| - (z_1\overline{z_2} + \overline{z_1}z_2) \\
&= 2|z_1||z_2| - 2\mathrm{Re}\ (\overline{z_1}z_2) \\
&= 2|\overline{z_1}||z_2| - 2\mathrm{Re}\ (\overline{z_1}z_2) \\
&= 2|\overline{z_1}z_2| - 2\mathrm{Re}\ (\overline{z_1}z_2) \\
&\geq 0
\end{aligned}$$

c This question simply requires a trick:

$$|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|.$$

Therefore,

$$|z_1 - z_2| \geq |z_1| - |z_2|.$$

6 $z = \mathrm{cis}\theta$

a $z + 1 = \text{cis}\theta + 1$

$$= \cos\theta + i\sin\theta + 1$$

$$= (1 + \cos\theta) + i\sin\theta$$

$$|z + 1| = \sqrt{(1 + \cos\theta)^2 + \sin^2\theta}$$

$$= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \sqrt{1 + 2\cos\theta + 1}$$

$$= \sqrt{2 + 2\cos\theta}$$

$$= \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}$$

$$= 2\cos\left(\frac{\theta}{2}\right) \text{ since } 0 \le \frac{\theta}{2} \le \frac{\pi}{2}.$$

To find the argument, we find that

$$\frac{\sin\theta}{1 + \cos\theta} = \frac{\sin\theta}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}$$

so that $\text{Arg}(z + 1) = \frac{\theta}{2}$.

b $z - 1 = \text{cis}\,\theta - 1$

$$= \cos\theta + i\sin\theta - 1$$

$$= (\cos\theta - 1) + i\sin\theta$$

$$|z - 1| = \sqrt{(\cos\theta - 1)^2 + \sin^2\theta}$$

$$= \sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$

$$= \sqrt{2 - 2\cos\theta}$$

$$= \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}$$

$$= 2\sin\left(\frac{\theta}{2}\right) \text{ since } 0 \le \frac{\theta}{2} \le \frac{\pi}{2}.$$

To find the argument, we evaluate

$$\frac{\sin\theta}{\cos\theta - 1} = -\frac{\sin\theta}{1 - \cos\theta}$$

$$= -\frac{\sin\theta}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$= -\cot\left(\frac{\theta}{2}\right)$$

$$= \tan\left(\frac{\theta}{2} + \frac{\pi}{2}\right)$$

so that $\text{Arg}(z - 1) = \frac{\pi}{2} + \frac{\theta}{2}$.

c $$\left|\frac{z-1}{z+1}\right| = \frac{|z-1|}{|z+1|}$$
$$= \frac{2\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}$$
$$= \tan\left(\frac{\theta}{2}\right)$$
$$\text{Arg}\left(\frac{z-1}{z+1}\right) = \text{Arg}(z-1) - \text{Arg}(z+1)$$
$$= \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}$$
$$= \frac{\pi}{2}$$

7 **a** $\Delta = b^2 - 4ac$

b The equation has no real solutions if and only if

$b^2 - 4ac < 0.$

c If $b^2 - 4ac$ then we can assume that

$$z_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } z_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a}.$$

It follows that P_1 has coordinates

$$\left(\frac{-b}{2a}, \frac{\sqrt{4ac - b^2}}{2a}\right)$$

and P_2 has coordinates

$$\left(\frac{-b}{2a}, -\frac{\sqrt{4ac - b^2}}{2a}\right).$$

i $$z_1 + z_2 = -\frac{b}{a}$$
$$|z_1| = |z_2| = \sqrt{\left(\frac{-b}{2a}\right)^2 + \left(\frac{\sqrt{4ac - b^2}}{2a}\right)^2}$$
$$= \sqrt{\frac{b^2}{4a^2} + \frac{4ac - b^2}{4a^2}}$$
$$= \sqrt{\frac{c}{a}}$$

ii To find $\angle P_1OP_2$ it will also help to find

$$z_1 - z_2 = \frac{i\sqrt{4ac - b^2}}{a}$$

$$|z_1 - z_2| = \frac{\sqrt{4ac - b^2}}{|a|}$$

Therefore, with reference to the diagram below, we use the cosine law to show that

$$P_1P_2 = OP_1^2 + OP_2^2 - 2 \cdot OP_1 \cdot OP_2 \cdot \cos\theta$$

$$\frac{4ac - b^2}{a^2} = \frac{c}{a} + \frac{c}{a} - 2\frac{c}{a}\cos\theta$$

$$\frac{4ac - b^2}{a^2} = \frac{2c}{a} - 2\frac{c}{a}\cos\theta$$

$$\frac{4ac - b^2}{a^2} = \frac{2c}{a}(1 - \cos\theta)$$

$$\frac{4ac - b^2}{a} = 2c(1 - \cos\theta)$$

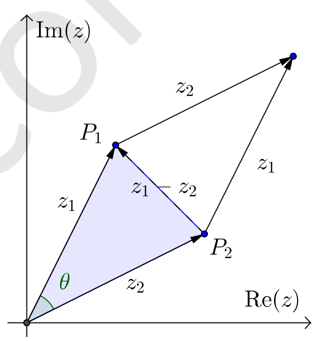
$$1 - \cos\theta = \frac{4ac - b^2}{2ac}$$

$$\cos\theta = 1 - \frac{4ac - b^2}{2ac}$$

$$\cos\theta = \frac{b^2 - 2ac}{2ac.}$$

Therefore

$$\cos(\angle P_1OP_2) = \frac{b^2 - 2ac}{2ac}.$$



8 a It's perhaps fastest to simply use the quadratic formula here:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2}$$
$$= \frac{-1 \pm \sqrt{-3}}{2}$$
$$= \frac{-1 \pm i\sqrt{3}}{2}$$

so that

$$z_1 = \frac{-1 + i\sqrt{3}}{2} \text{ and } z_2 = \frac{-1 - i\sqrt{3}}{2}.$$

b We prove the first equality. The proof for the second is similar. We have

$$z_2^2 = \left(\frac{-1 - i\sqrt{3}}{2}\right)^2$$
$$= \frac{1}{4}(1 + i\sqrt{3})^2$$
$$= \frac{1}{4}(1 + 2i\sqrt{3} + 3i^2)$$
$$= \frac{1}{4}(-2 + 2i\sqrt{3})$$
$$= \frac{-1 + i\sqrt{3}}{2}$$
$$= z_1,$$

as required.

c First consider $z_1 = \dfrac{-1 + i\sqrt{3}}{2}$. The point is in the second quadrant.

$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)}$$
$$= \sqrt{\frac{1}{4} + \frac{3}{4}}$$
$$= 1$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}$$

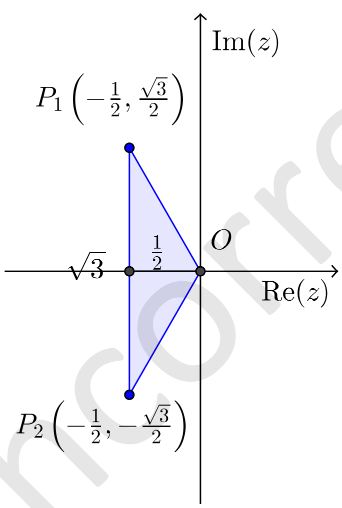
$$\therefore \quad \frac{-1 + i\sqrt{3}}{2} = 1 \operatorname{cis}\left(\frac{2\pi}{3}\right).$$

Now consider $z_2 = \dfrac{-1 - i\sqrt{3}}{2}$. The point is in the third quadrant.

$$
\begin{aligned}
r &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)} \\
&= \sqrt{\frac{1}{4} + \frac{3}{4}} \\
&= 1 \\
\cos\theta &= -\frac{1}{2} \\
\theta &= -\frac{2\pi}{3} \\
\therefore \quad \frac{-1 - i\sqrt{3}}{2} &= 1 \operatorname{cis}\left(-\frac{2\pi}{3}\right).
\end{aligned}
$$

d Plot points O, P_1 and P_2. From this, you will see that

$$
\begin{aligned}
A &= \frac{bh}{2} \\
&= \frac{\sqrt{3} \times \frac{1}{2}}{2} \\
&= \frac{\sqrt{3}}{4}.
\end{aligned}
$$



Chapter 18 – Matrices

Solutions to Exercise 18A

1 a Number of rows × number of columns = 2×2

b Number of rows × number of columns = 2×3

c Number of rows × number of columns = 1×4

d Number of rows × number of columns = 4×1

2 a There will be 5 rows and 5 columns to match the seating. Every seat of both diagonals is occupied, and so the diagonals will all be ones, and the rest of the numbers, representing unoccupied seats, will all be 0.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b If all seats are occupied, then every number in the matrix will be 1.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

3 a $i = j$ for the leading diagonal only, so the leading diagonal will be all ones, and the rest of the numbers 0.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b $$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

c $$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

4 We can present this as a table with the girls on the top row, and the boys on the bottom row, in order of year level, i.e. years 7, 8, 9, 10, 11 and 12 going from left to right.

$$\begin{bmatrix} 200 & 180 & 135 & 110 & 56 & 28 \\ 110 & 117 & 98 & 89 & 53 & 33 \end{bmatrix}$$

Alternatively, girls and boys could be the two columns, and year levels could run down from year 7 to 12, in order. This would give:

$$\begin{bmatrix} 200 & 110 \\ 180 & 117 \\ 135 & 98 \\ 110 & 89 \\ 56 & 53 \\ 28 & 33 \end{bmatrix}$$

5 a Matrices are equal only if they

have the same number of rows and columns, and all pairs of corresponding entries are equal. The first two matrices have the same dimensions, but the top entries are not equal, so the matrices cannot be equal.
The last two matrices have the same dimensions and equal first (left) entries, so they will be equal if $x = 4$.
Thus, $\begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$ if $x = 4$.

b The first two matrices cannot be equal because corresponding entries are not equal, nor can the second and third for the same reason.
The last matrix cannot equal any of the others because it has different dimensions. The only two that can be equal are the first and third.
$\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$ if $x = 4$

c All three matrices have the same dimensions and all corresponding numerical entries are equal. They could all be equal.

$$\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$$
if $x = 0, y = 2$

6 a The entry corresponding to x is 2, and the entry corresponding to y is 3, so $x = 2$ and $y = 3$.

b The entry corresponding to x is 3, and the entry corresponding to y is 2, so $x = 3$ and $y = 2$.

c The entry corresponding to x is 4, and the entry corresponding to y is −3, so $x = 4$ and $y = -3$.

d The entry corresponding to x is 3, and the entry corresponding to y is −2, so $x = 3$ and $y = -2$.

7 Write it as set out, with each row representing players A, B, C, D and E respectively, and columns showing points, rebounds and assists respectively.

$$\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 60 \\ 0 & 1 & 2 \end{bmatrix}$$

Solutions to Exercise 18B

1 Add the corresponding entries.

$$\mathbf{X}+\mathbf{Y}=\begin{bmatrix}1+3\\-2+0\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}$$

Double each entry.

$$2\mathbf{X}=\begin{bmatrix}2\times 1\\2\times -2\end{bmatrix}=\begin{bmatrix}2\\-4\end{bmatrix}$$

Multiply each entry in **Y** by 4 and add the corresponding entry for **X**.

$$4\mathbf{Y}+\mathbf{X}=\begin{bmatrix}4\times 3+1\\4\times 0+-2\end{bmatrix}=\begin{bmatrix}13\\-2\end{bmatrix}$$

Subtract corresponding entries.

$$\mathbf{X}-\mathbf{Y}=\begin{bmatrix}1-3\\-2-0\end{bmatrix}=\begin{bmatrix}-2\\-2\end{bmatrix}$$

Multiply each entry by −3.

$$-3\mathbf{A}=\begin{bmatrix}-3\times 1 & -3\times -1\\-3\times 2 & -3\times 3\end{bmatrix}$$

$$=\begin{bmatrix}-3 & 3\\-6 & -9\end{bmatrix}$$

Add **B** to the previous answer.

$$-3\mathbf{A}+\mathbf{B}=\begin{bmatrix}-3 & 3\\-6 & -9\end{bmatrix}+\begin{bmatrix}4 & 0\\-1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1 & 3\\-7 & -7\end{bmatrix}$$

2

$$2\mathbf{A}=\begin{bmatrix}2 & -2\\0 & 4\end{bmatrix}$$

$$-3\mathbf{A}=\begin{bmatrix}-3 & 3\\0 & -6\end{bmatrix}$$

$$-6\mathbf{A}=\begin{bmatrix}-6 & 6\\0 & -12\end{bmatrix}$$

3 **a** As the matrices have the same dimensions, corresponding terms can be added. They will simply be added in the opposite order.
Since the commutative law holds true for numbers, all corresponding entries in the added matrices terms will be equal, so the matrices will be equal.

b As the matrices have the same dimensions, corresponding terms can be added. The first matrix will add the first two numbers, then the third, and the second matrix will add the second and third numbers frst, then add the result to the first number.
Since the associative law holds true for numbers, all corresponding entries in the added matrices terms will be equal, so the matrices will be equal.

4 **a** Multiply each entry by 2.

$$2\mathbf{A}=\begin{bmatrix}6 & 4\\-4 & -4\end{bmatrix}$$

b Multiply each entry by 3.

$$3\mathbf{B}=\begin{bmatrix}0 & -9\\12 & 3\end{bmatrix}$$

c Add answers to **a** and **b**.

$$2\mathbf{A}+3\mathbf{B}=\begin{bmatrix}6 & 4\\-4 & -4\end{bmatrix}+\begin{bmatrix}0 & -9\\12 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5\\8 & -1\end{bmatrix}$$

d Subtract **a** fom **b**.

$$3\mathbf{B}-2\mathbf{A}=\begin{bmatrix}0 & -9\\12 & 3\end{bmatrix}-\begin{bmatrix}6 & 4\\-4 & -4\end{bmatrix}$$

$$=\begin{bmatrix}-6 & -13\\16 & 7\end{bmatrix}$$

5 **a** Add corresponding entries.

$$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}+\begin{bmatrix}-1 & 1\\2 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$$

b Triple entries in **Q**, then add to corresponding entries in **P**.

$$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}+\begin{bmatrix}-3 & 3\\6 & 0\end{bmatrix}=\begin{bmatrix}-2 & 3\\6 & 3\end{bmatrix}$$

c Double entries in **P**, then subtract **Q** and add **R**.

$$\begin{bmatrix}2 & 0\\0 & 6\end{bmatrix}-\begin{bmatrix}-1 & 1\\2 & 0\end{bmatrix}+\begin{bmatrix}0 & 4\\1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3 & 3\\-1 & 7\end{bmatrix}$$

6 a If $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$, then $2\mathbf{A} - \mathbf{B} = 3\mathbf{X}$

$3\mathbf{X} = 2\mathbf{A} - \mathbf{B}$

$$\mathbf{X} = \frac{2}{3}\mathbf{A} - \frac{1}{3}\mathbf{B}$$

$$= \frac{2}{3}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix} - \frac{1}{3}\begin{bmatrix}0 & -10\\-2 & 17\end{bmatrix}$$

$$= \begin{bmatrix}\frac{2}{3}\times 3 - \frac{1}{3}\times 0 & \frac{2}{3}\times 1 - \frac{1}{3}\times -10\\ \frac{2}{3}\times -1 - \frac{1}{3}\times 2 & \frac{2}{3}\times 4 - \frac{1}{3}\times -17\end{bmatrix}$$

$$= \begin{bmatrix}2 & 4\\0 & -3\end{bmatrix}$$

b If $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$ then $2\mathbf{Y} = 2\mathbf{B} - 3\mathbf{A}$

$$\mathbf{Y} = \mathbf{B} - 1\frac{1}{2}\mathbf{A}$$

$$= \begin{bmatrix}0 & -10\\-2 & 17\end{bmatrix} - 1\frac{1}{2}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix}$$

$$= \begin{bmatrix}0 - \frac{3}{2}\times 3 & -10 - \frac{3}{2}\times 1\\ -2 - \frac{3}{2}\times -1 & 17 - \frac{3}{2}\times 4\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{9}{2} & -\frac{23}{2}\\ -\frac{1}{2} & 11\end{bmatrix}$$

7

$\underset{\sim}{X} + \underset{\sim}{Y}$

$$= \begin{bmatrix}150+160 & 90+90 & 100+120 & 50+40\\100+100 & 0+0 & 75+50 & 0+0\end{bmatrix}$$

$$= \begin{bmatrix}310 & 180 & 220 & 90\\200 & 0 & 125 & 0\end{bmatrix}$$

The matrix represents the total production at two factories in two successive weeks.

Solutions to Exercise 18C

1 $\mathbf{AX} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 2 + -2 \times -1 \\ -1 \times 2 + 3 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

$\mathbf{BX} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 3 \times 2 + 2 \times -1 \\ 1 \times 2 + 1 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\mathbf{AY} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 1 + -2 \times 3 \\ -1 \times 1 + 3 \times 3 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 2 + 0 \times -1 \\ 0 \times 2 + 1 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\mathbf{AC} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 2 + -2 \times 1 & 1 \times 1 + -2 \times 1 \\ -1 \times 2 + 3 \times 1 & -1 \times 1 + 3 \times 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\mathbf{CA} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 1 + 1 \times -1 & 2 \times -2 + 1 \times 3 \\ 1 \times 1 + 1 \times -1 & 1 \times -2 + 1 \times 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Use $\mathbf{AC} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$(\mathbf{AC})\mathbf{X} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 0 \times 2 + -1 \times -1 \\ 1 \times 2 + 2 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Use $\mathbf{BX} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\mathbf{C}(\mathbf{BX}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 4 + 1 \times 1 \\ 1 \times 4 + 1 \times 1 \end{bmatrix}$

$= \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

$\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 1 + -2 \times 0 & 1 \times 0 + -2 \times 1 \\ -1 \times 1 + 3 \times 0 & -1 \times 0 + 3 \times 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 3 + 0 \times 1 & 1 \times 2 + 0 \times 1 \\ 0 \times 3 + 1 \times 1 & 0 \times 2 + 1 \times 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$$\mathbf{AB} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1\times 3 + -2\times 1 & 1\times 2 + -2\times 1 \\ -1\times 3 + 3\times 1 & -1\times 2 + 3\times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{BA} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3\times 1 + 2\times -1 & 3\times -2 + 2\times 3 \\ 1\times 1 + 1\times -1 & 1\times -2 + 1\times 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1\times 1 + -2\times -1 & 1\times -2 + -2\times 3 \\ -1\times 1 + 3\times -1 & -1\times -2 + 3\times 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$$
$$\mathbf{B}^2 = \mathbf{BB} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3\times 3 + 2\times 1 & 3\times 2 + 2\times 1 \\ 1\times 3 + 1\times 1 & 1\times 2 + 1\times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$
Use $\mathbf{CA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\mathbf{A}(\mathbf{CA})$
$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1\times 1 + -2\times 0 & 1\times -1 + -2\times 1 \\ -1\times 1 + 3\times 0 & -1\times -1 + 3\times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$
Use $\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

$$\mathbf{A}^2\mathbf{C} = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3\times 2 + -8\times 1 & 3\times 1 + -8\times 1 \\ -4\times 2 + 11\times 1 & -4\times 1 + 11\times 1 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -5 \\ 3 & 7 \end{bmatrix}$$

2 a A product is defined only if the number of columns in the first matrix equals the number of rows of the second.
A has 2 columns and **Y** has 2 rows, so **AY** is defined.
Y has 1 column and **A** has 2 rows, so **YA** is not defined.
X has 1 column and **Y** has 2 rows, so **XY** is not defined.
X has 1 column and 2 rows, so $\mathbf{X}^2$ is not defined.
C has 2 columns and **I** has 2 rows, so **CI** is defined.
X has 1 column and **I** has 2 rows, so **XI** is not defined.

3 $$\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2\times 0 + 0\times -3 & 2\times 0 + 0\times 2 \\ 0\times 0 + 0\times -3 & 0\times 0 + 0\times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4 $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5 No, because **Q.2** part **b** shows that **AB** can equal **O**, and $\mathbf{A} \neq \mathbf{O}, \mathbf{B} \neq \mathbf{O}$.

6 One possible answer is $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

7 $\mathbf{LX} = [2 \;\; -1]\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$= [2 \times 2 + -1 \times -3] = [7]$

$\mathbf{XL} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}[2 \;\; -1]$

$= \begin{bmatrix} 2 \times 2 & 2 \times -1 \\ -3 \times 2 & -3 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

8 A product is defined only if the number of columns in the first matrix equals the number of rows of the second.
This can only happen if $m = n$, in which case both products will be defined.

9

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \begin{bmatrix} a \times d + b \times -c & a \times -b + b \times a \\ c \times d + d \times -c & c \times -b + d \times a \end{bmatrix}$

$= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For the equations to be equal, all corresponding entries must be equal, therefore $ad - bc = 1$.
When written in reverse order, we get

$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} d \times a + -b \times c & d \times b + -b \times d \\ -c \times a + a \times c & -c \times b + a \times d \end{bmatrix}$

$= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

since $ad - bc = 1$.

10 We can use any values of a, b, c and d as long as $ad - bc = 1$.
For example, $a = 5, d = 2, b = 3, c = 3$ satisfy $ad - bc = 1$ and give

$\mathbf{AB} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Other values could be chosen.

11 One possible answer.

$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$

$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 + 0 & 2 + 1 \\ 4 + 2 & 3 + 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 6 & 6 \end{bmatrix}$

$\mathbf{B} + \mathbf{C} = \begin{bmatrix} 0 + -1 & 1 + 2 \\ 2 + -2 & 3 + 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 \times -1 + 2 \times 0 & 1 \times 3 + 2 \times 4 \\ 4 \times -1 + 3 \times 0 & 4 \times 3 + 3 \times 4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$

$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 0 + 2 \times 2 & 1 \times 1 + 2 \times 3 \\ 4 \times 0 + 3 \times 2 & 4 \times 1 + 3 \times 4 \end{bmatrix}$

$= \begin{bmatrix} 4 & 7 \\ 6 & 13 \end{bmatrix}$

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times -1 + 2 \times -2 & 1 \times 2 + 2 \times 1 \\ 4 \times -1 + 3 \times -2 & 4 \times 2 + 3 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 \\ -10 & 11 \end{bmatrix}$$

$$\mathbf{AB} + \mathbf{AC} = \begin{bmatrix} 4 & 7 \\ 6 & 13 \end{bmatrix} + \begin{bmatrix} -5 & 4 \\ -10 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + -5 & 7 + 4 \\ 6 + -10 & 13 + 11 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$$

$(\mathbf{B} + \mathbf{C})\mathbf{A}$

$$= \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \times 1 + 3 \times 4 & -1 \times 2 + 3 \times 3 \\ 0 \times 1 + 4 \times 4 & 0 \times 2 + 4 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$$

12 For example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and

$$\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \; \begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \times 1 + 12 \times 2 \\ 2.50 \times 1 + 3.00 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$$

1 × 5 min plus 2 × 12 min means 29 min for one milkshake and two banana splits. The total cost is \$8.50.

$$\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \times 1 + 12 \times 2 & 5 \times 2 + 12 \times 1 & 5 \times 0 + 12 \times 1 \\ 2.5 \times 1 + 3 \times 2 & 2.5 \times 2 + 3 \times 1 & 2.5 \times 0 + 3 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$$

The matrix shows that John spent 29 min and \$8.50, one friend spent 22 min and \$8.00 (2 milkshakes and 1 banana split) while the other friend spent 12 min and \$3.00 (no milkshakes and 1 banana split).

13 $\mathbf{A}^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix}$,

$\mathbf{A}^8 = \begin{bmatrix} -527 & 336 \\ -336 & -527 \end{bmatrix}$

14 $\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$,

$\mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

Solutions to Exercise 18D

1 a $\det(\mathbf{A}) = 2 \times 2 - 1 \times 3$

$= 1$

b $\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

c $\det(\mathbf{B}) = -2 \times 2 - -2 \times 3$

$= 2$

d $\mathbf{B}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$

2 a Determinant $= 3 \times -1 - -1 \times 4 = 1$

$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$

b Determinant $= 3 \times 4 - 1 \times -2 = 14$

$\mathbf{A}^{-1} = \frac{1}{14}\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{7} & -\frac{1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$

c Determinant $= 1 \times k - 0 \times 0 = k$

$\mathbf{A}^{-1} = \frac{1}{k}\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$

d Determinant $= \cos\theta \times \cos\theta$

$- -\sin\theta \times \sin\theta$

$= 1$

since $\cos^2\theta + \sin^2\theta = 1$

$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

3 Suppose $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$

and $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$

Then

$\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}$

4 $\det(\mathbf{A}) = 2 \times -1 - 1 \times 0 = -2$

$\mathbf{A}^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$

$\det(\mathbf{B}) = 1 \times 1 - 0 \times 3 = 1$

$\mathbf{B}^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 1 + 1 \times 3 & 2 \times 0 + 1 \times 1 \\ 0 \times 1 + -1 \times 3 & 0 \times 0 + -1 \times 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$

$\det(\mathbf{AB}) = 5 \times -1 - 1 \times -3 = -2$

$(\mathbf{AB})^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 3 & 5 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

$\mathbf{A}^{-1}\mathbf{B}^{-1}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} \times 1 + \frac{1}{2} \times -3 & \frac{1}{2} \times 0 + \frac{1}{2} \times 1 \\ 0 \times 1 + -1 \times -3 & 0 \times 0 + -1 \times 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times \frac{1}{2} + 0 \times 0 & 1 \times \frac{1}{2} + 0 \times -1 \\ -3 \times \frac{1}{2} + 1 \times 0 & -3 \times \frac{1}{2} + 1 \times -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

5 a $\det(\mathbf{A}) = 4 \times 1 - 3 \times 2 = -2$

$$\mathbf{A}^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$$

b If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, multiply both sides from the left by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$\therefore \mathbf{IX} = \mathbf{X}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \times 3 + \frac{3}{2} \times 1 & -\frac{1}{2} \times 4 + \frac{3}{2} \times 6 \\ 1 \times 3 + -2 \times 1 & 1 \times 4 + -2 \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$$

c If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, multiply both sides from the right by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}\mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y}$$

$$= \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times -\frac{1}{2} + 4 \times 1 & 3 \times \frac{3}{2} + 4 \times -2 \\ 1 \times -\frac{1}{2} + 6 \times 1 & 1 \times \frac{3}{2} + 6 \times -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$$

6 a If $\mathbf{AX} + \mathbf{B} = \mathbf{C}$ then $\mathbf{AX} = \mathbf{C} - \mathbf{B}$

$$\therefore \quad \mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$\det(\mathbf{A}) = 3 \times 6 - 2 \times 1 = 16$

$$\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}$$

If $\mathbf{AX} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$, multiply both sides from the left by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$\therefore \mathbf{IX} = \mathbf{X}$

$$= \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} \times -1 + -\frac{1}{8} \times 0 & \frac{3}{8} \times 5 + -\frac{1}{8} \times 4 \\ -\frac{1}{16} \times -1 + \frac{3}{16} \times 0 & -\frac{1}{16} \times 5 + \frac{3}{16} \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix}$$

b IF **YA** + **B** = **C** then **YA** = **C** − **B**

$$\therefore \mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

From part **a**, $\mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}$

If $\mathbf{YA} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$, multiply both sides from the right by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix} \mathbf{A}^{-1}$$

$$\mathbf{YI} = \mathbf{Y}$$

$$= \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}$$

$$= \begin{bmatrix} -1 \times \frac{3}{8} + 5 \times -\frac{1}{16} & -1 \times -\frac{1}{8} + 5 \times \frac{3}{16} \\ 0 \times \frac{3}{8} + 4 \times -\frac{1}{16} & 0 \times -\frac{1}{8} + 4 \times \frac{3}{16} \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} -\frac{11}{16} & \frac{17}{16} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

7 **A** must be $\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$.

det(**A**) $= a_{11} \times a_{22} - 0 \times 0 = a_{11}a_{22}$

det(**A**) $\neq 0$ since $a_{11} \neq 0$ and $a_{22} \neq 0$ and the product of two non-zero numbers cannot be zero.

∴ **A** is regular.

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22}} \begin{bmatrix} a_{22} & 0 \\ 0 & a_{11} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{bmatrix}$$

8 If **A** is invertible, it will have an inverse, $\mathbf{A}^{-1}$. Multiply both sides of the equation **AB** = 0 from the left by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}0$$

$$\therefore \quad \mathbf{IB} = 0$$

$$\mathbf{B} = 0$$

9 Let **A** be any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If the determinant is n, then the inverse of **A** is given by $\frac{1}{n}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{n}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$a = \frac{d}{n} \text{ and } d = \frac{a}{n}$$

Substituting for d, $a = \dfrac{a \div n}{n} = \dfrac{a}{n^2}$

This gives $n^2 = 1$, or $n = \pm 1$.

If $n = 1$, $a = d$ and $-b = b$, which gives $b = 0$ and similarly $c = 0$.

det(**A**) $= ad = a^2 = 1$

This leads to two matrices, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

If $n = -1$, $a = -d$; there are no restrictions on b and c but the determinant $= ad - bc = -1$.

$\therefore \quad a^2 + bc = 1$ (since $a = -d$)

If $b = 0$, $a = \pm 1$, giving $\begin{bmatrix} \pm 1 & 0 \\ c & \mp 1 \end{bmatrix}$,

which can be written $\begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}$ or

$\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}$.

If $b \neq 0$, $a^2 + bc = 1$ gives $c = \dfrac{1 - a^2}{b}$,

giving $\begin{bmatrix} a & b \\ \dfrac{1 - a^2}{b} & -a \end{bmatrix}$, which includes

the cases $\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix}$ when

$a = \pm 1$.

10 $a = \pm\sqrt{2}$

11

$$\det(\mathbf{A}) = \frac{1}{n^2 + 2n} - \frac{1}{n^2 + 2n + 1}$$
$$= \frac{1}{n(n+1)^2(n+2)}$$

Therefore

$$\mathbf{A}^{-1} = n(n+1)^2(n+2)\begin{bmatrix} \frac{1}{n+2} & -\frac{1}{n} \\ -\frac{1}{n+1} & \frac{1}{n} \end{bmatrix}$$
$$= \begin{bmatrix} n(n+1)^2 & -(n+1)^2(n+2) \\ -n(n+1)(n+2) & (n+1)^2(n+2 \end{bmatrix}$$

All the entries are integers

Solutions to Exercise 18E

1 First find the inverse of **A**.

det(**A**) $= 3 \times -1 - -1 \times 4 = 1$

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$$

a If $\mathbf{AX} = \mathbf{K}$ then $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K}$

$\therefore \mathbf{IX} = \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \times -1 + 1 \times 2 \\ -4 \times -1 + 3 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

b If $\mathbf{AX} = \mathbf{K}$ then $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K}$

$\therefore \mathbf{IX} = \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \times -2 + 1 \times 3 \\ -4 \times -2 + 3 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 17 \end{bmatrix}$$

2 a $\begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

Determinant $= -2 \times 1 - 4 \times 3 = -14$

$$\text{Inverse} = \frac{1}{-14}\begin{bmatrix} 1 & -4 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{14} \times 6 + \frac{2}{7} \times 1 \\ \frac{3}{14} \times 6 + \frac{1}{7} \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{7} \\ \frac{10}{7} \end{bmatrix}$$

$$x = -\frac{1}{7},\ y = \frac{10}{7}$$

b $\begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Determinant $= -1 \times 4 - 2 \times -1 = -2$

$$\text{Inverse} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \times -1 + 1 \times 2 \\ -\frac{1}{2} \times -1 + \frac{1}{2} \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ \frac{3}{2} \end{bmatrix}$$

$$x = 4,\ y = \frac{3}{2} \text{ or } 1.5$$

3 Solve the simultaneous equations

$2x - 3y = 7$

$3x + y = 5$

$$\begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

Determinant $= 2 \times 1 - -3 \times 3 = 11$

$$\text{Inverse} = \frac{1}{11}\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 1 \times 7 + 3 \times 5 \\ -3 \times 7 + 2 \times 5 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 22 \\ -11 \end{bmatrix}$$

$x = 2,\ y = -1$

The point of intersection is $(2,\ -1)$.

4 If x is the number of books they are buying and y is the number of CDs they are buying, then the following equations apply.

$4x + 4y = 120$

$5x + 3y = 114$

$$\begin{bmatrix} 4 & 4 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 120 \\ 114 \end{bmatrix}$$

Determinant $= 4 \times 3 - 4 \times 5 = -8$

$$\text{Inverse} = \frac{1}{-8}\begin{bmatrix} 3 & -4 \\ -5 & 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3 & 4 \\ 5 & -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3 & 4 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 120 \\ 114 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -3 \times 120 + 4 \times 114 \\ 5 \times 120 + -4 \times 114 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 96 \\ 144 \end{bmatrix}$$

$x = 12,\ y = 18$

One book costs \$12, a CD costs \$18.

5 **a** $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

b det(**A**) $= 2 \times -6 - -3 \times 4 = 0$, so the matrix is singular.

c Yes. For example $x = 0,\ y = -1$ is an obvious solution.

d You should notice that the second equation is simply the first with both sides multiplied by 2.

There is an infinite number of solutions to these equations, just as there is an infinite number of ordered pairs that make $2x - 3y = 3$ a true equation.

6 **a** $\mathbf{A}^{-1}\mathbf{C}$

b $\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{C}$

c $\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$

d $\mathbf{A}^{-1}\mathbf{C} - \mathbf{B}$

e $\mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

f $(\mathbf{A} - \mathbf{B})\mathbf{A}^{-1} = \mathbf{I} - \mathbf{B}\mathbf{A}^{-1}$

Solutions to technology-free questions

1 a

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 2 & 4\end{bmatrix}$$

$$\mathbf{A}-\mathbf{B}=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}-\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 2 & 2\end{bmatrix}$$

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})=\begin{bmatrix}0 & 0\\ 2 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\ 2 & 2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 12 & 8\end{bmatrix}$$

b

$$\mathbf{A}^2=\mathbf{AA}=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 8 & 9\end{bmatrix}$$

$$\mathbf{B}^2=\mathbf{BB}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\mathbf{A}^2-\mathbf{B}^2=\begin{bmatrix}1 & 0\\ 8 & 9\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 8 & 8\end{bmatrix}$$

2 Find the inverse of $\begin{bmatrix}3 & 4\\ 6 & 8\end{bmatrix}$.

Determinant $= 3\times 8 - 4\times 6 = 0$

This is a singular matrix.

If $\mathbf{A}=\begin{bmatrix}x\\ y\end{bmatrix}$, then this corresponds to the simultaneous equations:

$3x + 4y = 8$

$6x + 8y = 16$

The second equation is equivalent to the first, as it is obtained by multiplying both sides of the first by 2.

Thus if $x = a$,

$3a + 4y = 8$

$4y = 8 - 3a$

$y = 2 - \dfrac{3a}{4}$

The matrices may be expressed as $\begin{bmatrix}a\\ 2-\dfrac{3a}{4}\end{bmatrix}$.

3 a For a product to exist, the number of columns of the first matrix must equal the number of rows of the second. This is true only for **AC**, **CD** and **BE**, so these products exist.

b

$$\mathbf{DA}=\begin{bmatrix}2 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}2\times 1+4\times 3 & 2\times 2+4\times -1\end{bmatrix}=\begin{bmatrix}14 & 0\end{bmatrix}$$

$\det(\mathbf{A}) = 1\times -1 - 2\times 3 = -7$

$$\mathbf{A}^{-1}=\frac{1}{-7}\begin{bmatrix}-1 & -2\\ -3 & 1\end{bmatrix}=\frac{1}{7}\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}\text{ or }\begin{bmatrix}\frac{1}{7} & \frac{2}{7}\\ \frac{3}{7} & -\frac{1}{7}\end{bmatrix}$$

4 $\mathbf{AB} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$

$= \begin{bmatrix} 1\times 1 + -2\times 1 + 1\times 3 & 1\times -4 + -2\times -6 + 1\times -8 \\ -5\times 1 + 1\times 1 + 2\times 3 & -5\times -4 + 1\times -6 + 2\times -8 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$

$\det(\mathbf{C}) = 1\times 4 - 2\times 3 = -2$

$\mathbf{C}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5 Find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Determinant $= 1\times 4 - 2\times 3 = -2$

$$\text{Inverse} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

Multiply by the inverse on the right:

$$\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$$

6 $\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

7 The determinant must be zero.

$1\times x - 2\times 4 = 0$

$x - 8 = 0$

$x = 8$

8 a i $\mathbf{MM} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$

ii $\mathbf{MMM} = \mathbf{MM}(\mathbf{M})$

$= \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix}$

iii

Determinant $= 2\times 3 - -1\times 1 = 7$

$\mathbf{M}^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

b $\mathbf{M}^{-1}\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{M}^{-1}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$= \frac{1}{7}\begin{bmatrix} 14 \\ 7 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$x = 2,\ y = 1$

Solutions to multiple-choice questions

1 B The dimension is number of rows by number of columns, i.e. 4×2.

2 E The matrices cannot be added as they have different dimensions.

3 C

$$\mathbf{D} - \mathbf{C} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -3--3 & 1-1 \\ 2-1 & 3-0 & -1--2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

4 E Multiply every entry by -1.

$$-\mathbf{M} = -\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$$

5 C

$$2\mathbf{M} - 2\mathbf{N} = 2 \times \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} - 2 \times \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 \\ -6 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 8 \\ 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$$

6 A $\mathbf{A} + \mathbf{B}$ will have the same dimension as $\mathbf{A}$ and $\mathbf{B}$, i.e. $m \times n$.

7 E The number of columns of **Q** is not the same as the number of rows of **P**, so they cannot be multiplied.

8 A

$$\text{Determinant} = 2 \times 1 - 2 \times -1$$

$$= 4$$

9 E

$$\text{Determinant} = 1 \times -2 - -1 \times 1$$

$$= -1$$

$$\text{Inverse} = \frac{1}{-1}\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

10 D

$$\mathbf{NM} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \times 0 + 2 \times -3 & 0 \times -2 + 2 \times 1 \\ 3 \times 0 + 1 \times -3 & 3 \times -2 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 2 \\ -3 & -5 \end{bmatrix}$$

Solutions to extended-response questions

1 **a** **i** The equations $2x - 3y = 3$ and $4x + y = 5$ can be written as

$$\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

ii
$$\begin{aligned} \text{Determinant of } \mathbf{A} &= 2 \times 1 - 4 \times (-3) \\ &= 2 + 12 \\ &= 14 \\ \therefore \quad \mathbf{A}^{-1} &= \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix} \end{aligned}$$

iii
$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 18 \\ -2 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 9 \\ -1 \end{bmatrix} \end{aligned}$$

Therefore $x = \frac{9}{7}$ and $y = -\frac{1}{7}$.

iv The two lines corresponding to the equations intersect at $\left(\frac{9}{7}, -\frac{1}{7}\right)$.

b **i** The equations $2x + y = 3$ and $4x + 2y = 8$ can be written as

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

ii
$$\begin{aligned} \text{Determinant of } \mathbf{A} &= 2 \times 2 - 4 \times 1 \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

Since the determinant of **A** equals zero, **A** is a singular matrix and the inverse $\mathbf{A}^{-1}$ does not exist.

c The two lines corresponding to the equations are parallel.

2 **a** The 2×3 matrix is: $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$

The rows correspond to the semesters and the columns to the forms of assessment.

b The percentages of the three components can be represented in the 3×1 matrix:
$$\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$$

c Multiplying the two matrices gives the semester scores.
$$\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 79.2 \\ 80.4 \end{bmatrix}$$
Notice that multiplication of a 2×3 matrix by a 3×1 matrix results in a 2×1 matrix.

d For Chemistry the result is given by the following multiplication.
$$\begin{bmatrix} 86 & 82 & 84 \\ 81 & 80 & 70 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 83.8 \\ 75.2 \end{bmatrix}$$

e The aggregate of the four marks is 318.6. This is below 320.

f Three marks will be required to obtain an aggregate of marks above 320.

3 a The part-time and full-time teachers required for the 4 terms can be shown in a 4×2 matrix. The columns are for the two types of teachers and the rows for the different terms. Hence the matrix is: $\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix}$

b The full-time teachers are paid \$70 an hour and the part-time teachers \$60. This can be represented in the 2×1 matrix: $\begin{bmatrix} 70 \\ 60 \end{bmatrix}$

c The product these two matrices gives the cost per hour for each term.
$$\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix}\begin{bmatrix} 70 \\ 60 \end{bmatrix} = \begin{bmatrix} 820 \\ 800 \\ 1040 \\ 1020 \end{bmatrix}$$
The cost per hour for term 1 is \$820.
The cost per hour for term 2 is \$800.
The cost per hour for term 3 is \$1040.
The cost per hour for term 4 is \$1020.

d For the technical, catering and cleaning staff, the matrix for the 4 terms is the 4×3 matrix: $\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}$

e The rate per hour can be represented in the 3×1 matrix: $\begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix}$

f The cost per hour is given by the product.

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix} = \begin{bmatrix} 270 \\ 270 \\ 480 \\ 480 \end{bmatrix}$$

The cost per hour for term 1 is \$270.
The cost per hour for term 2 is \$270.
The cost per hour for term 3 is \$480.
The cost per hour for term 4 is \$480.

g The total cost per hour is given by the sum of the matrices.

$$\begin{bmatrix} 820 \\ 800 \\ 1040 \\ 1020 \end{bmatrix} + \begin{bmatrix} 270 \\ 270 \\ 480 \\ 480 \end{bmatrix} = \begin{bmatrix} 1090 \\ 1070 \\ 1520 \\ 1500 \end{bmatrix}$$

The cost per hour for term 1 is \$1090.
The cost per hour for term 2 is \$1070.
The cost per hour for term 3 is \$1520.
The cost per hour for term 4 is \$1500.

4 **a** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

$\det(\mathbf{A}) = ad - bc$ and $\det(\mathbf{B}) = eh - fg$.

Then $\det(\mathbf{A})\det(\mathbf{B}) = (ad - bc)(eh - fg)$

$$= adeh + bcfg - adfg - bceh$$

Furthermore $\mathbf{AB} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$.

and $\det(\mathbf{AB}) = adeh + bcfg - adfg - bceh$

$\therefore \det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

b A 2×2 matrix is invertible if and only if its determinant is non-zero. Hence if **A** and **B** are invertible then so is **AB**

Chapter 19 – Transformations of the plane

Solutions to Exercise 19A

1 **a** $(2, -4) \to (2 + (-4), 2 - (-4)) = (-2, 6)$

b $(2, -4) \to (2(2) + 3(-4), 3(2) - 4(-4)) = (-8, 22)$

c $(2, -4) \to (3(2) - 5(-4), 2) = (26, 2)$

d $(2, -4) \to (-4, -2)$

2 **a** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \times 2 + 1 \times 3 \\ 1 \times 2 + 0 \times 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
Therefore $(2, 3) \to (3, 2)$.

b $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \times 2 + 0 \times 3 \\ 0 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$
Therefore $(2, 3) \to (-4, 9)$.

c $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 2 \times 3 \\ 0 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$
Therefore $(2, 3) \to (8, 3)$.

d $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 1 \times 3 \\ 1 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$
Therefore $(2, 3) \to (7, 11)$.

3 **a** The linear transformation can be written as

$$x' = 2x + 3y$$
$$y' = 4x + 5y$$

so the transformation matrix is

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

b The linear transformation can be written as

$$x' = 11x - 3y$$
$$y' = 3x - 8y$$

so the transformation matrix is

$$\begin{bmatrix} 11 & -3 \\ 3 & -8 \end{bmatrix}.$$

c The linear transformation can be written as

$$x' = 2x + 0y$$
$$y' = x - 3y$$

so the transformation matrix is

$$\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}.$$

d The linear transformation can be written as

$$x' = 0x + 1y$$
$$y' = -1x + 0y$$

so the transformation matrix is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

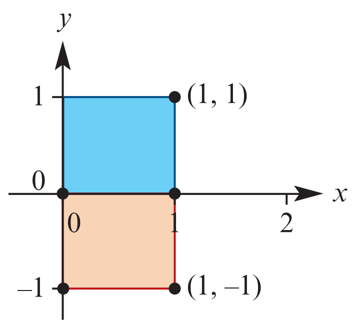
4 For each of these questions we multiply the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex.

a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \end{bmatrix}$

The columns then give the required points:

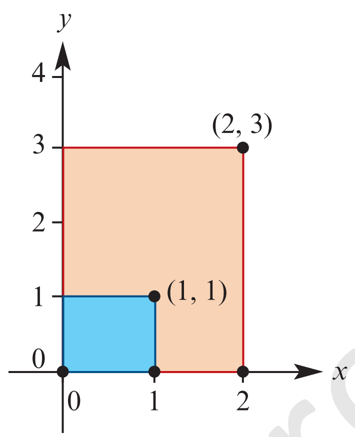
$$(0, 0), (0, -1), (1, 0), (1, -1).$$

The square is shown in blue, and its image in red.



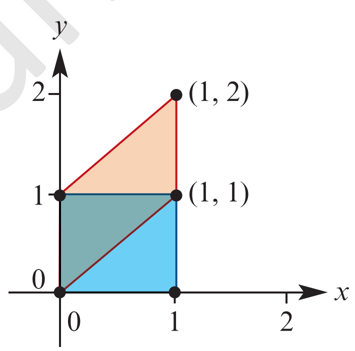
b $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

The columns then give the required points:
$(0, 0), (2, 0), (0, 3), (2, 3)$. The square is shown in blue, and its image in red.



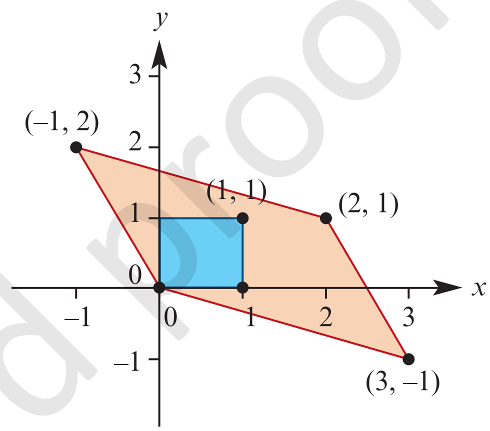
c $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

The columns then give the required points:
$(0, 0), (1, 1), (0, 1), (1, 2)$. The square is shown in blue, and its image in red.

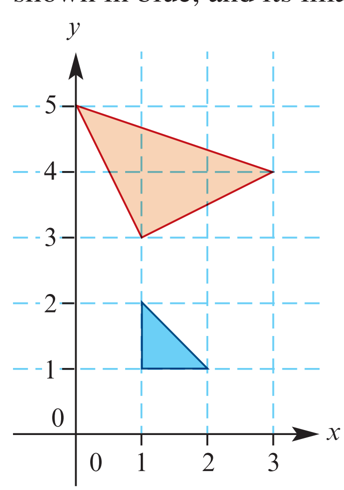


d $\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =$
$\begin{bmatrix} 0 & -1 & 3 & 2 \\ 0 & 2 & -1 & 1 \end{bmatrix}$

The columns then give the required points:
$(0, 0), (-1, 2), (3, -1), (2, 1)$. The original triangle is shown in blue, and its image in red.



5 We multiply the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex. This gives, $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 5 & 4 \end{bmatrix}$
The columns then give the required points:
$(1, 3), (0, 5), (3, 4)$. The square is shown in blue, and its image in red.



6 The image of $(1, 0)$ is $(3, 4)$. The image of $(0, 1)$ is $(5, 6)$. Write these images as the column of a matrix, $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

Therefore $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$
so that $(-2, 4) \to (14, 16)$.

7 The image of $(1, 0)$ is $(-3, 2)$. The image of $(0, 1)$ is $(1, -1)$. Write these images as the column of a matrix, $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}$.

Therefore $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$
so that $(2, 3) \to (-3, 1)$.

8 **a** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$.

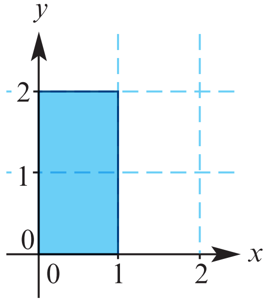
b $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$.

c $\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix}$.

Solutions to Exercise 19B

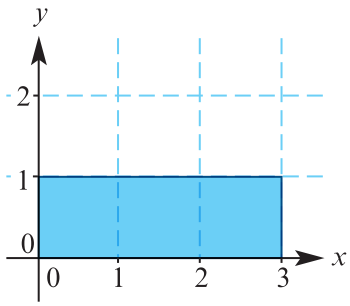
1 a i $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

ii



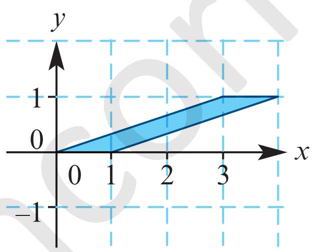
b i $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

ii



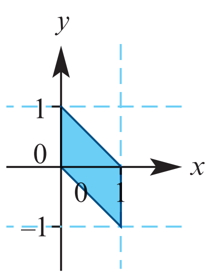
c i $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

ii



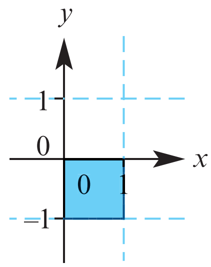
d i $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

ii



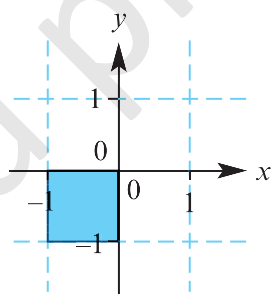
e i $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

ii



f i $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

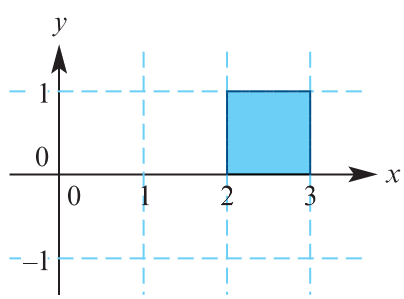
ii



2 a i

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix},$$

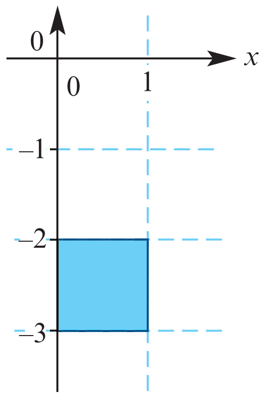
ii



b i

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y-3 \end{bmatrix},$$

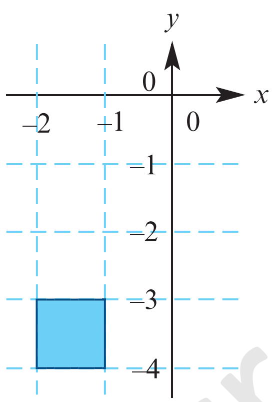
ii



c **i**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} x-2 \\ y-4 \end{bmatrix},$$

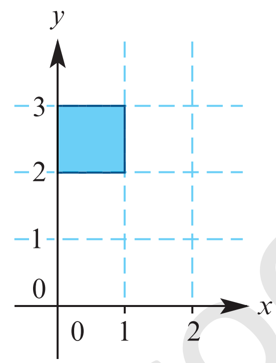
ii



d **i**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y+2 \end{bmatrix},$$

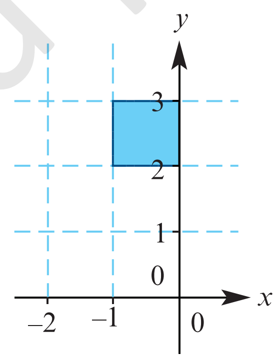
ii



e **i**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+2 \end{bmatrix},$$

ii



Solutions to Exercise 19C

1 a $\begin{bmatrix}\cos 270 & -\sin 270\\ \sin 270 & \cos 270\end{bmatrix}$

$= \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

b $\begin{bmatrix}\cos 30° & -\sin 30°\\ \sin 30° & \cos 30°\end{bmatrix}$

$= \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$

c $\begin{bmatrix}\cos(-60°) & -\sin(-60°)\\ \sin(-60°) & \cos(-60°)\end{bmatrix}$

$= \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

d $\begin{bmatrix}\cos(-135°) & -\sin(-135°)\\ \sin(-135°) & \cos(-135°)\end{bmatrix}$

$= \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$

2 a The rotation matrix is
$\begin{bmatrix}\cos 90° & -\sin 90°\\ \sin 90° & \cos 90°\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$
Therefore

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2\\ 3\end{bmatrix} = \begin{bmatrix}-3\\ 2\end{bmatrix}$$

so that $(2, 3) \to (-3, 2)$.

b The rotation matrix is

$\begin{bmatrix}\cos(-45°) & -\sin(-45°)\\ \sin(-45°) & \cos(-45°)\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

Therefore

$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}2\\ 3\end{bmatrix}$

$= \begin{bmatrix}\frac{5\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}$

so that $(2, 3) \to \left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

3 a $\begin{bmatrix}\cos 90° & \sin 90°\\ \sin 90° & -\cos 90°\end{bmatrix}$

$= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

b $\begin{bmatrix}\cos 120° & \sin 120°\\ \sin 120° & -\cos 120°\end{bmatrix}$

$= \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

c $\begin{bmatrix}\cos(-60°) & \sin(-60°)\\ \sin(-60°) & -\cos(-60°)\end{bmatrix}$

$= \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$

d $\begin{bmatrix}\cos 30° & \sin 30°\\ \sin 30° & -\cos 30°\end{bmatrix}$

$= \begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{\sqrt{3}}{2}\end{bmatrix}$

4 a Since

$$\tan\theta = 3 = \frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 3 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{10}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = \frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1\\ &= \frac{2}{10} - 1\\ &= -\frac{4}{5},\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}}\\ &= \frac{3}{5}.\end{aligned}$$

Therefore the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}-\frac{4}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{bmatrix}.$$

b Since

$$\tan\theta = 5 = \frac{5}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 5 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{26}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{26}} \text{ and } \sin\theta = \frac{5}{\sqrt{26}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{1}{\sqrt{26}}\right)^2 - 1\\ &= \frac{2}{26} - 1\\ &= -\frac{12}{13},\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\frac{1}{\sqrt{26}}\frac{5}{\sqrt{26}}\\ &= \frac{5}{13}.\end{aligned}$$

Therefore the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}-\frac{12}{13} & \frac{5}{13}\\ \frac{5}{13} & \frac{12}{13}\end{bmatrix}.$$

c Since $\tan\theta = \frac{2}{3}$,
we draw a right angled triangle with opposite and adjacent lengths 2 and 3 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{13}$. Therefore

$$\cos\theta = \frac{3}{\sqrt{13}} \text{ and } \sin\theta = \frac{2}{\sqrt{13}}.$$

We then use the double angle formulas to show that
$\cos 2\theta = 2\cos^2\theta - 1$ There-

$$\begin{aligned}&= 2\left(\frac{3}{\sqrt{13}}\right)^2 - 1\\ &= \frac{18}{13} - 1\\ &= \frac{5}{13},\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\frac{2}{\sqrt{13}}\frac{3}{\sqrt{13}}\\ &= \frac{12}{13}.\end{aligned}$$

fore the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}.$$

d Since

$$\tan\theta = -3 = -\frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 3 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{10}$. Therefore, since $-90° < \theta < 0°$,

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = -\frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that $\cos 2\theta = 2\cos^2\theta - 1$ There-

$$= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1$$

$$= \frac{2}{10} - 1$$

$$= -\frac{4}{5},$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= -2\frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}}$$

$$= -\frac{3}{5}.$$

fore the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

5 a Since

$$\tan\theta = m = \frac{m}{1},$$

we draw a right angled triangle with opposite and adjacent lengths m and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{m^2+1}$. Therefore,

$$\cos\theta = \frac{1}{\sqrt{m^2+1}}$$ We then use the

$$\sin\theta = \frac{m}{\sqrt{m^2+1}}.$$

double angle formulas to show that $\cos 2\theta = 2\cos^2\theta - 1$

$$= 2\left(\frac{1}{\sqrt{m^2+1}}\right)^2 - 1$$

$$= \frac{2}{m^2+1} - 1$$

$$= \frac{1-m^2}{m^2+1},$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\frac{m}{\sqrt{m^2+1}}\frac{1}{\sqrt{m^2+1}}$$

$$= \frac{2m}{m^2+1}.$$

Therefore the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & -\frac{m^2-1}{m^2+1} \end{bmatrix}.$$

b The gradient of the line is $m = 6$. Substituting this into the matrix found above, the reflection matrix is

$$\begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & -\frac{m^2-1}{m^2+1} \end{bmatrix} = \begin{bmatrix} \frac{1-6^2}{6^2+1} & \frac{2\times 6}{6^2+1} \\ \frac{2\times 6}{6^2+1} & -\frac{6^2-1}{6^2+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-35}{37} & \frac{12}{37} \\ \frac{12}{37} & -\frac{35}{37} \end{bmatrix}$$

$$= \frac{1}{37}\begin{bmatrix} -35 & 12 \\ 12 & 35 \end{bmatrix}$$

Therefore the image of (1, 1) can be found by evaluating,

$$\frac{1}{37}\begin{bmatrix} -35 & 12 \\ 12 & 35 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} -23 \\ 47 \end{bmatrix}$$

so that

$$(1,1) \to \left(\frac{-23}{37}, \frac{47}{37}\right).$$

6 a $\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

b To find the image of the unit square we evaluate

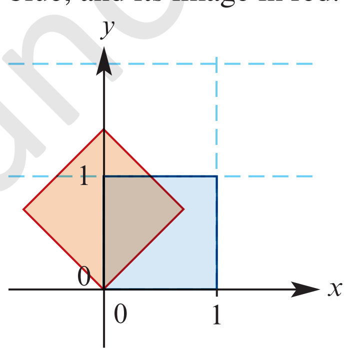
$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2} \end{bmatrix}.$$

The columns then give the required points:

$$(0,0), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), (0, \sqrt{2}).$$

The square is shown in blue, and its image in red.



c To find the overlapping region, we subtract the area of the small upper isosceles triangle from the right half of the red square. The base and height of the small isosceles triangle is $\sqrt{2} - 1$ so that the overlapping area is $A = \frac{1}{2} - \frac{1}{2}(\sqrt{2} - 1)^2$

$$= \frac{1}{2} - \frac{1}{2}(2 - 2\sqrt{2} + 1)$$

$$= \frac{1}{2} - \frac{1}{2}(3 - 2\sqrt{2})$$

$$= \frac{1}{2} - \frac{3}{2} + \sqrt{2}$$

$$= \sqrt{2} - 1.$$

7 a There is no real need to use the rotation matrix for this question. Let O be the origin. We know that length $OA = 1$. Therefore lengths $OB = 1$ and $OC = 1$. Therefore,

$$B = (\cos 120°, \sin 120°) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$C = (\cos 240°, \sin 240°) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

b Triangle ABC is clearly equilateral.

c Its lines of symmetry will be

$y = x \tan 60° = \sqrt{3}x$

$y = 0$

$y = x \tan 300° = -\sqrt{3}x$

Solutions to Exercise 19D

1 The matrix that will reflect the plane in the y-axis is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix that will dilate the result by a factor of 3 from the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Therefore the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$

2 The matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The matrix that will reflect the result in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

3 a The matrix that will reflect the plane in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The matrix that will reflect the plane in the y-axis is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore the matrix of the composition transformation is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b The matrix that will rotate the plane by 180° clockwise is given by

$$\begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the same as the matrix found above.

4 a T_1 : The matrix that will reflect the plane in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

T_2 : The matrix that will dilate the result by a factor of 2 from the y-axis is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore the matrix of T_1 followed by T_2 will be

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

b The matrix of T_2 followed by T_1 will be

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

c No. The order of transformation does not matter in this instance, since the two matrices are the same.

5 a T_1 : The matrix that will rotate the plane by 90° clockwise is given by

$$\begin{bmatrix}\cos(-90°) & -\sin(-90°)\\ \sin(-90°) & \cos(-90°)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

T_2 : The matrix that will reflect the plane in the line $y = x$ is given by

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

Therefore the matrix of T_1 followed by T_2 will be

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.$$

b The matrix of T_2 followed by T_1 will be

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

c Yes. The order of transformation does matter in this instance, as the two matrices are different (the first gives the reflection in the y-axis, the second a reflection in the x-axis).

6 **a**

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-3\\ 5\end{bmatrix}$$

$$= \begin{bmatrix}-x\\ y\end{bmatrix} + \begin{bmatrix}-3\\ 5\end{bmatrix}$$

$$= \begin{bmatrix}-x-3\\ y+5\end{bmatrix}$$

Therefore the transformation is $(x, y) \to (-x - 3, y + 5)$.

b

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-3\\ 5\end{bmatrix}\right)$$

$$= \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x-3\\ y+5\end{bmatrix}$$

$$= \begin{bmatrix}-x+3\\ y+5\end{bmatrix}$$

Therefore the transformation is $(x, y) \to (-x + 3, y + 5)$.

c Yes. The order of transformation does matter in this instance, as the rule for the each composition is different.

7 **a** This is a reflection in the x-axis followed by a dilation from the y-axis by a factor of 2 (or visa versa):

$$\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}.$$

b This is a reflection in the x-axis followed by a dilation from the x-axis by a factor of 3 (or visa versa):

$$\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -3\end{bmatrix}.$$

c This is a reflection in the line $y = x$ followed by a dilation from the x-axis by a factor of 2:

$$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix}.$$

Alternatively, it is a dilation from the y-axis by a factor of 2 followed by a reflection in the line $y = x$:

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix}.$$

d This is a reflection in the line $y = -x$ followed by a dilation from the y-axis by a factor of 2:

$$\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}0 & -2\\ -1 & 0\end{bmatrix}.$$

Alternatively, it is a dilation from the x-axis by a factor of 2 followed by a reflection in the line $y = -x$:

$$\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}0 & -2\\ -1 & 0\end{bmatrix}.$$

8 a The required matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

b The above matrix corresponds to a rotation by angle $\theta = 90°$ since

$$\begin{bmatrix} \cos 90° & \sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

9 We require that:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & sin\theta \end{bmatrix}$$

For these two matrices to be equal, we required that

$$-\sin\theta = \sin\theta$$

$$2\sin\theta = 0$$

$$\sin\theta = 0$$

$\theta = 180°k,$ where $k \in \mathbb{Z}$.

10 a Matrix A^2 will rotate the plane by angle 2θ.

b $$A^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

c Since A^2 will rotate the plane by angle 2θ, another expression for A^2 is

$$A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

Equating the two expressions for A^2 gives

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}.$$

Therefore

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$.

11 a $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} y+1 \\ x+2 \end{bmatrix}$$

Therefore the rule can be written in the form $(x, y) \to (y + 1, x + 2)$ or in the form
$$x' = y + 1$$
$$y' = x + 2.$$

b We have,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y+1 \\ x+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x+2 \\ y+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

This shows the the transformation can be expressed as a reflection in the line $y = x$ followed by a translation in the direction of vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

12 a $$\begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b $\begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

c A 60° rotation followed by a −45° rotation will give a 15° rotation. Therefore, the required matrix is.

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$ The

$$= \begin{bmatrix} \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{2} \end{bmatrix}$$

rotation matrix of 15° is also given by the expression

$$\begin{bmatrix} \cos(15) & -\sin(15) \\ \sin(15) & \cos(15) \end{bmatrix}.$$

Comparing the entries of these two matrices gives

$$\cos 15° = \frac{\sqrt{2}+\sqrt{6}}{4},$$

$$\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4},$$

13 The matrix that will reflect the plane in the line $y = x \tan\phi$ is

$$\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}.$$

The matrix that will reflect the plane in the line $y = x\tan\theta$ is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Therefore the matrix of the composition transformation is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta\cos 2\phi + \sin 2\theta\sin 2\phi & \cos 2\theta\sin 2\phi - \sin 2\theta\cos 2\phi \\ \sin 2\theta\cos 2\phi - \cos 2\theta\sin 2\phi & \cos 2\theta\cos 2\phi + \sin 2\theta\sin 2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta\cos 2\phi + \sin 2\theta\sin 2\phi & -(\cos 2\theta\sin 2\phi - \cos 2\theta\sin 2\phi) \\ \cos 2\theta\sin 2\phi - \cos 2\theta\sin 2\phi & \cos 2\theta\cos 2\phi + \sin 2\theta\sin 2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta - 2\phi) & -\sin(2\theta - 2\phi) \\ \sin(2\theta - 2\phi) & \cos(2\theta - 2\phi). \end{bmatrix}$$

This is a rotation matrix corresponding to angle $2\theta - 2\phi$.

Solutions to Exercise 19E

1 a $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{4\times 1 - 1\times 3}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$

$= \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$

b $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{3\times(-4) - 2\times 1}\begin{bmatrix} -4 & -1 \\ -2 & 3 \end{bmatrix}$

$= -\frac{1}{14}\begin{bmatrix} -4 & -1 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{14} \end{bmatrix}.$

c $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{0\times 4 - 3\times(-2)}\begin{bmatrix} 4 & -3 \\ 2 & 0 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 4 & -3 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}$

d $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{(-1)\times 5 - 3\times(-4)}\begin{bmatrix} 5 & -3 \\ 4 & -1 \end{bmatrix}$

$= \frac{1}{7}\begin{bmatrix} 5 & -3 \\ 4 & -1 \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$

2 a Since the matrix of this linear transformation is

$$A = \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix},$$

the inverse transformation will have matrix

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{5\times(-1) - (-2)\times 2}\begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$

$= \frac{1}{-1}\begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ 2 & -5 \end{bmatrix}$

Therefore the rule of the inverse transformation is

$(x, y) \to (x - 2y, 2x - 5y)$

b Since the matrix of this linear transformation is

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},$$

the inverse transformation will have matrix

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1 \times 0 - (-1) \times 1}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Therefore the rule of the inverse transformation is $(x, y) \to (y, -x + y)$.

3 **a** We need to solve the following equation for X. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore $(-1, 1) \to (1, 1)$.

b We need to solve the following equation for X.

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$X = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Therefore $(-\frac{1}{2}, 1) \to (1, 1)$.

4 We need to find a matrix A such that

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This can be written as a single equation, which we then solve to give

$$A\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}.$$

5 This can be solved in one step by solving the following equation for X.

$$\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} X = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$X = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & -1 \end{bmatrix}$$

The vertices are then given by the columns of matrix X. These are $(0, 0)$, $(-1, -2)$, $(1, 1)$ and $(0, -1)$.

6 **a** The dilation matrix is $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$.

b The inverse transformation will have matrix $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \frac{1}{k \times 1 - 0 \times 0}\begin{bmatrix} 1 & -0 \\ -0 & k \end{bmatrix}$$
$$= \frac{1}{k}\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$
$$= \begin{bmatrix} 1/k & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix will dilate each point from the y-axis by a factor of $1/k$.

7 a The shear matrix is
$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

b The inverse transformation will have matrix
$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1 \times 1 - k \times 0}\begin{bmatrix} 1 & -k \\ -0 & 1 \end{bmatrix}$$
$$= \frac{1}{1}\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}.$$
This matrix will shear each point from the x-direction by a factor of $-k$.

8 a The reflection matrix is
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b The inverse transformation will have matrix
$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1 \times (-1) - 0 \times 0}\begin{bmatrix} -1 & -0 \\ -0 & 1 \end{bmatrix}$$
$$= -1\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= A.$$
This is expected, since two reflections in the same axis will take the return the point (x, y) to its original position.

9 a The reflection matrix is
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

b The inverse transformation will have matrix
$$A^{-1}$$
$$= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{-\cos^2\theta - \sin^2\theta}\begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$
$$= \frac{1}{-1}\begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$
$$= A.$$
This is expected, since two reflections in the same axis will take the return the point (x, y) to its original position.

Solutions to Exercise 19F

1 **a** The matrix of the transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Therefore that $x' = x$ and $y' = -y$. Rearranging gives $x = x'$ and $y = -y''$. Therefore $y = 3x + 1$ becomes, $-y'' = 3x' + 1$ We now

$$y' = -3x' - 1.$$

ignore the apostrophes, so that the transformed equation is

$$y = -3x - 1.$$

b The matrix of the transformation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}.$$

Therefore that $x' = 2x$ and $y' = y$.
Rearranging gives $x = \frac{x'}{2}$ and $y = y''$.
Therefore $y = 3x + 1$ becomes,

$$y' = \frac{x'}{2} + 1.$$

We now ignore the apostrophes, so that the transformed equation is

$$y = \frac{x}{2} + 1.$$

c The matrix of the transformation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}.$$

Therefore that $x' = 2x$ and $y' = 3y$.
Rearranging gives $x = \frac{x'}{2}$ and $y = \frac{y'}{3}$.
Therefore $y = 3x + 1$ becomes,
$\frac{y'}{3} = 3\left(\frac{x'}{2}\right) + 1$ We now ignore the

$$y' = \frac{9x'}{2} + 3.$$

apostrophes, so that the transformed equation is

$$y = \frac{9x}{2} + 3.$$

d The matrix of the transformation is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

Therefore that $x' = -x$ and $y' = -y$.
Rearranging gives $x = -x'$ and $y = -y'$. Therefore $y = 3x + 1$ becomes, $-y' = 3(-x') + 1$ We now

$$-y' = -3x' + 1$$

$$y' = 3x' - 1.$$

ignore the apostrophes, so that the transformed equation is

$$y = 3x - 1.$$

e The matrix of the transformation is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ 3y \end{bmatrix}.$$

Therefore that $x' = -x$ and $y' = 3y$. Rearranging gives $x = -x'$ and $y = \frac{y'}{3}$. Therefore $y = 3x + 1$ becomes, $\frac{y'}{3} = 3(-x') + 1$ We now

$$\frac{y'}{3} = -3x' + 1$$

$$y' = -9x' + 3.$$

ignore the apostrophes, so that the transformed equation is

$$y = -9x + 3.$$

f The matrix of the transformation is

$$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Therefore that $x' = -y$ and $y' = x$. Rearranging gives $x = y'$ and $y = -x'$. Therefore $y = 3x + 1$ becomes, $-x' = 3y' + 1$ We now ignore the

$$3y' = -x' - 1$$

$$y' = \frac{-x' - 1}{3}.$$

apostrophes, so that the transformed equation is

$$y = \frac{-x - 1}{3}.$$

g Firstly, the rotation matrix is

$$\begin{bmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The reflection matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

Therefore that $x' = y$ and $y' = x$. Therefore $y = 3x + 1$ becomes, $x' = 3y' + 1$ We now ignore the

$$3y' = x' - 1$$

$$y' = \frac{x' - 1}{3}.$$

apostrophes, so that the transformed equation is

$$y = \frac{x - 1}{3}.$$

2 a If (x', y') be the coordinates of the image of (x, y), then $x' = 2x$ and $y' = 3y$.

Rearranging gives $x = \frac{x'}{2}$ and $y = \frac{y'}{3}$. Therefore $y = 2 - 3x$ becomes, $\frac{y'}{3} = 2 - 3\left(\frac{x'}{2}\right)$ We now ignore the

$$y' = 6 - \frac{9x'}{2}.$$

apostrophes, so that the transformed equation is

$$y = 6 - \frac{9x}{2}.$$

b If (x', y') be the coordinates of the image of (x, y), then $x' = -y$ and $y' = x$.

Rearranging gives $x = y'$ and $y = -x'$. Therefore $y = 2 - 3x$ becomes, $-x' = 2 - 3y'$ We now ignore the

$$3y' = x' + 2$$

$$y' = \frac{x' + 2}{3}.$$

apostrophes, so that the transformed equation is

$$y = \frac{x + 2}{3}.$$

c Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} x' + 2y' \\ y' \end{bmatrix}$$

so that $x = x' + 2y'$ and $y = y'$. Therefore $y = 2 - 3x$ becomes $y' = 2 - 3(x' + 2y')$. We solve the equation for for y' in terms of x', $y' = 2 - 3(x' + 2y')$ The

$$y' = 2 - 3x' - 6y'$$

$$7y' = 2 - 3x'$$

$$y' = \frac{2 - 3x'}{7}.$$

transformed equation is

$$y = \frac{2 - 3x}{7}.$$

d Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 2x' - 5y' \\ -x' + 3y' \end{bmatrix}$$

so that $x = 2x' - 5y'$ and $y = -x' + 3y'$. Therefore $y = 2 - 3x$ becomes $x' + 3y' = 2 - 3(2x' - 5y')$. We solve the equation for for y' in terms of x', $x' + 3y' = 2 - 3(2x' - 5y')$ The

$$x' + 3y' = 2 - 6x' + 15y'$$

$$12y' = 7x' - 2$$

$$y' = \frac{7x' - 2}{12}.$$

transformed equation is

$$y = \frac{7x - 2}{12}.$$

3 There are many answers. We find a matrix that maps the x-intercept of the first line to the x-intercept of the second line, and likewise for the y-intercepts. Then

$$(1, 0) \to (2, 0) \text{ and } (0, 1) \to (0, 2).$$

Since we have found the images of the standard unit vectors, the matrix that will achieve this results is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

4 There are many answers. Let's find the matrix that maps the x-intercept of the first line to the x-intercept of the second line, and likewise for the y-intercepts. Then

$$(-1,0) \to (3,0) \text{ and } (0,1) \to (0,6).$$

The matrix that will achieve this results is

$$\begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix}.$$

5 Let (x', y') be the coordinates of the image of (x, y). Then the rule for the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$$
$$= \begin{bmatrix} x-1 \\ -y-2 \end{bmatrix}$$

Therefore,
$x' = x - 1$ and $y' = -y - 2$ so that
$x = x' + 1$ and $y = -y' - 2$.
Therefore, the equation $y = x^2 - 1$ becomes $-y' - 2 = (x' + 1)^2 - 1$.
Therefore, t $-y' - 2 = (x' + 1)^2 - 1$

$$-y' = (x' + 1)^2 + 1$$
$$y' = -(x' + 1)^2 - 1$$

The transformed equation is

$$y = -(x + 1)^2 - 1.$$

6 Let (x', y') be the coordinates of the image of (x, y). Then the rule for the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} \text{ Therefore,}$$
$$= \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$
$$= \begin{bmatrix} -x+2 \\ y-3 \end{bmatrix}$$
$$x' = -x + 2 \text{ and} y' = y - 3$$

so that

$$x = -x' + 2 \text{ and } y = y' + 3.$$

Therefore, the equation $y = (x - 1)^2$ becomes $y' + 3 = (-x' + 2 - 1)^2$.
Therefore,
$y' + 3 = (-x' + 2 - 1)^2$

$$y' = (-x' + 1)^2 - 3$$
$$= (-(x' - 1))^2 - 3$$
$$= (x' - 1)^2 - 3$$

The transformed equation is

$$y = (x - 1)^2 - 3.$$

7 The dilation matrix is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

The rotation matrix is given by

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let (x', y') be the coordinates of the image of (x, y). Then the rule for the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3y \\ x \end{bmatrix}.$$

Therefore,

$$x' = -3y \text{ and} y' = x,$$

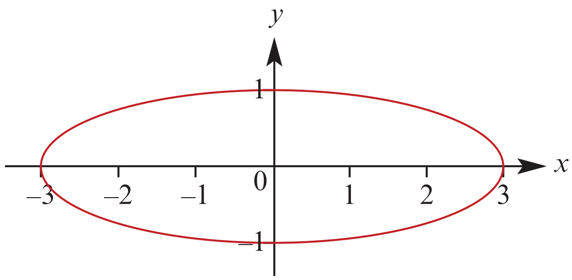
so that

$$y = -\frac{x'}{3} \text{ and } x = y'.$$

Therefore, the equation $x^2 + y^2 = 1$ becomes $(y')^2 + \left(-\frac{x'}{3}\right)^2 = 1$. Ignoring the apostrophes gives,

$$\frac{x^2}{3^2} + y^2 = 1,$$

which is the equation of an ellipse, shown below.



8 Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix}$

$$= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{dx' - by'}{ad - bc} \\ \frac{-cx' + ay'}{ad - bc} \end{bmatrix}$$

so that

$$x = \frac{dx' - by'}{ad - bc} \text{ and } y = \frac{-cx' + ay'}{ad - bc}.$$

Therefore $px + qy = r$ becomes,

$$p\frac{dx' - by'}{ad - bc} + q\frac{-cx' + ay'}{ad - bc} = r.$$

which, although horribly ugly, is most definitely the equation of a line.

9 The matrix of the transformation is

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} x' + y' \\ -x' + y' \end{bmatrix}$$

so that

$$x = \frac{1}{\sqrt{2}}(x' + y'),$$

$$y = \frac{1}{\sqrt{2}}(-x' + y').$$

Therefore $y = \frac{1}{x}$ becomes,

$$\frac{1}{\sqrt{2}}(-x' + y') = \frac{1}{\frac{1}{\sqrt{2}}(x' + y')}.$$

Ignoring the apostrophes, and simplifying this expression gives,

$\frac{1}{\sqrt{2}}(x + y)\frac{1}{\sqrt{2}}(-x + y) = 1$ This is the

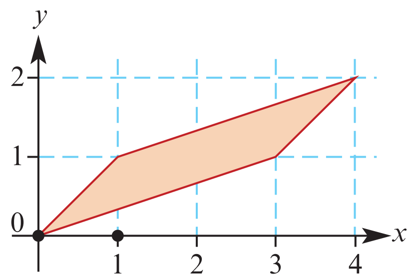
$$\frac{1}{2}(x + y)(y - x) = 1$$

$$y^2 - x^2 = 2$$

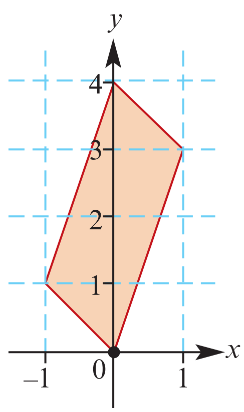
required equation.

Solutions to Exercise 19G

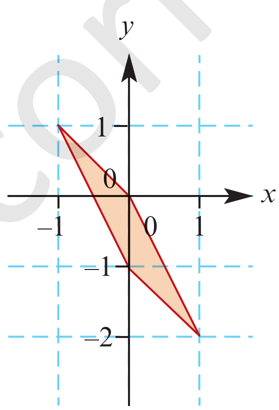
1 **a** The area will be given by
$|\det B| = |3 \times 1 - 1 \times 1| = |2| = 2.$



b The area will be given by
$|\det B| = |(-1) \times 3 - 1 \times 1| = |-4| = 4.$

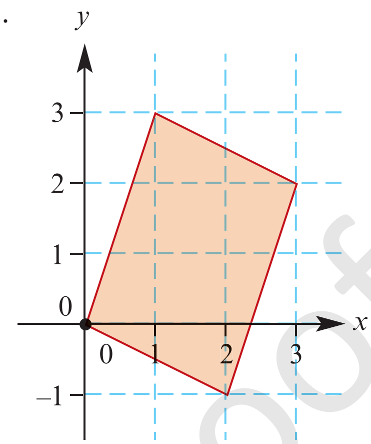


c The area will be given by
$|\det B| = |1 \times 1 - (-1) \times (-2)| =$
$|-1| = 1.$

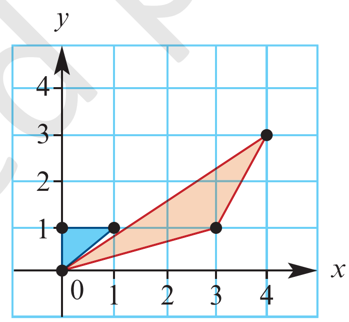


d The area will be given by
$|\det B| = |2 \times 3 - 1 \times (-1)| = |6 + 1| =$

7.



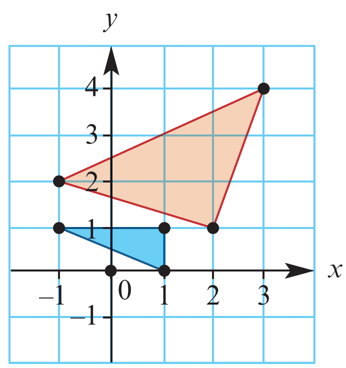
2 **a** The original triangle is shown in blue, and its image is in red.



b The area of the original triangle is $\frac{1}{2}$. Therefore the area of the image will be given by,

$$\text{Area of Image} = |\det B| \times \text{Area of Region}$$
$$= |1 \times 1 - 3 \times 2| \times \frac{1}{2}$$
$$= |-5| \times \frac{1}{2}$$
$$= 2.5.$$

3 **a** The original triangle is shown in blue, and its image is in red.



b The area of the original triangle is 1. Therefore the area of the image will be given by,

$$\begin{aligned}\text{Area of Image} &= |\det B| \times \text{Area of Region}\\ &= |2 \times 3 - 1 \times 1| \times 1\\ &= 5.\end{aligned}$$

4 Since the original area is 1 and the area of the image is 6, we have,

$$\begin{aligned}|\det B| \times 1 &= 6\\ |m \times m - 2 \times (-1)| &= 6\\ |m^2 + 2| &= 6\\ m^2 + 2 &= 6 \text{ (since } m^2 + 2 > 0)\\ m^2 &= 4\\ m &= \pm 2.\end{aligned}$$

5 The original area is 1 and the area of the image is 2. Therefore,

$$\begin{aligned}\text{Area of Image} &= |\det B| \times \text{Area of region}\\ 2 &= |m \times m - m \times 1| \times 1\\ 2 &= |m^2 - m|\end{aligned}$$

Therefore either

$$m^2 - m = 2 \text{ or } m^2 - m = -2.$$

In the first case, we have

$$\begin{aligned}m^2 - m - 2 &= 0\\ (m-2)(m+1) &= 0\\ m &= -1, 2.\end{aligned}$$

In the second case, we have $m^2 - m + 2 = 0$. This has no solutions since the discriminant of the quadratic equation is

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= 1^2 - 4(1)(2)\\ &= 1 - 8 < 0.\end{aligned}$$

6 a i If

$$B = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

then

$$|\det B| = |1 \times 1 - k \times 0| = 1.$$

ii If

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

then

$$\begin{aligned}|\det B| &= |\cos\theta\cos\theta - (-\sin\theta)\sin\theta|\\ &= |\cos^2\theta + \sin^2\theta|\\ &= 1.\end{aligned}$$

iii If

$$B = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

then

$$\begin{aligned}|\det B| &= |\cos 2\theta(-\cos 2\theta) - \sin 2\theta \sin 2\theta|\\ &= |-(\cos^2 2\theta + \sin^2 2\theta)|\\ &= |-1|\\ &= 1\end{aligned}$$

b i This transformation is a dilation by a factor k away from the y-axis and a factor of $1/k$ away from the x-axis.

ii We have,

$$\begin{aligned}|\det B| &= |k \times 1/k - 0 \times 0|\\ &= 1\end{aligned}$$

7 **a** We have,

$$|\det B| = |x \times (x+2) - 1 \times (-2)|$$
$$= |x^2 + 2x + 2|$$
$$= |(x^2 + 2x + 1) + 1|$$

(completing the square)

$$= |(x+1)^1 + 1|$$
$$= (x+1)^2 + 1.$$

b The area will be a minimum at the turning point the parabola whose equation is $y = (x+1)^2 + 1$. This occurs when $x = -1$.

8 We require that $|\det B| > 2$ Therefore

$$|4m - 6| > 2.$$

either $4m - 6 > 2$ or $4m - 6 < -2$. In the first case, $m > 2$. In the second case, $m < 1$. Therefore $m > 2$ or $m < 1$.

9 Since $(1, 0) \to (1, 0)$ we can assume that the matrix is of the form $\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$. Since the area is $\frac{1}{2}$, we know that

$$|1 \times c - b \times 0| = \frac{1}{2}$$
$$|c| = \frac{1}{2}$$
$$c = \pm\frac{1}{2}$$

Since $(0, 1) \to (b, c)$, one corner of the rhombus will be given by the second column (written as a coordinate). Moreover, since it is a rhombus, the distance from $(0, 0)$ to (b, c) is 1. Therefore

$$b^2 + c^2 = 1^2$$
$$b^2 + \left(\frac{1}{2}\right)^2 = 1$$
$$b^2 = \frac{3}{4}$$
$$b = \pm\frac{\sqrt{3}}{2}$$

so that the required matrix is

$$\begin{bmatrix} 1 & \pm\frac{\sqrt{3}}{2} \\ 0 & \pm\frac{1}{2} \end{bmatrix}.$$

10 **a** We can assume that $(1, 0) \to (a, c)$ and $(0, 1) \to (b, d)$. Therefore, the required matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

b The area of the original triangle is $\frac{1}{2}$. Therefore the area of the image will be given by,

$$\text{Area of Image} = |\det B| \times \text{Area of Region}$$
$$= |a \times d - b \times c| \times \frac{1}{2}$$
$$= \frac{1}{2}|ad - bc|$$

c If a, b, c, d are all rational numbers then so too is $\frac{1}{2}|ad - bc|$.

d We will assume that the triangle has vertices $O(0, 0)$, $A(a, c)$ and $B(b, d)$. Then the area of the triangle is

$$\frac{1}{2}|ad - bc|. \quad (1)$$

We will find another expression for the area. Since the triangle is equilateral,

$$OB = OA = \sqrt{a^2 + c^2}$$

Using Pythagoras' Theorem, we can show that

$$MB^2 + OM^2 = OB^2$$

$$MB^2 + \left(\frac{1}{2}OA\right)^2 = OA^2$$

$$MB^2 + \frac{1}{4}OA^2 = OA^2$$

$$MB^2 = \frac{3}{4}OA^2$$

$$MB = \frac{\sqrt{3}}{2}OA$$

$$= \frac{\sqrt{3}\sqrt{a^2 + c^2}}{2}.$$

Therefore, another expression for the area is

$$A = \frac{1}{2} \times OA \times MB$$

$$= \frac{1}{2} \times \sqrt{a^2 + c^2} \times \frac{\sqrt{3}\sqrt{a^2 + c^2}}{2}$$

$$= \frac{\sqrt{3}(a^2 + b^2)}{4} \qquad (2)$$

Equating equations (1) and (2) gives,

$$\frac{\sqrt{3}(a^2 + b^2)}{4} = \frac{1}{2}|ad - bc|$$

$$\sqrt{3} = \frac{2|ad - bc|}{a^2 + b^2}$$

Since a, b, c and d are all rational numbers, the right hand side of the above expression is rational. This contradicts the fact that $\sqrt{3}$ is irrational.

Solutions to Exercise 19H

1 Firstly, the matrix that will rotate the plane by 90° clockwise is given by

$$\begin{bmatrix}\cos(-90^\circ) & -\sin(-90^\circ)\\ \sin(-90^\circ) & \cos(-90^\circ)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

Therefore the required transformation is

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x-2\\ y-2\end{bmatrix} + \begin{bmatrix}2\\ 2\end{bmatrix}$$

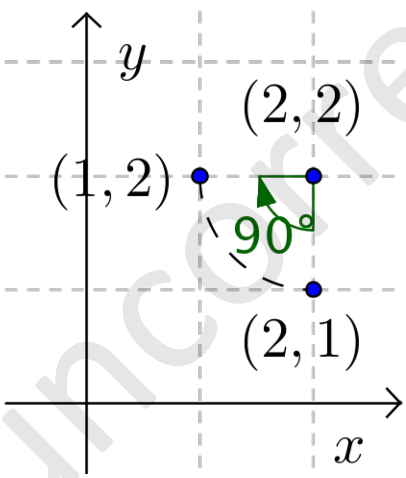
$$= \begin{bmatrix}y-2\\ -x+2\end{bmatrix} + \begin{bmatrix}2\\ 2\end{bmatrix}$$

$$= \begin{bmatrix}y\\ -x+4\end{bmatrix}.$$

We check our answer by finding the image of the point (2, 1). Let $x = 2$ and $y = 1$ so that

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}1\\ -2+4\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix}.$$

Therefore, $(2, 1) \to (1, 2)$, as expected from the diagram shown below.



2 Firstly, the matrix that rotates the plane by 180° about the origin is

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}.$$

Therefore the required transformation is

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x+1\\ y-1\end{bmatrix} + \begin{bmatrix}-1\\ 1\end{bmatrix}$$

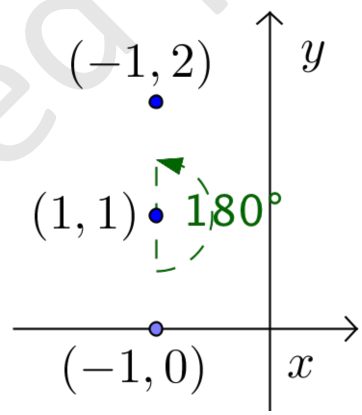
$$= \begin{bmatrix}-x-1\\ -y+1\end{bmatrix} + \begin{bmatrix}-1\\ 1\end{bmatrix}$$

$$= \begin{bmatrix}-x-2\\ -y+2\end{bmatrix}.$$

We check our answer by finding the image of the point $(-1, 0)$. Let $x = -1$ and $y = 0$ so that

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}-(-1)-2\\ -0+2\end{bmatrix} = \begin{bmatrix}-1\\ 2\end{bmatrix}.$$

Therefore, $(-1, 0) \to (-1, 2)$, as expected from the diagram shown below.



3 **a** Firstly, the matrix that reflects the plane in the line $y = x$ is given by

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

Therefore the required transformation is

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y+1\end{bmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix}$$

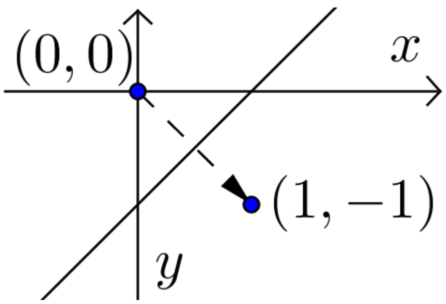
$$= \begin{bmatrix}y+1\\ x\end{bmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix}$$

$$= \begin{bmatrix}y+1\\ x-1\end{bmatrix}.$$

We check our answer by finding the image of the point $(0,0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore, $(0,0) \to (1,-1)$, as expected from the diagram shown below.



b Firstly, the matrix that reflects the plane in the line $y = -x$ is given by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

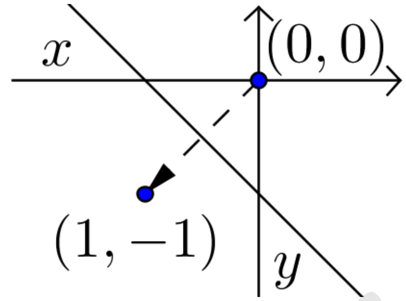
Therefore the required transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} -y-1 \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} -y-1 \\ -x-1 \end{bmatrix}.$$

We check our answer by finding the image of the point $(0,0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0-1 \\ -0-1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Therefore, $(0,0) \to (-1,-1)$, as expected from the diagram shown below.



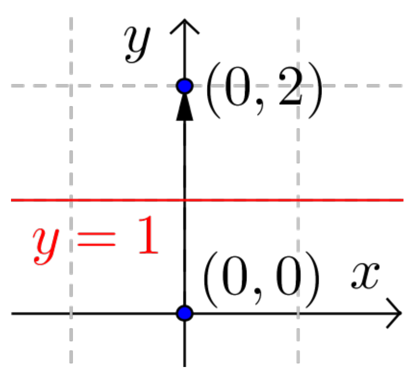
c We will translate the plane 1 unit down so that we can then reflect the plane in the line $y = 0$, that is, the x-axis. We then return the plane to its original position by translating the plane 1 unit up.
Therefore, the required transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} x \\ -y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} x \\ -y+2 \end{bmatrix}.$$

We check our answer by finding the image of the point $(0,0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ -0+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Therefore, $(0,0) \to (0,2)$, as expected from the diagram shown below.



d We will translate the plane 2 units

right so that we can then reflect the plane in the line $x = 0$, that is, the y-axis. We then return the plane to its original position by translating the plane 2 units left.
Therefore, the required transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x+2 \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

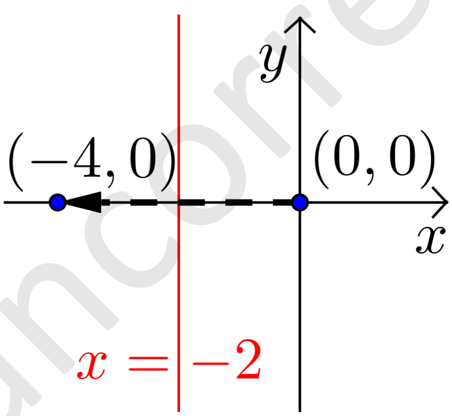
$$= \begin{bmatrix} -x-2 \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -x-4 \\ y \end{bmatrix}.$$

We check our answer by finding the image of the point $(0, 0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0-4 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

Therefore, $(0, 0) \to (-4, 0)$, as expected from the diagram shown below.



4 We will rotate the plane clockwise by angle θ, dilate the point (x, y) by a factor of k from the y-axis, then return the plane to its original position by rotating by angle θ anticlockwise.
Therefore, the required matrix will be

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} k\cos^2\theta + \sin^2\theta & k\cos\theta\sin\theta - \cos\theta\sin\theta \\ k\cos\theta\sin\theta - \cos\theta\sin\theta & k\sin^2\theta + \cos^2\theta \end{bmatrix}.$$

5 We will rotate the plane clockwise by angle θ, project the point (x, y) onto the x-axis, then return the plane to its original position by rotating by angle θ anticlockwise.
Therefore, the required matrix will be

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.$$

6 The transformation that reflects the plane in the line $y = x + 1$ is given by,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} y-1 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} y-1 \\ x+1 \end{bmatrix}.$$

If we then want to reflect the result in the the line $y = x$ we would multiply by the reflection matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This give a transformation whose rule is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y-1 \\ x+1 \end{bmatrix}$$

$$= \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}.$$

This is corresponds to a translation defined by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solutions to Technology-free questions

1 a We let $x = 2$ and $y = 3$ so that

$(2,3) \to (2\times2+3, -2+2\times3) = (7,4)$.

b The matrix of the transformation is given by the coefficients in the rule, that is,

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

c The fastest way to find the image of the unit square is to evaluate

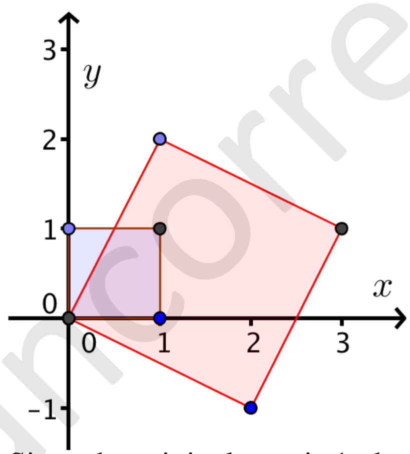
$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & -1 & 2 & 1 \end{bmatrix}.$$

The columns then give the required points:

$(0,0), (2,-1), (1,2), (3,1)$

The square is shown in blue, and its image in red.



Since the original area is 1, the area of the image will be Area $= |ad - bc| = |2\times2 - 1\times(-1)| = 5$

d Since the matrix of this linear transformation is

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$

the inverse transformation will have matrix

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{2\times2 - 1\times(-1)}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

Therefore the rule of the inverse transformation is

$$(x,y) \to \left(\frac{2}{5}x - \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y\right)$$

2 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

c $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

e $\begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

f $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

3 a Since

$$\tan\theta = 3 = \frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 5 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{3^2 + 1^2} = \sqrt{10}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = \frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1\\ &= \frac{2}{10} - 1\\ &= -\frac{4}{5},\\ \sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\frac{1}{\sqrt{10}}\frac{3}{\sqrt{10}}\\ &= \frac{3}{5}.\end{aligned}$$

Therefore the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}-\frac{4}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{bmatrix}.$$

b The image of the point $(2, 4)$ can be found by evaluating,

$$\frac{1}{5}\begin{bmatrix}-4 & 3\\ 3 & 4\end{bmatrix}\begin{bmatrix}2\\ 4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4\\ 22\end{bmatrix}.$$

Therefore, $(2, 4) \to \left(\frac{4}{5}, \frac{22}{5}\right)$.

4 a The matrix that will reflect the plane in the x-axis is given by

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

The matrix that will reflect the plane in the line $y = -x$ is given by

$$\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

Therefore the matrix of the composition transformation is

$$\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

b The matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

The dilation matrix by a factor of 2 from the x-axis is given by

$$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}.$$

Therefore the matrix of the composition transformation is

$$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}.$$

c The matrix that will reflect the plane in the line $y = x$ is given by

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

The matrix that will skew the result by a factor of 2 from the x-axis is given by

$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}.$$

Therefore the matrix of the composition transformation is

$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}.$$

5 **a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} x-3 \\ -y+4 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (x - 3, -y + 4)$.

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x-3 \\ y+4 \end{bmatrix}$$

$$= \begin{bmatrix} x-3 \\ -y-4 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (x - 3, -y - 4)$.

6 **a** The required matrix is

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

b The inverse transformation will have matrix

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

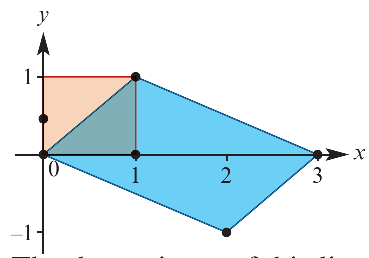
$$= \frac{1}{1\times 1 - 0\times k}\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}.$$

This matrix will shear each point in the y-direction by a factor of $-k$.

7 **a** The unit square is shown in red, and its image in blue.



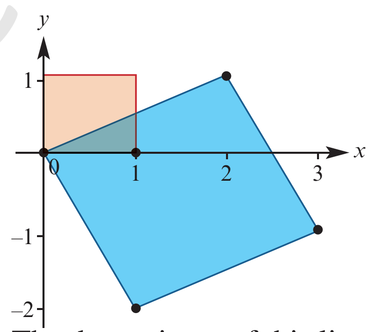
The determinant of this linear transformation is

$$\det B = 2 \times 1 - 1 \times (-1) = 2 + 1 = 3.$$

The unit square has area 1 square unit, so to find the area of its image we evaluate:

$$\begin{aligned} \text{Area of Image} &= |\det B| \times \text{Area of Region} \\ &= 3 \times 1 \\ &= 3 \text{ square units.} \end{aligned}$$

b The unit square is shown in red, and its image in blue.



The determinant of this linear transformation is

$$\det B = 2\times(-2) - 1\times 1 = -4 - 1 = -5.$$

The unit square has area 1 square unit, so to find the area of its image we evaluate:

$$\begin{aligned} \text{Area of Image} &= |\det B| \times \text{Area of Region} \\ &= |-5| \times 1 \\ &= 5 \text{ square units.} \end{aligned}$$

8 **a** We do this is a sequence of three steps:

- translate the plane so that the origin is the centre of rotation.
- rotate the plane about the origin by 90° anticlockwise.
- translate the plane back to its original position.

Firstly the rotation matrix is

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the overall transformation of

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y+1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -y-1 \\ x-1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -y \\ x-2 \end{bmatrix}$$

b To find the image of the point $(2, -1)$.

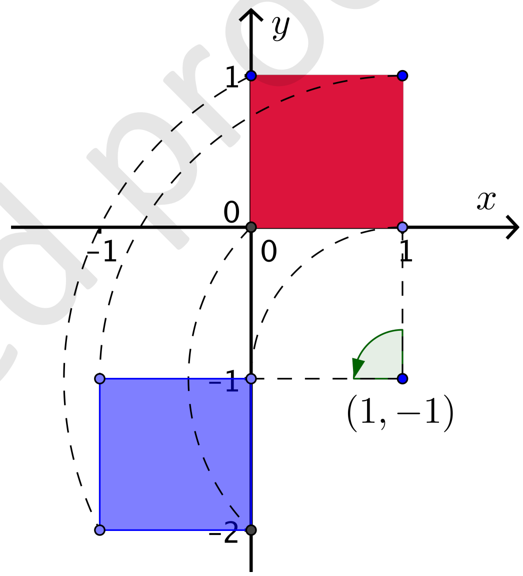
Let $x = 2$ and $y = -1$ so that $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y \\ x-2 \end{bmatrix}$ Therefore

$$= \begin{bmatrix} -(-1) \\ 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$(2, -1) \rightarrow (1, 0)$.

c The unit square is shown in red, and its image after the rotation is in blue.



Solutions to multiple-choice questions

1 B The point $(2, -1)$ maps to the point $(2\times2-3\times(-1), -2+4\times(-1)) = (7, -6)$.

2 D The required transformation is $(x, y) \to (-y, -x)$, which corresponds to matrix

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

3 A The matrix that will dilate the plane by a factor of 2 from the y-axis is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix that will reflect the result in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

4 D The location of the negative entry suggests that this should be a reflection matrix. Indeed, if $\theta = 30°$ then,

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

This corresponds to a reflection in the line $y = x \tan 30°$.

5 C Firstly, matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the required transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x+2 \\ y-3 \end{bmatrix}$$

$$= \begin{bmatrix} -y+3 \\ x+2 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (-y + 3, x + 2)$.

6 A Note that this matrix equal to the product:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This corresponds to a rotation by 180° (or, equivalently, a reflection through the origin) followed by a dilation by a factor of 2 from the x-axis.

7 D Note that this matrix corresponds to a reflection in both the x and y axes. So we draw the graph of $y = (x-1)^2$, then reflect this in each axis. Alternatively, you can show that the transformed graph has equation $y = -(x+1)^2$.

8 E We simply need to find the matrix that has a determinant of 2. Only the last matrix has this property.

9 D Matrix R is a rotation matrix of 40°. Therefore, matrix R^n is a rotation matrix of $40m°$. Since a rotation by any multiple of 360° corresponds to the identity matrix, we need to find the smallest value of m such that $40m$ is a multiple of 360°. Therefore, $m = 9$.

Solutions to extended-response questions

1 a The required rotation matrix is

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

b The required rotation matrix is

$$\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

c A 45° rotation followed by a 30° rotation will give a 75° rotation. Therefore, the required matrix is

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} \end{bmatrix}.$$

d The rotation matrix of 75° is also given by the expression

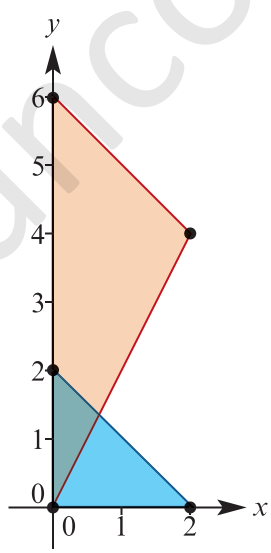
$$\begin{bmatrix} \cos 75° & -\sin 75° \\ \sin 75° & \cos 75° \end{bmatrix}.$$

Comparing the entries of these two matrices gives

$$\cos 75° = \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}+\sqrt{6}}{4},$$

$$\sin 75° = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

2 a The triangle is shown in blue and its image in red.



b The area of the original triangle is

$$\frac{bh}{2} = \frac{2 \times 2}{2} = 2.$$

Therefore the area of the image will be given by,

$$\begin{aligned}\text{Area of Image} &= |\det B| \times \text{Area of Region}\\ &= |1 \times 3 - 0 \times 2| \times 2\\ &= 3 \times \frac{1}{2}\\ &= 6 \text{ square units.}\end{aligned}$$

c When the red figure is revolved around the y-axis, we obtain a figure that is the compound of two cones. The upper cone has base radius $r_1 = 2$ and height $h_1 = 2$. The lower cone has base radius $r = 2$ and height $h = 4$. Therefore the total volume will be

$$\begin{aligned}V &= \frac{1}{3}\pi r_1^2 h_1 + \frac{1}{3}\pi r_2^2 h_2\\ &= \frac{1}{3} \times \pi 2^2 \times 2 + \frac{1}{3} \times \pi 2^2 \times 4\\ &= 8\pi \text{ cubic units.}\end{aligned}$$

3 **a** The matrix of the transformation is obtained by reading off the coefficients in the rule for the linear transformation. That is,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

b This transformation is a shear by a factor of 1 in the x direction.

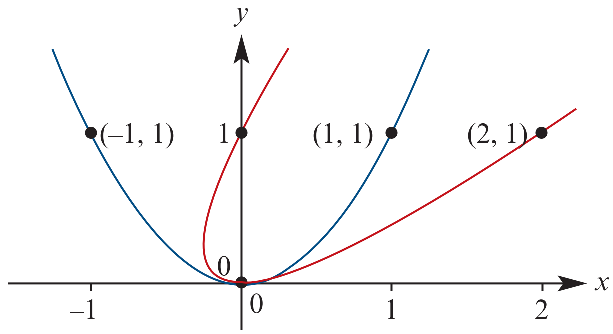
c The image of the points can be found in one step by evaluating,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The columns then give the required points:

$$(0, 0), (2, 1), (0, 1).$$

d The image will be a sheared parabola, shown in red. The original parabola is shown in blue.



4 a The matrix of the transformation is

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

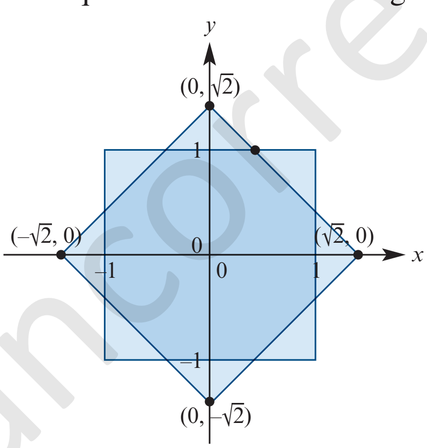
To find the image of the point (1, 1) we multiply,

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}.$$

Therefore $(1, 1) \to (0, \sqrt{2})$. Since this matrix will rotate the square by 45° anticlockwise, the four points must be:

$$(0, \sqrt{2}), (\sqrt{2}, 0), (0, -\sqrt{2}), (-\sqrt{2}, 0).$$

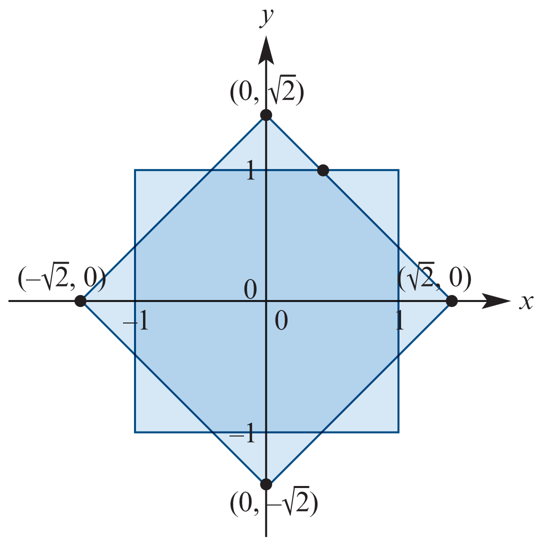
b The square and its rotated image are shown below.



c The area of the shape can be found in many ways. We will find the coordinates of point A shown the in the above diagram. Point A is the intersection of the lines

$$y = 1 \text{ and } x + y = \sqrt{2}.$$

Solving this pair of equations gives $x = \sqrt{2} - 1$ and $y = 1$ so that the required point is $A(\sqrt{2} - 1, 1)$. The area of the figure is the sum of one square and four triangles, one of which is indicated in red below.



Since point A has coordinates $(\sqrt{2} - 1, 1)$, the area of each triangle is

$$A = \frac{bh}{2}$$
$$= \frac{(2\sqrt{2} - 2)(\sqrt{2} - 1)}{2}$$
$$= 3 - 2\sqrt{2}.$$

Therefore, the total area will be $A = 1 + 4 \times (3 - 2\sqrt{2})$

$$= 13 - 8\sqrt{2} \text{ square units.}$$

5 a i $\text{Rot}(\theta)\text{Rot}(\phi)$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\sin\theta\cos\phi + \cos\theta\sin\theta) \\ \sin\theta\cos\phi + \cos\theta\sin\theta & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix}$$

$$= \text{Rot}(\theta + \phi)$$

ii Ref(θ)Ref(ϕ)

$$=\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\theta\cos 2\phi + \sin 2\theta\sin 2\phi & -(\sin 2\theta\cos 2\phi - \cos 2\theta\sin 2\theta) \\ \sin 2\theta\cos 2\phi - \cos 2\theta\sin 2\theta & \cos 2\theta\cos 2\phi + \sin 2\theta\sin 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\theta - 2\phi) & -\sin(2\theta - 2\phi) \\ \sin(2\theta - 2\phi) & \cos(2\theta - 2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\theta - \phi)) & -\sin(2(\theta - \phi)) \\ \sin(2(\theta - \phi)) & \cos(2(\theta - \phi))\end{bmatrix}$$

=Rot($2(\theta - \phi)$)

iii Rot(θ)Ref(ϕ)

$$=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta\cos 2\phi - \sin\theta\sin 2\phi & \sin\theta\cos 2\phi + \cos\theta\sin 2\theta \\ \sin\theta\cos 2\phi + \cos\theta\sin 2\theta & -(\cos\theta\cos 2\phi - \sin\theta\sin 2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\theta + 2\phi) & \sin(\theta + 2\phi) \\ \sin(\theta + 2\phi) & -\cos(\theta + 2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\phi + \theta/2)) & \sin(2(\phi + \theta/2)) \\ \sin(2(\phi + \theta/2)) & -\cos(2(\phi + \theta/2))\end{bmatrix}$$

=Ref($\phi + \theta/2$)

iv Ref(θ)Rot(ϕ)

$$=\begin{bmatrix}\cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\theta\cos\phi + \sin 2\theta\sin\phi & \sin 2\theta\cos\phi - \cos 2\theta\sin\theta \\ \sin 2\theta\cos\phi - \cos 2\theta\sin\theta & -(\cos 2\theta\cos\phi + \sin 2\theta\sin\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\theta - \phi) & \sin(2\theta - \phi) \\ \sin(2\theta - \phi) & -\cos(2\theta - \phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\theta - \phi/2)) & \sin(2(\theta - \phi/2)) \\ \sin(2(\theta - \phi/2)) & -\cos(2(\theta - \phi/2))\end{bmatrix}$$

=Ref($\theta - \phi/2$)

b **i** The composition of two rotation is a rotation.

ii The composition of a two reflections is a rotation.

iii The composition of a reflection followed by a rotation is a reflection.

iv The composition of a rotation followed by a reflection is a reflection.

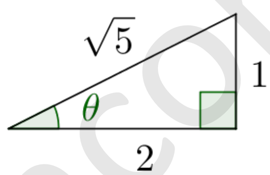
c Evaluating from left to right we have,

$$\begin{aligned}
&\text{Rot}(60^\circ)\text{Ref}(60^\circ)\text{Ref}(60^\circ)\text{Rot}(60^\circ)\\
&= (\text{Rot}(60^\circ)\text{Ref}(60^\circ))\,\text{Ref}(60^\circ)\text{Rot}(60^\circ)\\
&= \text{Ref}(60^\circ + 30^\circ)\text{Ref}(60^\circ)\text{Rot}(60^\circ)\\
&= \text{Ref}(60^\circ)\text{Ref}(60^\circ)\text{Rot}(60^\circ)\\
&= (\text{Ref}(60^\circ)\text{Ref}(60^\circ))\,\text{Rot}(60^\circ)\\
&= \text{Rot}(2(90^\circ - 60^\circ))\text{Rot}(60^\circ)\\
&= \text{Rot}(60^\circ)\text{Rot}(60^\circ)\\
&= \text{Rot}(120^\circ)\\
&= \begin{bmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{bmatrix}\\
&= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

6 a Since

$$\tan\theta = \frac{1}{2},$$

we draw a right angled triangle with opposite and adjacent lengths 1 and 2 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{5}$.



Therefore,

$$\cos\theta = \frac{2}{\sqrt{5}} \text{ and } \sin\theta = \frac{1}{\sqrt{5}}.$$

We then use the double angle formulas to show that

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{2}{\sqrt{5}}\right)^2 - 1 = \frac{8}{5} - 1 = \frac{3}{5},$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{1}{\sqrt{5}}\frac{2}{\sqrt{5}} = \frac{4}{5}.$$

Therefore the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

b The image of the point $A(-3, 1)$ is found be evaluating the matrix product,

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.$$

Therefore, the required point is $A'(-1, -3)$.

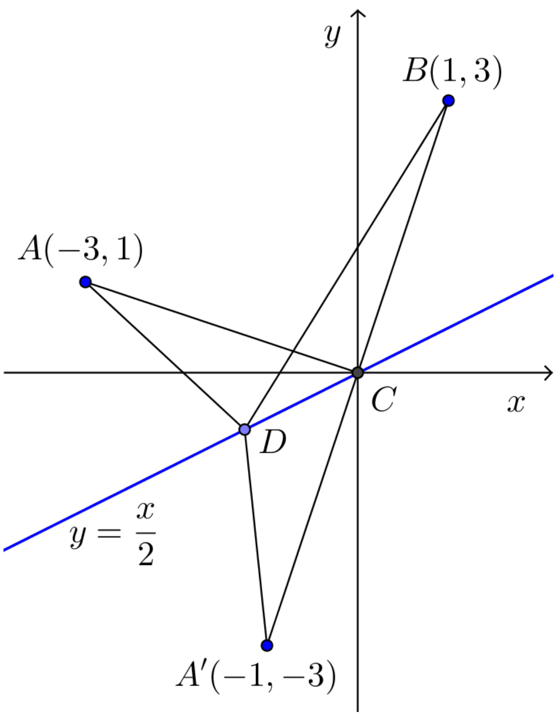
c Using the distance formula we find that

$$\begin{aligned} A'B &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1 - (-1))^2 + (3 - (-3))^2} \\ &= \sqrt{2^2 + 6^2} \\ &= \sqrt{40} \\ &= 2\sqrt{10}. \end{aligned}$$

d The line $y = \frac{x}{2}$ is the perpendicular bisects of line AA'. Therefore, $CA = CA'$, so that triangle ACA' is isosceles.

e Referring to the diagram below we have:

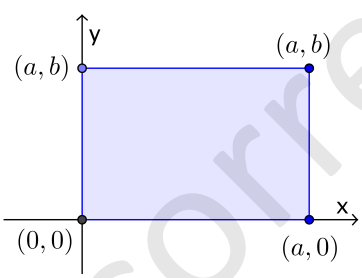
$$\begin{aligned} AD + DB &= A'D + DB \quad \text{(triangle } ADA' \text{ is isosceles)} \\ &> A'B \quad \text{(the side length of a triangle is always less than the sum of the other two)} \\ &= A'C + CB \\ &= AC + CB \quad \text{(triangle } ACA' \text{ is isosceles)} \end{aligned}$$



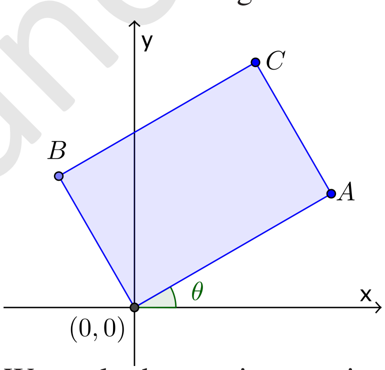
f The above calculation show that $AC + CB$ is the shortest distance from A to B via the line. Therefore the shortest distance is

$$AC + CB = A'C + CB = A'B = 2\sqrt{10}.$$

7 **a** The rectangle is shown below.



b The rotated rectangle is shown below.



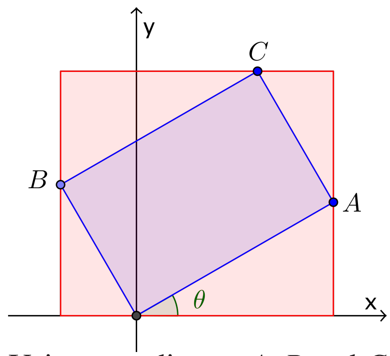
We apply the rotation matrix to the coordinate of the original rectangle to find the following co-ordinates:

$$A(a\cos\theta, a\sin\theta),$$

$$B(-b\sin\theta, b\cos\theta),$$

$$C(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta).$$

c The rectangle described is shown in red in the diagram below.



Using coordinates A, B and C found in the previous question, we can find the area of the triangle. It's width is equal to

$$a\cos\theta + b\sin\theta,$$

and its height is equal to

$$a\sin\theta + b\cos\theta.$$

Therefore, its area is

$$\begin{aligned} A &= (a\cos\theta + b\sin\theta)(a\sin\theta + b\cos\theta) \\ &= a^2\cos\theta\sin\theta + ab\cos^2\theta + ab\sin^2\theta + b^2\cos\theta\sin\theta \\ &= (a^2+b^2)\cos\theta\sin\theta + ab(\cos^2\theta + \sin^2\theta) \\ &= (a^2+b^2)\cos\theta\sin\theta + ab(\cos^2\theta + \sin^2\theta) \\ &= (a^2+b^2)\cos\theta\sin\theta + ab \\ &= \frac{(a^2+b^2)}{2}\sin 2\theta + ab \end{aligned}$$

d For θ between 0 and 90°, the maximum value of $\sin 2\theta$ occurs when $\theta = \frac{\pi}{4}$. Therefore, the maximum area will be

$$\begin{aligned} A &= \frac{(a^2+b^2)}{2} + ab \quad \text{as required.} \\ &= \frac{(a^2+2ab+b^2)}{2} \\ &= \frac{(a+b)^2}{2}, \end{aligned}$$

8 a Line L_1 is perpendicular to the line $y = mx$ and so has gradient $-\frac{1}{m}$. Moreover, it goes through the point $(1, 0)$. Therefore, its equation can be easily found:

$$y - 0 = -\frac{1}{m}(x - 1)$$
$$y = -\frac{x}{m} + \frac{1}{m}$$
$$= \frac{1}{m} - \frac{x}{m}.$$

To find where the line intersects the unit circle, we substitute $y = \frac{1}{m} - \frac{x}{m}$ into the equation for the circle, $x^2 + y^2 = 1$ and solve. This gives,

$$x^2 + y^2 = 1$$
$$x^2 + \left(\frac{1}{m} - \frac{x}{m}\right)^2 = 1$$
$$x^2 + \frac{1}{m^2} - \frac{2x}{m^2} + \frac{x^2}{m^2} = 1$$
$$m^2x^2 + 1 - 2x + x^2 = m^2$$
$$(m^2 + 1)x^2 - 2x + (1 - m^2) = 0.$$

Since we already know that $(x - 1)$ is a factor of this polynomial, we can find the other factor by inspection. This gives,

$$(x - 1)\left((m^2 + 1)x - (1 - m^2)\right) = 0$$

so that

$$x = 1 \text{ or } x = \frac{1 - m^2}{1 + m^2}.$$

Substituting $x = \frac{1 - m^2}{1 + m^2}$ into the equation of the line gives

$$y = \frac{1}{m} - \frac{x}{m}$$
$$= \frac{1}{m} - \frac{1 - m^2}{m(1 + m^2)}$$
$$= \frac{1 + m^2}{m(1 + m^2)} - \frac{1 - m^2}{m(1 + m^2)}$$
$$= \frac{2m^2}{m(1 + m^2)}$$
$$= \frac{2m}{1 + m^2}$$

Therefore the other point of intersection is

$$\left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right).$$

b Line L_2 is perpendicular to the line $y = mx$ and so has gradient $-\frac{1}{m}$. Moreover, it goes through the point $(0, 1)$. Therefore, its equation can be easily found:

$$y - 1 = -\frac{1}{m}(x - 0)$$
$$y = 1 - \frac{x}{m}$$

To find where the line intersects the unit circle, we substitute $y = 1 - \dfrac{x}{m}$ into the equation for the circle, $x^2 + y^2 = 1$ and solve. This gives,

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(1 - \frac{x}{m}\right)^2 &= 1 \\ x^2 + 1 - \frac{2x}{m} + \frac{x^2}{m^2} &= 1 \\ m^2x^2 + m^2 - 2mx + x^2 &= m^2 \\ (1 + m^2)x^2 - 2mx &= 0. \end{aligned}$$

We factorise this expression to give

$$x\big((1 + m^2)x - 2m\big) = 0$$

so that

$$x = 0 \text{ or } x = \frac{2m}{1 + m^2}.$$

Substituting $x = \dfrac{2m}{1 + m^2}$ into the equation of the line gives

$$\begin{aligned} y &= 1 - \frac{x}{m} \\ &= 1 - \frac{2m}{m(1 + m^2)} \\ &= 1 - \frac{2}{(1 + m^2)} \\ &= \frac{1 + m^2}{1 + m^2} - \frac{2}{(1 + m^2)} \\ &= \frac{m^2 - 1}{1 + m^2} \end{aligned}$$

Therefore the other point of intersection is

$$\left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{1 + m^2}\right).$$

c When reflected in the line $y = mx$, the point $(1, 0)$ maps to

$$\left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right)$$

while the point $(0, 1)$ maps to

$$\left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{1 + m^2}\right).$$

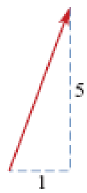
We write these points as the columns of a matrix to give,

$$\begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} \\ \dfrac{2m}{1+m^2} & \dfrac{m^2-1}{1+m^2} \end{bmatrix}.$$

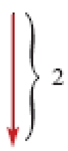
Chapter 20 – Vectors

Solutions to Exercise 20A

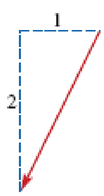
1 a $\begin{bmatrix}1\\5\end{bmatrix}$ is the vector "1 across to the right and 5 up."



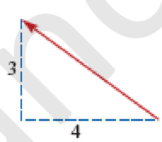
b $\begin{bmatrix}0\\-2\end{bmatrix}$ is the vector "2 down."



c $\begin{bmatrix}-1\\-2\end{bmatrix}$ is the vector "1 across to the left and 2 down."



d $\begin{bmatrix}-4\\3\end{bmatrix}$ is the vector "4 across to the left and 3 up."



2 $\boldsymbol{u} = \begin{bmatrix}6-1\\6-5\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}$

$a = 5, b = 1$

3 $\boldsymbol{v} = \begin{bmatrix}2--1\\-10-5\end{bmatrix} = \begin{bmatrix}-3\\15\end{bmatrix}$

$a = 3, b = -15$

4 a $\overrightarrow{OA} = \begin{bmatrix}1-0\\-2-0\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$

b $\overrightarrow{AB} = \begin{bmatrix}3-1\\0--2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

c $\overrightarrow{BC} = \begin{bmatrix}2-3\\-3-0\end{bmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix}$

d $\overrightarrow{CO} = -\overrightarrow{OC} = \begin{bmatrix}-2\\3\end{bmatrix}$

e $\overrightarrow{CB} = -\overrightarrow{BC} = \begin{bmatrix}1\\3\end{bmatrix}$

5 a i $\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\-3\end{bmatrix}$

$= \begin{bmatrix}1+1\\2+-3\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$

ii $2\boldsymbol{c} - \boldsymbol{a} = 2 \times \begin{bmatrix}-2\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}$

$= \begin{bmatrix}-4-1\\2-2\end{bmatrix} = \begin{bmatrix}-5\\0\end{bmatrix}$

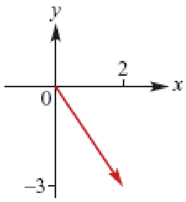
iii $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c} = \begin{bmatrix}2\\-1\end{bmatrix} - \begin{bmatrix}-2\\1\end{bmatrix}$

$= \begin{bmatrix}2--2\\-1-1\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$

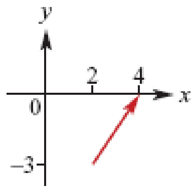
b $\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix}2\\-1\end{bmatrix} = -\boldsymbol{c}$

$\therefore$ $\boldsymbol{a}+\boldsymbol{b}$ is parallel to $\boldsymbol{c}$.

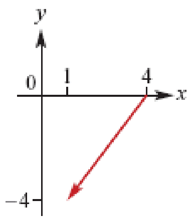
6 **a**



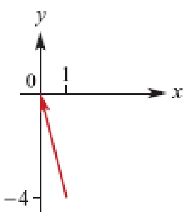
b



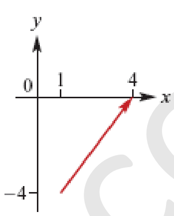
c



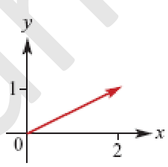
d



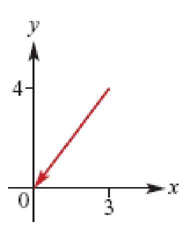
e



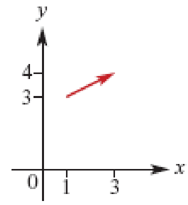
7 **a**



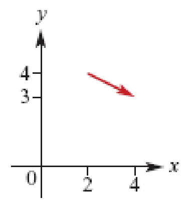
b



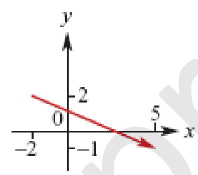
c



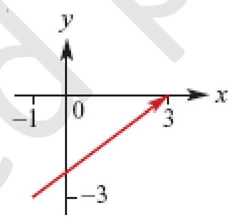
d



e



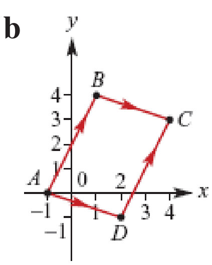
f



8 From the graphs above it can be seen that **a** and **c** are parallel.

9

a & b



c **i** $\overrightarrow{AB} = \begin{bmatrix} 1 - -1 \\ 4 - 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} 4 - 2 \\ 3 - -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$

ii $\overrightarrow{BC} = \begin{bmatrix} 4 - -1 \\ 3 - 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\overrightarrow{AD} = \begin{bmatrix} 2 - -1 \\ -1 - 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\therefore\ \overrightarrow{BC} = \overrightarrow{AD}$

d *ABCD* is a parallelogram.

10 $m\begin{bmatrix} 3 \\ -3 \end{bmatrix} + n\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3m \\ -3m \end{bmatrix} + \begin{bmatrix} 2n \\ 4n \end{bmatrix}$

$= \begin{bmatrix} 3m & +2n \\ -3m & +4n \end{bmatrix} = \begin{bmatrix} -19 \\ 61 \end{bmatrix}$

$3m + 2n = -19$

$6m + 4n = -38$ ①

$-3m + 4n = 61$ ②

① − ②:

$9m = -99$

$m = -11$

$-33 + 2n = -19$

$2n = -19 + 33$

$= 14$

$n = 7$

11 a i $\overrightarrow{MD} = \overrightarrow{MA} + \overrightarrow{AD}$

$= \frac{1}{2}\overrightarrow{BA} + \boldsymbol{b}$

$= -\frac{1}{2}\overrightarrow{AB} + \boldsymbol{b}$

$= \boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

ii $\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AD} + \overrightarrow{DN}$

$= \frac{1}{2}\overrightarrow{BA} + \boldsymbol{b} + \frac{1}{2}\overrightarrow{DN}$

$= -\frac{1}{2}\overrightarrow{AB} + \boldsymbol{b} + \frac{1}{2}\overrightarrow{DC}$

$= -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$

$= \boldsymbol{b}$

b $\overrightarrow{MN} = \overrightarrow{AD}$
(both are equal to $\boldsymbol{b}$)

12 a $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$

$= -\boldsymbol{b} + \boldsymbol{a} = \boldsymbol{a} - \boldsymbol{b}$

$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN}$

$= -\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

b $\overrightarrow{MN}$ is half the length of $\overrightarrow{CB}$, is parallel to $\overrightarrow{CB}$ and in the opposite direction to $\overrightarrow{CB}$.

13 a $\overrightarrow{CD} = \overrightarrow{AF} = \boldsymbol{a}$

b $\overrightarrow{ED} = \overrightarrow{AB} = \boldsymbol{b}$

c The regular hexagon can be divided into equilateral triangles, showing that
$\overrightarrow{BE} = 2\overrightarrow{AF} = 2\boldsymbol{a}$.

d Likewise, $\overrightarrow{FC} = 2\overrightarrow{AB} = 2\boldsymbol{b}$

e $\overrightarrow{FA} = -\overrightarrow{AF} = -\boldsymbol{a}$

f $\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB}$

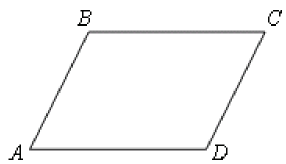
$= -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$

g $\overrightarrow{FE} = \overrightarrow{FA} + \overrightarrow{AB} + \overrightarrow{BE}$

$= -\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{a}$

$= \boldsymbol{a} + \boldsymbol{b}$

14



a $\overrightarrow{DC} = \overrightarrow{AB} = \boldsymbol{a}$

b $\overrightarrow{DA} = -\overrightarrow{BC} = -\boldsymbol{b}$

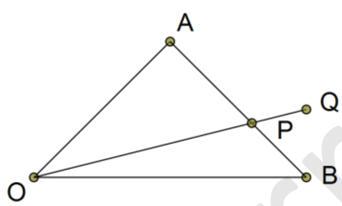
c $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$

d $\overrightarrow{CA} = -\overrightarrow{AC} = -\boldsymbol{a} - \boldsymbol{b}$

e $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$

$= -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$

15



a $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = \boldsymbol{a} - \boldsymbol{b}$

b $\overrightarrow{AB} = -\overrightarrow{BA} = \boldsymbol{b} - \boldsymbol{a}$

$\overrightarrow{PB} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})$

c $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB} = \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$

$= \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

d $\overrightarrow{PQ} = \frac{1}{3}\overrightarrow{OP}$

$= \frac{1}{3} \times \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

$= \frac{1}{9}(\boldsymbol{a} + 2\boldsymbol{b})$

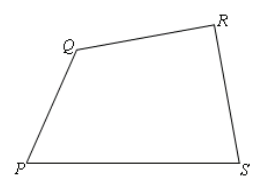
e $\overrightarrow{BP} = -\overrightarrow{PB} = \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$

$\overrightarrow{BQ} = \overrightarrow{BP} + \overrightarrow{PQ}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{1}{9}(\boldsymbol{a} + 2\boldsymbol{b})$

$= \frac{1}{9}(4\boldsymbol{a} - \boldsymbol{b})$

16



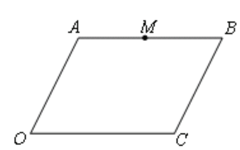
a $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \boldsymbol{u} + \boldsymbol{v}$

b $\overrightarrow{QS} = \overrightarrow{QR} + \overrightarrow{RS} = \boldsymbol{v} + \boldsymbol{w}$

c $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS}$

$= \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$

17



a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \boldsymbol{u} + \boldsymbol{v}$

$\overrightarrow{AM} = \overrightarrow{MB}$

$= \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\boldsymbol{v}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \boldsymbol{u} + \frac{1}{2}\boldsymbol{v}$

b $\overrightarrow{CM} = \overrightarrow{CB} + \overrightarrow{BM}$

$= \boldsymbol{u} + \frac{1}{2}\overrightarrow{BA}$

$= \boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$

c $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$

$= \frac{2}{3}\left(\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}\right)$

$= \frac{2}{3}\boldsymbol{u} - \frac{1}{3}\boldsymbol{v}$

d $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$

$= \boldsymbol{v} + \left(\frac{2}{3}\boldsymbol{u} - \frac{1}{3}\boldsymbol{v}\right)$

$= \frac{2}{3}\boldsymbol{u} + \frac{2}{3}\boldsymbol{v}$

$= \frac{2}{3}(\boldsymbol{u} + \boldsymbol{v}) = \frac{2}{3}\overrightarrow{OB}$

Since OP is parallel to OB and they share a common point O, OP must be on the line OB. Hence P is on $\overrightarrow{OB}$

e Using the result from part **d**, $OP : PB = 2 : 1$.

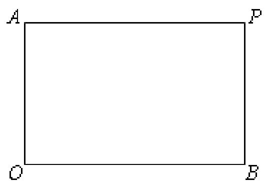
Solutions to Exercise 20B

1 $\overrightarrow{AB} = (3\boldsymbol{i} - 5\boldsymbol{j}) - (\boldsymbol{i} + 2\boldsymbol{j})$

$= 3\boldsymbol{i} - 5\boldsymbol{i} - \boldsymbol{i} - 2\boldsymbol{j}$

$= 2\boldsymbol{i} - 7\boldsymbol{j}$

2



a $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= 5\boldsymbol{i} + 6\boldsymbol{j}$

b $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -5\boldsymbol{i} + 6\boldsymbol{j}$

c $\overrightarrow{BA} = -\overrightarrow{AB}$

$= 5\boldsymbol{i} - 6\boldsymbol{j}$

3 a $|5\boldsymbol{i}| = \sqrt{5^2} = 5$

b $|-2\boldsymbol{j}| = \sqrt{(-2)^2} = 2$

c $|3\boldsymbol{i} + 4\boldsymbol{j}| = \sqrt{3^2 + 4^2}$

$= \sqrt{9 + 16} = 5$

d $|-5\boldsymbol{i} + 12\boldsymbol{j}| = \sqrt{(-5)^2 + 12^2}$

$= \sqrt{25 + 144} = 13$

4 a $\boldsymbol{u} - \boldsymbol{v} = (7\boldsymbol{i} + 8\boldsymbol{j}) - (2\boldsymbol{i} - 4\boldsymbol{j})$

$= 7\boldsymbol{i} + 8\boldsymbol{j} - 2\boldsymbol{i} + 4\boldsymbol{j}$

$= 5\boldsymbol{i} + 12\boldsymbol{j}$

$|\boldsymbol{u} - \boldsymbol{v}| = |5\boldsymbol{i} + 12\boldsymbol{j}|$

$= \sqrt{25 + 144}$

$= 13$

b $x\boldsymbol{u} + y\boldsymbol{v} = x(7\boldsymbol{i} + 8\boldsymbol{j}) + y(2\boldsymbol{i} - 4\boldsymbol{j})$

$= 7x\boldsymbol{i} + 8x\boldsymbol{j} + 2y\boldsymbol{i} - 4y\boldsymbol{j}$

$= 44\boldsymbol{j}$

$7x + 2y = 0$

$14x + 4y = 0$ ①

$8x - 4y = 44$ ②

① + ② :

$22x = 44$

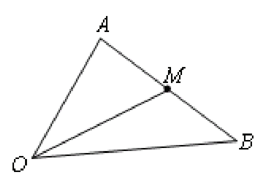
$x = 2$

$7 \times 2 + 2y = 0$

$2y = -14$

$y = -7$

5



$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -10\boldsymbol{i} + (4\boldsymbol{i} + 5\boldsymbol{j})$

$= -6\boldsymbol{i} + 6\boldsymbol{j}$

$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$

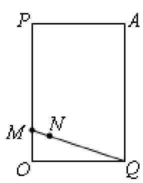
$= -3\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= 10\boldsymbol{i} + \left(-3\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}\right)$

$= 7\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$

6



a **i** $\overrightarrow{OM} = \frac{1}{5}\overrightarrow{OP}$

$= \frac{2}{5}\boldsymbol{i}$

ii $\overrightarrow{MQ} = \overrightarrow{MO} + \overrightarrow{OQ}$

$= -\overrightarrow{OM} + \overrightarrow{OQ}$

$= -\frac{2}{5}\boldsymbol{i} + \boldsymbol{j}$

iii $\overrightarrow{MN} = \frac{1}{6}\overrightarrow{MQ}$

$= \frac{1}{6}\left(-\frac{2}{5}\boldsymbol{i} + \boldsymbol{j}\right)$

$= -\frac{1}{15}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$

iv $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$= \frac{2}{5}\boldsymbol{i} + \left(-\frac{1}{15\boldsymbol{i}} + \frac{1}{6}\boldsymbol{j}\right)$

$= \frac{1}{3}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$

v $\overrightarrow{OA} = \overrightarrow{OP} + \overrightarrow{PA}$

$= 2\boldsymbol{i} + \boldsymbol{j}$

b **i** $\overrightarrow{ON} = \frac{1}{3}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$

$= \frac{1}{6}(2\boldsymbol{i} + \boldsymbol{j})$

$= \frac{1}{6}\overrightarrow{OA}$

Since *ON* is parallel to *OA* and they share a common point *O*, *ON* must be on the line *OA*. Hence *N* is on *OA*.

ii 1:5

7 $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \boldsymbol{i} + 3\boldsymbol{j}$

$\overrightarrow{OB} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} = 5\boldsymbol{i} - \boldsymbol{j}$

$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= -\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{i} - \boldsymbol{j}$

$= 4\boldsymbol{i} - 4\boldsymbol{j}$

$|\overrightarrow{AB}| = \sqrt{4^2 + (-4)^2}$

$= \sqrt{16 + 16}$

$= \sqrt{32} = 4\sqrt{2}$ units

8 a $2\boldsymbol{i} + 3\boldsymbol{j} = 2l\boldsymbol{i} + 2k\boldsymbol{j}$

$$2\boldsymbol{j} = 2$$
$$l = 1$$
$$2k = 3$$
$$k = \frac{3}{2}$$

b $x - 1 = 5$

$$x = 6$$
$$y = x - 4$$
$$= 2$$

c

$$x + y = 6 \quad ①$$
$$x - y = 0 \quad ②$$

① + ② :

$$2x = 6$$
$$x = 3$$
$$3 + y = 6$$
$$y = 3$$

d

$$k = 3 + 2l$$
$$k = -2 - l$$
$$3 + 2l = -2 - l$$
$$3l = -5$$
$$l = -\frac{5}{3}$$
$$k = -2 - -\frac{5}{3}$$
$$= -2 + \frac{5}{3}$$
$$= -\frac{1}{3}$$

9 a $\overrightarrow{AB} = \begin{bmatrix} 5-2 \\ 1-3 \end{bmatrix}$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$
$$= 3\boldsymbol{i} - 2\boldsymbol{j}$$

b $|\overrightarrow{AB}| = \sqrt{3^2 + (-2)^2}$

$$= \sqrt{9+4}$$
$$= \sqrt{13}$$

10 a $\overrightarrow{AB} = \boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{i}$

$$= -2\boldsymbol{i} + 4\boldsymbol{j}$$

b $\overrightarrow{AC} = -3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{i}$

$$= -6\boldsymbol{i} + \boldsymbol{j}$$

c $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$$= -6\boldsymbol{i} + \boldsymbol{j} - (-2\boldsymbol{i} + 4\boldsymbol{j})$$
$$= -4\boldsymbol{i} - 3\boldsymbol{j}$$
$$|\overrightarrow{BC}| = \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{16+9}$$
$$= 5$$

11 a Let $D = (a, b)$.

$$\overrightarrow{AB} = -5\boldsymbol{i} + 3\boldsymbol{j}$$
$$\overrightarrow{CD} = (a+1)\boldsymbol{i} + b\boldsymbol{j}$$
$$a + 1 = -5$$
$$a = -6$$
$$b = 3$$

D is $(-6, 3)$.

b Let $F = (c, d)$.

$\overrightarrow{BC} = -\boldsymbol{i} - 4\boldsymbol{j}$

$\overrightarrow{AF} = (c - 5)\boldsymbol{i} + (d - 1)\boldsymbol{j}$

$c - 5 = -1$

$c = 4$

$d - 1 = -4$

$d = -3$

F is $(4, -3)$.

c Let $G = (e, f)$.

$\overrightarrow{AB} = -5\boldsymbol{i} + 3\boldsymbol{j}$

$2\overrightarrow{GC} = 2(-1 - e)\boldsymbol{i} + 2(-f)\boldsymbol{j}$

$2(-1 - e) = -5$

$e = \frac{3}{2}$

$-2f = 3$

$f = -\frac{3}{2}$

G is $\left(\frac{3}{2}, -\frac{3}{2}\right)$.

12 $\overrightarrow{OA} = -\overrightarrow{AO}$

$= -\boldsymbol{i} - 4\boldsymbol{j}$

A is $(-1, -4)$.

B is $(-2, 2)$.

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

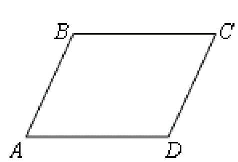
$\overrightarrow{OC} = \overrightarrow{BC} + \overrightarrow{OB}$

$= 2\boldsymbol{i} + 8\boldsymbol{j} + (-2\boldsymbol{i} + 2\boldsymbol{j})$

$= 10\boldsymbol{j}$

C is $(0, 10)$

13



a **i** $2\boldsymbol{i} - \boldsymbol{j}$

ii $-5\boldsymbol{i} + 4\boldsymbol{j}$

iii $\boldsymbol{i} + 7\boldsymbol{j}$

iv $6\boldsymbol{i} + 3\boldsymbol{j}$

v $\overrightarrow{AD} = \overrightarrow{BC}$

$= 6\boldsymbol{i} + 3\boldsymbol{j}$

b $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$\overrightarrow{OD} = \overrightarrow{AD} + \overrightarrow{OA}$

$= 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{i} - \boldsymbol{j}$

$= 8\boldsymbol{i} + 2\boldsymbol{j}$

D is $(8, 2)$.

14 **a** $\overrightarrow{OP} = 12\boldsymbol{i} + 5\boldsymbol{j}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= 18\boldsymbol{i} + 13\boldsymbol{j} - 12\boldsymbol{i} - 5\boldsymbol{j}$

$= 6\boldsymbol{i} + 8\boldsymbol{j}$

b $|\overrightarrow{RQ}| = |\overrightarrow{OP}|$

$= \sqrt{12^2 + 5^2}$

$= 13$

$|\overrightarrow{OR}| = |\overrightarrow{PQ}|$

$= \sqrt{6^2 + 8^2}$

$= 10$

15 **a** **i** $|\overrightarrow{AB}| = |2\boldsymbol{i} - 5\boldsymbol{j}|$

$= \sqrt{2^2 + 5^2} = \sqrt{29}$

ii $|\overrightarrow{BC}| = |10\boldsymbol{i} + 4\boldsymbol{j}|$

$= \sqrt{10^2 + 4^2}$

$= \sqrt{116} = 2\sqrt{29}$

iii $|\overrightarrow{CA}| = |12\boldsymbol{i} - \boldsymbol{j}|$

$= \sqrt{12^2 + 1^2} = \sqrt{145}$

b $AB^2 + BC^2 = 29 + 116$

$= 145 = AC^2$

$\therefore$ *ABC* is a right-angled triangle.

16 a i $\overrightarrow{AB} = -\boldsymbol{i} - 3\boldsymbol{j}$

ii $\overrightarrow{BC} = 4\boldsymbol{i} + 2\boldsymbol{j}$

iii $\overrightarrow{CA} = -3\boldsymbol{i} + \boldsymbol{j}$

b i $|\overrightarrow{AB}| = \sqrt{1^2 + 3^2}$

$= \sqrt{10}$

ii $|\overrightarrow{BC}| = \sqrt{4^2 + 2^2}$

$= \sqrt{20} = 2\sqrt{5}$

iii $|\overrightarrow{CA}| = \sqrt{3^2 + 1^2}$

$= \sqrt{10}$

c $AB = CA$

$= \sqrt{10}$

$AB^2 + CA^2 = 10 + 10$

$= 20 = BC^2$

$\therefore$ *ABC* is an isosceles right-angled triangle.

17 a i $\overrightarrow{OA} = -3\boldsymbol{i} + 2\boldsymbol{j}$

ii $\overrightarrow{OB} = 7\boldsymbol{j}$

iii $\overrightarrow{BA} = -3\boldsymbol{i} - 5\boldsymbol{j}$

iv $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BA}$

$= \frac{1}{2}(-3\boldsymbol{i} - 5\boldsymbol{j})$

b $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$

$\overrightarrow{OD} = 7\boldsymbol{j} + -\frac{3}{2}\boldsymbol{i} - \frac{5}{2}\boldsymbol{j}$

$= -\frac{3}{2}\boldsymbol{i} + \frac{9}{2}\boldsymbol{j}$

$M = \left(-\frac{3}{2}, -\frac{9}{2}\right)$

18 a $a = 3\boldsymbol{i} + 4\boldsymbol{j}$

$|a| = \sqrt{3^2 + 4^2}$

$= 5$

$\hat{a} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

b $b = 3\boldsymbol{i} - \boldsymbol{j}$

$|b| = \sqrt{3^2 + (-1)^2}$

$= \sqrt{10}$

$\hat{b} = \frac{1}{\sqrt{10}}(3\boldsymbol{i} - \boldsymbol{j})$

c $c = -\boldsymbol{i} + \boldsymbol{j}$

$|c| = \sqrt{(-1)^2 + 1^2}$

$= \sqrt{2}$

$\hat{c} = \frac{1}{\sqrt{2}}(-\boldsymbol{i} + \boldsymbol{j})$

d $d = \boldsymbol{i} - \boldsymbol{j}$

$\hat{d} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

e

$$e = \frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}$$

$$|e| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{9}}$$

$$= \sqrt{\frac{13}{36}}$$

$$= \frac{\sqrt{13}}{6}$$

$$\hat{e} = \frac{6}{\sqrt{13}}\left(\frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}\right)$$

$$= \frac{1}{\sqrt{13}}(3\boldsymbol{i} + 2\boldsymbol{j})$$

f

$$f = 6\boldsymbol{i} - 4\boldsymbol{j}$$

$$|f| = \sqrt{6^2 + (-4)^2}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

$$\hat{f} = \frac{1}{2\sqrt{13}}(6\boldsymbol{i} - 4\boldsymbol{j})$$

$$= \frac{1}{\sqrt{13}}(3\boldsymbol{i} - 2\boldsymbol{j})$$

Solutions to Exercise 20C

1 Let $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j}, \boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$

a $\boldsymbol{a} \cdot \boldsymbol{a} = 17$

b $\boldsymbol{b} \cdot \boldsymbol{b} = 13$

c $\boldsymbol{c} \cdot \boldsymbol{c} = 8$

d $\boldsymbol{a} \cdot \boldsymbol{b} = -10$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{i} - 4\boldsymbol{j}) \cdot (\boldsymbol{j}) = -4$

f $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

$$= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{c}$$
$$= 17 + 6 - 10 - 10$$
$$= 3$$

g $\boldsymbol{a} + 2\boldsymbol{b} = 5\boldsymbol{i} + 2\boldsymbol{j}$

$3\boldsymbol{c} - \boldsymbol{b} = -8\boldsymbol{i} - 9\boldsymbol{j}$

$\therefore (\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b}) = -58$

2 Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}, \boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j}$

a $\boldsymbol{a} \cdot \boldsymbol{a} = 5$

b $\boldsymbol{b} \cdot \boldsymbol{b} = 13$

c $\boldsymbol{a} \cdot \boldsymbol{b} = 8$

d $\boldsymbol{a} \cdot \boldsymbol{c} = -5$

e $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) = 13$

3 $|\boldsymbol{a}| = 5$ and $|\boldsymbol{b}| = 6$

a $\boldsymbol{a} \cdot \boldsymbol{b} = 5 \times 6 \cos 45°$

$$= 30 \times \frac{1}{\sqrt{2}}$$
$$= 15\sqrt{2}$$

b $\boldsymbol{a} \cdot \boldsymbol{b} = 5 \times 6 \cos 135°$

$$= 30 \times -\frac{1}{\sqrt{2}}$$
$$= -15\sqrt{2}$$

4 **a**

$$(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 3\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{a} + 4\boldsymbol{a} \cdot \boldsymbol{b} + 4(\boldsymbol{b} \cdot \boldsymbol{b})$$
$$= |\boldsymbol{a}|^2 + 4\boldsymbol{a} \cdot \boldsymbol{b} + 4|\boldsymbol{b}|^2$$

b

$$|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$$
$$= (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) + (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$$
$$= (\boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{a} - 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b})$$
$$= 4\boldsymbol{a} \cdot \boldsymbol{b}$$

c

$$\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{a}$$
$$= |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2$$

d

$$\frac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} = \frac{|\boldsymbol{a}|^2 + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$$
$$= |\boldsymbol{a}|$$

5 **a** $\overrightarrow{AB} = -2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{i} + 3\boldsymbol{j}$

$$= -3\boldsymbol{i} + \boldsymbol{j}$$

b $|\overrightarrow{AB}| = \sqrt{9 + 1} = \sqrt{10}$

c $\boldsymbol{a} \cdot \overrightarrow{AB} = |\boldsymbol{a}||\overrightarrow{AB}| \cos\theta$

$$\therefore -4 = \sqrt{10} \times 2\sqrt{2} \cos\theta$$
$$\therefore \cos\theta = -\frac{4}{2\sqrt{20}}$$
$$\therefore \theta = 116.57°$$

6 $\overrightarrow{CD} = -\boldsymbol{c} + \boldsymbol{d}$

Let θ be the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$

$\boldsymbol{c} \cdot \boldsymbol{d} = |\boldsymbol{c}||\boldsymbol{d}| \cos\theta$

$\therefore \cos\theta = \dfrac{4}{5 \times 7}$

Using the cosine rule.

$$|\overrightarrow{CD}|^2 = 5^2 + 7^2 - 2 \times 5 \times 7 \cos\theta$$

$$= 25 + 49 - 2 \times 5 \times 7 \times \frac{4}{35}$$

$$= 66$$

$\therefore |\overrightarrow{CD}| = \sqrt{66}$

7 **a** $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (5\boldsymbol{i} + x\boldsymbol{j}) = -6$

$5 + 2x = -6$

$2x = -11$

$x = -\dfrac{11}{2}$

b $(x\boldsymbol{i} + 7\boldsymbol{j}) \cdot (-4\boldsymbol{i} + x\boldsymbol{j}) = 10$

$-4x + 7x = 10$

$3x = 10$

$x = \dfrac{10}{3}$

c $(x\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 3\boldsymbol{j}) = x$

$-2x - 3 = x$

$-3 = 3x$

$-1 = x$

d $x(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} + x\boldsymbol{j}) = 6$

$x(2 + 3x) = 6$

$2x + 3x^2 = 6$

$3x^2 + 2x - 6 = 0$

$x = \dfrac{-2 \pm \sqrt{76}}{6}$

8 **a** $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$= -4\boldsymbol{i} - 4\boldsymbol{j} + q(2\boldsymbol{i} + 5\boldsymbol{j})$

$= (2q - 1)\boldsymbol{i} + (5q - 4)\boldsymbol{j}$

b

$\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$

$\Rightarrow ((2q - 1)\boldsymbol{i} + (5q - 4)\boldsymbol{j}) \cdot (2\boldsymbol{i} + 5\boldsymbol{j}) = 0$

$\Rightarrow 4q - 2 + 25q - 20 = 0$

$\Rightarrow 29q - 22 = 0$

$\Rightarrow q = \dfrac{22}{29}$

c $\overrightarrow{OP} = q\boldsymbol{b} = \dfrac{22}{9}(2\boldsymbol{i} + 5\boldsymbol{j})$

Cooordinates of P are $\left(\dfrac{44}{29}, \dfrac{110}{29}\right)$

9 **a** $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (\boldsymbol{i} - 4\boldsymbol{j}) = \sqrt{5} \times \sqrt{17} \cos\theta$

$-7 = \sqrt{85} \cos\theta$

$\cos\theta = -\dfrac{7}{\sqrt{85}}$

$\theta = 139.40°$

b $-2\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 2\boldsymbol{j}) = \sqrt{5} \times \sqrt{8} \cos\theta$

$2 = \sqrt{40} \cos\theta$

$\cos\theta = \dfrac{2}{\sqrt{40}}$

$\theta = 71.57°$

c $2\boldsymbol{i} - \boldsymbol{j}) \cdot (4\boldsymbol{i} = \sqrt{5} \times 4 \cos\theta$

$8 = 4\sqrt{5} \cos\theta$

$\cos\theta = \dfrac{2}{\sqrt{5}}$

$\theta = 26.57°$

d $7\boldsymbol{i}+\boldsymbol{j})\cdot(-\boldsymbol{i}+\boldsymbol{i})+ = \sqrt{50}\times\sqrt{2}\cos\theta$

$$-6 = 10\cos\theta$$

$$\cos\theta = -\frac{3}{5}$$

$$\theta = 126.87^\circ$$

10 $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

If $\boldsymbol{a}$ and $\boldsymbol{a}$ are non-zero vectors, then

$\boldsymbol{a}\cdot\boldsymbol{b} = 0 \Leftrightarrow \cos\theta = 0$

11 a $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$$

$$= \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$$

$$= \frac{3}{2}\boldsymbol{i}$$

b $\boldsymbol{a}\cdot\overrightarrow{OM} = |\boldsymbol{a}|\overrightarrow{OM}|\cos(\angle AOM)$

$$\cos(\angle AOM) = \frac{\frac{3}{2}}{\sqrt{2}\times\frac{3}{2}}$$

$$\therefore \angle AOM = 45^\circ$$

c $\overrightarrow{MB}\cdot\overrightarrow{MO} = |\overrightarrow{MB}||\overrightarrow{MO}|\cos(\angle BMO)$

$$\cos(\angle BMO) = \frac{-\frac{3}{4}}{\frac{\sqrt{5}}{2}\times\frac{3}{2}}$$

$$\cos(\angle BMO) = -\frac{1}{\sqrt{5}}$$

$$\angle BMO = 116.57^\circ$$

12 a i $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$$

$$= \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$$

$$= \frac{1}{2}(3\boldsymbol{i}+4\boldsymbol{j})$$

ii $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$

$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{c}-\boldsymbol{a})$$

$$= \frac{1}{2}(\boldsymbol{c}+\boldsymbol{b})$$

$$= \frac{1}{2}(\boldsymbol{i}+6\boldsymbol{j})$$

b

$$\overrightarrow{OM}\cdot\overrightarrow{ON} = |\overrightarrow{ON}||\overrightarrow{OM}|\cos(\angle MON)$$

$$\cos(\angle MON) = \frac{\frac{27}{4}}{\frac{5}{2}\times\frac{\sqrt{37}}{2}}$$

$$\cos(\angle MON) = \frac{1}{\sqrt{5}}$$

$$\angle BMO = 27.41^\circ$$

c $\overrightarrow{OM}\cdot\overrightarrow{OC} = |\overrightarrow{OM}||\overrightarrow{OC}|\cos(\angle MOC)$

$$\cos(\angle MOC) = \frac{9}{\frac{5}{2}\times\sqrt{40}}$$

$$\cos(\angle MOC) = \frac{9}{5\sqrt{10}}$$

$$\angle BMO = 55.30^\circ$$

Solutions to Exercise 20D

1 a $|\boldsymbol{a}| = \sqrt{1+9} = \sqrt{10}$

$\therefore \hat{\boldsymbol{a}} = \frac{1}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$

b $|\boldsymbol{b}| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\therefore \hat{\boldsymbol{a}} = \frac{1}{2\sqrt{2}}(2\boldsymbol{i} + 2\boldsymbol{j}) = \frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j})$

c $\boldsymbol{c} = \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \boldsymbol{i} - \boldsymbol{j}$

$\therefore \hat{\boldsymbol{c}} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

2 a **i** $\hat{\boldsymbol{a}} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

ii $|\boldsymbol{b}| = \sqrt{2}$

b $\frac{\sqrt{2}}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

3 a **i** $\hat{\boldsymbol{a}} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

ii $\hat{\boldsymbol{b}} = \frac{1}{13}(5\boldsymbol{i} + 12\boldsymbol{j})$

b Let $\overrightarrow{OA'} = \hat{\boldsymbol{a}}$ and $\overrightarrow{OB'} = \hat{\boldsymbol{b}}$
Then $\triangle A'OB'$ is isosceles. Therefore the angle bisector of $\angle AOB$ passes through the midpoint of $A'B'$.
Let M be the midpoint of $A'B'$
Then

$\overrightarrow{OM} = \frac{1}{2}(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}})$

$= \frac{1}{2}(\frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j}) + \frac{1}{13}(5\boldsymbol{i} + 12\boldsymbol{j}))$

$= \frac{8}{65}(4\boldsymbol{i} + 7\boldsymbol{j})$

$\therefore$the unit vector in the direction of $\overrightarrow{OM}$ is:

$= \frac{1}{\sqrt{65}}(4\boldsymbol{i} + 7\boldsymbol{j})$

4 a $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

$\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} = \frac{1-12}{17}(\boldsymbol{i} - 4\boldsymbol{j})$

$= -\frac{11}{17}(\boldsymbol{i} - 4\boldsymbol{j})$

b $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

$\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} = \frac{1+12}{17}(\boldsymbol{i} - 4\boldsymbol{j})$

$= \frac{13}{17}(\boldsymbol{i} - 4\boldsymbol{j})$

c The vector resolute is $\boldsymbol{b}$

5 a $\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|} = \frac{2}{1} = 2$

b $\frac{\boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{c}|} = \frac{3-2}{\sqrt{5}} = \frac{1}{\sqrt{5}}$

c $\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|} = \frac{2\sqrt{3}}{\sqrt{7}}$

d $\frac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{c}|} = \frac{-1-4\sqrt{5}}{\sqrt{17}}$

6 a $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = 2\boldsymbol{i}$ and $\boldsymbol{w} = \boldsymbol{j}$

b $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{w} = \boldsymbol{i} - \boldsymbol{j}$

c $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{w} = -\boldsymbol{i} + \boldsymbol{j}$

7 a $\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} = 2(\boldsymbol{i} + \boldsymbol{j})$

b Let $\overrightarrow{OC} = 2(\boldsymbol{i} + \boldsymbol{j})$
$\overrightarrow{OC}$ is the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$
$\therefore$ $\overrightarrow{CA}$ is a vector perpendiculr to $\overrightarrow{OB}$

$\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA}$

$= -2(\boldsymbol{i} + \boldsymbol{j}) + (\boldsymbol{i} + 3\boldsymbol{j})$

$= -\boldsymbol{i} + \boldsymbol{j}$

Therefore the unit vector is $\frac{1}{\sqrt{2}}(-\boldsymbol{i} + \boldsymbol{j})$

8 a $\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}} \boldsymbol{b} = \frac{3}{2}(\boldsymbol{i} - \boldsymbol{j})$

b $\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}} \boldsymbol{b} = 4\boldsymbol{i} + \boldsymbol{j} - \frac{3}{2}(\boldsymbol{i} - \boldsymbol{j})$

$= \frac{1}{2}(8\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{i} + 3\boldsymbol{j})$

$= \frac{1}{2}(5\boldsymbol{i} + 5\boldsymbol{j})$

9 $\overrightarrow{OA} = \boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{OB} = \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$

$\overrightarrow{OC} = \boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$

a i $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{i} + \boldsymbol{j}$

$= \boldsymbol{i} - \boldsymbol{j}$

ii $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{i} - 3\boldsymbol{j}$

$= \boldsymbol{i} - 5\boldsymbol{j}$

b The vector resolute $= \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\overrightarrow{AC} \cdot \overrightarrow{AC}} \overrightarrow{AC}$

$= \frac{1 + 5}{26}(\boldsymbol{i} - 5\boldsymbol{j})$

$= \frac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$

c

The shortest distance $= \overrightarrow{AB} - \frac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$

$= \frac{3}{13}(10\boldsymbol{i} + 2\boldsymbol{j})$

The shortest distance is the height of triangle *ABC* wherethe base is taken as *AC*

Therefore height= $|\frac{3}{13}(10\boldsymbol{i} + 2\boldsymbol{j})| = \frac{1}{13}\sqrt{104}$

The area of the triangle

$= \frac{1}{2} \times \frac{1}{13}\sqrt{104} \times \sqrt{26}$

$=2$

Solutions to Exercise 20E

1 **a** **i** $\overrightarrow{OR} = \frac{4}{5}\overrightarrow{OP}$

$= \frac{4}{5}\mathbf{p}$

ii $\overrightarrow{RP} = \frac{1}{5}\overrightarrow{OP}$

$= \frac{1}{5}\mathbf{p}$

iii $\overrightarrow{PO} = -\mathbf{p}$

iv $\overrightarrow{PS} = \frac{1}{5}\overrightarrow{PQ}$

$= \frac{1}{5}(\mathbf{q} - \mathbf{p})$

v $\overrightarrow{RS} = \overrightarrow{RP} + \overrightarrow{PS}$

$= \frac{1}{5}\mathbf{p} + \frac{1}{5}(\mathbf{q} - \mathbf{p})$

$= \frac{1}{5}\mathbf{q}$

b They are parallel (and $OQ = 5RS$).

c A trapezium (one pair of parallel lines).

d The area of triangle POQ is 25 times the area of $PRS = 125 \text{ cm}^2$.

$\therefore$ area of $ORSQ = 125 - 5$

$= 120 \text{ cm}^2$

2 **a** $AP = \frac{2}{3}AB$ and $CQ = \frac{6}{7}CB$.

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$= \mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a})$

$= \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

$\overrightarrow{OQ} = \overrightarrow{OC} + \overrightarrow{CQ}$

$= \overrightarrow{OC} + \frac{6}{7}\overrightarrow{CB}$

$= k\mathbf{a} + \frac{6}{7}(\mathbf{b} - k\mathbf{a})$

$= \frac{k}{7}\mathbf{a} + \frac{6}{7}\mathbf{b}$

b **i** OPQ is a straight line if $OP = nOQ$.

$\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b} = n\left(\frac{k}{7}\mathbf{a} + \frac{6}{7}\mathbf{b}\right)$

$= \frac{nk}{7}\mathbf{a} + \frac{6n}{7}\mathbf{b}$

$\frac{2}{3} = \frac{6n}{7}$

$n = \frac{14}{18} = \frac{7}{9}$

$\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b} = \frac{7}{9}\left(\frac{k}{7}a + \frac{6}{7}b\right)$

$= \frac{k}{9}\mathbf{a} + \frac{2}{3}\mathbf{b}$

$\frac{k}{9} = \frac{1}{3}$

$k = 3$

ii From part i

$$\overrightarrow{OP} = \frac{7}{9}\overrightarrow{OQ}$$

$$= \frac{7}{9}(OP + PQ)$$

$$= \frac{7}{9}OP + \frac{7}{9}PQ$$

$$\frac{2}{9}OP = \frac{7}{9}PQ$$

$$2OP = 7PQ$$

$$\frac{OP}{PQ} = \frac{7}{2}$$

c $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$$= -\boldsymbol{b} + k\boldsymbol{a}$$

$= 3\boldsymbol{a} - \boldsymbol{b}$, since $k = 3$

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$$

$$= -\frac{1}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b} + \frac{7}{3}\boldsymbol{a}$$

$$= 2\boldsymbol{a} - \frac{2}{3}\boldsymbol{b}$$

$$= \frac{2}{3}(3\boldsymbol{a} - \boldsymbol{b})$$

$$= \frac{2}{3}\overrightarrow{BC}$$

Hence PR is parallel to BC

3 a i $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$

$$= \frac{1}{3}(6\boldsymbol{i} - 1.5\boldsymbol{j})$$

$$= 2\boldsymbol{i} - 0.5\boldsymbol{j}$$

$$\overrightarrow{AB} = 3\boldsymbol{i} - 6\boldsymbol{j}$$

$$\overrightarrow{AE} = \frac{1}{4}(3\boldsymbol{i} - 5\boldsymbol{j})$$

$$= -0.75\boldsymbol{i} - 1.25\boldsymbol{j}$$

$$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$$

$$= 3\boldsymbol{i} + 3.5\boldsymbol{j} + 0.75\boldsymbol{i} - 1.25\boldsymbol{j}$$

$$= 3.75\boldsymbol{i} + 2.25\boldsymbol{j}$$

$$= \frac{15}{4}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}$$

ii $\overrightarrow{ED} = 2\boldsymbol{i} - 0.5\boldsymbol{j} - \left(\frac{15}{4}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}\right)$

$$= -\frac{6}{4}\boldsymbol{i} - \frac{11}{4}\boldsymbol{j}$$

$$|\overrightarrow{ED}| = \sqrt{\left(\frac{7}{4}\right)^2 + \left(\frac{11}{4}\right)^2}$$

$$= \sqrt{\frac{49 + 121}{16}}$$

$$= \sqrt{\frac{170}{16}}$$

$$= \frac{\sqrt{170}}{4}$$

b i $\overrightarrow{OX} = \frac{15p}{4}\boldsymbol{i} + \frac{9p}{4}\boldsymbol{j}$

ii $\overrightarrow{AD} = 2\boldsymbol{i} - 0.5\boldsymbol{j} - (3\boldsymbol{i} + 3.5\boldsymbol{j})$

$$= -\boldsymbol{i} - 4\boldsymbol{j}$$

$$\overrightarrow{XD} = -q\boldsymbol{i} - 4q\boldsymbol{j}$$

$$\overrightarrow{OD} = \overrightarrow{OX} + \overrightarrow{OD}$$

$$\overrightarrow{OX} = \overrightarrow{OD} - \overrightarrow{XD}$$

$$= 2\boldsymbol{i} = 0.5\boldsymbol{j} - (-q\boldsymbol{i} - 4q\boldsymbol{j})$$

$$= (q + 2)\boldsymbol{i} + (4q - 0.5)\boldsymbol{j}$$

c $(q + 2)\boldsymbol{i} + (4q - 0.5)\boldsymbol{j} = \frac{15p}{4}\boldsymbol{i} + \frac{9p}{4}\boldsymbol{j}$

$$q + 2 = \frac{15p}{4}$$

$$4q + 8 = 15p \quad ①$$

$$4q - 0.5 = \frac{9p}{4} \quad ②$$

$$① - ②:\ 8.5 = \frac{51p}{4}$$

$$p = \frac{8.5 \times 4}{51}$$

$$= \frac{2}{3}$$

$$q + 2 = \frac{15p}{4}$$

$$= \frac{10}{4} = \frac{5}{2}$$

$$q = \frac{1}{2}$$

4 a $\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p}$

$$= \overrightarrow{PM} + \overrightarrow{MQ}$$

$$\overrightarrow{MQ} = \frac{\beta}{\alpha}\overrightarrow{PM}$$

$$\therefore\ \overrightarrow{PQ} = \overrightarrow{PM} + \frac{\beta}{\alpha}\overrightarrow{PM}$$

$$= \frac{\alpha + \beta}{\alpha}\overrightarrow{PM}$$

$$\overrightarrow{PM} = \frac{\alpha}{\alpha + \beta}\overrightarrow{PQ}$$

$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$$

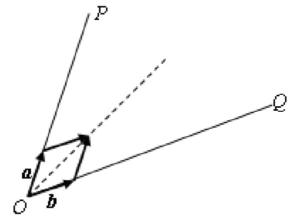
$$= \boldsymbol{p} + \frac{\alpha}{\alpha + \boldsymbol{ba}}(\boldsymbol{q} - \boldsymbol{p})$$

$$= \frac{\alpha + \beta}{\alpha + \beta}\boldsymbol{p} + \frac{\alpha}{\alpha + ba}(\boldsymbol{q} - \boldsymbol{p})$$

$$= \frac{\alpha + \beta - \alpha}{\alpha + \beta}\boldsymbol{p} + \frac{\alpha}{\alpha + \beta}\boldsymbol{q}$$

$$= \frac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha + \beta}$$

b i



It can be seen from the parallelogram formed by adding $\boldsymbol{a}$ and $\boldsymbol{b}$ that $\boldsymbol{a} + \boldsymbol{b}$ will lie on the bisector of angle POQ.
Hence any multiple, $\lambda(\boldsymbol{a} + \boldsymbol{b})$, will also lie on this bisector.

ii If $\boldsymbol{p} = k\boldsymbol{a}$ and $\boldsymbol{q} = l\boldsymbol{b}$, then

$$\overrightarrow{OM} = \frac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha + \beta}$$

$$= \frac{\beta k\boldsymbol{a} + \alpha l\boldsymbol{b}}{\boldsymbol{a}n + \beta}$$

If M is the bisector of $\angle POQ$,

$$OM = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$$

$$\therefore\ \alpha l = \beta k$$

Divide both sides by βl:

$$\frac{\alpha}{\beta} = \frac{k}{l}$$

5 Let $OABC$ be a rhombus.
Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$
We note that $|\boldsymbol{a}| = |\boldsymbol{c}|$

a i $\overrightarrow{AB} = \boldsymbol{c}$

ii $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \boldsymbol{a} + \boldsymbol{c}$

iii $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\boldsymbol{a} + \boldsymbol{c}$

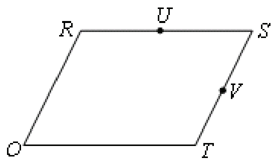
b $\overrightarrow{OB} \cdot \overrightarrow{AC} = (\boldsymbol{a} + \boldsymbol{c}) \cdot (-\boldsymbol{a} + \boldsymbol{c})$

$$= -\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{c}$$

$$= -|\boldsymbol{a}|^2 + |\boldsymbol{c}|^2$$

$$= 0$$

6



a $s = \overrightarrow{OS}$

$= \overrightarrow{OR} + \overrightarrow{RS}$

$= \overrightarrow{OR} + \overrightarrow{OT}$

$= \boldsymbol{r} + \boldsymbol{t}$

b $\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS}$

$= \boldsymbol{t} - \boldsymbol{s}$

$\boldsymbol{v} = \overrightarrow{OV}$

$= \overrightarrow{OS} + \overrightarrow{SV}$

$= \overrightarrow{OS} + \frac{1}{2}\overrightarrow{ST}$

$= \boldsymbol{s} - \frac{1}{2}(\boldsymbol{t} - \boldsymbol{s})$

$= \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

c Similarly:

$\boldsymbol{u} = \overrightarrow{OU}$

$= \overrightarrow{OS} + \overrightarrow{SU}$

$= \overrightarrow{OS} + \frac{1}{2}\overrightarrow{SR}$

$= \boldsymbol{s} - \frac{1}{2}(\boldsymbol{r} - \boldsymbol{s})$

$= \frac{1}{2}(\boldsymbol{s} + \boldsymbol{r})$

$\therefore\ \boldsymbol{u} + \boldsymbol{v} = \frac{1}{2}(\boldsymbol{s} + \boldsymbol{r}) + \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

$= \frac{1}{2}(2\boldsymbol{s} + \boldsymbol{r} + \boldsymbol{t})$

$2\boldsymbol{u} + 2\boldsymbol{v} = 2\boldsymbol{s} + \boldsymbol{r} + \boldsymbol{t}$

We may also express $\boldsymbol{u}$ as

$\boldsymbol{u} = \overrightarrow{OR} + \overrightarrow{RU}$

$= \overrightarrow{OR} + \frac{1}{2}\overrightarrow{RS}$

$= \overrightarrow{OR} + \frac{1}{2}\overrightarrow{OT}$

$= \boldsymbol{r} + \frac{1}{2}\boldsymbol{t}$

$\therefore\ \boldsymbol{u} + \boldsymbol{v} = 1 + \frac{1}{2}\boldsymbol{t} + \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

$= \frac{1}{2}(\boldsymbol{s} + 2\boldsymbol{r} + 2\boldsymbol{t})$

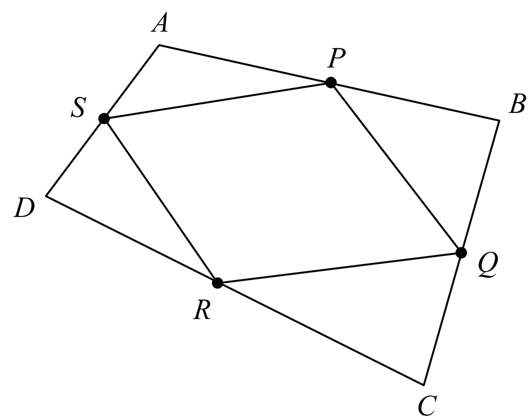
$2\boldsymbol{u} + 2\boldsymbol{v} = \boldsymbol{s} + 2\boldsymbol{r} + 2\boldsymbol{t}$

Add the two expressions for $2\boldsymbol{u} + 2\boldsymbol{v}$:

$4\boldsymbol{u} + 4\boldsymbol{v} = 3\boldsymbol{s} + 3\boldsymbol{r} + 3\boldsymbol{t}$

$= 3(\boldsymbol{s} + \boldsymbol{r} + \boldsymbol{t})$

7 Required to prove that if the midpoints of the sides of a quadrilateral are joined then a parallelogram if formed.



ABCD is a quadrilateral. *P*, *Q*, *R* and *S* are the midpoints of the sides *AB*, *BC*, *CD* and *DA* respectively.

$$\overrightarrow{AS} = \frac{1}{2}\overrightarrow{AD}$$

$$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB}$$

$$\overrightarrow{SP} = \overrightarrow{AP} - \overrightarrow{AS}$$

$$= \frac{1}{2}\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AD}$$

$$= \frac{1}{2}(\overrightarrow{AB} - \overrightarrow{AD})$$

$$= \frac{1}{2}\overrightarrow{DB}$$

$$\therefore \overrightarrow{SP} = \frac{1}{2}\overrightarrow{DB}$$

Similarly,

$$\overrightarrow{CR} = \frac{1}{2}\overrightarrow{CD}$$

$$\overrightarrow{CQ} = \frac{1}{2}\overrightarrow{CB}$$

$$\overrightarrow{RQ} = \overrightarrow{RC} + \overrightarrow{CQ}$$

$$= \frac{1}{2}\overrightarrow{CB} - \frac{1}{2}\overrightarrow{CD}$$

$$= \frac{1}{2}(\overrightarrow{CB} - \overrightarrow{CD})2$$

$$= \frac{1}{2}\overrightarrow{DB}$$

$$\therefore \overrightarrow{RQ} = \frac{1}{2}\overrightarrow{DB}$$

Thus $\overrightarrow{SP} = \overrightarrow{RQ}$ meaning $SP \parallel RQ$ and $SP = RQ$

Hence $PQRS$ is a parallelogram.

8 Consider the square $OACB$.

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

They are of equal magnitude. That is, $|\boldsymbol{a}| = |\boldsymbol{b}|$.

The diagonals are $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$

$$|\boldsymbol{a} + \boldsymbol{b}|^2 = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b})$$

$$= \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}$$

$$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$$

$$|\boldsymbol{a} - \boldsymbol{b}|^2 = (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$$ The

$$= \boldsymbol{a} \cdot \boldsymbol{a} - 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}$$

$$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$$

diagonals are of equal length

Let M be the midpoint of diagonal $\overrightarrow{OC}$.

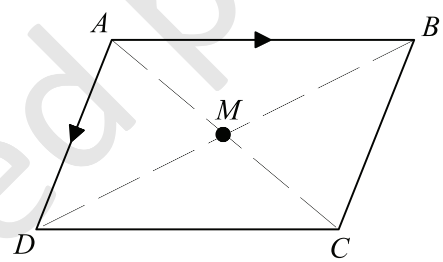
Then $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.

Let N be the midpoint of diagonal $\overrightarrow{BA}$.

Then $\overrightarrow{ON} = \overrightarrow{OB} + \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.

Therefore $M = N$. The diagonals bisect each other

9 Required to prove that the diagonals of a parallelogram bisect each other.



$ABCD$ is a parallelogram.

Let $\overrightarrow{AD} = \boldsymbol{a}$

Let $\overrightarrow{AB} = \boldsymbol{b}$

Let M be the midpoint of AC.

$$\overrightarrow{AC} = \boldsymbol{b} + \boldsymbol{a}$$

$$\Rightarrow \overrightarrow{AM} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

$$\overrightarrow{BM} = -\overrightarrow{AB} + \overrightarrow{AM}$$

$$= -\boldsymbol{b} + \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

$$= \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$$

$$\overrightarrow{MD} = -\overrightarrow{AM} + \overrightarrow{AD}$$

$$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{a}$$

$$= \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$$

$$= \overrightarrow{BM}$$

Thus M is the midpoint BD.
Therefore the diagonals of a parallelogram bisect each other.

10 Consider $\triangle ABC$. Let the altitudes from A to BC and B to AC meet at O.
Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.
Then
$(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0 \ldots (1)$.
$(\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} = 0 \ldots (2)$. Subtract (1) from (2)

$$\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} - (\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0$$

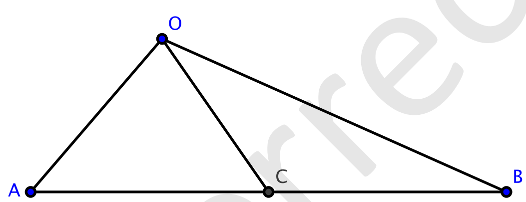
$$\therefore \boldsymbol{c} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{a} = 0$$

$$\therefore \boldsymbol{c} \cdot \boldsymbol{b} - \boldsymbol{c} \cdot \boldsymbol{a} = 0$$

$$\therefore \boldsymbol{c} \cdot (\boldsymbol{b} - \boldsymbol{a}) = 0$$

Therefore OC is the altitude from C to AB

11



$$\overrightarrow{OC} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OB})$$

$$= \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b})$$

$$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$

$$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$

$$4\overrightarrow{OC} \cdot \overrightarrow{OC} = \boldsymbol{b} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{b} \cdot \boldsymbol{a}$$

$$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\boldsymbol{b}.\boldsymbol{a}$$

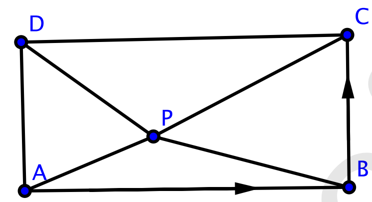
$$4\overrightarrow{AC} \cdot \overrightarrow{AC} = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{b}.\boldsymbol{a}$$

Therefore

$$4|\overrightarrow{OC}|^2 + 4|\overrightarrow{AC}|^2 = 2|\boldsymbol{a}|^2 + 2|\boldsymbol{b}|^2$$

$$\therefore 2|\overrightarrow{OC}|^2 + 2|\overrightarrow{AC}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$$

12



For rectangle $ABCD$
Let $\overrightarrow{AB} = \boldsymbol{x}$ and $\overrightarrow{BC} = \boldsymbol{y}$
Then there exist real numbers $0 < \lambda < 1$ and $0 < \mu < 1$ such that:

$$\overrightarrow{PB} = \lambda\boldsymbol{x} + \mu\boldsymbol{y}$$

$$\overrightarrow{PC} = \lambda\boldsymbol{x} + (1 - \mu)\boldsymbol{y}$$

$$\overrightarrow{PD} = -(1 - \lambda)\boldsymbol{x} + (1 - \mu)\boldsymbol{y}$$

$$\overrightarrow{PA} = -(1 - \lambda)\boldsymbol{x} - \mu\boldsymbol{y}$$

$$|\overrightarrow{PB}|^2 + |\overrightarrow{PD}|^2$$

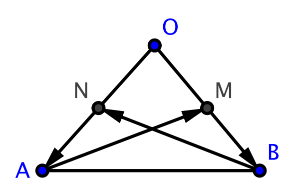
$$= \lambda^2|\boldsymbol{x}|^2 + \mu^2|\boldsymbol{y}|^2 + (1 - \lambda)^2|\boldsymbol{x}|^2 + (1 - \mu)^2|\boldsymbol{y}|^2$$

$$|\overrightarrow{PA}|^2 + |\overrightarrow{PC}|^2$$

$$= (1 - \lambda)^2|\boldsymbol{x}|^2 + \mu^2|\boldsymbol{y}|^2 + \lambda^2|\boldsymbol{x}|^2 + (1 - \mu)^2|\boldsymbol{y}|^2$$

$$\therefore |\overrightarrow{PB}|^2 + |\overrightarrow{PD}^2| = |\overrightarrow{PA}|^2 + |\overrightarrow{PC}|^2$$

13



Let $OA = OB$
Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$
Let M be the midpoint of OB and N be the midpoint of OA.

$\overrightarrow{AM} = \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OB}$

$= -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

$\overrightarrow{BN} = \overrightarrow{BO} + \frac{1}{2}\overrightarrow{OA}$

$= -\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$

$|\overrightarrow{AM}|^2 = (-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}) \cdot (-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b})$

$= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} + \frac{1}{4}\boldsymbol{b} \cdot \boldsymbol{b}$

$= |\boldsymbol{a}|^2 + \frac{1}{4}|\boldsymbol{b}|^2$

$|\overrightarrow{BN}|^2 = (\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}) \cdot (\frac{1}{2}\boldsymbol{a} - \boldsymbol{b})$

$= \frac{1}{4}\boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}$

$= \frac{1}{4}|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$

But $|\boldsymbol{a}|| = |\boldsymbol{b}|$.

Hence $|\overrightarrow{BN}| = |\overrightarrow{AM}|$

14 See question 10

Solutions to Exercise 20F

1 **a** $\boldsymbol{a} - \boldsymbol{b} = (\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$= -\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

b $3\boldsymbol{b} - 2\boldsymbol{a} + \boldsymbol{c} = 3(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$- 2(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

$+ (-\boldsymbol{i} + \boldsymbol{k})$

$= 6\boldsymbol{i} - 3\boldsymbol{j} + 9\boldsymbol{k} - 2\boldsymbol{i} - 2\boldsymbol{j}$

$- 4\boldsymbol{k} - \boldsymbol{i} + \boldsymbol{k}$

$= 3\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$

c $|\boldsymbol{b}| = \sqrt{2^2 + (-1)^2 + 3^2}$

$= \sqrt{4 + 1 + 9}$

$= \sqrt{14}$

d $|\boldsymbol{b} + \boldsymbol{c}| = |(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) + (-\boldsymbol{i} + \boldsymbol{k})|$

$= |\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}|$

$= \sqrt{1^2 + (-1)^2 + 4^2}$

$= \sqrt{18} = 3\sqrt{2}$

e $3(\boldsymbol{a} - \boldsymbol{b}) + 2\boldsymbol{c} = 3((\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

$- (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}))$

$+ 2(-\boldsymbol{i} + \boldsymbol{k})$

$= 3(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$- 2\boldsymbol{i} + 2\boldsymbol{k}$

$= -3\boldsymbol{i} + 6\boldsymbol{j} - 3k$

$- 2\boldsymbol{i} + 2\boldsymbol{k}$

$= -5\boldsymbol{i} + 6\boldsymbol{j} - \boldsymbol{k}$

2 **a** $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$

$= 2\boldsymbol{j} + 2\boldsymbol{k}$

b $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{OD}$

$= \boldsymbol{i} + 2\boldsymbol{j}$

c $\overrightarrow{OG} = \overrightarrow{OC} + \overrightarrow{OD}$

$= \boldsymbol{i} + 2\boldsymbol{k}$

d $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OD}$

$= \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

e $\overrightarrow{ED} = -\overrightarrow{OA}$

$= -2\boldsymbol{j}$

f $\overrightarrow{EG} = -\overrightarrow{OA} + \overrightarrow{OC}$

$= -2\boldsymbol{j} + 2\boldsymbol{k}$

g $\overrightarrow{CE} = -\overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{OD}$

$= \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

h $\overrightarrow{BD} = -\overrightarrow{OC} - \overrightarrow{OA} + \overrightarrow{OD}$

$= \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$

3 **a** **i** $|\boldsymbol{a}| = \sqrt{3^2 + 1^2 + 1^2}$

$= \sqrt{11}$

$\hat{\boldsymbol{a}} = \dfrac{1}{\sqrt{11}}(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$= \dfrac{3}{\sqrt{11}}\boldsymbol{i} + \dfrac{1}{\sqrt{11}}\boldsymbol{j} - \dfrac{1}{\sqrt{11}}\boldsymbol{k}$

ii $-2\hat{\boldsymbol{a}} = -\dfrac{6}{\sqrt{11}}\boldsymbol{i} - \dfrac{2}{\sqrt{11}}\boldsymbol{j} + \dfrac{2}{\sqrt{11}}\boldsymbol{k}$

b $5\hat{\boldsymbol{a}} = \dfrac{15}{\sqrt{11}}\boldsymbol{i} + \dfrac{5}{\sqrt{11}}\boldsymbol{j} - \dfrac{5}{\sqrt{11}}\boldsymbol{k}$

4 $|\boldsymbol{a}| = \sqrt{1^2 + 1^2 + 5^2}$

$= \sqrt{27} = 3\sqrt{3}$

$|\boldsymbol{b}| = \sqrt{2^2 + 1^2 + 3^2}$

$= \sqrt{14}$

$\boldsymbol{c} = \frac{|\boldsymbol{a}|}{|\boldsymbol{b}|}\boldsymbol{a}$

$= \frac{\sqrt{14}}{3\sqrt{3}}(\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k})$

$= \frac{\sqrt{42}}{9}(\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k})$

5 a $\overrightarrow{PQ} = \boldsymbol{i} - 3\boldsymbol{j}$

b $|\overrightarrow{PQ}| = \sqrt{1^2 + 3^2 + 0^2}$

$= \sqrt{10}$

c $\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$

$= \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$

$= \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + \frac{1}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j}$

$= \frac{3}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} - \boldsymbol{k}$

6 a $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$

$= \boldsymbol{i} + 3\boldsymbol{j}$

$\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OE}$

$= \frac{1}{3}\boldsymbol{i} + \boldsymbol{j}$

$\overrightarrow{BF} = \overrightarrow{OD}$

$= \boldsymbol{i}$

$\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BF}$

$= \frac{1}{2}\boldsymbol{i}$

$\overrightarrow{ON} = \overrightarrow{OC} + \overrightarrow{CB} + \overrightarrow{BN}$

$= \frac{1}{2}\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$

$= \frac{1}{2}\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k} - \left(\frac{1}{3}\boldsymbol{i} + \boldsymbol{j}\right)$

$= \frac{1}{6}\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

b $|\overrightarrow{MN}| = \sqrt{\left(\frac{1}{6}\right)^2 + 2^2 + 2^2}$

$= 4\sqrt{\frac{1 + 144 + 144}{36}}$

$= \sqrt{\frac{289}{36}}$

$= \frac{17}{6}$

Solutions to technology-free questions

1 **a** $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ if $\boldsymbol{a} = k\boldsymbol{b}$, where k is a constant.

$$7\boldsymbol{i} + 6\boldsymbol{j} = k(2\boldsymbol{i} + x\boldsymbol{j})$$

$$2k = 7$$

$$k = \frac{7}{2}$$

$$kx = 6$$

$$\frac{7x}{2} = 6$$

$$x = \frac{12}{7}$$

b

$$|\boldsymbol{a}| = \sqrt{7^2 + 6^2} = \sqrt{85}$$

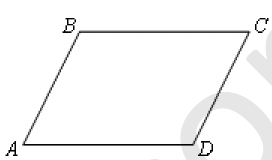
$$|\boldsymbol{b}| = \sqrt{2^2 + x^2} = |\boldsymbol{a}| = \sqrt{85}$$

$$\therefore\ x^2 + 4 = 85$$

$$x^2 = 81$$

$$x = \pm 9$$

2



$$A = (2, -1)$$

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = 5\boldsymbol{i} + 3\boldsymbol{j}$$

$$B = (5, 3)$$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \boldsymbol{i} + 9\boldsymbol{j}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{i} + 9\boldsymbol{j} = 3\boldsymbol{i} + 8\boldsymbol{j}$$

$$C = (3, 8)$$

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = 4\boldsymbol{j}$$

$$D = (0, 4)$$

3

$$\boldsymbol{a} + p\boldsymbol{b} + q\boldsymbol{c} = (2 + 2p - q)\boldsymbol{i} + (-3 - 4p - 4q)\boldsymbol{j} + (1 + 5p + 2q)\boldsymbol{k}$$

To be parallel to the x-axis,

$$\boldsymbol{a} + p\boldsymbol{b} + q\boldsymbol{c} = k\boldsymbol{i}$$

$$1 + 5p + 2q = 0$$

$$2 + 10p + 4q = 0 \quad ①$$

$$-3 - 4p - 4q = 0 \quad ②$$

① + ②:

$$-1 + 6p = 0$$

$$p = \frac{1}{6}$$

$$1 + \frac{5}{6} + 2q = 0$$

$$2q = -\frac{11}{6}$$

$$q = -\frac{11}{12}$$

4 **a**

$$\overrightarrow{PQ} = (3\boldsymbol{i} - 7\boldsymbol{j} + 12\boldsymbol{k}) - (2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}) = \boldsymbol{i} - 5\boldsymbol{j} + 8\boldsymbol{k}$$

$$|\overrightarrow{PQ}| = \sqrt{1^2 + 5^2 + 8^2} = \sqrt{90} = 3\sqrt{10}$$

b $\dfrac{1}{3\sqrt{10}}(\boldsymbol{i} - 5\boldsymbol{j} + 8\boldsymbol{k})$

5 $\overrightarrow{AB} = 4\boldsymbol{i} + 8\boldsymbol{j} + 16\boldsymbol{k}$

$\overrightarrow{AC} = x\boldsymbol{i} + 12\boldsymbol{j} + 24\boldsymbol{k}$

For A, B and C to be collinear, we need $\overrightarrow{AC} = k\overrightarrow{AB}$.

$$x\boldsymbol{i} + 12\boldsymbol{j} + 24\boldsymbol{k} = k(4\boldsymbol{i} + 8\boldsymbol{j} + 16\boldsymbol{k})$$

$$8k = 12$$

$$k = 1.5$$

$$x = 4k$$

$$= 6$$

6 a $\overrightarrow{OA} = \sqrt{4^2 + 3^2}$

$= 5$

Unit vector $= \dfrac{1}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$

b $\overrightarrow{OC} = \dfrac{16}{5}\overrightarrow{OA}$

$= \dfrac{16}{5} \times \dfrac{1}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$

$= \dfrac{16}{25}(4\boldsymbol{i} + 3\boldsymbol{j})$

7 a i $\overrightarrow{SQ} = \boldsymbol{b} + \boldsymbol{a} = \boldsymbol{a} + \boldsymbol{b}$

ii $\overrightarrow{TQ} = \dfrac{1}{3}\overrightarrow{SQ}$

$= \dfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

iii $\overrightarrow{RQ} = -2\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{a} = \boldsymbol{b} - \boldsymbol{a}$

iv $\overrightarrow{PT} = \overrightarrow{PQ} + \overrightarrow{QT}$

$= \overrightarrow{PQ} - \overrightarrow{TQ}$

$= \boldsymbol{a} - \dfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

$= \dfrac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$

v $\overrightarrow{TR} = \overrightarrow{TQ} + \overrightarrow{QR}$

$= \overrightarrow{TQ} - \overrightarrow{RQ}$

$= \dfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b}) - (\boldsymbol{b} - \boldsymbol{a})$

$= \dfrac{1}{3}(4\boldsymbol{a} - 2\boldsymbol{b})$

$= \dfrac{2}{3}(2\boldsymbol{a} - \boldsymbol{b})$

b $2\overrightarrow{PT} = \overrightarrow{TR}$

P, T and R are collinear.

8 $\boldsymbol{a} = \boldsymbol{b}$

a i $-s\boldsymbol{j} = 2\boldsymbol{j}$

$s = -2$

ii $5\boldsymbol{i} = t\boldsymbol{i}$

$t = 5$

iii $2\boldsymbol{k} = u\boldsymbol{k}$

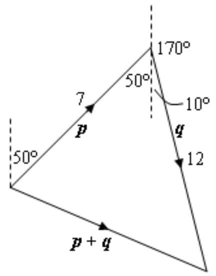
$u = 2$

b $\hat{\boldsymbol{a}} = \sqrt{5^2 + 2^2 + 2^2}$

$= \sqrt{25 + 4 + 4}$

$= \sqrt{33}$

9



Use the cosine rule

$|\boldsymbol{p} + \boldsymbol{q}|^2 = 7^2 + 12^2$

$\quad - 2 \times 7 \times 12 \times \cos 60°$

$\quad = 109$

$|\boldsymbol{p} + \boldsymbol{q}| = \sqrt{109}$

10 a $\boldsymbol{a} + 2\boldsymbol{b} = (5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

$\quad + 2 \times (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$\quad = 11\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

b $|\boldsymbol{a}| = \sqrt{5^2 + 2^2 + 1^2}$

$\quad = \sqrt{30}$

c $\hat{\boldsymbol{a}} = \frac{1}{\sqrt{30}}(5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

d $\boldsymbol{a} - \boldsymbol{b} = (5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) - (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$\quad = 2\boldsymbol{i} + 4\boldsymbol{j}$

11 a $\overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB}$

$\quad = (3\boldsymbol{i} + 4\boldsymbol{j}) - (4\boldsymbol{i} - 6\boldsymbol{j})$

$\quad = -\boldsymbol{i} + 10\boldsymbol{j}$

$C = (-1, 10)$

b $\boldsymbol{i} + 24\boldsymbol{j} = h(3\boldsymbol{i} + 4\boldsymbol{j}) + k(4\boldsymbol{i} - 6\boldsymbol{j})$

$3h + 4k = 1$

$4h - 6k = 24$

Multiply the first equation by 3 and the second equation by 2.

$9h + 12k = 3 \quad ①$

$8h - 12k = 48 \quad ②$

① + ②:

$17h = 51$

$h = 3$

$9 + 4k = 1$

$k = -2$

12 $m\boldsymbol{p} + n\boldsymbol{q} = 3m\boldsymbol{i} + 7m\boldsymbol{j} + 2n\boldsymbol{i} - 5n\boldsymbol{j}$

$\quad = 8\boldsymbol{i} + 9\boldsymbol{j}$

$3m + 2n = 8$

$7m - 5n = 9$

Multiply the first equation by 5 and the second equation by 2.

$15m + 10n = 40 \quad ①$

$14m - 10n = 18 \quad ②$

① + ② :

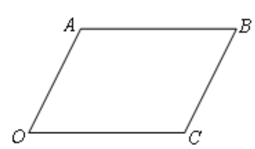
$29m = 58$

$m = 2$

$6 + 2n = 8$

$n = 1$

13 a



$\boldsymbol{b} = \overrightarrow{OB}$

$= \overrightarrow{OA} + \overrightarrow{AB}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= \boldsymbol{a} + c$

b $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$BC = \boldsymbol{c} - \boldsymbol{b}$

$AB : BC = 3 : 2$

$\frac{AB}{BC} = \frac{3}{2}$

$2AB = 3BC$

$2(\boldsymbol{b} - \boldsymbol{a}) = 3(\boldsymbol{c} - \boldsymbol{b})$

$2\boldsymbol{b} - 2\boldsymbol{a} = 3\boldsymbol{c} - 3\boldsymbol{b}$

$5\boldsymbol{b} = 2\boldsymbol{a} + 3\boldsymbol{c}$

$\boldsymbol{b} = \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$

14 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}, \boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$

a $\boldsymbol{a} \cdot \boldsymbol{a} = 13$

b $\boldsymbol{b} \cdot \boldsymbol{b} = 10$

c $\boldsymbol{c} \cdot \boldsymbol{c} = 8$

d $\boldsymbol{a} \cdot \boldsymbol{b} = -11$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = (2\boldsymbol{i} - 3\boldsymbol{j}) \cdot (-3\boldsymbol{i} + \boldsymbol{j}) = -9$

f

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{c}$$
$$= 13 + 2 - 11 - 4$$
$$= 0$$

g $\boldsymbol{a} + 2\boldsymbol{b} = 3\boldsymbol{j}$

$3\boldsymbol{c} - \boldsymbol{b} = -5\boldsymbol{i} - 9\boldsymbol{j}$

$\therefore (\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b}) = -27$

15 $\overrightarrow{OA} = \boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$

$\overrightarrow{OB} = \boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j}$

$\overrightarrow{OC} = \boldsymbol{c} = -5\boldsymbol{i} + 3\boldsymbol{j}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{i} + 5\boldsymbol{j}$

$= -\boldsymbol{i} + 4\boldsymbol{j}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -3\boldsymbol{i} - 5\boldsymbol{j} - 5\boldsymbol{i} + 3\boldsymbol{j}$

$= -8\boldsymbol{i} - 2\boldsymbol{j}$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 8 - 8 = 0.$

Hence there is a right angle at B.

16 $\boldsymbol{p} = 5\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} + t\boldsymbol{j}$

a If $\boldsymbol{p} + \boldsymbol{q}$ is parallell to $\boldsymbol{p} + \boldsymbol{q}$ there exists a non-zeo real number k such that.

$k(\boldsymbol{p} + \boldsymbol{q}) = \boldsymbol{p} - \boldsymbol{q}$

That is,

$k(7\boldsymbol{i} + (3 + t)\boldsymbol{j} = 3\boldsymbol{i} + (3 - t)\boldsymbol{j}$

Hence $7k = 3$

$k = \frac{3}{7}$

$k(3 + t) = (3 - t)$

$\therefore 3(3 + t) = 7(3 - t)$

$\therefore 9 + 3t = 21 - 7t$

$10t = 12$

$t = \frac{6}{5}$

b $\boldsymbol{p} - 2\boldsymbol{q} = 5\boldsymbol{i} + 3\boldsymbol{j} - 2(2\boldsymbol{i} + t\boldsymbol{j})$

$= \boldsymbol{i} + (3 - 2t)\boldsymbol{j}$

$\boldsymbol{p} + 2\boldsymbol{q} = 5\boldsymbol{i} + 3\boldsymbol{j} + 2(2\boldsymbol{i} + t\boldsymbol{j})$

$= 9\boldsymbol{i} + (3 + 2t)\boldsymbol{j}$

Since the vectors are perpendicular

$$(\boldsymbol{i} + (3 - 2t)\boldsymbol{j}) \cdot (9\boldsymbol{i} + (3 + 2t)\boldsymbol{j}) = 0$$
$$9 + (3 - 2t)(3 + 2t) = 0$$
$$9 + (9 - 4t^2) = 0$$
$$4t^2 = 18$$
$$t^2 = \frac{9}{2}$$
$$t = \pm\frac{3}{\sqrt{2}}$$

c
$$|\boldsymbol{p} - \boldsymbol{q}| = |3\boldsymbol{i} + (3 - t)\boldsymbol{j}|$$
$$= \sqrt{9 + (3 - t)^2}$$
$$|\boldsymbol{q}| = |2\boldsymbol{i} + t\boldsymbol{j}|$$
$$= \sqrt{4 + t^2}$$

If $|\boldsymbol{p} - \boldsymbol{q}| = |\boldsymbol{q}|$

then $9 + (3 - t)^2 = 4 + t^2$

$\therefore 9 + 9 - 6t + t^2 = 4 + t^2$

$14 - 6t = 0$

$t = \frac{7}{3}$

17 $\overrightarrow{OA} = \boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{OB} = \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{OC} = \boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$

a **i** $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{i}$

ii $\overrightarrow{AC} = -\boldsymbol{a} + \boldsymbol{c} = -5\boldsymbol{j}$

b The vector resolute $= \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\overrightarrow{AC} \cdot \overrightarrow{AC}} \overrightarrow{AC}$

$= 0$

c 1

Solutions to multiple-choice questions

1 C $v = \begin{bmatrix} 3-1 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$a = 2, b = 4$

2 C $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$

$= -\overrightarrow{AC} + \overrightarrow{AB}$

$= \boldsymbol{u} - \boldsymbol{v}$

3 E $\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} 1+2 \\ -2+3 \end{bmatrix}$

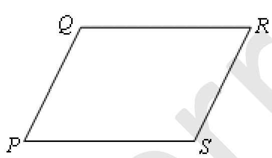
$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

4 A $2\boldsymbol{a} - 3\boldsymbol{b} = 2\begin{bmatrix} 3 \\ -2 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 6 - -3 \\ -4 - 9 \end{bmatrix}$

$= \begin{bmatrix} 9 \\ -13 \end{bmatrix}$

5 B



$\overrightarrow{SQ} = \overrightarrow{SR} + \overrightarrow{RQ}$

$= \overrightarrow{PQ} + -\overrightarrow{QR}$

$= \boldsymbol{p} - \boldsymbol{q}$

6 B $|3\boldsymbol{i} - 5\boldsymbol{j}| = \sqrt{3^2 + (-5)^2}$

$= \sqrt{9 + 25}$

$= \sqrt{34}$

7 A $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= (\boldsymbol{i} - 2\boldsymbol{j}) - (2\boldsymbol{i} + 3\boldsymbol{j})$

$= -\boldsymbol{i} - 5\boldsymbol{j}$

8 C $|\overrightarrow{AB}| = |-\boldsymbol{i} - 5\boldsymbol{j}|$

$= \sqrt{(-1)^2 + (-5)^2}$

$= \sqrt{1 + 25}$

$= \sqrt{26}$

9 D $|\boldsymbol{a}| = \sqrt{2^2 + 3^2}$

$= \sqrt{13}$

$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{13}}(2\boldsymbol{i} + 3\boldsymbol{j})$

10 C $|\boldsymbol{a}| = \sqrt{3^2 + 1^2 + 3^2}$

$= \sqrt{19}$

$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{19}}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$

Solutions to extended-response questions

1 $\begin{bmatrix}1\\0\end{bmatrix}$ is in the east direction and $\begin{bmatrix}0\\1\end{bmatrix}$ in the north direction.

a $\overrightarrow{OP} = -32\begin{bmatrix}0\\1\end{bmatrix} - 31\begin{bmatrix}1\\0\end{bmatrix}$

$= \begin{bmatrix}-31\\-32\end{bmatrix}$

b The ship is travelling parallel to the vector $\boldsymbol{u} = \begin{bmatrix}4\\3\end{bmatrix}$ with speed 20 km/h.

The unit vector in the direction of $\boldsymbol{u}$ is $\frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix}$.

The vector $\overrightarrow{PR} = \frac{20}{5}\begin{bmatrix}4\\3\end{bmatrix}$

$= \begin{bmatrix}16\\12\end{bmatrix}$

The position vector ofthe ship is

$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= \begin{bmatrix}-31\\-32\end{bmatrix} + \begin{bmatrix}16\\12\end{bmatrix}$

$= \begin{bmatrix}-15\\-20\end{bmatrix}$

$= -5\begin{bmatrix}3\\4\end{bmatrix}$

c $|\overrightarrow{OR}| = 5\sqrt{3^2 + 4^2}$

$= 25$

When the ship reaches R, it is 25 km from the lighthouse, and therefore the lighthouse is visible from the ship.

2 $\boldsymbol{p} = 3\boldsymbol{i} + \boldsymbol{j}$ and $q = -2\boldsymbol{i} + 4\boldsymbol{j}$

a $\therefore\ |\boldsymbol{p} - \boldsymbol{q}| = |3\boldsymbol{i} + \boldsymbol{j} - (-2\boldsymbol{i} + 4\boldsymbol{j})|$

$= |5\boldsymbol{i} - 3\boldsymbol{j}|$

$= \sqrt{25 + 9}$

$= \sqrt{34}$

b $|\boldsymbol{p}| = \sqrt{9+1}$

$= \sqrt{10}$

and $|\boldsymbol{q}| = \sqrt{4+16}$

$= 2\sqrt{5}$

$\therefore\ |\boldsymbol{p}| - |\boldsymbol{q}| = \sqrt{10} - 2\sqrt{5}$

c $3\boldsymbol{i} + \boldsymbol{j} + 2(-2\boldsymbol{i} + 4\boldsymbol{j}) + \boldsymbol{r} = \boldsymbol{0}$

$3\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{i} + 8\boldsymbol{j} + \boldsymbol{r} = \boldsymbol{0}$

$-\boldsymbol{i} + 9\boldsymbol{j} + \boldsymbol{r} = \boldsymbol{0}$

Hence $\boldsymbol{r} = \boldsymbol{i} - 9\boldsymbol{j}$

3 $\boldsymbol{a} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 7 \\ 9 \\ 7 \end{bmatrix}$ and $\boldsymbol{d} = \begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$

a $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = k\boldsymbol{d}$

$$\therefore \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \\ 7 \end{bmatrix} = k\begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 13 \\ 6 \\ 1 \end{bmatrix} = k\begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$$

Therefore $k = \frac{1}{2}$ and $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = \frac{1}{2}\boldsymbol{d}$

b $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{d}$

$$\therefore\ x\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$$

The following equations are formed:

$-2x + 11y = 26$ . . . ①

$x + 7y = 12$ . . . ②

$2x + 3y = 2$ . . . ③

Add ① and ③

$14y = 28$

$\therefore\ y = 2$

Substitute in ③

$2x + 6 = 2$

$\therefore\ x = -2$

Equation ② must be checked

$-2 + 14 = 12$

Therefore $-2\boldsymbol{a} + 2\boldsymbol{b} = \boldsymbol{d}$.

c $p\boldsymbol{a} + q\boldsymbol{b} - r\boldsymbol{c} = \mathbf{0}$

From parts **a** and **b**

$$\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = \frac{1}{2}\boldsymbol{d} \qquad \ldots ①$$

$$-2\boldsymbol{a} + 2\boldsymbol{b} = \boldsymbol{d} \qquad \ldots ②$$

From ① $2\boldsymbol{a} + 4\boldsymbol{b} - 2\boldsymbol{c} = \boldsymbol{d}$

Therefore from ②

$$-2\boldsymbol{a} + 2\boldsymbol{b} = 2\boldsymbol{a} + 4\boldsymbol{b} - 2\boldsymbol{c}$$

$\therefore\ 4\boldsymbol{a} + 2\boldsymbol{b} - 2\boldsymbol{c} = \mathbf{0}$

Hence $p = 4, q = 2$ and $r = 2$. (Other answers are possible e.g. $p = 2, q = 1,$ $r = -1$)

4 a $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= \begin{bmatrix} 5 \\ 8 \end{bmatrix} + \begin{bmatrix} 20 \\ -15 \end{bmatrix}$$

$$= \begin{bmatrix} 25 \\ -7 \end{bmatrix}$$

The coordinates of Q are $(25, -7)$.

$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

$$= \begin{bmatrix} -25 \\ 7 \end{bmatrix} + \begin{bmatrix} 32 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 24 \end{bmatrix}$$

b $\overrightarrow{RS} = \overrightarrow{QP}$

$$= \begin{bmatrix} -20 \\ 15 \end{bmatrix}$$

$\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{RS}$

$$= \begin{bmatrix} 32 \\ 17 \end{bmatrix} + \begin{bmatrix} -20 \\ 15 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 32 \end{bmatrix}$$

Hence the coordinates of S are $(12, 32)$.

5 **a** $\overrightarrow{OP} = 4\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 12 \\ 4 \end{bmatrix}$$

The coordinates of P are $(12, 4)$.

b $\overrightarrow{PM} = \overrightarrow{PO} + \overrightarrow{OM}$

$$= \begin{bmatrix} -12 \\ -4 \end{bmatrix} + \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} k - 12 \\ -4 \end{bmatrix}$$

c $|\overrightarrow{OP}| = \sqrt{12^2 + 4^2}$

$$= \sqrt{160}$$

$$= 4\sqrt{10}$$

Now $|\overrightarrow{OM}| = k$

and, from part **b**, $\overrightarrow{PM} = \begin{bmatrix} k - 12 \\ -4 \end{bmatrix}$

$$\therefore\ |\overrightarrow{PM}| = \sqrt{(k-12)^2 + 16}$$

For triangle OPM to be right-angled at P, Pythagoras' theorem has to be satisfied.

$$\text{i.e. } |\overrightarrow{OP}|^2 + |\overrightarrow{PM}|^2 = |\overrightarrow{OM}|^2$$

$$\therefore\ 160 + (k - 12)^2 + 16 = k^2$$

$$\therefore\ 160 + k^2 - 24k + 160 = k^2$$

$$\therefore\ 24k = 320$$

$$\therefore\ 3k = 40$$

$$\therefore\ k = \frac{40}{3}$$

d If M has coordinates $(9, 0)$ then,
if $\angle OPX = \alpha°, \tan\alpha° = 3$
and if $\angle MPX = \beta°, \tan\beta° = \frac{3}{4}$
$\therefore$ Angle $\theta = \alpha - \beta$

$$= \tan^{-1}(3) - \tan^{-1}\left(\frac{3}{4}\right)$$

$$= 34.7°, \text{ correct to one decimal place}$$

Chapter 22 – Kinematics

Solutions to Exercise 22A

1 a When $t = 0, x = 12$.
12 cm to the right of O

b When $t = 5, x = 5^2 - 7 \times 5 + 12$
$= 2$
2 cm to the right of O

c $v = \frac{dx}{dt}$
$= 2t - 7$
When $t = 0, v = -7$.
7cm/s to the left

d $v = 0$ when $2t - 7 = 0$
$t = 3.5$
When $t = 3.5$,
$x = 3.5^2 - 7 \times 3.5 + 12$
$= -0.25$
$t = 3.5$; the particle is 0.25 cm to the left of O.

e Average velocity
$= \frac{\text{change in position}}{\text{change in time}}$
$= \frac{2 - 12}{5}$
$= -2$ cm/s

f Average speed $= \frac{\text{distance travelled}}{\text{change in time}}$
For the first 3.5 s, the particle has travelled 12.25 cm.
From 3.5 s to 5 s, the particle has travelled $2 - (-0.25) = 2.25$ cm.

Average speed $= \frac{12.25 + 2.25}{5}$
$= \frac{14.5}{5}$
$= 2.9$ cm/s

2 a $v = \frac{dx}{dt}$
$= 2t - 7$
$v = 0$ when $2t - 7 = 0$
$t = 3.5$ s

b $a = \frac{dv}{dt}$
$= 2$ m/s^2

c When $t = 0, x = 10$.
When $t = 3.5, x = 3.5^2 - 7 \times 3.5 + 10$
$= -2.25$
For the first 3.5 s, the particle has travelled 12.25 m.
When $t = 5, x = 5^2 - 7 \times 5 + 10$
$= 0$
From 3.5 s to 5 s, the particle has travelled 2.25 m.
Distance travelled $= 12.25 + 2.25$
$= 14.5$ m

d $v = 2t - 7 = -2$
$2t = 5$
$t = 2.5$
$x = 2.5^2 - 7 \times 2.5 + 19$
$= -1.25$
After 2.5 s, when the particle is 1.25 m left of O.

3 **a** When $t = 0, x = -3$.

$v = \frac{dx}{dt}$

$= 3t^2 - 22t + 24$

When $t = 0, v = 24$.

3 cm to the left of O and moving at 24 cm/s to the right.

b $v = \frac{dx}{dt}$

$= 3t^2 - 22t + 24$

c $v = 0$ when

$3t^2 - 22t + 24 = 0$

$(3t - 4)(t - 6) = 0$

$t = \frac{4}{3}$ or 6

After $\frac{4}{3}$ s and after 6 s

d When $t = \frac{4}{3}$,

$x = \left(\frac{4}{3}\right)^3 - 11 \times \left(\frac{4}{3}\right)^2 + 24 \times \left(\frac{4}{3}\right) - 3$

$= \frac{64}{27} - \frac{176}{9} \times \frac{3}{3} + 32 - 3$

$= -\frac{464}{27} + 29$

$= 11\ \frac{22}{27}$

When $t = 6$,

$x = 6^3 - 11 \times 6^2 \times 6 - 3$

$= -39$

39 cm to the left of O and $11\ \frac{22}{27}$ cm to the right of O

e $v < 0$ when $(3t - 4)(t - 6) = 0$

This is a parabola with a minimum value.

$\therefore \quad v < 0$ when $\frac{4}{3} < t < 6$

Length of time $= 6 - \frac{4}{3}$

$= \frac{14}{3}$

$= 4\frac{2}{3}$ s

f $a = \frac{dv}{dt}$

$= 6t - 22$ m/s^2

g $6t - 22 = 0$

$t = \frac{22}{6} = \frac{11}{3}$

$v = 3t^2 - 22t + 24$

$= 3 \times \left(\frac{11}{3}\right)^2 - 22 \times \frac{11}{3} + 24$

$= \frac{121}{3} - \frac{242}{3} + 24$

$= 16\ \frac{2}{3}$

$x = \left(\frac{11}{3}\right)^3$

$= 11 \times \left(\frac{11}{3}\right)^2 + 24 \times \frac{11}{3} - 3$

$= \frac{1331}{27} - \frac{1331}{9} \times \frac{3}{3} + 88 - 3$

$= -13\ \frac{16}{27}$

The acceleration is zero after $\frac{11}{3}$ s, when the velocity is $16\ \frac{1}{3}$ cm/s to the left and its position is $13\ \frac{16}{27}$ cm left of O.

4 a $v = 6t^2 - 10t + 4$

When $v = 0$:

$$6t^2 - 10t + 4 = 0$$

$$3t^2 - 5t + 2 = 0$$

$$(3t - 2)(t - 1) = 0$$

$$t = \frac{2}{3} \text{ or } 1$$

$$a = 12t - 10$$

$t = \frac{2}{3}$:

$$a = 12 \times \frac{2}{3} - 10$$

$$= -2$$

$t = 1$:

$$a = 12 \times 1 - 10$$

$$= 2$$

Velocity is zero after $\frac{2}{3}$ s when the acceleration is 2 cm/s^2 to the left, and after 1 s when the acceleration is 2 cm/s^2 to the right.

b $a = 12t - 10$

$$= 0$$

$$t = \frac{10}{12} = \frac{5}{6}$$

Find v when $a = \frac{5}{6}$:

$$v = 6t^2 - 10t + 4$$

$$= 6 \times \left(\frac{5}{6}\right)^2 - 10 \times \frac{5}{5} + 4$$

$$= \frac{25}{6} - \frac{50}{6} + 4 = -\frac{1}{6}$$

Acceleration is zero after $\frac{5}{6}$ s, at which time the velocity is $\frac{1}{6}$ cm/s to the left.

5 The particle passes through O when $x = 0$.

$$t^3 - 13t^2 + 46t - 48 = 0$$

Trial and error will give $x = 0$ when $t = 2$.

This means $(t - 2)$ is a factor of $t^3 - 13t^2 + 46t - 48$.

$$t^3 - 13t^2 + 46t - 48$$

$$= (t - 2)(t^2 - 11t + 24)$$

$$= 0$$

Factorising the quadratic gives

$$(t - 2)(t - 3)(t - 8) = 0$$

$$t = 2, 3 \text{ or } 8$$

$$v = \frac{dx}{dt}$$

$$= 3t^2 - 26t + 46$$

$$a = \frac{dv}{dt}$$

$$= 6t - 26$$

$t = 2$:

$$v = 3 \times 4 - 26 \times 2 + 46$$

$$= 6 \text{ cm/s}$$

$$a = 6 \times 2 - 26$$

$$= -14 \text{ cm/s}$$

$t = 3$:

$$v = 3 \times 9 - 26 \times 3 + 46$$

$$= -5 \text{ cm/s}$$

$$a = 6 \times 3 - 26$$

$$= -8 \text{ cm/s}$$

$t = 8$:

$$v = 3 \times 64 - 26 \times 8 + 46$$

$$= 30 \text{ cm/s}$$

$$a = 6 \times 8 - 26$$

$$= -22 \text{ cm/s}$$

6 **a** They will be at the same position when

$$t^2 - 2t - 2 = t + 2$$

$$t^2 - 3t - 4 = 0$$

$$(t - 4)(t + 1) = 0$$

$$t = 4 \text{ or } -1$$

After 4 s, or 1 s before the start. (Note: In some cases, motion is not considered before $t = 0$, and negative values of t may be discarded.)

b The velocities are 1 cm/s and $2t - 2$ cm/s.

$$2t - 2 = 1$$

$$2t = 3$$

$$t = \frac{3}{2}$$

After $\frac{3}{2}$ s.

Solutions to Exercise 22B

1 a $x = 2t^2 - 6t + c$

When $t = 0$, $x = 0$.

$\therefore \quad 0 = 0 - 0 + c$

$c = 0$

$x = 2t^2 - 6t$

b $t = 3$

$x = 2 \times 3^2 - 6 \times 3$

$= 0$

It will be at the origin, O.

c Consider when $v = 0$:

$4t - 6 = 0$

$t = \frac{3}{2}$

$x = 2 \times \left(\frac{3}{2}\right)^2 - 6 \times \frac{3}{2}$

$= -4\frac{1}{2}$

The particle will travel $4\frac{1}{2}$ cm to the left of the origin and back, for a total of 9 cm.

d Average velocity

$= \frac{\text{change in position}}{\text{change in time}}$

$= \frac{0}{3} = 0 \text{ cm/s}$

e Average speed $= \frac{\text{distance travelled}}{\text{change in time}}$

$= \frac{9}{3} = 3 \text{ cm/s}$

2 a $x = t^3 - 4t^2 + 5t + c$

When $t = 0$, $x = 4$.

$\therefore \quad 4 = 0 - 0 + 0 + c$

$c = 4$

$x = t^3 - 4t^2 + 5t + 4$

$a = \frac{dv}{dt}$

$= 6t - 8$

b $3t^2 - 8t + 5 = 0$

$(3t - 5)(t - 1) = 0$

$t = \frac{5}{3}$ or 1

When $t = \frac{5}{3}$,

$x = \left(\frac{5}{3}\right)^3 - 4 \times \left(\frac{5}{3}\right)^2 + 5 \times \frac{5}{3} + 4$

$= \frac{125}{27} - \frac{100}{9} + \frac{25}{3} + 4$

$= 5\frac{23}{27}$

When $t = 1$,

$x = 1^3 - 4 \times 1^2 + 5 \times 1 + 4$

$= 6$

c When $t = \frac{5}{3}$,

$a = 6 \times \frac{5}{3} - 8$

$= 2 \text{ cm/s}^2$

When $t = 1$,

$a = 6 \times 1 - 8$

$= -2 \text{ cm/s}^2$

3 $v = 10t + c$

$x = 5t^2 + ct + d$

When $t = 2$:

$x = 5 \times 2^2 + 3c + d = 0$

$2c + d = -20$ ①

When $t = 3$:

$x = 5 \times 3^2 + 3c + 2 = 25$

$3c + d = -20$ ②

② − ① : $c = 0$

$d = -20$

$x = 5t^2 - 20$

When $t = 0, x = 20$

20 m to the right of O

4 $a = 2t - 3$

$v = t^2 - 3t + c$

When $t = 0, v = 3$.

$3 = 0 - 0 + c$

$c = 3$

$v = t^2 - 3t + 3$

$$x = \frac{t^3}{3} - \frac{3t^2}{2} + 3t + d$$

When $t = 0, x = 2$.

$2 = 0 - 0 + 0 + d$

$d = 2$

$$x = \frac{t^3}{3} - \frac{3t^2}{2} + 3t + 2$$

When $t = 10$,

$$x = \frac{10^3}{3} - \frac{3 \times 10^2}{2} + 3 \times 10 + 2$$

$$= \frac{2000 - 900}{6} + 32$$

$$= 215\,\frac{1}{3}$$

$v = t^2 - 3t - 3$

$= 10^2 - 3 \times 10 + 3$

$= 73$

5 **a** $a = -10$

$v = -10t + c$

When $t = 0, v = 25$.

$25 = 0 + c$

$c = 25$

$v = -10t + 25$

b $v = -10t + 25$

$x = -5t^2 + 25t + d$

When $t = 0, x = 0$.

(Define the point of projection as $x = 0$, the origin.)

$0 = 0 + 0 + d$

$d = 0$

$x = -5t^2 + 25t$

c Maximum height occurs when $v = 0$.

$v = -10t + 25 = 0$

$$t = \frac{25}{10} = \frac{5}{2}$$

2.5 s after projection

d When $t = 2.5$,

$x - 5t^2 + 25t$

$= -5 \times 2.5^2 + 25 \times 2.5$

$= 31.25$ m

e $x = -5t^2 + 25t = 0$

$-5t(t - 5) = 0$

$t = 5$ ($t = 0$ is the start)

6 Define $t = 0$ as the moment the lift passes the 50th floor.

$a = \frac{1}{9}t - \frac{5}{9}$

$v = \frac{1}{18}t^2 - \frac{5}{9}t + c$

$-8 = 0 - 0 + c$

$c = -8$

$v = \frac{1}{18}t^2 - \frac{5}{9} - 8$

$x = \frac{1}{54}t^3 - \frac{5}{18}t^2 - 8t + d$

$50 \times 6 = 0 - 0 - 0 + d$

$d = 300$

$v = 0$ when

$\frac{1}{18}t^2 - \frac{5}{9}t - 8 = 0$

$t^2 - 10t - 8 \times 18 = 0$

$(t - 18)(t + 8) = 0$

$t = 18$

$x = \frac{1}{54}t^3 - \frac{5}{18}t^2$

$- 8t + 300$

$= \frac{1}{54} \times 18^3 - \frac{5}{18}18^2$

$- 8 \times 18 + 300$

$= 174$

$\frac{174}{6} = 29$

It will stop on the 29th floor.

Solutions to Exercise 22C

1 $s = 30, u = 0, a = 1.5$

$$s = ut + \frac{1}{2}at^2$$

$$30 = \frac{1}{2} \times 1.5 \times t^2$$

$$t^2 = 40$$

$$t = \sqrt{40}$$

$$= 2\sqrt{10} \text{ s}$$

2 $u = 25, v = 0, t = 3$

$$s = \frac{1}{2}(u + v)t$$

$$= \frac{1}{2}(25 + 0) \times 3$$

$$= 37.5 \text{ m}$$

3 **a** For constant acceleration,

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

$$= \frac{27}{9} = 3 \text{ m/s}^2$$

b $u = 30, v = 50, a = 3$

$$v = u + at$$

$$50 = 30 + 3t$$

$$3t = 20$$

$$t = \frac{20}{3} = 6\frac{2}{3} \text{ s}$$

c $s = ut + \frac{1}{2}at^2$

$$= \frac{1}{2} \times 3 \times 15^2$$

$$= 337.5 \text{ m}$$

d $200 \text{ km/h} = 200 \div 3.6$

$$= \frac{500}{9} \text{ m/s}$$

$$u = 0, v = \frac{500}{9}, a = 3$$

$$v = u + at$$

$$\frac{500}{9} = 0 + 3t$$

$$3t = \frac{500}{9}$$

$$t = \frac{500}{27}$$

$$= 18\frac{14}{27} \text{ s}$$

4 **a** $45 \text{ km/h} = 45 \div 3.6$

$$= 12.5 \text{ m/s}$$

For constant acceleration,

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

$$= \frac{12.5}{5} = 2.5 \text{ m/s}^2$$

b $s = ut + \frac{1}{2}at^2$

$$= \frac{1}{2} \times 2.5 \times 5^2$$

$$= 31.25 \text{ m}$$

5 **a** $90 \text{ km/h} = 90 \div 3.6$

$$= 25 \text{ m/s}$$

$$u = 0, v = 25, a = 0.5$$

$$v = u + at$$

$$25 = 0 + 0.5t$$

$$0.5t = 25$$

$$t = \frac{2.5}{05} = 50 \text{ s}$$

b $s = ut + \frac{1}{2}at^2$

$= \frac{1}{2} \times 0.5 \times 50^2$

$= 625$ m

6 a 54 km/h = 54 ÷ 3.6

$= 15$ m/s

$u = 15, a = -0.25, s = 250$

$s = ut + \frac{1}{2}at^2$

$250 = 15t + \frac{1}{2} \times -0.25t^2$

Multiply both sides by 8:

$2000 = 120t - t^2$

$t^2 - 120t + 2000 = 0$

$(t - 20)(t - 100) = 0$

$t = 100$ represents the train changing velocity and retuming to this point.

$\therefore \quad t = 20$ s

b $v = u + at$

$= 15 + -0.25 \times 20$

$= 10$ m/s

$= 10 \times 3.6 = 36$ km/h

7 a $v = u + at$

$= 20 + -9.8 \times 4$

$= -19.2$ m/s

b $s = ut + \frac{1}{2}at^2$

$= 20 \times 4 + \frac{1}{2} \times -9.8 \times 4^2$

$= 1.6$ m

8 a $v = u + at$

$= -20 + -9.8 \times 4$

$= -59.2$ m/s

b $s = ut + \frac{1}{2}at^2$

$= -20 \times 4 + \frac{1}{2} \times -9.8 \times 4^2$

$= -158.4$ m

9 a $u = 49, s = 0, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$0 = 49t + \frac{1}{2} \times -9.8 \times t^2$

$0 = 49t - 4.9t^2$

$0 = 4.9t(10 - t)$

$t = 10$ s

b $u = 49, s = 102.9, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$102.9 = 49t + \frac{1}{2} \times -9.8 \times t^2$

$102.9 = 49t - 4.9t^2$

$0 = 4.9t^2 - 49t + 102.9$

Divide by 4.9:

$t^2 - 10t + 21 = 0$

$(t - 3)(t - 7) = 0$

At both 3 s (going up) and 7 s (going down).

10 a $v = u + at$

$= 4.9 - 9.8t$

$= 4.9(1 - 2t)$

b $s = ut + \frac{1}{2}at^2$

$= 4.9t + \frac{1}{2} \times -9.8 \times t^2$

$= 4.9t - 4.9t^2$

$= 4.9t(1 - t)$

This is his displacement from the initial 3 m height.

$\therefore h = 4.9t(1 - t) + 3$

c From part a, the diver's velocity is zero

when

$4.9(1 - 2t) = 0$

$t = \frac{1}{2} = 0.5$

The maximum height reached is

$h = 4.9(0.5)(1 - 0.5) + 3$

$= 4.9 \times 0.25 + 3$

$= 4.225$

d The diver reaches the water when $h = 0$, so:

$4.9t(1 - t) + 3 = 0$

$49t - 49t^2 + 30 = 0$

$49t^2 - 49t - 30 = 0$

$(7t + 3)(7t - 10) = 0$

$t = \frac{10}{7}$ s

Since $t > 0$

11 **a** Maximum height occurs when $v = 0$.

$u = 19.6, a = -9.8, v = 0$

$v = u + at$

$0 = 19.6 - 9.8t$

$t = \frac{19.6}{98} = 2$ s

b $s = ut + \frac{1}{2}at^2$

$= 19.6 \times 2 + \frac{1}{2} \times -9.8 \times 2^2$

$= 19.6$ m

So the maximum height from the foot of the cliff is $19.6 + 24.5 = 44.1$ m.

c $u = 19.6, s = 0, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$0 = 19.6t + \frac{1}{2} \times -9.8 \times t^2$

$0 = 19.6t - 4.9t^2$

$0 = 4.9t(4 - t)$

$t = 4$ s

d $u = 19.6, s = -24.5, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$-24.5 = 19.6t + \frac{1}{2} \times -9.8 \times t^2$

$-24.5 = 19.6t - 4.9t^2$

$0 = 4.9t^2 - 19.6t - 24.5$

Divide by 4.9:

$t^2 - 4t - 5 = 0$

$(t - 5)(t + 1) = 0$

$t = 5$ s

12 Let the distance between P and Q be x m.

$$u = 20, v = 40, s = x$$

$$v^2 = u^2 + 2as$$

$$1600 = 400 + 2ax$$

$$2ax = 1200$$

$$a = \frac{1200}{2x}$$

$$= \frac{600}{x}$$

At the halfway mark,

$$u = 20, a = \frac{600}{x}, s = \frac{x}{2}$$

$$v^2 = u^2 + 2as$$

$$= 400 + 2 \times \frac{600}{x} \times \frac{x}{2}$$

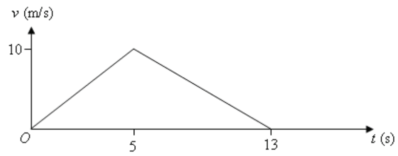
$$= 1000$$

$$v = \sqrt{1000}$$

$$= 10\sqrt{10} \text{ m/s}$$

Solutions to Exercise 22D

1 Draw the velocity–time graph.

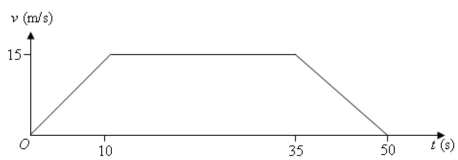


Distance travelled = area under graph

$$= \frac{1}{2} \times 10 \times 13$$

$$= 65 \text{ m}$$

2 Draw the velocity–time graph.



a The area can be calculated using the trapezium formula, or as the sum of two triangles and a rectangle.

$$A = \frac{1}{2}(a + b)h$$

$$= \frac{1}{2} \times (25 + 50) \times 15$$

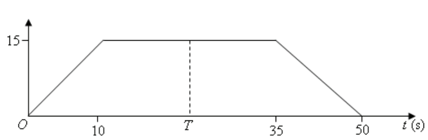
$$= 562.5 \text{ m}$$

b $A = \frac{1}{2}(a + b)h$

$$= \frac{1}{2} \times (25 + 35) \times 15$$

$$= 450 \text{ m}$$

c Let the halfway point be at time T as below.



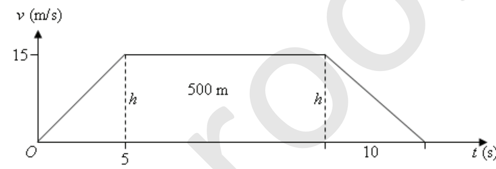
$$\frac{1}{2} \times 10 \times 15 + 15(T - 10) = \frac{562.5}{2}$$

$$75 + 15T - 150 = 281.25$$

$$15T = 356.25$$

$$T = 23.75 \text{ s}$$

3



Since the total distance travelled is 1 km or 1000 m, the combined areas of the two triangles will equal a distance of 500 m.

$$\frac{1}{2} \times 5 \times h + \frac{1}{2} \times 10 \times h = 500$$

$$5h + 10h = 1000$$

$$15h = 1000$$

$$h = \frac{1000}{15}$$

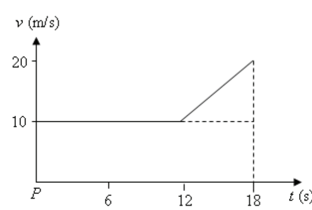
$$= 66\frac{2}{3}$$

Maximum speed = $66\frac{2}{3}$ m/s

4 36 km/h = 36 ÷ 3.6

= 10 m/s.

72 km/h = 20 m/s.

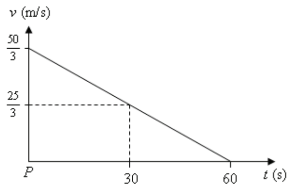


$$\text{Distance} = A = 18 \times 10 + \frac{1}{2} \times 6 \times 10$$

$$= 210 \text{ m}$$

5 60 km/h = 60 ÷ 3.6

$= \frac{50}{3}$ m/s



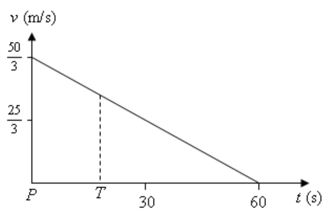
a Distance $= A = \frac{1}{2} \times 60 \times \frac{50}{3}$

$= 500$ m

b Distance $= A = \frac{1}{2} \times \left(\frac{50}{3} + \frac{25}{3}\right) \times 30$

$= 375$ m

c Let the required time be T s.



It is easier to work with the triangle on the right.

This triangle will have area

$= 500 \div 2$

$= 250$

Its base $= (60 - T)$

The sloping line has gradient

$= -\frac{50}{3} \div 60$

$= -\frac{50}{180} = \frac{5}{18}$

$\therefore$ the triangle's height $= \frac{5}{18}(60 - T)$

$$\frac{1}{2} \times (60 - T) \times \frac{5}{18}(60 - T) = 250$$

$$\frac{5}{36}(60 - T)^2 = 250$$

$$(60 - T)^2 = 250 \times \frac{36}{5}$$

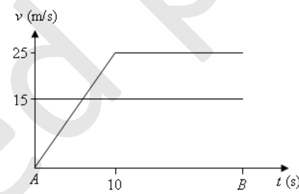
$$= 1800$$

$$60 - T = \sqrt{1800}$$

$$\approx 42.43$$

$$T \approx 17.57 \text{ s}$$

6 Let the common time be T s and the distance x m.



a For the first car, $x = 15t$

For the second car,

$x = \frac{1}{2} \times 10 \times 25 + 25(t - 10)$

$= 125 + 25t - 250$

$= 25t - 125$

$= 15t$

$10t = 125$

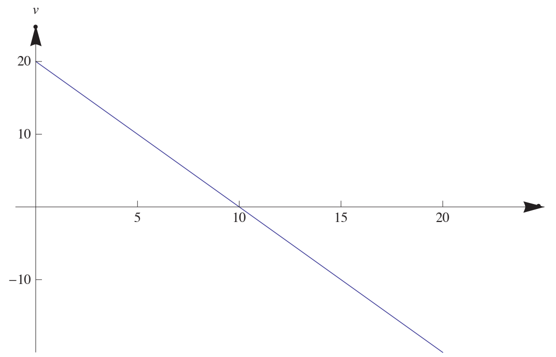
$t = 12.5$ s

b $x = 15t$

$= 15 \times 12.5$

$= 187.5$ m

7 **a**



b The particle moves to the right for the first 10 seconds. Its position at time t is given by
$s = 20t - t^2$
It slows for the first ten seconds. At time $t = 10$, it is 100 m to the right of its starting point. It then heads to the right for 4 seconds. When $t = 14$ it is 84 m from its starting point.

c Total distance travelled = 100 + 16 = 116m.

d It is 84m to the right of its starting point.

8 **a** For the first ten seconds of motion

$$\text{acceleration} = \frac{10 - 0}{10 - 0} = 1 \text{ m/s}^2$$

b From $t = 20$ to $t = 30$ the

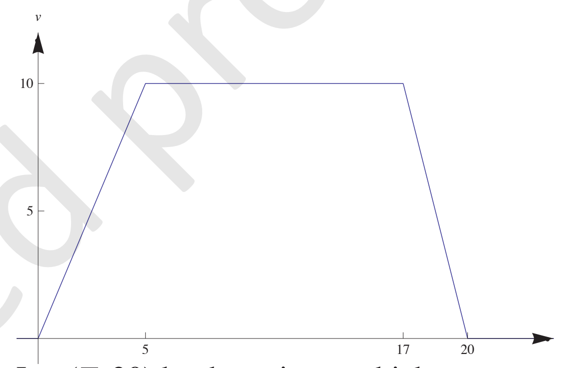
$$\text{acceleration} = \frac{-15 - 10}{30 - 20} = -\frac{5}{2} \text{ m/s}^2$$

c The equation of the line through $(20, 10)$ and $(30, -15)$ is
$v - 10 = -\frac{5}{2}(t - 20)$ which can be written as $v = -\frac{5}{2}t + 60$.
When $v = 0, t = 24$

Distance travelled in the first 24 s $= 5(10 + 24)$
$= 170$m
Distance travelled in next 6 s $= 3 \times 15$
$= 45$m
$\therefore$ total distance $= 45 + 170$
$= 215$m

d Displacement= 170 − 45 = 125 m to the right of its starting point.

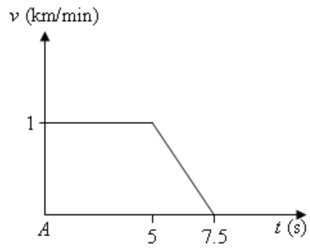
9 **a**



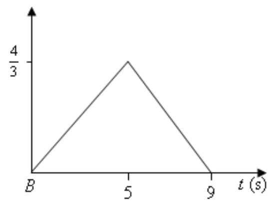
Let $(T, 20)$ be the point at which the constant acceleration ends. The motion ends at $(20, 0)$.
Considering the area of the trapezium:
$5(20 + (T - 5)) = 160$
$\therefore T - 15 = 32$
$\therefore T = 17$

b $\text{acceleration} = \frac{10}{17 - 20} = -\frac{10}{3}$ m/s^2

10 Convert the speeds to km/min.
60 km/h = 1 km/min
80 km/h $= \frac{4}{3}$ km/min
Treat each train separately.
The first train:



The second train:



First train distance

$= 5 \times 1 + \frac{1}{2} \times 2.5 \times 1$

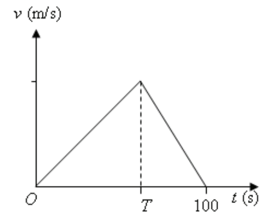
$= 6.25$ km

Second train distance $= \frac{1}{2} \times 9 \times \frac{4}{3}$

$= 6$ km

Since the trains have together travelled less than 14 km, they will not crash.

11 **a** The maximum speed will be the height of the triangle.



$\frac{1}{2} \times 100 \times h = 800$

$50h = 800$

$h = 16$

Maximum speed = 16 m/s

$= 16 \times 3.6$

$= 57.6$ km/h

b The slope of the deceleration is twice as steep as the slope of the acceleration.
Since the heights are equal, the acceleration run will be twice as long as the deceleration run.

$T = \frac{2}{3} \times 100$

$= 66\frac{2}{3}$ s

$= 1$ min $6\frac{2}{3}$ seconds

c Taking the acceleration section,
the gradient $= a = 16 \div 66\frac{2}{3}$

$= \frac{48}{200}$

$= 0.24$ m/s^2

Solutions to technology-free questions

1 a When $t = 0, x = -5$.
5 cm to the left of O

b When $t = 3, x = 3^2 - 4 \times 3 - 5$
$= -8$
8 cm to the left of O

c $v = \frac{dx}{dt}$
$= 2t - 4$
When $t = 0$, -4 cm/s

d $v = 0$ when $2t - 4 = 0$
$t = 2$
When $t = 2, x = 2^2 - 4 \times 2 - 5$
$= -9$
At 2 s, 9 cm to the left of O

e Average velocity
$= \frac{\text{change in position}}{\text{change in time}}$
$= \frac{-8 - (-5)}{3} = -1$ cm/s
1 cm/s to the left

f Distance travelled = distance from $t = 0$ to $t = 2$ (when $v = 0$), plus distance from $t = 2$ to $t = 3$
So distance travelled = 4 + 1
= 5 cm
Average speed $= \frac{\text{distance travelled}}{\text{change in time}}$
$= \frac{5}{3} = 1\frac{2}{3}$ cm/s
(Note: Average velocity has a direction and hence a sign, but average speed does not.)

2 a $v = \frac{dx}{dt}$
$= 3t^2 - 4t$
$a = \frac{dv}{dt} = 6t - 4$
When $t = 0, x = 8, v = 0$ and $a = -4$.
8 cm to the right of O, stationary and accelerating at 4 m/s^2 to the left.

b $v = 0$ when
$3t^2 - 4t = 0t(3t - 4) \quad = 0$
$t = 0$ or $\frac{4}{3}$
$t = 0 : x = 8$ and $a = -4$
So 8 cm to the right, -4 cm/s^2
$t = \frac{4}{3} : x = \frac{64}{27} - \frac{32}{9} + 8 = 6\frac{22}{27}$
$a = 8 - 4 = 4$
So $6\frac{22}{27}$ cm to the right, 4 cm/s^2

3 a Solve $-2t^3 + 3t^2 + 12t + 7 = 0$
Using factors of 7, $t = -1$ gives
$-2 \times (-1)^3 + 3 \times (-1)^2 + 12 \times -1$
$+7 = 0$
Dividing by $(t + 1)$,
$-2t^3 + 3tg^2 + 12t + 7$
$= -(t + 1)(2t^2 - 5t - 7)$
$= -(t + 1)(t + 1)(2t - 7)$
$= 0$
$t = 3.5$, as $t = -1$ is usually discarded.
$v = \frac{dx}{dt}$
$= -6t^2 + 6t + 12$
$a = \frac{dv}{dt} = -12t + 6$
When $t = 3.5$

$v = -6 \times 3.5^2 + 6 \times 3.5 + 12$

$= -40.5$ cm/s

$a = \frac{dv}{dt}$

$= -12 \times 3.5 + 6$

$= -36$ cm/s^2

b $v = 0$

$-6t^2 + 6t + 12 = 0$

$t^2 - t - 2 = 0$

$(t + 1)(t - 2) = 0$

$t = 2$

After 2s (discarding $t = -1$)

c Distance travelled in first 2 seconds

$= (-2 \times 2^3 + 3 \times 2^2 + 12 \times 2 + 7)$

$- (-0 + 0 + 0 + 7)$

$= 20$ cm

Distance travelled from $t = 2$ to $t = 3$ is

$|(-2 \times 3^2 + 3 \times 32 + 12 \times 3 + 7)$

$- (-2 \times 2^3 + 3 \times 2^2 + 12 \times 2 + 7)|$

$= |16 - 27|$

$= 11$ cm

Distance travelled in first 3 s

$= 20 + 11$

$= 31$ cm

4 a i $x_1\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2$

$= \frac{1}{8} - \frac{1}{4}$

$= -\frac{1}{8}$

$\frac{1}{8}$ cm to the left

ii $a_1(t) = \frac{d^2x}{dt^2}$

$= 6t - 2$

$a_1\left(\frac{1}{2}\right) = 6 \times \frac{1}{2} - 2$

$= 1$ cm/s^2

iii $v_2(t) = \frac{dx}{dt} = 2t$

$v_2 = 2 \times \frac{1}{2}$

$= 1$ cm/s

b i $x_1(t) = x_2(t)$

$t^3 - t^2 = t^2$

$t^3 - 2t^2 = 0$

$t^2(t - 2) = 0$

$t = 0$ and 2

The particles will have the same position at the start and after 2 s.

ii Let the distance between the particles be $y = |t^3 - 2t^2|$.

Define $y = t^3 - 2t^2$:

$\frac{dy}{dt} = 3t^2 - 4t$

$= t(3t - 4)$

$= 0$ when $t = 0$ and $\frac{4}{3}$

When $t = 0, y = 0$.

When $t = \frac{4}{3}, y = \frac{64}{27} - \frac{32}{9}$

$= -1\frac{5}{27}$

When $t = 2, y = 8 - 2 \times 4$

$= 0$

The maximum distance the particles are apart in the first 2 s is $\frac{32}{27} = 1\frac{5}{27}$ cm

5 **a** $a = 6t$

$v = 3t^2 + c$

When $t = 0, v = 0$.

$0 = 0 + c$

$c = 0$

$\therefore \quad v = 3t^2$

When $t = 2, v = 3 \times 4$

$= 12$ m/s

b $v = 3t^2$

$x = t^3 + d$

When $t = 0, x = 0$.

$0 = 0 + d$

$d = 0$

$x = t^3$

Since the particle starts at the origin, its displacement is $s = x = t^3$.

6 **a** $a = 3 - 2t$

$v = 3t^2 - t^2 + c$

When $t = 0, v = 4$.

$4 = 0 - 0 + c$

$c = 4$

$v = 3t - t^2 + 4 = 0$

$-(t^2 - 3t - 4) = 0$

$-(t - 4)(t + 1) = 0$

$t = 4$

After 4 s

b $v = 3t - t^2 + 4$

$x = \dfrac{3t^2}{2} - \dfrac{t^3}{3} + 4t + d$

When $t = 0, x = 0$.

$0 = 0 - 0 + 0 + d$

$d = 0$

$x = \dfrac{3t^2}{2} - \dfrac{t^3}{3} + 4t$

When $t = 4$, $x = \dfrac{3 \times 4^2}{2} - \dfrac{4^3}{3} + 4 \times 4$

$= 18\dfrac{2}{3}$

$18\dfrac{2}{3}$ m to the right

c When $t = 4, a = 3 - 2 \times 4$

$= -5$ m/s^2

d $a = 3 - 2t = 0$

$t = 1.5$ s

e When $t = 1.5$,

$v = 3t - t^2 + 4$

$= 3 \times 1.5 - 1.5^2 + 4$

$= 6.25$ m/s

7 **a** $s = \dfrac{2t^3}{3} - \dfrac{3t^4}{4} + c$

When $t = 0, s = 0$.

$0 = 0 - 0 + c$

$c = 0$

$s = \dfrac{2t^3}{3} - \dfrac{3t^4}{4}$

When $t = 1$, $x = \dfrac{2 \times 1^3}{3} - \dfrac{3 \times 1^4}{4}$

$= \dfrac{2}{3} - \dfrac{3}{4} = \dfrac{1}{12}$

$\dfrac{1}{12}$ m to the left.

b When $t = 1, v = 2 - 3$

$= -1$ m/s

c $a = \dfrac{dv}{dt}$

$= 4t - 9t^2$

When $t = 1, a = 4 \times 1 - 9 \times 1^2$

$= -5$ m/s^2

8 a $v = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$

$a = \frac{dv}{dt}$

$= \frac{1}{2} \times (-2t^{-3}) = -\frac{1}{t^3}$

b $v = \frac{1}{2}t^{-2}$

$s = -\frac{1}{2}t^{-1} + c$

When $t = 1$, $s = 0$.

$0 = -\frac{1}{2} \times 1^{-1} + c$

$0 = -\frac{1}{2} + c$

$c = \frac{1}{2}$

$s = \frac{1}{2} - \frac{1}{2t}$

9 a $a = \frac{dv}{dt}$

$= 3t^2 - 22t + 24$

b Solve for $v = 0$.

$t^3 - 11t + 24t = t(t - 3)(t - 8)$

Since motion is only defined for $t \geq 0$, it cannot be said to change direction at $t = 0$.

$\therefore \quad t = 3$

$a = 3 \times 3^2 - 22 \times 3 + 24$

$= -15 \text{ m/s}^2$

c $v = t^3 - 11t^2 + 24t$

$x = \frac{t^4}{4} - \frac{11t^3}{3} + 12t^2 + c$

When $t = 0$, $x = 0$

$0 = 0 - 0 + 0 + c$

$c = 0$

$x = \frac{t^4}{4} - \frac{11t^3}{3} + 12t^2$

When $t = 5$, $x = \frac{5^4}{4} - \frac{11 \times 5^3}{3} + 12 \times 5^2$

$= -2\frac{1}{2}$

When $t = 3$, $x = \frac{3^4}{4} - \frac{11 \times 3^3}{3} + 12 \times 3^2$

$= 29\frac{1}{4}$

When $t = 0$, $x = 0$.

Total distance

$= 29\frac{1}{4} + \left(29\frac{1}{4} + 2\frac{1}{12}\right)$

$= 60\frac{7}{12}$ m

$2\frac{1}{12}$ m left of O, $60\ \frac{7}{12}$ m

10 $u = 20, v = 0, t = 4$

$s = \frac{1}{2}(u + v)t$

$= \frac{1}{2} \times 20 \times 4$

$= 40$ m

11 a $u = 0, v = 30, t = 12$

$v = u + at$

$30 = 12a$

$a = \frac{30}{12}$

$= 2.5 \text{ m/s}^2$

b $u = 30, v = 50, a = 2.5$

$v = u + at$

$50 = 30 + 2.5t$

$2.5t = 20$

$t = 8$ s

c $s = ut + \frac{1}{2}at^2$

$= 0 + \frac{1}{2} \times 2.5 \times 20^2$

$= 500$ m

d 100 km/h $= 100 \div 3.6$

$= \frac{250}{9}$ m/s

$u = 0, v = \frac{250}{9}, a = 2.5$

$v = u + at$

$\frac{250}{9} = 2.5t$

$t = \frac{250}{9 \times 2.5}$

$= 11\frac{1}{9}$ s

12 a 100 km/h $= 100 \div 3.6$

$= \frac{50}{3}$ m/s

$u = 0, v = \frac{50}{3}, a = 0.4$

$v = u + at$

$\frac{50}{3} = 0.4t$

$t = \frac{50}{3 \times 0.4}$

$= 41\frac{2}{3}$ s

b $s = \frac{1}{2}(u + v)t$

$= \frac{1}{2} \times \frac{50}{3} \times \frac{125}{3}$

$= 347\frac{2}{9}$ m

13 a $u = 35, s = 0, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$0 = 3.5t - 4.9t^2$

$0.7t(50 - 7t) = 0$

$t = \frac{50}{7} = 7\frac{1}{7}$ s

≈ 7.143 s

b $u = 35, s = 60, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$60 = 35t - 49t^2$

$4.9t^2 - 35t + 60 = 0$

$49t - 250t + 600 = 0$

$(7t - 20)(7t - 30) = 0$

$t = 2\frac{6}{7}$ or $4\frac{2}{7}$

After $2\frac{6}{7}$ s (going up) and $4\frac{2}{7}$ s (going down)

14 a Maximum height occurs when $v = 0$.

$u = 19.6, a = -9.8, v = 0$

$v = u + at$

$0 = 19.6 - 9.8t$

$t = \frac{19.6}{9.8} = 2$ s

b $s = ut + \frac{1}{2}at^2$

$= 19.6 \times 2 + \frac{1}{2} \times -9.8 \times 2^2$

$= 19.6$ m

With respect to ground level,
height = 19.6 + 20 = 39.6 m

c $u = 19.6, s = 0, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$0 = 19.6t + \frac{1}{2} \times -9.8 \times t^2$

$0 = 19.6t - 4.9t^2$

$0 = 4.9t(4 - t)$

$t = 4$ s

d $u = 19.6, s = -20, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$-20 = 19.6t + \frac{1}{2} \times -9.8 \times t^2$

$-20 = 19.6t - 4.9t^2$

$4.9t^2 - 19.t - 20 = 0$

$49t^2 - 196t - 200 = 0$

$\triangle = b^2 - 4ac$

$= 196^2 - 4 \times 49 \times -200$

$= 77\,616$

$\sqrt{\triangle} \approx 278.596$

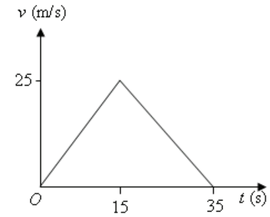
Since the discriminant is irrational, solve using the quadratic formula:

$t = \frac{196 \pm 278.596}{98}$

≈ 4.84 or -0.84

≈ 4.84 s (since $t > 0$)

15

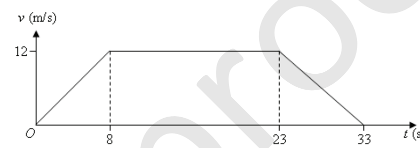


Distance = area

$= \frac{1}{2} \times 35 \times 25$

$= 437.5$ m

16



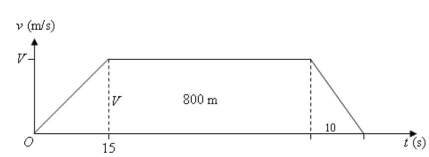
a Distance = trapezium area

$= \frac{1}{2} \times (33 + 15) \times 12$

$= 288$ m

b Halfway point is 144 m.
The car has travelled
$\frac{1}{2} \times 8 \times 12 = 48$ m in the first 8 s.
It must travel 144 − 48 = 96 m at 12 m/s.
This will take 96 ÷ 12 = 8 s.
Total of 16 s.

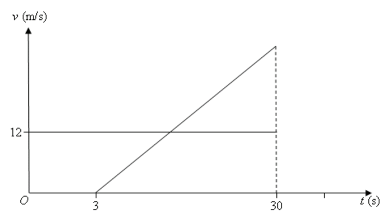
17



Since the vehicle travels 1 km = 1000 m, adding the two triangles together should give an area equal to a distance of 200 m. The triangles have a combined base of 25.

$$A = \frac{1}{2} \times 25 \times V$$
$$= 200$$
$$V = \frac{200 \times 2}{25}$$
$$= 16 \text{ m/s}$$

18 After 3 s, the first car has travelled $12 \times 3 = 36$ m.



Let the second car's final velocity be V m/s. The two areas will be equal.

$$\frac{1}{2} \times 27 \times V = 12 \times 30$$
$$= 360$$
$$V = \frac{2 \times 360}{27}$$
$$= \frac{80}{3}$$

For constant acceleration,

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$
$$= \frac{80}{3 \times 27} = \frac{80}{81} \text{ m/s}^2$$

19 **a** $v = \frac{10^2}{4} - 3 \times 10 + 5$

$$= 0 \text{ m/s}$$

b $a = \frac{dv}{dt}$

$$= \frac{2t}{4} - 3$$
$$= \frac{t}{2} - 3$$

When $t = 0$, $a = -3$ m/s^2.

c Minimum velocity occurs when $a = 0$.

$$\frac{t}{2} - 3 = 0$$
$$t = 6$$

When $t = 6$,

$$v = \frac{6^2}{4} - 3 \times 6 + 5$$
$$= -4 \text{ m/s}$$

d $v = \frac{t^2}{4} - 3t + 5$

$$x = \frac{t^3}{12} - \frac{3t^2}{2} + 5t + c$$

When $t = 0$, $x = 0$.

$$0 = 0 - 0 + 0 + c$$
$$c = 0$$
$$x = \frac{t^3}{12} - \frac{3t^2}{2} + 5t$$

Check for change of direction of velocity.

$$v = 0 \text{ if } \frac{t^2}{4} - 3t + 5 = 0$$
$$t^2 - 12t + 20 = 0$$
$$(t - 2)(t - 10) = 0$$
$$t = 2 \text{ or } 10$$

There will be no change of direction of velocity in the first 2 s.

When $t = 2$,

$$x = \frac{2^3}{12} - \frac{3 \times 2^2}{2} + 5 \times 2$$
$$= \frac{2}{3} - 6 + 10$$
$$= 4\frac{2}{3} \text{ m}$$

e When $t = 3$,

$$x = \frac{3^3}{12} - \frac{3 \times 3^3}{2} + 5 \times 3$$
$$= \frac{9}{4} - \frac{27}{2} + 15$$
$$= 3\frac{3}{4} \text{ m}$$

Distance travelled in the third second

$= 4\frac{2}{3} - 3\frac{3}{4}$

$= \frac{11}{12}$ m (to the left)

20 a $a = 2 - 2t$

$v = 2t = t^2 + c$

When $t = 3, v = 5$.

$5 = 2 \times 3 - 3^2 + c$

$5 = -3 + c$

$c = 8$

$v = 2t - t^2 + 8$

b $v = 2t - t^2 + 8$

$x = t^2 - \frac{t^3}{3} + 8t + d$

When $t = 0, x = 0$.

$0 = 0 - 0 + 0 + d$

$d = 0$

$x = t^2 - \frac{t^3}{3} + 8t$

21 a $a = 4 - 4t$

$v = 4t - 2t^2 + c$

When $t = 0, v = 6$.

$6 = 0 - 0 + c$

$c = 6$

$v = 4t - 2t^2 + 6$

$= 6 + 4t - 2t^2$

b Minimum velocity occurs when $a = 0$.

i $4 - 4t = 0$

$t = 1$

$v = 6 + 4t - 2t^2$

$= 6 + 4 \times 1 - 2 \times 1^2$

$= 8$ m/s

ii $6 + 4t - 2t^2 = 6$

$4t - 2t^2 = 0$

$2t(2 - t) = 0$

So the velocity of P is again 6 m/s after 2 s.

iii $6 + 4t - 2t^2 = 0$

$-2t^2 + 4t + 6 = 0$

$t^2 - 2t - 3 = 0$

$(t - 3)(t + 1) = 0$

$t = 3$

$x = -\frac{2t^3}{3} + 2t^2 + 6t + d$

$x = 0$ when $t = 0$

$\therefore d = 0$

$x = -\frac{2t^3}{3} + 2t^2 + 6t$

When $t = 3$,

$x = -\frac{2 \times 3^3}{3} + 2 \times 3^2 + 6 \times 3$

$= 18$ m

22 a When $t = 0, a = 27$ m/s^2.

b $a = 27 - 4t^2$

$v = 27t - \frac{4t^3}{3} + c$

When $t = 0, v = 5$.

$5 = 0 - 0 + c$

$c = 5$

$v = 27t - \frac{4t^3}{3} + 5$

When $t = 3, v = 27 \times 3 - \frac{4 \times 3^3}{3} + 5$

$= 50$ m/s

c $v = 27t - \frac{4t^3}{3} + 5 = 5$

$27t - \frac{4t^3}{3} = 0$

$81t - 4t^3 = 0$

$t(81 - 4t^2) = 0$

$t(9 - 2t)(9 + 2t) = 0$

$t = 4.5$ s

23 a $a = 3 - 3t$

$v = 3t = \frac{3t^2}{2} + c$

When $t = 0, v = 2$.

$2 = 0 - 0 + c$

$c = 2$

$v = 3t - \frac{3t^2}{2} + 2$

When $t = 4, v = 3 \times 4 - \frac{3 \times 4^2}{2} + 2$

$= -10$ m/s

b $v = 3t - \frac{3t^2}{2} + 2$

$x = \frac{3t^2}{2} - \frac{t^3}{2} + 2t + d$

When $t = 0, x = 0$.

$0 = 0 - 0 + 0 + d$

$d = 0$

$x = \frac{3t^2}{2} - \frac{t^3}{2} + 2t$

When $t = 4, x = \frac{3 \times 4^2}{2} - \frac{4^3}{2} + 24$

$= 24 - 32 + 8$

$= 0$

24 a $t^2 - 10t + 24 = 0$

$(t - 4)(t - 6) = 0$

$t = 4$ and 6

b $v = t^2 - 10t + 24$

$x = \frac{t^3}{5} - 5t^2 + 24t + c$

When $t = 0, x = 0$.

$0 = 0 - 0 + 0 + c$

$c = 0$

$x = \frac{t^3}{3} - 5t^2 + 24t$

When $t = 3, x = \frac{3^3}{3} - 5 \times 3^2$

$+ 24 \times 3$

$= 36$ m

c $a = 2t - 10 < 0$

$2t < 10$

$t < 5$

Since $t \geq 0, 0 \leq t < 5$

Solutions to multiple-choice questions

1 A When $t = 0, x = 0$

2 E When $t = 0, x = 0$.

When $t = 2, x = -2^3 + 7 \times 2^2 - 12 \times 2$

$= -4$

Average velocity

$$= \frac{\text{change in position}}{\text{change in time}}$$

$$= -\frac{4}{2}$$

$= -2$ cm/s

3 C $v = 4t - 3t^2 + c$

When $t = 0, v = -1$

$-1 = 0 - 0 + c$

$c = -1$

$v = 4t - 3t^2 - 1$

When $t = 1, v = 4 \times 1 - 3 \times 1^2 - 1$

$= 0$ m/s

4 C $u = 0, s = 90, a = 1.8$

$$s = ut + \frac{1}{2}at^2$$

$$90 = \frac{1}{2} \times 1.8 \times t^2$$

$90 = 0.9t^2$

$t^2 = 100$

$t = 10$ s

5 E 60 km/h $= 60 \div 3.6$

$$= \frac{50}{3} \text{ m/s}$$

$$u = 0, v = \frac{50}{3}, t = 4$$

$v = u + at$

$$\frac{50}{3} = 4a$$

$$a = \frac{50}{12} = \frac{25}{6} \text{ m/s}^2$$

6 C 60 km/h $= 60 \div 3.6$

$$= \frac{50}{3} \text{ m/s}$$

$$u = 0, v = \frac{50}{3}, t = 4$$

$$s = \frac{1}{2}(u + v)t$$

$$= \frac{1}{2} \times \frac{50}{3} \times 4$$

$$= \frac{100}{3} \text{ m}$$

7 D Distance

= area under graph

= triangle + trapezium + triangle

$$= \frac{1}{2} \times 4 \times 10 + \frac{1}{2} \times (10 + 25) \times 2 + \frac{1}{2} \times 9 \times 25$$

$= 20 + 25 + 112.5$

$= 167.5$ m

8 E $u = 0, a = 9.8, s = 40$

$v^2 = u^2 + 2as$

$= 0 + 2 \times 9.8 \times 40$

$= 784$

$v = \sqrt{784} = 28$ m/s

9 A $u = 20, v = 0, a = -4$

$v = u + at$

$0 = 20 - 4t$

$t = 5$

$s = \frac{1}{2}(u + v)t$

$= \frac{1}{2} \times 20 \times 5$

$= 50$ m

10 D $v = 6t^2 - 5t + c$

When $t = 0, v = 1$.

$1 = 0 - 0 + c$

$c = 1$

$v = 6t^2 - 5t + 1$

When $t = 1, v = 6 \times 1^2 - 5 \times 1 + 1$

$= 2$ m/s

Solutions to extended-response questions

1 a When $t = 0$, $x = -\frac{7}{3}$

Initial displacement is $\frac{7}{3}$ cm to the left of O.

b $v = t^2 - 4t + 4$

When $t = 0$, $v = 4$

Initial velocity is 4 cm/s.

c $a = 2t - 4$

When $t = 3$, $a = 2(3) - 4 = 2$

Acceleration after three seconds is 2 cm/s^2.

d When $v = 0$,

$$t^2 - 4t + 4 = 0$$

$$\therefore (t-2)^2 = 0$$

$$\therefore t = 2$$

Velocity is zero after two seconds.

e When $v = 0$, $t = 2$

$$\begin{aligned}\therefore x &= \frac{1}{3}(2)^3 - 2(2)^2 + 4(2) - \frac{7}{3}\\ &= \frac{8}{3} - 8 + 8 - \frac{7}{3}\\ &= \frac{1}{3}\end{aligned}$$

When the velocity is zero, the particle is $\frac{1}{3}$ cm to the right of O.

f When $x = 0$, $\frac{1}{3}t^3 - 2t^2 + 4t - \frac{7}{3} = 0$

Try $t = 1$

$$\begin{aligned}\text{LHS} &= \frac{1}{3}(1)^3 2(1)^2 + 4(1) - \frac{7}{3}\\ &= \frac{1}{3} - 2 + 4 - \frac{7}{3}\\ &= 0\end{aligned}$$

$\therefore$ LHS = RHS and $t = 1$

The displacement is zero after one second.

Also $3P(t) = t^3 - 6t^2 + 12t - 7 = (t-1)(t^2 - 5t + 7)$

and $t^2 - 5t + 7$ is irreducible since $\Delta = 25 - 4 \times 7 < 0$

2 a $x = t^4 + 2t^2 - 8t$

$$v = \frac{dx}{dt}$$

$$= 4t^3 + 4t - 8$$

When $t = 0, v = -8$

Since the initial velocity is negative, the particle moves first to the left.

b When $v = 0, 4t^3 + 4t - 8 = 0$ After one second, the particle is instantaneously at rest.

$$4(t^3 + t - 2) = 0$$

$$\therefore 4(t - 1)(t^2 + t + 2) = 0$$

$$\therefore t = 1$$

For $t > 1, t - 1 > 0$ and $t^2 + t + 2 > 0$

$$\therefore 4(t - 1)(t^2 + t + 2) > 0$$

$$\therefore v > 0$$

Hence at one second the particle has travelled the greatest distance to the left.

c As $v > 0$ when $t > 1$, the particle always moves to the right for $t > 1$.

3 a The rockets crashes when $h = 0$

i.e.$6t^2 - t^3 = 0$

$$t^2(6 - t) = 0$$

$$t = 0 \text{ or } 6$$

$t = 6$ since $t = 0$ represents take-off.

$$v = \frac{dv}{dh}$$

$$= 12t - 3t^2$$

When $t = 6, v = 12(6) - 3(6)^2$

$$= 72 - 108$$

$$= -36$$

The rocket crashes after six seconds with a velocity of -36 m/s.

b When $v = 0, 12t - 3t^2 = 0$

$\therefore 3t(4 - t) = 0$

$\therefore t = 0 \text{ or } 4$

When $t = 4$, $h = 6(4)^2 - (4)^3$

$= 96 - 64$

$= 32$

The speed of the rocket is zero at take-off and after four seconds. The maximum height of the rocket is 32 metres after four seconds.

c $a = \dfrac{dv}{dt}$

$= 12 - 6t$

When $a < 0, 12 - 6t < 0$

$\therefore 12 < 6t$

$\therefore 2 < t$

The acceleration becomes negative after two seconds.

4
- $x(1) - x(0) = 15, 1$
- $x(2) - x(1) = 5.3$ difference -9.8
- $x(3) - x(2) = -4.5$ difference -9.8
- $x(4) - x(3) = -14.3$ difference -9.8
- $x(5) - x(4) = -24.1$ difference -9.8
- $x(6) - x(5) = -33.9$ difference -9.8
- $x(7) - x(6) = -43.7$ difference -9.8
- $x(8) - x(7) = -53.5$ difference -9.8
- $x(9) - x(8) = -63.3$ difference -9.8
- $x(10) - x(9) = -73.1$ difference -9.8

The body has a constant acceleration of 9.8 m/s^2 which is the acceleration due to gravity.

5 a Let $a = -g$ (m/s^2), $v = 0$ (m/s)

Using $v = u + at$,

$$t = \frac{v - u}{a}$$
$$= \frac{0 - u}{-g}$$
$$= \frac{u}{g},$$

as required.

b When $t = \frac{u}{g}, v = 0$

$$s = \frac{1}{2}(u + v)t$$
$$= \frac{1}{2}(u + 0)\frac{u}{g}$$
$$= \frac{u^2}{2g}$$

The particle will have travelled $\frac{2u^2}{2g} = \frac{u^2}{g}$ metres to return to its point of projection.

Consider the path of the particle from its highest point when its velocity is zero, until it returns to the point of projection $\frac{u^2}{2g}$ downwards.

Then $u = 0, s = \frac{u^2}{2g}, a = g$

and $s = ut + \frac{1}{2}at^2$

$$\therefore = 0 \times t + gt^2$$
$$\therefore \frac{u^2}{2g} = gt^2$$
$$\therefore t^2 = \frac{u^2}{g^2}$$
$$\therefore t = \frac{u}{g}(t = -\frac{u}{g} \quad \text{is discounted as } t > 0)$$

Hence the total time taken is $\frac{u}{g} + \frac{u}{g} = \frac{2u}{g}$ seconds, as required.

c For the return downwards, $u = 0, t = \frac{u}{g}, a = g$

$$v = u + at$$
$$= 0 + g \times \frac{u}{g}$$
$$= u$$

Hence the speed of returning to the point of projection is u m/s.

6 Consider the throw of the stone to its maximum height. $u = 14, a = 9.8, v = 0$

$$t = \frac{v - u}{a}$$
$$= \frac{0 - 14}{-9.8}$$
$$= \frac{10}{7}$$

It therefore takes $2 \times \frac{10}{7} = \frac{20}{7}$ seconds for the stone to reach the top of the mine shaft on its descent.

From this point,

$u = -14, a = -9.8, s = ut + \frac{1}{2}at^2$

$\therefore s = 14t - 4.9t^2 \ldots (1)$

When the stone reaches the top of the mine shaft, the lift has been descending for $\frac{20}{7} + 5 = \frac{55}{7}$ seconds and has travelled $\frac{55}{7} \times 3.5 = 27.5$ metres.

From this point,

$s = -27.5 - 3.5t \qquad (\text{for the lift}) \ldots (2)$

Equating (1) and (2) to find the point of impact.

$$-14t - 4.9t^2 = 27.5 - 3.5t$$
$$\therefore 4.9t^2 + 10.5t - 27.5 = 0$$
$$\therefore t = \frac{-10.5 \pm \sqrt{10.5^2 - 4 \times 4.9 \times (-27.5)}}{2 \times 4.9}$$
$$= 1.42857\ldots$$

(the negative solution is not practical)

$$\text{When } t = 1.42857\ldots,$$
$$s = -27.5 - 3.5 \times 1.42857$$
$$= -32.85013$$

Hence the depth of the lift when the stone hits it is 33 metres, to the nearest metre.

7 a $90 \text{ km/h} = 90 \times \frac{5}{18} \text{ m/s}$

$$= 25 \text{ m/s}$$
$$v = -\frac{25}{5}t + 25$$
$$\therefore v = -5t + 25, \qquad 0 \le t \le 5$$

b Distance travelled $= \text{area under the graph}$

$$= \frac{1}{2} \times 25 \times 5$$

$$= 62.5$$

The distance travelled in five seconds is 62.5 metres.

8 $$x = 3t^4 - 4t^3 + 24t^2 - 48t$$

$$v = \frac{dx}{dt}$$

$$= 12t^3 - 12t^2 + 48t - 48$$

When $t = 0, v = -48$

Since $v < 0$, the particle moves at first to the left.

When $v = 0, 12t^3 - 12t^2 + 48t - 48 = 0$

$$\therefore 12(t^3 - t^2 + 4t - 4) = 0$$

$$\therefore 12(t - 1)(t^2 + 4) = 0$$

$$\therefore t = 1$$

When $t = 1, x = 3(1)^4 - 4(1)^3 + 24(1)^2 - 48(1)$

$$= 3 - 4 + 24 - 48$$

$$= -25$$

The particle comes to rest at $(1, -25)$

When $t > 1, t - 1 > 0$ and $t^2 + 4 > 0$

$\therefore 12(t - 1)(t^2 + 4) > 0$

$\therefore v > 0$

Since $v > 0$, the particle always moves to the right for $t > 1$.

9 For the first particle, $s = ut - \frac{1}{2}gt^2$ where $a = -g$

For the second particle, $s = u(t - T) - \frac{1}{2}g(t - T)^2$

The particles collide when

a **i**

$$ut - gt^2 = u(t - T) - \frac{1}{2}g(t - T)^2$$

$$= ut - uT - \frac{1}{2}gt^2 + gtT - \frac{1}{2}gT^2$$

$$\therefore 0 = -uT + gtT - \frac{1}{2}gT^2$$

$$= T(-u + gt - \frac{1}{2}gT)$$

$$\therefore -u + gt - \frac{1}{2}gT = 0 \qquad (T \neq 0)$$

$$\therefore gt = u + \frac{1}{2}gT$$

$$\therefore t = \frac{u}{g} + \frac{T}{2} \text{as required.}$$

ii When $t = \dfrac{u}{g} + \dfrac{T}{2}$

$$s = u(\frac{u}{g} + \frac{T}{2}) - g\left(\frac{u}{g} + \frac{T}{2}\right)^2$$

$$= \frac{u^2}{g} + \frac{uT}{2} - \frac{1}{2}g\left(\frac{u^2}{g^2} + \frac{uT}{g} + \frac{T^2}{4}\right)$$

$$= \frac{u^2}{g} + \frac{uT}{2} - \frac{u^2}{2g} - \frac{uT}{2} - \frac{gT^2}{8}$$

$$= \frac{u^2}{2g} - \frac{gT^2}{8}$$

$$= \frac{4u^2 - g^2T^2}{8g}, \text{ as required.}$$

$$\text{When } T = \frac{2u}{g}, s = \frac{4u^2 - g^2\left(\frac{2u}{g}\right)^2}{8g}$$

$$= \frac{4u^2 - 4u^2}{8g}$$

$$= 0$$

b This is the case when the second particle is projected upward at the instant the first particle lands. Hence there is no collision.

c If $T > \dfrac{2u}{g}$, the second particle is projected upward after the first particle has landed, hence no collision.

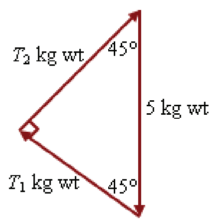
Chapter 23 – Statics of a particle

Solutions to Exercise 23A

1 $T_1 = 3$ kg wt

$T_2 = T_1 + 4 = 7$ kg wt

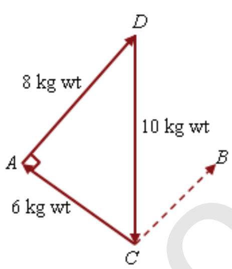
2 Rearrange into a triangle of forces.



Using trigonometry,

$$T_1 = T_2$$
$$= 5\sin 45^\circ$$
$$= \frac{5\sqrt{2}}{2} \text{ kg wt}$$

3 Rearrange into a triangle of forces.

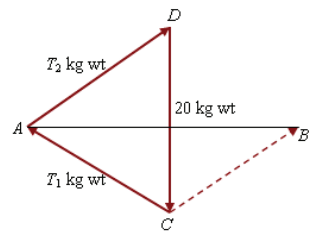


$\angle ACB = \angle ACD + \angle ADC$

These angles can be calculated using the cosine rule, but the student should notice that ΔACD is a 'doubled' 3-4-5 triangle with $\angle CAD = 90^\circ$.

$$\therefore ACB = \angle ACD + \angle ADC$$
$$= 180 - 90 = 90^\circ$$

4 Rearrange into a triangle of forces.



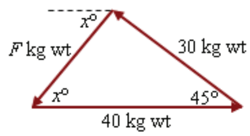
Using the cosine rule in the triangle in the original diagram, it is clear that:

$$\cos\angle CAB = \frac{15^2 + 10^2 - 12^2}{2\times 15\times 10}$$
$$= 0.6033$$
$$\angle CAB = 52.89^\circ$$
$$\angle ADC = 90 - \angle CAB$$
$$= 37.11^\circ$$
$$\cos\angle CBA = \frac{15^2 + 12^2 - 10^2}{2\times 15\times 12}$$
$$= 0.7472$$
$$\angle CBA = 41.65^\circ$$
$$\angle ACD = 90 - \angle CBA$$
$$= 48.35^\circ$$
$$\angle CAD = 180 - 37.11 - 48.35$$
$$= 94.54^\circ$$

Use the sine rule to find T_1 and T_2.

$$\frac{T_1}{\sin\angle ACD} = \frac{20}{\sin\angle CAD}$$
$$T_1 = \frac{20\times\sin 48.35^\circ}{\sin 94.54^\circ}$$
$$\approx 14.99 \text{ kg wt}$$
$$\frac{T_2}{\sin\angle ADC} = \frac{20}{\sin\angle CAD}$$
$$T_2 = \frac{20\times\sin 37.11^\circ}{\sin 94.54^\circ}$$
$$\approx 12.10 \text{ kg wt}$$

5 Rearrange into a triangle of forces.



Using the cosine rule,

$$F^2 = 40^2 + 30^2 - 2 \times 30 \times 40 \times \cos 45^\circ$$
$$= 802.94$$
$$F \approx 28.34 \text{ kg wt}$$

Using the cosine rule,

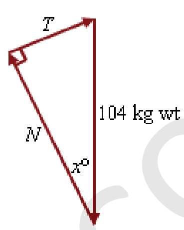
$$\cos x = \frac{F^2 + 40^2 - 30^2}{2 \times F \times 40}$$
$$= 0.663$$
$$x \approx 48.5^\circ$$

W 48.5° S or S 41.5° W

6 The angle between the plane and the horizontal is given by

$$\tan x = \frac{5}{12}$$
$$= 0.4167$$
$$x \approx 22.619^\circ$$

Rearrange into a triangle of forces.



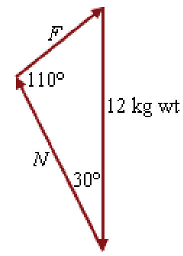
$$T = 104 \sin x$$
$$= 40 \text{ kg wt}$$

Note: The hypotenuse is 13, so $\sin x = \frac{5}{13}$ and $\cos x = \frac{12}{13}$.

$$N = 104 \cos x$$
$$= 96 \text{ kg wt}$$

7 Note that F will be acting at 50° to the horizontal and 70° to N, which becomes 110° when the force vectors joined head to tail.

Rearrange into a triangle of forces.

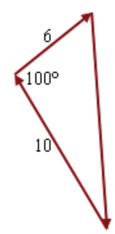


Use the sine rule.

$$\frac{F}{\sin 30^\circ} = \frac{12}{\sin 110^\circ}$$
$$F = \frac{12 \times \sin 30^\circ}{\sin 110^\circ} \approx 6.39 \text{ kg wt}$$

8 In each case, the particle will be in equilibrium if the forces add to zero. Draw the first two forces, and calculate the third force required for equilibrium.

a

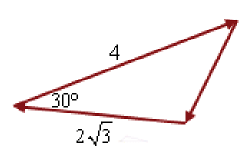


Use the cosine rule to calculate the magnitude of the third force.

$$F^2 = 10^2 + 6^2 - 2 \times 10 \times 6 \cos 100^\circ$$
$$= 156.837$$
$$F \approx 12.52 \text{ kg wt}$$

This is not the force in the diagram, so these forces will not be in equilibrium.

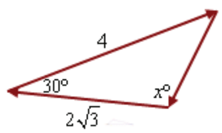
b



Use the cosine rule to calculate the magnitude of the third force.

$$F^2 = 4^2 + (2\sqrt{3})^2 - 2 \times 4 \times 2\sqrt{3} \times \cos 30^\circ$$
$$= 4$$
$$F = 2 \text{ kg wt}$$

It has the same magnitude as the third force in the diagram.



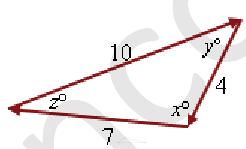
Use the sine rule to find x.

$$\frac{\sin x}{4} = \frac{\sin 30^\circ}{2}$$
$$\sin x = \frac{0.5 \times 4}{2} = 1$$
$$x = 90^\circ$$

This vector is at the same angle with the $2\sqrt{3}$ vector as in the original diagram.

∴ the vectors will be in equilibrium.

9 Draw the triangle of forces and use the cosine rule to find the three angles. When the vectors are placed tail to tail, the angles between them will be the supplements of the angles in the triangle.



$$\cos x = \frac{7^2 + 4^2 - 10^2}{2 \times 7 \times 4}$$
$$= 0.625$$
$$x \approx 128^\circ 41'$$

Angle between vectors is $180^\circ - 128^\circ 41' = 51^\circ 19'$

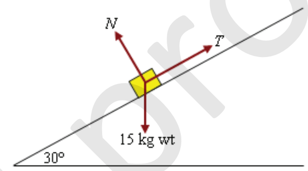
$$\cos y = \frac{10^2 + 4^2 - 7^2}{2 \times 10 \times 4}$$
$$= 0.8375$$
$$y \approx 33^\circ 7'$$

Angle between vectors is $180^\circ - 33^\circ 7' = 146^\circ 53'$

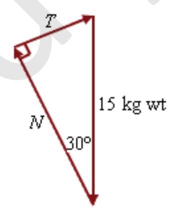
$$z \approx 180^\circ - 128^\circ 41' - 33^\circ 7'$$
$$= 18^\circ 12'$$

Angle between vectors is $180^\circ - 18^\circ 12' = 161^\circ 48'$

10 a



Draw the triangle of forces.

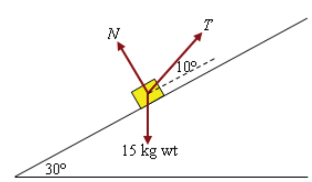


$$T = 15 \sin 30^\circ$$
$$= 7.5 \text{ kg wt}$$

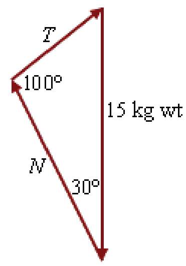
b The situation will be the same, except that the 30° angle will now be 40°.

$$T = 15 \sin 40^\circ$$
$$\approx 9.64 \text{ kg wt}$$

c The angle between T and N is now 80°.



Draw the triangle of forces.



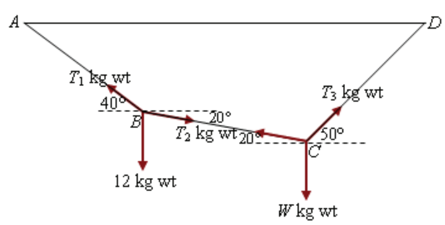
Use the sine rule.

$$\frac{T}{\sin 30^\circ} = \frac{15}{\sin 100^\circ}$$

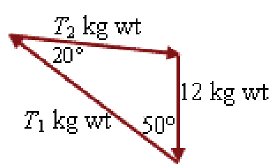
$$T = \frac{15 \times 0.5}{\sin 100^\circ}$$

$$\approx 7.62 \text{ kg wt}$$

11



Draw the triangle of forces for point *B*.



Use the sine rule to find T_1 and T_2.

$$\frac{T_1}{\sin 110^\circ} = \frac{12}{\sin 20^\circ}$$

$$T_1 = \frac{12 \times \sin 110^\circ}{\sin 20^\circ}$$

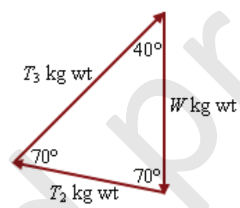
$$\approx 32.97 \text{ kg wt}$$

$$\frac{T_2}{\sin 50^\circ} = \frac{12}{\sin 20^\circ}$$

$$T_2 = \frac{12 \times \sin 50^\circ}{\sin 20^\circ}$$

$$\approx 26.88 \text{ kg wt}$$

Now draw the triangle of forces for point *C*.



Use the sine rule to find T_3.

$$\frac{T_3}{\sin 70^\circ} = \frac{T_2}{\sin 40^\circ}$$

$$T_3 = \frac{26.88 \times \sin 70^\circ}{\sin 40^\circ}$$

$$\approx 39.29 \text{ kg wt}$$

Since the triangle is isosceles,

$W = T_3 \approx 39.29$ kg wt

The mass of *W* is 39.29 kg.

Solutions to Exercise 23B

1 $F\cos 40^\circ = 10$ kg wt

$$F = \frac{10}{\cos 40^\circ}$$

$$\approx 13.05 \text{ kg wt}$$

2 Resolve in the direction of F.

$F - 10\cos 55^\circ = 0$

$$F = 5.74 \text{ kg wt}$$

3 First resolve vertically to find N.

$N\cos 25^\circ - 8 = 0$

$$N = \frac{8}{\cos 25^\circ}$$

$$\approx 8.83 \text{ kg wt}$$

Keep the exact value of N in your calculator.

Resolve horizontally.

$F - N\sin 25^\circ = 0$

$$F = N\sin 25^\circ$$

$$\approx 3.73 \text{ kg wt}$$

$F - N\sin 25^\circ = 0$

$F = N\sin 25^\circ \approx 3.73$ kg wt

4 Resolve parallel to the plane, i.e. perpendicular to N.

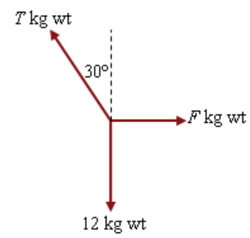
F is at an angle of 34° to the plane.

$F\cos 34^\circ - 10\sin 20^\circ = 0$

$$F = \frac{10\sin 20^\circ}{\cos 34^\circ}$$

$$\approx 4.13 \text{ kg wt}$$

5



Resolve vertically:

$T\cos 30^\circ - 12 = 0$

$$T = \frac{12}{\cos 30^\circ}$$

Resolve horizontally:

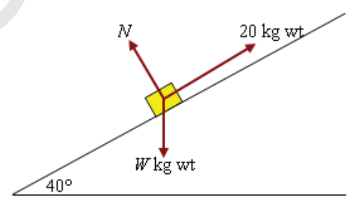
$F - T\sin 30^\circ = 0$

$$F = T\sin 30^\circ$$

$$= \frac{12\sin 30^\circ}{\cos 30^\circ}$$

$$\approx 6.93 \text{ kg wt}$$

6



Resolve parallel to the plane.

$20 - W\sin 40^\circ = 0$

$$W = \frac{20}{\sin 40^\circ}$$

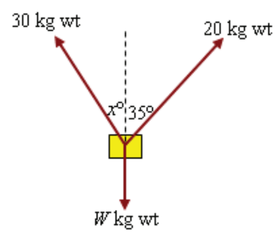
$$\approx 31.11 \text{ kg wt}$$

The force W exerts on the plane is the part of its weight resolved perpendicular to the plane.

$F = W\cos 40^\circ$

$$= \frac{20\cos 40^\circ}{\sin 40^\circ}$$

$$= 23.84 \text{ kg wt}$$

7 

First resolve horizontally so only one unknown is involved.

$$30\sin x - 20\sin 35^\circ = 0$$

$$\sin x = \frac{20\sin 35^\circ}{30} = 0.382$$

$$x \approx 22^\circ 29'$$

Keep the exact value in your calculator and resolve vertically.

$$0 = W - 20\cos 35^\circ - 30\cos 22.481^\circ$$

$$W = 20\cos 35^\circ + 30\cos 22.481^\circ \approx 44.10 \text{ kg wt}$$

8 Pressure of body on plane

$$= 10\cos 50^\circ \approx 6.43 \text{ kg wt}$$

Resolve parallel to the plane.

$$T - 10\sin 50^\circ = 0$$

$$T = 10\sin 50^\circ \approx 7.66 \text{ kg wt}$$

Resolve parallel to the second plane.

$$T - W\sin 40^\circ = 0$$

$$W = \frac{T}{\sin 40^\circ} = \frac{10\sin 50^\circ}{\sin 40^\circ} \approx 11.92 \text{ kg wt}$$

9 First find the angle between the string and the vertical.

$$\sin x = \frac{9}{15+9} = 0.375$$

$$x = 22.024^\circ$$

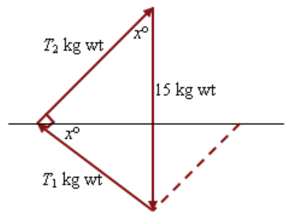
Resolve vertically.

$$T\cos x - 3 = 0$$

$$T = \frac{3}{\cos 22.024^\circ} \approx 3.24 \text{ kg wt}$$

Solutions to technology-free questions

1 Note that the two strings form a 3-4-5 triangle. Draw the triangle of forces.

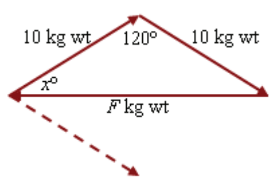


Note:

$\sin x = \frac{6}{10} = \frac{3}{5}; \cos x = \frac{8}{10} = \frac{4}{5}$

$$T_1 = 15 \sin x$$
$$= 15 \times \frac{3}{5} = 9 \text{ kg wt}$$
$$T_2 = 15 \cos x$$
$$= 15 \times \frac{4}{5} = 12 \text{ kg wt}$$

2 Draw the triangle of forces.



Use the cosine rule.

$$F^2 = 10^2 + 10^2 - 2 \times 10 \times 10 \times \cos 120°$$
$$= 100 + 100 - 200 \times -\frac{1}{2}$$
$$= 300$$
$$F = \sqrt{300}$$
$$= 10\sqrt{3} \text{ kg wt}$$

Since the triangle is isosceles,

$$x = \frac{180 - 120}{2}$$
$$= 30°$$

$10\sqrt{3}$ kg wt, at 150° to each 10 kg wt force.

3 The force exerted on the body by the plane will be perpendicular to the plane. Resolve parallel to the plane, so the component this force will be zero.

The hypotenuse of the marked triangle is

$$h = \sqrt{12^2 + 6^2}$$
$$= \sqrt{180} = 6\sqrt{5} \text{ cm}$$

If x is the angle of the plane to the horizontal,

$$\sin x = \frac{6}{6\sqrt{5}} = \frac{1}{\sqrt{5}}$$
$$\cos x = \frac{12}{6\sqrt{5}} = \frac{2}{\sqrt{5}}$$

Resolving,

$$T - 70 \sin x = 0$$
$$T = 70 \sin x$$
$$= 70 \times \frac{1}{\sqrt{5}}$$
$$= \frac{70\sqrt{5}}{5} = 14\sqrt{5} \text{ kg wt}$$

Resolving perpendicular to the plane,

$$N - 70 \cos x = 0$$
$$N = 70 \cos x$$
$$= 70 \times \frac{2}{\sqrt{5}}$$
$$= \frac{140\sqrt{5}}{5} = 28\sqrt{5} \text{ kg wt}$$

4 The force exerted on the body by the plane will be perpendicular to the plane. Resolve parallel to the plane, so the component this force will be zero.

$$F\cos 30^\circ - 15\sin 30^\circ = 0$$

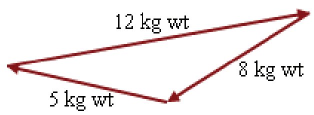
$$\frac{F\sqrt{3}}{2} = 15 \times \frac{1}{2}$$

$$F = \frac{15}{\sqrt{3}}$$

$$= \frac{15\sqrt{3}}{3}$$

$$= 5\sqrt{3} \text{ kg wt}$$

5 Draw the triangle of forces and use the cosine rule.



$$\cos x = \frac{12^2 + 5^2 - 8^2}{2 \times 15 \times 5}$$

$$= \frac{105}{120} = \frac{7}{8}$$

Since the required angle is $180^\circ - x$, the cosine is $-\frac{7}{8}$.

6 $F\cos 30^\circ = 20$

$$\frac{F\sqrt{3}}{2} = 20$$

$$F = \frac{20 \times 2}{\sqrt{3}}$$

$$= \frac{40\sqrt{3}}{3} \text{ kg wt}$$

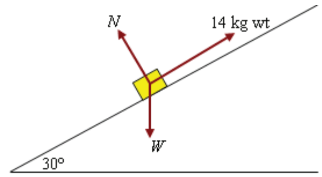
7 Resolve parallel to the plane.

$F - 15\sin 45^\circ = 0$

$$F = 15\sin 45^\circ$$

$$= \frac{15\sqrt{2}}{2} \text{ kg wt}$$

8



Resolve parallel to the plane.

$W\sin 30^\circ - 14 = 0$

$$W = \frac{14}{\sin 30^\circ}$$

$$= \frac{14}{0.5} = 28 \text{ kg wt}$$

Resolve perpendicular to the plane.

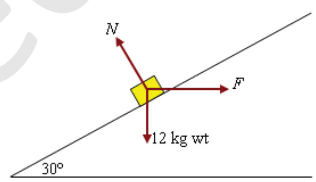
$N - 28\cos 30^\circ = 0$

$$N = 28\cos 30^\circ$$

$$= \frac{28\sqrt{3}}{2}$$

$$= 14\sqrt{3} \text{ kg wt}$$

9



Calculate F by resolving parallel to the plane.

$F\cos 30^\circ - 12\sin 30^\circ = 0$

$$\frac{F\sqrt{3}}{2} = 12 \times \frac{1}{2}$$

$$F = 6 \times \frac{2}{\sqrt{3}}$$

$$= \frac{12\sqrt{3}}{3}$$

$$= 4\sqrt{3} \text{ kg wt}$$

Solutions to multiple-choice questions

1 E $50\cos 60^\circ = 50 \times \frac{1}{2}$

$= 25\text{ N}$

2 C Use Pythagoras' theorem.

Resultant $= \sqrt{5^2 + 4^2}$

$= \sqrt{41}$ kg wt

3 E Resolve perpendicular to the plane.

$N - 20\cos 30^\circ = 0$

$N = 20\cos 30^\circ$

$= 20 \times \frac{\sqrt{3}}{2}$

$= 10\sqrt{3}$ kg wt

4 A Resolve parallel to the plane.

$F - 20\sin 30^\circ = 0$

$F = 20\sin 30^\circ$

$= 20 \times \frac{1}{2}$

$= 10$ kg wt

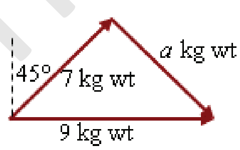
5 C For the particle to be in equilibrium, B must equal the sum of the forces on A and C.

$\therefore B = A\cos 60^\circ + C\cos 30^\circ$

(since $180 - 150 = 30$).

As this is true, **C** cannot be true.

6 B



Since the forces are perpendicular,

$a^2 + 7^2 = 9^2$

$a^2 = 81 - 49$

$= 32$

$a = \sqrt{32} = 4\sqrt{2}$

7 B The angle between the forces when they are head to tail will be 120°.

Use the cosine rule.

$F^2 = 20^2 + 20^2 - 2 \times 20 \times 20 \times \cos 120^\circ$

$= 400 + 400 - 800 \times -\frac{1}{2}$

$= 1200$

$F = \sqrt{1200}$

$= 20\sqrt{3}$ kg wt

8 A The angle between the forces when they are head to tail will be 120°.

Use the cosine rule.

$F^2 = 300^2 + 200^2 - 2 \times 300 \times 200 \times \cos 120^\circ$

$= 90\,000 + 40\,000 - 120\,000 \times -\frac{1}{2}$

$= 190\,000$

$F = \sqrt{190\,000}$

$= 100\sqrt{19}$ kg wt

9 C $R = \sqrt{16^2 + 30^2}$

$= \sqrt{1156}$

$= 34$ kg wt

10 B The forces will be at right angles to each other.

$a^2 + 8^2 = 12^2$

$a^2 = 144 - 64$

$= 80$

$a = \sqrt{80} = 4\sqrt{5}$